Local Tb theorems and Hardy type inequalities

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Théorèmes Tb locaux et inégalités de type Hardy

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Résumé

On étudie dans cette thèse les théorèmes Tb locaux pour les opérateurs d'intégrale singulière, dans le cadre des espaces de type homogène. On donne une preuve directe du théorème Tb local avec hypothèses d'intégrabilité L^2 sur le système pseudo-accrétif. Notre argument repose sur l'algorithme Beylkin-Coifman-Rokhlin, appliqué dans des bases d'ondelettes de Haar adaptées, et sur des résultats de temps d'arrêt. Motivés par une question posée par S. Hofmann, on étend notre résultat au cas où les conditions d'intégrabilité sont inférieures à 2, avec une hypothèse supplémentaire de type faible bornitude, qui incorpore des inégalités de type Hardy. On étudie la possibilité d'affaiblir les conditions de support du système pseudo-accrétif en l'autorisant à être défini sur un petit élargissement des cubes dyadiques. On donne également un résultat dans le cas où, pour des raisons pratiques, les hypothèses sur le système pseudo-accrétif sont faites sur les boules au lieu des cubes dyadiques. Enfin, on s'intéresse au cas des opérateurs parfaitement dyadiques pour lesquels la démonstration est grandement simplifiée.

Notre argument nous donne l'opportunité de nous intéresser aux inégalités de type Hardy. Ces estimations sont bien connues des spécialistes dans le cadre Euclidien, mais elles ne semblent pas avoir été étudiées dans les espaces de type homogène. On montre qu'elles sont vérifiées sans restriction dans le cadre dyadique. Dans le cas plus général d'une boule B et de sa couronne $2B\backslash B$, elles peuvent être déduites de certaines conditions géométriques de distribution des points dans l'espace de type homogène. Par exemple, on prouve qu'une condition de petite couche relative est suffisante. On montre aussi que cette propriété est impliquée par la propriété de monotonie géodésique de Tessera. Enfin, on présente quelques exemples et contre-exemples explicites dans le plan complexe, afin d'illustrer le lien entre la géométrie de l'espace de type homogène et la validité des inégalités de type Hardy.

Mots-clefs : Théorèmes Tb locaux, opérateurs d'intégrale singulière, géométrie des espaces de type homogène, inégalités de type Hardy, propriétés de petite couche.

Local Tb theorems and Hardy type inequalities

Abstract

In this thesis, we study local Tb theorems for singular integral operators in the setting of spaces of homogeneous type. We give a direct proof of the local Tb theorem with L^2 integrability on the pseudo-accretive system. Our argument relies on the Beylkin-Coifman-Rokhlin algorithm applied in adapted Haar wavelet basis and some stopping time results. Motivated by questions of S. Hofmann, we extend it to the case when the integrability conditions are lower than 2, with an additional weak boundedness type hypothesis, which incorporates some Hardy type inequalities. We study the possibility of relaxing the support conditions on the pseudo-accretive system to a slight enlargement of the dyadic cubes. We also give a result in the case when, for practical reasons, hypotheses on the pseudo-accretive system are made on balls rather than dyadic cubes. Finally we study the particular case of perfect dyadic operators for which the proof gets much simpler.

Our argument gives us the opportunity to study Hardy type inequalities. The latter are well known in the Euclidean setting, but seem to have been overlooked in spaces of homogeneous type. We prove that they hold without restriction in the dyadic setting. In the more general case of a ball B and its corona $2B\backslash B$, they can be obtained from some geometric conditions relative to the distribution of points in the homogeneous space. For example, we prove that some relative layer decay property suffices. We also prove that this property is implied by the monotone geodesic property of Tessera. Finally, we give some explicit examples and counterexamples in the complex plane to illustrate the relationship between the geometry of the homogeneous space and the validity of the Hardy type inequalities.

Keywords : Local Tb theorems, singular integral operators, geometry in spaces of homogeneous type, Hardy type inequalities, layer decay properties.

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Table des matières

Introduction

L'objectif principal de cette thèse est de répondre à une question soulevée par S. Hofmann dans [Ho2]. Cette question est la suivante : est-il possible, dans les théorèmes $T(b)$ locaux existants, d'affaiblir les hypothèses d'intégrabilité portant sur les systèmes pseudo-accrétifs à des hypothèses de type L^p pour p quelconque strictement supérieur à 1 ? Jusqu'à maintenant, les arguments connus, disons, pour simplifier, concernant des noyaux anti-symétriques [Ch1], [Ho1], [AY], [TY], [HM2] traitent le cas d'exposants supérieurs ou égaux à 2, mais jamais plus petits, en dehors du cas modèle des opérateurs parfaitement dyadiques présenté dans [AHMTT]. Les motivations derrière cette question sont à trouver dans [HM1] pour montrer l'uniforme rectifiabilité d'ensembles Ahlfors-David réguliers qui sont vus comme frontières de domaines vérifiant certaines conditions d'accès intérieur à la frontière dont le noyau de Poisson vérifie des estimations L^p invariantes par échelle pour un p petit proche de 1. L'intérêt des théorèmes $T(b)$ locaux par rapport au théorème $T(b)$ "global" de G. David, J. Journé et S. Semmes [DJS] est qu'il n'est pas nécessaire de produire une fonction para-accrétive définie sur l'espace tout entier pour appliquer le théorème, ce qui est inévitable avec les formulations classiques du $T(b)$. En pratique, la formulation locale est donc souvent plus facile à appliquer. Le cadre de notre étude se limitera aux opérateurs d'intégrale singulière à valeurs scalaires dans des espaces de type homogène, bien que d'autres théories aient récemment vu le jour, notamment dans le cadre inhomogène ou concernant des opérateurs à valeurs vectorielles (voir la Section 1.1 pour plus de précisions).

On apportera une réponse, au moins partielle, à la question de S. Hofmann, avec notre Théorème 3.1.2. On donnera en effet un argument qui fonctionne pour tous les exposants d'intégrabilité, y compris les petits exposants proches de 1, ce qui n'avait encore jamais été réalisé. Cet argument est basé sur l'algorithme Beylkin-Coifman-Rokhlin [BCR], appliqué dans des bases d'ondelettes de Haar adaptées au système pseudo-accrétif. Cet algorithme fournit un certain nombre de coefficients matriciels à estimer, ce qui sera fait en appliquant les hypothèses de contrôle du théorème, les propriétés de régularité du noyau, et des inégalités dites de type Hardy. Cependant, l'impossibilité d'appliquer ces inégalités de type Hardy dans le cas de petits exposants proches de 1 oblige à supposer alors une hypothèse technique de faible bornitude (voir l'énoncé de notre théorème, Section 3.1). Nous sommes d'ailleurs à présent convaincus que le théorème évoqué dans [Ho2] ne peut être valide en général pour des exposants proches de 1. Il est donc intéressant d'isoler cette hypothèse technique supplémentaire, mais elle pourrait être difficile à vérifier en pratique. On donnera toutefois l'exemple explicite d'un système pseudo-accrétif de faible intégrabilité vérifiant précisément cette hypothèse. La compréhension de ce problème demeure incomplète, et elle nécessitera probablement encore quelques efforts et idées nouvelles.

Le deuxième objectif de cette thèse est l'étude, dans les espaces de type homogène, des inégalités dites de type Hardy évoquées plus haut. Ces inégalités sont de la forme

$$\left| \int_I \int_J \frac{f(y)g(x)}{\mathrm{Vol}(B(x, \mathrm{dist}(x,y)))} \mathrm{d}\mu(x)\mathrm{d}\mu(y) \right| \leq C \|f\|_{L^p(I,\mathrm{d}\mu)} \|g\|_{L^{p'}(J,\mathrm{d}\mu)}$$

pour I, J ensembles disjoints et $1 < p < +\infty$, $1/p + 1/p' = 1$. Notre terminologie vient du fait que dans le cadre Euclidien, ces inégalités sont conséquence de la continuité bien connue de l'opérateur de Hardy en dimension 1 (voir Section 1.3). Il apparaît qu'elles sont vérifiées sans aucune restriction lorsque I, J sont des cubes dyadiques de Christ (voir Section 1.2.1), ce qui semble ne jamais avoir été remarqué dans la littérature. C'est notre Proposition 2.1.1. L'argument repose en particulier sur la propriété de petite frontière des cubes dyadiques de Christ. Par contre, lorsque I est une boule B, et J, disons, la couronne $2B \backslash B$, alors le résultat n'est pas vrai en général et dépend fortement de la façon dont les points sont distribués à la frontière commune des ensembles B et $2B \backslash B$. On montrera que des propriétés géométriques de distribution des points, comme les propriétés de relative petite couche ou de relative petite couronne sont suffisantes. On montrera que ces propriétés sont par exemple vérifiées par n'importe quelle variété Riemanienne complète doublante, ainsi que par n'importe quel espace métrique doublant vérifiant la propriété de monotonie géodésique (voir Section 2.4). Cette dernière notion est récemment apparue en théorie géométrique de la mesure dans les travaux de R. Tessera [Te]. On étudiera les relations d'implication existant entre ces différentes propriétés, notamment au moyen de quelques exemples et contre-exemples explicites dans le plan complexe. Cette étude est synthétisée par notre Théorème 2.1.3.

Ce manuscrit est organisé comme suit. Le Chapitre 1 est consacré à un état de l'art général des théorèmes $T(b)$ et des inégalités de type Hardy. On commencera par retracer un historique des différents théorèmes $T(b)$ et de leurs applications notables (Section 1.1). On rappellera ensuite dans la Section 1.2 quelques définitions et généralités concernant les espaces de type homogène, notamment les cubes dyadiques de Christ, la façon de définir un opérateur d'intégrale singulière sur un tel espace, ou encore les estimations standards de Calderón-Zygmund dont on se servira par la suite. Enfin on terminera ce chapitre par une présentation des inégalités de type Hardy dans le cadre Euclidien (Section 1.3). Le Chapitre 2 est quant à lui consacré à l'étude des inégalités de type Hardy et des propriétés de distribution des points évoquées plus haut dans un espace de type homogène. Après avoir présenté nos principaux résultats, c'est-à-dire les Proposition 2.1.1 et Théorème 2.1.3, on commencera par démontrer l'inégalité de type Hardy dyadique (Section 2.2), avant de chercher des conditions suffisantes pour l'inégalité de type Hardy pour les boules (Section 2.3). Cela nous amènera à définir un certain nombre de propriétés géométriques (Section 2.4), et on consacrera la fin de ce chapitre à l'étude des liens et des implications qu'elles entretiennent. Enfin, le Chapitre 3 sera consacré à notre théorème $T(b)$ local, le Théorème 3.1.2. Après avoir présenté et commenté nos principaux résultats, c'est-à-dire le Théorème 3.1.2 (hypothèses dyadiques), le Théorème 3.1.5 (hypothèses sur les boules), et le Théorème 3.1.7 (hypothèses avec support élargi pour le système pseudo-accrétif), on donnera la démonstration de ces deux derniers résultats dans la Section 3.2. On donnera ensuite Section 3.3 un exemple explicite de système pseudo-accrétif avec faible intégrabilité, vérifiant les hypothèses du Théorème 3.1.2 mais pas davantage. La démonstration du Théorème 3.1.2 sera développée sur les trois sections suivantes : on présentera quelques résultats préliminaires dans la Section 3.4, notamment l'existence de bases d'ondelettes de Haar adaptées ; on expliquera comment procéder à d'importantes réductions dans la Section 3.5, et on détaillera finalement notre algorithme et les estimations de tous les termes obtenus dans la Section 3.6. La Section 3.7 sera, pour finir, consacrée à l'étude de deux cas particuliers pour lesquels la démonstration est beaucoup plus facile : les opérateurs parfaitement dyadiques d'une part, et, d'autre part, le cas d'exposants d'intégrabilité supérieurs ou égaux à 2.

Chapitre 1

État de l'art

Sommaire

1.1 Historique des théorèmes $T(b)$

Les théorèmes $T(b)$ sont des théorèmes ayant pour objet la continuité des opérateurs d'intégrale singulière. L'étude de cette classe d'opérateurs, dont on rappellera la définition dans la Section 1.2.1, et de leur continuité sur L^2 a, ces dernières décennies, été à l'origine de nombreuses applications en analyse harmonique, théorie géométrique de la mesure ou à certains problèmes d'équations aux dérivées partielles. Rappelons tout de suite quelques exemples fondamentaux d'opérateurs d'intégrale singulière pour motiver leur étude. La transformée de Hilbert H est donnée sur $\mathbb{R}$ par

$$Hf(x) = \frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{|y-x|>\varepsilon} \frac{f(y)}{x-y} \mathrm{d}y := \frac{1}{\pi} \operatorname{vp} \int_{\mathbb{R}} \frac{f(y)}{x-y} \mathrm{d}y,$$

pour toute fonction $f \in C_0^1(\mathbb{R})$, l'ensemble des fonctions C^1 à support compact. Si $n \geq 2$ et $1 \leq j \leq n$, les transformées de Riesz R_j sont données sur $\mathbb{R}^n$ par

$$R_j f(x) = c_n \operatorname{vp} \int_{\mathbb{R}^n} \frac{x_j - y_j}{|x-y|^{n+1}} \mathrm{d}y,$$

pour $f \in C_0^1(\mathbb{R}^n)$, c_n étant une constante dimensionnelle connue. Elles généralisent la transformée de Hilbert en dimension supérieure et constituent d'autres exemples bien connus d'opérateurs d'intégrale singulière. Enfin, donnons l'exemple fondamental de la transformée de Cauchy

$\mathcal{C}_\Gamma$ sur une courbe Lipschitz Γ du plan complexe, ou plus généralement sur un compact de $\mathbb{C}$ de longueur finie, définie formellement par

$$\mathcal{C}_\Gamma f(z) = \int_\Gamma \frac{f(\xi)}{\xi - z} \mathrm{d}\Lambda(\xi),$$

Λ étant la longueur de Hausdorff dans $\mathbb{C}$ (voir [Ch2] pour une définition plus rigoureuse). Cet exemple est capital car il existe un lien très fort entre la continuité sur L^2 de cet opérateur intégral de Cauchy $\mathcal{C}_\Gamma$ et les propriétés géométriques de l'ensemble Γ comme sa rectifiabilité ou sa capacité analytique. On renvoie à [P], [Ch2] pour des définitions de ces notions et plus de détails. Ce lien entre continuité L^2 de l'opérateur intégral de Cauchy et propriétés géométriques de l'ensemble considéré a naturellement motivé la recherche de critères de continuité pour ce type d'opérateurs. Un remarquable théorème, mis au point par G. David et J. Journé en 1984, donne une première condition nécessaire et suffisante de continuité L^2 pour les opérateurs d'intégrale singulière. C'est le théorème $T(1)$ que l'on rappelle ici.

Théorème 1.1.1 (David et Journé ; 1984 [DJ])**.** *Soit T un opérateur d'intégrale singulière sur $\mathbb{R}^n$. T admet une extension bornée à $L^2(\mathbb{R}^n)$ si et seulement si les trois conditions suivantes sont vérifiées :*

1. *T est faiblement borné*
2. $T(1) \in \mathrm{BMO}$
3. $T^*(1) \in \mathrm{BMO}$

Ce théorème, sa preuve, et les idées s'y rattachant, ont été fondamentaux dans la compréhension de ce sujet et de ses applications. On ne rappellera pas la définition d'un opérateur faiblement borné ici, puisque cela n'interviendra pas dans la suite de notre travail, et on renvoie par exemple à [D3] ou [Ch2] pour plus de détails. C'est notamment le cas de n'importe quel opérateur d'intégrale singulière canoniquement associé à un noyau anti-symétrique.

Un des principaux inconvénients de ce critère de continuité est qu'il peut être en pratique difficile voire impossible de vérifier les principales hypothèses (2) et (3). Cela a motivé une généralisation du théorème $T(1)$, où la fonction 1 est remplacée par n'importe quel élément b d'une large classe de fonctions, dites para-accrétives. Rappelons qu'une fonction $b : \mathbb{R}^n \to \mathbb{C}$ est dite para-accrétive s'il existe $\varepsilon > 0$ tel que pour tous $x \in \mathbb{R}^n, r > 0$, il existe $y \in \mathbb{R}^n$, et $\rho \geq \varepsilon r > 0$ tels que $B = B(y,\rho) \subset B(x,r)$ et $|B|^{-1} \left|\int_B b\right| \geq \varepsilon$. Ce théorème est le célèbre théorème $T(b)$ que l'on rappelle également ici.

Théorème 1.1.2 (David, Journé et Semmes ; 1985 [DJS])**.** *Soient b_1 et b_2 deux fonctions para-accrétives. Soit T un opérateur d'intégrale singulière sur $\mathbb{R}^n$, défini de $b_1\mathcal{D}$ dans $(b_2\mathcal{D})'$. Notons M_b l'opérateur de multiplication par b. Supposons que les trois conditions suivantes soient vérifiées :*

1. *L'opérateur $M_{b_2}TM_{b_1}$ est faiblement borné.*
2. $T(b_1) \in \mathrm{BMO}$.
3. $T^*(b_2) \in \mathrm{BMO}$.

Alors T admet une extension bornée à $L^2(\mathbb{R}^n)$.

Le cas $T(b) = 0$ avait déjà été traité par A. McIntosh et Y. Meyer dans [MM]. Dans le cas général, on se ramène au cas $T(b) = 0$ en faisant intervenir des termes de paraproduits qui se contrôlent assez facilement. La démonstration dans le cas $T(b) = 0$ repose ensuite sur la construction d'une base d'ondelettes de Haar adaptées aux fonctions b_i sur laquelle on décompose l'opérateur T. Il reste alors un peu d'estimations techniques pour parvenir à contrôler les coefficients matriciels obtenus et pouvoir appliquer un Lemme de Shur pour obtenir la continuité L^2 de T. On renvoie également aux travaux de Y.-S. Han et E. Sawyer [HS] concernant le théorème $T(b)$ et les fonctions para-accrétives.

Le théorème $T(b)$ de G. David, J. Journé et S. Semmes a une application directe à l'opérateur intégral de Cauchy puisqu'il fournit de façon immédiate sa continuité L^2 sur n'importe quelle courbe Lipschitz du plan complexe. Ce résultat était bien sûr déjà connu (voir [Ca] pour le cas de petites constantes de Lipschitz et [CMM] pour le cas général). Les idées au coeur du théorème $T(b)$ ont également permis la mise au point d'un théorème similaire pour les fonctionnelles quadratiques, qui a, entre autres, conduit à la résolution de la conjecture de Kato pour les opérateurs elliptiques sur $\mathbb{R}^n$. Ce vieux problème d'équations aux dérivées partielles, consistant à montrer que le domaine de la racine carrée d'un opérateur elliptique complexe $L = -\mathrm{div}(A\nabla)$ à coefficients mesurables bornés dans $\mathbb{R}^n$ est l'espace de Sobolev $H^1(\mathbb{R}^n)$, avait été essentiellement posé par T. Kato [K] en 1961. Il a été complètement résolu en 2002 par P. Auscher *et al.* dans [AHLMT] (le cas de la dimension 1 avait déjà été traité dans [CMM]) en utilisant des méthodes propres au $T(b)$.

En pratique, il peut cependant s'avérer encore difficile de vérifier les hypothèses assez rigides du théorème $T(b)$ classique : il faut être capable d'exhiber une fonction para-accrétive b, définie sur l'espace tout entier, et satisfaisant les hypothèses de contrôle BMO du théorème. Pour gagner en souplesse, et, motivé entre autres par de possibles applications à l'étude de la capacité analytique, M. Christ a l'idée en 1990 de non plus considérer une fonction b fixée sur l'espace tout entier, mais plutôt une collection de fonctions définies et supportées chacune sur les cubes dyadiques de l'espace considéré. C'est le premier théorème $T(b)$ local.

Théorème 1.1.3 (Christ ; 1990 [Ch1]). *Soit T un opérateur d'intégrale singulière sur $\mathbb{R}^n$ associé à un noyau localement borné K. Supposons qu'il existe $C > 0$ et deux systèmes de fonctions $(b_Q^1)_Q, (b_Q^2)_Q$ définies sur les cubes dyadiques de $\mathbb{R}^n$ tels que pour tout cube dyadique Q, b_Q^1, b_Q^2 soient supportées sur Q et*

1. $\int_Q b_Q^1 = |Q| = \int_Q b_Q^2$,
2. $\|b_Q^1\|_\infty + \|b_Q^2\|_\infty \leq C$,
3. $\|T(b_Q^1)\|_\infty + \|T^*(b_Q^2)\|_\infty \leq C$.

Alors T admet une extension bornée à $L^2(\mathbb{R}^n)$, de norme indépendante de $\|K\|_{\infty,\mathrm{loc}}$.

Un mot sur l'hypothèse K localement borné que l'on verra dans tous les énoncés ultérieurs. Elle simplifie la définition des objets liés à la preuve mais n'est pas essentielle. Un tel système $(b_Q)_Q$ est dit pseudo-accrétif. Remarquer qu'on suppose ici un contrôle de type L^∞ uniforme et non plus BMO des $b_Q^i, T(b_Q^i)$. Ce sont donc des hypothèses pouvant être plus souples d'utilisation du fait de leur caractère local, mais le contrôle requis est beaucoup plus fort. Pour démontrer ce théorème, M. Christ met au point un algorithme permettant de construire une fonction b para-accrétive vérifiant les hypothèses du théorème $T(b)$ classique à partir du système pseudo-accrétif donné. Cet algorithme fait appel à un argument de temps d'arrêt pour lequel les hypothèses de contrôle L^∞ jouent justement un rôle central et ne peuvent, semble-t-il, être affaiblies. Une question naturelle est donc de savoir si l'on peut, avec un algorithme différent, affaiblir le contrôle L^∞ en un contrôle L^p pour $1 < p < +\infty$. On reviendra un peu plus tard à cette question, qui est le principal objectif de cette thèse. On peut notamment citer les travaux de M. Alfonseca, P. Auscher, A. Axelsson, S. Hofmann et S. Kim [AAAHK] comme exemple d'application de ce type de théorème $T(b)$ local avec contrôle L^p. M. Christ illustre également bien la pertinence de ce théorème $T(b)$ local dans [Ch1] en l'appliquant à l'étude de la capacité analytique, ou pour obtenir une nouvelle démonstration de la continuité de l'opérateur intégral de Cauchy sur des courbes rectifiables dites Ahlfors-régulières du plan complexe, résultat déjà obtenu par G. David dans [D1].

Un autre axe de recherche, initié par F.Nazarov, S. Treil et A. Volberg en 2002, est la généralisation de ces théorèmes $T(b)$ au cas non homogène. Comme on le verra en Section 1.2, le cadre naturel d'étude pour les théorèmes $T(b)$ est le cadre des espaces métriques doublants dits espaces de type homogène, dans lesquels tous les résultats précédents se généralisent sans grande difficulté. Le cadre non homogène est, quant à lui, beaucoup plus complexe. F. Nazarov, S. Treil et A. Volberg parviennent cependant dans [NTV2] à généraliser le $T(b)$ classique à des sous-ensembles de $\mathbb{R}^n$ munis d'une mesure vérifiant uniquement une condition de croissance polynômiale, mais aucune condition de doublement. Leur démonstration repose sur un remarquable argument probabiliste, et sur la notion de réseau dyadique aléatoire. Leur idée est de reprendre la démonstration du $T(b)$ classique, sauf qu'au lieu de construire une base de Haar adaptée à b sur le réseau dyadique canonique de $\mathbb{R}^n$, ils le font sur un réseau dyadique aléatoire, translaté du réseau canonique. Ils montrent ensuite que, bien que ponctuellement les estimations des coefficients matriciels requises ne soient pas forcément respectées, elles le sont au moins en moyenne. Cela permet de prouver l'existence d'au moins un réseau dyadique pour lequel toutes les estimations requises sont vérifiées, et de prouver ainsi le $T(b)$. F. Nazarov *et al.* parviennent d'ailleurs avec ces méthodes probabilistes à étendre également le théorème $T(b)$ local de M. Christ au cadre non homogène dans [NTV1].

Ces théorèmes $T(b)$ non homogène et les idées s'y rattachant sont ainsi à l'origine de la résolution du problème de Painlevé par X. Tolsa en 2003. Ce vieux problème d'analyse complexe, consistant à caractériser de manière géométrique les ensembles effaçables du plan complexe pour les fonctions analytiques bornées, avait été partiellement résolu, notamment par G. David dans [D4], puis J. Mateu, X. Tolsa et J. Verdera dans [MTV]. Dans ce dernier article, tous les principaux ingrédients, et en particulier ces arguments de type $T(b)$ non homogène appliqués à la capacité analytique, sont alors présents pour permettre à X. Tolsa d'achever la résolution du problème de Painlevé dans [To1]. On renvoie à [To2] pour une synthèse des derniers résultats concernant justement le problème de Painlevé et la capacité analytique qui est une notion étroitement liée.

Ce concept de réseau dyadique aléatoire est également au coeur des récents et remarquables travaux de T. Hytönen, notamment concernant l'étude des opérateurs d'intégrale singulière à valeur vectorielle. Cette théorie avait déja été étudiée à la fin des années 1980 dans les travaux de T. Figiel [F], mais c'est en employant le même type d'arguments probabilistes que Nazarov *et al.* que T. Hytönen est parvenu à généraliser le théorème $T(b)$ aux opérateurs d'intégrale singulière ayant pour valeur des vecteurs d'espaces de Banach, à la fois dans le cadre homogène [Hy1], et non homogène [Hy2]. On ne s'intéressera cependant pas à ces aspects de la théorie $T(b)$ dans cette thèse et on se limitera aux opérateurs d'intégrale singulière à valeur scalaire, dans le cadre homogène.

Comme cela a déjà été mentionné, l'objectif principal de cette thèse est l'étude des théorèmes $T(b)$ locaux dans le cadre homogène, et l'affaiblissement du contrôle L^∞ du Théorème 1.1.3 à un contrôle L^p. Une grande avancée dans ce sens a été réalisée en 2002 par P. Auscher, S. Hofmann, C. Muscalu, T. Tao et C. Thiele [AHMTT]. Leur résultat ne s'applique cependant qu'à des opérateurs modèles, dits opérateurs parfaitement dyadiques (voir Section 3.7.1 pour la définition précise de ces opérateurs), dont la singularité est en quelque sorte adaptée à la grille dyadique. Rappelons qu'on appelle exposant conjugué d'un réel $p > 0$ le réel p' tel que $1/p + 1/p' = 1$.

Théorème 1.1.4 (Auscher, Hofmann, Muscalu, Tao, Thiele ; 2002 [AHMTT])**.** *Soit T un opérateur d'intégrale singulière parfaitement dyadique sur $\mathbb{R}^n$. Soient $1 < p, q < \infty$, d'exposants conjugués p', q'. Supposons qu'il existe $C > 0$ et deux systèmes de fonctions $(b_Q^1)_Q, (b_Q^2)_Q$ tels que pour tout cube dyadique Q, b_Q^1, b_Q^2 soient supportées sur Q et*

1. $\int_Q b_Q^1 = |Q| = \int_Q b_Q^2$,

2. $\int_Q (|b_Q^1|^p + |b_Q^2|^q) \leq C|Q|$,
3. $\int_Q (|Tb_Q^1|^{q'} + |T^*b_Q^2|^{p'}) \leq C|Q|$.

Alors T admet une extension bornée à $L^2(\mathbb{R}^n)$.

La démonstration de ce théorème repose une nouvelle fois sur la décomposition de l'opérateur T sur une base d'ondelettes de Haar adaptées aux b^i, et surtout sur un remarquable lemme de temps d'arrêt permettant d'éliminer les termes matriciels pour lesquels les bonnes estimations ne sont pas vérifiées. L'argument est considérablement simplifié par le caractère parfaitement dyadique de l'opérateur, qui entraîne l'annulation d'une grande partie des coefficients matriciels non diagonaux.

La généralisation du Théorème 1.1.4 aux opérateurs d'intégrale singulière standards pose malheureusement problème. En effet, le fait que l'opérateur ne soit plus parfaitement dyadique fait apparaître un grand nombre de termes de bord, des coefficients matriciels non diagonaux qu'on ne sait pas forcément tous estimer à l'aide des seules propriétés de régularité du noyau et des hypothèses L^p. S. Hofmann parvient cependant en 2007 [Ho1] à mener les calculs à leur terme et il obtient un théorème valide à condition de se limiter à une plage d'exposants suffisamment grands.

Théorème 1.1.5 (Hofmann ; 2007 [Ho1])**.** *Soit T un opérateur d'intégrale singulière sur $\mathbb{R}^n$ associé à un noyau standard K localement borné. Supposons qu'il existe $C > 0$ et deux systèmes de fonctions $(b_Q^1)_Q, (b_Q^2)_Q$ tels que pour tout cube dyadique Q, b_Q^1, b_Q^2 soient supportées sur Q et*

1. $\int_Q b_Q^1 = |Q| = \int_Q b_Q^2$,
2. $\int_Q (|b_Q^1|^s + |b_Q^2|^s) \leq C|Q|$, *pour un $s > 2$,*
3. $\int_Q (|Tb_Q^1|^2 + |T^*b_Q^2|^2) \leq C|Q|$.

Alors T admet une extension bornée à $L^2(\mathbb{R}^n)$, de norme indépendante de $\|K\|_{\infty,\mathrm{loc}}$.

La démonstration reprend celle du Théorème 1.1.4 quasiment à l'identique, utilisant le même argument de temps d'arrêt, avec le souci de pouvoir contrôler tous les coefficients matriciels supplémentaires. Ceci n'est techniquement possible qu'à condition de supposer une intégrabilité L^s des b_Q pour un $s > 2$, et une intégrabilité au moins L^2 des $T(b_Q)$. Cette méthode ne permet donc pas de conclure dans le cas fondamental d'exposants $p = q = 2$, et encore moins pour des petits exposants p, q tels que $1/p + 1/q > 1$. Le problème restait donc ouvert en ce qui concerne les petits exposants.

Une nouvelle avancée a été réalisée en 2009 par P. Auscher et Q. Yang dans [AY]. L'argument employé dans [Ho1] ne fonctionnant pas pour les petits exposants, les auteurs ont eu une approche différente et ont mis au point une nouvelle méthode, consistant essentiellement à se ramener au cas parfaitement dyadique. Cette méthode a permis de conclure dans le cas $p = q = 2$, mais toujours pas pour des exposants plus petits. C'est le Théorème 1.1.6 que voici.

Théorème 1.1.6 (Auscher, Yang ; 2009 [AY])**.** *Soit T un opérateur d'intégrale singulière sur $\mathbb{R}^n$ associé à un noyau standard K localement borné. Soient $1 < p, q < \infty$ tels que $1/p + 1/q \leq 1$, d'exposants conjugués p', q'. Supposons qu'il existe $C > 0$ et deux systèmes de fonctions $(b_Q^1)_Q, (b_Q^2)_Q$ tels que pour tout cube dyadique Q, b_Q^1, b_Q^2 soient supportées sur Q et*

1. $\int_Q b_Q^1 = |Q| = \int_Q b_Q^2$,
2. $\int_Q (|b_Q^1|^p + |b_Q^2|^q) \leq C|Q|$, ,
3. $\int_Q (|Tb_Q^1|^{q'} + |T^*b_Q^2|^{p'}) \leq C|Q|$.

Alors T admet une extension bornée à $L^2(\mathbb{R}^n)$, de norme indépendante de $\|K\|_{\infty,\mathrm{loc}}$.

La démonstration repose sur la décomposition d'un opérateur d'intégrale singulière standard comme somme d'un opérateur borné sur tous les L^p et d'un opérateur parfaitement dyadique. Pour montrer ce résultat, P. Auscher et Q. Yang appliquent l'algorithme Beylkin-Coifman-Rokhlin introduit dans [BCR] (voir aussi le travail de T. Figiel [F]). Bien que permettant de conclure dans le cas $p = q = 2$, cette méthode présente l'inconvénient de n'apporter aucun éclairage sur ce qu'il se passe pour les petits exposants $(1/p + 1/q > 1)$. La restriction $1/p + 1/q \leq 2$ apparaît uniquement comme une barrière technique, sans permettre de comprendre ce qui fait véritablement obstacle. Citons également la contribution de C. Tan et L. Yan [TY] qui ont étendu cet argument aux espaces de type homogène, modulo un petit complément concernant l'inégalité de type Hardy dyadique dans un espace de type homogène général, voir Remarque 3.1.6, Section 3.1.

Parallèlement, T. Hytönen et H. Martikainen sont parvenus à appliquer les arguments probabilistes de réseau dyadique aléatoire évoqués plus haut au problème du théorème $T(b)$ local, et ils ont récemment réussi à obtenir un résultat très proche du Théorème 1.1.5.

Théorème 1.1.7 (Hytönen, Martikainen ; 2010 [HM2])**.** *Soit T un opérateur d'intégrale singulière sur $\mathbb{R}^n$ associé à un noyau standard K localement borné. Supposons qu'il existe $C > 0$ et deux systèmes de fonctions $(b_Q^1)_Q, (b_Q^2)_Q$ tels que pour tout cube dyadique Q, b_Q^1, b_Q^2 soient supportées sur Q et*

1. $\int_Q b_Q^1 = |Q| = \int_Q b_Q^2$,
2. $\int_Q (|b_Q^1|^2 + |b_Q^2|^2) \leq C|Q|$,,
3. $\int_Q (|Tb_Q^1|^s + |T^* b_Q^2|^s) \leq C|Q|$ *pour un* $s > 2$.

Alors T admet une extension bornée à $L^2(\mathbb{R}^n)$, de norme indépendante de $\|K\|_{\infty,\mathrm{loc}}$.

La seule différence par rapport au Théorème 1.1.5 est que la condition L^s pour $s > 2$ n'est plus placée sur les b_Q mais sur les $T(b_Q)$, permettant d'avoir une intégrabilité un peu plus faible pour les b_Q. En fait, leur étude se fait dans un cadre un peu plus général, puisqu'ils s'intéressent également au cas non homogène de mesures dites supérieurement doublantes, pour lesquelles des hypothèses de type L^∞ sont requises. On renvoie à [HM2] pour plus de précisions. Leur travail a l'avantage de présenter un autre point de vue et de fournir une toute autre approche du théorème $T(b)$ local. Malheureusement, la barrière des petits exposants subsiste et ne semble pas pouvoir être franchie par cette méthode plus que par les précédentes.

Voici donc l'objectif principal de cette thèse : parvenir à obtenir un théorème $T(b)$ local dans le cas d'hypothèses de type L^p avec p petit proche de 1, ou à défaut comprendre pourquoi un tel théorème ne serait pas vrai dans le cas des petits exposants. Comme dit en préambule, ce problème a été soulevé par S. Hofmann dans [Ho2] (question 3.3.1). Il est motivé par une possible application dans [HM1], dans le but d'obtenir l'uniforme rectifiabilité d'ensembles n-dimensionnels Ahlfors-David réguliers qui sont vus comme frontières de domaines vérifiant une condition d'accès intérieur à la frontière, et dont le noyau de Poisson vérifie une estimation L^p invariante par échelle pour p plus grand que 1 et p proche de 1.

Ce sera l'objet du Chapitre 3 de ce manuscrit, et le Théorème 3.1.2 répond, au moins partiellement, à cet objectif. Notre argument utilise l'algorithme BCR, mais, contrairement à [AY] ou [TY], appliqué avec des ondelettes de Haar adaptées au système pseudo-accrétif des b_Q, ce qui permet d'obtenir une preuve directe, sans utiliser la décomposition d'un opérateur d'intégrale singulière en somme d'un opérateur borné et d'un opérateur parfaitement dyadique. Ce travail fournit donc une démonstration directe du cas d'exposants $p = q = 2$, ainsi qu'un résultat dans le cas de petits exposants $1/p + 1/q > 1$, à condition toutefois de supposer une hypothèse technique supplémentaire, nécessaire pour contrôler certains termes de paraproduits apparaissant dans notre décomposition. On donnera d'ailleurs l'exemple explicite d'un système pseudo-accrétif avec faible intégrabilité vérifiant cette hypothèse technique (voir Section 3.3). Il nous semble aujourd'hui peu probable qu'un théorème comme le Théorème 1.1.6 puisse être étendu au cas d'exposants p, q tels que $1/p + 1/q > 1$ sans hypothèse de contrôle supplémentaire. On renvoie à la Section 3.1 pour plus de commentaires sur le Théorème 3.1.2, principal résultat de cette thèse, et sur notre argument, qui ont fait l'objet d'un article [AR]. Un mot également sur la condition de support $\operatorname{supp} b_Q \subset Q$ qui semble rigide. En fait, elle est inessentielle à condition de la remplacer par $\operatorname{supp} b_Q \subset \widehat{Q}$, réunion de Q et de ses cubes dyadiques voisins, voir Théorème 3.1.7. De même, il est possible d'obtenir un énoncé avec un système pseudo-accrétif défini sur les boules de l'espace au lieu des cubes, voir Théorème 3.1.5. Un tel énoncé peut être préférable dans le cadre d'un espace de type homogène général, étant donné que les cubes dyadiques sont alors définis comme des ensembles assez abstraits, voir Section 1.2.1.

Avant de passer à la suite, rajoutons quelques mots concernant la démonstration de notre théorème. Comme on l'a déjà dit ci-dessus, l'algorithme BCR va nous fournir un certain nombre de coefficients matriciels à estimer. Ces coefficients sont typiquement de la forme $\langle b^2_{Q_0}\theta_Q, T(b^1_{Q_0}\theta_R)\rangle$, où les fonctions θ_Q, θ_R sont supportées respectivement sur les cubes $Q, R \subset Q_0$. Pour les coefficients diagonaux, c'est-à-dire lorsque $Q = R$, le contrôle est assuré par les hypothèses du théorème, ainsi que par des résultats de troncation obtenus par argument de temps d'arrêt. Pour les coefficients non diagonaux à interaction éloignée, c'est-à-dire lorsque les cubes Q et R sont loin l'un de l'autre, c'est la régularité du noyau et les estimations standards de Calderón-Zygmund (voir Section 1.2.3) qui jouent. Enfin, pour les coefficients non diagonaux à interaction proche, c'est-à-dire lorsque les cubes Q et R sont voisins, le contrôle est assuré par des inégalités dites de type Hardy (voir Section 1.3). Cependant, lorsque les exposants sont petits ($1/p+1/q > 1$) les inégalités de type Hardy connues ne s'appliquent pas, et c'est là qu'intervient une hypothèse technique de type faible bornitude, sans laquelle ces termes ne semblent pas pouvoir être contrôlés. Faisons cependant observer que dans [HM2], les inégalités de type Hardy ne sont pas utilisées et elles sont remplacées par un argument probabiliste reposant sur la notion de réseau dyadique aléatoire évoquée plus haut. Tout comme les inégalités de type Hardy, cet argument ne semble malheureusement pas pouvoir s'étendre au cas des petits exposants.

Précisons que tout notre travail est réalisé dans le cadre général des espaces de type homogène, dont on va rappeler la définition et les propriétés générales dans la prochaine section.

1.2 Notations et généralités dans les espaces de type homogène

Avant toute chose, commençons par rappeler la définition d'un espace de type homogène.

Définition 1.2.1. Soit X un ensemble, ρ une distance sur X et μ une mesure borélienne positive localement finie. On dit que (X, ρ, μ) est un espace de type homogène s'il existe une constante $C_D < +\infty$ telle que toutes les boules $B(x, r) = \{y \in X; \rho(x, y) < r\}$ satisfassent la propriété de doublement suivante

$$0 < \mu(B(x, 2r)) \leq C_D\mu(B(x, r)) < \infty$$

pour tout $x \in X$ et $r > 0$. On dit alors que la mesure μ est doublante. On parle aussi d'espaces métriques doublants.

Remarques 1.2.2.
- On se contente parfois d'avoir une quasi-distance ρ sur X dans la définition d'un espace de type homogène (c'est par exemple le cas dans [Ch1]). On se limitera dans ce travail au cas métrique et on n'étudiera pas le cas général des espaces quasi-métriques. Faisons remarquer d'une part que la plupart des espaces naturels, au sens de l'analyse géométrique, qui supportent des mesures doublantes comme les groupes de Carnot ou les variétés riemanniennes à courbure de Ricci positive sont de vrais espaces métriques. D'autre part, nous sommes confiants quant à la possibilité d'étendre les résultats de ce manuscrit au cas quasi-métrique, en particulier ceux du Chapitre 3. Cela reste cependant à vérifier.
- On autorise ici la présence d'atomes dans X, c'est-à-dire de points $x \in X$ tels que $\mu(\{x\}) \neq 0$. En réalité, tous les points peuvent très bien être des atomes si on le souhaite.
- Enfin, $\mu(X) \in]0, +\infty]$.

On utilisera à de nombreuses reprises dans ce travail la notation $A \lesssim B$ (resp. $A \approx B$) pour écrire l'estimation $A \leq CB$ (resp. $(1/C)B \leq A \leq CB$) pour une certaine constante uniforme C pouvant varier ligne à ligne. Le support d'une fonction f définie sur X sera noté $\operatorname{supp} f$, le diamètre d'un sous-ensemble $E \subset X$ sera noté $\operatorname{diam} E$, la distance entre deux sous-ensembles $E, F \subset X$ sera notée $\rho(E, F) = \inf_{x \in E, y \in F} \rho(x, y)$. Enfin, le cardinal d'un ensemble fini I sera noté $\operatorname{Card} I$, et la mesure de Lebesgue d'un ensemble $E \subset \mathbb{R}^n$ sera notée $|E|$.

1.2.1 Cubes dyadiques

Il est possible de munir un espace de type homogène quelconque d'une structure dyadique analogue à la structure dyadique Euclidienne. Ce résultat bien connu a été obtenu pour la première fois dans [D2] par G. David dans le cas de sous-ensembles Ahlfors-réguliers de l'espace Euclidien (voir aussi [D3]). Citons également une construction obtenue par S. Semmes et G. David [DS], toujours dans le cas Ahlfors-régulier, par utilisation du théorème de plongement d'Assouad [As], qui permet de se ramener au cas Euclidien. On donne ici le résultat d'une construction très élégante, valide dans un espace de type homogène quelconque, dûe à M. Christ ([Ch1]).

Lemme 1.2.3. *Soit (X, ρ, μ) un espace de type homogène. Il existe une collection de sous-ensembles ouverts $\{Q_\alpha^j \subset X : j \in \mathbb{Z}, \alpha \in I_j\}$, où I_j est un ensemble d'indices (éventuellement fini) dépendant de j, et des constantes $0 < \delta < 1$, $a_0 > 0$, $\eta > 0$, et $C_1, C_2 < +\infty$ telles que*

1. *Pour tout $j \in \mathbb{Z}$, $\mu(\{X \backslash \bigcup_{\alpha \in I_j} Q_\alpha^j\}) = 0$.*
2. *Si $j < j'$, alors soit $Q_\beta^{j'} \subset Q_\alpha^j$, soit $Q_\beta^{j'} \cap Q_\alpha^j = \varnothing$.*
3. *Pour chaque couple (j, α) et chaque $j' < j$, il existe un unique β tel que $Q_\alpha^j \subset Q_\beta^{j'}$.*
4. *Pour chaque couple (j, α), on a $\operatorname{diam}(Q_\alpha^j) \leq C_1 \delta^j$.*
5. *Chaque Q_α^j contient une boule $B(z_\alpha^j, a_0 \delta^j)$. On dit alors que z_α^j est le centre du cube Q_α^j.*
6. *Condition de petite frontière :*

$$\mu\left(\left\{x \in Q_\alpha^j : \rho(x, X \backslash Q_\alpha^j) \leq t\delta^j\right\}\right) \leq C_2 t^\eta \mu(Q_\alpha^j) \quad \forall j, \alpha, \quad \forall t > 0. \tag{1.2.1}$$

On appelle ces ensembles ouverts Q_α^j cubes dyadiques de l'espace de type homogène X. Pour un cube dyadique $Q = Q_\alpha^j$, j est appelée génération de Q, et on pose $l(Q) = \delta^j$. Par (4) et (5), $l(Q)$ est comparable au diamètre de Q, et, par analogie avec le cadre euclidien, on appelle cette quantité la longueur du cube Q. Toujours par analogie, lorsque $Q_\alpha^{j+1} \subset Q_\beta^j$, on dira que Q_α^{j+1} est un fils du cube Q_β^j, et Q_β^j l'unique père du cube Q_α^{j+1}. Pour tout cube dyadique Q, on note $\widetilde{Q}$ la collection de tous les cubes fils du cube Q. Il est facile de vérifier que chaque cube dyadique a un nombre maximal de fils uniformément borné. Remarquer que tous les atomes de l'espace X sont nécessairement des points isolés par la condition de doublement. Si x est un atome, il existe alors un plus petit entier j tel que $\{x\}$ constitue un cube dyadique Q_α^j de génération j. Dans ce cas, Q_α^j a un unique cube fils $Q_\beta^{j+1} = Q_\alpha^j$, égalité en tant qu'ensembles, Q_β^{j+1} a à son tour un unique cube fils, etc.

On appelle cube dyadique voisin d'un cube dyadique Q tout cube dyadique Q' de même génération que Q satisfaisant la condition $\rho(Q, Q') < l(Q)$. On notera $\widehat{Q}$ la réunion de Q et de tous ses cubes dyadiques voisins. Il est clair que Q et $\widehat{Q}$ ont une mesure comparable. Il est également facile de vérifier que tout cube dyadique Q a un nombre maximal de cubes dyadiques voisins uniformément borné.

1.2.2 Opérateurs d'intégrale singulière

Donnons maintenant la définition d'un opérateur d'intégrale singulière sur un espace de type homogène. Soit (X, ρ, μ) un espace de type homogène, pour $1 \leq p \leq \infty$, on note $L^p(X)$ l'espace des fonctions à valeurs complexes sur X p-intégrables par rapport à μ, $\|f\|_p$ la norme d'une fonction $f \in L^p(X)$. La dualité est donnée par la forme bilinéaire $\langle f, g\rangle = \int_X fg \mathrm{d}\mu$. Enfin, la moyenne d'une fonction f sur un ensemble E est notée $[f]_E = \mu(E)^{-1} \int_E f \mathrm{d}\mu$. Pour tout $x, y \in X$, posons

$$\lambda(x, y) = \mu(B(x, \rho(x, y))).$$

Il est immédiat que $\lambda(x, y)$ est comparable à $\lambda(y, x)$, uniformément par rapport à x, y.

Définition 1.2.4. Un noyau standard de Calderón-Zygmund sur X est une fonction continue $K : X \times X \backslash \{x = y\} \to \mathbb{C}$ telle qu'il existe une constante $\alpha > 0$ pour laquelle

$$|K(x, y)| \lesssim \frac{1}{\lambda(x, y)} \tag{1.2.2}$$

et

$$|K(x, y) - K(x', y)| + |K(y, x) - K(y, x')| \lesssim \left(\frac{\rho(x, x')}{\rho(x, y)}\right)^\alpha \frac{1}{\lambda(x, y)} \tag{1.2.3}$$

lorsque $\rho(x, x') \leq \rho(x, y)/2$, et $\rho(x, y) > 0$.

Soit Λ_α l'espace des fonctions bornées et α-hölderiennes sur X avec $\alpha \in]0, 1]$, espace de Banach pour la norme

$$\|f\|_{\Lambda_\alpha} = \|f\|_\infty + \sup_{x \neq y} \frac{|f(x) - f(y)|}{\rho(x, y)^\alpha}.$$

Soit $\mathcal{D}_\alpha$ le sous-espace de Λ_α constitué des fonctions à support compact. Tout comme l'espace des distributions $\mathcal{D}'$ est défini dans le cas Euclidien, introduisons l'espace dual $\mathcal{D}'_\alpha$ constitué des formes linéaires $l : \mathcal{D}_\alpha \to \mathbb{C}$, satisfaisant la propriété de continuité suivante : pour tout ensemble borné $E \subset X$, il existe une constante $C_E < +\infty$ telle que pour toute fonction $f \in \mathcal{D}_\alpha$, avec $\operatorname{supp} f \subset E$, on ait $|l(f)| \leq C_E \|f\|_{\Lambda_\alpha}$.

Notons $\langle h, g\rangle$ le crochet de dualité naturelle entre des éléments $h \in \mathcal{D}'_\alpha$, $g \in \mathcal{D}_\alpha$. Un opérateur d'intégrale singulière (sio) T sur X est un opérateur continu de $\mathcal{D}_\alpha$ dans $\mathcal{D}'_\alpha$ qui est associé à un noyau standard de Calderón-Zygmund $K(x, y)$, au sens où

$$\langle Tf, g\rangle = \iint K(x, y) f(y) g(x) \mathrm{d}\mu(x) \mathrm{d}\mu(y)$$

lorsque $f, g \in \mathcal{D}_\alpha$ à supports disjoints. Des estimations standards et un argument de densité classique montrent que l'on peut étendre T de $L^p(K)$ à $L^\infty_{loc}(K^c)$ pour tout compact K, si bien que

$$Tf(x) = \int K(x,y)f(y)\mathrm{d}\mu(y) \tag{1.2.4}$$

pour toute fonction $f \in L^p(X)$ avec $\operatorname{supp} f \subset K$ et μ-p.t. $x \notin K$. Dans la suite, on notera T^* l'opérateur adjoint de T. Moyennant ces définitions, et le cadre dyadique présenté ci-dessus, tous les théorèmes $T(b)$ donnés en Section 1.1 peuvent être généralisés au cadre des espaces de type homogène.

1.2.3 Estimations standards

Nous allons rappeler dans cette section les estimations standards de Calderón-Zygmund. Comme on l'a déjà dit, ces estimations liées à la régularité du noyau nous permettront de contrôler les coefficients matriciels éloignés de la diagonale qui apparaîtront dans notre argument.

Proposition 1.2.5. *Soit (X, ρ, μ) un espace de type homogène. Soit T un opérateur d'intégrale singulière sur X. Soient $r > 0, y_B \in X$ et soit B la boule ouverte de centre y_B et rayon r dans X. Soit f une fonction de moyenne nulle supportée sur B, et g une fonction supportée sur le complémentaire de $2B$. On a alors l'estimation standard de Calderón-Zygmund*

$$|\langle Tf, g\rangle| \lesssim \|f\|_1 \int_{(2B)^c} \left(\frac{r}{\rho(y_B, x)}\right)^\alpha \lambda(y_B, x)^{-1}|g(x)|\mathrm{d}\mu(x). \tag{1.2.5}$$

De même, dans le cadre dyadique, si Q est un cube dyadique dans X de centre z_Q, f une fonction de moyenne nulle supportée sur Q et g une fonction supportée sur le complémentaire de $\widehat{Q}$, on a

$$|\langle Tf, g\rangle| \lesssim \|f\|_1 \int_{\widehat{Q}^c} \left(\frac{l(Q)}{\rho(z_Q, x)}\right)^\alpha \lambda(z_Q, x)^{-1}|g(x)|\mathrm{d}\mu(x). \tag{1.2.6}$$

Démonstration. Montrons par exemple (1.2.5). Appliquons le fait que f soit de moyenne nulle sur B et écrivons

$$\begin{aligned}\langle Tf, g\rangle &= \int_{(2B)^c}\int_B K(x,y)f(y)g(x)\mathrm{d}\mu(y)\mathrm{d}\mu(x)\\ &= \int_{(2B)^c}\int_B (K(x,y) - K(x,y_B))f(y)g(x)\mathrm{d}\mu(y)\mathrm{d}\mu(x)\end{aligned}$$

On applique la régularité Hölderienne du noyau (1.2.3) pour obtenir

$$\begin{aligned}|\langle Tf, g\rangle| &\lesssim \int_{(2B)^c}\int_B |K(x,y) - K(x,y_B))|\,|f(y)|\,|g(x)|\,\mathrm{d}\mu(y)\mathrm{d}\mu(x)\\ &\lesssim \int_{(2B)^c}\int_B \left(\frac{\rho(y,y_B)}{\rho(x,y_B)}\right)^\alpha \frac{1}{\lambda(x,y_B)}\,|f(y)|\,|g(x)|\,\mathrm{d}\mu(y)\mathrm{d}\mu(x)\\ &\lesssim \|f\|_1 \int_{(2B)^c}\left(\frac{r}{\rho(y_B,x)}\right)^\alpha \lambda(y_B,x)^{-1}|g(x)|\mathrm{d}\mu(x).\end{aligned}$$

La démonstration de (1.2.6) est identique. $\square$

Introduisons maintenant une notation dont on se servira par la suite : pour Q, R deux cubes dyadiques de l'espace de type homogène X, posons

$$\mu(Q,R) = \inf_{x \in Q, y \in R} \{\mu(B(x,\rho(x,y))), \mu(B(y,\rho(x,y)))\}\,.$$

La Proposition 1.2.5 implique directement le résultat suivant.

