On fillability of contact manifolds

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Mémoire d'habilitation à diriger des recherches

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Abstract: The aim of this text is to give an accessible overview to some recent results concerning contact manifolds and their symplectic fillings. In particular, we work out the weakest compatibility conditions between a symplectic manifold and a contact structure on its boundary to still be able to obtain a sensible theory (Chapter II), furthermore we prove two results (Theorem A and B in Section I.4) that show how certain submanifolds inside a contact manifold obstruct the existence of a symplectic filling or influence its topology. We conclude by giving several constructions of contact manifolds that for different reasons do not admit a symplectic filling.

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Introduction

The aim of this work will be to explain and develop the following paragraph in Mikhail Gromov's famous '85 paper [**Gro85**] on pseudo-holomorphic curves:

> **2.4.** D_2' **(c)** The above results generalizes to submanifolds W of dimension n in those $(2n-1)$–dimensional contact manifolds X which appear as J–convex boundaries of almost complex $2n$–dimensional manifolds (V, J), where J can be tamed by a symplectic form on V. The submanifolds $W \subset X = \partial V$ in question are quite special: The space $\Theta \cap T_w(W)$ must be of dimension $n-1$ for all $w \in W$ outside a codimension two submanifold $W_0 \subset W$ and the hyperplane field $\Theta \cap T(W)$ on $W \setminus W_0$ must be integrable. One shows in certain cases (see [Gr02][1] the existence of "sufficiently many" J–holomorphic disks $(D^2, \partial D^2) \mapsto (V, W \setminus W_0)$. This imposes a non-trivial global condition on the geometry of the foliation on $W \setminus W_0$ tangent to the field $\Theta \cap T(W)$ on $W \setminus W_0$. Then one easily produces examples of submanifolds W in some contact manifolds X diffeomorphic to $\mathbb{R}^{2n-1}$ where this condition is not met; this prevents any contact embedding of such an X into $\mathbb{R}^{2n-1}$ with the standard contact structure (given by the form $\sum_{i=1}^{n-1} x_i\, dy_i + dz$).

In his article from '85 [**Gro85**], Gromov introduced the study of pseudo-holomorphic curves that made symplectic topology as we know it today only possible. Using these techniques, Gromov presented many spectacular results in this initial paper, and soon many other people started using these methods to settle questions that before had been out of reach [**Eli90a, McD90, McD91, Hof93, Eli96, Abr98**] and many others; for more recent results in this vein we refer to [**Wen10b, OV12**].

While the references above rely on studying the topology of the moduli space itself, Gromov's J-holomorphic methods have also been used to develop powerful algebraic theories like Floer Homology, Gromov-Witten Theory, Symplectic Field Theory, Fukaya Theory etc. that basically rely on counting rigid holomorphic curves (that means holomorphic curves that are isolated). In my own work I have mostly ignored such algebraic techniques, and they will not be mentioned in this text.

Gromov's approach for studying a symplectic manifold (W, ω) consists in choosing an *auxiliary* almost complex structure J on W that is compatible with ω in a certain way. This auxiliary structure allows us to study so called J-holomorphic curves, that means, equivalence classes of maps

$$u\colon (\Sigma, j) \to (W, J)$$

[1] The reference given is an article that to my knowledge has never been published:
[Gro2] Gromov, M.: Pseudo-holomorphic curves in symplectic manifolds, II. Berlin-Heidelberg-New York-Tokyo: Springer (In press)

from a Riemann surface (Σ, j) to W whose differential at every point $x \in \Sigma$ is a (j, J)-complex map

$$Du_x \colon T_x\Sigma \to T_{u(x)}W \ .$$

Conceivable generalizations of such a theory based on studying J-holomorphic surfaces or even higher dimensional J-complex manifolds only work for *integrable* complex structures; otherwise generically such submanifolds do not exist. A different approach has been developed by Donaldson [**Don96, Don99**], and consists in studying approximately holomorphic sections in a line bundle over W. This theory yields many important results, but has a very different flavor than the one discussed here by Gromov.

The J-holomorphic curves are relatively rare and usually come in finite dimensional families. Technical problems aside, one tries to understand the symplectic manifold (W, ω) by studying how these curves move through W.

Let us illustrate this strategy with the well-known example of $\mathbb{C}P^n$. We know that there is exactly one complex line through any two points of $\mathbb{C}P^n$. We fix a point $z_0 \in \mathbb{C}P^n$, and study the space of all holomorphic lines going through z_0. It follows directly that $\mathbb{C}P^n \setminus \{z_0\}$ is foliated by these holomorphic lines, and every line with z_0 removed is a disk. Using that the lines are parametrized by the corresponding complex line in $T_{z_0}\mathbb{C}P^n$ that is tangent to them, we see that the space of holomorphic lines is diffeomorphic to $\mathbb{C}P^{n-1}$, and that $\mathbb{C}P^n \setminus \{z_0\}$ will be a disk bundle over $\mathbb{C}P^{n-1}$.

In this example, we have used an ambient manifold that we understand rather well, $\mathbb{C}P^n$, to compute the topology of the space of complex lines. So far, it might seem unclear how one could obtain information about the topology of the space of complex lines in an ambient space that we do not understand equally well, to then be able to extract in a second step missing information about the ambient manifold, which we would not be able to read off directly.

The common strategy is to assume that the almost complex manifold we want to study already contains a family of holomorphic curves. We then observe how this family evolves, hoping that it will eventually "fill up" the entire symplectic manifold (or produce other interesting effects).

To briefly sketch the type of arguments used in general, consider now a symplectic manifold W with a compatible almost complex structure, and suppose that it contains an open subset U diffeomorphic to a neighborhood of $\mathbb{C}P^1 \times \{0\}$ in $\mathbb{C}P^1 \times \mathbb{C}$ (see [**McD90**]). In this neighborhood we find a family of holomorphic spheres $\mathbb{C}P^1 \times \{z\}$ parametrized by the points z. We can explicitly write down the holomorphic spheres that lie completely inside U, but Gromov compactness tells us that as the holomorphic curves approach the boundary of U, they cannot just cease to exist but instead there is a well understood way in which they can degenerate, which is called *bubbling*. Bubbling means that a family of holomorphic curves decomposes in the limit into several smaller ones. Sometimes bubbling can be controlled or even excluded by imposing technical conditions, and in those cases, the limit curve will just be a regular holomorphic curve.

In the example we were sketching above, this means that if no bubbling can happen, there will be regular holomorphic spheres (partially) outside U that are obtained by pushing the given ones towards the boundary of U. This limit curve is also part of the 2-parameter space of spheres, and thus it will be surrounded by other holomorphic spheres of the same family. As long as we do not have any bubbling, we can thus extend the family by pushing the spheres to the limit and then obtain a new regular sphere, which again is surrounded by other holomorphic spheres. This way, we can eventually show that the whole symplectic manifold is filled up by a 2-dimensional family of holomorphic spheres. Furthermore the holomorphic spheres do not intersect each other (in dimension 4), and this way we obtain a 2-sphere fibration of the symplectic manifold.

In conclusion, we obtain in this example just from the existence of the chart U, and the conditions that exclude bubbling that the symplectic manifold needs to be a 2-sphere bundle over a compact surface (the space of spheres).

Note that many arguments in the example above (in particular the idea that the moduli spaces foliate the ambient manifold) do not hold in general, that means for generic almost complex structures in manifolds of dimension more than 4. Either one needs to weaken the desired statements or find suitable work-arounds. The principle that is universal is the use of a well understood local model in which we can detect a family of holomorphic curves. If bubbling can be excluded, this family extends into the unknown parts of the symplectic manifold, and can be used to understand certain topological properties of this manifold.

Fillability in dimension 3. The main question I have studied during my career is the one of fillability. A common point of view is to consider contact manifolds as hypersurfaces or as boundaries of a symplectic manifold requiring a certain type of compatibility between the contact and the symplectic structures. An important observation is that such a contact hypersurface splits its tubular neighborhood into a *convex* and a *concave* component, and that these have very different properties.

The fillability problem then consists in asking if a given contact manifold (M, ξ) can be realized as the boundary of a compact symplectic manifold (usually without other boundary components), and again, it is a fundamentally different question of being a convex or a concave boundary: It has been shown by Eliashberg [**Eli04**] and Etnyre [**Etn04**] independently that every contact 3-manifold is the concave boundary of some symplectic manifold.

For convex fillings the situation is radically different. The first known examples of non-convexly fillable contact manifolds were the so-called *overtwisted* contact 3-manifolds. The proof of this fact by Eliashberg and Gromov [**Gro85, Eli90a**] was highly non-trivial and was based on the same J-holomorphic curve ideas that we are presenting in this text. A contact manifold is overtwisted if it contains an *overtwisted disk*. It had already been shown by Bennequin [**Ben83**] that the standard contact structure on $\mathbb{S}^3$ is not overtwisted using only topological techniques, but J-holomorphic curve methods gave via the non-fillability result an easy criterion to test if a much wider class of contact manifolds is *tight*, that means, not overtwisted.

Overtwisted contact manifolds are also interesting for other more important reasons, mainly because Eliashberg gave a complete classification [**Eli89**], which implies that an oriented 3-manifold admits in every homotopy class of plane fields up to isotopy a unique overtwisted contact structure. In this sense overtwisted contact manifolds are *flexible*, and distinguish little more than basic topological properties, while the tight contact structures depend in subtle ways on the 3-manifold itself.

In subsequent years, many refinements of the fillability question arose, but the most important one was if being overtwisted is equivalent to being non-fillable. The answer to this was given by Eliashberg and Giroux [**Gir94, Eli96**], where they showed that the 3-torus admits many contact structures that are tight, but that are not *strongly* fillable (even though they are *weakly* fillable). In [**EH02**], it was then shown that there are also tight contact structures that are not even weakly fillable, confirming again that the properties of tight contact structures were hard to decipher.

In retrospect it was understood that the reason why many of the contact structures on the 3-torus are not strongly fillable is because they have *positive Giroux torsion*. As was shown by Gay [**Gay06**] using gauge theoretic arguments, and Ghiggini and Honda [**GH08**] using Heegaard-Floer methods, positive Giroux torsion contradicts strong fillability and sometimes also weak fillability. In [**Wen10b**] and [**NW11**], these results were reproved only using J-holomorphic curve techniques. Apart from the arguable advantage of using more classical and basic methods, another important point is that J-holomorphic curve techniques partially generalize to higher dimensions (as we will show in this text), while the other methods are more specialized for the 3- and 4-dimensional situation. Note also that nowadays the easiest proof of the non-fillability of contact manifolds with positive Giroux torsion is via a cobordism construction between the manifold with torsion and an overtwisted contact manifold. This then contradicts the existence of any type of filling of the first manifold that could be glued to the cobordism, because otherwise this would yield a filling of an overtwisted manifold [**Wen10a**].

We should also mention that even though Giroux torsion does not share the flexibility of overtwisted contact structures, they play an important role in the *coarse classification* of tight contact structures [**CGH03, CGH09**].

There are many other filling obstructions for manifolds that don't have positive Giroux torsion, but we will not try to list them here. Instead going back to the initial notion of overtwistedness, we want to mention that overtwistedness of a contact manifold can be stated in terms of existence of certain open book decompositions [**Gir02**]. Given the contact open book decomposition of a 3-manifold, it can be negatively stabilized; the contact structure corresponding to this negatively stabilized open book is always overtwisted. Conversely, there exists for every overtwisted 3-manifold an open book decomposition that is the negative stabilization of another one.

Before discussing the situation in higher dimensions, a final remark about overtwisted contact manifolds is that they have vanishing contact homology [**Yau06**], and it is an open conjecture that vanishing of SFT invariants might also be equivalent to having an overtwisted contact structure.

Fillability in higher dimension. The paragraph from [**Gro85**] which we reproduced at the beginning of this introduction, indicates that Gromov already had formulated a fillability obstruction in higher dimension that was similar to an overtwisted disk in dimension 3. We believe what he had in mind is what we called in [**Nie06**] a plastikstufe, the product of an overtwisted disk with some closed submanifold, foliated in a trivial way[2]. Unfortunately, at the time our article was published, no closed contact manifold was known that contained such an object. Francisco Presas was the first one to find such manifolds [**Pre07**] by a beautiful construction that combined contact fibrations and fiber connected sums. It was then not very difficult to surger down these first examples and reduce them to smooth spheres [**KN07**], obtaining as a result PS-overtwisted contact structures on spheres of any dimension. Using connected sums, it followed that every contact structure on a given manifold could be converted into one that is PS-overtwisted. A different construction given in [**EP09**] was a bit more explicit, and implied that every contact structure can be modified into a PS-overtwisted one even without changing its homotopy class as almost contact structure. A further very geometric construction of PS-overtwisted contact structures in dimension 5 is the one given by Atsuhide Mori [**Mor09**] that generalizes Lutz twists in a convincing way to dimension 5.

[2] Yuri Chekanov had an unpublished proof of the same statement based on Lagrangian tori.

In dimension 3, it had been shown by Hofer [**Hof93**] that setting up the non-fillability proof of overtwisted contact structures given by Eliashberg and Gromov in the context of a symplectization, implied the existence of a closed contractible Reeb orbit (independently of the chosen contact form). As it was shown in [**AH09**], the same argument also works for PS-overtwisted contact structures. Together with Frédéric Bourgeois, we realized how the Hofer proof could be used to show vanishing of contact homology [**Obe07**, pp. 1945–1949] (and in fact of all SFT [**BN10**]) for PS-overtwisted contact structures. Unfortunately, we have not written down the details yet, but little doubt exists that this line of proof should work out (we have reproduced the relevant pages from [**Obe07**] in the appendix).

Meanwhile the initial definition of the plastikstufe has been generalized to the one of *bordered Legendrian open book* (bLob), see [**MNW13**]. It is more general than a plastikstufe, but in particular it clarifies the structure of the Legendrian foliation for the holomorphic curve argument by not depending on an explicit product structure. bLobs are the main object that we will be discussing in this text.

Emmanuel Giroux has proposed a possible definition of overtwistedness in higher dimension using negatively stabilized open books. As it has been shown in [**BvK10**], all such manifolds have vanishing contact homology – and in fact, the same proof implies non-existence of a weak filling, and existence of closed contractible Reeb orbits. Unfortunately, even though in dimension 3 it is relatively straight forward to find an overtwisted disk in a negatively stabilized manifold, so far no direct links between bLobs and negative stabilizations have been established in higher dimensions. A fundamental question is if there is a unique notion of overtwistedness in higher dimension or if it might be that there are non-equivalent classes of overtwisted contact manifolds.

Initially meant as a method to find the link between negative stabilizations and the plastikstufe, together with Francisco Presas we studied tubular neighborhoods of overtwisted contact manifolds [**NP10**]. We realized that from some critical size on, these neighborhoods contain a bLob, but that underneath this threshold these neighborhoods do not obstruct fillability. This has lead us to speculate that the most general notion of PS-overtwistedness in high dimension should be formulated as admitting a chart that is contactomorphic to an overtwisted contact $\mathbb{R}^3$ and a large neighborhood in $\mathbb{R}^{2n}$ with the standard Liouville form.

In the discussion of overtwisted 3-manifolds, we explained that their main feature is not the non-fillability, but rather their flexibility. No such universal property has been established for PS-overtwisted contact structures, but a first indication that flexibility might partially hold has been given in [**MNPS13**]. There it has been shown that in a contact manifold containing a certain type of plastikstufe, every Legendrian submanifold not intersecting this plastikstufe is necessarily *loose*, which implies by Emmy Murphy's results [**Mur12**] that the Legendrian is flexible. As a consequence, it follows that many exotic Stein structures [**SS05, McL09**] can be undone by performing a connected sum with a PS-overtwisted contact manifold.

Before finishing this short overview, we would like to come back to the discovery that many tight contact 3-manifolds are not fillable. The most basic examples for this were manifolds with positive Giroux torsion. In [**MNW13**], Patrick Massot, Chris Wendl and I gave first examples of higher-dimensional contact manifolds that are non-fillable, but that are certainly not overtwisted either; even though the final definition of overtwistedness has not been settled, there are many properties that one would expect from such manifolds. For example, after introducing the notion of a *weak filling* in higher dimension, we showed that some of the examples constructed do admit a weak filling even though they do not have strong fillings. Furthermore they do not have contractible Reeb orbits or vanishing contact homology.

The examples were all obtained by a generalization of a positive torsion domain to higher dimensions that has similarities to the Lutz tube in [**Mor09**].

Topology of fillings. Above we have mostly discussed the question whether a contact manifold is or is not the convex boundary of a symplectic manifold. Assuming that a contact manifold *is* fillable, a further question could be to determine the class of all such potential fillings.

In dimension 3, many contact manifolds imply surprisingly restrictive conditions on their filling. Using the J-holomorphic curve techniques from [**Gro85**], it has been proved for example that the only symplectic filling of the standard sphere is up to blow-up a standard symplectic 4-ball [**Eli90a**], that lens spaces have essentially unique fillings [**McD90**], that a filling of the connected sum of two contact manifolds is the boundary connected sum of two fillings, or that $\mathbb{S}^1 \times \mathbb{S}^2$ has up to blow up a unique filling [**Eli90a**].

For more recent results in dimension 3, we would like to mention that every symplectic filling of a contact manifold that admits a planar open book decomposition needs to be essentially equivalent to a Lefschetz fibration [**Wen10b**]. In this large class of examples, understanding the potential fillings reduces then to a combinatorial problem. We should also mention that there are many results that have been obtained via Seiberg-Witten theory, but since these methods work exclusively in dimension 4, and we are mainly interested in higher dimensions, we will not discuss them here any further.

In higher dimensions, J-holomorphic curve techniques cannot yield comparable results, mostly due to the lack of the adjunction inequality. The results are hence only topological and do not give unfortunately any information about the symplectic structure of the filling. Among the few known results, the most prominent one is a theorem by Eliashberg-Floer-McDuff that states that every symplectically aspherical filling of the standard sphere is diffeomorphic to a smooth ball [**McD91**]. This result is obtained by showing that the filling needs to be simply connected and have vanishing homology, and then applying the h-cobordism theorem. An important implication of this fact was that in contrast to dimension 3, higher dimensional spheres do admit exotic contact structures that are (even Stein) fillable. Nowadays there exist much better techniques that show that there are infinitely many non-equivalent Stein fillable contact structures on the sphere [**Ust99**], or that there are even exotic contact structures bounding Stein manifolds that are diffeomorphic (but not symplectomorphic) to a smooth ball [**SS05, McL09**] (and in particular cannot hence be distinguished from the standard sphere using the Eliashberg-Floer-McDuff Theorem).

The techniques from [**McD91**] have been pushed in [**OV12**] to the limit, showing many homological implications for the fillings of contact manifolds that embed into $\mathbb{C}^n$ or that admit a subcritically Stein filling (possibly different from the considered filling).

A completely different direction has been taken in [**Eli90b, CE12**] by studying Stein manifolds from a homotopical point of view. This produces very strong results, including a Stein h-cobordism result, but for one, these methods work in the Stein category which is different from the symplectic one, and they mostly give information about a given filling, and not so much about the class of *all* possible fillings.

Outline of the notes. This text is based to a large extend on the lecture notes [**Nie13**] of a course I held in Nantes in June 2011, and on the article [**MNW13**] with Patrick Massot and Chris Wendl. I have tried to give a relatively self-contained overview on how to deduce fillability properties using J-holomorphic curves with boundary lying in a submanifold with Legendrian foliation.

Chapter I gives an overview on submanifolds with a singular Legendrian foliation. The initial aim while writing up this chapter was to give a general theory on such submanifolds. The main difficulty in understanding such manifolds in higher dimensions compared to dimension 3 is that typical codimension-1 distributions aren't foliations, and hence the property of a submanifold of having a Legendrian foliation is not automatic anymore. This makes it more subtle to apply many of the arguments that in dimension 3 can be taken for granted, and it is not possible to treat such submanifolds in a purely topological way.

Unfortunately due to lack of time, I have not succeeded in developing the general theory I had in mind. Still I show many preliminary results, which would be unavoidable for anyone trying to pursue the initial project (see also [**Hua13**]). Among them it is shown that a Legendrian foliation determines the germ of the contact structure in its neighborhood, see Theorem I.1.3. This statement is very easy to see both in the neighborhood of the singularities and away from the singularities, but gluing these models is surprisingly tedious. I also show that there are very few obstructions for a given foliation to be realizable as a Legendrian foliation under some embedding, see Theorem I.1.5. We spend some time understanding the local shape of codimension-1 and codimension-2 singularities, and we finish the chapter by explaining in Section I.4 what a Legendrian open book (Lob), and what a bordered Legendrian open book (bLob) is, and stating the main consequences for fillability of contact manifolds (Theorem A and B).

Chapter II gives first a basic introduction to the constraints imposed by J-plurisubharmonic functions on holomorphic curves, and how the imposed behavior can be understood geometrically. This is then compared in Section II.2 to weak fillability, in particular we explain that the notion of weak fillability given in [**MNW13**] is equivalent to the existence of almost complex structures that are tamed and that make the contact boundary J-convex. We finish this chapter by giving local models for almost complex structures living close to the singularities of a Legendrian foliation. These are then used to describe the behavior of J-holomorphic disks. In particular we show that elliptic codimension-2 singularities emit a Bishop family of holomorphic disks, and that there are no other holomorphic disks nearby, and that codimension-1 singularities can be used as walls, blocking holomorphic disks from escaping the enclosed domain. These properties will be fundamental in the proof of Theorem A and B.

Chapter III finally contains the proof of Theorem A and B. These results had been stated first in a special form in [**Nie06**], and had then been adapted in subsequent publications to various situations. Here I try to give a self-contained account for a reasonably general case. The only part that is missing is the behavior of holomorphic disks for families of Lobs that we will use in our future research, see Chapter V (or for example also the results in [**Eli90a**] or the overtwisted annulus argument in [**NW11**]).

Chapter IV deals with the construction of $\mathbb{S}^1$-invariant contact manifolds given in [**MNW13**]. In some special cases, the $\mathbb{S}^1$-action allows us to read off PS-overtwistedness, but more interesting it may prove non-fillability for certain non-overtwisted manifolds, analogously to the case of 3-manifolds with positive Giroux torsion. We only explain here the non-fillability results without constructing explicit examples (which was a crucial part of [**MNW13**]), as this would have taken us too far from the main subject of this work. Initially, we would have liked to generalize the results to non-trivial circle bundles hoping that this would give a large source of new examples, but again time-constraints prevented us from doing so.

In a second part of that chapter, we reprove the main result of [**NP10**] about large neighborhoods of overtwisted contact manifolds, but instead of using the initial proof of the statement that was based on analyzing the bubbling behavior of holomorphic disks with boundary on an immersed submanifold, we use the methods presented in the first part of this chapter.

Chapter V explains the research directions that I'm currently studying (in collaboration with Paolo Ghiggini and Chris Wendl). Our main aim is to understand the topology of symplectic fillings. Since this is work in progress, the contents of this chapter are rather vague.

We first show how to reprove the Eliashberg-Floer-McDuff Theorem [**McD91**] using holomorphic disks and families of Lobs. Similarly, we can produce families of Lobs for subcritical surgeries, and we would like to use these to show that the belt sphere of a subcritical contact surgery will be contractible in any symplectically aspherical filling of the contact manifold. An important detail in this approach is to understand the tangent bundle of the space of holomorphic disks.

A second very preliminary idea consists in proving that certain contact manifolds admit a strong filling but not an exact one. The obstruction to the exact fillability could be a family of Lobs producing a moduli space whose boundary has non-vanishing Stiefel-Whitney numbers. The moduli space itself cannot be a smooth compact manifold, hence there need to be holomorphic spheres bubbling off.

In the Appendix we have added the notes Frédéric Bourgeois and I submitted for the report [**Obe07**]. They explain why contact homology vanishes for PS-overtwisted manifolds. Unfortunately, we still haven't worked out the technical details of the argument, but we feel that the idea is very intuitive and it does not rely on choosing a particular contact form to work with explicitly.

Acknowledgments. Many people have contributed over the years to develop the ideas described in this work. Among them were several of my collaborators which I would like to thank here: Frédéric Bourgeois, Otto van Koert, Patrick Massot, Francisco Presas, and Chris Wendl.

I would also like to thank many of the institutions that have employed me over the years: *Universiteit Leiden* and the *Universität zu Köln*, the *Université Libre de Bruxelles* and the *Fonds National de la Recherche Scientifique* (FNRS), the *École normale supérieure de Lyon* and the ANR *Symplexe*, and finally the *Université Paul Sabatier – Toulouse III.*

I would also like to thank the members of the jury for their time and effort.

Notation

We assume throughout a certain working knowledge on contact topology (for a reference see for example [**MS98**, Chapter 3.4] and [**Gei08**]) and on holomorphic curves [**AL94, MS04**]. Contact structures considered in this text are always *cooriented* (unless explicitly stated otherwise). Remember that by choice of a coorientation, (M, ξ) always obtains a natural orientation and its contact structure ξ carries a natural *conformal* symplectic structure. For both, it suffices to choose a **positive** contact form α, that means, a 1-form with $\xi = \ker \alpha$ that evaluates positively on vectors that are positively transverse to the contact structure. The orientation on M is then given by the volume form

$$\alpha \wedge d\alpha^n ,$$

where $\dim M = 2n+1$, while the conformal symplectic structure is represented by $d\alpha|_\xi$.

One can easily check that these notions are well-defined by choosing any other positive contact form α', because there exists a smooth function $f\colon M \to \mathbb{R}$ such that $\alpha' = e^f\,\alpha$.

We denote the **1-jet space** of a manifold M by

$$\mathcal{J}^1(M) := \mathbb{R} \times T^*M$$

and its canonical contact structure is $\xi_{\mathrm{can}} := \ker\bigl(dz - \lambda_{\mathrm{can}}\bigr)$, where z is the coordinate on $\mathbb{R}$ and λ_{can} is the canonical 1-form that satisfies $\sigma^*\lambda_{\mathrm{can}} = \sigma$.

Further conventions. Note that $\mathbb{D}^2$ denotes in this text the *closed* unit disk.

I owe it to Patrick Massot to have been converted to the following *jargon*.

Definition. The term **regular equation** can refer in this text to any of the following objects:

(1) When Σ is a cooriented hypersurface in a manifold M, then we call a smooth function $h\colon M \to \mathbb{R}$ a **regular equation for** Σ, if 0 is a regular value of h and $h^{-1}(0) = \Sigma$.

(2) When $\mathcal{D} \le TM$ is a singular codimension-1 distribution, then we say that a 1-form β is a **regular equation for** $\mathcal{D}$, if $\mathcal{D} = \ker\beta$ and if $d\beta \neq 0$ at singular points of $\mathcal{D}$.

In particular, we can call contact forms for a given contact structure ξ a (regular) equations of ξ.

Let E be a vector bundle over a manifold M, and assume that E is equipped with a metric g. We use the notation

$$E_{<R} = \bigl\{v \in E \bigm| g(v,v) < R^2\bigr\} .$$

for the **open disk bundle of size** R **in** E, and use similar subscripts to refer to other subsets of the bundle E.

CHAPTER I

Legendrian foliations

I.1. General facts about Legendrian foliations

Let (M, ξ) be a contact manifold that contains a submanifold N. Generically, if we look at any point $p \in N$ the intersection between ξ_p and the tangent space T_pN will be a codimension-1 hyperplane. Generally though, the distribution $\mathcal{D} = \xi \cap TN$ may be singular, because there can be points $p \in N$ where $T_pN \leq \xi_p$, and equally important the distribution $\mathcal{D}$ will only be in very rare cases a foliation. In fact, if we choose a contact form α for ξ, then we obtain by the Frobenius theorem that $\mathcal{D}$ will be a (singular) foliation if and only if

$$\big(\alpha \wedge d\alpha\big)\big|_{TN} \equiv 0 \ .$$

Another way to state this condition is to say that we have $d\alpha|_{\mathcal{D}_p} = 0$ at every regular point $p \in N$ of $\mathcal{D}$, so that $\mathcal{D}_p$ has to be an isotropic subspace of $(\xi_p, d\alpha_p)$. In particular, this shows that the induced distribution $\mathcal{D}$ can never be integrable if $\dim \mathcal{D} > \frac{1}{2} \dim \xi$.

We will usually denote the distribution $\xi \cap TN$ by $\mathcal{F}$ whenever it is a singular foliation. Furthermore, we will call such an $\mathcal{F}$ a **Legendrian foliation** if $\dim \mathcal{F} = \frac{1}{2} \dim \xi$, which implies that N has to be a submanifold of dimension $n + 1$ in an ambient contact manifold of dimension $2n + 1$. For reasons that we will briefly sketch below, but that will be treated extensively from Chapter III on, we will be mostly interested in this monograph in submanifolds carrying such a Legendrian foliation. Note in particular that in a contact 3-manifold every hypersurface N carries automatically a Legendrian foliation.

Denote the set of points $p \in N$ where $\mathcal{F}$ is singular by $\operatorname{Sing}(\mathcal{F})$. One of the basic properties of a Legendrian foliation is that for any contact form α, the restriction $d\alpha|_{TN}$ does not vanish on $\operatorname{Sing}(\mathcal{F})$, because otherwise $T_pN \subset \xi_p$ would be an isotropic subspace of $(\xi_p, d\alpha_p)$ which is impossible for dimensional reasons. Since $d\alpha|_{TN}$ does not vanish on $\operatorname{Sing}(\mathcal{F})$, we deduce in particular that $N \setminus \operatorname{Sing}(\mathcal{F})$ is a dense and open subset of N.

Remark I.1.1. The main reason, why we are interested in submanifolds that have a Legendrian foliation is that they often allow us to successfully use J-holomorphic curve techniques. On one side, such submanifolds will be automatically totally real for any suitable almost complex structure on a symplectic filling, thus posing a good boundary condition for the Cauchy-Riemann equation: The solution space of a Cauchy-Riemann equation with totally real boundary condition is generically a finite dimensional smooth manifold, so that it follows that the moduli spaces of J-holomorphic curves whose boundaries lie in a submanifold with a Legendrian foliation will have a nice local structure. A second important property is that the topology of the Legendrian foliation controls the behavior of J-holomorphic curves, and will allow us to obtain many results in contact and symplectic topology. Elliptic codimension 2 singularities of the Legendrian foliation “emit” families of holomorphic disks; suitable codimension 1 singularities form “walls” that cannot be crossed by holomorphic disks.

In the rest of this section, we will study some general properties of Legendrian foliations. Theorem I.1.3 shows that a manifold with a Legendrian foliation determines the germ of the contact structure on its neighborhood. This allows us to describe small deformations of the Legendrian foliation, and study almost complex structures more explicitly (see Chapter II). Theorem I.1.5 gives a precise characterization of the foliations that can be realized as Legendrian ones. Note that a similar study to ours has also been carried out by Yang Huang [**Hua13**].

I.1.1. Neat singular foliations. The principal aim of this section will be to explain a result due to Kupka [**Kup64**] that tells us that the behavior of a Legendrian foliation close to a singular point can always be reduced to the 2-dimensional situation (see Fig. 1). We will furthermore define the *secondary foliation.* I thank Nguyen Tien Zung for pointing out Kupka's result to me.

Definition. Let N be a compact manifold possibly with boundary. A **neat singular foliation** $\mathcal{F}$ on N is a singular foliation that satisfies the following conditions:

- $\mathcal{F}$ is of codimension-1 and admits a regular equation. In particular it follows that regular leaves are cooriented.
- If a boundary component $\partial_j N$ contains a singular point $p \in \operatorname{Sing}(\mathcal{F})$, then $\partial_j N$ will be tangent to $\mathcal{F}$ in a small neighborhood of p.

Many results stated in this chapter can be proved for more general types of foliations, but for simplicity we will mostly assume that all foliations are neat.

Theorem I.1.2. *Let N be a manifold with a neat singular foliation $\mathcal{F}$ given by the regular equation β.*

(a) *If p is a singular point of the foliation lying in the interior of N, then we find a chart centered at p with coordinates $(s, t, x_1, \dots, x_{n-2}) \in \mathbb{R}^n$ such that β is represented by the 1-form*
$$a(s,t)\, ds + b(s,t)\, dt$$
with some smooth functions a and b that only depend on the s- and t-coordinates.

(b) *If p is a singular point on the boundary of N, then we find a chart U centered at p with coordinates $(s, t, x_1, \dots, x_{n-2}) \in [0,\infty) \times \mathbb{R}^{n-1}$ such that the boundary $U \cap \partial N$ corresponds to the subset $\{s = 0\}$ and such that β is represented by the 1-form*
$$a(s,t)\, ds + b(s,t)\, dt$$
for smooth functions a and b. Since $\mathcal{F}$ is neat, b vanishes along the boundary $\{s = 0\}$.

We will call any chart of $(N, \mathcal{F})$ of the form described in the theorem a **Kupka chart**.

Proof. (a) If $\dim N = 2$, then there is nothing to show, hence assume that $\dim N \geq 3$. The linear bundle map $F \colon TN \to T^*N$, $v \mapsto d\beta(v, \cdot)$ has at every point in a neighborhood of p at least rank 2, but from the Frobenius condition $\beta \wedge d\beta \equiv 0$, it follows that $d\beta^2 = 0$. Hence $\ker F$ defines a codimension-2 subbundle of TN, and we may choose a non-vanishing vector field X on a neighborhood of p with $d\beta(X, \cdot) = 0$. We can also easily see that $X \in \ker \beta$ and $\mathcal{L}_X \beta = 0$, because
$$0 = \iota_X\bigl(\beta \wedge d\beta\bigr) = \beta(X)\, d\beta - \beta \wedge \bigl(\iota_X d\beta\bigr) = \beta(X)\, d\beta\ ,$$
and $d\beta$ does not vanish on a neighborhood of p.

Translating a small hypersurface Σ that is transverse to X with the flow Φ_t^X of X, we can define a diffeomorphism

$$\Psi\colon \Sigma \times (-\varepsilon, \varepsilon) \to N,\ (p,t) \mapsto \Phi_t^X(p)\ .$$

The pull-back of the 1-form β to $\Sigma\times(-\varepsilon,\varepsilon)$ reduces to $\beta|_{T\Sigma}$, and by repeating this construction the necessary number of times we obtain the desired statement.

(b) Choose a smooth chart around p diffeomorphic to $[0,\infty) \times \mathbb{R}^{n-1}$ with coordinates $(y_1,\dots,y_n)$. The 1-form β is represented by

$$\beta = \sum_{j=1}^n f_j(y_1,\dots,y_n)\,dy_j$$

with a set of smooth functions f_j, and because $\mathcal{F}$ is neat, the boundary component of N needs to be tangent to $\mathcal{F}$, so that $f_2(0,y_2,\dots,y_n) = \dots = f_{n+1}(0,y_2,\dots,y_n) = 0$.

Along ∂N we easily compute

$$d\beta = \sum_{j=2}^n \bigl(\partial_1 f_j - \partial_j f_1\bigr)\,dy_1\wedge dy_j = dy_1 \wedge \Bigl(\sum_{j=2}^n \bigl(\partial_1 f_j - \partial_j f_1\bigr)\,dy_j\Bigr)\ ,$$

and it follows that if $\dim(\partial N) \geq 2$, there is a vector field X tangent to ∂N that lies in the kernel of $d\beta$. The rest of the proof is identical to case (a). $\square$

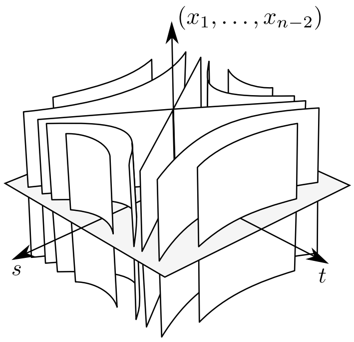


FIGURE 1. The singularities of a neat singular foliation look locally like the product of $\mathbb{R}^{n-2}$ with a foliation in the plane. The directions $\partial_{x_1},\dots,\partial_{x_{n-2}}$ correspond to the secondary foliation associated to a contact form.

Definition. Let $\mathcal{F}$ be a singular foliation on a manifold N given by the regular equation β. We call

$$\mathcal{F}_\beta := \bigl\{v \in \mathcal{F} \bigm| d\beta(v,\cdot) = 0\bigr\}$$

the **secondary foliation associated to** β. The singular points $\operatorname{Sing}(\mathcal{F}_\beta)$ of $\mathcal{F}_\beta$ are those at which $d\beta$ vanishes.

It is easy to see that the secondary foliation really is a foliation: Let X and Y be any two vector fields in $\mathcal{F}_\beta$, and let Z be any other vector field. It suffices to show that $[X,Y]$ also lies in $\mathcal{F}_\beta$. This is true by the Leibniz rule

$$d\beta([X,Y],Z) = \mathcal{L}_X\bigl(d\beta(Y,Z)\bigr) - d\beta(Y,[X,Z]) = 0\ .$$

Note however that the secondary foliation depends on the regular equation chosen and that it is not intrinsic to $\mathcal{F}$. It is however well-defined at singular points $\operatorname{Sing}(\mathcal{F})$, as $d\bigl(f\,\alpha\bigr) = f\,d\alpha + df\wedge\alpha$ reduces at such points to the first term only.

Comparing with Theorem I.1.2, it is also clear that close to singular points of $\mathcal{F}$, $\mathcal{F}_\beta$ is the codimension-2 foliation that corresponds in the Kupka chart to the directions spanned by $\partial_{x_1},\dots,\partial_{x_{n-2}}$.

I.1.2. Local behavior of Legendrian foliations. The first major aim in this section will be to prove the following result, which states that the Legendrian foliation determines in its neighborhood the germ of the contact structure.

A **neat Legendrian foliation** is a Legendrian foliation that is a neat singular foliation.

Theorem I.1.3. *Let N be a compact manifold (possibly with boundary) that has been embedded into two contact manifolds (M_1,ξ_1) and (M_2,ξ_2). Assume that ξ_1 and ξ_2 induce the same neat Legendrian foliation $\mathcal{F}$ on N. Then there are neighborhoods $U_1\subset M_1$ and $U_2\subset M_2$ of N and a contactomorphism*

$$\Psi\colon (U_1,\xi_1)\to(U_2,\xi_2)$$

such that $\Psi|_N = \mathrm{id}_N$.

The proof of the theorem is based on the following lemma.

Lemma I.1.4. *Let (M_1,ξ_1) and (M_2,ξ_2) be two contact manifolds with contact forms α_1 and α_2, and let J_1 and J_2 be tame complex structures on $\bigl(\xi_1,\,d\alpha_1|_{\xi_1}\bigr)$ and $\bigl(\xi_2,\,d\alpha_2|_{\xi_2}\bigr)$ respectively. Suppose that N is a compact manifold (possibly with boundary) that has been embedded into M_1 and into M_2 in such a way that ξ_1 and ξ_2 induce the same neat Legendrian foliation $\mathcal{F}$ on N. Let β be a regular equation for $\mathcal{F}$ and assume additionally that $\alpha_1|_{TN} = \alpha_2|_{TN} = \beta$.*

Then there is a bundle isomorphism

$$\Phi\colon TM_1|_N\to TM_2|_N$$

such that:

(i) *Φ restricts to the identity on TN, that is, $\Phi|_{TN} = \mathrm{id}_{TN}$.*
(ii) *The pull-back $\alpha_2\circ\Phi$ is equal to the restriction of α_1 to $TM_1|_N$.*
(iii) *Let $\mathcal{F}_\beta$ be the secondary foliation of β. The restriction of Φ to $\mathcal{F}_\beta\oplus J_1\,\mathcal{F}_\beta$ is a (J_1,J_2)-linear map, that means,*

$$\Phi(J_1X) = J_2\,\Phi(X)$$

for every $X\in\mathcal{F}_\beta\oplus J_1\,\mathcal{F}_\beta$.
(iv) *Φ is orientation preserving.*

Proof. To construct the isomorphism Φ, we need to distinguish two types of neighborhoods on N: Write $N_{\mathrm{reg}} := N\setminus\operatorname{Sing}(\mathcal{F})$, then we see that for every $p\in N_{\mathrm{reg}}$ and $j=1,2$

$$\bigl(J_j\,\mathcal{F}_p\bigr)\cap\mathcal{F}_p = \{0\}\ ,$$

because $\bigl(d\alpha_j\bigr)\bigr|_{\mathcal{F}_p} = 0$. For dimensional reasons it then follows that

$$TM_j|_{N_{\mathrm{reg}}} = TN|_{N_{\mathrm{reg}}}\oplus\bigl(J_j\,\mathcal{F}\bigr)\bigr|_{N_{\mathrm{reg}}}\ .$$

We define a first bundle isomorphism

$$\Phi_{\mathrm{reg}}\colon TM_1|_{N_{\mathrm{reg}}}\to TM_2|_{N_{\mathrm{reg}}}$$

by setting $\Phi_{\mathrm{reg}}(v_N + J_1w) := v_N + J_2w$ for every $v_N\in TN$ and $w\in\mathcal{F}$.

Next, we consider the open set

$$N_{\mathrm{sing}} = \bigl\{\text{points } p \text{ in } N \text{ where } d\beta_p \neq 0\bigr\} = N \setminus \operatorname{Sing}(\mathcal{F}_\beta) .$$

Since $d\beta$ does not vanish in a neighborhood of the singular points of the Legendrian foliation, we obtain $\operatorname{Sing}(\mathcal{F}) \subset N_{\mathrm{sing}}$, so that the union of N_{sing} and N_{reg} covers all of N.

Note that with the same type of argument as above, it follows that $J_j\, \mathcal{F}_\beta \cap TN = \{0\}$ on N_{sing}, because otherwise there would be a non-vanishing vector $v \in \mathcal{F}_\beta$ such that $J_j v \in \mathcal{F}$. But this would imply that $0 = d\beta(v, J_j v) = d\alpha_j(v, J_j v) > 0$.

Furthermore, the Reeb field R_j of α_j is also transverse to $TN|_{N_{\mathrm{sing}}} \oplus J_j\, \mathcal{F}_\beta|_{N_{\mathrm{sing}}}$. Assume this were false, then we could write $R_j(p)$ as $v + J_j w$ for vectors $v \in T_pN$ and $w \in \mathcal{F}_\beta$. It is easy to see that w needs to be 0, because $d\alpha_j(R_j, \cdot) = 0$ implies

$$0 = d\alpha_j(R_j, w) = d\alpha_j(v, w) + d\alpha_j(J_j w, w) = d\beta(v, w) + d\alpha_j(J_j w, w) = d\alpha_j(J_j w, w) .$$

Now if there were a point $p \in N_{\mathrm{sing}}$ for which $R_j(p) \in T_pN$, we could further deduce from $d\alpha_j(R_j, \cdot) = 0$ in particular that $d\beta(R_j, \cdot) = 0$, and so it would be obvious in a Kupka chart that $R_j(p) \in \mathcal{F}_\beta \subset \xi_j$, which cannot be true.

Together, these arguments yield

$$TM_j|_{N_{\mathrm{sing}}} = TN|_{N_{\mathrm{sing}}} \oplus (J_j\, \mathcal{F}_\beta)|_{N_{\mathrm{sing}}} \oplus \operatorname{span}\langle R_j\rangle|_{N_{\mathrm{sing}}}$$

for $j = 1, 2$, and we can define a second bundle isomorphism

$$\Phi_{\mathrm{sing}} \colon TM_1|_{N_{\mathrm{sing}}} \to TM_2|_{N_{\mathrm{sing}}}$$

mapping $\Phi_{\mathrm{sing}}(v_N + J_1 w + z\, R_1) := v_N + J_2 w + z\, R_2$ for every $v_N \in TN$, every $w \in \mathcal{F}_\beta$ and $z \in \mathbb{R}$.

Now let $h \colon N \to [0, 1]$ be a smooth function that has support in N_{sing}, and is equal to 1 in a smaller neighborhood of $\operatorname{Sing}(\mathcal{F})$. We define the desired bundle map Φ on all of N by interpolating between Φ_{reg} and Φ_{sing}:

$$\begin{array}{cccc} \Phi\colon & TM_1|_N & \to & TM_2|_N \\ & v_p & \mapsto & h(p)\, \Phi_{\mathrm{sing}}(v_p) + \bigl(1 - h(p)\bigr)\, \Phi_{\mathrm{reg}}(v_p) \end{array} .$$

It is easy to see that Φ restricts to the identity on TN, and we will show next that Φ is in fact a bundle isomorphism.

Outside the support $\overline{\operatorname{supp} h} \cap \overline{\operatorname{supp}(1 - h)}$, this is obvious, so we only need to study $N_{\mathrm{sing}} \cap N_{\mathrm{reg}}$: Note that the two maps Φ_{sing} and Φ_{reg} agree at all points of $N_{\mathrm{sing}} \cap N_{\mathrm{reg}}$ on the subspaces $T_pN \oplus J_1\, \mathcal{F}_\beta$ so that

$$\Phi(v_N + J_1 w) = v_N + J_2 w \tag{I.1.1}$$

for every $v_N \in TN$ and every $w \in \mathcal{F}_\beta$. We have in particular that $\Phi\bigl(T_pN \oplus J_1\, \mathcal{F}_\beta\bigr) = T_pN \oplus J_2\, \mathcal{F}_\beta$, and to show that Φ is an isomorphism, it suffices hence to show that the image $\Phi\bigl(R_1(p)\bigr)$ of the Reeb field never lies in $T_pN \oplus J_2\, \mathcal{F}_\beta$.

Since this is a pointwise property, we consider a neighborhood of $p \in N_{\mathrm{sing}} \setminus \operatorname{Sing}(\mathcal{F})$, and choose a Kupka chart $U_K \subset N_{\mathrm{sing}} \cap N_{\mathrm{reg}}$ with coordinates $(s, t, x_1, \dots, x_{n-1})$ such that β takes the form

$$a(s, t)\, ds + b(s, t)\, dt .$$

The subbundle $\mathcal{F}_\beta$ is spanned by $\partial_{x_1}, \dots, \partial_{x_{n-1}}$, and we can choose the basis

$$\operatorname{span}\langle \partial_s, \partial_t, \partial_{x_1}, \dots, \partial_{x_{n-1}}, \partial_{y_1}, \dots, \partial_{y_{n-1}}, \partial_z\rangle$$

for T_pM_1, where $\partial_{y_j} := J_1\,\partial_{x_j}$ and $\partial_z := R_1(p)$. Using the analogous basis for T_pM_2 it follows Φ_{sing} is represented with respect to these two basis by the unit matrix.

The contact form simplifies in the chosen basis to

$$\alpha_1|_{T_pM_1} = a\,ds + b\,dt + dz$$

with kernel

$$\xi_1 = \operatorname{span}\langle X_1, X_2\rangle \oplus \mathcal{F}_\beta \oplus J_1\,\mathcal{F}_\beta\ ,$$

where $X_1 := b\,\partial_s - a\,\partial_t$ and $X_2 := (a^2+b^2)\,\partial_z - a\,\partial_s - b\,\partial_t$.

To describe the map Φ_{reg} it is necessary to better understand J_1X_1, hence write J_1X_1 as

$$J_1X_1 = A\,X_1 + B\,X_2 + v + J_1w$$

with smooth functions $A, B\colon U_K \to \mathbb{R}$ and vector fields $v, w \in \mathcal{F}_\beta$. A short computation shows that for every $p \in U_K$

$$0 < d\alpha_1\big(J_1\,(X_1 - w), X_1 - w\big) = B\,d\alpha_1(X_2, X_1) = B\,(a^2+b^2)\,\left(\frac{\partial b}{\partial s} - \frac{\partial a}{\partial t}\right)\ ,$$

where we have used that $d\alpha_1$ does not contain any dz-terms, or any terms of the form $ds\wedge dx_j$, $dt\wedge dx_j$ or $dx_j\wedge dx_k$. Using the analogous basis in T_pM_2, we may write J_2X_1 as

$$J_2X_1 = A'\,X_1 + B'\,X_2 + v' + J_2w'$$

with functions $A', B'\colon U_K \to \mathbb{R}$ and vector fields $v', w' \in \mathcal{F}_\beta$, and an identical computation as the previous one yields that the coefficients B and B' must have the same sign on U_K.

Using that Φ maps $TN \oplus J_1\,\mathcal{F}_\beta$ over $U_K \subset N_{\mathrm{sing}} \setminus \mathrm{Sing}(\mathcal{F})$ onto $TN \oplus J_2\,\mathcal{F}_\beta$, we can define an induced bundle map

$$\widehat{\Phi}\colon TM_1/\big(TN \oplus J_1\,\mathcal{F}_\beta\big) \to TM_2/\big(TN \oplus J_2\,\mathcal{F}_\beta\big)\ .$$

Showing that $\widehat{\Phi}$ is a bundle isomorphism over U_K is equivalent to showing that Φ is an isomorphism. We obtain in the quotient space $T_pM_1/\big(T_pN \oplus J_1\,\mathcal{F}_\beta\big)$ over $U_K \subset N_{\mathrm{sing}} \setminus \mathrm{Sing}(\mathcal{F})$ the equation

$$J_1X_1\ +\ T_pN \oplus J_1\,\mathcal{F}_\beta = (a^2+b^2)\,B\,R_1\ +\ T_pN \oplus J_1\,\mathcal{F}_\beta\ ,$$

and similarly it follows that

$$J_2X_1\ +\ T_pN \oplus J_2\,\mathcal{F}_\beta = (a^2+b^2)B'\,R_2\ +\ T_pN \oplus J_2\,\mathcal{F}_\beta\ .$$

An easy computation then shows that

$$\begin{aligned}
\widehat{\Phi}\big(R_1\ + T_pN \oplus J_1\,\mathcal{F}_\beta\big) &= \Phi(R_1)\ + T_pN \oplus J_2\,\mathcal{F}_\beta \\
&= h\,\Phi_{\mathrm{sing}}(R_1) + (1-h)\,\Phi_{\mathrm{reg}}(R_1)\ + T_pN \oplus J_2\,\mathcal{F}_\beta \\
&= \left(h + (1-h)\,\frac{B'}{B}\right) R_2\ + T_pN \oplus J_2\,\mathcal{F}_\beta\ .
\end{aligned}$$

Since we have shown that B' and B have the same sign, it follows that the R_2-component will never vanish and so we conclude the proof that $\widehat{\Phi}$ and hence also Φ defines a bundle isomorphism.

We only need to verify properties (i)–(iv).

(i) Since both Φ_{reg} and Φ_{sing} restrict to the identity on TN, statement (i) follows immediately.

(ii) Using the splitting $T_pM_j = T_pN \oplus J_j\,\mathcal{F}$ at a point $p \in N_{\mathrm{reg}}$, it is easy to check that $\alpha_2 \circ \Phi_{\mathrm{reg}}|_{T_pM_1} = \alpha_1|_{T_pM_1}$. Similarly we obtain at every $q \in N_{\mathrm{sing}}$ that $\alpha_2 \circ \Phi_{\mathrm{sing}}|_{T_qM_1} = \alpha_1|_{T_qM_1}$ by using the decomposition $T_qM_j = T_qN \oplus J_j\,\mathcal{F}_\beta \oplus \operatorname{span}\langle R_j(q)\rangle$. Combining these results, it follows for every $p \in N$ that

$$\alpha_2 \circ \Phi|_{T_pM_1} = h(p)\,\alpha_2 \circ \Phi_{\mathrm{reg}} + \bigl(1 - h(p)\bigr)\,\alpha_2 \circ \Phi_{\mathrm{sing}} = \alpha_1|_{T_pM_1}\;.$$

(iii) Recall that by equation (I.1.1) we have for every pair of vectors $v, w \in \mathcal{F}_\beta$ the relation $\Phi(v + J_1w) = v + J_2w$. This implies that Φ is (J_1, J_2)-linear on $\mathcal{F}_\beta \oplus J_1\,\mathcal{F}_\beta$, as can be seen by decomposing a vector $X \in \mathcal{F}_\beta \oplus J_1\,\mathcal{F}_\beta$ into $X = v + J_1w$, because then $\Phi(J_1X) = \Phi(J_1v - w) = J_2v - w = J_2\,\Phi(X)$.

(iv) From property (iii) it follows that Φ preserves the coorientations of the contact structures, hence it is sufficient to study the restriction of Φ to the contact structures themselves. Note that Φ_{reg} restricts to a (J_1, J_2)-linear map on ξ_1, and hence it is orientation preserving. If there is a $p \in N$ at which $h(p) = 0$, this finishes the proof, otherwise choose any point $p \in N_{\mathrm{sing}} \cap N_{\mathrm{reg}}$, and consider the 1-parameter family of linear maps

$$\Phi_\tau \colon T_pM_1 \to T_pM_2$$

defined by $\Phi_\tau = \tau\,\Phi_{\mathrm{sing}} + (1 - \tau)\,\Phi_{\mathrm{reg}}$ for every $\tau \in [0, 1]$. By the discussion above the Φ_τ are isomorphisms, and $\Phi_0 = \Phi_{\mathrm{reg}}$, hence all Φ_τ and hence also Φ itself are orientation preserving. □

Proof of Theorem I.1.3. Choose contact forms α_1 and α_2 for ξ_1 and ξ_2 respectively. Using Lemma I.6.3, we find a function $F \colon N \to \mathbb{R}$ such that

$$\alpha_1|_{TN} = F\,\alpha_2|_{TN}\;.$$

Extending this function from N to all of M_1, and then denoting $F\,\alpha_1$ again by α_1, we may assume that

$$\alpha_1|_{TN} = \alpha_2|_{TN}\;.$$

Let J_1 be a complex structure on ξ_1 that is tamed by $d\alpha_1|_{\xi_1}$ and let J_2 be one on ξ_2 tamed by $d\alpha_2|_{\xi_2}$.

As the proof of the theorem will be based on some variation of Gray stability and the Moser trick, we will first need to identify the normal bundles of N in M_1 and in M_2. Applying Lemma I.1.4 to $(M_1, \xi_1, \alpha_1, J_1)$, $(M_2, \xi_2, \alpha_2, J_2)$ and $(N, \mathcal{F})$, we obtain a bundle isomorphism

$$\Phi \colon TM_1|_N \to TM_2|_N\;.$$

We will now find neighborhoods U_1 of N in M_1 and U_2 of N in M_2 together with a diffeomorphism

$$\Psi \colon U_1 \to U_2$$

that restricts on N to the identity, and whose differential $D\Psi$ coincides with the bundle map Φ, that means

$$D\Psi \colon TM_1|_N \to TM_2|_N$$

is equal to Φ.

If N has boundary, extend it by a small open collar and denote the resulting manifold by $\widehat{N}$. This enlarged manifold also embeds into M_1 and into M_2 extending the given embeddings of N, and (after possibly shrinking $\widehat{N}$) we find a bundle isomorphism $TM_1|_{\widehat{N}} \to TM_2|_{\widehat{N}}$ that restricts over N to Φ. For simplicity we denote this new bundle isomorphism also by Φ.

