



Some applications of vector bundles in algebraic geometry

Daniele Faenzi

► To cite this version:

Daniele Faenzi. Some applications of vector bundles in algebraic geometry. Algebraic Geometry [math.AG]. Université de Pau et des Pays de l'Adour, 2013. tel-01062906v1

HAL Id: tel-01062906

<https://theses.hal.science/tel-01062906v1>

Submitted on 14 May 2019 (v1), last revised 15 May 2019 (v2)

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Habilitation à Diriger des Recherches

Daniele FAENZI

*Some applications of vector bundles
in algebraic geometry*

Mémoire de synthèse

Exposé de recherche, travaux en cours

ÉQUIPE D'ALGÈBRE ET GÉOMÉTRIE DE PAU

LMA UMR CNRS 5142

UNIVERSITÉ DE PAU ET DES PAYS DE L'ADOUR

Thématiques de recherche

Mathématique

Géométrie algébrique - Géométrie complexe - Singularités

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Some applications of vector bundles in algebraic geometry

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To my family

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Introduction

Algebraic geometry studies algebraic varieties, which are the loci of points in space satisfying a set of polynomial equations in several variables. This is an ancient and popular subject of mathematics, connected to many other areas such as algebraic topology, singularity theory, representation theory, combinatorics, commutative algebra, and perhaps even theoretical physics and computational complexity.

Vector bundles can be thought of as vector spaces varying continuously along a given variety. These geometric objects offer a valuable point of view on algebraic geometry, and represent the technical core of this work.

The first example of vector bundles that we will encounter are logarithmic vector fields along a (reduced) divisor D in projective space $\mathbb{P}^n$.

This bundle, denoted by $\mathcal{T}_{\mathbb{P}^n}(-\log D)$, was originally introduced by Deligne and Saito (cf. [97, 277]) to study the Hodge theory of the complement of D . It is actually not a bundle in general but only a reflexive sheaf, obtained as a modification of the tangent bundle along the Jacobian ideal J_D of D . More explicitly, J_D is generated by the $n + 1$ partial derivatives of a defining polynomial f of D , i.e. J_D is the image of the gradient map $\nabla(f)$, and the restriction of J_D to D is the equisingular normal sheaf of D . The sheaf $\mathcal{T}_{\mathbb{P}^n}(-\log D)$ is the kernel of $\nabla(f)$, or in other words the first (sheafified) syzygy of J_D .

The ideal J_D and the sheaf $\mathcal{T}_{\mathbb{P}^n}(-\log D)$ carry deep geometric information on the hypersurface D and the embedding j of D in $\mathbb{P}^n$. Work in this direction was started by Griffiths [154], and we know that $\mathcal{T}_{\mathbb{P}^n}(-\log D)$ captures part of the primitive cohomology of the normalization of D , and controls the deformations of j , cf. [288]. The Chern classes of $\mathcal{T}_{\mathbb{P}^n}(-\log D)$ give invariants of the polarity map associated with D , well-studied especially for homaloidal polynomials (i.e. when the polarity map is birational). This ties in with classical geometry of Cremona transformations, and also with the theory of prehomogeneous vector spaces, invariant hypersurfaces and Severi varieties. In a different direction, relying on projective duality and on Kempf-Lascoux-Weyman's method for studying syzygies, one can compute in some cases the graded free resolution of $\mathcal{T}_{\mathbb{P}^n}(-\log D)$, and this is also relevant in the calculation of classical invariants such as discriminants, resultants etc.

In spite of these nice properties, several features of the sheaf of logarithmic derivations remain quite mysterious. One of them will attract our attention, namely the observation that for special choices of D , the sheaf $\mathcal{T}_{\mathbb{P}^n}(-\log D)$ “splits”, i.e., it is a direct sum of line bundles; or in other words the associated graded module is free - the hypersurface D is thus called *free*. In this nice case, the divisor is expressed as determinant of the square matrix presenting the logarithmic bundle.

We should be aware that this phenomenon is quite rare. In fact, the singularities of D have to be very rich to force freeness (the singular locus has to be maximal Cohen-Macaulay of codimension 1). On the other hand, free hypersurfaces are very interesting and arise naturally in several contexts. Formal free divisors occur, for instance, as discriminants of versal deformations of singularities (cf. [61, 226], see also [93] for a survey).

For projective divisors, freeness has been extensively studied for hyperplane arrangements, i.e., divisors consisting of the union of finitely many hyperplanes in $\mathbb{P}^n$. An important instance of free arrangements arises when taking the set of reflecting hyperplanes of a Coxeter group W . The quotient X_W of the complement of this hypersurface by the action of W is an Eilenberg-Mac Lane space whose fundamental group is an Artin-Tits group, as shown by fundamental work of Deligne [98] and Brieskorn [53]. From the point of view of derivations, this has been widely studied cf. [250, 298], and interesting free arrangements arise when adding integral translates to the reflecting hyperplanes, [318]. A tool to study freeness in this framework is given by Ziegler’s multiarrangements, where logarithmic derivations are asked to be tangent with a certain multiplicity to each hyperplane.