Corollaire 1.2.6. *Soit (X,ρ,μ) un espace de type homogène. Soit T un opérateur d'intégrale singulière sur X. Soient Q, R deux cubes dyadiques de X tels que $\rho(Q,R) \geq l(R) \geq l(Q)$. Soient f supportée sur Q, de moyenne nulle, $f \in L^1(Q)$, et g supportée sur R, $g \in L^1(R)$. On a alors*

$$|\langle Tf, g\rangle| \lesssim \|f\|_{L^1(Q)}\|g\|_{L^1(R)} \left(\frac{\inf(l(Q),l(R))}{\rho(Q,R)}\right)^{\alpha} \frac{1}{\mu(Q,R)}. \tag{1.2.7}$$

1.3 Inégalités de type Hardy dans le cadre Euclidien

Comme on l'a dit précédemment, le principal outil qui sera utilisé pour estimer les coefficients matriciels proches de la diagonale obtenus dans notre argument sont les inégalités dites de type Hardy. Le prototype d'une inégalité de type Hardy est l'inégalité suivante en dimension 1 :

$$\left|\int_I\int_J \frac{f(y)g(x)}{x-y}\mathrm{d}y\mathrm{d}x\right| \leq C\left(\int_I |f(y)|^{\nu}\mathrm{d}y\right)^{\frac{1}{\nu}}\left(\int_J |g(x)|^{\nu'}\mathrm{d}x\right)^{\frac{1}{\nu'}}, \tag{1.3.1}$$

où I, J sont des intervalles adjacents, $\operatorname{supp} f \subset I$, $\operatorname{supp} g \subset J$, et $1 < \nu < \infty$, $\nu' = \frac{\nu}{\nu-1}$. L'intégrale est ici absolument convergente, et c'est une conséquence immédiate de la continuité de l'opérateur de Hardy $H(f)(x) = \frac{1}{x}\int_0^x f(t)\mathrm{d}t$ (d'où notre terminologie), qui n'utilise pas la régularité du noyau $\frac{1}{x-y}$: on se ramène par translation au cas $I =]-1,0[$, $J =]0,1[$, et il suffit alors d'écrire

$$\begin{aligned}
\left|\int_0^1\int_{-1}^0 \frac{f(y)g(x)}{x-y}\mathrm{d}y\mathrm{d}x\right| &\leq \int_{\mathbb{R}_+^*}\int_{\mathbb{R}_+^*} \frac{|f(y)|\,|g(x)|}{x+y}\mathrm{d}y\mathrm{d}x \\
&\leq \Big(\int_{\mathbb{R}_+^*} \frac{|g(x)|}{x}\int_0^x |f(y)|\mathrm{d}y\mathrm{d}x + \int_{\mathbb{R}_+^*}\frac{|f(y)|}{y}\int_0^y |g(x)|\mathrm{d}x\mathrm{d}y\Big) \\
&\leq \Big(\int_{\mathbb{R}_+^*} |g(x)|H(|f|)(x)\mathrm{d}x + \int_{\mathbb{R}_+^*} |f(y)|H(|g|)(y)\mathrm{d}y\Big) \\
&\leq C\|f\|_{L^{\nu}(]-1,0[)}\|g\|_{L^{\nu'}(]0,1[)}.
\end{aligned}$$

(1.3.1) se généralise aisément en dimension supérieure comme le montre la proposition suivante.

Proposition 1.3.1. *Pour tout $1 < \nu < +\infty$, il existe $C < +\infty$ telle que pour tous cubes dyadiques Q, Q' disjoints de $\mathbb{R}^n$, pour toutes fonctions $f \in L^{\nu}(Q)$ supportée sur Q, $g \in L^{\nu'}(Q')$ supportée sur Q', l'intégrale ci-dessous est convergente et on a*

$$\left|\int_Q\int_{Q'} \frac{f(y)g(x)}{|x-y|^n}\mathrm{d}x\mathrm{d}y\right| \leq C\|f\|_{L^{\nu}(Q)}\|g\|_{L^{\nu'}(Q')}. \tag{1.3.2}$$

En particulier, si $\alpha > 1$, il existe $C_\alpha < +\infty$ telle que pour toutes fonctions f supportée sur un cube dyadique Q, $f \in L^{\nu}(Q)$, et g supportée sur $\alpha Q \backslash Q$, $g \in L^{\nu'}(\alpha Q\backslash Q)$, on ait

$$\left|\int_Q\int_{\alpha Q\backslash Q} \frac{f(y)g(x)}{|x-y|^n}\mathrm{d}x\mathrm{d}y\right| \leq C_\alpha\|f\|_{L^{\nu}(Q)}\|g\|_{L^{\nu'}(\alpha Q\backslash Q)}. \tag{1.3.3}$$

Ce résultat est bien connu des spécialistes, mais on en trouve rarement la démonstration dans la littérature, c'est pourquoi on choisit de la redonner succinctement ici. On reprend l'argument utilisé dans [AT2] pour prouver (1.3.2). Soient donc Q, Q' deux cubes dyadiques disjoints de $\mathbb{R}^n$. Supposons que Q et Q' sont deux cubes voisins. Soient $f \in L^\nu(Q)$ supportée sur Q, $g \in L^{\nu'}(Q')$ supportée sur Q', et notons $I(f,g)$ le membre de gauche de (1.3.2). Soit Σ_1 la face du cube Q tangente à Q', et soit Σ_2 la face du cube Q' tangente à Q. Soit $e_n \in \mathbb{R}^n$ normal à Σ_1 tel que $Q \subset \{\bar{y} - y_n e_n \mid \bar{y} \in \Sigma_1, y_n > 0\}$ et $Q' \subset \{\bar{x} + x_n e_n \mid \bar{x} \in \Sigma_2, x_n > 0\}$. Notons σ la mesure de Lebesgue dans $\mathbb{R}^{n-1}$. Écrivons

$$\begin{aligned}
I(f,g) &\leq \int_Q \int_{Q'} \frac{|f(y)|\,|g(x)|}{|x-y|^n} \mathrm{d}x\mathrm{d}y \\
&\leq \int_{\bar{y}\in\Sigma_1} \int_{y_n>0} \int_{\bar{x}\in\Sigma_2} \int_{x_n>0} \frac{|f(\bar{y} - y_n e_n)|\,|g(\bar{x} + x_n e_n)|}{|\bar{x} - \bar{y} + (x_n + y_n)e_n|^n} \mathrm{d}x_n \mathrm{d}\sigma(\bar{x}) \mathrm{d}y_n \mathrm{d}\sigma(\bar{y}).
\end{aligned}$$

Posons $t = x_n + y_n$. On a

$$\begin{aligned}
\int_{\Sigma_2} \frac{\mathrm{d}\sigma(\bar{x})}{|\bar{x} - \bar{y} + (x_n + y_n)e_n|^n} &= \int_{\Sigma_2} \frac{\mathrm{d}\sigma(\bar{x})}{(|\bar{x} - \bar{y}|^2 + (x_n + y_n)^2)^{n/2}} \\
&= \int_{\Sigma_2} \frac{\mathrm{d}\sigma(\bar{x})}{(|\bar{x} - \bar{y}|^2 + t^2)^{n/2}} \\
&\leq \int_{\mathbb{R}^{n-1}} \frac{\mathrm{d}\sigma(v)}{(1 + |v|^2)^{n/2}} \frac{1}{t} \leq \frac{C_n}{t} = \frac{C_n}{x_n + y_n},
\end{aligned}$$

où l'on a fait le changement de variable $v = \frac{\bar{x} - \bar{y}}{t}$ dans la dernière ligne. Posons pour $x_n, y_n > 0$

$$\begin{aligned}
F(y_n) &= \Big(\int_{\Sigma_1} |f(\bar{y} - y_n e_n)|^\nu \mathrm{d}\sigma(\bar{y}) \Big)^{\frac{1}{\nu}} \in L^\nu(\mathbb{R}_+^*), \quad \text{avec } \|F\|_{L^\nu(\mathbb{R}_+^*)} = \|f\|_{L^\nu(Q)}, \\
G(y_n) &= \Big(\int_{\Sigma_2} |g(\bar{x} + x_n e_n)|^{\nu'} \mathrm{d}\sigma(\bar{x}) \Big)^{\frac{1}{\nu'}} \in L^{\nu'}(\mathbb{R}_+^*), \quad \text{avec } \|G\|_{L^{\nu'}(\mathbb{R}_+^*)} = \|g\|_{L^{\nu'}(Q')}.
\end{aligned}$$

Il vient, en appliquant Fubini et l'inégalité de Hölder,

$$\begin{aligned}
I(f,g) &\leq \int_{x_n>0} \int_{y_n>0} \int_{\bar{y}\in\Sigma_1} \int_{\bar{x}\in\Sigma_2} \frac{|f(\bar{y} - y_n e_n)|\,|g(\bar{x} + x_n e_n)|}{|\bar{x} - \bar{y} + (x_n + y_n)e_n|^n} \mathrm{d}\sigma(\bar{x})\mathrm{d}\sigma(\bar{y})\mathrm{d}y_n\mathrm{d}x_n \\
&\leq \int_{\mathbb{R}_+^*}\int_{\mathbb{R}_+^*}\Big(\int_{\Sigma_1}\int_{\Sigma_2} \frac{|f(\bar{y} - y_n e_n)|^\nu}{|x-y|^n} \mathrm{d}\sigma(\bar{x})\mathrm{d}\sigma(\bar{y})\Big)^{\frac{1}{\nu}}\Big(\int_{\Sigma_1}\int_{\Sigma_2} \frac{|g(\bar{x} + x_n e_n)|^{\nu'}}{|x-y|^n} \mathrm{d}\sigma(\bar{x})\mathrm{d}\sigma(\bar{y})\Big)^{\frac{1}{\nu'}}\mathrm{d}y_n\mathrm{d}x_n \\
&\leq C_n \int_{\mathbb{R}_+^*} \int_{\mathbb{R}_+^*} \frac{F(y_n)\,G(x_n)}{x_n + y_n} \mathrm{d}y_n\mathrm{d}x_n \\
&\leq C_n' \|F\|_{L^\nu(\mathbb{R}_+^*)} \|G\|_{L^{\nu'}(\mathbb{R}_+^*)} = C_n' \|f\|_{L^\nu(Q)} \|g\|_{L^{\nu'}(Q')},
\end{aligned}$$

où l'on a obtenu la dernière ligne en appliquant l'inégalité de type Hardy en dimension 1 (1.3.1) démontrée plus haut. Lorsque les cubes Q et Q' ne sont plus voisins, c'est évidemment la même chose, il suffit de prendre pour Σ un plan affine qui sépare les deux cubes, parallèle à l'une de leurs faces. On a donc démontré (1.3.2). Ensuite, remarquer qu'un cube dyadique de $\mathbb{R}^n$ a un nombre invariant de cubes dyadiques voisins et de même taille, il suffit donc de paver $\alpha Q \backslash Q$ par des cubes dyadiques de même taille que Q et d'appliquer (1.3.2) à chacun de ces cubes, dont le nombre est uniquement fonction de α, pour obtenir (1.3.3).

Appliquant l'inégalité (1.3.3) aux opérateurs d'intégrale singulière, on obtient immédiatement la propriété suivante sur $\mathbb{R}^n$.

Corollaire 1.3.2. *Soit $\alpha > 1$. Soit T un opérateur d'intégrale singulière sur $\mathbb{R}^n$. Soit $1 < \nu < +\infty$. Il existe $C_\alpha < +\infty$ telle que pour tout cube dyadique Q, et toute fonction f supportée sur Q avec $f \in L^\nu(Q)$, on ait*

$$\|Tf\|_{L^\nu(\alpha Q \setminus Q)} \leq C_\alpha \|f\|_{L^\nu(Q)}.$$

Notons que la démonstration de la Proposition 1.3.1 n'est pas directement généralisable au cadre des espaces de type homogène étant donné qu'il n'existe pas alors de direction transverse comme dans $\mathbb{R}^n$. Cependant, comme cet argument n'utilise pas la régularité du noyau, on peut quand même espérer étendre (1.3.1) au cadre homogène lorsque I, J sont des sous-ensembles disjoints raisonnables de l'espace.

Curieusement, un tel résultat ne semble être présent nulle part dans la littérature, en dehors du cas particulier où l'une au moins des deux fonctions dans (1.3.2) est dans L^∞, traité dans [Ch1] (Lemme 18). C'est donc l'objet du Chapitre 2 de cette thèse, motivé par la nécessité d'avoir ce type d'estimations dans notre argument pour le $T(b)$ local du Chapitre 3. On prouve le résultat souhaité dans le cadre dyadique, c'est notre Proposition 2.1.1. Motivé par la recherche d'un théorème Tb local avec un système pseudo-accrétif supporté sur les boules de l'espace au lieu des cubes dyadiques, on étudiera ensuite ce type d'inégalités pour une boule B et une couronne $2B \setminus B$. Le résultat dépend alors étroitement de la façon dont les points sont distribués à proximité de la frontière commune de ces deux ensembles. Cela nous amènera à considérer des propriétés géométriques exprimant la distribution des points dans un espace de type homogène, comme les propriétés de petite couche et de petite couronne, introduites par S. Buckley ([B2], [B1]), Section 2.3, ou la propriété de monotonie géodésique, étudiée par R. Tessera dans [Te] ainsi que H. Lin *et al.* dans [LNY], Section 2.4. On étudiera les liens entre ces différentes propriétés, et ce qu'elles impliquent en ce qui concerne les inégalités de type Hardy. Le résultat principal du Chapitre 2 est le Théorème 2.1.3, qui synthétise notre étude du lien entre la géométrie des espaces de type homogène et les inégalités de type Hardy.

Chapter 2

Distribution of points and Hardy type inequalities

Contents

2.1 Main results

The first objective of this chapter is to obtain a Hardy type inequality analogous to the Euclidean one (1.3.2), valid in general spaces of homogeneous type, to be able to estimate some of the coefficient estimates appearing in our argument for the local *Tb* theorem. It appears that such a result is true, as expressed by the following proposition.

Proposition 2.1.1. *Let (X, ρ, μ) be a space of homogeneous type. Let Q, Q' be two disjoint dyadic cubes in X. Let $1 < \nu < +\infty$, with dual exponent ν'. There exists $C < +\infty$ such that for every function f supported on Q, $f \in L^\nu(Q)$, and every function g supported on Q', $g \in L^{\nu'}(Q')$, we have*

$$\int_{Q'} \int_Q \frac{|f(y)g(x)|}{\lambda(x,y)} \mathrm{d}\mu(x)\mathrm{d}\mu(y) \leq C\|f\|_\nu \|g\|_{\nu'}. \tag{2.1.1}$$

The constant C only depends on C_D and ν.

We defer the proof of Proposition 2.1.1 to Section 2.2.

As it has been explained in Chapter 1, we also would like to know if we can get a similar result outside of the dyadic setting, in order to be able to give a local *Tb* theorem with hypotheses on balls rather than cubes. Let us precise what we mean with the following definition.

Definition 2.1.2. Hardy property.
Let (X, ρ, μ) be a space of homogeneous type. We say that X has the Hardy property (HP) if for every $1 < \nu < +\infty$, with dual exponent ν', there exists $C < +\infty$ such that for every ball B in X, with $2B$ denoting the concentric ball with double radius, and all functions f supported on B, $f \in L^\nu(B)$, g supported on $2B \backslash B$, $g \in L^{\nu'}(2B\backslash B)$, we have

$$\int_B \int_{2B\backslash B} \frac{|f(y)g(x)|}{\lambda(x,y)} \mathrm{d}\mu(y)\mathrm{d}\mu(x) \leq C\|f\|_\nu \|g\|_{\nu'}. \tag{H}$$

Obviously, if X has the Hardy property, (H) will remain true with $2B$ replaced by cB for any $c > 1$, with a different constant.

Our main result is the following:

Theorem 2.1.3. *Let (X, ρ, μ) be a space of homogeneous type. We have the following diagram of implications in X:*

$$\begin{tikzcd}
 & & \text{(HB)} \arrow[d, Rightarrow, shift left] \arrow[dr, Rightarrow, shift left] & \\
\text{(M)} \arrow[r, Rightarrow, shift left] & \text{(RAD)} \arrow[l, Rightarrow, shift left, "/" marking] \arrow[r, Rightarrow, shift left] \arrow[d, Rightarrow] & \text{(RLD)} \arrow[l, Rightarrow, shift left, "/" marking] \arrow[u, Rightarrow, shift left, "/" marking] \arrow[r, Rightarrow] \arrow[dl, Rightarrow, "/" marking] \arrow[d, Rightarrow, shift left] & \text{(HP)} \arrow[ul, Rightarrow, shift left, "/" marking] \\
 & \text{(AD)} \arrow[r, Rightarrow, shift left] & \text{(LD)} \arrow[l, Rightarrow, shift left, "/" marking] \arrow[u, Rightarrow, shift left, "/" marking] &
\end{tikzcd}$$

Let us comment. Theorem 2.1.3 sums up our study of sufficient conditions for the Hardy property (HP), and of the relationships between these conditions. Every property in this diagram except (HP) is related to the geometry and the distribution of points in the homogeneous space. (M) is a purely metric property, while the others are geometric conditions both metric and measure related. The properties (LD),(RLD), (AD) and (RAD) will be introduced in Section 2.3 in which we will prove that (RAD) $\Rightarrow$ (RLD) $\Rightarrow$ (HP). Next, the properties (HB) and (M) will be defined in Section 2.4, in which we will prove that (M) and (HB) both imply (RLD). Section 2.5 will be devoted to the presentation of a counterexample for some of the false implications in Theorem 2.1.3, while we will give in Section 2.6 examples of spaces not satisfying (HP) and complete the proof of the theorem. Next, in Section 2.7, we will give a partial answer to the natural question whether (H) for a particular couple of exponents (p,p') implies (HP). Finally, we will present some open questions we still have not managed to answer.

2.2 Proof of Proposition 2.1.1

Proof. [1] Let I be the integral in (2.1.1). For a locally integrable function f, we will denote by $M_\mu(f)$ the centered maximal function

$$M_\mu(f)(x) = \sup_{\tau>0} \frac{1}{\mu(B(x,\tau))} \int_{B(x,\tau)} |f(y)| \mathrm{d}\mu(y).$$

Recall that by the Hardy-Littlewood maximal theorem, M_μ is of strong type (p,p) for any $1 < p < +\infty$, and of weak type $(1,1)$. We refer for example to [S] for further details. We prove that for all $1 < r, s < \infty$, we have

$$I \lesssim \left\langle (M_\mu(|f|^r))^{1/r}, |g| \right\rangle + \left\langle |f|, (M_\mu(|g|^s))^{1/s} \right\rangle, \tag{2.2.1}$$

and the Hardy-Littlewood maximal theorem then gives the desired result, choosing $1 < r < \nu$ and $1 < s < \nu'$. Without loss of generality, we can assume $f, g \geq 0$. We have

$$I = \int_{\substack{x\in Q' \\ \rho(x,Q)>0}} g(x) \int_{\substack{y\in Q \\ \rho(y,Q')\leq\rho(x,Q)}} \frac{f(y)}{\lambda(x,y)} \mathrm{d}\mu(y)\mathrm{d}\mu(x) + \int_{\substack{y\in Q \\ \rho(y,Q')>0}} f(y) \int_{\substack{x\in Q' \\ \rho(y,Q')>\rho(x,Q)}} \frac{g(x)}{\lambda(x,y)} \mathrm{d}\mu(x)\mathrm{d}\mu(y)$$

Indeed, $\mu\left(\{x \in Q' \,|\, \rho(x,Q) = 0\}\right) = \mu\left(\{y \in Q \,|\, \rho(y,Q') = 0\}\right) = 0$, because of (1.2.1): we have for example

$$\{x \in Q' \,|\, \rho(x,Q) = 0\} \subset \bigcap_{j\in\mathbb{N}} \{x \in Q' \,|\, \rho(x, Q'^c) \leq 2^{-j}\delta^k\},$$

and all those sets have their measure bounded by $\mu(Q')2^{-j\eta} \underset{j\to\infty}{\longrightarrow} 0$. By symmetry, it is enough to estimate the first integral for example, which we call I_1. For $x \in Q'$, let $E_x = \{y \in Q \,|\, \rho(y, Q^c) \leq \rho(x,Q)\}$. For a fixed x in Q', we prove that for all $1 < r < \infty$,

$$I_1(x) = \int_{y\in E_x} \frac{f(y)}{\lambda(x,y)} \mathrm{d}\mu(y) \lesssim M_\mu(f^r)(x)^{1/r},$$

and (2.2.1) will follow. Consider the dyadic subcubes Q_α of Q, which are maximal for the relation $l(Q_\alpha) \leq \rho(Q_\alpha, x)$. Call them $Q^l_\alpha(x)$, where l denotes their generation. They realize what we call a Whitney partition of the cube Q, as for μ-almost every $y \in Q$, there exists a sufficiently small cube Q_y containing y such that $l(Q_y) \leq \rho(x,Q) \leq \rho(x,Q_y)$, and Q_y is then included in one of those maximal $Q^l_\alpha(x)$. The case where there is a unique cube $Q^l_\alpha(x) = Q$ means that $l(Q) \leq \rho(x,Q)$. Let $n \in \mathbb{N}$ be such that $2^n l(Q) \leq \rho(x,Q) < 2^{n+1} l(Q)$. Then

1. We thank J.M. Martell for the suggestion of using Whitney coverings that led to an improvement of our earlier argument.

$\lambda(x,y) \geq \mu(B(x,2^n l(Q)))$ while $Q \subset B(x,(C_1+2^{n+1})l(Q))$ for the dimensional constant C_1 of Lemma 1.2.3. Hence, by the doubling property,

$$I_1(x) \leq \frac{1}{\mu(B(x,2^n l(Q)))} \int_{B(x,(C_1+2^{n+1})l(Q))} |f(y)| \mathrm{d}\mu(y) \lesssim M_\mu f(x).$$

The second case is when Q is not a maximal cube. By maximality, if $Q_\beta^{l-1} \subset Q$ is the unique parent of a $Q_\alpha^l(x)$, then we have $\rho(x,Q_\beta^{l-1}) < l(Q_\beta^{l-1})$, and thus $\rho(x,Q_\alpha^l(x)) \lesssim \rho(x,Q_\beta^{l-1}) + l(Q_\beta^{l-1}) \lesssim l(Q_\beta^{l-1}) = \delta^{-1} l(Q_\alpha^l(x))$. Note that this implies that for every l such that there exists a maximal cube $Q_\alpha^l(x)$, we have $\rho(x,Q) \leq \rho(x,Q_\alpha^l(x)) \lesssim \delta^l$ with implicit constant independent of x and l. For a fixed l as above, let

$$C^l(x) = \bigcup_{\alpha: Q_\alpha^l(x) \cap E_x \neq \varnothing} Q_\alpha^l(x).$$

Observe that if $y \in Q_\alpha^l(x) \cap E_x$, then $\rho(y,(Q_\alpha^l(x))^c) \leq \rho(y,Q^c) \leq \rho(x,Q)$. Thus, $\mu(Q_\alpha^l(x) \cap E_x) \lesssim \left(\frac{\rho(x,Q)}{\delta^l}\right)^\eta \mu(Q_\alpha^l(x))$ by the small boundary property (1.2.1) of Christ's dyadic cubes. Summing over the cubes in $C^l(x)$, we have

$$\mu(C^l(x)) \lesssim \left(\frac{\rho(x,Q)}{\delta^l}\right)^\eta \mu(C^l(x)) \lesssim \left(\frac{\rho(x,Q)}{\delta^l}\right)^\eta \mu(B(x,\delta^l)),$$

the last inequality being a consequence of doubling and the fact that each of the cubes $Q_\alpha^l(x)$ in $C^l(x)$ is of length δ^l and at a distance comparable to δ^l from x, so that $C^l(x) \subset B(x, C\delta^l)$ for some C independent of l and x. Now, write

$$\begin{aligned}
I_1(x) &\lesssim \sum_l \int_{C^l(x)\cap E_x} \frac{|f(y)|}{\mu(B(x,\rho(x,y)))} \mathrm{d}\mu(y) \\
&\lesssim \sum_l \frac{1}{\mu(B(x,\delta^l))} \int_{B(x,C\delta^l)} |f| 1_{C^l(x)\cap E_x} \mathrm{d}\mu \\
&\lesssim \sum_l \left(\frac{1}{\mu(B(x,\delta^l))} \int_{B(x,C\delta^l)} |f|^r \mathrm{d}\mu\right)^{1/r} \left(\frac{\mu(C^l(x)\cap E_x)}{\mu(B(x,\delta^l))}\right)^{\frac{1}{r'}} \\
&\lesssim M_\mu(|f|^r)(x)^{1/r} \sum_l \left(\frac{\rho(x,Q)}{\delta^l}\right)^{\frac{\eta}{r'}},
\end{aligned}$$

where the last inequality is obtained by applying the Hölder inequality with $r > 1$. Now, observe that every integer l intervening in the above sum is such that $\rho(x,Q) \lesssim \delta^l$. Summing over such integers provides us with a geometric sum, uniformly bounded with respect to x. Hence,

$$I_1(x) \lesssim M_\mu(|f|^r)(x)^{1/r}.$$

This concludes our proof. □

2.3 Sufficient conditions for (HP)

2.3.1 Layer decay properties (LD), (RLD)

It is not clear when (H) is true in a space of homogeneous type for a given ball B. In fact, it is false in general, as will be illustrated by some counterexamples in the following sections. It obviously depends on how the sets B and B^c see each other in X. By analogy with Christ's dyadic cubes, natural objects are the outer and inner layers $\{x \in B | \rho(x, B^c) \leq \varepsilon\}$ and $\{y \in B^c | \rho(y, B) \leq \varepsilon\}$. We shall assume they tend to zero in measure as $\varepsilon \to 0$, and in a scale invariant way, as expressed by the following definition.

Definition 2.3.1. Layer decay and relative layer decay properties.
Let (X, ρ, μ) be a space of homogeneous type. For a ball B in X, set $B_\varepsilon = \{x \in B | \rho(x, B^c) \leq \varepsilon\} \cup \{y \in B^c | \rho(y, B) \leq \varepsilon\}$ the union of the inner and outer layers.

– We say that X has the **layer decay property** if there exist constants $\eta > 0$, $C < +\infty$ such that for every ball $B = B(z, r)$ in X and every $\varepsilon > 0$, we have

$$\mu(B_\varepsilon) \leq C \left(\frac{\varepsilon}{r}\right)^\eta \mu(B(z, r)). \tag{LD}$$

– We say that X has the **relative layer decay property** if there exist constants $\eta > 0$, $C < +\infty$ such that for every ball $B = B(z, r)$ in X, every ball $B(w, R)$ with $R \leq 2r$, and every $\varepsilon > 0$, we have

$$\mu\left(B_\varepsilon \cap B(w, R)\right) \leq C \left(\frac{\varepsilon}{R}\right)^\eta \mu(B(w, R)). \tag{RLD}$$

The layer decay property already appeared in [B1] (with only $\mu\left(\{x \in B | \rho(x, B^c) \leq \varepsilon\}\right)$ in the left hand side of (LD)). One can note that while (LD) is a global condition, a sort of averaging property over the whole layer, (RLD) is a rather local condition. Remark also that (RLD) implies (LD). As a matter of fact, let $z \in X, r > 0, 0 < \varepsilon < r$, denote $B = B(z, r)$ and suppose that $\varepsilon < r/2$ (else the inequality is trivial). By the well known Vitali covering lemma (see [S] for example), if B_ε is non empty, there exist points $w_k \in B_\varepsilon$ such that the balls $B(w_k, r/10)$ are mutually disjoint and B_ε is covered by the union $\cup_k B(w_k, r/2)$. Now, write

$$\begin{aligned}
\mu(B_\varepsilon) = \mu(B_\varepsilon \cap (\cup_k B(w_k, r/2))) &\leq \sum_k \mu(B_\varepsilon \cap B(w_k, r/2)) \\
\lesssim \sum_k \left(\frac{\varepsilon}{r}\right)^\eta \mu(B(w_k, r/2)) &\lesssim \left(\frac{\varepsilon}{r}\right)^\eta \sum_k \mu(B(w_k, r/10)) \\
\lesssim \left(\frac{\varepsilon}{r}\right)^\eta \mu(B),
\end{aligned}$$

where the second inequality comes from (RLD), the third from the doubling property of μ, and the last from the disjointness of the balls $B(w_k, r/10)$ and the fact that they are all inside a ball of measure comparable to $\mu(B)$. Because of the opposed local and global nature of (RLD) and (LD) though, it is sensible to think that these properties should not be equivalent. We will prove it in Section 2.6.2.

It turns out that the relative layer decay property constitutes a sufficient condition for the Hardy property (HP):

Proposition 2.3.2. *Let (X, ρ, μ) be a space of homogeneous type, and suppose that X has the relative layer decay property* (RLD). *Then the Hardy property* (HP) *is satisfied on X.*

Proof. The proof is quite similar to the proof of Proposition 2.1.1, with modifications owing to the fact one cannot use exact coverings with balls as for dyadic cubes. Let again I denote the integral in (H). Fix a ball B of center z and radius $r > 0$, and functions f, g respectively supported on $2B\backslash B$ and B with $f \in L^{\nu}$, $g \in L^{\nu'}$. As before, we can assume that $f, g \geq 0$ and it suffices to prove that for all $1 < \sigma, s < \infty$, we have

$$I \lesssim \left\langle (M_\mu(f^\sigma))^{1/\sigma}, g \right\rangle + \left\langle f, (M_\mu(g^s))^{1/s} \right\rangle. \tag{2.3.1}$$

The hypotheses imply $\mu(\overline{B}\backslash B) = 0$ and allow us to write

$$I = \int_{\substack{x\in B \\ \rho(x,B^c)>0}} g(x) \int_{\substack{y\in 2B\backslash B \\ \rho(y,B)\leq\rho(x,B^c)}} \frac{f(y)}{\lambda(x,y)} \mathrm{d}\mu(y)\mathrm{d}\mu(x) + \int_{\substack{y\in 2B\backslash B \\ \rho(y,B)>0}} f(y) \int_{\substack{x\in B \\ \rho(y,B)>\rho(x,B^c)}} \frac{g(x)}{\lambda(x,y)} \mathrm{d}\mu(x)\mathrm{d}\mu(y).$$

Let us begin by estimating the first term. Fix $x \in B$. As before, let

$$E_x = \{y \in 2B\backslash B \mid \rho(y,B) \leq \rho(x,B^c)\}.$$

We decompose the integral in y over coronae at distance $2^j\rho(x,B^c)$ from x:

$$\begin{aligned}
I_1(x) &= \int_{y\in E_x} \frac{f(y)}{\lambda(x,y)} \mathrm{d}\mu(y) \\
&\lesssim \sum_{j\geq 0} \int_{\substack{y\in E_x \\ 2^j\rho(x,B^c)\leq\rho(x,y)<2^{j+1}\rho(x,B^c)}} \frac{f(y)}{\mu(B(x,\rho(x,y)))} \mathrm{d}\mu(y) \\
&\lesssim \sum_{\substack{j\geq 0 \\ z\notin B(x,2^{j+1}\rho(x,B^c))}} M_\mu(f^\sigma)(x)^{1/\sigma} \left(\frac{\mu(E_x \cap B(x,2^{j+1}\rho(x,B^c)))}{\mu(B(x,2^{j+1}\rho(x,B^c)))} \right)^{\frac{1}{\sigma'}} \\
&+ \sum_{\substack{j\geq 0 \\ z\in B(x,2^{j+1}\rho(x,B^c)))}} \frac{1}{\mu(B(x,2^{j+1}\rho(x,B^c)))} \int_{\substack{y\in E_x \\ 2^j\rho(x,B^c)\leq\rho(x,y)<2^{j+1}\rho(x,B^c)}} f\mathrm{d}\mu,
\end{aligned}$$

where the last inequality is obtained applying the Hölder inequality with $\sigma > 1$. Observe that there are at most four integers $j \geq 0$ such that $z \in B(x, 2^{j+1}\rho(x,B^c)))$ and $E_x \cap \{y \in B^c | 2^j\rho(x,B^c) \leq \rho(x,y) < 2^{j+1}\rho(x,B^c)\} \neq \varnothing$. Indeed, let j_0 be the first such integer, which implies $\rho(x,z) \leq 2^{j_0+1}\rho(x,B^c)$, and let $j \geq j_0$ be another one. Let $y \in E_x \cap \{y \in B^c | 2^j\rho(x,B^c) \leq \rho(x,y) < 2^{j+1}\rho(x,B^c)\}$. Using $y \in E_x$, we have $\rho(z,y) \leq r + \rho(y,B) \leq r + \rho(x,B^c)$. Also, $r \leq \rho(x,z) + \rho(x,B^c)$. Hence $\rho(z,y) \leq \rho(x,z) + 2\rho(x,B^c)$. Using $2^j\rho(x,B^c) \leq \rho(x,y)$, we obtain

$$2^j\rho(x,B^c) \leq \rho(x,y) \leq \rho(x,z) + \rho(z,y) \leq 2\rho(x,z) + 2\rho(x,B^c) \leq (2^{j_0+2}+2)\rho(x,B^c),$$

hence $j \leq j_0 + 3$.
Moreover, if $z \notin B(x, 2^{j+1}\rho(x,B^c))$, since $x \in B$, we have $2^{j+1}\rho(x,B^c) \leq 2r$. Consequently, we can apply (RLD) and we get

$$I_1(x) \lesssim M_\mu(f^\sigma)(x)^{1/\sigma} \sum_{j\geq 0} \left(\frac{\rho(x,B^c)}{2^{j+1}\rho(x,B^c)} \right)^{\frac{\eta}{\sigma'}} + 4M_\mu f(x) \lesssim M_\mu(f^\sigma)(x)^{1/\sigma},$$

so the first integral is controlled by $\langle g, (M_\mu(f^\sigma))^{1/\sigma}\rangle$. The argument for the second integral is entirely similar using the symmetry of our assumptions (and 4 above becomes 3). This proves (2.3.1). ☐

Remark 2.3.3. An interesting remark is that if f and g are taken in $L^{\nu_1}(X)$ and $L^{\nu_2}(X)$ with $1/\nu_1 + 1/\nu_2 < 1$ (and g no longer in $L^{\nu_1'}$), then if I denotes the integral in (H), the normalised inequality

$$\frac{1}{\mu(B)} I \lesssim \|f\|_{L^{\nu_1}(2B\backslash B, \frac{\mathrm{d}\mu}{\mu(B)})} \|g\|_{L^{\nu_2}(B, \frac{\mathrm{d}\mu}{\mu(B)})}$$

is true in any space of homogeneous type only satisfying (LD). The proof is much easier: let $B = B(z,r)$ be a ball in X, f be supported on $2B\backslash B$, with $f \in L^{\nu_1}(2B\backslash B)$, g be supported on B with $g \in L^{\nu_2}(B)$, denote $1/\nu = 1 - 1/\nu_1 - 1/\nu_2$, $S_j = \{x \in B : \rho(x, B^c) \leq 2^{-j}r\}$. Observe that by (LD), $\mu(S_j) \lesssim 2^{-j\eta}\mu(B)$, and splitting the integral in y over coronae at distance $2^{-j}r$ from x, write

$$\begin{aligned}
I &\lesssim \sum_{j\geq -2} \int_{x\in B} |g(x)| \int_{\substack{y\in 2B\backslash B \\ 2^{-j-1}r<\rho(x,y)\leq 2^{-j}r}} \frac{|f(y)|}{\mu(B(x, 2^{-j}r))} \mathrm{d}\mu(y)\mathrm{d}\mu(x) \\
&\lesssim \sum_{j\geq -2} \int_{x\in S_j} |g(x)| M_\mu f(x) \mathrm{d}\mu(x) \lesssim \sum_{j\geq -2} \int_{x\in X} |g(x)| M_\mu f(x) 1_{S_j}(x) \mathrm{d}\mu(x) \\
&\lesssim \|f\|_{\nu_1} \|M_\mu g\|_{\nu_2} \sum_{j\geq -2} \mu(S_j)^{1/\nu},
\end{aligned}$$

where we have applied the Hölder inequality to get the last line. It remains to sum in j to obtain the desired estimate. The same remark with identical proof holds if $B, 2B\backslash B$ are replaced by $Q, \widehat{Q}\backslash Q$ for Q a dyadic cube.

2.3.2 Annular decay properties (AD), (RAD)

Interestingly, another very similar condition already appeared in the literature (see for example [MS], [DJS], [B2], [Te]). It is the notion of annular decay that we recall now.

Definition 2.3.4. Annular decay and relative annular decay properties.
Let (X, ρ, μ) be a space of homogeneous type. For $z \in X$ and $r > 0, 0 < s < r$, set $C_{r,r-s}(z) = B(z,r)\backslash B(z, r-s)$.

- We say that X has the **annular decay property** if there exist constants $\eta > 0$ and $C < +\infty$ such that for every $z \in X, r > 0, 0 < s < r$, we have

$$\mu(C_{r,r-s}(z)) \leq C \left(\frac{s}{r}\right)^\eta \mu(B(z,r)). \tag{AD}$$

- We say that X has the **relative annular decay property** if there exist constants $\eta > 0$ and $C < +\infty$ such that for every $z \in X$, $r > 0, 0 < s < r$, and every ball $B(w, R)$ with $R \leq 2r$, we have

$$\mu(C_{r,r-s}(z) \cap B(w,R)) \leq C \left(\frac{s}{R}\right)^\eta \mu(B(w,R)). \tag{RAD}$$

Note that this condition (AD) was made an assumption in [DJS] for the first proof of the global Tb theorem in a space of homogeneous type. Again, (AD) is a global property while (RAD) is a local one. Similarly as for layer decay properties, we have that (RAD) implies (AD). Observe that for a ball $B = B(x,r)$, with $x \in X$, $r > 0$, we have, if $\varepsilon > 0$,

$$B_\varepsilon = \{y \in C_{r,r-\varepsilon}(x) \mid \rho(y, B^c) \leq \varepsilon\} \cup \{y \in C_{r+2\varepsilon,r}(x) \mid \rho(y, B) \leq \varepsilon\}.$$

It follows that $B_\varepsilon \subset C_{r+2\varepsilon,r-\varepsilon}(x)$ and thus (RAD) (respectively (AD)) implies (RLD) (respectively (LD)). In particular, (RAD) is a sufficient condition for the Hardy property (HP) because of Proposition 2.3.2.

2.4 Sufficient conditions for (RLD)

2.4.1 Monotone geodesic property (M)

Now, we would like to find some more geometric conditions ensuring that (RLD) is satisfied.

Definition 2.4.1. Let (X, ρ) be a metric space. We say that X has the **monotone geodesic property** (M) if there exists a constant $0 < C < +\infty$ such that for all $u > 0$ and all $x, y \in X$ with $\rho(x, y) \geq u$, there exists a point $z \in X$ such that

$$\rho(z, y) \leq Cu \quad \text{and} \quad \rho(z, x) \leq \rho(y, x) - u. \tag{M}$$

Remark that C must satisfy $C \geq 1$. Remark also that iterating this property, one gets that for every $x, y \in X$ with $\rho(x, y) \geq u$, there exists a sequence of points $y_0 = y, y_1, ..., y_m = x$ such that for every $i \in \{0, ..., m-1\}$

$$\rho(y_{i+1}, y_i) \leq Cu \quad \text{and} \quad \rho(y_{i+1}, x) \leq \rho(y_i, x) - u.$$

In [B2], Buckley introduces the notion of chain ball spaces and proves that under that condition, a doubling metric measure space satisfies (AD). Colding and Minicozzi II already had proved that this property was satisfied by doubling complete riemannian manifolds in [CM]. Tessera introduced a notion of monotone geodesic property in [Te], and proved that this property also implies (AD) (called there the Føllner property for balls) in a doubling metric measure space. Lin, Nakai and Yang recently showed in [LNY] that chain ball and a slightly stronger scale invariant version of the monotone geodesic are equivalent. This is the latter that we introduced above. Observe that this is a purely metric property of the homogeneous space. It appears that (M) not only yields the annular decay property, but also, as we next show, the stronger relative annular decay property, thus improving [Te].

Proposition 2.4.2. *Let (X, ρ, μ) be a space of homogeneous type, and suppose that X has the monotone geodesic property* (M)*. Then X has the relative annular decay property* (RAD)*.*

It immediately yields the following corollary.

Corollary 2.4.3. *Let (X, ρ, μ) be a space of homogeneous type. If X has the monotone geodesic property* (M)*, then X satisfies the Hardy property* (HP)*.*

Proof of Corollary 2.4.3. This is a straightforward application of Proposition 2.3.2 and Proposition 2.4.2. □

Proof of Proposition 2.4.2. Let $C_0 \geq 1$ be the constant intervening in the monotone geodesic property of the homogeneous type space X. We already know that X has the annular decay property (see [Te] for example), say with constants $C_1, \eta_1 > 0$. Note that (AD) is still valid for all $\eta \in [0, \eta_1]$, so that one can make η_1 as small as we wish. We prove (RAD). The argument adapts the one in [Te], with more care on localization. Fix $z \in X$, $r > 0, 0 < s < r$, and $w \in X$, $0 < R \leq 2r$.

We suppose first that $r/4 \le R \le 2r$. If $C_{r,r-s}(z) \cap B(w,R) = \varnothing$, the result is trivial, so we assume this intersection to be non empty. This implies that $B(z,r) \subset B(w,R{+}2r) \subset B(w,9R)$. Then, applying (AD), we have

$$\begin{aligned}\mu(C_{r,r-s}(z) \cap B(w,R)) &\le \mu(C_{r,r-s}(z)) \le C_1 \left(\frac{s}{r}\right)^{\eta_1} \mu(B(z,r)) \\ &\le C_1 \left(\frac{s}{R}\right)^{\eta_1} \left(\frac{R}{r}\right)^{\eta_1} \mu(B(w,9R)) \\ &\lesssim \left(\frac{s}{R}\right)^{\eta_1} \mu(B(w,R)).\end{aligned}$$

Assume now that $R < r/4$. Observe that if $6C_0 s \ge R$, (RAD) is trivial, so that we can freely assume $6C_0 s < R$ in the following. This implies in particular that $s < R < r/4$. Once again, if $C_{r,r-s}(z) \cap B(w,R) = \varnothing$, the result is trivial, so we assume this intersection to be non empty. This implies that $z \notin B(w,R)$ because if $y \in C_{r,r-s}(z) \cap B(w,R)$, we have $\rho(w,z) \ge \rho(z,y) - \rho(y,w) \ge r - s - R \ge 4R - R - R \ge 2R$. We will prove (RAD) in two steps. We first show that there exist $\theta < 1$ and $C < +\infty$ such that for all $\sigma > 0$ with $6C_0\sigma < R$ (which implies $\sigma < r/2$), we have

$$\mu(C_{r,r-\sigma}(z) \cap B(w,R)) \le \theta\mu(C_{r,r-2\sigma}(z) \cap B(w,R)) + C\left(\frac{\sigma}{R}\right)^{\eta_1} \mu(B(w,R)). \tag{2.4.1}$$

Assuming these conditions on σ are met, we begin by establishing the existence of a dimensional constant $A > 0$ such that

$$\mu(C_{r,r-\sigma}(z) \cap B(w, R - 6C_0\sigma) \le A\mu(C_{r-\sigma,r-2\sigma} \cap B(w,R)).$$

If $C_{r,r-\sigma}(z) \cap B(w, R - 6C_0\sigma) = \varnothing$, there is nothing to do, so we will suppose that this intersection is not empty. Let $y \in C_{r,r-\sigma}(z) \cap B(w, R - 6C_0\sigma)$, and let $0 < u < \min(\sigma/4, C_0(r-\sigma))$. Since $\rho(z,y) \ge r - \sigma > u/C_0$, by monotone geodesicity of X there exist points $y_0 = y, y_1, ..., y_m = z$ such that for every $i \in \{0, ..., m-1\}$,

$$\rho(z, y_{i+1}) \le \rho(z, y_i) - \frac{u}{C_0} \quad \text{and} \quad \rho(y_i, y_{i+1}) \le u.$$

Let i_0 be the first integer such that $y_{i_0} \in B(z, r - 3\sigma/2)$. Note that $u < \sigma/4$ implies $i_0 > 2$. Write

$$r - 3\sigma/2 \le \rho(z, y_{i_0-1}) \le \rho(z,y) - \frac{(i_0-1)u}{C_0} \le r - \frac{(i_0-1)u}{C_0},$$

so that $\frac{3\sigma}{2} \ge \frac{(i_0-1)u}{C_0}$, and

$$i_0 u \le \frac{i_0}{i_0-1}\frac{3C_0\sigma}{2} < 3C_0\sigma,$$

because $i_0 > 2$. Thus $\rho(y, y_{i_0}) \le i_0 u < 3C_0\sigma$ and it follows that $y_{i_0} \in B(w, R - 3C_0\sigma)$. Furthermore,

$$\rho(z, y_{i_0}) \ge \rho(z, y_{i_0-1}) - \rho(y_{i_0-1}, y_{i_0}) \ge r - \frac{3\sigma}{2} - u \ge r - \frac{7\sigma}{4}.$$

It means that for every $y \in C_{r,r-\sigma}(z) \cap B(w, R - 6C_0\sigma)$, there exists $y' \in C_{r-3\sigma/2, r-7\sigma/4}(z) \cap B(w, R - 3C_0\sigma)$ such that $\rho(y,y') < 3C_0\sigma$. Then,

$$B_y = B(y', \sigma/4) \subset C_{r-\sigma,r-2\sigma}(z) \cap B(w,R).$$

This construction is illustrated in Figure 2.1.

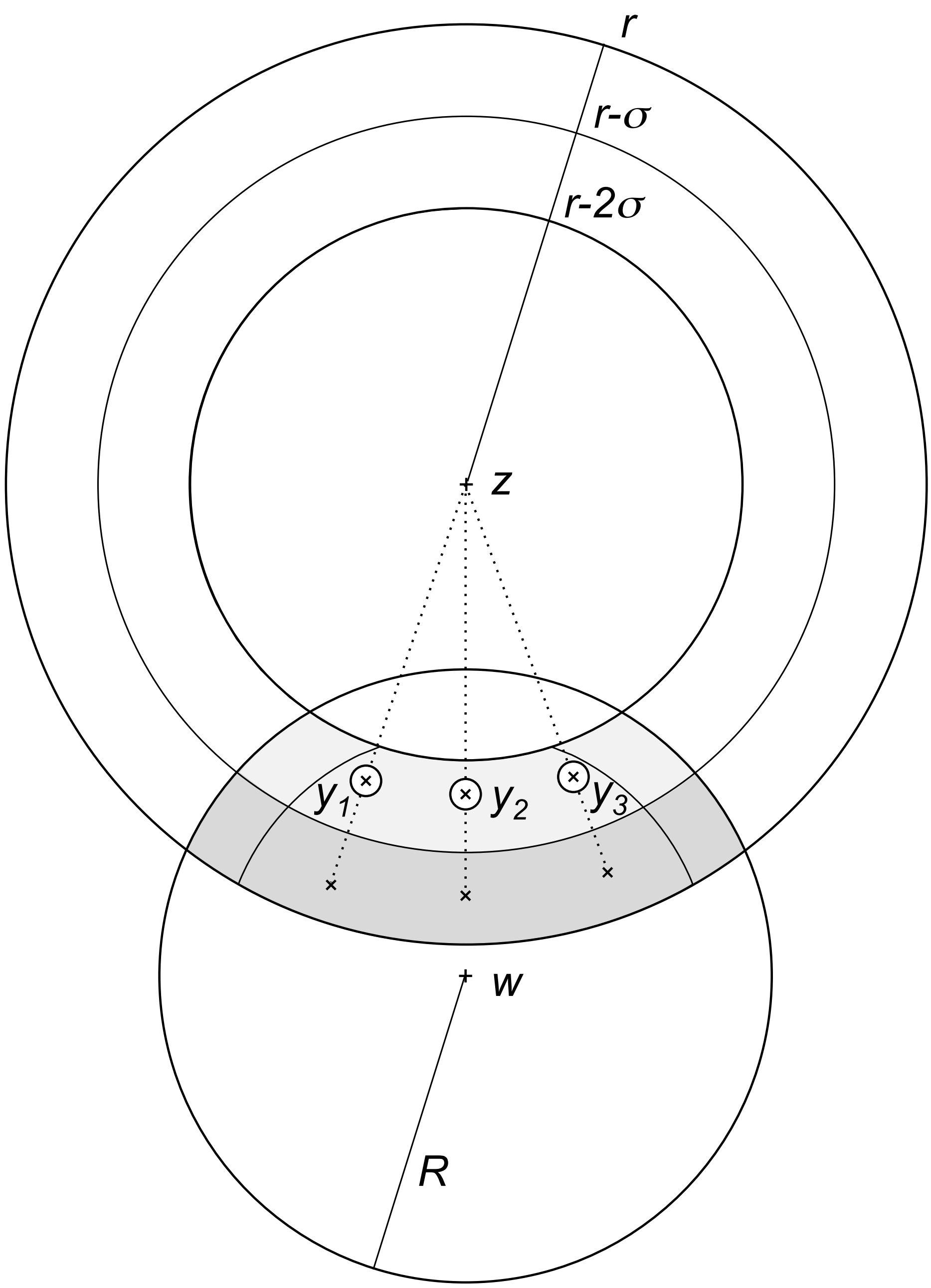

Figure 2.1: Use of the monotone geodesic property (M) to prove (RAD).

