CONTRIBUTIONS À LA THÉORIE DE MORSE DISCRÈTE ET À L'HOMOLOGIE DE HEEGAARD-FLOER COMBINATOIRE

Étienne Gallais

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Contents

Chapter 1

Introduction

Introduction

Le travail effectué dans cette thèse s'articule autour de la description combinatoire d'objets mathématiques à travers deux théories : l'homologie de Heegaard–Floer combinatoire des entrelacs et la théorie de Morse discrète. Ces deux théories combinatoires sont issues de la théorie de Morse : théorie de Morse en dimension infinie dans le premier cas, théorie de Morse classique dans le second cas.

Les raisons de s'intéresser à la combinatoire sont nombreuses : calculer des invariants pour une grande classe d'exemples (par exemple pour l'homologie de Heegaard-Floer des entrelacs voir [BG06]), avoir de nouvelles versions de théorèmes existants (somme d'états pour les polynômes de Jones et d'Alexander d'un noeud ou [OSS07] plus récemment pour les noeuds singuliers), définir de nouveaux invariants (voir [OST06] pour de nouveaux invariants pour les noeuds legendriens et transverses et [NOT07] pour un exemple de noeuds transverses distingués par ce nouvel invariant ; l'homologie de Khovanov voir par exemple [BN02] ou [Vir02] ; invariant des variétés de dimension 3 définis par des sommes d'états voir see [TV92]). On voit donc que comprendre une théorie combinatoire ou réécrire de manière combinatoire une théorie est une démarche très utile.

Le chapitre 2 est consacré à l'homologie de Heegaard-Floer combinatoire des entrelacs. Une description combinatoire de l'homologie de Heegaard–Floer des entrelacs a récemment été donnée par C. Manolescu, P. Ozsváth et S. Sarkar [MOS06] et son invariance a été démontrée de manière purement combinatoire par C. Manolescu, P. Ozsváth, Z. Szabó et D. Thurston [MOST06]. L'homologie de Heegaard-Floer est un invariant des variétés de dimension 3 défini en utilisant des disques holomorphes et des diagrammes de Heegaard. Dans [OS04b] et [Ras03], cette construction est étendue pour donner un nouvel invariant, l'homologie de Floer de noeuds $\widehat{HFK}$ pour des

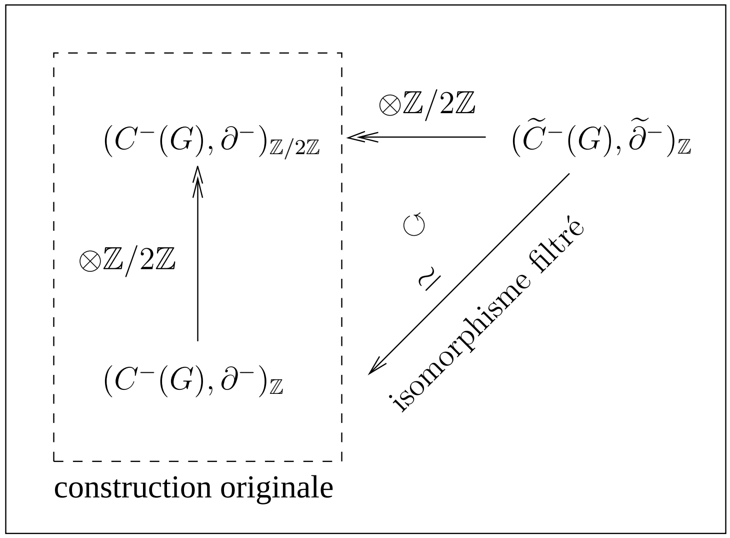

FIG. 1.1 – **Organisation du chapitre 2.** Tout d'abord, on construit un complexe de chaînes filtrées $(\widetilde{C}^-(G), \widetilde{\partial}^-)_{\mathbb{Z}}$ qui relève le complexe de chaînes original sur les entiers (Proposition 2.4.1). Ensuite, on montre que l'homologie filtrée de ce complexe est un invariant de l'entrelacs (Théorème 2.4.3). Finalement, on construit un isomorphisme de complexes de chaînes qui respecte la filtration entre $(\widetilde{C}^-(G), \widetilde{\partial}^-)_{\mathbb{Z}}$ et le relèvement original du complexe sur les entiers $(C^-(G), \partial^-)_{\mathbb{Z}}$ (Proposition 2.5.3) qui rend le diagramme ci-dessus commutatif.

noeuds nuls homologues plongés dans des variétés closes orientées de dimension 3. Cette construction a ensuite été généralisée dans [OS05] au cas des entrelacs orientés. Elle permet aussi de définir de nouveaux invariants pour les noeuds legendriens et transverses (voir [NOT07]). Cette invariant est très riche (voir section 2.1) mais se heurte à une difficulté : en général il est dur à calculer, la principale difficulté venant du décompte des disques holomorphes. L'idée de la description combinatoire de l'homologie de Heegaard-Floer des entrelacs est de choisir un diagramme de Heegaard de genre 1 de S^3 avec des cercles attachants redondants et de *voir* directement quels sont les disques holomorphes. On associe alors à cette présentation particulière un complexe gradué et filtré, le plus dur étant de montrer que l'homologie filtrée est un invariant de l'entrelacs.

Au début, la description combinatoire de l'homologie de Heegaard-Floer a été faite à coefficients dans $\mathbb{Z}/2\mathbb{Z}$ et dans le second papier, elle a été étendue au cas des coefficients entiers par un procédé appelé raffinement de signes. Comme pour l'homologie de Khovanov (voir par exemple [BN02]) le point clé pour étendre un complexe de chaînes à coefficients dans $\mathbb{Z}/2\mathbb{Z}$ en un complexe de chaînes à coefficients entiers est de faire anticommuter des diagrammes (le carré de la différentielle doit être égal à zéro). Dans notre cas, la différentielle est définie en comptant des rectangles sur un diagramme en grille (ils représentent les disques holomorphes). Le point de vue original adopté dans [MOST06] est d'attacher à chaque rectangle un signe et sous certaines conditions ce signe permet de définir un complexe de chaînes à coefficients dans $\mathbb{Z}$. Manolescu, Ozsváth, Szabó et D. Thurston donnent une formule explicite pour le signe d'un rectangle et vérifient que cette formule satisfait les conditions requises. La preuve est élémentaire et assez technique. On adopte un autre point de vue qui est plus algébrique pour définir le complexe de chaînes à coefficients entiers. L'idée cette fois-ci n'est pas d'attacher un signe aux rectangles mais plutôt de doubler l'ensemble des générateurs en leur attachant un signe $\pm$. L'ensemble des générateurs est l'extension spin $\widetilde{\mathfrak{S}}_n$ du groupe $\mathfrak{S}_n$ des permutations d'un ensemble à n éléments. Dans le cas où cette extension est une extension centrale non triviale de $\mathfrak{S}_n$ $(n \geq 4)$ on définit alors une extension du complexe de chaînes à coefficients dans $\mathbb{Z}$ i.e. en tensorisant par $\mathbb{Z}/2\mathbb{Z}$, on retrouve le complexe de chaînes original (Proposition 2.4.1). La preuve de l'invariance de l'homologie combinatoire de Heegaard-Floer des entrelacs s'adapte à cette nouvelle description. On montre alors que notre construction permet de définir un raffinement de signes au sens de [MOST06] et grâce à cela on montre que le complexe de chaînes à coefficients entiers est isomorphe (par une application qui respecte la filtration) au complexe de chaînes de [MOST06] (Lemme 2.5.2 et Proposition 2.5.3). Finalement, on donne un algorithme récursif qui permet de calculer le raffinement de signe avec notre approche. Récemment, une description du complexe de chaîne original à coefficients dans $\mathbb{Z}/2\mathbb{Z}$ avec moins de générateurs a été donnée par A. Beliakova [Bel07]. Il est donc naturel de se demander si cette construction s'adapte à cette nouvelle description.

Dans le chapitre 3, on s'intéresse à la théorie de Morse discrète définie par R. Forman [For98a], [For98b]. Cette théorie permet d'obtenir sur des objets combinatoires (des complexes simpliciaux ou plus généralement des CW-complexes) des résultats dans l'esprit de la théorie de Morse. Une fonction de Morse discrète sur un CW-complexe X est une donnée combinatoire sur l'ensemble des cellules de X (on assigne à chaque cellule une hauteur de sorte qu'elle vérifie certaines conditions vis à vis des hauteurs des cellules voisines). Étant donnée une fonction de Morse discrète, on peut travailler avec son champ de vecteurs combinatoire de Morse : c'est l'analogue combinatoire d'un champ de vecteurs (pseudo)-gradient pour une fonction de Morse. Nous allons faire ce choix qui permet de s'affranchir de l'ambiguïté de la hauteur des cellules pour ne conserver que la combinatoire entre les cellules (cette ambiguïté est la même que lorsque que l'on fait une homotopie d'une fonction de Morse sans changer le complexe de Thom-Smale). Par ailleurs les points critiques sont remplacés par les cellules critiques et la notion d'indice d'un point critique est remplacée par la notion de dimension d'une cellule critique. Soit V un tel champ de vecteurs combinatoire c'est-à-dire un couplage sur le diagramme de Hasse de X qui vérifie certaines propriétés. On notera

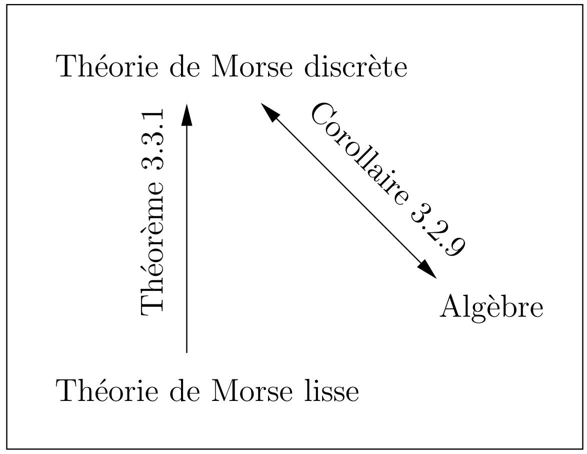

FIG. 1.2 – **Organisation du chapitre 3.** Tout d'abord, on montre le corollaire 3.2.9 qui établit le lien entre la théorie de Morse discrète et l'algèbre sur les complexes de chaînes. Ensuite, on construit une réalisation combinatoire du complexe de Thom-Smale (Théorème 3.3.1).

$Crit_k(V)$ l'ensemble des cellules critiques de V de dimension k. Les deux résultats suivants illustrent la similitude existant entre la théorie de Morse lisse et la théorie de Morse discrète.

Théorème 1 (Forman [For98b])**.** *Soit X un CW-complexe fini et V un champ de vecteurs combinatoire de Morse défini sur X. Alors X est homotopiquement équivalent à un CW-complexe avec exactement $|Crit_k(f)|$ cellules de dimension k pour tout $k \in \mathbb{N}$.*

Corollaire 2. (Inégalités de Morse fortes [For98b]) *Soient $b_i(X)$ le i-ème nombre de Betti de X et $m_k = |Crit_k(f)|$ le nombre de cellules critiques de dimension k. Pour tout $n \geq 0$,*

$$m_n - m_{n-1} + \ldots \pm m_0 \geq b_n - b_{n-1} + \ldots \pm b_0$$

À l'aide de la notion de V-orbite (qui est l'analogue combinatoire des lignes de gradient à renormalisation près) Forman définit une version combinatoire du complexe de chaînes de Thom-Smale.

Définition 3. On suppose que chaque cellule de X est munie d'une orientation. Le complexe de Thom-Smale combinatoire associé à (X, V) est $(C_*^V, \partial V)$ où :

1. $C_k^V = \bigoplus_{\sigma \in Crit_k(V)} \mathbb{Z}.\sigma$,
2. si $\tau \in Crit_{k+1}(V)$ alors

$$\partial^V \tau = \sum_{\sigma \in Crit_k(V)} n(\tau, \sigma).\sigma$$

où $n(\tau, \sigma)$ est la somme algébrique des V-orbites entre les cellules critiques σ et τ.

Forman montre alors le théorème suivant :

Théorème 4 (Forman [For98b])**.** *Le complexe de Thom-Smale combinatoire* $(C_*^V, \partial V)$ *est un complexe de chaînes dont l'homologie est l'homologie cellulaire de* X*.*

La preuve originale donnée par Forman est une preuve indirecte : il construit un complexe de Morse ([For98b] section 7) et montre qu'il est en fait engendré par les cellules critiques, que sa différentielle s'exprime alors comme ci-dessus et que l'homologie est l'homologie cellulaire grâce à des homotopies de complexes de chaînes.

Lorsque X est un complexe simplicial, on donne deux nouvelles preuves du fait que le complexe de Thom-Smale est un complexe de chaînes. La première est géométrique et s'intéresse à la manière dont les V-orbites contribuent dans l'expression de $\partial^V \circ \partial^V$. La seconde est une preuve algébrique qui montre aussi que l'homologie du complexe de Thom-Smale est l'homologie cellulaire (simpliciale dans ce cas). Elle utilise pour cela l'élimination gaussienne qui est un procédé algébrique qui permet de retirer à un complexe de chaînes des complexes de chaînes acycliques courts du type

$$0 \to \mathbb{Z}.a \xrightarrow{\pm 1} \mathbb{Z}.b \to 0$$

(cette idée est apparue pour la première fois dans [Cha00], voir aussi [Skö06]). On montre le résultat suivant :

Corollaire 5 (Corollaire 3.2.9)**.** *Soit* X *un complexe simplicial fini,* $\mathcal{C} = (C_*, \partial)$ *le complexe de chaînes simpliciales correspondant et* $\mathcal{M}$ *un couplage* $(\sigma_i < \tau_i)_{i \in I}$ *sur le diagramme de Hasse de* X*. Alors les deux propriétés suivantes sont équivalentes :*

1. $\mathcal{M}$ *définit un champ de vecteurs combinatoire de Morse* $V \iff$*,*
2. *pour toute suite* $(\sigma_{i_1} < \tau_{i_1}), (\sigma_{i_2} < \tau_{i_2}), \ldots, (\sigma_{i_{|I|}} < \tau_{i_{|I|}})$ *telles que* $i_j \neq i_k$ *si* $j \neq k$*, les éliminations gaussiennes peuvent être effectuées dans cet ordre.*

En particulier, toute suite d'éliminations gaussiennes correspondant à $\mathcal{M}$ *aboutit au même complexe de chaînes, le complexes de Thom-Smale combinatoire de* V*.*

Ainsi, un procédé purement algébrique sur les complexes de chaînes (enlever des complexes de chaînes acycliques courts) a une interprétation géométrique dans l'esprit de la théorie de Morse.

Puisque le lien entre l'algèbre et la combinatoire est fait, on souhaite aller un cran plus loin et comprendre le lien entre la théorie de Morse lisse et discrète. Étant donnée une fonction de Morse $f : M \to \mathbb{R}$ définie sur une variété lisse, on dira qu'elle est générique si un gradient de Morse–Smale v pour f a été choisi (voir section 3.3). On montre le théorème suivant :

Théorème 6 (Théorème 3.3.1)**.** *Soit M une variété riemannienne lisse close et $f : M \to \mathbb{R}$ une fonction de Morse générique. On suppose que chaque variété stable est orientée de sorte que le complexe de Thom-Smale lisse est défini. Alors, il existe une C^1-triangulation T de M et un champ de vecteurs combinatoire de Morse V qui réalisent le complexe de Thom-Smale lisse (après choix d'orientation des cellules de T) au sens suivant :*

1. *il existe une bijection entre l'ensemble des cellules critiques et l'ensemble des points critiques,*
2. *pour chaque paire de points critiques σ_p et σ_q telle que $dim(\sigma_p) = dim(\sigma_q) + 1$, les V-orbites partant des hyperfaces de σ_p et aboutissant à σ_q sont en bijection avec les lignes de gradient de v à renormalisation près reliant q à p,*
3. *cette bijection induit un isomorphisme entre les complexes de Thom-Smale lisse et combinatoire.*

Ce théorème dit que les données combinatoires provenant d'une fonction de Morse lisse (i.e. le complexe de Thom-Smale) peuvent être réalisées par des objets combinatoires : une triangulation et un champ de vecteurs combinatoire de Morse. On peut donc espérer transporter certains résultats ou questions dans le monde de la combinatoire par ce biais. Nous utilisons cela pour donner une description combinatoire des structures d'Euler sur les variétés de dimension 3, closes, orientées comme des couplages complets sur le diagramme de Hasse d'une triangulation. On montre en particulier que toute structure d'Euler peut être réalisé de cette manière (Théorème 3.4.6).

Malgré le fait que les chapitres 2 et 3 semblent éloignés l'un de l'autre, une problématique les relie : l'homologie de Heegaard-Floer. La première étape pour définir l'homologie de Heegaard-Floer consiste à choisir une fonction de Morse générique sur une variété de dimension 3 et travailler sur le diagramme de Heegaard induit (par exemple de genre g). Puis, on munit ce diagramme de Heegaard d'une structure Spin^c en choisissant un point base en dehors du jeu de cercles attachant et un g-uplet de points d'intersection entre les deux ensembles de cercles attachants. En fait, se donner une structure Spin^c revient au même que se donner une structure d'Euler (voir Turaev [Tur97], [Tur89]). Donc le théorème de réalisation combinatoire 3.3.1 ainsi que le théorème 3.4.6 donnent une réalisation combinatoire d'un diagramme de Heegaard muni d'une telle structure Spin^c. De la même manière que pour la description combinatoire de l'homologie de Heegaard-Floer, il faut maintenant comprendre combinatoirement les disques holomorphes (pas uniquement en genre 1) qui contribue à la différentielle du complexe de chaînes.

Toujours en dimension 3, une autre problématique intéressante est l'obtention d'une description combinatoire d'un champ de repères. On sait maintenant qu'une description combinatoire d'un champ de vecteurs non singulier est, si elle existe, un couplage couplet sur le diagramme de Hasse d'une triangulation de la variété (voir le corollaire 3.4.2). La question revient donc à comprendre ce qu'est un champ de repères combinatoire en terme de couplages complets.

Introduction

The main work of this thesis is around the combinatorial description of mathematical objects. It deals with two distinct theories which are combinatorial link Floer homology and discrete Morse theory. These two combinatorial theories are coming from Morse theory: Morse theory in infinite dimension in the first case, classical Morse theory in the second case.

Interests of combinatorial point of view are numerous : actually computing invariants for a large class of examples (for Heegaard-Floer homology of links see [BG06]), having new versions of theorems (e.g. state sum formula for Jones and Alexander polynomial or [OSS07] more recently for singular knots), defining new invariants (see [OST06] for new invariants for legendrian and transverse knots, [NOT07] for an example of transverse knots distinguished by this new invariant; Khovanov homology see e.g [BN02] or [Vir02]; state sum invariants for 3-manifolds see [TV92]). Therefore, understanding a combinatorial theory or rewriting combinatorially a theory is very useful.

In chapter 2, we focus on combinatorial link Floer homology. A combina-

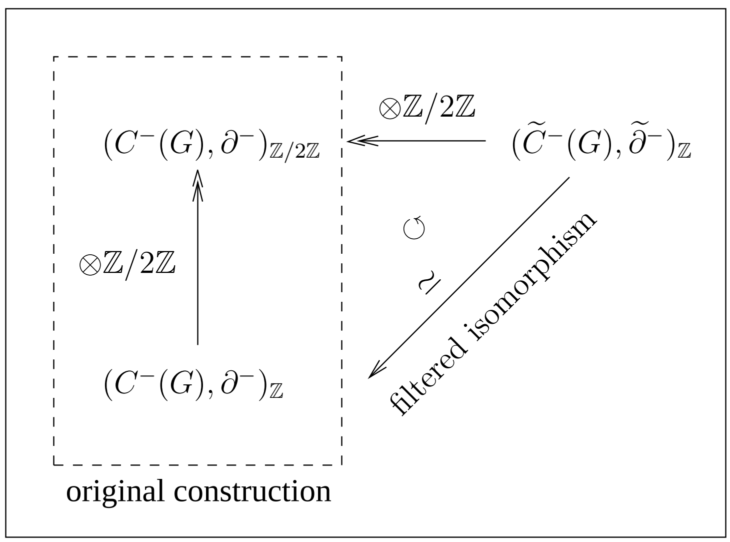

Figure 1.3: **Schematic view of chapter 2.** First, we construct a filtered chain complex $(\widetilde{C}^-(G),\widetilde{\partial}^-)_{\mathbb{Z}}$ which extends the original chain complex over $\mathbb{Z}$ (Proposition 2.4.1). Then, we prove that its filtered homology is an invariant of the link (Theorem 2.4.3). Finally we construct a filtered isomorphism between $(\widetilde{C}^-(G),\widetilde{\partial}^-)_{\mathbb{Z}}$ and the original chain complex $(C^-(G),\partial^-)_{\mathbb{Z}}$ (Proposition 2.5.3) which makes the above diagram commuting.

torial description of link Floer homology was recently given by C. Manolescu, P. Ozsváth and S. Sarkar [MOS06] and its invariance was proved in a purely combinatorial way by C. Manolescu, P. Ozsváth, Z. Szabó and D. Thurston

[MOST06]. Heegaard-Floer homology is an invariant for three-manifolds, defined using holomorphic disks and Heegaard diagrams. In [OS04b] and [Ras03], this construction is extended to give an invariant, knot Floer homology $\widehat{HFK}$ for null-homologous knots in a closed, oriented three-manifold. This construction is further generalized in [OS05] to the case of oriented links. It also gives new invariants for legendrian and transverse knots (see [NOT07]). While this is a powerfull invariant (see section 2.1) it remains rather challenging to compute. The idea of the combinatorial description of link Floer homology is to consider a suitable Heegaard diagram of genus one with redundant attaching circles and *seeing* all the holomorphic disks directly. One associates with this particular presentation a chain complex (which is filtrated) and the remaining hard part is to prove that its (filtered) homology is an invariant of the oriented link. Finally, everything becomes combinatorial and easier to compute in general.

At the beginning, the combinatorial description of link Floer homology was done with coefficients modulo two and in the second paper it was extended to the case of integer coefficients by a process called a sign refinement. Similarly to Khovanov homology (see e.g. [BN02]), the key point to give an extension of a chain complex over $\mathbb{Z}$ is to make diagrams anticommute (the square of the differential must be equal to zero). The differential is defined by counting rectangles which are in fact holomorphic disks. The original point of view in [MOST06] is to attach to each rectangle a sign and under some conditions, this sign defines a chain complex over $\mathbb{Z}$. Manolescu, Ozsváth, Szabó and D. Thurston give an explicit formula for the sign of a rectangle and check that it satisfies the desired properties. The proof is elementary but in the same time technical. We give another construction of a sign refinement for this chain complex which adopts a more algebraic point of view. The idea is doubling the set of generators (attach a sign $\pm$) instead of attaching signs to rectangles. The set of generators is the spin extension $\widetilde{\mathfrak{S}}_n$ of the permutation group of n elements. In the case it is non-trivial ($n \geq 4$), it gives an extension of the original chain complex over $\mathbb{Z}$ (Proposition 2.4.1) that is tensoring the chain complex by $\mathbb{Z}/2\mathbb{Z}$ we get the original chain complex. The proof of the combinatorial invariance of combinatorial link Floer homology adapts directly to our description. We also prove that our construction defines a sign refinement in the sense of [MOST06] and so our chain complex is (filtered) isomorphic to the chain complex of [MOST06] (Lemma 2.5.2 and Proposition 2.5.3). Finally, we describe an algorithm to compute signs (section 2.6).

Recently, a description of the original chain complex over $\mathbb{Z}/2\mathbb{Z}$ with less generator has been given by A. Beliakova [Bel07]. It is natural to ask wether our construction extends to Beliakova's construction.

In chapter 3, we focus on discrete Morse theory. Discrete Morse theory was defined by R. Forman [For98a], [For98b] and gives results over combinatorial objects (simplicial complexes, CW-complexes) in the spirit of Morse theory. A discrete Morse function on a CW-complex X is a combinatorial

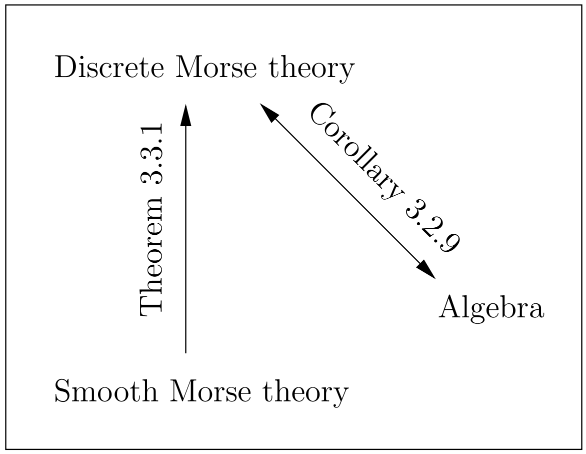

Figure 1.4: **Schematic view of chapter 3.** First, we prove corollary 3.2.9 which is the link between discrete Morse theory and algebra over chain complexes. Then, we construct a combinatorial realization of the smooth Thom-Smale complex (Theorem 3.3.1).

data on the set of cells of X: assign to every cell an height such that it satisfies some compatibility conditions with respect to neighbour cells. Given a discrete Morse function, one can consider the induced combinatorial Morse vector field, which is the combinatorial analog of gradient-like vector field. Similarly to Morse theory, one can make homotopies between Morse functions without changing the combinatorial data. This ambiguity remains for discrete Morse functions but it is dropped if we consider combinatorial Morse vector fields (we drop the height of each cell and only keep the combinatorial data). So, we will only work with this last notion. Let V be a combinatorial Morse vector field that is a matching over edges of the Hasse diagram of X which satisfies a particular property. One associates with V the set of its critical cells (which is the anolog of the set of critical points for a Morse function) where the dimension of a critical cell plays the role of the index of a critical point. Let $Crit_k(V)$ be the set of critical cells of dimension k and m_k be equal to $|Crit_k(V)|$. The next two results illustrate the similarity between discrete and smooth Morse theories.

Theorem 1 (Forman [For98b])**.** *X is homotopy equivalent to a CW-complex with exactly $|Crit_k(f)|$ cells of dimension k for all $k \in \mathbb{N}$.*

Corollary 2. (Strong Morse Inequalities [For98b]) *Let $b_i(X)$ be the i-th Betti number of X. For any $n \geq 0$,*

$$m_n - m_{n-1} + \ldots \pm m_0 \geq b_n - b_{n-1} + \ldots \pm b_0$$

Thanks to V-paths (which is the combinatorial analog of gradient lines up to renormalization), Forman defines a combinatorial version of the Thom-Smale complex.

Definition 3. Suppose every cell of X has been given an orientation. The combinatorial Thom-Smale complex associated with (X, V) is (C_*^V, ∂^V) where:

1. $C_k^V = \bigoplus_{\sigma \in Crit_k(V)} \mathbb{Z}.\sigma$,
2. if $\tau \in Crit_{k+1}(V)$ then

$$\partial^V \tau = \sum_{\sigma \in Crit_k(V)} n(\tau, \sigma).\sigma$$

   where $n(\tau, \sigma)$ is the algebraic sum of V-paths between the critical cells σ and τ.

Thus, Forman proves the following theorem.

Theorem 4 (Forman [For98b])**.** *The combinatorial Thom-Smale complex (C_*^V, ∂^V) is a chain complex whose homology is the cellular homology.*

The original proof of this theorem is an indirect proof: Forman constructed a Morse complex (see [For98b] section 7) and proved that it is generated by critical cells, its differential can be expressed as above and its homology is equal to the cellular homology thanks to a chain homotopy.

In the case where X is a simplicial complex, we give two other proofs of the fact that the Thom-Smale complex is a chain complex. One stresses on the geometric aspect that is how V-paths contribute to $\partial^V \circ \partial^V$. The other one is an algebraic proof which also gives an homotopy equivalence with the simplicial chain complex (this idea first appeared in [Cha00], see also [Skö06]). It uses Gaussian elimination to remove from the original simplicial complex short acyclic chain complexes:

$$0 \to \mathbb{Z}.a \xrightarrow{\pm 1} \mathbb{Z}.b \to 0$$

We prove the following:

Corollary 5 (Corollary 3.2.9)**.** *Let X be a finite simplicial complex, $\mathcal{C} = (C_*, \partial)$ be the corresponding simplicial chain complex and $\mathcal{M}$ be a matching $(\sigma_i < \tau_i)_{i \in I}$ on its Hasse diagram. Then the following properties are equivalent:*

1. *$\mathcal{M}$ defines a combinatorial Morse vector field V,*
2. *for any sequence $(\sigma_{i_1} < \tau_{i_1}), (\sigma_{i_2} < \tau_{i_2}), \ldots, (\sigma_{i_{|I|}} < \tau_{i_{|I|}})$ such that $i_j \neq i_k$ if $j \neq k$, Gaussian eliminations can be performed in this order.*

In particular, any sequence of Gaussian eliminations corresponding to $\mathcal{M}$ *lead to the same chain complex which is the combinatorial Thom-Smale complex of* V*.*

Therefore, one can see that a purely algebraic process (removing short acyclic complexes) has a geometrical interpretation (under some conditions) in the spirit of Morse theory.

Since the link between algebra and combinatorial data is done, we wish to push one step forward our understanding of discrete Morse theory by having a bridge between smooth and discrete Morse theories. Given a Morse function $f : M \to \mathbb{R}$ on a smooth manifold, we say that it is generic if a Morse–Smale gradient v for f has been chosen (see section 3.3). We prove the following theorem:

Theorem 6 (Theorem 3.3.1)**.** *Let* M *be a smooth closed Riemannian manifold and* $f : M \to \mathbb{R}$ *be a generic Morse function. Suppose that every stable manifold has been given an orientation so that the smooth Thom-Smale complex is defined. Then, there exists a* C^1*-triangulation* T *of* M *and a combinatorial Morse vector field* V *on it which realize the smooth Thom-Smale complex (after a choice of orientation of each cells of* T*) in the following sense:*

1. *there is a bijection between the set of critical cells and the set of critical points,*

2. *for each pair of critical cells* σ_p *and* σ_q *such that* $dim(\sigma_p) = dim(\sigma_q) + 1$*,* V*-paths from hyperfaces of* σ_p *to* σ_q *are in bijection with gradient lines of* v *up to renormalization connecting* q *to* p*,*

3. *this bijection induce an isomorphism between the smooth and the combinatorial Thom-Smale complexes.*

This theorem states that combinatorial data arising from a smooth Morse function (that is the Thom-Smale complex) can be realized by combinatorial objects: a triangulation and a combinatorial Morse vector field. One can rewrite and reformulate with the combinatorial point of view a large class of problems and theorems coming from the smooth Morse theory. We use this to give a combinatorial description of Euler structures on closed, oriented 3-manifolds as complete matchings on the Hasse diagram of a given triangulation. We prove in particular that any Euler structure can be realized this way (Theorem 3.4.6).

Chapter 2 and chapter 3 seem far one from each other but there is a connection. The first step to define Heegaard-Floer homology is taking a Morse function on a three-manifold and working on the induced Heegaard diagram (say of genus g). Then, we endow this Heegaard diagram with a

Spin^c structure by choosing a basepoint outside the set of attaching circles and a g-uplet intersection point between the two sets of attaching circles. In fact, Spin^c structures are exactly the same as Euler structures (see Turaev [Tur97], [Tur89]). Thus, the combinatorial realization theorem 3.3.1 together with Theorem 3.4.6 give a combinatorial description of Heegaard diagram equipped with Spin^c structures. In the spirit of combinatorial link Floer homology, it remains to understand combinatorially holomorphic disks.