Choose a Riemannian metric on $\widehat{N}$, and extend it to a Riemannian metric on all of M_2. Use Φ to pull this metric back to the bundle $TM_1|_{\widehat{N}}$ and extend it also there to a metric on all of M_1. By the tubular neighborhood theorem, we find open neighborhoods U_1 of $\widehat{N}$ in M_1 and U_2 of $\widehat{N}$ in M_2 and neighborhoods V_1 and V_2 of the 0-sections in the normal bundles $\nu_1(\widehat{N})$ and $\nu_2(\widehat{N})$ respectively, such that the exponential normal maps

$$\exp_j \colon V_j \subset \nu_j(\widehat{N}) \to U_j \subset M_j$$

define for both $j = 1, 2$ diffeomorphisms. By our construction, $\Phi\colon TM_1|_N \to TM_2|_N$ is an isometry, hence it maps $\nu_1(N)$ isomorphically onto $\nu_2(N)$. After restricting the neighborhoods V_1 and V_2 suitably, we may assume that $\Phi(V_1) = V_2$, so that we obtain a diffeomorphism Ψ from U_1 to U_2 by setting $\Psi(p) := \exp_2 \circ \Phi \circ \exp_1^{-1}(p)$ for every $p \in U_1$. Using that the differential of the exponential map is the identity along the 0-section, we obtain that $D\Psi = \Phi$ at every $p \in N$.

Using the Moser trick, we will now isotope Ψ to the desired contactomorphism. Let β_τ for $\tau \in [0, 1]$ be the linear interpolation

$$\beta_\tau := (1 - \tau)\, \alpha_1 + \tau\, \Psi^* \alpha_2$$

on U_1. Using Lemma I.6.4, we will show that (after possibly decreasing U_1) all of the 1-forms in the family β_τ are contact.

Write α_N for $\alpha_1|_N$ and $J_N = J_1|_N$. The bundle map Φ obtained by Lemma I.1.4 pulls-back α_2 to α_1 along N, so that $\beta_\tau|_N = \alpha_N$ for all $\tau \in [0, 1]$. Since Φ is (J_1, J_2)-linear on $\mathcal{F}_{\alpha_N} \oplus J_N\, \mathcal{F}_{\alpha_N}$, it follows that

$$(\Phi^* d\alpha_2)(J_1 X, X) = d\alpha_2\big(J_2 \Phi(X), \Phi(X)\big) > 0$$

for every non-vanishing vector $X \in \mathcal{F}_{\alpha_N} \oplus J_N\, \mathcal{F}_{\alpha_N}$, so that the restriction of J_1 to $\mathcal{F}_{\alpha_N} \oplus J_N\, \mathcal{F}_{\alpha_N}$ is tamed both by α_1 as well as by $\Psi^*\alpha_2$. It follows that both α_1 and $\Psi^*\alpha_2$ are elements of the convex set $\mathcal{A}(\alpha_N, J_N)$ defined in Lemma I.6.4.

We will now finish the proof by applying the Moser trick: Lemma I.6.4 states that these forms all satisfy the contact condition on a sufficiently small neighborhood of N, and that the β_τ are constant on $TM_1|_N$.

The Moser trick uses the unique vector field Y_τ on U_1 defined by the equations

$$\big(\iota_{Y_\tau} d\beta_\tau\big)\big|_{\ker \beta_\tau} = -\dot{\beta}_\tau\Big|_{\ker \beta_\tau} \qquad \text{and} \qquad \beta_\tau(Y_\tau) = 0\,.$$

The 1-form $\dot{\beta}_\tau$ vanishes along N, so that $Y_\tau(p) = 0$ for every point $p \in N$. This allows us to integrate the flow $\Phi_\tau^{Y_\tau}$ on a smaller neighborhood of N up to time 1 and the submanifold N remains pointwise fixed by this flow. Furthermore we have $\big(\Phi_1^{Y_\tau}\big)^* \beta_1 = F\, \beta_0$ for some positive function F, so that $\Psi' = \Psi \circ \Phi_1^{Y_\tau}$ is the desired contactomorphism between neighborhoods of N in M_1 and in M_2, finishing the proof. □

Another useful fact is the following theorem that tells us that the singular foliations that can be realized as Legendrian ones are precisely those that admit a regular equation (in accordance with the 3-dimensional situation [**Gir91**], where this property was called "*non isochore*").

Theorem I.1.5. *Let N be a manifold with a neat singular foliation $\mathcal{F}$. There is an (open) cooriented contact manifold (M, ξ) that contains N as a submanifold such that ξ induces $\mathcal{F}$ as Legendrian foliation on N.*

PROOF. Assume for now that N does not have boundary.

Choose a regular equation β for $\mathcal{F}$. As in the proof of Lemma I.1.4, we will cover N with the open set of regular points $N_{\mathrm{reg}} = N \setminus \operatorname{Sing}(\mathcal{F})$, where β does not vanish, and the set of points $N_{\mathrm{sing}} := \bigl\{p \in N \bigm| d\beta_p \neq 0\bigr\}$. We will construct a vector bundle over each of these two regions and show that we find suitable contact structures close to the 0-sections in both total spaces. In a final step, we glue the two models together to obtain the desired manifold M.

Choose any Riemannian metric g on N, then we can define the orthogonal projection

$$\pi_{\mathcal{F}} \colon TN \to \mathcal{F}$$

of the tangent space onto the foliation. Let $\mathcal{F}^*|_{N_{\mathrm{reg}}}$ be the dual vector bundle of $\mathcal{F}|_{N_{\mathrm{reg}}}$, and write the bundle projection as $\pi\colon \mathcal{F}^*|_{N_{\mathrm{reg}}} \to N_{\mathrm{reg}}$.

Using these two maps, we can construct a 1-form on the total space of $\mathcal{F}^*|_{N_{\mathrm{reg}}}$ that resembles the canonical 1-form on a cotangent bundle. Let (p,ν) be a point of $\mathcal{F}^*$, that means, p is a point of N, and ν is a covector on $\mathcal{F}_p$, and assume that $v \in T_{(p,\nu)}\mathcal{F}^*$ is any vector at (p,ν). We can project v with $D\pi$ first into T_pN, and then with $\pi_{\mathcal{F}}$ into $\mathcal{F}_p$. Then we may just plug $\pi_{\mathcal{F}}\bigl(D\pi(v)\bigr)$ into $\nu \in \mathcal{F}_p^*$ to obtain a real number. Clearly this map is fiberwise linear, and so we can define a 1-form $\lambda_{\mathcal{F}}(v) = \nu\bigl(\pi_{\mathcal{F}}\bigl(D\pi(v)\bigr)\bigr)$ on the total space $\mathcal{F}^*|_{N_{\mathrm{reg}}}$. We claim that

$$\alpha_{\mathrm{reg}} := \pi^*\beta + \lambda_{\mathcal{F}}$$

is a contact form with the desired properties. The canonical embedding of N_{reg} as the 0-section preserves the Legendrian foliation $\mathcal{F}$, and so we only need to check that α_{reg} really is contact. Let $U \subset N$ be a standard chart of the foliation with coordinates $(q_0, q_1, \dots, q_n)$, in which the subsets $\{q_0 = \mathrm{const}\}$ correspond to the leaves of $\mathcal{F}$. We obtain an induced chart for $\mathcal{F}^*|_U$ with coordinates $(q_0, q_1, \dots, q_n; p_1, \dots, p_n)$ which represent the 1-form $p_1\, q_1^* + \dots + p_n\, q_n^*$, where $\langle q_1^*, \dots, q_n^*\rangle$ is the dual basis of $\langle \partial_{q_1}, \dots, \partial_{q_n}\rangle$. Note that the q_j^* are elements of $\mathcal{F}^*$ so that plugging in ∂_{q_0} into any of them is not defined. Write a vector $v \in T\mathcal{F}^*$ as $(\dot q_0, \dot q_1, \dots, \dot q_n; \dot p_1, \dots, \dot p_n)$, then we have $\pi_{\mathcal{F}}\bigl(D\pi(v)\bigr) = \pi_{\mathcal{F}}(\dot q_0, \dot q_1, \dots, \dot q_n) = (\dot q_1, \dots, \dot q_n) + \pi_{\mathcal{F}}(\dot q_0, 0, \dots, 0)$. Combining these results, we obtain that the 1-form $\lambda_{\mathcal{F}}$ can be written in these coordinates as

$$\lambda_{\mathcal{F}} = p_1\, dq_1 + \dots + p_n\, dq_n + F(q_0, q_1, \dots, q_n; p_1, \dots, p_n)\, dq_0$$

with a smooth function F that is linear in the p_j-coordinates. Since $\lambda_{\mathcal{F}}$ vanishes along the 0-section $(q_0, q_1, \dots, q_n; 0, \dots, 0)$, we compute there that

$$\alpha_{\mathrm{reg}} \wedge d\alpha_{\mathrm{reg}}^n = (\pi^*\beta) \wedge d\lambda_{\mathcal{F}}^n = n!\, (\pi^*\beta) \wedge dp_1 \wedge dq_1 \wedge \dots \wedge dp_n \wedge dq_n\ ,$$

so that α_{reg} is a contact form on a neighborhood of N_{reg}. We will denote the total space of $\mathcal{F}^*|_{N_{\mathrm{reg}}}$ by M_{reg} so that we obtain the manifold with 1-form $\bigl(M_{\mathrm{reg}}, \alpha_{\mathrm{reg}}\bigr)$ that is close to the 0-section N_{reg} contact.

Now we will construct a neighborhood for the set N_{sing}. Remember that $d\beta^2 = d\bigl(\beta\wedge d\beta\bigr) \equiv 0$, so that there is a well-defined codimension-2 foliation $\mathcal{F}_\beta$, the secondary foliation, spanned by the vectors on which $d\beta$ vanishes. In the same way as before, we use the metric g to define a generalized "canonical" 1-form $\lambda_{\mathcal{F}_\beta}$ on the total space of the bundle $\mathcal{F}_\beta^*$. Add a trivial $\mathbb{R}$-factor with coordinate z to obtain the bundle $\mathcal{F}_\beta^* \oplus \mathbb{R}$ over N_{sing}, and equip the total space with the 1-form

$$\alpha_{\mathrm{sing}} = \pi^*\beta + \lambda_{\mathcal{F}_\beta} + dz\ .$$

To see that α_{sing} is a contact form, choose a Kupka chart U with coordinates $(s, t, q_1, \dots, q_{n-1})$ where β takes the form $a(s,t)\,ds + b(s,t)\,dt$, and extend it to a bundle chart of $\left(\mathcal{F}_\beta^* \oplus \mathbb{R}\right)\Big|_U$ with coordinates $(s, t, q_1, \dots, q_{n-1}; p_1, \dots, p_{n-1}, z)$. The p_j represent the dual form $p_1\, q_1^* + \dots + p_{n-1}\, q_{n-1}^* \in \mathcal{F}_\beta^*$ as above. The "canonical" 1-form simplifies on this chart to

$$\lambda_{\mathcal{F}_\beta} = p_1\, dq_1 + \dots + p_{n-1}\, dq_{n-1} + F_s\, ds + F_t\, dt\ ,$$

where F_s and F_t are functions that vanish on the 0-section $(s, t, q_1, \dots, q_{n-1}; 0, \dots, 0)$. It is easy to see that

$$\alpha_{\mathrm{sing}} \wedge d\alpha_{\mathrm{sing}}^n = dz \wedge \bigl(n\, d(\pi^*\beta) + d\lambda_{\mathcal{F}_\beta}\bigr) \wedge d\lambda_{\mathcal{F}_\beta}^{n-1}\ ,$$

and furthermore since the F_s- and the F_t-coefficients do not vary along the 0-section in any of the q_j-directions, it follows that

$$\alpha_{\mathrm{sing}} \wedge d\alpha_{\mathrm{sing}}^n = n!\, dz \wedge d(\pi^*\beta) \wedge dp_1 \wedge dq_1 \wedge \dots \wedge dp_{n-1} \wedge dq_{n-1} \neq 0$$

on the 0-section, and thus there exists a small neighborhood of N_{sing} on which α_{sing} will be contact. We denote the total space of $\mathcal{F}_\beta^* \oplus \mathbb{R}$ by M_{sing} and so we obtain the manifold $\bigl(M_{\mathrm{sing}}, \alpha_{\mathrm{sing}}\bigr)$ that is contact close to the 0-section N_{sing}. Furthermore note that $\mathcal{F}_\beta$ is the secondary foliation both for α_{sing} and for α_{reg}.

We finally need to glue both parts M_{reg} and M_{sing}. As a preparation, we will first construct a bundle isomorphism

$$\Phi\colon\ TM_{\mathrm{reg}}|_{(N_{\mathrm{reg}} \cap N_{\mathrm{sing}})} \to TM_{\mathrm{sing}}|_{(N_{\mathrm{reg}} \cap N_{\mathrm{sing}})}\ .$$

Choose a complex structure J_{reg} on $\xi_{\mathrm{reg}} = \ker \alpha_{\mathrm{reg}}$ compatible with $d\alpha_{\mathrm{reg}}|_{\xi_{\mathrm{reg}}}$, and a complex structure J_{sing} on $\xi_{\mathrm{sing}} = \ker \alpha_{\mathrm{sing}}$ compatible with $d\alpha_{\mathrm{sing}}|_{\xi_{\mathrm{sing}}}$. We set

$$E := \bigl(\mathcal{F}_\beta \oplus J_{\mathrm{reg}}\, \mathcal{F}_\beta\bigr)\big|_{(N_{\mathrm{reg}} \cap N_{\mathrm{sing}})} \subset TM_{\mathrm{reg}}|_{(N_{\mathrm{reg}} \cap N_{\mathrm{sing}})}\ ,$$

so that we may decompose ξ_{reg} along the 0-section $N_{\mathrm{reg}} \cap N_{\mathrm{sing}}$ into the two J_{reg}-invariant subbundles

$$\xi_{\mathrm{reg}}|_{(N_{\mathrm{reg}} \cap N_{\mathrm{sing}})} = E \oplus E^\perp\ ,$$

where $E^\perp$ is the $d\alpha_{\mathrm{reg}}$-symplectic complement of the subbundle E. The intersection $E^\perp \cap TN$ is a (real) line bundle, and we will next choose a non-vanishing vector field $X^\perp$ in $E^\perp \cap TN$. For this let X be any vector field in $TN|_{N_{\mathrm{reg}}}$ such that $\beta(X) = 1$. The 1-form $d\beta(X, \cdot)$ does not vanish anywhere on $N_{\mathrm{reg}} \cap N_{\mathrm{sing}}$, and so it follows that there is a vector field $X^\perp \in E^\perp \cap TN$ with $d\beta(X, X^\perp) = 1$. The vector field $X^\perp$ is in fact unique and does not depend on the choice of X. Hence we can split $TM_{\mathrm{reg}}|_{(N_{\mathrm{reg}} \cap N_{\mathrm{sing}})}$ into

$$TN|_{(N_{\mathrm{reg}} \cap N_{\mathrm{sing}})} \oplus \bigl(J_{\mathrm{reg}}\, \mathcal{F}_\beta\bigr)\big|_{(N_{\mathrm{reg}} \cap N_{\mathrm{sing}})} \oplus \operatorname{span}\langle J_{\mathrm{reg}} X^\perp\rangle\ ,$$

and using this decomposition, we define a bundle isomorphism

$$\Phi\colon\ TM_{\mathrm{reg}}|_{(N_{\mathrm{reg}} \cap N_{\mathrm{sing}})} \to TM_{\mathrm{sing}}|_{(N_{\mathrm{reg}} \cap N_{\mathrm{sing}})}$$

by mapping any vector $v + J_{\mathrm{reg}} w + c\, J_{\mathrm{reg}} X^\perp$ with $v \in TN$, $w \in \mathcal{F}_\beta$ and $c \in \mathbb{R}$ to the vector $v + J_{\mathrm{sing}} w + c\,(X - R_{\mathrm{sing}})$, where X is the vector field in TN chosen above for which $\beta(X) \equiv 1$, and R_{sing} is the Reeb field of α_{sing}. The map Φ is clearly injective, because $\Phi\bigl(J_{\mathrm{reg}} X^\perp\bigr) = X - R_{\mathrm{sing}} = X - \partial_z$ does not lie in $\Phi\bigl(TN \oplus J_{\mathrm{reg}}\, \mathcal{F}_\beta\bigr) = TN \oplus J_{\mathrm{sing}}\, \mathcal{F}_\beta$, and Φ is hence an isomorphism.

We will use Φ to glue M_{reg} and M_{sing} in a neighborhood of N to each other. Choose a Riemannian metric on N and extend it to all of M_{sing}. Let $N_- \subset N_{\mathrm{reg}}$ and $N_+ \subset N_{\mathrm{sing}}$ be open sets such that $N_- \cup N_+$ still covers all of N, and such that $\overline{N_-} \subset N_{\mathrm{reg}}$ and $\overline{N_+} \subset N_{\mathrm{sing}}$. From now on let Φ always denote the restriction

$$\Phi \colon TM_{\mathrm{reg}}|_{(N_- \cap N_+)} \to TM_{\mathrm{sing}}|_{(N_- \cap N_+)} \; .$$

Use this map to pull-back the metric from $TM_{\mathrm{sing}}|_{(N_- \cap N_+)}$ to $TM_{\mathrm{reg}}|_{(N_- \cap N_+)}$, and extend it then to a metric on all of M_{reg} that coincides with the chosen metric on N.

By our construction, Φ is an isometry, and hence, if we further restrict Φ to the normal bundle $\nu_{\mathrm{reg}}(N_- \cap N_+)$ of $(N_- \cap N_+)$ in M_{reg}, the image of Φ will be the normal bundle $\nu_{\mathrm{sing}}(N_- \cap N_+)$ of $(N_- \cap N_+)$ in M_{sing}. We use Φ to glue $\nu_{\mathrm{reg}}(N_-)$ and $\nu_{\mathrm{sing}}(N_+)$ over $N_- \cap N_+$ via a clutching construction together and obtain this way a vector bundle

$$\nu(N) := \nu_{\mathrm{reg}}(N_-) \cup_\Phi \nu_{\mathrm{sing}}(N_+)$$

over all of N. The total space of this bundle is a manifold M that contains N as a submanifold.

By the tubular neighborhood theorem (and because we have decreased the size of N_- and N_+), there is a small neighborhood U_- of the 0-section in $\nu_{\mathrm{reg}}(N_-)$ that is via the exponential map diffeomorphic to an open set in M_{reg} containing N_- and a neighborhood U_+ of the 0-section in $\nu_{\mathrm{sing}}(N_+)$ that can be identified with an open set in M_{sing}. This way, after possibly decreasing the size of U_- and U_+ further, we can pull-back α_{reg} to U_-, and α_{sing} to U_+.

Using Lemma I.6.4, we will show that the interpolation between these two forms gives a globally defined contact form on a neighborhood of N. The following remarks will allow us to verify more easily the necessary conditions of the lemma. Remember that the differential of the exponential map

$$D\exp_+ \colon T\nu_{\mathrm{reg}}(N_-) \to TM_{\mathrm{reg}}$$

can be naturally identified along the 0-section with the identity map on $\bigl(TM_{\mathrm{reg}}\bigr)\big|_{N_-}$ by representing the domain in the form

$$\bigl(T\nu_{\mathrm{reg}}(N_-)\bigr)\big|_{N_-} \cong TN_- \oplus \nu_{\mathrm{reg}}(N_-) = \bigl(TM_{\mathrm{reg}}\bigr)\big|_{N_-} \; .$$

A similar representation exists for $\bigl(T\nu_{\mathrm{sing}}(N_+)\bigr)\big|_{N_+}$, and together they allow us to reduce the differential

$$D\Phi \colon \bigl(T\nu_{\mathrm{reg}}(N_-)\bigr)\big|_{(N_- \cap N_+)} \to \bigl(T\nu_{\mathrm{sing}}(N_+)\bigr)\big|_{(N_- \cap N_+)}$$

naturally to $\Phi \colon TM_{\mathrm{reg}}|_{(N_- \cap N_+)} \to TM_{\mathrm{sing}}|_{(N_- \cap N_+)}$.

We construct the 1-form α on a neighborhood of N in $M = \nu(N)$ by using a partition of unity $\{\rho_-, \rho_+\}$ subordinate to $\{N_-, N_+\}$ and setting

$$\alpha := \rho_- \cdot \alpha_- + \rho_+ \cdot \alpha_+ \; ,$$

where $\alpha_- := (\exp_-)^* \alpha_{\mathrm{reg}}$ and $\alpha_+ := (\exp_+)^* \alpha_{\mathrm{sing}}$. We will prove by applying Lemma I.6.4 that α is close to the 0-section N a contact form. Note that by the previous remarks, α can be written at points of $N_- \cap N_+$ as $\alpha := \rho_- \cdot \alpha_{\mathrm{reg}} + \rho_+ \cdot (\alpha_{\mathrm{sing}} \circ \Phi)$, and it can be easily checked that the pull-back $\alpha_{\mathrm{sing}} \circ \Phi$ is identical to α_{reg} along $N_- \cap N_+$ by evaluating both forms on the basis of TM_{reg} used above. Similarly, we obtain that J_{reg} is tamed on $\bigl(\mathcal{F}_\beta^* \oplus J_{\mathrm{reg}}\, \mathcal{F}_\beta^*\bigr)\Big|_{(N_{\mathrm{reg}} \cap N_{\mathrm{sing}})}$ by $\Phi^* d\alpha_+ = d\alpha_{\mathrm{sing}} \circ \Phi$. To see that $\Phi^* \alpha_+$ lies along $N_- \cap N_+$ in the space $\mathcal{A}(\alpha_-, J_{\mathrm{reg}})$, it only remains to see that $\Phi^* \alpha_+$ induces the same orientation as α_-.

We can evaluate for the basis chosen above

$$\alpha_{\mathrm{reg}} \wedge d\alpha_{\mathrm{reg}}^n\bigl(X, J_{\mathrm{reg}}X^\perp, X^\perp, \partial_{y_1}, \partial_{x_1}, \dots, \partial_{y_{n-1}}, \partial_{x_{n-1}}\bigr) \\ = n\, d\alpha_{\mathrm{reg}}(J_{\mathrm{reg}}X^\perp, X^\perp) \cdot d\alpha_{\mathrm{reg}}^{n-1}\bigl(\partial_{y_1}, \partial_{x_1}, \dots, \partial_{y_{n-1}}, \partial_{x_{n-1}}\bigr)$$

using that X is the only vector that does in the kernel of α_{reg}, and that $X^\perp$ lies in the $d\alpha_{\mathrm{reg}}$-symplectic complement of $\mathcal{F}_\beta \oplus J_{\mathrm{reg}}\,\mathcal{F}_\beta$.

If we repeat the same computation for $\Phi^*\alpha_{\mathrm{sing}}$, we obtain

$$\begin{aligned} &\Phi^*\bigl(\alpha_{\mathrm{sing}} \wedge d\alpha_{\mathrm{sing}}^n\bigr)\bigl(X, J_{\mathrm{reg}}X^\perp, X^\perp, \partial_{y_1}, \partial_{x_1}, \dots, \partial_{y_{n-1}}, \partial_{x_{n-1}}\bigr) \\ &= \alpha_{\mathrm{sing}} \wedge d\alpha_{\mathrm{sing}}^n\bigl(X, \Phi(J_{\mathrm{reg}}X^\perp), X^\perp, \Phi(\partial_{y_1}), \partial_{x_1}, \dots, \Phi(\partial_{y_{n-1}}), \partial_{x_{n-1}}\bigr) \\ &= \alpha_{\mathrm{sing}} \wedge d\alpha_{\mathrm{sing}}^n\bigl(X, X - R_{\mathrm{sing}}, X^\perp, J_{\mathrm{sing}}\,\partial_{x_1}, \partial_{x_1}, \dots, J_{\mathrm{sing}}\,\partial_{x_{n-1}}, \partial_{x_{n-1}}\bigr) \\ &= d\alpha_{\mathrm{sing}}^n\bigl(X, X^\perp, J_{\mathrm{sing}}\,\partial_{x_1}, \partial_{x_1}, \dots, J_{\mathrm{sing}}\,\partial_{x_{n-1}}, \partial_{x_{n-1}}\bigr)\ . \end{aligned}$$

Here we have that $d\alpha_{\mathrm{sing}}(X, \partial_{x_j})$, $d\alpha_{\mathrm{sing}}(X^\perp, \partial_{x_j})$, and $d\alpha_{\mathrm{sing}}(\partial_{x_i}, \partial_{x_j})$ vanish, and so that we finally obtain

$$\Phi^*\bigl(\alpha_{\mathrm{sing}} \wedge d\alpha_{\mathrm{sing}}^n\bigr)\bigl(X, J_{\mathrm{reg}}X^\perp, X^\perp, \partial_{y_1}, \partial_{x_1}, \dots, \partial_{y_{n-1}}, \partial_{x_{n-1}}\bigr) \\ = n\, d\alpha_{\mathrm{sing}}(X, X^\perp) \cdot d\alpha_{\mathrm{sing}}^{n-1}\bigl(J_{\mathrm{sing}}\,\partial_{x_1}, \partial_{x_1}, \dots, J_{\mathrm{sing}}\,\partial_{x_{n-1}}, \partial_{x_{n-1}}\bigr)$$

yielding the desired orientation.

We can hence interpolate between the two contact forms to obtain a global contact form on a neighborhood of N.

If N has boundary, the proof is more tiresome, but not fundamentally different. We attach in a first step a small open collar to the boundary of N and denote the resulting manifold by $\widehat{N}$. Bundles always extend naturally over collars, and we obtain neighborhoods $\widehat{M}_{\mathrm{reg}}$ and $\widehat{M}_{\mathrm{sing}}$. We can also extend the two forms α_{reg} and α_{sing} smoothly to these two subsets. If necessary, we need to shrink the size of $\widehat{M}_{\mathrm{reg}}$ and $\widehat{M}_{\mathrm{sing}}$ to guarantee the contact properties. Also in general the contact structures will not induce a Legendrian foliation on $\widehat{N} \setminus N$.

Finally we extend also the gluing map Φ, and note again that reducing the size of the collars further, we can make sure that $\widehat{\Phi}$ will be an isomorphism. For the rest of the proof we can proceed as in the case without boundary. $\square$

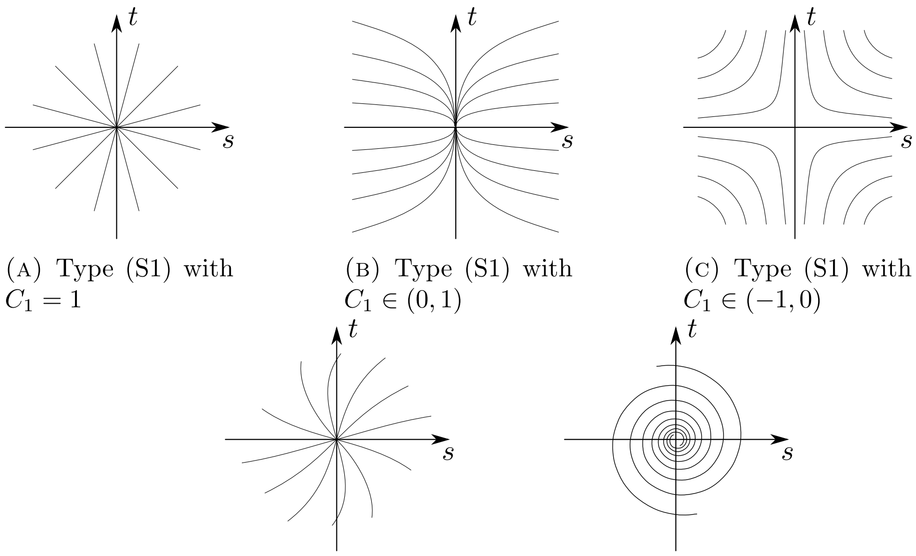
I.2. Singularities of the Legendrian foliation

The singular set of a Legendrian foliation $\mathcal{F}$ can be extremely complicated. We will briefly comment on some general properties of such points, but we will soon need to specialize to singularities forming isolated codimension-2 and codimension-1 submanifolds.

Let N have a singular foliation $\mathcal{F}$ given by a regular equation β, and let $p \in \operatorname{Sing}(\mathcal{F})$ be a singular point of $\mathcal{F}$. Choose a Kupka chart U with coordinates $(s, t, x_1, \dots, x_{n-1})$ centered at p. In this chart β is represented by

$$a(s,t)\, ds + b(s,t)\, dt$$

with two smooth functions $a, b\colon U \to \mathbb{R}$ that only depend on the s- and t-coordinates, and that vanish at the origin.



(A) Type (S1) with $C_1 = 1$

(B) Type (S1) with $C_1 \in (0, 1)$

(C) Type (S1) with $C_1 \in (-1, 0)$

(D) Type (S2)

(E) Type (S3)

FIGURE 2. The linearized singularities correspond to one of the five possible cases sketched here (or to type (S1) with $C_1 = 0$ which we have omitted because the linearization does not necessarily correspond to the actual shape).

Remark I.2.1. To understand the shape of the foliation depending on the functions a and b, we might study trajectories of the vector field

$$X = b(s,t)\,\frac{\partial}{\partial s} - a(s,t)\,\frac{\partial}{\partial t}$$

that spans the projection of the foliation to the (s,t)-plane. Its divergence $\operatorname{div} X = \partial b/\partial s - \partial a/\partial t$ does not vanish, since $d\beta \neq 0$. Up to a genericity condition, we know by the Grobman-Hartman theorem that the flow of X is C^0-equivalent to the flow of its linearization (see [**PdM82**]). In dimension 2, the Grobman-Hartman theorem even yields a C^1-equivalence, but this does not suffice for our purposes, so we will use a more hands-on approach.

After a change of coordinates, we can assume according to Lemma I.6.5 that β agrees with one of the following three models around p

$$\beta = s\,dt - C_1 t\,ds + \mathcal{O}^2(s,t)\ , \tag{S1}$$

$$\beta = (s + \varepsilon t)\,dt - t\,ds + \mathcal{O}^2(s,t)\ , \tag{S2}$$

or

$$\begin{aligned} \beta &= (s - C_2 t)\,ds + (C_2 s + t)\,dt + \mathcal{O}^2(s,t) \\ &= C_2 r^2\,d\varphi + \frac{1}{2}\,d(r^2) + \mathcal{O}^2(r,\varphi)\ , \end{aligned} \tag{S3}$$

where $\mathcal{O}^2(s,t)$ stands for a 1-form of order 2, that means, a form whose coefficient functions can be written as $s^2 f(s,t) + stg(s,t) + t^2 h(s,t)$; C_1, C_2, and ε are real constants with $C_1 \in (-1, 1]$, $C_2 \neq 0$, and ε arbitrarily small.

In every 2-dimensional slice with fixed $(x_1, \dots, x_{n-1})$-coordinates in a Kupka chart, the origin is an isolated singularity (see also Figure 2), with the possible exception of case (S1) when $C_1 = 0$, where the linearization is not enough to determine anymore the topology of the foliation. In the section below, we will study codimension 1 singularities which of course correspond always to this exceptional case. From then on, we will suppose that every singular point of type (S1) with $C_1 = 0$ lies in a codimension 1 component of $\operatorname{Sing}(\mathcal{F})$, so that all the remaining singularities will be collections of isolated codimension-2 submanifolds. We will study those in Section I.2.2.

I.2.1. Singularities of codimension-1. Legendrian foliations with singular codimension-1 sets are rather ungeneric, but can often be found through explicit constructions (as in Example I.3.2). We will show in this section that by slightly deforming the foliated submanifold one can sometimes modify the Legendrian foliation in a controlled way that turns the singular set into a regular compact leaf (see Fig. 3).

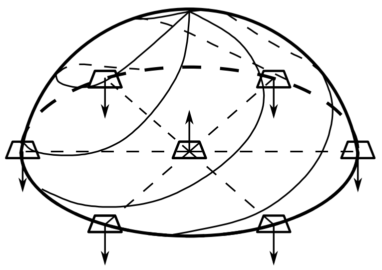


FIGURE 3. In dimension 3, it is well-known that we can get rid of 1-dimensional singular sets of a Legendrian foliation by slightly tilting the surface along the singular set. The figure depicts an overtwisted disk with singular boundary. By pushing the interior of the disk up keeping its boundary fixed, we obtain the standard form of an overtwisted disk whose boundary is a compact leaf of the foliation.

Lemma I.2.2. *Let N be a manifold with a singular codimension-1 foliation $\mathcal{F}$ given by the regular equation β. Assume that there is a closed codimension-1 submanifold $S \subset N$ that is cooriented and lies in the singular set $\operatorname{Sing}(\mathcal{F})$ of the foliation.*

Then we can find a tubular neighborhood of S diffeomorphic to $(-\varepsilon, \varepsilon) \times S$ such that β pulls back to

$$s \cdot \widetilde{\beta}\,,$$

where s denotes the coordinate on $(-\varepsilon, \varepsilon)$, and $\widetilde{\beta}$ is a non-vanishing 1-form on S that defines a regular codimension-1 foliation $\mathcal{F}_S$ on S.

Remark I.2.3. The foliation $\mathcal{F}_S$ agrees on S with the secondary foliation $\mathcal{F}_\beta$. The lemma then says that there is a neighborhood diffeomorphic to $(-\varepsilon, \varepsilon) \times S$ for which $\mathcal{F}$ is the prolongation of the secondary foliation $\mathcal{F}_\beta$ in $(-\varepsilon, \varepsilon)$-direction, and $\mathcal{F}$ becomes singular along $\{0\} \times S$ by flipping its coorientations.

Remark I.2.4. In case the hypersurface $S \subset \operatorname{Sing}(\mathcal{F})$ is not coorientable, the results of the previous lemma can be adapted by constructing the *orientation double cover* $\widetilde{S} \xrightarrow{\pi} S$ of the normal bundle $\nu(S)$ of S in N. We find a natural cover

$$\widetilde{S} \times \mathbb{R} \cong \pi^*\nu(S) \to \nu(S) \cong \widetilde{S} \times_{\mathbb{Z}_2} \mathbb{R} ,$$

and we can apply Lemma I.2.2 to the pull-back of $\mathcal{F}$ to $\widetilde{S} \times \mathbb{R}$, and then quotient out the model situation by the $\mathbb{Z}_2$-action.

Proof of Lemma I.2.2. Choose a coorientation for S. We will first construct on a neighborhood of S a vector field X that is transverse to S and lies in the kernel of β. Study the local situation in a Kupka chart U around a point $p \in S$ with coordinates $(s, t, x_1, \dots, x_{n-1})$. Assume that β restricts to

$$a(s,t)\, ds + b(s,t)\, dt ,$$

such that $S \cap U$ corresponds to the subset $\{s = 0\}$, and such that s increases in direction of the chosen coorientation.

Since a and b vanish along $S \cap U$, we can write this form according to Corollary I.6.2 also as

$$s\, a_s(s,t)\, ds + s\, b_s(s,t)\, dt = s\, \bigl(a_s(s,t)\, ds + b_s(s,t)\, dt\bigr)$$

with smooth functions a_s and b_s that satisfy the conditions

$$a_s(0,t) = \frac{\partial a}{\partial s}(0,t) \quad \text{and} \quad b_s(0,t) = \frac{\partial b}{\partial s}(0,t) .$$

The function b_s does not vanish in a small neighborhood of $S \cap U$, because $0 \neq d\beta = \partial_s b\, ds \wedge dt$. Choose then on the Kupka chart U the smooth vector field

$$X_U\bigl(s, t, x_1, \dots, x_{n-1}\bigr) := \partial_s - \frac{a_s(s,t)}{b_s(s,t)}\, \partial_t = \partial_s - \frac{a(s,t)}{b(s,t)}\, \partial_t .$$

This field lies in $\mathcal{F}$, and is positively transverse to $S \cap U$

The desired vector field X is now obtained by covering the singular set S with a finite number of Kupka charts $U_1, \dots, U_N$, constructing on each of them a vector field X_{U_j} using the method described above, and gluing the X_{U_j} together using a partition of unity $\{\rho_j\}$ subordinate to the cover, so that

$$X := \sum_{j=1}^{N} \rho_j \cdot X_{U_j} .$$

The flow of X allows us to define a tubular neighborhood of S that is diffeomorphic to $(-\varepsilon, \varepsilon) \times S$, where $\{0\} \times S$ corresponds to the submanifold S, and X corresponds to the field ∂_s, where s is the coordinate on the interval $(-\varepsilon, \varepsilon)$. Since $\beta(X) \equiv 0$, it follows that β does not contain any ds-terms in this model.

Let γ be the 1-form $\iota_X d\beta$. A short computation shows that

$$0 \equiv \iota_X\bigl(\beta \wedge d\beta\bigr) = \beta(X)\, d\beta - \beta \wedge \bigl(\iota_X d\beta\bigr) = -\beta \wedge \gamma ,$$

and since γ does not vanish on a neighborhood of the singular set S (because $d\beta \neq 0$, while $\beta|_{TS} \equiv 0$), there is a smooth function $F\colon (-\varepsilon, \varepsilon) \times S \to \mathbb{R}$ with $F|_S = 0$ such that $\beta = F\gamma$. In particular, it follows that γ defines a regular foliation $\widetilde{\mathcal{F}}$ that agrees outside $\operatorname{Sing}(\mathcal{F}) = \{0\} \times S$ with the initial one. Furthermore, using that F vanishes along S, but that γ does not, we obtain from

$$\gamma = \iota_X d\beta = dF(X)\, \gamma + F\, \iota_X d\gamma$$

that $dF(X) = 1$ on S, and S is hence a regular zero level set of the function F.

Finally, we can use $\iota_X\gamma \equiv 0$ to see that

$$0 \equiv \iota_X\bigl(\gamma \wedge d\gamma\bigr) = -\gamma \wedge (\iota_X d\gamma) ,$$

and hence there is a smooth function $f\colon (-\varepsilon,\varepsilon) \times S \to \mathbb{R}$ such that $\mathcal{L}_X\gamma = \iota_X d\gamma = f\,\gamma$. The flow in s-direction only rescales the 1-form γ, but it leaves its kernel invariant, thus the foliation $\widetilde{\mathcal{F}}$ is tangent to the s-direction and s-invariant. We can hence represent $\widetilde{\mathcal{F}}$ on $(-\varepsilon,\varepsilon)\times S$ as the kernel of the 1-form $\widetilde{\beta} = \gamma|_{TS}$ that is independent of the s-coordinate, and does not have any ds-terms. It follows that γ is equal to $\widetilde{F}\,\gamma|_{TS}$ with a function $\widetilde{F}$ such that $\widetilde{F}(S) = \{1\}$. Set $h := F\widetilde{F}$.

For the initial 1-form β this means that $\beta = h\widetilde{\beta}$, and h is a smooth function for which $\{0\}\times S$ is the (regular) 0-level set. Rearranging the model $(-\varepsilon,\varepsilon)\times S$ by using the flow of a vector field $G^{-1}\,\partial_s$ with $G = \partial_s h$, we obtain a new model for which β simplifies to $s\,\widetilde{\beta}$. □

Suppose that we are in the model situation of Lemma I.2.2, that means, we have a singular foliation $\mathcal{F}$ on the space $N := (-\varepsilon,\varepsilon)\times S$ given as the kernel of the 1-form $s\,\widetilde{\beta}$, where S is a closed manifold, s is the coordinate on $(-\varepsilon,\varepsilon)$, and S has a regular codimension-1 foliation $\mathcal{F}_S$ given as the kernel of the 1-form $\widetilde{\beta}\in\Omega^1(S)$.

The canonical 1-form λ_{can} on T^*S has the important property that $\sigma^*\lambda_{\mathrm{can}} = \sigma$ for every 1-form σ on S. If we embed $N = (-\varepsilon,\varepsilon)\times S$ via the map (the map is an embedding, because β does not vanish anywhere)

$$(s,p) \mapsto (0, -s\,\widetilde{\beta})$$

into the 1-jet space $\mathcal{J}^1(S) = \mathbb{R}\times T^*S$, the contact structure $\xi_{\mathrm{can}} = \ker(dz - \lambda_{\mathrm{can}})$ induces a singular Legendrian foliation on N that agrees with the given one. According to Theorem I.1.3, every embedding of N into a contact manifold, inducing $\mathcal{F}$ as Legendrian foliation, will be equivalent to this model.

Sometimes (and in the cases that are important for us), a slight perturbation of the embedding removes the codimension-1 singularities of the Legendrian foliation and replaces them by closed leaves.

Proposition I.2.5. *Let S be a closed manifold with a non-singular codimension-1 foliation $\mathcal{F}_S$ defined by a* closed, *nowhere vanishing 1-form λ. Note that by a result of Tischler* [**Tis70**], *the manifold S always fibers over the circle.*

Let $\mathcal{J}^1(S)$ be the corresponding 1-jet space $\mathbb{R}\times T^*S$ with its natural contact structure $\xi = \ker(dz - \lambda_{\mathrm{can}})$. Consider the embedding

$$\mathbb{R}\times S \hookrightarrow \mathcal{J}^1(S)$$

given by the map $(s,p)\mapsto(0,-s\lambda)$. The contact structure ξ induces a singular Legendrian foliation $\mathcal{F}$ on $\mathbb{R}\times S$ that agrees outside its singular set $\mathrm{Sing}(\mathcal{F}) = \{0\}\times S$ with $\mathbb{R}\times\mathcal{F}_S$.

We can define a C^∞-small perturbation of this embedding by choosing a smooth odd function $f\colon\mathbb{R}\to\mathbb{R}$ with compact support such that the derivative $f'(0)$ equals -1, and setting

$$\mathbb{R}\times S\hookrightarrow\mathbb{R}\times T^*S,\quad (s,p)\mapsto\bigl(\delta f(s), s\lambda\bigr)$$

for small $\delta>0$. This new map is clearly an embedding that differs only in an arbitrarily small neighborhood of S from the initial one. The perturbed submanifold carries a non-singular Legendrian foliation $\mathcal{F}'$ for which $\{0\}\times S$ is a regular closed leaf.

PROOF. The pull-back of $ds + \lambda_{\mathrm{can}}$ to the perturbed submanifold yields the 1-form $\lambda' = \delta f'\, ds + s\lambda$, which never vanishes. It is easy to check the Frobenius condition

$$\lambda' \wedge d\lambda' = \big(\delta f'\, ds + s\lambda\big) \wedge \big(ds \wedge \lambda + s\, d\lambda\big) = 0 ,$$

and to see that $\{0\} \times S$ is a closed leaf of $\mathcal{F}' = \ker \lambda'$. □

I.2.2. Non-degenerate singularities. From now on, we will assume that the singular foliation $\mathcal{F}$ on N is of a "Morse-Bott"-kind. We mean by this that all singular points $p \in \mathrm{Sing}(\mathcal{F})$ whose Kupka chart can be written in the form (S1) with $C_1 = 0$, lie on codimension-1 singular set as described in the previous section. In particular it follows that all other singular points admit a Kupka chart with coordinates $(s, t; x_1, \dots, x_{n-1})$, where the singular points are isolated in each 2-dimensional slice with constant $(x_1, \dots, x_{n-1})$-coordinates. Correspondingly, these singular components are embedded codimension-2 submanifolds, and we will from now on assume additionally that none of these components intersect the boundary ∂N.

Let from now on, S be such a connected codimension-2 component of $\mathrm{Sing}(\mathcal{F})$. We will try to understand the foliation in a neighborhood of S and describe in some specific cases how we may homotope the foliation to an easier form.

We will briefly explain what the holonomy of a regular foliation is. Let $\mathcal{F}_0$ be a regular codimension-k foliation on a manifold M. Choose a point $p \in M$ and a small k-disk D_p through p that is transverse to the leaf F_p of p. Since the foliation is transverse, D_p will also be transverse to every leaf passing through a small neighborhood of p. For every closed loop $\gamma \subset F_p$ with $\gamma(0) = \gamma(1) = p$, we can define a diffeomorphism on a small neighborhood of the origin of D_p in the following way: Cover the loop with foliation charts $U_1, \dots, U_N$, in which the leaves are parallel hyperplanes $\big(\{s_1, \dots, s_k\} \times \mathbb{R}^{n-k}\big) \cap U_j$, and such that F_p lies always in the slice $\{s_1 = \dots = s_k = 0\}$. We assume for simplicity that $D_p \cap U_1$ and $D_p \cap U_N$ lies in the $\mathbb{R}^k \times \{(0, \dots, 0)\}$-slice.

Choose a point $q \in D_p$ that corresponds in the chart U_1 to a point $(s_1, \dots, s_k; 0, \dots, 0)$, and follow the parallel copy of the path γ in the $(s_1, \dots, s_k)$-leaf until the translated path enters chart U_2. In U_2 we follow the parallel copy of γ in the corresponding leaf until we arrive in chart U_N, where the lifted path will end at some point of D_k (that will usually be different from the initial one).

It is not hard to see that this construction defines a diffeomorphism $\Phi_\gamma \colon V_0 \to V_1$ between sufficiently small open neighborhoods V_0, V_1 of the origin in D_p. It is also not difficult to see that a different choice of charts or a slight deformation of the loop γ in F_p yield always the same map Φ_γ, which implies that Φ_γ does not depend on the loop γ itself, but only on its homotopy class (it may be necessary though to decrease the domain when using a homotopic path). The correct way to formalize this notion, consists in saying that the loop γ defines the *germ* of diffeomorphisms on $D_p \cong \mathbb{R}^k$ mapping the origin to itself.

Definition. The **holonomy** at a point $p \in N$ is the map

$$\pi_1(F_p, p) \to \mathrm{Diff}_{\mathrm{germ}}(\mathbb{R}^k, 0)$$

from the fundamental group of the leaf F_p to the group of germs of diffeomorphisms of $\mathbb{R}^k$ that keep the origin fixed.

Up to conjugation the holonomy only depends on the leaf $F \subset \mathcal{F}$ and neither on the choice of the point $p \in F$ or the disk D_p used in the construction. It is also not difficult to see that the holonomy defines a group homomorphism.

The following theorem is a well-known standard result from foliation theory.

Theorem I.2.6. *Let F be a closed manifold, and let Φ be a group homomorphism*

$$\Phi\colon \pi_1(F) \to \mathrm{Diff}_{\mathrm{germ}}(\mathbb{R}^k, 0) .$$

Then we can construct an (open) manifold M with a regular codimension k-foliation $\mathcal{F}$ that has a compact leaf diffeomorphic to F whose holonomy is isomorphic to Φ. If Φ was the holonomy of a codimension k foliation with compact leaf F, then $(M, \mathcal{F})$ constructed as above will be on some neighborhood of F diffeomorphic to the initial foliation.

Proof. We only give a short sketch of the construction. Denote the universal cover of F by $\widetilde{F}$, then we can choose a foliation $\widetilde{\mathcal{F}}$ on $\widetilde{F} \times \mathbb{R}^k$ whose leaves are the sections $\widetilde{F} \times \{(x_1, \dots, x_k)\}$ for $x_1, \dots, x_k$ fixed. Next we choose a finite set of generators $\gamma_1, \dots, \gamma_N$ of $\Gamma = \pi_1(F)$, acting on an open neighborhood $U \subset \mathbb{R}^k$. We define M to be the quotient

$$M := \widetilde{F} \times_\Gamma U$$

by the equivalence relation $(p; x_1, \dots, x_k) \sim \big(pg^{-1}; \Phi(g) \cdot (x_1, \dots, x_k)\big)$ for g in the system of generators. □

Let us now come back to the initial situation of a compact manifold N carrying a neat singular foliation $\mathcal{F}$ with a closed component $S \subset \mathrm{Sing}(\mathcal{F})$ of codimension 2. Assume that $\mathcal{F}$ is given by a regular equation β. The secondary foliation $\mathcal{F}_\beta$ is a regular codimension 2 foliation around S, and S is one of its leaves. Hence, $\mathcal{F}_\beta$ determines a holonomy

$$\Phi\colon \pi_1(S) \to \mathrm{Diff}_{\mathrm{germ}}(\mathbb{R}^2, 0) ,$$

and this holonomy encodes the local shape of $\mathcal{F}_\beta$.

Note that the secondary foliation $\mathcal{F}_\beta$ is everywhere tangent to the original one $\mathcal{F}$. Hence the holonomy Φ preserves the 1-dimensional foliation induced by $\mathcal{F}$ on the transverse 2-disk D which we have used in the construction of Φ. In fact, after a moment's thought, we see that the holonomy even preserves the restriction of β to D: This is because we can cover the whole singular leaf S with Kupka charts with coordinates $\{(s, t; \mathbf{x}) = (s, t; x_1, \dots, x_{n-1})\}$ where β takes the form

$$a(s,t)\, ds + b(s,t)\, dt .$$

The coordinate transformations on the overlap of two charts are of the form

$$(s, t; \mathbf{x}) \mapsto \big(\widetilde{s}(s,t), \widetilde{t}(s,t); \widetilde{\mathbf{x}}(s,t,\mathbf{x})\big) ,$$

because they preserve the secondary foliation. The holonomy of a loop can be recovered only from the concatenation of the coordinate transformations of the first two coordinates, but these transformations need to preserve the defining 1-form β.

Proposition I.2.7. *Let $(N, \mathcal{F})$ be a compact manifold with a neat foliation that is given by a regular equation β. The foliation on a neighborhood of a submanifold $S \subset \mathrm{Sing}(\mathcal{F})$ is determined by the restriction $\beta|_D$ to a 2-dimensional disk D that is transverse to S, and by the holonomy*

$$\Phi\colon \pi_1(S) \to \mathrm{Diff}^{\beta}_{\mathrm{germ}}(D, 0)$$

taking values in the germs of diffeomorphisms on D that preserve the 1-form $\beta|_D$, and that map the origin to itself.

Note that we immediately obtain the following easy case.

Corollary I.2.8. *If $S \subset \operatorname{Sing}(\mathcal{F})$ is simply connected, then the neighborhood of S is diffeomorphic to $S \times \mathbb{R}^2$, and the foliation is the product of a foliation on $\mathbb{R}^2$ with S.*

In general though, the foliation might be very complicated, and further results would require us to understand the deformations of the holonomy.

After having studied the shape of the codimension-2 singularities themselves, we are going to describe the neighborhoods of such singularities inside a contact manifold. Our aim will be to slightly perturb them to an easier form corresponding to the models we use in the next chapter, where we study holomorphic disks.

Remember that the holonomy map Φ allows us to reconstruct the singular foliation in a neighborhood of S. For this we used a product between the universal cover of S and a small 2-disk

$$\widetilde{S} \times D^2$$

foliated by the constant sections $\widetilde{S} \times \{z_0\}$. The desired model neighborhood was build by identify pairs $(p, z) \sim \big(pg^{-1}, \Phi_g(z)\big)$ (see the proof of Theorem I.2.6) for elements $g \in \Gamma = \pi_1(S)$.

In our case, D^2 is equiped with a 1-form $\widetilde{\beta}$ that defines a singular codimension-1 foliation $\mathcal{F}_{D^2}$ on D^2. We extend $\mathcal{F}_{D^2}$ to all of $\widetilde{S} \times D^2$ simply by taking the product of $\mathcal{F}_{D^2}$ with $\widetilde{S}$, and then we quotient again by the action of Γ as before. If the holonomy Φ preserves $\mathcal{F}_{D^2}$, the foliation $\widetilde{S} \times \mathcal{F}_{D^2}$ will descend to the quotient. Similarly, since Γ also preserves $\widetilde{\beta}$, the defining 1-form induces a 1-form on $\widetilde{S} \times_\Gamma D^2$ that defines the foliation on the quotient space.

We can now define a contact manifold

$$\left(T^*\widetilde{S} \times D^3, \ker(dz + \beta + \lambda_{\mathrm{can}})\right) ,$$

where λ_{can} is the canonical 1-form on the cotangent bundle $T^*\widetilde{S}$, and where we denote the coordinates on D^3 by (s, t, z). The contact form $\widetilde{\alpha} = dz + \beta + \lambda_{\mathrm{can}}$ induces the given singular foliation on $\widetilde{S} \times D^2$. We let $\Gamma = \pi_1(S)$ act on the cotangent bundle by the canonical diffeomorphisms induced from the action on $\widetilde{S}$, and we let it act on the D^3-factor by keeping the z-coordinate invariant, and using the holonomy on the (x, y)-coordinates.

The quotient of $T^*\widetilde{S} \times D^3$ by Γ is a contact manifold, and it contains the embedding of $\widetilde{S} \times_\Gamma D^2$ with the desired singular foliation. Hence we have found by Theorem I.1.3 a model contact manifold containing the given singular Legendrian foliation.

It is easy to convince oneself that we can modify a submanifold with a singular Legendrian foliation looking in a neighborhood of a codimension-2 singularity like a product foliation by a C^∞-small perturbation into one that looks like the linearization of this singularity.

I.3. Examples

The following example will be the first one that relates Legendrian foliations to Lagrangian submanifolds.

Example I.3.1. Let P be a principal circle bundle over a base manifold B, and suppose that ξ is a contact structure on P that is transverse to the $\mathbb{S}^1$-fibers and invariant under the action. It is well-known that by averaging, we can choose an $\mathbb{S}^1$-invariant contact form α for ξ and that there exists a symplectic form ω on B such that $\pi^*\omega = d\alpha$, where π is the bundle projection $\pi\colon P \to B$. The symplectic form ω represents the image of the Euler class $e(P)$ in $H^2(B, \mathbb{R})$, and hence P cannot be a trivial bundle (see [**BW58**]). The manifold (P, α) is usually called the **pre-quantization of the symplectic manifold** (B, ω) (or the **Boothby-Wang manifold**).

Let L be a Lagrangian submanifold in (B, ω), and let $P_L := \pi^{-1}(L)$ be the fibration over L. Note first that in this situation, we have $\omega|_{TL} = 0$, so that $e(P_L) = e(P)|_L$ will automatically either vanish or be a torsion class. If $e(P_L) = 0$, the fibration P_L will be trivial, and we can find a genuine section $\sigma\colon L \to P_L$, otherwise let k be the order $e(P_L)$, then we find a k-fold multi-section in P_L: By basic computation rules of characteristic classes, we know that the k-fold tensor product $P_L^k = P_L \otimes \dots \otimes P_L$ has Euler class $e\big(P_L^k\big) = k\, e(P_L) = 0$, and hence it follows that P_L^k is a trivial bundle over B. We find a global section σ_0 of P_L^k, which allows us to write fiberwise equations $\nu^k = \nu \otimes \dots \otimes \nu = \sigma_0(p)$ where $\nu \in P_L$ is an element in the fiber over $p \in B$. In a bundle chart, one sees easily that there are k solutions to this equation and that $\mathbb{Z}_k \subset \mathbb{S}^1$ acts on the possible solutions. In fact we obtain as solution space a smooth submanifold of P_L that intersects every fiber k-times transversely, and this submanifold is hence a k-fold covering of the base manifold L.

In both cases, we have $\big(\alpha \wedge d\alpha\big)\big|_{TP_L} = \big(\alpha \wedge \pi^*\omega\big)\big|_{TP_L} \equiv 0$, so that ξ induces a Legendrian foliation $\mathcal{F}$ on P_L. Furthermore, since the infinitesimal generator X_φ of the circle action satisfies $\alpha(X_\varphi) \equiv 1$, it follows that $\mathcal{F}$ is everywhere regular. Assume for simplicity that the fibration is trivial, so that we can identify P_L with $\mathbb{S}^1 \times L$. Then we can write $\alpha|_{TP_L}$ as

$$d\varphi + \beta \ ,$$

where φ is the coordinate on the circle and β is a closed 1-form on L. The leaves of the foliation $\mathcal{F}$ are local sections, but they need not be global ones, and usually these leaves will not even be compact. Instead the proper way to think of them is as the horizontal lift of the flat connection 1-form $\alpha|_{TP_L}$.

