



Dynatomic curve and core entropy for iteration of polynomials

Yan Gao

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Thèse de Doctorat

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Dynatomic curve and core entropy for iteration of polynomials

Courbes dynatomiques et entropie noyau de polynômes itérés

JURY

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Chapter 1

Introduction

In this thesis, I study three problems of complex dynamics:

- Dynatomic periodic and preperiodic curve for polynomials $f_c(z) = z^d + c$. Such a curve is a combination of parameter space and dynamical plane which is compatible with dynamics. So it give us a chance to study the structure of the curve by means of the method of dynamics.
- Core entropy of polynomials. It is a new field opened by William Thurston in recent years to study the parameter space of polynomials. Many interesting and basic problems that seems true are to be resolved. We are following his idea and try to move as far as possible.
- Wandering continuum of post critical finite rational map. It is also an example of the common phenomenon in complex dynamics that the conclusion is well known for polynomial but almost vacuity for rational map. Few existing tools and method can be used to deal with this problem because it is very difficult to find a partition on Julia set of rational map.

In this chapter, I will simply introduce the background, development and important conclusion about these three problem respectively. At the same time, I will give the main results of this thesis.

1.1 Dynatomic periodic and preperiodic curve for polynomi-

als $f_c(z) = z^d + c$

Fix $d \geq 2$. For $c \in \mathbb{C}$, set $f_c(z) = z^d + c$. For $p \geq 1$, define

$$\check{\mathcal{X}}_{0,p} := \{(c, z) \in \mathbb{C}^2 \mid f_c^p(z) = z \text{ and for all } 0 < k < p, f_c^k(z) \neq z\}.$$

$$\mathcal{X}_{0,p} := \text{the closure of } \check{\mathcal{X}}_{0,p} \text{ in } \mathbb{C}^2.$$

It is known that each $\mathcal{X}_{0,p}$ is an affine algebraic curve. It is called the periodic dynatomic curve. It has been the subject of several studies in algebraic and holomorphic dynamical systems.

In case $d = 2$, Douady-Hubbard proved the smoothness of $\mathcal{X}_{0,p}$ by the technique of parabolic implosion. The irreducibility is proved by Morton [Mo] using a combinatorics of algebraic arguments, by Bousch [B] using combiantion of algebraic and dynamical method and by Lau and Schleicher [LS] [Sch] using dynamical arguments only. Bousch [B] also calculated the Galois group and the genus of some kind of compactification of $\mathcal{X}_{0,p}$. Recently, Buff and Tan Lei reproves the smoothness and irreducibility of $\mathcal{X}_{0,p}$ with a different method ([BT]).

For the general case $d \geq 2$, we generalized the method in [BT] and prove the following theorem:

Theorem 1.1.1. *For any $p \geq 1$, the periodic dynatomic curve $\mathcal{X}_{0,p}$ satisfies:*

1. $\mathcal{X}_{0,p}$ is an affine algebraic curve;
2. $\mathcal{X}_{0,p}$ is smooth and irreducible;
3. The Galois group $G_{0,p}$ for the defining polynomial of $\mathcal{X}_{0,p}$ consists of the permutations on roots of the defining polynomial that commute with f_c .

Next, we consider the preperiodic case.

Definition. For $n \geq 0$, $p \geq 1$, a point z is called a (n, p) -preperiodic point of f_c if $f_c^{on+p}(z) = f_c^n(z)$ and $f_c^{l+k}(z) \neq f_c^l(z)$ for any $0 \leq l \leq n$, $0 \leq k \leq p$ with $(l, k) \neq (n, p)$.

Now, for any $n \geq 1, p \geq 1$, define

$$\check{\mathcal{X}}_{n,p} = \{(c, z) \in \mathbb{C}^2 \mid z \text{ is a } (n, p)\text{-preperiodic point of } f_c\}$$

$$\mathcal{X}_{n,p} := \text{the closure of } \check{\mathcal{X}}_{n,p} \text{ in } \mathbb{C}^2 .$$

In fact, as we shall see below, each $\mathcal{X}_{n,p}$ is also an affine algebraic curve. These curves are called preperiodic dynatomic curves. There are much less studies about them. The known results include the connectivity of $\check{\mathcal{X}}_{n,p}$ and the computation of the Galois group of its defining polynomial ([B], [Sch]) in case $d = 2$.

In Chapter 4, we will give a more detailed description of $\mathcal{X}_{n,p}$ (for any degree d) from both algebraic and topology point of view. We summarize our main result below. This result is to be compared with results on periodic dynatomic curves.

For $\nu_d(p)$ the unique sequence of positive integers satisfying the recursive relation

$$d^p = \sum_{k|p} \nu_d(k)$$

and for $\varphi(m)$ the Euler totient function (i.e. the number of positive integers less than m and co-prime with m), set

$$g_p(d) = 1 + \frac{(d-1)(p-1)}{2d} \nu_d(p) - \frac{d-1}{2d} \sum_{k|p, k < p} \varphi\left(\frac{p}{k}\right) k \cdot \nu_d(k) .$$

$$g_{n,p}(d) = 1 + \frac{1}{2} \nu_d(p) d^{n-2} [(d-1)(n+p) - 2d] - \frac{1}{2} d^{n-2} (d-1) \sum_{k|p, k < p} \varphi\left(\frac{p}{k}\right) k \cdot \nu_d(k)$$

Theorem 1.1.2. *For any $d \geq 2$, $n, p \geq 1$, the preperiodic dynatomic curve $\mathcal{X}_{n,p}$ has the following properties :*

1. *The set $\mathcal{X}_{n,p}$ is an affine algebraic curve. It has $d-1$ irreducible components and each one is smooth. Moreover, every pair of these components intersect transversally at the singular points of $\mathcal{X}_{n,p}$. The set $\check{\mathcal{X}}_{n,p}$ has $d-1$ connected components.*
2. *In particular, if $d = 2$, the curve $\mathcal{X}_{n,p}$ is smooth and irreducible, and the set $\check{\mathcal{X}}_{n,p}$ is connected.*
3. *The genus of the compactification of every irreducible component of $\mathcal{X}_{n,p}$ is $g_{n,p}(d)$. Furthermore, all irreducible components are mutually homeomorphic.*
4. *The Galois group of the defining polynomial of $\mathcal{X}_{n,p}$ consists of all permutations on its roots which commute with f_c and d -th rotation.*

Here is a tableau comparing these various curves :

periodic $\mathcal{X}_{0,p}$	$d = 2$	$d > 2$
	irreducible	irreducible
	smooth	smooth
# ideal points	$\nu_2(p)/2$	$\nu_d(p)/d$
genus	$g_p(2)$	$g_p(d)$
Galois group	$\text{sym}(\nu_2(p)/p) \ltimes \mathbb{Z}_p^{\nu_2(p)/p}$	$\text{sym}(\nu_d(p)/p) \ltimes \mathbb{Z}_p^{\nu_d(p)/p}$

preperiodic $\mathcal{X}_{n,p}, n \geq 1$	$d = 2$	$d > 2$
	irreducible	$d - 1$ irreducible components
	smooth	each component is smooth
component-wise genus	$g_{n,p}(2)$	$g_{n,p}(d)$
Galois group	$G_{n,p}(2)$	$G_{n,p}(d)$
component-wise Galois group	$G_{n,p}(2)$	$G_{n,p}^j$
pairwise intersection	empty	$C_{n,p,4}$

1.2 Core entropy of polynomais

A central theme of research in dynamical systems is the variation of dynamics along parametrized families. There are many dynamical properties that are interesting to study within a parameter family. We are mainly concerned with families of complex polynomials or rational maps. In this case, each map induces a fractal set, the Julia set, on which the dynamics behave chaotically. One may for example study the variation of the topology and the geometry of the Julia set. On a more statistic level, the variation of the topological entropy provides also important dynamical/parameter information.

For a continuous map f acting on a compact set X , we can define the topological entropy $h(X, f)$ (see Section 2.5 for its definition). Topological entropy is a quantity that measures the complexity of the induced dynamical system.

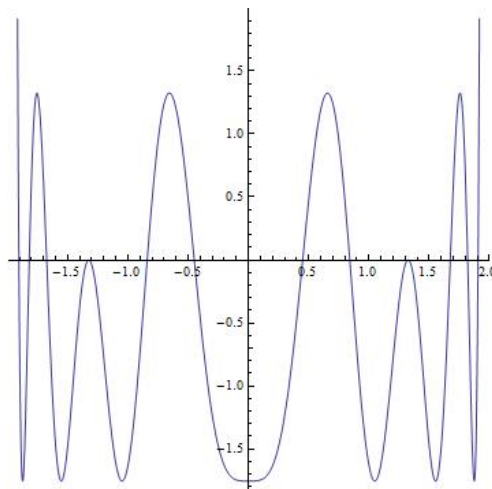
The pictures in this Section are most due to Tiozzo and TanLei.

Core entropy of real polynomials

Let $f : I \rightarrow I$ continuous (I is denoted by the unit interval), by Misiurewicz-Szlenk

$$h(I, f) := \lim_{n \rightarrow \infty} \frac{\log\{\#\text{laps of } f^n\}}{n}$$

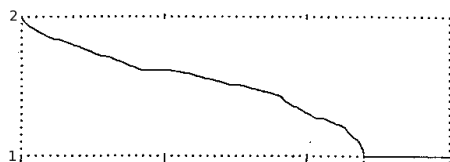
where $\#\text{laps of } f^n$ is the number of monotone intervals of f^n , or equivalently, the number of critical points of f^n .



Now consider the quadratic family $f_c(z) = z^2 + c$. For $c \in [-2, 1/4]$, f_c has a invariant interval $I_c = [-\beta_c, \beta_c]$ where β_c is the landing point of external ray $R_c(0)$. The **core entropy** of f_c is defined by $h(I_c, f_c)$, the topological entropy of f_c on I_c . A natural question is:

how does it change with the parameter c ?

Theorem. (Milonr-Thurston (77), Douady (93)) As $c \in [-2, 1/4]$, $h(I_c, f_c)$ is continuous and monotone decreasing from $\log 2$ to 0.

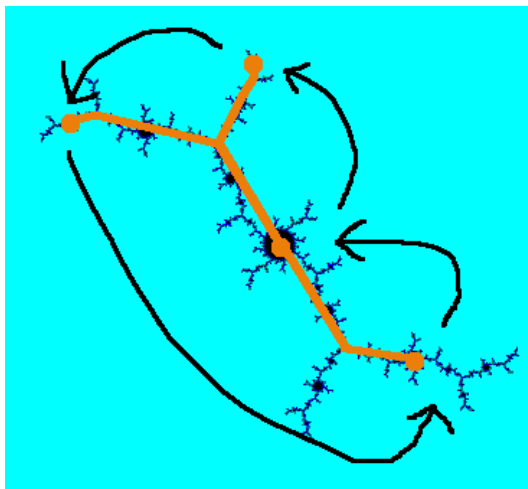


Core entropy of complex polynomials

Thurston's idea of core entropy: (The entropy for a polynomial map acting on its entire Julia set is always $\log d$, where d is the degree. what's tricky is figuring out the best

definition that filters out the way the polynomial acts on the tips of the leaves of the Julia set, where the invariant measure for entropy is concentrated, filtering this behavior out and leaving the action on the "interior" of the Julia set in some sense.)

The "interior" of the Julia set are the points that at least two external rays land on, these points will be finally attracted by the Hubbard tree (i.e. the convex hull of the postcritical orbits within the (filled-in) Julia set) under the iteration of f . At the same time, the Hubbard tree is a natural object to replace the real trace segment in the real case, so it is reasonable to define the core entropy of a polynomial as the topological entropy of f on its Hubbard tree (if it exists). In many cases, including the case that f is postcritical finite, the Hubbard tree H_f is a finite tree and is forward invariant by f .



So for a polynomial f , if its Hubbard tree exists and is a finite tree, we define by $h(H_f, f)$ the **core entropy** of f . Note that this definition is compatible to the real case..

In parameter space of quadratic polynomials, the segment $[-2, 0]$ can be homeomorphic embedded to any limb of Mandelbrot set by Branner-Hubbard surgery. Its image is called the **main vein** of this limb. In any main vein of Mandelbrot set, Hubbard trees H_c are finite trees and have the same shape. For example, any polynomial f_c which is on the main vein of $1/3$ -limb has the Hubbard tree with a shape of "Y".

Tiozzo generalizes the Theorem of Milnor-Thurston to any main vein:

Theorem.(Tiozzo) As c moves along a main vein from 0 to its tip, the core entropy $h(H_c, f_c)$ is continuous and increasing.

Bill Thurston has made important progress in the study of core entropy during the last two years of his life. He used a quite different approach to the computation of the core

entropy, by following an outer contour of the Mandelbrot set, as well as its higher degree substitutes, as opposite to following the veins from inside. Such a contour can be seen combinatorially as the space of critical portraits (or external angles of \mathcal{M}), or analytically as the space of polynomials all of whose critical points escape to infinity with the same rate. Thurston established an effective algorithm computing the core entropy for the rational critical portraits without proof (see the degree 2 plot below).

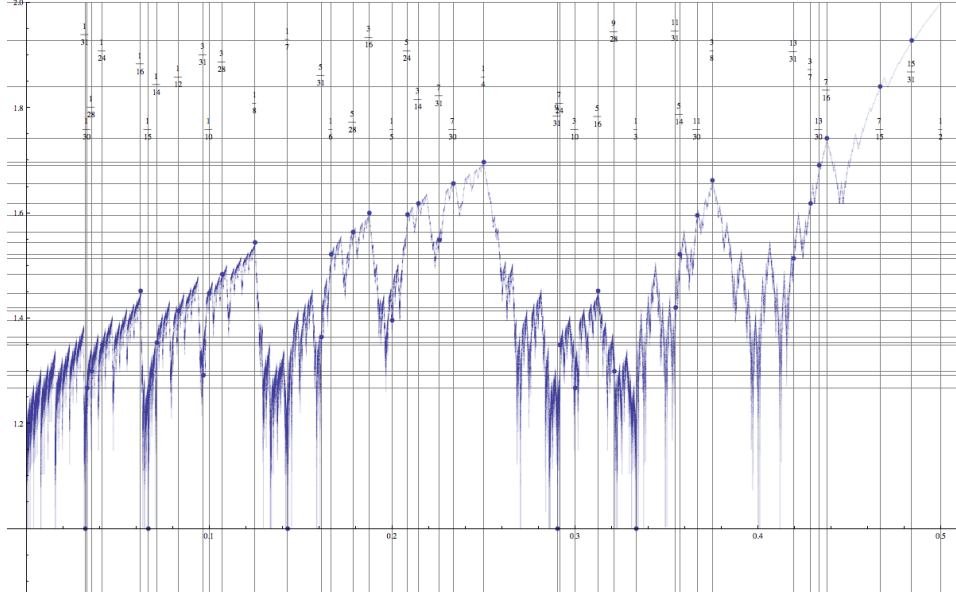


Figure 1-1: core entropy of quadratic polynomials

We will describe Thurston's entropy algorithm and prove this algorithm in Section 5.1

Conjecture (Thurston): The core entropy is a continuous function on the boundary of the Mandelbrot set.

To prove the continuity of core entropy, Thurston suggests the torus model of polynomials (see Section 2.2.3 for the construction). Roughly speaking, the degree d polynomials can be characterized combinatorially by the degree d critical portraits (or called primitive majors by Thurston). Stating from any degree d primitive major m , we can obtain a degree d invariant lamination $L(m)$ by pulling back m under the map τ_d . The set $\overline{b_\infty(m)}$ is exactly the preimage of $L(m)$ under the map \wp from \mathbb{T}^2 to the set of all hyperbolic geodesics in $\overline{\mathbb{D}}$ by mapping $(x, y) \in \mathbb{T}^2$ to the geodesic $\overline{xy} \in \overline{\mathbb{D}}$.

Definition. Let $m = \{\Theta_1, \dots, \Theta_s\}$ be a primitive major. We say that m is a rational primitive major, if all angles in $\cup_{j=1}^s \Theta_j$ are rational numbers.

Denote by $F : \mathbb{T}^2 \longrightarrow \mathbb{T}^2$ mapping (x, y) to (dx, dy) . If m is a rational primitive major,

we can find a compact F -invariant set $TH_1(m) \subset \overline{b_\infty}(m)$, called combinatorial Hubbard tree. We define by $h(TH_1(m), F)$ the **core entropy** of primitive major m .

Theorem 1.2.1. *Let m be any rational primitive major.*

1. $h(TH_1(m), F) = \log d \cdot \text{H.dim } (TH_1(m))$.
2. *We can find a transition matrix $M(m)$ such that*

$$h(TH_1(m), F) = \begin{cases} 0 & \text{if } M(m) \text{ is nilpotent} \\ \log \rho(M(m)) & \text{otherwise} \end{cases}$$

3. *If m can be realized by a postcritical finite polynomial f , then*

$$h(TH_1(m), F) = h(H_f, f).$$

In fact, there is a strong dynamical background of torus model $\overline{b_\infty}(m)$ and combinatorial Hubbard tree $TH_1(m)$. If m can be realized by a postcritical finite polynomial f , each point in $TH_1(m)$ corresponds to a ray pair landing on the same point of H_f . One can see Section 5.2.4 for more details.

For any critical portrait, there exists a stretching ray $R(m)$ in shift locus \mathcal{S}_d of parameter space of degree d polynomials realizing it. In other words, any polynomial in $R(m)$ has m as its critical portrait and all critical point have the same escaping speed. Denote the stretching ray $R(m)$ by $\{f_{m,t}\}_{t>0}$ where t is the escaping speed of the critical points of $f_{m,t}$. For any $t > 0$, we can define a continuous map

$$\pi_{m,t} : b'_\infty(m) \longrightarrow J(f_{m,t})$$

mapping (x, y) to the common landing point of $R_{f_{m,t}}(x)$ and $R_{f_{m,t}}(y)$.

If m is a rational critical portrait, denote by $H_{m,t} \subset J(f_{m,t})$ the image of $TH_1(m)$ under $\pi_{m,t}$. Then $H_{m,t}$ is a compact $f_{m,t}$ -invariant set. We define by $h(H_{m,t}, f_{m,t})$ the **core entropy** of $f_{m,t}$. It is not difficult to prove that for any $t > 0$,

$$h(H_{m,t}, f_{m,t}) = h(TH_1(m), F).$$

So along a stretching ray of rational critical portrait, the core entropy of all polynomials

are equal. By term 3 of Theorem 1.2.1, if $\{f_{m,t}\}_{t>0}$ lands on a polynomial $f_{m,0}$ with critical portrait m , then $h(H_{f_{m,0}}, f_{m,0}) = h(H_{m,t}, f_{m,t})$ for any $t > 0$.

Applying the discussion above to the quadratic case, for any $c \in \mathbb{C} \setminus \mathcal{M}_2$, c can be labeled by (θ, R) where $\theta \in \mathbb{S}^1$ is the angle of c and $R > 1$ is the potential of c . Then we denote $c \in \mathbb{C} \setminus \mathcal{M}_2$ by $c_{\theta,R}$. If $R_{\mathcal{M}_2}(\theta)$ lands, denote by $c_{\theta,1}$ the landing parameter of the ray. Let θ be a rational angle, by the discussion above, we have that the core entropy of $f_{c_{\theta,R}}$ with $R \geq 1$ is unrelated with R . So the rational parameter rays that land on the same parameter together with the landing parameter form a contour of core entropy.

Motivated by term 1 of Theorem 1.2.1, we have an alternate definition of the core entropy of primitive majors by Hausdorff dimension. For any primitive major m , the set $b'_\infty(m)$ contains the diagonal Δ of \mathbb{T}^2 . We define the **core entropy** of m by

$$\log d \cdot \text{H.dim } (b'_\infty(m)' \setminus \Delta).$$

Note that the preimage of Δ in $b'_\infty(m) \setminus \Delta$ under F is a countable set. So for a rational primitive major, almost every point of $b'_\infty(m) \setminus \Delta$ is mapped to $TH_1(m)$ by the iteration of F , then we have

$$\text{H.dim } (b'_\infty(m) \setminus \Delta) = \text{H.dim } (TH_1(m))$$

According to term 1 of Theorem 1.2.1, in the case of rational primitive majors, the definition of core entropy of primitive major by Hausdorff dimension is the same as that by the topological entropy of F on $TH_1(m)$.

1.3 Wandering continuum of postcritical finite rational map

Let f be a rational map. By a **wandering continuum** we means a non-degenerate continuum $K \in J_f$ (i.e. K is a connected compact set consisting more than one point) such that $f^n(K) \cap f^m(K) = \emptyset$ for any $n > m \geq 0$.

For a polynomial, the existence or not of wandering continuum has been studied by many authors. It is proved that for a polynomial without irrational indifferent periodic cycles, there is no wandering continuum if and only if the Julia set is locally connected ([K]).

For a non-polynomial rational map, as far as I know, few results about the wandering

continuum is known. Recently, Cui, Peng and Tan make a big progress about this problem for postcritical finite rational map. They point out that for a postcritical finite rational map f , there are two kinds of possible wandering continuum: **full** and **separate** (see definition below). They prove that f admits a separate wandering continuum if and only if f has a **Contor multicurve** ([CPT]). The remaining question is that

$$\text{In what condition does } f \text{ admit a full wandering continuum?} \quad (1)$$

Definition. (wandering continuum) Let f be a postcritical finite rational map, E be a non-degenerate continuum in $\hat{\mathbb{C}}$. E is called **full** if $\hat{\mathbb{C}} \setminus E$ is simply connected and called **separate** if there is an annulus $A \in \hat{\mathbb{C}} \setminus P_f$ (P_f means the postcritical set of f , see §2) such that $E \subset A$ and each components of $\hat{\mathbb{C}} \setminus A$ contains at least two points of P_f . We call K a **full** wandering continuum of f if $f^n(K)$ is full for all $n > 0$ and call K a **separate** wandering continuum if there exists an integer $N > 0$, such that $f^n(K)$ is separate for all $n > N$.

In the study of postcritical finite rational map dynamics, the objective **stable multicurve** (Definition 2.3.4) plays a very important role. It was first used in complex dynamics by Thurston to study the topological characterization of postcritical finite rational map (see [DH2]). Then it is widely used to study the combinatorics of rational dynamics, see for example [CPT], [CT1], [CT2], [Pil], [T]. A basic problem about stable multicurve is that for a postcritical finite rational map f ,

$$\text{In what condition does } f \text{ admits a stable multicurve ?} \quad (2)$$

In this thesis, we answer questions (1) and (2) in the simplest case: f is a postcritical finite non-polynomial rational map with parabolic orbifold (see Definition 2.3.1). Throughout this thesis, we call such a map a rational map with parabolic orbifold for convenience.

Let f be a rational map with parabolic orbifold \mathcal{O}_f . The map f can be lifted to be a holomorphic map between torus. Then we can study the dynamics of rational map by means of the holomorphic dynamics on torus. Such a dynamics is very simple because the holomorphic map between torus has the form $z \mapsto \alpha z + \beta \pmod{\Lambda}$ for some complex number α , β and lattice Λ . In this case, the possible signature of \mathcal{O}_f are $(2, 2, 2, 2)$, $(3, 3, 3)$, $(2, 4, 4)$, or $(2, 3, 6)$ (see 2.3.1 for details).

Definition.(Lattes map) If \mathcal{O}_f has the signature $(2, 2, 2, 2)$, the rational map f is called **Lattes map**. For a Lattes map f , it is called **flexible** if L_f , the lift of f on torus, has the form $z \mapsto mz + \beta \pmod{\Lambda}$ for some $m \in \mathbb{Z}$ and called **non-flexible** otherwise.

For a rational map with parabolic orbifold, $\#P_f = 4$ if f is a Lattes map and $\#P_f = 3$ otherwise. So if f has a stable multicurve, the multicurve contains only one curve.

Theorem 1.3.1. *Let f be a rational map with parabolic orbifold, then*

1. *f admits a full wandering C^1 arc K if and only if f is a flexible Lattes map and K is a segment with irrational slop.*
2. *f has a stable multicurve if and only if f is a flexible Lattes map. Furthermore, if f is a flexible Lattes map, any non-peripheral simple closed curve is stable.*

The thesis is organized as follows:

In Chapter 2, we give some basic acknowledge and notations used in this thesis; In Chapter 3, we study the periodic dynatomic curve for f_c and prove Theorem 1.1.1; In Chapter 4, we describe the preperiodic curve for f_c and prove Theorem 1.1.2; In Chapter 5, we study the core entropy of polynomials from the view of primitive majors and prove Theorem 1.2.1; In Chapter 6, we deal with the wandering continuum problem and prove Theorem 1.3.1.

Chapter 2

Basic knowledge and results

In this chapter, we will give some knowledge and results about complex dynamics and other fields that will be used below. The pictures in Section 2.2.3 are due to W. Thurston and TanLei.

2.1 General theory of complex dynamics

Let $f : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ be a rational map. That means f can be expressed as

$$f(z) = \frac{P(z)}{Q(z)},$$

where P, Q are two polynomials without common factor. Denote by $\deg P$ the degree of polynomial P , define

$$\deg f = \max\{\deg P, \deg Q\}.$$

It is called the degree of f and is equal to the number of roots (counting with multiplicity) of equation $f(z) = a \in \widehat{\mathbb{C}}$

Throughout this thesis, we only consider the rational map with $\deg f \geq 2$, because when $\deg f = 1$, f is a Mobius transformation which has a simple dynamics.

Now, we give some notation used in this thesis:

Definition 2.1.1. Let $f : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ be a rational map and $z_0 \in \widehat{\mathbb{C}}$. We call sequence $\{z_0, z_1 = f(z_0), \dots, z_n = f^n(z_0), \dots\}$ the (post) orbit of z_0 under f , denoted by $O^+(z_0)$; and call the set $\{z_0, f^{-1}(z_0), \dots, f^{-n}(z_0), \dots\}$ the back forward orbit of z_0 under f , denoted

by $O^-(z_0)$.

Two kinds of important orbit are periodic orbit and preperiodic orbit, they are defined as follows:

Definition 2.1.2. Let $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ be a rational map. For $p \geq 1$, a point $z \in \widehat{\mathbb{C}}$ is called a p periodic point of f if $f^p(z) = z$ and $f^k(z) \neq z$ for any $k < p$; For $n \geq 0$, $p \geq 1$, a point z is called a (n, p) -preperiodic point of f if $f_c^{n+p}(z) = f_c^n(z)$ and $f_c^{l+k}(z) \neq f_c^l(z)$ for any $0 \leq l \leq n$, $0 \leq k \leq p$ with $(l, k) \neq (n, p)$.

It is obviously that the orbit of a periodic point or preperiodic point is finite and all points in a periodic orbit have the same period.

Definition 2.1.3. Let z_0 be a p periodic point of f . We call $\lambda = (f^p)'(z_0)$ the multiplier of z_0 . If the orbit of z_0 is denoted by $O(z_0) = \{z_0, z_1, \dots, z_{p-1}\}$, then $\lambda = \prod_{j=0}^{p-1} f'(z_j)$.

In the following, we give the classification of periodic points:

Definition 2.1.4. Let z_0 be a p periodic point of f with multiplier $\lambda = (f^p)'(z_0)$, then

1. If $0 < |\lambda| < 1$, the point z_0 is called attractive periodic point;
2. If $\lambda = 0$, the point z_0 is called super attractive periodic point;
3. If $|\lambda| > 1$, the point z_0 is called repelling periodic point;
4. If $|\lambda| = 1$, the point z_0 is called indifferent periodic point. In this case, $\lambda = e^{2\pi i \alpha}$, $\alpha \in \mathbb{R}$. Furthermore, if α is a irrational number, then z_0 is called irrational indifferent periodic point, otherwise z_0 is called a parabolic point.

In complex dynamics, Fatou set and Julia set are two most important set. They are disjoint and form the entire Riemann sphere together. The dynamics of a rational map in the two set are quite different. In order to give the definition of these two sets, we will introduce the theory of Montel normal family.

Definition 2.1.5. Let $U \subset \widehat{\mathbb{C}}$ be a domain and \mathcal{F} is consisting of all holomorphic map from U to $\widehat{\mathbb{C}}$. If any sequence in \mathcal{F} has a subsequence with local uniform convergence, then \mathcal{F} is called a normal family. Equivalently, \mathcal{F} is compact under the compact-open topology.

The definition of normal family is local. Let \mathcal{F} be the family of holomorphic map from U to $\widehat{\mathbb{C}}$. We call \mathcal{F} is normal at a point $z \in U$ if there is a neighborhood V_z of z such that \mathcal{F} is a normal family on V_z . Obviously, if \mathcal{F} is normal at any point of U , then \mathcal{F} is a normal family on U .

Now we can define the Fatou set and Julia set.

Definition 2.1.6. Let $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ be a rational map. If $\{f^n\}$ is normal at $z_0 \in \widehat{\mathbb{C}}$, then z_0 is called a normal point of f . The set of all normal point of f is called the Fatou set of f , denoted by $F(f)$. The complementary set of $F(f)$ is called the Julia set of f , denoted by $J(f)$. By the definition, $F(f)$ is open and $J(f)$ is compact. A connected component of $F(f)$ is called a Fatou component.

Remark 2.1.7. According to the definition, it is not difficult to see that for any $n \geq 1$,

$$F(f) = F(f^n), \quad J(f) = J(f^n).$$

and $F(f)$, $J(f)$ are both completely invariant, that is

$$f^{-1}(F(f)) = F(f) = f(F(f));$$

$$f^{-1}(J(f)) = J(f) = f(J(f)).$$

Another important concept in complex dynamics is critical point.

Definition 2.1.8. Let $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ be a rational map and $c \in \widehat{\mathbb{C}}$. If $f'(c) = 0$, then c is called a critical point of f . The image of a critical point $v = f(c)$ is called a critical value. The orbit of a critical point is called a post critical orbit. We denote by $C = \{c \in \widehat{\mathbb{C}} \mid f'(c) = 0\}$ the set of critical point of f , and by $V = f(C)$ the set of critical value of f . We call $P_f = \overline{\bigcup_{n \geq 0} f^n(C)}$ the post critical set of f .

For more detail of Fatou set and Julia set, one can refers to [Mil3].

In the following, we state two important theorem that completely characterize the dynamics on Fatou set.

Definition 2.1.9. Let $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ be a rational map. A Fatou component D is called p periodic if p is the smallest integer satisfying $f^p(D) = D$. If there exists $k \geq 0$ such that

$f^k(D)$ is periodic, then D is called preperiodic. If D is preperiodic but not periodic, then D is called strictly preperiodic.

A non-preperiodic Fatou component is called wandering. Equivalently, a wandering Fatou component can be defined as for any $i, j \geq 0$, $i \neq j$, $f^i(D) \cap f^j(D) = \emptyset$.

Theorem 2.1.10. ([Mil2]) *Let f be a rational map. Then each Fatou component is preperiodic.*

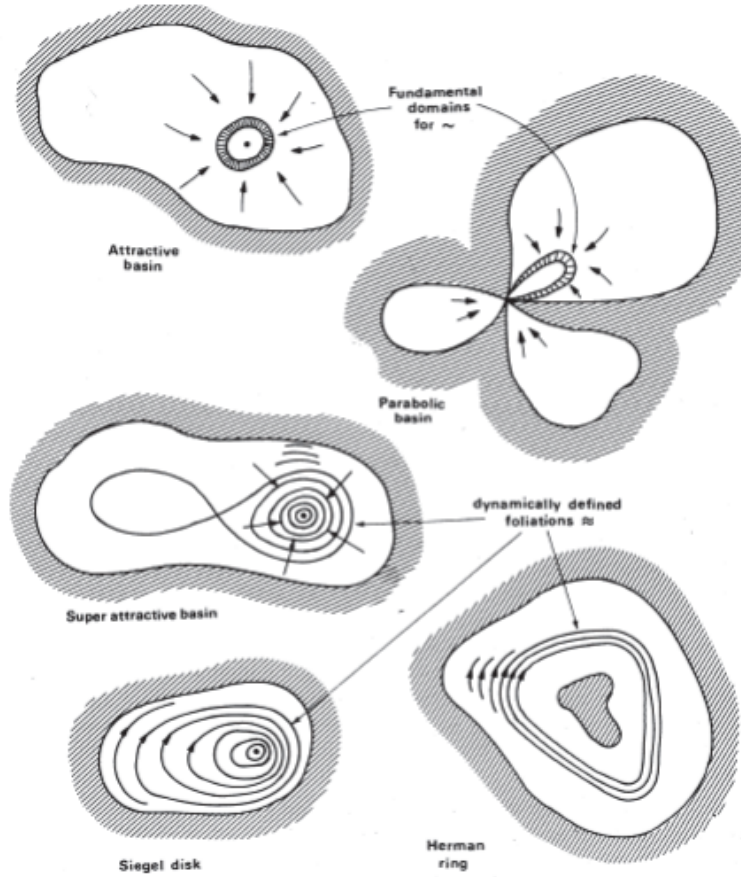


Figure 2-1: Five kinds of different Fatou component

Theorem 2.1.11. ([Mil2]) *Let f be a rational map. If $f(D) = D$, then D is one of the following 4 exclusive cases:(see Figure 2-1)*

1. ((super) attractive basin): *There exists a attractive fixed point $z_0 \in D$ with multiplier $0 \leq |\lambda| < 1$. $\{f^n\}$ locally uniform converge to z_0 in D .*

2. (*parabolic basin*): There exists a parabolic fixed point $z_0 \in \partial D$ with multiplier $\lambda = 1$. $\{f^n\}$ locally uniform converge to z_0 in D .
3. (*Siegel disk*): D is conformal to the unit disk \mathbb{D} and f conformally conjugate to an irrational rotation in $\overline{\mathbb{D}}$;
4. (*Herman ring*): D is conformal to an annulus $A = \{z \in \mathbb{C} \mid 0 < r < |z| < 1\}$ and f conformally conjugate to an irrational rotation in A .

2.2 Dynamics of polynomial

2.2.1 Filled in Julia set and Multibrot set

Let us recall some results about filled-in Julia set and Multibrot set that will be used following. These can be found in [DH1], [Mil2] and [DE].

For $c \in \mathbb{C}$, we denote by K_c the filled-in Julia set of f_c , that is the set of points $z \in \mathbb{C}$ whose orbit under f_c is bounded. We denote by M_d the Multibrot set for $f_c(z) = z^d + c$, that is the set of parameters $c \in \mathbb{C}$ for which the critical point 0 belongs to K_c .

If $c \in M_d$, then K_c is connected. There is a conformal isomorphism $\phi_c : \mathbb{C} \setminus \overline{K_c} \rightarrow \mathbb{C} \setminus \overline{\mathbb{D}}$ which satisfies $\phi_c \circ f_c = (\phi_c)^d$ and $\phi'_c(\infty) = 1$. The dynamical ray of angle $\theta \in \mathbb{T}$ is

$$R_c(\theta) := \{z \in \mathbb{C} \setminus K_c \mid \arg(\phi_c(z)) = 2\pi\theta\}.$$

If θ is rational, then as r tends to 1 from above, $\phi_c^{-1}(re^{2\pi i\theta})$ converges to a point $\gamma_c(\theta) \in K_c$. We say that $R_c(\theta)$ lands at $\gamma_c(\theta)$. We have $f_c \circ \gamma_c = \gamma_c \circ \tau$ on \mathbb{Q}/\mathbb{Z} . In particular, if θ is periodic under τ , then $\gamma_c(\theta)$ is periodic under f_c . In addition, $\gamma_c(\theta)$ is either repelling (its multiplier has modulus > 1) or parabolic (its multiplier is a root of unity).

If $c \notin M_d$, then K_c is a Cantor set. There is a conformal isomorphism $\phi_c : U_c \rightarrow V_c$ between neighborhoods of ∞ in \mathbb{C} , which satisfies $\phi_c \circ f_c = (\phi_c)^d$ on U_c . We may choose U_c so that U_c contains the critical value c and V_c is the complement of a closed disk. For each $\theta \in \mathbb{T}$, there is an infimum $r_c(\theta) \geq 1$ such that ϕ_c^{-1} extends analytically along $R_0(\theta) \cap \{z \in \mathbb{C} \mid r_c(\theta) < |z|\}$. We denote by ψ_c this extension and by $R_c(\theta)$ the dynamical ray

$$R_c(\theta) := \psi_c\left(R_0(\theta) \cap \{z \in \mathbb{C} \mid r_c(\theta) < |z|\}\right).$$

As r tends to $r_c(\theta)$ from above, $\psi_c(re^{2\pi i\theta})$ converges to a point $x \in \mathbb{C}$. If $r_c(\theta) > 1$, then $x \in \mathbb{C} \setminus K_c$ is an iterated preimage of 0 and we say that $R_c(\theta)$ bifurcates at x . If $r_c(\theta) = 1$, then $\gamma_c(\theta) := x$ belongs to K_c and we say that $R_c(\theta)$ lands at $\gamma_c(\theta)$. Again, $f_c \circ \gamma_c = \gamma_c \circ \tau$ on the set of θ such that $R_c(\theta)$ does not bifurcate. In particular, if θ is periodic under τ and $R_c(\theta)$ does not bifurcate, then $\gamma_c(\theta)$ is periodic under f_c .

The Multibrot set is connected. The map

$$\phi_{M_d} : \mathbb{C} \setminus M_d \ni c \mapsto \phi_c(c) \in \mathbb{C} \setminus \overline{\mathbb{D}}$$

is a conformal isomorphism. For $\theta \in \mathbb{T}$, the parameter ray $R_{M_d}(\theta)$ is

$$R_{M_d}(\theta) := \{c \in \mathbb{C} \setminus M_d \mid \arg(\phi_{M_d}(c)) = 2\pi\theta\}.$$

It is known that if θ is rational, then as r tends to 1 from above, $\phi_{M_d}^{-1}(re^{2\pi i\theta})$ converges to

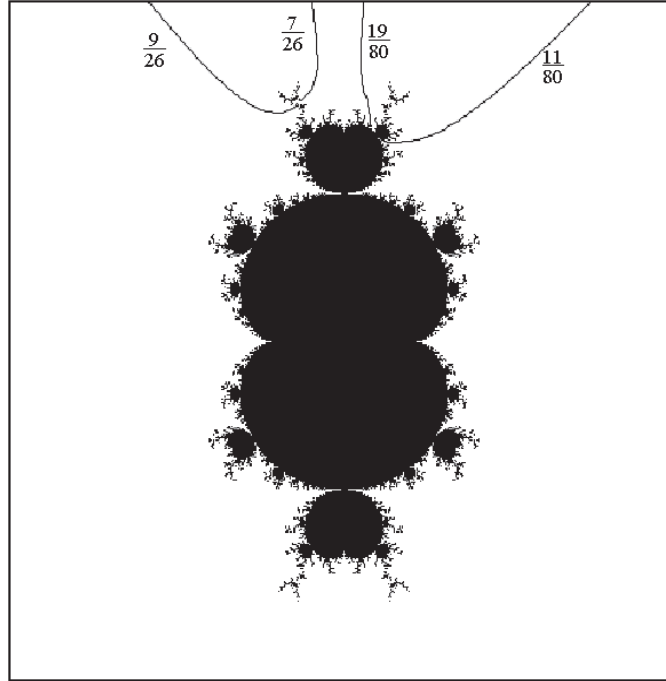


Figure 2-2: The parameter rays $R_{M_3}(7/26)$ and $R_{M_3}(9/26)$ land on a common root of a primitive hyperbolic component while $R_{M_3}(19/80)$ and $R_{M_3}(11/80)$ land on a common root of a satellite hyperbolic component. Only angles of rays are labelled in the graph.

a point of M_d . We say that $R_{M_d}(\theta)$ lands at this point. A dynamical ray or parameter ray

is called (n, p) -preperiodic if its angle is (n, p) -periodic under $\tau : \theta \rightarrow d\theta \pmod{\mathbb{Z}}$. There are three kinds of important parameters:

- c is called a *parabolic parameter* if f_c has a parabolic periodic point. Furthermore, if the multiplier is 1, the parameter c is called a *primitive* parabolic parameter, otherwise c is called a *satellite* parabolic parameter.
- c is called a *hyperbolic parameter* if f_c has an *attracting* periodic point.
- c is called a *Misiurewicz parameter* if c is a (n, p) -preperiodic point of f_c for some $n, p \geq 1$.

Parabolic parameters and Misiurewicz parameters lie on the boundary of M_d .

The set of hyperbolic parameters forms an open and closed subset of the interior of M_d . Each connected component is called a *hyperbolic component*. Within a hyperbolic component, the period of the attracting periodic point for any parameter is the same and this number is called the period of the hyperbolic component.

If θ is periodic for τ of exact period n and if $c_0 := \gamma_{M_d}(\theta)$, then the point $\gamma_{c_0}(\theta)$ is periodic for f_{c_0} with period p dividing n ($ps = n$, $s \geq 1$) and multiplier a s -th root of unity. If the period of $\gamma_{c_0}(\theta)$ for f_{c_0} is exactly n then the multiplier is 1, c_0 is called primitive parabolic parameter, otherwise c_0 is called satellite parabolic parameter.

Lemma 2.2.1 (near parabolic map). *c_0 is defined as above. When we make a small perturbation to c_0 in parameter space, If c_0 is a primitive parabolic parameter, then the parabolic orbit of f_{c_0} is splitted into a pair of nearby periodic orbits of f_c , both have length n ; If c_0 is a satellite parabolic parameter, then the parabolic orbit of f_{c_0} is splitted into a pair of nearby periodic orbits of f_c , one has length p and the other has length $sp = n$.*

This lemma was proved by Milnor in [Mil1] lemma 4.2 for the case $d = 2$, but we can translate the proof word by word to the general case.

Let H be a p -periodic ($p \geq 1$) hyperbolic component. For every parameter $c \in H$, the polynomial f_c has an attracting periodic orbit $\{ z(c), \dots, f_c^{p-1}(z(c)) \}$. Its multiplier defines a map

$$\lambda_H : H \rightarrow D, \quad c \mapsto \frac{\partial}{\partial z} f_c^{\circ n}(z) \Big|_{z=z(c)}.$$

Then $\lambda_H : H \rightarrow D$ is a branched covering of degree $d - 1$ with only one branched point which is the preimage of 0. This branched point is called the *center* of H . The map λ_H

can be extended continuously to the closure \overline{H} . Considering parameter $c \in \partial H$ such that $\lambda_H(c) = 1$, Eberlein proved the following results:

- For $p \geq 2$, among these points, there is exactly one c which is the landing point of two parameter rays of period p , this point is called the *root* of H . Any one of the other $d-2$ points is the landing point of only one parameter ray of period p . They are called the *co-roots* of H . The component H is said of *primitive* or *satellite* type according to whether its root is a primitive or satellite parabolic parameter. All co-roots of H are primitive parabolic parameters.
- For $p = 1$, any one of these $d-1$ points is the landing point of only one fixed parameter ray and hence a primitive parabolic parameter.

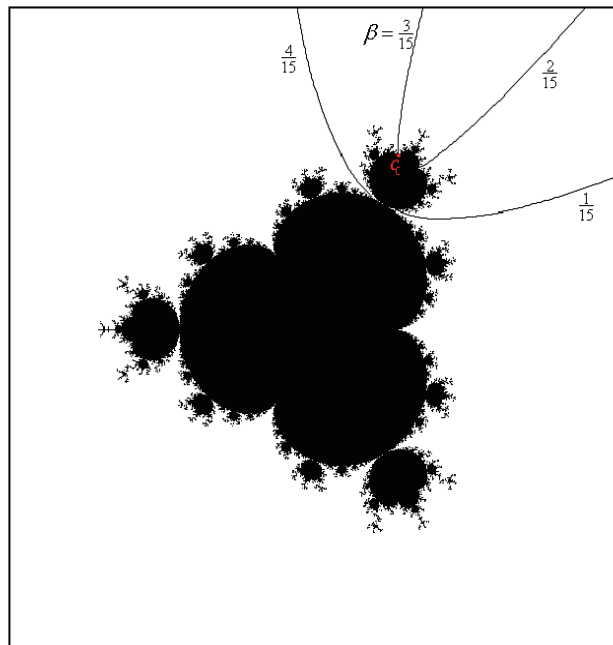


Figure 2-3: Multibrot set M_4 . The parameter rays $R_{M_4}(1/15)$ and $R_{M_4}(4/15)$ land on the root of some hyperbolic component. $R_{M_4}(2/15)$ and $R_{M_4}(1/5)$ land on two co-root of this hyperbolic component respectively.

2.2.2 Hubbard tree

In the work [DH1], Douady and Hubbard suggested a combinatorial description of the dynamics of a post critical finite polynomial using a tree-like structure. It is called Hubbard

tree. In this section, we state some basic result without proof. One can refer to [DH1] for the proof and more material of Hubbard tree.

Definition 2.2.2. *A finite connected tree T is a topological space which satisfies the following to properties:*

- *For any two points of T , there exists unique Jordan arc in T connecting the two points.*
- *T is homeomorphism to the union of finite closed segments.*

Let T be a finite connected tree. A point p in T is called *endpoint* of T if $T \setminus p$ is connected, p is called a *branching point* of T if the number of connected component of $T \setminus p$ is no less than 3.

Let f be a critically finite polynomial. Its filled Julia K_f set is connected, locally connected and arc connected. Given two points in the closure of a bounded Fatou component, they can be joined in a unique way by a Jordan arc consisting of (at most two) segments of internal rays. We call such arcs **regulated** (following Douady and Hubbard). Since K_f is arc connected, given two points $z_1, z_2 \in K_f$, there is an arc $\gamma : [0, 1] \rightarrow K_f$ such that $\gamma(0) = z_1$ and $\gamma(1) = z_2$. In general, we will not distinguish between the map and its image. Such arcs can be chosen in a unique way so that the intersection with the closure of a Fatou component is regulated. We still call such arcs regulated and denoted them by $[z_1, z_2]_{K_f}$.

Definition 2.2.3. *We say that a subset $X \subset K_f$ is **regulated connected** if every $z_1, z_2 \in X$ we have $[z_1, z_2]_{K_f} \subset X$. We define the **regulated hull** $[X]_{K_f}$ of $X \subset K_f$ as the minimal closed regulated connected subset of K_f containing X .*

Proposition 2.2.4. *If z_1, \dots, z_n are points in K_f , the regulated hull $[z_1, \dots, z_n]_{K_f}$ of $\{z_1, \dots, z_n\}$ is a finite tree.*

Definition 2.2.5. *Let f be a post critical finite polynomial. The Hubbard tree H_f of f is defined by $[C_f \cup P_f]_{K_f}$.*

The vertex set $V(H_f)$ of H_f is the union of C_f , P_f and all branching points of H_f . The closure of a connected component of $H_f \setminus V(H_f)$ is called an *edge*.

Lemma 2.2.6. *H_f is invariant under f ($f(H_f) \subset H_f$) and f maps each edge of H_f homeomorphic to the union of some edges of H_f .*

2.2.3 Invariant lamination and torus model

Let f be a degree d polynomial with connected Julia set and K_f be its filled-in Julia set. The Bottcher theorem give the commutative graph:

$$\begin{array}{ccc} \mathbb{C} \setminus K_f & \xrightarrow{f} & \mathbb{C} \setminus K_f \\ \phi \downarrow & & \downarrow \phi \\ \mathbb{C} \setminus \overline{\mathbb{D}} & \xrightarrow{z \mapsto z^d} & \mathbb{C} \setminus \overline{\mathbb{D}} \end{array}$$

In addition that f is locally connected, the conformal map ϕ^{-1} can be extended continuously to \mathbb{S}^1 and then each external ray land on a point in J_f . If we define a equivalent relation $\lambda(f)$ on \mathbb{S}^1 by

$$\theta \sim \eta \text{ iff } \theta = \eta \text{ or } R_f(\theta), R_f(\eta) \text{ land on the same point.}$$

then $\phi^{-1}/\lambda(f) : \mathbb{S}^1/\lambda(f) \rightarrow J(f)$ is a homeomorphism and the dynamic of f on $J(f)$ is topologically conjugate to that of $z \rightarrow z^d$ on $\mathbb{S}^1/\lambda(f)$. It means that the dynamic on f on its Julia set is completely characterized by equivalent relation $\lambda(f)$.

We abstract the core properties that $\lambda(f)$ satisfies:

- (R1) $\lambda(f)$ is a closed relation in \mathbb{S}^1 ;
- (R2) Each equivalence class is a finite subset of \mathbb{S}^1 ;
- (R3) If A is an equivalence class, then $d \cdot A$ is an equivalence class;
- (R4) The equivalence classes are un-linked, i.e the hull of any two equivalence classes in $\overline{\mathbb{D}}$ are disjoint.

Then we can imagine using this kind of equivalent relation, or more geometric view of the equivalent relation called invariant lamination, as the combinatorial model of polynomials. The following definitions about lamination are due to Thurston ([Th])

Definition 2.2.7 (lamination). *A **lamination** is a set L of hyperbolic chords in the closed unit disk $\overline{\mathbb{D}}$, called **leaves of L** , satisfying the following conditions:*

- (L1) *elements of L are disjoint, except possibly at their endpoints;*

(L2) *the union of L is closed.*

A **gap** of a lamination L is the closure of a component of complement of $\cup L$.

Definition 2.2.8 (invariant lamination). *A lamination L is a degree d **invariant** for the map*

$$\tau_d : \mathbb{S}^1 \longrightarrow \mathbb{S}^1, \quad \tau_d(z) = z^d,$$

if it satisfies the following conditions:

(L3) *Forward invariant: If any leaf \overline{pq} is in L , then either $\tau_d(p) = \tau_d(q)$, or $\overline{\tau_d(p)\tau_d(q)}$ is in L .*

(L4) *Backward invariance: if any leaf \overline{pq} is in L , then there exists a collection of d disjoint leaves, each joining a preimage of p to a preimage of q .*

(L5) *Gap invariant: for any gap G , the convex hull of the image of $G_0 = G \cap \mathbb{S}^1$ is either a gap or a leaf or a single point.*

Definition 2.2.9. *Let L be a degree d invariant lamination. A gap G of L is called **critical** if the restriction of τ_d on G_0 is not injective; a leaf $l \in L$ is called **critical** if the image of its two endpoints are the same.*

*The **major** of a degree- d -invariant lamination is the set of critical leaves and critical gaps.*

We'll say that the major of a lamination is **primitive** if each critical gap is a polygon whose vertices are all identified by $z \mapsto z^d$. Even without a predefined lamination, we can define a **primitive degree- d major** to be a collection of disjoint leaves and polygons each of whose vertices are identified under $z \mapsto z^d$, with total criticality $d - 1$ (see right of Figure 2.2.3, 2.2.3). Denote by $PM(d)$ the space of all degree- d -primitive major.

In the following, we give a description of Thurston's torus model to show how to construct a degree- d -invariant lamination whose major is a given primitive major.

Denote by $\mathbb{T} = \mathbb{S}^1 \times \mathbb{S}^1$ the torus. For $x, y \in \mathbb{S}^1$, we use \overline{xy} to represent the hyperbolic geodesic in the unit disc linking $e^{2\pi ix}$ and $e^{2\pi iy}$. Each leaf \overline{xy} is represented twice on the torus, as (x, y) and (y, x) .

If a lamination contains a leaf $l = \overline{xy}$, then a certain set $X(l)$ of other leaves are excluded from the lamination because they cross \overline{xy} . On the torus, if you draw the horizontal and

vertical circles through the two points (x, y) and (y, x) , they subdivide the torus into four rectangles having the same vertex set; the remaining two common vertices are (x, x) and (y, y) . The leaves represented by points in the interior of two of the rectangles constitute $X(l)$, while the leaves represented by the closures of the other two rectangles are all compatible with the given leaf $l = \overline{xy}$. We will call this good, compatible region $G(l)$. The compatible rectangles are actually squares, of sidelengths $a - b \bmod 1$ and $b - a \bmod 1$. They form a checkerboard pattern, where the two squares of $G(l)$ are bisected by the diagonal.

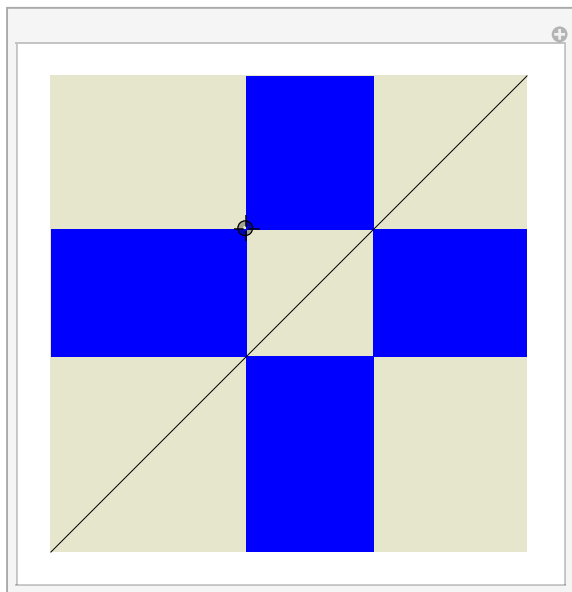


Figure 2-4: A leaf \overline{xy} of a lamination can be represented by a pair of points $\{(x, y), (y, x)\}$ on a torus. Leaves that are excluded by \overline{xy} because they intersect it are represented in two shaded rectangles, and leaves compatible with it are represented in two squares of sidelength $b - a \bmod 1$ and $a - b \bmod 1$.