Another problem arising in dimension 3 is having a nice combinatorial description of frame fields. We know a nice combinatorial description of a non-singular vector field which is a complete matching on the Hasse diagram (see corollary 3.4.2). The question is now to understand what would be a combinatorial frame field in terms of complete matchings.

Chapter 2

Combinatorial link Floer homology

Homologie de Heegaard-Floer combinatoire des entrelacs

L'homologie de Heegaard-Floer [OS04c] est un invariant des variétés closes orientées de dimension 3 qui a été étendu en un invariant pour les entrelacs nuls homologues dans de telles variétés : on l'appelle alors homologie de Heegaard-Floer des entrelacs [OS04b], [OS05] [Ras03]. Elle permet de calculer le genre de Seifert $g(K)$ d'un noeud K [OS04a] c'est-à-dire le genre minimale d'une surface plongée dans S^3 qui borde le noeud K. Elle détecte les noeuds fibrés ([Ghi06] dans le cas où $g(K) = 1$ et [Ni06] en général) c'est-à-dire si le complémentaire du noeud $K \hookrightarrow S^3$ est un fibré en cercles. L'homologie de Heegaard-Floer des entrelacs est une homologie graduée par le degré de Maslov et filtrée par le degré d'Alexander qui catégorifie le polynôme d'Alexander : sa caractéristique d'Euler graduée donne le polynôme d'Alexander symétrisé de l'entrelacs ([OS04b], [Ras03]). Récemment, une description combinatoire de l'homologie de Heegaard-Floer des entrelacs a été donnée [MOS06] et son invariance topologique a été prouvée de manière purement combinatoire [MOST06]. D'abord définie à coefficient dans $\mathbb{Z}/2\mathbb{Z}$ l'homologie est définie à coefficients dans $\mathbb{Z}$ par un procédé appelé relèvement de signes. Cette description combinatoire a permis de définir de nouveaux invariants pour les noeuds legendriens et transverses (voir [NOT07]). Dans ce chapitre, nous donnons une description alternative de l'homologie de Heegaard-Floer combinatoire des entrelacs à coefficients entiers.

Rappelons tout d'abord ce qu'est l'homologie de Heegaard-Floer combinatoire des entrelacs, pour cela on suit les conventions de [MOST06]. Un diagramme en grille planaire G est un carré dans le plan découpé de manière régulière en $n \times n$ carrés. On appelle n la complexité du diagramme en grille G. Chaque carré est décoré avec un X, un O ou alors rien de telle manière que chaque colonne horizontale et verticale contienne exactement un X et un O. On numérote les X's et les O's de 1 à n et on note $\mathbb{X}$ l'ensemble $\{X_i\}_{i=1}^n$ et $\mathbb{O}$ l'ensemble $\{O_i\}_{i=1}^n$.

Étant donné un diagramme en grille planaire G, on le positionne de manière standard dans le plan de la manière suivante : le coin inférieur gauche est à l'origine et chaque carré est de longueur 1. On construit une projection planaire d'un entrelacs en reliant par des segments orientés les O's aux X's le long des colonnes verticales et les X's aux O's le long des colonnes horizontales. À chaque point d'intersection, on fait passer le brin vertical au dessus du brin horizontal. On obtient ainsi un entrelacs orienté $\overrightarrow{L}$ dans S^3 et on dit que $\overrightarrow{L}$ a une présentation en grille donnée par G.

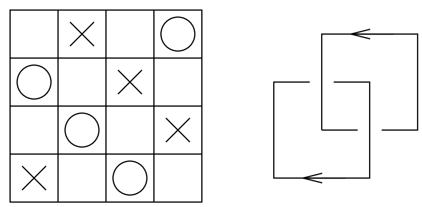

FIG. 2.1 – **Présentation en grille de l'entrelacs de Hopf.**

On place le diagramme en grille sur le tore $\mathcal{T}$ en faisant les identifications usuelles du bord du carré et on munit le tore de l'orientation induite par l'orientation du plan. En particulier, cela fournit un scindement de Heegaard de S^3 de genre 1 avec deux jeux de cercles attachants. On note α l'ensemble des cercles horizontaux et β l'ensemble des cercles verticaux. On associe à G un complexe de chaînes $(C^-(G), \partial^-)$. $C^-(G)$ est l'anneau du groupe $\mathfrak{S}_n$ sur l'anneau $\mathbb{Z}/2\mathbb{Z}[U_{O_1}, \ldots, U_{O_n}]$ où $\mathfrak{S}_n$ est le groupe des permutatiosn d'un ensemble à n éléments. Un générateurs $\mathbf{x} \in \mathfrak{S}_n$ est représenté sur G par son graphe : on marque par les points $(i, x(i))$ for $i = 0, \ldots, n-1$.

Pour A, B deux ensembles finis de points dans le plan on définit $\mathcal{I}(A, B)$ comme étant le nombre de paires de couple $(a_1, a_2) \in A$ et $(b_1, b_2) \in B$ telles $a_1 < b_1$ et $a_2 < b_2$. Soit $\mathcal{J}(A, B) = (\mathcal{I}(A, B) + \mathcal{I}(B, A))/2$.

On munit l'ensemble des générateurs d'un degré de Maslov M défini par

$$M(\mathbf{x}) = \mathcal{J}(\mathbf{x} - \mathbb{O}, \mathbf{x} - \mathbb{O}) + 1$$

où l'on étend $\mathcal{J}$ par bilinéarité à toutes les sommes (ou différences) formelles d'ensemble. Chaque variable U_{O_i} a un degré de Maslov égal à -2 et les constantes ont un degré de Maslov nul. Plus généralement, on note $M_S(\mathbf{x}) = \mathcal{J}(\mathbf{x} - S, \mathbf{x} - S) + 1$. L'ensemble des générateurs est filtré par le degré d'Alexander donné par $A(\mathbf{x}) = (A_1(\mathbf{x}), \ldots, A_\ell(\mathbf{x}))$ avec

$$A_i(\mathbf{x}) = \mathcal{J}(\mathbf{x} - \frac{1}{2}(\mathbb{X} + \mathbb{O}), \mathbb{X}_i - \mathbb{O}_i) - \frac{n_i - 1}{2}$$

où lorsque l'on numérote les composante de $\overrightarrow{L}$ de 1 à ℓ, $\mathbb{O}_i \subset \mathbb{O}$ (resp. $\mathbb{X}_i \subset \mathbb{X}$) est le sous-ensemble de $\mathbb{O}$ (resp. $\mathbb{X}$) qui appartient à la i-ème composante de $\overrightarrow{L}$ et n_i est le nombre de segment horizontaux de la i-ème composante (en fait, c'est la complexité du diagramme en grille représentant la i-ème composante). De plus, $A(U_{O_j}) = (0, \dots, 0, -1, 0, \dots, 0)$ où -1 se trouve à la i-ème coordonnée si O_j appartient à la i-ème composante de l'entrelacs.

Étant donnés deux généréteurs $\mathbf{x}$ et $\mathbf{y}$ et un rectangle r plongé dans le tore dont les arêtes sont sur les cercles α et β, on dit que r relie $\mathbf{x}$ à $\mathbf{y}$ si

- $\mathbf{y}.\mathbf{x}^{-1}$ est une transposition,
- les quatre coins du rectangle r sont des points appartenant aux graphes de $\mathbf{x}$ et $\mathbf{y}$,
- l'orientation du tore induit une orientation du bord de r de sorte que le long des arêtes horizontales de r on aille d'un point de $\mathbf{x}$ à un point de $\mathbf{y}$.

On note $\mathrm{Rect}(\mathbf{x}, \mathbf{y})$ l'ensemble des rectangles reliant $\mathbf{x}$ à $\mathbf{y}$: c'est soit l'ensemble vide, soit un ensemble composé de deux rectangles.

Un rectangle $r \in \mathrm{Rect}(\mathbf{x}, \mathbf{y})$ est dit vide si il n'y a pas de point du graphe de $\mathbf{x}$ à l'intérieur de r. On note $\mathrm{Rect}^\circ(\mathbf{x}, \mathbf{y})$ l'ensemble des rectangles vides reliant $\mathbf{x}$ à $\mathbf{y}$.

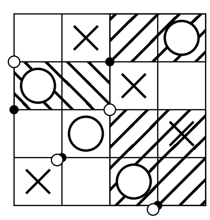

FIG. 2.2 – **Rectangles.** On a représenté le graphe de $\mathbf{x}$ par des points noirs et celui de $\mathbf{y}$ par des points blancs. Il y a deux rectangles dans $\mathrm{Rect}(\mathbf{x}, \mathbf{y})$ mais seul celui de gauche est vide.

La différentielle $\partial^- : C^-(G) \to C^-(G)$ est définie sur les générateurs par

$$\partial^- \mathbf{x} = \sum_{\mathbf{y} \in \mathfrak{S}_n} \sum_{r \in \mathrm{Rect}^\circ(\mathbf{x}, \mathbf{y})} U_{O_1}^{O_1(r)} \dots U_{O_n}^{O_n(r)}.\mathbf{y}$$

où $O_i(r)$ est le nombre de fois que O_i apparaît dans l'intérieur de r.

Théorème 2.0.1 (Manolescu-Ozsváth-Sarkar [MOS06])**.** $(C^-(G), \partial^-)$ *est un complexe de chaîne pour* $CF^-(S^3)$ *avec un degré homologique induit par* M *et une filtration induite par* A *qui coïncide avec la filtration des entrelacs de* $CF^-(S^3)$*.*

Dans [MOST06], les auteurs définissent une assignation de signes pour les rectangles vides $\mathbf{S} : \mathrm{Rect}^\circ \to \{\pm 1\}$. Alors, en considérant $C^-(G)$ l'anneau du groupe de $\mathfrak{S}_n$ sur $\mathbb{Z}[U_{O_1},\dots,U_{O_n}]$ et la différentielle $\partial^- : C^-(G) \to C^-(G)$ définie par

$$\partial^- \mathbf{x} = \sum_{\mathbf{y}\in\mathfrak{S}_n} \sum_{r\in \mathrm{Rect}^\circ(\mathbf{x},\mathbf{y})} \mathbf{S}(r).U_{O_1}^{O_1(r)}\dots U_{O_n}^{O_n(r)}.\mathbf{y}$$

ils obtiennent le résultat suivant :

Théorème 2.0.2 (Manolescu-Ozsváth-Szabó-D. Thurston [MOST06])**.** *Soit $\overrightarrow{L}$ un entrelacs orienté avec ℓ composantes. On numérote l'ensemble des O's de sorte que $O_1,\dots,O_\ell$ correspondent aux différentes composantes de l'entrelacs $\overrightarrow{L}$. Alors le type de quasi-isomorphisme filtré de $(C^-(G),\partial^-)$ sur $\mathbb{Z}[U_{O_1},\dots,U_{O_\ell}]$ est un invariant de l'entrelacs.*

Dans ce chapitre, nous donnons une manière de relever le complexe à coefficients dans $\mathbb{Z}/2\mathbb{Z}$ en un complexe à coefficients entiers grâce à $\widetilde{\mathfrak{S}}_n$ l'extension spin de $\mathfrak{S}_n$. L'extension spin $\widetilde{\mathfrak{S}}_n$ admet une présentation par générateurs et relations. L'ensemble des générateurs est $\{z\}\cup\{\widetilde{\tau}_{i,j}\}_{i\neq j}$ où $0\leq i,j\leq n-1$ ($\widetilde{\tau}_{i,j}$ représente la permutation (ij)) et les relations sont

$$\begin{array}{l} z.z=\widetilde{\mathbf{1}} \qquad z\widetilde{\tau}_{i,j}=\widetilde{\tau}_{i,j}z \qquad \widetilde{\tau}_{i,j}=z\widetilde{\tau}_{j,i} \qquad \widetilde{\tau}_{i,j}.\widetilde{\tau}_{i,j}=z \qquad \text{pour tout } i,j \\ \widetilde{\tau}_{i,j}.\widetilde{\tau}_{k,l}=z\widetilde{\tau}_{k,l}.\widetilde{\tau}_{i,j} \qquad \text{pour tout } i,j,k,l \text{ si } \{i,j\}\cap\{k,l\}=\emptyset \\ \widetilde{\tau}_{i,j}.\widetilde{\tau}_{j,k}.\widetilde{\tau}_{i,j}=\widetilde{\tau}_{j,k}.\widetilde{\tau}_{i,j}.\widetilde{\tau}_{j,k}=\widetilde{\tau}_{i,k} \qquad \text{pour tout } i,j,k \end{array}$$

Cette extension est une extension centrale non triviale lorsque $n\geq 4$. Ainsi, au lieu d'attacher un signe aux rectangles, on munit l'ensemble des générateurs d'un signe. Pour cela, on considère l'anneau $\Lambda=\mathbb{Z}[U_{O_1},\dots,U_{O_n}]$ et on définit le complexe de chaînes filtrées $(\widetilde{C}^-(G),\widetilde{\partial}^-)$ où $\widetilde{C}^-(G)$ est le module quotient du Λ-module libre de base $\widetilde{\mathfrak{S}}_n$ par le sous-module engendré par $\{z+1\}$. Cela permet de penser à z comme un signe. En particulier, les rectangles intervenant dans la différentielles sont maintenant vu des éléments du type $\widetilde{\tau}_{i,j}$ où i est l'abscisse du coin inférieur gauche du rectangle et j est l'abscisse du coin supérieur droit du rectangle. Dans la section 2.4 on montre que l'homologie filtrée de $(\widetilde{C}^-(G)\otimes\mathbb{Z}/2\mathbb{Z},\widetilde{\partial}^-)$ coïncide avec l'homologie filtrée du complexe de chaînes $(C^-(G),\partial^-)$ à coefficients dans $\mathbb{Z}/2\mathbb{Z}$ (Proposition 2.4.1). En particulier, cela signifie que l'on a bien défini un relèvement du complexe à coefficients dans $\mathbb{Z}/2\mathbb{Z}$ en un complexe à coefficients entiers. Alors, en adaptant la démonstration faite dans [MOST06] on obtient

Théorème 2.0.3 (Theorem 2.4.3)**.** *Soit $\overrightarrow{L}$ un entrelacs orienté avec ℓ composantes. On numérote l'ensemble des O's de sorte que $O_1,\dots,O_\ell$ correspondent aux différentes composantes de l'entrelacs $\overrightarrow{L}$. Alors le type de quasi-isomorphisme filtré de $(\widetilde{C}^-(G),\widetilde{\partial}^-)$ sur $\mathbb{Z}[U_{O_1},\dots,U_{O_\ell}]$ est un invariant de l'entrelacs.*

Dans la section 2.5, on montre que le complexe de chaînes définit une assignation de signes au sens de [MOST06] (Lemme 2.5.2) et que $(\widetilde{C}^-(G), \widetilde{\partial}^-)$ est isomorphe par un isomorphisme qui respecte la filtration au complexe de chaînes $(C^-(G), \partial^-)$ à coefficients dans $\mathbb{Z}$ (Proposition 2.5.3).

Finalement la section 2.6 est dévolue à l'explication d'un algorithme permettant de calculer le relèvement de signe correspondant à la description faite ci-dessus.

2.1 Introduction

Heegaard-Floer homology [OS04c] is an invariant for closed oriented 3-manifolds which was extended to give an invariant for null-homologous oriented links in such manifolds called link Floer homology [OS04b], [OS05] [Ras03]. It gives the Seifert genus $g(K)$ of a knot K [OS04a], detects fibered knots ([Ghi06] in the case where $g(K) = 1$ and [Ni06] in general) and its graded Euler characteristic gives the Alexander polynomial ([OS04b], [Ras03]). Recently, a combinatorial description of link Floer homology was given [MOS06] and its topological invariance was proved in a purely combinatorial way [MOST06]. This combinatorial description also gives new invariants for legendrian and transverse knots (see [NOT07]). In this chapter we give an alternative description of combinatorial link Floer homology with $\mathbb{Z}$ coefficients. This point of view was recently used by Audoux [Aud07] to describe combinatorial Heegaard–Floer homology for singular knots.

Let first recall the context of combinatorial link Floer homology: we follow conventions of [MOST06]. A **planar grid diagram** G lies in a square on the plane with $n \times n$ squares where n is the complexity of G. Each square is decorated with an X, an O or nothing in such a way that each row and each column contains exactly one X and one O. We number the X's and the O's from 1 to n and denote $\mathbb{X}$ the set $\{X_i\}_{i=1}^n$ and $\mathbb{O}$ the set $\{O_i\}_{i=1}^n$.

Given a grid diagram G, we place it in standard position on the plane as follows: the bottom left corner is at the origin and each cell is a square of length one. We construct a planar link projection by drawing horizontal segments from the O's to the X's in each row and vertical segments from the X's to the O's in each column. At each intersection point, the vertical segment is over the horizontal one. This gives an oriented link $\overrightarrow{L}$ in S^3 and we say that $\overrightarrow{L}$ has a **grid presentation** given by G.

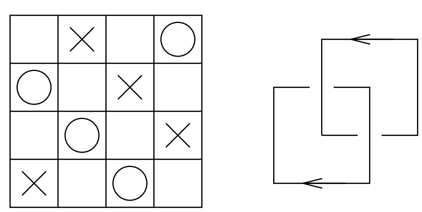

Figure 2.3: **Grid presentation of the Hopf link.**

We place the grid diagram on the oriented torus $\mathcal{T}$ by making the usual identification of the boundary of the square. We endow $\mathcal{T}$ with the orientation induced by the planar orientation. Let α be the collection of the horizontal circles and β the collection of the vertical ones. We associate with G a chain complex (C^-, ∂^-): it is the group ring of $\mathfrak{S}_n$ over $\mathbb{Z}/2\mathbb{Z}[U_{O_1}, \dots, U_{O_n}]$ where $\mathfrak{S}_n$ is the permutation group of n elements. A generator $\mathbf{x} \in \mathfrak{S}_n$ is given on G by its graph: we place dots in points $(i, x(i))$ for $i = 0, \ldots, n-1$.

For A, B two finite sets of points in the plane we define $\mathcal{I}(A, B)$ to be the number of pairs $(a_1, a_2) \in A$ and $(b_1, b_2) \in B$ such that $a_1 < b_1$ and $a_2 < b_2$. Let $\mathcal{J}(A, B) = (\mathcal{I}(A, B) + \mathcal{I}(B, A))/2$. We provide the set of generators with a **Maslov degree** M given by

$$M(\mathbf{x}) = \mathcal{J}(\mathbf{x} - \mathbb{O}, \mathbf{x} - \mathbb{O}) + 1$$

where we extend $\mathcal{J}$ by bilinearly over formal sums (or differences) of subsets. Each variable U_{O_i} has a Maslov degree equal to -2 and constants have Maslov degree equal to zero. Let $M_S(\mathbf{x})$ be the same as $M(\mathbf{x})$ with the set S playing the role of $\mathbb{O}$.

We provide the set of generators with an **Alexander filtration** A given by $A(\mathbf{x}) = (A_1(\mathbf{x}), \ldots, A_l(\mathbf{x}))$ with

$$A_i(\mathbf{x}) = \mathcal{J}(\mathbf{x} - \frac{1}{2}(\mathbb{X} + \mathbb{O}), \mathbb{X}_i - \mathbb{O}_i) - \frac{n_i - 1}{2}$$

where when we number the components of $\overrightarrow{L}$ from 1 to ℓ, $\mathbb{O}_i \subset \mathbb{O}$ (respect. $\mathbb{X}_i \subset \mathbb{X}$) is the subset of $\mathbb{O}$ (resp. $\mathbb{X}$) wich belongs to the i-th component of $\overrightarrow{L}$ and n_i is the number of horizontal segments which belongs to the i-th component. We let $A(U_{O_j}) = (0, \ldots, -1, 0, \ldots, 0)$ where -1 corresponds to the i-th coordonate if O_j belongs to the i-th component of $\overrightarrow{L}$.

Given two generators $\mathbf{x}$ and $\mathbf{y}$ and an immersed **rectangle** r in the torus whose edges are arcs in the horizontal and vertical circles, we say that r connects $\mathbf{x}$ to $\mathbf{y}$ if $\mathbf{y}.\mathbf{x}^{-1}$ is a transposition, if all four corners of r are intersection points in $\mathbf{x} \cup \mathbf{y}$, and if we traverse each horizontal boundary component of r in the direction dictated by the orientation of r induced by $\mathcal{T}$, then the arc is oriented from a point in $\mathbf{x}$ to the point in $\mathbf{y}$. Let $\mathrm{Rect}(\mathbf{x}, \mathbf{y})$ be the set of rectangles connecting $\mathbf{x}$ to $\mathbf{y}$: either it is the empty set or it consists of exactly two rectangles. Here a rectangle $r \in \mathrm{Rect}(\mathbf{x}, \mathbf{y})$ is said to be **empty** if there is no point of $\mathbf{x}$ in its interior. Let $\mathrm{Rect}^\circ(\mathbf{x}, \mathbf{y})$ be the set of empty rectangles connecting $\mathbf{x}$ to $\mathbf{y}$.

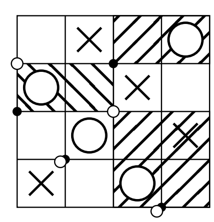

Figure 2.4: **Rectangles.** We mark with black dots the generator $\mathbf{x}$ and with white dots the generator $\mathbf{y}$. There are two rectangles in $\mathrm{Rect}(\mathbf{x}, \mathbf{y})$ but only the left one is in $\mathrm{Rect}^\circ(\mathbf{x}, \mathbf{y})$.

The differential $\partial^- : C^-(G) \to C^-(G)$ is given on the set of generators by

$$\partial^- \mathbf{x} = \sum_{\mathbf{y}\in\mathfrak{S}_n} \sum_{r\in\mathrm{Rect}^\circ(\mathbf{x},\mathbf{y})} U_{O_1}^{O_1(r)} \dots U_{O_n}^{O_n(r)}.\mathbf{y}$$

where $O_i(r)$ is the number of times O_i appears in the interior of r.

Theorem 2.1.1 (Manolescu-Ozsváth-Sarkar [MOS06])**.** *$(C^-(G), \partial^-)$ is a chain complex for $CF^-(S^3)$ with homological degree induced by M and filtration level induced by A which coincides with the link filtration of $CF^-(S^3)$.*

In [MOST06], the authors define a sign assigment for empty rectangles $\mathbf{S} : \mathrm{Rect}^\circ \to \{\pm 1\}$. Then, by considering $C^-(G)$ the group ring of $\mathfrak{S}_n$ over $\mathbb{Z}[U_{O_1}, \dots, U_{O_n}]$ and the differential $\partial^- : C^-(G) \to C^-(G)$ given by

$$\partial^- \mathbf{x} = \sum_{\mathbf{y}\in\mathfrak{S}_n} \sum_{r\in\mathrm{Rect}^\circ(\mathbf{x},\mathbf{y})} \mathbf{S}(r).U_{O_1}^{O_1(r)} \dots U_{O_n}^{O_n(r)}.\mathbf{y}$$

they obtain the following result:

Theorem 2.1.2 (Manolescu-Ozsváth-Szabó-D. Thurston [MOST06])**.** *Let $\overrightarrow{L}$ be an oriented link with ℓ components. We number the $\mathbb{O}$ so that $O_1, \dots, O_\ell$ correspond to the different components of $\overrightarrow{L}$. Then the filtered quasi-isomorphism type of $(C^-(G), \partial^-)$ over $\mathbb{Z}[U_{O_1}, \dots, U_{O_\ell}]$ is an invariant of the link.*

We give a way to refine the complex over $\mathbb{Z}$ thanks to $\widetilde{\mathfrak{S}}_n$ the spin extension of $\mathfrak{S}_n$ which is a non-trivial central extension of $\mathfrak{S}_n$ by $\mathbb{Z}/2\mathbb{Z}$. This chapter is organised as follow. In section 2.2 we define the spin extension $\widetilde{\mathfrak{S}}_n$ and make some algebraic calculus. Let z be the unique non-trivial central element of $\widetilde{\mathfrak{S}}_n$ and $\Lambda = \mathbb{Z}[U_{O_1}, \dots, U_{O_n}]$. In section 2.3 we define a filtered chain complex $(\widetilde{C}^-(G), \widetilde{\partial}^-)$ where $\widetilde{C}^-(G)$ is the quotient module of the free Λ-module with generating set $\widetilde{\mathfrak{S}}_n$ by the submodule generated by $\{z+1\}$. In section 2.4 we prove that the filtered homology of $(\widetilde{C}^-(G) \otimes \mathbb{Z}/2\mathbb{Z}, \widetilde{\partial}^-)$ coincides with the filtered homology of the chain complex $(C^-(G), \partial^-)$ with coefficients in $\mathbb{Z}/2\mathbb{Z}$. Then, following the proof in [MOST06] we obtain

Theorem 2.1.3. *Let $\overrightarrow{L}$ be an oriented link with ℓ components. We number the $\mathbb{O}$ so that $O_1, \dots, O_\ell$ correspond to the different components of $\overrightarrow{L}$. Then the filtered quasi-isomorphism type of $(\widetilde{C}^-(G), \widetilde{\partial}^-)$ over $\mathbb{Z}[U_{O_1}, \dots, U_{O_\ell}]$ is an invariant of the link.*

In section 2.5, we prove that our chain complex defines a sign assignment in the sense of [MOST06] and that $(\widetilde{C}^-(G), \widetilde{\partial}^-)$ is filtered isomorphic to $(C^-(G), \partial^-)$ with coefficients in $\mathbb{Z}$. Finally in section 2.6 we explain how to compute signs.

2.2 Algebraic preliminaries

Let $\mathfrak{S}_n$ be the group of bijections of a set with n elements numbered from 0 to $n-1$. It is given in terms of generators and relations where the set of generators is $\{\tau_i\}_{i=0}^{n-2}$ with τ_i the transposition which exchanges i and $i+1$ and relations are

$$\tau_i^2 = \mathbf{1} \qquad 0 \le i \le n-2$$
$$\tau_i.\tau_j = \tau_j.\tau_i \qquad |i-j| > 1, \quad 0 \le i,j \le n-2$$
$$\tau_i.\tau_{i+1}.\tau_i = \tau_{i+1}.\tau_i.\tau_{i+1} \qquad 0 \le i \le n-3$$

Proposition 2.2.1. *The group given by generators and relations*

$$\widetilde{\mathfrak{S}}_n = < \widetilde{\tau}_0, \ldots, \widetilde{\tau}_{n-2}, z | \quad \begin{array}{l} z^2 = \widetilde{\mathbf{1}}, z\widetilde{\tau}_i = \widetilde{\tau}_i z, \widetilde{\tau}_i^2 = z, \quad 0 \le i \le n-2; \\ \widetilde{\tau}_i.\widetilde{\tau}_j = z\widetilde{\tau}_j.\widetilde{\tau}_i \qquad |i-j| > 1, \quad 0 \le i,j \le n-2; \\ \widetilde{\tau}_i.\widetilde{\tau}_{i+1}.\widetilde{\tau}_i = \widetilde{\tau}_{i+1}.\widetilde{\tau}_i.\widetilde{\tau}_{i+1} \qquad 0 \le i \le n-3 \ > \end{array}$$

is a non-trivial central extension ($n \ge 4$) of $\mathfrak{S}_n$ by $\mathbb{Z}/2\mathbb{Z}$ called the spin extension of $\mathfrak{S}_n$.

Remark. The terminology *spin extension* of $\mathfrak{S}_n$ is inspired by the fact that $\widetilde{\mathfrak{S}}_n$ can be viewed as a subgroup of Spin(n) (see also [Bes94], [Dij99]).

$$\begin{array}{c} \widetilde{\mathfrak{S}}_n \subset \mathrm{Spin(n)} \\ \downarrow \\ \mathfrak{S}_n \subset O(n) \end{array}$$

Proof. Let $p : \widetilde{\mathfrak{S}}_n \to \mathfrak{S}_n$ be the morphism given on generators by $p(\widetilde{\tau}_i) = \tau_i$ and $p(z) = \mathbf{1}$: it is onto. The central subgroup $H = \{\widetilde{\mathbf{1}}, z\} \subset \widetilde{\mathfrak{S}}_n$ is such that $H \subset \ker p$. The group $\widetilde{\mathfrak{S}}_n/H$ admits a presentation by generators and relations which coincides with the one for $\mathfrak{S}_n$ and so $\overline{p} : \widetilde{\mathfrak{S}}_n/H \to \mathfrak{S}_n$ is an isomorphism. In particular $|\widetilde{\mathfrak{S}}_n| = 2n!$ and $\widetilde{\mathfrak{S}}_n$ is the following central extension of $\mathfrak{S}_n$ by $\mathbb{Z}/2\mathbb{Z}$:

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xhookrightarrow{i} \widetilde{\mathfrak{S}}_n \xrightarrow{p} \mathfrak{S}_n \longrightarrow 1$$

We prove that it is non-trivial. Let $\mathbb{Q}_8$ be the subgroup of $\widetilde{\mathfrak{S}}_n$ generated by $\widetilde{\tau}_0, \widetilde{\tau}_2, z$. Then $\mathbb{Q}_8$ is isomorphic to the unit sphere in the space of quaternions intersected with the lattice $\mathbb{Z}^4$ by a morphism Φ such that $\Phi(\widetilde{\tau}_0) = i$, $\Phi(\widetilde{\tau}_2) = j$, $\Phi(\widetilde{\tau}_0.\widetilde{\tau}_2) = k$ and $\Phi(z) = -1$. Therefore $\widetilde{\mathfrak{S}}_n$ is non-trivial. □

For $i < j$, define

$$\widetilde{\tau}_{i,j} = \widetilde{\tau}_i.\widetilde{\tau}_{i+1}. \ldots .\widetilde{\tau}_{j-2}.\widetilde{\tau}_{j-1}.\widetilde{\tau}_{j-2}. \ldots .\widetilde{\tau}_{i+1}.\widetilde{\tau}_i$$

and $\widetilde{\tau}_{j,i} = z\widetilde{\tau}_{i,j}$.

Let $\varepsilon : \mathfrak{S}_n \to \{0,1\}$ be the signature morphism.

Lemma 2.2.2. *Let* $\widetilde{\mathbf{x}} = \widetilde{\tau}_{i_1}.\widetilde{\tau}_{i_2}.\dots.\widetilde{\tau}_{i_k}$ *be an element in* $\widetilde{\mathfrak{S}}_n$ *and* $\mathbf{x} = p(\widetilde{\mathbf{x}}) \in \mathfrak{S}_n$*. Then for any* $0 \leq i \neq j \leq n-1$

$$\widetilde{\mathbf{x}}.\widetilde{\tau}_{i,j}.\widetilde{\mathbf{x}}^{-1} = z^{\varepsilon(\mathbf{x})}\widetilde{\tau}_{\mathbf{x}(i),\mathbf{x}(j)}$$

Proof. Since $\widetilde{\mathbf{x}} = \widetilde{\tau}_{i_1}.\widetilde{\tau}_{i_2}.\dots.\widetilde{\tau}_{i_k}$, $\widetilde{\mathbf{x}}^{-1} = z^{\varepsilon(\mathbf{x})}\widetilde{\tau}_{i_k}.\dots.\widetilde{\tau}_{i_1}$. We prove by induction on $k \geq 1$ that for any $i,j \in \{0,\dots,n-1\}$ we have $\widetilde{\mathbf{x}}.\widetilde{\tau}_{i,j}.\widetilde{\mathbf{x}}^{-1} = z^{\varepsilon(\mathbf{x})}\widetilde{\tau}_{\mathbf{x}(i),\mathbf{x}(j)}$.