Choose any loop $\gamma \subset L$ based at a point $p_0 \in L$. To better understand the Legendrian leaves, we lift $\gamma(t)$ to a path $\widetilde{\gamma}(t) = \big(e^{i\varphi(t)}, \gamma(t)\big)$ in $P_L \cong \mathbb{S}^1 \times L$ that is always tangent to a leaf of $\mathcal{F}$, so that

$$\widetilde{\gamma}'(t) = \big(-\beta(\gamma'(t)), \gamma'(t)\big) \ .$$

In particular start and end point of $\widetilde{\gamma}$ are related by the monodromy

$$C_\gamma := -\int_\gamma \beta \ ,$$

that means, if $\widetilde{\gamma}$ starts at $\big(e^{i\varphi_0}, p_0\big) \in \mathbb{S}^1 \times L$, then its end point will be $\big(e^{i(\varphi_0 + C_\gamma)}, p_0\big)$.

Note that since the connection is flat, that means, β is closed, and two homologous paths from p_0 to p_1 in L will lift the end point in the same way. Thus we have a well-defined group homomorphism

$$H_1(L, \mathbb{Z}) \to \mathbb{S}^1 \ .$$

The leaves of the Legendrian foliation will only be compact, if the image of this map is discrete. Later, in particular in Section IV.1, we use a similar construction to work with holomorphic curves. But there it will be important that the Legendrian foliation over P_L is given by a fibration over $\mathbb{S}^1$. We will briefly show that by slightly perturbing L this can always be achieved in the considered example.

The embedding of $H^1(L, \mathbb{Q}) \to H^1(L, \mathbb{R})$ is dense, and hence we find a closed 1-form β' on L, arbitrarily close to β, that represents a rational homology class. Let $\delta := \beta' - \beta$, and extend δ to a neighborhood of L in B by pulling it back to the normal bundle of L in B and multiplying it with a bump function ρ with small support around the 0-section. If δ is sufficiently small, then the 1-form $\alpha' = \alpha + \rho\delta$ defines a contact structure that is isotopic to the one given by α, and P_L will still carry a Legendrian foliation, but now its foliation is given by $d\varphi + \beta'$, and all the leaves are compact submanifolds. In fact, since $H_1(L,\mathbb{Z})$ is finitely generated, we find a number $c \in \mathbb{Q}$ such that all possible values of the monodromy are a multiple of c, and we obtain a regular Legendrian foliation on P_L, with compact leaves.

The second example gives a Legendrian foliation with a codimension-1 singular set.

Example I.3.2. Let L be any smooth $(n+1)$-dimensional manifold with a Riemannian metric g. It is well-known that the unit cotangent bundle $\mathbb{S}\big(T^*L\big)$ carries a contact structure given as the kernel of the canonical 1-form λ_{can}. The fibers of this bundle are Legendrian spheres, hence if we choose any smooth regular loop $\gamma\colon \mathbb{S}^1 \to L$, and if we study the fibers lying over this path, we obtain the submanifold $N_\gamma := \pi^{-1}(\gamma)$ that has a singular Legendrian foliation.

In fact, we can naturally decompose $\mathbb{S}\big(T^*L\big)\big|_\gamma$ into the two subsets U_+ and U_- defined as

$$U_\pm = \big\{\nu \in N_\gamma \;\big|\; \pm\nu(\gamma') \geq 0\big\} \;.$$

These sets correspond in each fiber of N_γ to opposite hemispheres. We claim that the singular set of the Legendrian foliation on N_γ is $U_+ \cap U_-$, and that the regular leaves correspond to the intersection of each fiber of N_γ with the interior of U_+ and U_-. In particular, if N_γ is orientable, we obtain that it can be written as

$$\big(\mathbb{S}^1 \times \mathbb{S}^n, x_0\, d\varphi\big) \;,$$

where φ is the coordinate on $\mathbb{S}^1$, and $(x_0,\dots,x_n)$ are the coordinates on $\mathbb{S}^n$.

Remember that the canonical 1-form λ_{can} is defined like this: Let $v \in T_\nu(T^*L)$ be a vector at a base point $\nu \in T^*L$. We can project v with the differential of the map $\pi\colon T^*L \to L$ into TL, and then evaluate it in ν, that means,

$$\lambda_{\mathrm{can}}(v) := \nu\big(D\pi \cdot v\big) \;.$$

It follows that the restriction of λ_{can} to N_γ vanishes only at points that lie in $U_+ \cap U_-$.

I.4. Legendrian open books

Even though we discussed Legendrian foliations quite generally, we will only be interested in two special types: *Legendrian open books* introduced in [NR11] and *bordered Legendrian open books* introduced in [MNW13]. Both objects were defined with the aim of generalizing results from 3-dimensional contact topology that hold for the 2-sphere with standard foliation and the overtwisted disk respectively [**BG83, Gro85, Eli90a, Hof93**].

Definition. Let N be a closed manifold. An **open book** on N is a pair (B,ϑ) where:

- The **binding** B is a nonempty codimension 2 submanifold in the interior of N with trivial normal bundle.
- $\vartheta\colon N\setminus B \to \mathbb{S}^1$ is a fibration, which coincides in a neighborhood $B\times\mathbb{D}^2$ of $B = B\times\{0\}$ with the normal angular coordinate.

Definition. If N is a compact manifold with nonempty boundary, then a **relative open book** on N is a pair (B,ϑ) where:

- The **binding** B is a nonempty codimension 2 submanifold in the interior of N with trivial normal bundle.

- $\vartheta\colon N \setminus B \to \mathbb{S}^1$ is a fibration whose fibers are transverse to ∂N, and which coincides in a neighborhood $B \times \mathbb{D}^2$ of $B = B \times \{0\}$ with the normal angular coordinate.

We are interested in studying contact manifolds with submanifolds with a Legendrian foliation that either define an open book or a relative open book.

Definition. A closed submanifold N carrying a Legendrian foliation $\mathcal{F}$ in a contact manifold (M, ξ) is a **Legendrian open book** (abbreviated **Lob**), if N admits an open book (B, ϑ), whose fibers are the regular leaves of the Legendrian foliation and whose binding is the singular set of $\mathcal{F}$.

Definition. A compact submanifold N with boundary in a contact manifold (M, ξ) is called a **bordered Legendrian open book** (abbreviated **bLob**), if N carries a Legendrian foliation $\mathcal{F}$ and if it has a relative open book (B, ϑ) such that:

(i) the regular leaves of $\mathcal{F}$ lie in the fibers of ϑ,
(ii) $\operatorname{Sing}(\mathcal{F}) = \partial N \cup B$.

A contact manifold that contains a bLob is called ***PS*-overtwisted**.

Example I.4.1. (i) Every Lob in a contact 3-manifold is diffeomorphic to a 2-sphere with the binding consisting of the north and south poles, and the fibers being the longitudes. This special type of Lob has been studied extensively and has given several important applications, see for example [**BG83, Gro85, Eli90a, Hof93**]. It is easy to find such Lobs locally, for example, the unit sphere in $\mathbb{R}^3$ with the standard contact structure $\xi = \ker\bigl(dz + x\,dy - y\,dx\bigr)$.

(ii) A bLob in a 3-dimensional contact manifold is an overtwisted disk (with singular boundary).

(iii) In higher dimensions, the plastikstufe had been introduced as a filling obstruction [**Nie06**], but note that a plastikstufe is just a specific bLob that is diffeomorphic to $\mathbb{D}^2 \times B$, where the fibration is the one of an overtwisted disk (with singular boundary) on the $\mathbb{D}^2$-factor, extended by a product with a closed manifold B, see Fig. 4b. Topologically a bLob might be *much* more general than the initial definition of the plastikstufe. For example, a plastikstufe in dimension 5 is always diffeomorphic to a solid torus $\mathbb{D}^2 \times \mathbb{S}^1$ while a 3-manifold admits a relative open book if and only if its boundary is a nonempty union of tori.

The importance of the previous definitions lie in the following two theorems, which will be proved in Chapter III.

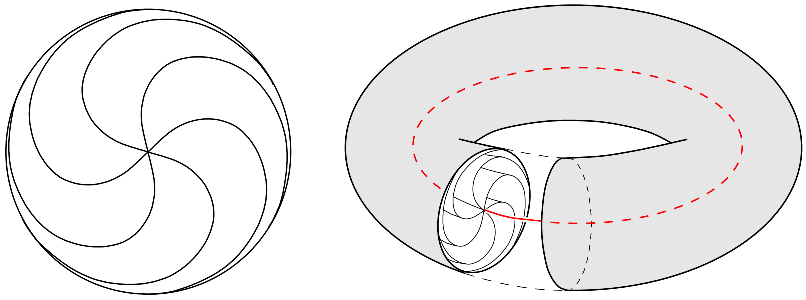
Theorem A. *Let (M, ξ) be a contact manifold that contains a bLob N, then M does not admit any semi-positive weak symplectic filling (W, ω) for which $\omega|_{TN}$ is exact.*

The statement above is a generalization of the analogous statement found first for the overtwisted disk in [**Gro85, Eli90a**].

Remark I.4.2. (i) In dimension 4 and 6, every symplectic manifold is automatically semi-positive.

(ii) A bLob always obstructs (semi-positive) *strong* symplectic filling, because in that case the restriction of ω to N is exact.

(iii) The condition that the restriction of the symplectic form ω should be exact is trivially satisfied in dimension 5 for the plastikstufes defined in [**Nie06**], which were all diffeomorphic to $\mathbb{S}^1 \times \mathbb{D}^2$. In general however this condition could fail, and we



(A) The overtwisted disk. (B) A plastikstufe in dimension 5.

FIGURE 4. The easiest two examples of bLobs are the overtwisted disk, and a plastikstufe in a 5-dimensional contact manifold. In both of the pictures above, the objects do not quite correspond to the definition of the bLob, because the boundary is a closed Legendrian leaf. We can imagine that these models have been obtain by deforming the embedding in a neighborhood of the boundary using Proposition I.2.5.

believe that this could provide a hint as to varying degrees of filling obstructions or overtwistedness. Though it is unknown whether there is a *unique* natural notion of overtwistedness beyond dimension 3, or whether the different definitions known thus far are nonequivalent, it would be interesting to speculate that a manifold can only be overtwisted in some "universal" sense if the bLob (or a similar object) can be *embedded into a ball* within the contact manifold. In this way the cohomological condition is satisfied automatically, thus defining an obstruction to weak fillings due to the above theorem. We will refer to any bLob that lies inside a ball in the contact manifold as a **small bLob** .

Theorem B. *Let (M, ξ) be a contact manifold of dimension $(2n+1)$ that contains a Lob N. If M has a weak symplectic filling (W, ω) that is symplectically aspherical, and for which $\omega|_{TN}$ is exact, then it follows that N represents a trivial class in $H_{n+1}(W, \mathbb{Z}_2)$. If the first and second Stiefel-Whitney classes $w_1(N)$ and $w_2(N)$ vanish, then we obtain that N must be a trivial class in $H_{n+1}(W, \mathbb{Z})$.*

Remark I.4.3. The methods from [**Hof93**] can be generalized for Theorem A, see [**AH09**], and for Theorem B, see [**NR11**], to find closed contractible Reeb orbits.

I.5. Examples of bLobs

The most important result in this section is the construction of PS-overtwisted manifolds in higher dimensions. Later, we will show a few other methods in Chapter IV. The first such manifolds were obtained by Presas in [**Pre07**], and modifying his examples it was soon possible to show that every contact structure can be converted into one that is PS-overtwisted [**KN07**].

This result was reproved and generalized in [**EP09**], where it was shown that we may modify a contact structure into one that is PS-overtwisted without changing the homotopy class of the underlying almost contact structure.

A very nice explicit construction in dimension 5 that is similar to the 3-dimensional Lutz twist was given in [**Mor09**]. In [**MNW13**] the construction was extended and produced examples that are not PS-overtwisted but share many properties with 3-manifold that have positive Giroux torsion.

The following unpublished construction is due to Francisco Presas who explained it to me during a stay in Madrid. It is probably the easiest way to produce a closed PS-overtwisted manifolds of arbitrary dimensions.

Theorem I.5.1 (Fran Presas)**.** *Let (M_1, ξ_1) and (M_2, ξ_2) be contact manifolds of dimension $2n+1$ that both contain a PS-overtwisted submanifold (N, ξ_N) of codimension 2 with trivial normal bundle. The **fiber sum** of M_1 and M_2 along N is a PS-overtwisted $(2n+1)$-manifold.*

Proof. Let α_N be a contact form for ξ_N. The manifold N has neighborhoods $U_1 \subset M_1$ and $U_2 \subset M_2$ that are contactomorphic to

$$\mathbb{D}^2_{\sqrt{\varepsilon}} \times N$$

with contact structure given as the kernel of the 1-form $\alpha_N + r^2\, d\varphi$ [**Gei08**, Theorem 2.5.15].

We can remove the submanifold $\{0\} \times N$ in this model, and do a reparametrization of the r-coordinate by $s = r^2$ to bring the neighborhood into the form

$$(0, \varepsilon) \times \mathbb{S}^1 \times N$$

with contact form $\alpha_N + s\, d\varphi$. We extend $M_1 \setminus N$ and $M_2 \setminus N$ by attaching the negative s-direction to the model collar, so that we obtain a neighborhood

$$\big((-\varepsilon, \varepsilon) \times \mathbb{S}^1 \times N, \quad \alpha_N + s\, d\varphi\big) .$$

Denote these extended manifolds by $(\widetilde{M}_1, \widetilde{\xi}_1)$ and $(\widetilde{M}_2, \widetilde{\xi}_2)$, and glue them together using the contactomorphism

$$\begin{aligned} (-\varepsilon, \varepsilon) \times \mathbb{S}^1 \times N &\to (-\varepsilon, \varepsilon) \times \mathbb{S}^1 \times N \\ (s, \varphi, p) &\mapsto (-s, -\varphi, p) . \end{aligned}$$

We call the contact manifold (M', ξ') that we have obtained this way the **fiber sum** of M_1 and M_2 along N.

If S is a bLob in N, then it is easy to see that $\{0\} \times \mathbb{S}^1 \times S$ is a bLob in the model neighborhood $(-\varepsilon, \varepsilon) \times \mathbb{S}^1 \times N$. □

With this proposition, we can now construct non-fillable contact manifolds of arbitrary dimension. Every oriented 3-manifold admits an overtwisted contact structure in every homotopy class of almost contact structures.

Let (M, ξ) be a compact manifold, let α_M be a contact form for ξ. A fundamental result due to Emmanuel Giroux gives the existence of a compatible open book decomposition for M [**Gir02**]. Using this open book decomposition, it is easy to find functions $f, g\colon M \to \mathbb{R}$ such that

$$\big(M \times \mathbb{T}^2, \ker(\alpha_M + f\, dx + g\, dy)\big)$$

is a contact structure, see [**Bou02**], where (x, y) denotes the coordinates on the 2-torus. The fibers $M \times \{z\}$ are contact submanifold with trivial normal bundle, so that in particular if (M, ξ) is PS-overtwisted, we can apply the construction above to glue two copies of $M \times \mathbb{T}^2$

along a fiber $M \times \{z\}$. This way, we obtain a PS-overtwisted contact structure on $M \times \Sigma_2$, where Σ_2 is a genus 2 surface.

Using this process inductively, we find closed PS-overtwisted contact manifolds of any dimension ≥ 3.

Note that in dimension 5, we can find more easily examples to which we can apply Theorem I.5.1, so that it is not necessary to rely on [**Bou02**]. Let (M, ξ) be an overtwisted 3-manifold with contact form α. After normalizing α with respect to a Riemannian metric, it describes a section

$$\sigma_\alpha \colon M \to \mathbb{S}(T^*M)$$

in the unit cotangent bundle. It satisfies the fundamental relation $\sigma_\alpha^* \lambda_{\mathrm{can}} = \alpha$, hence it gives a contact embedding of (M, ξ) into $\big(\mathbb{S}(T^*M), \ker \lambda_{\mathrm{can}}\big)$.

For trivial normal bundle, this allows us to glue with Theorem I.5.1 two copies together and obtain a PS-overtwisted 5-manifold.

I.6. Appendix: Technical lemmas

The aim of this appendix is to state several lemmas which were needed in previous sections, but which are relatively technical so that we decided to exclude them from the main text for the sake of readability. Some of the results are well known, and we only present them here for completeness and as reference.

Throughout this text we often use the following standard result (see for example [**Mil63**, Lemma 2.1]).

Proposition I.6.1. *Let U be an open convex neighborhood of $\mathbf{0}$ in $\mathbb{R}^n$. For every smooth function $f \colon U \to \mathbb{R}$ we find smooth functions $g_1, \dots, g_n \colon U \to \mathbb{R}$ with*

$$g_j(\mathbf{0}) = \frac{\partial f}{\partial x_j}(\mathbf{0})$$

such that f may be written in the form

$$f(x_1, \dots, x_n) = f(\mathbf{0}) + \sum_{j=1}^{n} x_j \, g_j(x_1, \dots, x_n) \; .$$

Corollary I.6.2. *Let f be as in the proposition above. Then we find a neighborhood V of $\mathbf{0}$ in U, and a smooth function $f_1 \colon V \to \mathbb{R}$ with*

$$f_1(0, x_2, \dots, x_n) = \frac{\partial f}{\partial x_1}(0, x_2, \dots, x_n)$$

such that f may be written in the form

$$f(x_1, \dots, x_n) = f(0, x_2, \dots, x_n) + x_1 \, f_1(x_1, \dots, x_n) \; .$$

Proof. Simply interpret f as a family of functions parametrized by $(x_2, \dots, x_n)$ depending only on x_1. $\square$

The following lemma has been taken from [**Mas08**] and is due to Giroux. We have used it as a preliminary step in the proof of Theorem I.1.3.

Lemma I.6.3. *Let N be a compact manifold that may have boundary, and let $\mathcal{F}$ be a neat singular foliation on N given by the regular equation β_1. If β_2 is any other regular equation of $\mathcal{F}$, then there exists a unique smooth and nowhere vanishing function $F \colon N \to \mathbb{R}$ such that $\beta_2 = F \, \beta_1$.*

PROOF. Neither of the two 1-forms β_1 and β_2 vanish at regular points of the foliation, and since both have equal kernel, it follows that β_1 is a non-zero multiple of β_2 over $N \setminus \operatorname{Sing}(\mathcal{F})$. This shows the existence of the desired function F over the regular points, and we are only left with studying the points $p \in \operatorname{Sing}(\mathcal{F})$, where $\beta_1 = \beta_2 = 0$, and prove that F extends smoothly to a non-vanishing function.

If p is a singular point, we work in a Kupka chart around p provided by Theorem I.1.2 with coordinates $\big\{(s, t, x_1, \dots, x_n)\big\}$ such that β_1 is given by

$$a(s,t)\, ds + b(s,t)\, dt \;,$$

and β_2 is of the form

$$f(s,t,x_1,\dots,x_n)\, ds + g(s,t,x_1,\dots,x_n)\, dt \;.$$

If p lies in the interior of N, then the Kupka chart will be an open set, if p lies on ∂N, then the set $\{s = 0\}$ corresponds to the boundary of the chart, and the functions b and g both vanish on $\{s = 0\}$.

Since the two forms have equal kernels, it follows that $\beta_1 \wedge \beta_2 \equiv 0$, which means that

$$a(s,t)\, g(s,t,x_1,\dots,x_n) - b(s,t)\, f(s,t,x_1,\dots,x_n) \equiv 0 \;. \tag{I.6.1}$$

If p is a point in the interior of N, and since $d\beta_1$ does not vanish in p, we may assume either directly or by permuting the s- and the t-coordinates that $\frac{\partial b}{\partial s}(0,0) \neq 0$. Then we can apply the implicit function theorem to find a coordinate system $(S, T, y_1, \dots, y_n)$ in which p corresponds to the origin, and such that

$$b(S,T,y_1,\dots,y_n) = S \;.$$

Equation (I.6.1) then simplifies to

$$a(S,T,y_1,\dots,y_n)\, g(S,T,y_1,\dots,y_n) - S\, f(S,T,y_1,\dots,y_n) \equiv 0 \;. \tag{I.6.2}$$

The function g has to vanish along the whole subset $\{S = 0\}$: If $(0, T, y_1, \dots, y_n)$ is a singular point of $\mathcal{F}$, this is true by definition, otherwise we have $a(0, T, y_1, \dots, y_n) \neq 0$. Using Corollary I.6.2, we can find a smooth function $g_S(S, T, y_1, \dots, y_n)$ so that we can write g as

$$g(S,T,y_1,\dots,y_n) = S\, g_S(S,T,y_1,\dots,y_n) \;.$$

This way, we may factor S out of equation (I.6.2) and we find

$$a(S,T,y_1,\dots,y_n)\, g_S(S,T,y_1,\dots,y_n) = f(S,T,y_1,\dots,y_n)$$

everywhere. Setting $F = g_S$, we obtain the desired result, because $g = S\, g_S = g_S\, b$, and $f = g_S\, a$, so that $\beta_2 = F\, \beta_1$ as we wanted to show.

If p lies on ∂N, slightly more care needs to be taken, because the s- and t-coordinates are not equivalent. If $\frac{\partial b}{\partial s}(0,0) \neq 0$, then the argument is similar to the previous one: Using the fact that b and g are 0 along the boundary $\{s = 0\}$, we find by Corollary I.6.2 smooth functions b_s and g_s such that

$$b(s,t) = s\, b_s(s,t) \quad \text{and} \quad g(s,t,x_1,\dots,x_n) = s\, g_s(s,t,x_1,\dots,x_n) \;.$$

Because $b_s(0,0) = \frac{\partial b}{\partial s}(0,0)$, it follows that $b_s(0,0)$ does not vanish, and we may set $F(s,t,x_1,\dots,x_n) := g_s(s,t,x_1,\dots,x_n)/b_s(s,t)$. After factoring out an s-factor, equation (I.6.1) reduces to

$$f(s,t,x_1,\dots,x_n) = F(s,t,x_1,\dots,x_n)\, a(s,t) \;,$$

and so we obtain $\beta_2 = F\, \beta_1$.

We still need to understand the case when $p \in \operatorname{Sing}(\mathcal{F})$ lies on the boundary of N, but $\frac{\partial b}{\partial s}(0,0) = 0$ in the Kupka chart. From $d\beta_1 \neq 0$, we see that $\frac{\partial a}{\partial t}(0,0) \neq 0$, so that the subset $\{a = 0\}$ is a smooth hypersurface transverse to ∂N. We can find a chart for the (s,t)-plane with coordinates (S,T), where $\{S = 0\}$ corresponds to ∂N and $\{T = 0\}$ corresponds to the subset $\{a = 0\}$. The rest of the proof is then analogous to the previous case: Using Corollary I.6.2, we can write

$$a(S,T) = T\, a_T(S,T)$$

with a non-vanishing smooth function $a_T(S,T)$. In equation (I.6.1)

$$T\, a_T(s,t)\, g(S,T,x_1,\dots,x_n) - b(S,T)\, f(S,T,x_1,\dots,x_n) \equiv 0 \;,$$

we see that f needs to vanish along $\{T = 0\}$; either because the considered point lies in $\operatorname{Sing}(\mathcal{F})$ or because otherwise b does not vanish. Applying Corollary I.6.2 to f, we conclude as above. □

The following lemma is used in the proofs of Theorem I.1.3 and Theorem I.1.5. It gives sufficient (but rather strong) conditions under which interpolations between two contact forms lie themselves in the space of contact forms.

Lemma I.6.4. *Let M be an oriented $(2n+1)$-manifold that contains an $(n+1)$-dimensional compact submanifold N (possibly with $\partial N \neq \emptyset$). Assume that*

- *ξ_N is a cooriented codimension-1 subbundle of $TM|_N$,*
- *α_N is a section of $T^*M|_N$ with $\xi_N = \ker \alpha_N$,*
- *ξ_N induces a (possibly singular) foliation $\mathcal{F} = \xi_N \cap TN$ on N,*
- *J_N is a complex structure on ξ_N.*

Denote by $\mathcal{A}(\alpha_N, J_N)$ the space of germs of 1-forms α defined on a neighborhood of N that satisfy the following properties:

(i) *The restriction of α to $TM|_N$ is equal to α_N.*
(ii) *Let $\mathcal{F}_\alpha$ be the secondary foliation (see page 13) on N associated to α. The restriction of J_N to $\mathcal{F}_\alpha \oplus J_N\, \mathcal{F}_\alpha$ is tamed by $d\alpha$.*
(iii) *α is a contact form compatible with the orientation on M.*

Then it follows that $\mathcal{A}(\alpha_N, J_N)$ is either empty or a pointwise convex set, that means, if α, α' are any two 1-forms in $\mathcal{A}(\alpha_N, J_N)$, and if we choose a smooth function $\rho\colon M \to [0,1]$, then the 1-form

$$\alpha_\rho := \rho\, \alpha + (1-\rho)\, \alpha'$$

will also lie in $\mathcal{A}(\alpha_N, J_N)$.

Proof. Choose two 1-forms α and α' in $\mathcal{A}(\alpha_N, J_N)$, and a smooth function $\rho\colon M \to [0,1]$, and let α_ρ be the corresponding interpolated form. It is directly clear that property (i) holds for the whole family α_ρ, and since $d\rho \wedge \bigl(\alpha' - \alpha\bigr)$ vanishes along N, it follows in particular that the differential $d\alpha_\rho$ simplifies along this submanifold to the interpolation between $d\alpha$ and $d\alpha'$

$$(1-\rho)\, d\alpha + \rho\, d\alpha' \;.$$

Both properties (ii) and (iii) only depend pointwise on α_ρ and $d\alpha_\rho$, hence we may from now on assume without loss of generality that $\rho \equiv C$ for a constant number C in $[0,1]$.

Property (ii) is now also obvious, because the secondary foliation $\mathcal{F}_\alpha$ does not depend on a 1-form itself, but only on its restriction to N. Since the 1-form α_C agrees along N with

α_N, it follows that the secondary foliation $\mathcal{F}_{\alpha_C}$ is equal to the one of α_N. If (ii) holds both for α and α', then we can easily check that

$$d\alpha_C(X, J_N X) = (1-C)\, d\alpha(X, J_N X) + C\, d\alpha'(X, J_N X) > 0$$

for every non-vanishing vector $X \in \mathcal{F}_\alpha \oplus J_N\, \mathcal{F}_\alpha$.

We are left with proving that α_C is a contact form. Assume first that $p \in N$ is a point at which $d\alpha_N|_{TN}$ vanishes. Then p cannot lie in $\operatorname{Sing}(\mathcal{F})$, because otherwise α could not be a contact form. The secondary foliation $\mathcal{F}_\alpha$ at p is equal to $\mathcal{F}$, and $\mathcal{F}_\alpha \cap J_N \mathcal{F}_\alpha = \{0\}$, because the restriction $d\alpha|_{T_pN}$ vanishes, and $d\alpha$ tames J_N on $\mathcal{F}_\alpha \oplus J_N\, \mathcal{F}_\alpha$. For dimensional reasons, we then obtain that $\mathcal{F}_\alpha \oplus J_N\, \mathcal{F}_\alpha = \xi_N$ at p, and we easily verify that

$$d\alpha_C(X_p, J_N X_p) = (1-C)\, d\alpha(X_p, J_N X_p) + C\, d\alpha'(X_p, J_N X_p) > 0$$

for all non-vanishing vectors $X_p \in \xi_N$, and hence α_C satisfies the contact condition at p.

Let $p \in N$ be now a point at which $d\alpha_N|_{TN} \neq 0$. We find a chart U of M around p with coordinates $(s, t, x_1, \dots, x_{n-1}, y_1, \dots, y_{n-1}, z)$ such that $N \cap U$ corresponds to the set $\big\{y_1 = \dots = y_{n-1} = z = 0\big\}$, and we may assume by Theorem I.1.2 that α_N restricts on $N \cap U$ to the Kupka form

$$\alpha_N|_{T(N\cap U)} = a(s,t)\, ds + b(s,t)\, dt\ .$$

The secondary foliation $\mathcal{F}_\alpha$ is spanned by $\{\partial_{x_1}, \dots, \partial_{x_{n-1}}\}$, and we may additionally suppose that $\partial_{y_j} = J_N\, \partial_{x_j}$ along $N \cap U$. Let ∂_z be the Reeb field R_α of α, which is along $N \cap U$ transverse to $TN \oplus J_N\, \mathcal{F}_\alpha$ as has already been shown at the beginning of the proof of Lemma I.1.4. The family α_C can be written on U as

$$\alpha_C = a_C\, ds + b_C\, dt + u_C\, dz + \sum_j f_j^C\, dx_j + \sum_j g_j^C\, dy_j$$

with coefficient functions $a_C, b_C, u_C, f_j^C, g_j^C \colon U \to \mathbb{R}$, and because α_C is equal to α_N along $N \cap U$, we have $f_j^C\Big|_{N\cap U} = g_j^C\Big|_{N\cap U} \equiv 0$ and $u_C|_{N\cap U} \equiv 1$, and consequently none of the coefficient functions depend along N on the x_j-coordinates, so that the only dx_j-terms in $d\alpha_C|_{N\cap U}$ will come from $df_j^C \wedge dx_j$. Now we will check the contact condition $\alpha_C \wedge d\alpha_C^n > 0$, which by the previous remark simplifies significantly along $N \cap U$. We obtain

$$\begin{aligned}\big(\alpha_C \wedge d\alpha_C^n\big)\big|_{N\cap U} = n!\, \alpha_N \wedge \Big(& da_C \wedge ds + db_C \wedge dt + du_C \wedge dz + \\ &+ \sum_j dg_j^C \wedge dy_j\Big) \wedge df_1^C \wedge dx_1 \wedge \dots \wedge df_{n-1}^C \wedge dx_{n-1}\ .\end{aligned}$$

On the other hand, we also know that the only contributions with a ds- or dt-term come from $a_C\, ds + b_C\, dt$ and its exterior derivative, so that, we may further simplify

$$\begin{aligned}\big(\alpha_C \wedge d\alpha_C^n\big)\big|_{N\cap U} &= n!\, \big(a\, ds + b\, dt + dz\big) \wedge \\ &\qquad \wedge \big(da_C \wedge ds + db_C \wedge dt\big) \wedge df_1^C \wedge dx_1 \wedge \dots \wedge df_{n-1}^C \wedge dx_{n-1} \\ &= n!\, \Big(b\, \frac{\partial a_C}{\partial z} - a\, \frac{\partial b_C}{\partial z} + \frac{\partial b}{\partial s} - \frac{\partial a}{\partial t}\Big)\, ds \wedge dt \wedge dz \wedge \\ &\qquad \wedge df_1^C \wedge dx_1 \wedge \dots \wedge df_{n-1}^C \wedge dx_{n-1}\ .\end{aligned}$$

By our assumptions, it is clear that the restriction of the 2-form

$$\Omega_C := \sum_{j=1}^{n-1} df_j^C \wedge dx_j$$

to $\mathcal{F}_\alpha \oplus J_N \mathcal{F}_\alpha = \operatorname{span}\langle \partial_{x_1}, \dots, \partial_{x_{n-1}}\rangle \oplus \operatorname{span}\langle \partial_{y_1}, \dots, \partial_{y_{n-1}}\rangle$ tames J_N and is hence also symplectic, so that the sign of $\alpha_C \wedge d\alpha_C^n$ depends only on the sign of the leading factor

$$b\,\frac{\partial a_C}{\partial z} - a\,\frac{\partial b_C}{\partial z} + \frac{\partial b}{\partial s} - \frac{\partial a}{\partial t}\ . \tag{I.6.3}$$

The functions a_C and b_C are the linear interpolation between the corresponding coefficient functions of α and α'. It follows that (I.6.3) is the linear interpolation between the corresponding factor of α and the one of α'. Since both α and α' are contact forms that induce the same orientation, both factors need to be positive, showing that α_C is a contact form. □

Lemma I.6.5 (Linearizated normal form for singularities)**.** *Let $\mathcal{F}$ be a singular foliation given by a regular equation β on an open subset $U \subset \mathbb{R}^2$. Assume that $(0,0)$ lies in U and that it is a singular point of $\mathcal{F}$. A linear diffeomorphism of $\mathbb{R}^2$, allows us to write β in one of the three following forms*

$$\beta = s\,dt - C_1 t\,ds + \mathcal{O}^2(s,t)\ ,$$
$$\beta = (s + \varepsilon t)\,dt - t\,ds + \mathcal{O}^2(s,t)\ ,$$

or

$$\begin{aligned}\beta &= (s - C_2 t)\,ds + (C_2 s + t)\,dt + \mathcal{O}^2(s,t)\\ &= C_2 r^2\,d\varphi + \frac{1}{2}\,d(r^2) + \mathcal{O}^2(r,\varphi)\end{aligned}$$

with real constants $\varepsilon, C_1, C_2 \in \mathbb{R}$ such that $C_1 \in (-1,1]$ and $C_2 \neq 0$, and an $\varepsilon \neq 0$ that can be chosen arbitrarily small. The term $\mathcal{O}^2(s,t)$ stands for 1-forms with coefficient functions of order 2, that means, smooth functions of type $s^2 f(s,t) + stg(s,t) + t^2 h(s,t)$.

Proof. Write β first as

$$\beta = a(x,y)\,dx + b(x,y)\,dy$$

with smooth functions a and b that vanish at the origin, and use Proposition I.6.1 to bring β into the form

$$\bigl(xa_x(x,y) + ya_y(x,y)\bigr)\,dx + \bigl(xb_x(x,y) + yb_y(x,y)\bigr)\,dy$$

with smooth function a_x, a_y, b_x, and b_y. By assumption β is a regular equation, hence

$$d\beta_{(0,0)} = \bigl(b_x(0,0) - a_y(0,0)\bigr)\,dx \wedge dy \neq 0\ .$$

We obtain the desired model by applying a suitable linear coordinate transformation to U. Define for two linearly independent vectors $v = (v_1, v_2)$ and $w = (w_1, w_2) \in \mathbb{R}^2$ a map

$$\Phi\colon (s,t) \mapsto (x,y) = s\,v + t\,w\ .$$

The pull-back of β by Φ yields

$$\Phi^*\beta = \bigl(s\widetilde{a}_s(s,t) + t\widetilde{a}_t(s,t)\bigr)\,ds + \bigl(s\widetilde{b}_s(s,t) + t\widetilde{b}_t(s,t)\bigr)\,dt$$

with coefficient functions

$$\begin{aligned}
\widetilde{a}_s(s,t) &= v_1\,(v_1 a_x + v_2 a_y) + v_2\,(v_1 b_x + v_2 b_y)\;,\\
\widetilde{a}_t(s,t) &= v_1\,(w_1 a_x + w_2 a_y) + v_2\,(w_1 b_x + w_2 b_y)\;,\\
\widetilde{b}_s(s,t) &= w_1\,(v_1 a_x + v_2 a_y) + w_2\,(v_1 b_x + v_2 b_y)\;,\ \text{and}\\
\widetilde{b}_t(s,t) &= w_1\,(w_1 a_x + w_2 a_y) + w_2\,(w_1 b_x + w_2 b_y)\;.
\end{aligned}$$

A compact way to express this transformation behavior is to arrange the coefficients in the following matrix

$$B = \begin{pmatrix} b_x(0,0) & b_y(0,0) \\ -a_x(0,0) & -a_y(0,0) \end{pmatrix}\;,$$

whose trace $\operatorname{tr} B = b_x(0,0) - a_y(0,0)$ is non-zero. If we then set

$$D\Phi = \begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \end{pmatrix}\;,$$

and if we suppose that $D\Phi$ is invertible, then we can easily check that the coefficients of $\Phi^*\beta$ fit into the matrix

$$\widetilde{B} = \begin{pmatrix} \widetilde{b}_s(0,0) & \widetilde{b}_t(0,0) \\ -\widetilde{a}_s(0,0) & -\widetilde{a}_t(0,0) \end{pmatrix} = (\det D\Phi)\cdot (D\Phi)^{-1}\cdot B\cdot D\Phi\;.$$

That means that the coefficients of β transform up to scaling like a matrix under conjugation.

We will now choose suitable vectors v, w to bring B and hence $\Phi^*\beta$ into the desired normal forms at the origin. There are three possible cases for the eigenvalues of B: If B has at least one purely real eigenvalue, the Jordan normal form of B will either be

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \qquad \text{or} \qquad \begin{pmatrix} \lambda_3 & 1 \\ 0 & \lambda_3 \end{pmatrix}$$

with $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ and $\lambda_1 + \lambda_2 \neq 0$ and $\lambda_3 \neq 0$. If the eigenvalues of B are not purely real, then they must be conjugate complex numbers $\lambda_4, \bar{\lambda}_4 \in \mathbb{C}\setminus\mathbb{R}$ and B will be diagonalizable over $\mathbb{C}$. We denote the eigenvectors in this case by v_λ and $v_{\bar\lambda}$ with $v_{\bar\lambda} = \bar{v}_\lambda$.

In the first case, where B was diagonizable over the reals, let λ_1 be the eigenvalue for which $|\lambda_1| \geq |\lambda_2|$, and choose for v the eigenvector of λ_1, and for w the eigenvector of λ_2. Additionally suppose that the vectors have been normalized in such a way that $v_1 w_2 - w_1 v_2 = 1/\lambda_1$. Then we can read off the coefficients of $\Phi^*\beta$ at the origin by looking at the matrix $\widetilde{B}$:

$$\begin{aligned} \widetilde{a}_s(0,0) &= 0 \\ \widetilde{a}_t(0,0) &= -\lambda_2/\lambda_1 \end{aligned} \qquad \text{and} \qquad \begin{aligned} \widetilde{b}_s(0,0) &= 1 \\ \widetilde{b}_t(0,0) &= 0 \end{aligned}\;.$$

We obtain the desired form

$$\Phi^*\beta = s\,dt - C_1 t\,ds + \mathcal{O}^2(s,t)$$

with $|C_1| \leq 1$ and $C_1 \neq -1$.

When the matrix B is not diagonizable, then choosing for v the eigenvector of λ_3, scaled in such a way that we can choose a second vector w such that $Bw = \lambda_3 w + \lambda_3 \varepsilon v$, and assuming that the vectors have been normalized in such a way that $v_1 w_2 - w_1 v_2 = 1/\lambda_3$, we find

$$\Phi^*\beta = (s + \varepsilon t)\,dt - t\,ds + \mathcal{O}^2(s,t)\;.$$

In case the eigenvalues are not purely real, we write $v = \frac{1}{2}\left(v_\lambda + \bar{v}_\lambda\right) = \operatorname{Re}(v)$ and $w = \frac{1}{2i}\left(v_\lambda - \bar{v}_\lambda\right) = \operatorname{Im}(v)$, and we suppose again that $v_1 w_2 - v_2 w_1 = 1/\operatorname{Im}\lambda_4$. The matrix B takes the form

$$\frac{1}{\operatorname{Im}\lambda_4}\begin{pmatrix} \operatorname{Re}\lambda_4 & \operatorname{Im}\lambda_4 \\ -\operatorname{Im}\lambda_4 & \operatorname{Re}\lambda_4 \end{pmatrix} = \begin{pmatrix} \frac{\operatorname{Re}\lambda_4}{\operatorname{Im}\lambda_4} & 1 \\ -1 & \frac{\operatorname{Re}\lambda_4}{\operatorname{Im}\lambda_4} \end{pmatrix}$$

with respect to this basis, and the coefficients at $(0,0)$ simplify to

$$\begin{aligned} \widetilde{a}_s(0,0) &= 1 \\ \widetilde{a}_t(0,0) &= -C_2 \end{aligned} \qquad \text{and} \qquad \begin{aligned} \widetilde{b}_s(0,0) &= C_2 \\ \widetilde{b}_t(0,0) &= 1 \end{aligned} \; .$$

The 1-form reduces then in this last case to

$$\Phi^*\beta = \big(s - C_2 t\big)\, ds + \big(C_2 s + t\big)\, dt + \mathcal{O}^2(s,t) = \frac{1}{2}\, d(r^2) + C_2 r^2\, d\varphi + \mathcal{O}^2(s,t)\; . \qquad \square$$

CHAPTER II

Almost complex structures and Legendrian foliations

The next sections only fix notation, and explains some well-known facts about J-convexity. With some basic knowledge on J-holomorphic curves, one can safely skip it and continue directly with Section II.3, which describes the local models around the binding and the boundary of the Lobs and bLobs and the behavior of holomorphic disks that lie nearby.

II.1. Preliminaries: J-convexity

II.1.1. The maximum principle. One of the basic ingredients in the theory of J-holomorphic curves with boundary is the maximum principle, which we will now briefly describe in the special case of Riemann surfaces. We assume in this section that (Σ, j) is a Riemann surface that does not need to be compact and may or may not have boundary. We define the differential operator d^j that associates to every smooth function $f\colon \Sigma \to \mathbb{R}$ a 1-form given by

$$\big(d^j f\big)(v) := df(j\,v)$$

for $v \in T\Sigma$.

Definition. We say that a function $f\colon (\Sigma, j) \to \mathbb{R}$ is

(a) **harmonic** if the 2-form $-dd^j f$ vanishes everywhere,

(b) it is **subharmonic** if the 2-form $-dd^j f$ is a positive volume form with respect to the orientation defined by $\langle v, j\,v\rangle$ for any non-vanishing vector $v \in T\Sigma$.

(c) If f only satisfies

$$-dd^j f\big(v, j\,v\big) \geq 0$$

then we call it **weakly subharmonic**.

In particular, if we choose a complex chart $\big(U \subset \mathbb{C}, \phi\big)$ for Σ with coordinate $z = x + iy$, we can represent f by $f_U := f \circ \phi^{-1}\colon U \to \mathbb{R}$. The 2-form $-dd^j f$ simplifies on this chart to $-dd^i f_U$, because ϕ is holomorphic with respect to j and i, and we can write $-dd^i f_U$ in the form $\big(\triangle f_U\big)\,dx \wedge dy$, where the Laplacian is defined as

$$\triangle f_U = \frac{\partial^2 f_U}{\partial x^2} + \frac{\partial^2 f_U}{\partial y^2}\;.$$

Note that f_U is subharmonic, if and only if $-dd^i f_U(\partial_x, \partial_y) > 0$, that means, $\triangle f_U > 0$.

For strictly subharmonic functions, it is obvious that they may not have any interior maxima, because the Hessian needs to be negative definite at any such point. We really need to consider both weakly subharmonic functions and the behavior at boundary points. To prove the maximum principle in this more general setup, we use the following technical result.

Lemma II.1.1. *Let $f\colon \mathbb{D}^2 \subset \mathbb{C} \to \mathbb{R}$ be a function that is C^1 on the closed unit disk, and both C^2 and weakly subharmonic on the interior of the disk. Assume that f takes its maximum at a boundary point $z_0 \in \partial\mathbb{D}^2$ and is everywhere else strictly smaller than $f(z_0)$. Choose an arbitrary vector $X \in T_{z_0}\mathbb{C}$ at z_0 pointing transversely out of $\mathbb{D}^2$.*

Then the derivative $\mathcal{L}_X f(z_0)$ in X-direction needs to be strictly *positive.*

Proof. We will perturb f to a *strictly* subharmonic function making use of the auxiliary function $g\colon \mathbb{D}^2 \to \mathbb{R}$ defined by

$$g(r) = r^4 - \frac{9}{4}\,r^2 + \frac{5}{4}\ .$$

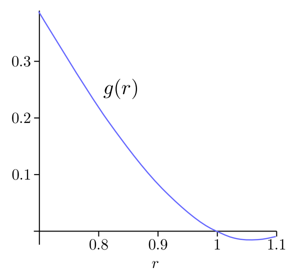


Figure 1. The function $g(r)$ is subharmonic, vanishes on the boundary, and has negative radial derivative.

The function g vanishes along the boundary $\partial\mathbb{D}^2$, and its derivative in any direction v that is positively transverse to the boundary $\partial\mathbb{D}^2$ is strictly negative, because $\partial_\varphi g = 0$ and because

$$r\,\partial_r g = \frac{1}{2}\,r^2\,(8r^2 - 9)\ .$$

Finally, we also see that g is strictly subharmonic on the open annulus $\mathring{\mathbb{A}} = \{z \in \mathbb{C} \,|\, 3/4 < |z| < 1\}$ as

$$\triangle g = \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = 16r^2 - 9\ .$$

We slightly perturb f by setting $f_\varepsilon = f + \varepsilon\, g$ for small $\varepsilon > 0$, and we additionally restrict f_ε to the closure of the annulus $\mathring{\mathbb{A}}$. Note in particular that f_ε must take its maximum on $\partial\mathbb{A}$, because f_ε is *strictly* subharmonic on the interior of $\mathbb{A}$ so that one of $\frac{\partial^2 f_\varepsilon}{\partial x^2}$ or $\frac{\partial^2 f_\varepsilon}{\partial y^2}$ must be strictly positive. This contradicts existence of possible interior maximum points. The functions f_ε are equal to f along the outer boundary of $\mathbb{A}$ so that the maximum of f_ε will either lie in z_0 or on the inner boundary of $\mathbb{A}$.

The initial function f is by assumption strictly smaller than $f(z_0)$ on the inner boundary of the annulus and by choosing ε sufficiently small, it follows that the perturbed function f_ε will still be strictly smaller than $f_\varepsilon(z_0) = f(z_0)$. Thus z_0 will also be the maximum of f_ε. Let X be a vector at z_0 that points transversely out of $\mathbb{D}^2$. The derivative $\mathcal{L}_X f_\varepsilon$ at z_0 cannot be strictly negative, because z_0 is a maximum, and so since

$$0 \le \mathcal{L}_X f_\varepsilon = \mathcal{L}_X f + \varepsilon\,\mathcal{L}_X g\ ,$$

the derivative of f in X-direction has to be *strictly* positive, yielding the desired result. $\square$

Now we are prepared to state and prove the maximum principle.

Theorem II.1.2 (Weak maximum principle)**.** *Let (Σ, j) be a connected compact Riemann surface. A weakly subharmonic function $f\colon \Sigma \to \mathbb{R}$ that attains its maximum at an interior point $z_0 \in \Sigma \setminus \partial\Sigma$ must be constant.*

PROOF. The proof is classical and holds in much greater generality (see for example [**GT01**]). Nonetheless we will explain it in the special case needed by us to show that it only uses elementary techniques. The strategy is simply to find a closed disk in the interior of the Riemann surface with the properties required by Lemma II.1.1. Then the function f increases in radial direction further, so that the maximum point was not really a maximum.

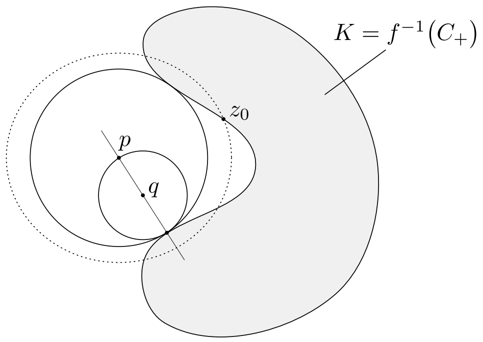


FIGURE 2. Constructing a disk that has a single maximum on its boundary.

More precisely, assume f not to be constant, and to have a maximum at an interior point $z_+ \in \Sigma \setminus \partial\Sigma$ with $C_+ := f(z_+)$. The subset $K := f^{-1}(C_+) \cap \mathring{\Sigma}$ is closed in $\mathring{\Sigma}$. For every point $z \in K$ and every complex chart $(U \subset \mathring{\Sigma}, \varphi)$ containing z_0, we find an $R_z > 0$ such that the open disk $D_{R_z}(z)$ lies in U. There must be a point $z_0 \in K$ for which the half sized disk $D_{\frac{1}{2}R_{z_0}}(z_0)$ intersects $\mathring{\Sigma} \setminus K$, for otherwise K would be open and hence as $\mathring{\Sigma}$ is connected, $K = \mathring{\Sigma}$.

Let p be a point in $D_{\frac{1}{2}R_{z_0}}(z_0) \setminus K$ (see Fig. 2). It lies so close to z_0 that the entire closed disk of radius $|p - z_0|$ lies in the chart $\varphi(U)$, and then we can choose first a disk $\mathbb{D}^2_R(p)$ centered at p, where R is the largest number for which the *open* disk does not intersect K. We are interested in finding a closed disk that intersects K at a *single* boundary point: For this let q be the mid point between p and one of the boundary points in $\partial\mathbb{D}^2_R(p) \cap K$. The disk $\mathbb{D}^2_{\frac{1}{2}R}(q)$ touches K at exactly one point.

This smaller disk satisfies the conditions of Lemma II.1.1, and so it follows that the derivative of f at the maximum is strictly positive in radial direction. But since this point lies in the interior of Σ, it follows that f still increases in that direction and hence this point cannot be the maximum. Of course, the whole existence of the disk was based on the assumption that f was not constant, so we obtain the statement of the theorem. $\square$

If Σ has boundary, we also get the following refinement.

Theorem II.1.3 (Boundary point lemma)**.** *Let $f\colon \Sigma \to \mathbb{R}$ be a weakly subharmonic function on a connected compact Riemann surface (Σ, j) with boundary. Assume f takes its maximum at a point $z_+ \in \partial\Sigma$, then f will either be constant or the derivative at z_+*

$$\mathcal{L}_X f(z_+) > 0$$

in any outward direction $X \in T_{z_+}\Sigma$ has to be strictly positive.

Proof. Denote the maximum value $f(z_+)$ by C_+. By the maximum principle, Theorem II.1.2, we know that f will be constant if there is a point $z \in \Sigma \setminus \partial\Sigma$ for which $f(z) = C_+$. We can thus assume that for all $z \notin \partial\Sigma$, we have $f < C_+$. Using a chart U around the point z_+, that represents an open set in $\mathbb{H} := \{z \in \mathbb{C} |\ \operatorname{Im} z \geq 0\}$, such that z_+ corresponds to the origin, we can easily find a small disk in $\mathbb{H}$ that touches $\partial\mathbb{H}$ only in 0, and hence allows us to directly apply Lemma II.1.1 to complete the proof. □

II.1.2. Plurisubharmonic functions. We will now explain the connection between the previous section and contact topology.

Let (W, J) be an almost complex manifold, that means that J is a section of the endomorphism bundle $\operatorname{End}(TM)$ with $J^2 = -\mathbf{1}$. Define the differential $d^J f$ of a smooth function $f\colon W \to \mathbb{R}$ as before by

$$\big(d^J f\big)(v) := df(J \cdot v)$$

for any vector $v \in TW$.

Definition. We say that a function $h\colon W \to \mathbb{R}$ is ***J*-plurisubharmonic**, if the 2-form

$$\omega_h := -dd^J h$$

evaluates positively on J-complex lines, that means that $\omega_h(v, Jv)$ is strictly positive for every non-vanishing vector $v \in TW$.

If ω_h vanishes, then we say that h is ***J*-harmonic**.

Remark II.1.4.

(1) If h is J-plurisubharmonic, then ω_h is an exact symplectic form that tames J.
(2) If ω_h is only non-negative, then we say that h is **weakly *J*-plurisubharmonic**. This notion might be for example interesting in the context of confoliations.

Let (Σ, j) be a Riemann surface that does not need to be compact, and may or may not have boundary. We say that a smooth map $u\colon \Sigma \to W$ is ***J*-holomorphic**, if its differential commutes with the pair (j, J), that means, at every $z \in \Sigma$ we have

$$J \cdot Du = Du \cdot j\ .$$

Using the commutation relation, we easily check for every J-holomorphic map u and every smooth function $f\colon U \to \mathbb{R}$ the formula

$$-u^* d^J f = -df \cdot J \cdot Du = -df \cdot Du \cdot j = -d(f \circ u) \cdot j = d^j(f \circ u) = -d^j u^* f\ . \tag{II.1.1}$$

Corollary II.1.5. *If $u\colon (\Sigma, j) \to (W, J)$ is J-holomorphic and $h\colon W \to \mathbb{R}$ is a J-plurisubharmonic function, then $h \circ u$ will be weakly subharmonic, because*

$$-dd^j(h \circ u) = -d\, u^* d^J h = -u^* dd^J h$$

and because the differential Du commutes with the complex structures, so that

$$-dd^j(h \circ u)\big(v, jv\big) = -dd^J h(Du \cdot v, J \cdot Du \cdot v) \geq 0$$

for every vector $v \in T\Sigma$. The function is strictly positive precisely at points $z \in \Sigma$, where Du_z does not vanish.

The maximum principle restricts severely the behavior of holomorphic maps:

Corollary II.1.6. *Let $u\colon (\Sigma, j) \to (W, J)$ be a J-holomorphic map and $h\colon W \to \mathbb{R}$ be a J-plurisubharmonic function. If u is not a constant map then $h \circ u\colon \Sigma \to \mathbb{R}$ will never take its maximum on the interior of Σ.*

Proof. Since $h \circ u$ is weakly subharmonic, it follows immediately from the maximum principle (Theorem II.1.2) that $h \circ u$ must be constant if it takes its maximum in the interior of Σ, and hence $d(h \circ u) = 0$. On the other hand, we know that if there were a point $z \in \Sigma$ with $D_z u \neq 0$, then $\omega_h(Du \cdot v, Du \cdot jv)$ would need to be strictly positive for non-vanishing vectors. This is not possible though, because $u^*\omega_h = -dd^j(h \circ u) = 0$. □

Corollary II.1.7. *Let (Σ, j) be a Riemann surface with boundary, $u\colon (\Sigma, j) \to (W, J)$ a J-holomorphic map and $h\colon W \to \mathbb{R}$ be a J-plurisubharmonic function. If $h \circ u\colon \Sigma \to \mathbb{R}$ takes its maximum at $z_0 \in \partial\Sigma$ then it follows either that $d(h \circ u)(v) > 0$ for every vector $v \in T_{z_0}\Sigma$ pointing transversely out of the surface, or u will be constant.*

Proof. The proof is analogous to the previous one, but uses the boundary point lemma (Theorem II.1.3) instead of the simple maximum principle. □

Remark II.1.8. Note that if h is only *weakly* plurisubharmonic, then we can only deduce in the two corollaries above that u has to lie in a level set of h, and not that u itself must be constant.

II.1.3. Contact structures as J-convex boundaries. Now we will finally explain the relation between plurisubharmonic functions and contact manifolds.

Definition. Let (W, J) be an almost complex manifold with boundary. We say that W has **J-convex boundary**, if there exists a smooth function $h\colon W \to (-\infty, 0]$ with the properties

- h is J-plurisubharmonic on a *neighborhood* of ∂W,
- h is a regular equation for ∂W, that means, 0 is a regular value of h and $\partial W = h^{-1}(0)$.

Note that the function h in the definition takes its maximum on ∂W, so that it must be strictly increasing in outward direction.