Given a set S of leaves, the excluded region $X(S)$ is the union of the excluded regions $X(l)$ for $l \in S$, and the good region $G(S)$ is the intersection of the good regions $G(l)$ for $l \in S$. If S is a finite lamination, then $G(S)$ is a finite union of rectangles that are disjoint except for corners.

In the particular case of a primitive major lamination $m \in PM(d)$, each region of the disk minus m touches S^1 in a union of one or more intervals $J_1 \cup \dots \cup J_k$ of total length $1/d$. This determines a finite union of rectangles $(J_1 \cup \dots \cup J_k) \times (J_1 \cup \dots \cup J_k)$ of $G(m)$ whose total area is $1/d^2$ that maps under the degree d^2 covering map $(x, y) \mapsto d \cdot (x, y)$ to the entire torus.

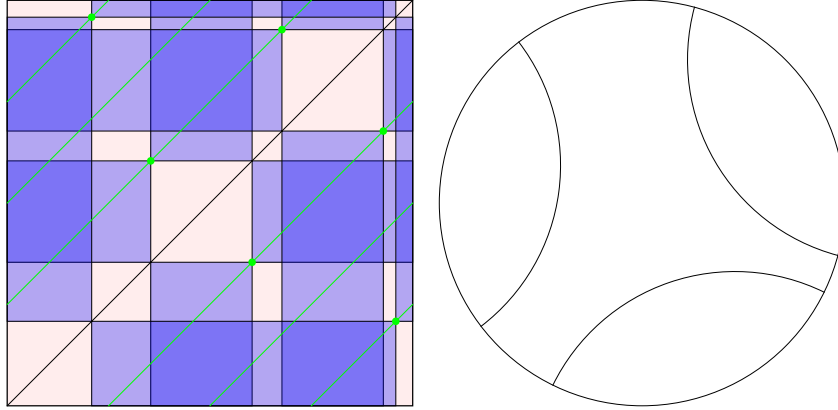


Figure 2-5: On the left is a plot of showing the excluded region $X(m)$, shaded, together with the compatible region $G(m)$ on the torus, where $m \in PM(4)$ is a primitive degree-4 major. The figure is symmetric by reflection in the diagonal. The quotient of the torus by this symmetry is a Moebius band. Note that $G(m)$ is made up of three $1/4 \times 1/4$ squares (one of them wrapped around) corresponding to the regions that touch the circle in only one edge, together with $3 \cdot 3 = 9$ additional rectangles with total area is $1/4^2$, corresponding to the region that touches S^1 in 3 intervals. The 6 green dots represent the leaves of the major, one dot for each orientation of the leaf.

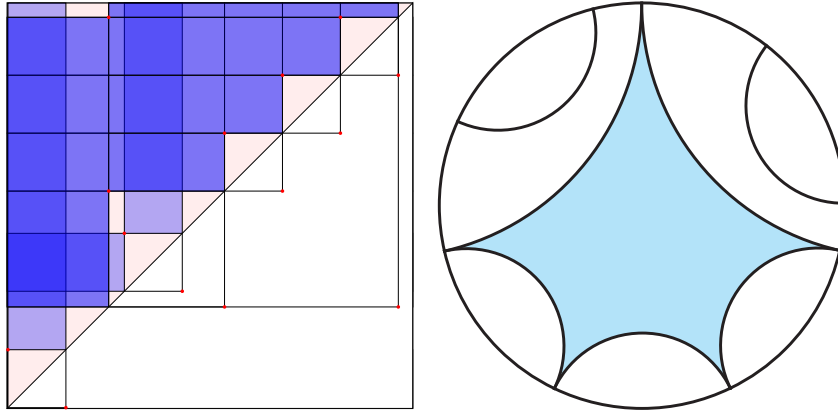


Figure 2-6: Here is a primitive heptic (degree-7) major with a pentagonal gap, shown in a variation of the torus plot, along with the standard Poincaré disk picture. On the left, half of the torus has been replaced by a drawing that indicates each leaf of the lamination by a path made up of a horizontal segment and a vertical segment. The lower right triangular picture transforms to the Poincaré disk picture by collapsing the horizontal and vertical edges of the triangle to a point, bending the collapsed triangle so that the it goes to the unit disk with the collapsed edges going to $1 \in \mathbb{C}$, then straightening each rectilinear path into the hyperbolic geodesic with the same endpoints.

For $m \in PM(d)$ we can now define a sequence of backward-image laminations $b_i(m)$. Let $b_0(m) = m$ and inductively define $b_{i+1}(m)$ is the union of m with the preimages under F_d of $b_i(m)$ that are in $G(m)$. Then For each i , $b_i(m)$ is a lamination.

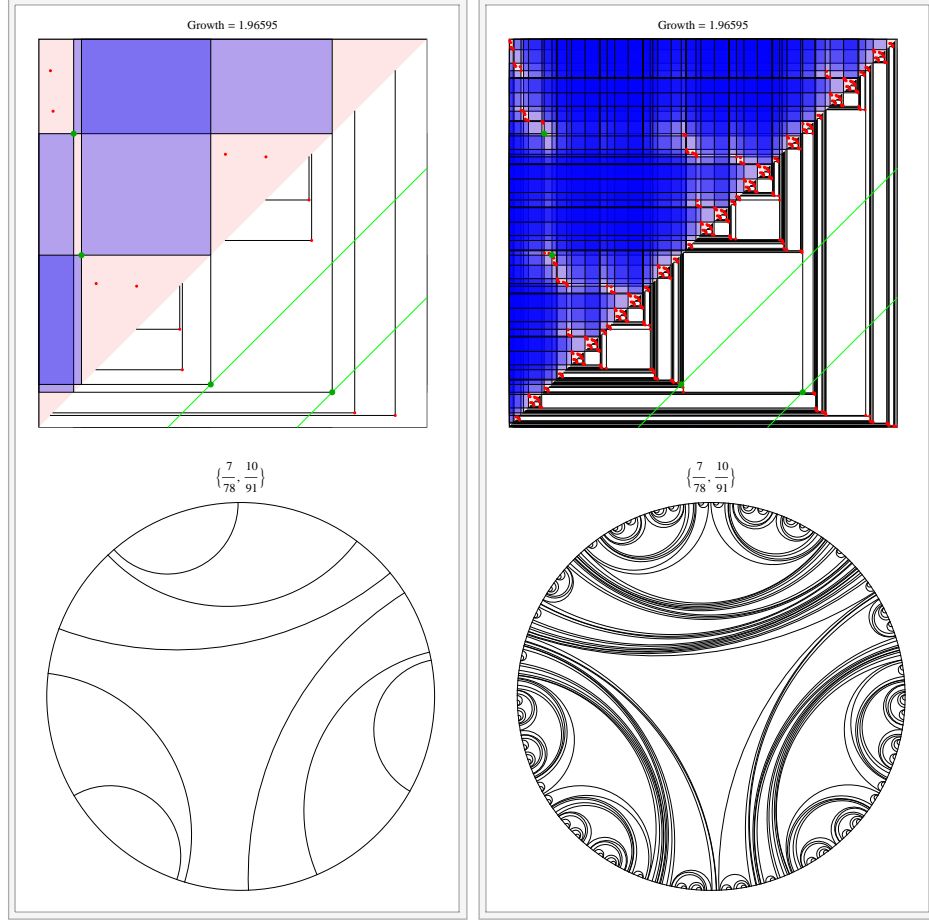


Figure 2-7: On the left is stage 1 ($b_1(m)$) in building a cubic-invariant lamination for $m = \{(\frac{10}{91}, \frac{121}{273}), (\frac{7}{78}, \frac{59}{78})\} \in PM(3)$. The two longer leaves of m subdivide the disk into 3 regions, each with two new leaves induced by the map f_3 . On the right is a later stage that gives a reasonable approximation of $b_\infty(m)$.

Note that this is an increasing sequence, $b_i(m) \subset b_{i+1}(m)$. By induction, the good region $G(b_i(m))$ has area $1/d^m$. Set $b_\infty(m) = \bigcup_{i \geq 0} b_i(m)$, $b'_\infty(m)$ is the cluster set of $b_\infty(m)$ and $\overline{b_\infty(m)} = b_\infty(m) \cup b'_\infty(m)$ is the closure of $b_\infty(m)$.

It follows readily that:

Theorem 2.2.10. *The closure $\overline{b_\infty(m)}$ is a degree- d -invariant lamination having m as its major.*

2.3 Orbifold and multicurve

Let $f : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ be a rational map. We denote by $\deg_z f$ the local degree of f at z . We will call

$$\Omega_f = \{z \mid \deg_z f > 1\}.$$

the critical set of f and

$$P_f = \bigcup_{n>0} \overline{f^n(\Omega_f)}$$

the postcritical set. The rational map f is called **postcritical finite** if $\# < \infty$.

Definition 2.3.1 (orbifold). *Let f be a postcritical finite rational map. A map $\nu_f : \hat{\mathbb{C}} \rightarrow \mathbb{N} \cup \{\infty\}$ is defined such that $\nu_f(z)$ is the least common multiple of the local degree $\deg_y f^n$ for all $n > 0$ and $y \in \hat{\mathbb{C}}$ with $f^n(y) = z$. The map ν_f is called **signature** of f . We call $\mathcal{O}_f = (\hat{\mathbb{C}}, \nu_f)$ the **orbifold** of f .*

Note that $\nu_f(z) > 1$ if and only if $z \in P_f$. The Euler Characteristic of \mathcal{O}_f is given by

$$\chi(\mathcal{O}_f) = 2 - \sum_{z \in \hat{\mathbb{C}}} \left(1 - \frac{1}{\nu_f(z)}\right).$$

Definition 2.3.2. *The orbifold \mathcal{O}_f is called **elliptic**, **parabolic** or **hyperbolic** according to $\chi(\mathcal{O}_f) > 0$, $\chi(\mathcal{O}_f) = 0$ or $\chi(\mathcal{O}_f) < 0$.*

The following theorem is a part of the Uniformization Theorem of orbifold. One can refer to appendix A.1 in [Mc] and appendix in [Mil2] for more details.

Theorem 2.3.3. *Let f be a rational map with parabolic orbifold, then we have*

1. \mathcal{O}_f is parabolic orbifold if and only if the signature of f is $(2, 2, 2, 2)$, $(3, 3, 3)$, $(2, 4, 4)$ or $(2, 3, 6)$.
2. The map f can be lifted to be a holomorphic map between torus along a branched covering $\wp_f : \mathbb{T}_\tau \rightarrow \hat{\mathbb{C}}$ for some $\tau \in \mathbb{H}$. That is

$$\begin{array}{ccc} \mathbb{T}_\tau & \xrightarrow{L_f = \alpha z \pmod{\Lambda_\tau}} & \mathbb{T}_\tau \\ \wp_f \downarrow & & \downarrow \wp_f \\ \hat{\mathbb{C}} & \xrightarrow{f} & \hat{\mathbb{C}} \end{array} \quad \mathbb{T}_\tau = \mathbb{C}/(\mathbb{Z} \oplus \tau \mathbb{Z})$$

3. There exists $\rho_n : \mathbb{T}_\tau \longrightarrow \mathbb{T}_\tau$, $z \mapsto e^{\frac{2\pi i}{n}}(z - z_0) + z_0 \pmod{\Lambda_\tau}$ (z_0 is a fixed point of ρ_n on \mathbb{T}_τ) such that

$$\wp_f(z_1) = \wp_f(z_2) \iff z_2 = \rho_n^k(z_1) \text{ for some } k \in \mathbb{Z}$$

The possible numbers n are 2, 3, 4, 6 and there is a equation $\deg_z(\wp_f) = \nu_f(\wp_f(z))$. The set of critical values of \wp_f is P_f . The relationship between n and the signature of f are given by the following table:

signature of f	(2, 2, 2, 2)	(3, 3, 3)	(2, 4, 4)	(2, 3, 6)
n	2	3	4	6
The Deck transformation group of \wp_f	$\langle \rho_2 \rangle$	$\langle \rho_3 \rangle$	$\langle \rho_4 \rangle$	$\langle \rho_6 \rangle$
degree of \wp_f	2	3	4	6

Now we turn to the definition of stable multicurve.

Let f be a postcritical finite rational map. We say that a Jordan curve γ on $\hat{\mathbb{C}} \setminus P_f$ is **non-peripheral** if any component of $\hat{\mathbb{C}} \setminus \gamma$ contains at least two points of P_f .

Definition 2.3.4. A **multicurve** of f is a finite non-empty collection of disjoint non-peripheral Jordan curves on $\hat{\mathbb{C}} \setminus P_f$ such that any two of them are not homotopic rel P_f . A multicurve Γ is called **stable** if each non-peripheral curve in $f^{-1}(\gamma)$ for $\gamma \in \Gamma$ is homotopic rel P_f to a curve in Γ .

Remark 2.3.5. Let S be any Riemann surface and γ_1, γ_2 be two curves in S . In this paper, we denote by $\gamma_1 = \gamma_2$ ($\gamma_1 \neq \gamma_2$) that the images of γ_1 and γ_2 in S are coincide (not coincide) as sets.

2.4 Algebraic curve and Galois group

The objective here is to give some definition and notation about affine algebraic curve and Galois group that will be used later. The material can be found in [G] and [H].

2.4.1 Affine algebraic curve, singular points and tangent

Definition 2.4.1. An affine algebraic curve over \mathbb{C} is defined as the set

$$\mathcal{C} = \{(a, b) \in \mathbb{C}^2 \mid f(a, b) = 0\}$$

for a non-constant squarefree polynomial $f(a, b) \in \mathbb{C}[x, y]$.

The polynomial f is called the defining polynomial of \mathcal{C} , the degree of f is called the *degree* of \mathcal{C} . We assume that all polynomials appearing in this section are squarefree.

If $f = \prod_{i=1}^m f_i$, where f_i are the irreducible factor of f , we say that the affine curve defined by f_i is a *component* of \mathcal{C} . Furthermore, the curve \mathcal{C} is said to be irreducible if the defining polynomial is irreducible.

Definition 2.4.2. Let \mathcal{C} be an affine algebraic curve for \mathbb{C} defined by $f \in \mathbb{C}[x, y]$, and let $P = (a, b) \in \mathcal{C}$. The multiplicity of \mathcal{C} at P , denoted by $\text{mult}_P(\mathcal{C})$, is defined as the order of the first non-vanishing term in the Taylor expansion of f at P , i.e.

$$f(x, y) = \sum_{s=0}^{\infty} \frac{1}{s!} \sum_{t=0}^s \binom{s}{t} (x-a)^t (y-b)^{s-t} \frac{\partial^s f}{\partial x^t \partial y^{s-t}}(a, b).$$

If $\text{mult}_P(\mathcal{C}) = 1$, the point P is called a *smooth point* of \mathcal{C} . If $\text{mult}_P(\mathcal{C}) = r > 1$, then we say that P is a *singular point* of multiplicity r . We say that \mathcal{C} or f is *smooth* if any point on \mathcal{C} is smooth.

The following theorem provides a topological interpretation of the irreducibility of polynomials.

Theorem 2.4.3. The polynomial f is irreducible if and only if the set of smooth points of f is connected.

Let $P = (a, b) \in \mathcal{C}$ be a point of multiplicity r ($r \geq 1$). Then the first non vanishing term in the Taylor expansion of f at P is

$$T_r(x, y) = \sum_{t=0}^r \binom{r}{t} (x-a)^t (y-b)^{r-t} \frac{\partial^r f}{\partial x^t \partial y^{r-t}}(a, b).$$

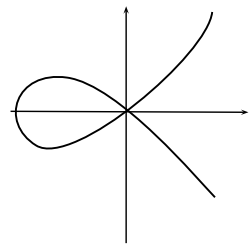
Note that $T_r(c, z)$ is a homogeneous polynomial about $x-a$ and $y-b$, so all its irreducible factors are linear and they will be called the tangents of \mathcal{C} at P .

Definition 2.4.4. Let \mathcal{C} , P , $T_r(c, z)$ be as above. Then the tangents of \mathcal{C} at P are the linear irreducible factors of $T_r(x, y)$ and the multiplicity of a tangent is the multiplicity of the corresponding factor.

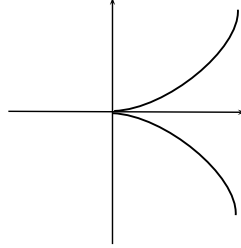
For analyzing a singular point P on a curve \mathcal{C} we need to know its multiplicity but also the multiplicities of the tangents. If all the r tangents at the point P are different, then this singularity is of well-behaved type.

Definition 2.4.5. A singular point P of multiplicity r on an affine plane curve \mathcal{C} is called ordinary if the r tangents to \mathcal{C} at P are distinct, and non-ordinary otherwise.

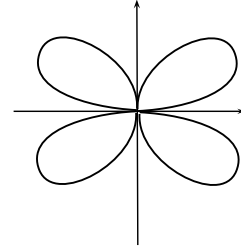
Example. Considering the following three curve in \mathbb{C}^2 . The pictures represent the projection of these curves to \mathbb{R}^2 . $(0, 0)$ is the unique singular point of \mathcal{A} , \mathcal{B} , \mathcal{C} . For \mathcal{A} , the origin



$\mathcal{A} : f_A(x, y) = y^2 - x^3 - x^2$



$\mathcal{B} : f_B(x, y) = y^2 - x^3$



$\mathcal{C} : f_C(x, y) = (x^2 + y^2)^3 - 4x^2y^2$

$\text{mult}_{(0,0)}(\mathcal{A}) = 2$ and there are two different tangents $x \pm y = 0$ at $(0, 0)$. So $(0, 0)$ is a double ordinary singular point of \mathcal{A} , called a “node”. For \mathcal{B} , $\text{mult}_{(0,0)}(\mathcal{B}) = 2$ and there is only one tangent $y = 0$ at $(0, 0)$ with multiplicity 2. So $(0, 0)$ is a double non-ordinary singular point of \mathcal{B} , called a “cusp”. For \mathcal{C} , $\text{mult}_{(0,0)}(\mathcal{C}) = 4$. There are two different tangents $x = 0, y = 0$ at $(0, 0)$ with multiplicity 2 both. So $(0, 0)$ is a non-ordinary singular point of \mathcal{C} with multiplicity 4. It can be seen as a mixing of node and cusp.

Definition 2.4.6. Let \mathcal{C}_1 , \mathcal{C}_2 be any two affine algebraic curves and P is an intersecting point of the two curves. We say that \mathcal{C}_1 , \mathcal{C}_2 intersect transversally at P if P is a smooth point for both \mathcal{C}_1 and \mathcal{C}_2 and the tangents of \mathcal{C}_1 , \mathcal{C}_2 at P are distinct.

2.4.2 The Galois group of a polynomial

Let M be a field, $f(x) \in M[x]$. If all roots of f are simple, then the splitting field of f over M , denoted by L , is called the Galois extension over M . The Galois group of f is

$$\text{Gal}(f) = \{\sigma \in \text{Aut}(L) \mid \sigma|_M = \text{id}_M\}.$$

Note that each $\sigma \in \text{Gal}(f)$ can be seen as a permutation on the roots of f and it is completely determined by this permutation, then $\text{Gal}(f)$ can be seen as a subgroup of $\text{sym}(\text{roots of } f)$. Usually, $\text{Gal}(f)$ doesn't equal to the symmetric group on all roots of f , so we can see some intrinsic structure and symmetry of polynomial f from its Galois group.

Now let $M = K = \mathbb{C}(c)$, and let $P(c, z) \in K[z]$ be a polynomial about c and z . If $P(c, z)$ has no multiple roots in $K[z]$, applying the statement above to $P(c, z)$, we obtain a Galois group $\text{Gal}(P)$ of $P(c, z)$. In fact, the Galois theory admits an interpretation in terms of the covering theory. Here we only state what we need as a theorem below.

Definition 2.4.7. *Let X, Y be two topological spaces, X is connected. Let $f : Y \rightarrow X$ be a covering. Fix any base point $x_0 \in X$, then the monodromy action of the elements in $\pi_1(X, x_0)$ gives a group morphism*

$$\Phi_f : \pi_1(X, x_0) \longrightarrow \text{sym}(f^{-1}(x_0))$$

The image of $\pi_1(X, x_0)$ under Φ_f is called the monodromy group of f , denoted by $\text{Mon}(f)$.

Let $P(c, z) \in \mathbb{C}[c, z] \subset K(z)$ be a polynomial, monic in z , and \mathcal{C} be the affine algebraic curve defined by $P(c, z) = 0$. Denote by $\pi_P : \mathcal{C} \rightarrow \mathbb{C}$, $(c, z) \mapsto c$ be the projection to the first parameter and by $C_P = \{(c, z) \in \mathcal{C} : \frac{\partial P}{\partial z}(c, z) = 0\}$ the set of critical points of π_P . If $C_P \neq \mathcal{C}$, then $\mathcal{C} \setminus \pi_P^{-1}(\pi_P(C_P))$ is a covering of $\mathbb{C} \setminus \pi_P(C_P)$. So for any $c_0 \in \mathbb{C} \setminus \pi_P(C_P)$, we obtain a group morphism

$$\Phi_P : \pi_1(\mathbb{C} \setminus \pi_P(C_P), c_0) \longrightarrow \text{sym}(\pi_P^{-1}(c_0))$$

whose image is the monodromy group $\text{Mon}(P)$.

In fact, the monodromy action of $\pi_1(\mathbb{C} \setminus \pi_P(C_P), c_0)$ on $\pi_P^{-1}(c_0)$ under the covering map π_P is induced by analytic continuations. By the Implicit Function Theorem, for $c_0 \in$

$\mathbb{C} \setminus \pi_P(C_P)$, there exist $\deg(P(c_0, z))$ local holomorphic solutions for $P(c, z) = 0$ at c_0 . These solutions accept analytic continuations along any curve in $\mathbb{C} \setminus \pi_P(C_P)$. So analytic continuations along the closed curves based on c_0 give a group morphism

$$\phi_P : \pi_1(\mathbb{C} \setminus \pi_P(C_P), c_0) \longrightarrow \text{sym}(Z_P)$$

where Z_P is the set of roots of $P(c_0, z) = 0$ (note that $\pi_P^{-1}(c_0) = \{(c_0, z) \in \mathcal{C} | z \in Z_P\}$). The monodromy action of $\gamma \in \pi_1(\mathbb{C} \setminus \pi_P(C_P), c_0)$ under $\pi_P : \mathcal{C} \setminus \pi_P^{-1}(\pi_P(C_P)) \longrightarrow \mathbb{C} \setminus \pi_P(C_P)$ is induced by the analytic continuation of local solutions of $P(c, z) = 0$ at c_0 along γ , that is

$$\Phi_P(\gamma)(c_0, z) = (c_0, \phi_P(\gamma)(z))$$

for all $(c_0, z) \in \pi_P^{-1}(c_0)$. So $\text{Mon}(P)$ is isomorphic to $AC(P) := \phi_P(\pi_1(\mathbb{C} \setminus \pi_P(C_P), c_0))$.

Theorem 2.4.8. *The Galois group of $P(c, z)$ is isomorphic to the monodromy group of $\pi_P : \mathcal{C} \setminus \pi_P^{-1}(\pi_P(C_P)) \rightarrow \mathbb{C} \setminus \pi_P(C_P)$, that is , $\text{Gal}(P) \cong \text{Mon}(P)$.*

This proposition is due to the correspondence between the Galois theory and the covering theory. One can refer to [H] for its proof.

2.5 Topological entropy

Let X be a compact topological space. For any open cover \mathcal{U} , let $N(\mathcal{U})$ denote the minimal cardinal of a sub cover of \mathcal{U} . Since X is compact, there always exists a finite subcover. For any two covers \mathcal{U}, \mathcal{V} of X , define

$$\mathcal{U} \vee \mathcal{V} = \{U \cap V | U \in \mathcal{U}, V \in \mathcal{V}\}.$$

Let f be a continuous map of X into itself. If \mathcal{U} is an open cover of X , we set

$$f^{-1}(\mathcal{U}) = \{f^{-1}(U) | U \in \mathcal{U}\}$$

and

$$\bigvee^n \mathcal{U} = \mathcal{U} \vee f^{-1}(\mathcal{U}) \vee \cdots \vee f^{-n}(\mathcal{U})$$

For two covers \mathcal{U} and \mathcal{V} , We denote by $\mathcal{U} \subset \mathcal{V}$ if \mathcal{U} is a sub-cover of \mathcal{V} . For the operators f^{-1} and \vee , there are two basic properties that will be used later: let $\mathcal{U}, \mathcal{V}, \mathcal{W}$ be open covers

$$\text{If } f \text{ is onto, then } f^{-1}(\mathcal{U} \vee \mathcal{V}) \subset f^{-1}(\mathcal{U}) \vee f^{-1}(\mathcal{V}) \quad (2.5.1)$$

$$\text{If } \mathcal{U} \subset \mathcal{V}, \text{ then } f^{-1}(\mathcal{U}) \subset f^{-1}(\mathcal{V}) \text{ and } \mathcal{U} \vee \mathcal{W} \subset \mathcal{V} \vee \mathcal{W}. \quad (2.5.2)$$

The *entropy* of f on X with respect to \mathcal{U} is defined by :

$$h(X, \mathcal{U}, f) = \lim_{n \rightarrow \infty} \frac{\log N(\bigvee^n \mathcal{U})}{n}.$$

This limit exists and it is also the infimum because the sequence $\{N(\bigvee^n \mathcal{U})\}$ satisfy sub-additive property. Finally, we define the *topological entropy* of f on X :

$$h(X, f) = \sup_{\mathcal{U}, \text{ open cover of } X} h(X, \mathcal{U}, f).$$

We will give three basic properties of topological entropy that will be used in the following. One can see [Do] for more details.

Let X be a compact set, $f : X \rightarrow X$ is a continuous map.

Proposition 2.5.1. ([Do]. Pro 2) *If $X = X_1 \cup X_2$, with X_1 and X_2 compact, $f(X_1) \subset X_1$ and $f(X_2) \subset X_2$, then $h(X, f) = \sup(h(X_1, f), h(X_2, f))$.*

Proposition 2.5.2. ([Do]. Pro 3) *Let Y be a closed subset of X such that $f(Y) \subset Y$. Suppose that , for any $x \in X$, the distance of $f^n(x)$ to Y tends to 0, uniformly on any compact set in $X - Y$. Then $h(X, f) = h(Y, f)$.*

Proposition 2.5.3. ([Do]. Pro 4) *Let X, Y be compact sets, $f : X \rightarrow X$, $g : Y \rightarrow Y$ and $\pi : Y \rightarrow X$ continuous maps with π surjective, and such that $f \circ \pi = \pi \circ g$. Then $h(X, f) \leq h(Y, g)$. Suppose all fibers $\pi^{-1}(x)$ have a cardinal bounded by a fixed finite number m . Then $h(X, f) = h(Y, g)$.*

From the definition, if we want to compute the topological entropy of a continuous map on a compact space, we should consider all the open covers of the space. It is very difficult for a general compact space. But if the space is a compact metric space, then there is a simpler method to compute the entropy.

Let X be a compact metric space with metric d . It can be shown that the topological entropy depends only on the topology induced by d .

Definition 2.5.4. The diameter $d(\mathcal{U})$ of a cover \mathcal{U} is defined by

$$d(\mathcal{U}) = \sup_{U \in \mathcal{U}} d(U)$$

where $d(U)$ is the diameter of set U .

Definition 2.5.5. A cover \mathcal{V} is said to be a refinement of a cover \mathcal{U} if every member V is a subset of some member of \mathcal{U} . It is denoted by $\mathcal{U} \prec \mathcal{V}$. Note that if $\mathcal{U} \prec \mathcal{V}$, then $h(X, f, \mathcal{U}) \leq h(X, f, \mathcal{V})$.

Proposition 2.5.6. If $\{\mathcal{U}_k\}$ is a sequence of open covers such that

$$(1) \quad \mathcal{U}_k \prec \mathcal{U}_{k+1},$$

$$(2) \quad d(\mathcal{U}_k) \rightarrow 0 \text{ as } k \rightarrow \infty$$

Then $h(X, f) = \lim_{k \rightarrow \infty} h(X, \mathcal{U}_k, f)$.

Proof. This proposition mainly relies on the Lebesgue's Covering Lemma:

Lebesgue's Covering Lemma. For every open cover \mathcal{U} of a compact metric space X , there exists $\delta > 0$ such that if V is a subset of X with $d(V) < \delta$, then V is contained in one of the members of \mathcal{U} . The supremum of all such numbers δ is called Lebesgue number of \mathcal{U} .

For any $\epsilon > 0$, there exists an open cover \mathcal{U} of X such that

$$0 < h(X, f) - h(X, f, \mathcal{U}) < \epsilon.$$

Since $d(\mathcal{U}_k) \rightarrow 0$ as $k \rightarrow \infty$, there is integer k_0 such that $d(\mathcal{U}_{k_0})$ is less than the Lebesgue number of \mathcal{U} . By Lebesgue's Covering Lemma and property (2), $\mathcal{U} \prec \mathcal{U}_{k_0} \prec \mathcal{U}_k$ for $k > k_0$. It follows that

$$0 \leq h(X, f) - h(X, f, \mathcal{U}_k) \leq h(X, f) - h(X, f, \mathcal{U}) < \epsilon \quad \text{for } k > k_0.$$

□

2.6 Matrices

Let $\mathcal{M} = \{k \times k \text{ square matrices}\}$. \mathcal{M} can be viewed as a vector space of dimension k^2 and it is isomorphic to the space $\{\text{linear maps of } \mathbb{R}^k \rightarrow \mathbb{R}^k\}$.

An **operator norm** on \mathcal{M} is defined as follows: Equip at first a norm on \mathbb{R}^k : $\|x\|_*$. Define then $\|D\|_* = \sup_{\|x\|_* \leq 1} \|Dx\|_*$. This type of norm satisfies a multiplicative inequality :

$$\forall A, B \in \mathcal{M}, \|A \cdot B\|_* \leq \|A\|_* \cdot \|B\|_*. \quad (2.6.1)$$

In particular,

$$\|D^n\|_* \leq \|D\|_*^n \quad (2.6.2)$$

A general norm $\|\cdot\|$ does not necessarily satisfy (2.6.1). On the other hand,

Lemma 2.6.1. *All norms in \mathcal{M} are equivalent.*

Lemma 2.6.2. *(spectral radius) There exists a map $\rho : \mathcal{M} \rightarrow [0, +\infty)$ so that for any norm $\|\cdot\|$ in \mathcal{M} , any $D \in \mathcal{M}$, $\|D^n\|_*^{\frac{1}{n}} \rightarrow \rho(D)$ as $n \rightarrow \infty$. Moreover, if $\|\cdot\|_*$ is an operator norm, $\rho(D) = \inf_n \|D^n\|_*^{\frac{1}{n}} = \liminf_n \|D^n\|_*^{\frac{1}{n}}$.*

For any $D \in \mathcal{M}$, $\rho(D)$ is called the spectral radius of D .

Proof. Denote the norm $\|\cdot\|_*$ be an operator norm. Then by (2.6.2), $\|D^n\|_*$ is sub multiplicative. Therefore $\|D^n\|_*^{\frac{1}{n}}$ has a well defined limit, denoted by $\rho(D)$, which is also equal to $\inf_n \|D^n\|_*^{\frac{1}{n}}$ and to $\liminf_n \|D^n\|_*^{\frac{1}{n}}$. But by Lemma 2.6.1, $C\|D^n\|_* \leq \|D^n\| \leq C'\|D^n\|_*$. \square

Lemma 2.6.3. *For any invertible matrix $P \in \mathcal{M}$, any $D \in \mathcal{M}$, $\rho(PDP^{-1}) = \rho(D)$, that is to say ρ is a conjugacy invariant.*

Proof. Choose $\|\cdot\|_*$ to be a norm of Type 1. Set $A = PDP^{-1}$. Then $\|A^n\|_* = \|PD^nP^{-1}\|_* \leq \|P\|_* \cdot \|D^n\|_* \cdot \|P^{-1}\|_*$. So $\rho(A) = \rho(D)$. By symmetry, $\rho(D) \leq \rho(A)$. So $\rho(A) = \rho(D)$. \square

Lemma 2.6.4. *For any $D \in \mathcal{M}$, $\rho(D) = \max\{|\lambda| \mid \lambda \text{ is an eigenvalue of } D\}$.*

Proof. By Lemma 2.6.3, one can replace D by its Jordan Form. We may assume $|\lambda_1| = \max\{|\lambda| \mid \lambda \text{ is an eigenvalue of } D\}$. Let $\|D\|_\infty$ be the norm associated to $\|x\|_\infty = \max_i |x_i|$. Then $\|D\|_\infty = \max_i \{\sum_j |D_{ij}|\}$, where D_{ij} denote the element in D of the i -th row and the j -th column. So $\|D^n\|_\infty \geq |\lambda_1^n|$ and $\rho(D) \geq |\lambda_1|$. Note that $\|D^n\|_\infty \leq |\lambda_1^{n-k}| \cdot \text{poly}(n)$,

where $\text{poly}(n)$ denotes a polynomial about n whose coefficients depend on λ_1 , and k denotes the maximal size of a Jordan block corresponding to λ_1 . So $\rho(D) \leq |\lambda_1|$. \square

We will state without proof the classical

Theorem 2.6.5. (*Perron-Frobenius*) Let $D \in \mathcal{M}$ and D_{ij} denote the element in D of the i -th row and the j -th column.

1. If $D > 0$ (i.e. for any pair $(i, j) \in \{1, \dots, k\}$, $D_{ij} > 0$), then $\rho(D) = \lambda(D)$ is itself an eigenvalue. Moreover, $\lambda(D)$ is a simple eigenvalue, there exists a unique (up to scaling) $v \geq 0$, such that $Dv = \lambda(D)v$ and necessarily $v > 0$. Furthermore for any other either value λ , we have $|\lambda| < \lambda(D)$, and λ does not have an eigenvector u with $u \geq 0$ and $u \neq 0$.

2. If $D \geq 0$ (i.e. for any pair $(i, j) \in \{1, \dots, k\}$, $D_{ij} \geq 0$), then $\rho(D) = \lambda(D)$ is itself an eigenvalue. Moreover if $\rho(D) > 0$, there exists a $v \geq 0$, $v \neq 0$ such that $Dv = \lambda(D)v$.

Theorem 2.6.6 (Due to TanLei). Denote by $v = (1, \dots, 1) \in \mathbb{R}^k$. Let A be a non-nilpotent $k \times k$ matrix whose entries are non-negative integers. Set $v_n = A^n v$. Then, for any norm $\|\cdot\|$ of \mathbb{R}^k ,

$$\lim_{n \rightarrow \infty} \frac{\|v_n\|^{1/n}}{(\|v_n\| + \dots + \|v_0\|)^{1/n}} = 1 .$$

Proof. The result is independent of the choice of the norm. We will use $\|\cdot\|$ to denote the L^1 norm on \mathbb{R}^k , and again $\|\cdot\|$ to denote the corresponding operator norm in \mathcal{M} .

In particular $\|A\|$ is equal to the L_∞ norm of Av . Set the spectral radius of A as λ . We claim that $\lambda \geq 1$. By the non-nilpotent assumption, for each n , A^n contains at least one entry that is a strictly positive integer, and the other entries are non-negative. It follows that $\|A^n\| = \|A^n v\|_\infty \geq 1$. So $\lambda \geq 1$.

For any $\varepsilon > 0$, there is $C_\varepsilon > 0$ such that

$$\lambda^n \leq \|A^n\| \leq C_\varepsilon (\lambda + \varepsilon)^n, \quad \|v_n\| \leq \|A^n\| \cdot \|v\| \leq k C_\varepsilon (\lambda + \varepsilon)^n .$$

Let $\epsilon > 0$ be arbitrary. Denote by D_ϵ the matrix obtained by adding ϵ to every entry of A . Denote by λ_ϵ the leading eigenvalue of D_ϵ .

Apply the Perron-Frobenius theorem to the transpose of D one sees that there is a positive horizontal vector μ_ϵ satisfying

$$\mu_\epsilon D = \lambda_\epsilon \mu_\epsilon . \tag{2.6.3}$$

One may normalize μ_ε to have L^1 norm equal to 1.

Letting $\varepsilon \rightarrow 0$, and by compacity, a sequence of the triple $(\mu_\varepsilon, D_\varepsilon, \lambda_\varepsilon)$ converges to (μ, A, λ) with $\mu A = \lambda \mu$, $\|\mu\| = 1$ and $\mu \geq 0$. Note that $\mu v = \|\mu\| = 1$. So

$$\lambda^n = \lambda^n \mu v = \mu A^n v \leq \|\mu\| \cdot \|A^n v\| = \|A^n v\| = \|v_n\| .$$

It follows from (2.6.3)

$$\forall n, \quad \lambda^n \leq \|v_n\| \leq kC_\varepsilon(\lambda + \varepsilon)^n .$$

Thus, as $\lambda \geq 1$,

$$\limsup_{n \rightarrow \infty} \left(\frac{\|v_n\|}{\|v_n\| + \dots + \|v_0\|} \right)^{1/n} \leq \lim_{n \rightarrow \infty} \left(\frac{kC_\varepsilon(\lambda + \varepsilon)^n}{\lambda^n + \dots + 1} \right)^{1/n} = \frac{\lambda + \varepsilon}{\lambda}$$

(to obtain the last equality one should treat separately the cases $\lambda > 1$ and $\lambda = 1$). But the left hand side is independent of ε . Arguing symmetrically, one obtains

$$1 \leq \liminf_{n \rightarrow \infty} \left(\frac{\|v_n\|}{\|v_n\| + \dots + \|v_0\|} \right)^{1/n} \leq \limsup_{n \rightarrow \infty} \left(\frac{\|v_n\|}{\|v_n\| + \dots + \|v_0\|} \right)^{1/n} \leq 1 .$$

□

Chapter 3

Dynatomic periodic curve

In this chapter, we will give a description of the dynatomic periodic curve for the family of unimodal polynomials, including the smoothness and irreducibility of this curve, and the Galois group of the defining polynomial of this curve.

3.1 Defining polynomial of $\mathcal{X}_{0,p}$

In this section, we define the dynatomic periodic polynomial $Q_{0,p} \in \mathbb{C}[c, z]$ and prove that

$$\mathcal{X}_{0,p} = \{(c, z) \in \mathbb{C}^2 \mid Q_{0,p}(c, z) = 0\}$$

For $p \geq 1$, let $\Phi_{0,p}(c, z) = f_c^{\circ p}(z) - z$. Then the solutions of $\Phi_{0,p}(c, z) = 0$ is consisting of all $(c, z) \in \mathbb{C}$ such that

$$z \text{ is a } k \text{ periodic point of } f_c, \ k|p$$

. With an abusing of notations, we can consider a polynomial in $\mathbb{C}[c, z]$ as a polynomial in $\mathbb{K}[z]$, where $\mathbb{K} = \mathbb{C}(c)$ is the field of rational function about c .

Definition 3.1.1. *A polynomial $g(c, z) \in \mathbb{C}[c, z]$ is called squarefree if it can't be division by $h(c, z)^2$ for any non-constant $h(c, z) \in \mathbb{C}[c, z]$.*

Lemma 3.1.2. *There exists an unique sequence of square-free polynomials $\{Q_{0,p}(c, z)\}_{p \geq 1} \subset$*

$\mathbb{C}[c, z] \subset K[z]$ monic about z such that

$$\Phi_{0,p}(c, z) = \prod_{k|p} Q_{0,k}(c, z)$$

Proof. At first, we claim: for any $c_0 \in \mathbb{C} \setminus M_d$, $p \geq 1$, all roots of $\Phi_{0,p}(c_0, z)$ are simple. (proof: in this case, all periodic points of f_{c_0} are repelling and the critical orbit converge to ∞ . Then for any root z_0 of $\Phi_{0,p}(c_0, z)$, we have

$$(\partial \Phi_{0,p} / \partial z)(c_0, z_0) = [f_{c_0}^{\circ p}]'(z_0) - 1 \neq 0$$

). By this claim and the fact that $\Phi_{0,p}(c, z)$ is monic about z , if we find a sequence of polynomials $\{Q_{0,p}(c, z)\}_{p \geq 1}$ satisfying the equation in the lemma, they are naturally square-free.

We will define this sequence of polynomials by the induction on p . As $p = 1$, we define

$$Q_{0,1} = \Phi_{0,1}(c, z) = z^2 - z + c.$$

The equation in Lemma 3.1.2 holds.

Now suppose for each $1 \leq k < p$, we have find the polynomial $Q_{0,k}(c, z)$ that satisfies the requirement of the lemma. Let c_0 be any parameter in $\mathbb{C} \setminus M_d$. If z_0 is a root of $Q_{0,k}(c_0, z)$, it is also a root of $\Phi_{0,k}(c_0, z)$. Then z_0 is a m periodic point of f_{c_0} with $m|k$. In fact, m must be equal to k . Otherwise $Q_{0,k}(c_0, z) \cdot \prod_{m'|m} Q_{0,m'}(c_0, z)$ would have a double root at z_0 . But according to the assumption of induction, $Q_{n,k}(c_0, z) \cdot \prod_{m'|m} Q_{0,m'}(c_0, z)$ divides $\Phi_{0,k}(c_0, z)$, it is a contradiction to the claim above.

So we can conclude that: in $\{Q_{0,k}(c_0, z)\}_{1 \leq k < p, k|p}$, any two polynomials have no common roots, and any polynomial divides $\Phi_{0,p}(c_0, z)$. Therefore, $\prod_{k|p, k < p} Q_{0,k}(c_0, z)$ divides $\Phi_{0,p}(c_0, z)$ in $\mathbb{C}[z]$. Since c_0 is any point of $\mathbb{C} \setminus M_d$, the polynomial $\prod_{k|p, k < p} Q_{0,k}(c, z)$ divides $\Phi_{0,p}(c, z)$ in $\mathcal{K}[z]$. So we can define

$$Q_{0,p}(c, z) = \Phi_{0,p}(c, z) / \left[\prod_{k|p, k < p} Q_{0,k}(c, z) \right].$$

It satisfies the requirement of this lemma. □

From the definition of integer $\nu_d(p)$ and polynomial $Q_{0,p}(c, z)$, it is easy to see that

$$\deg_z Q_{0,p} = \deg Q_{0,p} = \nu_d(p).$$

Now, for any parameter $c_0 \in \mathbb{C}$, we will study the properties of the roots of $Q_{0,p}(c_0, z)$.

Proposition 3.1.3. *Let $p \geq 1$ be any integer and $c_0 \in \mathbb{C}$ be any parameter. Then $z_0 \in \mathbb{C}$ is a root of $Q_{0,p}(c_0, z)$ iff it satisfies the following 3 exclusive properties:*

- (1) z_0 is a p periodic point of f_{c_0} and $[f_{c_0}^{\circ p}]'(z_0) \neq 1$, in this case z_0 is a simple root of $Q_{0,p}(c_0, z)$;
- (2) z_0 is a p periodic point of f_{c_0} and $[f_{c_0}^{\circ p}]'(z_0) = 1$, in this case z_0 is a double root of $Q_{0,p}(c_0, z)$;
- (3) z_0 is a k periodic point of f_{c_0} where k is a proper factor of p and $[f_{c_0}^{\circ k}]'(z_0)$ is $\frac{p}{k}$ -th primitive root of unit, in this case z_0 is a root of $Q_{0,p}(c_0, z)$ with multiplicity $\frac{p}{k}$.

Proof. If $Q_{0,p}(c_0, z_0) = 0$ then $\Phi_{0,p}(c_0, z_0) = 0$, so z_0 is a k periodic point of f_{c_0} with $k|p$. On the contrary, if z_0 is a k periodic point of f_{c_0} , then $\Phi_{0,m}(c_0, z_0) = 0$ iff m is a multiple of k . In particular, if m is not a multiple of k , then $Q_{0,m}(c_0, z_0) \neq 0$. Since

$$0 = \Phi_{0,k}(c_0, z_0) = \prod_{k'|k} Q_{0,k'}(c_0, z_0)$$

we obtain $Q_{0,k}(c_0, z_0) = 0$.

Case 1. If the multiplier ρ of z_0 as a fixed point of $f_{c_0}^{\circ k}$ is not a root of unit, when p is a multiple of k , z_0 is a simple root of $\Phi_{0,p}(c_0, z)$. In this case, $Q_{0,k}(c_0, z)$ is a factor of $\Phi_{0,p}(c_0, z)$, so no other factors of $\Phi_{0,p}(c_0, z)$ vanish at z_0 . Then $Q_{0,p}(c_0, z_0) = 0$ iff $p = k$. Moreover, $Q_{0,p}(c_0, z) \in \mathbb{C}[z]$ has a simple root at z_0 .

If the multiplier ρ of z_0 as a fixed point of $f_{c_0}^{\circ k}$ is a s -th root of unit, then the multiplier of z_0 as a fixed point of $f_{c_0}^{\circ km}$ is ρ^m . It is equal to 1 iff m is a multiple of s . In this case, z_0 is a root of $\Phi_{0,km}(c_0, z)$ with multiplicity $s+1$. In fact, f_{c_0} has only one attractive petal cycle.

Case 2. If $s = 1$, when p is a multiple of k , z_0 is a double root of $\Phi_{0,p}(c_0, z)$. As the discussion above, $Q_{0,p}(c_0, z_0) = 0$ iff $p = k$, but in this case $Q_{0,p}(c_0, z) \in \mathbb{C}[z]$ has a double

root at z_0 .

Case 3. If $s \geq 2$, when p is a multiple of k but not a multiple of ks , z_0 is a simple root of $\Phi_{0,p}(c_0, z)$; when p is a multiple of ks , z_0 is a root of $\Phi_{0,p}(c_0, z)$ with multiplicity $s + 1$. Therefore, $Q_{0,p}(c_0, z_0) = 0$ iff $p = k$ or $p = ks$; z_0 is a simple root of $Q_{0,k}(c_0, z) \in \mathbb{C}[z]$ and a root of $Q_{0,ks}(c_0, z)$ with multiplicity s . \square

Remark 3.1.4. By the Proposition 3.1.3, we have

$$\check{\mathcal{X}}_{0,p} = \{(c, z) \in \mathbb{C}^2 \mid (c, z) \text{ satisfies the property (1) in Proposition 3.1.3}\}$$

For $\alpha = 2, 3$, define

$$C_{0,p,\alpha} = \{(c, z) \in \mathbb{C}^2 \mid (c, z) \text{ satisfies the property } (\alpha) \text{ in Proposition 3.1.3}\}$$

Then $C_{0,p,2}$, $C_{0,p,3}$ are finite set and $\mathcal{X}_{0,p} = \check{\mathcal{X}}_{0,p} \cup C_{0,p,2} \cup C_{0,p,3}$. So

$$\mathcal{X}_{0,p} = \{(c, z) \mid Q_{0,p}(c, z) = 0\}.$$

3.2 The smoothness of periodic curve

In this section, we will prove the smoothness of $\mathcal{X}_{0,p}$. The idea is to prove that some partial derivative of some defining function of $\overline{X_n}$ is non vanishing. Following A. Epstein, we will express this derivative as the coefficient of a quadratic differential of the form $(f_c)_* \mathcal{Q} - \mathcal{Q}$. Thurston's contraction principle gives $(f_c)_* \mathcal{Q} - \mathcal{Q} \neq 0$, therefore the non-nullness of our partial derivative.

3.2.1 Quadratic differentials and contraction principle

A meromorphic quadratic differential (or in short, a quadratic differential) \mathcal{Q} on \mathbb{C} takes the form $\mathcal{Q} = q \, dz^2$ with q a meromorphic function on \mathbb{C} .

We use $\mathcal{Q}(\mathbb{C})$ to denote the set of meromorphic quadratic differentials on \mathbb{C} whose poles (if any) are all simple. If $\mathcal{Q} \in \mathcal{Q}(\mathbb{C})$ and U is a bounded open subset of \mathbb{C} , the norm

$$\|\mathcal{Q}\|_U := \iint_U |q|$$

is well defined and finite.

For example

$$\left\| \frac{dz^2}{z} \right\|_{\{|z| < R\}} = \int_0^{2\pi} \int_0^R \frac{1}{r} dr d\theta = 2\pi R .$$

For $f : \mathbb{C} \rightarrow \mathbb{C}$ a non-constant polynomial and $\mathcal{Q} = q dz^2$ a meromorphic quadratic differential on \mathbb{C} , the pushforward $f_*\mathcal{Q}$ is defined by the quadratic differential

$$f_*\mathcal{Q} := Tq dz^2 \quad \text{with} \quad Tq(z) := \sum_{f(w)=z} \frac{q(w)}{f'(w)^2} .$$

If $\mathcal{Q} \in \mathcal{Q}(\mathbb{C})$, then $f_*\mathcal{Q} \in \mathcal{Q}(\mathbb{C})$ also.

The following lemma is a weak version of Thurston's contraction principle.

Lemma 3.2.1 (contraction principle). *For a non-constant polynomial f and a round disk V of radius large enough so that $U := f^{-1}(V)$ is relatively compact in V , we have*

$$\|f_*\mathcal{Q}\|_V \leq \|\mathcal{Q}\|_U < \|\mathcal{Q}\|_V, \quad \forall \mathcal{Q} \in \mathcal{Q}(\mathbb{C}).$$

Proof. The strict inequality on the right is a consequence of the fact that U is relatively compact in V . The inequality on the left comes from

$$\begin{aligned} \|f_*\mathcal{Q}\|_V &= \iint_{z \in V} \left| \sum_{f(w)=z} \frac{q(w)}{f'(w)^2} \right| |dz|^2 \\ &\leq \iint_{z \in V} \sum_{f(w)=z} \left| \frac{q(w)}{f'(w)^2} \right| |dz|^2 \\ &= \iint_{w \in U} |q(w)| |dw|^2 = \|\mathcal{Q}\|_U. \end{aligned}$$

□

Corollary 3.2.2. *If $f : \mathbb{C} \rightarrow \mathbb{C}$ is a polynomial and if $\mathcal{Q} \in \mathcal{Q}(\mathbb{C})$, then $f_*\mathcal{Q} \neq \mathcal{Q}$.*

Remark 3.2.1. Thurston's contraction principal says that if \mathcal{Q} is a meromorphic quadratic differential on \mathbb{P}^1 and $f : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ is a rational function, if one requires $f_*\mathcal{Q} = \mathcal{Q}$ with $\mathcal{Q} \neq 0$, then f is necessarily a Lattès example.

The formulas below appeared in [L] chapter 2, we write them together as a lemma.

Lemma 3.2.3 (Levin). *For $f = f_c$, we have*

$$\begin{cases} f_* \left(\frac{dz^2}{z} \right) = 0 \\ f_* \left(\frac{dz^2}{z-a} \right) = \frac{1}{f'(a)} \left(\frac{dz^2}{z-f(a)} - \frac{dz^2}{z-c} \right) & \text{if } a \neq 0 \\ f_* \left(\frac{dz^2}{(z-a)^2} \right) = \frac{dz^2}{(z-f(a))^2} - \frac{d-1}{af'(a)} \left(\frac{dz^2}{z-f(a)} - \frac{dz^2}{z-c} \right) & \text{if } a \neq 0. \end{cases} \quad (3.2.1)$$

3.2.2 The proof of smoothness

Lemma 3.2.4. *Given $z \in \mathbb{C}$, for $n \geq 0$ and $d \geq 2$, define $z_p : c \mapsto f_c^{\circ p}(z)$ and $\delta_p = f'_c(z_p) = dz_p^{d-1}$. Then*

$$\frac{dz_p}{dc} = 1 + \delta_{p-1} + \delta_{p-1}\delta_{p-2} + \dots + \delta_{p-1}\delta_{p-2} \cdots \delta_1.$$

Proof. From $z_p = z_{p-1}^d + c$, $d \geq 2$, we obtain

$$\frac{dz_p}{dc} = 1 + \delta_{p-1} \frac{dz_{p-1}}{dc} \quad \text{with} \quad \frac{dz_0}{dc} = 0.$$

The result follows by induction. □

The proof of smoothness of $\mathcal{X}_{0,p}$.