- **Initialization.** Let $\widetilde{\mathbf{x}} = \widetilde{\tau}_l$ and $0 \leq i < j \leq n-1$. So $\widetilde{\tau}_l^{-1} = z\widetilde{\tau}_l$ and $\varepsilon(\mathbf{x}) = 1$. There are several cases.
  - **Case 1:** $l < i-1$ **or** $l > j$**.** $\widetilde{\mathbf{x}}.\widetilde{\tau}_{i,j}.z\widetilde{\mathbf{x}} = z\tau_{i,j}$.
  - **Case 2:** $l = i-1$**.** $\widetilde{\mathbf{x}}.\widetilde{\tau}_{i,j}.z\widetilde{\mathbf{x}} = z\widetilde{\tau}_{i-1}.\widetilde{\tau}_{i,j}.\widetilde{\tau}_{i-1} = z\widetilde{\tau}_{i-1,j}$ by definition.
  - **Case 3:** $l = i$**.** $\widetilde{\tau}_i.\widetilde{\tau}_{i,j}.z\widetilde{\tau}_i = z\widetilde{\tau}_{i+1,j}$.
  - **Case 4:** $i < l < j-1$**.** We prove by induction on $l - i \geq 1$ for i,j fixed that $\widetilde{\tau}_l.\widetilde{\tau}_{i,j}.z\widetilde{\tau}_l = z\widetilde{\tau}_{\tau(i),\tau(j)}$. For $l = i+1$ then we have

$$\begin{aligned}\widetilde{\tau}_{i+1}.\widetilde{\tau}_{i,j}.z\widetilde{\tau}_{i+1} &= z\widetilde{\tau}_i.\widetilde{\tau}_{i+1}.\widetilde{\tau}_i.\widetilde{\tau}_{i+2,j}.\widetilde{\tau}_i.\widetilde{\tau}_{i+1}.\widetilde{\tau}_i \\ &= z\widetilde{\tau}_i.\widetilde{\tau}_{i+1}.\widetilde{\tau}_{i+2,j}.\widetilde{\tau}_{i+1}.\widetilde{\tau}_i \\ &= z\widetilde{\tau}_{i,j}\end{aligned}$$

    Suppose it is proved until rank $(l-1)-i$. Then for $\widetilde{\mathbf{x}} = \widetilde{\tau}_l$ with $l < j-1$ we have

$$\begin{aligned}\widetilde{\mathbf{x}}.\widetilde{\tau}_i.z\widetilde{\mathbf{x}} &= z\widetilde{\tau}_l.\widetilde{\tau}_{i,j}.\widetilde{\tau}_l \\ &= z(\widetilde{\tau}_i.\dots.\widetilde{\tau}_{l-2}).(\widetilde{\tau}_l.\widetilde{\tau}_{l-1}.\widetilde{\tau}_l).\widetilde{\tau}_{l-1,j}.(\widetilde{\tau}_l.\widetilde{\tau}_{l-1}.\widetilde{\tau}_l).(\widetilde{\tau}_{l-2}.\dots.\widetilde{\tau}_i) \\ &= z(\widetilde{\tau}_i.\dots.\widetilde{\tau}_{l-2}).(\widetilde{\tau}_{l-1}.\widetilde{\tau}_l.\widetilde{\tau}_{l-1}).\widetilde{\tau}_{l-1,j}.(\widetilde{\tau}_{l-1}.\widetilde{\tau}_l.\widetilde{\tau}_{l-1}).(\widetilde{\tau}_{l-2}.\dots.\widetilde{\tau}_i) \\ &= z(\widetilde{\tau}_i.\dots.\widetilde{\tau}_{l-1}.\widetilde{\tau}_l).\widetilde{\tau}_{l-1,j}.(\widetilde{\tau}_l.\widetilde{\tau}_{l-1}.\dots.\widetilde{\tau}_i) \text{ by induction} \\ &= z(\widetilde{\tau}_i.\dots.\widetilde{\tau}_{l-1}).\widetilde{\tau}_{l,j}.(\widetilde{\tau}_{l-1}.\dots.\widetilde{\tau}_i) \text{ by induction} \\ &= z\widetilde{\tau}_{i,j} \text{ by case 2}\end{aligned}$$

  - **Case 5:** $l = j-1$**.**

$$\begin{aligned}\widetilde{\tau}_{j-1}.\widetilde{\tau}_{i,j}.z\widetilde{\tau}_{j-1} &= z(\widetilde{\tau}_i.\dots.\widetilde{\tau}_{j-3}).\widetilde{\tau}_{j-1}.\widetilde{\tau}_{j-2}.\widetilde{\tau}_{j-1}.\widetilde{\tau}_{j-2}.\widetilde{\tau}_{j-1}.(\widetilde{\tau}_{j-3}.\dots.\widetilde{\tau}_i) \\ &= z\widetilde{\tau}_{i,j-1}\end{aligned}$$

  - **Case 6:** $l = j$**.**

$$\begin{aligned}\widetilde{\tau}_j.\widetilde{\tau}_{i,j}.z\widetilde{\tau}_j &= z(\widetilde{\tau}_i.\dots.\widetilde{\tau}_{j-2}).\widetilde{\tau}_j.\widetilde{\tau}_{j-1}.\widetilde{\tau}_j.(\widetilde{\tau}_{j-2}.\dots.\widetilde{\tau}_i) \\ &= z\widetilde{\tau}_{i,j+1}\end{aligned}$$

- **Heredity.** Suppose the property is true until rank k. Let $\widetilde{\mathbf{x}} = \widetilde{\tau}_{i_1}.\widetilde{\tau}_{i_2}.\dots.\widetilde{\tau}_{i_k}$ and $\widetilde{\tau}_{i,j}$ be two elements in $\widetilde{\mathfrak{S}}_n$. Denote $\widetilde{\mathbf{y}} = \widetilde{\tau}_{i_2}.\dots.\widetilde{\tau}_{i_k}$. Then $\widetilde{\mathbf{x}}.\widetilde{\tau}_{i,j}.\widetilde{\mathbf{x}}^{-1} = \widetilde{\tau}_{i_1}.\widetilde{\mathbf{y}}.\widetilde{\tau}_{i,j}.\widetilde{\mathbf{y}}^{-1}.z\widetilde{\tau}_{i_1}$. By induction hypothesis,

$$\widetilde{\mathbf{y}}.\widetilde{\tau}_{i,j}.\widetilde{\mathbf{y}}^{-1} = z^{\varepsilon(\mathbf{y})}.\widetilde{\tau}_{\mathbf{y}(i),\mathbf{y}(j)}$$

So, $\widetilde{\mathbf{x}}.\widetilde{\tau}_{i,j}.\widetilde{\mathbf{x}}^{-1} = \widetilde{\tau}_{i_1}.z^{\varepsilon(\mathbf{y})}.\widetilde{\tau}_{\mathbf{y}(i),\mathbf{y}(j)}.z\widetilde{\tau}_{i_1}$. By induction hypothesis one more time,

$$\widetilde{\mathbf{x}}.\widetilde{\tau}_{i,j}.\widetilde{\mathbf{x}}^{-1} = z^{\varepsilon(\mathbf{y})+1}\widetilde{\tau}_{\tau_{i_1}.\mathbf{y}(i),\tau_{i_1}.\mathbf{y}(j)} = z^{\varepsilon(\mathbf{x})}.\widetilde{\tau}_{\mathbf{x}(i),\mathbf{x}(j)}$$

□

The group $\widetilde{\mathfrak{S}}_n$ has another presentation in terms of generators and relations. Take $\{z'\} \cup \{\widetilde{\tau}'_{i,j}\}_{i\neq j}$ where $0 \leq i,j \leq n-1$ as the set of generators with the following relations:

$$z'.z' = \widetilde{\mathbf{1}}' \qquad z'\widetilde{\tau}'_{i,j} = \widetilde{\tau}'_{i,j}z' \qquad \widetilde{\tau}'_{i,j} = z'\widetilde{\tau}'_{j,i} \qquad \widetilde{\tau}'_{i,j}.\widetilde{\tau}'_{i,j} = z' \qquad \text{for any } i,j \tag{2.1}$$

$$\widetilde{\tau}'_{i,j}.\widetilde{\tau}'_{k,l} = z'\widetilde{\tau}'_{k,l}.\widetilde{\tau}'_{i,j} \qquad \text{for any } i,j,k,l \text{ if } \{i,j\}\cap\{k,l\} = \emptyset \tag{2.2}$$

$$\widetilde{\tau}'_{i,j}.\widetilde{\tau}'_{j,k}.\widetilde{\tau}'_{i,j} = \widetilde{\tau}'_{j,k}.\widetilde{\tau}'_{i,j}.\widetilde{\tau}'_{j,k} = \widetilde{\tau}'_{i,k} \qquad \text{for any } i,j,k \tag{2.3}$$

Proof. Let $\widetilde{\mathfrak{S}}_n$ the group with z and $\widetilde{\tau}_i$ as generators and $\widetilde{\mathfrak{S}}'_n$ the other one. Define $\phi : \widetilde{\mathfrak{S}}_n \to \widetilde{\mathfrak{S}}'_n$ given on generators by $\phi(\widetilde{\tau}_i) = \widetilde{\tau}'_{i,i+1}$, $\phi(z) = z'$. For $i < j$, let $\phi(\widetilde{\tau}_{i,j}) = \widetilde{\tau}'_{i,j}$. By definition, (2.1) is verified. Lemma 2.2.2 gives equations (2.2) and (2.3). So the map ϕ extends to a group isomorphism. □

In what follows, we drop the prime exponent and only reffer to $\widetilde{\tau}_{i,j}$ and z ($\widetilde{\tau}_i$ means $\widetilde{\tau}_{i,i+1}$).

2.3 The chain complex

Let G be a grid presentation with complexity n of the link $\overrightarrow{L}$. Let Λ denote the ring $\mathbb{Z}[U_{O_1},\ldots,U_{O_n}]$. We define $\widetilde{C}^-(G)$ to be the free Λ-module with generating set $\widetilde{\mathfrak{S}}_n$ quotiented by the submodule generated by $\{z+1\}$ i.e.

$$\widetilde{C}^-(G) = \Lambda[\widetilde{\mathfrak{S}}_n]/<z+1>$$

Considered as module, $\widetilde{C}^-(G)$ coincides with the free Λ-module with generating set $\mathfrak{S}_n$. But we can also consider the structure of algebra of $\widetilde{C}^-(G)$ over Λ. In this case, one can think of $\widetilde{C}^-(G)$ as the group algebra of $\mathfrak{S}_n$ over Λ where the product is twisted by a non-trivial 2-cocycle (see section 2.5).

We endow the set of generators with a Maslov grading M and an Alexander filtration A given by:

$$M(\widetilde{\mathbf{x}}) = M(\mathbf{x})$$

$$A(\widetilde{\mathbf{x}}) = A(\mathbf{x})$$

Let $\widetilde{\mathbf{x}}$ and $\widetilde{\mathbf{y}}$ be two elements of $\widetilde{\mathfrak{S}}_n$. The set of rectangles $\mathrm{Rect}(\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}})$ connecting $\widetilde{\mathbf{x}}$ to $\widetilde{\mathbf{y}}$ is the empty set if $\widetilde{\mathbf{y}} \neq \widetilde{\mathbf{x}}.\widetilde{\tau}_{i,j}$ for all $i \neq j$, else it is $\widetilde{\tau}_{i,j}$.

If we consider the set $\mathrm{Rect}(\mathbf{x}, \mathbf{y})$ of rectangles connecting $\mathbf{x}$ to $\mathbf{y}$ as in [MOST06], either it is the empty set, or it consists of two rectangles. We interpret the rectangle $\widetilde{\tau}_{i,j}$ in the oriented torus $\mathcal{T}$ as the rectangle whose bottom left corner belongs to the i-th vertical circle. So in the case where $\mathrm{Rect}(\mathbf{x}, \mathbf{y}) = \{r_1, r_2\}$ the two corresponding rectangles are $\widetilde{\tau}_{i,j}$ and $\widetilde{\tau}_{j,i}$ and remember that $\widetilde{\tau}_{i,j} = z\widetilde{\tau}_{j,i}$. Let r be the rectangle of $\mathrm{Rect}(\mathbf{x}, \mathbf{y})$ corresponding to $\widetilde{r}$. A rectangle $\widetilde{r} \in \mathrm{Rect}(\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}})$ is said to be empty if $r \in \mathrm{Rect}^\circ(\mathbf{x}, \mathbf{y})$. The set of empty rectangles connecting $\widetilde{\mathbf{x}}$ to $\widetilde{\mathbf{y}}$ is denoted $\mathrm{Rect}^\circ(\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}})$.

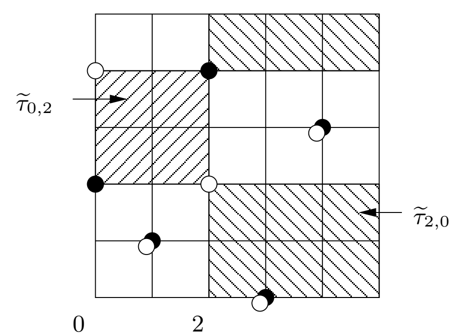

Figure 2.5: **Rectangles.** Black dots represent $\mathbf{x}$ and white dots $\mathbf{y}$. The two hatched regions correspond to rectangles $\widetilde{\tau}_{0,2} \in \mathrm{Rect}(\widetilde{\mathbf{x}}, \widetilde{\mathbf{x}}.\widetilde{\tau}_{0,2})$ and $\widetilde{\tau}_{2,0} \in \mathrm{Rect}(\widetilde{\mathbf{x}}, \widetilde{\mathbf{x}}.\widetilde{\tau}_{2,0})$. The rectangle $\widetilde{\tau}_{0,2}$ is an empty rectangle while $\widetilde{\tau}_{2,0}$ is not.

We endow $\widetilde{C}^-(G)$ with a differential $\widetilde{\partial}^-$ given on elements of $\widetilde{\mathfrak{S}}_n$ by:

$$\widetilde{\partial}^- \widetilde{\mathbf{x}} = \sum_{\widetilde{\mathbf{y}} \in \widetilde{\mathfrak{S}}_n} \sum_{\widetilde{r} \in \mathrm{Rect}^\circ(\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}})} U_{O_1}^{O_1(\widetilde{r})} \dots U_{O_n}^{O_n(\widetilde{r})} . \widetilde{\mathbf{y}}$$

where $O_k(\widetilde{r})$ is the number of times O_k appears in the interior of r.

Proposition 2.3.1. *The differential $\widetilde{\partial}^-$ drops the Maslov degree by one and respect the Alexander filtration.*

Proof. It is a straightforward consequence of calculus done in [MOST06]. □

Proposition 2.3.2. *The endomorphism $\widetilde{\partial}^-$ of $\widetilde{C}^-(G)$ is a differential, i.e.*

$$\widetilde{\partial}^- \circ \widetilde{\partial}^- = 0$$

Proof. Let $\widetilde{\mathbf{x}} = s(\mathbf{x}) \in \widetilde{\mathfrak{S}}_n$, viewed as a generator of $\widetilde{C}^-(G)$. Then

$$\widetilde{\partial}^- \circ \widetilde{\partial}^- (\widetilde{\mathbf{x}}) = \sum_{(\widetilde{\mathbf{y}}, \widetilde{\mathbf{z}}) \in \widetilde{\mathfrak{S}}_n} \sum_{\widetilde{r}_2 \in \mathrm{Rect}^\circ(\widetilde{\mathbf{y}}, \widetilde{\mathbf{z}})} \sum_{\widetilde{r}_1 \in \mathrm{Rect}^\circ(\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}})} U_{O_1}^{O_1(\widetilde{r}_1) + O_1(\widetilde{r}_2)} \dots U_{O_n}^{O_n(\widetilde{r}_1) + O_n(\widetilde{r}_2)} . \widetilde{\mathbf{z}}$$

There are different cases which are illustrated by figure 2.6.

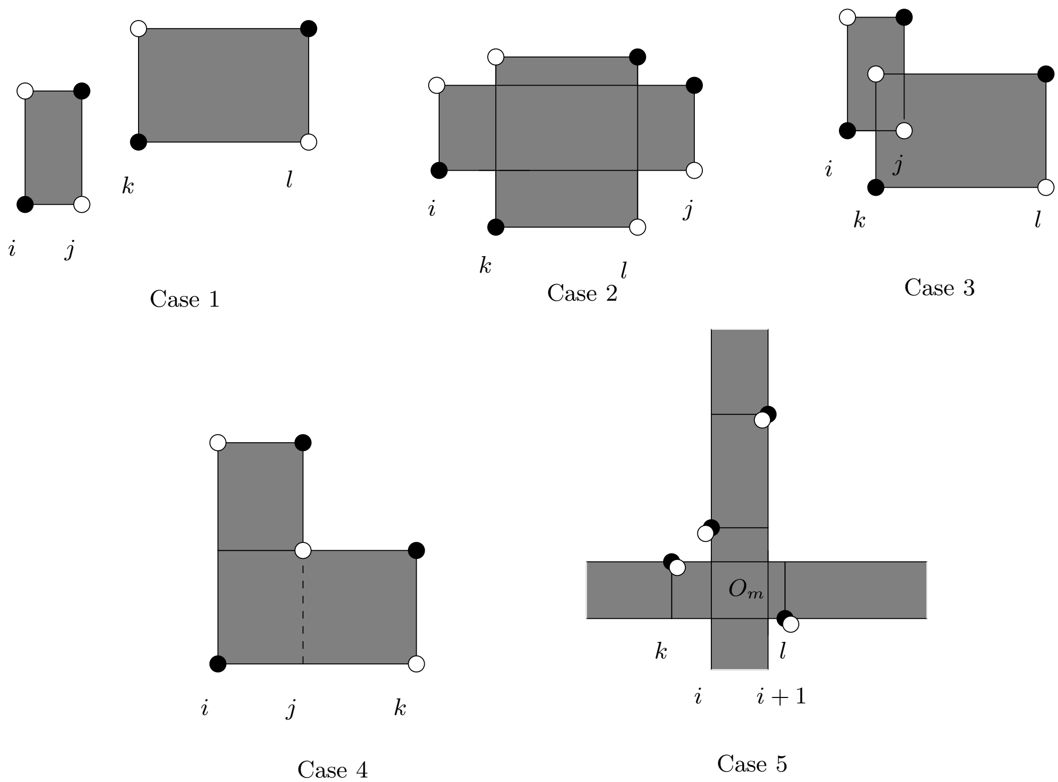

Figure 2.6: $\widetilde{\partial}^- \circ \widetilde{\partial}^- = 0$.

Cases 1,2,3. The rectangles corresponding to $\widetilde{\tau}_{i,j}$ and $\widetilde{\tau}_{k,l}$ give the elements $\widetilde{\mathbf{z}}_1 = \widetilde{\mathbf{x}}.\widetilde{\tau}_{k,l}.\widetilde{\tau}_{i,j}$ and $\widetilde{\mathbf{z}}_2 = \widetilde{\mathbf{x}}.\widetilde{\tau}_{i,j}.\widetilde{\tau}_{k,l}$. By equation (2.2) contribution to $\widetilde{\partial}^- \circ \widetilde{\partial}^-(\widetilde{\mathbf{x}})$ is null.

Case 4. Supports of the rectangles have a common edge. The two corresponding elements are $\widetilde{\mathbf{z}}_1 = \widetilde{\mathbf{x}}.\widetilde{\tau}_{i,j}.\widetilde{\tau}_{j,k}$ and $\widetilde{\mathbf{z}}_2 = \widetilde{\mathbf{x}}.\widetilde{\tau}_{i,k}.\widetilde{\tau}_{i,j}$ with $i < j < k$. By equation (2.3), $\widetilde{\mathbf{z}}_1 = z\widetilde{\mathbf{z}}_2$ and so the contribution is null. Other cases work in a similar way.

Case 5. The vertical annulus is of width 1 and corresponds to $\widetilde{\mathbf{z}}_1 = U_{O_m}.\widetilde{\mathbf{x}}.\widetilde{\tau}_i.\widetilde{\tau}_i$ (it is a consequence of the condition on rectangles to be empty).

To this vertical annulus corresponds the horizontal annulus of height 1 which contains O_m. This horizontal annulus contributes for $U_{O_m}.\widetilde{\mathbf{x}}.\widetilde{\tau}_{l,k}.\widetilde{\tau}_{k,l} = U_{O_m}.\widetilde{\mathbf{x}}$ for a pair $k < l \in \{0, \dots, n-1\}$. So, the contribution of each vertical annulus is canceled by the corresponding horizontal annulus. The global contribution to $\widetilde{\partial}^- \circ \widetilde{\partial}^-(\widetilde{\mathbf{x}})$ is null. $\square$

2.4 Properties of the chain complex

Proposition 2.4.1. *The tensor product of the filtered chain complex $\widetilde{C}^-(G)$ with $\mathbb{Z}/2\mathbb{Z}$ over $\mathbb{Z}$ is isomorphic to the filtered chain complex $C^-(G)$ with coefficients in $\mathbb{Z}/2\mathbb{Z}$. In particular*

$$H_*(\widetilde{C}^-(G) \otimes \mathbb{Z}/2\mathbb{Z}) \cong H_*(C^-(G); \mathbb{Z}/2\mathbb{Z})$$

Proof. It is a consequence of the construction of the complex $\widetilde{C}^-(G)$. □

Lemma 2.4.2. *Suppose that O_i and O_j belongs to the same component of $\overrightarrow{L}$. Then multiplication by U_{O_i} is filtered chain homotopic to multiplication by U_{O_j}.*

Proof. The proof is the same as in Proposition 2.9 [MOST06]. □

Theorem 2.4.3. *Let $\overrightarrow{L}$ be an oriented link with ℓ components. Number the set $\mathbb{O} = \{O_i\}_{i=1}^n$ such that $O_1, \ldots, O_\ell$ correspond to the different components of $\overrightarrow{L}$. Then, the filtered quasi-isomorphism type of $(\widetilde{C}^-(G), \widetilde{\partial}^-)$ viewed over $\mathbb{Z}[U_{O_1}, \ldots, U_{O_\ell}]$ is an invariant of the link.*

Proof. Multiplication by U_{O_i} is filtered homotopic to multiplication by U_{O_j} if O_i and O_j belong to the same component of the link (lemma 2.4.2). It allows us to take for each homology class a representative in the variables $U_{O_1}, \ldots, U_{O_l}$.

Different grid diagrams can lead to the same link. However, given a link $\overrightarrow{L}$ and two grid diagrams G and H of $\overrightarrow{L}$, one can obtain H starting with G by a finite sequence of elementary moves which are **cyclic permutation**, **commutation** and **(de)-stabilization** ([Cro95], [Dyn06]). We now prove that the filtered quasi-isomorphism type is unchanged after making any elementary move.

Cyclic permutation. Let $\widetilde{\sigma} = \widetilde{\tau}_0.\widetilde{\tau}_1.\ldots.\widetilde{\tau}_{n-2}$ be an element in $\widetilde{\mathfrak{S}}_n$. So σ is the cyclic permutation $(01\ldots(n-1))$. There are two cases:

1. **Vertical cyclic permutation.** Suppose that the grid is moved one step upper. Denote G the grid diagram before the move and H after the move. Define
$$\Phi : \widetilde{C}^-(G) \to \widetilde{C}^-(H)$$
given on generators by
$$\Phi(\widetilde{\mathbf{x}}) = \widetilde{\sigma}.\widetilde{\mathbf{x}}$$
The application Φ induces an application on rectangles $\Phi_{\mathrm{Rect}} : \mathrm{Rect}_G \to \mathrm{Rect}_H$ which is the identity i.e. if $\widetilde{r} \in \mathrm{Rect}(\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}})$ then $\widetilde{r} \in \mathrm{Rect}(\widetilde{\sigma}.\widetilde{\mathbf{x}}, \widetilde{\sigma}.\widetilde{\mathbf{y}})$.

Then Φ is a filtered isomorphism of chain complexes:

$$\begin{array}{rcl}\Phi\circ\widetilde{\partial}^-\widetilde{\mathbf{x}} & = & \Phi(\sum_{\widetilde{\mathbf{y}}\in\widetilde{\mathfrak{S}}_n}\sum_{\widetilde{r}\in\mathrm{Rect}^\circ(\widetilde{\mathbf{x}},\widetilde{\mathbf{y}})}U_{O_1}^{O_1(\widetilde{r})}\dots U_{O_n}^{O_n(\widetilde{r})}.\widetilde{\mathbf{y}}) \\ & = & \sum_{\widetilde{\mathbf{y}}\in\widetilde{\mathfrak{S}}_n}\sum_{\widetilde{r}\in\mathrm{Rect}^\circ(\widetilde{\mathbf{x}},\widetilde{\mathbf{y}})}U_{O_1}^{O_1(\widetilde{r})}\dots U_{O_n}^{O_n(\widetilde{r})}.\widetilde{\sigma}.\widetilde{\mathbf{y}} \\ & = & \sum_{\widetilde{\mathbf{y}}\in\widetilde{\mathfrak{S}}_n}\sum_{\widetilde{r}'\in\mathrm{Rect}^\circ(\widetilde{\sigma}.\widetilde{\mathbf{x}},\widetilde{\sigma}.\widetilde{\mathbf{y}})}U_{O_1}^{O_1(\widetilde{r}')}\dots U_{O_n}^{O_n(\widetilde{r}')}.\widetilde{\sigma}.\widetilde{\mathbf{y}} \\ & = & \sum_{\widetilde{\mathbf{y}}'\in\widetilde{\mathfrak{S}}_n}\sum_{\widetilde{r}'\in\mathrm{Rect}^\circ(\widetilde{\sigma}.\widetilde{\mathbf{x}},\widetilde{\mathbf{y}}')}U_{O_1}^{O_1(\widetilde{r}')}\dots U_{O_n}^{O_n(\widetilde{r}')}.\widetilde{\mathbf{y}}' \\ & = & \widetilde{\partial}^-\circ\Phi(\widetilde{\mathbf{x}})\end{array}$$

2. **Horizontal cyclic permutation.** Let G the initial diagram and H the one obtained after horizontal cyclic permutation (we consider the case where the grid is moved left, the other case is similar). Define $\Phi:\widetilde{C}^-(G)\to\widetilde{C}^-(H)$ given on generators by

$$\Phi(\widetilde{\mathbf{x}})=(-1)^{\varepsilon(\sigma).\varepsilon(\mathbf{x})}.\widetilde{\mathbf{x}}.\widetilde{\sigma}^{-1}$$

where the signature maps to the group $\{0,1\}$.

Lemma 2.4.4. Φ *is a filtered isomorphism of chain complexes.*

Proof. The map Φ induces on rectangles the map $\Phi_{\mathrm{Rect}}:\mathrm{Rect}_G\to\mathrm{Rect}_H$ given by

$$\Phi_{\mathrm{Rect}}(\widetilde{\tau}_{i,j})=(-1)^{\varepsilon(\sigma)}.\widetilde{\sigma}.\widetilde{\tau}_{\sigma(i),\sigma(j)}.\widetilde{\sigma}^{-1}$$

(see lemma 2.2.2). Consider a summand in $\widetilde{\partial}^-\circ\Phi(\widetilde{\mathbf{x}})$ (we forget the corresponding variables U_k):

$$\begin{array}{rcl}(-1)^{\varepsilon(\sigma).\varepsilon(\mathbf{x})}.\widetilde{\mathbf{x}}.\widetilde{\sigma}^{-1}.\widetilde{\tau}_{i,j} & = & (-1)^{\varepsilon(\sigma).\varepsilon(\mathbf{x})}.\widetilde{\mathbf{x}}.\widetilde{\sigma}^{-1}.(-1)^{\varepsilon(\sigma)}.\widetilde{\sigma}.\widetilde{\tau}_{\sigma(i),\sigma(j)}.\widetilde{\sigma}^{-1} \\ & = & (-1)^{\varepsilon(\sigma).(\varepsilon(\mathbf{x})+1)}.\widetilde{\mathbf{x}}.\widetilde{\tau}_{\sigma(i),\sigma(j)}.\widetilde{\sigma}^{-1}\end{array}$$

The corresponding summand in $\Phi\circ\widetilde{\partial}^-\widetilde{\mathbf{x}}$ is

$$(-1)^{\varepsilon(\sigma).\varepsilon(\mathbf{x}.\tau_{\sigma(i),\sigma(j)})}.\widetilde{\mathbf{x}}.\widetilde{\tau}_{\sigma(i),\sigma(j)}.\widetilde{\sigma}^{-1}$$

Since $\varepsilon(\mathbf{x}.\tau_{\sigma(i),\sigma(j)})=\varepsilon(\mathbf{x})+1$, we obtain

$$(-1)^{\varepsilon(\sigma).\varepsilon(\mathbf{x}.\tau_{\sigma(i),\sigma(j)})}.\widetilde{\mathbf{x}}.\widetilde{\tau}_{\sigma(i),\sigma(j)}.\widetilde{\sigma}^{-1}=(-1)^{\varepsilon(\sigma).\varepsilon(\mathbf{x})}.\widetilde{\mathbf{x}}.\widetilde{\sigma}^{-1}.\widetilde{\tau}_{i,j}$$

□

Commutation. There are two cases which are commutation of columns and commutation of rows. We deal in detail the case of commutation of columns.

- **Commutation of columns.** Let G be a grid presentation of $\overrightarrow{L}$ and H the grid diagram obtained from G after commutation. Let β be the vertical circle of G and α be the one for H. We represent the two diagrams G and H on the same torus (see figure 2.7).

  Now, we define a filtered chain morphism

  $$\Phi_{\beta\gamma} : \widetilde{C}^-(G) \to \widetilde{C}^-(H)$$

  by counting pentagons.

  We recall the definition in [MOST06]. Let $\mathbf{x} \in C^-(G)$ and $\mathbf{y} \in C^-(H)$ be two generators. We denote $\mathrm{Pent}_{\beta\gamma}(\mathbf{x}, \mathbf{y})$ the set of immersed pentagons connecting $\mathbf{x}$ to $\mathbf{y}$. It is the empty set if $\mathbf{x}$ and $\mathbf{y}$ differ by more than two points. An element $p \in \mathrm{Pent}_{\beta\gamma}(\mathbf{x}, \mathbf{y})$ is an immersed disk in $\mathcal{T}$, whose boundary consists of 5 arcs, each of them is contained in a vertical or a horizontal circle. Moreover, with the orientation induced by $\mathcal{T}$ of the boundary of p, we begin by the point in $\mathbf{x}$ on the β circle, we go through a horizontal arc to a point in $\mathbf{y}$, we follow a vertical arc to the corresponding point of $\mathbf{x}$, we go through a horizontal arc to a point in $\mathbf{y}$ and we go back to the beginning by going first on γ and then on β through one intersection point between α and β. Moreover, we require that each angle of the pentagon be acute (see figure 2.7). Let $\mathrm{Pent}^\circ_{\beta\gamma}(\mathbf{x}, \mathbf{y})$ be the set of empty pentagons i.e. if $p \in \mathrm{Pent}^\circ_{\beta\gamma}(\mathbf{x}, \mathbf{y})$ then $\mathbf{x} \cap Int(p) = \emptyset$.