We will show that the boundary of an almost complex manifold is J-convex if and only if it carries a natural cooriented contact structure (whose conformal symplectic structure tames J). Remember that we are always assuming our contact manifolds to be cooriented. Hence the manifold is oriented, and its contact structure will have a natural conformal symplectic structure.

Definition. Let M be a codimension-1 submanifold in an almost complex manifold (W, J). The **subbundle of complex tangencies** of M is the J-complex subbundle

$$\xi := TM \cap (J \cdot TM) .$$

Proposition II.1.9. *Let (W, J) be an almost complex manifold with boundary $M := \partial W$ and let ξ be the subbundle of complex tangencies of M. We have the following equivalence:*

1. *The boundary M is J-convex.*
2. *The subbundle ξ is a cooriented contact structure whose natural orientation is compatible with the boundary orientation of M, and whose natural conformal symplectic structure tames $J|_\xi$.*

Proof. To prove the direction "(1) ⇒ (2)", let h be the J-plurisubharmonic equation of M that exists by assumption. A straight forward calculation shows that the kernel of the

1-form $\alpha := -d^J h\big|_{TM}$ is precisely ξ, and in particular that α does not vanish. Furthermore $d\alpha|_{TM} = \omega_h|_{TM}$ is a symplectic structure on ξ that tames $J|_\xi$, so that α is a contact form. To check that $\alpha \wedge d\alpha^{n-1}$ is a positive volume form with respect to the boundary orientation induced on M by (W, J), let R_α be the Reeb field of α, and define a vector field $Y = -J\,R_\alpha$. The field Y is positively transverse to ∂W, because $\mathcal{L}_Y h = dh(Y) = -d^J h(R_\alpha) = \alpha(R_\alpha) = 1$ is positive. Choosing a basis $\langle v_1, \dots, v_{2n-2}\rangle$ for ξ at a point $p \in M$, we compute

$$\alpha \wedge d\alpha^{n-1}\big(R_\alpha, v_1, \dots, v_{2n-2}\big) = d\alpha^{n-1}\big(v_1, \dots, v_{2n-2}\big) = \omega_h^{n-1}\big(v_1, \dots, v_{2n-2}\big)\ .$$

Similarly, we obtain

$$\begin{aligned}\omega_h^n\big(Y, R_\alpha, v_1, \dots, v_{2n-2}\big) &= n\,\omega_h(Y, R_\alpha)\cdot \omega_h^{n-1}\big(v_1, \dots, v_{2n-2}\big)\\ &= n\,\omega_h(R_\alpha, JR_\alpha)\cdot \omega_h^{n-1}\big(v_1, \dots, v_{2n-2}\big)\ ,\end{aligned}$$

where we have used that $\omega_h(R_\alpha, v_j) = d\alpha(R_\alpha, v_j) = 0$ for all $j \in \{1, \dots, n-1\}$. The first term $\omega_h(R_\alpha, JR_\alpha)$ is positive, and hence $\alpha \wedge d\alpha^{n-1}$ and $\iota_Y \omega_h^n$ induce identical orientations on M.

To prove the direction "(2) $\Rightarrow$ (1)", choose any collar neighborhood $(-\varepsilon, 0] \times M$ for the boundary, and let t be the coordinate on $(-\varepsilon, 0]$. First note that $\alpha = -d^J t\big|_{TM}$ is a non-vanishing 1-form with kernel ξ, so in particular it will be contact. Let R_α be the Reeb field of α, and set $Y := -J\,R_\alpha$. As before, the field Y is positively transverse to M, because of $\mathcal{L}_Y t = -dt(J\,R_\alpha) = \alpha(R_\alpha) = 1$.

Let C be a large constant, whose size will be determined below, and set $h(t,p) := e^{Ct} - 1$. Clearly, h is a regular equation for M.

The 1–form $\alpha_C = -d^J h\big|_{TM} = Ce^{Ct}\,\alpha$ is a contact form that represents the same coorientation as α. We claim that for sufficiently large C, h will be a J-plurisubharmonic function.

Let $v \in T_pW$ be any non-vanishing vector at $p \in M$ and represent it as

$$v = aY + bR_\alpha + cZ\ ,$$

where Y and R_α were defined above, and $Z \in \xi$ is a vector in the contact structure that has been normalized such that $d\alpha(Z, JZ) = \omega_t(Z, JZ) = 1$. Note that the 1-form $\alpha_C = -d^J h\big|_{TM} = Ce^{Ct}\,\alpha$ is a contact form that represents the same coorientation as α.

We compute $\omega_h = -dd^J h = Ce^{Ct}\big(\omega_t - C\,dt \wedge d^J t\big)$, which simplifies for $t = 0$ further to $\omega_h = C\big(\omega_t - C\,dt \wedge d^J t\big)$ and so we have

$$\omega_h(R_\alpha, \cdot) = C\big(\omega_t(R_\alpha, \cdot) - C\,dt\big) \quad \text{and} \quad \omega_h(Y, \cdot) = C\big(\omega_t(\cdot, J\,R_\alpha) - C\,d^J t\big)$$

This implies $\omega_h(R_\alpha, Z) = \omega_h(R_\alpha, JZ) = 0$ for all $Z \in \xi$, and $\omega_h(Y, R_\alpha) = C^2 + C\,\omega_t(R_\alpha, JR_\alpha)$ can be made arbitrarily large by increasing the size of C. With these relations we obtain

$$\begin{aligned}\omega_h(v, Jv) &= \omega_h(aY + bR_\alpha + cZ, aR_\alpha - bY + cJZ)\\ &= \big(a^2 + b^2\big)\,\omega_h(Y, R_\alpha) + c^2\,\omega_h(Z, JZ) + ac\,\omega_h(Y, JZ) + bc\,\omega_h(Y, Z)\\ &= \big(a^2 + b^2\big)\,\big(C^2 + \mathcal{O}^1(C)\big) + C\,\big(c^2\,\omega_t(Z, JZ) + ac\,\omega_t(Y, JZ) + bc\,\omega_t(Y, Z)\big)\end{aligned}$$

and setting $A_a = \omega_t(Y, JZ)$ and $A_b = \omega_t(Y, Z)$ and using that $\omega_t(Z, JZ) = 1$

$$\begin{aligned}
&= \left(a^2+b^2\right)\left(C^2 + \mathcal{O}^1(C)\right) + C\left(c^2 + A_a ac + A_b bc\right) \\
&= \left(a^2+b^2\right)\left(C^2 + \mathcal{O}^1(C)\right) + \frac{C}{2}\left(\left(c + aA_a\right)^2 - a^2A_a^2 + \left(c + bA_b\right)^2 - b^2A_b^2\right) \\
&= a^2\left(C^2 + \mathcal{O}^1(C)\right) + b^2\left(C^2 + \mathcal{O}^1(C)\right) + \frac{C}{2}\left((c + aA_a)^2 + (c + bA_b)^2\right) .
\end{aligned}$$

By choosing C large enough, we can ensure that the a^2- and b^2-coefficients are both positive. Then it is obvious from the computation above that ω_h tames J, and hence h is J-plurisubharmonic. □

II.1.4. Legendrian foliations in J-convex boundaries.

Definition. A **totally real submanifold** N of an almost complex manifold (W, J) is a submanifold of dimension $\dim N = \frac{1}{2}\dim W$ that is not tangent to any J-complex line, that means, $TN \cap (J\,TN) = \{0\}$, which is equivalent to requiring

$$TW|_N = TN \oplus (J\,TN) .$$

Proposition II.1.10. *Let (W, J) be an almost complex manifold with J-convex boundary (M, ξ). Assume N is a submanifold of M for which the complex tangencies ξ induce the Legendrian foliation $\mathcal{F} = TN \cap \xi$. Then it is easy to check that $N \setminus \operatorname{Sing}(\mathcal{F})$ is totally real.*

Proof. If $X \in TN$ is a non-vanishing vector with JX also in TN, then in particular

$$X \in TN \cap (JTN) \subset TM \cap (JTM) = \xi ,$$

so that X and JX have to lie in $\mathcal{F}$. The 2-form $d\alpha$ tames $J|_\xi$ so that $d\alpha(X, JX) > 0$, but $d\alpha|_{\mathcal{F}}$ vanishes at regular points of the foliation, and hence X must be 0. □

We will next study the restrictions imposed by a Legendrian foliation on J-holomorphic curves. Let (Σ, j) be a compact Riemann surface with boundary, and let A be a subset of an almost complex manifold (W, J). We introduce for J-holomorphic maps $u\colon \Sigma \to W$ with $u(\partial\Sigma) \subset A$ the notation

$$u\colon (\Sigma, \partial\Sigma, j) \to (W, A, J) .$$

Note that we are always supposing that u is at least C^1 along the boundary.

Corollary II.1.11. *Let (W, J) be an almost complex manifold with convex boundary (M, ξ). Let $N \hookrightarrow M$ be a submanifold with an induced Legendrian foliation $\mathcal{F}$, and let u be a J-holomorphic map*

$$u\colon (\Sigma, \partial\Sigma, j) \to (W, N \setminus \operatorname{Sing}(\mathcal{F}), J) .$$

If there is an interior point $z_0 \in \Sigma \setminus \partial\Sigma$ at which u touches M, or if ∂u is not positively transverse to $\mathcal{F}$, then u is a constant map.

Proof. Choose a J-plurisubharmonic function $h\colon W \to \mathbb{R}$ that is a regular equation for M. The first implication follows directly from Corollary II.1.6, because z_0 would be an interior maximum for $h \circ u$.

For the second implication note first that $h \circ u$ takes its maximum on $\partial\Sigma$ so that if u is not constant, we have by Corollary II.1.7 that the derivative $\mathcal{L}_v(h \circ u)$ is strictly positive for every point $z_1 \in \partial\Sigma$ and every vector $v \in T_{z_1}\Sigma$ pointing out of Σ. Now if $w \in T\Sigma$ is a vector that is tangent to $\partial\Sigma$ such that jw points inward (so that w corresponds to the boundary orientation of $\partial\Sigma$, because $\langle -jw, w\rangle$ is a positive basis of $T\Sigma$), we obtain

$$\alpha(Du \cdot w) = -dh(JDu \cdot w) = -dh(Du \cdot jw) = -d(h \circ u)(jw) > 0 \,.$$

The boundary of ∂u has thus to be positively transverse to ξ, and so it is in particular positively transverse to the Legendrian foliation $\mathcal{F}$. ☐

Note that the result above applies only for holomorphic maps that are C^1 along the boundary.

II.2. Preliminaries: ω-convexity

Above we have explained the notion of J-convexity, and the relevant relationship between contact and almost complex structures. In this section, we want to discuss the notion of ω-convexity, that means the relationship between an (almost) symplectic and a contact structure.

In fact, we are not interested in studying almost complex manifolds for their own sake, but we would like to use the almost complex structure to understand instead a symplectic manifold (W, ω). As initiated by Gromov, we introduce an auxiliary almost complex structure to be able to study J-holomorphic curves in the hope that even though the J-holomorphic curves depend very strongly on the almost complex structure chosen, we'll be able to extract interesting information about the initial symplectic structure.

For this strategy to work, we need the almost complex structure to be **tamed** by ω, that means, we want

$$\omega(X, JX) > 0$$

for every non-vanishing vector $X \in TW$. This tameness condition is important, because it allows us to control the limit behavior of sequences of holomorphic curves (see Section III.3).

As explained in the previous section, J-convexity is a property that greatly helps us in understanding holomorphic curves in ambient manifolds that have boundary. When (W, ω) is a symplectic manifold with boundary $M = \partial W$, we would thus like to chose an almost complex structure J that is

- tamed by ω, and
- that makes the boundary J-convex.

In particular, if such a J exists, we know that the boundary admits an induced contact structure

$$\xi = TM \cap \bigl(J \cdot TM\bigr) \,.$$

From the symplectic or contact topological view point, the opposite setup would be more natural though: given a symplectic manifold (W, ω) with contact boundary (M, ξ), can we choose an almost complex structure J that is tamed by ω, and that makes the boundary J-convex such that ξ is the bundle of J-complex tangencies?

The general answer to that question was given in [MNW13], and we will rediscuss it here:

Definition. Let (M, ξ) be a cooriented contact manifold of dimension $2n-1$, and let (W, ω) be a symplectic manifold whose boundary is M. Let α be a positive contact form for ξ, and assume that the orientation induced by $\alpha \wedge d\alpha^{n-1}$ on M agrees with the boundary orientation of (W, ω). We call (W, ω) a **weak symplectic filling** of (M, ξ), if

$$\alpha \wedge \bigl(T\, d\alpha + \omega\bigr)^{n-1} > 0$$

for every $T \in [0, \infty)$.

The proofs of the following statements are quite lengthy and have been postponed to Section II.4.

Theorem II.2.1. *Let (M, ξ) be a cooriented contact manifold, and let (W, ω) be a symplectic manifold with boundary $M = \partial W$. The following two statements are equivalent*

(i) *(W, ω) is a weak symplectic filling of (M, ξ).*

(ii) *There exists an almost complex structure J on W that is tamed by ω and that makes M a J-convex boundary whose J-complex tangencies are ξ.*

A weak filling is a notion that is relatively recent in higher dimensions; traditionally it is the concept of a strong symplectic filling that has been studied for a much longer time. Let (W, ω) be a symplectic manifold. A vector field X_L is called a **Liouville vector field**, if it satisfies the equation

$$\mathcal{L}_{X_L} \omega = \omega .$$

Definition. Let (M, ξ) be a cooriented contact manifold, and let (W, ω) be a symplectic manifold whose boundary is M. We call (W, ω) a **strong symplectic filling** of (M, ξ), if there exists a Liouville vector field X_L on a neighborhood of M such that $\lambda := \bigl(\iota_{X_L} \omega\bigr)\bigr|_{TM}$ is a positive contact form for ξ.

It is easy to see that a strong filling is in particular a weak filling. Note that the symplectic form of a strong filling becomes always exact when restricted to the boundary, but that this needs not be true for a weak filling; if it is then it will usually still not be a strong symplectic filling, but by Corollary II.2.4 it can deformed into one.

Lemma II.2.2. *Let (W, ω) be a symplectic manifold and let M be a hypersurface (possibly a boundary component of W) together with a non-vanishing 1-form λ. Assume that the restriction of ω to $\ker \lambda$ is symplectic.*

Then there is a tubular neighborhood of M in W that is symplectomorphic to the model

$$\bigl((-\varepsilon, \varepsilon) \times M,\, d(t\,\lambda) + \omega|_{TM}\bigr) ,$$

where t is the coordinate on the interval $(-\varepsilon, \varepsilon)$. The 0-slice $\{0\} \times M$ corresponds in this identification to the hypersurface M. If M is a boundary component of W then of course we need to replace the model by $(-\varepsilon, 0] \times M$ or by $[0, \varepsilon) \times M$ depending on whether $\lambda \wedge \omega^{n-1}$ is oriented as the boundary of (W, ω) or not.

Proof. In a first step we define a collar neighborhood of M by choosing a vector field that is transverse to M. In a second we then deform the collar to the desired shape. Let $E \subset TW|_M$ be the ω-orthogonal complement of ξ along M. The intersection of E with TM is a 1-dimensional subbundle, and we can uniquely define a *Reeb-like* vector field X_ω by taking the section in $E \cap TM$ that satisfies $\lambda(X_\omega) \equiv 1$. By our definition, $\omega(X_\omega, \cdot)|_{TM}$ vanishes. Choose now a second section Y in E that is transverse to M, and normalize it such that $\omega(Y, X_\omega) \equiv 1$. Note that if such a section is already given near some subset of M, then we can choose Y to be an extension of that section. We now have $\omega(Y, \cdot)|_{TM} = \lambda$, since both forms vanish on ξ and agree on X_ω.

Extend Y to a smooth vector field in a neighborhood of M, and use the flow Φ^Y of this vector field to define a smooth diffeomorphism

$$\Phi \colon (-\varepsilon, \varepsilon) \times M \hookrightarrow W,\ (t, p) \mapsto \Phi_t^Y(p) ,$$

which agrees with the canonical identification on $\{0\} \times M$. Next, compare the 2-forms $\Phi^*\omega$ and $\omega_M + d(t\lambda)$ on $(-\varepsilon, \varepsilon) \times M$. Both forms coincide along $\{0\} \times M$, thus the linear interpolation of these forms is a path of symplectic structures (decreasing $\varepsilon > 0$ if necessary). We can then use the Moser trick to show that they are all symplectomorphic to each other (perhaps in a smaller neighborhood) by an isotopy that keeps the level set $\{0\} \times M$ fixed. □

Proposition II.2.3. *Let (W, ω) be a weak filling of a contact manifold (M, ξ), and let Ω be a 2-form on M that is cohomologous to $\omega|_{TM}$. Choose a positive contact form α for (M, ξ). Then if we allow $C > 0$ to be sufficiently large, we can attach a collar $[0, C] \times M$ to W with a symplectic form ω_C that agrees close to $\{C\} \times M$ with $d(t\alpha) + \Omega$, and such that the new manifold is a weak filling of $(\{t_0\} \times M, \xi)$ for every $t_0 \in [0, C]$.*

The proof can be found in [**MNW13**, Lemma 2.10].

Corollary II.2.4. *Let (W, ω) be a weak symplectic filling of (M, ξ) and assume that the restriction of ω to M is exact. Then we may deform ω on a small neighborhood of M such it becomes a strong symplectic filling.*

Proof. Since $\omega|_{TM}$ is exact, we can apply the proposition above with $\Omega = 0$. Afterward we can isotope the collar back into the neighborhood of the boundary of W. □

Note that two contact structures that are strongly filled by the same symplectic manifold are isotopic, while a symplectic manifold may be a weak filling of two different contact manifolds. This is true even when the restriction of the symplectic structure to the boundary is exact, see [**MNW13**, Remark 2.11].

II.3. Local models for maximally foliated submanifolds and J-holomorphic curves

Let (W, J) be an almost complex manifold with J-convex boundary (M, ξ), and let $N \subset M$ be a submanifold carrying a Legendrian foliation $\mathcal{F}$. The aim of this section will be to better understand the behavior of J-holomorphic maps

$$u \colon (\Sigma, \partial\Sigma, j) \to (W, N, J) ,$$

that lie close to a singular point $p \in \operatorname{Sing}(\mathcal{F})$ of the Legendrian foliation. For this we will assume that J is of a very specific form in a neighborhood of the point p. With this choice of the almost complex structure, elliptic singularities (see Fig. 2.(A) and (B)) give birth to a family of holomorphic disks. Apart from these disks and their branched covers, no other holomorphic disks may get close to the elliptic singularities (see Sections II.3.2 and II.3.3). Similarly certain codimension-1 singularities may be used as barriers for holomorphic curves, preventing them to get nearby (see Section II.3.4).

II.3.1. Existence of J-convex functions close to totally real submanifolds. As a preliminary tool, we will need the following result.

Proposition II.3.1. *Let (W, J) be an almost complex manifold that contains a closed totally real submanifold L. Then there exists a smooth function $f \colon W \to [0, \infty)$ with $L = f^{-1}(0)$ that is J-plurisubharmonic on a neighborhood of L. In particular, it follows that $df_p = 0$ at every point $p \in L$.*

PROOF. We will first show that we find around every point $p \in L$ a chart U with coordinates $\{(x_1, \dots, x_n; y_1, \dots, y_n)\} \subset \mathbb{R}^{2n}$ such that $L \cap U = \{y_1 = \dots = y_n = 0\}$ and

$$J \, \frac{\partial}{\partial x_j}\bigg|_{L\cap U} = \frac{\partial}{\partial y_j}\bigg|_{L\cap U} .$$

For this, start by choosing coordinates $\{(x_1, \dots, x_n)\} \subset \mathbb{R}^n$ for the submanifold L around the point p, and consider the associated vector fields

$$Y_1 = J \, \frac{\partial}{\partial x_1}, \dots, Y_n = J \, \frac{\partial}{\partial x_n}$$

along L. These vector fields are everywhere linearly independent and transverse to L, hence, we can define a smooth map from a small ball around 0 in $\mathbb{R}^{2n} = \{(x_1, \dots, x_n; y_1, \dots, y_n)\}$ to W by

$$y_1 \, Y_1(x_1, \dots, x_n) + \dots + y_n \, Y_n(x_1, \dots, x_n) \mapsto \exp\bigl(y_1 \, Y_1 + \dots + y_n \, Y_n\bigr) ,$$

where exp is the exponential map for an arbitrary Riemannian metric on W. If the ball is chosen sufficiently small, the map will be a chart with the desired properties.

For such a chart U, we will choose a function

$$f_U \colon U \to [0, \infty), \, (x_1, \dots, x_n; y_1, \dots, y_n) \mapsto \frac{1}{2} \left(y_1^2 + \dots + y_n^2\right) .$$

It is obvious that both the function itself, and its differential vanish along $L \cap U$. Furthermore f is plurisubharmonic close to $L \cap U$, because

$$\begin{aligned} -dd^J f_U &= -d\bigl(y_1 \, d^J y_1 + \dots + y_n \, d^J y_n\bigr) \\ &= -dy_1 \wedge d^J y_1 - \dots - dy_n \wedge d^J y_n - y_1 \, dd^J y_1 - \dots - y_n \, dd^J y_n \end{aligned}$$

simplifies at $L \cap U$ to

$$-\, dd^J f_U\big|_{L\cap U} = dx_1 \wedge dy_1 + \dots + dx_n \wedge dy_n ,$$

where we have used that all y_j vanish, and that $J \, \frac{\partial}{\partial x_j} = \frac{\partial}{\partial y_j}$ and $J \, \frac{\partial}{\partial y_j} = J^2 \, \frac{\partial}{\partial x_j} = -\frac{\partial}{\partial x_j}$. It is easy to check that this 2-form evaluates positively on complex lines along $L \cap U$, and hence also in a small neighborhood of p.

Now to obtain a global plurisubharmonic function as stated in the proposition, cover L with finitely many charts $U_1, \dots, U_N$, each with a function $f_1, \dots, f_N$ according to the construction given above. Choose a subordinate partition of unity $\rho_1, \dots, \rho_N$, and define

$$f = \sum_{j=1}^N \rho_j \cdot f_j .$$

The function f and its differential $df = \sum_{j=1}^N \bigl(\rho_j \, df_j + f_j \, d\rho_j\bigr)$ vanish along L so that the only term in

$$\begin{aligned} -dd^J f &= -d \sum_{j=1}^N \bigl(\rho_j \, d^J f_j + f_j \, d^J \rho_j\bigr) \\ &= -\sum_{j=1}^N \bigl(\rho_j \, dd^J f_j + d\rho_j \wedge d^J f_j + f_j \, dd^J \rho_j + df_j \wedge d^J \rho_j\bigr) \end{aligned}$$

that survives along L is the first one, giving us along L

$$-dd^J f = -\sum_{j=1}^{N} \rho_j \, dd^J f_j \; .$$

This 2-form is positive on J-complex lines, and hence there is a small neighborhood of L on which f is plurisubharmonic. Finally, we modify f to be positive outside this small neighborhood so that we have $L = f^{-1}(0)$ as required. □

Corollary II.3.2. *Let (W, J) be an almost complex structure that contains a closed totally real submanifold L. Then we find a small neighborhood U of L for which every J-holomorphic map*

$$u\colon (\Sigma, \partial\Sigma, j) \to \big(W, L, J\big)$$

from a compact Riemann surface needs to be constant if $u(\Sigma) \subset U$.

Proof. Let $f\colon W \to [0, \infty)$ be the function constructed in Proposition II.3.1, and let $U \subset (W, J)$ be the neighborhood of L, where f is J-plurisubharmonic. Because $u(\Sigma) \subset U$, we obtain from Corollary II.1.6 that $f \circ u$ must take its maximum on the boundary of Σ, but because $f \circ u$ is zero on all of $\partial\Sigma$, it follows that $f \circ u$ will vanish on the whole surface Σ. The image $u(\Sigma)$ lies then in the totally real submanifold L, and this implies that the differential of u vanishes everywhere. Hence there is a $\mathbf{q}_0 \in L$ with $u(z) = \mathbf{q}_0$ for all $z \in \Sigma$. □

II.3.2. J-holomorphic curves close to codimension-2 singularities: The 4-dimensional situation. Before studying the higher dimensional case in the next section, we will first construct a model situation for a 4-dimensional almost complex manifold with convex boundary. The planar singularities we are interested in, were all described in Section I.2 (see also Figure 2).

Consider $\mathbb{C}^2$ with its standard complex structure i. Then it is easy to check that $h_0(z_1, z_2) = \frac{1}{2}\left(|z_1|^2 + |z_2|^2\right)$ is a plurisubharmonic function whose regular level sets are the concentric spheres around the origin. The level set $M_0 = h^{-1}(1/2)$ is the unit sphere $\mathbb{S}^3$ which is the i-convex boundary of the closed unit ball $W_0 := h_0^{-1}\big([0, 1/2]\big)$, and has the induced contact form

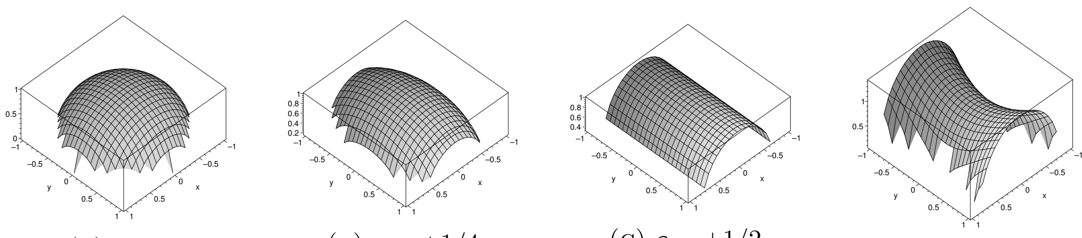
$$\alpha_0 = -\,d^i h_0\big|_{TM_0} = x_1\, dy_1 - y_1\, dx_1 + x_2\, dy_2 - y_2\, dx_2 \; .$$

It is easy to check that the embedding of a small disk by the map

$$\Phi\colon D^2 \to M_0,\; z \mapsto \left(z, \sqrt{1 - |z|^2}\right)$$

is foliated by $\xi_0 = \ker \alpha_0$ with a singularity in $\Phi(0) = (0, 1)$. In fact, the pull-back of α_0 to the disk is just $\Phi^*\alpha_0 = x\, dy - y\, dx$, hence the foliation corresponds to one with radial leaves depicted in Fig. 2.(A).

For all applications in this text, it is sufficient to use the model just described. If the reader is only interested in that situation, he can just study case (S1) below, setting everywhere $\varepsilon = 0$. We will keep the situation for now slightly more general by considering the case (S1), with arbitrary ε-coefficients (but still assuming that the model has been linearized). For this, we will modify the plurisubharmonic function h_0 by adding the real part of a suitable holomorphic function f. The function $h_f := h_0 + \mathrm{Re}(f)$ is still plurisubharmonic, because the real part of a holomorphic map is always harmonic. The chosen functions we will use will all



(A) $\varepsilon = 0$ (B) $\varepsilon = +1/4$ (C) $\varepsilon = +1/2$ (D) $\varepsilon = +1$

FIGURE 3. As a model situation, we study a 1-parameter family of plurisubharmonic functions in $\mathbb{C}^2$, and consider for each one the level set $1/2$. The surfaces represented here are the intersection of these level sets with the 3-dimensional slice $\mathbb{C} \times \mathbb{R}$. The induced Legendrian foliation for each of those intersections can be seen in Fig. 2.(A), and Fig. 2.(B) that correspond to the graph (A) and (B) respectively. The hyperbolic case in Fig. 2.(C) corresponds to graph (D). The holomorphic disks attached to these surfaces are just the subsets of the horizontal flat planes lying beneath these graphs.

vanish at $(0,1)$ and also have vanishing differential at that point so that the hypersurfaces $M_f = h_f^{-1}(1/2)$ all intersect tangentially in the point $(0,1)$.

Let f be now the function $f(z_1, z_2) = \varepsilon z_1^2$, then we set $M_\varepsilon := h_f^{-1}(1/2)$. The natural contact structure induced by i on M_ε is the kernel of the contact form

$$\alpha_\varepsilon = -d^i h_f\big|_{TM_\varepsilon} = (1+2\varepsilon)\, x_1\, dy_1 - (1-2\varepsilon)\, y_1\, dx_1 + x_2\, dy_2 - y_2\, dx_2 \,.$$

We can embed a small disk by the map

$$\Phi_\varepsilon \colon D^2 \to M_f, \quad z \mapsto \left(z, \sqrt{1 - |z|^2 - 2\varepsilon \operatorname{Re}(z^2)}\right)$$

into the level set M_f. We denote the image of Φ_ε by N_ε (see Fig 3 for a sketch of the graph of Φ_ε for different values of ε). The pull-back of α_ε by Φ_ε is

$$\Phi_\varepsilon^* \alpha_\varepsilon = (1+2\varepsilon)\, x\, dy - (1-2\varepsilon)\, y\, dx \,, \tag{II.3.1}$$

and the Legendrian foliation is thus singular at the origin. Assuming without loss of generality that $\varepsilon \geq 0$ (otherwise permute the x and the y-coordinates), the singularity corresponds to class (S1) on page 23 with $C_1 = (1-2\varepsilon)/(1+2\varepsilon)$. When $\varepsilon = 0$, we have $C_1 = 1$, and we are in the standard situation described at the beginning of this section and depicted in Fig. 2.(A); when ε lies in the range $(0, 1/2)$, the constant C_1 lies in $(0,1)$, hence we have an elliptic singularity as the one drawn in Fig. 2.(B); and when $\varepsilon > 1/2$, then we have $C_1 \in (-1, 0)$ and thus there is a hyperbolic singularity as in Fig. 2.(C). We exclude $\varepsilon = 1/2$, because in that case the singularity won't be isolated.

Let U_ε be the subset

$$U_\varepsilon = \left\{(z_1, z_2) \in \mathbb{C}^2 \;\middle|\; \operatorname{Re}(z_2) \geq 1 - \delta\right\} \cap h_\varepsilon^{-1}\big((-\infty, 1/2]\big)$$

for small $\delta > 0$, that means, we just cut off the points under a certain x_2-height. In the elliptic case when $|\varepsilon| < 1/2$, the subset U_ε will be compact. Decreasing δ, we can make U_ε arbitrarily small, and we will use this subset later as our almost complex model of a neighborhood of an elliptic singularity.

The following two lemmas explain that in the elliptic case there is for any point $(z_1, z_2) \in U_\varepsilon$ essentially a unique holomorphic disk passing through (z_1, z_2) that is *entirely contained in* U_ε and whose boundary lies in N_ε. All other holomorphic curves with the same boundary condition will either be constant or will be (branched) covers of that disk.

Lemma II.3.3. *Assume that* $0 \le \varepsilon < 1/2$*. For every* $x_2 \in [1-\delta, 1)$*, there exists a unique injective holomorphic map*

$$u_{x_2} \colon (\mathbb{D}^2, \partial\mathbb{D}^2) \to \bigl(U_\varepsilon, N_\varepsilon\bigr)$$

with $u(\mathbb{D}^2) \subset \bigl(\mathbb{C} \times \{x_2\}\bigr) \cap U_\varepsilon$*, with* $u_{x_2}(0) = (0, x_2)$ *and* $u_{x_2}(1) \in \{(x_1, x_2) \in U_\varepsilon |\, x_1 > 0\}$*.*

The last two conditions only serve to fix a parametrization of a given geometric disk by setting the center point and a point on the boundary of the disk.

Proof. Let u_{x_2} be a map as in the lemma. To prove uniqueness assume that there were a second holomorphic map

$$\tilde{u}_{x_2} \colon (\mathbb{D}^2, \partial\mathbb{D}^2) \to \bigl(U_\varepsilon, N_\varepsilon\bigr)$$

with the required properties. Denote the intersection of U_ε with the complex plane $\mathbb{C} \times \{x_2\}$ by L_{x_2}. It is easy to check that $L_{x_2} = \{(x + iy, x_2) \in \mathbb{C}^2 \bigm| (1 + 2\varepsilon)\, x^2 + (1 - 2\varepsilon)\, y^2 \le 1 - x_2^2\}$ is a planar domain bounded by an ellipse.

By Corollary II.1.11, the restriction $u_{x_2}|_{\partial\mathbb{D}^2}$ of the map to the boundary has non-vanishing derivative, and since it is by assumption injective, it is a diffeomorphism onto ∂L_{x_2}. For topological reasons, u_{x_2} has to be surjective on L_{x_2} (because otherwise we could construct a retract of the disk onto its boundary). Note also that the germ of a holomorphic map around the origin in $\mathbb{C}$ is always biholomorphic to $z \mapsto z^k$ for some integer $k \in \mathbb{N}_0$, so that the differential of u_{x_2} may not vanish anywhere, because otherwise u_{x_2} could not be injective.

Together this allows us to define a biholomorphism

$$\varphi := u_{x_2}^{-1} \circ \tilde{u}_{x_2} \colon (\mathbb{D}^2, \partial\mathbb{D}^2) \to (\mathbb{D}^2, \partial\mathbb{D}^2)$$

with $\varphi(0) = 0$ and $\varphi(1) = 1$, but the only automorphism of the disk with these properties is the identity, thus showing that $u_{x_2} = \tilde{u}_{x_2}$.

To see existence of the map u_{x_2}, note that for $\varepsilon = 0$, the map can be explicitly written down as

$$u_{x_2}(z) = \bigl(Cz, x_2\bigr)$$

with $C = \sqrt{1 - x_2^2}$. It is probably not very hard to find an explicit formula for u_{x_2} for general ε, but instead we would like to appeal to the Riemann mapping theorem. The Riemann theorem itself only guarantees us that the interior of the domain $U_\varepsilon \cap \bigl(\mathbb{C} \times \{x_2\}\bigr)$ is biholomorphic to the an open unit disk, so we use a strengthening by Painlevé which extends to a smooth map on the closed unit disk [**Pai87**] (see [**BK87**] for a modern account). By using Möbius transformations, we can now arrange for the map to be of the required form. □

Lemma II.3.4. *Let*

$$u \colon (\Sigma, \partial\Sigma, j) \to \bigl(U_\varepsilon, N_\varepsilon, i\bigr)$$

be any holomorphic map from a connected compact Riemann surface (Σ, j) *to* U_ε *with* $u(\partial\Sigma) \subset N_\varepsilon$*.*

If the singularity in N_ε *is hyperbolic, that means, if* $|\varepsilon| > 1/2$*, then* u *is a constant map. If on the other hand,* $|\varepsilon| < 1/2$ *so that the singularity is elliptic, then the image of* u *has to lie in one of the slices* $L_{x_2} = U_\varepsilon \cap \bigl(\mathbb{C} \times \{x_2\}\bigr)$*. If* u *is injective at one of the points of its boundary,* *then Σ will be a disk, and u will be up to reparametrization by a Möbius transformation the map u_{x_2} described in Lemma II.3.3.*

Proof. Note that we are supposing that u is at least C^1 on the boundary so that independently of the value of ε, the map u will be constant if it touches the singularity in N_ε. This is in general important, because in the hyperbolic case there may be non-constant holomorphic curves that have a boundary cusp that touches the singular point of N_ε [**BK91, Hin97**] (the curves cannot be entirely contained in U_ε though).

The proof of the proposition will be based on the harmonicity of the coordinate functions x_1, y_1, x_2, and y_2. Let $\pi_{y_2} : U \to \mathbb{R}$ be the function $(z_1, z_2) \mapsto y_2 = \mathrm{Im}(z_2)$. As functions on a compact domain, both $\pi_{y_2} \circ u$ and $-\pi_{y_2} \circ u$ must have a maximum, and by the maximum principle (Corollary II.1.6), these points have to lie on $\partial\Sigma$. But since $u(\partial\Sigma) \subset N_\varepsilon$ has vanishing imaginary z_2-part, it follows that $\pi_{y_2} \circ u \equiv 0$ on the whole surface. Using now the Cauchy-Riemann equations, it immediately follows that the real part of the z_2-coordinate of u has to be everywhere constant. Hence the first conclusion is that the image of u has to lie in one of the slices $L_{x_2} = \mathbb{C} \times \{x_2\}$, and the image of $u(\partial\Sigma)$ has to lie in the subset $\partial L_{x_2} = \{(x + iy, x_2) \in \mathbb{C}^2 \bigm| (1 + 2\varepsilon)\, x^2 + (1 - 2\varepsilon)\, y^2 = 1 - x_2^2\}$.

For a hyperbolic singularity with $|\varepsilon| > 1/2$, the boundary of the slice ∂L_{x_2} has two disconnected components that are both diffeomorphic to $\mathbb{R}$. Let $z_0 \in \partial\Sigma$ be the point for which $u(z_0) \in \partial L_{x_2}$ lies farthest away from the origin. At that point, $u|_{\partial\Sigma}$ will have vanishing derivative, and hence it follows by Corollary II.1.11 that u has to be constant.

Assume on the other hand from now on that $|\varepsilon| < 1/2$, and that u is not constant. Since u lies in L_{x_2}, we can choose the map u_{x_2} from Lemma II.3.3, to define a holomorphic map

$$\varphi := u_{x_2}^{-1} \circ u \colon (\Sigma, \partial\Sigma) \to (\mathbb{D}^2, \partial\mathbb{D}^2) .$$

If u were not surjective on L_{x_2}, we could suppose (after a Möbius transformation) that the image of φ did not contain 0. The function $h(z) = \ln|z|$ is the real part of a holomorphic one, and hence it is harmonic on $\mathbb{D}^2 \setminus \{0\}$. It follows that $h \circ \varphi$ takes its minimum on the interior of Σ, which implies that the image of φ has to be a single point in $\partial\mathbb{D}^2$, and hence in contraction to our assumptions, u will also be constant.

By Corollary II.1.11, the restriction $u|_{\partial\Sigma} \colon \partial\Sigma \to \partial L_{x_2}$ may not have vanishing derivatives, hence every boundary component of Σ covers ∂L_{x_2} at least once. If u is injective at one of its boundary points, then $\partial\Sigma$ must be connected, and $u|_{\partial\Sigma} \colon \partial\Sigma \to \partial L_{x_2}$ must be a diffeomorphism. Furthermore, there must be a small collar neighborhood of ∂L_{x_2} on which u will also be injective, for otherwise we could find two sequences $(z_k)_k$ and $(\tilde{z}_k)_k$ getting arbitrarily close to $\partial\Sigma$ with $u(z_k) = u(\tilde{z}_k)$, but with $z_k \neq \tilde{z}_k$ for every k. Possibly reducing to subsequences, we may assume that they both converge with $z_\infty := \lim z_k$ and $\tilde{z}_\infty := \lim \tilde{z}_k$, and by continuity it follows that $u(z_\infty) = u(\tilde{z}_\infty)$. Since z_∞ and $\tilde{z}_\infty$ lies both in $\partial\Sigma$ where u is injective, we can conclude that $\lim z_k = \lim \tilde{z}_k$. Using that the differential of Du is not singular in z_∞, we obtain that for k sufficiently large, we will always have $z_k = \tilde{z}_k$ showing that u is indeed injective on a small neighborhood of $\partial\Sigma$.

Assume $z_0 \in \Sigma$ is a point at which the differential $D\varphi$ vanishes. Then we know that φ can be represented in suitable charts as $z \mapsto z^k$ for some $k \in \mathbb{N}$ with $k > 1$. This however yields a contradiction, because we know that φ is a biholomorphism on a neighborhood of $\partial\Sigma$, and hence its degree must be 1. Since φ is holomorphic, it preserves orientations, so that on the other hand, we would have that the degree would need to be *at least* k, if there were such a critical point.

We obtain that φ has nowhere vanishing differential, and hence it must be a regular cover, but since it is of degree 1, it is in fact a biholomorphism, and Σ must be a disk. □

II.3.3. J-holomorphic curves close to elliptic singularities: The higher dimensional situation. In this section, L will always be a closed manifold, and we will choose for T^*L an almost complex structure J_L for which the 0-section L is totally real. Hence there is by Proposition II.3.1 a function $f_L\colon T^*L \to [0,\infty)$ that vanishes on L (and only on L) and that is plurisubharmonic on a small neighborhood of L.

Also, we will only treat elliptic singularities, that means, we will use the level sets of the function $h_f(z_1,z_2) = \frac{1}{2}\left(|z_1|^2 + |z_2|^2\right) + \varepsilon\operatorname{Re}(z_1^2)$ described in the previous section, always supposing that $|\varepsilon| < 1/2$. Note also that we are only studying the product case, where the holonomy of the foliation around the codimension-2 singular set is trivial.

We will now describe an explicit manifold that will serve as a model for the neighborhood of an elliptic singularity. Let $\mathbb{C}^2 \times T^*L$ be the almost complex manifold with almost complex structure $J = i \oplus J_L$, where i is the standard complex structure on $\mathbb{C}^2$. We define a function $F\colon \mathbb{C}^2 \times T^*L \to [0,\infty)$ by

$$F(z_1,z_2,\mathbf{q},\mathbf{p}) = h_f(z_1,z_2) + f_L(\mathbf{q},\mathbf{p})$$

If we stay in a sufficiently small neighborhood of the 0-section of T^*L, this function is clearly J-plurisubharmonic and we denote its regular level set $F^{-1}\big(1/2\big)$ by M; its contact form is given by

$$\alpha := -\,d^J F\big|_{TM} = -\left(d^i h_f + d^{J_L} f_L\right)\big|_{TM} .$$

Let $N_\varepsilon \subset \mathbb{C}^2$ be the submanifold introduced in the previous section as image of the map Φ_ε. The product manifold $N_\varepsilon \times L \subset \mathbb{C}^2 \times T^*L$, where we consider L as the 0-section of T^*L, embeds into M and it carries a Legendrian foliation $\mathcal{F}$ induced by

$$\alpha|_{T(N_\varepsilon\times L)} = -\,d^J F\big|_{T(N_\varepsilon\times L)} = -\,d^i h_f\big|_{TN_\varepsilon} .$$

In particular, the leaves of the foliation are parallel to the L-factor in $N_\varepsilon \times L$ and $\mathcal{F}$ has an elliptic singularity in $\Phi_\varepsilon(0) \times L$.

Note that both the almost complex structure as well as the submanifold $N_\varepsilon \times L$ split as a product, thus if we consider a holomorphic map

$$u\colon (\Sigma,\partial\Sigma;j) \to \big(\mathbb{C}^2 \times T^*L, N_\varepsilon \times L; J\big) ,$$

we can decompose it into $u = (u_1,u_2)$ with

$$\begin{aligned} u_1\colon (\Sigma,\partial\Sigma;j) &\to (\mathbb{C}^2, N_\varepsilon; i) \\ u_2\colon (\Sigma,\partial\Sigma;j) &\to (T^*L, L; J_L) . \end{aligned}$$

This allows us to treat each factor independently from the other one, and we will easily be able to apply the results from the previous section.

Since we are interested in finding a local model, we will first restrict our situation to the following subset

$$U := \big\{(z_1,z_2;\mathbf{q},\mathbf{p}) \;\big|\; \operatorname{Re}(z_2) \ge 1-\delta\big\} \cap F^{-1}\big((-\infty,1/2]\big) \tag{II.3.2}$$

that is, for δ sufficiently small (and $\varepsilon < 1/2$), a compact neighborhood of $\{(0,1)\}\times L \subset N_\varepsilon \times L$ in $F^{-1}\big((-\infty,1/2]\big)$, because the points $(z_1,z_2;\mathbf{q},\mathbf{p})$ in U satisfy

$$0 \le \frac{1}{2}\,|z_1|^2 + f_L(\mathbf{q},\mathbf{p}) + \varepsilon\operatorname{Re}(z_1^2) \le \frac{1}{2}\left(1 - |z_2|^2\right) \le \delta - \frac{1}{2}\,\delta^2 \le \delta$$

so that all coordinates are bounded. Note in particular, that this bound on the $\mathbf{p}$-coordinates guarantees that F will be J-plurisubharmonic on U.

The submanifold $(N_\varepsilon \times L) \cap U$ can also be written in the following easy form

$$\bigl\{(z, x_2) \bigm| \; x_2 \geq 1 - \delta \text{ and } (1 + 2\varepsilon)\, x^2 + (1 - 2\varepsilon)\, y^2 = 1 - x_2^2\bigr\} \times L \; .$$

Corollary II.3.5. *Let*

$$u\colon (\Sigma, \partial\Sigma, j) \to \bigl(U, (N_\varepsilon \times L) \cap U; J\bigr)$$

be any holomorphic map from a connected compact Riemann surface (Σ, j) to U with $u(\partial\Sigma) \subset N_\varepsilon \times L$.

Either u is constant or its image is one of the slices $L_{x_2,\mathbf{q}_0} = \Bigl(\mathbb{C} \times \{x_2\} \times \{\mathbf{q}_0\}\Bigr) \cap U$ with $x_2 \in [1 - \delta, 1)$ and $\mathbf{q}_0$ a point on the 0-section of T^*L. If u is injective at one of its boundary points, then Σ will be a disk, and u is equal to

$$u(z) = \bigl(u_{x_2} \circ \varphi(z); \mathbf{q}_0, \mathbf{0}\bigr) \; ,$$

where u_{x_2} is the map given in Lemma II.3.3, and φ is a Möbius transformation of the unit disk.

Proof. Let u be a J-holomorphic map as in the statement. We will study u by decomposing it into $u = \bigl(u_{\mathbb{C}^2}, u_{T^*L}\bigr)$ with

$$u_{\mathbb{C}^2}\colon (\Sigma, \partial\Sigma, j) \to (\mathbb{C}^2, N_\varepsilon, i)$$
$$u_{T^*L}\colon (\Sigma, \partial\Sigma, j) \to (T^*L, L, J_L) \; .$$

Using that f_L is J_L-plurisubharmonic on the considered neighborhood of the 0-section contained in U, it follows from Corollary II.3.2 that u_{T^*L} is constant. Once we know that u_{T^*L} is constant, the situation for $u_{\mathbb{C}^2}$ is identical to the one in Lemma II.3.4, so that we obtain the desired result. □

The results obtained so far only explain the behavior of holomorphic curves that are completely contained in the model neighborhood U. Next we will extend this result to show that a holomorphic curve is either disjoint from the subset U or is lies completely inside U.

Assume (W, J) is a compact almost complex manifold with convex boundary $M = \partial W$. Let N be a submanifold of M diffeomorphic to $N_\varepsilon \times L$, and assume that there is a compact subset U in W such that U is diffeomorphic to the model above, with $M \cap U$, $N \cap U$ and $J|_U$ all being equal to the corresponding objects in our model neighborhood.

Proposition II.3.6. *Let*

$$u\colon (\Sigma, \partial\Sigma; j) \to (W, N; J)$$

be a holomorphic map, and let U be a compact subset of W that agrees with the model described above.

If $u(\Sigma)$ intersects U, then it has to lie entirely in U, and it will be consequently of the form given by Corollary II.3.5.

Proof. Assume u to be a holomorphic map whose image lies partially in U. The set U is a compact manifold with corners, and we write $\partial U = \partial_M U \cup \partial_W U$, where

$$\partial_M U = U \cap M$$

is the upper boundary of U given by M, and

$$\partial_W U = \bigl\{(z_1, z_2; \mathbf{q}, \mathbf{p}) \bigm| \operatorname{Re}(z_2) \geq 1 - \delta\bigr\} \cap F^{-1}\bigl((-\infty, 1/2]\bigr)$$

is the boundary obtained by cutting off U at some chosen height (see also Fig. 4). We will show that the real part of the z_2-coordinate of u needs to be constant. This then proves the proposition, because it prevents u from leaving U.

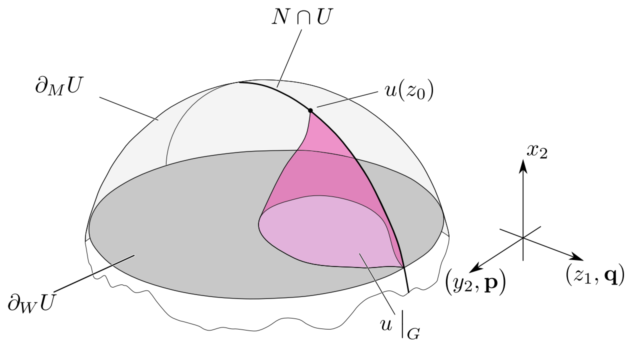


FIGURE 4

If the image of u lies only partially in U, then we may assume that u has varying real part for z_2. Slightly decreasing the cut-off level δ in (II.3.2) and using Sard's theorem, the holomorphic map u will intersect $\partial_W U$ transversely, so that $u^{-1}\big(\partial_W U\big)$ will be a properly embedded submanifold of Σ. We will restrict u to the compact subset $G = u^{-1}\big(U\big)$, and denote the boundary components of this domain by $\partial_M G = u^{-1}\big(N \cap U\big)$ and $\partial_W G = u^{-1}\big(\partial_W U\big)$. We thus have a holomorphic map

$$u|_G\colon (G, \partial G; j) \to \big(U, \partial U; J\big)$$

with $u\big(\partial_M G\big) \subset N \cap U$ and $u\big(\partial_W G\big) \subset \partial_W U$.

The coordinate maps $\pi_{x_2}\colon (z_1, z_2; \mathbf{q}, \mathbf{p}) \mapsto \mathrm{Re}(z_2)$ and $\pi_{y_2}\colon (z_1, z_2; \mathbf{q}, \mathbf{p}) \mapsto \mathrm{Im}(z_2)$ are harmonic, and by the maximum principle it follows that for each component of G the maximum of $\pi_{x_2} \circ u|_G$ needs to lie on the boundary of that component.

Furthermore the maximum of $\pi_{x_2} \circ u|_G$ cannot lie on $\partial_W G$, because by our assumption $u|_G$ is transverse to $\partial_W U$. It follows that the maximum of $\pi_{x_2} \circ u|_G$ will be a point $z_0 \in \partial_M G$; in particular z_0 does not lie on one of the edges of G. By the boundary point lemma, either $\pi_{x_2} \circ u|_G$ is constant or the outward derivative of this function at z_0 must be strictly positive. On the other hand, the function $\pi_{y_2} \circ u|_G$ is equal to 0 all along the boundary $\partial_M G$ so that the derivatives of $\pi_{x_2} \circ u|_G$ and $\pi_{y_2} \circ u|_G$ vanish both at z_0 in directions that are tangent to the boundary. Using the Cauchy-Riemann equation we see that this implies that the derivatives of these two functions at z_0 vanish in *every direction*, in particular this implies that the function $\pi_{x_2} \circ u|_G$ needs to be constant.

In either case, we have proved that the image of u lies completely inside U. □

The conclusion of this section is that by choosing a particular almost complex structure, the elliptic singularities will have a neighborhood for which every holomorphic curve is either completely contained within or it lies completely outside this neighborhood. All holomorphic curves lying inside this neighborhood can be explicitly specified.

II.3.4. J-holomorphic curves close to codimension-1 singularities. Let $(N, \mathcal{F})$ be a submanifold with Legendrian foliation in a contact manifold (M, ξ). We will show in this section that certain codimension-1 components $S \subset \mathrm{Sing}(\mathcal{F})$ in N exclude that any holomorphic curves get close to this component. This way, if S is for example a boundary component of N (as in the case of a bLob), it may block any holomorphic disks from escaping the submanifold N. The argument is similar to that of the previous section, where we constructed an almost complex manifold that served as a model for the neighborhood of the singular set.

Assume in this section that the Legendrian foliation $\mathcal{F}$ is given on a manifold N of the form $(-\varepsilon, \varepsilon) \times S$, where S is a closed manifold, and that $\{0\} \times S$ is a component of singular points of codimension-1. Let r denote the coordinate on $(-\varepsilon, \varepsilon)$. By Lemma I.2.2, we know there is up to diffeomorphism a 1-form $\widetilde{\beta}$ on S, such that $\beta = r\widetilde{\beta}$ is the regular equation determining $\mathcal{F}$ on N. We will assume that $\ker \widetilde{\beta}$ determines a foliation $\mathcal{F}_0$ in $\{0\} \times S$ that *fibers over the circle* hence we can write $\widetilde{\beta}$ as the pull-back of an angular 1-form on the base circle.

We will now find a suitable model neighborhood in the case, where S lies in the interior of N. The following steps can all be easily adapted to the situation, where $S \subset \mathrm{Sing}(\mathcal{F})$ is a boundary component of N (as is for example the case for the boundary of a bLob).

The model neighborhood for $\{0\} \times S$ in a symplectic manifold will be a bundle with exact symplectic fibers and holomorphic projection map. Let F_0 be a leaf of the foliation $\mathcal{F}_0$, then S is the mapping torus of some diffeomorphism $\psi\colon F_0 \to F_0$. We consider the T^*F_0-fibration

$$\begin{array}{rcl} \pi\colon \mathbb{C} \times \mathbb{R} \times \big(\mathbb{R} \times T^*F_0\big)/\sim & \to & \mathbb{C} \times T^*\mathbb{S}^1 \\ (z, r; s; \mathbf{q}, \mathbf{p}) & \mapsto & (z; s, r) \end{array},$$

where we use the equivalence relation $(z, r; s; \mathbf{q}, \mathbf{p}) \sim \big(z, r; s+1; \psi(\mathbf{q}), (D\psi^{-1})^*\mathbf{p}\big)$ on the total space. The submanifold S can be naturally embedded into this model as $S' := \{1\} \times \{0\} \times \big(\mathbb{R} \times F_0\big)/\sim$, where F_0 denotes the 0-section in T^*F_0. The foliation $\mathcal{F}_0$ corresponds under this identification to the fibers with constant s-value.

Remark II.3.7. Note that the construction of the local model includes the case of contact 3-manifolds (for example the boundary of a “flat” overtwisted disk), because we may choose F_0 to just be a point.

Since $(\psi, (D\psi^{-1})^*)\colon T^*F_0 \to T^*F_0$ is symplectic, we get a symplectic structure $d\lambda_{\mathrm{can}}$ on the vertical bundle $\ker D\pi$. Let $J_{\mathcal{F}}$ be a compatible complex structure on this bundle. Note that the directions ∂_r, and ∂_s are well defined, so that we can extend $J_{\mathcal{F}}$ to an almost complex structure $J = i \oplus i \oplus J_{\mathcal{F}}$ on the total space, where $i\,\partial_r = \partial_s$, and $i\,\partial_s = -\partial_r$. By construction, π is holomorphic with respect to J upstairs and $i \oplus i$ on $\mathbb{C} \times T^*\mathbb{S}^1$ downstairs.

