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Présentée en vue d'obtenir le grade de Docteur,
spécialité Informatique

par

Mohammed Amin TAHRAOUI

Coloring, Packing and Embedding of graphs

Thèse soutenue le 04/12/2012 devant le jury composé de:

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Chapter 1
Introduction

Graph theory is an important field of mathematics and computer science. Originally, it started in 1735 with the problem of the Seven Bridges of Königsberg. The city of Königsberg in Prussia (modern Kaliningrad, Russia) was set on both sides of the Pregel River, and included two islands connected to each other and the mainland by seven bridges. The problem was to find a continuous tour through the city that would cross each bridge exactly once, ending up at the point from which it began. The Swiss mathematician, Leonard Euler, demonstrated that no such tour was possible. This result is often referred to as the first theorem in graph theory [Eul41]. Graphs are useful mathematical tools for modeling the relationships among objects, which are represented by vertices. In their turn, relationships between vertices are represented by edges. In this context, graph theory received considerable attention, not only from the mathematical community, but also from the whole scientific community. Over the years, there have been a large number of significant and authoritative publications in biochemistry, computer science, genetics or in sociology where there is an interesting connection with graph theory.

These problems are most often modeled by finding an optimal structure in a graph, generally a structure that must satisfy some constraints. For example, some problems can be modeled by finding a *maximum matching* (*i.e.,* a largest set of edges without common vertices), a *maximum stable set* (*i.e.,* a largest set of vertices such that no two of them are linked by an edge), or a *minimum coloring* (*i.e.,* a minimum-cardinality partition of the vertices of a graph into stable sets).

Some problems can be solved efficiently, *i.e.,* with a polynomial-time algorithm (an algorithm is polynomial if its running time is bounded by a polynomial function according to the number and size of the inputs). For example, the problem of computing a maximum matching can be solved using very efficient polynomial-time algorithms. Unfortunately, many graph problems are hard in the sense that there is (probably) no polynomial algorithm which solves these problems. More formally, these problems are proved to be NP-hard (this is the case for minimum coloring and maximum stable set problem). Another class of problems concerns those that have not been classified yet as either polynomial or NP-hard. Isomorphism between two graphs is one of the very few natural problems in this class. We say that two graphs are isomorphic if there exists a permutation of the vertices that makes both graphs identical.

Recently, graph theoretical concepts were widely used to study and model various computer applications such as image segmentation, information retrieval, networking, clustering, etc. For example in XML information retrieval, XML document and query data can be represented by a graph model, and hence, retrieving XML documents can be considered as a graph matching problem between the query tree and the document trees (*i.e.,* document/query isomorphism).

The three major problems considered in this thesis are the *graph coloring problem*, the *graph packing problem* and the *tree pattern matching.* The common point between these three problems is that they deal with labeled graphs. We thus divided the thesis into three main parts, each containing two chapters, numbered from 3 to 8, where **Chapter two** is devoted to some basic graph theory concepts and preliminary definitions, that are needed for the understanding of the results exposed in this document.

Part 1: Graph coloring problem

Born with the famous Four Color Problem in 1852, the field of graph coloring has become one of the most popular areas of graph theory. Graph coloring problems come in many varieties but in general they consist in partitioning the objects (vertices, edges, faces, etc.) of a graph into different classes so that given constraints are satisfied. In its basic form, graph coloring is a way of coloring the vertices of a given graph such that adjacent vertices get different colors; this is called the *proper vertex coloring problem.* Similarly, *a proper edge coloring problem* assigns a color to each edge so that every two adjacent edges receive different colors. About forty years ago, many researchers considered the problem of assigning colors to the edges of a graph G such that any two vertices of G are uniquely identified either by sets, multisets or sums of their incident colors. The literature about these variants is reviewed in **Chapter three**. **Chapter four** is intended to define a new graph coloring parameter called the *gap vertex distinguishing edge coloring* number, that combines many features of the previously introduced methods. It consists in an edge-coloring of a graph G which induces a vertex distinguishing labeling of G such that the label of each vertex is given by the difference between the highest and the lowest colors of its adjacent edges. In particular, we will study this problem for various families of graphs and we will also show how this new parameter is strongly correlated to the vertex distinguishing problem by sets and multisets. As a consequence, we will provide bounds on these two problems.

Part 2: Graph packing problem

Graphs $G_1, G_2, \ldots, G_k$ (on n vertices each) pack, if there exists an edge disjoint placement of them into the complete graph K_n. This problem is a classical one in graph theory and has been extensively studied since the early 70's. However, the majority of existing works focuses on unlabeled graphs. In **Chapter five**, we introduce for the first time the packing problem for a vertex labeled graph. Roughly speaking, it consists in a classical graph packing which preserves the labels of the vertices. Then, we study the corresponding optimization parameter for k copies of cycles. In **Chapter six**, we study the labeled packing of two copies of trees and of all graphs of order n and size $n-2$. As an application, we will show that this new problem can be used to solve the matching problem between two labeled graphs.

Part 3: XML Tree Pattern Matching Problem

With the increasing number of available XML documents, numerous approaches for retrieval have been proposed in the literature. They usually use the tree representation of documents and queries to process them in an implicit or explicit way. Although retrieving XML documents can be considered as a tree matching problem between the query tree and the document trees, we consider in this part the matching problem from an exact point of view. In **Chapter seven**, we outline and compare the various features of different tree pattern algorithms. Most of these algorithms find twig pattern matching in two steps. In the first one, a query tree is decomposed into a set of binary patterns or single paths and then search for matches for these individual patterns/paths. Finally, these matches are stitched together to form the answers to the twig query. In **Chapter eight**, we propose a novel holistic twig join algorithm, called *TwigStack++*, which features two main improvements in the decomposition and matching phase. The proposed solutions are shown to be efficient and scalable, and should be helpful for the future research on efficient query processing in a large XML database.

Finally, in **chapter nine**, we summarize the results presented in this thesis, and we give some remarks and directions for further research.

List of publications arising from this thesis

International journals

1. M. A. Tahraoui and H. Kheddouci. TwigStack++. A New Efficient Holistic Twig Join Algorithm, In International Journal of Information-Interaction-Intelligence (I3) N:02, 2011.
2. M. A. Tahraoui, E. Duchêne and H. Kheddouci. Gap vertex-distinguishing edge colorings of graphs, Discrete Mathematics Volume 312(20):3011-3025. 2012.

International conferences

3. E. Duchêne, H. Kheddouci, R. J. Nowakowski and M. A. Tahraoui. Labeled Embeddings of Graphs. In SIAM Conference on Discrete Mathematics, Canada, 2012.

4. M. A. Tahraoui, E. Duchêne, H. Kheddouci and R. J. Nowakowski. Labeled packing of Graphs. In the proceedings of BCC 2011 - 23rd British Combinatorial Conference. University of Exeter, Exeter, UK, 2011.
5. M. A Tahraoui and H. Kheddouci. An evolutionary algorithm for the k-b-coloring problem. In the proceedings of 3nd International Conference on Metaheuristics and Nature Inspired Computing META'10, Hammamet, Tunisia, 2010.
6. M. A Tahraoui, E. Duchêne and H. Kheddouci. Max-Min vertex distinguishing edge colorings of graphs. In the proceedings of 8th French Combinatorial Conference, Orsay, France, 2010.

International Journal, accepted with minor revision

7. M. A. Tahraoui, K. P. Sauvagnat, L. Ning, C. Lattang, M. Boughanem and H. Kheddouci. A survey on tree matching and XML retrieval, Computer Science Review.
8. E. Duchêne, H. Kheddouci, R. J. Nowakowski and M. A. Tahraoui. Labeled packing of graphs, Australasian Journal of Combinatorics.

Submitted papers

9. M. A. Tahraoui, E. Duchêne and H. Kheddouci. Labeled embedding of trees, Submitted to Discrete Mathematics.

CHAPTER 2

Preliminaries

Contents

We assume the reader is familiar with basic concepts of graph theory and theoretical computer science. We will thus give a short overview of the terminology used in this thesis. More background information can be found in [CZ05], [Har69] and [Wes96].

2.1 Basic notations

Graph: a graph $G = (V(G), E(G))$ consists of two finite sets: $V(G)$, the *vertex set* of G, which is a non-empty set of elements called *vertices* and $E(G) \subseteq V(G) \times V(G)$ is called set of *edges*. The cardinality of the vertex set $V(G)$ is called the *order* of G, commonly denoted by $|V(G)|$. The cardinality of the edge set $E(G)$ is the *size* of G, denoted by $|E(G)|$. A graph of order n and size m is often called an (n, m)-*graph*. Two distinct vertices u and v are *adjacent* (or *neighbors*) if there exists an edge uv that connects them. An edge uv is said to be incident to the vertices u and v. Two edges are adjacent if they are incident to a same vertex. A graph is *simple* if there is at most one edge between every two vertices. In this document, unless it is specified, the graphs which are considered will be finite simple graphs.

Degree and Neighborhood: the set of all neighbors of a vertex v in a graph G is denoted by $N(v)$. The number of neighbors of v is called the degree of v in G, denoted by $d(v)$. If $d(v) = 0$, it means that v is not adjacent to any other vertex, then v is called an *isolated vertex*. A vertex of degree one is called an *endpoint* or a *pendant vertex* or a *leaf*. The minimum degree of a graph G is $\delta(G) = min\{d(v) : v \in V(G)\}$ and the maximum degree of a graph G is denoted by $\triangle(G) = max\{d(v) : v \in V(G)\}$.

Independent sets and Cliques: an independent set of G is a subset of vertices $U \subseteq V$, such that no two vertices in U are adjacent. An independent set is said to be maximal if no independent set properly contains it. An independent set of maximum cardinality is called a maximum independent set. A clique in a graph G

is a subset C of $V(G)$ such that every two vertices in C are adjacent in G. The clique number $\omega(G)$ is the order of a maximum clique of G.

Distance and Connectivity: a path in an undirected graph G is a sequence of vertices $(v_1, v_2, \ldots, v_k)$ such that each pair v_i, v_{i+1} is an edge in $E(G)$. A path is called *simple* if all its vertices are distinct. A graph G is *connected* if there exists a path between any two distinct vertices of G. Otherwise, the graph G is disconnected. The distance between two vertices u, v in G, denoted by $dist(u, v)$, is the minimum number of edges in a path connecting them. The *diameter* of G is the maximum distance between any two vertices of G. A graph is k-*edge-connected* if there are k edge-disjoint paths between each pair of vertices or, equivalently, no two vertices can be separated by removing less than k edges.

Isomorphism of graphs: two graphs G and H are said to be *isomorphic* (written $G \cong H$) if there is a one-to-one correspondence between their vertex-sets which preserves the adjacency of vertices. We provide an example in Figure 2.1.

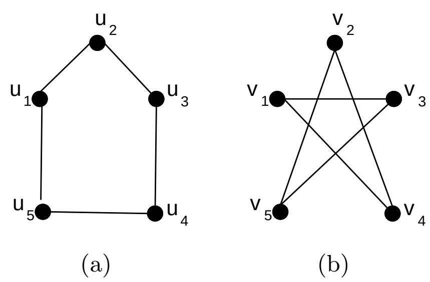


Figure 2.1: Example of isomorphic graphs: $f(u1) = v1$, $f(u_2) = v_3$, $f(u_3) = v_5$, $f(u_4) = v_2$ and $f(u_5) = v4$.

Subgraphs: a graph H is a subgraph of G, written $H \subseteq G$, if every vertex of H is a vertex of G and every edge of H is an edge of G. In other words, $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. We say that a subgraph H is a *spanning subgraph*, or a *factor*, of G if H contains all the vertices of G. Given $V' \subseteq V$, the subgraph $G[V'] = (V', E')$ denotes the subgraph of G *induced* by V', *i.e.,* E' contains all the edges of E which have both endpoints in V'.

2.2 Some graph operations

In the following definitions, we detail some well known graph operations.

Complement graph: if G is a simple graph with vertex set $V(G)$, its complement $\overline{G}$ is the simple graph with vertex set $V(G)$ in which two vertices are adjacent if and only if they are not adjacent in G. Figure 2.2 shows a graph and its complement.

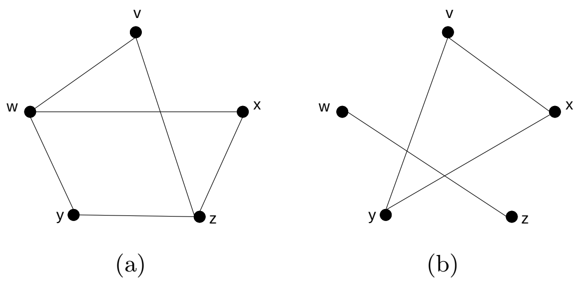


Figure 2.2: (a) G, (b) $\overline{G}$.

Power of graph: the *k-th* power G^k of a graph G is the supergraph of G formed by adding an edge between all pairs of vertices of G with distance at most k. The second power of a graph is also called its *square*.

Vertex removal: if v is a vertex of the graph $G = (V, E)$, then $G - v$ is the subgraph of G induced by the vertex set $V \setminus \{v\}$.

Edge removal: similarly to the previous operation, if e is an edge of the graph $G = (V, E)$, then $G - e$ is the graph (V, E'), where E' is obtained by removing e from E. Note that the endpoints of e are not removed from G.

Graph union: the union $G = G_1 \cup G_2$ of graphs G_1 and G_2 with disjoint vertex sets V_1 and V_2 and edge sets E_1 and E_2 is the graph with $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$. This operation is sometimes also known explicitly as the *graph disjoint union*.

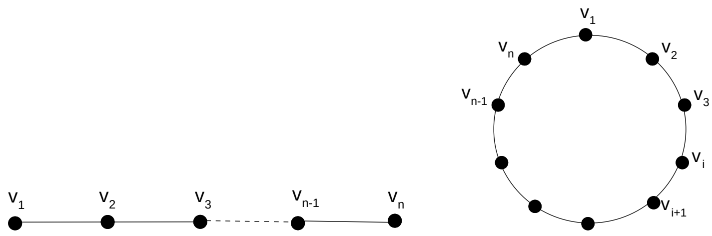
Cartesian product of graphs: the Cartesian product of two graphs G_1 and G_2, denoted by $G_1 \square G_2$, is the simple graph with $V(G_1) \times V(G_2)$ as its vertex set and two vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G_1 \square G_2$ if and only if either $u_1 = u_2$ and v_1, v_2 are adjacent in G_2, or u_1, u_2 are adjacent in G_1 and $v_1 = v_2$.

2.3 Some families of graphs

We review in this section several classes of graphs that will be considered in this document.

Path graph: a graph G is called path, denoted by P_n, if it is of the form: $V(P) = \{v_0, v_1, \ldots, v_n\}$, $E(P) = \{v_0v_1, v_1v_2, \ldots v_{n-1}v_n\}$ (see Figure 2.3(a)).

Cycle graph: a graph G is called cycle, denoted by C_n, if it is of the form: $V(P) = \{v_0, v_1, \ldots, v_n\}$, $E(P) = \{v_0v_1, v_1v_2, \ldots v_{n-1}v_n, v_nv_1\}$ (see Figure 2.3(b)).

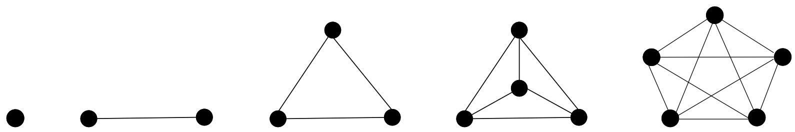


(a) (b)

Figure 2.3: (a) Path graph, (b) Cycle graph.

Hamiltonian graph: a *Hamilton cycle* is a cycle containing every vertex of the graph. A graph is *Hamiltonian* if it has an Hamilton cycle.

Complete graph: a complete graph is a graph in which every two distinct vertices are joined by exactly one edge. The complete graph with n vertices is denoted by K_n. In Figure 2.4, we give five examples of complete graphs.



(a) (b) (c) (d) (e)

Figure 2.4: Complete graphs (a) K_1, (b) K_2, (c) K_3, (d) K_4, (e) K_5.

Planar graph: a graph is planar if it can be drawn in a plane without edge crossing (*i.e.,* edges intersect only at their common vertices). For example, the complete graph K_4 is a planar graph.

Tree: a tree is a connected graph which has no cycle. An example of tree is given in Figure 2.5. A tree is called a *rooted tree* if one of its nodes is distinguished as the root, in which case the edges have a natural orientation, towards or away from the root. A vertex v in a rooted tree is a descendant of a vertex u if u lies on the unique path from the root to v. The parent of a vertex v is the last vertex before v in a path from the root to v. The depth of a vertex v in a rooted tree is the length of the path from the root to v. Thus, the depth of the root is 0.

Star graph: the star graph S_n, is a tree with n vertices such that one vertex (called the center) has degree $n-1$ and the other $n-1$ vertices have degree 1 (see Figure 2.6(a)).

Caterpillar: a caterpillar is a tree which becomes a path when all its endpoints are removed, as shown in Figure 2.6(b).

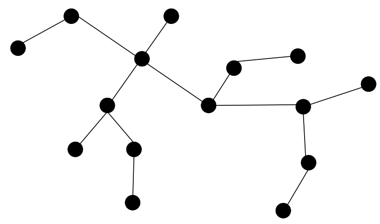
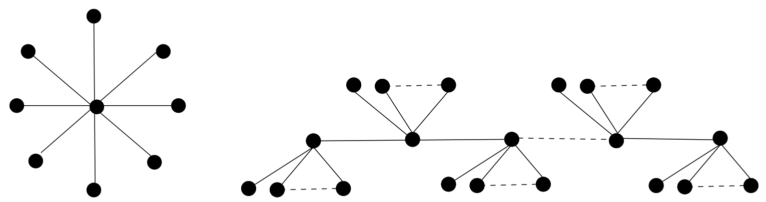


Figure 2.5: Tree.



(a) (b)

Figure 2.6: (a) Star graph, (b) Caterpillar.

Bipartite graph: a bipartite graph $G = (V, E)$ is a graph whose vertex set V can be divided into two disjoint subsets U and W, such that each edge of G has one endpoint in U and one endpoint in W, as shown in Figure 2.7. Equivalently, a bipartite graph is a graph that does not contain any odd-length cycle. The complete bipartite graph on n and m vertices, denoted by $K_{n,m}$ is the bipartite graph $G = (U, W, E)$, where U and W are disjoint sets of size n and m, respectively, and E connects every vertex in V with every vertice in U. It follows that $K_{n,m}$ has nm edges.

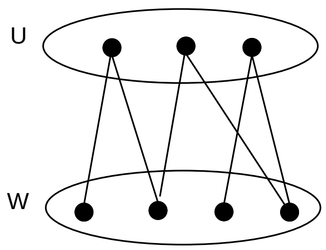


Figure 2.7: bipartite graph

Regular graph: a regular graph is a graph whose vertices all have equal degree. A k-regular graph is a regular graph whose common degree is k. A *cubic graph* is a 3-regular graph.

d-degenerate graph: a d-degenerate graph is a graph in which every induced subgraph has a vertex with degree at most d.

Part I

Graph Coloring Problem

Chapter 3

Vertex-Distinguishing Edge Coloring of Graphs

Contents

There have been several studies using a variety of methods for the purpose of uniquely identifying (or distinguishing) the vertices of a graph. Many of these methods involve graph labelings or graph colorings. In several variants, certain edge colorings have given rise to vertex-distinguishing labelings. These are often referred to as *vertex-distinguishing edge colorings*, which is the subject of this chapter.

3.1 Introduction to graph coloring

Graph coloring is one of the oldest areas in graph theory. It is widely believed that the graph coloring problem was born in 1852 when Guthrie asked whether it was possible to color the countries of any geographical map with four or fewer colors, so that every two countries sharing a common boundary are colored differently. This is the famous *Four Color Problem*. The first proof of this problem was given by Kempe in 1879 and accepted for more than ten years until Heawood in 1890 found a flaw in Kempe's reasoning using a map with 18 countries. By a revision of Kempe's proof, Heawood was able to show that five colors are always sufficient. Graph coloring has become a subject of great interest for researchers, mainly because of its diverse theoretical results and unsolved problems. Furthermore, graph colorings have been proved to be paramount in other domains of graph theory (for instance in the study of connectivity, matchings, Hamilton cycles) and also in real world applications in many engineering fields, including register allocation [CH90], timetabling [Wer85], frequency assignment [Gam86], scheduling [Lei79] and communication networks [WSW02].

3.1.1 Vertex coloring problem

Graph coloring problems that received the most attention deal with the vertices of a graph. A proper vertex coloring of a graph G is a mapping $f : V(G) \longrightarrow \mathbb{N}$ (where $\mathbb{N}$ is the set of positive integers) such that $f(u) \neq f(v)$ if u and v are adjacent in G. If $f(V)$ has size at most k, then we refer to the coloring as a *k-coloring.* A subset of vertices assigned with the same color is called a *color class*, every such class being an independent set. Thus, a *k-coloring* is equivalent to a partition of $V(G)$ into k independent sets, and the terms *k-partite* and *k-colorable* have the same meaning. The following figure illustrates an example of a proper vertex coloring with three colors.

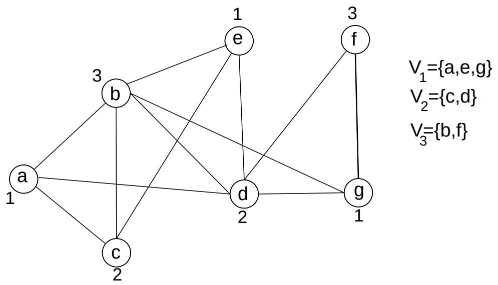


Figure 3.1: A graph with a proper 3-coloring

The chromatic number $\chi(G)$ of G is the minimum positive integer k for which G has a *k-coloring.* Determining the chromatic number of a general graph G is well-known to be a NP-hard problem [GJ79]. Consequently, much work has been devoted to (1) determining bounds on the chromatic number of general graphs and (2) determining the chromatic number of some families of graphs. It is quite easy to see that for every graph G with maximum degree $\triangle(G)$, we have $\chi(G) \leq \triangle(G)+1$. However, Brooks has improved this bound as follows [Bro41].

Theorem 3.1 ([Bro41]) *Let G be a connected graph. Then $\chi(G) \leq \triangle(G)$ unless G is a complete graph or an odd cycle.*

A rather obvious, but often useful, lower bound for the chromatic number of a graph involves the chromatic numbers of its subgraphs.

Theorem 3.2 *If H is a subgraph of a graph G, then $\chi(H) \leq \chi(G)$.*

The following result is an immediate consequence of the previous theorem.

Corollary 3.3 *For every graph G, $\chi(G) \geq \omega(G)$.*

A graph G is *perfect* if every induced subgraph H of G satisfies $\chi(H) = \omega(H)$. In the early 1960's, Berge observed [Ber60, Ber61] that several classical families of graphs (e.g. bipartite graphs, trees, interval graphs, chordal graphs) are perfect. In [L7́2], the author showed that a complement of a perfect graph is also perfect. However, there are also many graphs whose chromatic number exceeds their clique number such as the Petersen graph and the odd cycles of length 5 or more. In [GLS84], Grötschel *et al.* showed that the chromatic number of a perfect graph can be determined in polynomial time. This is important as perfect graphs have many connections to other combinatorial problems.

In addition to vertex coloring problems, many other types of graph coloring problems have been formulated in the literature. However, the significance of vertex coloring problems is often emphasized since many of these new problems can be reformulated in terms of vertex colorings, or strongly depend on the chromatic number. For more details on those variants, we refer the reader to the book of de Werra and Hertz [WH89].

3.1.2 Edge coloring problem

After vertex colorings, it makes sense to study edge colorings. In this problem we are looking for the minimum number of colors necessary to be assigned to the edges of a graph such that any two incident edges are colored with different colors. This parameter is called the *chromatic index* and is denoted by $\chi'(G)$. In 1965, Vizing [Viz65] proved the famous Vizing's theorem.

Theorem 3.4 ([Viz64]) *For every nonempty graph G,*

$$\triangle(G) \leq \chi'(G) \leq \triangle(G) + 1$$

In [Hol81], Holyer proved that it is NP-complete to decide whether $\chi'(G) = \triangle(G)$ or $\chi'(G) = \triangle(G) + 1$. The chromatic index has been determined for several classes of graphs. Indeed, bipartite graphs satisfy $\chi'(G) = \triangle(G)$, while for odd cycles, we have $\chi'(G) = \triangle(G) + 1$. For planar graphs, Vizing proposed the following conjecture:

Conjecture 3.5 ([Viz65]) *For every planar graph G with maximum degree $\triangle(G) =$ 6 or 7, we have*

$$\chi'(G) = \triangle(G)$$

This conjecture has been confirmed in [SZ01, Zha00] for $\triangle(G) = 7$. But the case $\triangle(G) = 6$ remains an open problem.

Various generalizations of edge-coloring have been introduced and investigated in the literature. In the 1970's, Hilton and de Werra obtained many results on *equitable and edge-balanced colorings* in which each color appears uniformly in G

[HW82, Wer74, Wer75]. In the 1980's Hakimi and Kariv [HK86] proposed the following f-coloring problem: let f be a function which assigns a positive integer $f(v)$ to each vertex $v \in V(G)$, an f-coloring of a graph G is a proper edge coloring of G such that for each vertex $v \in V$, at most $f(v)$ edges incident to v are colored with the same color. The minimum number of colors needed to f-color G is called the *f-chromatic index* $\chi'_f(G)$ of G. Since the proper edge-coloring problem is NP-complete, the f-coloring problem is also NP-complete in general. Various upper bounds on $\chi'_f(G)$ have been provided in [HK86, NNS88, Sey90].

3.1.3 Total coloring problem

We now consider colorings that assign colors to both vertices and edges of a graph. *A total coloring* of a graph G is an assignment of colors to the vertices and edges of G such that distinct colors are assigned to (i) every two adjacent vertices, (ii) every two adjacent edges, and (iii) every incident vertex and edge. The total chromatic number $\chi''(G)$ of a graph G is the least number of colors needed in any total coloring of G. Immediately, we have that $\chi''(G) \geq \triangle(G) + 1$. Sánchez-Arroyo [SA89] shown that deciding whether a given graph has $\chi''(G) = \triangle(G) + 2$ is an NP-complete problem. The total coloring conjecture proposed independently by Behzad [Beh65] and Vizing [Viz65] claims that for every simple graph G, $\chi''(G) \leq \triangle(G) + 2$. This conjecture has been verified for a few important classes of graphs such as all bipartite graphs and most planar graphs except those with maximum degree 6.

3.2 Vertex-distinguishing edge coloring

A problem in graph theory that has received considerable attention in the literature concerns methods to uniquely identify the vertices of a connected graph. One of the most popular methods is due to Entringer and Gassman [EG74] and Sumner [Sum73]. They characterized the graphs G such that every two non-adjacent vertices of G have distinct open neighborhoods. Harary and Melter [HM76] introduced the idea of selecting a subset $A = \{u_1, u_2, \dots, u_k\}$ and labeling each vertex of v of G with the ordered k-tuple $l(v) = (d_1, d_2, \dots, d_k)$ called distance label of v, where $d_i = dist(v, u_i)$ for $1 \leq i \leq k$. In this case, the vertices of G are distinguishable if distinct vertices of G have distinct distance labels. The *symmetry breaking* method was introduced by Harary [Har96, Har01] and, independently, by Albertson and Collins [AC96]. In this method, the distinction of the vertices of G is realized with the aid of a certain automorphism of G.

In this thesis, we are interested in another distinguishing method that is related to the graph coloring problem. Indeed, many researchers investigated the question of finding an edge coloring inducing a vertex distinguishing labeling. This is often referred to as *vertex-distinguishing edge coloring.* This problem has received increasing attention in the literature during the past forty years. To the best of our knowledge, four main different functions have been proposed to label each vertex v

of G according to the colors of its incident edges. A vertex labeling l induced by an edge-coloring f is said to be:

1. vertex-labeling by sum if $l(v) = \sum_{v \ni e} f(e), \forall v \in V$ (see [CJL+88, BKT05]).
2. vertex-labeling by sets if $l(v) = \bigcup_{v \ni e} f(e), \forall v \in V$ (see [BS97, CHS96, HP85]).
3. vertex-labeling by multiset if $l(v) = \biguplus_{v \ni e} f(e), \forall v \in V$ (see [ATT92, Bur95, CEOZ06]).
4. vertex-labeling by product if $l(v) = \prod_{v \ni e} f(e), \forall v \in V$ (see [Kaz12]).

The problem of vertex-distinguishing edge colorings offers many variants and received a great interest during these last years. We refer the interested reader to Chapter 13 of Chartrand and Zhang's book [CZ08]. In the next section, we give some results and open questions about this area of research.

3.2.1 Variations of the problem

Over the years, several vertex distinguishing edge coloring problems have been defined. In many variants, some authors used the same terminology to describe different concepts. In Table 3.1, we describe a summary of these variants depending on whether:

1. the edge coloring is proper or not,
2. the vertex labeling is induced by sum; product; set or multiset (if the edge-coloring is not proper),
3. the edge coloring must distinguish all vertices (global) or only adjacent vertices (local).

Hence, we potentially obtain fourteen variants of vertex distinguishing edge colorings, among which ten were studied in the literature (to the best of our knowledge). Each row of Table 3.1 presents a vertex-distinguishing variant, for example, the first row means that Burris and Schelp [BS97] defined a **strong edge coloring** of G as a **proper** edge coloring that induces a **vertex-distinguishing** labeling by **sum**. In Figure 3.2, we provide a coloring scheme of C_7 for each variant appearing in Table 3.1. Since the literature on these topics is very rich, we are not able to give an exhaustive list of all the relevant references. In what follows, we only mention three variants of vertex distinguishing colorings that will have a strong relation with our coloring parameter introduced in the next chapter.

3.2.2 Strong edge coloring

Let $\chi'_s(G)$ denote the minimum number of colors required to have a proper edge coloring of G that induces a vertex-distinguishing labeling by sets. This coloring number was introduced and studied by Burris and Schelp in [Bur93, BS97], and

	Edge coloring	Labeling operation	vertex distinguishing	Reference	*Example*
Strong edge coloring	proper	set	global	[BS97]	Fig 3.2(a)
Point-distinguishing edge coloring	improper	set	global	[HP85]	Fig 3.2(b)
Adjacent strong edge coloring	proper	set	local	[ZLW02]	Fig 3.2(c)
Neighbour-distinguishing coloring	improper	set	local	[GHPW08]	Fig 3.2(d)
Detectable coloring	improper	multiset	global	[ATT92]	Fig 3.2(e)
Vertex colouring edge partitions	improper	multiset	local	[AADR05]	Fig 3.2(f)
Irregular weighting	improper	sum	global	[CJL+88]	Fig 3.2(g)
Vertex-colouring edge-weighting	improper	sum	local	[KLT04]	Fig 3.2(h)
Neighbor sum distinguishing edge colorings	proper	sum	local	[FMP+12]	Fig 3.2(j)
Multiplicative vertex-colouring weightings	improper	product	local	[Kaz12]	Fig 3.2(i)

Table 3.1: Summary of vertex distinguishing edge coloring of graphs

was independently called *observability* of a graph by Cerny *et al* [CHS96]. In their original article on this topic, Burris and Schelp [BS97] gave values of χ'_s for some families of graphs such as complete graphs, bipartite complete graphs, paths and cycles. Let $n_d(G)$ be the number of vertices of degree d in G, it is clear that $\binom{\chi'_s(G)}{d} \geq n_d$ for all d with $\delta(G) \leq d \leq \triangle(G)$. In [BS97], Burris and Schelp posed the following conjecture.

Conjecture 3.6 ([BS97]) *Let G be a graph with no isolated vertices or edges and let k be the minimum integer such that $\binom{k}{d} \geq n_d$ for all d with $\delta(G) \leq d \leq \triangle(G)$. Then $\chi'_s(G) = k$ or $k+1$.*

This conjecture remains open in general and has only been determined for some classes of regular graphs [RS08] and graphs with small components [BBS02].

The following result has been conjectured by Burris and Schelp [BS97] and proved in [BBLW99].

Theorem 3.7 ([BBLW99]) *A graph G with n vertices, without isolated edges and with at most one isolated vertex satisfies $\chi'_s(G) \leq n+1$.*

This upper bound is the best possible since it is achieved for the complete graph K_n where n is even. For some families of graphs, the parameter $\chi'_s(G)$ is close to the maximum degree than to the order of the graph as shown by the following theorem.

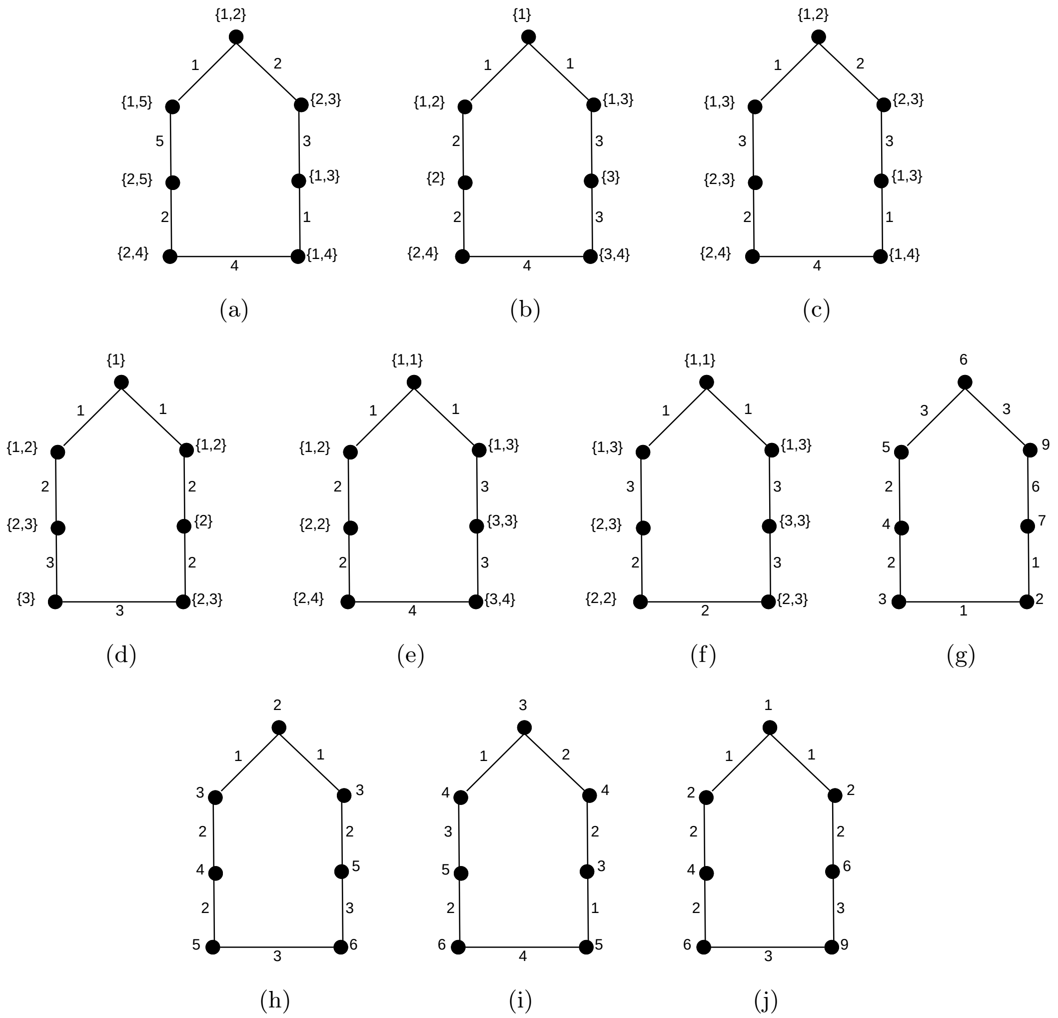


Figure 3.2: Coloring scheme of C_7 for each variant in Tabel 3.1.

Theorem 3.8 ([BBLW01]) *Let G be a graph of order $n \geq 3$ without isolated edges and with at most one isolated vertex. If $\delta(G) > \frac{n}{3}$, then*

$$\chi'_s(G) \leq \triangle(G) + 5$$

There have been many bounds related to some specific families of graphs. In the case of trees, Burris and Schelp have obtained the following upper bound: for any tree $T \neq K_2$, $\chi'_s(T) \leq max\{n_1 + 1, 6.35n_2^{1/2}, 21\}$. Recently, Jun-qiao and Yue-hua [JY12] have found bounds on the vertex distinguishing chromatic index of 3-regular Halin graphs and Halin graphs with $\triangle(G) \geq 4$, respectively.

3.2.3 Detectable edge coloring

Let $c(G)$ (called also *detection number*) denote the minimum number of colors required to have an edge coloring (not necessarily proper) of G that induces a vertex-distinguishing labeling by multisets. The following result has been stated in [ATT92].

Theorem 3.9 ([ATT92]) *Let c be a k-coloring of the edges of a graph G. The maximum number of different labels of the vertices of degree r in G is $\binom{r+k-1}{r}$.*

In other words, this means the following:

Corollary 3.10 *If c is a detectable k-coloring of a connected graph G of order at least 3, then G contains at most $\binom{r+k-1}{r}$ vertices.*

Since vertices with distinct degrees in a connected graph always have distinct labels, then, it seemed most challenging to study the graphs having many vertices of the same degree. Indeed, the parameter $c(G)$ of complete graphs and complete bipartite graphs have been determined and detectable colorings of connected r-regular graphs and trees have been studied as well (see [ATT92], [Bur94],[Bur95], [CEOZ06]).

The detection number of cycles and paths have been determined in [CEOZ06] and [EP05], respectively.