Now if we consider the union of all those balls B_y for $y \in C_{r,r-\sigma}(z) \cap B(w, R - 6C_0\sigma)$, by the Vitali covering lemma we can find points y_k such that the balls B_{y_k} are mutually disjoint and $\bigcup_y B_y \subset \bigcup_k 5B_{y_k}$. But since for every k, $\rho(y_k, y_k') < 3C_0\sigma$,

$$C_{r,r-\sigma}(z) \cap B(w, R - 6C_0\sigma) \subset \bigcup_k CB_{y_k},$$

with $C = 5 + 12C_0$. Applying the doubling measure μ and remembering that the B_{y_k} are mutually disjoint sets contained in $C_{r-\sigma,r-2\sigma}(z) \cap B(w, R)$, we obtain

$$\mu(C_{r,r-\sigma}(z) \cap B(w, R - 6C_0\sigma)) \leq \sum_k \mu(CB_{y_k}) \leq A \sum_k \mu(B_{y_k}) \leq A\mu(C_{r-\sigma,r-2\sigma}(z) \cap B(w, R)).$$

Applying the annular decay property inside the ball $B(w, R)$, write then

$$\begin{aligned}
&\mu(C_{r,r-\sigma}(z) \cap B(w, R)) = \mu(C_{r,r-\sigma}(z) \cap B(w, R - 6C_0\sigma)) + \mu(C_{R,R-6C_0\sigma}(w)) \\
&\leq A\mu(C_{r-\sigma,r-2\sigma}(z) \cap B(w, R)) + C_1 \left(\frac{6C_0\sigma}{R}\right)^{\eta_1} \mu(B(w, R)) \\
&\leq A(\mu(C_{r,r-2\sigma}(z) \cap B(w, R)) - \mu(C_{r,r-\sigma}(z) \cap B(w, R))) + C_1 \left(\frac{6C_0\sigma}{R}\right)^{\eta_1} \mu(B(w, R)),
\end{aligned}$$

so that (2.4.1) holds for $\theta = \frac{A}{1+A} < 1$ and $C = \frac{C_1(6C_0)^{\eta_1}}{1+A} < +\infty$.

The second step in the proof now consists in iterating this inequality. Let $d = \rho(w, B(z, r)) \geq 0$. Let us distinguish two cases. If $d \geq \left(1 - (\frac{s}{R})^{\frac{1}{2}}\right) R$, then $C_{r,r-s}(z) \cap B(w, R) \subset C_{R,R-(\frac{s}{R})^{1/2}R}(w)$ as one easily checks. Applying the annular decay property in the ball $B(w, R)$, we have

$$\mu(C_{r,r-s}(z) \cap B(w, R)) \leq C_1 \left(\frac{s}{R}\right)^{\frac{\eta_1}{2}} \mu(B(w, R)),$$

so that there is no iteration to make.
Suppose now that $d < \left(1 - (\frac{s}{R})^{\frac{1}{2}}\right) R$. Observe that $R \leq \rho(w, z) \leq r + d$, so that $R - d \leq r$. Thus, if n is an integer such that $2^n s < \frac{R-d}{3C_0}$, we also have $2^n s < \frac{r}{3C_0} < r$, so that we can apply (2.4.1) with $\sigma = 2^{k-1}s$ for $k = 1, ..., n$, and iterate to obtain

$$\mu(C_{r,r-s}(z) \cap B(w, R)) \leq \theta^n \mu(C_{r,r-2^n s}(z) \cap B(w, R)) + \left(\sum_{k=0}^{n-1} \theta^k 2^{k\eta_1}\right) C \left(\frac{s}{R}\right)^{\eta_1} \mu(B(w, R)).$$

Consequently, let $n_0 = [\log_2(\frac{R-d}{3C_0 s})] - 1$, where $[x]$ denotes the integer part of x, note that $R - d > (\frac{s}{R})^{1/2}R$, and write

$$\begin{aligned}
\mu(C_{r,r-s}(z) \cap B(w, R)) &\leq \theta^{n_0} \mu(C_{r,r-2^{n_0}s}(z) \cap B(w, R)) + \left(\sum_{k=0}^{n_0-1} \theta^k 2^{k\eta_1}\right) C \left(\frac{s}{R}\right)^{\eta_1} \mu(B(w, R)) \\
&\leq \frac{1}{\theta^2} \left(\frac{3C_0 s}{(\frac{s}{R})^{1/2}R}\right)^{-\log_2\theta} \mu(B(w, R)) + \frac{C}{1 - \theta 2^{\eta_1}} \left(\frac{s}{R}\right)^{\eta_1} \mu(B(w, R)) \\
&\lesssim \left(\frac{s}{R}\right)^{\frac{\eta_1}{2}} \mu(B(w, R)),
\end{aligned}$$

provided $\eta_1 < -\log_2\theta$, which we may assume by a previous remark, and because we have supposed $6C_0 s < R$. $\square$

Remarks 2.4.4. 1. We have proven the desired result with $\eta = \eta_1/2$ and $\eta_1 < -\log_2 \theta$, but remark that we can clearly get $\eta = \eta_1 - \varepsilon$ for $\varepsilon > 0$ as small as desired (provided $\eta_1 < -\log_2 \theta$).

2. We have obtained a rather satisfying geometric condition ensuring that (HP) is satisfied: this monotone geodesic property is obviously satisfied by complete doubling riemannian manifolds. It is also satisfied by any geodesic space or length space (see [BBI] for a definition). We have also shown the stronger relative annular decay property in such cases.
3. Observe that conversely, neither (HP) nor (RAD) imply (M). Let us give two examples to illustrate this. First consider the space formed by the real line from which an arbitrary interval has been withdrawn, equipped with the Euclidean distance and Lebesgue measure. This space obviously does not have the monotone geodesic property, as, to put it roughly, there is a hole in it. On the other hand, this space clearly satisfies the Hardy property, as a consequence of (HP) on the real line, as well as (RAD). The second example is a connected one: consider the space made of the three edges of an arbitrary triangle in the plane, again equipped with the induced Euclidean distance and Lebesgue measure. This space has the Hardy property, once again as a straightforward consequence of the fact that the unit circle has it and easy change of variables. It easily follows from the fact that one of the angles must be less than $\pi/2$ that it does not have the monotone geodesic property: one of the pairs (x, y) with x a vertex and y its orthogonal projection on the opposite side cannot meet condition (M). In passing, it proves that this property is not stable under bi-Lipschitz mappings (see also [Te]).

2.4.2 Homogeneous balls property (HB)

Studying in detail the proof of Proposition 2.1.1, one realizes that if B and $X\backslash B$ are themselves spaces of homogeneous types with uniform constants, then there will be no difficulty to prove that (HP) is satisfied, as one can then adapt the proof using Christ's dyadic cubes both on B and $2B\backslash B$. This motivates the following definition.

Definition 2.4.5. Let (X, ρ, μ) be a space of homogeneous type. Let B be a ball in X, and suppose that B and B^c are themselves spaces of homogeneous type with doubling constant C_B, *i.e.* for all $x \in B$, $y \in B^c$, and all $r > 0$, we have

$$\mu(B(x, 2r) \cap B) \leq C_B \;\; \mu(B(x, r) \cap B), \qquad \mu(B(y, 2r) \cap B^c) \leq C_B \;\; \mu(B(y, r) \cap B^c).$$

We say that X has the **homogeneous balls property** (HB) if this is satisfied by all the balls in X and if

$$\sup_{B \subset X} C_B < +\infty.$$

Proposition 2.4.6. *Let* (X, ρ, μ) *be a space of homogeneous type.*

1. *If* X *has the homogeneous balls property* (HB)*, then* X *has the relative layer decay property* (RLD).
2. *If* X *has the homogeneous balls property* (HB)*, then* X *has the Hardy property* (HP).

Proof. (1) If X has the homogeneous balls property, then every ball in X as well as its complement in X can be partitioned into dyadic cubes, with uniform constants (see [Ch1]). But these cubes themselves have the layer decay property, and it is easy to see that this property transposes to the balls. Let $x \in X$, $r > 0$, $B = B(x,r)$, $w \in X$, $0 < R \le 2r$, and fix $\varepsilon > 0$. Let us estimate for example the measure of the inner layer $C_\varepsilon = \{x \in B : \rho(x, B^c) \le \varepsilon\}$. We want to prove that there exists $\eta' > 0$ such that $\mu(C_\varepsilon \cap B(w,R))) \lesssim (\frac{\varepsilon}{R})^{\eta'} \mu(B(w,R))$. As B constitutes by itself a space of homogeneous type with uniform doubling constant, there exists at every scale a partitioning of B into Christ's dyadic cubes (with uniform constants for these cubes). The idea is to pave B by dyadic cubes of a well chosen generation. Let $N, m, l \in \mathbb{Z}$ be such that $\delta^{N+1} < r \le \delta^N$, $\delta^{N+m+1} < \varepsilon \le \delta^{N+m}$, and $\delta^{N+l+1} < R \le \delta^{N+l}$. We can assume that $\varepsilon \le R$, since otherwise the result is trivial, which means $m \ge l$. We look at the cubes of generation $j = N + [\frac{m+l}{2}] \ge N$ because $R \le 2r$ (otherwise there would be no such cubes) : for all $y \in C_\varepsilon$, there exists a unique β such that $y \in Q^j_\beta$, and $\rho(y, (Q^j_\beta)^c) \le \rho(y, B^c) \le \varepsilon \le \delta^{m+N}$. By the small boundary property (1.2.1) of Christ's dyadic cubes, one gets that for all β

$$\mu(C_\varepsilon \cap Q^j_\beta) \lesssim \left(\frac{\delta^{n+m}}{\delta^j}\right)^\eta \mu(Q^j_\beta) \lesssim \left(\frac{\varepsilon}{R}\right)^{\eta/2} \mu(Q^j_\beta).$$

On the other hand, if $C_\varepsilon \cap Q^j_\beta \cap B(w,R) \neq \varnothing$ and if $y \in C_\varepsilon \cap Q^j_\beta \cap B(w,R)$, $x \in Q^j_\beta$, then, using the fact that since $m \ge l$, $\delta^j \le \delta^{n+l} \lesssim R$, we have

$$\rho(x,w) \le \rho(x,y) + \rho(y,w) \le C_1\delta^j + R \le CR$$

for a dimensional constant $C > 0$, and then $Q^j_\beta \subset B(w, CR)$. Finally, we get

$$\begin{aligned}
\mu(C_\varepsilon \cap B(w,R)) &= \sum_{\beta : C_\varepsilon \cap Q^j_\beta \cap B(w,R) \neq \varnothing} \mu(C_\varepsilon \cap Q^j_\beta \cap B(w,R)) \\
&\lesssim \sum_{\beta : C_\varepsilon \cap Q^j_\beta \cap B(w,R) \neq \varnothing} \left(\frac{\varepsilon}{R}\right)^{\eta/2} \mu(Q^j_\beta) \\
&\lesssim \left(\frac{\varepsilon}{R}\right)^{\eta/2} \mu(B(w,CR)) \lesssim \left(\frac{\varepsilon}{R}\right)^{\eta/2} \mu(B(w,R)),
\end{aligned}$$

where the last line is obtained using the disjointness of the cubes Q^j_β and then the doubling property. We can do the same for the outer layer as B^c also constitutes a space of homogeneous type, with uniform doubling constant. It proves (RLD).

(2) It is a direct consequence of (1) and Proposition 2.3.2. But it can also be proven directly slightly adapting the proof of Proposition 2.1.1. Indeed, observe that the homogeneous balls property allows to use exact coverings of B and $2B \backslash B$ by Christ's dyadic cubes as above, and then everything works out as in the dyadic setting. $\square$

It is easy to see that a space of homogeneous type does not satisfy the homogeneous balls property (HB) in general. Let us give a counterexample.

Example 2.4.7. Consider the real line, from which one has withdrawn the interval $I_\varepsilon =]1-\varepsilon, 1-\varepsilon^2[$, with ε a fixed small constant. Consider the ball in this set of center 1/2 and of radius 1/2. It is easy to see that it has, as a space of homogeneous type, a doubling constant of at least $1/\varepsilon$: inside this ball, consider the ball of center $1-\varepsilon^2$ and radius $\varepsilon - \varepsilon^2$, and its concentric double. Now, set

$$I_{n,\varepsilon} =]n - \varepsilon/2^{n-1}, n - (\varepsilon/2^{n-1})^2[,$$

and consider the space $X = \mathbb{R}\setminus \cup_{n\geq 1} I_{n,\varepsilon}$, equipped with the Euclidean distance and the Lebesgue measure. It is clear that X does not satisfy the homogeneous balls property since the doubling constants explode. Neither does X satisfy the monotone geodesic property (M) as, to put it roughly, it has holes in it. However, observe that X satisfies both (RLD) and (HP). As a matter of fact, proving (HP) on X is exactly the same as proving it for the real line, and this is trivial (it is the same for (RLD)). In particular, this example shows that neither the monotone geodesic property nor the homogeneous balls property are necessary conditions for (HP).

Remark 2.4.8. We have given two sufficient conditions for (RLD), one which is purely metric, the monotone geodesic property, while the other is rather a measure property. Let us examine how these two properties are connected. It is clear that (HB) does not imply (M), as is shown by the trivial example of the real line from which an arbitrary interval has been withdrawn. Now, if we suppose that X satisfies (M), we have a partial result regarding the homogeneous balls property. As a matter of fact, let $B = B(x, r_0)$, and $y \in B$. We prove that for all $r > 0$,

$$\mu(B(y, 2r) \cap B) \lesssim \mu(B(y, r) \cap B).$$

First, if $r \geq 2r_0$, then $B \subset B(y, r) \subset B(y, 2r)$ and $B(y, 2r) \cap B = B(y, r) \cap B = B$. Then, if $r \leq 2r_0$ and $\rho(x, y) \leq r_0/2$, we have $B(y, r/4) \subset (B \cap B(y, r))$ and by the doubling property, $\mu(B(y, 2r)) \lesssim \mu(B(y, r/4)) \lesssim \mu(B(y, r) \cap B)$. It only remains to study the case $r \leq 2r_0$ and $r_0/2 < \rho(x, y) < r_0$. Let $\alpha < \min(\frac{2}{1+C}, \frac{1}{2})$, and $\beta < \alpha/2$. We have $\rho(x, y) \geq r_0/2 \geq r\alpha/2$ and by (M), there exists a point z in X such that

$$\rho(y, z) \leq Cr\frac{\alpha}{2} \quad \text{and} \quad \rho(x, z) \leq \rho(y, x) - r\frac{\alpha}{2}.$$

Consider the ball $B(z, \beta r)$. If $w \in B(z, \beta r)$, then

$$\rho(w, x) \leq \rho(w, z) + \rho(z, x) \leq \beta r + \rho(y, x) - r\frac{\alpha}{2} \leq r_0.$$

Thus $B(z, \beta r) \subset B$. Furthermore,

$$\rho(w, y) \leq \rho(w, z) + \rho(z, y) \leq \beta r + Cr\frac{\alpha}{2} < r,$$

since $\alpha < \frac{2}{1+C}$. Thus $B(z, \beta r) \subset B(y, r)$. Finally,

$$\mu(B(y, 2r) \cap B) \lesssim \mu(B(y, r)) \lesssim \mu(B(z, 2r)) \lesssim \mu(B(z, \beta r)) \lesssim \mu(B(y, r) \cap B).$$

This means that every ball in X is itself a space of homogeneous type, with uniform constant. However, we cannot obtain the same result for the complement of balls, and (HB) cannot be inferred.

With the results of these first sections, we have already proven a part of Theorem 2.1.3: for now, we know that we have the following

$$\begin{tikzcd}
 & & \text{(HB)} \arrow[d, Rightarrow, shift left] \arrow[dr, Rightarrow, shift right] & \\
\text{(M)} \arrow[r, Rightarrow, shift left] & \text{(RAD)} \arrow[l, Rightarrow, shift left, "/" marking] \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{(RLD)} \arrow[u, Rightarrow, shift left, "/" marking] \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{(HP)} \arrow[ul, Rightarrow, shift right, "/" marking] \\
 & \text{(AD)} \arrow[r, Rightarrow] & \text{(LD)} &
\end{tikzcd}$$

Before going further, let us make one more comment. The study of Example 2.4.7 motivates the following observation: to prove the Hardy property (HP) on a space of homogeneous type X, it is enough to prove it on a larger space X_0 of which X is a subset, provided that X_0 does not have too much weight compared to X. Let us precise this. Let (X_0, ρ, μ) be a space of homogeneous type, and let $X \subset X_0$, equipped with the distance $\rho' = \rho|_X$ and the measure $\nu = \mu|_X$. Suppose furthermore that the following measure compatibility condition is satisfied: for every $x \in X$ and $r > 0$,

$$\mu(B_{X_0}(x,r)) \lesssim \mu(B_{X_0}(x,r) \cap X) = \nu(B_X(x,r)). \tag{2.4.2}$$

Then X inherits (HP) from X_0. Indeed, under (2.4.2), the kernels taken respectively on X and X_0, which we denote K_X and K_{X_0}, have comparable size: if $x, y \in X$,

$$K_X(x,y) \approx \sup_{B_X \ni x,y} \frac{1}{\nu(B_X)} = \sup_{\substack{B_{X_0} \ni x,y \\ B \text{ centered in } X}} \frac{1}{\mu(B_{X_0} \cap X)} \lesssim \sup_{B_{X_0} \ni x,y} \frac{1}{\mu(B_{X_0})} \approx K_{X_0}(x,y).$$

It is clear then that the Hardy inequality is in that sense stable by restriction, provided one does not withdraw too much from the initial space. To check that (HP) is satisfied by a space of homogeneous type X, it is thus enough to see if X can be seen as a subset of a bigger space, satisfying (2.4.2), on which (HP) is known to be true, or on which one can prove the monotone geodesic property for example.

2.5 An example in the complex plane: the space of Tessera

To further understand these properties of the homogeneous space, we will study in this section an explicit example inspired by Tessera in [Te]. This example is a connected linear space, given by a stairway-like curve in the complex plane, starting from 0, and containing for every $k \in \mathbb{N}$ a half-circle of center 0 and radius 2^k. More precisely, consider in the complex plane the parametric curve $\gamma(t)$ defined for $t \geq 0$, and constructed as follows with $|\gamma'(t)| = 1$ for every $t \geq 0$

- $\gamma(0) = 0$.
- $\{\gamma(t) \mid 0 \leq t \leq t_1\}$ is the segment $[0, 1]$.
- $\{\gamma(t) \mid t_1 \leq t \leq t_2\}$ is the half-circle of center 0 and radius 1 in the half-plan $\{\Im z \geq 0\}$.
- By induction, for $k \geq 1$, $\{\gamma(t) \mid t_{2k} \leq t \leq t_{2k+1}\}$ is the segment $[2^{k-1}, 2^k]$, and $\{\gamma(t) \mid t_{2k+1} \leq t \leq t_{2k+2}\}$ is the half-circle of center 0 and radius 2^k in the half-plan $\{\Im z \geq 0\}$ if k is even, in the half-plan $\{\Im z \leq 0\}$ if k is odd.

Set $t_0 = 0$. An easy computation shows that we have for all $k \geq 1$,

$$t_{2k} = 1 + \pi + (1 + 2\pi)(2^{k-1} - 1) = -\pi + (1 + 2\pi)2^{k-1},$$

$$t_{2k+1} = 1 + \pi + (1 + 2\pi)(2^{k-1} - 1) + 2^{k-1} = -\pi + (2 + 2\pi)2^{k-1}.$$

Set $X_T = \{\gamma(t) \mid t \geq 0\}$, equipped with the Euclidean distance d in $\mathbb{C}$ and the Hausdorff length Λ. See Figure 2.3 for a representation of X_T. For $x, y \in X_T$, denote by (x, y) the arc in X_T between x and y. For $z \in X_T$, and $r > 0$, we denote by $B(z, r)$ the open ball of center z and radius r in X_T : $B(z,r) = \{x \in X_T \mid |z - x| < r\}$. We recall that a bounded set $E \subset \mathbb{C}$ is said to be Ahlfors-David regular (of dimension 1) when there exists a constant $0 < C < +\infty$ such that for every $z \in \mathbb{C}$ and $0 < r \leq 1$, $\frac{1}{C} r \leq \Lambda(E \cap B(x,r)) \leq Cr$ (see [Ah], [D3]).

Proposition 2.5.1. *1. $(X_T, \mathrm{d}, \Lambda)$ is an Ahlfors-David set of dimension 1, and thus $(X_T, \mathrm{d}, \Lambda)$ can be seen as a space of homogeneous type.*

2. *X_T does not satisfy the annular decay property* (AD) *(nor the relative annular decay property* (RAD)*).*
3. *X_T does not satisfy the homogeneous balls property* (HB)*, nor the monotone geodesic property* (M)*.*
4. *X_T satisfies the relative layer decay property* (RLD)*.*
5. *X_T satisfies the Hardy property* (HP)*.*

Proof. (1) Let us first make a preliminary observation : as $|\gamma'(t)| = 1$ for every $t \geq 0$, we have $\Lambda((\gamma(\alpha), \gamma(\beta))) = |\alpha - \beta|$. But there exists a dimensional constant $1 < C_{AD} < +\infty$ such that for all $\alpha, \beta > 0$

$$\frac{1}{C_{AD}} \mathrm{d}(\gamma(\alpha), \gamma(\beta)) \leq \Lambda((\gamma(\alpha), \gamma(\beta))) \leq C_{AD}\, \mathrm{d}(\gamma(\alpha), \gamma(\beta)). \tag{2.5.1}$$

The left inequality is trivial. For the right inequality, let $n, m \in \mathbb{N}$ such that $t_n \leq \alpha < t_{n+1}$, $t_m \leq \beta < t_{m+1}$. If $n = m$, then $\gamma(\alpha)$ and $\gamma(\beta)$ are either on the same half-circle, or on the same segment, and the result is clear. Assume that $|n - m| = 1$, then one of these two points is on a segment, and the other on a connected half-circle. Suppose for example that $t_{2k} \leq \alpha < t_{2k+1}$ and $t_{2k+1} \leq \beta < t_{2k+2}$, so that $\gamma(\alpha)$ is on the segment $[2^{k-1}, 2^k]$ and $\gamma(\beta)$ on a half-circle of center 0 and radius 2^k. Let $\omega = \gamma(t_{2k+1}) = 2^k$. Set $a = \mathrm{d}(\gamma(\alpha), \omega)$, $c = \mathrm{d}(\gamma(\beta), \omega)$ and $b = \mathrm{d}(\gamma(\alpha), \gamma(\beta))$. Applying elementary triangle geometry (see Figure 2.2), and the fact that $a \leq b$, $c \leq a + b \leq 2b$, write

$$\Lambda((\gamma(\alpha), \gamma(\beta)))^2 \lesssim (a + c)^2 \lesssim b^2 = \mathrm{d}(\gamma(\alpha), \gamma(\beta))^2.$$

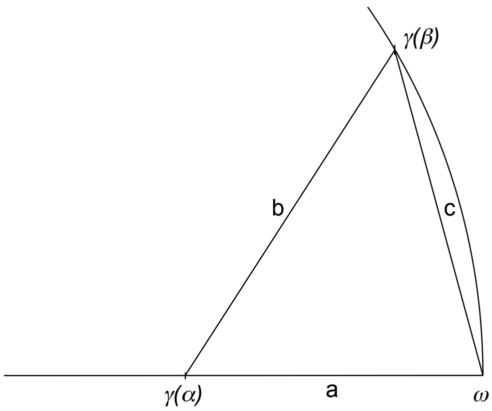

Figure 2.2: X_T is an Ahlfors-David set of dimension 1.

Finally, if $|n - m| \geq 2$, assume that for example $n \geq m$, then $\mathrm{d}(\gamma(\alpha), \gamma(\beta)) \geq |\gamma(\alpha)| - |\gamma(\beta)| \geq 2^{\frac{n}{2}-1} - 2^{\frac{m}{2}}$. On the other hand, $\Lambda((\gamma(\alpha), \gamma(\beta))) = \alpha - \beta \leq t_{n+1} - t_m \lesssim 2^{n/2} - 2^{m/2}$. The result follows. Now, let us check that $(X_T, \mathrm{d}, \Lambda)$ is Ahlfors-David. Let $x \in X_T$, $r > 0$, and set $B = B(x, r)$. As X_T is connected and not bounded, the connected component of x in B, denoted by $C(x)$, is at distance zero from the complement of B. Thus, there exists $y \in C(x)$ such that

$\mathrm{d}(x,y) \geq r/2$. But then, by (2.5.1), we have $\Lambda(B) \geq \Lambda((x,y)) \geq C_{AD}^{-1}\mathrm{d}(x,y) \geq C_{AD}^{-1}(r/2)$. Moreover, set $t_0 = \inf\{t \geq 0 \mid \gamma(t) \in B\} \geq 0$ and $t_1 = \sup\{t \geq 0 \mid \gamma(t) \in B\} < +\infty$ because X_T is unbounded. Then $\Lambda(B) \leq \Lambda((\gamma(t_0), \gamma(t_1))) \leq C_{AD}\mathrm{d}(\gamma(t_0), \gamma(t_1)) \leq C_{AD}\, 2r$. This proves that $(X_T, \mathrm{d}, \Lambda)$ is an Ahlfors-David set of dimension 1.

(2) As a consequence of (1), observe that for any $k \geq 1$, $\Lambda(B(0, 2^k)) \approx 2^k$. However, for every $\varepsilon > 0$, we have

$$\Lambda(B(0, 2^k + \varepsilon)\backslash B(0, 2^k)) \geq \pi 2^k.$$

If X_T had the annular decay property, the measure of this set would be going to 0 with ε. We thus have a contradiction. Hence, X_T cannot satisfy (AD), nor (RAD).

(3) It is obvious that X_T does not satisfy the monotone geodesic property (pick any two points on different half circles in X_T). For the homogeneous balls property, fix $\varepsilon > 0$ and consider the ball in X_T of center $x = (0,4)$ and radius $r = 3 + \varepsilon$. Let $y = (0,1)$ and $\rho = 3$. The set $B(x,r) \cap B(y,\rho)$ has only one connected component, containing y, and its length is comparable to ε. On the other hand, the set $B(x,r) \cap B(y, 2\rho)$ has two connected components, one of which containing x and of length comparable to 1. Thus, the doubling constant of the ball $B(x,r)$ seen as a space of homogeneous type exceeds C/ε. It follows that X_T cannot satisfy (HB), as the doubling constants of the balls cannot be uniform.

(4) Let $0 < \alpha < 1$ and let $c_1 > 0$ be such that $\forall t < c_1,\ \ t^{1-\alpha}|\ln t| \leq 1$. We first prove that X_T satisfies the layer decay property (LD). Fix $z \in X_T, r > 0, 0 < \varepsilon \leq r$, set $B = B(z,r)$. Set as before $B_\varepsilon = \{x \in B | \mathrm{d}(x, B^c) \leq \varepsilon\} \cup \{y \in B^c | \mathrm{d}(y, B) \leq \varepsilon\}$ the union of the inner and outer layers. Let $\theta = \varepsilon/r$. We show that there exists a dimensional constant $C < +\infty$ such that

$$\Lambda(B_\varepsilon) \leq C\theta^\alpha \Lambda(B). \tag{2.5.2}$$

Observe that if $c_1 \leq \theta \leq 1$, the result is trivial :

$$\Lambda(B_\varepsilon) \leq \Lambda(B) \leq \left(\frac{1}{c_1}\right)^\alpha \theta^\alpha \Lambda(B).$$

So assume now that $\theta < c_1$. Observe that the points in B_ε are elements of X_T at distance less or equal to ε from $\overline{B}\backslash B$, where $\overline{B}$ is the adherence of B in X_T: $\gamma(t) \in \overline{B}\backslash B$ if $\gamma(t) \in B^c$ and either for every $s > 0$ small enough $\gamma(t+s) \in B$ or for every $s > 0$ small enough $\gamma(t-s) \in B$. If $r \leq 1$, remark that B and B^c are connected sets, so that there are less than two points in $\overline{B}\backslash B$. But since X_T is Ahlfors-David, we get $\Lambda(B_\varepsilon) \lesssim 2\varepsilon$, and also $\Lambda(B) \approx 1$. (2.5.2) follows.

We suppose now $r > 1$. Assume first that $0 \notin B$, that is $|z| > r$. Denote by C_i, respectively C'_j, $0 \leq i \leq p$, $0 \leq j \leq p+1$, the different connected components of B, respectively B^c, starting from the one closest to the origin. Because of (2.5.1), it is easy to see that each C_i, C'_j will roughly contribute to ε towards $\Lambda(B_\varepsilon)$. More precisely, set $B_\varepsilon^i = B_\varepsilon \cap (C_i \cup C'_i)$ for $0 \leq i \leq p$, and $B_\varepsilon^{p+1} = B_\varepsilon \cap C'_{p+1}$. Then, it follows from (2.5.1) that we have for every $0 \leq i \leq p+1$,

$$\Lambda(B_\varepsilon^i) \leq 4C_{AD}\,\varepsilon \lesssim \varepsilon. \tag{2.5.3}$$

Now, the idea is to estimate the number of components C_i, and to take care of the fact that some of them can contribute to $\Lambda(B_\varepsilon)$ for less than ε, as their length can be less than that if r and ε are large enough. It is easy to see that p can be roughly bounded by $\ln r$. Indeed, let $k_0 \geq 0$ be such that $2^{k_0} \leq \mathrm{dist}(C_0, 0) < 2^{k_0+1}$ (remember that we have assumed $|z| > r$), and observe (see Figure 2.3) that we have, for $0 \leq i \leq p-1$, $\mathrm{d}(C_i, C_{i+1}) \geq 2^{k_0+2i} \geq 2^{2i}$. Consequently, we must have

$$\sum_{i=0}^{p-1} 2^{2i} \leq 2r \Rightarrow 4^p \lesssim r \Rightarrow p \lesssim \ln r.$$

On the other hand, observe that for every $0 \le i \le p$, $(C_i \cup C_i') \subset B(0, 2^{k_0+2i+2})$. Since X_T is Ahlfors-David, it follows that

$$\Lambda(C_i \cup C_i') \lesssim 2^{k_0+2i+2} \lesssim 4^{k_0+i}. \tag{2.5.4}$$

Applying (2.5.3), we get $\Lambda(B_\varepsilon^i) \lesssim \min(\varepsilon, 4^{k_0+i})$. Finally, we obtain

$$\Lambda(B_\varepsilon) = \sum_{i=0}^{p+1} \Lambda(B_\varepsilon^i) \lesssim \sum_{i \ge 0 : 4^{k_0+i} \le \varepsilon} 4^{k_0+i} + \varepsilon \operatorname{Card}\{0 \le i \le p+1 \mid \varepsilon < 4^{k_0+i}\}.$$

But $\operatorname{Card}\{0 \le i \le p+1 \mid \varepsilon < 4^{k_0+i}\} \le \operatorname{Card}\{0 \le i \le p+1 \mid \varepsilon < 4^i\}$. Remark that if $4^i > \varepsilon$, then $i \ge \frac{\ln \varepsilon}{\ln 4}$. The cardinal intervening in the second term is thus bounded by $C(\ln r - \ln \varepsilon)$ where C is an absolute constant. Consequently, we have

$$\Lambda(B_\varepsilon) \lesssim \varepsilon + \varepsilon \ln\left(\frac{r}{\varepsilon}\right) \lesssim \varepsilon(1 - \ln\theta) \lesssim \theta^\alpha r,$$

because $\varepsilon(1-\ln\theta)\theta^{-\alpha} r^{-1} = \theta^{1-\alpha} - \theta^{1-\alpha}\ln\theta \le 2$ as $\theta < c_1$. But since X_T is Ahlfors-David, we have $\Lambda(B) \eqsim r$ and (2.5.2) follows.
It remains to consider the case when $0 \in B$. But the same argument still works, only the notations have to be slightly modified because, this time, the origin belongs to the connected component C_0 of B. Thus, there is in this case the same number of components C_i and C_i', and it is the distance $\mathrm{d}(C_0', 0)$ that plays a role in the argument instead of $\mathrm{d}(C_0, 0)$. Apart from this, the argument is mostly unchanged, so we do not elaborate on it here.

It remains to prove (RLD). Let $B = B(z, r)$ be a ball in X_T as before. Let $0 < R \le 2r$, $w \in X_T$. We prove that

$$\Lambda(B_\varepsilon \cap B(w, R)) \lesssim \left(\frac{\varepsilon}{R}\right)^\alpha \Lambda(B(w, R)), \tag{2.5.5}$$

Once again, when $\varepsilon/R \ge c_1$, the result is trivial, so we can assume that $\varepsilon/R < c_1$. Now, observe that we can apply exactly the same argument as above. The only difference is that instead of estimating the total number p of connected components C_i of B, we now have to estimate the number of these connected components that intersect $B(w, R)$. But by the same argument as before, this number is bounded by $(\ln R)$ as soon as $R > 1$ (and the result is trivial when $R \le 1$). Going through with the argument, this provides the bound

$$\Lambda(B_\varepsilon \cap B(w, R)) \lesssim \left(\frac{\varepsilon}{R}\right)^\alpha R.$$

But since X_T is Ahlfors-David, we have $\Lambda(B(w, R)) \eqsim R$ and (2.5.5) follows.

(5) Applying Proposition 2.3.2, (5) is a direct consequence of (4), but to better understand this example, we will give a direct proof here. Fix $z \in X_T$, $r > 0$, $1 < \nu < +\infty$, set $B = B(z, r)$, and let $f \in L^\nu(B)$, f supported on B, $g \in L^{\nu'}(2B \backslash B)$, g supported on $2B \backslash B$. Assume as before that for example $0 \notin B$, the argument is unchanged when $0 \in B$, only the notations have to be adapted. We adopt the same notations as in (4): denote by C_i, respectively C_j', $0 \le i \le p$, $0 \le j \le p+1$, the different connected components of B, respectively B^c, starting from the one closest to the origin. Set $I_{2i} = \{t \ge 0 \mid \gamma(t) \in C_{i-1}\}$ for $1 \le i \le p+1$, $I_{2j+1} = \{t \ge 0 \mid \gamma(t) \in C_j' \cap (2B \backslash B)\}$ for $0 \le j \le p+1$. We want to estimate the following quantity

$$H(f, g) = \int_B \int_{2B \backslash B} \frac{f(x) g(y)}{\Lambda(B(x, \mathrm{d}(x, y)))} \mathrm{d}\Lambda(x) \mathrm{d}\Lambda(y).$$

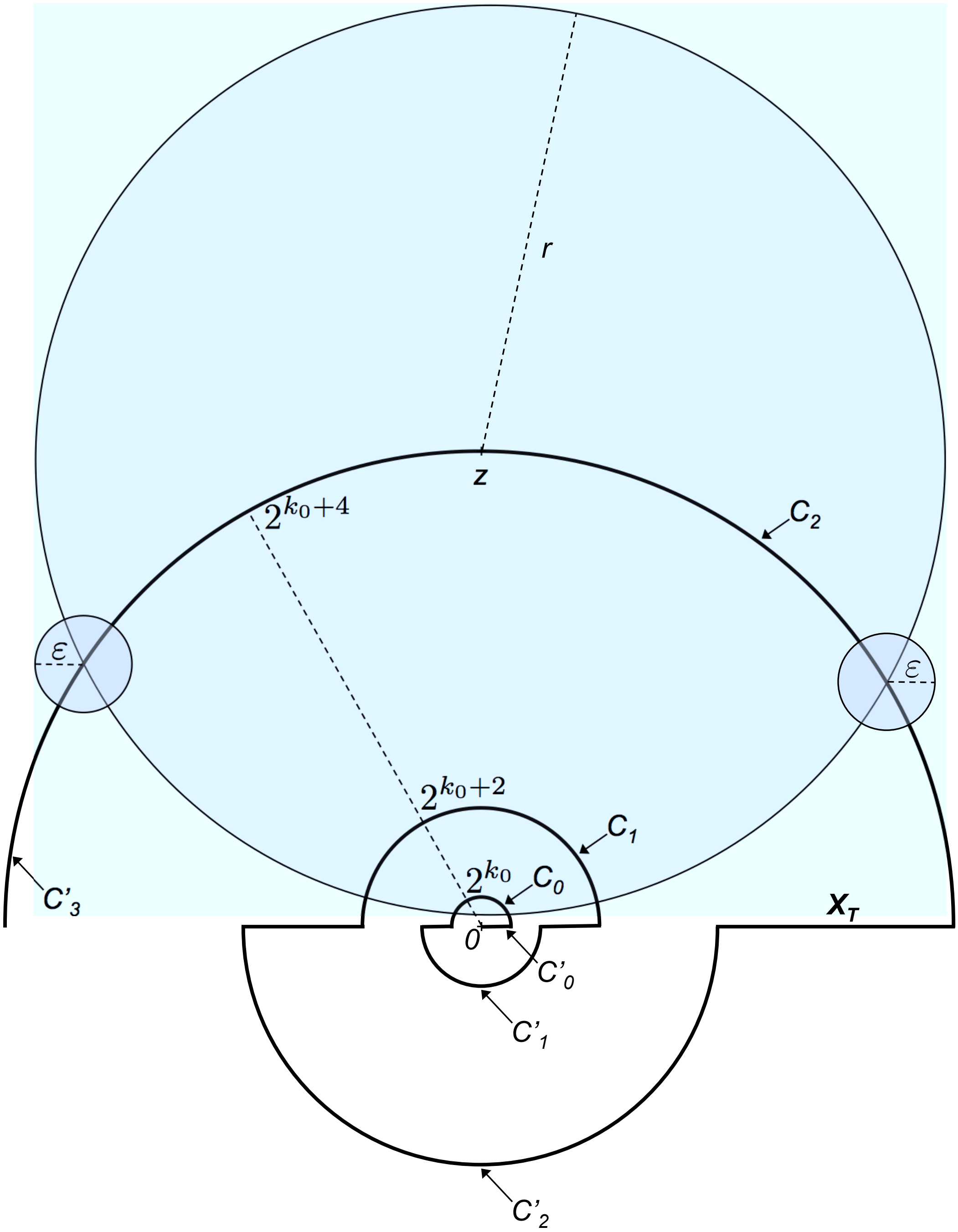

Figure 2.3: Layer decay property (LD) in the space X_T.

As X_T is Ahlfors-David, and applying (2.5.1), we have

$$H(f,g) \approx \int_B \int_{2B\setminus B} \frac{f(x)g(y)}{\mathrm{d}(x,y)} \mathrm{d}\Lambda(x)\mathrm{d}\Lambda(y) \approx \sum_{i=0}^{p} \sum_{j=-1}^{p} \int_{I_{2i}} \int_{I_{2j+1}} \frac{f(\gamma(t))g(\gamma(s))}{|s-t|} \mathrm{d}t\mathrm{d}s, \quad (2.5.6)$$

because $|\gamma'(t)| = 1$ for every $t \geq 0$. Set $\widetilde{f} = f \circ \gamma$, $\widetilde{g} = g \circ \gamma$, $\widetilde{f} \in L^\nu$, supported on $\bigcup_{i=1}^{p+1} I_{2i}$, $\widetilde{g} \in L^{\nu'}$, supported on $\bigcup_{j=0}^{p+1} I_{2j+1}$, with $\|\widetilde{f}\|_\nu = \|f\|_\nu$, $\|\widetilde{g}\|_{\nu'} = \|g\|_{\nu'}$. Let again $k_0 \geq 0$ be such that $2^{k_0} \leq \mathrm{dist}(C_0, 0) < 2^{k_0+1}$ (remember that we have assumed $|z| > r$). Because of the fact that $|\gamma'(t)| = 1$ for every $t \geq 0$, observe that we have, by (2.5.4), for every $1 \leq i \leq p+1$, $0 \leq j \leq p$,

$$|I_{2i}| = \Lambda(C_{i-1}) \leq \Lambda(C_{i-1} \cup C'_{i-1}) \lesssim 2^{k_0} 4^i, \qquad |I_{2j+1}| \leq \Lambda(C'_j) \leq \Lambda(C_j \cup C'_j) \lesssim 2^{k_0} 4^j.$$

For $j = p+1$, remark that as for $0 \leq i \leq p$, $C_i \subset B(0, 2^{k_0+2i+2})$, we have $B \subset B(0, 2^{k_0+2p+2})$. It follows that $2B \subset B(0, 3 \times 2^{k_0+2p+2})$. As a matter of fact, if $\mathrm{d}(z,x) < 2r$, then $|x| \leq \mathrm{d}(x,z) + |z| < 2r + |z| < 3|z| < 3 \times 2^{k_0+2p+2}$, because we have assumed $|z| > r$. Thus, since X_T is Ahlfors-David, it follows that

$$|I_{2p+3}| \leq \Lambda(2B\setminus B) \leq \Lambda(B(0, 3 \times 2^{k_0+2p+2})) \lesssim 2^{k_0} 4^p.$$

Finally, it is easy to see that if $j \notin \{i-1, i\}$, we have $\mathrm{dist}(I_{2i}, I_{2j+1}) \gtrsim 2^{k_0}|4^i - 4^j|$. Now, set

$$f_i = \left(\int_{I_{2i}} |\widetilde{f}(t)|^\nu \mathrm{d}t \right)^{1/\nu} \in \ell^\nu(\{1, ..., p+1\}), \quad \text{with} \quad \|(f_i)_i\|_{\ell^\nu} = \|\widetilde{f}\|_\nu = \|f\|_\nu,$$

$$g_j = \left(\int_{I_{2j+1}} |\widetilde{g}(s)|^{\nu'} \mathrm{d}s \right)^{1/\nu'} \in \ell^{\nu'}(\{0, ..., p+1\}), \quad \text{with} \quad \|(g_j)_j\|_{\ell^{\nu'}} = \|\widetilde{g}\|_{\nu'} = \|g\|_{\nu'}.$$

Split the sum in (2.5.6) for the neighboring I_k and the ones that are far from one another: we have

$$\begin{aligned} H(f,g) &\lesssim \sum_{i=1}^{p+1} \sum_{j=i-1}^{i} \int_{I_{2i}} \int_{I_{2j+1}} \frac{|\widetilde{f}(t)||\widetilde{g}(s)|}{|s-t|} \mathrm{d}t\mathrm{d}s + \sum_{i=1}^{p+1} \sum_{j \notin \{i-1,i\}} \int_{I_{2i}} \int_{I_{2j+1}} \frac{|\widetilde{f}(t)||\widetilde{g}(s)|}{|s-t|} \mathrm{d}t\mathrm{d}s \\ &= H_1(f,g) + H_2(f,g). \end{aligned}$$

For H_1, apply (H) on $\mathbb{R}$ and then the Cauchy-Schwarz inequality to get

$$\begin{aligned} H_1(f,g) &\lesssim \sum_{i=1}^{p+1} \sum_{j \in \{i-1,i\}} f_i g_j \lesssim \left(\sum_{i=1}^{p+1} \sum_{j \in \{i-1,i\}} f_i^\nu \right)^{1/\nu} \left(\sum_{j=0}^{p+1} \sum_{\substack{1 \leq i \leq p+1 \\ i \in \{j+1,j\}}} g_j^{\nu'} \right)^{1/\nu'} \\ &\lesssim \|f\|_\nu \|g\|_{\nu'}. \end{aligned}$$

To estimate H_2, write

$$\begin{aligned} H_2(f,g) &\lesssim \sum_{i=1}^{p+1} \sum_{j \notin \{i-1,i\}} \frac{1}{\mathrm{dist}(I_{2i}, I_{2j+1})} f_i |I_{2i}|^{\frac{1}{\nu'}} g_j |I_{2j+1}|^{\frac{1}{\nu}} \\ &\lesssim \sum_{i=1}^{p+1} \sum_{j \notin \{i-1,i\}} \frac{4^{\frac{i}{\nu'}} 4^{\frac{j}{\nu}}}{|4^i - 4^j|} f_i g_j \end{aligned}$$

By symmetry, we will be done if we can bound for example the sum for $j > i$. But if $j > i$ observe that $|4^i - 4^j| = 4^i(4^{j-i} - 1) \gtrsim 4^j$. Applying the Cauchy-Schwarz inequality, write then

$$\begin{aligned}
\sum_{1\le i\le p+1,\, j>i} \frac{4^{\frac{i}{\nu'}}4^{\frac{j}{\nu}}}{|4^i - 4^j|} f_i g_j &\lesssim \sum_{1\le i\le p+1,\, j>i} f_i g_j 4^{-\frac{j-i}{\nu'}} \\
&\lesssim \left(\sum_{1\le i\le p+1,\, j>i} f_i^\nu 4^{-\frac{j-i}{\nu'}} \right)^{\frac{1}{\nu}} \left(\sum_{1\le i\le p+1,\, j>i} g_j^{\nu'} 4^{-\frac{j-i}{\nu'}} \right)^{\frac{1}{\nu'}} \\
&\lesssim \left(\sum_{1\le i\le p+1} f_i^\nu \sum_{j>i} 4^{-\frac{j-i}{\nu'}} \right)^{\frac{1}{\nu}} \left(\sum_{2\le j\le p+1} g_j^{\nu'} 4^{-\frac{j}{\nu'}} \sum_{1\le i<j} 4^{\frac{i}{\nu'}} \right)^{\frac{1}{\nu'}} \\
&\lesssim \left(\sum_{1\le i\le p+1} f_i^\nu \right)^{\frac{1}{\nu}} \left(\sum_{2\le j\le p+1} g_j^{\nu'} \right)^{\frac{1}{\nu'}} \lesssim \|f\|_\nu \|g\|_{\nu'}.
\end{aligned}$$

Finally, one gets $H(f,g) \lesssim \|f\|_\nu \|g\|_{\nu'}$, and thus X_T satisfies the Hardy property (HP).

□

Consequently, the space X_T satisfies (RLD) and (HP), but neither (HB), (M) nor (RAD). As we have already pointed out before, there is a tangible difference between the definitions of layer decay and annular decay properties. Thus, it is not surprising to find out that these properties are not equivalent. We now have a counterexample for most of the false implications in Theorem 2.1.3. It only remains to build a counterexample to prove that (LD) $\not\Rightarrow$ (RLD) to complete the proof.

Remark 2.5.2. Slightly modifying the construction, basically just by truncating the space X_T, one can get a much simpler example of a space satisfying (RLD) but not (RAD). Indeed, let $X'_T = \{\gamma(t) \mid 0 \le t \le t_8\} \cup [8, +\infty[$ where $[8, +\infty[$ denotes the half-line on the real axis. Then the argument in (3) obviously still holds and X'_T does not satisfy (RAD). On the other hand, it is immediate to see that for any given ball B of X'_T, $\mathrm{Card}(\overline{B}\backslash B) \le 6$, and (RLD) follows easily.

2.6 Example of spaces not satisfying (HP) and end of the proof of Theorem 2.1.3

We now present some variations of the space X_T in order to provide an example where the Hardy property cannot be satisfied. It was originally inspired from the example of Tessera, but is actually in the end only marginally connected to it. Still in the complex plane, consider the space formed by the union of the segment $[0,1]$ on the real axis, the half circle of center 0 and radius 1 in the half-plan $\Im z \ge 0$, and the half-line $]-\infty, -1]$ on the real axis. Parameter this curve so that it is traveled at constant speed. This is someway a truncation of the space X_T. Now introduce a small perturbation ϵ of the half-circle : for $t \in [0,\pi]$, set $\rho(t) = 1 + \epsilon(t)$, where ϵ is a rapidly oscillating function in the neighborhood of the origin, and set

$$X_\epsilon = \{\gamma(t) \mid t \ge -1\}, \quad \text{with} \quad \gamma(t) = \begin{cases} (t+1, 0) & \text{for} \quad -1 \le t \le 0, \\ (\rho(t)\cos t, \rho(t)\sin t) & \text{for} \quad 0 < t < \pi, \\ (\pi - t, 0) & \text{for} \quad t \ge \pi. \end{cases}$$

Choose an oscillating function ϵ, ensuring that X_ϵ keeps finite arclength: $\epsilon(t) = a(t)\sin(b(t))$ for functions a, b appropriately chosen.

2.6.1 Exponential oscillation

Let

$$\epsilon_1(t) = e^{-\frac{1}{t^2}} \sin(\pi e^{\frac{1}{t} - \frac{1}{\pi}}),$$

and set $X_1 = X_{\epsilon_1}$. See Figure 2.4 for a representation of the space X_1.

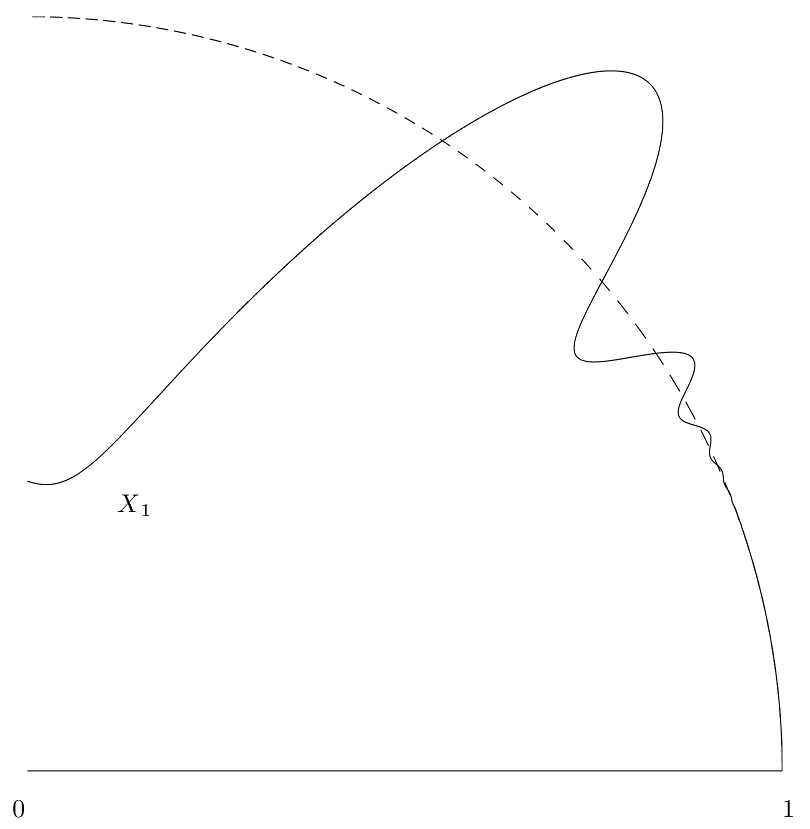

Figure 2.4: A representation of the space X_1.

Remark that choosing $\frac{1}{t^\alpha}$ instead of $\frac{1}{t}$ would not change anything in the following. Observe that with this choice of ϵ, γ is C^1 and $|\gamma'|$ is uniformly bounded below and above, which obviously makes of $(X_1, \mathrm{d}, \Lambda)$ an Ahlfors-David space. In the following of this section, $B(z,r)$ will always denote the ball in $\mathbb{R}^2$ of center z and radius r. For a ball B in $\mathbb{R}^2$ centered at a point of $X_1 \subset \mathbb{R}^2$, we will denote by B^{X_1} the corresponding ball in the space X_1. Let $B^0 = B(0,1)$.

Proposition 2.6.1. *1. X_1 does not satisfy the layer decay property* (LD)*.*

2. X_1 does not satisfy the Hardy property (HP)*.*

Proof. We prove that neither of these properties are satisfied for the ball B^{0,X_1}.