  Let $\widetilde{\mathbf{x}} \in \widetilde{C}^-(G)$ and $\widetilde{\mathbf{y}} \in \widetilde{C}^-(H)$ be two generators. We consider $\widetilde{\mathbf{x}}$ and $\widetilde{\mathbf{y}}$ as elements of the same group $\widetilde{\mathfrak{S}}_n$ where we identify the β circle with the γ circle. The set of **pentagons** $\mathrm{Pent}_{\beta\gamma}(\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}})$ connecting $\widetilde{\mathbf{x}}$ to $\widetilde{\mathbf{y}}$ is the empty set if $\mathrm{Pent}_{\beta\gamma}(\mathbf{x}, \mathbf{y}) = \emptyset$. Otherwise $\mathrm{Pent}_{\beta\gamma}(\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}}) = \widetilde{\tau}_{i,j}$ where the bottom left corner of the pentagon is in the i-th circle and if $\widetilde{\mathbf{y}} = \widetilde{\mathbf{x}}.\widetilde{\tau}_{i,j}$. A pentagon $\widetilde{p} \in \mathrm{Pent}_{\beta\gamma}(\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}})$ is said **empty** if $p \in \mathrm{Pent}^\circ_{\beta\gamma}(\mathbf{x}, \mathbf{y})$. Let $\mathrm{Pent}^\circ_{\beta\gamma}(\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}})$ be the set of empty pentagons connecting $\widetilde{\mathbf{x}}$ to $\widetilde{\mathbf{y}}$.

  Define

  $$\varepsilon_{\beta\gamma} : \mathrm{Pent}^\circ_{\beta\gamma}(\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}}) \to \{\pm 1\}$$

  by $\varepsilon_{\beta\gamma}(\widetilde{p}) = +1$ if p is a left pentagon and -1 if it is a right pentagon.

  $$\Phi_{\beta\gamma}(\widetilde{\mathbf{x}}) = \sum_{\widetilde{\mathbf{y}} \in \widetilde{\mathfrak{S}}_n} \sum_{\widetilde{p} \in \mathrm{Pent}^\circ_{\beta\gamma}(\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}})} (-1)^{M(\widetilde{\mathbf{x}})} \varepsilon_{\beta\gamma}(\widetilde{p}).U_{O_1}^{O_1(\widetilde{p})} \dots U_{O_n}^{O_n(\widetilde{p})}.\widetilde{\mathbf{y}}$$

  where $O_m(\widetilde{p})$ is the number of times O_m appears in the interior of p.

  Define $\varepsilon_{\gamma\beta}$ with the same convention: left pentagons takes value $+1$ and right pentagons takes value -1.

  Remark. Instead of taking $(-1)^{M(\widetilde{\mathbf{x}})}$, one can take $(-1)^{\varepsilon(\mathbf{x})}$ as in the proof of cyclic permutation. In the following, we take the Maslov degree but we can think of it as the signature.

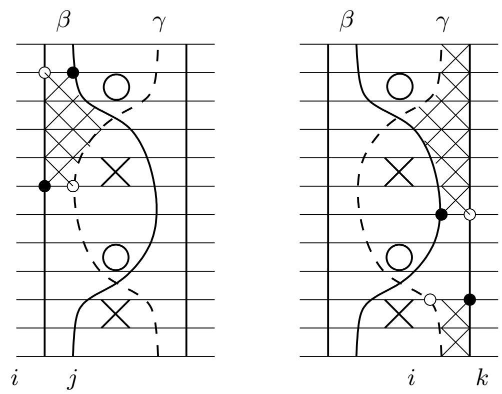

Figure 2.7: **Pentagons.** We mark with black dots $\mathbf{x}$ and with white dots $\mathbf{y}$. The pentagon $\widetilde{\tau}_{i,j} \in \mathrm{Pent}^{\circ}_{\beta\gamma}(\widetilde{\mathbf{x}}, \widetilde{\mathbf{x}}.\widetilde{\tau}_{i,j})$ is a left pentagon and so $\varepsilon_{\beta\gamma}(\widetilde{\tau}_{i,j}) = +1$. The pentagon $\widetilde{\tau}_{i,k}) \in \mathrm{Pent}^{\circ}_{\beta\gamma}(\widetilde{\mathbf{x}}, \widetilde{\mathbf{x}}.\widetilde{\tau}_{i,k})$ is a right pentagon and so $\varepsilon_{\beta\gamma}(\widetilde{\tau}_{i,k}) = -1$.

Lemma 2.4.5. $\Phi_{\beta\gamma}$ *is a filtered chain morphism.*

Proof. Summands belonging to $\widetilde{\partial}^- \circ \Phi_{\beta\gamma}(\widetilde{\mathbf{x}})$ are $\varepsilon_{\beta\gamma}(\widetilde{p})(-1)^{M(\widetilde{\mathbf{x}})}.\widetilde{\mathbf{x}}.\widetilde{p}.\widetilde{r}$, and those in $\Phi_{\beta\gamma} \circ \widetilde{\partial}^-\widetilde{\mathbf{x}}$ are $\varepsilon_{\beta\gamma}(\widetilde{p}')(-1)^{M(\widetilde{\mathbf{x}}.\widetilde{r}')}.\widetilde{\mathbf{x}}.\widetilde{r}'.\widetilde{p}'$. Each term in $\widetilde{\partial}^- \circ \Phi_{\beta\gamma}(\widetilde{\mathbf{x}})$ corresponds to a term in $\Phi_{\beta\gamma}(\widetilde{\partial}^-\widetilde{\mathbf{x}})$. In case of figure 2.8, we have in fact $\widetilde{r}'.\widetilde{p}' = \widetilde{p}.\widetilde{r}$. Nevertheless, there are a left pentagon (say $\widetilde{p}'$) and a right pentagon (say $\widetilde{p}$) so via $\varepsilon_{\beta\gamma}$ they take different values which are compensated by the Maslov degree $M(\widetilde{\mathbf{x}}) = M(\widetilde{\mathbf{x}}.\widetilde{r}) + 1$.

In other cases except the special case illustrated by figure 2.9, the two pentagons are on the same side but $\widetilde{p}.\widetilde{r} = z\widetilde{r}'.\widetilde{p}'$ and it works similary.

There is a special case illustrated by figure 2.9 and which appears once in $\widetilde{\partial}^- \circ \Phi_{\beta\gamma}(\widetilde{\mathbf{x}})$ and once in $\Phi_{\beta\gamma} \circ \widetilde{\partial}^-\widetilde{\mathbf{x}}$. With the same discussion as above we check that signs behave well.

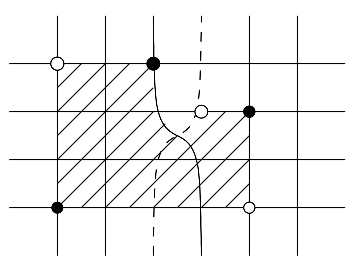

Figure 2.8: $\Phi_{\beta\gamma}$ **filtered chain morphism.** The hatched domain is decomposed either as a left pentagon followed by a rectangle, or a rectangle followed by a right pentagon. The first decomposition appears in $\widetilde{\partial}^- \circ \Phi_{\beta\gamma}$ and the second one appears in $\Phi_{\beta\gamma} \circ \widetilde{\partial}^-$.

□

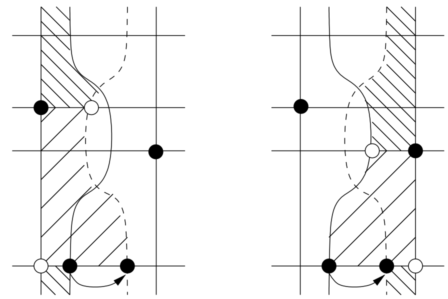

Figure 2.9: $\Phi_{\beta\gamma}$ **filtered chain morphism: special case.** Generators marked by black dots differ only in one point. On the left, we have a left pentagon followed by a rectangle, on the right we have a rectangle followed by a right pentagon.

To prove that $\Phi_{\beta\gamma}$ is a filtered homotopy equivalence we define an application which raises the maslov degree by one and counts empty hexagons.

First, we recall the definition of **hexagons** in the sense of [MOST06]. Let $\mathbf{x}, \mathbf{y} \in C^-(G)$ be two generators. Denote $\mathrm{Hex}_{\beta\gamma\beta}(\mathbf{x}, \mathbf{y})$ the set of immersed hexagons connecting $\mathbf{x}$ to $\mathbf{y}$. It is the empty set if $\mathbf{y}.\mathbf{x}^{-1}$ is not a transposition. An element $h \in \mathrm{Hex}_{\beta\gamma\beta}(\mathbf{x}, \mathbf{y})$ is an immersed disk in $\mathcal{T}$ with the orientation induced by $\mathcal{T}$.

Following the oriented boundary of h, we begin with the point of $\mathbf{x}$ which is on β, we go through a horizontal arc to a point of $\mathbf{y}$, through a vertical arc to a point of $\mathbf{x}$, through a horizontal arc to the point of $\mathbf{y}$ in β and finally go back to the initial point of $\mathbf{x}$ through the vertical arc β and γ and the two intersection points between β and γ (see figure 2.10). We require that all angles of the hexagon be acute. The set $\mathrm{Hex}^\circ_{\beta\gamma\beta}$ of **empty hexagons** is those for which the interior doesn't contain any point of the corresponding generators.

The set of hexagons $\mathrm{Hex}_{\beta\gamma\beta}(\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}})$ connecting $\widetilde{\mathbf{x}}$ to $\widetilde{\mathbf{y}}$ is the empty set if $\mathrm{Hex}_{\beta\gamma\beta}(\mathbf{x}, \mathbf{y}) = \emptyset$. Otherwise $\mathrm{Hex}_{\beta\gamma}(\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}}) = \widetilde{\tau}_{i,j}$ where the bottom left corner of the pentagon is in the i-th circle and if $\widetilde{\mathbf{y}} = \widetilde{\mathbf{x}}.\widetilde{\tau}_{i,j}$. A hexagon $\widetilde{h} \in \mathrm{Hex}_{\beta\gamma\beta}(\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}})$ is said empty if $h \in \mathrm{Hex}^\circ_{\beta\gamma\beta}(\mathbf{x}, \mathbf{y})$. Let $\mathrm{Hex}^\circ_{\beta\gamma\beta}(\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}})$ be the set of empty hexagons connecting $\widetilde{\mathbf{x}}$ to $\widetilde{\mathbf{y}}$.

The morphism $H_{\beta\gamma\beta} : \widetilde{C}^-(G) \to \widetilde{C}^-(G)$ is given on generators by

$$H_{\beta\gamma\beta}(\widetilde{\mathbf{x}}) = \sum_{\widetilde{\mathbf{y}} \in \widetilde{\mathfrak{S}}_n} \sum_{\widetilde{h} \in \mathrm{Hex}^\circ_{\beta\gamma\beta}(\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}})} U_{O_1}^{O_1(\widetilde{h})} \dots U_{O_n}^{O_n(\widetilde{h})} . \widetilde{\mathbf{y}}$$

where $O_m(\widetilde{h})$ is the number of times O_m appears in the interior of h. In particular $H_{\beta\gamma\beta}$ raise the Maslov degree by 1.

Lemma 2.4.6. *The application $\Phi_{\beta\gamma}$ induces a filtered isomorphism in homology; more precisely*

$$\Phi_{\gamma\beta} \circ \Phi_{\beta\gamma} + id = \widetilde{\partial}^{-} \circ H_{\beta\gamma\beta} + H_{\beta\gamma\beta} \circ \widetilde{\partial}^{-}$$

$$\Phi_{\beta\gamma} \circ \Phi_{\gamma\beta} + id = \widetilde{\partial}^{-} \circ H_{\gamma\beta\gamma} + H_{\gamma\beta\gamma} \circ \widetilde{\partial}^{-}$$

Proof. By juxtaposing two pentagons which appears in $\Phi_{\gamma\beta} \circ \Phi_{\beta\gamma}$, we get a domain which has a unique alternative decomposition as a rectangle followed by a hexagon or a hexagon followed by a rectangle, counted in $\widetilde{\partial}^{-} \circ H_{\beta\gamma\beta}$ or in $H_{\beta\gamma\beta} \circ \widetilde{\partial}^{-}$ (it is the case where the width of the rectangle is > 1).

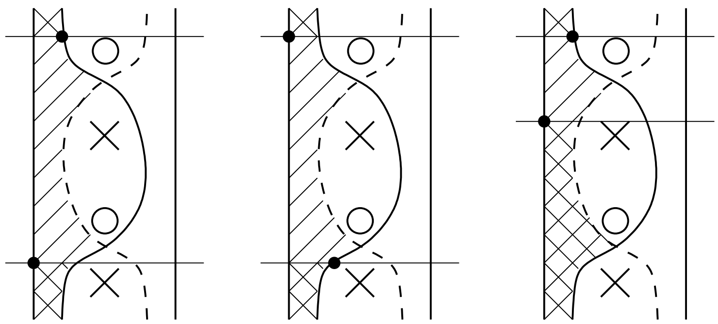

Figure 2.10: **Decomposing the identity map.** Consider the 3 configurations in $\widetilde{C}^{-}(G)$ marked by dark dots. On the left, we have a hexagon followed by a rectangle: it appears in $\widetilde{\partial}^{-} \circ H_{\beta\gamma\beta}$. In the middle, we have a rectangle followed by a hexagon: it appears in $H_{\beta\gamma\beta} \circ \widetilde{\partial}^{-}$. On the right, we have a pentagon followed by a pentagon: it appears in $\Phi_{\gamma\beta} \circ \Phi_{\beta\gamma}$.

However, there is a domain which has a unique decomposition (see figure 2.10): it corresponds to a vertical annulus without any X. It counts for $-id$ and admits a unique alternative decomposition.

When it is a pentagon followed by a pentagon, we start with $\widetilde{\mathbf{x}}$ and arrive to

$$\varepsilon_{\beta\gamma}(\widetilde{p}).\varepsilon_{\gamma\beta}(\widetilde{p}').(-1)^{M_G(\widetilde{\mathbf{x}})+M_H(\widetilde{\mathbf{x}}.\widetilde{\tau}_i)}\widetilde{\mathbf{x}}.\widetilde{\tau}_i.\widetilde{\tau}_i = z\widetilde{\mathbf{x}}$$

where M_G (resp. M_H) is the Maslov degree on G (resp. on H). Since the two pentagons are on the same side, we have $\varepsilon_{\beta\gamma}(\widetilde{p}).\varepsilon_{\gamma\beta}(\widetilde{p}') = +1$. Moreover we have $M_G(\widetilde{\mathbf{x}}) = M_H(\widetilde{\mathbf{x}}.\widetilde{\tau}_i)$.

While starting with a hexagon and composing with a rectangle or starting with a rectangle and composing with a hexagon, we start with $\widetilde{\mathbf{x}}$ and arrive to $\widetilde{\mathbf{x}}.\widetilde{\tau}_i.\widetilde{\tau}_i = z\widetilde{\mathbf{x}}$.

In every case, starting with $\widetilde{\mathbf{x}}$ we get $z\widetilde{\mathbf{x}}$. ∎

- **Commutation of rows.** We deal the commutation of rows exactly the same way except that we must change the notions of *left* and *right* by *top* and *bottom*.

Stabilisation. We follow the proof of invariance under stabilization done in [MOST06], updating it to our context. Let G be the initial diagram of complexity n and H the one of complexity $n+1$ obtained after one stabilization move along the row containing O_2. The set of $\mathbb{O}$ in G is numbered from 2 to $n+1$. We can suppose (after some cyclic permutations) that the new row and column are the $n+1$-th row and the $n+1$-th column. We deal with the case drawn in figure 2.11.

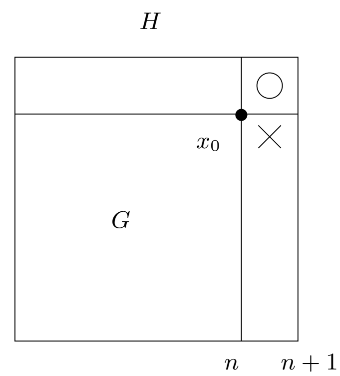

Figure 2.11: **Stabilisation.** The point x_0 is the point belonging to all elements of $\mathbf{I}$ when we consider the set of generators of $\widetilde{C}^-(G)$ as a subset of the set of generators of $\widetilde{C}^-(H)$. The O and the X are numbered O_1 and X_1.

Let $B = \widetilde{C}^-(G)$ and $C = \widetilde{C}^-(H)$ the corresponding complexes. First, we recall that C' is the mapping cone of

$$U_{O_1} - U_{O_2} : B[U_{O_1}] \to B[U_{O_1}]$$

and the differential is given by $\partial'(a,b) = (\widetilde{\partial}^- a, (U_{O_1} - U_{O_2}).a - \widetilde{\partial}^- b)$. Let $\mathcal{L}$ et $\mathcal{R}$ be the submodules of C such that $C' = B[U_{O_1}] \bigoplus B[U_{O_1}] = \mathcal{L} \bigoplus \mathcal{R}$. The module $\mathcal{R}$ heritates a Maslov degree and an Alexander filtration which is the one of $B[U_{O_1}]$ whereas $\mathcal{L}$ heritates the ones of $B[U_{O_1}]$ dropped by 1.

Lemma 2.4.7. *C' is filtered quasi-isomorphic to B.*

Proof. The proof is the same as in lemma 3.3 [MOST06]. □

It remains to prove that C' is filtered quasi-isomorphic to C. We define an injection which maps each generators of B to a subset $\mathbf{I}$ of the set of generators of C. It is $\phi : \widetilde{\mathfrak{S}}_n \hookrightarrow \widetilde{\mathfrak{S}}_{n+1}$ which maps $\widetilde{\tau}_i$ to $\widetilde{\tau}_i$ for all $i \in \{0, \dots, n-2\}$. We extend it on all elements of $\widetilde{\mathfrak{S}}_n$ by making it compatible with the normal form previously described in section 2.2. In particular, if $\widetilde{\mathbf{y}} \in \mathbf{I}$ then $\mathbf{x}_0$ is a point of $\mathbf{y}$. We have the following equalities for $\widetilde{\mathbf{x}} \in B$:

$$M_B(\widetilde{\mathbf{x}}) = M_C(\phi(\widetilde{\mathbf{x}})) + 1 = M_{C'}(0, \phi(\widetilde{\mathbf{x}})) = M_{C'}(\phi(\widetilde{\mathbf{x}}), 0) + 1$$

$$A_B(\widetilde{\mathbf{x}}) = A_C(\phi(\widetilde{\mathbf{x}})) + A(U_1) = A_{C'}(0, \phi(\widetilde{\mathbf{x}})) = A_{C'}(\phi(\widetilde{\mathbf{x}}), 0) + A(U_1)$$

We recall the definition of domains of type L and R (see [MOST06]). Let $\mathbf{x}, \mathbf{y}$ be two generators in $\mathfrak{S}_n$. A path from $\mathbf{x}$ to $\mathbf{y}$ is an oriented closed path γ made of arcs in the circles α and β which angles are points in $\mathbf{x} \cup \mathbf{y}$, oriented so that $\partial(\gamma \cap \alpha) = \mathbf{x} - \mathbf{y}$. A **domain** p connecting $\mathbf{x}$ to $\mathbf{y}$ is a two chain in $\mathcal{T}$ whose boundary ∂p is a path from $\mathbf{x}$ to $\mathbf{y}$ and denote $\pi(\mathbf{x}, \mathbf{y})$ the set of domains connecting $\mathbf{x}$ to $\mathbf{y}$.

Definition 2.4.8. Let $\mathbf{x} \in \mathfrak{S}_n$ and $\mathbf{y} \in \mathfrak{S}_n \subset \mathfrak{S}_{n+1}$ (the inclusion is given by ϕ). A domain $p \in \pi(\mathbf{x}, \mathbf{y})$ is said to have type L (resp. type R) if it is trivial (in this case it has type L) or if it satisfies the following conditions:

- p has no negative local multiplicity,
- for each $c \in \mathbf{x} \cup \mathbf{y}$ other than x_0, at least three of the four adjoining squares have local multiplicity equal to zero,
- in a neighbourhood of x_0 the local multiplicity in three of the adjoining squares are k. When p has type L, the lower left corner has local multiplicity $k-1$, while for p of type R, the lower right corner has multiplicity $k+1$.
- ∂p is connected.

The complexity of the trivial domain is 1 and for any other domains it is the number of horizontal arcs of its boundary. The set of domains of type L (resp. R) is denoted $\pi^L(\mathbf{x}, \mathbf{y})$ (resp. $\pi^R(\mathbf{x}, \mathbf{y})$). We denote $\pi^F(\mathbf{x}, \mathbf{y}) = \pi^L(\mathbf{x}, \mathbf{y}) \cup \pi^R(\mathbf{x}, \mathbf{y})$ and called its elements of type F.

We wish to define a map $F : C \to C'$ which maps generators of C to generators of B through domains of type R or L. To do this, we define type F domains $\pi^F(\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}})$ for $\widetilde{\mathbf{x}} \in C$ and $\widetilde{\mathbf{y}} \in \mathbf{I}$.

To a domain $p \in \pi^F(\mathbf{x}, \mathbf{y})$ of complexity m we associate a **standard decomposition** in a finite sequence of rectangles. Since ∂p is a connected oriented curve, we number the β-circles by $\{v_i\}_{i=1}^m$ in such a way that they inherits the cyclic order given by ∂p, and such that v_m does not contain x_0. We decompose p in a sequence of rectangles $\{r_i\}_{i=1}^{m-1}$ so that r_i is the rectangle between the vertical circles v_i and v_m (see figure 2.12).

We define $\pi^F(\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}})$ as the empty set if $\pi^F(\mathbf{x}, \mathbf{y}) = \emptyset$, otherwise $\widetilde{p} \in \pi^F(\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}})$ is

$$\widetilde{r}_1 . \dots . \widetilde{r}_{m-1}$$

corresponding to the rectangles $\{r_i\}_{i=1}^{m-1}$ if $\mathbf{y} = \mathbf{x}.r_1.\dots.r_{m-1}$.

Define on the set of generators of C the two applications $F^L : C \to \mathcal{L}$ and $F^R : C \to \mathcal{R}$ by

$$F^L(\widetilde{\mathbf{x}}) = \sum_{\widetilde{\mathbf{y}} \in \mathbf{I}} \sum_{\widetilde{p} \in \pi^L(\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}})} U_{O_2}^{O_2(\widetilde{p})} \dots U_{O_n}^{O_n(\widetilde{p})}.\widetilde{\mathbf{y}}$$

$$F^R(\widetilde{\mathbf{x}}) = \sum_{\widetilde{\mathbf{y}} \in \mathbf{I}} \sum_{\widetilde{p} \in \pi^R(\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}})} U_{O_2}^{O_2(\widetilde{p})} \dots U_{O_n}^{O_n(\widetilde{p})}.\widetilde{\mathbf{y}}$$

Putting those two applications together we get

$$F = \begin{pmatrix} F^L \\ F^R \end{pmatrix} : C \to C'$$

Before proving that F is a filtered chain morphism, we need to deal with the differential in C'. The differential in C' counts rectangles which are either empty (type 1) or empty except of the point x_0 (type 2). For type 2 rectangles the variable U_1 doesn't appear in the differential. Let $\widetilde{r}$ be a rectangle in B. Viewed as a rectangle in C', if it is of type 1 we note $\widetilde{r}'$ the corresponding rectangle (under the injection ϕ) and its standard decomposition is $D_0(\widetilde{r}') = \widetilde{r}'$. If it is of type 2, it corresponds to $\widetilde{r}'$ and admits a unique decomposition shown in figure 2.12 in three rectangles. $D_0(\widetilde{r}') = \widetilde{r}'_1.\widetilde{r}'_2.\widetilde{r}'_3$ is its standard decomposition.

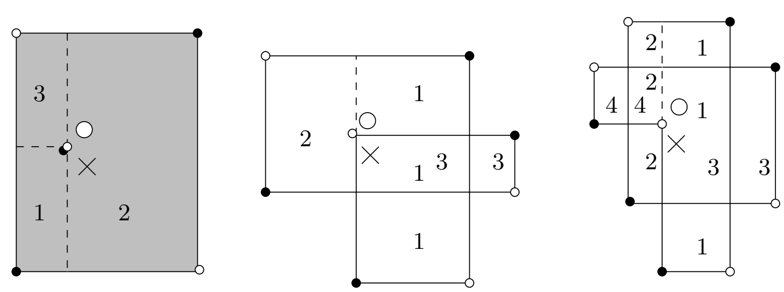

Figure 2.12: **Decomposing polygons.** On the left, we have the standard decomposition of a rectangle of type 2 as $\widetilde{r}_1.\widetilde{r}_2.\widetilde{r}_3$. In the middle, we have the standard decomposition of a polygonal domain of type R of complexity 4 as $\widetilde{r}_1.\widetilde{r}_2.\widetilde{r}_3$. On the right, we have the standard decomposition of a polygonal domain of type L of complexity 5 as $\widetilde{r}_1.\widetilde{r}_2.\widetilde{r}_3.\widetilde{r}_4$. Some regions are in the supports of more than one rectangle and are labeled with the number of each corresponding rectangles.

Lemma 2.4.9. *Let $\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}}$ be in B and $\widetilde{\mathbf{x}}', \widetilde{\mathbf{y}}' \in \mathbf{I}$ their image in C. Suppose there exists $\widetilde{r}' \in \mathrm{Rect}(\widetilde{\mathbf{x}}', \widetilde{\mathbf{y}}')$ a rectangle of type 1 or 2. Then the corresponding rectangle $\widetilde{r} \in \mathrm{Rect}^\circ(\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}})$ satisfies:*

1. *$\widetilde{r} = \widetilde{r}'$ if $\widetilde{r}'$ has type 1.*

2. *$\widetilde{r} = \widetilde{r}'_1.\widetilde{r}'_2.\widetilde{r}'_3$ if $\widetilde{r}'$ has type 2 and $D_0(\widetilde{r}') = \widetilde{r}'_1.\widetilde{r}'_2.\widetilde{r}'_3$.*

Proof. For rectangles of type 1, there is nothing to prove since $\mathbf{x}_0$ has coordonates (n, n).

For rectangles of type 2, we must prove that for all $0 \leq i, j < n-1$

$$\widetilde{\tau}_{i,j} = \widetilde{\tau}_{i,n}.\widetilde{\tau}_{n,j}.\widetilde{\tau}_{i,n}$$

but this is equation (2.3). □

Lemma 2.4.10. *The map $F : C \to C'$ is a filtered chain morphism which respects the Maslov degree and preserves the Alexander filtration.*

Proof. The proof is by cutting in two different ways domains appearing in $F \circ \widetilde{\partial}^-_C$ and $\widetilde{\partial}^-_{C'} \circ F$.

Definition 2.4.11. Let $\widetilde{p} \in \pi^F(\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}})$ be a domain decomposed as $\widetilde{p} = \widetilde{r}_1.\ldots.\widetilde{r}_n$ with $\widetilde{r}_i \in \text{Rect}^\circ$ for all i. If for some i we have $r_i * r_{i+1} = r'_i * r'_{i+1}$ with $r'_i, r'_{i+1} \in \text{Rect}^\circ$ distinct from r_i and r_{i+1} then we say that the two decompositions of the domain $\widetilde{r}_1.\ldots.\widetilde{r}_i.\widetilde{r}_{i+1}.\ldots.\widetilde{r}_n$ and $\widetilde{r}_1.\ldots.\widetilde{r}'_i.\widetilde{r}'_{i+1}.\ldots.\widetilde{r}_n$ differ by an elementary move (here $*$ is the juxtaposition of rectangle). In particular $\widetilde{r}_1.\ldots.\widetilde{r}_i.\widetilde{r}_{i+1}.\ldots.\widetilde{r}_n = z\widetilde{r}_1.\ldots.\widetilde{r}'_i.\widetilde{r}'_{i+1}.\ldots.\widetilde{r}_n$

With this definition, the proof of the lemma 2.4.10 is a rewriting of the proof of lemma 3.5 [MOST06]. □

Proposition 2.4.12. *F is a filtered quasi-isomorphism.*

Proof. The proof is in four steps which we recall here:

1. Define a filtration $\mathcal{F}$ on the graded object associated with the Alexander filtration A on complexes $\widetilde{C}$ and $\widetilde{C}'$ obtained from C and C'.

2. Describing the homology of the graded object associated with the filtration $\mathcal{F}$ and A.

3. Proving that $\widetilde{F}_{gr|A,\mathcal{F}}$ is an isomorphism on the homology.

4. Using a principle of homological algebra to conclude that F is a filtered quasi-isomorphism.

Step 1. First, remark that the graded object associated with the Alexander filtration is just given by considering rectangles empty of $X \in \mathbb{X}$. Let $\widetilde{C}$ be equal to $C_{|U_i=0\forall i}$ (in particular, the corresponding differential counts only rectangles which do no contain any $\mathbb{O}$). Let $\widetilde{C}_{gr|A}$ be the associated graded object and $\widetilde{C}_{gr|A=h}$ the summand generated by generators $\widetilde{\mathbf{x}} \in \widetilde{\mathfrak{S}}_n$ where $A(\widetilde{\mathbf{x}}) = h \in \mathbb{Z}^l$.

Let $\mathcal{Q}$ be the $n \times n$ square corresponding to $G \subset H$ (see figure 2.11): in particular, the row and column containing O_1 is not in $\mathcal{Q}$. For any $\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}} \in \widetilde{\mathfrak{S}}_{n+1}$ and any domain $\widetilde{p} \in \pi(\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}})$ such that $O_i(\widetilde{p}) = X_i(\widetilde{p}) = 0$ for all i then

$$\mathcal{F}(\widetilde{\mathbf{x}}) - \mathcal{F}(\widetilde{\mathbf{y}}) = \#(\mathcal{Q} \cap p)$$

is well defined (see p. 23 [MOST06]) and determines a filtration on $\widetilde{C}_{gr|A=h}$. Let $C_{gr|A,\mathcal{F}}$ be the graded object associated.

Step 2. Let $\widetilde{\mathfrak{S}}_{n+1} = \mathbf{I} \cup \mathbf{NI} \cup \mathbf{NN}$, where $\mathbf{NI}$ is the subset of $\widetilde{\mathfrak{S}}_{n+1}$ whose elements $\widetilde{\mathbf{x}} \in \mathbf{NI}$ are such that $\mathbf{x}(n+1) = n$ and $\mathbf{NN}$ is the complement in $\widetilde{\mathfrak{S}}_{n+1}$ of $\mathbf{I} \cup \mathbf{NI}$.

Lemma 2.4.13 (lemma 3.7 [MOST06])**.** *$H_*(\widetilde{C}_{gr|A,\mathcal{F}})$ is isomorphic to the free $\mathbb{Z}$-module generated by elements of $\mathbf{I}$ and $\mathbf{NI}$.*

Step 3. Let $\widetilde{F}_{gr|A,\mathcal{F}}$ be the map induced by F:

$$\widetilde{F}_{gr|A,\mathcal{F}} : \widetilde{C}_{gr|A,\mathcal{F}} \to \widetilde{C}'_{gr|A,\mathcal{F}}$$

where $\widetilde{C}'_{gr|A,\mathcal{F}}$ splits as the direct sum $\widetilde{\mathcal{L}}_{gr|A,\mathcal{F}} \oplus \widetilde{\mathcal{R}}_{gr|A,\mathcal{F}}$ (the map $\times(U_{O_1} - U_{O_2})$ is equal to 0 in $\widetilde{C}$), both of which are freely generated by elements in $\mathbf{I}$. Then $\widetilde{F}_{gr|A,\mathcal{Q}}$ is a quasi-isomorphism (Proposition 3.8 [MOST06]).