Theorem 3.11 ([CEOZ06]) *Let $n \geq 3$ be an integer and let $l = \lceil\sqrt{\frac{n}{2}}\rceil$. Then*

$$c(C_n) = \begin{cases} 2l-1 & \text{if } 2(l-1)^2+1 \leq n \leq 2l^2-l \\ 2l & \text{if } 2l^2-l+1 \leq n \leq 2l^2 \end{cases}$$

Theorem 3.12 ([EP05]) *Let $n \geq 3$ be an integer and let $l = \lceil\sqrt{\frac{n}{2}}\rceil$. Then*

$$c(P_n) = \begin{cases} 2l & \text{if } 2l^2-l+1 \leq n \leq 2l^2+3 \\ 2l+1 & \text{if } 2l^2+4 \leq n \leq 2l^2+3l+2 \end{cases}$$

The following result, stated in [CEOZ06], gives an upper bound for $c(G)$ on a connected graph.

Theorem 3.13 ([CEOZ06]) *If G is a connected graph of order $n \geq 4$, then $c(G) \leq n-1$.*

This upper bound is obviously reached since there are many families of graphs G with the property that $c(G) = n-1$ such as $K_{1,n}$ for $n \geq 2$, K_3, K_4, P_3 and C_4.

3.2.4 Point-distinguishing edge coloring

Let $\chi'_0(G)$ denote the minimum number of colors required to have an edge coloring (not necessarily proper) of a graph G that induces a vertex distinguishing labeling by sets. Harary and Plantholt [HP85] referred to this type of coloring as the *point-distinguishing edge coloring*. They proved, among other things, the exact value of $\chi'_0(P_n)$, $\chi'_0(C_n)$, $\chi'_0(Q_n)$ and $\chi'_0(K_n)$ for $n \geq 3$. For bipartite graphs it seems that the problem of determining $\chi'_0(K_{m,n})$ is not easy (see [HS97, HS06, Sal90]).

Clearly we have $c(G) \leq \chi'_0(G) \leq \chi'_s(G)$, and the following result follows from Theorem 3.7.

Theorem 3.14 *A graph G with n vertices, without isolated edges and with at most one isolated vertex satisfies $\chi'_0(G) \leq n + 1$.*

During the analysis of this problem we have observed that there is no connected graph G of order n with $\chi'_0(G) = n + 1$. Thus, we propose the following conjecture.

Conjecture 3.15 *For every connected graph G of order $n \geq 3$, $\chi'_0(G) \leq n$.*

If this conjecture is true, then the resulting theorem cannot be improved in general (clearly $\chi'_0(Q_n) = n$). In the next chapter, we will prove that Conjecture 3.15 holds for several families of graphs.

3.2.5 Our observations

Derived from our study of the above literature, the following three observations can be made.

1. If the problems of determining $\chi'_0(G)$, $\chi'_s(G)$ and $c(G)$ are compared, their complexity depends on the structure of G and none of them can be stated to be more difficult than another one. For example, the exact value of $\chi_0(Q_n)$ has been determined in [HP85], while $\chi'_s(Q_n)$ is computed only for $n \leq 5$. On the other hand, $\chi'_s(K_{n,n}) = n + 2$, while the exact value of $\chi_0(K_{n,n})$ is not easy to get. In Table 3.2, we give a summary of some important results about these three parameters.

2. The problem of vertex-distinguishing edge coloring has been the subject of a renewed interest in recent years. Perhaps the only downside of this area is the lack of applications. But do not despair, this type of coloring could be simpler and still easier to use in some areas such as the identification of nodes in networks, routing problem, etc.

3. Note that from the point of view of computational complexity, we know almost nothing about these problems (only the problem of proper coloring that distinguishes adjacent vertices by sets is known to be NP-complete, even for regular bipartite graphs).

In order to simplify the study of $\chi'_0(G)$, $\chi'_s(G)$ and $c(G)$, we introduce the following notation: given a set S of positive integers, we denote by $diam(S)$ the diameter of S, where $diam(S) = max\{x - y : x, y \in S\}$. It is clear that for every two sets S_1 and S_2, if $diam(S_1) \neq diam(S_2)$, then $S_1 \neq S_2$. Hence, we can conclude that the diameter of sets may help the vertex distinguishing property by sets or multisets. Hence, we look to extend the vertex distinguishing edge coloring problems by introducing the notion of diameter of sets, which is the subject of the next chapter.

Parameter	Exact values	Upper bounds
$\chi'_s(G)$	paths, cycles, complete graphs, bipartite graphs, wheel graphs, chordal graphs, union of paths and union of cycles.	**General graphs:** $\chi'_s(G) \leq n+1$, **Some graph families:** trees, graphs with minimum degree $\sigma(G) > \frac{n}{3}$, graphs with $\sigma \geq 5$ and $\triangle < \frac{n(2c-1)-4}{3}$ where c is a constant with $\frac{1}{2} < c \leq 1$, cubic graphs, graphs contains both paths and cycles.
$c(G)$	paths, cycles, wheel graphs, complete graphs, complete bipartite graphs and union of paths	**General graphs:** $c(G) \leq n - 1$, **Some graph families:** trees and r-regular graphs.
$\chi'_0(G)$	paths, cycles, complete graphs, union of paths, bipartite graphs $K_{m,n}$ with $m \leq 10$ or $n \geq 8m^2 - 2m + 1$ and wheel graphs.	**General graphs:** $\chi'_s(G) \leq n+1$, **Some graph families:** bipartite graphs and complete bipartite graphs.

Table 3.2: Summary of some important results about $\chi'_s(G)$, $c(G)$ and $\chi'_0(G)$.

3.3 Conclusion

In this chapter, we have presented a short survey on vertex distinguishing edge coloring. In particular, we detailed three parameters that will be useful to study the coloring problem that will be detailed the next chapter.

Chapter 4

Gap Vertex-Distinguishing Edge Colorings of Graphs

Contents

In this chapter, we define and study a new variation of the vertex-distinguishing edge coloring problem. It consists in an edge-coloring of a graph G which induces a vertex distinguishing labeling of G such that the label of each vertex is given by the difference between the highest and the lowest colors of its adjacent edges. The minimum number of colors required for a gap vertex-distinguishing edge coloring of G is called the *gap chromatic number* of G and is denoted by $gap(G)$.

We study in this chapter the gap chromatic number for a large set of graphs G of order n and we even prove that $gap(G) \in \{n-1, n, n+1\}$.

4.1 Definitions and preliminary results

As mentioned in the previous chapter, the problem of vertex-distinguishing edge coloring offers many variants and received a great interest during these last years. The aim of this chapter is to introduce a new variant of vertex-distinguishing edge coloring called *gap vertex-distinguishing edge coloring*, which is defined as follows:

Definition 4.1 *Let G be a graph, k be a positive integer and f be a mapping from $E(G)$ to the set $\{1, 2, ..., k\}$. For each vertex v of G, the label of v is defined as*

$$l(v) = \begin{cases} f(e)_{e \ni v} & \text{if } d(v) = 1 \\ \max_{e \ni v} f(e) - \min_{e \ni v} f(e) & \text{otherwise} \end{cases}$$

The mapping f is called a gap vertex-distinguishing edge coloring if distinct vertices have distinct labels. Such a coloring is called a gap-k-coloring.

The minimum positive integer k for which G admits a gap-k-coloring is called the *gap chromatic number* of G and is denoted by $gap(G)$. Necessary and sufficient conditions for the existence of such a coloring are given by the following proposition:

Proposition 4.1 *A graph G admits a gap vertex-distinguishing edge coloring if and only if it has no connected component isomorphic to K_1 or K_2.*

Proof. Since no isolated vertex of a graph G is assigned a label in an edge coloring of G, we may assume that G has no isolated vertex. Furthermore, if G contains a connected component K_2, then the two vertices of K_2 are assigned the same label in any edge coloring of G. Hence, when considering the gap vertex-distinguishing edge coloring of G, we may assume that the order of every connected component of G is at least 3. Let G be such a graph and let $E(G) = \{e_1, e_2, ..., e_m\}$. The following edge coloring function: $f(e_i) = 2^{i-1}$ for $1 \leqslant i \leqslant m$ clearly induces a gap vertex-distinguishing edge coloring of G. □

The following lemma gives a lower bound on the gap chromatic number.

Lemma 4.2 *A graph G of order n and without connected component isomorphic to K_1 or K_2 satisfies $gap(G) \geq n-1$. Moreover, if $\delta(G) \geq 2$ or if any vertex of degree greater than 1 has at least two neighbors of degree one, then $gap(G) \geq n$.*

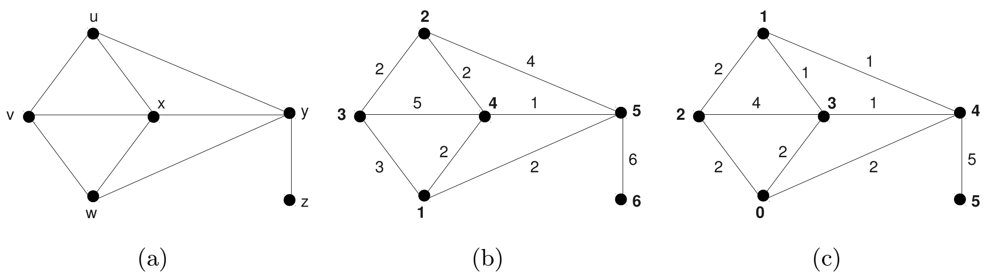


Figure 4.1: A gap vertex-distinguishing edge coloring of a graph.

To illustrate these concepts, consider the graph G shown in Figure 4.1(a). A 6-edge coloring f_1 of G is given in Figure 4.1(b) and a 5-edge coloring f_2 of G is given in Figure 4.1(c). For example, in Figure 4.1(b), the vertex w is incident to two edges colored 2 and one edge colored 3, then $l_1(w) = 1$, while the vertex z is incident with one edge colored 6, then $l_1(z) = 6$. The resulting vertex labels are distinct for both figures. By Lemma 4.2, we have $gap(G) \geq 5$, hence we can immediately conclude that $gap(G) = 5$.

After a strong analysis of this problem, we raised the conjecture asserting that there is no graph G of order n with $gap(G) > n + 1$.

Conjecture 4.3 *For every connected graph G of order $n \geq 3$, we have $gap(G) \in \{n-1, n, n+1\}$.*

In the following sections, we prove this conjecture for a large set of graphs and we even decide the exact value of $gap(G)$. The rest of this chapter is organized as follows: first, we give in Section 4.2 some motivations to investigate this new parameter. The results of Section 4.3 will confirm our conjecture for a large part of graphs with minimum degree at least 2. In Section 4.4, we prove our conjecture for some classes of graphs with minimum degree 1, such as paths, complete binary trees and all trees with at least two leaves at distance 2. This classification of our results according to $\delta(G)$ is due to the definition of our parameter, especially to the definition of labels of vertices of degree one. Finally, concluding remarks are given in the last section.

4.2 Motivation

In this section, we describe the motivation to study the gap coloring problem. First, we give the following proposition, where we recall that the diameter of a set S, denoted by $diam(S)$ is the largest distance between any two points of the set.

Proposition 4.4 *Let S_1 and S_2 be two sets of positive integers, if $diam(S_1) \neq diam(S_2)$, then $S_1 \neq S_2$.*

From the gap vertex labeling function (Definition 4.1), we observe that the label of every vertex v with degree at least 2 is the diameter of the set of colors incident to v. Note that this is not the case for the vertices of degree 1. Then, the gap labeling of a graph G can be seen as a strong version of set and multisets labelings (defined in the previous chapter). Indeed, according to Proposition 4.4, a gap distinguishing labeling of a graph G is also a multiset distinguishing labeling of G and a set distinguishing labeling (if $\delta(G) > 1$). Hence, we now characterize the relationship between our coloring parameter and the two coloring parameters $\chi'_0(G)$ and $c(G)$ defined previously. The following results follows from Proposition 4.4 and the definitions of $\chi'_0(G)$ and $c(G)$.

Lemma 4.5 *For every graph G without components isomorphic to either K_1 or K_2 and with minimum degree at least 2, we have*

$$\chi'_0(G) \leq gap(G)$$

Lemma 4.6 *For every graph G, without components isomorphic to either K_1 or K_2, we have*

$$c(G) \leq gap(G)$$

We will see in Corollary 4.17 how the results of our parameter can be connected to the study of $\chi'_0(G)$.

4.3 Gap chromatic number of graphs with minimum degree at least two

The main result of this section is the following:

Theorem 4.7 *For every m-edge-connected graph G of order n with $m \geq 2$,*

$$gap(G) = \begin{cases} n & \text{if } G \text{ is not a cycle of length} \equiv 2, 3 (mod\ 4) \\ n+1 & \text{otherwise} \end{cases}$$

The proof of Theorem 4.7 is the combination of several results detailed below.

4.3.1 Gap chromatic number of cycles

Theorem 4.8 *Let C_n be a cycle of order n, then*

$$gap(C_n) = \begin{cases} n & \text{if } n \equiv 0, 1 (mod\ 4) \\ n+1 & \text{otherwise} \end{cases}$$

Proof. Let $C_n = (v_1, v_2, \ldots, v_n, v_{n+1} = v_1)$. For each integer i with $1 \leq i \leq n$, let $e_i = v_i v_{i+1}$. We consider two cases as follows:

Case 1: $n \equiv 0, 1(mod\ 4)$. By Lemma 4.2, we have $gap(C_n) \geq n$, it then suffices to prove that C_n admits a gap-n-coloring. Two subcases are considered:
Subcase 1.1: $n \equiv 0(mod\ 4)$. A mapping f from $E(C_n)$ to $\{1, 2, \ldots, n\}$ is defined as follows (see Figure 4.2(a)).

$$\text{For } 1 \leq i \leq n, f(e_i) = \begin{cases} n+1-i & \text{if } i \text{ is odd} \\ 1 & \text{if } i \equiv 2(mod\ 4) \\ 2 & \text{if } i \equiv 0(mod\ 4) \end{cases}$$

This mapping induces the following gap vertex labeling function:

$$\text{For } 1 \leq i \leq n, l(v_i) = \begin{cases} n-i+1 & \text{if } i \equiv 2(mod\ 4) \\ n-i & \text{if } i \equiv 0, 3(mod\ 4) \\ n-i-1 & \text{if } i \equiv 1(mod\ 4) \end{cases}$$

Then, it is easy to check that l is a bijection from $V(C_n)$ to $\{0, 1, \ldots, n-1\}$. Hence $gap(C_n) = n$.
Subcase 1.2: $n \equiv 1(mod\ 4)$. A mapping f from $E(C_n)$ to $\{1, 2, \ldots, n\}$ is defined as follows (see Figure 4.2(b)):

$$\text{For } 1 \leq i \leq n, f(e_i) = \begin{cases} i & \text{if } i \text{ is odd} \\ n-1 & \text{if } i \equiv 2(mod\ 4) \\ n & \text{if } i \equiv 0(mod\ 4) \end{cases}$$

This mapping induces the following gap vertex labeling function:

$$\text{For } 1 \leq i \leq n, l(v_i) = \begin{cases} n-i & \text{if } i \equiv 1,2 (mod\ 4) \\ n-i+1 & \text{if } i \equiv 0 (mod\ 4) \\ n-i-1 & \text{if } i \equiv 3 (mod\ 4) \end{cases}$$

Then, it is easy to check that l is a bijection from $V(C_n)$ to $\{0, 1, \ldots, n-1\}$. Hence $gap(C_n) = n$.

Case 2: $n \equiv 2, 3(mod\ 4)$. We first prove that $gap(C_n) > n$. Let $f : V(C_n) \longrightarrow \{1, 2, \ldots, n\}$ be any edge-coloring of C_n which induces a gap vertex-distinguishing function l. Now note that:

$$\sum_{i=1}^{n} l(v_i) = \mid f(e_1) - f(e_n) \mid + \sum_{i=2}^{n} \mid f(e_i) - f(e_{i-1}) \mid = \frac{n(n-1)}{2}$$

In this formula, each term $f(e_i)$ appears twice with opposite (or same) signs, hence $\frac{n(n-1)}{2}$ is even. But this latter value is odd if $n \equiv 2, 3(mod\ 4)$, which is a contradiction. Thus, $gap(C_n) \geq n+1$. It then remains to show that $gap(C_n) \leq n+1$. Two subcases are considered according to whether $n\ mod\ 4 = 2$ or 3.
Subcase 2.1: $n \equiv 3(mod\ 4)$. We know that C_{n+1} admits a gap-$(n+1)$-coloring. Necessarily, C_{n+1} must contain two successive edges of same color j where $1 \leq j \leq n+1$. By merging these two edges into a single edge colored by j, we obtain a gap-$(n+1)$-coloring of C_n (see Figure 4.2(c)).
Subcase 2.2: $n \equiv 2(mod\ 4)$. In this subcase, we define an edge coloring f from $E(C_n)$ to $\{1, 2, \ldots, n, n+1\}$ by (see Figure 4.2(d)) : $f(e_n) = f(e_{n-1}) = 2$, $f(e_{n-2}) = 3$ and

$$\text{For } 1 \leq i \leq n-3, f(e_i) = \begin{cases} n+2-i & \text{if } i \text{ is odd} \\ 1 & \text{if } i \equiv 2(mod\ 4) \\ 2 & \text{if } i \equiv 0(mod\ 4) \end{cases}$$

This mapping induces the following gap vertex distinguishing labeling:
$l(v_{n-2}) = 2$, $l(v_{n-1}) = 1$, $l(v_n) = 0$ and

$$\text{For } 1 \leq i \leq n-3, l(v_i) = \begin{cases} n-i & \text{if } i \equiv 1(mod\ 4) \\ n+2-i & \text{if } i \equiv 2(mod\ 4) \\ n+1-i & \text{if } i \equiv 0,3(mod\ 4) \end{cases}$$

Then, it is easy to check that l is a bijection from the vertex set of C_n to the set $\{0, 1, \ldots, n\} \setminus \{3\}$. Hence $gap(C_n) = n+1$. □

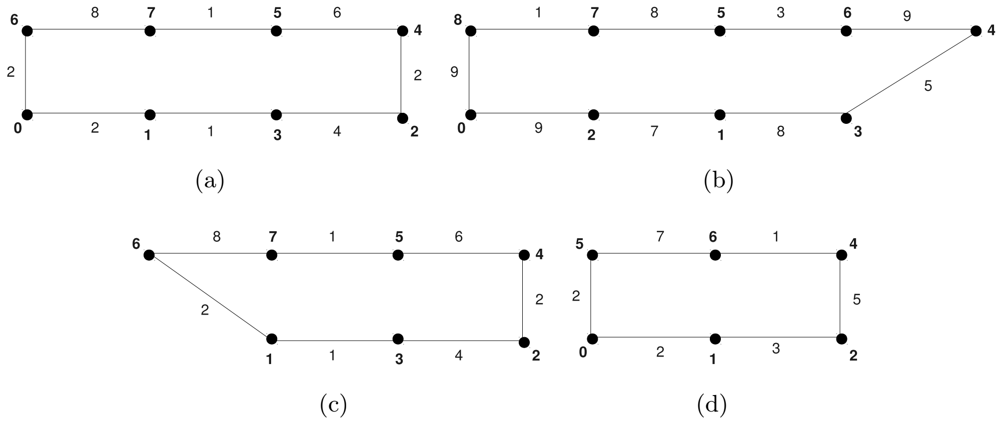


Figure 4.2: A gap vertex-distinguishing edge coloring of C_n: (a) $n = 8$, (b) $n = 9$, (c) $n = 7$, (d) $n = 6$.

4.3.2 Gap chromatic number of m-edge-connected graphs

We first introduce a definition which plays a pervasive role in this section.

Definition 4.2 *Let G be a graph of order n with $V(G) = \{v_1, v_2, \ldots, v_n\}$ and let f be an edge coloring of G. For every vertex v of G, we define the interval $I(v) = [minf(e)_{e \ni v}, maxf(e)_{e \ni v}]$. We say that f is balanced if $I(v_1) \cap I(v_2) \cap \cdots \cap I(v_n) \neq \emptyset$.*

The following proposition summarizes an important property of our coloring parameter.

Proposition 4.9 *Let G be a graph with $\delta(G) \geq 2$. If there exists a spanning subgraph H of G with $\delta(H) \geq 2$ and there exists a gap vertex-distinguishing balanced edge coloring f of H with k colors, then $gap(G) \leq k$.*

Proof. Under the stated hypothesis, the gap vertex-distinguishing labeling of H is induced by a balanced edge coloring f with k colors. Therefore, there exists at least an integer j where $1 \leq j \leq k$ such that $\forall v \in V$, we have $j \in I(v)$. By coloring the edges of $G \setminus H$ with the color j, we obtain a gap-k-coloring of G. Hence $gap(G) \leq k$. □

We illustrate the interest of Proposition 4.9 by considering the following example: let G be a Hamiltonian graph of order $n \equiv 0(mod\ 4)$. In the proof of Theorem 4.8 (Subcase 1.1), it is easy to check that the proposed edge coloring of C_n is balanced. Indeed, for each vertex v in G, we have $2 \in I(v)$. Hence, We can extend the edge-coloring of C_n to an edge-coloring of G by weighting the added edges with color 2 without affecting the gap chromatic number of C_n. Thus, for every Hamiltonian graph G of order $n \equiv 0(mod\ 4)$, we have $gap(G) = n$.

The following proposition will be useful for proving Theorem 4.7. Furthermore, it provides a useful tool for proving other results.

Proposition 4.10 *If $G = (V, E)$ is an m-edge-connected graph of order n (with $m \geq 2$), different from a cycle of length $\equiv 1, 2$ or $3(mod\ 4)$, then for every integer $a \geq 0$, there exists an $(a+n)$-edge-coloring f which induces a gap vertex-distinguishing labeling $l : V \to \{a, a+1, \dots, a+n-1\}$.*

Proof. The proof of this proposition is done by giving a polynomial-time coloring algorithm. Let us begin with some definitions and notations. For every subset S of V, let N_S denote the set of neighboring vertices of S, not included in S.

$$N_S = \{u \in V \setminus S : \exists v \in S \text{ for which } (v, u) \in E\}$$

For every two adjacent vertices u and v of G such that $v \in S$ and $u \in N_S$, let $P(v, u)$ be a function which returns a path (or cycle) from v to a vertex $w \in S$ that passes through u, such that the set of vertices between v and w does not belong to S.
Let f be an edge coloring of G. For every subgraph R of G, let $g(R)$ be a function defined on the set $E(R)$ as follows:

$$g(R) = \min\{f(E(R) \setminus \{1, 2\}\}$$

We denote by Q the set of all graphs that are isomorphic to a cycle of order multiple of 4 or to two cycles having at least one vertex in common.

Observation Every m-edge-connected graph G (with $m \geq 2$), different from a cycle of length $\equiv 1, 2$ *or* $3(mod\ 4)$ contains at least one subgraph $H \in Q$.

It is clear that if G is a 2-edge-connected graph, different from a cycle, then $\triangle(G) \geq 3$. Hence, the subgraph H can always be obtained from G. The basic idea of our algorithm is to find a balanced $(a+n)$-edge-coloring f of a 2-edge-connected spanning subgraph $G' = (V', E')$ of G. Initially, both sets V' and E' are empty. During the algorithm, the updating of V' and E' is done gradually through a specific edge coloring procedure (which is explained in more detail below). When an edge of G is colored by this procedure it is inserted into E'. A vertex $v \in V$ is inserted into V' if and only if it is incident with at least two colored edges ($e, s \in E'$). Note that when a vertex v is inserted in V', we set the label $l(v)$ as $l(v) = |f(e) - f(s)|$ and the interval $I(v)$ at $[\min(f(e), f(s)), \max(f(e), f(s))]$. Such an edge coloring will ensure that for every interval $I(v)$, we have $2 \in I(v)$.

In more details, the proposed algorithm starts by coloring the edges of a subgraph $H \in Q$ of G of order k which induces a gap vertex-distinguishing labeling of H, where the vertices of H are labeled by distinct numbers ranging from $n+a-k$ to $n+a-1$. We can easily establish this labeling structure for every subgraph H of G which is isomorphic to a member of Q. Then, we propose four edge-coloring functions to color the set of edges which constructs a cycle that has an unique vertex in V' or a path between two vertices of V'. This last step is iterated until all vertices are labeled ($i.e.$, $|V'| = |V|$).

In order to color the subgraph H, we need to define several edge-coloring functions. For a proper understanding of our algorithm, we are going to present the algorithm for a graph G which contains at least one cycle of length multiple of 4. Otherwise, all other edge-coloring functions of H are described in detail in the Appendix of [TDK12]. The different steps of the algorithm are illustrated in the example of Figure 4.3, where $a = 12$.

Algorithm 1

Input: An integer $a \geq 0$ and a m-edge-connected graph $G = (V, E)$ of order n, such that $m \geq 2$ and G is not isomorphic to a cycle of length $\equiv 1, 2$ *or* $3 (mod\ 4)$.

Output: A balanced $(a + n)$-edge-coloring f of G which induces a gap vertex-distinguishing function $l : V \rightarrow \{a, a + 1, \ldots, a + n - 1\}$.

Begin of Algorithm

Step 1: $V' \leftarrow \emptyset, E' \leftarrow \emptyset$. Let an index $t = 2$.

Step 2: Take any subgraph $H = R_1 \in Q$ of G.

2.1 If (R_1 is a cycle of length $k \equiv 0(mod\ 4)$) **Then**
Let $H = (v_1, v_2, \ldots, v_k, v_{k+1} = v_1)$. For each integer i with $1 \leq i \leq k$, let $e_i = v_i v_{i+1}$. A mapping f from $E(R_1)$ to $\{1, 2, \ldots, a+n\}$ is defined as follows:

$$\text{For } 1 \leq i \leq k, f(e_i) = \begin{cases} n + a - i + 1 & \text{if } i \text{ is odd} \\ 1 & \text{if } i \equiv 2(mod\ 4) \\ 2 & \text{if } i \equiv 0(mod\ 4) \end{cases}$$

This mapping induces the following vertex labeling of R_1:

$$\text{For } 1 \leq i \leq k, l(v_i) = \begin{cases} n + a - i + 1 & \text{if } i \equiv 2(mod\ 4) \\ n + a - i & \text{if } i \equiv 0, 3(mod\ 4) \\ n + a - i - 1 & \text{if } i \equiv 1(mod\ 4) \end{cases}$$

Then, it is easy to check that l is a bijection from the vertex set of R_1 to the set $\{n + a - 1, n + a - 2, \ldots, n + a - k\}$.
Otherwise all other edge-coloring functions of R_1 are described in detail in the Appendix of [TDK12].
2.2 $V' \leftarrow V(R_1)$, $E' \leftarrow E(R_1)$ and set $z = g(R_1)$.

Step 3: While ($V' \neq V$) **do**
Begin while
3.1 Take any two adjacent vertices u and v such that $v \in V'$ and $u \in N_{V'}$.
3.2 Let $R_t = P(v, u)$, we represent the obtained subgraph R_t by the walk $(v_1 = v, v_2 = u, \ldots, v_{k-1}, v_k)$. For each integer i with $1 \leq i \leq k - 1$, let

$e_i = v_iv_{i+1}$. We now define an edge coloring f of R_t. Four cases are considered according to the value of $k \ mod \ 4$.

Case 1: $k \equiv 0(mod\ 4)$. A mapping f from $E(R_t)$ to $\{1, 2, \ldots, a+n\}$ is defined as follows: $f(e_{k-1}) = z - k + 2$ and

$$\text{For } 1 \le i \le k-2, f(e_i) = \begin{cases} z - i & \text{if } i \text{ is odd} \\ 1 & \text{if } i \equiv 0(mod\ 4) \\ 2 & \text{if } i \equiv 2(mod\ 4) \end{cases}$$

This mapping induces the following gap vertex labeling of R_t: $l(v_{k-1}) = z - k$ and

$$\text{For } 2 \le i \le k-2, l(v_i) = \begin{cases} z - i - 1 & \text{if } i \equiv 1, 2(mod\ 4) \\ z - i - 2 & \text{if } i \equiv 3(mod\ 4) \\ z - i & \text{if } i \equiv 0(mod\ 4) \end{cases}$$

Case 2: $k \equiv 2(mod\ 4)$. A mapping f from $E(R_t)$ to $\{1, 2, \ldots, a+n\}$ is defined as follows:

$$\text{For } 1 \le i \le k-1, f(e_i) = \begin{cases} z - i & \text{if } i \text{ is even} \\ 1 & \text{if } i \equiv 3(mod\ 4) \\ 2 & \text{if } i \equiv 1(mod\ 4) \end{cases}$$

This mapping induces the following gap vertex labeling of R_t.

$$\text{For } 2 \le i \le k-1, l(v_i) = \begin{cases} z - i - 1 & \text{if } i \equiv 0, 1(mod\ 4) \\ z - i - 2 & \text{if } i \equiv 2(mod\ 4) \\ z - i & \text{if } i \equiv 3(mod\ 4) \end{cases}$$

Case 3: $k \equiv 1(mod\ 4)$. A mapping f from $E(R_t)$ to $\{1, 2, \ldots, a+n\}$ is defined as follows: $f(e_1) = z - 2$ and

$$\text{For } 2 \le i \le k-1, f(e_i) = \begin{cases} z - i & \text{if } i \text{ is odd} \\ 1 & \text{if } i \equiv 2(mod\ 4) \\ 2 & \text{if } i \equiv 0(mod\ 4) \end{cases}$$

This mapping induces the following gap vertex labeling of R_t: $l(v_2) = z - 3$ and

$$\text{For } 3 \le i \le k-1, l(v_i) = \begin{cases} z - i - 1 & \text{if } i \equiv 0, 3(mod\ 4) \\ z - i - 2 & \text{if } i \equiv 1(mod\ 4) \\ z - i & \text{if } i \equiv 2(mod\ 4) \end{cases}$$

Case 4: $k \equiv 3(mod\ 4)$. A mapping f from $E(R_t)$ to $\{1, 2, \ldots, a+n\}$ is defined as follows: $f(e_{k-1}) = z - k + 2$ and

$$\text{For } 1 \le i \le k-2, f(e_i) = \begin{cases} z - i & \text{if } i \text{ is even} \\ 1 & \text{if } i \equiv 3(mod\ 4) \\ 2 & \text{if } i \equiv 1(mod\ 4) \end{cases}$$

This mapping induces the following gap vertex labeling of R_t: $l(v_{k-1}) = z-k$, and

$$\text{For } 2 \leq i \leq k-2, l(v_i) = \begin{cases} z-i-1 & \text{if } i \equiv 0,1(mod\, 4) \\ z-i-2 & \text{if } i \equiv 2(mod\, 4) \\ z-i & \text{if } i \equiv 3(mod\, 4) \end{cases}$$

Observation: In the previous four cases, it is easy to check that l is a bijection from the vertex set $V(R_t) - \{v_1, v_k\}$ to $\{z-3, z-4, \ldots, z-k\}$.

2.3 $V' \leftarrow V' \cup V(R_t)$, $E' \leftarrow E' \cup E(R_t)$. Set $z = g(R_t)$ and $t = t+1$.
End while

Step 4: For all edges $e \in E \setminus E'$, set $f(e) = 2$.

End of algorithm.

We now present the proof of correctness of the above algorithm. We first show that this algorithm achieves its goal without blocking, *i.e.*, both actions in Step 3 (3.1 and 3.2) satisfy the following assertions:

$$\text{If } |V'| < |V| \text{ then } N_{V'} \neq \emptyset. \tag{4.1}$$

$$\begin{array}{c}\text{For every vertex } u \in N_{V'} \text{ there exists a path} \\ \text{from } u \text{ to a vertex } v \in V' \text{of order at least 2.}\end{array} \tag{4.2}$$

The assertion (1) follows from the connectivity hypothesis on G. For a vertex $u \in N_{V'}$ there exists, at last, an edge $(u, v) \in E$ such that $v \in V'$. The 2-edge-connectivity hypothesis of G implies that every edge of G belongs to a cycle, then the two vertices u and v belong to the same cycle. Therefore, the assertion (2) also holds.

We now prove that our coloring algorithm gives a gap vertex-distinguishing function $l : V' \to \{a, a+1, \ldots, a+n-1\}$ of G' induced by a balanced edge coloring f with $a+n$ colors. At the end of the loop of Step 3, we obtain a bijection l from the set V' to the set $\{a, a+1, ..., a+n-1\}$, *i.e.*, for any two vertices u, v of V', we have $l(u) \neq l(v)$. It then remains to show that f is a balanced edge-coloring and for every vertex v of V', we have $l(v)$ equal to $\max_{e \ni v} f(e) - \min_{e \ni v} f(e)$ in G'. By considering the degree in G' of each vertex v, we have two cases.
Case 1. $d(v) = 2$: from the algorithm, it is clear that the label of each vertex v of degree 2 which is incident with two edges e and s of E' is equal to $|f(e) - f(s)|$.
Case 2. $d(v) > 2$: let $R(v) = \{R_{d_1}, R_{d_2}, \ldots, R_{d_p}\}$ denote the set of all subgraphs having a common vertex v, where $d_1 \leq d_2 \leq \cdots \leq d_p$. From the algorithm, we can observe that (see Figure 4.3(g)):

- for any two subgraphs R_i and R_j of $R(v)$, we have $E(R_i) \cap E(R_j) = \emptyset$.
- v is incident with exactly two edges e_{d_1} and s_{d_1} of $E(R_{d_1})$. Let $f(e_{d_1}) \geq f(s_{d_1})$, then the label of v is fixed as $l(v) = f(e_{d_1}) - f(s_{d_1})$.

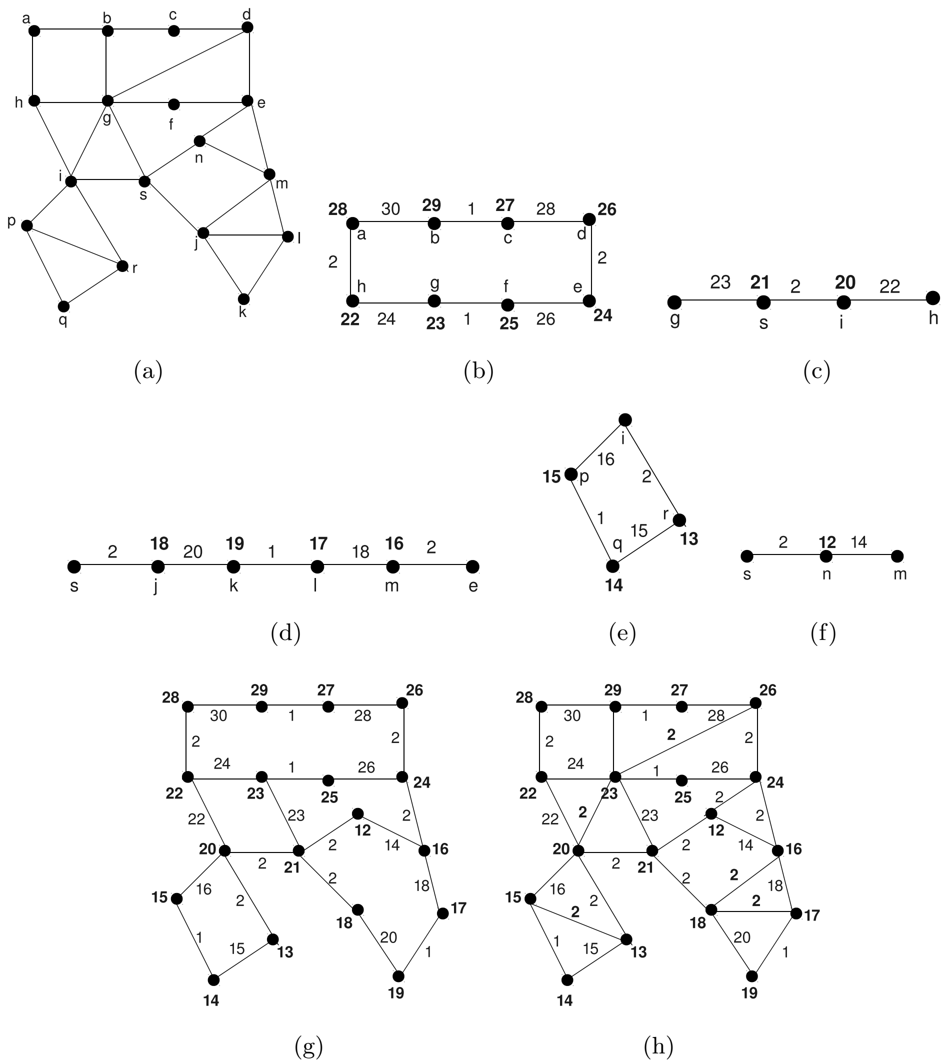


Figure 4.3: Illustration of Algorithm 1 (a=12): (a) A 2-edge-connected graph G. (b) Coloring of R_1. (c),(d),(e),(f) illustrates the coloring of R_2, R_3, R_4, R_5, respectively. (g) A balanced gap-30-coloring of a spanning subgraph G' of G. (h) A gap-30-coloring of G which induces a gap vertex-distinguishing function $l : V \to \{12, 13, \ldots, 29\}$.

- for every subgraph R_i of $R(v)$, where $i \geq d_2$, we have v is incident with one or two edges of $E(R_i)$.

Furthermore, according to the edge coloring f, we can easily see that:

- for every vertex v of G' , we have $2 \in I(v)$.
- $1 \leq f(s_{d_1}) \leq 2$ and $f(e_{d_1}) \geq g(R_{d_1}) \geq 2$.
- for every subgraph R_i of $R(v)$, where $i \geq d_2$ then $\forall e \in E(R_i)$ with $v \in e$, we have $2 \leq f(e) \leq g(R_{d_1})$.

From these observations we can conclude the following:

- the edge-coloring f is balanced.
- for every vertex v of V', $\max_{e \ni v} f(e) = f(e_{d_1})$ and $\min_{e \ni v} f(e) = f(s_{d_1})$.

At Step 4 of the algorithm, we know that the obtained edge coloring f of G' is balanced. Hence, we can extend G' to G by coloring the added edges with color 2 without affecting the vertex labeling function $l : V \to \{a, a+1, \ldots, a+n-1\}$. $\square$

Now, we can state the proof of Theorem 4.7. To proceed, we introduce the following result.

