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Laboratoire Jacques-Louis Lions (LJLL)

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# Derivation and analysis of reduced models for Type-I superconductors

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Par ALESSANDRO COSENZA

Thèse de doctorat de MATHÉMATIQUES

Dirigée par MICHAEL GOLDMAN

Et co-encadrée par ALESSANDRO ZILIO

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# Abstract (French)

**Titre:** Dérivation et analyse de modèles réduits pour les supraconducteurs de type-I.

**Mots clefs:** Gamma Convergence; Transport optimal branché ; Ginzburg-Landau ; Supraconducteurs de type-I ; Transitions de phase

**Résumé:** L'objectif de cette thèse est l'étude de modèles réduits pour les supraconducteurs de type-I. Elle se concentre à la fois sur la dérivation rigoureuse de ces modèles et sur l'étude des propriétés qualitatives de leurs minimiseurs.

Les supraconducteurs sont des matériaux qui, en dessous d'une température critique, présentent une perte soudaine de résistivité et expulsent les champs magnétiques. Nous nous concentrons sur le modèle de Ginzburg-Landau pour les supraconducteurs de type-I. Ces matériaux, lorsqu'ils sont soumis à un champ magnétique externe, montrent l'apparition de motifs complexes, ce qui suggère la présence de motifs de ramification à l'intérieur de l'échantillon. D'un point de vue mathématique, la présence de ramifications a d'abord été étudiée sous la forme de lois d'échelle, et deux régimes ont été distingués. Un régime de ramification uniforme, où le champ magnétique se répartit uniformément à la frontière de l'échantillon, et un régime de ramification non uniforme, où le champ magnétique s'écarte de l'uniformité à la frontière. Dans le premier régime et pour un paramètre de Ginzburg-Landau  $\kappa$  tendant vers zéro, Conti, Goldman, Otto et Serfaty ont dérivé un modèle réduit pour la magnetization (dans le sens de la  $\Gamma$ -convergence). Le modèle résultant est un problème de transport optimal branché non standard, dans lequel le champ magnétique se concentre sur une structure arborescente unidimensionnelle, qui s'affine vers la frontière pour satisfaire aux conditions de bord uniformes. Ce manuscrit est divisé en cinq

chapitres, le premier servant d'introduction. Les quatre chapitres suivantes contiennent les contributions originales de cette thèse.

Dans le deuxième chapitre, nous étendons les résultats de Conti, Goldman, Otto et Serfaty au régime intermédiaire (crossover) entre branchement uniforme et non-uniforme, en prouvant la  $\Gamma$ -convergence de la fonctionnelle de Ginzburg-Landau vers un problème de transport branché avec des conditions aux limites faiblement imposées.

Dans le troisième chapitre, nous considérons une fonction non standard de type Modica-Mortola en 2D, qui découle de la fonction de Ginzburg-Landau dans le régime où la longueur de pénétration et la longueur de cohérence sont comparables. Nous prouvons que la fonctionnelle  $\Gamma$  converge vers la fonctionnelle de périmètre.

Dans le quatrième chapitre, nous étudions le problème de transport branché dérivé dans le chapitre 1, pour lequel Conti, Otto et Serfaty ont conjecturé que la dimension de la mesure limite n'est pas entière. Nous prouvons cette conjecture dans un cadre simplifié en 2D, sous l'hypothèse (forte) de Ahlfors régularité de la mesure irriguée.

Enfin, dans le dernier chapitre, nous étudions la fonctionnelle obtenue au chapitre 1 dans le cas où l'échantillon est un demi-plan. Nous étendons une partie des résultats du chapitre 3 à cette fonctionnelle 3D. En particulier, nous prouvons que la mesure limite ne peut pas être une fonction bornée absolument continue.



# Abstract (English)

**Title:** Derivation and analysis of reduced models for type-I superconductors.

**Keywords:** Gamma convergence; Branched optimal transport; Ginzburg-Landau; Type-I superconductors; Phase transitions

**Abstract:** The goal of this thesis is the study of reduced models for type-I superconductors. It focuses both on the rigorous derivation of these models and on the study of qualitative properties of their minimizers.

Superconductors are materials that, below a critical temperature, exhibit a sudden loss of resistivity and expel magnetic fields. We focus on the Ginzburg-Landau model for type-I superconductors. Such materials, when subject to an external magnetic field, show the appearance of complex patterns, a suggestion of the presence of branching patterns inside the sample. Mathematically, the presence of branching was first studied in the form of scaling laws, and two regimes were distinguished: a regime of uniform branching, where the magnetic field distributes uniformly at the boundary on the sample, and a regime of non uniform branching, where the magnetic field deviates from uniformity at the boundary. Deep inside the first regime and for vanishing Ginzburg-Landau parameter  $\kappa$ , Conti, Goldman, Otto and Serfaty derived a reduced model for the magnetization (in the sense of  $\Gamma$ -convergence). The resulting model is a non standard branched optimal transport problem, in which the magnetic field concentrates on a one dimensional tree structure, which refines towards the boundary to match uniform boundary conditions.

This manuscript is divided into five chapters, the first one serving as an introduction.

The other four chapters contain the original contributions of this thesis.

In the second chapter we extend the results of Conti, Goldman, Otto and Serfaty to the crossover regime between uniform and non-uniform branching, proving  $\Gamma$ -convergence of the Ginzburg-Landau functional to a branched transport problem with weakly imposed boundary conditions.

In the third chapter we consider a 2D non-standard Modica-Mortola type functional, which arises from the Ginzburg-Landau functional in the regime of comparable penetration length and coherence length. We prove that the functional  $\Gamma$ -converges to the perimeter functional.

In the fourth chapter we study the branched transport problem which was derived in Chapter 1, for which Conti, Otto and Serfaty conjectured that the dimension of the boundary measure is non-integer. We prove this conjecture in a simplified 2D setting, under the (strong) assumption of Ahlfors regularity of the irrigated measure.

Finally in the last chapter we study the functional derived in Chapter 1 in the case where the sample is a half-plane. We extend part of the results of Chapter 3 to this 3D functional. In particular we prove that the boundary measure cannot be a bounded absolutely continuous function.



# Acknowledgments



# Structure of the manuscript

This manuscript contains the results obtained during my PhD thesis conducted at the Laboratoire Jacques-Louis Lions (LJLL) in Université Paris Cité, under the supervision of Michael Goldman and Alessandro Zilio. This manuscript contains five chapters. The first chapter serves as an introduction and contains the mathematical context of the thesis and an overview of the results obtained in the thesis. The other four chapters are based on works which are either submitted or to be submitted. Each of the other four chapters is self contained and carries its own internal notation. The following works are part of the thesis:

- Chapter 2: " $\Gamma$ -convergence of the Ginzburg-Landau functional to a branched transport problem with weakly imposed boundary conditions".
- Chapter 3: "A  $\Gamma$ -convergence result for 2D type-I superconductors", in collaboration with Michael Goldman and Alessandro Zilio, available at <https://arxiv.org/abs/2504.19587> [42] (Accepted in "Interfaces and Free Boundaries").
- Chapter 4: "New dimensional bounds for a branched transport problem", in collaboration with Michael Goldman and Melanie Koser, available at <https://arxiv.org/abs/2411.14547>[41] (Accepted in "Siam Journal of Mathematical Analysis").
- Chapter 5: "Concentration phenomena in a branched transport problem in the half space", in collaboration with Michael Goldman and Felix Otto.

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# Chapter 1

## General Introduction

### 1.1 Aperçu de la thèse en français

L'objectif de cette thèse est l'étude de modèles réduits pour les supraconducteurs de type-I. Elle se concentre à la fois sur la dérivation rigoureuse de ces modèles et sur l'étude des propriétés qualitatives de leurs minimiseurs.

Les supraconducteurs sont des matériaux qui, en dessous d'une température critique, présentent une perte soudaine de résistivité et expulsent les champs magnétiques. Le modèle Ginzburg-Landau est un modèle phénoménologique pour les supraconducteurs qui a été proposé par V. Ginzburg et L. Landau en 1950. Nous nous concentrons sur une classe spécifique de supraconducteurs, connus sous le nom de supraconducteurs de type-I.

Il a été observé expérimentalement que les supraconducteurs de type-I, lorsqu'ils sont soumis à un champ magnétique externe, montrent l'apparition de motifs complexes à la surface de l'échantillon. Ce comportement suggère que des motifs de branchement sont présents à l'intérieur de l'échantillon. Une première justification mathématique de la présence de ramifications est donnée dans [26, 28, 39] sous la forme de lois d'échelle. En particulier, [26, 28, 39] distingue deux régimes de paramètres, correspondant à une ramification uniforme et non uniforme. Dans le premier régime, le champ magnétique se répartit uniformément à la frontière de l'échantillon, alors que, étonnamment, dans le second régime, il est favorable du point de vue de l'énergie pour le champ magnétique de s'écarter de l'uniformité à la frontière. Allant au-delà des lois d'échelle, dans [37] un modèle réduit est dérivé (dans le sens de  $\Gamma$ -convergence [12, 43]) à l'intérieur du premier régime et pour un paramètre de Ginzburg-Landau qui tend vers zero. Le modèle résultant est un problème de transport optimal branché non standard, dans lequel le champ magnétique se concentre sur une structure arborescente unidimensionnelle, qui s'affine vers la frontière pour correspondre à des conditions de frontière uniformes.

Outre ce chapitre, le manuscrit contient quatre autres chapitres :

- Dans le deuxième chapitre, nous étendons les résultats de [37] au régime de croisement entre les ramifications uniformes et non uniformes décrit dans [39]. Nous prouvons la  $\Gamma$ -convergence de la fonctionnelle de Ginzburg-Landau pour un problème de transport branché similaire à celui étudié dans [37], mais pour lequel des conditions aux limites uniformes ne sont pas imposées, mais plutôt pénalisées par une norme faible. La fonctionnelle limite que nous dérivons est celle étudiée dans [48], pour laquelle a été conjecturé que le champ magnétique présente un comportement fractal à la frontière du supraconducteur [38]. Ce chapitre est basé sur un travail en cours.
- Dans le troisième chapitre, nous considérons une fonctionnelle non standard de type Modica-Mortola en 2D. Cette fonctionnelle découle de la fonctionnelle de Ginzburg-

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Landau dans le cas d'un échantillon infiniment long et dans le régime où la longueur de pénétration et la longueur de cohérence sont comparables. Nous prouvons que la fonctionnelle  $\Gamma$ -converge vers la fonctionnelle périmètre. Ce résultat est une première étape dans la compréhension de l'extension des résultats de [37] au régime du paramètre de Ginzburg-Landau  $\kappa$  qui ne tend pas vers zero. Ce chapitre est basé sur un travail avec Michael Goldman et Alessandro Zilio [42].

- Dans le quatrième chapitre, nous étudions le problème de transport branché qui a été dérivé dans le chapitre 1, pour lequel il est conjecturé dans [38] que la dimension de la mesure limite est non entière, précisément  $8/5$ . Nous prouvons cette conjecture dans un cadre 2D simplifié, sous l'hypothèse (forte) de régularité Ahlfors de la mesure irriguée. Ce résultat est la première preuve rigoureuse d'un comportement singulier pour les mesures irriguées résultant de la minimalité. Ce chapitre est basé sur un travail avec Michael Goldman et Melanie Koser [41].
- Dans le dernier chapitre, nous étudions la fonctionnelle dérivée du Chapitre 1 dans le cas où l'échantillon est un demi-plan. Nous étendons une partie des résultats du Chapitre 3 à cette fonction 3D. En particulier, nous prouvons que la mesure limite ne peut pas être une fonction absolument continue et bornée. Ce chapitre est basé sur un travail en cours avec Felix Otto.

Dans ce chapitre, nous rappelons d'abord quelques résultats sur la  $\Gamma$ -convergence, le transport optimal et le transport optimal branché, qui seront utilisés tout au long du manuscrit. Ensuite, nous donnons une introduction à la théorie de Ginzburg-Landau de la supraconductivité et définissons la fonctionnelle de Ginzburg-Landau. Nous nous concentrons sur un type spécifique de supraconducteurs, appelés supraconducteurs de type-I. Nous passons en revue la littérature sur les modèles réduits pour les supraconducteurs de type-I, rappelons les travaux [37, 39] et présentons les différentes fonctionnelles que nous étudions. En particulier, nous soulignons comment elles se rapportent à la fonction de Ginzburg-Landau à différentes échelles de longueur. Enfin, nous résumons le contenu des contributions originales de cette thèse et énonçons quelques problèmes ouverts.

### 1.1.1 Contexte mathématique

Avant de passer à l'introduction, nous soulignons brièvement le contexte mathématique général de cette thèse.

**Problèmes de transition de phase,  $\Gamma$ -convergence et modèles réduits.** Dans les supraconducteurs de type-I, une énergie de surface positive pénalise les interfaces entre deux régions, la région normale dans laquelle le champ magnétique externe pénètre l'échantillon et la région supraconductrice dans laquelle aucun champ magnétique n'est présent. Ce comportement est typique des problèmes de transition de phase. Dans le contexte des modèles variationnels pour les transitions de phase, un résultat clé est le théorème de Modica-Mortola [85, 84] (voir la Section 1.3.1). En utilisant le langage de la  $\Gamma$ -convergence (voir [46, 43, 12]), ce théorème énonce que le modèle de Cahn-Hilliard (ici  $W(u) = u^2(1-u)^2$ )

$$F_\varepsilon(u) = \int_\Omega \varepsilon |\nabla u|^2 + \frac{1}{\varepsilon} W(u) dx,$$

qui décrit un système de deux fluides se mélangeant dans une couche de transition de largeur  $\varepsilon > 0$ ,  $\Gamma$ -converge vers un multiple de la fonctionnelle périmètre

$$P(\{u = 1\}, \Omega),$$

qui décrit plutôt un système avec des interfaces nettes entre les deux fluides.

La fonctionnelle du périmètre peut être considérée comme un modèle réduit de l'énergie de Cahn-Hilliard lorsque le paramètre  $\varepsilon$  tend vers zero, en respectant la « bonne » notion de convergence variationnelle fournie par la  $\Gamma$ -convergence. Dans notre cadre, on peut interpréter l'énergie de Cahn-Hilliard comme une version simplifiée du modèle complet de Ginzburg-Landau que nous considérerons et pour lequel nous dériverons des modèles réduits.

Ces dernières années, plusieurs généralisations du théorème de Modica-Mortola ont été envisagées. Parmi les nombreux modèles, nous mentionnons les matériaux du second ordre [25] (dérivés de l'analyse des membranes bicouches à deux composants) et les biomembranes [64], pour lesquelles le problème du profil optimal associé (voir la Section 1.3.1) n'est pas unidimensionnel, ainsi que les transitions de phase solide-solide [36, 40, 45, 44], qui sont des problèmes de transition de phase vectorielle. Dans ce manuscrit, nous traiterons d'un problème de transition de phase vectorielle dont le profil optimal n'est pas forcément unidimensionnel, voir le Chapitre 3.

**Transport optimal et transport optimal branché.** Les fonctionnelles que nous dérivons au Chapitre 2 et étudions aux Chapitres 4 et 5 sont des problèmes de transport optimal branché non standard.

Dans sa formulation moderne, le problème du transport optimal est le suivant (voir la Section 1.3.2) : étant donné deux mesures de probabilité  $\mu$  et  $\nu$  sur  $\mathbb{R}^d$ , considérons le problème variationnel

$$\inf_{\pi} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 d\pi,$$

où l'infimum est pris parmi toutes les mesures de probabilité  $\pi$  sur  $\mathbb{R}^d \times \mathbb{R}^d$  dont les marginales sont  $\mu$  et  $\nu$ .

La théorie du transport optimal est aujourd'hui largement étudiée et utilisée dans plusieurs domaines des mathématiques pures et appliquées. Conscients qu'un aperçu exhaustif de l'état actuel de la recherche dépasse le cadre de ce manuscrit, nous mentionnons que le transport optimal est utilisé dans l'analyse des espaces de mesure métriques [4, 78], dans l'étude des solutions de flux de gradient des équations d'évolution [87, 70], dans la dérivation des inégalités fonctionnelles [55, 86], dans les statistiques [94] et l'analyse de la régularité des la mappe de transport elle-même a été un domaine de recherche actif récemment [47]. Le transport optimal a également été utilisé pour modéliser les mouvements collectifs, le trafic et les réseaux, ce qui a conduit à des modèles généralisés pour les embouteillages [22], modèles pour les jeux à champ moyen [9] et en particulier, ce qui nous intéresse, le *transport branché*.

L'une des idées clés de la théorie du transport optimal est que le coût du déplacement d'une particule de masse  $m$  sur une distance  $l$  est proportionnel à  $m \times l^2$ . Les particules se déplacent en suivant des géodésiques (segments dans  $\mathbb{R}^d$ ), de sorte que leur coût ne dépend que de leur point de départ et de leur point d'arrivée, plutôt que de sur l'ensemble de la trajectoire. Dans de nombreuses applications réelles, cette hypothèse n'est pas satisfaisante. Dans les situations où un principe d'économie d'échelle est présent, comme les réseaux, les pipelines ou les vaisseaux sanguins, il peut être pratique pour les particules de voyager ensemble. Pour modéliser ces phénomènes, il est logique de considérer des fonctions de coût concaves de la forme

$$c = m^\alpha \times l,$$

avec  $\alpha < 1$ , qui favorisent les masses voyageant ensemble autant que possible. Dans ce cas, l'énergie doit tenir compte de l'ensemble de la trajectoire d'une particule donnée, et non pas seulement du point de départ et du point d'arrivée. Les problèmes de ce type relèvent de la théorie du transport optimal branché (voir la Section 1.3.3).

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Dans les dernières années, le transport optimal branché a été un domaine de recherche actif, avec des résultats concernant le fait que le problème soit bien posé [30, 31, 32], des généralisations à de plus grandes classes d'énergies [17] et l'apparition de structures auto-similaires ou fractales [91, 90, 81, 15]. Nous mentionnons également [89], où un résultat de Modica-Mortola pour le transport branché a été prouvé, similaire dans l'esprit aux résultats de [37].

*Formation de motifs en sciences des matériaux.* Remarquons enfin que des lois d'échelle pour les microstructures branchées, telles que celles prouvées dans [39], ont été obtenues pour plusieurs fonctionnelles apparaissant dans les sciences des matériaux. Nous mentionnons les alliages à mémoire de forme [72, 33, 35, 98, 99], les ferromagnétiques uniaxiaux [27, 98] et les couches minces élastiques [7]. Les résultats de [37] pour les supraconducteurs de type-I prouvent en fait un résultat plus fort, à savoir la convergence des minima vers des microstructures branchées, comme pour [88, 35]. Nous remarquons que pour le modèle de Kohn-Müller [72], il est en outre connu que les minimiseurs sont asymptotiquement auto-similaires [33, 34].

## 1.2 Brief overview of the thesis

The goal of this thesis is the study of reduced models for type-I superconductors. It focuses both on the rigorous derivation of these models and on the study of qualitative properties of their minimizers.

Superconductors are materials that, below a critical temperature, exhibit a sudden loss of resistivity and expel magnetic fields. The Ginzburg-Landau model is a phenomenological model for superconductors which was proposed by V. Ginzburg and L. Landau in 1950. We focus on a specific class of superconductors, known as type-I superconductors. It has been observed experimentally [95, 96, 97] that type-I superconductors, when subject to an external magnetic field, show the appearance of complex patterns at the surface of the sample. This behavior suggests that branching patterns are present inside the sample. A first mathematical justification of the presence of branching is given in [26, 28, 39] in the form of scaling laws. In particular, [26, 28, 39] distinguish two regimes of parameters, corresponding to uniform and non uniform branching. In the first regime, the magnetic field distributes uniformly at the boundary on the sample, whereas, surprisingly, in the second regime it is energetically favorable for the magnetic field to deviate from uniformity at the boundary. Going beyond scaling laws, in [37] a reduced model is derived (in the sense of  $\Gamma$ -convergence [12, 43]) deep inside the first regime and for vanishing Ginzburg-Landau parameter  $\kappa$ . The resulting model is a non standard branched optimal transport problem, in which the magnetic field concentrates on a one dimensional tree structure, which refines towards the boundary to match uniform boundary conditions.

Apart from this chapter, the manuscript contains four other chapters:

- In the second chapter we extend the results of [37] to the crossover regime between uniform and non-uniform branching described in [39]. We prove  $\Gamma$ -convergence of the Ginzburg-Landau functional to a branched transport problem similar to the one studied in [37], but for which uniform boundary conditions are not imposed, but rather penalized with a weak norm. The limit functional we derive is the one studied in [48], for which the magnetization is conjectured to exhibit fractal behavior at the boundary of the superconductor [38]. This chapter is based on an ongoing work.
- In the third chapter we consider a 2D non-standard Modica-Mortola type functional. This functional arises from the Ginzburg-Landau functional in the case of an infinitely

long sample and in the regime of comparable penetration length and coherence length. We prove that the functional  $\Gamma$ -converges to the perimeter functional. This result is a first step in understanding how to extend the results of [37] to the regime of non vanishing Ginzburg-Landau parameter  $\kappa$ . This chapter is based on a joint work with Michael Goldman and Alessandro Zilio [42].

- In the fourth chapter we study the branched transport problem which was derived in Chapter 1, for which it is conjectured in [38] that the dimension of the boundary measure is non integer, precisely  $8/5$ . We prove this conjecture in a simplified 2D setting, under the (strong) assumption of Ahlfors regularity of the irrigated measure. This result is the first rigorous proof of a singular behavior for irrigated measures resulting from minimality. This chapter is based on a joint work with Michael Goldman and Melanie Koser [41].
- In the last chapter we study the functional derived in Chapter 1 in the case where the sample is a half-plane. We extend part of the results of Chapter 3 to this 3D functional. In particular we prove that the boundary measure cannot be a bounded absolutely continuous function. This chapter is based on an ongoing work with Felix Otto.

In this chapter, we first recall some results on  $\Gamma$ -convergence, optimal transport and branched optimal transport, which will be used throughout the manuscript. Later, we give an introduction to the Ginzburg-Landau theory of superconductivity and define the Ginzburg-Landau functional. We focus on a specific type of superconductors, called type-I superconductors. We review the literature on reduced models for type-I superconductors, recall the works [37, 39] and introduce the several functionals we study. In particular we underline how they relate to the Ginzburg-Landau functional at different lengthscales. Finally we summarize the content of the original contributions of this thesis and state some open problems.

### 1.2.1 General mathematical context

Before moving on with the introduction, we briefly highlight the general mathematical context of this thesis.

**Phase transition problems,  $\Gamma$ -convergence and reduced models.** In type-I superconductors, a positive surface energy penalizes the interfaces between two regions, the normal region in which the external magnetic field penetrates the sample, and the superconducting region in which no magnetic field is present. Such a behavior is typical of *phase transition* problems. In the context of variational models for phase transitions, a key result is the Modica-Mortola theorem [85, 84] (see Section 1.3.1). Using the language of  $\Gamma$ -convergence (see [46, 43, 12]), this theorem states that the Cahn-Hilliard model (here  $W(u) = u^2(1 - u)^2$ )

$$F_\varepsilon(u) = \int_\Omega \varepsilon |\nabla u|^2 + \frac{1}{\varepsilon} W(u) dx,$$

which describes a system of two fluids mixing in a transition layer of width  $\varepsilon > 0$ ,  $\Gamma$ -converges towards a multiple of the perimeter functional

$$P(\{u = 1\}, \Omega),$$

which instead describes a system with sharp interfaces between the two fluids. The perimeter functional can be seen as a *reduced model* for the Cahn-Hilliard energy as the parameter  $\varepsilon$  goes to zero, with respect to the "good" notion of variational convergence provided by  $\Gamma$ -convergence. In our setting, one may interpret the Cahn-Hilliard energy as a simplified version of the full Ginzburg-Landau model we will consider and for which we will derive reduced models.

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In recent years, several generalizations of the Modica-Mortola theorem have been considered. Among many models, we mention second order material [25] (derived from the analysis of two-component bilayer membranes) and biomembranes [64], for which the associated optimal profile problem (see Section 1.3.1) is not one dimensional, as well as solid-solid phase transitions [36, 40, 45, 44], which are vectorial phase transition problems. In this manuscript, we will deal with a vectorial phase transition problem whose optimal profile is not known to be one dimensional, see Chapter 3.

**Optimal transport and branched optimal transport.** The functionals which we derive in Chapter 2 and study in Chapters 4 and 5 are non standard branched optimal transport problems.

In its modern formulation, the *optimal transport* problem is the following (see Section 1.3.2): given two probability measures  $\mu$  and  $\nu$  on  $\mathbb{R}^d$ , consider the variational problem

$$\inf_{\pi} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 d\pi,$$

where the infimum is taken among all probability measures  $\pi$  on  $\mathbb{R}^d \times \mathbb{R}^d$  whose marginals are  $\mu$  and  $\nu$ . Optimal transport theory is nowadays widely studied and employed in several areas of pure and applied mathematics. Aware that providing an exhaustive overview of the current state of research is beyond the scope of this manuscript, we mention that optimal transport is used in the analysis of metric measure spaces [4, 78], in the study of gradient flow solutions of evolution equations [87, 70], in the derivation of functional inequalities [55, 86], in statistics [94] and the analysis of the regularity of transport maps itself has been an active field of research recently [47]. Optimal transport has also been used for modeling collective motions, traffic and networks, leading to generalized models for traffic congestion [22], models for mean field games [9] and in particular, which is of interest for us, *branched transport*.

One key insight of optimal transport theory is that the cost of moving a particle of mass  $m$  for a distance  $l$  is proportional to  $m \times l^2$ . Particles move following geodesics (segments in  $\mathbb{R}^d$ ), so that their cost depends only on their starting and ending point, rather than on the whole trajectory. In many real world applications, this assumption is not satisfactory. In situations in which some economy of scale principle is present, such as networks, pipelines or blood vessels, it may be convenient for particles to travel together. To model these phenomena, it makes sense to consider concave cost functions of the form

$$c = m^\alpha \times l,$$

with  $\alpha < 1$ , which favor masses traveling together as much as possible. In this case, the energy should account for the whole trajectory of any given particle, instead of only the starting and ending point. Problems of this type fall within the theory of branched optimal transport (see Section 1.3.3). In recent years branched optimal transport has been an active field of research, with results concerning well-posedness [30, 31, 32], generalizations to larger classes of energies [17] and the appearance of self similar or fractal structures [91, 90, 81, 15]. We also mention [89], where a Modica-Mortola result for branched transport was proved, similar in spirit to the results of [37].

**Pattern formation in material sciences.** Let us finally remark that scaling laws for branching microstructures, as the ones proved in [39], have been obtained for several functionals appearing in material sciences. We mention shape memory alloys [72, 33, 35, 98, 99], uniaxial ferromagnets [27, 98] and elastic thin films [7]. The results of [37] for type-I superconductors actually prove a stronger result, namely convergence of minima to branched microstructures, similarly to [88, 35]. We remark that for the Kohn-Müller model [72] it is furthermore known that minimizers are asymptotically self similar [33, 34].

## 1.3 Preliminaries

We introduce in this section notions from  $\Gamma$ -convergence theory, optimal transport and branched optimal transport.

### 1.3.1 $\Gamma$ -convergence and the Modica-Mortola Theorem

We recall here the notion of  $\Gamma$ -convergence and some properties, see [43, 12] for details on the topic. We also introduce the Modica-Mortola theorem, a classical example of  $\Gamma$ -convergence.

$\Gamma$ -convergence was introduced by De Giorgi [46] as an appropriate notion of convergence for variational problems. Let  $X$  be a metric space and for  $\varepsilon > 0$  let  $F_\varepsilon, F : X \rightarrow [0, +\infty]$ .

**Definition 1.3.1.** We say that  $F_\varepsilon$   $\Gamma$ -converges to  $F$  ( $F_\varepsilon \xrightarrow{\Gamma} F$ ) as  $\varepsilon \rightarrow 0$  if:

1. For every  $\varepsilon_n \rightarrow 0$ ,  $u \in X$  and for every  $\{u_n\}_{n \in \mathbb{N}}$  such that  $u_n \rightarrow u$

$$\liminf_n F_{\varepsilon_n}(u_n) \geq F(u).$$

2. For every  $\varepsilon_n \rightarrow 0$  and  $u \in X$  there exists  $\{u_n\}_{n \in \mathbb{N}}$  such that  $u_n \rightarrow u$  and

$$\limsup_n F_{\varepsilon_n}(u_n) \leq F(u).$$

The notion of  $\Gamma$ -convergence is usually paired with a compactness property of the following type.

**Property 1.3.2.** Let  $\varepsilon_n \rightarrow 0$  and consider  $\{u_n\}_{n \in \mathbb{N}}$  such that  $F_{\varepsilon_n}(u_n) \leq C$  for some  $C > 0$ . Then,  $\{u_n\}_{n \in \mathbb{N}}$  is precompact.

$\Gamma$ -convergence and Property 1.3.2 imply the following important result.

**Theorem 1.3.3** (Fundamental Theorem of  $\Gamma$ -convergence). *Suppose that  $F_\varepsilon \xrightarrow{\Gamma} F$  and that Property 1.3.2 holds. Let  $\varepsilon_n \rightarrow 0$ . Then, if for  $n \in \mathbb{N}$   $m_n$  is a minimizer of  $F_{\varepsilon_n}$ , then  $(m_n)_{n \in \mathbb{N}}$  is precompact in  $X$  and every limit point  $m \in X$  is a minimizer for  $F$ .*

The theorem above informally states that, if  $F_\varepsilon \xrightarrow{\Gamma} F$  and the compactness property holds, one may interpret the functional  $F$  as "reduced model" for  $F_\varepsilon$  as  $\varepsilon \rightarrow 0$ , whose minimizers are cluster points of minimizers of  $F_\varepsilon$ .

When proving  $\Gamma$ -convergence results, it is often useful to restrict the class of points  $u$  for which condition 2 in Definition 1.3.1 is satisfied. We have the following proposition, which follows by a diagonal procedure.

**Proposition 1.3.4.** *Let  $D \subset X$  be such that for any  $u \in X$  there exists  $\{u_n\}_{n \in \mathbb{N}} \subset D$  such that  $F(u_n) \rightarrow F(u)$ . Then, condition 2 in Definition 1.3.1 is equivalent to the following statement: for all  $u \in D$ ,  $\eta > 0$  and  $\varepsilon_n \rightarrow 0$  there exists  $\{u_n\}_{n \in \mathbb{N}}$  such that  $u_n \rightarrow u$  and*

$$\limsup_n F_{\varepsilon_n}(u_n) \leq F(u) + \eta.$$

We now present a classical example of  $\Gamma$ -convergence. First, we recall the definition of the set  $BV(\Omega)$  of functions  $u \in L^1(\Omega)$  such that

$$|Du|(\Omega) = \sup \left\{ \int_{\Omega} u \operatorname{div} \varphi \, dx, \varphi \in C_c^1(\Omega), \|\varphi\|_{\infty} \leq 1 \right\} < +\infty.$$

We call a set  $E \subset \mathbb{R}^d$  a set of finite perimeter if  $\chi_E \in BV(\Omega)$  and we denote by  $P(E, \Omega) = |D\chi_E|(\Omega)$  its perimeter. For properties of  $BV$  functions and sets of finite perimeter we refer to [3, 51, 80]. Consider an open bounded set  $\Omega \subset \mathbb{R}^d$  representing a container in which two immiscible fluids are present. We describe the system by a function  $u \in BV(\Omega, \{0, 1\})$ , with the values  $\{0, 1\}$  indicating the presence of one or the other fluid. Assume that the volume of both fluids is fixed, that is

$$\int_{\Omega} u \, dx = |\{u = 1\}| = V < |\Omega|.$$

Assuming that the system minimizes the area of the interface between the two fluids, a simple model of the equilibrium configuration is given by the celebrated isoperimetric problem

$$\inf \left\{ \sigma P(\{u = 1\}, \Omega), \int_{\Omega} u \, dx = V \right\},$$

where  $\sigma > 0$  is a constant representing surface tension. Consider now two fluids for which mixing between the two is allowed in a layer of width  $\varepsilon > 0$ . We can describe the configuration with a function  $u \in H^1(\Omega; [0, 1])$  representing the concentration of one of the two fluids. The Cahn-Hilliard model [21] associates to the system the energy

$$E_{\varepsilon}(u) = \int_{\Omega} \varepsilon^2 |\nabla u|^2 + W(u) \, dx,$$

where  $W(u) = u^2(1 - u)^2$  is a double well potential, favoring the phases  $u = 0$  and  $u = 1$ , while the gradient term penalizes transitions between the two phases. The Modica-Mortola theorem [85, 84] (see also [2]) states that an appropriate rescaling of the Cahn-Hilliard energy  $\Gamma$ -converges to the perimeter functional.

**Theorem 1.3.5** (Modica-Mortola). *Let  $\Omega \subset \mathbb{R}^d$  be a bounded regular domain and let  $0 < V < |\Omega|$ . Let  $X = \{u : \Omega \rightarrow [0, 1], \int_{\Omega} u = V\}$ , equipped with the  $L^1$  metric. Let  $W(t) = t^2(1 - t)^2$  and  $\sigma = 2 \int_0^1 \sqrt{W(t)} \, dt$ . Define*

$$F_{\varepsilon}(u) = \frac{E_{\varepsilon}(u)}{\varepsilon} = \begin{cases} \int_{\Omega} \varepsilon |\nabla u|^2 + \frac{1}{\varepsilon} W(u) \, dx & \text{if } u \in H^1(\Omega) \cap X \\ +\infty & \text{otherwise} \end{cases}$$

and

$$F(u) = \begin{cases} \sigma P(\{u = 1\}, \Omega) & \text{if } u \in BV(\Omega; \{0, 1\}) \cap X \\ +\infty & \text{otherwise.} \end{cases}$$

Then  $F_{\varepsilon} \xrightarrow{\Gamma} F$  and Property 1.3.2 holds.

The proof of Theorem 1.3.5 uses the auxiliary minimization problem

$$\sigma = \inf \left\{ \int_{-\infty}^{+\infty} \left( |\dot{u}(t)|^2 + W(u(t)) \right) \, dt, u : \mathbb{R} \rightarrow [0, 1], u(-\infty) = 0, u(+\infty) = 1 \right\}. \quad (1.3.1)$$

Problem (1.3.1) is called the optimal profile problem associated with the energy  $F_{\varepsilon}$ , and, roughly speaking, represents the minimal cost that the energy  $F_1$  spends transitioning from one phase to another. Heuristically, an opportune rescaling of (1.3.1) should represent the behavior of a minimizer of  $F_{\varepsilon}$  near the transition between the two phases. In general, in phase transition problems, the identification of the optimal rescaling and the characterization of the optimal profile are important steps in the analysis of the model.

### 1.3.2 Optimal transport

We recall some basic facts from optimal transport theory. We refer to [4, 101, 108] for a broader overview of optimal transport, and to [56] for an introduction.

For a measurable function  $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$  ( $f : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ ) and a measure  $\mu \in \mathcal{P}(\mathbb{R}^d)$  ( $\mu \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ ), we denote by  $f\#\mu \in \mathcal{P}(\mathbb{R}^d)$  the pushforward measure of  $\mu$  by  $f$ .

Let  $d \in \mathbb{N}$  and  $\mu_1, \mu_2 \in \mathcal{P}(\mathbb{R}^d)$ . For  $i = 1, 2$ , we consider the projections  $e_i : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  defined by  $(x_1, x_2) \rightarrow x_i$  and for  $\pi \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$  we call  $\pi_i = e_i\#\pi \in \mathcal{P}(\mathbb{R}^d)$  its marginals. Let  $c : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^+$  be a continuous function. The classical Monge-Kantorovich problem is the following:

$$\inf \left\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} c(x, y) d\pi : \pi \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d), \pi_i = \mu_i \text{ for } i = 1, 2 \right\}. \quad (1.3.2)$$

For  $\mu_1, \mu_2 \in \mathcal{P}(\mathbb{R}^d)$  we call any  $\pi \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$  such that  $\pi_i = \mu_i$  for  $i = 1, 2$  a transport plan between  $\mu_1$  and  $\mu_2$ . Classic optimal transport theory states that the infimum in (1.3.2) is attained with minimal assumptions on  $c$ . We are mainly interested in the case  $c(x, y) = |x - y|^2$ . We define the (2-)Wasserstein distance

$$W^2(\mu_1, \mu_2) = \min \left\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 d\pi : \pi \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d), \pi_i = \mu_i \text{ for } i = 1, 2 \right\}. \quad (1.3.3)$$

If we set

$$\mathcal{P}_2(\mathbb{R}^d) = \left\{ \mu \in \mathcal{P}(\mathbb{R}^d), \int_{\mathbb{R}^d} |x|^2 d\mu < +\infty \right\},$$

then  $W$  is a distance on  $\mathcal{P}_2(\mathbb{R}^d)$ . For  $\{\mu_n\}_{n \in \mathbb{N}} \subset \mathcal{P}_2(\mathbb{R}^d)$ ,  $\mu \in \mathcal{P}_2(\mathbb{R}^d)$  we denote by  $\mu_n \xrightarrow{*} \mu$  the weak convergence of probability measures. It holds

$$W(\mu_n, \mu) \rightarrow 0 \iff \begin{cases} \mu_n \xrightarrow{*} \mu, \\ \int_{\mathbb{R}^d} |x|^2 d\mu_n \rightarrow \int_{\mathbb{R}^d} |x|^2 d\mu. \end{cases}$$

In particular on compact subsets of  $\mathbb{R}^d$  one has that  $W$  metrizes weak convergence of probability measures. By Brenier's theorem [20], if  $\mu_1 \in \mathcal{P}(\mathbb{R}^d)$  is absolutely continuous with respect to the Lebesgue measure, then the minimum  $\pi$  of (1.3.3) is unique and there exists a map  $T : \mathbb{R}^d \rightarrow \mathbb{R}^d$  such that  $\pi = (I_{\mathbb{R}^d}, T)\#\mu_1$ . We call such a map  $T$  an optimal transport map.

Of particular interest for us will be the Benamou-Brenier formula [8], which gives an eulerian interpretation to the problem (1.3.3). For  $\mu_1, \mu_2 \in \mathcal{P}(\mathbb{R}^d)$  and  $a, b \in \mathbb{R}$  with  $a < b$  we have

$$\begin{aligned} & \frac{1}{b-a} W^2(\mu_1, \mu_2) \\ &= \min_{(\mu, m)} \left\{ \int_{\mathbb{R}^d \times (a, b)} \left| \frac{dm}{d\mu} \right|^2 d\mu dt : m \ll \mu, \partial_t m + \nabla \cdot m = 0, \mu_a = \mu_1, \mu_b = \mu_2 \right\}. \end{aligned} \quad (1.3.4)$$

We finally recall the construction of constant speed geodesics in  $\mathcal{P}_2(\mathbb{R}^d)$ . Given  $\mu_a, \mu_b \in \mathcal{P}_2(\mathbb{R}^d)$  and  $a \leq t \leq b$ , we define the interpolated measure (McCann's interpolant) as

$$\int_{\mathbb{R}^d} \psi d\mu^t = \int_{\mathbb{R}^d \times \mathbb{R}^d} \psi \left( \frac{b-t}{b-a} x + \frac{t-a}{b-a} y \right) d\pi \quad (1.3.5)$$

where  $\pi$  is an optimal transport plan for  $W(\mu_a, \mu_b)$ . By [101, Theorem 5.27], for  $a \leq s \leq t \leq b$

$$W(\mu^t, \mu^s) = \left( \frac{t-s}{b-a} \right) W(\mu_a, \mu_b).$$

### 1.3.3 Branched optimal transport

A first simple model for branched optimal transport was proposed by Gilbert in [62]. Informally the problem is the following: we consider a compact set  $K \subset \mathbb{R}^d$ , and for two measures with finitely many atoms  $\mu_{\pm} \in \mathcal{P}(K)$ , we want to find a graph  $G$  "connecting"  $\mu_+$  to  $\mu_-$ .

We can characterize a graph  $G$  by a triple  $(V, E, \varphi)$ , where  $V$  is a finite sets of vertices in  $K$ ,  $E$  is a finite set of oriented straight edges  $e \subset K$  connecting two vertices in  $V$  and  $\varphi : E \rightarrow (0, +\infty)$  is a function assigning a weight to the edges. Denoting by  $\tau_e$  the unit tangent vector to  $e$ , the graph  $G$  is defined as the vector measure

$$G = \sum_{e \in E} \varphi(e) \tau_e \mathcal{H}^1 \llcorner e.$$

In the language of Geometric Measure Theory, such an object can be interpreted as a 1-dimensional normal current [53]. We say that  $G$  irrigates  $(\mu_+, \mu_-)$  if

$$\operatorname{div} G = \mu_+ - \mu_-$$

distributionally. In particular, this divergence condition enforces Kirchhoff's law on the vertices  $V$ . For  $\alpha \in [0, 1]$  we define the Gilbert energy as

$$M^\alpha(G) = \sum_{e \in E} \varphi(e)^\alpha \mathcal{H}^1(e)$$

and consider the problem

$$\inf\{M^\alpha(G), G \text{ irrigates } (\mu_+, \mu_-)\}. \quad (1.3.6)$$

If  $\alpha < 1$ , the subadditivity of  $\varphi(e)^\alpha$  favors masses traveling together on branches. In the extreme case  $\alpha = 0$  the mass does not play a role and the problem coincides with the Steiner problem of finding a 1-dimensional set with minimal length that connects two finite sets of points [63]. If instead  $\alpha = 1$ , mass concentration is no longer favored and the problem reduces to the Monge-Kantorovich problem with linear cost.

To deal with non atomic irrigated measures, problem (1.3.6) was generalized in several ways. In [109], Xia introduced the following eulerian model based on a relaxation of (1.3.6). Given  $\mu_{\pm} \in \mathcal{P}(K)$  and a graph  $G \in \mathcal{M}(K; \mathbb{R}^d)$  such that  $\operatorname{div} G = \mu_+ - \mu_-$ , we consider any sequences  $\mu_{\pm}^n \in \mathcal{P}(K)$  and  $G^n \in \mathcal{M}(K; \mathbb{R}^d)$  such that  $\mu_{\pm}^n \xrightarrow{*} \mu_{\pm}$ ,  $G^n \xrightarrow{*} G$  and  $\operatorname{div} G^n = \mu_+^n - \mu_-^n$ . We define the relaxed energy

$$M^\alpha(G) = \inf \liminf_n M^\alpha(G^n), \quad (1.3.7)$$

where the infimum is taken on all such sequences.

A lagrangian approach to the problem was instead proposed by Maddalena, Morel and Solimini in [79], which was later studied in [10]. In their approach, a network is represented by a traffic plan, that is a measure  $\eta \in P(\Gamma(K))$ , where  $\Gamma(K)$  is the set of 1-Lipschitz curves in  $K$ . The measure  $\eta$  weights the quantity of mass which is transported along each curve  $\gamma \in \Gamma(K)$ . For  $\gamma \in \Gamma(K)$ , we define  $e_0(\gamma) = \gamma(0)$  and  $e_{+\infty}(\gamma) = \lim_{t \rightarrow +\infty} \gamma(t)$ . We say that a traffic plan  $\eta$  connects two measures  $\mu_{\pm} \in \mathcal{P}(K)$  if  $e_0 \# \eta = \mu_-$  and  $e_{+\infty} \# \eta = \mu_+$ . For any traffic plan  $\eta$  and point  $x \in K$ , we define the multiplicity function  $\theta_\eta(x)$ , representing the quantity of mass passing from  $x \in K$ ,

$$\theta_\eta(x) = \eta(\{\gamma \in \Gamma(K), x \in \gamma\}).$$

For  $\alpha \in [0, 1]$ , we integrate  $\theta_\eta(x)^{\alpha-1}$  along every curve  $\gamma \in \Gamma(K)$  and integrate with respect to  $\eta$  to obtain the total cost

$$I_\alpha(\eta) = \int_{\Gamma(K)} \int_\gamma \theta_\eta^{\alpha-1} d\gamma d\eta(\gamma), \quad (1.3.8)$$

which we minimize among all  $\eta \in \mathcal{P}(\Gamma(K))$  that connect two measures  $\mu_\pm \in \mathcal{P}(K)$ . Under rectifiability hypothesis on  $\eta$ , the above energy takes the more intuitive form

$$I_\alpha(\eta) = \int_K \theta_\eta(x)^\alpha d\mathcal{H}(x).$$

The two models (1.3.7) and (1.3.8) were later proved to be equivalent [90].

We finally mention a dynamical approach to branched transportation, proposed by Brasco, Buttazzo and Santambrogio in [19] and inspired by the Benamou-Brenier formula (1.3.4). Informally, in their approach they consider couples  $(\mu, m) \in C([0, 1], \mathcal{P}(K)) \times L^1([0, 1], \mathcal{M}(K, \mathbb{R}^d))$  which solve the continuity equation

$$\partial_t \mu + \operatorname{div} m = 0$$

and such that  $\mu_0 = \mu_-$  and  $\mu_1 = \mu^+$ . They moreover impose that  $m \ll \mu$  and that for a.e.  $t \in [0, 1]$  the measure  $\mu_t$  is atomic, so that there exist some numbers  $\rho_{i,t} \in (0, 1]$ , points  $X_{i,t} \in K$  and vectors  $v_{i,t} \in \mathbb{R}^2$  such that

$$\mu_t = \sum_{i \in \mathbb{N}} \rho_{i,t} \delta_{X_{i,t}} \quad \text{and} \quad m_t = \sum_{i \in \mathbb{N}} v_{i,t} \delta_{X_{i,t}}.$$

They finally define the energy

$$\mathcal{F}_\alpha(\mu, m) = \int_0^1 \left[ \sum_{i \in \mathbb{N}} v_{i,t} \rho_{i,t}^\alpha \right] dt \quad (1.3.9)$$

and minimize among all couples  $(\mu, m)$  with the above properties. Also this last model is equivalent to (1.3.7) and (1.3.8).

## 1.4 General context for type-I superconductors

We introduce in this section the mathematical and physical context for superconductors. First, we briefly introduce the Ginzburg-Landau theory and type-I superconductors. Later, we present some results of [39, 37], highlighting the different regimes and lengthscales present in the Ginzburg-Landau functional. We moreover derive from this framework the functionals which we study in the following chapters. Finally we address the reduced models obtained, comparing them with classical branched transport problems.

### 1.4.1 Superconductivity and Ginzburg-Landau theory

In 1911, K. Onnes observed [71] that several materials, such as mercury, lead and tin, showed complete loss of electrical resistance at a low critical temperature. This was evidenced by currents in the sample with extremely large decay time. Such materials are nowadays called superconductors. In 1933, another key feature of superconductivity was discovered: W. Meissner found that magnetic fields were expelled from the sample as the material was cooled below its critical temperature, a phenomenon which is now known as the Meissner effect [83]. This phenomenon differentiates superconductors from idealized classical "perfect

conductors", and can only be explained by quantum effects. Meissner effect also implies that superconductivity is destroyed if the material is subject to a critical large magnetic field. Models for these two phenomena were proposed first by the brothers F. and H. London in 1935 [77] and later by Bardeen, Cooper and Schrieffer in 1957 [6]. In the so-called BCS theory, charges in superconducting currents are carried by couples of weakly interacting electrons, known as Cooper pairs.

The nowadays universally accepted model for superconductivity was proposed by V. Ginzburg and L. Landau. Actually developed as a phenomenological model in 1950 [65], it was fully appreciated in 1959 when Gorkov formally derived the Ginzburg-Landau theory as a microscopic limit of the BCS theory (see also [60] for a more rigorous derivation). In their model, they introduce a complex valued pseudowave function  $u$  (which physicists would call  $\psi$ ) describing the state of superconducting electrons, with their density given by  $|u|^2$ . This model was able to appropriately predict the behavior of a superconductor in the so-called intermediate state. In this state two regions coexist: the *normal region*, that is the set in which  $|u|$  vanishes and some magnetic field  $B$  is present, and the *superconducting region*, in which no magnetic field is present and  $|u| \simeq 1$ . In this framework, the expulsion of the magnetic field observed by Meissner is now explained as  $|u|B \simeq 0$ . Moreover, Ginzburg-Landau theory features two characteristic parameters,  $\lambda$  and  $\xi$ . The parameter  $\lambda$  is called *penetration length* and represents the typical length of penetration of the magnetic field in the superconducting region, whereas the parameter  $\xi$  is called *coherence length* and represents the typical scale at which  $\rho$  varies. The number  $\kappa = \lambda/\xi$  is called the Ginzburg-Landau parameter. If  $\kappa < 1/\sqrt{2}$ , it was shown that a positive surface energy penalizes the interface between the normal region and the superconducting region. These materials are nowadays referred to as type-I superconductors. Remarkably, in 1957 Abrikosov showed that, if  $\kappa > 1/\sqrt{2}$ , a negative surface tension is present and the magnetic field penetrates the sample through vortices which are now known as Abrikosov lattices [1], see Figure 1.1. These materials are instead called type-II

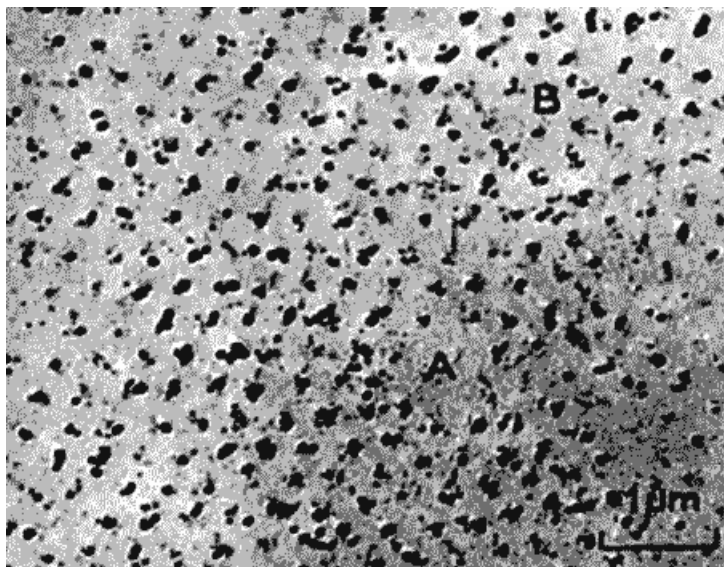


Figure 1.1: Abrikosov lattices, experiments from [50].

superconductors.

Among the scientists mentioned in this paragraph, Onnes, Bardeen, Cooper, Ginzburg, Landau and Abrikosov all won the Nobel prize in Physics for their pioneering works on superconductivity. Superconductors enjoy numerous possible applications, such as MRI machines, mass spectrometers, particle accelerators and levitating trains, see Figure 1.2. For



Figure 1.2: A maglev train runs along a test track in Yamanashi Prefecture, west of Tokyo, picture from Takashi Kurokawa.

an introduction to the topic of superconductors, we refer to [105].

We now give more details on the Ginzburg-Landau model. A superconductor is represented by a sample  $\Omega \subset \mathbb{R}^3$  and its state is given by the order parameter function  $u : \Omega \rightarrow \mathbb{C}$ , so that the density of superconducting electrons is  $\rho = |u|^2$ , the normal region is the region in which  $\rho \simeq 0$  and the superconducting region is the region in which  $\rho \simeq 1$ . We consider a magnetic potential  $A : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ , so that the magnetic field is  $B = \nabla \times A$ . Denoting by  $i$  the imaginary unit, we define the covariant derivative

$$\nabla_A u = \nabla u - iAu.$$

We assume that the sample is subject to an external magnetic field  $B_{\text{ext}}$ . The Ginzburg-Landau functional is defined as

$$E_{GL}(u, A) = \int_{\Omega} |\nabla_A u|^2 + \frac{\kappa^2}{2} (1 - |u|^2)^2 dx + \int_{\mathbb{R}^3} |\nabla \times A - B_{\text{ext}}|^2 dx. \quad (1.4.1)$$

The first term in (1.4.1) penalizes oscillations of the order parameter, while the second and third term favor the superconducting phase and alignment with the external magnetic field respectively. The value of  $\kappa$  heavily influences the behavior of the superconductor. We are interested in the case  $\kappa < 1/\sqrt{2}$ , that is type-I superconductors, where a positive surface energy penalizes the interface between the normal region and the superconducting region. We mention that the opposite regime,  $\kappa > 1/\sqrt{2}$ , has been actively studied in recent years. In this case the interesting question is the appearance of vortices, that is singularities for  $u$  of codimension 2. We mention [11] for results on vortices for the Ginzburg-Landau model without magnetic field, and [100, 104] for results for the full model. We also refer to [100, Chapter 14] for a guide to the literature of type-II superconductors.

It has been observed experimentally [95, 96, 97] that type-I superconductors, when subject to an external magnetic field, exhibit the appearance of complex patterns at the surface of the sample, see Figure 1.3. Deep inside the superconductor, because of the positive surface tension, large normal and superconducting region should be present. The complex patterns

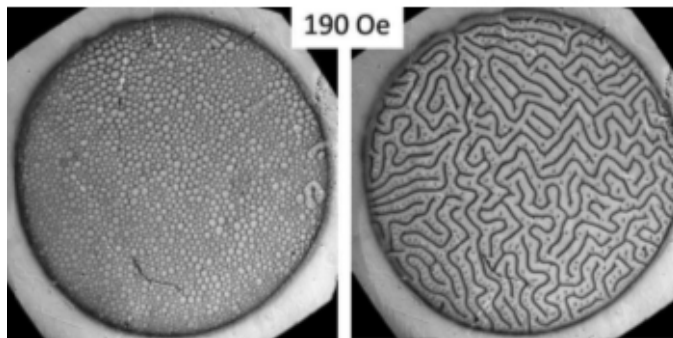


Figure 1.3: Complex patterns at the surface of a type-I superconductor, experiments from [95].

at the boundary could thus be attained by branching patterns inside the sample. A first mathematical justification of the presence of branching structures was given in [26, 28], where scaling laws for a simplified sharp interface model were proven. Later, these scaling laws were extended to the full Ginzburg-Landau model in [39]. In particular, [26, 28, 39] distinguish two regimes of parameters, corresponding to uniform and non uniform branching. In the first regime, the magnetic field distributes uniformly at the boundary of the sample, whereas in the second regime it is energetically favorable for the magnetic field to deviate from uniformity at the boundary. Going beyond scaling laws, in [37] a reduced model was derived (in the sense of  $\Gamma$ -convergence) deep inside the first regime. The resulting model is a non standard branched transport problem, where the magnetic field concentrates on a one dimensional tree structure, which refines towards the boundary to match uniform boundary conditions. We explain in more detail the works [37, 39] in the next section, and the limit functional in Section 1.4.4.

#### 1.4.2 The various lengthscales of the Ginzburg-Landau functional for type-I superconductors

In this section we recall the results of [39, 37], examining the various terms in the Ginzburg-Landau functional for type-I superconductors and underlining the various lengthscales that appear. To this extent, we manipulate the functional (1.4.1), isolating a bulk term and performing several rescalings.

For  $L_0 > 0$  we consider the two dimensional torus  $\mathbb{T}_{L_0}^2 = \mathbb{R}^2/(L_0\mathbb{Z}^2)$ . For  $T > 0$  we consider as material sample  $\Omega$  a box  $\mathbb{T}_{L_0, T}^2 = \mathbb{T}_{L_0}^2 \times [-T, T]$ . We also write  $\mathbb{T}^2 = \mathbb{T}_1^2$ . We consider periodic lateral boundary conditions on the physical quantities  $\rho$  and  $B$ , that is we consider functions  $u : \mathbb{R}^2 \times [-T, T] \rightarrow \mathbb{C}$  and  $A : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  such that the density  $\rho = |u|^2$  and the magnetic field  $B = \nabla \times A$  are  $L_0$ -periodic in the first two variables, so that they can be thought of as functions on  $\mathbb{T}_{L_0, T}^2$  and  $\mathbb{T}_{L_0}^2 \times \mathbb{R}$  respectively (see Chapter 2 or [39] for precise assumptions on the function spaces we use). As above, if  $\nabla_A u = \nabla u - iAu$  and  $\kappa \in (0, 1/\sqrt{2})$ , the Ginzburg-Landau functional is

$$E_{GL}(u, A) = \int_{\mathbb{T}_{L_0, T}^2} |\nabla_A u|^2 + \frac{\kappa^2}{2}(1 - |u|^2)^2 dx + \int_{\mathbb{T}_{L_0}^2 \times \mathbb{R}} |\nabla \times A - B_{\text{ext}}|^2 dx. \quad (1.4.2)$$

We first remark that an important property of (1.4.2) is gauge invariance. We say that two configurations  $(u, A)$  and  $(\hat{u}, \hat{A})$  are gauge equivalent if there exists  $\phi$  such that

$$\begin{cases} u = \hat{u}e^{i\phi} \\ A = \hat{A} + \nabla\phi. \end{cases} \quad (1.4.3)$$

Notice that the observable quantities  $\rho$  and  $B$ , as well as  $|\nabla_A u|$ , are invariant under the gauge transformation (1.4.3). For a real number  $b_{\text{ext}} > 0$  and denoting by  $e_3$  the third vector of the canonical basis of  $\mathbb{R}^3$ , we now consider an external applied magnetic field  $B_{\text{ext}} = b_{\text{ext}}e_3$  perpendicular to the sample. In the sequel, we write  $x = (x', x_3)$ ,  $B = (B', B_3)$  and  $\nabla', \text{div}'$  for the derivative in the first two variables. Since we have periodic boundary condition, the Maxwell equation  $\text{div} B = 0$  implies that for any  $x_3 \in \mathbb{R}$  one has

$$\int_{\mathbb{T}_{L_0}^2 \times \{x_3\}} B_3 = L_0^2 b_{\text{ext}}. \quad (1.4.4)$$

Defining

$$\mathcal{D}_A^3 u = (\nabla_A u)_2 - i(\nabla_A u)_1$$

(which is called Bogomol'nyi operator), it is possible to isolate a bulk term in (1.4.2) (see [37, Chapter 3]). One has

$$E_{GL}(u, A) = 2TL_0^2(\kappa\sqrt{2}b_{\text{ext}} - b_{\text{ext}}^2) + F(u, A),$$

where

$$\begin{aligned} F(u, A) = & \int_{\mathbb{T}_{L_0, T}^2} |\nabla_A u|^2 + \kappa\sqrt{2}(|\mathcal{D}_A^3 u|^2 - |\nabla'_A u|^2) dx \\ & + \int_{\mathbb{T}_{L_0, T}^2} \left( B_3 - \frac{\kappa}{\sqrt{2}}(1 - \rho) \right)^2 dx \\ & + \int_{\mathbb{T}_{L_0}^2 \times \mathbb{R}} |B'|^2 dx + \int_{\mathbb{T}_{L_0}^2 \times (\mathbb{R}/(-T, T))} (B_3 - b_{\text{ext}})^2 dx. \end{aligned} \quad (1.4.5)$$

For  $f \in L^2(\mathbb{T}_{L_0}^2)$  with  $\int_{\mathbb{T}_{L_0}^2} f dx' = 0$  we define the weak norm

$$\|f\|_{H^{-1/2}(\mathbb{T}_{L_0}^2)}^2 = \inf \left\{ \int_{\mathbb{T}_{L_0}^2 \times [0, +\infty]} |B|^2 dx, \text{div} B = 0, B(\cdot, 0) = f \right\}. \quad (1.4.6)$$

The norm (1.4.6) actually coincides with the homogeneous Sobolev norm defined via Fourier modes (see [28, Proposition A.1]). One has

$$\|f\|_{H^{-1/2}(\mathbb{T}_{L_0}^2)}^2 = \frac{1}{2\pi} \sum_{k \in \mathbb{Z}^2} |k|^{-1} |\widehat{f}_k|^2 < \infty,$$

where  $\widehat{f}_k$  is the Fourier coefficient for  $k \in \mathbb{Z}^2$  given by

$$\widehat{f}_k = \int_{\mathbb{T}_{L_0}^2} f(x') \exp(2\pi i(k \cdot x')) dx'.$$

By (1.4.6), minimizing (1.4.5) is equivalent to minimizing

$$\begin{aligned} E(u, A) = & \int_{\mathbb{T}_{L, T}^2} |\nabla_A u|^2 + \kappa\sqrt{2}(|\mathcal{D}_A^3 u|^2 - |\nabla'_A u|^2) + \left( B_3 - \frac{\kappa}{\sqrt{2}}(1 - \rho) \right)^2 \\ & + |B'|^2 dx + \|B_3 - b_{\text{ext}}\|_{H^{-1/2}(\{x_3 = \pm T\})}^2, \end{aligned} \quad (1.4.7)$$

where for any  $f(x', x_3) \in L^2(\mathbb{T}_{L_0, T}^2)$  we use the convention

$$\|f\|_{H^{-1/2}(\{x_3=\pm T\})}^2 = \|f(\cdot, T)\|_{H^{-1/2}(\mathbb{T}_{L_0}^2)}^2 + \|f(\cdot, -T)\|_{H^{-1/2}(\mathbb{T}_{L_0}^2)}^2.$$

Let us examine the various terms in (1.4.7). If  $u = \rho^{1/2}e^{i\theta}$ , notice that

$$|\nabla_A u|^2 = \left| \nabla \rho^{1/2} \right|^2 + \rho |\nabla \theta - A|^2. \quad (1.4.8)$$

Formally, (1.4.8) means that the functional penalizes oscillations of  $\rho$  and imposes  $A \simeq \nabla \theta$ , so that, since  $\nabla \times \nabla = 0$ ,  $B \simeq 0$  when  $\rho > 0$ , hence the Meissner condition  $\rho B \simeq 0$  holds. If  $\rho B \simeq 0$ , the term

$$\left( B_3 - \frac{\kappa}{\sqrt{2}}(1 - \rho) \right)^2 \quad (1.4.9)$$

can be interpreted as a double well term in  $\rho$  and  $B$ , so that

$$\int_{\mathbb{T}_{L, T}^2} |\nabla_A u|^2 + \kappa \sqrt{2} (|\mathcal{D}_A^3 u|^2 - |\nabla'_A u|^2) + \left( B_3 - \frac{\kappa}{\sqrt{2}}(1 - \rho) \right)^2 dx \quad (1.4.10)$$

can be thought of as a Cahn-Hilliard type functional in the physical quantities  $\rho$  and  $B$ , subject to the "volume" constraint (1.4.4). As for the classical Modica-Mortola functional (recall Theorem 1.3.5), we expect that an appropriate rescaling of (1.4.10) acts as a perimeter term for the normal and superconducting regions. Notice that positivity of (1.4.10) is ensured by the condition  $\kappa < 1/\sqrt{2}$ , coherently with the physical interpretation of the term as a positive "surface energy", as expected in type-I superconductors.

Regarding the term

$$\int_{\mathbb{T}_{L_0, T}^2} |B'|^2 dx, \quad (1.4.11)$$

we notice that we can rewrite the condition  $\operatorname{div} B = 0$  as

$$\partial_{x_3} B_3 + \operatorname{div}' B' = 0.$$

If the term (1.4.9) is small, then  $B_3 \sim (1 - \rho)$ , which paired with the Meissner condition implies

$$|B'|^2 \sim \frac{1}{B_3} |B|^2.$$

If we think of the  $x_3$  direction as a time variable, the term (1.4.11) can be thus interpreted as a transport cost similarly to the Benamou-Brenier formulation of optimal transport (1.3.4).

Finally, the last term

$$\|B_3 - b_{\text{ext}}\|_{H^{-1/2}(\{x_3=\pm T\})}^2$$

penalizes deviation from the applied uniform magnetic field at the boundary.

The combination of these three contributions means that inside the sample we expect that the superconducting phase  $\rho \simeq 1, B \simeq 0$  alternates with the normal one  $\rho \simeq 0, B_3 \simeq \kappa/\sqrt{2}$ , minimizing the surface area between the two. However, normal domains need to refine close to the surface of the sample to match the external magnetic field. The interaction of these two effects and of the transport term gives rise to the observed complex patterns.

This behavior was mathematically justified in [39], where the following scaling law was proven (we use the notation  $a \sim b$  if there exists a constant  $C > 0$  such that  $C^{-1}a \leq b \leq Ca$ ).

**Theorem 1.4.1.** *Assume*

$$8b_{\text{ext}} \leq \kappa \leq 1/2 \quad b_{\text{ext}}L_0^2 \in 2\pi\mathbb{N}, \quad \kappa T \geq 1, \quad L_0 \gg 1. \quad (1.4.12)$$

Then,

$$\min E(u, A) \sim \min\{b_{\text{ext}}^{2/3}\kappa^{2/3}T^{1/3}, b_{\text{ext}}\kappa^{3/7}T^{3/7}\}L_0^2. \quad (1.4.13)$$

Let us comment briefly on the hypothesis (1.4.12). The condition  $\kappa \leq 1/2$ ,  $\kappa T \geq 1$  and  $L_0 \gg 1$  mean that the superconductor is well inside the type-I regime and the extensive behavior is studied, while the condition  $b_{\text{ext}} < \kappa/8$  is a small external field hypothesis. Indeed, the energy of a superconducting configuration grows with the intensity of the external magnetic field, so that for large fields the normal solution  $u = 0$ ,  $A = b_{\text{ext}}x_1e_2$  (which means  $B = b_{\text{ext}}e_3$ ) with energy  $E_{GL}(u, A) = L_0^2T\kappa^2$  is eventually favored and superconductivity is broken. Finally  $b_{\text{ext}}L_0^2 \in 2\pi\mathbb{N}$  is a flux quantization hypothesis, and is a consequence of the fact that the phase circulation on the order parameter  $u$  is naturally quantized (see [105]).

The two regimes in (1.4.13) correspond to different branching constructions. The first regime corresponds to uniform branching patterns, while the second one corresponds to non-uniform branching, that is the magnetic field deviates from a uniform distribution at the boundary. This interesting phenomenon was captured, still in the form of scaling laws, first in [26, 28] for a simplified sharp interface model.

We now present the results of [37]. To better understand the different lengthscales present in the model, we set

$$\alpha = \frac{\kappa T}{\sqrt{2}}, \quad \beta = \frac{b_{\text{ext}}\sqrt{2}}{\kappa}, \quad L = \frac{L_0}{T}$$

and consider the setting

$$T \gg 1, \quad \alpha \gg 1, \quad \beta \ll 1, \quad \alpha T^{-1} \ll 1, \quad \alpha\beta^{7/2} \sim \lambda^{1/3} \quad (1.4.14)$$

for some  $\lambda \geq 0$ . This means in particular that the term

$$\kappa\sqrt{2} \int_{\mathbb{T}_{L,T}^2} (|\mathcal{D}_A^3 u|^2 - |\nabla'_A u|^2) dx \quad (1.4.15)$$

in (1.4.7) is negligible since  $\kappa \sim \alpha T^{-1} \ll 1$ . We now rescale so that

$$\hat{x} = \frac{x}{T}, \quad \hat{u}(\hat{x}) = u(x), \quad \hat{A}(\hat{x}) = A(x) \quad \text{and} \quad E_T(\hat{u}, \hat{A}) = \frac{1}{TL^2}E(u, A).$$

Dropping hats, we obtain

$$E_T(u, A) = \frac{1}{L^2} \int_{\mathbb{T}_{L,1}^2} |\nabla_{TA} u|^2 + (B_3 - \alpha(1 - \rho))^2 + |B'|^2 dx + \|B_3 - \alpha\beta\|_{H^{-1/2}(\{x_3=\pm 1\})}^2, \quad (1.4.16)$$

and the scaling (1.4.13) becomes

$$\min E_T(u, A) \sim \min\{\alpha^{4/3}\beta^{2/3}, \alpha^{10/7}\beta\}.$$

Recalling the definition of the Ginzburg-Landau parameter, the regime (1.4.14) can be interpreted as the regime of vanishing penetration length,  $T^{-1}$ , vanishing coherence length,  $\alpha^{-1}$ , vanishing Ginzburg-Landau parameter,  $\alpha T^{-1}$ , and vanishing external field  $\beta$ . The branching constructions in (1.4.13) moreover suggest that the width of the branches is of the order

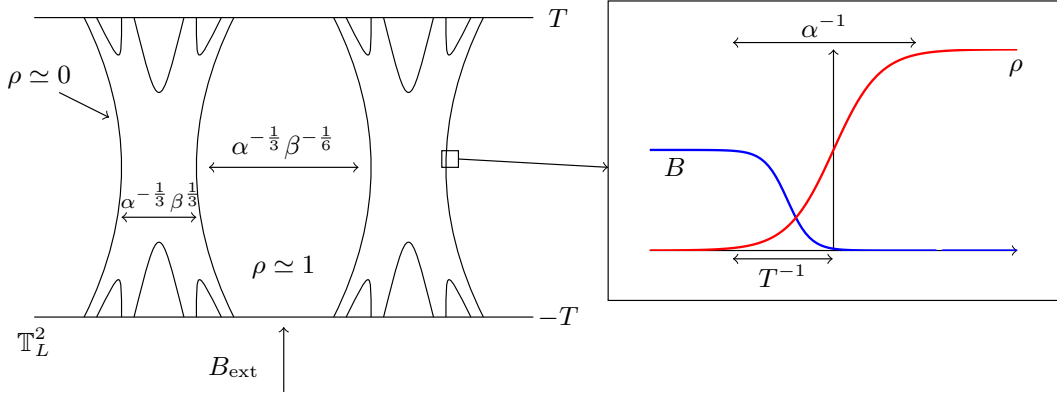


Figure 1.4: Sketch of the different lengthscales involved in the functional. The penetration length has order  $T^{-1}$ , the coherence length has order  $\alpha^{-1}$ , the width of the threads is of order  $\alpha^{-1/3}\beta^{1/3}$  and the distance between the threads is of order  $\alpha^{-1/3}\beta^{-1/6}$ .