Step 4. Using the fact that a filtered chain map which induces an isomorphism on the homology of the associated graded objects is a filtered quasi-isomorphism gives that $\widetilde{F}$ is a quasi-isomorphism between $\widetilde{C}$ and $\widetilde{C}'$. Using this fact one more time gives that F is a quasi-isomorphism between C and C': $\widetilde{C}$ is just the graded object associated with C with the filtration counting the number of variables U_i's. $\square$

Thanks to proposition 2.4.12, invariance under stabilization is proved. $\square$

2.5 Sign assignment induced by the complex

In this section we prove that the chain complex $\widetilde{C}^-(G)$ coincides with the chain complex $C^-(G)$ over $\mathbb{Z}$ after a choice of a sign assignment.

Definition 2.5.1. A sign assignment is a function $\mathbf{S} : \mathrm{Rect}^\circ \to \{\pm 1\}$ such that

(Sq) for any distincts $r_1, r_2, r'_1, r'_2 \in \mathrm{Rect}^\circ$ such that $r_1 * r_2 = r'_1 * r'_2$ we have

$$\mathbf{S}(r_1).\mathbf{S}(r_2) = -\mathbf{S}(r'_1).\mathbf{S}(r'_2)$$

(V) if $r_1, r_2 \in \mathrm{Rect}^\circ$ are such that $r_1 * r_2$ is a vertical annulus then

$$\mathbf{S}(r_1).\mathbf{S}(r_2) = -1$$

(H) if $r_1, r_2 \in \mathrm{Rect}^\circ$ are such that $r_1 * r_2$ is a horizontal annulus then

$$\mathbf{S}(r_1).\mathbf{S}(r_2) = +1$$

To define a sign assignment, we must define a map $s : \mathfrak{S}_n \to \widetilde{\mathfrak{S}}_n$ such that $p \circ s = id_{\mathfrak{S}_n}$

$$\begin{tikzcd} 1 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r, hook, "i"] & \widetilde{\mathfrak{S}}_n \arrow[r, "p"] & \mathfrak{S}_n \arrow[l, bend left, "s"] \arrow[r] & 1 \end{tikzcd}$$

Define $s : \mathfrak{S}_n \to \widetilde{\mathfrak{S}}_n$ by setting:

- $s(\mathbf{1}) = \widetilde{\mathbf{1}}$ and for all $0 \leq i \leq n-2$, $s(\tau_i) = \widetilde{\tau}_i$.
- Let $\tau_{i,j}$ be the transposition which exchanges i and j with $j - i > 1$. Then it can be writing in a unique way like that:

$$\tau_{i,j} = \tau_i.\tau_{i+1}. \ldots .\tau_{j-2}.\tau_{j-1}.\tau_{j-2}. \ldots .\tau_{i+1}.\tau_i$$

  We define

$$s(\tau_{i,j}) = \widetilde{\tau}_i.\widetilde{\tau}_{i+1}. \ldots .\widetilde{\tau}_{j-2}.\widetilde{\tau}_{j-1}.\widetilde{\tau}_{j-2}. \ldots .\widetilde{\tau}_{i+1}.\widetilde{\tau}_i$$

- Let $\mathbf{x} \in \mathfrak{S}_n$ and $i_{n-1} \in \{0, \ldots, n-1\}$ be such that $\mathbf{x}(i_{n-1}) = n-1$. Then $\mathbf{x}.\tau_{i_{n-1},n-1}(n-1) = n-1$. We think of $\mathbf{x}.\tau_{i_{n-1},n-1}$ as an element of $\mathfrak{S}_{n-1}$. We go on to obtain this kind of writing:

$$\mathbf{x} = \tau_{i_0,0}.\tau_{i_1,1}. \ldots .\tau_{i_{n-1},n-1}$$

  Remark. In general $x(i_k) \neq k$, except for $k = n-1$.

  This writing is unique and we let

$$s(\mathbf{x}) = \widetilde{\tau}_{i_0,0}. \ldots .\widetilde{\tau}_{i_{n-1},n-1}$$

  With this section s, every element $\widetilde{\mathbf{x}} \in \widetilde{\mathfrak{S}}_n$ has a unique writing called normal form which is

$$\widetilde{\mathbf{x}} = z^u s(\mathbf{x})$$

  where $u \in \{0, 1\}$ and $\mathbf{x} = p(\widetilde{\mathbf{x}})$.

To define the sign assignment induced by $\widetilde{C}^-(G)$ we need the 2-cocycle $c \in C^2(\mathfrak{S}_n, \mathbb{Z}/2\mathbb{Z})$ associated with the map s given by

$$s(\mathbf{x}).s(\mathbf{y}) = (i \circ c(\mathbf{x}, \mathbf{y}))s(\mathbf{x}.\mathbf{y}) \tag{2.4}$$

The cohomological class of c measures how s fails to be a group morphism. In particular, it is non-trivial ($n \geq 4$) since $\widetilde{\mathfrak{S}}_n$ is a non-trivial central extension of $\mathfrak{S}_n$ by $\mathbb{Z}/2\mathbb{Z}$.

We say that a rectangle r is **horizontally torn** if given the coordonates (i_{bl}, j_{bl}) of its bottom left corner and (i_{tr}, j_{tr}) of its top right corner then $i_{bl} > i_{tr}$. Otherwise, r is said to be not horizontally torn.

Lemma 2.5.2. *The complex $(\widetilde{C}^-(G), \widetilde{\partial}^-)$ induces a sign assignment in the sense of definition 2.5.1: for all $(\mathbf{x}, \mathbf{y}) \in \mathfrak{S}_n^2$ and all $r \in \mathrm{Rect}^\circ(\mathbf{x}, \mathbf{y})$*

$$\mathbf{S}(r) = \varepsilon(r).c(\mathbf{x}^{-1}.\mathbf{y}, \mathbf{x}) \tag{2.5}$$

where $\varepsilon(r) = +1$ if r is a rectangle not horizontally torn and $\varepsilon(r) = -1$ otherwise.

Remark. The sign assignment in the sense of definition 2.5.1 is unique up to a 1-coboundary: if $\mathbf{S}_1$ and $\mathbf{S}_2$ are two sign assignments then there exists an application $f : \mathfrak{S}_n \to \{\pm 1\}$ such that for all rectangles $r \in \mathrm{Rect}^\circ(\mathbf{x}, \mathbf{y})$, $\mathbf{S}_1(r) = f(\mathbf{x}).f(\mathbf{y}).\mathbf{S}_2(r)$. It is a consequence of the fact that the central extension corresponds to a 2-cohomological class in $H^2(\mathfrak{S}_n, \mathbb{Z}/2\mathbb{Z})$ (compare with theorem 4.2 [MOST06]). Here, we construct explicitely a map $s : \mathfrak{S}_n \to \widetilde{\mathfrak{S}}_n$ such that $p \circ s = id$ which means making a choice of a representative of this class, another choice must differ by a 1-coboundary.

Proof. Since c is 2-cocycle we have $\delta c = 1$ i.e. for all $(\mathbf{x}, \mathbf{y}, \mathbf{z}) \in \mathfrak{S}_n^3$

$$\delta c(\mathbf{x}, \mathbf{y}, \mathbf{z}) = c(\mathbf{y}, \mathbf{z}).c(\mathbf{x}.\mathbf{y}, \mathbf{z}).c(\mathbf{x}, \mathbf{y}.\mathbf{z}).c(\mathbf{x}, \mathbf{y}) = 1$$

By definition we have $c(\mathbf{x}, \mathbf{1}) = c(\mathbf{1}, \mathbf{x}) = 1$ and $c(\tau_{i,j}, \tau_{i,j}) = -1$. Let's prove that $\mathbf{S}$ satisfy properties (Sq), (V) et (H).

(Sq) Let any four distincts rectangles S $r_1, r_2, r_1', r_2' \in \mathrm{Rect}^\circ$ such that $r_1 * r_2 = r_1' * r_2'$. Suppose

$$\widetilde{r}_1 \in \mathrm{Rect}^\circ(\widetilde{\mathbf{x}}, \widetilde{\mathbf{x}}.\widetilde{\tau}_{i,j})$$

corresponds to r_1 and

$$\widetilde{r}_2 \in \mathrm{Rect}^\circ(\widetilde{\mathbf{x}}.\widetilde{\tau}_{i,j}, \widetilde{\mathbf{x}}.\widetilde{\tau}_{i,j}.\widetilde{\tau}_{k,l})$$

corresponds to r_2. Then

$$\widetilde{r}_1' \in \mathrm{Rect}^\circ(\widetilde{\mathbf{x}}, \widetilde{\mathbf{x}}.\widetilde{\tau}_{k,l})$$

corresponds to r'_1 and

$$\widetilde{r}'_2 \in \mathrm{Rect}^\circ(\widetilde{\mathbf{x}}.\widetilde{\tau}_{k,l}, \widetilde{\mathbf{x}}.\widetilde{\tau}_{k,l}.\widetilde{\tau}_{i,j})$$

corresponds to r'_2. There are several cases to verify, as for the proof of $\widetilde{\partial}^- \circ \widetilde{\partial}^- = 0$ but all cases can be verified in a similar way. We verify the case $i < j < k < l$. We calculate $\delta c(\tau_{k,l}, \tau_{i,j}, \mathbf{x})$ and $\delta c(\tau_{i,j}, \tau_{k,l}, \mathbf{x})$. With equalities $c(\tau_{i,j}.\tau_{k,l}, \mathbf{x}) = c(\tau_{k,l}.\tau_{i,j}, \mathbf{x})$ and $c(\tau_{i,j}, \tau_{k,l}) = -c(\tau_{k,l}, \tau_{i,j})$ we get

$$\mathbf{S}(r_1).\mathbf{S}(r_2) = -\mathbf{S}(r'_1).\mathbf{S}(r'_2)$$

(V) Let $r_1, r_2 \in \mathrm{Rect}^\circ$ such that $r_1 * r_2$ is a vertical annulus. Suppose that $\widetilde{r}_1 \in \mathrm{Rect}^\circ(\widetilde{\mathbf{x}}, \widetilde{\mathbf{x}}.\widetilde{\tau}_i)$ corresponds to r_1 and $\widetilde{r}_2 \in \mathrm{Rect}^\circ(\widetilde{\mathbf{x}}.\widetilde{\tau}_i, \widetilde{\mathbf{x}}.\widetilde{\tau}_i.\widetilde{\tau}_{i,j})$ corresponds to r_2. We calculate $\delta c(\tau_i, \tau_i, \mathbf{x})$ and with equalities $c(\mathbf{x}, \mathbf{1}) = 1$, $c(\tau_i, \tau_i) = -1$ we get

$$\mathbf{S}(r_1).\mathbf{S}(r_2) = -1$$

(H) Let $r_1, r_2 \in \mathrm{Rect}^\circ$ such that $r_1 * r_2$ is a horizontal annulus (of height one). Suppose $\widetilde{r}_1 \in \mathrm{Rect}^\circ(\widetilde{\mathbf{x}}, \widetilde{\mathbf{x}}.\widetilde{\tau}_{i,j})$ corresponds to r_1 and $\widetilde{r}_2 \in \mathrm{Rect}^\circ(\widetilde{\mathbf{x}}.\widetilde{\tau}_{i,j}, \widetilde{\mathbf{x}}.\widetilde{\tau}_{i,j}.\widetilde{\tau}_{j,i})$ corresponds to r_2. We calculate $\delta c(\tau_{i,j}, \tau_{i,j}, \mathbf{x})$ and with equalities $c(\mathbf{x}, \mathbf{1}) = 1$, $c(\tau_{i,j}, \tau_{i,j}) = -1$ we get

$$\mathbf{S}(r_1).\mathbf{S}(r_2) = +1$$

□

Proposition 2.5.3. *The filtered chain complex $(\widetilde{C}^-(G), \widetilde{\partial}^-)$ is filtered isomorphic to the filtered chain complex $(C^-(G), \partial^-)$.*

Proof. The map $s : \mathfrak{S}_n \to \widetilde{\mathfrak{S}}_n$ extends linearly with respect to $\mathbb{Z}[U_1, \ldots, U_n]$ uniquely to a map $s : C^-(G) \to \widetilde{C}^-(G)$ which is an isomorphism of modules. It commutes with the differentials i.e. $s \circ \partial^- = \widetilde{\partial}^- \circ s$ where the sign assignment $\mathbf{S}$ is given by equation (2.5). By definition, s respects the Alexander filtration and the Maslov grading. So s defines a filtered isomorphism between the complexes $(C^-(G), \partial^-)$ and $(\widetilde{C}^-(G), \widetilde{\partial}^-)$. □

2.6 Computation of signs

In this section, we explain the way one can compute the 2-cocycle given in section 2.5. We recall lemma 2.2.2 which plays a key role:

Lemma 2.6.1. *Let $\widetilde{\mathbf{x}}$ be an element in $\widetilde{\mathfrak{S}}_n$ and $\mathbf{x} = p(\widetilde{\mathbf{x}}) \in \mathfrak{S}_n$. Then for any $0 \leq i \neq j \leq n-1$*

$$\widetilde{\mathbf{x}}.\widetilde{\tau}_{i,j}.\widetilde{\mathbf{x}}^{-1} = z^{\varepsilon(\mathbf{x})}\widetilde{\tau}_{\mathbf{x}(i),\mathbf{x}(j)}$$

In equation (2.5), we need to calculate this kind of expression

$$c(\tau_{i,j},\mathbf{x}) \in \{1,z\}$$

To every element $\mathbf{x} \in \mathfrak{S}_n$ we associate $s(\mathbf{x}) = \prod_{l=1}^{l(x)} \widetilde{\tau}_{i_l,j_l}$ and we suppose that for all $l \in \{1,\dots,l(x)\}$, $i_l < j_l$ (we use the section defined in section 2.5). We would like to compute $s(\mathbf{x}).\widetilde{\tau}_{i,j}$ and compare it with $s(x.\tau_{i,j}) = s(\mathbf{y})$ for $i<j$. Write $s(\mathbf{y}) = \prod_{l=1}^{l(y)} \widetilde{\tau}_{i_l,j_l}$ we obtain $s(\mathbf{x}).\widetilde{\tau}_{i,j} = c(\tau_{i,j},\mathbf{x})s(\mathbf{y})$. That is

$$\big(\prod_{l=1}^{l(x)} \widetilde{\tau}_{i_l,j_l}\big).\widetilde{\tau}_{i,j}.\big(z^{\varepsilon(y)}\prod_{l=1}^{l(y)} \widetilde{\tau}_{i_{l(y)+1-l},j_{l(y)+1-l}}\big) = c(\tau_{i,j},\mathbf{x}) \tag{2.6}$$

To compute the sign we need to take care of the way we reduce this word to the trivial word.

The problem is now this one: given a word in $\widetilde{\mathfrak{S}}_n$ of the type $\prod_{l=1}^{r} \widetilde{\tau}_{i_l,j_l}$ where for all $l \in \{1,\dots,r\}$, $i_l < j_l$ and whose projection in $\mathfrak{S}_n$ is trivial, is it equal to $\widetilde{\mathbf{1}}$ or to z?

Let us describe an algorithm which computes the sign of this reduction by induction on n.

- For $n=4$, compute *by hand* the sign using the relations in $\widetilde{\mathfrak{S}}_4$.
- Suppose the reduction problem is solved until rank n and let $\widetilde{\mathbf{x}} = \prod_{l=1}^{r} \widetilde{\tau}_{i_l,j_l} \in \widetilde{\mathfrak{S}}_{n+1}$ be a word whose projection in $\mathfrak{S}_{n+1}$ is trivial. Let $J = max(\{j_l\}_{l=1}^{r})$. If $J < n+1$, then $\widetilde{\mathbf{x}}$ is in fact an element of $\widetilde{\mathfrak{S}}_n$ and the reduction problem of $\widetilde{\mathbf{x}}$ is known. Otherwise, the index J appears at least two times in the writing of $\mathbf{x}$. Let $\widetilde{\tau}_{i_0,J}$ be the most right transposition of $\mathbf{x}$ which contains J as an index. We go back to the left to find the first transposition $\widetilde{\tau}_{(i_0,i_1)}$ which contains i_0 (here (i_0,i_1) means that the two elements are already ordered).

$$\mathbf{x} = \widetilde{u}.\widetilde{\tau}_{(i_0,i_1')}.\widetilde{v_0}.\widetilde{\tau}_{i_0,J}.\widetilde{v}$$

Lemma 2.2.2 gives

$$\widetilde{\tau}_{(i_0,i_1')}.\widetilde{v} = z^{\varepsilon(v_0)}\widetilde{v_0}.\widetilde{\tau}_{(v_0^{-1}(i_0),v_0^{-1}(i_1'))} = z^{\varepsilon(v_0)}\widetilde{v_0}.\widetilde{\tau}_{(i_0,i_1))}$$

where $i_1 = v_0^{-1}(i_1')$. We remark here that $\widetilde{v}$ is written as a product of transpositions therefore $\varepsilon(v_0) = l(v_0) - 1 \in \mathbb{Z}/2\mathbb{Z}$ where $l(v_0)$ is the length of v.

$$\mathbf{x} = \widetilde{u}.z^{l(v_0)-1}\widetilde{v_0}.\widetilde{\tau}_{(i_0,i_1)}.\widetilde{\tau}_{i_0,J}.\widetilde{w} = z^{l(v_0)-1}\widetilde{u}.\widetilde{v_0}.\widetilde{\tau}_{(i_0,i_1)}.\widetilde{\tau}_{i_0,J}.\widetilde{w}$$

We go back again to the left to find the first transposition $\widetilde{\tau}_{(i_1,i_2')}$ which contains i_1 and do the same computation. We stop the first time $i_l = J$ (it must exists since $\widetilde{\mathbf{x}}$ projects to the identity). So, the writing of $\widetilde{\mathbf{x}}$ is

$$\widetilde{\mathbf{x}} = z^{\sum_{k=0}^{l-1} l(v_k)-1}.\widetilde{t}.\big(\prod_{k=0}^{l-1} \widetilde{\tau}_{(i_{l-k-1},i_{l-k})}\big).\widetilde{\tau}_{i_0,J}.\widetilde{w}$$

Denote $\prod_{k=0}^{l-1} \widetilde{\tau}_{(i_{l-k-1},i_{l-k})}$ by $\widetilde{\nu}$. Using again lemma 2.2.2 we get

$$\begin{aligned}
\widetilde{\nu}.\widetilde{\tau}_{i_0,J} &= z^{l(\nu)-1}\widetilde{\tau}_{\nu(i_0),\nu(J)}.\widetilde{\nu} \\
&= z^{l(\nu)-1}\widetilde{\tau}_{J,i_{l-1}}.\widetilde{\nu} \\
&= z^{l(\nu)-1+\delta}.\prod_{k=1}^{l-1} \widetilde{\tau}_{(i_{l-k-1},i_{l-k})}
\end{aligned}$$

where $\delta = 0$ if $\widetilde{\tau}_{(i_{l-1},i_l)} = \widetilde{\tau}_{i_{l-1},J}$ and 1 otherwise. We go on until there is no transposition with index $J = N+1$ remaining. Thus, $\widetilde{\mathbf{x}}$ is seen as a word in $\widetilde{\mathfrak{S}}_n$ and we use the induction to calculate the sign.

Chapter 3

Discrete Morse theory

Théorie de Morse discrète

R. Forman a défini un analogue combinatoire de la théorie de Morse lisse dans [For98b], [For98b], [For98c] pour les complexes simpliciaux et plus généralement pour les CW-complexes.

La théorie de Morse discrète a de nombreuses applications (dessin assisté par ordinateur [LLT04], théorie des graphes [FS05]). Un problème important est la recherche de fonctions de Morse discrètes optimales au sens où elles ont le nombre minimal de cellules critiques ([JP04], [LLT03],[SS07] pour la minimalité d'arrangements d'hyperplans).

Grâce à un champ de vecteurs combinatoire de Morse V, Forman construit un complexe de Thom-Smale combinatoire (C^V, ∂^V) dont l'homologie est l'homologie simpliciale du complexe simplicial. La différentielle est définie comme la somme algébrique de V-orbites entre cellules critiques. Néanmoins, la preuve de $\partial^V \circ \partial^V = 0$ est une preuve indirecte (voir [For98b]). Nous donnons deux nouvelles preuves de $\partial^V \circ \partial^V = 0$, une qui se focalise sur la géométrie et une autre qui s'appuie sur l'élimination gaussienne (voir Lemme 3.2.8) qui est une opération algébrique sur les complexes de chaînes qui ne change pas l'homologie (voir [Cha00] et [Skö06]). En fait, la preuve algébrique montre aussi que le complexe de Thom-Smale combinatoire est un complexe de chaînes et qu'il est homotopiquement équivalent au complexe de chaînes simpliciales. Le corollaire 3.2.9 donne le lien exact entre la théorie de Morse discrète et l'algèbre des complexes de chaînes.

Après cela, nous nous intéressons au lien existant entre la théorie de Morse lisse et discrète. Nous montrons tout complexe de Thom-Smale lisse est réalisable par un complexe de Thom-Smale combinatoire. Grâce à cela, nous montrons que toute structure Spin^c sur une variété de dimension 3 close orientée peut être réalisée par un couplage complet sur une triangulation de cette variété.

Champ de vecteurs combinatoire

Tout d'abord, la théorie de Morse peut s'aborder de deux manière différentes : soit on s'intéresse à la fonction à valeur dans $\mathbb{R}$, soit on s'intéresse au champ de vecteurs (pseudo)-gradient associé. En théorie de Morse discrète, on a exactement le même phénomène (voir Théorème 9.3 [For98b]). Nous allons exposer la théorie de Morse discrète uniquement du point de vue champ de vecteurs combinatoire ce qui possède l'avantage d'être plus simple et de mettre en avant la dynamique sur les objets combinatoires considérés.

Dans tout ce qui suit, X est un complexe simplicial fini (voir [Hud69], [RS72]). On note K l'ensemble des cellules ouvertes de X et on munit K d'un ordre partiel $<$ donné par $\sigma < \tau$ si $\sigma \subset \overline{\tau}$. Étant donné un complexe simplicial X, on définit son **diagramme de Hasse** comme suit : l'ensemble des sommets est l'ensemble des cellules K, un arêtes relie deux cellules σ et τ si $\sigma < \tau$ et $dim(\tau) = dim(\sigma)+1$ (dans ce cas σ est appelée une **hyperface** de τ).

Définition 3.0.1. Un **champ de vecteurs combinatoire** V sur X est la donnée d'un **couplage** orienté sur le diagramme d'incidence de X c'est-à-dire un ensemble d'arêtes tel quel

- deux arêtes distinctes du couplage sont disjointes,
- une arête du couplage reliant σ à τ est orientée de σ vers τ si $\sigma < \tau$.

Une cellule qui n'appartient pas à une arête du couplage est appelée une **cellule critique**.

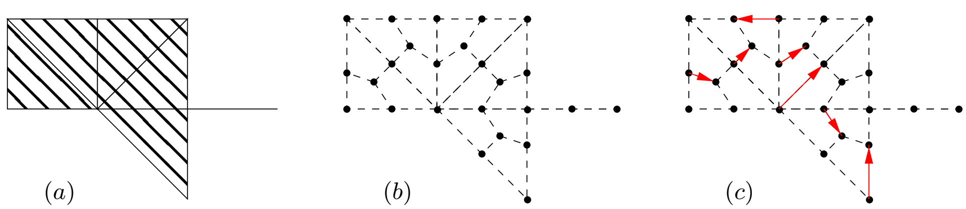

FIG. 3.1 – **D'un complexe simplicial à un champ de vecteurs combinatoire.** (a) est le complexe simplicial X. (b) représente le diagramme de Hasse associé. (c) est un exemple de champ de vecteurs combinatoire sur X.

Remarque. La définition originale d'un champ de vecteurs combinatoire est la suivante : étant donné un couplage sur le diagramme d'incidence de X, on définit

$$\begin{array}{rccl} V: & K & \to & K \cup \{0\} \\ & \sigma & \mapsto & V(\sigma) = \begin{cases} \tau & \text{ssi } (\sigma,\tau) \text{ est une arête du couplage et } \sigma < \tau, \\ 0 & \text{sinon.} \end{cases} \end{array}$$

Nous utiliserons les deux points de vue par la suite.

Une V**-orbite** de dimension k est une suite de cellules $\gamma : \sigma_0, \sigma_1, \ldots, \sigma_r$ de dimension k telle que

1. $\sigma_i \neq \sigma_{i+1}$ pour tout $i \in \{0, \ldots, r-1\}$,
2. pour tout $i \in \{0, \ldots, r-1\}$, $\sigma_{i+1} < V(\sigma_i)$.

Une V-orbite γ est dite **fermée** si $\sigma_0 = \sigma_r$, et **non stationnaire** si $r > 0$.

Définition 3.0.2. Un champ de vecteurs combinatoire V est appelé **champ de vecteurs combinatoire de Morse** ssi V n'a pas d'orbite fermée non stationnaire. Dans ce cas, le couplage correspondant est appelé couplage de Morse.

La terminologie *couplage de Morse* est apparue pour la première dans [Cha00]. Par exemple, le champ de vecteurs combinatoire de la Fig.3.1 (c) est de Morse alors que celui de la Fig.3.2 ne l'est pas.

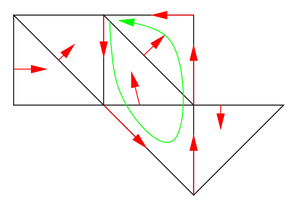

FIG. 3.2 – **Exemple de champ de vecteurs combinatoire qui n'est pas de Morse.** La V-orbite de dimension 0 dessinée en vert est fermée et non stationnaire.

Complexe de Thom-Smale combinatoire

Nous avons besoin des données suivantes pour définir le complexe de Thom-Smale combinatoire (cf [For98b]). Tout d'abord, soit X un complexe simplicial, K l'ensemble de ses cellules ouvertes et V un champ de vecteurs combinatoire de Morse. On suppose que chaque cellule $\sigma \in K$ est orientée.

Soit $\gamma : \sigma_0, \sigma_1, \ldots, \sigma_r$ une V-orbite. Alors la **multiplicité** de γ est définie par la formule

$$m(\gamma) = \prod_{i=0}^{r-1} - < \partial V(\sigma_i), \sigma_i >< \partial V(\sigma_i), \sigma_{i+1} > \qquad \in \{\pm 1\}$$

où pour toutes cellules σ, τ, $< \sigma, \tau > \in \{-1, 0, 1\}$ est le nombre d'incidence entre les cellules σ et τ (cf [LW69]) et ∂ est l'opérateur de bord du complexe simplicial associé à X. En fait, on peut penser à la multiplicité comme une comparaison entre l'orientation de la cellule initiale σ_0 transportée par γ jusqu'en σ_r et l'orientation de la cellule finale σ_r. Si les deux orientations coïncident, on obtient $+1$, sinon on obtient -1.

Soient $\Gamma(\sigma, \sigma')$ l'ensemble des V-orbites commençant en σ et aboutissant en σ' et $Crit_k(V)$ l'ensemble des cellules critiques de dimension k.

Définition 3.0.3 (Forman)**.** Le complexe de Thom-Smale combinatoire associé à la paire (X, V) est $\mathcal{M}^V = (C_*^V, \partial^V)$ où :

1. $C_k^V = \bigoplus_{\sigma \in Crit_k(V)} \mathbb{Z}.\sigma$,
2. si $\tau \in Crit_{k+1}(V)$ alors
$$\partial^V \tau = \sum_{\sigma \in Crit_k(V)} n(\tau, \sigma).\sigma$$
où
$$n(\tau, \sigma) = \sum_{\widetilde{\sigma} < \tau} < \partial\tau, \widetilde{\sigma} > \sum_{\gamma \in \Gamma(\widetilde{\sigma}, \sigma)} m(\gamma)$$

Ainsi, ce complexe est un analogue combinatoire du complexe de Thom-Smale. Il est engendré par les cellules critiques et la différentielle s'exprime en comptant algébriquement des V-orbites reliant des (hyperfaces des) cellules critiques de dimension $k+1$ à des cellules critiques de dimension k pour tout k.

Sous les hypothèses ci-dessus on a

Théorème 3.0.4 (Forman)**.** (C_*^V, ∂^V) *est un complexe de chaînes.*

Théorème 3.0.5 (Forman)**.** *L'homologie du complexe* (C_*^V, ∂^V) *est l'homologie simpliciale du complexe simplicial.*

Nous donnons une preuve de ces deux théorèmes qui s'appuie sur l'élimination gaussienne (voir par exemple [BN06]). Lorsque $V(\sigma) = \tau$, on souhaite enlever au complexe simplicial le complexe acyclique suivant :

$$0 \to \mathbb{Z}.\tau \xrightarrow{<\partial\tau, \sigma>} \mathbb{Z}.\sigma \to 0$$

Lorsque l'on peut le faire, la différentielle s'en trouve changée et des paires de cellules critiques sont supprimées : ces modifications définissent le complexe de Thom-Smale combinatoire. On peut donc réinterpréter un champ de vecteurs combinatoire de Morse comme une instruction de suppression de complexe de chaînes acycliques *courts* au complexe de chaînes simpliciales original. Cette interprétation de la théorie de Morse discrète est donnée pour la première fois par Chari [Cha00] (voir aussi [Skö06]).

Corollaire 3.0.6. *Soit X un complexe simplicial fini, $\mathcal{C} = (C_*, \partial)$ le complexe de chaînes simplicial associé et $\mathcal{M}$ un couplage $(\sigma_i < \tau_i)_{i \in I}$ sur son diagramme de Hasse définissant un champ de vecteurs combinatoire V. Alors les deux propriétés suivantes sont équivalentes :*

1. *$\mathcal{M}$ est un couplage de Morse,*
2. *pour toute suite $(\sigma_{i_1} < \tau_{i_1}), (\sigma_{i_2} < \tau_{i_2}), \ldots, (\sigma_{i_{|I|}} < \tau_{i_{|I|}})$ telle que $i_j \neq i_k$ si $j \neq k$, les éliminations gaussiennes peuvent être réalisées dans cet ordre.*

En particulier, toute suite d'éliminations gaussiennes correspondant à $\mathcal{M}$ aboutit au même complexe de chaînes qui est le complexe combinatoire de Thom-Smale associé à V.

Théories de Morse lisse et discrète

Étant donnée une fonction de Morse $f : M \to \mathbb{R}$ définie sur une variété lisse, on dira qu'elle est générique si un gradient parfait v pour f a été choisi ([Paj95] v est gradient parfait pour f si v est un champ de vecteurs pseudo-gradient pour f tel que les variétés stables et instables soient transverses).