Theorem 4.11 *For every m-edge-connected graph G of order n (with $m \geq 2$), different from a cycle of length $n \equiv 2$ or $3 (mod\, 4)$, we have*

$$gap(G) = n$$

Proof. By Lemma 4.2, we have $gap(G) \geq n$. It then suffices to prove that G admits a gap-n-coloring. We know by Theorem 4.8 that if G is a cycle of length $n \equiv 0, 1 (mod\, 4)$, then $gap(G) = n$. Otherwise, it is clear by Proposition 4.10 that if we set the integer parameter a at 0, we obtain a gap-n-coloring of G induced by a balanced edge coloring. Hence $gap(G) = n$. $\square$

We can now conclude that the result of Theorem 4.7 is a direct consequence of Theorem 4.8 and Theorem 4.11.
Here we generalize the previous results to a special case of disconnected graphs as follows:

Theorem 4.12 *If G is a graph of order n with connected components $G_1, \ldots, G_t$, such that for every component G_i of G, G_i is different from a cycle of length $\equiv 1, 2, 3 (mod\, 4)$ and the edge connectivity of each component of G is at least 2, then*

$$gap(G) = n$$

Proof. Let n_i be the order of G_i $(1 \leq i \leq t)$. The proof is essentially due to Proposition 4.10. The idea is to provide a gap vertex distinguishing edge coloring for each component of G_i according to the parameter a of Proposition 4.10 as follows: by applying this proposition in sequence to $G_1, G_2, \ldots, G_t$, we can obtain the labeling function $l : V(G_i) \to \{a, a+1, \ldots, a+n_i-1\}$ induced by an edge coloring f with $a + n_i$ colors such that $a = n - \sum_{j=1}^{i} n_j$. From this, it is easy to check that l is a bijection from the vertex set of G to the set $\{0, 1, 2, \ldots, n-1\}$. Thus $gap(G) = n$. $\square$

We believe that the result of Theorem 4.7 can be extended to all graphs of minimum degree at least 2. But we have not been able to prove it. We suggest the following conjecture:

Conjecture 4.13 *For every connected graph G of order n with minimum degree $\delta(G) \geq 2$, we have*

$$gap(G) = \begin{cases} n+1 & \text{if } G \text{ is a cycle of length} \equiv 2, 3 (mod\ 4) \\ n & \text{otherwise} \end{cases}$$

4.4 Gap chromatic number of graphs with minimum degree one

In this section we give the value of $gap(G)$ for some classes of graphs with $\delta(G) = 1$.

4.4.1 Gap chromatic number of paths

Theorem 4.14 *Let P_n be the path of order n. Then*

$$gap(P_n) = \begin{cases} n-1 & \text{if } n \equiv 0, 1 (mod\ 4) \\ n & \text{otherwise} \end{cases}$$

Proof. The proof of this theorem is similar to the one of Theorem 4.8. Let $V(P_n) = (v_1, v_2, \ldots, v_n)$. For each integer i with $1 \leq i \leq n-1$, let $e_i = v_i v_{i+1}$. We consider two cases as follows:

Case 1: $n \equiv 0, 1(mod\ 4)$. By Lemma 4.2, we have $gap(P_n) \geq n-1$, it then suffices to prove that P_n admits a gap-$(n-1)$-coloring. Two subcases are considered:

Subcase 1.1: $n \equiv 0(mod\ 4)$. A mapping f from $E(P_n)$ to $\{1, 2, \ldots, n-1\}$ is defined as follows (see Figure 4.4(a)).

$$\text{For } 1 \leq i \leq n-1, f(e_i) = \begin{cases} \frac{i}{2} & \text{if } i \text{ even} \\ \frac{n-2}{2} & \text{if } i \equiv 3(mod\ 4) \\ n-1 & \text{if } i \equiv 1(mod\ 4) \end{cases}$$

This mapping induces the following vertex labeling function: $l(v_n) = \frac{n-2}{2}$

$$\text{and for } 1 \leq i \leq n-1, l(v_i) = \begin{cases} \frac{n-i-2}{2} & \text{if } i \equiv 0(mod\ 4) \\ n-1-\frac{i-1}{2} & \text{if } i \equiv 1(mod\ 4) \\ n-1-\frac{i}{2} & \text{if } i \equiv 2(mod\ 4) \\ \frac{n-i-1}{2} & \text{if } i \equiv 3(mod\ 4) \end{cases}$$

Then, it is easy to check that l is a bijection from $V(P_n)$ to $\{0, 1, \ldots, n-1\}$. Hence $gap(P_n) = n-1$.

Subcase 1.2: $n \equiv 1(mod\ 4)$. A mapping f from $E(P_n)$ to $\{1, 2, \ldots, n-1\}$ is defined as follows (see Figure 4.4(b)).

$$\text{For } 1 \leq i \leq n-1, f(e_i) = \begin{cases} \frac{i}{2} & \text{if } i \text{ even} \\ \frac{n-1}{2} & \text{if } i \equiv 3(mod\ 4) \\ n-1 & \text{if } i \equiv 1(mod\ 4) \end{cases}$$

This mapping induces the following vertex labeling function:

$$\text{and for } 1 \leq i \leq n-1, l(v_i) = \begin{cases} \frac{n-1-i}{2} & \text{if } i \equiv 0(mod\ 4) \\ n-1-\frac{i-1}{2} & \text{if } i \equiv 1(mod\ 4) \\ n-1-\frac{i}{2} & \text{if } i \equiv 2(mod\ 4) \\ \frac{n-i}{2} & \text{if } i \equiv 3(mod\ 4) \end{cases}$$

Then, it is easy to check that l is a bijection from $V(P_n)$ to $\{0, 1, \ldots, n-1\}$. Hence $gap(P_n) = n-1$.

Case 2: $n \equiv 2, 3(mod\ 4)$. We first prove that $gap(P_n) > n-1$. Let $f : V(P_n) \longrightarrow \{1, 2, \ldots, n-1\}$ be any edge-coloring of P_n which induces a gap vertex-distinguishing labeling l. We note that:

$$\sum_{i=1}^{n} l(v_i) = f(e_1) + f(e_{n-1}) + \sum_{i=2}^{n-1} \mid f(e_i) - f(e_{i-1}) \mid = \frac{n(n-1)}{2}$$

In this formula, each term $f(e_i)$ appears twice with opposite (or same) signs, hence $\frac{n(n-1)}{2}$ is even. But this latter value is odd if $n \equiv 2, 3(mod\ 4)$, which is a contradiction. Thus, $gap(P_n) \geq n$. It then remains to show that $gap(P_n) \leq n$. Two subcases are considered according to whether $n\ mod\ 4 = 2$ or 3.

Subcase 2.1: $n \equiv 3(mod\ 4)$. We know that P_{n+1} admits a gap-n-coloring. Necessarily P_{n+1} must contain two successive edges of same color j where $1 \leq j \leq n$. By merging these two edges into a single edge colored by j, we obtain a gap-n-coloring of P_n (see Figure 4.4(c)).

Subcase 2.2: $n \equiv 2(mod\ 4)$ In this subcase, we define an edge coloring f from $E(P_n)$ to $\{1, 2, \ldots, n\}$ (see Figure 4.4(d)) by $f(e_{n-1}) = n-1$ and

$$\text{For } 1 \leq i \leq n-2, f(e_i) = \begin{cases} \frac{i}{2}+1 & \text{if } i \text{ even} \\ \frac{n}{2} & \text{if } i \equiv 3(mod\ 4) \\ n & \text{if } i \equiv 1(mod\ 4) \end{cases}$$

This mapping induces the following gap vertex distinguishing labeling: $l(v_1) = n$, $l(v_{n-1}) = \frac{n}{2} - 1$,$l(v_n) = n-1$ and

$$\text{for } 2 \leq i \leq n-2, l(v_i) = \begin{cases} \frac{n-i}{2} - 1 & \text{if } i \equiv 0(mod\ 4) \\ n - \frac{i+1}{2} & \text{if } i \equiv 1(mod\ 4) \\ n - \frac{i}{2} - 1 & \text{if } i \equiv 2(mod\ 4) \\ \frac{n-i-1}{2} & \text{if } i \equiv 3(mod\ 4) \end{cases}$$

Then, it is easy to check that l is a bijection from $V(P_n)$ to $\{0, 1, \ldots, n\} \setminus \{\frac{n}{2}\}$. Hence $gap(P_n) = n$. □

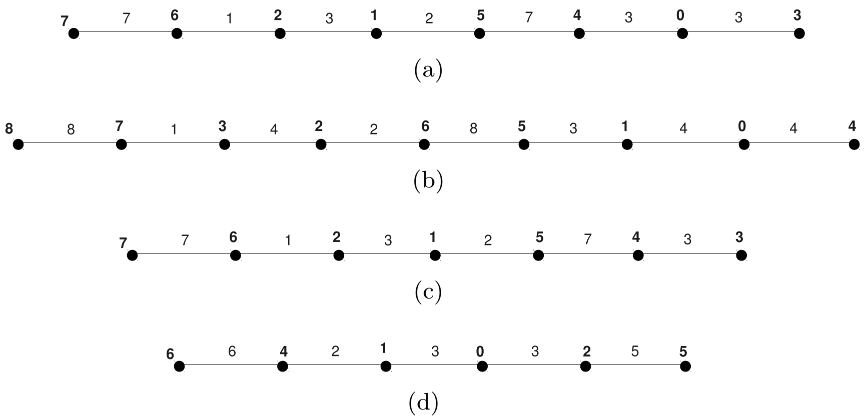


Figure 4.4: A gap-coloring of P_n: (a) $n = 8$, (b) $n = 9$, (c) $n = 7$, (d) $n = 6$.

4.4.2 Gap chromatic number of trees

We first determine the gap chromatic number of complete binary trees. We denote by BT_h a complete binary tree of height $h > 0$, note that BT_h has exactly $2^{h+1} - 1$ vertices.

Theorem 4.15 *For any complete binary tree BT_h of order $n = 2^{h+1}$, we have*

$$gap(BT_h) = n - 1$$

Proof. By Theorem 4.14, we have $gap(BT_1) = gap(P_3) = 3$. Then, we may restrict our attention to $h \geq 2$. By Lemma 4.2, we have $gap(BT_h) \geq n - 1$, it then suffices to prove that BT_h admits a gap-$(n-1)$-coloring. We define the level $l(u)$ of vertex u of BT_h as the number of edges along the unique path between it and the root. Similarly, the level of an edge $e = (u, v)$ of BT_h is $l(e) = max\{l(u), l(v)\}$. We represent the vertices and the edges of BT_h, level by level, left to right by the sequence $v_1, v_2, \ldots, v_n$ and $e_1, e_2, \ldots, e_{n-1}$, respectively (see Figure 4.5(a)). We now define a mapping f from $E(BT_h)$ to $\{1, 2, \ldots, n-1\}$ as follows.

$$\text{For } 1 \leq i \leq n-1, f(e_i) = \begin{cases} 2h & \text{if } i \leq 2 \\ i + 2(h - l(e_i)) & \text{if } i \geq 3 \end{cases}$$

This mapping induces the following gap vertex labeling: $l(v_i) = i - 1$ for $1 \leq i \leq n$. Then it is easy to check that l is a bijection from $V(BT_h)$ to $\{0, 1, \ldots, n-1\}$. Thus $gap(BT_h) = n - 1$. □

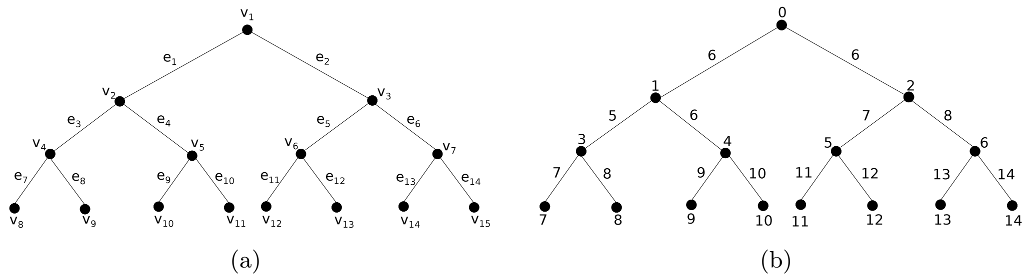


Figure 4.5: (a) Notation of BT_3, (b) A gap-14-coloring of BT_3

The following theorem gives an upper bound on the gap chromatic number of general trees.

Theorem 4.16 *Let $T = (V, E)$ be a tree of order n which has at least two leaves u and v at distance two, then*

$$gap(T) \leq n$$

Proof. The proof of this theorem is done by giving a polynomial-time algorithm. We first start with some definitions used in the following. Let $R_1 = (u, w, v)$ be a path of T (where u, v are leaves) and let R be the subtree of T rooted at w and induced by the set $V \setminus \{u, v\}$ (see Figure 4.6(a)). Let h be the depth of R. For every level i of R, let L_i denote the set of leaves at level i. Let S be a subset of $V(R)$ and for every vertex x of $V(R) \setminus S$, let $P(x, S)$ be the function which returns a path from x to a vertex $y \in S$, such that the set of vertices between x and y does not belong to S.

Let l be a vertex labeling of $V(T)$. For every path P of T, let $g(P)$ be the function defined as follows:

$$g(P) = \min\{l(v) : \forall v \in V(P)\}$$

The different steps of Algorithm 2 are illustrated in the example of Figure 4.6.

Algorithm 2

Input: A tree $T = (V, E)$ of order n with two leaves u and v at distance 2.

Output: A gap-n-coloring of T.

Begin of Algorithm

Set $f : E(R_1) \to \{1, n\}$ as follows: $f(vw) = n$, $f(uw) = 1$.

This mapping induces the following gap vertex labeling of R_1: $l(v) = n$, $l(w) = n-1$ and $l(u) = 1$.

Let $S = \{w\}$, $z = n - 1$ and $t = 2$.

For $i = 1$ to h **do**

Begin For

For every vertex x of L_i in the subtree R **do**

Begin For

Let $R_t = P(x, S)$. We denote the path R_t by the sequence of vertices $v_1 = x, v_2, \ldots v_{k-1}, v_k$. For each integer i with $1 \leq i \leq k-1$, let $e_i = v_i v_{i+1}$. Set an edge coloring f of R_t as follows:

$$\text{For } 1 \le i \le k-1, f(e_i) = \begin{cases} z - \frac{i+1}{2} & \text{if } i \text{ odd} \\ \frac{i}{2} & \text{otherwise} \end{cases}$$

This mapping induces the following gap vertex labeling of R_t.

$$\text{For } 1 \le i \le k-1, l(v_i) = z - i$$

$S \leftarrow S \cup V(R_t)$, $z \leftarrow g(R_t)$, $t \leftarrow t+1$.

End for

End for

End of Algorithm

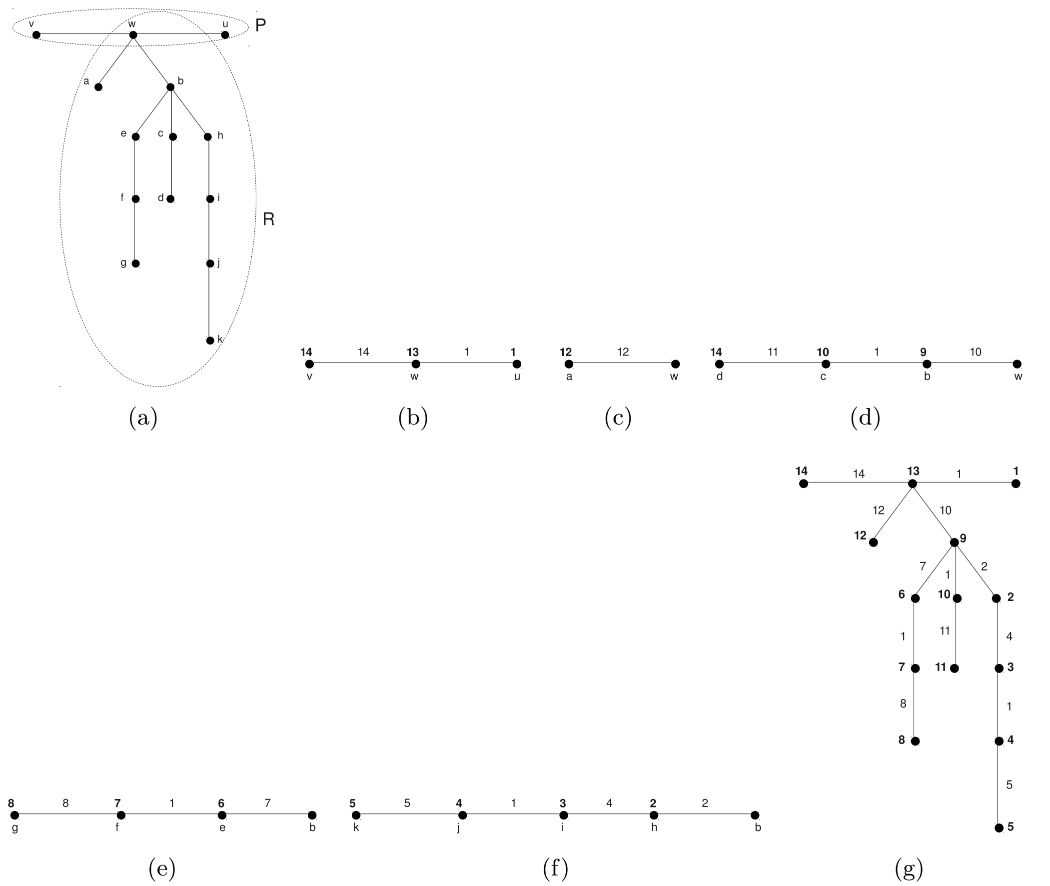


Figure 4.6: Illustration of Algorithm 2: (a) A tree T. (b) Coloring of R_1. (c),(d),(e),(f) illustrates the coloring of R_2, R_3, R_4, R_5, respectively. (g) A gap-14-coloring of T.

Now, we present the proof of correctness for the above algorithm. At the end of this algorithm, we obtain a bijection l from V to the set $\{1, 2, ..., n\}$. It then remains

to show the property of our coloring parameter. By considering the degree of each vertex v of T, we have three cases:
Case 1. $d(v) = 1$: from the algorithm, it is clear that $l(v) = f(e)_{e \ni v}$.
Case 2. $d(v) = 2$: from the algorithm, it is clear that the label of vertex v of degree 2 which is incident with two edges e and s of E equals $|f(e) - f(s)|$.
Case 3. $d(v) > 2$: let $R(v) = \{R_{d_1}, R_{d_2}, \ldots, R_{d_p}\}$ denote the set of all paths having the vertex v in common, where $d_1 \leq d_2 \leq \cdots \leq d_p$. We represent the distance between two vertices $x, y \in V$ by $dist(x, y)$. From the algorithm, we can observe that:

- every path R_i of $R(v)$ contains a leaf l_i of T which is an endpoint of R_i. We can see that $dist(v, l_{d_1}) \leq dist(v, l_{d_2}) \leq \cdots \leq dist(v, l_{d_p})$.
- for any two paths R_i and R_j of $R(v)$, $E(R_i) \cap E(R_j) = \emptyset$.
- the vertex v is incident with exactly two edges e_{d_1} and s_{d_1} of $E(R_{d_1})$. Let $f(e_{d_1}) \geq f(s_{d_1})$, then the label of v is fixed as $l(v) = f(e_{d_1}) - f(s_{d_1})$.
- for every path R_i of $R(v)$, where $i \geq d_2$, the vertex v is incident to exactly one edge e_i of $E(R_i)$.

Furthermore, according to the edge-coloring f, we can see that:

- $f(s_{d_1}) = \lceil \frac{dist(v, l_{d_1})}{2} \rceil$.
- for every path R_i of $R(v)$, where $i \geq d_2$, we consider two cases for the value of $f(e_i)$ (with $v \in e_i$) according to the distance between v and l_i:
  - *$dist(v, l_i)$ is even.* We have $f(e_i) = \frac{dist(v, l_i)}{2}$. Hence $f(s_{d_1}) \leq f(e_i) \leq g(R_{d_1}) \leq f(e_{d_1})$.
  - *$dist(v, l_i)$ is odd.* We have $f(e_i) = g(R_i) + \frac{dist(v, l_i) - 1}{2}$. Hence $f(s_{d_1}) \leq f(e_i) \leq g(R_{d_1}) \leq f(e_{d_1})$.

From these observations, we can conclude that for every vertex v of V, $f(e_{d_1}) = \max_{e \ni v} f(e)$ and $f(s_{d_1}) = \min_{e \ni v} f(e)$. Hence, T admits a gap-n-coloring. □

4.5 Conclusion

In this chapter, we studied a new variant of graph edge coloring that induces a vertex distinguishing labeling. Exact results are given for paths, cycles, some trees and all m-edge-connected graphs with $m \geq 2$. Our results are summarized in the follwoing table.

Graph classes	**gap(G)**
Cycle C_n of order $n \equiv 2, 3(mod\ 4)$	$gap(C_n) = n + 1$
Cycle C_n of order $n \equiv 0, 1(mod\ 4)$	$gap(C_n) = n$
m-edge connected graph G (with $m \geq 2$) different from a cycle	$gap(G) = n$
Path P_n of order $n \equiv 2, 3(mod\ 4)$	$gap(P_n) = n$
Path P_n of order $n \equiv 0, 1(mod\ 4)$	$gap(P_n) = n - 1$
complete binary tree BT_h of order $n = 2^{h+1}$	$gap(BT_h) = n - 1$
Tree T of order n which has at least two leaves u and v at distance two	$gap(G) \leq n$

Table 4.1: Our summary results on the gap chromatic number

The study of the relationships between our parameter and the point distinguishing problem gives the following result which is a direct consequence of Lemma 4.5 and Theorem 4.12.

Corollary 4.17 *If G is a graph of order n with connected components $G_1, \ldots, G_t$, such that for every component G_i of G, G_i is different from a cycle of length $\equiv 1, 2, 3(mod\ 4)$ and the edge connectivity of each component of G is at least 2, then $\chi'_0(G) \leq n$.*

Part II

Graph Packing Problem

Chapter 5
Labeled Packing of Graphs

Contents

In this chapter, we focus on graph packing problems, which are a well-known field of graph theory for over 30 years. We start in Section 5.1 by presenting a variety of results, some quite recent, concerning the packing of unlabeled graphs. We then discuss the relationship between the embedding of graphs and the structure of the embedding permutations. In Section 5.2, we introduce and study a new variant of the graph packing problem, called the *labeled packing of graphs*. In particular, we present some results and conjectures about cycles.

5.1 Packing and embedding of unlabeled graphs

5.1.1 Basic definitions

We here consider only finite simple graphs and use standard terminology notation from Chapter 2 except when indicated. We recall that if a graph G has order n and size m, we say that G is an (n,m)*-graph*. For more brevity, we will call an unlabeled graph simply a graph.

A *permutation* σ is an one-to-one mapping of a set S into itself. We say that an element e of S is a *fixed point* of a given permutation σ if $\sigma(e) = e$. Then, a permutation σ is called *fixed-point-free* if σ has no fixed point. Any permutation σ of a finite set can be written as the disjoint union of cycles (two cycles are disjoint if they do not have any common element). Here, a cycle $(a_1, a_2, \ldots, a_n)$ is a permutation sending a_i to a_{i+1} for $1 \leq i \leq n-1$ and a_n to a_1. This representation is called the *cyclic decomposition of* σ and is denoted by $C(\sigma)$. In this context, the cycles of length one correspond to fixed points of σ.

The study of graph packing was initiated in the 1970s by Bollobás and Eldridge [Bol78, BE78], Sauer and Spencer [SS78], and Catlin [Cat74].

Definition 5.1 *Let $G_1, G_2, \ldots, G_k$ be k graphs of order n. We say that there is a packing of $G_1, \ldots, G_k$ (into the complete graph K_n) if there exist k permutations $\sigma_i : V(G_i) \longrightarrow V(K_n)$, $i = 1, \ldots, k$, such that $\sigma_i^*(E(G_i)) \cap \sigma_j^*(E(G_j)) = \emptyset$ for $i \neq j$, where the mapping $\sigma_i^* : E(G_i) \longrightarrow E(K_n)$ is the one induced by σ_i.*

By definition, two graphs, G_1 and G_2, of the same order *pack* if G_1 is a subgraph of the complement $\overline{G_2}$ of G_2, or, equivalently, if G_2 is a subgraph of the complement $\overline{G_1}$ of G_1. A packing of k copies of the same graph G into K_n will be called a k*-placement* of G. A packing of two copies of G in K_n (*i.e.,* a 2-placement) is called an *embedding* of G (into its complement $\overline{G}$). In other words, an embedding of a graph G is a permutation σ on $V(G)$ such that if an edge vu belongs to $E(G)$ then $\sigma(v)\sigma(u)$ does not belong to $E(G)$. If there exists an embedding of G into K_n, we say that G is *embeddable.* For example, the following figure presents a packing of two copies of $P_4 \cup P_3$ into K_7, where the dotted edges represent the second copie of $P_4 \cup P_3$. This embedding can also be represented by the following cyclic decomposition: $(x_2)(x_7)(x_4)(x_1, x_6)(x_3, x_5)$.

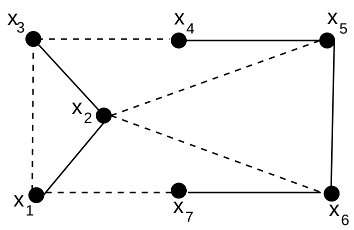


Figure 5.1: A packing of two copies of $P_4 \cup P_3$ into K_7.

Many problems in graph theory could be formulated as graph packing problems. Here are some relevant examples.

Example 5.1 *The question of the existence of an Hamiltonian cycle in a graph G of order n is equivalent to the question whether the cycle C_n and $\overline{G}$ are packable into K_n.*

Example 5.2 *Given two graphs $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ with $|V(G_1)| = |V(G_2)|$, G_1 is isomorphic to G_2 if and only if (1) G_1 and $\overline{G}_2$ are packable into K_n and (2) $\overline{G}_1$ and G_2 are packable into K_n.*

Example 5.3 *The independence number $\alpha(G)$ ($\alpha(G)$ represents the size of the largest independent set of G) of a graph G of order n is at least p if and only if G and the graph $K_p \cup (n-p)K_1$ (i.e., the graph consisting of the k-clique and $n-p$ isolated vertices) are packable into K_n.*

In 1959, Erdős and Gallai [EG59] proved that every graph G with n vertices and more than $\frac{n(k-1)}{2}$ edges contains a path of length k. Motivated by this result, Erdős and Sós made the following conjecture in 1963.

Conjecture 5.1 *If G is a graph of order n and $E(G) > \frac{n(k-1)}{2}$, then G contains every tree T of size k.*

Example 5.4 *We can easily see that if $|E(G)| > \frac{n(k-1)}{2}$ then $|E(\overline{G})| < \frac{n(n-k)}{2}$. Thus, Conjecture 5.1 can be formulated as follows: let G be a graph of order n and T any tree of order k, with $k < n$. If $|E(G)| < \frac{n(n-k)}{2}$ then there exists a packing of G and T into K_n.*

Packing of graphs is a heavily studied subject having many applications in computer science (see e.g. [BE78, Haj91]). The reader can find a good survey on packing of graphs in [Woz04] and [Yap88]. In general, the review of existing literature identifies three different lines of research. The first one is about packing of general graphs having small size in K_n(e.g. (n, m)-graphs with $m \leq n$). The second one deals with packing of graphs with bounded maximum degree and bounded girth in K_n. The last one considers some additional properties of permutations in the packing construction. We present in the following some important results in these areas.

5.1.2 Packing of graphs of small size in a complete graph

In 1978, Bollobás and Eldridge [BE78] made the following conjecture about the existence of a k-packing of graphs of small size, which is generally considered to be one of the most important open problems in graph packing theory.

Conjecture 5.2 ([BE78]) *Let $G_1, G_2 \ldots G_k$ be k graphs of order n. If $|E(G_i)| \leq n-k$, $i = 1, 2, \ldots k$, then $G_1, G_2, \ldots G_k$ are packable into K_n.*

So far only the cases $k = 2$ and $k = 3$ of this conjecture were proved in [SS78] and [KMSW01], respectively. This shows the hardness of the problem and motivates future research. In what follows, we survey some of the well known results relative to the number of graphs to pack.

<u>**Packing of two graphs**</u>

The following theorem was independently proposed in [Bol78, BS77, SS78].

Theorem 5.3 ([Bol78, BS77, SS78]) *Let $G = (V, E)$ be a graph of order n. If $|E(G)| \leq n-2$ then G can be embedded in its complement.*

The example of the star S_n shows that Theorem 5.3 cannot be improved by raising the size of G even in the case when G is a tree. However, in this case we have the following theorem, proved in [BS78], which completely characterizes the embedding of $(n, n-1)$ graphs.

Theorem 5.4 ([BS78]) *Let $G = (V, E)$ be a graph of order n. If $|E(G)| \leq n-1$ then either G is embeddable in its complement or G is isomorphic to one of the following graphs: $K_{1,n-1}, K_{1,n-4} \cup K_3$ with $n \geq 8$, $K_1 \cup K_3$, $K_2 \cup K_3$, $K_1 \cup 2K_3$, $K_1 \cup C_4$.*

Among the exceptions mentioned in this theorem, the star $K_{1,n-1}$ is the only graph which is connected. Therefore, we have the following immediate consequence (proved also in [HHS81]).

Theorem 5.5 ([HHS81]) *Let T be a tree of order n, $T \neq K_{1,n-1}$. Then T is contained in its own complement.*

These results has been improved in many ways. In Section 5.1.4, we are interested in improvements that deal with some additional properties of the permutation structure.

Packing of three graphs

We are now going to consider the problem of packing of three graphs. We begin with a theorem about the 3-placement of $(n, n-2)$-graphs, proved in [WW93], which can be considered as an extension of Theorem 5.3.

Theorem 5.6 ([WW93]) *Let $G = (V, E)$ be a graph of order n. If $|E(G)| \leq n-2$ then either there exists a 3-placement of G or G is isomorphic to $K_3 \cup 2K_1$ or to $K_4 \cup 4K_1$.*

The following theorem refines Theorem 5.6 by specifying the permutation structure of the 3-placement of G (see [Woz94]).

Theorem 5.7 ([Woz94]) *Let $G = (V, E)$ be a graph of order n, $G \neq K_3 \cup 2K_1, K_4 \cup 4K_1$. If $|E(G)| \leq n-2$, then there exists a permutation σ on $V(G)$ such that σ^1, σ^2, σ^3 define a 3-placement of G. Moreover, all cycles of σ have length 3, except for 1 of length one if $n \equiv 1 (mod\ 3)$ or two of length 1 if $n \equiv 2 (mod\ 3)$.*

Now, consider the packing of three copies of a tree. Obviously, if there is a packing of three copies of T into K_n, then we have

$$3(n-1) \leq \frac{n(n-1)}{2}$$

which implies $n \geq 6$. Furthermore, since every vertex v of T with degree $d(v) = \triangle(T)$ must be mapped with two other vertices with degree at least one, then, we must assume that $\triangle(T) \leq n-3$. However, Huang and Rosa [HR78] observed that these two necessary conditions are not sufficient as shown by the graph of Figure 5.2. In [WS93], Wang and Sauer characterized all the trees T such that T is 3-placeable. The packing of three different trees into K_n has been considered by Wozniak [Woz96], who proved the following theorem.

Theorem 5.8 ([Woz96]) *Let T_1, T_2, T_3 be three trees of order n. Then there is a packing of T_1, T_2, T_3 into K_{n+1}.*

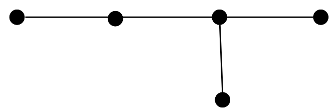


Figure 5.2: Example of a not 3-placeable tree

Packing of k graphs

To the best of our knowledge, only two graph packing results are known for the general case [BW04, Zak11]. In [BW04], Brandt and Wozniak considered another special case of Conjecture 5.2. In particular, they considered the packing of k copies of a tree into K_n, where $k = \lfloor \frac{n}{2} \rfloor$.

Theorem 5.9 ([BW04]) *Let T be a tree of size $\lceil \frac{n}{2} \rceil$. Then there exists a packing of $\lfloor \frac{n}{2} \rfloor$ copies of T into K_n.*

Recently, Zak [Zak11] studied for the first time the packing problem for all k, and proved the following result.

Theorem 5.10 ([Zak11]) *Let k be a positive integer and G be a graph of order $n \geq 2(k-1)^3$. If $|E(G)| \leq n - 2(k-1)^3$, then G is k-placeable.*

On the basis of Conjecture 5.2, the bound on the size of a graph in Theorem 5.10 is rather far from what can be expected. However, to the best of our knowledge, this work is the first attempt that consider the general case of Conjecture 5.2.

5.1.3 Packing graphs with bounded maximum degree and bounded girth in a complete graph

The graph packing theory also investigates two important directions concerning graphs with bounded maximum degree and bounded girth. The two main conjectures in this area were given by Bollobás and Eldridge [BS78] and Faudree *et al.* [FRSS81], respectively.

Conjecture 5.11 ([BE78]) *Let G_1 and G_2 be two graphs of order n with maximum degrees $\triangle_1$ and $\triangle_2$. If $(\triangle_1+1)(\triangle_2+1) \leq n+1$, then there is a packing of G_1 and G_2.*

Conjecture 5.12 ([FRSS81]) *If a graph G is a non-star graph without cycles of length $m \leq 4$, then G is embeddable.*

If Conjecture 5.11 is true, then Conjecture 5.11 would be sharp, and considered as an extension of the Hajnal-Szemerédi theorem [HS70] on equitable colorings. Indeed, the Hajnal-Szemerédi theorem is a special case of Conjecture 5.11 when G_2 is the disjoint union of cliques of the same size. This conjecture has been proved in the case $\triangle_1 \leq 2$ by Aigner and Brandt [AB93], and independently by Alon and Fischer [AF96]. The case when $\triangle_1 = 3$ and n is huge has been proved by Csaba *et al.* in [CSS03]. Bollobás *et al.* [BKN08] proved a strengthening of the conjecture when G_1 is d-degenerate and $d < \frac{\triangle_1}{40}$. Eaton [Eat00] showed that under the given condition, there is a near-packing of degree 1 of G_1 and G_2, that is, an embedding of the two graphs into a common vertex set such that the maximum degree of the subgraph defined by the edges common to both copies is at most 1. Recently, Kaul *et al.* [KKY08] proved Conjecture 5.11 for graphs with large maximum degrees. They proved in particular that for $\triangle_1, \triangle_2 \geq 300$, if $(\triangle_1 + 1)(\triangle_2 + 1) \leq 0.6n + 1$, then G_1 and G_2 pack.

In [FRSS81], Conjecture 5.12 was proved for every graph G with n vertices and no more than $\frac{6}{5}n - 2$ edges. Wozniak [Woz91] proved that a graph G without cycles of length at most 7 is embeddable. This result was improved later by Brandt [Bur95] who showed that a graph G without cycles of length at most 6 is embeddable. Recently, Görlich and Zak [GZ09] proved that a graph G without cycles of length at most 5 is embeddable.

5.1.4 Graph packing and permutation structure

In this section, we review some results concerning the relationship between the embedding of graphs and the structure of the embedding permutations. This study allows us to obtain some results concerning our graph packing problem which will be introduced in Section 5.2. Let us start by the permutations with no fixed point. In [Sch78], Schuster proved the following theorem.

Theorem 5.13 ([Sch78]) *Let $G = (V, E)$ be a graph of order n. If $|E(G)| \leq n - 2$ then there exists an embedding σ of G such that σ has no fixed point.*

The above theorem cannot be improved by increasing the number of edges as shown by the graphs $K_{1,2} \cup K_3$ and $K_{1,3} \cup K_3$. However, all the other $(n, n-1)$-graphs that are contained in their complements can be embedded without fixed points. More precisely, we have the following theorem (proved also in [Sch78]).

Theorem 5.14 ([Sch78]) *Let $G = (V, E)$ be a graph of order n with $E(G) \leq n-1$. If (i) G is not an exceptional graph of Theorem 5.4 and (ii) $G \neq K_{1,2} \cup K_3$ and $G \neq K_{1,3} \cup K_3$, then there exists a fixed-point-free embedding of G.*

Other work deal with the study of families of embeddable graphs such that the corresponding permutation is cyclic. First of all, we formulate some results proved in [Woz99] in the following theorem.

Theorem 5.15 ([Woz99]) *The following graphs are cyclically embeddable:*

1. *non-star trees.*
2. $(n, n-2)$ *graphs.*
3. *cycles* C_n *for* $n \geq 6$.
4. *unicyclic graphs, except graphs that are not embeddable at all (See [FRSS81]) and five graphs given in Figure 5.3.*

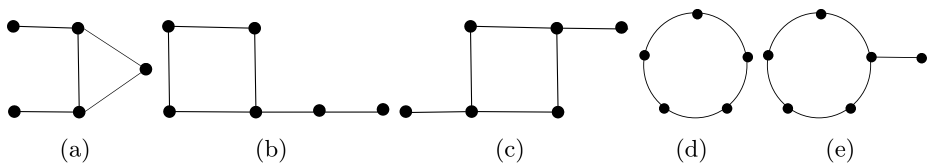


Figure 5.3: Five embeddable unicyclic graphs which are not cyclically embeddable.

Theorem 5.16 ([Woz99]) *Let G be a graph of order n with $|E(G)| \leq n-1$ and such that (i) G is not an exceptional graph of Theorem 5.4 and (ii) $G \neq K_{1,2} \cup K_3, K_{1,3} \cup K_3, K_1 \cup C_5$. Then there exists a cyclic embedding of G.*

Assume that an embedding σ of G is expressed as the product of disjoint cycles (consedring σ as a permutation). The previous results can be improved in another direction by specifying the size and the number of cycles occurring in the embedding permutation. We begin with a theorem about embedding of $(n, n-1)$-graphs, proved in [BW85], which can be considered as an improvement of Theorem 5.14.