The next step consists in finding a J-plurisubharmonic function on a neighborhood of S'. Define a function h on $\mathbb{C} \times \mathbb{R} \times \big(\mathbb{R} \times T^*F_0\big)/\sim$ by using a metric on the *vector bundle* $\mathbb{C} \times \mathbb{R} \times \big(\mathbb{R} \times T^*F_0\big)/\sim$ over $\mathbb{C} \times \mathbb{R} \times \big(\mathbb{R} \times F_0\big)/\sim$, and defining $h(v) = \|v\|^2/2$ for every vector v in this bundle. In a bundle chart, we obtain

$$h(z, r; s; \mathbf{q}, \mathbf{p}) = \frac{1}{2} \sum_{i,j} g_{i,j}(z, r; s; \mathbf{q})\, p_i p_j \;,$$

and it follows that $-dd^J h = -d(dh \circ J)$ simplifies on the 0-section of this bundle to

$$-dd^J h = -\sum_{i,j} g_{i,j}\, dp_i \wedge (dp_j \circ J) \;.$$

We claim now that the function

$$F\colon \mathbb{C}\times\mathbb{R}\times\big(\mathbb{R}\times T^*F_0\big)/\sim\;\to[0,\infty),$$
$$(z,r;s;\mathbf{q},\mathbf{p})\mapsto |z|^2+r^2+h(z,r;s;\mathbf{q},\mathbf{p})$$

is J-plurisubharmonic in a neighborhood of S'. Here one just needs to check that $-dd^JF$ simplifies near $\{1\}\times\mathbb{R}\times\big(\mathbb{R}\times F_0\big)/\sim$ to

$$-dd^JF = 4\,dx\wedge dy + 2\,dr\wedge ds - dd^J h\ ,$$

where $x=\operatorname{Re} z$ and $y=\operatorname{Im} z$. This 2-form is positive on complex lines.

We find a neighborhood of $\{z=1\}$ in the level set $F^{-1}(1)$, where the restriction of the 1-form $\alpha := -d^JF$ defines a contact structure. Furthermore, the submanifold $N'\subset F^{-1}(1)$ given by the embedding

$$\begin{array}{ccc}(-\varepsilon,\varepsilon)\times\big(\mathbb{R}\times F_0\big)/\sim & \hookrightarrow & \mathbb{C}\times\mathbb{R}\times\big(\mathbb{R}\times T^*F_0\big)/\sim\\ (r;s;\mathbf{q}) & \mapsto & \big(\sqrt{1-r^2},r;s;\mathbf{q},\mathbf{0}\big)\end{array}$$

inherits a singular Legendrian foliation given by the form $r\,ds$, whose singular set is as desired the submanifold $S'=N'\cap\{r=0\}$. This foliation is diffeomorphic to the foliation $\mathcal{F}$ on N which was our point of departure. By Theorem I.1.3, a small neighborhood of N' in $F^{-1}(1)$ may be regarded as a model for an embedding of $(N,\mathcal{F})$ in (M,ξ)

Choose a small relatively open set $U\subset F^{-1}\big((0,1]\big)$ containing S' in its boundary, such that $\partial_+U := U\cap F^{-1}(1)$ with contact form α is contactomorphic to a neighborhood of S in M. If $\delta>0$ is a sufficiently small number, the level set $\{x=1-\delta\}$ is a compact hypersurface with boundary in ∂_+U, and we will set $\partial_-U := \{x=1-\delta\}\cap U$, redefining U to be the compact set $U\cap\{x\ge 1-\delta\}$.

Assume from now on that (W,J) is an almost complex manifold with convex boundary, that N is a submanifold in $M=\partial W$ with a singular Legendrian foliation, and that there is a compact subset $U_W\subset W$ diffeomorphic to U such that the almost complex structures agree, and such that $N\cap U_W$ is identified with $U\cap N'$.

Proposition II.3.8. *If the image of a J-holomorphic map*

$$u\colon(\Sigma,\partial\Sigma,j)\to(W,N,J)$$

intersects the neighborhood U_W, then its image must be contained in U_W. In fact, in the identification of U_W with the model neighborhood U, the image of u lies in a slice $U_W\cap\{z=x_0\}$ for a fixed real value x_0.

Proof. Let $u\colon(\Sigma,\partial\Sigma)\to(W,N)$ be any J-holomorphic curve that intersects the neighborhood U_W. Our aim is to show that u must be completely contained in U_W. Define $G:=u^{-1}(U_W)$ and write $u|_G$ for the restriction of u. Perturbing δ slightly, we can assume that $u^{-1}\big(\partial_-U\big)\subset G$ is a properly embedded submanifold so that G has piecewise smooth boundary. Project the holomorphic map $u|_G$ via

$$\pi\colon \mathbb{C}\times\mathbb{R}\times\big(\mathbb{R}\times T^*F_0\big)/\sim\quad\to\quad\mathbb{C}\times T^*\mathbb{S}^1\ ,$$

and note that $\pi\circ u|_G$ is a holomorphic map with respect to the standard complex structure. The boundary $\pi\circ u(\partial G)$ lies in the union of

$$\pi\big(\partial_+U\cap N'\big)=\left\{(z;s,r)\;\middle|\;\operatorname{Re} z\ge 1-\delta,\,\operatorname{Im} z=0,\,r=-\sqrt{1-|z|^2}\right\}$$

and

$$\pi(\partial_- U) = \left\{(z; s, r) \;\middle|\; \operatorname{Re} z = 1 - \delta,\, |z|^2 + r^2 \leq 1\right\} .$$

Since both coordinate functions $x = \operatorname{Re} z$ and $y = \operatorname{Im} z$ are harmonic, it follows that their maxima and minima are both attained on ∂G, so that if we assume y is not everywhere equal to 0, then u must intersect $\pi(\partial_- U)$, and in particular the minimum value of x needs to be $1 - \delta$. Let $z_0 \in \partial G$ be a point for which $u(z_0)$ has both minimal x-coordinate and extremal y-coordinate. At z_0, the derivative of $\pi \circ u|_G$ along the boundary direction ∂G has vanishing x and y-coordinates. Using the Cauchy-Riemann equation at the point z_0, we then see that the derivatives also vanish in the radial direction, thus contradicting the boundary point lemma, making both x and y constant on G.

Using that the boundary of u lies in N, it follows now that u is completely contained in the slice $U_W \cap \{z = x_0\}$ for $x_0 \in \mathbb{R}$. □

The previous proposition showed that holomorphic curves intersecting the neighborhood U need in fact to be entirely contained in U. Note that the singular set S splits the foliated submanifold locally into two components $(N \setminus S) \cap U$.

Proposition II.3.9. *A J-holomorphic map*

$$u\colon (\Sigma, \partial\Sigma, j) \to (U, N \cap U, J)$$

from a compact Riemann surface into U must be constant, if its boundary is mapped only into one component of $(N \setminus S) \cap U$.

This proposition ensures in particular that no holomorphic curves may get close to the boundary of a bLob.

PROOF. From Proposition II.3.8, we immediately recover that the boundary $u|_{\partial\Sigma}$ needs to have r-coordinate equal to $\pm\sqrt{1 - x_0^2}$, where the x_0-value fixes the slice that contains the image of u. Since we are assuming that the boundary of u lies on one side of S, it must have constant r-coordinate, and using that $(z; s, r) \mapsto r$ is a harmonic function, we obtain that $\pi \circ u$ must have constant r-coordinate everywhere, because its maximum and its minimum values are equal. The Cauchy-Riemann equation then implies that the s-coordinate is also constant.

This finishes the proof, because it follows that the projection $\pi \circ u$ is constant, so that u is completely contained in a fiber of π, which we will denote by W_0. By construction, W_0 is diffeomorphic to T^*F_0 and J restricts on W_0 to $J_{\mathcal{F}}$, so that W_0 is an almost complex submanifold equipped with a plurisubharmonic function $F|_{W_0} = h|_{W_0} + 1$. Since the boundary of u lies in W_0 along the 0-section of T^*F_0, where h attains its minimum, it follows that u must be constant. □

Remark II.3.10. Note that when the codimension 1 singular set lies in the *interior* of the maximally foliated submanifold, the almost complex structure J that we have chosen above allows us to easily write down holomorphic annuli with one boundary component on each side of the singular set. These annuli cannot be stable under perturbations of the Cauchy-Riemann operator, because the expected dimension of the moduli space (given by the index of the linearized CR-problem) does not agree with the observed dimension for J. In particular, the holomorphic annuli for this J do not give rise to a Bishop family. In [NW11], we increased the index of the considered problem by choosing a family of foliated submanifolds, and allowing each boundary of the holomorphic annuli to sit on *different* foliated submanifolds. This approach has allowed us to give an alternate proof for the non-fillability of contact 3-manifolds that have positive Giroux torsion, using only classical holomorphic curve methods.

II.4. Appendix: Cotamed complex structures – Existence and contractability

The aim of this appendix is to give the proof of Theorem II.2.1. All of the statements given here can also be found in the appendix of [MNW13], only Proposition II.4.5 has been mildly modified to increase its clarity.

Theorem II.2.1. Let (M, ξ) be a cooriented contact manifold, and let (W, ω) be a symplectic manifold with boundary $M = \partial W$. The following two statements are equivalent

(i) (W, ω) is a weak symplectic filling of (M, ξ).
(ii) There exists an almost complex structure J on W that is tamed by ω and that makes M a J-convex boundary whose J-complex tangencies are ξ.

Proof. Suppose (W, ω) is a weak filling of (M, ξ), and choose a positive contact form α for ξ. Let $p \in M$ be an arbitrary point. Since all the 2-forms $T\,d\alpha + \omega$ on ξ_p for $T \geq 0$ are symplectic, we may use Proposition II.4.4 to find a complex structure $J_\xi(p)$ on ξ_p that is cotamed by $\omega|_{\xi_p}$ and $d\alpha|_{\xi_p}$. Then using the fact that the cotaming property is open, it follows that every point in the manifold M has a small neighborhood on which there exists a complex structure J_ξ on ξ tamed by both $\omega|_\xi$ and $d\alpha|_\xi$. The contractability of the space of all such $J_\xi(p)$'s (Proposition II.4.1), allows us to replace the local complex structures with a global one defined on all of ξ that is cotamed by both 2-forms.

Choose now any vector field X on M that spans $\ker \omega|_{TM}$, and extend it to a collar neighborhood U of M. Let Y be a vector field on U that lies along M in the ω-orthogonal complement of ξ and that satisfies $\omega(X, Y) > 0$. We extend J_ξ to an almost complex structure J on U by setting $JX = Y$. Clearly, J is tamed by ω on a small neighborhood of M, and we can then extend J to the interior of W to obtain the desired tamed almost complex structure on the entire filling W. By construction, $\xi = TM \cap (JTM)$, and M is strictly J-pseudoconvex since J_ξ is tamed by $d\alpha|_\xi$.

Conversely, assume W has an almost complex structure J that is tamed by ω and makes the boundary strictly pseudoconvex, with ξ being the field of complex tangencies $TM \cap (JTM)$. We can then write ξ as the kernel of a nonvanishing 1-form α, and pseudoconvexity implies that we can choose the sign of α in such a way that $d\alpha|_\xi$ tames $J|_\xi$, and such that the natural orientation of ξ together with its co-orientation defined via α is compatible with the boundary orientation of W. Since ω tames J, ω_ξ also tames $J|_\xi$. We therefore have cotaming 2-forms on ξ, so the easy implication (3) $\Longrightarrow$ (2) of Proposition II.4.4 guarantees that (W, ω) is a weak filling of (V, ξ). □

II.4.1. Contractibility of the space of cotamed almost complex structures.

To go from the linear situation to global existence results on a manifold we used the following result.

Proposition II.4.1 (Sévennec)**.** *The space of complex structures on a vector space tamed by two given symplectic forms is either empty or contractible.*

Using the fact that the space of complex structures tamed by a symplectic form is nonempty (which follows for instance by the linear Darboux theorem), and applying the proposition above twice to the same symplectic form, we recover as a special case the classical result of Gromov that states that the space of tamed complex structures is contractible. The proof of the proposition uses the following two lemmas, of which the first is more or less standard.

Lemma II.4.2 (Cayley, Sévennec)**.** *Let V be a real vector space and $\mathcal{J}(V)$ the space of complex structures on V. We can define for any fixed $J_0 \in \mathcal{J}(V)$ a map*

$$\mu_{J_0} \colon J \mapsto (J+J_0)^{-1} \cdot (J-J_0)$$

which is a diffeomorphism from

$$\mathcal{J}^*_{J_0}(V) := \bigl\{J \in \mathcal{J}(V) \mid J + J_0 \in \mathrm{GL}(V)\bigr\}$$

to

$$\mathcal{A}^*_{J_0}(V) := \bigl\{A \in \mathrm{End}(V) \mid AJ_0 = -J_0 A \text{ and } A - I \in \mathrm{GL}(V)\bigr\} \; .$$

The inverse of this map is given by $\mu_{J_0}^{-1} \colon A \mapsto (A-I)J_0(A-I)^{-1}$.

Proof. One can view $\mathcal{A}^*_{J_0}(V)$ as the set of J_0-complex antilinear maps that do not have any eigenvalue equal to 1. Using the equations $(J-J_0)\,J_0 = -J\,(J-J_0)$ and $(J+J_0)\,J_0 = J\,(J+J_0)$, one sees that the image of μ_{J_0} consists of J_0-complex antilinear maps, and $\mu_{J_0}(J) - I = -2\,(J+J_0)^{-1}\,J_0$ is invertible. □

Lemma II.4.3 (Sévennec)**.** *Let (V,ω) be a symplectic vector space and denote by $\mathcal{J}_t(\omega) \subset \mathcal{J}(V)$ the space of complex structures tamed by ω. Choosing any $J_0 \in \mathcal{J}_t(\omega)$, it follows that $\mathcal{J}_t(\omega)$ lies in $\mathcal{J}^*_{J_0}(V)$, and the image of $\mathcal{J}_t(\omega)$ under the associated map μ_{J_0} is a convex domain in $\mathcal{A}^*_{J_0}(V)$.*

We first explain how to prove Proposition II.4.1 using the above lemma. Suppose there is a complex structure J_0 tamed by ω_0 and ω_1. The space of cotamed complex structures $\mathcal{J}_t(\omega_0) \cap \mathcal{J}_t(\omega_1)$ is then diffeomorphic under the map μ_{J_0} to the intersection of the convex subsets given by the lemma. This intersection is again convex and hence contractible.

Proof of Lemma II.4.3. For any complex structure J tamed by ω, the endomorphism $J+J_0$ is invertible because for any nonzero w, $\omega\bigl(w, (J+J_0)\,w\bigr) > 0$, so in particular $(J+J_0)\,w$ is not zero. This proves the first part of the lemma.

Now fix a nonzero vector $v \in V$, and let C_v be the set of $A \in \mathrm{End}(V)$ that anticommute with J_0, and that satisfy

$$\omega\bigl((A-I)\,v, (A-I)\,J_0 v\bigr) = -\omega\bigl((A-I)\,v, J_0(A+I)\,v\bigr) > 0 \; .$$

We now prove that $C_v \subset \mathrm{End}(V)$ is convex. Every segment $A_s = (1-s)\,A_0 + s\,A_1$ with $s \in [0,1]$ for arbitrary $A_0, A_1 \in C_v$ defines a polynomial of degree 2

$$P(s) = -\omega\bigl((A_s - I)\,v, J_0(A_s + I)\,v\bigr) \; ,$$

and the above inequality corresponds to checking that $P(s)$ is positive for all values $s \in [0,1]$. The leading coefficient $-\omega\bigl((A_1 - A_0)\,v, J_0(A_1 - A_0)\,v\bigr)$ of $P(s)$ is never positive, because J_0 tames ω, so that $P(s)$ is either a line or a parabola facing downward. In both cases $P(s) \geq \min\{P(0), P(1)\} > 0$ for all $s \in (0,1)$ so the inequality holds for the whole segment A_s.

Note that $C_v \neq \emptyset$ since $\mathbf{0} \in C_v$. Define the intersection

$$C^* := \bigcap_{v \neq 0} C_v \; ,$$

which is a nonempty convex subset of $\operatorname{End}(V)$. In fact, one has $C^* \subset \mathcal{A}^*_{J_0}(V)$, because if there were a matrix $A \in C^*$ with $\det(A - I) = 0$, then A would have an eigenvector $w \in V$ with eigenvalue 1, but then $-\omega\big((A-I)\,w, J_0(A+I)\,w\big) = 0$ so that $A \notin C_w$.

Since C^* lies in the domain of $\mu_{J_0}^{-1}$ and $\mathcal{J}_t(\omega)$ lies in the domain of μ_{J_0}, we have $C^* = \mu_{J_0}\big(\mathcal{J}_t(\omega)\big)$, so that the image of the complex structures tamed by ω is convex as we wanted to show. □

II.4.2. Existence of a cotamed complex structure on a vector space.

Proposition II.4.4. *Let V be a real vector space equipped with two symplectic forms ω_0 and ω_1. The following properties are equivalent:*

(1) *the segment between ω_0 and ω_1 consists of symplectic forms*
(2) *the ray starting at ω_0 and directed by ω_1 consists of symplectic forms*
(3) *there is a complex structure J on V tamed both by ω_0 and by ω_1.*

The equivalence between (1) and (3) was explained to us by Jean-Claude Sikorav. It relies on the simultaneous reduction of symplectic forms. Specifically, we need [**LR05**, Theorem 9.1] which we shall state (in a slightly weakened form) and reprove (in its full force) below as Proposition II.4.5, since the very general context of [**LR05**] makes it hard to read for people interested only in the symplectic case.

Recall that according to the linear Darboux theorem, any symplectic form on a $2n$-dimensional vector space is represented in some basis by the standard matrix

$$\Omega_{2n} = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix} .$$

We now want to understand what can be said for a pair of symplectic structures. Below we give an approximate normal form which is sufficient for our purposes and more pleasant to state than the precise result (cf. [**LR05**, Theorem 9.1]), though the precise result can also be extracted from the proof that we will give at the end of this section.

Proposition II.4.5. *Let ω_0 and ω_1 be symplectic forms on a vector space V. There exists a matrix A_1 that is block diagonal with blocks of the form*

$$\begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix} \in \operatorname{Mat}\big(\mathbb{R}^{2\times 2}\big) \text{ and } \begin{pmatrix} 0 & 0 & \mu & \nu \\ 0 & 0 & -\nu & \mu \\ -\mu & \nu & 0 & 0 \\ -\nu & -\mu & 0 & 0 \end{pmatrix} \in \operatorname{Mat}\big(\mathbb{R}^{4\times 4}\big)$$

for $\lambda, \nu \neq 0$. Using A_1 we define a second matrix A_0 by removing in A_1 the blocks on the diagonal and replacing them either with Ω_2 or with Ω_4 depending on the size of the erased block.

For any $\varepsilon > 0$, there is a basis of V such that ω_0 is represented with respect to this basis by A_0, and ω_1 is represented by a matrix which is ε-close to A_1. If the linear segment between ω_0 and ω_1 consists of symplectic forms, then the coefficients λ in the 2×2-blocks of A_1 described above cannot be negative.

The relation with cotamed complex structures will come from the following.

Proposition II.4.6.

(a) *Let $V = \mathbb{R}^2$ with two antisymmetric bilinear forms ω_0 and ω_1 defined by $\omega_j(v,w) = v^t A_j w$, where*

$$A_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad A_1 = \begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix} .$$

If $\lambda > 0$, then $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is tamed by both forms.

(b) *Let $V = \mathbb{R}^4$, and let ω_0 and ω_1 be antisymmetric bilinear forms defined by the matrices*

$$A_0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad A_1 = \begin{pmatrix} 0 & 0 & \lambda & \mu \\ 0 & 0 & -\mu & \lambda \\ -\lambda & \mu & 0 & 0 \\ -\mu & -\lambda & 0 & 0 \end{pmatrix} ,$$

with $\mu \neq 0$. Then there exists a complex structure J on $\mathbb{R}^4$ that is tamed by both forms.

Proof. We only need to prove (b). For simplicity write V as $\mathbb{C}^2$, and the matrices A_0 and A_1 as

$$A_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad A_1 = \begin{pmatrix} 0 & z \\ -\bar{z} & 0 \end{pmatrix}$$

with $z = \lambda + i\mu = re^{i\psi}$. The matrices

$$J_\phi = \begin{pmatrix} 0 & e^{i\phi} \\ -e^{-i\phi} & 0 \end{pmatrix}$$

define complex structures on V, and it follows that $A_0 J_\phi = -\left(\begin{smallmatrix} e^{-i\phi} & 0 \\ 0 & e^{i\phi} \end{smallmatrix}\right)$ is positive definite if $\cos\phi < 0$, and $A_1 J_\phi = -r\left(\begin{smallmatrix} e^{i(\psi-\phi)} & 0 \\ 0 & e^{i(\phi-\psi)} \end{smallmatrix}\right)$ is positive definite if $\cos(\psi - \phi) < 0$. As long as $\psi \neq \pi$ (which we have excluded by requiring that $\mu \neq 0$), it follows that we can choose ϕ such that $\phi \in (\pi/2, 3\pi/2)$ and $\phi - \psi \in (\pi/2, 3\pi/2) + 2\pi\mathbb{Z}$. □

Proof of Proposition II.4.4. We first explain the easy equivalence between (1) and (2). The (open) ray starting at ω_0 and directed by ω_1 and the open interval between ω_0 and ω_1 span the same cone in the space of skew symmetric bilinear forms. Since being symplectic is invariant under nonzero scalar multiplication, we have the equivalence.

The implication (3) $\implies$ (1) is also direct because, for any $t \in [0,1]$, we have

$$\big((1-t)\,\omega_0 + t\,\omega_1\big)(v, Jv) = (1-t)\,\omega_0(v, Jv) + t\,\omega_1(v, Jv) ,$$

which is positive whenever $v \in V$ is nonzero. So in particular, no such v can be in the kernel of an element of the segment between ω_0 and ω_1.

To prove (1) $\implies$ (3), we use the fact that by Proposition II.4.5, there is a matrix A_1' that splits into certain standard blocks, such that we can find for any $\varepsilon > 0$ a basis of V for which ω_0 is in canonical form, and for which ω_1 is represented by a matrix that is ε-close to A_1'.

If condition (1) holds, then the blocks of A_1' correspond to the ones described in Proposition II.4.6, and we obtain the existence of a complex structure J on V that is tamed both by the standard symplectic form and by A_1'. By choosing $\varepsilon > 0$ sufficiently small, it follows that J is also tamed by ω_0 and ω_1, because tameness is an open condition. □

Proof of Proposition II.4.5. The proof will proceed in several steps.

Decomposition into generalized eigenspaces. In the first step we shall decompose V into suitable subspaces that are both ω_0- and ω_1-orthogonal.

Let $\varphi_r \colon V \to V^*$ for $r = 0, 1$ be the isomorphisms defined by $\varphi_r(v) := \omega_r(v, \cdot)$. We consider the endomorphism $B = \varphi_0^{-1} \circ \varphi_1$ of V so that $\omega_1(v, w) = \omega_0(Bv, w)$. The endomorphism B is invertible and it is ω_0-symmetric since:

$$\omega_0(Bv, w) = \omega_1(v, w) = -\omega_1(w, v) = -\omega_0(Bw, v) = \omega_0(v, Bw) .$$

To define the generalized eigenspaces of B, complexify the vector space V to obtain $V^{\mathbb{C}}$, and extend the ω_r to sesquilinear forms $\omega_r^{\mathbb{C}}$. A computation analogous to the preceding one shows that B is $\omega_0^{\mathbb{C}}$-symmetric and we still have $\omega_0^{\mathbb{C}}(v, Bw) = \omega_1^{\mathbb{C}}(v, w)$.

The characteristic polynomial of B splits over $\mathbb{C}$ as $P(X) = \prod_\lambda (X - \lambda)^{m_\lambda}$, so we can decompose $V^{\mathbb{C}}$ into generalized eigenspaces

$$V^{\mathbb{C}} = \bigoplus_{\lambda \in Sp(B)} E_\lambda^{\mathbb{C}} \quad ; \quad E_\lambda^{\mathbb{C}} = \ker(B - \lambda)^{m_\lambda} .$$

Lemma II.4.7. *If λ and μ are eigenvalues of B such that $\lambda \neq \bar{\mu}$, then $E_\lambda^{\mathbb{C}}$ and $E_\mu^{\mathbb{C}}$ are both $\omega_0^{\mathbb{C}}$- and $\omega_1^{\mathbb{C}}$-orthogonal.*

Proof. We prove by induction on k and l that $\ker(B-\lambda)^k$ and $\ker(B-\mu)^l$ are orthogonal.

To start the induction, note that if $v_\lambda \in \ker(B - \lambda)$, and $v_\mu \in \ker(B - \mu)$, then

$$(\bar{\lambda} - \mu)\, \omega_0^{\mathbb{C}}(v_\lambda, v_\mu) = \omega_0^{\mathbb{C}}\big((B - \bar{\mu})\, v_\lambda, v_\mu\big) = \omega_0^{\mathbb{C}}\big(v_\lambda, (B - \mu)\, v_\mu\big) = 0 ,$$

thus since $\lambda \neq \bar{\mu}$, it follows that $\omega_0^{\mathbb{C}}(v_\lambda, v_\mu) = 0$. Similarly, $\omega_1^{\mathbb{C}}(v_\lambda, v_\mu) = \omega_0^{\mathbb{C}}(v_\lambda, Bv_\mu) = \mu\, \omega_0^{\mathbb{C}}(v_\lambda, v_\mu) = 0$.

Assume now it has already been shown for the integers k and l that $\ker(B - \lambda)^k$ and $\ker(B-\mu)^l$ are both $\omega_0^{\mathbb{C}}$- and $\omega_1^{\mathbb{C}}$-orthogonal. Choose a vector $v'_\lambda \in \ker(B-\lambda)^{k+1}$ and use the fact that $Bv'_\lambda = \lambda\, v'_\lambda + w$ for some $w \in \ker(B-\lambda)^k$. Then we obtain for any $v_\mu \in \ker(B-\mu)^l$,

$$\begin{aligned}(\bar{\lambda} - \mu)^l\, \omega_0^{\mathbb{C}}(v'_\lambda, v_\mu) &= (\bar{\lambda} - \mu)^{l-1}\, \omega_0^{\mathbb{C}}\big((B - \bar{\mu})\, v'_\lambda - w, v_\mu\big) \\ &= (\bar{\lambda} - \mu)^{l-1}\, \omega_0^{\mathbb{C}}\big((B - \bar{\mu})\, v'_\lambda, v_\mu\big) = \omega_0^{\mathbb{C}}\big(v'_\lambda, (B - \mu)^l\, v_\mu\big) = 0 ,\end{aligned}$$

and also $\omega_1^{\mathbb{C}}(v'_\lambda, v_\mu) = \omega_0^{\mathbb{C}}(Bv'_\lambda, v_\mu) = \bar{\lambda}\, \omega_0^{\mathbb{C}}(v'_\lambda, v_\mu) + \omega_0^{\mathbb{C}}(w, v_\mu) = 0$, which proves the induction step from (k, l) to $(k+1, l)$. Since λ and μ have completely symmetric roles, this also explains how to go to $(k, l+1)$. □

We now relate this decomposition of $V^{\mathbb{C}}$ to the initial real vector space V. For a real eigenvalue λ, the intersection $V \cap E_\lambda^{\mathbb{C}}$ defines a real subspace E_λ with $\dim_{\mathbb{R}} E_\lambda = \dim_{\mathbb{C}} E_\lambda^{\mathbb{C}}$. Complex conjugation defines an isomorphism $E_\lambda^{\mathbb{C}} \to E_{\bar{\lambda}}^{\mathbb{C}}, v_\lambda \mapsto \bar{v}_\lambda$, and we can write $V \cap \big(E_\lambda^{\mathbb{C}} \oplus E_{\bar{\lambda}}^{\mathbb{C}}\big)$ for $\lambda \in \mathbb{C} \setminus \mathbb{R}$ as the direct sum of real subspaces $E_{\{\lambda, \bar{\lambda}\}} = \big\{v + \bar{v} \bigm| v \in E_\lambda^{\mathbb{C}}\big\} \oplus \big\{i\,(v - \bar{v}) \bigm| v \in E_\lambda^{\mathbb{C}}\big\}$.

This way we find a decomposition of V into pairwise ω_0- and ω_1-orthogonal subspaces

$$E_{\mu_1} \oplus \cdots \oplus E_{\mu_k} \oplus E_{\{\lambda_1, \bar{\lambda}_1\}} \oplus \cdots \oplus E_{\{\lambda_l, \bar{\lambda}_l\}}$$

with $\mu_1, \ldots, \mu_k \in \mathbb{R} \setminus \{0\}$, and $\lambda_1, \ldots, \lambda_l \in \mathbb{C} \setminus \mathbb{R}$.

Blocks with real eigenvalue.

For the following considerations, we restrict to one of the subspaces E_{λ_j} with $\lambda_j \in \mathbb{R}$, and denote λ_j for simplicity just by λ. We will construct a basis of E_λ such that ω_0 and ω_1

have the particularly nice form described in the proposition. Note that ω_0 and ω_1 are both nondegenerate on E_λ.

Let $k+1$ be the nilpotency index of $B-\lambda$, i.e. $(B-\lambda)^{k+1}=0$ and $(B-\lambda)^k \neq 0$. Let v_0 be an element of E_λ not in $\ker(B-\lambda)^k$. We set $v_j := \varepsilon^{-j}(B-\lambda)^j v_0$ to define a collection of vectors $v_0,\dots,v_k$. Choose now a vector $w_k \in E_\lambda$ with $\omega_0(v_k,w_k)=1$ and $\omega_0(v_j,w_k)=0$ for every $j\neq k$, and define inductively $w_{j-1} := \varepsilon^{-1}(B-\lambda)\,w_j$, or equivalently

$$Bw_j = \lambda\, w_j + \varepsilon\, w_{j-1}$$

for $j \geq 1$.

Lemma II.4.8. *The vectors $v_0,\dots,v_k,w_0,\dots,w_k$ are linearly independent and satisfy the relations $\omega_r(v_j,v_{j'}) = \omega_r(w_j,w_{j'}) = 0$ for all $r=0,1$, and j,j', and*

$$\omega_0(v_j,w_{j'}) = \delta_{j,j'} \qquad \textit{and} \qquad \omega_1(v_j,w_{j'}) = \lambda\,\delta_{j,j'} + \varepsilon\,\delta_{j,j'-1}\ .$$

PROOF. We start by proving $\omega_r(v_j,v_{j'})=0$. For this we will use an induction on $|j-j'|$. If $j-j'=0$ then the statement follows directly from the antisymmetry of ω_r. Suppose that the claim is true for $j-j'\leq m$ and consider any j and j' with $j-j'=m+1$ (in particular $j\geq 1$). We have

$$\varepsilon\,\omega_0(v_j,v_{j'}) = \omega_0\big((B-\lambda)\,v_{j-1},v_{j'}\big) = \omega_1(v_{j-1},v_{j'}) - \lambda\,\omega_0(v_{j-1},v_{j'}) = 0$$

by the induction hypothesis. Using the fact that $Bv_{j'} = \varepsilon\, v_{j'+1} + \lambda\, v_{j'}$, we compute

$$\omega_1(v_j,v_{j'}) = \omega_0(v_j,Bv_{j'}) = \varepsilon\,\omega_0(v_j,v_{j'+1}) + \lambda\,\omega_0(v_j,v_{j'})\ .$$

The first term is zero by the induction hypothesis and the second one is zero because of the preceding computation. The proof of $\omega_r(w_j,w_{j'})=0$ follows the same lines, and will be omitted.

Note that

$$\begin{aligned}\omega_0(v_j,w_{j'}) &= \varepsilon^{j'-k}\,\omega_0\big(v_j,(B-\lambda)^{k-j'}\,w_k\big)\\ &= \varepsilon^{j'-k}\,\omega_0\big((B-\lambda)^{k-j'}\,v_j,w_k\big) = \omega_0(v_{k+j-j'},w_k) = \delta_{j,j'}\ ,\end{aligned}$$

and in particular this implies that $v_0,\dots,v_k,w_0,\dots,w_k$ are linearly independent vectors with respect to which ω_0 has standard form.

The remaining relation for ω_1 can be obtained by

$$\omega_1(v_j,w_{j'}) = \omega_0(v_j,Bw_{j'}) = \lambda\,\omega_0(v_j,w_{j'}) + \varepsilon\,\omega_0(v_j,w_{j'-1}) = \lambda\,\delta_{j,j'} + \varepsilon\,\delta_{j,j'-1}\ . \qquad \square$$

If we restrict ω_0 and ω_1 to the subspace $E=\langle v_0,\dots,v_k,w_0,\dots,w_k\rangle$ and represent them in this basis, we now find that ω_0 is in standard form Ω_{2k} and ω_1 is represented by a matrix ε-close to $\lambda\,\Omega_{2k}$.

To continue the proof, restrict ω_0, ω_1, and B to the ω_0-symplectic complement E' of the space E. Note that E' is stable under B because for $u\in E'$,

$$\omega_0(v_j,Bu) = \omega_0(Bv_j,u) = \lambda\,\omega_0(v_j,u) + \varepsilon\,\omega_0(v_{j-1},u) = 0\ ,$$

and similarly for $\omega_0(w_j,Bu)=0$. We can thus proceed as before to reduce all eigenspaces E_λ with $\lambda\in\mathbb{R}$ to ω_0-symplectic blocks in normal form.

Blocks with complex eigenvalue. We proceed now to the generalized complex eigenspace $E_\lambda^{\mathbb{C}}$ with $\lambda\in\mathbb{C}\setminus\mathbb{R}$. Let k be the largest integer for which $E_\lambda^{\mathbb{C}} \neq \ker(B-\lambda)^k$, and construct as before a chain of vectors $v_0, \dots, v_k \in E_\lambda^{\mathbb{C}}$ by starting with an element $v_0 \in E_\lambda^{\mathbb{C}} \setminus \ker(B-\lambda)^k$, and defining inductively

$$v_{j+1} := \varepsilon^{-1}\,(B-\lambda)\,v_j \;.$$

Using complex conjugation, we also find a chain $\bar v_0, \dots, \bar v_k$ that lies in $E_{\bar\lambda}^{\mathbb{C}}$. Since B is the complexification of a real linear map, $\bar v_{j+1} := \varepsilon^{-1}\,(B-\bar\lambda)\,\bar v_j$ holds.

Next, we define two chains $w_0, \dots, w_k$ in $E_{\bar\lambda}^{\mathbb{C}}$ and $\bar w_0, \dots, \bar w_k$ in $E_\lambda^{\mathbb{C}}$ by starting with a vector $w_k \in E_{\bar\lambda}^{\mathbb{C}}$ with $\omega_0^{\mathbb{C}}(v_k, w_k) = 1$ and $\omega_0^{\mathbb{C}}(v_j, w_k) = 0$ for every $j \neq k$, and defining $w_{j-1} := \varepsilon^{-1}\,(B-\bar\lambda)\,w_j$, or equivalently

$$Bw_j = \bar\lambda\, w_j + \varepsilon\, w_{j-1}$$

for $j \geq 1$. Similarly, we obtain $\bar w_{j-1} = \varepsilon^{-1}\,(B-\lambda)\,\bar w_j$.

Lemma II.4.9.

(a) *The space spanned by $v_0, \dots, v_k, \bar v_0, \dots, \bar v_k$ and the one spanned by $w_0, \dots, w_k, \bar w_0, \dots, \bar w_k$ are each isotropic with respect to both ω_0 and ω_1.*

(b) *The $\omega_0^{\mathbb{C}}$-pairings for these vectors are given by*

$$\begin{aligned} \omega_0^{\mathbb{C}}(v_j, \bar w_{j'}) &= 0 \;, & \omega_0^{\mathbb{C}}(v_j, w_{j'}) &= \delta_{j,j'} \;, \\ \omega_0^{\mathbb{C}}(\bar v_j, w_{j'}) &= 0 \;, & \omega_0^{\mathbb{C}}(\bar v_j, \bar w_{j'}) &= \delta_{j,j'} \;. \end{aligned}$$

(c) *The $\omega_1^{\mathbb{C}}$-pairings for these vectors are given by*

$$\begin{aligned} \omega_1^{\mathbb{C}}(v_j, \bar w_{j'}) &= 0 \;, & \omega_1^{\mathbb{C}}(v_j, w_{j'}) &= \lambda\, \delta_{j,j'} + \varepsilon\, \delta_{j,j'-1} \;, \\ \omega_1^{\mathbb{C}}(\bar v_j, w_{j'}) &= 0 \;, & \omega_1^{\mathbb{C}}(\bar v_j, \bar w_{j'}) &= \bar\lambda\, \delta_{j,j'} + \varepsilon\, \delta_{j,j'-1} \;. \end{aligned}$$

Proof. To prove (a) note that since $\lambda \neq \bar\lambda$, the spaces $E_\lambda^{\mathbb{C}}$ and $E_{\bar\lambda}^{\mathbb{C}}$ are both $\omega_0^{\mathbb{C}}$- and $\omega_1^{\mathbb{C}}$-isotropic, so we only need to show that $\omega_r^{\mathbb{C}}(\bar v_j, v_{j'}) = \omega_r^{\mathbb{C}}(\bar w_j, w_{j'}) = 0$ for all j, j', and for $r = 0, 1$. If $j = j'$, we write v_j as $v_x + iv_y$, and we use sesquilinearity as follows:

$$\begin{aligned} \omega_0^{\mathbb{C}}(\bar v_j, v_j) &= \omega_0^{\mathbb{C}}(v_x, v_x) + \omega_0^{\mathbb{C}}(v_x, iv_y) - \omega_0^{\mathbb{C}}(iv_y, v_x) - \omega_0^{\mathbb{C}}(iv_y, iv_y) \\ &= \omega_0(v_x, v_x) + i\omega_0(v_x, v_y) + i\omega_0(v_y, v_x) - \omega_0(v_y, v_y) \\ &= 0 \;. \end{aligned}$$

By the same computation, $\omega_1^{\mathbb{C}}(\bar v_j, v_j) = 0$.

If the statement is true for $j' - j = m \geq 0$, then

$$\begin{aligned} \varepsilon\, \omega_0^{\mathbb{C}}(\bar v_j, v_{j'+1}) &= \omega_0^{\mathbb{C}}\big(\bar v_j, (B-\lambda)\, v_{j'}\big) = \omega_1^{\mathbb{C}}(\bar v_j, v_{j'}) - \lambda\, \omega_0^{\mathbb{C}}(\bar v_j, v_{j'}) \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} \omega_1^{\mathbb{C}}(\bar v_j, v_{j'+1}) &= \omega_0^{\mathbb{C}}(B\bar v_j, v_{j'+1}) = \omega_0^{\mathbb{C}}(\bar\lambda\, \bar v_j + \varepsilon\, \bar v_{j+1}, v_{j'+1}) \\ &= 0 \;, \end{aligned}$$

which finishes the induction. The argument for $\omega_r^{\mathbb{C}}(\bar w_j, w_{j'})$ is identical.

To prove (b), note first that the second two equations are the complex conjugate of the first two. Since $v_j, \bar w_{j'} \in E_\lambda^{\mathbb{C}}$, it also follows immediately that $\omega_0^{\mathbb{C}}(\bar v_j, \bar w_{j'}) = 0$, so that we are only left with showing $\omega_0^{\mathbb{C}}(v_j, w_{j'}) = \delta_{j,j'}$, but the required computation is identical to the one used to show the analogous relation in the proof of Lemma II.4.8.

The equalities for (c) follow similarly. □

We will now intersect the complex subspace spanned by the chains defined above with the initial real vector space V to finish the proof of the proposition. For this, define for all $j \leq k$ the real vectors

$$v_j^+ = \frac{1}{\sqrt{2}}\,(v_j + \bar{v}_j), \qquad v_j^- = \frac{i}{\sqrt{2}}\,(v_j - \bar{v}_j)$$

and

$$w_j^+ = \frac{1}{\sqrt{2}}\,(w_j + \bar{w}_j), \qquad w_j^- = \frac{i}{\sqrt{2}}\,(w_j - \bar{w}_j)$$

which all lie in $E_{\lambda,\bar{\lambda}}$. Using the results deduced above, we obtain for all $r = 0, 1$, and j, j' the equations $\omega_r(v_j^+, v_{j'}^\pm) = \omega_r(v_j^-, v_{j'}^\pm) = 0$ and $\omega_r(w_j^+, w_{j'}^\pm) = \omega_r(w_j^-, w_{j'}^\pm) = 0$, and finally

$$\begin{aligned}
2\,\omega_0(v_j^+, w_{j'}^+) &= \omega_0^{\mathbb{C}}(v_j, w_{j'} + \bar{w}_{j'}) + \omega_0^{\mathbb{C}}(\bar{v}_j, w_{j'} + \bar{w}_{j'}) = 2\,\delta_{j,j'}\ , \\
2\,\omega_0(v_j^+, w_{j'}^-) &= i\,\omega_0^{\mathbb{C}}(v_j, w_{j'} - \bar{w}_{j'}) + i\,\omega_0^{\mathbb{C}}(\bar{v}_j, w_{j'} - \bar{w}_{j'}) = 0\ , \\
2\,\omega_1(v_j^+, w_{j'}^+) &= \omega_1^{\mathbb{C}}(v_j, w_{j'} + \bar{w}_{j'}) + \omega_1^{\mathbb{C}}(\bar{v}_j, w_{j'} + \bar{w}_{j'}) \\
&= \omega_0^{\mathbb{C}}(v_j, Bw_{j'}) + \omega_0^{\mathbb{C}}(\bar{v}_j, B\bar{w}_{j'}) = \bar{\lambda}\,\omega_0^{\mathbb{C}}(v_j, w_{j'}) + \varepsilon\,\omega_0^{\mathbb{C}}(v_j, w_{j'-1}) \\
&+ \lambda\,\omega_0^{\mathbb{C}}(\bar{v}_j, \bar{w}_{j'}) + \varepsilon\,\omega_0^{\mathbb{C}}(\bar{v}_j, \bar{w}_{j-1}) \\
&= (\lambda + \bar{\lambda})\,\delta_{j,j'} + 2\varepsilon\,\delta_{j,j'-1}
\end{aligned}$$

and similar computations for the other matrix elements, which prove the desired result with $\mu = \operatorname{Re}\lambda$ and $\nu = \operatorname{Im}\lambda$.

Sign of real eigenvalues. Assume that all 2-forms in the family

$$\omega_t := (1-t)\,\omega_0 + t\,\omega_1$$

for $t \in [0,1]$ are nondegenerate. The λ-coefficients in the 2×2-blocks of A_1' correspond to the real eigenvalues of the map B, so that if $\lambda < 0$ with eigenvector v, then we have $\omega_1(v,\cdot) = \omega_0(Bv,\cdot) = \lambda\,\omega_0(v,\cdot)$, and it follows that $\omega_t(v,\cdot) = (1-t+t\lambda)\,\omega_0(v,\cdot)$ has to vanish for a certain value $t_0 \in (0,1)$, so that ω_{t_0} is degenerate. $\square$

CHAPTER III

Moduli spaces of holomorphic disks and filling obstructions

III.1. The moduli space of holomorphic disks

Let us assume again that (W, J) is an almost complex manifold, and that $N \subset W$ is a totally real submanifold. We want to study the space of maps

$$u\colon (\mathbb{D}^2, \partial\mathbb{D}^2) \to (W, N; J)$$

that are J-holomorphic (strictly speaking they are (i, J)-holomorphic), meaning that we want the differential of u to be complex linear, so that it satisfies at every $z \in \mathbb{D}^2$ the equation

$$Du_z \cdot i = J\big(u(z)\big) \cdot Du_z \ .$$

Note that J depends on the point $u(z)$!

A different way to state this equation is by introducing the Cauchy-Riemann operator

$$\bar{\partial}_J u = J(u) \cdot Du - Du \cdot i \ ,$$

and writing $\bar{\partial}_J u = 0$, so that the space of J-holomorphic maps, we are interested in then becomes

$$\widetilde{\mathcal{M}}\big(\mathbb{D}^2, N; J\big) = \big\{u\colon \mathbb{D}^2 \to W \;\big|\; \bar{\partial}_J u = 0 \text{ and } u(\partial\mathbb{D}^2) \subset N\big\} \ .$$

Remark III.1.1. The situation of holomorphic disks is a bit special compared to the one of general holomorphic maps, because all complex structures on the disk are equivalent. If Σ were a smooth compact surface of higher genus, we would usually need to study the space of pairs (u, j), where j is a complex structure on Σ, and u is a map $u\colon (\Sigma, \partial\Sigma) \to (W, N)$ that should be (j, J)-holomorphic, that means, $J(u) \cdot Du - Du \cdot j = 0$.

To be a bit more precise, we do not choose pairs (u, j) with arbitrary complex structures j on Σ, but we only allow for j a single element in each equivalence class of complex structures: If $\varphi\colon \Sigma \to \Sigma$ is a diffeomorphism, and j is some complex structure, then of course $\varphi^* j$ will generally be a complex structure different from j, but we usually identify all complex structures up to isotopy, and use that the space of equivalence classes of complex structures can be represented as a smooth finite dimensional manifold (see [**Hum97**] for a nice introduction to this theory).

Fortunately, these complications are not necessary for holomorphic disks (or spheres), and it is sufficient for us to work with the standard complex structure i on $\mathbb{D}^2$.

In this section, we want to explain the topological structure of the space $\widetilde{\mathcal{M}}\big(\mathbb{D}^2, N; J\big)$ without entering into too many technical details. Instead of starting directly with our particular case, we will try to argue on an intuitive level by considering a finite dimensional situation that has strong analogies with the problem we are dealing with.

Let us consider a vector bundle E of rank r over a smooth n-manifold B. Choose a section $\sigma\colon B \to E$, and let $M = \sigma^{-1}(0)$ be the set of points at which σ intersects the 0-section. We would "expect" M to be a smooth submanifold of dimension $\dim M = n - r$ (if

$n - r < 0$, we could hope not to have any intersections at all); unfortunately, this intuitive expectation might very well be false. A sufficient condition under which it holds, is when σ is transverse to the 0-section, that means, for every $x \in M$, the tangent space to the 0-section T_xB in T_xE spans together with the image $D\sigma \cdot T_xB$ the whole tangent space T_xE. It is well-known that when the transversality condition is initially not true, it can be achieved by slightly perturbing the section σ.

Let us now come again to the Cauchy-Riemann problem. The role of B will be taken by the space of all maps $u\colon \big(\mathbb{D}^2, \partial\mathbb{D}^2\big) \to (W, N)$, which we will denote by $\mathcal{B}\big(\mathbb{D}^2; N\big)$. We do not want to spend any time thinking about the regularity of the maps and point instead to [**MS04**] as reference. It is sufficient for us to observe that the space $\mathcal{B}\big(\mathbb{D}^2; N\big)$ is a Banach manifold, that means, an infinite dimensional manifold modeled on a Banach space.

The section σ will be replaced by the Cauchy-Riemann operator $\bar\partial_J$, and before pursuing this analogy further, we want first to specify the target space of this operator. In fact, $\bar\partial_J$ associates to every map $u \in \mathcal{B}\big(\mathbb{D}^2; N\big)$ a 1-form on Σ with values in TW. The formal way to state this is that we have for every map u a vector bundle u^*TW over $\mathbb{D}^2$, which allows us to construct

$$\mathrm{Hom}\big(T\mathbb{D}^2, u^*TW\big) \; .$$

The sections in $\mathrm{Hom}\big(T\mathbb{D}^2, u^*TW\big)$ form a vector space, and if we look at all sections for *all* maps u, we obtain a vector bundle over $\mathcal{B}\big(\mathbb{D}^2; N\big)$, whose fiber over a point u are all sections in $\mathrm{Hom}\big(T\mathbb{D}^2, u^*TW\big)$. We denote this bundle by $\mathcal{E}\big(\mathbb{D}^2; N\big)$.

The operator $\bar\partial_J$ associates to every u, that means, to every point of $\mathcal{B}\big(\mathbb{D}^2; N\big)$ an element in $\mathcal{E}\big(\mathbb{D}^2; N\big)$ so that we can think of $\bar\partial_J$ as a section in the bundle $\mathcal{E}\big(\mathbb{D}^2; N\big)$. The J-holomorphic maps are the points of $\mathcal{B}\big(\mathbb{D}^2; N\big)$ where the section $\bar\partial_J$ intersects the 0-section. In fact, $\bar\partial_J u$ is always anti-holomorphic, because

$$J(u) \cdot \bar\partial_J u = -Du - J(u) \cdot Du \cdot i = \big(Du \cdot i - J(u) \cdot Du\big) \cdot i = -\big(\bar\partial_J u\big) \cdot i \; ,$$

and for analytical reasons we will only consider sections in $\mathrm{Hom}\big(T\Sigma, u^*TW\big)$ taking values in the subbundle $\overline{\mathrm{Hom}_{\mathbb{C}}}\big(T\Sigma, u^*TW\big)$ of anti-holomorphic homomorphisms. We denote the subbundle of sections taking values in $\overline{\mathrm{Hom}_{\mathbb{C}}}\big(T\Sigma, u^*TW\big)$ by $\bar{\mathcal{E}}_{\mathbb{C}}\big(\mathbb{D}^2; N\big)$.

III.1.1. The expected dimension of $\widetilde{\mathcal{M}}\big(\mathbb{D}^2, N; J\big)$. The rank of $\bar{\mathcal{E}}_{\mathbb{C}}\big(\mathbb{D}^2; N\big)$ and the dimension of $\mathcal{B}\big(\mathbb{D}^2; N\big)$ are both infinite, hence we cannot compute the expected dimension of the solution space $\widetilde{\mathcal{M}}\big(\mathbb{D}^2, N; J\big)$ as in the finite dimensional case, where it was just the difference $\dim B - \operatorname{rank} E$. Nonetheless we can associate a so-called Fredholm index to a Cauchy-Riemann problem. We will later give some more details about how the index is actually defined, for now we just note that it is an integer that determines the expected dimension of the space $\widetilde{\mathcal{M}}\big(\mathbb{D}^2, N; J\big)$.

For a Cauchy-Riemann problem with totally real boundary condition the index has an easy explicit formula (see for example [**MS04**, Theorem C.1.10]) that simplifies in our specific case of holomorphic disks to

$$\operatorname{ind}_u \bar\partial_J = \frac{1}{2} \dim W + \mu\big(u^*TW, u^*TN\big) \; , \tag{III.1.1}$$

where we have used that the Euler characteristic of a disk is $\chi\big(\mathbb{D}^2\big) = 1$.

Remark III.1.2. We would like to warn the reader that the dimension of a moduli space of holomorphic disks or holomorphic spheres tends to increase, if we increase the dimension of the symplectic ambient manifold. Unfortunately, the opposite is true for a higher genus curve Σ: The formula above becomes

$$\operatorname{ind}_u \bar\partial_J = \frac{1}{2}\,\chi(\Sigma)\dim W + \mu\bigl(u^*TW, u^*TN\bigr) ,$$

and since the Euler characteristic is negative, it is harder to find curves with genus in high dimensional spaces than in lower dimensional ones.

The Maslov index μ is an integer that classifies loops of totally real subspaces up to homotopy:

Definition. Let $E_{\mathbb{C}}$ be a complex vector bundle over the closed 2-disk $\mathbb{D}^2$ and let $E_{\mathbb{R}}$ be a totally real subbundle of $E_{\mathbb{C}}|_{\partial\mathbb{D}^2}$ defined only over the boundary of the disk. The **Maslov index** $\mu(E_{\mathbb{C}}, E_{\mathbb{R}})$ is an integer that is computed by trivializing $E_{\mathbb{C}}$ over the disk, and choosing a continuous frame $A(e^{i\phi}) \in \mathrm{GL}(n,\mathbb{C})$ over the boundary $\partial\mathbb{D}^2$ representing $E_{\mathbb{R}}$ with respect to the chosen trivialization. We then set

$$\mu(E_{\mathbb{C}}, E_{\mathbb{R}}) := \deg \frac{\det A^2}{\det(A^*A)} ,$$

where $\deg(f)$ is the degree of a continuous map $f\colon \mathbb{S}^1 \to \mathbb{S}^1$.

In these notes, we will compute the Maslov index only once, in Section III.1.3, but note that the index $\operatorname{ind}_u \bar\partial_J$ depends on the holomorphic disk u in $\widetilde{\mathcal{M}}\bigl(\mathbb{D}^2, N; J\bigr)$, we are considering; this should not confuse us however, because it only means that the space of disks may have different components and the expected dimensions of the different components do not need to agree.

We will now briefly explain how the index of $\bar\partial_J$ is defined. We have a map $\bar\partial_J\colon \mathcal{B}\bigl(\mathbb{D}^2; N\bigr) \to \bar{\mathcal{E}}_{\mathbb{C}}\bigl(\mathbb{D}^2; N\bigr)$, and we need to compute the linearization of $\bar\partial_J$ at a point of $u \in \mathcal{B}\bigl(\mathbb{D}^2; N\bigr)$, that means, we have to compute the differential

$$\bar D_J(u)\colon T_u\mathcal{B}\bigl(\mathbb{D}^2; N\bigr) \to T_{\bar\partial_J u}\bar{\mathcal{E}}_{\mathbb{C}}\bigl(\mathbb{D}^2; N\bigr) .$$

To find $\bar D_J(u)$, choose a smooth path u_t of maps in $\mathcal{B}\bigl(\mathbb{D}^2; N\bigr)$ with $u_0 = u$, then we can regard the image $\bar\partial_J u_t$, and take its derivative with respect to t in $t = 0$. If we set $\dot u_0 = \frac{d}{dt}\bigr|_{t=0} u_t$, this allows us to obtain a linear operator $\bar D_J(u)$ by

$$\bar D_J(u)\cdot \dot u_0 = \left.\frac{d}{dt}\right|_{t=0} \bar\partial_J u_t .$$

It is a good exercise to determine the domain and target space of this operator, and find a way to describe them.

The index of $\bar\partial_J$ at u is defined as

$$\operatorname{ind}_u \bar\partial_J := \dim\ker \bar D_J(u) - \dim\operatorname{coker} \bar D_J(u) .$$

It is a remarkable fact that the index is finite and determined by formula (III.1.1) above. Also note that the index is constant on each connected component of $\mathcal{B}\bigl(\mathbb{D}^2; N\bigr)$.

III.1.2. Transversality of the Cauchy-Riemann problem. Just as in the finite dimensional analogue, it may happen that the formal dimension we have computed does not correspond to the dimension we are observing in an actual situation. In fact, if the section σ (or in our infinite dimensional case, $\bar\partial_J$) are not transverse to the 0-section, there is no reason why M or $\widetilde{\mathcal{M}}\bigl(\mathbb{D}^2, N; J\bigr)$ would need to be smooth manifolds at all.

On the other hand, if σ is transverse to the 0-section, then $M = \sigma^{-1}(0)$ is a smooth submanifold of dimension $\dim B - \operatorname{rank} E$, and the analogue result is also true for the Cauchy-Riemann problem: If $\bar\partial_J$ is at every point of $\widetilde{\mathcal{M}}\big(\mathbb{D}^2, N; J\big)$ transverse to 0 (or said equivalently, if the cokernel of the linearized operator is trivial for every holomorphic disk), then $\widetilde{\mathcal{M}}\big(\mathbb{D}^2, N; J\big)$ will be a smooth manifold whose dimension is given by the index of $\bar\partial_J$.

In the finite dimensional situation, we can often achieve transversality by a small perturbation of σ, but of course, this might require a subtle analysis of the situation, when we want to perturb σ only within a space of sections satisfying certain prescribed properties.

Definition. Let $u\colon \Sigma \to W$ be a holomorphic map from a Riemann surface with or without boundary. We call u **somewhere injective**, if there exists a point $z \in \Sigma$ with $Du_z \neq 0$, and such that z is the only point that is mapped by u to $u(z)$.