À partir de la donnée d'une fonction de Morse générique et d'une orientation des variétés stables et instables correspondantes on définit le complexes de Thom-Smale (lisse). Nous montrons le théorème suivant, qui est le résultat principal de ce chapitre :

Théorème 3.0.7 (Théorème 3.3.1)**.** *Soit M une variété riemannienne lisse close et $f : M \to \mathbb{R}$ une fonction de Morse générique. On suppose que chaque variété stable est orientée de sorte que le complexe de Thom-Smale lisse est défini. Alors, il existe une C^1-triangulation T de M et un champ de vecteurs combinatoire de Morse V qui réalisent le complexe de Thom-Smale lisse (après choix d'orientation des cellules de T) au sens suivant :*

1. *il existe une bijection entre l'ensemble des cellules critiques et l'ensemble des points critiques,*
2. *pour chaque paire de points critiques σ_p et σ_q telle que $dim(\sigma_p) = dim(\sigma_q) + 1$, les V-orbites partant des hyperfaces de σ_p et aboutissant à σ_q sont en bijection avec les lignes de gradient de v à renormalisation près reliant q à p,*
3. *cette bijection induit un isomorphisme entre le complexe de Thom-Smale lisse et combinatoire.*

Un **couplage complet** sur un graphe est un couplage tel que chaque sommet du graphe appartienne à une arête du couplage. Nous obtenons alors le corollaire suivant :

Corollaire 3.0.8 (Corollaire 3.4.2)**.** *Soit M une variété lisse close de dimension 3. Alors il existe une C^1-triangulation de M de sorte qu'un couplage complet sur son diagramme de Hasse existe.*

On peut se demander si toute C^1-triangulation d'une telle variété possède un couplage complet. On ne sait pas si cette propriété est vraie en général mais en adaptatant l'algorithme de Ford-Fulkerson, on a écrit un algorithme pour *Regina* [Bur05] qui recherche les couplages complets sur une triangulation d'une variété de dimension de dimension 3 donnée. Sur tous les exemples, on a toujours trouvé un couplage complet.

Les couplages complets s'interprètent comme des structures d'Euler. Une structure d'Euler a une $\mathcal{M}$-réalisation si il existe une triangulation et un couplage complet dessus qui la réalise.

Théorème 3.0.9 (Théorème 3.4.6)**.** *Toute structure d'Euler sur une variété de dimension 3 lisse, close, orientée a une $\mathcal{M}$-réalisation.*

À l'aide d'un théorème de Turaev, on sait que les structures d'Euler sont en bijection avec les structures Spin^c sur les variétés de dimension 3, closes, orientées. La donnée d'une variété de dimension 3, close, orientée munie d'une structure Spin^c et d'un scindement de Heegaard (i.e. d'une fonction de Morse générique) sont les données essentielles pour définir l'homologie de Heegaard-Floer de telles variétés (voir [OS04c]). Le théorème 3.3.1 et le théorème 3.4.6 permettent de réaliser ces données de manière combinatoire.

3.1 Introduction

R. Forman defines a combinatorial analog of the smooth Morse theory in [For98a], [For98b], [For98c] for simplicial complexes and more generally for CW-complexes.

Discrete Morse theory has many applications (computer graphics [LLT04], graph theory [FS05]). An important problem is the research of optimal discrete Morse functions in the sense that they have the minimal number of critical cells ([JP04], [LLT03],[SS07] for the minimality of hyperplane arrangements).

Thanks to a combinatorial Morse vector field V, Forman constructs a combinatorial Thom-Smale complex (C^V, ∂^V) whose homology is the simplicial homology of the simplicial complex. The differential is defined by counting algebraically V-paths between critical cells. Nevertheless, the proof of $\partial^V \circ \partial^V = 0$ is an indirect proof (see [For98b]). We give two proofs of $\partial^V \circ \partial^V = 0$, one which focusses on the geometry and another one which focusses on the algebraic point of view (compare with [Cha00] and [Skö06]). In fact, the algebraic proof gives also the property that the combinatorial Thom-Smale complex is a chain complex homotopy equivalent to the simplicial chain complex. After that, we investigate one step forward the relation between the smooth Morse theory and the discrete Morse theory. We prove that any Thom-Smale complex has a combinatorial realization. We use this to prove that any Spin^c-structures on a closed oriented 3-manifold can be realized by a complete matching on a triangulation of this manifold.

This chapter is organised as follow. In section 3.2, we recall the discrete Morse theory from the viewpoint of combinatorial Morse vector field. Section 3.2.2 is devoted to the proofs of $\partial^V \circ \partial^V = 0$ and that the Thom-Smale complex is a chain complex homotopy equivalent to the simplicial chain complex. In section 3.3 we prove that any combinatorial Thom-Smale complex is realizable as a combinatorial Thom-Smale complex. In section 3.4, we obtain as a corollary the existence of triangulations with complete matchings on their Hasse diagram and prove that any Spin^c-structures on a closed oriented 3-manifold can be realized by such complete matchings. Finallly we explain how we compute that on *Regina* [Bur05].

3.2 Discrete Morse theory

3.2.1 Combinatorial Morse vector field

First of all, instead of considering discrete Morse functions on a simplicial complex, we will only consider combinatorial Morse vector fields. In fact, working with discrete Morse functions or combinatorial Morse vector fields is exactly the same (see Theorem 9.3 [For98b]).

In the following, X is finite simplicial complex and K is the set of cells of

X. A cell $\sigma \in K$ of dimension k is denoted $\sigma^{(k)}$. Let $<$ be the partial order on K given by $\sigma < \tau$ iff $\sigma \subset \overline{\tau}$. Given a simplicial complex, one associates its **Hasse diagram**: the set of vertices is the set of cells K, an edge joins two cells σ and τ if $\sigma < \tau$ and $dim(\sigma) + 1 = dim(\tau)$ (see Fig.3.3 (b)).

Definition 3.2.1. A combinatorial vector field V on X is an oriented matching on the associated Hasse diagram of X that is a set of edges $\mathcal{M}$ such that

1. any two distincts edges of $\mathcal{M}$ do not share any common vertex,

2. every edge belonging to $\mathcal{M}$ is oriented toward the top dimensional cell.

A cell which does not belong to any edge of the matching is said to be **critical**.

Remark 3.2.2. The original definition of a combinatorial vector field is the following one: given a matching on the Hasse diagram define

$$\begin{array}{rccl} V: & K & \to & K \cup \{0\} \\ & \sigma & \mapsto & V(\sigma) = \begin{cases} \tau & \text{iff } (\sigma,\tau) \text{ is an edge of the matching and } \sigma < \tau, \\ 0 & \text{otherwise.} \end{cases} \end{array}$$

We will use both these points of view in the following.

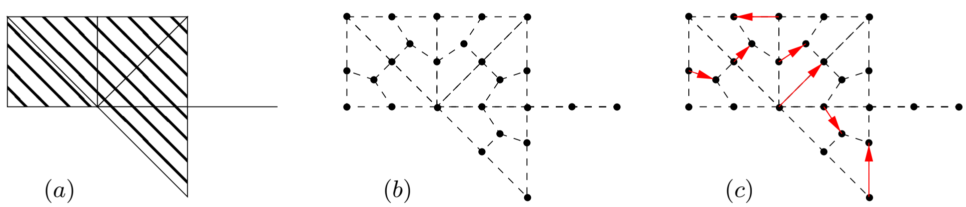

Figure 3.3: **From a simplicial complex to a combinatorial vector field.** (a) is the simplicial complex X. (b) represents the associated Hasse diagram. (c) shows an example of combinatorial vector field on the X.

A V**-path** of dimension k is a sequence of cells $\gamma : \sigma_0, \sigma_1, \ldots, \sigma_r$ of dimension k such that

1. $\sigma_i \neq \sigma_{i+1}$ for all $i \in \{0, \ldots, r-1\}$,

2. for every $i \in \{0, \ldots, r-1\}$, $\sigma_{i+1} < V(\sigma_i)$.

A V-path γ is said to be **closed** if $\sigma_0 = \sigma_r$, and **non-stationary** if $r > 0$.

Definition 3.2.3. A combinatorial vector field V which has no non-stationary closed path is called a combinatorial Morse vector. In this case, the corresponding matching is called a Morse matching.

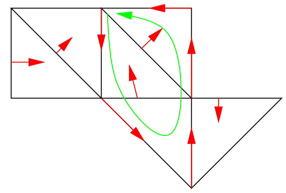

Figure 3.4: **Example of a non combinatorial Morse vector field.** The V-path drawn in green is a non-stationary closed path of dimension 0.

The terminology *Morse matching* first appeared in [Cha00]. For instance, the combinatorial vector field drawn on Fig.3.3 (c) is Morse while the one of Fig.3.4 is not.

Remark 3.2.4. Let V be a combinatorial (resp. combinatorial Morse) vector field. It we remove an edge from the underlying matching, it remains a combinatorial (resp. combinatorial Morse) vector field (there are two extra critical cells).

3.2.2 The combinatorial Thom-Smale complex

Definition of the combinatorial Thom-Smale complex

The following data are necessary to define the combinatorial Thom-Smale complex (see [For98b]). First, let X be a finite simplicial complex, K its set of cells and V a combinatorial Morse vector field. Suppose that every cell $\sigma \in K$ is oriented.

Let $\gamma : \sigma_0, \sigma_1, \dots, \sigma_r$ be a V-path. Then the **multiplicity** of γ is given by the formula

$$m(\gamma) = \prod_{i=0}^{r-1} - < \partial V(\sigma_i), \sigma_i >< \partial V(\sigma_i), \sigma_{i+1} > \qquad \in \{\pm 1\}$$

where for every cell σ, τ, $< \sigma, \tau > \in \{-1, 0, 1\}$ is the incidence number between the cells σ and τ (see [LW69]) and ∂ is the boundary map when we consider X as a CW-complex. In fact, one can consider the multiplicity as checking if the orientation of the first cell σ_0 moved along γ coincides or not with the orientation of the last cell σ_r.

Let $\Gamma(\sigma, \sigma')$ be the set of V-paths starting at σ and ending at σ' and $Crit_k(V)$ be the set of critical cells of dimension k.

Definition 3.2.5. The combinatorial Thom-Smale complex associated with (X, V) is (C_*^V, ∂^V) where:

1. $C_k^V = \bigoplus_{\sigma \in Crit_k(V)} \mathbb{Z}.\sigma$,

2. if $\tau \in Crit_{k+1}(V)$ then

$$\partial^V \tau = \sum_{\sigma \in Crit_k(V)} n(\tau,\sigma).\sigma$$

where

$$n(\tau,\sigma) = \sum_{\widetilde{\sigma}<\tau} < \partial\tau, \widetilde{\sigma} > \sum_{\gamma \in \Gamma(\widetilde{\sigma},\sigma)} m(\gamma)$$

Thus, this complex is exactly in the same spirit as the Thom-Smale complex for smooth Morse functions (see section 3.3): it is generated by critical cells and the differential is given by counting algebraically V-paths.

Theorem 3.2.6 (Forman [For98b])**.** $\partial^V \circ \partial^V = 0$.

Theorem 3.2.7 (Forman [For98b])**.** (C_*^V, ∂^V) *is homotopy equivalent to the simplicial chain complex. In particular, its homology is equal to the simplicial homology.*

We will give a direct proof of both of these theorems. The proof of Theorem 3.2.6 is by looking at V-paths and understanding their contribution to $\partial^V \circ \partial^V$. Then, we prove Theorem 3.2.7 (which gives another proof of Theorem 3.2.6) using Gaussian elimination (this idea first appeared in [Cha00], see also [Skö06]).

Proof of Theorem 3.2.6

Let X be a simplicial complex, K be the set of its cells and K_n the set of cells of dimension n. The proof is by induction on the number on edges belonging to the matching: we always suppose that this matching gives rise to a combinatorial Morse vector field (i.e. there is no non-stationary closed path).

Initialization: matching with no edge. In this case, every cell is critical and the combinatorial Thom-Smale complex coincides with the well-known simplicial chain complex. Therefore, $\partial^V \circ \partial^V = 0$.

Heredity: suppose the property is true for every matching with at most k edges defining a combinatorial Morse vector field. Let V be a combinatorial Morse vector field with corresponding matching consisting of $k+1$ edges. Let (σ,τ) be an edge of this matching with $\sigma < \tau$ and let $\overline{V}$ be the combinatorial Morse vector field corresponding to the original matching with the edge (σ,τ) removed. By induction hypothesis $\partial^{\overline{V}} \circ \partial^{\overline{V}} = 0$. In particular, for every $n \in \mathbb{N}$, every $\tau \in K_{n+1}$ and every $\nu \in K_{n-1}$ when there is a cell $\sigma_1 \in K_n$ such that there is a $\overline{V}$-path from an hyperface of τ to σ_1 and another $\overline{V}$-path from an hyperface of σ_1 to ν there exists another cell

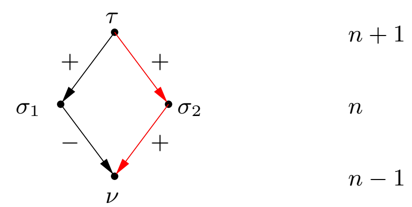

Figure 3.5: $\partial^{\overline{V}} \circ \partial^{\overline{V}} = 0$. V-paths cancel by pair.

$\sigma_2 \in K_n$ with the same property so that their contribution to $\partial^{\overline{V}} \circ \partial^{\overline{V}}$ are opposite (see Fig.3.5).

First, we will prove that $\partial^V \circ \partial^V = 0$ when the chain complex is with coefficients in $\mathbb{Z}/2\mathbb{Z}$ and after we will take care of signs.

Suppose that the distinguished edge of the matching (σ, τ) is such that $dim(\sigma) + 1 = dim(\tau) = n + 1$. Therefore, $C_i^V = C_i^{\overline{V}}$ for $i \neq n, n+1$ and $\partial^V_{|C_i^V} = \partial^{\overline{V}}_{|C_i^V}$ for $i \notin \{n, n+1, n+2\}$. So we have $\partial^V \circ \partial^V(\mu) = 0$ for all $\mu \in K - (K_n \cup K_{n+1} \cup K_{n+2})$.

Remark that it is also true for every $\sigma' \in Crit_n(V)$ that $\partial^V \circ \partial^V(\sigma') = 0$ (since with respect to $\overline{V}$ it is true and $\sigma' \neq \sigma$).

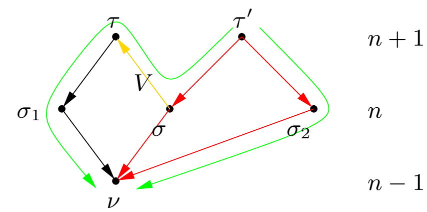

Figure 3.6: **Case 1.** The $\overline{V}$-path which disappears is replaced by exactly a new one which goes through the edge (σ, τ). The new V-paths are drawn in green.

There are two cases left.

Case 1. Let $\tau' \in Crit_{n+1}(V)$. To see that $\partial^V \circ \partial^V(\tau') = 0$ we must consider two cases. First case is when the two $\overline{V}$-paths which annihilates don't go through σ. Then, nothing is changed and contributions to $\partial^V \circ \partial^V(\tau')$ cancel by pair. The second case is when at least one the $\overline{V}$-path which cancel by pair for $\partial^{\overline{V}}$ go through σ. This case is illustrated by Fig.3.6. The $\overline{V}$-paths which go from τ' to ν are of two types: those who go via σ and the others. Let $\tau' \to \sigma_2 \to \nu$ be a juxtaposition of two $\overline{V}$-paths which cancel with the juxtaposition of $\overline{V}$-path $\tau' \to \sigma \to \nu$. Since $\partial^{\overline{V}} \circ \partial^{\overline{V}}(\tau) = 0$, there must be a critical cell σ_1 such that the juxtaposition of $\overline{V}$-paths $\tau \to \sigma \to \nu$ and $\tau \to \sigma_1 \to \nu$ cancels. Therefore, when considering ∂^V, three juxtapositions of $\overline{V}$-

paths disappear and one is created: $\tau' \to (\sigma \to \tau) \to \sigma_1 \to \nu$. It cancels with $\tau' \to \sigma_2 \to \nu$.

It may happens that two juxtapositions of $\overline{V}$-paths go through σ but this case works exactly in the same way.

Case 2. This case is similar to the previous case. Let ς be a cell in K_{n+2}. There are two cases to see that $\partial^V \circ \partial^V(\varsigma) = 0$. The first case is when the two $\overline{V}$-paths whose contributions are opposite don't go through τ. Then, nothing is changed and contributions to $\partial^V \circ \partial^V(\tau')$ cancel by pair. The second case is illustrated by Fig.3.7.

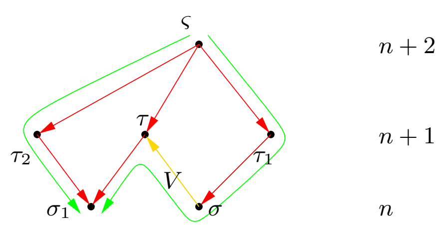

Figure 3.7: **Case 2.** The $\overline{V}$-path which disappears is replaced by exactly a new one which goes through the edge (σ, τ). The new V-paths are drawn in green.

Now, let's deal with the signs. We will only consider the case illustrated in Fig.3.6, the other cases work similarly. Denote $n(\alpha \to \beta)$ (resp. $\overline{n}(\alpha \to \beta)$) the sign of the contribution in the differential ∂^V (resp. $\partial^{\overline{V}}$) of a path going from α to β where both cells are critical of consecutive dimension. While considering $\overline{V}$, we have by induction hypothesis

$$\overline{n}(\tau' \to \sigma_2).\overline{n}(\sigma_2 \to \nu) = -\overline{n}(\tau' \to \sigma).\overline{n}(\sigma \to \nu) \tag{3.1}$$

and

$$\overline{n}(\tau \to \sigma_1).\overline{n}(\sigma_1 \to \nu) = -\overline{n}(\tau \to \sigma).\overline{n}(\sigma \to \nu) \tag{3.2}$$

Since the juxtaposition of the $\overline{V}$-paths $\tau' \to \sigma_2 \to \nu$ don't go through σ we have that

$$\overline{n}(\tau' \to \sigma_2).\overline{n}(\sigma_2 \to \nu) = n(\tau' \to \sigma_2).n(\sigma_2 \to \nu) \tag{3.3}$$

By definition of the multiplicity of paths we have

$$n(\tau' \to \sigma_1) = \overline{n}(\tau' \to \sigma).(- < \partial\tau, \sigma >).\overline{n}(\tau \to \sigma_1) \tag{3.4}$$

Combining equations (3.1)-(3.4) we obtain the following equalities

$$\begin{aligned}
n(\tau' \to \sigma_1).n(\sigma_1 \to \nu) &= \overline{n}(\tau' \to \sigma).(- < \partial\tau, \sigma >).\overline{n}(\tau \to \sigma_1).n(\sigma_1 \to \nu) \quad (3.4) \\
&= \overline{n}(\tau' \to \sigma).(- < \partial\tau, \sigma >).\overline{n}(\tau \to \sigma_1).\overline{n}(\sigma_1 \to \nu) \\
&= \overline{n}(\tau' \to \sigma). < \partial\tau, \sigma > .\overline{n}(\tau \to \sigma).\overline{n}(\sigma \to \nu) \quad (3.2) \\
&= (< \partial\tau, \sigma > .\overline{n}(\tau \to \sigma)).\overline{n}(\tau' \to \sigma).\overline{n}(\sigma \to \nu) \\
&= \overline{n}(\tau' \to \sigma).\overline{n}(\sigma \to \nu) \quad \text{by definition} \\
&= -\overline{n}(\tau' \to \sigma_2).\overline{n}(\sigma_2 \to \nu) \quad (3.1) \\
&= -n(\tau' \to \sigma_2).n(\sigma_2 \to \nu) \quad (3.3)
\end{aligned}$$

which concludes the proof of the theorem.

Remark. We see that at level 0 i.e. matching with no edge, $\partial^V \circ \partial^V = 0$ corresponds to the simplicial case. In particular, V-paths which cancel by pair are boundary of disks: consider the Hasse diagram and take edges used by V-paths. This is a phenomenum similar to the smooth case (see [Wit82]). Making the induction on the number of edges has an effect on disks: either they are not changed or two disks are glued together along a common edge to create a new disk. Then the property $\partial^V \circ \partial^V = 0$ can be thought as *"V-paths bound disks"*.

Proof of Theorem 3.2.7

The main ingredient of the proof is thinking about combinatorial Morse vector field as an instruction to remove acyclic complexes from the original simplicial chain complex as done by Chari (Proposition 3.3 [Cha00]) or Sköldberg [Skö06]. Given a matching between two cells $\sigma < \tau$, we would like to remove the following short complex (which is acyclic)

$$0 \to \mathbb{Z}.\tau \xrightarrow{<\partial\tau,\sigma>} \mathbb{Z}.\sigma \to 0$$

where ∂ is the boundary operator of the simplicial chain complex. To do this, we use **Gaussian elimination** (see e.g. [BN06]):

Lemma 3.2.8. *[Gaussian elimination] Let $\mathcal{C} = (C_*, \partial)$ be a chain complex over $\mathbb{Z}$ freely generated. Let $b_1 \in C_i$ (resp. $b_2 \in C_{i-1}$) be such that $C_i = \mathbb{Z}.b_1 \oplus D$ (resp. $C_{i-1} = \mathbb{Z}.b_2 \oplus E$). If $\phi : \mathbb{Z}.b_1 \to \mathbb{Z}.b_2$ is an isomorphism of $\mathbb{Z}$-modules, then the four term complex segment of $\mathcal{C}$*

$$\ldots \to [C_{i+1}] \xrightarrow{\begin{pmatrix}\alpha\\ \beta\end{pmatrix}} \begin{bmatrix}b_1\\ D\end{bmatrix} \xrightarrow{\begin{pmatrix}\phi & \delta\\ \gamma & \varepsilon\end{pmatrix}} \begin{bmatrix}b_2\\ E\end{bmatrix} \xrightarrow{\begin{pmatrix}\mu & \nu\end{pmatrix}} [C_{i-2}] \to \ldots \tag{3.5}$$

is isomorphic to the following chain complex segment

$$\ldots \to [C_{i+1}] \xrightarrow{\begin{pmatrix}0\\ \beta\end{pmatrix}} \begin{bmatrix}b_1\\ D\end{bmatrix} \xrightarrow{\begin{pmatrix}\phi & 0\\ 0 & \varepsilon - \gamma\phi^{-1}\delta\end{pmatrix}} \begin{bmatrix}b_2\\ E\end{bmatrix} \xrightarrow{\begin{pmatrix}0 & \nu\end{pmatrix}} [C_{i-2}] \to \ldots \tag{3.6}$$

Both these complexes are homotopy equivalent to the complex segment

$$\ldots \to \begin{bmatrix} C_{i+1} \end{bmatrix} \xrightarrow{\begin{pmatrix} \beta \end{pmatrix}} \begin{bmatrix} D \end{bmatrix} \xrightarrow{\begin{pmatrix} \varepsilon - \gamma\phi^{-1}\delta \end{pmatrix}} \begin{bmatrix} E \end{bmatrix} \xrightarrow{\begin{pmatrix} \nu \end{pmatrix}} \begin{bmatrix} C_{i-2} \end{bmatrix} \to \ldots \tag{3.7}$$

Here we used matrix notation for the differential ∂.

Proof. Since $\partial^2 = 0$ in $\mathcal{C}$, we obtain $\phi\alpha + \delta\beta = 0$ and $\mu\phi + \nu\gamma = 0$. By doing the following change of basis $A = \begin{pmatrix} 1 & \phi^{-1}\delta \\ 0 & 1 \end{pmatrix}$ on $\begin{bmatrix} b_1 \\ D \end{bmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ -\gamma\phi^{-1} & 1 \end{pmatrix}$ on $\begin{bmatrix} b_2 \\ E \end{bmatrix}$ we see that the complex segments 3.5 and 3.6 are isomorphic. Then, we remove the short complex $0 \to \begin{bmatrix} b_1 \end{bmatrix} \xrightarrow{\phi} \begin{bmatrix} b_2 \end{bmatrix} \to 0$ which is acyclic. $\square$

Now, we are ready to prove Theorem 3.2.7.

Proof of Theorem 3.2.7. Like for the proof of Theorem 3.2.6, we make an induction on the number of edges belonging to the matching defining the combinatorial Morse vector field. Let X be a simplicial complex, K be the set of its cells. The property that V is a combinatorial Morse vector field implies that there is a path of k gaussian eliminations corresponding to the Morse matching such that the incidence number between the corresponding matched cells is unchanged.

Initialization: matching with no edge. In this case, there is nothing to prove since the combinatorial Thom-Smale complex is exactly the simplicial chain complex.

Heredity: suppose the property is true for every matching with at most k edges defining a combinatorial Morse vector field. Let V be a combinatorial Morse vector field whose underlying matching consists of $k+1$ edges. Let $\sigma^{(n)} < \tau^{(n+1)}$ be an element of this matching and $\overline{V}$ be the combinatorial Morse vector field equal to V with the matching $\sigma < \tau$ removed (it is actually a combinatorial Morse vector field, see Remark 3.2.4). So, $C_i^V = C_i^{\overline{V}}$ for all $i \neq n, n+1$ and $\partial^V = \partial^{\overline{V}}$ when restricted to C_i^V for all $i \notin \{n, n+1, n+2\}$. Moreover, we have the following equalities: $(\partial^{\overline{V}})^{|C_{n+1}^V} = (\partial^V)^{|C_{n+1}^V}$ and $\partial^{\overline{V}}_{|C_n^V} = \partial^V_{|C_n^V}$. By induction hypothesis, the combinatorial Thom-Smale complex $(C_*^{\overline{V}}, \partial^{\overline{V}})$ is a chain complex homotopy equivalent to the simplicial chain complex of X. Thus, the combinatorial Thom-Smale complex associated with $\overline{V}$ is equal to the one of V except on the following chain segment (where $\varepsilon = (\partial^{\overline{V}})^{|C_n^V}_{|C_{n+1}^V}$):

$$\ldots \to [C_{n+2}^V] \xrightarrow{\begin{pmatrix} \alpha \\ \partial^V \end{pmatrix}} \begin{bmatrix} \tau \\ C_{n+1}^V \end{bmatrix} \xrightarrow{\begin{pmatrix} < \partial\tau, \sigma > & \delta \\ \gamma & \varepsilon \end{pmatrix}} \begin{bmatrix} \sigma \\ C_n^V \end{bmatrix} \xrightarrow{\begin{pmatrix} \mu & \partial^V \end{pmatrix}} [C_{n-1}^V] \to \ldots \tag{3.8}$$

Since X is a simplicial complex we have $< \partial\tau, \sigma > \in \{\pm 1\}$. Applying lemma 3.2.8, we obtain the following new combinatorial chain complex which is homotopic to the combinatorial Thom-Smale complex of $\overline{V}$

$$\ldots \to [C_{n+2}^V] \xrightarrow{\left(\partial^V\right)} [C_{n+1}^V] \xrightarrow{\left(\alpha\right)} [C_n^V] \xrightarrow{\left(\partial^V\right)} [C_{n-1}^V] \to \ldots \tag{3.9}$$

where $\alpha = (\partial^{\overline{V}})^{|C_n^V}_{|C_{n+1}^V} - \gamma < \partial\tau, \sigma > \delta$. Thus, the only thing to prove is that $\alpha = \partial^V$ over C_{n+1}^V. To do this, we investigate ∂^V. There are two types of contributions. First type correspond to V-paths which do not go through σ, and they are counted in $\partial^{\overline{V}}_{|C_{n+1}^V}$. Second type are V-paths which go through σ. They begin at an hyperface of a critical cell τ' and go through σ: this is the contribution of δ. Then, they jump to τ: this is the contribution of $< \partial\tau, \sigma >$. Finally they begin at an hyperface of τ and go to a critical cell in C_n^V: this is the contribution of γ. It remains to check that the sign is correct, but this is exactly the same as in the first proof of Theorem 3.2.6. $\square$

Corollary 3.2.9. *Let X be a finite simplicial complex, $\mathcal{C} = (C_*, \partial)$ be the corresponding simplicial chain complex and $\mathcal{M}$ be a matching $(\sigma_i < \tau_i)_{i \in I}$ on its Hasse diagram defining a combinatorial vector field V. Then the following properties are equivalent:*

1. *$\mathcal{M}$ is a Morse matching,*
2. *for any sequence $(\sigma_{i_1} < \tau_{i_1}), (\sigma_{i_2} < \tau_{i_2}), \ldots, (\sigma_{i_{|I|}} < \tau_{i_{|I|}})$ such that $i_j \neq i_k$ if $j \neq k$, Gaussian eliminations can be performed in this order.*

In particular, any sequence of Gaussian eliminations corresponding to $\mathcal{M}$ lead to the same chain complex which is the combinatorial Thom-Smale complex of V.

Proof.

1. $\Rightarrow$ 2. This is an immediate consequence of the proof of Theorem 3.2.7 and the fact that it leads to the Thom-Smale complex of V.

2. $\Rightarrow$ 1. It is enough to show that there is no non-stationary closed path under the hypothesis. Suppose there is a closed V-path $\gamma : \sigma_1, \ldots, \sigma_r, \sigma_1$ and consider any sequence of Gaussian elimination which coincides with

$(\sigma_j < V(\sigma_j))$ until step r. In particular, $r \geq 3$ since X is a simplicial complex. Let V be the corresponding combinatorial vector field. Let $\gamma' : \sigma_1, \ldots, \sigma_r$ be the V-path with length decrease by one. Then

$$< \partial^{r-1} V(\sigma_r), \sigma_r >=< \partial V(\sigma_r), \sigma_r > +m(\gamma')$$

Since $m(\gamma') = \pm 1$, $< \partial^{r-1} V(\sigma_r), \sigma_r >$ is not invertible over $\mathbb{Z}$ and the Gaussian elimination cannot be performed (see lemma 3.2.8). This is a contradiction.

□

Remark 3.2.10. Here is an example of a sequence of Gaussian elimination which can be performed but which does not define a combinatorial Morse vector field.

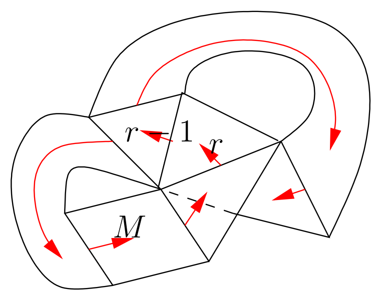

Figure 3.8: **Example of a Gaussian elimination which does not define a combinatorial Morse vector field.** The block M means that there is a twisted ribbon.

Consider the case illustrated by Fig.3.8. The two numbers appearing are the index of the Gaussian elimination corresponding to the matching. When there is no number, it means that Gaussian elimination have been performed previously. Then, until the $r-1$-th Gaussian elimination, we have defined a combinatorial Morse vector field. Since on one side there is a Moebius ribbon and on the other side just an annular the contribution of the right and the left V-paths cancel and we can again perform a Gaussian elimination. At step r, this is not any more a combinatorial Morse vector field.

3.3 Relation between smooth and discrete Morse theories

In this section, we investigate the link between smooth and discrete Morse theories. We first recall briefly the main ingredients of smooth Morse theory. In particular, we describe the Thom-Smale complex and prove the following:

Theorem 3.3.1 (Combinatorial realization)**.** *Let M be a smooth closed Riemannian manifold and $f : M \to \mathbb{R}$ be a generic Morse function. Suppose that every stable manifold has been given an orientation so that the smooth Thom-Smale complex is defined. Then, there exists a C^1-triangulation T of M and a combinatorial Morse vector field V on it which realize the smooth Thom-Smale complex (after a choice of orientation of each cells of T) in the following sense:*

1. *there is a bijection between the set of critical cells and the set of critical points,*

2. *for each pair of critical cells σ_p and σ_q such that $dim(\sigma_p) = dim(\sigma_q) + 1$, V-paths from hyperfaces of σ_p to σ_q are in bijection with gradient lines of v up to renormalization connecting q to p,*

3. *this bijection induces an isomorphism between the smooth and the combinatorial Thom-Smale complexes.*

Throughout this section, we follow conventions of Milnor ([Mil63], [Mil65]).