(1) Let $\varepsilon > 0$. Remark that if $\gamma(t) \in \overline{B^{0,X_1}} \backslash B^{0,X_1}$, then $\epsilon(t) = 0$. Denote by t_k these points with $(t_k)_{k\in\mathbb{N}}$ a sequence decreasing to zero, and $t_0 = \pi$. Then we have

$$B_\varepsilon^{0,X_1} = \{\gamma(t) \mid \exists k \in \mathbb{N} \ \ \mathrm{d}(\gamma(t), \gamma(t_k)) < \varepsilon\}.$$

Because of the uniform boundedness of $|\gamma'|$ above and below, observe that we have

$$\Lambda(B_\varepsilon^{0,X_1}) \approx |\{t \in [0,\pi] \mid \exists k \ \ |t_k - t| < \varepsilon\}|,$$

On the other hand, for $k \geq 1$, we have

$$\pi e^{-\frac{1}{\pi}} e^{\frac{1}{t_k}} = k\pi \Rightarrow t_k = \frac{1}{\ln k + 1/\pi} \Rightarrow t_k - t_{k+1} \lesssim \frac{1}{k}.$$

Thus, if $k \geq \frac{1}{\varepsilon}$, and $t \leq t_k \lesssim \frac{1}{-\ln\varepsilon}$, then $t_k - t_{k+1} \lesssim \varepsilon$. It implies that $\gamma(t)$ stays inside B_ε^{0,X_1} for all the $t \leq \frac{C}{-\ln\varepsilon}$ for some constant $C < +\infty$. We obtain

$$\Lambda(B_\varepsilon^{0,X_1}) \gtrsim \left|\left\{t \in [0,\pi] \mid \exists k \geq \frac{1}{\varepsilon} \ \ |t_k - t| < \varepsilon\right\}\right| \geq \left|\left[0, \frac{C}{-\ln\varepsilon}\right]\right| \gtrsim \frac{1}{-\ln\varepsilon}.$$

It follows that there cannot be any upper bound of the form ε^η for $\Lambda(B_\varepsilon^{0,X_1})$, and (LD) cannot be satisfied.

(2) Let $f \in L^\nu(B^{0,X_1})$, f supported on B^{0,X_1}, $g \in L^{\nu'}(2B^{0,X_1}\backslash B^{0,X_1})$, g supported on $2B^{0,X_1}\backslash B^{0,X_1}$. Denote by $I_{2k}, k \geq 0$ the connected sets of t for which $\gamma(t) \in B^{0,X_1}$, $I_{2k} =]t_{2k}, t_{2k+1}[$, and $I_{2p+1}, p \geq 0$ the connected sets of t for which $\gamma(t) \in 2B^{0,X_1}\backslash B^{0,X_1}$, $I_{2p+1} = [t_{2p+1}, t_{2p+2}]$. By the Ahlfors-David property of X_1, we have

$$|H(f,g)| = \left|\int_{B^{0,X_1}} \int_{2B^{0,X_1}\backslash B^{0,X_1}} \frac{f(x)g(y)}{\lambda(x,y)} \mathrm{d}\Lambda(y)\mathrm{d}\Lambda(x)\right| \approx \left|\sum_{k,p\geq 0} \int_{t\in I_{2k}} \int_{s\in I_{2p+1}} \frac{\widetilde{f}(t)\widetilde{g}(s)}{|t-s|} \mathrm{d}s\mathrm{d}t\right|,$$

where $\widetilde{f} = f \circ \gamma$, $\widetilde{g} = g \circ \gamma$, $\widetilde{f} \in L^\nu$, supported on $\cup_{k\geq 0} I_{2k}$, $\widetilde{g} \in L^{\nu'}$, supported on $\cup_{p\geq 0} I_{2p+1}$, and $\|\widetilde{f}\|_\nu \approx \|f\|_\nu$, $\|\widetilde{g}\|_{\nu'} \approx \|g\|_{\nu'}$, because $|\gamma'| \approx 1$. Now, assume that $\widetilde{f}, \widetilde{g}$ are constant and positive on each I_{2k}, I_{2p+1}. Set

$$f_k = \left(\int_{I_{2k}} |\widetilde{f}(t)|^\nu \mathrm{d}t\right)^{1/\nu} \in \ell^\nu(\mathbb{N}), \quad \text{with} \quad \|(f_k)_k\|_{\ell^\nu} = \|\widetilde{f}\|_\nu,$$

$$g_p = \left(\int_{I_{2p+1}} |\widetilde{g}(s)|^{\nu'} \mathrm{d}s\right)^{1/\nu'} \in \ell^{\nu'}(\mathbb{N}), \quad \text{with} \quad \|(g_p)_p\|_{\ell^{\nu'}} = \|\widetilde{g}\|_{\nu'}.$$

Remark that if $p, k \geq 1$ with $p \notin \{k, k-1\}$, and $t \in I_{2k}$, $s \in I_{2p+1}$, then

$$|s-t| \lesssim \left|\frac{1}{\ln p} - \frac{1}{\ln k}\right| \lesssim \frac{|\ln(p/k)|}{\ln p \ln k}.$$

We thus have

$$\begin{aligned}
|H(f,g)| &\gtrsim \sum_{\substack{k,p\geq 1\\ p\notin\{k,k-1\}}} \frac{\ln p \ln k}{|\ln(p/k)|} \int_{t\in I_{2k}} |\widetilde{f}(t)|\mathrm{d}t \int_{s\in I_{2p+1}} |\widetilde{g}(s)|\mathrm{d}s \\
&= \sum_{\substack{k,p\geq 1\\ p\notin\{k,k-1\}}} f_k g_p \frac{\ln p \ln k}{|\ln(p/k)|} |I_{2k}|^{\frac{1}{\nu'}} |I_{2p+1}|^{\frac{1}{\nu}}.
\end{aligned}$$

But since

$$|I_l| = |t_{l+1} - t_l| \underset{l\to+\infty}{\sim} \frac{1}{l(\ln l)^2},$$

there exists $N \in \mathbb{N}^*$ such that if $l \geq N$, then $|I_l| \geq \frac{1}{2l(\ln l)^2}$. It follows that

$$|H(f,g)| \gtrsim \sum_{\substack{k,p\geq N\\ p\notin\{k,k-1\}}} f_k g_p \frac{(\ln p)^{1-\frac{2}{\nu}}(\ln k)^{1-\frac{2}{\nu'}}}{k^{\frac{1}{\nu'}}p^{\frac{1}{\nu}}|\ln(p/k)|}.$$

It is easy to see that this is an unbounded operator. Fix $0 < \eta < 1/2$, and set for example for $k, p \geq N$

$$f_k = \frac{1}{k^{\frac{1}{\nu}}(\ln k)^{\frac{1}{\nu}+\eta}}, \quad g_p = \frac{1}{p^{\frac{1}{\nu'}}(\ln p)^{\frac{1}{\nu'}+\eta}}.$$

Then

$$\begin{aligned}\sum_{p\geq k+1} g_p \frac{1}{p^{\frac{1}{\nu}}}\frac{(\ln p)^{1-\frac{2}{\nu}}}{|\ln(p/k)|} &= \sum_{p\geq k+1}\frac{1}{p(\ln p)^{\frac{1}{\nu}+\eta}\ln(p/k)}\\
&\geq \int_{k+1}^{+\infty}\frac{\mathrm{d}t}{t(\ln t)^{\frac{1}{\nu}+\eta}\ln(t/k)} = \int_{1+\frac{1}{k}}^{+\infty}\frac{\mathrm{d}u}{u\ln u\ln(ku)^{\frac{1}{\nu}+\eta}}\\
&\geq \frac{1}{\ln(3k)^{\frac{1}{\nu}+\eta}}\int_{1+\frac{1}{k}}^{3}\frac{\mathrm{d}u}{u\ln u}\\
&\gtrsim \frac{|\ln(\ln(1+1/k))|}{(\ln k)^{\frac{1}{\nu}+\eta}} \underset{k\to+\infty}{\sim} (\ln k)^{\frac{1}{\nu'}-\eta}.\end{aligned}$$

It follows that

$$|H(f,g)| \gtrsim \sum_{k\geq N} f_k \frac{(\ln k)^{1-\frac{2}{\nu'}}}{k^{\frac{1}{\nu'}}}(\ln k)^{\frac{1}{\nu'}-\eta} = \sum_{k\geq N}\frac{1}{k(\ln k)^{2\eta}} = +\infty.$$

Thus (H) cannot be satisfied for any $1 < \nu < +\infty$. □

Remark 2.6.2. Observe that although the layer decay property is not satisfied here, we still have $\Lambda(B_\varepsilon^{0,X_1}) \underset{\varepsilon\to0}{\to} 0$. Indeed, $\bigcap_{n\geq1} B_{1/n}^{0,X_1} = \overline{B^{0,X_1}}\backslash B^{0,X_1}$, but this set is of measure 0 as it is countable. Thus, we have

$$\Lambda(\overline{B^{0,X_1}}\backslash B^{0,X_1}) = 0 = \lim_{n\to+\infty}\Lambda(B_{1/n}^{0,X_1}).$$

Observe furthermore that it is always the case in this kind of example with a continuous function ϵ, because $\overline{B^{0,X_\epsilon}}\backslash B^{0,X_\epsilon}$ must necessarily be a countable set, hence of measure zero. Indeed, $\overline{B^{0,X_\epsilon}}\backslash B^{0,X_\epsilon}$ is the set of points $\gamma(t_0)$ for which $\epsilon(t_0) = 0$ and for every $\eta > 0$, there exists $0 < \varepsilon < \eta$ such that $\epsilon(t_0 \pm \varepsilon) < 0$. By continuity of ϵ, we deduce from this that for every point $\gamma(t_0) \in \overline{B^{0,X_\epsilon}}\backslash B^{0,X_\epsilon}$, there exists $q_0 \in \mathbb{Q}$ with $\epsilon(q_0) < 0$ and $|t_0 - q_0|$ as small as one wants. We can thus construct an injection from $\overline{B^{0,X_\epsilon}}\backslash B^{0,X_\epsilon}$ to $\mathbb{Q}$ and it follows that $\overline{B^{0,X_\epsilon}}\backslash B^{0,X_\epsilon}$ is countable, hence of measure zero.

2.6.2 Polynomial oscillation

This time, let $b(t) = \frac{\pi^2}{t}$: choose

$$\epsilon_2(t) = A_0 \left(\frac{t}{\pi}\right)^3 \sin\left(\frac{\pi^2}{t}\right),$$

with A_0 a sufficiently small constant to be specified later, and construct a space X_2 as before. Again, $(X_2, \mathrm{d}, \Lambda)$ is an Ahlfors-David space (and thus a space of homogeneous type).

Proposition 2.6.3. *1. X_2 does not satisfy the Hardy property* (HP)*.*

2. X_2 does satisfy the layer decay inequality (LD)*, but not the relative layer decay inequality* (RLD)*.*

Proof. (1) We prove again that (H) is not satisfied for the unit ball B^{0,X_2}. Let us use the same notations as before. For functions $f \in L^2(B^{0,X_2})$, f supported inside B^{0,X_2}, and $g \in L^2(2B^{0,X_2} \backslash B^{0,X_2})$, g supported inside $2B^{0,X_2} \backslash B^{0,X_2}$, we have

$$|H(f,g)| \approx \left| \sum_{k,p\geq 0} \int_{t\in I_{2k}} \int_{s\in I_{2p+1}} \frac{\widetilde{f}(t)\widetilde{g}(s)}{|t-s|} \mathrm{d}s\mathrm{d}t \right|.$$

This time, we have for $k \geq 1$, $t_k = \frac{\pi}{k}$, and thus for $l \geq 1$, we have $|I_l| = \frac{\pi}{l(l+1)} \gtrsim \frac{1}{l^2}$. Besides, if $p, k \geq 1$ with $p \notin \{k, k-1\}$, and $t \in I_{2k}$, $s \in I_{2p+1}$, then

$$|s-t|^{-1} \gtrsim \left|\frac{1}{p} - \frac{1}{k}\right|^{-1} \gtrsim \frac{pk}{|p-k|}.$$

Assume again that $\widetilde{f}, \widetilde{g}$ are constant and positive on each I_{2k}, I_{2p+1}, and set

$$f_k = \left(\int_{I_{2k}} |\widetilde{f}(t)|^2 \mathrm{d}t\right)^{1/2} \in \ell^2(\mathbb{N}), \quad \text{with} \quad \|(f_k)_k\|_{\ell^2} = \|\widetilde{f}\|_2,$$

$$g_p = \left(\int_{I_{2p+1}} |\widetilde{g}(s)|^2 \mathrm{d}s\right)^{1/2} \in \ell^2(\mathbb{N}), \quad \text{with} \quad \|(g_p)_p\|_{\ell^2} = \|\widetilde{g}\|_2.$$

Then, we have

$$|H(f,g)| \gtrsim \sum_{\substack{k,p\geq 1 \\ p\notin\{k,k-1\}}} \frac{f_k g_p}{|p-k|}.$$

But it is well known that this operator is unbounded on ℓ^2. It follows that (H) cannot be satisfied for $\nu = 2$. Thus, X_2 does not satify (HP), nor (RLD) because of Proposition 2.3.2.

(2) Let $\varepsilon > 0$. We are going to prove (LD) for all the balls $B^{X_2}(z,r)$ centered at a point $z \in X_2$ of radius $r > 0$. We classify these balls in three categories, each of which will be taken care of differently: first there are the balls $B(z,r)$ of radius $r \geq 1/2$, then there are the balls $B(z,r)$ of radius $0 < r < 1/2$ tangential to the ball B^0 at the point of affix 1, and finally the balls $B(z,r)$ of radius $0 < r < 1/2$ non tangential to the ball B^0 at the point of affix 1. We begin by taking care of the first category. We first show that X_2 satisfies (LD) for the unit ball B^{0,X_2}, for some exponent $\eta < 1$. Indeed, remark that if $k \geq \varepsilon^{-1/2}$, then $|I_k| = |t_k - t_{k+1}| \lesssim \frac{1}{k^2} \lesssim \varepsilon$,

and it implies that $\gamma(t)$ stays inside B_ε^{0,X_2} for all the $t \leq C\varepsilon^{1/2}$ for some uniform constant $0 < C < +\infty$. Besides, observe that we have

$$\operatorname{Card}\left\{k \in \mathbb{N} \mid k \lesssim \varepsilon^{-1/2}\right\} \lesssim \varepsilon^{-1/2},$$

and that for each one of the corresponding t_k, there is a contribution of at most ε to $\Lambda(B_\varepsilon^{0,X_2})$. Thus, we have

$$\Lambda(B_\varepsilon^{0,X_2}) \lesssim |[0, C\varepsilon^{1/2}]| + \varepsilon \times \operatorname{Card}\left\{k \in \mathbb{N} \mid k \lesssim \varepsilon^{-1/2}\right\} \lesssim \varepsilon^{1/2} + \varepsilon \times \varepsilon^{-1/2} \lesssim \varepsilon^{1/2}.$$

Since X_2 is Ahlfors-David, we have $\Lambda(B^{0,X_2}) \approx 1$, and (LD) follows.

Now, observe that this extends to all the balls $B = B(z,r)$ of radius $r \geq 1/2$. As a matter of fact, remark that we necessarily have $\Lambda(B_\varepsilon^{X_2}) \leq \Lambda(B_\varepsilon^{0,X_2}) \lesssim \varepsilon^{1/2}$. Indeed, there are at most two elements in $\overline{B^{X_2}} \backslash B^{X_2}$ outside of $\{\gamma(t) \mid 0 \leq t \leq \pi\}$. And it is also easy to see that $\overline{B^{X_2}} \backslash B^{X_2} \cap \{\gamma(t) \mid 0 \leq t \leq \pi\}$ can be injected inside $\overline{B^{0,X_2}} \backslash B^{0,X_2}$. Thus the preceding argument still applies. It follows that

$$\Lambda(B_\varepsilon^{X_2}) \lesssim \left(\frac{\varepsilon}{r}\right)^{1/2} r^{1/2} \lesssim \left(\frac{\varepsilon}{r}\right)^{1/2} r,$$

because $r \geq 1/2$. But since X_2 is Ahlfors-David, we have $\Lambda(B^{X_2}) \approx r$ and (LD) follows.

Now, we consider the balls $B = B(z,r)$ with $z = 1-r$ and $0 < r < 1/2$, tangential to B^0 at the point of affix 1. Let $\mathcal{C}$ denote in $\mathbb{R}^2$ the circle of center z and radius r. Switching to polar coordinates, for $t > 0$ sufficiently small ($t \leq t_{\max} = \arctan \frac{r}{1-r}$), denote by $M(t) = (t, v(t))$ the point of the circle $\mathcal{C}$ farthest from the origin, and let $u(t) = 1 - v(t)$ (see Figure 2.5). Then $v(t)$ satisfies the following equation

$$v(t)^2 - 2v(t)(1-r)\cos t + (1-r)^2 = r^2,$$

so that we have

$$u(t)^2 + 2u(t)[(1-r)\cos t - 1] + 2(1-r)[1 - \cos t] = 0,$$

hence

$$u(t) = 1 - (1-r)\cos t - [(1-r)^2 \cos^2 t + 1 - 2(1-r)]^{1/2}.$$

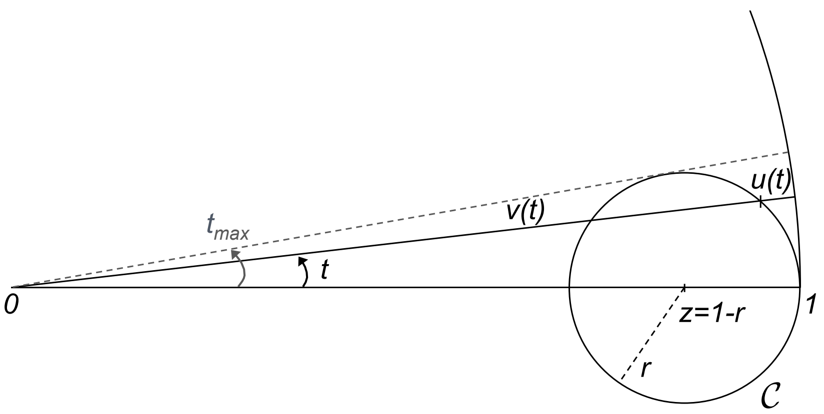

Figure 2.5: Existence of a separation between $\mathcal{C}$ and the unit circle.

There exists $t_0 > 0$ such that for every $0 < t < t_0$, $\cos t \leq 1 - \frac{t^2}{4}$ and $t^4 \leq t^2$. Then, for every $0 < t < t_0$ and $0 < r < 1/2$, we have

$$\begin{aligned}
u(t) &\geq 1 - (1-r)(1 - \frac{t^2}{4}) - \left[(1-r)^2(1-\frac{t^2}{4})^2 + 2r - 1\right]^{1/2} \\
&\geq r + \frac{1-r}{4}t^2 - \left[(1-r)^2(1 - \frac{t^2}{2} + \frac{t^4}{16}) + 2r - 1\right]^{1/2} \\
&\geq r + \frac{1-r}{4}t^2 - r\left[1 - \frac{(1-r)^2}{2r^2}t^2 + \frac{(1-r)^2}{16r^2}t^4\right]^{1/2} \\
&\geq r + \frac{1-r}{4}t^2 - r\left[1 - \frac{7(1-r)^2}{16r^2}t^2\right]^{1/2} \geq r + \frac{1-r}{4}t^2 - r\left[1 - \frac{1}{2}\frac{7(1-r)^2}{16r^2}t^2\right] \\
&\geq \frac{7-6r-r^2}{32r}t^2 \geq \frac{3}{16}t^2.
\end{aligned}$$

Furthermore, since $u(t)$ clearly is an increasing function, if $t_0 \leq t \leq t_{\max}$, we have $u(t) \geq \frac{3}{16}t_0^2$. Now, as $|\epsilon(t)| \leq A_0(t/\pi)^3$, it is clear that if $A_0 < \frac{3}{16}t_0^2$, then $\gamma(t)$ stays outside of B for every $0 < t \leq \pi$. Thus, B^{X_2} reduces to the open segment $]1-2r, 1[$ which is connected. It implies that $\mathrm{Card}(\overline{B^{X_2}} \backslash B^{X_2}) \leq 2$, and consequently, we have

$$\Lambda(B_\varepsilon^{X_2}) \lesssim 2\varepsilon \lesssim \left(\frac{\varepsilon}{r}\right)\Lambda(B^{X_2}),$$

as, once again, by the Ahlfors-David property of X_2 we have $\Lambda(B^{X_2}) \simeq r$.

It remains only to consider the balls $B = B(z,r)$ of radius $0 < r < 1/2$ non tangential to the ball B^0 at the point of affix 1. But it is immediate to see that the number of connected components of such a ball B^{X_2} is at most 2: B^{X_2} is a connected set in most cases, but there can be two connected components if $z = \gamma(\tau)$ with $0 < \tau \leq \pi$ is small and r is small enough (then $1 \notin B^{X_2}$ but $\gamma(t) \in B^{X_2}$ for some $-1 < t < 0$). Thus, as for the balls of the previous category, we have $\mathrm{Card}(\overline{B^{X_2}} \backslash B^{X_2}) \leq 4$, and

$$\Lambda(B_\varepsilon^{X_2}) \lesssim 4\varepsilon \lesssim \left(\frac{\varepsilon}{r}\right)\Lambda(B^{X_2}).$$

Putting all this together, we have proven that X_2 satisfies (LD) for $\eta = 1/2$. $\square$

The space X_2 is thus a counterexample to the implication (LD) $\Rightarrow$ (RLD). The proof of Theorem 2.1.3 is now complete.

Remarks 2.6.4. – Choosing $b(t) = \frac{\pi^{1+\alpha}}{t^\alpha}$ with $\alpha > 0$, and $a(t)$ accordingly, would give a space very similar to X_2, satisfying the same properties. Actually, the choice of a does not really matter as long as it ensures that γ keeps finite arclength.

- One could pick similar examples for other types of decreasing functions b. This range of examples shows that the Hardy property, as well as the relative and non relative layer decay properties, are very unstable, as it suffices to apply a slight perturbation to the initial space, where they were satisfied, to lose them.

2.7 Does (H) for a particular couple (p, p') imply (HP) ?

A natural question regarding these Hardy inequalities is whether proving (H) for a particular couple (p, p') is enough to have (HP). The following result gives a partial answer.

Proposition 2.7.1. *Let (X, ρ, μ) be a space of homogeneous type. Assume that X satisfies the annular decay property* (AD)*. Then, if X satisfies* (H) *for one particular couple (p, p'), X has the Hardy property* (HP)*.*

Proof. The argument uses the standard Calderón-Zygmund decomposition. We first prove that the kernel $\lambda(x,y)^{-1}$ satisfies the Hölder standard estimate (1.2.3). Because of the annular decay property (AD), we have

$$\begin{aligned}\left|\frac{1}{\lambda(x,y)}-\frac{1}{\lambda(x,y')}\right| &= \left|\frac{\mu(B(x,\rho(x,y')))-\mu(B(x,\rho(x,y)))}{\mu(B(x,\rho(x,y')))\mu(B(x,\rho(x,y)))}\right| \\ &\lesssim \left|\frac{\rho(x,y)-\rho(x,y')}{\rho(x,y)}\right|^{\eta}\frac{\mu(B(x,\rho(x,y)))}{\mu(B(x,\rho(x,y)))^2} \\ &\lesssim \frac{1}{\lambda(x,y)}\left(\frac{\rho(y,y')}{\rho(x,y)}\right)^{\eta},\end{aligned}$$

if $\rho(y,y') \le \frac{1}{2}\rho(x,y)$, and it is obviously symmetric in x and y. The kernel $\frac{1}{\lambda(x,y)}$ is thus a standard Calderón-Zygmund kernel.

Now, fix a ball $B = B(z_B, r) \subset X$. We assume that (H) holds for the couple (p, p'). By Fubini's theorem, this implies that one can define $T : L^p(B) \to L^p(2B\backslash B)$ by the absolutely convergent integral

$$(Tf)(y) = \int_B \frac{f(x)}{\lambda(x,y)}\mathrm{d}\mu(x) \quad \text{for almost every} \quad y \in 2B\backslash B.$$

We show that T is of weak type $(1,1)$: we prove that for all $f \in L^1(B)$, with $\operatorname{supp} f \subset B$

$$\mu(\{x \in 2B\backslash B \mid |Tf(x)| > \alpha\}) \lesssim \frac{1}{\alpha}\|f\|_{L^1(B)}.$$

The idea is to write a Calderón-Zygmund decomposition of f on X. However, we have to be a bit careful, because if we write the standard decomposition $f = g + \sum_{i\in I} b_i$ directly on the ball B, g will not be supported inside B. To avoid this problem, let us use a Whitney partition of the ball B, as in Section 2.2: consider the dyadic cubes $Q \subset B$ which are maximal for the relation $l(Q) \le \rho(Q, B^c)$. Call them Q_j, $j \in J$. They are mutually disjoint and they realize a partition of the ball B but for a set of measure zero. Now, for $f \in L^1(B)$, with $\operatorname{supp} f \subset B$, we have $f = \sum_j f 1_{Q_j}$ $\mu\, a.e.$ Set $f_j = f 1_{Q_j}$, and for every fixed j, write a Calderón-Zygmund decomposition of f_j on Q_j: $f_j = g_j + \sum_{i\in I} b_{i,j}$ with

- $g_j = 1_{\Omega^c_{j,\alpha}} f_j$ where $\Omega_{j,\alpha} = \{x \in Q_j \mid |f_j^*(x)| > \alpha\}$, f_j^* denoting the dyadic maximal function of f_j on Q_j. Thus $\operatorname{supp} g_j \subset \overline{Q_j}$, $g_j \in L^\infty$, $\|g_j\|_\infty \le \alpha$, and $\|g_j\|_p \le (\|g_j\|_\infty^{p-1}\|g_j\|_1)^{1/p} \le \alpha^{1/p'}\|f_j\|_1^{1/p}$.
- $\operatorname{supp} b_{i,j} \subset \overline{Q_{i,j}}$ where the sets $Q_{i,j}$ are dyadic subcubes of Q_j of center $z_{Q_{i,j}}$, realizing in turn a Whitney partition of the open set $\Omega_{j,\alpha}$: $\mu(\Omega_{j,\alpha}\backslash \cup_i Q_{i,j}) = 0$, $Q_{i,j}$ are mutually disjoint, and there exists a dimensional constant $C > C_1$ such that for every i, $B(z_{Q_{i,j}}, Cl(Q_{i,j})) \cap \Omega^c_{j,\alpha} \neq \varnothing$. In addition, we have $[|b_{i,j}|]_{Q_{i,j}} \lesssim \alpha$ and $[b_{i,j}]_{Q_{i,j}} = 0$.
- $\|f_j\|_1 = \sum_i \|b_{i,j}\|_1 + \|g_j\|_1$.

Now, set $g = \sum_j g_j$. Observe that we have $\operatorname{supp} g \subset B$, and $f = g + \sum_{i,j} b_{i,j}$. Applying (H) for the couple (p, p'), and the disjointness of the dyadic cubes Q_j, we have

$$\begin{aligned}\mu(\{x \in 2B\backslash B \mid |Tg(x)| > \alpha/2\}) &\lesssim \frac{1}{\alpha^p}\int_{2B\backslash B} |Tg|^p \mathrm{d}\mu \lesssim \frac{1}{\alpha^p}\int_B |g|^p \mathrm{d}\mu \\ &= \frac{1}{\alpha^p}\sum_j \int_{Q_j} |g_j|^p \mathrm{d}\mu = \frac{1}{\alpha^p}\sum_j \|g_j\|_p^p \\ &\leq \frac{1}{\alpha}\sum_j \|f_j\|_1 = \frac{1}{\alpha}\|f\|_{L^1(B)}.\end{aligned}$$

Also,

$$\begin{aligned}\mu(\{x \in 2B\backslash B \mid |T(\sum_{i,j} b_{i,j})(x)| > \alpha/2\}) &\leq \mu(\bigcup_{i,j}\widehat{Q_{i,j}}) + \mu\Big((\bigcup_{i,j}\widehat{Q_{i,j}})^c \cap \{x \in X \mid \sum_{i,j}|Tb_{i,j}(x)| > \alpha/2\}\Big) \\ &\lesssim \sum_{i,j}\mu(Q_{i,j}) + \mu(\{x \in X \mid \sum_{i,j}|Tb_{i,j}(x)| 1_{\widehat{Q_{i,j}}^c} > \alpha/2\}) \\ &\lesssim \sum_{i,j}\mu(Q_{i,j}) + \sum_{i,j}\frac{1}{\alpha}\int_{\widehat{Q_{i,j}}^c} |Tb_{i,j}| \mathrm{d}\mu.\end{aligned}$$

Since $b_{i,j}$ is of mean 0 on $Q_{i,j}$, and because the kernel $\lambda(x,y)^{-1}$ satisfies (1.2.3), we can apply the standard estimate (1.2.6) to obtain

$$\begin{aligned}\int_{\widehat{Q_{i,j}}^c} |Tb_{i,j}| \mathrm{d}\mu &\lesssim \|b_{i,j}\|_1 \int_{\widehat{Q_{i,j}}^c} \frac{1}{\lambda(z_{Q_{i,j}}, y)}\left(\frac{l(Q_{i,j})}{\rho(z_{Q_{i,j}}, y)}\right)^\alpha \mathrm{d}\mu(y) \\ &\lesssim \|b_{i,j}\|_1 \sum_{k\geq 0}\int_{2^k l(Q_{i,j}) \leq \rho(y, z_{Q_{i,j}}) < 2^{k+1} l(Q_{i,j})} \frac{2^{-k\alpha}}{\mu(B(z_{Q_{i,j}}, 2^k l(Q_{i,j})))} \mathrm{d}\mu(y) \\ &\lesssim \|b_{i,j}\|_1 \sum_{k\geq 0} 2^{-k\alpha} \lesssim \alpha\mu(Q_{i,j}).\end{aligned}$$

Finally, we have

$$\mu(\{x \in 2B\backslash B \mid |T(\sum_{i,j} b_{i,j})(x)| > \alpha/2\}) \lesssim \sum_{i,j}\mu(Q_{i,j}) \lesssim \sum_j \mu(\Omega_{j,\alpha}) \lesssim \sum_j \frac{1}{\alpha}\|f_j\|_1 = \frac{1}{\alpha}\|f\|_{L^1(B)}.$$

Thus, we get as expected

$$\mu(\{x \in 2B\backslash B \mid |Tf(x)| > \alpha\}) \lesssim \frac{1}{\alpha}\|f\|_{L^1(B)}.$$

By interpolation, it follows that we have $Tf \in L^q(2B\backslash B)$ for every $1 < q \leq p$ and $f \in L^q(B)$, with $\operatorname{supp} f \subset B$.

It remains to apply some duality argument to conclude. (H) for the couple (p, p') implies that T^* is bounded from $L^{p'}(2B\backslash B)$ to $L^{p'}(B)$. We show again that T^* is of weak type $(1,1)$: we prove that for all $f \in L^1(2B\backslash B)$, with $\operatorname{supp} f \subset 2B\backslash B$,

$$\mu(\{x \in B \mid |T^* f(x)| > \alpha\}) \lesssim \frac{1}{\alpha}\|f\|_{L^1(2B\backslash B)}.$$

The idea is to use as before a Whitney partition of the open set $\overline{B}^c$: consider the dyadic cubes $Q_j \subset \overline{B}^c$ that are maximal for the relation $l(Q) \leq \rho(Q, B)$. They partition $\overline{B}^c$ but for a set of measure 0. Of course, there are some of these Q_j that intersect $(2B)^c$. To overcome this problem, let us keep only the cubes $Q_j, j \in J$, intersecting the set $cB\backslash B$, with $1 < c < \frac{C_1+2}{C_1+1}$. These cubes satisfy, for every $j \in J$, $l(Q_j) \leq \rho(Q_j, B) \leq (c-1)r$. Because of property (4) of Christ's dyadic cubes (Lemma 1.2.3), it implies that for every $x \in Q_j$, $\rho(x, z_B) \le cr + \operatorname{diam} Q_j \le cr + C_1(c-1)r < 2r$, so that $Q_j \subset 2B\backslash B$. Now, for $f \in L^1(2B\backslash B)$, with $\operatorname{supp} f \subset 2B\backslash B$, write

$$f = \sum_{j\in J} f 1_{Q_j} + f 1_{(2B\backslash B)\backslash \cup_{j\in J} Q_j} = \sum_{j\in J} f_j + \tau \quad \mu\, a.e.$$

Apply the same argument as above to f_j, for every $j \in J$, to get

$$\mu(\{x \in B \mid |T^*(\sum_{j\in J} f_j)(x)| > \alpha/2\}) \lesssim \frac{1}{\alpha}\|f\|_{L^1(2B\backslash B)}.$$

For the remaining term τ, observe that since $\Big((2B\backslash B)\backslash \cup_{j\in J} Q_j\Big) \subset (cB)^c$, $\operatorname{supp}\tau \subset (cB)^c$, and we trivially have $T^*\tau \in L^1(B)$. Indeed, by the doubling property, we have

$$\int_B |T^*\tau| \mathrm{d}\mu \le \int_B \int_{(cB)^c} \frac{|\tau(y)|}{\lambda(x,y)} \mathrm{d}\mu(y)\mathrm{d}\mu(x) \le \|\tau\|_1 \frac{\mu(B(x,r))}{\mu(B(x,cr))} \lesssim \|f\|_{L^1(2B\backslash B)}.$$

Hence,

$$\mu(\{x \in B \mid |T^*\tau(x)| > \alpha\}) \lesssim \frac{1}{\alpha}\|f\|_{L^1(2B\backslash B)},$$

and T^* is of weak type $(1,1)$. By interpolation, it follows that $T^* f \in L^q(B)$ for every $1 < q \le p'$ and $f \in L^q(2B\backslash B)$, with $\operatorname{supp} f \subset 2B\backslash B$. By duality, the Hardy property (HP) is satisfied on X. □

Remarks 2.7.2.
1. This result is interesting because, although we did not manage to produce a counterexample yet, we think that, similarly to (LD) and (RLD), (AD) and (RAD) are not equivalent. Under (RAD), we already know that the Hardy property (HP) is satisfied. What we have proven is that if we only assume (AD), (HP) is satisfied provided it is true for one particular couple (p, p').
2. It would be more satisfying to prove this result under the weaker layer decay property (LD). Indeed, we have seen that it is the key property in our study, and it is the one which truly expresses the distribution of points at the border of balls in the homogeneous space. But the regularity of the kernel $\frac{1}{\lambda(x,y)}$ only comes at the expense of (AD).

2.8 Open questions

We end this chapter by giving a few open questions we did not manage to answer yet.

1. Are (RAD) and (AD) equivalent ? As we mentioned earlier, we do not know the answer to this question yet. Because of the similarity with (RLD) and (LD), we do think that it is not the case and that the implication (AD) $\Rightarrow$ (RAD) is false. We have not been able to produce a counterexample though. It might be possible to use again the space X_2 presented in Section 2.6.2, and to refine the argument we gave to prove that (LD) is satisfied by this space. However, our proof does not work directly as there can very well be points in a corona $C_{r+\varepsilon,r}(z)$ even if $\gamma(t)$ stays outside of $B = B(z,r)$ all the time, and thus $B_\varepsilon = \varnothing$. Some additional work is needed.

2. Which properties are necessary conditions for the Hardy property (HP) ? In particular, does (HP) imply that for every ball B of the homogeneous space, $\mu(\overline{B}\backslash B) = 0$? We think that the answer to the latter is affirmative, but we have been unable to prove it. Similarly, does (HP) imply (RLD) ? We think this has to be false, but again, we have not come up with a counterexample yet.
3. Is the result of Proposition 2.7.1 true under the weaker property (LD) ? See Remarks 2.7.2.

Chapter 3

Local Tb theorems

Contents

This chapter is mostly taken from an article written with Pascal Auscher, to appear in the Journal of Geometric Analysis [AR].

3.1 Main results and comments

Throughout this section, we assume that (X, ρ, μ) is a space of homogeneous type. First, let us state an immediate corollary to Proposition 2.1.1, a Hardy type inequality which we will frequently apply throughout this chapter.

Proposition 3.1.1. *Let T be a singular integral operator on X. Let $1 < \nu < +\infty$ with dual exponent ν'.*

- *There exists $C < +\infty$ such that for every disjoint dyadic cubes Q,Q', and every functions f, g respectively supported on Q, Q', with $f \in L^\nu(Q)$, $g \in L^{\nu'}(Q')$, we have*

$$|\langle Tf, g\rangle| \leq C\|f\|_{L^\nu(Q)}\|g\|_{L^{\nu'}(Q')}. \tag{3.1.1}$$

- *There exists $C < +\infty$ such that for every dyadic cube Q, and every function f supported on Q with $f \in L^\nu(Q)$, we have*

$$\|Tf\|_{L^\nu(\widehat{Q}\setminus Q)} \leq C\|f\|_{L^\nu(Q)}. \tag{3.1.2}$$

Proof. Use (1.2.2), Proposition 2.1.1, and for the second part the fact that the number of neighbors of any given cube is uniformly bounded. □

Now, our main result is the following.

Theorem 3.1.2. *Let $1 < p, q < +\infty$. Let T be a singular integral operator with locally bounded kernel. Assume that there exists a (p, q) dyadic pseudo-accretive system adapted to T. Then T extends to a bounded operator on $L^2(X)$, with bounds independent of $\|K\|_{\infty,loc}$.*

Let us explain what is a (p, q) dyadic pseudo-accretive system adapted to T.

Definition 3.1.3. (p, q) dyadic pseudo-accretive system.
Let $1 < p, q < +\infty$ with dual exponents p', q', and let T be a singular integral operator on X. We say that a collection of functions $(\{b^1_Q\}_Q, \{b^2_Q\}_Q)$ is a (p, q) dyadic pseudo-accretive system adapted to T if there exists a constant $C_A < +\infty$, such that for each dyadic cube Q, b^1_Q, b^2_Q are supported on Q with

$$\int_Q b^1_Q \mathrm{d}\mu = \mu(Q) = \int_Q b^2_Q \mathrm{d}\mu, \tag{3.1.3}$$

$$\int_Q \left(|b^1_Q|^p + |b^2_Q|^q\right) \mathrm{d}\mu \leq C_A\, \mu(Q), \tag{3.1.4}$$

$$\int_{\widehat{Q}} \left(|T(b^1_Q)|^{q'} + |T^*(b^2_Q)|^{p'}\right) \mathrm{d}\mu \leq C_A\, \mu(Q), \tag{3.1.5}$$

Furthermore, the functions b^i_Q are required to satisfy the following properties: for all $C < +\infty$, there exists $C_H < +\infty$ and $1 < \nu < +\infty$ such that for every (Q, Q') dyadic cubes, for every dyadic cubes $R' \subset Q'$, $R_n \subset (\widehat{R'}\setminus R') \cap Q$, R_n mutually disjoint, with $\rho(R', R_n) < l(R_n)$, $[|b^1_{Q'}|^p]_{R'} \leq C$, $[|b^2_Q|^q]_{R_n} \leq C$, and for every set of coefficients $(\alpha_n)_n$, we have

$$\left|\left\langle b^2_Q \left(\sum_n \alpha_n 1_{R_n}\right), T(b^1_{Q'} 1_{R'})\right\rangle\right| \leq C_H \left\|\sum_n \alpha_n 1_{R_n}\right\|_\nu \mu(R')^{\frac{1}{\nu'}}. \tag{3.1.6}$$

We also need a control of the diagonal terms: for all $C < +\infty$, there exists $C_{WBP} < +\infty$ such that for every (Q, Q') dyadic cubes, for every dyadic cube $R \subset Q \cap Q'$ with $[|b^1_{Q'}|^p]_R \leq C$, $[|b^2_Q|^q]_R \leq C$, we have

$$\left|\left\langle b^2_Q 1_R, T(b^1_{Q'} 1_R)\right\rangle\right| \leq C_{WBP}\, \mu(R). \tag{3.1.7}$$

We require the b_Q^i to satisfy also the symmetric properties, with respectively b^1 instead of b^2, p instead of q, q instead of p, and T^* instead of T.

We remark that the statement has a converse. If T is bounded, then the collection $(\{1_Q\}_Q, \{1_Q\}_Q)$ is a (p,q) accretive system adapted to T for any exponents. Several comments are in order. Let us begin with the case $1/p + 1/q > 1$.

1. In this case, we cannot use the Hardy inequality (2.1.1) with exponents p and q replacing ν and ν'. So our hypotheses (3.1.6) and (3.1.7) are a substitute for the missing (2.1.1). In practice, they could very well hold due to specific relations or cancellations between the b_Q^i and T. See Section 3.3 for an explicit example of pseudo-accretive system with low integrability.
2. Observe that, while being a rather unsatisfactory condition, (3.1.6) and (3.1.7) imply the following weaker statement: for every dyadic subcubes R, R' of cubes Q, Q', such that R, R' are neighbors, and $b^1_{Q'}$, b^2_Q satisfy the same size estimates as above on R, R', we have
$$|\langle b_Q^2 1_R, T(b^1_{Q'} 1_{R'})\rangle| \lesssim \mu(R'). \tag{3.1.8}$$
This weaker property is more satisfactory as it is a lot closer to what we are used to calling a weak boundedness property. Unfortunately, (3.1.8) suffices for all but one term we could not estimate otherwise than assuming the stronger property (3.1.6).
3. Note that it is not clear whether for any systems $(b_Q^1), (b_Q^2)$ satisfying (3.1.3) and (3.1.4), if T is bounded on $L^2(X)$ then (3.1.5), (3.1.6), (3.1.7) hold. But a (p,q) accretive system adapted to T is, as its name indicates, not any pair of systems.

Let us now assume $1/p + 1/q \leq 1$. First, we show that (3.1.6) is necessarily satisfied as a consequence of (3.1.4) and (3.1.5), and it is an application of Proposition 3.1.1. This is stated in the following proposition.

Proposition 3.1.4. *Let $1 < p, q < +\infty$ with dual exponents p', q', and such that $1/p+1/q \leq 1$. Let T be a singular integral operator on X. Suppose that $(b_Q^1), (b_Q^2)$ constitute a collection of functions supported on Q, satisfying* (3.1.3)*,* (3.1.4)*, and the following weaker form of* (3.1.5)*,*
$$\int_Q \left(|T(b_Q^1)|^{q'} + |T^*(b_Q^2)|^{p'}\right) \mathrm{d}\mu \leq C_A\, \mu(Q). \tag{3.1.9}$$
Then the functions b_Q^i satisfy (3.1.5) *and* (3.1.6)*.*

Proof. We first prove (3.1.5). By (3.1.2), we have
$$\begin{aligned}\int_{\widehat{Q}\setminus Q} \left(|T(b_Q^1)|^{q'} + |T^*(b_Q^2)|^{p'}\right) \mathrm{d}\mu &\lesssim \int_Q (|b_Q^1|^{q'} + |b_Q^2|^{p'})\mathrm{d}\mu \\ &\lesssim \left(\int_Q |b_Q^1|^p \mathrm{d}\mu\right)^{\frac{q'}{p}} \mu(Q)^{1-\frac{q'}{p}} \quad + \quad \left(\int_Q |b_Q^2|^q \mathrm{d}\mu\right)^{\frac{p'}{q}} \mu(Q)^{1-\frac{p'}{q}} \\ &\lesssim \mu(Q),\end{aligned}$$
where we have applied the Hölder inequality to get the second inequality, which is made possible only because $1/p+1/q \leq 1$ implies that $p/q', q/p' \geq 1$. The inequality (3.1.5) follows. Now we prove (3.1.6) with $\nu = q$ (and the symmetrical inequality will hold for $\nu = p$). Write
$$\left|\left\langle b_Q^2 \left(\sum_n \alpha_n 1_{R_n}\right), T(b^1_{Q'} 1_{R'})\right\rangle\right| \lesssim \left\| b_Q^2 \left(\sum_n \alpha_n 1_{R_n}\right)\right\|_q \|T(b^1_{Q'} 1_{R'})\|_{L^{q'}(\widehat{R'}\setminus R')}$$

By (3.1.2), and using again the fact that $1/p + 1/q \leq 1$,

$$\|T(b^1_{Q'} 1_{R'})\|_{L^{q'}(\widehat{R'}\setminus R')} \lesssim \|b^1_{Q'}\|_{L^{q'}(R')} \lesssim \mu(R')^{\frac{1}{q'}}.$$

Moreover, because the R_n are disjoint,

$$\left\| b^2_Q \left(\sum_n \alpha_n 1_{R_n} \right) \right\|_q \lesssim \left(\sum_n |\alpha_n|^q \int_{R_n} |b^2_Q|^q \mathrm{d}\mu \right)^{\frac{1}{q}} \lesssim \left(\sum_n |\alpha_n|^q \mu(R_n) \right)^{\frac{1}{q}} \lesssim \left\| \sum_n \alpha_n 1_{R_n} \right\|_q.$$

□

As formulated, the inequality (3.1.7) is not a direct consequence of (3.1.3), (3.1.4) and (3.1.9). For this, one needs further control for $b^1_{Q'}, T(b^1_{Q'}), b^2_Q, T^*(b^2_Q)$ on R than the one written in (3.1.9). This can be achieved (see Section 3.7.2). In other words, when $1/p + 1/q \leq 1$, a possible definition of a (p, q) accretive system adapted to T to prove Theorem 3.1.2 is (3.1.3), (3.1.4) and (3.1.9). In particular, this covers the case $p = q = 2$.

Our argument to prove Theorem 3.1.2 involves using the BCR algorithm introduced in [BCR], applied with Haar wavelets adapted to the dyadic pseudo-accretive system (b^i_Q), which allows us to obtain a direct proof without having to use the decomposition of a singular integral operator as the sum of a bounded operator and a perfect dyadic singular integral operator used in [AY] and [TY]. We will develop this in the following sections.

Since dyadic cubes can be ugly sets in practice, on which checking (3.1.5), (3.1.6) or (3.1.7) might be difficult, a natural question is whether or not one can switch from dyadic hypotheses in Theorem 3.1.2 to hypotheses made on balls. The following Theorem 3.1.5 deals with this concern.

Theorem 3.1.5. *Let $1 < p, q < +\infty$ with dual exponents p', q', such that $1/p + 1/q \leq 1$. Let T be a singular integral operator with locally bounded kernel. Assume that there exists a collection of functions $(\{b^1_B\}_B, \{b^2_B\}_B)$, such that there exists a constant $C < +\infty$ such that for every ball B in X, b^i_B is supported on B, and we have*

$$\int_B b^1_B \mathrm{d}\mu = \mu(B) = \int_B b^2_B \mathrm{d}\mu, \tag{3.1.10}$$

$$\int_B \left(|b^1_B|^p + |b^2_B|^q \right) \mathrm{d}\mu \leq C\, \mu(B), \tag{3.1.11}$$

$$\int_X \left(|T(b^1_B)|^{q'} + |T^*(b^2_B)|^{p'} \right) \mathrm{d}\mu \leq C\, \mu(B), \tag{3.1.12}$$

Then T extends to a bounded operator on $L^2(X)$, with bounds independent of $\|K\|_{\infty,loc}$. Furthermore, if (3.1.12) *is replaced by the weaker uniform bound*

$$\int_B \left(|T(b^1_B)|^{q'} + |T^*(b^2_B)|^{p'} \right) \mathrm{d}\mu \leq C\, \mu(B), \tag{3.1.13}$$

then the conclusion still holds provided X has the Hardy property (HP).

Remark 3.1.6. 1. In [TY], the authors state Theorem 3.1.5, with hypothesis (3.1.13), but they do not assume X satisfies (HP). They justify their statement by reducing to a dyadic pseudo-accretive system as in our proof of Theorem 3.1.5 (see the following section). It might have been a bit over-optimistic, as we do not see how to obtain (3.1.5) from (3.1.13) without using a Hardy type inequality, and the Hardy property (HP) might not always be satisfied in a general space of homogeneous type.

2. We had to assume that $1/p + 1/q \leq 1$. Indeed, when $1/p + 1/q > 1$, the incompatibility of exponents p, q makes things tricky, and we cannot see a way of adapting (3.1.6) and (3.1.7) to the balls setting.
3. We will prove in the next section that integrability over X is equivalent to integrability over CB for any $C > 1$, and it obviously implies integrability over B. Conversely though, integrability over B does not necessarily imply integrability over X. It does when a Hardy type inequality on balls is satisfied in the space X. Some particular relation between T and the b^i_B could also be a substitute.

Finally, a natural question is whether one can relax the support condition on the accretive system and impose that the b^i_Q are only supported on a slight enlargment, say $\widehat{Q}$, of Q. We answer this question with following Theorem 3.1.7 [1].

Theorem 3.1.7. *Let $1 < p, q < +\infty$ with dual exponents p', q'. Let T be a singular integral operator with locally bounded kernel. Assume that there exists a collection of functions $(\{b^1_Q\}_Q, \{b^2_Q\}_Q)$, and a constant $C < +\infty$ such that for every dyadic cube Q in X, b^i_Q is supported on $\widehat{Q}$, and we have*

$$\int_{\widehat{Q}} b^1_Q \mathrm{d}\mu = \mu(Q) = \int_{\widehat{Q}} b^2_Q \mathrm{d}\mu, \tag{3.1.14}$$

$$\int_{\widehat{Q}} \left(|b^1_Q|^p + |b^2_Q|^q\right) \mathrm{d}\mu \leq C\, \mu(Q), \tag{3.1.15}$$

$$\int_X \left(|T(b^1_Q)|^{q'} + |T^*(b^2_Q)|^{p'}\right) \mathrm{d}\mu \leq C\, \mu(Q). \tag{3.1.16}$$

Suppose as well that for all $C < +\infty$, there exists $C_H < +\infty$ and $1 < \nu < +\infty$ such that for every (Q, Q') dyadic cubes, for every dyadic cubes $R' \subset \widehat{Q'}$, $R_n \subset (\widehat{R'} \backslash R') \cap \widehat{Q}$, R_n mutually disjoint, with $\rho(R', R_n) < l(R_n)$, $[|b^1_{Q'}|^p]_{R'} \leq C$, $[|b^2_Q|^q]_{R_n} \leq C$, and for every set of coefficients $(\alpha_n)_n$, we have

$$\left|\left\langle b^2_Q \left(\sum_n \alpha_n 1_{R_n}\right), T(b^1_{Q'} 1_{R'})\right\rangle\right| \leq C_H \left\|\sum_n \alpha_n 1_{R_n}\right\|_\nu \mu(R')^{\frac{1}{\nu'}}. \tag{3.1.17}$$

Suppose also that for all $C < +\infty$, there exists $C_{WBP} < +\infty$ such that for every (Q, Q') dyadic cubes, for every dyadic cube $R \subset \widehat{Q} \cap \widehat{Q'}$ with $[|b^1_{Q'}|^p]_R \leq C$, $[|b^2_Q|^q]_R \leq C$, we have

$$\left|\left\langle b^2_Q 1_R, T(b^1_{Q'} 1_R)\right\rangle\right| \leq C_{WBP}\, \mu(R). \tag{3.1.18}$$

We naturally require the b^i_Q to satisfy the symmetric properties, with respectively b^1 instead of b^2, p instead of q, q instead of p, and T^* instead of T.

Then T extends to a bounded operator on $L^2(X)$, with bounds independent of $\|K\|_{\infty,loc}$.