$\alpha^{-1/3}\beta^{1/3}$  and the distance between the branches is of order  $\alpha^{-1/3}\beta^{-1/6}$ , so that the separation of scales

$$T^{-1} \ll \alpha^{-1} \ll \alpha^{-1/3}\beta^{1/3} \ll \alpha^{-1/3}\beta^{-1/6} \quad (1.4.17)$$

holds (see Figure 1.4). Finally, the condition

$$\alpha\beta^{7/2} \sim \lambda$$

is concerned with the scaling law (1.4.13). If  $\lambda = +\infty$ , then  $\alpha^{-2/7} \ll \beta$ , corresponding to the first regime in (1.4.13) (uniform branching). The case  $\lambda < \infty$  instead corresponds to the crossover regime  $\alpha^{-2/7} \sim \beta$ . In the regime (1.4.14), based on the separation of scales (1.4.17), in [37] a further anisotropic rescaling is introduced, which, setting  $\tilde{L} = L\alpha^{1/3}\beta^{1/6}$ , leads to the functional

$$\begin{aligned} \tilde{E}_T(u, A) = & \frac{1}{\tilde{L}^2} \left[ \int_{\mathbb{T}_{\tilde{L},1}^2} \alpha^{-2/3}\beta^{-1/3} \left| \nabla'_{\alpha^{1/3}\beta^{-1/3}TA} u \right|^2 + \alpha^{-4/3}\beta^{-2/3} \left| \left( \nabla_{\alpha^{1/3}\beta^{-1/3}TA} u \right)_3 \right|^2 \right. \\ & \left. + \alpha^{2/3}\beta^{-2/3} (B_3 - (1 - \rho))^2 + \beta^{-1} |B'|^2 dx + \alpha^{1/3}\beta^{7/6} \left\| \beta^{-1}B_3 - 1 \right\|_{H^{-1/2}(x_3=\pm 1)}^2 \right]. \quad (1.4.18) \end{aligned}$$

See Section 1.4.3 for the details and motivation of this anisotropic rescaling. We consider now the limit behavior of (1.4.18) in the regime (1.4.14), that is  $T, \alpha, \beta^{-1}, T\alpha^{-1} \rightarrow +\infty$  and  $\alpha\beta^{7/2} \rightarrow \lambda^{1/3}$ . In the case  $\lambda = +\infty$ , it is proved in [37] that the functional (1.4.18)  $\Gamma$ -converges towards a functional  $I$  defined on measures  $\mu$  living on one-dimensional trees. In particular, if  $x = (x', x_3) \in \mathbb{T}_{\tilde{L},1}^2$ , the normal phase can be represented by measures  $\mu = \mu_{x_3} \otimes dx_3$  where, for a.e.  $x_3 \in (-1, 1)$ ,  $\mu_{x_3} = \sum_i \varphi_i \delta_{X_i}$ . The curves  $x_3 \rightarrow X_i(x_3) \in \mathbb{T}_{\tilde{L}}^2$  represent the branches of the tree and the weights  $\varphi_i > 0$  represent the flux carried by each branch. Let  $K^* = 8\sqrt{\pi}/3$ . The functional  $I$  is defined as

$$I(\mu) = \frac{1}{\tilde{L}^2} \int_{-1}^1 K^* \sum_i \varphi_i^{1/2} + \varphi_i \left| \dot{X}_i \right|^2 dx_3, \quad (1.4.19)$$

where  $\dot{X}_i(x_3)$  is the derivative of the map  $x_3 \rightarrow X_i(x_3)$ . Moreover, since  $\alpha^{1/3}\beta^{7/6} \gg 1$ , the last term in (1.4.18) forces  $\mu_{\pm 1} = dx'$  in an appropriate trace sense.

The functional (1.4.18) (see Section 1.4.4) is studied in Chapter 2, where we extend the results of [37] to the case  $\lambda < +\infty$ , that is the crossover regime  $\alpha^{-2/7} \sim \beta$ . In this case, since

the last term in (1.4.18) stays bounded, the limit model we consider is

$$\mathcal{E}(\mu) = I(\mu) + \lambda \|\mu_{\pm 1} - 1\|_{H^{-1/2}(\mathbb{T}_L^2)}^2, \quad (1.4.20)$$

where boundary conditions on  $\mu$  are not enforced, but rather penalized with a weak norm. The functional (1.4.20) (see Section 1.4.4) is instead studied in Chapters 4, while in 5 a similar "blow up" version on the half plane is considered.

### 1.4.3 A hierarchy of models

In this section we better explain the intermediate functionals that appear in the study of (1.4.16) in the regime (1.4.14), leading to the rescaling (1.4.18). For the full computations of the rescaling and further information on the intermediate functionals we consider, we refer to [37, Section 3 and 4].

We want to heuristically send in order  $T \rightarrow \infty, \alpha \rightarrow \infty$  and  $\beta \rightarrow 0$ , obtaining intermediate functionals at each step from the energy (1.4.16) up to the limit energy (1.4.19), highlighting the different phenomena at play at different lengthscales. In the case of Chapter 2, the assumption  $\lambda < +\infty$  means that we cannot actually send  $\alpha$  and  $\beta$  to zero independently and the rescaling needs to be performed jointly in  $\alpha$  and  $\beta$ . Since  $\alpha\beta^{7/2}$  stays bounded, also the boundary term in (1.4.18) is bounded, leading instead to the functional (1.4.20). For simplicity, we consider the case  $\lambda = +\infty$  in (1.4.14), so that the separation of scales holds we can send in order  $T \rightarrow \infty, \alpha \rightarrow \infty$  and  $\beta \rightarrow 0$ . In any case, we remark that the results of [37] and of Chapter 2 are convergence results of the rescaled energy (1.4.18) to the limit energy  $I(\mu)$  and  $\mathcal{E}(\mu)$  respectively, skipping the convergence of the intermediate functionals we will define.

We first send  $T \rightarrow +\infty$  in (1.4.16). Looking at the separation of scales (1.4.17), this means that penetration length is sent to zero and Meissner condition  $\rho B = 0$  is enforced. The functional  $E_T$  approximates

$$F_{\alpha,\beta}(\rho, B) = \frac{1}{L^2} \int_{\mathbb{T}_{L,1}^2} |\nabla \rho|^2 + (B_3 - \alpha(1 - \rho))^2 + |B'|^2 dx + \|B_3 - \alpha\beta\|_{H^{-1/2}(\{x_3=\pm 1\})}^2, \quad (1.4.21)$$

subject to

$$\rho B = 0, \quad \operatorname{div} B = 0.$$

Since  $\rho B = 0$ , the term

$$\int_{\mathbb{T}_{L,1}^2} |\nabla \rho|^2 + (B_3 - \alpha(1 - \rho))^2 dx \quad (1.4.22)$$

acts as a Cahn-Hilliard type functional in  $\rho$  and  $B$ , with typical transition length given by  $\alpha$ , the coherence length. We recall that the coherence length represents the typical length at which the order parameter transition from normal to superconducting regions, that is the "thickness" of the walls of the superconducting region (see Figure 1.4). We now want to send this thickness to zero keeping  $\beta$  and the size of the superconducting domains fixed. Since by (1.4.17) the typical horizontal width of the superconducting domain is of the order  $\alpha^{-1/3}\beta^{1/3}$ , it is natural to set  $\hat{L} = \alpha^{1/3}L$  and perform a rescaling of (1.4.21) of the order  $\alpha^{1/3}$  in the horizontal variables, so that in the limit for  $\alpha \rightarrow +\infty$  the term (1.4.22) behaves like a perimeter term for  $\rho$  in the horizontal variables. In the limit the superconducting regions may be represented by a function  $\chi : \mathbb{T}_{\hat{L},1}^2 \rightarrow \{0, 1\}$ , with  $\chi(\cdot, x_3) \rightarrow \beta dx'$  as  $x_3 \rightarrow \pm 1$ . An opportune anisotropic rescaling of (1.4.21) (see [37, Section 4]) approximates

$$G_\beta(\chi, B') = \frac{1}{\hat{L}} \left( \int_{\mathbb{T}_{\hat{L},1}^2} \frac{4}{3} |D'\chi| + |B'|^2 dx \right) \quad (1.4.23)$$

with

$$\partial_3 \chi + \operatorname{div}' B' = 0 \quad \text{and} \quad \chi B' = B',$$

where

$$\int_{\mathbb{T}_{\tilde{L},1}^2} |D' \chi| = \int_{-1}^1 P(\{\chi(\cdot, x_3) = 1\}, \mathbb{T}_{\tilde{L}}^2) dx_3$$

denotes the perimeter in the horizontal variables. Functional (1.4.23) is similar to the sharp interface model studied in [28, 26]. At this scale, the width of the superconducting regions is of order  $\beta^{1/3}$  and their separation is of order  $\beta^{-1/6}$ . If we now want to send the width of the regions to zero keeping their separation fixed, we should set  $\tilde{L} = \beta^{1/6} \hat{L}$  and again rescale anisotropically in the horizontal directions (see [37, Section 4]). Letting  $\beta \rightarrow 0$  the superconducting phase is now represented by a measure living on a one dimensional tree, and an opportune rescaling of  $G_\beta$  converges to the functional (1.4.19).

#### 1.4.4 The limit problem

In this section we provide a description of the class of functionals to which (1.4.19) and (1.4.20) belong.

Let  $d \in \mathbb{N}$  and consider a measure  $\mu \in \mathcal{M}^+(\mathbb{T}_{1,T}^d)$  such that  $\mu = \mu_t \otimes dt$  where, for a.e.  $t \in (-T, T)$ ,  $\mu_t = \sum_i \varphi_i \delta_{X_i}$ , with  $X_i \in \mathbb{T}^d$ ,  $\varphi_i > 0$  and  $\sum_i \varphi_i = 1$ . We consider for  $-T \leq a < b \leq T$  the internal energy

$$I(\mu, (a, b)) = \int_a^b \sum_i \left[ \varphi_i^{\frac{d-1}{d}}(t) + \varphi_i(t) |\dot{X}_i(t)|^2 \right] dt, \quad (1.4.24)$$

where  $\dot{X}_i(t)$  is the derivative of the map  $t \mapsto X_i(t) \in \mathbb{T}^d$  (see Chapter 4 for the precise definition). Since  $\varphi \rightarrow \varphi^{\frac{d-1}{d}}$  is subadditive, the first term in (1.4.24) favors masses traveling together. Coherently with the isoperimetric effects that come into play in its derivation in [37], we interpret the term as a "perimeter" term. The second term instead penalizes the kinetic energy associated to the curve  $t \rightarrow \mu_t$ , exactly as in Benamou-Brenier's formula (4.2.3), and is thus a transport term. Functional (1.4.24) can be interpreted as a (non-standard) branched transport energy, see Section 1.3.3. Given two measures  $\bar{\mu}_\pm \in \mathcal{M}^+(\mathbb{T}^d)$  such that  $\bar{\mu}_+(\mathbb{T}^d) = \bar{\mu}_-(\mathbb{T}^d)$  consider the problem

$$\inf_{\mu} \{I(\mu, (-T, T)) : \mu_{\pm T} = \bar{\mu}_\pm\}. \quad (1.4.25)$$

The case  $d = 2$  is studied in [37, Chapter 5], and corresponds to the limit problem (1.4.19) in the case  $\bar{\mu}_\pm = 1$  and  $T = 1$ , up to a normalization of the constants  $\tilde{L}$  and  $K^*$ . In particular, it is shown that a minimum  $\mu$  exists and for any  $t \in (-T, T)$  the measure  $\mu_t$  is composed of finitely many Dirac deltas. Moreover, minimizers contain no loops and can be thought of as branching trees that refine towards the boundaries  $t = \pm T$ . These results translate directly to any dimension  $d \in \mathbb{N}$ . A stronger result is known for  $d = 1$  [67]. For  $\bar{\mu}_- = \delta_0$  and  $\bar{\mu}_+ = 1$  the minimizer is unique and completely characterized: it is a self-similar branching tree, see Figure 1.5.

We now define for  $\lambda > 0$  and  $s \in (0, 1)$  the full energy

$$\mathcal{E}(\mu) = I(\mu, (-T, T)) + \lambda \|\mu_{\pm T} - 1\|_{\dot{H}^{-s}(\mathbb{T}_1^d)}^2 \quad (1.4.26)$$

and consider the problem

$$\inf_{\mu} \mathcal{E}(\mu). \quad (1.4.27)$$

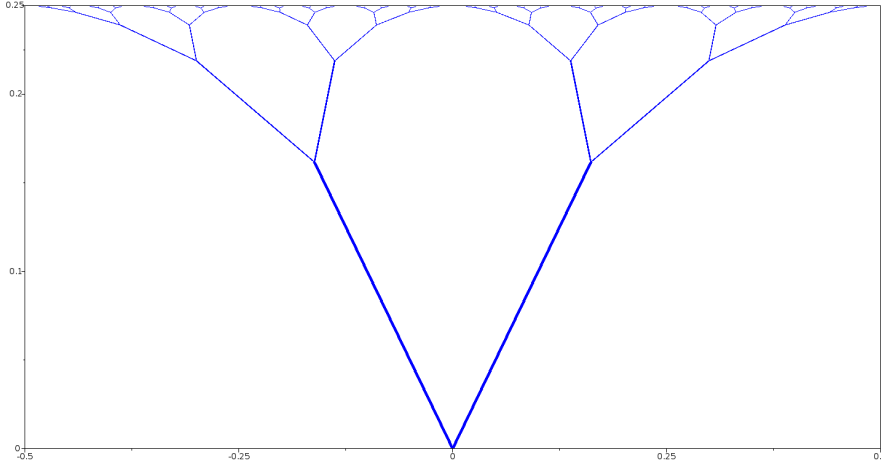


Figure 1.5: Minimizer for the 2D problem, time in the vertical axis. Picture from [67].

The case  $d = 2$ ,  $s = 1/2$  and  $T = 1$  corresponds to the limit model (1.4.20), which we derive from the full Ginzburg-Landau model in Chapter 2. In this case, boundary conditions are not enforced on  $\mu$ , but rather penalized with a weak norm. While the internal energy  $\mu$  favors concentration of the measure  $\mu$ , the boundary term favors diffused boundary measures  $\mu_{\pm T}$ . Based on the constructions performed in [26, 39], Conti, Otto and Serfaty conjectured the following [38] (see Chapter 4 for the definition of dimension of a measure):

**Conjecture 1.4.2.** *Let  $d = 2$ ,  $s = 1/2$  and  $\mu$  be a minimizer of  $\mathcal{E}(\mu)$ . Then*

$$\dim \mu_T = \frac{8}{5}.$$

In [48], bounds on the lower and upper dimension (see Chapter 4) of a minimizer  $\mu$  were related to local in time scaling laws. The following result was proved.

**Proposition 1.4.3.** *Let  $d = 2$ ,  $s = 1/2$  and  $\mu$  be a minimizer of  $\mathcal{E}(\mu)$ . Assume that for some  $\beta \in (0, 1)$*

$$\limsup_{\varepsilon \rightarrow 0} \varepsilon^{-\beta} I(\mu, (T - \varepsilon, T)) < \infty. \quad (1.4.28)$$

*Then*

$$f(\beta) = 1 + \frac{2\beta}{1 + \beta} \leq \underline{\dim} \mu_T \leq \overline{\dim} \mu_T \leq \frac{4(1 - \beta)}{1 + \beta} = g(\beta). \quad (1.4.29)$$

The bounds in (1.4.29) are sharp. Indeed proving (1.4.28) for  $\beta = 3/7$  would imply

$$f\left(\frac{3}{7}\right) = g\left(\frac{3}{7}\right) = \underline{\dim} \mu_T = \overline{\dim} \mu_T = \dim \mu_T = \frac{8}{5}.$$

The upper bound (1.4.28) for  $\beta = 3/7$  is the local counterpart of the global bound  $\mathcal{E}(\mu) \lesssim T^{3/7}$  which one has for  $0 < T \leq 1$  and  $\lambda = 1$ , see [48, Proposition 3.1]. Moreover, it was proved that (1.4.28) holds for  $\beta = 1/3$ , giving the diffusion bound

$$\underline{\dim} \mu_T \geq \frac{3}{2}.$$

Finally, in [48] a toy model was considered, fixing  $d = 1$  in (1.4.26) and replacing the boundary penalization by the Wasserstein distance (which morally behaves like a  $H^{-1}$  penalization, see

for instance [93])

$$\mathcal{E}^{1d}(\mu) = \int_{-T}^T \sum_i \varphi_i(t) |\dot{X}_i(t)|^2 dt + \lambda W^2(\mu_{\pm T}, 1).$$

In this case, it was proved that the measure  $\mu_T$  is atomic, as stated in the following proposition.

**Proposition 1.4.4.** *Let  $\mu$  be a minimizer of  $\mathcal{E}^{1d}(\mu)$ . Then  $\mu_T$  is a finite sum of Dirac deltas.*

As anticipated, the energy (1.4.24) shares some similarities with branched transport (or irrigation) problems. To explain these similarities, for simplicity we focus on the setting of atomic irrigated measures and compare the problems (1.4.25) and (1.3.6). In this case, by [37, Propositions 5.9 and 5.19], a minimum  $\mu$  for (1.4.25) is concentrated on a finite number  $M$  of segments  $\Gamma_i = \{(X_i(t), t), t \in [a_i, b_i]\}$  with mass  $\varphi_i$ , so that

$$\mu = \sum_{i=1}^M \frac{\varphi_i}{\sqrt{1 + |\dot{X}_i|^2}} \mathcal{H} \llcorner \Gamma_i \quad (1.4.30)$$

and the energy (1.4.24) takes the form

$$I(\mu, (-T, T)) = \sum_{i=1}^M \int_{a_i}^{b_i} \varphi_i^{\frac{d-1}{d}} + \varphi_i |\dot{X}_i|^2 dt. \quad (1.4.31)$$

To draw an analogy with (1.3.6) the third direction the graph (1.4.30) needs to be interpreted as a time direction. If we consider the graph  $G$  composed of the segments  $\tilde{\Gamma}_i = \{(X_i(t), t) \in [a_i, b_i]\}$  with weight  $\varphi_i$  and notice that  $\mathcal{H}^1(\tilde{\Gamma}_i) = \int_{a_i}^{b_i} |\dot{X}_i| dt$ , one has

$$M^\alpha(G) = \sum_{i=1}^M \int_{a_i}^{b_i} \varphi_i^\alpha |\dot{X}_i| dt. \quad (1.4.32)$$

Comparing (1.4.31) and (1.4.32), one may interpret (1.4.25) as a branched transport problem in which the "perimeter" term and the "velocity" term are decoupled. In particular, if  $\mu_+ = \mu_-$  the problem (1.4.32) is trivially solved by a "constant" graph  $G$  with null energy, while this does not happen in (1.4.31). In other terms, stationary particles in (1.4.31) still pay energy. In general, looking at (1.4.31) and (1.4.32),  $I(\mu)$  resembles the dynamical formulation of classical branched transport (1.3.9), given in [19] (see also [14], where a similar problem is formulated in terms of "path functionals"). Several tools and questions are shared between  $I(\mu)$  and the classical branched transport functional (no-loop condition, density of "regular" graphs). In particular, a key object which has proved useful lately is the so called landscape function, introduced in [103] and based on the lagrangian formulation of branched transport. A lagrangian reformulation of  $I(\mu)$  is also available (see [48, Section 4] and Proposition 4.2.5 in Chapter 4) and used to some extent in this manuscript. However, the precise analysis of such an object in our context is still an interesting topic for future work, see Section 1.6.

## 1.5 Summary of the original contributions

We now summarize the content of the next chapters.

### 1.5.1 Contributions of Chapter 2

In this chapter we study the rescaled functional introduced in Section 1.4.2

$$\begin{aligned} \tilde{E}_T(u, A) = & \frac{1}{\tilde{L}^2} \left[ \int_{\mathbb{T}_{\tilde{L},1}^2} \alpha^{-2/3} \beta^{-1/3} \left| \nabla'_{\alpha^{1/3} \beta^{-1/3} T A} u \right|^2 + \alpha^{-4/3} \beta^{-2/3} \left| \left( \nabla_{\alpha^{1/3} \beta^{-1/3} T A} u \right)_3 \right|^2 \right. \\ & \left. + \alpha^{2/3} \beta^{-2/3} \left( B_3 - (1 - |u|^2) \right)^2 + \beta^{-1} |B'|^2 dx + \alpha^{1/3} \beta^{7/6} \left\| \beta^{-1} B_3 - 1 \right\|_{H^{-1/2}(x_3=\pm 1)}^2 \right], \end{aligned}$$

in the regime

$$T, \alpha, \beta^{-1}, T\alpha^{-1} \rightarrow +\infty, \quad \alpha\beta^{7/2} \rightarrow \lambda^{1/3}.$$

Motivated by the discussions in Sections 1.4.2 and 1.4.3, the limit functional should be

$$\mathcal{E}(\mu) = I(\mu) + \lambda \left\| \mu_{\pm 1} - 1 \right\|_{H^{-1/2}(\mathbb{T}_{\tilde{L}}^2)},$$

which was presented in Section 1.4.4. As explained, in the case  $\lambda = +\infty$ , it was shown in [37] that the functional  $\tilde{E}_T$   $\Gamma$ -converges to the functional  $I(\mu)$  with uniform boundary conditions. We extend this result to the case  $\lambda < \infty$ , so that boundary conditions on  $\mu$  are not enforced, but rather penalized with a weak norm.

For a sequence  $(u_n, A_n)$ , denote  $B_n = \nabla \times A_n$  and  $B_3^n = (B_n)_3$ . Our main result is the following.

**Theorem 1.5.1.** *Consider  $\lambda, \tilde{L} > 0$ ,  $T_n, \alpha_n, \beta_n^{-1} \rightarrow +\infty$  with  $\alpha_n \beta_n^{7/2} \rightarrow \lambda^{1/3}$ . Let  $\mathcal{K}$  as in Section 3.2.2 and  $\mathcal{A}^*(\mathbb{T}_{\tilde{L}}^2, (-1, 1))$  as in Section 2.3. The following facts hold:*

1. *Let  $(u_n, A_n) \in \mathcal{K}$  such that  $\sup_{n \in \mathbb{N}} \tilde{E}_{T_n}(u_n, A_n) < +\infty$ . Then, there exists a measure  $\mu \in \mathcal{A}^*(\mathbb{T}_{\tilde{L}}^2, (-1, 1))$  of the form  $\mu = \mu_{x_3} \otimes dx_3$  where, for almost every  $x_3 \in (-1, 1)$  there holds  $\mu_{x_3} = \sum_i \varphi_i(x_3) \delta_{X_i(x_3)}$  and such that, up to a subsequence,  $\beta_n^{-1} B_3^n \rightharpoonup \mu$ ,  $\beta_n^{-1} (1 - |u_n|^2) \rightharpoonup \mu$ ,  $\beta_n^{-1} B_3^n(\cdot, \pm 1) \xrightarrow{H^{-1/2}} \mu_{\pm 1}$  and*

$$\liminf_{n \rightarrow +\infty} \tilde{E}_{T_n}(u_n, A_n) \geq \mathcal{E}(\mu).$$

2. *Assume in addition  $L_n^2 \alpha_n \beta_n T_n \in 2\pi\mathbb{Z}$ , where  $L_n = \tilde{L} \alpha_n^{-1/3} \beta_n^{-1/6}$ . For every measure  $\mu \in \mathcal{A}^*(\mathbb{T}_{\tilde{L}}^2, (-1, 1))$  such that  $\mathcal{E}(\mu) < +\infty$ , there exists  $(u_n, A_n) \in \mathcal{K}$  such that  $\beta_n^{-1} B_3^n \rightharpoonup \mu$ ,  $\beta_n^{-1} (1 - |u_n|^2) \rightharpoonup \mu$ ,  $\beta_n^{-1} B_3^n(\cdot, \pm 1) \xrightarrow{H^{-1/2}} \mu_{\pm 1}$  and*

$$\limsup_{n \rightarrow +\infty} \tilde{E}_{T_n}(u_n, A_n) \leq \mathcal{E}(\mu).$$

Let us comment on the proof of Theorem 1.5.1. The compactness and lower bound (Theorem 1.5.1-1) follow from a minor modification of [37, Proposition 6.1]. Setting

$$\tilde{E}_T(u, A) = \tilde{E}_T^{\text{int}}(u, A) + \alpha^{1/3} \beta^{7/6} \left\| \beta^{-1} B_3 - 1 \right\|_{H^{-1/2}(x_3=\pm 1)}^2,$$

that proposition gives compactness of the magnetic field in the sample and the lower bound

$$\liminf_{n \rightarrow +\infty} \tilde{E}_{T_n}^{\text{int}}(u_n, A_n) \geq I(\mu).$$

under the hypothesis  $\lambda = +\infty$ . With a small adaptation, we extend it so that it holds for any  $\lambda > 0$ . Theorem 1.5.1-1 then follows by weak compactness and weak lower semicontinuity of the negative Sobolev norm.

To tackle Theorem 1.5.1-2, as usual with  $\Gamma$ -convergence results, we prove first a density result, recall Proposition 1.3.4. In [37, Section 5], it was proved that "regular" measures with atomic boundary conditions well approximate measures with  $I(\mu) < +\infty$  and uniform boundary conditions with respect to the topology given by  $I(\mu)$ , which was a key factor in the construction of the recovery sequence. In our case, measures such that  $\mathcal{E}(\mu) < +\infty$  do not enjoy in general uniform boundary conditions, thus a new density argument is needed. Using an interpolation argument we prove that measures with boundary trace in  $H^{-1/2}$  can be approximated by measures whose trace is piecewise constant on cubes. For this last type of measure we can adapt the density arguments of [37, Section 5], to further approximate any measure with  $I(\mu) < \infty$  with "regular" measures in the "bulk".

With these density results at hand we can prove Theorem 1.5.1-2 only for measures  $\mu$  as described above. As in [37], we construct the density  $\rho$  and the magnetic field  $B$  separately in two regions. In the bulk, the measure  $\mu$  is regular and we can directly plug in the bulk construction from [37, Proposition 7.9]. In the boundary layer, we plug the construction from [39, Lemma 4.7], adapting the argument from [37, Proposition 7.7] to account for non uniform boundary conditions. Finally we can reconstruct the couple  $(u, A)$  by applying [37, Proposition 7.8].

## 1.5.2 Contributions of Chapter 3

This chapter is based on [42]. The starting point is the remark that the results of [37] are obtained under the hypothesis of vanishing Ginzburg-Landau parameter  $\kappa$ , that is under the hypothesis of penetration length much smaller than coherence length. In this chapter we want to examine the case of comparable coherence and penetration length, that is  $\kappa \in (0, 1/\sqrt{2})$ . We derive a functional which can be interpreted as a blow-up of the Ginzburg-Landau functional at the scale of coherence/penetration length.

We start from (1.4.5) and notice that for  $\kappa \in (0, 1/\sqrt{2})$  the term (1.4.15) can no longer be neglected. We consider the simplified setting  $T = +\infty$ . By symmetry, this implies that no quantity should depend on  $x_3$ . We can thus impose  $u(x) = u(x')$  and  $A(x) = (A_1(x'), A_2(x'), 0)$ , which implies  $B(x) = (0, 0, B_3(x'))$ . We can now consider the energy density on a slice  $\mathbb{T}_L^2 \times \{x_3\}$ , which with abuse of notation we still call  $F(u, A)$ . Plugging in (1.4.5) these assumptions on  $u$  and  $A$  we get

$$F(u, A) = \int_{\mathbb{T}_L^2 \times \{x_3\}} (1 - \kappa\sqrt{2}) |\nabla'_A u|^2 + \kappa\sqrt{2} |\mathcal{D}_A^3 u|^2 + \left( B_3 - \frac{\kappa}{\sqrt{2}}(1 - |u|^2) \right)^2 dx.$$

We can see that  $F(u, A)$  only encapsulates the "Modica-Mortola" part of the Ginzburg-Landau functional. Since we expect the functional to behave like a perimeter term in 2D, we rescale it by the typical length  $L$ . Dropping the prime superscripts and calling  $B = B_3$  we get

$$\mathcal{E}(u, A) = \frac{1}{L} F(u, A) = \frac{1}{L} \int_{\mathbb{T}_L^2} (1 - \kappa\sqrt{2}) |\nabla_A u|^2 + \kappa\sqrt{2} |\mathcal{D}_A^3 u|^2 + \left( B - \frac{\kappa}{\sqrt{2}}(1 - |u|^2) \right)^2 dx$$

under the constraint

$$\frac{1}{L^2} \int_{\mathbb{T}_L^2} B dx = b_{\text{ext}}.$$

Finally, we consider the rescaling  $x = L\hat{x}$ ,  $A(x) = \kappa L \hat{A}(\hat{x})$ ,  $u(x) = \hat{u}(\hat{x})$  and we set  $\varepsilon = (\kappa L)^{-1}$ ,

so that  $\nabla_A u = (L)^{-1} \hat{\nabla}_{\kappa^{-1}\varepsilon^{-2}\hat{A}} \hat{u}$  and  $B = \kappa \hat{B}$ . We find (dropping hats)

$$\begin{aligned} \mathcal{E}_\varepsilon(u, A) = \int_{\mathbb{T}^2} \varepsilon((1 - \kappa\sqrt{2}) |\nabla_{\kappa^{-1}\varepsilon^{-2}A} u|^2 + \kappa\sqrt{2} |\mathcal{D}_{\kappa^{-1}\varepsilon^{-2}A}^3 u|^2) \\ + \frac{1}{\varepsilon} \left( B - \frac{1}{\sqrt{2}}(1 - |u|^2) \right)^2 dx, \end{aligned} \quad (1.5.1)$$

with

$$\int_{\mathbb{T}^2} B dx = \frac{b_{\text{ext}}}{\kappa}.$$

In this chapter, to align with the notation of [42], the density of superconducting electrons is  $\rho = |u|$  (instead of  $\rho = |u|^2$ ). As explained, this functional can be understood as a blow-up at the scale of coherence and penetration length in the regime where they are comparable, that is for  $\kappa \in (0, 1/\sqrt{2})$  fixed. Similarly to Chapter 2, we expect the functional to act as a perimeter term as  $\varepsilon \rightarrow 0$ . However, differently from Chapter 2, coherence and penetration length being comparable means that the scale at which the  $u$  transitions is comparable to the scale from which the Meissner condition  $\rho B \simeq 0$  holds, see the discussion in Section 1.4.3. Recalling that, if  $u = \rho e^{i\theta}$ ,

$$|\nabla_{\alpha A} u|^2 = |\nabla \rho|^2 + \rho^2 |\nabla \theta - \alpha A|^2, \quad (1.5.2)$$

the second term in (1.5.2) acts as global coupling between  $\theta$  and  $A$  which, differently from Chapter 2, plays a role in the phase transition. This also makes the problem of vectorial nature and similar to generalizations of the standard Modica-Mortola theorem which have been studied in recent years [25, 36, 64, 59, 40, 44, 45]. Our main result is the following.

**Theorem 1.5.2.** *Let  $\kappa \in (0, 1/\sqrt{2})$ ,  $b_{\text{ext}} < \kappa/\sqrt{2}$  and consider  $\varepsilon_n \rightarrow 0$ . Denoting by  $\sigma_\kappa$  the constant defined in Section 3.4 and by  $\mathcal{A}_{\varepsilon_n}$  the space of admissible configurations defined in Section 3.2.2, then the following holds:*

1. *Let  $(u_n, A_n) \in \mathcal{A}_{\varepsilon_n}$  be a sequence such that  $\sup_n \mathcal{E}_{\varepsilon_n}(u_n, A_n) < \infty$ . Then, up to a subsequence,  $|u_n| = \rho_n \rightarrow \rho$  in  $L^p(\mathbb{T}^2)$  for all  $1 \leq p < \infty$ , with  $\rho \in BV(\mathbb{T}^2, \{0, 1\})$  and  $B_n = \nabla \times A_n \rightarrow (1 - \rho)/\sqrt{2}$  in  $L^2(\mathbb{T}^2)$ .*
2. *Let  $(u_n, A_n) \in \mathcal{A}_{\varepsilon_n}$  be a sequence such that  $|u_n| = \rho_n \rightarrow \rho$  in  $L^p(\mathbb{T}^2)$  for all  $1 \leq p < \infty$ , with  $\rho \in BV(\mathbb{T}^2, \{0, 1\})$  and  $B_n = \nabla \times A_n \rightarrow (1 - \rho)/\sqrt{2}$  in  $L^2(\mathbb{T}^2)$ . Then*

$$\liminf_n \mathcal{E}_{\varepsilon_n}(u_n, A_n) \geq \sigma_\kappa P(\{\rho = 0\}, \mathbb{T}^2).$$

3. *Suppose  $\varepsilon_n^{-2} \kappa^{-2} b_{\text{ext}} \in 2\pi\mathbb{Z}$ . Let  $\rho \in BV(\mathbb{T}^2, \{0, 1\})$  with  $\int_{Q_1} (1 - \rho)/\sqrt{2} dx = b_{\text{ext}}/\kappa$ . Then there exists  $(u_n, A_n) \in \mathcal{A}_{\varepsilon_n}$  sequences such that  $|u_n| = \rho_n \rightarrow \rho$  in  $L^p(\mathbb{T}^2)$  for all  $1 \leq p < \infty$ ,  $B_n = \nabla \times A_n \rightarrow (1 - \rho)/\sqrt{2}$  in  $L^2(\mathbb{T}^2)$ ,  $\int_{Q_1} B_n dx = b_{\text{ext}}/\kappa$  and*

$$\limsup_n \mathcal{E}_{\varepsilon_n}(u_n, A_n) \leq \sigma_\kappa P(\{\rho = 0\}, \mathbb{T}^2).$$

Let us now comment on the strategy of the proof. Compactness does not vary significantly from [37] and Chapter 2. We exploit a weak version of the Meissner effect (Proposition 3.3.2) to prove that the functional (3.1.1) can be estimated from below by a classical Modica-Mortola type functional in  $\rho$ .

For the  $\Gamma$ -lim inf, we can no longer use the classical Modica-Mortola trick, and we resort to a blow up procedure [58, 13]. We define  $\sigma_\kappa$  implicitly as a minimization problem on a unit cell where the interface is a vertical segment.

Next, we prove a characterization for  $\sigma_\kappa$  which will be useful in the construction of the recovery sequence. This is the main part of the paper and is divided in two steps.

In the first step, as in typical Modica-Mortola type results, we show that for the minimization problem on the cell  $\sigma_\kappa$  can be interpreted as an optimal profile problem  $\sigma_{\kappa,\delta}$  with fixed vertical boundary conditions at a distance  $\delta$  from the transition. The proof of this result is however delicate in our setting. In particular, extra care is necessary because of the global coupling term in (1.5.2). Furthermore, we account for the gauge invariance of the functional by fixing an appropriate gauge for which the result holds.

In the second step, we prove that, up to a small cost, we can impose horizontal periodic boundary conditions. In particular, we impose that  $u$  is equal to a real profile  $U$  near the top and bottom boundaries. We actually conjecture that the optimal transition profile itself is one dimensional, in analogy with some previous results [24, 23]. Finally, we prove the  $\Gamma$ -lim sup inequality. By density of polyhedral sets in the class of finite perimeter sets [80], we can suppose that the set  $E = \{\rho = 0\}$  is polyhedral (recall Proposition 1.3.4). Again in analogy with Modica-Mortola, we "glue" a rescaled competitor  $(u_0, A_0)$  of  $\sigma_\kappa$  on a  $\varepsilon_n$ -neighborhood of  $\partial E$ . Then, we globally construct the physical quantities  $\rho_n$  and  $B_n$  so that they coincide with the wells of the functional  $\mathcal{E}_{\varepsilon_n}$  far from  $\partial E$ . The main difficulty of the construction is then to appropriately define the phase  $\theta_n$  and the potential  $A_n$  so that  $u_n = \rho_n e^{i\theta_n}$  and  $A_n$  match the local values  $u_0, A_0$  on an  $\varepsilon_n$  neighborhood of  $E$ . In particular the connected components of  $E$  act as "poles" for the phase  $\theta_n$ , and force the quantization condition  $\varepsilon_n^{-2} \kappa^{-2} b_{\text{ext}} \in 2\pi\mathbb{Z}$ . As explained in Section 1.4.2, the quantization condition is a consequence of the fact that the phase  $\theta$  in the full Ginzburg-Landau model is naturally quantized. Let us remark that the condition is less and less stringent as  $\varepsilon_n \rightarrow 0$ .

### 1.5.3 Contributions of Chapter 4

This chapter is based on [41]. We consider a minimum  $\mu$  of (1.4.26). Our goal is to prove results on the dimension of  $\mu$ . If  $d = 2$  and  $s = 1/2$ , by Proposition 1.4.3, in order to prove Conjecture 1.4.2 it is in theory enough to prove that (1.4.28) holds for  $\beta = 3/7$ . However, a direct proof of a sharp upper bound seems, to the best of our knowledge, too optimistic. Calling  $\bar{\alpha}$  the conjectured dimension, the strategy we propose is the following (see Section 4.3 for the definition of the dimension of a measure):

- $\dim \mu_T \leq \bar{\alpha}$  should result from the combination of an interpolation argument reminiscent of the proof of the lower bound in (1.4.13) and of the upper bound in (1.4.29), together with a first variation argument, see Theorem 1.5.5 below.
- $\dim \mu_T \geq \bar{\alpha}$  should instead result from a combination of a construction, reminiscent of the argument for the upper bound in (1.4.13), see Theorem 1.5.6 below, together with another first variation argument, similar to the one used in the proof of the lower bound in (1.4.29).

We perform our argument for  $d = 1$  but for any  $s \in (0, 1)$ . In this case, based on considerations analog to the case  $d = 2$ , see Remark 4.3.11, the conjectured dimension is

$$\bar{\alpha}(s) = \min \left( \frac{4}{3}(1-s), 1 \right).$$

Notice that the case  $s = 1$  is coherent with Proposition 1.4.4. Intuitively, if  $s$  large, the boundary penalization is weak and the boundary measure is concentrated, whereas for  $s$  small ( $s < 1/4$ ) the boundary penalization is strong hence the boundary measure is diffused.

Since our arguments do not localize, we need to further assume that the irrigated measure  $\mu_T$  is Ahlfors regular. For  $\alpha \in [0, d]$ , we say that a measure  $\sigma$  is upper  $\alpha$ -Ahlfors regular if for some  $M > 0$ ,

$$\sigma(B_r(x)) \leq Mr^\alpha \quad \text{for } x \in \text{supp } \sigma \text{ and } r \in (0, 1).$$

Similarly we say that it is lower  $\alpha$ -Ahlfors regular if

$$\sigma(B_r(x)) \geq Mr^\alpha \quad \text{for } x \in \text{supp } \sigma \text{ and } r \in (0, 1).$$

If both conditions hold (possibly for different  $M, m > 0$  respectively), we say that  $\sigma$  is  $\alpha$ -Ahlfors regular.

Our main result is the following.

**Theorem 1.5.3.** *Let  $d = 1$ ,  $s \in (0, 1)$  and  $\mu$  be a minimizer of  $E_{s,T}$ . If  $s \leq 1/4$  then (see Section 4.3 for precise definitions)  $\underline{\dim} \mu_T = 1$ . If instead  $s > 1/4$  and  $\mu_T$  is upper  $\alpha$ -Ahlfors regular, then*

$$\alpha \leq \bar{\alpha} \tag{1.5.3}$$

whereas if  $\mu_T$  is lower  $\alpha$ -Ahlfors regular, then

$$\alpha \geq \bar{\alpha}. \tag{1.5.4}$$

**Remark 1.5.4.** *Theorem 1.5.3 implies in particular that if  $\mu_T$  is  $\alpha$ -Ahlfors regular for  $\alpha \in (0, 1)$ , then*

$$\dim \mu_T = \bar{\alpha}.$$

*In particular,  $\mu_T$  cannot be smooth i.e. we cannot have  $\mu_T = f dx$  for some  $f \in L^\infty$ .*

Let us now comment on the proof of Theorem 1.5.3. As explained above, the proof of (1.5.3) is a combination of an interpolation argument and a first variation argument: the interpolation argument is related to the proof of the lower bound in the global scaling law (1.4.13), which we operatively apply using the upper bound in (1.4.29). The first variation argument is contained in the proof of Theorem 1.5.5 below. To prove (1.5.4) we combine instead the first variation argument which gives the lower bound in (1.4.29), together with a construction which is reminiscent of the upper bound construction for the global scaling law (1.4.13). This construction is contained in Theorem 1.5.6.

For both Theorem 1.5.5 and Theorem 1.5.6 we need to assume  $d = 1$ . However since large portions of the proofs are valid for arbitrary dimensions, we decided when possible to keep the presentation general. Let us now comment on the two bounds separately. The first result that we prove, in order to prove (1.5.3), is the following local scaling law.

**Theorem 1.5.5.** *Let  $d = 1$  and  $s > 1/4$ . Let  $\mu$  be a minimizer of  $E_{s,T}$  and assume that  $\mu_T$  is upper  $\alpha$ -Ahlfors regular for some  $\alpha > 1 - 2s$ . Then, for any  $\varepsilon \in (0, T)$  and*

$$\beta < \beta_{\text{reg}}(s, \alpha) = \frac{2s - 1 + \alpha}{2(1 - s) + 1 - \alpha}$$

*there holds*

$$I(\mu, (T - \varepsilon, T)) \lesssim_{\mu, \beta} \varepsilon^\beta.$$

To prove Theorem 1.5.5, we first establish some properties of upper  $\alpha$ -Ahlfors regular measures. We prove that this class of measures embeds into some negative Sobolev space. Then, we prove that  $\alpha$ -Ahlfors regularity is preserved by McCann interpolation when one of the two interpolated measures enjoys the property. This result, which is a generalization of

$L^\infty$  bounds on interpolated measures (see [108, Theorem 8.7 & Corollary 19.5]), could be of independent interest. Then, we build a competitor by taking a minimizer  $\mu$  and shrinking by a factor its branches in the interval  $[T - \varepsilon, T]$ . This keeps the perimeter part unchanged and reduces the kinetic energy. Since we assume  $d = 1$ , the transport plan induced by  $\mu$  is the optimal one and the measure that this new competitor irrigates is the McCann interpolant of  $\mu_{T-\varepsilon}$  and  $\mu_T$ , recall (1.3.5). Thus, we can estimate the boundary energy by the Wasserstein distance using interpolation inequalities and the additional regularity properties of upper  $\alpha$ -Ahlfors regular measures.

Let us now get to the proof of (1.5.4). In this case we have the following local scaling law.

**Theorem 1.5.6.** *Let  $d = 1$  and  $s > 1/4$ . Let  $\mu$  be a minimizer of  $E_{s,T}$  and assume that  $\mu_T$  is lower  $\alpha$ -Ahlfors regular. Then, for any  $\varepsilon \in (0, T)$  and for*

$$\beta_{\text{con}}(\alpha) = \frac{2 - \alpha}{2 + \alpha}$$

there holds

$$I(\mu, (T - \varepsilon, T)) \lesssim_{M, \alpha} \varepsilon^{\beta_{\text{con}}}.$$

The proof of Theorem 1.5.6 is based on the construction of a competitor in  $[T - \varepsilon, T]$ . To this extent we notice that lower  $\alpha$ -Ahlfors regular measure are supported on a  $\alpha$  dimensional set. Moreover, if  $d = 1$ , minimizers enjoy the so-called "cone property" (see Section 4.2.4). This property guarantees a localization property of the branched transport plan. This allows us to make a local construction in the spirit of [48, Theorem 1.3], from which Theorem 1.5.6 follows.

#### 1.5.4 Contributions of Chapter 5

The goal of Chapter 5 is to study an energy similar to (1.4.26), in the idealized case of a superconducting sample which is the half space  $\mathbb{R}^2 \times (0, +\infty)$ . We represent the magnetization by a measure  $\mu = \mu_t \otimes dt$  where, for a.e.  $t \in (0, +\infty)$ ,  $\mu_t = \sum_i \varphi_i \delta_{X_i}$ , with  $X_i \in \mathbb{R}^2$ ,  $\varphi_i > 0$  and  $\sum_i \varphi_i = \Phi > 0$ . We consider the internal energy (see Section 5.2.3 for the precise definition)

$$I(\mu) = \int_0^{+\infty} \sum_i \varphi_i^{1/2}(t) - \Phi^{1/2} + \sum_i \varphi_i(t) |\dot{X}_i(t)|^2 dt, \quad (1.5.5)$$

where  $\dot{X}_i(t)$  is the derivative of the map  $t \mapsto X_i(t) \in \mathbb{R}^2$ . Energy (1.5.5) may be interpreted as a "blow-up" of energy (1.4.26). Similarly to (1.4.26), the first term in (1.5.5) can be seen as a perimeter term that favors mass traveling together. Since we deal with an unbounded domain, we renormalize the energy by the perimeter of a Dirac delta with the total mass of  $\mu_t$ , which is  $\Phi^{1/2}$ . The last term in (1.5.5) is the kinetic energy, exactly like (1.4.26). We impose a weak boundary penalization with a negative Sobolev norm (see Section 5.2.2) and consider the energy

$$\mathcal{E}(\mu) = I(\mu) + \|\mu_0\|_{H^{-1/2}(\mathbb{R}^2)}^2. \quad (1.5.6)$$

Since the energy (1.5.6) is translation invariant, we may assume  $\int_{\mathbb{R}^2} x d\mu_0 = 0$ . Moreover, we fix the mass  $\mu_0(\mathbb{R}^2) = 1$  and consider the minimization problem

$$e = \inf \left\{ \mathcal{E}(\mu) : \int_{\mathbb{R}^2} x d\mu_0 = 0, \mu_0(\mathbb{R}^2) = 1 \right\}. \quad (1.5.7)$$

Problem (1.5.7) is an analogue to problem (1.4.27) in an unbounded domain. The internal energy  $I(\mu)$  favors measures which are concentrated, while the boundary term  $\|\mu_0\|_{H^{-1/2}(\mathbb{R}^2)}$

favors a diffused boundary measure. Based on the same considerations as in [38], we may thus formulate the following analogue of Conjecture 1.4.2 (See Chapter 4 for the definition of dimension of a measure).

**Conjecture 1.5.7.** *Let  $\mu$  be a minimizer of (1.5.7). Then,*

$$\dim \mu_0 = \frac{8}{5}.$$

For the bounded problem (1.4.27) the above conjecture was linked to local scaling laws of the internal energy in [48], and to Ahlfors regularity in Chapter 4 for an equivalent 2D model. In this chapter, we prove a result similar to the concentration result (1.5.3) in Theorem 1.5.3, in the case of energy (1.5.6), which is a 3D model (in the simplified setting of the half space). We recall that a measure  $\sigma \in \mathcal{M}^+(\mathbb{R}^2)$  is called upper  $\alpha$ -Ahlfors regular for  $\alpha \in [0, 2]$  if there exists  $M > 0$  such that

$$\sigma(B(x, r)) \leq Mr^\alpha \quad \text{for all } x \in \text{supp } \sigma \text{ and } r \in (0, 1].$$

Our main result is the following.

**Theorem 1.5.8.** *Let  $\mu$  be a minimizer of (1.5.7). Suppose that there exist  $x_0 \in \mathbb{R}^2$  and  $r_0 > 0$  such that  $\mu_0(B(x_0, r_0/2)) > 0$  and  $\mu_0 \llcorner B(x_0, r_0)$  is upper  $\alpha$ -Ahlfors regular for some  $\alpha > 1, M > 0$ . Then  $\alpha \leq 8/5$ .*

**Remark 1.5.9.** *An immediate consequence of Theorem 1.5.8 is that  $\mu_0$  cannot be upper  $\alpha$ -Ahlfors regular for  $\alpha > 8/5$ . In particular,  $\mu_0 \notin L^\infty(\mathbb{R}^2)$ .*

Theorem 1.5.8 states, roughly speaking, that in any ball in which some mass of the boundary measure is present, some part of such measure needs to concentrate up to the conjectured dimension  $8/5$ . If we compare with (1.5.3) in Theorem 1.5.3, our result excludes that there exist large portions of the support of  $\mu_0$  in which  $\mu_0$  is diffused, which a priori cannot be excluded in the case of Theorem 1.5.3.

We now comment on the strategy of the proof. While the setting of the half space offers some advantages with respect to the periodic setting considered in [37, 48, 41], namely an easier understanding of the weak boundary norm, this advantage comes with a price: loss of compactness. We thus cannot directly generalize the proof of [37] to obtain existence of minimizers. To overcome this problem, we proceed as follows. First, we fix  $R, T > 0$  and consider the following minimization problem in a compact setting:

$$e(R, T) = \inf \left\{ \mathcal{E}(\mu) : \int_{\mathbb{R}^2} x \, d\mu_0 = 0, \text{supp } \mu_0 \subset [-R/2, R/2]^2, \mu_T = \delta_0 \right\}. \quad (1.5.8)$$

For (1.5.8), existence is easily adapted from [37]. Then, with the help of a Lagrangian reformulation of the problem (Proposition 5.3.8), we are able to prove a priori bounds (Lemma 5.3.12) for a minimizer of (1.5.8) which do not depend on  $R$  and  $T$ . This enables us to prove that there exists  $R_0, T_0 > 0$  such that a minimum of (1.5.8) is a minimum of (1.5.7) (Lemma 5.3.15). We thus show that (1.5.7) admits at least one minimum  $\mu$  which is compact in space and for which  $\mu_t = \delta_0$  for all  $t \geq T_0$ .

With the existence of a minimizer at hand, we now comment on the proof of Theorem 1.5.8. We consider a ball  $B(x_0, r_0/2)$  of positive mass on which the boundary measure is upper  $\alpha$ -Ahlfors regular. We focus on a subsystem (Proposition 5.2.3) contained in the ball  $B(x_0, r_0/2)$ , which can be found by the a priori bounds in Lemma 5.3.12. Up to dealing with the barycenter contribution of the energy (Lemma 5.3.1), we construct a competitor similar

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to the one used in the proof of (1.5.3) in Theorem 1.5.3: we shrink the subsystem around its barycenter. This keeps the perimeter unchanged and reduces the kinetic energy. For the boundary term we make use of the following crucial observation: if a measure  $\sigma \in \mathcal{M}^+(\mathbb{R}^2)$  is upper  $\alpha$ -Ahlfors regular for  $\alpha \in [1, 2]$ , then its electrostatic potential  $u : \mathbb{R}^2 \rightarrow \mathbb{R}$ , defined as

$$u(x) = \int_{\mathbb{R}^2} \frac{1}{|x - y|} d\mu(y),$$

must be  $(\alpha - 1)$ -Hölder continuous (Proposition 5.4.2). Such a result may be interpreted as a sort of Schauder estimate, see Remark 5.4.3. The regularity of the potential is then used in arguing that the local structure of the minimizer is not much affected by the non-local interactions of the boundary penalization. Finally, the contradiction with  $\alpha > 8/5$  arises by comparing the mass of the subsystem, which is arbitrarily small, and the first moment of the local boundary measure.

## 1.6 Open problems

To finish this introduction, we review some interesting questions and open problems.

**Different regimes and profiles in the  $\Gamma$ -convergence results.** Concerning the results of Chapters 2 and 3, a natural question which is left open is whether it is possible to extend the results of Chapter 2 to the regime considered in Chapter 3, that is the regime of comparable penetration and coherence length. As explained, in this setting the failure of the Meissner condition plays a bigger role. Looking at the proof of the upper bound in Chapter 2, the main difference would be that the optimal transition profile which we plug at the scale of coherence length would no longer be explicit, but rather one should be able to use the recovery sequence constructed in Chapter 3. Also the proof of the  $\Gamma$ -liminf should differ; instead of the Modica-Mortola trick, one should probably employ a blow-up procedure as in Chapter 2.

As for the results of Chapter 3 itself, the main open question is whether the optimal profile associated with the energy (1.5.1) is one dimensional. Let  $\rho, A : \mathbb{R} \rightarrow \mathbb{R}$  and consider for  $a < b \in \mathbb{R} \cup \{\pm\infty\}$

$$\mathcal{E}^{1d}(\rho, A, (a, b)) = \int_a^b |\dot{\rho}|^2 + \kappa^{-2} \rho^2 A^2 + \left( \dot{A} - \frac{1}{\sqrt{2}}(1 - \rho^2) \right)^2 dt.$$

In [23, 24], the following one dimensional phase transition problem for type-I superconductors is studied:

$$\sigma_\kappa^{1d} = \inf \left\{ \mathcal{E}^{1d}(\rho, A, (-\infty, +\infty)), (\rho, A)(-\infty) = (1, 0), (\rho, \dot{A})(+\infty) = \left( 0, \frac{1}{\sqrt{2}} \right) \right\}. \quad (1.6.1)$$

Notice that if we consider  $\tilde{u}(x_1, x_2) = \rho(x_1)$  and  $\tilde{A}(x_1, x_2) = [0, A(x_1)]$ , comparing with (1.5.1) one has

$$\mathcal{E}_1(\tilde{u}, \tilde{A}) = \mathcal{E}^{1d} \left( \rho, A, \left( -\frac{1}{2}, \frac{1}{2} \right) \right).$$

We conjecture that the optimal profile associated with (1.5.1) is actually the one given by (1.6.1), which would thus imply that the transition in our problem is one dimensional. However due to the vectorial nature of the problem, this seems to be a difficult question.

**The 8/5 conjecture, the landscape function and other "fractal" problems.** Several questions are open for the limit functional (1.4.26). Of course, Conjecture 1.4.2 is still open. Our results of Chapters 4 and 5 are effective only under some hypothesis of Ahlfors regularity, which is a strong assumption. In particular our analysis is not able to treat measures which

have two parts of "different dimension". For instance, for  $\alpha \neq \beta \in (3/2, 2)$ , a boundary measure

$$\mu_T = \mu_1 + \mu_2 \tag{1.6.2}$$

with  $\dim \mu_1 = \alpha$  and  $\dim \mu_2 = \beta$  is a priori not excluded. In general, performing variations in which only one of the two "dimensional parts" in (1.6.2) varies seems difficult.

A key object which has not been studied deeply in this manuscript is the so called landscape function. Consider the problem (1.4.25) with  $\mu_{\pm} = \bar{\mu} \in \mathcal{P}(\mathbb{T}^2)$ . With the help of a Lagrangian reformulation (see Section 5.3.3 and also Proposition 4.2.5), it is possible to prove that the energy of a minimum  $\mu$  can be written as

$$I(\mu, (-T, T)) = \int_{\mathbb{T}^2} z_{\mu}(x) d\bar{\mu}(x),$$

where we call  $z_{\mu} : \text{supp } \bar{\mu} \rightarrow [0, +\infty]$  the landscape function associated with  $\mu$ . An analogue of such an object is now a key tool in the domain of classical branched transport. Introduced in [103], its regularity has been used to prove results on questions which are similar in spirit to Conjecture 1.4.2. In particular we mention [92, 91]. In [92] a "unit ball" for a branched transport problem is studied and its boundary is conjectured to have a fractal behavior. Similarly in [91] the concept of Voronoi cells for branched transport is introduced. Again, interfaces between cells are conjectured to have fractal behavior. Motivated by these results, we think that the study of the landscape function could be an important tool in trying to solve Conjecture 1.4.2. We also recall [16], in which the regularity of the landscape function is linked with the Ahlfors regularity of the irrigated measure. Finally, the problem of the "unit ball" and of Voronoi cells are natural questions which could be interesting to formulate and study also for our functional.



## Chapter 2

# $\Gamma$ -convergence of the Ginzburg-Landau functional to a branched transport problem with weakly imposed boundary conditions

### 2.1 Introduction

The goal of this chapter is to prove the  $\Gamma$ -convergence of the functional (1.4.18) to the functional (1.4.26). We recall briefly the setting introduced in Section 1.4.2. We consider for  $L_0, T > 0$  the setting of a sample  $\Omega = \mathbb{T}_{L_0, T}^2 = \mathbb{T}_{L_0}^2 \times (-T, T)$  with  $\mathbb{T}_{L_0}^2 = \mathbb{R}^2 / (L_0 \mathbb{Z}^2)$ , that is a box with lateral periodic boundary conditions and thickness  $2T$ , and of an external applied magnetic field  $B_{\text{ext}} = b_{\text{ext}} e_3$  perpendicular to the sample (see Section 2.2.3 for the precise functional setting). After rescaling and subtracting a bulk term and a lower order term (see Section 1.4.2), minimizing the full Ginzburg-Landau functional is equivalent to minimizing

$$E_T(u, A) = \frac{1}{L^2} \int_{\mathbb{T}_{L,1}^2} |\nabla_{TA} u|^2 + (B_3 - \alpha(1 - \rho))^2 + |B'|^2 dx + \|B_3 - \alpha\beta\|_{H^{-\frac{1}{2}}(x_3=\pm 1)}^2, \quad (2.1.1)$$

where  $\rho = |u|$ ,  $B = (B', B_3) = \nabla \times A$  and

$$\alpha = \frac{\kappa T}{\sqrt{2}}, \quad \beta = \frac{b_{\text{ext}} \sqrt{2}}{\kappa}, \quad L = \frac{L_0}{T}. \quad (2.1.2)$$

As explained in Section 1.4.2, the term

$$\frac{1}{L^2} \int_{\mathbb{T}_{L,1}^2} |\nabla_{TA} u|^2 + (B_3 - \alpha(1 - \rho))^2 dx, \quad (2.1.3)$$

paired with the Meissner condition  $\rho B = 0$ , may be interpreted as a non standard Modica Mortica type functional in  $\rho$  and  $B$ , the term

$$\int_{\mathbb{T}_{L,1}^2} |B'|^2 dx,$$

paired with the condition  $\text{div } B = 0$  can be seen as a transport cost similar to the Benamou-Brenier Formula (1.3.4) and the term

$$\|B_3 - \alpha\beta\|_{H^{-1/2}(x_3=\pm 1)}^2$$

favors a uniform distribution of the magnetic field at the boundary. Inside the sample we expect the superconducting phase  $\rho \simeq 1$ ,  $B \simeq 0$  to alternate with the normal one  $\rho \simeq 0$ ,  $B_3 \simeq \alpha$  and refine towards the boundary, leading to complex patterns. We recall that by Theorem 1.4.1 and the discussion below, one has for  $T \gg 1$ ,  $\alpha \gg 1$  and  $\beta \ll 1$

$$\min E_T(u, A) \sim \min\{\alpha^{4/3}\beta^{2/3}, \alpha^{10/7}\beta\}. \quad (2.1.4)$$

The first regime in (2.1.4) corresponds to uniform branching patterns whereas the second one corresponds to non uniform branching patterns.

We put ourselves in the setting

$$T \gg 1, \quad \alpha \gg 1, \quad \beta \ll 1, \quad \alpha T^{-1} \ll 1, \quad \alpha\beta^{7/2} \sim \lambda^{1/3} \quad (2.1.5)$$

for some  $\lambda > 0$ . Motivated by the discussion in Section 1.4.3, by the same anisotropic rescaling as [37] we may reduce our analysis, setting

$$\tilde{L} = L\alpha^{1/3}\beta^{1/6}, \quad (2.1.6)$$

to the functional

$$\begin{aligned} \tilde{E}_T(u, A) = & \frac{1}{\tilde{L}^2} \left[ \int_{\mathbb{T}_{\tilde{L},1}^2} \alpha^{-2/3}\beta^{-1/3} \left| \nabla'_{\alpha^{1/3}\beta^{-1/3}TA} u \right|^2 + \alpha^{-4/3}\beta^{-2/3} \left| \left( \nabla_{\alpha^{1/3}\beta^{-1/3}TA} u \right)_3 \right|^2 \right. \\ & \left. + \alpha^{2/3}\beta^{-2/3} \left( B_3 - (1 - |u|^2) \right)^2 + \beta^{-1} |B'|^2 dx + \alpha^{1/3}\beta^{7/6} \left\| \beta^{-1} B_3 - 1 \right\|_{H^{-1/2}(\{x_3=\pm 1\})}^2 \right], \end{aligned} \quad (2.1.7)$$

where for any  $f(x', x_3) \in L^2(\mathbb{T}_{\tilde{L},1}^2)$  we use the convention (see Section 2.2.2)

$$\|f\|_{H^{-1/2}(\{x_3=\pm 1\})}^2 = \|f(\cdot, 1)\|_{H^{-1/2}(\mathbb{T}_{\tilde{L}}^2)}^2 + \|f(\cdot, -1)\|_{H^{-1/2}(\mathbb{T}_{\tilde{L}}^2)}^2.$$

In [37], the case  $\lambda = +\infty$  in (2.1.5) is studied. This implies  $\alpha^{-2/7} \ll \beta$  so that, by (2.1.4),  $\min E_T(u, A) \sim \alpha^{4/3}\beta^{2/3}$ , corresponding to the uniform branching regime. It is proved that the functional (2.1.7)  $\Gamma$ -converges towards the functional  $I$  introduced in Section 1.4.4. If  $x = (x', x_3) \in \mathbb{T}_{\tilde{L},1}^2$ , the normal phase can be represented by measures  $\mu = \mu_{x_3} \otimes dx_3$  where, for a.e.  $x_3 \in (-1, 1)$ ,  $\mu_{x_3} = \sum_i \varphi_i \delta_{X_i}$ . The curves  $x_3 \rightarrow X_i(x_3) \in \mathbb{T}_{\tilde{L}}^2$  represent the branches of the tree and the weights  $\varphi_i > 0$  represent the flux carried by each branch. Let  $K^* = 8\sqrt{\pi}/3$ . The functional  $I$  is defined as

$$I(\mu) = \frac{1}{\tilde{L}^2} \int_{-1}^1 K^* \sum_i \varphi_i^{1/2} + \varphi_i \left| \dot{X}_i \right|^2 dx_3, \quad (2.1.8)$$

where  $\dot{X}_i(x_3)$  is the derivative of the map  $x_3 \rightarrow X_i(x_3)$  (see Section 2.3 for the precise definition). Moreover, since  $\alpha^{1/3}\beta^{7/6} \gg 1$ , the last term in (2.1.7) forces  $\mu_{\pm 1} = dx'$  in an appropriate trace sense.

We consider the case  $\lambda < \infty$ , corresponding to the crossover regime  $\alpha^{-2/7} \sim \beta$  in (2.1.4). Since now the quantity  $\alpha^{1/3}\beta^{7/6}$  stays bounded, the last term in (2.1.7) suggests that the limit functional should be

$$\mathcal{E}(\mu) = I(\mu) + \lambda \left\| \mu_{\pm 1} - 1 \right\|_{H^{-1/2}(\mathbb{T}_{\tilde{L}}^2)}^2,$$

where boundary conditions on  $\mu$  are not enforced, but rather penalized with a weak norm.

For a sequence  $(u_n, A_n)$ , denote  $B_n = \nabla \times A_n$  and  $B_3^n = (B_n)_3$ . Our main result is the following.

**Theorem 2.1.1.** *Consider  $\lambda, \tilde{L} > 0$ ,  $T_n, \alpha_n, \beta_n^{-1} \rightarrow +\infty$  with  $\alpha_n \beta_n^{7/2} \rightarrow \lambda^{1/3}$ . Let  $\mathcal{K}$  as in Section 2.2.3 and  $\mathcal{A}^*(\mathbb{T}_{\tilde{L}}^2, (-1, 1))$  as in Section 2.3. The following facts hold:*

1. *Let  $(u_n, A_n) \in \mathcal{K}$  be such that  $\sup_{n \in \mathbb{N}} \tilde{E}_{T_n}(u_n, A_n) < +\infty$ . Then, there exists a measure  $\mu \in \mathcal{A}^*(\mathbb{T}_{\tilde{L}}^2, (-1, 1))$  of the form  $\mu = \mu_{x_3} \otimes dx_3$  where, for almost every  $x_3 \in (-1, 1)$  there holds  $\mu_{x_3} = \sum_i \varphi_i(x_3) \delta_{X_i(x_3)}$  and such that, up to a subsequence,  $\beta_n^{-1} B_3^n \rightharpoonup \mu$ ,  $\beta_n^{-1}(1 - |u_n|^2) \rightharpoonup \mu$ ,  $\beta_n^{-1} B_3^n(\cdot, \pm 1) \xrightarrow{H^{-1/2}} \mu_{\pm 1}$  and*

$$\liminf_{n \rightarrow +\infty} \tilde{E}_{T_n}(u_n, A_n) \geq \mathcal{E}(\mu).$$

2. *Assume in addition  $L_n^2 \alpha_n \beta_n T_n \in 2\pi\mathbb{Z}$ , where  $L_n = \tilde{L} \alpha_n^{-1/3} \beta_n^{-1/6}$ . For every measure  $\mu \in \mathcal{A}^*(\mathbb{T}_{\tilde{L}}^2, (-1, 1))$  such that  $\mathcal{E}(\mu) < +\infty$ , there exists  $(u_n, A_n) \in \mathcal{K}$  such that  $\beta_n^{-1} B_3^n \rightharpoonup \mu$ ,  $\beta_n^{-1}(1 - |u_n|^2) \rightharpoonup \mu$ ,  $\beta_n^{-1} B_3^n(\cdot, \pm 1) \xrightarrow{H^{-1/2}} \mu_{\pm 1}$  and*

$$\limsup_{n \rightarrow +\infty} \tilde{E}_{T_n}(u_n, A_n) \leq \mathcal{E}(\mu).$$

**Remark 2.1.2.** *By scaling, in the rest of the chapter we may assume  $\tilde{L} = 1$ .*

Let us now comment on the proof of Theorem 2.1.1. The compactness and lower bound (Theorem 2.1.1-1) follow from a minor modification of [37, Proposition 6.1]. Setting

$$\tilde{E}_T(u, A) = \tilde{E}_T^{\text{int}}(u, A) + \alpha^{1/3} \beta^{7/6} \|\beta^{-1} B_3 - 1\|_{H^{-1/2}(\{x_3 = \pm 1\})}^2,$$

that proposition gives compactness of the magnetic field in the sample and the lower bound

$$\liminf_{n \rightarrow +\infty} \tilde{E}_{T_n}^{\text{int}}(u_n, A_n) \geq I(\mu)$$

under the hypothesis  $\lambda = +\infty$ . With a small adaptation, we extend it so that it holds for any  $\lambda > 0$ . Theorem 2.1.1-1 then follows by weak compactness and weak lower semicontinuity of the negative Sobolev norm.

To tackle Theorem 2.1.1-2, as usual with  $\Gamma$ -convergence results, we prove first a density result. In [37, Section 5], it was proved that "regular" measures with atomic boundary conditions (see Definition 2.3.2) well approximate measures with  $I(\mu) < +\infty$  and uniform boundary conditions with respect to the topology given by  $I(\mu)$ , which was a key factor in the construction of the recovery sequence. In our case, measures such that  $\mathcal{E}(\mu) < +\infty$  do not enjoy in general uniform boundary conditions, thus a new density argument is needed. Using an interpolation argument we prove that measures with boundary trace in  $H^{-1/2}$  can be approximated by measures whose trace is piecewise constant on cubes (Lemma 2.3.4). For this last type of measure we can adapt the density arguments of [37, Section 5], to further approximate any measure with  $I(\mu) < \infty$  with "regular" measures in the "bulk" (Lemma 2.3.6).

With Lemma 2.3.6 at hand we can prove Theorem 2.1.1-2 restricting to measures  $\mu$  with the above properties. As in [37], we construct the density  $\rho$  and the magnetic field  $B$  separately in two regions. In the bulk, the measure  $\mu$  is regular and we can directly plug in the bulk construction from [37, Proposition 7.9]. In the boundary layer, we plug the construction from [39, Lemma 4.7], adapting the argument from [37, Proposition 7.7] to account for non uniform boundary conditions. Finally we can reconstruct the couple  $(u, A)$  by applying [37, Proposition 7.8].

The rest of the chapter is structured as follows: In Section 2.2, we set the notation, recall some results on optimal transport and Sobolev spaces and introduce the rescaled Ginzburg-Landau functional. In Section 2.3, we prove the density result for  $\mathcal{E}(\mu)$ . In Section 2.4, we prove the lower bound and finally in Section 2.5 we prove the upper bound.

## 2.2 Preliminaries

In this section we fix the necessary notation, recall some results on optimal transport and norms for measure and introduce the rescaled Ginzburg-Landau functional.

### 2.2.1 Notation

We use the symbols  $\lesssim$  and  $\gtrsim$  to indicate that there exists  $C > 0$  such that  $a \leq Cb$  or  $a \geq Cb$  respectively. Additionally, we denote by  $a \sim b$  a relation that satisfies  $a \lesssim b$  and  $a \gtrsim b$ . We write  $a \lesssim_p b$  to indicate  $a \leq Cb$  for some constant  $C = C(p) > 0$ , and we do similarly for  $a \gtrsim_p b$  and  $a \sim_p b$ . We use  $A \ll B$  as an hypothesis. It means that there exists an  $\varepsilon_0 > 0$  such that for  $A \leq \varepsilon_0 B$  the conclusion holds. For any vector  $v \in \mathbb{R}^3$ , we write  $v' = (v_1, v_2, 0)$ . We denote by  $e_1, e_2, e_3$  the canonical base of  $\mathbb{R}^3$ . For  $L > 0$  we consider the torus  $\mathbb{T}_L^2 = (\mathbb{R}/(L\mathbb{Z}))^2$ , endowed with the periodic distance  $|\cdot| = \min\{|\cdot - Lz| : z \in \mathbb{Z}^d\}$ . We set  $\mathbb{T}^2 = \mathbb{T}_1^2$ . For  $T, L > 0$ , we set  $\mathbb{T}_{L,T}^2 = \mathbb{T}_L^2 \times (-T, T)$ . For  $x = (x', x_3) \in \mathbb{T}_{L,T}^2$  and  $r > 0$  we write  $B(x, r)$  for a ball with respect to the periodic distance in  $x'$  and the Euclidean distance in  $x_3$ , and  $B'(x', r)$  for a two dimensional ball centered in  $x'$  with respect to the periodic distance. In the sequel, we call a set  $Q \subset \mathbb{T}^2$  a cube if it can be identified with  $x_0 + (-L/2, L/2)^2$  for some  $x_0 \in \mathbb{T}^2$  and  $0 < L < 1$ . We denote by  $\mathcal{M}(\mathbb{T}^2)$  the set of finite measures on  $\mathbb{T}^2$ , by  $\mathcal{M}^+(\mathbb{T}^2)$  the set of positive measures on  $\mathbb{T}^2$  and by  $\mathcal{P}(\mathbb{T}^2)$  the set of probability measures on  $\mathbb{T}^2$ . We define analogously for  $a < b \in \mathbb{R}$  the sets  $\mathcal{M}(\mathbb{T}^2 \times (a, b))$  and  $\mathcal{M}^+(\mathbb{T}^2 \times (a, b))$ . We denote by  $BV(\mathbb{T}^2)$  the space of periodic bounded variation functions. We let  $\{\rho_\varepsilon : \varepsilon \in (0, 1)\}$  be a family of convolution kernels that satisfy for all  $\varepsilon \in (0, 1)$

$$\rho_\varepsilon \in C^\infty(\mathbb{T}^2), \quad \text{supp}(\rho_\varepsilon) \subseteq B'(0, \varepsilon), \quad \rho_\varepsilon \geq 0, \quad \int_{\mathbb{T}^2} \rho_\varepsilon dx = 1 \quad \text{and} \quad \sup_{x \in \mathbb{T}^2} \rho_\varepsilon(x) \lesssim \varepsilon^{-2}. \quad (2.2.1)$$

For  $\mu \in \mathcal{P}(\mathbb{T}^2)$ , we set

$$(\rho_\varepsilon * \mu)(x) = \int_{\mathbb{T}^2} \rho_\varepsilon(x - y) d\mu(y) \quad \text{for } x \in \mathbb{T}^2.$$

### 2.2.2 Optimal transport and norms for measures

Let us recall some facts about optimal transport and norms for measures. We refer to [4, 101] for a broader overview of optimal transport.