Theorem 5.17 ([BW85]) *Let $G = (V, E)$ be a graph of order n, $n \equiv 0, 1 (mod 4)$. If $E(G) \leq n-1$ then either there exists an embedding σ of G having all its cycles of length 4, except for one of length 1 if n is odd, or G is isomorphic to one of the following graphs: $K_{1,n-1}, K_{1,n-4} \cup K_3$, $n \geq 8$, $K_1 \cup C_4$, $K_1 \cup K_3$, $K_2 \cup K_3$.*

The other cases (where $n \equiv 2, 3 (\text{mod } 4)$) have been investigated by Wozniak in [Woz94]. Another possibility for improvement of Theorem 5.13 is the following theorem proved in [Woz94].

Theorem 5.18 ([Woz94]) *Let $G = (V, E)$ be a graph of order n, $n \geq 3$. If $|E(G)| \leq n-2$ then there exists an embedding σ of G such that each cycle of σ has length 3 except for one of length 1 if $n \equiv 1 (mod\ 3)$ or two of length 1 if $n \equiv 2 (mod\ 3)$.*

Finally, it is also interesting to note that another way to study the graph packing problem is to consider the placement of graphs into a (partial) subgraph of a complete graph. In this case, it is necessary to consider more information about the permutation structure. We can mention for example the paper of Kheddouci [Khe03] that considers the packing of trees in their third power.

5.1.5 Our observations

The graph packing problem has attracted considerable attention from many researchers. However, the most of existing work focuses on unlabeled graphs. Therefore, it is interesting to consider this problem on labeled graphs, which we will be done in the rest of this part. To the best of our knowledge, this variant has not been studied so far.

5.2 Labeled Packing Problem

In this section, we introduce and study the packing problem for a vertex labeled graph.

5.2.1 Definitions and general results

We first give the formal definition of our packing problem. Roughly speaking, it consists of a graph packing which preserves the labels of the vertices.

Definition 5.2 *Let p be an integer greater than one and let $G_1, G_2, \ldots, G_k$ be k copies of a graph $G = (V, E)$. Let f be a mapping from $V(G)$ to the set of labels $\{1, 2, \ldots, p\}$. The mapping f is called a p-labeled packing of k copies of G into K_n if there exist permutations $\sigma_i : V(G_i) \longrightarrow V(K_n)$, where $i = 1, \ldots, k$, such that:*

1. *$\sigma_i^*(E(G_i)) \cap \sigma_j^*(E(G_j)) = \emptyset$ for all $i \neq j$.*
2. *for every vertex v of G, we have $f(v)=f(\sigma_1(v))=f(\sigma_2(v)) \cdots =f(\sigma_k(v))$.*

For example, the following figure presents a 5-labeled packing of two copies of $P_4 \cup P_3$, where the dotted edges represent the second copie of $P_4 \cup P_3$.

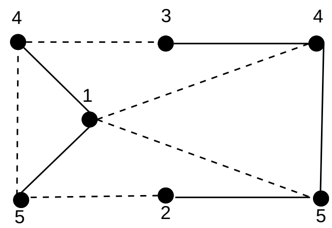


Figure 5.4: A 5-labeled packing of two copies of $P_4 \cup P_3$.

From the previous definition, we define our packing parameter as follows:

Definition 5.3 *The maximum positive integer p for which G admits a p-labeled packing of k copies of G is called the k-labeled packing number of G and is denoted by $\lambda^k(G)$.*

Naturally, the existence of a packing of k copies of a graph G is a necessary and sufficient condition for the existence of a *p-labeled packing* of k copies of G (where $p \geq 1$). Indeed, it suffices to choose the mapping $f : V(G) \longrightarrow \{1\}$. Then, we obtain the following proposition.

Proposition 5.19 *For every graph G, if there exists a k-placement of G into K_n, then*

$$\lambda^k(G) \geq 1.$$

The following lemma is a direct consequence of Definition 5.2.

Lemma 5.20 *Let G be a graph of order n and let H be a spanning subgraph of G. If there exists a packing of k copies of G into K_n, then*

$$\lambda^k(H) \geq \lambda^k(G).$$

Let us introduce the following notations: let G be a graph of order n and f be a mapping from $V(G)$ to $\{1, 2, \ldots, p\}$. We define $S(f)$ as the subset of labels that occur only once in $f(V(G))$ and let $V_{S(f)}$ consist of the vertices colored with the labels of $S(f)$. Thus it is obvious that $|V_{S(f)}| = |S(f)|$. Throughout this part, a labeled packing of two copies of G will be called a *labeled embedding* of G.

The following lemma gives an upper bound on the labeled embedding number $\lambda^2(G)$.

Lemma 5.21 *Let G be a graph of order n and let I be a maximum independent set of G. If there exists an embedding of G, then*

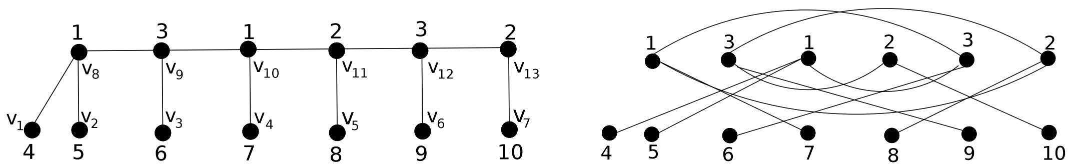
$$\lambda^2(G) \leq |I| + \lfloor \frac{n - |I|}{2} \rfloor$$

Proof. Let f be a mapping from $V(G)$ to $\{1, 2 \ldots, p\}$ corresponding to a *p-labeled embedding* of G with p maximum, (*i.e.*, $p = \lambda^2(G)$). A necessary condition for the existence of a *p-labeled embedding* of G is that for every two adjacent vertices of G, one of their labels must occur at least twice in $f(V(G))$. Hence, it is easy to see that all vertices of $V_{S(f)}$ form an independent set in G. Then, we have $|V_{S(f)}| \leq |I|$ and $p \leq |V_{S(f)}| + \lfloor \frac{n-|V_{S(f)}|}{2} \rfloor$. Since $g(x) = x + \lfloor \frac{n-x}{2} \rfloor$ is an increasing function of x, it follows that $p \leq |I| + \lfloor \frac{n-|I|}{2} \rfloor$, giving the desired result. $\square$

Using the same sketch of proof as before, we can show a similar upper bound on $\lambda^3(G)$ as follows:

Lemma 5.22 *Let G be a graph of order n and let I be a maximum independent set of G. If there exists a 3-placement of G into K_n, then*

$$\lambda^3(G) \leq |I| + \lfloor \frac{n - |I|}{3} \rfloor$$



(a) (b)

Figure 5.5: (a) A caterpillar T, (b) A 10-labeled embedding of T

It is clear that any labeled embedding construction (the labeling function f and the permutation σ) that achieves the upper bound of Lemma 5.21 must consider the vertices of $V_{S(f)}$ as fixed-points under the permutation σ, which then requires to define a $\lfloor \frac{|V(G) \setminus V_{S(f)}|}{2} \rfloor$-labeled embedding without fixed points for the subgraph induced by the set of vertices $V(G) \setminus V_{S(f)}$. For example, let us consider the caterpillar T of Figure 5.5(a). From Lemma 5.21, we have $\lambda^2(T) \leq 10$. This upper bound can be reached by finding a 3-labeled embedding without fixed points for the central path of T (Figure 5.5(b)).

Remark We explain in this observation a fundamental congruence relation between the labeled embedding number and the permutation structure. For any labeled embedding of a graph G induced by a permutation σ, we can see that the vertices of every cycle of $C(\sigma)$ share the same label. Hence, we can say that the labeled embedding number of G denotes the maximum number of cycles induced by a permutation of G. The following lemma is thus obtained.

Lemma 5.23 *Let G be a graph. If G has an embedding σ with p cycles, then*

$$\lambda^2(G) \geq p$$

Therefore, we can observe that the graph packing can be used to distinguish some vertices of G as follows: let f be a mapping from $V(G)$ to $\{1, 2, \ldots, p\}$ and let σ be a permutation embedding on $V(G)$, where for every two vertices $u, v \in V(G)$, we have

$$\begin{array}{ll} f(u) = f(v) & \text{if } u, v \text{ belong to the same cycle in } \sigma(T) \\ f(u) \neq f(v) & \textit{otherwise.} \end{array}$$

In Section 5.1.4, we have presented some results on graph embedding that deal with some additional properties of the embedding permutation. These results can be exploited to obtain a lower bound on $\lambda^2(G)$ for some families of graphs. For example, from the statements of Theorem 5.18 and Lemma 5.23, we can deduce the following corollary:

Corollary 5.24 *Let $G = (V, E)$ be a graph of norder n. If $|E(G)| \leq n - 2$ then*

$$\lambda^2(G) \geq \lfloor \frac{n}{3} \rfloor$$

In the same way, the following corollary is an immediate consequence of Theorem 5.17 and Lemma 5.23.

Corollary 5.25 *Let $G = (V, E)$ be a graph of order $n \equiv 0, 1 (mod\ )4$. If $|E(G)| \leq n-1$ then $\lambda^2(G) \geq \lfloor \frac{n}{4} \rfloor$ or G is isomorphic to one of the following graphs $K_{1,n-1}$, $K_{1,n-4} \cup K_3$, $n \geq 8$, $K_1 \cup C_4$, $K_1 \cup K_3$, $K_2 \cup K_3$, $K_1 \cup 2K_3$.*

As seen previously, graph packing is a very hard problem; the main results concern graphs of small size ($(n, n - q)$ graphs where $q \geq 0$). Two studies are possible for the labeled graph packing problem: (i) packing an important number of sample graphs (or copies of a given graph) in a complete graph; or (ii) packing a fixed number (usually two) of non trivial graphs in a subgraph of K_n. For the first study of labeled packing of graphs, we opt for the first possibility. We will give some results concerning the labeled packing number of k copies of cycles (where $k \geq 2$).

Given a graph G, the generalization of Lemma 5.21 for any k is not trivial. Yet, it becomes easier when G is a cycle. We know that any complete graph K_n can be decomposed into $\frac{n-1}{2}$ Hamilton cycles if n is odd and $\frac{n-2}{2}$ Hamilton cycles plus a perfect matching if n is even (see [Luc92]). Thus, C_n cannot admit a packing of k copies in K_n if $n \leq 2k$.

Lemma 5.26 *For every cycle C_n of order $n \geq 2k + 1$, we have*

$$\lambda^k(C_n) \leq \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n - \lfloor \frac{n}{2} \rfloor}{k} \rfloor$$

Proof. Let f be a mapping from $V(G)$ to $\{1, 2, \ldots, p\}$ corresponding to a p-labeled packing of k copies of C_n with p maximum. Let us introduce the following notation: for a set $S \subseteq V(G)$, we denote by $N(S)$ the neighborhood of S, *i.e.,* the set of all vertices of G which have a neighbor in S. Then, two necessary conditions for the existence of *p-labeled packing* of k copies of C_n are that:

1. the labels of $N(V_{S(f)})$ must occur at least k times in $f(V(C_n))$.
2. the labels of $V \setminus (V_{S(f)} \cup N(V_{S(f)}))$ must occur at least twice in $f(V(C_n))$.

Hence, we have

$$p \leq |V_{S(f)}| + \lfloor \frac{|N(V_{S(f)})|}{k} \rfloor + \lfloor \frac{n - |V_{S(f)}| - |N(V_{S(f)})|}{2} \rfloor$$

Since $k \geq 2$, the maximum value p is obtained when $|V_{S(f)}| = \lfloor \frac{n}{2} \rfloor$ and $|N(V_{S(f)})| = n - \lfloor \frac{n}{2} \rfloor$, giving the desired result. □

5.2.2 Labeled-packing of k copies of cycles with order at least $4k$

In this section, we determine the exact value of the labeled packing number of k copies of cycles of order $n \geq 4k$. It is done by proving that the upper bound of Lemma 5.26 is reached for all $n \neq 4k + x$, where $k > 2$ and $x \bmod 4 = 2$.

Theorem 5.27 *For every cycle C_n of order $n = 2km + x$, where $k, m \geq 2$ and $x < 2k$, we have*

$$\lambda^k(C_n) = \begin{cases} \frac{n}{2} + 1 & \text{if } (x \bmod 4, m) = (2,2) \text{ and } k > 2. \\ \lfloor \frac{n}{2} \rfloor + m + 1 & \text{if } x = 2k - 1. \\ \lfloor \frac{n}{2} \rfloor + m & \text{otherwise.} \end{cases}$$

Proof. Several cases are considered in this proof according to the values of x and m. In Figure 5.6, we outline the general scheme of our proof.

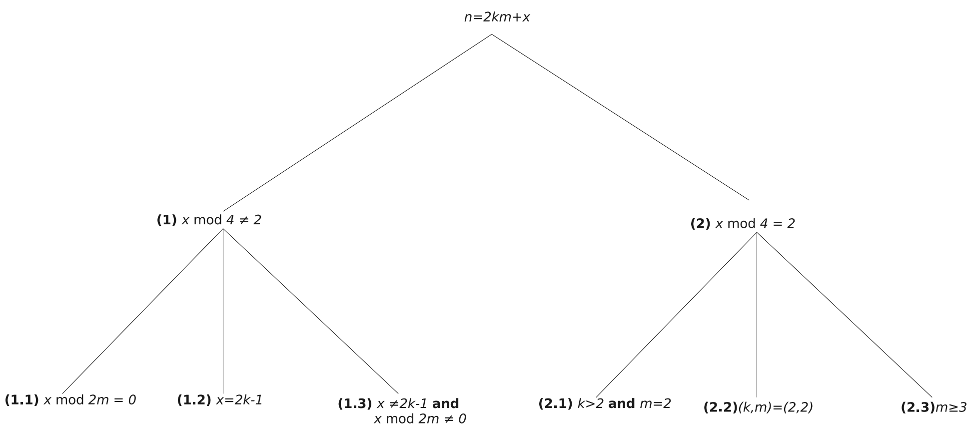


Figure 5.6: Proof structure.

Case 1 : $x \bmod 4 \neq 2$. Let $C_n = (v_1, v_2, \ldots, v_n)$ be a cycle of order n, where $n \geq 4k$. From Lemma 5.26, we have $\lambda^k(C_n) \leq \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n - \lfloor \frac{n}{2} \rfloor}{k} \rfloor$, it then suffices to prove that C_n admits a $(\lfloor \frac{n}{2} \rfloor + \lfloor \frac{n - \lfloor \frac{n}{2} \rfloor}{k} \rfloor)$-labeled packing of k copies. In order to simplify our proof, we consider three subcases according to the values of m and x as follows:

Subcase 1.1: $x \bmod 2m = 0$. Let $B = \{b_0, b_1, \ldots, b_{p-1}\}$ be a partition of $V(C_n)$ into p sets, where all sets b_i of B have the same cardinalty $2m$. Then, we have $p = k + \frac{x}{2m}$ such that for $0 \le i \le p-1$, $b_i = \{v_1^i, v_2^i, \ldots, v_{2m}^i\}$, where $v_j^i = v_{2im+j}$. These notations are illustrated in the example of Figure 5.7 , where $n = 16$ and $k = 3$.

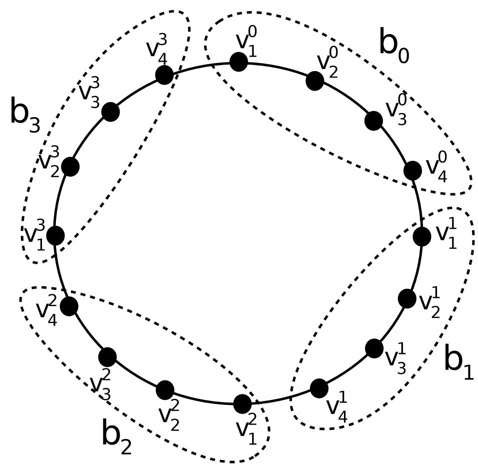


Figure 5.7: Partition of $V(C_{16})$ for $k = 3$.

A mapping f from $V(C_n)$ to $\{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor + m\}$ is now defined as follows: for every set b_i in B, let

$$\begin{aligned} f(v_{2j+1}^i) &= im + j + 1 && \text{for } 0 \le j \le m-1. \\ f(v_{2j}^i) &= \tfrac{n}{2} + j && \text{for } 1 \le j \le m. \end{aligned}$$

Let $\sigma^0(C_n)$, $\sigma^1(C_n)$, $\sigma^2(C_n), \ldots, \sigma^{k-1}(C_n)$ be a packing of k copies of C_n into K_n under the permutations $\sigma^0, \sigma^1, \ldots, \sigma^{k-1}$, respectively, where for $0 \le t \le k-1$,

$$\begin{aligned} \sigma^t(v_{2j+1}^i) &= v_{2j+1}^i && \text{for all } (i,j) \text{ satisfying } 0 \le i \le p-1 \text{ and } 0 \le j \le m-1. \\ \sigma^t(v_{2j}^i) &= v_{2j}^{(i+t) \bmod p} && \text{for all } (i,j) \text{ satisfying } 0 \le i \le p-1 \text{ and } 1 \le j \le m. \end{aligned}$$

Figure 5.8 presents a 10-labeled-packing of three copies of C_{16}.

From the set of permutations, we can easily see that for every vertex v_i in $V(C_n)$, we have $f(v_i) = f(\sigma^1(v_i)) = f(\sigma^2(v_i)) = \cdots = f(\sigma^{k-1}(v_i))$. It then remains to show that the set of cycles $\sigma^0(C_n) = C_n, \sigma^1(C_n), \sigma^2(C_n) \ldots, \sigma^{k-1}(C_n)$ are edge-disjoint into K_n. To prove this, it suffices to show that for every vertex v_i of $V(C_n)$, we have:

For $0 \le t \ne l \le k-1$, if $\sigma^t(v_i) = \sigma^l(v_i)$, then $\sigma^t(v_{i+1})$ and $\sigma^t(v_{i-1})$ must be different from both $\sigma^l(v_{i+1})$ and $\sigma^l(v_{i-1})$.

From the set of permutations, we can observe that for every two distinct integers $l, t \in \{0, 1, \ldots, k-1\}$ and for every set b_i of B, we have $\sigma^l(v_{2j+1}^i) = \sigma^t(v_{2j+1}^i) =$

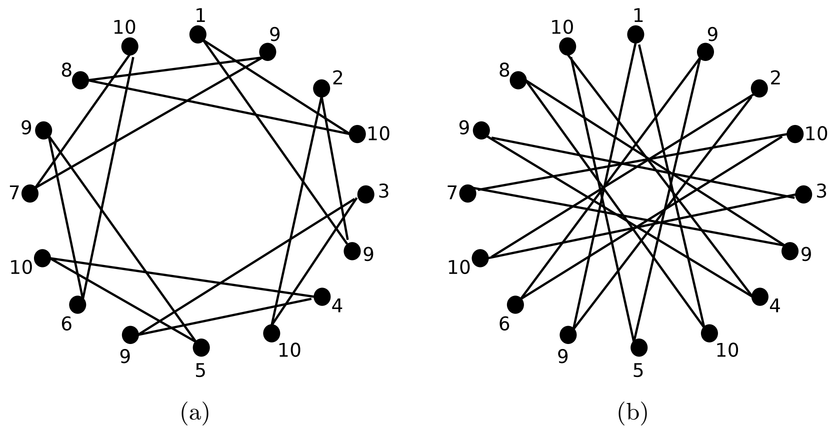


Figure 5.8: A 10-labeled-packing of three copies of C_{16}: (a) $\sigma^1(C_{16})$, (b) $\sigma^2(C_{16})$.

v^i_{2j+1}, where $0 \leq j \leq m-1$. Then for every set b_i of B, we consider two cases according to whether $j \neq 1$ or $j = 1$.

1. for $j = 3, 5 \ldots 2m-1$, we have $\sigma^l(v^i_{j+1}), \sigma^l(v^i_{j-1}) \in b^{(l+i) \bmod p}$ and $\sigma^t(v^i_{j+1})$, $\sigma^t(v^i_{j-1}) \in b^{(t+i) \bmod p}$. It follows that $(l+i) \bmod p \neq (t+i) \bmod p \neq i$.

2. for $j = 1$, we can easily see that $\sigma^l(v^i_2)$ and $\sigma^l(v^{i-1}_{2m})$ are both different from $\sigma^t(v^i_2)$ and $\sigma^t(v^{i-1}_{2m})$.

Hence, according to the two previous observations, we can see that C_n, $\sigma^1(C_n)$, $\sigma^2(C_n)$, … , $\sigma^{k-1}(C_n)$ are k pairwise edge-disjoint cycles, thus the mapping f is a $(\lfloor \frac{n}{2} \rfloor + m)$-labeled-packing of k copies of C_n.

Subcase 1.2: $x = 2k-1$. In this case, the partition $B = \{b_0, b_1, \ldots, b_{k-1}\}$ of $V(C_n)$ is defined as follows: for $0 \leq i \leq k-2$, $b_i = \{v^i_1, v^i_2, \ldots, v^i_{2(m+1)}\}$ and $b_{k-1} = \{v^{k-1}_1, v^{k-1}_2, \ldots, v^{k-1}_{2m+1}\}$, where $v^i_j = v_{2i(m+1)+j}$.

A mapping f from $V(C_n)$ to $\{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor + m + 1\}$ is now defined as follows: for every set b_i in B, let

$$
\begin{array}{ll}
f(v^i_{2j+1}) = i(m+1) + j + 1 & \text{for } 0 \leq j \leq m \text{ and } (i,j) \neq (k-1, m). \\
f(v^{k-1}_{2m+1}) = \frac{n-1}{2} + m + 1. & \\
f(v^i_{2j}) = \frac{n-1}{2} + j & \text{for } 1 \leq j \leq m+1 \text{ and } (i,j) \neq (k-1, m+1).
\end{array}
$$

Let $\sigma^0(C_n)$, $\sigma^1(C_n), \ldots, \sigma^{k-1}(C_n)$ be a packing of k copies of C_n into K_n under the permutations $\sigma^0, \sigma^1, \ldots, \sigma^{k-1}$, respectively, where $\sigma^0(C_n) = C_n$ and for $1 \leq t \leq$

$k-1$, let

$\sigma^t(v^i_{2j+1}) = v^i_{2j+1}$	for all (i,j) satisfying $0 \le i \le k-1, 0 \le j \le m$ and $(i,j) \ne (k-1,m)$.
$\sigma^t(v^i_{2j}) = v^{(i+t) \bmod k}_{2j}$	for all (i,j) satisfying $0 \le i \le k-1, 1 \le j \le m+1$ and $(i,j) \ne (k-t-1,m+1)$.
$\sigma^t(v^{k-1}_{2m+1}) = v^{t-1}_{2m+2}$.	
$\sigma^t(v^{k-t-1}_{2m+2}) = v^{k-1}_{2m+1} = v_n$.	

Figure 5.9 shows a 11-labeled-packing of three copies of C_{17}.

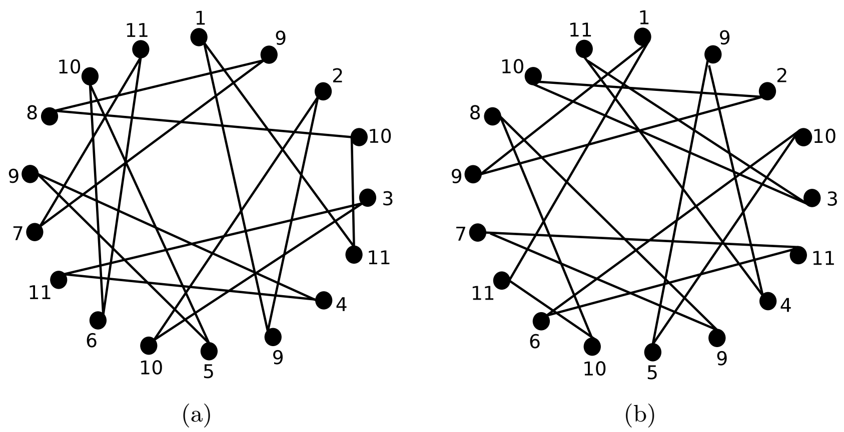


Figure 5.9: A 11-labeled-packing of three copies of C_{17}: (a) $\sigma^1(C_{17})$, (b) $\sigma^2(C_{17})$.

Using the same proof as in Subcase 1.1, we can show that the mapping f is a $(\lfloor \frac{n}{2} \rfloor + m + 1)$-labeled-packing of k copies of C_n.

Subcase 1.3: $x \bmod 2m \ne 0$ and $x \ne 2k-1$. Let $B = \{b_0, b_1, \dots, b_{p-1}\}$ be a partition of $V(C_n)$ into $p = k + \lceil \frac{x}{2m} \rceil$ sets such that for $0 \le i \le p-2$, $b_i = \{v^i_1, v^i_2, \dots, v^i_{2m}\}$ and $b_{p-1} = \{v^{p-1}_1, v^{p-1}_2, \dots, v^{p-1}_{x \bmod 2m}\}$, where $v^i_j = v_{2im+j}$. We let $r = x \bmod 2m$.

A mapping f from $V(C_n)$ to $\{1, 2, \dots, \lfloor \frac{n}{2} \rfloor + m\}$ is defined as follows:

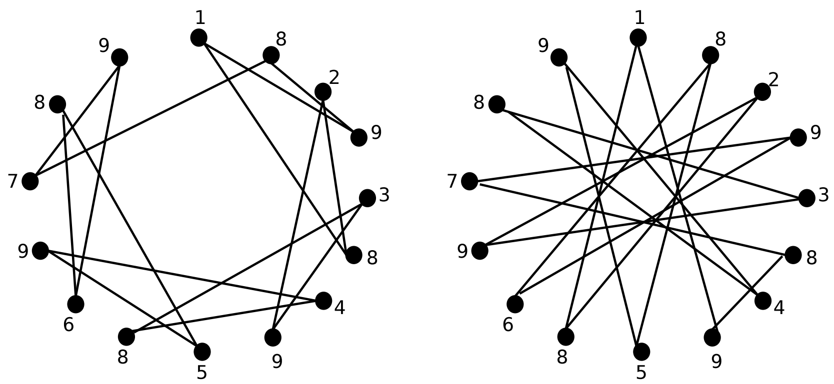
$f(v^i_{2j+1}) = im + j + 1$	for all (i,j) satisfying $0 \le i \le p-2, 0 \le j \le m-1$.
$f(v^i_{2j}) = \lfloor \frac{n}{2} \rfloor + j$	for all (i,j) satisfying $0 \le i \le p-2, 1 \le j \le m$.
$f(v^{p-1}_{2j+1}) = (p-1)m + j + 1$	for $0 \le j \le \lceil \frac{r-1}{2} \rceil - 1$.
$f(v^{p-1}_{2j}) = \lfloor \frac{n}{2} \rfloor + j$	for $1 \le j \le \lfloor \frac{r-1}{2} \rfloor$.
$f(v^{p-1}_r) = \lfloor \frac{n}{2} \rfloor + m$.	

Let $\sigma^0(C_n), \sigma^1(C_n), \dots, \sigma^{k-1}(C_n)$ be a packing of k copies of C_n into K_n under the permutations $\sigma^0, \sigma^1, \dots, \sigma^{k-1}$, respectively, where $\sigma^0(C_n) = C_n$ and for $1 \le t \le$

$k-1$, let

$$
\begin{array}{ll}
\sigma^t(v_{2j+1}^i) = v_{2j+1}^i & \text{for all } (i,j),\ 0 \le i \le p-2 \text{ and } 0 \le j \le m-1. \\
\sigma^t(v_{2j}^i) = v_{2j}^{(i+t) \bmod p} & \text{for all } (i,j),\ 0 \le i \le p-2 \text{ and } 1 \le j \le \lfloor \frac{r-1}{2} \rfloor. \\
\sigma^t(v_{2j}^i) = v_{2j}^{(i+t) \bmod (p-1)} & \text{for all } (i,j),\ 0 \le i \le p-2 \text{ and } \lfloor \frac{r-1}{2} \rfloor + 1 \le j \le m-1. \\
\sigma^t(v_{2m}^i) = v_{2m}^{(i+t) \bmod p} & \text{for } 0 \le i \ne p-t-1 \le p-2. \\
\sigma^t(v_{2m}^{p-t-1}) = v_r^{p-1} = v_n. & \\
\sigma^t(v_{2j+1}^{p-1}) = v_{2j+1}^{p-1} & \text{for } 0 \le j \le \lceil \frac{r-1}{2} \rceil - 1. \\
\sigma^t(v_{2j}^{p-1}) = v_{2j}^{t-1} & \text{for } 1 \le j \le \lfloor \frac{r-1}{2} \rfloor. \\
\sigma^t(v_r^{p-1}) = v_{2m}^{t-1}. &
\end{array}
$$

Figure 5.11 shows a 9-labeled-packing of three copies of C_{15}.



(a) (b)

Figure 5.10: A 9-labeled-packing of three copies of C_{15}: (a) $\sigma^1(C_{15})$, (b) $\sigma^2(C_{15})$.

Using the same proof as in Subcase 1.1, we can show that the mapping f is a $(\lfloor \frac{n}{2} \rfloor + m)$-labeled-packing of k copies of C_n.

Case 2: $x \bmod 4 = 2$. We consider three subcases according to the value of k and m as follows:
Subcase 2.1: $k > 2$ **and** $m = 2$. We first prove that $\lambda^k(C_{4k+x}) < \frac{n}{2} + 2$. From Lemma 5.26, it follows that the value $\frac{n}{2} + 2$ is an upper bound of $\lambda^k(C_{4k+x})$. We assume that there exists a mapping $f : V(C_{4k+x}) \longrightarrow L = \{1, 2, \ldots, \frac{n}{2} + 2\}$ which is a $(\frac{n}{2} + 2)$-labeled-packing of C_{4k+x}. Let $\sigma^0(C_{4k+x}), \sigma^1(C_{4k+x}), \ldots, \sigma^{k-1}(C_{4k+x})$ be a packing of k copies of C_{4k+x} into K_{4k+x} under the permutations $\sigma^0, \sigma^1, \ldots,$ σ^{k-1}, respectively. We know that the vertices colored with the labels of S_f form an independent set in C_{4k+x} (implying that $|S_f| \le \frac{n}{2}$). We consider two cases according to the cardinality of S_f as follows:
(a): $|S_f| < \frac{n}{2}$. We know that each label of $L \backslash S_f$ occurs at least k times in $f(C_{4k+x})$. This requires at least $5k + \frac{x}{2} - 1$ vertices in C_{4k+x}, which is a contradiction since $k > 2$.

(b): $|S_f| = \frac{n}{2}$. It means that the vertices colored with the labels of S_f form a maximum independent set in C_{4k+x}. Then the two labels of $L \setminus S_f$ are attributed to the remaining vertices such that each label occurs at least k times. With this labeling scheme, there exist necessarily three vertices v_{i-1}, v_i and v_{i+1} such that $f(v_i) \in S_f$ and $f(v_{i-1}), f(v_{i+1}) = c \in L \setminus S_f$. This requires that the label c must occur at least $2k$ times in C_{4k+x}, yielding a contradiction. Hence $\lambda^k(C_{4k+x}) < \frac{n}{2}+2$.

We now prove that $\lambda^k(C_{4k+x}) = \frac{n}{2} + 1$. Let f be the mapping from $V(C_n)$ to $\{1, 2, \ldots, \frac{n}{2} + 1\}$ defined as follows:

$$\begin{array}{ll} f(v_{2j+1}) = j + 1 & \text{for } j = 0, 1, \ldots, \frac{n}{2} - 1. \\ f(v_{2j}) = \frac{n}{2} + 1 & \text{for } j = 1, 2 \ldots, \frac{n}{2}. \end{array}$$

Let $\sigma^0(C_n)$, $\sigma^1(C_n), \ldots, \sigma^{k-1}(C_n)$ be a packing of k copies of C_n into K_n under the permutations $\sigma^0, \sigma^1, \ldots, \sigma^{k-1}$, respectively, where $\sigma^0(C_n) = C_n$ and for $1 \leq t \leq k-1$:

$$\begin{array}{ll} \sigma^t(v_{2j+1}) = v_{2j+1} & \text{for } j = 0, 1, \ldots, \frac{n}{2} - 1. \\ \sigma^t(v_{2j}) = v_{2(j+2t) \bmod n} & \text{for } j = 1, 2 \ldots, \frac{n}{2}. \end{array}$$

According to this scheme, Figure 5.11 presents a 8-labeled-packing of three copies of C_{14}.

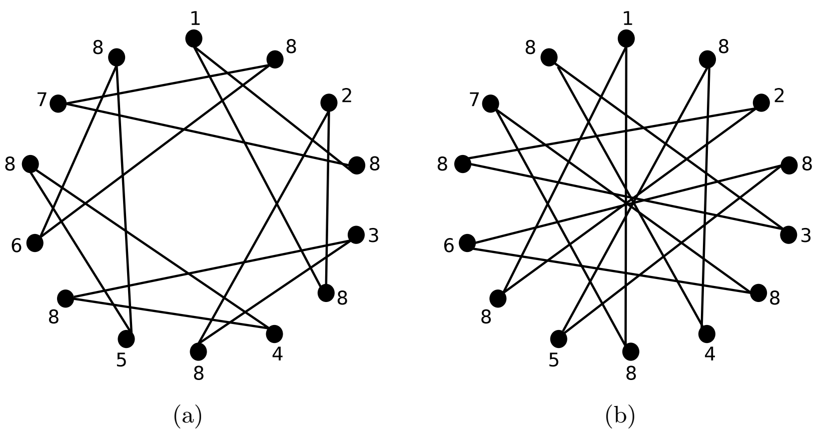


Figure 5.11: A 8-labeled-packing of three copies of C_{14}: (a) $\sigma^1(C_{14})$, (b) $\sigma^2(C_{14})$.

Subcase 2.2: $(k, m) = (2, 2)$. From Lemma 5.26, we have $\lambda^2(C_{10}) \leq 7$. The following figure gives a 7-labeled-embedding of C_{10}.

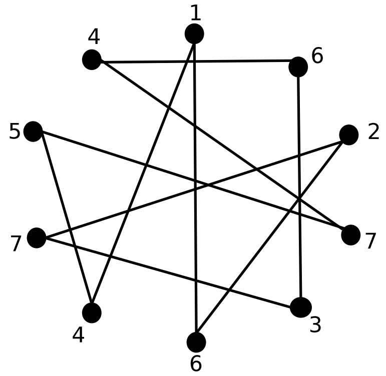


Figure 5.12: A 7-labeled-embedding of C_{10}

Subcase 2.3: $m \geq 3$. Let $B = \{b_0, b_1, \ldots, b_{p-1}\}$ be a partition of $V(C_n) \setminus \{v_{n-1}, v_n\}$ into p sets, where all sets b_i of B have the same cardinalty $2m$. Then, we have $p = k + \frac{x}{2m}$ such that for $0 \leq i \leq p-1$, $b_i = \{v_1^i, v_2^i, \ldots, v_{2m}^i\}$, where $v_j^i = v_{2im+j}$.

A mapping f from $V(C_n)$ to $\{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor + m\}$ is defined as follows: for every set b_i in B, let

$$
\begin{array}{ll}
f(v_{2j+1}^i) = im + j + 1 & \text{for } 0 \leq j \leq m-1. \\
f(v_{2j}^i) = \frac{n}{2} + j & \text{for } 1 \leq j \leq m. \\
f(v_{n-1}) = \frac{n}{2}. & \\
f(v_n) = \frac{n}{2} + m - 1. &
\end{array}
$$

Let $\sigma^0(C_n), \sigma^1(C_n), \ldots, \sigma^{k-1}(C_n)$ be a packing of k copies of C_n into K_n under the permutations $\sigma^0, \sigma^1, \ldots, \sigma^{k-1}$, respectively, where $\sigma^0(C_n) = C_n$ and for $1 \leq t \leq p-1$:

$$
\begin{array}{ll}
\sigma^t(v_{2j+1}^i) = v_{2j+1}^i & \text{for all } (i,j) \text{ satisfying } 0 \leq i \leq p-1, 0 \leq j \leq m-1. \\
\sigma^t(v_{2j}^i) = v_{2j}^{(i+t) \bmod p} & \text{for all } (i,j) \text{ satisfying } 0 \leq i \leq p-1, 1 \leq j \leq m \\
& \text{and } j \neq m-1. \\
\sigma^t(v_{2m-2}^i) = v_{2m-2}^{(i+t) \bmod (p+1)} & \text{for } 0 \leq i \leq p-1 \text{ and } i \neq p-t. \\
\sigma^t(v_{2m-2}^{p-t}) = v_n. & \\
\sigma^t(v_{n-1}) = v_{n-1}. & \\
\sigma^t(v_n) = v_{2m-2}^{t-1}. &
\end{array}
$$

Using the same proof as in Subcase 1.1, we can show that the mapping f is a $(\lfloor \frac{n}{2} \rfloor + m)$-labeled-packing of k copies of C_n. To illustrate this case, Figure 5.13 presents a 13-labeled-packing of three copies of C_{20}. □

5.2.3 Labeled packing of k copies of cycles of order at most $4k-1$

In the case where n is relatively small compared to k, some additional difficulties arise naturally, and $\lambda^k(C_n)$ has to be estimated differently. We know that any

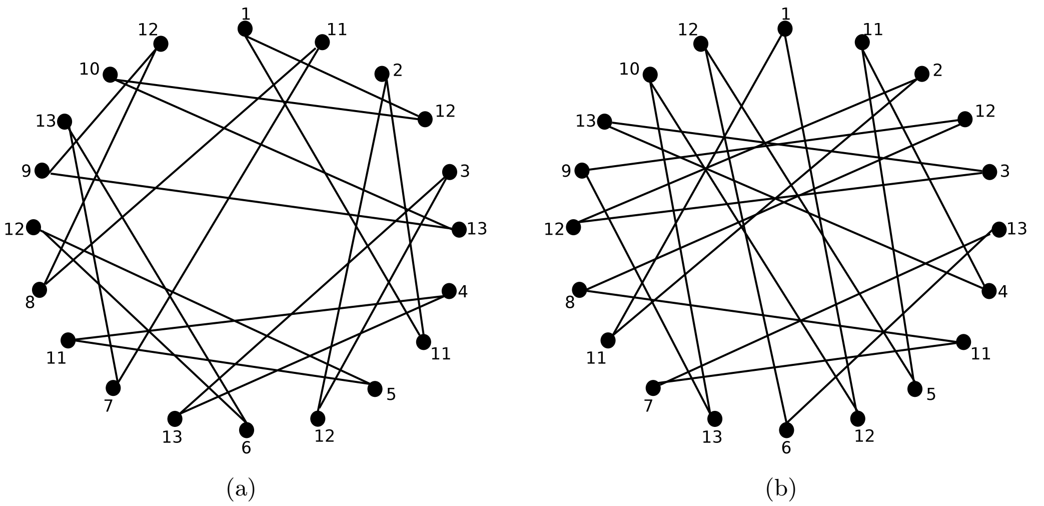


Figure 5.13: A 13-labeled-packing of three copies of C_{20}: (a) $\sigma^1(C_{20})$, (b) $\sigma^2(C_{20})$.

complete graph K_n can be decomposed into $\frac{n-1}{2}$ Hamilton cycles if n is odd and $\frac{n-2}{2}$ Hamilton cycles plus a perfect matching if n is even. Thus, C_n cannot admit a packing of k copies if $n \leq 2k$. After deep analysis, we raise the following conjecture.