Remark 3.1.8. As for Theorem 3.1.2, when $1/p + 1/q \leq 1$, we have a simpler formulation because one needs not (3.1.17) and (3.1.18) and the result holds only assuming (3.1.14), (3.1.15) and (3.1.16).

1. We thank T. Hytönen for his suggestion which led us to the formulation of this theorem.

3.2 Proofs of Theorem 3.1.5 and Theorem 3.1.7

Proof of Theorem 3.1.5. Theorem 3.1.5 is a direct consequence of Theorem 3.1.2 in the particular case when $1/p + 1/q \leq 1$. In this case, one needs not check hypotheses (3.1.6) and (3.1.7) as we remarked earlier (see Section 3.7.2 for the detail). We begin by proving the first part of Theorem 3.1.5. Assuming there exists a pseudo-accretive system $(\{b_B^1\}, \{b_B^2\})$ on the balls B of X satisfying (3.1.10), (3.1.11) and (3.1.12), for $i = 1, 2$, and Q a dyadic cube, let us consider the functions $b_{B_Q}^i$ where B_Q is a ball contained in Q of radius comparable to the diameter of Q. The existence of such a ball is given by property (5) of Lemma 1.2.3. Then, normalizing the $b_{B_Q}^i$ so that they have mean 1 on Q, we obtain a collection of functions b_Q^i, supported on Q, and that satisfy

$$[b_Q^i]_Q = 1,$$

$$\int_Q (|b_Q^1|^p + |b_Q^2|^q) \mathrm{d}\mu \lesssim \int_B (|b_{B_Q}^1|^p + |b_{B_Q}^2|^q) \mathrm{d}\mu \lesssim \mu(Q),$$

$$\int_{\widehat{Q}} \left(|T(b_Q^1)|^{q'} + |T^*(b_Q^2)|^{p'}\right) \mathrm{d}\mu \lesssim \int_X \left(|T(b_{B_Q}^1)|^{q'} + |T^*(b_{B_Q}^2)|^{p'}\right) \mathrm{d}\mu \lesssim \mu(Q).$$

Thus, applying Theorem 3.1.2 in the particular case when $1/p + 1/q \leq 1$, we obtain the boundedness of T on $L^2(X)$. For the second part of Theorem 3.1.5, we prove that if X has the Hardy property then (3.1.13) implies (3.1.12). Observe first that as a consequence of (H) and of the fact that $1/p + 1/q \leq 1$, we have

$$\int_{2B \backslash B} \left(|T(b_B^1)|^{q'} + |T^*(b_B^2)|^{p'}\right) \mathrm{d}\mu \lesssim \mu(B).$$

The result is then a direct application of the following lemma.

Lemma 3.2.1. *Let T be a singular integral operator on X. Let $\alpha > 1$, let B be a ball in X, and f_B a function supported on B such that $\|f_B\|_{L^1(B)} \lesssim \mu(B)$. Then $T(f_B) \in L^\nu(X \backslash \alpha B)$ and $\|T(f_B)\|_{L^\nu(X\backslash \alpha B)}^\nu \lesssim \mu(B)$ for all $1 < \nu < +\infty$.*

Admitting Lemma 3.2.1, one only has to apply it to the functions b_B^i, with $\alpha = 2$, and $\nu_1 = q', \nu_2 = p'$, to obtain

$$\int_{X \backslash 2B} \left(|T(b_B^1)|^{q'} + |T^*(b_B^2)|^{p'}\right) \mathrm{d}\mu \lesssim \mu(B).$$

Summing up, we obtain (3.1.12) as desired. □

Proof of Lemma 3.2.1. For $x \in X \backslash 2B$, applying (1.2.4) and (1.2.2), write

$$|T(f_B)(x)| \lesssim \int_B \frac{|f_B(y)|}{\lambda(x,y)} \mathrm{d}\mu(y) \lesssim \frac{\mu(B)}{\mu(B(x, \rho(x,B)))}.$$

Now, for $j \geq 0$, set $B_j = 2^{j+1}\alpha B$ and $C_j = B_j \backslash B_{j-1}$, and split the integral over $X \backslash \alpha B$ as

$$\int_{X\backslash \alpha B} |T(f_B)|^\nu \mathrm{d}\mu \lesssim \mu(B)^\nu \sum_{j \geq 0} \frac{\mu(C_j)}{\mu(B_j)^\nu}.$$

Splitting this sum into bundles for which $\mu(B_j)$ is comparable, we have

$$\begin{aligned}\sum_{j\geq 0}\frac{\mu(C_j)}{\mu(B_j)^{\nu}} &= \sum_{k\geq 0}\sum_{j:2^k\mu(B)\leq\mu(B_j)<2^{k+1}\mu(B)}\frac{\mu(B_j)-\mu(B_{j-1})}{\mu(B_j)^{\nu}}\\ &\leq \sum_{k\geq 0}\frac{1}{(2^k\mu(B))^{\nu}}\sum_{j:2^k\mu(B)\leq\mu(B_j)<2^{k+1}\mu(B)}(\mu(B_j)-\mu(B_{j-1}))\\ &\leq \sum_{k\geq 0}\frac{2^{k+1}\mu(B)}{(2^k\mu(B))^{\nu}}\lesssim \mu(B)^{1-\nu}.\end{aligned}$$

Finally, we obtain as desired

$$\int_{X\setminus\alpha B}|T(f_B)|^{\nu}\mathrm{d}\mu\lesssim\mu(B),$$

and Lemma 3.2.1 follows. □

Proof of Theorem 3.1.7. The proof uses the same idea as in the previous argument. We define new systems from the given ones and check the hypotheses of Theorem 3.1.2. These verifications can be long and technical to write but not difficult. They only use basic Calderón-Zygmund estimates for singular integrals (1.2.6) and Hardy inequalities on dyadic cubes (3.1.1). Details are as follows.

Let $0 < C < +\infty$, and suppose that there exists a dyadic pseudo-accretive system $(\{b_Q^1\},\{b_Q^2\})$ on X satisfying (3.1.14), (3.1.15), (3.1.16), (3.1.18) and (3.1.17) for some $1<\sigma<+\infty$. For $i=1,2$, and Q a dyadic cube, set $\beta_Q^i=\lambda_Q b_{Q^k}^i$, where Q^k is a dyadic subcube of Q such that $l(Q^k)=\delta^k l(Q)$, $\rho(Q^k,Q^c)\geq(1+C_1)l(Q^k)$ where C_1 is the constant intervening in property (4) of Lemma 1.2.3 (that is, Q^k is far enough from the border of Q), and λ_Q is such that $[\beta_Q^i]_Q=1$ for every cube Q. That Q^k can be so chosen follows from the small boundary condition (1.2.1) because such cubes occupy a set of measure larger than $\frac{1}{2}\mu(Q)$ if k is taken large enough (independently of Q). We prove that $\{\beta_Q^1\}_Q,\{\beta_Q^2\}_Q$ form a (p,q) dyadic pseudo-accretive system adapted to T. Remark first that because Q^k is taken sufficiently far from the boundary of Q, the β_Q^i are always supported inside Q, so that (3.1.3) is satisfied. Next, observe that the $(\lambda_Q)_Q$ form a set of uniformly bounded coefficients : as a consequence of (3.1.14), we have for every dyadic cube Q,

$$\lambda_Q=\frac{\mu(Q)}{\mu(Q^k)}\lesssim 1,$$

by the doubling property. We have by (3.1.15)

$$\int_Q(|\beta_Q^1|^p+|\beta_Q^2|^q)\mathrm{d}\mu\lesssim\int_{\widehat{Q^k}}(|b_{Q^k}^1|^p+|b_{Q^k}^2|^q)\mathrm{d}\mu\lesssim\mu(Q^k)\lesssim\mu(Q),$$

so that (3.1.4) is satisfied. Similarly, for (3.1.5), we have by (3.1.16)

$$\int_{\widehat{Q}}(|T(\beta_Q^1)|^{q'}+|T^*(\beta_Q^2)|^{p'})\mathrm{d}\mu\lesssim\int_X(|T(b_{Q^k}^1)|^{q'}+|T^*(b_{Q^k}^2)|^{p'})\mathrm{d}\mu\lesssim\mu(Q^k)\lesssim\mu(Q).$$

Let us now prove (3.1.7). Let Q_1,Q_2 be two dyadic cubes and let $R\subset Q_1\cap Q_2$ be such that $[|\beta_{Q_1}^1|^p]_R\leq C$, $[|\beta_{Q_2}^2|^q]_R\leq C$. First, if $R\subset\widehat{Q_1^k}\cap\widehat{Q_2^k}$, (3.1.7) is a direct consequence of (3.1.18). Suppose now that for example $R\subset\widehat{Q_2^k}$, $R\not\subset\widehat{Q_1^k}$. Write

$$\langle\beta_{Q_2}^2 1_R,T(\beta_{Q_1}^1 1_R)\rangle=\langle b_{Q_2^k}^2 1_R,T(b_{Q_1^k}^1 1_{R\cap\widehat{Q_1^k}})\rangle,$$

because $b^1_{Q_1^k}$ is supported on $\widehat{Q_1^k}$. If $R \cap \widehat{Q_1^k} \neq \varnothing$, then we must have $l(R) > l(Q_1^k)$. Remark that since $l(R) \leq l(Q_1)$ this implies that R and Q_1^k have comparable measure. Write

$$R \cap \widehat{Q_1^k} = \bigcup_j Q_{1,j}^k,$$

where the $Q_{1,j}^k$ are dyadic neighbors of Q_1^k, and each one of them is strictly contained inside R. Remark that there is a finite number (uniformly bounded) of such cubes, and write

$$\langle b^2_{Q_2^k} 1_R, T(b^1_{Q_1^k} 1_{Q_{1,j}^k})\rangle = \langle b^1_{Q_1^k} 1_{Q_{1,j}^k}, T^*(b^2_{Q_2^k} 1_R)\rangle = \Sigma_1 + \Sigma_2 + \Sigma_3. \tag{3.2.1}$$

where we have applied the decomposition $1_R = 1_{Q_{1,j}^k} + 1_{R \cap (\widehat{Q_{1,j}^k} \backslash Q_{1,j}^k)} + 1_{R \backslash \widehat{Q_{1,j}^k}}$. Applying (3.1.18) on $Q_{1,j}^k \subset \widehat{Q_2^k} \cap \widehat{Q_1^k}$, one gets $|\Sigma_1| \lesssim \mu(Q_{1,j}^k)$. Decompose further Σ_2 in a sum on dyadic neighbors of $Q_{1,j}^k$ (they have their measure comparable to that of $Q_{1,j}^k$), and apply (3.1.17) in the particular case when there is only one cube R_n to each one of those terms to get $|\Sigma_2| \lesssim \mu(Q_{1,j}^k)^{\frac{1}{\sigma}} \mu(Q_{1,j}^k)^{\frac{1}{\sigma'}} \lesssim \mu(Q_{1,j}^k)$. To bound the last term, it remains to apply the standard estimates : write

$$b^1_{Q_1^k} 1_{Q_{1,j}^k} = \left(b^1_{Q_1^k} 1_{Q_{1,j}^k} - [b^1_{Q_1^k}]_{Q_{1,j}^k} 1_{Q_{1,j}^k}\right) + [b^1_{Q_1^k}]_{Q_{1,j}^k} 1_{Q_{1,j}^k} = f + g,$$

where both functions are supported on $Q_{1,j}^k$ and f has mean zero. Remark that by (3.1.15)

$$|[b^1_{Q_1^k}]_{Q_{1,j}^k}| \lesssim \frac{1}{\mu(Q_1^k)} \int_{\widehat{Q_1^k}} |b^1_{Q_1^k}| \mathrm{d}\mu \lesssim \frac{\mu(\widehat{Q_1^k})^{\frac{1}{p'}}}{\mu(Q_1^k)} \left(\int_{\widehat{Q_1^k}} |b^1_{Q_1^k}|^p \mathrm{d}\mu\right)^{\frac{1}{p}} \lesssim C^{\frac{1}{p}}.$$

Thus $\int_{Q_{1,j}^k} |f| \mathrm{d}\mu \lesssim \mu(Q_{1,j}^k)$. Remark also that $|[b^2_{Q_2^k}]_R| \lesssim C^{\frac{1}{q}}$ because of the size estimate of $\beta^2_{Q_2}$ on R. For $x \in Q_{1,j}^k$ and $y \in R \backslash \widehat{Q_{1,j}^k}$, we have $\lambda(x,y) \gtrsim \mu(Q_1^k) \gtrsim \mu(R)$ because as we have already remarked these two cubes have comparable measure. By (1.2.6), we thus have

$$|\langle b^2_{Q_2^k} 1_{R \backslash \widehat{Q_{1,j}^k}}, Tf\rangle| \lesssim \int_{Q_{1,j}^k} |f| \mathrm{d}\mu \int_R |b^2_{Q_2^k} 1_{R \backslash \widehat{Q_{1,j}^k}}| \mathrm{d}\mu \left(\frac{l(Q_1^k)}{\rho(Q_{1,j}^k, R \backslash \widehat{Q_{1,j}^k})}\right)^\alpha \frac{1}{\mu(R)} \lesssim \mu(Q_{1,j}^k).$$

For the second term, we have

$$|\langle b^2_{Q_2^k} 1_{R \backslash \widehat{Q_{1,j}^k}}, Tg\rangle| \lesssim |\langle b^2_{Q_2^k} 1_{R \backslash \widehat{Q_{1,j}^k}}, T(1_{Q_{1,j}^k})\rangle| \lesssim \mu(R)^{\frac{1}{q}} \mu(Q_{1,j}^k)^{\frac{1}{q'}} \lesssim \mu(R),$$

by the Hardy inequality (3.1.1). Thus $|\Sigma_3| \lesssim \mu(R)$. As we have a uniformly bounded number of j, we have shown in this case

$$|\langle \beta^2_{Q_2} 1_R, T(\beta^1_{Q_1} 1_R)\rangle| \lesssim \mu(R).$$

Suppose now that $R \not\subset \widehat{Q_1^k}$, $R \not\subset \widehat{Q_2^k}$. Write again

$$\langle \beta^2_{Q_2} 1_R, T(\beta^1_{Q_1} 1_R)\rangle = \langle b^2_{Q_2^k} 1_{R \cap \widehat{Q_2^k}}, T(b^1_{Q_1^k} 1_{R \cap \widehat{Q_1^k}})\rangle.$$

For this to be different from zero, we must have $l(R) > \max(l(Q_1^k), l(Q_2^k))$. Write then as before

$$R \cap \widehat{Q_1^k} = \bigcup_j Q_{1,j}^k, \qquad R \cap \widehat{Q_2^k} = \bigcup_i Q_{2,i}^k,$$

with $Q^k_{1,j}$ (resp $Q^k_{2,i}$) dyadic neighbors of Q^k_1 (resp Q^k_2), and decompose

$$\langle b^2_{Q^k_2} 1_{R\cap\widehat{Q^k_2}}, T(b^1_{Q^k_1} 1_{R\cap\widehat{Q^k_1}})\rangle = \sum_{i,j} \langle b^2_{Q^k_2} 1_{Q^k_{2,i}}, T(b^1_{Q^k_1} 1_{Q^k_{1,j}})\rangle.$$

Fix i and j. If $Q^k_{1,j} \cap Q^k_{2,i} \neq \varnothing$, then we estimate the above term by decomposing it as in (3.2.1). If $Q^k_{1,j} \cap Q^k_{2,i} = \varnothing$ and $\rho(Q^k_{1,j}, Q^k_{2,i}) < \min(l(Q^k_1), l(Q^k_2))$, then apply (3.1.17) in the particular case when there is only one cube R_n to get the expected bound. Finally, if $Q^k_{1,j} \cap Q^k_{2,i} = \varnothing$ and $\rho(Q^k_{1,j}, Q^k_{2,i}) \geq \min(l(Q^k_1), l(Q^k_2))$, a standard computation using (1.2.6) also gives the expected bound, that is

$$|\langle \beta^2_{Q_2} 1_R, T(\beta^1_{Q_1} 1_R)\rangle| \lesssim \mu(R),$$

and (3.1.7) is proved.

We now prove (3.1.6). Let Q_1, Q_2 be two dyadic cubes, N a set of integers, and $R' \subset Q_1$, for every $n \in N$, $R_n \subset (\widehat{R'} \backslash R') \cap Q_2$, R_n mutually disjoint, with $\rho(R', R_n) < l(R_n)$, $[|\beta^1_{Q_1}|^p]_{R'} \leq C$, $[|\beta^2_{Q_2}|^q]_{R_n} \leq C$, and let $(\alpha_n)_{n\in N}$ be a set of coefficients. We distinguish two cases. Suppose first that $R' \subset \widehat{Q^k_1}$. If for all $n \in N$, $R_n \subset \widehat{Q^k_2}$, then (3.1.6) is a direct consequence of (3.1.17). Else, for the dyadic cubes R_n which are not contained inside $\widehat{Q^k_2}$ but intersect $\widehat{Q^k_2}$ (and thus satisfy $l(R_n) > l(Q^k_2)$), write as before $R_n \cap \widehat{Q^k_2} = \cup_i R_{n,i}$, where the $R_{n,i}$ are dyadic neighbors of Q^k_2. Denote by N_0 the set of integers n such that R_n satisfy the previous property. Remark that N_0 is a finite set, and that its cardinal is uniformly bounded : indeed, for $n \in N_0$, the dyadic cubes R_n satisfy $l(Q^k_2) < l(R_n) \leq l(Q_2)$ and they are disjoint with respect to one another. Remark also that we have for every $n \in N_0$ and every i, $[|b^2_{Q_2}|^q]_{R_{n,i}} \lesssim C$. As a matter of fact, since $R_{n,i}$ is a dyadic neighbor of Q^k_2, its measure is comparable to the measure of Q^k_2, so that since $R_n \subset Q_2$, we have $\frac{\mu(R_n)}{\mu(R_{n,i})} \lesssim \frac{\mu(Q_2)}{\mu(Q^k_2)} \lesssim 1$ and

$$[|b^2_{Q_2}|^q]_{R_{n,i}} \leq [|b^2_{Q_2}|^q]_{R_n} \times \frac{\mu(R_n)}{\mu(R_{n,i})} \lesssim C.$$

For every $n \in N_0$, and every i such that $\rho(R_{n,i}, R') \geq l(R_{n,i})$, apply as before the standard Calderón-Zygmund estimates (1.2.6) and the Hardy inequality (3.1.1) to get

$$\begin{aligned}|\langle b^2_{Q^k_2} 1_{R_{n,i}}, T(b^1_{Q^k_1} 1_{R'})\rangle| &\leq |\langle T^*(b^2_{Q^k_2} 1_{R_{n,i}} - [b^2_{Q^k_2}]_{R_{n,i}} 1_{R_{n,i}}), b^1_{Q^k_1} 1_{R'}\rangle| + |[b^2_{Q^k_2}]_{R_{n,i}}| |\langle T^*(1_{R_{n,i}}), b^1_{Q^k_1} 1_{R'}\rangle| \\ &\lesssim \mu(R_{n,i}) l(R_{n,i})^\alpha \int_{R'} \frac{|b^1_{Q^k_1}(y)|}{\rho(y, z_{n,i})^\alpha} \frac{\mathrm{d}\mu(y)}{\mu(B(z_{n,i}, \rho(y, z_{n,i})))} + \mu(R_{n,i})^{\frac{1}{p'}} \mu(R')^{\frac{1}{p}},\end{aligned}$$

where $z_{n,i}$ is the center of the dyadic cube $R_{n,i}$. To estimate this integral, split it onto coronae for which $2^l l(R_{n,i}) \leq \rho(y, z_{n,i}) < 2^{l+1} l(R_{n,i})$: let $B_l = B(z_{n,i}, 2^{l+1} l(R_{n,i}))$ and applying Hölder's inequality, write

$$\begin{aligned}\int_{R'} \frac{|b^1_{Q^k_1}(y)|}{\rho(z_{n,i}, y)^\alpha} \frac{\mathrm{d}\mu(y)}{\mu(B(y, \rho(z_{n,i}, y)))} &\lesssim \sum_{l\geq l_0} \frac{1}{2^{l\alpha} l(R_{n,i})^\alpha} \frac{1}{\mu(B_l)} \int_{R'\cap B_l} |b^1_{Q^k_1}| \mathrm{d}\mu \\ &\lesssim \frac{1}{l(R_{n,i})^\alpha} \sum_{l\geq l_0} \frac{1}{2^{l\alpha}} \frac{\mu(B_l)^{\frac{1}{p'}}}{\mu(B_l)} \left(\int_{R'} |b^1_{Q^k_1}|^p \mathrm{d}\mu\right)^{\frac{1}{p}} \\ &\lesssim \frac{\mu(R')^{\frac{1}{p}}}{l(R_{n,i})^\alpha \mu(R_{n,i})^{\frac{1}{p}}} \sum_{l\geq l_0} \frac{1}{2^{l\alpha}},\end{aligned}$$

where l_0 is a fixed integer which only depends on the dimensional constant a_0 of Lemma 1.2.3. We have applied the Hölder inequality to get the second line, and the fact that for every $l \geq l_0$, $\mu(B_l) \gtrsim \mu(R_{n,i})$ to get the last line. Consequently, since $\mu(R_{n,i}) \leq \mu(R_n)$, we obtain

$$|\langle b^2_{Q^k_2} 1_{R_{n,i}}, T(b^1_{Q^k_1} 1_{R'})\rangle| \lesssim \mu(R_n)^{\frac{1}{p'}} \mu(R')^{\frac{1}{p}}.$$

For every $n \in N_0$ and every i such that $\rho(R_{n,i}, R') < l(R_{n,i})$, apply (3.1.17) in the particular case when there is only one cube involved to get

$$|\langle b^2_{Q^k_2} 1_{R_{n,i}}, T(b^1_{Q^k_1} 1_{R'})\rangle| \lesssim \mu(R_{n,i})^{\frac{1}{\sigma}} \mu(R')^{\frac{1}{\sigma'}} \lesssim \mu(R_n)^{\frac{1}{\sigma}} \mu(R')^{\frac{1}{\sigma'}}.$$

Since $\mu(R_n) \lesssim \mu(R')$ and since there is a finite uniformly bounded number of i, we have

$$|\langle b^2_{Q^k_2} 1_{R_n}, T(b^1_{Q^k_1} 1_{R'})\rangle| \lesssim \mu(R_n)^{\frac{1}{\nu}} \mu(R')^{\frac{1}{\nu'}},$$

with $\nu = \max(\sigma, p')$. Now, applying (3.1.17) to the terms for which $n \in N \backslash N_0$, and the fact that the cardinal of N_0 is uniformly bounded, write

$$\begin{aligned}
\left|\left\langle \beta^2_{Q_2}\left(\sum_{n\in N} \alpha_n 1_{R_n}\right), T(\beta^1_{Q_1} 1_{R'})\right\rangle\right| &= \left|\left\langle b^2_{Q^k_2}\left(\sum_{n\in N} \alpha_n 1_{R_n}\right), T(b^1_{Q^k_1} 1_{R'})\right\rangle\right| \\
&\lesssim \left(\sum_{n\in N\backslash N_0} |\alpha_n|^\sigma \mu(R_n)\right)^{\frac{1}{\sigma}} \mu(R')^{\frac{1}{\sigma'}} + \sum_{n\in N_0} |\alpha_n| \mu(R_n)^{\frac{1}{\nu}} \mu(R')^{\frac{1}{\nu'}} \\
&\lesssim \left(\sum_{n\in N\backslash N_0} |\alpha_n|^\sigma \mu(R_n)\right)^{\frac{1}{\sigma}} \mu(R')^{\frac{1}{\sigma'}} + \left(\sum_{n\in N_0} |\alpha_n|^\nu \mu(R_n)\right)^{\frac{1}{\nu}} \mu(R')^{\frac{1}{\nu'}}
\end{aligned}$$

Remark that using Hölder's inequality with $\sum \mu(R_n) \lesssim \mu(R')$ we can raise σ to ν if $\sigma < p'$. Thus, we have

$$\left|\left\langle \beta^2_{Q_2}\left(\sum_{n\in N} \alpha_n 1_{R_n}\right), T(\beta^1_{Q_1} 1_{R'})\right\rangle\right| \lesssim \left\|\sum_{n\in N} \alpha_n 1_{R_n}\right\|_\nu \mu(R')^{\frac{1}{\nu'}}.$$

It remains only to see what happens when R' is not contained inside $\widehat{Q^k_1}$ and is not disjoint with $\widehat{Q^k_1}$. In this case, write again $R' \cap \widehat{Q^k_1} = \cup_{j\in J} R'_j$, where the R'_j are dyadic neighbors of Q^k_1. Distinguish the j for which $\rho(R'_j, R'^c) \geq l(R'_j)$ (let J_0 be the set composed of such j) and those for which $\rho(R'_j, R'^c) < l(R'_j)$ (remark that there can very well be no such j). For the latter, apply the same argument as above : the only difference is that there can be a finite (uniformly bounded) number of n for which $l(R_n) > l(R'_j)$ (and then since $l(R_n) \leq l(R')$ they have comparable measure) but one only has to apply either the dual estimate of (3.1.17) if $\rho(R_n, R'_j) < l(R'_j)$, or the usual decomposition and (1.2.6) if $\rho(R_n, R'_j) \geq l(R'_j)$, to get the expected bound for each one of those terms. We obtain for every $j \in J \backslash J_0$

$$\left|\left\langle b^2_{Q^k_2}\left(\sum_n \alpha_n 1_{R_n}\right), T(b^1_{Q'} 1_{R'_j})\right\rangle\right| \lesssim \left\|\sum_n \alpha_n 1_{R_n}\right\|_{\nu_1} \mu(R')^{\frac{1}{\nu'_1}},$$

with $\nu_1 = \max(\sigma, p')$. For $j \in J_0$, apply once again the same kind of decomposition as before : setting $f = \sum_n \alpha_n 1_{R_n}$, we have

$$\begin{aligned}
\left\langle b^2_{Q^k_2} f, T(b^1_{Q^k_1} 1_{R'_j})\right\rangle &= \left\langle b^2_{Q^k_2} f, T(b^1_{Q^k_1} 1_{R'_j} - [b^1_{Q^k_1}]_{R'_j} 1_{R'_j})\right\rangle + [b^1_{Q^k_1}]_{R'_j} \left\langle b^2_{Q^k_2} f, T(1_{R'_j})\right\rangle \\
&= \Sigma_1 + \Sigma_2
\end{aligned}$$

Remark that $|[b^1_{Q^k_1}]_{R'_j}| \leq \frac{\mu(R')}{\mu(R'_j)}|[b^1_{Q^k_1}]_{R'}| \lesssim \frac{\mu(Q_1)}{\mu(Q^k_1)} \lesssim 1$. Apply (1.2.6) and the disjointness of the cubes R_n to get

$$\begin{aligned}
|\Sigma_1| &\lesssim \int_{\cup R_n} |b^2_{Q^k_2} f| \mathrm{d}\mu \int_{R'_j} |b^1_{Q^k_1} 1_{R'_j} - [b^1_{Q^k_1}]_{R'_j} 1_{R'_j}| \mathrm{d}\mu \frac{1}{\mu(R'_j)} \\
&\lesssim \|f\|_{q'} \left(\int_{\cup R_n} |b^2_{Q^k_2}|^q \mathrm{d}\mu \right)^{\frac{1}{q}} \lesssim \|f\|_{q'} \quad \mu(R')^{\frac{1}{q}}.
\end{aligned}$$

Apply the Hardy inequality (3.1.1) to Σ_2 for some $1 < r < q$ to get

$$|\Sigma_2| \lesssim \left(\int_{\cup R_n} |f b^2_{Q^k_2}|^r \mathrm{d}\mu \right)^{\frac{1}{r}} \mu(R'_j)^{\frac{1}{r'}} \lesssim \left(\int_{\cup R_n} |f|^{r\theta'} \mathrm{d}\mu \right)^{\frac{1}{r\theta'}} \left(\int_{\cup R_n} |b^2_{Q^k_2}|^q \mathrm{d}\mu \right)^{\frac{1}{r\theta}} \mu(R')^{\frac{1}{r'}},$$

where the last inequality is obtained by applying Hölder's inequality with the exponent $\theta = \frac{q}{r}$. Since $\sum \mu(R_n) \lesssim \mu(R')$, we have $\int_{\cup R_n} |b^2_{Q^k_2}|^q \mathrm{d}\mu \lesssim C\mu(R')$, and thus

$$|\Sigma_2| \lesssim \|f\|_{\nu_2} \mu(R')^{\frac{1}{\nu'_2}},$$

with $\nu_2 = r\theta' = \frac{rq}{q-r}$. Finally, as there is a finite uniformly bounded number of j in J, we obtain as expected the bound

$$\left| \left\langle \beta^2_{Q_2} \left(\sum_n \alpha_n 1_{R_n} \right), T(\beta^1_{Q_1} 1_{R'}) \right\rangle \right| \lesssim \left\| \sum_n \alpha_n 1_{R_n} \right\|_{\nu} \mu(R')^{\frac{1}{\nu'}},$$

with $\nu = \max(\nu_1, \nu_2, q')$, and thus (3.1.6) is proved. The dual estimates obviously follow by symmetry of our hypotheses. It remains only to apply Theorem 3.1.2 to get the boundedness of T. □

3.3 An explicit example of pseudo-accretive system with low integrability

It is natural to wonder if the integrability exponents in the hypotheses of Theorem 3.1.2 self improve. The following example answers this negatively. Let $X = \mathbb{R}^2$, equipped with the Euclidean distance and the Lebesgue measure, and let $T = \partial_1/(-\Delta)^{1/2}$ the Riesz transform in $\mathbb{R}^2$. If Q is a dyadic cube in X (respectively I an interval in $\mathbb{R}$), the length of Q (respectively I) will be denoted $\ell(Q)$ (respectively $\ell(I)$).

Proposition 3.3.1.[2] *Let $\phi \in C^1_0(]0,1[)$ and $h \in L^{4/3}(]0,1[)$ be nonnegative functions with $\int \phi = \int h = 1$, and define for dyadic cubes $Q = I \times J$*

$$b_Q(x,y) = \phi_I(x) h_J(y) = \phi\left(\frac{x - \inf I}{\ell(I)}\right) h\left(\frac{y - \inf J}{\ell(J)}\right).$$

Then $(b_Q)_Q$ forms a $(4/3, 4/3)$ dyadic pseudo-accretive system adapted to T.

2. This example follows from a suggestion of T. Hytönen whom we thank for letting us present it.

Remark 3.3.2. Clearly, b_Q does not satisfy any integrability condition higher than $4/3$ if h does not. This example shows that hypotheses (3.1.6), (3.1.7) cannot be dropped in our argument. More precisely, there is no hope of self improvement in the hypotheses of Theorem 3.1.2: it is not possible to prove that any system satisfying (3.1.3), (3.1.4), (3.1.5), (3.1.6) and (3.1.7) actually satisfies higher integrability conditions, allowing to apply Theorem 3.1.2 in the case $1/p + 1/q \leq 1$.

Proof. Set $p = 4/3$. First, we clearly have

$$[b_Q]_Q = 1, \qquad [|b_Q|^p]^{1/p} = \|\phi\|_p \|h\|_p.$$

Observe that there is the additional property

$$\left(\frac{1}{|Q|} \int_Q |\partial_1 b_Q|^p \mathrm{d}x \right)^{1/p} = \frac{1}{\ell(Q)} \|\phi'\|_p \|h\|_p.$$

Now we estimate Tb_Q. We will need the well known fractional integration result : $(-\Delta)^{-1/2}$ is bounded from $L^p(\mathbb{R}^2)$ to $L^q(\mathbb{R}^2)$ for $1/q = 1/p - 1/2$, $1 < p, q < +\infty$. Thus, $(-\Delta)^{-1/2}$ is bounded from $L^p(\mathbb{R}^2)$ to $L^{p'}(\mathbb{R}^2)$ for $p = 4/3$ and $p' = 4$. It follows that

$$\|Tb_Q\|_{p'} = \|(-\Delta)^{-1/2} \partial_1 b_Q\|_{p'} \lesssim \|\partial_1 b_Q\|_p \lesssim \frac{1}{\ell(Q)} |Q|^{\frac{1}{p}} = |Q|^{\frac{1}{p} - \frac{1}{2}} = |Q|^{\frac{1}{p'}}.$$

Thus, $(b_Q)_Q$ satisfies (3.1.3), (3.1.4) and (3.1.5). It remains to prove (3.1.6) and (3.1.7). We begin with (3.1.7). We actually prove the stronger property

$$|\langle b_Q 1_R, T(b_{Q'} 1_{R'}) \rangle| \lesssim |R|^{1/2} |R'|^{1/2}, \tag{3.3.1}$$

for dyadic subcubes $R = K \times L \subset Q = I \times J$ and $R' = K' \times L' \subset Q' = I' \times J'$ such that the function $b_Q = \phi_I \otimes h_J$ satisfies the following non-stopping criteria on R

$$\int_R |b_Q|^p \lesssim |R|, \tag{3.3.2}$$

and similarly for $b_{Q'}$ on R'. It implies that we have

$$\left(\frac{1}{\ell(K)} \int_K |\phi_I|^p \right)^{1/p} \lesssim 1, \qquad \left(\frac{1}{\ell(L)} \int_L |h_J|^p \right)^{1/p} \lesssim 1,$$

and similarly for $\phi_{I'}, h_{J'}$ on K', L'. Observe that since $\phi \in C_0^1(]0,1[)$, we have $\|\phi_I\|_\infty \lesssim 1$, $\|\phi_I'\|_\infty \lesssim \frac{1}{\ell(I)}$. It follows that

$$\left(\frac{1}{\ell(K)} \int_K |\phi_I'|^p \right)^{1/p} \lesssim \frac{1}{\ell(I)}.$$

With $T = (-\Delta)^{-1/2} \partial_1$, and $K' = [a', b'[$, we have

$$\partial_1 (b_{Q'} 1_{R'}) = \left(\phi_{I'}' 1_{K'} + \phi_{I'}(a') \delta_{a'} - \phi_{I'}(b') \delta_{b'} \right) \otimes h_{J'} 1_{L'},$$

where δ denotes the Dirac delta function. By the $L^p \to L^{p'}$ boundedness of $(-\Delta)^{-1/2}$, the first term is immediately handled by

$$\begin{aligned} |\langle b_Q 1_R, (-\Delta)^{-1/2} ((\phi_{I'}' \otimes h_{J'}) 1_{R'}) \rangle| &\lesssim \|b_Q 1_R\|_p \|(\phi_{I'}' \otimes h_{J'}) 1_{R'}\|_p \lesssim |R|^{1/p} \|\phi_{I'}' 1_{K'}\|_p \|h_{J'} 1_{L'}\|_p \\ &\lesssim |R|^{1/p} \frac{1}{\ell(I')} \ell(K')^{1/p} \ell(L')^{1/p} = |R|^{1/p} \frac{1}{\ell(Q')} |R'|^{1/p} \\ &\leq |R|^{1/p} |R'|^{1/p - 1/2} = |R|^{1/p} |R'|^{1/p'}. \end{aligned}$$

As we will next show, the remaining δ terms will give the bound $|R|^{1/2}|R'|^{1/2}$, leading to the overall estimate $|R|^{1/p}|R'|^{1/p'} + |R|^{1/2}|R'|^{1/2}$. Then, using the symmetry of R and R' with T^*, and taking the minimum of the two bounds thus obtained, we will get (3.3.1). We now handle one of the δ terms; no cancellation between them will be used. In fact, all the rest will be handled by a positive kernel estimate, bringing absolute values inside the integration. The bounded factor $\phi_{I'}(a')$ will be discarded. Then

$$\begin{aligned}|(-\Delta)^{-1/2}(\delta_{a'}\otimes h_{J'}(y')1_{L'})(x,y)| &= \left|c\iint \frac{\mathrm{d}\delta_{a'}(x')h_{J'}(y')1_{L'}(y')\mathrm{d}y'}{[(x-x')^2+(y-y')^2]^{1/2}}\right| \\ &\lesssim \int \frac{h_{J'}(y')1_{L'}(y')\mathrm{d}y'}{|x-a'|+|y-y'|}.\end{aligned} \tag{3.3.3}$$

We still have to integrate this against $b_Q 1_R(x,y)$ with respect to $\mathrm{d}x\mathrm{d}y$. Let us carry out the x integration first, recalling that $\phi_I 1_K(x) \lesssim 1_K(x)$. Then

$$\int \frac{1_K(x)\mathrm{d}x}{|x-a'|+|y-y'|} \lesssim \int_0^{\ell(K)} \frac{\mathrm{d}t}{t+|y-y'|} = \ln\left(\frac{\ell(K)+|y-y'|}{|y-y'|}\right). \tag{3.3.4}$$

Thus, we are left with

$$\begin{aligned}&|\langle b_Q 1_R, (-\Delta)^{-1/2}((\delta_{a'}\otimes h_{J'}1_{L'})\rangle| \\ &\lesssim \iint h_J(y)1_L(y)\ln\left(1+\frac{\ell(K)}{|y-y'|}\right)h_{J'}(y')1_{L'}(y')\mathrm{d}y'\mathrm{d}y \\ &\leq \|h_J 1_L\|_p \left(\iint_{L\times L'} \ln^{p'}\left(1+\frac{\ell(K)}{|y-y'|}\right)\mathrm{d}y\mathrm{d}y'\right)^{1/p'} \|h_{J'}1_{L'}\|_p \\ &\lesssim \ell(L)^{1/p}\left(\int_0^{\ell(L')}\int_0^{\ell(L)} \ln^{p'}\left(1+\frac{\ell(L)}{t}\right)\mathrm{d}t\mathrm{d}s\right)^{1/p'} \ell(L')^{1/p} \\ &\lesssim \ell(L)^{1/p}\left(\int_0^{\ell(L')} \ell(L)\mathrm{d}s\right)^{1/p'} \ell(L')^{1/p} = \ell(L)\ell(L') = |R|^{1/2}|R'|^{1/2},\end{aligned} \tag{3.3.5}$$

and (3.3.1) follows.

We now turn to the most technical part of the weak boundedness property, the verification of (3.1.6):

$$\left|\left\langle b_Q \sum_n \alpha_n 1_{R_n}, T(b_{Q'}1_{R'})\right\rangle\right| \lesssim \left\|\sum_n \alpha_n 1_{R_n}\right\|_{p'} |R'|^{1/p},$$

where $R_n \subset R \subset Q$, $R' \subset Q'$, R and R' neighbors, $\partial R_n \cap \partial R' \neq \varnothing$, and $b_Q, b_{Q'}$ satisfy the non stopping criteria (3.3.2) on respectively R_n, R'. As before, set $R = K\times L, R_n = K_n \times L_n, Q = I\times J$, $R' = K'\times L', Q' = I'\times J'$, and $K' = [a', b'[$. If the sum has only one term, we can apply (3.3.1) to get

$$|\langle b_Q\alpha_n 1_{R_n}, T(b_{Q'}1_{R'})\rangle| \lesssim |\alpha_n||R_n|^{1/2}|R'|^{1/2} \leq |\alpha_n||R_n|^{1/p'}|R'|^{1/p} = \|\alpha_n 1_{R_n}\|_{p'}|R'|^{1/p},$$

where the second inequality is due to $|R_n| \leq |R'|$. If R only touches R' at a corner, then there can be at most one cube R_n, and we are done. So assume that R and R' have an edge in common. Still, there are at most two R_n which touch a corner of R', and these can be individually handled as above. Hence, we are left with the main case where R is a neighbor of R' having a common edge with R', and all R_n are separated from the corners of R'. Since all cubes are dyadic, this means that $d(R_n, c) \geq \ell(R_n)$ when c is a corner of R'.

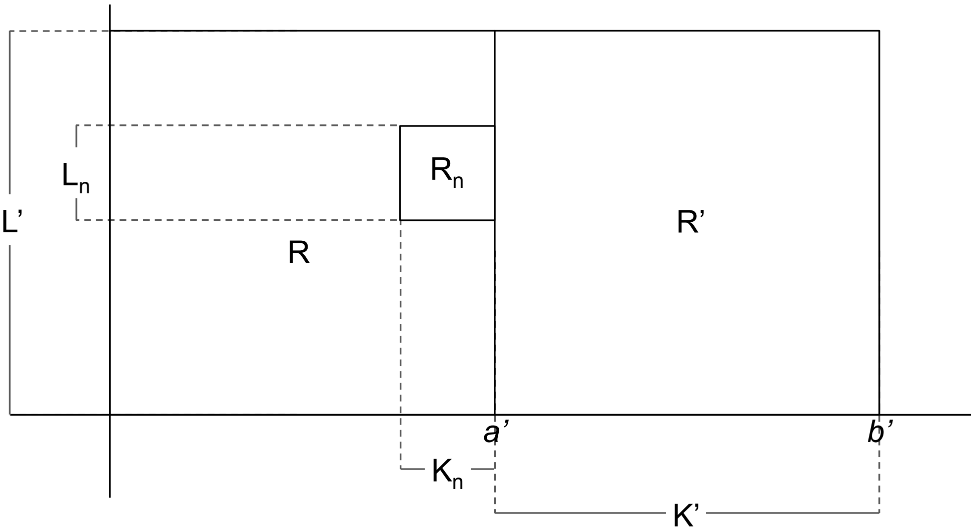

Figure 3.1: Weak boundedness property (3.1.6), case of vertical common edge.

Case of vertical common edge. It is important that in this case the vertical components L_n of the cubes R_n are disjoint, see Figure 3.1.

We use the bound (3.3.3), which now should be integrated against

$$b_Q(x,y)\sum_n \alpha_n 1_{R_n}(x,y) = \sum_n \alpha_n \phi_I(x) 1_{K_n}(x) h_J(y) 1_{L_n}(y).$$

Again, $\|\phi_I\|_\infty \lesssim 1$, and as in (3.3.4), we get

$$\int \frac{1_{K_n}(x)\mathrm{d}x}{|x-a'|+|y-y'|} \lesssim \ln\Big(\frac{\ell(K_n)+|y-y'|}{|y-y'|}\Big). \tag{3.3.6}$$

Instead of (3.3.5), we then have

$$\begin{aligned}
&\Big|\Big\langle b_Q \sum_n \alpha_n 1_{R_n}, (-\Delta)^{-1/2}((\delta_{a'}\otimes h_{J'}1_{L'})\Big\rangle\Big| \\
&\lesssim \sum_n |\alpha_n| \iint h_J(y)1_{L_n}(y) \ln\Big(1+\frac{\ell(K_n)}{|y-y'|}\Big) h_{J'}(y')1_{L'}(y')\mathrm{d}y'\mathrm{d}y \\
&\lesssim \iint \Big(\sum_n h_J(y)1_{L_n}(y)\Big)\Big\{\sum_n |\alpha_n| 1_{L_n}(y)\ln\Big(1+\frac{\ell(K_n)}{|y-y'|}\Big)1_{L'}(y')\Big\} h_{J'}(y')1_{L'}(y')\mathrm{d}y'\mathrm{d}y \\
&\le \Big\|\sum_n h_J 1_{L_n}\Big\|_p \Big(\iint_{L\times L'} \Big|\sum_n \alpha_n 1_{L_n}(y)\ln\Big(1+\frac{\ell(L_n)}{|y-y'|}\Big)\Big|^{p'} \mathrm{d}y\mathrm{d}y'\Big)^{1/p'} \|h_{J'}1_{L'}\|_p.
\end{aligned}$$

This is where we use the disjointness of the vertical components L_n: we can continue with

$$
\begin{aligned}
&= \Big(\sum_n \|h_J 1_{L_n}\|_p^p\Big)^{1/p} \Big(\sum_n |\alpha_n|^{p'} \iint_{L_n \times L'} \ln^{p'}\Big(1 + \frac{\ell(L_n)}{|y-y'|}\Big) \mathrm{d}y\mathrm{d}y'\Big)^{1/p'} \|h_{J'} 1_{L'}\|_p \\
&\lesssim \Big(\sum_n \ell(L_n)\Big)^{1/p} \Big(\sum_n |\alpha_n|^{p'} \Big[\int_{L_n}\int_{3L_n} \ln^{p'}\Big(1 + \frac{\ell(L_n)}{|y-y'|}\Big) \mathrm{d}y\mathrm{d}y' \\
&\qquad\qquad + \int_{L_n}\int_{(3L_n)^c} \frac{\ell(L_n)^{p'}}{|y-y'|^{p'}} \mathrm{d}y\mathrm{d}y'\Big)^{1/p'} \ell(L')^{1/p} \\
&\lesssim \ell(L)^{1/p} \Big(\sum_n |\alpha_n|^{p'} \ell(L_n)^2\Big)^{1/p'} \ell(L')^{1/p} \\
&\lesssim \Big(\sum_n |\alpha_n|^{p'} |R_n|\Big)^{1/p'} \ell(L')^{2/p} = \Big\|\sum_n \alpha_n 1_{R_n}\Big\|_{p'} |R'|^{1/p}.
\end{aligned}
$$

Case of horizontal common edge. Here we use the condition that $d(R_n, c) \geq \ell(R_n)$ for every corner c of R'. In the case of common horizontal edge of R_n and R, this means that $d(K_n, a') \geq \ell(K_n)$. Let K_n' be a reflection of K_n about the point a', see Figure 3.2.

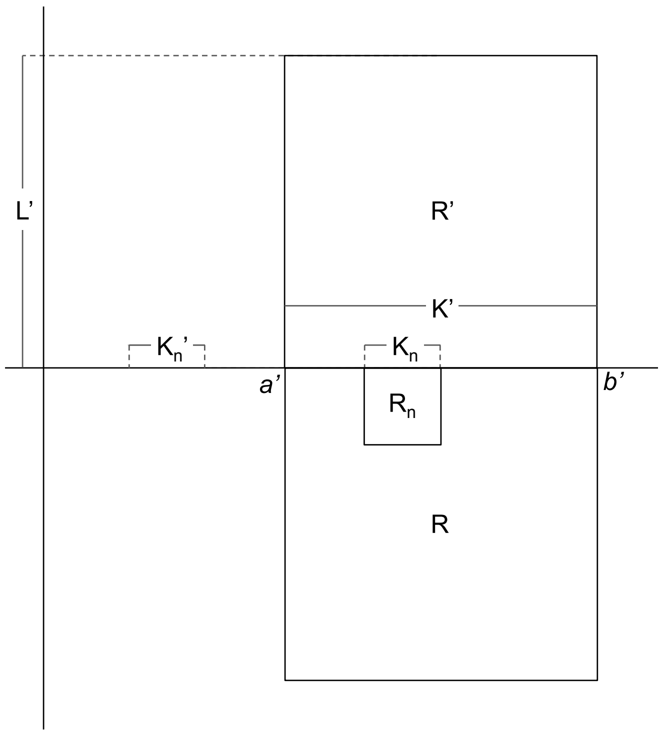

Figure 3.2: Weak boundedness property (3.1.6), case of horizontal common edge.

If $x \in K_n$ and $x' \in K_n'$, then

$$d(K_n, a') \leq |x - a'| \leq d(K_n, a') + \ell(K_n) \leq 2d(K_n, a'),$$

and also

$$d(K_n, a') \leq |x - x'| \leq 2d(K_n, a') + 2\ell(K_n) \leq 4d(K_n, a').$$

Thus, $|x - a'|$ is comparable to $|x - x'|$. This leads to the following variant of (3.3.6):

$$\int \frac{1_{K_n}(x)\mathrm{d}x}{|x-a'| + |y-y'|} \eqsim \iint \frac{1_{K_n}(x)}{|x-x'| + |y-y'|} \frac{1_{K_n'}(x')}{\ell(K_n)} \mathrm{d}x\mathrm{d}x',$$

and thus, in place of (3.3.5):

$$\begin{aligned}
&\Big|\Big\langle b_Q \sum_n \alpha_n 1_{R_n}, (-\Delta)^{-1/2}((\delta_{a'} \otimes h_{J'} 1_{L'})\Big\rangle\Big| \\
&\lesssim \sum_n |\alpha_n| \iint (h_J 1_{L_n})(y) \int \frac{(\phi_I 1_{K_n})(x)\mathrm{d}x}{|x-a'|+|y-y'|}(h_{J'}1_{L'})(y')\mathrm{d}y'\mathrm{d}y \\
&\eqsim \sum_n |\alpha_n| \iint (\phi_I 1_{K_n})(x)(h_J 1_{L_n})(y) \iint \frac{1_{K'_n}(x')(h_{J'}1_{L'})(y')}{\ell(K_n)(|x-x'|+|y-y'|)}\mathrm{d}x'\mathrm{d}y'\mathrm{d}x\mathrm{d}y \\
&\eqsim \sum_n |\alpha_n| \Big\langle b_Q 1_{R_n}, (-\Delta)^{-1/2}\Big(\frac{1_{K'_n}}{\ell(K_n)} \otimes h_{J'}1_{L'}\Big)\Big\rangle \\
&\lesssim \sum_n |\alpha_n| \|b_Q 1_{R_n}\|_p \Big\|\frac{1_{K'_n}}{\ell(K_n)}\|_p \|h_{J'}1_{L'}\|_p \\
&\lesssim \sum_n |\alpha_n||R_n|^{1/p}\frac{\ell(K'_n)^{1/p}}{\ell(K_n)}\ell(L')^{1/p} \\
&= \sum_n |\alpha_n||R_n|^{3/4+3/8-1/2}|R'|^{3/8} = \sum_n |\alpha_n||R_n|^{5/8}|R'|^{3/8}.
\end{aligned}$$

It remains to see that

$$\begin{aligned}
\sum_n |\alpha_n||R_n|^{5/8} = \sum_n |\alpha_n||R_n|^{1/4}|R_n|^{3/8} &\le \Big(\sum_n |\alpha_n|^4|R_n|\Big)^{1/4}\Big(\sum_n |R_n|^{3/8\times 4/3}\Big)^{3/4} \\
&\le \Big\|\sum_n \alpha_n 1_{R_n}\|_4 \Big(\sum_n \ell(R_n)\Big)^{3/4},
\end{aligned}$$

where $\sum_n \ell(R_n) \le \ell(R) = \ell(R') = |R'|^{1/2}$. Finally,

$$\sum_n |\alpha_n||R_n|^{5/8}|R'|^{3/8} \le \Big\|\sum_n \alpha_n 1_{R_n}\|_4 |R'|^{3/4} = \Big\|\sum_n \alpha_n 1_{R_n}\|_{p'}|R'|^{1/p}.$$

This concludes the proof of (3.1.6) and we are done. □

3.4 Preliminaries to the proof of Theorem 3.1.2

We shall use the language of dyadic cubes rather than tiles used in [AHMTT].

3.4.1 Adapted martingale difference operators

Let Q be a dyadic cube of the space of homogeneous type X. For every $f \in L^2(X)$ we define

$$E_Q(f) = [f]_Q 1_Q \quad \text{and} \quad \Delta_Q(f) = \sum_{Q' \in \widetilde{Q}} E_{Q'}(f) - E_Q(f).$$

If Q has only one child, then $\Delta_Q = 0$. It is easy to check that if $\mu(X) = +\infty$, for any $f \in L^2(X)$ we have the representation formula

$$f = \sum_{Q \subset X} \Delta_Q f,$$

with $\|f\|_2^2 = \sum_{Q\subset X} \|\Delta_Q f\|_2^2$. If $\mu(X) < +\infty$, replace f by $f - [f]_X$ on the left hand sides. We omit the details and refer to [AHMTT] for further explanation.