If  $\mu_1, \mu_2 \in \mathcal{M}^+(\mathbb{T}^2)$  with  $\mu_1(\mathbb{T}^2) = \mu_2(\mathbb{T}^2)$ , we define the Wasserstein distance

$$W_{\text{per}}^2(\mu_1, \mu_2) = \min \left\{ \int_{\mathbb{T}^2 \times \mathbb{T}^2} |x - y|^2 d\pi : \pi \in \mathcal{M}^+(\mathbb{T}^2 \times \mathbb{T}^2), \pi_i = \mu_i \text{ for } i = 1, 2 \right\}.$$

We define analogously the Wasserstein distance between two measures  $\mu_1, \mu_2 \in \mathcal{M}^+(\mathbb{T}_{L,T}^2)$ . Let us recall the Benamou-Brenier formula: for  $a < b \in \mathbb{R}$  we have

$$\begin{aligned} & \frac{1}{b-a} W_{\text{per}}^2(\mu_1, \mu_2) \\ &= \min_{(\mu, m)} \left\{ \int_{\mathbb{T}^2 \times (a,b)} \left| \frac{dm}{d\mu} \right|^2 d\mu dt : m \ll \mu, \partial_t \mu + \nabla \cdot m = 0, \mu_a = \mu_1, \mu_b = \mu_2 \right\}. \end{aligned} \quad (2.2.2)$$

For  $\psi : \mathbb{T}^2 \rightarrow \mathbb{R}$  Lipschitz we define  $\|\psi\|_{Lip} = \|\psi\|_{L^\infty(\mathbb{T}^2)} + \|\nabla \psi\|_{L^\infty(\mathbb{T}^2)}$ . For a measure  $\sigma \in \mathcal{M}(\mathbb{T}^2)$  we define the Bounded-Lipschitz norm

$$\|\sigma\|_{BL} = \sup_{\|\psi\|_{Lip} \leq 1} \int_{\mathbb{T}^2} \psi d\sigma.$$

For  $s \in \mathbb{R}$ , we define  $\dot{H}^{-s}(\mathbb{T}^2)$  (see [5, 107]) as the set of all measures  $\sigma \in \mathcal{M}(\mathbb{T}^2)$  such that

$$\|\sigma\|_{\dot{H}^{-s}}^2 = \frac{1}{2\pi} \sum_{k \in \mathbb{Z}^2} |k|^{-2s} |\hat{\sigma}_k|^2 < \infty, \quad (2.2.3)$$

where  $\hat{\sigma}_k$  is the Fourier coefficient for  $k \in \mathbb{Z}^2$  given by

$$\hat{\sigma}_k = \int_{\mathbb{T}^2} \exp(2\pi i(k \cdot x)) d\sigma(x).$$

Notice that with this notation,  $\dot{H}^{-0}(\mathbb{T}^2) = L^2(\mathbb{T}^2)$  and, for  $s < 0$ ,  $\sigma(\mathbb{T}^d) = 0$  is enforced. Moreover,  $\dot{H}^{-s}(\mathbb{T}^2) = (\dot{H}^s(\mathbb{T}^2))^*$ . In the physical case  $s = 1/2$ , (2.2.3) can be interpreted as a trace norm, indeed by [28, Proposition A.1] one has for any  $f \in L^2(\mathbb{T}^2)$  with  $\int_{\mathbb{T}^2} f dx' = 0$

$$\|f\|_{\dot{H}^{-1/2}(\mathbb{T}^2)}^2 = \inf \left\{ \int_{\mathbb{T}^2 \times [0, +\infty]} |B|^2 dx, \operatorname{div} B = 0, B(\cdot, 0) = f \right\}.$$

We recall the following proposition (see [76, 93]).

**Proposition 2.2.1.** *Let  $\sigma_1, \sigma_2 \in L^\infty(\mathbb{T}^2) \cap \mathcal{P}(\mathbb{T}^2)$ . Then,*

$$\|\sigma_1 - \sigma_2\|_{\dot{H}^{-1}(\mathbb{T}^2)} \leq \max\{\|\sigma_1\|_{L^\infty(\mathbb{T}^2)}, \|\sigma_2\|_{L^\infty(\mathbb{T}^2)}\}^{\frac{1}{2}} W_{\text{per}}(\sigma_1, \sigma_2).$$

For  $u \in BV(\mathbb{T}^2)$  we denote its  $BV$  norm (see [3]) by

$$\int_{\mathbb{T}^2} |Du| = |Du|(\mathbb{T}^2).$$

For any  $u \in L^1(\mathbb{T}_{1,T}^2)$  such that for a.e.  $x_3 \in (-T, T)$  one has  $u(\cdot, x_3) \in BV(\mathbb{T}^2)$ , we set

$$\int_{\mathbb{T}_{1,T}^2} |D'u| = \int_{-1}^1 \left( \int_{\mathbb{T}^2} |D(u(\cdot, x_3))| \right) dx_3 = \int_{-1}^1 |D(u(\cdot, x_3))|(\mathbb{T}^2) dx_3,$$

so that by [3, Theorem 3.40] it holds

$$\int_{\mathbb{T}_{1,T}^2} |D'u| = \int_{\mathbb{R}} \int_{-1}^1 \mathcal{H}^1(\partial\{u(\cdot, x_3) > s\}) dx_3 ds, \quad (2.2.4)$$

where  $\mathcal{H}^1$  denotes the periodic Hausdorff measure on  $\mathbb{T}^2$  and  $\partial\{u(\cdot, x_3) > s\}$  is the reduced boundary of the set of finite perimeter  $\{u(\cdot, x_3) > s\}$  (see [3]).

### 2.2.3 The rescaled Ginzburg-Landau functional

In this section we precise the functional setting of the rescaled Ginzburg-Landau model  $\tilde{E}(u, A)$ . In [37, Section 3], the functional  $\tilde{E}(u, A)$  was derived from (2.1.1) in the regime (2.1.5) by performing an anisotropic rescaling.

Let  $\alpha, \beta, T > 0$  (which correspond the parameters introduced in (2.1.2)). Let

$$\mathcal{K}_0 = \{(u, A) \in H_{\text{loc}}^1(\mathbb{R}^2 \times [-1, 1]); \mathbb{C}\} \times L_{\text{loc}}^2(\mathbb{R}^2 \times [-1, 1]; \mathbb{R}^3), \quad \nabla \times A \in L_{\text{loc}}^2(\mathbb{R}^2 \times [-1, 1])\}.$$

For  $(u, A) \in \mathcal{K}_0$ , the complex function  $u$  is called the order parameter,  $A$  is the magnetic potential,  $\rho = |u|^2$  is the density of superconducting electrons,  $B = \nabla \times A$  is the magnetic field and

$$\nabla_A u = \nabla u - iAu$$

is the covariant derivative. We say that two configurations  $(u, A)$  and  $(\hat{u}, \hat{A}) \in \mathcal{K}_0$  are gauge equivalent if there exists  $\phi \in H_{\text{loc}}^1(\mathbb{R}^2 \times [-1, 1])$  such that

$$\begin{cases} u = \hat{u}e^{i\phi} \\ A = \hat{A} + \frac{1}{\alpha^{1/3}\beta^{-1/3}T} \nabla\phi. \end{cases} \quad (2.2.5)$$

Notice that the physical quantities  $\rho$  and  $B$ , and  $|\nabla_{\alpha^{1/3}\beta^{-1/3}TA}u|$  are invariant under the gauge transformation (2.2.5). We consider a sample with periodic boundary conditions, that is  $\mathbb{T}_{1,1}^2 = \mathbb{T}^2 \times (-1, 1)$ . We introduce the set of functions

$$\mathcal{K} = \mathcal{K}(\alpha, \beta, T) = \{(u, A) \in \mathcal{K}_0 : (u(\cdot + k), A(\cdot + k)) \text{ is gauge equivalent to } (u, A) \forall k \in \mathbb{Z}^2\}.$$

The set  $\mathcal{K}$  depends on  $\alpha, \beta$  and  $T$  since it depends on the gauge equivalence (2.2.5). In the set  $\mathcal{K}$  the physical quantities  $\rho$  and  $B$  are 1-periodic and we can think of them as defined on  $\mathbb{T}_{1,1}^2$ , whereas in general  $u$  and  $A$  will not be periodic. We also observe that  $(u, A) \in \mathcal{K}$  implies  $\rho \in H^1(\mathbb{T}_{1,1}^2)$  and  $B \in L^2(\mathbb{T}_{1,1}^2)$ . For  $(u, A) \in \mathcal{K}$  we consider the functional

$$\begin{aligned} \tilde{E}_T(u, A) = & \int_{\mathbb{T}_{1,1}^2} \alpha^{-2/3}\beta^{-1/3} \left| \nabla'_{\alpha^{1/3}\beta^{-1/3}TA} u \right|^2 + \alpha^{-4/3}\beta^{-2/3} \left| \left( \nabla_{\alpha^{1/3}\beta^{-1/3}TA} u \right)_3 \right|^2 \\ & + \alpha^{2/3}\beta^{-2/3} \left( B_3 - (1 - |u|^2) \right)^2 + \beta^{-1} |B'|^2 dx + \alpha^{1/3}\beta^{7/6} \|\beta^{-1}B_3 - 1\|_{H^{-1/2}(\{x_3=\pm 1\})}^2. \end{aligned} \quad (2.2.6)$$

We will consider sequences  $\alpha_n, \beta_n, T_n$  for which there exists  $\lambda \geq 0$  such that

$$T_n \rightarrow +\infty, \quad \alpha_n \rightarrow +\infty, \quad \beta_n \rightarrow 0, \quad T_n \alpha_n^{-1} \rightarrow +\infty, \quad \alpha_n \beta_n^{7/2} \rightarrow \lambda^{1/3}. \quad (2.2.7)$$

## 2.2.4 The limit functional

**Definition 2.2.2.** For  $-T \leq a < b \leq T$ , we denote by  $\mathcal{A}(\mathbb{T}^2, (a, b))$  the set of measures  $\mu \in \mathcal{M}^+(\mathbb{T}^2 \times (a, b))$ ,  $m \in \mathcal{M}(\mathbb{T}^2 \times (a, b), \mathbb{R}^d)$ , with  $m \ll \mu$  satisfying in the sense of distributions the continuity equation

$$\partial_t \mu + \nabla \cdot m = 0 \quad \text{in } \mathbb{T}^2 \times (a, b), \quad (2.2.8)$$

and such that  $\mu = \mu_t \otimes dt$  where, for almost every  $t \in (a, b)$  there holds  $\mu_t = \sum_i \varphi_i(t) \delta_{X_i(t)}$ , for some  $t \mapsto \varphi_i(t) > 0$  and  $t \mapsto X_i(t) \in \mathbb{T}^2$ . We denote by  $\mathcal{A}^*(\mathbb{T}^2, (a, b)) = \{\mu : \exists m, (\mu, m) \in \mathcal{A}(\mathbb{T}^2, (a, b))\}$  the set of admissible  $\mu$ . We use the notation  $\mathcal{A}_T = \mathcal{A}(\mathbb{T}^2, (-T, T))$  and  $\mathcal{A}_T^* = \mathcal{A}^*(\mathbb{T}^2, (-T, T))$ . We define the functional  $I : \mathcal{A}(\mathbb{T}^2, (a, b)) \rightarrow [0, +\infty]$  by

$$I(\mu, m, (a, b)) = \int_a^b \sum_{x' \in \mathbb{T}^2} (\mu_t(\{x'\}))^{\frac{1}{2}} dt + \int_{\mathbb{T}^2 \times (a, b)} \left| \frac{dm}{d\mu} \right|^2 d\mu.$$

We also define (with abuse of notation) the functional  $I : \mathcal{A}^*(\mathbb{T}^2, (a, b)) \rightarrow [0, +\infty]$

$$I(\mu, (a, b)) = \min\{I(\mu, m, (a, b)) : (\mu, m) \in \mathcal{A}_T\}.$$

We use the notation  $I(\mu) = I(\mu, (-T, T))$ .

Notice that  $\mu \in \mathcal{A}_T^*$  implies that  $\Phi = \mu_t(\mathbb{T}^2)$  does not depend on  $t \in (-T, T)$ .

Given two measures  $\bar{\mu}_\pm \in \mathcal{M}^+(\mathbb{T}^2)$  such that  $\bar{\mu}_+(\mathbb{T}^2) = \bar{\mu}_-(\mathbb{T}^2)$  consider the problem

$$\inf\{I(\mu) : \mu \in \mathcal{A}_T^*, \mu_{\pm T} = \bar{\mu}_\pm\}.$$

By (2.2.2), for  $-T < a < b < T$

$$W_{\text{per}}(\mu_a, \mu_b) \leq I(\mu)^{\frac{1}{2}}(b-a)^{\frac{1}{2}}.$$

This means that, if  $I(\mu) < \infty$ ,  $t \rightarrow \mu_t$  is a  $\frac{1}{2}$ -Hölder continuous curve in the Wasserstein space, hence  $\mu_T$  and  $\mu_{-T}$  are well defined.

For  $T > 0$  and  $\lambda > 0$  we consider the functional

$$\mathcal{E}(\mu) = I(\mu) + \lambda \|\mu_{\pm T} - 1\|_{\dot{H}^{-1/2}(\mathbb{T}^2)}^2,$$

where

$$\|\mu_{\pm T} - 1\|_{\dot{H}^{-s}(\mathbb{T}^2)}^2 = \|\mu_T - 1\|_{\dot{H}^{-s}(\mathbb{T}^2)}^2 + \|\mu_{-T} - 1\|_{\dot{H}^{-1/2}(\mathbb{T}^2)}^2.$$

If  $\mathcal{E}(\mu) < +\infty$ , then necessarily  $\Phi = \mu_t(\mathbb{T}^2) = 1$  for every  $t > 0$ .

## 2.3 A density result

In this section we prove a density result for the limit energy, which will be necessary for the proof of Theorem 2.1.1-2.

We start with the following result, which is a minor adaptation of [48, Proposition 3.9].

**Lemma 2.3.1.** *Let  $\varepsilon \in (0, 1)$ ,  $\Phi > 0$ ,  $Q \subset \mathbb{T}^2$  be a cube and  $\mu^\pm \in \mathcal{M}^+(\mathbb{T}^2)$  be such that  $\mu^\pm(\mathbb{T}^2) = \Phi$  and  $\text{supp } \mu^\pm \subset Q$ . Then, there exists  $\mu \in \mathcal{A}^*(\mathbb{T}^2, (-\varepsilon, \varepsilon))$  such that for all  $t \in (-\varepsilon, \varepsilon)$  one has  $\text{supp } \mu_t \subset Q$ , that  $\mu_{\pm\varepsilon} = \mu^\pm$  and*

$$I(\mu, (-\varepsilon, \varepsilon)) \lesssim \frac{1}{\varepsilon} W_{\text{per}}^2(\mu^-, \mu^+) + \left(\Phi + \Phi^{\frac{1}{2}}\right) \varepsilon^{\frac{1}{3}}.$$

Moreover, if we let  $N_0 = \lfloor \varepsilon^{-2/3} \rfloor$  and partition  $Q$  into  $N_0^2$  cubes  $Q_i$  of sidelength  $\sqrt{|Q|}/N_0$  and center  $X_i$ , then  $\mu_0 = \sum_{i=1}^{N_0^2} \varphi_i \delta_{X_i}$  for some  $\varphi_i \geq 0$ . Furthermore, if  $\mu^+ = \mu^-$ , then  $\varphi_i = \mu^\pm(Q_i)$ .

Let us recall the definition of regular measure from [37].

**Definition 2.3.2.** We say that a measure  $\mu \in \mathcal{A}_T^*$  is regular if the following facts hold:

1. There exists  $N > 0$  such that

$$\mu = \sum_{i=1}^N \frac{\varphi_i}{\sqrt{1 + |\dot{X}_i|^2}} \mathcal{H}^1 \llcorner \Gamma_i,$$

where for  $i = 1 \dots N$ ,  $\varphi_i > 0$ ,  $X_i : [a_i, b_i] \rightarrow \mathbb{T}^2$  are affine maps for some  $-T < a_i < b_i < T$  and  $\Gamma_i = \{(X_i(x_3), x_3) : x_3 \in [a_i, b_i]\}$  are pairwise disjoint segments.

2. Each branching point is a triple point, that is any  $x \in \mathbb{T}_{1,T}^2$  is in the closure of at most three segments.

We denote by  $\mathcal{M}_{\text{reg}}(\mathbb{T}_{1,T}^2)$  the set of regular measures.

For measures whose boundary trace is a finite sum of Dirac deltas, the following result was proved in [37, Lemma 5.16].

**Lemma 2.3.3.** *Let  $\mu \in \mathcal{A}_T^*$  with  $I(\mu) < \infty$  and such that  $\mu_{\pm T}$  are finite sums of Dirac deltas. Then, there exists a sequence of measures  $\mu^N \in \mathcal{M}_{\text{reg}}(\mathbb{T}_{1,T}^2)$  such that  $\mu^N \rightharpoonup \mu$ ,  $\mu_{\pm T}^N = \mu_{\pm T}$  and  $\limsup_{N \rightarrow +\infty} I(\mu^N) \leq I(\mu)$ .*

Let us define the set

$$\mathcal{P}^c(\mathbb{T}^2) = \left\{ \sigma \in \mathcal{P}(\mathbb{T}^2), \sigma = \sum_{j=1}^m a_j \chi_{Q_j} \text{ for some } m \in \mathbb{N}, a_j \in \mathbb{R} \text{ and cubes } Q_j \subset \mathbb{T}^2, Q_i \cap Q_j = \emptyset \right\}.$$

The main result of this section is the following lemma.

**Lemma 2.3.4.** *Let  $\mu \in \mathcal{A}_T^*$  be such that  $\mathcal{E}(\mu) < \infty$ . Then, there exists a sequence  $(\mu^n)_{n \in \mathbb{N}} \in \mathcal{A}_T^*$  such that  $\mu_{\pm T}^n \in \mathcal{P}^c(\mathbb{T}^2)$ ,  $\mu^n \rightharpoonup \mu$ ,  $\mu_{\pm T}^n \xrightarrow{H^{-1/2}} \mu_{\pm T}$  and*

$$\limsup_{n \rightarrow +\infty} \mathcal{E}(\mu^n) \leq \mathcal{E}(\mu).$$

To prove Lemma 2.3.4, we need the following intermediate result.

**Lemma 2.3.5.** *Let  $\sigma \in \mathcal{P}(\mathbb{T}^2)$  with  $\sigma - 1 \in \dot{H}^{-1/2}(\mathbb{T}^2)$ . Then, there exists a sequence  $(\sigma_n)_{n \in \mathbb{N}}$  such that  $\sigma_n \in \mathcal{P}^c(\mathbb{T}^2)$ ,  $\|\sigma_n - \sigma\|_{\dot{H}^{-1/2}(\mathbb{T}^2)} \rightarrow 0$  and  $W_{\text{per}}(\sigma_n, \sigma) \rightarrow 0$ .*

*Proof.* The proof is divided in two steps: we first prove that the statement holds for  $\sigma \in L^\infty(\mathbb{T}^2)$  and then extend it to any  $\sigma \in \mathcal{P}(\mathbb{T}^2)$  with  $\sigma \in \dot{H}^{-1/2}(\mathbb{T}^2)$ .

Suppose first that  $\sigma \in L^\infty(\mathbb{T}^2)$ . Consider  $n \in \mathbb{N}$  and consider a grid of  $n^2$  cubes  $Q_j \subset \mathbb{T}^2$ ,  $j = 1, \dots, n^2$ , of side  $1/n$ . Define

$$\sigma^n = n^2 \sum_{j=1}^{n^2} \sigma(Q_j) \chi_{Q_j}. \quad (2.3.1)$$

One has that  $\sigma^n \in \mathcal{P}^c(\mathbb{T}^2)$ ,  $\sigma^n(\mathbb{T}^2) = \sigma(\mathbb{T}^2)$  and  $\|\sigma^n\|_{L^\infty(\mathbb{T}^2)} \leq \|\sigma\|_{L^\infty(\mathbb{T}^2)}$ . For  $j = 1, \dots, n^2$ , consider an optimal transport plan  $\pi_j \in \mathcal{M}^+(\mathbb{T}^2 \times \mathbb{T}^2)$  between  $\sigma|_{Q_j}$  and  $n^2 \sigma(Q_j) \chi_{Q_j}$  and let  $\pi = \sum_{j=1}^{n^2} \pi_j$ . We have

$$W_{\text{per}}(\sigma^n, \sigma) \leq \sum_{j=1}^{n^2} \int_{\mathbb{T}^2 \times \mathbb{T}^2} |x - y|^2 d\pi_j \leq \frac{2}{n^2}. \quad (2.3.2)$$

Moreover, by Proposition 2.2.1 and interpolation (see [5, Proposition 1.32])

$$\|\sigma^n - \sigma\|_{\dot{H}^{-1/2}(\mathbb{T}^2)} \leq \|\sigma^n - \sigma\|_{L^2(\mathbb{T}^2)}^{1/2} \|\sigma^n - \sigma\|_{\dot{H}^{-1}(\mathbb{T}^2)}^{1/2} \lesssim \|\sigma\|_{L^\infty(\mathbb{T}^2)}^{3/4} W_{\text{per}}^{1/2}(\sigma^n, \sigma). \quad (2.3.3)$$

Inequalities (2.3.2) and (2.3.3) imply the claim.

Let us now consider any  $\sigma \in \mathcal{P}(\mathbb{T}^2)$  with  $\sigma \in \dot{H}^{-1/2}(\mathbb{T}^2)$ . Fix  $\varepsilon > 0$  and consider a family of convolution kernels  $(\rho_\varepsilon)_\varepsilon$  as in (2.2.1). Define  $\sigma_\varepsilon = \sigma * \rho_\varepsilon$ . It holds  $\sigma_\varepsilon \in C^\infty(\mathbb{T}^2)$ ,  $\|\sigma_\varepsilon - \sigma\|_{\dot{H}^{-1/2}(\mathbb{T}^2)} \rightarrow 0$  (see for instance [5, 106]) and  $W_{\text{per}}(\sigma_\varepsilon, \sigma) \rightarrow 0$  (see [101, Lemma 5.2]). Moreover, we have  $\sigma_\varepsilon \in L^\infty(\mathbb{T}^2)$  with

$$\|\sigma_\varepsilon\|_{L^\infty(\mathbb{T}^2)} \lesssim \frac{1}{\varepsilon^2}. \quad (2.3.4)$$

Consider  $\sigma_\varepsilon^n = (\sigma_\varepsilon)^n$  as given by (2.3.1). By triangle inequality, (2.3.3), (2.3.2) and (2.3.4) we have

$$\|\sigma - \sigma_\varepsilon^n\|_{\dot{H}^{-1/2}(\mathbb{T}^2)} \leq \|\sigma - \sigma_\varepsilon\|_{\dot{H}^{-1/2}(\mathbb{T}^2)} + \|\sigma_\varepsilon - \sigma_\varepsilon^n\|_{\dot{H}^{-1/2}(\mathbb{T}^2)} \lesssim \|\sigma - \sigma_\varepsilon\|_{\dot{H}^{-1/2}(\mathbb{T}^2)} + \frac{1}{\varepsilon^{3/2}n},$$

which implies that the sequence  $\sigma_n = \sigma_{n^{-1/2}}$  is such that  $\|\sigma_n - \sigma\|_{\dot{H}^{-1/2}(\mathbb{T}^2)} \rightarrow 0$ . Triangle inequality and (2.3.2) imply that also  $W_{\text{per}}(\sigma_n, \sigma) \rightarrow 0$ .  $\square$

*Proof of Lemma 2.3.4.* We present the construction only on the interval  $(0, T)$ , an analogous construction can be performed in the interval  $(-T, 0)$ . Fix  $\varepsilon > 0$  and  $n \in \mathbb{N}$ . Let  $\mu \in \mathcal{A}_T^*$  such that  $\mathcal{E}(\mu) < \infty$ . Consider the measure  $\mu_{T,n} \in \mathcal{P}^c(\mathbb{T}^2)$  given by Lemma 2.3.5 applied to  $\mu_T$ . Then, take  $\hat{\mu} \in \mathcal{A}^*(\mathbb{T}^2, (0, \varepsilon))$  given by Lemma 2.3.1 applied to  $\mu_T$  and  $\mu_{T,n}$ . For  $t \in (0, T)$ , we define

$$\mu_t^n = \begin{cases} \mu_{\frac{t}{1-\varepsilon}} & \text{for } t < (1-\varepsilon)T, \\ \hat{\mu}_{\frac{t}{T} - (1-\varepsilon)} & \text{for } (1-\varepsilon)T \leq t \leq T. \end{cases}$$

Clearly  $\mu^n \in \mathcal{A}^*(0, T)$  and  $\mu_T^n = \mu_{T,n} \in \mathcal{P}^c(\mathbb{T}^2)$ . We have

$$\mathcal{E}(\mu^n) = I(\mu^n, (0, (1-\varepsilon)T)) + I(\mu^n, ((1-\varepsilon)T, T)) + \|\mu_T^n - 1\|_{H^{-1/2}(\mathbb{T}^2)}^2. \quad (2.3.5)$$

Let us examine each term of (2.3.5) separately. A direct computation gives

$$I(\mu^n, (0, (1-\varepsilon)T)) \leq \frac{1}{1-\varepsilon} I(\mu, (0, T)). \quad (2.3.6)$$

Moreover, using Lemma 2.3.1,

$$I(\mu^n, ((1-\varepsilon)T, T)) \lesssim \frac{1}{\varepsilon} W_{\text{per}}^2(\mu_T^n, \mu_T) + \varepsilon^{1/3}. \quad (2.3.7)$$

Furthermore, one has  $\mu(\mathbb{T}^2 \times ((-T, -T + \varepsilon) \cup (T - \varepsilon, T))) \lesssim \varepsilon$  and

$$W_{\text{per}}^2(\mu \llcorner (-T + \varepsilon, T - \varepsilon), \mu^n \llcorner (-T + \varepsilon, T - \varepsilon)) \lesssim \varepsilon,$$

which imply

$$W_{\text{per}}^2(\mu^n, \mu) \lesssim \varepsilon.$$

Finally, by triangle inequality,

$$\|\mu_T^n - 1\|_{H^{-1/2}(\mathbb{T}^2)}^2 \leq \left( \|\mu_T^n - \mu_T\|_{H^{-1/2}(\mathbb{T}^2)} + \|\mu_T - 1\|_{H^{-1/2}(\mathbb{T}^2)} \right)^2. \quad (2.3.8)$$

Combining (2.3.6), (2.3.7) and (2.3.8) gives for some positive constant  $C > 0$

$$\limsup_{n \rightarrow +\infty} \mathcal{E}(\mu^n) \leq \frac{1}{1-\varepsilon} I(\mu, (0, T)) + C\varepsilon^{1/3} + \|\mu_T - 1\|_{H^{-\frac{1}{2}}(\mathbb{T}^2)}^2,$$

and the conclusion follows by a diagonal procedure in  $\varepsilon$ .  $\square$

We now show that, for  $N \gg 1$ , we can approximate any measure such that  $\mu_{\pm T} \in \mathcal{P}^c(\mathbb{T}^2)$  with a measure  $\mu^N$  which is regular in  $(-T + N^{-3/2}, T - N^{-3/2})$  (see Figure 2.1).

**Lemma 2.3.6.** *Let  $N \gg 1$  and consider  $\mu \in \mathcal{A}_T^*$  such that  $\mu_{\pm T} \in \mathcal{P}^c(\mathbb{T}^2)$  and  $I(\mu) < +\infty$ . Consider  $\mu_{\pm T} = \sum_{j=1}^{m^\pm} a_j^\pm \chi_{Q_j^\pm}$  and for  $j = 1 \dots m^\pm$  divide each cube  $Q_j^\pm$  into  $N^2$  cubes of sidelength  $\sqrt{|Q_j^\pm|}/N$  (see Figure 2.1). For  $i = 1 \dots N^2$  we denote by  $Q_{i,j}^\pm$  such cubes and call  $X_{i,j}^\pm$  their centers. Setting  $\varepsilon_N = N^{-3/2}$ , there exist a measure  $\mu^N \in \mathcal{A}_T^*$  such that  $\mu_{\pm T}^N = \mu_{\pm T}$ ,  $\mu^N \in \mathcal{M}_{\text{reg}}(\mathbb{T}_{1,T-\varepsilon_N}^2)$  and*

$$\mu_{\pm(T-\varepsilon_N)}^N = \sum_{j=1}^m \sum_{i=1}^{N^2} \frac{a_j^\pm |Q_j^\pm|}{N^2} \delta_{X_{i,j}^\pm}. \quad (2.3.9)$$

Moreover, there exist a constant  $C = C(\mu) > 0$  such that

$$I(\mu^N) \leq \frac{T}{T - \varepsilon_N} I(\mu) + CN^{-1/2} \quad (2.3.10)$$

and

$$W_{\text{per}}^2(\mu^N, \mu) \leq CN^{-3/2}. \quad (2.3.11)$$

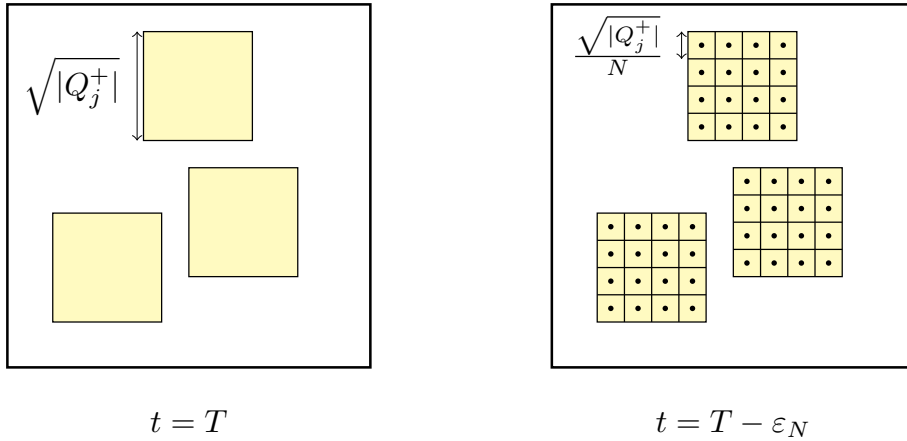


Figure 2.1: Details of  $\mu^N$  at times  $t = T$  and  $t = T - \varepsilon_N$  for  $m^+ = 3$  and  $N = 4$ . In each of the  $m^+$  cubes (in yellow) of sidelength  $\sqrt{|Q_j^+|}$  we put  $N^2$  Dirac deltas in a uniform grid of sidelength  $\sqrt{|Q_j^+|}/N$ .

*Proof.* Let us first focus on the interval  $(0, T)$ . For each  $j = 1 \dots m^+$ , we apply Lemma 2.3.1 with  $\varepsilon = \varepsilon_N$  and  $\mu_\pm = a_j^+ \chi_{Q_j^+}$ . For any  $j = 1 \dots m^+$  there exists  $\mu^j \in \mathcal{A}^*(-\varepsilon_N, \varepsilon_N)$  such that  $\mu_{\pm \varepsilon_N}^j = a_j^+ \chi_{Q_j^+}$  and  $\mu_0^j = \sum_{i=1}^{N^2} \frac{a_j^+}{N^2} \delta_{X_{i,j}^+}$ , where the points  $X_{i,j}^+ \in Q_j^+$  are the centers of the  $N^2$  cubes  $Q_{i,j}^+$  of sidelength  $\sqrt{|Q_j^+|}/N$  composing  $Q_j^+$ . Moreover,

$$I(\mu, (-\varepsilon, \varepsilon)) \lesssim_\mu \varepsilon_N^{\frac{1}{3}} = N^{-1/2}. \quad (2.3.12)$$

We now define for  $t \in (0, T)$  an auxiliary measure  $\hat{\mu}^N$  as

$$\hat{\mu}_t^N = \begin{cases} \mu_{tT/(T-2\varepsilon_N)} & \text{for } 0 \leq t \leq T - 2\varepsilon_N \\ \sum_{j=1}^m \mu_{t-(T-\varepsilon_N)}^j & \text{for } T - 2\varepsilon_N \leq t \leq T. \end{cases}$$

Similarly, we can repeat the same construction to define  $\hat{\mu}^N$  on the interval  $(-T, 0)$ . Notice that by Lemma 2.3.1, we have

$$\hat{\mu}_{\pm(T-\varepsilon_N)}^N = \sum_{j=1}^m \sum_{i=1}^{N^2} \frac{a_j^\pm |Q_j^\pm|}{N^2} \delta_{X_{i,j}^\pm}.$$

In particular,  $\hat{\mu}_{T-\varepsilon_N}^N$  are finite sums of Dirac deltas. Moreover,

$$I(\hat{\mu}^N, (-T + \varepsilon_N, T - \varepsilon_N)) \leq \frac{T}{T - \varepsilon_N} I(\mu). \quad (2.3.13)$$

By Lemma 2.3.3, there exist  $\tilde{\mu}^N \in \mathcal{M}_{\text{reg}}(\mathbb{T}_{1, T-\varepsilon_N}^2)$  such that  $\tilde{\mu}_{\pm(T-\varepsilon_N)}^N = \hat{\mu}_{\pm(T-\varepsilon_N)}^N$ ,

$$I(\tilde{\mu}^N, (-T + \varepsilon_N, T - \varepsilon_N)) \leq I(\hat{\mu}^N, (-T + \varepsilon_N, T - \varepsilon_N)) + N^{-1/2} \quad (2.3.14)$$

and

$$W_{\text{per}}^2(\tilde{\mu}^N, \hat{\mu}^N \llcorner (-T + \varepsilon_N, T - \varepsilon_N)) \leq N^{-3/2}. \quad (2.3.15)$$

We define for  $t \in (-T, T)$

$$\mu_t^N = \begin{cases} \tilde{\mu}_t & \text{for } |t| \leq T - \varepsilon_N \\ \hat{\mu}_t & \text{for } |t| \geq T - \varepsilon_N. \end{cases}$$

The measure  $\mu_N$  satisfies the boundary conditions and (2.3.9). Moreover, (2.3.12), (2.3.13) and (2.3.14) imply (2.3.10). Finally, one has  $\mu(\mathbb{T}^2 \times ((-T, -T + \varepsilon_N) \cup (T - \varepsilon_N, T))) \lesssim \varepsilon_N$  and

$$W_{\text{per}}^2(\mu \llcorner (-T + \varepsilon_N, T - \varepsilon_N), \hat{\mu}^N \llcorner (-T + \varepsilon_N, T - \varepsilon_N)) \lesssim \varepsilon_N,$$

which, combined with (2.3.15), give (2.3.11). □

We finally address quantization.

**Definition 2.3.7.** Let  $k > 0$ . We say that  $\mu \in \mathcal{M}_{\text{reg}}(\mathbb{T}_{1, T}^2)$  is  $k$ -quantized if for all  $(x, t) \in \mathbb{T}_{1, T}^2$  one has  $k\mu_t(\{x'\}) \in 2\pi\mathbb{N}$ .

We recall the following Lemma, which is minor adaptation of [37, Lemma 5.18].

**Lemma 2.3.8.** *Let  $\mu \in \mathcal{M}_{\text{reg}}(\mathbb{T}_{1, T}^2)$  such that  $\mu_T(\mathbb{T}^2) = \Phi$ . For all  $k > 0$  such that  $k\Phi \in 2\pi\mathbb{N}$ , there exists a  $k$ -quantized regular measure  $\mu^k \in \mathcal{M}_{\text{reg}}(\mathbb{T}_{1, T}^2)$  and  $C = C(\mu) > 0$  such that  $\mu^k(\mathbb{T}_{1, T}^2) = \mu(\mathbb{T}_{1, T}^2)$  and*

$$\left(1 - \frac{C(\mu)}{k}\right) \mu \leq \mu^k \leq \mu \left(1 + \frac{C(\mu)}{k}\right).$$

In particular,  $\mu^k \ll \mu$ ,  $\mu^k \rightarrow \mu$  strongly,

$$W(\mu, \mu^k) \lesssim C(\mu)/k \quad (2.3.16)$$

and

$$I(\mu^k) \rightarrow I(\mu). \quad (2.3.17)$$

## 2.4 Lower bound

The goal of this section is to prove the following proposition regarding the energy (2.2.6).

**Proposition 2.4.1.** *Consider  $T_n, \alpha_n, \beta_n$  as in (2.2.7). Let  $(u_n, A_n) \in \mathcal{K}$  such that*

$$\sup_{n \in \mathbb{N}} \tilde{E}_{T_n}(u_n, A_n) < +\infty.$$

*Then, there exist  $(\mu, m) \in \mathcal{A}_{1,1}$  such that, up to a subsequence, the following facts hold:*

(i)  $\beta_n^{-1} B_3^n \rightharpoonup \mu$ ,  $\beta_n^{-1}(1 - |u_n|^2) \rightharpoonup \mu$  and  $\beta_n^{-1} B'_n \rightharpoonup m$ .

(ii)  $\beta_n^{-1} B_3^n(\cdot, \pm 1) \xrightarrow{H^{-1/2}} \mu_{\pm 1}$ .

(iii) *One has*

$$\liminf_{n \rightarrow +\infty} \tilde{E}_{T_n}(u_n, A_n) \geq I(\mu, m) + \lambda \|\mu_{\pm 1} - 1\|_{H^{-1/2}(\mathbb{T}^2)}^2.$$

Define

$$\begin{aligned} \tilde{E}_T^{\text{int}}(u, A) = & \int_{\mathbb{T}_{1,1}^2} \alpha^{-2/3} \beta^{-1/3} \left| \nabla'_{\alpha^{1/3} \beta^{-1/3} T_A} u \right|^2 + \alpha^{-4/3} \beta^{-2/3} \left| \left( \nabla_{\alpha^{1/3} \beta^{-1/3} T_A} u \right)_3 \right|^2 \\ & + \alpha^{2/3} \beta^{-2/3} \left( B_3 - (1 - |u|^2) \right)^2 + \beta^{-1} |B'|^2 dx. \end{aligned}$$

Let us start by recalling some lemmas from [37]. We start from [37, Lemma 6.5].

**Lemma 2.4.2.** *For  $\varepsilon > 0$ , denote*

$$W_\varepsilon(\rho) = \min \left\{ \frac{\rho}{\varepsilon}, 1 \right\} (1 - \rho)^2. \quad (2.4.1)$$

*There exist  $C_\varepsilon > 0$  such that for every  $(u, A) \in \mathcal{K}$  with  $\|\rho\|_{L^\infty} \leq 1$  it holds*

$$\int_{\mathbb{T}_{1,1}^2} \alpha^{2/3} \beta^{-2/3} W_\varepsilon(\rho) dx \leq \int_{\mathbb{T}_{1,1}^2} \alpha^{2/3} \beta^{-2/3} (B_3 - (1 - \rho))^2 + C_\varepsilon \tilde{E}_T(u, A) \frac{\alpha}{T}.$$

We also need [37, Lemma 6.3].

**Lemma 2.4.3.** *Let  $(u, A) \in \mathcal{K}$  with  $\|\rho\|_{L^\infty(\mathbb{T}^2)} \leq 1$  and denote by  $E = \tilde{E}_T^{\text{int}}(u, A)$ . For every  $x_3, \tilde{x}_3 \in (-1, 1)$ , it holds*

$$\left\| \beta^{-1} B_3(\cdot, x_3) - \beta^{-1} B_3(\cdot, \tilde{x}_3) \right\|_{BL} \lesssim E^{1/2} |x_3 - \tilde{x}_3|^{\frac{1}{2}} + \sigma(\alpha, \beta, T)(E^{1/2} + E), \quad (2.4.2)$$

*where  $\sigma(\alpha, \beta, T)$  is a constant which goes to zero in the regime (2.2.7).*

Let us finally recall [37, Lemma 6.6].

**Lemma 2.4.4.** *Let  $\beta_n \rightarrow 0$ . Let  $\chi_n \in BV(\mathbb{T}^2, \{0, \beta_n^{-1}\})$  be such that  $\lim_{n \rightarrow +\infty} \int_{\mathbb{T}^2} \chi_n dx' = 1$  and*

$$\sup_n \int_{\mathbb{T}^2} \beta_n^{1/2} |D' \chi_n| < +\infty.$$

*Then, up to a subsequence,  $\chi_n \rightharpoonup \sum_i \varphi_i \delta_{X_i}$  for some at most countable family  $\varphi_i > 0$  and  $X_i \in \mathbb{T}^2$ , and*

$$\liminf_{n \rightarrow +\infty} \int_{\mathbb{T}^2} \beta_n^{1/2} |D' \chi_n| \geq 2\sqrt{\pi} \sum_i \sqrt{\varphi_i}. \quad (2.4.3)$$

The proof of Proposition 2.4.1 follows from the following lemma, which is a minor modification of [37, Proposition 6.1].

**Lemma 2.4.5.** *Consider  $T_n, \alpha_n, \beta_n, \lambda > 0$  as in (2.2.7). Let  $(u_n, A_n) \in \mathcal{K}$  be such that  $\sup_{n \in \mathbb{N}} \tilde{E}_{T_n}^{\text{int}}(u_n, A_n) < +\infty$ . Then, there exist  $(\mu, m) \in \mathcal{A}_{1,1}$  such that, up to a subsequence, the following facts hold:*

(i)  $\beta_n^{-1} B_3^n \rightharpoonup \mu$ ,  $B_3^n(\cdot, x_3) \rightharpoonup \mu_{x_3}$  for a.e.  $x \in (-1, 1)$ ,  $\beta_n^{-1}(1 - |u_n|^2) \rightharpoonup \mu$  and  $\beta_n^{-1} B'_n \rightharpoonup m$ .

(ii) One has

$$\liminf_{n \rightarrow +\infty} \tilde{E}_{T_n}^{\text{int}}(u_n, A_n) \geq I(\mu, m).$$

*Proof.* As said, the proof is an adaptation of [37, Proposition 6.1]. The proof of [37, Proposition 6.1] is divided in two steps, and uses the additional hypothesis  $\lambda = +\infty$ .

In Step 1, without assumptions on  $\lambda$ , it is shown the existence of  $\mu \in \mathcal{M}^+(\mathbb{T}_{1,1}^2)$  and  $m \in \mathcal{M}(\mathbb{T}_{1,1}^2, \mathbb{R}^2)$  such that  $m \ll \mu$ , (2.2.8) is satisfied in the sense of distributions,  $\mu = \mu_{x_3} \otimes dx_3$  and, up to a subsequence, (i) holds with

$$\liminf_{n \rightarrow \infty} \int_{\mathbb{T}_{1,1}^2} \beta_n^{-1} |B'_n|^2 dx \geq \int_{\mathbb{T}_{1,1}^2} \left| \frac{dm}{d\mu} \right|^2 d\mu. \quad (2.4.4)$$

Thus, we need to prove that for a.e.  $x_3 \in (-1, 1)$  it holds  $\mu_{x_3} = \sum_i \varphi_i X_i$  for some  $\varphi_i > 0$  and  $X_i \in \mathbb{T}^2$ . Moreover, we need to prove (ii). Both facts are proved in Step 2 of [37, Proposition 6.1] with the extra assumption  $\lambda = +\infty$ . Thus, in the rest of the proof we adapt Step 2 of [37, Proposition 6.1] to the case  $0 < \lambda < +\infty$ . Even though most parts of the reasoning stay the same, for convenience of the reader we repeat the entire argument.

First, notice that

$$\int_{\mathbb{T}_{1,1}^2} \beta_n^{-2} (B_3^n - (1 - \rho_n))^2 dx \leq \alpha_n^{-2/3} \beta_n^{-4/3} \tilde{E}_{T_n}(u_n, A_n) \rightarrow 0. \quad (2.4.5)$$

Using [39, Lemma 3.7], there exists  $\hat{u}_n$  with  $\hat{\rho}_n = |\hat{u}_n|^2 = \min\{\rho_n, 1\}$  such that  $\tilde{E}_{T_n}(\hat{u}_n, A_n) \leq (1 + 2\alpha_n/T_n) \tilde{E}_{T_n}(u_n, A_n)$ . Moreover, by (2.2.7) also

$$\int_{\mathbb{T}_{1,1}^2} \beta_n^{-2} (B_3^n - (1 - \hat{\rho}_n)) \rightarrow 0$$

and  $\alpha_n/T_n \rightarrow 0$ , hence we may prove (ii) for  $(\hat{u}_n, A_n)$  instead of  $(u_n, A_n)$ . To ease notation, we drop the hats in the rest of the proof. Fix  $\varepsilon > 0$ . Using Lemma 2.4.2 and  $|\nabla' \rho^{1/2}| \leq \left| \nabla'_{\alpha^{1/3} \beta^{-1/3} T A} u \right|$  we obtain

$$\left(1 + C_\varepsilon \frac{\alpha_n}{T_n}\right) \tilde{E}_{T_n}(u_n, A_n) \geq \beta_n^{-1/2} \int_{\mathbb{T}_{1,1}^2} \alpha_n^{-2/3} \beta_n^{1/6} \left| \nabla' \rho_n^{1/2} \right|^2 + \alpha_n^{2/3} \beta_n^{-1/6} W_\varepsilon(\rho_n) dx. \quad (2.4.6)$$

Moreover, using Cauchy-Schwarz,

$$\begin{aligned} \beta_n^{-1/2} \int_{\mathbb{T}_{1,1}^2} \alpha_n^{-2/3} \beta_n^{1/6} \left| \nabla' \rho_n^{1/2} \right|^2 + \alpha_n^{2/3} \beta_n^{-1/6} W_\varepsilon(\rho_n) dx \\ \geq \beta_n^{-1/2} \int_{\mathbb{T}_{1,1}^2} 2\sqrt{W_\varepsilon(\rho_n)} \left| \nabla' \rho_n^{1/2} \right| dx. \end{aligned} \quad (2.4.7)$$

Fix  $0 < \delta < \varepsilon$ . By using the coarea formula (2.2.4),

$$\int_{\mathbb{T}_{1,1}^2} 2\sqrt{W_\varepsilon(\rho_n)} \left| \nabla' \rho_n^{1/2} \right| dx \geq \int_\delta^{1-\delta} \int_{-1}^1 2\sqrt{W_\varepsilon(s^2)} \mathcal{H}^1(\partial\{\rho_n(\cdot, x_3) > s^2\}) dx_3 ds.$$

Moreover, using the mean value theorem, there exists  $s_n \in [\delta, 1 - \delta]$  such that

$$\begin{aligned} \int_\delta^{1-\delta} \int_{-1}^1 2\sqrt{W_\varepsilon(s^2)} \mathcal{H}^1(\partial\{\rho_n(\cdot, x_3) > s^2\}) dx_3 ds \\ \geq \frac{1}{1-2\delta} \int_\delta^{1-\delta} 2\sqrt{W_\varepsilon(s^2)} ds \int_{-1}^1 \mathcal{H}^1(\partial\{\rho_n(\cdot, x_3) > s_n^2\}) dx_3. \end{aligned}$$

We introduce  $\chi_n(x', x_3) = \beta_n^{-1}(1 - \chi_{\{\rho_n(\cdot, x_3) > s_n^2\}})$ , so that we can infer

$$\int_{\mathbb{T}_{1,1}^2} \beta_n^{-1/2} 2\sqrt{W_\varepsilon(\rho_n)} \left| \nabla' \rho_n^{1/2} \right| dx \geq C_{\delta,\varepsilon} \int_{\mathbb{T}_{1,1}^2} \beta_n^{1/2} |D'\chi_n| dx_3, \quad (2.4.8)$$

where  $C_{\delta,\varepsilon} = \frac{1}{1-2\delta} \int_\delta^{1-\delta} 2\sqrt{W_\varepsilon(s^2)} ds$ . We now claim that

$$\lim_{n \rightarrow +\infty} \int_{\mathbb{T}_{1,1}^2} |\chi_n - \beta_n^{-1}(1 - \rho_n)| dx = 0, \quad (2.4.9)$$

which we prove in the sequel. Using (2.4.5), (2.4.9) implies that

$$\lim_{n \rightarrow +\infty} \int_{\mathbb{T}_{1,1}^2} |\chi_n - \beta_n^{-1} B_3^n| dx = 0.$$

Using Fubini, this implies that, up to a subsequence, for a.e.  $x \in (-1, 1)$  it holds  $\chi_n(\cdot, x_3) \rightharpoonup \mu_{x_3}$ . Furthermore,  $\int_{\mathbb{T}^2} \beta_n^{-1} B_3^n(x', x_3) dx' = 1$  for a.e.  $x \in (-1, 1)$  implies

$$\lim_{n \rightarrow +\infty} \int_{\mathbb{T}^2} \chi_n(x', x_3) dx' = 1$$

for a.e.  $x \in (-1, 1)$ . Using Lemma 2.4.4, we deduce that  $\mu_{x_3} = \sum_i \varphi_i \delta_{X_i}$  for some at most countable family  $\varphi_i > 0$  and  $X_i \in \mathbb{T}^2$ . Moreover by (2.4.3), using Fatou's Lemma,

$$\liminf_{n \rightarrow +\infty} \int_{\mathbb{T}_{1,1}^2} \beta_n^{1/2} |D'\chi_n| dx_3 \geq 2\sqrt{\pi} \int_{-1}^1 \sum_i \sqrt{\varphi_i} dx_3. \quad (2.4.10)$$

Combining (2.4.4), (2.4.7), (2.4.8) and (2.4.10), we deduce for any  $\varepsilon, \delta > 0$

$$\liminf_{n \rightarrow +\infty} \tilde{E}_{T_n}^{\text{int}}(u_n, A_n) \geq 2\sqrt{\pi} C_{\delta,\varepsilon} \int_{-1}^1 \sum_i \sqrt{\varphi_i} dx_3 + \int_{\mathbb{T}_{1,1}^2} \left| \frac{dm}{d\mu} \right|^2 d\mu.$$

Since

$$\lim_{\varepsilon \rightarrow 0} \lim_{\delta \rightarrow 0} C_{\delta,\varepsilon} = 2 \int_0^1 (1 - s^2) ds = \frac{4}{3},$$

the claim follows. Let us finally prove (2.4.9) This is the part of the proof that departs from [37]. Let  $\gamma_n \rightarrow 0$  to be chosen later. By the definitions of  $\chi_n$  and  $s_n$ ,  $\rho_n \leq \delta$  implies  $\chi_n = \beta_n^{-1}$ ,

while  $\rho_n \geq 1 - \delta$  implies  $\chi_n = 0$ . We thus can estimate for  $n$  large enough

$$\begin{aligned} \int_{\mathbb{T}_{1,1}^2} |\chi_n - \beta_n^{-1}(1 - \rho_n)| dx &\leq \beta_n^{-1} \int_{\{\rho_n \leq \delta\}} \rho_n dx + \beta_n^{-1} \int_{\{\rho_n \geq 1 - \gamma_n\}} (1 - \rho_n) dx \\ &\quad + \beta_n^{-1} \int_{\{1 - \delta \leq \rho_n \leq 1 - \gamma_n\}} (1 - \rho_n) dx + \beta_n^{-1} |\{\delta \leq \rho_n \leq 1 - \delta\}| \\ &\leq \beta_n^{-1} \gamma_n + \beta_n^{-1} \int_{\{\rho_n \leq \delta\}} \rho_n dx + \beta_n^{-1} \int_{\{1 - \delta \leq \rho_n \leq 1 - \gamma_n\}} (1 - \rho_n) dx \\ &\quad + \beta_n^{-1} |\{\delta \leq \rho_n \leq 1 - \delta\}|. \end{aligned} \quad (2.4.11)$$

Since  $\delta < \varepsilon$ , by (2.4.1) we have that  $\min_{[\delta, 1 - \delta]} W_\varepsilon = \delta^2$ . Hence using (2.4.6) and  $\tilde{E}_{T_n}(u_n, A_n) \lesssim 1$ , we find

$$|\{\delta \leq \rho_n \leq 1 - \delta\}| \leq \delta^{-2} \int_{\mathbb{T}_{1,1}^2} W_\varepsilon(\rho_n) dx \lesssim_\delta \alpha_n^{-2/3} \beta_n^{2/3}. \quad (2.4.12)$$

Moreover, using again (2.4.1), (2.4.6) and  $\tilde{E}_{T_n}(u_n, A_n) \lesssim 1$ ,

$$\int_{\{\rho_n \leq \delta\}} \rho_n dx \lesssim_{\varepsilon, \delta} \int_{\mathbb{T}_{1,1}^2} W_\varepsilon(\rho_n) dx \lesssim_{\varepsilon, \delta} \alpha_n^{-2/3} \beta_n^{2/3}. \quad (2.4.13)$$

Finally, once more by (2.4.1), (2.4.6) and  $\tilde{E}_{T_n}(u_n, A_n) \lesssim 1$ ,

$$\begin{aligned} \int_{\{1 - \delta \leq \rho_n \leq 1 - \gamma_n\}} (1 - \rho_n) dx &\leq \gamma_n^{-1} \int_{\{1 - \delta \leq \rho_n \leq 1 - \gamma_n\}} (1 - \rho_n)^2 dx \\ &\lesssim_{\varepsilon, \delta} \gamma_n^{-1} \int_{\mathbb{T}_{1,1}^2} W_\varepsilon(\rho_n) dx \lesssim_{\varepsilon, \delta} \gamma_n^{-1} \alpha_n^{-2/3} \beta_n^{2/3}. \end{aligned} \quad (2.4.14)$$

Combining (2.4.11), (2.4.12), (2.4.13), (2.4.14), we find

$$\int_{\mathbb{T}_{1,1}^2} |\chi_n - \beta_n^{-1}(1 - \rho_n)| dx \lesssim_{\delta, \varepsilon} \beta_n^{-1} (\gamma_n + \gamma_n^{-1} \alpha_n^{-2/3} \beta_n^{2/3} + \alpha_n^{-2/3} \beta_n^{2/3}).$$

We then optimize by choosing  $\gamma_n = \alpha_n^{-1/3} \beta_n^{1/3}$ , which entails

$$\int_{\mathbb{T}_{1,1}^2} |\chi_n - \beta_n^{-1}(1 - \rho_n)| dx \lesssim_{\delta, \varepsilon} \alpha_n^{-1/3} \beta_n^{-2/3} + \alpha_n^{-2/3} \beta_n^{-1/3},$$

from which (2.4.9) holds thanks to (2.2.7). □

*Proof of Proposition 2.4.1.* Applying Lemma 2.4.5, we get that there exist  $(\mu, m) \in \mathcal{A}_{1,1}$  such that, up to a subsequence, (i) holds and

$$\liminf_{n \rightarrow +\infty} \tilde{E}_{T_n}^{\text{int}}(u_n, A_n) \geq I(\mu, m). \quad (2.4.15)$$

Since  $E_{T_n}(u_n, A_n) \lesssim 1$  and  $\alpha_n \beta_n^{7/2} \rightarrow \lambda^{1/3}$ , we have

$$\|\beta^{-1} B_3^n - 1\|_{H^{-1/2}(\{x_3 = \pm 1\})}^2 \lesssim_\lambda 1. \quad (2.4.16)$$

By weak compactness, (2.4.16) implies that there exist  $\sigma_\pm \in H^{-1/2}(\mathbb{T}^2)$  such that

$$\beta^{-1} B_3^n(\cdot, \pm 1) \xrightarrow{H^{-1/2}} \sigma_\pm.$$

Now take  $\psi \in C^\infty(\mathbb{T}^2)$ . For a.e.  $x_3 \in (-1, 1)$ , using (i) we have

$$\begin{aligned} & \left| \int_{\mathbb{T}^2} \psi d(\sigma_+ - \mu_{x_3}) \right| \leq \limsup_{n \rightarrow +\infty} \left| \int_{\mathbb{T}^2} \psi d(\sigma_+ - \beta_n^{-1} B_3^n(\cdot, 1)) \right| \\ & + \limsup_{n \rightarrow +\infty} \left| \int_{\mathbb{T}^2} \psi d(\beta_n^{-1} B_3^n(\cdot, 1) - \beta_n^{-1} B_3^n(\cdot, x_3)) \right| = \limsup_{n \rightarrow +\infty} \left| \int_{\mathbb{T}^2} \psi d(\beta_n^{-1} B_3^n(\cdot, 1) - \beta_n^{-1} B_3^n(\cdot, x_3)) \right| \\ & \leq \limsup_{n \rightarrow +\infty} \left\| \beta_n^{-1} B_3^n(\cdot, 1) - \beta_n^{-1} B_3^n(\cdot, x_3) \right\|_{BL} \|\psi\|_{Lip}. \end{aligned} \quad (2.4.17)$$

We now estimate using (2.4.2) and  $\tilde{E}_{T_n}(u_n, A_n) \lesssim 1$

$$\limsup_{n \rightarrow +\infty} \left\| \beta_n^{-1} B_3^n(\cdot, 1) - \beta_n^{-1} B_3^n(\cdot, x_3) \right\|_{BL} \lesssim |1 - x_3|^{1/2}. \quad (2.4.18)$$

Pairing (2.4.17) and (2.4.18), we deduce  $\sigma_+ = \mu_1$ . Similarly,  $\sigma_- = \mu_{-1}$ , thus proving (ii). Finally, by weak lower semicontinuity of the  $H^{-1/2}$  norm and (2.4.15), also (iii) holds.  $\square$

## 2.5 Upper bound

In this section we consider sequences  $\alpha_n, \beta_n, T_n$  such that (2.2.7) and the quantization condition

$$L_n^2 T_n \alpha_n \beta_n \in 2\pi\mathbb{N} \quad (2.5.1)$$

hold (recall (2.1.6)). If we set  $k_n = \alpha_n^{1/3} \beta_n^{2/3} T_n$ , the condition (2.5.1) reduces to  $k_n \in 2\pi\mathbb{N}$ . Let us set for  $-1 < a < b < 1$

$$\begin{aligned} \tilde{F}^{(a,b)}(\rho, B) &= \int_{\mathbb{T}^2 \times (a,b)} \alpha^{-2/3} \beta^{-1/3} \left| \nabla' \rho^{1/2} \right|^2 + \alpha^{-4/3} \beta^{-2/3} \left| \partial_3 \rho^{1/2} \right|^2 \\ &+ \alpha^{2/3} \beta^{-2/3} (B_3 - (1 - \rho))^2 + \beta^{-1} |B'|^2 dx. \end{aligned}$$

We write  $\tilde{F}(\rho, B) = \tilde{F}^{(-1,1)}(\rho, B)$ . The goal of this section is to prove the following proposition.

**Proposition 2.5.1.** *Consider  $\alpha_n, \beta_n, T_n$  as in (2.2.7) and such that (2.5.1) holds. For every  $\mu \in \mathcal{A}_1^*$  with  $\mathcal{E}(\mu) < \infty$  there exists  $(u_n, A_n) \in \mathcal{K}$  such that  $\beta_n^{-1} B_3^n \rightharpoonup \mu$ ,  $\beta_n^{-1} (1 - |u_n|^2) \rightharpoonup \mu$ ,  $\beta_n^{-1} B_3^n(\cdot, \pm 1) \xrightarrow{H^{-1/2}} \mu_{\pm 1}$  and*

$$\limsup_{n \rightarrow +\infty} \tilde{E}_{T_n}(u_n, A_n) \leq \mathcal{E}(\mu). \quad (2.5.2)$$

Moreover,  $\beta_n^{-1} B_n' \rightharpoonup m$ , where  $m$  is such that  $(\mu, m) \in \mathcal{A}(\mathbb{T}^2, (-1, 1))$  and  $I(\mu) = I(\mu, m)$ .

The construction of the recovery sequence proceeds as follows. By Lemma 2.3.4 we may restrict to proving Proposition 2.5.1 for measures  $\mu$  such that  $\mu_{\pm 1} \in \mathcal{P}^c(\mathbb{T}^2)$ . Furthermore, if we fix  $N \gg 1$  and set  $\varepsilon_N = N^{-3/2}$ , we can further approximate  $\mu$  by a measure  $\mu^N$  which is regular on  $\mathbb{T}^2 \times [-1 + N^{-3/2}, 1 - N^{-3/2}]$ . Then, as in [37], we construct the density  $\rho$  and the magnetic field  $B$  separately in two regions. In the bulk (that is, in  $\mathbb{T}^2 \times [-1 + N^{-3/2}, 1 - N^{-3/2}]$ ), the measure  $\mu^N$  is regular and we can plug in the bulk construction from [37, Proposition 7.9]. In the boundary layer, we adapt the construction from [39, Lemma 4.7]. Finally we can reconstruct the couple  $(u, A) \in \mathcal{K}$  by applying [37, Proposition 7.8].

The construction in the bulk and the reconstruction of the couple  $(u, A)$  do not differ significantly from [37]. For the convenience of the reader, we give a sketch of the proof of the bulk

construction. On the contrary, the boundary layer construction from [37] needs to be adapted to our case of non-uniform boundary conditions.

We start by defining for  $a < b \in \mathbb{R}$  and  $\beta > 0$  the set

$$\begin{aligned} \mathcal{K}^*(a, b) &= \mathcal{K}_\beta^*(a, b) \\ &= \left\{ (\rho, B) \in H^1(\mathbb{T}^2 \times (a, b)) \times L^2(\mathbb{T}^2 \times (a, b)), \operatorname{div} B = 0, \int_{\mathbb{T}^2} B(x', x_3) dx' = \beta \quad \forall x_3 \in (a, b) \right\}. \end{aligned}$$

**Definition 2.5.2.** Let  $k > 0$ ,  $a < b \in \mathbb{R}$  and  $\beta > 0$ . A couple  $(\rho, B) \in \mathcal{K}^*(a, b)$  is  $k$ -quantized if there exists a closed set  $\omega \subset \mathbb{T}^2 \times (a, b)$  such that  $B = 0$  in  $\omega^c$ ,  $\rho = 0$  in  $\omega$  and for all  $z \in (a, b)$  the flux of  $\beta^{-1}B_3$  over every connected component of  $\omega \cap \{x_3 = z\}$  is an integer multiple of  $2\pi/k$ .

We recall [37, Proposition 7.8].

**Proposition 2.5.3.** Let  $\alpha, \beta, T > 0$ ,  $k = \alpha^{1/3}\beta^{2/3}T$ ,  $(\rho, B) \in \mathcal{K}^*(-1, 1)$  be  $k$ -quantized with  $\rho B = 0$ . Set  $\omega = \{\rho = 0\}$  and, for any  $z \in (-1, 1)$ ,  $\omega_z = \omega \cap \{x_3 = z\}$ . Assume that  $\omega$  is closed,  $\omega^c$  is connected and, for any  $z \in (-1, 1)$ ,  $\omega_z^c$  is also connected. Then there exists  $(u, A) \in \mathcal{K}$  such that  $\rho = |u|^2$ ,  $B = \nabla \times A$  and

$$\tilde{E}_T(u, A) = \tilde{F}(\rho, B). \quad (2.5.3)$$

The following proposition is just a rewriting of the bulk construction performed in [37, Proposition 7.9].

**Proposition 2.5.4.** Let  $\delta > 0$ ,  $R \gg 1$  and  $t \leq 1$ . Consider  $\alpha_n, \beta_n, T_n$  and  $k_n = \alpha_n^{1/3}\beta_n^{2/3}T_n$  satisfying (2.2.7) and (2.5.1). Consider  $\mu \in \mathcal{M}_{\text{reg}}(\mathbb{T}_{1,t}^2)$  such that  $\mu_{\pm t} = \sum_{i=1}^{m^\pm} \varphi_i^\pm \delta_{X_i^\pm}$ , with  $m^\pm, \varphi_i^\pm > 0$  and  $X_i^\pm \in \mathbb{T}^2$ . Let  $I(\mu) = I(\mu, (-t, t))$ . Then, there exist  $C > 0$ ,  $K_R \xrightarrow{R \rightarrow +\infty} K^*$  and  $k_n$ -quantized  $(\rho_n, B_n) \in \mathcal{K}^*(-t, t)$  such that  $\rho_n B_n = 0$ ,

$$\limsup_{n \rightarrow +\infty} \tilde{F}^{(-t, t)}(\rho_n, B_n) \leq \frac{t}{t - \delta} \frac{K_R}{K^*} I(\mu) + C\delta \max\{m^+, m^-\}^{1/2} (1 + I(\mu)) \quad (2.5.4)$$

and

$$\limsup_{n \rightarrow +\infty} W_{\text{per}}^2(\beta_n^{-1}B_3^n, \mu) \lesssim \delta. \quad (2.5.5)$$

Moreover, the set  $\omega_n = \{\rho_n = 0\}$  is closed,  $\omega_n^c$  is connected and for any  $z \in (-t, t)$  the set  $\omega_{n,z} = \omega_n \cap \{x_3 = z\}$  is such that  $\omega_{n,z}^c$  is connected. Finally, for  $i = 1 \dots m^\pm$ , there exist some values  $\varphi_{i,n}^\pm$  with

$$\left| \varphi_{i,n}^\pm - \varphi_i^\pm \right| \lesssim_\mu k_n^{-1}, \quad (2.5.6)$$

$k_n \varphi_{i,n}^\pm \in 2\pi\mathbb{N}$  and  $\sum_i \varphi_{i,n}^\pm = 1$  such that, if we denote by  $Q_{n,i}^\pm$  the cubes centered in the points  $X_i^\pm$  with  $|Q_{n,i}^\pm| = \beta_n \varphi_{i,n}^\pm$ , it holds

$$B_3^n(x, \pm t) = \sum_{i=1}^{m^\pm} \chi_{Q_{n,i}^\pm}(x') \quad \text{and} \quad \rho_n^{1/2}(x', \pm t) = \min\{1, \eta_n^{-1} \operatorname{dist}(x', \cup_i Q_{n,i}^\pm)\}, \quad (2.5.7)$$

where  $\eta_n = \alpha_n^{-2/3}\beta_n^{1/6}$ .

**Remark 2.5.5.** *The values  $K_R$  and  $K^*$  are related to the optimal problem given by (2.1.3) at the scale of coherence length,  $\alpha^{-2/3}\beta^{1/6}$  in our units. Since the penetration length  $\alpha^{-2/3}\beta^{1/6}T^{-1}$  is much smaller than the coherence length, by the Meissner condition (2.1.3) reduces to a degenerate Modica-Mortola type functional (see [37, Section 4]). The constant  $K_R$  is the one defined in [37, Lemma 7.2], while  $K^* = 8\sqrt{\pi}/3$  as in (2.1.8). Consider  $G(v) = 2\sqrt{\pi} \int_0^{+\infty} |\dot{v}|^2 + (1 - v^2)^2 dt$ . The constant  $K^*$  is characterized by the optimal profile problem*

$$K^* = \inf\{G(v) : v(0) = 0, \lim_{t \rightarrow +\infty} v(t) = 1\}. \quad (2.5.8)$$

Let  $v$  be a minimizer of (2.5.8). The constant  $K_R$  satisfies  $K_R = G(v_R)$ , where  $v_R$  is an appropriate cut version of  $v$  at the scale  $R \gg 1$  (see [37, Lemma 7.2]). The profile  $v_R$  is the one we will plug near the boundary of the flux tubes at the scale of coherence length in the following construction, and is the one which allows us to apply [37, Lemma 7.3] and [37, Lemma 7.6].

*Proof (sketch).* The key construction is described in [37, Proposition 7.9]. For convenience of the reader, we give a sketch of the proof. Let us start by defining for  $s \in (-t, t)$

$$\hat{\mu}_s = \begin{cases} \mu_{s/(t-\delta)} & \text{for } |s| \leq t - \delta, \\ \mu_t & \text{for } s \geq t - \delta, \\ \mu_{-t} & \text{for } s \leq -t + \delta. \end{cases}$$

We have

$$I(\hat{\mu}) \leq \frac{t}{t-\delta} I(\mu) + 2K^*\delta \max\{m^+, m^-\}^{1/2},$$

thus we may perform the construction on  $\hat{\mu}$ .

We first characterize the geometry of the construction, which does not depend on  $n$ . Consider first the region  $\{|s| \leq t - \delta\}$ . Since  $\hat{\mu} \in \mathcal{M}_{\text{reg}}(\mathbb{T}_{1,t}^2)$ , the measure  $\hat{\mu}$  is supported on finitely many polygonal curves  $X_i : [a_i, b_i] \rightarrow \mathbb{T}^2$ , each with a flux  $\varphi_i > 0$ . We call  $y_j = (Y_j, z_j) \in \mathbb{T}_{1,1}^2$  the internal endpoints of the curves, which correspond to triple points. By fixing  $l > 0$  small enough, it is possible to find some disjoint cylinders  $\omega_j$  and "tubular" sets  $\tau_i$  with the properties as follows.

Each  $\omega_j$  is centered in an internal endpoint  $y_j = (Y_j, z_j)$  and is of the form

$$\omega_j = B'(Y_j, c_1 l) \times (z_j - c_2 l, z_j + c_2 l)$$

for some  $c_1, c_2 > 0$  depending only on  $\hat{\mu}$ , and

$$\text{supp } \hat{\mu} \cap (\partial B'(Y_j, c_1 l) \times (z_j - c_2 l, z_j + c_2 l)) = \emptyset,$$

that is the support of  $\hat{\mu}$  intersects only the top and bottom boundaries of  $\omega_j$ . Each  $\tau_i$  is of the form  $\tau_i = X_i^l \cap (\cup_j \omega_j)^c$ , where  $X_i^l$  is  $c_3 l$ -neighborhood of a curve  $X_i$ , with  $c_3 > 0$  depending only on  $\hat{\mu}$ . Moreover,

$$\text{supp } \hat{\mu} \cap \{|s| \leq t - \delta\} \subset (\cup_j \omega_j) \cup (\cup_i \tau_i).$$

The construction of  $(\rho, B)$  near a curve  $X_i$  is done in [37, Lemma 7.3], while the construction around branching points is done in [37, Lemma 7.6]. For  $n$  large enough, by (2.2.7) the hypothesis of [37, Lemma 7.3] and [37, Lemma 7.6] are satisfied. If we momentarily ignore the quantization requirement, we can thus define  $(\rho, B)$  in each  $\tau_i$  using [37, Lemma 7.3] and in each  $\omega_j$  using [37, Lemma 7.6]. We moreover define  $(\rho, B) = (1, 0)$  outside of  $(\cup_j \omega_j) \cup (\cup_i \tau_i)$ . In the region  $\{|s| \geq t - \delta\}$ , the measure  $\hat{\mu}$  is supported on constant curves, and we may apply [37, Lemma 7.4] on each curve so that the boundary conditions (2.5.7) are attained. With this

construction, the sets  $\omega_n = \{\rho_n = 0\}$  and  $\omega_{n,z} = \omega_n \cap \{x_3 = z\}$  as above satisfy the required hypothesis.

To account for quantization, let  $\hat{\mu}^n$  be the  $k_n$ -quantized approximation of  $\hat{\mu}$  given by Lemma 2.3.8. For  $n$  large enough, it holds

$$\frac{1}{2}\hat{\mu} \leq \hat{\mu}^n \leq 2\hat{\mu}. \quad (2.5.9)$$

Moreover, if  $\hat{\mu}_{\pm t}^n = \sum_{i=1}^{m_{\pm}} \varphi_{i,n}^{\pm} \delta_{X_i^{\pm}}$ , the fluxes  $\varphi_{n,i}^{\pm}$  satisfy the required hypothesis. Because of (2.5.9), we can repeat for  $\hat{\mu}^n$  the same construction of  $(\rho_n, B_n)$  which we described for  $\hat{\mu}$ , with a geometry which does not depend on  $n$ . Finally, the same computations as [37, Proposition 7.9], paired with (2.3.16) and (2.3.17), give (2.5.4) and (2.5.5).  $\square$

Let us now address the boundary layer. The following proposition is a variation of [37, Proposition 7.7].