In section 3.1, we have seen $\mathcal{X}_{0,p} = \check{\mathcal{X}}_{0,p} \cup C_{0,p,2} \cup C_{0,p,3}$. So we will check case by case the smoothness of $\mathcal{X}_{0,p}$ on the points of the three sets.

Case 1. Firstly, consider a point $(c_0, z_0) \in \check{\mathcal{X}}_{0,p}$.

By term 1 of Proposition 3.1.3, $\frac{\partial Q_{0,p}}{\partial z}(c_0, z_0) \neq 0$. So $\mathcal{X}_{0,p}$ is smooth at this point.

Case 2. Consider a point $(c_0, z_0) \in C_{0,p,2}$.

By term 2 of Proposition 3.1.3, $\frac{\partial Q_{0,p}}{\partial z}(c_0, z_0) = 0$, so we must prove $\frac{\partial Q_{0,p}}{\partial c}(c_0, z_0) \neq 0$.

Since

$$\frac{\Phi_{0,p}}{\partial c}(c_0, z_0) = \frac{\partial Q_{0,p}}{\partial c}(c_0, z_0) \cdot \prod_{k|p, k < p} Q_{0,k}(c_0, z_0)$$

and in this case $\prod_{k|p, k < p} Q_{0,k}(c_0, z_0) \neq 0$, so we must prove:

$$\frac{\partial \Phi_{0,p}}{\partial c}(c_0, z_0) \neq 0.$$

For $m \geq 0$, inductively define $z_{m+1} = f_{c_0}(z_m)$ and define $\delta_m := f'_{c_0}(z_m)$. By lemma 3.2.4, we have

$$\frac{\partial \Phi_{0,p}}{\partial c}(c_0, z_0) = \frac{d}{dc}(f_c^{\circ p}(z_0) - z_0) \Big|_{c_0} = 1 + \delta_{p-1} + \delta_{p-1}\delta_{p-2} + \dots + \delta_{p-1}\delta_{p-2} \cdots \delta_1.$$

Now consider the following quadratic differential $\mathcal{Q} \in \mathcal{Q}(\mathbb{C})$

$$\mathcal{Q}(z) := \sum_{m=0}^{p-1} \frac{\rho_m}{z - z_m} dz^2, \quad \text{where } \rho_m = \delta_{p-1}\delta_{p-2} \cdots \delta_m.$$

Applying Lemma 3.2.3, and let $f := f_{c_0}$, we obtain

$$f_*\mathcal{Q}(z) = \sum_{m=0}^{p-1} \frac{\rho_m}{\delta_m} \left(\frac{dz^2}{z - z_{m+1}} - \frac{dz^2}{z - c_0} \right) = \mathcal{Q}(z) - \frac{\partial \Phi_p}{\partial c}(c_0, z_0) \cdot \frac{dz^2}{z - c_0}.$$

By Corollary 3.2.2, we have $f_*\mathcal{Q} \neq \mathcal{Q}$. So

$$\frac{\partial \Phi_{0,p}}{\partial c}(c_0, z_0) \neq 0.$$

Case 3. Finally, consider $(c_0, z_0) \in C_{0,p,3}$.

By term 3 of Proposition 3.1.3, $\frac{\partial Q_{0,p}}{\partial z}(c_0, z_0) = 0$, so we must prove $\frac{\partial Q_{0,p}}{\partial c}(c_0, z_0) \neq 0$. Let z_0 be a k periodic point of f_{c_0} ($k < p, k|p$), then the point (c_0, z_0) belongs to both $\mathcal{X}_{0,p}$ and $\mathcal{X}_{0,k}$. Therefore

$$\frac{\partial \Phi_{0,p}}{\partial z}(c_0, z_0) = \frac{\partial \Phi_{0,p}}{\partial c}(c_0, z_0) = 0.$$

we can't prove $Q_{0,p}$ has a non-trivial partial derivative at (c_0, z_0) by the partial derivative of $\Phi_{0,p}$.

Now we write $\Phi_{0,p}$ as

$$\Phi_{0,p}(c, z) = \Phi_{0,k}(c, z) \cdot P(c, z). \tag{3.2.2}$$

where $P(c, z) = \prod_{m|p, m \nmid k} Q_{0,m}(c, z)$. On one hand, since $Q_{0,p}(c_0, z_0) = 0$, we have

$$\frac{\partial P}{\partial c}(c_0, z_0) = \frac{\partial Q_{0,p}}{\partial c} \cdot \prod_{m|p, m \nmid k, m < p} Q_{0,m}(c_0, z_0).$$

On the other hand, for all $m < n, m \neq k$, $Q_{0,m}(c_0, z_0) \neq 0$, so we only need to prove

$$\frac{\partial P}{\partial c}(c_0, z_0) \neq 0.$$

Note that at this time, $\rho = [f_{c_0}^{\circ k}]'(z_0) \neq 1$. By the Implicitly Function Theorem, there exists holomorphic germ $\zeta : (\mathbb{C}, c_0) \rightarrow (\mathbb{C}, z_0)$ such that $Q_{0,k}(c, \zeta(c)) = 0$. In other words, $\zeta(c)$ is a k periodic point of f_c . Let ρ_c be the multiplier of f_c at $\zeta(c)$ and set

$$\dot{\rho} := \frac{d\rho_c}{dc} \Big|_{c_0}.$$

Lemma 3.2.5. *We have*

$$\frac{\partial P}{\partial c}(c_0, z_0) = \frac{s \cdot \dot{\rho}}{\rho(\rho - 1)}.$$

Proof. Differentiating the equation (??) with respect to z , and then evaluating at $(c, \zeta(c))$, we get:

$$\rho_c^s - 1 = (\rho_c - 1) \cdot P(c, \zeta(c)) + \underbrace{(f_c^k(\zeta(c)) - \zeta(c))}_{=0} \cdot \frac{\partial P}{\partial z}(c, \zeta(c)) = (\rho_c - 1) \cdot P(c, \zeta(c)).$$

Setting

$$R(c) := P(c, \zeta(c)) = \frac{\rho_c^s - 1}{\rho_c - 1},$$

we have

$$R'(c_0) = \frac{\partial P}{\partial c}(c_0, z_0) + \underbrace{\frac{\partial P}{\partial z}(c_0, z_0)}_{=0} \cdot \zeta'(c_0) = \frac{\partial P}{\partial c}(c_0, z_0).$$

Using $\rho^s = 1$ and $\rho^{s-1} = 1/\rho$, we deduce that

$$\frac{\partial P}{\partial c}(c_0, z_0) = \frac{d}{dc} \left(\frac{\rho_c^s - 1}{\rho_c - 1} \right) \Big|_{c_0} = \left(\frac{s\rho^{s-1}}{\rho - 1} - \frac{\rho^s - 1}{(\rho - 1)^2} \right) \frac{d\rho_c}{dc} \Big|_{c_0} = \frac{s \cdot \dot{\rho}}{\rho(\rho - 1)}. \quad \square$$

Therefore, we only need to prove $\dot{\rho} \neq 0$. The proof of this fact need to use a meromorphic quadratic differential with double poles along the orbit of z_0 .

Set $f := f_{c_0}$,

$$z_m := f^m(z_0), \quad \delta_m := dz_m^{d-1} = f'(z_m), \quad \zeta_m(c) := f_c^m(\zeta(c)) \quad \text{and} \quad \dot{\zeta}_m := \zeta'_m(c_0).$$

Then

$$\zeta_{m+1}(c) = f_c(\zeta_m(c)) \quad \text{and} \quad \zeta_k = \zeta_0.$$

Since

$$\delta_0 \delta_1 \cdots \delta_{k-1} = \rho \neq 0,$$

there is a unique k -tuple $(\mu_0, \dots, \mu_{k-1})$ such that

$$\mu_{m+1} = \frac{\mu_m}{dz_m^{d-1}} - \frac{d-1}{dz_m^d},$$

where the indices are considered to be modulo k .

Now consider the quadratic differential \mathcal{Q} (with double poles) defined by

$$\mathcal{Q} := \sum_{m=0}^{k-1} \left(\frac{1}{(z - z_m)^2} + \frac{\mu_m}{z - z_m} \right) dz^2.$$

Lemma 3.2.6 (Compare with [L]). *We have*

$$f_* \mathcal{Q} = \mathcal{Q} - \frac{\dot{\rho}}{\rho} \cdot \frac{dz^2}{z - c_0}.$$

Proof. By construction of \mathcal{Q} and the calculation of $f_* \mathcal{Q}$ in Lemma 3.2.3, the polar parts of \mathcal{Q} and $f_* \mathcal{Q}$ along the cycle of z_0 are identical. But $f_* \mathcal{Q}$ has an extra simple pole at the critical value c_0 with coefficient

$$\sum_{m=0}^{k-1} \left(-\frac{\mu_m}{dz_m^{d-1}} + \frac{d-1}{dz_m^d} \right) = -\sum_{m=0}^{k-1} \mu_{m+1}.$$

We need to show that this coefficient is equal to $-\frac{\dot{\rho}}{\rho}$.

Using $\zeta_{m+1}(c) = \zeta_m(c)^d + c$, we get

$$\dot{\zeta}_{m+1} = dz_m^{d-1} \dot{\zeta}_m + 1.$$

It follows that

$$\dot{\zeta}_{m+1} \mu_{m+1} - \mu_{m+1} = dz_m^{d-1} \dot{\zeta}_m \mu_{m+1} = \dot{\zeta}_m \mu_m - \frac{(d-1) \dot{\zeta}_m}{z_m}.$$

Therefore

$$\sum_{m=0}^{k-1} \mu_{m+1} = \sum_{m=0}^{k-1} \left(\dot{\zeta}_{m+1} \mu_{m+1} - \dot{\zeta}_m \mu_m + \frac{(d-1)\dot{\zeta}_m}{z_m} \right) = (d-1) \sum_{m=0}^{k-1} \frac{\dot{\zeta}_m}{z_m} = \frac{\dot{\rho}}{\rho},$$

where last equality is obtained by evaluating at c_0 of the logarithmic derivative of

$$\rho_c := \prod_{m=0}^{k-1} d\zeta_m^{d-1}(c). \quad \square$$

Lemma 3.2.7 (Epstein[E]). *We have $f_*\mathcal{Q} \neq \mathcal{Q}$.*

Proof. The proof rests again on the contraction principle, but we can not apply directly Lemma 3.2.1 since \mathcal{Q} is not integrable near the cycle $\langle z_0, \dots, z_{m-1} \rangle$. Consider a sufficiently large round disk V so that $U := f^{-1}(V)$ is relatively compact in V . Given $\varepsilon > 0$, we set

$$V_\varepsilon := \bigcup_{k=1}^m f^k(D(z_0, \varepsilon)) \quad \text{and} \quad U_\varepsilon := f^{-1}(V_\varepsilon).$$

When ε tends to 0, we have

$$\|f_*\mathcal{Q}\|_{V-V_\varepsilon} \leq \|\mathcal{Q}\|_{U-U_\varepsilon} = \|\mathcal{Q}\|_{V-V_\varepsilon} - \|\mathcal{Q}\|_{V-U} + \|\mathcal{Q}\|_{V_\varepsilon-U_\varepsilon} - \|\mathcal{Q}\|_{U_\varepsilon-V_\varepsilon}.$$

If we had $f_*\mathcal{Q} = \mathcal{Q}$, we would have

$$0 < \|\mathcal{Q}\|_{V-U} \leq \|\mathcal{Q}\|_{V_\varepsilon-U_\varepsilon}.$$

However, $\|\mathcal{Q}\|_{V_\varepsilon-U_\varepsilon}$ tends to 0 as ε tends to 0, which is a contradiction. Indeed, $\mathcal{Q} = q(z)dz^2$, the meromorphic function q is equivalent to $\frac{1}{(z-z_0)}$ as z tends to z_0 . In addition, since the multiplier of z_0 has modulus 1,

$$D(z_0, \varepsilon) \subset U_\varepsilon - V_\varepsilon \subset D(z_0, \varepsilon') \quad \text{with} \quad \frac{\varepsilon'}{\varepsilon} \xrightarrow{\varepsilon \rightarrow 0} 1.$$

Therefor,

$$\|\mathcal{Q}\|_{V_\varepsilon-U_\varepsilon} \leq \int_0^{2\pi} \int_\varepsilon^{\varepsilon'} \frac{1+o(1)}{r^2} r dr d\theta = 2\pi(1+o(1)) \log \frac{\varepsilon'}{\varepsilon} \xrightarrow{\varepsilon \rightarrow 0} 0$$

\square

The fact $\dot{\rho} \neq 0$ follows from the above two lemmas.

Remark 3.2.8. Define a projection $\pi_{0,p} : \mathcal{X}_{0,p} \longrightarrow \mathbb{C}$ mapping (c, z) to c . According to the smoothness of $\mathcal{X}_{0,p}$, $\pi_{0,p}$ is a branched cover of degree $\nu_d(p)$. Proposition 3.1.3 tell us $\pi_{0,p}$ has two kinds of critical points: $C_{0,p,2}$ and $C_{0,p,3}$. Each point in $C_{0,p,2}$ is a simple critical point of $\pi_{0,p}$. Let (c, z) be a point of $C_{0,p,3}$, where z is a k periodic point of f_c ($k|p, k < p$). Then (c, z) is a critical point of $\pi_{0,p}$ with multiplicity $\frac{p}{k} - 1$. Therefore, the set of critical value $V_{0,p}$ of $\pi_{0,p}$ is consisting of roots and co-roots of all hyperbolic components of period p .

3.3 The irreducibility of the periodic curves

Recall that f_c denote the polynomial $z \mapsto z^d + c$, where $d \geq 2$, and we have defined

$$\check{\mathcal{X}}_{0,p} := \{(c, z) \in \mathbb{C}^2 \mid z \text{ is a } p \text{ periodic point of } f_c \text{ and } [f_c^p]'(z) \neq 1\}.$$

The objective here is to prove:

Theorem 3.3.1. *For every $p \geq 1$, the set $\check{\mathcal{X}}_{0,p}$ is connected.*

It follows immediately that the closure of X_n in \mathbb{C}^2 is irreducible.

3.3.1 Kneading sequences

Set $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and let $\tau : \mathbb{T} \rightarrow \mathbb{T}$ be the angle map

$$\tau : \mathbb{T} \ni \theta \mapsto d\theta \in \mathbb{T}, d \geq 2.$$

We shall often make the confusion between an angle $\theta \in \mathbb{T}$ and its representative in $[0, 1[$. In particular, the angle $\theta/d \in \mathbb{T}$ is the element of $\tau^{-1}(\theta)$ with representative in $[0, 1/d[$ and the angle $(\theta + (d-1))/d$ is the element of $\tau^{-1}(\theta)$ with representative in $[(d-1)/d, 1[$.

Every angle $\theta \in \mathbb{T}$ has an associated kneading sequence $\nu(\theta) = \nu_1\nu_2\nu_3 \dots$ defined by

$$\nu_k = \begin{cases} 1 & \text{if } \tau^{k-1}(\theta) \in \left[\frac{\theta}{d}, \frac{\theta+1}{d} \right[, \\ 2 & \text{if } \tau^{k-1}(\theta) \in \left[\frac{\theta+1}{d}, \frac{\theta+2}{d} \right[, \\ \cdot & \\ \cdot & \\ \cdot & \\ d-1 & \text{if } \tau^{k-1}(\theta) \in \left[\frac{\theta+(d-2)}{d}, \frac{\theta+(d-1)}{d} \right[, \\ 0 & \text{if } \tau^{k-1}(\theta) \in \mathbb{T} \setminus \left[\frac{\theta}{d}, \frac{\theta+(d-1)}{d} \right] , \\ \star & \text{if } \tau^{k-1}(\theta) \in \left\{ \frac{\theta}{d}, \frac{\theta+1}{d}, \dots, \frac{\theta+(d-2)}{d}, \frac{\theta+(d-1)}{d} \right\} . \end{cases}$$

For example,

- as $d = 3$, $\nu(\frac{1}{7}) = \overline{12102\star}$ and $\nu(\frac{27}{28}) = \overline{22200\star}$;

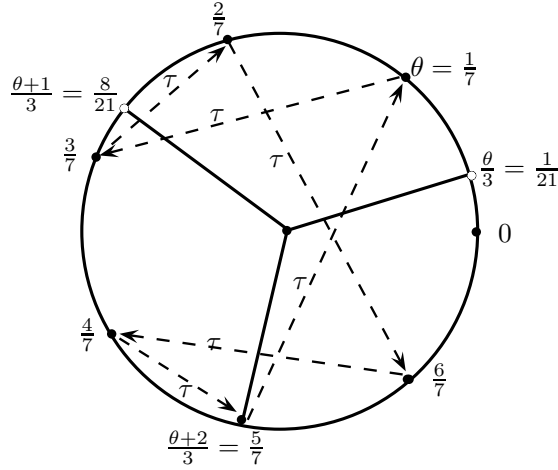


Figure 3-1: As $d = 3$, the kneading sequence of $\theta = 1/7$ is $\nu(1/7) = \overline{12102\star}$

We shall say that an angle $\theta \in \mathbb{T}$, periodic under τ , is *maximal in its orbit* if its representative in $[0, 1)$ is maximal among the representatives of $\tau^j(\theta)$ in $[0, 1)$ for all $j \geq 1$. If the period is n and the d-expansion ($d \geq 2$) of θ is $.\overline{\varepsilon_1 \dots \varepsilon_p}$, then θ is maximal in its orbit if and only if the periodic sequence $\overline{\varepsilon_1 \dots \varepsilon_p}$ is maximal (in the lexicographic order) among its shifts. For example, as $d = 4$, $\frac{5}{31} = .\overline{02211}$ is not maximal in its orbit but $\frac{20}{31} = .\overline{22110}$ is maximal in the same orbit.

The following lemma indicates cases where the d-expansion ($d \geq 2$) and the kneading sequence coincide.

Lemma 3.3.2 (Realization of kneading sequences). *Let $\theta \in \mathbb{T}$ be a periodic angle which is maximal in its orbit and let $\overline{\varepsilon_1 \dots \varepsilon_p}$ be its d-expansion ($d \geq 2$). Then, $\varepsilon_p \in \{0, 1, 2, \dots, d-2\}$ and the kneading sequence $\nu(\theta)$ is equal to $\overline{\varepsilon_1 \dots \varepsilon_{p-1} \star}$.*

For example,

- as $d = 3$ $\frac{13}{14} = \overline{.221001}$ and $\nu(\theta) = \overline{22100\star}$.
- as $d = 4$ $\frac{28}{31} = \overline{.32130}$ and $\nu(\theta) = \overline{3213\star}$.

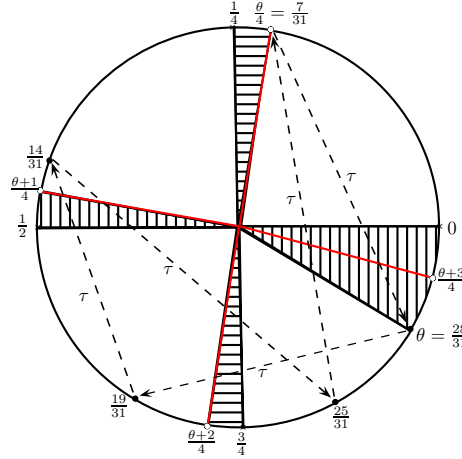


Figure 3-2: As $d = 4$, the kneading sequence of $\theta = 28/31$ is $\nu(28/31) = \overline{3213\star}$

Proof. Since θ is maximal in its orbit under τ , the orbit of θ is disjoint from $] \frac{\theta}{d}, \frac{1}{d}] \cup] \frac{\theta+1}{d}, \frac{2}{d}] \cup \dots \cup] \frac{\theta+(d-2)}{d}, \frac{d-1}{d}] \cup] \theta, 1]$. It follows that the orbit $\tau^j(\theta)$, $j = 0, 1, \dots, n-2$ have the same itinerary relative to the two partitions $\mathbb{T} - \{0, \frac{1}{d}, \frac{2}{d}, \dots, \frac{d-2}{d}, \frac{d-1}{d}\}$ and $\mathbb{T} - \{\frac{\theta}{d}, \frac{\theta+1}{d}, \dots, \frac{\theta+(d-2)}{d}, \frac{\theta+(d-1)}{d}\}$ (see Figure 5-2). The first one gives the d-expansion ($d \geq 2$) whereas the second gives the kneading sequence. Therefore, the kneading sequence of θ is $\overline{\varepsilon_1 \dots \varepsilon_{p-1} \star}$. Since $\tau^{p-1}(\theta) \in \tau^{-1}(\theta) = \{\frac{\theta}{d}, \frac{\theta+1}{d}, \dots, \frac{\theta+(d-1)}{d}\}$ and since $\frac{\theta+(d-1)}{d} \in]\theta, 1]$, we must have $\tau^{p-1}(\theta) = \{\frac{\theta}{d}, \frac{\theta+1}{d}, \dots, \frac{\theta+(d-2)}{d}\} < \frac{d-1}{d}$. So ε_p , as the first digit of $\tau^{p-1}(\theta)$, must be in $\{0, 1, 2, \dots, d-2\}$. \square

3.3.2 Cyclic expression of kneading sequence

$X = \{0, 1, \dots, d-1\} (d \geq 2)$ is an alphabet. X^* is the set of all sequence of symbols from X with finite length, that is,

$$X^* = \{\nu_1 \dots \nu_t | \nu_i \in X, t \in \mathbb{N}^*\}.$$

The element of X^* is called word, its length is denoted by $|\cdot|$. For any $w \in X^*$, w can be written as $u^n := \underbrace{u \dots u}_n$ with $u \in X^*$ and $n \geq 1$.

For example: $121212 = 12^3$, $1234 = 1234$.

Definition 3.3.3. A word is called primitive if it is not the form u^n for any $n > 1, u \in X^*$.

The following lemma is a basic result about primitive words due to F.W.Levi. One can refer to [KM] for the proof.

Lemma 3.3.4 (F.W.Levi). For each $w \in X^*$, there exists an unique primitive word $a(w)$ such that $w = a(w)^n$ for some $n \geq 1$.

$a(w)$ is called the primitive root of w , this lemma means the primitive root of a word is unique. Let w be a word, we denote by L_w the set of all words different from w only at the last digit.

Lemma 3.3.5. If w is a non-primitive word, then any word in L_w is primitive.

Proof. As w is not primitive, then $w = a^m$ where a is the primitive root of w and $m > 1$. w' is any element of L_w , then $w' = a^{m-1}a'$ for some $a' \in L_a$. Now assume w' is not primitive, then $w' = z^n$ where z is the primitive root of w' and $n > 1$. Obviously $|z| \neq |a|$.

If $|z| < |a|$, then $n > m \geq 2$ and $a = zb$ for some $b \in X^*$.

$$a^{m-1}a' = z^n \implies za^{m-1}a' = a^{m-1}a'z \implies za^{m-1}a' = zba^{m-2}a'z \implies$$

$$\exists v \in X^*, s.t. a = bv, |v| = |z| \implies a^{m-1}bv' = ba^{m-2}a'z(a' = bv') \implies$$

$$v' = z \text{ and } a^{m-1}b = ba^{m-2}a' \implies a^{m-2}bvb = ba^{m-2}a' \implies a' = vb.$$

It is a contradiction to $a = zb$.

If $|z| > |a|$, then there exists $z' \in L_z$ such that $z^{n-1}z' = a^m = w$ with $m > n \geq 2$. It reduces to the case above.

□

Now, let θ be a periodic angle with period $p \geq 2$. $\nu(\theta)$ is the kneading sequence of θ .

Definition 3.3.6. *If there is a word $w = \nu_1 \dots \nu_t$ such that $\nu(\theta) = \overline{w^{s-1}w_\star} := \underbrace{\overline{w \dots w}}_{s-1} w_\star$, where $w_\star = \nu_1 \dots \nu_{t-1}\star$ and t is a proper factor of p with $ts = p$, then $\nu(\theta)$ is called cyclic, otherwise $\nu(\theta)$ is called acyclic.*

Definition 3.3.7. $\nu(\theta) = \overline{w^{s-1}w_\star}$ is cyclic. If w is a primitive word, we call $\overline{w^{s-1}w_\star}$ a cyclic expression of $\nu(\theta)$.

The following proposition is a corollary of Lemma 3.3.4 and 3.3.5.

Proposition 3.3.8. *If $\nu(\theta)$ is cyclic, then its cyclic expression is unique.*

Proof. Assume $\overline{w^{s-1}w_\star}$ and $\overline{u^{l-1}u_\star}$ are two cyclic expression of $\nu(\theta)$ where $w = \nu_1 \dots \nu_t$ and $u = \epsilon_1 \dots \epsilon_m$. If $\nu_t = \epsilon_m$, then $w^s = u^l$. By Lemma 3.3.4, we have $w = u$. If $\nu_t \neq \epsilon_m$, then $w^s = u^{l-1}u'$ with some $u' \in L_u$, but this is a contradiction to Lemma 3.3.5. □

3.3.3 Dynamics of parabolic unimodel polynomial

Let $f_c(z) = z^d + c$ ($d \geq 2$) and M_d be the Multibrot set for this family of unimodel polynomials (Section 2.2.1).

If c is the root of some hyperbolic component and $c \neq \gamma_{M_d}(0)$, then two periodic parameter rays $R_{M_d}(\theta)$ and $R_{M_d}(\eta)$ land on c , we say θ and η are companion angles, and θ, η have the same period under τ . c is primitive if and only if the orbit of $R_{M_d}(\theta)$ and $R_{M_d}(\eta)$ under τ are distinct. In dynamical plane, the dynamic rays $R_c(\theta)$ and $R_c(\eta)$ land at a common point $x_1 := \gamma_c(\theta) = \gamma_c(\eta)$. This point is on the parabolic orbit of f_c with its immediate basin containing the critical value. $R_c(\theta)$ and $R_c(\eta)$ are adjacent to the Fatou component containing c and the curve $R_c(\theta) \cup R_c(\eta) \cup \{x_1\}$ is a Jordan curve that cuts the plane into two connected components: one component, denoted by V_1 , contains the critical value c ; the other component, denoted by V_0 , contains $R_c(0)$ and all points of parabolic cycle except x_1 . Since V_1 contains the critical value, its preimage $U_\star = f_c^{-1}(V_1)$ is connected and contains the critical point 0. It is bounded by the dynamical rays $R_c(\theta/d), \dots, R_c((\theta + d - 1)/d); R_c(\eta/d), \dots, R_c((\eta + d - 1)/d)$. Suppose $\theta > \eta$, and since each component of $\mathbb{C} \setminus \overline{U_\star}$ is conformally mapped to V_0 which is bounded by $R_c(\theta)$

and $R_c(\eta)$, it is easy to see that $R_c((\theta + m - 1)/d)$ and $R_c((\eta + m)/d)$ land on a common point which is one of the preimage of x_1 for $m \in \mathbb{Z}_d$. Denote U_m the component of $\mathbb{C} \setminus R_c((\theta + m - 1)/d) \cup \{\gamma_c((\eta + m)/d)\} \cup R_c((\eta + m)/d)$ disjoint with U_\star . See Figure 3-3 (primitive case) and Figure 3-4 (satellite case). Note that $f_c : U_m \rightarrow V_0$ is conformal.

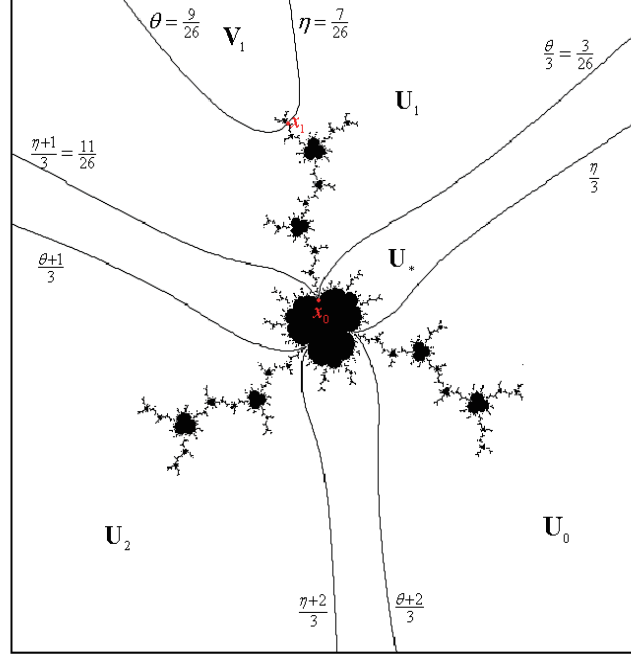


Figure 3-3: The dynamical plane of f_{c_0} . $c_0 := \gamma_{M_3}(7/26) = \gamma_{M_3}(9/26)$ is the root of some primitive hyperbolic component as illustrated in Figure 2-2. The dynamical rays $R_{c_0}(7/26)$ and $R_{c_0}(9/26)$ land on a common parabolic point of f_{c_0} with period 3.

If c is a co-root of some hyperbolic component, then exactly one period parameter ray $R_{M_d}(\beta)$ land on it (see Figure 2-3). In dynamical plane, $R_c(\beta)$ is the unique dynamical ray landing on a parabolic periodic point $\gamma_c(\beta) := x_1$, whose immediate basin contains the critical value c . The parameter c is a primitive parabolic parameter. Denote V_1 the union of Fatou component containing c and external ray $R_c(\beta)$, $V_0 = \mathbb{C} \setminus \overline{V_1}$, $U_\star = f_c^{-1}(V_1)$. U_{km} is the component of $f_c^{-1}(V_0)$ adjacent with $R_c((\beta + m - 1)/d)$ and $R_c((\beta + m)/d)$, $m \in \mathbb{Z}_d$. (see Figure 3-5).

Remark: in our paper, if c is a parabolic parameter, then f_c has unique parabolic orbit, denoted by $\{x_0, x_1, \dots, x_{k-1}\}$. x_1 is the point whose immediate basin contains critical value c .

The following lemma provides a criterion for θ such that $\gamma_{M_d}(\theta)$ is a primitive parabolic parameter.

Definition 3.3.9. Let θ be a periodic angle of period p and the d -expansion of θ be $\overline{\epsilon_1 \dots \epsilon_p}$.

We call $\epsilon_1 \dots \epsilon_p$ the periodic part of the d -expansion of θ .

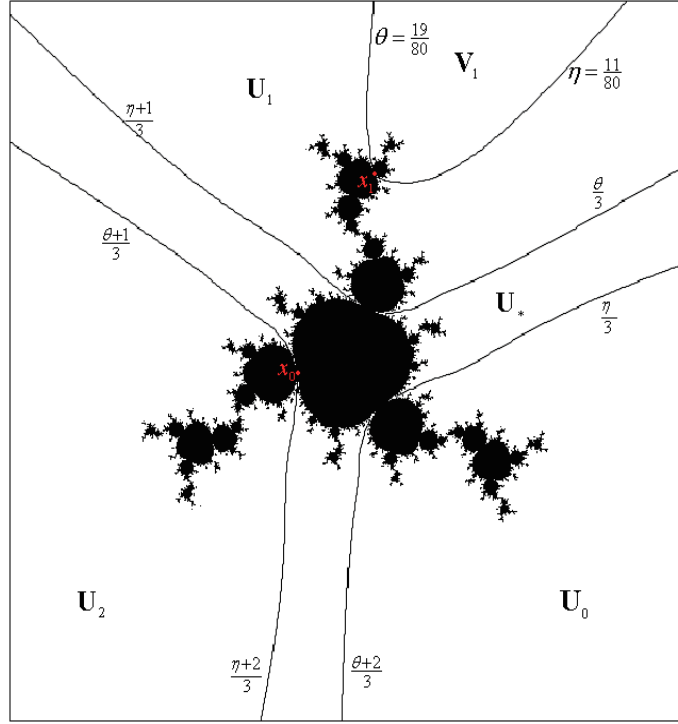


Figure 3-4: The dynamical plane of f_{c_1} . $c_1 := \gamma_{M_3}(11/80) = \gamma_{M_3}(19/80)$ is the root of some satellite hyperbolic component as illustrated in Figure 2-2. The dynamical rays $R_{c_1}(11/80)$ and $R_{c_1}(19/80)$ land on a common parabolic point of f_{c_1} with period 2.

Lemma 3.3.10. θ is periodic under τ with period $p \geq 2$. If $c_0 := \gamma_{M_d}(\theta)$ is the root of some satellite hyperbolic component, then θ satisfies the following properties:

- (1) $\nu(\theta)$ is cyclic.
- (2) Denote by $\overline{w^{s-1}w_\star}$ the cyclic expression of $\nu(\theta)$ where $w = \nu_1 \dots \nu_t$, t is a proper factor of n and $ts = p$. Then the last digit of the period part of the d -expansion of θ is ν_t or $\nu_t - 1$.

Moreover, if θ is maximal in its orbit, then $\nu(\theta)$ also satisfies

(3) t is the length of parabolic orbit and the last digit of the period part of the d -expansion of θ must be $\nu_t - 1 \in [0, d - 2]$.

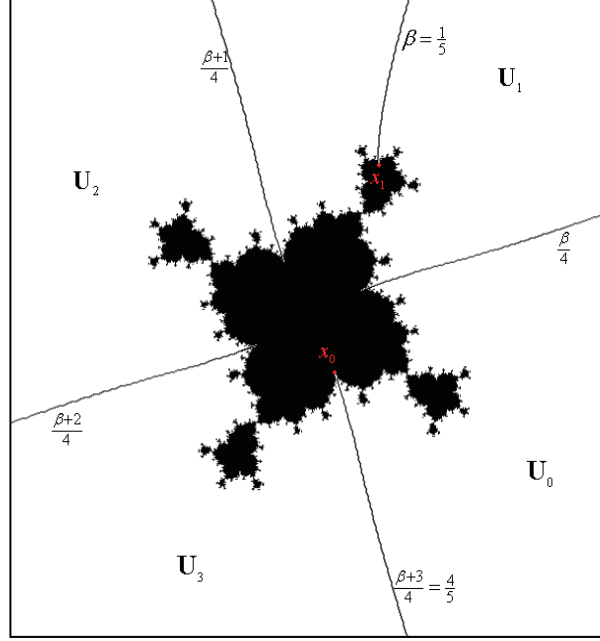


Figure 3-5: The dynamical plane of f_{c_0} . $c_0 := \gamma_{M_4}(1/5)$ is a co-root of the hyperbolic component illustrated in Figure 2-3. $R_{c_0}(1/5)$ is the unique dynamical ray landing on $\gamma_{c_0}(1/5)$ which is the parabolic point of f_{c_0} with period 2.

Proof. Let η be the companion angle of θ , then in dynamical plane of f_{c_0} , $R_{c_0}(\theta)$ and $R_{c_0}(\eta)$ land on x_1 (see Figure 3-4). As V_1 contains no points and external rays of the parabolic orbit, then $\{x_0, x_1, \dots, x_{k-1}\}$ together with their external rays belong to $\bigcup_{m=0}^{d-1} \overline{U}_m$.

For c_0 is satellite parabolic parameter, the length p of parabolic orbit is a proper factor of n and f_{c_0} acts on the rays of the orbit transitively. Then we have, in $\nu(\theta) = \overline{\nu_1 \dots \nu_{p-1} \star}$, $\nu_j = \nu_{j(\text{mod})k}$ for $1 \leq j \leq p-1$, that is, $\nu(\theta) = \overline{u^{l-1} u_\star}$ where $u = \nu_1 \dots \nu_k$. By definition of kneading sequence, we can see $\tau^{o(k-1)}(\theta) \in ((\theta + \nu_k - 1)/d, (\theta + \nu_k)/d)$. It follows x_0 together with its external rays belong to \overline{U}_{ν_k} . Then $\tau^{p-1}(\theta)$ is either $(\theta + \nu_k - 1)/d$ ($\theta > \eta$) or $(\theta + \nu_k)/d$ ($\theta < \eta$) (see Figure 3-6). So the last digit of d -expansion of θ is either $\nu_k - 1$ ($\theta > \eta$) or ν_k ($\theta < \eta$). Let $w = \nu_1 \dots \nu_t$ be the primitive root of u , then $u = w^{k/t}$. We have $\overline{w^{s-1} w_\star}$ is the cyclic expression of $\nu(\theta)$ (Proposition 3.3.8) and $\nu_t = \nu_k$, so θ satisfies property (1) and (2).

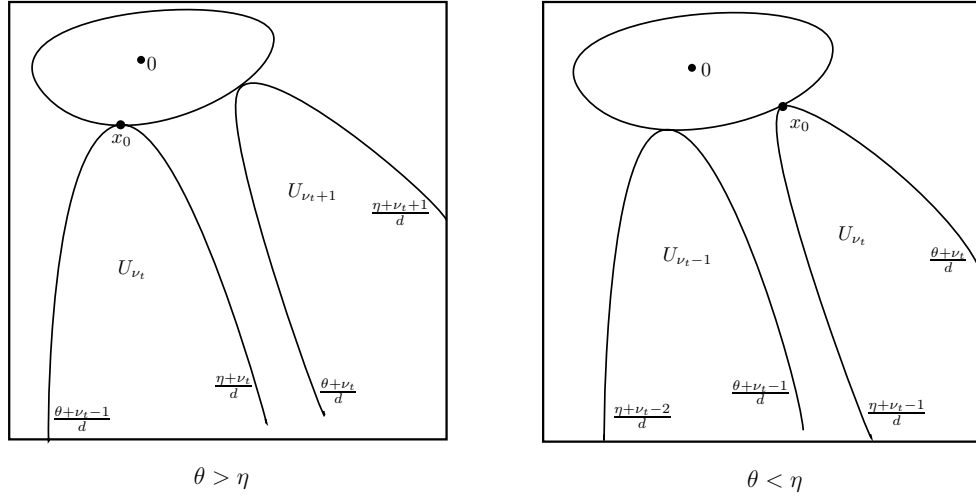


Figure 3-6:

Furthermore, if θ is maximal in its orbit, then $\theta > \eta$, so the last digit of the period part of the d -expansion of θ must be $\nu_t - 1$. By lemma 3.3.2, $\theta = \overline{.w^{s-1}\nu_1 \dots \nu_{t-1}(\nu_t - 1)}$ and $0 \leq \nu_t - 1 \leq d - 2$. Note that the angles of external rays belonging to x_1 are $\theta, \tau^k(\theta), \dots, \tau^{(s-1)k}(\theta)$ with the order $\theta > \tau^p(\theta) > \dots > \tau^{(s-1)p}(\theta)$. The maximum of θ implies η is the second largest angle in orbit of θ , then $\eta = \tau^k(\theta) = \overline{.u^{l-2}\nu_1 \dots \nu_{k-1}(\nu_k - 1)u}$. If u is not primitive, then $k/t > 1$. It follows $\tau^t(\theta) > \tau^k(\theta) = \eta$, a contradiction to that η is the second largest angle in orbit of θ . So u is a primitive word and hence $t = k$ is length of parabolic orbit.

□

Then once θ doesn't satisfy the property in this lemma, we have $\gamma_{M_d}(\theta)$ is a primitive parabolic parameter. The lemma below can be seen as an application of lemma 3.3.10.

Lemma 3.3.11. *Assume $\theta = \overline{.w^{s-1}\nu_1 \dots \nu_{t-1}(\nu_t - 1)}$ is maximal in its orbit, where $w = \nu_1 \dots \nu_t$ is primitive with $\nu_t \in [1, d - 1]$ and t is a proper factor of p with $ts = p$. Let*

$$\beta_{\nu_t-i} = \overline{.w^{s-1}\nu_1 \dots \nu_{t-1}(\nu_t - i)} \quad \text{for } 2 \leq i \leq \nu_t$$

$$\beta_{-1} = \begin{cases} \overline{.w^{s-1}\nu_1 \dots (\nu_{t-1} - 1)(d - 1)} & \text{as } t \geq 2 \\ \overline{.m \dots m(m - 1)(d - 1)} & \text{as } t = 1 \end{cases}$$

Then $\gamma_{M_d}(\beta_{\nu_t-i})$ is a primitive parabolic parameter for any $2 \leq i \leq \nu_t$. $\gamma_{M_d}(\beta_{-1})$ is a

satellite parabolic parameter for $\theta = \overline{.(d-1)\cdots(d-1)(d-2)}$ and a primitive parabolic parameter for any other case.

Proof. Let $\beta = \overline{.w^{s-1}\nu_1\cdots\nu_{t-1}j}$ be any angle among $\{\beta_{\nu_t-i}\}_{2\leq i\leq \nu_t}$, then $0 \leq j \leq \nu_t - 2$. The maximum of θ implies the maximum of β in its orbit. Since w is primitive, by lemma 3.3.2, we have $\overline{w^{s-1}w_\star}$ is the cyclic expression of $\nu(\beta)$. As $j \leq \nu_t - 2 < \nu_t - 1$, with the maximum of β , the property (3) in lemma 3.3.10 is not satisfied. So $\gamma_{M_d}(\beta)$ is a primitive parabolic parameter.

For β_{-1} , the maximum of θ implies β_{-1} is greater than $\tau(\beta_{-1}), \tau^2(\beta_{-1}), \dots, \tau^{p-2}(\beta_{-1})$ but less than $\tau^{p-1}(\beta_{-1})$. It follows $\nu(\beta) = \begin{cases} \overline{w^{s-1}\nu_1\cdots\nu_{t-1}\star} & \text{as } t \geq 2 \\ \overline{m\cdots mm\star} & \text{as } t = 1 \end{cases} = \overline{w^{s-1}w_\star}$. It is the cyclic expression of $\nu(\beta)$, then if β satisfies the property in lemma 3.3.10, ν_t is either 0 or $d-1$. Since $1 \leq \nu_t \leq d-1$, we have ν_t must be $d-1$, then the maximum of θ implies $\theta = \overline{.(d-1)\cdots(d-1)(d-2)}$. So $\gamma_{M_d}(\beta_{-1})$ is a primitive parabolic parameter as long as $\theta \neq \overline{.(d-1)\cdots(d-1)(d-2)}$. In the case of $\theta = \overline{.(d-1)\cdots(d-1)(d-2)}$, we will see in lemma 3.3.13 that $\gamma_{M_d}(\theta)$ is the root of a hyperbolic component attached to the main cardioid and β_{-1} is the companion angle of θ . In this case, $\gamma_{M_d}(\beta_{-1})$ is a satellite parabolic parameter.

□

Remark. In this lemma, we distinguish β_{-1} according to whether $t \geq 2$ or $t = 1$. It is because that we don't find a uniform expression of β_{-1} for the two cases rather than the case of $t = 1$ is special.

3.3.4 Itineraries outside the Multibrot set

If $c \in \mathbb{C} \setminus M_d$, the Julia set of f_c is a Cantor set. If $c \in R_{M_d}(\theta)$ with $\theta \neq 0$ not necessarily periodic, then the dynamical rays $R_c(\theta/d) \dots R_c((\theta + d - 1)/d)$ bifurcate on the critical point. The set $R_c(\theta/d) \cup \dots \cup R_c((\theta + d - 1)/d) \cup \{0\}$ separates the complex plane in d connected components. We denote by U_0 the component containing the dynamical ray $R_c(0)$ and by U_1, \dots, U_{d-1} the other component in counterclockwise (see Figure 3-7).

The orbit of a point $x \in K_c$ has an itinerary with respect to this partition. In other words, to each $x \in K_c$, we can associate a sequence in $\mathbb{Z}_d^{\mathbb{N}}$ whose j -th entry is equal to k

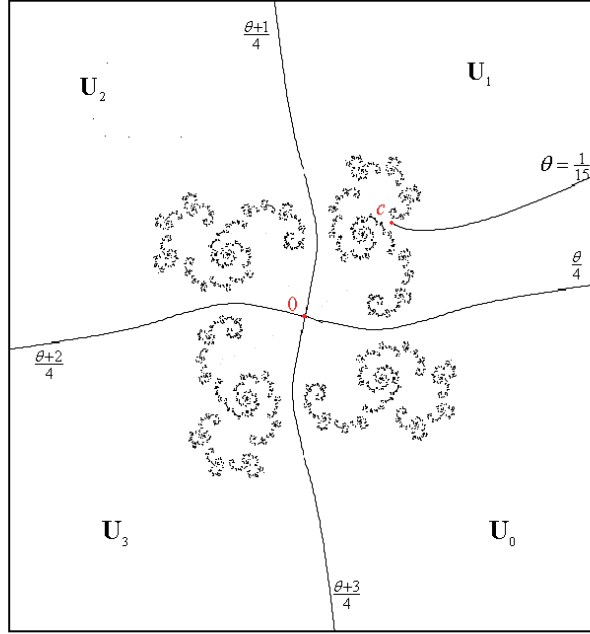


Figure 3-7: The regions U_0 , U_1 , U_2 , U_3 for a parameter c belonging to $R_{M_4}(1/15)$.

if $f_c^{\circ j-1}(x) \in U_k$. This gives a map $\iota_c : K_c \rightarrow \mathbb{Z}_d^{\mathbb{N}}$. Moreover, ι_c is a bijection for any $c \in \mathbb{C} \setminus M_d$.

In $\mathbb{Z}_d^{\mathbb{N}}$, we can define a *shift* which maps $\epsilon_1 \epsilon_2 \epsilon_3 \dots$ to $\epsilon_2 \epsilon_3 \epsilon_4 \dots$. A sequence in $\mathbb{Z}_d^{\mathbb{N}}$ is called (n, p) -preperiodic if it is preperiodic under shift with preperiod n and period p . It is known that for c outside Multibrot set M_d , the dynamic of f_c on K_c is conjugate to shift on $\mathbb{Z}_d^{\mathbb{N}}$ via the map ι_c . In particular, z is a (n, p) -preperiodic point of f_c if and only if $\iota_c(z)$ is a (n, p) -preperiodic sequence in $\mathbb{Z}_d^{\mathbb{N}}$.

Proposition 3.3.12. *Let $\overline{\epsilon_1 \dots \epsilon_{p-1} \star}$ be the kneading sequence of a periodic angle θ with period $n \geq 2$. If $c_0 := \gamma_{M_d}(\theta)$ is a primitive parabolic parameter and if one follows continuously the periodic points of period n of f_c as c makes a small turn around c_0 , then the periodic points with itineraries $\overline{\epsilon_1 \dots \epsilon_{p-1} m}$ and $\overline{\epsilon_1 \dots \epsilon_{p-1} (m+1)}$ get exchanged where $m \in \mathbb{Z}_d$ is the last digit of the period part of the d -expansion of θ .*

Proof. Since c_0 is a primitive parabolic parameter, then the periodic point $x_1 := \gamma_{c_0}(\theta)$ has period n and multiplier 1. According to Case 2 in the proof of smoothness and lemma 2.2.1, the projection from a small neighborhood of (c_0, x_1) in X_n to the first coordinate is a degree

2 covering. So the neighborhood of (c_0, x_1) in $\overline{X_n}$ can be written as

$$\{(c_0 + \delta^2, x(\delta)), (c_0 + \delta^2, x(-\delta)) \mid |\delta| < \varepsilon\}$$

where $x : (\mathbb{C}, 0) \rightarrow (\mathbb{C}, x_1)$ is a holomorphic germ with $x'(0) \neq 0$. In particular, the pair of periodic points for f_c which are splitted from x_1 get exchanged when c makes a small turn around c_0 . So, using analytic continuation on $\mathbb{C} \setminus (M_d \cup R_{M_d}(0))$, it is enough to show that there exists a $c \in \mathbb{C} \setminus M_d$ close to c_0 such that $x(\pm\sqrt{c - c_0})$ have itineraries $\overline{\varepsilon_1 \dots \varepsilon_{p-1} m}$ and $\overline{\varepsilon_1 \dots \varepsilon_{p-1} (m+1)}$ where $m \in \mathbb{Z}_d$ is the last digit of the period part of the d -expansion of θ .

Let us denote by $V_0(c_0)$, $V_1(c_0)$, $U_0(c_0), \dots, U_{d-1}(c_0)$ and $U_\star(c_0)$ the sets defined in the previous section. For $j \geq 0$, set $x_j := f_{c_0}^j(x_0)$ and observe that for $j \in [1, n-1]$, we have $x_j \in U_{\varepsilon_j}(c_0)$.

For $c \in R_{M_d}(\theta)$, consider the following compact subsets of the Riemann sphere :

$$R(c) := R_c(\theta) \cup \{c, \infty\} \quad \text{and} \quad S(c) := R_c(\theta/d) \cup \dots \cup R_c((\theta + d - 1)/d) \cup \{0, \infty\}.$$

Denote by $U_0(c)$ the component of $\mathbb{C} \setminus S(c)$ containing $R_c(0)$ and by $U_1(c), \dots, U_{d-1}(c)$ the other component in counterclockwise. From any sequence $\{c_n\} \subset R_{M_d}(\theta)$ converging to c_0 , by extracting a subsequence if necessary, we can assume $R(c_n)$ and $S(c_n)$ converge respectively, for the Hausdorff topology on compact subsets of $\mathbb{C} \cup \{\infty\}$, to connected compact sets R and S . Since $S(c) = f_c^{-1}(R(c))$, we have $S = f_{c_0}^{-1}(R)$. According to [PR, Section 2 and 3], $R \cap (\mathbb{C} \setminus K_{c_0}) = R_{c_0}(\theta)$, the intersection of R with the boundary of K_{c_0} is reduced to $\{x_1\}$ and the intersection of R with the interior of K_{c_0} is contained in the immediate basin of x_1 , whence in V_1 . It follows $R \subset \overline{V_1}(c_0)$ and $S \subset \overline{U}_\star(c_0)$, that means any compact subset of $\mathbb{C} \setminus \overline{U}_\star(c_0)$ is contained in $\mathbb{C} \setminus S(c_n)$ for n sufficiently large .

For $j \in [1, p-1]$ and let D_j be a sufficiently small disk around x_j so that

$$\overline{D_j} \subset U_{\varepsilon_j}(c_0) \subset \mathbb{C} \setminus \overline{U}_\star(c_0).$$

According to the previous discussion, if n is sufficiently large, we have

$$f_{c_n}^{j-1}(x(\pm\sqrt{c_n - c_0})) \subset D_j \subset U_{\varepsilon_j}(c_n).$$

So the first $p-1$ symbols of the itineraries of $x(\pm\sqrt{c_n - c_0})$ are all $\epsilon_1, \dots, \epsilon_{p-1}$. As $x(\sqrt{c_n - c_0})$ and $x(-\sqrt{c_n - c_0})$ are different p periodic points of f_{c_n} , their itineraries must be different. It follows $f_{c_n}^{p-1}(x(\pm\sqrt{c_n - c_0}))$, which are splitted from x_0 , lie in different component of $\mathbb{C} \setminus S(c_n)$. Combining with the fact that $R_{c_0}((\theta + m)/d)$ lands on x_0 (m is the last digit of the period part of the d -expansion of θ), we have $f_{c_n}^{p-1}(x(\pm\sqrt{c_n - c_0}))$ belong to $U_m(c_n)$ and $U_{m+1}(c_n)$ respectively, then $x(\pm\sqrt{c_n - c_0})$ have itineraries $\overline{\epsilon_1 \dots \epsilon_{p-1} m}$ and $\overline{\epsilon_1 \dots \epsilon_{p-1} (m+1)}$ respectively.

□

Lemma 3.3.13. *For $\theta = 1 - 1/(d^p - 1) = \overline{(d-1) \dots (d-1)(d-2)}$ ($p \geq 2$), we have $\gamma_{M_d}(\theta)$ is the root of some periodic p hyperbolic component attached to the main cardioid. If η is denoted the companion angle of θ , then $\eta = d\theta - d + 1$.*

Proof. Let $c_0 := \gamma_{M_d}(\theta)$, then $x_1 := \gamma_{c_0}(\theta)$ is the parabolic periodic point of f_{c_0} as previous. By lemma 3.3.2, $\nu(\theta) = \overline{(d-1) \dots (d-1)\star}$, so $\overline{(d-1) \dots (d-1)\star}$ is the cyclic expression of $\nu(\theta)$. If $x_0 \neq x_1$, then the length of parabolic orbit is greater than 1. It implies the property (3) in lemma 3.3.10 is not satisfied, so c_0 is a primitive parabolic parameter. According to proposition 3.3.12, when $c \in \mathbb{C} \setminus M_d$ is close to c_0 , x_1 splits into two n periodic point y, z of f_c with itineraries $\overline{(d-1) \dots (d-1)(d-2)}$ and $\overline{(d-1) \dots (d-1)(d-1)}$. It leads to a contradiction to the period n of y and z . So $x_0 = x_1$ and then c_0 is the root of some periodic n satellite hyperbolic component attached to the main cardioid.