Conjecture 5.28 *For every cycle C_n of order $n = 2k + x$, where $k \geq 2$ and $1 \leq x \leq 2k - 1$, then*

$$\lambda^k(C_n) = \begin{cases} 2 & \text{if } x = 1 \text{ and } k \text{ is even.} \\ x + 2 & \text{otherwise .} \end{cases}$$

This conjecture asserts that the upper bound of Lemma 5.26 is reached for all $n = 4k - 1, 4k - 2$ and $4k - 3$, where $(k, n) \neq (2, 5)$. We report in the following the results of our attempt to give some support to Conjecture 5.28 for some particular cases.

Theorem 5.29 *For every cycle C_n of order $n = 2k + x$, where $k \geq 2$, $2k - 3 \leq x \leq 2k - 1$ and $(k, n) \neq (2, 5)$, we have*

$$\lambda^k(C_n) = x + 2$$

Proof. Let $C_n = (v_1, v_2, \ldots, v_n)$ be a cycle of order $n = 2k + x$, where $2k - 3 \leq x \leq 2k - 1$ and $(k, n) \neq (2, 5)$. We know that a maximum independent set of C_n has size $\lfloor \frac{n}{2} \rfloor$. Then, from Lemma 5.26, we have $\lambda^k(C_n) \leq \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n - \lfloor \frac{n}{2} \rfloor}{k} \rfloor = x + 2$, it then suffices to prove that C_n admits a $(x + 2)$-labeled packing of k copies. In what follows, we give the proof only for the case $x = 2k - 1$, since the proofs of the other cases are similar.

Assume that $x = 2k - 1$ and let $B = \{b_0, b_1, \ldots, b_{k-1}\}$ be a partition of $V(C_n)$ into k sets, such that for $0 \leq i \leq k - 2$, $b_i = \{v_1^i, v_2^i, v_3^i, v_4^i\}$, and $b_{k-1} =$

$\{v_1^{k-1}, v_2^{k-1}, v_3^{k-1}\}$, where $v_j^i = v_{4i+j}$.

A mapping f from $V(C_n)$ to $\{1, 2, \ldots, 2k+1\}$ is defined as follows:

$$
\begin{array}{ll}
f(v_{2j+1}^i) = 2i + j + 1 & \text{for all } (i,j) \text{ where } 0 \le j \le 1,\ 0 \le i \le k-2. \\
f(v_2^i) = 2k & \text{for } 0 \le i \le k-2. \\
f(v_4^i) = 2k+1 & \text{for } 0 \le i \le k-2. \\
f(v_1^{k-1}) = 2k-1. & \\
f(v_2^{k-1}) = 2k. & \\
f(v_3^{k-1}) = 2k+1. &
\end{array}
$$

Let $\sigma^0(C_n)$, $\sigma^1(C_n), \ldots, \sigma^{k-1}(C_n)$ be a packing of k copies of C_n into K_n under the permutations $\sigma^0, \sigma^1, \ldots, \sigma^{k-1}$, respectively, where $\sigma^0(C_n) = C_n$ and for $1 \le t \le k-1$, let

$$
\begin{array}{ll}
\sigma^t(v_{2j+1}^i) = v_{2j+1}^i & \text{for all } (i,j) \text{ satisfying } 0 \le i \le k-2, 0 \le j \le 1. \\
\sigma^t(v_{2j}^i) = v_{2j}^{(i+t) \bmod k} & \text{for all } (i,j) \text{ satisfying } 0 \le i \le k-2, 1 \le j \le 2. \\
 & \text{and } (i,j) \ne (k-t-1, 2). \\
\sigma^t(v_4^{k-t-1}) = v_3^{(k-1)}. & \\
\sigma^t(v_1^{k-1}) = v_1^{(k-1)}. & \\
\sigma^t(v_2^{k-1}) = v_2^{(t-1)}. & \\
\sigma^t(v_3^{k-1}) = v_4^{(t-1)}. &
\end{array}
$$

Figure 5.14 presents a 7-labeled packing of three copies of C_{11}. $\square$

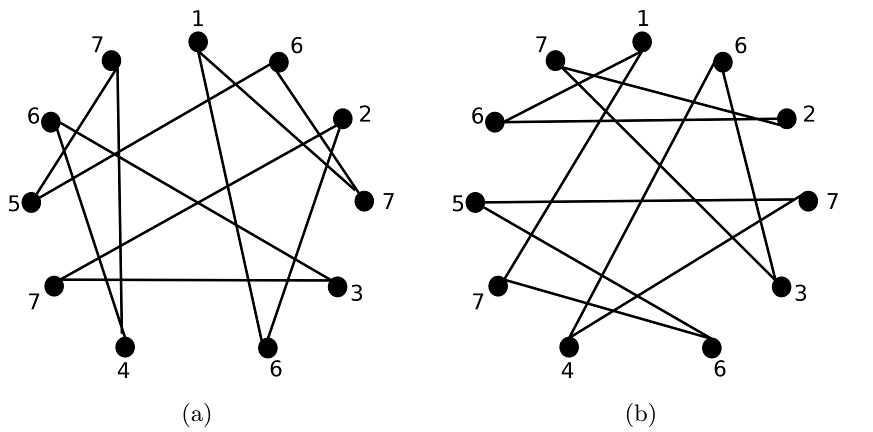


Figure 5.14: A 7-labeled packing of three copies of C_{11}: (a) $\sigma^1(C_{11})$, (b) $\sigma^2(C_{11})$.

We now present our results concerning the cycles of order $2k+1$ and $2k+2$.

Theorem 5.30 *For every prime number $k > 2$, we have $\lambda^k(C_{2k+1}) < 4$.*

Proof. Let $C_{2k+1} = (v_1, v_2, \dots, v_{2k+1}, v_{2k+1} = v_1)$. We assume that there exists a mapping $f : V(C_{2k+1}) \longrightarrow L = \{1, 2, 3, 4\}$ which is a 4-labeled packing of C_{2k+1}. Let $\sigma^0(C_{2k+1})$, $\sigma^1(C_{2k+1}), \dots, \sigma^{k-1}(C_{2k+1})$ be a packing of k copies of C_{2k+1} into K_{2k+1} under the permutations $\sigma^0, \sigma^1, \dots, \sigma^{k-1}$, respectively. We consider two cases according to the cardinality of S_f (which is defined in Section 5.2.1).

Case 1. $|S_f| = 0$. In this case, each label occurs at least twice in $f(V(C_n))$. Let us introduce the following notation: for every label i of L, let V_i be the subset of vertices with color i. We know that $\sigma^0(C_{2k+1}) \cup \sigma^1(C_{2k+1}) \cup \cdots \cup \sigma^{k-1}(C_{2k+1}) = K_{2k+1}$. We know that all vertices of the complete graph are neighbors. Then, for every label i of L, the number of edges in the subgraph of K_{2k+1} induced by the set V_i is $\frac{|V_i|(|V_i|-1)}{2}$. Then, for every cycle $\sigma^i(C_{2k+1})$ of the packing, we must have exactly $\frac{|V_i|(|V_i|-1)}{2k} = p_i$ edges between the vertices colored with i in the cycle $\sigma^i(C_{2k+1})$. Hence, the following system of equations must have at least one solution.

$$\begin{cases} |V_i|(|V_i|-1) = 2kp_i \quad \text{for } 1 \le i \le 4 & (1) \\ |V_1| + |V_2| + |V_3| + |V_4| = 2k+1 & (2) \end{cases}$$

From hypothesis, it is clear that the equation (1) has a solution if and only if $|V_i|$ or $|V_i| - 1$ are multiples of k (where $1 \le i \le 4$) since k is prime. Then we have $|V_1| + |V_2| + |V_3| + |V_4| > 2k + 1$, therefore this system of equations has no solution which is a contradiction for $k > 2$.

Case 2. $|S_f| \ge 1$. Let v_i be any vertex of $V_{S(f)}$. Then we consider two cases according to the labels of the neighbors vertices of v_i.

Subcase 1. $f(v_{i-1}) = f(v_{i+1}) = c$. In this case, we must have at least $2k$ vertices colored with c. This requires at least $2k + 3$ vertices in C_{2k+1}, which is a contradiction.

Subcase 2. $f(v_{i-1}) \ne f(v_{i+1})$. From hypothesis, each of the two labels $f(v_{i-1})$ and $f(v_{i+1})$ must occur at least k times. This requires at least $2k + 2$ vertices in C_{2k+1}. Hence, we also reached a contradiction and the theorem is proved. □

Using the same proof as in the above theorem, we obtain the following results.

Theorem 5.31 *For every even number $k \ge 2$, where k is a power of 2, we have $\lambda^k(C_{2k+1}) = 2$.*

Theorem 5.32 *For every even number $k \ge 2$, we have $\lambda^k(C_{2k+2}) < 5$.*

5.2.4 Summary and concluding remarks

In this section, we have proved the exact value of $\lambda^k(C_n)$ for all $k \ge 2$ and $n \ge 4k-3$. Our results are summarized in the follwoing table.

value of n	$\lambda^2(C_n)$
$n \leq 4$	no packing
$n = 5$	2
$n \geq 6$	$\lfloor \frac{3n}{4} \rfloor$
$n = 2km + x$, where $x < 2k$ and $k > 2$	$\lambda^k(C_n)$
$n \in \{3, 4, \ldots, 2k\}$	no packing
$n \in \{2k+1, 2k+2, \ldots, 4k-4\}$	remains to prove
$n \geq 4k-3$ and (x mod $4 \neq 2$ or $m \geq 3$)	$\lfloor \frac{n}{2} \rfloor + \lfloor \frac{n - \lfloor \frac{n}{2} \rfloor}{k} \rfloor$
x mod $4 = 2$ and $m = 2$	$\lfloor \frac{n}{2} \rfloor + \lfloor \frac{n - \lfloor \frac{n}{2} \rfloor}{k} \rfloor - 1$

As a corollary, the exact value of the 2-labeled packing number of cycles is a direct consequence of Theorem 5.27, Theorem 5.29 and Theorem 5.31.

Corollary 5.33 *Let C_n be a cycle of order at least 5. Then*

$$\lambda^2(C_n) = \begin{cases} \lfloor \frac{3n}{4} \rfloor & \text{if } n > 5. \\ 2 & n = 5. \end{cases}$$

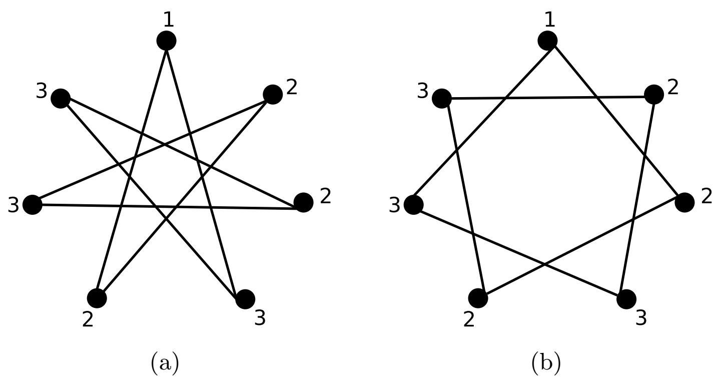


Figure 5.15: A 3-labeled packing of three copies of C_7: (a) $\sigma^1(C_7)$, (b) $\sigma^2(C_7)$.

Similarly, from Theorem 5.27 and Theorem 5.29, we immediately obtain the exact value of $\lambda^3(C_n)$ for all $n \geq 9$. Then, it follows from Theorem 5.30 and Theorem 5.32 that $\lambda^3(C_7) < 4$ and $\lambda^3(C_8) < 5$. The 3-labeled packing and 4-labeled packing of three copies of C_7 and C_8 are shown in Figure 5.15 and Figure 5.16, respectively. Hence, we obtain the following corollary.

Corollary 5.34 *Let C_n be a cycle of order at least 7. Then*

$$\lambda^3(C_n) = \begin{cases} 3 & \text{if } n = 7. \\ 4 & \text{if } n = 8. \\ 8 & \text{if } n = 14. \\ \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n - \lfloor \frac{n}{2} \rfloor}{3} \rfloor & \text{otherwise.} \end{cases}$$

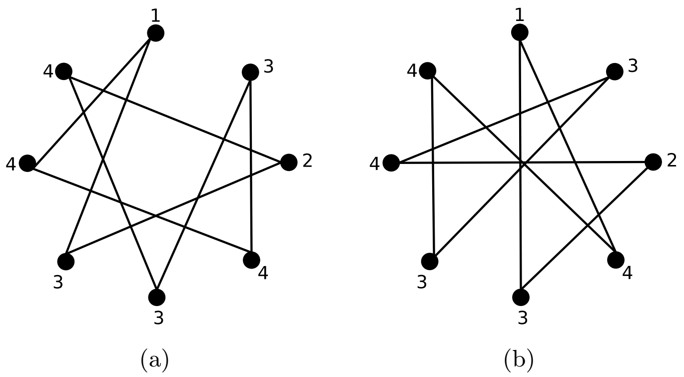


Figure 5.16: A 4-labeled packing of three copies of C_8: (a) $\sigma^1(C_8)$, (b) $\sigma^2(C_8)$.

5.3 Conclusion

In this chapter, we have introduced and studied a new graph packing problem, called labeled packing of graphs. One may consider two important issues: (i) packing important number of sample graphs (or copies of a given graph) in a complete graph; or (ii) packing a fixed number (usually two) of non trivial graph in a subgraph of K_n. Our first contribution opted for the first possibility. In particular, we have studied the labeled packing number of k copies of cycles, in which the exact value of $\lambda^k(C_n)$ have been shown for all $n \geq 4k-3$. A conjecture was given for the remaining cases. In the next chapter, we will continue to study the labeled packing of graphs of small size such as trees and $(n, n-2)$ graphs.

Chapter 6

Labeled Embedding of Trees and $(n, n-2)$-graphs

Contents

This chapter deals with the labeled embedding of trees and $(n, n-2)$-graphs. In Section 6.1, we will introduce a new labeled embedding problem called *labeled fixed-point-free embedding* of graphs (*i.e.,* a labeled embedding without fixed vertices). Then we will show that the correlation between this problem and the labeled embedding problem allows to derive some results on $\lambda^2(G)$. In Section 6.2, we present some results on the labeled embedding number of a tree. Finally, in Section 6.3, an upper bound on the labeled embedding number of $(n, n-2)$-graphs is proposed.

6.1 Labeled fixed-point-free embedding of graphs

In correlation with the labeled embedding number, we introduce a new embedding problem called *labeled fixed-point-free embedding* of graphs.

Definition 6.1 *Let f be a mapping from $V(G)$ to $L = \{1, 2, \ldots, p\}$, which is a labeled embedding of G with p labels under the permutation σ. We say that f is a labeled fixed-point-free embedding if σ is a fixed-point-free permutation.*

The maximum positive integer p for which G admits a labeled fixed-point-free embedding of G is called the *labeled fixed-point-free embedding number* and is denoted by $\alpha^2(G)$.

The following lemma shows an upper bound on $\alpha^2(G)$.

Lemma 6.1 *Let G be a graph of order n. If there exists an embedding of G into K_n, then*

$$\alpha^2(G) \leq \lfloor \frac{n}{2} \rfloor.$$

Proof. We can easily verify that a necessary condition for the existence of *p-labeled fixed-point-free embedding* of G is that every label of L occurs at least twice in G, *i.e.,* the upper bound of $\alpha^2(G)$ is reached if and only if we find an embedding of G having all its cycles of length two (except one of length three if n is odd). Hence, $\alpha^2(G) \leq \lfloor \frac{n}{2} \rfloor$. □

Given a graph G, the following lemma shows a lower bound on $\lambda^2(G)$ in terms of $\alpha^2(G)$ as follows:

Lemma 6.2 *Let G be a graph of order n and let X be any set of end vertices of G. If there exists a fixed-point-free embedding of $G[V \setminus X]$ with p labels into K_n, then*

$$\lambda^2(G) \geq |X| + p$$

Proof. By hypothesis, there exists a permutation σ' on $V(T)$ such that σ' is a fixed-point-free embedding with p labels. This permutation can be extended to a permutation σ on G as follows:

$$\sigma(x) = x \text{ for all } x \in X \text{ and } \sigma(y) = \sigma'(y) \text{ for all } y \in V \setminus X$$

From this permutation function, it is clear that for every edge $xy \in E(G)$ such that $x \in X$ and $y \in V \setminus X$, we have $\sigma(x)\sigma(y) \notin E(G)$. Hence $\lambda^2(G) \geq |X| + p$. □

We can conclude that the correlation between the two parameters $\lambda^2(G)$ and $\alpha^2(G)$ allows to reach the upper bound of Lemma 5.21 if there is a maximum independent set in G with only vertices of degree 1.

As we have seen in Lemma 5.23, we can obtain the following lower bound on $\alpha^2(G)$ which is related to the permutation structure.

Lemma 6.3 *Let G be a graph. If G has a fixed-point-free embedding σ with p cycles, then*

$$\alpha^2(G) \geq p.$$

We note that the parameter $\alpha^2(G)$ will be used frequently to prove the main results of the next section.

6.1.1 Labeled-embedding and labeled fixed-point-free embedding of paths

We first determine the exact value of the labeled embedding number of paths. It is done by proving that the upper bound of Lemma 5.21 is reached for all paths of order at least 5.

Theorem 6.4 *Let P_n be a path of order at least 4. Then*

$$\lambda^2(P_n) = \begin{cases} \lfloor \frac{3n}{4} \rfloor & \text{if } n \bmod 4 = 0,2,3 \text{ and } n > 4, \\ \lfloor \frac{3n}{4} \rfloor + 1 & \text{if } n \bmod 4 = 1 \text{ and } n > 5, \\ 3 & \text{if } n = 5, \\ 1 & \text{if } n = 4. \end{cases}$$

Proof. Let $P_n = (v_1, v_2, \dots, v_n)$ be a path of order n where $n \geq 4$. We consider four cases as follows:

Case 1: $n \bmod 4 = 0,2,3$ *and* $n > 4$. From Lemma 5.21, we have $\lambda^2(P_n) \leq \lfloor \frac{3n}{4} \rfloor$, it then suffices to prove that P_n admits a $\lfloor \frac{3n}{4} \rfloor$-labeled embedding. Since P_n is a spanning subgraph of C_n, By putting together Lemma 5.20 and Corollary 5.33, we get $\lambda^2(P_n) \geq \lambda^2(C_n) = \lfloor \frac{3n}{4} \rfloor$.

Case 2: $n = 4m + 1$ *and* $m \geq 2$. A mapping $f : V(P_n) \longrightarrow \{1, 2, \dots, \lfloor \frac{3n}{4} \rfloor + 1\}$ is defined as follows:

$$\begin{array}{ll} f(v_{2i+1}) = i + 1 & \text{for } i = 0 \dots 2m. \\ f(v_{2i}) = 2m + i + 1 & \text{for } i = 1 \dots m. \\ f(v_{2i}) = m + i + 1 & \text{for } i = m + 1 \dots 2m. \end{array}$$

Let $\sigma(P_n)$ be the embedding of P_n in K_n under the permutation σ where

$$\begin{array}{ll} \sigma(v_{2i+1}) = v_{2i+1} & \text{for } i = 0 \dots 2m. \\ \sigma(v_{2i}) = v_{2(m+i)} & \text{for } i = 1 \dots m. \\ \sigma(v_{2i}) = v_{2(i-m)} & \text{for } i = m + 1 \dots 2m. \end{array}$$

This permutation induces the following path:

$$\sigma(P_n) = v_1, v_{2m+2}, v_3, v_{2m+4} \dots v_{2m+1}, v_2, v_{2m+3}, v_4, v_{2m+5} \dots v_{2m}, v_{4m+1}$$

According to the permutation σ, it is clear that for every vertex v_i in $V(P_n)$, we have $f(v_i) = f(\sigma(v_i))$. It now remains to show that P_n and $\sigma(P_n)$ are edge-disjoint. To do so, it suffices to show that:

for every vertex v_i of P_n, if $\sigma(v_i) = v_i$ then $\sigma(v_{i+1}) \notin \{v_{i-1}, v_{i+1}\}$.

From the permutation σ, we have $\sigma(v_i) = v_i$ if and only if i is odd. Then we consider four cases for the vertices $\sigma(v_{i+1})$ and $\sigma(v_{i-1})$:

1. for $i = 1$, we have $\sigma(v_2) = v_{2m+2}$.
2. for $i = 3, 5, \dots 2m-1$, we have $v_{i-1}, v_{i+1} \in \{v_2, v_4, \dots, v_{2m}\}$ and $\sigma(v_{i-1}), \sigma(v_{i+1}) \in \{v_{2m+2}, v_{2m+4}, \dots, v_{4m}\}$.
3. for $i = 2m + 1$, we have $\sigma(v_{2m}) = v_{4m}$ and $\sigma(v_{2m+2}) = v_2$.

4. for $i = 2m+3, \ldots 4m-3$, we have $v_{i-1}, v_{i+1} \in \{v_{2m+2}, v_{2m+4}, \ldots, v_{4m-2}\}$ and $\sigma(v_{i-1}), \sigma(v_{i+1}) \in \{v_2, v_4, \ldots, v_{2m-2}\}$.

5. for $i = 4m-1$, we have $\sigma(v_{4m}) = v_{2m}$.

According to the previous observations, we get that $E(P_n) \cap E(\sigma(P_n)) = \emptyset$ for $m \geq 2$, thus the mapping f is a $(\lfloor \frac{3n}{4} \rfloor + 1)$-labeled-embedding of P_n.

Case 3: $n = 5$. We first prove that $\lambda^2(P_5) < 4$. We assume that there exists a mapping $f : V(P_5) \longrightarrow L = \{1, 2, 3, 4\}$ which is 4-labeled embedding of P_5. We can easily see that the existence of this 4-labeled embedding requires the existence of three independent vertices v_1, v_3 and v_5 colored with three different labels $\{c_1, c_2, c_3\} \subset L$, then the two vertices v_2 and v_4 are colored with the same label of $L \setminus \{c_1, c_2, c_3\}$. We can easily see that this coloring scheme is not 4-labeled embedding of P_5 which is a contradiction with the hypothesis. We conclude that $\lambda^2(P_n) < 4$. It then remains to show that $\lambda^2(P_5) = 3$. This is proved by Figure 6.1.

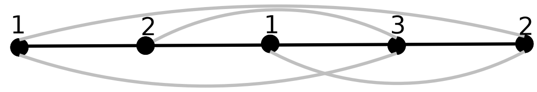


Figure 6.1: A 3-labeled embedding of P_5.

Case 4: $n = 4$. We can easily see that there is an unique embedding of P_4 into K_4 (Figure 6.2). This embedding scheme gives $\lambda^2(P_4) = 1$. $\square$

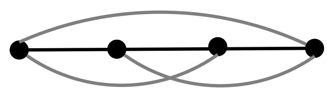


Figure 6.2: A 1-labeled embedding of P_4.

We now study the labeled fixed-point-free embedding number of paths. This result is crucial for the proof of our result in the next section.

Theorem 6.5 *Let P_n be a path of order at least 4, then*

$$\alpha^2(P_n) = \begin{cases} \lfloor \frac{n}{2} \rfloor & \textit{if } n > 5, \\ 1 & \textit{if } n = 4, 5. \end{cases}$$

Proof. Let $P_n = (v_1, v_2, \ldots, v_n)$ be a path of order n where $n \geq 4$. We consider two cases depending on whether $n > 5$ or $n \leq 5$.

Case 1: $n > 5$. From Lemma 6.1, we know that $\alpha(P_n) \leq \lfloor \frac{n}{2} \rfloor$. It now suffices to prove that P_n admits a $\lfloor \frac{n}{2} \rfloor$-labeled fixed-point-free embedding into K_n. We consider two subcases as follows:
Subcase 1: *n is even.* We divide P into three subpaths as follows:

$$P^1 = (v_1, v_2, \ldots, v_{\frac{n}{2}-3}) = (x_1, x_2, \ldots, x_{\frac{n}{2}-3}).$$
$$P^2 = (v_{\frac{n}{2}-2}, v_{\frac{n}{2}-1}, \ldots, v_{\frac{n}{2}+3}) = (y_1, y_2, \ldots, y_6).$$
$$P^3 = (v_{\frac{n}{2}+4}, v_{\frac{n}{2}+5}, \ldots, v_n) = (z_1, z_2, \ldots, z_{\frac{n}{2}-3}).$$

We define the permutation σ of P as follows:

$$\sigma(P) = (y_1, y_3)(y_2, y_5)(y_4, y_6)(x_1, z_1)(x_2, z_2)(x_3, z_3) \ldots (x_{\frac{n}{2}-3}, z_{\frac{n}{2}-3}).$$

It is clear that all cycles of σ have length two. Hence, we have $\alpha^2(P_n) = \frac{n}{2}$ if n is even.

Subcase 2: *n is odd.* In this case the path P is divided as follows:

$$P^1 = (v_1, v_2, \ldots, v_{\frac{n-7}{2}}) = (x_1, x_2, \ldots, x_{\frac{n-7}{2}}).$$
$$P^2 = (v_{\frac{n-7}{2}+1}, v_{\frac{n-7}{2}+2}, \ldots, v_{\frac{n-7}{2}+7}) = (y_1, y_2, \ldots, y_7).$$
$$P^3 = (v_{\frac{n-7}{2}+8}, v_{\frac{n-7}{2}+9}, \ldots, v_n) = (z_1, z_2, \ldots, z_{\frac{n-7}{2}-3}).$$

We define the permutation σ of P as follows:

$$\sigma(P) = (y_1, y_3)(y_2, y_5)(y_4, y_6, y_7)(x_1, z_1)(x_2, z_2)(x_3, z_3) \ldots (x_{\frac{n-7}{2}}, z_{\frac{n-7}{2}}).$$

Hence, we have $\alpha^2(P_n) = \frac{n-1}{2}$ if n is odd.

Case 2: $n = 4, 5$. Using the same proof idea as in the proof of Theorem 6.4 (Case 4), we show that $\alpha^2(P_4) = \alpha^2(P_5) = 1$. □

6.1.2 Labeled embedding of caterpillars

A caterpillar is a tree which becomes a path when all its end vertices are removed. The following theorem shows the exact value of the embedding number of caterpillars.

Theorem 6.6 *Let C be a non-star caterpillar of order n with $n \geq 4$ and let I be a maximum independent set of C. Then*

$$\lambda^2(C) = \begin{cases} |I| + \lfloor \frac{n-|I|}{2} \rfloor & \text{if } |V \setminus I| \geq 5 \text{ or if } C \text{ is isomorphic to one of the graphs} \\ & \text{of Figure 6.3,} \\ |I| + \lfloor \frac{n-|I|}{2} \rfloor - 1 & \text{if } C \text{ is isomorphic to one of the graphs of Figure 6.4,} \\ |I| + \lfloor \frac{n-|I|}{2} \rfloor - 2 & \text{if } C \text{ is isomorphic to one of the graphs of Figure 6.5.} \end{cases}$$

Proof. Let C be a non-star caterpillar; a maximum independent set I of C can be computed as follows: let $I = S_1 \cup S_2$, where S_1 is the set of vertices with degree one in C and S_2 is a maximum independent set of the subgraph of C induced by the vertices of degree two. Then, it follows from this construction that the vertices of $V \setminus I$ belong to the central path of C. Hence, we can deduce the following observation:

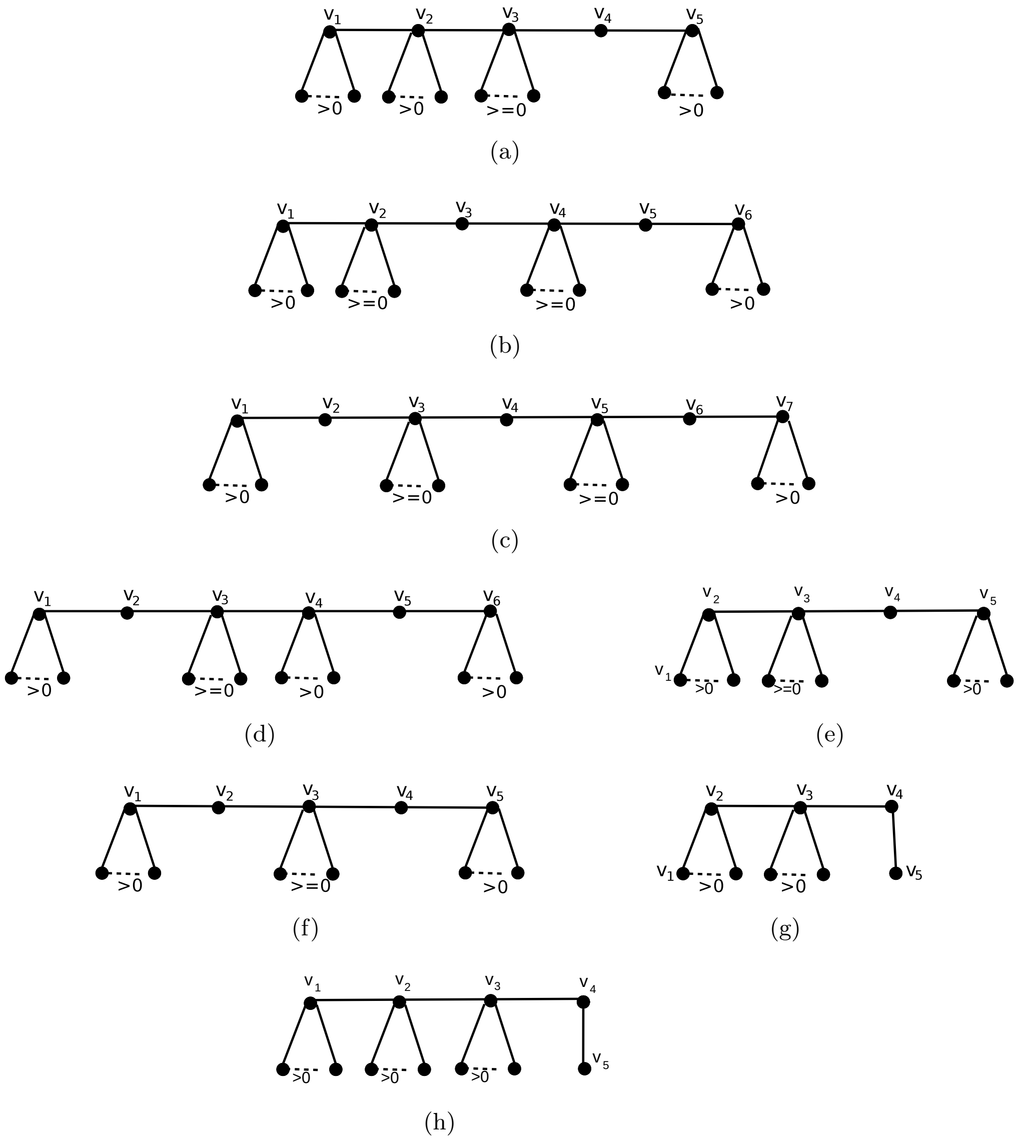


Figure 6.3: Caterpillars with $\lambda^2(G) = |I| + \lfloor \frac{n-|I|}{2} \rfloor$

Observation 6.7 *The induced subgraph $G[V \setminus I]$ is a set of paths.*

Let f be a vertex labeling of C and let $\sigma(C)$ be an embedding of C into K_n under the permutation σ. We distinguish three cases according to the cardinality of $V \setminus I$ as follows:

Case 1: $|V \setminus I| \geq 5$ *or C is isomorphic to one of the graphs of Figure. 6.3.* From Lemma 5.21, we have $\lambda^2(C) \leq |I| + \lfloor \frac{n-|I|}{2} \rfloor$, it then suffices to prove that C admits a $(|I| + \lfloor \frac{n-|I|}{2} \rfloor)$-labeled embedding. Three subcases are considered as follows:

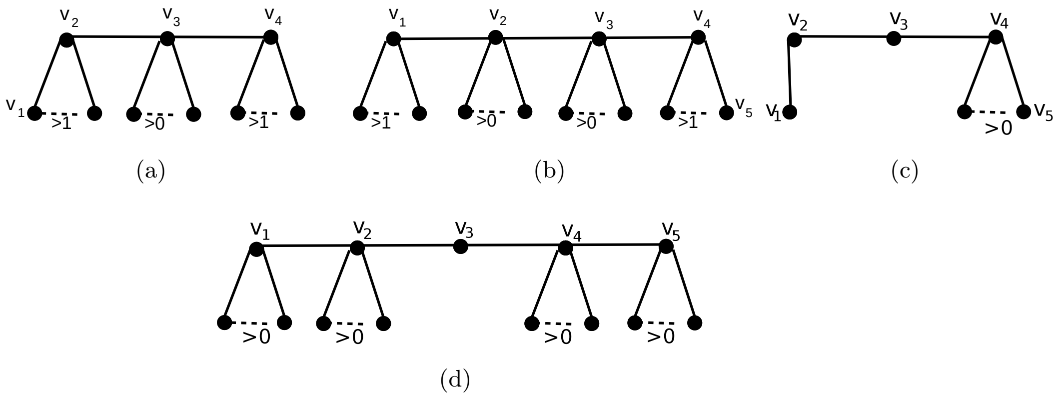


Figure 6.4: Caterpillars with $\lambda^2(G) = |I| + \lfloor \frac{n-|I|}{2} - 1 \rfloor$

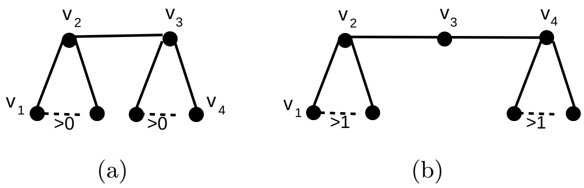


Figure 6.5: Caterpillars with $\lambda^2(G) = |I| + \lfloor \frac{n-|I|}{2} - 2 \rfloor$

Subcase 1: $|V \setminus I| \geq 6$. Let $H = C[V \setminus I]$. The vertices of I are considered as fixed points under σ, *i.e.,* for every vertex v of I, $\sigma(v) = v$ and $f(v) = f(\sigma(v))$. Let $P = (v_1, v_2, \ldots, v_m)$ be the central path of C. We augment the edge set $E(H)$ by connecting the two vertices v_{i-1}, v_{i+1} for all i such that $v_i \in I$. Since v_i is a fixed point, the edge v_{i-1}, v_{i+1} allows to avoid the following situation: $\sigma(v_{i-1}) = v_{i+1}$ or $\sigma(v_{i+1}) = v_{i-1}$. From Observation 6.7, the induced subgraph H is a path of order at least 6. Then, we can derive from Theorem 6.5 a $\lfloor \frac{|V \setminus I|}{2} \rfloor$-labeled fixed-point-free embedding of H. Therefore, we obtain by Theorem 6.2 that $\lambda^2(T) = |I| + \lfloor \frac{n-|I|}{2} \rfloor$.

Subcase 2: $|V \setminus I| = 5$. From Observation 6.7, we have that $C[V \setminus I]$ is a set of paths. We create a path H as follows: let $H = C[V \setminus I] \cup \{v\}$, where v is a vertex of I which is adjacent to an end-vertex of $C[V \setminus I]$. Then, the subgraph H is a set of paths of order 6. From Theorem 6.5, we know that $\alpha^2(C[V \setminus I]) = 3$. Hence we obtain from Lemma 6.2 that $\lambda^2(C) = |I| + 2$.

Subcase 3: *C is isomorphic to one of the graphs of Figure. 6.3.* We consider eight cases depending on the structure of C. Let L be the set of end vertices in C.

1. **Figure 6.3(a)**: let $S = L \cup \{v_4\} = \{u_1, u_2, \ldots, u_m\}$,

   For every vertex u_i of S, $f(u_i) = i$ and $\sigma(u_i) = u_i$.
   $f(v_1) = f(v_3) = m + 1$ and $f(v_2) = f(v_5) = m + 2$,

$\sigma(v_1) = v_3$, $\sigma(v_3) = v_1$, $\sigma(v_2) = v_5$ and $\sigma(v_5) = v_2$.

2. **Figure 6.3(b)**: let $S = L \cup \{v_3, v_5\} = \{u_1, u_2, \ldots, u_m\}$,

   For every vertex u_i of S, $f(u_i) = i$ and $\sigma(u_i) = u_i$.
   $f(v_1) = f(v_4) = m + 1$ and $f(v_2) = f(v_6) = m + 2$.
   $\sigma(v_1) = v_4$, $\sigma(v_4) = v_1$, $\sigma(v_2) = v_6$ and $\sigma(v_6) = v_2$.