**Proposition 2.5.6.** *Let  $N \in \mathbb{N}$ ,  $\alpha, \beta, T, t > 0$ . Consider  $m \in \mathbb{N}$ , for  $j = 1 \dots m$  consider  $a_j > 0$  and pairwise disjoint cubes  $Q_j \subset \mathbb{T}^2$  such that  $\sum_j a_j |Q_j| = 1$ . Define*

$$\sigma = \sum_{j=1}^m a_j \chi_{Q_j}. \quad (2.5.10)$$

Set  $k = \alpha^{1/3} \beta^{2/3} T$  and consider for  $i = 1 \dots N^2, j = 1 \dots m$  positive numbers  $\varphi_{i,j}$  such that

$$k\varphi_{i,j} \in 2\pi\mathbb{N} \quad \text{and} \quad \sum_{i=1}^{N^2} \sum_{j=1}^m \varphi_{i,j} = 1.$$

Assume that  $\beta \leq \alpha$ ,  $\alpha\sqrt{2}/t \leq 1$ ,  $\alpha^{-1} \ll t$  and that, for all  $i = 1 \dots N^2$  and  $j = 1 \dots m$ , it holds  $\alpha^{-2/3} \beta^{-1/3} \lesssim \sqrt{|Q_j|}/N$  and  $\varphi_{i,j} \sim a_j |Q_j| N^2$ . Then there exist  $k$ -quantized  $(\rho, B) \in \mathcal{K}^*(0, t)$  such that

$$F^{(0,t)}(\rho, B) \lesssim_{\sigma} tN + \frac{1}{N^2 t} \quad (2.5.11)$$

and

$$\alpha^{1/3} \beta^{7/6} \left\| \beta^{-1} B_3 - \sigma \right\|_{H^{-1/2}(x_3=0)}^2 \lesssim_{\sigma} \frac{\beta^{1/3}}{\alpha^{1/3}} + \frac{\beta^{1/2}}{N^2 t} + \frac{\alpha^{1/3} N^4}{\beta^{5/6}} \sup_{i,j} \left| \varphi_{i,j} - \frac{a_j |Q_j|}{N^2} \right|^2. \quad (2.5.12)$$

Moreover, if for  $j = 1 \dots m$  we denote by  $Q_{i,j}$  the  $N^2$  cubes centered in an uniform grid of spacing  $\sqrt{|Q_j|}/N$  inside  $Q_j$ , with  $|Q_{i,j}| = \beta\varphi_{i,j}$ , it holds

$$B_3(x', t) = \sum_{i=1}^{N^2} \sum_{j=1}^m \chi_{Q_{i,j}}(x') \quad \text{and} \quad \rho_n^{1/2}(x', \pm t) = \min\{1, \eta^{-1} \text{dist}(x', \cup_i Q_{i,j}^{\pm})\}, \quad (2.5.13)$$

where  $\eta = \alpha^{-2/3} \beta^{1/6}$ .

*Proof.* The proof is similar to [37, Proposition 7.7] and is based on a careful application of [39, Lemma 4.7]. However, differently from [37, Proposition 7.7], we apply [39, Lemma 4.7] in the case of non-uniform branching, that is when the measure at the boundary is not the Lebesgue measure.

For  $j = 1 \dots m$  we let  $r_{i,j}$  be the  $N^2$  cubes of side  $\sqrt{|Q_j|}/N$  forming a uniform grid inside  $Q_j$ ,  $Q_{i,j}$  be the cubes with the same center as  $r_{i,j}$  and size  $|Q_{i,j}| = \beta\varphi_{i,j}$ , and  $b_{i,j} =$

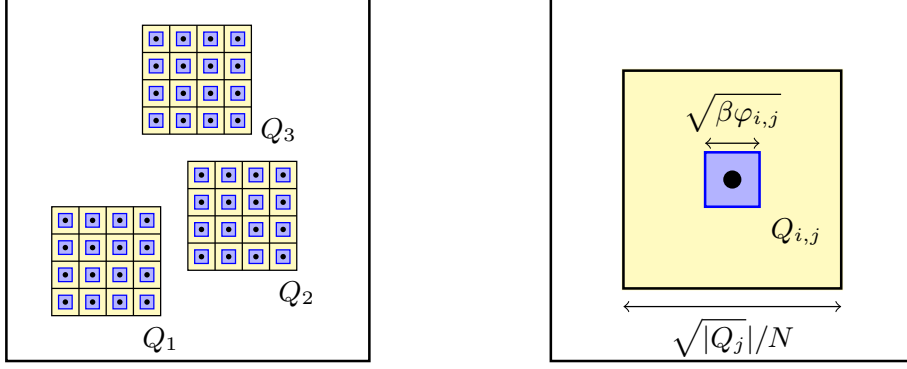


Figure 2.2: Details of the construction of Proposition 2.5.6 at time  $t$  for a measure  $\sigma$  concentrated on three cubes  $Q_1, Q_2$  and  $Q_3$ , in yellow. On the left, the uniform grid of squares  $Q_{i,j}$  in blue. On the right, a zoom on one square  $Q_{i,j}$ . In the blue squares, the magnetic field  $B$  is uniformly distributed while the density  $\rho$  vanishes.

$\beta\varphi_{i,j}N^2/|Q_j|$  (see Figure 2.2). Let us recall the rescaling performed in [37, Proposition 7.7]. We denote with a star the quantities from [39, Lemma 4.7] and set  $T_* = Tt/2$ ,  $\kappa_* = \alpha\sqrt{2}/T$ ,  $L_* = T\alpha^{-1/3}\beta^{-1/6}$ ,  $r_{i,j}^* = L_*r_{i,j}$ ,  $\hat{r}_{i,j}^* = L_*Q_{i,j}$ ,  $d_0^* = L_*/N$ ,  $\rho_0^* = \beta^{1/2}L_*/N$  (recall that  $\varphi_{i,j} \sim_\sigma 1/N^2$  so that  $|\hat{r}_{i,j}^*| \sim_\sigma (\rho_0^*)^2$ ),  $N^* = \sqrt{m}N$  and  $b_{i,j}^* = (|\hat{r}_{i,j}^*|/|r_{i,j}^*|)k^*/\sqrt{2} = b_{i,j}\kappa^*/\sqrt{2}$ . By [39, Lemma 4.7], there exist  $Q_{L_*}$ -periodic fields  $\chi^* \in BV_{\text{loc}}(\mathbb{R}^2 \times (0, T_*); \{0, 1\})$  and  $B_* = (B'_*, B_3^*) \in L_{\text{loc}}^2(\mathbb{R}^3; \mathbb{R}^3)$  such that  $\text{div } B_* = 0$ ,  $B_*(1 - \chi_*) = 0$  and

$$\chi_*(x', T_*) = \sum_{i,j} \chi_{\hat{r}_i^*}(x') \quad B_3^*(x', T_*) = \frac{\kappa_*}{\sqrt{2}} \sum_{i,j} \chi_{\hat{r}_{i,j}^*}(x') \quad \text{for } x' \in \mathbb{T}_{L_*}^2.$$

Moreover, it holds

$$\begin{aligned} \frac{1}{L_*^2} \int_{\mathbb{T}_{L_*}^2 \times (0, T_*)} \kappa_* |D\chi_*| + |B'_*| + \chi_* \left( B_3^* - \frac{\kappa_*}{\sqrt{2}} \right)^2 dx \\ \lesssim E_*^{\text{int}} = \left( \frac{\kappa_* \rho_0^* T_*}{\sqrt{2}} + \frac{\kappa_*^2 (\rho_0^* d_0^*)^2}{2T_*} \right) \left( \frac{N^*}{L_*} \right)^2 \end{aligned} \quad (2.5.14)$$

and

$$\frac{1}{L_*^2} \left\| B_3^* - \sum_{i,j} b_{i,j}^* \chi_{r_{i,j}^*} \right\|_{H^{-1/2}(\mathbb{T}_{L_*}^2)}^2 \lesssim E_*^{\text{ext}} = \left( \kappa_*^* (\rho_0^*)^2 + \frac{\kappa_*^2 (\rho_0^*)^3 d_0^*}{T_*} \right) \left( \frac{N^*}{L_*} \right)^2.$$

As in [37, Proposition 7.7], we define  $\chi_*$  and  $B_*$  in  $\mathbb{T}_{L_*}^2 \times (0, 2T_*)$  by

$$\chi_*(x) = \chi_*(x', T_*) \quad \text{and} \quad B_*(x) = (0, B_3^*(x', T_*)) \quad \text{for } x' \in \mathbb{T}_{L_*}^2 \times (T_*, 2T_*),$$

so that estimate (2.5.14) holds in  $\mathbb{T}_{L_*}^2 \times (0, 2T_*)$ . We now define  $\omega_* = \{x : \chi_*(x) = 1\}$  and, denoting by  $d(\cdot, \omega_*)$  the 3D distance periodic in the horizontal directions, we define

$$\rho_*(x) = \min\{1, \kappa_*^2 d(x, \omega_*)\}.$$

The set  $\omega_*$  is invariant in the  $x_3$  direction in  $(T_*, 2T_*)$ , thus, since  $t \gg \alpha^{-1}$  gives  $T_* \gg \kappa_*^{-1}$ , we obtain for  $x_3 = 2T_*$

$$\min\{1, \kappa_*^2 d(x, \omega_*)\} = \min\{1, \kappa_*^2 d(x', \omega_* \cap \{x_3 = 2T_*\})\} \quad \text{for } x = (x', 2T_*).$$

Therefore,

$$\rho_*(x', 2T_*) = \min_{i,j} \{ \min(1, \kappa_*^2 d(x', \hat{r}_{i,j}^*)) \}.$$

Arguing as in [39, Theorem 4.9], we get

$$\frac{1}{L_*^2} \int_{\mathbb{T}_{L_*}^2 \times (0, 2T_*)} \left| \nabla \rho_*^{1/2} \right|^2 + |B_*'|^2 + \left( B_3^* - \frac{\kappa_*}{\sqrt{2}} (1 - \rho_*) \right)^2 dx \lesssim E_*^{\text{int}}.$$

We scale back and set

$$\rho(x', x_3) = \rho^*(L_* x', x_3) \quad \text{and} \quad (B', B_3)(x', x_3) = \frac{\sqrt{2}}{\kappa} \left( \frac{B_*'}{L_*}, B_3^* \right) (L_* x', x_3).$$

In particular,  $(\rho, B) \in \mathcal{K}_t^*$  and the boundary conditions (2.5.13) hold. Since  $\nabla' \rho^{1/2}(x) = L_* \nabla' \rho_*^{1/2}(L_* x', x_3)$  and  $\partial_3 \rho^{1/2}(x) = \partial_2 \rho_*^{1/2}(L_* x', x_3)$ , we have

$$\begin{aligned} \tilde{F}^{(0,t)}(\rho, B) &\leq \alpha^{-4/3} \beta^{-2/3} \frac{1}{L_*^2} \int_{\mathbb{T}_{L_*}^2 \times (0, 2T_*)} \left| \nabla \rho_*^{1/2} \right|^2 + |B_*'|^2 + \left( B_3^* - \frac{\kappa_*}{\sqrt{2}} (1 - \rho_*) \right)^2 dx \\ &\lesssim \alpha^{-4/3} \beta^{-2/3} E_*^{\text{int}}, \end{aligned}$$

which implies (2.5.11) since  $E_*^{\text{int}} \lesssim_\sigma \alpha^{-4/3} \beta^{-2/3} (Nt + 1/(N^2t))$ . Let us now turn to (2.5.12). By the embedding of  $L^\infty$  into  $H^{-1/2}$  and recalling (2.5.10)

$$\begin{aligned} \left\| \sum_{i,j} b_{i,j} \chi_{r_{i,j}} - \beta \sigma \right\|_{H^{-1/2}(\mathbb{T}^2)}^2 &= \left\| \sum_{i,j} (b_{i,j} - \beta a_j) \chi_{r_{i,j}} \right\|_{H^{-1/2}(\mathbb{T}^2)}^2 \\ &\lesssim \sup_{i,j} |b_{i,j} - \beta a_j|^2 \lesssim_\sigma \beta^2 N^4 \sup_{i,j} \left| \varphi_{i,j} - \frac{a_j |Q_j|}{N^2} \right|^2. \end{aligned} \quad (2.5.15)$$

Finally, we rescale the estimate for  $B_3^*$  to obtain

$$\left\| B_3 - \sum_{i,j} b_{i,j} \chi_{r_{i,j}} \right\|_{H^{-1/2}(\mathbb{T}^2)}^2 \lesssim \frac{2}{L_*^3 \kappa_*^2} \left\| B_3^* - \sum_{i,j} b_{i,j}^* \chi_{r_{i,j}^*} \right\|_{H^{-1/2}(\mathbb{T}^2)}^2 \lesssim_\sigma \frac{\beta^{7/6}}{\alpha^{2/3}} + \frac{\beta^{4/3}}{\alpha^{1/3} N^2 t},$$

which, paired with (2.5.15), gives (2.5.12).  $\square$

*Proof of Proposition 2.5.1.* Let  $\mu \in \mathcal{A}_1^*$  with  $\mathcal{E}(\mu) < \infty$ . By Lemma 2.3.4, we may suppose  $\mu_{\pm 1} \in \mathcal{P}^c(\mathbb{T}^2)$ . Let  $N, R \gg 1$  and define  $\varepsilon_N = N^{-3/2}$ . Consider  $\mu^N$  given by Lemma 2.3.6 applied to  $\mu$ . By (2.3.9), we may apply Proposition 2.5.4 to  $\mu^N \llcorner (-1 + \varepsilon_N, 1 - \varepsilon_N)$  with  $\delta = \varepsilon_N$  to obtain fields  $(\rho_n, B_n) \in \mathcal{K}^*(-1 + \varepsilon_N, 1 - \varepsilon_N)$ . In particular, by (2.3.10) and (2.5.4), if we let  $I(\hat{\mu}) = I(\hat{\mu}, (-1 + \varepsilon_N, 1 - \varepsilon_N))$ , it holds

$$\begin{aligned} \limsup_{n \rightarrow +\infty} \tilde{F}^{(-1+\varepsilon_N, 1-\varepsilon_N)}(\rho_n, B_n) &\leq \frac{1 - \varepsilon_N}{1 - 2\varepsilon_N} \frac{K_R}{K^*} I(\hat{\mu}) + C\varepsilon_N \max\{m^+ N^2, m^- N^2\}^{1/2} (1 + I(\hat{\mu})) \\ &\leq \frac{1}{1 - 2\varepsilon_N} \frac{K_R}{K^*} I(\mu) + C(\mu) N^{-1/2}. \end{aligned} \quad (2.5.16)$$

For  $|x_3| > 1 - \varepsilon_N$ , by (2.5.7) we may extend the fields  $(\rho, B)$  using Proposition 2.5.6. For simplicity let us focus on the interval  $(1 - \varepsilon_N, 1)$ . Condition (2.5.11) implies

$$F^{(1-\varepsilon_N, 1)}(\rho_n, B_n) \lesssim_\mu N^{-1/2}, \quad (2.5.17)$$

while, recalling (2.3.9) and (2.5.6), (2.5.12) gives

$$\alpha_n^{1/3} \beta_n^{7/6} \|\beta_n^{-1} B_3 - \mu_1\|_{H^{-1/2}(x_3=1)}^2 \lesssim \mu \frac{\beta_n^{1/3}}{\alpha_n^{1/3}} + \frac{\beta_n^{1/2}}{N^{1/2}} + \frac{\alpha_n^{1/3} N^4}{\beta_n^{5/6} k_n}. \quad (2.5.18)$$

Because of (2.2.7), condition (2.5.18) gives  $\beta_n^{-1} B_3^n(\cdot, 1) \xrightarrow{H^{-1/2}} \mu_1$ . Moreover, for any  $\xi > 0$ , by Young's inequality

$$\limsup_{n \rightarrow +\infty} \alpha_n^{1/3} \beta_n^{7/6} \|\beta_n^{-1} B_3 - \mu_1\|_{H^{-1/2}(x_3=1)}^2 \leq \lambda(1 - \xi) \|\mu_1 - 1\|_{H^{-1/2}(\mathbb{T}^2)}^2. \quad (2.5.19)$$

A similar reasoning applies for the interval  $(-1, -1 + \varepsilon_N)$ . Finally, define  $(u_n, A_n) \in \mathcal{K}$  by applying Proposition 2.5.3. By combining (2.5.3), (2.5.16), (2.5.17) and (2.5.19), we obtain

$$\limsup_{n \rightarrow +\infty} \tilde{E}_{T_n}(u_n, A_n) \leq \frac{1}{1 - 2\varepsilon_N} \frac{K_R}{K^*} I(\mu) + C(\mu) N^{-1/2} + \lambda(1 - \xi) \|\mu_{\pm 1} - 1\|_{H^{-1/2}(\mathbb{T}^2)}^2. \quad (2.5.20)$$

Moreover, notice that by (2.5.5)

$$\begin{aligned} & \limsup_{n \rightarrow +\infty} W_{\text{per}}^2(\beta_n^{-1} B_3^n, \mu^N) \\ & \lesssim \limsup_{n \rightarrow +\infty} W_{\text{per}}^2(\beta_n^{-1} B_3^n \lfloor (-1 + \varepsilon_N, 1 - \varepsilon_N), \mu^N \lfloor (-1 + \varepsilon_N, 1 - \varepsilon_N)) + N^{-3/2} \stackrel{(2.5.5)}{\lesssim} N^{-3/2}, \end{aligned}$$

which paired with (2.3.11) gives

$$\limsup_{n \rightarrow +\infty} W_{\text{per}}^2(\beta_n^{-1} B_3^n, \mu) \lesssim N^{-3/2}. \quad (2.5.21)$$

By a diagonal procedure in  $\xi, N$  and  $R$ , (2.5.20) and (2.5.21) give (2.5.2) and  $\beta_n^{-1} B_3^n \rightharpoonup \mu$ . Finally, from the compactness result in Proposition 2.4.1 and the uniqueness of  $m$  we obtain  $\beta_n^{-1}(1 - |u_n|^2) \rightharpoonup \mu$  and  $\beta_n^{-1} B'_n \rightharpoonup m$ .  $\square$

*Proof of Theorem 2.1.1.* The theorem follows by Proposition 2.4.1 and Proposition 2.5.1.  $\square$

## Chapter 3

# A $\Gamma$ -convergence result for 2D type-I superconductors

We consider a 2D non-standard Modica-Mortola type functional. This functional arises from the Ginzburg-Landau theory of type-I superconductors in the case of an infinitely long sample and in the regime of comparable penetration and coherence lengths. We prove that the functional  $\Gamma$ -converges to the perimeter functional. This result is a first step in understanding how to extend the results of [37] to the regime of non vanishing Ginzburg-Landau parameter  $\kappa$ .

### 3.1 Introduction

Superconductivity was discovered in 1911 by K. Onnes, who observed that certain materials at low temperature showed complete loss of resistivity. In 1933 W. Meissner discovered that this loss of resistivity was related to the expulsion of the magnetic field outside the sample. In 1950, V. Ginzburg and L. Landau proposed a phenomenological model for superconductivity. In their model, a superconductor is represented by a sample  $\Omega$  and its state is given through an order parameter  $u : \Omega \rightarrow \mathbb{C}$ , so that  $\rho = |u|$  is the density of superconducting electrons. The magnetic field then lives in the normal region, that is the set in which  $\rho$  vanishes, as opposed to the superconducting region,  $\rho \simeq 1$ , in which no magnetic field is present. This phenomenon is the so-called Meissner effect. Moreover, a superconductor is characterized by two physical parameters,  $\lambda$  and  $\xi$ . The parameter  $\lambda$  is called penetration length and represents the typical length of penetration of the magnetic field in the superconducting region, whereas the parameter  $\xi$  is called coherence length and represents the typical scale at which  $\rho$  varies. The number  $\kappa = \lambda/\xi$  is called the Ginzburg-Landau parameter, and its value dictates the behavior of the superconductor. If  $\kappa < 1/\sqrt{2}$  a positive surface energy penalizes the interface between the normal region and the superconducting region. These materials are referred to as type-I superconductors. If instead  $\kappa > 1/\sqrt{2}$ , the surface tension is negative and the magnetic field penetrates the sample through vortices (see [11, 100]). These are instead called type-II superconductors. We are interested in the former type. Type-I superconductors, when subject to an external magnetic field, experimentally exhibit the appearance of complex patterns at the surface [95, 96, 97]. This behavior hints at the presence of branching patterns inside the sample. A first mathematical justification of this fact was given in [26, 28], where scaling laws for a simplified model were proved. These scaling laws were later extended to the full Ginzburg-Landau model in [39]. In particular, [26, 28, 39] distinguish two regimes of parameters, corresponding to uniform and non uniform branching. In the first regime, the magnetic field distributes uniformly at the boundary on the sample, whereas, surprisingly, in the second regime it is energetically favorable for the magnetic field to deviate from uniformity

at the boundary. Going beyond the scaling law, in [37] a reduced model was derived (in the sense of  $\Gamma$ -convergence [12, 43]). The resulting model is a non standard branched transport problem, where the magnetic field concentrates on a one dimensional tree structure. Minima of this reduced functional were studied in [67, 48, 41]. The result of [37] was proved in the uniform regime and for vanishing applied magnetic field and  $\kappa$ .

The goal of this chapter is to better understand the limit behavior of type-I superconductors in the case when  $\kappa$  and the external magnetic field do not vanish. Recalling the definition of  $\kappa$ , this is a regime in which penetration length and coherence length are comparable. In the full Ginzburg-Landau model, various lengthscales are involved, see [37]. We consider a simplified 2D setting, which can be understood informally as a blow-up at the scale of coherence and penetration length. In this setting the Ginzburg-Landau model resembles a non-standard Modica-Mortola type functional [85, 84]. We prove that this functional  $\Gamma$ -converges to the perimeter functional. Since  $\kappa$  does not vanish and due to the vectorial nature of the problem, several difficulties arise in the proof. In particular, the main difficulty is given by the fact that the functional pairs a local coupling between  $\rho$  and  $B$  with a global coupling between  $\theta$ , the phase of  $u$ , and the magnetic potential  $A$ . This result is a first step in understanding how to extend the results of [37] to the regime of non-vanishing  $\kappa$ .

To present the main result of the chapter, let us introduce the functional we study (see Section 3.2.2). Consider the torus  $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ . Let  $\varepsilon > 0$ ,  $\kappa \in (0, 1/\sqrt{2})$  and  $\alpha = \kappa^{-1}\varepsilon^{-2}$ . The model we consider is

$$\mathcal{E}_\varepsilon(u, A) = \int_{\mathbb{T}^2} \varepsilon((1 - \kappa\sqrt{2})|\nabla_{\alpha A}u|^2 + \kappa\sqrt{2}|\mathcal{D}_{\alpha A}^3u|^2) + \frac{1}{\varepsilon} \left( B - \frac{1}{\sqrt{2}}(1 - |u|^2) \right)^2 dx \quad (3.1.1)$$

with the constraint

$$\int_{\mathbb{T}^2} B dx = \frac{b_{\text{ext}}}{\kappa}.$$

Here,  $u$  is the complex valued order parameter representing the state of the material,  $\rho^2 = |u|^2$  is the density of superconducting electrons,  $A$  is the vector valued magnetic potential,  $B = \nabla \times A$  is the magnetic field in the sample and  $b_{\text{ext}}$  is the external magnetic field. The quantity  $\nabla_A u = \nabla u - iAu$  is the covariant derivative of  $u$  and  $\mathcal{D}_A^3 u = (\nabla_A u)_2 - i(\nabla_A u)_1$  is the so-called Bogomol'nyi operator. We consider the setting of periodic boundary conditions on the physical values  $\rho$  and  $B$  (see Section 3.2.2 for the precise assumptions on the functional spaces we use). The functional (3.1.1) can be derived from the full Ginzburg-Landau model in the case of an infinitely long sample (see Chapter 1). Notice that when  $\rho > 0$ , if  $u = \rho e^{i\theta}$ , then

$$|\nabla_{\alpha A}u|^2 = |\nabla\rho|^2 + \rho^2 |\nabla\theta - \alpha A|^2. \quad (3.1.2)$$

For  $\varepsilon \simeq 0$  and  $\rho \simeq 1$ , the second term in (3.1.2) forces  $A \simeq \nabla(\theta/\alpha)$ , so that  $B = 0$ . This means that for  $\varepsilon \simeq 0$ , the Meissner condition  $\rho^2 B \simeq 0$  is enforced. Such condition implies that the last term in (3.1.1) can be interpreted as a double well potential in  $\rho$  and  $B$  with the two phases  $(\rho, B) = (1, 0)$  and  $(\rho, B) = (0, 1/\sqrt{2})$ . Complementarily, the first term in (3.1.2) penalizes oscillations of  $\rho$ . Thus, we can interpret the functional as a Modica-Mortola type functional [85, 84] in  $\rho$  and  $B$ . In particular, our functional is of vectorial nature, similarly to generalizations of the standard Modica-Mortola which have been studied in recent years [25, 36, 64, 59, 40, 44, 45]. Our main result is the following.

**Theorem 3.1.1.** *Let  $\kappa \in (0, 1/\sqrt{2})$ ,  $b_{\text{ext}} < \kappa/\sqrt{2}$  and consider  $\varepsilon_n \rightarrow 0$ . Denoting by  $\sigma_\kappa$  the constant defined in Section 3.4 and by  $\mathcal{A}_{\varepsilon_n}$  the space of admissible configurations defined in Section 3.2.2, then the following holds:*

1. Let  $(u_n, A_n) \in \mathcal{A}_{\varepsilon_n}$  be a sequence such that  $\sup_n \mathcal{E}_{\varepsilon_n}(u_n, A_n) < \infty$ . Then, up to a subsequence,  $|u_n| = \rho_n \rightarrow \rho$  in  $L^p(\mathbb{T}^2)$  for all  $1 \leq p < \infty$ , with  $\rho \in BV(\mathbb{T}^2, \{0, 1\})$  and  $B_n = \nabla \times A_n \rightarrow (1 - \rho)/\sqrt{2}$  in  $L^2(\mathbb{T}^2)$ .

2. Let  $(u_n, A_n) \in \mathcal{A}_{\varepsilon_n}$  be a sequence such that  $|u_n| = \rho_n \rightarrow \rho$  in  $L^p(\mathbb{T}^2)$  for all  $1 \leq p < \infty$ , with  $\rho \in BV(\mathbb{T}^2, \{0, 1\})$  and  $B_n = \nabla \times A_n \rightarrow (1 - \rho)/\sqrt{2}$  in  $L^2(\mathbb{T}^2)$ . Then

$$\liminf_n \mathcal{E}_{\varepsilon_n}(u_n, A_n) \geq \sigma_\kappa P(\{\rho = 0\}, \mathbb{T}^2).$$

3. Suppose  $\varepsilon_n^{-2} \kappa^{-2} b_{\text{ext}} \in 2\pi\mathbb{Z}$ . Let  $\rho \in BV(\mathbb{T}^2, \{0, 1\})$  with  $\int_{Q_1} (1 - \rho)/\sqrt{2} dx = b_{\text{ext}}/\kappa$ . Then, there exists  $(u_n, A_n) \in \mathcal{A}_{\varepsilon_n}$  such that  $|u_n| = \rho_n \rightarrow \rho$  in  $L^p(\mathbb{T}^2)$  for all  $1 \leq p < \infty$ ,  $B_n = \nabla \times A_n \rightarrow (1 - \rho)/\sqrt{2}$  in  $L^2(\mathbb{T}^2)$ ,  $\int_{Q_1} B_n dx = b_{\text{ext}}/\kappa$  and

$$\limsup_n \mathcal{E}_{\varepsilon_n}(u_n, A_n) \leq \sigma_\kappa P(\{\rho = 0\}, \mathbb{T}^2).$$

Let us now comment on the strategy of the proof. Regarding compactness (which we essentially take from [37]), we exploit a weak version of the Meissner effect (Proposition 3.3.2) to prove that the functional (3.1.1) can be estimated from below by a classical Modica-Mortola type functional in  $\rho$ . Then, the compactness follows by the standard Modica-Mortola argument for compactness [84, 2].

For the  $\Gamma$ -lim inf, we use a blow up procedure [58, 13]. We define  $\sigma_\kappa$  implicitly as the minimal value of a minimization problem on a unit cell where the interface is a vertical segment.

Next, we prove a characterization for  $\sigma_\kappa$  which will be useful in the construction of the recovery sequence. This is the main part of the chapter and is divided in two steps.

In the first step (Section 3.5.1), as in typical Modica-Mortola type results, we show that for the minimization problem on the cell  $\sigma_\kappa$  can be interpreted as an optimal profile problem  $\sigma_{\kappa, \delta}$  with fixed vertical boundary conditions at a distance  $\delta$  from the transition (Proposition 3.5.1). The proof of this result is however delicate in our setting. In particular, extra care is necessary because of the global coupling term in (3.1.2). Furthermore, we account for the gauge invariance of the functional (see Section 3.2.2) by fixing an appropriate gauge for which the result holds.

In the second step (Section 3.5.2), we prove that, up to a small cost, we can impose horizontal periodic boundary conditions (Proposition 3.5.7). In particular, we impose that  $u$  is equal to a real profile  $U$  near the top and bottom boundary. We actually conjecture that the optimal transition profile itself is one dimensional, in analogy with some previous results [24, 23], see Remark 3.5.8. Finally, we prove the  $\Gamma$ -lim sup inequality. By density of polyhedral sets in the class of finite perimeter sets [80], we can suppose that the set  $E = \{\rho = 0\}$  is polyhedral. Again in analogy with Modica-Mortola, we "glue" a rescaled competitor  $(u_0, A_0)$  of  $\sigma_\kappa$  on a  $\varepsilon_n$ -neighborhood of  $\partial E$ . Then, we globally construct the physical quantities  $\rho_n$  and  $B_n$  so that they coincide with the wells of the functional  $\mathcal{E}_{\varepsilon_n}$  far from  $\partial E$ . The main difficulty of the construction is then to appropriately define the phase  $\theta_n$  and the potential  $A_n$  so that  $u_n = \rho_n e^{i\theta_n}$  and  $A_n$  match the local values  $u_0, A_0$  on an  $\varepsilon_n$  neighborhood of  $E$ . In particular the connected components of  $E$  act as "poles" for the phase  $\theta_n$ , and force the quantization condition  $\varepsilon_n^{-2} \kappa^{-2} b_{\text{ext}} \in 2\pi\mathbb{Z}$ . The quantization condition is a consequence of the fact that the phase  $\theta$  in the full Ginzburg-Landau model is naturally quantized (see [37, 39]). Let us remark that the condition is less and less stringent as  $\varepsilon_n \rightarrow 0$ .

The rest of the chapter is organized as follows. In Section 3.2, we introduce the notation and the precise hypothesis on the model. In Section 3.3, we prove some a priori estimates and the compactness result. In Section 3.4 we prove the  $\Gamma$ -lim inf result. Section 3.5 is devoted to characterizing the constant  $\sigma_\kappa$ . Finally, in Section 3.6 we prove the  $\Gamma$ -lim sup inequality.

## 3.2 Preliminaries

In this section, we fix the notation and precise the assumptions we make on the energy functional.

### 3.2.1 Notation

We use the notation  $a \lesssim b$  to indicate  $a \leq Cb$  for some universal constant  $C > 0$  and  $a \lesssim_p b$  to indicate  $a \leq Cb$  for some constant  $C = C(p) > 0$ . For  $L, S > 0$  we set  $Q_L = (-L/2, L/2)^2$ ,  $Q_{L,S} = (-L/2, L/2) \times (-S/2, S/2)$  and  $H_{L,S}^- = (-L/2, 0) \times (-S/2, S/2)$ . We call  $H^- = H_{1/2,1/2}^- = (-1/2, 0) \times (-1/2, 1/2)$  and for any direction  $\nu \in \mathbb{S}^1$  we call  $H_\nu^- = R_\nu H^-$  where  $R_\nu$  is the rotation matrix such that  $R_\nu e_1 = \nu$ , ( $e_i$  being the canonical base of  $\mathbb{R}^2$ ). Moreover, let us denote by  $Q_r^\nu(x_0)$  the cube centered in a point  $x_0 \in Q_1$  of scale  $r \in (0, 1)$  and of direction  $\nu \in \mathbb{S}^1$ , namely  $Q_r^\nu(x_0) = x_0 + rR_\nu Q_1$ . We also abbreviate  $Q_r^\nu = Q_r^\nu(0)$ . We define the 2D curl  $\nabla \times v = \partial_1 v_2 - \partial_2 v_1$ . For any vector  $v \in \mathbb{R}^2$ , we define  $v^\perp = (-v_2, v_1)$ . We identify periodic functions on  $Q_1$  as functions on the torus  $\mathbb{T}^2 = \mathbb{R}^2/(\mathbb{Z}^2)$  equipped with the distance  $|\cdot| = \min_{z \in \mathbb{Z}^2} |\cdot - z|$ . We denote  $BV(\mathbb{T}^2)$  the space of periodic bounded variation functions on  $\mathbb{T}^2$ . For a finite perimeter set  $\Omega \subset \mathbb{T}^2$  (see [3] for details), we denote by  $P(\Omega, \mathbb{T}^2)$  its periodic perimeter and by  $\partial^* \Omega$  its reduced boundary. We denote by  $\mathcal{H}^1$  the periodic Hausdorff measure on  $\mathbb{T}^2$ . For any measure  $\mu$ , we denote by  $\text{supp } \mu$  its support. For any curve  $\Gamma \subset \mathbb{T}^2$  whose parametrization is  $\gamma : [0, T] \rightarrow \mathbb{T}^2$  and for any field  $A : \mathbb{T}^2 \rightarrow \mathbb{R}^2$ , we use the notation  $\int_\Gamma A \cdot dl = \int_0^T A(\gamma(t)) \cdot \dot{\gamma}(t) dt$ . We always omit relabeling subsequences.

### 3.2.2 The 2D model

We detail the functional setting that will be assumed throughout the chapter. For  $\varepsilon > 0$  and  $\kappa \in (0, 1/\sqrt{2})$  we let  $\alpha = \kappa^{-1}\varepsilon^{-2}$ . Let  $\Omega \subset \mathbb{R}^2$  be a Lipschitz domain (the material sample). We consider a complex function  $u : \Omega \rightarrow \mathbb{C}$  which represents the state of the material (the order parameter) and  $\rho^2 = |u|^2$  the density of superconducting electrons, the vector field  $A : \Omega \rightarrow \mathbb{R}^2$  describing the magnetic vector potential associated to the magnetic field  $B = \nabla \times A$  in the sample and  $b_{\text{ext}} \in \mathbb{R}$  the external magnetic field. We denote by

$$\nabla_A u = \nabla u - iAu$$

the covariant derivative of  $u$  and by

$$\mathcal{D}_A^3 u = (\nabla_A u)_2 - i(\nabla_A u)_1$$

the so-called Bogomol'nyi operator. The scalar field

$$j = \text{Im}(u \overline{\nabla_{\alpha A} u}) = \text{Re}(-i\bar{u} \nabla_{\alpha A} u), \quad (3.2.1)$$

describes the superconducting current.

We introduce the set of functions

$$\mathcal{A}(\Omega) = \{(u, A) \in H_{\text{loc}}^1(\Omega; \mathbb{C}) \times L_{\text{loc}}^2(\Omega; \mathbb{R}^2), \rho = |u| \leq 1, B = \nabla \times A \in L_{\text{loc}}^2(\Omega)\}$$

and denote  $\mathcal{A} = \mathcal{A}(Q_1)$ . We impose the condition  $\rho \leq 1$  since it is always true for minimizers of the functional (3.1.1) (see [100, Proposition 3.9]).

We say that two configurations  $(u, A)$  and  $(\hat{u}, \hat{A}) \in \mathcal{A}(\Omega)$  are gauge equivalent if there exists  $\varphi \in H_{\text{loc}}^1(\Omega)$  such that

$$\begin{cases} u = \hat{u} e^{i\varphi} \\ A = \hat{A} + \frac{1}{\alpha} \nabla \varphi. \end{cases} \quad (3.2.2)$$

Notice that the physical quantities  $\rho$ ,  $B$ ,  $j$ , and the terms  $|\nabla_A u|$  and  $|\mathcal{D}_A^3 u|$  are invariant under the gauge transformation (3.2.2).

We now specify the space of admissible configurations. We define

$$\mathcal{A}_\varepsilon = \left\{ (u, A) \in \mathcal{A}(\mathbb{R}^2) \mid \begin{array}{l} (u(\cdot + k), A(\cdot + k)) \text{ is gauge equivalent to } (u, A) \forall k \in \mathbb{Z}^2 \\ \int_{Q_1} B \, dx = b_{\text{ext}}/\kappa \end{array} \right\}.$$

Here the condition  $\int_{Q_1} B \, dx = b_{\text{ext}}/\kappa$  is a consequence of Maxwell law for the magnetic field  $B$  (see Chapter 1). Notice that the set depends on  $\varepsilon$  since it depends on the gauge equivalence (3.2.2), which itself depends on  $\alpha = \kappa^{-1}\varepsilon^{-2}$ . In the set  $\mathcal{A}_\varepsilon$  the physical quantities  $\rho$ ,  $B$  and  $j$  are  $Q_1$ -periodic and we can think of them as defined on the torus  $\mathbb{T}^2$ , whereas in general  $u$  and  $A$  will not be periodic. We also observe that  $(u, A) \in \mathcal{A}_\varepsilon$  implies  $\rho \in H^1(\mathbb{T}^2)$ ,  $B \in L^2(\mathbb{T}^2)$  and  $j \in L^2(\mathbb{T}^2; \mathbb{R}^2)$ .

Let  $\Omega \subset \mathbb{R}^2$  be a bounded Lipschitz domain. For  $(u, A) \in \mathcal{A}(\Omega)$ , we introduce the functional

$$\mathcal{E}_\varepsilon(u, A, \Omega) = \int_\Omega \varepsilon \left( (1 - \kappa\sqrt{2}) |\nabla_{\alpha A} u|^2 + \kappa\sqrt{2} |\mathcal{D}_{\alpha A}^3 u|^2 \right) + \frac{1}{\varepsilon} \left( B - \frac{1}{\sqrt{2}}(1 - |u|^2) \right)^2 \, dx.$$

Notice that  $\mathcal{E}_\varepsilon$  is invariant under change of gauge. For brevity, we denote

$$|\nabla_A^\kappa u|^2 = (1 - \kappa\sqrt{2}) |\nabla_A u|^2 + \kappa\sqrt{2} |\mathcal{D}_A^3 u|^2,$$

so that

$$\mathcal{E}_\varepsilon(u, A, \Omega) = \int_\Omega \varepsilon |\nabla_{\alpha A}^\kappa u|^2 + \frac{1}{\varepsilon} \left( B - \frac{1}{\sqrt{2}}(1 - |u|^2) \right)^2 \, dx.$$

We remark that, when  $\rho > 0$ , letting  $u = \rho e^{i\theta}$ , we can write

$$|\nabla_{\alpha A} u|^2 = |\nabla \rho|^2 + \rho^2 |\nabla \theta - \alpha A|^2. \quad (3.2.3)$$

Moreover, by a direct computation (see [39, Lemma 2.1]), we have

$$|\nabla_{\alpha A} u|^2 = |\mathcal{D}_{\alpha A}^3 u|^2 + \alpha \rho^2 B + \nabla \times j. \quad (3.2.4)$$

We are interested in studying the limit behavior as  $\varepsilon \rightarrow 0$  of the functional  $\mathcal{E}_\varepsilon(u, A) = \mathcal{E}_\varepsilon(u, A, Q_1)$  on the set of admissible configurations  $\mathcal{A}_\varepsilon$  (see Theorem 3.1.1). For this problem to be meaningful we assume

$$\frac{b_{\text{ext}}}{\kappa} = \int_{\mathbb{T}^2} B \, dx < \frac{1}{\sqrt{2}}.$$

Indeed, if  $b_{\text{ext}}/\kappa = 1/\sqrt{2}$  the trivial minimizer given by  $(\rho, B) = (0, b_{\text{ext}}/\kappa)$  is admissible and no phase transition occurs, whereas for  $b_{\text{ext}}/\kappa > 1/\sqrt{2}$  the energy is unbounded in  $\varepsilon$  for any  $B$ .

### 3.3 A priori estimates and compactness

The goal of this section is to derive some a priori estimates and prove the following theorem (Theorem 3.1.1-1). The proof is essentially contained in the proof of [37, Proposition 6.1], but we repeat it for the reader's convenience.

**Theorem 3.3.1.** *Let  $\kappa \in (0, 1/\sqrt{2})$ ,  $b_{\text{ext}} < \kappa/\sqrt{2}$  and consider  $\varepsilon_n \rightarrow 0$ . Let  $(u_n, A_n) \in \mathcal{A}_{\varepsilon_n}$  be a sequence such that  $\sup_n \mathcal{E}_{\varepsilon_n}(u_n, A_n) < \infty$ . Then, up to a subsequence,  $|u_n| = \rho_n \rightarrow \rho$  in  $L^q(\mathbb{T}^2)$  for all  $1 \leq q < \infty$ , with  $\rho \in BV(\mathbb{T}^2, \{0, 1\})$  and  $B_n = \nabla \times A_n \rightarrow (1 - \rho)/\sqrt{2}$  in  $L^2(\mathbb{T}^2)$ .*

We start by proving a weak version of the Meissner effect. Informally, the Meissner effect states that, for  $\varepsilon$  small,  $\rho B \simeq 0$ , that is the magnetic field  $B$  is expelled from the region of superconductivity  $\{\rho > 0\}$ .

**Proposition 3.3.2.** *Let  $\Omega \subset Q_1$  be a Lipschitz domain. For any  $\phi \in H^1(\Omega) \cap L^\infty(\Omega)$  and  $(u, A) \in \mathcal{A}(\Omega)$ , we have*

$$\left| \int_{\Omega} \rho^2 B \phi \, dx \right| \lesssim_{\kappa} \varepsilon \left( \mathcal{E}_{\varepsilon}(u, A, \Omega) \|\phi\|_{L^\infty(\Omega)} + \varepsilon^{\frac{1}{2}} \mathcal{E}_{\varepsilon}(u, A, \Omega)^{\frac{1}{2}} \|\nabla \phi\|_{L^2(\Omega)} + \varepsilon \left| \int_{\partial\Omega} \phi j \cdot dl \right| \right).$$

*Proof.* The proof is essentially the same as [37, Lemma 6.2], but without assuming periodic boundary conditions. We start by noticing that

$$\int_{\Omega} (\nabla \times j) \phi \, dx = \int_{\Omega} \nabla \times (j \phi) \, dx - \int_{\Omega} j^{\perp} \cdot \nabla \phi \, dx = \int_{\partial\Omega} \phi j \cdot dl - \int_{\Omega} j^{\perp} \cdot \nabla \phi \, dx.$$

Then, multiplying (3.2.4) by  $\phi$  and integrating over  $\Omega$ , we have

$$\begin{aligned} \left| \int_{\Omega} \rho^2 B \phi \right| &\leq \varepsilon^2 \kappa \int_{\Omega} (|\nabla_{\alpha A} u|^2 + |\mathcal{D}_{\alpha A}^3 u|^2) |\phi| \, dx + \varepsilon^2 \kappa \left| \int_{\Omega} (\nabla \times j) \phi \, dx \right| \\ &\lesssim_{\kappa} \varepsilon \mathcal{E}_{\varepsilon}(u, A) \|\phi\|_{L^\infty(\Omega)} + \varepsilon^2 \int_{\Omega} |j| |\nabla \phi| \, dx + \varepsilon^2 \left| \int_{\partial\Omega} \phi j \cdot dl \right| \\ &\lesssim_{\kappa} \varepsilon \left( \mathcal{E}_{\varepsilon}(u, A) \|\phi\|_{L^\infty(\Omega)} + \varepsilon^{\frac{1}{2}} \mathcal{E}_{\varepsilon}(u, A)^{\frac{1}{2}} \|\nabla \phi\|_{L^2(\Omega)} + \varepsilon \left| \int_{\partial\Omega} \phi j \cdot dl \right| \right), \end{aligned}$$

where, in the last step, we have exploited the fact that  $\rho \leq 1$  implies  $|j| \leq |\nabla_{\alpha A} u|$ .  $\square$

We now prove that the energy (3.1.1) can be estimated from below by a classical Modica-Mortola type functional. The following lemma is an adaptation of [37, Lemma 6.5], see also [66] where a similar idea is used.

**Lemma 3.3.3.** *Let  $\Omega \subset Q_1$  be a Lipschitz domain,  $(u, A) \in \mathcal{A}(\Omega)$  and consider the functions*

$$W(\rho) = \frac{1}{2} \min\{2\rho^2, 1\} (1 - \rho^2)^2,$$

and

$$\psi = \min\left\{2, \frac{1}{\rho^2}\right\} (1 - \rho^2). \quad (3.3.1)$$

Then we have the inequality

$$M_{\varepsilon}(\rho, \Omega) = \int_{\Omega} \left( \varepsilon |\nabla \rho|^2 + \frac{1}{\varepsilon} W(\rho) \right) dx \lesssim_{\kappa} \mathcal{E}_{\varepsilon}(u, A, \Omega) + \varepsilon \left| \int_{\partial\Omega} \psi j \cdot dl \right|. \quad (3.3.2)$$

*Proof.* Let us start by noticing that, since

$$|\nabla_{\alpha A}^{\kappa} u|^2 \gtrsim_{\kappa} |\nabla \rho|^2,$$

it is sufficient to prove

$$\int_{\Omega} \frac{1}{\varepsilon} W(\rho) \, dx \lesssim_{\kappa} \mathcal{E}_{\varepsilon}(u, A) + \varepsilon \left| \int_{\partial\Omega} \psi j \cdot dl \right|.$$

Young's inequality gives the relation

$$\frac{1}{2} (1 - \rho^2)^2 \leq (B - \frac{1}{\sqrt{2}} (1 - \rho^2))^2 + \sqrt{2} B (1 - \rho^2).$$

Multiplying both sides by  $\min\{2\rho^2, 1\}/\varepsilon$ , we obtain

$$\frac{1}{\varepsilon}W(\rho) \leq \frac{1}{\varepsilon}\left(B - \frac{1}{\sqrt{2}}(1 - \rho^2)\right)^2 + \frac{\sqrt{2}}{\varepsilon}\min\{2\rho^2, 1\}B(1 - \rho^2). \quad (3.3.3)$$

Integrating the previous inequality over  $\Omega$ , we are left with bounding the last term in (3.3.3). By applying Proposition 3.3.2 with  $\phi = \psi = \min\{2, 1/\rho^2\}(1 - \rho^2)$ , we have

$$\int_{Q_1} \frac{1}{\varepsilon}\rho^2 B\psi \lesssim_{\kappa} \mathcal{E}_{\varepsilon}(u, A) \|\psi\|_{L^{\infty}(\Omega)} + \varepsilon^{\frac{1}{2}}\mathcal{E}_{\varepsilon}(u, A)^{\frac{1}{2}} \|\nabla\psi\|_{L^2(\Omega)} + \varepsilon \left| \int_{\partial\Omega} \psi j \cdot dl \right|.$$

Moreover, direct computations give  $\|\psi\|_{L^{\infty}(\Omega)} \lesssim 1$  and  $\|\nabla\psi\|_{L^2(\Omega)} \lesssim \|\nabla\rho\|_{L^2(\Omega)}$ . Finally, we have

$$\varepsilon^{\frac{1}{2}}\mathcal{E}_{\varepsilon}(u, A)^{\frac{1}{2}} \|\nabla\psi\|_{L^2(\Omega)} \lesssim \mathcal{E}_{\varepsilon}(u, A) + \varepsilon \|\nabla\rho\|_{L^2(\Omega)}^2 \lesssim_{\kappa} \mathcal{E}_{\varepsilon}(u, A),$$

where in the last step we used again that  $|\nabla\rho|^2 \lesssim_{\kappa} |\nabla_{\alpha A}^{\kappa} u|^2$ .  $\square$

We are now ready to prove the first statement of Theorem 3.1.1.

*Proof of Theorem 3.3.1.* Let  $(u_n, A_n) \in \mathcal{A}_{\varepsilon_n}$  be a sequence such that  $\sup_n \mathcal{E}_{\varepsilon_n}(u_n, A_n) < \infty$ . We apply Lemma 3.3.3. Since, for  $(u_n, A_n) \in \mathcal{A}_{\varepsilon_n}$ ,  $\psi$  and  $j$  are periodic, the boundary term in (3.3.2) vanishes and we find that

$$\sup_n M_{\varepsilon_n}(\rho_n, \mathbb{T}^2) \lesssim \sup_n \mathcal{E}_{\varepsilon_n}(u_n, A_n) < \infty.$$

Then, by the standard Modica-Mortola argument for compactness (see for instance [2]) we have, up to a subsequence,  $\rho_n \rightarrow \rho$  in  $L^p(\mathbb{T}^2)$  for all  $1 \leq p < \infty$ , with  $\rho \in BV(\mathbb{T}^2, \{0, 1\})$ . Moreover,  $\sup_n \mathcal{E}_{\varepsilon_n}(u_n, A_n) < \infty$  implies

$$\int_{\mathbb{T}^2} \left( B_n - \frac{1}{\sqrt{2}}(1 - \rho_n^2) \right)^2 dx \lesssim \varepsilon.$$

This, paired with the  $L^p$  convergence of  $\rho_n$  to  $\rho$ , implies that  $B_n \rightarrow (1 - \rho)/\sqrt{2}$  in  $L^2(\mathbb{T}^2)$ .  $\square$

### 3.4 $\Gamma$ -lim inf inequality

We introduce the set of sequences

$$\mathcal{K} = \left\{ (u_n, A_n)_{n \in \mathbb{N}} \subset \mathcal{A} \left| \begin{array}{l} \rho_n \rightarrow \chi_{H^-} \text{ in } L^p(Q_1) \text{ for } p \geq 1 \\ B_n \rightarrow \frac{1}{\sqrt{2}}(1 - \chi_{H^-}) \text{ in } L^2(Q_1) \end{array} \right. \right\} \quad (3.4.1)$$

and the constant

$$\sigma_{\kappa} = \inf \left\{ \liminf_n \mathcal{E}_{\varepsilon_n}(u_n, A_n), \varepsilon_n \rightarrow 0, (u_n, A_n)_{n \in \mathbb{N}} \in \mathcal{K} \right\}.$$

**Remark 3.4.1.** Let  $\varepsilon_n \rightarrow 0$ . Consider any smooth function  $v : \mathbb{R} \rightarrow [0, 1]$  such that  $v(-\infty) = 1$  and  $v(x) = 0$  for  $x > 0$ . Define  $(u_n, A_n) \in \mathcal{K}$  for  $(x_1, x_2) \in Q_1$  as

$$u_n(x_1, x_2) = v\left(\frac{x_1}{\varepsilon_n}\right) = v_n(x_1), \quad A_n(x_1, x_2) = \left(0, \frac{(x_1)_+}{\sqrt{2}}\right).$$

Notice that  $\rho_n B_n = 0$  and  $|\nabla_{\alpha A_n}^\kappa u_n|^2 = |\dot{v}_n|^2$ . This means that, using a change of variable,

$$\liminf_{n \rightarrow \infty} \mathcal{E}_{\varepsilon_n}(u_n, A_n) = \liminf_{n \rightarrow \infty} \int_{-\frac{1}{2}}^0 \varepsilon_n |\dot{v}_n|^2 + \frac{1}{2\varepsilon_n} (1 - v_n^2)^2 dx_1 = \int_{-\infty}^0 |\dot{v}|^2 + \frac{1}{2} (1 - v^2)^2 dt = G(v).$$

Hence

$$\sigma_\kappa \leq \sigma_0 = \inf\{G(v) : v(-\infty) = 1, v(0) = 0\}.$$

The minimization problem  $\sigma_0$  is, up to a multiplicative constant, the same problem as the one defined in [37, Lemma 7.2], and it admits the explicit solution  $v(t) = \tanh(t/\sqrt{2})$ . In particular this implies  $\sigma_\kappa < \infty$ . Moreover, arguing as in [37, Lemma 6.5], it is possible to prove that  $\sigma_\kappa \rightarrow \sigma_0$  as  $\kappa \rightarrow 0$ .

**Remark 3.4.2.** Since  $\sigma_\kappa < \infty$ , using a diagonal argument it is always possible to find a sequence  $(u_n, A_n)_{n \in \mathbb{N}} \in \mathcal{K}$  such that  $\mathcal{E}_{\varepsilon_n}(u_n, A_n) \rightarrow \sigma_\kappa$ .

The goal of this section is to prove the following theorem (Theorem 3.1.1-2).

**Theorem 3.4.3.** Let  $\kappa \in (0, \frac{1}{\sqrt{2}})$ ,  $b_{\text{ext}} < \kappa/\sqrt{2}$  and consider  $\varepsilon_n \rightarrow 0$ . Let  $(u_n, A_n) \in \mathcal{A}_{\varepsilon_n}$  be a sequence such that  $|u_n| = \rho_n \rightarrow \rho$  in  $L^p(\mathbb{T}^2)$  for all  $1 \leq p < \infty$ , with  $\rho \in BV(\mathbb{T}^2, \{0, 1\})$  and  $B_n = \nabla \times A_n \rightarrow (1 - \rho)/\sqrt{2}$  in  $L^2(\mathbb{T}^2)$ . Then

$$\liminf_n \mathcal{E}_{\varepsilon_n}(u_n, A_n) \geq \sigma_\kappa P(\{\rho = 0\}, \mathbb{T}^2).$$

*Proof of Theorem 3.4.3.* We use a blow-up argument. Let  $(u_n, A_n) \in \mathcal{A}_{\varepsilon_n}$  be a sequence as in the assumption. We consider a sequence of positive measures  $\{\mu_n\}_{n \in \mathbb{N}}$  on  $\mathbb{T}^2$  defined on every Borel set  $F \subset \mathbb{T}^2$  as

$$\mu_n(F) = \mathcal{E}_{\varepsilon_n}(u_n, A_n, F) = \int_F \varepsilon_n |\nabla_{\alpha A_n}^\kappa u_n|^2 + \frac{1}{\varepsilon_n} \left( B_n - \frac{1}{\sqrt{2}} (1 - |u_n|^2) \right)^2 dx.$$

We may assume without loss of generality that  $\sup_n \mathcal{E}_{\varepsilon_n}(u_n, A_n) < \infty$ , and we can extract a (not relabelled) subsequence such that  $\lim_n \mathcal{E}_{\varepsilon_n}(u_{\varepsilon_n}, A_{\varepsilon_n})$  exists and such that  $\mu_n$  converges weakly\* to a positive finite measure  $\mu$ . Now, since  $\rho \in BV(\mathbb{T}^2, \{0, 1\})$ ,  $\rho = \chi_E$  for some  $E \subset \mathbb{T}^2$  with  $P(E, \mathbb{T}^2) = \mathcal{H}^1(\partial^* E) < +\infty$ . By the Radon-Nikodym theorem, we can write

$$\mu = \frac{d\mu}{d\mathcal{H}^1} \mathcal{H}^1|_{\partial^* E} + \mu_c$$

for some  $\mu_c \perp \mathcal{H}^1|_{\partial^* E}$ . We claim that  $\frac{d\mu}{d\mathcal{H}^1}(x) \geq \sigma_\kappa$  for  $\mathcal{H}^1$ -a.e.  $x \in \partial^* E$ . Under such claim, by the semicontinuity of the total variation with respect to weak\* convergence (see [3, Theorem 1.59]) we would get

$$\begin{aligned} \liminf_n \mathcal{E}_n(u_{\varepsilon_n}, A_n) &= \liminf_n \mu_n(\mathbb{T}^2) \geq \mu(\mathbb{T}^2) = \int_{\mathbb{T}^2} \frac{d\mu}{d\mathcal{H}^1} d\mathcal{H}^1|_{\partial^* E} + \mu_c(\mathbb{T}^2) \\ &\geq \int_{\partial^* E} \sigma_\kappa d\mathcal{H}^1 = \sigma_\kappa \mathcal{H}^1(\partial^* E) = P(E, \mathbb{T}^2), \end{aligned}$$

thus proving the statement.

Let us now prove the claim. We use the blow-up method (introduced in [58], see also [13]). We know that at  $\mathcal{H}^1$ -a.e.  $x_0 \in \partial^* E$  the set  $\partial^* E$  admits an outward normal  $\nu = \nu(x_0)$ . Using Besicovitch derivation theorem (which we apply using squares instead of balls, see for instance [57, Theorem 1.153]), the derivative is given by

$$\frac{d\mu}{d\mathcal{H}^1|_{\partial^* E}}(x_0) = \lim_{r \rightarrow 0} \frac{\mu(Q_r^\nu(x_0))}{\mathcal{H}^1(Q_r^\nu(x_0) \cap \partial^* E)}.$$

We can assume without loss of generality that  $x_0 = 0$ . Moreover by the invariance of the energy under rotation we may also assume that  $\nu = e_1$ . By the properties of sets of finite perimeter we know that (see [3, Theorems 3.59 and 3.61])

$$\lim_{r \rightarrow 0} \frac{\mathcal{H}^1(Q_r \cap \partial^* E)}{r} = 1,$$

hence

$$\frac{d\mu}{d\mathcal{H}^1|_{\partial^* E}}(0) = \lim_{r \rightarrow 0} \frac{\mu(Q_r)}{r}.$$

Since  $\mu$  is finite,  $\mu(\partial Q_r) = 0$  for a.e.  $r$ . Thus, by standard weak\* convergence arguments (see [3, Proposition 1.62]), we have  $\mu_n(Q_r) \rightarrow \mu(Q_r)$  for a.e.  $r > 0$ . Writing for every  $\varepsilon_n \rightarrow 0$

$$\frac{d\mu}{d\mathcal{H}^1|_{\partial^* E}}(0) = \lim_{r \rightarrow 0} \frac{\mu(Q_r)}{\mu_n(Q_r)} \frac{\mu_n(Q_r)}{r},$$

we can use a diagonal argument to find  $\varepsilon_j, r_j$  such that

$$\frac{d\mu}{d\mathcal{H}^1|_{\partial^* E}}(0) = \lim_j \frac{\mu_{\varepsilon_j}(Q_{r_j})}{r_j}. \quad (3.4.2)$$

Moreover, we can also assume that  $\varepsilon_j/r_j \rightarrow 0$  and

$$\frac{1}{r_j^2} \int_{Q_{r_j}} |\rho_{\varepsilon_j} - \chi_E| dx \rightarrow 0. \quad (3.4.3)$$

By (3.4.2), we have

$$\frac{d\mu}{d\mathcal{H}^1|_{\partial^* E}}(0) = \lim_j \frac{1}{r_j} \int_{Q_{r_j}^\nu} \varepsilon_j \left| \nabla_{\alpha_j A_{\varepsilon_j}}^\kappa u_{\varepsilon_j} \right|^2 + \frac{1}{\varepsilon_j} \left( B_{\varepsilon_j} - \frac{1}{\sqrt{2}} (1 - |u_{\varepsilon_j}|^2) \right)^2 dx.$$

We change variable  $x = r_j y$  and define  $\hat{A}_j(y) = r_j^{-1} A_{\varepsilon_j}(r_j y)$  and  $\hat{u}_j(y) = u_{\varepsilon_j}(r_j y)$  (so that  $\hat{\rho}_j = |\hat{u}_j|$  and  $\hat{B}_j = \nabla \times \hat{A}_j$ ). We find

$$\frac{d\mu}{d\mathcal{H}^1|_{\partial^* E}}(0) = \lim_j \int_{Q_1} \frac{\varepsilon_j}{r_j} \left| \nabla_{r_j^2 \alpha_j \hat{A}_j}^\kappa \hat{u}_j \right|^2 + \frac{r_j}{\varepsilon_j} \left( \hat{B}_j - \frac{1}{\sqrt{2}} (1 - |\hat{u}_j|^2) \right)^2 dy.$$

By the properties of sets of finite perimeter (see [3, Theorem 3.59]), we can show that for all  $p \geq 1$ ,  $\hat{\rho}_j \rightarrow \chi_{H^-}$  in  $L^p(Q_1)$  and  $\hat{B}_j \rightarrow (1 - \chi_{H^-})/\sqrt{2}$  in  $L^2(Q_1)$ . Indeed, we have

$$\begin{aligned} \int_{Q_1} |\hat{\rho}_j - \chi_{H^-}| dy &= \frac{1}{r_j^2} \int_{Q_{r_j}} \left| \rho_j(x) - \chi_{H^-} \left( \frac{x}{r_j} \right) \right| dx \\ &\leq \frac{1}{r_j^2} \int_{Q_{r_j}} |u_{\varepsilon_j}(x) - \chi_E(x)| + \left| \chi_E(x) - \chi_{H^-} \left( \frac{x}{r_j} \right) \right| dx, \end{aligned} \quad (3.4.4)$$

where the first term on the right of (3.4.4) goes to zero by (3.4.3) and the second one by [3, Theorem 3.59]. Since  $\hat{\rho}_j \leq 1$  this implies also the convergence in  $L^p$  for every  $p \geq 1$ . Similarly to the proof of Theorem 3.1.1-1, the convergence of  $\hat{\rho}_j$  and the boundedness of the energy imply the convergence of  $\hat{B}_j$ . Hence, setting  $\hat{\varepsilon}_j = \varepsilon_j/r_j$ , we find

$$\frac{d\mu}{d\mathcal{H}^1|_{\partial^* E}}(0) = \lim_j \mathcal{E}_{\hat{\varepsilon}_j}(\hat{u}_j, \hat{A}_j, Q_1^\nu).$$

This concludes the proof of the claimed

$$\frac{d\mu}{d\mathcal{H}^1|_{\partial^* E}}(0) \geq \sigma_\kappa. \quad \square$$

### 3.5 Characterization of $\sigma_\kappa$

In this section we prove a characterization for  $\sigma_\kappa$ , which will help with the construction of the recovery sequence. This is done in two steps. First, we show that, at a distance  $0 < \delta < 1/2$  from the interface, we can impose vertical Dirichlet boundary conditions for competitors of  $\sigma_\kappa$ . In analogy with the study of Modica Mortola type functionals, that means that we think of  $\sigma_\kappa$  as an optimal profile problem on the unit cell. Then, we prove that we can also impose horizontal periodic boundary conditions, up to a small cost which depends on  $\delta$ . That is, we prove that for any  $0 < \delta < 1/2$ , we can define a minimization problem  $\sigma_{\kappa,\delta}^{\text{per}}$ , see (3.5.36), such that

$$\sigma_{\kappa,\delta}^{\text{per}} \leq \sigma_\kappa + C\delta$$

and whose competitors enjoy horizontal periodic boundary conditions (see Figure 3.1).

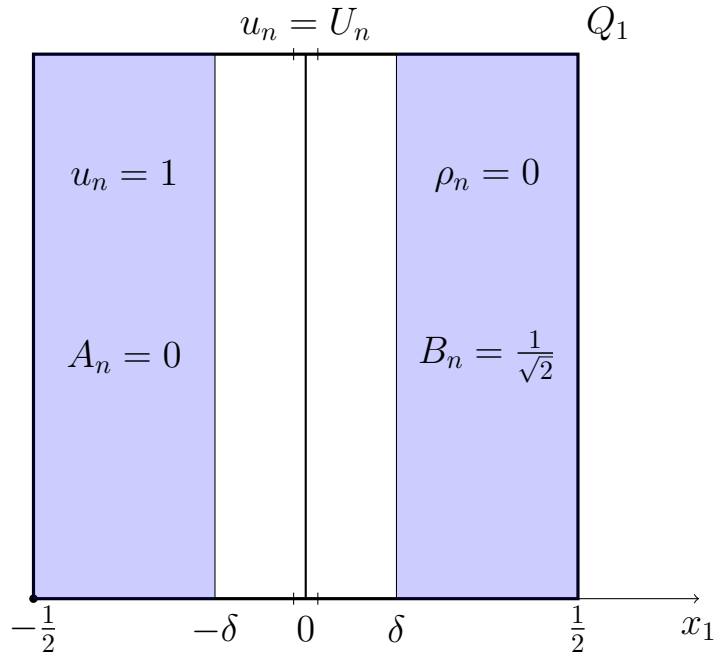


Figure 3.1: A typical competitor  $(u_n, A_n)_n$  for  $\sigma_\kappa$ . Competitors enjoy fixed boundary conditions at distance  $\delta$  from the transition (the zone in blue). Competitors for  $\sigma_{\kappa,\delta}^{\text{per}}$  moreover are such that  $u_n$  coincides with a given real profile  $U_n$  near the top and bottom boundaries of  $Q_1$ .

#### 3.5.1 Characterization of $\sigma_\kappa$ : vertical Dirichlet conditions

Define, for  $0 < \delta < 1/2$ , the set of functions

$$\mathcal{A}^\delta = \left\{ (u, A) \in \mathcal{A} \left| \begin{array}{l} (u, A) = (1, 0) \text{ for } x_1 < -\delta \\ (u, B) = (0, \frac{1}{\sqrt{2}}) \text{ for } x_1 > \delta \\ A_1 = 0 \end{array} \right. \right\},$$

the set of sequences

$$BC^\delta = \left\{ (u_n, A_n)_n \in \mathcal{K} \mid (u_n, A_n) \in \mathcal{A}^\delta \right\} \quad (3.5.1)$$

and the constant

$$\sigma_{\kappa,\delta} = \inf \left\{ \liminf_n \mathcal{E}_{\varepsilon_n}(u_n, A_n) \mid \varepsilon_n \rightarrow 0, (u_n, A_n)_n \in BC^\delta \right\}. \quad (3.5.2)$$

The goal of this section is to prove the following proposition.

**Proposition 3.5.1.** *For every  $0 < \delta < 1/2$ ,*

$$\sigma_\kappa = \sigma_{\kappa,\delta}.$$

**Remark 3.5.2.** *The conditions on  $(u_n, A_n)$  in (3.5.1) come from two different requirements. On the one hand, as in usual Modica-Mortola type arguments, we want to define the constant  $\sigma_{\kappa,\delta}$  as a minimization problem with fixed boundary conditions at some distance  $\delta$  from the transition line, as this simplifies the construction of a recovery sequence. This explains the conditions on  $\rho_n$  and  $B_n$ . On the other hand, the conditions  $u_n = 1$  for  $x_1$  near  $-1/2$  and  $A_{n,1} = 0$  fix the gauge of the minimizing sequence. This is instead helpful for adding horizontal periodic boundary conditions (see Section 3.5.2), which in turn helps again with the construction of the recovery sequence.*

Let us start by understanding the scaling properties of the constants  $\sigma_\kappa$  and  $\sigma_{\kappa,\delta}$ . First, we make explicit their dependence on the domain. Recalling the notation of Section 3.2.1, for  $a, b > 0$  we consider the sets

$$\mathcal{A}_{a,b} = \{(u, A) \in H^1(Q_{a,b}; \mathbb{C}) \times L^2(Q_{a,b}; \mathbb{R}^2) \mid \rho \leq 1, \nabla \times A \in L^2(Q_{a,b})\}$$

and

$$\mathcal{K}_{a,b} = \left\{ (u_n, A_n)_n \subset \mathcal{A}_{a,b} \mid \begin{array}{l} \rho_n \rightarrow \chi_{H_{a,b}^-} \text{ in } L^p(Q_{a,b}) \text{ for } p \geq 1 \\ B_n \rightarrow \frac{1}{\sqrt{2}}(1 - \chi_{H_{a,b}^-}) \text{ in } L^2(Q_{a,b}) \end{array} \right\}.$$

For  $0 < \delta < a/2$ , we also consider the sets

$$\mathcal{A}_{a,b}^\delta = \left\{ (u, A) \in \mathcal{A}_{a,b} \mid \begin{array}{l} (u, A) = (1, 0) \text{ for } x_1 < -\delta \\ (u, B) = (0, \frac{1}{\sqrt{2}}) \text{ for } x_1 > \delta \\ A_1 = 0 \end{array} \right\}$$

and

$$BC_{a,b}^\delta = \left\{ (u_n, A_n)_n \in \mathcal{K}_{a,b} \mid (u_n, A_n) \in \mathcal{A}_{a,b}^\delta \right\}.$$

We define the domain-dependent constants

$$\sigma_\kappa(a, b) = \inf \left\{ \liminf_n \mathcal{E}_{\varepsilon_n}(u_n, A_n, Q_{a,b}), \varepsilon_n \rightarrow 0, (u_n, A_n)_n \in \mathcal{K}_{a,b} \right\}$$

and

$$\sigma_{\kappa,\delta}(a, b) = \inf \left\{ \liminf_n \mathcal{E}_{\varepsilon_n}(u_n, A_n, Q_{a,b}), \varepsilon_n \rightarrow 0, (u_n, A_n)_n \in BC_{a,b}^\delta \right\}.$$

Notice that  $\sigma_\kappa = \sigma_\kappa(1, 1)$ . We prove the following scaling property for  $\sigma_\kappa$ .