By the maximum of θ , we have U_{d-1} is bounded by $R_{c_0}((\theta + d - 2)/d)$ and $R_{c_0}((\eta + d - 1)/d)$. $\nu(\theta) = \overline{(d-1) \dots (d-1)\star}$ implies $R_{c_0}(\theta) \subset \overline{U}_{d-1}$, then $\theta \leq (\eta + d - 1)/d$ and x_0 is on the boundary of U_{d-1} . On the other hand, $(\eta + d - 1)/d$ is in the orbit of θ , so $\theta \geq (\eta + d - 1)/d$. Then we have $\eta = d\theta - d + 1$.

□

Remark. The dynamical rays $R_{c_0}(\theta)$ and $R_{c_0}(\eta)$ are consecutive among the rays landing at x_0 . Lemma 3.3.13 implies $R_{c_0}(\theta)$ is mapped to $R_{c_0}(\eta)$. It follows that each dynamical ray landing at x_0 is mapped to the one which is once further clockwise.

Proposition 3.3.14. *Let $\theta = 1 - 1/(d^p - 1) = \overline{(d-1) \dots (d-1)(d-2)}$ be periodic with period $n \geq 2$. If one follows continuously the periodic points of period n of f_c as c makes a*

small turn around $\gamma_{M_d}(\theta)$, then the periodic points in the cycle of $\iota_c^{-1}((d-1) \cdots (d-1)(d-2))$ get permuted cyclically.

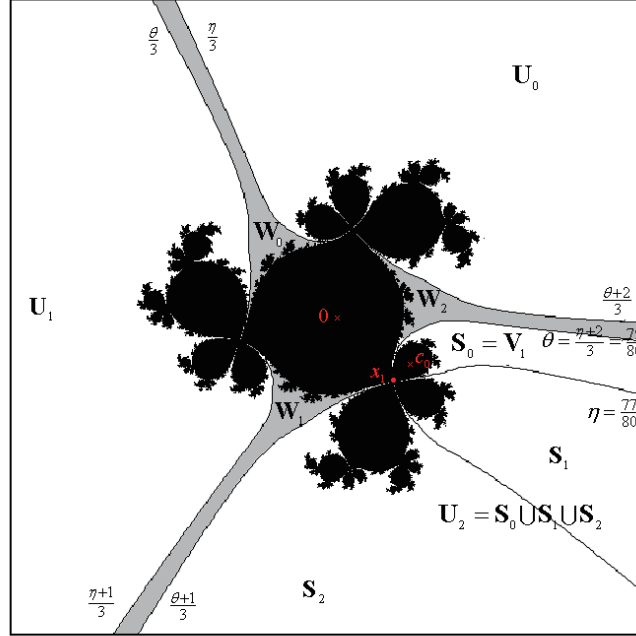


Figure 3-8: The dynamical plane of f_{c_0} . $c_0 := \gamma_{M_3}(\theta)$ with $\theta = .\overline{2221}$

Proof. Set $c_0 := \gamma_{M_d}(\theta)$. By Lemma 3.3.13, all the dynamical rays $R_{c_0}(\tau^j(\theta))$ land on a common fixed point x_0 . This fixed point is parabolic and the companion angle of θ , denoted by η , equals to $d\theta - (d-1) \equiv d\theta \pmod{\mathbb{Z}}$. $V_1(c_0) \subset U_{d-1}(c_0)$ which is bounded by $R_{c_0}((\theta + d - 2)/d)$ and $R_{c_0}(\theta)$.

According to Case 3 in the proof of smoothness and lemma 2.2.1, we have the projection from a small neighborhood of (c_0, x_0) in X_p to the parameter plane is a degree p covering. Then the neighborhood of (c_0, x_0) in $\mathcal{X}_{0,p}$ can be written as

$$\{(c_0 + \delta^p, x(\delta)), (c_0 + \delta^p, x(\omega\delta)), \dots, (c_0 + \delta^p, x(\omega^{p-1}\delta)) \mid |\delta| < \varepsilon\}$$

where $x : (\mathbb{C}, 0) \rightarrow (\mathbb{C}, x_0)$ is a holomorphic germ satisfying $x'(0) \neq 0$. So, for c close to c_0 , the set $x\{\sqrt[p]{c - c_0}\}$ is a cycle of period p of f_c , and when c makes a small turn around c_0 , the periodic points in the cycle $x\{\sqrt[p]{c - c_0}\}$ get permuted cyclically. So, combining with analytic continuation on $\mathbb{C} \setminus (M_d \cup R_{M_d}(0))$, it is enough to show there exists a $c \in \mathbb{C} \setminus M_d$

close enough to c_0 such that the point $\iota_c^{-1}(\overline{(d-1) \cdots (d-1)(d-2)})$ belongs to $x\{\sqrt[p]{c-c_0}\}$. Equivalently, we must show that there is a sequence $\{c_j\} \subset \mathbb{C} \setminus M_d$ converging to c_0 , such that the periodic point $y_j := \iota_{c_j}^{-1}(\overline{(d-1) \cdots (d-1)(d-2)})$ converges to x_0 .

Let $\{c_j\} \subset R_{M_d}(\theta)$ converge to c_0 as $j \rightarrow \infty$. Without loss of generality, we may assume that the sequence y_j converges to a point z , $R(c_j)$ converges to R and $S(c_j)$ converges to S in Hausdoff topology. The definition of $R(c)$, $S(c)$, $U_0(c), \dots, U_{d-1}(c)$ are in the proof of proposition 3.3.12. As (c_0, z) is on $\mathcal{X}_{0,p}$, then z is either the parabolic fixed point or repelling p periodic point of f_{c_0} .

Suppose z is a repelling p periodic point, set $z_i := f_{c_0}^i(z)$. Now we will define a new sequence of open domain $\{W_m(c_0)\}$. $W_m(c_0)$ is the connected component of $U_\star(c_0) \setminus$ the closure of Fatou component containing 0, adjacent with $U_m(c_0), U_{m+1}(c_0)$ (see Figure 3-8). According to [PR, Section 2 and 3], $R \cap (\mathbb{C} \setminus K_{c_0}) = R_{c_0}(\theta)$, the intersection of R with the boundary of K_{c_0} is reduced to $\{x_0\}$ and the intersection of R with the interior of K_{c_0} is contained in the immediate basin of x_0 . It follows $\{z_0, \dots, z_{p-1}\} \cap S = \emptyset$. Then for j sufficiently large, $\{z_0, \dots, z_{p-1}\} \subset \mathbb{C} \setminus S_{c_j}$. As y_j has itineraries $\overline{(d-1) \cdots (d-1)(d-2)}$, we have $\{z_0, \dots, z_{p-2}\} \subset U_{d-1}(c_0) \cup \overline{W_{d-1}(c_0)}$, $z_{p-1} \in U_{d-2}(c_0) \cup \overline{W_{d-2}(c_0)}$.

Claim 1. $z_{p-1} \notin \overline{W_{d-2}(c_0)}$.

Proof. In $J(f_{c_0})$, x_0 is the unique periodic point with more than one external rays landing on it (refer to [Poi2, proposition 3.3]). So there is exactly one external ray landing on z_{p-1} with period n . Its angle is denoted by $\frac{a}{d^p - 1}$, a is a integer. If $z_{p-1} \in \overline{W_{d-2}}$, the angle of external ray belonging to z_{p-1} satisfy

$$\frac{\eta + d - 2}{d} < \frac{a}{d^n - 1} < \frac{\theta + d - 2}{d} \quad (\theta = 1 - \frac{1}{d^p - 1}, \eta = d\theta - d + 1).$$

by simple computation, we have

$$\frac{k(d^p - 1)}{d - 1} - d^{p-1} - 1 + \frac{1}{d} < a < \frac{k(d^p - 1)}{d - 1} - d^{p-1},$$

a contradiction to a is an integer. This ends the proof of claim 1.

Claim 2. $z_{p-1} \notin U_{d-2}(c_0)$.

Proof. If $z_{p-1} \in U_{d-2}(c_0)$, we label the sectors at x_0 by $S_i (0 \leq i \leq p-1)$ clockwise with

$S_0 = V_1(c_0)$. The dynamics between these sectors satisfy

$$V_1(c_0) = S_0 \xrightarrow{f_{c_0}} S_1 \xrightarrow{f_{c_0}} \cdots \xrightarrow{f_{c_0}} S_{p-2} \xrightarrow{f_{c_0}} S_{p-1} = \mathbb{C} \setminus \overline{U}_{d-1}(c_0)$$

As $\{z_0, \dots, z_{p-2}\} \subset U_{d-1}(c_0) \cup \overline{W}_{d-1}(c_0)$, we have $z_0 = f_{c_0}(z_{p-1})$ belongs to the union of $\overline{W}_{d-1}(c_0)$ and $\bigcup_{i=1}^{p-2} S_i$. If $z_0 \in S_{i_0}$ ($1 \leq i_0 \leq p-2$), then $f_{c_0}^{(p-2-i_0)}(z_0) = z_{p-2-i_0} \in f_{c_0}^{(p-2-i_0)}(S_{i_0}) = S_{p-2}$. It follows $f_{c_0}(z_{p-2-i_0}) = z_{p-1-i_0}$ must belong to $\overline{W}_{d-1}(c_0)$. So $z_{p-i_0} \in S_0$ and $f_{c_0}^{(i_0-1)}(z_{p-i_0}) = z_{p-1} \in f_{c_0}^{(i_0-1)}(S_0) = S_{i_0-1}$, contradiction to $z_{p-1} \in U_{d-2}$. If $z_0 \in \overline{W}_{d-1}(c_0)$, then $z_1 \in S_0$. We have $f_{c_0}^{(p-2)}(z_1) = z_{p-1} \in f_{c_0}^{(p-2)}(S_0) = S_{p-2}$, also a contradiction to $z_{p-1} \in U_{d-2}(c_0)$. This ends the proof of claim 2.

The two claim imply the assumption that z is repelling p periodic point is false and then z must be a parabolic fixed point of f_{c_0} , that is $z = x_0$. \square

3.3.5 Proof of Theorem 3.3.1

Fix $n > 1$ (the case $n = 1$ has been treated directly at the beginning). We proceed to show that X_n is connected.

Set $X := \mathbb{C} \setminus (M_d \cup R_{M_d}(0))$ and $F_p := \mathbb{C} \setminus V_{0,p}$. Take any pair of points $(a, w), (a', w')$ in $\check{\mathcal{X}}_{0,p}$. By analytic continuation, we may assume $a, a' \in X$. Again by analytic continuation on simply connected open set X , we may assume $a = a'$. Thus it is enough to show that there exists a loop in F_p based on a such that the analytic continuation along the loop connects w and w' . We will give a algorithm to find such a loop.

Let z be any p periodic point of f_a .

step 1 In the orbit of z , there is a point with maximal itineraries among the shift of $\iota_a(z)$ in the lexicograph order, denoted by $\overline{\epsilon_1 \dots \epsilon_p}$. Set $\theta = \overline{\epsilon_1 \dots \epsilon_p}$ (θ is maximal in its orbit). If θ satisfies the properties in lemma 3.3.10, do step 2 below. Otherwise, $\gamma_{M_d}(\theta)$ is a primitive parabolic parameter. According to lemma 3.3.2 and proposition 3.3.12, when a makes a turn around $\gamma_{M_d}(\theta)$, the periodic point of f_a with itineraries $\overline{\epsilon_1 \dots \epsilon_p}$ and $\overline{\epsilon_1 \dots (\epsilon_p + 1)}$ get changed. Then z is connected to a new orbit containing $\iota_a^{-1}(\overline{\epsilon_1 \dots (\epsilon_p + 1)})$. For this new orbit, repeat doing step 1.

step 2 $\theta = \overline{\epsilon_1 \dots \epsilon_p}$ is maximal in its orbit and satisfies the properties in lemma 3.3.10. If $\theta = \overline{(d-1) \dots (d-1)(d-2)}$, step 2 ends. Otherwise, let $\overline{w^{s-1}w_\star}$ be the cyclic expression of $\nu(\theta)$ where $w = \nu_1 \dots \nu_t$, $\nu_t \in [1, d-1]$. As in lemma 3.3.11, we obtain

a sequence of angles $\{\beta_{\nu_t-2}, \dots, \beta_0, \beta_{-1}\}$ and know that $\gamma_{M_d}(\beta_{\nu_t-i})$ is a primitive parabolic parameter with $\nu(\theta) = \overline{\epsilon_1 \dots \epsilon_{p-1} \star}$ for any $i \in [2, \nu_t+1]$. Then by proposition 3.3.12 again, as a makes a turn around $\gamma_{M_d}(\beta_{\nu_t-i})$ ($2 \leq i \leq \nu_t+1$), the periodic points of f_a with itineraries $\overline{\epsilon_1 \dots \epsilon_{p-1}(\nu_t-i)}$ and $\overline{\epsilon_1 \dots \epsilon_{p-1}(\nu_t-i+1)}$ get changed. Then let a makes turns around from $\gamma_{M_d}(\beta_{\nu_t-2})$ to $\gamma_{M_d}(\beta_{-1})$ one by one, we have $\iota_a^{-1}(\overline{\epsilon_1 \dots \epsilon_{p-1} \epsilon_p})$ are connected with $\iota_a^{-1}(\overline{\epsilon_1 \dots \epsilon_{p-1}(d-1)})$ by analytic continuation through the points $\iota_a^{-1}(\overline{\epsilon_1 \dots \epsilon_{p-1}(\epsilon_p-1)}), \dots, \iota_a^{-1}(\overline{\epsilon_1 \dots \epsilon_{p-1}0})$. For the new periodic point $\iota_a^{-1}(\overline{\epsilon_1 \dots \epsilon_{p-1}(d-1)})$, do step 1.

Every time a n periodic point of f_a passes through step 1 or step 2, the sum of all digits in the itineraries of the output periodic point is greater than that of the input one. For fixed p , this sum is bounded (the bound is $(d-1)p-1$), then each p periodic point z can be connected to the orbit containing $\iota_a^{-1}(\overline{(d-1) \dots (d-1)(d-2)})$.

In our case, applying the procedure above to w and w' , we have w and w' are connected to two points of the periodic orbit containing $\iota_a^{-1}(\overline{(d-1) \dots (d-1)(d-2)})$. Proposition 3.3.14 tells us, by analytic continuation, any two point in this orbit can be connected as long as a makes the appropriate number of turns around $\gamma_{M_d}(1 - \frac{1}{d^p-1})$. Thus w and w' are connected. \square

3.4 The Galois group of $Q_{0,p}$

We apply the discussion in Section 2.4.2 to polynomial $Q_{0,p}$. The first term of Proposition 3.1.3 ensure that $Q_{0,p}(c, z)$, as a polynomial of $\mathbb{K}[z]$, has only simple roots. Then the splitting of $Q_{0,p}(c, z)$ over \mathbb{K} is a Galois extension over \mathbb{K} . So we can define the Galois group of $Q_{0,p}$, denoted by $G_{0,p}$.

It is easy to see that any element of $G_{0,p}$ commutes with f_c . We want to prove no other restrictive properties for $G_{0,p}$, in other words,

Theorem 3.4.1. *The Galois group $G_{0,p}$ of $Q_{0,p}(c, z)$ is consisting all permutations between the roots of $Q_{0,p}(c, z) \in \mathbb{K}[z]$ that can commute with f_c .*

Proof. It is enough to prove that any permutation between the roots of $Q_{0,p}(c, z)$ that commute with f_c must belong to $G_{0,p}$. We will use the equivalent expression of $G_{0,p}$ stated

in Section 2.4.2:

$$AC_{0,p} := AC(f_{c_0}) \cong G_{0,p}. \quad c_0 \in \mathbb{C} \setminus M_d$$

Let $V_{0,p} = \{c_i \mid 1 \leq i \leq \nu_d(p)(d-1)/d\}$. For $1 \leq i \leq \nu_d(p)(d-1)/d$, denote by γ_i a closed Jordan curve based at c_0 which rotates around c_i counter clockwise. If we don't distinguish a curve and the homotopy class of the curve, then

$$\pi_1(\mathbb{C} \setminus V_{0,p}, c_0) = \langle \gamma_i \mid 1 \leq i \leq \nu_d(p)(d-1)/d \rangle.$$

So by the notation and discussion in Section 2.4.2,

$$AC_{0,p} = \langle \sigma_{0,p}^{\gamma_i} := \phi_{0,p}(\gamma_i) \mid 1 \leq i \leq \nu_d(p)(d-1)/d \rangle.$$

Note that each element of $AC_{0,p}$ commute with f_{c_0} , so each element of $AC_{0,p}$ can be seen a permutation between the p periodic orbits of f_{c_0} .

By Remark 3.2.8, Lemma 2.2.1 and analytic continuation, we can obtain

- (a) If consider $\sigma_{0,p}^{\gamma_i}$ as the permutation between the p periodic orbits of f_{c_0} , then as c_i is a primitive parabolic parameter, $\sigma_{0,p}^{\gamma_i}$ is a exchange of two orbits; as c_i is a satellite parabolic parameter, $\sigma_{0,p}^{\gamma_i}$ keep every orbit fixed.

From the proof of Theorem 3.3.1, we can see that

- (b) The action of $AC_{0,p}$ on the p periodic orbits of f_{c_0} is transitive.

with (a), (b), and by the following result in group theory (Lemma 3.4.2), we can obtain

Claim 1. $AC_{0,p}$ is the symmetric group of the p periodic orbits of f_{c_0} .

Lemma 3.4.2. *Let G be a subgroup of symmetric group S_n . If it is generated by exchanges and acts transitively on $\{1, \dots, n\}$, then $G = S_n$.*

By Proposition 3.3.14, there exists a permutation in $AC_{0,p}$ such that it cyclicly acts on one p periodic orbit of f_{c_0} and keep other p periodic points of f_{c_0} fixed. Since also $AC_{0,p}$ is the symmetric group of the p periodic orbits of f_{c_0} , we can deduce

Claim 2. For any p periodic orbit of f_{c_0} , there exists a permutation in $AC_{0,p}$ such that it cyclicly acts on this p periodic orbit of f_{c_0} and keep other p periodic points of f_{c_0} fixed.

Now choose any permutation σ between the p periodic points of f_{c_0} satisfying $\sigma \circ f = f \circ \sigma$. Then σ can be seen firstly as a permutation between the p periodic orbits of f_{c_0} , and then

as a cyclic action on each orbit. According to Claim 1 and Claim 2, it is easy to deduce $\sigma \in AC_{0,p}$. \square

Chapter 4

Dynatomic preperiodic curve

In this chapter, we will give a description of dynatomic preperiodic curve from the view of algebraic and topology. In Section 4.1, we will prove that every $\mathcal{X}_{n,p}$ is an affine algebraic curve and find its defining polynomial $Q_{n,p}(c, z)$. In Section 4.2, we give the irreducible factorization of $Q_{n,p}(c, z)$ and prove that each irreducible factor is smooth. Then each irreducible component of $\mathcal{X}_{n,p}$ is a Riemann surface. We will show that these Riemann surfaces intersect pairwise transversally at the singular points of $Q_{n,p}(c, z)$. In Section 4.3, we will give a kind of compactification for each irreducible component of $\mathcal{X}_{n,p}$ by adding some ideal boundary points such that it becomes a compact Riemann surface and then calculate the genus of this compact Riemann surface. In Section 4.4, we will describe $\mathcal{X}_{n,p}$ from the algebraic point of view by calculating the Galois group of $Q_{n,p}(c, z)$.

4.1 The defining polynomial for $\mathcal{X}_{n,p}$

The objective of this section is to show that $\mathcal{X}_{n,p}$ is an affine algebraic curve and find its defining polynomial.

Let $\Phi_{n,p}(c, z) = f_c^{\circ(n+p)}(z) - f_c^{\circ n}(z)$ ($n \geq 1, p \geq 1$). Then the solutions of the equation $\Phi_{n,p}(c, z) = 0$ consist of all $(c, z) \in \mathbb{C}^2$ such that z is a preperiodic point of f_c with preperiod l and period k where $0 \leq l \leq n$ and $k|p$. By abuse of notation, we will consider a polynomial in $\mathbb{C}[c, z]$ as a polynomial in $\mathbb{K}[z]$ where $\mathbb{K} = \mathbb{C}(c)$ is the field of rational functions about c .

Lemma 4.1.1. *There exists a unique double indexed sequence of squarefree polynomials*

$\{Q_{n,p}(c, z)\}_{n \geq 1, p \geq 1} \subset \mathbb{C}[c, z] \subset \mathbb{K}[z]$ monic about z such that

$$\Phi_{n,p}(c, z) = \Phi_{n-1,p}(c, z) \prod_{k|p} Q_{n,k}(c, z) \text{ for all } n \geq 1, p \geq 1.$$

Proof. Fix any $n \geq 1$. We claim: for any $c_0 \in \mathbb{C} \setminus M_d$, $p \geq 1$, all roots of $\Phi_{n,p}(c_0, z)$ are simple. (Demonstration: In this case, all periodic points of f_{c_0} are repelling and the critical orbit escapes to ∞ . Then for any root z_0 of $\Phi_{n,p}(c_0, z)$, $(\partial \Phi_{n,p} / \partial z)(c_0, z_0) = [f_{c_0}^{pn}]'(z_0)([f_{c_0}^{op}]'(z_0) - 1) \neq 0$). From this claim and the fact that $\Phi_{n,p}(c, z)$ is monic about z , it deduce that if we can find a sequence of polynomials $\{Q_{n,p}(c, z)\}_{n \geq 1, p \geq 1}$ which satisfy the equation in the lemma, they are naturally squarefree.

Let $c_0 \in \mathbb{C} \setminus M_d$ be arbitrarily. The fact that z_0 is a root of $\Phi_{n-1,p}(c_0, z)$ implies z_0 is a root of $\Phi_{n,p}(c_0, z)$. By the claim above, we have $\Phi_{n-1,p}(c_0, z) \mid \Phi_{n,p}(c_0, z)$ in $\mathbb{C}[z]$. Since c_0 is any point of $\mathbb{C} \setminus M_d$, we also have $\Phi_{n-1,p}(c, z) \mid \Phi_{n,p}(c, z)$ in $\mathbb{K}[z]$.

We proceed by induction on p . For $p = 1$, we define $Q_{n,1} = \Phi_{n,1}(c, z) / \Phi_{n-1,1}(c, z)$. It satisfies the requirement of the lemma.

Assume now that for every $1 \leq k < p$, the polynomial $Q_{n,k}(c, z)$ is defined and satisfies the requirement of the lemma. Let c_0 be any parameter in $\mathbb{C} \setminus M_d$. Note that for $1 \leq k < p$, the two polynomials $\Phi_{n-1,k}(c_0, z)$ and $Q_{n,k}(c_0, z)$ don't have a common root (by the claim above). Thus, if z_0 is a root of $Q_{n,k}(c_0, z)$, then it is a preperiodic point of f_{c_0} with preperiod n and period m (and $m \mid k$). In fact, m must be equal to k . (Otherwise, $Q_{n,k}(c_0, z) \cdot \prod_{m' \mid m} Q_{n,m'}(c_0, z)$ would have a double root at z_0 by induction, but would at the same time divide $\Phi_{n,k}(c_0, z)$, a contradiction to the claim above). Then we can conclude that any two polynomials among $\{ \Phi_{n-1,p}(c_0, z), Q_{n,k}(c_0, z) \}_{1 \leq k < p}$ have no common roots and any one among $\{ \Phi_{n-1,p}(c_0, z), Q_{n,k}(c_0, z) \}_{1 \leq k < p}$ divide $\Phi_{n,p}(c_0, z)$. Hence $\Phi_{n-1,p}(c_0, z) \cdot \prod_{k|p, k < p} Q_{n,k}(c_0, z)$ divides $\Phi_{n,p}(c_0, z)$ in $\mathbb{C}[z]$. As c_0 is any point of $\mathbb{C} \setminus M_d$, the polynomial $\Phi_{n-1,p}(c, z) \cdot \prod_{k|p, k < p} Q_{n,k}(c, z)$ divides $\Phi_{n,p}(c, z)$ in $\mathbb{K}[z]$. We can then define

$$Q_{n,p}(c, z) = \Phi_{n,p}(c, z) / [\Phi_{n-1,p}(c, z) \cdot \prod_{k|p, k < p} Q_{n,k}(c, z)].$$

It satisfies the requirement of the lemma. □

Recall that $\{\nu_d(p)\}_{p \geq 1}$ is the unique sequence of positive integers satisfying the recursive relation $d^p = \sum_{k|p} \nu_d(k)$. It is easy to see that the degree of $Q_{0,p}$ is $\nu_d(p)$ and the degree of

$Q_{n,p}$ is $\nu_d(p)(d-1)d^{n-1}$ for $n \geq 1$.

Remark 4.1.2. Note that $\Phi_{n,p}(c, z) = \Phi_{n-1,p}(c, f_c(z))$ for any $n \geq 1$, $p \geq 1$. By the definition of $Q_{n,p}$, we have

$$\begin{cases} \prod_{k|p} Q_{n-1,p}(c, f_c(z)) = \prod_{k|p} Q_{n,p}(c, z) & n \geq 2 \\ \prod_{k|p} Q_{0,p}(c, f_c(z)) = \prod_{k|p} Q_{0,p}(c, z) \prod_{k|p} Q_{1,p}(c, z) & n = 1 \end{cases}$$

for any $p \geq 1$. By induction on p , it follows

$$\begin{cases} Q_{n-1,p}(c, f_c(z)) = Q_{n,p}(c, z) & n \geq 2 \\ Q_{0,p}(c, f_c(z)) = Q_{0,p}(c, z) Q_{1,p}(c, z) & n = 1 \end{cases}$$

for any $p \geq 1$. This equation implies that we can obtain the properties of $Q_{n,p}$ by induction on n .

Proposition 4.1.3. Let $n \geq 1$, $p \geq 1$ be any pair of integers and $c_0 \in \mathbb{C}$ be any parameter. Then $z_0 \in \mathbb{C}$ is a root of $Q_{n,p}(c_0, z)$ if and only if one of the following 5 mutually exclusive conditions holds:

- (0) z_0 is a (n, p) -preperiodic point of f_{c_0} such that $f_{c_0}^l(z_0) \neq 0$ for any $0 \leq l < n$ and $[f_{c_0}^{op}]'(f_{c_0}^{on}(z_0)) \neq 1$.
- (1) z_0 is a (n, p) -preperiodic point of f_{c_0} such that $f_{c_0}^l(z_0) = 0$ for some $0 \leq l < n$.
- (2) z_0 is a (n, p) -preperiodic point of f_{c_0} and $[f_{c_0}^{op}]'(f_{c_0}^{on}(z_0)) = 1$.
- (3) z_0 is a (n, m) -preperiodic point of f_{c_0} such that m is a proper factor of n and $[f_{c_0}^{om}]'(f_{c_0}^{on}(z_0))$ is a primitive n/m -th root of unity.
- (4) z_0 is a $(n-1, p)$ -preperiodic point of f_{c_0} and $f_{c_0}^{o(n-1)}(z_0) = 0$.

Proof. The proof goes by induction on n . As $n = 1$, $Q_{0,p}(c, f_c(z)) = Q_{0,p}(c, z) \cdot Q_{1,p}(c, z)$. For any $c_0 \in \mathbb{C}$, z_0 is a multiplier root of $Q_{0,p}(c_0, f_{c_0}(z)) \iff (c_0, f_{c_0}(z_0)) \in C_{0,p,2} \cup C_{0,p,3}$ (case 1) or $z_0 = 0$ and c_0 is the center of hyperbolic component with period p (case 2).

In case 1, if z_0 is periodic, then z_0 is a root of $Q_{0,p}(c_0, z)$. Moreover, $Q_{0,p}(c_0, f_{c_0}(z))$ and $Q_{0,p}(c_0, z)$ have the same multiplicity at z_0 , so z_0 is not a root of $Q_{1,p}(c_0, z)$. If z_0 is not periodic, by Proposition 3.1.3, z_0 is not a root of $Q_{0,p}(c_0, z)$. So z_0 is not a common root

of $Q_{0,p}(c_0, z)$ and $Q_{1,p}(c_0, z)$. In case 2, by (1) of Proposition 3.1.3, z_0 is simple root of $Q_{0,p}(c_0, z)$, hence z_0 is also a root of $Q_{0,p}(c_0, f_{c_0}(z))$. In any other situation, $Q_{0,p}(c_0, f_{c_0}(z))$ only has simple root. Then z_0 is a common root of $Q_{0,p}(c_0, z)$ and $Q_{1,p}(c_0, z)$ if and only if $z_0 = 0$ and c_0 is the center of some hyperbolic component with period p (condition (4)). Except this case, z_0 is a root of $Q_{1,p}(c_0, z) \iff f_{c_0}(z_0)$ is a root of $Q_{0,p}(c_0, z)$ but z_0 is not a root of $Q_{0,p}(c_0, z)$. Then Proposition 3.1.3 implies that z_0 satisfies one of the conditions (0), (1), (2), (3) in Proposition 4.1.3.

Assume that the proposition is established for $1 \leq l < n$. At this time, $Q_{n,p}(c, z) = Q_{n-1,p}(c, f_c(z))$. So for any $c_0 \in \mathbb{C}$, z_0 is a root of $Q_{n,p}(c_0, z)$ if and only if $f_{c_0}(z_0)$ is a root of $Q_{n-1,p}(c_0, z)$. Then by the inductive assumption, the point z_0 satisfies one of the 5 exclusive conditions in Proposition 4.1.3. \square

Now we set

$$\check{\mathcal{X}}_{n,p,0} = \{(c, z) \in \mathbb{C}^2 \mid (c, z) \text{ satisfies Condition (0) in Proposition 4.1.3}\}$$

and for $1 \leq \alpha \leq 4$, set

$$C_{n,p,\alpha} = \{(c, z) \in \mathbb{C}^2 \mid (c, z) \text{ satisfies Condition } (\alpha) \text{ in Proposition 4.1.3}\}$$

It is easy to see that $\check{\mathcal{X}}_{n,p,0} \cup C_{n,p,1} \cup C_{n,p,2} = \check{\mathcal{X}}_{n,p}$ and $C_{n,p,1} \cup C_{n,p,2} \cup C_{n,p,3} \cup C_{n,p,4}$ is a finite set. Then we have

$$\mathcal{X}_{n,p} = \{(c, z) \mid Q_{n,p}(c, z) = 0\}.$$

4.2 The irreducible factorization of $Q_{n,p}$

In the periodic case ($n = 0$), we have proved that $Q_{0,p}$ is smooth and irreducible in Chapter 1. But in the preperiodic case ($n \geq 1$), the polynomial $Q_{n,p}$ displays a very different behavior: for $d = 2$, it is smooth and irreducible as the periodic case, however, for $d \geq 3$, it is neither smooth nor irreducible. In this section, we will find its irreducible factorization and prove the smoothness for each irreducible component. We will also show that these components pairwise intersect transversally at the singular points of $\mathcal{X}_{n,p}$.

4.2.1 Factorization of $Q_{n,p}$ and the behavior at its singular points

Fix any $n \geq 1$, $p \geq 1$. We have the following factorization result.

Lemma 4.2.1. (Algebraic version) *There exists a unique sequence of monic polynomials $\{q_{n,p}^j(c, z)\}_{1 \leq j \leq d-1} \subset \mathbb{C}[c, z] \subset \mathbb{K}[z]$ such that*

$$Q_{n,p}(c, z) = \prod_{j=1}^{d-1} q_{n,p}^j(c, z).$$

The degree of $q_{n,p}^j$ is $d^{n-1}\nu_d(p)$. All points in $C_{n,p,4}$ are the roots of $q_{n,p}^j$ for any $1 \leq j \leq d-1$, and there are no other common roots for $q_{n,p}^i$ and $q_{n,p}^j$ with $1 \leq i \neq j \leq d-1$.

(Topological version) *Let $\mathcal{V}_{n,p}^j = \{(c, z) \in \mathbb{C}^2 \mid q_{n,p}^j(c, z) = 0\}$ ($1 \leq j \leq d-1$). Then $C_{n,p,4} \subset \mathcal{V}_{n,p}^j$ for any $1 \leq j \leq d-1$ and $\{\mathcal{V}_{n,p}^j \setminus C_{n,p,4}\}_{1 \leq j \leq d-1}$ are pairwise disjoint.*

Proof. Let \overline{K} be a fixed algebraic closure of K . Set $\omega = e^{\frac{2\pi i}{d}}$.

Let Δ be a root of $Q_{0,p}(c, z)$ in \overline{K} . Then $\omega\Delta, \dots, \omega^{d-1}\Delta$ are roots of $Q_{1,p}(c, z)$ in \overline{K} .

Let us factorize $Q_{0,p}(c, z)$ in \overline{K} by

$$Q_{0,p}(c, z) = \prod_{s=1}^{\nu_d(p)} (z - \Delta_s)$$

($\Delta_{s_1} \neq \Delta_{s_2}$ for $s_1 \neq s_2$). Then $Q_{1,p}(c, z)$ can be expressed as

$$Q_{1,p}(c, z) = \prod_{s=1}^{\nu_d(p)} (z - \omega\Delta_s) \cdots (z - \omega^{d-1}\Delta_s) = \prod_{j=1}^{d-1} \prod_{s=1}^{\nu_d(p)} (z - \omega^j\Delta_s) = \prod_{j=1}^{d-1} (\omega^j)^{\nu_d(p)} \prod_{s=1}^{\nu_d(p)} (\omega^{-j}z - \Delta_s)$$

Note that $d \mid \nu_d(p)$ so $(\omega^j)^{\nu_d(p)} = 1$. For $1 \leq j \leq d-1$, set

$$q_{1,p}^j(c, z) = \prod_{s=1}^{\nu_d(p)} (z - \omega^j\Delta_s) = Q_{0,p}(c, \omega^{-j}z) \in \mathbb{C}[c, z] \subset \mathbb{K}[z].$$

Then $q_{1,p}^j(c, z)$ is a polynomial in (c, z) and is monic in $\mathbb{K}[z]$, satisfying

$$Q_{1,p}(c, z) = \prod_{j=1}^{d-1} q_{1,p}^j(c, z).$$

This gives a factorization of $Q_{1,p}$ in $\mathbb{K}[z]$. For $n \geq 2$, we can define $q_{n,p}^j(c, z)$ inductively by

$q_{n,p}^j(c, z) = q_{n-1,p}^j(c, f_c(z))$. As $Q_{n,p}(c, z) = Q_{n-1,p}(c, f_c(z))$, we have

$$Q_{n,p}(c, z) = \prod_{j=1}^{d-1} q_{1,p}^j(c, z).$$

This is a factorization of $Q_{n,p}(c, z)$ in $\mathbb{K}[z]$.

We are left to prove that each $q_{n,p}^j(c, z)$ satisfies the remaining properties announced in the lemma. For $n = 1$, since $q_{1,p}^j(c, z) = Q_{0,p}(c, \omega^{-j}z)$, then (c_0, z_0) is a common root of $q_{1,p}^i(c, z)$ and $q_{1,p}^j(c, z)$ for some $1 \leq i \neq j \leq d-1 \iff (c_0, \omega^{-i}z_0)$ and $(c_0, \omega^{-j}z_0)$ are all roots of $Q_{0,p}(c, z) \iff (c_0, z_0) = (c_0, 0) \in C_{1,p,4}$. For $n \geq 2$, the conclusion can be deduced from that of case $n = 1$ and the definition of $q_{n,p}(c, z)$. □

From Lemma 4.2.1, we can see that $Q_{n,p}$ is reducible and non-smooth ($C_{n,p,4}$ belongs to the set of singular points of $Q_{n,p}$) for $d \geq 3$. Let us now concentrate our study on the factors $q_{n,p}^j(c, z)$, $j \in [1, d-1]$.

For $n \geq 2$, $q_{n,p}^j(c, z)$ is defined by $q_{n,p}^j(c, z) = q_{n-1,p}^j(c, f_c(z))$. Interpret these equations by topological view, we obtain a sequence of maps

$$\left\{ \wp_{n,p}^j : \mathcal{V}_{n,p}^j \longrightarrow \mathcal{V}_{n-1,p}^j, \quad (c, z) \mapsto (c, f_c(z)) \mid n \geq 2, p \geq 1, 1 \leq j \leq d-1 \right\}.$$

Note that for $n = 1$, we can also define a map $\wp_{1,p}^j : \mathcal{V}_{1,p}^j \rightarrow \mathcal{X}_{0,p}$ by $\wp_{1,p}^j(c, z) = (c, f_c(z))$.

Lemma 4.2.2. *For any $p \geq 1$, $1 \leq j \leq d-1$,*

- *the map $\wp_{1,p}^j : \mathcal{V}_{1,p}^j \rightarrow \mathcal{X}_{0,p}$ is a homeomorphism.*
- *for $n \geq 2$, the map $\wp_{n,p}^j : \mathcal{V}_{n,p}^j \rightarrow \mathcal{V}_{n-1,p}^j$ is of degree d with critical set*

$$D_{n,p}^j = \{(c, 0) \in \mathcal{V}_{n,p}^j \mid q_{n,p}^j(c, 0) = 0\}.$$

Moreover, each critical point has multiplicity $d-1$.

Proof. It can be deduced directly from the definition of $q_{n,p}^j(c, z)$, $n, p \geq 1$, $1 \leq j \leq d-1$. □

The following proposition is the core of this section.

Proposition 4.2.3. *For any $n, p \geq 1$, $1 \leq j \leq d-1$, the polynomial $q_{n,p}^j(c, z)$ is smooth and irreducible.*

The proof of this proposition will be postponed to 4.2.2 .

By Proposition 4.2.3, we can restate Lemma 4.2.2 as follows.

Lemma 4.2.4. *For any $p \geq 1$, $1 \leq j \leq d-1$,*

- *the maps $\wp_{1,p}^j : \mathcal{V}_{1,p}^j \rightarrow \mathcal{X}_{0,p}$ is a conformal homeomorphism.*
- *For $n \geq 2$, the map $\wp_{n,p}^j : \mathcal{V}_{n,p}^j \rightarrow \mathcal{V}_{n-1,p}^j$ is a holomorphic branched covering of degree d with critical set*

$$D_{n,p}^j = \{(c, 0) \in \mathcal{V}_{n,p}^j \mid q_{n,p}^j(c, 0) = 0\}$$

Moreover, each critical point has multiplier $d-1$.

Remark 4.2.5. *By the definition of $q_{1,p}^j(c, z)$, we have $q_{1,p}^i(c, z) = q_{1,p}^j(c, \omega^{i-j}z)$ for any $1 \leq i \neq j \leq d-1$. Then we obtain a “rotation” r_{ij} between $\mathcal{V}_{1,p}^i$ and $\mathcal{V}_{1,p}^j$ for any $1 \leq i \neq j \leq d-1$, defined by $r_{ij}(c, z) = (c, \omega^{i-j}z)$. It is obviously a conformal map, then all $\{\mathcal{V}_{1,p}^j\}_{1 \leq j \leq d-1}$ are conformal equivalent (also equivalent to $\mathcal{X}_{0,p}$). But unfortunately, the map r_{ij} can't be lifted along $\wp_{2,p}^i$ and $\wp_{2,p}^j$ because r_{ij} doesn't map the critical values of $\wp_{2,p}^i$ to that of $\wp_{2,p}^j$, so we can not prove that $\mathcal{V}_{2,p}^i$ is conformal equivalent to $\mathcal{V}_{2,p}^j$ by simply lifting r_{ij} .*

Q: *Are $\{\mathcal{V}_{n,p}^j\}_{1 \leq j \leq d-1}$ conformal equivalent for fixed $n \geq 2$, $p \geq 1$?*

Now, we will provide some discussion about the singular points of on $\mathcal{X}_{n,p}$. Following the definition and notation in 6.1.1, we have the proposition below.

Proposition 4.2.6. *For $n, p \geq 1$, each singular point of $\mathcal{X}_{n,p}$ is ordinary and has multiplicity $d-1$.*

Proof. For any $(c_0, 0) \in C_{1,p,4}$, 0 is a simple root of $Q_{0,p}(c_0, z)$, then $\partial Q_{0,p}/\partial z(c_0, 0) \neq 0$. By the Implicit Function theorem, there exists a local holomorphic function $z(c)$ in a neighborhood of c_0 such that $z(c_0) = 0$ and $z(c)$ is the attracting p periodic point of f_c . Then as illustrated in section ??, we have a local holomorphic function

$$\mu(c) = [f_c^{\circ p}]'(z(c)) = f'_c(f_c^{\circ(p-1)}(z(c))) \cdots f'_c(f_c^{\circ(p-1)}(z(c))) \quad (4.2.1)$$

which has local degree $d - 1$ at c_0 .

Let $z(c) = a_k(c - c_0)^k + O((c - c_0)^{k+1})$ be the Taylor expansion of $z(c)$ at c_0 . Note that $f_{c_0}^{\circ p-1}(0), \dots, f_{c_0}(0)$ are all distinct from 0. Then substitute the expansion of $z(c)$ into (5.2.10), we have

$$\mu(c) = \lambda(c - c_0)^{(d-1)k} + \text{higher order terms}$$

in a small neighborhood of c_0 with $\lambda \neq 0$. Thus k must be 1. Then $Q_{0,p}(c, z)$ can be expressed as

$$Q_{0,p}(c, z) = a_{0,p}(c - c_0) + b_{0,p}z + \text{higher order terms}$$

at $(c_0, 0)$ with $a_{0,p} \neq 0$, $b_{0,p} \neq 0$. And hence

$$q_{1,p}^j(c, z) = Q_{0,p}(c, \omega^{-j}z) = a_{0,p}(c - c_0) + b_{0,p}\omega^{-j}z + \text{higher order terms}$$

Therefore the tangents of $\{\mathcal{V}_{1,p}^j\}_{1 \leq j \leq d-1}$ at $(c_0, 0)$ are pairwise distinct.

Now assume that for $1 \leq l < n$, the tangents of $\{\mathcal{V}_{l,p}^j\}_{1 \leq l \leq d-1}$ are pairwise different at each point of $C_{l,p,4}$. Let (c_0, z_0) be any point in $C_{n,p,4}$, then $(c_0, f_{c_0}(z_0)) = (c_0, w_0) \in C_{n-1,p,4}$. Denote the Taylor expansion of $q_{n-1,p}^j(c, z)$ at (c_0, w_0) by

$$q_{n-1,p}^j(c, z) = a_{n-1,p}^j(c - c_0) + b_{n-1,p}^j(w - w_0) + \text{higher order terms} \quad (4.2.2)$$

where $b_{n-1,p}^j \neq 0$ (w_0 is a simple root of $q_{n-1,p}^j(c_0, z)$). Since $[\partial f_c / \partial c](c_0, z_0) = 1$, $[\partial f_c / \partial z](c_0, z_0) = dz_0^{d-1}$, the Taylor expansion of $f_c(z)$ at (c_0, z_0) is

$$f_c(z) = w_0 + (c - c_0) + dz_0^{d-1}(z - z_0) + \text{higher order terms} \quad (4.2.3)$$

Substituting (4.2.3) into (4.2.2), we obtain

$$q_{n,p}^j(c, z) = (a_{n-1,p}^j + b_{n-1,p}^j)(c - c_0) + dz_0^{d-1} \cdot b_{n-1,p}^j(z - z_0) + \text{higher order terms}$$

By the inductive assumption, the tangents of $\{\mathcal{V}_{n,p}^j\}_{1 \leq j \leq d-1}$ at (c_0, z_0) are pairwise distinct.

So for $n, p \geq 1$ and any point $(c_0, z_0) \in C_{n,p,4}$, the first non vanishing term of $Q_{n,p}(c, z)$ at (c_0, z_0) is

$$\prod_{j=1}^{d-1} (a_{n,p}^j(c - c_0) + b_{n,p}^j(z - z_0))$$

with $\{(a_{n,p}^j, b_{n,p}^j) \neq (0, 0)\}_{1 \leq j \leq d-1}$ pairwise different. Then (c_0, z_0) is an ordinary singular point with multiplicity $d - 1$.

□

Remark 4.2.7. Lemma 4.2.1, Proposition 4.2.3 and Proposition 4.2.6 provide us a clear topological picture of $\mathcal{X}_{n,p}$:

- $C_{n,p,4}$ is exactly the set of singular points of $Q_{n,p}$;
- $\mathcal{X}_{n,p}$ is the union of the Riemann surfaces $\{\mathcal{V}_{n,p}^j\}_{1 \leq j \leq d-1}$ and any two of them intersect transversally at the singular points of $\mathcal{X}_{n,p}$.

4.2.2 Proof of the smoothness and the irreducibility of $q_{n,p}^j$

The objective here is to prove Proposition 4.2.3.

The approach to prove the smoothness is similar to that of proving the smoothness of $\mathcal{X}_{0,p}$ in Section smoothness.

The approach to the irreducibility is based on the connectivity of periodic curve $\mathcal{X}_{0,p}$. Then we will show the connectivity of $\mathcal{V}_{n,p}^j$ using the branched covering $\wp_{n,p}^j$ by induction on n .

Proof of Proposition 4.2.3. The proof goes by induction on n .

For $n = 1$, as $q_{1,p}^j(c, z) = Q_{0,p}(c, \omega^{-j}z)$ and $Q_{0,p}(c, z)$ is smooth and irreducible, we know that $q_{1,p}^j(c, z)$ is smooth and irreducible for $1 \leq j \leq d - 1$. Assume that for $1 \leq l < n$, $1 \leq j \leq d - 1$, the polynomial $q_{l,p}^j(c, z)$ is smooth and irreducible. Then we will show that $q_{n,p}^j(c, z)$ is smooth and irreducible. Now fix any $j_0 \in [1, d - 1]$.

Smoothness of $q_{n,p}^{j_0}$: As $q_{n,p}^{j_0}(c, z) = q_{n-1,p}^{j_0}(c, f_c(z))$, for any (c_0, z_0) a root of $q_{n,p}^{j_0}(c, z)$, we have

$$\begin{cases} \frac{\partial q_{n,p}^{j_0}}{\partial c}(c_0, z_0) = \frac{\partial q_{n-1,p}^{j_0}}{\partial c}(c_0, w_0) + \frac{\partial q_{n-1,p}^{j_0}}{\partial z}(c_0, w_0) \\ \frac{\partial q_{n,p}^{j_0}}{\partial z}(c_0, z_0) = \frac{\partial q_{n-1,p}^{j_0}}{\partial z}(c_0, w_0) \cdot f'_{c_0}(z_0) \end{cases} \quad (4.2.4)$$

where $w_0 = f_{c_0}(z_0)$. Then if $z_0 \neq 0$, by assumption of induction of smoothness, $[\partial q_{n,p}^{j_0}/\partial c](c_0, z_0)$ and $[\partial q_{n,p}^{j_0}/\partial c](c_0, z_0)$ can not equal to 0 simultaneously, it follows that $q_{n,p}^{j_0}(c, z)$ is smooth at (c_0, z_0) . So we are left to prove that $q_{n,p}^{j_0}(c, z)$ is smooth at $(c_0, 0) \in \mathcal{V}_{n,p}^{j_0}$. In this case, c_0 is a Misiurewicz parameter with preperiod $n - 1$ and period p . Note that

$[\partial q_{n,p}^{j_0}/\partial z](c_0, 0) = 0$, then we have to show $[\partial q_{n,p}^{j_0}/\partial c](c_0, 0) \neq 0$. Since

$$\frac{\partial Q_{n,p}}{\partial c}(c_0, 0) = \prod_{1 \leq j \neq j_0 \leq d-1} q_{n,p}^j(c_0, 0) \cdot \frac{\partial q_{n,p}^{j_0}}{\partial c}(c_0, 0)$$

and by Lemma 4.2.1, the point $(c_0, 0)$ is not a root of $\prod_{j \neq j_0} q_{n,p}^j(c, z)$, we only have to show $[\partial Q_{n,p}/\partial c](c_0, 0) \neq 0$. Furthermore,

$$\frac{\partial \Phi_{n,p}}{\partial c}(c_0, 0) = \Phi_{n-1,p}(c_0, 0) \cdot \prod_{k|p, k < p} Q_{n,k}(c_0, 0) \cdot \frac{\partial Q_{n,p}}{\partial c}(c_0, 0)$$

and it is known that $\Phi_{n-1,p}(c_0, 0) \cdot \prod_{k|p, k < p} Q_{n,k}(c_0, 0) \neq 0$. So we only have to show $[\partial \Phi_{n,p}/\partial c](c_0, 0) \neq 0$. We shall choose a meromorphic quadratic differential with simple poles such that

$$(f_{c_0})_* \mathcal{Q} = \mathcal{Q} + \frac{\partial \Phi_{n,p}}{\partial c}(c_0, 0) \cdot \frac{dz^2}{z - c_0}.$$

Then with Lemma ??, we obtain $[\partial \Phi_{n,p}/\partial c](c_0, 0) \neq 0$.

We shall use the following notations:

$$\begin{aligned} z_k &:= f_{c_0}^{cn+k}(0), & \delta_k &:= f'_{c_0}(z_k) = dz_k^{d-1}, & 0 \leq k \leq p-1 \\ y_l &:= f_{c_0}^l(0), & \varepsilon_l &:= f'_{c_0}(y_l) = dy_l^{d-1}, & 1 \leq l \leq n-1 \end{aligned}$$

With these notations and a bit of calculations, we get

$$\begin{aligned} \frac{\partial \Phi_{n,p}}{\partial c}(c_0, 0) &= \frac{\partial f_c^{c(n+p)}}{\partial c}(c_0, 0) - \frac{\partial f_c^{cn}}{\partial c}(c_0, 0) \\ &= (\delta_0 \cdots \delta_{p-1} - 1)(\varepsilon_{n-1} \cdots \varepsilon_1 + \cdots + \varepsilon_{n-1} \varepsilon_{n-2} + \varepsilon_{n-1} + 1) \\ &\quad + \delta_{p-1} \cdots \delta_1 + \cdots + \delta_{p-1} + 1 \end{aligned}$$

Denote $(\delta_0 \cdots \delta_{p-1} - 1)(\varepsilon_{n-1} \cdots \varepsilon_1 + \cdots + \varepsilon_{n-1} \varepsilon_{n-2} + \varepsilon_{n-1} + 1)$ by α . Let

$$\mathcal{Q} = \sum_{k=0}^{p-1} \frac{\rho_k}{z - z_k} dz^2 + \sum_{l=1}^{n-1} \frac{\lambda_l}{z - y_l} dz^2$$

be a quadratic differential in $\mathcal{Q}(\mathbb{C})$. Here ρ_k ($0 \leq k \leq p-1$), λ_l ($1 \leq l \leq n-1$) are undetermined coefficients (note that $y_1 = c_0$). Applying Lemma ?? and writing f for f_{c_0} ,

is

$$\left\{ \begin{array}{lcl} \rho_0 & = & \delta_0 \cdots \delta_{p-1} \\ \rho_1 & = & \delta_1 \cdots \delta_{p-1} \\ \vdots & & \\ \rho_{p-1} & = & \delta_{p-1} \\ \lambda_1 & = & (\delta_0 \cdots \delta_{p-1} - 1) \cdot \varepsilon_{n-1} \cdots \varepsilon_1 \\ \vdots & & \\ \lambda_{n-2} & = & (\delta_0 \cdots \delta_{p-1} - 1) \cdot \varepsilon_{n-1} \varepsilon_{n-2} \\ \lambda_{n-1} & = & (\delta_0 \cdots \delta_{p-1} - 1) \cdot \varepsilon_{n-1} \end{array} \right. \quad (4.2.5)$$

Therefore, for $(\rho_0, \dots, \rho_{p-1}, \lambda_1, \dots, \lambda_{n-1})$ satisfies (4.2.5), we have

$$f_* \mathcal{Q} - \mathcal{Q} = - \left(\alpha + \sum_{k=0}^{p-1} \frac{\rho_k}{\delta_k} \right) \frac{dz^2}{z - c_0} = - \frac{\partial \Phi_{n,p}}{\partial c}(c_0, 0) \cdot \frac{dz^2}{z - c_0}$$

As a consequence $[\partial \Phi_{n,p} / \partial c](c_0, 0) \neq 0$.