Let $\mathbf{P}$ be a collection of cubes in X. We will say that a locally integrable function b is pseudo-accretive on $\mathbf{P}$ if $|[b]_Q| \gtrsim 1$ for all $Q \in \mathbf{P}$, and is strongly pseudo-accretive on $\mathbf{P}$ if we have $|[b]_Q| \gtrsim 1$ and $|[b]_{Q'}| \gtrsim 1$ for all $Q \in \mathbf{P}$, $Q' \in \widetilde{Q}$. Conversely, we say that such cubes Q are respectively pseudo-accretive (*pa*), strongly pseudo-accretive (*spa*) with respect to b. A cube that is pa but not spa with respect to b is called degenerate pseudo-accretive (*dpa*) with respect to b. Given a function b strongly pseudo-accretive on Q (that is, on $\mathbf{P} = \{Q\}$), we define

$$E_Q^b(f) = \frac{[f]_Q}{[b]_Q} 1_Q,$$

$$\Delta_Q^b(f) = \sum_{Q'\in\widetilde{Q}} E_{Q'}^b(f) - E_Q^b(f).$$

Again, if Q has only one child then $\Delta_Q^b = 0$. A straightforward computation shows that

$$\int_Q b\Delta_Q^b f \mathrm{d}\mu = 0.$$

Note that the operators Δ_Q^b introduced here are not the same ones than those defined by S. Hofmann in [Ho1]. However, if D_Q^b is the operator defined in [Ho1], we have $D_Q^b = b\Delta_Q^b$. It will be easier to work with the operators Δ_Q^b introduced here because we do not assume $b \in L^2(Q)$. The following proposition gives a wavelet representation of theses operators.

Proposition 3.4.1. *Let Q be a dyadic cube and N_Q the number of children of Q. Assume $N_Q \geq 2$. If $b \in L^1(Q)$ is strongly pseudo-accretive on Q, then there exist $0 < C < \infty$ and functions $\phi_Q^{b,s}, \widetilde{\phi}_Q^{b,s}$, $1 \leq s \leq N_Q - 1$ with the following properties*

1. *The functions $\phi_Q^{b,s}, \widetilde{\phi}_Q^{b,s}$ are supported on Q and constant on each child of Q.*
2. $\int_Q b\phi_Q^{b,s} d\mu = \int_Q \widetilde{\phi}_Q^{b,s} b \mathrm{d}\mu = 0$.
3. $\|\phi_Q^{b,s}\|_2 + \|\widetilde{\phi}_Q^{b,s}\|_2 \leq C$.
4. $\int_Q \widetilde{\phi}_Q^{b,s} b \phi_Q^{b,s'} \mathrm{d}\mu = \delta_{s,s'}$.
5. *For all $f \in L^1(X)$*
$$\Delta_Q^b f = \sum_{s=1}^{N_Q-1} \langle f, \phi_Q^{b,s}\rangle \widetilde{\phi}_Q^{b,s}.$$
6. *For all $f \in L^1(X)$*
$$\frac{1}{C} \sum_{s=1}^{N_Q-1} |\langle f, \phi_Q^{b,s}\rangle|^2 \leq \|\Delta_Q^b f\|_{L^2(Q)}^2 \leq C \sum_{s=1}^{N_Q-1} |\langle f, \phi_Q^{b,s}\rangle|^2.$$

Moreover, C depends only on $C(b, Q) = \sup_{Q'\in\widetilde{Q}}([|b|]_Q, |[b]_Q^{-1}|, |[b]_{Q'}^{-1}|)$.

The proof is postponed to the Appendix. Let us comment. If $N_Q = 1$, then there is nothing to do as $\Delta_Q^b f = 0$ anyway. On the other hand, N_Q is uniformly bounded as mentioned earlier. Recall here that $\langle f, g\rangle = \int_X fg \mathrm{d}\mu$. Next, these functions are adapted Haar functions as in [AHMTT] with a biorthogonality property. Note that (1) and (2) imply that

$$\|\phi_Q^{b,s}\|_1 + \|\widetilde{\phi}_Q^{b,s}\|_1 \leq C\mu(Q)^{\frac{1}{2}} \tag{3.4.1}$$

and

$$\|\phi_Q^{b,s}\|_\infty + \|\widetilde{\phi}_Q^{b,s}\|_\infty \leq C_X^{\frac{1}{2}} C \mu(Q)^{-\frac{1}{2}}, \tag{3.4.2}$$

with

$$C_X = \sup_{Q \subset X, Q' \in \widetilde{Q}} \frac{\mu(Q)}{\mu(Q')}. \tag{3.4.3}$$

As long as $C(b,Q)$ remain uniform as Q and b vary, we obtain uniform estimates. As this shall be the case throughout, and as s will play no role in our analysis, we shall make the standing assumption that, unless it is 0, $\Delta_Q^b f$ rewrites as

$$\Delta_Q^b f = \langle f, \phi_Q^b \rangle \phi_Q^b.$$

Given a dyadic cube $Q \subset X$, set

$$R_Q = \{Q' \,\text{dyadic}\,;\, Q' \subset Q\}.$$

Now, if $\{P_i\}$ is a collection of non overlapping dyadic subcubes of a cube Q_0, set

$$\Omega(Q_0, \{P_i\}) = R_{Q_0} \setminus \bigcup_i R_{P_i},$$

and assume that $\Omega(Q_0, \{P_i\}) \subset \{Q \subset Q_0 \,:\, Q \,\text{spa}\, b\}$. Then we have the following results.

Lemma 3.4.2. *Let $1 < q < +\infty$, let $C_{\{P_i\}}(b) = \sup_i [|b|^q]_{P_i}^{1/q}$. We have, with $\Omega := \Omega(Q_0, \{P_i\})$,*

$$\sum_{Q \in \Omega} |\langle f, \phi_Q^b \rangle|^2 \lesssim C_{\{P_i\}}(b)^2 \, \|f\|^2_{L^2(Q_0)}.$$

Proof. We refer to Lemma 6.7 of [AHMTT] for the detail. The only difference is that we use here the language of cubes rather than tiles. ☐

Lemma 3.4.3. *Suppose that $C_{\{P_i\}}(b) < +\infty$, $b 1_{Q_0 \setminus \cup P_i} \in L^\infty(Q_0)$, and $b \in L^q(Q_0)$ for some $1 < q < +\infty$. Let c_Q be a set of uniformly bounded coefficients, and define, with $\Omega := \Omega(Q_0, \{P_i\})$, the bounded bilinear form*

$$\langle Lf, g \rangle = \sum_{Q \in \Omega} c_Q \langle f, \phi_Q^b \rangle \langle \phi_Q^b, g \rangle, \quad \text{for}\, f, g \in L^2(Q_0).$$

Then L is bounded on $L^\nu(Q_0)$ for all $1 < \nu < +\infty$, with

$$\|Lf\|_{L^\nu(Q_0)} \lesssim \|\{c_Q\}\|_{l^\infty} \left(C_{\{P_i\}}(b) + 1 \right) \|f\|_{L^\nu(Q_0)}.$$

Furthermore, $\sum_{Q \in \Omega} \langle f, \phi_Q^b \rangle b \phi_Q^b$ unconditionally converges in $L^\nu(Q_0)$ if $1 < \nu \leq q$ and $f \in L^\nu(Q_0)$, with

$$\left\| \sum_{Q \in \Omega} c_Q \langle f, \phi_Q^b \rangle b \phi_Q^b \right\|_{L^\nu(Q_0)} \leq \|L\|_{L^\nu \to L^\nu} \left(C_{\{P_i\}}(b) + \|b 1_{Q_0 \setminus \cup P_i}\|_\infty \right) \|f\|_{L^\nu(Q_0)}.$$

The implicit constants in the $\lesssim$ depend only on the doubling constant C_D and ν.

Proof. Let $F = Q_0 \setminus \cup P_i$. First, L is bounded on $L^2(Q_0)$ by Lemma 3.4.2 and the Cauchy-Schwarz inequality. We now prove that L is of weak type $(1,1)$. Indeed, let $f \in L^1(Q_0) \cap L^2(Q_0)$, $\lambda > 0$, and write a classical Calderón-Zygmund decomposition of f: there exist functions h and $(\beta_i)_i$ supported on mutually disjoint dyadic cubes Q_i such that

$f = h + \sum_i \beta_i$, with $h \in L^1(Q_0) \cap L^\infty(Q_0)$, $\|h\|_{L^1(Q_0)} \lesssim \|f\|_{L^1(Q_0)}$, $\|h\|_{L^\infty(Q_0)} \lesssim \lambda$, and for all i, $\|\beta_i\|_{L^1(Q_0)\cap L^2(Q_0)} < +\infty$, $\int_{Q_i} \beta_i \mathrm{d}\mu = 0$.
Observe that $L\beta = \sum_i \sum_{Q\in\Omega} c_Q \langle \beta_i, \phi_Q^b\rangle \phi_Q^b$, so that if Q strictly contains Q_i then $\langle \beta_i, \phi_Q^b\rangle = 0$ since $\int_{Q_i} \beta_i = 0$ and ϕ_Q^b constant on the children of Q. It is thus clear that $L\beta$ is supported on $\cup_i \overline{Q_i}$, where $\overline{Q_i}$ denotes the closure of Q_i. Hence $Lf = Lh$ on $Q_0 \setminus \cup_i \overline{Q_i}$. Now, write

$$\mu(\{x \in Q_0 : |Lf(x)| > \lambda\}) \leq \mu(\{x \in Q_0 \setminus \cup_i \overline{Q_i} : |Lh(x)| > \lambda\}) + \sum_i \mu(\overline{Q_i}),$$

and note that $\mu(\overline{Q_i}) = \mu(Q_i)$ by (1.2.1). Since $h \in L^1(Q_0) \cap L^\infty(Q_0)$, $h \in L^2(Q_0)$ with $\|h\|^2_{L^2(Q_0)} \lesssim \lambda \|f\|_{L^1(Q_0)}$, so that, using the boundedness of L on $L^2(Q_0)$,

$$\mu(\{x \in Q_0 \setminus \cup_i \overline{Q_i} : |Lh(x)| > \lambda\}) \leq \frac{1}{\lambda^2} \int_{Q_0} |Lh|^2 \mathrm{d}\mu \lesssim \frac{\|L\|^2_{L^2\to L^2}}{\lambda} \|f\|_{L^1(Q_0)}.$$

Furthermore, we have by the maximal theorem for dyadic averages

$$\sum_i \mu(Q_i) \leq \frac{1}{\lambda} \|f\|_{L^1(Q_0)}.$$

This proves that L is of weak type $(1,1)$. By duality and interpolation, L is thus bounded on $L^\nu(Q_0)$ for all $1 < \nu < +\infty$, with the required control on its norm.
The second part of Lemma 3.4.3 amounts to proving that for all $1 < \nu \leq q$, and $f \in L^\nu(Q_0), g \in L^{\nu'}(Q_0)$,

$$|\langle Lf, bg\rangle| \leq \|L\|_{L^\nu \to L^\nu} \left(C_{\{P_i\}}(b) + \|b 1_{Q_0 \setminus \cup P_i}\|_\infty \right) \|f\|_{L^\nu(Q_0)} \|g\|_{L^{\nu'}(Q_0)}.$$

Write

$$\langle Lf, bg\rangle = \sum_{Q\in\Omega} c_Q \langle f, \phi_Q^b\rangle \langle \phi_Q^b, bg1_F\rangle + \sum_{Q\in\Omega} c_Q \langle f, \phi_Q^b\rangle \langle \phi_Q^b, bg1_{\bigcup P_i}\rangle = \text{(i)} + \text{(ii)}.$$

First, as a consequence of what precedes, we have

$$|\text{(i)}| \leq \|L\|_{L^\nu \to L^\nu} \|f\|_{L^\nu(Q_0)} \|b1_F g\|_{L^{\nu'}(Q_0)} \leq \|L\|_{L^\nu \to L^\nu} \|b1_F\|_\infty \|f\|_{L^\nu(Q_0)} \|g\|_{L^{\nu'}(Q_0)}.$$

For the second term, write

$$\text{(ii)} = \sum_{Q\in\Omega} \sum_i c_Q \langle f, \phi_Q^b\rangle \langle \phi_Q^b, 1_{P_i}\rangle [bg]_{P_i} + \sum_{Q\in\Omega} \sum_i c_Q \langle f, \phi_Q^b\rangle \langle \phi_Q^b, (bg - [bg]_{P_i}) 1_{P_i}\rangle.$$

Observe that for any $Q \in \Omega$, either Q and P_i are disjoint, or ϕ_Q^b is constant over P_i, which yields in any case

$$\langle \phi_Q^b, (bg - [bg]_{P_i}) 1_{P_i}\rangle = [\phi_Q^b]_{P_i} \int_{P_i} (bg - [bg]_{P_i}) \mathrm{d}\mu = 0.$$

Therefore,

$$\text{(ii)} = \sum_{Q\in\Omega} \sum_i c_Q \langle f, \phi_Q^b\rangle \langle \phi_Q^b, 1_{P_i}\rangle [bg]_{P_i},$$

and

$$|\text{(ii)}| \leq \|L\|_{L^\nu \to L^\nu} \|f\|_{L^\nu(Q_0)} \left\| \sum_i [bg]_{P_i} 1_{P_i} \right\|_{L^{\nu'}(Q_0)} \leq \|L\|_{L^\nu \to L^\nu} \|f\|_{L^\nu(Q_0)} \left(\sum_i \mu(P_i) |[bg]_{P_i}|^{\nu'} \right)^{1/\nu'}.$$

Observe that for all i, applying the Hölder inequality, we have

$$\begin{aligned}|[bg]_{P_i}| &\leq \frac{1}{\mu(P_i)}\int_{P_i}|bg|\mathrm{d}\mu \leq \left(\frac{1}{\mu(P_i)}\int_{P_i}|b|^{\nu}\mathrm{d}\mu\right)^{1/\nu}\left(\frac{1}{\mu(P_i)}\int_{P_i}|g|^{\nu'}\mathrm{d}\mu\right)^{1/\nu'}\\ &\leq C_{\{P_i\}}(b)\left(\frac{1}{\mu(P_i)}\int_{P_i}|g|^{\nu'}\mathrm{d}\mu\right)^{1/\nu'},\end{aligned}$$

because $\nu \leq q$. Consequently, we get

$$\begin{aligned}|(\mathrm{ii})| &\leq \|L\|_{L^\nu\to L^\nu}\, C_{\{P_i\}}(b)\, \|f\|_{L^\nu(Q_0)}\left(\sum_i\int_{P_i}|g|^{\nu'}\mathrm{d}\mu\right)^{1/\nu'}\\ &\leq \|L\|_{L^\nu\to L^\nu}\, C_{\{P_i\}}(b)\|f\|_{L^\nu(Q_0)}\, \|g\|_{L^{\nu'}(Q_0)},\end{aligned}$$

where the last inequality comes from the fact that the P_i are non overlapping cubes. As this argument is valid for any set of uniformly bounded coefficients $(c_Q)_Q$, this proves the unconditional convergence of the sum $\sum_{Q\in\Omega}\langle f,\phi_Q^b\rangle b\phi_Q^b$ in $L^\nu(Q_0)$, and we are done. □

Remark 3.4.4. The operator L is a sample of a perfect dyadic singular integral operator as in [AHMTT]. See Section 3.7.1.

3.4.2 A stopping time decomposition

We now state a stopping time lemma, already seen in [AHMTT] and [Ho1], and which is central in our argument.

Lemma 3.4.5. *Let $1 < p, q < +\infty$ with dual exponents p', q'. Assume that T is a singular integral operator with locally bounded kernel. Suppose that there exists a (p, q) dyadic pseudo-accretive system $(\{b_Q^1\}, \{b_Q^2\})$ adapted to T. Then there exist $0 < \varepsilon \ll 1$ and $C < +\infty$ depending only on the doubling constant C_D, the implicit constants in* (1.2.2) *and* (1.2.3), *and the constant C_A from* (3.1.4) *and* (3.1.9), *such that for each fixed dyadic cube Q_0, R_{Q_0} has a partition*

$$R_{Q_0} = \Omega^1 \cup \Omega^1_{buffer} \cup (\cup R_{P_n^1}),$$

adapted to $b = b^1_{Q_0}$ as follows

1. *The tops $\{P_n^1\}$ are non overlapping dyadic subcubes of Q_0 with*
$$\sum_n \mu(P_n^1) \leq (1-\varepsilon)\mu(Q_0). \tag{3.4.4}$$
We say that they realize a $(1-\varepsilon)$-packing of Q_0.
2. *b is strongly pseudo-accretive on Ω^1, and degenerate pseudo-accretive on Ω^1_{buffer}.*
3. *We have for all $Q \in \Omega^1 \cup \Omega^1_{buffer} \cup \{P_n\}_n$ the mean bounds*
$$\int_Q \left(|b|^p + |T(b)|^{q'}\right)\mathrm{d}\mu \leq C\mu(Q). \tag{3.4.5}$$
4. *$\Omega^1_{buffer} = \{Q \in R_{Q_0}\setminus \cup R_{P_n^1} \mid \exists n : P_n^1 \in \widetilde{Q}\}$, that is Ω^1_{buffer} is composed of the dpa cubes with respect to b, and we have, with C_X as in* (3.4.3), *the C_X-packing property*
$$\sum_{Q\in\Omega^1_{buffer}} \mu(Q) \leq C_X\mu(Q_0). \tag{3.4.6}$$

5. *We have the decomposition*

$$f = [f]_{Q_0} b + \sum_{Q \in \Omega^1} b \Delta_Q^b f + \sum_{Q \in \Omega^1_{buffer}} \xi_Q^b f + \sum_n (f 1_{P_n^1} - [f]_{P_n^1} b^1_{P_n^1}), \tag{3.4.7}$$

where the "buffer functions" ξ_Q^b are supported on Q, have mean zero, and take the form

$$\xi_Q^b f = \sum_{\substack{Q' \in \widetilde{Q} \\ Q' \in \Omega^1 \cup \Omega^1_{buffer}}} a_{Q'} b 1_{Q'} + \sum_{P_n^1 \in \widetilde{Q}} (a'_{P_n^1} b^1_{P_n^1} - a''_{P_n^1} b 1_{P_n^1}),$$

where the coefficients a_Q, a'_Q, a''_Q depend on f and b, and obey the bounds

$$\sum_{Q' \in \widetilde{Q}} \left(|a_{Q'}| + |a'_{Q'}| + |a''_{Q'}| \right) \lesssim \|f\|_{L^\infty(Q_0)}.$$

A similar statement holds with b^1_Q, Tb^1_Q and p, q' replaced by b^2_Q, $T^* b^2_Q$ and q, p'.

Proof. Only minor modifications have to be made to adapt the proof to the homogeneous setting. The stopping time is simply on whether

$$(i) \quad [|b|]_Q < \delta$$

or

$$(ii) \quad [|b|^p]_Q + [|T(b)|^{q'}]_Q > C$$

for dyadic subcubes Q of Q_0 for appropriately chosen $\delta > 0$ and $C < +\infty$. We refer to [Ho1] for a proof that can be followed line by line (we do not need the control with the maximal function used there). The fact that (3.4.5) holds also for $Q = P_n$ comes from the doubling property. The decomposition formula (3.4.7) is identical to Lemma 6.11 of [AHMTT]. ☐

Observe that this lemma does not require (3.1.6) nor (3.1.7) to be satisfied. Remark as well that with our previous notation, we have $\Omega^i \cup \Omega^i_{buffer} = \Omega(Q_0, \{P_n^i\})$. We now introduce the stopping-time projections which are fundamental to our analysis. For $f \in L^1(Q_0)$, and with the notation of Lemma 3.4.5, for $i \in \{1, 2\}$, set

$$\Pi^{b^i_{Q_0}} f = f - \sum_n (f 1_{P_n^i} - [f]_{P_n^i} b^i_{P_n^i}) = f 1_{Q_0 \setminus \cup P_n^i} + \sum_n [f]_{P_n^i} b^i_{P_n^i}. \tag{3.4.8}$$

This operator not only depends on $b^i_{Q_0}$ but also on the stopping-time cubes P_n^i and the corresponding functions $b^i_{P_n^i}$. Thus, the notation is not completely accurate but simplifies the understanding.

Lemma 3.4.6. *The stopping-time projections $\Pi^{b^i_{Q_0}}$ have the following properties: for all $f \in L^\infty(Q_0)$, we have*

$$\int_{Q_0} |\Pi^{b^i_{Q_0}} f|^{p_i} \mathrm{d}\mu \leq (1 + C_A) \|f\|^{p_i}_{L^\infty(Q_0)} \mu(Q_0), \tag{3.4.9}$$

with C_A the constant in (3.1.4) *and $(p_1, p_2) = (p, q)$.*

$$[\Pi^{b^i_{Q_0}} f]_Q = [f]_Q, \tag{3.4.10}$$

for all $Q \in \Omega^i \cup \Omega^i_{buffer} \cup \{P_n^i\}$.

$$\Pi^{b^i_{Q_0}} (\Pi^{b^i_{Q_0}} f) = \Pi^{b^i_{Q_0}} f. \tag{3.4.11}$$

$$(\Pi^{b^i_{Q_0}} f)1_Q = \Pi^{b^i_{Q_0}}(f1_Q), \tag{3.4.12}$$

for all $Q \in \Omega^i \cup \Omega^i_{buffer} \cup \{P^i_n\}$.

$$\int_Q |\Pi^{b^i_{Q_0}} f|^{p_i} \mathrm{d}\mu \le (1 + C_A)\|f\|^{p_i}_{L^\infty(Q)}\mu(Q), \tag{3.4.13}$$

for all $Q \in \Omega^i \cup \Omega^i_{buffer} \cup \{P^i_n\}$.

Proof. We prove first (3.4.9) and (3.4.13). By (3.4.8) we have $|\Pi^{b^i_{Q_0}} f|^{p_i} = |f|^{p_i} 1_{Q_0 \setminus \cup P^i_n} + \sum_n |[f]_{P^i_n} b^i_{P^i_n}|^{p_i}$. By (3.1.4) for $b^i_{P^i_n}$ and (3.4.4), we have

$$\begin{aligned}\int_{Q_0} |\Pi^{b^i_{Q_0}} f|^{p_i} \mathrm{d}\mu &\le \|f\|^{p_i}_{L^\infty(Q_0)}\mu(Q_0) + \|f\|^{p_i}_{L^\infty(Q_0)} \sum_n \int_{Q_0} |b^i_{P^i_n}|^{p_i} \mathrm{d}\mu \\ &\le \|f\|^{p_i}_{L^\infty(Q_0)}\mu(Q_0) + \|f\|^{p_i}_{L^\infty(Q_0)} C_A \sum_n \mu(P^i_n) \le (1 + C_A)\|f\|^{p_i}_{L^\infty(Q_0)}\mu(Q_0).\end{aligned}$$

If $Q \in \Omega^i \cup \Omega^i_{buffer} \cup \{P^i_n\}$, then Q cannot be contained in any P^i_n. Thus, applying again (3.1.4) for $b^i_{P^i_n}$ and the disjointness of the cubes P^i_n, we have

$$\int_Q |\Pi^{b^i_{Q_0}} f|^{p_i} \mathrm{d}\mu \le \|f\|^{p_i}_{L^\infty(Q_0)}\mu(Q \setminus \cup P^i_n) + \|f\|^{p_i}_{L^\infty(Q_0)} C_A \sum_{n; P^i_n \subset Q} \mu(P^i_n) \le (1 + C_A)\|f\|^{p_i}_{L^\infty(Q_0)}\mu(Q).$$

For (3.4.10), observe that because of (3.1.3) for $b^i_{P^i_n}$,

$$\int_Q \Pi^{b^i_{Q_0}} f \mathrm{d}\mu = \int_{Q \setminus \cup P^i_n} f \mathrm{d}\mu + \sum_{n; P^i_n \subset Q} [f]_{P^i_n} \mu(P^i_n) = \int_Q f \mathrm{d}\mu,$$

and (3.4.10) follows. Note that property (3.4.11) is a direct consequence of (3.4.10). Only the property (3.4.12) thus remains to be proved. Observe that for a cube $Q \in \Omega^i \cup \Omega^i_{buffer} \cup \{P^i_n\}$, and for any n, $[f1_Q]_{P^i_n} = 0$ if $Q \cap P^i_n = \varnothing$ and $[f1_Q]_{P^i_n} = [f]_{P^i_n}$ if $P^i_n \subset Q$. As we have already remarked, there is no other possibility, so that

$$\Pi^{b^i_{Q_0}}(f1_Q) = f1_{Q \setminus \cup P^i_n} + \sum_{n; P^i_n \subset Q} [f]_{P^i_n} b^i_{P^i_n} = (\Pi^{b^i_{Q_0}} f)1_Q.$$

$\square$

3.4.3 Sketch of the proof of Theorem 3.1.2

To prove Theorem 3.1.2, we will begin by showing that we can reduce to proving that for every f, g supported on a reference cube Q_0 and uniformly bounded by 1, we have $|\langle \Pi^{b^2_{Q_0}} f, T(\Pi^{b^1_{Q_0}} g)\rangle| \le C\mu(Q_0)$. This is done in Section 3.5 and involves estimating the error terms; we will see that stopping-time Lemma 3.4.5 plays a central part in this, and particularly (3.4.4) and (3.4.5). To prove the above inequality, we apply (3.4.7) of Lemma 3.4.5 to decompose the stopping-time projectors Π on our adapted Haar wavelet basis given by Proposition 3.4.1. This gives us a number of matrix coefficients to estimate, which, to put it briefly, is done using the decay of the kernel of T as in Proposition 1.2.6, and hypotheses (3.1.4), (3.1.5), (3.1.6) and (3.1.7) of Theorem 3.1.2.

3.5 Reductions

We now proceed to the proof of Theorem 3.1.2. Let T be a singular integral operator with locally bounded kernel on X, and assume that there exists a (p,q) dyadic pseudo-accretive system $(\{b_Q^1\},\{b_Q^2\})$ adapted to T. It is of classical knowledge that to prove the L^2 boundedness of T, it suffices to show that there exists a constant $C<+\infty$ such that for every dyadic cube Q,

$$\|T1_Q\|_{L^1(Q)} \leq C\mu(Q) \quad \text{and} \quad \|T^*1_Q\|_{L^1(Q)} \leq C\mu(Q). \tag{3.5.1}$$

By symmetry of our hypotheses, it is enough to prove the second assertion. We give ourselves a reference cube Q_0, and we do this for all the cubes Q contained in Q_0. Since Q_0 is arbitrary, the general case follows, as long as our constants are independent of Q_0. As a matter of fact, let us introduce the equivalence relation on the dyadic cubes "$Q_1 \sim Q_2 \Leftrightarrow \exists Q$ dyadic cube such that $Q_1 \cup Q_2 \subset Q$". It is easy to see that in a space of homogeneous type, there must be a finite number of equivalence classes for this relation. Thus, the argument which follows will give us a constant C_i for each one of those equivalent classes, by letting Q_0 grow into one such equivalent class. But since they come in a finite number, (3.5.1) follows.

Let us now proceed to our argument. For a dyadic cube Q, we denote by E_Q the set of all functions supported on Q and uniformly bounded by 1. By duality, it suffices to prove that

$$A = \sup_{\substack{Q\subset Q_0 \\ f\in E_Q}} \left|\frac{1}{\mu(Q)}\langle f, T1_Q\rangle\right| \lesssim 1.$$

Note that $A<+\infty$ with our a priori assumption $C=\|K\|_{L^\infty(Q_0\times Q_0)}<+\infty$ and $\mu(Q_0)<+\infty$. Indeed, $|\langle f,T1_Q\rangle| \leq C\|f\|_\infty \mu(Q)^2$ so $A\leq C\mu(Q_0)<+\infty$. Of course, this is not the bound we are after.

3.5.1 First reduction

Let

$$B = \sup_{\substack{Q\subset Q_0 \\ f\in E_Q}} \left|\frac{1}{\mu(Q)}\langle \Pi^{b_Q^2} f, T1_Q\rangle\right|.$$

Lemma 3.5.1. *We have $A \lesssim B+1$.*

Proof. Let us fix a cube Q in X, a function $f\in E_Q$, and for the sake of simplicity, denote b_Q^2 by b^2 and $\Pi^{b_Q^2}$ by Π^2. We write

$$f = \Pi^2 f + \sum_j f_j, \quad \text{with } f_j = f1_{P_j^2} - [f]_{P_j^2} b_{P_j^2}^2.$$

We thus have $\langle f, T1_Q\rangle = \langle \Pi^2 f, T1_Q\rangle + \Sigma$, with

$$\Sigma = \sum_j \langle f_j, T1_Q\rangle,$$

which we decompose in two parts

$$\Sigma_1 = \sum_j \langle f_j, T1_{P_j^2}\rangle \quad \text{and} \quad \Sigma_2 = \sum_j \langle f_j, T1_{Q\setminus P_j^2}\rangle.$$

By definition of the quantity A, the fact that $f1_{P_j^2} \in E_{P_j^2}$, and (3.4.4), we have

$$\sum_j |\langle f1_{P_j^2}, T1_{P_j^2}\rangle| \leq A \sum_j \mu(P_j^2) \leq A(1-\varepsilon)\mu(Q).$$

By (3.1.5) for $T^*(b^2_{P_j^2})$, we also have

$$\sum_j |\langle [f]_{P_j^2} b^2_{P_j^2}, T1_{P_j^2}\rangle| \lesssim \sum_j \mu(P_j^2) \lesssim \mu(Q),$$

which takes care of the sum Σ_1. For the sum Σ_2, we write

$$\Sigma_2 = \sum_j \langle f_j, T1_{Q\setminus \widehat{P_j^2}}\rangle + \sum_j \langle f_j, T1_{\widehat{P_j^2}\setminus P_j^2}\rangle.$$

As the functions f_j have mean zero, a standard computation using (1.2.6) gives the bound $\sum \mu(P_j^2) \lesssim \mu(Q)$ for the first sum. For the second sum, (3.1.1) applied to $f_j \in L^q(Q)$ by (3.1.4) for $b^2_{P_j^2}$, and $1_Q \in L^{q'}(Q)$ gives us the same bound. We have obtained

$$|\langle f, T1_Q\rangle| \leq B\mu(Q) + A(1-\varepsilon)\mu(Q) + C\mu(Q),$$

and Lemma 3.5.1 follows using $A < +\infty$. □

3.5.2 Second reduction

This is where our argument departs from the ones in [AHMTT] or [Ho1]. Let Q be a fixed dyadic subcube of Q_0. Let us consider

$$A_Q = \sup \left| \frac{1}{\mu(Q')} \langle \Pi^{b_Q^2} f, T1_{Q'}\rangle \right|,$$

$$B_Q = \sup \left| \frac{1}{\mu(Q')} \langle \Pi^{b_Q^2} f, T\Pi^{b^1_{Q'}} 1_{Q'}\rangle \right|,$$

where the suprema are taken over all the b_Q^2 pseudo-accretive subcubes Q' of Q, and all the functions $f \in E_{Q'}$. As Q is itself b_Q^2 pseudo-accretive, it is clear that $B \leq \sup_Q A_Q$. Again $A_Q < +\infty$ for each Q by our qualitative assumptions.

Lemma 3.5.2. *We have $A_Q \lesssim B_Q + 1$.*

Proof. As before, let us fix a pseudo-accretive subcube Q' of Q, a function $f \in E_{Q'}$, and let us forget Q and Q' in our notations for the sake of simplicity. We write

$$1_{Q'} - \Pi^1 1_{Q'} = \sum_i g_i, \quad \text{with } g_i = 1_{P_i^1} - b^1_{P_i^1},$$

where we recall that the P_i^1 are given by the decomposition of Q' with respect to $b^1_{Q'}$ as in Lemma 3.4.5 and $\Pi^1 = \Pi^{b^1_{Q'}}$. We thus have $\langle \Pi^2 f, T1_{Q'}\rangle = \langle \Pi^2 f, T(\Pi^1 1_{Q'})\rangle + \Sigma$, with

$$\Sigma = \sum_i \langle \Pi^2 f, Tg_i\rangle.$$

We first consider the sum Σ_1 running over the indices i such that the cubes P_i^1 are pa 2 (here, P_i^1 pa 2 means that $P_i^1 \in \Omega^2 \cup \Omega^2_{buffer}$), and we write

$$\Pi^2 f = \Pi^2(f1_{P_i^1}) + \Pi^2(f1_{Q'\setminus P_i^1}).$$

We then have
$$\langle \Pi^2(f\mathbb{1}_{P_i^1}), Tg_i\rangle = \langle \Pi^2(f\mathbb{1}_{P_i^1}), T\mathbb{1}_{P_i^1}\rangle - \langle \Pi^2(f\mathbb{1}_{P_i^1}), Tb^1_{P_i^1}\rangle$$
By definition of A_Q, the fact that $f\mathbb{1}_{P_i^1} \in E_{P_i^1}$, and (3.4.4), we have
$$\sum_{P_i^1\,pa\,2} |\langle \Pi^2(f\mathbb{1}_{P_i^1}), T\mathbb{1}_{P_i^1}\rangle| \leq A_Q \sum_{P_i^1\,pa\,2} \mu(P_i^1) \leq A_Q(1-\varepsilon)\mu(Q').$$
On the other hand, by (3.4.12), (3.4.13), and (3.1.5) for $T(b^1_{P_i^1})$ on P_i^1, we have
$$\sum_{P_i^1\,pa\,2} |\langle \Pi^2(f\mathbb{1}_{P_i^1}), Tb^1_{P_i^1}\rangle| \lesssim \sum_{P_i^1\,pa\,2} \|\Pi^2 f\|_{L^q(P_i^1)} \|T(b^1_{P_i^1})\|_{L^{q'}(P_i^1)} \lesssim \sum_{P_i^1\,pa\,2} \mu(P_i^1) \lesssim \mu(Q').$$
Finally, again for P_i^1 pa 2, we have by Lemma 3.4.6,
$$\Pi^2(f\mathbb{1}_{Q'\setminus P_i^1}) = f\mathbb{1}_{F\cap(Q'\setminus P_i^1)} + \sum_{P_j^2\cap P_i^1=\varnothing} [f]_{P_j^2} b^2_{P_j^2} = f\mathbb{1}_{Q'\setminus P_i^1} - \sum_{P_j^2\cap P_i^1=\varnothing} f_j,$$
with $f_j = f\mathbb{1}_{P_j^2} - [f]_{P_j^2} b^2_{P_j^2}$ as before. The sum $\sum_{P_i^1\,pa\,2} |\langle f\mathbb{1}_{Q'\setminus P_i^1}, Tg_i\rangle|$ can be estimated in the same way we treated the sum Σ_2 in Lemma 3.5.1. It thus remains to estimate the term
$$\Sigma_1' = -\sum_{P_i^1\,pa\,2}\ \sum_{P_j^2\cap P_i^1=\varnothing} \langle f_j, Tg_i\rangle,$$
which we defer for now. We now estimate the sum Σ_2 running over those indices i such that P_i^1 is not pa 2, which means that there exists an index k (unique), with $P_i^1 \subset P_k^2$. For a fixed index k, denote
$$G_k = \sum_{P_i^1\subset P_k^2} g_i.$$
Observe that G_k is supported on P_k^2, with mean zero and that $\int_{P_k^2} |G_k|^p \mathrm{d}\mu \lesssim \mu(P_k^2)$ by (3.1.5) for the $b^1_{P_i^1}$ on P_i^1 and $P_i^1 \subset P_k^2$. Write
$$\Pi^2 f = f\mathbb{1}_{Q'\setminus P_k^2} + [f]_{P_k^2} b^2_{P_k^2} - \sum_{j\neq k} f_j,$$
which allows us to decompose Σ_2 into three parts: $\Sigma_2 = \Sigma_2' + \Sigma_2'' + \Sigma_2'''$. Once more, we estimate the sum $\Sigma_2' = \sum_k \langle f\mathbb{1}_{Q'\setminus P_k^2}, TG_k\rangle$ as we estimated the sum Σ_2 in Lemma 3.5.1. The sum $\Sigma_2'' = \sum_k [f]_{P_k^2} \langle T^*(b^2_{P_k^2}), G_k\rangle$ is also easy to manage using the hypothesis (3.1.5) for $T^*(b^2_{P_k^2})$ on P_k^2 and the support and size properties of the functions G_k. Overall, we are left to estimate the term
$$\begin{aligned}\Sigma_3 = \Sigma_1' + \Sigma_2''' &= -\sum_{P_i^1\,pa\,2}\ \sum_{P_j^2\cap P_i^1=\varnothing} \langle f_j, Tg_i\rangle - \sum_{P_i^1\,non\,pa\,2}\ \sum_{P_j^2\cap P_i^1=\varnothing} \langle f_j, Tg_i\rangle \\ &= -\sum_{(i,j):P_j^2\cap P_i^1=\varnothing} \langle f_j, Tg_i\rangle.\end{aligned}$$

We split the sum into two parts, depending on whether (i,j) is such that $l(P_j^2) \leq l(P_i^1)$ or not. Let us consider for example the sum for the indices (i,j) satisfying that condition, the other sum can be treated in a symmetric way. We want to estimate
$$\Sigma_3' = -\sum_{\substack{(i,j):P_j^2\cap P_i^1=\varnothing\\ l(P_j^2)\leq l(P_i^1)}} \langle f_j, Tg_i\rangle.$$

Fix the index i. Then either $P_j^2 \subset \widehat{P_i^1} \backslash P_i^1$ or $P_j^2 \cap \widehat{P_i^1} = \varnothing$ because of the relative sizes of those cubes. Set $F_i = \sum f_j$ where the sum runs over all the indices j such that $P_j^2 \subset \widehat{P_i^1} \backslash P_i^1$. Lemma 3.1.1 and the use of hypothesis (3.1.5) for $T(b^1_{P_i^1})$ on $\widehat{P_i^1}$ assure that

$$|\langle F_i, Tg_i\rangle| \leq |\langle F_i, T1_{P_i^1}\rangle| + |\langle F_i, Tb^1_{P_i^1}\rangle| \lesssim \mu(P_i^1).$$

For the indices j such that $P_j^2 \cap \widehat{P_i^1} = \varnothing$ we have to work a little bit more. Using that g_i is of mean 0 and applying (1.2.6), we obtain for each concerned couple (i,j)

$$\begin{aligned}
|\langle f_j, Tg_i\rangle| &\lesssim \int_{P_i^1} |g_i(y)| \int_{P_j^2} |f_j(x)| \left(\frac{l(P_i^1)}{\rho(x,y)}\right)^\alpha \frac{1}{\lambda(x, z_{P_i^1})} \mathrm{d}\mu(x)\mathrm{d}\mu(y) \\
&\lesssim \int_{P_i^1} |g_i(y)| \sum_{k\geq 1} \int_{P_j^2 \cap C_k(P_i^1)} |f_j(x)| \left(\frac{l(P_i^1)}{\delta^{-k+1} l(P_i^1)}\right)^\alpha \frac{1}{\mu(B(z_{P_i^1}, \delta^{-k+1} l(P_i^1)))} \mathrm{d}\mu(x)\mathrm{d}\mu(y) \\
&\lesssim \mu(P_i^1) \sum_{k\geq 1} \delta^{\alpha k} \mu(B(z_{P_i^1}, \delta^{-k+1} l(P_i^1)))^{-1} \int_{P_j^2 \cap C_k(P_i^1)} |f_j| \mathrm{d}\mu,
\end{aligned}$$

where $z_{P_i^1}$ denotes the center of the cube P_i^1 and

$$C_k(P_i^1) = \{x \in X | \delta^{-k+1} l(P_i^1) \leq \rho(x, z_{P_i^1}) < \delta^{-k} l(P_i^1)\}.$$

Now, remark that since $l(P_j^2) \leq l(P_i^1)$, for each j there are at most two sets $C_k(P_i^1)$ non disjoint with P_j^2. Thus we can sum over j with k fixed, and as the functions f_j are supported on the cubes P_j^2 with $\int |f_j| \lesssim \mu(P_j^2)$, we get

$$\sum_j \int_{P_j^2 \cap C_k(P_i^1)} |f_j| \mathrm{d}\mu \lesssim \sum_{P_j^2 \cap C_k(P_i^1) \neq \varnothing} \mu(P_j^2) \lesssim \mu(C_k(P_i^1)).$$

Then, summing over k, we have

$$\begin{aligned}
\sum_{j: P_j^2 \cap \widehat{P_i^1} = \varnothing} |\langle f_j, Tg_i\rangle| &\lesssim \sum_{k\geq 1} \delta^{\alpha k} \mu(B(z_{P_i^1}, \delta^{-k+1} l(P_i^1)))^{-1} \mu(C_k(P_i^1)) \mu(P_i^1) \\
&\lesssim \sum_{k\geq 1} \delta^{\alpha k} \mu(P_i^1) \lesssim \mu(P_i^1).
\end{aligned}$$

It remains only to sum over i to get $|\Sigma_3| \lesssim \mu(Q')$. To summarize, we have obtained

$$|\langle \Pi^2 f, T1_{Q'}\rangle| \leq B_Q \mu(Q') + A_Q(1-\varepsilon)\mu(Q') + C\mu(Q'),$$

and Lemma 3.5.2 follows using $A_Q < \infty$. $\square$

Remark 3.5.3. Observe that this argument does not require the properties (3.1.6) and (3.1.7) to be satisfied. We used (3.1.5) (and not (3.1.9)) once. No further conditions on p, q are required.

As a consequence of what precedes, we will be done if we prove there exists a constant $C < +\infty$ such that

$$|\langle \Pi^{b_Q^2} f, T(\Pi^{b^1_{Q'}} g)\rangle| \leq C\mu(Q')$$

for all dyadic subcube Q of Q_0, all b_Q^2 pseudo-accretive subcube Q' of Q, and all $f, g \in E_{Q'}$. By (3.4.12) of Lemma 3.4.6, if Q' is a b_Q^2 pseudo-accretive subcube of Q and $f, g \in E_{Q'}$, we have

$$\Pi^{b_Q^2} f = \Pi^{b_Q^2}(f 1_{Q'}) = (\Pi^{b_Q^2} f) 1_{Q'} = f 1_{Q' \backslash (\cup P_j^2)} + \sum_{P_j^2 \subset Q'} [f]_{P_j^2} b_{P_j^2}^2,$$

and

$$\Pi^{b_{Q'}^1} g = g 1_{Q' \backslash (\cup P_i^1)} + \sum_{P_i^1 \subset Q'} [g]_{P_i^1} b_{P_i^1}^1.$$

Thus, the expressions reduce to partitions of Q' by dyadic subcubes and possibly the complement of their union. From this point on, we no longer use the $(1-\varepsilon)$-packing property (3.4.4) in Lemma 3.4.5 and the 1-packing property suffices, that is we do not care if $Q' \backslash (\cup P_j^2) = \varnothing$, which is a possibility. This means that Q' can become our reference cube, and b_Q^2 could be as well replaced by any function b^2 for which the non pseudo-accretive cubes are the $P_j^2 \subset Q'$ and Lemma 3.4.5 holds with $(1-\varepsilon)$ replaced by 1 in (3.4.4). To simplify notation, we shall do this and set $Q' = Q_0$, and even further assume it is of generation 0. We have to prove

$$|\langle \Pi^2 f, T(\Pi^1 g) \rangle| \leq C\mu(Q_0) \tag{3.5.2}$$

for any $f, g \in E_{Q_0}$ where $\Pi^i = \Pi^{b_{Q_0}^i}$, and Lemma 3.4.5 holds for both with the $(1-\varepsilon)$ of (3.4.4) replaced by 1. For simplicity, we denote $b_{Q_0}^i$ by b^i. We shall also say Q spa i (resp. dpa i) if $Q \in \Omega^i$ (resp. $Q \in \Omega_{buffer}^i$). We also set $\Delta_Q^i = \Delta_Q^{b^i}$ and $\xi_Q^i = \xi_Q^{b^i}$.

3.6 BCR algorithm and end of the proof

We intend to prove (3.5.2) with the simplification of notation.

3.6.1 Representation of projections

By the decomposition formula (3.4.7) of Lemma 3.4.5, we have

$$\Pi^i f = [f]_{Q_0} b^i + \sum_{Q \,\mathrm{spa}\, i} b^i \Delta_Q^i f + \sum_{Q \,\mathrm{dpa}\, i} \xi_Q^i f.$$

Set $E_0^i f = [f]_{Q_0} b^i$, and $\forall j \geq 0$,

$$D_j^i f = \sum_{\substack{Q \,\mathrm{spa}\, i \\ l(Q)=\delta^j}} b^i \Delta_Q^i f + \sum_{\substack{Q \,\mathrm{dpa}\, i \\ l(Q)=\delta^j}} \xi_Q^i f = \sum_{\substack{Q \,\mathrm{spa}\, i \\ l(Q)=\delta^j}} \langle f, \phi_Q^i \rangle b^i \phi_Q^i + \sum_{\substack{Q \,\mathrm{dpa}\, i \\ l(Q)=\delta^j}} \xi_Q^i f$$

$$E_j^i f = E_0^i f + \sum_{l=0}^{j-1} D_l^i f.$$

Lemma 3.6.1. *With the previous notation, we have*

$$E_j^i f = \sum_{\substack{Q \,\mathrm{pa}\, i \\ l(Q)=\delta^j}} \frac{[f]_Q}{[b^i]_Q} b^i 1_Q + \sum_{\substack{P_n^i \\ l(P_n^i) \geq \delta^j}} [f]_{P_n^i} b_{P_n^i}^i. \tag{3.6.1}$$

Furthermore, we have when $f \in L^\infty(Q_0)$

$$\Pi^i f = \lim_{j \to +\infty} E_j^i f, \tag{3.6.2}$$

with the convergence taking place in $L^1(Q_0)$.

Proof. The first part of Lemma 3.6.1 is a linear algebra computation, we will omit the detail. The second assertion is an application of the dominated convergence theorem in $L^1(Q_0)$. Observe first that, by (3.6.1), we have

$$|E_j^i f| \leq C^{-1}|b^i| + \sum_n |b^i_{P_n^i}|,$$

where $C = \inf |[b]_Q|$, the infimum taken over the pa i cubes Q. To see that $E_j^i f$ converges μ-a.e. on Q_0 towards Π^i, use again (3.6.1) and observe that for every n, $E_j^i f$ is constant on P_n^i for j large enough (depending on n), and equal to $[f]_{P_n^i} b^i_{P_n^i}$. If $x \in F^i = Q_0 \backslash \cup P_n^i$, then by the Lebesgue differentiation theorem $\frac{[f]_Q}{[b]_Q}$ tends μ-a.e. towards $\frac{f(x)}{b(x)}$ when Q tends towards x. Thus $E_j^i f$ tends μ-a.e. towards f on F^i, and, by (3.4.8), $E_j^i f$ tends μ-a.e. towards $\Pi^i f$ on Q_0. It remains only to apply the dominated convergence theorem in $L^1(Q_0)$ to get (3.6.2).

□

Now, because of (3.6.2), and because we assumed the kernel of the operator T to be locally bounded, we can take the limit and write

$$\begin{aligned}
\langle \Pi^2 f, T\Pi^1 g\rangle &= \lim_{j\to+\infty} \langle E_j^2 f, TE_j^1 g\rangle \\
&= \langle E_0^2 f, TE_0^1 g\rangle + \sum_{j\geq 0} \Big(\langle D_j^2 f, TD_j^1 g\rangle + \langle D_j^2 f, TE_j^1 g\rangle + \langle E_j^2 f, TD_j^1 g\rangle\Big) \\
&= \langle E_0^2 f, TE_0^1 g\rangle + \langle f, Ug\rangle + \langle f, Vg\rangle + \langle f, Wg\rangle.
\end{aligned}$$

This is the so called BCR algorithm, introduced in [BCR] for the classical martingale differences. Here we use the adapted martingale differences. Let us remark that the first term is easy to estimate because $f, g \in E_{Q_0}$, and because of (3.1.4), (3.1.5). Remark also that the two last terms V, W can be treated in the same way by duality. They lead to paraproduct type operators. The diagonal term U is relatively easy to treat. With this decomposition and the wavelet representation, we will obtain matrix coefficients of the form $\langle b^2\theta_Q, T(b^1\theta_R)\rangle$, where the functions θ_Q, θ_R are supported on the cubes Q, R respectively. The idea is mostly to use the kernel decay and the standard Calderón-Zygmund estimates (1.2.7) when the cubes Q, R are far away, and to use the weak boundedness properties (3.1.6), (3.1.7) when they are close. Before going on, let us give all the coefficient estimates we will need to bound these terms.