**Proposition 3.5.3.** *For any  $a, b > 0$ ,  $\mathbb{N}$  and  $0 < \delta < 1/2$*

$$\sigma_\kappa(a, b) = b\sigma_\kappa, \quad (3.5.3)$$

and

$$\sigma_{\kappa,\delta}(1, N) \geq N\sigma_{\kappa,\delta}. \quad (3.5.4)$$

*Proof.* Let us start with (3.5.3). Fix  $L > 0$  and consider a sequence  $(u_n, A_n)_n \in \mathcal{K}_{La, Lb}$  such that

$$\mathcal{E}_{\varepsilon_n}(u_n, A_n, Q_{La, Lb}) \rightarrow \sigma_\kappa(La, Lb).$$

Consider, for  $S > 0$ , the change of variable  $x = (L/S)y$ , and define the sequence  $\hat{\varepsilon}_n = (S/L)\varepsilon_n$ , and the functions  $\hat{u}_n(y) = u_n(L/Sy)$  and  $\hat{A}_n(y) = (S/L)A_n(L/Sy)$ . We find that

$$\mathcal{E}_{\varepsilon_n}(u_n, A_n, Q_{La, Lb}) = \frac{L}{S} \mathcal{E}_{\hat{\varepsilon}_n}(\hat{u}_n, \hat{A}_n, Q_{Sa, Sb}).$$

This inequality implies, letting  $n \rightarrow \infty$ ,

$$\sigma_\kappa(La, Lb) \geq \frac{L}{S} \sigma_\kappa(Sa, Sb),$$

from which we deduce that, for any  $L > 0$ ,

$$\sigma_\kappa(La, Lb) = L\sigma_\kappa(a, b). \quad (3.5.5)$$

The energy density is positive, thus, for  $(u_n, A_n)_n \in \mathcal{K}_{a,b}$  and any  $c, d > 0$  with  $Q_{c,d} \subset Q_{a,b}$ , we have

$$E_{\varepsilon_n}(u_n, A_n, Q_{a,b}) \geq E_{\varepsilon_n}(u_n, A_n, Q_{c,d}),$$

which implies

$$\sigma_\kappa(a, b) \geq \sigma_\kappa(c, d). \quad (3.5.6)$$

Moreover, for any  $c, d > 0$  and  $(u_n, A_n)_n \in \mathcal{K}_{a,c+d}$ , by the translation invariance of the energy we have

$$E_{\varepsilon_n}(u_n, A_n, Q_{a,c+d}) = E_{\varepsilon_n}(u_n, A_n, Q_{a,c}) + E_{\varepsilon_n}(u_n, A_n, Q_{a,d}),$$

which implies

$$\sigma_\kappa(a, c+d) \geq \sigma_\kappa(a, c) + \sigma_\kappa(a, d). \quad (3.5.7)$$

We now claim that

$$\sigma_\kappa(1, b) = b\sigma_\kappa. \quad (3.5.8)$$

Then, combining (3.5.8) and (3.5.5), we would get

$$\sigma_\kappa(a, b) = a\sigma_\kappa\left(1, \frac{b}{a}\right) = b\sigma_\kappa$$

and the result would follow. Let us now prove (3.5.8). First, we show that (3.5.8) holds for  $b \in \mathbb{N}$ . If  $b \in \mathbb{N}$ , from (3.5.7) we find immediately that  $\sigma_\kappa(1, b) \geq b\sigma_\kappa$ . For the opposite inequality, using (3.5.5) and (3.5.6) we have

$$b\sigma_\kappa = \sigma_\kappa(b, b) \geq \sigma_\kappa(1, b).$$

Next, we show that (3.5.8) holds for  $b = p/q$ , with  $p, q \in \mathbb{N}$ . By (3.5.6), we have that  $\sigma_\kappa(q, p) \geq \sigma_\kappa(1, p)$ . Hence, using (3.5.5) we find

$$\sigma_\kappa(1, b) = \frac{1}{q} \sigma_\kappa(q, p) \geq \frac{1}{q} \sigma_\kappa(1, p) = b\sigma_\kappa.$$

Now, if  $b > 1$ , we have using (3.5.5) and (3.5.6)

$$\sigma_\kappa(1, b) = b\sigma_\kappa\left(\frac{1}{b}, 1\right) \leq b\sigma_\kappa$$

and (3.5.8) holds. If  $0 < b < 1$ , by (3.5.7) we have

$$\sigma_\kappa(1, b+1) \geq \sigma_\kappa + \sigma_\kappa(1, b). \quad (3.5.9)$$

Using (3.5.8) for  $b+1 > 1$ , we have

$$\sigma_\kappa(1, b+1) = (b+1)\sigma_\kappa. \quad (3.5.10)$$

Combining (3.5.9) and (3.5.10) we can conclude that  $b\sigma_\kappa \geq \sigma_\kappa(1, b)$ , hence (3.5.8) holds also for  $b < 1$ .

Finally, the case  $b \in \mathbb{R}^+$  follows by density and monotonicity of the function  $b \mapsto \sigma_\kappa(1, b)$ .

The proof of (3.5.4) follows from the same reasoning as (3.5.7). For any  $a, c, d > 0$ , by the translation invariance of the energy we have

$$\sigma_{\kappa, \delta}(a, c+d) \geq \sigma_{\kappa, \delta}(a, c) + \sigma_{\kappa, \delta}(a, d),$$

which implies (3.5.4).  $\square$

The goal now is to construct an appropriate minimizing sequence for  $\sigma_\kappa$  such that we can compare it with a competitor for  $\sigma_{\kappa, \delta}$ . To do so, the following two lemmas are essential. Informally, we want to show that, given  $\delta > 0$  and  $\varepsilon_n \rightarrow 0$ , we can “cut” an appropriate competitor  $(u_n, A_n)$  for  $\sigma_\kappa$  in two points  $t_1, t_2 \in [-\delta, \delta]$  and “glue” the prescribed boundary conditions of  $\sigma_{\kappa, \delta}$ . Such points need to be chosen such that  $E_{\varepsilon_n}(u_n, A_n)$  is small on the segments  $\{t_i\} \times (-1/2, 1/2)$ ,  $i = 1, 2$ . This is similar to standard Modica-Mortola type arguments. However, in our setting extra care is necessary to handle the second term in (3.2.3). In particular, we would like that on the segments  $\{t_1\} \times (-1/2, 1/2)$  the phase  $\theta$  of  $u$  is well defined. Since we are not able to ensure that this happens on a cube of height 1, we perform a similar construction on a larger domain  $Q_{1, N}$ , with  $N \in \mathbb{N}$  and  $N > 2$ . Using the scaling properties of  $\sigma_\kappa$ , this is enough to prove Proposition 3.5.1.

We start by constructing a minimizing sequence for which an appropriate Modica-Mortola type functional in  $\rho$  vanishes away from the boundary and the interface.

**Lemma 3.5.4.** *Let  $0 < \delta < 1/2$ ,  $\varepsilon_n \rightarrow 0$  and  $N \in \mathbb{N}$ ,  $N > 2$ . Then, there exist a sequence of smooth functions  $(u_n, A_n)_n \in \mathcal{K}_{1, N}$  and values  $a_n, b_n, c_n, d_n \in \mathbb{R}$  such that  $a_n \in (-\delta, -3\delta/4)$ ,  $b_n \in (-\delta/2, -\delta/4)$ ,  $c_n \in (-N/2 + \delta, -N/2 + 2\delta)$ ,  $d_n \in (N/2 - \delta, N/2 - 2\delta)$ ,  $\mathcal{E}_{\varepsilon_n}(u_n, A_n) \rightarrow \sigma_\kappa(1, N)$ ,*

$$M_{\varepsilon_n}(\rho_n, R_n) = \int_{R_n} \varepsilon_n |\nabla \rho_n|^2 + \frac{1}{\varepsilon_n} W(\rho_n) dx \rightarrow 0, \quad (3.5.11)$$

where  $R_n = [a_n, b_n] \times [c_n, d_n]$  and  $W(\rho) = \frac{1}{2} \min(2\rho^2, 1)(1 - \rho^2)$ , and

$$\mathcal{E}_{\varepsilon_n}(u_n, A_n, R_n) \rightarrow 0 \quad (3.5.12)$$

**Remark 3.5.5.** *The lemma follows from Lemma 3.3.3 and from the idea that no energy is present far from the interface. The main difficulty is the absence of periodic boundary conditions in this case, so we need to restrict the energy to a “good” rectangle  $R$  to control the boundary term in (3.3.2). The rectangle  $R$  in principle depends on  $n$ , but can be chosen such that it never degenerates for any  $n \in \mathbb{N}$  and such that it never touches the boundary or the interface of  $Q_{1, N}$ , see Figure 3.2. This motivates the assumptions on the values  $a, b, c, d$ .*

*Proof.* In this proof all implicit constants may depend on  $N$  and  $\kappa$ . Let  $(u_n, A_n)_n \in \mathcal{K}_{1, N}$  be such that  $\mathcal{E}_{\varepsilon_n}(u_n, A_n) \rightarrow \sigma_\kappa(1, N)$ . By density of smooth functions in  $\mathcal{A}(Q_{1, N})$ , we can assume



Using Lemma 3.5.4, for any  $\delta > 0$ ,  $\varepsilon_n \rightarrow 0$  and  $N \in \mathbb{N}$  with  $N > 2$  we now find an appropriate competitor  $(u_n, A_n)$  for  $\sigma_\kappa(1, N)$  and two values  $t_1, t_2 \in \mathbb{R}$  such that the energy is controlled on the segments  $\{t_i\} \times (-N/2 + 1, N/2 - 1)$ ,  $i = 1, 2$ .

**Lemma 3.5.6.** *Let  $0 < \delta < 1/2$ ,  $\varepsilon_n \rightarrow 0$  and  $N \in \mathbb{N}$ ,  $N > 2$ . Then, there exist a sequence of smooth functions  $(u_n, A_n)_n \in \mathcal{K}_{1,N}$  and values  $t_1 = t_1(n) \in \mathbb{R}$ ,  $t_2 = t_2(n) \in \mathbb{R}$  such that  $t_1 \in (-\delta, -\delta/4)$ ,  $t_2 \in (\delta/2, \delta)$ ,  $\mathcal{E}_{\varepsilon_n}(u_n, A_n) \rightarrow \sigma_\kappa(1, N)$  and*

(i) on  $[-N/2 + 1, N/2 - 1]$ ,  $\rho_n(t_1, \cdot) \rightarrow 1$  uniformly and  $\theta_n$  is well defined,

$$(ii) \int_{-N/2+1}^{N/2-1} \varepsilon_n |\nabla \rho_n(t_1, x_2)|^2 dx_2 \rightarrow 0,$$

$$(iii) \int_{-N/2+1}^{N/2-1} \varepsilon_n |\partial_2 \theta_n(t_1, x_2) - \alpha A_{n,2}(t_1, x_2)|^2 dx_2 \rightarrow 0,$$

$$(iv) \int_{-N/2+1}^{N/2-1} \varepsilon_n |\nabla_{\alpha A_n} u_n(t_2, x_2)|^2 dx_2 \rightarrow 0,$$

$$(v) \int_{-N/2+1}^{N/2-1} |\rho_n(t_2, x_2)|^p dx_2 \rightarrow 0 \quad \forall p \geq 1.$$

*Proof.* We start by taking the sequence  $(u_n, A_n)_n \in \mathcal{K}_{1,N}$  and the values  $a_n, b_n, c_n, d_n \in \mathbb{R}$  constructed in Lemma 3.5.4. Arguing as for (3.5.12), we have

$$\limsup_{n \rightarrow \infty} \mathcal{E}_{\varepsilon_n}(u_n, A_n, (\delta/2, \delta) \times (c_n, d_n)) = 0.$$

Moreover, for all  $p > 1$

$$\int_{Q_{1,N} \cap \{x_1 \geq 0\}} |\rho_n|^p \rightarrow 0,$$

thus by the mean value theorem we can find that  $t_2 \in (\delta/2, \delta)$  such that (iv) and (v) hold. Now, from Lemma 3.5.4 we know that (3.5.11) holds. Hence, again by the mean value theorem, we can find  $t_1 \in (a_n, b_n) \subset (-\delta, -\delta/4)$  such that (ii) holds,

$$\int_{-N/2+1}^{N/2-1} \varepsilon_n |\partial_2 \rho_n(t_1, x_2)|^2 + \frac{1}{\varepsilon_n} W(\rho_n(t_1, x_2)) dx_2 \rightarrow 0, \quad (3.5.14)$$

$$\int_{-N/2+1}^{N/2-1} \varepsilon_n |\nabla_{\alpha A_n} u_n(t_1, x_2)|^2 dx_2 \rightarrow 0 \quad (3.5.15)$$

and

$$\int_{-N/2+1}^{N/2-1} |\rho_n(t_1, x_2) - 1| dx_2 \rightarrow 0. \quad (3.5.16)$$

Concerning (i), take any  $s, t \in (-N/2 + 1, N/2 - 1)$ . We consider a real function  $\eta$  such that  $\eta' = 2\sqrt{W}$ . Using the usual Modica-Mortola trick and (3.5.14) we find that

$$|\eta(\rho_n(t_1, t)) - \eta(\rho_n(t_1, s))| \leq \int_s^t \varepsilon_n |\partial_2 \rho_n(t_1, x_2)|^2 + \frac{1}{\varepsilon_n} W(\rho_n(t_1, x_2)) dx_2 \rightarrow 0. \quad (3.5.17)$$

Since  $\eta$  is continuous and strictly increasing, (3.5.17) and (3.5.16) imply (i).

Finally, by (i), the phase  $\theta$  is well defined for  $x_1 = t_1$ , so that by (3.5.15)

$$\int_{c_n}^{d_n} \varepsilon_n \rho_n(t_1, x_2)^2 |\partial_2 \theta_n(t_1, x_2) - \alpha A_{n,2}(t_1, x_2)|^2 dx_2 \rightarrow 0. \quad (3.5.18)$$

(i) and (3.5.18) together imply that (iii) holds.  $\square$

Finally, we can prove Proposition 3.5.1.

*Proof of Proposition 3.5.1.* Let  $0 < \delta < 1/2$ ,  $\varepsilon_n \rightarrow 0$  and  $N \in \mathbb{N}$  with  $N > 2$ . Consider  $(u_n, A_n)_n \in \mathcal{K}_{1,N}$ ,  $t_1, t_2 \in \mathbb{R}$  as given by Lemma 3.5.6. We claim that we can find a pair  $(\hat{u}_n, \hat{A}_n) \in BC_{1,N-2}^\delta$  such that

$$\mathcal{E}_{\varepsilon_n}(\hat{u}_n, \hat{A}_n, Q_{1,N-2}) \leq \mathcal{E}_{\varepsilon_n}(u_n, A_n, Q_{1,N}) + o_n(1). \quad (3.5.19)$$

Then, by taking the liminf for  $n \rightarrow +\infty$  in (3.5.19) we find  $\sigma_{\kappa,\delta}(1, N-2) \leq \sigma_\kappa(1, N)$ , so that by (3.5.3) and (3.5.4) we have

$$\sigma_{\kappa,\delta} \leq \frac{N}{N-2} \sigma_\kappa$$

and the result follows by taking the limit for  $N \rightarrow +\infty$ . Let us now construct a pair of functions that satisfy (3.5.19). To simplify notation, we drop the dependency on  $n$ . By Lemma 3.5.6 (i), for  $n$  large enough on the segment  $t_1 \times (-N/2 + 1, N/2 - 1)$  we can write  $u = \rho e^{i\theta}$  for a well-defined phase  $\theta$ . By fixing an appropriate gauge (recall (3.2.2)) we can assume that  $A = (0, A_2)$  and that on the segment  $t_1 \times (-N/2 + 1, N/2 - 1)$  we have  $\theta = 0$  and thus  $u = \rho$ . Indeed, it suffices to take the change of gauge  $\varphi(x_1, x_2) = \eta(x_1, x_2) - \theta(t_1, x_2)$  where  $\eta$  is such that  $\partial_1 \eta = -\alpha A_1$ . We observe that the properties in Lemma 3.5.6 are gauge invariant.

We thus define for  $x \in (-1/2, 1/2) \times (-N/2 + 1, N/2 - 1)$

$$\hat{u}(x_1, x_2) = \begin{cases} 1 & \text{if } -\frac{1}{2} \leq x_1 \leq t_1 - \varepsilon, \\ 1 - \frac{x_1 - (t_1 - \varepsilon)}{\varepsilon} (1 - \rho(t_1, x_2)) & \text{if } t_1 - \varepsilon \leq x_1 \leq t_1, \\ u(x_1, x_2) & \text{if } t_1 \leq x_1 \leq t_2, \\ u(t_2, x_2) (1 - \frac{x_1 - t_2}{\varepsilon}) & \text{if } t_2 \leq x_1 \leq t_2 + \varepsilon, \\ 0 & \text{if } t_2 + \varepsilon \leq x_1 \leq \frac{1}{2} \end{cases}$$

and  $\hat{A} = (0, \hat{A}_2)$  where

$$\hat{A}_2(x_1, x_2) = \begin{cases} 0 & \text{if } -\frac{1}{2} \leq x_1 \leq t_1 - \varepsilon, \\ A_2(t_1, x_2) \left( \frac{x_1 - (t_1 - \varepsilon)}{\varepsilon} \right) & \text{if } t_1 - \varepsilon \leq x_1 \leq t_1, \\ A_2(x_1, x_2) & \text{if } t_1 \leq x_1 \leq t_2, \\ A_2(t_2, x_2) + \frac{1}{\sqrt{2}}(x_1 - t_2) & \text{if } x_1 \geq t_2 \end{cases}$$

so that

$$\hat{B}(x_1, x_2) = \nabla \times \hat{A}(x_1, x_2) = \begin{cases} 0 & \text{if } -\frac{1}{2} \leq x_1 \leq t_1 - \varepsilon, \\ \frac{1}{\varepsilon} A_2(t_1, x_2) & \text{if } t_1 - \varepsilon \leq x_1 \leq t_1, \\ B(x_1, x_2) & \text{if } t_1 \leq x_1 \leq t_2, \\ \frac{1}{\sqrt{2}} & \text{if } x_1 \geq t_2. \end{cases}$$

By construction, we have  $(\hat{u}, \hat{A}) \in BC_{1,N-2}^\delta$ . Denoting

$$f(x_1) = \int_{-N/2+1}^{N/2-1} \left[ \varepsilon \left| \nabla_{\alpha \hat{A}}^\kappa \hat{u} \right|^2 + \frac{1}{\varepsilon} \left( \hat{B} - \frac{1}{\sqrt{2}} (1 - |\hat{u}|^2) \right)^2 \right] (x_1, x_2) dx_2,$$

we have

$$\begin{aligned} \mathcal{E}_\varepsilon \left( \hat{u}, \hat{A}, \left( -\frac{1}{2}, \frac{1}{2} \right) \times \left( -\frac{N}{2} + 1, \frac{N}{2} - 1 \right) \right) &= \int_{-\frac{1}{2}}^{t_1 - \varepsilon} f(x_1) dx_1 + \int_{t_1 - \varepsilon}^{t_1} f(x_1) dx_1 \\ &+ \int_{t_1}^{t_2} f(x_1) dx_1 + \int_{t_2}^{t_2 + \varepsilon} f(x_1) dx_1 + \int_{t_2 + \varepsilon}^{\frac{1}{2}} f(x_1) dx_1. \end{aligned} \quad (3.5.20)$$

Let us examine individually each integral. We notice that the first and the last integral in (3.5.20) are equal to 0. In the third integral  $(\hat{u}, \hat{A}) = (u, A)$ , hence we can estimate it by  $\mathcal{E}_\varepsilon(u, A, Q_{1,N})$ . The rest of the proof is dedicated to estimating the second and the fourth integral in (3.5.20) using Lemma 3.5.6. Since  $|\nabla_{\alpha\hat{A}}^\kappa \hat{u}|^2 \lesssim_\kappa |\nabla_{\alpha\hat{A}} \hat{u}|^2$ , we estimate the latter. We analyze the second term in (3.5.20). Recall that in  $[t_1 - \varepsilon, t_1] \times (-1/2, 1/2)$  it holds  $\hat{u} = \hat{\rho}$  so that

$$|\nabla_{\alpha\hat{A}} \hat{u}|^2 = |\nabla \hat{\rho}|^2 + \alpha^2 \hat{\rho}^2 |\hat{A}_2|^2.$$

First, we have that for  $x_1 \in [t_1 - \varepsilon, t_1]$

$$\hat{\rho}(x_1, x_2) = 1 - \frac{x_1 - (t_1 - \varepsilon)}{\varepsilon} (1 - \rho(t_1, x_2))$$

so that

$$\partial_1 \hat{\rho}(x_1, x_2) = -\frac{1}{\varepsilon} (1 - \rho(t_1, x_2)), \quad (3.5.21)$$

$$\partial_2 \hat{\rho}(x_1, x_2) = \frac{x_1 - (t_1 - \varepsilon)}{\varepsilon} \partial_2 \rho(t_1, x_2). \quad (3.5.22)$$

Using (3.5.21), (3.5.22) and Lemma 3.5.6 (i) and (ii) we find

$$\begin{aligned} & \int_{t_1 - \varepsilon}^{t_1} \int_{-N/2+1}^{N/2-1} \varepsilon |\nabla \hat{\rho}|^2 dx \\ & \lesssim \int_{-N/2+1}^{N/2-1} |1 - \rho(t_1, x_2)|^2 dx_2 + \varepsilon^2 \int_{-N/2+1}^{N/2-1} |\partial_2 \rho(t_1, x_2)|^2 dx_2 = o_\varepsilon(1). \end{aligned} \quad (3.5.23)$$

Furthermore,

$$\begin{aligned} & \varepsilon \alpha^2 \int_{t_1 - \varepsilon}^{t_1} \int_{-N/2+1}^{N/2-1} \hat{\rho}^2 |\hat{A}_2|^2 dx \\ & \lesssim \varepsilon \alpha^2 \int_{t_1 - \varepsilon}^{t_1} \int_{-N/2+1}^{N/2-1} |A_2(t_1, x_2)|^2 dx = \varepsilon^2 \alpha^2 \int_{-N/2+1}^{N/2-1} |A_2(t_1, x_2)|^2 dx_2. \end{aligned}$$

Thus, by Lemma 3.5.6 (iii) and our choice of gauge,

$$\varepsilon \alpha^2 \int_{t_1 - \varepsilon}^{t_1} \int_{-N/2+1}^{N/2-1} \hat{\rho}^2 |\hat{A}_2|^2 dx = o_\varepsilon(\varepsilon). \quad (3.5.24)$$

Regarding the double-well part, using Young's inequality and  $\hat{\rho} \leq 1$  we have

$$\begin{aligned} & \int_{t_1 - \varepsilon}^{t_1} \int_{-N/2+1}^{N/2-1} \frac{1}{\varepsilon} \left( \hat{B} - \frac{1}{\sqrt{2}} (1 - \hat{\rho}^2) \right)^2 dx \\ & \lesssim_\kappa \int_{t_1 - \varepsilon}^{t_1} \int_{-N/2+1}^{N/2-1} \varepsilon \alpha^2 |A_2(t_1, x_2)|^2 + \frac{1}{\varepsilon} (1 - \rho(t_1, x_2))^2 dx \\ & = \int_{-N/2+1}^{N/2-1} \varepsilon^2 \alpha^2 |A_2(t_1, x_2)|^2 + (1 - \rho(t_1, x_2))^2 dx_2 = o_\varepsilon(1), \end{aligned} \quad (3.5.25)$$

where in the last step we used Lemma 3.5.6 (i) and (iii). By (3.5.23),(3.5.24) and (3.5.25) we can conclude that the second term in (3.5.20) is  $o_\varepsilon(1)$ . We finally argue similarly for the fourth term in (3.5.20). In this case for  $x_1 \in [t_2, t_2 + \varepsilon]$  and  $x_2 \in (-N/2 + 1, N/2 - 1)$ , we have

$$\partial_1 \hat{u}(x_1, x_2) = -\frac{1}{\varepsilon} u(t_2, x_2) \quad (3.5.26)$$

$$\partial_2 \hat{u}(x_1, x_2) = \partial_2 u(t_2, x_2) \left(1 - \frac{x_1 - t_2}{\varepsilon}\right). \quad (3.5.27)$$

Lemma 3.5.6 (v) and (3.5.26) imply that

$$\int_{t_2}^{t_2+\varepsilon} \int_{-N/2+1}^{N/2-1} \varepsilon |\partial_1 \hat{u}|^2 dx = o_\varepsilon(1). \quad (3.5.28)$$

Now, (3.5.27) implies that for  $x_1 \in [t_2, t_2 + \varepsilon]$  and  $x_2 \in (-N/2 + 1, N/2 - 1)$ ,

$$\begin{aligned} \left| \partial_2 \hat{u} - i\alpha \hat{A}_2 \hat{u} \right|^2(x_1, x_2) &= \left(1 - \frac{x_1 - t_2}{\varepsilon}\right)^2 \left| \partial_2 u(t_2, x_2) - i\alpha \hat{A}_2(x_1, x_2) u(t_2, x_2) \right|^2 \\ &\lesssim |\partial_2 u(t_2, x_2) - i\alpha A_2(t_2, x_2) u(t_2, x_2)|^2 + \alpha^2 (x_1 - t_2)^2 |u(t_2, x_2)|^2 \\ &\lesssim_\kappa |\partial_2 u(t_2, x_2) - i\alpha A_2(t_2, x_2) u(t_2, x_2)|^2 + \varepsilon^{-2} |u(t_2, x_2)|^2. \end{aligned} \quad (3.5.29)$$

Hence (3.5.29) and Lemma 3.5.6 (iv) and (v) imply that

$$\int_{t_2}^{t_2+\varepsilon} \int_{-N/2+1}^{N/2-1} \varepsilon \left| \partial_2 \hat{u} - i\alpha \hat{A}_2 \hat{u} \right|^2 dx = o_\varepsilon(1). \quad (3.5.30)$$

Moreover, notice that using Lemma 3.5.6 (v) we find

$$\int_{t_2}^{t_2+\varepsilon} \int_{-N/2+1}^{N/2-1} \frac{1}{\varepsilon} \left( \hat{B} - \frac{1}{\sqrt{2}} (1 - |\hat{u}|^2) \right)^2 dx = o_\varepsilon(1). \quad (3.5.31)$$

Combining (3.5.28),(3.5.30) and (3.5.31) we find that the fourth term in (3.5.20) is  $o_\varepsilon(1)$  as desired.  $\square$

### 3.5.2 Characterization of $\sigma_\kappa$ : horizontal periodic boundary conditions

We now prove that up to a small cost we can impose periodic boundary conditions for  $u$  on the sides orthogonal to the transition line. In particular we want  $u$  to coincide with a real valued one dimensional profile. To this extent, consider the profile (this choice is relatively arbitrary)

$$U(t) = \begin{cases} 1 & \text{if } t \leq -1, \\ -t & \text{if } -1 < t < 0, \\ 0 & \text{if } t \geq 0. \end{cases}$$

Now, given  $\varepsilon > 0$ , let us define  $U_\varepsilon : (-1/2, 1/2) \rightarrow [0, 1]$  as

$$U_\varepsilon(t) = U\left(\frac{t}{\varepsilon}\right). \quad (3.5.32)$$

Notice that  $U_\varepsilon$  satisfies  $U_\varepsilon(t) = 1$  for  $t < -\varepsilon$ ,  $U_\varepsilon(t) = 0$  for  $t > 0$ ,

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \varepsilon \left| \dot{U}_\varepsilon(t) \right|^2 dt = 1, \quad (3.5.33)$$

and

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} |U_\varepsilon(t) - 1|^2 dt \leq \varepsilon. \quad (3.5.34)$$

**Proposition 3.5.7.** *Let  $0 < \delta < 1/2$ , then*

$$\sigma_{\kappa,\delta}^{per} \leq \sigma_\kappa + C_\kappa \delta, \quad (3.5.35)$$

where

$$\sigma_{\kappa,\delta}^{per} = \inf \left\{ \liminf_n \mathcal{E}_{\varepsilon_n}(u_n, A_n) \mid \varepsilon_n \rightarrow 0, (u_n, A_n)_n \in BC^\delta, u_n = U_{\varepsilon_n} \text{ for } x_2 = \pm \frac{1}{2} \right\}. \quad (3.5.36)$$

*Proof.* In this proof all implicit constants depend on  $\kappa$ . Let  $\delta \in (0, 1/4)$  and  $\varepsilon_n \rightarrow 0$ . By (3.5.2), we can find a sequence  $(u_n, A_n)_n \in BC^\delta$  such that  $E_{\varepsilon_n}(u_n, A_n) \rightarrow \sigma_{\kappa,\delta}$ . We prove that there exists a sequence  $(\hat{u}_n, \hat{A}_n)_n \in BC^{2\delta}$  with  $\hat{u}_n = U_{\varepsilon_n}$  for  $x_2 = \pm 1/2$  and such that

$$\mathcal{E}_{\varepsilon_n}(\hat{u}_n, \hat{A}_n) \leq \mathcal{E}_{\varepsilon_n}(u_n, A_n) + C \left( \delta + \frac{\varepsilon_n}{\delta} \right), \quad (3.5.37)$$

which, combined with Proposition 3.5.1, implies (3.5.35). We focus on the top edge  $x_2 = 1/2$ , the same reasoning applies for the bottom edge. Since  $\mathcal{E}_{\varepsilon_n}(u_n, A_n) \lesssim 1$ , by the mean value theorem we can find  $1/2 - \delta < \bar{t} < 1/2$  such that

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \varepsilon_n |\partial_1 u_n(x_1, \bar{t})|^2 dx_1 \lesssim \frac{1}{\delta}. \quad (3.5.38)$$

We can further assume that  $\varepsilon_n < \delta$  and that  $\bar{t} + \varepsilon_n < 1/2$ . We define the function  $\hat{u}_n$  as

$$\hat{u}_n(x_1, x_2) = \begin{cases} u_n(x_1, x_2) & \text{if } x_2 \leq \bar{t}, \\ u_n(x_1, \bar{t}) \frac{\bar{t} + \varepsilon_n - x_2}{\varepsilon_n} + U_{\varepsilon_n}(x_1 + \delta) \frac{x_2 - \bar{t}}{\varepsilon_n} & \text{if } \bar{t} \leq x_2 \leq \bar{t} + \varepsilon_n, \\ U_{\varepsilon_n}(x_1 + \delta) \frac{\bar{t} + 2\varepsilon_n - x_2}{\varepsilon_n} + U_{\varepsilon_n}(x_1) \frac{x_2 - \bar{t} - \varepsilon_n}{\varepsilon_n} & \text{if } \bar{t} + \varepsilon_n \leq x_2 \leq \bar{t} + 2\varepsilon_n, \\ U_{\varepsilon_n}(x_1) & \text{if } x_2 \geq \bar{t} + 2\varepsilon_n. \end{cases}$$

Similarly, we define the vector field  $\hat{A}_n = (0, \hat{A}_{n,2})$  where

$$\hat{A}_{n,2}(x_1, x_2) = \begin{cases} A_{n,2}(x_1, x_2) & \text{if } x_2 \leq \bar{t}, \\ \max\left\{\frac{1}{\sqrt{2}}(x_1 - \delta), 0\right\} & \text{if } \bar{t} \leq x_2 \leq \bar{t} + 2\varepsilon_n, \\ \max\left\{\frac{1}{\sqrt{2}}x_1, 0\right\} & \text{if } x_2 \geq \bar{t} + 2\varepsilon_n. \end{cases}$$

Notice that  $\hat{A}_{n,2}$  has a discontinuity, but still  $\nabla \times \hat{A}_n = \partial_1 \hat{A}_{n,2} \in L^2(Q_1)$ . As a result, it holds  $(\hat{u}_n, \hat{A}_n) \in BC^{2\delta}$ . We now prove estimate (3.5.37). Letting  $F_1 = \{x_2 \leq \bar{t}\}$ ,  $F_2 = \{\bar{t} \leq x_2 \leq \bar{t} + \varepsilon_n\}$ ,  $F_3 = \{\bar{t} + \varepsilon_n \leq x_2 \leq \bar{t} + 2\varepsilon_n\}$  and  $F_4 = \{x_2 \geq \bar{t} + 2\varepsilon_n\}$ , we have

$$\begin{aligned} \mathcal{E}_{\varepsilon_n}(\hat{u}_n, \hat{A}_n, Q_1) &= \mathcal{E}_{\varepsilon_n}(\hat{u}_n, \hat{A}_n, F_1) + \mathcal{E}_{\varepsilon_n}(\hat{u}_n, \hat{A}_n, F_2) \\ &\quad + \mathcal{E}_{\varepsilon_n}(\hat{u}_n, \hat{A}_n, F_3) + \mathcal{E}_{\varepsilon_n}(\hat{u}_n, \hat{A}_n, F_4). \end{aligned} \quad (3.5.39)$$

We estimate each term in (3.5.39) separately. On  $F_1$  it holds  $(\hat{u}_n, \hat{A}_n) = (u_n, A_n)$ , hence

$$\mathcal{E}_{\varepsilon_n}(\hat{u}_n, \hat{A}_n, F_1) \leq \mathcal{E}_{\varepsilon_n}(u_n, A_n, Q_1). \quad (3.5.40)$$

On  $F_2$ , using the fact that  $|\nabla_{\hat{A}}^{\kappa} \hat{u}|^2 \lesssim |\nabla_{\hat{A}} \hat{u}|^2$ , we have

$$\mathcal{E}_{\varepsilon_n}(\hat{u}_n, \hat{A}_n, F_2) \lesssim \int_{\bar{t}}^{\bar{t}+\varepsilon_n} \int_{-\delta-\varepsilon_n}^{\delta} \varepsilon_n \left( |\partial_1 \hat{u}_n|^2 + |\partial_2 \hat{u}_n|^2 \right) + \frac{1}{2\varepsilon_n} (1 - |\hat{u}_n|^2)^2 dx_1 dx_2. \quad (3.5.41)$$

By definition of  $\hat{u}_n$ , on  $F_2$  we find

$$\partial_1 \hat{u}_n(x_1, x_2) = \partial_1 u_n(x_1, \bar{t}) \frac{\bar{t} + \varepsilon_n - x_2}{\varepsilon_n} + \dot{U}_{\varepsilon_n}(x_1 + \delta) \frac{x_2 - \bar{t}}{\varepsilon_n}, \quad (3.5.42)$$

$$\partial_2 \hat{u}_n(x_1, x_2) = \frac{1}{\varepsilon_n} (U_{\varepsilon_n}(x_1 + \delta) - u_n(x_1, \bar{t})). \quad (3.5.43)$$

Equations (3.5.42), (3.5.33) and (3.5.38) imply that

$$\begin{aligned} & \int_{\bar{t}}^{\bar{t}+\varepsilon_n} \int_{-\delta-\varepsilon_n}^{\delta} \varepsilon_n |\partial_1 \hat{u}_n|^2 dx_1 dx_2 \\ & \lesssim \int_{\bar{t}}^{\bar{t}+\varepsilon_n} \int_{-\delta-\varepsilon_n}^{\delta} \varepsilon_n \left( |\partial_1 u_n(x_1, \bar{t})|^2 + |\dot{U}_{\varepsilon_n}(x_1)|^2 \right) dx_1 dx_2 \lesssim \frac{\varepsilon_n}{\delta} + \varepsilon_n \lesssim \frac{\varepsilon_n}{\delta}. \end{aligned} \quad (3.5.44)$$

Furthermore, (3.5.43) implies that

$$\begin{aligned} & \int_{\bar{t}}^{\bar{t}+\varepsilon_n} \int_{-\delta-\varepsilon_n}^{\delta} \varepsilon_n |\partial_2 \hat{u}_n|^2 dx_1 dx_2 \\ & \lesssim \int_{\bar{t}}^{\bar{t}+\varepsilon_n} \int_{-\delta-\varepsilon_n}^{\delta} \frac{1}{\varepsilon_n} (|u_n(x_1, \bar{t})|^2 + |U_{\varepsilon_n}(x_1 + \delta)|^2) dx_1 dx_2 \lesssim \delta. \end{aligned} \quad (3.5.45)$$

Moreover, since  $|\hat{u}_n| \leq 1$ ,

$$\int_{\bar{t}}^{\bar{t}+\varepsilon_n} \int_{-\delta-\varepsilon_n}^{\delta} \frac{1}{2\varepsilon_n} (1 - |\hat{u}_n|^2)^2 dx_1 dx_2 \lesssim \delta. \quad (3.5.46)$$

Plugging (3.5.44), (3.5.45) and (3.5.46) in (3.5.41) we get

$$\mathcal{E}_{\varepsilon_n}(\hat{u}_n, \hat{A}_n, F_2) \lesssim \delta + \frac{\varepsilon_n}{\delta}. \quad (3.5.47)$$

Let us now consider  $F_3$ . With a similar computation as before

$$\begin{aligned} & \int_{\bar{t}+\varepsilon_n}^{\bar{t}+2\varepsilon_n} \int_{-\delta-\varepsilon_n}^{\delta} \varepsilon_n |\partial_1 \hat{u}_n|^2 dx_1 dx_2 \\ & \lesssim \int_{\bar{t}+\varepsilon_n}^{\bar{t}+2\varepsilon_n} \int_{-\delta-\varepsilon_n}^{\delta} \varepsilon_n \left( |\dot{U}_{\varepsilon_n}(x_1 + \delta)|^2 + |\dot{U}_{\varepsilon_n}(x_1)|^2 \right) dx_1 dx_2 \lesssim \varepsilon_n \end{aligned}$$

and

$$\int_{\bar{t}+\varepsilon_n}^{\bar{t}+2\varepsilon_n} \int_{-\delta-\varepsilon_n}^{\delta} \varepsilon_n |\partial_2 \hat{u}_n|^2 + \frac{1}{2\varepsilon_n} (1 - |\hat{u}_n|^2)^2 dx_1 dx_2 \lesssim \delta$$

so that

$$\mathcal{E}_{\varepsilon_n}(\hat{u}_n, \hat{A}_n, F_3) \lesssim \delta + \varepsilon_n. \quad (3.5.48)$$

Finally we address  $F_4$ . By (3.5.32), (3.5.33), (3.5.34) and the definitions of  $\hat{u}_n$  and  $\hat{A}_n$  we have

$$\mathcal{E}_{\varepsilon_n}(\hat{u}_n, \hat{A}_n, F_4) = \int_{\bar{t}+\varepsilon_n}^{\bar{t}} \int_{-\varepsilon_n}^0 \varepsilon_n |\dot{U}_{\varepsilon_n}|^2 + \frac{1}{2\varepsilon_n} (1 - |U_{\varepsilon_n}|^2)^2 dx_1 dx_2 \lesssim \delta. \quad (3.5.49)$$

Combining (3.5.40), (3.5.49), (3.5.47) and (3.5.48) gives (3.5.37).  $\square$

**Remark 3.5.8.** Let  $\rho, A : \mathbb{R} \rightarrow \mathbb{R}$  and consider for  $a < b \in \mathbb{R} \cup \{\pm\infty\}$

$$\mathcal{E}^{1d}(\rho, A, (a, b)) = \int_a^b |\dot{\rho}|^2 + \kappa^{-2} \rho^2 A^2 + \left( \dot{A} - \frac{1}{\sqrt{2}}(1 - \rho^2) \right)^2 dt.$$

In [23, 24], the following one dimensional phase transition problem for type-I superconductors is studied:

$$\sigma_\kappa^{1d} = \inf \left\{ E^{1d}(\rho, A, (-\infty, +\infty)), (\rho, A)(-\infty) = (1, 0), (\rho, \dot{A})(+\infty) = \left( 0, \frac{1}{\sqrt{2}} \right) \right\}.$$

Notice that if we consider  $\tilde{u}(x_1, x_2) = \rho(x_1)$  and  $\tilde{A}(x_1, x_2) = [0, A(x_1)]$ , then

$$\mathcal{E}_1(\tilde{u}, \tilde{A}, Q_1) = \mathcal{E}^{1d} \left( \rho, A, \left( -\frac{1}{2}, \frac{1}{2} \right) \right),$$

which implies  $\sigma_\kappa \leq \sigma_\kappa^{1d}$ . We conjecture that  $\sigma_\kappa = \sigma_\kappa^{1d}$ , which would thus imply that the transition in our problem is one dimensional. However due to the vectorial nature of our problem, this seems to be a difficult question.

### 3.6 $\Gamma$ -limsup inequality

The goal of this section is to prove the following theorem (Theorem 3.1.1-3).

**Theorem 3.6.1.** Let  $\varepsilon_n \rightarrow 0$  such that  $\varepsilon_n^{-2} \kappa^{-2} b_{ext} \in 2\pi\mathbb{Z}$ . Let  $\rho \in BV(\mathbb{T}^2, \{0, 1\})$  be such that  $\int_{\mathbb{T}^2} (1 - \rho)/\sqrt{2} dx = b_{ext}/\kappa \in (0, 1)$ . Then, there exists a sequence  $(u_n, A_n) \in \mathcal{A}_{\varepsilon_n}$  such that  $|u_n| = \rho_n \rightarrow \rho$  in  $L^p(\mathbb{T}^2)$  for all  $p \geq 1$ ,  $B_n = \nabla \times A_n \rightarrow (1 - \rho)/\sqrt{2}$  in  $L^2(\mathbb{T}^2)$ ,  $\int_{\mathbb{T}^2} B_n dx = b_{ext}/\kappa$  and

$$\limsup_n \mathcal{E}_{\varepsilon_n}(u_n, A_n) \leq \sigma_\kappa P(\{\rho = 0\}, \mathbb{T}^2). \quad (3.6.1)$$

A function  $\rho \in BV(\mathbb{T}^2, \{0, 1\})$  is characterized by the set of finite perimeter  $E = \{\rho = 0\}$ . To understand the main ideas behind the proof, we first start by the easier case where  $E$  is a simply connected closed polyhedral set. We then discuss the generalization to the case where  $E$  is a connected closed polyhedral set, then finally we examine the construction on a generic closed polyhedral set. The result for a set of finite perimeter follows by a standard density argument. In all these arguments the fundamental building block for the construction is given by the following proposition, which follows immediately from Proposition 3.5.1, (3.5.35) and the density of continuous functions in (3.1.1).

**Proposition 3.6.2.** Let  $\xi > 0$ . Then, there exist  $0 < \varepsilon_0 < \delta_0 < 1/2$  and  $(u_0, A_0) \in \mathcal{A}^{\delta_0}$  such that  $u_0 = U_{\varepsilon_0}$  for  $x_2 = \pm 1/2$  and

$$\mathcal{E}_{\varepsilon_0}(u_0, A_0, Q_1) \leq \sigma_\kappa + \xi.$$

We write  $B_0 = \nabla \times A_0$  and  $\rho_0 = |u_0|$ . Moreover,  $u_0$  is continuous and there exists a function  $\theta_0 \in H_{loc}^1(\{\rho_0 > 0\})$  such that in the set  $\{\rho_0 > 0\}$  we have  $u_0 = \rho_0 e^{i\theta_0}$ . In particular, on  $\{x_1 = -1/2\}$  and on  $\{x_1 \leq 0\} \cap \{x_2 = \pm 1/2\}$  it holds  $\rho_0 > 0$  and  $\theta_0 = 0$ . Furthermore,

$$\frac{1}{4\sqrt{2}} < \int_{Q_1} B_0 dx \leq \frac{3}{4\sqrt{2}}. \quad (3.6.2)$$

**Remark 3.6.3.** Condition (3.6.2) comes from the convergence constraint on the magnetic field in (3.4.1).

*Proof of Theorem 3.6.1.* Let  $E = \{\rho = 0\}$ . In this proof all implicit constants may depend on  $E$ . By [80, Remark 13.13], closed polyhedral sets are dense in finite perimeter sets with respect to the  $L^1$  topology. Hence, as usual in  $\Gamma$ -convergence results (see [12, 43, 25]), by a diagonal argument we may restrict the construction of the recovery sequence to closed polyhedral sets.

**Case 1:  $E$  simply connected closed polyhedral set.** Let  $E \subset \mathbb{T}^2$  be a closed simply connected polyhedral set with  $N$  edges. That is  $\partial E = \cup_{i=1}^N e_i$  where the sets  $e_i$  are segments intersecting at their endpoints. We denote, for  $i = 1 \dots N$ , by  $L_i$  the length of  $e_i$  and by  $\nu_i$  the inward normal vector of  $E$  on  $e_i$ . Let us also call  $c_i^\pm$  the two endpoints of  $e_i$ .

We proceed as follows: first, we define  $\rho_n$  and  $B_n$  that satisfy the required convergence assumptions and the mass constraint and then, we construct an appropriate couple  $(u_n, A_n) \in \mathcal{A}_{\varepsilon_n}$  that satisfies the energy bound.

Let  $\xi > 0$ . Consider  $(u_0, A_0)$ ,  $\varepsilon_0$ , and  $\delta_0$  as given by Proposition 3.6.2. The idea is to cover each edge of  $E$  with mutually disjoint rectangular sets  $\Omega_i$  of width  $\varepsilon_n/\varepsilon_0$  and length almost equal to  $L_i$ . Moreover, if  $e_i \subset \partial E_k$ , we shift the sets  $\Omega_i$  in the direction given by  $\nu_i$  by a factor  $\zeta = \zeta(\varepsilon_n) \rightarrow 0$  to be chosen later. To do so, for any set  $F \subset \mathbb{T}^2$  consider the signed distance function

$$sd_F(x) = \begin{cases} d(x, \partial F) & \text{if } x \in F, \\ -d(x, \partial F) & \text{if } x \in F^c. \end{cases}$$

Let  $\zeta \in (-\varepsilon_n/(2\varepsilon_0), \varepsilon_n/(2\varepsilon_0))$  to be chosen later and define for any  $F \subset \mathbb{T}^2$

$$T(\varepsilon_n, \zeta, F) = \left\{ x \in Q_1, -\frac{\varepsilon_n}{2\varepsilon_0} + \zeta < sd_F(x) < \frac{\varepsilon_n}{2\varepsilon_0} + \zeta \right\}. \quad (3.6.3)$$

Notice that  $T(\varepsilon_n, 0, E)$  is just an  $(\varepsilon_n/\varepsilon_0)$ -neighborhood of  $\partial E$ . For  $i = 1 \dots N$ , we set

$$\Omega_i(\varepsilon_n, \zeta) = \left\{ x \in T(\varepsilon_n, \zeta, E), d(x, e_i + \zeta\nu_i) < \frac{\varepsilon_n}{2\varepsilon_0}, |P_{e_i}x - c_i^+| > l_i^+, |P_{e_i}x - c_i^-| > l_i^- \right\},$$

where  $P_{e_i}$  is the projection on the line containing  $e_i$  and  $l_i^\pm = l_i^\pm(\varepsilon_n) \rightarrow 0$  are such that  $\Omega_i(2\varepsilon_n, 0) \cap \Omega_j(2\varepsilon_n, 0) = \emptyset$ . This last condition implies that for any  $\zeta \in (-\varepsilon_n/(2\varepsilon_0), \varepsilon_n/(2\varepsilon_0))$  it holds  $\Omega_i(\varepsilon_n, \zeta) \cap \Omega_j(\varepsilon_n, \zeta) = \emptyset$ . Moreover, it is possible to assume that  $L_i - l_i^- - l_i^+ = N_i\varepsilon_n/\varepsilon_0$ , with  $N_i = N_i(\varepsilon_n) \in \mathbb{N}$ , so that each  $\Omega_i(\varepsilon_n, \zeta)$  consists of  $N_i$  squares of side length  $\varepsilon_n/\varepsilon_0$ , see Figure 3.3. For  $j = 1 \dots N_i(\varepsilon)$ , let us call  $x_{i,j}$  the center of the square  $j$  constituting  $\Omega_i$ . For simplicity, we call  $T = T(\varepsilon_n, \zeta, E)$  and  $\Omega_i = \Omega_i(\varepsilon_n, \zeta)$ . Notice that, by construction,

$$\left| T \setminus \bigcup_i \Omega_i \right| \lesssim \varepsilon_n^2. \quad (3.6.4)$$

We can now define  $\rho_n$  and  $B_n$  by "gluing" an appropriate rescaling of the building blocks  $(u_0, A_0)$  on the sets  $\Omega_i$ . Recalling (3.5.32), let

$$\rho_n(x) = \begin{cases} 0 & \text{in } E \setminus T, \\ \rho_0\left(\frac{\varepsilon_0}{\varepsilon_n} R_{\nu_i}(x - x_{i,j})\right) & \text{in } Q_{\frac{\varepsilon_n}{\varepsilon_0}}(x_{i,j}), \\ U_{\frac{\varepsilon_n}{\varepsilon_0}}(sd_E(x) - \zeta) & \text{in } T \setminus \bigcup_i \Omega_i, \\ 1 & \text{elsewhere} \end{cases} \quad (3.6.5)$$

and

$$B_n(x) = \begin{cases} \frac{1}{\sqrt{2}} & \text{in } E \setminus T, \\ B_0\left(\frac{\varepsilon_0}{\varepsilon_n} R_{\nu_i}(x - x_{i,j})\right) & \text{in } Q_{\frac{\varepsilon_n}{\varepsilon_0}}(x_{i,j}), \\ 0 & \text{elsewhere.} \end{cases} \quad (3.6.6)$$

The definition of  $\rho_n$  on  $T \setminus \bigcup_i \Omega_i$  is just a rescaled Lipschitz extension of  $U_{\varepsilon_0}$ , which is the horizontal boundary condition of  $u_0$  (compare with (3.5.32)). Hence,  $\rho_n$  is continuous. A

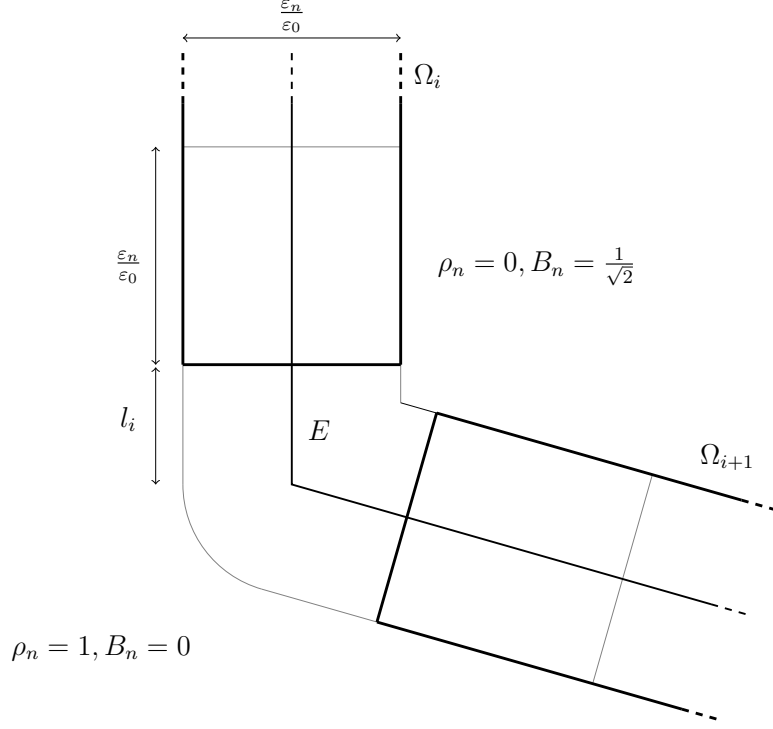


Figure 3.3: Sketch of the construction near a corner of  $\partial E$ . For simplicity in this picture  $\zeta = 0$ .

computation shows that  $\rho_n$  and  $B_n$  satisfy

$$\int_{\mathbb{T}^2} |\rho_n - \rho| + \left| B_n - \frac{1}{\sqrt{2}}(1 - \rho) \right|^2 dx \lesssim |T| \lesssim \varepsilon_n,$$

which implies  $\rho_n \rightarrow \rho \in L^1(\mathbb{T}^2)$  (hence in  $L^p(\mathbb{T}^2)$  for all  $1 \leq p < \infty$  since, by construction,  $\rho_n \leq 1$ ) and  $B_n \rightarrow (1 - \rho)/\sqrt{2} \in L^2(\mathbb{T}^2)$ , as required.

Regarding the flux constraint of  $B_n$ , we have

$$\int_{\mathbb{T}^2} B_n dx = \int_{E \cup T} B_n dx = \frac{1}{\sqrt{2}} |E \setminus T| + \frac{\varepsilon_n^2}{\varepsilon_0^2} \sum_i N_i \int_{Q_1} B_0 dx = h(\zeta).$$

By construction we have that

$$|E \setminus T| = |E| - P(E) \left( \zeta + \frac{\varepsilon_n}{2\varepsilon_0} \right) + o(\varepsilon_n)$$

and  $\sum_i N_i \varepsilon_n / \varepsilon_0 \rightarrow P(E)$ . Thus,

$$h(\zeta) = \frac{1}{\sqrt{2}} |E| - P(E) \left[ \frac{1}{\sqrt{2}} \left( \zeta + \frac{\varepsilon_n}{2\varepsilon_0} \right) - \frac{\varepsilon_n}{\varepsilon_0} \int_{Q_1} B_0 dx \right] + o(\varepsilon_n). \quad (3.6.7)$$

By (3.6.2), for  $n$  large enough  $h(-\varepsilon_n/(2\varepsilon_0)) > |E|/\sqrt{2}$  and  $h(\varepsilon_n/(2\varepsilon_0)) < |E|/\sqrt{2}$ . Since  $h \in C^0$ , we can choose  $\zeta \in (-\varepsilon_n/(2\varepsilon_0), \varepsilon_n/(2\varepsilon_0))$  such that

$$\int_{\mathbb{T}^2} B_n dx = \frac{1}{\sqrt{2}} |E| = \frac{b_{\text{ext}}}{\kappa}. \quad (3.6.8)$$

Let us now construct  $(u_n, A_n) \in \mathcal{A}_{\varepsilon_n}$  that satisfies the energy bound.

Since we have defined  $B_n$  everywhere in  $\mathbb{T}^2$ , we can find a global vector potential  $A_n$  such that  $\nabla \times A_n = B_n$ . For instance we can take  $A_n = (0, A_{n,2})$  with

$$A_{n,2}(x_1, x_2) = \int_{-\frac{1}{2}}^{x_1} B_n(t, x_2) dt. \quad (3.6.9)$$

We now construct an appropriate phase  $\theta_n$  such that the non local term  $\nabla\theta_n - \alpha A_n$  is controlled. To this extent, we introduce an auxiliary vector field by setting for  $x \in E^c \cup T$

$$A_n^0(x) = \begin{cases} \frac{\varepsilon_n}{\varepsilon_0} R_{\nu_i}^{-1} A_0(\frac{\varepsilon_0}{\varepsilon_n} R_{\nu_i}(x - x_{i,j})) & \text{in } Q_{\frac{\varepsilon_n}{\varepsilon_0}}(x_{i,j}), \\ 0 & \text{elsewhere.} \end{cases} \quad (3.6.10)$$

Notice that  $A_n^0$  has discontinuities but still  $\nabla \times A_n^0 \in L^2(E^c \cup T)$ . We also define an auxiliary phase function. For  $x \in \{\rho_n > 0\}$ , let

$$\theta_n^0(x) = \begin{cases} \theta_0(\frac{\varepsilon_0}{\varepsilon_n} R_{\nu_i}(x - x_{i,j})) & \text{in } Q_{\frac{\varepsilon_n}{\varepsilon_0}}(x_{i,j}), \\ 0 & \text{elsewhere.} \end{cases} \quad (3.6.11)$$

The function  $\theta_n^0$  is continuous in  $\{\rho_n > 0\}$  since  $u_0 = U_{\varepsilon_0}$  on the horizontal boundaries of  $Q_1$  and  $u = 1$  on the left boundary of  $Q_1$ , hence on  $\partial Q_1 \cap \{\rho_0 > 0\}$   $u_0$  is real and  $\theta_0 = 0$ . We remark that up to this point we have not used the hypothesis that  $E$  is simply connected. Since  $E$  is closed and simply connected, the set  $E^c \cup T$  is open and connected. Let  $x_0 \in E^c \cup T$  and for  $x \in \{\rho_n > 0\}$  let  $\Gamma_{x,x_0} \subset E^c \cup T$  be any Lipschitz curve connecting  $x$  to  $x_0$ . We define for  $x \in \{\rho_n > 0\}$

$$\theta_n(x) = \theta_n^0(x) + \alpha \int_{\Gamma_{x,x_0}} (A_n - A_n^0) \cdot dl. \quad (3.6.12)$$

Let us prove that  $\theta_n$  is well defined up to  $2\pi\mathbb{Z}$ . Let  $\Gamma$  be a simple closed Lipschitz curve in  $E^c \cup T$  enclosing  $E \setminus T$ . Since in  $E^c \cup T$  the term  $A_n - A_n^0$  is curl free, by Stokes' theorem its path integral is invariant for homotopies of  $\Gamma$  and in particular

$$\int_{\Gamma} (A_n - A_n^0) \cdot dl = \int_{\partial(E \cup T)} (A_n - A_n^0) \cdot dl.$$

Moreover, since  $A_n^0 = 0$  on  $\partial(E \cup T)$ , we have that

$$\alpha \int_{\partial(E \cup T)} (A_n - A_n^0) \cdot dl = \alpha \int_{\partial(E \cup T)} A_n \cdot dl = \alpha \int_{E \cup T} B_n dx = \alpha \frac{b_{\text{ext}}}{\kappa} = \varepsilon_n^{-2} \kappa^{-2} b_{\text{ext}}. \quad (3.6.13)$$

Since by assumption  $\varepsilon_n^{-2} \kappa^{-2} b_{\text{ext}} \in 2\pi\mathbb{Z}$ , (3.6.13) implies that  $\theta_n(x)$  is defined up to a multiple of  $2\pi$ , and thus we can uniquely define  $u_n = \rho_n e^{i\theta_n}$ . Moreover, we remark that on  $\{\rho_n > 0\}$  it holds

$$\nabla\theta_n - \alpha A_n = \nabla\theta_n^0 - \alpha A_n^0, \quad (3.6.14)$$

and in particular (3.6.14) is zero outside of  $\bigcup_i \Omega_i$ . We now prove (3.6.1). Notice that, because of (3.6.14) and the definitions (3.6.5), (3.6.6), (3.6.9) and (3.6.12), we have

$$\mathcal{E}_{\varepsilon_n}(u_n, A_n, T^c) = 0.$$

We just have to focus on the squares  $Q_{\frac{\varepsilon_n}{\varepsilon_0}}(x_{i,j})$  composing the sets  $\Omega_i$  and on the ‘‘corners’’ (that is  $T \setminus \bigcup_i \Omega_i$ ). By a change of variable and using Proposition 3.6.2, we have for every  $i = 1 \dots N$  and  $j = 1 \dots N_i$

$$\mathcal{E}_{\varepsilon_n}(u_n, A_n, Q_{\frac{\varepsilon_n}{\varepsilon_0}}(x_{i,j})) = \frac{\varepsilon_n}{\varepsilon_0} \mathcal{E}_{\varepsilon_0}(u_0, A_0, Q_1) \leq \frac{\varepsilon_n}{\varepsilon_0} (\sigma_\kappa + \xi). \quad (3.6.15)$$

We now show that the energy near the corners tends to 0 as  $\varepsilon_n \rightarrow 0$ . By (3.5.33) and since (3.6.14) and  $B_n$  are zero on  $T \setminus \bigcup_i \Omega_i$ , we have

$$\mathcal{E}_{\varepsilon_n}(u_n, A_n, T \setminus \bigcup_i \Omega_i) = \int_{T \setminus \bigcup_i \Omega_i} \varepsilon_n |\nabla \rho_n|^2 + \frac{1}{2\varepsilon_n} (1 - \rho_n^2)^2 dx \lesssim \frac{1}{\varepsilon_n} \left| T \setminus \bigcup_i \Omega_i \right| \stackrel{(3.6.4)}{\lesssim} \varepsilon_n. \quad (3.6.16)$$

Putting (3.6.15) and (3.6.16) together we get,

$$\mathcal{E}_{\varepsilon_n}(u_n, A_n) = \sum_{i,j} \mathcal{E}_{\varepsilon_n}(u_n, A_n, Q_{\frac{\varepsilon_n}{\varepsilon_0}}(x_{i,j})) + \mathcal{E}_{\varepsilon_n}(u_n, A_n, T \setminus \bigcup_i \Omega_i) \leq \sum_i N_i \frac{\varepsilon_n}{\varepsilon_0} (\sigma_\kappa + \xi) + C\varepsilon_n.$$

Since  $N_i \varepsilon_n / \varepsilon_0 \rightarrow L_i$  as  $\varepsilon_n \rightarrow 0$ , taking the lim sup in the previous inequality yields

$$\limsup_{\varepsilon_n} \mathcal{E}_{\varepsilon_n}(u_n, A_n) \leq \sum_i L_i (\sigma_\kappa + \xi) = P(\{\rho = 0\}, Q_1) (\sigma_\kappa + \xi).$$

The conclusion then follows by a diagonal procedure with respect to  $\xi$ .

**Case 2:  $E$  connected closed polyhedral set.** We now assume that  $E$  is a connected closed polyhedral set. We can repeat the constructions of Case 1 for  $T$ ,  $\rho_n$ ,  $B_n$ ,  $A_n$ ,  $A_n^0$ ,  $\theta_n^0$  and  $A_n$ , as in (3.6.3), (3.6.5), (3.6.6), (3.6.9), (3.6.10) and (3.6.11) respectively, since they do not use the fact that  $E$  is simply connected. We can also choose the value  $\zeta$  as in (3.6.7) so that (3.6.8) holds.

We are left with the definition of an appropriate phase  $\theta_n$ . Since  $E$  is not simply connected,  $E^c \cup T$  is not connected in general. Let us call  $S > 0$  the number of connected components of  $E^c \cup T$  and  $F_k$ ,  $k = 1 \dots S$  each component. Since  $E$  is connected, at most one connected component of  $E^c \cup T$  is not simply connected and encloses  $E$ . Without loss of generality, we assume that  $F_1$  encloses  $E$  and  $F_k$ ,  $k = 2 \dots S$  are simply connected. We now define  $\theta_n$  separately on each connected component of  $E^c \cup T$ . For every  $k = 1 \dots S$ , we define  $\theta_n(x)$  for  $x \in F_k \cap \{\rho > 0\}$  as

$$\theta_n(x) = \theta_n^0(x) + \alpha \int_{\Gamma_{x, x_0^k}} (A_n - A_n^0) \cdot dl$$

where  $x_0^k \in F_k$  is arbitrary and  $\Gamma_{x, x_0^k}$  is any curve contained in  $F_k$  connecting  $x$  to  $x_0^k$ . For  $k = 1$ , the same computations as (3.6.13) give that  $\theta_n$  is defined up to a multiple of  $2\pi$ . Then, if  $k > 1$ , the set  $F_k$  is simply connected. Since  $A_n - A_n^0$  is irrotational, the definition of  $\theta_n$  does not depend on the choice of the curve  $\Gamma_{x, x_0^k}$ . This means that as before  $u_n = \rho_n e^{i\theta_n}$  is uniquely defined, and the same computations as Case 1 apply for the proof of the  $\Gamma$ -limsup inequality.

**Case 3:  $E$  closed polyhedral set.** Let us finally assume that  $E \subset \mathbb{T}^2$  is any arbitrary closed polyhedral set. In particular  $E$  has a finite number of sides and thus a finite number  $M$

of connected components. We denote  $E_j$  with  $j = 1 \dots M$  these sets. We now consider for  $j = 1 \dots M$  values  $\zeta_j \in (-\varepsilon_n/(2\varepsilon_0), \varepsilon_n/(2\varepsilon_0))$  to be chosen later. We set  $T_j = T(\varepsilon_n, \zeta_j, E_j)$  and  $T = \cup_{j=1}^M T_j$ . For  $\varepsilon_n$  small enough we can suppose that the sets  $T^j$  are disjoint. For each set  $T_j$  and value  $\zeta_j$  consider the constructions of Case 1 (3.6.5), (3.6.6), (3.6.10) and (3.6.11) which we denote by  $\rho_{n,j}, B_{n,j}, A_{n,j}^0$  and  $\theta_{n,j}^0$ . We set

$$\rho_n(x) = \begin{cases} 0 & \text{in } E_j \setminus T_j, \\ \rho_j(x) & \text{in } T_j, \\ 1 & \text{elsewhere} \end{cases}$$

and

$$B_n(x) = \begin{cases} \frac{1}{\sqrt{2}} & \text{in } E_j \setminus T_j, \\ B_j(x) & \text{in } T_j, \\ 0 & \text{elsewhere.} \end{cases}$$

Arguing as in (3.6.7), we have for  $j = 1 \dots M$

$$\int_{E_j \cup T^j} B_n dx = \frac{1}{\sqrt{2}} |E_j| - P(E_j) \left[ \frac{1}{\sqrt{2}} \left( \zeta_j + \frac{\varepsilon_n}{2\varepsilon_0} \right) + \frac{\varepsilon_n}{\varepsilon_0} \int_{Q_1} B_0 dx \right] + o(\varepsilon_n).$$

We now choose in a similar way as before in the values  $\zeta_j \in (-\varepsilon_n/(2\varepsilon_0), \varepsilon_n/(2\varepsilon_0))$  such that for all  $j = 1 \dots M - 1$  we have (recall that  $\alpha = \kappa \varepsilon_n^{-2}$ )

$$\int_{E^j \cup T_n^j} B_n dx = \frac{2\pi}{\alpha} \left[ \frac{\alpha}{2\pi} \frac{|E_j|}{\sqrt{2}} \right],$$

so that

$$\alpha \int_{E^j \cup T_n^j} B_n dx \in 2\pi\mathbb{Z}. \quad (3.6.17)$$

Moreover, we choose  $\zeta_M \in (-\varepsilon_n/(2\varepsilon_0), \varepsilon_n/(2\varepsilon_0))$  such that

$$\int_{\mathbb{T}^2} B_n dx = b_{\text{ext}}/\kappa.$$

The assumption  $\varepsilon_n^{-2} \kappa^{-2} b_{\text{ext}} \in 2\pi\mathbb{Z}$  then ensures that (3.6.17) holds also for  $j = M$ . As before, we define the auxiliary field for  $x \in E^c \cup T$

$$A_n^0(x) = \begin{cases} A_{n,j}^0(x) & \text{in } T_j, \\ 0 & \text{elsewhere} \end{cases}$$

and for  $x \in \{\rho_n > 0\}$

$$\theta_n^0(x) = \begin{cases} \theta_{n,j}^0(x) & \text{in } T_j, \\ 0 & \text{elsewhere.} \end{cases}$$

We then define  $A_n$  as a global vector potential on  $Q_1$  as in (3.6.9). Now consider the set  $E^c \cup T$ . Let us call  $S > 0$  the number of its connected components and denote by  $F_k$ ,  $k = 1 \dots S$  each component. As in Case 2, we define the phase  $\theta_n$  locally on each connected component of  $E^c \cup T$ . For every  $k = 1 \dots S$ , we define  $\theta_n(x)$  for  $x \in F_k \cap \{\rho > 0\}$  as

$$\theta_n(x) = \theta_n^0(x) + \alpha \int_{\Gamma_{x,x_0^k}} (A_n - A_n^0) \cdot dl$$

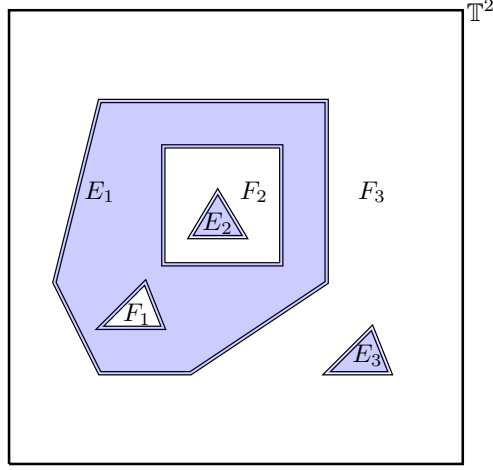


Figure 3.4: Example of a polyhedral set  $E$  (in blue) such that both  $E$  and  $E^c$  have multiple connected components,  $E_1, E_2, E_3$ , and  $F_1, F_2, F_3$  respectively. The set  $T$  is represented by the double line around  $\partial E$ . The phase  $\theta_n$  is defined locally on each  $F_k$ ,  $k = 1 \dots 3$ . On any  $F_k$ ,  $k = 1 \dots 3$ , to ensure that the phase is defined up to a multiple of  $2\pi$ , each connected component of  $E$  which is enclosed by  $F_k$  is subject to the flux quantization condition (3.6.17). In particular, in this example  $F_3$  forces a quantization condition on  $E_1, E_2$  and  $E_3$ , while  $F_2$  forces a quantization condition on  $E_2$ . Since  $F_1$  is simply connected, its phase is already quantized and no quantization condition is enforced.

where  $x_0^k \in F_k$  is arbitrary and  $\Gamma_{x, x_0^k}$  is any curve contained in  $F_k$  connecting  $x$  to  $x_0^k$ . Differently to Case 2, each  $F_k$  may now enclose multiple connected components of  $E$ , see Figure 3.4. To prove that  $\theta_n$  is defined up to a factor of  $2\pi$ , for any  $k = 1 \dots S$  consider a closed curve  $\Gamma$  in  $F_k$ . With the same computation as (3.6.13) we get

$$\alpha \int_{\Gamma} (A - A_0) \cdot dl = \alpha \sum_j a_j \int_{E^j \cup T_n^j} B_n dx, \quad (3.6.18)$$

for some  $a_j \in \{-1, 1\}$ , where the sum on the right hand side of (3.6.18) is performed on all  $j$  such that  $E_j$  is enclosed by  $\Gamma$ . By (3.6.17) all the terms in the sum are multiples of  $2\pi$ , hence also the sum is a multiple of  $2\pi$ . Thus,  $u_n = \rho_n e^{i\theta_n}$  is uniquely defined. To conclude, the same computations as in Case 1 apply for the proof of the  $\Gamma$ -limsup inequality.  $\square$



## Chapter 4

# New dimensional bounds for a branched transport problem

We consider a branched transport problem with weakly imposed boundary conditions. This problem arises as a reduced model for pattern formation in type-I superconductors. For this model, it is conjectured that the dimension of the boundary measure is non-integer. We prove this conjecture in a simplified 2D setting, under the (strong) assumption of Ahlfors regularity of the irrigated measure. This work is the first rigorous proof of a singular behaviour for irrigated measures resulting from minimality.

### 4.1 Introduction

Ginzburg-Landau theory [73] predicts two types of superconductors, usually named type-I and type-II. While in type-II superconductors vortices appear [11, 100], in type-I superconductors a positive surface energy penalizes the interface between normal and superconducting regions. Such superconductors, when subject to an applied external magnetic field, exhibit complex patterns [95, 96, 97]. This behaviour suggests that branching patterns should appear inside the sample. Results in this direction in the form of scaling laws were first proved for some simplified models in [26, 28] and were more recently extended to the full Ginzburg-Landau model in [39]. Besides the anticipated uniform branching regime where, as its name suggests, the magnetization is (approximately) uniform at the boundary of the sample, a surprising finding of [26, 28, 39] is the presence of a regime where it is energetically favourable for the magnetic field to be far from uniform at the boundary.

The aim of this chapter is to better understand this non-uniform branching regime and more specifically the magnetization pattern at the boundary of the sample. Due to the various lengthscales involved, the full Ginzburg-Landau model is too complicated to analyze, see [37]. We thus make a few simplifications. First, we consider the asymptotic regime of very small external magnetic field in the cross-over between uniform and non-uniform branching regimes. In this regime, formal computations suggest that the normal region collides into a one-dimensional structure. The magnetization, which lives inside the normal region by the Meissner condition, can then be represented as a measure. The reduced model to study should be a branched transport problem inside the sample complemented by a penalization of the deviation from uniformity at the boundary. Because of the magnetic field energy outside the sample, this penalization comes through a Sobolev norm of negative exponent (thus in a weak norm). Let us point out that deep inside the uniform branching regime, a similar model has been rigorously derived in [37]. In that case, the internal energy remains the same but the boundary condition is rather imposed as a hard constraint. The rigorous derivation

of the model considered in this chapter is work in progress. A second simplification we will make in this chapter is to consider a 2D model rather than the original 3D model. Actually, since vertical and horizontal variables play a different role, we rather think of our 2D setting as being a one-dimensional but time dependent branched transport problem. As in [67], where a similar 2D model for the uniform branching regime was studied, the simple geometry of  $\mathbb{R}$  as opposed to  $\mathbb{R}^2$  turns out to be very useful. In particular, the no-loop property of minimizers, see Lemma 4.2.3, implies in dimension one that the branched transport plan is monotone and thus also optimal for the standard optimal transport problem, see Lemma 4.2.7.