Irreducibility of $q_{n,p}^{j_0}$: By the smoothness of $q_{n,p}^j(c, z)$ ($n, p \geq 1, 1 \leq j \leq d-1$) and the inductive assumption of irreducibility, we know that $\varphi_{n,p}^{j_0} : \mathcal{V}_{n,p}^{j_0} \rightarrow \mathcal{V}_{n-1,p}^{j_0}$ is a branched covering of degree d , $\mathcal{V}_{n-1,p}^{j_0}$ and each connected component of $\mathcal{V}_{n,p}^{j_0}$ is a Riemann surface. Then it is easy to prove that the restriction of $\varphi_{n,p}^{j_0}$ to any connected component of $\mathcal{V}_{n,p}^{j_0}$ is also a branched covering. Since the set of critical points of $\varphi_{n,p}^{j_0}$

$$D_{n,p}^{j_0} = \{(c, 0) \mid q_{n,p}^{j_0}(c, 0) = 0\}$$

is non-empty and each critical point has multiplicity $d-1$, the set $\mathcal{V}_{n,p}^{j_0}$ must be connected. By Theorem 2.4.3 and the smoothness of $q_{n,p}^{j_0}$, we conclude that $q_{n,p}^{j_0}(c, z)$ is irreducible in $\mathbb{C}[c, z]$. □

4.3 Genus of the compactification of $\mathcal{V}_{n,p}^j$

In the previous section, we have seen that $\mathcal{X}_{n,p}$ is the union of $d-1$ Riemann surfaces $\{\mathcal{V}_{n,p}^j\}_{1 \leq j \leq d-1}$ and any two of them intersect transversally at the singular points of $\mathcal{X}_{n,p}$. In order to give a complete topological description of $\mathcal{X}_{n,p}$, we also need the topological characterization of each $\mathcal{V}_{n,p}^j$.

In fact, by adding an ideal boundary point at each end of $\mathcal{V}_{n,p}^j$, we obtain a compactification of $\mathcal{V}_{n,p}^j$, denoted by $\widehat{\mathcal{V}}_{n,p}^j$, such that $\widehat{\mathcal{V}}_{n,p}^j$ is a compact Riemann surface (in section 4.3.1). We will also calculate the genus of $\widehat{\mathcal{V}}_{n,p}^j$ (in section 4.3.2). Topologically, $\mathcal{X}_{n,p}$ is completely determined by the number of its singular points, the genus of $\widehat{\mathcal{V}}_{n,p}^j$ and the number of ideal boundary points added on $\mathcal{V}_{n,p}^j$ (or the number of ends of $\mathcal{V}_{n,p}^j$) for $1 \leq j \leq d-1$. So we can give a complete topological description of $\mathcal{X}_{n,p}$ (Lemma 4.3.4).

4.3.1 Compactification of $\mathcal{V}_{n,p}^j$

Denote by $\pi_{n,p}^j: \mathcal{V}_{n,p}^j \rightarrow \mathbb{C}$ the projection from $\mathcal{V}_{n,p}^j$ to the parameter plane. It is easy to see

$$\pi_{n,p}^j = \pi_{0,p} \circ \wp_{1,p}^j \circ \cdots \circ \wp_{n-1,p}^j \circ \wp_{n,p}^j$$

where $\pi_{0,p}$ is the projection from $\mathcal{X}_{0,p}$ to the parameter plane. By Theorem ?? and Lemma 4.2.4, the map $\pi_{n,p}^j: \mathcal{V}_{n,p}^j \rightarrow \mathbb{C}$ is a degree $\nu_d(p)d^{n-1}$ branched covering whose set of critical points equals to $C_{n,p,1}^j \cup C_{n,p,2}^j \cup C_{n,p,3}^j$, where $C_{n,p,\alpha}^j := C_{n,p,\alpha} \cap \mathcal{V}_{n,p}^j$ for all $n, p \geq 1$, $1 \leq j \leq d-1$, $0 \leq \alpha \leq 4$ (note that $C_{1,p,1} = \emptyset$). Let $V_{n,p,\alpha}^j := \pi_{n,p}^j(C_{n,p,\alpha}^j)$ ($\alpha = 1, 2, 3$). Then the critical value set of $\pi_{n,p}^j$, denoted by $V_{n,p}^j$, is equal to $V_{n,p,1}^j \cup V_{n,p,2}^j \cup V_{n,p,3}^j$.

Lemma 4.3.1. (1) *For any $1 \leq i \neq j \leq d-1$, we have $V_{n,p,1}^i \cap V_{n,p,1}^j = \emptyset$. The set*

$\bigcup_{j=1}^{d-1} V_{n,p,1}^j$ consists of all Misiurewicz parameters such that c is (l, p) -preperiodic point of f_c for some $0 < l \leq n-1$.

(2) *For any $1 \leq j \leq d-1$, $V_{n,p,2}^j \cup V_{n,p,3}^j$ consists of roots and co-roots of all hyperbolic components of period p .*

Proof. (1) By Proposition 4.1.3, the set $\bigcup_{j=1}^{d-1} V_{n,p,1}^j$ consists of all Misiurewicz parameters such that c is (l, p) -preperiodic point of f_c for some $0 < l \leq n-1$. If $c_0 \in V_{n,p,1}^i \cap V_{n,p,1}^j$ for some $1 \leq i \neq j \leq d-1$, then c_0 is a (l, p) -preperiodic point of f_{c_0} for some $l \in [1, n-1]$ and there are two points $(c_0, z_1), (c_0, z_2)$ belonging to $\mathcal{V}_{n,p}^i$ and $\mathcal{V}_{n,p}^j$ respectively. It follows $(c_0, 0) \in \mathcal{V}_{l+1,p}^i \cap \mathcal{V}_{l+1,p}^j$, a contradiction to Lemma 4.2.1.

(2) follows directly from $\pi_{n,p}^j = \pi_{0,p} \circ \wp_{1,p}^j \circ \cdots \circ \wp_{n-1,p}^j \circ \wp_{n,p}^j$ and Remark 3.2.8. \square

Set $\bigcup_{i=1}^{m_{n,p}^j} \mathcal{E}_{n,p,i}^j := (\pi_{n,p}^j)^{-1}(\mathbb{C} \setminus M_d)$, where $\mathcal{E}_{n,p,i}^j$ is a connected component of $(\pi_{n,p}^j)^{-1}(\mathbb{C} \setminus M_d)$, called an end of $\mathcal{V}_{n,p}^j$. Fix any $i_0 \in [1, m_{n,p}^j]$. Since $\mathbb{C} \setminus M_d$ contains no critical values

of $\pi_{n,p}^j$, the map

$$\pi_{n,p}^j|_{\mathcal{E}_{n,p,i_0}^j} : \mathcal{E}_{n,p,i_0} \rightarrow \mathbb{C} \setminus M_d$$

is a covering whose degree is denoted by d_{n,p,i_0}^j . Note that $\mathbb{C} \setminus M_d$ is conformal to $\mathbb{C} \setminus \overline{\mathbb{D}}$, it follows that \mathcal{E}_{n,p,i_0}^j is also conformal to $\mathbb{C} \setminus \overline{\mathbb{D}}$. So if we add a point ∞_{n,p,i_0}^j to the infinite far boundary of \mathcal{E}_{n,p,i_0}^j , then we get a new set $\widehat{\mathcal{E}}_{n,p,i_0}^j := \mathcal{E}_{n,p,i_0}^j \cup \{\infty_{n,p,i_0}^j\}$ and it is conformal to $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$. The point ∞_{n,p,i_0}^j is called the *ideal boundary point* of \mathcal{E}_{n,p,i_0}^j and d_{n,p,i_0}^j is called the *multiplicity* of \mathcal{E}_{n,p,i_0}^j . In this case, $\pi_{n,p}^j|_{\mathcal{E}_{n,p,i_0}^j}$ can be extended to

$$\widehat{\pi}_{n,p}^j|_{\widehat{\mathcal{E}}_{n,p,i_0}^j} : \widehat{\mathcal{E}}_{n,p,i_0} \rightarrow \widehat{\mathbb{C}} \setminus M_d$$

by setting $\widehat{\pi}_{n,p}^j(\infty_{n,p,i_0}^j) = \infty$. This map becomes a branched covering of degree d_{n,p,i_0}^j with a unique branched point ∞_{n,p,i_0}^j .

Adding the ideal boundary point at each end of $\mathcal{V}_{n,p}^j$, we obtain a compact Riemann surface $\widehat{\mathcal{V}}_{n,p}^j := \mathcal{V}_{n,p}^j \cup \{\infty_{n,p,i}^j\}_{i=1}^{m_{n,p}^j}$ and an extended branched covering $\widehat{\pi}_{n,p}^j : \widehat{\mathcal{V}}_{n,p}^j \rightarrow \widehat{\mathbb{C}}$. We are left to calculate the number $m_{n,p}^j$ of ends for $\mathcal{V}_{n,p}^j$ and the multiplicity $d_{n,p,i}^j$ of each end of $\mathcal{V}_{n,p}^j$.

Lemma 4.3.2. *For any $n, p \geq 1$, $1 \leq j \leq d-1$, $1 \leq i \leq m_{n,p}^j$, we have $d_{n,p,i}^j = d$ and $m_{n,p}^j = \nu_d(p)d^{n-2}$.*

Proof. The map $\pi_{n,p}^j|_{\mathcal{E}_{n,p,i}^j} : \mathcal{E}_{n,p,i}^j \rightarrow \mathbb{C} \setminus M_d$ is a covering. Fix $c_0 \in \mathbb{C} \setminus (M_d \cup R_{M_d}(0))$, $d_{n,p,i}^j = \# \left(\pi_{n,p}^j|_{\mathcal{E}_{n,p,i}^j} \right)^{-1}(c_0)$. Since $\mathcal{E}_{n,p,i}^j$ is connected, $\text{Mon} \left(\pi_{n,p}^j|_{\mathcal{E}_{n,p,i}^j} \right)$ acts on $\left(\pi_{n,p}^j|_{\mathcal{E}_{n,p,i}^j} \right)^{-1}(c_0)$ transitively. Then fixing any point $(c_0, z_0) \in \left(\pi_{n,p}^j|_{\mathcal{E}_{n,p,i}^j} \right)^{-1}(c_0)$, the set $\left(\pi_{n,p}^j|_{\mathcal{E}_{n,p,i}^j} \right)^{-1}(c_0)$ is exactly the orbit of (c_0, z_0) under $\text{Mon} \left(\pi_{n,p}^j|_{\mathcal{E}_{n,p,i}^j} \right)$.

Let $c_t : [0, 1] \rightarrow \mathbb{C} \setminus M_d$ be a oriented simple closed curve based at c_0 such that c_t intersects $R_{M_d}(0)$ at only one point c_{t_0} . Let z_t be the (n, p) -preperiodic point of f_{c_t} obtained from the analytic continuation of z_0 along c_t . Note that as c varies in $\mathbb{C} \setminus (M_d \cup R_{M_d}(0))$, the (n, p) -preperiodic points of f_c , the dynamical rays $R_c(0)$ and $R_c((\theta_c + s)/d)$ ($s \in \mathbb{Z}_d$) move continuously. Consequently, we have

$$\begin{cases} \iota_{c_t}(z_t) = \iota_{c_0}(z_0) & \text{for } t \in [0, t_0) \\ \iota_{c_t}(z_t) = \iota_{c_0}(z_1) & \text{for } t \in (t_0, 1] \end{cases}$$

Furthermore, on one hand, z_t and $R_{c_t}(0)$ move continuously for $t \in [0, 1]$. On the other hand, when c_t passes through $R_{M_d}(0)$, the dynamical rays $R_{c_t}((\theta_t + s)/d)$ ($s \in \mathbb{Z}_d$) move discontinuously and jump from $R_{c_{t_-}}((\theta_{t_-} + s)/d)$ to $R_{c_{t_+}}((\theta_{t_+} + s + 1)/d)$, $t_- < t_0 < t_+$. So if $\iota_{c_0}(z_0) = \beta_n \dots \beta_1 \overline{\epsilon_1 \dots \epsilon_p}$, then

$$\iota_{c_0}(z_1) = (\beta_n + 1) \dots (\beta_1 + 1) \overline{(\epsilon_1 + 1) \dots (\epsilon_p + 1)} \quad (4.3.1)$$

Hence the map $\Phi_{\pi_{n,p}^j}(c_t)$ maps (c_0, z_0) to (c_0, z_1) with z_1 satisfying (4.3.1). Since $\pi_1(\mathbb{C} \setminus M_d, c_0) = \langle c_t \rangle$, then we have

$$\left(\pi_{n,p}^j|_{\mathcal{E}_{n,p,i}^j} \right)^{-1}(c_0) = \{ (c_0, z) \mid \iota_{c_0}(z) = (\beta_n + s) \dots (\beta_1 + s) \overline{(\epsilon_1 + s) \dots (\epsilon_p + s)}, s \in \mathbb{Z}_d \}$$

As a consequence, $d_{n,p,i}^j = d$ and $m_{n,p}^j = \nu_d(p)d^{n-2}$. \square

We can also use the itinerary to label the ends of $\mathcal{V}_{n,p}^j$. The open set $W := \mathbb{C} \setminus (M_d \cup R_{M_d}(0))$ is simply connected. Let $\mathcal{W}_{n,p}^j \subset \mathcal{V}_{n,p}^j$ be the preimage of W by $\pi_{n,p}^j : \mathcal{V}_{n,p}^j \rightarrow \mathbb{C}$. Since W is simply connected, each connected component of \mathcal{W} maps isomorphically to W by $\pi_{n,p}^j$ (so there are $\nu_d(p)d^{n-1}$ components in $\mathcal{W}_{n,p}^j$).

Define $\iota_{n,p}^j : \mathcal{W}_{n,p}^j \rightarrow \mathbb{Z}_d^{\mathbb{N}}$ by $\iota_{n,p}^j(c, z) = \iota_c(z)$. As c varies in W , the (n, p) -preperiodic points of f_c , the dynamical rays $R_c(0)$ and $R_c((\theta_c + s)/d)$ ($s \in \mathbb{Z}_d$) move continuously. As a consequence, the map $\iota_{n,p}^j : \mathcal{W}_{n,p}^j \rightarrow \mathbb{Z}_d^{\mathbb{N}}$ is locally constant, whence constant on each connected component of $\mathcal{W}_{n,p}^j$. Since $\iota_c : K_c \rightarrow \mathbb{Z}_d^{\mathbb{N}}$ is bijective, distinct components have distinct itineraries, so each connected component of $\mathcal{U}_{n,p,t}^j$ of $\mathcal{W}_{n,p}^j$ can be labelled by its itinerary $\iota_{n,p}^j(\mathcal{U}_{n,p,t}^j)$.

According to the proof of Lemma 4.3.2, each end of $\mathcal{V}_{n,p}^j$ contains d components of $\mathcal{W}_{n,p}^j$ and they are labelled by

$$\{ (\beta_n + s) \dots (\beta_1 + s) \overline{(\epsilon_2 + s) \dots (\epsilon_p + s)(\epsilon_1 + s)} \}_{s \in \mathbb{Z}_d}$$

for some (n, p) -sequence $\beta_n \dots \beta_1 \overline{\epsilon_2 \dots \epsilon_p \epsilon_1} \in \mathbb{Z}_d^{\mathbb{N}}$. We define an equivalence relationship in all (n, p) -preperiodic sequences in $\mathbb{Z}_d^{\mathbb{N}}$ such that $\beta_n \dots \beta_1 \overline{\epsilon_2 \dots \epsilon_p \epsilon_1} \sim \beta'_n \dots \beta'_1 \overline{\epsilon'_2 \dots \epsilon'_p \epsilon'_1}$ if and only if

$$\beta'_n \dots \beta'_1 \overline{\epsilon'_2 \dots \epsilon'_p \epsilon'_1} = (\beta_n + s) \dots (\beta_1 + s) \overline{(\epsilon_2 + s) \dots (\epsilon_p + s)(\epsilon_1 + s)}$$

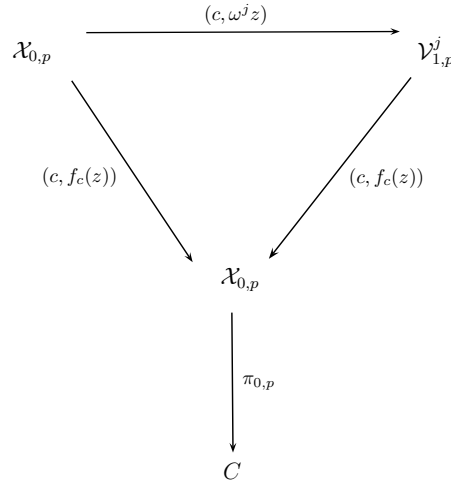
for some $s \in \mathbb{Z}_d$. The equivalence class of $\beta_n \dots \beta_1 \overline{\epsilon_2 \dots \epsilon_p \epsilon_1}$ is denoted by $[\beta_n \dots \beta_1 \overline{\epsilon_2 \dots \epsilon_p \epsilon_1}]$.

Then each $\mathcal{E}_{n,p,i}^j$ can be labelled by $[\iota_{n,p}^j(\mathcal{U}_{n,p,i,s}^j)]$ where $\mathcal{U}_{n,p,i,s}^j$ is a component of $\mathcal{W}_{n,p}^j$ contained in $\mathcal{E}_{n,p,i}^j$.

Proposition 4.3.3. *All ends of $\mathcal{V}_{n,p}^j$ can be labelled by $\{[\beta_n \dots \beta_2(\epsilon_1 + j) \overline{\epsilon_2 \dots \epsilon_p \epsilon_1}]\}$, where $\beta_m \in \mathbb{Z}_d$ for $2 \leq m \leq n$ and $\overline{\epsilon_2 \dots \epsilon_p \epsilon_1} \in \mathbb{Z}_d^{\mathbb{N}}$ is any p -periodic sequence under shift.*

Proof. Let $\mathcal{E}_{1,p,i}^j$ be any end of $\mathcal{V}_{1,p}^j$. Let (c_0, w) be a point of $\mathcal{E}_{1,p,i}^j$ with $\iota_{c_0}(w) = \beta \overline{\epsilon_2 \dots \epsilon_p \epsilon_1}$.

By the following commutative graph, We have $\beta_1 = \epsilon_1 + j$. Then for (c_0, z) belonging to



any end of $\mathcal{V}_{n,p}^j$,

$$\iota_{c_0}(z) = \beta_n \dots \beta_2(\epsilon_1 + j) \overline{\epsilon_2 \dots \epsilon_p \epsilon_1}$$

which is some (n, p) -sequence in $\mathbb{Z}_d^{\mathbb{N}}$. So each end of $\mathcal{V}_{n,p}^j$ can be labelled by $[\beta_n \dots \beta_2(\epsilon_1 + j) \overline{\epsilon_2 \dots \epsilon_p \epsilon_1}]$ for some (n, p) -preperiodic sequence $\beta_n \dots \beta_2(\epsilon_1 + j) \overline{\epsilon_2 \dots \epsilon_p \epsilon_1}$. Besides, the number of all equivalence classes with the form $[\beta_n \dots \beta_2(\epsilon_1 + j) \overline{\epsilon_2 \dots \epsilon_p \epsilon_1}]$ is $\nu_d(p)d^{n-2}$, the same with the number of ends on $\mathcal{V}_{n,p}^j$ (Lemma 4.3.2). So all ends of $\mathcal{V}_{n,p}^j$ are labelled by

$$\{[\beta_n \dots \beta_2(\epsilon_1 + j) \overline{\epsilon_2 \dots \epsilon_p \epsilon_1}] \mid \beta_n \dots \beta_2(\epsilon_1 + j) \overline{\epsilon_2 \dots \epsilon_p \epsilon_1} \text{ is a } (n, p)\text{-preperiodic sequence} \}$$

□

4.3.2 Calculation of the genus of $\widehat{\mathcal{V}}_{n,p}^j$

Now, for any $n, p \geq 1$, $1 \leq j \leq d-1$, we have obtained a branched covering $\widehat{\pi}_{n,p}^j: \widehat{\mathcal{V}}_{n,p}^j \rightarrow \widehat{\mathbb{C}}$ of degree $\nu_d(p)d^{n-1}$ between two compact Riemann surface. By the Riemann-Hurwitz formula,

we have

$$2 - 2g_{n,p}^j + \text{total number of critical points of } \widehat{\pi}_{n,p}^j = 2\nu_d(p)d^{n-1}.$$

where $g_{n,p}^j$ denotes the genus of $\widehat{\mathcal{V}}_{n,p}^j$. So in order to calculate the genus of $\widehat{\mathcal{V}}_{n,p}^j$, we only need to count the number of critical points of $\pi_{n,p}^j$ counting with multiplicity. It is known that the set of critical points for $\widehat{\pi}_{n,p}^j$ is

$$C_{n,p,1}^j \cup C_{n,p,2}^j \cup C_{n,p,3}^j \cup \{\infty_{n,p,i}^j\}_{i=1}^{\nu_d(p)d^{n-2}}$$

We will count the critical points class by class.

- Counting the points in $C_{n,p,1}^j$.

By the definition of $C_{n,p,1}^j$, we have

$$C_{n,p,1}^j = \bigcup_{s=2}^n (\wp_{n,p}^j)^{-1} \circ \dots \circ (\wp_{s+1,p}^j)^{-1} (D_{s,p}^j).$$

Recall that $D_{s,p}^j = \{(c, 0) \in \mathbb{C}^2 \mid q_{s,p}^j(c, 0) = 0\}$ is the set of critical points of $\wp_{s,p}^j$. Fix any $s \in [2, n]$. Firstly, we claim $\#D_{s,p}^j = \nu_d(p)d^{s-2}$: on one hand, $q_{s,p}^j(c, 0) = 0 \iff q_{s-1,p}^j(c, c) = 0$. So $\deg(q_{s-1,p}^j(c, c)) = \nu_d(p)d^{s-2}$ implies $\#D_{s,p}^j \leq \nu_d(p)d^{s-2}$. On the other hand, by the smoothness of $\mathcal{V}_{s,p}^j$ at $(c, 0) \in D_{s,p}^j$, we have $[\partial q_{s,p}^j / \partial c](c, 0) \neq 0$. It follows that each root of $q_{s,p}^j(c, 0)$ is simple, and $\#D_{s,p}^j = \nu_d(p)d^{s-2}$.

Next, consider the map

$$h_{n,s,p}^j := \wp_{s+1,p}^j \circ \dots \circ \wp_{n,p}^j : \mathcal{V}_{n,p}^j \longrightarrow \mathcal{V}_{s,p}^j$$

It is easy to see that the set of critical points of $h_{n,s,p}^j$ is disjoint from $(h_{n,s,p}^j)^{-1}(D_{s,p}^j)$. It follows that $\#(h_{n,s,p}^j)^{-1}(D_{s,p}^j) = \nu_d(p)d^{s-2} \cdot d^{n-s}$ and each point in $(h_{n,s,p}^j)^{-1}(D_{s,p}^j)$ is a critical point of $\widehat{\pi}_{n,p}^j$ with multiplicity $d-1$. Therefore the total number of critical points of $\widehat{\pi}_{n,p}^j$ in $C_{n,p,1}^j$ is equal to

$$\sum_{s=2}^n \nu_d(p)d^{s-2} \cdot d^{n-s} \cdot (d-1) = (n-2)\nu_d(p)d^{n-2}(d-1).$$

- Counting the points in $C_{n,p,3}^j$

By definition, $C_{n,p,3}^j = (\wp_{n,p}^j)^{-1} \circ \dots \circ (\wp_{1,p}^j)^{-1}(C_{0,p,3}) = (h_{n,0,p}^j)^{-1}(C_{0,p,3})$. It is obvious that $C_{0,p,3}$ is disjoint from the set of critical values for $h_{n,0,p}^j$, then each point P in $C_{0,p,3}$ has exactly d^{n-1} pre-image under $h_{n,0,p}^j$ and each pre-image of P , considered as critical point of $\hat{\pi}_{n,p}^j$, has the same multiplicity as that of P considered as critical point of $\pi_{0,p}$. So the total number of critical points of $\hat{\pi}_{n,p}^j$ in $C_{n,p,3}^j$ is d^{n-1} times the number of critical points of $\pi_{0,p}$ in $C_{0,p,3}$. Now we are only left to count the number of critical points of $\pi_{0,p}$ in $C_{0,p,3}$.

Let $C_{0,p,3,k} = \{(c, z) \in C_{0,p,3} \mid z \text{ is } k \text{ periodic point of } f_c\}$, then $C_{0,p,3}$ is the disjoint union of all $C_{0,p,3,k}$ with $k|p$, $k < p$. Fix any k_0 satisfying $k_0|p$, $k_0 < p$. Note that there are $\nu_d(k_0)/d$ hyperbolic components in M_d of period k_0 and on the boundary of any such component, there are $(d-1)\varphi(p/k_0)$ parameters with their parabolic periodic points satisfying the property of $C_{0,p,3,k_0}$. (For any $t \in \mathbb{N}$, $\varphi(t)$ is the number of all numbers among $\{1, \dots, t-1\}$ which are co-prime with t). Then $\#C_{0,p,3,k_0} = (\nu_d(k_0)/d)(d-1)\varphi(p/k_0)k_0$. Moreover, each point in $C_{0,p,3,k_0}$ has multiplicity $(p/k_0)-1$ as a critical point of $\pi_{0,p}$. So the number of critical points of $\pi_{0,p}$ in $C_{0,p,3}$ is equal to

$$\sum_{k|p, k < p} (\nu_d(k)/d)(d-1)\varphi(p/k)k(p/k-1)$$

Hence, the number of critical points of $\hat{\pi}_{n,p}^j$ in $C_{n,p,3}^j$ is

$$d^{n-1} \sum_{k|p, k < p} (\nu_d(k)/d)(d-1)\varphi(p/k)k(p/k-1).$$

- Counting the points in $C_{n,p,2}^j$

By definition, $C_{n,p,2}^j = (\wp_{n,p}^j)^{-1} \circ \dots \circ (\wp_{1,p}^j)^{-1}(C_{0,p,2}) = (h_{n,0,p}^j)^{-1}(C_{0,p,2})$. It is very similar to the case above. With the same reason, we can also conclude that the total number of critical points of $\hat{\pi}_{n,p}^j$ in $C_{n,p,2}^j$ is d^{n-1} times of the number of critical points of $\pi_{0,p}$ in $C_{0,p,2}$.

Now we begin to count the number of critical points of $\pi_{0,p}$ in $C_{0,p,2}$. By definition of $C_{0,p,2}$, the parameter set $\pi_{0,p}(C_{0,p,2})$ consists of co-roots and primitive roots of all hyperbolic components of period p . From Section 2.2.1, the number of co-roots and roots for all hyperbolic component of period p is $(d-1)\nu_d(p)/d$. Moreover, in the

calculation of number of critical points in $C_{0,p,3}$, we have actually got the number of root of the satellite components with period p , that is $\sum_{k|p, k < p} (\nu_d(k)/d)(d-1)\varphi(p/k)$. Then

$$\#C_{0,p,2} = p[(d-1)\nu_d(p)/d - \sum_{k|p, k < p} (\nu_d(k)/d)(d-1)\varphi(p/k)].$$

By Remark 3.2.8, each critical point of $\pi_{0,p}$ in $C_{0,p,2}$ is simple, then the number of all critical points of $\pi_{0,p}$ in $C_{0,p,2}$ is

$$p[(d-1)\nu_d(p)/d - \sum_{k|p, k < p} (\nu_d(k)/d)(d-1)\varphi(p/k)].$$

Hence the number of all critical points of $\widehat{\pi}_{n,p}^j$ in $C_{n,p,2}^j$ is

$$d^{n-1}p[(d-1)\nu_d(p)/d - \sum_{k|p, k < p} (\nu_d(k)/d)(d-1)\varphi(p/k)].$$

- Counting the points in $\{\infty_{n,p,i}^j\}_{i=1}^{\nu_d(p)d^{n-2}}$

By Lemma 4.3.2, there are $\nu_d(p)d^{n-2}$ ideal boundary points on $\widehat{\mathcal{V}}_{n,p}^j$ and each one is a critical point of $\widehat{\pi}_{n,p}^j$ with multiplicity $d-1$. So the number of critical points of $\widehat{\pi}_{n,p}^j$ in $\{\infty_{n,p,i}^j\}_{i=1}^{\nu_d(p)d^{n-2}}$ is equal to $\nu_d(p)d^{n-2}(d-1)$.

By the Riemann-Hurwitz formula, we have

$$g_{n,p}^j = 1 + \frac{1}{2}\nu_d(p)d^{n-2}[(d-1)(n+p) - 2d] - \frac{1}{2}d^{n-2}(d-1) \sum_{k|p, k < p} k\nu_d(k)\varphi(p/k).$$

From the formula of genus and Lemma 4.3.2, it is known that both $g_{n,p}^j$ (genus of $\widehat{\mathcal{V}}_{n,p}^j$) and $m_{n,p}^j$ (the number of ends of $\widehat{\mathcal{V}}_{n,p}^j$) are independent of j . So we can omit j for simplicity. The following lemma implies a complete topological description of $\mathcal{V}_{n,p}^j$ ($j \in [1, d-1]$) and $\mathcal{X}_{n,p}$.

Lemma 4.3.4. (1) S_1, S_2 are two compact Riemann surface with the same genus. $X_1 \subset S_1, X_2 \subset S_2$ are two finite set with $\#X_1 = \#X_2$. Then there exists a homeomorphism $h : S_1 \rightarrow S_2$ such that $h(X_1) = h(X_2)$.

(2) S is a compact Riemann surface and $X \subset S$ is a finite set. Then for any $\sigma \in \text{sym}(X)$, there exist a homeomorphism $h : S \rightarrow S$ such that $h(X) = X$ and $h|_X = \sigma$.

They are classical results of topology of surface, then we omit the proof.

Corollary 4.3.5 (Topological description of $\mathcal{X}_{n,p}$). *Topologically, $\mathcal{V}_{n,p}^j$ is determined by $g_{n,p}$ and $m_{n,p}$. $\mathcal{X}_{n,p}$ is determined by $g_{n,p}$, $m_{n,p}$ and $\#C_{n,p,4}$.*

Proof. It follows directly from Lemma 4.3.4 and remark 6.2.2. \square

4.4 The Galois group of $Q_{n,p}(c, z)$

The objective here is to study $Q_{n,p}(c, z)$ from the algebraic point of view by calculating its Galois group.

We apply the discussion in 4.4.1 to $Q_{n,p}$ ($n, p \geq 1$). In proof of Lemma 4.2.1, we have seen that $Q_{n,p}(c, z)$ has no multiple roots as a polynomial in $\mathbb{K}[z]$. So the splitting field of $Q_{n,p}(c, z)$ over K is a Galois extension over K and then we obtain the Galois group of $Q_{n,p}$, denoted by $G_{n,p}$

Let $\pi_{n,p} : \mathcal{X}_{n,p} \rightarrow \mathbb{C}$ be the projection from $\mathcal{X}_{n,p}$ to parameter space. From the previous content, we know that $\pi_{n,p} = \bigcup_{j=1}^{d-1} \pi_{n,p}^j$ and the set of critical points is

$$C_{n,p} := C_{n,p,1} \cup C_{n,p,2} \cup C_{n,p,3} \cup C_{n,p,4}.$$

The set of critical values $V_{n,p} = \pi_{n,p}(C_{n,p})$ is equal to the union of $\bigcup_{j=1}^{d-1} V_{n,p}^j$ together with the center of the hyperbolic components of period p (Lemma 4.3.1).

According to the discussion in 4.4.1 and Theorem 2.4.8, fixing any $c_0 \in \mathbb{C} \setminus M_d \subset \mathbb{C} \setminus V_{n,p}$, we have two group morphisms:

$$\Phi_{n,p} : \mathbb{C} \setminus V_{n,p} \rightarrow \text{sym}(\pi_{n,p}^{-1}(c_0)) \text{ and } \phi_{n,p} : \mathbb{C} \setminus V_{n,p} \rightarrow \text{sym}(Z_{n,p}).$$

($Z_{n,p}$ consists of all (n, p) -preperiodic points of f_{c_0}) and three kinds of expressions of the Galois group of $Q_{n,p} \in \mathbb{K}[z]$:

$$G_{n,p} \cong \text{Mon}_{n,p} \cong AC_{n,p}.$$

We will compute the Galois group of $Q_{n,p}(c, z)$ in terms of the expression $AC_{n,p}$.

Firstly, we will find some necessary properties that $AC_{n,p}$ should satisfy.

$$AC_{n,p} = \{\sigma_{n,p}^\gamma \mid \sigma_{n,p}^\gamma \text{ is the permutation on } Z_{n,p} \text{ induced by } \gamma \in \pi_1(\mathbb{C} \setminus V_{n,p})\}.$$

Choose any $\sigma_{n,p}^\gamma \in AC_{n,p}$. Note that γ can be seen as the element of $\pi_1(\mathbb{C} \setminus V_{l,p})$ for any $0 \leq l \leq n$. Then by the monodromy theorem of analytic continuation, we have

$$f_{c_0} \circ \sigma_{n,p}^\gamma = \sigma_{n-1,p}^\gamma \circ f_{c_0} \text{ on } Z_{n,p} \quad (4.4.1)$$

Now we turn to the expression $G_{n,p}$. For $\sigma \in G_{n,p}$, let Δ be any root of $Q_{n,p}$ in \overline{K} . If $\omega\Delta$ is also a root of $Q_{n,p}$, then $\sigma(\omega\Delta) = \omega\sigma(\Delta)$ is a root of $Q_{n,p}$, that is, σ commutes with d -th rotation. (In case $n \geq 2$, if Δ is a root of $Q_{n,p}$, then $\omega\Delta$ is always a root of $Q_{n,p}$. But this fails in case $n = 1$). Interoperating of this property in term of the expression $AC_{n,p}$, we have

$$\text{For any } \sigma_{n,p}^\gamma \in AC_{n,p}, z \in Z_{n,p}, \text{ if } \omega z \in Z_{n,p}, \text{ then } \sigma_{n,p}^\gamma(\omega z) = \omega \sigma_{n,p}^\gamma(z) \quad (4.4.2)$$

So we have had two necessary properties that $AC_{n,p}$ should satisfy. What we would like to prove is that no other restrictions are imposed on $AC_{n,p}$. Set $H_{0,p} := AC_{0,p}$, consisting of all permutations of $Z_{0,p}$ which commute with f_{c_0} (5 of Theorem 3.4.1). For $n \geq 1$, inductively define by $H_{n,p}$ the subgroup of $\text{sym}(Z_{n,p})$ consisting of all permutations satisfying (4.4.1) and (4.4.2), that is

$$\bullet \text{ For each } \sigma \in H_{n,p}, \text{ there is an unique } \sigma' \in H_{n-1,p} \text{ such that } f_{c_0} \circ \sigma = \sigma' \circ f_{c_0} \quad (4.4.1')$$

$$\bullet \text{ For each } \sigma \in H_{n,p}, z \in Z_{n,p}, \text{ if } \omega z \in Z_{n,p}, \text{ then } \sigma(\omega z) = \omega \sigma(z) \quad (4.4.2')$$

σ' is called the restriction of σ on $Z_{n-1,p}$, denoted by $\sigma|_{Z_{n-1,p}}$.

Proposition 4.4.1. *For $n \geq 1$, $AC_{n,p} = H_{n,p}$, or equivalently, $G_{n,p}$ consists of all permutations on roots of $Q_{n,p}$ which commute with f_c and the d -th rotation.*

Proof. The inclusion “ \subseteq ” is obvious. We will show “ \supseteq ” by induction on n .

As $n = 1$, suppose $Z_{0,p} = \{z_s\}_{1 \leq s \leq \nu_d(p)}$, then $Z_{1,p} = \{\omega^j z_s\}_{1 \leq s \leq \nu_d(p)}^{1 \leq j \leq d-1}$. Choose any $\sigma_1 \in H_{1,p}$, properties (4.4.1') and (4.4.2') imply that

$$\sigma_1(\omega^j z_s) = \omega^j \sigma_0(z_s). \quad (4.4.3)$$

where $\sigma_0 := \sigma|_{Z_{0,p}}$, $j \in [1, d-1]$, $s \in [1, \nu_d(p)]$. Since $H_{0,p} = AC_{0,p}$, there exists $\gamma_0 \in \pi_1(\mathbb{C} \setminus V_{0,p}, c_0)$ with $\sigma_{0,p}^{\gamma_0} = \sigma_0$. We can find $\gamma_1 \in \pi_1(\mathbb{C} \setminus V_{1,p}, c_0)$ such that $\gamma_1|_{\mathbb{C} \setminus V_{0,p}} = \gamma_0$. By the following commutative diagram

$$\begin{array}{ccc}
 \mathcal{X}_{0,p} & \xrightarrow{(c, \omega^j z)} & \mathcal{V}_{1,p}^j \\
 & \searrow (c, f_c(z)) & \swarrow (c, f_c(z)) \\
 & \mathcal{X}_{0,p} & \\
 & \downarrow \pi_{0,p} & \\
 & C &
 \end{array}$$

we have $\sigma_{1,p}^{\gamma_1} = \sigma_1$, and hence $G_{1,p} = H_{1,p}$. From (4.4), we can also see $AC_{1,p} \cong AC_{0,p}$.

Assume now $AC_{l,p} = H_{l,p}$ for $1 \leq l < n$ ($n \geq 2$). Denote $\kappa_n := \nu_d(p)(d-1)d^{n-2}$, $Z_{n-1,p} = \{w_i\}_{1 \leq i \leq \kappa_n}$. Then $Z_{n,p} = \{\omega^j z_i\}_{1 \leq i \leq \kappa_n}^{1 \leq j \leq d-1}$, where $f_{c_0}(\omega^j z_i) = w_i$. Let σ_n be any element of $H_{n,p}$. By property (4.4.1') and assumption of induction,

$$\sigma_{n-1} := \sigma_n|_{Z_{n-1,p}} \in H_{n-1,p} = AC_{n-1,p}$$

Then there exists $\gamma_{n-1} \in \pi_1(\mathbb{C} \setminus V_{n-1,p}, c_0)$ with $\sigma_{n-1,p}^{\gamma_{n-1}} = \sigma_{n-1}$. Consider $\gamma_n \in \pi_1(\mathbb{C} \setminus V_{n,p})$ such that $\gamma_n|_{\mathbb{C} \setminus V_{n-1,p}} = \gamma_{n-1,p}$, then we have

$$\sigma_{n,p}^{\gamma_n}|_{Z_{n-1,p}} = \sigma_{n-1,p}^{\gamma_{n-1}} = \sigma_{n-1}.$$

Set $\delta = (\sigma_{n,p}^{\gamma_n})^{-1} \circ \sigma_n$, then $\delta|_{Z_{n-1,p}} = id$ and by properties (4.4.2), (4.4.2'),

$$\delta = \prod_{i=1}^{\kappa_n} (j_i \ (j_i + 1) \cdots (d-1) \ 1 \cdots (j_i - 1))$$

where $(j_i \ (j_i + 1) \cdots (d-1) \ 1 \cdots (j_i - 1))$ is the cyclical permutation on $\{z_i, \omega z_i, \dots, \omega^{d-1} z_i\}$ mapping z_i to $\omega^{j_i-1} z_i$. To finish the proof of Proposition 4.4.1, we only need to find an element $\lambda \in \pi_1(\mathbb{C} \setminus V_{n,p}, c_0)$ such that $\sigma_{n,p}^\lambda = \delta$. In fact, we will show a stronger result: for any $i \in [1, \kappa_n]$, we can find $\lambda_i \in \pi_1(\mathbb{C} \setminus V_{n,p}, c_0)$ with $\sigma_{n,p}^{\lambda_i} = (j_i \ (j_i + 1) \cdots (j_i - 1))$.

Fix any $i_0 \in [1, \kappa_n]$. Suppose $\{z_{i_0}, \omega z_{i_0}, \dots, \omega^{d-1} z_{i_0}\} \subset \mathcal{V}_{n,p}^{j_0}$ for some $j_0 \in [1, d-1]$.

Let $(\hat{c}, 0) \in \mathcal{V}_{n,p}^{j_0}$ be a critical point of $\pi_{n,p}$. Then by the Implicit Function Theorem, a neighborhood of $(\hat{c}, 0)$ in $\check{\mathcal{X}}_{n,p}$ can be written as

$$\{(c, z_c) \cup (c, \omega z_c) \cup \cdots \cup (c, \omega^{d-1} z_c) \mid |c - \hat{c}| < \epsilon\}$$

where z_c is a (n, p) preperiodic point of f_c nearby 0 for $c \neq \hat{c}$ and $z_{\hat{c}} = 0$. The map $\pi_{n,p}$ is a degree d branched covering in a neighborhood of $(\hat{c}, 0)$ with only one branched point $(\hat{c}, 0)$. As c make a small turn around \hat{c} , the set $\{z_c, \omega z_c, \dots, \omega^{d-1} z_c\}$ gets a cyclical permutation with $\omega^j z_c$ mapped to $\omega^{j+1} z_c$ and other (n, p) preperiodic points of f_c get fixed. It follows that for $\gamma \in \pi_1(\mathbb{C} \setminus V_{n,p}, c_0)$ homotopic to \hat{c} , $\sigma_{n,p}^\gamma = (2 \cdots d \ 1)$ acts on some $\{z_{i_*}, \dots, \omega^{d-1} z_{i_*}\}$ such that $\{(c_0, \omega^j z_{i_*}) \mid 0 \leq j \leq d-1\} \subset \mathcal{V}_{n,p}^{j_0}$. Now we connect (c_0, z_{i_0}) and (c_0, z_{i_*}) by a curve from (c_0, z_{i_0}) to (c_0, z_{i_*}) on $\mathcal{V}_{n,p}^{j_0} \setminus \pi_{n,p}^{-1}(V_{n,p})$ and denote its projection under $\pi_{n,p}$ by β . With an abuse of notation of curves and their homotopy classes, we have $\beta \in \pi_1(\mathbb{C} \setminus V_{n,p}, c_0)$. Then

$$\lambda_i = \beta \cdot \gamma^{j_{i_0}-1} \cdot \beta^{-1}$$

satisfies our requirement. □

By Theorem 3.4.1, we have known the Galois group $G_{0,p}$. Then with Proposition 4.4.1, we can calculate $G_{n,p}$ by induction on n . In the proof of this proposition, we obtain $G_{1,p} = G_{0,p}$ and a short exact sequence

$$0 \longrightarrow \mathbb{Z}_d^{\kappa_n} \longrightarrow G_{n,p} \longrightarrow G_{n-1,p} \longrightarrow 0 \quad n \geq 2 \quad (\kappa_n = \nu_d(p)(d-1)d^{n-2})$$

We will show that $G_{n,p}$ can be expressed as the wreath product of $\mathbb{Z}_d^{\kappa_n}$ and $G_{n-1,p}$ for $n \geq 2$.

Definition 4.4.2. Let G be a group and Σ be a subgroup of $\text{sym}(\mathbb{Z}_d)$. Denote by $\Sigma \ltimes G^d$ the wreath product of G and Σ . As a set, it consists of $g = \sigma_g(g_1, \dots, g_d)$ where $g_i \in G$ and $\sigma_g \in \Sigma$. The multiplication is defined by

$$g \cdot h = \sigma_g(g_1, \dots, g_d) \cdot \sigma_h(h_1, \dots, h_d) = \sigma_g \circ \sigma_h(g_{\sigma_h(1)} \cdot h_1, \dots, g_{\sigma_h(d)} \cdot h_d).$$

Under this multiplicity, $\Sigma \ltimes G^d$ is a group with $g^{-1} = \sigma_g^{-1}(g_{\sigma_g^{-1}(1)}^{-1}, \dots, g_{\sigma_g^{-1}(d)}^{-1})$ and unit element $(0, \dots, 0)$.

Corollary 4.4.3. For $n \geq 2$, $G_{n,p} \cong G_{n-1,p} \ltimes \mathbb{Z}_d^{\nu_d(p)(d-1)d^{n-2}}$.

Proof. A nice way to visualize the action of $G_{n,p}$ on points $Z_{n,p}$ is to consider the following table:

w_1	w_2	\cdots	w_{κ_n-1}	w_{κ_n}
z_1	z_2	\cdots	z_{κ_n-1}	z_{κ_n}
ωz_1	ωz_2	\cdots	ωz_{κ_n-1}	ωz_{κ_n}
\vdots	\vdots		\vdots	\vdots
$\omega^{d-1} z_1$	$\omega^{d-1} z_2$	\cdots	$\omega^{d-1} z_{\kappa_n-1}$	$\omega^{d-1} z_{\kappa_n}$

where $Z_{n-1,p} = \{w_i\}_{i=1}^{\kappa_n}$, $\omega^j z_i$ ($0 \leq j \leq d-1$) are the preimages of w_i under f_{c_0} . An element of $G_{n,p}$ permutes the columns and within each column, it performs a cyclic shift. Algebraically, let σ be any element of $G_{n,p}$, by Proposition 4.4.1, for $1 \leq i \leq \kappa_n$,

$$\sigma(z_i) = \omega^{c_\sigma(i)} z_{s_\sigma(i)} \text{ where } c_\sigma \in \mathbb{Z}_d^{\kappa_n} \text{ and } s_\sigma \in G_{n-1,p}$$

and σ is completely determined by c_σ and s_σ . We obtain a map

$$\psi : G_{n,p} \rightarrow G_{n-1,p} \ltimes \mathbb{Z}_d^{\kappa_n} \text{ with } \psi(\sigma) = s_\sigma(c_\sigma(1), \dots, c_\sigma(\kappa_n))$$

For any $\sigma, \tau \in G_{n,p}$, $i \in [1, \kappa_n]$.

$$\begin{aligned} \sigma \cdot \tau(z_i) &= \sigma(\tau(z_i)) = \sigma(\omega^{c_\tau(i)} \cdot z_{s_\tau(i)}) = \omega^{c_\tau(i)} \cdot \sigma(z_{s_\tau(i)}) \\ &= \omega^{c_\tau(i)} \cdot \omega^{c_\sigma(s_\tau(i))} \cdot z_{s_\sigma(s_\tau(i))} = \omega^{c_\sigma(s_\tau(i)) + c_\tau(i)} \cdot z_{s_\sigma \cdot s_\tau(i)} \end{aligned}$$

Then we have

$$\begin{aligned} \psi(\sigma \cdot \tau) &= s_\sigma \cdot s_\tau(c_\sigma(s_\tau(1)) + c_\tau(1), \dots, c_\sigma(s_\tau(\kappa_n)) + c_\tau(\kappa_n)) \\ &= s_\sigma(c_\sigma(1), \dots, c_\sigma(\kappa_n)) \cdot s_\tau(c_\tau(1), \dots, c_\tau(\kappa_n)) \\ &= \psi(\sigma) \cdot \psi(\tau). \end{aligned}$$

Thus ψ is a group isomorphism. The injectivity is obvious and surjectivity is ensured by proposition 4.4.1. \square

To end this manuscript, we will provide some simple remarks on the relationship between

the Galois group of $Q_{n,p}$ and the Galois group of $q_{n,p}^j$ ($1 \leq j \leq d-1$). For $n \geq 1$, $p \geq 1$, denote by $G_{n,p}^j$ the Galois group of $q_{n,p}^j$. Note that the splitting field of $q_{n,p}^j$ are all the same as that of $Q_{0,p}$, then

$$G_{0,p} = G_{1,p}^j = G_{1,p} \text{ for } 1 \leq j \leq d-1.$$

For $n \geq 2$, by the same reason as that of Proposition 4.4.1 and corollary 4.4.3, we have $G_{n,p}^j \cong G_{n-1,p}^j \ltimes \mathbb{Z}_d^{\nu_d(p)d^{n-2}}$. There are two natural group morphisms:

$$G_{n,p} \xrightarrow{s_{n,p}^j} G_{n,p}^j \longrightarrow 0 \text{ such that } s_{n,p}^j(\sigma_{n,p}^\gamma) = \sigma_{n,p}^{\gamma_j}$$

where γ_j is the image of γ under the canonical map from $\pi_1(\mathbb{C} \setminus V_{n,p})$ to $\pi_1(\mathbb{C} \setminus V_{n,p}^j)$, and

$$0 \longrightarrow G_{n,p} \xrightarrow{i_{n,p}} G_{n,p}^1 \times \cdots \times G_{n,p}^{d-1} \text{ such that } i_{n,p}(\sigma_{n,p}^\gamma) = (\sigma_{n,p}^{\gamma_1}, \dots, \sigma_{n,p}^{\gamma_{d-1}}).$$

However, we have $G_{n,p} \not\cong G_{n,p}^1 \times \cdots \times G_{n,p}^{d-1}$ for $n \geq 1$, $d \geq 3$. Note that for any $n \geq 2$,

$$G_{n,p}^1 \times \cdots \times G_{n,p}^{d-1} \cong \prod_{j=1}^{d-1} (G_{n-1,p}^j \ltimes \mathbb{Z}_d^{\nu_d(p)d^{n-2}}) \cong (G_{n-1,p}^1 \times \cdots \times G_{n-1,p}^{d-1}) \ltimes \mathbb{Z}_d^{\nu_d(p)(d-1)d^{n-2}}$$

and

$$G_{n,p} \cong G_{n-1,p} \ltimes \mathbb{Z}_d^{\nu_d(p)(d-1)d^{n-2}}$$

By an induction on n , it reduces to show $G_{1,p} \not\cong G_{1,p}^1 \times \cdots \times G_{1,p}^{d-1}$. This is obvious because

$$G_{1,p} \cong G_{1,p}^1 \cong \cdots \cong G_{1,p}^{d-1} \cong G_{0,p}.$$

Chapter 5

Core entropy of polynomials

In this chapter, we will study the core entropy of complex polynomials and primitive major. In Section 5.1, we describe Thurston's entropy algorithm and prove the correctness of this algorithm. In Section 5.2, we give a algorithm of core entropy for rational primitive major and prove Theorem 1.2.1.

5.1 Thurston's entropy algorithm

Here we only discuss the algorithm for quadratic polynomial, the algorithm for polynomials of higher degree is similar.

Let $f_c = z^2 + c$ be a post critical finite quadratic polynomial. By the contents in Section 2.2.2, we obtain a Hubbard tree H_f and a dynamics $f_c|_{H_c} : H_c \rightarrow H_c$. Moreover, f_c maps each edge of H_f homeomorphic to the union of some edges of H_f , so it induces an *incidence matrix* D_c as follows:

Numerate the edges of H by γ_i , $i = 1, \dots, k$. Set $D_c = (a_{ij})_{k \times k}$ with $a_{ij} = 1$ if $f_c(\gamma_j) \supset \gamma_i$ and $a_{ij} = 0$ otherwise.

We denote by $\rho(D_c)$ the leading eigenvalue of D_c . By Perron-Frobenius theorem, $\rho(D_c)$ is a non-negative real number and it is also the growth rate of $\|D_c^n\|$ for any matrix norm.

5.1.1 Thurston's entropy algorithm

Since $f_c(H_c) \subset H_c$, we can define the topological entropy $h(H_c, f_c)$ as that defined in Section 2.5. It is well known that

$$h(H_c, f_c) = \log \rho(D_c).$$

In this algorithm, we need to know the structure of Hubbard tree H_c and the action of f_c on H_c , so this algorithm can't be realized in the computer. Thurston provides a method to compute the topological entropy $h(H_c, f_c)$ only by the angle of a parameter ray landing on c and draw a picture of entropy of quadratic polynomials with this algorithm in computer (see Figure 1-1)

In the following, we will describe Thurston's entropy algorithm and give a proof of this algorithm. Firstly, we give some notation:

Definition 5.1.1. *The map $\tau : \mathbb{R}/\mathbb{Z} \longrightarrow \mathbb{R}/\mathbb{Z}$ defined by $\tau(z) = 2z \pmod{\mathbb{Z}}$. If a angle θ is periodic (strictly preperiodic) under τ , we simply call θ periodic (strictly preperiodic). Let θ be any rational angle, if θ is strictly periodic, then let $c = \gamma_M(\theta)$; if θ is periodic, then let c be the center of the hyperbolic component which has $\gamma_M(\theta)$ as its root.*

Let $\theta \neq 0$ be any angle, we separate the circle into two halves by $\{\frac{\theta}{2}, \frac{\theta+1}{2}\}$, each half is a closed segment in the circle (so the boundary belong to both halves). In the whole algorithm, we assume the following two properties:

- All angles below are mod 1;
- If $\frac{\theta+1}{2}$ is periodic, replace $\frac{\theta}{2}$ by $\frac{\theta+1}{2}$.

Thurston's entropy algorithm: Define

$$Y_\theta = \{\{2^n\theta, 2^l\theta\} \mid l, n \geq -1 \text{ and } 2^n\theta \neq 2^l\theta\}$$

1. Let Σ_θ is the abstract linear space over \mathbb{R} (real number field) generated by the elements of Y_θ .
2. Define a linear map $\mathcal{A}_\theta : \Sigma_\theta \longrightarrow \Sigma_\theta$: for any basis $\{2^n\theta, 2^l\theta\} \in Y_\theta$, if $2^n\theta, 2^l\theta$ in the same half-circle, then \mathcal{A}_θ maps $\{2^n\theta, 2^l\theta\}$ to $\{2^{n+1}\theta, 2^{l+1}\theta\}$; otherwise \mathcal{A}_θ maps $\{2^n\theta, 2^l\theta\}$ to $\{2^{n+1}\theta, \theta\} + \{\theta, 2^{l+1}\theta\}$???
3. Denote by A_θ the matrix of \mathcal{A}_θ under the basis Y_θ . Compute the leading eigenvalue $\rho(A_\theta)$ of A_θ .