We call a holomorphic curve that is not the multiple cover of any other holomorphic curve a **simple holomorphic curve**. Simple holomorphic curves are somewhere injective.

It is a non-trivial result that by perturbing the almost complex structure J, we can achieve transversality of the Cauchy-Riemann operator for every *somewhere injective* disk in W with boundary in a totally real submanifold N. We could hope that this theoretical result would be sufficient for us, because the considered disks are injective along their boundaries, but we have chosen a very specific almost complex structure in Section II.3, and perturbing this J would destroy the results obtained in that section. Below, we will prove by hand that $\bar\partial_J$ is transverse to 0 for the holomorphic disks in our model neighborhood.

Remark III.1.3. Note that often it is not possible to work only with somewhere injective holomorphic curves, and perturbing J will in that case not be sufficient to obtain transversality for holomorphic curves. Sometimes one can work around this problem by requiring that W is semi-positive, see Section III.3. Unfortunately, there are many situations where this approach won't work either, as is the case of SFT, where transversality has been one of the most important outstanding technical problems.

III.1.3. The Bishop family. In this section, we will show that the disks that we have found in Section II.3.3 lying in the model neighborhood are regular solutions of the Cauchy-Riemann problem.

Before starting the actual proof of our claim, we will briefly recapitulate the situation described in Section II.3.3. Let (W, J) be an almost complex manifold of dimension $2n$ with boundary that contains a model neighborhood U of the desired form. Remember that U was a subset of $\mathbb{C}^2 \times T^*L$ with almost complex structure $i \oplus J_L$, that we had a function $f\colon \mathbb{C}^2 \times T^*L \to [0, \infty)$ given by

$$f(z_1, z_2, \mathbf{q}, \mathbf{p}) = \frac{1}{2}\left(|z_1|^2 + |z_2|^2\right) + f_L(\mathbf{q}, \mathbf{p}) ,$$

and that the model neighborhood U was the subset

$$U := \big\{(z_1, z_2; \mathbf{q}, \mathbf{p}) \bigm| \operatorname{Re}(z_2) \geq 1 - \delta\big\} \cap f^{-1}\big([0, 1/2]\big) .$$

The totally real manifold N is the image of the map

$$(z; \mathbf{q}) \in \mathbb{D}^2_\varepsilon \times L \mapsto \left(z, \sqrt{1 - |z|^2}; \mathbf{q}, \mathbf{0}\right) \subset \partial U .$$

For every pair $(s,\mathbf{q}) \in [1-\delta,1)\times L$, we find a holomorphic map of the form

$$\begin{aligned} u_{s,\mathbf{q}}\colon \big(\mathbb{D}^2,\partial\mathbb{D}^2\big) &\to U\\ z &\mapsto \big(C_s z, s; \mathbf{q}, \mathbf{0}\big)\end{aligned}$$

with $C_s = \sqrt{1-s^2}$. We call this map a **(parametrized) Bishop disk**, and we call the collection of these disks, the **Bishop family**. Sometimes we will not be precise about whether the disks are parametrized or not, and whether we speak about disks with or without a marked point (see Section III.2), but we hope that in each situation it will be clear what is meant.

To check that a given Bishop disk $u_{s,\mathbf{q}}$ is regular, we will first compute the index of the linearized Cauchy-Riemann operator that gives us the expected dimension for the space of holomorphic disks containing the Bishop family. Note that the observed dimension is $1+\dim L+3 = 1+(n-2)+3 = n+2$. The first part, $1+\dim L$ corresponds to the s- and $\mathbf{q}$-parameters of the family; the three corresponds to the dimension of the group of Möbius transformations acting on the complex unit disk: If $u_{s,\mathbf{q}}$ is a Bishop disk, and if $\varphi\colon \mathbb{D}^2\to\mathbb{D}^2$ is a Möbius transformation, then of course $u_{s,\mathbf{q}}\circ\varphi$ will also be a holomorphic map with admissible boundary condition. On the other hand we showed in Corollary II.3.5 that every holomorphic disk that lies in U is up to a Möbius transformation one of the Bishop disks.

For the index computations, it suffices by Section III.1.1 to trivialize the bundle $E_\mathbb{C} := u_{s,\mathbf{q}}^*TW$ over $\mathbb{D}^2$, and study the topology of the totally real subbundle $E_\mathbb{R} = u_{s,\mathbf{q}}^*TN$ over $\partial\mathbb{D}^2$.

Before starting any concrete computations, we will significantly simplify the setup by choosing a particular chart: Note that the T^*L-part of a Bishop disk $u_{s,\mathbf{q}}$ is constant, we can hence choose a chart diffeomorphic to $\mathbb{R}^{2n-4} = \{(x_1,\dots,x_{n-2};y_1,\dots,y_{n-2})\}$ for T^*L with the properties

- the point $(\mathbf{q},\mathbf{0})$ corresponds to the origin,
- the almost complex structure J_L is represented at the origin by the standard i,
- the intersections of the 0-section L with the chart corresponds to the subspace $(x_1,\dots,x_{n-2};0,\dots,0)$.

In the chosen chart, we write $u_{s,\mathbf{q}}$ as

$$u_{s,\mathbf{q}}(z) = \big(C_s z, s; 0,\dots,0\big) \in \mathbb{C}^2\times\mathbb{R}^{2n-4}$$

with $C_s = \sqrt{1-s^2}$. By our assumption, the complex structure on the second factor is at the origin of $\mathbb{R}^{2n-4}$ equal to i, and there is then a direct identification of $u_{s,\mathbf{q}}^*TW$ with $\mathbb{C}^2\times\mathbb{C}^{n-2}$. The submanifold N corresponds in the chart to

$$\big\{(z_1,z_2;x_1,\dots,x_{n-2},0,\dots,0)\in\mathbb{C}^2\times\mathbb{R}^{2n-4}\;\big|\;\operatorname{Im} z_2 = 0,\ |z_1|^2+|z_2|^2 = 1\big\}\ .$$

The boundary of $u_{s,\mathbf{q}}$ is given by $e^{i\varphi}\mapsto\big(\sqrt{1-s^2}\,e^{i\varphi}, s; 0,\dots,0\big)$, and the tangent space of TN over this loop is spanned over $\mathbb{R}$ by the vector fields

$$\Big(ie^{i\varphi},0;0,\dots,0\Big),\ \Big(-\frac{s}{\sqrt{1-s^2}}\,e^{i\varphi},1;0,\dots,0\Big),\ \big(0,0;1,0,\dots,0\big),\dots,\big(0,0;0,\dots,0,1,0,\dots,0\big)\ .$$

We can now easily compute the Maslov index $\mu(E_\mathbb{C},E_\mathbb{R})$ as

$$\deg\frac{\det A^2}{\det(A^*A)} = \deg\frac{-e^{2i\varphi}}{1} = 2\ ,$$

where A is the matrix composed by the vector fields given above. Hence we obtain for the index

$$\operatorname{ind}_u \bar\partial_J = \frac{1}{2}\dim W + \mu\bigl(u_{s,\mathbf{q}}^* TW, u_{s.\mathbf{q}}^* TN\bigr) = n+2 \;,$$

which corresponds to the observed dimension computed above.

We will now show that the linearized operator $\bar D_J$ is surjective. We do not do this directly, but we compute instead the dimension of its kernel, and show that it is equal (and not larger than) the Fredholm index. From the definition of the index

$$\operatorname{ind}_u \bar\partial_J := \ker \bar D_J(u) - \operatorname{coker} \bar D_J(u) \;,$$

we see that the cokernel needs to be trivial, and this way the surjectivity result follows.

We now compute the linearized Cauchy-Riemann operator at a Bishop disk $u_{s,\mathbf{q}}$. Let v_t be a smooth family of maps

$$v_t \colon \bigl(\mathbb{D}^2, \partial\mathbb{D}^2\bigr) \to (U, N)$$

with $v_0 = u_{s,\mathbf{q}}$ (think of each v_t as a smooth map, but for an analytically correct study, we would need to allow here for Sobolev maps).

In this chart, we can write the family v_t as

$$v_t(z) = \bigl(z_1(z,t), z_2(z,t); \mathbf{x}(z,t), \mathbf{y}(z,t)\bigr) \in \mathbb{C}^2 \times \mathbb{R}^{2n-4} \;,$$

where we have set $\mathbf{x}(z,t) = \bigl(x_1(z,t), \dots, x_{n-2}(z,t)\bigr)$ and $\mathbf{y}(z,t) = \bigl(y_1(z,t), \dots, y_{n-2}(z,t)\bigr)$, and we require that the boundary of each of the v_t has to lie in N. When we now take the derivative of v_t with respect to t at $t=0$, we obtain a vector in $T_{u_{s,\mathbf{q}}}\mathcal{B}$ that is represented by a map

$$\dot v_0 \colon \mathbb{D}^2 \to \mathbb{C}^2 \times \mathbb{R}^{2\,(n-2)}, z \mapsto \bigl(\dot z_1(z), \dot z_2(z); \dot{\mathbf{x}}(z), \dot{\mathbf{y}}(z)\bigr)$$

with boundary conditions $\dot{\mathbf{y}}(z) = \mathbf{0}$ and $\operatorname{Im} \dot z_2(z) = 0$ for every $z \in \partial\mathbb{D}^2$. Furthermore taking the derivative of $\lvert z_1(z,t)\rvert^2 + \lvert z_2(z,t)\rvert^2 = 1$ for every $z \in \partial\mathbb{D}^2$ with respect to t, we obtain the condition

$$\bar z_1(z,0)\cdot \dot z_1(z) + z_1(z,0)\cdot \dot{\bar z}_1(z) + \bar z_2(z,0)\cdot \dot z_2(z) + z_2(z,0)\cdot \dot{\bar z}_2(z) = 0 \;,$$

which simplifies by using the explicit form of $\bigl(z_1(z,0), z_2(z,0)\bigr)$ to

$$C_s \bar z \cdot \dot z_1(z) + C_s z \cdot \dot{\bar z}_1(z) + s\dot z_2(z) + s\dot{\bar z}_2(z) = 0$$

for every $z \in \partial\mathbb{D}^2$.

The linearization of the Cauchy-Riemann operator $\bar\partial_J$ at $u_{s,\mathbf{q}}$ given by

$$\bar D_J \cdot \dot v_0 := \left.\frac{d}{dt}\right|_{t=0} \bar\partial_J v_s$$

decomposes into the $\mathbb{C}^2$-part

$$\bigl(i d\dot z_1 - d\dot z_1 i,\; i d\dot z_2 - d\dot z_2 i\bigr)$$

and the $\mathbb{R}^{2\,(n-2)}$-part

$$\left.\frac{d}{dt}\right|_{t=0} \Bigl(J_L\bigl(\mathbf{x}(z,t), \mathbf{y}(z,t)\bigr)\cdot\bigl(d\mathbf{x}(z,t), d\mathbf{y}(z,t)\bigr) - \bigl(d\mathbf{x}(z,t)\cdot i, d\mathbf{y}(z,t)\cdot i\bigr)\Bigr) \;.$$

The second part can be significantly simplified by using first the product rule, and applying then that $\mathbf{x}(z,0) = \mathbf{0}$ and $\mathbf{y}(z,0) = \mathbf{0}$ are constant so that their differentials vanish. We obtain then

$$J_L\bigl(\mathbf{0},\mathbf{0}\bigr)\cdot\bigl(d\dot{\mathbf{x}}, d\dot{\mathbf{y}}\bigr) - \bigl(d\dot{\mathbf{x}}\cdot i, d\dot{\mathbf{y}}\cdot i\bigr) \;,$$

and using that $J_L(\mathbf{0},\mathbf{0}) = i$, it finally reduces to

$$\bigl(d\dot{\mathbf{y}} - d\dot{\mathbf{x}}\cdot i, -d\dot{\mathbf{x}} - d\dot{\mathbf{y}}\cdot i\bigr) \;.$$

We have shown that linearized Cauchy-Riemann operator simplifies for all coordinates to the standard Cauchy-Riemann operator, so that if $\dot{v}_0(z) = \bigl(\dot{z}_1(z), \dot{z}_2(z); \dot{\mathbf{x}}(z), \dot{\mathbf{y}}(z)\bigr)$ lies in the kernel of $\bar{D}_J$ then the coordinate functions $\dot{z}_1(z), \dot{z}_2(z)$ and $\dot{\mathbf{x}}(z) + i\dot{\mathbf{y}}(z)$ need all to be holomorphic in the classical sense.

Now using the boundary conditions, we easily deduce that $\dot{\mathbf{y}}(z)$ needs to vanish, because it is a harmonic function, and it takes both maximum and minimum on $\partial\mathbb{D}^2$. A direct consequence of $\dot{\mathbf{y}} \equiv \mathbf{0}$ and the Cauchy-Riemann equation is that $\dot{\mathbf{x}}(z)$ will be everywhere constant. We get the analogous result for the function $\dot{z}_2(z)$, so that we can write

$$\dot{v}_0(z) = \bigl(\dot{z}_1(z), \dot{s};\, \dot{\mathbf{q}}_0, \mathbf{0}\bigr) \;,$$

where $\dot{s}$ is a real constant, and $\dot{\mathbf{q}}_0$ is a fixed vector in $\mathbb{R}^{2\,(n-2)}$, and we only need to still understand the holomorphic function $\dot{z}_1(z)$.

The boundary condition for $\dot{z}_1(z)$ is $\bar{z}\cdot\dot{z}_1(z) + z\cdot\dot{\bar{z}}_1(z) = -\frac{2\,s\dot{s}}{C_s}$ for every $z \in \partial\mathbb{D}^2$. Using that the function $\dot{z}_1(z)$ is holomorphic, we can write it as power series in the form

$$\dot{z}_1(z) = \sum_{k=0}^{\infty} a_k\, z^k$$

and we get at $e^{i\varphi} \in \partial\mathbb{D}^2$

$$\dot{z}_1\bigl(e^{i\varphi}\bigr) = \sum_{k=0}^{\infty} a_k\, e^{ik\varphi} \;.$$

Plugging these series into the equation of the boundary condition, we find

$$e^{-i\varphi}\cdot\sum_{k=0}^{\infty} a_k\, e^{ik\varphi} + e^{i\varphi}\cdot\sum_{k=0}^{\infty} \bar{a}_k\, e^{-ik\varphi} = -\frac{2\,s\dot{s}}{C_s}$$

so that

$$\sum_{k=0}^{\infty}\bigl(a_k\, e^{(k-1)\,i\varphi} + \bar{a}_k\, e^{-(k-1)\,i\varphi}\bigr) = -\frac{2\,s\dot{s}}{C_s}$$

and by comparing coefficients we see that

$$a_1 + \bar{a}_1 = -\frac{2\,s\dot{s}}{C_s}, \quad a_0 + \bar{a}_2 = 0, \quad a_k = 0 \text{ for all } k \geq 3.$$

This means that the three (real) parameters we can choose freely are a_0 and $\operatorname{Im} a_1$.

Concluding, we have found that the dimension of the kernel of $\bar{D}_J$ is equal to $3+1+n-2 = n+2$ which corresponds to the Fredholm index of our problem. Thus there is no need to perturb J on the neighborhood of the Bishop family to obtain regularity.

Corollary III.1.4. *Let (W,ω) be a compact symplectic manifold that is a weak symplectic filling of a contact manifold (M,ξ). Suppose that N is either a* Lob *or a* **bLob** *in M, then we can choose close to the binding and to the boundary of N the almost complex structure described in the previous sections, and extend it to an almost complex structure J that is tamed by ω, whose bundle of complex tangencies along M is ξ and that makes M J-convex. By a generic perturbation away from the binding and the boundary of N, we can achieve that all somewhere injective holomorphic curves become regular.*

We call a J with these properties an ***almost complex structure adapted to*** *N.*

The argument in the proof of the corollary above is that the Bishop disks are already regular, and that all other simple holomorphic curves have to lie outside the neighborhood where we require an explicit form for J. Thus it suffices to perturb outside these domains to obtain regularity for every other simple curve.

III.2. The moduli space of holomorphic disks with a marked point

Until now, we only have studied the space of certain J-holomorphic *maps*

$$\widetilde{\mathcal{M}}\big(\mathbb{D}^2, N; J\big) = \big\{u\colon \mathbb{D}^2 \to W \;\big|\; \bar\partial_J u = 0 \text{ and } u(\partial\mathbb{D}^2) \subset N\big\} ,$$

but many maps correspond to different parametrizations of the same geometric disk. To get rid of this ambiguity (and to obtain compactness), we quotient the space of maps by the biholomorphic reparametrizations of the unit disk, that means, by the Möbius transformations, but we will also add a marked point $z_0 \in \mathbb{D}^2$ to preserve the structure of the geometric disk. To simplify the notation, we will also omit the almost complex structure J in $\widetilde{\mathcal{M}}\big(\mathbb{D}^2, N\big)$.

From now on let

$$\widetilde{\mathcal{M}}\big(\mathbb{D}^2, N; z_0\big) = \big\{(u, z_0) \;\big|\; z_0 \in \mathbb{D}^2,\, \bar\partial_J u = 0 \text{ and } u(\partial\mathbb{D}^2) \subset N\big\} = \widetilde{\mathcal{M}}\big(\mathbb{D}^2, N\big) \times \mathbb{D}^2$$

be the space of holomorphic maps together with a special point $z_0 \in \mathbb{D}^2$ that will be called the **marked point**. The **moduli space** we are interested in is the space of equivalence classes

$$\mathcal{M}\big(\mathbb{D}^2, N; z_0\big) = \widetilde{\mathcal{M}}\big(\mathbb{D}^2, N; z_0\big)/\sim$$

where we identify two elements (u, z_0) and (u', z_0'), if and only if there is a biholomorphism $\varphi\colon \mathbb{D}^2 \to \mathbb{D}^2$ such that $u = u' \circ \varphi^{-1}$ and $z_0 = \varphi(z_0')$. The map $(u, z) \mapsto u(z)$ descends to a well defined map

$$\begin{aligned} \mathrm{ev}\colon \mathcal{M}\big(\mathbb{D}^2, N; z_0\big) &\to W \\ [u, z_0] &\mapsto u(z_0) \end{aligned}$$

on the moduli space, which we call the **evaluation map**.

Let N be a Lob or a bLob, and assume that B_0 is one of the components of the binding of N. Since this is the only situation, we are really interested in in these notes, we introduce the notation $\widetilde{\mathcal{M}}_0\big(\mathbb{D}^2, N\big)$ for the connected component in $\widetilde{\mathcal{M}}\big(\mathbb{D}^2, N\big)$ that contains the Bishop family around B_0. When adding a marked point, we write $\widetilde{\mathcal{M}}_0\big(\mathbb{D}^2, N; z_0\big)$ and $\mathcal{M}_0\big(\mathbb{D}^2, N; z_0\big)$ for the corresponding subspaces.

It is easy to see that $\mathcal{M}_0\big(\mathbb{D}^2, N; z_0\big)$ is a smooth (non-compact) manifold with boundary. Note first that $\widetilde{\mathcal{M}}_0\big(\mathbb{D}^2, N; z_0\big)$ is also a smooth and non-compact manifold with boundary: If J is regular, we know that $\widetilde{\mathcal{M}}_0\big(\mathbb{D}^2, N\big)$ is a smooth manifold, and so the boundary of the product manifold $\widetilde{\mathcal{M}}_0\big(\mathbb{D}^2, N; z_0\big)$ is

$$\partial\widetilde{\mathcal{M}}_0\big(\mathbb{D}^2, N; z_0\big) = \widetilde{\mathcal{M}}_0\big(\mathbb{D}^2, N\big) \times \partial\mathbb{D}^2 .$$

Passing to the quotient preserves this structure, because the boundary of the maps in $\widetilde{\mathcal{M}}_0\big(\mathbb{D}^2, N\big)$ intersects each of the pages of the open book exactly once (this is a consequence of Corollary II.1.11 and Section II.3.3), and hence each of the disks is injective along its boundary.

The only Möbius transformation that preserves the boundary pointwise is the identity, hence it follows that the group of Möbius transformations acts smoothly, freely and properly on $\widetilde{\mathcal{M}}_0\big(\mathbb{D}^2, N; z_0\big)$, and hence the quotient will be a smooth manifold of dimension

$$\dim \mathcal{M}_0\big(\mathbb{D}^2, N; z_0\big) = \dim \widetilde{\mathcal{M}}_0\big(\mathbb{D}^2, N; z_0\big) - 3 = \operatorname{ind}_u \bar{\partial}_J + 2 - 3 = n + 1 \ .$$

As before the points on the boundary of $\mathcal{M}_0\big(\mathbb{D}^2, N; z_0\big)$ are the classes $[u, z]$ with $z \in \partial\mathbb{D}^2$. It is also clear that the evaluation map $\operatorname{ev}_{z_0} \colon \mathcal{M}_0\big(\mathbb{D}^2, N; z_0\big) \to W$ is smooth.

Remember that the Bishop disks contract to points as they approach the binding B_0. We will show that we incorporate B_0 into the moduli space $\mathcal{M}_0\big(\mathbb{D}^2, N; z_0\big)$ and that the resulting space carries a natural smooth structure that corresponds to the intuitive picture of disks collapsing to one point.

The neighborhood of the binding B_0 in W is diffeomorphic to the model

$$U = \big\{(z_1, z_2; \mathbf{q}, \mathbf{p}) \in \mathbb{C}^2 \times T^*B_0 \ \big|\ \operatorname{Re}(z_2) > 1 - \delta\big\} \cap h^{-1}\big((-\infty, 1/2]\big)$$

for small $\delta > 0$ with the function

$$h(z_1, z_2; \mathbf{q}, \mathbf{p}) = \frac{1}{2}\left(|z_1|^2 + |z_2|^2\right) + f_{B_0}(\mathbf{q}, \mathbf{p}) \ ,$$

see Section II.3.3.

The content of Proposition II.3.6 and of Corollary II.3.5 is that for every point

$$(z, s; \mathbf{q}_0, \mathbf{0}) \in U$$

with $s \in (1 - \delta, 1)$ and $\mathbf{q}_0$ in the 0-section of T^*B_0,

- there is up to a Möbius transformation a unique holomorphic map $u \in \widetilde{\mathcal{M}}_0\big(\mathbb{D}^2, N\big)$ containing that point in its image, and
- $\widetilde{\mathcal{M}}\big(\mathbb{D}^2, N\big)$ does not contain any holomorphic maps whose image is not entirely contained in $U \cap (\mathbb{C} \times \mathbb{R} \times B_0)$.

As a result, it follows that $V = \operatorname{ev}_{z_0}^{-1}(U)$ is an open subset of $\mathcal{M}_0\big(\mathbb{D}^2, N; z_0\big)$, and that the restriction of the evaluation map

$$\operatorname{ev}_{z_0}\big|_V \colon V \to U$$

is a diffeomorphism onto $U \cap \big(\mathbb{C} \times (1 - \delta, 1) \times B_0\big)$. The closure of this subset is the smooth submanifold

$$U \cap \big(\mathbb{C} \times \mathbb{R} \times B_0\big) \ ,$$

which we obtain by including the binding $\{0\} \times \{1\} \times B_0$ of N.

Using the evaluation map, we can identify V with its image in U, and this way glue B_0 to the moduli space $\mathcal{M}_0\big(\mathbb{D}^2, N; z_0\big)$. The new space is also a smooth manifold with boundary, and the evaluation map extends to it, and is a diffeomorphism onto its image in U so that we can effectively identify U with a subset of the moduli space. In particular, it follows that B_0 is a submanifold that is of codimension 2 in the boundary of the moduli space.

The aim of the next section will consist in studying the Gromov compactification of $\mathcal{M}_0\big(\mathbb{D}^2, N; z_0\big)$.

III.3. Compactness

Gromov compactness is a result that describes the possible limits of a sequence of holomorphic curves, and ensures under certain conditions that every such sequence contains a converging subsequence. In the limit, a given sequence of holomorphic curves may break into several components, called **bubbles**, each of which is again a holomorphic curve. We will not describe in detail what "convergence" in this sense really means, but we only sketch the idea: The holomorphic curves in a moduli space can be represented by holomorphic maps, and in the optimal case, one could hope that by choosing for each curve in the given sequence a suitable representative, we might have uniform convergence of the maps, and this way we would find the limit of the sequence as a proper holomorphic curve. Unfortunately, this is usually wrong, but it might be true that for the correct choice of parametrization we have convergence on subdomains. Choosing different reparametrizations, we then obtain convergence on different domains, and each such domain gives then rise to a bubble, that means, a holomorphic curve that represents one component of the Gromov limit.

Theorem III.3.1 (Gromov compactness)**.** *Let (W,J) be a compact almost complex manifold (with or without boundary), and assume that J is tamed by a symplectic form ω. Let L be a compact totally real submanifold. Choose a sequence of J-holomorphic maps $u_k\colon (\mathbb{D}^2,\partial\mathbb{D}^2)\to(W,L)$ whose ω-**energy***

$$E(u_k) := \int_{\mathbb{D}^2} u_k^*\omega$$

is bounded by a constant $C>0$.

Then there is a subsequence of $\big(u_{k_l}\big)_l$ that converges in the Gromov sense to a bubble tree composed of a finite family of non-constant holomorphic disks $u_\infty^{(1)},\dots,u_\infty^{(K)}$ whose boundary lies in L, and a finite family of non-constant holomorphic spheres $v_\infty^{(1)},\dots,v_\infty^{(K')}$. The total energy is preserved so that

$$\lim_{l\to\infty} E\big(u_{k_l}\big) = \sum_{j=1}^{K} E\big(u_\infty^{(j)}\big) + \sum_{j=1}^{K'} E\big(v_\infty^{(j)}\big)\ .$$

If each of the disks u_k is equipped with a marked point $z_k\in\mathbb{D}^2$, then after possibly reducing to a another subsequence, there is a marked point z_∞ on one of the components of the bubble tree such that $\lim_k z_k = z_\infty$ in a suitable sense.

The ω-energy is fundamental in the proof of the compactness theorem to limit the number of possible bubbles: By [**MS04**, Proposition 4.1.4], there exists in the situation of Theorem III.3.1 a constant $\hbar>0$ that bounds the energy of every holomorphic sphere or every holomorphic disk $u_k\colon(\mathbb{D}^2,\partial\mathbb{D}^2)\to(W,L)$ from below. Since every bubble needs to have at least an $\hbar$-quantum of energy, and since the total energy of the curves in the sequence is bounded by C, the limit curve will never break into more than $C/\hbar$ bubbles (the upper bound of the energy is also used to make sure that each bubble is a compact surface).

We will show in the rest of this section that we can apply Gromov compactness to sequences of holomorphic disks lying in the moduli space $\mathcal{M}_0\big(\mathbb{D}^2,N\big)$ studied in the previous section, and how we can incorporate these limits into $\mathcal{M}_0\big(\mathbb{D}^2,N;z_0\big)$ to construct the compactification $\overline{\mathcal{M}}_0\big(\mathbb{D}^2,N;z_0\big)$.

Proposition III.3.2. *Let N be a* Lob *or a* bLob *in the contact boundary (M,ξ) of a symplectic filling (W,ω), and assume that we find a contact form α for ξ such that $\omega|_{TN}=d\alpha|_{TN}$.*

There is a global energy bound $C > 0$ for all holomorphic disks in $\widetilde{\mathcal{M}}_0\big(\mathbb{D}^2, N\big)$.

Proof. There is a slight complication in our proof, because we may not assume that ω is globally exact, which would allow us to obtain the energy of a holomorphic disk by integrating over the boundary of the disk. To prove the desired statement, proceed as follows: Let $u\colon (\mathbb{D}^2, \partial\mathbb{D}^2) \to (W, N)$ be any element in $\widetilde{\mathcal{M}}_0\big(\mathbb{D}^2, N\big)$. By our assumption, there exists a smooth path of maps u_t that starts at the constant map $u_0(z) \equiv b_0 \in B_0$ in the binding and ends at the chosen map $u_1 = u$. This family of disks may be interpreted as a map from the 3-ball into W. The boundary consists of the image of u_1, and the union of the boundary of all disks $u_t|_{\partial\mathbb{D}^2}$.

Using Stokes' theorem, we get

$$0 = \int_{[0,1]\times\mathbb{D}^2} u_t^* d\omega = \int_{\mathbb{D}^2} u_1^*\omega + \int_{[0,1]\times\partial\mathbb{D}^2} u_t^*\omega$$

so that $E(u) = -\int_{[0,1]\times\partial\mathbb{D}^2} u_t^*\omega$.

By our assumption, we have a contact form on the contact boundary M for which $\omega|_{TN} = d\alpha|_{TN}$, so that using Stokes' theorem a second time (and that $u_0(z) = b_0$) we get

$$E(u) = \int_{\partial\mathbb{D}^2} u^*\alpha\ .$$

The Legendrian foliation on N is an open book whose pages are fibers of a fibration $\vartheta\colon N \setminus B \to \mathbb{S}^1$. Hence the 1-form $d\vartheta$ and $\alpha|_{TN}$ have the same kernel, and it follows that there exists a smooth function $f\colon N \to [0,\infty)$ such that

$$\alpha|_{TN} = f\, d\vartheta\ .$$

The function f vanishes on the binding and on the boundary of a bLob, and f is hence bounded on N so that we define $C := 2\pi \max_{x\in N} |f(x)|$.

Using that the boundary of u intersects every leaf of the open book exactly once, we obtain for the energy of u the estimate

$$E(u) = \int_{\partial\mathbb{D}^2} u^*\alpha \le \max_{x\in N} |f(x)| \int_{\partial\mathbb{D}^2} u^* d\vartheta \le 2\pi \max_{x\in N} |f(x)| = C\ . \qquad \square$$

With the given energy bound, we obtain now Gromov compactness in form of the following corollary.

Corollary III.3.3. *Let N be a Lob or a bLob in the contact boundary (M, ξ) of a symplectic filling (W, ω), and assume that we find a contact form α for ξ such that $\omega|_{TN} = d\alpha|_{TN}$. Let $(u_k)_k$ be a sequence of holomorphic maps in $\widetilde{\mathcal{M}}_0\big(\mathbb{D}^2, N\big)$.*

There exists a subsequence $\big(u_{k_l}\big)_l$ that converges either

- *uniformly up to reparametrizations of the domain to a J-holomorphic map $u_\infty \in \widetilde{\mathcal{M}}_0\big(\mathbb{D}^2, N\big)$,*
- *to a constant disk $u_\infty(z) \equiv b_0$ lying in the binding of N,*
- *or to a bubble tree composed of a single holomorphic disk $u_\infty\colon (\mathbb{D}^2, \partial\mathbb{D}^2) \to (W, N)$ and a finite family of non-constant holomorphic spheres $v_1, \dots, v_j$ with $j \ge 1$.*

Proof. We will apply Theorem III.3.1. The submanifold N is not totally real along the binding B and ∂N, but we simply remove a small open neighborhood of both sets. By Proposition II.3.9, none of the holomorphic disks u_k may get close to ∂N, and by Proposition II.3.6 we know precisely how the curves look like that intersect a neighborhood of B. If we find disks in $(u_k)_k$ that get arbitrarily close to the binding of N, then using that B is compact, we may choose a subsequence that converges to a single point in the binding. If $(u_k)_k$ stays at finite distance from B, we may assume that the neighborhood, we have removed from N is so small that the holomorphic disks we are studying all lie inside.

If the sequence $(u_k)_k$ does not contain any subsequence that can be reparametrized in such a way that it converges to a single non-constant disk u_∞, we use Gromov compactness to obtain a subsequence that splits into a finite collection of holomorphic spheres and disks. But as a consequence from Corollary II.1.11, we see that non-constant holomorphic disks attached to N need to intersect the pages of the open book transversely in positive direction. A sequence of holomorphic disks that intersects every page of the open book exactly once, cannot split into several disks intersecting pages several times. In particular possible bubble trees contain by this argument a single disk in its limit. ☐

Above, we have obtained compactness for a sequence of disks, but we would like to understand how these limits can be incorporated into the moduli space. Adding the bubble trees to the space of parametrized maps does not give rise to a valid topology, because the bubbling phenomenon can only be understood by using different reparametrizations of the disk to recover all components of the bubble tree.

We will denote the compactification of $\mathcal{M}_0\big(\mathbb{D}^2, N; z_0\big)$ by $\overline{\mathcal{M}}_0\big(\mathbb{D}^2, N; z_0\big)$. For us, it is not necessary to understand the topology of $\overline{\mathcal{M}}_0\big(\mathbb{D}^2, N; z_0\big)$ in detail, but it will be sufficient to see that bubbling is a "codimension-2 phenomenon". In fact, it is not the topology of the moduli space itself we are interested in, but our aim is to obtain information about the symplectic manifold. For this we want to make sure that the image under the evaluation map of all bubble trees that appear in the limit, that means, of $\overline{\mathcal{M}}_0\big(\mathbb{D}^2, N; z_0\big) \setminus \mathcal{M}_0\big(\mathbb{D}^2, N; z_0\big)$ is contained in the image of a smooth map defined on a finite union of manifolds each of dimension at most

$$\dim \mathcal{M}_0\big(\mathbb{D}^2, N; z_0\big) - 2 \ .$$

For this to be true, we need to impose additional conditions for (W, ω).

Definition. A $(2n)$-dimensional symplectic manifold (W, ω) is called

- **symplectically aspherical**, if $\omega([A])$ vanishes for every $A \in \pi_2(W)$.
- It is called **semipositive** if every $A \in \pi_2(W)$ with $\omega([A]) > 0$ and $c_1(A) \geq 3 - n$ has non-negative Chern number.

Note that every symplectic 4- or 6-manifold is obviously semipositive.

In a symplectically aspherical manifold no J-holomorphic spheres exist, because their energy would be zero. So in particular they may not appear in any bubble tree and Corollary III.3.3 implies in our situation that every sequence of holomorphic disks contains a subsequence that either collapses into the binding or that converges to a single disk in $\mathcal{M}_0\big(\mathbb{D}^2, N; z_0\big)$. Using the results of Section III.2, we obtain the following corollary.

Corollary III.3.4. *Let (W, ω) be a compact symplectically aspherical manifold that is a weak filling of a contact manifold (M, ξ). Let N be a* Lob *or a* bLob *in M, and assume that we find a contact form for ξ such that $\omega|_{TN} = d\alpha|_{TN}$. Choose an almost complex structure J that is adapted to N (as in Corollary III.1.4).*

Then the compactification of the moduli space $\mathcal{M}_0\big(\mathbb{D}^2, N; z_0\big)$ is a smooth compact manifold

$$\overline{\mathcal{M}}_0\big(\mathbb{D}^2, N; z_0\big) = \mathcal{M}_0\big(\mathbb{D}^2, N; z_0\big) \cup \big(\textit{binding of } N\big)$$

with boundary. The binding of N *is a submanifold of codimension* 2 *in the boundary* $\partial\overline{\mathcal{M}}_0\bigl(\mathbb{D}^2, N; z_0\bigr)$.

The condition of asphericity is very strong, and we will obtain more general results by studying instead semipositive manifolds. The important point here is that a generic almost complex structure only ensure transversality for somewhere injective holomorphic curves, see Section III.1.2. Even though the holomorphic disks in $\mathcal{M}_0\bigl(\mathbb{D}^2, N; z_0\bigr)$ are simple, it could happen that once the disks bubble, there appear spheres that are multiple covers. For these, we cannot guarantee transversality, and hence we cannot directly predict if the compactification of $\mathcal{M}_0\bigl(\mathbb{D}^2, N; z_0\bigr)$ consists of adding "codimension-2 strata" or if we will be forced to include too many bubble trees

Still, we know that every sphere that is not simple is the multiple cover of a simple one (by the Riemann-Hurwitz formula a sphere can only multiply cover a sphere), we can hence compute the dimension of the moduli space of the underlying simple spheres, and use this information as an upper bound for the dimension of the spheres that appear in the bubble tree.

Let $v\colon \mathbb{S}^2 \to W$ be a holomorphic sphere that is a k-fold cover of a sphere $\widetilde{v}$ representing a homology class $[v]$ and $[\widetilde{v}] \in H_2(W, \mathbb{Z})$ respectively with $[v] = k[\widetilde{v}]$ and with $\omega\bigl([\widetilde{v}]\bigr) > 0$. The expected dimension of the space of maps containing v is by an index formula

$$\operatorname{ind}_v \bar{\partial}_J = 2n + 2\, c_1\bigl([v]\bigr) = 2n + 2k\, c_1\bigl([\widetilde{v}]\bigr) \ .$$

The space of biholomorphisms of $\mathbb{S}^2$ has dimension 6, and hence the expected dimension of the moduli space of unparametrized spheres that contain $[v]$ is $\operatorname{ind}_v \bar{\partial}_J - 6 = 2\,(n-3) + 2k\, c_1([\widetilde{v}])$.

As we explained above and in Section III.1.2, this expected dimension does not correspond in general to the observed dimension of the bubble trees, instead we study the expected dimension of the underlying simple spheres. The dimension of the space containing $\widetilde{v}$ is given by $\operatorname{ind}_{\widetilde{v}} \bar{\partial}_J - 6 = 2\,(n-3) + 2\, c_1\bigl([\widetilde{v}]\bigr)$. If $c_1\bigl([\widetilde{v}]\bigr) < 3 - n$, then the expected dimension will be negative, and since we obtain regularity of all simple holomorphic curves by choosing a generic almost complex structure, it follows that the moduli space containing $\widetilde{v}$ is generically empty. As a consequence bubble trees appearing as limits do not contain any component that is the k-fold cover of a simple sphere representing the homology class $[\widetilde{v}]$.

If $c_1\bigl([\widetilde{v}]\bigr) \geq 3 - n$, the definition of semipositivity implies that $c_1\bigl([\widetilde{v}]\bigr) \geq 0$. When we compare the expected dimension of the moduli space containing v with the one of the underlying disk $\widetilde{v}$, we observe that $\operatorname{ind}_v \bar{\partial}_J - 6 = 2\,(n-3) + 2k\, c_1\bigl([\widetilde{v}]\bigr) \geq 2\,(n-3) + 2\, c_1\bigl([\widetilde{v}]\bigr) = \operatorname{ind}_{\widetilde{v}} \bar{\partial}_J - 6$.

Consider now the image in W of all spheres in the moduli space of v that are k-fold multiple covers of some simple sphere. Their image is contained in the image of the simple spheres lying in the same moduli space as $\widetilde{v}$. The dimension of this second moduli space is smaller or equal than the expected dimension of the initial moduli space containing v, and even though we cannot ensure regularity for v, we have an estimate on the dimension of the subset containing all singular spheres.

The following result allows us to find the desired bound for the dimension of the image of complete bubble trees.

Proposition III.3.5. *Assume that* (W, ω) *is semipositive. To compactify the moduli space* $\mathcal{M}_0(W, N, z_0)$, *one has to add bubbled curves. We find a finite set of manifolds* $X_1, \dots, X_N$ *with* $\dim X_j \leq \dim \mathcal{M}_0(W, N, z_0) - 2$ *and smooth maps* $f_j\colon X_j \to W$ *such that the image of the bubbled curves under the evaluation map* ev_{z_0} *is contained in*

$$\cup f_j(X_j) \ .$$

When we consider instead the compactification of the boundary $\partial\mathcal{M}_0(W, N, z_0)$*, that means the space of holomorphic disks with a marked point on the boundary of the disk only, then we obtain the analogue result, only that the manifolds* $X_1, \dots, X_N$ *have dimension* $\dim X_j \leq \dim \partial\mathcal{M}_0(W, N, z_0) - 2 = \dim \mathcal{M}_0(W, N, z_0) - 3$.

Proof. The standard way to treat bubbled curves consists in considering them as elements in a bubble tree: Here such a tree is composed by a simple holomorphic disk $u_0 \colon (\mathbb{D}^2, \partial\mathbb{D}^2) \to (W, N)$ and holomorphic spheres $u_1, \dots, u_{k'} \colon \mathbb{S}^2 \to W$. These holomorphic curves are connected to each other in a certain way. We formalize this relation by saying that the holomorphic curves are vertices in a tree, i.e. in a connected graph without cycles. We denote the edges of this graph by $\{u_i, u_j\}$, $0 \leq i < j \leq k'$.

Now we assign to any edge two nodal points z_{ij} and z_{ji}, the first one in the domain of the bubble u_i, the other one in the domain of u_j, and we require that $\mathrm{ev}_{z_{ij}}(u_i) = \mathrm{ev}_{z_{ji}}(u_j)$. For technical reasons, we also require nodal points on each holomorphic curve to be pairwise distinct. To include into the theory, trees with more than one bubble connected at the same point to a holomorphic curve, we add “ghost bubbles”. These are constant holomorphic spheres inserted at the point where several bubbles are joined to a single curve. Now all the links at that point are opened and reattached at the ghost bubble. Ghost bubbles are the only constant holomorphic spheres we allow in a bubble tree.

The aim is to give a manifold structure to these bubble trees, but unfortunately this is not always possible, when multiply covered spheres appear in the bubble tree.

Instead, we note that the image of every bubble tree is equal to the image of a simple bubble tree, that means, to a tree, where every holomorphic sphere is simple and any two spheres have different image. Since we are only interested in the image of the evaluation map on the bubble trees, it is for our purposes equivalent to consider the simple bubble tree instead of the original one. The disk u_0 is always simple, and does not need to be replaced by another simple curve.

Let $u_0, u_1, \dots, u_{k'}$ be the holomorphic curves composing the original bubble tree, and let $A_i \in H_2(W)$ be the homology class represented by the holomorphic sphere u_i. The simple tree is composed by $u_0, v_1, \dots, v_k$ such that for every u_j there is a bubble sphere v_{i_j} with equal image

$$u_j(\mathbb{S}^2) = v_{i_j}(\mathbb{S}^2)$$

and in particular $A_j = m_j B_{i_j}$, where $B_{i_j} = [v_{i_j}] \in H_2(W)$ and $m_j \geq 1$ is an integer. Write also $\mathbf{A}$ for the sum $\sum_{j=1}^{k'} A_j$ and $\mathbf{B}$ for the sum $\sum_{i=1}^{k} B_i$. Below we will compute the dimension of this simple bubble tree.

The initial bubble tree $u_0, u_1, \dots, u_{k'}$ is the limit of a sequence in the moduli space $\mathcal{M}_0(W, N, z_0)$. Hence the connected sum $u_\infty := u_0 \# \cdots \# u_{k'}$ is, as element of $\pi_2(W, N)$, homotopic to a disk u in the bishop family, and the Maslov indeces

$$\mu(u) := \mu(u^*TW, u^*TN) \ \text{ and } \ \mu(u_\infty) := \mu(u_\infty^*TW, u_\infty^*TN)$$

have to be equal. With the standard rules for the Maslov index (see for example [**MS04**, Appendix C.3]), we obtain

$$2 = \mu(u) = \mu(u_\infty) = \mu(u_0) + \sum_{j=1}^{k'} 2c_1([u_j]) = \mu(u_0) + 2c_1(\mathbf{A}) \,.$$

The dimension of the unconnected set of holomorphic curves $\widetilde{\mathcal{M}}_{[u_0]}(W, N, z_0) \times \prod_{j=1}^k \widetilde{\mathcal{M}}_{B_j}(W)$ for the simple bubble tree is

$$\begin{aligned}\bigl(n + \mu(u_0)\bigr) + \sum_{j=1}^k 2\left(n + c_1(B_j)\right) &= n + 2 - 2c_1(A) + 2nk + \sum_{j=1}^k 2c_1(B_j) \\ &= n + 2 + 2nk + 2\left(c_1(\mathbf{B}) - c_1(\mathbf{A})\right) .\end{aligned}$$

In the next step, we want to consider the subset of connected bubbles, i.e. we choose a total of k pairs of nodal points, which then have to be pairwise equal under the evaluation map. The nodal points span a manifold

$$Z(2k) \subset \left\{(1, \dots, 2k) \to \mathbb{D}^2 \amalg \mathbb{S}^2 \cdots \amalg \mathbb{S}^2\right\}$$

of dimension $4k$. The dimension reduction comes from requiring that the evaluation map

$$\mathrm{ev}\colon \widetilde{\mathcal{M}}_{[u_0]}(W, N, z_0) \times \prod_{j=1}^k \widetilde{\mathcal{M}}_{B_j}(W) \times Z(2k) \to W^{2k}$$

sends pairs of nodal points to the same image in the symplectic manifold. By regularity and transversality of the evaluation map to the diagonal submanifold $\triangle(k) \hookrightarrow W^{2k}$, the dimension of the space of holomorphic curves is reduced by the codimension of $\triangle(k)$, which is $2nk$.

As a last step, we have to add the marked point z_0 used for the evaluation map ev_{z_0}, this way increasing the dimension by 2, and then we take the quotient by the automorphism group to obtain the moduli space. The dimension of the automorphism group is $6k + 3$. Hence the dimension of the total moduli space is

$$\begin{aligned}&n + 2 + 2nk + 2\left(c_1(\mathbf{B}) - c_1(\mathbf{A})\right) + 4k - 2nk + 2 - (6k + 3) \\ &\qquad = n + 1 - 2k + 2\left(c_1(\mathbf{B}) - c_1(\mathbf{A})\right) \le n + 1 - 2k .\end{aligned}$$

The inequality holds because by the assumption of semipositivity, all the Chern classes are non-negative on holomorphic spheres, and all coefficients n_j in the difference $c_1(\mathbf{B}) - c_1(\mathbf{A}) = \sum_j c_1(B_j) - \sum_i c_1(A_i) = \sum_j c_1(B_j) - \sum_i m_i c_1(B_{j_i}) = \sum_j n_j c_1(B_j)$ are non-positive integers.

The computations for the disks in $\partial\mathcal{M}_0\bigl(\mathbb{D}^2, N; z_0\bigr)$ only differs by the requirement that the marked point needs to lie on the boundary of the disk u_0 instead of moving freely on the bubble tree. Instead of having two degrees of freedom for this choice, we thus only add one extra dimension. □

III.4. Proof of the non-fillability Theorem A

Theorem A. *Let (M, ξ) be a contact manifold that contains a* bLob *N, then M does not admit any semi-positive weak symplectic filling (W, ω) for which $\omega|_{TN}$ is exact.*

Assume there were a semi-positive symplectic filling (W, ω) for which $\omega|_{TN}$ is exact. Let α be a positive contact form for ξ. By Proposition II.2.3, we can extend (W, ω) with a collar in such a way that we have $\omega|_{TN} = d\alpha|_{TN}$, which will allow us to use the energy estimates of the previous section. Now we choose an almost complex structure that is adapted to the bLob N as in Corollary III.1.4, and we will study the moduli space $\mathcal{M}_0\bigl(\mathbb{D}^2, N; z_0\bigr)$ defined in Section III.2 of holomorphic disks with one marked point lying in the same component as the Bishop family around a chosen component B_0 of the binding of N.

Trace a smooth path $\gamma\colon [0,1] \to N$ that starts at $\gamma(0) \in B_0$ and ends on the boundary ∂N. Assume further that γ is a regular curve, and that it intersects the binding and ∂N only on the endpoints of $[0,1]$. We want to select a 1-dimensional moduli space in $\mathcal{M}_0\big(\mathbb{D}^2, N; z_0\big)$ by only considering

$$\mathcal{M}^\gamma := \mathrm{ev}_{z_0}^{-1}\big(\gamma(I)\big) .$$

It will be important for us that $\gamma(I)$ does not intersect the image of any bubble trees in $\overline{\mathcal{M}}_0\big(\mathbb{D}^2, N; z_0\big) \setminus \mathcal{M}_0\big(\mathbb{D}^2, N; z_0\big)$.

By Proposition III.3.5, we have that the bubble trees in $\overline{\partial\mathcal{M}}_0\big(\mathbb{D}^2, N; z_0\big)$ lie in the image of a finite union of smooth maps defined on manifolds of dimension $\dim \partial\mathcal{M}_0\big(\mathbb{D}^2, N; z_0\big) - 2 = \dim N - 2$. The subset $N \setminus \mathrm{ev}_{z_0}(\text{bubble trees})$ is connected and we can deform γ keeping the endpoints fixed so that it does not intersect any of the bubble trees.

For a small perturbation of J (away from the binding and the boundary of N), we can make sure that the evaluation map ev_{z_0} is transverse to the path $\gamma(I)$. If the perturbed J lies sufficiently close to the old one, then γ will also not intersect any bubble trees for this new J, for otherwise we could choose a sequence of almost complex structures J_k converging to the unperturbed J such that for everyone there existed a bubble tree v_k intersecting γ. We would find a converging subsequence of v_k yielding a bubble tree v_∞ for the unperturbed almost complex structure intersecting γ, which contradicts our assumption.

It follows that $\mathcal{M}^\gamma$ is a collection of compact 1-dimensional submanifolds of $\partial\mathcal{M}_0\big(\mathbb{D}^2, N; z_0\big)$. There is one component in $\mathcal{M}^\gamma$, which we will denote by $\mathcal{M}_0^\gamma$ that contains the Bishop disks that intersect $\gamma\big([0,\varepsilon)\big)$. We know that the Bishop disks are the only disks close to the binding, and hence it follows that $\mathcal{M}_0^\gamma$ cannot be a loop that closes up, but must be instead a closed interval.

The first endpoint of $\mathcal{M}_0^\gamma$ is the constant disk with image $\gamma(0) \in B_0$, and we will deduce a contradiction by showing that no holomorphic disk can be the second endpoint of $\mathcal{M}_0^\gamma$.

By Proposition II.3.9, there is a small neighborhood of ∂N that cannot be entered by any holomorphic disk. By our construction the endpoint of $\mathcal{M}_0^\gamma$ cannot be any bubble tree either. It follows that the endpoint needs to be a regular disk $[u, z_0] \in \partial\mathcal{M}_0\big(\mathbb{D}^2, N; z_0\big)$ for which the boundary of u lies in $N \setminus \big(\partial N \cup B\big)$ and whose interior points cannot touch ∂W either, because we are assuming that the boundary of W is convex.

It follows that this regular disk cannot really be the endpoint of $\mathcal{M}_0^\gamma$, because the evaluation map ev_{z_0} will also be transverse to γ at $[u, z_0]$ so that we can extend $\mathcal{M}_0^\gamma$ further.

This leads to a contradiction that shows that the assumption that the boundary of W is everywhere convex cannot hold.

III.5. Proof of Theorem B

For the proof, we first recall the definition of the degree of a map.

Definition. Let X and Y be closed oriented n-manifolds. The **degree** of a map $f\colon X \to Y$ is the integer $d = \deg(f)$ such that

$$f_\#[X] = d \cdot [Y] ,$$

where $[X] \in H_n(X, \mathbb{Z})$ and $[Y] \in H_n(Y, \mathbb{Z})$ are the fundamental classes of the corresponding manifolds. When the manifolds X and Y are not orientable, we define the degree to be an element of $\mathbb{Z}_2$ using the same formula, where the fundamental classes are elements in $H_n(X, \mathbb{Z}_2)$ and $H_n(Y, \mathbb{Z}_2)$.

Note that we can easily compute the degree of a smooth map f between smooth manifolds by considering a regular value $y_0 \in Y$ of f (which by Sard's theorem exist in abundance), and adding

$$\deg f = \sum_{x \in f^{-1}(y_0)} \operatorname{sign} Df_x \;,$$

where the point x contributes to the sum with $+1$, whenever Df_x is orientation preserving, and contributes with -1 otherwise. In case the manifolds are not orientable, we can always add $+1$ in the above formula, but need to take sum over $\mathbb{Z}_2$.

Theorem B. *Let (M, ξ) be a contact manifold of dimension $(2n+1)$ that contains a* Lob *N. If M has a weak symplectic filling (W, ω) that is symplectically aspherical, and for which $\omega|_{TN}$ is exact, then it follows that N represents a trivial class in $H_{n+1}(W, \mathbb{Z}_2)$. If the first and second Stiefel-Whitney classes $w_1(N)$ and $w_2(N)$ vanish, then we obtain that $[N]$ must be a trivial class in $H_{n+1}(W, \mathbb{Z})$.*

Using Proposition II.2.3 we can assume that $\omega|_{TN} = d\alpha|_{TN}$ for a chosen contact form α. Choose an almost complex structure J on W that is adapted to the Lob N, and let $\mathcal{M}_0\big(\mathbb{D}^2, N; z_0\big)$ be the moduli space of holomorphic disks with one marked point lying in the same component as the Bishop family around a chosen component of the binding of N.

Since W is symplectically aspherical, we obtain by Corollary III.3.4 that $\overline{\mathcal{M}}_0\big(\mathbb{D}^2, N; z_0\big)$ is a compact smooth manifold with boundary. It was shown in [**Geo11**] that $\mathcal{M}_0\big(\mathbb{D}^2, N; z_0\big)$ is orientable if the first and second Stiefel-Whitney classes of $N \setminus B$ vanish. With our assumptions this is the case, because $w_j\big(N \setminus B\big) = w_j(N)|_{(N \setminus B)}$. If $\mathcal{M}_0\big(\mathbb{D}^2, N; z_0\big)$ is orientable then $\overline{\mathcal{M}}_0\big(\mathbb{D}^2, N; z_0\big)$ will also be orientable: If there were an orientation reversing loop γ in the compactified moduli space (which is obtained from $\mathcal{M}_0\big(\mathbb{D}^2, N; z_0\big)$ by gluing in B as codimension 3 submanifold), then due to the large codimension we could easily push γ completely into the regular part of the moduli space, where it would still need to be orientation reversing.

It follows that the boundary $\partial\overline{\mathcal{M}}_0\big(\mathbb{D}^2, N; z_0\big)$ is also homologically a boundary (either with $\mathbb{Z}$- or $\mathbb{Z}_2$-coefficients depending on the orientability of the considered spaces).

Denote the restriction of the evaluation map

$$\mathrm{ev}_{z_0}|_{\partial\overline{\mathcal{M}}_0\left(\mathbb{D}^2, N; z_0\right)} \colon \partial\overline{\mathcal{M}}_0\big(\mathbb{D}^2, N; z_0\big) \to N \;,$$

by f. We know that close to the binding every point is covered by a unique Bishop disk, this implies by the remarks made above that the degree $\deg(f)$ needs to be ± 1.

We have the following obvious equation

$$\mathrm{ev}_{z_0} \circ \iota_{\partial\overline{\mathcal{M}}} = \iota_N \circ f \;,$$

where $\iota_{\partial\overline{\mathcal{M}}}$ denotes the embedding of $\partial\overline{\mathcal{M}}_0\big(\mathbb{D}^2, N; z_0\big)$ in $\overline{\mathcal{M}}_0\big(\mathbb{D}^2, N; z_0\big)$ and ι_N the embedding of N in W. The homomorphism induced by $\iota_{\partial\overline{\mathcal{M}}}$ is the trivial map on the $(n+1)$-st homology group, so that the left side of the equation gives rise to the 0-map

$$H_{n+1}\Big(\partial\overline{\mathcal{M}}_0\big(\mathbb{D}^2, N; z_0\big), R\Big) \to H_{n+1}\big(W, R\big)$$

with R being either $\mathbb{Z}$ or $\mathbb{Z}_2$. Since $f_\#$ is $\pm$ identity, it follows that ι_N has to induce the trivial map on homology, which implies that N is homologically trivial in W.