In order to present our main results, let us briefly introduce the model (see Section 4.2.3 for a precise definition). Motivated by the discussion above, we give the definition in arbitrary space dimension. Let  $s \in (0, 1)$ ,  $d \in \mathbb{N}$  and  $T > 0$ . Let  $\mathbb{T}_T^d = \mathbb{T}^d \times (-T, T)$  with  $\mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$ , be a sample with thickness  $2T$  and periodic boundary conditions. We represent the normal phase, or equivalently the magnetic field as a measure  $\mu = \mu_t \otimes dt$  where, for a.e.  $t \in (-T, T)$ ,  $\mu_t = \sum_i \varphi_i \delta_{X_i}$ , with  $X_i \in \mathbb{T}^d$ ,  $\varphi_i > 0$  and  $\sum_i \varphi_i = 1$ . We consider for  $-T \leq a < b \leq T$  the internal energy

$$I(\mu, (a, b)) = \int_a^b \sum_i \left[ \varphi_i^{\frac{d-1}{d}}(t) + \varphi_i(t) |\dot{X}_i(t)|^2 \right] dt, \quad (4.1.1)$$

where  $\dot{X}_i(t)$  is the derivative of the map  $t \mapsto X_i(t) \in \mathbb{T}^d$ . The first term in (4.1.1) can be seen as a perimeter term that favors mass travelling together, while the second term penalizes the kinetic energy (it is a transport term). Hence, this functional is a (non-standard) branched transport energy, see for instance [90, 92, 103, 91, 10, 30, 109, 18].

Let us now define the full energy

$$E_{s,T}(\mu, (-T, T)) = I(\mu, (-T, T)) + \|\mu_{\pm T} - 1\|_{\dot{H}^{-s}}^2,$$

where  $\|\mu_{\pm T} - 1\|_{\dot{H}^{-s}}^2 = \|\mu_T - 1\|_{\dot{H}^{-s}}^2 + \|\mu_{-T} - 1\|_{\dot{H}^{-s}}^2$ . The physical case corresponds to  $d = 2$  and  $s = 1/2$ , see in particular [37, 48], while we mostly focus here on the case  $d = 1$  but with arbitrary  $s \in (0, 1)$ .

Our aim is to study the dimension of the irrigated measures  $\mu_{\pm T}$ . By symmetry we will focus on  $\mu_T$ . Based on the construction of low energy states, it was conjectured in [38] that for  $d = 2$  and  $s = 1/2$  we should have  $\dim \mu_T = 8/5$ , see also [48] for a heuristic explanation. In [48] it was proven that this conjecture may be reduced to the proof of local scaling laws matching the global scaling laws. Let us recall that by global scaling law we mean the existence of  $\beta > 0$  such that for a minimizer  $\mu$ ,

$$E_{s,T}(\mu, (-T, T)) \sim T^\beta. \quad (4.1.2)$$

Similarly, by a local (upper) scaling law we mean a statement of the form

$$I(\mu, (T - \varepsilon, T)) \lesssim_\mu \varepsilon^\beta \quad \text{for } \varepsilon \in (0, T). \quad (4.1.3)$$

By [26, 28, 39], (4.1.2) holds with  $\beta = 3/7$  so that by [48], in order to prove the conjecture from [38] it is in theory enough to prove that (4.1.3) holds for  $\beta = 3/7$ , see also Remark 4.3.11. The main insight of this chapter is that it is probably too optimistic to hope for a direct proof of a sharp upper bound of the form (4.1.3) and that [48] contained only half of the puzzle. We now believe that (see Section 4.3 for the definition of the dimension of a measure):

- $\dim \mu_T \leq \alpha$  should result from the combination of an interpolation argument reminiscent of the proof of the lower bound in (4.1.2), see [48, Theorem 3.7], together with a first variation argument, see Theorem 4.1.5.

- $\dim \mu_T \geq \alpha$  should instead result from a combination of a construction, reminiscent of the argument for the upper bound in (4.1.2), see Theorem 4.1.6, together with another first variation argument, see [48, Corollary 3.6] and Lemma 4.3.8.

Unfortunately, and contrary to [48], our argument is so far only effective for  $d = 1$ . In this case, based on considerations analog to the case  $d = 2$ , see Remark 4.3.11, the conjectured dimension is

$$\bar{\alpha}(s) = \min\left(\frac{4}{3}(1-s), 1\right). \quad (4.1.4)$$

Notice that this is coherent with [48, Section 4], where it is proven that in the case of a Wasserstein penalization (which morally behaves like a  $H^{-1}$  penalization),  $\mu_T$  is purely atomic. Since our arguments do not localize, we need to further assume that  $\mu_T$  is Ahlfors regular. For  $\alpha \in [0, d]$ , we say that a measure  $\sigma$  is upper  $\alpha$ -Ahlfors regular if for some  $M > 0$ ,

$$\sigma(B_r(x)) \leq Mr^\alpha \quad \text{for } x \in \text{supp } \sigma \text{ and } r \in (0, 1).$$

Similarly we say that it is lower  $\alpha$ -Ahlfors regular if

$$\sigma(B_r(x)) \geq Mr^\alpha \quad \text{for } x \in \text{supp } \sigma \text{ and } r \in (0, 1).$$

If both conditions hold (possibly for different  $M, m > 0$  respectively), we say that  $\sigma$  is  $\alpha$ -Ahlfors regular.

Our main result is the following.

**Theorem 4.1.1.** *Let  $d = 1$ ,  $s \in (0, 1)$  and  $\mu$  be a minimizer of  $E_{s,T}$ . If  $s \leq 1/4$  then (see Section 4.3 for precise definitions)  $\underline{\dim} \mu_T = 1$ . If instead  $s > \frac{1}{4}$  and  $\mu_T$  is upper  $\alpha$ -Ahlfors regular, then*

$$\alpha \leq \bar{\alpha} \quad (4.1.5)$$

whereas if  $\mu_T$  is lower  $\alpha$ -Ahlfors regular, then

$$\alpha \geq \bar{\alpha}. \quad (4.1.6)$$

**Remark 4.1.2.** *Theorem 4.1.1 implies in particular that if  $\mu_T$  is  $\alpha$ -Ahlfors regular for  $\alpha \in (0, 1)$ , then*

$$\dim \mu_T = \bar{\alpha}.$$

Another important implication of Theorem 4.1.1 is that  $\mu_T$  cannot be smooth i.e. we cannot have  $\mu_T = f dx$  for some  $f \in L^\infty$ .

**Remark 4.1.3.** *The fact that for  $s \leq 1/4$ ,  $\underline{\dim} \mu_T = 1$  is a combination of Lemma 4.3.8 together with the general upper bound Proposition 4.3.7 yielding that (4.1.3) holds with  $\beta = 1/3$ . It thus holds for every  $d \geq 1$ .*

**Remark 4.1.4.** *For a positive measure  $\sigma$ , let us introduce the quantities*

$$\begin{aligned} \overline{\dim}_M \sigma &= \inf\{\alpha \in [0, d] : \sigma \text{ is lower } \alpha\text{-Ahlfors regular}\}, \\ \dim_F \sigma &= \sup\{\alpha \in [0, d] : \sigma \text{ is upper } \alpha\text{-Ahlfors regular}\}. \end{aligned}$$

*In the language of [52], these are the upper Minkowski dimension and the Frostman dimension of  $\sigma$ . Theorem 4.1.1 may be rephrased as saying that if  $s > \frac{1}{4}$  and  $\mu$  is a minimizer of  $E_{s,T}$ , then we have (unconditionally)*

$$\dim_F \mu_T \leq \bar{\alpha} \leq \overline{\dim}_M \mu_T.$$

Let us observe that (4.1.4) as a conjecture is similar in spirit to some questions studied in [92, 91]. In [92] a "unit ball" for a branched transport problem is studied and its boundary is conjectured to have a fractal behavior. Similarly, in [91] the concept of Voronoi cells for branched transport is introduced. Again, interfaces between cells are conjectured to have fractal behaviour. It would be interesting to see if the ideas developed here could prove useful also in that context.

Let us now comment on the proof of Theorem 4.1.1. As explained above, the proof of (4.1.5) is a combination of an interpolation argument related to the proof of the lower bound in the global scaling law (4.1.2) together with a first variation argument. The interpolation argument can be found in [48] and is reported in the upper bound in (4.3.4). The first variation argument is contained in the proof of Theorem 4.1.5 below. To prove (4.1.6) we combine instead the first variation argument from [48] (which we extend here to arbitrary  $d$  and  $s$  in Lemma 4.3.8) together with a construction which is reminiscent of the upper bound construction for the global scaling law (4.1.2). This construction is contained in Theorem 4.1.6. For both Theorem 4.1.5 and Theorem 4.1.6 we need to assume  $d = 1$ . However since large portions of the proofs are valid for arbitrary dimensions, we decided when possible to keep the presentation general. Let us now comment on the two bounds separately. The first result that we prove, in order to prove (4.1.5), is the following local scaling law.

**Theorem 4.1.5.** *Let  $d = 1$  and  $s > 1/4$ . Let  $\mu$  be a minimizer of  $E_{s,T}$  and assume that  $\mu_T$  is upper  $\alpha$ -Ahlfors regular for some  $\alpha > 1 - 2s$ . Then, for any  $\varepsilon \in (0, T)$  and*

$$\beta < \beta_{\text{reg}}(s, \alpha) = \frac{2s - 1 + \alpha}{2(1 - s) + 1 - \alpha} \quad (4.1.7)$$

there holds

$$I(\mu, (T - \varepsilon, T)) \lesssim_{\mu, \beta} \varepsilon^\beta.$$

To prove Theorem 4.1.5, we first establish in Section 4.4.1 some properties of upper  $\alpha$ -Ahlfors regular measures. We prove that this class of measures embeds into some negative Sobolev space. Then, we prove that  $\alpha$ -Ahlfors regularity is preserved by McCann interpolation when one of the two interpolated measures enjoys the property, see Lemma 4.4.1. This result, which is a generalization of  $L^\infty$  bounds on interpolated measures (see [108, Theorem 8.7 & Corollary 19.5]), could be of independent interest. Then, we build a competitor by taking a minimizer  $\mu$  and shrinking by a factor its branches in the interval  $[T - \varepsilon, T]$ . This keeps the perimeter part unchanged and reduces the kinetic energy. Since we assume  $d = 1$ , the transport plan induced by  $\mu$  is the optimal one and the measure that this new competitor irrigates is the McCann interpolant of  $\mu_{T-\varepsilon}$  and  $\mu_T$ . Thus, we can estimate the boundary energy by the Wasserstein distance (Lemma 4.4.5) using interpolation inequalities and the additional regularity properties of upper  $\alpha$ -Ahlfors regular measures.

Let us now get to the proof of (4.1.6). In this case we have the following local scaling law.

**Theorem 4.1.6.** *Let  $d = 1$  and  $s > \frac{1}{4}$ . Let  $\mu$  be a minimizer of  $E_{s,T}$  and assume that  $\mu_T$  is lower  $\alpha$ -Ahlfors regular. Then, for any  $\varepsilon \in (0, T)$  and for*

$$\beta_{\text{con}}(\alpha) = \frac{2 - \alpha}{2 + \alpha} \quad (4.1.8)$$

there holds

$$I(\mu, (T - \varepsilon, T)) \lesssim_{M, \alpha} \varepsilon^{\beta_{\text{con}}}. \quad (4.1.9)$$

The proof of Theorem 4.1.6 is based on the construction of a competitor in  $[T - \varepsilon, T]$ . To this extent we notice that lower  $\alpha$ -Ahlfors regular measures are supported on a  $\alpha$  dimensional set. Moreover, if  $d = 1$ , minimizers enjoy the so-called "cone property" (see Section 4.2.4). This property guarantees a localization property of the branched transport plan. This allows us to make a local construction in the spirit of [48, Theorem 1.3] (see Lemma 4.5.1), from which Theorem 4.1.6 follows.

The chapter is organized as follows: In Section 4.2 we recall some facts about Wasserstein distance and Sobolev spaces and we introduce the energy functional. Afterwards, we recall some already known properties of the functional and we highlight some properties which are specific to  $d = 1$ . We then prove some global upper bounds. In Section 4.3 we introduce the concepts of dimension of a measure and of upper and lower  $\alpha$ -Ahlfors regular measures. We then introduce the results of [48] which we generalize to any  $d \in \mathbb{N}$  and  $s \in (0, 1)$ . Afterwards, assuming Theorems 4.1.5 and 4.1.6, we prove Theorem 4.1.1. Section 4.4 is devoted to the study of upper  $\alpha$ -Ahlfors regular measures, followed by the proof of Theorem 4.1.5. Finally in Section 4.5 we adapt the construction of [48] to prove Theorem 4.1.6.

## 4.2 Preliminaries

In this section, we introduce the mathematical model and the necessary notation. We start by recalling basic facts about optimal transport and the Wasserstein distance. Afterward, we introduce the internal energy and recall some results of [37], highlighting results which hold only for  $d = 1$ . Then, we recall some properties of fractional Sobolev spaces. Finally we introduce the full energy functional and derive global scaling laws.

### 4.2.1 Notation

We use the symbols  $\lesssim$  and  $\gtrsim$  to indicate that there exists  $C > 0$  depending only on  $d$  and  $s$  such that  $a \leq Cb$  or  $a \geq Cb$  respectively. Additionally, we denote by  $a \sim b$  a relation that satisfies  $a \lesssim b$  and  $a \gtrsim b$ . We write  $a \lesssim_p b$  to indicate  $a \leq Cb$  for some constant  $C = C(p) > 0$ , and we do similarly for  $a \gtrsim_p b$  and  $a \sim_p b$ . We use  $A \ll B$  as an hypothesis. It means that there exists an  $\varepsilon > 0$  such that for  $A \leq \varepsilon B$  the conclusion holds. For  $x \in \mathbb{R}^d$ , we set  $|x|_{per} = \min\{|x - z| : z \in \mathbb{Z}^d\}$  for the distance on the torus  $\mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$ . We write  $B_r(x)$  for a ball with respect to the Euclidean distance. We also write  $Q_r(x)$  for a square of side  $r$  centered in  $x$ , parallel to the usual basis of  $\mathbb{R}^d$ . For  $z \in \mathbb{R}^{d+1}$  we write  $z = (x, t) \in \mathbb{R}^d \times \mathbb{R}$ . The  $\nabla$  operator is always meant with respect to the variable  $x$  and  $\partial_t$  with respect to  $t$ . For  $T > 0$ , we set  $\mathbb{T}_T^d = \mathbb{T}^d \times (-T, T)$ . We denote by  $\mathcal{B}(\mathbb{T}^d)$  the Borel  $\sigma$ -algebra of  $\mathbb{T}^d$ . Let  $A \subseteq \mathbb{R}^d$  be a Borel set. We denote by  $\mathcal{M}(A)$  the set of finite measures on  $A$ , by  $\mathcal{M}^+(A)$  the set of positive measures, and by  $\mathcal{P}(A)$  the set of probability measures on  $A$ . If  $f$  is a function and  $\mu$  a measure, we denote  $f\#\mu$  the pushforward of  $\mu$  by  $f$ .

Let  $\rho_1 \in C_c^\infty(B_1)$  denote a radially symmetric function such that  $\rho_1 \geq 0$  and  $\int_{\mathbb{R}^d} \rho_1 dx = 1$ . Set  $\rho_\varepsilon(x) = \varepsilon^{-d} \rho_1(x/\varepsilon)$  so that for some  $\varepsilon_0 > 0$  small enough  $\{\rho_\varepsilon : \varepsilon \in (0, \varepsilon_0)\}$  is a family of convolution kernels (extended periodically) that satisfy

$$\rho_\varepsilon \in C^\infty(\mathbb{T}^d), \quad \text{supp}(\rho_\varepsilon) \subseteq B_\varepsilon, \quad \rho_\varepsilon \geq 0, \quad \text{and} \quad \int_{\mathbb{T}^d} \rho_\varepsilon dx = 1 \quad \text{for all } \varepsilon \in (0, \varepsilon_0). \quad (4.2.1)$$

For  $\mu \in \mathcal{P}(\mathbb{T}^d)$ , we set

$$(\rho_\varepsilon * \mu)(x) = \int_{\mathbb{T}^d} \rho_\varepsilon(x - y) d\mu(y) \quad \text{for } x \in \mathbb{T}^d.$$

### 4.2.2 Optimal transport

In this section, we recall some basic facts and definitions from optimal transport. We refer to [4, 101] for a broader overview of optimal transport.

If  $\mu_1, \mu_2 \in \mathcal{M}^+(\mathbb{R}^d)$  with  $\mu_1(\mathbb{R}^d) = \mu_2(\mathbb{R}^d)$ , we define

$$W^2(\mu_1, \mu_2) = \min \left\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 d\pi : \pi \in \mathcal{M}^+(\mathbb{R}^d \times \mathbb{R}^d), \pi_i = \mu_i \text{ for } i = 1, 2 \right\}. \quad (4.2.2)$$

Analogously, if  $\mu_1, \mu_2 \in \mathcal{M}^+(\mathbb{T}^d)$  with  $\mu_1(\mathbb{T}^d) = \mu_2(\mathbb{T}^d)$ , we define

$$W_{\text{per}}^2(\mu_1, \mu_2) = \min \left\{ \int_{\mathbb{T}^d \times \mathbb{T}^d} |x - y|_{\text{per}}^2 d\pi : \pi \in \mathcal{M}^+(\mathbb{T}^d \times \mathbb{T}^d), \pi_i = \mu_i \text{ for } i = 1, 2 \right\}.$$

Let us recall the Benamou-Brenier formula: for  $\mathbb{X} \in \{\mathbb{T}^d, \mathbb{R}^d\}$  and  $a, b \in \mathbb{R}$  with  $a < b$  we have

$$\frac{1}{b-a} W_*^2(\mu_1, \mu_2) = \min_{(\mu, m)} \left\{ \int_{\mathbb{X} \times (a, b)} \left| \frac{dm}{d\mu} \right|^2 d\mu dt : m \ll \mu, \partial_t \mu + \nabla \cdot m = 0, \mu_a = \mu_1, \mu_b = \mu_2 \right\}, \quad (4.2.3)$$

where  $W_* = W$  if  $\mathbb{X} = \mathbb{R}^d$  and  $W_* = W_{\text{per}}$  if  $\mathbb{X} = \mathbb{T}^d$ . Given  $\mu_a, \mu_b \in \mathcal{M}^+(\mathbb{R}^d)$  with  $\mu_a(\mathbb{R}^d) = \mu_b(\mathbb{R}^d)$  and  $a \leq t \leq b$ , we define the interpolated measure (McCann's interpolant) as

$$\int_{\mathbb{R}^d} \psi d\mu^t = \int_{\mathbb{R}^d \times \mathbb{R}^d} \psi \left( \frac{b-t}{b-a} x + \frac{t-a}{b-a} y \right) d\pi$$

where  $\pi$  is an optimal transport plan for  $W(\mu_a, \mu_b)$ . This is the same as saying  $\mu^t = F_t \# \pi$ , where  $F_t : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  is defined as

$$F_t(x, y) = \frac{b-t}{b-a} x + \frac{t-a}{b-a} y.$$

By [101, Theorem 5.27], for  $a \leq s \leq t \leq b$

$$W(\mu^t, \mu^s) = \left( \frac{t-s}{b-a} \right) W(\mu_a, \mu_b). \quad (4.2.4)$$

### 4.2.3 The internal energy functional

Following the notation from [48], let us define the internal energy.

**Definition 4.2.1.** We denote by  $\mathcal{A}(\mathbb{T}^d, (a, b))$  for  $-T \leq a < b \leq T$  the set of measures  $\mu \in \mathcal{M}^+(\mathbb{T}^d \times (a, b))$ ,  $m \in \mathcal{M}(\mathbb{T}^d \times (a, b), \mathbb{R}^d)$ , with  $m \ll \mu$  satisfying in the sense of distributions the continuity equation

$$\partial_t \mu + \nabla \cdot m = 0 \quad \text{in } \mathbb{T}^d \times (a, b),$$

and such that  $\mu = \mu_t \otimes dt$  where, for almost every  $t \in (a, b)$  there holds  $\mu_t = \sum_i \varphi_i(t) \delta_{X_i(t)}$ , for some  $t \mapsto \varphi_i(t) > 0$  and  $t \mapsto X_i(t) \in \mathbb{T}^d$ . We denote by  $\mathcal{A}^*(\mathbb{T}^d, (a, b)) = \{\mu : \exists m, (\mu, m) \in \mathcal{A}(\mathbb{T}^d, (a, b))\}$  the set of admissible  $\mu$ . For  $\mathbb{T}_T^d = \mathbb{T}^d \times (-T, T)$ , we use the notation  $\mathcal{A}_T = \mathcal{A}(\mathbb{T}^d, (-T, T))$  and  $\mathcal{A}_T^* = \mathcal{A}^*(\mathbb{T}^d, (-T, T))$ . We define the functional  $I : \mathcal{A}(\mathbb{T}^d, (a, b)) \rightarrow [0, +\infty]$  by

$$I(\mu, m, (a, b)) = \int_a^b \sum_{x \in \mathbb{T}^d} (\mu_t(\{x\}))^{\frac{d-1}{d}} dt + \int_{\mathbb{T}^d \times (a, b)} \left| \frac{dm}{d\mu} \right|^2 d\mu.$$

We also define (with abuse of notation) the functional  $I : \mathcal{A}^*(\mathbb{T}^d, (a, b)) \rightarrow [0, +\infty]$

$$I(\mu, (a, b)) = \min_m \{I(\mu, m, (a, b)) : (\mu, m) \in \mathcal{A}_T\}.$$

We use the notation  $I(\mu) = I(\mu, (-T, T))$ . We then set for  $-T \leq a < t < b \leq T$

$$\dot{P}(\mu, t) = \sum_{x \in \mathbb{T}^d} (\mu_t(x))^{\frac{d-1}{d}}, \quad \dot{E}_{\text{kin}}(\mu, t) = \int_{\mathbb{T}^d} \left| \frac{dm_t}{d\mu_t} \right|^2 d\mu_t$$

and

$$P(\mu, (a, b)) = \int_a^b \dot{P}(\mu, t) dt, \quad E_{\text{kin}}(\mu, (a, b)) = \int_a^b \dot{E}_{\text{kin}}(\mu, t) dt,$$

so that  $I(\mu, (a, b)) = P(\mu, (a, b)) + E_{\text{kin}}(\mu, (a, b))$ .

Notice that for  $-T \leq a < b \leq T$  and  $\mu \in \mathcal{A}_T^*$ , by (4.2.3) we have

$$E_{\text{kin}}(\mu, (a, b)) \geq \frac{1}{b-a} W_{\text{per}}^2(\mu_a, \mu_b). \quad (4.2.5)$$

Given two measures  $\bar{\mu}_{\pm} \in \mathcal{M}^+(\mathbb{T}^d)$  such that  $\bar{\mu}_+(\mathbb{T}^d) = \bar{\mu}_-(\mathbb{T}^d)$  consider the problem

$$\inf \{I(\mu) : \mu \in \mathcal{A}_T^*, \mu_{\pm T} = \bar{\mu}_{\pm}\}. \quad (4.2.6)$$

By (4.2.5), if  $I(\mu) < \infty$  then for  $-T < a < b < T$

$$W_{\text{per}}(\mu_a, \mu_b) \leq I(\mu)^{\frac{1}{2}} (b-a)^{\frac{1}{2}}.$$

This means that  $t \mapsto \mu_t$  is a  $\frac{1}{2}$ -Hölder continuous curve in the Wasserstein space, hence  $\mu_T$  and  $\mu_{-T}$  are well-defined.

Let us now recall some results of [37, 48] for this functional. First, arguing as in [37, Proposition 5.5] we have existence of minimizers (in [37] the discussion is carried for  $d = 2$ , but the argument extends to any dimension). Moreover, when  $\bar{\mu}_+ = \bar{\mu}_-$ , by a symmetrization argument we can concentrate on minimizers which are symmetric with respect to  $t = 0$ . Let us recall the notion of subsystems, see [48, Proposition 2.6] and [37, Proposition 5.7].

**Proposition 4.2.2** (Subsystems). *Let  $t \in [-T, T]$  and  $(\mu, m) \in \mathcal{A}_T$  with  $I(\mu, m) < \infty$ . Set  $v = dm/d\mu$ . Then, for every measure  $\sigma \leq \mu_t$  there exists a measure  $\tilde{\mu} \in \mathcal{A}_T^*$  such that*

1.  $\mu' = \mu - \tilde{\mu} \in \mathcal{M}^+(\mathbb{T}_T^d)$ ;
2.  $\tilde{\mu}_t = \sigma$ ;
3.  $\tilde{\mu}$  satisfies the continuity equation  $\partial_t \tilde{\mu} + \nabla \cdot (v \tilde{\mu}) = 0$ .

*In particular  $(\tilde{\mu}, v \tilde{\mu}) \in \mathcal{A}_T$  and  $I(\tilde{\mu}, v \tilde{\mu}) \leq I(\mu, m)$ . Similarly  $(\mu', v \mu') \in \mathcal{A}_T$  and  $I(\mu', v \mu') \leq I(\mu, m)$ .*

If  $\mu_t = \sum_i \varphi_i \delta_{X_i}$  and  $\sigma = \varphi_i \delta_{X_i}$  for some  $i$ , we call  $\mu_+^{t,i} = \tilde{\mu} \llcorner (t, T) \times \mathbb{T}^d$  the forward subsystem emanating from  $X_i$  and  $\mu_-^{t,i} = \tilde{\mu} \llcorner (-T, t) \times \mathbb{T}^d$  the backward subsystem emanating from  $X_i$ .

Let us now recall the no-loop property of minimizers, see [37, Lemma 5.8].

**Lemma 4.2.3** (No-loop condition). *Let  $\mu$  be a minimizer of (4.2.6) and  $t \in (-T, T)$ . Let  $X_1, X_2 \in \mathbb{T}^d$  with  $\mu_t(X_i) \neq 0$  for  $i = 1, 2$  and consider the forward and backward subsystems  $\mu_{\pm}^{t,i}$  emanating from  $X_i$ . If there exist  $-T \leq t_- < t < t_+ \leq T$  such that for  $* \in \{\pm\}$ ,  $\mu_*^{t_*,1}$  and  $\mu_*^{t_*,2}$  are not mutually singular, then  $X_1 = X_2$ .*

Let us also recall the following representation formula, see [37, Proposition 5.10 and Proposition 5.11].

**Proposition 4.2.4.** *Let  $\bar{\mu} \in \mathcal{M}^+(\mathbb{T}^d)$  and  $\mu$  be a symmetric minimizer of (4.2.6) with  $\mu_{\pm} = \bar{\mu}$ . Then, for every  $T' < T$ ,  $\mu$  is a finite graph in  $\mathbb{T}_{T'}^d$ , without loops. In particular we have*

$$I(\mu) = \int_{-T}^T \sum_i \left[ \varphi_i^{\frac{d-1}{d}}(t) + \varphi_i(t) |\dot{X}_i(t)|^2 \right] dt \quad (4.2.7)$$

where the sum is locally finite. Moreover, between two branching points  $t \mapsto \varphi_i(t)$  is constant while  $t \mapsto X_i(t)$  is affine.

In the rest of the chapter we will always use the representation formula (4.2.7) for  $I$ . Let us now introduce a Lagrangian reformulation of the problem. The proof of the following proposition is essentially the same as [48, Theorem 4.3] so we omit it. We just remark that it is essential that the no-loop property holds so that we can parametrize the curves induced by a minimizer  $\mu \in \mathcal{A}_T^*$  by their initial point. This avoids the introduction of traffic plans as in [92, 103]. Let

$$C_T = \{X : \mathbb{T}^d \times [-T, T] \mapsto \mathbb{T}^d : X(x, -T) = X(x, T) = x \text{ and for a.e. } x \ t \mapsto X(x, t) \text{ is AC}\},$$

where AC denotes absolutely continuous curves. Let  $X \in C_T$ ,  $\bar{\mu} \in \mathcal{P}(\mathbb{T}^d)$ ,  $y \in \mathbb{T}^d$  and  $t \in (-T, T)$ . We define the multiplicity function as

$$\varphi_{X, \bar{\mu}}(y, t) = \bar{\mu}(\{x \in \text{supp } \bar{\mu} : X(x, t) = y\}).$$

We then set

$$\mathcal{L}(X, \bar{\mu}) = \int_{\mathbb{T}^d} \left( \varphi_{X, \bar{\mu}}(X(x, t), t)^{-\frac{1}{d}} + |\partial_t X(x, t)|^2 \right) d\bar{\mu}(x) dt.$$

**Proposition 4.2.5.** *For every  $T > 0$  and  $\bar{\mu} \in \mathcal{P}(\mathbb{T}^d)$*

$$\min_{\mu \in \mathcal{A}_T^*} \{I(\mu, (-T, T)), \mu_{\pm T} = \bar{\mu}\} = \min_{X \in C_T} \mathcal{L}(X, \bar{\mu}).$$

Moreover, for every  $X$  minimizing  $\mathcal{L}$ ,  $\mu_t = X(\cdot, t) \# \bar{\mu}$  is a minimizer for  $I$ . Viceversa, we can associate to any minimizer  $\mu \in \mathcal{A}_T^*$  a collection of curves  $X \in C_T$  (defined for  $\bar{\mu}$ -a.e.  $x \in \mathbb{T}^d$ ) such that  $X$  is a minimizer for  $\mathcal{L}$ .

Let us finally recall the equipartition of energy [48, Proposition 2.11].

**Proposition 4.2.6** (Equipartition of Energy). *Let  $\bar{\mu} \in \mathcal{M}^+(\mathbb{T}^d)$  and consider a symmetric minimizer of (4.2.6) with  $\mu_{\pm} = \bar{\mu}$ . There exists  $\Lambda \in \mathbb{R}$  with*

$$|\Lambda| \lesssim \frac{I(\mu)}{T}$$

such that for every  $t \in (-T, T)$

$$\dot{P}(\mu, t) = \dot{E}_{kin}(\mu, t) + \Lambda.$$

#### 4.2.4 The case $d = 1$

When  $d = 1$ , minimizers of (4.2.6) satisfy additional properties. A first simple but useful implication of the no-loop condition for  $d = 1$  is that the coupling between  $\mu_T$  and  $\mu_t$  induced by  $\mu$  is monotone, see [48, Lemma 4.1].

**Lemma 4.2.7.** *Let  $\bar{\mu} \in \mathcal{M}^+(\mathbb{T})$  and  $\mu$  be a symmetric minimizer of (4.2.6) with  $\mu_{\pm} = \bar{\mu}$ . For every  $t \in [0, T)$ , if  $\mu_t = \sum_i \varphi_i \delta_{X_i}$  and if we denote by  $\mu^{t,i} = \mu_+^{t,i}$  the forward subsystems emanating from  $X_i$ , then considering  $\mu_t$  as a periodic measure on  $\mathbb{R}$  we have*

$$X_i < X_j \quad \implies \quad \sup \text{supp } \mu_T^{t,i} \leq \inf \text{supp } \mu_T^{t,j}.$$

Moreover for  $i \neq j$ ,

$$\mu_T^{t,i}(\{x\})\mu_T^{t,j}(\{x\}) = 0 \quad \forall x \in \mathbb{T}.$$

As a consequence, the map  $X(\cdot, t)$  given by Proposition 4.2.5 (considered as a map on  $\mathbb{R}$ ) is monotone.

We now recall from [48] the cone property. Let us consider the parametric representation  $X$  of a minimizer  $\mu$  of (4.2.16) given by Proposition 4.2.5, such that for any  $x \in \mathbb{T}$ ,  $X(x, T) = X(x, -T) = x$  and for any  $t \in (-T, T)$  it holds  $X(\cdot, t) \# \mu_T = \mu_t$ . Let us define the irrigated set of a given point  $\bar{x}$  at time  $t \geq 0$  as

$$J(\bar{x}, t) = \{x \in \mathbb{T} \mid X(x, t) = \bar{x}\}.$$

Let  $C(\bar{x}, t)$  be the cone defined by the convex hull in space and time of the point  $(\bar{x}, t)$  and the set  $J(\bar{x}, t) \times \{T\}$ . The following was proved in [48, Proposition 4.7].

**Proposition 4.2.8** (Cone property). *Let  $x \in J(\bar{x}, t)$ . Then  $X(x, s) \in C(\bar{x}, t)$  for any  $0 < s < t$ .*

We now observe, as in [67] and [48, Proposition 4.10 & Remark 4.11], that for  $t = 0$ , if  $\mu_0 = \sum_i \varphi_i \delta_{X_i}$  then  $X_i$  must coincide with the barycenter of the corresponding irrigated measure  $\mu_T^i = (\mu_+^{0,i})_T$ .

**Lemma 4.2.9.** *Let  $\bar{\mu} \in \mathcal{M}^+(\mathbb{T})$  and  $\mu$  be a symmetric minimizer of (4.2.6). Extending  $\mu$  by periodicity on  $\mathbb{R}$ , letting  $\mu_0 = \sum_i \varphi_i \delta_{X_i}$  and  $\mu_T^i = (\mu_+^{0,i})_T$ , we have for every  $i$ ,*

$$\varphi_i X_i = \int_{\mathbb{R}} y d\mu_T^i.$$

**Remark 4.2.10.** *As a consequence of Lemma 4.2.9 and the cone property, up to choosing appropriately the origin, a periodic minimizer  $\mu$  of  $I$  can be identified with a minimizer  $\tilde{\mu}$  of  $I$  on  $[0, 1] \times [-T, T]$  without boundary conditions. This in particular implies that (4.2.5) holds also for the Euclidean Wasserstein distance (4.2.2). Moreover, if we now consider a parametric representation  $\tilde{X}$  of  $\tilde{\mu}$  as given by Proposition 4.2.5, by Lemma 4.2.7 we have that  $\tilde{X}(\cdot, t)$  is a monotone map on  $\text{supp } \tilde{\mu}_T$ . Since  $d = 1$ , this implies that for all  $t \in (0, T)$   $\tilde{X}(\cdot, t)$  is an optimal transport map for  $W(\tilde{\mu}_T, \tilde{\mu}_t)$ . In the rest of the chapter we will omit distinguishing between  $\mu$  and  $\tilde{\mu}$ .*

*Proof.* Let  $X$  be a parametric representation of  $\mu$  as given by Proposition 4.2.5. As in [48, Proposition 4.10] we define

$$\hat{X}(x, t) = X(x, t) - \frac{T-t}{T} \left( X_i - \frac{1}{\varphi_i} \int_{\mathbb{R}} y d\mu_T^i \right) \quad \text{for } t \in [0, T] \text{ and } x \in \text{supp } \mu_T^i.$$

We then extend  $\widehat{X}$  to  $[-T, T]$  by symmetry. Notice that by the cone property this construction leaves the perimeter unchanged. Hence by a direct computation

$$\mathcal{L}(\widehat{X}, \bar{\mu}) = \mathcal{L}(X, \bar{\mu}) - \frac{\varphi_i}{T} \left| X_i - \frac{1}{\varphi_i} \int_{\mathbb{R}} y d\mu_T^i \right|^2.$$

This concludes the proof.  $\square$

We now want to show that given an interval  $J = (x^+, x^-) \subset \mathbb{T}$ , it can be irrigated by (or rather it "splits in") at most 3 disjoint intervals. We let

$$\begin{aligned} J^1 &= \{x \in J \mid \exists t \in (0, T) \text{ s.t. } X(x, t) = X(x^+, t)\}, \\ J^2 &= \{x \in J \mid X(x, t) \neq X(x^\pm, t) \forall t \in (0, T)\}, \\ J^3 &= \{x \in J \mid \exists t \in (0, T) \text{ s.t. } X(x, t) = X(x^-, t)\}. \end{aligned}$$

Clearly  $J^2$  does not intersect with  $J^1$  and  $J^3$ , and may be empty. By the no-loop condition, if  $J^1$  and  $J^3$  intersect, then they must coincide and in that case  $J^2$  must be empty. In the case where the three sets are disjoint and non-empty, again by the no loop condition they must satisfy  $\sup J^1 \leq \inf J^2$  and  $\sup J^2 \leq \inf J^3$ , see Figure 4.1. Notice that if  $\bar{x}$  is such that  $J(\bar{x}, 0) \subset J^2$ , by Lemma 4.2.9, we must have  $\bar{x} \in J^2$  and thus by Proposition 4.2.8 for  $t = 0$ , also  $C(\bar{x}, 0) \subset J^2 \times [0, T]$ . This implies in particular:

**Proposition 4.2.11.** *For every  $x \in J^2$  and every  $t \in [-T, T]$ ,  $X(x, t) \in J^2$ .*

Moreover, by the cone property, we have that for all  $x \in J^1$ ,  $X(x, t) \in C(X(x^+, T), T)$  and for all  $x \in J^3$ ,  $X(x, t) \in C(X(x^-, T), T)$ . This gives us the following corollary, see Figure 4.1.

**Corollary 4.2.12.** *Let  $J \subset \mathbb{T}$  be an interval. Then, for any  $t \in [0, T]$  there exist at most three disjoint intervals  $J^i(t)$ ,  $i = 1, 2, 3$ , with  $|J^i(t)| \leq |J|$  and such that for  $\mu_T$ -a.e.  $x \in J$  we have  $X(x, t) \in J^i(t)$  for some  $i = 1, 2, 3$ .*

## 4.2.5 Negative Sobolev spaces

Let us recall some facts regarding homogeneous negative Sobolev spaces  $\dot{H}^{-s}(\mathbb{T}^d)$  for  $s \in [0, 1]$ , see [5, 107]. For  $s \in \mathbb{R}$ , we define  $\dot{H}^{-s}(\mathbb{T}^d)$  as the set of all measures  $\sigma \in \mathcal{M}(\mathbb{T}^d)$  such that

$$\|\sigma\|_{\dot{H}^{-s}}^2 = \sum_{k \in \mathbb{Z}^d} |k|^{-2s} |\widehat{\sigma}_k|^2 < \infty, \quad (4.2.8)$$

where  $\widehat{\sigma}_k$  is the Fourier coefficient for  $k \in \mathbb{Z}^d$  given by

$$\widehat{\sigma}_k = \int_{\mathbb{T}^d} \exp(2\pi i(k \cdot x)) d\sigma(x).$$

Notice that with this notation,  $\dot{H}^{-0}(\mathbb{T}^d) = L^2(\mathbb{T}^d)$  and that for  $s < 0$  (4.2.8) implies  $\sigma(\mathbb{T}^d) = 0$ . We recall that  $\dot{H}^{-s}(\mathbb{T}^d) = (\dot{H}^s(\mathbb{T}^d))^*$  and that for  $s \in (0, 1)$ ,  $\dot{H}^s(\mathbb{T}^d)$  coincides with the completion of the set of  $C^\infty(\mathbb{T}^d)$  functions with zero mean with respect to the norm

$$\|\psi\|_{\dot{H}^s(\mathbb{T}^d)}^2 = \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \frac{|\psi(x) - \psi(y)|^2}{|x - y|_{\text{per}}^{d+2s}} dx dy. \quad (4.2.9)$$

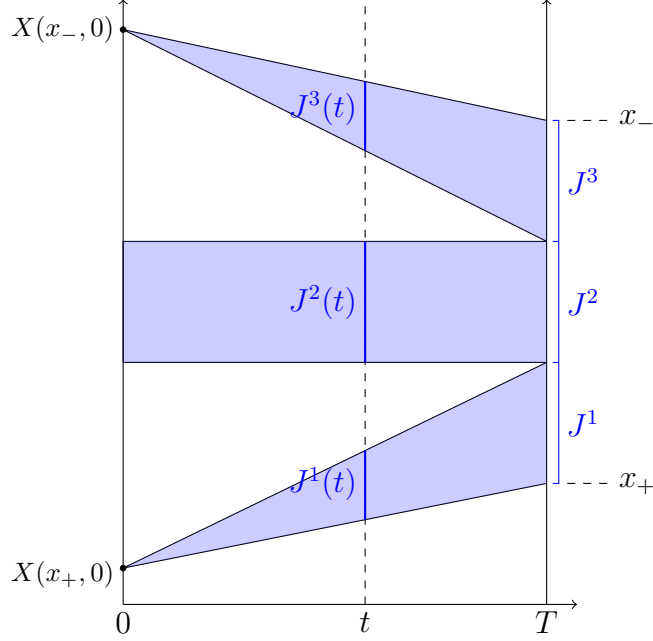


Figure 4.1: Representation of the typical case for the 3 intervals  $J^1(t)$ ,  $J^2(t)$  and  $J^3(t)$  for  $t \in [0, T]$ .

We will also need the space  $\dot{H}^s(\Omega)$  for a bounded Lipschitz domain  $\Omega \subset \mathbb{R}^d$ , defined as the completion of zero mean  $C^\infty(\Omega)$  functions with respect to the norm

$$\|\psi\|_{\dot{H}^s(\Omega)}^2 = \int_{\Omega} \int_{\Omega} \frac{|\psi(x) - \psi(y)|^2}{|x - y|^{d+2s}} dx dy. \quad (4.2.10)$$

Let us present some characterizations of the negative Sobolev norm.

**Proposition 4.2.13.** *Let  $s \in (0, 1)$  and  $(\rho_\varepsilon)_{\varepsilon > 0}$  be a family of regular symmetric convolution kernels with  $\varepsilon \in (0, \varepsilon_0)$  as in (4.2.1). For all  $\sigma \in \mathcal{M}(\mathbb{T}^d)$  with  $\sigma(\mathbb{T}^d) = 0$ , and  $\gamma > 0$  there holds*

$$\|\sigma\|_{\dot{H}^{-s}}^2 \lesssim_{\gamma} \int_0^1 \eta^{2s+\gamma} \left( \sum_{|k| \leq \eta^{-1}} |k|^\gamma |\widehat{\sigma}_k|^2 \right) \frac{d\eta}{\eta} \quad (4.2.11)$$

and

$$\|\sigma\|_{\dot{H}^{-s}}^2 \lesssim \int_0^{\varepsilon_0} \varepsilon^{2s} \int_{\mathbb{T}^d} |\rho_\varepsilon * \sigma|^2 dx \frac{d\varepsilon}{\varepsilon}. \quad (4.2.12)$$

*Proof.* For a proof of the first estimate with  $\gamma = 1$ , we refer to [48, Proposition 2.13]. The proof is essentially the same for a generic  $\gamma > 0$ . Let us prove the second estimate, see also [68, Theorem 2.7] for a variant. First, notice that by Plancherel's identity and properties of Fourier coefficients we have

$$\int_{\mathbb{T}^d} |\rho_\varepsilon * \sigma|^2 dx = \sum_{k \in \mathbb{Z}^d \setminus \{0\}} |\widehat{\sigma}_k|^2 |\widehat{(\rho_\varepsilon)}_k|^2.$$

Thus, to conclude we just need to show that

$$\int_0^{\varepsilon_0} \varepsilon^{2s} |\widehat{(\rho_\varepsilon)}_k|^2 \frac{d\varepsilon}{\varepsilon} \gtrsim |k|^{-2s}.$$

With a change of variable we have (recall (4.2.1))

$$\begin{aligned} \widehat{(\rho_\varepsilon)}_k &= \int_{\mathbb{T}^d} \rho_\varepsilon(x) \exp(2\pi i(k \cdot x)) dx = \int_{B_\varepsilon} \frac{1}{\varepsilon^d} \rho_1\left(\frac{x}{\varepsilon}\right) \exp(2\pi i(k \cdot x)) dx \\ &= \int_{\mathbb{R}^d} \rho_1(x) \exp(2\pi i(\varepsilon k \cdot x)) dx = F(\varepsilon k). \end{aligned}$$

Let us notice that, since  $\rho_1$  is radially symmetric,  $F(v) = \tilde{F}(|v|)$  for  $v \in \mathbb{R}^d$ . Finally we have

$$\int_0^{\varepsilon_0} \varepsilon^{2s} |F(\varepsilon k)|^2 \frac{d\varepsilon}{\varepsilon} = |k|^{-2s} \int_0^{|k|\varepsilon_0} t^{2s} |F(tk/|k|)|^2 \frac{dt}{t} \geq |k|^{-2s} \int_0^{\varepsilon_0} t^{2s} |\tilde{F}(t)|^2 \frac{dt}{t}.$$

Since  $\tilde{F}(0) = 1$  and  $\tilde{F}$  is continuous on the integration domain, the last integral is bounded from below.  $\square$

We will use the following rather standard result.

**Proposition 4.2.14.** *Let  $s \in (0, 1)$ ,  $\gamma \in (0, s]$ , and  $(\rho_\varepsilon)_{\varepsilon>0}$  be a family of regular convolution kernels with  $\varepsilon \in (0, \varepsilon_0)$  as in (4.2.1). For every  $\sigma \in \mathcal{M}(\mathbb{T}^d)$  with  $\sigma(\mathbb{T}^d) = 0$*

$$\|\sigma - \sigma * \rho_\varepsilon\|_{\dot{H}^{-s}} \lesssim \varepsilon^{s-\gamma} \|\sigma\|_{\dot{H}^{-\gamma}}.$$

*Proof.* We have for  $k \neq 0$  and using  $(\widehat{\sigma * \rho_\varepsilon})_k = \widehat{\sigma}_k (\widehat{\rho_\varepsilon})_k$

$$\begin{aligned} \|\sigma - \sigma * \rho_\varepsilon\|_{\dot{H}^{-s}}^2 &= \sum_{|k| \leq 1/\varepsilon} |k|^{-2s} |\widehat{\sigma}_k|^2 |1 - (\widehat{\rho_\varepsilon})_k|^2 \\ &\quad + \sum_{|k| > 1/\varepsilon} |k|^{-2s} |\widehat{\sigma}_k|^2 |1 - (\widehat{\rho_\varepsilon})_k|^2 = T_1 + T_2. \end{aligned} \quad (4.2.13)$$

We estimate the two terms on the right-hand side of (4.2.13) separately. We start with  $T_2$ . If  $|k| \geq 1/\varepsilon$  then  $|k|^{-2(s-\gamma)} \leq \varepsilon^{2(s-\gamma)}$ . Moreover, for every  $k \in \mathbb{N}$ ,

$$|(\widehat{\rho_\varepsilon})_k| = \left| \int_{\mathbb{T}^d} \exp(2\pi i k \cdot x) \rho_\varepsilon(x) dx \right| \leq \int_{\mathbb{T}^d} \rho_\varepsilon(x) dx = 1.$$

Hence  $T_2$  is bounded by

$$T_2 = \sum_{|k| > 1/\varepsilon} |k|^{-2s} |\widehat{\sigma}_k|^2 |1 - (\widehat{\rho_\varepsilon})_k|^2 \lesssim \varepsilon^{2(s-\gamma)} \sum_{|k| > 1/\varepsilon} |k|^{-2\gamma} |\widehat{\sigma}_k|^2 \lesssim \varepsilon^{2(s-\gamma)} \|\sigma\|_{\dot{H}^{-\gamma}}^2. \quad (4.2.14)$$

For  $T_1$ , we recall that for  $|z| \lesssim 1$  we have  $|1 - \exp(z)| \lesssim |z|$ . Hence for  $|k| < 1/\varepsilon$  we estimate

$$\begin{aligned} |1 - (\widehat{\rho_\varepsilon})_k| &= \left| \int_{\mathbb{T}^d} (1 - \exp(2\pi i k \cdot x)) \rho_\varepsilon(x) dx \right| \leq \int_{\mathbb{T}^d} |1 - \exp(2\pi i \varepsilon k \cdot x)| \rho_1(x) dx \\ &\lesssim \int_{\mathbb{T}^d} |\varepsilon k \cdot x| \rho_1(x) dx \lesssim \varepsilon |k|. \end{aligned}$$

This leads to the bound

$$T_1 = \sum_{|k| \leq 1/\varepsilon} |k|^{-2s} |\widehat{\sigma}_k|^2 |1 - (\widehat{\rho_\varepsilon})_k|^2 \leq \varepsilon^2 \sum_{|k| \leq 1/\varepsilon} |k|^{2-2s+2\gamma} |k|^{-2\gamma} |\widehat{\sigma}_k|^2 \leq \varepsilon^{2(s-\gamma)} \|\sigma\|_{\dot{H}^{-\gamma}}^2. \quad (4.2.15)$$

Combining (4.2.15) and (4.2.14) implies the desired inequality.  $\square$

Finally, for  $s = 1$ , the  $\dot{H}^{-1}$  norm morally behaves like the Wasserstein distance for measures. We recall the following inequality, which is a special case of [76, Proposition 2.8], see also [93].

**Proposition 4.2.15.** *Let  $\sigma_1, \sigma_2 \in L^\infty(\mathbb{T}^d) \cap \mathcal{M}^+(\mathbb{T}^d)$ . Then,*

$$\|\sigma_1 - \sigma_2\|_{\dot{H}^{-1}} \leq \max\{\|\sigma_1\|_{L^\infty}, \|\sigma_2\|_{L^\infty}\}^{\frac{1}{2}} W_{\text{per}}(\sigma_1, \sigma_2).$$

#### 4.2.6 The main functional and global scaling laws

In this section, we introduce the full energy functional and recall some results on the global scaling law. For  $s \in (0, 1)$  and  $T > 0$  we define

$$E_{s,T}(\mu) = I(\mu) + \|\mu_{\pm T} - 1\|_{\dot{H}^{-s}}^2, \quad (4.2.16)$$

where

$$\|\mu_{\pm T} - 1\|_{\dot{H}^{-s}}^2 = \|\mu_T - 1\|_{\dot{H}^{-s}}^2 + \|\mu_{-T} - 1\|_{\dot{H}^{-s}}^2.$$

By a symmetrization argument, we may restrict the problem (4.2.16) to minimizers that are symmetric with respect to  $t = 0$ .

Let us now turn to global upper bounds for the energy.

**Proposition 4.2.16** (Global upper bounds). *Let  $T \geq 1$ . We have*

$$\min\{E_{s,T}(\mu) : \mu \in \mathcal{A}_T^*\} \lesssim T. \quad (4.2.17)$$

Let  $0 < T \leq 1$  and  $d > 2s$ . For  $s > \frac{1}{4}$  we have

$$\min\{E_{s,T}(\mu) : \mu \in \mathcal{A}_T^*\} \lesssim T^{\frac{d+2s}{3d+2(1-s)}} \quad (4.2.18)$$

and for  $s \leq \frac{1}{4}$  we have

$$\min\{E_{s,T}(\mu) : \mu \in \mathcal{A}_T^*\} \lesssim T^{\frac{1}{3}}. \quad (4.2.19)$$

Finally if  $0 < T \leq 1$ ,  $d < 2s$  we have

$$\min\{E_{s,T}(\mu) : \mu \in \mathcal{A}_T^*\} \lesssim T^{\frac{2s}{2s+1}}. \quad (4.2.20)$$

**Remark 4.2.17.** *While we do not use these bounds directly, they give indication on the behaviour of the functional in the different parameter regimes and generalize [48, Proposition 3.1]. In particular, it is interesting to compare these global bounds with the local bounds which we find in Section 3. The most interesting regime is (4.2.18). Notice that (4.2.18) coincides with the conjectured optimal local scaling (4.3.5). Notice also that if  $d = 2$  and  $s = 1/2$ , it coincides with the results of [28] and [39, Theorem 4.1]. The regime (4.2.17) corresponds to samples which are too thick so that the details of the branching are not relevant. The regime (4.2.19) instead corresponds to strong boundary penalizations, which force the boundary measure to have full dimension, see Remark 4.3.10. Indeed the scaling (4.2.19) matches the local scaling given by Proposition 4.3.7, which is the scaling of a uniformly distributed measure. Finally, in the regime (4.2.20) Dirac deltas have finite  $H^{-s}$  norm. In this case we do not expect that global and local scaling laws coincide. We also remark that, as in [37, 39, 29], it should be possible to find matching lower bounds, but this goes beyond the scope of the thesis.*

*Proof.* Let  $N \in \mathbb{N}$  and  $r > 0$ . Consider  $N$  equidistributed squares of side length  $r$ , which we denote by  $Q_r(X_i)$  where for  $i = 1, \dots, N$   $X_i$  denotes the centers. Let

$$\mu_T = \sum_i \frac{1}{Nr^d} \chi_{Q_r(X_i)}.$$

Following the strategy in [39] and [37, Proposition 5.2], it is possible to construct  $\mu \in \mathcal{A}_T^*$  such that

$$E_{s,T}(\mu) \lesssim TN^{\frac{1}{d}} + \frac{r^2}{T} + \|\mu_T - 1\|_{H^{-s}}^2. \quad (4.2.21)$$

If  $T \gtrsim 1$  we can just chose  $N = r = 1$  which gives  $\mu_T = 1$  and

$$E_{s,T}(\mu) \lesssim T + \frac{1}{T} \lesssim T.$$

From now on we assume that  $T \ll 1$ . Using Proposition 4.2.18 below we get for  $d > 2s$

$$E_{s,T}(\mu) \lesssim TN^{\frac{1}{d}} + \frac{r^2}{T} + \frac{1}{Nr^{d-2s}}.$$

Optimizing in  $r$  we find the optimal value

$$r = T^{\frac{1}{d+2(1-s)}} N^{-\frac{1}{d+2(1-s)}} \quad (4.2.22)$$

that gives

$$E_{s,T}(\mu) \lesssim TN^{\frac{1}{d}} + T^{-\frac{d-2s}{d+2(1-s)}} N^{-\frac{2}{d+2(1-s)}}.$$

Optimizing now in  $N$  we find

$$N = T^{-\frac{2d(1+d-2s)}{3d+2(1-s)}} \quad (4.2.23)$$

which gives the bound

$$E_{s,T}(\mu) \lesssim T^{\frac{d+2s}{3d+2(1-s)}}.$$

Notice that since  $T \ll 1$ , the choice (4.2.23) of  $N$  yields  $N \gg 1$  which is compatible with the requirement  $N \in \mathbb{N}$ . Plugging this choice of  $N$  in (4.2.22) we get

$$r = T^{\frac{1+2d}{3d+2(1-s)}}, \quad (4.2.24)$$

and comparing (4.2.23) and (4.2.24) the condition  $r \ll N^{-1/d}$  becomes

$$s > \frac{1}{4}.$$

If instead  $s \leq \frac{1}{4}$ , we choose  $r^d = \frac{1}{N}$  in (4.2.21), so that the boundary term is zero. This implies

$$E_{s,T}(\mu) \lesssim TN^{\frac{1}{d}} + \frac{1}{TN^{\frac{2}{d}}}.$$

Optimizing in  $N$  we find the desired bound.

Finally suppose that  $d < 2s$ . This can happen only for  $d = 1$  since we assume that  $s \in (0, 1)$ . In this case, Dirac deltas are in  $H^{-s}(\mathbb{T})$ . Consider  $\mu_T$  to be a sum of  $N$  equispaced Dirac deltas with weight  $1/N$  and let  $\mu_t = \mu_T$  for all  $t \in [-T, T]$ . We find

$$E_{s,T}(\mu) = 2TN + \|\mu_T - 1\|_{H^{-s}}^2.$$

By Proposition 4.2.19 below, we can estimate the boundary term to find

$$E_{s,T}(\mu) \lesssim TN + \frac{1}{N^{2s}}.$$

Optimizing in  $N$  completes the proof.  $\square$

Let us estimate the boundary term in the following propositions. The first one is a slight variation of [39, Lemma 4.4].

**Proposition 4.2.18.** *Let  $d > 2s$ ,  $N \gg 1$  and  $0 < r \leq N^{-1/d}$ . Consider  $N$  equidistributed squares of side length  $r$ , which we denote by  $Q_r(X_i)$  where  $X_i$ ,  $i = 1, \dots, N$  are the centers and let*

$$\mu = \sum_i \frac{1}{Nr^d} \chi_{Q_r(X_i)}.$$

Then

$$\|\mu - 1\|_{\dot{H}^{-s}}^2 \lesssim \frac{1}{Nr^{d-2s}}.$$

*Proof.* Set  $\sigma = \mu - 1$  and let  $\psi \in L^1(Q_{N^{-1/d}}(X_i))$ . Denote by  $\psi_0$  the mean of  $\psi$  on  $Q_{N^{-1/d}}(X_i)$  with respect to the Lebesgue measure. Since  $\sigma(Q_{N^{-1/d}}(X_i)) = 0$ , we have

$$\int_{Q_{N^{-1/d}}(X_i)} \psi d\sigma = \int_{Q_{N^{-1/d}}(X_i)} (\psi - \psi_0) d\sigma = \int_{Q_r(X_i)} \frac{1}{Nr^d} (\psi - \psi_0) dx \leq \frac{1}{Nr^d} \|\psi - \psi_0\|_{L^1(Q_r(X_i))}.$$

By Sobolev embedding, for every  $\ell > 0$ , if  $Q_\ell$  is a cube of side length  $\ell$ ,  $\dot{H}^s(Q_\ell) \hookrightarrow L^{p^*}(Q_\ell)$  where  $p^* = 2d/(d - 2s)$ , see for instance [49]. Moreover, the norm of the embedding does not depend on  $\ell$ . Hence, by Hölder's inequality,

$$\begin{aligned} \|\psi - \psi_0\|_{L^1(Q_r(X_i))} &\leq \|\psi - \psi_0\|_{L^{p^*}(Q_r(X_i))} |Q_r(X_i)|^{1 - \frac{1}{p^*}} \\ &= r^{\frac{d+2s}{2}} \|\psi - \psi_0\|_{L^{p^*}(Q_r(X_i))} \leq r^{\frac{d+2s}{2}} \|\psi - \psi_0\|_{L^{p^*}(Q_{N^{-1/d}}(X_i))} \lesssim r^{\frac{d+2s}{2}} \|\psi\|_{\dot{H}^s(Q_{N^{-1/d}}(X_i))}. \end{aligned}$$

Thus,

$$\int_{Q_{N^{-1/d}}(X_i)} \psi d\sigma \lesssim \frac{1}{Nr^{\frac{d-2s}{2}}} \|\psi\|_{\dot{H}^s(Q_{N^{-1/d}}(X_i))}.$$

Now if we take any  $\psi \in \dot{H}^s(\mathbb{T}^d)$  we have

$$\begin{aligned} \left| \int_{\mathbb{T}^d} \psi d\sigma \right| &\leq \sum_i \left| \int_{Q_{N^{-1/d}}(X_i)} \psi d\sigma \right| \leq \sum_i \frac{1}{Nr^{\frac{d-2s}{2}}} \|\psi\|_{\dot{H}^s(Q_{N^{-1/d}}(X_i))} \\ &\lesssim \left( \frac{1}{Nr^{d-2s}} \right)^{\frac{1}{2}} \left( \sum_i \|\psi\|_{\dot{H}^s(Q_{N^{-1/d}}(X_i))}^2 \right)^{\frac{1}{2}}. \end{aligned}$$

To conclude, just notice that by  $N \gg 1$ , (4.2.9) and (4.2.10),

$$\left( \sum_i \|\psi\|_{\dot{H}^s(Q_{N^{-1/d}}(X_i))}^2 \right) \leq \|\psi\|_{\dot{H}^s(\mathbb{T}^d)}^2.$$

□

**Proposition 4.2.19.** *Let  $d = 1$  and  $s > \frac{1}{2}$ . Let  $\mu \in \mathcal{M}^+(\mathbb{T})$  be a sum of  $N$  equispaced Dirac deltas with weight  $1/N$ . Then*

$$\|\mu - 1\|_{\dot{H}^{-s}}^2 \lesssim \frac{1}{N^{2s}}.$$

*Proof.* We may assume without loss of generality that

$$\mu = \frac{1}{N} \sum_{j=0}^{N-1} \delta_{X_j},$$

where  $X_j = j/N$ . We want to compute

$$\|\mu - 1\|_{\dot{H}^{-s}}^2 = \sum_{k \in \mathbb{Z} \setminus \{0\}} |k|^{-2s} |\hat{\mu}_k|^2.$$

If  $k = nN$  for some  $n \in \mathbb{Z}$ , then

$$\hat{\mu}_k = \int_{\mathbb{T}} \exp(2\pi i k x) d\mu = \frac{1}{N} \sum_{j=0}^{N-1} \exp(2\pi i k j / N) = \frac{1}{N} \sum_{j=0}^{N-1} \exp(2\pi i n j) = 1. \quad (4.2.25)$$

Otherwise,

$$\hat{\mu}_k = \int_{\mathbb{T}} \exp(-2\pi i k x) d\mu = \frac{1}{N} \sum_{j=0}^{N-1} \exp(2\pi i k j / N) = \frac{1 - \exp(2\pi i k)}{1 - \exp(2\pi i k / N)} = 0, \quad (4.2.26)$$

where in the last passage we used the equality  $\sum_{i=0}^N x^i = \frac{1-x^{N+1}}{1-x}$  for  $x \neq 1$ . Combining (4.2.25) and (4.2.26) we get

$$\|\mu - 1\|_{\dot{H}^{-s}}^2 = \sum_{k \in \mathbb{Z} \setminus \{0\}} |k|^{-2s} |\hat{\mu}_k|^2 = \frac{1}{N^{2s}} \sum_{n \in \mathbb{Z} \setminus \{0\}} |n|^{-2s} \lesssim \frac{1}{N^{2s}}.$$

□

### 4.3 From local scaling laws to bounds on the lower and upper dimension of $\mu_T$

In this section we recall the notion of dimension of a measure and of Ahlfors regular measures. Then, we present some results of [48] that, given a minimizer  $\mu \in \mathcal{A}_T^*$ , link local bounds for its energy with the dimension of the boundary measure  $\mu_T$ . In [48] these results were established for  $s = \frac{1}{2}$  and  $d = 2$ , here we present the results for any  $d \geq 1$  and  $s \in (0, 1)$ .

#### 4.3.1 Dimension of a measure

Let  $\alpha \in [0, d]$ . If we denote by  $\mathcal{H}^\alpha$  the  $\alpha$ -dimensional Hausdorff measure, the Hausdorff dimension of a set  $A \subseteq Q$  is defined as

$$\dim A = \sup\{\alpha \in [0, d]: \mathcal{H}^\alpha(A) > 0\} = \inf\{\alpha \in [0, d]: \mathcal{H}^\alpha(A) = 0\}.$$

There are many ways to define lower and upper dimension of a measure  $\mu_T$  (see [82]). We mainly focus on the Hausdorff dimension, which is the notion used in [48].

**Definition 4.3.1** (Dimension of a measure). Let  $\sigma \in \mathcal{M}^+(\mathbb{T}^d)$ . The lower and upper Hausdorff dimension of  $\sigma$  are defined by

$$\underline{\dim} \sigma = \sup\{\alpha \in [0, d]: \sigma \ll \mathcal{H}^\alpha\} \quad \text{and} \quad \overline{\dim} \sigma = \inf\{\alpha \in [0, d]: \sigma \perp \mathcal{H}^\alpha\}.$$

If the lower and upper Hausdorff dimension of  $\sigma$  coincide we define the Hausdorff dimension of  $\sigma$  to be the common value  $\dim \sigma = \underline{\dim} \sigma = \overline{\dim} \sigma$ .

Let us introduce the notion of upper  $\alpha$ -Ahlfors regular and lower  $\alpha$ -Ahlfors regular measures for  $\alpha \in [0, d]$ .

**Definition 4.3.2.** Let  $\sigma \in \mathcal{M}^+(\mathbb{T}^d)$ , and  $\alpha \in [0, d]$ .

(i) A measure is called upper  $\alpha$ -Ahlfors regular if there exists  $M > 0$  such that

$$\sigma(B_r(x)) \leq Mr^\alpha \quad \text{for all } x \in \text{supp } \sigma \text{ and } r \in (0, 1].$$

(ii) A measure is called lower  $\alpha$ -Ahlfors regular if there exists  $M > 0$  such that

$$\sigma(B_r(x)) \geq Mr^\alpha \quad \text{for all } x \in \text{supp } \sigma \text{ and } r \in (0, 1]. \quad (4.3.1)$$

If both conditions hold (possibly for different  $M, m > 0$  respectively), the measure is called  $\alpha$ -Ahlfors regular.

**Remark 4.3.3.** Notice that if  $\alpha = d$  upper  $\alpha$ -Ahlfors regularity is equivalent to  $\sigma = f dx$  for  $f \in L^\infty$ . Notice also that if  $\alpha' < \alpha$  then

$$\sigma \text{ is upper } \alpha\text{-Ahlfors regular} \Rightarrow \sigma \text{ is upper } \alpha'\text{-Ahlfors regular}$$

and

$$\sigma \text{ is lower } \alpha'\text{-Ahlfors regular} \Rightarrow \sigma \text{ is lower } \alpha\text{-Ahlfors regular.}$$

By [3, Theorem 2.5.6] we have

**Proposition 4.3.4.** Let  $\sigma \in \mathcal{M}^+(\mathbb{T}^d)$ . The following implications hold:

$$\begin{aligned} \sigma \text{ is upper } \alpha\text{-Ahlfors regular} &\Rightarrow \sigma \ll \mathcal{H}^\alpha, \\ \sigma \text{ is lower } \alpha\text{-Ahlfors regular} &\Rightarrow \sigma \perp \mathcal{H}^{\alpha'} \quad \forall \alpha' > \alpha. \end{aligned}$$

**Remark 4.3.5.** By a direct application of the previous proposition, for an upper  $\alpha$ -Ahlfors regular measure  $\sigma$  we have

$$\underline{\dim} \sigma \geq \alpha,$$

whereas for a lower  $\alpha$ -Ahlfors regular measure we have

$$\overline{\dim} \sigma \leq \alpha.$$

Intuitively, this means that upper  $\alpha$ -Ahlfors regular measures are measures that are "diffused at least  $\alpha$ ", whereas lower  $\alpha$ -Ahlfors regular measures are measures that are "diffused at most  $\alpha$ ".

Let us also recall the following characterization of the lower dimension of a measure, see [48, Lemma 2.17].

**Lemma 4.3.6.** Let  $\sigma \in \mathcal{M}^+(\mathbb{T}^d)$  with  $\sigma(\mathbb{T}^d) = 1$ . Then

$$\underline{\dim} \sigma \geq \sup\{\alpha : \alpha \leq d, \|\sigma - 1\|_{\dot{H}^{-\frac{1}{2}(d-\alpha)}} < \infty\}.$$

### 4.3.2 Local scaling laws and dimensional estimates

Let us now recall some results from [48]. We consider a minimizer  $\mu$  of  $E_{s,T}$ . In [48, Theorem 1.3] a first local upper bound on  $I(\mu, (T - \varepsilon, T))$  is proved for  $d = 2$ . This upper bound corresponds to the worst possible scaling. It is attained when the irrigated measure is uniform, see [48, Theorem 3.7]. The extension to arbitrary dimension is straightforward, see also the proof of Theorem 4.1.6.

**Proposition 4.3.7.** *Let  $\mu \in \mathcal{A}_T^*$  be a minimizer of  $E_{s,T}$  and  $\varepsilon \ll 1$ . Then*

$$I(\mu, (T - \varepsilon, T)) \lesssim \varepsilon^{\frac{1}{3}}.$$

We now extend the proof of [48, Theorem 3.5] to arbitrary  $d$  and  $s$ .

**Lemma 4.3.8.** *Let  $\mu \in \mathcal{A}_T^*$  be a symmetric minimizer of  $E_{s,T}$  and assume that there exists  $\beta \in (0, 1)$  such that  $\limsup_{\varepsilon \rightarrow 0} \varepsilon^{-\beta} I(\mu, (T - \varepsilon, T)) < \infty$ . Then,*

$$\|\mu_T - 1\|_{\dot{H}^{-\frac{1}{2}(2-\alpha)}} < \infty \text{ for all } \alpha < 2(1-s) + \frac{2\beta}{1+\beta}.$$

*Proof.* Let  $\eta \in (0, 1)$ ,  $\varepsilon \in (0, T)$  and consider  $\mu_t = \sum_i \varphi_i \delta_{X_i(t)}$ . Let  $e_j$  for  $j = 1, \dots, d$ , denote the canonical base of  $\mathbb{R}^d$ . Let us build a competitor for  $\mu$ . Since  $\mu$  is symmetric we can restrict to  $(0, T)$ . Let

$$X_i^\pm(t) = \begin{cases} X_i(t) & \text{if } t \in (0, T - \varepsilon), \\ X_i(t) \pm \frac{\eta}{\varepsilon}(t - (T - \varepsilon))e_j & \text{if } t \in (T - \varepsilon, T). \end{cases}$$

We set  $\mu_t^\pm = \sum_i \varphi_i \delta_{X_i^\pm(t)}$  and define  $\tilde{\mu} = \frac{1}{2}(\mu^+ + \mu^-)$ . We have by a direct computation

$$I(\tilde{\mu}, (T - \varepsilon, T)) \lesssim I(\mu, (T - \varepsilon, T)) + \frac{\eta^2}{\varepsilon}.$$

By minimality we therefore conclude

$$\begin{aligned} \|\mu_T - 1\|_{\dot{H}^{-s}}^2 - \|\tilde{\mu}_T - 1\|_{\dot{H}^{-s}}^2 &\leq I(\tilde{\mu}, (T - \varepsilon, T)) - I(\mu, (T - \varepsilon, T)) \\ &\lesssim I(\mu, (T - \varepsilon, T)) + \frac{\eta^2}{\varepsilon} \lesssim_\mu \varepsilon^\beta + \frac{\eta^2}{\varepsilon} \lesssim_\mu \eta^{\frac{2\beta}{1+\beta}} \end{aligned} \quad (4.3.2)$$

choosing  $\varepsilon = \eta^{2/(1+\beta)}$  to minimize the right-hand side of (4.3.2). Now, we turn our attention to the left-hand side of (4.3.2). We set  $\sigma = \mu_T - 1$  and  $\tilde{\sigma} = \tilde{\mu}_T - 1$ . Notice that if we set  $\tau_\eta(x) = x - \eta e_j$  we have

$$\tilde{\sigma} = \frac{1}{2}(\tau_\eta \# \sigma + \tau_{-\eta} \# \sigma).$$

This implies

$$\widehat{\tilde{\sigma}}_k = \int_{\mathbb{T}^d} \exp(i2\pi k \cdot x) d\tilde{\sigma}(x) = \cos(2\pi\eta k \cdot e_j) \widehat{\sigma}_k.$$

This leads us to

$$\|\mu_T - 1\|_{\dot{H}^{-s}}^2 - \|\tilde{\mu}_T - 1\|_{\dot{H}^{-s}}^2 = \sum_{k \in \mathbb{Z}^d} |k|^{-2s} \sin^2(2\pi k \cdot e_j \eta) |\widehat{\sigma}_k|^2 \gtrsim \eta^2 \sum_{|k| \lesssim \eta^{-1}} |k \cdot e_j|^2 |k|^{-2s} |\widehat{\sigma}_k|^2.$$

Since this argument holds for any  $j = 1, \dots, d$ , it follows from (4.3.2) that

$$\eta^{\frac{2\beta}{1+\beta}} \gtrsim_\mu \eta^2 \sum_{|k| \lesssim \eta^{-1}} |k|^{2(1-s)} |\widehat{\sigma}_k|^2.$$

Using the characterization (4.2.11) and this last estimate yields for  $\alpha > 0$ ,

$$\begin{aligned} \|\sigma\|_{\dot{H}^{-\frac{1}{2}(2-\alpha)}}^2 &\lesssim_{\mu} \int_0^1 \eta^{2\frac{1}{2}(2-\alpha)+2(1-s)} \left( \sum_k |k|^{2(1-s)} |\widehat{\sigma}(k)|^2 \right) \frac{d\eta}{\eta} \\ &\lesssim_{\mu} \int_0^1 \eta^{2(1-s)-\alpha} \eta^{\frac{2\beta}{1+\beta}} \frac{d\eta}{\eta}. \end{aligned} \quad (4.3.3)$$

The integral on the right-hand side of (4.3.3) is finite if and only if  $2(1-s) - \alpha + \frac{2\beta}{1+\beta} > 0$ . Hence, the claim follows.  $\square$

We can now apply the strategy from [48, Corollary 3.6 & Theorem 3.7] to obtain the following relation between local scaling laws and the dimension of  $\mu_T$ .