3.6.2 Coefficient estimates

Let Q and R be two dyadic cubes. Remember that we introduced in Section 1.2.3 the quantity

$$\mu(Q,R) = \inf_{x\in Q, y\in R} \{\mu(B(x,\rho(x,y))), \mu(B(y,\rho(x,y)))\}.$$

Now, set

$$\alpha_{Q,R} = \alpha_{R,Q} = \begin{cases} \mu(Q)^{\frac12}\mu(R)^{\frac12}\left(\frac{\inf(l(Q),l(R))}{\rho(Q,R)}\right)^\alpha \frac{1}{\mu(Q,R)} & \text{if } \ \rho(Q,R) \geq \sup(l(Q),l(R)) \\ 1 & \text{if } \ l(Q) = l(R), \ \text{ and } \ \rho(Q,R) < l(Q). \end{cases}$$

Of course $\alpha_{Q,R}$ is not defined on all pairs of cubes (Q,R), but that does not matter as we will only use this notation when we are in one of the above cases. We state the following lemma:

Lemma 3.6.2. *Let Q, R be two pseudo-accretive dyadic cubes in X, either strongly pseudo-accretive or degenerate pseudo-accretive with respect to b^1, b^2 respectively, and depending on the nature of the inequality below. Let also P_i^1 be one of the stopping cubes with respect to b^1. Then we have the following coefficient estimates: whenever $l(Q) = l(R)$,*

$$\left|\langle b^2\phi_Q^2, T(b^1\phi_R^1)\rangle\right| \lesssim \alpha_{Q,R}, \tag{3.6.3}$$

$$\left|\langle b^2\phi_Q^2, T(\xi_R^1 g)\rangle\right| \lesssim \alpha_{Q,R}\, \mu(R)^{\frac{1}{2}}, \tag{3.6.4}$$

$$\left|\langle \xi_Q^2 f, T(\xi_R^1 g)\rangle\right| \lesssim \alpha_{Q,R}\, \mu(Q)^{\frac{1}{2}}\, \mu(R)^{\frac{1}{2}}, \tag{3.6.5}$$

$$\left|\langle \xi_Q^2 f, T(b^1 1_R)\rangle\right| \lesssim \alpha_{Q,R}\, \mu(Q)^{\frac{1}{2}}\, \mu(R)^{\frac{1}{2}}, \tag{3.6.6}$$

$$\left|\langle b^2\phi_Q^2, T(b^1 1_R)\rangle\right| \lesssim \alpha_{Q,R}\, \mu(R)^{\frac{1}{2}}, \tag{3.6.7}$$

and when $\rho(Q, P_i^1) \geq l(P_i^1) \geq l(Q)$,

$$\left|\langle \xi_Q^2 f, T(b^1_{P_i^1})\rangle\right| \lesssim \alpha_{Q,P_i^1}\, \mu(Q)^{\frac{1}{2}}\, \mu(P_i^1)^{\frac{1}{2}}, \tag{3.6.8}$$

$$\left|\langle b^2\phi_Q^2, T(b^1_{P_i^1})\rangle\right| \lesssim \alpha_{Q,P_i^1}\, \mu(P_i^1)^{\frac{1}{2}}, \tag{3.6.9}$$

$$\left|\langle b^2\phi_Q^2, T(b^1 1_{P_i^1})\rangle\right| \lesssim \alpha_{Q,P_i^1}\, \mu(P_i^1)^{\frac{1}{2}}, \tag{3.6.10}$$

Proof. We detail the proof of the first inequality. It is mostly an application of (1.2.7) and (3.1.8). Suppose first that we have $\rho(Q,R) \geq l(Q)$. Since $b^1\phi_R^1$ has mean 0, we can apply (1.2.7) to get the bound

$$\left|\langle b^2\phi_Q^2, T(b^1\phi_R^1)\rangle\right| \lesssim \|b^2\phi_Q^2\|_{L^1(Q)}\|b^1\phi_R^1\|_{L^1(R)} \left(\frac{\inf(l(Q), l(R))}{\rho(Q,R)}\right)^\alpha \frac{1}{\mu(Q,R)} \lesssim \alpha_{Q,R},$$

the last inequality being a consequence of the L^∞ estimates of the functions ϕ_Q^2, ϕ_R^1. Indeed, Lemma 3.4.1 entails that $\|\phi_Q^2\|_\infty \lesssim \mu(Q)^{-1/2}, \|\phi_R^1\|_\infty \lesssim \mu(R)^{-1/2}$. Thus $\|b^2\phi_Q^2\|_{L^1(Q)} \lesssim \mu(Q)^{-1/2}\|b^2\|_{L^1(Q)} \lesssim \mu(Q)^{1/2}$ since Q is pa 2, and similarly for $b^1\phi_R^1$.
Now, assume that $\rho(Q,R) < l(Q)$. Write

$$\langle b^2\phi_Q^2, T(b^1\phi_R^1)\rangle = \sum_{Q' \in \widetilde{Q}, R' \in \widetilde{R}} [\phi_Q^2]_{Q'} \langle b^2 1_{Q'}, T(b^1 1_{R'})\rangle [\phi_R^1]_{R'}.$$

If Q' and R' are neighbors, by the weak boundedness property (3.1.8) (we do not need the stronger property (3.1.6) here), we have

$$|\langle b^2 1_{Q'}, T(b^1 1_{R'})\rangle| \lesssim \mu(R').$$

If $\rho(Q', R') \geq l(R')$, then write

$$\begin{aligned}
|\langle b^2 1_{Q'}, T(b^1 1_{R'})\rangle| &\leq |\langle b^2 1_{Q'}, T(b^1 1_{R'} - [b^1]_{R'} 1_{R'})\rangle| + |[b^1]_{R'}||\langle b^2 1_{Q'}, T(1_{R'})\rangle| \\
&\lesssim \mu(Q')\mu(R') \left(\frac{l(R')}{\rho(Q', R')}\right)^\alpha \frac{1}{\mu(Q', R')} + \mu(R') \\
&\lesssim \mu(R').
\end{aligned}$$

To get the second inequality, we have applied (1.2.6) to the first term (which is possible because $\int (b^1 1_{R'} - [b^1]_{R'} 1_{R'}) \mathrm{d}\mu = 0$), and the Hardy inequality (3.1.1) to the second term. Then the last inequality comes from the fact that $\rho(Q', R') \geq l(R')$, and $\mu(Q', R') \approx \mu(\widehat{Q}) \approx \mu(Q')$ as both Q' and R' are children of the neighbor cubes Q, R. Finally, using again the L^∞ estimates of the functions ϕ, and the fact that Q', R' have comparable measure, we have

$$\left|\langle b^2 \phi_Q^2, T(b^1 \phi_R^1)\rangle\right| \lesssim \sum_{Q' \in \widetilde{Q}, R' \in \widetilde{R}} \mu(Q')^{-\frac{1}{2}} \mu(R') \mu(R')^{-\frac{1}{2}} \lesssim 1,$$

which was the desired estimate.

For the other estimates we asserted, it is exactly the same idea. Indeed, observe that we always have at least one of the functions in the pairing which has mean zero, allowing us to apply (1.2.7) when the cubes Q and R (or P_i^1) are far away. Remark that the only change is the normalization of those functions: we have

$$\|\xi_R^1 g\|_{L^1(R)} \lesssim \mu(R), \quad \|\xi_Q^2 f\|_{L^1(Q)} \lesssim \mu(Q), \quad \|b^1 1_R\|_{L^1(R)} \lesssim \mu(R), \quad \|b_{P_i^1}^1\|_{L^1(P_i^1)} \lesssim \mu(P_i^1).$$

It gives all the desired estimates when the cubes are far away. Now when Q and R are neighbors, it is also the same idea as above, and one only has to apply the weak boundedness property (3.1.8) to get the desired estimates. We therefore omit the detail here. □

Now, we can proceed to our argument.

3.6.3 Analysis of U

First, let us decompose

$$\langle f, Ug\rangle = \sum_{j \geq 0} \langle D_j^2 f, T D_j^1 g\rangle = \langle f, U_1 g\rangle + \langle f, U_2 g\rangle + \langle f, U_3 g\rangle + \langle f, U_4 g\rangle,$$

with

$$\begin{aligned}
\langle f, U_1 g\rangle &= \sum_{\substack{Q \text{ spa } 2; R \text{ spa } 1 \\ l(R) = l(Q)}} \langle f, \phi_Q^2\rangle \langle b^2 \phi_Q^2, T(b^1 \phi_R^1)\rangle \langle g, \phi_R^1\rangle, \\
\langle f, U_2 g\rangle &= \sum_{\substack{Q \text{ spa } 2; R \text{ dpa } 1 \\ l(R) = l(Q)}} \langle f, \phi_Q^2\rangle \langle b^2 \phi_Q^2, T(\xi_R^1 g)\rangle, \\
\langle f, U_3 g\rangle &= \sum_{\substack{Q \text{ dpa } 2; R \text{ spa } 1 \\ l(R) = l(Q)}} \langle \xi_Q^2 f, T(b^1 \phi_R^1)\rangle \langle g, \phi_R^1\rangle, \\
\langle f, U_4 g\rangle &= \sum_{\substack{Q \text{ dpa } 2; R \text{ dpa } 1 \\ l(R) = l(Q)}} \langle \xi_Q^2 f, T(\xi_R^1 g)\rangle.
\end{aligned}$$

Observe that the sums with respect to j have disappeared once we notice they force the cubes Q and R to have equal lengths.

Estimate of $\langle f, U_1 g\rangle$

Applying the coefficient estimate (3.6.3), we have

$$|\langle f, U_1 g\rangle| \lesssim \sum_{\substack{Q \text{ spa } 2; R \text{ spa } 1 \\ l(R) = l(Q)}} \alpha_{Q,R} |\langle f, \phi_Q^2\rangle| \, |\langle g, \phi_R^1\rangle|.$$

Remember that by Lemma 3.4.2

$$\sum_{Q\,\mathrm{spa}\,2} |\langle f, \phi_Q^2\rangle|^2 \lesssim \|f\|_2^2 \lesssim \mu(Q_0),$$

and similarly for $\langle g, \phi_R^1\rangle$. Therefore, we have by the Cauchy-Schwarz inequality

$$\begin{aligned}
|\langle f, U_1 g\rangle| &\lesssim \sum_{\substack{Q\,\mathrm{spa}\,2;R\,\mathrm{spa}\,1\\ l(R)=l(Q)}} \alpha_{Q,R}\,|\langle f, \phi_Q^2\rangle|\,|\langle g, \phi_R^1\rangle| \\
&\lesssim \Big(\sum_{\substack{Q\,\mathrm{spa}\,2;R\,\mathrm{spa}\,1\\ l(R)=l(Q)}} \alpha_{Q,R}\Big(\frac{\mu(R)}{\mu(Q)}\Big)^{1/2}|\langle f, \phi_Q^2\rangle|^2\Big)^{1/2}\Big(\sum_{\substack{Q\,\mathrm{spa}\,2;R\,\mathrm{spa}\,1\\ l(R)=l(Q)}} \alpha_{Q,R}\Big(\frac{\mu(Q)}{\mu(R)}\Big)^{1/2}|\langle g, \phi_R^1\rangle|^2\Big)^{1/2}
\end{aligned}$$

Let us state a lemma that will handle those sums.

Lemma 3.6.3. *Let R be a fixed dyadic cube of the space of homogeneous type X. We have the following summing property*

$$\sum_{\substack{Q;l(Q)=l(R)\\ \rho(Q,R)\geq l(R)}} \mu(Q)\left(\frac{l(Q)}{\rho(Q,R)}\right)^{\alpha}\frac{1}{\mu(Q,R)} \lesssim 1.$$

Furthermore, if $p\in\mathbb{N}$, we have the stronger summing property

$$\sum_{\substack{Q;l(Q)=\delta^p l(R)\\ \rho(Q,R)\geq l(R)}} \mu(Q)\left(\frac{l(Q)}{\rho(Q,R)}\right)^{\alpha}\frac{1}{\mu(Q,R)} \lesssim \delta^{p\alpha},$$

which, by summing over $p\in\mathbb{N}$, yields

$$\sum_{\substack{Q;l(Q)\leq l(R)\\ \rho(Q,R)\geq l(R)}} \mu(Q)\left(\frac{l(Q)}{\rho(Q,R)}\right)^{\alpha}\frac{1}{\mu(Q,R)} \lesssim 1.$$

The exponent α can be replaced by any $\alpha' > 0$.

Proof. It is enough to prove the second inequality. The idea is to split the sum for $\delta^{-m}l(R) \leq \rho(Q,R) < \delta^{-m-1}l(R), m\in\mathbb{N}$ (here and subsequently, we then write that $\rho(Q,R)\sim\delta^{-m}l(R)$): write

$$\begin{aligned}
\sum_{\substack{Q;l(Q)=\delta^p l(R)\\ \rho(Q,R)\geq l(R)}} \mu(Q)\left(\frac{l(Q)}{\rho(Q,R)}\right)^{\alpha}\frac{1}{\mu(Q,R)} &= \delta^{p\alpha}\sum_{m\geq 0}\sum_{\substack{Q;l(Q)=\delta^p l(R)\\ \rho(Q,R)\sim\delta^{-m}l(R)}} \mu(Q)\left(\frac{l(R)}{\rho(Q,R)}\right)^{\alpha}\frac{1}{\mu(Q,R)} \\
&\lesssim \delta^{p\alpha}\sum_{m\geq 0}\delta^{m\alpha}\sum_{\substack{Q;l(Q)=\delta^p l(R)\\ \rho(Q,R)\sim\delta^{-m}l(R)}} \frac{\mu(Q)}{\mu(Q,R)} \\
&\lesssim \delta^{p\alpha}\sum_{m\geq 0}\delta^{m\alpha}\sum_{\substack{Q;l(Q)=\delta^p l(R)\\ \rho(Q,R)\sim\delta^{-m}l(R)}} \frac{\mu(Q)}{\mu(B(z_R,\delta^{-m}l(R)))},
\end{aligned}$$

where z_R denotes the center of the cube R. All those cubes Q are non overlapping as they are of the same generation, and they are all contained in a ball of comparable measure to the ball $B(z_R,\delta^{-m}l(R))$. We therefore have

$$\sum_{\substack{Q;l(Q)=\delta^p l(R)\\ \rho(Q,R)\sim\delta^{-m}l(R)}} \frac{\mu(Q)}{\mu(B(z_R,\delta^{-m}l(R)))} \lesssim 1,$$

and summing then over $m \in \mathbb{N}$, we get the desired estimate. Moreover, it is clear that α can be replaced by any $\alpha' > 0$. $\square$

Now let us go back to our argument. For a fixed cube R, we write

$$\sum_{\substack{Q \text{ spa } 2\\ l(Q)=l(R)}} \alpha_{Q,R} \left(\frac{\mu(Q)}{\mu(R)}\right)^{1/2} = \sum_{\substack{Q \text{ spa } 2; l(Q)=l(R)\\ \rho(Q,R)\geq l(R)}} \mu(Q) \left(\frac{l(Q)}{\rho(Q,R)}\right)^{\alpha} \frac{1}{\mu(Q,R)} + \sum_{\substack{Q \text{ spa } 2; l(Q)=l(R)\\ \rho(Q,R)< l(R)}} \left(\frac{\mu(Q)}{\mu(R)}\right)^{1/2}.$$

Lemma 3.6.3 insures that the first sum is uniformly bounded, and the second sum is bounded as well, because such cubes Q and R have comparable measure (they are neighbors), and any given cube R has a uniformly bounded number of neighbors. We thus have for any given cube R,

$$\sum_{\substack{Q \text{ spa } 2\\ l(Q)=l(R)}} \alpha_{Q,R} \left(\frac{\mu(Q)}{\mu(R)}\right)^{1/2} \lesssim 1,$$

and similarly, for any given cube Q,

$$\sum_{\substack{R \text{ spa } 1\\ l(R)=l(Q)}} \alpha_{Q,R} \left(\frac{\mu(R)}{\mu(Q)}\right)^{1/2} \lesssim 1.$$

Therefore,

$$|\langle f, U_1 g\rangle| \lesssim \Big(\sum_{Q \text{ spa } 2} |\langle f, \phi_Q^2\rangle|^2\Big)^{1/2} \Big(\sum_{R \text{ spa } 1} |\langle g, \phi_R^1\rangle|^2\Big)^{1/2} \lesssim \mu(Q_0).$$

Estimate of $\langle f, U_2 g\rangle$

Remark that by duality, $\langle f, U_3 g\rangle$ is estimated in the same way. Applying the coefficient estimate (3.6.4), we have

$$|\langle f, U_2 g\rangle| \lesssim \sum_{\substack{Q \text{ spa } 2; R \text{ dpa } 1\\ l(R)=l(Q)}} \alpha_{Q,R}\, \mu(R)^{\frac{1}{2}}\, |\langle f, \phi_Q^2\rangle|.$$

We split this sum into two parts, depending on whether $\rho(Q,R) \geq l(Q)$ or not. By the Cauchy-Schwarz inequality and Lemma 3.6.3, we have

$$\begin{aligned}
\sum_{\substack{Q \text{ spa } 2; R \text{ dpa } 1\\ l(R)=l(Q)\\ \rho(Q,R)\geq l(Q)}} \alpha_{Q,R}\, \mu(R)^{\frac{1}{2}}\, |\langle f, \phi_Q^2\rangle| &\lesssim \Big(\sum_{Q \text{ spa } 2} |\langle f, \phi_Q^2\rangle|^2 \sum_{\substack{R \text{ dpa } 1; l(R)=l(Q)\\ \rho(Q,R)\geq l(Q)}} \mu(R) \Big(\frac{l(R)}{\rho(Q,R)}\Big)^{\alpha} \frac{1}{\mu(Q,R)}\Big)^{1/2} \\
&\quad \times \Big(\sum_{R \text{ dpa } 1} \mu(R) \sum_{\substack{Q \text{ spa } 2; l(Q)=l(R)\\ \rho(Q,R)\geq l(R)}} \mu(Q) \Big(\frac{l(Q)}{\rho(Q,R)}\Big)^{\alpha} \frac{1}{\mu(Q,R)}\Big)^{1/2} \\
&\lesssim \Big(\sum_{Q \text{ spa } 2} |\langle f, \phi_Q^2\rangle|^2\Big)^{1/2} \times \Big(\sum_{R \text{ dpa } 1} \mu(R)\Big)^{1/2} \lesssim \mu(Q_0),
\end{aligned}$$

the last inequality being a consequence of the fact that $\sum_{R \text{ dpa } 1} \mu(R) \lesssim \mu(Q_0)$. The remaining sum is easy to estimate: since for a fixed Q there is a uniformly bounded number of neighbors R, one can write

$$\begin{aligned}
\sum_{\substack{Q \text{ spa } 2; R \text{ dpa } 1\\ l(R)=l(Q)\\ \rho(Q,R)< l(Q)}} \mu(R)^{\frac{1}{2}}\, |\langle f, \phi_Q^2\rangle| &\lesssim \Big(\sum_{Q \text{ spa } 2} \sum_{\substack{R \text{ dpa } 1\\ R\subset \widehat{Q}; l(R)=l(Q)}} |\langle f, \phi_Q^2\rangle|^2\Big)^{1/2} \Big(\sum_{R \text{ dpa } 1} \sum_{\substack{Q \text{ spa } 2\\ Q\subset \widehat{R}; l(Q)=l(R)}} \mu(R)\Big)^{1/2} \\
&\lesssim \Big(\sum_{Q \text{ spa } 2} |\langle f, \phi_Q^2\rangle|^2\Big)^{1/2} \Big(\sum_{R \text{ dpa } 1} \mu(R)\Big)^{1/2} \lesssim \mu(Q_0).
\end{aligned}$$

Estimate of $\langle f, U_4 g\rangle$

This term is pretty easy to handle. Indeed, applying the coefficient estimate (3.6.5), we have

$$\begin{aligned}
|\langle f, U_4 g\rangle| &\lesssim \sum_{\substack{Q \,\mathrm{dpa}\, 2; R \,\mathrm{dpa}\, 1\\ l(R)=l(Q)}} \alpha_{Q,R}\, \mu(Q)^{\frac{1}{2}}\, \mu(R)^{\frac{1}{2}} \\
&\lesssim \sum_{Q \,\mathrm{dpa}\, 2} \Big(\sum_{\substack{R \,\mathrm{dpa}\, 1; l(R)=l(Q)\\ \rho(Q,R)<l(Q)}} \mu(Q)^{\frac{1}{2}}\, \mu(R)^{\frac{1}{2}} + \mu(Q) \sum_{\substack{R \,\mathrm{dpa}\, 1; l(R)=l(Q)\\ \rho(Q,R)\geq l(Q)}} \mu(R) \Big(\frac{l(R)}{\rho(Q,R)}\Big)^{\alpha} \frac{1}{\mu(Q,R)}\Big) \\
&\lesssim \sum_{Q \,\mathrm{dpa}\, 2} \mu(Q) \lesssim \mu(Q_0),
\end{aligned}$$

where we have used Lemma 3.6.3 and once again the fact that any fixed cube Q has a uniformly bounded number of neighbors.

3.6.4 Analysis of V

As in the previous subsection, we write

$$\langle f, V g\rangle = \sum_{j\geq 0} \langle D_j^2 f, T E_j^1 g\rangle = \langle f, V_1 g\rangle + \langle f, V_2 g\rangle + \langle f, V_3 g\rangle + \langle f, V_4 g\rangle,$$

with

$$\begin{aligned}
\langle f, V_1 g\rangle &= \sum_{\substack{Q \,\mathrm{spa}\, 2; R \,\mathrm{pa}\, 1\\ l(R)=l(Q)}} \langle f, \phi_Q^2\rangle \langle b^2 \phi_Q^2, T(b^1 1_R)\rangle \frac{[g]_R}{[b^1]_R}, \\
\langle f, V_2 g\rangle &= \sum_{Q \,\mathrm{spa}\, 2} \sum_{l(P_i^1)\geq l(Q)} \langle f, \phi_Q^2\rangle \langle b^2 \phi_Q^2, T(b^1_{P_i^1})\rangle [g]_{P_i^1}, \\
\langle f, V_3 g\rangle &= \sum_{\substack{Q \,\mathrm{dpa}\, 2; R \,\mathrm{pa}\, 1\\ l(R)=l(Q)}} \langle \xi_Q^2 f, T(b^1 1_R)\rangle \frac{[g]_R}{[b^1]_R}, \\
\langle f, V_4 g\rangle &= \sum_{Q \,\mathrm{dpa}\, 2} \sum_{l(P_i^1)\geq l(Q)} \langle \xi_Q^2 f, T(b^1_{P_i^1})\rangle [g]_{P_i^1}.
\end{aligned}$$

The term V_1 will be the most difficult term, it is where a parapoduct appears, which will require the use of hypothesis (3.1.6) to control.

Estimate of $\langle f, V_4 g\rangle$

We split the sum into two parts, as before, depending on the distance between P_i^1 and Q. When $\rho(P_i^1, Q) \geq l(P_i^1)$, we have, by the coefficient estimate (3.6.8),

$$\begin{aligned}
\sum_{Q \,\mathrm{dpa}\, 2} \sum_{\substack{l(P_i^1)\geq l(Q)\\ \rho(P_i^1,Q)\geq l(P_i^1)}} |\langle \xi_Q^2 f, T(b^1_{P_i^1})\rangle [g]_{P_i^1}| &\lesssim \sum_{P_i^1} \sum_{\substack{Q \,\mathrm{dpa}\, 2; l(Q)\leq l(P_i^1)\\ \rho(Q,P_i^1)\geq l(P_i^1)}} \mu(P_i^1)^{\frac12} \mu(Q)^{\frac12} \alpha_{Q,P_i^1}\\
&\lesssim \sum_{P_i^1} \mu(P_i^1) \sum_{\substack{Q \,\mathrm{dpa}\, 2; l(Q)\leq l(P_i^1)\\ \rho(Q,P_i^1)\geq l(P_i^1)}} \mu(Q) \left(\frac{l(Q)}{\rho(Q,R)}\right)^{\alpha} \frac{1}{\mu(Q,R)}\\
&\lesssim \sum_{P_i^1} \mu(P_i^1) \lesssim \mu(Q_0),
\end{aligned}$$

where we have once again used Lemma 3.6.3 to obtain the third inequality, and the $1-$packing property of the P_i^1 to get the last one. It then remains to estimate the sum

$$\langle f, V_{4,1} g\rangle = \sum_{P_i^1} \sum_{\substack{Q \,\mathrm{dpa}\, 2; l(Q)\leq l(P_i^1)\\ \rho(Q,P_i^1)< l(P_i^1)}} \langle \xi_Q^2 f, T(b^1_{P_i^1})\rangle [g]_{P_i^1} = \sum_{P_i^1} \sum_{\substack{P\subset \widehat{P_i^1}\\ l(P)=l(P_i^1)}} \sum_{\substack{Q \,\mathrm{dpa}\, 2\\ Q\subset P}} \langle \xi_Q^2 f, T(b^1_{P_i^1})\rangle [g]_{P_i^1},$$

which we will do below, when we take care of the term $V_{2,1}$.

Estimate of $\langle f, V_3 g\rangle$

This term is easy to handle. Indeed, applying the coefficient estimate (3.6.6), and the fact that the mean of b^1 is bounded below uniformly on the pa 1 cubes R, we have

$$\begin{aligned}
|\langle f, V_3 g\rangle| &\lesssim \sum_{Q \,\mathrm{dpa}\, 2} \sum_{\substack{R \,\mathrm{pa}\, 1\\ l(R)=l(Q)}} |\langle \xi_Q^2 f, T(b^1 1_R)\rangle| \frac{|[g]_R|}{|[b^1]_R|}\\
&\lesssim \sum_{Q \,\mathrm{dpa}\, 2} \sum_{\substack{R \,\mathrm{pa}\, 1\\ l(R)=l(Q)}} \alpha_{Q,R}\, \mu(Q)^{\frac12}\, \mu(R)^{\frac12}\\
&\lesssim \sum_{Q \,\mathrm{dpa}\, 2} \mu(Q) \sum_{\substack{R \,\mathrm{pa}\, 1; l(R)=l(Q)\\ \rho(R,Q)\geq l(R)}} \mu(R) \left(\frac{l(R)}{\rho(Q,R)}\right)^{\alpha} \frac{1}{\mu(Q,R)} + \sum_{Q \,\mathrm{dpa}\, 2} \mu(Q)^{\frac12} \sum_{\substack{R \,\mathrm{pa}\, 1; l(R)=l(Q)\\ \rho(R,Q)< l(R)}} \mu(R)^{\frac12}\\
&\lesssim \sum_{Q \,\mathrm{dpa}\, 2} \mu(Q) + \sum_{Q \,\mathrm{dpa}\, 2} \mu(Q) \lesssim \mu(Q_0),
\end{aligned}$$

where once more the last line is a consequence of Lemma 3.6.3, the fact that neighbor cubes have comparable measure, that any dyadic cube Q has a uniformly bounded number of such neighbors, and the C_X-packing property of the dpa 2 cubes (3.4.6).

Estimate of $\langle f, V_2 g\rangle$

First, let us examine the part of the sum when the cubes P_i^1 and Q are close, that is

$$\begin{aligned}\langle f, V_{2,1} g\rangle &= \sum_{P_i^1} \sum_{\substack{Q \text{ spa } 2; l(Q)\le l(P_i^1)\\ \rho(Q,P_i^1)<l(P_i^1)}} \langle f, \phi_Q^2\rangle\langle b^2\phi_Q^2, T(b^1_{P_i^1})\rangle [g]_{P_i^1}\\ &= \sum_{P_i^1} \sum_{\substack{P\subset \widehat{P_i^1}\\ l(P)=l(P_i^1)}} \sum_{\substack{Q \text{ spa } 2\\ Q\subset P}} \langle f, \phi_Q^2\rangle\langle b^2\phi_Q^2, T(b^1_{P_i^1})\rangle [g]_{P_i^1}.\end{aligned}$$

We put this sum together with the term left to estimate $\langle f, V_{4,1} g\rangle$ to get

$$\Sigma = \sum_{P_i^1} [g]_{P_i^1} \sum_{\substack{P\subset \widehat{P_i^1}\\ l(P)=l(P_i^1)}} \Big(\sum_{\substack{Q \text{ spa } 2\\ Q\subset P}} \langle f, \phi_Q^2\rangle\langle b^2\phi_Q^2, T(b^1_{P_i^1})\rangle + \sum_{\substack{Q \text{ dpa } 2\\ Q\subset P}} \langle \xi_Q^2 f, T(b^1_{P_i^1})\rangle\Big).$$

By definition, we have

$$\sum_{\substack{Q \text{ spa } 2\\ Q\subset P}} \langle f, \phi_Q^2\rangle\langle b^2\phi_Q^2, T(b^1_{P_i^1})\rangle + \sum_{\substack{Q \text{ dpa } 2\\ Q\subset P}} \langle \xi_Q^2 f, T(b^1_{P_i^1})\rangle = \langle \Pi^2(f1_P), T(b^1_{P_i^1})\rangle - [f]_P\langle b^2 1_P, T(b^1_{P_i^1})\rangle.$$

We can assume that P is pa 2 (else the above sum reduces to 0), so that, using (3.1.5) for $T(b^1_{P_i^1})$ on $P\subset \widehat{P_i^1}$, we get

$$|\langle b^2 1_P, T(b^1_{P_i^1})\rangle| \lesssim \|b^2\|_{L^q(P)}\|T(b^1_{P_i^1})\|_{L^{q'}(P)} \lesssim \mu(P).$$

Observe that since the P_j^2 contained in P are non overlapping dyadic subcubes of P,

$$\|\Pi^2(f1_P)\|^q_{L^q(P)} \lesssim \|f\|^q_{L^q(P)} + \sum_{P_j^2\subset P} \|b^2_{P_j^2}\|^q_{L^q(P)} \lesssim \mu(P) + \sum_{P_j^2\subset P}\mu(P_j^2) \lesssim \mu(P),$$

and thus, applying again (3.1.5) for $T(b^1_{P_i^1})$ on P, we have $|\langle \Pi^2(f1_P), T(b^1_{P_i^1})\rangle| \lesssim \mu(P)$, giving the expected bound on $\langle f, (V_{2,1}+V_{4,1})g\rangle$.

We now estimate the sum in $\langle f, V_2 g\rangle$ running over the pairs of cubes Q and P_i^1 which are far away from one another. Applying the coefficient estimate (3.6.9) and then the Cauchy-Schwarz inequality, one gets

$$\sum_{P_i^1} \sum_{\substack{Q \text{ spa } 2; l(Q)\le l(P_i^1)\\ \rho(Q,P_i^1)\ge l(P_i^1)}} |\langle f, \phi_Q^2\rangle|\,|\langle b^2\phi_Q^2, T(b^1_{P_i^1})\rangle [g]_{P_i^1}| \lesssim \sum_{P_i^1} \sum_{\substack{Q \text{ spa } 2; l(Q)\le l(P_i^1)\\ \rho(Q,P_i^1)\ge l(P_i^1)}} |\langle f, \phi_Q^2\rangle|\, \mu(P_i^1)^{\frac12}\, \alpha_{Q,P_i^1}$$

$$\begin{aligned}&\lesssim \sum_{P_i^1} \sum_{\substack{Q \text{ spa } 2; l(Q)\le l(P_i^1)\\ \rho(Q,P_i^1)\ge l(P_i^1)}} \Big\{ |\langle f, \phi_Q^2\rangle| \frac{\mu(P_i^1)^{1/2}}{\mu(Q,P_i^1)^{1/2}} \frac{l(Q)^{\alpha/2}}{\rho(Q,P_i^1)^{\alpha/2}}\Big\}\Big\{\mu(P_i^1)^{\frac12} \frac{\mu(Q)^{1/2}}{\mu(Q,P_i^1)^{1/2}} \frac{l(Q)^{\alpha/2}}{\rho(Q,P_i^1)^{\alpha/2}}\Big\}\\ &\lesssim S_1^{1/2}\,.\,S_2^{1/2},\end{aligned}$$

with

$$S_1 = \sum_{Q\,\mathrm{spa}\,2} |\langle f, \phi_Q^2\rangle|^2 \sum_{\substack{P_i^1; l(P_i^1)\geq l(Q) \\ \rho(P_i^1,Q)\geq l(P_i^1)}} \frac{\mu(P_i^1)}{\mu(Q,P_i^1)} \left(\frac{l(Q)}{\rho(Q,P_i^1)}\right)^{\alpha},$$

$$S_2 = \sum_{P_i^1} \mu(P_i^1) \sum_{\substack{Q\,\mathrm{spa}\,2; l(Q)\leq l(P_i^1) \\ \rho(Q,P_i^1)\geq l(P_i^1)}} \frac{\mu(Q)}{\mu(Q,P_i^1)} \left(\frac{l(Q)}{\rho(Q,P_i^1)}\right)^{\alpha}.$$

By Lemma 3.6.3 and the 1−packing property of the P_i^1, we have already $S_2 \lesssim \sum_i \mu(P_i^1) \lesssim \mu(Q_0)$. For the first sum, we have to work a bit more. The result is obtained applying another summing lemma:

Lemma 3.6.4. *Let Q be a fixed dyadic cube of the space of homogeneous type X. Let $p \in \mathbb{N}$. We have the following summing property*

$$\sum_{\substack{R; l(R)=\delta^{-p}l(Q) \\ \rho(R,Q)\geq l(R)}} \mu(R) \left(\frac{l(Q)}{\rho(Q,R)}\right)^{\alpha} \frac{1}{\mu(Q,R)} \lesssim \delta^{p\alpha},$$

which, by summing over $p \in \mathbb{N}$, immediately yields

$$\sum_{\substack{R; l(R)\geq l(Q) \\ \rho(R,Q)\geq l(R)}} \mu(R) \left(\frac{l(Q)}{\rho(Q,R)}\right)^{\alpha} \frac{1}{\mu(Q,R)} \lesssim 1.$$

Proof. Observe that since $\rho(Q,R) \geq l(R) \geq l(Q)$, we have $\mu(Q,R) \approx \mu(B(z_Q, \rho(z_Q,y)))$ for all $y \in R$, with z_Q denoting the center of the cube Q. In the same way, $\rho(Q,R) \approx \rho(z_Q,y)$ for all $y \in R$. Therefore, we can write

$$\begin{aligned}
\sum_{\substack{R; l(R)=\delta^{-p}l(Q) \\ \rho(R,Q)\geq l(R)}} \left(\frac{l(Q)}{\rho(Q,R)}\right)^{\alpha} \frac{\mu(R)}{\mu(Q,R)} &\lesssim \sum_{\substack{R; l(R)=\delta^{-p}l(Q) \\ \rho(R,Q)\geq l(R)}} \int_R \left(\frac{l(Q)}{\rho(z_Q,y)}\right)^{\alpha} \frac{\mathrm{d}\mu(y)}{\mu(B(z_Q,\rho(z_Q,y)))} \\
&\lesssim \int_{\rho(z_Q,y)>l(R)} \left(\frac{l(Q)}{\rho(z_Q,y)}\right)^{\alpha} \frac{\mathrm{d}\mu(y)}{\mu(B(z_Q,\rho(z_Q,y)))} \\
&\lesssim \sum_{m>0} \int_{\rho(z_Q,y)\sim\delta^{-m}l(R)} \left(\frac{l(Q)}{\delta^{-m}l(R)}\right)^{\alpha} \frac{\mathrm{d}\mu(y)}{\mu(B(z_Q,\delta^{-m}l(R)))} \\
&\lesssim \left(\frac{l(Q)}{l(R)}\right)^{\alpha} \sum_{m>0} \delta^{m\alpha} \frac{\mu(B(z_Q,\delta^{-m-1}l(R)))}{\mu(B(z_Q,\delta^{-m}l(R)))} \lesssim \delta^{p\alpha}.
\end{aligned}$$

The second inequality comes from the fact that the cubes R are non overlapping as they are all of the same generation. Then we obtained the third inequality by splitting the integral over the coronae $\rho(z_Q,y) \sim \delta^{-m}l(R)$, and the last line follows easily. □

Going back to our argument, Lemma 3.6.4 ensures that $S_1 \lesssim \sum_{Q\,\mathrm{spa}\,2} |\langle f, \phi_Q^2\rangle|^2 \lesssim \mu(Q_0)$, and it thus concludes the estimation of the term $\langle f, V_2 g\rangle$.

Estimate of $\langle f, V_1 g\rangle$

This is the most difficult term. Recall that

$$\langle f, V_1 g\rangle = \sum_{\substack{Q \text{ spa } 2; R \text{ pa } 1\\ l(R)=l(Q)}} \langle f, \phi_Q^2\rangle \langle b^2\phi_Q^2, T(b^1 1_R)\rangle \frac{[g]_R}{[b^1]_R}.$$

We split it into three parts: write

$$\langle f, V_1 g\rangle = \langle f, V_{1,1} g\rangle + \langle f, V_{1,2} g\rangle + \langle f, V_{1,3} g\rangle,$$

with

$$\langle f, V_{1,1} g\rangle = \sum_{\substack{Q \text{ spa } 2 \text{ and non pa } 1\\ R \text{ pa } 1\\ l(R)=l(Q)}} \langle f, \phi_Q^2\rangle \langle b^2\phi_Q^2, T(b^1 1_R)\rangle \frac{[g]_R}{[b^1]_R},$$

$$\langle f, V_{1,2} g\rangle = \sum_{\substack{Q \text{ spa } 2\\ \text{and pa } 1}} \Big(\langle f, \phi_Q^2\rangle \langle b^2\phi_Q^2, T(b^1 1_Q)\rangle \frac{[g]_Q}{[b^1]_Q} - \langle f, \phi_Q^2\rangle \langle b^2\phi_Q^2, T(b^1 (1_Q - \sum_{\substack{R' \text{ pa } 1\\ l(R')=l(Q)}} 1_{R'}))\rangle \frac{[g]_Q}{[b^1]_Q}\Big),$$

$$\langle f, V_{1,3} g\rangle = \sum_{\substack{Q \text{ spa } 2\\ \text{and pa } 1}} \Big(\sum_{\substack{R \text{ pa } 1; R\neq Q\\ l(R)=l(Q)}} \langle f, \phi_Q^2\rangle \langle b^2\phi_Q^2, T(b^1 1_R)\rangle \frac{[g]_R}{[b^1]_R} + \langle f, \phi_Q^2\rangle \langle b^2\phi_Q^2, T(b^1 (1_Q - \sum_{\substack{R' \text{ pa } 1\\ l(R')=l(Q)}} 1_{R'}))\rangle \frac{[g]_Q}{[b^1]_Q}\Big).$$

Estimate of $\langle f, V_{1,1} g\rangle$. Remark first that Q non pa 1 means $Q \subset P_i^1$ for some i. Hence,

$$\langle f, V_{1,1} g\rangle = \sum_{P_i^1} \sum_{\substack{Q \text{ spa } 2\\ Q\subset P_i^1}} \sum_{\substack{R \text{ pa } 1\\ l(R)=l(Q)}} \langle f, \phi_Q^2\rangle \langle b^2\phi_Q^2, T(b^1 1_R)\rangle \frac{[g]_R}{[b^1]_R}.$$

Now, applying the coefficient estimate (3.6.7) and the fact that the mean of b^1 is uniformly bounded below on the pa 1 cubes, we obtain

$$|\langle f, V_{1,1} g\rangle| \lesssim \sum_{P_i^1} \sum_{\substack{Q \text{ spa } 2\\ Q\subset P_i^1}} \sum_{\substack{R \text{ pa } 1\\ l(R)=l(Q)}} \alpha_{Q,R}\, |\langle f, \phi_Q^2\rangle|\, \mu(R)^{\frac12}.$$

Fix P_i^1, and observe that the cubes R involved in this sum are necessarily outside of P_i^1. Split the sum for the cubes R that are contained in $\widehat{P_i^1}$, and those that are outside of $\widehat{P_i^1}$. We first show that

$$\mathrm{I} = \sum_{\substack{Q \text{ spa } 2\\ Q\subset P_i^1}} \sum_{\substack{R \text{ pa } 1; l(R)=l(Q)\\ R\cap \widehat{P_i^1}=\varnothing}} \alpha_{Q,R}\, |\langle f, \phi_Q^2\rangle|\, \mu(R)^{\frac12} \lesssim \mu(P_i^1).$$

For such pairs of cubes (Q,R), as $Q \subset P_i^1$ we necessarily have $\rho(Q,R) \geq \rho(P_i^1, R) \geq l(P_i^1)$, and $\mu(Q,R) \gtrsim \mu(P_i^1, R)$, so that, applying the Cauchy-Schwarz inequality, one gets

$$\begin{aligned}
\mathrm{I} &\lesssim \sum_{\substack{Q \text{ spa } 2\\ Q\subset P_i^1}} \sum_{\substack{R \text{ pa } 1; l(R)=l(Q)\\ \rho(R,P_i^1)\geq l(P_i^1)}} |\langle f, \phi_Q^2\rangle|\, \mu(R)^{\frac12} \frac{l(R)^\alpha}{\rho(P_i^1,R)^\alpha} \frac{\mu(Q)^{\frac12}\mu(R)^{\frac12}}{\mu(P_i^1,R)} \\
&\lesssim \Big(\sum_{\substack{Q \text{ spa } 2\\ Q\subset P_i^1}} \sum_{\substack{R \text{ pa } 1\\ l(R)=l(Q)\\ \rho(R,P_i^1)\geq l(P_i^1)}} |\langle f, \phi_Q^2\rangle|^2 \frac{l(R)^\alpha}{\rho(P_i^1,R)^\alpha} \frac{\mu(R)}{\mu(P_i^1,R)}\Big)^{1/2} \Big(\sum_{\substack{Q \text{ spa } 2\\ Q\subset P_i^1}} \sum_{\substack{R \text{ pa } 1\\ l(R)=l(Q)\\ \rho(R,P_i^1)\geq l(P_i^1)}} \mu(Q) \frac{l(R)^\alpha}{\rho(P_i^1,R)^\alpha} \frac{\mu(R)}{\mu(P_i^1,R)}\Big)^{1/2}
\end{aligned}$$

By Lemma 3.6.3, we get

$$\sum_{\substack{R \text{ pa } 1; l(R)=l(Q) \\ \rho(R,P_i^1)\geq l(P_i^1)}} \left(\frac{l(R)}{\rho(P_i^1,R)}\right)^{\alpha} \frac{\mu(R)}{\mu(P_i^1,R)} \lesssim \left(\frac{l(Q)}{l(P_i^1)}\right)^{\alpha} \lesssim 1.$$

Therefore,

$$\begin{aligned} \mathrm{I} &\lesssim \Big(\sum_{\substack{Q \text{ spa } 2 \\ Q\subset P_i^1}} |\langle f,\phi_Q^2\rangle|^2\Big)^{1/2} \Big(\sum_{\substack{Q \text{ spa } 2 \\ Q\subset P_i^1}} \mu(Q)\Big(\frac{l(Q)}{l(P_i^1)}\Big)^{\alpha}\Big)^{1/2} \\ &\lesssim \|f\|_{L^2(P_i^1)}\Big(\sum_{m\geq 0}\delta^{m\alpha} \sum_{\substack{Q\subset P_i^1 \\ l(Q)=\delta^m l(P_i^1)}} \mu(Q)\Big)^{1/2} \lesssim \mu(P_i^1). \end{aligned}$$

We are left to show that

$$\mathrm{II} = \sum_{\substack{Q \text{ spa } 2 \\ Q\subset P_i^1}} \sum_{\substack{R \text{ pa } 1; l(R)=l(Q) \\ R\subset \widehat{P_i^1}}} \alpha_{Q,R}\, |\langle f,\phi_Q^2\rangle|\, \mu(R)^{\frac12} \lesssim \mu(P_i^1).$$

We again split this sum for the cubes Q that are at a distance less than $l(Q)$ to the complement of P_i^1, and for those that are not. In the first case, for a fixed such cube Q, there are the cubes R which are neighbors of Q - they come in a uniformly bounded number and they have a measure comparable to that of Q - and there are the cubes R at a distance greater than $l(Q)$ of Q. Write

$$\begin{aligned} \mathrm{II}_1 = &\sum_{\substack{Q \text{ spa } 2; Q\subset P_i^1 \\ \rho(Q,(P_i^1)^c)<l(Q)}} \sum_{\substack{R \text{ pa } 1; l(R)=l(Q) \\ R\subset \widehat{P_i^1}}} \alpha_{Q,R}\, |\langle f,\phi_Q^2\rangle|\, \mu(R)^{\frac12} \\ \lesssim &\sum_{\substack{Q \text{ spa } 2; Q\subset P_i^1 \\ \rho(Q,(P_i^1)^c)<l(Q)}} |\langle f,\phi_Q^2\rangle| \Big\{\mu(Q)^{\frac12} + \sum_{\substack{R \text{ pa } 1; l(R)=l(Q) \\ R\subset \widehat{P_i^1}; \rho(R,Q)\geq l(Q)}} \mu(Q)^{\frac12}\frac{\mu(R)}{\mu(Q,R)}\left(\frac{l(R)}{\rho(Q,R)}\right)^{\alpha}\Big\} \\ \lesssim &\sum_{\substack{Q \text{ spa } 2; Q\subset P_i^1 \\ \rho(Q,(P_i^1)^c)<l(Q)}} |\langle f,\phi_Q^2\rangle| \mu(Q)^{\frac12} \lesssim \Big\{\sum_{\substack{Q \text{ spa } 2; Q\subset P_i^1 \\ \rho(Q,(P_i^1)^c)<l(Q)}} \mu(Q)\Big\}^{1/2} \Big\{\sum_{\substack{Q \text{ spa } 2; Q\subset P_i^1 \\ \rho(Q,(P_i^1)^c)<l(Q)}} |\langle f,\phi_Q^2\rangle|^2\Big\}^{1/2}, \end{aligned}$$

applying Lemma 3.6.3 to get the second inequality and then the Cauchy-Schwarz inequality to get the last one. By Lemma 3.4.2, $\sum_{Q\subset P_i^1} |\langle f,\phi_Q^2\rangle|^2 \lesssim \|f\|^2_{L^2(P_i^1)} \lesssim \mu(P_i^1)$. Furthermore,

$$\begin{aligned} \sum_{\substack{Q\subset P_i^1 \\ \rho(Q,(P_i^1)^c)<l(Q)}} \mu(Q) &= \sum_{m\geq 0} \sum_{\substack{Q\subset P_i^1; l(Q)=\delta^m l(P_i^1) \\ \rho(Q,(P_i^1)^c)<\delta^m l(P_i^1)}} \mu(Q) \\ &\lesssim \sum_{m\geq 0} \mu\left(\{x\in P_i^1 : \rho(x,(P_i^1)^c)\lesssim \delta^m l(P_i^1)\}\right) \\ &\lesssim \sum_{m\geq 0} \delta^{m\eta}\mu(P_i^1) \lesssim \mu(P_i^1), \end{aligned}$$

by the small boundary condition (1.2.1), so that, as desired, we have $\mathrm{II}_1 \lesssim \mu(P_i^1)$. The sum left to estimate is

$$\mathrm{II}_2 = \sum_{\substack{Q \text{ spa } 2; Q\subset P_i^1 \\ \rho(Q,(P_i^1)^c)\geq l(Q)}} \sum_{\substack{R \text{ pa } 1; l(R)=l(Q) \\ R\subset \widehat{P_i^1}}} |\langle f,\phi_Q^2\rangle| \mu(Q)^{\frac12} \frac{\mu(R)}{\mu(Q,R)}\left(\frac{l(Q)}{\rho(Q,R)}\right)^{\alpha}.$$

For such couples of cubes (Q,R), we have necessarily $\rho(Q,R)\geq\rho(Q,(P_i^1)^c)\geq l(Q)$ as R and P_i^1 are disjoint. For a fixed cube Q, we thus see that

$$\sum_{\substack{R\subset\widehat{P_i^1};l(R)=l(Q)\\ \rho(R,Q)\geq l(Q)}}\frac{\mu(R)}{\mu(Q,R)}\left(\frac{l(Q)}{\rho(Q,R)}\right)^{\alpha}\lesssim\left(\frac{l(Q)}{\rho(Q,(P_i^1)^c)}\right)^{\alpha/2}\sum_{\substack{R\subset\widehat{P_i^1};l(R)=l(Q)\\ \rho(R,Q)\geq l(Q)}}\frac{\mu(R)}{\mu(Q,R)}\left(\frac{l(R)}{\rho(Q,R)}\right)^{\alpha/2}$$
$$\lesssim\left(\frac{l(Q)}{\rho(Q,(P_i^1)^c)}\right)^{\alpha/2},$$

by Lemma 3.6.3. Applying the Cauchy-Schwarz inequality, we then have

$$\begin{aligned}\mathrm{II}_2&\lesssim\sum_{\substack{Q\,\mathrm{spa}\,2;Q\subset P_i^1\\ \rho(Q,(P_i^1)^c)\geq l(Q)}}|\langle f,\phi_Q^2\rangle|\mu(Q)^{\frac12}\left(\frac{l(Q)}{\rho(Q,(P_i^1)^c)}\right)^{\frac{\alpha}{2}}\\&\lesssim\Big\{\sum_{\substack{Q\,\mathrm{spa}\,2;Q\subset P_i^1\\ \rho(Q,(P_i^1)^c)\geq l(Q)}}|\langle f,\phi_Q^2\rangle|^2\Big\}^{1/2}\Big\{\sum_{\substack{Q\,\mathrm{spa}\,2;Q\subset P_i^1\\ \rho(Q,(P_i^1)^c)\geq l(Q)}}\left(\frac{l(Q)}{\rho(Q,(P_i^1)^c)}\right)^{\alpha}\mu(Q)\Big\}^{1/2}\end{aligned}$$

Lemma 3.4.2 ensures that the first term is bounded by $\mu(P_i^1)^{1/2}$. For the second term, observe that

$$\sum_{\substack{Q\subset P_i^1\\ \rho(Q,(P_i^1)^c)\geq l(Q)}}\left(\frac{l(Q)}{\rho(Q,(P_i^1)^c)}\right)^{\alpha}\mu(Q)=\sum_{\substack{m\in\mathbb{N};j\in\mathbb{Z}\\ m\geq j>j_0}}\delta^{\alpha(m-j)}\sum_{Q\in E_{m,j}}\mu(Q),$$

where, for $m\in\mathbb{N},j\in\mathbb{Z}$, with $m\geq j>j_0$,

$$E_{m,j}=\{Q\subset P_i^1:\frac{l(Q)}{l(P_i^1)}=\delta^m\text{ and }\delta^j l(P_i^1)\leq\rho(Q,(P_i^1)^c)<\delta^{j-1}l(P_i^1)\},$$

and j_0 is such that $\delta^{j_0}\geq C_1>\delta^{j_0+1}$, with C_1 the constant intervening in Lemma 1.2.3. Indeed, for the cubes Q intervening in the above sum, we always have $\rho(Q,(P_i^1)^c)<\mathrm{diam}(P_i^1)\leq C_1 l(P_i^1)$, which forces $j>j_0$. Observe that if $x\in Q\subset E_{m,j}$, then $\rho(x,(P_i^1)^c)\leq\rho(Q,(P_i^1)^c)+l(Q)\leq\delta^{j-1}l(P_i^1)+\delta^m l(P_i^1)\lesssim\delta^j l(P_i^1)$ since $m\geq j$. Thus the cubes of $E_{m,j}$ pave a corona $\{x\in P_i^1:\rho(x,(P_i^1)^c)\lesssim\delta^j l(P_i^1)\}$. But by the small boundary condition (1.2.1), this corona has its measure bounded by $\delta^{j\eta}\mu(P_i^1)$. Therefore,

$$\sum_{\substack{Q\subset P_i^1\\ \rho(Q,(P_i^1)^c)\geq l(Q)}}\left(\frac{l(Q)}{\rho(Q,(P_i^1)^c)}\right)^{\alpha}\mu(Q)\lesssim\sum_{\substack{m\in\mathbb{N};j\in\mathbb{Z}\\ m\geq j>j_0}}\delta^{\alpha m}\delta^{(\eta-\alpha)j}\mu(P_i^1)\lesssim\sum_{j>j_0}\delta^{(\eta-\alpha)j}\delta^{\alpha j}\mu(P_i^1)\lesssim\mu(P_i^1),$$

which gives us the bound $\mathrm{II}_2\lesssim\mu(P_i^1)$. Summing then all these terms in i, and applying the $1-$packing property of the P_i^1, we obtain the expected bound $|\langle f,V_{1,1}g\rangle|\lesssim\mu(Q_0)$.