3.3.1 Smooth Morse theory

Let M be a smooth closed riemannian manifold of dimension n. Given a smooth function $f : M \to \mathbb{R}$, a point $p \in M$ is said **critical** if $Df(p) = 0$. Let $Crit(f)$ be the set of critical points. At a critical point p, we consider the bilinear form $D^2f(p)$. The number of negative eighenvalues of $D^2f(p)$ is called the **index** of p (denoted $ind(p)$). We denote $Crit_k(f)$ the set of critical points of index k.

Definition 3.3.2. A smooth map $f : M \to \mathbb{R}$ is called a Morse function if at each critical point p of f, $D^2f(p)$ is non-degenerate.

More generally, a Morse function on a (smooth) cobordism $(M; M_0, M_1)$ is a smooth map $f : M \to [a, b]$ such that

1. $f^{-1}(a) = M_0$, $f^{-1}(b) = M_1$,

2. all critical points of f are interior (lie in $M - (M_0 \cup M_1)$) and are non-degenerate.

For technical reasons, we must consider the following object:

Definition 3.3.3. Let f be a Morse function on a cobordism $(M^n; M_0, M_1)$. A vector field v on M^n is a gradient-like vector field for f if

1. $v(f) > 0$ throughout the complement of the set of critical points of f,

2. given any critical point p of f there is a Morse chart in a neighbourhood U of p so that
$$f(x) = f(p) - \sum_{i=1}^{k} x_i^2 + \sum_{i=k+1}^{n} x_i^2$$
and v has coordinates $v(x) = (-x_1, \dots, -x_k, x_{k+1}, \dots, x_n)$.

Given any Morse function, there always exists a gradient-like vector field (see [Mil65]). In the following, we shall abreviate "*gradient like vector field*" by "***gradient***". Thus, when needed, we will assume that we have chosen one.

Given any $x_0 \in M$, we consider the following Cauchy problem
$$\begin{cases} \gamma'(t) & = & v(\gamma(t)) \\ \gamma(0) & = & x_0 \end{cases}$$
and call integral curve (denoted γ_{x_0}) the solution of this Cauchy problem. The **stable manifold** of a critical point p is by definition the set $\{x \in M | \lim_{t \to +\infty} \gamma_x(t) = p\}$. We denote it $W^s(p, v)$. The **unstable manifold** of a critical point p is by definition the set $\{x \in M | \lim_{t \to -\infty} \gamma_x(t) = p\}$. When stable and unstable manifolds are transverse (this is called Morse-Smale condition), we called v a **Morse–Smale gradient**: such gradient always exists in a neighbourhood of a gradient (see e.g. [Paj95]). We shall call a Morse function f **generic** if we have chosen for f a Morse–Smale gradient.

To define the smooth Thom-Smale complex we need the following data:

- a generic Morse function f,
- an orientation of each stable manifold.

Under these conditions, the number of integral curves of v up to renormalization (that is $\gamma_x \sim \gamma_y$ iff there exists $t \in \mathbb{R}$ such that $\gamma_x(t) = y$) connecting two critical points of consecutive index is finite. Moreover, when we consider an integral curve from q to p where $ind(p) = ind(q) + 1$, it carries a coorientation induced by the orientation of the stable manifold and the orientation of the integral curve line. One can move this coorientation from p to q along the integral curve and compare it with the orientation of the stable manifold of q. This gives the sign which is carried by the integral curve connecting q to p.

The **Thom-Smale complex** (C^f_*, ∂^f) is defined as:

1. $C^f_k = \bigoplus_{p \in Crit_k(f)} \mathbb{Z}.p$,
2. if $p \in Crit_k(f)$ then $\partial p = \sum_{q \in Crit_{k-1}(f)} n(p, q).q$ where $n(p, q)$ is the algebraic number of integral curves up to renormalization connecting q to p.

Theorem 3.3.4. *The homology of the Thom-Smale complex is equal to the singular homology of M.*

The proof of this theorem can be extracted from [Mil65].

3.3.2 Elementary cobordisms

In this subsection, we will prove that we can realize combinatorially the smooth Thom-Smale complex of any elementary cobordisms. Thus, by cutting the manifold M into elementary cobordism we will obtain the first part of Theorem 3.3.1: there exists a bijection between the set of critical cells and the set of critical points.

We will only consider C^1-triangulation of manifolds for technical reasons (see [Whi40]). So, whenever we use the word triangulation it means C^1-triangulation. A **triangulation of a $n+1$-cobordism** $(M^{n+1}; M_0, M_1)$ is a triplet $(T; T_0, T_1)$ such that T is a C^1-triangulation of M, T_0 (resp. T_1) is a subcomplex of T which is a C^1-triangulation of M_0 (resp. M_1).

A combinatorial Morse vector field V on a triangulated $n+1$-cobordism $(T; T_0, T_1)$ is a combinatorial Morse vector field on T such that no cells of T_1 is critical and every cell of T_0 is critical.

Definition 3.3.5. Let V be a combinatorial Morse vector field on a triangulated $n+1$-cobordism $(T; T_0, T_1)$. V satisfies the ancestor's property if given any n-cell $\sigma_0 \in T_0$, there exists an n-cell $\sigma_1 \in T_1$ and a V-path starting at σ_1 and ending at σ_0.

Remark. There is a key difference between gradient lines of a gradient v and V-paths. Given a point $x \in M$, there is only one solution to the Cauchy problem. Moreover, the past and the future of a point pushed along the flow is uniquely determined. *A contrario* given a cell σ, there are (in general) many V-paths starting at σ. The ancestor's property caracterises $n+1$-cobordism equipped with a combinatorial Morse vector field which knows its history in maximal dimension $n, n+1$.

To prove that elementary cobordism can be realized , we need a combinatorial description of being a deformation retract. Let X be a simplicial complex and σ be an hyperface of τ which is free (that is σ is a face of no other cell). In this case, we say that X collapses to $X - (\sigma \cup \tau)$ by an **elementary collapsing** and write $X \searrow X - (\sigma \cup \tau)$. A collapsing is a finite sequence of such elementary collapsings. In particular, a collapsing defines a matching on the Hasse diagram of the simplicial complex. Moreover, one can prove that $X - (\sigma \cup \tau)$ is a deformation retract of X.

Proposition 3.3.6. *Let X be a simplicial complex and X_0 be a subcomplex. Suppose $X \searrow X_0$. Then the matching given by this collapsing defines a combinatorial Morse vector field whose set of critical cells is the set of cells of X_0.*

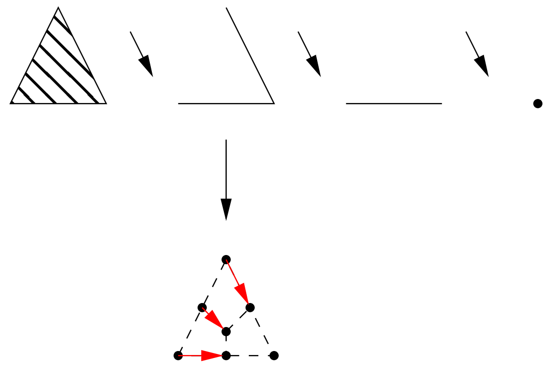

Figure 3.9: **Examples of collapsings and the combinatorial Morse vector induced.**

Proof. The only thing to check is that there is no non-stationary closed path. Since elementary collapsings are performed by choosing a free hyperface of a cell, there is no non-stationary closed path. $\square$

Let $\Delta^m = (a_0, \ldots, a_m)$ be the standard simplex of dimension m. The **cartesian product** $X = \Delta^m \times \Delta^n$ is the cellular complex whose set of cells is $\{\mu \times \nu\}$ where μ (resp. ν) is a cell of Δ^m (resp. Δ^n) (see [Whi57]).

Proposition 3.3.7 (Prop. 2.9 [RS72]). *The cartesian product $\Delta^m \times \Delta^n$ has a simplicial subdivision without any new vertex. More generally, the cartesian product of two simplicial complexes has a simplicial subdivision without any new vertex.*

Lemma 3.3.8. *Let $X_1 = \Delta^k$ be the standard simplicial complex of dimension k and X_0 be a simplicial subdivision of X_1. Consider the CW-complex which is equal to the cartesian product $\Delta^k \times \Delta^1$ and where we subdivide $\Delta^k \times \{0\}$ so that it is equal to X_0. Then, there exists a simplicial subdivision X of this CW-complex such that $X_{|\Delta \times \{i\}} = X_i$ for $i \in \{0, 1\}$.*

Moreover for $i \in \{0, 1\}$ there exists a collapsing $X \searrow X_i$ and the combinatorial Morse vector field associated V_i satifies the ancestor's property on $(X; X_{\overline{i}}, X_{\overline{i+1}})$ ($\overline{j}$ is the class in $\mathbb{Z}/2\mathbb{Z}$).

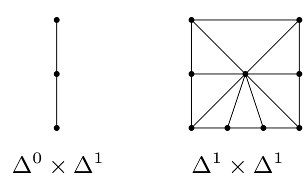

Figure 3.10: **Examples of simplicial subdivision given by lemma 3.3.8.**

Proof. The simplicial subdivision and the collapsing is constructed by induction on k. If $k = 0$, choose a new vertex in the interior of the simplex $\Delta^0 \times \Delta^1$ and the elementary collapsing $\Delta^1 \searrow \{i\}$ gives the two collapsing $\Delta^0 \times \Delta^1 \searrow \Delta^0 \times \{i\}$ for $i \in \{0,1\}$ (see Fig.3.10 left). In particular, the corresponding combinatorial Morse vector field satifies the ancestor's property.

Suppose the lemma is true until rank $k-1$. At rank k, let Y be the corresponding CW-complex and x be a point in the interior of the cell of dimension $k+1$. By induction hypothesis $Y_{|\partial\Delta^k \times \Delta^1}$ admits a simplicial subdivision. Therefore, $Y_{|\partial(\Delta^k \times \Delta^1)}$ admits a simplicial subdivision denoted Z (just add the simplexes $\Delta^k \times \{i\}$ which are equal to X_i for $i \in \{0,1\}$). The simplicial subdivision X is given by making the join of the simplicial subdivision of the boundary over $\{x\}$: $X = Z * \{x\}$.

Now, the collapsing $X \searrow X_0$ is performed in three steps.

Step 1. The cell $\sigma \in X_1$ of dimension k is the free hyperface of the cell $\sigma * \{x\}$. We do the following elementary collapsing:

$$X \searrow X - (\sigma \cup \sigma * \{x\}) \tag{3.10}$$

Step 2. By induction hypothesis, $X_{|\partial\Delta^k \times \Delta^1} \searrow X_{|\partial\Delta^k \times \{0\}}$. Performing the join over x induces the following collapsing:

$$X_{|\partial\Delta^k \times \Delta^1} * \{x\} \searrow X_{|\partial\Delta^k \times \{0\}} * \{x\} \tag{3.11}$$

Step 3. It remains to collapse $X_0 * \{x\}$ on X_0. Let y be a vertex in X_0 which is a vertex of the original simplex X_1. Since X_0 is a simplicial subdivision of Δ^k, there exists a collapsing $X_0 \searrow \{y\}$. This collapsing gives the following collapsing (see Fig.3.11):

$$X_0 * \{x\} \searrow X_0 \cup (\{y\} \times \{0\}) * \{x\} \searrow X_0 \tag{3.12}$$

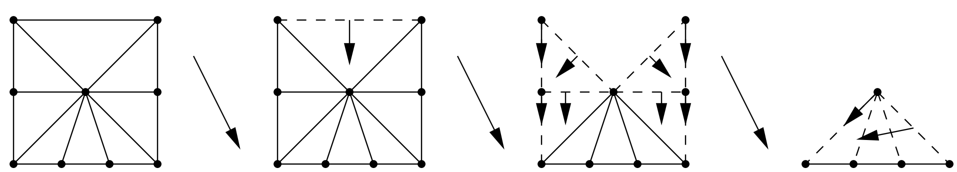

Figure 3.11: **The collapsing $X \searrow X_0$.** The first collapsing is collapsing (3.10). The second is collapsing (3.11). The last collapsing is collapsing (3.12).

Combining collapsings (3.10), (3.11) and (3.12) gives $X \searrow X_0$. The corresponding combinatorial Morse vector field satisfies the ancestor's property by construction.

The collapsing $X \searrow X_1$ is constructed in the same way and conclusions of lemma follows. □

Remark 3.3.9. The proof of lemma 3.3.8 is by induction. Let $\delta^{(j)}$ be the j-th skeleton of Δ^k. Denote by $X^{(j)}$ (resp. $X_i^{(j)}$) the simplicial complex $X_{|\delta^{(j)}\times\Delta^1}$ (resp. $(X_i)_{|\delta^{(j)}\times\Delta^1}$). For $i \in \{0,1\}$, the collapsing $X \searrow X_i$ can be restricted to $X^{(j)} \searrow X_i^{(j)}$ for any $0 \leq j \leq k$ and the induced combinatorial Morse vector field satisfies the ancestor's property.

The next two lemmas are technical lemmas. The first one is the basic tool to glue together triangulated cobordisms. The second one will be useful to construct a combinatorial realization of a cobordism with exactly one critical point and is a generalization of lemma 3.3.8.

Lemma 3.3.10. *Let (T_i^M, T_i^N) be two C^1-triangulations of the pair (M,N) where N^k is a submanifold (possibly with boundary) of M^n ($k \leq n$). Then, there exists a C^1-triangulation T of $(M\times[0,1], N\times[0,1])$ such that*

$$(T_{|M\times\{i\}}, T_{|N\times\{i\}}) = (T_i^M, T_i^N)$$

for $i \in \{0,1\}$ and 2 collapsings

$$T \searrow T_0^M \cup T_{|N\times[0,1]} \tag{3.13}$$

$$T_{|N\times[0,1]} \searrow T_0^N \tag{3.14}$$

Moreover, the induced combinatorial Morse vector fields V satifies the ancestor's property on each corresponding cobordisms.

Proof. First, suppose $N = \emptyset$. Both triangulations T_0 and T_1 are C^1-triangulation of the same manifold therefore they have a common simplicial subdivision $T^{1/2}$ (see [Whi40], they are P.L. isomorphic) (this is where we use the fact that triangulations are C^1-triangulations). Subdivide $\Delta^1 = [0,1]$ in two standard simplexes $[0,1/2]$ and $[1/2,1]$. Lemma 3.3.8 gives a C^1-triangulation of $M \times [0,1/2]$ (resp. $M \times [1/2,1]$) denoted $T^{[0,1/2]}$ (resp. $T^{[1/2,1]}$). The union $T^{[0,1/2]} \cup T^{[1/2,1]}$ is a triangulation of $M\times[0,1]$ denoted T. By construction, $T_{|M\times\{i\}} = T_i^M$ for $i \in \{0,1\}$ and we have the two following collapsings:

$$T^{[1/2,1]} \searrow T^{1/2}$$

$$T^{[0,1/2]} \searrow T^0$$

Composing these two collapsings give the desired collapsing and lemma 3.3.8 give the ancestor's property.

In the case where the submanifold N is non-empty, the construction above gives a triangulation T of the pair $(M\times[0,1], N\times[0,1])$ and we have $(T_{|M\times\{i\}}, T_{|N\times\{i\}}) = (T_i^M, T_i^N)$ for $i \in \{0,1\}$. The collapsing $T \searrow T_0$ can be restricted to $T_{|N\times[0,1]}$. We remove from the matching edges corresponding to $T_{N\times[0,1]} \searrow T_{N\times\{0\}}$ to obtain the desired collapsing. Again lemma 3.3.8 give the ancestor's property. $\square$

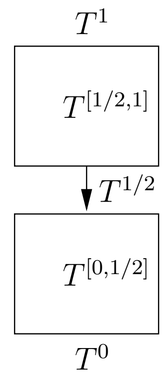

Figure 3.12: **Gluing cobordisms along a common subdivision**.

Lemma 3.3.11. *Let (m,n) be a pair of positive integers. Let $\Delta^n = (a_0, \dots, a_n)$ be the standard simplex of dimension n and $\delta^{n-1} = (\widehat{a}_0, \dots, a_n)$ be the hyperface which do not contain a_0. In particular $\Delta^n = \{a_0\} * \delta^{n-1}$. Then there exists a simplicial subdivision X of the cartesian product $\Delta^m \times \Delta^n$ such that*

- *$X_{|\Delta^m \times \delta^{n-1}}$ is a simplicial subdivision without any new vertex given by lemma 3.3.7,*
- *$X_{|\Delta^m \times \{a_0\}} = \Delta^m$,*
- *$X \searrow X_{|(\partial\Delta^m \times \Delta^n) \cup (\Delta^m \times \{a_0\})}$.*

Moreover, for each simplex $\Delta^1_i = (a_0, a_i)$ $(i \neq 0)$,

- *$X_{|\Delta^m \times \Delta^1_i}$ coincides with the simplicial complex given by lemma 3.3.8,*
- *the collapsing $X \searrow X_{|(\partial\Delta^m \times \Delta^n) \cup (\Delta^m \times \{a_0\})}$ restricted to $X_{|\Delta^m \times \Delta^1_i}$ coincides with the collapsing of lemma 3.3.8,*
- *the induced combinatorial Morse vector field satisfies the ancestor's property on $(X_{|\Delta^m \times \Delta^1_i}; X_{|\Delta^m \times \{a_0\}}, X_{|\Delta^m \times \{a_i\}})$.*

Proof. The proof is by induction on $k = m+n > 0$. At rank $k = 1$ there are two cases. The case $m = 0$ and $n = 1$ is trivial: there is nothing to prove. The case $m = 1$ and $n = 0$ is given by lemma 3.3.8.

Suppose the lemma is true until rank $k-1$. Let $(m,n) \in \mathbb{N}^2$ be such that $m + n = k$. We will first subdivide the boundary of $\Delta^m \times \Delta^n$. Since

$$\begin{aligned}
\partial(\Delta^m \times \Delta^n) &= (\partial\Delta^m \times \Delta^n) \cup (\Delta^m \times \partial\Delta^n) \\
&= (\partial\Delta^m \times \Delta^n) \cup (\Delta^m \times (\{a_0\} * \partial\delta^{n-1})) \cup (\Delta^m \times \delta^{n-1})
\end{aligned}$$

we define for each cellular complex above a simplicial subdivision.

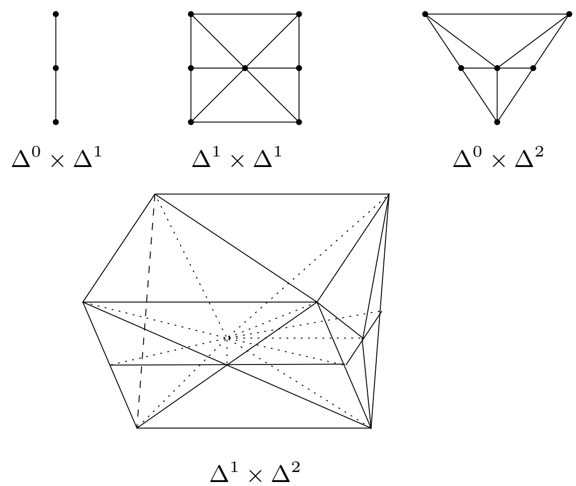

Figure 3.13: **Examples of simplicial subdivisions.** The four picture are the simplicial subdivision given by lemma 3.3.11. In the fourth picture, we only have drawn the simplicial subdivision over three faces (the top, the front and the right ones) of $\Delta^1 \times \Delta^2$.

- The simplicial subdivision of $\Delta^m \times \delta^{n-1}$ is given by Proposition 3.3.7: in particular, we do not create any new vertex.
- The induction hypothesis gives a simplicial subdivision of $(\partial\Delta^m \times (\{a_0\} * \delta^{n-1})) \cup (\Delta^m \times (\{a_0\} * \partial\delta^{n-1}))$.

Let x be a point which is in the interior of the $(m+n)$-cell of $\Delta^m \times \Delta^n$. The simplicial subdivision X of $\Delta^m \times \Delta^n$ is given by making the cone over $\{x\}$ of the simplicial subdivision of the boundary of $\Delta^m \times \Delta^n$.

By construction we have the following collapsing

$$X_{|\{x\}*(\Delta^m \times \delta^{n-1})} \searrow X_{|\{x\}*\partial(\Delta^m \times \delta^{n-1})} \tag{3.15}$$

which is realized by a downward induction on the dimension of cells of $\Delta^m \times (\delta^{n-1} - \partial\delta^{n-1})$: every cell $\sigma \in \Delta^m \times (\delta^{n-1} - \partial\delta^{n-1})$ is a free hyperface of $\{x\} * \sigma$.

The induction hypothesis says that there exits a simplicial subdivision Y of $\Delta^u \times \Delta^v$ such that $Y \searrow Y_{|(\partial\Delta^u \times \Delta^v) \cup (\Delta^u \times \{a_0\})}$ whenever $u + v < k$ (a_0 is the first vertex of Δ^v). In fact, we have also the following collapsing since the construction is made by induction:

$$Y \searrow Y_{|\Delta^u \times \{a_0\}}$$

Therefore, we have the following collapsings

$$X_{|\partial\Delta^m \times (\{a_0\} * \delta^{n-1})} \searrow X_{|\partial\Delta^m \times \{a_0\}} \tag{3.16}$$

$$X_{|\Delta^m \times \{a_0\} * \partial\delta^{n-1}} \searrow X_{|\Delta^m \times \{a_0\}} \tag{3.17}$$

Collapsing (3.15) followed by the cone over x of the collapsing (3.16) and the cone over x of the collapsing (3.17) give the following collapsing:

$$X \searrow X_{|(\partial\Delta^m \times \Delta^n)\cup(\{x\}*(\Delta^m\times\{a_0\}))}$$

Finally there exists a collapsing $\{x\}*(\Delta^m \times \{a_0\}) \searrow \Delta^m \times \{a_0\}$ (by choosing a vertex $y \in \Delta^m$ and considering the collapsing $\Delta^m \searrow \{y\}$) which gives the result.

In case $n = 1$, this construction is the same as the one of lemma 3.3.8. □

Theorem 3.3.12. *Let f be a generic Morse function on a cobordism $(M^n; M_0, M_1)$ with exactly one critical point p of index k. Then, there exists a C^1-triangulation of the cobordism $(T; T_0, T_1)$ such that*

1. *the stable manifold of p is a subcomplex of T denoted T^s_p and $T \searrow T^s_p \cup T_0$,*
2. *there is a cell σ_p of dimension $ind(p)$ such that $p \in \sigma_p \subset T^s_p$ and $T^s_p - \sigma_p \searrow (T^s_p \cap T_0)$*

In particular, the combinatorial Morse vector field given by these two collapsings has exactly one critical cell σ_p outside cells of T_0.

Proof. Suppose a Morse–Smale gradient v for f is fixed. Let $W^s(p, v)$ be the corresponding stable manifold of p. We follow the proof of Milnor which proves that $M_0 \cup W^s(p, v)$ is a deformation retract of M (see the proof of Theorem 3.14 [Mil63]). Let C be a (small enough) tubular neighbourhood of $W^s(p, v)$. The original proof consits of two steps. First, $M_0 \cup C$ is a deformation retract of M: this is done by pushing along the gradient lines of v. Then, $M_0 \cup W^s(p, v)$ is a deformation retract of $M_0 \cup C$. We prove the theorem in two steps.

First step: construction of a good triangulation of C.
The tubular neighbourhood C is diffeomorphic to $D^k \times D^{n-k}$ (for $i \in \mathbb{N}^*$, D^i is the unit disk in $\mathbb{R}^i$). Thanks to this diffeomorphism, the stable manifold is identified with $D^k \times \{0\}$ and the adherence of the unstable manifold is identified with $\{0\} \times D^{n-k}$. Triangulate the stable manifold by the standard simplex Δ^k and denote σ_p its interior (so $T^s_p = \overline{\sigma}_p$).

Remark. In fact, this is not a C^1 triangulation of the disk D^{n-k} but by performing a small isotopy of D^{n-k} it becomes a C^1-triangulation. Therefore, we suppose that first we push a little M_0 (denote this level M'_0) inside the cobordism so that the stable manifold can be C^1-triangulated by a the standard simplex. Then, we can make the construction described below on the cobordism $(M; M'_0, M_1)$. Then, we push the C^1-triangulation constructed on M'_0 on M_0 along the gradient lines and lemma 3.3.10 give the desired C^1-triangulation on $(M; M_0, M_1)$.

We triangulate D^{n-k} by choosing an arbitrary triangulation of $\partial D^{n-k} = S^{n-k-1}$ and considering D^{n-k} as the cone over its center $\{0\}$: this gives a triangulation of D^{n-k}. The triangulation of $\overline{\sigma}_p \times D^{n-k}$ is the following one: choose a simplicial subdivision of $\overline{\sigma}_p \times \partial D^{n-k}$ without any new vertex given by proposition 3.3.7. Then, triangulate the cartesian product $\overline{\sigma}_p \times D^{n-k}$ with the triangulation of $\overline{\sigma}_p \times \partial D^{n-k}$ already fixed thanks to lemma 3.3.11:

- for each simplex $\nu \in \partial D^{n-k}$, the lemma constructs a triangulation of $\overline{\sigma}_p \times (\{0\} * \nu)$,
- for each pair of simplexes $(\nu_0, \nu_1) \in (\partial D^{n-k})^2$, the simplicial subidivisions of $\overline{\sigma}_p \times (\{0\} * \nu_i)$ coincides over $\overline{\sigma}_p \times (\{0\} * (\nu_0 \cap \nu_1))$.

Let T^C be the triangulation of $\overline{\sigma}_p \times D^{n-k}$ constructed above. By construction, we have the following collapsing

$$T^C \searrow T_p^s \cup T^C_{|\partial\overline{\sigma}_p \times D^{n-k}} \tag{3.18}$$

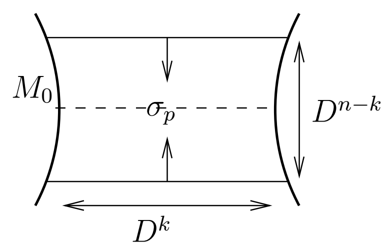

Figure 3.14: **C^1-triangulation of the tubular neighbourhood of C.**

Second step: combinatorial realization of the first retraction.
Let T_0 be a triangulation of M_0 which coincides over $M_0 \cap C$ with the triangulation above (this is an extension of the triangulation to M_0 and it exists, see [Mun66, Chap. II]). Consider the following submanifolds with boundary: $\partial C_- = M_0 \cap C$, $M_0^{\overline{C}} = M_0 - Int(\partial C_-)$ and $\partial C_+ = \partial C - Int(\partial C_-)$ (see Fig.3.15). Let V be $\partial C_- \cap \partial C_+$: it is diffeomorphic to $\partial D^k \times \partial D^{n-k} = S^{k-1} \times S^{n-k-1}$ a manifold of dimension $n-2$.

The manifold $(\partial C_+, V)$ is a manifold with boundary which is triangulated. The gradient lines of v starting at any point of this manifold are transverse to it: we push along the gradient lines of v the triangulation until it meets M_1. It gives a triangulation of $(M_1^{\partial C_+}, M_1^V)$ which is a submanifold of M_1 with boundary. This triangulation is C^1 since pushing along the flow in this case is a diffeomorphism. Then, we get a product cobordism (with boundary) with triangulation of the top and the bottom already fixed: lemma 3.3.10 gives a triangulation of this cobordism with the desired collapsing.

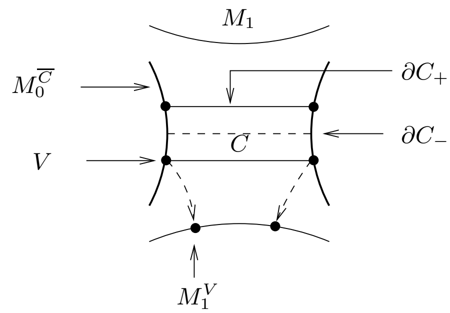

Figure 3.15: C^1**-triangulation of the intermediary cobordism.**

The same construction holds for $(M_0^{\overline{C}}, V)$ (we suppose that the triangulation of $V \times [0,1]$ is the same as the one given above). Let T be the corresponding triangulation of M. Then, we have the following collapsing

$$T \searrow T_0 \cup T^C \tag{3.19}$$

Conclusion. The composition of collapsings (3.19) and (3.18) give

$$T \searrow T_0 \cup T_p^s$$

Since $T_p^s = \overline{\sigma}_p$ we get the following collapsing: $T_p^s - \sigma_p \searrow \partial T_p^s$. Thus a combinatorial Morse vector field which satifies the conclusion of the theorem has been constructed. □

Corollary 3.3.13. *Let f be a generic Morse function on a Riemannian closed manifold M. Then, there exists T a C^1-triangulation of M and a combinatorial Morse vector field V defined on T such that for every $k \in \mathbb{N}$ the set of critical points of index k is in bijection with the set of critical cells of dimension k.*

Proof. Since the Morse function f is generic, we have that for any critical points $p \neq q$, $f(p) \neq f(q)$. Let $a_1 < a_2 < \ldots < a_l$ be the ordered set of critical values of f. For each $k \in \{1, \ldots, l\}$, let $\varepsilon_k > 0$ be small enough so that the cobordism

$$(M^{a_k}; M_-^{a_k}, M_+^{a_k}) = \big(f^{-1}([a_k - \varepsilon_k, a_k + \varepsilon_k]); f^{-1}(a_k - \varepsilon_k), f^{-1}(a_k + \varepsilon_k)\big)$$

is a cobordism with exactly one critical point. Define for $k \in \{1, \ldots, l-1\}$ the product cobordisms

$$(M^{b_k}; M_+^{a_{k-1}}, M_-^{a_k}) = \big(f^{-1}([a_{k-1}+\varepsilon_{k-1}, a_k-\varepsilon_k]); f^{-1}(a_{k-1}+\varepsilon_{k-1}), f^{-1}(a_k-\varepsilon_k)\big)$$

The manifold M is equal to:

$$M^{a_1} \cup M^{b_1} \cup \ldots \cup M^{b_{l-1}} \cup M^{a_l}$$

Theorem 3.3.12 gives for $k = 1, \dots, l$ a combinatorial realization of the cobordism $(M^{a_k}; M_-^{a_k}, M_+^{a_k})$. Lemma 3.3.10 gives a combinatorial realization of each cobordism $(M^{b_k}; M_+^{a_{k-1}}, M_-^{a_k})$ for $k = 1, \dots, l-1$ (with the convention that $M^{a_0} = \emptyset$). Then, we construct a C^1-triangulation of M and define on it a combinatorial vector field. It is in fact a combinatorial Morse vector field since along V-paths we only can go down and the conclusion of the corollary follows. □

3.3.3 Proof of Theorem 3.3.1

Since f is generic, we use the rearrangement Theorem (Theorem 4.8 [Mil65]) to consider g a generic self-indexed Morse function such that

- the set of critical points of index k of g coincides with the one of f for every $k \in \mathbb{N}$,
- for each pair of critical points p and q of successive index, the set of integral curves up to renormalization connecting q to p for g is in bijection with the corresponding set for f (we suppose here that Morse–Smale gradients have been chosen for f and for g),
- this bijection induces an isomorphism between the Thom-Smale complexes of f and g (we suppose that orientations of stable manifolds have been chosen).

Thus, we suppose that $f : M^n \to \mathbb{R}$ is a generic self-indexed Morse function i.e. for every $k \in \mathbb{N}$, for every $p \in Crit_k(f)$, $f(p) = k$. In particular $f(M) = [0, n]$. We suppose whenever we need it that a Morse–Smale gradient v for f is given.