**Proposition 4.3.9.** *Let  $s \in (0, 1)$  and  $\mu \in \mathcal{A}_T^*$  be a symmetric minimizer of  $E_{s,T}$ . Assume that for some  $\beta \in (0, 1)$  with  $s > \beta/(1+\beta)$*

$$\limsup_{\varepsilon \rightarrow 0} \varepsilon^{-\beta} I(\mu, (T - \varepsilon, T)) < \infty.$$

Then

$$d - 2s + \frac{2\beta}{1+\beta} \leq \underline{\dim} \mu_T \leq \overline{\dim} \mu_T \leq \frac{2d(1-\beta)}{1+\beta}. \quad (4.3.4)$$

*Proof.* The proof is basically the same as in [48], in which the result is proved for  $s = \frac{1}{2}$  and  $d = 2$ . In particular, the first estimate is a direct consequence of Lemma 4.3.6 combined with Lemma 4.3.8. The second estimate does not depend on  $s$  and the proof found in [48, Theorem 3.7] generalizes directly to any dimension.  $\square$

**Remark 4.3.10.** *In the last proposition we assume that*

$$s > \frac{\beta}{1+\beta}.$$

*If this does not hold, Lemma 4.3.8 together with Lemma 4.3.6 directly imply that*

$$\underline{\dim} \mu_T = \overline{\dim} \mu_T = \dim \mu_T = d.$$

*In particular, since by Proposition 4.3.7, we have  $\beta \geq 1/3$ , we may restrict ourselves to  $s > 1/4$ .*

**Remark 4.3.11.** *Equality of both sides in (4.3.4) is achieved for*

$$\beta_c(s, d) = \frac{d+2s}{3d+2(1-s)}. \quad (4.3.5)$$

*This would correspond to*

$$\dim \mu_T = \frac{2d(d+1-2s)}{2d+1}.$$

*The critical exponent (4.3.5) is equal to the exponent of the global scaling law for  $0 < T \leq 1$  and  $d > 2s$ , see Proposition 4.2.16. In particular, for  $s = 1/2$  and  $d = 2$  we get  $\beta_c(1/2, 2) = 3/7$ , corresponding to the conjectured  $\dim \mu_T = 8/5$ . For  $d = 1$  both sides of (4.3.4) coincide for  $\beta_c(s, 1) = (1+2s)/(5-2s)$  and correspondingly,  $\dim \mu_T = 4/3(1-s)$ .*

In the subsequent sections we will prove Theorem 4.1.5 and Theorem 4.1.6, which relates upper  $\alpha$ -Ahlfors regularity and lower  $\alpha$ -Ahlfors regularity of  $\mu_T$  with local scaling laws. We postpone the proofs of these results and first show how they combine with Proposition 4.3.9 to prove Theorem 4.1.1.

*Proof of Theorem 4.1.1.* Assume that  $\mu_T$  is upper  $\alpha$ -Ahlfors regular. We can suppose  $s > \frac{1-\alpha}{2}$  otherwise there is nothing to prove. Then, by Theorem 4.1.5, for every  $\delta \in (0, \beta_{\text{reg}})$  (recall the definition of  $\beta_{\text{reg}}$  (4.1.7))

$$I(\mu, (T - \varepsilon, T)) \lesssim_{\delta} \varepsilon^{\beta_{\text{reg}}(s, \alpha) - \delta}.$$

By Proposition 4.3.9 and Remark 4.3.5 we have

$$\alpha \leq \underline{\dim} \mu_T \leq \overline{\dim} \mu_T \leq \frac{2(1 - \beta_{\text{reg}}(s, \alpha) + \delta)}{1 + \beta_{\text{reg}}(s, \alpha) - \delta}.$$

This yields, recalling (4.1.4)

$$\alpha \leq \bar{\alpha} + C(s, \alpha)\delta$$

for some  $C(s, \alpha) > 0$  and all  $\delta \in (0, \beta_{\text{reg}})$ . Letting  $\delta \rightarrow 0$ , this gives (4.1.5). Analogously, assume that  $\mu_T$  is lower  $\alpha$ -Ahlfors regular. Then, by Theorem 4.1.6 we have (recall the definition (4.1.8) of  $\beta_{\text{con}}(\alpha)$ )

$$I(\mu, (T - \varepsilon, T)) \lesssim_{\mu} \varepsilon^{\beta_{\text{con}}(\alpha)}.$$

Thus, by Proposition 4.3.9 we have

$$1 - 2s + \frac{2\beta_{\text{con}}(\alpha)}{1 + \beta_{\text{con}}(\alpha)} \leq \underline{\dim} \mu_T \leq \overline{\dim} \mu_T \leq \alpha$$

which yields  $\alpha \geq \bar{\alpha}$  using the definitions (4.1.4) and (4.1.8) of  $\bar{\alpha}$  and  $\beta_{\text{con}}$  respectively.  $\square$

## 4.4 Local scaling laws for minimizers with upper $\alpha$ -Ahlfors regular boundary measures

The goal of this section is to prove Theorem 4.1.5. This section is divided in two parts. In the first part (which holds for any  $d \geq 1$ ), we prove that the class of upper  $\alpha$ -Ahlfors regular measures is stable with respect to McCann interpolation. This result could be of independent interest. We then show that upper  $\alpha$ -Ahlfors regular measures are embedded in some Sobolev space of negative order. In the second part we fix  $d = 1$  and consider a minimizer  $\mu$  of  $E_{s, T}$ . We prove that if  $\mu_T$  is upper  $\alpha$ -Ahlfors regular, then  $\mu$  satisfies a certain local scaling law.

### 4.4.1 Interpolated measures, upper $\alpha$ -Ahlfors regularity and Sobolev regularity

**Lemma 4.4.1** (Upper  $\alpha$ -Ahlfors regularity of displacement measures). *Let  $\pi \in \mathcal{M}_{\text{loc}}^+(\mathbb{R}^d \times \mathbb{R}^d)$  be a monotone transport plan i.e. for every  $(x_1, y_1), (x_2, y_2) \in \text{supp } \pi$*

$$(x_1 - x_2) \cdot (y_1 - y_2) \geq 0 \tag{4.4.1}$$

*and for  $\lambda \in (0, 1)$  let  $\mu^\lambda = F_\lambda \# \pi$  where  $F_\lambda(x, y) = (1 - \lambda)x + \lambda y$ . If  $\mu^0 = \pi_1$  is upper  $\alpha$ -Ahlfors regular for some  $M > 0$  then  $\mu^\lambda$  is upper  $\alpha$ -Ahlfors regular for some constant  $M_\lambda \lesssim (1 - \lambda)^{-\alpha} M$ .*

*Proof.* We claim that for every  $r \in (0, 1)$  and  $x_0 \in \mathbb{R}^d$  there exists  $x_{\lambda, r}$  such that

$$F_\lambda^{-1}(B_r(x_0)) \cap \text{supp } \pi \subset B_{2r/(1-\lambda)}(x_{\lambda, r}) \times \mathbb{R}^d. \quad (4.4.2)$$

Indeed, this would imply

$$\begin{aligned} \mu^\lambda(B_r(x_0)) &= \pi(F_\lambda^{-1}(B_r(x_0))) \leq \pi(B_{2r/(1-\lambda)}(x_{\lambda, r}) \times \mathbb{R}^d) \\ &= \mu^0(B_{2r/(1-\lambda)}(x_{\lambda, r})) \leq M2^\alpha(1-\lambda)^{-\alpha}r^\alpha, \end{aligned}$$

which would conclude the proof of the upper  $\alpha$ -Ahlfors regularity of  $\mu^\lambda$ . We now prove claim (4.4.2). To this aim, it is enough to show that for any  $(x_1, y_1), (x_2, y_2) \in F_\lambda^{-1}(B_r(x_0)) \cap \text{supp } \pi$  we have  $|x_1 - x_2| \leq 2r/(1-\lambda)$ . By definition, for any  $(x_1, y_1), (x_2, y_2) \in F_\lambda^{-1}(B_r(x_0))$  we have

$$|(1-\lambda)(x_1 - x_2) + \lambda(y_1 - y_2)| = |F_\lambda(x_1, y_1) - F_\lambda(x_2, y_2)| \leq 2r.$$

By (4.4.1), we then have

$$2r \geq |(1-\lambda)(x_1 - x_2) + \lambda(y_1 - y_2)| \geq (1-\lambda)|x_1 - x_2|$$

and the claim is proved.  $\square$

**Remark 4.4.2.** Notice that the measure  $\mu^\lambda$  can be in general much less concentrated than  $\mu^0$  (if for instance  $\mu^1 = \pi_2$  is upper  $\alpha'$ -Ahlfors regular for some  $\alpha' > \alpha$ ).

**Remark 4.4.3.** The arguments in the proof of Lemma 4.4.1 are reminiscent of the ones of [108, Theorem 8.7] where it is shown that absolute continuity with respect to the Lebesgue measure is preserved by displacement interpolation. Lemma 4.4.1 can also be seen as a generalization of the  $L^\infty$  bounds on the McCann interpolant, see [108, Corollary 19.5]. Our proof is however quite elementary compared to the standard proof based on the Monge-Ampère equation and the log-concavity of the determinant. Let us however point out that as opposed to [108, Theorem 19.4], we do not get sharp constants (which are essential in many geometric applications of [108, Theorem 19.4]). See also [102], where similar ideas are used for proving estimates for transport densities.

**Proposition 4.4.4** (Sobolev embedding). Assume that  $\mu \in \mathcal{P}(\mathbb{T}^d)$  is upper  $\alpha$ -Ahlfors regular for some  $M > 0$ . Then  $\mu - 1 \in \dot{H}^{-\gamma}(\mathbb{T}^d)$  for every  $\gamma$  such that

$$\gamma > \frac{d - \alpha}{2}. \quad (4.4.3)$$

Moreover,

$$\|\mu - 1\|_{\dot{H}^{-\gamma}}^2 \lesssim_\gamma M + 1.$$

*Proof.* Consider a family of convolution kernels  $(\rho_\varepsilon)_\varepsilon$ ,  $\varepsilon > 0$ . We want to use (4.2.12). For every  $x \in \mathbb{T}^d$  we estimate

$$\rho_\varepsilon * \mu(x) = \int_{B_\varepsilon(x)} \rho_\varepsilon(x - y) d\mu(y) \lesssim \frac{\mu(B_\varepsilon(x))}{\varepsilon^d}. \quad (4.4.4)$$

Moreover,

$$\int_{\mathbb{T}^d} \mu(B_\varepsilon(x))^2 dx \lesssim \sup_{x \in \mathbb{T}^d} \mu(B_\varepsilon(x)) \int_{\mathbb{T}^d} \mu(B_\varepsilon(x)) dx \lesssim M\varepsilon^\alpha \varepsilon^d, \quad (4.4.5)$$

where in the last passage we used upper  $\alpha$ -Ahlfors regularity and Fubini's Theorem. Notice finally that

$$|\rho_\varepsilon * (\mu - 1)(x)|^2 \lesssim |\rho_\varepsilon * \mu(x)|^2 + 1. \quad (4.4.6)$$

Plugging (4.4.4), (4.4.5) and (4.4.6) in (4.2.12) we obtain

$$\|\mu - 1\|_{\dot{H}^{-\gamma}}^2 \lesssim (M+1) \int_0^1 \frac{\varepsilon^{2\gamma}}{\varepsilon^{2d}} \varepsilon^\alpha \varepsilon^d \frac{d\varepsilon}{\varepsilon} = (M+1) \int_0^1 \varepsilon^{2\gamma+\alpha-d-1} d\varepsilon.$$

This integral is finite if and only if

$$2\gamma + \alpha > d,$$

which yields the condition (4.4.3).  $\square$

#### 4.4.2 Local scaling laws for minimizers with upper $\alpha$ -Ahlfors regular boundary measure

In this section we fix  $s > \frac{1}{4}$ . Let  $\mu$  be a minimizer of  $E_{s,T}$ . Goal of this section is to prove Theorem 4.1.5. We start with the following lemma.

**Lemma 4.4.5.** *Let  $\mu, \nu \in \mathcal{P}(\mathbb{T})$  be such that  $\mu$  is upper  $\alpha$ -Ahlfors regular for some  $M > 0$  and  $\alpha > d - 2s$ . Let  $\pi$  be an optimal transport plan for  $W(\mu, \nu)$ . For  $\lambda \in (0, 1)$  let  $\mu^\lambda = F_\lambda \# \pi$  where  $F_\lambda(x, y) = (1 - \lambda)x + \lambda y$ . Then, letting*

$$\bar{\theta}(s, \alpha) = s - \frac{d - \alpha}{2},$$

we have for every  $\theta < \bar{\theta}(s, \alpha)$ ,

$$\|\mu - \mu^\lambda\|_{\dot{H}^{-s}} \lesssim_{\lambda, \theta, \mu} W^\theta(\mu, \nu). \quad (4.4.7)$$

*Proof.* Fix  $\varepsilon \in (0, \varepsilon_0)$  and consider a regular convolution kernel  $\rho_\varepsilon$ , recall (4.2.1). Using the triangle inequality we have

$$\|\mu - \mu^\lambda\|_{\dot{H}^{-s}} \leq \|\mu - \mu * \rho_\varepsilon\|_{\dot{H}^{-s}} + \|\mu * \rho_\varepsilon - \mu^\lambda * \rho_\varepsilon\|_{\dot{H}^{-s}} + \|\mu^\lambda - \mu^\lambda * \rho_\varepsilon\|_{\dot{H}^{-s}}. \quad (4.4.8)$$

Let us estimate the first term on the right-hand side of the previous inequality. By Proposition 4.4.4 we have  $\|\mu - 1\|_{\dot{H}^{-\gamma}}^2 \lesssim_\gamma M + 1$  for any  $\gamma > \frac{d-\alpha}{2}$ . Moreover by Proposition 4.2.14 we have

$$\|\mu - \mu * \rho_\varepsilon\|_{\dot{H}^{-s}} \lesssim \varepsilon^{s-\gamma} \|\mu - 1\|_{\dot{H}^{-\gamma}} \lesssim_{\gamma, \mu} \varepsilon^{s-\gamma} (M + 1)^{\frac{1}{2}}.$$

We now turn to the third right-hand side term in (4.4.8). Since  $\pi$  is monotone, by Lemma 4.4.1,  $\mu^\lambda$  is upper  $\alpha$ -Ahlfors regular for some  $M_\lambda \lesssim_\lambda M$ , so that arguing exactly as above,

$$\|\mu^\lambda - \mu^\lambda * \rho_\varepsilon\|_{\dot{H}^{-s}} \lesssim_\lambda \varepsilon^{s-\gamma} \|\mu - 1\|_{\dot{H}^{-\gamma}} \lesssim_{\lambda, \gamma, \mu} \varepsilon^{s-\gamma} (M + 1)^{\frac{1}{2}}.$$

Let us now focus on the second right-hand side term in (4.4.8). Using Hölder inequality with  $p = 1/(1-s)$  and  $q = 1/s$  we find that for any  $\sigma \in \dot{H}^{-s}(\mathbb{T})$

$$\|\sigma\|_{\dot{H}^{-s}} = \left( \sum_{k \in \mathbb{Z}} |k|^{-2s} |\widehat{\sigma}_k|^2 \right)^{\frac{1}{2}} = \left( \sum_{k \in \mathbb{Z}} |\widehat{\sigma}_k|^{2(1-s)} |k|^{-2s} |\widehat{\sigma}_k|^{2s} \right)^{\frac{1}{2}} \leq \|\sigma\|_{L^2}^{1-s} \|\sigma\|_{\dot{H}^{-1}}^s.$$

Hence,

$$\begin{aligned} \|\mu * \rho_\varepsilon - \mu^\lambda * \rho_\varepsilon\|_{\dot{H}^{-s}} &\leq \|\mu * \rho_\varepsilon - \mu^\lambda * \rho_\varepsilon\|_{L^2}^{1-s} \|\mu * \rho_\varepsilon - \mu^\lambda * \rho_\varepsilon\|_{\dot{H}^{-1}}^s \\ &\leq (\|\mu * \rho_\varepsilon\|_{L^2} + \|\mu^\lambda * \rho_\varepsilon\|_{L^2})^{1-s} \|\mu * \rho_\varepsilon - \mu^\lambda * \rho_\varepsilon\|_{\dot{H}^{-1}}^s. \end{aligned}$$

Moreover, by Proposition 4.2.15,

$$\|\mu * \rho_\varepsilon - \mu^\lambda * \rho_\varepsilon\|_{\dot{H}^{-1}} \lesssim \max\{\|\mu * \rho_\varepsilon\|_{L^\infty}^{\frac{1}{2}}, \|\mu^\lambda * \rho_\varepsilon\|_{L^\infty}^{\frac{1}{2}}\} W_{\text{per}}(\mu * \rho_\varepsilon, \mu^\lambda * \rho_\varepsilon).$$

Now, arguing as in the proof of Proposition 4.4.4 we get

$$\rho_\varepsilon * \mu(x) \lesssim \frac{\mu(B_\varepsilon(x))}{\varepsilon^d} \leq \frac{M}{\varepsilon^{d-\alpha}}.$$

Hence

$$\|\mu * \rho_\varepsilon\|_{L^2}^2 \leq \|\mu * \rho_\varepsilon\|_{L^\infty} \int_{\mathbb{T}} (\mu * \rho_\varepsilon)(x) dx \lesssim \frac{M}{\varepsilon^{d-\alpha}}.$$

Moreover, again by Lemma 4.4.1 the same estimates hold also for  $\mu^\lambda$ , that is

$$\|\mu^\lambda * \rho_\varepsilon\|_{L^2}^2 \leq \|\mu^\lambda * \rho_\varepsilon\|_{L^\infty} \lesssim \lambda \frac{M}{\varepsilon^{d-\alpha}}.$$

Finally, by [101, Lemma 5.2],

$$W_{\text{per}}(\mu * \rho_\varepsilon, \mu^\lambda * \rho_\varepsilon) \leq W_{\text{per}}(\mu, \mu^\lambda) \leq W(\mu, \mu^\lambda) \stackrel{(4.2.4)}{=} \lambda W(\mu, \nu).$$

Putting all the estimates together we get

$$\|\mu - \mu^\lambda\|_{\dot{H}^{-s}} \lesssim_{\lambda, \gamma, \mu} \varepsilon^{s-\gamma} + \varepsilon^{\frac{\alpha-d}{2}} W^s(\mu, \nu).$$

We optimize in  $\varepsilon$  choosing

$$\varepsilon = W^{\frac{s}{s-\beta}}(\mu, \nu) \quad \text{where} \quad \beta = \gamma - \frac{d-\alpha}{2}$$

if  $W^{\frac{s}{s-\beta}}(\mu, \nu) \leq \varepsilon_0$  and  $\varepsilon = \varepsilon_0$  otherwise. In both cases this implies

$$\|\mu - \mu^\lambda\|_{\dot{H}^{-s}} \lesssim_{\lambda, \theta, \mu} W^{s-\frac{s(d-\alpha)}{2(s-\beta)}}(\mu, \nu).$$

Since this holds for any  $\beta > 0$ , we conclude the proof of (4.4.7).  $\square$

*Proof of Theorem 4.1.5.* Let  $\pi$  be an optimal transport plan for  $W(\mu_{T-\varepsilon}, \mu_T)$ .

**Step 1: Construction of a competitor.** Let  $X$  be the Lagrangian representation of  $\mu$  given by Proposition 4.2.5. We first notice that since  $d = 1$  by Remark 4.2.10 and Lemma 4.2.7, the map  $\psi = X(\cdot, T - \varepsilon)$  is monotone and thus  $\pi = (\text{Id} \times \psi) \# \mu_T$  as well. Since  $d = 1$ , it is thus also an optimal transport plan for  $W^2(\mu_{T-\varepsilon}, \mu_T)$ . We then set

$$Y(x, t) = \begin{cases} X(x, t) & \text{for } t \leq T - \varepsilon \\ \frac{1}{2}(X(x, T - \varepsilon) + X(x, t)) & \text{for } T - \varepsilon < t \leq T, \end{cases}$$

and  $\nu_t = Y(\cdot, t) \# \mu_T$ . With the notation of Lemma 4.4.5,  $\nu_T = F_{1/2} \# \pi = \mu_T^{1/2}$ . Notice that if  $\mu_t = \sum_i \varphi_i \delta_{X_i(t)}$ , this construction is equivalent to setting  $\nu_t = \sum_i \varphi_i \delta_{Y_i(t)}$  where

$$Y_i(t) = \begin{cases} X_i(t) & \text{if } t \leq T - \varepsilon \\ \frac{1}{2}(X_i(t) + X_i(T - \varepsilon)) & \text{if } T - \varepsilon < t \leq T. \end{cases}$$

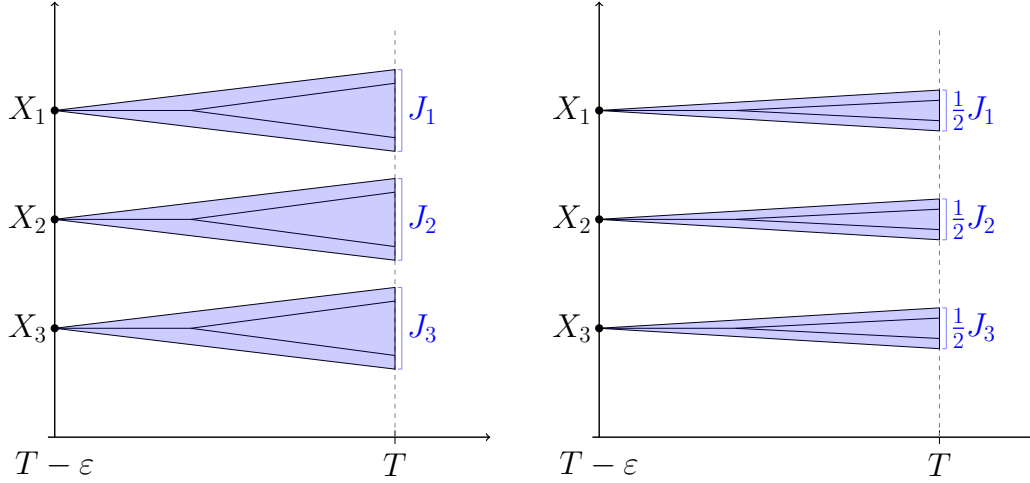


Figure 4.2: On the left a sketch of  $\mu$  on  $[T - \varepsilon, T]$ , on the right a sketch of  $\nu$  on  $[T - \varepsilon, T]$ .

**Step 2: Estimate of the internal energy of  $\mu_T$ .**

Let us compare the minimizer  $\mu$  to the competitor we constructed in Step 1. By construction we have (notice that for  $t \geq T - \varepsilon$ ,  $\dot{Y}_i(t) = \dot{X}_i(t)/2$ )

$$\begin{aligned} I(\nu, (0, T - \varepsilon)) &= I(\mu, (0, T - \varepsilon)), & P(\nu, (T - \varepsilon, T)) &= P(\mu, (T - \varepsilon, T)) \\ \text{and } E_{\text{kin}}(\nu, (T - \varepsilon, T)) &= \frac{1}{4} E_{\text{kin}}(\mu, (T - \varepsilon, T)). \end{aligned}$$

The minimality of  $\mu$  yields

$$\frac{3}{4} E_{\text{kin}}(\mu, (T - \varepsilon, T)) \leq \|\nu_T - 1\|_{\dot{H}^{-s}}^2 - \|\mu_T - 1\|_{\dot{H}^{-s}}^2 = \|\mu_T - \nu_T\|_{\dot{H}^{-s}}^2 + 2\langle \mu_T - \nu_T, \mu_T - 1 \rangle_{\dot{H}^{-s}}. \quad (4.4.9)$$

Using Lemma 4.4.5 we get

$$\|\mu_T - \nu_T\|_{\dot{H}^{-s}} \lesssim_{\theta, \mu} W^\theta(\mu_T, \mu_{T-\varepsilon}) \quad \text{for } \theta < s - \frac{1 - \alpha}{2}.$$

Let us now focus on the scalar product in (4.4.9). By Proposition 4.4.4 we have  $\mu_T - 1 \in \dot{H}^{-\gamma}(\mathbb{T})$  for all  $\gamma > \frac{1 - \alpha}{2}$  with  $\|\mu_T - 1\|_{\dot{H}^{-\gamma}}^2 \lesssim_\gamma M + 1$ . We then have by Hölder inequality

$$\begin{aligned} \langle \mu_T - \nu_T, \mu_T - 1 \rangle_{\dot{H}^{-s}} &= \sum_{k \in \mathbb{Z}} |k|^{-2s} (\widehat{\mu_T - \nu_T})_k \overline{(\widehat{\mu_T - 1})_k} = \sum_{k \in \mathbb{Z}} |k|^{-2s + \gamma} (\widehat{\mu_T - \nu_T})_k \cdot |k|^{-\gamma} \overline{(\widehat{\mu_T - 1})_k} \\ &\leq \left( \sum_{k \in \mathbb{Z}} |k|^{-4s + 2\gamma} |\widehat{\mu_T - \nu_T}|_k^2 \right)^{\frac{1}{2}} \left( \sum_{k \in \mathbb{Z}} |k|^{-2\gamma} |\widehat{\mu_T - 1}|_k^2 \right)^{\frac{1}{2}} = \|\mu_T - \nu_T\|_{\dot{H}^{-(2s-\gamma)}} \|\mu_T - 1\|_{\dot{H}^{-\gamma}}. \end{aligned}$$

From (4.4.7) again we obtain

$$\|\mu_T - \nu_T\|_{\dot{H}^{-(2s-\gamma)}} \lesssim_{\tilde{\theta}, \mu} W^{\tilde{\theta}}(\mu_{T-\varepsilon}, \mu_T) \quad \text{for } \tilde{\theta} < 2s - \gamma - \frac{1 - \alpha}{2}.$$

Now fix  $\delta > 0$  and choose

$$\begin{aligned}\gamma &= \frac{1-\alpha}{2} + \delta, \\ \theta &= s - \frac{1-\alpha}{2} - \frac{\delta}{2}, \\ \tilde{\theta} &= 2s - \gamma - \frac{1-\alpha}{2}.\end{aligned}$$

Putting all the estimates together we arrive at

$$\begin{aligned}E_{\text{kin}}(\mu, (T-\varepsilon, T)) &\lesssim_{\delta, \mu} (W^{s-\frac{1-\alpha}{2}-\frac{\delta}{2}}(\mu_T, \mu_{T-\varepsilon}))^2 + \|\mu_T - 1\|_{\dot{H}^{-\gamma}} W^{2s-\gamma-\frac{1-\alpha}{2}-\delta}(\mu_T, \mu_{T-\varepsilon}) \\ &\lesssim_{\delta, \mu} W^{2s-(1-\alpha)-\delta}(\mu_T, \mu_{T-\varepsilon}).\end{aligned}\quad (4.4.10)$$

Since  $E_{\text{kin}}(\mu, (T-\varepsilon, T)) \geq \frac{1}{\varepsilon} W^2(\mu_T, \mu_{T-\varepsilon})$  by (4.2.5) (see Remark 4.2.10) and since (4.4.10) holds for any  $\delta > 0$ , we obtain after simplification that for every  $\beta < \beta_{\text{reg}}$  (recall (4.1.7)),

$$E_{\text{kin}}(\mu, (T-\varepsilon, T)) \lesssim_{\beta, \mu} \varepsilon^\beta.$$

Finally, by Proposition 4.2.6 integrated on  $[T-\varepsilon, T]$  we have

$$P(\mu, (T-\varepsilon, T)) = E_{\text{kin}}(\mu, (T-\varepsilon, T)) + \Lambda\varepsilon.$$

This implies

$$P(\mu, (T-\varepsilon, T)) \lesssim_{\beta, \mu} \varepsilon^\beta + \Lambda\varepsilon \lesssim_{\beta, \mu} \varepsilon^\beta,$$

thus the estimate holds for  $I(\mu, (T-\varepsilon, T))$  as well. This concludes the proof of Theorem 4.1.5.  $\square$

**Remark 4.4.6.** *Most of the arguments of the proof also hold for  $d > 1$ . The hypothesis  $d = 1$  plays a role only to ensure that the coupling  $\pi$  between  $\mu_{T-\varepsilon}$  and  $\mu_T$  induced by a minimizer is the optimal transport plan for  $W$ , so that Lemma 4.4.5 can be applied.*

## 4.5 Local scaling laws for minimizers with lower $\alpha$ -Ahlfors regular boundary measures

The aim of this section is to prove Theorem 4.1.6. We start by adapting the construction from [48] to measures with small support. This is then used to construct a competitor that irrigates with small energy any boundary measure  $\mu_T$  which is lower  $\alpha$ -Ahlfors regular (recall Definition 4.3.2). Let us insist on the fact that as opposed to the previous section, we do not modify the boundary condition  $\mu_T$  for  $t = T$ .

### 4.5.1 Local construction for concentrated boundary measures

In this section, we provide a local construction for boundary measures with small support. We make the construction for arbitrary  $d \geq 1$ .

**Lemma 4.5.1.** *Let  $\varepsilon, \Phi > 0$ ,  $r \in (0, 1)$  and  $\mu^\pm \in \mathcal{M}^+(\mathbb{R}^d)$  be such that  $\mu^\pm(\mathbb{R}^d) = \Phi$ . Assume that there exist  $x_\pm \in \mathbb{R}^d$  such that  $\text{supp}(\mu^\pm) \subseteq B_r(x_\pm)$ . Then, there is  $\mu \in \mathcal{A}^*(\mathbb{R}^d, (0, \varepsilon))$  and  $C > 0$  such that*

$$I(\mu, (0, \varepsilon)) \leq \frac{1}{\varepsilon} W^2(\mu^-, \mu^+) + C \left( \frac{r^2}{\varepsilon} \Phi + \varepsilon \Phi^{\frac{d-1}{d}} \right) \quad (4.5.1)$$

and

$$\text{supp}(\mu_t) \subseteq \frac{\varepsilon-t}{\varepsilon} B_r(x_-) + \frac{t}{\varepsilon} B_r(x_+) \quad \text{for all } t \in [0, \varepsilon]. \quad (4.5.2)$$

*Proof.* The construction follows the lines of [48, Proposition 3.9]. The main difference is that in our case we have a bound on the diameter of the support of the measures.

**Step 1: Construction on  $[0, 2r]^d$**  Suppose first  $x_- = x_+$ . Up to translation, we may assume that  $x_{\pm} = 0$  and thus  $\text{supp } \mu^{\pm} \subseteq [0, 2r]^d$ . Since  $\mu^+(\mathbb{R}^d) = \mu^-(\mathbb{R}^d)$ , consider an optimal transport plan  $\bar{\pi}$  for  $W(\mu^-, \mu^+)$  and set

$$\int_{\mathbb{R}^d} \psi d\nu^t = \int_{\mathbb{R}^d \times \mathbb{R}^d} \psi \left( \frac{\varepsilon - t}{\varepsilon} x + \frac{t}{\varepsilon} y \right) d\bar{\pi}(x, y) \quad \text{for } \psi \in C^0(\mathbb{R}^d) \text{ and } t \in [0, \varepsilon].$$

We only perform the construction on  $[0, \varepsilon/2]$ . The construction on  $[\varepsilon/2, \varepsilon]$  is done analogously. Let  $\delta \in (1/4, 1/2)$  and for every  $k \geq 0$  let  $N_k = 2^k$ ,

$$t_k = \delta^k \frac{\varepsilon}{2} \quad \text{so that} \quad t_k - t_{k+1} = \delta^k (1 - \delta) \frac{\varepsilon}{2}. \quad (4.5.3)$$

We divide  $[0, 2r]^d$  into  $N_k^d$  cubes  $Q_{i,k}$  of side length  $l_k = 2rN_k^{-1}$ . We denote by  $X_{i,k}$  the center of  $Q_{i,k}$ . We discretize the displacement measure at time  $t_k$  by defining for  $k \in \mathbb{N}$

$$\nu_{t_k} = \sum_{i=0}^{N_k^d} \nu^{t_k}(Q_{i,k}) \delta_{X_{i,k}}.$$

To complete the construction, consider an optimal transport plan  $\pi^k$  for  $W(\nu_{t_k}, \nu_{t_{k+1}})$  and set

$$\int_{\mathbb{R}^d} \psi d\nu_t = \int_{\mathbb{R}^d} \psi \left( \frac{t - t_{k+1}}{t_k - t_{k+1}} x + \frac{t_k - t}{t_k - t_{k+1}} y \right) d\pi^k \quad \text{for } t \in [t_{k+1}, t_k].$$

Since  $\nu_{t_k}$  and  $\nu_{t_{k+1}}$  are discrete so is  $\nu_t$  for any  $t \in [t_{k+1}, t_k]$ . Thus, if we set  $M(t) = \mathcal{H}^0(\text{supp}(\nu_t))$ , there are  $t \mapsto \psi_i(t)$  and  $t \mapsto X_i(t)$  such that

$$\nu_t = \sum_{i=1}^{M(t)} \psi_i(t) \delta_{X_i(t)}.$$

As remarked in [48], the optimal transport problem between two sums of Dirac deltas is a linear programming problem. This means that the transport plan  $\pi^k$  can be chosen to be *sparse*, in the sense that we have for  $t \in [t_{k+1}, t_k]$

$$M(t) \leq M(t_k) + M(t_{k+1}).$$

By construction,  $M(t_k) = \mathcal{H}^0(\text{supp}(\nu_{t_k})) = 2^{dk}$ , thus for  $t \in [t_{k+1}, t_k]$

$$M(t) \lesssim 2^{dk}. \quad (4.5.4)$$

Finally let us show that  $t \mapsto \nu_t$  satisfies  $\nu_t \rightarrow \mu^-$  as  $t \rightarrow 0$ . If  $t \in (t_{k+1}, t_k)$ , then

$$W(\nu_t, \mu^-) \leq W(\nu_t, \nu_{t_k}) + W(\nu_{t_k}, \nu^{t_k}) + W(\nu^{t_k}, \mu^-).$$

By (4.2.4) and triangle inequality again

$$W(\nu_t, \nu_{t_k}) \leq W(\nu_{t_{k+1}}, \nu_{t_k}) \leq W(\nu_{t_{k+1}}, \nu^{t_{k+1}}) + W(\nu^{t_{k+1}}, \nu^{t_k}) + W(\nu^{t_k}, \nu_{t_k}).$$

Moreover, since we can construct a competitor for  $W(\nu^{t_k}, \nu_{t_k})$  by simply sending all the mass of  $\nu^{t_k}$  inside  $Q_{i,k}$  to  $X_{i,k}$ , we have

$$W(\nu^{t_k}, \nu_{t_k}) \lesssim l_k \sqrt{\Phi} = r2^{-k} \sqrt{\Phi}. \quad (4.5.5)$$

Using (4.2.4) again we thus obtain

$$W(\nu_t, \mu^-) \lesssim W(\nu_{t_{k+1}}, \nu^{t_{k+1}}) + W(\nu^{t_{k+1}}, \nu^{t_k}) + W(\nu^{t_k}, \nu_{t_k}) + W(\nu^{t_k}, \mu^-) \lesssim r2^{-k}\sqrt{\Phi} + \left(\frac{t_k - t_{k+1}}{\varepsilon} + \frac{t_k}{\varepsilon}\right) W(\mu^-, \mu^+)$$

and the conclusion follows letting  $k \rightarrow \infty$ .

**Step 2: Energy estimate for  $\nu$ .** Let us estimate the energy  $I(\nu)$ . Let us start with the perimeter. By (4.5.4) and Hölder's inequality we get for  $t \in [t_{k+1}, t_k]$

$$\dot{P}(\nu, t) = \sum_{i=1}^{M(t)} \varphi_i^{\frac{d-1}{d}}(t) \leq \left( \sum_{i=1}^{M(t)} \varphi_i(t) \right)^{\frac{d-1}{d}} M(t)^{\frac{1}{d}} \lesssim \Phi^{\frac{d-1}{d}} 2^k.$$

This implies

$$P(\nu, (t_{k+1}, t_k)) = \int_{t_{k+1}}^{t_k} \left( \sum_{i=1}^{M(t)} \varphi_i^{\frac{d-1}{d}}(t) \right) dt \lesssim \Phi^{\frac{d-1}{d}} 2^k (t_k - t_{k+1}).$$

Summing up all the terms and recalling (4.5.3) we can bound the perimeter by

$$P(\nu, (0, \varepsilon)) \lesssim \Phi^{\frac{d-1}{d}} \sum_{k \in \mathbb{N}} (t_k - t_{k+1}) 2^k \lesssim \Phi^{\frac{d-1}{d}} \varepsilon \sum_{k \in \mathbb{N}} (2\delta)^k \lesssim \Phi^{\frac{d-1}{d}} \varepsilon, \quad (4.5.6)$$

where in the last passage we used that  $\delta < \frac{1}{2}$ . Let us now focus on the kinetic energy. By the definition of  $\nu_t$ , triangle and Young's inequality we get

$$\begin{aligned} E_{kin}(\nu, (t_{k+1}, t_k)) &= \left( \frac{1}{t_k - t_{k+1}} \right) W^2(\nu_{t_{k+1}}, \nu_{t_k}) \\ &\leq \frac{1}{t_k - t_{k+1}} (W(\nu_{t_{k+1}}, \nu^{t_{k+1}}) + W(\nu^{t_{k+1}}, \nu^{t_k}) + W(\nu_{t_k}, \nu^{t_k}))^2 \\ &\lesssim \frac{1}{t_k - t_{k+1}} (W^2(\nu^{t_{k+1}}, \nu^{t_k}) + W^2(\nu_{t_{k+1}}, \nu^{t_{k+1}}) + W^2(\nu_{t_k}, \nu^{t_k})). \end{aligned}$$

By (4.2.4)

$$W^2(\nu^{t_{k+1}}, \nu^{t_k}) = \left( \frac{t_k - t_{k+1}}{\varepsilon} \right)^2 W^2(\mu^-, \mu^+) \leq \left( \frac{t_k - t_{k+1}}{\varepsilon} \right)^2 r^2 \Phi.$$

Combining this with (4.5.5) yields

$$E_{kin}(\nu, (t_{k+1}, t_k)) \lesssim r^2 \Phi \left( \frac{t_k - t_{k+1}}{\varepsilon^2} + \frac{4^{-k}}{t_k - t_{k+1}} \right).$$

This gives the total estimate

$$\begin{aligned} E_{kin}(\nu, (0, \varepsilon)) &\lesssim r^2 \Phi \left( \sum_{k \in \mathbb{N}} \frac{t_k - t_{k+1}}{\varepsilon^2} + \left( \sum_{k \in \mathbb{N}} \frac{1}{4^k (t_k - t_{k+1})} \right) \right) \\ &\lesssim \frac{r^2}{\varepsilon} \Phi \left( \sum_{k \in \mathbb{N}} \delta^k + \left( \sum_{k \in \mathbb{N}} (4\delta)^{-k} \right) \right) \lesssim \frac{r^2}{\varepsilon} \Phi, \quad (4.5.7) \end{aligned}$$

where in the last passage we use that  $\delta \in (1/4, 1/2)$ . Combining (4.5.7) with (4.5.6) we finally obtain the energy estimate

$$I(\nu, (0, \varepsilon)) \lesssim \frac{r^2}{\varepsilon} \Phi + \Phi^{\frac{d-1}{d}} \varepsilon. \quad (4.5.8)$$

**Step 3: General construction.** Let us now construct a competitor for generic  $x_{\pm}$ . For  $x_0 \in \mathbb{R}^d$ , let  $\tau_{x_0}(x) = x - x_0$ . Set  $h = x_+ - x_-$  and consider

$$\nu^- = \tau_h \# \mu^-.$$

We can apply Step 1 to  $\nu^-, \mu^+$  and obtain a competitor  $\nu \in \mathcal{A}^*(\mathbb{R}^d, (0, \varepsilon))$  such that  $\nu_0 = \nu^-$ ,  $\nu_{\varepsilon} = \mu^+$  and (4.5.1) holds. For  $t \in (0, \varepsilon)$ , we then set

$$\mu_t = \tau_{-\frac{\varepsilon-t}{\varepsilon}h} \# \nu_t.$$

Notice that  $\mu \in \mathcal{A}^*(\mathbb{R}^d, (0, \varepsilon))$ ,

$$\mu_0 = \tau_{-h} \# \nu_0 = \mu^- \quad \text{and} \quad \mu_{\varepsilon} = \tau_0 \# \nu_{\varepsilon} = \mu^+.$$

Moreover (see Figure 4.3),

$$\text{supp}(\mu_t) \subseteq \frac{\varepsilon-t}{\varepsilon} B_r(x_-) + \frac{t}{\varepsilon} B_r(x_+) \quad \text{for all } t \in [0, \varepsilon].$$

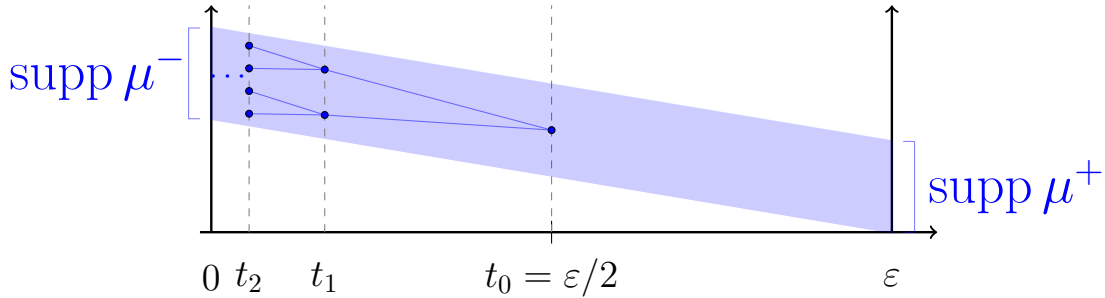


Figure 4.3: Illustration of the construction on one single branch for  $d = 1$ . The support of  $\mu_t$  is contained in the blue domain.

Let us estimate the energy of  $\mu$ . Clearly

$$P(\mu, (0, \varepsilon)) = P(\nu, (0, \varepsilon)). \quad (4.5.9)$$

Let us now compute the kinetic energy. If  $\mu_t = \sum_i \varphi_i(t) \delta_{\tilde{X}_i(t)}$  and  $\nu_t = \sum_i \varphi_i(t) \delta_{X_i(t)}$ , then by construction

$$\tilde{X}_i(t) = X_i(t) - \frac{\varepsilon-t}{\varepsilon} h.$$

This implies

$$\left| \dot{\tilde{X}}_i(t) \right|^2 = \left| \dot{X}_i(t) \right|^2 + \frac{|h|^2}{\varepsilon^2} + \frac{2}{\varepsilon} h \cdot \dot{X}_i(t),$$

which gives

$$E_{kin}(\mu, (0, \varepsilon)) = E_{kin}(\nu, (0, \varepsilon)) + \frac{|h|^2}{\varepsilon} \Phi + \frac{2}{\varepsilon} h \cdot \int_0^{\varepsilon} \sum_i \varphi_i \dot{X}_i(t) dt. \quad (4.5.10)$$

Take  $\pi$  to be an optimal transport plan for  $W(\mu^-, \mu^+)$  and let  $F(x, y) = (x - h, y)$ . Then,  $\pi$  can be chosen such that

$$\pi = F\#\bar{\pi}$$

where  $\bar{\pi}$  is the optimal transport plan for  $W(\nu^-, \mu^+)$ . This implies that

$$\begin{aligned} W^2(\mu^-, \mu^+) &= \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 d\pi = \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - h - y|^2 d\bar{\pi} \\ &= W^2(\nu^-, \mu^+) + |h|^2 \Phi + 2h \cdot \int_{\mathbb{R}^d \times \mathbb{R}^d} (y - x) d\bar{\pi}. \end{aligned} \quad (4.5.11)$$

We claim that

$$\int_0^\varepsilon \sum_i \varphi_i \dot{X}_i dt = \int_{\mathbb{R}^d \times \mathbb{R}^d} (y - x) d\bar{\pi}. \quad (4.5.12)$$

Then by (4.5.9), (4.5.10), (4.5.11) and (4.5.12), we would get

$$I(\mu) = I(\nu) + \frac{1}{\varepsilon} (W^2(\mu^-, \mu^+) - W^2(\nu^-, \mu^+)) \leq I(\nu) + \frac{1}{\varepsilon} W^2(\mu^-, \mu^+).$$

Combining this with (4.5.8) would conclude the proof of (4.5.1). We thus prove (4.5.12). The computation is similar to [67, Lemma 3.1]. By definition,

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} (y - x) d\bar{\pi} = \int_{\mathbb{R}^d} y d\mu^+ - \int_{\mathbb{R}^d} x d\nu^-.$$

Let now  $\delta < \frac{\varepsilon}{2}$  and split

$$\int_0^\varepsilon \sum_i \varphi_i \dot{X}_i dt = \int_0^\delta \sum_i \varphi_i \dot{X}_i dt + \int_\delta^{\varepsilon-\delta} \sum_i \varphi_i \dot{X}_i dt + \int_{\varepsilon-\delta}^\varepsilon \sum_i \varphi_i \dot{X}_i dt. \quad (4.5.13)$$

In  $[\delta, \varepsilon - \delta]$ ,  $\nu$  has by construction a finite number of branches, hence applying [37, Lemma 5.9], see also [67, Lemma 2.6], we find

$$\int_\delta^{\varepsilon-\delta} \sum_i \varphi_i \dot{X}_i(t) dt = \sum_i \varphi_i(\varepsilon - \delta) X_i(\varepsilon - \delta) - \sum_i \varphi_i(\delta) X_i(\delta) = \int_{\mathbb{R}^d} y d\mu_{\varepsilon-\delta} - \int_{\mathbb{R}^d} x d\mu_\delta.$$

Sending  $\delta \rightarrow 0$ , we find

$$\lim_{\delta \rightarrow 0} \int_\delta^{\varepsilon-\delta} \sum_i \varphi_i \dot{X}_i(t) dt = \int_{\mathbb{R}^d} y d\mu^+ - \int_{\mathbb{R}^d} x d\nu^-.$$

To conclude, let us show that the other terms in (4.5.13) are vanishing when  $\delta \rightarrow 0$ . By Hölder inequality applied twice and since  $\sum_i \varphi_i = 1$

$$\int_0^\delta \sum_i \varphi_i \dot{X}_i dt \leq \int_0^\delta \left( \sum_i \varphi_i |\dot{X}_i|^2 \right)^{\frac{1}{2}} dt \leq \delta^{\frac{1}{2}} \left( \int_0^\delta \sum_i \varphi_i |\dot{X}_i|^2 dt \right)^{\frac{1}{2}} \xrightarrow{\delta \rightarrow 0} 0.$$

Arguing similarly on the interval  $[\varepsilon - \delta, \varepsilon]$  concludes the proof.  $\square$

### 4.5.2 Local scaling laws for concentrated boundary measures $\mu_T$

In this section we fix  $d = 1$ .

*Proof of Theorem 4.1.6.* Let  $r \in (0, 1)$ . We start by covering the support of  $\mu_T$  with pairwise disjoint intervals of radius proportional to  $r$ . Using the uniform lower bound (4.3.1), we have that the number of these intervals is at most of the order of  $r^{-\alpha}$ . Next, by the cone property, at  $t = T - \varepsilon$  the set of points that irrigate one of these intervals can be decomposed into at most three intervals of length proportional to  $r$ . Having the decomposition at  $t = T - \varepsilon$  and  $t = T$  we construct the competitor with respect to each of the intervals on the left boundary by applying Lemma 4.5.1.

**Step 1. Decomposition of left and right boundaries.** Since  $\mu_T$  is lower  $\alpha$ -Ahlfors regular, for every  $x \in \text{supp}(\mu_T)$  and  $r \in (0, 1)$  we have  $\mu_T(B_r(x)) \gtrsim_M r^\alpha$ . Let us fix  $r \in (0, 1)$ . By Vitali's covering theorem [51, Theorem 1.24], there exists  $x_i \in \text{supp}(\mu_T)$  for  $i = 1, \dots, N$  such that  $B_r(x_i) \cap B_r(x_j) = \emptyset$  for  $i \neq j$  and  $\text{supp}(\mu_T) \subset \cup_{i=1}^N B_{5r}(x_i)$ . By hypothesis we have

$$1 = \mu_T(\mathbb{T}) \geq \mu_T(\cup_{i=1}^N B_r(x_i)) = \sum_{i=1}^N \mu_T(B_r(x_i)) \gtrsim_M Nr^\alpha. \quad (4.5.14)$$

We can increase the radius of the balls  $B_r(x_i)$  until they either touch or reach the radius  $5r$ . Rigorously, consider the collection  $(\bar{B}_{r_i}(x_i))_i$  with  $r_i$  defined as follows:

$$r_1 = \min\{\sup\{s : B_s(x_1) \cap B_r(x_j) = \emptyset, j = 1 \dots N\}, 5r\}$$

and for  $i > 1$

$$r_i = \min\{\sup\{s : B_s(x_i) \cap B_{r_j}(x_j) = \emptyset, j = 1 \dots i-1, B_s(x_i) \cap B_r(x_j) = \emptyset, j = i \dots N\}, 5r\}.$$

We thus find a collection of at most  $N$  disjoint intervals  $J_i(T)$  of length at most  $10r$  that cover  $\text{supp}(\mu_T)$ . Consider the parametric representation  $X(x, t)$  of  $\mu$  given by Proposition 4.2.5. By Corollary 4.2.12, for each of the intervals  $J_i$  and for all  $t \in [T - \varepsilon, T]$  there exist at most three intervals  $J_i^j(t)$  such that for  $\mu_T$ -a.e.  $x \in J_i$ ,  $X(x, t) \in J_i^j(t)$  for some  $j = 1, 2, 3$ , see Figure 4.4. Moreover, the three intervals satisfy  $|J_i^j(t)| \leq |J_i| \leq 10r$  for  $j = 1, 2, 3$ .

**Step 2. Construction on  $J_i^j$**

We introduce the restricted measure  $\mu_T^{i,j}$  by setting

$$\mu_T^{i,j} = \mu_T \llcorner J_i^j(T) \quad \text{for } i = 1, \dots, N \text{ and } j = 1, 2, 3.$$

Let us define

$$\mu_{T-\varepsilon}^{i,j} = X(\cdot, T - \varepsilon) \# \mu_T^{i,j} \quad \text{for } i = 1, \dots, N \text{ and } j = 1, 2, 3.$$

In the language of Proposition 4.2.2, this is the backward subsystem starting from  $\mu_T^{i,j}$ . Since the supports of  $\mu_T^{i,j}$  are disjoint,

$$\sum_{i,j} \mu_{T-\varepsilon}^{i,j} = \mu_{T-\varepsilon}.$$

Moreover, by the decomposition in Step 1, we have

$$\text{supp}(\mu_{T-\varepsilon}^{i,j}) \subseteq J_i^j(T - \varepsilon).$$

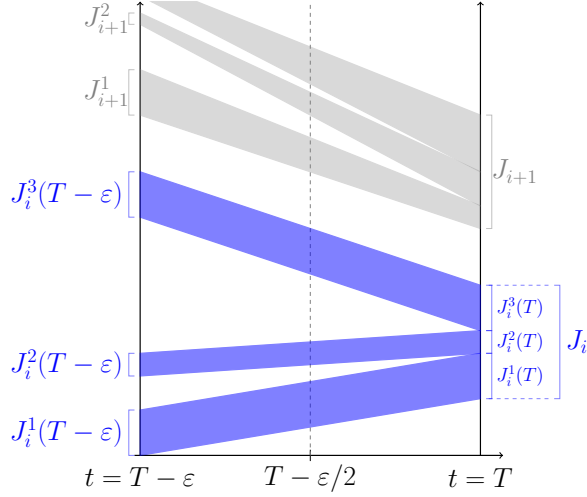


Figure 4.4: Sketch of the described decomposition with respect to  $J_i$ . The part of the minimizer which irrigates  $J_i$  is contained in the blue domain.

Hence, for  $t \in \{T - \epsilon, T\}$

$$|\text{supp}(\mu_t^{i,j})| \leq |J_i^j(t)| \lesssim r.$$

Finally, notice that since by Lemma 4.2.7,  $X(\cdot, T - \epsilon)$  is monotone (see also Remark 4.2.10), it is an optimal transport map for both  $W^2(\mu_T, \mu_{T-\epsilon})$  and  $W^2(\mu_T^{i,j}, \mu_{T-\epsilon}^{i,j})$  so that

$$\sum_{i,j} W^2(\mu_{T-\epsilon}^{i,j}, \mu_T^{i,j}) = W^2(\mu_{T-\epsilon}, \mu_T). \quad (4.5.15)$$

We now apply the previous lemma within the cones, i.e. the blue marked domains in Figure 4.5.

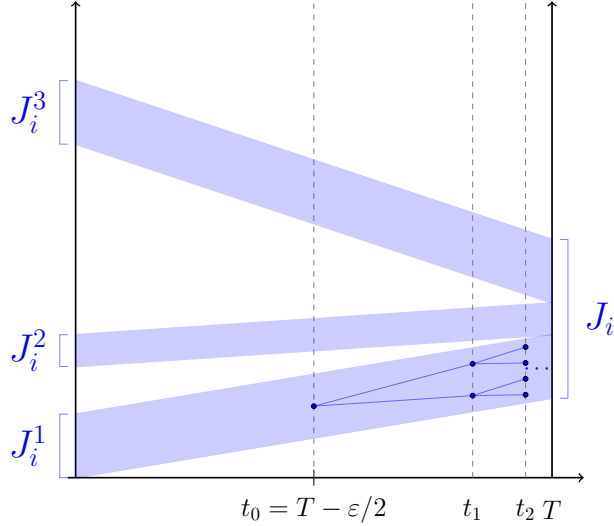


Figure 4.5: Sketch of the local construction within the cones.

By Lemma 4.5.1, for every  $i, j$ , there is an admissible construction  $\nu^{i,j} \in \mathcal{A}^*(\mathbb{T}, (T - \epsilon, T))$  such that  $\nu_T^{i,j} = \mu_T^{i,j}$ ,  $\nu_{T-\epsilon}^{i,j} = \mu_{T-\epsilon}^{i,j}$  and

$$I(\nu^{i,j}, (T - \epsilon, T)) \leq \frac{1}{\epsilon} W^2(\mu_{T-\epsilon}^{i,j}, \mu_T^{i,j}) + C \left( \frac{r^2}{\epsilon} \mu_T(J_i^j(T)) + \epsilon \right). \quad (4.5.16)$$

Notice that by (4.5.2), the supports of  $\nu^{i,j}$  are pairwise disjoint. We finally set  $\nu = \sum_{i,j} \nu^{i,j}$ . We have  $\nu \in \mathcal{A}^*(\mathbb{T}, (T - \varepsilon, T))$ ,  $\nu_T = \mu_T$  and  $\nu_{T-\varepsilon} = \mu_{T-\varepsilon}$ . The minimality of  $\mu$  together with estimates (4.5.16), (4.5.15) and subadditivity of the energy imply that for some  $C = C(M) > 0$ ,

$$\begin{aligned} I(\mu, (T - \varepsilon, T)) &\leq I(\nu, (T - \varepsilon, T)) \leq \sum_{i,j} I(\nu^{i,j}, (T - \varepsilon, T)) \\ &\leq \frac{1}{\varepsilon} W^2(\mu_{T-\varepsilon}, \mu_T) + C \left( \frac{r^2}{\varepsilon} \sum_{i,j} \mu_T(J_i^j(T)) + N\varepsilon \right) \stackrel{(4.5.14)}{\leq} \frac{1}{\varepsilon} W^2(\mu_{T-\varepsilon}, \mu_T) + C \left( \frac{r^2}{\varepsilon} + r^{-\alpha} \varepsilon \right). \end{aligned}$$

**Step 3. Equipartition of energy and conclusion .** By the previous step,

$$I(\mu, (T - \varepsilon, T)) \leq \frac{1}{\varepsilon} W^2(\mu_{T-\varepsilon}, \mu_T) + C \left( \frac{r^2}{\varepsilon} + r^{-\alpha} \varepsilon \right).$$

Choosing  $r = \varepsilon^{2/(2+\alpha)} \in (0, 1)$  to optimize the right-hand side, we obtain

$$I(\mu, (T - \varepsilon, T)) \leq \frac{1}{\varepsilon} W^2(\mu_{T-\varepsilon}, \mu_T) + C \varepsilon^{\frac{2-\alpha}{2+\alpha}}.$$

Since (4.2.5) holds, we obtain

$$P(\mu, (T - \varepsilon, T)) \lesssim_M \varepsilon^{\frac{2-\alpha}{2+\alpha}}.$$

Finally, by the equipartition of energy, see Proposition 4.2.6, there is a constant  $\Lambda$  with  $\Lambda \leq I(\mu, (0, \varepsilon))/T$  such that

$$P(\mu, (T - \varepsilon, T)) = E_{kin}(\mu, (T - \varepsilon, T)) + \Lambda\varepsilon,$$

from which the claimed (4.1.9) follows. □

## Chapter 5

# Concentration phenomena in a branched transport problem in the half space

### 5.1 Introduction

The goal of this chapter is to study an energy similar to (1.4.26), in the idealized case of a superconducting sample which is the half space  $\mathbb{R}^2 \times (0, +\infty)$ . We represent the magnetization by a Radon measure  $\mu = \mu_t \otimes dt$  where, for a.e.  $t \in (0, +\infty)$ ,  $\mu_t = \sum_i \varphi_i \delta_{X_i}$ , with  $X_i \in \mathbb{R}^2$ ,  $\varphi_i > 0$  and  $\sum_i \varphi_i = \Phi > 0$ . We consider the internal energy (see Section 5.2.3 for the precise definition)

$$I(\mu) = \int_0^{+\infty} \left( \sum_i \varphi_i^{1/2}(t) \right) - \Phi^{1/2} + \sum_i \varphi_i(t) |\dot{X}_i(t)|^2 dt, \quad (5.1.1)$$

where  $\dot{X}_i(t)$  is the derivative of the map  $t \mapsto X_i(t) \in \mathbb{R}^2$ . Energy (5.1.1) may be interpreted as a "blow-up" of energy (1.4.26). Similarly to (1.4.26), the first term in (5.1.1) can be seen as a perimeter term that favors mass traveling together. Since we deal with an unbounded domain, we renormalize the energy by the perimeter of a Dirac delta with the total mass of  $\mu_t$ , which is  $\Phi^{1/2}$ . The last term in (5.1.1) is the kinetic energy, exactly like (1.4.26). We impose a weak boundary penalization with a negative Sobolev norm (see Section 5.2.2) and consider the energy

$$\mathcal{E}(\mu) = I(\mu) + \|\mu_0\|_{H^{-1/2}(\mathbb{R}^2)}^2. \quad (5.1.2)$$

Since the energy (5.1.2) is translation invariant, we may assume  $\int_{\mathbb{R}^2} x d\mu_0 = 0$ . Moreover, we fix the mass  $\mu_0(\mathbb{R}^2) = \Phi = 1$  and consider the minimization problem

$$e = \inf \left\{ \mathcal{E}(\mu) : \int_{\mathbb{R}^2} x d\mu_0 = 0, \mu_0(\mathbb{R}^2) = 1 \right\}. \quad (5.1.3)$$

Problem (5.1.3) is an analogue to problem (1.4.27) in an unbounded domain. The internal energy  $I(\mu)$  favors measures which are concentrated, while the boundary term  $\|\mu_0\|_{H^{-1/2}(\mathbb{R}^2)}$  favors a diffused boundary measure. Based on the same considerations as in [38], we may thus formulate the following analogue of Conjecture 1.4.2 (See Chapter 4 for the definition of dimension of a measure).

**Conjecture 5.1.1.** *Let  $\mu$  be a minimizer of (5.1.3). Then,*

$$\dim \mu_0 = \frac{8}{5}.$$

For the bounded problem (1.4.27) the above conjecture was linked to local scaling laws of the internal energy in [48], and to Ahlfors regularity in Chapter 4 for an equivalent 2D model. In this chapter, we prove a result similar to the concentration result (4.1.5) in Theorem 4.1.1, in the case of energy (5.1.2), which is a 3D model (in the simplified setting of the half space). We recall that a measure  $\sigma \in \mathcal{M}^+(\mathbb{R}^2)$  is called upper  $\alpha$ -Ahlfors regular for  $\alpha \in [0, 2]$  if there exists  $M > 0$  such that

$$\sigma(B(x, r)) \leq Mr^\alpha \quad \text{for all } x \in \text{supp } \sigma \text{ and } r \in (0, 1].$$

Our main result is the following.

**Theorem 5.1.2.** *Let  $\mu$  be a minimizer of (5.1.3) as provided by Theorem 5.3.15 below. Suppose that there exist  $x_0 \in \mathbb{R}^2$  and  $r_0 > 0$  such that  $\mu_0(B(x_0, r_0/2)) > 0$  and  $\mu_0 \llcorner B(x_0, r_0)$  is upper  $\alpha$ -Ahlfors regular for some  $\alpha > 1, M > 0$ . Then  $\alpha \leq 8/5$ .*

**Remark 5.1.3.** *An immediate consequence of Theorem 5.1.2 is that  $\mu_0$  cannot be upper  $\alpha$ -Ahlfors regular for  $\alpha > 8/5$ . In particular,  $\mu_0 \notin L^\infty(\mathbb{R}^2)$ .*

Theorem 5.1.2 states, roughly speaking, that in any ball in which some mass of the boundary measure is present, some part of such measure needs to concentrate up to the conjectured dimension  $8/5$ . If we compare with (4.1.5) in Theorem 4.1.1, our result excludes that there exist large portions of the support of  $\mu_0$  in which  $\mu_0$  is diffused, which a priori cannot be excluded in the case of Theorem 4.1.1.

We now comment on the strategy of the proof. While the setting of the half space offers some advantages with respect to the periodic setting considered in [37, 48, 41], namely an easier understanding of the weak boundary norm, this advantage comes with a price: loss of compactness. We thus cannot directly generalize the proof of [37] to obtain existence of minimizers. To overcome this problem, we proceed as follows. First, we fix  $R, T > 0$  and consider the following minimization problem in a compact setting:

$$e(R, T) = \inf \left\{ \mathcal{E}(\mu) : \int_{\mathbb{R}^2} x d\mu_0 = 0, \text{supp } \mu_0 \subset [-R/2, R/2]^2, \mu_T = \delta_0 \right\}. \quad (5.1.4)$$

For (5.1.4), existence is easily adapted from [37]. Then, with the help of a Lagrangian reformulation of the problem (Proposition 5.3.8), we are able to prove a priori bounds (Lemma 5.3.12) for a minimizer of (5.1.4) which do not depend on  $R$  and  $T$ . This enables us to prove that there exists  $R_0, T_0 > 0$  such that a minimum of (5.1.4) is a minimum of (5.1.3) (Lemma 5.3.15). We thus show that (5.1.3) admits at least one minimum  $\mu$  which is compact in space and for which  $\mu_t = \delta_0$  for all  $t \geq T_0$ .

With the existence of a minimizer at hand, we now comment on the proof of Theorem 5.1.2. We consider a ball  $B(x_0, r_0/2)$  of positive mass on which the boundary measure is upper  $\alpha$ -Ahlfors regular. We focus on a subsystem (Proposition 5.2.3) contained in the ball  $B(x_0, r_0/2)$ , which can be found by the a priori bounds in Lemma 5.3.12. Up to dealing with the barycenter contribution of the energy (Lemma 5.3.1), we construct a competitor similar to the one used in the proof of (4.1.5) in Theorem 4.1.1: we shrink the subsystem around its barycenter (compare in particular with the variation argument in Theorem 4.1.5). This keeps the perimeter unchanged and reduces the kinetic energy. For the boundary term we make use of the following crucial observation: if a measure  $\sigma \in \mathcal{M}^+(\mathbb{R}^2)$  is upper  $\alpha$ -Ahlfors regular for  $\alpha \in [1, 2]$ , then its electrostatic potential  $u : \mathbb{R}^2 \rightarrow \mathbb{R}$ , defined as

$$u(x) = \int_{\mathbb{R}^2} \frac{1}{|x - y|} d\mu(y),$$

must be  $(\alpha - 1)$ -Hölder continuous (Proposition 5.4.2). Such a result may be interpreted as a sort of Schauder estimate, see Remark 5.4.3. The regularity of the potential is then used in arguing that the local structure of the minimizer is not much affected by the non-local interactions of the boundary penalization. Finally, the contradiction with  $\alpha > 8/5$  arises by comparing the mass of the subsystem, which is arbitrarily small, and the first moment of the local boundary measure.

The rest of the chapter is structured as follows: In Section 5.2 we set notation, recall some notions from optimal transport theory and Sobolev norms and define the energy functional. Section 5.3 is dedicated to the existence of minimizers. We start by dealing with shifts and constructions for the energy  $\mathcal{E}$  in Section 5.3.1. Then we introduce the problems (5.1.4) in Section 5.3.2, for which we prove a priori bounds in Section 5.3.3 thanks to a Lagrangian reformulation of the problem. We later prove existence for problem (5.1.3) in Section 5.3.4, and characterize some properties of subsystems of minimizers in Section 5.3.5. Finally, in Section 5.4.1 we prove Hölder continuity of the potential  $u$  and in Section 5.4.2 we prove the concentration result.

## 5.2 Preliminaries

### 5.2.1 Notation

We use the symbols  $\lesssim$  and  $\gtrsim$  to indicate that there exists  $C > 0$  such that  $a \leq Cb$  or  $a \geq Cb$  respectively. Additionally, we denote by  $a \sim b$  a relation that satisfies  $a \lesssim b$  and  $a \gtrsim b$ . We write  $a \lesssim_p b$  to indicate  $a \leq Cb$  for some constant  $C = C(p) > 0$ , and we do similarly for  $a \gtrsim_p b$  and  $a \sim_p b$ . We use  $A \ll B$  as an hypothesis. It means that there exists an  $\varepsilon > 0$  such that for  $A \leq \varepsilon B$  the conclusion holds. We write  $B(x, r)$  for a ball centered in  $x \in \mathbb{R}^2$  of radius  $r > 0$ . We denote by  $\mathcal{M}(\mathbb{R}^2)$  the set of finite measures on  $\mathbb{R}^2$  (respectively  $\mathcal{M}(\mathbb{R}^2 \times \mathbb{R}^2)$  for measures on  $\mathbb{R}^2 \times \mathbb{R}^2$ ), by  $\mathcal{M}^+(\mathbb{R}^2)$  the set of positive measures, and by  $\mathcal{P}(\mathbb{R}^2)$  the set of probability measures on  $\mathbb{R}^2$ . For a compact set  $K \subset \mathbb{R}$ , we denote by  $\mathcal{M}(\mathbb{R}^2 \times K)$  the set of finite measures on  $\mathbb{R}^2 \times K$  and by  $\mathcal{M}^+(\mathbb{R}^2 \times K)$  the set of positive measures on  $\mathbb{R}^2 \times K$ . For  $0 \leq a < b \leq \infty$ , we denote by  $\mathcal{M}_{\text{loc}}(\mathbb{R}^2 \times (a, b))$  (respectively  $\mathcal{M}_{\text{loc}}^+(\mathbb{R}^2 \times (a, b))$ ) the set of Radon measures  $\mu$  such that for any compact set  $K \subset (a, b)$  one has  $\mu \in \mathcal{M}(\mathbb{R}^2 \times K)$  (respectively  $\mu \in \mathcal{M}^+(\mathbb{R}^2 \times K)$ ). We denote by  $C_0(\mathbb{R}^2)$  the set of continuous functions on  $\mathbb{R}^2$  vanishing at infinity. If  $f$  is a function and  $\mu \in \mathcal{M}^+(\mathbb{R}^2)$  a measure, we denote  $f\#\mu$  the pushforward of  $\mu$  by  $f$ . We denote by  $\mathcal{H}^1$  the 1-Hausdorff measure. For a measure  $\mu \in \mathcal{M}(\mathbb{R}^2)$  and a measurable set  $A$ , we denote by  $\mu \llcorner A$  the measure such that  $\mu \llcorner A(B) = \mu(A \cap B)$  for all measurable sets  $B \subset \mathbb{R}^2$ .

### 5.2.2 Optimal transport and Sobolev norms for measures

We recall some basic facts and definitions about optimal transport and negative Sobolev norms.