3. **Figure 6.3(c)**: let $S = L \cup \{v_2, v_4, v_6\} = \{u_1, u_2, \ldots, u_m\}$,

   For every vertex u_i of S, $f(u_i) = i$ and $\sigma(u_i) = u_i$.
   $f(v_1) = f(v_5) = m + 1$ and $f(v_3) = f(v_7) = m + 2$.
   $\sigma(v_1) = v_5$, $\sigma(v_5) = v_1$, $\sigma(v_3) = v_7$ and $\sigma(v_7) = v_3$.

4. **Figure 6.3(d)**: let $S = L \cup \{v_2, v_5\} = \{u_1, u_2, \ldots, u_m\}$,

   For every vertex u_i of S, $f(u_i) = i$ and $\sigma(u_i) = u_i$.
   $f(v_1) = f(v_4) = m + 1$, $f(v_3) = f(v_6) = m + 2$.
   $\sigma(v_1) = v_4$, $\sigma(v_4) = v_1$, $\sigma(v_3) = v_6$ and $\sigma(v_6) = v_3$.

5. **Figure 6.3(e)**: let $S = (L \setminus \{v_1\}) \cup \{v_4\} = \{u_1, u_2, \ldots, u_m\}$,

   For every vertex u_i of S, $f(u_i) = i$ and $\sigma(u_i) = u_i$.
   $f(v_1) = f(v_3) = m + 1$ and $f(v_2) = f(v_5) = m + 2$.
   $\sigma(v_1) = v_3$, $\sigma(v_3) = v_1$, $\sigma(v_2) = v_5$ and $\sigma(v_5) = v_2$.

6. **Figure 6.3(f)**: let $S = L \cup \{v_4\} = \{u_1, u_2, \ldots, u_m\}$,

   For every vertex u_i of S, $f(u_i) = i$ and $\sigma(u_i) = u_i$.
   $f(v_1) = f(v_3) = m + 1$ and $f(v_2) = f(v_5) = m + 2$.
   $\sigma(v_1) = v_3$, $\sigma(v_3) = v_1$, $\sigma(v_2) = v_5$, and $\sigma(v_5) = v_2$.

7. **Figure 6.3(g)**: let $S = (L \setminus \{v_1, v_5\}) \cup \{v_4\} = \{u_1, u_2, \ldots, u_m\}$,

   For every vertex u_i of S, $f(u_i) = i$ and $\sigma(u_i) = u_i$.
   $f(v_1) = f(v_3) = m + 1$ and $f(v_2) = f(v_5) = m + 2$.
   $\sigma(v_1) = v_3$, $\sigma(v_3) = v_1$,$\sigma(v_2) = v_5$, and $\sigma(v_5) = v_2$.

8. **Figure 6.3(h)**: let $S = (L \setminus \{v_5\}) \cup \{v_4\} = \{u_1, u_2, \ldots, u_m\}$,

$$\begin{gathered}\text{For every vertex } u_i \text{ of } S,\ f(u_i) = i \text{ and } \sigma(u_i) = u_i.\\ f(v_1) = f(v_3) = m+1 \text{ and } f(v_2) = f(v_5) = m+2.\\ \sigma(v_1) = v_3,\ \sigma(v_2) = v_5,\ \sigma(v_3) = v_1 \text{ and } \sigma(v_5) = v_2.\end{gathered}$$

Case 2: *C is isomorphic to one of the graphs of Figure 6.4.* We can easily check that $\lambda^2(C) \leq |I| + \lfloor \frac{n-|I|}{2} \rfloor - 1$. It then suffices to prove that $\lambda^2(C) \geq |I| + \lfloor \frac{n-|I|}{2} \rfloor - 1$ for every graph of Figure 6.4.

1. **Figure 6.4(a)**: let $S = L \setminus \{v_1\} = \{u_1, u_2, \ldots, u_m\}$,

$$\begin{gathered}\text{For every vertex } u_i \text{ of } S,\ f(u_i) = i \text{ and } \sigma(u_i) = u_i.\\ f(v_1) = f(v_2) = f(v_3) = f(v_4) = m+1.\\ \sigma(v_1) = v_3,\ \sigma(v_2) = v_1,\ \sigma(v_3) = v_4 \text{ and } \sigma(v_4) = v_2.\end{gathered}$$

2. **Figure 6.4(b)**: let $S = L\{u_1, u_2, \ldots, u_m\}$,

$$\begin{gathered}\text{For every vertex } u_i \text{ of } S,\ f(u_i) = i \text{ and } \sigma(u_i) = u_i.\\ f(v_1) = f(v_2) = f(v_3) = f(v_4) = m+1.\\ \sigma(v_1) = v_3,\ \sigma(v_2) = v_1,\ \sigma(v_3) = v_4 \text{ and } \sigma(v_4) = v_2.\end{gathered}$$

3. **Figure 6.4(c)**: let $S = (L \cup \{v_1, v_5\}) \cup \{v_2\} = \{u_1, u_2, \ldots, u_m\}$,

$$\begin{gathered}\text{For every vertex } u_i \text{ of } S,\ f(u_i) = i \text{ and } \sigma(u_i) = u_i.\\ f(v_1) = f(v_4) = m+1,\ f(v_3) = f(v_5) = m+2.\\ \sigma(v_1) = v_4,\ \sigma(v_4) = v_1,\ \sigma(v_3) = v_5 \text{ and } \sigma(v_5) = v_3.\end{gathered}$$

4. **Figure 6.4(d)**: let $S = (L \setminus \cup \{v_3\} = \{u_1, u_2, \ldots, u_m\}$,

$$\begin{gathered}\text{For every vertex } u_i \text{ of } S,\ f(u_i) = i \text{ and } \sigma(u_i) = u_i.\\ f(v_1) = f(v_2) = f(v_3) = f(v_4) = m+1.\\ \sigma(v_1) = v_3,\ \sigma(v_2) = v_1,\ \sigma(v_3) = v_4 \text{ and } \sigma(v_4) = v_2.\end{gathered}$$

Case 3: *C is isomorphic to one of the graphs of Figure 6.5.* In this case, we can also check that $\lambda^2(C) \leq |I| + \lfloor \frac{n-|I|}{2} \rfloor - 2$. It then suffices to prove that $\lambda^2(T) \geq |I| + \lfloor \frac{n-|I|}{2} \rfloor - 2$ for every graph of Figure 6.5.

1. **Figure 6.5(a)**: let $S = L \setminus \{v_1, v_4\} = \{u_1, u_2, \ldots, u_m\}$,

$$\begin{gathered}\text{For every vertex } u_i \text{ of } S,\ f(u_i) = i \text{ and } \sigma(u_i) = u_i.\\ f(v_1) = f(v_2) = f(v_3) = f(v_4) = m+1.\\ \sigma(v_1) = v_2,\ \sigma(v_2) = v_4,\ \sigma(v_3) = v_1 \text{ and } \sigma(v_4) = v_3.\end{gathered}$$

2. **Figure 6.5(b)**: let $S = L \setminus \{v_1\} = \{u_1, u_2, \ldots, u_m\}$,

$$\begin{array}{c} \text{For every vertex } u_i \text{ of } S,\ f(u_i) = i \text{ and } \sigma(u_i) = u_i. \\ f(v_1) = f(v_2) = f(v_3) = f(v_4) = m+1. \\ \sigma(v_1) = v_2,\ \sigma(v_2) = v_4,\ \sigma(v_3) = v_1 \text{ and } \sigma(v_4) = v_3. \end{array}$$

□

6.2 Labeled embedding of trees

In this section, we intend to prove a lower bound on the labeled embedding number of general trees.

Theorem 6.8 *Let T be a non-star tree of order n and let X be the set of end-vertices of T, then*

$$\begin{array}{ll} \lambda^2(T) \geq max(|X| + \lfloor \frac{n-|X|}{2} \rfloor - 1, \lfloor \frac{3n}{4} \rfloor - 2) & \textit{if } T \setminus X \neq K_{1,n-1}, \\ \lambda^2(T) \geq \lfloor \frac{3n}{4} \rfloor - 2 & \textit{otherwise.} \end{array}$$

The lower bound of the previous theorem relies on two main theorems (Theorem 6.10 and Theorem 6.11). We first introduce the following notation to simplify our proofs. Let $T = (V, E)$ be a graph of order n. For a vertex $v \in V$, let $d(v)$ denote its degree and let $diam(T)$ denote the diameter of T. A permutation σ of $V(T)$ will be called *good* for T if and only if:

1. σ is an embedding of T.
2. σ has at least $\lfloor \frac{3n}{4} \rfloor - 2$ cycles.
3. for every pair of end-vertices (u, v) in T, we have $\sigma(u) \neq v$ and $\sigma(v) \neq u$.

Thus, to prove that $\lambda^2(T) \geq \lfloor \frac{3n}{4} \rfloor - 2$, it suffices to show that T admits a good permutation. Finally, for a vertex v of T, let Tv be the subtree rooted at v and let $V(Tv)$ denote the set of vertices of Tv. At the beginning, we prove the following theorem which will be useful for the proof of Theorem 6.10.

Theorem 6.9 *Let T be a tree of order n and diameter 4, then there exists an embedding σ of T such that σ is good.*

Proof. If T is a caterpillar, then we have by Theorem 6.6 that $\lambda^2(T) \geq \lfloor \frac{3n}{4} \rfloor - 2$. In the other case, the level 2 of T contains at least three vertices of degree greater than 1 (see Figure 6.6). Let $U = \{u_1, u_2, \ldots, u_m\}$ be the set of vertices of T with degree at least 2 except x and y. Let $\sigma(T)$ be an embedding of T into K_n under the permutation σ where

$\sigma(x) = t$, $\sigma(t) = x$, $\sigma(y) = v$ and $\sigma(v) = y$.
For every vertex u of degree 1 in T, where $u \notin \{t, v\}$, $\sigma(u) = u$.
For every vertex $u_i \in U$, $\sigma(u_i) = \begin{cases} u_{(i+1) \bmod m} & \textit{if } i \textit{ is odd}, \\ u_{i-1} & \textit{otherwise}. \end{cases}$

From this permutation scheme, we can see that the number of cycles of σ is at least $|X| + \lfloor \frac{n-|X|}{2} \rfloor - 1$, where X is the set of end-vertices of T. Since for a tree of diameter 4, $|X| \geq \lfloor \frac{n-1}{2} \rfloor$, we get $|X| + \lfloor \frac{n-|X|}{2} \rfloor - 1 \geq \lfloor \frac{3n}{4} \rfloor - 2$. Hence, σ is good permutation of T. □

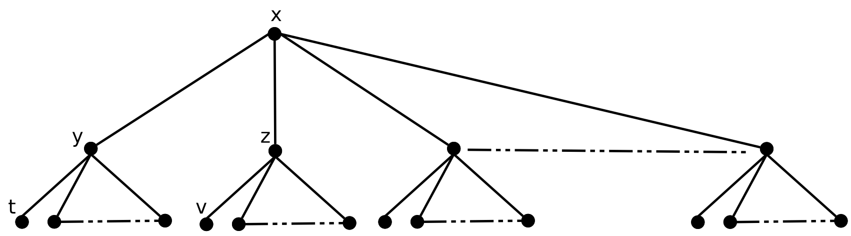


Figure 6.6: A tree of diameter 4.

The following theorem gives our first lower bound on the labeled embedding number of general trees.

Theorem 6.10 *Let $T = (V, E)$ be a tree of order n. If $T \neq K_{1,n-1}$, then there exists an embedding σ of T such that σ is good.*

Proof. The proof is done by induction on the number of vertices n. For $n = 4$, there is only one tree which is not a star, namely P_4. By Theorem 6.4, we have $\lambda^2(P_4) = 1$. Then, our theorem is true for $n = 4$. Let $n \geq 5$ and suppose that the theorem has been proved for all $n' < n$. We shall consider two main cases as follows:
Case 1: $diam(T) = 3$. We choose a set of vertices $X \subset V$ such that $T \setminus X = P_4$. The induction hypothesis guarantees the existence of a good permutation σ' on P_4. We can easily see that σ' is fixed-point-free permutation. Then, σ' can be extended to a good permutation σ of T as follows: for every vertex $x \in X, \sigma(x) = x$. Hence σ is a good permutation of T.
Case 2: $diam(T) = 4$. It follows by Theorem 6.9 that $\lambda^2(T) \geq \lfloor \frac{3n}{4} \rfloor - 2$.
Case 3: $diam(T) \geq 5$. We choose $D = (u_1, u_1, \ldots, u_m)$ as a diameter of T and we root T at a vertex $r \notin (V(Tu_2) \cup V(Tu_{m-1}))$ as shown in Figure 6.7.
Two main subcases are considered as follows:
Subcase 3.1: $d(u_2) = 2$ or $d(u_{m-1}) = 2$. Wlog, assume that $d(u_2) = 2$. We consider the tree $T' = T \setminus (V(Tu_2) \cup V(Tu_{m-1}))$. We can easily see that if T' is a star then T is a caterpillar. Hence we have by Theorem 6.6 that $\lambda^2(T) \geq \lfloor \frac{3n}{4} \rfloor - 2$. We Suppose now that $T' \neq K_{1,m-1}$, then the induction hypothesis guarantees the existence of a good permutation σ' on $V(T')$. Let us we consider the following

proposition: $*: \sigma(u_3) = u_{m-2}$ *or* $\sigma(u_{m-2}) = u_3$. The permutation σ' can be extended to a good permutation σ of T as follows:
for every vertex $v \in (V(Tu_2) \cup V(Tu_{m-1})) \setminus \{u_1, u_2, u_{m-1}\}$, $\sigma(v) = v$.
For every vertex $v \in V(T')$, $\sigma(v) = \sigma'(v)$.

$$\sigma(u_2) = \begin{cases} u_2 & \text{if } * \textit{ holds}, \\ u_{m-1} & \textit{otherwise}. \end{cases} \quad \sigma(u_{m-1}) = \begin{cases} u_1 & \text{if } * \textit{ holds}, \\ u_2 & \textit{otherwise}. \end{cases}$$

$$\sigma(u_1) = \begin{cases} u_{m-1} & \text{if } * \textit{ holds}, \\ u_1 & \textit{otherwise}. \end{cases}$$

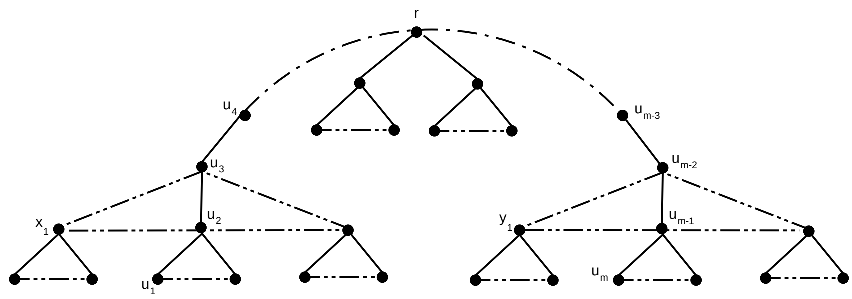
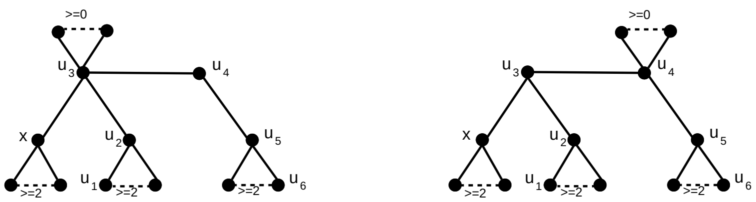


Figure 6.7: Tree notation.

From this permutation scheme, we obtain the following equation:
$\lambda^2(T) \geq \lambda^2(T') + |V(Tu_2)| + |V(Tu_{m-1})| - 1$.
By hypothesis, we have that $\lambda^2(T') \geq \lfloor \frac{3n'}{4} \rfloor - 2$, and thus
$\lambda^2(T) \geq \lfloor \frac{3(n-|V(Tu_2)|-|V(Tu_{m-1})|)}{4} \rfloor + |V(Tu_2)| + |V(Tu_{m-1})| - 3$.
Since $|V(Tu_2)| + |V(Tu_{m-1})| \geq 4$, we get $\lambda^2(T) \geq \lfloor \frac{3n}{4} \rfloor - 2$. Hence σ is a good permutation of T.

Subcase 3.2: $d(u_2) \geq 3$ and $d(u_{m-1}) \geq 3$. Three cases are considered as follows:
(a) $diam(Tu_3) = 4$ or $diam(Tu_{m-2}) = 4$. Wlog, assume that $diam(Tu_3) = 4$. We consider the tree $T' = T \setminus (V(Tu_2) \cup V(Tx_1) \cup V(Tu_{m-1}))$. If T' is a star, then T is isomorphic to one of the graphs of Figure 6.8. In this case, we choose a set of vertices $X \subset V$ such that $T \setminus X = (u_1, u_2, u_3, u_4)$. The induction hypothesis guarantees the existence of a good permutation σ' on (u_1, u_2, u_3, u_4). We can easily see that σ' is fixed-point-free permutation. Then, σ' can be extended to a good permutation σ of T as follows: $\sigma(x) = u_5$, $\sigma(u_5) = x$ and for every vertex $v \in X \setminus \{x, u_5\}, \sigma(v) = v$. Hence σ is a good permutation of T.

We suppose now that $T' \neq K_{1,m-1}$, then the induction hypothesis guarantees the existence of a good permutation σ' on $V(T')$. The permutation σ' can be extended to a good permutation σ of T as follows:
for every vertex $v \in (V(Tu_2) \cup V(Tx_1) \cup V(Tu_{m-1})) \setminus \{u_1, u_2, x_1, u_{m-1}\}$, $\sigma(v) = v$.
For every vertex $v \in V(T')$, $\sigma(v) = \sigma'(v)$.



(a) (b)

Figure 6.8: Forbidden graphs

$$\sigma(u_2) = \begin{cases} x_1 & \text{if } * \ holds, \\ u_{m-1} & otherwise. \end{cases} \quad \sigma(u_{m-1}) = \begin{cases} u_1 & \text{if } * \ holds, \\ u_2 & otherwise. \end{cases}$$
$$\sigma(x_1) = \begin{cases} u_2 & \text{if } * \ holds, \\ u_1 & otherwise. \end{cases} \quad \sigma(u_1) = \begin{cases} u_{m-1} & \text{if } * \ holds, \\ x_1 & otherwise. \end{cases}$$

From this permutation scheme, we obtain the following equation:
$\lambda^2(T) \geq \lambda^2(T') + |V(Tu_2)| + |V(Tx_1)| + |V(Tu_{m-1})| - 2$.
By hypothesis, we have that $\lambda^2(T') \geq \lfloor \frac{3n'}{4} \rfloor - 2$, and thus
$\lambda^2(T) \geq \lfloor \frac{3(n-|V(Tu_2)|-|V(Tx_1)|-|V(Tu_{m-1})|)}{4} \rfloor + |V(Tu_2)| + |V(Tx_1)| + |V(Tu_{m-1})| - 4$.
Since $|V(Tu_2)| + |V(Tx_1)| + |V(Tu_{m-1})| \geq 8$, we get $\lambda^2(T) \geq \lfloor \frac{3n}{4} \rfloor - 2$. Hence σ is a good permutation of T.
(b) $diam(Tu_3) = 3$ or $diam(Tu_{m-2}) = 3$. Wlog, assume that $diam(Tu_3) = 3$. We consider the tree $T' = T \setminus (V(Tu_3) \cup V(Tu_{m-1}))$. We can easily see that if T' is a star then T is a caterpillar. Hence we have by Theorem 6.6 that $\lambda^2(T) \geq \lfloor \frac{3n}{4} \rfloor - 2$. We Suppose now that $T' \neq K_{1,m-1}$, then the induction hypothesis guarantees the existence of a good permutation σ' on $V(T')$. The permutation σ' can be extended to a good permutation σ of T as follows:

$$\sigma(u_1) = u_3,\ \sigma(u_3) = u_1,\ \sigma(u_2) = u_{m-1} \text{ and } \sigma(u_{m-1}) = u_2.$$
$$\text{for every vertex } v \in (V(Tu_3) \cup V(Tu_{m-1})) \setminus \{u_1, u_2, u_3, u_{m-1}\},\ \sigma(v) = v.$$
$$\text{for every vertex } v \in V(T'),\ \sigma(v) = \sigma'(v).$$

From this permutation scheme, we obtain the following equation:
$\lambda^2(T) \geq \lambda^2(T') + |V(Tu_3)| + |V(Tu_{m-1})| - 2$.
By hypothesis, we have that $\lambda^2(T') \geq \lfloor \frac{3n'}{4} \rfloor - 2$ and thus
$\lambda^2(T) \geq \lfloor \frac{3(n-|V(Tu_3)|-|V(Tu_{m-1})|)}{4} \rfloor + |V(Tu_3)| + |V(Tu_{m-1})| - 4$.
Since $|V(Tu_3)| + |V(Tu_{m-1})| \geq 8$, we get $\lambda^2(T) \geq \lfloor \frac{3n}{4} \rfloor - 2$. Hence σ is a good permutation of T.
(c) $diam(Tu_3) = 2$ and $diam(Tu_{m-2}) = 2$. Consider the tree $T' = T \setminus (V(Tu_2) \cup V(Tu_{m-1}))$. We can easily see that if T' is a star then T is a caterpillar. Hence we have by Theorem 6.6 that $\lambda^2(T) \geq \lfloor \frac{3n}{4} \rfloor - 2$. We Suppose now that $T' \neq K_{1,m-1}$, then the induction hypothesis guarantees the existence of a good permutation σ' on $V(T')$. Since u_2 and u_{m-1} are two end-vertices in T', we get by the property (4) of σ' that $\sigma'(u_2) \neq u_{m-1}$ and $\sigma'(u_{m-1}) \neq u_2$. Then, the permutation σ' can be extended to a good permutation σ of T as follows:

$$\sigma(u_2) = u_{m-1} \text{ and } \sigma(u_{m-1}) = u_2.$$
$$\text{for every vertex } v \in (V(Tu_2) \cup V(Tu_{m-1})) \setminus \{u_2, u_{m-1}\},\ \sigma(v) = v.$$
$$\text{for every vertex } v \in V(T'),\ \sigma(v) = \sigma'(v).$$

From this permutation scheme, we obtain the following equation:
$\lambda^2(T) \geq \lambda^2(T') + |V(Tu_2)| + |V(Tu_{m-1})| - 1$.
By hypothesis, we have that $\lambda^2(T') \geq \lfloor \frac{3n'}{4} \rfloor - 2$ and thus
$\lambda^2(T) \geq \lfloor \frac{3(n-|V(Tu_2)|-|V(Tu_{m-1})|}{4} \rfloor + |V(Tu_2)| + |V(Tu_{m-1})| - 3$.
Since $|V(Tu_2)| + |V(Tu_{m-1})| \geq 6$, we get $\lambda^2(T) \geq \lfloor \frac{3n}{4} \rfloor - 2$. So σ is a good permutation of T. □

We now give a second lower bound on $\lambda^2(T)$ that depends on the number of end-vertices of T.

Theorem 6.11 *Let T be a tree of order n and let X be the set of end-vertices of T. If $T \setminus X \neq K_{1,n-1}$, then*

$$\lambda^2(T) \geq |X| + \lfloor \frac{n - |X|}{2} \rfloor - 1.$$

We can clearly see that Theorem 6.11 can be proved as a consequence of Lemma 6.2 and the following theorem.

Theorem 6.12 *Let $T = (V, E)$ be a non star tree of order n, then*

$$\alpha^2(T) \geq \lfloor \frac{n}{2} \rfloor - 1.$$

In the following, a permutation σ of $V(T)$ will be called *good* for T if and only if:

1. σ is an fixed-point-free embedding of T.
2. σ has at least $\lfloor \frac{n}{2} \rfloor - 1$ cycles.

Using the same proof technique as in Theorem 6.9, we can show the following:

Theorem 6.13 *Let T be a tree of order n and diameter 4, then*

$$\alpha^2(T) \geq \lfloor \frac{n}{2} \rfloor - 1.$$

Proof of Theorem 6.12. The proof is done by induction on the number of vertices n. The theorem is true for $n = 4, 5$. Let $n \geq 6$ and suppose that the theorem has been proved for all $n' < n$. We shall consider four main cases as follows:
Case 1: *$diam(T) = 3$.* We use the notation of Figure 6.9 which represents a double star with two non leaf vertices. Three subcases are considered as follows:
Subcase 1: *$d(v_2) \geq 4$ or $d(v_3) \geq 4$.* Wlog, assume that $d(v_2) \geq 4$. Then, let x_1 and x_2 be two end-vertices adjacent to v_2. Consider the tree $T' = T \setminus \{x_1, x_2\}$. Thus, the induction hypothesis guarantees the existence of a good permutation σ' on $V(T')$. This permutation can be extended to a good permutation σ of T as

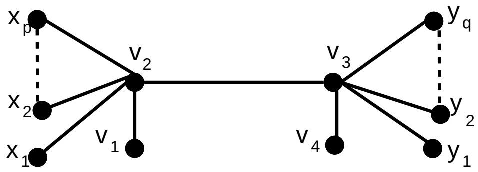


Figure 6.9: A double star notation.

follows: $\sigma(x_1) = x_2$, $\sigma(x_2) = x_1$ and for every vertex $v \in V(T')$, $\sigma(v) = \sigma'(v)$.
Subcase 2: $d(v_2) = 3$ *and* $d(v_3) = 3$. We consider the tree $T' = T \setminus \{x_1, y_1\} = P_4$. We know that there exists a good permutation σ' on $V(T')$ such that $\sigma(v_2) \neq v_3$ and $\sigma(v_3) \neq v_2$. This permutation can be extended to a good permutation σ of T as follows: $\sigma(x_1) = y_1$, $\sigma(y_1) = x_1$ and for every vertex $v \in V(T')$, $\sigma(v) = \sigma'(v)$.
Case 2: $diam(T) = 4$. It follows by Theorem 6.13 that $\alpha^2(T) \geq \lfloor \frac{n}{2} \rfloor - 1$.
Case 3: $diam(T) \geq 5$. Two subcases are considered as follows:
Subcase 3.1: *there exists a vertex v in T such that v is adjacent to at least two end vertices of T.* Let x and y be two end vertices of T which is adjacent to v. Consider the tree $T' = T \setminus \{x, y\}$. Thus, the induction hypothesis guarantees the existence of a good permutation σ' on $V(T')$. This permutation can be extended to a good permutation σ of T as follows: $\sigma(x) = y$ and $\sigma(y) = x$. Hence σ is a good permutation of T.
Subcase 3.2: *For every vertex v of T, v is adjacent is adjacent to at most one end vertex of T.* We choose $D = (u_1, u_1, \ldots, u_m)$ as a diameter of T and we root T at a vertex $r \notin (V(Tu_2) \cup V(Tu_{m-1}))$. Three cases are considered according to whether $diam(Tu_3) = 4, 3$ or 2.
(a): $diam(Tu_3) = 4$. The vertex u_3 has at least two children vertices of degree 2 (see Figure 6.10). Consider the tree $T' = T \setminus \{x_1, y_1, x_2, y_2, u_{m-1}, u_m\}$. Suppose that

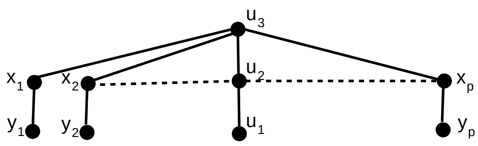


Figure 6.10: Tu_3

$T' \neq K_{1,m-1}$, the case where $T' = K_{1,m-1}$ is left to the reader. Thus, the induction hypothesis guarantees the existence of a good permutation σ' on $V(T')$. This permutation can be extended to a good permutation σ of T as follows: $\sigma(x_1) = x_2$, $\sigma(x_2) = x_1$, $\sigma(y_1) = u_m$, $\sigma(u_m) = y_1$, $\sigma(y_2) = u_{m-1}$, $\sigma(u_{m-1}) = y_2$ and for every vertex $v \in V(T')$, $\sigma(v) = \sigma'(v)$. Hence σ is a good permutation of T.
(b): $diam(Tu_3) = 3$. In this case, the vertex u_3 is adjacent to two children vertices: an internal vertex u_2 and an end vertex x. Consider the tree $T' = T \setminus \{u_1, u_2, x, u_m\}$. Suppose that $T' \neq K_{1,m-1}$, the case where $T' = K_{1,m-1}$ is left to the reader. Thus, the induction hypothesis guarantees the existence of a good permutation σ' on $V(T')$. This permutation can be extended to a good permutation σ of T as

follows: $\sigma(u_1) = u_m$, $\sigma(u_m) = u_1$, $\sigma(x) = u_2$, $\sigma(u_2) = x$ and for every vertex $v \in V(T')$, $\sigma(v) = \sigma'(v)$. Hence σ is a good permutation of T.
(c): $diam(Tu_3) = 2$. Consider the tree $T' = T \setminus \{u_1, u_2, u_3, u_m\}$. Suppose that $T' \neq K_{1,m-1}$, the case where $T' = K_{1,m-1}$ is left to the reader. Thus, the induction hypothesis guarantees the existence of a good permutation σ' on $V(T')$. This permutation can be extended to a good permutation σ of T as follows: $\sigma(u_1) = u_3$, $\sigma(u_3) = u_1$, $\sigma(u_2) = u_m$, $\sigma(u_m) = u_2$ and for every vertex $v \in V(T')$, $\sigma(v) = \sigma'(v)$. Hence σ is a good permutation of T.
□

6.3 Labeled embedding of $(n, n-2)$ graphs

In this section, a permutation σ on $V(G)$ is said to be a *good permutation* for G if σ is an embedding of G and the number of its cycle is at least $\lfloor \frac{2n}{3} \rfloor$. So we will prove that if $|E(G)| \leq n-2$ then there exists a good permutation for G. More formally, we have to prove that.

Theorem 6.14 *Let G be a $(n, n-2)$ graph, then $\lambda^2(G) \geq \lfloor \frac{2n}{3} \rfloor$.*

Proof. The proof is by induction on n. Without loss of generality, we can assume that $|E(G)| = n-2$, the theorem is true for $n = 3, 4, 5$. Assume it is true for every graph of order $n' < n$, where $n' \geq 6$. The components of G must include at least two non-trivial trees T and H. Let $D_1 = (x_1, x_2, \ldots, x_{p-1}, x_p)$ and $D_2 = (y_1, y_2, \ldots, y_{q-1}, y_q)$ be the diameter of T and H, respectively. We root T and H at the leaves x_p and y_q, respectively. We shall consider four main cases as follows:

Case 1: $p, q \geq 4$. We consider three subcases as follows:

Subcase 1.1: *$d(x_2) = 2$ or $d(y_2) = 2$.* Wlog, assume that $d(x_2) = 2$. Consider the graph $G' = G \setminus (V(Tx_2) \cup V(Hy_2))$. So the induction hypothesis guarantees the existence of a good permutation σ' on $V(G')$. The permutation σ' can be extended to a good permutation σ of G as follows:
for every vertex $v \in (V(Tx_2) \cup V(Hy_2)) \setminus \{x_1, x_2, y_2\}$, $\sigma(v) = v$.
For every vertex $v \in V(G')$, $\sigma(v) = \sigma'(v)$.

$$\sigma(x_2) = \begin{cases} x_2 & \text{if } \sigma'(x_3) = y_3, \\ y_2 & otherwise. \end{cases} \quad \sigma(y_2) = \begin{cases} x_1 & \text{if } \sigma'(x_3) = y_3, \\ x_2 & otherwise. \end{cases}$$

$$\sigma(x_1) = \begin{cases} y_2 & \text{if } \sigma'(x_3) = y_3, \\ x_1 & otherwise. \end{cases}$$

From this permutation scheme, we obtain the following equation:
$\lambda^2(G) \geq \lambda^2(G') + |V(Tx_2)| + |V(Hy_2)| - 1$.
By hypothesis, we have that $\lambda^2(G') \geq \lfloor \frac{2n'}{3} \rfloor$, then
$\lambda^2(G) \geq \lfloor \frac{2(n - |V(Tx_2)| - |V(Hy_2)|)}{3} \rfloor + |V(Tx_2)| + |V(Hy_2)| - 1$.

By hypothesis, we have that $|V(Tx_2)| + |V(Hy_2)| \geq 4$, hence $\lambda^2(G) \geq \lfloor \frac{2n}{3} \rfloor$. So σ is a good permutation of G.

Subcase 1.2: *$d(x_2) \geq 3$ and $d(y_2) \geq 3$.* Three cases are considered as follows:
(a) $diam(x_3) = 4$ or $diam(y_3) = 4$. Wlog, assume that $diam(Tx_2) = 4$ (see Figure 6.11). Consider the graph $G' = G \setminus (V(Tx_2) \cup V(Tz_1) \cup V(Hy_2))$. So the induction hypothesis guarantees the existence of a good permutation σ' on $V(G')$. The permutation σ' can be extended to a good permutation σ of G as follows:
for every vertex $v \in (V(Tx_2) \cup V(Tz_1) \cup V(Hy_2)) \setminus \{x_1, x_2, z_1, y_2\}$, $\sigma(v) = v$.
For every vertex $v \in V(G')$, $\sigma(v) = \sigma'(v)$.

$$\sigma(x_2) = \begin{cases} z_1 & \text{if } \sigma'(x_3) = y_3, \\ y_2 & otherwise. \end{cases} \quad \sigma(y_2) = \begin{cases} x_1 & \text{if } \sigma'(x_3) = y_3, \\ x_2 & otherwise. \end{cases}$$
$$\sigma(x_1) = \begin{cases} y_2 & \text{if } \sigma'(x_3) = y_3, \\ z_1 & otherwise. \end{cases} \quad \sigma(z_1) = \begin{cases} x_2 & \text{if } \sigma'(x_3) = y_3, \\ x_1 & otherwise. \end{cases}$$

From this permutation scheme, we obtain the following equation:
$\lambda^2(G) \geq \lambda^2(G') + |V(Tx_2)| + |V(Tz_1)| + |V(Hy_2)| - 2$.
By hypothesis, we have that $\lambda^2(G') \geq \lfloor \frac{2n'}{3} \rfloor$, then
$\lambda^2(G) \geq \lfloor \frac{2(n - |V(Tx_2)| - |V(Tz_1)| - |V(Hy_2)|)}{3} \rfloor + |V(Tx_2)| + |V(Tz_1)| + |V(Hy_2)| - 2$.
By hypothesis, we have that $|V(Tx_2)| + |V(Tz_1)| + |V(Hy_2)| \geq 8$, hence $\lambda^2(G) \geq \lfloor \frac{2n}{3} \rfloor$. So σ is a good permutation of G.

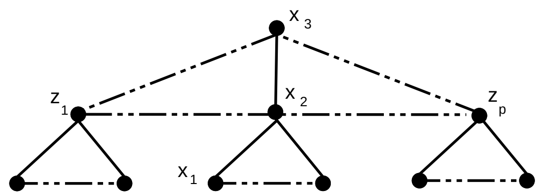


Figure 6.11: Tx_3.

(b) $diam(x_3) = 3$ or $diam(y_3) = 3$. Wlog, assume that $diam(Tx_2) = 3$. Consider the graph $G' = G \setminus (V(Tx_3) \cup V(Hy_2))$. So the induction hypothesis guarantees the existence of a good permutation σ' on $V(G')$. The permutation σ' can be extended to a good permutation σ of G as follows:

$$\sigma(x_1) = x_3,\ \sigma(x_3) = x_1,\ \sigma(x_2) = y_2 \text{ and } \sigma(y_2) = x_2.$$
$$\text{for every vertex } v \in (V(Tx_3) \cup V(Hy_2)) \setminus \{x_1, x_2, x_3, y_2\},\ \sigma(v) = v.$$
$$\text{for every vertex } v \in V(G'),\ \sigma(v) = \sigma'(v).$$

From this permutation scheme, we obtain the following equation:
$\lambda^2(G) \geq \lambda^2(G') + |V(Tx_3)| + |V(Hy_2)| - 2$.
By hypothesis, we have that $\lambda^2(G') \geq \lfloor \frac{2n'}{3} \rfloor$, then
$\lambda^2(T) \geq \lfloor \frac{2(n - |V(Tx_3)| - |V(Hy_2)|)}{3} \rfloor + |V(Tx_3)| + |V(Hy_2)| - 2$.
By hypothesis, we have that $|V(Tx_3)| + |V(Hy_2)| \geq 8$, hence $\lambda^2(G) \geq \lfloor \frac{2n}{3} \rfloor$. So σ

is a good permutation of G.
(c) $diam(x_3) = 2$ and $diam(y_3) = 2$. consider the graph $G' = G \backslash (V(Tx_3) \cup V(Hy_3))$. So the induction hypothesis guarantees the existence of a good permutation σ' on $V(G')$. The permutation σ' can be extended to a good permutation σ of G as follows:

$$\sigma(x_3) = y_1,\ \sigma(y_1) = x_3,\ \sigma(x_1) = y_3 \text{ and } \sigma(y_3) = x_1.$$
$$\text{for every vertex } v \in (V(Tx_3) \cup V(Hy_3)) \setminus \{x_1, x_3, y_1, y_3\},\ \sigma(v) = v.$$
$$\text{for every vertex } v \in V(G'),\ \sigma(v) = \sigma'(v).$$

From this permutation scheme, we obtain the following equation:
$\lambda^2(G) \geq \lambda^2(G') + |V(Tx_3)| + |V(Hy_3)| - 2$.
By hypothesis, we have that $\lambda^2(G') \geq \lfloor \frac{2n'}{3} \rfloor$, then
$\lambda^2(G) \geq \lfloor \frac{2(n - |V(Tx_3)| - |V(Hy_3)|)}{3} \rfloor + |V(Tx_3)| + |V(Hy_3)| - 2$.
By hypothesis, we have that $|V(Tx_3)| + |V(Hy_3)| \geq 8$, hence $\lambda^2(G) \geq \lfloor \frac{2n}{3} \rfloor$. So σ is a good permutation of G.