One more time, we will cut M in cobordisms (almost) elementary and control combinatorially the behavior of V-paths. For $i \in \{0, \dots, n\}$ choose $0 < \varepsilon_i < 1/2$. For $i \in \{0, \dots, n\}$, let $(M^i; M_{-,}^i, M_+^i)$ be the cobordism

$$(f^{-1}([i - \varepsilon_i, i + \varepsilon_i]); f^{-1}(i - \varepsilon_i), f^{-1}(i + \varepsilon_i))$$

Similarly, define $(M^{i,i+1}; M_+^i, M_-^{i+1})$ the product cobordism equal to

$$(f^{-1}([i + \varepsilon_i, i + 1 - \varepsilon_{i+1}]); M_+^i, M_-^{i+1})$$

Then

$$M = M^0 \cup M^{0,1} \cup M^1 \cup \ldots \cup M^{n-1,n} \cup M^n$$

For all $i \in \{0, \dots, n\}$, $(M^i; M_-^i, M_+^i)$ is a cobordism with $|Crit_i(f)|$ critical points of index i (maybe there is no critical point). The triangulation of M is constructed in the following way:

1. triangulation of cobordisms $(M^i; M_-^i, M_+^i)$ for all $i \in \{0, \dots, n\}$ given by Theorem 3.3.12,

2. triangulation of cobordisms $(M^{i,i+1}; M^i_+, M^{i+1}_-)$ for all $i \in \{0, \ldots, n\}$ given by lemma 3.3.10.

Remark. Theorem 3.3.12 is proved in the case where there is exactly one critical point. This proof extends directly to the case of k critical points of the same index under the condition that tubular neighbourhoods of stable manifolds are chosen to be disjoint one from each other.

Let p be a critical point of index k and $C(p)$ be a tubular neighbourhood (small enough) of the stable manifold of p in the corresponding cobordism. Denote $\partial C_-(p)$ (resp. $\partial C_+(p)$) the submanifold diffeomorphic to $\partial D^k \times D^{n-k}$ (resp. $D^k \times \partial D^{n-k}$) (see Fig.3.15). Denote σ_p the critical cell of dimension k corresponding to p (see Theorem 3.3.12).

Hypothesis on the triangulation of $\partial C_+(p)$.

1. stable manifolds of critical points of index $k+1$ intersect $\partial C_+(p)$ along a subcomplex of dimension k and intersect $\sigma_p \times \partial D^{n-k}$ along cells of dimension k of the type $\sigma \times \{a_i\}$ where a_i is a vertex of ∂D^{n-k},

2. each gradient line up to renormalization γ from p to $q \in Crit_{k+1}(f)$ intersects $\partial C_+(p)$ in the interior of a k-cell $\sigma_\gamma \in \sigma_p \times \partial D^{n-k}$,

3. given two distinct gradient lines up to renormalization γ and γ' from p to critical points of index $k+1$ then $\sigma_\gamma \neq \sigma_{\gamma'}$.

Remark. The first hypothesis is satisfied by choosing small enough ε_k and since stable and unstable manifolds are transverse. For such a small enough ε_k the last hypothesis will be satisfied. The second hypothesis is automatically satisfied if the first hypothesis is satisfied.

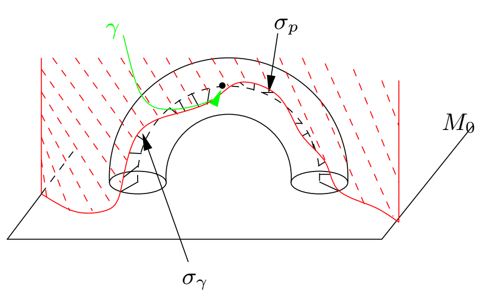

Figure 3.16: **Example of a triangulation which satisfies the hypothesis.**

In each triangulated cobordism $(M^k; M^k_-, M^k_+)$, stable manifolds of critical points of index k are subcomplexes. Following notations of Theorem 3.3.12, we have the following collapsing

$$T^s_p - \sigma_p \searrow \partial T^s_p$$

Using lemma 3.3.10, we obtain the following collapsing

$$M^{k-1,k} \searrow M_+^{k-1}$$

which can be restricted to the stable manifold of p since it is a submanifold of M_-^k. With respect to the stable manifold, the combinatorial Morse vector field satisfies the ancestor's property.

Let γ be a gradient line of v up to renormalization from $q \in Crit_{k-1}(f)$ to p. It intersects $\partial C_+(q)$ in a point which by hypothesis belongs to a cell $\sigma_q \times \{a_\gamma\}$. There is a $1-1$ correspondance between gradient lines up to renormalization from q to p (with $ind(p) = ind(q) + 1$) and V-paths from hyperfaces of σ_p to σ_q given by $\gamma \longleftrightarrow \sigma_\gamma$.

From σ_γ, there is a unique V-path ending at σ_q. Since V satisfies the ancestor's property in the stable manifold and σ_γ is a cell of dimension $k-1$ there is an ancestor of σ_γ which is an hyperface of σ_p. This gives a V-path between an hyperface of σ_p to σ_q which corresponds to γ.

We endow each critical cell with the orientation of the corresponding stable manifold and every other cell is endowed with an arbitrary orientation.

By construction, the multiplicity of V-path coincides with the sign of the corresponding gradient path and the theorem follows.

3.3.4 An example: the 2-dimensional torus.

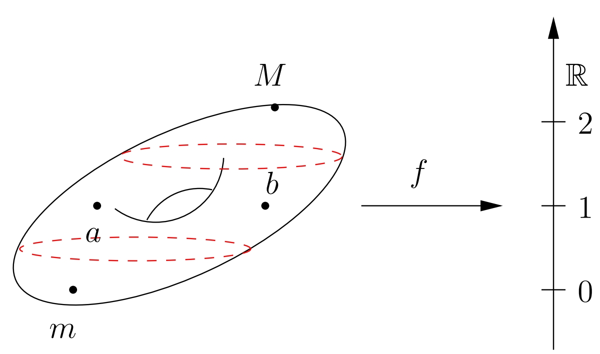

Figure 3.17: **The torus $\mathbb{T}^2$ and a generic Morse function on it.** Points M, a, b and m are the critical points of f with index 2, 1, 1 and 0 respectively. The torus is cut along two circles to obtain three cobordisms.

We describe an example of the triangulation and the combinatorial Morse vector field given by Theorem 3.3.1. First, we choose a generic self-indexed Morse function on the torus $\mathbb{T}^2$ with one maximum, one minimum and two index one critical points.

With the common identifications of the boundary of the square, we draw the torus as follows (we cut the torus along the 4 gradient lines coming from m and going to a or b).

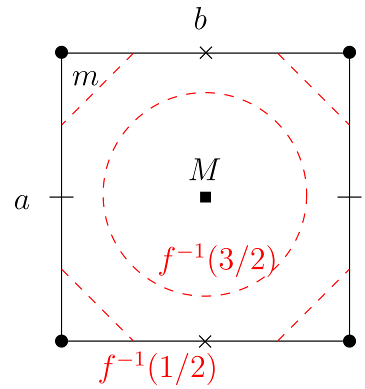

Figure 3.18: **The torus drawn in the plan.**

We apply each step of the proof of Theorem 3.3.1 to obtain Fig.3.19.

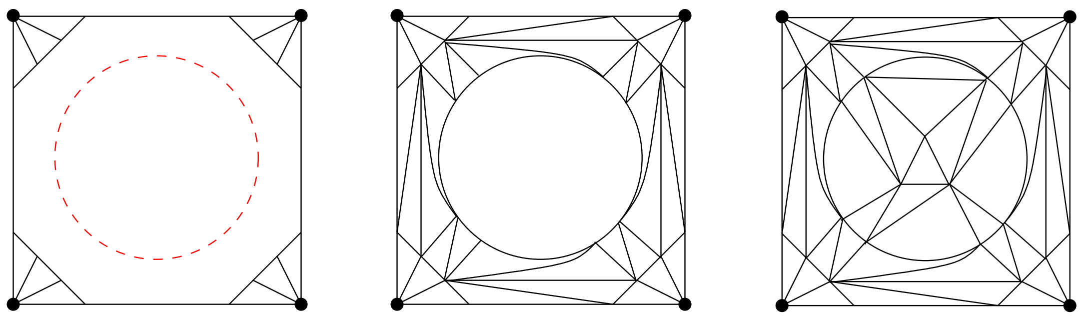

Figure 3.19: **The construction of the triangulation of $\mathbb{T}^2$.**

Then, the combinatorial Morse vector field is given by Fig.3.20.

3.4 Complete matchings and Euler structures

In this section, we use Theorem 3.3.1 to prove the following: given a closed oriented 3-manifold then any Euler structure can be actually realized by a complete matching on the Hasse diagram of a given triangulation. We first prove the existence of complete matchings and then we prove this result. Finally, we explain an algorithm to find complete matchings, which we compute using Regina [Bur05].

3.4.1 Complete matchings

Definition 3.4.1. A **complete matching** on a graph is a matching such that every vertex belongs to an edge of the matching.

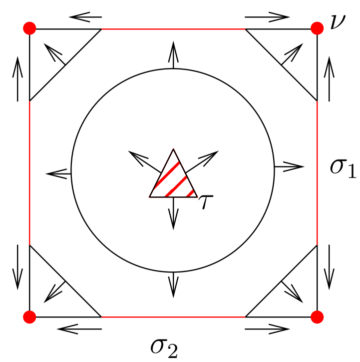

Figure 3.20: **The combinatorial Morse vector field.** Critical cells are $\nu^{(0)}$, $\sigma_1^{(1)}$, $\sigma_2^{(1)}$ and $\tau^{(2)}$. They correspond respectively to m, a, b and M.

As a corollary of theorem 3.3.1 we obtain:

Corollary 3.4.2. *Let M be a closed smooth manifold of dimension 3. Then there exits a C^1-triangulation of M such that a complete matching on its Hasse diagram exists.*

Proof. Since M is a closed smooth manifold of dimension 3 we have $\chi(M) = 0$ where χ denotes the Euler characteristic. Take a pointed Heegaard splitting of M $(\Sigma_g; \underline{\alpha} = (\alpha_1, \dots, \alpha_g), \underline{\beta} = (\beta_1, \dots, \beta_g); z)$ of genus g so that there is an n-uplets of intersection points x between the α's and the β's which defines a bijection between the sets $\underline{\alpha}$ and $\underline{\beta}$. It is always possible to find such a pointed Heegaard splitting after a finite number of isotopies of the α's and β's curves (see [GS99]). The Morse function f corresponding to this Heegaard splitting has one critical point of index 0 and 3 and g critical points of index 1 and 2. Denote the set of index 1 (resp. 2) critical points by $\{q_i\}_{i=1}^g$ (resp. $\{p_i\}_{i=1}^g$) where q_i (resp. p_i) corresponds to α_i (resp. β_i) for all $i \in \{1, \dots, g\}$. The n-uplet of intersection points $x = (x_{1,i_1}, \dots, x_{n,i_n})$ gives for each $j \in \{1, \dots, g\}$ a gradient line connecting q_j to p_{i_j}. The point z gives a gradient line connecting the index 0 critical point to the index 3 critical point.

Take a combinatorial realization (T, V) as given by Theorem 3.3.1 of (M, f). Then to each point x_{i,j_i} correspond now a V-path γ from an hyperface of the critical cell $\sigma_{p_{i_j}}$ to σ_{q_i}: we can change the matching along this path so that both $\tau_{p_{i_j}}$ and σ_{q_i} are no more critical cells. If $\gamma : \sigma_0, \dots, \sigma_r = \sigma_{q_i}$ then change the matching along γ this way:

- match σ_0 with τ,
- for every $i \in \{1, \dots, r\}$ match σ_i with $V(\sigma_{i-1})$.

Then suppose that z belongs to the interior of a 2-cell τ_z (if not, subdivide T). Denote by ς the critical cell of dimension 3 and by υ the critical cell of dimension 0. There is by construction a V-path $\gamma : \tau_0, \dots, \tau_r = \tau_z$ from an hyperface of ς to τ_z since z is in the stable manifold of the index 3 critical point. We modify the matching along γ this way:

- match τ_0 with ς,
- for every $i \in \{1, \dots, r\}$ match τ_i with $V(\tau_{i-1})$.

In fact, it is no more a matching since τ_z belongs to two edges of the matching.

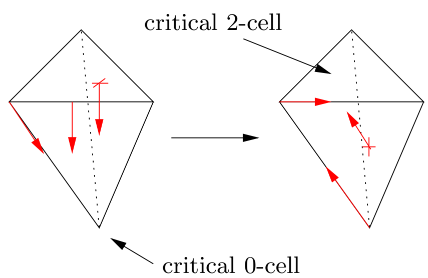

Figure 3.21: **Move of the critical cell.** In the original matching (on the left), the bottom vertex is critical. After the modification, the top face is critical.

Nevertheless, τ_z belongs to the unstable manifold of υ the critical cell of dimension 0. By the construction done in Theorem 3.3.1, the tubular neighbourhood of the critical point of index 0 is equal to $D^3 = \partial D^3 * \{0\}$. The triangulation of this tubular neighbourhood is given by making the cone over $\{0\} = \upsilon$ of a triangulation of ∂D^3. We modify the matching as illustrated in Fig.3.21 to move the critical cell until τ_z. This gives a complete matching over T. $\square$

3.4.2 Euler structures and homologous vector fields

Throughout this subsection we use conventions of Turaev [Tur89].

Combinatorial Euler structures

Complete matchings have an interpretation as Euler chains. First, we recall Euler structures as defined by Turaev [Tur89]. Let $(M, \partial M)$ be a smooth manifold of dimension n and T be a C^1-triangulation of M.

Suppose $\partial M = \partial_0 M \coprod \partial_1 M$ be such that $\chi(M, \partial_0 M) = 0$ and let T_i be equal to $T_{|\partial_i M}$ for $i \in \{0, 1\}$.

Denote K the set of cells of T and K_i the set of cells of T_i for $i \in \{0, 1\}$. For each cell $\sigma \in K$, let $sgn(\sigma)$ be equal to $(-1)^{dim(\sigma)}$ and pick a_σ a point in the interior of σ. An **Euler chain** in (T, T_0) is a one-dimensional singular chain ξ in T with the boundary of the form $\sum_{\sigma \in K - K_0} sgn(\sigma) a_\sigma$. Since $\chi(M, \partial_0 M) = 0$, the set of Euler chains is non-empty. Given two Euler chains ξ and η, the difference $\xi - \eta$ is a cycle. If $\xi - \eta = 0 \in H_1(M)$ then we say that ξ and η are **homologous**. A class of homologous Euler chains in (T, T_0) is called a **combinatorial Euler structure** on (T, T_0). Let $Eul(T, T_0)$ be the set of Euler structures on (T, T_0). If ξ is an Euler chain, denote by $[\xi]$ its class as a combinatorial Euler structure. Euler chains behave well with respect to the subdivision of a triangulation: this allows us to consider the set $Eul(M, \partial_0 M)$ of Euler structures on $(M, \partial_0 M)$. Taking ξ an element of this set means choosing a triangulation (T, T_0) of $(M, \partial_0 M)$ and considering an Euler chain on (T, T_0).

Remark 3.4.3. Let $\mathcal{C}$ be a complete matching on a C^1-triangulation (T, T_0) of $(M, \partial_0 M)$. Then it defines an Euler chain $[\xi_c] \in Eul(T, T_0)$: orient every edge of the matching from odd dimensional cells to even dimensional cells. Complete matchings are special Euler chains: we do not pass through a cell more than one time.

Homologous vector fields

By a vector field on $(M, \partial_0 M)$ we mean (except in clearly mentioned case) a non-singular continuous vector field of tangent vectors on M directed into M on $\partial_0 M$ and directed outwards on $\partial_1 M$. Since $\chi(M, \partial_0 M) = 0$, there exists such vector fields on $(M, \partial_0 M)$.

Vector fields u and v on $(M, \partial_0 M)$ are called **homologous** if for some closed ball $B \subset Int(M)$ the restriction of the fields u and v are homotopic in the class of non-singular vector fields on $M - Int(B)$ directed into M on $\partial_0 M$, outwards on $\partial_1 M$, and arbitrarily on ∂B. Denote by $vect(M, \partial_0 M)$ the set of homologous vector fields on $(M, \partial_0 M)$ and the class of a vector field u is denoted by $[u]$.

The canonical bijection

Turaev proved the following:

Theorem 3.4.4 (Turaev [Tur89]). *Let $(M, \partial_0 M)$ be a smooth pair such that $dim(M) \geq 2$. For each C^1-triangulation (T, T_0) of the pair $(M, \partial_0 M)$ there exists a bijection*

$$\rho : Eul(T, T_0) \to vect(M, \partial_0 M)$$

Remark. In fact, this bijection is an $H_1(T)$-isomorphism, but we'll make no use of it.

Let us describe the construction of Turaev in the case $\partial M = \emptyset$. Let T be a C^1-triangulation of M and T' be the first barycentric subdivision of T. We recall the definition of the vector field F_1 with singularities on M. For a simplex a of the triangulation T, let $\underline{a}$ denotes its barycenter. If $A =< \underline{a}_0, \underline{a}_1, \ldots, \underline{a}_p >$ is a simplex of the triangulation T', where $a_0 < a_1 <$

$\ldots < a_p$ are simplexes of T, then, at a point $x \in Int(A)$,

$$F_1(x) = \sum_{0 \le i < j \le p} \lambda_i(x)\lambda_j(x)(\underline{a}_j - x)$$

Here $\lambda_0, \lambda_1, \ldots, \lambda_p$ are barycentric coordinates in A and $\underline{a}_j - x$ is the image of the tangent vector $\underline{a}_j - x \in T_x A$ via the homeomorphism between T and M.

Remark. The vector field F_1 is smooth on each simplex but only continuous on M.

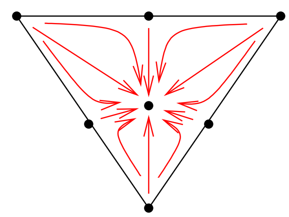

Figure 3.22: **The vector field F_1 on a triangle.** Every black point is a singular point of the vector field F_1.

Every barycenter of each cell of T is a singular point of F_1. Let $\mathcal{M}$ be a matching on the Hasse diagram of T: in particular, every edge of the matching connects two singular points. Turaev proved that the index of F_1 in a neighbourhood of every edge of the Hasse diagram (thought as embedded in M in the obvious way) is equal to zero (see Fig. 3.23). Thus, if we think about combinatorial (not necessarily Morse) vector field on T as a Matching on its Hasse diagram, it encodes a desingularization of the vector field F_1 (where critical points of F_1 remain if the corresponding cell is critical). This can be done in the case of Euler chain. More precisely, if ξ is an Euler chain, let F_ξ denote an extension of F_1 to a non-singular vector field. Turaev proved that the homotopy $[F_\xi] \in Vect(M)$ depends only on $[\xi] \in Eul(T)$. The map ρ is defined by $\rho([\xi]) = [F_\xi]$.

In the general case where ξ corresponds to a matching (instead of a complete matching), we still denote by ρ the map which assigns to ξ the vector field F_ξ given by the construction above.

We refer to [Tur89] in the case $\partial M \neq \emptyset$.

3.4.3 $\mathcal{M}$-realization of Euler structures

Definition 3.4.5. Let $[\xi] \in Eul(M, \partial_0 M)$. We say that $[\xi]$ has a $\mathcal{M}$-realization if there exits a C^1-triangulation T of $(M, \partial M)$ and a matching η on the corresponding Hasse diagram such that $[\eta] = [\xi]$.

The question arising is wether any Euler structure has a $\mathcal{M}$-realization. We prove the following:

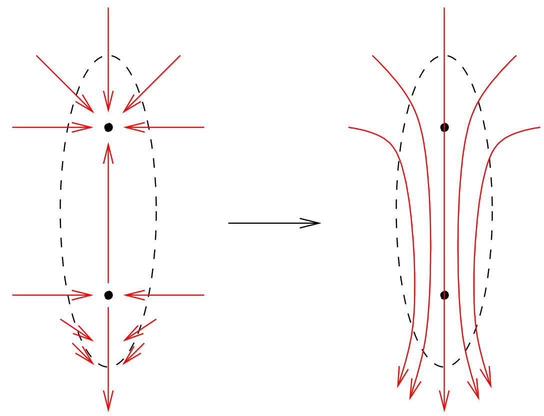

Figure 3.23: **Modifying F_1 in a neighbourhood of an edge of a matching.** The pair of critical points disappears. Notice that there are many way to modify F_1.

Theorem 3.4.6. *Any Euler structure on a smooth oriented closed riemannian 3-manifold has an $\mathcal{M}$-realization.*

This theorem establish a connection with Heegaard-Floer theory, not only for links but more generally for 3-manifolds. Recall the first steps of the construction of Heegaard-Floer homology for closed, oriented 3-manifolds. Choose a pointed Heegaard splitting $M = U_0 \cup_\Sigma U_1$ (that is choose a Morse function) and fix a Spin^c-structure on M given by an n-uplets of intersection points between the α's and the β's. Turaev proved that Spin^c-structures are in bijection with Euler structures (and so the set $vect(M)$). Thus, Theorem 3.4.6 together with Theorem 3.3.1 give for a closed oriented 3-manifold a combinatorial realization of both the Spin^c-structure and the pointed Heegaard splitting. Next step is now to understand combinatorially holomorphic disks which contribute to the differential of the Heegaard-Floer chain complex.

The following two lemmas will be useful to prove this theorem.

Lemma 3.4.7. *Let $(M \times [0,1]; M \times \{0\}, M \times \{1\})$ be a smooth $n+1$ product cobordism such that T_i is a C^1-triangulation of $M \times \{i\}$ for $i \in \{0,1\}$. Let $(T; T_0, T_1)$ be any triangulation given by lemma 3.3.10 and $[\xi] \in Eul(M \times [0,1], M \times \{0\})$ the Euler structure given by the combinatorial Morse vector field induced by $T \searrow T_0$. Then $\rho([\xi]) \in vect(M \times [0,1], M \times \{0\})$ is homologous to the vector field $v : M \times [0,1] \to T(M \times [0,1])$ defined by $v(x,t) = ((x,t), dt)$.*

Proof. The construction of the collapsing $T \searrow T_0$ defines a combinatorial Morse vector field pointing downwards as illustrated by Fig.3.24. Let $\xi \in Eul(T, T_0)$ be the corresponding Euler chain. The map $\rho : Eul(M \times [0,1], M \times \{0\}) \to vect(M \times [0,1], M \times \{0\})$ sends ξ to a non-singular vector

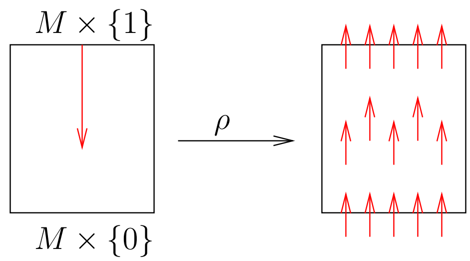

Figure 3.24: **The combinatorial Morse vector field on T and the induced non-singular vector field given by ρ.**

field on $(M \times [0,1], M \times \{0\})$ which is, by definition of ρ, by the construction of the triangulation and by the definition of the combinatorial Morse vector field, homologous to the desired vector field. □

Lemma 3.4.8. *Let f be a generic Morse function on a cobordism $(M; M_0, M_1)$ with exactly one critical point p of index k and $(T; T_0, T_1)$ be a C^1-triangulation of the cobordism $(T; T_0, T_1)$ given by Theorem 3.3.12. Let ξ be the one singular chain corresponding to the matching given by the combinatorial Morse vector field on T. Then $\rho(\xi)$ is homologous to the chosen Morse–Smale gradient of f outside a small ball neighbourhood of the critical point p. Moreover, the index of $\rho(\xi)$ at p is equal to k.*

Proof. We follow notations of Theorem 3.3.12. The two collapsings $T \searrow T_p^s \cup T_0$ and $T_p^s - \sigma_p \searrow (T_p^s \cap T_0)$ define a combinatorial Morse vector field with only one critical cell. Let ξ be the corresponding one singular chain. Thus, the barycenter of this cell (which is p) must be a critical point of $\rho(\xi)$. It remains to check that outside a small neighbourhood of p, $\rho(\xi)$ is homologous to the Morse–Smale gradient v. Since outside the tubular neighbourhood C of the stable manifold the triangulation is constructed by pushing it along gradient lines of v, we can use lemma 3.4.7 to see that $\rho(\xi)$ is homologous to v outside the tubular neighbourhood C of the stable manifold. In a small ball neighbourhood (which is a Morse chart at p) of the critical point p, the vector field F_1 coincides with the Morse–Smale gradient v. Since $T^C \searrow \sigma_p \cup (T_0 \cap T^C)$, $\rho(\xi)$ is homologous to v outside the Morse chart of p . The fact that the index of $\rho(\xi)$ at p is equal to k is a consequence of the definition of F_1. □

Proof of Theorem 3.4.6. We apply Theorem 3.3.1 to obtain a C^1-triangulation of the 3-manifold and a combinatorial Morse vector field which realizes combinatorially the Thom-Smale complex. Then, the construction done in corollary 3.4.2 defines a matching which in turns defines an Euler chain ξ. The map ρ sends ξ to a non-singular vector field which is by construction homologous to the Morse–Smale of f. Thus, to prove the theorem

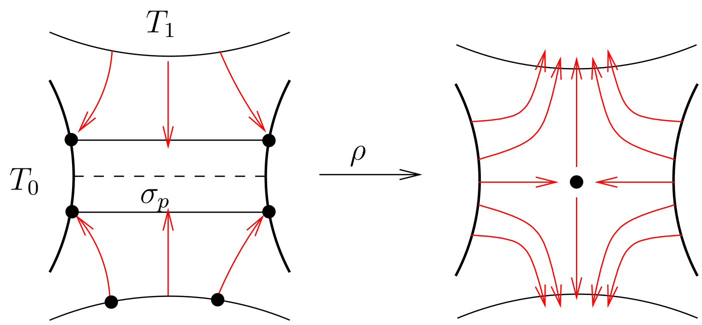

Figure 3.25: **The combinatorial Morse vector field on T and the induced singular vector field given by ρ.** There is only one singular point which is the barycenter of the critical cell.

for any $[v] \in vect(M)$, it remains to find a pointed Heegaard splitting $(\Sigma_g; \underline{\alpha} = (\alpha_1, \dots, \alpha_g), \underline{\beta} = (\beta_1, \dots, \beta_g); z)$ of the 3-manifold M such that an n-uplet of intersection points x corresponds to a given $[v] \in vect(M)$. Lemma 5.2 in [OS04c] tells that any $[v] \in vect(M)$ can be realized in such a way. This concludes the proof. $\square$

3.4.4 An algorithm to find maximum matchings

We use the Ford-Fulkerson algorithm to find a matching which is maximum in terms of number of edges.

We consider a **directed graph**: each edge has a source and a target. A graph is **simple** if there is at most one edge between a given source and tarjet. A simple graph is **antisymetric** is for each edge $u \to v$ there is no edge $v \to u$.

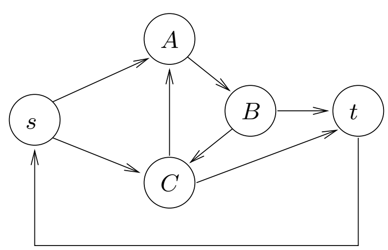

Figure 3.26: **Example of a directed graph which is simple and antisymetric.**

The max-flow problem

Definition 3.4.9. A flow on a directed graph is a valuation of each edge so that it respects the Kirchhoff law: for every vertex u, the sum of valuations of edges having u as tarjet is equal to the sum of edges having u as source.

Let Γ be a **bounded graph** that is a simple antisymetric directed graph whose edges are marked with closed intervals of type $[m, n]$ with $-\infty \leq m \leq n \leq +\infty$. In case when it is $[0, m]$, we just denote it $[m]$ and m is called the **capacity**. We suppose that Γ has two distinguished vertices, the source s and the tarjet t, and that there is an edge $t \to s$ called **edge of return** marked $[\infty, +\infty]$.

The problem of the compatible flow is the problem of the existence of a flow with valuation over edges in the interval given by their mark. The **max-flow problem** is the problem of finding a compatible flow with maximum valuation on the edge of return.

An edge $u \xrightarrow{[m,n]} v$ is said **satured** by a compatible flow if its valuation is n and it is said **antisatured** if its valuation is m.

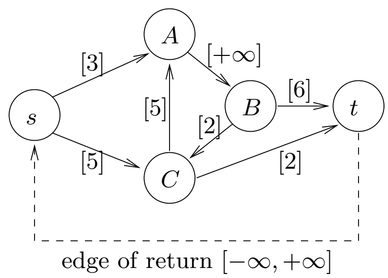

Figure 3.27: **Example of a bounded graph.**

Definition 3.4.10. A minimal cut of a bounded graph associated with a flow is a partition of the set vertices in two sets M and N so that the source s is in M, the tarjet t is in N and every edges directed from M to N is satured and every edge from N to M is antisatured.

Theorem 3.4.11 (caracterisation of a maximum flow)**.** *If a flow admits a minimal cut then the flow is maximal.*

So the max-flow problem on a bounded graph reduces to the problem of finding a minimal cut. The Ford-Fulkerson algorithm (see [CLR01] Section 26.2: The Ford-Fulkerson method, pp.651-664) constructs a cut until it is a minimal cut.

Reduction to the max-flow problem

Let T be a triangulation of a manifold M such that $\chi(M) = 0$. The Hasse diagram Γ of T is in fact a **bipartite graph**:

- the set of vertices is divided in two sets O and E, the set of odd and the set of even dimensional cells,
- each edge join a vertex of O to a vertex of E.

We orient each edge from O to E.

1. **Construction of the flow**
   Starting with Γ, we construct another bipartite graph Γ'

   (a) the set of vertices of Γ' equal the set of vertices of Γ with a vertex source s and a vertex tarjet t added,

   (b) the set of edges of Γ' is the same as the one of Γ with one more edge $s \to o$ for every $o \in O$, one more edge $e \to t$ for every $e \in E$, and an edge of return $t \to s$,

   (c) each edge has capacity 1 and we put the valuation 0 on each edge.

2. **Resolution of the max-flow problem**
   We use the Ford-Fulkerson algorithm on Γ'.

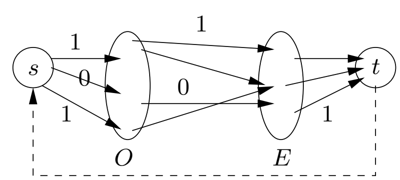

Figure 3.28: **The graph Γ'.**

What is the benefit?

- the flow on each edge is 0 or 1 by construction,
- for each vertex o of O, there is at most one edge having o as source with valuation 1,
- for each vertex e of E, there is at most one edge having e as tarjet with valuation 1.

A maximum matching over Γ is given by edges with valuation 1. The valuation of the edge of return is equal to the number of edges of the maximum matching.

In our case, we compute the Ford-Fulkerson algorithm with the graph described above to find maximum matching over triangulation of 3-manifold with Euler characteristic equal to zero. We write a module in Python for the software *Regina* [Bur05]. What is quite surprising is that even if is not in general a triangulation on examples (it can be a CW-complex) there is always a complete matching.

Thus, the question is wether given any C^1-triangulation of a 3-manifold M such that $\chi(M) = 0$ there always exists a complete matching on its Hasse diagram? It seems true but we don't know how to prove it.

Index

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