We refer to [4, 101] for a broader overview of optimal transport. If  $\mu_1, \mu_2 \in \mathcal{M}^+(\mathbb{R}^2)$  with  $\mu_1(\mathbb{R}^2) = \mu_2(\mathbb{R}^2)$ , we define

$$W^2(\mu_1, \mu_2) = \min \left\{ \int_{\mathbb{R}^2 \times \mathbb{R}^2} |x - y|^2 d\pi : \pi \in \mathcal{M}^+(\mathbb{R}^2 \times \mathbb{R}^2), \pi_i = \mu_i \text{ for } i = 1, 2 \right\}.$$

Let us recall the Benamou-Brenier formula [8]: for  $a, b \in \mathbb{R}$  with  $a < b$  we have

$$\frac{1}{b-a} W^2(\mu_1, \mu_2) = \min_{(\mu, m)} \left\{ \int_{\mathbb{R}^2 \times (a, b)} \left| \frac{dm}{d\mu} \right|^2 d\mu dt : m \ll \mu, \partial_t \mu + \nabla \cdot m = 0, \mu_a = \mu_1, \mu_b = \mu_2 \right\}. \quad (5.2.1)$$

For  $\mu \in \mathcal{M}^+(\mathbb{R}^2)$ , we define the weak Sobolev norm (see [5, 107])

$$\|\mu\|_{H^{-1/2}(\mathbb{R}^2)}^2 = \int_{\mathbb{R}^2} |\xi|^{-1} |\mathcal{F}(\mu)(\xi)|^2 d\xi \quad (5.2.2)$$

where for  $\xi \in \mathbb{R}^2$

$$\mathcal{F}(\mu)(\xi) = \int_{\mathbb{R}^2} e^{-2\pi i \xi \cdot x} d\mu(x).$$

We make use of the following characterization of (5.2.2). For any  $\mu \in \mathcal{M}^+(\mathbb{R}^2)$ , we define its electrostatic potential  $u : \mathbb{R}^2 \rightarrow \mathbb{R}$  as

$$u(x) = \int_{\mathbb{R}^2} \frac{1}{|x-y|} d\mu(y). \quad (5.2.3)$$

It is known (see for instance [49, 75, 74]) that there exists  $C > 0$  such that

$$\int_{\mathbb{R}^2} u(x) d\mu(x) = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{1}{|x-y|} d\mu(x) d\mu(y) = C \|\mu\|_{H^{-1/2}(\mathbb{R}^2)}^2. \quad (5.2.4)$$

### 5.2.3 The energy functional

**Definition 5.2.1.** We denote by  $\mathcal{A}(\mathbb{R}^2, (a, b))$  for  $0 \leq a < b \leq +\infty$  the set of measures  $\mu \in \mathcal{M}_{\text{loc}}^+(\mathbb{R}^2 \times (a, b))$ ,  $m \in \mathcal{M}_{\text{loc}}(\mathbb{R}^2 \times (a, b); \mathbb{R}^2)$ , with  $m \ll \mu$  satisfying the continuity equation

$$\int_{\mathbb{R}^2 \times (a, b)} \partial_t \phi(x, t) d\mu + \int_{\mathbb{R}^2 \times (a, b)} \nabla_x \phi(x, t) \cdot dm = 0 \quad \text{for any } \phi \in C_c^\infty(\mathbb{R}^2 \times (a, b)), \quad (5.2.5)$$

and such that  $\mu = \mu_t \otimes dt$  and  $m = m_t \otimes dt$  where, for almost every  $t \in (0, +\infty)$  there holds  $\mu_t = \sum_i \varphi_i(t) \delta_{X_i(t)}$ , for some  $t \mapsto \varphi_i(t) > 0$  and  $t \mapsto X_i(t) \in \mathbb{R}^2$ . We denote by  $\mathcal{A}^*(\mathbb{R}^2, (a, b)) = \{\mu : \exists m, (\mu, m) \in \mathcal{A}(\mathbb{R}^2, (a, b))\}$  the set of admissible  $\mu$ . We use the notation  $\mathcal{A} = \mathcal{A}(\mathbb{R}^2, (0, +\infty))$  and  $\mathcal{A}^* = \mathcal{A}^*(\mathbb{R}^2, (0, +\infty))$ . Notice that by (5.2.5),  $\mu_t(\mathbb{R}^2)$  is independent of  $t$ , and we denote it by  $\Phi$ . We define the functional  $I : \mathcal{A}(\mathbb{R}^2, (a, b)) \rightarrow [0, +\infty]$  by

$$I(\mu, m, (a, b)) = \int_a^b \left( \sum_{x' \in \mathbb{R}^2} (\mu_t(\{x'\}))^{\frac{1}{2}} \right) - \Phi^{\frac{1}{2}} dt + \int_{\mathbb{R}^2 \times (a, b)} \left| \frac{dm}{d\mu} \right|^2 d\mu. \quad (5.2.6)$$

We also define (with abuse of notation) the functional  $I : \mathcal{A}^*(\mathbb{R}^2, (a, b)) \rightarrow [0, +\infty]$

$$I(\mu, (a, b)) = \min\{I(\mu, m, (a, b)) : (\mu, m) \in \mathcal{A}\}. \quad (5.2.7)$$

We use the notation  $I(\mu) = I(\mu, (0, +\infty))$  and  $I(\mu, m) = I(\mu, m, (0, +\infty))$ . We then set for  $0 \leq a < t < b < +\infty$

$$\dot{P}(\mu, t) = \left( \sum_{x \in \mathbb{R}^2} (\mu_t(x)) \right)^{\frac{1}{2}} - \Phi^{\frac{1}{2}}, \quad \dot{E}(\mu, t) = \int_{\mathbb{R}^2} \left| \frac{dm_t}{d\mu_t} \right|^2 d\mu_t$$

and for  $0 \leq a < t < b \leq +\infty$

$$P(\mu, (a, b)) = \int_a^b \dot{P}(\mu, t) dt, \quad E(\mu, (a, b)) = \int_a^b \dot{E}_{\text{kin}}(\mu, t) dt,$$

so that  $I(\mu, (a, b)) = P(\mu, (a, b)) + E(\mu, (a, b))$ . We also use the notation  $P(\mu) = P(\mu, (0, +\infty))$  and  $E(\mu) = E(\mu, (0, +\infty))$ .

As in [48, 41], for  $0 \leq a < b < +\infty$  and  $\mu \in \mathcal{A}^*$ , by Benamou-Brenier formula (5.2.1) we have

$$E(\mu, (a, b)) \geq \frac{1}{b-a} W_2^2(\mu_a, \mu_b). \quad (5.2.8)$$

By (5.2.8), if  $I(\mu) < \infty$  then for  $0 < a < b < +\infty$

$$W_2(\mu_a, \mu_b) \leq I(\mu)^{1/2} (b-a)^{1/2} \lesssim_\mu (b-a)^{1/2}.$$

This means that  $t \rightarrow \mu_t$  is a 1/2-Hölder continuous curve in the Wasserstein space, hence in particular  $\mu_0$  is well defined.

**Remark 5.2.2.** *By a regularization argument (see [4, Remark 8.1.1]), it is sufficient to satisfy (5.2.5) for test functions  $\phi \in C^1(\mathbb{R}^2 \times (a, b))$ , bounded, with bounded gradient and whose support has compact projection on  $(a, b)$  (in particular the support may not be compact in  $x$ ).*

Let us recall the notion of subsystems (see [48, Proposition 2.6] and [37, Proposition 5.7]).

**Proposition 5.2.3** (Subsystems). *Let  $t \in [0, +\infty)$  and  $(\mu, m) \in \mathcal{A}$  with  $I(\mu, m) < \infty$ . Set  $v = dm/d\mu$ . Then for every measure  $\sigma \leq \mu_t$  there exists a measure  $\tilde{\mu} \in \mathcal{A}$  such that*

1.  $\mu' = \mu - \tilde{\mu} \in \mathcal{M}_{\text{loc}}^+(\mathbb{R}^2 \times (0, +\infty))$ ;
2.  $\tilde{\mu}_t = \sigma$ ;
3.  $(\tilde{\mu}, v\tilde{\mu})$  satisfy the continuity equation (5.2.5).

*In particular  $(\tilde{\mu}, v\tilde{\mu}) \in \mathcal{A}$  and  $I(\tilde{\mu}, v\tilde{\mu}) \leq I(\mu, m)$ . Similarly  $(\mu', v\mu') \in \mathcal{A}$  and  $I(\mu', v\mu') \leq I(\mu, m)$ . If  $\mu_t = \sum_i \varphi_i \delta_{X_i}$  and  $\sigma = \varphi_i \delta_{X_i}$  for some  $i$ , we call  $\mu_+^{t,i} = \tilde{\mu} \llcorner (\mathbb{R}^2 \times (t, +\infty))$  the forward subsystem emanating from  $X_i$  and  $\mu_-^{t,i} = \tilde{\mu} \llcorner (\mathbb{R}^2 \times (0, t))$  the backward subsystem emanating from  $X_i$ .*

For  $\mu \in \mathcal{A}^*$ , we are interested in minimizing the energy

$$\mathcal{E}(\mu) = I(\mu) + \|\mu_0\|_{H^{-1/2}(\mathbb{R}^2)}^2$$

with mass  $\mu_0(\mathbb{R}^2) = 1$ . We thus consider the minimization problem

$$e = \inf \left\{ \mathcal{E}(\mu) : \mu \in \mathcal{A}^*, \int_{\mathbb{R}^2} x d\mu_0 = 0, \mu_0(\mathbb{R}^2) = 1 \right\}. \quad (5.2.9)$$

### 5.3 Existence and properties of minimizers

In this section, we prove existence of minimizers for (5.2.9) and several properties of such minimizers. In particular we prove that (5.2.9) admits a minimizer which is compact in space and such that there exists some  $T_0 > 0$  for which  $\mu_t = \delta_0$  for  $t \geq T_0$ . In this way, we inherit most of the properties which were developed in [37] for the same problem in a bounded domain.

### 5.3.1 Barycenter shift and constructions

In this section we prove that, for the energy (5.2.6), we may isolate the kinetic contribution of the barycenter. We later recall a construction from [48].

**Lemma 5.3.1.** *Let  $\mu \in \mathcal{A}^*$ . Assume that  $I(\mu) < +\infty$  and that  $t \rightarrow \int_{\mathbb{R}^2} |x| d\mu_t \in L^1_{\text{loc}}(0, +\infty)$ . For  $t \geq 0$  let  $X(t) = \frac{1}{\Phi} \int_{\mathbb{R}^2} x d\mu_t$  and define  $\nu_t = (x - X(t))\#\mu_t$ . Then,*

$$I(\mu) = I(\nu) + \Phi \int_0^\infty |\dot{X}(t)|^2 dt.$$

*Proof.* Let us start by noticing that the perimeter part does not change:  $P(\mu) = P(\nu)$ . By a global shift we may also assume without loss of generality that  $\int_{\mathbb{R}^2} x d\mu_0 = 0$ . Let  $m \in \mathcal{M}_{\text{loc}}(\mathbb{R}^2 \times (0, +\infty); \mathbb{R}^2)$  be a minimum for (5.2.7). In particular  $(\mu, m)$  solves the continuity equation (5.2.5) and  $m = m_t \otimes dt$ . Since  $\nu_t = (x - X(t))\#\mu_t$ , the measure

$$n_t = (x - X(t))\#m_t - \dot{X}(t)\nu_t$$

is such that  $(\nu, n)$  solves the continuity equation. Moreover for any  $x \in \mathbb{R}^2$  and  $t \geq 0$

$$\frac{dm_t}{d\mu_t}(x) = \frac{dn_t}{d\nu_t}(x + X(t)) + \dot{X}(t).$$

Hence,

$$\begin{aligned} I(\mu, m) &= P(\mu) + \int_0^\infty \int_{\mathbb{R}^2} \left| \frac{dm_t}{d\mu_t} \right|^2 d\mu_t dt \\ &= P(\nu) + \int_0^\infty \int_{\mathbb{R}^2} \left| \frac{dn_t}{d\nu_t} \right|^2 d\nu_t dt + \Phi \int_0^\infty |\dot{X}(t)|^2 dt + 2 \int_0^\infty \dot{X}(t) \cdot \int_{\mathbb{R}^2} \frac{dn_t}{d\nu_t} d\nu_t dt \\ &= I(\nu, n) + \Phi \int_0^\infty |\dot{X}(t)|^2 dt + 2 \int_0^\infty \dot{X}(t) \cdot n_t(\mathbb{R}^2) dt. \end{aligned}$$

The conclusion follows if we can show that for a.e.  $t \in (0, +\infty)$  one has  $n_t(\mathbb{R}^2) = 0$ . By the hypothesis  $t \rightarrow \int_{\mathbb{R}^2} |x| d\mu_t \in L^1_{\text{loc}}(0, +\infty)$  and the construction of  $\nu$ , we know that

$$\int_{\mathbb{R}^2} x d\nu_t = 0 \tag{5.3.1}$$

and

$$\int_{\mathbb{R}^2} |x| d\nu_t \lesssim 1 + |X(t)| \in L^1_{\text{loc}}(0, +\infty). \tag{5.3.2}$$

Consider now  $\xi(t) \in C_c^\infty(0, +\infty)$ . For  $j = 1, 2$  and  $R > 0$  consider the canonical base  $e_j$  of  $\mathbb{R}^2$  and define

$$\chi_{R,j}(x) = \begin{cases} x \cdot e_j & \text{for } |x \cdot e_j| \leq R, \\ R & \text{for } x \cdot e_j \geq R, \\ -R & \text{for } x \cdot e_j \leq -R \end{cases}$$

and test equation (5.2.5) with  $\phi = \xi(t)\chi_{R,j}(x)$  (recall Remark 5.2.2). We find

$$\int_0^\infty \dot{\xi}(t) \int_{\mathbb{R}^2} \chi_{R,j}(x) d\nu_t dt + \int_0^\infty \xi(t) e_j \cdot n_t(\{|x \cdot e_j| \leq R\}) dt = 0. \tag{5.3.3}$$

Let us examine both terms in (5.3.3) as  $R \rightarrow +\infty$ . For the term

$$\int_0^\infty \dot{\xi}(t) \int_{\mathbb{R}^2} \chi_{R,j}(x) d\nu_t dt,$$

by (5.3.2) we can apply dominated convergence and by (5.3.1) we find that it vanishes. For the other term notice that, since  $I(\nu, n) < +\infty$ ,

$$|\xi(t)e_j \cdot n_t(\{|x \cdot e_j| \leq R\})| \leq |\xi(t)| \int_{\mathbb{R}^2} \left| \frac{dn_t}{d\nu_t} \right| d\nu_t \leq \Phi^{1/2} |\xi(t)| \left( \int_{\mathbb{R}^2} \left| \frac{dn_t}{d\nu_t} \right|^2 d\nu_t \right)^{1/2} \in L^1(0, +\infty).$$

Hence, since for  $R \rightarrow +\infty$  one has  $n_t(\{|x \cdot e_j| \leq R\}) \rightarrow n_t(\mathbb{R}^2)$  for a.e.  $t \in (0, +\infty)$ , again by dominated convergence we find

$$\int_0^\infty \xi e_j \cdot n_t(\mathbb{R}^2) dt = 0.$$

Since  $\xi$  and  $j = 1, 2$  are arbitrary, we find that  $n_t(\mathbb{R}^2) = 0$  for a.e.  $t \in (0, +\infty)$ , which implies the claim.  $\square$

Let us now recall the following proposition, which is a minor modification of [48, Proposition 3.9].

**Proposition 5.3.2.** *Let  $T, \Phi, R > 0$ ,  $\mu^\pm \in \mathcal{M}^+(\mathbb{R}^2)$  be such that  $\mu^\pm(\mathbb{R}^2) = \Phi$  and  $\text{supp } \mu^\pm \subset [-R/2, R/2]^2$ . Then, there exists  $\mu \in \mathcal{A}^*(\mathbb{R}^2, (0, T))$  such that for all  $t \in (0, T)$  one has  $\text{supp } \mu_t \subset [-R/2, R/2]^2$ ,  $\mu_0 = \mu^-$ ,  $\mu_T = \mu^+$  and*

$$I(\mu, (0, T)) \lesssim \frac{1}{T} W^2(\mu^-, \mu^+) + (\Phi + \Phi^{\frac{1}{2}}) \max \left\{ T^{\frac{1}{3}}, T \right\}. \quad (5.3.4)$$

Combining Proposition 5.3.2 and Lemma 5.3.1, we have the following corollary.

**Corollary 5.3.3.** *There exists a measure  $\mu \in \mathcal{A}^*$  such that*

$$\mu_0 = \chi_{[-1/2, 1/2]^2} dx,$$

for all  $t \geq 1$   $\mu_t = \delta_0$ , for all  $t > 0$   $\text{supp } \mu_t \subset [-1/2, 1/2]^2$  and

$$\int_{\mathbb{R}^2} x d\mu_t = 0.$$

Moreover,

$$\mathcal{E}(\mu) \lesssim 1.$$

*Proof.* It is sufficient to apply Proposition 5.3.2 with  $\Phi = R = T = 1$ ,  $\mu_- = \chi_{[-1/2, 1/2]^2} dx$  and  $\mu^+ = \delta_0$ . By Lemma 5.3.1, we may assume  $\int_{\mathbb{R}^2} x d\mu_t = 0$  for all  $t > 0$ .  $\square$

### 5.3.2 Existence in bounded domains

We now introduce some auxiliary problems on bounded domains, for which most of the theory was developed in [37].

We fix throughout the section  $R, T > 0$ . Define the set

$$\mathcal{A}^*(R, T) = \{\mu \in \mathcal{A}^*(\mathbb{R}^2, (0, T)), \text{supp } \mu_t \subset [-R/2, R/2]^2 \forall t \in [0, T]\}.$$

Let  $\bar{\mu}_\pm \in \mathcal{M}^+(\mathbb{R}^2)$  with  $\text{supp } \bar{\mu}_\pm \subset [-R/2, R/2]^2$  and  $\mu_-(\mathbb{R}^2) = \mu_+(\mathbb{R}^2)$ . Consider the minimization problems

$$\inf \{I(\mu, (0, T)) : \mu \in \mathcal{A}^*(R, T), \mu_0 = \bar{\mu}_-, \mu_T = \bar{\mu}_+\} \quad (5.3.5)$$

and

$$e(R, T) = \inf \{ \mathcal{E}(\mu) : \mu \in \mathcal{A}^*(R, T), \text{supp } \mu_0 \subset [-R/2, R/2]^2, \mu_T = \delta_0 \}. \quad (5.3.6)$$

Problem (5.3.5) is an equivalent problem to the one studied in [37], while problem (5.3.6) is a one sided version of the minimization problem studied in [48, 41, 37]. Most of the theory developed in those papers can be extended with minor modifications. First, arguing like [37, Proposition 5.5], minimizers for both problems exist. Clearly, a minimum  $\mu$  of (5.3.6) is a minimum of (5.3.5) with  $\bar{\mu}_- = \mu_0$  and  $\bar{\mu}_+ = \delta_0$ . Notice that the condition  $\mu_T = \delta_0$  in (5.3.6) fixes the mass  $\mu_0(\mathbb{R}^2) = 1$  and, because of Lemma 5.3.1,  $\int_{\mathbb{R}^2} x d\mu_t = 0$  for all  $0 \leq t \leq T$ . Minimizers of (5.3.5) enjoy the no-loop property (see [37, Lemma 5.8]).

**Lemma 5.3.4** (No-loop condition). *Let  $\mu$  be a minimizer of (5.3.5) and  $t \in (0, T)$ . Let  $X_1, X_2 \in \mathbb{R}^2$  with  $\mu_t(\{X_i\}) \neq 0$  for  $i = 1, 2$  and consider the forward and backwards subsystems  $\mu_{\pm}^{t,i}$  emanating from  $X_i$ . If there exist  $0 \leq t_- < t < t_+ < T$  such that for  $* \in \{\pm\}$ ,  $\mu_*^{t_*,1}$  and  $\mu_*^{t_*,2}$  are not mutually singular, then  $X_1 = X_2$ .*

**Definition 5.3.5.** We say that a measure  $\mu \in \mathcal{A}^*(R, T)$  is locally polygonal if

$$\mu = \sum_i \frac{\varphi_i}{\sqrt{1 + |\dot{X}_i|^2}} \mathcal{H}^1 \llcorner \Gamma_i,$$

where the sum is countable,  $\Gamma_i = \{(X_i(t), t) : t \in [a_i, b_i]\}$  are segments disjoint up to the endpoints and for any  $0 < s < t < T$  only finitely many segments intersect  $[-R/2, R/2]^2 \times (s, t)$ . For a locally polygonal measure  $\mu \in \mathcal{A}^*(R, T)$  with  $\mu_0(\mathbb{R}^2) = \Phi$  the following representation formula holds:

$$I(\mu, (0, T)) = \int_0^T \sum_i \left[ \varphi_i^{1/2}(t) + \varphi_i(t) |\dot{X}_i(t)|^2 \right] dt - \Phi^{1/2} T \quad (5.3.7)$$

where the sum is locally finite. Moreover, between two branching points  $t \rightarrow \varphi_i(t)$  is constant while  $t \rightarrow X_i(t)$  is affine.

By arguing as in [37, Proposition 5.10 and Proposition 5.11], we have the following proposition.

**Proposition 5.3.6.** *Let  $\mu$  be a minimizer of (5.3.5). Then,  $\mu$  is locally polygonal.*

In the rest of the chapter we will always use the representation formula (5.3.7) for minimizers of (5.3.5). For locally polygonal measures, we prove the following first moment bound.

**Lemma 5.3.7.** *Let  $\mu \in \mathcal{A}^*(R, T)$  be locally polygonal with*

$$\int_{\mathbb{R}^2} x d\mu_t = 0 \quad (5.3.8)$$

for all  $0 \leq t \leq T$  and such that  $\mu_T = \Phi \delta_0$  for some  $\Phi > 0$ . Then for all  $0 \leq t \leq T$

$$\Phi^{-1/4} \int_{\mathbb{R}^2} |x| d\mu_t \lesssim I(\mu, (0, T)).$$

*Proof.* Fix  $0 < t < T$ . We know that  $\mu_t = \sum_i \varphi_i(t) \delta_{X_i(t)}$ . Let us suppose without loss of generality  $\varphi_0(t) \geq \varphi_i(t)$  for all  $i \in \mathbb{N}$ . We start by arguing that

$$\sum_i \varphi_i^{1/2}(t) - \Phi^{1/2} \gtrsim \Phi^{-1/2} \sum_{i \neq 0} \varphi_i(t). \quad (5.3.9)$$

If  $\sum_i \varphi_i^{1/2}(t) - \Phi^{1/2} \gtrsim \Phi^{1/2}$  the validity of (5.3.9) is immediate, so we can assume  $\sum_i \varphi_i^{1/2}(t) - \Phi^{1/2} \ll \Phi^{1/2}$ . In this case we have that  $\Phi - \varphi_0(t) \ll 1$ , hence (since  $\sqrt{x} - 1 > \frac{3}{4}(x - 1)$  for  $1/2 < x < 1$ )

$$\varphi_0^{1/2}(t) - \Phi^{1/2} \geq \frac{3\Phi^{-1/2}}{4}(\varphi_0(t) - \Phi) = -\frac{3\Phi^{-1/2}}{4} \sum_{i \neq 0} \varphi_i(t). \quad (5.3.10)$$

Moreover, since  $\varphi_i(t) \leq \Phi$ , one has

$$\sum_{i \neq 0} \left( \frac{\varphi_i(t)}{\Phi} \right)^{1/2} \geq \sum_{i \neq 0} \frac{\varphi_i(t)}{\Phi},$$

which implies

$$\sum_{i \neq 0} \varphi_i^{1/2}(t) \geq \Phi^{-1/2} \sum_{i \neq 0} \varphi_i(t). \quad (5.3.11)$$

Putting (5.3.10) and (5.3.11) together, we find (5.3.9). Using Hölder's inequality and (5.3.9),

$$\begin{aligned} \Phi^{-1/4} \sum_{i \neq 0} \varphi_i(t) \left| \dot{X}_i(t) \right| &\leq \Phi^{-1/2} \sum_{i \neq 0} \varphi_i(t) + \sum_{i \neq 0} \varphi_i(t) \left| \dot{X}_i(t) \right|^2 \\ &\lesssim \sum_i \varphi_i^{1/2}(t) - \Phi^{1/2} + \sum_i \varphi_i(t) \left| \dot{X}_i(t) \right|^2 = \dot{P}(\mu, t) + \dot{E}(\mu, t). \end{aligned} \quad (5.3.12)$$

Now, since  $\mu$  is locally polygonal, we may find a countable union of open intervals  $U_j$  on which no branching occurs and such that  $\left| [0, T] / \bigcup_j U_j \right| = 0$ . In particular, on each  $U_j$ ,  $\mu_t = \sum_i \varphi_i \delta_{X_i(t)}$ , with weights  $\varphi_i$  which are constant in  $t$ . If  $U_j = (a_j, b_j)$  integrating (5.3.12) on  $U_j$  we find

$$\begin{aligned} \Phi^{-1/4} \sum_{i \neq 0} \varphi_i |X_i(b_j) - X_i(a_j)| &\leq \Phi^{-1/4} \sum_{i \neq 0} \varphi_i \int_{a_j}^{b_j} \left| \dot{X}(t) \right| dt \\ &\stackrel{(5.3.12)}{\lesssim} E(\mu, U_j) + P(\mu, U_j) = I(\mu, U_j). \end{aligned} \quad (5.3.13)$$

Let us now notice that (5.3.8) implies

$$\sum_i \varphi_i X_i(a_j) = \sum_i \varphi_i X_i(b_j)$$

which can be rewritten as

$$\varphi_0(X_i(b_j) - X_i(a_j)) = - \sum_{i \neq 0} \varphi_i (X_i(b_j) - X_i(a_j)). \quad (5.3.14)$$

Hence, putting (5.3.13) and (5.3.14) together we find

$$\Phi^{-1/4} \left| \int_{\mathbb{R}^2} |x| d\mu_{b_j} - \int_{\mathbb{R}^2} |x| d\mu_{a_j} \right| \lesssim I(\mu, U_j).$$

Now, for any  $0 < t < T$ , we obtain

$$\begin{aligned} \Phi^{-1/4} \left| \int_{\mathbb{R}^2} |x| d\mu_t - \int_{\mathbb{R}^2} |x| d\mu_T \right| &\leq \Phi^{-1/4} \sum_j \left| \int_{\mathbb{R}^2} |x| d\mu_{b_j} - \int_{\mathbb{R}^2} |x| d\mu_{a_j} \right| \\ &\lesssim \sum_j I(\mu, U_j) = I(\mu, (0, T)). \end{aligned}$$

Since we know that  $\mu_T = \Phi\delta_0$ , this implies the claim. Finally, the result holds also for  $t = 0$  by lower semicontinuity of the integral with respect to weak\* convergence of measures, and for  $t = T$  trivially.  $\square$

### 5.3.3 Lagrangian reformulation and apriori bounds

In this section we prove apriori bounds for minima of (5.3.6) which are independent of  $R, T$ . These bounds will be crucial for establishing existence of a minimum for (5.2.9) in the next section.

Let  $R, T > 0$ . We introduce, as in [48, 41], a Lagrangian reformulation of the minimization problem (5.3.5) when  $\bar{\mu}^+ = \delta_0$ . The proof of the following proposition is essentially the same as [48, Theorem 4.3], hence we omit it. We just remark that it crucially uses the no-loop condition. Let

$$\mathcal{C} = \{X : \mathbb{R}^2 \times [0, T] \rightarrow \mathbb{R}^2 : X(x, 0) = x, X(x, T) = 0 \text{ and for a.e. } x \ t \rightarrow X(x, t) \text{ is AC}\}$$

where AC denotes absolutely continuous curves. Let  $X \in \mathcal{C}$ ,  $\bar{\mu} \in \mathcal{P}(\mathbb{R}^2)$  with  $\text{supp } \bar{\mu}_0 \subset [-R/2, R/2]^2$ ,  $y \in \mathbb{R}^2$  and  $t \in [0, T]$ . We define the multiplicity function as

$$\varphi_{X, \bar{\mu}}(y, t) = \bar{\mu}(\{x \in \text{supp } \bar{\mu} : X(x, t) = y\}). \quad (5.3.15)$$

We set

$$\mathcal{L}(X, \bar{\mu}) = \int_{\mathbb{R}^2 \times (0, T)} \left( \varphi_{X, \bar{\mu}}(X(x, t), t)^{-1/2} - 1 + |\partial_t X(x, t)|^2 \right) dt d\bar{\mu}(x). \quad (5.3.16)$$

**Proposition 5.3.8.** *Let  $R, T > 0$ . For every  $\bar{\mu} \in \mathcal{P}(\mathbb{R}^2)$  with  $\text{supp } \bar{\mu} \subset [-R/2, R/2]^2$  one has*

$$\min_{\mu \in \mathcal{A}^*(R, T)} \{I(\mu, (0, T)), \mu_0 = \bar{\mu}, \mu_T = \delta_0\} = \min_{X \in \mathcal{C}} \mathcal{L}(X, \bar{\mu}).$$

Moreover, for every  $X$  minimizing  $\mathcal{L}$ ,  $\mu_t = X(\cdot, t) \# \bar{\mu}$  is a minimizer for  $I$ . Vice versa, we can associate to any minimizer  $\mu \in \mathcal{A}^*$  a collection of curves  $X \in \mathcal{C}$  such that  $X$  is a minimizer for  $\mathcal{L}$ .

**Definition 5.3.9.** Let  $R, T > 0$ . For every  $\bar{\mu} \in \mathcal{P}(\mathbb{R}^2)$  with  $\text{supp } \bar{\mu} \subset [-R/2, R/2]^2$  we consider a minimum  $\mu^{R, T}$  of

$$\min_{\mu \in \mathcal{A}^*(R, T)} \{I(\mu, (0, T)), \mu_0 = \bar{\mu}, \mu_T = \delta_0\},$$

consider the set of curves  $X \in \mathcal{C}$  associated to  $\mu$  provided by Proposition 5.3.8 and define the landscape function associated to  $\mu^{R, T}$  as the function  $z_{R, T} : \text{supp } \bar{\mu} \rightarrow [0, +\infty]$  such that

$$z_{R, T}(x) = \int_0^T \left( \varphi_{X, \bar{\mu}}(X(x, t), t)^{-1/2} - 1 + |\partial_t X(x, t)|^2 \right) dt, \quad (5.3.17)$$

so that

$$\mathcal{L}(X, \bar{\mu}) = \int_{\mathbb{R}^2} z_{R, T}(x) d\bar{\mu}(x).$$

Such an object is an analogue to the landscape function as defined in classical branched optimal transport problems (see [103]), where it is a key tool to prove properties of irrigated measures [91, 92]. In this chapter we limit ourself to exploiting the following first variation.

**Proposition 5.3.10.** *Let  $\mu \in \mathcal{A}^*(R, T)$  be a minimum for  $e(R, T)$  and consider the landscape function  $z_{R,T}$  associated to  $\mu$  and the potential  $u$  of  $\mu_0$  as defined in (5.2.3). Then, for  $\mu_0$ -a.e.  $x \in \mathbb{R}^2$ ,*

$$z_{R,T}(x) + 2u(x) = I(\mu) + 2 \|\mu_0\|_{H^{-1/2}(\mathbb{R}^2)}^2.$$

*Proof.* For a minimum  $\mu \in \mathcal{A}^*(R, T)$  consider the set of curves  $X \in \mathcal{C}$  associated to  $\mu$  as provided by Proposition 5.3.8. Let  $\psi \in C(\mathbb{R}^2)$  with  $\int_{\mathbb{R}^2} \psi d\mu_0 = 0$  and  $\text{supp } \psi \subset [-R/2, R/2]^2$ . Consider for  $\varepsilon \in \mathbb{R}$

$$\tilde{\mu}_0 = (1 + \varepsilon\psi)\mu_0.$$

For  $|\varepsilon|$  small enough,  $\tilde{\mu}_0 \in \mathcal{M}^+(\mathbb{R}^2)$ . Using Proposition 5.3.8 we have

$$\mathcal{L}(X, \tilde{\mu}_0) \geq \min_{X' \in \mathcal{C}} \mathcal{L}(X', \tilde{\mu}) = \min_{\mu' \in \mathcal{A}^*(R, T)} \{I(\mu'), (0, T), \mu'_0 = \tilde{\mu}_0, \mu'_T = \delta_0\},$$

which implies by minimality of  $\mu$

$$\mathcal{L}(X, \tilde{\mu}_0) + \|\tilde{\mu}_0\|_{H^{-1/2}(\mathbb{R}^2)}^2 \geq \mathcal{E}(\mu) = \mathcal{L}(X, \mu_0) + \|\mu_0\|_{H^{-1/2}(\mathbb{R}^2)}^2. \quad (5.3.18)$$

Recalling (5.3.16), (5.3.17) and (5.2.4), sending  $\varepsilon \rightarrow 0$  in (5.3.18) we obtain

$$\int_{\mathbb{R}^2} (z_{R,T}(x) + 2u(x)) \psi d\mu_0 = 0.$$

Since  $\psi$  was arbitrary with  $\int_{\mathbb{R}^2} \psi d\mu_0 = 0$ , this last equality implies that there exists a constant  $K \in \mathbb{R}$  such that  $z_{R,T}(x) + 2u(x) = K$  for  $\mu_0$ -a.e.  $x \in \mathbb{R}^2$ . Finally, integrating on  $\mathbb{R}^2$  we find  $K = I(\mu) + 2 \|\mu_0\|_{H^{-1/2}(\mathbb{R}^2)}^2$ .  $\square$

We now deduce boundedness of the landscape function when  $R$  and  $T$  are far from 0.

**Lemma 5.3.11.** *Let  $R, T > 1$ . Let  $\mu \in \mathcal{A}^*$  be a minimum for  $e(R, T)$  and consider the landscape function  $z_{R,T}$  associated to  $\mu$ . Then for  $\mu_0$ -a.e.  $x \in \mathbb{R}^2$ ,*

$$z_{R,T}(x) \lesssim 1.$$

*Proof.* It is sufficient to combine Proposition 5.3.10 and Corollary 5.3.13. One has for  $\mu_0$ -a.e.  $x \in \mathbb{R}^2$

$$z_{R,T}(x) \leq z_{R,T}(x) + 2u(x) = I(\mu) + 2 \|\mu_0\|_{H^{-1/2}(\mathbb{R}^2)}^2 \lesssim \mathcal{E}(\mu) \lesssim 1,$$

where in the last passage we used the construction of Corollary 5.3.13, since  $R, T > 1$ .  $\square$

Lemma 5.3.11 implies the following important a priori bounds.

**Lemma 5.3.12.** *Let  $T, R > 1$ ,  $\mu \in \mathcal{A}^*$  be a minimum for  $e(R, T)$  and  $0 < \tilde{T} < T$  be such that  $\mu_{\tilde{T}} \neq \delta_0$ . Then*

$$\tilde{T} \lesssim 1. \quad (5.3.19)$$

Moreover let  $\bar{X} \in \mathbb{R}^2$  and  $0 < \bar{T} \leq T$  such that  $\mu_{\bar{T}}(\{\bar{X}\}) > 0$  and consider the backwards subsystem  $\mu_{\bar{T}}^{\bar{X}, \bar{T}}$ . Then for  $\mu_0^{\bar{X}, \bar{T}}$ -a.e.  $x \in \mathbb{R}^2$  one has

$$|x - \bar{X}| \lesssim \bar{T}^{1/2}. \quad (5.3.20)$$

In particular, (5.3.19) and (5.3.20) imply

$$\text{supp } \mu_0 \subset B(0, \tilde{R}) \quad (5.3.21)$$

for some  $\tilde{R} \lesssim 1$ .

*Proof.* Consider the the landscape function  $z_{R,T}$  associated to  $\mu$ . We prove (5.3.19) first. Suppose  $\tilde{T} > 0$  is such that  $\mu_{\tilde{T}} \neq \delta_0$ . This means there exist  $x \in \text{supp } \mu_0$  such that (recall (5.3.15))

$$\varphi_{X,\mu_0}(X(x, \tilde{T}), \tilde{T}) \leq \frac{1}{2}.$$

By Lemma 5.3.11,

$$\begin{aligned} 1 \gtrsim z_{R,T}(x) &\geq \int_0^{\tilde{T}} \left( \varphi_{X,\mu_0}(X(x, t), t)^{-1/2} - 1 \right) dt \\ &\geq \tilde{T} \left( \varphi_{X,\mu_0}(X(x, \tilde{T}), \tilde{T})^{-1/2} - 1 \right) \geq \tilde{T}(\sqrt{2} - 1), \end{aligned}$$

where we used the fact that  $\varphi_{X,\mu_0}(X(x, \cdot), \cdot)$  is increasing by the no-loop condition. This concludes the proof of (5.3.19). Let us now tackle (5.3.20). Again by Lemma 5.3.11, for  $\mu_0^{\bar{X}, \bar{T}}$ -a.e.  $x \in \mathbb{R}^2$  one has

$$|\bar{X} - x| = \left| \int_0^{\bar{T}} \partial_t X(x, t) dt \right| \leq \bar{T}^{1/2} \left( \int_0^{\bar{T}} |\partial_t X(x, t)|^2 dt \right)^{1/2} \leq \bar{T}^{1/2} z_{R,T}^{1/2}(x) \lesssim \bar{T}^{1/2}.$$

□

The apriori bounds (5.3.19) and (5.3.21) imply the following corollary.

**Corollary 5.3.13.** *There exist  $T_0, R_0 > 1$  such that for all  $T > T_0$  and  $R > R_0$  one has*

$$e(R, T) = e(R_0, T_0).$$

### 5.3.4 Existence in the half space

We now tackle existence for problem (5.2.9). Existence will be inherited by the existence for problems (5.3.6). By Lemma 5.3.12 and Corollary 5.3.13, we prove that at least one minimum for (5.2.9) is compact in space and has the property  $\mu_t = \delta_0$  for all  $t \geq T_0$  for some  $T_0 > 0$ . We start with the following lemma.

**Lemma 5.3.14.** *Let  $\mu \in \mathcal{A}^*$  with  $\mathcal{E}(\mu) < \infty$ . There exist sequences  $R_n, T_n \rightarrow \infty$  and  $\mu^n \in \mathcal{A}^*(R_n, T_n)$  such that  $\mu_T^n = \delta_{X_n}$  for some  $X_n \in \mathbb{R}^2$  and, extending  $\mu_t^n = \delta_{X_n}$  for  $t > T_n$ ,*

$$\limsup_n \mathcal{E}(\mu_n) \leq \mathcal{E}(\mu). \quad (5.3.22)$$

*Proof.* The proof is divided in two steps.

**Step 1.** Let  $\mu \in \mathcal{A}^*$  with  $\mathcal{E}(\mu) < \infty$  and  $R_n \rightarrow +\infty$ . We prove that there exists  $\mu^n \in \mathcal{A}^*$  such that for every  $t \geq 0$  one has  $\text{supp } \mu_t^n \subset [-R_n, R_n]^2$  and such that

$$\limsup \mathcal{E}(\mu_n) \leq \mathcal{E}(\mu). \quad (5.3.23)$$

To this extent we let for  $x, v \in \mathbb{R}^2$  and  $n \in \mathbb{N}$

$$\pi_n(x) = \begin{cases} x & \text{if } |x| \leq R_n, \\ R_n \frac{x}{|x|} & \text{if } |x| > R_n \end{cases}$$

and

$$\Psi_n(x, v) = \begin{cases} v & \text{if } |x| \leq R_n, \\ \frac{R_n}{|x|} \left( v - \frac{v \cdot x}{|x|^2} x \right) & \text{if } |x| > R_n. \end{cases}$$

Let  $m \in \mathcal{M}_{\text{loc}}(\mathbb{R}^2 \times (0, +\infty); \mathbb{R}^2)$  be a minimum for (5.2.7) so that  $m = m_t \otimes dt$ . For  $t \geq 0$  define the measure

$$\hat{\mu}_t^n = \pi_n \# \mu_t$$

and the measure  $\hat{m}_t^n \in \mathcal{M}(\mathbb{R}^2; \mathbb{R}^2)$  as

$$\int_{\mathbb{R}^2} \xi \cdot d\hat{m}_t^n = \int_{\mathbb{R}^2} \xi(\pi_n(x)) \cdot \Psi_n \left( x, \frac{dm_t}{d\mu_t} \right) d\mu_t.$$

where  $\xi \in C_0(\mathbb{R}^2, \mathbb{R}^2)$ . By construction  $(\hat{\mu}, \hat{m})$  solves (5.2.5) and

$$I(\hat{\mu}, \hat{m}) \leq I(\mu, m). \quad (5.3.24)$$

We now apply Proposition 5.3.2 twice as follows. We let  $\gamma_n \rightarrow 0$  to be chosen later and consider  $\nu^n \in \mathcal{A}^*(\mathbb{R}^2, (0, \gamma_n))$  such that

$$\nu_0^n = \frac{\mu_0(B(0, R_n)^c)}{(2\pi R_n - \pi)} \chi_{B(0, R_n) \setminus B(0, R_n - 1)} dx \quad \text{and} \quad \nu_{\gamma_n}^n = \hat{\mu}_0^n \llcorner \partial B(0, R_n)$$

as given by Proposition 5.3.2 (notice that the two boundary measures have the same mass since  $\hat{\mu}_0^n(\partial B(0, R_n)) = \mu_0(B(0, R_n)^c)$ ). Since  $\mu_0(B(0, R_n)^c) \rightarrow 0$ , for  $n$  large enough by (5.3.4) it holds

$$I(\nu^n, (0, \gamma_n)) \lesssim \frac{\mu_0(B(0, R_n)^c)}{\gamma_n} + \mu_0(B(0, R_n)^c)^{1/2} \gamma_n^{1/3}. \quad (5.3.25)$$

We thus choose  $\gamma_n$  such that  $\mu_0(B(0, R_n)^c)/\gamma_n \rightarrow 0$ , so that the right hand side of (5.3.25) vanishes. We moreover let  $\tilde{\mu}^n \in \mathcal{A}^*(\mathbb{R}^2, (0, \gamma_n))$  such that

$$\tilde{\mu}_0^n = \tilde{\mu}_{\gamma_n}^n = \mu_0 \llcorner B(0, R_n),$$

again as given by Proposition 5.3.2. In this case (5.3.4) implies

$$I(\tilde{\mu}^n, (0, \gamma_n)) \lesssim \gamma_n^{1/3}. \quad (5.3.26)$$

We finally define for  $t \geq 0$

$$\mu_t^n = \begin{cases} \tilde{\mu}_t^n + \nu_t^n & \text{if } t < \gamma_n, \\ \hat{\mu}_{t-\gamma_n}^n & \text{if } t > \gamma_n. \end{cases}$$

Notice that  $\mu^n \in \mathcal{A}^*$  and for every  $t \geq 0$  one has  $\text{supp } \mu_t \subset [-R_n, R_n]^2$ . By (5.3.24), (5.3.25) and (5.3.26) one finds

$$\limsup_n I(\mu^n) \leq I(\mu). \quad (5.3.27)$$

Finally, one has

$$\|\mu_0^n\|_{H^{-1/2}(\mathbb{R}^2)}^2 \leq \left( \|\nu_0^n\|_{H^{-1/2}(\mathbb{R}^2)} + \|\tilde{\mu}_0^n\|_{H^{-1/2}(\mathbb{R}^2)} \right)^2 \leq \left( \|\nu_0^n\|_{H^{-1/2}(\mathbb{R}^2)} + \|\mu_0\|_{H^{-1/2}(\mathbb{R}^2)} \right)^2. \quad (5.3.28)$$

Since  $\|\nu_0^n\|_{H^{-1/2}(\mathbb{R}^2)} \rightarrow 0$ , combining (5.3.27) and (5.3.28) gives (5.3.23).

**Step 2.** Let  $\mu \in \mathcal{A}^*$  with  $\mathcal{E}(\mu) < \infty$  and such that there exists an  $R > 0$  such that for every  $t \geq 0$  one has  $\text{supp } \mu_t \subset [-R, R]^2$ . We prove that there exists  $T_n \rightarrow +\infty$  and  $\mu^n \in \mathcal{A}^*(R, T_n)$  such that  $\mu_{T_n}^n = \delta_0$  and

$$\limsup \mathcal{E}(\mu_n) \leq \mathcal{E}(\mu). \quad (5.3.29)$$

Since  $\int_0^{+\infty} \dot{P}(\mu, t) dt < +\infty$ , given  $n \in \mathbb{N}$  there exists  $X_n \in \mathbb{R}^2$  and  $T_n > 0$  such that  $\Phi_n = \mu_{T_n}(\mathbb{R}^2 \setminus \{X_n\}) \leq 1/n$ . We consider the measure  $\nu^n \in \mathcal{A}^*(\mathbb{R}^2, (0, 1))$  such that

$$\nu_0^n = \mu_{T_n} \llcorner (\mathbb{R}^2 \setminus \{X_n\}) \quad \text{and} \quad \nu_1^n = \Phi_n \delta_{X_n}$$

as given by Proposition 5.3.2. By (5.3.4) we have

$$I(\nu^n, (0, 1)) \lesssim W^2(\mu_{T_n} \llcorner (\mathbb{R}^2 \setminus \{X_n\}), \Phi_n \delta_{X_n}) + \Phi_n^{1/2} + \Phi_n \lesssim_R n^{-1/2}. \quad (5.3.30)$$

We define

$$\mu_t^n = \begin{cases} \mu_t & \text{if } t \leq T_n, \\ \nu_{t-T_n}^n + \mu(\{X_n\})\delta_{X_n} & \text{if } T_n \leq t \leq T_n + 1, \\ \delta_{X_n} & \text{if } t > T_n + 1. \end{cases}$$

One has

$$\mathcal{E}(\mu_n) \leq \mathcal{E}(\mu) + I(\mu_n, (T_n, T_n + 1)), \quad (5.3.31)$$

so that, to show (5.3.29), it is sufficient to show that the last term in (5.3.31) vanishes as  $n \rightarrow +\infty$ . By subadditivity and (5.3.30) we find

$$I(\mu^n, (T_n, T_n+1)) \leq I(\nu^n, (0, 1)) + I(\mu(\{X_n\})\delta_{X_n}, (T_n, T_n+1)) + (1 - \Phi_n)^{1/2} + \Phi_n^{1/2} - 1 \lesssim_R n^{-1/2}.$$

**Conclusion.** The theorem follows by combining Step 1 and Step 2 and a diagonal procedure.  $\square$

**Theorem 5.3.15.** *There exist  $R_0, T_0 > 0$  such that any minimum  $\mu$  of  $e(R_0, T_0)$  is a minimum of (5.2.9), up to extending  $\mu_t = \delta_0$  for  $t > T_0$ .*

*Proof.* Let  $\nu \in \mathcal{A}^*$  with  $\mathcal{E}(\nu) < \infty$  and  $\int_{\mathbb{R}^2} x d\nu_t = 0$  for every  $t \geq 0$ . By Lemma 5.3.14, we can find  $R_n, T_n \rightarrow +\infty$  and  $\nu^n \in \mathcal{A}^*(R_n, T_n)$  such that (5.3.22) holds. By Lemma 5.3.1, we may find  $\tilde{\nu}^n \in \mathcal{A}^*(R_n, T_n)$  with  $\mathcal{E}(\tilde{\nu}^n) \leq \mathcal{E}(\nu^n)$  that  $\int_{\mathbb{R}^2} x d\nu_t^n = 0$  for every  $t \geq 0$ . We thus find by minimality

$$\mathcal{E}(\nu) \geq \limsup_n \mathcal{E}(\nu^n) \geq \limsup_n \mathcal{E}(\tilde{\nu}^n) \geq \limsup_n e(R_n, T_n) = e(R_0, T_0),$$

where in the last passage we used Corollary 5.3.13. This last inequality means that any minimum  $\mu$  of  $e(R_0, T_0)$  is a minimum for (5.2.9), up to setting  $\mu_t = \delta_0$  for  $t > T_0$ .  $\square$

### 5.3.5 Properties of subsystems

We finally address some properties of subsystems of minimizers of (5.2.9).

**Lemma 5.3.16.** *Let  $\mu$  be a minimizer of (5.2.9). For any  $T > 0$  and  $X \in \mathbb{R}^2$  with  $\mu_T(\{X\}) = \Phi > 0$ , consider the backwards subsystem  $\mu^{X,T}$ . For  $t \geq 0$  call  $X(t) = \frac{1}{\Phi} \int_{\mathbb{R}^2} x d\mu_t^{X,T}$ . Then,  $X(t) = X(0) + \frac{t}{T}(X - X(0))$  and it holds*

$$I(\mu^{X,T}, (0, T)) = I(\nu, (0, T)) + \Phi \frac{|X - X(0)|^2}{T},$$

where  $\nu_t = (\cdot - X(t))\#\mu_t$ . Moreover,  $\nu$  is a minimum of

$$\min\{I(\nu, (0, T)) : \nu \in \mathcal{A}^*(\mathbb{R}^2, (0, T)), \nu_0 = (\cdot - X(0))\#\mu_0, \nu_T = \Phi\delta_0\} \quad (5.3.32)$$

which satisfies  $\int_{\mathbb{R}^2} x d\nu_t = 0$  for any  $t \in [0, T]$ .

*Proof.* Without loss of generality we may assume  $X = 0$ . By the no loop condition and minimality,  $\mu^{0,T}$  is a solution of the minimization problem

$$\min\{I(\mu, (0, T)) : \mu \in \mathcal{A}^*(0, T), \mu_0 = \mu_0^{0,T}, \mu_T = \Phi\delta_0\}.$$

The conclusion then follows by Lemma 5.3.1.  $\square$

We also prove a refined version of the equipartition of energy [48, Proposition 2.11].

**Proposition 5.3.17.** *Let  $\mu$  be a minimizer of (5.2.9). For any  $T > 0$  and  $X \in \mathbb{R}^2$  with  $\mu_T(\{X\}) = \Phi > 0$ , consider the backwards subsystem  $\mu^{X,T}$  and call  $X_0 = \frac{1}{\Phi} \int_{\mathbb{R}^2} x d\mu_0^{X,T}$ . Then, there exists  $\Lambda = \Lambda(\mu, X, T)$  such that*

$$P(\mu^{X,T}, (0, T)) = E(\mu^{X,T}, (0, T)) - \Phi \frac{|X - X_0|^2}{T} + \Lambda. \quad (5.3.33)$$

Moreover,  $\Lambda \leq 0$ .

*Proof.* Again, without loss of generality we may assume  $X = 0$ . Let us call  $\bar{\mu} = \mu^{0,T}$  the backwards subsystem at time  $T > 0$ . If we let  $X(t) = \int_{\mathbb{R}^2} x d\bar{\mu}_t$  and  $\bar{\nu}_t = (\cdot - X(t))\#\bar{\mu}_t$ , by Lemma 5.3.16  $\bar{\nu}$  is a minimum of (5.3.32) and

$$I(\bar{\mu}, (0, T)) = I(\bar{\nu}, (0, T)) + \Phi \frac{|X_0|^2}{T}. \quad (5.3.34)$$

By [48, Proposition 2.11], there exists  $\Lambda = \Lambda(\mu, X, T)$  such that for any  $t \in (0, T)$

$$\dot{P}(\bar{\nu}, t) = \dot{E}(\bar{\nu}, t) + \Lambda. \quad (5.3.35)$$

Integrating (5.3.35) and combining with (5.3.34) and the fact that  $P(\bar{\mu}, (0, T)) = P(\bar{\nu}, (0, T))$  we find (5.3.33). Let us prove that  $\Lambda \leq 0$ . Fix  $0 < \delta < 1$ . Consider the competitor  $\tilde{\nu}$  given by  $\tilde{\nu}_t = \bar{\nu}_{\frac{t}{1-\delta}}$  for  $t < T(1-\delta)$  and  $\tilde{\nu}_t = \Phi\delta_0$  for  $t \geq T(1-\delta)$ . By the no loop condition and minimality for  $\bar{\nu}$  we find

$$I(\bar{\nu}, (0, T)) \leq I(\tilde{\nu}, (0, T)) = I(\tilde{\nu}, (0, T(1-\delta))), \quad (5.3.36)$$

where in the last passage we used the fact that by definition  $\dot{E}(\tilde{\nu}, t) = \dot{P}(\tilde{\nu}, t) = 0$  for  $t > T(1-\delta)$ .

By scaling, we know that

$$\begin{aligned} P(\tilde{\nu}, (0, T(1-\delta))) &= (1-\delta)P(\bar{\nu}, (0, T)) \\ E(\tilde{\nu}, (0, T(1-\delta))) &= \frac{1}{1-\delta}E(\bar{\nu}, (0, T)). \end{aligned}$$

Plugging into (5.3.36) we obtain

$$\delta \int_0^T \dot{P}(\bar{\nu}, t) dt + \left(1 - \frac{1}{1-\delta}\right) \int_0^T \dot{E}(\bar{\nu}, t) dt \leq 0.$$

Sending  $\delta \rightarrow 0$  we obtain

$$\int_0^T (\dot{P}(\bar{\nu}, t) - \dot{E}(\bar{\nu}, t)) dt \leq 0,$$

which by (5.3.35) implies  $\Lambda \leq 0$ . □

## 5.4 Fractal behavior of minimizers

In this section we prove a concentration result for minima of (5.2.9). First, we recall the notion of Ahlfors regularity of a measure and prove a sort of Schauder estimate for the potential (5.2.3) of upper Ahlfors regular measures. This estimate is then crucially used in the first variation argument which leads to the concentration result.

### 5.4.1 Alfhors regularity and Hölder continuity of the potential

We want to establish a connection between Alfhors regularity of a measure and the regularity of its electrostatic potential. Let us start by recalling the notion of upper Alfhors regularity.

**Definition 5.4.1.** Let  $\sigma \in \mathcal{M}^+(\mathbb{R}^2)$ , and  $\alpha \in [0, 2]$ . A measure is called upper  $\alpha$ -Alfhors regular if there exists  $M > 0$  such that

$$\sigma(B(x, r)) \leq Mr^\alpha \quad \text{for all } x \in \text{supp } \sigma \text{ and } r \in (0, 1]. \quad (5.4.1)$$

The key result of this section is the following proposition.

**Proposition 5.4.2.** Let  $\mu \in \mathcal{M}^+(\mathbb{R}^2)$  be upper  $\alpha$ -Alfhors regular for some  $\alpha > 1$  and  $M > 0$ . Then for all  $x, y \in \mathbb{R}^2$

$$|u(x) - u(y)| \lesssim_\alpha M |x - y|^{\alpha-1}.$$

**Remark 5.4.3.** We may interpret Proposition 5.4.2 as a sort of Schauder estimate in negative Hölder spaces. Indeed, rescaling both sides in (5.4.1) by  $r^2$ , we may write

$$\sup_{x \in \mathbb{R}^2} \frac{1}{r^2} \sigma(B(x, r)) \leq Mr^{\alpha-2}. \quad (5.4.2)$$

Condition (5.4.2) resembles the definition of a negative Hölder norm of exponent  $\alpha - 2$  (see for instance [61, 69]). At the same time, the potential  $u$  solves in the sense of distributions (see for instance [49, 75, 74, 54] for the definition of fractional Laplacian)

$$(-\Delta)^{1/2} u = \sigma.$$

Since the half laplacian can be interpreted as a Dirichlet-to-Neumann operator of order 1, the regularity we expect for  $u$  is indeed  $(\alpha - 1)$ -Hölder continuity.

*Proof.* The proof could be derived from general Schauder estimates in [69, 61], for simplicity we give a self contained proof. Let us assume without loss of generality that  $y = 0$  and set  $R = |x|$ . We consider the difference  $u(x) - u(0)$  and split into a near-field contribution and a far-field contribution

$$\begin{aligned} u(x) - u(0) &= \left( \int_{B(0,2R)} \frac{1}{|z-x|} d\mu(z) - \int_{B(0,2R)} \frac{1}{|z|} d\mu(z) \right) \\ &\quad + \int_{B(0,2R)^c} \left( \frac{1}{|z-x|} - \frac{1}{|z|} \right) d\mu(z) = A + B. \end{aligned} \quad (5.4.3)$$

Let us estimate each term in (5.4.3) separately. Notice that  $B(0, 2R) \subset B(x, 3R)$ . Hence,

$$\begin{aligned} |A| &\leq \int_{B(0,2R)} \frac{1}{|z-x|} d\mu(z) + \int_{B(0,2R)} \frac{1}{|z|} d\mu(z) \\ &\leq \int_{B(x,3R)} \frac{1}{|z-x|} d\mu(z) + \int_{B(0,3R)} \frac{1}{|z|} d\mu(z) \lesssim \sup_{w \in \mathbb{R}^2} \int_{B(w,3R)} \frac{1}{|z-w|} d\mu(z). \end{aligned}$$

Notice that

$$|z-w|^{-1} = \int_0^\infty \chi_{\{|z-w| < r\}}(r) r^{-2} dr.$$

We thus find using Fubini, upper  $\alpha$ -Ahlfors regularity and the hypothesis  $\alpha > 1$

$$\begin{aligned} \int_{B(w,3R)} \frac{1}{|z-w|} d\mu(z) &= \int_0^{3R} r^{-2} \mu(B(w,3R)) dr + \int_{3R}^{\infty} r^{-2} \mu(B(w,3R)) dr \\ &\lesssim M \int_0^{3R} r^{\alpha-2} dr + MR^\alpha \int_{3R}^{\infty} r^{-2} dr \lesssim_\alpha MR^{\alpha-1}, \end{aligned}$$

which implies

$$|A| \lesssim_\alpha MR^{\alpha-1}. \quad (5.4.4)$$

Let us now focus on  $B$ . Notice that for  $z \in B(0,2R)^c$  it holds  $|z| - |x| > |z|/2$ . Hence, using the triangle inequality we get

$$\begin{aligned} \left| \int_{B(0,2R)} \frac{1}{|z-x|} - \frac{1}{|z|} d\mu(z) \right| &= \left| \int_{B(0,2R)} \frac{|z| - |z-x|}{|z||z-x|} d\mu(z) \right| \\ &\leq \int_{B(0,2R)} \frac{|x|}{||z|-|x|| |z|} d\mu(z) \lesssim R \int_{B(0,2R)} \frac{1}{|z|^2} d\mu(z). \end{aligned}$$

Let us now remark that

$$|z|^{-2} = 2 \int_0^{\infty} \chi_{\{|z|<r\}}(r) r^{-3} dr.$$

Thus, using again Fubini and upper  $\alpha$ -Ahlfors regularity, we find

$$R \int_{B(0,2R)} \frac{1}{|z|^2} d\mu(z) = 2MR \int_{2R}^{\infty} r^{-3} \mu(B_r(0)) dr \lesssim MR \int_{2R}^{\infty} r^{\alpha-3} dr \lesssim_\alpha MR^{\alpha-1},$$

which implies

$$|B| \lesssim_\alpha MR^{\alpha-1}. \quad (5.4.5)$$

Combining (5.4.4) and (5.4.5) gives the claim.  $\square$

We also establish a local version of Proposition 5.4.2.

**Proposition 5.4.4.** *Let  $B \subset \mathbb{R}^2$  be an open set and  $\mu \in \mathcal{M}^+(\mathbb{R}^2)$  be such that  $\mu \llcorner B$  is upper  $\alpha$ -Ahlfors regular for some  $\alpha > 1$  and  $M > 0$ . Then for all  $x, y \in B$*

$$|u(x) - u(y)| \lesssim_\mu |x - y|^{\alpha-1}.$$

*Proof.* By definition

$$u = \int_B \frac{1}{|\cdot - y|} d\mu(y) + \int_{B^c} \frac{1}{|\cdot - y|} d\mu(y) = u_B + u_{B^c}.$$

By definition  $u_{B^c} \in C^\infty(B)$ , while by Proposition 5.4.2 it holds  $u_B \in C^{\alpha-1}(B)$ .  $\square$

We now prove quantitative estimates on the Sobolev norm of  $\alpha$ -Ahlfors regular measure.

**Proposition 5.4.5.** *Let  $\mu \in \mathcal{M}^+(\mathbb{R}^2)$  be upper  $\alpha$ -Ahlfors regular for some  $\alpha > 1$  and  $M > 0$ , with  $\mu(\mathbb{R}^2) = \Phi$ . Then*

$$\|\mu\|_{H^{-1/2}(\mathbb{R}^2)}^2 \lesssim_\alpha M \Phi^{2-\frac{1}{\alpha}}. \quad (5.4.6)$$

*Proof.* Let us start by noticing that, since by (5.2.4)

$$\|\mu\|_{H^{-1/2}(\mathbb{R}^2)}^2 = C \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{1}{|x-y|} d\mu(x)d\mu(y),$$

it suffices to show that for any  $y \in \mathbb{R}^2$

$$\int_{\mathbb{R}^2} \frac{1}{|x-y|} d\mu(x) \lesssim M\Phi^{1-\frac{1}{\alpha}}.$$

Recall as before that for any  $x, y \in \mathbb{R}^2$

$$|x-y|^{-1} = \int_0^\infty \chi_{\{|x-y|<r\}}(r)r^{-2} dr$$

Let  $R > 0$  to be chosen below. We have

$$\int_{\mathbb{R}^2} \frac{1}{|x-y|} d\mu(x) = \int_0^R \mu(B(y,r))r^{-2} dr + \int_R^\infty \mu(B(y,r))r^{-2} dr. \quad (5.4.7)$$

For the first term in (5.4.7) we have using the hypothesis

$$\int_0^R \mu(B(y,r))r^{-2} dr \lesssim M \int_0^R r^{\alpha-2} dr \lesssim_\alpha MR^{\alpha-1}, \quad (5.4.8)$$

whereas for the second term in (5.4.7) using  $\mu(\mathbb{R}^2) = \Phi$

$$\int_R^\infty \mu(B(0,r))r^{-2} dr \leq \Phi \int_R^\infty r^{-2} dr \lesssim M\Phi R^{-1}. \quad (5.4.9)$$

Combining (5.4.8) and (5.4.9)

$$\int_{\mathbb{R}^2} \frac{1}{|x-y|} d\mu(x) \lesssim_\alpha M(R^{\alpha-1} + \Phi R^{-1}),$$

which gives the conclusion choosing  $R = \Phi^{\frac{1}{\alpha}}$ .  $\square$

For a measure  $\mu \in \mathcal{M}^+(\mathbb{R}^2)$ , we now denote its barycenter

$$X = \frac{1}{\Phi} \int_{\mathbb{R}^2} x d\mu(x) \quad (5.4.10)$$

and its spreading scale

$$r = \frac{1}{\Phi} \int_{\mathbb{R}^2} |x-X| d\mu(x). \quad (5.4.11)$$

**Corollary 5.4.6.** *Let  $\mu \in \mathcal{M}^+(\mathbb{R}^2)$  be upper  $\alpha$ -Ahlfors regular for some  $\alpha > 1$  and  $M > 0$ , with  $\mu(\mathbb{R}^2) = \Phi$ . Then*

$$Mr \gtrsim_\alpha \Phi^{\frac{1}{\alpha}}. \quad (5.4.12)$$

*Proof.* Using Hölder's inequality we have

$$\Phi^2 = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} 1 d\mu(x)d\mu(y) \lesssim \left( \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |x-y| d\mu(x)d\mu(y) \right)^{1/2} \|\mu\|_{H^{-1/2}(\mathbb{R}^2)}. \quad (5.4.13)$$

By (5.4.6),

$$\|\mu\|_{H^{-1/2}(\mathbb{R}^2)} \lesssim_\alpha M^{1/2} \Phi^{1-\frac{1}{2\alpha}}. \quad (5.4.14)$$

Moreover by triangle inequality

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |x-y| d\mu(x)d\mu(y) \lesssim \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |x-X| d\mu(x)d\mu(y) = \Phi^2 r. \quad (5.4.15)$$

Plugging (5.4.14) and (5.4.15) in (5.4.13) we find the claim.  $\square$

### 5.4.2 Concentration of minimizers

In this section we prove that minimizers of (5.2.9) cannot be upper  $\alpha$ -Ahlfors regular for  $\alpha > 8/5$ . More generally we prove the following.

**Theorem 5.4.7.** *Let  $\mu$  be a minimizer of (5.2.9) as provided by Theorem 5.3.15. Suppose that there exist  $x_0 \in \mathbb{R}^2$  and  $r_0 > 0$  such that  $\mu_0(B(x_0, r_0/2)) > 0$  and  $\mu_0 \llcorner B(x_0, r_0)$  is upper  $\alpha$ -Ahlfors regular for some  $\alpha > 1, M > 0$ . Then  $\alpha \leq 8/5$ .*

*Proof.* Fix  $T \ll 1$  and choose  $x \in B(x_0, 3r_0/4)$  such that  $\mu_T(\{x\}) = \Phi > 0$ . Such a point can be found for any  $T \ll 1$  since  $\mu_0(B(x_0, r_0/2)) > 0$ . We can assume that  $\Phi \ll 1$  for  $T$  small enough, otherwise  $\mu_0$  would be atomic. Let us call  $\mu'$  the backwards subsystem emanating from  $x$  at time  $T$  and  $\mu'' = \mu - \mu'$ . Furthermore, by Lemma 5.3.12, for  $T$  small enough  $\text{supp } \mu'_0 \subset B(x_0, r_0)$ . In particular this implies that  $\mu'_0$  is upper  $\alpha$ -Ahlfors regular. Without loss of generality in the rest of the proof we assume  $x = 0$ . Our contradiction will arise at the level of the subsystem  $\mu'$ .

To simplify the notation we set  $I(\mu) = I(\mu, (0, T))$ ,  $E(\mu) = E(\mu, (0, T))$  and  $P(\mu) = P(\mu, (0, T))$ . Let us consider the boundary measure of the subsystem  $\mu'_0$  and let us denote by  $X$  and  $r$  its barycenter and first moment respectively, as defined in (5.4.10) and (5.4.11). Let us prove a bound for  $r$ . Let  $X(t)$  be the barycenter of  $\mu'_t$  so that  $X = X(0)$ . By Lemma 5.3.16 we have that  $X(t) = \frac{T-t}{T}X$  and  $\nu'_t = (\cdot - X(t))\#\mu'_t$  satisfies

$$I(\nu') = I(\mu') - \Phi \frac{|X|^2}{T}. \quad (5.4.16)$$

By Lemma 5.3.7 we have

$$\Phi^{\frac{3}{4}}r = \Phi^{-1/4} \int_{\mathbb{R}^2} |x - X| d\mu'_0(x) = \Phi^{-1/4} \int_{\mathbb{R}^2} |x| d\nu'_0(x) \lesssim I(\nu')$$

which implies

$$\Phi^{\frac{3}{4}}r \lesssim I(\mu') - \Phi \frac{|X|^2}{T}. \quad (5.4.17)$$

We now define  $\tilde{\nu}' \in \mathcal{A}^*$  as

$$\tilde{\nu}'_t = (\cdot/2)\#\nu'_t$$

and

$$\tilde{\mu}'_t = (\cdot + X(t))\#\tilde{\nu}'_t.$$

Since  $\int_{\mathbb{R}^2} x d\tilde{\nu}'_t = 0$ , we can use Lemma 5.3.1 to find

$$I(\tilde{\mu}') - \Phi \frac{|X|^2}{T} = I(\tilde{\nu}'). \quad (5.4.18)$$

A direct computation shows that

$$E(\tilde{\nu}') = 1/4E(\nu') \quad \text{and} \quad P(\tilde{\mu}') \leq P(\mu'). \quad (5.4.19)$$

Moreover,

$$\int_{\mathbb{R}^2} |x - X| d\tilde{\mu}'_0 \lesssim \Phi r \quad (5.4.20)$$

and

$$\|\tilde{\mu}'_0\|_{H^{-1/2}(\mathbb{R}^2)}^2 \lesssim \|\mu'_0\|_{H^{-1/2}(\mathbb{R}^2)}^2. \quad (5.4.21)$$

Let us finally define  $\tilde{\mu} = \mu - \mu' + \tilde{\mu}'$ . By linearity

$$E(\tilde{\mu}) = E(\mu) - E(\mu') + E(\tilde{\mu}') \quad (5.4.22)$$

and by the no loop condition,

$$P(\tilde{\mu}) - P(\mu) = P(\tilde{\mu}') - P(\mu'). \quad (5.4.23)$$

Hence, by minimality, (5.4.22) and (5.4.23) we find

$$I(\mu') \leq I(\tilde{\mu}') + \|\tilde{\mu}_0\|_{H^{-1/2}(\mathbb{R}^2)}^2 - \|\mu_0\|_{H^{-1/2}(\mathbb{R}^2)}^2. \quad (5.4.24)$$

Exploiting (5.4.16), (5.4.18) and (5.4.19), (5.4.24) reduces to

$$E(\mu') - \Phi \frac{|X|^2}{T} \lesssim \|\tilde{\mu}_0\|_{H^{-1/2}(\mathbb{R}^2)}^2 - \|\mu_0\|_{H^{-1/2}(\mathbb{R}^2)}^2. \quad (5.4.25)$$

Let us now recall (5.2.4) and let us call  $u$  the potential of  $\mu_0$  as in (5.2.3). Expanding the square we find

$$\begin{aligned} \|\tilde{\mu}_0\|_{H^{-1/2}(\mathbb{R}^2)}^2 &= \|\mu_0\|_{H^{-1/2}(\mathbb{R}^2)}^2 + \int_{\mathbb{R}^2} u(x) (d\tilde{\mu}'_0 - d\mu'_0)(x) \\ &\quad + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{1}{|x-y|} (d\tilde{\mu}'_0 - d\mu'_0)(x) (d\tilde{\mu}'_0 - d\mu'_0)(y) \\ &\leq \|\mu_0\|_{H^{-1/2}(\mathbb{R}^2)}^2 + \int_{\mathbb{R}^2} u(x) (d\tilde{\mu}'_0 - d\mu'_0)(x) + \|\tilde{\mu}'_0\|_{H^{-1/2}(\mathbb{R}^2)}^2 + \|\mu'_0\|_{H^{-1/2}(\mathbb{R}^2)}^2, \end{aligned} \quad (5.4.26)$$

where in the last passage we neglected the negative mixed terms between  $\tilde{\mu}'_0$  and  $\mu'_0$ . Putting (5.4.26), (5.4.25) and (5.4.21) together we find

$$E(\mu') - \Phi \frac{|X|^2}{T} \lesssim \int_{\mathbb{R}^2} u(x) (d\tilde{\mu}'_0 - d\mu'_0)(x) + \|\mu'_0\|_{H^{-1/2}(\mathbb{R}^2)}^2. \quad (5.4.27)$$

Let us now estimate the first term in the right hand side. Using Proposition 5.4.4 and Hölder's inequality we find

$$\begin{aligned} \left| \int_{\mathbb{R}^2} u(x) (d\tilde{\mu}'_0 - d\mu'_0)(x) \right| &\leq \int_{\mathbb{R}^2} |u(x) - u(X)| (d\tilde{\mu}'_0 + d\mu'_0)(x) \\ &\lesssim M \int_{\mathbb{R}^2} |x - X|^{\alpha-1} (d\tilde{\mu}'_0 + d\mu'_0)(x) \lesssim M \Phi^{2-\alpha} \left( \int_{\mathbb{R}^2} |x - X| (d\tilde{\mu}'_0 + d\mu'_0)(x) \right)^{\alpha-1}. \end{aligned}$$

Using now the definition of  $r$  and (5.4.20) we obtain

$$\left| \int_{\mathbb{R}^2} u(x) (d\tilde{\mu}'_0 - d\mu'_0)(x) \right| \lesssim_M \Phi r^{\alpha-1}. \quad (5.4.28)$$

If we now plug (5.4.6), (5.4.18) and (5.4.28) in (5.4.27) we obtain

$$E(\mu') - \Phi \frac{|X|^2}{T} \lesssim_M \Phi^{2-\frac{1}{\alpha}} + \Phi r^{\alpha-1}.$$

By equipartition of energy (5.3.33) we find

$$I(\mu') - \Phi \frac{|X|^2}{T} \lesssim_M \Phi^{2-\frac{1}{\alpha}} + \Phi r^{\alpha-1}. \quad (5.4.29)$$

---

Combining (5.4.17) and (5.4.29) we find

$$\Phi^{\frac{3}{4}} r \lesssim_M \Phi^{2-\frac{1}{\alpha}} + \Phi r^{\alpha-1}.$$

Finally, by Young's inequality, for any  $\varepsilon > 0$

$$\Phi r^{\alpha-1} \lesssim \varepsilon \Phi^{\frac{3}{4}} r + C(\varepsilon) \Phi^{\frac{7-3\alpha}{8-4\alpha}},$$

for some  $C(\varepsilon) > 0$ . Choosing  $\varepsilon$  small enough, we can conclude

$$\Phi^{\frac{3}{4}} r \lesssim_M \Phi^{2-\frac{1}{\alpha}} + \Phi^{\frac{7-3\alpha}{8-4\alpha}}.$$

Combining with (5.4.12), we find

$$\Phi^{\frac{3}{4}+\frac{1}{\alpha}} \lesssim_{M,\alpha} \Phi^{2-\frac{1}{\alpha}} + \Phi^{\frac{7-3\alpha}{8-4\alpha}}.$$

Since for  $\alpha > 8/5$  one has  $3/4 + 1/\alpha < 2 - 1/\alpha < (7 - 3\alpha)/(8 - 4\alpha)$ , the last inequality gives a contradiction with  $\Phi \ll 1$ .  $\square$



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