Case 2: ($p = 3$ and $q \geq 3$) or ($p \geq 3$ and $q = 3$). Wlog, assume that $p = 3$. Two subcases are considered as follows:

Subcase 2.1: $d(x_2) = 2$ *or* $d(y_2) = 2$. This case follows easily using the same induction scheme as in Subcase 1.1.

Subcase 2.2: $d(x_2) \geq 3$ *and* $d(y_2) \geq 3$. Consider the graph $G' = G \setminus (T \cup \{y_1, y_2\})$. So the induction hypothesis guarantees the existence of a good permutation σ' on $V(G')$. The permutation σ' can be extended to a good permutation σ of G as follows:

$$\sigma(x_3) = x_1,\ \sigma(x_1) = x_3,\ \sigma(x_2) = y_2 \text{ and } \sigma(y_2) = x_2.$$
$$\text{for every vertex } v \in V(T) \setminus \{x_1, x_2, x_3, y_1, y_2\},\ \sigma(v) = v.$$
$$\text{for every vertex } v \in V(G'),\ \sigma(v) = \sigma'(v).$$

From this permutation scheme, we obtain the following equation:
$\lambda^2(G) \geq \lambda^2(G') + |V(T)|$.
By hypothesis, we have that $\lambda^2(G') \geq \lfloor \frac{2n'}{3} \rfloor$, then
$\lambda^2(G) \geq \lfloor \frac{2(n - |V(T)| - 2)}{3} \rfloor + |V(T)|$.
By hypothesis, we have that $|V(T)| \geq 5$, hence $\lambda^2(G) \geq \lfloor \frac{2n}{3} \rfloor$. So σ is a good permutation of G.

Case 3: *G has a star S_m.* Let v be the root vertex of S_m and let $u \neq v$ be a vertex of degree 2 of G. Consider the graph $G' = G \setminus (V(S_m) \cup \{u\})$. Then there exists a good permutation for G', say σ'. Putting $\sigma(u) = v$, $\sigma(v) = u$, for every leaf vertex $x \in V(S_m)$, $\sigma(x) = x$ and for $x \in V(G')$, $\sigma(x) = \sigma'(x)$ we get a good permutaion for G.

Case 4: *all previous cases are not satisfied.* In this case G contains only isolated vertices (at least two) and non-tree components. Two subcases are considered as follows:

Subcase 4.1: *G has a vertex x of degree at least* 3. Let u, v be two isolated vertices of G. Consider the graph $G' = G \setminus \{u, v, x\})$. Then there exists a good permutation for G', say σ'. Putting $\sigma(u) = u$, $\sigma(v) = x$, $\sigma(x) = v$ and for $y \in V(G')$, $\sigma(y) = \sigma'(y)$ we get a good permutaion for G.

Subcase 4.2: *all the components of G are cycles and isolated vertices.* In this case we can easily show that G has a good permutation.

From the proof above, we can conclude that $\lambda^2(G) \geq \lfloor \frac{2n}{3} \rfloor$. □

To improve Theorem 6.14, we propose the following conjecture which is probably true but it seems difficult to prove it!

Conjecture 6.15 *Let G be a graph of order n, if $|E(G)| \leq n-2$, then*

$$\lambda^2(G) \geq \lfloor \frac{3n}{4} \rfloor - 2$$

6.4 Conclusion

In this chapter, we have studied the labeled embedding of trees and $(n, n-2)$ graphs. In particular, we have proved the exact value of the labeled embedding number of paths and caterpillars. We have also shown the interest to study the fixed-point-free labeled embedding in order to improve the lower bound of the labeled embedding number of general trees and $(n, n-2)$ graphs.

We end this part by reflecting on the applicability of labeled packing to the graph matching problem. If we consider the example of XML documents that are modeled as ordered labeled trees, we can see that querying an XML document is equivalent to searching an embedding of the query in the data tree. Therefore, it would be interesting to exploit tree embedding techniques in order to define a coherent similarity measure between two labeled trees. The XML pattern matching is the subject of the next part of this thesis.

Part III

XML Tree Pattern Matching Problem

Chapter 7

Tree Matching and XML Retrieval

Contents

Among all the available formats used to represent information (from text to video or audio), the XML (*eXtensible Markup Language*) [W3C98b] format is now extensively used. The simple, self-description nature of the XML standard promises to enable a broad suite of next-generation Internet applications, ranging from intelligence Web searching and querying to e-commerce.

The growing number of XML documents leads to the need for appropriate **retrieval methods** which are able to exploit the specific features of this type of documents. Indeed, in XML documents, textual content (data) is hierarchically structured with tags. As opposed to other markup languages (like HTML for example), tags are used to specify semantic information about the content, and not for presentation purposes. Although structure allows to organize content in XML documents, it is not always used with the same intent. XML documents can be either **data-oriented**, where structure is intensively used to organize data, and where XML components can be seen as database records, or **text-oriented**, where structure is irregular and documents are designed to be used by humans.

Many approaches have been proposed in the literature for XML retrieval. However, most of them propose ab nihilo solutions, whereas the use of graph theoretical concepts could be of interest. Indeed, the underlying data model of XML documents allows to consider them as a particular kind of graphs, *i.e.,* trees [W3C98a]. More precisely, they can be considered as labeled trees of element nodes, where each element can be either an atomic data item or a composite data collection consisting of references (represented as edges) to child elements in the XML tree. The same representation can be used for structured queries. Considering this data model for retrieval, the retrieval process can thus be seen as a matching problem between query and document trees.

In this chapter, we aim at studying and discussing what are the solutions proposed by graph theory for the tree-matching problem in a general perspective, and how they have been exploited in XML retrieval. In particular, we will focus our attention on the exact tree matching problem.

7.1 Querying XML Data: Key points

Before giving some algorithms used in graph theory for tree matching, it seems of great importance to recall some background about XML documents and query languages, and to detail issues behind the retrieval of XML information.

7.1.1 Tree representation of XML documents

In XML documents, tags are used to hierarchically and semantically organize information. In the document presented in Figure 7.1 for example, content is organized within a *header* and a *body* tag. The *body* tag contains *section* elements, which are in turn composed of *title* and *paragraph* elements, etc.

An element begins with an opening tag <tag> and ends with a closing tag </tag>. It may contain atomic data (as for example *author* element), other elements (as for example one of the *section* elements) or a mixture of both (one talk about mixed content). Elements are also called components.

Thanks to this data model, XML documents can be represented as labeled trees [W3C98a]. In an XML tree, the whole document is represented by the *root node*, elements are represented by *internal nodes* (*i.e.,* non terminal nodes) and the content itself is in *leaf nodes* (*i.e.,* terminal nodes). These nodes are linked with edges showing their hierarchical relations. The tree representation of the document in Figure 7.1 is given in Figure 7.2.

As previously mentioned, XML documents can be classified into two groups of documents: *data-oriented documents* or *text-oriented documents*

Documents of the first category have a fine granularity, are highly structured and do not contain mixed contents. The order of the children of a given element is often without any importance. Elements can be considered as database records, there are

```
<article year="2001">
   <header>
      <title>Information retrieval on the Web </title>
      <author>A. Dupont</author>
   </header>
   <body>
      <section>
            <title> The history of hypertext</title>
            <paragraph>In order to describe what hypertext is…</paragraph>
            <paragraph>…</paragraph>
      </section>
      <section>
            <title>Different kinds of Web Search engines </title>
            <paragraph>Plain-text search engines…</paragraph>
            <paragraph>…</paragraph>
            <paragraph>…</paragraph>
      </section>
      <section>
      …
      </section>
   </body>
<article>
```

Figure 7.1: An example of XML document

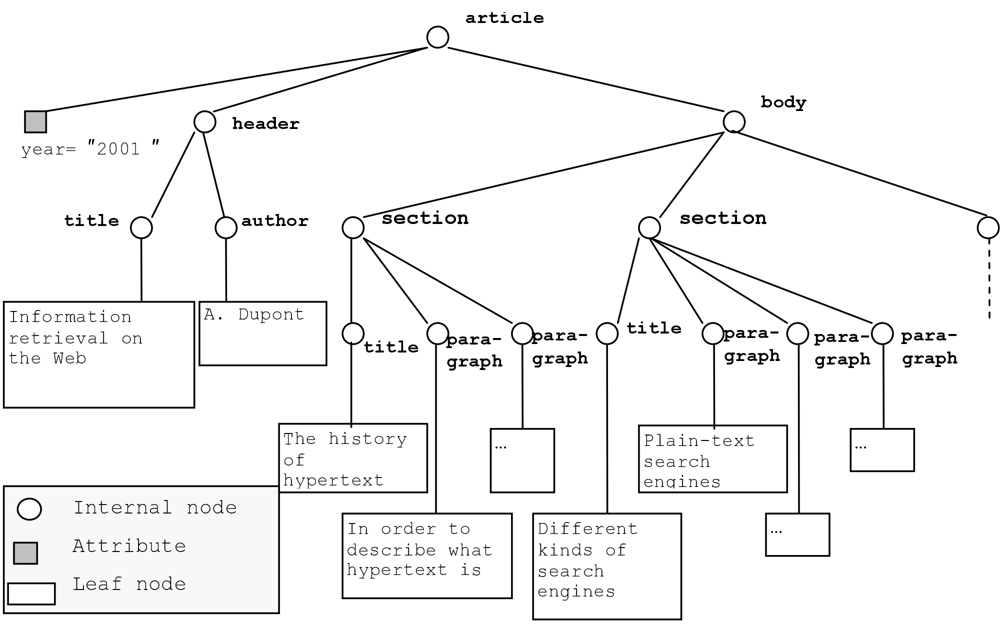


Figure 7.2: XML tree associated with the document of Figure 7.1

not so much information content. The document in Figure 7.3 is an example of a data-oriented document.

On the other hand, text-oriented documents are loosely structured. They are designed to be used by humans; books, articles, electronic messages are good examples of this type of documents. Their structure is irregular, and they can contain many mixed contents. Moreover, the order of elements is very paramount for the understanding of documents. For example, in the text-oriented document of Figure 7.1, *section* elements should be read in the good order to make the whole article

```
<publications>
   <inproceedings>
      <title>Information retrieval on the Web </title>
      <author>A. Dupont</author>
      <year>2001</year>
      <conference>WWW Conference</conference>
   </inproceedings>
   <inproceedings>
      <author>J. Allan</author>
      <title>Information retrieval and graphs</title>
      <conference>AOC 2008</conference>
      <year>2008</year>
   </inproceedings>
   …
<publications>
```

Figure 7.3: Example of a data-oriented document

understandable.

Boundaries between the two types of approaches are however nowadays not so strict. As explained by Lalmas and Baeza-Yates in [LB09]:

> "From a terminology point of view, structured text retrieval and querying semi-structured data, in terms of end goals, are the same, *i.e.,* finding elements answering a given query. The difference comes from the fact that in information retrieval, the structure is added, and in databases, the structure is loosened."

In the following, we will thus present only approaches from the database community (exact tree matching, section 7.3). Whatever the considered approach, the problem is to match a document tree with a query tree. The following paragraph presents the different query languages proposed in the literature for XML retrieval.

7.1.2 Query languages

Queries for XML retrieval can be classified into *content-only* and *content-and-structure queries.*

Content-only queries are composed of simple keywords terms, and have historically been used in traditional information retrieval. They are also suitable for XML retrieval, in retrieval scenarios where the user does not know the structure of documents he/she is querying. In this thesis, we are not interested in such queries, since open issues when considering this type of queries are different. Indeed, with content-only queries, the main problem is to find the good granularity of information to be returned to users, and not to match documents and queries trees.

In content and structure queries, users provide content conditions linked with structure conditions. They are two reasons a user might add structural constraints to a query [Tro09]:

- the first is to constraint the size of the result,

◦ the second is to restrict the search to document parts that are supposed to be appropriate.

According to [AYL06], there are three main categories of content and structure query languages:

◦ *tag-based* queries allow users to express very simple conditions concerning the tag of the element in which the required content should be. They are of the form: "tag: content condition". For example the query *"section: search engines"* means that the user is looking for a section element about "search engines".

◦ *path-based* queries are based on the XPath [W3C07b] syntax. They include content conditions in a XPath-based syntax. Examples of languages allowing path-based queries are the NEXI language [TS04] or FuzzyXPath [CDG+09].

◦ *clause-based* queries have a structure similar to the one of SQL. They contain nested clauses that allow to express the used need. In XML retrieval, one can cite XQuery [W3C07a] or XQuery full-text [W3C11] as examples for clause-based queries.

If we now purely consider the structure conditions of queries, XQuery and XPath [W3C07a], [W3C07b] queries can be basically divided into two main groups: *path queries* and *twig queries*. Path queries are simple queries against XML document, which contain path expressions, *i.e.,* child axis "/" and descendant axis "//". Twig queries are represented as node-labeled twig patterns, *i.e.,* small trees. They are also named *tree pattern queries*.

An example of a twig query is represented in Figure 7.4.

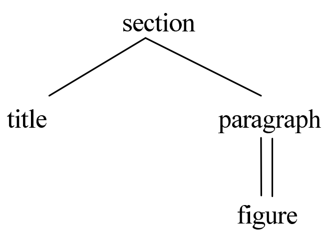


Figure 7.4: A twig query

Whatever the query language used, content and structure queries, in the same manner than XML documents, can be represented as labeled trees. The retrieval process can thus be summarized to a tree matching process. In the following section, we give a short survey for exact tree matching problem. For more information about approximate tree matching we refer to our survey paper [TPSN+12], which was submitted to ACM Computing Surveys.

7.2 Algorithms for exact tree pattern matching

This section describes state-of-the-art algorithms for exact tree matching. In order to state the problem, the terminology of labeled tree and their components has to be defined first.

7.2.1 Notation and terminology

A rooted labeled tree is denoted by $T = (V, E, r, \mu)$, where $V = \{v_1, v_2, \ldots, v_n\}$ is a finite set of nodes, $E \subseteq \{(u, v)|u, v \in V\}$ is a set of edges. The node r is a distinguished node called the root, and μ is a labeling function which maps each node of T to a label in a finite set $L = \{l_1, l_2, ..., l_k\}$. For brevity, in the remaining of this chapter, we call a rooted labeled tree simply a tree. We denote the level of a vertex v by $l(v)$. We recall that an ordered tree is a rooted tree for which an ordering is specified for the children of every node, and it is unordered otherwise.

We give below the formal definition of the exact tree matching problem.

Definition 7.1 *Let target $T_1 = (V_1, E_1, r_1, \mu_1)$ and pattern $T_2 = (V_2, E_2, r_2, \mu_2)$ be two ordered labeled trees. T_2 matches T_1 at node $x_1 \neq r_1$ if there exists a one-to-one injective function from the nodes of T_2 into the nodes of T_1 such that:*

1. *the root of T_2 maps to x_1,*
2. *if $v_2 \in V_2$ maps to $v_1 \in V_1$, then $\mu_1(v_1) = \mu_2(v_2)$,*
3. *if $v_2 \in V_2$ maps to $v_1 \in V_1$ and v_1 is not a leaf, then each child of v_2 maps to some child of v_1.*

Figure 7.5 shows an example of a tree pattern matching between pattern tree P and target tree T, where the one-to-one mappings is represented by dotted lines. The tree pattern matching problem has been largely studied and has many important applications such as term rewriting systems, transformational programming systems, code-generator generators, theorem provers, and a variety of functional languages with equational function definitions (see [HO82]). The following is a brief overview of general exact tree pattern matching algorithms.

7.2.2 Algorithms

Given a tree pattern P and a target T of order m and n, respectively, where $m \leq n$, a naive tree pattern matching algorithm takes $O(mn)$ in the worst case. The basic idea is to visit all nodes of T in a pre-order walk, for each visited node a special recursive procedure is applied to check for possible occurrence of P at that node v. The matching procedure is terminated as soon as a mismatch is detected.

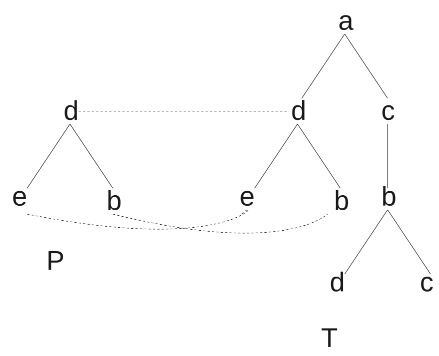


Figure 7.5: An example of tree pattern matching

Several attempts were made to improve the naive $O(mn)$ steps algorithm. Hoffmann and O'Donnell [HO82] proposed two approaches, called *bottom-up* and *top-down* matching algorithms with the same worst case bound. *Top-down* algorithm is efficient in space, but matching time is non-linear. The main idea of this algorithm is to encode the root-to-leaf paths as strings. It then finds all occurrences of each string in the target tree according to the string pattern matching algorithm of Aho and Crasick [AC75]. On the contrary, the key idea of the *bottom-up* technique is to find, for each node in the target tree, all patterns and all parts of patterns that match this node. *Bottom-up* techniques achieve matching in linear time. However, auxiliary space is required to store and establish the table encoding of all nodes in the target tree.

Hoffmann and O'Donnell's work stimulated a number of additional studies offering heuristic for space improvements. Among these improvements, we can mention the following significant approaches: [Cha87], [Bur88], [Kos89], [DGM94], [CH97] and [CH03]. Chase's method [Cha87] received considerable attention in the literature. This method was able to improve *bottom-up* solutions presented by Hoffmann and O'Donnell by exploiting the deeper structure of the pattern P to avoid some useless operations of the bottom-up algorithm. Chase proved that this transition map utilizes space much better than Hoffmann and O'Donnells. In [Bur88], Burghardt proposed a tree pattern algorithm, which consists of two phases. In the first phase a matching automaton is built from a given pattern set. For this, it generalizes the string matching algorithm of Aho and Corasick [AC75] such that each pattern is linearized according to the position ordering relation. Then these pattern strings are merged into an automaton by sharing as long initial parts as possible. In the second phase, one or more target trees are put into the automaton, by walking through the automaton graph according to the target tree and collecting the match information.

Kosaraju [Kos89] gave a new algorithm with improved complexity bound from $O(nm)$ to $(nm^{\frac{3}{4}})$. Kosaraju introduced three new techniques. Dubiner *et al.* [DGM94] improved Kosaraju's algorithm by discovering and exploiting periodical strings appearing in the pattern. They obtained a bound of $(nm^{\frac{1}{2}})$. Cole and Hariharan [CH97] introduced and gave an efficient algorithm for the *subset Matching problem*. The goal of this problem is to find all occurrences of a pattern string P of length m in a text string T of length n. Then the tree pattern matching has been reduced to a number of instances of the subset matching problem. Combining this reduction with the subset matching algorithm take an $O(n \log^3 m)$ time randomized algorithm for the tree pattern matching problem. Later the same authors improved the reduction time complexity [CH03]. They obtained an $O(n \log^2 m + m)$ time deterministic algorithm and an $O(n \log n + m)$ time Monte Carlo algorithm for the tree pattern matching problem.

7.3 Exact tree pattern matching for XML retrieval

As we have seen in Section 7.1, XML uses a tree-structured model for representing data and twig patterns for expressing queries. Finding all occurrences of such a twig pattern in an XML database is clearly a core operation in XML query processing. There has been a great deal of interest in recent years to overcome efficiently this problem. Most of the proposed approaches in the literature are interested in structural properties of queries and can be classified into four groups:

- structural join approaches,
- holistic twig join approaches,
- sequence matching approaches,
- other important exact XML tree matching algorithms.

In the following, we present an overview of the main algorithms and results available in each group.

7.3.1 Structural join approaches

In this section, we review the join based approach, a very important native idea, which usually includes three parts: (1) decomposition, (2) matching and (3) merging. Firstly, a twig pattern is decomposed into a set of basic parent-child and ancestor-descendant relationships between pairs of nodes. In the second phase, each binary relationship is separately executed using structural join techniques and its intermediate results are stored for further processing. The final result is formed by merging the intermediate results. We describe in the following the two main structural join algorithms proposed in the literature.

(i) *Multi-predicate merge join algorithm*

The key to the efficiency of this algorithm is in the use of a *containment labeling scheme* [ZND+01] that encodes each element in an XML database by its positional information. The main purpose of this representation is that the structural relationship between two vertices u, v can be determined easily without knowledge of the intermediate nodes on the path between u, v. For more information, this labeling scheme (called also *region encoding*) will be described in detail in the next chapter.

Based on the containment labeling scheme, Zhang *et al.* [ZND+01] proposed the multi-predicate merge join (*MPMGJN*) algorithm, which is the first structural join to find all occurrences of the basic structural relationships. This mechanism is an extension of the classical merge-join algorithm developed in relational query optimizers for equi-joins. The results of Zhang showed that for many XML queries, *MPMGJN* is more than an order-of-magnitude faster than the standard Relational Database Management System (RDBMS) join implementations.

(ii) *Tree-Merge and Stack-Tree algorithm:*

Al-Khalifa *et al.* [AJK+02] took advantage of the containment labeling scheme of XML elements to decrease the time of join processing. They developed two algorithms for matching parent-child and ancestor-descendant structural relationships efficiently: *Tree-Merge* and *Tree-Stack*.

The behavior of these two algorithms is explained by analytical results. In particular, the *Stack-Tree* algorithm was shown to be both I/O and CPU optimal, and practically as efficient as *Tree-Merge* family for evaluating containment queries.

7.3.2 Holistic twig join approaches

Until now, the holistic twig join approach was regarded as the most efficient family in the literature. Bruno *et al.* [BKS02] proposed the first holistic XML twig pattern matching algorithm to avoid producing large intermediate results. This approach constituted the major attempt for several subsequent works in order to make the twig pattern matching very efficient. In the following, we present an overview of the basic ideas and results of the main holistic twig join algorithms available in the literature.

(i) *TwigStack algorithm*

The main disadvantage of twig query decomposition into multiple binary relationships is that this approach generates a large amount of intermediate query results even when the input and output size are more manageable. Frequently, such intermediate results cannot be held in main memory and must be stored on disk. This will result in high disk I/O cost. In order to overcome this weakness, Bruno *et al.* [BKS02] proposed a novel holistic twig join algorithm, called *TwigStack*, wherein no large intermediate results are created.

The idea behind this approach is to use a chain of linked stacks to compactly represent partial results of individual query root-to-leaf paths, which are then composed to produce the final solutions. This decomposition method avoids computing large redundant intermediate results. This algorithm will be described in detail in the next chapter. The main advantage of this approach is to avoid storing intermediate results unless they contribute to the final results when the query twig has only ancestor-descendant edges. The analytical results of this approach demonstrate that *TwigStack* is I/O and CPU optimal among all sequential algorithms that read the entire input. This analysis is confirmed by experimental results on a range of real synthetic data and query twig patterns. The experimental results suggest that the holistic approach has more than six-fold faster query processing performance than the *Stack-Tree* approach coupled with the optimal join order.

(ii) *Improvements on TwigStack*

The idea of holistic twig join has been adopted in several works in order to make the structural join algorithm very efficient. This section is devoted to a structured review of these advances.

Improvement 1: efficient processing of parent-child edge query. As mentioned before, when all edges in query patterns are ancestor-descendant queries, *TwigStack* ensures that each root-to-leaf intermediate solution contributes to the final results. However, this algorithm still cannot control a large number of intermediate results for parent-child edge query. In the first improvement attempt, many efficient algorithms for XML twig pattern have been proposed to efficiently handle twig queries with parent-child relationships. Among them, Lu *et al.* [LCL04] extended *TwigStack* by proposing *TwigStackList* algorithm. This algorithm has the same performance than *TwigStack* for query patterns with only ancestor-descendant edges, but also produces much less useless intermediate solutions than *TwigStack* for queries with parent-child relationships. The main technique of *TwigStackList* algorithm is to look-ahead read more elements in the input streams and saves some of them (only those that might contribute to final answers) into lists in the main memory, so that we can make a more accurate decision to determine whether an element can contribute to the final solution or not. The experimental results obtained by authors demonstrate the significant superiority of *TwigStackList* on *TwigStack* according to the size of intermediate results for queries with parent-child edges. Chen *et al.* [CLL05] suggested another algorithm, called *iTwigJoin*, which can be used on various data streaming strategies (e.g. *Tag+Level Streaming* and *Prefix Path Streaming*). *Tag+Level streaming* can be optimal for both ancestor-descendant and parent-child only twig patterns whereas *Prefix Path streaming* could be optimal for ancestor-descendant only, parent-child relationship only. The experiments in [CLL05] show that *Tag+Level* streaming guarantees very few useless intermediate solutions in most case tested. None of the previous algorithms can avoid useless intermediate solutions completely. Li and Wang [LW08b] proposed the first algorithm, called *TwigBuffer* that completely avoids the useless partial solutions for arbitrary twig patterns. *TwigBuffer* use the same idea of buffering like *TwigStackList* to buffers some elements from the input streams in order to check the parent-child relationships. The experimental results show that *TwigStackList* and *TwigBuffer* have the same performance when the queries that do not have parent-child edges or parent-child edges happen under non-branching nodes. However *TwigBuffer* performs better than *TwigStackList* when the queries have parent-child edges under branching nodes.

Improvement 2: elimination of redundant computations. The existing holistic twig join algorithms may perform many redundant checks of XML elements. Indeed, an improvement strategy consists in avoiding these unnecessary computations. *TSGeneric+* [JWLY03] makes improvements on *TwigStack* by using *XR-Tree* to effectively skip some useless elements that do not contribute to the final results. The motivation to use *XR-tree* is that, the ancestors (or descendants) of any XML element indexed by an *XR-tree* can be derived with optimal worst case I/O cost. However, *TSGeneric+* may still output many useless intermediate path solutions like *TwigStack* for queries with parent-child relationship. Guoliang *et al.* [LFZZ07] proposed the *TJEssential* algorithm based on three optimization rules to avoid some unnecessary computations. They presented two algorithms incorporated with these optimization rules to effectively answer twig patterns in leaf-to-root combining with root-to-leaf way. A novel holistic twig join algorithm, called *TwigStack+* is proposed in [ZXM07], it is based on holistic twig join guided by extended solution extension to avoid many redundant computations. It significantly improves the query processing cost, simply because it can check whether other elements can be processed together with current one.

Improvement 3: eliminate the merging phase. Another improvement consists in reducing the cost of queries execution. Indeed there exist very interesting approaches that eliminate the second phase of merging of individual solutions [CLT+06], [QYD07], [JLH+07], [LW08a]. However, these algorithms need to load the entire document tree in memory. Chen *et al.* [CLT+06] proposed the first tree pattern solution, called *Twig2Stack* that avoids any post path join. *Twig2Stack* algorithm uses a hierarchical-stack to capture the ancestor-descendant relationships for the elements in the same query node. In this way, *Twig2Stack* can process path matching efficiently without a redundant relationships checking. *Twig2Stack* algorithm has shown a better performance in query processing than *TwigStack* and *TJ-Fast* [LLCC05] (see below). However, maintaining the hierarchical structure among stacks in *Twig2Stack* algorithm has a critical impact on the performance processing of twig query. Aiming to avoid this complex hierarchical-stacks, Qin *et al.* [QYD07] proposed an algorithm, called *TwigList*, which is a refined version of *Twig2Stack*. In fact, the main difference between both methods is that *TwigList* has only changed the complicated stacks used in *Twig2Stack* to simple list structures. The experimental studies of Qin *et al.* [QYD07] show that *TwigList* algorithm outperforms *Twig2Stack* as well as *TwigStack*. In the same context a novel algorithm, called *HolisticTwigStack* was developed in [JLH+07]. The authors proposed a novel complex stack structure like *Twig2Stack* to preserve the holisticity of the twig matches, without generating intermediate solutions. However, a considerable amount of time is taken to maintain the stack structure. *HolisticTwigStack* were evaluated better than both *TwigStack* and *Twig2Stack* in terms of time complexity, memory space and I/O. complexity. Dao and Gao [DC08] reviewed and analyzed both of the *HolisticTwigStack* and *TwigList* algorithm on processing for XML twig pattern matching. The statistics from this analysis clearly indicate that the *TwigList* algorithm seems to be significantly more efficient in the most tested cases.

Improvement 4: benefit the properties of some labeling schemes. Most of the existing holistic twig join algorithms are based on region encoding scheme (which will be explained in detail in the following chapter) to capture the structural relationship between XML elements. However, there exist other approaches which exploit others labeling scheme. Among them, *Dewey labeling* scheme has been widely used in XML query processing. A Dewey label of a node v represents the path from the document root to v. Based on the Dewey labeling, Lu *et al.* [LLCC05] proposed *TJFast* algorithm which uses a new encoding scheme, called the *extended Dewey code*, which allows to combine effectively the types and identifiers of elements in a label. The main advantage of this labeling is that, the ancestors of any XML element can be derived from its label alone. Thus, *TJFast* typically access much less elements than algorithms based on region encoding and can efficiently process queries with wildcards in internal nodes. More recently, based on the preliminary idea of extended Dewey labeling scheme and *TJFast* algorithm, Lu *et al.* [LML11] proposed three novel holistic twig join algorithms *GTJFast*, *TJFastTL* and *GTJFastTL*. The first algorithm *GTJFast* allows to compactly represent the intermediate matching solutions and avoid the output of non-return nodes to reduce the I/O cost. Both *TJFastTL* and *GTJFastTL* algorithms are proposed by extending *TJFast* and *GTJFast* algorithm based on *tag +level* streaming scheme. In *tag +level* streaming, XML elements with the same label but different level numbers have been separated to different streams. This level information is used to prune some useless streams. The authors proved that *TJFastTL* and *GTJFastTL* guarantee the I/O optimality for queries with only parent-child relationships. The experimental results reported in [LML11] show that these algorithms are superior to existing approaches in terms of the number of scanned elements, the size of intermediate solutions and query performance.

7.3.3 Sequence matching approaches

As opposed to the holistic twig join algorithm, the sequence matching approaches use an indexing method to transform both XML documents and queries into sequences and evaluate queries based on sequence matching. Querying XML data is equivalent to find subsequence matches. In [ZMM04] and [ZADR03], the authors use the *pre-order* and *post-order* ranks to linearize the tree structures and apply the sequence inclusion algorithms for strings. They proposed a novel strategy that include three parts. Firstly, a query decomposition process is applied to transform the query twig into a set of root-to-leaf paths so that the ordered tree inclusion can be safely applied. It has to be evaluated against the data signature in the second phase. Finally, the set of answers is determined by joining compatible intermediate solutions. The experiments in [ZADR03] demonstrate the efficiency of the decomposition approach, which is especially advantages for the large query trees, and for trees with highly selective predicates.

Two other sequence matching algorithms, *ViST* [WPFY03] and *PRIX* [RM04] were proposed to avoid expensive join operations. *ViST* method represents a major departure from previous XML indexing approaches. In fact, unlike classical index methods that decompose a query into multiple sub-queries, and then join the solutions of these sub-queries to provide the final answers, *ViST* uses tree structures as the basic unit of query to avoid expensive join operations. This method introduced a new sequential representation that transforms XML data trees and twig queries into structure-encoded sequences. *ViST* performs efficient subsequence matching algorithms on the structure-encoded sequences to find twig patterns in XML documents. However, the query processing in *ViST* may result in false alarms and false dismissals (these two problems are explained in more detail in [WM05]). In [RM04], Rao and Moon proposed a new indexing XML documents and processing twig patterns in an XML database. This indexing approach transforms XML documents and twig query into sequences by *prüfer* method that constructs an one-to-one correspondence between trees and sequences. Based on this transformation, a query execution system, called *PRIX* (*Prüfer* sequences for indexing XML) is developed for indexing XML documents and processing twig queries. The matching phase is achieved by applying subsequence matching on the set of sequences in XML database.

7.3.4 Other important exact XML tree algorithms

In this section, we take a quick look over some other well-known query tree pattern processing algorithms which have been developed in recent years. Among them, some approaches have benefited from fundamental progress in node labeling schemes of XML documents (for a survey, see [HL09]).

In [WL08], a twig pattern matching algorithm, called *TwigVersion* is proposed. the key idea of this approach is to compress both structural index and numbering schemes technique. *TwigVersion* is based on new version-labeling scheme that encodes all repetitive structures in XML documents. The identification of these repetitive structures matching allows to avoid a large amount of unnecessary computations. The experimental results reported in [WL08] show that *TwigVersion* significantly outperforms *TwigStack*, *TJFast* and *Twig2Stack* algorithms.

Recently, Izadi *et al.* [IHH09] proposed a novel method, called S^3, which can selectively process the document nodes. In S^3, unlike all previous methods, path expressions are not directly executed on the XML document, but they first evaluated against a guidance structure, called *QueryGuide*. The information extracted from the *QueryGuide* is an abstraction of the XML document. It describes the structure of XML document by its paths, such as the nodes of XML data are labeled by *Dewey labeling scheme*. The experimental results of [IHH09] confirm that S^3 substantially outperforms *TwigStack , TJFast, and TwigList* in terms of response time, I/O overhead, and memory consumption-critical parameters.

7.3.5 Summary

Table 7.1 depicts the various approaches presented in this section. These approaches are categorized into four classes by considering three basic features: the decomposition technique, the labeling technique, and a boolean parameter which indicates whether the merging phase is used in the query processing or not.

7.4 Conclusion

Much research effort has been devoted on efficient XML query processing. As a core operation in XML data, finding all occurrences of a query pattern in XML documents attracted more and more attention. In this chapter, we gave a structured overview of the numerous recent advances in the exact tree matching for querying XML data. As mentioned before, a very large volume of algorithms were proposed in the literature, and it seems nearly impossible to consider all related works. It is however difficult to decide what is the best algorithm that performs the query processing efficiently. For example, the one-phase holistic twig join algorithm eliminates completely the second phase of merging at the expense of huge memory requirement. Of course, holistic twig join algorithms are good candidates for physical operators supporting query evaluation in XDBMSs. Therefore, the development of new structural join algorithms is still valuable, the next chapter follows this line of research, In particular, we will propose a new efficient holistic twig join algorithm, called *TwigStack++*, which can greatly improves query processing performance of XML documents.

Approach	Structural Joins algorithms		
	Decomposition	**Merging Phase**	**Labeling technique**
MPMGJN [ZND+01]	binary relationship	Yes	Region Encoding
Tree-Merge [AJK+02]	binary relationship	Yes	Region Encoding
Stack-Tree [AJK+02]	binary relationship	Yes	Region Encoding
	Holistic twig Join Approach		
	Decomposition	**Merging Phase**	**Labeling technique**
TwigStack [BKS02]	root-to-leaf paths	Yes	Region Encoding
TwigStackList [LCL04]	root-to-leaf paths	Yes	Region Encoding
iTwigJoin [CLL05]	root-to-leaf paths	Yes	Region Encoding
TwigBuffer [LW08b]	root-to-leaf paths	Yes	Region Encoding
TSGeneric+ [JWLY03]	root-to-leaf paths	Yes	Region Encoding
TJEssential [LFZZ07]	root-to-leaf paths	Yes	Region Encoding
TwigStack+ [ZXM07]	root-to-leaf paths	Yes	Region Encoding
Twig2Stack [CLT+06]	without decomposition	No	Region Encoding
TwigList [QYD07]	without decomposition	No	Region Encoding
Holistictwigstack [JLH+07]	without decomposition	No	Region Encoding
TwigMix [LW08a]	without decomposition	No	Region Encoding
TwigFast [LW08a]	without decomposition	No	Region Encoding
TJFast [LLCC05]	root-to-leaf paths	Yes	Extended Dewey Labeling
TJFastTL [LML11]	root-to-leaf paths	Yes	Extended Dewey Labeling
GTJFast [LML11]	root-to-leaf paths	Yes	Extended Dewey Labeling
GTJFastTL [LML11]	root-to-leaf paths	Yes	Extended Dewey Labeling
	Sequence Matching Approach		
	Decomposition	**Merging Phase**	**Indexing technique**
PRIX [RM04]	Without decomposition	No	prüfer Sequences
Tree signature [ZMM04], [ZADR03]	root-to-leaf paths	Yes	Pre-order, Post-order
ViST [WPFY03]	without decomposition	Yes	structure-encoded sequences
Work of [WM05]	without decomposition	No	path labeling
	Other exact XML tree algorithms		
	Decomposition	**Merging Phase**	**Labeling technique**
TwigVersion [WL08]	root-to-leaf paths	Yes	Dewey ID labeling
S^3 [IHH09]	set of match patterns	Yes	Dewey ID labeling

Table 7.1: Summary of XML retrieval approaches using XML exact tree matching

Chapter 8

TwigStack++: A New Efficient Holistic Twig Join Algorithm

Contents

8.1 Introduction

As mentioned in the previous chapter, a typical approach of twig pattern matching is to first decompose the twig query into a set of binary (parent-child or ancestor-descendant) relationships. Thus, each binary relationship is separately executed and its intermediate results are stored for further processing, and the final solution is formed by merging these intermediate results. The main disadvantage of such a decomposition is that intermediate result sizes can become very large, even if the input and the final result sizes are much more manageable. In order to overcome this problem, Bruno *et al.* [BKS02] introduced the first holistic twig join algorithm, called *TwigStack*. It answers the twig query holistically and avoids huge intermediate results. The holistic twig join algorithm forms an important branch in XML retrieval. However, it may perform a lot of unnecessary computation when processing the queries and disk I/O in the merging phase.

In order to address the above weaknesses, we propose in this chapter a new efficient holistic twig join algorithm, which can greatly improve query processing performance. Our contributions can be summarized as follows:

1. We first introduce a new buffering technique that allows to eliminate many unnecessary computations in the matching phase.

2. We propose a new structural decomposition of a twig query, called *star-path twig decomposition* which decomposes the twig query into a set of star and path queries. The main objective of this decomposition is to decrease the number of join operations and the number of join attributes in the merging phase.