Here is a variant algorithm that should be faster than the original Thurston's entropy algorithm, as the matrix is considerably smaller.

Variant algorithm: Define

$$Y'_\theta = \{ \{2^n\theta, 2^l\theta\} \mid l, n \geq -1, 2^n\theta, 2^l\theta \text{ in the same half-circle } 2^n\theta \neq 2^l\theta \}$$

1'. Let Σ'_θ is the abstract linear space over \mathbb{R} generated by the elements of Y'_θ .

2'. Define a linear map $\mathcal{A}'_\theta : \Sigma'_\theta \longrightarrow \Sigma'_\theta$: for any basis $\{2^n\theta, 2^l\theta\} \in Y'_\theta$, if $2^{n+1}\theta, 2^{l+1}\theta$ in the same half-circle, then \mathcal{A}'_θ maps $\{2^n\theta, 2^l\theta\}$ to $\{2^{n+1}\theta, 2^{l+1}\theta\}$; otherwise \mathcal{A}'_θ maps $\{2^n\theta, 2^l\theta\}$ to $\{2^{n+1}\theta, \frac{\theta}{2}\} + \{\frac{\theta}{2}, 2^{l+1}\theta\}$.

3'. Denote by A_θ the matrix of \mathcal{A}'_θ under the basis Y'_θ . Compute the leading eigenvalue $\rho(A'_\theta)$ of A'_θ .

Note that A_θ and A'_θ are both non-negative matrix, so there leading eigenvalue are non-negative real number.

Theorem 5.1.2. *Let θ, c be defined as that in Definition 5.1.1, then $\log \rho(A_\theta) = \log \rho(A'_\theta)$ is the core entropy of f_c . More precise, the spectral radii $\rho(D_c)$, $\rho(A_\theta)$ and $\rho(A'_\theta)$ are all equal and furthermore, the non-trivial eigenvalues of D_c, A_θ, A'_θ off S^1 are equal with identical multiplicity.*

Proof. For a quadratic polynomial $f_c = z^2 + c$, it has unique critical point 0 and unique critical value c .

Denote by H^+ the closure of the connected component of $H \setminus \{0\}$ that contains c , $H^- = \overline{H \setminus H^+}$. Let Ω_c be the abstract linear space over R generated by the elements of

$$L_c = \{l_{p,q} \mid p, q \in P_f, p \neq q, l_{p,q} \text{ is the unique road in } H_c \text{ that connect } p, q\}$$

and Ω'_c be the abstract linear space over R generated by the elements of

$$L'_c = \{l'_{p,q} \mid p, q \in P_f \cap H^+ \text{ or } P_f \cap H^-, p \neq q, l'_{p,q} \text{ is the unique road in } H_c \text{ that connect } p, q\}$$

By the action on the basis L_c and L'_c , the map f_c induce two linear maps on Ω_c and Ω'_c respectively. Denote by F_c and F'_c the transition matrix of the two linear maps under the

basis L_c and L'_c respectively. Choosing an appropriate order of the basis, we have

$$A_c = \begin{pmatrix} F_c & 0 \\ 0 & B_c \end{pmatrix}, \quad A'_c = \begin{pmatrix} F'_c & 0 \\ 0 & B_c \end{pmatrix}, \quad B_c = \begin{pmatrix} & & J_c \\ I_c & & \\ & \ddots & \\ & & I_c \end{pmatrix}$$

where I_c is the $t_c \times t_c$ unit matrix, J_c is a $t_c \times t_c$ square matrix that has only one 1 at each row and column, and has 0 at other positions (it is a so called permutation matrix). In fact, the matrix B_c is the restrictive matrix of maps \mathcal{A}_c and \mathcal{A}'_c to the invariant subspace generated by

$$\{\{2^n\theta, 2^l\theta\} | R_f(2^n\theta) \text{ and } R_f(2^l\theta) \text{ land on a same point}\}$$

It is known that all eigenvalue of a permutation matrix are on \mathbb{S}^1 , so we only need to prove that the non-trivial eigenvalues of D_c , F_c and F'_c are equal with the identical multiplicity.

Now let $\{l_1, \dots, l_n\}$ be the basis of linear space Ω_c (resp. Ω'_c). Following the previous notation, $\gamma_1, \dots, \gamma_k$ is denoted by the edges of H_c . So we obtain a transition matrix C ($k \times n$ matrix) such that

$$\{l_1, \dots, l_n\} = \{\gamma_1, \dots, \gamma_k\}C$$

where $c_{ij} = 1$ if l_j contains γ_i and $c_{ij} = 0$ otherwise.

Lemma 5.1.3. *We have $k \leq n$ and rank $C = k$.*

The poof of this lemma will be postponed to Section 5.1.2

Therefor, by adjusting the order of $\{l_j\}$?? $\{\gamma_i\}$, the matrix C can be written as $(C_1 \ C_2)$ where C_1 is a $k \times k$ invertible matrix. So C has a right inverse $\begin{pmatrix} C_1^{-1} \\ 0 \end{pmatrix}$ and $\{\gamma_1, \dots, \gamma_k\} = \{l_1, \dots, l_n\} \begin{pmatrix} C_1^{-1} \\ 0 \end{pmatrix}$. Let f_c act on $\{l_1, \dots, l_n\}$, we have

$$\begin{aligned} \{l_1, \dots, l_n\} F_c (\text{resp. } F'_c) &= f_c \{l_1, \dots, l_n\} = f \{\gamma_1, \dots, \gamma_k\} C = \{\gamma_1, \dots, \gamma_k\} D_c C \\ &= \{l_1, \dots, l_n\} \begin{pmatrix} C_1^{-1} \\ 0 \end{pmatrix} D_c (C_1 \ C_2) \\ &= \{l_1, \dots, l_n\} \begin{pmatrix} C_1^{-1} D_c C_1 & C_1^{-1} D_c C_2 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

So we obtain $F_c(\text{resp. } F'_c) = \begin{pmatrix} C_1^{-1}D_cC_1 & C_1^{-1}D_cC_2 \\ 0 & 0 \end{pmatrix}$. It follows that the non-trivial eigenvalues of $F_c(\text{resp. } F'_c)$ and D_c are equal with identical multiplicity.

□

5.1.2 The proof of Lemma 5.1.3

Let T be a finite connected tree. Denote by $E(T)$ the set of end points of T , by $B(T)$ the set of branching points of T . The set $V(T) = E(T) \cup B(T)$ is called the set of *vertices* of T . We always assume $\#V(T) \geq 2$ (otherwise T is a point). The closure of a connected component of $T \setminus V(T)$ is called an *edge*. We numerate the edges of T by e_i , $i = 1, \dots, m$. For any $p, q \in E(T)$, there is a unique road $l_{p,q}$ in T connecting p, q , it is called a *path*. Numerate the paths by l_j , $j = 1, \dots, n$. We obtain a transition matrix $M = (a_{ij})_{m \times n}$, $a_{ij} = 1$ if l_j contains e_i and $a_{ij} = 0$ otherwise. Therefore

$$(l_1, \dots, l_n) = (e_1, \dots, e_m)M$$

For this matrix, we have the following lemma:

Lemma 5.1.4. $m \leq n$ and $\text{rank } M = m$.

Proof. The proof goes by induction on the number of branching points.

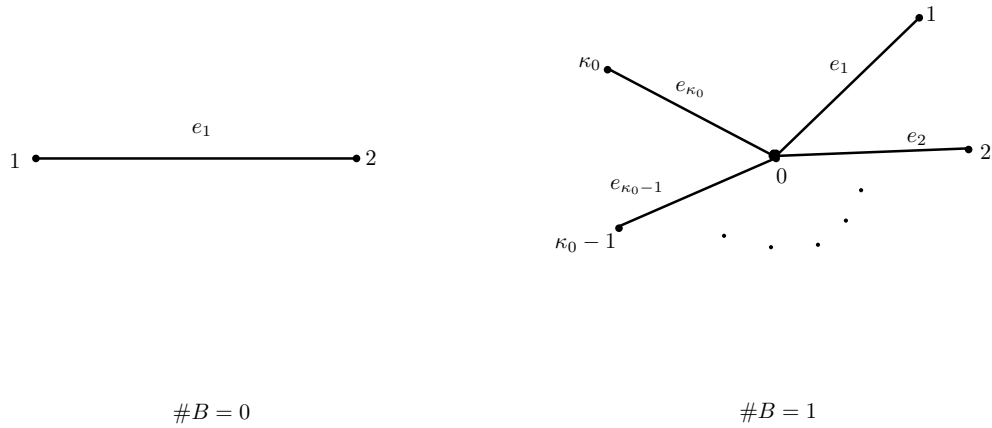


Figure 5-1:

If $\#B(T) = 0$, T is a segment with two points (left of Figure 5-1). The conclusion holds obviously.

If $\#B(T) = 1$, label the branching point by 0, label the end points by $1, \dots, \kappa_0$ ($\kappa_0 \geq 3$) clockwise starting from any point of $E(T)$ (right of Figure 5-1). We can see $m_1 = \kappa_0$, $n_1 = \binom{\kappa_0}{2} = \frac{\kappa_0(\kappa_0-1)}{2} \geq \kappa_0$. If $\kappa_0 = 3$,

$$(l_{1,2} \ l_{2,3} \ l_{1,3}) = (e_1 \ e_2 \ e_3) \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

Rank $M_1 = 3$. If $\kappa_0 > 3$,

$$\begin{aligned} (l_{1,2} \ l_{2,3} \ \cdots \ l_{\kappa_0,1} \ l_{2,\kappa_0}) &= (e_1 \ \cdots \ e_{\kappa_0}) \begin{pmatrix} 1 & & & & 1 & & \\ & 1 & 1 & & & & 1 \\ & & 1 & \ddots & & & * \\ & & & \ddots & 1 & & \\ & & & & 1 & 1 & 1 \end{pmatrix} \\ &= (e_1 \ \cdots \ e_{\kappa_0})(M'_1 *) \end{aligned}$$

M'_1 is a $\kappa_0 \times (\kappa_0 + 1)$ matrix, it is easy to see $\text{rank } M'_1 = \kappa_0$.

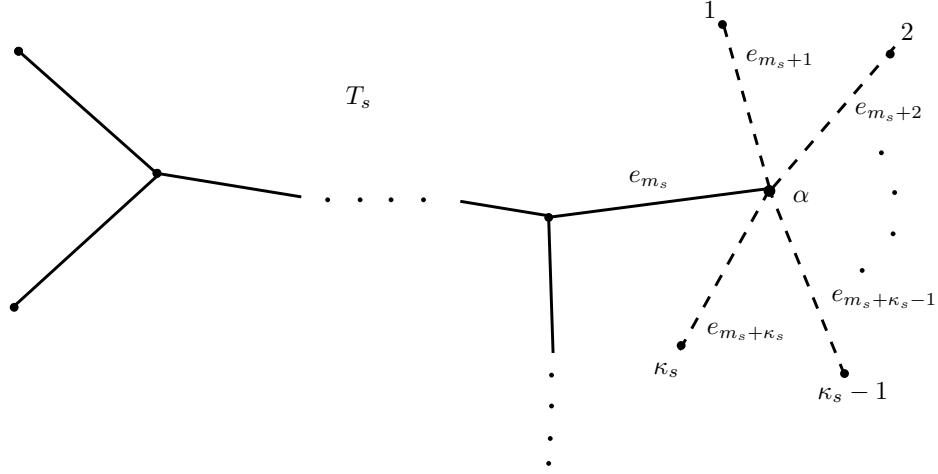
Now suppose the conclusion is established for $B(T) \leq s$ ($s \geq 1$). Let T_{s+1} be a tree with $\#B(T_{s+1}) = s + 1$. Then T_{s+1} can be obtained by adding κ_s ($\kappa_s \geq 2$) edges on some end point of a tree T_s with $B(T_s) = s$. Label the new branching point by α , the edge of T_s containing α by e_{m_s} , the new edges in T_{s+1} by $e_{m_s+1}, \dots, e_{m_s+\kappa_s}$ clockwise (see Figure 5-2). Then $m_{s+1} = m_s + \kappa_s$, $n_{s+1} > n_s + \kappa_s + \binom{\kappa_s}{2}$. Since $n_s \geq m_s$ (by the inductive assumption), it follows $n_{s+1} > m_{s+1}$. Let l_1, \dots, l_{d_s} be the paths of T_s that do not contain e_{m_s} .

$$(l_1 \ \cdots \ l_{d_s} \ l_{d_s+1} \ \cdots \ l_{n_s}) = (e_1 \ \cdots \ e_{m_s})M_s$$

By the inductive assumption, $\text{rank } M_s = m_s$. Note that l_1, \dots, l_{d_s} are also paths of T_{s+1} . Set

$$l_j^i = l_j \cup e_{m_s+i} \ (d_s + 1 \leq j \leq n_s, \ 1 \leq i \leq \kappa_s), \quad l_{u,v} = e_{m_s+u} \cup e_{m_s+v} \ (1 \leq u, v \leq s)$$

Then $\{l_1, \dots, l_{d_s}\} \cup \{l_j^i\} \cup \{l_{u,v}\}$ form the set of paths of T_{s+1} .



$$T_{s+1} = T_s \cup e_{m_s+1} \cup \dots \cup e_{m_s+\kappa_s}$$

Figure 5-2:

In case $\kappa_s = 2$,

$$\begin{aligned} (l_1 \dots l_{d_s} \ l_{d_s+1}^1 \dots l_{n_s}^1 \ l_{d_s+1}^2 \dots l_{n_s}^2 \ l_{1,2}) &= (e_1 \dots e_{m_s+2}) M_{s+1} \\ &= (e_1 \dots e_{m_s+2}) \begin{pmatrix} M_{s,1} & M_{s,2} & M_{s,2} \\ & 1 \dots 1 & 1 \\ & & 1 \dots 1 & 1 \end{pmatrix} \end{aligned}$$

where $M_{s,1}$ is a $m_s \times d_s$ matrix and $(M_{s,1} \ M_{s,2}) = M_s$. A little argument shows that the line vectors of M_{s+1} are linear independence, so $\text{rank } M_{s+1} = m_s + 2 = m_{s+1}$.

In case $\kappa_s = 3$,

$$\begin{aligned} (l_1 \dots l_{d_s} \ l_{d_s+1}^1 \dots l_{n_s}^1 \ l_{1,2} \ l_{2,3} \ l_{1,3} \dots) &= (e_1 \dots e_{m_s} \ e_{m_s+1} \ e_{m_s+2} \ e_{m_s+3}) M_{s+1} \\ &= (e_1 \dots e_{m_s} \ e_{m_s+1} \ e_{m_s+2} \ e_{m_s+3}) \begin{pmatrix} M_s & & * \\ * & M'_s & * \end{pmatrix} \\ &= (e_1 \dots e_{m_s} \ e_{m_s+1} \ e_{m_s+2} \ e_{m_s+3}) \begin{pmatrix} M_s & & & * \\ & 1 & & 1 \\ & * & 1 & 1 & * \\ & & 1 & 1 \end{pmatrix} \end{aligned}$$

$\text{Rank } M_{s+1} = \text{rank } M_s + 3 = m_s + 3 = m_{s+1}.$

In case $\kappa_s > 3$,

$$\begin{aligned}
& (l_1 \cdots l_{d_s} \ l_{d_s+1}^1 \cdots l_{n_s}^1 \ l_{1,2} \ l_{2,3} \cdots l_{\kappa_s,1} \ l_{2,\kappa_s} \cdots) \\
&= (e_1 \cdots e_{m_s} \ e_{m_s+1} \cdots e_{m_s+\kappa_s}) M_{s+1} \\
&= (e_1 \cdots e_{m_s} \ e_{m_s+1} \cdots e_{m_s+\kappa_s}) \begin{pmatrix} M_s & & * \\ * & M'_s & * \end{pmatrix} \\
&= (e_1 \cdots e_{m_s} \ e_{m_s+1} \cdots e_{m_s+\kappa_s}) \begin{pmatrix} M_s & & & & & * \\ & 1 & & & 1 & \\ & & 1 & 1 & & 1 \\ & & & 1 & \ddots & \\ * & & & & \ddots & 1 \\ & & & & & 1 & 1 & 1 \end{pmatrix}
\end{aligned}$$

M'_s is a $\kappa_s \times (\kappa_s + 1)$ matrix. It is easy to check that $\text{rank} M'_s = s$, then we have

$$\text{rank} M_{s+1} = \text{rank } M_s + \text{rank } M'_s = m_s + \kappa_s = m_{s+1}.$$

□

We will use this lemma to prove Lemma 5.1.3

Proof of Lemma 5.1.3. Here we only treat the case that $\{l_1 \dots, l_n\}$ is the basis of linear space Ω_c . For the case that $\{l_1 \dots, l_n\}$ is the basis of linear space Ω'_c , the proof is completely the same.

Numerate by S_i , $i = 1, 2, \dots, i_0$ the closure of connected components of $H \setminus P_f$ such that $B(S_i) \leq B(S_{i+1})$. Set

$$\begin{aligned}
J_i &= \{l \in L_A \mid l \text{ connects two points of } E(S_i)\} \\
\Gamma_i &= \{\gamma \in \{\gamma_j\}_{1 \leq j \leq k} \mid \gamma \subset S_i\}
\end{aligned}$$

with $\#J_i = n_i$ and $\#\Gamma_i = m_i$. Then we have

$$(J_i \cdots J_{i_0} \cdots) = (\Gamma_1 \cdots \Gamma_{i_0}) \begin{pmatrix} C_1 & & \\ & \ddots & \\ & & C_{i_0} \end{pmatrix} \begin{pmatrix} * \\ \\ \end{pmatrix}$$

where C_i is a $m_i \times n_i$ matrix. By lemma 5.1.4, $\text{rank } C_i = m_i$, it follows

$$\text{rank } C = \sum_{i=1}^{i_0} m_i = k .$$

□

5.2 Core entropy of rational primitive major in torus model

The objective here is to prove Theorem 1.2.1. In Section 5.2.1 and 5.2.2, we establish the linear relationship between Hausdorff dimension and topological entropy of a general compact F -invariant set in torus. This implies term 1 of Theorem 1.2.1. In section 5.2.3, we define the combinatorial Hubbard tree for rational primitive majors and provide a algorithm to compute the core entropy for such a major. Finally, we give the dynamical explanation of the invariant lamination and combinatorial Hubbard tree, then finish the proof of Theorem 1.2.1.

5.2.1 The Hausdorff dimension

Let $A \subset \mathbb{T}^2$. Its Hausdorff dimension is defined as follows:

Fix $s > 0$. Let $\mathcal{U} = \{B_i, i \in \mathbb{N}\}$ be a countable cover (not necessarily closed or open) of A . Denote by $|B_i|$ the diameter of B_i and by $|\mathcal{U}|$ the supremum of $|B_i|$.

For each \mathcal{U} , we consider $\sum_{B_i \in \mathcal{U}} |B_i|^s$ as an approximation of the 'size' of A in dimension s . In order to be accurate, it's better to refine the cover by taking smaller pieces.

Fix $r > 0$. We may thus consider \mathcal{U} with $|\mathcal{U}| < r$. Again to be accurate, it's better to throw away pieces in \mathcal{U} that do not touch A , or better, it's better to consider covers that are as economic as possible. Thus we consider $\inf_{\mathcal{U}, |\mathcal{U}| < r} \sum_{B_i \in \mathcal{U}} |B_i|^s$ as a further approximation of the 'size' of A in dimension s .

When we let $r \searrow 0$, we have less choices of \mathcal{U} , so the above quantity increases. We may

then consider

$$H^s(A) := \lim_{r \searrow 0} \inf_{\mathcal{U}, |\mathcal{U}| < r} \sum_{B_i \in \mathcal{U}} |B_i|^s = \sup_{r > 0} \inf_{\mathcal{U}, |\mathcal{U}| < r} \sum_{B_i \in \mathcal{U}} |B_i|^s$$

as the actual 'size' of A in dimension s . This is called 'the Hausdorff measure' of A in dimension s .

What is then the best dimension in which to measure A ? When s is too small, $H^s(A) = +\infty$ (a square has an infinite 1-dimensional length). When s is too big, $H^s(A) = 0$ (a line has zero 2-dimensional area). It's easy to prove that the transition from ∞ to 0 happens at a single value of s , this is the Hausdorff dimension of A .

Replacing a random cover by a cover with closed squares.

For our $A \subset \mathbb{T}^2$, and \mathcal{U} a countable cover with $|\mathcal{U}| < 1/2$, note that if we replace each B_i in \mathcal{U} by the smallest closed square \widehat{B}_i containing it, we have $\text{size}(\widehat{B}_i) \leq |B_i| \leq |\widehat{B}_i| = \sqrt{2} \text{size}(\widehat{B}_i)$, and $\widehat{\mathcal{U}} = \{\widehat{B}_i, i \in \mathbb{N}\}$ is again a countable cover of A .

So

$$\sum_{\widehat{B}_i \in \widehat{\mathcal{U}}} \text{size}(\widehat{B}_i)^s \leq \sum_{B_i \in \mathcal{U}} |B_i|^s \leq \sqrt{2}^s \sum_{\widehat{B}_i \in \widehat{\mathcal{U}}} \text{size}(\widehat{B}_i)^s$$

So the quantity $\widehat{H}^s(A) := \sup_{r > 0} \inf_{\widehat{\mathcal{U}}, |\widehat{\mathcal{U}}| < r} \sum_{\widehat{B}_i \in \widehat{\mathcal{U}}} \text{size}(\widehat{B}_i)^s$ satisfies $\widehat{H}^s(A) \leq H^s(A) \leq \sqrt{2}^s \widehat{H}^s(A)$.

We may thus use the transition s -value of $\widehat{H}^s(A)$ as the Hausdorff dimension of A . It is denoted by $\text{H.dim}(A)$.

The standard covers

Fix now an integer d . We will consider the torus expansion $F : \mathbb{T}^2 \rightarrow \mathbb{T}^2, \begin{pmatrix} x \\ y \end{pmatrix} \mapsto d \begin{pmatrix} x \\ y \end{pmatrix}$. We want to measure 'sizes' and dimensions of invariant subsets of F . We will adapt the argument of Furstenberg to our situation ([Fu], Prop. III.1).

For each n , we will cover the torus by the standard level- n closed squares $\left[\frac{p}{d^n}, \frac{p+1}{d^n}\right] \times \left[\frac{q}{d^n}, \frac{q+1}{d^n}\right]$ with $p, q \in \{0, 1, \dots, d^n - 1\}$. There are d^{2n} such squares each of size $1/d^n$.

Fix $A \subset \mathbb{T}^2$. For each n , denote by $\nu(A, n)$ the number of standard level- n squares

intersecting A and, whenever the limit exist,

$$\mathcal{C}(A) := \lim_{n \rightarrow \infty} \frac{\log \nu(A, n)}{n}, \quad d(A) := \lim_{n \rightarrow \infty} \frac{\log \nu(A, n)}{\log \frac{1}{\text{size}(\text{level-}n \text{ square})}} = \lim_{n \rightarrow \infty} \frac{\log \nu(A, n)}{n \log d} = \frac{\mathcal{C}(A)}{\log d}$$

Proposition 5.2.1. *If $A \subset \mathbb{T}^2$ satisfies that $F(A) \subset A$, then $\mathcal{C}(A), d(A)$ exist and $d(A) \geq \text{H.dim}(A)$. If furthermore A is closed, then $d(A) = \text{H.dim}(A)$.*

Proof. If B_{n+m} is a level- $n+m$ standard square intersecting A , then B_{n+m} is contained in a level- n square B_n intersecting A , the image $F^n(B_{n+m}) =: B_m$ is a level- m square intersecting $F^n(A) \subset A$, and $B_{n+m} \mapsto (B_n, B_m)$ is injective (as $F^n : B^n \rightarrow \mathbb{T}^2$ is injective in the interior). See the following schematic picture.

$$\begin{array}{ccc} & & \mathbb{T}^2 \\ & \nearrow_{F^n} & \cup \\ B_n & & B_m \\ \cup & \nearrow_{F^n} & \\ B_{n+m} & & \end{array}$$

So

$$\nu(A, n+m) \leq \nu(A, n) \cdot \nu(A, m).$$

This submultiplicativity implies that $\mathcal{C}(A)$ therefore $d(A)$ exist.

Let $s > d(A)$ be arbitrary. Then for n large enough, we have

$$\frac{\log \nu(A, n)}{\log \frac{1}{\text{size}(\text{level-}n \text{ square})}} = \frac{\log \nu(A, n)}{n \log d} < s, \quad \text{so } \nu(A, n) < d^{ns}.$$

Let \mathcal{U}_n be the cover of A consisting of the set of level- n standard squares intersecting A . Then $\sum_{B_i \in \mathcal{U}_n} \text{size}(B_i)^s = \frac{\nu(A, n)}{d^{ns}} < 1$. It follows that

$$\inf_{\hat{\mathcal{U}}, |\hat{\mathcal{U}}| < 1/d^n} \sum_{\hat{B}_i \in \hat{\mathcal{U}}} \text{size}(\hat{B}_i)^s < 1, \quad \text{therefore} \quad \hat{H}^s(A) := \sup_{r > 0} \inf_{\hat{\mathcal{U}}, |\hat{\mathcal{U}}| < r} \sum_{\hat{B}_i \in \hat{\mathcal{U}}} \text{size}(\hat{B}_i)^s \leq 1.$$

Therefore the transition value of $\hat{H}^s(A)$ happens at at most s . So $\text{H.dim}(A) \leq s$.

But $s > d(A)$ were arbitrary, so $\text{H.dim}(A) \leq d(A)$.

Let now $\delta < d(A)$ be arbitrary. We want to show $\text{H.dim}(A) \geq \delta$.

For n large enough, we have

$$\frac{\log \nu(A, n)}{n \log d} > \delta, \quad \text{so } \nu(A, n) > d^{n\delta} \quad \text{and} \quad \sum_n \frac{\nu(A, n)}{d^{n\delta}} = +\infty.$$

Geometrically, this means that the sum over n of the δ -dimensional volume of the level- n standard cover of A is equal to the infinity. In other words,

$$\sum_{U \text{ a standard square, } U \cap A \neq \emptyset} \text{size}(U)^\delta = +\infty. \quad (5.2.1)$$

We need to show that, there is a constant $c > 0$, such that for any $r > 0$ sufficiently small, and for any cover \mathcal{U} of A with $|\mathcal{U}| < r$, we have

$$\sum_{B_i \in \mathcal{U}} |B_i|^\delta \geq c > 0.$$

By the discussion in the previous section we may assume B_i to be of square shape and $|B_i|$ to be size of B_i . For each B_i , let n_i be maximal so that $|B_i| < 1/d^{n_i}$. Then

$$\frac{1}{d^{n_i+1}} \leq |B_i| < \frac{1}{d^{n_i}}$$

and there are at most 4 level- n_i standard squares whose union contains $\overline{B_i}$ in the interior. Replacing each B_i by the collection of these standard squares, we get a new cover \mathcal{V} of A . Furthermore

$$\overline{A} \subset \text{interior}\left(\bigcup_{B \in \mathcal{V}} B\right). \quad (5.2.2)$$

This condition will play an important role in the following. Then

$$\sum_{B \in \mathcal{V}} \text{size}(B)^\delta \leq 4 \sum_{B_i \in \mathcal{U}} \left(\frac{1}{d^{n_i}}\right)^\delta = 4d^\delta \sum_{B_i \in \mathcal{U}} \left(\frac{1}{d^{n_i+1}}\right)^\delta \leq 4d^\delta \sum_{B_i \in \mathcal{U}} |B_i|^\delta.$$

It thus suffices to prove that, for any $r > 0$ sufficiently small, and for any cover \mathcal{V} of A by standard squares of various levels with $|\mathcal{V}| < r$ and satisfying (5.2.2), we have

$$\sum_{B \in \mathcal{V}} \text{size}(B)^\delta \geq c' = c/(4d^\delta) > 0.$$

We now assume that A is compact. It suffices then to show that, for any finite cover \mathcal{V} of A by standard squares satisfying (5.2.2), the δ -dimensional volume of \mathcal{V} must be at least 1.

Choose now such a cover \mathcal{V} , with pieces B_1, \dots, B_k . Set $\lambda_i = \text{size}(B_i)^\delta$, $i = 1, \dots, k$.

Suppose to the contrary that $\sum_{i=1}^k \lambda_i < 1$, then $\sum_n (\sum_{i=1}^k \lambda_i)^n < \infty$. So

$$\sum_n \sum_{(i_1, \dots, i_n) \in \{1, \dots, k\}^n} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_n} < \infty. \quad (5.2.3)$$

Considering \mathcal{V} as a depth-1 puzzle. For each $B_i \in \mathcal{V}$, say of size $\frac{1}{d^{n_i}}$, the map F^{n_i} sends B_i affinely onto \mathbb{T}^2 . The function system $\{(B_i, F^{n_i})\}$ generates puzzle pieces of deeper levels. The ordered strings $(i_1, \dots, i_n) \in \{1, \dots, k\}^n$ are in 1-1 correspondence with the depth- n puzzle pieces. The δ -dimensional volume of the piece with string (i_1, \dots, i_n) is $\lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_n}$. Note that we do not require that these deeper puzzle pieces all intersect A , although the union of the pieces of the same depth contain A .

So the inequality (5.2.3) means that the δ -dimensional volume of the puzzle pieces sum up to a finite value.

Let N be the highest level of the finitely many pieces in \mathcal{V} . Let now U a standard square intersecting A . We claim that if U is of level greater than N then U is contained in a \mathcal{V} -piece.

Proof. If there is $x \in \text{interior}(U) \cap A$ then $x \in B_i \in \mathcal{V}$ for some B_i . As $\text{size}(U) < \text{size}(B_i)$ we have $U \subset B_i$. Assume now $x \in U \cap A \subset \partial U$ and no B_i in \mathcal{V} contains U . Then there is an arc in the interior of U ending at x and being outside of $\bigcup_i B_i$. It follows that $x \in (\partial \bigcup_i B_i) \cap A$, contradicting (5.2.2). The claim is thus proved.

Let now U be a standard square intersecting A . Let us associate U to V , the smallest puzzle piece of pullbacks of \mathcal{V} such that V contains U . We claim then $\text{level}(U) - \text{level}(V) \leq N$. This is because $F^{\text{level}(V)}$ maps V onto \mathbb{T}^2 and U to a square U' of level $\text{level}(U) - \text{level}(V)$, and $U' \cap A \neq \emptyset$. Now U' is not contained in any puzzle piece of \mathcal{V} . So it has level at most N by the above claim.

There is thus an integer $C > 0$ such that the map $U \mapsto V$ is at most C to 1.

It follows that

$$\sum_{U \text{ a standard square}, U \cap A \neq \emptyset} \text{size}(U)^\delta \leq C^\delta \sum_{V \text{ a puzzle piece}} \text{size}(V)^\delta \stackrel{(5.2.3)}{<} \infty.$$

This contradicts (5.2.1).

5.2.2 Topological entropy

We will use Proposition 2.5.6 to prove the following dimension formula.

Proposition 5.2.2. *Let $F : \mathbb{T}^2 \rightarrow \mathbb{T}^2$, $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto d \begin{pmatrix} x \\ y \end{pmatrix}$. For A a closed subset of \mathbb{T}^2 such that $F(A) \subset A$, then we have*

$$h(A, F) = \mathcal{C}(A) = (\log d) \cdot \text{H.dim}(A).$$

Proof. The right side equality is proved by Proposition 5.2.1. So what we need to prove is $h(A, F) = \mathcal{C}(A)$.

To prove this equation, we will construct a special sequence of open covers $\{\mathcal{U}_l\}_{l \geq 1}$ of A such that it satisfies three properties:

1. This sequence of covers of A satisfies the two property of Proposition 2.5.6.
2. For any $l \geq 1$, $h(A, \mathcal{U}_l, F) = h(A, \mathcal{U}_0, F)$.
3. There exists a constant C such that $\frac{1}{C}N(\bigvee^n \mathcal{U}_0) \leq \nu(A, n) \leq C \cdot N(\bigvee^n \mathcal{U}_0)$

Then by Proposition 2.5.6, we obtain $h(A, F) = \mathcal{C}(A)$. So in the following, we only need to construct such a sequence of open covers of A and check that it satisfies the three properties above.

The construction of $\{\mathcal{U}_l\}_{l \geq 1}$ and Property 1.

Now we construct a sequence of open covers $\{\tilde{\mathcal{U}}_l\}$ of \mathbb{T}^2 such that $\tilde{\mathcal{U}}_l$ consists of four kinds of open sets:

- $\left(\frac{p}{d^l}, \frac{p+1}{d^l}\right) \times \left(\frac{q}{d^l}, \frac{q+1}{d^l}\right)$, $0 \leq p, q \leq d^l - 1$; these are the interior of level- l standard squares;
- $\left(\frac{p}{d^l} - \frac{1}{d^{l+1}}, \frac{p}{d^l} + \frac{1}{d^{l+1}}\right) \times \left(\frac{q}{d^l}, \frac{q+1}{d^l}\right)$, $0 \leq p, q \leq d^l - 1$; these are open rectangles covering vertical open segments of the level- l grid;

- $\left(\frac{p}{d^l}, \frac{p+1}{d^l}\right) \times \left(\frac{q}{d^l} - \frac{1}{d^{l+1}}, \frac{q}{d^l} + \frac{1}{d^{l+1}}\right)$, $0 \leq p, q \leq d^l - 1$; these are open rectangles covering horizontal open segments of the level- l grid;
- $\left(\frac{p}{d^l} - \frac{1}{d^{l+1}}, \frac{p}{d^l} + \frac{1}{d^{l+1}}\right) \times \left(\frac{q}{d^l} - \frac{1}{d^{l+1}}, \frac{q}{d^l} + \frac{1}{d^{l+1}}\right)$, $0 \leq p, q \leq d^l - 1$, these are small open squares covering the nodes of the level- l grid.

From the construction of $\tilde{\mathcal{U}}_l$ ($l \geq 1$), it is easy to see that $\tilde{\mathcal{U}}_{l+1} \prec \tilde{\mathcal{U}}_l$ and for any $U \in \tilde{\mathcal{U}}_{l+1}$, $F(U) \in \tilde{\mathcal{U}}_l$.

For $l \geq 1$, set

$$\mathcal{U}_l = \{U \in \tilde{\mathcal{U}}_l \mid U \cap A \neq \emptyset\}$$

Then the sequence $\{\mathcal{U}_l\}_{l \geq 1}$ is just what we need. The property 1 follows directly from the construction of $\{\mathcal{U}_l\}_{l \geq 1}$.

Check property 2.

Lemma 5.2.3. *For each $l \geq 1$, the open cover \mathcal{U}_l of A satisfies the following properties:*

- (1) *For any $l \geq 1, n \geq 0$, $\mathcal{U}_{l+n} \subset F^{-n}(\mathcal{U}_l) \vee \mathcal{U}_l$.*
- (2) *For $n \geq 0$, let V be any member of $\bigvee^n \mathcal{U}_l$. Then V is a rectangle and the restriction of F^{n+l} on V is injective.*

Proof. (1) For any $U_{n+l} \in \mathcal{U}_{l+n}$ with $l \geq 1, n \geq 0$, $F^n(U_{n+l}) \subset \mathcal{U}_l$ and there exists $U_l \subset \mathcal{U}_l$ such that $U_{n+l} \subset U_l$. It implies $U_{n+l} \in F^{-n}(\mathcal{U}_l) \vee \mathcal{U}_l$, and hence $\mathcal{U}_{l+n} \subset F^{-n}(\mathcal{U}_l) \vee \mathcal{U}_l$.

- (2) Fix $l \geq 1$. Any member of \mathcal{U}_l is rectangle and according to the construction of \mathcal{U}_l , the restriction of F^l to any component of \mathcal{U}_l is injective. So the property (2) holds as $n = 0$.

Now suppose the property (2) holds for $n \geq 0$. Let V_l^{n+1} be a member of $\bigvee^{n+1} \mathcal{U}_l$. Then there exist a member V_l^n of $\bigvee^n \mathcal{U}_l$ and a member V_l^0 of \mathcal{U}_l such that $V_l^{n+1} = F^{-(n+1)}(V_l^0) \cap V_l^n$. By the inductive assumption, V_l^n is a rectangle and the restriction of F^{l+n} to V_l^n is injective. It follows that $F^{n+1}(V_l^n) \cap V_l^0$ is a rectangle and there is a unique connected component U of $F^{-(n+1)}(V_l^0)$ that intersects V_l^n . Then $V_l^{n+1} = U \cap V_l^n$ and $F^{n+1} : V_l^{n+1} \rightarrow V_l^0$ is injective. Since V_l^0 is a member of \mathcal{U}_l , the restriction of F^l to it is injective, the restriction of F^{n+l+1} to V_l^{n+1} is injective. Note that U is a rectangle, thus $V_l^{n+1} = U \cap V_l^n$ is a rectangle.

□

Lemma 5.2.4. *For $l \geq 1$, $n \geq 0$, we have*

- (1) $\mathcal{U}_{l+n} \subset \bigvee^n \mathcal{U}_l$.
- (2) *There exists a constant K such that any member of $\bigvee^n \mathcal{U}_l$ that intersects A intersects at most K members of \mathcal{U}_{l+n} .*

Proof. (1) Fix $l \geq 1$. Set $\mathcal{V}_{l,0} := \mathcal{U}_l$, then $\mathcal{U}_l \subset \mathcal{V}_{l,0} \subset \bigvee^0 \mathcal{U}_l$. For $n \geq 1$, We inductively define $\mathcal{V}_{l,n} = F^{-1}(\mathcal{V}_{l,n-1}) \vee \mathcal{U}_l$ and suppose

$$\mathcal{U}_{l+n} \subset \mathcal{V}_{l,n} \subset \bigvee^n \mathcal{U}_l \quad \text{for some } n \geq 0.$$

Then by (2.5.2) we have

$$F^{-1}(\mathcal{U}_{l+n}) \vee \mathcal{U}_l \subset F^{-1}(\mathcal{V}_{l,n}) \vee \mathcal{U}_l = \mathcal{V}_{l,n+1} \subset F^{-1}(\bigvee^n \mathcal{U}_l) \vee \mathcal{U}_l.$$

According to term (1) of Lemma 5.2.3 and (2.5.2),

$$\mathcal{U}_{l+n+1} \subset F^{-1}(\mathcal{U}_{l+n}) \vee \mathcal{U}_{l+n} \subset F^{-1}(\mathcal{U}_{l+n}) \vee \mathcal{U}_l$$

and by (2.5.1),

$$F^{-1}(\bigvee^n \mathcal{U}_l) \vee \mathcal{U}_l \subset \bigvee^{n+1} \mathcal{U}_l$$

So we have

$$\mathcal{U}_{l+n+1} \subset \mathcal{V}_{l,n+1} \subset \bigvee^{n+1} \mathcal{U}_l$$

- (2) We can choose an integer K such that any member of \mathcal{U}_0 intersects at most K members of \mathcal{U}_l . For $n \geq 0, l \geq 1$, let

$$V_l^n = F^{-n}(V^{l,n}) \vee F^{-(n-1)}(V^{l,n-1}) \vee \dots \vee V^{l,0} \text{ where } V^{l,n}, \dots, V^{l,0} \in \mathcal{U}_l$$

be a component of $\bigvee^n \mathcal{U}_l$ intersecting A and set $V^{n+l,1}, \dots, V^{n+l,\alpha}$ are all members of \mathcal{U}_{n+l} that intersect V_l^n . Then each element of $\{F^{n+l-1}(V^{n+l,i})\}_{i=1}^\alpha$ is a member of \mathcal{U}_0 and $F^{n+l-1}(V_l^n) \subset F^{l-1}(V^{l,n})$ intersects $F^{n+l-1}(V^{n+l,i})$ for $i \in [1, \alpha]$. By term (2) of Lemma 5.2.3, $F^{n+l-1} : V_l^n \rightarrow F^{n+l-1}(V_l^n)$ is a homeomorphism, so the members of $\{F^{n+l-1}(V^{n+l,i})\}_{i=1}^\alpha$ are pairwise different. It follows $\alpha \leq K$.

□

Now for any $l \geq 1, n \geq 1$, denote by $\mathcal{V}_{l,n}$ a sub-cover of A belonging to $\bigvee^n \mathcal{U}_l$ with minimal cardinality. By Lemma 5.2.4, each member of $\mathcal{V}_{l,n}$ intersects at most K members of \mathcal{U}_{n+l} . So the number of members of \mathcal{U}_{n+l} that intersect some element of $\mathcal{V}_{l,n}$ is less than $K \cdot \#\mathcal{V}_{l,n}$. Since each member of \mathcal{U}_{n+l} intersects A , each member of \mathcal{U}_{n+l} intersects some member of $\mathcal{V}_{l,n}$. By (1) of Lemma 5.2.4, the cover \mathcal{U}_{l+n} is a sub-cover of A belonging to $\bigvee^n \mathcal{U}_l$, thus we have

$$\#\mathcal{V}_{l,n} \leq \#\mathcal{U}_{n+l} \leq K \cdot \#\mathcal{V}_{l,n} \quad (5.2.4)$$

It follows that for any $l \geq 1$

$$\begin{aligned} h(A, \mathcal{U}_l, F) &= \lim_{n \rightarrow \infty} \frac{1}{n} \log(\#\mathcal{V}_{l,n}) \stackrel{(5.2.4)}{=} \lim_{n \rightarrow \infty} \frac{1}{n} \log(\#\mathcal{U}_{l+n}) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n+l} \log(\#\mathcal{U}_{n+l}) \cdot \lim_{n \rightarrow \infty} \frac{n+l}{n} = h(A, \mathcal{U}_0, F). \end{aligned} \quad (5.2.5)$$

Check Property 3

On one hand, there are $\nu(A, n)$ level- n standard squares intersecting A , and each square is covered by exactly 9 pieces in \mathcal{U}_n . The collection of these $9\nu(A, n)$ (not necessarily distinct) pieces covers A and contains every piece of \mathcal{U}_n , so $\#\mathcal{U}_n \leq 9\nu(A, n)$. On the other hand, each $U \in \mathcal{U}_n$ intersects at most four level- n standard squares. The collection of these $4\#\mathcal{U}_n$ (not necessarily distinct) level- n standard squares covers A and contains all level- n standard squares that intersect A , so $\nu(A, n) \leq 4\#\mathcal{U}_n$. It follows that

$$\frac{1}{4}\nu(A, n) \leq \#\mathcal{U}_n \leq 9\nu(A, n), \quad n \in \mathbb{N}.$$

Combining also (5.2.4), we have

$$\frac{1}{9}N(\bigvee^n \mathcal{U}_0) \leq \nu(A, n) \leq 4KN(\bigvee^n \mathcal{U}_0)$$

□

5.2.3 Rational primitive majors

Thurston gave a torus model to construct an invariant lamination from a given degree d primitive major. Let

$$m = \{ \Theta_1, \dots, \Theta_s \}$$

be any degree d primitive major (or critical portrait). Remembering the notation and construction in Section 2.2.3, we obtain an increasing sequence of sets $\{b_i(m)\}_{i \geq 0}$ belonging to the closed good region of m and a degree- d invariant lamination $\overline{b_\infty}(m) = b_\infty(m) \cup b'_\infty(m)$. The map \wp defined in Section 1.2 maps $\overline{b_\infty}$ to $L(m)$ which is a degree d invariant lamination having m as its major.

The objective of this section is to define a transition matrix whose leading eigenvalue encodes the core entropy of $L(m)$.

The combinatorial Hubbard tree

Assume that m is a rational primitive major of degree d . We will define a compact invariant set $TH(m) \subset \overline{b_\infty}(m)$ corresponding to the status of the Hubbard tree of a postcritical finite polynomial.

Set

$$P(m) = \{d^n \theta \mid n \geq 0, \theta \in \bigcup_{j=1}^s \Theta_j\} \quad \text{and} \quad TP(m) = \{(\theta, \theta) \mid \theta \in P(m)\}.$$

(These sets correspond to the postcritical set of a polynomial). The torus can be regarded as the quotient of closed unit square by identifying $(0, y)$ with $(1, y)$ and $(x, 0)$ with $(x, 1)$ where $x, y \in [0, 1]$. As m is a rational critical portrait, the set $TP(m)$ is a finite F -forward invariant set in the diagonal of the torus.

For any point $(1, 0) \neq (x, y) \in \mathbb{T}^2$, denote by $l_{x,y}$ the path made up of a horizontal segment and a vertical segment connecting (x, y) to the diagonal points (x, x) and (y, y) . Define $l_{1,0} = (1, 0)$.

For any point $(x, y) \in \mathbb{T}^2$ not on the diagonal, there are four horizontal and vertical circles in \mathbb{T}^2 passing through (x, y) and (y, x) . These circles partition \mathbb{T}^2 into four rectangles. Two of them intersect the diagonal and they are squares. If (x, y) is not on the boundary of unit square, only one of the above two squares can be seen in the closed unit square. It

is denoted by $S_{x,y}^+$ and the other is denoted by $S_{x,y}^-$. If (x,y) is on the boundary of unit square, both of the above two squares belong to the unit square. In this case, we denote the left-lower square by $S_{x,y}^+$ and the other by $S_{x,y}^-$. The boundary curves of $S_{x,y}^+$ and $S_{x,y}^-$ are denoted by $C_{x,y}^+$ and $C_{x,y}^-$ respectively. If (x,y) belongs to the diagonal of \mathbb{T}^2 , we define $C_{x,y}^+ = C_{x,y}^- = (x,y)$.

For $(x,y) \in \mathbb{T}^2, \delta \in \{\pm\}$, we say that $C_{x,y}^\delta$ **separates** $TP(m)$ if both components of $\mathbb{T}^2 \setminus C_{x,y}^\delta$ contains points of $TP(m)$. It is obvious that $C_{x,y}^\delta$ separates or intersects $TP(m)$ iff $C_{x,y}^{\bar{\delta}}$ separates or intersects $TP(m)$, where $\bar{\delta}$ is the opposite symbol of δ . Now define

$$TH_0(m) = \{ (x,y) \in b_\infty(m) \mid C_{x,y}^+ \text{ separates or intersects } TP(m) \} \cup TP(m).$$

Denote by $TH_1(m)$ the cluster set of $TH_0(m)$ and $TH(m) := TH_0(m) \cup TH_1(m) = \overline{TH_0(m)}$.

Lemma 5.2.5. *For any $(x,y) \in TH_1(m)$, $C_{x,y}^+$ also intersects or separates $TP(m)$.*

Proof. Choose a sequence of points $\{(x_n, y_n)\} \subset TH_0(m)$ such that $\lim_{n \rightarrow \infty} (x_n, y_n) = (x, y)$. It is equivalent that C_{x_n, y_n}^+ converge to $C_{x,y}^+$ in the Hausdorff topology as $n \rightarrow \infty$.

Assume at first that there is a sub-sequence, denoted also by $\{(x_n, y_n)\}$, such that each C_{x_n, y_n}^+ intersects $TP(m)$. Since $TP(m)$ is finite, without loss of generalization, we may assume that all C_{x_n, y_n}^+ contain a common point $a \in TP(m)$. Since C_{x_n, y_n}^+ belongs to the ϵ_n neighborhood of $C_{x,y}^+$ with $\epsilon_n \rightarrow 0$ as $n \rightarrow \infty$, the point a must belong to $C_{x,y}^+$. So $C_{x,y}^+$ intersects $TP(m)$ as well.

Assume now that we are in the remaining case, i.e. for sufficiently large n , C_{x_n, y_n}^+ separates but does not intersect $TP(m)$. Since $TP(m)$ is finite set, we can assume that there exist two points $a, b \in TP(m)$ such that every C_{x_n, y_n}^+ separates the points a and b . In this case, $C_{x,y}^+$ must separate or intersect a, b . Otherwise, there exist sufficiently small ϵ such that the ϵ neighborhood doesn't separate nor intersect a, b . It follows that for sufficiently large n , C_{x_n, y_n}^+ does not separate a, b . It leads to a contradiction. \square

Proposition 5.2.6. *The sets $TH(m)$ and $TH_1(m)$ are all compact F -forward invariant sets.*

Proof. The compactness of $TH(m)$ and $TH_1(m)$ is obvious, so we only need to prove that they are F -forward invariant.

We treat the set $TH_0(m)$ first. Note that for any $(x, y) \in TH_0(m)$, $(dx, dy) \in b_\infty(m) \cup TP(m)$. So we are left to prove that $S_{dx, dy}$ separates or intersects $TP(m)$.

For any two points $x, y \in \mathbb{S}^1$, denote by \overline{xy} the hyperbolic chord in \mathbb{D} connecting x, y (if $x = y$, \overline{xx} means the point x). Consider the upper left triangle of \mathbb{T}^2 (containing the diagonal). If we collapse the horizontal and vertical edges of the upper left triangle to a point such that the triangle goes to the unit disk with the collapsed edges going to 1 and straighten each path $l_{x, y}$ to the chord \overline{xy} , then this right triangular picture transforms to the Poincare disk picture. It follows that

$$\begin{aligned} & C_{x, y}^+ \text{ separates or intersects } TP(m) \text{ in } \mathbb{T}^2 \\ \iff & l_{x, y} \text{ separates or intersects } TP(m) \text{ in the upper left triangle of unit square} \\ \iff & \overline{xy} \text{ separates or intersects } P(m) \text{ in } \overline{\mathbb{D}} \end{aligned}$$

If \overline{xy} intersects $P(m)$, the chord $\overline{(dx)(dy)}$ must intersect $P(m)$; otherwise, \overline{xy} belongs to a component W of $\overline{\mathbb{D}} \setminus m$. Notice that z^d is injective on $W \cap S^1$ and maps each boundary leaf of W to a single point in $P(m)$. Since \overline{xy} separates $P(m)$, either \overline{xy} separates two boundary leaves of W or separates a point of $P(m)$ and a boundary leaf of W . In both cases $\overline{(dx)(dy)}$ must separate $P(m)$.

Second, we will show $TH_1(m)$ is F -forward invariant. Suppose $\{(x_n, y_n)\}$ is a sequence of points in $TH_0(m)$ such that $\lim_{n \rightarrow \infty} (x_n, y_n) = (x, y) \in TH_1(m)$. According to the F -invariant property of $TH_0(m)$ proved above, the sequence of points $\{F(x_n, y_n)\}$ belong to $TH_0(m)$. Since $\lim_{n \rightarrow \infty} F(x_n, y_n) = F(x, y)$, we have $F(x, y) \in TH_1(m)$.

Finally, the F -forward invariant property of $TH(m)$ follows directly from the F -forward invariant of $TH_0(m)$ and $TH_1(m)$. \square

Since $TH(m)$ and $TH_1(m)$ are compact F -forward invariant sets, we can define the topological entropy $h(TH(m), F)$ and $h(TH_1(m), F)$ as that in Section 3. Note that $TH_0(m)$ is a countable set, so $\text{H.dim}(TH(m)) = \text{H.dim}(TH_1(m))$. By Proposition 5.2.2, we have

$$h(TH(m), F) = \log d \cdot \text{H.dim}(TH(m)) = \log d \cdot \text{H.dim}(TH_1(m)) = h(TH_1(m), F)$$

We establish first a proposition computing $h(TH_1(m), F)$.

Invariant sets admitting a Markov partition

All intervals below are intervals in the natural numbers \mathbb{N} .

Let $A \in \mathbb{T}^2$ be a compact F -forward invariant set satisfying the following property:

(\star) There exists a finite collection of closed sets of \mathbb{T}^2 :

$$E_1, \dots, E_p, E_{p+1}, \dots, E_q, E_{q+1}, \dots, E_r$$

with $p > 0$ (but we allow $q = p$, or $r = p$ or $p < q = r$) such that

- (1) For $i \in [1, p]$, E_i is a closed rectangle or a horizontal or vertical closed segment and for $s \in [p+1, r]$, E_s is a point.
- (2) $\{E_t, t \in [1, r]\}$ is a cover of A and $A \cap E_t \neq \emptyset$ for $1 \leq t \leq r$;
- (3) For any $i \in [1, p]$ and any $t \in [1, r]$ with $t \neq i$, the set E_i does not contain E_t and the interior of E_i is disjoint from E_t . Furthermore, the points in $\{E_s\}_{s=p+1}^r$ are pairwise distinct.
- (4) For $t \in [1, r]$, $F : E_t \rightarrow F(E_t)$ is a homeomorphism. Moreover, for $i \in [1, p]$, $t \in [1, r]$, either $F(E_i)$ contains E_t or the interior of $F(E_i)$ is disjoint from E_t . For $s \in [p+1, q]$, $F(E_s)$ is a member of $\{E_s\}_{s=p+1}^r$. For $s \in [q+1, r]$, $F(E_s)$ is contained in E_i for some $i \in [1, p]$.
- (5) For $i \in [1, p]$, $F : E_i \cap A \rightarrow F(E_i) \cap A$ is a homeomorphism.