CHAPTER IV

Constructions of contact manifolds

IV.1. $\mathbb{S}^1$-invariant contact manifolds

Lutz classified in '77 3-dimensional contact manifolds with free $\mathbb{S}^1$-actions [**Lut77**]. Even though such manifolds are topologically rather exceptional, they form a very rich class of examples. Most of the relevant contact properties up to this day can be studied on these structures.

The aim of this section is to show how one can construct higher dimensional contact $\mathbb{S}^1$-manifolds starting from certain symplectic domains, and to explain how we can extract relevant contact information. For this we start with a complete symplectic domain, and construct from it in Section IV.1.2, a compact $\mathbb{S}^1$-principal bundle with non-empty boundary and with an $\mathbb{S}^1$-invariant contact structure. We call these building blocks *Giroux domains*. Giroux domains can be glued along their boundaries, and boundary components may be blown down, see Section IV.1.3. Every $\mathbb{S}^1$-invariant contact manifold with only free orbits is either a Boothby-Wang manifold or it can be obtained by gluing Giroux domains [**DG12**] together; it is easy to convince oneself that every $\mathbb{S}^1$-invariant contact manifold with only free orbits or with fixed point components of codimension 2 can be obtained by gluing and blowing down Giroux domains.

In a final step, we explain how we can read off in very special cases properties like PS-overtwistedness or fillability of such manifolds. The first non-fillability result we give, see Section IV.1.3, is a generalization of the following idea in dimension 3: If we have a contact 3-manifold with an $\mathbb{S}^1$-action that has both fixed points and Legendrian orbits, then we can find an overtwisted disk simply by connecting a fixed point to a point on a Legendrian orbit using an embedded segment γ that is tangent to the contact structure. The union of the $\mathbb{S}^1$-orbits passing through γ is then an overtwisted disk.

The core results of this section were first described in [**MNW13**]. We state them here without too many modifications. We would have liked to extend those techniques by studying also non-trivial circle bundles (for related results see [**DG12**] and [**CDv12**]), but time did not allow me to arrive to analogous results as for trivial bundles.

IV.1.1. Complete symplectic domains.

In this section we will discuss a definition of a complete symplectic domain that is due to Giroux. This definition adds precision to the classical notion by keeping in an elegant way account of the "structure at infinity". We use this as a preparation for the next section, where we will construct contact manifolds starting from such symplectic domains. A written account for exact symplectic manifolds (called ideal Liouville domains) can be found in [**MNW13**]. This notion has been extended in [**DG12**] to non-exact structures. We thank Sylvain Courte for his generous help with many problems we encountered, and for explaining to us many implications of the definition.

Definition. Let Σ be a compact $(2n)$-manifold with boundary, and let $\omega \in \Omega^2(\mathring{\Sigma})$ be a symplectic form defined only on the *interior* $\mathring{\Sigma} = \Sigma \setminus \partial\Sigma$.

We say that Σ is a **complete symplectic domain** , if we can choose a 1-form $\lambda \in \Omega^1(\mathring{\Sigma})$ and a 2-form $\omega_0 \in \Omega^2(\Sigma)$ that is defined on all of Σ with the following properties:

- The restriction to $\mathring{\Sigma}$ is cohomologous to ω, and λ is a primitive of $\omega - \omega_0|_{\mathring{\Sigma}}$ so that
$$\omega = d\lambda + \omega_0|_{\mathring{\Sigma}} .$$
- If we choose a function $f\colon \Sigma \to [0,\infty)$ for which $\partial\Sigma = f^{-1}(0)$ is a regular level set, then
$$f\,\lambda$$
can be extended to a smooth 1-form λ_f on all of Σ, and the restriction of λ_f to $\partial\Sigma$ is a contact form compatible with the positive boundary orientation.

We denote the cooriented contact structure induced on the boundary by ξ_ω.

When ω is exact, we call (Σ, ω) an **ideal Liouville domain** , and if we choose additionally $\omega_0 = 0$, then we call any primitive λ as above with $d\lambda = \omega$ a **Liouville form**.

Remark IV.1.1. It is easy to extend the definition also for negative ends by requiring that the auxiliary 2-form ω_0 vanishes at negative ends, and that $\frac{1}{f}\,\lambda$ extends to a contact form at every negative ends.

This definition may seem at first very abstract compared to the standard definition of a symplectic manifold with positive cylindrical ends, so let us briefly discuss its advantages. As we will prove below a symplectic manifold with positive cylindrical ends is symplectomorphic to the interior of a complete symplectic domain, but the power of the definition lies in the fact that it captures the precise behavior "at infinity" of a symplectic structure.

Sylvain Courte [**Cou14**] showed that there exist contact manifolds of dimension 5 and higher, that are not even *diffeomorphic*, but that have symplectomorphic symplectizations. This implies that the symplectic structure alone is not sufficient to understand behavior at infinity if we do not fix an additional Liouville vector field. On the other hand, there are different Liouville vector fields that lead to the same structure, which would make it difficult to state when two Liouville manifolds are equivalent. A complete symplectic domain fixes the behavior at infinity without depending on additional choices, and gives a compact way of defining precisely the cylindrical structure one has in mind.

We should of course also mention that Giroux's initial motivation was to describe several operations in contact topology in a compact way, without the "fiddling" that is usually necessary. Among them, there is for example the construction of a contact manifold from an abstract open book, where the classical method requires several choices of cut-off parameters and other data, which makes it afterward necessary to show manually that the resulting manifold does not depend on them.

Before proving any further properties about complete symplectic domains, we will first show that the contact structure ξ_ω on $\partial\Sigma$ is completely determined by ω and does not depend on any of the auxiliary choices in the definition.

Remark IV.1.2. For any two smooth functions $f_0, f_1\colon \Sigma \to [0,\infty)$ for which $f_j^{-1}(0) = \partial\Sigma$ is a regular level set, there exists a smooth positive function g with $f_1 := g f_0$. It is clear that the contact forms induced by λ_{f_0} and λ_{f_1} only differ by multiplication with g, and hence define the same contact structure.

Fix a smooth function f as above, then we can show that $f^2\omega$ extends from $\mathring{\Sigma}$ to a smooth 2-form $\widehat{\omega}$ defined on all Σ. Choose for this a 2-form $\omega_0 \in \Omega^2(\Sigma)$ and a primitive $\lambda \in \Omega^1(\mathring{\Sigma})$ with

$$\omega = d\lambda + \omega_0|_{\mathring{\Sigma}}$$

as in the definition. We can easily compute on $\mathring{\Sigma}$

$$f\omega = f\omega_0 + f\,d\lambda = f\omega_0 + d(f\lambda) - df \wedge \lambda \tag{IV.1.1}$$

so that $f^2\omega$ extends smoothly to $\widehat{\omega} := f\,d\lambda_f - df \wedge \lambda_f + f^2\omega_0$ on Σ. Along $\partial\Sigma$, this 2-form simplifies to $-df \wedge \lambda_f$, which means that we recover the contact structure ξ_ω at every point $p \in \partial\Sigma$ as the subspace

$$\xi_\omega = \left\{v \in T_p\Sigma \,\middle|\, \widehat{\omega}(v,\cdot) = 0\right\} .$$

Clearly, the extension $\widehat{\omega}$ does not depend on the choice of the primitive λ or the choice of the 2-form ω_0.

Lemma IV.1.3. *Let $f\colon \Sigma \to [0,\infty)$ be a regular equation for the boundary of a complete symplectic domain (Σ,ω). The volume form*

$$\mu_f := f^{n+1}\omega^n$$

on $\mathring{\Sigma}$ extends to a smooth volume form $\widehat{\mu}_f$ on all of Σ.

Proof. Compute μ_f by taking the maximum exterior power of Equation (IV.1.1), and multiplying the result with f. We obtain

$$\mu_f = f\left(f\omega_0 + d(f\lambda)\right)^n - n\,df \wedge (f\lambda) \wedge \left(f\omega_0 + d(f\lambda)\right)^{n-1} ,$$

which we can extend smoothly over all of Σ by replacing the $f\lambda$-terms by λ_f. This way we find the desired extension to all of Σ as

$$\widehat{\mu}_f := f\left(f\omega_0 + d\lambda_f\right)^n - n\,df \wedge \lambda_f \wedge \left(f\omega_0 + d\lambda_f\right)^{n-1} .$$

It is clear, that $\widehat{\mu}_f$ is a volume form on $\mathring{\Sigma}$, and along the boundary, $\widehat{\mu}_f$ simplifies to $-n\,df \wedge \lambda_f \wedge d\lambda_f^{n-1}$, which by the contact condition does not vanish either. □

The following lemma will be important for technical reasons, but also because it shows that a complete symplectic domain as defined by Giroux has cylindrical ends like the ones of the classical definition.

Lemma IV.1.4. *Let (Σ,ω) be a complete symplectic domain, and let $\alpha \in \Omega^1(\partial\Sigma)$ be a positive contact form for ξ_ω. There is a collar neighborhood of $\partial\Sigma$ diffeomorphic to $(-\varepsilon,0] \times \partial\Sigma$ such that the symplectic form ω is given by*

$$\omega|_{(-\varepsilon,0)\times\partial\Sigma} = -d\big(\frac{1}{s}\,\alpha\big) + \omega_\partial ,$$

where s denotes the coordinate on $(-\varepsilon,0]$, and $\omega_\partial \in \Omega^2(\partial\Sigma)$ is a closed 2-form on $\partial\Sigma$.

Proof. Choose a function $f\colon \Sigma \to [0,\infty)$ that is a regular equation for the boundary, and let λ be a primitive of ω such that $f\lambda$ extends on Σ to a 1-form λ_f whose restriction

$$\alpha := \lambda_f|_{T\partial\Sigma}$$

is a contact form of ξ_ω. If desired we can arrange the given situation to find instead any contact form α' of our choice on $\partial\Sigma$. Simply let $g\colon \partial\Sigma \to (0,\infty)$ be a smooth positive function such that $\alpha' = g\alpha$, and extend g to a positive function on all of Σ, then replacing f by $h := gf$, we obtain a 1-form $h\lambda$ that will induce the contact form α' on $\partial\Sigma$.

The standard method of constructing a collar neighborhood, is to follow the flow of a vector field transverse to the boundary. The **Liouville field** X_L of λ is the vector field on $\mathring{\Sigma}$ defined by the equation

$$\iota_{X_L}\omega = \lambda \; . \tag{IV.1.2}$$

Unfortunately X_L is not defined along $\partial\Sigma$ (in fact, its extension vanishes along the boundary), but it is easy to see from Equation (IV.1.2) that the vector field $X_f := \frac{1}{f}\, X_L$ is the unique solution of

$$\iota_{X_f}\bigl(f^{n+1}\omega^n\bigr) = nf\lambda \wedge (f\omega)^{n-1} \; . \tag{IV.1.3}$$

We will show below that X_f extends to a field on all of Σ that is positively transverse to $\partial\Sigma$. This will allow us to use its flow to construct a collar neighborhood of $\partial\Sigma$.

Using Lemma IV.1.3, we can replace the left hand side of Equation (IV.1.3) by $\iota_{X_f}\widehat{\mu}_f$, and for the right hand side, we find

$$nf\lambda \wedge (f\omega)^{n-1} = nf\lambda \wedge \bigl(f\omega_0 + d(f\lambda) - df \wedge \lambda\bigr)^{n-1} = n\lambda_f \wedge (f\omega_0 + d\lambda_f)^{n-1} \; ,$$

which also extends smoothly to all of Σ. Using that $\widehat{\mu}_f$ is a volume form, we can define X_f equivalently as the unique solution of

$$\iota_{X_f}\widehat{\mu}_f = n\lambda_f \wedge \bigl(f\omega_0 + d\lambda_f\bigr)^{n-1} \; ,$$

which shows that X_f is defined on all of Σ. To see that X_f is positively transverse to $\partial\Sigma$, note that the previous equation reduces on the boundary to

$$-n\,\iota_{X_f}\bigl(df \wedge \lambda_f \wedge d\lambda_f^{n-1}\bigr) = n\lambda_f \wedge d\lambda_f^{n-1} \; .$$

Clearly, plugging X_f a second time into the equation makes the left side vanish, so that $\iota_{X_f}\bigl(\lambda_f \wedge d\lambda_f^{n-1}\bigr) = 0$, and in particular we obtain that $df(X_f) = -1$ along $\partial\Sigma$.

Let $(-\infty, 0] \times \partial\Sigma$ be the collar neighborhood of the boundary obtained by following the flow of X_f in negative time direction starting from $\partial\Sigma$. We denote the $(-\infty, 0]$-coordinate by s, and write X_f as ∂_s.

We will now deform the symplectic structure to bring it in the collar neighborhood into the desired form, and then show that there is an isotopy of Σ that fixes the boundary pointwise and whose restriction to $\mathring{\Sigma}$ is a symplectomorphism between the old and the new form. We may write the given symplectic structure on the collar $(-\infty, 0) \times \partial\Sigma$ as

$$\omega = d\lambda + \omega_\partial + d\gamma \; ,$$

where we have set $\omega_\partial := \omega_0|_{T\partial\Sigma}$, and where γ is a smooth 1-form such that $\omega_0 = \omega_\partial + d\gamma$ on $(-\infty, 0] \times \partial\Sigma$. We would like to deform ω to

$$-d\bigl(\frac{1}{s}\,\alpha\bigr) + \omega_\partial \; .$$

Let ρ_0 be a smooth function $\rho_0\colon (-\infty, 0] \to [0,1]$ that vanishes on $(-\infty, -1]$ and that is equal to 1 in a small neighborhood of 0. We may assume that the derivative of ρ_0 only takes values in the interval $[0, 2]$. Using ρ_0, we can define cut-off functions with arbitrarily small support by setting $\rho(s) := \rho_0(Cs)$ for large constants C. Independently of the choice of C, we obtain for all $s \in (-\infty, 0]$ the uniform bound

$$-2 \le s\rho'(s) \le 0 \; .$$

Let λ_s be the extension of the 1-form $s\,\lambda$ to all of $(-\varepsilon, 0] \times \partial\Sigma$. Note that the restrictions of λ_s is equal to $-\alpha$, because

$$s\,\lambda = \frac{s}{f}\,\lambda_f$$

on $(-\varepsilon, 0) \times \partial\Sigma$, and $df(X_f) = -1$. Define now a family of 2-forms parametrized by $T \in [0,1]$

$$\Omega_T := -d\Big(\frac{T\rho(s)}{s}\,\alpha + \frac{1 - T\rho(s)}{s}\,\lambda_s\Big) + \omega_\partial + d\big((1 - T\rho(s))\,\gamma\big)$$

and extend it outside the collar neighborhood to ω. The family Ω_T is a homotopy of 2-forms between the given structure $\omega = \Omega_0$ and a 2-form Ω_1 that has the desired shape in a small neighborhood of $\partial\Sigma$.

To check that the Ω_T are symplectic forms on $\mathring{\Sigma}$, simply compute $s^{n+1}\Omega_T^n$. Outside the support of ρ all the forms agree with $s^{n+1}\omega^n$, and hence there is nothing to show. Using that $s\Omega_T$ can be written on the collar neighborhood as

$$\begin{aligned} s\Omega_T = -T\rho'\,ds \wedge \big(\alpha - \lambda_s + s\gamma\big) + T\rho\,\Big(\frac{1}{s}\,ds \wedge \alpha - d\alpha + s\omega_\partial\Big) \\ + \big(1 - T\rho\big)\,\Big(\frac{1}{s}\,ds \wedge \lambda_s - d\lambda_s + s\omega_0\Big)\;, \end{aligned}$$

we obtain

$$\begin{aligned} s^{n+1}\Omega_T^n = (-1)^{n-1} n\,ds \wedge \Big(T\rho\,\alpha + \big(1 - T\rho\big)\,\lambda_s\Big) \wedge \Big(T\rho\,d\alpha + \big(1 - T\rho\big)\,d\lambda_s\Big)^{n-1} \\ - nTs\rho'\,ds \wedge \big(\alpha - \lambda_s\big) \wedge \Big(-T\rho\,d\alpha - \big(1 - T\rho\big)\,d\lambda_s\Big)^{n-1} + \mathcal{O}^1(s)\;, \end{aligned}$$

where we have grouped all terms whose order in s is 1 or higher in $\mathcal{O}^1(s)$. The right hand side is also defined on the boundary of Σ, and we write μ_T for the smooth extension of $s^{n+1}\Omega_T^n$ to all of Σ. If we choose the support of ρ sufficiently small, then the terms of order one composing $\mathcal{O}^1(s)$ can be neglected, and $s\rho'\,ds \wedge \big(\alpha - \lambda_s\big)$ can also be made arbitrarily small, because $s\rho'(s)$ is bounded and because λ_s agrees along $\partial\Sigma$ with α. The only the term on the right hand side relevant for the sign of μ_T is thus the first, which simplifies on $\partial\Sigma$ to $(-1)^{n-1} n\,ds \wedge \alpha \wedge d\alpha^{n-1}$. It follows that μ_T is a family of volume forms on Σ, and hence the Ω_T are symplectic forms on $\mathring{\Sigma}$.

Now we are ready to apply the Moser trick to show that Ω_1 and ω are isotopic. As explained for example in [**MS98**, Section 3.2], we can obtain the desired isotopy if we can integrate the vector field Y_T given as solution of the equation

$$\iota_{Y_T}\Omega_T = \lambda_T\;,$$

where λ_T is a 1-form with $d\lambda_T = \frac{d}{dT}\Omega_T$ on $\mathring{\Sigma}$.

We can choose

$$\lambda_T = -\rho(s)\,\Big(\frac{1}{s}\,\alpha - \frac{1}{s}\,\lambda_s + \gamma\Big)$$

which vanishes outside the collar neighborhood. Since the Ω_T are all symplectic forms, the vector field Y_T is uniquely determined on $\mathring{\Sigma}$. By repeating the strategy applied previously, that means by finding an alternate equation that is defined on all of Σ for which Y_T is the unique solution, we can show that Y_T extends to a smooth vector field on all of Σ that vanishes along the boundary $\partial\Sigma$ (so that we may integrate its flow).

A short computation shows that Y_T can be equivalently defined by the equation

$$\iota_{Y_T}\mu_T = \iota_{Y_T}\big(s^{n+1}\Omega_T^n\big) = ns^{n+1}\lambda_T \wedge \Omega_T^{n-1}\;.$$

The right hand side is everywhere defined, and vanishes along $\partial\Sigma$ as can be seen by a short computation for $s = 0$

$$s^{n+1}\lambda_T \wedge \Omega_T^{n-1} = (n-1)\, ds \wedge \big(\alpha - \lambda_s\big) \wedge \big(T\,\alpha + (1-T)\,\lambda_s\big) \wedge \Big(-T\,d\alpha - \big(1-T\big)\,d\lambda_s\Big)^{n-2} .$$

The vector field Y_T is hence also defined on all of Σ, and we can integrate its flow obtaining the desired isotopy that keeps the boundary pointwise fixed, and maps the deformed symplectic structure to the original one. □

Remark IV.1.5. Let (Σ, ω) be a complete symplectic domain, and let $\Sigma_\delta = \Sigma \setminus (-\delta, 0] \times \Sigma$ be a compact subdomain, where we have cut-off a cylindrical end given by Lemma IV.1.4. It is easy to see that (Σ_δ, ω) is a weak filling of $(\partial\Sigma_\delta, \alpha)$ if $\delta > 0$ has been chosen small enough. Conversely, if (W, ω) is a weak filling of a contact manifold (M, α) then we can easily attach a collar as in Proposition II.2.3, that compactifies to a complete symplectic domain.

IV.1.2. Giroux domains. Consider first an ideal Liouville domain (Σ, ω) with Liouville form λ, then one can endow $\Sigma \times \mathbb{R}$ with the contact structure $\ker(f\,dt + f\lambda)$ for any smooth function $f\colon \Sigma \to [0, \infty)$ with regular level set $f^{-1}(0) = \partial\Sigma$. Over the interior of Σ, $\ker(f\,dt + f\,\lambda) = \ker(dt + \lambda)$, so one recovers the standard notion of the *contactization* of the Liouville manifold defined by λ. On the boundary we have $f\,dt = 0$, so the contact hyperplanes are $\xi_\omega \oplus T\mathbb{R}$. Since the contact forms constructed on $\Sigma \times \mathbb{R}$ are $\mathbb{R}$-invariant, one can just as well replace $\mathbb{R}$ by $\mathbb{S}^1$.

Suppose now more generally that (Σ, ω) is a complete symplectic domain, but that ω represents a class in $H^2(\mathring{\Sigma}, \mathbb{R})$ that is not necessarily trivial, but that is *integer valued.* We can choose a 2-form $\omega_0 \in \Omega^2(\Sigma)$ whose restriction to $\mathring{\Sigma}$ is cohomologous to ω and a primitive λ as in the definition of a complete symplectic domain such that

$$\omega = d\lambda + \omega_0|_{\mathring{\Sigma}} .$$

By the classification of circle bundles, we find an $\mathbb{S}^1$-bundle E_ω over Σ whose Euler class $e\big(E_\omega\big)$ is equal to $[\omega]$, and we can actually chose a connection 1-form A_0 on the total space of E_ω such that $dA_0 = \pi^*\omega_0$. To simplify notation, we will just write f, λ and ω_0 instead of $f \circ \pi$, $\pi^*\lambda$ and $\pi^*\omega_0$, when doing computations on E_ω.

This allows us to define a contact structure on E_ω given as the kernel of the 1-form

$$\alpha := f A_0 + \lambda_f .$$

Note first that this form is globally defined and invariant under the natural circle action on the fibers. Over $\mathring{\Sigma}$, we have

$$\begin{aligned} \alpha \wedge d\alpha^n &= \big(fA_0 + f\lambda\big) \wedge \big(f\omega + df \wedge (A_0 + \lambda)\big)^n = \big(fA_0 + f\lambda\big) \wedge (f\omega)^n \\ &= A_0 \wedge \big(f^{n+1}\omega^n\big) , \end{aligned}$$

and we had already seen in Lemma IV.1.3, $f^{n+1}\omega^n$ extends to a volume form $\widehat{\mu}_f$ defined on all of Σ. This shows that α is a contact form on E_ω.

Definition. We refer to E_ω with the contact structure defined by $\ker\big(fA_0 + \lambda_f\big)$ as the **Giroux domain associated to** (Σ, ω).

Example IV.1.6. We consider

$$\Sigma = \mathbb{S}^1 \times [0, \pi], \quad \omega = \frac{1}{\sin^2 s}\, d\theta \wedge ds$$

where s is the coordinate in $[0, \pi]$ and θ the coordinate in $\mathbb{S}^1$, carrying the trivial contact structure $\ker \pm d\theta$. One can take as a Liouville form $\beta = \cot s\; d\theta$. Setting $f(\theta, s) = \sin s$, we get the contact form $f(\theta, s) \cdot (\beta + dt) = \cos s\; d\theta + \sin s\; dt$ on $\Sigma \times \mathbb{S}^1$. Thus the Giroux domain associated to this ideal Liouville domain is a Giroux π-torsion domain.

Observe that the contact form α simplifies over the boundary $\partial E_\omega = E_\omega|_{\partial\Sigma}$ to $\pi^*\lambda_f$ so that the contact structure intersects ∂E_ω in $(D\pi)^{-1}(\xi_\omega)$. In particular by Remark IV.1.2, this intersection does not depend on the choice of the connection 1-form A_0, on λ, or on f.

The properties of the boundary of a Giroux domain will allow us to glue different Giroux domains or to blow down their boundary components by a technique called *contact cuts* [**Ler01**]. This motivates the following definition.

Definition. Let P be a circle bundle $\pi\colon P \to B$ over a closed contact manifold (B, ξ_B). We call an embedding of P into a contact manifold (M, ξ) a **contact cutting hypersurface** if it is of codimension 1, and if the intersection

$$\xi \cap TP$$

projects by $D\pi$ onto ξ_B.

Observe that in dimension three, a cutting hypersurface is simply a pre-Lagrangian torus with closed characteristic leaves.

Lemma IV.1.7. *The Giroux domains associated to (Σ, ω) are up isotopy independent of the auxiliary choices of the connection 1-form A_0, of λ, or of f.*

PROOF. Different regular equations f_1 and f_2 for $\partial\Sigma$ only rescale the contact form but do not change the contact structure, hence we may fix a function f for the rest of the proof. The admissible choices of the connection 1-form and of the primitive λ, form a convex set, and hence it is sufficient to show the statement for paths of contact forms

$$\alpha_\tau = f\left(A_\tau + \lambda_\tau\right) .$$

The Gray stability theorem, e.g. [**Gei08**, Theorem 2.2.2], provides the desired isotopy by integrating the vector field Y_τ defined by the two equations

$$\alpha_\tau(Y_\tau) = 0 \qquad \text{and} \qquad \left(\iota_{Y_\tau} d\alpha_\tau\right)\big|_{\ker\alpha_\tau} = -\dot\alpha_\tau|_{\ker\alpha_\tau} .$$

By Remark IV.1.2, $f^2\omega$ extends to a smooth 2-form on all of Σ that can be written along the boundary as $-df \wedge \lambda_f$, but in particular it is independent of the choice of λ, and all of the α_τ agree when restricted to $T\partial E_\omega$. Hence we obtain for every $v \in (T\partial E_\omega) \cap \ker\alpha_\tau$ that $d\alpha_\tau(Y_\tau, v) = -\dot\alpha_\tau(v) = 0$, and it follows that Y_τ needs to be parallel to the boundary of the Giroux domain, because every vector in $\ker\alpha_\tau$ transverse to ∂E_ω would pair positively with some v.

The flow of Y_τ exists and defines the desired isotopy on E_ω. □

Remark IV.1.8. A Giroux domain is an $\mathbb{S}^1$-bundle over a symplectic manifold (Σ, ω). A different construction, also due to Giroux, known as the *mapping torus* of a symplectomorphism φ with compact support in $\mathring{\Sigma}$ produces instead a Σ-bundle over $\mathbb{S}^1$. When the boundary of the resulting bundle is blown down as describe in the next section below, we obtain a contact manifold associated to the abstract open book $(\Sigma, \omega, \varphi)$. Observe that unlike Giroux's original construction of the contact structure associated to an open book (see e.g. in [**Gei08**, Section 7.3]), the construction based on ideal Liouville domains does not require any tweaking near the binding.

IV.1.3. Blowing down. In the previous section, we defined contact cutting hypersurfaces. Typical examples are the boundary of a Giroux domain, but not all contact cutting hypersurfaces arise as boundaries of such domains. What is true tough is that the $\mathbb{S}^1$-fibration on such a cutting surface can always be extended to a circle action on a small tubular neighborhood, that preserves the contact structure. In this section, we will show that if the boundary component of a contact manifold is a cutting hypersurface, we may collapse the $\mathbb{S}^1$-fibers using the contact cut construction [**Ler01**], and produce a smooth contact manifold without boundary. In case the hypersurface lies in the interior of the contact manifold, we may split the contact manifold along a contact cutting hypersurface, and apply the same construction.

Topologically the contact cut just consists in removing the boundary component, and gluing a certain disk bundle in an $\mathbb{S}^1$-invariant way to the collar of the hypersurface.

Lemma IV.1.9. *Let P be a ξ-cutting hypersurface that fibers over the contact manifold (B,ξ_B) and that lies in the interior (or boundary) of (M,ξ). Then it has a neighborhood $(-\varepsilon,\varepsilon)\times P$ (or $[0,\varepsilon)\times P$ respectively) on which ξ can be defined by the contact form $\pi^*\alpha_B + s\,A_0$ where s is the coordinate on the interval, A_0 is a connection 1-form of P, and α_B is a contact form for ξ_B. In particular, there is a free contact action on $\mathbb{S}^1$ on this neighborhood that is outside P transverse to the contact structure.*

Proof. Fix any tubular neighborhood (or collar neighborhood) of P with coordinate s. The 1-form given in the lemma defines a contact structure near P which induces the same hyperplane field on P as ξ, hence by the same argument as in the proof of Lemma IV.1.7, they are isotopic keeping P fixed as subset. $\square$

Suppose P is a cutting hypersurface in a boundary component of (M,ξ). We will now explain how to modify (M,ξ) by **blowing down** P to B.

The disk bundle, we want to glue in is obtained in the following way: Consider the (complex) line bundle E_P associated to P, i.e. the bundle obtained from $P\times\mathbb{C}$ by identifying (p,z) with $\left(e^{-i\phi}\cdot p, e^{i\phi}\,z\right)$ for every $e^{i\phi}\in\mathbb{S}^1$. The base manifold of this bundle is the space $B = P/\mathbb{S}^1$, and P embeds naturally into E_P as its unit bundle via the map

$$P\hookrightarrow E_P, \quad p\mapsto [p,1]\ .$$

Define on $P\times\mathbb{C}$ the 1-form

$$\widehat{\alpha} := \pi^*\alpha_B + |z|^2 A_0 + x\,dy - y\,dx\ ,$$

where A_0 is a connection 1-form on E_P, and where $z = x+iy$ denote the coordinates on the $\mathbb{C}$-factor. It is easy to check that $\widehat{\alpha}$ descends to a well-defined 1-form α on E_P, because it is invariant under the circle action on the product, and because $\widehat{\alpha}(Z_{P\times\mathbb{C}}) = 0$ for the infinitesimal generator $Z_{P\times\mathbb{C}} = -Z_P + x\,\partial_y - y\,\partial_x$. We can see that

$$\widehat{\alpha}\wedge d\widehat{\alpha}^n := \left(\pi^*\alpha_B + |z|^2 A_0 + x\,dy - y\,dx\right)\wedge\left(\pi^*d\alpha_B + |z|^2\,dA_0 + d|z|^2\wedge A_0 + 2\,dx\wedge dy\right)^n$$

simplifies at $P\times\{0\}$ to

$$\widehat{\alpha}\wedge d\widehat{\alpha}^n := \pi^*\alpha_B\wedge\left(\pi^*d\alpha_B + 2\,dx\wedge dy\right)^n = 2n\,dx\wedge dy\wedge\pi^*\left(\alpha_B\wedge d\alpha_B^{n-1}\right)\ ,$$

which has exactly 1-dimensional kernel. Hence the 1-form α will be (at least in a neighborhood of the 0-section of E_P) a contact form.

Let us now come back to the contact manifold (M,ξ) whose boundary has a component P that is a contact cutting hypersurface. Choose a collar neighborhood $P\times[0,\varepsilon)$ given by Lemma IV.1.9 assuming that the contact form is $\pi^*\alpha_B + s\,A_0$ with the same connection 1-form A_0 as the one used in the line bundle construction above.

Remove the hypersurface $P = P \times \{0\}$ from M, and glue a small disk bundle $E_{<\delta}$ of E_P with $\delta^2 < \varepsilon$ onto the remaining collar neighborhood using the diffeomorphism

$$\begin{aligned}\Psi\colon \qquad & E_{<\delta} \setminus B \to P \times (0,\varepsilon) \subset M \setminus P \\ & [p,z] = \left[\frac{z}{|z|}\cdot p, |z|\right] \mapsto \left(\frac{z}{|z|}\cdot p, |z|^2\right) .\end{aligned}$$

The pull-back of $\pi^*\alpha_B + s\,A_0$ is $\pi^*\alpha_B + |z|^2\,A_0$. To recognize that this is just α, pull back the form a from E_P to $P \times \mathbb{C}$. We see that $|z|^2\,A_0$ pulls up to $|z|^2 A_0 + x\,dy - y\,dx$ as we wanted to show, and hence $\pi^*\alpha_B + |z|^2\,A_0$ is equal to α.

Note that gluing map Ψ is $\mathbb{S}^1$-equivariant with respect to the circle action on E_P that multiplies the fibers with complex numbers of modulus 1, and the given action on $P \times (0,\varepsilon)$, because $e^{i\theta}\cdot[p,z] := [p, e^{i\theta}z] = \left[\frac{e^{i\theta}z}{|z|}\cdot p, |z|\right]$ so that

$$\Psi\big(e^{i\theta}\cdot[p,z]\big) = \left(\frac{e^{i\theta}z}{|z|}\cdot p, |z|^2\right) = e^{i\theta}\cdot\Psi([p,z]) .$$

Thus we can glue $E_{<\delta}$ to $M \setminus P$ to get a new contact manifold in which P has been replaced by B, and where the $\mathbb{S}^1$-action has fixed points along B.

Let $P_1 \cup \cdots \cup P_N$ be a union of connected components of the boundary of a Giroux domain E_ω. These components are contact cutting hypersurfaces and can thus be blown down as just described. We shall denote the resulting manifold by $E_\omega /\!/ \big(P_1 \cup \cdots \cup P_N\big)$. It inherits a natural contact structure for which each of the blown down boundary components becomes a codimension two contact submanifold. The $\mathbb{S}^1$-action on E_ω is not modified outside $P_1 \cup \cdots \cup P_N$, but the blown down boundary components convert into fixed points of the action.

Example IV.1.10. Continuing the annulus example, a Giroux π-torsion domain with one boundary component blown down is a so-called *Lutz tube*, i.e. the solid torus that results from performing a Lutz twist along a transverse knot. With both boundary components blown down, it is the standard contact structure on $\mathbb{S}^2 \times \mathbb{S}^1$.

In the above example, when one boundary component is blown down but not the other, the resulting domain contains an overtwisted disk. We now generalize this to higher dimensions.

Proposition IV.1.11. *Suppose (M,ξ) is a contact manifold containing a subdomain G with nonempty boundary, obtained from a Giroux domain by blowing down at least one boundary component (but not all!).*

If the blown down Giroux domain was constructed from an ideal Liouville domain (Σ,ω), then (M,ξ) contains a small bLob *(cf. Remark I.4.2.(iii)).*

The bLob in the above proposition will come from a Lagrangian submanifold in a complete symplectic domain (Σ,ω). We first need a technical definition describing how these submanifolds will be allowed to approach the boundary. We say that a submanifold L properly embedded inside Σ and transverse to the boundary is a **Lagrangian with cylindrical end** if:

- $\mathring{L}$ is Lagrangian in $\mathring{\Sigma}$.
- ∂L is Legendrian in $\partial\Sigma$.

- We find a boundary collar $(-\varepsilon, 0] \times \partial\Sigma$ of Σ with symplectic form
$$-d\Big(\frac{\alpha}{s}\Big) + \omega_\partial \;,$$
as in Lemma IV.1.4 in which L is parallel to the s-direction, that means,
$$L \cap \big((-\varepsilon, 0] \times \partial\Sigma\big) = (-\varepsilon, 0] \times \partial L \;,$$
and ω_∂ vanishes on a neighborhood of ∂L.

Next we show that a Lagrangian with cylindrical ends inside a complete symplectic manifold determines a submanifold with a Legendrian foliation in the corresponding Giroux domain (cf. Example I.3.1). If additionally a certain homological condition holds, then the Legendrian foliation will be a fibration over $\mathbb{S}^1$.

Lemma IV.1.12. *Let (Σ, ω) be an ideal Liouville domain. If L is a Lagrangian with cylindrical end in Σ, then $\widehat{L} := L \times \mathbb{S}^1$ inside the contactization $E_\omega = \Sigma \times \mathbb{S}^1$ is isotopic to a maximally foliated submanifold whose singular set is its boundary and whose foliation is otherwise defined via a fibration*
$$\vartheta \colon \widehat{L} \to \mathbb{S}^1, \quad (l, t) \mapsto F(l) + t \;,$$
for some smooth function $F \colon L \to \mathbb{S}^1$ that is constant on a neighborhood of ∂L.

Proof. We first assume that there is a Liouville form λ adapted to L which induces a rational cohomology class on L. This implies there is a real number $\hbar > 0$ such that $\hbar^{-1}$ times the cohomology class of the restriction of λ to L is integral: $\hbar^{-1}\,[i^*\lambda] \in H^1(L; \mathbb{Z})$. First note that $\hbar f\,dt + \lambda_f$ defines a contact structure isotopic to $\ker(f\,dt + \lambda_f)$ relative to the boundary of the Giroux domain $G := \Sigma \times \mathbb{S}^1$ (compare this to the situation in Section IV.3, where we need sufficient "space" to realize the isotopy). Furthermore, the vector field constructed in the standard proof of Gray's theorem vanishes along $\partial\Sigma \times \mathbb{S}^1$, so this isotopy is actually tangent to the identity along the boundary. We shall now prove the lemma using this contact form (and no further isotopy of $L \times \mathbb{S}^1$).

In the interior of G, the contact structure is defined by $\hbar\,dt + \lambda$, which restricts to $\eta = \hbar\,dt + i^*\lambda$ on $\widehat{L}$. Since η is closed, $\widehat{L}$ is foliated. Moreover, $\hbar\,dt$ never vanishes in $\widehat{L}$, so there is no singularity there. Along the boundary, the contact structure is defined by λ_f, whose restriction to $\widehat{L}$ vanishes, thus the singularities are as claimed.

We now define the fibration ϑ using Tischler's construction (cf. [**Tis70**]). Let (l_0, t_0) be any base point in the interior of $\widehat{L}$. We define $\vartheta(l, t) = \frac{1}{\hbar}\int_\gamma \eta$, where γ is any path from (l_0, t_0) to (l, t). Since η is closed, Stokes' theorem guarantees that this is well defined modulo the integral of η along loops based at (l_0, t_0). If (γ_L, γ_t) is such a loop, then the integral over it is $\big\langle [\lambda], [\gamma_L] \big\rangle + \hbar \big\langle [dt], [\gamma_t] \big\rangle$, which belongs to $\hbar\mathbb{Z} + \hbar\mathbb{Z} = \hbar\mathbb{Z}$, thus ϑ has a well-defined value in $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$. Observe that $\vartheta(l, t) = \vartheta(l, 0) + t$, and two points (l_1, t_1) and (l_2, t_2) lie in the same connected component of a fiber of ϑ if and only if they lie on the same leaf of the Legendrian foliation. On a suitable collar neighborhood of the boundary, the 1-form η simplifies to $\hbar\,dt$, so the behavior of ϑ is also as claimed.

We now explain how to enforce the rationality assumption by perturbation of the Liouville structure. Suppose λ_0 is any Liouville form adapted to L, in which case $\lambda_0|_{TL}$ is a closed 1-form that vanishes on a collar neighborhood of ∂L. For every $\varepsilon > 0$, we will find a closed 1-form λ_L on L with compact support in $\mathring{L}$ and $\|\lambda_L\| < \varepsilon$ (in the $\mathcal{C}^0$-norm with respect to a fixed auxiliary metric on L) such that $i^*\lambda_0 + \lambda_L$ represents a rational cohomology class on L. Since the restriction of λ_0 to L vanishes near ∂L, its cohomology class belongs to the kernel K of the map $H^1_{\mathrm{dR}}(L) \to H^1_{\mathrm{dR}}(\partial L)$ induced by inclusion. Let $\beta_1, \dots, \beta_p$ be a set of closed 1-forms representing a basis of the image in K of $H^1(L;\mathbb{Z})$. By the definition of K, we can assume that all these 1-forms vanish near the boundary of L. The restriction of λ_0 to L can be written as $\sum c_i\beta_i + df$ for some real coefficients c_i and some function f. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, one can find arbitrarily small numbers ε_i such that $c_i + \varepsilon_i$ is rational for all i and then set $\lambda_L = \sum \varepsilon_i\beta_i$.

We extend λ_L to a tubular neighborhood U of L in Σ by pulling it back to the normal bundle, and multiply it by a fixed cutoff function $\rho\colon U \to [0,1]$ that has compact support on U and equals 1 on L. In this way we obtain a 1-form λ_0' given by $\lambda_0 + \rho\lambda_L$ on U that extends smoothly to β_0 on $\Sigma \setminus U$, and whose restriction to L yields the desired closed 1-form with compact support in $\mathring{L}$ that represents a rational cohomology class. We can choose ε above arbitrarily small, hence we can assume that all forms in the segment between $d\beta_0$ and $d\beta_0'$ are symplectic. The corresponding contact structures are then isotopic relative to $\Sigma \setminus U$ and $\partial\Sigma$. $\square$

Proof of Proposition IV.1.11. Let (Σ, ω) denote the ideal Liouville domain used to construct G. We will construct a Lagrangian $L \subset \Sigma$ with cylindrical end and blow down the foliated submanifold of Lemma IV.1.12 to find the desired bLob. If $\dim \Sigma = 2$, it suffices to take for L an embedded path between two distinct boundary components of Σ, where one corresponds to a blown down boundary component of G and the other does not. More generally, choose two disjoint boundary parallel Lagrangian disks L_{bd} and L_{p} with cylindrical ends in Σ such that ∂L_{bd} is a Legendrian sphere in one of the blown down boundary components of $\partial\Sigma$, and ∂L_{p} is a Legendrian sphere in another boundary component that is not blown down. By a symplectic isotopy supported in a tube connecting them, we can deform L_{p} away from ∂L_{p} so that it intersects L_{bd} transversely.

One can remove transverse self-intersection points between two Lagrangians L and L' using [**Pol91**]. This construction works by removing for each intersection two small balls from L and L' containing this point, and gluing in a tube diffeomorphic to $[-\varepsilon, \varepsilon] \times \mathbb{S}^{n-1}$ joining the boundaries of the two balls. In fact, the construction is explicit: choose a Darboux chart around the intersection point such that L and L' are represented by the n-planes $\{(x_1, \dots, x_n, 0, \dots, 0)\}$ and $\{(0, \dots, 0, y_1, \dots, y_n)\}$ respectively. Remove a disk of radius ε around 0 in both planes and glue in the tube

$$(-\varepsilon, \varepsilon) \times \mathbb{S}^{n-1} \hookrightarrow \mathbb{R}^{2n},\ (t; x_1, \dots, x_n) \mapsto \big(\rho_1(t) \cdot (x_1, \dots, x_n); \rho_2(t) \cdot (x_1, \dots, x_n)\big)$$

for a smooth function $\rho_1\colon (-\varepsilon, \varepsilon) \to [0,1]$ that is 0 for values between $-\varepsilon$ and $-\varepsilon/2$, has positive derivative for $t > -\varepsilon/2$ and is the identity close to $+\varepsilon$. Define $\rho_2(t) := \rho_1(-t)$. This defines a Lagrangian manifold that glues well to $L \setminus \varepsilon \cdot \mathbb{D}^n$ for t close to ε and to $L' \setminus \varepsilon \cdot \mathbb{D}^n$ for t close to $-\varepsilon$.

The symplectic isotopy and the surgery process both took place far away from the boundary, so we obtain by this construction a Lagrangian that still has cylindrical ends. Lemma IV.1.12 then produces a foliated submanifold which becomes a bLob in the blown down Giroux domain. This bLob also embeds into a ball, because L is obtained from two Lagrangian disks parallel to the boundary and a thin tube that lies in the neighborhood of an embedded path, so that L lies in a ball of the form $[0,1] \times \mathbb{D}^{2n-1} \subset \Sigma$. Moving to the contactization and blowing down the corresponding boundary components then gives a neighborhood diffeomorphic to a ball $\mathbb{D}^2 \times \mathbb{D}^{2n-1}$ that contains the bLob. $\square$

Remark IV.1.13. As in [MNW13], we have only stated Proposition IV.1.11 for Giroux domains constructed over ideal Liouville domains (Σ, ω), even though we have also introduced in this section the blow down operation for more general domains, where ω is not exact. The difficulty of carrying out the proof above for these more general examples is that we are not able to rescale the connection form A_0 used in the construction of the contact form, because a non-trivial curvature would change by this same scaling factor, and this would result in a different contact manifold.

One of the main difficulties for applying Proposition IV.1.11 is that it is extremely hard to find examples of Giroux domains with disconnected boundary, and a large part of [MNW13] was dedicated to the construction of such examples. One could hope that generalizing Proposition IV.1.11 for general Giroux domains, one would be able to find much more easily examples where one could apply the theory (for example, it is easy to see that $T^*\mathbb{S}^1 \times \mathbb{T}^2$ is a weak filling of two 3-tori [**Gir94**]).

IV.1.4. Obstructions to fillability. We now want to state a non-fillability result. As preparation, note that any embedding of the interior of a Giroux domain $I_\Sigma := \mathring{\Sigma} \times \mathbb{S}^1$ into a contact manifold (M, ξ) determines a distinguished subspace $H_1(\Sigma; \mathbb{R}) \otimes H_1(\mathbb{S}^1; \mathbb{R}) \subset H_2(M; \mathbb{R})$. We call its annihilator in $H^2_{\mathrm{dR}}(M)$ the space of cohomology classes **obstructed** by I_Σ, and we denote it by $\mathcal{O}(I_\Sigma)$. Classes in $\mathcal{O}(I_\Sigma)$ are exactly those whose restriction to I_Σ can be represented by closed 2-forms pulled back from the interior of Σ. If $N \subset (M, \xi)$ is any subdomain resulting from gluing together a collection of Giroux domains $I_{\Sigma_1}, \dots, I_{\Sigma_k}$, then we define its obstructed subspace $\mathcal{O}(N) \subset H^2_{\mathrm{dR}}(M)$ to be $\mathcal{O}(I_{\Sigma_1}) \cap \dots \cap \mathcal{O}(I_{\Sigma_k})$. We will say that such a domain is **fully obstructing** if $\mathcal{O}(N) = H^2_{\mathrm{dR}}(M)$.

Theorem IV.1.14. *Suppose (M, ξ) is a closed contact manifold containing a subdomain N with nonempty boundary, which is obtained by gluing and blowing down Giroux domains constructed from ideal Liouville domains.*

(a) *If N has at least one blown down boundary component then it contains a small* bLob*, hence (M, ξ) does not have any (semipositive) weak filling.*
(b) *If N contains two Giroux domains $\Sigma^+ \times \mathbb{S}^1$ and $\Sigma^- \times \mathbb{S}^1$ glued together such that Σ^- has a boundary component not touching Σ^+, then (M, ξ) has no (semipositive) weak filling (W, ω) with $[\omega_M] \in \mathcal{O}(\Sigma^+ \times \mathbb{S}^1)$.*

In particular (M, ξ) has no (semipositive) strong filling in either case.

The first statement in this theorem follows immediately from Proposition IV.1.11 and Theorem A. We will prove the second in Section IV.2.3, essentially by using the symplectic cobordism construction explained below to reduce it to the first statement, though some care must be taken because the filling obtained by attaching our cobordism to a given semipositive filling need not always be semipositive.

IV.2. Surgery along Giroux domains

IV.2.1. A handle attachment theorem. In Section IV.1.3, we explained a method for blowing down the boundary of a Giroux domain. The surgery procedure which we will study in this section consists in removing the interior of a Giroux domain from a contact manifold and then blowing down any resulting boundary component. The main result will be that if the Giroux domain is constructed from an ideal Liouville domain, we can realize this surgery by a symplectic cobordism that can be glued on top of any weak filling satisfying suitable cohomological conditions, leading to a proof of Theorem IV.1.14. The construction in this section has been taken from [MNW13].

Suppose (M, ξ) is a $(2n-1)$-dimensional contact manifold without boundary, containing a Giroux domain $G \subset M$. Removing the interior of G, the boundary of $\overline{M \setminus G}$ is then a ξ-cutting hypersurface

$$\partial(\overline{M \setminus G}) = P \ ,$$

that fibers over a (possibly disconnected) closed contact manifold (B, ξ_B). We can thus blow it down as described in Section IV.1.3, producing a new manifold

$$M' := (\overline{M \setminus G})//\partial G$$

without boundary, which inherits a natural contact structure ξ'.

Topologically, the surgery taking (M, ξ) to (M', ξ') can be understood as a certain handle attachment. We now give a point-set description of this handle attachment which is sufficient to state the theorem below, and postpone the smooth description to the next subsection. Assume that G is a Giroux domain over an ideal Liouville domain $\big(\Sigma, \omega\big)$ with boundary $\partial\Sigma$ so that $G = \Sigma \times \mathbb{S}^1$.

We want to glue the disk bundle $E_\Sigma = \Sigma \times D^2$. Note that the boundary of E_Σ consists of G and the product $\partial\Sigma \times D^2$ over $\partial\Sigma$. Then we can consider the manifold with boundary and corners defined by

$$\big([0,1] \times M\big) \cup_{\{1\} \times G} \Sigma \times D^2 \ .$$

After smoothing the corners, this becomes a smooth oriented cobordism W with boundary

$$\partial W = -M \sqcup M' \ .$$

We can now state the main theorem of this section.

Theorem IV.2.1. *Suppose W denotes the $2n$-dimensional smooth cobordism described above, and Ω is a closed 2-form on M such that:*

- *Ω weakly dominates ξ*
- *the cohomology class of Ω belongs to the obstructed subspace $\mathcal{O}(G)$, i.e. for every 1-cycle Z in Σ,*

$$\int_{Z \times \mathbb{S}^1} \Omega = 0 \ .$$

Then W admits a symplectic structure ω_W with the following properties:

1. *$\omega_W|_{TM} = \Omega$.*
2. *The co-core $\Sigma \times \{0\}$ embedded into $\Sigma \times D^2 \subset W$ is a symplectic submanifold weakly filling $(\partial\Sigma \times \{0\}, \xi_\Sigma)$.*
3. *(M', ξ') is a weakly filled boundary component of (W, ω_W) that is contactomorphic to the blown down manifold $(\overline{M \setminus G})//\partial G$.*

Remark IV.2.2. Recall that due to Lemma II.2.2, a pair of weak symplectic cobordisms can be smoothly glued together along a positive/negative pair of contactomorphic boundary components whenever the symplectic forms restricted to these boundary components match. Thus the symplectic cobordism of the above theorem can be glued on top of any weak filling (W, ω) of (M, ξ) for which $[\omega|_{TM}] \in \mathcal{O}(G)$.

IV.2.2. Construction of the symplectic cobordism. In this section we will give the proof of Theorem IV.2.1. The proof will consist of the following five steps:

(1) Find a standardized model with a special contact form α_M for tubular neighborhoods of ∂G.
(2) Construct a symplectic form on our proto-cobordism $[0,1] \times M$ that is well adjusted to both Ω and λ.
(3) Carve out the interior of $\{1\} \times G$ from $[0,1] \times M$. This creates a notch with corners along its edges, and we will then smoothly glue the handle $\Sigma \times D^2$ into the cavity, creating a smooth manifold.
(4) Study the symplectic form induced from the proto-cobordism on the glued part of the handle and extend it to the whole handle.
(5) Check that the new boundary of the cobordism has the desired properties.

Step 1: Neighborhoods and contact form for G
Since (Σ, ω) is an ideal Liouville domain, there is a 1-form λ on $\mathring{\Sigma}$ such that $\omega = d\lambda$. Furthermore by Lemma IV.1.4, we can assume that for a chosen contact form α on $(\partial\Sigma, \xi_\omega)$, we find a collar neighborhood $(-\varepsilon, 0] \times \partial\Sigma$, on which λ restricts to $-\alpha/s$, so that

$$\omega|_{(-\varepsilon,0)\times\partial\Sigma} = -d\big(\frac{1}{s}\,\alpha\big)\;.$$

We denote by u a smooth function $\Sigma \to [0,1]$ which has the boundary $\partial\Sigma = u^{-1}(0)$ as a regular level set, equals $-s$ in the region $s \geq -\varepsilon/3$ and 1 in the region $s \leq -\varepsilon/2$ and outside the collar, and satisfies $u' \leq 0$ everywhere on the collar (see Fig. 1). We set $\lambda_u = u\lambda$. The contact form on G associated to (Σ, ω) can be chosen to be $\alpha_G := u\,d\theta + \lambda_u$. In the collar, one can set α_G to be $u\,d\theta + f\alpha$, where $f(s)$ is the function $\frac{u(s)}{s}$. The contact condition implies

$$uf' - fu' \neq 0\;. \tag{IV.2.1}$$

Appealing to Lemma IV.1.9, we can slightly extend our collar neighborhood embedded in (M, ξ) to one of the form $(-\varepsilon, \varepsilon') \times \partial G$, with α_G being of the form $-s\,d\theta + \alpha$ also for $s > 0$.

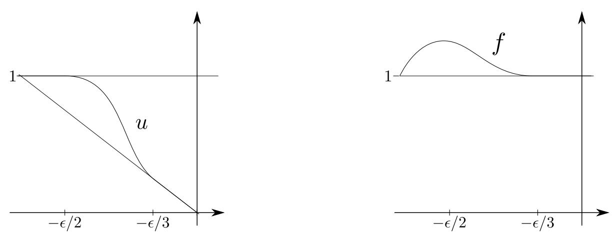


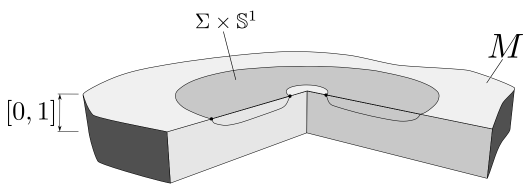
FIGURE 1. The functions u and f.

Step 2: The symplectic form on $[0,1] \times M$
The assumption that Ω weakly dominates ξ implies that the 2-form $\omega_W = d(t\alpha_G) + \Omega$ is symplectic on $(-\delta, 1] \times M$ for some small positive constant δ. The cohomological assumption $[\Omega] \in \mathcal{O}(G)$ implies that Ω is cohomologous to some 2-form Ω_0 such that $\Omega_0|_G$ is the pull back of a 2-form on Σ. In addition, since the collar neighborhood $(-\varepsilon, 0] \times \partial\Sigma$ retracts to $\partial\Sigma$, we can assume that $\iota_{\partial_s}\Omega_0 = 0$ when $s \geq -\varepsilon/2$.

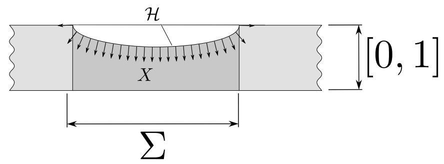
Lemma IV.2.3. *We can modify the form ω_W defined above to a new symplectic form on $(-\delta, 1] \times M$, keeping the assumption that ω_W restricts to Ω on $\{0\} \times M$ and ξ be weakly dominated by ω_W on each slice $\{t\} \times M$, but asking in addition that ω_W restrict to $C\, d(t\alpha_M) + \Omega_0$ on $[1/2, 1] \times M$ for some large constant $C > 0$.*

PROOF. Using Lemma II.2.3, we find a symplectic form ω' on $(-\delta, \infty) \times M$ such that each $\{t\} \times M$ is still weakly filled and ω' restricts to $d(t\lambda) + \Omega_0$ for t greater than some large constant $C/2$. The scaling diffeomorphism $(t, v) \mapsto (t/C, v)$ pulls back ω' to the desired symplectic form. $\square$

Step 3: Handle attachment
We now give a smooth description of the handle attachment which is compatible with the smooth description of the blow-down process for ξ-cutting hypersurfaces. For this, we will first create a small basin in the top of $[0, 1] \times M$ to which we can glue in the handle.



(A) The precise construction of the handle attachment works by creating a trench on the top side of the cobordism $[0, 1] \times M$ to which we can glue in the handle. In the picture above we need to remove the hatched area under the Giroux domain $\Sigma \times \mathbb{S}^1$.