One of the main issues related to freeness of hyperplane arrangements is whether it depends only on combinatorics, i.e., on the intersection lattice of the hyperplanes under consideration: this is *Terao’s conjecture*. We will briefly discuss this conjecture in Chapter 1. More generally, the problem of determining the projective dimension of the sheaf of logarithmic derivations is far from being understood. Freeness has been examined also for other divisors such as unions of lines and conics in the plane, [282]. In this case, strictly speaking, freeness is not combinatorial, although it might be so if combinatorics were to be taken in a broader sense (cf. the discussion below of Milnor vs. Tjurina numbers). However, irreducible projective free divisors are, in general, not quite easy to find (cf. [289, 290]).

To study freeness, we carried out in [139] an approach based on vector bundles and projective duality. It turns out that, if Z is the set of points dual to the arrangement given by a divisor D , then $\mathcal{T}_{\mathbb{P}^n}(-\log D)$ is the direct image of $\mathcal{I}_{Z/\mathbb{P}^n}(1)$ under the standard point-hyperplane incidence. This method allows

to quickly reprove many known results on (multi)arrangements (Saito's criterion, addition-deletion etc), and to show that freeness of a divisor D given by $2k + r + 1$ lines in $\mathbb{P}^2$ is automatic in case $c_2(\mathcal{T}_{\mathbb{P}^2}(-\log D)) = k(k + r)$ and D has a point of multiplicity $h \in [k, k + r + 1]$, see Theorem II.9 of Chapter 1. So in this case a strong version of Terao's conjecture holds, for c_2 is a very weak combinatorial invariant of D .

The second main feature of the sheaf of logarithmic derivations (or dually, logarithmic differentials) that we will be interested in, is a version of the *Torelli problem*, namely the question of whether, or when, this sheaf determines the divisor D we started with. We will give a brief survey of what is known about this question, rooted in work of Clemens, Griffiths, Donagi, up to Comessatti and Torelli himself.

Although a general answer to the Torelli problem is not known, several important cases are indeed well-understood, for instance smooth hypersurfaces and, again, hyperplane arrangements $\mathcal{A}$. The case when $\mathcal{A}$ is generic (or normal crossing) was solved in [107, 306]. To tackle the general case, one first has to adopt a modified version, due to Dolgachev [105] (cf. also [76]), of the sheaf of logarithmic differentials. This new sheaf is not even reflexive in general (its double dual is indeed the classical sheaf of logarithmic differentials), but carries a much richer information on the arrangement $\mathcal{A}$. Having this in mind, we gave in [138] a general Torelli theorem for arrangements $\mathcal{A}$, that states that $\mathcal{A}$ satisfies the Torelli property if and only if the dual set of points Z in $\check{\mathbb{P}}^n$ does not lie in what we called a Kronecker-Weierstrass variety. These are defined by the 2×2 minors of a $2 \times n$ matrix of linear forms, and in this sense they constitute a possibly degenerate version of rational normal curves. We refer to Theorem IV.6 of Chapter 1 for a more precise statement. We will also sketch a proof of this result, based on integral functors and unstable hyperplanes.

Next we outline a *generic Torelli* theorem, asserting that for a general choice of sufficiently many hypersurfaces of various degrees in $\mathbb{P}^n$, our sheaf of logarithmic derivations determines the union of such hypersurfaces. Although this result is far from being sharp, it provides a wide region where the Torelli property does hold, and leaves to test only finitely many cases for each choice of degrees of the components of D . It is the content of Theorem IV.8 of Chapter 1.

The next topic that we will treat is a particular class of vector bundles (or sheaves), namely *arithmetically Cohen-Macaulay (ACM)* and *Ulrich sheaves*, traditionally important in commutative algebra and representation theory of rings, and recently much studied also in view of their connection with Boij-Söderberg theory, cf. [122, 285], derived categories of singularities, cf. [253, 254] and non-commutative desingularization [64].

ACM sheaves over an embedded variety $X \subset \mathbb{P}^n$ are characterized by the vanishing of intermediate cohomology, or equivalently by fact that the minimal graded free resolution of the associated module of global sections over the polynomial ring is as short as possible, namely of length equal to the codimension of X . For Ulrich sheaves, one further requires that the resolution is linear. ACM sheaves are the object of Chapter 2.

These sheaves offer a tight connection with determinantal representations. Indeed, an ACM sheaf F on an integral hypersurface D of $\mathbb{P}^n$ defined as vanishing of a polynomial f provides a matrix factorization, [118] i.e. a square matrix M whose determinant is a power of f , and the entries of M is linear if F is Ulrich. One may further require the matrix to be symmetric or skew-symmetric, in which case ACM bundles are related to theta characteristics and Pfaffian representations. A survey of results on existence and classification of ACM and Ulrich bundles on hypersurfaces will be given in Chapter 2. We will also mention some speculations on the minimal rank of these bundles, related to the conjectural minimality attained by the rank of the terms of the Koszul complex among Betti numbers of an ACM sheaf.