Estimate of $\langle f,V_{1,2}g\rangle$. This is where the paraproduct appears, and unfortunately, a non trivial error term. Write

$$\begin{aligned}\langle f,V_{1,2}g\rangle&=\sum_{Q\,\mathrm{spa}\,2\,\mathrm{and}\,\mathrm{pa}\,1}\langle f,\phi_Q^2\rangle\langle b^2\phi_Q^2,T(b^1)\rangle\frac{[g]_Q}{[b^1]_Q}-\sum_{Q\,\mathrm{spa}\,2\,\mathrm{and}\,\mathrm{pa}\,1}\langle f,\phi_Q^2\rangle\langle b^2\phi_Q^2,T(b^1 1_{\bigcup_{l(P_i^1)\geq l(Q)}P_i^1})\rangle\frac{[g]_Q}{[b^1]_Q}\\&=\mathrm{I}+\mathrm{II}.\end{aligned}$$

Observe that for the pa 1 cubes Q, the set of coefficients formed by $\left(\frac{[g]_Q}{[b^1]_Q}\right)_Q$ is uniformly bounded. It is easy to see that the hypotheses of Lemma 3.4.3 are verified (with $b = b^2$). Indeed, as a consequence of (3.4.5), we have $C_{\{P_j^2\}}(b^2) \lesssim 1$. Moreover, on $F = Q_0 \setminus \cup P_j^2$, it is clear that b^2 is uniformly bounded as a consequence of Lebesgue differentiation theorem. We can thus apply Lemma 3.4.3 to the term I, with $\nu = q$, and it gives us the bound

$$|\mathrm{I}| \lesssim \|f\|_{L^q(Q_0)} \|T(b^1)\|_{L^{q'}(Q_0)} \lesssim \mu(Q_0).$$

Next, we can rewrite the second term involving a sum over the cubes P_i^1. We split it into two parts, depending as usual on whether or not the cubes Q and P_i^1 are at a distance less than $l(P_i^1)$. Applying the coefficient estimate (3.6.10), one gets

$$\Big|\sum_{P_i^1} \sum_{\substack{Q \text{ spa 2 and pa 1} \\ l(Q)\le l(P_i^1);\rho(Q,P_i^1)\ge l(P_i^1)}} \langle f, \phi_Q^2\rangle \langle b^2\phi_Q^2, T(b^1 1_{P_i^1})\rangle \frac{[g]_Q}{[b^1]_Q}\Big| \lesssim \sum_{P_i^1} \sum_{\substack{Q \text{ spa 2 and pa 1} \\ l(Q)\le l(P_i^1);\rho(Q,P_i^1)\ge l(P_i^1)}} |\langle f, \phi_Q^2\rangle|\, \mu(P_i^1)^{\frac12}\, \alpha_{Q,P_i^1}.$$

Remark that we already estimated this sum when we were taking care of the term $\langle f, V_2 g\rangle$. We have seen that it is bounded by $\mu(Q_0)$. For the remaining sum, observe that since a pa 1 cube Q is disjoint with any P_i^1, such a cube which is at a distance less than $l(P_i^1)$ to P_i^1 is necessarily contained inside one of the neighbors of P_i^1. We thus have left to estimate the term

$$\sum_{P_i^1} \sum_{\substack{P \subset \widehat{P_i^1} \setminus P_i^1 \\ l(P)=l(P_i^1)}} \sum_{\substack{Q \text{ spa 2 and pa 1} \\ Q\subset P}} \langle f, \phi_Q^2\rangle \langle b^2\phi_Q^2, T(b^1 1_{P_i^1})\rangle \frac{[g]_Q}{[b^1]_Q}.$$

Fix P_i^1 and P. Set $F = P \setminus \cup P_j^2$, and decomposing

$$b^2 = b^2 1_F + \sum_{P_j^2 \subset P : \rho(P_j^2, P_i^1)\ge l(P_j^2)} b^2 1_{P_j^2} + \sum_{P_j^2 \subset P : \rho(P_j^2, P_i^1) < l(P_j^2)} b^2 1_{P_j^2},$$

write

$$\sum_{\substack{Q \text{ spa 2 and pa 1} \\ Q\subset P}} \langle f, \phi_Q^2\rangle \langle b^2\phi_Q^2, T(b^1 1_{P_i^1})\rangle \frac{[g]_Q}{[b^1]_Q} = \Sigma_1 + \Sigma_2 + \Sigma_3.$$

As in Lemma 3.4.3, denote by L the operator appearing in these sums. For the first term, apply the boundedness of L given by Lemma 3.4.3 to get

$$\begin{aligned} |\Sigma_1| = \Big|\langle Lf, b^2 1_F T(b^1 1_{P_i^1})\rangle\Big| &\lesssim \|f\|_{L^{p'}(P)} \|b^2 1_F T(b^1 1_{P_i^1})\|_{L^p(P)} \\ &\lesssim \mu(P)^{1/p'} \|b^1\|_{L^p(P_i^1)} \lesssim \mu(P_i^1), \end{aligned}$$

where the second inequality comes from the Hardy inequality (3.1.2), and the fact that $b^2 1_F \in L^\infty(P)$, and the last inequality from the fact that by (3.4.5), $b^1 1_{P_i^1} \in L^p(P_i^1)$, and P, P_i^1 have comparable measure. For the second term Σ_2, remark that for any fixed j, as P_j^2 is strictly contained in any spa 2 cube, Lf is constant on P_j^2. Thus we can write

$$\begin{aligned} |\Sigma_2| &= \Big| \sum_{\substack{P_j^2 \subset P \\ \rho(P_j^2, P_i^1)\ge l(P_j^2)}} [Lf]_{P_j^2} \langle b^2 1_{P_j^2}, T(b^1 1_{P_i^1})\rangle \Big| \\ &\lesssim \sum_{\substack{P_j^2 \subset P \\ \rho(P_j^2, P_i^1)\ge l(P_j^2)}} \int_{P_i^1}\int_{P_j^2} \frac{|[Lf]_{P_j^2} b^2(x) b^1(y)|}{\lambda(x,y)} \mathrm{d}\mu(x)\mathrm{d}\mu(y). \end{aligned}$$

For a fixed j such that $\rho(P_j^2, P_i^1) \geq l(P_j^2)$, as $l(P_i^1) \geq l(P_j^2)$, observe that for a fixed $y \in P_i^1$, $\sup_{x\in P_j^2} \lambda(x,y) \approx \inf_{x\in P_j^2} \lambda(x,y)$ uniformly in y and j. Therefore, we can integrate in x to get

$$\int_{P_j^2} |[Lf]_{P_j^2}||b^2(x)|\mathrm{d}\mu(x) \leq |[Lf]_{P_j^2}|\mu(P_j^2) \leq \int_{P_j^2} |Lf(x)|\mathrm{d}\mu(x),$$

and we obtain

$$\begin{aligned}|\Sigma_2| &\lesssim \sum_{P_j^2\subset P} \int_{P_j^2}\int_{P_i^1} \frac{|[Lf(x)||b^1(y)|}{\lambda(x,y)}\mathrm{d}\mu(y)\mathrm{d}\mu(x) \lesssim \int_P\int_{P_i^1} \frac{|[Lf(x)||b^1(y)|}{\lambda(x,y)}\mathrm{d}\mu(y)\mathrm{d}\mu(x)\\ &\lesssim \|Lf\|_{L^{p'}(P)}\|b^1\|_{L^p(P_i^1)} \lesssim \mu(P_i^1),\end{aligned}$$

where the last line is obtained applying the Hardy inequality (2.1.1), the boundedness of L, and (3.4.5) for b^1 on P_i^1. Only the third term Σ_3 remains. This is the term for which we will need to use the stronger weak boundedness property (3.1.6) (and it is the only time in the proof we use it!). We do not see how to estimate this term without this extra property. As for Σ_2, write

$$\Sigma_3 = \sum_{\substack{P_j^2\subset P\\ \rho(P_j^2,P_i^1)<l(P_j^2)}} [Lf]_{P_j^2}\langle b^2 1_{P_j^2}, T(b^1 1_{P_i^1})\rangle = \left\langle b^2\Big(\sum_{\substack{P_j^2\subset P\\ \rho(P_j^2,P_i^1)<l(P_j^2)}} [Lf]_{P_j^2} 1_{P_j^2}\Big), T(b^1 1_{P_i^1})\right\rangle.$$

Apply (3.1.6) to get

$$\begin{aligned}|\Sigma_3| &\lesssim \Big\|\sum_{P_j^2\subset P} [Lf]_{P_j^2} 1_{P_j^2}\Big\|_\nu \mu(P_i^1)^{\frac{1}{\nu'}} \lesssim \Big(\sum_{P_j^2\subset P} \mu(P_j^2)[|Lf|^\nu]_{P_j^2}\Big)^{1/\nu} \mu(P_i^1)^{\frac{1}{\nu'}}\\ &\lesssim \|Lf\|_{L^\nu(P)}\ \ \mu(P_i^1)^{\frac{1}{\nu'}} \lesssim \mu(P_i^1),\end{aligned}$$

where once again we have used the boundedness of L given by Lemma 3.4.3. It remains only to sum over the neighbors P of P_i^1, which come in a uniformly bounded number, and then over the P_i^1 (recall that they realize a 1-packing of Q_0), to obtain as desired

$$|\mathrm{II}| \lesssim \sum_i \mu(P_i^1) \lesssim \mu(Q_0).$$

Estimate of $\langle f, V_{1,3}g\rangle$. This is the only term left to estimate to complete the proof. For Q spa 2 and pa 1, and R pa 1 of the same generation, set

$$\beta_{Q,R} = \begin{cases} \langle f, \phi_Q^2\rangle\langle b^2\phi_Q^2, T(b^1 1_R)\rangle & \text{if } Q \neq R\\ \langle f, \phi_Q^2\rangle\langle b^2\phi_Q^2, T(b^1(1_Q - \sum_{\substack{R' \text{ pa } 1\\ l(R')=l(Q)}} 1_{R'}))\rangle & \text{if } Q = R,\end{cases}$$

so that the definition of $V_{1,3}$ rewrites

$$\langle f, V_{1,3}g\rangle = \sum_{\substack{Q \text{ spa } 2\\ \text{and pa } 1}} \sum_{\substack{R \text{ pa } 1\\ l(R)=l(Q)}} \beta_{Q,R}\ \frac{[g]_R}{[b^1]_R}.$$

Observe that for any fixed spa 2 and pa 1 cube Q,

$$\sum_{\substack{R \text{ pa } 1\\ l(R)=l(Q)}} \beta_{Q,R} = 0.$$

Recall that for any pa 1 cube R, $[g]_R = [\Pi^1 g]_R$ by (3.4.10) of Lemma 3.4.6, allowing us to replace $[g]_R$ by $[\Pi^1 g]_R$ in the above sum. Recall also that

$$\Pi^1 g = E_0^1 g + \sum_{l \geq 0} D_l^1 g.$$

Fix a pa 1 cube R. As $b^1 \Delta_S^1 g$ and $\xi_S^1 g$ have mean 0, it is clear that $[D_l^1 g]_R = 0$ as soon as $l \leq l(R)$, which implies $[\Pi^1 g]_R = [E_0^1 g]_R + \sum_{l > l(R)} [D_l^1 g]_R$. Note that $[E_0^1 g]_R = [g]_{Q_0}[b^1]_R$. Furthermore, if $l > l(R)$, there is a unique cube S_R of generation l such that $R \subset S_R$. If S_R is spa 1, then $[D_l^1 g]_R = [b^1 \Delta_{S_R}^1 g]_R = [b^1]_R \langle g, \phi_{S_R}^1 \rangle [\phi_{S_R}^1]_R$, since $\Delta_{S_R}^1 g$ is constant on the children of S_R and therefore on R as well. Else, S_R is dpa 1 (R being pa 1 it cannot be contained in a P_i^1), and then $[D_l^1 g]_R = [\xi_{S_R}^1 g]_R$. Consequently, we can write

$$\begin{aligned}
\langle f, V_{1,3} g \rangle &= \sum_{\substack{Q \text{ spa } 2 \\ \text{and pa } 1}} \sum_{\substack{R \text{ pa } 1 \\ l(R)=l(Q)}} \beta_{Q,R} \frac{[\Pi^1 g]_R}{[b^1]_R} \\
&= \sum_{\substack{Q \text{ spa } 2 \\ \text{and pa } 1}} \sum_{\substack{R \text{ pa } 1 \\ l(R)=l(Q)}} \beta_{Q,R} \frac{1}{[b^1]_R} \Big\{ \sum_{\substack{S \text{ spa } 1; l(S) > l(Q) \\ S \supset R, S \neq R}} [b^1]_R \langle g, \phi_S^1 \rangle [\phi_S^1]_R \Big\} \\
&+ \sum_{\substack{Q \text{ spa } 2 \\ \text{and pa } 1}} \sum_{\substack{R \text{ pa } 1 \\ l(R)=l(Q)}} \beta_{Q,R} \frac{1}{[b^1]_R} \Big\{ \sum_{\substack{S \text{ dpa } 1; l(S) > l(Q) \\ S \supset R, S \neq R}} [\xi_S^1 g]_R \Big\} + \sum_{\substack{Q \text{ spa } 2 \\ \text{and pa } 1}} [g]_{Q_0} \sum_{\substack{R \text{ pa } 1 \\ l(R)=l(Q)}} \beta_{Q,R}.
\end{aligned}$$

The last sum is equal to zero because of the summing property of the $\beta_{Q,R}$ we stated above. We have thus decomposed our term into two parts: $\langle f, V_{1,3} g \rangle = \langle f, V_{1,3,1} g \rangle + \langle f, V_{1,3,2} g \rangle$.

Estimate of $\langle f, V_{1,3,1} g \rangle$. Inverting the sums over R and S, one gets

$$\langle f, V_{1,3,1} g \rangle = \sum_{\substack{Q \text{ spa } 2 \\ \text{and pa } 1}} \sum_{\substack{S \text{ spa } 1 \\ l(S) > l(Q)}} \sum_{\substack{R \text{ pa } 1; l(R)=l(Q) \\ R \subset S, R \neq S}} \beta_{Q,R} \langle g, \phi_S^1 \rangle [\phi_S^1]_R.$$

Next, we want to use the pseudo-annular decomposition used in [AY]. To achieve this, let us introduce some notation. For $N \in \mathbb{N}$, Q spa 2, S spa 1 with $l(S) > l(Q)$, and R pa 1 with $l(R) = l(Q)$ and $R \subset S, R \neq S$, set

$$\beta_{Q,S,R}^N = \begin{cases} \beta_{Q,R} & \text{if } \rho(Q,R) \sim \delta^{-N+1} l(R) \\ -\sum_{\substack{R \text{ pa } 1; l(R)=l(Q) \\ R \subset S; R \neq S \\ \rho(R,Q) \sim \delta^{-N+1} l(R)}} \beta_{Q,R} & \text{if } Q = R \\ 0 & \text{else.} \end{cases}$$

where as before $\rho(R,Q) \sim \delta^{-N+1} l(R)$ means $\delta^{-N+1} l(R) \leq \rho(R,Q) < \delta^{-N} l(R)$. Obviously, for $N = 0$, the cubes R such that $\rho(Q,R) \sim \delta l(R)$ simply are the cubes R at distance less than $l(R)$ of Q. Remark that everything has been done so that for all $N \in \mathbb{N}$, for fixed Q and S,

$$\sum_{\substack{R \text{ pa } 1; l(R)=l(Q) \\ R \subset S; R \neq S}} \beta_{Q,S,R}^N = 0.$$

Now, set

$$\gamma_{Q,S}^N = \sum_{\substack{R \text{ pa } 1; l(R)=l(Q) \\ R \subset S; R \neq S}} \beta_{Q,S,R}^N \langle g, \phi_S^1 \rangle [\phi_S^1]_R,$$

and

$$\langle f, V_{1,3,1}^N g\rangle = \sum_{\substack{Q \text{ spa } 2 \\ \text{and pa } 1}} \sum_{\substack{S \text{ spa } 1 \\ l(S)>l(Q)}} \gamma_{Q,S}^N,$$

so that $V_{1,3,1} = \sum_{N\in\mathbb{N}} V_{1,3,1}^N$. We will prove that for all $N \geq 0$

$$|\langle f, V_{1,3,1}^N g\rangle| \lesssim (N+1)\delta^{N\alpha}\mu(Q_0),$$

and it will give us the expected bound for $V_{1,3,1}$ by summing over $N \in \mathbb{N}$. To prove this result, we split the sum into two parts depending on the relative sizes of Q and S:

$$\langle f, V_{1,3,1}^N g\rangle = \sum_{\substack{Q \text{ spa } 2 \\ \text{and pa } 1}} \sum_{\substack{S \text{ spa } 1 \\ \delta^{N+3} < \frac{l(Q)}{l(S)} \leq \delta}} \gamma_{Q,S}^N + \sum_{\substack{Q \text{ spa } 2 \\ \text{and pa } 1}} \sum_{\substack{S \text{ spa } 1 \\ \frac{l(Q)}{l(S)} \leq \delta^{N+3}}} \gamma_{Q,S}^N = \mathrm{I} + \mathrm{II}.$$

For the first term, fix cubes Q and S. Remark that for a cube $R \neq Q$, $\beta_{Q,S,R}^N \neq 0 \Rightarrow \rho(Q,R) \geq \delta^{-N+1} l(R)$, so that applying the coefficient estimate (3.6.7), one gets

$$\begin{aligned}|\beta_{Q,S,R}^N| \lesssim |\langle f, \phi_Q^2\rangle|\, \alpha_{Q,R}\, \mu(R)^{\frac{1}{2}} &\lesssim |\langle f, \phi_Q^2\rangle|\, \mu(Q)^{\frac{1}{2}} \frac{\mu(R)}{\mu(Q,R)} \left(\frac{l(Q)}{\rho(Q,R)}\right)^{\alpha} \\ &\lesssim |\langle f, \phi_Q^2\rangle|\, \mu(Q)^{\frac{1}{2}} \delta^{N\alpha} \frac{\mu(R)}{\mu(Q,R)}.\end{aligned}$$

Remark also that we have $|\langle g, \phi_S^1\rangle [\phi_S^1]_R| \lesssim |\langle g, \phi_S^1\rangle| \mu(S)^{-\frac{1}{2}}$ as a consequence of properties (1) and (2) of Lemma 3.4.1. Finally, observe that $\gamma_{Q,S}^N \neq 0 \Rightarrow \rho(Q,S) \leq \delta^{-N} l(Q)$, because else all the $\beta_{Q,S,R}^N$ are equal to zero. Now, combining all this, write

$$\begin{aligned}|\mathrm{I}| &\lesssim \delta^{N\alpha} \sum_{Q \text{ spa } 2} \sum_{\substack{S \text{ spa } 1 \\ \delta^{N+3} < \frac{l(Q)}{l(S)} \leq \delta \\ \rho(Q,S) \leq \delta^{-N} l(Q)}} |\langle f, \phi_Q^2\rangle| |\langle g, \phi_S^1\rangle| \left(\frac{\mu(Q)}{\mu(S)}\right)^{\frac{1}{2}} \sum_{\substack{R \text{ pa } 1 \\ l(R)=l(Q) \\ R\subset S \\ \rho(R,Q)\sim\delta^{-N} l(Q)}} \frac{\mu(R)}{\mu(Q,R)} \\ &\lesssim \delta^{N\alpha}\, S_1^{1/2} \times S_2^{1/2},\end{aligned}$$

where

$$S_1 = \sum_{Q \text{ spa } 2} \sum_{\substack{S \text{ spa } 1 \\ \delta^{N+3} < \frac{l(Q)}{l(S)} \leq \delta \\ \rho(Q,S) \leq \delta^{-N} l(Q)}} |\langle f, \phi_Q^2\rangle|^2 \sum_{\substack{R \text{ pa } 1 \\ l(R)=l(Q) \\ R\subset S \\ \rho(R,Q)\sim\delta^{-N} l(Q)}} \frac{\mu(R)}{\mu(Q,R)}$$

$$S_2 = \sum_{Q \text{ spa } 2} \sum_{\substack{S \text{ spa } 1 \\ \delta^{N+3} < \frac{l(Q)}{l(S)} \leq \delta \\ \rho(Q,S) \leq \delta^{-N} l(Q)}} |\langle g, \phi_S^1\rangle|^2 \frac{\mu(Q)}{\mu(S)} \sum_{\substack{R \text{ pa } 1 \\ l(R)=l(Q) \\ R\subset S \\ \rho(R,Q)\sim\delta^{-N} l(Q)}} \frac{\mu(R)}{\mu(Q,R)}.$$

Let us first consider S_1. Observe that for a fixed cube Q of center z_Q, if $\rho(Q,R) \sim \delta^{-N} l(Q)$, then $\mu(Q,R) \approx \mu(B(z_Q, \delta^{-N} l(Q)))$. Therefore

$$\sum_{\substack{R \text{ pa } 1 \\ l(R)=l(Q) \\ R\subset S \\ \rho(R,Q)\sim\delta^{-N} l(Q)}} \frac{\mu(R)}{\mu(Q,R)} \lesssim \frac{1}{\mu(B(z_Q, \delta^{-N} l(Q)))} \sum_{\substack{R\subset S \\ l(R)=l(Q)}} \mu(R) \lesssim \frac{\mu(S)}{\mu(B(z_Q, \delta^{-N} l(Q)))}.$$

Then, the cubes S of a fixed generation, at a distance less than $\delta^{-N}l(Q)$ of Q, and of a diameter not exceeding $\delta^{-N-3}l(Q)$, pave a ball of a measure comparable to $\mu(B(z_Q, \delta^{-N}l(Q)))$. So, summing over those cubes S, and then summing over those $N+3$ generations, we get

$$S_1 \lesssim (N+1) \sum_{Q \text{ spa } 2} |\langle f, \phi_Q^2 \rangle|^2 \lesssim (N+1)\|f\|_{L^2(Q_0)}^2 \lesssim (N+1)\mu(Q_0).$$

For S_2, it is the same idea. Observe that for a fixed cube S of center z_S, if $\rho(Q,R) \sim \delta^{-N}l(Q)$, then $\mu(Q,R) \approx \mu(B(z_R, \delta^{-N}l(Q)))$, where z_R is the center of the cube R. But since S has a diameter not exceeding $\delta^{-N-3}l(Q)$,, we even have $\mu(Q,R) \approx \mu(B(z_S, \delta^{-N}l(Q)))$. Therefore

$$\frac{\mu(Q)}{\mu(S)} \sum_{\substack{R \text{ pa } 1 \\ l(R)=l(Q) \\ R \subset S \\ \rho(R,Q) \sim \delta^{-N}l(Q)}} \frac{\mu(R)}{\mu(Q,R)} \lesssim \frac{\mu(Q)}{\mu(B(z_S, \delta^{-N}l(Q)))}.$$

Now, summing once more over cubes Q of the same generation, then summing over those $N+3$ generations, and at last summing over the cubes S spa 1, we also get

$$S_2 \lesssim (N+1)\mu(Q_0),$$

which finally gives us the bound

$$|\mathrm{I}| \lesssim (N+1)\delta^{N\alpha}\mu(Q_0),$$

for all $N \in \mathbb{N}$. We now estimate the term

$$\mathrm{II} = \sum_{\substack{Q \text{ spa } 2 \\ \text{and pa } 1}} \sum_{\substack{S \text{ spa } 1 \\ \frac{l(Q)}{l(S)} \leq \delta^{N+3}}} \gamma_{Q,S}^N.$$

Fix such cubes Q and S, and note that $l(Q) < l(S)$. For any cube R strictly contained inside S, we know that ϕ_S^1 is constant over R. In the following, let us denote by S_Q the unique cube of length $\delta l(S)$ containing Q. First, if $Q \cap S = \varnothing$, and $\rho(Q,S) > \delta^{-N-1}l(Q)$, then it is immediate to see that all the $\beta_{Q,S,R}^N$ are equal to zero, and $\gamma_{Q,S}^N$ as well. Next, if $Q \subset S$ and $\rho(Q, {S_Q}^c) > \delta^{-N-1}l(Q)$, then if $\beta_{Q,S,R}^N \neq 0$, it implies $\rho(Q,R) \leq \delta^{-N}l(Q)$, and thus R is in the same child of S as Q. Therefore, for all the $\beta_{Q,S,R}^N$ non equal to zero, $[\phi_S^1]_R$ has the same value, and $\gamma_{Q,S}^N = [\phi_S^1]_Q \sum_R \beta_{Q,S,R}^N = 0$. Finally, the only cubes Q for which $\gamma_{Q,S}^N \neq 0$ are the cubes Q disjoint from S, at distance less than $\delta^{-N-1}l(Q) \leq \delta^{-N-1}\delta^{N+3}l(S) < l(S)$ from S (which implies $\rho(Q, {S_Q}^c) \leq \delta^{-N-1}l(Q)$), and the cubes $Q \subset S$ such that $\rho(Q, {S_Q}^c) \leq \delta^{-N-1}l(Q)$. Now, for such cubes Q, applying the coefficient estimate (3.6.7), we have

$$|\beta_{Q,S,R}^N| \lesssim |\langle f, \phi_Q^2 \rangle| \, \mu(Q)^{\frac{1}{2}} \delta^{N\alpha} \frac{\mu(R)}{\mu(Q,R)},$$

and

$$\sum_{\substack{R \subset S; l(R)=l(Q) \\ \rho(R,Q) \sim \delta^{-N}l(Q)}} |\beta_{Q,S,R}^N| |[\Delta_S^1 g]_R| \lesssim |\langle f, \phi_Q^2 \rangle| \, |\langle g, \phi_S^1 \rangle| \left(\frac{\mu(Q)}{\mu(S)} \right)^{\frac{1}{2}} \delta^{N\alpha} \sum_{\substack{l(R)=l(Q) \\ \rho(R,Q) \sim \delta^{-N}l(Q)}} \frac{\mu(R)}{\mu(B(z_Q, \delta^{-N}l(Q)))}$$

$$\lesssim \delta^{N\alpha} |\langle f, \phi_Q^2 \rangle| \, |\langle g, \phi_S^1 \rangle| \left(\frac{\mu(Q)}{\mu(S)} \right)^{\frac{1}{2}}.$$

Therefore, if we introduce the set $E_{t,S}^N = \bigcup_{P\in\theta_{t,S}^N} P$, where

$$\theta_{t,S}^N = \{P \subset \widehat{S}; l(P) = tl(S) \quad \text{and} \quad \rho(P, S_P{}^c)) \leq (\delta^{-N-1}t)l(S)\},$$

we have

$$|\gamma_{Q,S}^N| \lesssim \delta^{N\alpha}|\langle f, \phi_Q^2\rangle| \ \ |\langle g, \phi_S^1\rangle| \ \ \left(\frac{\mu(Q)}{\mu(S)}\right)^{\frac{1}{2}} 1_{Q\subset E^N_{\frac{l(Q)}{l(S)},S}}.$$

Remark that if $x \in P$, with $P \in \theta_{t,S}^N$, then $\rho(x, S_P{}^c)) \lesssim \rho(P, S_P{}^c) + l(P) \lesssim (\delta^{-N-1}t)l(S) + tl(S) \lesssim (\delta^{-N-1}t)l(S)$ as $N \geq 0$. Thus, by the small boundary condition (1.2.1), we have

$$\mu(E_{t,S}^N) \lesssim (\delta^{-N-1}t)^\eta \mu(S),$$

as there is a uniformly bounded number of cubes of length $\delta l(Q)$ contained inside $\widehat{S}$, each of measure comparable to the measure of S. Let $\beta > 0$ be such that $\eta - 2\beta > 0$, and let $\lambda_{\frac{l(Q)}{l(S)}} = (\delta^{-N-1+j})^\beta$ whenever $\frac{l(Q)}{l(S)} = \delta^j$. Now, by the Cauchy-Schwarz inequality, write

$$|\mathrm{II}| \lesssim \delta^{N\alpha}\Big\{\sum_{\substack{Q \,\text{spa}\, 2\\ \text{and pa}\, 1}} \sum_{\substack{S\,\text{spa}\,1\\ \frac{l(Q)}{l(S)}\leq\delta^{N+3}}} \big|\langle f, \phi_Q^2\rangle\big|^2 \lambda^2_{\frac{l(Q)}{l(S)}} 1_{Q\subset E^N_{\frac{l(Q)}{l(S)},S}}\Big\}^{\frac{1}{2}} \Big\{\sum_{\substack{Q \,\text{spa}\, 2\\ \text{and pa}\, 1}} \sum_{\substack{S\,\text{spa}\,1\\ \frac{l(Q)}{l(S)}\leq\delta^{N+3}}} \big|\langle g, \phi_S^1\rangle\big|^2 \lambda^{-2}_{\frac{l(Q)}{l(S)}} \frac{\mu(Q)}{\mu(S)} 1_{Q\subset E^N_{\frac{l(Q)}{l(S)},S}}\Big\}^{\frac{1}{2}}$$

Observe that for a fixed cube Q and a fixed value of $l(S)$, the number of cubes S such that $Q \subset E^N_{\frac{l(Q)}{l(S)},S}$ is uniformly bounded. Indeed, as we have seen, either S is the parent of Q, or S is a neighbor of the latter. Therefore

$$\begin{aligned}
\sum_{\substack{Q \,\text{spa}\, 2\\ \text{and pa}\, 1}} \sum_{\substack{S\,\text{spa}\,1\\ \frac{l(Q)}{l(S)}\leq\delta^{N+3}}} \big|\langle f, \phi_Q^2\rangle\big|^2 \lambda^2_{\frac{l(Q)}{l(S)}} 1_{Q\subset E^N_{\frac{l(Q)}{l(S)},S}} &\lesssim \sum_{Q\,\text{spa}\,2} \big|\langle f, \phi_Q^2\rangle\big|^2 \sum_{j\geq N+3} (\delta^{-N-1+j})^{2\beta} \sum_{\substack{S\,\text{spa}\,1\\ \frac{l(Q)}{l(S)}=\delta^j}} 1_{Q\subset E^N_{\frac{l(Q)}{l(S)},S}}\\
&\lesssim \sum_{Q\,\text{spa}\,2} \big|\langle f, \phi_Q^2\rangle\big|^2 \lesssim \mu(Q_0).
\end{aligned}$$

We can similarly bound the second sum: we have

$$\begin{aligned}
\sum_{\substack{Q \,\text{spa}\, 2\\ \text{and pa}\, 1}} \sum_{\substack{S\,\text{spa}\,1\\ \frac{l(Q)}{l(S)}\leq\delta^{N+3}}} \big|\langle g, \phi_S^1\rangle\big|^2 \lambda^{-2}_{\frac{l(Q)}{l(S)}} \frac{\mu(Q)}{\mu(S)} 1_{Q\subset E^N_{\frac{l(Q)}{l(S)},S}} &\lesssim \sum_{S\,\text{spa}\,1} \big|\langle g, \phi_S^1\rangle\big|^2 \sum_{j\geq N+3} (\delta^{-N-1+j})^{-2\beta} \sum_{\substack{Q\,\text{spa}\,2\\ l(Q)=\delta^j l(S)\\ Q\subset E^N_{\delta^j,S}}} \frac{\mu(Q)}{\mu(S)}\\
&\lesssim \sum_{S\,\text{spa}\,1} \big|\langle g, \phi_S^1\rangle\big|^2 \sum_{j\geq N+3} (\delta^{-N-1+j})^{-2\beta} \ \frac{\mu(E^N_{\delta^j,S})}{\mu(S)}\\
&\lesssim \sum_{S\,\text{spa}\,1} \big|\langle g, \phi_S^1\rangle\big|^2 \sum_{j\geq N+3} (\delta^{-N-1+j})^{\eta-2\beta}\\
&\lesssim \sum_{S\,\text{spa}\,1} \big|\langle g, \phi_S^1\rangle\big|^2 \lesssim \mu(Q_0).
\end{aligned}$$

We thus proved that $|\mathrm{II}| \lesssim \delta^{N\alpha}\mu(Q_0)$, and therefore for all $N \in \mathbb{N}$, we have as expected $|\langle f, V_{1,3,1}^N g\rangle| \lesssim (N+1)\delta^{N\alpha}\mu(Q_0)$. Summing over $N \in \mathbb{N}$, this handles the term $\langle f, V_{1,3,1}g\rangle$.

Estimate of $\langle f, V_{1,3,2}g\rangle$. Remember that by definition of the "buffer" functions $\xi^1_S g$, if $R \subset S'$, $S' \in \widetilde{S}$, we have $[\xi^1_S g]_R = a_{S'}[b^1]_R$, where $a_{S'}$ denotes the coefficient uniformly bounded by $\|g\|_\infty \leq 1$ introduced in Lemma 3.4.5. Therefore

$$\begin{aligned}\langle f, V_{1,3,2}g\rangle &= \sum_{\substack{Q \text{ spa } 2 \\ \text{and pa } 1}} \sum_{\substack{S \text{ dpa } 1 \\ l(S)>l(Q)}} \sum_{S' \in \widetilde{S}} \sum_{\substack{R \text{ pa } 1; l(R)=l(Q) \\ R \subset S'}} \beta_{Q,R} \ a_{S'} \\ &= \sum_{\substack{Q \text{ spa } 2 \\ \text{and pa } 1}} \sum_{\substack{S \text{ dpa } 1 \\ l(S)>l(Q)}} \sum_{S' \in \widetilde{S}} a_{S'} \ \mu(S)^{\frac{1}{2}} \sum_{\substack{R \text{ pa } 1; l(R)=l(Q) \\ R \subset S'}} \beta_{Q,R} \ \mu(S)^{-\frac{1}{2}}.\end{aligned}$$

Remark that by the C_X-packing property (3.4.6) of the dpa 1 cubes,

$$\sum_{S \text{ dpa } 1} \sum_{S' \in \widetilde{S}} \left| a_{S'} \ \mu(S)^{\frac{1}{2}} \right|^2 \lesssim \sum_{S \text{ dpa } 1} \mu(S) \lesssim \mu(Q_0).$$

That is all we need to apply exactly the same argument we applied to $\langle f, V_{1,3,1}g\rangle$, *i.e.* applying the pseudo-annular decomposition of $V_{1,3,2}$ into operators $V^N_{1,3,2}$, estimating those operators, and then summing over $N \in \mathbb{N}$. Indeed, the crucial point was that we could bound the coefficients $|\langle g, \phi^1_S\rangle[\phi^1_S]_R|$ by $|\langle g, \phi^1_S\rangle| \, \mu(S)^{-\frac{1}{2}}$, and then that the $|\langle g, \phi^1_S\rangle|$ were square summable over the spa 1 cubes S. We have the same square summability property here, and everything works out in the same way, so that though we will omit the detail, we obtain the expected estimate $|\langle f, V_{1,3,2}g\rangle| \lesssim \mu(Q_0)$. This completes the proof of Theorem 3.1.2.

3.7 Two particular cases

3.7.1 Perfect dyadic operators

The notion of perfect dyadic operators was introduced in [AHMTT]. We recall the definition.

Definition 3.7.1. Perfect dyadic singular integral operators (pdsio).
We say that T is a perfect dyadic singular integral operator on X if it satisfies the following properties:

1. T is a linear continuous operator from $\mathcal{D}_\alpha$ to $\mathcal{D}'_\alpha$.
2. T has a kernel K satisfying the size condition (1.2.2).
3. $\langle g, Tf\rangle$ is well defined for pairs of functions $(f, g) \in L^p_c(X) \times L^{p'}_c(X)$ for $1 < p < +\infty$ and if, furthermore, they are integrable with support on disjoint dyadic cubes (up to a set of measure 0)
$$\langle g, Tf\rangle = \iint_{X \times X} g(x)K(x,y)f(y)\mathrm{d}\mu(x)\mathrm{d}\mu(y).$$
4. For all (f, g) as above, if f has support in a dyadic cube Q and mean 0, then $\langle g, Tf\rangle = 0$ whenever the support of g does not meet Q (up to a set of measure 0).

A pdsio has in a sense its singularity adapted to the dyadic grid. As was proven in [AHMTT] in the setting of the real line (but there is no difficulty adapting it to a space of homogeneous type), pdsio satisfy Theorem 3.1.2 without having to suppose conditions (3.1.6), (3.1.7), and with (3.1.9) instead of (3.1.5). Looking at our proof, it is easy to recover this result. Indeed, most of the terms we had to estimate simply vanish if one considers pdsio, and only the diagonal terms remain, which makes the proof a lot more straightforward. Here is how one estimates those diagonal terms. Consider for example the term

$$\langle b^2\phi_Q^{b2}, T(b^1\phi_Q^{b1})\rangle = \sum_{Q'\in\widetilde{Q}, R'\in\widetilde{Q}} [\phi_Q^{b2}]_{Q'}\langle b^2 1_{Q'}, T(b^1 1_{R'})\rangle[\phi_Q^{b1}]_{R'},$$

when Q is a spa 1 and spa 2 cube. It remains to compute $\langle b^2 1_{Q'}, T(b^1 1_{R'})\rangle$. If $Q' \neq R'$, one cannot use the Hardy type inequality (3.1.1) directly because the exponents p, q can happen to be uncompatible if $1/p + 1/q > 1$, but one can circumvent this as follows: observe that $\int (b^1 1_{R'} - [b^1]_{R'} 1_{R'})\mathrm{d}\mu = \int (b^2 1_{Q'} - [b^2]_{Q'} 1_{Q'})\mathrm{d}\mu = 0$, so that

$$\langle b^2 1_{Q'}, T(b^1 1_{R'})\rangle = \langle b^2 1_{Q'}, T(1_{R'})\rangle[b^1]_{R'} = [b^2]_{Q'}\langle 1_{Q'}, T(1_{R'})\rangle[b^1]_{R'}. \tag{3.7.1}$$

Now, we can apply (3.1.1) because $1_{Q'}, 1_{R'} \in L^\infty(X)$. If $Q' = R'$, using the fact that T is perfect dyadic, write

$$\begin{aligned}\langle b^2 1_{Q'}, T(b^1 1_{Q'})\rangle = &\ \langle b^2 1_{Q'} - [b^2]_{Q'} b^2_{Q'}, T(b^1 1_{Q'})\rangle + [b^2]_{Q'}\langle b^2_{Q'}, T(b^1 1_{Q'} - [b^1]_{Q'} b^1_{Q'})\rangle \\ &+ [b^2]_{Q'}[b^1]_{Q'}\langle b^2_{Q'}, T(b^1_{Q'})\rangle \\ = &\ \langle b^2 1_{Q'} - [b^2]_{Q'} b^2_{Q'}, T(b^1)\rangle + [b^2]_{Q'}\langle T^*(b^2_{Q'}), b^1 1_{Q'} - [b^1]_{Q'} b^1_{Q'}\rangle \\ &+ [b^2]_{Q'}[b^1]_{Q'}\langle b^2_{Q'}, T(b^1_{Q'})\rangle,\end{aligned}$$

and all those terms are easily estimated applying (3.1.4), (3.1.5) (on same Q') and the fact that Q' is pa 1 and pa 2. We thus always have

$$|\langle b^2 1_{Q'}, T(b^1 1_{R'})\rangle| \lesssim \mu(Q')^{1/2}\mu(R')^{1/2} \lesssim \mu(Q'),$$

because these cubes have comparable measure as they are children of the same cube. Consequently, applying the L^∞ estimates (3.4.2) of the functions ϕ_Q^{b2}, ϕ_Q^{b1}, we still have the estimate

$$\left|\langle b^2\phi_Q^{b2}, T(b^1\phi_Q^{b1})\rangle\right| \lesssim 1,$$

and all this argument remains valid when $1/p + 1/q > 1$. All the other terms can be treated in the same way: to put it roughly, when considering pdsio and estimating terms where one had to use (3.1.8), one can always reduce to considering L^∞ functions by substracting the mean value of the functions involved, and then apply the Hardy inequality (3.1.1) to those L^∞ functions. Actually, here a weaker L^∞ version of (3.1.1) would suffice (see [Ch1]). Similarly, one needs not suppose an integrability of $T(b^i_Q)$ over $\widehat{Q}$, an integrability condition over Q suffices. Putting all this together, we recover Theorem 6.8 of [AHMTT], with a different proof.

Remark 3.7.2. The identity (3.7.1) shows that, whenever T is a pdsio, if $Q \neq R$ are neighbors and $\operatorname{supp} f \subset Q, \operatorname{supp} g \subset R$, then for all $1 \leq p, q \leq +\infty$, we have

$$|\langle Tf, g\rangle| \leq |\langle T1_Q, 1_R\rangle|\,|[f]_Q|\,|[g]_R| \leq C\mu(Q)^{1-\frac{1}{p}-\frac{1}{q}}\|f\|_p\|g\|_q.$$

Such an inequality is wrong when $1/p + 1/q > 1$ for standard sio. This indicates that pdsio do not approximate well standard sio in this range of exponents.

3.7.2 The case $1/p + 1/q \leq 1$

We have stated in Section 3.1 that when $1/p + 1/q \leq 1$, Theorem 3.1.2 is valid under the hypotheses (3.1.3), (3.1.4) and (3.1.9). This is Theorem 3.4 of [AY]. Let us prove this statement. We already pointed out that in that particular case, hypotheses (3.1.5) and (3.1.6) come as a consequence of (3.1.3), (3.1.4) and (3.1.9) (see Proposition 3.1.4). However we cannot directly prove (3.1.7), which we obviously need in order to estimate all the diagonal terms. The proof thus has to be adapted slightly. More precisely, we need to include a stronger control in stopping time Lemma 3.4.5: we need to control also the maximal function (non dyadic) of b^1, as we want the means $[|Mb^1|^p]_{\widehat{Q}}$ to be uniformly bounded on the pa 1 cubes Q. To achieve this, the stopping time has to be on whether

$$(i) \quad [|b^1|]_Q < \delta$$

or

$$(ii) \quad [|T(b^1)|^{q'}]_Q + \sup_E [|Mb^1|^p]_E > C,$$

for dyadic subcubes Q of Q_0 for appropriately chosen $\delta > 0$ and $C < +\infty$, and where the supremum runs over the unions E of dyadic cubes such that $Q \subset E \subset \widehat{Q}$. This can easily be done and we refer to [Ho1] for the detail. Once having this additional control, the diagonal terms of (3.1.7) can be estimated as follows. If R is a pa 1 and pa 2 dyadic cube, write

$$\langle b^2 1_R, T(b^1 1_R)\rangle = [b^2]_R \langle T^*(b^2_R), b^1 1_R\rangle + \langle h, T(b^1)\rangle - \langle h, T(b^1 1_{\widehat{R}\setminus R})\rangle - \langle h, T(b^1 1_{\widehat{R}^c})\rangle,$$

where $h = b^2 1_R - [b^2]_R b^2_R$. Remark that by the controls given by the stopping time, we have $\|h\|_q^q \lesssim \mu(R)$. For the first term, use (3.1.5) for $T^*(b^2_R)$ on R, (3.4.5) for b^1 and b^2 on R, and $p' \leq q$:

$$|[b^2]_R \langle T^*(b^2_R), b^1 1_R\rangle| \lesssim \|T^*(b^2_R)\|_{L^{p'}(R)} \|b^1\|_{L^p(R)} \lesssim \mu(R).$$

For the same reasons, the second term is also bounded:

$$|\langle h, T(b^1)\rangle| \lesssim \|h\|_q \|T(b^1)\|_{L^{q'}(R)} \lesssim \mu(R).$$

Apply (3.1.1) to the third term, $q' \leq p$, and the control on Mb^1 given by the stopping-time to get

$$\begin{aligned}|\langle h, T(b^1 1_{\widehat{R}\setminus R})\rangle| &\lesssim \|h\|_q \|T(b^1 1_{\widehat{R}\setminus R})\|_{L^{q'}(R)} \lesssim \|h\|_q \|b^1\|_{L^{q'}(\widehat{R})} \\ &\lesssim \|h\|_q \|Mb^1\|_{L^{q'}(\widehat{R})} \lesssim \mu(R).\end{aligned}$$

Finally, for the last term, since h has mean 0, we can apply (1.2.6) to get

$$\begin{aligned}|\langle h, T(b^1 1_{\widehat{R}^c})\rangle| &\lesssim \int_{x\in R} |h(x)| \int_{y\in \widehat{R}^c} |b^1(y)| \left(\frac{l(R)}{\rho(x,y)}\right)^\alpha \frac{1}{\lambda(x,y)} \mathrm{d}\mu(y)\mathrm{d}\mu(x) \\ &\lesssim \int_{x\in R} |h(x)| \sum_{j\geq 0} 2^{-j\alpha} \int_{2^j l(R) \leq \rho(x,y) < 2^{j+1} l(R)} |b^1(y)| \frac{1}{\lambda(x,y)} \mathrm{d}\mu(y)\mathrm{d}\mu(x) \\ &\lesssim \int_{x\in R} |h(x)| \sum_{j\geq 0} 2^{-j\alpha} \frac{1}{\mu(B(x, 2^{j+1} l(R)))} \int_{B(x, 2^{j+1} l(R))} |b^1(y)| \mathrm{d}\mu(y)\mathrm{d}\mu(x) \\ &\lesssim \int_{x\in R} |h(x)|\, |Mb^1(x)| \mathrm{d}\mu(x) \lesssim \|h\|_{L^q(R)} \|Mb^1\|_{L^{q'}(R)} \lesssim \mu(R),\end{aligned}$$

where once again we have used the control on Mb^1 given by the stopping time and $q' \leq p$. Summing up, we obtain as expected

$$|\langle b^2 1_R, T(b^1 1_R)\rangle| \lesssim \mu(R),$$

for the pa 1 and pa 2 cubes R. This allows us to estimate all the diagonal terms appearing in our argument, which we previously controlled with hypothesis (3.1.7). It means that when $1/p + 1/q \leq 1$, we do not need to assume (3.1.7), nor (3.1.6), and this proves Theorem 3.1.2 under (3.1.3), (3.1.4) and (3.1.9).

Appendix : Wavelet representation of the adapted martingale difference operators

We give here the proof of Proposition 3.4.1.

Proof of Proposition 3.4.1. It follows essentially from results on pseudo-accretive sesquilinear forms developed in [AT1]. It is here simpler as everything is finite dimensional. Still, the uniform control of constants must be achieved. Let V_Q be the space of complex-valued functions which are constant on each dyadic child of Q (recall we assume there are at least two children), seen as a subspace of $L^2(X)$ and equipped with the complex L^2 inner product. On V_Q consider the bilinear form

$$B(f,g) = \int_Q f\, b\, g \,\mathrm{d}\mu = \sum_{Q' \in \widetilde{Q}} [f]_{Q'} [b]_{Q'} [g]_{Q'} \mu(Q').$$

Clearly, since for $f \in V_Q$, $\|f\|_2^2 = \sum [|f|^2]_{Q'}\mu(Q')$ and $[|f|^2]_{Q'} = [f]_{Q'}\overline{[f]_{Q'}}$, we have

$$|B(f,g)| \le \sup_{Q' \in \widetilde{Q}} |[b]_{Q'}| \;\; \|f\|_2 \;\; \|g\|_2,$$

and

$$\inf_{\|g\|_2 = 1} |B(f,g)| \ge \inf_{Q' \in \widetilde{Q}} |[b]_{Q'}| \;\; \|f\|_2$$

by taking $g \in V_Q$ with

$$[g]_{Q'} = \frac{\overline{[f]_{Q'}}}{\|f\|_2} \frac{\overline{[b]_{Q'}}}{|[b]_{Q'}|}.$$

Consider $W_Q \subset V_Q$ equal to the orthogonal complement of $\mathbb{C}1_Q$ and P_{W_Q} the orthogonal projection of V_Q onto W_Q. Let also $X_Q = \{g \in V_Q; B(1_Q, g) = 0\}$. Then $\mathbb{C}1_Q \oplus X_Q = V_Q$ is a topological sum. Indeed, let $c_Q \in \mathbb{C}$ such that $c_Q^2 \frac{B(1_Q,1_Q)}{\mu(Q)} = 1$ which is possible because $|B(1_Q, 1_Q)| = |[b]_Q \mu(Q)| \neq 0$. Then the splitting is given by $f = \frac{c_Q^2}{\mu(Q)} B(f, 1_Q) 1_Q + g$, and

$$\left\| \frac{c_Q^2}{\mu(Q)} B(f,1_Q) 1_Q \right\|_2 \le |c_Q^2| \sup_{Q' \in \widetilde{Q}} |[b]_{Q'}| \|f\|_2 \|g\|_2 \le (1 + |c_Q^2| \sup_{Q' \in \widetilde{Q}} |[b]_{Q'}|) \|f\|_2.$$

Let P_{X_Q} be the projector on X_Q associated to this splitting. Then it can be checked that $P_{X_Q} : W_Q \to X_Q$ is an isomorphism with inverse $P_{W_Q} : X_Q \to W_Q$. Set $\phi_Q^0 = \frac{1_Q}{\mu(Q)^{1/2}}$ and complete it to an orthonormal basis of W_Q, $\phi_Q^1, ..., \phi_Q^{N_Q - 1}$ for $\|.\|_2$. Define $\widetilde{\phi}_Q^{b,s}$ by $\widetilde{\phi}_Q^{b,0} = c_Q \phi_Q^0$, $\widetilde{\phi}_Q^{b,s} = P_{X_Q}(\phi_Q^s)$ for $s \ge 1$. For $f \in X_Q \backslash \{0\}$, let $g \in V_Q$, $\|g\|_2 = 1$, such that $B(f,g) \ge \inf_{Q' \in \widetilde{Q}} |[b]_{Q'}| \|f\|_2$ as before. Let $h = \frac{P_{X_Q} g}{\|P_{X_Q} g\|_2}$, as $P_{X_Q} g \neq 0$ because otherwise $g \in \mathbb{C}1_Q$ and $B(f,g) = 0$. Thus $h \in X_Q$, $\|h\|_2 = 1$, and

$$B(f,h) = \frac{B(f,g)}{\|P_{X_Q} g\|_2} \ge \frac{\inf_{Q' \in \widetilde{Q}} |[b]_{Q'}|}{\|P_{X_Q} g\|_2} \|f\|_2.$$

By the Riesz representation theorem, there exists $A \in \mathcal{L}(X_Q)$, $\|A\| \le \sup_{Q' \in \widetilde{Q}} |[b]_{Q'}|$, such that for all $f, g \in X_Q$

$$B(f,g) = \langle Af, \overline{g} \rangle = \int_Q Af\, g \,\mathrm{d}\mu$$

and the above argument says that $A^{-1} \in \mathcal{L}(X_Q)$, and

$$\|A^{-1}\| \leq \frac{\|P_{X_Q} g\|_2}{\inf_{Q' \in \widetilde{Q}} |[b]_{Q'}|}.$$

Set then $\phi_Q^{b,s} = {}^t A^{-1} \widetilde{\phi}_Q^{b,s}$ for $1 \leq s \leq N_Q - 1$ where ${}^t A$ is the real transpose of A. It is quite clear that $\phi_Q^{b,s}$ and $\widetilde{\phi}_Q^{b,s}$ satisfy $(1), (2), (3), (4)$. It remains to prove (5) and (6). Remark that for $f \in X_Q$, we have obtained

$$f = \sum_{s=1}^{N_Q - 1} B(f, \phi_Q^{b,s}) \widetilde{\phi}_Q^{b,s}$$

and $\|f\|_2 \eqsim \sum |B(f, \phi_Q^{b,s})|^2$ (use $f = P_{X_Q}(\sum B(f, \phi_Q^{b,s}) \phi_Q^s)$ and the bounds for the operators P_{X_Q} and P_{W_Q}). Now, let $f \in L^1(X)$ and observe that $\Delta_Q^b f \in V_Q$ and $\int_Q b \Delta_Q^b f \mathrm{d}\mu = 0$. Hence, $\Delta_Q^b f \in X_Q$. It remains to show that $B(\Delta_Q^b f, \phi_Q^{b,s}) = \int_Q f \phi_Q^{b,s} \mathrm{d}\mu$ for $s \geq 1$. Indeed, $B(E_Q^b f, \phi_Q^{b,s}) = 0$, so that

$$B(\Delta_Q^b f, \phi_Q^{b,s}) = \sum_{Q' \in \widetilde{Q}} B\left(\frac{[f]_{Q'}}{[b]_{Q'}} 1_{Q'}, \phi_Q^{b,s}\right) = \sum_{Q' \in \widetilde{Q}} [f]_{Q'} [\phi_Q^{b,s}]_{Q'} \mu(Q') = \int_Q f \phi_Q^{b,s} \mathrm{d}\mu,$$

where we have used the fact that $\phi_Q^{b,s}$ is constant on each child Q' of Q. $\square$

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