3. We implemented our proposed method and conducted an extensive performance evaluation using real and synthetic datasets. The results showed that our algorithm achieved high efficiency and outperformed existing proposals.

The rest of this chapter is organized as follows. We start describing the containment labeling scheme in the next section. Section 8.3 is dedicated to present the sub-optimality of *TwigStack*. Then, *TwigStack++* algorithm is proposed in detail in Section 8.4. Section 8.5 reports experimental results, and we conclude the chapter in Section 8.6.

8.2 Containment labeling scheme of an XML document

Containment labeling scheme is one of the most popular XML labeling schemes [ZND+01]. This labeling scheme (called *region encoding*) uses a textual-position of *start* and *end* tags in XML document such that the position is represented by the 4-tuple $(docId, start, end, level)$ for an XML element and $(docId, start, level)$ for a string value, where (i) $docId$ is the identifier of the document; (ii) $start$ and end can be generated by counting word numbers from the beginning of the XML document with identifier $docId$ until the start of the element and the end of the element, respectively. And (iii) $level$ is the nesting depth of the element (or string value) in the document. For example, Figure 8.1 shows an example of tree data with containment labels.

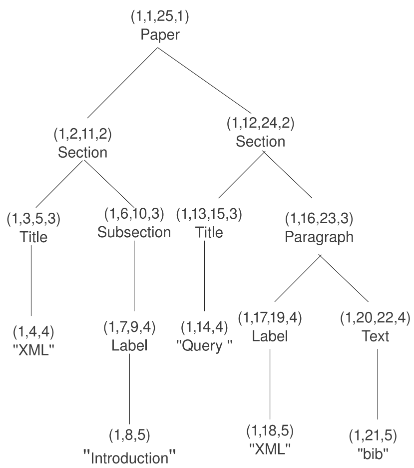


Figure 8.1: A sample XML document with containment labels

The structural relationship between tree nodes in this representation can be determined easily without knowledge of the intermediate nodes on the path. Formally, let two nodes x_1 and x_2 whose positions in the XML database are encoded as (d_1, s_1, e_1, l_1) and (d_2, s_2, e_2, l_2), respectively. We have

1. *Ancestor-Descendant relationship:* an element x_1 is an ancestor of another element x_2 if and only if $d_1 = d_2$, $s_1 < s_2$ and $e_2 < e_1$.

2. *Parent-Child relationship:* an element x_1 is a parent of another element x_2 if and only if $d_1 = d_2$, $s_1 < s_2$, $e_2 < e_1$ and $l_2 = l_1 + 1$.

8.3 Sub-optimality of TwigStack

TwigStack algorithm constituted the major attempt for several subsequent holistic twig join algorithms. The idea behind this approach is to use a chain of linked stacks to compactly represent partial results of individual query root-leaf paths, which are then composed to produce the final solutions.

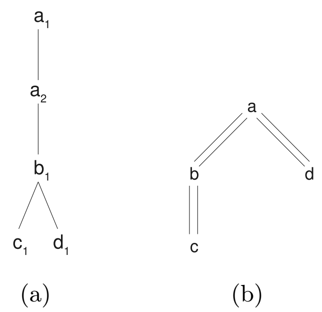


Figure 8.2: (a) D: An XML data tree and (b) Q: A twig query pattern

The main technique of *TwigStack* is to make use of two data structures: stack S_q and stream T_q for each node q in twig query. The stream T_q contains the positional representation of all elements of tag q. At any time during the algorithm, the elements in stack S_q are guaranteed to lie on a root-to-leaf path in the XML document and the chain of stacks contains a compact encoding of partial results to the twig query. The insertion and deletion elements over stacks are operated in the following way:

1. An element x_q from stream T_q is pushed to the stack S_q if and only if (i) x_q has a descendant x_{q_i} in each T_{q_i}, where q_i is a child of q and (ii) each node e_{q_i} recursively has the first property.

2. An element x_q is popped out from its stack if all matches involving it have been output.

For example, consider the query and data tree in Figure 8.2. The stacks encoding and the varying positions of cursors in data streams are shown in Figure 8.3(a)-(e).

At the beginning (in Figure 8.3(a)), all cursors point to the first nodes a_1, b_1, c_1, d_1. Since a_1 is an ancestor of b_1 and d_1, a_1 is inserted to stack S_a in Figure 8.3(b). Then we advance T_a to read a_2 which is also pushed to S_a (Figure 8.3(c)). Next, b_1 and c_1 are pushed into corresponding stacks. At this point, the algorithm outputs two intermediate paths (a_1, b_1, c_1), (a_2, b_1, c_1). Finally, d_1 is pushed to S_d (in Figure 8.3(e)) and output another two intermediate paths $(a_1, d_1), (a_2, d_2)$. Hence, the final results is obtained by merging the obtained intermediate paths.

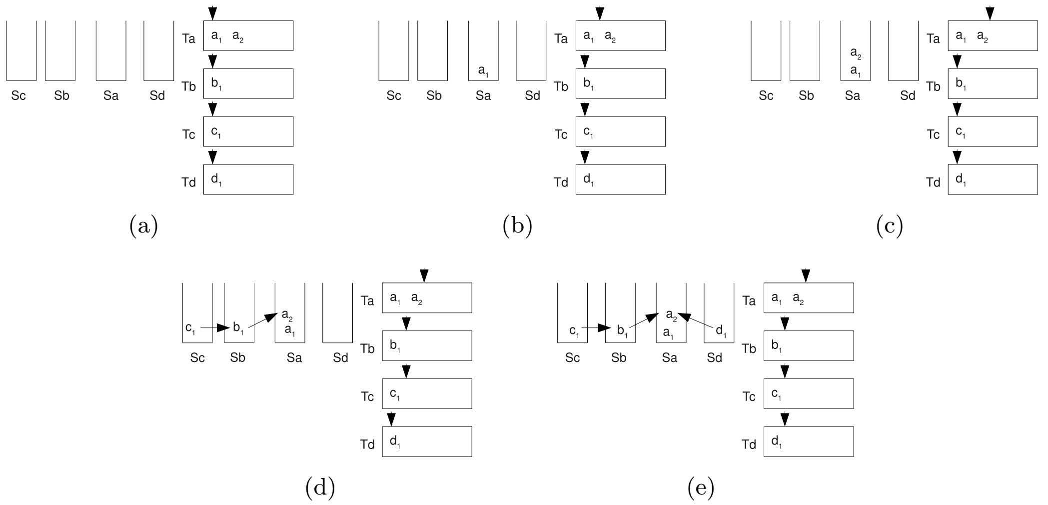


Figure 8.3: Illustrate to stack operations

Our observation: existing holistic twig join algorithms avoid storing intermediate results unless they contribute to the final results when the query twig has only ancestor-descendant edges. However, they have to recursively call the *getNext* function (or other functions based on the *getNext*) many times, which is a core function for the holistic algorithms. Each time, *getNext* returns a node q and ensures that: (i) the current element in stream q has a descendant element in each stream q_i, for every child q_i of q, and (ii) each current element in stream q_i recursively satisfies the first property. But, most of these calls are unnecessary and could be avoided. Hence, they involve many unnecessary computations. In the previous example, *TwigStack* calls *getNext(a)*, *getNext(b)*, *getNext(c)*, *getNext(d)*, many times respectively (see Table 8.1), but only a few times are useful and pivotal, and other times can be pruned.

	getNext(a)	getNext(b)	getNext(c)	getNext(d)
Number of calls	5	5	5	5

Table 8.1: Number of calls to *getNext* durning *TwigStack*

In the next section, we will demonstrate how to avoid those unnecessary computations using a new buffering technique. The main idea is to look-ahead read some matching nodes in input data steams and buffer a limited number of them in the main memory.

Another overhead incurred in the merging phase of holistic twig join algorithms could be large because the cost to input and output of the individual matches and finally merge them to form twig matches can be substantial, especially when the number of matching paths is large. To address this problem, we propose a new structural decomposition of the twig query which reduces significantly the number of pieces to be joined.

8.4 A new algorithm: TwigStack++

In this section, we present a new holistic twig join algorithm, we start with some notations and definitions that will be used later.

8.4.1 Notations

Consider a tree T, we use $V(T)$ to denote the vertex set of T. Let x be an internal node of T and let N be a subset of vertices in $V(T)$, we define the function $star(T, x, N)$ which returns a star S consisting of a rooted node x and all of its children nodes which are not in the set N. For each vertex x of T, let $subtree(T, x)$ to be the function which returns a subtree of T consisting of a rooted node x and all of its descendants in T.

Given a twig pattern query Q and XML document D conforming to XML schemas. For better understanding of our algorithm, we make use of the following node operations. The self-explaining functions $isRoot(q)$ and $isLeaf(q)$ are used to examine whether q is a root node or a leaf node. The function $children(q)$ returns all child nodes of q and $parent(q)$ gets the parent node of q. The function $end(Cq)$ returns a boolean value that indicates whether the current position Cq is at the end of the stream Tq. We use $Cq.element$ to denote the element pointed by Cq. We can access the attribute values of Cq by $Cq.start$, $Cq.end$ and $Cq.level$. The cursor can be forwarded to the next element in Tq with the procedure $advance(Tq)$. Initially, all the cursors point to the head of the corresponding stream. Similar to *TwigStack*, we make use a chain of linked stacks to compactly represent partial results. Indeed, each internal query node q in a twig pattern Q is associated with a stack Sq. The operations over stacks are: *pop*, *push*, *top* and *empty*.

We associate a list Fq instead of stack with every leaf query node q in Q. Each data node in the list consists of a pair: (positional representation of an element from Tq, pointer to *Sparent(q)*), where *Sparent(q)* is the stack associated with the parent of q. The operations over list Fq are: Fq.*getLastElement()*, Fq.*append(e, pointer to Sparent(q))* and Fq.*clearList()*. The first operation returns the element at the end of Fq. The second operation appends pair information at the end of Fq. The last operation removes all elements from Fq.

The main technique of *TwigStack++* is to make use of buffer Lq for each node q in twig pattern query. For each node q, the operations over Lq are *Lq.remove* and *Lq.append(e)*. The first operation removes and returns the element at the end of Lq and the last operation appends element e at the end of Lq.

8.4.2 *TwigStack++* description

We now present our algorithm *TwigStack++* for processing twig pattern matching queries. *TwigStack++* preserves the main idea of *TwigStack*. The objective here is to reduce the amount of I/O and CPU cost by introducing two major improvements as follows.

Star-path decomposition

As a first improvement, we propose a new structural decomposition of a twig query, called *star-path twig decomposition*, which decomposes the twig query into a set of star and path queries. This decomposition strategy is shown in Algorithm 1 given below, which takes as input a twig query Q. The variable SP is used to save progressively the set of components to be generated from Q. Initially, the forest SP is a null graph (a graph which has no vertices). For example, the structural *star-path decomposition* corresponding to the query of Figure 8.4(a) is shown in Figure 8.4(b). Our decomposition algorithm consists of two main steps as follows: for every vertex u of degree greater than 2:

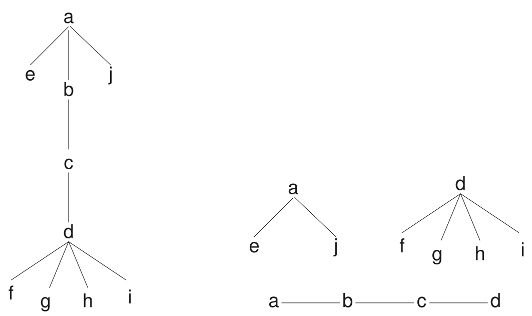
1. *path decomposition:* at line 3-6 in Algorithm 1, the algorithm provides a path for every edge (u, v) where v has degree 2. Then from Figure 8.4(a), we can see that only one path $P = (a, b, c, d)$ is returned for the edge (a, b).

2. *star decomposition:* at line 7-9, only one star is generated if and only if the vertex u has at least one child node with degree $\neq 2$. Then from Figure 8.4(a), two stars are generated for the vertices a and d.

The following theorem shows the advantage of our decomposition approach compared to the root-to-leaf paths decomposition of classical holistic twig join algorithms.

Theorem 8.1 *The total number of join operations required for star-path twig decomposition is less than or equal to the number of join operations of the root-to-leaf paths decomposition.*

Proof. In *TwigStack*, a (k-1)-way merge is performed in the merging phase, where k is the number of leaf nodes in the twig query. From Algorithm 1, we can see that for every node u of degree at least 3, we have:



(a) (b)

Figure 8.4: (a) Tree pattern, (b) Structural star-path decomposition

Algorithm 1 Decomposition

Input: twig pattern query Q.

Output: star-path twig decomposition of Q.

1: **Forest** SP =**null graph**;
2: **for** every node u with degree more than 2 in Q **do**
3:    **for** every child v of u with degree exactly 2 **do**
4:       Let $P = u, v, x_1, \cdots, x_m$ be an induced path of Q where the nodes from x_1 to x_{m-1} has degree 2 in Q and the degree of x_m is different from 2.
5:       $SP = SP \cup \{P\}$.
6:    **end for**
7:    **if** u has at least one child of degree $\neq 2$ **then**
8:       $SP = SP \cup \{star(Q, u, V(SP))\}$.
9:    **end if**
10: **end for**
11: return SP.

1. star-path twig decomposition approach provides a path for every edge (u, v) where v has degree 2.

2. only one star is generated if and only if the vertex v has at least one child node with degree $\neq 2$ (at line 7-9).

According to these two observations, we can see that the number of generated components passing through the node u is less or equal to the number of children of u which is less than or equal to the number of leaf nodes of $subtree(Q, u)$. Therefore, the total number of join operations of *star-path twig decomposition* is less than or equal to the number of leaf nodes of any tree, giving the desired result. □

From Algorithm 1, we can easily obtain the following theorem.

Theorem 8.2 *For every two different components C_1 and C_2 of SP, we have*

$$|V(C_1) \cap V(C_2)| \leq 1$$

Merging individual query solutions to form twig matches is simple, at least, conceptually, but it may turn out to be a very expensive process in reality since the number of join operations can be very large, rendering the merge phase cumbersome and time consuming. By reducing the number of join operations, the performance of XML query processing can be greatly improved. Note that our structural decomposition will not only reduce the number of join operations but will also allow us to decrease a large number of join attributes in the merging phase. So from Theorem 8.2, we can see that the number of join attributes is restricted to one attribute.

Improvement of *getNext* function

Our second improvement consists essentially in rewriting of *TwigStack* to eliminate many unnecessary computations in the matching phase. As mentioned earlier, *TwigStack* works by recursively calling a method called *getNext* to efficiently filter useless elements in order to return the next node for processing. The existing holistic twig join algorithms may perform many redundant checks in the calls to *getNext*.

We propose here a new strategy guided by containment labeling scheme to avoid the redundant function calls. *TwigStack++* changes the *getNext* to *getNextSet* function (Algorithm 2 given below). Implicitly, the *getNextSet* function does not return only one node (like *getNext*) but a set of nodes at once. The core of this procedure is more complex than the original algorithm but gives much better results.

In what follows, we first explain the *getNextSet* function and then present the main algorithm in details. *getNextSet(q)* is a procedure called in the main algorithm of *TwigStack++*. It updates the list Lq and returns a query node q' (possibly $q = q'$) with the following properties: at every point during computation:

1. the elements in sequence Lq (from the beginning of L_q to the end) are guaranteed to lie on a root-to-leaf path in the XML document.
2. for all ancestor-descendant pairs $(q//q')$ in the twig query, we have $\forall(eq, eq') \in Lq \times Lq'$, then eq' is descendant of eq in XML data tree.
3. the number of buffered elements in any list is bounded by the depth of the data tree. Note that the maximal depth of XML documents is usually very limited.

At line 6-18 in Algorithm 2, we recursively invoke *getNextSet*(q_i) for each $q_i \in children(q)$. In this computation step, we can avoid many unnecessary computations. Unlike *TwigStack*, it is unnecessary to recursively check whether q_i and their corresponding descendants have solution extensions if one of the following conditions holds (this is because, the query node q_i is processed already).

1. The existing top element of Sq_i is decendant of the current element in the data stream of node q.

2. The list Lq_i is not empty.

Algorithm 2 Function getNextSet(q)

1: **if** *isLeaf(q)* **then**
2:    Lq.append($Cq.element$);
3:    advance(Tq);
4:    return q;
5: **end if**
6: **for** q_i in *children(q)* **do**
7:    **if not** isLeaf (q_i) **then**
8:       $e = top(Sq_i)$;
9:    **else**
10:       $e = Fq_i$.*getLastElement()*;
11:    **end if**
12:    **if** (Cq is not an ancestor of e)**and**($Lq_i = \emptyset$) **then**
13:       q_j =getNextSet(q_i);
14:       **if** $q_j \neq q_i$ **then**
15:          return q_j;
16:       **end if**
17:    **end if**
18: **end for**
19: let h_x be the head element of the list Lx;
20: n_{min}=minarg$_{q_i}\{h_{q_i}.star : Lq_i \neq \emptyset\} \cup \{top(Sq_i).star : Lq_i = \emptyset\}$;
21: n_{max}=maxarg$_{q_i}\{h_{q_i}.star : Lq_i \neq \emptyset\} \cup \{top(Sq_i).star : Lq_i = \emptyset\}$;
22: **while** $Cq.end < maxStar$ **do**
23:    advance(Tq):
24: **end while**
25: **while** ($Cq.star < n_{min}.star$)**and**($Cq.end > n_{max}.star$) **do**
26:    Lq.append($Cq.element$);
27:    advance(Tq);
28: **end while**
29: **if** $Lq = \emptyset$ **then**
30:    return n_{min};
31: **else**
32:    return q;
33: **end if**

In order for node q to be returned, we make sure that the node q has a solution extension as well by advancing Cq (line 23). At line 25-28, we load all common ancestors from n_{min} to n_{max} cursors in Lq. If no such ancestor exists, we return the child node with the smallest start value (n_{min}).

Algorithm *TwigStack++*, which computes answers to a query pattern Q is outlined in Algorithm 3. Similar to *TwigStack*, our algorithm operates in two phases.

Algorithm 3 TwigStack++

```
1: while not (end(Q)) do
2:   q=getNextSet(Q.root);
3:   for all nodes q_i in the subtreeNodes(q) do
4:     for all elements e of Lq_i do
5:       e = Lq_i.remove();
6:       updateStackList(e, q_i);
7:     end for
8:   end for
9: end while
10: mergeAllSolutions();

11: procedure updateStackList(e, q)
12: if not isRoot(q) then
13:   cleanStack(Sparent(q), e);
14: end if
15: if isRoot(q) or not empty(Sparent(q)) then
16:   cleanStack (Sq, e);
17:   if not isLeaf (q) then
18:     push(Sq, e, top(Sparent(q)));
19:   else
20:     Fq.append(e, top(Sparent(q));
21:   end if
22: end if
23: end procedure

24: procedure cleanStack(S, e)
25: while not empty(S) and (top(S).end < e.start) do
26:   pop(S);
27:   showSolution(top(S));
28: end while
29: if emty(Sq) then
30:   for all leaves qi of q do
31:     Fq_i.clearList();
32:   end for
33: end if
34: end procedure
```

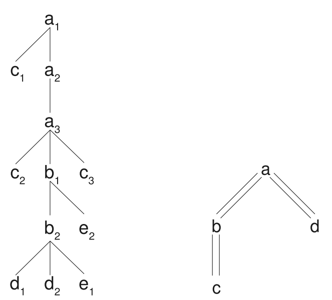
In the first phase (line 1-9), it repeatedly calls the *getNextSet* function (line 2). In this phase, the matching solutions are computed according to the star-path twig decomposition proposed previously. In the second phase, these solutions are merge-joined to compute the answers of the query twig pattern (line 10). The procedure call *updateStackList*(eq) (line 11-23) plays the role of updating the stack (or list if

q is a leaf) structure. The update of stacks in this procedure is the same as the one used in the original *TwigStack*. The following property shows the conditions for a list Fq (q is a leaf) to be cleared (line 29-33): a list Fq is cleared if and only if $Sparent(q)$ is empty. At line 27, *showSolution(eq)* is called to output all answers (star and path) matches passing through the XML element eq.

We present now an illustrative example of our approach. Consider the twig query and data tree in Figure 8.5(a-b). There are five buffer lists L_a, L_b, L_c, L_d and L_e. Initially, $getNextSet(a)$ recursively calls $getNextSet(b)$ and $getNext(c)$ (for $b, c \in children(a)$ in Q). Since c is a leaf node in Q, the new element c_1 is inserted into the list L_c. Next, $getNextSet(b)$ recursively calls $getNextSet(d)$ and $getNextSet(e)$. Since both d and e are leaf nodes in Q, we obtain $getNextSet(d) = d$, $getNext(e) = e$, $L_d = \{d_1\}$ and $L_e = \{e_1\}$. Therefore, $getNextSet(b) = b$, $getNextSet(a) = a$, $L_b = \{b_1, b_2\}$ and $L_a = \{a_1\}$. Figure 8.6(a) shows the state of the algorithm after the previous computation has been completed.

In the second call of $getNextSet(a)$, the top element of S_b is descendant of the current element in the data stream of node a. Therefore, it is unnecessary to recursively check whether b and their corresponding descendants (d and e) have solution extensions (see Figure 8.6(b)).

Figure 8.6(c) shows the state of the algorithm when evaluating the query Q, right after node e_2 has been processed. When e_2 is inserted to S_2, b_2 is popped out (because b_2 is not an ancestor of e_2). At this point, the algorithm outputs all solutions (star or path) matches passing through b_2 (Figure 8.6(d)).



(a) (b)

Figure 8.5: (a) D: An XML Data Tree and (b) Q: A Twig Query Pattern

In order to demonstrate our improvement compared to *TwigStack*, we consider Q and D in Figure 8.2 of Section 8.3. Initially, $getNextSet(a)$ recursively calls $getNextSet(b)$ and $getNext(d)$. In the same way, $getNextSet(b)$ calls $getNextSet(c)$. Since both c and d are leaf nodes in Q, c_1 and d_1 are inserted into L_c and L_d, respectively. Therefore, $getNextSet(b)$ inserts b_1 to L_b. Finally, a_1 and a_2 are common ancestors to b_1 and c_1. Then we insert a_1 and a_2 to L_a. At this point, the algorithm outputs directly all intermediate solutions :(a_1, b_1, c_1), (a_2, b_1, c_1), (a_1, d_1) and

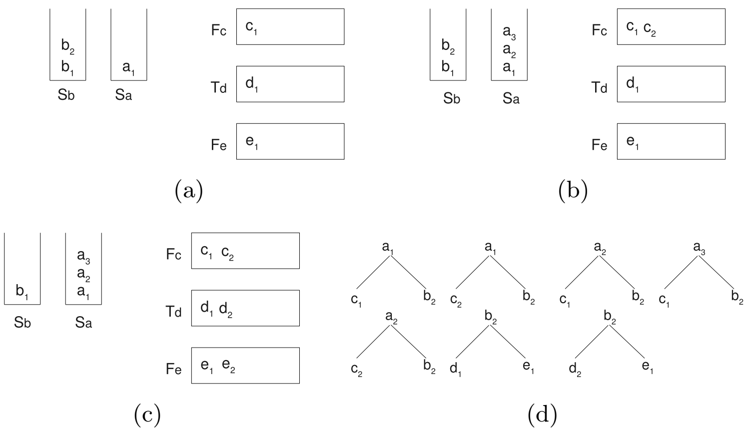


Figure 8.6: Illustration to *TwigStack++*.

(a_2, d_1). In this example, we have used only four calls to *getNextSet* to returns the final solutions: *getNextSet(a), getNextSet(b), getNextSet(c), getNextSet(d)*. However, From Table 8.1 of Section 4, *TwigStack* requires 20 calls to *getNext* in order to return the final solution.

Finally, it is clear that our algorithm follows the same line of reasoning of *TwigStack*. Hence, it is easy to show the correctness and the completeness of *TwigStack++* by directly deriving the proofs of *TwigStack* [BKS02]. Thus, we obtain the following theorem.

Theorem 8.3 *Given a twig query Q and an XML database D, the TwigStack++ algorithm correctly returns all the answers for Q on D.*

8.5 Experimental result

In this section, we present the experiments conducted to evaluate the efficiency of our algorithm and report some of the results obtained. We compared *TwigStack++* with two other twig join algorithms: *TwigStack* [BKS02] and *TJEssential* [LFZZ07] when the twig query pattern only involves ancestor-descendant relationships. The reason for which we choose these two existing algorithms for comparisons is that *TwigStack* and *TJEssential* are very efficient when the query contains only ancestor-descendant relationships. Since our improvements do not depend on the structure of the twig pattern, our algorithm can be easily adapted to other holistic twig join algorithms like *TwigStackList* [LCL04] for efficiently processing twig queries with parent-child edges.

All the algorithms were coded using Java 1.4.2, and performed experiments on a PC with a Intel(R) Core(TM) 2 Duo T7250-2GHz processor and 2GB of main memory. We used real-world (TreeBank) and synthetic data sets (XMark) for the experimental evaluation. The comparison between the three algorithms is based on running time of both CPU and I/O cost. In this experimentation, We use five queries from the standard TreeBank and XMark respectively as shown in Table 8.2. These queries have different twig structures.

Label	Query
TreeBank1	//S[//NP[//DT][//NN]]//PP[//IN]//NN
TreeBank2	//S[//NP][//NONE]//VP//PP[//IN]//DT
TreeBank3	//S[//VP[//NN][//VBD]]//NP[//IN]//DT
TreeBank4	//S//NP[//PP//TO][//VP//NONE]//JJ
TreeBank5	//S//VP//PP[//NP//VBN]//IN
XMark1	//person[//profile[//age][//interest][//education][//gender][//business]][//address]//email
XMark2	//person[//emailaddress][//homepage][//name]//address[//country]//city
XMark3	//site[//person[//homepage][//emailaddress]][//open-auction[//bidder] [//reserve]]//price
XMark4	//closed-auction[//annotation[//description]][//price][//date][//buyer]//seller
XMark5	//open-auction[//bidder[//personref][//time][//date]][//quantity][//reserve]//current

Table 8.2: Queries used in our experiments

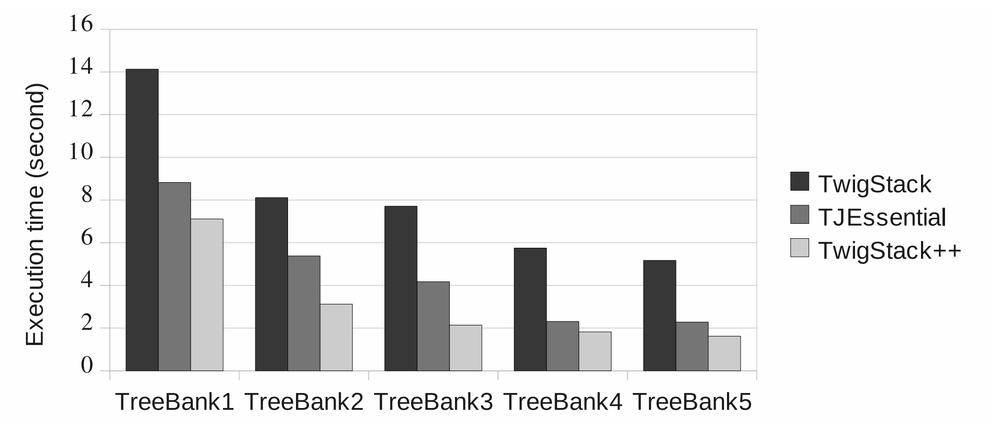


Figure 8.7: Running time of CPU and I/O on TreeBank

Figure 8.7 and Figure 8.8 summarize the processing time of four twig queries for the datasets, TreeBank (Figure 8.7), and XMark (Figure 8.8). The queries give a comprehensive comparison of the three algorithms. Overall, our algorithm outperforms the level of *TJEssential* and *TwigStack* in query processing time. This can be explained by the fact that *TwigStack++* reduces the cost (I/O and CPU time) of *TwigStack* and *TJEssential* by reducing the number of join operations and the

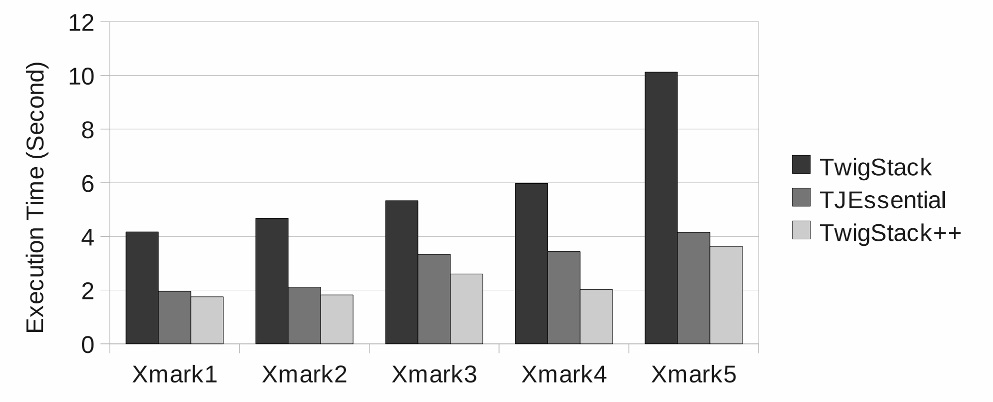


Figure 8.8: Running time of CPU and I/O on XMark

number of join attributes in the merging phase and the unnecessary computations in the matching phase.

8.6 Conclusion

In this chapter, we have proposed a new holistic twig join algorithm, called *TwigStack++* to efficiently process XML twig patterns. Our approach is based on two main improvements in the matching and merging phase. Experimental results show that our approach achieves high efficiency and outperforms existing proposals.

CHAPTER 9

Conclusion and Future Work

In this thesis, we discussed three graph theory problems applied to different classes of labeled graphs. In this chapter, we present several open questions and directions for future research in these areas.

******

Over the years, several vertex-distinguishing edge coloring variants have been defined in the literature. We gave in **Chapter three** a survey of various methods, recent results and open conjectures from this area of research. In particular, we have detailed the vertex-distinguishing edge coloring problem by sets and multisets. Originally motivated by these two variants, we investigated in **Chapter four** a new variant of edge coloring that induces a vertex distinguishing labeling where the label of each vertex is given by the difference between the highest and the lowest colors of its adjacent edges. In view of our results, we observed that our parameter can be used to get a bound on the detection number. In addition, several results have been obtained for various classes of graphs.

As future works, we plan to focus on the following three issues.

1. We left as an open question to show that the gap chromatic number of a graph of order n is always in $\{n-1, n, n+1\}$.
2. The computational complexity of the gap chromatic number is still an open problem (this is the case of most variants of vertex distinguishing problems derived from an improper edge coloring).
3. As for the other distinguishing parameters, it would be interesting to consider the variant of the gap coloring problem that distinguishes adjacent vertices only.

******

The purpose of the second part of this thesis was to study the graph packing problem for labeled graphs. Much work has been done on the unlabeled graph packing problem and we felt that investigating the labeled case might yield interesting results. The first section of **Chapter five** was dedicated to giving the reader an introduction to some relevant results on the packing of unlabeled graphs. Then, we studied the labeled graph packing problems for k copies of cycles in which we have proved the exact value of $\lambda^k(G)$ for all $n \geq 4k-3$. In **Chapter six**, we computed upper bounds for the labeled embedding number of trees and $(n, n-2)$-graphs. From these results, we can consider several directions for future work:

1. We would like to prove Conjecture 5.28 (in Chapter 5) that deals with the labeled packing of k copies of cycles of order n, where $2k+1 \leq n \leq 4k-3$.

2. we hope that the work in Chapter 6 provides a helpful beginning to the study of the labeled packing of three copies of trees and $(n, n-2)$-graphs.

3. Characterize all trees T for which the labeled embedding number of T reaches the upper bound of Lemma 5.21.

4. We would also like to improve the lower bound of the labeled embedding number of $(n, n-2)$-graphs from $\lfloor \frac{2n}{3} \rfloor$ to $\lfloor \frac{3n}{4} \rfloor - 2$ (as shown in Conjecture 6.15).

******

The last part of this thesis addresses the problem of query processing in XML databases, namely *XML twig queries*. An XML twig query, represented as a labeled tree, is essentially a complex selection on the structure of an XML document. Thus, XML tree patten matching is widely regarded as a new challenge for efficient XML query processing. Until now, the holistic twig join approach was regarded as the most efficient family in the literature. In **Chapter seven**, we provided a comprehensive survey of the basic ideas and results of the main holistic twig join algorithms available in the literature. In **Chapter eight**, we proposed a novel holistic twig join algorithm, called *TwigStack++*. This algorithm is based on two main improvements in the decomposition and matching phases. We analytically and experimentally shown that *TwigStack++* can efficiently avoid many unnecessary computation and I/O cost. While this part has proposed an efficient algorithm for XML tree pattern matching, a number of issues need to be further investigated.

1. We would like to combine our decomposition technique with other efficient holistic twig join algorithms. Furthermore it would also be interesting to study the parallelism of these algorithms on a multi-core system in order to maximize the computing performance of query processing in a large XML database.

2. In the literature, quantum algorithms have attracted considerable attention in the theoretical computer science community because of the considerable speed up over classical algorithms they achieve. For a practical perspective of tree matching problem, several aspects remains to be investigated. Indeed, quantum versions of matching approaches based on decomposition and combinatorics on words can be developed. Nevertheless, the formalization of such

solutions may be very difficult, but very promising. One can address that in the future.

3. As mentioned in Part two, the graph matching problem can be naturally stated as a packing of labeled graphs. Therefore, it would be interesting to exploit tree packing techniques in order to define a coherent similarity measure between XML query and tree data.

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Coloring, Packing and Embedding of graphs

Abstract: In this thesis, we investigate some problems in graph theory, namely the graph coloring problem, the graph packing problem and tree pattern matching for XML query processing. The common point between these problems is that they use labeled graphs.

In the first part, we study a new coloring parameter of graphs called the *gap vertex-distinguishing edge coloring*. It consists in an edge-coloring of a graph G which induces a vertex distinguishing labeling of G such that the label of each vertex is given by the difference between the highest and the lowest colors of its adjacent edges. The minimum number of colors required for a gap vertex-distinguishing edge coloring of G is called the *gap chromatic number* of G and is denoted by $gap(G)$. We will compute this parameter for a large set of graphs G of order n and we even prove that $gap(G) \in \{n-1, n, n+1\}$.

In the second part, we focus on graph packing problems, which is an area of graph theory that has grown significantly over the past several years. However, the majority of existing works focuses on unlabeled graphs. In this thesis, we introduce for the first time the packing problem for a vertex labeled graph. Roughly speaking, it consists of graph packing which preserves the labels of the vertices. We study the corresponding optimization parameter on several classes of graphs, as well as finding general bounds and characterizations.

The last part deal with the query processing of a core subset of XML query languages: XML twig queries. An XML twig query, represented as a small query tree, is essentially a complex selection on the structure of an XML document. Matching a twig query means finding all the occurrences of the query tree embedded in the XML data tree. Many holistic twig join algorithms have been proposed to match XML twig pattern. Most of these algorithms find twig pattern matching in two steps. In the first one, a query tree is decomposed into smaller pieces, and solutions against these pieces are found. In the second step, all of these partial solutions are joined together to generate the final solutions. In this part, we propose a novel holistic twig join algorithm, called *TwigStack++*, which features two main improvements in the decomposition and matching phase. The proposed solutions are shown to be efficient and scalable, and should be helpful for the future research on efficient query processing in a large XML database.

Keywords: graph theory, labeled graph, vertex-distinguishing edge coloring, labeled packing of graphs, XML tree pattern matching.

Coloration, Placement and Plongement de graphes

Résumé:

Cette thèse se situe dans le domaine de graphes et de leurs applications, Elle est constitué de trois grandes parties, la première est consacrée à l'étude d'un nouveau type de coloration sommets distinguantes, les arête-colorations sommets-distinguantes par écarte. Il consiste de trouver une valuation des arêtes qui permette de distinguer les sommets de graphes telle que chaque sommet v du graphe est identifié de façon unique par la différence entre la plus grande et la plus petite des valeurs incidentes à v. Le plus entier pour lequel le graphe G admet une arête-coloration sommets-distinguantes par écarte est le nombre chromatique par écart de G, noté $gap(G)$. Nous avons étudié ce paramètre pour diverses familles de graphes. Une conjecture intéressante, proposée dans cette partie, suggère que le nombre chromatique par écart de tout graphe connexe d'ordre $n > 2$ vaut $n-1$, n ou $n+1$. La deuxième partie du manuscrit concerne le problème du placement de graphes. Nous proposons un état de l'art des problèmes de placement de graphes, puis nous introduisons la nouvelle notion de placement de graphes étiquetés. Il s'agit d'un placement de graphes qui préserve les étiquettes des sommets. Ensuite, nous proposons des encadrements de ce nouveau paramètre pour plusieurs classes de graphes. La troisième partie de la thèse s'intéresse au problème d'appariement d'arbres dans le cadre de la recherche d'information dans des documents structurés de type XML. Les algorithmes holistique de jointure structurelle est l'une des premières méthodes proposées pour résoudre l'appariement exact des documents XML. Ces algorithmes sont souvent divisés en deux grandes étapes. La première étape permet de décomposer l'arbre de la requête en un ensemble de petites composantes connexes. Ensuite, des solutions intermédiaires pour chaque composante de la requête sont trouvées, ces résultats intermédiaires sont joints pour obtenir la solution finale. Nous proposons dans cette partie un nouvel algorithme appelé *TwigStack++* qui vise principalement à diminuer le coût de la jointure et le calcule inutile recherche. Notre algorithme obtient de meilleurs résultats en comparaison avec deux autres méthodes de l'état de l'art.

Keywords: théorie des graphes, graphes étiquetés, colorations sommets distinguantes, placement de graphes étiquetés, l'appariement exact des documents XML.