The sequence of sets $\{E_s\}_{s=1}^r$ does not form a Markov partition for (F, A) in the traditional sense. But we can still define a transition matrix $M = (a_{s,t})_{r \times r}$ as follows : For $s, t \in [1, r]$, $a_{st} = 1$ if $F(E_s)$ contains E_t and $a_{st} = 0$ otherwise. Note that $a_{st} = 0$ for any $s > q$ and any t , and for any $s \in [p+1, q]$, there is a unique $t \in [p+1, r]$ such that $a_{st} = 1$.

Denote the spectral radius of M by $\rho(M)$. Our main result here is :

Proposition 5.2.7. *If a compact F -forward invariant set $A \subset \mathbb{T}^2$ satisfies Property (\star), then*

$$h(A, F) = \begin{cases} 0 & \text{if } M \text{ is nilpotent} \\ \log \rho(M) & \text{otherwise} \end{cases}$$

The proof of this proposition will go by 3 steps. At first, we construct a puzzle $\mathcal{Q} = \{\mathcal{Q}_n\}_{n \geq 0}$ according to the property (\star). Secondly, we use Proposition 5.2.2 to prove

$h(A, F) = \lim_{n \rightarrow \infty} \frac{1}{n} \log(\#\mathcal{Q}_n)$. Finally, we estimate $\#\mathcal{Q}_n$ by the modules of a sequence of vectors $\{v_n\}$ in the normed vector space $(\mathbb{R}^r, |\cdot|)$ for some norm $|\cdot|$ of \mathbb{R}^r . After these three steps, we can prove Proposition 5.2.7.

Step 1. Construction of puzzles \mathcal{P} , \mathcal{Z} , \mathcal{Q} .

Puzzle \mathcal{P} :

0-level puzzle \mathcal{P}_0 : E_1, \dots, E_r .

1-level puzzle \mathcal{P}_1 : It is the collection of the unique component of $F^{-1}(P_0^*)$ contained in E_s for any $s, t \in [1, r]$ with $a_{st} = 1$ and any 0-level puzzle piece P_0^* in \mathcal{P}_0 contained in E_t .

\vdots

n-level puzzle \mathcal{P}_n : It is the collection of the unique component of $F^{-1}(P_{n-1}^*)$ contained in E_s for any $s, t \in [1, r]$ with $a_{st} = 1$ and any (n-1)-level puzzle piece P_{n-1}^* in \mathcal{P}_{n-1} contained in E_t .

\vdots

Puzzle \mathcal{Z} :

0-level puzzle \mathcal{Z}_0 : \emptyset .

1-level puzzle \mathcal{Z}_1 : E_{q+1}, \dots, E_r .

2-level puzzle \mathcal{Z}_2 : It is the collection of the components of $F^{-1}(Z_1^*)$ contained in $\bigcup_{s=1}^q E_s$ where Z_1^* is a member of \mathcal{Z}_1 .

\vdots

n-level puzzle \mathcal{Z}_n : It is the collection of the components of $F^{-1}(Z_{n-1}^*)$ contained in $\bigcup_{s=1}^q E_s$ where Z_{n-1}^* is a member of \mathcal{Z}_{n-1} .

\vdots

From the construction of \mathcal{P} and \mathcal{Z} , we can see that for any $n \geq 1$, any n-level puzzle piece of \mathcal{Z} is a (n-1)-level puzzle piece of \mathcal{P} . That is $\mathcal{Z}_n \subset \mathcal{P}_{n-1}$.

Puzzle \mathcal{Q} :

0-level puzzle \mathcal{Q}_0 : E_1, \dots, E_r .

1-level puzzle \mathcal{Q}_1 : The unique component of $F^{-1}(Q_0^*)$ contained in E_s for any $s, t \in [1, r]$ with $a_{st} = 1$ and any 0-level puzzle piece Q_0^* in \mathcal{Q}_0 contained in E_t , together with the puzzle pieces in \mathcal{Z}_1 .

\vdots

n-level puzzle \mathcal{Q}_n : The unique component of $F^{-1}(Q_{n-1}^*)$ contained in E_s for any $s, t \in [1, r]$ with $a_{st} = 1$ and any (n-1)-level puzzle piece Q_{n-1}^* in \mathcal{Q}_{n-1} contained in E_t , together with the puzzle pieces in \mathcal{Z}_1 .

\vdots

By induction on n , it is not difficult to see

$$\mathcal{Q}_n = \mathcal{P}_n \cup \mathcal{Z}_n \cup \dots \cup \mathcal{Z}_1.$$

Lemma 5.2.8. *For each $n \geq 0$, \mathcal{Q}_n satisfies the following properties:*

- (1) *If a point $x \in A$ belongs to E_s for some $s \in [1, r]$, then there exists a puzzle piece Q_n in \mathcal{U}_n such that $Q_n \subset E_s$ and $a \in Q_n$. Consequently \mathcal{Q}_n is a cover of A .*
- (2) *If Q_{n+1} is a puzzle piece in \mathcal{Q}_{n+1} belonging to E_s for some $s \in [1, r]$, then there exists a puzzle piece Q_n in \mathcal{Q}_n such that $Q_{n+1} \subset Q_n \subset E_s$. Consequently, we have $\mathcal{Q}_n \prec \mathcal{Q}_{n+1}$.*
- (3) *Each puzzle piece in \mathcal{Q}_n contains some point of A .*

Proof. (1) For $n = 0$, \mathcal{Q}_0 is a cover of A , so the result holds. Suppose for $n \geq 0$, the term (1) in Lemma 5.2.8 holds. Let x be a point of A belonging to E_s for some $s \in [1, r]$. If $s \in [q+1, r]$, $E_s \in \mathcal{Z}_1 \subset \mathcal{Q}_{n+1}$ contains x . If $s \in [1, q]$, there exists $t \in [1, r]$ such that $a_{st} = 1$ and $F(x) \in E_t$. By the assumption of induction, there exists a puzzle piece Q_n in \mathcal{Q}_n contained in E_t . The unique component of $F^{-1}(Q_n)$ contained in E_s is a puzzle piece in \mathcal{Q}_{n+1} and must contains x .

- (2) For $n = 1$, the construction of \mathcal{Q}_1 shows that each puzzle piece in \mathcal{Q}_1 belongs to a puzzle piece in \mathcal{Q}_0 . So the result holds.

Now suppose for $n \geq 1$, the term (2) in Lemma 5.2.8 holds. Let Q_{n+1} be a puzzle piece in \mathcal{Q}_{n+1} contained in E_s . If $s \in [q+1, r]$, by the construction of \mathcal{Q} , $Q_{n+1} = E_s \in \mathcal{Q}_n$.

If $s \in [1, q]$, then there exist $t \in [1, r]$ with $a_{st} = 1$ and a puzzle piece Q_n in \mathcal{Q}_n such that $Q_n \subset E_t$ and Q_{n+1} is the unique component of $F^{-1}(Q_n)$ that contained in E_s . By the assumption of induction, there exist a puzzle piece Q_{n-1} in \mathcal{Q}_{n-1} that belongs to E_t and contains E_n . The unique component of $F^{-1}(Q_{n-1})$ belonging to E_s is a puzzle piece in \mathcal{Q}_n and contains Q_{n+1} .

- (3) For $n = 0$, the property $(*)$ shows that each puzzle piece in \mathcal{Q}_0 contains some point of A and $F : Q_0^* \cap A \longrightarrow F(Q_0^*) \cap A$ is a homeomorphism, where Q_0^* is any puzzle piece in \mathcal{Q}_0 . Now for $n \geq 0$, assume that each puzzle piece of \mathcal{Q}_n contains some point of A and $F : Q_n^* \cap A \longrightarrow F(Q_n^*) \cap A$ is a homeomorphism, where Q_n^* is any puzzle piece in \mathcal{Q}_n .

Let Q_{n+1}^* be a puzzle piece in \mathcal{Q}_{n+1} . Since $\mathcal{Q}_n \prec \mathcal{Q}_{n+1}$, there exists a puzzle piece Q_n^* of \mathcal{Q}_n with $Q_{n+1}^* \subset Q_n^*$. By the assumption of induction, we have $Q_n^* \cap A \neq \emptyset$, $F(Q_{n+1}^*) \cap A \neq \emptyset$ and $F : Q_n^* \cap A \longrightarrow F(Q_n^*) \cap A$ is a homeomorphism. It follows that the homeomorphism $F^{-1} : F(Q_n^*) \cap A \longrightarrow Q_n^* \cap A$ maps $F(Q_{n+1}^*) \cap A$ homeomorphic to $Q_{n+1}^* \cap A$. So we have $Q_{n+1}^* \cap A \neq \emptyset$ and $F : Q_{n+1}^* \cap A \longrightarrow F(Q_{n+1}^*) \cap A$ is a homeomorphism.

□

Step 2. $h(A, F) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \#\mathcal{Q}_n$

In fact, this result follows from the following lemma and Proposition 5.2.2.

Lemma 5.2.9. *There exists a constant K such that for $n \geq 0$, each puzzle piece in \mathcal{Q}_n intersects at most K level- $(n+1)$ standard squares and each level- $(n+1)$ standard squares intersect at most $K + n(r - q)$ puzzle pieces in \mathcal{Q}_n .*

Proof. This proof is similar with that of term (2) of Lemma 5.2.4. For $n = 0$, we can choose an integer $K \geq 4$ such that each puzzle piece in \mathcal{Q}_0 (resp. each level-1 standard square) intersects at most K level-1 standard squares (resp. puzzle pieces in \mathcal{Q}_0).

Suppose for $n \geq 0$, each puzzle piece in \mathcal{Q}_n (resp. each level- $(n+1)$ standard square) intersects at most K level- $(n+1)$ standard squares (resp. $K + n(r - q)$ puzzle pieces in \mathcal{Q}_n). Let Q_{n+1}^* (resp. B_{n+2}^*) be any puzzle piece in \mathcal{Q}_{n+1} (resp. level- $(n+2)$ standard square) and

$B_{n+2}^1, \dots, B_{n+2}^\alpha$ (resp. $Q_{n+1}^1, \dots, Q_{n+1}^\beta$) are all level-(n+2) standard squares that intersect Q_{n+1}^* (resp. puzzle pieces in \mathcal{Q}_{n+1} that intersect B_{n+2}^*).

If $Q_{n+1}^* \in \mathcal{Z}_1$, it intersects at most $4 \leq K$ level-(n+2) standard squares. Otherwise, $F(Q_{n+1}^*)$ is a puzzle piece in \mathcal{Q}_n and $F(B_{n+2}^1), \dots, F(B_{n+2}^\alpha)$ are level-(n+1) standard squares which intersect $F(Q_{n+1}^*)$. Since $F : Q_{n+1}^* \rightarrow F(Q_{n+1}^*)$ is a homeomorphism, the level-(n+1) standard squares $F(B_{n+2}^1), \dots, F(B_{n+2}^\alpha)$ are pairwise different. Then by the assumption of induction, $\alpha \leq K$. (resp. If none of these puzzle pieces belong to \mathcal{Z}_1 , then $F(Q_{n+1}^1), \dots, F(Q_{n+1}^\beta)$ are puzzle pieces in \mathcal{Q}_n and they intersect $F(B_{n+2}^*)$ which is a level-(n+1) standard square. Since $F : B_{n+2}^* \rightarrow F(B_{n+2}^*)$ is a homeomorphism, the puzzle pieces $F(Q_{n+1}^1), \dots, F(Q_{n+1}^\beta)$ are pairwise different. Then by the assumption of induction, $\beta \leq K + (r - q)n$. Since there are at most $r - q$ members of $\mathcal{Z}_1 \subset \mathcal{Q}_{n+1}$ that can intersect B_{n+2}^* , we have $\beta \leq K + (r - q)(n + 1)$). \square

For $n \geq 1$, denote by \mathcal{B}_n the set consisting of all level-n standard squares that intersect A , so $\#\mathcal{B}_n = \nu(A, n)$.

By Lemma 5.2.9, each member of \mathcal{B}_{n+1} (resp. puzzle piece in \mathcal{Q}_n) intersects at most $K + (r - q)n$ puzzle pieces in \mathcal{Q}_n (resp. K members of \mathcal{B}_{n+1}). It means that the number of puzzle pieces in \mathcal{Q}_n (resp. members of \mathcal{B}_{n+1}) that intersect some member of \mathcal{B}_{n+1} (resp. some puzzle piece in \mathcal{Q}_n) is less than $[K + (r - q)n] \cdot \nu(A, n + 1)$ (resp. $K \cdot \#\mathcal{Q}_n$). Since each puzzle piece in \mathcal{Q}_n (resp. member of \mathcal{B}_{n+1}) intersects some member of \mathcal{B}_{n+1} (resp. a puzzle piece in \mathcal{Q}_n), we have

$$\#\mathcal{Q}_n \leq [K + (r - q)n] \cdot \nu(A, n + 1) \quad (\text{resp. } \nu(A, n + 1) \leq K \cdot \#\mathcal{Q}_n).$$

It follows

$$\frac{1}{K} \nu(A, n + 1) \leq \#\mathcal{Q}_n \leq [K + (r - q)n] \cdot \nu(A, n + 1)$$

Then by Proposition 5.2.2,

$$h(A, F) = \mathcal{C}(A) = \lim_{n \rightarrow \infty} \frac{1}{n} \log(\nu(A, n)) = \lim_{n \rightarrow \infty} \frac{1}{n} \log(\#\mathcal{Q}_n). \quad (5.2.6)$$

Step 3. Estimate $\#\mathcal{Q}_n$

Since $\mathcal{Q}_n = \mathcal{P}_n \cup \mathcal{Z}_n \cup \dots \cup \mathcal{Z}_1$, then $\#\mathcal{Q}_n$ can be estimated by means of $\#\mathcal{P}_n$ and $\#\mathcal{Z}_j$

($j \in [1, n]$). So we start to estimate $\#\mathcal{P}_n$, $\#\mathcal{Z}_n$ and $\#\mathcal{Q}_n$.

Equip \mathbb{R}^r a norm $\|\cdot\|$ so that for a vector $v = (x_1, \dots, x_r)^T \in \mathbb{R}^r$,

$$\|v\| = |x_1| + \dots + |x_r|.$$

Denote by $v_0 = (1, \dots, 1)^T \in \mathbb{R}^r$, $v_n = M^n \cdot v_0 = (v_{n,1}, \dots, v_{n,r})^T$. It is known that

Lemma 5.2.10. *If M is nilpotent, then $\rho(M)=0$; otherwise $\log \rho(M) = \lim_{n \rightarrow \infty} \frac{\log \|v_n\|}{n}$.*

1. Estimate $\#\mathcal{P}_n$:

In the property (*), consider each element of $\{E_{p+1}, \dots, E_r\}$ as a small square and keep other properties invariant. Denote the new sequence of rectangles by $\{\tilde{E}_s\}_{s=1}^r$. With the same definition as that of the transition matrix M and puzzle \mathcal{P} , we obtain a transition matrix \tilde{M} and a puzzle $\tilde{\mathcal{P}}$ corresponding to $\{\tilde{E}_1, \dots, \tilde{E}_r\}$. It is easy to see $M = \tilde{M}$.

For any $n \geq 1$, the difference between \mathcal{P}_n and $\tilde{\mathcal{P}}_n$ lies that some puzzle piece E_* in \mathcal{P}_n may belong to different puzzle pieces in \mathcal{P}_{n-1} (since any two puzzle piece of \mathcal{P} with the same level intersect at most at their boundary, this case happens only if E_* is a point) but any puzzle piece in $\tilde{\mathcal{P}}_n$ belongs to the unique puzzle piece in $\tilde{\mathcal{P}}_{n-1}$.

For $n \geq 1$, let E_* be a puzzle piece in \mathcal{P}_n . Denote by $\kappa(E_*)$ the number of puzzle pieces in \mathcal{P}_{n-1} that contain E_* . Since each puzzle piece in \mathcal{P}_{n-1} is a rectangle or a point and any two members of \mathcal{P}_{n-1} intersect at most at their boundary, we have $1 \leq \kappa(E_*) \leq 4$.

Lemma 5.2.11. *For any $n \geq 0$, $\#\tilde{\mathcal{P}}_n = \sum_{E_* \in \mathcal{P}_n} \kappa(E_*)$ and consequently, $\#\mathcal{P}_n \leq \#\tilde{\mathcal{P}}_n \leq 4\#\mathcal{P}_n$.*

Proof. For any $n \geq 0$, we can denote a puzzle piece in \mathcal{P}_n (resp. $\tilde{\mathcal{P}}_n$) by $E_{s_0 \dots s_n}$ (resp. $\tilde{E}_{s_0 \dots s_n}$) such that

$$F^j(E_{s_0 \dots s_n}) \subset E_{s_j} \quad (\text{resp. } F^j(\tilde{E}_{s_0 \dots s_n}) \subset \tilde{E}_{s_j}) \quad \text{for } 0 \leq j \leq n$$

Then

$$\mathcal{P}_n = \{E_{s_0 \dots s_n} \mid (s_0, \dots, s_n) \in I_n\} \quad (\text{resp. } \tilde{\mathcal{P}}_n = \{\tilde{E}_{s_0 \dots s_n} \mid (s_0, \dots, s_n) \in I_n\})$$

where the set of subscript

$$I_n = \{(s_0, \dots, s_n) \in [1, r]^{n+1} \mid a_{s_j s_{j+1}} = 1 \text{ for } j \in [0, n-1]\}$$

Note that for $j \in [0, n-1]$, $F^j : E_{s_0 \dots s_n}$ (resp. $\tilde{E}_{s_0 \dots s_n}$) \longrightarrow $E_{s_j \dots s_n}$ (resp. $\tilde{E}_{s_j \dots s_n}$) is a homeomorphism.

Since $\tilde{E}_{s_0 \dots s_n} \neq \tilde{E}_{t_0 \dots t_n}$ as $(s_0, \dots, s_n) \neq (t_0, \dots, t_n)$, we can define a surjection $\wp_n : \tilde{\mathcal{P}}_n \longrightarrow \mathcal{P}_n$ mapping $\tilde{E}_{s_0 \dots s_n}$ to $E_{s_0 \dots s_n}$. It is easy to see

$$\wp_n(\tilde{E}_{s_0 \dots s_n}) = \wp_n(\tilde{E}_{t_0 \dots t_n}) \iff (s_0, \dots, s_n) = (t_0, \dots, t_n) \text{ or } E_{s_0 \dots s_n} = E_{t_0 \dots t_n} \subset E_{s_0 \dots s_{n-1}} \cap E_{t_0 \dots t_{n-1}}$$

It follows that $E_{s_0 \dots s_n}$ has $\kappa(E_{s_0 \dots s_n})$ preimages under \wp_n . So $\#\tilde{\mathcal{P}}_n = \sum_{E_* \in \mathcal{P}_n} \kappa(E_*)$ \square

Lemma 5.2.12. For $n \geq 0$, $\#\tilde{\mathcal{P}}_n = \|v_n\|$.

Proof. For any $n \geq 0, s \in [1, r]$, denote by $\tilde{\mathcal{P}}_{n,s}$ the puzzle pieces in $\tilde{\mathcal{P}}_n$ that are contained in \tilde{E}_s . Since $\tilde{\mathcal{P}}_{n,s_1}$ and $\tilde{\mathcal{P}}_{n,s_2}$ don't have a common puzzle piece, it is enough to prove that

$$\#\tilde{\mathcal{P}}_{n,s} = v_{n,s} \text{ for any } n \geq 0, s \in [1, r].$$

As $n = 0$, $\#\tilde{\mathcal{P}}_{0,s} = 1 = v_{0,s}$ for $s \in [1, r]$. For $n \geq 0$, suppose $\#\tilde{\mathcal{P}}_{n,s} = v_{n,s}$ for $s \in [1, r]$. Then by the construction of puzzle $\tilde{\mathcal{P}}$, for any $s \in [1, r]$,

$$\#\tilde{\mathcal{P}}_{n+1,s} = a_{s1} \cdot \#\tilde{\mathcal{P}}_{n,1} + \dots + a_{sr} \cdot \#\tilde{\mathcal{P}}_{n,r} = a_{s1} \cdot v_{n,1} + \dots + a_{sr} \cdot v_{n,r} = v_{n+1,s}$$

The second “=” holds according to the assumption of induction. \square

By Lemma 5.2.11, 5.2.12, we obtain the estimation of \mathcal{P}_n as

$$\frac{1}{4}\|v_n\| \leq \#\mathcal{P}_n \leq \|v_n\| \quad (5.2.7)$$

2. Estimate $\#\mathcal{Z}_n$: Since $\mathcal{Z}_n \subset \mathcal{P}_{n-1}$, we have

$$\#\mathcal{Z}_n \leq \#\mathcal{P}_{n-1} \text{ for } n \geq 1. \quad (5.2.8)$$

3. Estimate $\#\mathcal{Q}_n$:

Since $\mathcal{Q}_n = \mathcal{P}_n \cup \mathcal{Z}_n \cup \cdots \cup \mathcal{Z}_1$, we have

$$\#\mathcal{P}_n \leq \#\mathcal{Q}_n \leq \#\mathcal{P}_n + \#\mathcal{Z}_n + \cdots + \#\mathcal{Z}_1$$

By (5.2.7) and (5.2.8), we obtained that

$$\frac{1}{4}\|v_n\| \leq \#\mathcal{Q}_n \leq \|v_n\| + \|v_{n-1}\| + \cdots + \|v_0\| \quad (5.2.9)$$

Now we have finished the preparation of three steps in the proof of Proposition 5.2.7.

Proof of Proposition 5.2.7. According to (5.2.6), we only need to prove

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log(\#\mathcal{Q}_n) = \begin{cases} 0 & \text{if } M \text{ is nilpotent} \\ \log \rho(M) & \text{otherwise} \end{cases}$$

If M is nilpotent, then $\|v_n\| = 0$ for sufficiently large n , By (5.2.9), $\#\mathcal{Q}_n$ is bounded as $n \rightarrow \infty$, so $\lim_{n \rightarrow \infty} \frac{1}{n} \log(\#\mathcal{Q}_n) = 0$.

If M is not nilpotent, by Theorem 2.6.6,

$$\lim_{n \rightarrow \infty} \left(\frac{1}{n} \log \|v_n\| - \frac{1}{n} \log(\|v_n\| + \cdots + \|v_0\|) \right) = 0$$

Since M is not nilpotent, by Lemma 5.2.10, $\lim_{n \rightarrow \infty} \frac{1}{n} \log \|v_n\| = \log \rho(M)$, so

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log(\|v_n\| + \cdots + \|v_0\|) = \log \rho(M)$$

Then by (5.2.9), $\lim_{n \rightarrow \infty} \frac{1}{n} \log(\#\mathcal{Q}_n) = \log \rho(M)$. □

Computing the entropy of a combinatorial Hubbard tree

For computing the topological entropy $h(TH(m), F)$, we should generalize the property (\star) to property (\star') as

(\star') There exist two sequence of closed sets $\{E_i\}_{i=1}^p$ and $\{E_s\}_{s=p+1}^r$ belonging to \mathbb{T}^2 such that

(1') For any $t \in [1, r]$, $E_t = E_t^+ \cup E_t^-$. Furthermore, as $i \in [1, p]$, E_i^+ and E_i^- are two closed rectangles or two horizontal or vertical segments that coincide or have disjoint

interior; as $s \in [p+1, r]$, E_s^+ and E_s^- are either the same point or different points.

(2') $\{E_t\}_{t=1}^r$ is a cover of A and $A \cap E_t^{\delta_t} \neq \emptyset$ for any $1 \leq t \leq r$ and $\delta_t \in \{\pm\}$.

(3') For any $i \in [1, p]$, $t \in [1, r]$, $\delta_i, \delta_t \in \{\pm\}$, $E_i^{\delta_i}$ doesn't contain $E_t^{\delta_t}$ and the interior of $E_i^{\delta_i}$ is disjoint with $E_t^{\delta_t}$. Besides, the members of $\{E_s\}_{s=p+1}^r$ are pairwise disjoint.

(4') For any $t \in [1, r]$, $\delta \in \pm$, $F : E_t^{\delta_t} \longrightarrow F(E_s^{\delta_s})$ is a homeomorphism. For $i \in [1, p]$, $t \in [1, r]$, $\delta_i, \delta_t \in \{\pm\}$, either $F(E_i^{\delta_i})$ contains $E_t^{\delta_t}$ or the interior of $F(E_i^{\delta_i})$ is disjoint with $E_t^{\delta_t}$; For $s \in [p+1, r]$, $\delta_s \in \{\pm\}$, $F(E_s^{\delta_s}) = E_t^{\delta_t}$ for some $t \in [p+1, r]$, $\delta_t \in \{\pm\}$ or $F(E_s)^{\delta_s}$ belongs to $E_i^{\delta_i}$ for some $i \in [1, p]$, $\delta_i \in \{\pm\}$.

(5') For $\delta \in \{\pm\}$, denote by $\bar{\delta}$ the opposite signal of δ . We require that for $s \in [1, r]$, $t \in [1, r]$, $F(E_s^{\delta_s})$ contains $E_t^{\delta_t}$ iff $F E_s^{\bar{\delta}_s}$ contains $E_t^{\bar{\delta}_t}$ and for $s \in [p+1, r]$, $i \in [1, p]$, $F(E_s^{\delta_s})$ belongs to $E_i^{\delta_i}$ iff $F E_s^{\bar{\delta}_s}$ belongs to $E_i^{\bar{\delta}_i}$.

(6') For $i \in [1, p]$, $\delta_i \in \{\pm\}$, the map $F : E_i^{\delta_i} \cap A \longrightarrow F(E_i^{\delta_i}) \cap A$ is a homeomorphism.

With the property (\star') , we can also define a transition matrix $M = (a_{st})_{r \times r}$ according to $\{E_s\}_{s=1}^r$: For $s, t \in [1, r]$, $a_{st} = 1$ if $F(E_s)$ contains E_t and $a_{st} = 0$ otherwise. Denote the spectral radius of M by $\rho(M)$, by the same proof as that of Proposition 5.2.7, we have

Proposition 5.2.7'. If a compact F -forward invariant set $A \subset \mathbb{T}^2$ satisfies property (\star') , then

$$h(A, F) = \begin{cases} 0 & \text{if } M \text{ is nilpotent} \\ \log \rho(M) & \text{otherwise} \end{cases}$$

Next, we will check that $TH_1(m)$ satisfies the property (\star') and then we can compute $h(TH_1(m), F)$ by Proposition 5.2.7'.

Denote by $\tau_d : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$, $z \mapsto z^d$. Let

$$m = \{\Theta_1, \dots, \Theta_{s_1}, \Theta_{s_1+1}, \dots, \Theta_{s_2}, \Theta_{s_2+1}, \dots, \Theta_s\}$$

be a degree d rational primitive major, indexed in a way so that Θ_j contains a periodic angle iff $j \in [1, s_1]$, and Θ_i doesn't contain a periodic angle but the orbit of the unique angle in $\tau_d(\Theta_i)$ passes through $\cup_{j=1}^{s_1} \Theta_j$ iff $i \in [s_1+1, s_2]$. Set $m_p = \{\Theta_1, \dots, \Theta_{s_2}\}$, then we

define a subset of $TP(m)$ as

$$TP(m_p) = \{(\theta, \theta) \mid \theta \in P(m_p)\} \quad \text{where} \quad P(m_p) = \{d^n \theta \mid n \geq 0, \theta \in \bigcup_{j=1}^{s_2} \Theta_j\}.$$

Now we will divide $TH_0(m)$ into a finite number of subsets as follows: set

$$A(m) = \{(x, y) \in TH_0(m) \mid C_{x,y}^+ \text{ separates but doesn't intersect } TP(m)\}.$$

and for $\alpha \in TP(m)$, set

$$B_\alpha(m) = \{(x, y) \in TH_0(m) \mid C_{x,y}^+ \text{ intersects } \alpha\}$$

$$\text{Then } TH_0(m) = A(m) \cup \bigcup_{\alpha \in TP(m)} B_\alpha.$$

We can define an equivalence relation in $A(m)$: two points $(x_1, y_1), (x_2, y_2) \in A(m)$ are called **parallel** if either C_{x_1, y_1}^+ and C_{x_2, y_2}^+ coincide or C_{x_1, y_1}^+ is homotopic to C_{x_2, y_2}^δ relative to $TP(m)$ in \mathbb{T}^2 . Note that C_{x_1, y_1}^+ coincide with C_{x_2, y_2}^+ iff $(x_1, y_1) = (x_2, y_2)$ or $(x_1, y_1) = (y_2, x_2)$ ((x_1, y_1) (x_2, y_2) are symmetric relative the diagonal) and $C_{x_1, y_1}^{\delta_1}$ is homotopic to $C_{x_2, y_2}^{\delta_2}$ relative $TP(m)$ in \mathbb{T}^2 iff $C_{x_1, y_1}^{\delta_1}$ is homotopic to $C_{x_2, y_2}^{\delta_2}$ relative $TP(m)$ in \mathbb{T}^2 . Then it is easy to check that the parallel relation is indeed a equivalence relation.

For any equivalence class A_* of $A(m)$, any $(x, y) \in A_*$, we can define $\gamma_{x,y} = C_{x,y}^+$ or $C_{x,y}^-$ such that all $\gamma_{x,y}$ with $(x, y) \in A_*$ are homotopic relative to $TP(m)$ in \mathbb{T}^2 . Then (x_1, y_1) is parallel to (x_2, y_2) iff γ_{x_1, y_1} is homotopic to γ_{x_2, y_2} relative $TP(m)$ in \mathbb{T}^2 . Since $TP(m)$ is finite and each $\gamma_{x,y}$ is the boundary of square, there are only finite equivalent classes of $A(m)$, denoted by $A_1(m), \dots, A_p(m)$

For $i \in [1, p]$, denote by $A'_i(m)$ the set of accumulation points of $A_i(m)$ and for $\alpha \in TP(m_p)$, denote by $B'_\alpha(m)$ the accumulation points of $B_\alpha(m)$. Without loss of generality, we assume $A'_i \neq \emptyset$ for $i \in [1, p]$.

Lemma 5.2.13. (1) B_α is an infinite set iff $\alpha \in TP(m_p)$ and in this case, B'_α is a finite set.

$$(2) \quad TH_1(m) = \bigcup_{i=1}^p A'_i \cup \bigcup_{\alpha \in TP(m_p)} B_\alpha.$$

Proof. (1) It follows directly from the construction of $b_\infty(m)$.

(2) Choose any point $(x, y) \in TH_1(m)$ such that $\{(x_n, y_n)\} \in TH_0$ and $\lim_{n \rightarrow \infty} (x_n, y_n) = (x, y)$. If there exist an infinite subsequence of $\{(x_n, y_n)\}$, also denoted by $\{(x_n, y_n)\}$, belonging to $A(m)$, since the number of equivalent classes is finite, there exist $t_0 \in [1, t]$ such that an infinite subsequence of $\{(x_n, y_n)\}$ belong to $A_{t_0}(m)$, hence $(x, y) \in \bar{A}_{t_0}(m)$; otherwise, for sufficient large n , $(x_n, y_n) \in \cup_{\alpha \in TP(m_p)} B_\alpha(m)$. By the same reason as above, $(x, y) \in B'_\alpha(m)$ for some $\alpha \in TP(m_p)$.

□

For $i \in [1, p]$, $(x, y) \in A_i(m)$, denote by $S(\gamma_{x,y})$ the unique square in \mathbb{T}^2 bounded by $\gamma_{x,y}$. Then we can define an order on

$$\Lambda_i = \{\gamma_{x,y} \mid (x, y) \in A_i(m)\}$$

such that $\gamma_{x_1, y_1} \geq \gamma_{x_2, y_2}$ iff $S(\gamma_{x_1, y_1})$ contains $S(\gamma_{x_2, y_2})$. If a sequence of points $\{x_n, y_n\} \subset A_i$ such that $\lim_{n \rightarrow \infty} (x_n, y_n) = (x, y)$, then γ_{x_n, y_n} converge to $\gamma_{x,y}$ in Hausdorff topology and $\gamma_{x,y}$ is exactly $C_{x,y}^+$ or $C_{x,y}^-$. So we can also define an order on

$$\bar{\Lambda}_i = \{\gamma_{x,y} \mid (x, y) \in \bar{A}_i(m)\}$$

as the same as that on Λ_i where $\bar{A}_i(m) = A_i(m) \cup A'_i(m)$. Since $\bar{\Lambda}_i$ is compact, there is a unique maximal curve γ_{a_i, b_i} and a unique minimal curve γ_{c_i, d_i} in $\bar{\Lambda}_i'$. Here we always require that (a_i, b_i) and (c_i, d_i) are on the lower-right triangle of square $S(\gamma_{a_i, b_i})$. Then all the curves in $\bar{\Lambda}_i$ are contained in

$$\Omega_i = S(\gamma_{a_i, b_i}) \setminus \text{the interior of } S(\gamma_{c_i, d_i})$$

We define

$$E_i^+ = [c_i, a_i] \times [b_i, d_i] \quad \text{and} \quad E_i^- = [b_i, d_i] \times [c_i, a_i]$$

Since we assume $A'_i \neq \emptyset$ for $i \in [1, p]$, then each \bar{A}_i is an infinite set. For any $(x, y) \in \bar{A}_i$, $\gamma_{x,y} \in \Omega_i$ implies $(x, y) \in E_i$. So we have $A'_i(m) \subset \bar{A}_i(m) \subset E_i$ for $i \in [1, p]$. There are the 4 cases for the shape of E_i^\pm :

1. (c_i, d_i) is in the interior of $S(\gamma_{a_i, b_i})$ and $c_i \neq d_i$. In this case, E_i^\pm are two disjoint squares.

2. (c_i, d_i) is in the interior of $S(\gamma_{a_i, b_i})$ and $c_i = d_i$. In this case, E_i^\pm are two squares intersecting at a point of the diagonal.
3. (c_i, d_i) is on γ_{a_i, b_i} and $c_i \neq d_i$. In this case E_i^\pm are two disjoint segments.
4. (c_i, d_i) is on γ_{a_i, b_i} and $c_i = d_i$. In this case E_i^\pm are two segments intersecting at a point of the diagonal.

We have known that B'_α is a finite set for $\alpha \in TP(m_p)$, denote

$$\bigcup_{\alpha \in TP(m_p)} B'_\alpha = \{(a_{p+1}, b_{p+1}), (a_{p+1}, b_{p+1}); \dots; (a_{r_1}, b_{r_1}), (a_{r_1}, b_{r_1})\}$$

where $(a_{p+1}, b_{p+1}), \dots, (a_{r_1}, b_{r_1})$ are points in the lower-right triangle of unit square. Then for $s \in [p+1, r_1]$, we define

$$E_s^+ = (a_s, b_s) \quad \text{and} \quad E_s^- = (a_s, b_s)$$

In the beginning of this section, we define a map $\wp : \overline{b_\infty}(m) \longrightarrow L(m)$ by mapping $(x, y) \in \mathbb{T}^2$ to the chord $\overline{xy} \in \overline{\mathbb{D}}$. A leaf of $L(m)$ has two preimages symmetric relative to the diagonal of \mathbb{T}^2 if it is not a point of \mathbb{S}^1 and one preimage on the diagonal of \mathbb{T}^2 otherwise. When we equip the Hausdorff topology to $L(m)$, the map \wp is continuous. Moreover, we have the commutative graph

$$\begin{array}{ccc} \overline{b_\infty}(m) & \xrightarrow{F} & \overline{b_\infty}(m) \\ \wp \downarrow & & \downarrow \wp \\ L(m) & \xrightarrow{\tau_d} & L(m) \end{array}$$

Now we can interpret the discussion above on the torus model to the unit disk model by the language of lamination and prove the result about the torus model by means of proving the corresponding result in the disk model. Usually, the proof in the disk model by lamination is simpler and immediate.

Interoperation from torus model to unit disk model:

1. $\wp(A(m))$ consists of all leaves in $L(m)$ that separate but do not intersect $P(m)$.

2. For $i \in [1, p]$, all leaves of $\wp(A_i(m))$ are homotopic relative $P(m)$ in $\overline{\mathbb{D}}$ and for $1 \leq i \neq j \leq p$, $(x_i, y_i) \in A_i(m)$, $(x_j, y_j) \in A_j(m)$, the chord $\overline{x_1 y_1}$ is not homotopic to $\overline{x_2 y_2}$ (rel $P(m)$). This conclusion follows directly from the fact that for any $(x_1, y_1), (x_2, y_2) \in A(m)$ belonging to the lower-right triangle of the unit square,

$(x_1, y_1), (x_2, y_2)$ are parallel

$\iff l_{x_1, y_1}, l_{x_2, y_2}$ are homotopic (rel $TP(m)$) in the lower-right triangle of the unit square

$\iff \overline{x_1 y_1}, \overline{x_2 y_2}$ are homotopic (rel $P(m)$) in $\overline{\mathbb{D}}$

3. $\wp(A'_i(m))$ consist of any leaf that is the Hausdorff limit of any sequence of leaves $\{\overline{x_n y_n}\} \subset A_i(m)$ and $\wp(\bar{A}_i(m)) = \wp(A_i(m)) \cup \wp(A'_i(m))$.
4. For any $i \in [1, p]$, the points b_i, d_i, c_i, a_i lie on \mathbb{S}^1 by the counter-clockwise order. It is denoted by $b_i \leq d_i \leq c_i \leq a_i$ where “=” holds if two adjacent points of “=” coincide. The chords $\overline{c_i d_i}$ and $\overline{a_i b_i}$ divide \mathbb{D} into 2 or 3 components. We denote by $D(a_i, b_i, c_i, d_i)$ the closure of the component whose boundary contain both $\overline{a_i b_i}$ and $\overline{c_i d_i}$. The chords $\overline{a_i b_i}, \overline{c_i d_i}$ are called the edges of $D(a_i, b_i, c_i, d_i)$. Then we have $\wp(\bar{A}_i(m)) \in D(a_i, b_i, c_i, d_i)$, any leaf of $\wp(\bar{A}_i(m))$ except $\overline{a_i b_i}, \overline{c_i d_i}$ separate the two edges of $D(a_i, b_i, c_i, d_i)$ and $\cup \wp(E_i) := \cup_{(x, y) \in E_i} \overline{xy} = D(a_i, b_i, c_i, d_i)$.

There are 4 cases for the shape of $D(a_i, b_i, c_i, d_i)$:

- (a) $b_i < d_i < c_i < a_i$, then $D(a_i, b_i, c_i, d_i)$ is a 4-gon with two opposite sides being disc chords and the other two arcs in the unit circle
- (b) $b_i < d_i = c_i < a_i$, then $D(a_i, b_i, c_i, d_i)$ is a 3-gon with only one disc chord;
- (c) $b_i = d_i < c_i < a_i$ or $b_i < d_i < c_i = a_i$, then $D(a_i, b_i, c_i, d_i)$ is a 3-gon with exactly two disc chords;
- (d) $b_i = d_i = c_i < a_i$ or $b_i < d_i = c_i = a_i$, then $D(a_i, b_i, c_i, d_i)$ is a 2-gon with only one disc chord.

For $i \in [1, p], \beta \in [1, 4]$, the map \wp maps E_i of case β to $D(a_i, b_i, c_i, d_i)$ of case β

Lemma 5.2.14. (1) *For $i \in [1, p]$, the possible points of intersection of $D(a_i, b_i, c_i, d_i)$ and $P(m)$ are a_i, b_i, c_i, d_i . If $\overline{a_i b_i}$ (resp. $\overline{c_i d_i}$) contains point of $P(m)$, then (a_i, b_i) (resp. (c_i, d_i)) belongs to $A'_i(m)$.*

(2) The case 4 of the shape of $D(a_i, b_i, c_i, d_i)$ never happen.

Proof. (1) Suppose there is a point $\theta \in P(m)$ such that $\theta \in D(a_i, b_i, c_i, d_i)$ but $\theta \notin \{a_i, b_i, c_i, d_i\}$. If $\overline{a_i b_i} \in \wp(A_i(m))$, we choose $\overline{a_i b_i}$ as a objective leaf; otherwise $\overline{a_i b_i} \in \wp(A'_i(m))$, then we can choose a leaf of $\wp(A_i(m))$ that separates $\overline{a_i b_i}$ and θ as a objective leaf. We do the same thing for $\overline{c_i d_i}$. So we obtain two objective leaves in $\wp(A_i(m))$ that are not homotopic relative $P(m)$. It is a contradiction.

(2) If case 4 happens, we assume $b_i = d_i = c_i < a_i$. Then $b_i = \lim_{n \rightarrow \infty} \overline{b_i x_n}$ with $\{(b_i, x_n)\} \subset A_i(m)$. If $b_i \notin P(m)$, $\overline{b_i x_n}$ doesn't separate $P(m)$ for sufficiently n ; if $b_i \in P(m)$, all $\overline{b_i x_n}$ contain the point of $P(m)$. Both of the case lead to a contradiction to the definition of $A(m)$.

□

Lemma 5.2.15. *If $(x, y) \in \cup_{\alpha \in TP(m_p)} B_\alpha$ belongs to E_i for some $i \in [1, p]$, then $(x, y) \in A'_i(m)$.*

Proof. If $(x, y) \in \cup_{\alpha \in TP(m_p)} B_\alpha$, $x = \alpha$ or $y = \alpha$ for some $\alpha \in TP(m_p)$. Assume $x = \alpha$, then $(\alpha, y) \in E_i$ implies $\overline{\alpha y} \subset D(a_i, b_i, c_i, d_i)$. By Lemma 5.2.14, one edge of $D(a_i, b_i, c_i, d_i)$ contains α and there exist a sequence of points $\{(x_n, y_n)\} \subset A_i(m)$ such that $\overline{x_n y_n}$ converge this edge α from $D(a_i, b_i, c_i, d_i)$. Then $\overline{\alpha y}$ must coincide with this edge, otherwise $\overline{x_n y_n}$ will intersect $\overline{\alpha y}$ transversally. So we have $(x, y) \in A'_i(m)$.

□

According to this lemma, if E_k belongs to E_i for some $k \in [p+1, r]$ and $i \in [1, p]$, we can remove E_k from $\{E_s\}_{s=1}^{r_1}$ without affecting the set $TH_1(m)$. Then after removing all such members in $\{E_s\}_{s=p+1}^{r_1}$ and rearrangement of the index of the left members, we obtain two sequences of sets $\{E_i\}_{i=1}^p$ and $\{E_s\}_{s=p+1}^r$.

Lemma 5.2.16. *The set $TH_1(m) \subset \mathbb{T}^2$ and the two sequences of sets $\{E_i\}_{i=1}^r$ and $\{E_s\}_{s=p+1}^r$ satisfy property (\star')*

Proof. We will check case by case that the properties of (\star') are satisfied by $TH_1(m)$ and $\{E_t\}_{t=1}^r$.

(1'), (2') follows from the construction of E_s^\pm ($s \in [1, r]$).

(3') It is enough to prove that for any $i \neq j \in [1, p]$, $\delta_i, \delta_2 \in \{\pm\}$, $E_i^{\delta_i}$ doesn't contain $E_j^{\delta_j}$ and the interior of $E_i \delta_i$ doesn't intersect $E_j^{\delta_j}$. Equivalently, we need to prove that for any $i \neq j \in [1, p]$, $D(a_i, b_i, c_i, d_i)$ doesn't contain $D(a_j, b_j, c_j, d_j)$ and the interior of $D(a_i, b_i, c_i, d_i)$ doesn't intersect $D(a_j, b_j, c_j, d_j)$. This result is true, because otherwise, by Lemma 5.2.14, there must exist a chord of $\wp(A_i(m))$ and a chord of $\wp(A_j(m))$ such that they are homotopic relative $P(m)$ on $\overline{\mathbb{D}}$. It is a contradiction to item 2 of the interoperation to disk model.

(4'), (5') For any $i \in [1, p]$, $D(a_i, b_i, c_i, d_i)$ belongs to the closure of a component of $\mathbb{D} \setminus m$. It implies $|[b_i, d_i]| + |[c_i, a_i]| \leq \frac{1}{d}$. Then $F : E_i^{\delta_i} \longrightarrow F(E_i^{\delta_i})$ is a homeomorphism. Since $F(E_i^+)$ and $F(E_i^-)$ are symmetric relative the diagonal, we have $F(E_i^+)$ contains $E_t^{\delta_t}$ iff $F(E_i^-)$ contains $E_t^{\bar{\delta}_t}$ for $\delta_t \in \{\pm\}$, $t \in [1, r]$. So we are left to prove that for $i \in [1, p]$, $t \in [1, r]$, $\delta_i, \delta_t \in \{\pm\}$, either $F(E_i^{\delta_i})$ contains $E_t^{\delta_t}$ or the interior of $F(E_i^{\delta_i})$ is disjoint with $E_t^{\delta_t}$.

Suppose this result is not true. Then there exist $i, j \in [1, p]$ such that $\tau_d(D(a_i, b_i, c_i, d_i)) \neq D(a_j, b_j, c_j, d_j)$ and the intersection of $D(a_j, b_j, c_j, d_j)$ and the interior of $\tau_d(D(a_i, b_i, c_i, d_i))$ is not empty. In this case, we can find a chord $\overline{z_j w_j} \in \wp(A_j(m))$ belonging to the exterior of $\tau_d(D(a_i, b_i, c_i, d_i))$ such that $\overline{z_j w_j}$ is homotopic to a chord of $\tau_d(\wp(A_i(m)))$. Then one of the preimage of $\overline{z_j w_j}$ under τ_d is homotopic to a chord of $\wp(A_i(m))$ and it must belong to the exterior of $D(a_i, b_i, c_i, d_i)$. It is a contradiction.

(6') Since $F : E_i^{\delta_i} \longrightarrow F(E_i^{\delta_i})$ is a homeomorphism, it is enough to prove that for $(z, w) \in F(E_i^{\delta_i}) \cap TH_1(m)$, the unique preimage of (z, w) under F contained in $E_i^{\delta_i}$, denoted by (x, y) , is in $TH_1(m)$.

correspondingly, in the unit disk model, the leaf $\overline{xy} \in D(a_i, b_i, c_i, d_i)$ and the leaf \overline{zw} belongs to $D(a_j, b_j, c_j, d_j)$ for some $j \in [1, p]$ or coincides with $\wp(E_s)$ for some $s \in [p+1, r]$.

In case of $\overline{zw} \subset D(a_j, b_j, c_j, d_j)$ for $j \in [1, p]$, $\tau_d(D(a_i, b_i, c_i, d_i))$ contains $D(a_j, b_j, c_j, d_j)$. There exists a sequence of chords $\{\overline{x_n y_n}\} \subset \wp(A_j(m)) \subset D(a_j, b_j, c_j, d_j)$ such that $\lim_{n \rightarrow \infty} \overline{x_n y_n} = \overline{zw}$. Denote by $\overline{x_n y_n}$ the preimage of $\overline{x_n y_n}$ in $D(a_i, b_i, c_i, d_i)$, then $\{\overline{x_n y_n}\} \subset \wp(A_i(m))$ and $\lim_{n \rightarrow \infty} \overline{x_n y_n} = \overline{xy}$. It implies $(x, y) \in A'_i(m) \subset TH_1(m)$.

In case of $\overline{zw} = \wp(E_s)$ for some $s \in [p+1, r]$, we can assume $z = \alpha$ for some $\alpha \in P(m_p)$.

Then there exist a sequence of leaves $\{\overline{\alpha w_n}\} \subset B_\alpha$ such that $\lim_{n \rightarrow \infty} \overline{\alpha w_n} = \overline{\alpha w}$. Denote by $\overline{xy_n}$ the preimage of $\overline{\alpha w_n}$ based at x . If $x, y \notin P(m)$, then for sufficiently large n , the leaf $\overline{xy_n}$ is homotopic to some leaf of $\wp(A_i(m))$ relative $P(m)$ in $\overline{\mathbb{D}}$. So once removing finite leaves, $\{\overline{xy_n}\} \subset \wp(A_i(m))$. It follows that $\lim_{n \rightarrow \infty} (x, y_n) = (x, y) \in A'_i$. If x or $y = \beta \in P(m)$, then \overline{xy} must be an edge of $D(a_i, b_i, c_i, d_i)$. For otherwise, there exist a sequence of leaves $\{\overline{x_n y_n}\} \subset \wp(A_i(m))$ that converge to an edge of $D(a_i, b_i, c_i, d_i)$ containing β and these leaves intersect \overline{xy} transversally. In this case, by Lemma 5.2.14, $(x, y) \in A'_i(m) \subset TH_1(m)$.

□

5.2.4 Hubbard trees for polynomials

In fact, the dynamical background of the invariant lamination generated by a primitive major is the polynomial. So for a degree d rational primitive major m that can be realized by a polynomial f , we can establish a correspond relationship between the points of $\overline{b_\infty}(m)$ and the ray pair of f . This provides a dynamical interpret of the points in $\overline{b_\infty}(m)$ and $TH(m)$.

Let f be a postcritical finite polynomial, the degree of f is d . The Böttcher theorem give the commutative graph:

$$\begin{array}{ccc}
 \mathbb{C} \setminus K_f & \xrightarrow{f} & \mathbb{C} \setminus K_f \\
 \phi \downarrow & & \downarrow \phi \\
 \mathbb{C} \setminus \overline{\mathbb{D}} & \xrightarrow{z \mapsto z^d} & \mathbb{C} \setminus \overline{\mathbb{D}}
 \end{array} \tag{5.2.10}$$

and the conformal map ϕ^{-1} can be extended continuously to \mathcal{S}^1 . Then each external ray land on a point in J_f .

Let γ be a Jordan curve in complex plane. The point $v \in \gamma$ is the unique possible critical value of f . Set

$$f^{-1}(v) = \{c_1, \dots, c_k\} \quad \text{and} \quad \deg_{c_i}(f) = d_i, \quad i = 1, \dots, k.$$

Then the connected component of $f^{-1}(\gamma)$ which contains c_i is consisting of d_i Jordan curve

intersecting only at point c_i pairwise. It can be seen as the union of $2d_i$ rays starting from c_i . We label these rays counter clockwise by $l_i^1, \dots, l_i^{2d_i}$ starting from any such ray. Define

$$\gamma_i^{m_i} = l_i^{m_i} \cup l_i^{m_i+1}, \quad 1 \leq i \leq k, \quad 1 \leq m_i \leq 2d_i.$$

Following the notation above, for any Jordan curve γ which contains at most one critical value of f , we define

$$f^*(\gamma) = \{ \gamma_i^{m_i} \mid 1 \leq i \leq k, \quad 1 \leq m_i \leq 2d_i \}$$

It is easy to see that for any $\beta \in f^*(\gamma)$, $f : \beta \rightarrow \gamma$ is a homeomorphism. If $\Gamma = \{\gamma_1, \dots, \gamma_n\}$, where $\gamma_1, \dots, \gamma_n$ are Jordan curves (these curve maybe intersect mutually) containing at most one critical value of f each, we define

$$f^*(\Gamma) = \bigcup_{i=1}^n f^*(\gamma_i)$$

For a postcritical finite polynomial f of degree d , we can define a degree d primitive major (or critical portrait)

$$m_f = \{\Theta_1, \dots, \Theta_s\}$$

associated to f (one can refer to [?] Chapter 1 for a concrete definition). Now, we begin to give the dynamical explanation of the points in $\overline{b_\infty}(m_f)$

Definition 5.2.17. *Two rays $R_f(\theta)$ and $R_f(\eta)$ are called **ray pair** if they land on the same point. For simplicity, we also call $\{\theta, \eta\}$ a ray pair of f .*

If $\{\theta, \eta\}$ is a ray pair of f , the curve $R_f(\theta) \cup R_f(\eta) \cup \gamma_f(\theta)$ divide the complex plane into two parts.