(B) The vector field X is tangent to the top face and transverse to the hypersurface $\mathcal{H}$, which is Σ pushed inside $[0, 1] \times \Sigma$ relative to its boundary. Everything above $\mathcal{H}$ has been discarded to make room for the handle.

FIGURE 2

Let h be a smooth function from Σ to $(1/2, \infty)$ such that

- h restricts on the special collar of Step 1 to a function only depending on s with nonnegative derivative $h'(s)$,
- outside the collar on $(-\varepsilon/2, 0] \times \partial\Sigma$, h is constant,
- for $s \geq -\varepsilon/3$, $h(s) = 1 + s$ near $\partial\Sigma$, so that in particular $\partial\Sigma = h^{-1}(1)$.

Let $h_G \colon G \to (1/2, \infty)$ be the $\mathbb{S}^1$-invariant extension of h from Σ to the Giroux domain $G = \Sigma \times \mathbb{S}^1$. We denote by $\mathcal{H} \subset [0, 1] \times M$ the graph of h_G over G, see Fig. 2. We discard the region $\{t \geq h_G\}$ from $[0, 1] \times M$ to get an open manifold, to which we will glue the "handle" $\Sigma \times D$. Here D denotes the disk of radius $\sqrt{\varepsilon}$. In the following, we will find a symplectic vector field X in a neighborhood of the hypersurface $\mathcal{H}$ in $[0, 1] \times M$ that is transverse to $\mathcal{H}$, never points in the positive t-direction, and is tangent to $\{1\} \times M$ near the boundary of $\mathcal{H}$. Shrinking ε if needed, we may assume that the flow of X starting from $\mathcal{H}$ embeds $\mathcal{H} \times [0, \varepsilon]$ into $[1/2, 1] \times M$. We denote by Φ^X_τ the flow of X at time τ. The manifold W' is obtained by attaching $\Sigma \times D$ to $\big([0, 1] \times M\big) \setminus \{t \geq h\}$ using the gluing map Ψ from $\Sigma \times D^*$ (where D^* the punctured disk $D \setminus \{0\}$) to $[0, 1] \times M$ defined by

$$\Psi(\sigma, re^{i\theta}) = \Phi^X_{r^2}\big(h(\sigma); \sigma, e^{i\theta}\big) .$$

Here $\Phi^X_{r^2}$ is the flow of the vector field X followed for time r^2. Note that as a point-set operation, the handle attachment reduces to the operation of adding the co-core Σ to the open manifold $\big([0,1]\times M\big)\setminus\{t\geq h_\Sigma\}$.

The vector field X that we will use below coincides with ∂_s near $\{1\}\times\partial G$. This implies that the attachment using Ψ restricts precisely to the gluing map used to blow down the cutting hypersurface ∂G.

As a gluing vector field X, we choose the ω_W-dual of $-C\,d\theta$, where C is the constant appearing in ω_W, and θ is the circle direction in the Giroux domain $\Sigma\times\mathbb{S}^1$. Since this 1-form is closed, X is a symplectic vector field.

Lemma IV.2.4. *The vector field X is transverse to the hypersurface $\mathcal{H}$ and coincides with ∂_s near $\{1\}\times\partial G$.*

Proof. Away from the special collar neighborhood considered in Step 1, $\alpha_M = d\theta+\lambda$, and computing

$$\omega_W(X,\partial_\theta) = -C\ ,$$

we see that $dt(X)=-1$. Elsewhere, on the collars $[0,1]\times\big([-\frac{\varepsilon}{2},0]\times\partial G\big)$, we use the ansatz $X = X^t\,\partial_t + X^s\,\partial_s$. Computing the interior product $\iota_X\omega_W$ using $\omega_W = C\,d\big(t(u\,d\theta+f\alpha)\big)+\Omega_0$ and $\iota_{\partial_s}\Omega_0=0$, we find that X is indeed ω_W-dual to $-C\,d\theta$ provided

$$\begin{aligned} uX^t + tu'X^s &= -1\ ,\\ f\,X^t + t\,f'\,X^s &= 0\ . \end{aligned}$$

This system is everywhere nonsingular due to the contact condition (IV.2.1). For $s\geq-\varepsilon/3$ and $t=1$, we have $X=\partial_s$ as promised. For $s<-\varepsilon/3$, the conditions $f(s)>0$ and $f'(s)>0$ imply $X^t<0$ and $X^s>0$, hence X is transverse to $\mathcal{H}$. □

Step 4: Symplectic form on the handle

Lemma IV.2.5. *The gluing map Ψ from $\Sigma\times D^*$ to $[0,1]\times M$ pulls back ω_W to*

$$\Psi^*\omega_W = 2C\omega_D + C\,d(hu)\wedge d\theta + \Omega_0$$

where $\omega_D := -r\,dr\wedge d\theta$ and Ω_0 is a symplectic form on Σ which weakly fills $(\partial\Sigma,\ker\lambda_u)$.

Proof. Let $j_{\mathcal{H}}$ denote the embedding $G\hookrightarrow\mathcal{H}\subset[0,1]\times M, (\sigma,e^{i\theta})\mapsto\big(h_\Sigma(\sigma);\sigma,e^{i\theta}\big)$. Then we can decompose Ψ as $\Psi=\Phi\circ P$, where P is the map from $\Sigma\times D^*$ to $G\times(0,\varepsilon]$ defined by $P(\sigma,re^{i\theta})=(\sigma,e^{i\theta};r^2)$ and

$$\Phi(\sigma,e^{i\theta};\tau) := \Phi^X_\tau\big(h(\sigma);\sigma,e^{i\theta}\big) = \Phi^X_\tau\big(j_{\mathcal{H}}(\sigma,e^{i\theta})\big)\ .$$

Using the fact that the flow of $\big(\Phi^X_\tau\big)_*\partial_\tau = X$ preserves ω_W and $\iota_X\omega_W=-C\,d\theta$, we obtain for the pull back

$$\Phi^*\omega_W = -C\,d\tau\wedge A_0 + j^*_{\mathcal{H}}\omega_W\ ,$$

and since the symplectic form ω_W is given in the range of $j_{\mathcal{H}}$ by $C\,d(t\alpha_M)+\Omega_0$ with $\alpha_M = u\,d\theta+\lambda_u$, we can compute

$$j^*_{\mathcal{H}}\omega_W = C\,d(h_\Sigma\alpha_M)+\omega_0 = C\,d(h_\Sigma u)\wedge d\theta+\omega_0\ ,$$

where we have set $\omega_0 = C\,d(h_\Sigma\lambda_u)+\Omega_0$ (which is a 2-form on Σ).

Now since $P^* d\tau = 2r\,dr$, the only thing left to prove is that ω_0 is a symplectic form which weakly fills $(\partial\Sigma, \ker\alpha)$. Since ω_D is the only term in $\Psi^*\omega_W$ that contains a dr-factor, and thus it follows that $(\Psi^*\omega_W)^n = 2nC\omega_D \wedge \omega_0^{n-1} \neq 0$, we deduce that ω_0 is symplectic.

The 2-form ω_0 restricts on $\partial\Sigma$ to $C\,d\alpha + \Omega_0$. Recall that the weakly dominating condition on $\{1\} \times M$ means that for any constant $\nu \geq 0$, $\alpha_M \wedge \big(\omega_W + \nu\,d\alpha_M\big)^{n-1} > 0$. Restricting to $\{1\} \times G$, where $\alpha_M = u\,d\theta + \lambda_u$ and $\omega_W = C\,d\alpha_M + \Omega_0$, this becomes:

$$(u\,d\theta + \lambda_u) \wedge \big[(C+\nu)\,du \wedge d\theta + \big(C\,d\lambda_u + \Omega_0 + \nu\,d\lambda_u\big)\big]^{n-1} > 0\ ,$$

which we expand along $\{1\} \times \partial\Sigma \times \mathbb{S}^1$ as

$$(n-1)(C+\nu)\,\alpha \wedge du \wedge d\theta \wedge \big(C\,d\alpha + \Omega_0 + \nu\,d\alpha\big)^{n-2} > 0\ .$$

In particular, this proves that $\alpha \wedge (\Omega_0 + \nu\,d\alpha)^{n-2}$ never vanishes. In order to check that it has the correct sign, it suffices to consider the case $\nu = 0$. □

To finish the construction of the symplectic cobordism, we want to define a symplectic structure on $\Sigma \times D$ that agrees in a neighborhood of the boundary $\Sigma \times \partial D$ with $\Psi^*\omega_W$, and that has a split form near $\Sigma \times \{0\}$. Let ρ_1 and ρ_2 be functions from $[0, \sqrt{\varepsilon}]$ to $\mathbb{R}$ (constraints will be added later). We set:

$$\begin{aligned}\widetilde{\omega} &:= 2C\rho_1\omega_D + C\,d(\rho_2 h_\Sigma u) \wedge d\theta + \omega_0\\ &= g\omega_D + C\rho_2\,d(h_\Sigma u) \wedge d\theta + \omega_0 \qquad \text{with } g := \left(2\rho_1 - \frac{hu\rho_2'}{r}\right)C.\end{aligned}$$

We choose $\rho_1(r) = \rho_2(r) = 1$ for r close to $\sqrt{\varepsilon}$, so that $\widetilde{\omega}$ extends $\Psi^*\omega_W$. Near 0, we choose ρ_1 to be a large positive constant and ρ_2 to vanish so that $\widetilde{\omega}$ makes sense near the center of D. One can compute $\widetilde{\omega}^n = ng\omega_D \wedge \omega_0^{n-1}$. Since ω_0 is symplectic on Σ, we see that $\widetilde{\omega}$ is symplectic as soon as g is positive. This condition is arranged by choosing ρ_1 sufficiently large away from $r = \sqrt{\varepsilon}$.

Step 5: Properties of the new boundary of W

We now consider the new boundary component M' resulting from the above construction. Since hu is constant on $\partial\Sigma$, the restriction of $\widetilde{\omega}$ to $\partial\Sigma \times D$ is $g\omega_D + \omega_0$. As we already noted, the gluing map Ψ extends the one used to define the blow-down, and the contact form on M' is $\lambda = \lambda_u - r^2 d\theta$. Thus in order to check the weak filling condition, we only need compute, for any constant $\nu \geq 0$,

$$\lambda \wedge (\widetilde{\omega} + \nu\,d\lambda)^{n-1} = (n-1)(g + 2\nu)\,\omega_D \wedge \gamma \wedge (\omega_0 + \nu\,d\lambda_u)^{n-2}.$$

This is indeed a positive volume form for any nonnegative ν because (Σ, ω_0) is a weak filling of $(\partial\Sigma, \ker\lambda_u)$ according to Lemma IV.2.5.

IV.2.3. Giroux domains and non-fillability. We now use the cobordism of the preceding section to prove Theorem IV.1.14 on filling obstructions.

IV.2.4. Proof of Theorem IV.1.14. Part (a) of the theorem follows immediately from the fact that if (M, ξ) contains a Giroux domain N that has some boundary components that are blown down and others that are not, then by Proposition IV.1.11 it contains a small bLob, so Theorem A implies that (M, ξ) does not admit any semipositive weak filling.

To prove part (b), suppose N has the form

$$N = (\Sigma^+ \times \mathbb{S}^1) \cup_{Y \times \mathbb{S}^1} (\Sigma^- \times \mathbb{S}^1)\ ,$$

where $\Sigma^{\pm}$ are ideal Liouville domains with boundary $\partial\Sigma^{\pm} = \partial_{\mathrm{glue}}\Sigma^{\pm} \sqcup \partial_{\mathrm{free}}\Sigma^{\pm}$, $Y := \partial_{\mathrm{glue}}\Sigma^{+} = \partial_{\mathrm{glue}}\Sigma^{-}$ carries the induced contact form α and $\partial_{\mathrm{free}}\Sigma^{-}$ is not empty. Arguing by contradiction, assume that (M,ξ) is weakly filled by a semipositive symplectic filling (W_0,ω) with $[\omega|_{TM}] \in \mathcal{O}(\Sigma^{+})$. This establishes the cohomological condition needed by Theorem IV.2.1 on $\Sigma^{+} \times \mathbb{S}^1$, so applying the theorem, we can enlarge (W_0,ω) by attaching $\Sigma^{+} \times \mathbb{D}^2$, producing a compact symplectic manifold (W_1,ω) whose boundary (M',ξ') supports a contact structure that is weakly filled.

Since the boundary M' of the new symplectic manifold (W_1,ω) is contactomorphic to $(\overline{M \setminus (\Sigma^{+} \times \mathbb{S}^1)})//Y$, we find in (M',ξ') a domain isomorphic to $(\Sigma^{-} \times \mathbb{S}^1)//Y$ that contains a small bLob. Unfortunately this does not directly obstruct the existence of the weak filling (W_1,ω), because even though W_0 was semipositive, W_1 might not be. We will follow the proof of Theorem A, with the difference that we need to reconsider compactness to make sure that bubbling is still a "codimension 2 phenomenon".

Choose an almost complex structure J on (W_1,ω) with the following properties:

(i) J is tamed by ω and makes (M',ξ') strictly J-convex,
(ii) J is adapted to the bLob in the standard way, i.e. it is chosen close to the boundary of the bLob as in Section II.3.4 and in a neighborhood of the binding according to Section II.3.3 (cf. the proof of Theorem A),
(iii) for some small radius $r > 0$, $J = J_{\Sigma^+} \oplus i$ on $\Sigma^{+} \times \mathbb{D}^2_r \subset W_1$, where J_{Σ^+} is a tamed almost complex structure on Σ^{+} for which $\partial\Sigma^{+}$ is J_{Σ^+}-convex.

The third condition uses the fact from Theorem IV.2.1 that the co-core $\mathcal{K}' := \Sigma^{+} \times \{0\}$ of the handle is a symplectic (and now also J-holomorphic) hypersurface weakly filling its boundary. The binding of the bLob lies in the boundary of the co-core $\mathcal{K}'_{+}$, and the normal form described in Section II.3.3 is compatible with the splitting $\Sigma^{+} \times \mathbb{D}^2_r$ so that (ii) and (iii) can be simultaneously achieved.

By choosing J_{Σ^+} generic, we can also assume that every somewhere injective J_{Σ^+}-holomorphic curve in Σ^{+} is Fredholm regular and thus has nonnegative index. Note that any closed J-holomorphic curve in $\Sigma^{+} \times \mathbb{D}^2_r$ is necessarily contained in $\Sigma^{+} \times \{z\}$ for some $z \in \mathbb{D}^2_r$, and the index of this curve differs from its index as a J_{Σ^+}-holomorphic curve in Σ^{+} by the Euler characteristic of its domain. This implies that every somewhere injective J-holomorphic sphere contained in $\Sigma^{+} \times \mathbb{D}^2_r$ has index at least 2. Likewise, by a generic perturbation of J outside of this neighborhood we may assume all somewhere injective curves that are *not* contained entirely in $\Sigma^{+} \times \mathbb{D}^2_r$ also have nonnegative index.

Now let $\mathcal{M}$ be the connected moduli space of holomorphic disks attached to the bLob that contains the standard Bishop family. We can cap off every holomorphic disk $u \in \mathcal{M}$ by attaching a smooth disk that lies in the bLob, producing a trivial homology class in $H_2(W_1)$. The cap and the co-core intersect exactly once, and it follows that u also must intersect the co-core $\mathcal{K}'_{+}$ exactly once, because u and $\mathcal{K}'_{+}$ are both J-complex.

To finish the proof, we have to study the compactness of $\mathcal{M}$ and argue that $\overline{\mathcal{M}} \setminus \mathcal{M}$ consists of strata of codimension at least 2. A nodal disk u_∞ lying in $\overline{\mathcal{M}} \setminus \mathcal{M}$ has exactly one disk component u_0, which is injective at the boundary, and one component u_{+} that intersects the co-core once; either $u_{+} = u_0$ or u_{+} is a holomorphic sphere. Every other nonconstant connected component v is a holomorphic sphere whose homology class has vanishing intersection with the relative class $[\mathcal{K}'_{+}]$. So either v does not intersect the J-complex submanifold $\mathcal{K}'_{+}$ at all or v is completely contained in $\mathcal{K}'_{+}$. In either case, v is homotopic to a sphere lying in W_0: indeed, if v does not intersect the co-core, we can move it out of the handle by pushing it radially from $\Sigma^+ \times (\mathbb{D}^2 \setminus \{0\})$ into the boundary $\Sigma^+ \times \mathbb{S}^1 \subset W_0$, and if $v \subset \mathcal{K}'_+ = \Sigma_+ \times \{0\}$, then we can simply shift it to $\Sigma_+ \times \{1\} \subset W_0$. Using the fact that u_0 and u_+ are both somewhere injective, together with the semipositivity and genericity assumptions, we deduce that every connected component of u_∞ has nonnegative index, thus $\overline{\mathcal{M}} \setminus \mathcal{M}$ has codimension at least two in $\overline{\mathcal{M}}$. The rest of the proof is the same as for Theorem A.

IV.3. Overtwisted charts

It is not difficult to equip $\mathbb{R}^{2n+1}$ with a contact structure that contains an embedded bLob. For this it suffices to embed a bLob N smoothly into $\mathbb{R}^{2n+1}$ and take a contact structure on a neighborhood of N (using for example Theorem I.1.5). Under weak topological conditions, the h-principle [**Gro86**], [**EM02**, Theorem 10.3.2] allows us to extend this contact structure to all of $\mathbb{R}^{2n+1}$. We presume this is the approach that Gromov had in mind in [**Gro85**, 2.4.D_2' (c)], when he wrote:

> Then one easily produces examples of submanifolds W in some contact manifolds X diffeomorphic to $\mathbb{R}^{2n-1}$ where this condition is not met; this prevents any contact embedding of such an X into $\mathbb{R}^{2n-1}$ with the standard contact structure (given by the form $\sum_{i=1}^{n-1} x_i\, dy_i + dz$).

See page 1 for the full quotation.

The aim of this section is to reprove the main result from [**NP10**]. Let α_{ot} be the standard overtwisted contact form

$$\alpha_{\mathrm{ot}} = \cos(r)\, dz + r \sin(r)\, d\phi$$

written in cylindrical coordinates on $\mathbb{R}^3$, and let

$$\lambda_{\mathrm{can}} = \sum_{j=1}^{n} x_j\, dy_j$$

be a Liouville form on $\mathbb{R}^{2n}$. We will study the open contact manifold

$$\big(\mathbb{R}^3 \times \mathbb{R}^{2n}, \ker \alpha\big)$$

equipped with the contact form $\alpha = \alpha_{\mathrm{ot}} + \lambda_{\mathrm{can}}$.

Write $\mathbb{D}_{\mathrm{ot}}$ for the overtwisted disk $\{z = 0,\, r \le \pi\}$ in $\big(\mathbb{R}^3, \alpha_{\mathrm{ot}}\big)$.

Proposition IV.3.1. *Let U_0 be an open neighborhood of $\mathbb{D}_{\mathrm{ot}}$ in $\mathbb{R}^3$, and let B_R be the neighborhood*

$$B_R := \big\{(\mathbf{x}, \mathbf{y}) \bigm| \|\mathbf{x}\| < R, \|\mathbf{y}\| < R\big\} \subset \mathbb{R}^{2n} .$$

If R is chosen sufficiently large, then

$$\big(U_0 \times B_R, \ker \alpha\big)$$

contains a bLob. In particular, $\big(\mathbb{R}^3 \times \mathbb{R}^{2n}, \ker \alpha\big)$ contains a bLob.

In comparison to the construction based on the h-principle, the proposition just stated has the advantage of giving a very explicit contact structure. Furthermore, it is interesting that the PS-overtwistedness depends on the size of R: It is easy to see that every Darboux ball contains subsets that are contactomorphic to $U_0 \times B_R$ for R small. In particular, this chart has a formal similarity to the *loose charts* discovered by Emmy Murphy [**Mur12**]. This loose charts characterize *loose Legendrian submanifolds*, a class of Legendrians that is flexible.

Proof of Proposition IV.3.1. Note that α is invariant under $\mathbb{S}^1$-rotations in the ϕ-coordinate. This allows us to consider $U_0 \times B_R$ as a blown down (non-compact) exact Giroux domain (see Section IV.1.3).

Let $(\Sigma, d\lambda)$ be the (non-compact) ideal Liouville

$$\Sigma = \left\{(r, z, x_1, \dots, x_n, y_1, \dots, y_n) \in \mathbb{R}^{2n+3} \;\middle|\; r \in [0, \pi], z \in (-\varepsilon, \varepsilon), \|\mathbf{x}\| < R', \|\mathbf{y}\| < R'\right\}$$

with the 1-form

$$\lambda := \frac{\cos r}{\sin r}\, dz + \sum_{j=1}^{n} \frac{x_j}{\sin r}\, dy_j \ .$$

defined only on $\mathring{\Sigma} = \Sigma \setminus \{r = 0 \text{ or } r = \pi\}$, and for an $R' \ll R$.

Taking the Giroux domain over Σ and blowing down the boundary component $\{r = 0\}$ will produce a subdomain of $U_0 \times B_R$.

As in the proof of Proposition IV.1.11, we choose to Lagrangian disks with cylindrical ends sitting on the boundaries $\{r = 0\}$ and $\{r = \pi\}$ respectively. Create two transverse intersections between the two Lagrangians, by pulling one with a Hamiltonian isotopy along a small path towards the other one, and then resolve the intersections by using a Lagrangian Polterovich surgery.

The symplectic isotopy and the surgery process both took place far away from the boundary, so we obtain by this construction a Lagrangian L that still has cylindrical ends. We have also created by the surgery a loop in L and λ will not vanish, when integrated over this loop, but we may assume that $\lambda|_{TL}$ represents a class in $H^1(L, \mathbb{Q})$. If we now construct the Giroux domain $\Sigma \times \mathbb{S}^1$ with contact form

$$\cos(r)\, dz + \hbar \sin(r)\, d\phi + \sum_{j=1}^{n} x_j\, dy_j$$

for suitable value of $\hbar$, we will obtain on the submanifold $L \times \mathbb{S}^1$ a Legendrian foliation fibering over $\mathbb{S}^1$. In the lemma below we show that we can choose an R sufficiently large so that we can embed

$$\Bigl(U_0 \times B_{R'}, \cos(r)\, dz + \hbar r \sin(r)\, d\phi + \sum_{j=1}^{n} x_j\, dy_j\Bigr)$$

into the domain

$$\Bigl(U_0 \times B_R, \cos(r)\, dz + r \sin(r)\, d\phi + \sum_{j=1}^{n} x_j\, dy_j\Bigr) \ ,$$

hence the bLob we have constructed will also live in the larger domain, and the proposition has been proved. $\square$

Lemma IV.3.2. *Let $U_0 \times B_{R'}$ be the open domain with contact structure given as kernel of*

$$\alpha_\hbar := \cos(r)\, dz + \hbar r \sin(r)\, d\phi + \sum_{j=1}^{n} x_j\, dy_j$$

for a positive $\hbar > 1$. For every $R', \hbar$, we find an $R(R', \hbar)$ such that we can embed $\bigl(U_0 \times B_{R'}, \ker \alpha_\hbar\bigr)$ by a contactomorphism into $\bigl(U_0 \times B_R, \ker \alpha\bigr)$ with the standard $\alpha = \alpha_1$ defined at the beginning of this section.

PROOF. The proof will of course be based on the Moser trick. Note that the reason, why we have a size condition in this section, which we did not have in Proposition IV.1.11 is that on the compact domain, it is easier to guarantee existence of the flow for the Moser trick.

Define on $\mathbb{R}^3 \times \mathbb{R}^{2n}$ a family of contact forms

$$\alpha_T := \cos(r)\, dz + (1 - T + T\hbar)\, r \sin(r)\, d\phi + \sum_{j=1}^{n} x_j\, dy_j$$

with $T \in [0,1]$. It is easy to check that the α_T are all contact forms since

$$\begin{aligned} \alpha_T \wedge d\alpha_T^{n+1} &= (n+1)!\, \alpha_T \wedge d\alpha_T \wedge dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n \\ &= (n+1)!\, (1 - T + T\hbar)^2 \alpha_{\mathrm{ot}} \wedge d\alpha_{\mathrm{ot}} \wedge dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n \ . \end{aligned}$$

To apply the Moser trick, one would use the flow of the vector field Y_T uniquely specified by the equations

$$\bigl(\iota_{Y_T} d\alpha_T\bigr)\big|_{\ker \alpha_T} = -\,\dot\alpha_T|_{\ker \alpha_T} \qquad \text{and} \qquad \alpha_\tau(Y_T) = 0\ .$$

The vector field Y_T can be computed to be of the form

$$Y_T = A_T(r)\, \partial_r + B_T(r) \sum x_j\, \partial_{x_j}\ ,$$

so the only coordinates that are not invariant under the flow are the r- and the x_j-coordinates. The A_T-function vanishes for $r = \pi$, and since the x_j-coefficients can be bounded by $x_j \max_r |B_T|$, it follows that the flow is bounded by an exponential function. Hence if we have enough space in the domain, the flow of a compact subset, will exist for all $T \in [0,1]$. $\square$

CHAPTER V

Outlook and future directions

In this chapter, we want to give an overview of several research projects I'm currently working on (with Paolo Ghiggini and Chris Wendl). Statements in this chapter may be vague as this is still work in progress.

V.1. The Eliashberg–Floer–McDuff Theorem revisited

We slightly modify the proof of the Eliashberg–Floer–McDuff Theorem [**McD91**, Theorem 1.5] by using holomorphic disks and Lobs instead of using holomorphic spheres and a symplectic cap as in the original paper. The main reason, why we give a sketch of the following statement is to illustrate how one can find a family of Lobs by deforming the contact boundary into a submanifold with edges.

Theorem V.1.1 (Eliashberg–Floer–McDuff)**.** *Let $\mathbb{S}^{2n-1} \subset \mathbb{C}^n$ be the unit sphere with its standard contact structure ξ_0 given by the complex tangencies to the sphere, that means,*

$$\xi_0 = T\mathbb{S}^{2n-1} \cap \left(i \cdot T\mathbb{S}^{2n-1}\right) .$$

Every symplectically aspherical filling of $\left(\mathbb{S}^{2n-1}, \xi_0\right)$ is diffeomorphic to the $2n$-ball.

Proof. As in the original paper, the aim of the proof consists in showing that the filling needs to be simply connected and have vanishing homology so that the desired result follows from the h-cobordism.

Let $\mathbf{z} = \mathbf{x} + i\mathbf{y} = \left(x_1 + iy_1, \dots, x_n + iy_n\right)$ be the coordinates of $\mathbb{C}^n$. The function $f\colon \mathbb{C}^n \to [0,\infty)$ given by

$$f(\mathbf{z}) = \sum_{j=1}^{n} (x_j^2 + y_j^2)$$

is plurisubharmonic, and the unit sphere is the J-convex boundary of the ball

$$B^{2n} = \left\{\mathbf{z} \in \mathbb{C}^n \;\middle|\; f(\mathbf{z}) \le 1\right\} .$$

We would like to find a family of Lobs in $\mathbb{S}^{2n-1}$ foliating the unit sphere. Unfortunately we have not succeeded and it seems unlikely that this is even possible. Instead we will deform the sphere into a shape that does make it possible for us to find a suitable family of Lobs. Define two functions $g, h\colon \mathbb{C}^n \to [0,\infty)$ by

$$\begin{aligned} g(\mathbf{z}) &= x_1^2 + \dots + x_{n-1}^2 + x_n^2 + y_n^2 \\ h(\mathbf{z}) &= y_1^2 + \dots + y_{n-1}^2 . \end{aligned}$$

Note that g is strictly plurisubharmonic and h is weakly plurisubharmonic as

$$\begin{aligned} -dd^{\mathbb{C}} g &= 2\, dx_1 \wedge dy_1 + \dots + 2\, dx_{n-1} \wedge dy_{n-1} + 4\, dx_n \wedge dy_n \\ -dd^{\mathbb{C}} h &= 2\, dx_1 \wedge dy_1 + \dots + 2\, dx_{n-1} \wedge dy_{n-1} . \end{aligned}$$

Additionally, we have $g(\mathbf{z}) + h(\mathbf{z}) = f(\mathbf{z})$.

We will now consider the subset

$$\widehat{B}^{2n} = \left\{\mathbf{z} \in \mathbb{C}^n \;\middle|\; g(\mathbf{z}) \le 2\right\} \cap \left\{\mathbf{z} \in \mathbb{C}^n \;\middle|\; h(\mathbf{z}) \le 2\right\} .$$

By reordering and rescaling the coordinates, we see that $\widehat{B}^{2n}$ is a bi-disk $B^{n+1} \times B^{n-1} \subset \mathbb{R}^{2n}$ that clearly contains the unit ball. It is also easy to see that $\widetilde{B}^{2n}$ deformation retracts to B^{2n} rel B^{2n}.

Unfortunately the boundary of $\widehat{B}^{2n}$

$$\partial\widehat{B}^{2n} \cong \left(\partial B^{n+1}\right) \times B^{n-1} \cup B^{n+1} \times \left(\partial B^{n-1}\right) = \mathbb{S}^n \times B^{n-1} \cup B^{n+1} \times \mathbb{S}^{n-2}$$

is not a smooth manifold, but it is nonetheless homeomorphic to the unit sphere (see Fig. 1).

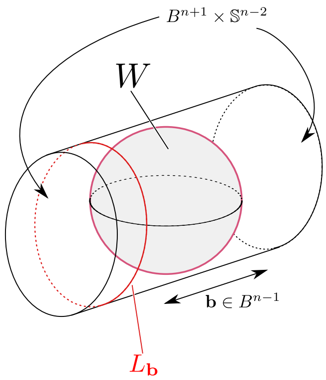


FIGURE 1. We find a family of Lobs by deforming the sphere to the boundary of a bi-disk. One of the two parts of the boundaries will then be foliated by Lobs.

Let now (W, ω) be a symplectic filling that is symplectically aspherical. If it is only a weak filling, we can extend it by attaching a symplectic collar to obtain a strong symplectic filling of the sphere, because $\omega|_{T\mathbb{S}^{2n-1}}$ is exact, see Corollary II.2.4 . This filling is diffeomorphic to the initial one, and it is also still symplectically aspherical, because any 2-sphere can just be pushed by a homotopy entirely into the old symplectic filling. After rescaling the symplectic form, the extended symplectic manifold will be a strong symplectic filling of the unit sphere.

Remove now the interior $\mathring{B}^{2n}$ of the unit ball from $\widehat{B}^{2n}$, and glue then $\widehat{B}^{2n} \setminus \mathring{B}^{2n}$ symplectically onto the filling W. Denote this new symplectic manifold by $(\widehat{W}, \widehat{\omega})$. To study $\widehat{W}$ using holomorphic curves, choose first an almost complex structure J on $\widehat{W}$ that is tamed by $\widehat{\omega}$ and that agrees on a small neighborhood of $\widehat{B}^{2n} \setminus \mathring{B}^{2n}$ in $\widehat{W}$ with the standard complex structure i on $\mathbb{C}^n$.

The moduli space we are interested in is linked to a family of Lobs, which we will introduce now. Embed $\mathbb{S}^n \times B^{n-1}$ via the diffeomorphism

$$\Psi\colon \big((a_1, a_2, \dots, a_{n+1}); (b_1, \dots, b_{n-1})\big) \mapsto \big(a_1 + ib_1, \dots, a_{n-1} + ib_{n-1}, a_n + ia_{n+1}\big)$$

into the boundary of $\widehat{W}$. The image of Ψ lies in the level set $g^{-1}(1) \subset \partial\widehat{W}$, and the J-complex tangencies on the corresponding part of $\partial\widehat{W}$ are the kernel of the 1-form

$$-d^{\mathbb{C}}g = 2x_1\, dy_1 + \dots + 2x_{n-1}\, dy_{n-1} + 2\left(x_n\, dy_n - y_n\, dx_n\right) .$$

Hence the restriction of $\Psi^*\bigl(-d^{\mathbb{C}}g\bigr)$ to each sphere with $\mathbb{S}^n \times \bigl\{(b_1, \dots, b_{n-1}) = \mathrm{const}\bigr\}$ gives

$$2\left(a_n\, da_{n+1} - a_{n+1}\, da_n\right) ,$$

that means, every sphere $\mathbb{S}^n \times \bigl\{(b_1, \dots, b_{n-1})\bigr\}$ is a Lob whose page is just an $(n-1)$-ball and that has trivial monodromy. We denote from now on the points in B^{n-1} by $\mathbf{b} = (b_1, \dots, b_{n-1})$, and denote the Lobs by

$$L_{\mathbf{b}} = \Psi\bigl(\mathbb{S}^n, \mathbf{b}\bigr) .$$

We study the space

$$\widetilde{\mathcal{M}} = \Bigl\{(\mathbf{b}, u, z_0) \Bigm| \mathbf{b} \in B^{n-1},\, u\colon (\mathbb{D}^2, \partial\mathbb{D}^2) \to \bigl(\widehat{W}, L_{\mathbf{b}}\bigr),\, z_0 \in \mathbb{D}^2\Bigr\}$$

of holomorphic maps with a marked point z_0 that map the boundary of the disk into any of the Lobs $L_{\mathbf{b}}$, and we quotient it out by the biholomorphisms of the disk as in Section III.2. Restrict to the component that contains the Bishop disks, then we have a smooth manifold $\mathcal{M}$ with boundary and a smooth evaluation map

$$\mathrm{ev}_{z_0}\colon \mathcal{M} \to \widehat{W},\ [\mathbf{b}, u, z_0] \mapsto u(z_0)$$

that restricts on the boundary to a degree 1 map.

Note that if we forget the marked point, the moduli space $\mathcal{M}$ projects as a 2-disk bundle onto a smooth moduli space $\mathcal{M}_0$ of unparametrized disks. In fact, the open book structure of the pages allows us trivialize this bundle, and it follows that $\mathcal{M}$ is diffeomorphic to $\mathcal{M}_0 \times \mathbb{D}^2$. This product structure is what will allow us to understand the topology of $\widehat{W}$.

Note that it is easy to see that $\widehat{W}$ is simply connected: Choose the base point of $\pi_1\bigl(\widehat{W}\bigr)$ in the domain of the Bishop family. Any class in the fundamental group can be realized by an embedded loop γ, and we may assume that γ is transverse to ev_{z_0}. It follows that $\mathrm{ev}_{z_0}^{-1}(\gamma)$ will be a collection of properly embedded 1-dimensional submanifold of $\mathcal{M}$, and by the uniqueness of the Bishop disks there will be a unique component that covers the whole loop γ. The image of this component is homotopic to γ, and it can be easily pushed into $\mathcal{M}_0 \times \mathbb{S}^1 \subset \partial\mathcal{M}$. Hence we can homotope γ to a loop in

$$\mathrm{ev}_{z_0}\bigl(\partial\mathcal{M}\bigr) \subset \partial\widehat{W} \cong \mathbb{S}^{2n-1} ,$$

and we have $\pi_1(\widehat{W}) = \{0\}$.

Careful use of the universal coefficient theorem, and the Leftschetz-Poincaré Theorem as in [**McD91**] and in [**OV12**] allow us to conclude that

$$\bigl(\mathrm{ev}_{z_0}\bigr)_*\colon H_*\bigl(\mathcal{M}, \mathbb{K}\bigr) \to H_*\bigl(\widehat{W}, \mathbb{K}\bigr)$$

needs to be surjective for field coefficients $\mathbb{K}$. But since the homology classes of the moduli space $\mathcal{M}$ can all be shifted into the boundary of $\mathcal{M}$, where they are then mapped to $\partial\widehat{W} \cong \mathbb{S}^{2n-1}$, we obtain that the homology with $\mathbb{K}$-coefficients vanishes. Since this is true for every field, it follows that homology with integer coefficients also needs to be trivial. The h-cobordism theorem then implies our claim. □

V.2. Subcritical surgery and the topology of potential fillings

Conjecture V.2.1. *Let (M', ξ') be a contact manifold that has been obtained by subcritical surgery from (M, ξ), and let $S \subset M'$ be the belt sphere of the surgery. If (W', ω') is a symplectically aspherical filling of M', then S will be contractible in W'.*

Idea of a possible proof. Attach an infinite cylindrical collar to W', and deform M' inside this collar in such a way that the belt sphere S contains a large family of Lobs as in the proof above. Now we study a family of holomorphic disks attached to these Lobs, and we can prove relatively easy that there is a smooth manifold $\mathcal{M}$ consisting of holomorphic disks with a marked point giving us a map

$$\mathrm{ev}_{z_0}\colon \bigl(\mathcal{M}, \partial\mathcal{M}\bigr) \to (W', S) \ .$$

The moduli space is again a product $\mathcal{M} = \mathcal{M}_0 \times \mathbb{D}^2$, and the restriction

$$\mathrm{ev}_{z_0}\big|_{\partial\mathcal{M}}\colon \partial\mathcal{M} \to S$$

of the evaluation map is of degree 1.

If $\mathcal{M}$ happened to be a ball, we would be done, because ev_{z_0} being of degree 1 could be deformed close to the boundary to a homeomorphism between $\partial\mathcal{M}$ and S. The idea is now to modify $\mathcal{M}$ by surgeries, handle attachments and maybe other operations to convert it into a ball. Assuming that this can be done on the level of abstract manifolds without difficulty, one needs to check at each step, if the evaluation map can be modified correspondingly. Attaching a handle to the boundary of $\mathcal{M}$ works for example without any problems, because $\partial\mathcal{M}$ is mapped into S, and there is no topological obstruction to extending a continuous map to the handle, because the image of the attaching sphere is contractible in S. We have checked that these techniques work in very easy cases (for example 1-subcritical surgery on 5-manifolds), but it is unclear if we will be able to work out the general situation. If our strategy proves succesful, we can modify the original map

$$\mathrm{ev}_{z_0}\colon \bigl(\mathcal{M}, \partial\mathcal{M}\bigr) \to (W', S)$$

into a map

$$F\colon \bigl(B^k, \mathbb{S}^{k-1}\bigr) \to (W', S)$$

with degree one along the boundary, and the desired statement would follow.

A preliminary approach for applying the sketched strategy is that surgeries for example can only be performed along embedded spheres that have trivial normal bundle. In our case, this leads to the following technical conjecture we would like to show. □

Conjecture V.2.2. *Let (W, J) be an almost complex manifold with J-convex boundary (M, ξ), and suppose that $L \subset (M, \xi)$ is a Lob. Let $\mathcal{M}$ be the space of J-holomorphic disks described in Section III.2, assume that no bubbling happens, and that J has been chosen generically, so that $\mathcal{M}$ is a smooth compact manifold with boundary.*

Let $B \subset L$ be the 0-page of the Lob, then the tangent bundle $T\mathcal{M}$ is stably isomorphic to the pull-back of $TB \oplus \mathbb{R}^2$.

Idea of a possible proof. The tangent bundle of $\mathcal{M}$ can be seen as the kernel of the linearized Cauchy-Riemann operators for each holomorphic disk $u \in \mathcal{M}$. More explicitly, if u_0 is a given disk in our moduli space, and if we fix a path $u(t, \cdot)$ of smooth disks with $u(0, \cdot) = u_0(\cdot)$, then we can set $\dot u_0 = \frac{d}{dt}\big|_{t=0} u(t, \cdot)$, and we obtain a linear operator $\bar D_J(u)$ by

$$\bar D_J(u_0) \cdot \dot u_0 = \frac{d}{dt}\bigg|_{t=0} \bar\partial_J u_t \ .$$

Choosing the correct range and domain, the kernel of this operator gives us the tangent space of $T_{u_0}\mathcal{M}$. In a model, we see roughly that the linearized operator has the form

$$\bar D_J(u_0) \cdot \dot u_0 = J(u_0) \cdot D\dot u_0 - D\dot u_0 \cdot j + \bigl(\mathcal{L}_{u_0} J\bigr) \cdot du_0 \ .$$

The first two terms on the right hand side are in our case just the standard Cauchy-Riemann operator, and it is the last term that complicates the situation significantly.

Now we can consider the map $\mathcal{M} \times [0,1]$ that just associates to every holomorphic disk u_0 and every $\tau \in [0,1]$, the linear Fredholm operator

$$J(u_0) \cdot D\dot{u}_0 - D\dot{u}_0 \cdot j + \tau \left(\mathcal{L}_{u_0} J\right) \cdot du_0 \ ,$$

that means we start for $\tau = 1$ with the given linearized Cauchy-Riemann operator and deform them continuously to one where the last term vanishes. Similarly to the appendix in [**Ati67**], there should be a well-defined element in K-theory over $\mathcal{M} \times [0,1]$ associated to this family of Fredholm operators. Then on one end of the interval, we find for $\mathcal{M} \times \{1\}$ the equivalence class of $[T\mathcal{M}]$, and hence the K-theory element over $\mathcal{M} \times \{0\}$ will be isomorphic to $[T\mathcal{M}]$.

We hope that we will be able to write down all the solutions for $\mathcal{M} \times \{0\}$ explicitly, in a similar way as we did for the Bishop family in Section III.1.3, and in fact we expect the solutions are essentially constant vector fields. If this were the case, we could identify the solution space with the vectors in TB at a point of the boundary, yielding the hoped result. □

V.3. Further filling obstructions

If the previous conjecture was true, we could hope to find a further more subtle filling obstruction based on Stiefel-Whitney numbers. A well-known result by Pontryagin states that a closed manifold that is boundary of a compact one has vanishing Stiefel-Whitney numbers [**Pon47**].

The conjecture above claims that the tangent space to certain moduli spaces are stably isomorphic to the pull-back bundles of Lobs. Additionally we know that the evaluation map on the boundary of the moduli space to the Lob has degree 1, hence the Stiefel-Whitney numbers of the Lob should correspond to the Stiefel-Whitney numbers of the boundary of the moduli space.

This might be exploited to find a contradiction: The moduli space cannot be smooth and compact, if its boundary has non-vanishing Stiefel-Whitney numbers, hence some type of bubbling needs to occur. If there are no disks that can bubble, then the symplectic manifold needs to contain holomorphic spheres, and hence it cannot be an exact symplectic manifold.

The aim would be to apply these methods to higher dimensional real projective spaces. It is known that they are strongly fillable, but not Stein fillable. This approach would give hope to show that they do not admit any exact symplectic filling either. Unfortunately, some technical problems related to bubbling of holomorphic disks have appeared, because we are not working directly with Lobs, but with rational open books.

APPENDIX A

PS-overtwisted manifolds are algebraically overtwisted

The notes printed below are taken from [**Obe07**]. They explain why contact homology has to vanish in the presence of a `bLob`.

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PS–overtwisted contact manifolds are algebraically overtwisted

Klaus Niederkrueger
(joint work with Frédéric Bourgeois)

The plastikstufe [Nie06] is an attempt to generalize the overtwisted disk to higher dimensional contact topology. Since it is unclear whether the notion is general enough, we call contact manifolds containing a plastikstufe *PS–overtwisted* (instead of just calling them *overtwisted*).

Over the last two years several indications have been collected that give some justifications for the definition of the plastikstufe: *PS*–overtwisted manifolds are non fillable [Nie06], and after the first closed higher dimensional examples of such manifolds were found [Pre06], it was not difficult to convert any contact structure into one that is *PS*–overtwisted [KN07]. Recently the Weinstein conjecture has been shown to hold for these structures [AH07].

In the work sketched here, we show that a PS–overtwisted manifold has vanishing contact homology (a manifold with trivial contact homology is called *algebraically overtwisted*). In fact, we seem to be able to prove that symplectic field theory vanishes for such manifolds, extending the well known result for dimension 3 [Yau06], and giving virtually the first explicit computations of symplectic field theory.

Sketch of the proof. Contact homology is the homology of a differential graded algebra $(\mathcal{A}, \partial)$ with $\mathbb{1}$–element. The vanishing of $H_*(\mathcal{A}, \partial)$ is equivalent to the exactness of the $\mathbb{1}$–element. The aim of our proof is thus to show that the $\mathbb{1}$–element of the differential graded algebra $\mathcal{A}$ is exact. Recall that the algebra $\mathcal{A}$ is generated by linear combinations of abstract products of closed Reeb orbits $\{\gamma_j\}$, and the boundary operator is given by

$$\partial\gamma = \sum \left(\#\mathcal{M}_0^A(\gamma; \gamma_{a_1}, \dots, \gamma_{a_m})\right) e^A \gamma_{a_1} * \cdots * \gamma_{a_m} \ ,$$

where $\mathcal{M}^A(\gamma; \gamma_{a_1}, \dots, \gamma_{a_m})$ is the moduli space of the $(n+1)$–times punctured holomorphic spheres in the symplectization W of the contact manifold such that the first puncture converges in a certain sense to the closed Reeb orbit $\{+\infty\} \times \gamma$, and for each orbit γ_{a_j} there is a puncture converging to $\{-\infty\} \times \gamma_{a_j}$. The symbol $\#\mathcal{M}_0^A(\gamma; \gamma_{a_1}, \dots, \gamma_{a_m})$ denotes a rational number that counts the 0–dimensional components of $\mathcal{M}^A(\gamma; \gamma_{a_1}, \dots, \gamma_{a_m})$ taking into account orientations and the order of the automorphism group. Note that the "empty product" of closed Reeb orbits corresponds to the $\mathbb{1}$–element in $\mathcal{A}$ and we also have to include in the definition of $\partial\gamma$ the term

$$\left(\#\mathcal{M}_0^A(\gamma; \emptyset)\right) \cdot \emptyset = \left(\#\mathcal{M}_0^A(\gamma; \emptyset)\right) \cdot \mathbb{1}$$

in the summation. The elements in $\mathcal{M}^A(\gamma; \emptyset)$ are called *finite energy planes*, and if such an element lies in a 0–dimensional moduli space, it is called a *rigid* finite energy plane.

We have to find a finite combination of closed Reeb orbits

$$\sigma = \sum_{j \in I} a_j \Gamma_j \ ,$$

where Γ_j is a formal product $\gamma_1 * \cdots * \gamma_m$ of closed Reeb orbits, such that

$$\partial\sigma = \mathbb{1} \ .$$

Our proof can now be sketched like this: In a first step we find a closed Reeb orbit γ_0 that bounds a rigid finite energy plane. Existence of such an orbit follows Hofer's argument in the proof of the Weinstein conjecture for overtwisted 3–manifolds [Hof93] (for higher dimensions [AH07] respectively). Regard the manifold M as the 0–level set in W. Then there is a 1–dimensional family of holomorphic disks, the so-called Bishop family, living in the "lower half" of the symplectization W, and having its boundary on the plastikstufe. The moduli space is a compact closed interval. On one of its ends, the disks collapse to a point on the singular set of $\mathcal{PS}(S)$, and on the other one some kind of bubbling has to occur. The only type of bubbling that is possible in this situation is that the disks grow deeper and deeper into the negative direction finally breaking as a punctured disk u_C in $\big(W, \mathcal{PS}(S)\big)$ that goes from the plastikstufe to a closed Reeb orbit $\{-\infty\} \times \gamma_0$, and a rigid finite energy plane u_0 in W that is bounded on the top by $\{+\infty\} \times \gamma_0$ (see Figure 1).

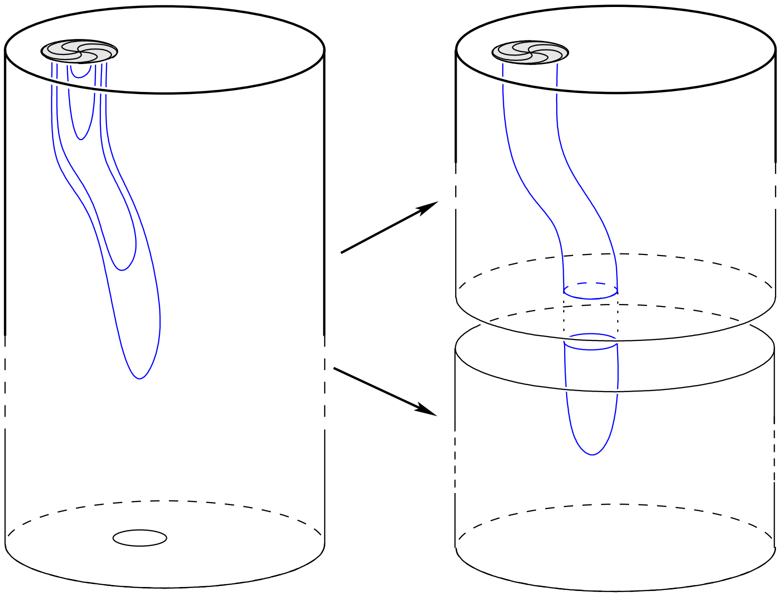

FIGURE 1. The disks in the Bishop family start as a point on the singular set S of the plastikstufe $\mathcal{PS}(S)$. They grow down into the symplectization until they finally break into a punctured disk u_C ending asymptotically at the Reeb orbit γ_0 and a finite energy plane u_0 having γ_0 as its top boundary.

If there was no other rigid punctured sphere having γ_0 as the only top puncture, then the proof would finish here, because then $\partial\gamma_0 = \pm\mathbb{1}$. Unfortunately, this is in general not the case. So assume there are other rigid punctured holomorphic spheres $u_1, \dots, u_N$ in W having γ_0 as the only top boundary (N is finite because the moduli space is a discrete compact set). Let u_1 have the closed Reeb orbits $\gamma_1, \dots, \gamma_m$ as bottom punctures. We can glue u_1 and u_C to obtain a 1–dimensional moduli space of punctured holomorphic disks in $\big(W, \mathcal{PS}(S)\big)$, whose boundary sits on the plastikstufe and whose punctures converge asymptotically to $\{-\infty\} \times \gamma_1, \dots, \{-\infty\} \times \gamma_m$ (see Figure 2). This moduli space can also be naturally compactified, and becomes this way a closed interval. Both of its ends correspond to breaking. The left boundary point of the interval represents the breaking into the curves we glued together, i.e., into u_C and the punctured sphere u_1. The other end breaks into a single punctured sphere u_1' and a collection of vertical cylinders in one level of W and a punctured disk u_C' lying one level higher. The boundary of the disk u'_C sits on the plastikstufe and its punctures converge to closed Reeb orbits $\{-\infty\} \times \gamma'_1, \ldots, \{-\infty\} \times \gamma'_k$ at the bottom. The vertical cylinders and the sphere u'_j connect at the orbits $\{+\infty\} \times \gamma'_1, \ldots, \{+\infty\} \times \gamma'_k$ to u'_C and converge at the bottom punctures to the orbits $\{-\infty\} \times \gamma_1, \ldots, \{-\infty\} \times \gamma_m$. The reason why the holomorphic curve in the lower part of the breaking consists of a single non trivial element is that otherwise the dimensions of the bubbled moduli space would be larger than 0, because disconnected components could be moved independently against each other increasing the dimension.

When applying the boundary operator ∂ to the sum of the element $\gamma_0 \in \mathcal{A}$ and the product $\gamma'_1 * \cdots * \gamma'_k$, we do not find terms of the form $\gamma'_1 * \cdots * \gamma'_k$ in the result, because the punctured spheres u_j and u'_j represent points with different orientation in the moduli space. By repeating the gluing argument first for all curves $u_1, \ldots, u_N$, and collecting the elements corresponding to the second boundary of the 1–dimensional moduli spaces, we obtain a term $\sigma_0 = \gamma_0 + \sum \gamma'_{j_1} * \cdots * \gamma'_{j_{k_j}}$. In the boundary $\partial\sigma_0$, we have succeeded in canceling out all the contributions from γ_0 with the exception of the $\mathbb{1}$–element. Unfortunately, the "correction terms" $\gamma'_{j_1} * \cdots * \gamma'_{j_{k_j}}$ may give new undesired terms in the boundary $\partial\sigma_0$. But each of these elements can be dealt with by repeating analogous steps as above, and after a finite number of applications of this method, we arrive at a collection of elements, whose boundary is finally just the $\mathbb{1}$–element.

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Think global, act local - a new aproach to Gromov compactness for pseudo-holomorphic curves.

Joel Fish

Since their introduction by Gromov, pseudo-holomorphic curves have been studied as maps from closed Riemann surfaces into almost complex manifolds with a taming symplectic form. This parameterized view has lead to a number of versions of Gromov compactness which are quite global in nature. For instance, in order to obtain convergence of a sequence of pseudo-holomorphic curves mapping into a family of symplectic manifolds, typically one must first assume the family

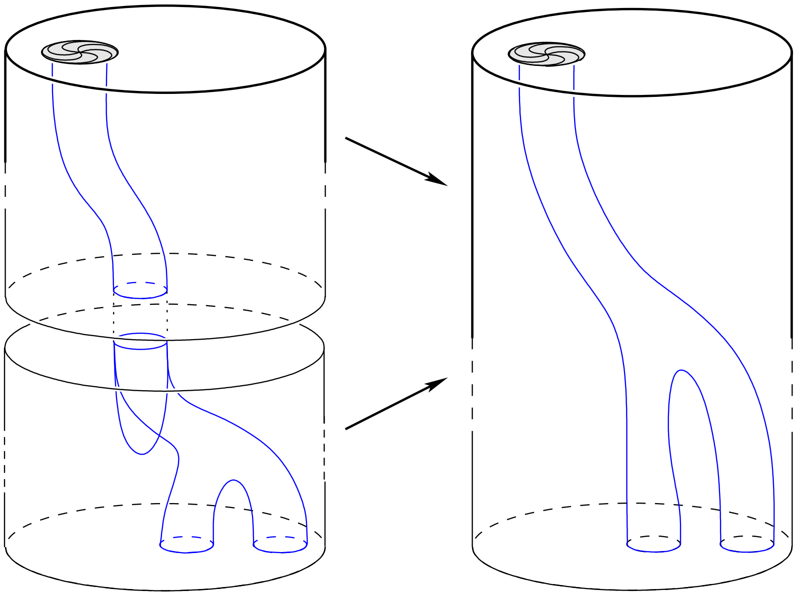

Figure 2. If γ_0 bounds several rigid punctured curves, we can perform gluing of any of these new planes to the punctured disk u_C to obtain a 1–dimensional moduli space of (punctured) holomorphic disks.

has uniform global bounds on geometric quantities like curvature, injectivity radius, energy threshold, etc. This talk will focus on a new approach to Gromov's compactness theorem, in which the curves are treated as generalized (unparameterized) surfaces. In particular, we prove a local compactness theorem which is useful when considering a family of target manifolds which develop unbounded geometry. This result recovers for instance compactness in the standard "stretching the neck" construction. Furthermore we will also provide applications of the local result to families of connected sums of contact manifolds in which the connecting handle degenerates to a point.

KAM-Liouville Theory for quasi-periodic cocycles

Raphael Krikorian

In this joint work with Bassam Fayad (CNRS, Paris 13) we extend the reducibility theory of cocycles of the form $(\alpha, A) : \mathbb{R}/\mathbb{Z} \times SL(2,\mathbb{R}) \to \mathbb{R}/\mathbb{Z} \times SL(2,\mathbb{R})$, $(\theta, y) \mapsto (\theta+\alpha, A(\theta)y)$, $A \in C^\infty(\mathbb{R}/\mathbb{Z}, SL(2,\mathbb{R}))$ to the case where α is of Liouville type ($q_{n+1} \geq q_n^n$ infinitely many times). We prove that such a C^∞ cocycle which is

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