Definition 5.2.18. *The ray pair $\{\theta, \eta\}$ is called **adjacent** if in one of the two parts, there are no external rays landing on $\gamma_f(\theta)$. Such a part is called a **wake** of $\{\theta, \eta\}$ (there may be two wake for an adjacent ray pair). A Fatou component is called **bounded** by an adjacent ray pair if the component belongs to a wake of the ray pair and the ray pair land on the boundary of this component.*

Following the Böttcher Theorem, we can define the internal rays in each Fatou compo-

ment of f such that the image of a internal ray under f_c is also a internal ray. Now we will define a kind of special Jordan curve in dynamical plane of f and then correspond a point in $b_\infty(m_f)$ to such a simple curve.

Definition of a simple curve $\overline{R}_f(x, y)$: If $R_f(x)$ and $R_f(y)$ land on the same point, set

$$\overline{R}_f(x, y) = R_f(x) \cup \gamma_f(x) \cup R_f(y).$$

In special,

$$\overline{R}_f(x, x) = R_f(x) \cup \gamma_f(x)$$

If $R_f(x)$ and $R_f(y)$ land on the same Fatou component U but do not land on the same point, then there are two internal rays $r_U(\alpha)$ and $r_U(\beta)$ that connect $z_0, \gamma_f(x)$ and $z_0, \gamma_f(y)$ respectively (z_0 is the center of U). In this case, we set

$$\overline{R}_f(x, y) = R_f(x) \cup \gamma_f(x) \cup r_U(\alpha) \cup r_U(\beta) \cup \gamma_f(y) \cup R_f(y)$$

According to the definition of m_f , for each point $(x, y) \in b_0(m_f)$, the Jordan curve $\overline{R}_f(x, y)$ always exists. Let

$$\Gamma_0(f) = \{ \overline{R}_f(x, y) \mid (x, y) \in b_0(m_f) \}$$

Inductively define

$$\Gamma_{i+1}(f) = f_c^*(\Gamma_i(f)) \cup \Gamma_0(f)$$

It follows easily that $\Gamma_i(f) \subset \Gamma_{i+1}(f)$. According to the commutative graph (5.2.10) (Böttcher Theorem), the construction of $b_i(m_f)$ and the definition of f^* , we have

- If $(x, y) \in b_i(m_f)$, then the simple curve $\overline{R}_f(x, y)$ exists and belong to $\Gamma_i(f)$
- We can define a sequence map $l_i : b_i(m_f) \longrightarrow \Gamma_i(f)$, mapping (x, y) to $\overline{R}_f(x, y)$. Every such map is a 2 to 1 onto map.

- There exist a sequence of commutative graph:

$$\begin{array}{ccc}
b_{i+1}(\theta) & \xrightarrow{F} & b_i(m_f) \cup \{(d\theta, d\theta) \mid \theta \in \cup_{j=1}^s \Theta_j\} \\
l_{i+1} \downarrow & & \downarrow l_i \\
\Gamma_{i+1}(c) & \xrightarrow{f_c} & \Gamma_i(f) \cup \{\overline{R}_f(\theta) \mid \theta \in \cup_{j=1}^s \Theta_j\}
\end{array}$$

where

$$\overline{R}_f(\theta) = \begin{cases} R_f(\theta) \cup \gamma_f(\theta) & \text{if } d\theta \text{ is strictly preperiodic} \\ R_f(\theta) \cup \gamma_f(\theta) \cup r_U(\alpha) & \text{if } d\theta \text{ is periodic (} U \text{ is the Fatou component containing critical value } \alpha \text{)} \end{cases}$$

The sequence map $\{l_i\}$ provide a interpret of the points of $b_\infty(m_f)$ in the view of dynamics:

Proposition 5.2.19. *a point $(x, y) \in b_\infty(m_f)$ if and only if $\overline{R}_f(x, y)$ exists and there is an integer n such that $f^n : \overline{R}_f(x, y) \rightarrow \overline{R}_f(d^n x, d^n y)$ is a homeomorphism, where $\overline{R}_f(d^n x, d^n y) \in \Gamma_0(m_f)$.*

In order to give an interpret of the points of $b'_\infty(m_f)$, we consider the limit set of $\cup_{i=0}^\infty \Gamma_i(f)$ in Hausdorff topology. Suppose there exist $\overline{R}_f(x_i, y_i) \in \Gamma_i(f)$ such that $\overline{R}_f(x_i, y_i) \rightarrow R$ in Hausdorff topology as $i \rightarrow \infty$, then R must be one of the following cases:

1. $R = R_f(x) \cup \gamma_f(x) \cup R_f(y) \cup r_U(\alpha)$, where $R_f(x)$ and $R_f(y)$ is an adjacent ray pair, U is a Fatou component bounded by $R_f(x)$ and $R_f(y)$ and $r_U(\alpha)$ is the internal ray in U landing at $\gamma_f(x)$.
2. $R = R_f(x) \cup \gamma_f(x) \cup r_U(\alpha)$, where $R_f(x)$ lands on the boundary of a Fatou component U and $r_U(\alpha)$ is the internal ray in U landing at $\gamma_f(x)$.
3. $R = R_f(x) \cup \gamma_f(x) \cup R_f(y)$, where $R_f(x)$ and $R_f(y)$ is an adjacent ray pair.
4. $R = R_f(x) \cup \gamma_f(x)$.

Note that if $\lim_{i \rightarrow \infty} (x_i, y_i) = (x, y)$, $(x_i, y_i) \in b_i(m_f)$, then $\lim_{i \rightarrow \infty} \overline{R}_f(x_i, y_i) = R$ in Hausdorff topology and R contains $R_f(x)$ and $R_f(y)$. So it is natural to image defining a map $l : b'_\infty(m_f) \rightarrow \Gamma'(f)$, mapping (x, y) to R , where $\tilde{\Gamma}'(f)$ is consisting of all sets R which is one of the cases listed above. This map is not well defined because it depends on

the choosing of a sequence $\{(x_i, y_i)\}$. However, if we forget the internal ray contained in R for $R \in \tilde{\Gamma}'(f)$, the map l is well defined. That is, set

$$\begin{aligned}\Gamma'(f) &= \{\check{R} \mid \check{R} = R \setminus \text{the internal ray contained in } R, R \in \tilde{\Gamma}'(f)\} \\ &= \{\overline{R}_f(x, y) \mid \{x, y\} \text{ is an adjacent ray pair of } f \text{ or } x = y \in \mathbb{S}^1\}\end{aligned}$$

The map

$$l : b'_\infty(m_f) \longrightarrow \Gamma', \text{ mapping } (x, y) \text{ to } \overline{R}_f(x, y)$$

is well defined. Moreover, it is not difficult to see that the map l is surjective; $\#l^{-1}(\overline{R}_f(x, y)) = 2$ if $x \neq y$ and $\#l^{-1}(\overline{R}_f(x, y)) = 1$ if $x = y$. Thus, we can give a dynamical interpret of the points of $b'_\infty(m_f)$:

Proposition 5.2.20. *A point $(x, y) \in b'_\infty(m_f)$ if and only if $x = y$ or $\{x, y\}$ is an adjacent ray pair of f .*

In Thurston's torus model, if we correspond a point $(x, y) \in \overline{b}_\infty(m_f)$ to a chord $\overline{xy} \in \overline{\mathbb{D}}$, then the set

$$L(m_f) = \{\overline{xy} \mid (x, y) \in \overline{b}_\infty(m_f)\} \quad (5.2.11)$$

is a degree d invariant lamination having m_f as its major. Similarly, the set $b'_\infty(m_f)$ gives a sub-lamination of $L(m_f)$ as

$$L_1(m_f) = \{\overline{xy} \mid (x, y) \in b_\infty(m_f)\} \quad (5.2.12)$$

Note that $L_1(m_f)$ is also a degree d invariant lamination, it is exactly the lamination defined by f .

Now we can use Proposition 5.2.20 to give an dynamical interpret of the points in $TH_1(m_f)$.

Let H_f be the Hubbard tree of f . Then for any $z \in H_f \cap J_f$, there exists an adjacent ray pair or a single ray landing on z that separates or intersects with P_f . On the contrary, an adjacent ray pair which separates or intersects P_f must land on a point of Hubbard tree. This separating property can be expressed by means of angles. For $\Theta_j \in m_f$, if it contains a periodic angle, set Θ_j^- the set consisting of the unique periodic angle in Θ_j ; otherwise set

$\Theta_j^- = \Theta_j$. Set

$$P^-(m_f) = \{d^n \theta \mid n \geq 0, \theta \in \bigcup_{j=1}^s \Theta_j^-\} \quad \text{and} \quad TP^-(m_f) = \{(\theta, \theta) \mid \theta \in P^-(m_f)\}.$$

Then it is easy to see that a ray pair $\{x, y\}$ of f separates or intersects P_f if and only if the chord \overline{xy} separates or intersects $P^-(m_f)$ in $\overline{\mathbb{D}}$. If we set

$$TH_1^-(m_f) = \{ (x, y) \in \overline{b'_\infty}(m_f) \mid C_{x,y}^+ \text{ separates or intersects } TP^-(m_f) \},$$

$$Q_f = \{\gamma_f(x) \mid x \in TH_1^-(m_f) \cap \text{diagonal of } \mathbb{T}^2\}.$$

then $Q_f \subset H_f$ is a f invariant finite set. By the proof of Proposition 5.2.6, the discussion above and Proposition 5.2.20, we have

1. A point $(x, y) \in TH_1^-(m_f)$ if and only if $x = y\gamma_f(x) \in Q_f$ or $x \neq y$, the adjacent ray pair $\{x, y\}$ lands on a point of H_f and separates or intersects P_f ;
2. The set $TH_1^-(m_f)$ is also a compact F -invariant set;
3. $TH_1^-(m_f) \subset TH_1(m_f)$ and $TH_1(m_f) \setminus TH_1^-(m_f)$ is a finite set.

Thus, this result gives the dynamical explanation of the point in $TH_1(m_f)$.

Let λ be the leading eigenvalue of the transition matrix on its Hubbard tree. Then we have

Proposition 5.2.21. $h(TH(m_f), F) = \log \lambda$.

Proof. Since $TH_1^-(m_f)$ is also a compact F -invariant set, we will compute $h(TH_1^-(m_f), F)$ at first. According to the definition of $TH_1^-(m_f)$, we can define a map

$$\pi : TH_1^-(m_f) \longrightarrow H_f$$

mapping $(x, y) \in TH_1^-(m_f)$ to the common landing point of $R_f(x)$ and $R_f(y)$. The image of the map is exactly $H_f \cap J_f$. If $\{(x_n, y_n)\} \subset TH_1^-(m_f)$ converge to (x, y) as $n \rightarrow \infty$, then ray pair $\{x, y\}$ converges to $\{x, y\}$ in Hausdorff topology, so the map π is continuous. Moreover, it satisfies the following two properties:

- For any $z \in H_f \cap J_f$, the fiber $\pi^{-1}(z)$ has a cardinal bounded by a fixed number M .

- π is a semi-conjugate from F to f . That is

$$\begin{array}{ccc}
TH_f & \xrightarrow{F} & TH_1^-(m_f) \\
\pi \downarrow & & \downarrow \pi \\
H_f \cap J_f & \xrightarrow{f} & H_f \cap J_f
\end{array}$$

By Proposition 2.5.1, 2.5.2 and 2.5.3, we have

$$h(TH_1^-(m_f), F) \stackrel{Pro\ 2.5.3}{=} h(H_f \cap J_f, f) \stackrel{Pro\ 2.5.1, 2.5.2}{=} h(H_f, f)$$

In [Do], Douady proves $h(I, f) = \log \lambda$ for postcritical finite real map. One can use the completely same argument in complex case to prove $h(H_f, f) = \log \lambda$ for postcritical finite polynomial f . Then it follows $h(TH_1^-(m_f), f) = \log \lambda$.

Since $TH_1(m_f) \setminus TH_1^-(m_f)$ is a finite set and $TH_0(m_f)$ is a countable set, we have

$$\text{H.dim } (TH_1^-(m_f)) = \text{H.dim } (TH_1(m_f)) = \text{H.dim } (TH(m_f)).$$

Then by Proposition 5.2.2, we have

$$\begin{aligned}
h(TH(m_f)) & \stackrel{Pro\ 5.2.2}{=} \log d \cdot \text{H.dim } (TH(m_f)) = \log d \cdot \text{H.dim } (TH_1^-(m_f)) \\
& \stackrel{Pro\ 5.2.2}{=} h(TH_1^-(m_f)) = \log \lambda.
\end{aligned}$$

□

Chapter 6

Wandering C^1 arc and stable multicurve of p.f rational map with parabolic orbifold

In this chapter, we study the wandering continuum problem for post critical finite rational map with parabolic orbifold (see the definition in Section 2.3.1). In Section 6.1, we study the holomorphic dynamics on torus and prove the wandering C^1 arc theorem for map of torus. In Section ??, we prove Theorem ?? by the results obtained in Section 6.1. In Section 6.3, we prove Theorem ??.

6.1 Wandering arc of holomorphic map on \mathbb{T}_τ

The objective here is to prove the following proposition which is used to prove Theorem ??.

Proposition 6.1.1. *Let $L : \mathbb{T}_\tau \rightarrow \mathbb{T}_\tau$, $L(z) = \alpha z \pmod{\Lambda_\tau}$ be a holomorphic map of torus. An arc K is wandering under L if and only if α is an integer and K is a short line segment with irrational slop.*

6.1.1 Holomorphic dynamics on torus

Here we give some basic description of holomorphic dynamics on torus without proof. One can refer to [Mil2] §6 for more details.

A Riemann surface is called **torus** if it is compact and of genus one. Any torus is conformal to $\mathbb{T}_\tau = \mathbb{C}/\Lambda_\tau = \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$ with $\tau \in \mathbb{H} = \{z \in \mathbb{C} \mid \Im z > 0\}$. A map $L : \mathbb{T}_\tau \rightarrow \mathbb{T}_\tau$ is holomorphic if and only if

$$L(z) = \alpha z + \beta \pmod{\Lambda_\tau} \text{ with } \alpha\Lambda_\tau \subset \Lambda_\tau.$$

It is easy to see that for any $\tau \in \mathbb{H}, m \in \mathbb{Z}, \alpha = m$ satisfies this condition. If α satisfies this property, then α is either an integer or a complex number ($\Im \alpha \neq 0$). The degree of L is equal to $|\alpha|^2$. The Julia set $J(L)$ is either the empty set or the entire torus according to whether $|\alpha| \leq 1$ or $|\alpha| > 1$.

If $\alpha \neq 1$, note that L has a fixed point $z_0 = \beta/(1 - \alpha)$ and hence is conjugate to the map $z \mapsto L(z + z_0) - z_0 = \alpha z$. In this paper, we only consider the map with $|\alpha| > 1$, so without loss of generality, we can assume $L(z) = \alpha z \pmod{\Lambda_\tau}$ and $J(L)$ is the entire torus.

6.1.2 Wandering arc on torus

Let \mathbb{T}_τ be a torus, $L(z) = \alpha z \pmod{\Lambda_\tau} (|\alpha| > 1)$ be a holomorphic map on \mathbb{T}_τ . We have the commutative graph

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{z \mapsto \alpha z} & \mathbb{C} \\ \pi \downarrow & & \downarrow \pi \\ \mathbb{T}_\tau & \xrightarrow{z \mapsto \alpha z \pmod{\Lambda_\tau}} & \mathbb{T}_\tau \end{array}$$

The complex plane can be seen as a real linear space based on $\{1, \tau\}$. So each point $z \in \mathbb{C}$ has a coordinate $(x, y) \in \mathbb{R}^2$ corresponding to the basis $\{1, \tau\}$, that is, $z = x + y\tau$. where $x := \Re_\tau z, y := \Im_\tau z$.

Let l be a line in \mathbb{C} . A The **slop** of l (corresponding to τ), denoted by k_l , is equal to $\frac{y_2 - y_1}{x_2 - x_1}$ where $x_1 + y_1\tau, x_2 + y_2\tau$ are any two points on the line l . It is well know that

- k_l is a rational number or $\infty \iff \exists z_1, z_2 \in l \text{ s.t } z_1 - z_2 \in \Lambda_\tau \iff \pi(l)$ is a closed curve.
- k_l is a irrational number $\Rightarrow \pi : l \rightarrow \mathbb{T}_\tau$ is injective and $\pi(l)$ is density on \mathbb{T}_τ .

Definition 6.1.2. Let $l \subset \mathbb{C}$ be a line. If l has irrational slop, we call $\pi(l)$ a line on \mathbb{T}_τ . The set $\pi(l)$ (a line or a closed curve on \mathbb{T}_τ) is called (p, q) -**preperiodic** under L if $L^{p+q}(\pi(l)) = L^p(\pi(l))$ and $L^{s+t}(\pi(l)) \neq L^s(\pi(l))$ for any $0 \leq s \leq p, 0 \leq t \leq q$ with

$(s, t) \neq (p, q)$. If $p = 0$, $\pi(l)$ is called q periodic. A full connected compact set in $\pi(l)$ is called a **segment**. A segment K is called **short segment** of L if either $\pi(l)$ is wandering under L or $\pi(l)$ is (p, q) -preperiodic and $L^p(K) \cap L^{p+q}(K) = \emptyset$.

With these notation, we can write the obvious result:

Lemma 6.1.3. *Let K be a line segment on \mathbb{T}_τ . Then K is wandering under L if and only if α is a integer and K is a short segment of L with irrational slop.*

Proof. Let $l \subset \mathbb{C}$ be the line such that $K \subset \pi(l)$.

“ \Rightarrow ” If K is wandering under L , then the argument of α is 0 and $\pi(l)$ is a line in \mathbb{T}_τ . It follows α is an integer and K has the irrational slop. The property that K is wandering also implies K is a short segment of L .

“ \Leftarrow ” In this case, $\pi(l)$ is a line in \mathbb{T}_τ . In order to prove K is wandering under L , we only need to prove $L^{np}(K) \cap L^{mp}(K) = \emptyset$ for $m \neq n \in \mathbb{N} \cup \{0\}$ in case that $\pi(l)$ is p periodic under L . For $s, t \in \mathbb{Z}$, denote by $l_{s,t}$ the translation of l along the vector $s + t\tau$ in \mathbb{C} . If $\pi(l)$ is p periodic, then there exist $s_0, t_0 \in \mathbb{Z}$ such that $\tilde{L}^p(l) = l_{s_0 t_0}$ where \tilde{L} is the lift of L along the projection π .

Let E be any segment of $\pi(l)$, \tilde{E} be the component of $\pi^{-1}(E)$ which belongs to l . Denote by $X(E)$ the projection of \tilde{E} to the x axle. Suppose $X(K) = [x_1, x_2]$, then $X(L^p(K)) = [\alpha^p x_1, \alpha^p x_2]$. The property that K is a short segment implies that $x_2 < \alpha^p x_1$ or $x_1 > \alpha^p x_2$. We assume $x_2 < \alpha^p x_1$. In this case, $X(L(K)) > X(K)$ (that means all points of $X(L(K))$ is on the right of $X(K)$). We have $X(L^{2p}(K)) = [\alpha^p(\alpha^p x_1) - s_0, \alpha^p(\alpha^p x_2) - s_0]$. The assumption $x_2 < \alpha^p x_1 \Rightarrow \alpha^p x_2 < \alpha^p(\alpha^p x_1) - s_0 \Rightarrow X(L^{2p}(K)) > X(L^p(K))$. With the same reason, we can prove $X(L^m(K)) < X(L^n(K))$ for $m < n$. Similarly, if we assume at the beginning that $x_1 > \alpha^p x_2$, we can prove $X(L^m(K)) < X(L^n(K))$ for $m < n$. In both case, we have $L^{np}(K) \cap L^{mp}(K) = \emptyset$ for $m \neq n \in \mathbb{N} \cup \{0\}$. \square

Definition 6.1.4. *Let S be any Riemann surface, The map $\lambda : [0, 1] \rightarrow S$ is called an C^1 arc on S if λ is C^1 and injective.*

Now we will give a lemma which is very useful in proof of Proposition 6.1.1, a C^1 arc λ can be written as $\lambda(t) = x(t) + y(t)\tau$, ($t \in [0, 1]$).

Lemma 6.1.5. *Suppose $\lambda \in \mathbb{C}$ is an C^1 arc which is parameterized by x , that is $\lambda = x + y(x)\tau$ ($x_1 \leq x \leq x_2$). Set $z_1 = \lambda(x_1)$, $z_2 = \lambda(x_2)$. Then there is a sequence integers*

$\{n_j\}$ such that $\Re_\tau(L^{n_j}(z_2 - z_1)) \rightarrow \infty$, $\arg(\alpha^{n_j}) \rightarrow 0^+$ and $L^{n_j}(\lambda) = x + y_{n_j}(x)\tau$ for $\Re_\tau(L^{n_j}(z_1)) \leq x \leq \Re_\tau(L^{n_j}(z_2))$.

Proof. • If α is an integer, the lemma is correct by choosing $n_j = j$.

- If α is not an integer and $\arg(\alpha)$ is a rational number, then there exists an integer k such that α^k is an integer. By choosing $n_j = kj$, the lemma is correct.
- If α is not an integer and $\arg(\alpha)$ is an irrational number, then the set $\{\arg(\alpha^n) \mid n \in \mathbb{N}\}$ is density on \mathbb{S}^1 . It follows there exists a sequence of integers $\{n_j\}$ such that $\arg(\alpha^{n_j}) \rightarrow 0$ as $j \rightarrow \infty$. Set $\tau = \tau_x + \tau_y i$, $\arg(\alpha) = \theta$ and

$$\lambda_{n_j} := L^{n_j}(\lambda) = \{\alpha^{n_j} z \mid z \in \lambda\} = x_{n_j}(x) + y_{n_j}(x)\tau \quad (x_1 \leq x \leq x_2)$$

By simple computation, we obtain the concrete parameter expression of λ_{n_j} as

$$\begin{cases} x_{n_j} &= \frac{|\alpha|^{n_j}}{\tau_y} [\cos(2\pi n_j \theta) \tau_y x - \sin(2\pi n_j \theta) (y(x) |\tau|^2 + \tau_x x)] \\ y_{n_j} &= \frac{|\alpha|^{n_j}}{\tau_y} [\cos(2\pi n_j \theta) \tau_y y(x) + \sin(2\pi n_j \theta) (x + y(x)) \tau_x] \end{cases} \quad (x_1 \leq x \leq x_2).$$

Since λ is an C^1 arc, it follows that

$$\frac{dx_{n_j}}{dx} = \frac{|\alpha|^{n_j}}{\tau_y} [\cos(2\pi n_j \theta) \tau_y - \sin(2\pi n_j \theta) (y'(x) |\tau|^2 + \tau_x)]$$

Note that $y'(x)$ is bounded for $x_1 \leq x \leq x_2$ and $\tau_y > 0$, so we can choose a subsequence of $\{n_j\}$, denoted also by $\{n_j\}$ for simplicity such that $\frac{dx_{n_j}}{dx} > 0$ ($x_1 \leq x \leq x_2$). Then x can be expressed as the function of x_{n_j} :

$$x = x(x_{n_j}) \quad \Re_\tau(L^{n_j}(z_1)) \leq x_{n_j} \leq \Re_\tau(L^{n_j}(z_2)). \quad (6.1.1)$$

Substitution refformula into $y_{n_j}(x)$, then the arc λ_{n_j} can be parameterized by x_{n_j} as

$$\lambda_{n_j} = x_{n_j} + y_{n_j}(x(x_{n_j})), \quad \Re_\tau(L^{n_j}(z_1)) \leq x_{n_j} \leq \Re_\tau(L^{n_j}(z_2))$$

□

Proposition 6.1.6. *If λ is an wandering C^1 arc under L , then λ must contains a segment of line.*

Proof. In fact, we will show that if λ is a arc that doesn't contain any segment of line, then there exists integer N such that $L^{\circ N}(\lambda)$ intersects itself transversally.

Let $\lambda(t) = x(t) + y(t)\tau$ be a C^1 arc on \mathbb{T}_τ that doesn't contain any segment of line. If $x'(t) = 0$ for all $t \in [0, 1]$, then λ is a line segment with direction τ . So there exists $t_0 \in [0, 1]$ such that $x'(t_0) \neq 0$. We can assume $x'(t_0) > 0$ (otherwise we can change the direction of parameter). By the Inverse Function Theorem, locally, we have $t = t(x)$ ($x_0 - \epsilon \leq x \leq x_0 + \epsilon$, $t(x_0) = t_0$) and then $y = y(t(x))$ for $x_0 - \epsilon \leq x \leq x_0 + \epsilon$. So there is a sub-arc λ_0 of λ that can be parameterized by x , that is $\lambda_0 = x + y(t(x))\tau$, $x \in [x_0 - \epsilon, x_0 + \epsilon]$.

Lift $\lambda_0 \subset \mathbb{T}_\tau$ to a arc $\tilde{\lambda}_0 \subset \mathbb{C}$. By Lemma 6.1.5, there exists n_0 such that

1. $L^{n_0}(\tilde{\lambda}_0)$ can be parameterized by x ;
2. Under the translation, $L^{n_0}(\tilde{\lambda}_0)$ intersects with two lines $\tau\mathbb{R}$ and $1 + \tau\mathbb{R}$ at z_1 and z_2 .

Without loss of generalization, we can assume $\Im_\tau z_1 \leq \Im_\tau z_2$ and $\Im_\tau z_1 \in [0, 1)$.

Now we denote by $\tilde{\lambda}(x) = x + y(x)\tau$ ($x \in [0, 1]$) the sub-arc of $L^{n_0}(\tilde{\lambda}_0)$ between $\tau\mathbb{R}$ and $1 + \tau\mathbb{R}$. Note that $y(1) \geq y(0)$. The projection of $\tilde{\lambda}$ to \mathbb{T}_τ is also denoted by λ . Since $\tilde{\lambda}$ contains no segment, there exist minimal integer p, q such that $p/2^q + y(p/2^q)\tau \notin [z_1, z_2]$. Without loss of generalization, we can assume this point is $1/2 + y(1/2)\tau$ and above on the segment $[z_1, z_2]$. Then we have

$$(y(1/2) - y(0)) - (y(1) - y(1/2)) = \epsilon_0 > 0.$$

We choose a sequence integers $\{n_j\}$ as that in the proof of Lemma 6.1.5, then we have

- $e^{2\pi n_j \theta i} \rightarrow 1^+$ as $j \rightarrow \infty$, where $\alpha = |\alpha|e^{2\pi \theta i}$;
- $\tilde{\lambda}_{n_j} := \tilde{L}^{n_j}$ is parameterized by x as $\tilde{\lambda}_{n_j} = x + y_{n_j}(x)$, $\Re_\tau(\alpha^{n_j} z_1) \leq x \leq \Re_\tau(\alpha^{n_j} z_2)$.

In the following, we will define three sequence of points $\{a_{n_j}\}$, $\{b_{n_j}\}$ and $\{c_{n_j}\}$.

Define $a'_{n_j} = \tau\mathbb{R} \cap \tilde{\lambda}_{n_j} = y_{n_j}(0)\tau$.

Since L is holomorphic on \mathbb{T}_τ , we have

$$\alpha \begin{pmatrix} 1 \\ \tau \end{pmatrix} = A \begin{pmatrix} 1 \\ \tau \end{pmatrix} \text{ for some } A = \begin{pmatrix} d_{1,1} & d_{2,1} \\ d_{3,1} & d_{4,1} \end{pmatrix} \in GL(2, \mathbb{Z})$$

Set $A^{n_j} = \begin{pmatrix} d_{1,n_j} & d_{2,n_j} \\ d_{3,n_j} & d_{4,n_j} \end{pmatrix}$. Denote by m the minimal positive integer such that $\tilde{\lambda}$ belongs to the parallelogram generated by 1 and $m\tau$. Then

$$\alpha^{n_j}(1 + m\tau) = \alpha^{n_j}(1, m) \begin{pmatrix} 1 \\ \tau \end{pmatrix} = (d_{1,n_j} + md_{3,n_j}, d_{2,n_j} + md_{4,n_j}) \begin{pmatrix} 1 \\ \tau \end{pmatrix}$$

Define

$$\begin{aligned} b'_{n_j} &= \begin{cases} (d_{1,n_j} + md_{3,n_j} + \tau\mathbb{R}) \cap \tilde{\lambda}_{n_j} = d_{1,n_j} + md_{3,n_j} + y_{n_j}(d_{1,n_j} + md_{3,n_j})\tau & \text{if } d_{1,n_j} + md_{3,n_j} \text{ is odd} \\ (d_{1,n_j} + md_{3,n_j} - 1 + \tau\mathbb{R}) \cap \tilde{\lambda}_{n_j} = d_{1,n_j} + md_{3,n_j} - 1 + y_{n_j}(d_{1,n_j} + md_{3,n_j} - 1)\tau & \text{otherwise} \end{cases} \\ c'_{n_j} &= \frac{xb'_{n_j}}{2} + y_{n_j} \left(\frac{xb'_{n_j}}{2} \right). \end{aligned}$$

Define

$$\begin{aligned} a_{n_j} &= \tilde{L}^{-n_j}(a'_{n_j}) \cap \tilde{\lambda} = x_{a_{n_j}} + y_{a_{n_j}}\tau \\ b_{n_j} &= \tilde{L}^{-n_j}(b'_{n_j}) \cap \tilde{\lambda} = x_{b_{n_j}} + y_{b_{n_j}}\tau \\ c_{n_j} &= \tilde{L}^{-n_j}(c'_{n_j}) \cap \tilde{\lambda} = x_{c_{n_j}} + y_{c_{n_j}}\tau \end{aligned}$$

It is easy to see that $x_{a_{n_j}} \rightarrow 0$ and $x_{b_{n_j}} \rightarrow 1$ as $j \rightarrow \infty$.

We have defined $\{a_{n_j}\}, \{b_{n_j}\}, \{c_{n_j}\}, \tilde{\lambda}_{n_j}, \lambda_{n_j}$

Since λ is wandering under L , the arc λ_{n_j} doesn't intersect itself. It follows that for any $j \geq 1$

$$\begin{aligned} & \arg(c'_{n_j} - (a'_{n_j} + \tau)) < \arg(b'_{n_j} - c'_{n_j}) < \arg(c'_{n_j} + \tau - a'_{n_j}) \\ \iff & \arg(\alpha^{n_j}c_{n_j} - (\alpha^{n_j}a_{n_j} + \tau)) < \arg(\alpha^{n_j}b_{n_j} - \alpha^{n_j}c_{n_j}) < \arg(\alpha^{n_j}c_{n_j} + \tau - \alpha^{n_j}a_{n_j}) \\ \iff & \arg\alpha^{n_j} + \arg(c_{n_j} - (a_{n_j} + \frac{\tau}{\alpha^{n_j}})) < \arg\alpha^{n_j} + \arg(b_{n_j} - c_{n_j}) < \arg\alpha^{n_j} + \arg(c_{n_j} + \frac{\tau}{\alpha^{n_j}} - a_{n_j}) \\ \iff & \arg(c_{n_j} - (a_{n_j} + \frac{\tau}{\alpha^{n_j}})) < \arg(b_{n_j} - c_{n_j}) < \arg(c_{n_j} + \frac{\tau}{\alpha^{n_j}} - a_{n_j}) \end{aligned}$$

As $j \rightarrow \infty$, it is easy to see that $a_{n_j} \rightarrow z_1$, $b_{n_j} \rightarrow z_2$, $\Re_\tau(c_{n_j}) = (\Re_\tau a_{n_j} + \Re_\tau b_{n_j})/2 \rightarrow 1/2$ and then $c_{n_j} \rightarrow 1/2 + y(1/2)\tau =: c_0$. Let $j \rightarrow \infty$, we have

$$\arg(c_0 - z_1) \leq \arg(z_2 - c_0) \leq \arg(c_0 - z_1)$$

It implies that c_0 is on the line segment $[z_1, z_2]$. With the same reason, we conclude that for any $q \geq 1$ and $1 \leq p \leq 2^q$, the point $p/2^q + y(p/2^q)\tau$ belongs to the segment $[z_1, z_2]$. It follows $\lambda = [z_1, z_2]$. \square

Proof of Theorem 6.1.1. It follows directly from Lemma 6.1.3 and Proposition 6.1.6. \square

6.2 Wandering arc of rational map with parabolic orbifold

In this section, we prove Theorem ?? with the help of Proposition 6.1.1

Let f be a rational map with parabolic orbifold, $L_f : \mathbb{T}_\tau \longrightarrow \mathbb{T}_\tau$ be the lift of f . Recall the two projections:

$$\pi : \mathbb{C} \longrightarrow \mathbb{T}_\tau \quad \text{and} \quad \wp_f : \mathbb{T}_\tau \longrightarrow \hat{\mathbb{C}}$$

Denote by $C(\wp_f)$ the set of critical points of \wp_f . The composition map $\wp_f \circ \pi : \mathbb{C} \longrightarrow \hat{\mathbb{C}}$ is a branched covering with the set of critical point $\tilde{C}(\wp_f) = \pi^{-1}(C(\wp_f))$. The Deck transformation group of $\wp_f \circ \pi$ is

$$H_n = \left\langle \begin{array}{l} \gamma_1 : z \rightarrow z + 1 \\ \gamma_2 : z \rightarrow z + \tau \\ \rho_n : z \rightarrow e^{\frac{2\pi i}{n}}(z - z_0) + z_0 \end{array} \right| \rho_n \langle \gamma_1, \gamma_2 \rangle = \langle \gamma_1, \gamma_2 \rangle \rho_n \right\rangle$$

where n is determined by the signature of f .

Lemma 6.2.1. *Let $l \subset \mathbb{C} \setminus \tilde{C}(\wp_f)$ be line with irrational slop. Then the map $\wp_f \circ \pi : l \longrightarrow \hat{\mathbb{C}}$ is injective if and only if f is a Lattes map.*

Proof. A little thought shows that If $\wp_f \circ \pi : l \longrightarrow \hat{\mathbb{C}}$ is injective, then the orbit of l under the group $G_f = \langle \rho_n \rangle$ is pairwise disjoint. It is only possible for some lines as $n = 2$. On the other hand, when f is a Lattes map ($n = 2$), for any $l \subset \mathbb{C} \setminus \tilde{C}(\wp_f)$ with irrational slop, we have $\pi(l) \cap \pi(-l) = \emptyset$ (otherwise l must pass through the points in $\tilde{C}(\wp_f)$). It follows the map $\wp_f : \pi(\pm l) \longrightarrow \hat{\mathbb{C}}$ are all injective, then $\wp_f \circ \pi : l \longrightarrow \hat{\mathbb{C}}$ is injective. \square

Remark 6.2.2. *Let l be a line in \mathbb{C} , f be a Lattes map, the following result is easy to check:*

1. *As the same as Definition 6.1.2, we can define the set $\wp_f \circ \pi(l)$ preperiodic or wandering under f and also define a segment in $\hat{\mathbb{C}}$ and a short segment of f .*

2. The map $\wp_f : \pi(l) \longrightarrow \hat{\mathbb{C}}$ is injective $\iff \pi(l) \cap C(\wp_f) = \emptyset \iff \pi(l) \cap \pi(-l) = \emptyset$.
3. If $\pi(l) \cap C(\wp_f) = \emptyset$ and the orbit of $\pi(l)$ contains the line $\pi(-l)$, then $\pi(l)$ has the period $2k$ ($k \in \mathbb{N}$) under L_f . In this case, the line $\wp_f \circ \pi(l)$ is k periodic under f .
4. In the case $\pi(l) \cap C(\wp_f) \neq \emptyset$. If l has a rational slop, the curve $\pi(l)$ contains 2 points of $C(\wp_f)$ and the image $\wp_f(\pi(l))$ is a segment obtained by folding $\pi(l)$ at the two points of $C(\wp_f) \cap \pi(l)$. If l has an irrational slop, the line $\pi(l)$ contains 1 points of $C(\wp_f)$ and the image $\wp_f(\pi(l))$ is a ray obtained by folding $\pi(l)$ at the point of $C(\wp_f) \cap \pi(l)$.

Proof of Theorem ??. Let f be a rational map with parabolic orbifold, $L_f : \mathbb{T}_\tau \xrightarrow{z \mapsto \alpha z} \mathbb{T}_\tau$ be the lift of f and the Deck transformation group of $\wp \circ \pi$ be n .

“ \Rightarrow ”. Suppose K is a full C^1 wandering arc under f , then $K \cap P_f = \emptyset$. It follows that $\wp_f^{-1}(K)$ is consisting of n pairwise disjoint full C^1 arc on \mathbb{T}_τ and $\wp_f^{-1}(K)$ is wandering under L_f . By Proposition 6.1.1, the number α is an integer and each component of $\wp_f^{-1}(K)$ is a short segment with irrational slop on \mathbb{T}_τ . For any component \tilde{K} of $\wp_f^{-1}(K)$, if f is not a Lattes map, Lemma 6.2.1 shows that there exists a sufficient large positive integer j_0 such that $\wp_f(L_f^{j_0}(\tilde{K}))$ intersects by itself, then $f^{j_0}(K)$ intersects by itself. So f must be a flexible Lattes map. By the wandering of $\wp_f^{-1}(K)$ and the definition of short segment in $\hat{\mathbb{C}}$, K must be a short segment.

“ \Leftarrow ”. Let K be a short segment of a flexible Lattes map f . The definition of a short segment in $\hat{\mathbb{C}}$ and Remark 6.2.2 show that the two components of $\wp_f(K)$ are all short segment of L_f in T_τ and $\wp_f^{-1}(K)$ is wandering under L_f . It follows K is wandering under f .

□

6.3 Stable multicurve for rational map with parabolic orbifold

The objective here is to prove Theorem ?? . Firstly, we study the stable multicurve on torus and then use this result to prove the main theorem. All notation is the same as before.

Note that $\pi(1)$, $\pi(\tau)$ are simple closed curve on \mathbb{T}_τ and they are the generators of the fundamental group of \mathbb{T}_τ . With an abuse of the notations, we also use 1 , τ to denote

these two curves. Then any simple closed curve $\gamma \subset \mathbb{T}_\tau$ is homotopic to $p + q\tau$ for some $p, q \in \mathbb{Z}$, $\gcd(p, q) = 1$ (greatest common divisor).

Lemma 6.3.1. *Let $L(z) = \alpha z \pmod{\Lambda_\tau}$ be a holomorphic self map on \mathbb{T}_τ and γ be a simple closed curve on \mathbb{T}_τ . Then the curves in $L^{-1}(\gamma)$ are pairwise homotopic. The degrees of the restriction of L on these curves are the same. The curves in $L^{-1}(\gamma)$ are homotopic to γ if and only if α is an integer.*

Proof. The map L is holomorphic if and only if

$$\alpha \begin{pmatrix} 1 \\ \tau \end{pmatrix} = A \begin{pmatrix} 1 \\ \tau \end{pmatrix} \text{ for some } A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in GL(2, \mathbb{Z}) \quad (6.3.1)$$

If α is an integer, then $A = \begin{pmatrix} \alpha & \\ & \alpha \end{pmatrix}$, otherwise the eigenvalue of A are conjugate complex numbers $\alpha, \bar{\alpha}$.

Let γ be a simple closed curve homotopic to $p + q\tau$ on \mathbb{T}_τ , then any connected components of $L^{-1}(\gamma)$ is a simple closed curve. Choose one curve γ_1 in $L^{-1}(\gamma)$. Suppose γ_1 is homotopic to $s + t\tau$ and $\deg(L|_{\gamma_1}) = k$. We have

$$s\alpha + t\alpha\tau = k(p + q\tau) \quad \gcd(s, t, k) = 1. \quad (6.3.2)$$

Substitution 6.3.1 into 6.3.2, we obtain

$$A^T \begin{pmatrix} s \\ t \end{pmatrix} = k \begin{pmatrix} p \\ q \end{pmatrix} \quad (6.3.3)$$

where A^T is the transposition of A . It follows

$$\begin{pmatrix} s \\ t \end{pmatrix} = \frac{k}{|\det A|} \begin{pmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \frac{k}{a_{11}a_{22} - a_{12}a_{21}} \begin{pmatrix} a_{22}p - a_{21}q \\ -a_{12}p + a_{11}q \end{pmatrix}$$

Set $\Delta = \gcd(a_{11}a_{22} - a_{12}a_{21}, a_{22}p - a_{21}q, -a_{12}p + a_{11}q)$, we have

$$k = \frac{a_{11}a_{22} - a_{12}a_{21}}{\Delta}, \quad s = \frac{a_{22}p - a_{21}q}{\Delta}, \quad t = \frac{-a_{12}p + a_{11}q}{\Delta}$$

It deduce that k, s, t is only determined by the map L and the homotopic class of γ . So all curves in $L^{-1}(\gamma)$ are pairwise homotopic and we can use d_r to denote the degree of the restriction of L to any curve in $L^{-1}(\gamma)$.

By Equation 6.3.3, the curve γ is stable on $\mathbb{T}_\tau \iff \gamma_1$ is homotopic to $\gamma \iff d_r$ (i.e k) is an eigenvalue of A^T and $(p, q)^T$ is an eigenvector corresponding to $d_r \iff \alpha = d_r$. \square

Proof of Theorem ??. Let $f : \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$ be a Lattes map. By Theorem 2.3.3, the map $\wp_f : \mathbb{T}_\tau \longrightarrow \hat{\mathbb{C}}$ is a branched covering of degree 2 satisfying $\wp_f(z) = \wp_f(-z)$. This map has 4 simple critical points. The set of critical points of \wp_f is denoted by $C(\wp_f)$ and the set of critical values of this map is P_f . For any non-peripheral simple closed curve $\gamma \subset \hat{\mathbb{C}} \setminus P_f$, each component of $\hat{\mathbb{C}} \setminus \gamma$ contains two points of P_f . We claim that $\wp^{-1}(\gamma)$ is consisting of two simple closed curves $\tilde{\gamma}, \tilde{\gamma}' \subset \mathbb{T}_\tau \setminus C(\wp_f)$, these two curves are homotopic in \mathbb{T}_τ and the restriction of \wp_f on either of the two curves is a homeomorphism to γ .

Proof of Claim 1: If $\wp^{-1}(\gamma)$ contains only one curve $\tilde{\gamma}$, then $\tilde{\gamma}$ must be trivial on \mathbb{T}_τ and $\deg(\wp_f|_{\tilde{\gamma}})=2$. The restriction of \wp_f on each component of $\mathbb{T}_\tau \setminus C(\wp_f)$ is a branched covering of degree 2 to one component of $\hat{\mathbb{C}} \setminus \gamma$. By the Riemann-Hurwitz formula, the simply connected component of \mathbb{T}_τ contains 1 point of $C(\wp_f)$, so its image contains only one point of P_f , contradiction to non-peripheral of γ . It follows $\wp^{-1}(\gamma) = \{\tilde{\gamma}, \tilde{\gamma}'\}$ and the restriction of \wp_f on $\tilde{\gamma}$ or $\tilde{\gamma}'$ is a homeomorphism to γ . With a similar reason as above, neither $\tilde{\gamma}$ nor $\tilde{\gamma}'$ is trivial on \mathbb{T}_τ , so $\tilde{\gamma} \cap \tilde{\gamma}' = \emptyset \Rightarrow \tilde{\gamma}$ and $\tilde{\gamma}'$ are homotopic on \mathbb{T}_τ . Then the two components of $\mathbb{T}_\tau \setminus \{\tilde{\gamma} \cup \tilde{\gamma}'\}$ are both annulus. The rest of the Claim follows directly from the Riemann-Hurwitz formula. This end the proof of the Claim.

Following the notation in Lemma 6.3.1, denoted by d_γ the degree of the restriction of L from a curve of $L^{-1}(\tilde{\gamma})$ to $\tilde{\gamma}$. By Lemma 6.3.1, $L^{-1}(\tilde{\gamma} \cup \tilde{\gamma}')$ is consisting of $2|\alpha|^2/d_\gamma$ simple closed curves on $\mathbb{T}_\tau \setminus C(\wp_f)$ and they are pairwise homotopic in \mathbb{T}_τ . These curve divide \mathbb{T}_τ into $2|\alpha|^2/d_\gamma$ annuluses. According to the commutative graph in Theorem ??, any such a annulus is a degree 1 or 2 branched covering under \wp_f to a component of $\hat{\mathbb{C}} \setminus f^{-1}(\gamma)$. The Riemann-Hurwitz fomula implies that each annulus contains 2 or no points of $C(\wp_f)$, then there are exactly 2 annulus in $\mathbb{T}_\tau \setminus L^{-1}(\tilde{\gamma} \cup \tilde{\gamma}')$, denoted by A_0, A_r ($r = |\alpha|^2/d_r$), which contains 2 points of $C(\wp_f)$ each. Set $U_0 = \wp_f(A_0)$, $U_r = \wp_f(A_r)$. We have U_0, U_r are simply connected, $\#U_0 \cap P_f = \#U_r \cap P_f = 2$ and $\deg(\wp_f|_{A_0}) = \deg(\wp_f|_{A_r}) = 2$. The other components of $\hat{\mathbb{C}} \setminus f^{-1}(\gamma)$ are all annuluses disjointed with P_f . So the curves in $f^{-1}(\gamma)$ are

pairwise homotopic in $\hat{\mathbb{C}} \setminus P_f$ (see Figure).

Since $\wp_f : \mathbb{T}_\tau \setminus C(\wp_f) \rightarrow \hat{\mathbb{C}} \setminus P_f$ is a covering, the curve γ is stable under $f \Rightarrow$ there exists a curve in $L^{-1}(\tilde{\gamma} \cap \tilde{\gamma}')$ which is homotopic to $\tilde{\gamma}$ or $\tilde{\gamma}'$ on $\mathbb{T}_\tau \setminus C(\wp_f) \Rightarrow$ there exists a curve in $L^{-1}(\tilde{\gamma} \cap \tilde{\gamma}')$ which is homotopic to $\tilde{\gamma}$ or $\tilde{\gamma}'$ on $\mathbb{T}_\tau \xrightarrow{\text{Lemma 6.3.1}} \alpha$ is an integer (i.e f is a flexible Lattes map).

Now we are left to prove that any non-peripheral simple closed curve is stable under a flexible Lattes map f . We consider a class of special simple closed curves on $\hat{\mathbb{C}} \setminus P_f$: for any $p, q \in \mathbb{Z}$, $\gcd(p, q) = 1$, let l be a long in $\mathbb{C} \setminus \pi^{-1}(C(\wp_f))$ with $\text{slop } k_l = q/p$. Define $\tilde{\gamma} = \pi(l)$, $\tilde{\gamma}' = \pi(-l)$, then $\tilde{\gamma}, \tilde{\gamma}'$ are homotopic simple closed curve in \mathbb{T}_τ and $\tilde{\gamma} \cap \tilde{\gamma}' = \emptyset$. It follows $\wp_f(\tilde{\gamma}) = \wp_f(\tilde{\gamma}') \triangleq \gamma$ and the restriction of \wp_f from either $\tilde{\gamma}$ or $\tilde{\gamma}'$ to γ is a homeomorphism. A simple argument by Riemann-Hurwitz formula shows that γ is a non-peripheral curve on $\mathbb{T} \setminus \hat{\mathbb{C}}$. In fact, the homotopic class of such γ is independent of the choosing of l . To prove this, let l' be another line with the same properties as l , $\tilde{\delta} = \pi(l')$, $\tilde{\delta}' = \pi(-l')$. Suppose $\tilde{\delta}$ doesn't coincide with $\tilde{\gamma}$ and $\tilde{\gamma}'$ (it is equivalent to $(\tilde{\delta} \cup \tilde{\delta}') \cap (\tilde{\gamma} \cup \tilde{\gamma}') = \emptyset$), we claim that $\tilde{\delta}$ is homotopic to either $\tilde{\gamma}$ or $\tilde{\gamma}'$ on $\mathbb{T}_\tau \setminus C(\wp_f)$. It follows that δ is homotopic to γ on $\hat{\mathbb{C}} \setminus P_f$, so we define such kind of simple closed curve in $\hat{\mathbb{C}}$ by $\gamma_{p,q}$. It is easy to see that $\gamma_{p,q}$ is not homotopic to $\gamma_{p',q'}$ for $(p, q) \neq (p', q')$.

Proof of Claim 2: Denote by A_0, A_1 the two annulus of \mathbb{T}_τ ($\tilde{\gamma} \cup \tilde{\gamma}'$). By Claim 1, the annulus A_i ($i = 0, 1$) contains 2 points of $C(\wp_f)$. Since $\tilde{\delta}$ is disjoint from $\tilde{\gamma} \cup \tilde{\gamma}'$, we can assume $\tilde{\delta} \subset A_0$. If $\tilde{\delta}$ is homotopic to neither $\tilde{\gamma}$ or $\tilde{\gamma}'$, then $\tilde{\delta}$ must separate the points of $C(\wp_f) \cap A_0$ (that is each of the two components $A_0 \setminus \tilde{\delta}$ contains a point of $C(\wp_f) \cap A_0$). Since the restriction of \wp_f from A_0 to a component of $\hat{\mathbb{C}}$ is a branched covering of degree 2, the curve δ (the image of $\tilde{\delta}$ under \wp_f) must intersect by itself (see Figure). It is a contradiction.

Let f be a flexible Lattes map, the lift $L(z) = mz \pmod{\Lambda_\tau}$. By Lemma 6.3.1 and a similar argument as proof of Claim 2, we can show that $\tilde{\gamma}_{p,q}$ is homotopic to a curve in $L^{-1}(\tilde{\gamma}_{p,q} \cup \tilde{\gamma}'_{p,q})$ on $\mathbb{T}_\tau \setminus C(\wp_f)$. It follows that $\gamma_{p,q}$ is homotopic to a curve $f^{-1}(\gamma_{p,q})$ on $\hat{\mathbb{C}} \setminus P_f$, that is, $\gamma_{p,q}$ is stable.

Proposition 2.6 in [FM] says that $\{\gamma_{p,q} \mid p, q \in \mathbb{Z}, \gcd(p, q) = 1\}$ are ?? of the homotopy classed of non-peripheral simple closed curves in $\hat{\mathbb{C}} \setminus P_f$. So for any non-peripheral simple closed curve $\gamma \subset \hat{\mathbb{C}} \setminus P_f$, we can choose $\gamma_{p,q}$ homotopic to γ in $\hat{\mathbb{C}} \setminus P_f$. The stability of $\gamma_{p,q}$ deduces the stability of γ . \square

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Thèse de Doctorat

YAN GAO

Dynatonic curve and core entropy for iteration of polynomials

Courbes dynatonomiques et entropie noyau de polynômes itérés

Résumé

Lorsqu'on étudie les systèmes dynamiques engendrés par une famille de polynômes, il apparaît naturellement des courbes algébriques de type cyclotomique, contenant des points périodiques ou prépériodiques. Dans le cas périodique de la famille $z^d + c$, le premier chapitre de cette thèse montre, en collaboration avec Ou, que ces courbes sont toutes lisses et irréductibles, généralisant les résultats connus au cas $d=2$. Dans le cas prépériodique de la même famille, le deuxième chapitre de la thèse montre, contre tout attendu, que ces courbes sont en général réductibles. En plus, il y contient une caractérisation des composantes irréductibles ainsi que leur relation géométrique et analytique.

Le deuxième thème de cette thèse concerne un nouveau sujet développé par W. Thurston, il s'agit d'entropie noyau des polynômes. Thurston a donné un algorithme, sans preuve, pour calculer ces entropies. La thèse contient une preuve rigoureuse de cet algorithme ainsi que des nouvelles méthodes pour étudier la variation de ces entropies en jonglant plusieurs points de vue.

Le dernier thème de cette thèse donne une condition nécessaire et suffisante pour qu'une fraction rationnelle possède un compact errant plein dans son ensemble de Julia. On savait que dans le cas particulier des polynômes ce genre de compact ne pouvait pas du tout exister.

Mots clés

courbes dynatonomiques, courbes irréductibles, itération des fractions rationnelles, entropie topologique, compact errant, ensemble de Julia.

Abstract

When studying dynamical systems generated by a family of polynomials, it arises naturally cyclotomic type algebraic curves containing periodic or preperiodic points. In the periodic case of the family $f_c(z) = z^d + c$, the first chapter of this thesis shows that all these curves are smooth and irreducible, generalizing the known results to the case $d = 2$. In the preperiodic case of the same family, the second chapter of this thesis shows, against all expected that these curves are in general reducible. In addition, there contains a characterization of irreducible components and their analytical and geometrical relationship.

The second theme of this thesis a new topic developed by W. Thurston, it is core entropy of polynomials. Thurston gave an algorithm, without proof, for compute these entropies. The thesis contains a rigorous proof of this algorithm and new methods to study the variation of these entropies from several views.

The last topic of this thesis gives a necessary and sufficient condition for a kind of rational map having a C^1 -arc in its Julia set.

Key Words

dynatonic curves, irreducible, iteration of rational functions, topological entropy, wandering continuum.