In a different direction, determinantal hypersurfaces are closely related to representations of a convex region in affine space as a *linear matrix inequality* (LMI), i.e. as the set of points where a given symmetric matrix M of affine linear forms is positive semidefinite. A beautiful theorem of Helton-Vinnikov provides an LMI for real plane curves with a maximum number of nested ovals; however existence of LMI's in higher dimension is the object of an important conjecture of P. Lax, see again Chapter 2 for details.

One main issue concerning ACM sheaves is to describe as completely as possible the class of ACM bundles on a given variety. For some particular varieties this boils down to an exhaustive classification, tightly related to splitting criteria. For instance on projective spaces and quadrics, ACM indecomposable bundles are either of rank 1 or, for quadrics, isomorphic to spinor bundles (cf. [173, 206], see also [258] for extensions of Horrocks' splitting criterion).

However, this extremely simplified behaviour is very rare, for few varieties X admit finitely many indecomposable isomorphism classes of ACM bundles up to twist by line bundles (X is called of *finite CM type*), and their classification is completed in [121]. Besides projective spaces, quadrics, and rational normal curves, only two more CM-finite varieties of positive dimension exist: a rational cubic scroll in $\mathbb{P}^4$ and the Veronese surface in $\mathbb{P}^5$. This classification, rooted in work of Auslander, Buchweitz, Greuel, Schreyer and others, ties in with Bertini's classification of varieties of minimal degree.

Some other varieties, for instance curves of genus ≥ 2 , admit families of arbitrarily large dimension of such bundles (the variety is thus called of *wild CM type*), which makes the classification a bit hopeless. We will give a brief

survey of the very large class of varieties which are known to be of this type, which include most Segre products (cf. [90]), the triple Veronese embedding of any variety (cf. [237]), and hypersurfaces of degree ≥ 4 in $\mathbb{P}^n$, with $n \geq 3$ (cf. [92]).

However, not all varieties are of finite or wild CM type. A smooth projective curve of genus 1, for instance, is of *tame CM type*, namely, although isomorphism classes of ACM indecomposable bundles are indeed infinitely many, any family parametrizing such classes has dimension 1 at most, [14]. Conjecturally this trichotomy exhausts all smooth varieties (not singular ones, however, as for instance quadric cones over a point support countably many indecomposable ACM sheaves).

In Theorem IV.2 of Chapter 2, we prove that $\mathbb{P}^1 \times \mathbb{P}^1$, embedded by $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 2)$, is of tame CM type. The proof, given in some detail, goes through a classification based on a semiorthogonal decomposition of the derived category of $\mathbb{P}^1 \times \mathbb{P}^1$ adapted to our purpose. This surface can also be seen as a scroll $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(2)^2)$, embedded by the relatively ample line bundle. Although we do not study this here, the scroll $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(3))$, also should be of tame CM type (this is actually work in progress). One should be warned that no other smooth projective variety of this kind is known today, besides the elliptic curve we already mentioned. We conjecture that this is for the good reason that there is actually no other such variety at least in positive dimension.

In a different direction, one can hope to classify ACM bundles with special properties. Among them, a notable one is rigidity (i.e. no non-trivial infinitesimal deformation of our sheaf exists). For some Veronese rings, this has been done in [189], see also [200], making use of cluster tilting in triangulated categories. We will give a different proof of this result based on vector bundles methods and classical results of Beilinson and Kac, [34, 193], see Theorem III.1 and Theorem III.3 of Chapter 2.

The last chapter of this work is devoted to another special class of vector bundles, namely *instantons*. These arose in the algebrization of solutions of the Yang-Mills differential equation, via the fundamental work of Penrose, Atiyah, and others [15, 16, 262]. In terms of algebraic geometry, they are defined as stable vector bundles E of rank 2 on the complex projective space $\mathbb{P}^3$, having $c_1(E) = 0$ and with the prescribed cohomology vanishing $H^1(\mathbb{P}^3, E(-2)) = 0$. We speak of k -instanton if $c_2(E) = k$.

The main questions on k -instantons concern geometric properties of their moduli space $\text{MI}_{\mathbb{P}^3}(k)$ such as smoothness, irreducibility and so forth. The analysis in this sense was arguably started by Barth, Hartshorne and others, [23, 24, 163], and has recently come to show smoothness [191] and irreducibility at least for odd k , cf. [300].

An extension of the notion of instanton to a slightly broader class of base manifolds has been proposed in [133, 220], namely Fano threefolds X of Picard number 1. In this case, the canonical bundle of X is of the form $\mathcal{O}_X(-i_X h)$, for some integer $1 \leq i_X \leq 4$ called the index of X , where h is the positive generator of the Picard group of X . Somehow these varieties provide a natural framework to extend notions typical of projective spaces; one advantage in dimension 3 is that they are completely classified (even neglecting the assumption on the Picard number), we refer to the book [188] for an extensive treatment.

For a given Fano threefold X of Picard number 1 and index i_X , we set $q = \lfloor i_X/2 \rfloor$ and define a k -instanton to be a stable vector bundle E of rank 2, such that $c_1(E(-q)) = -i_X$ and $H^1(X, E(-q)) = 0$. Indeed, for $i_X = 4$ (so $q = 2$), if $c_1(E) = 0$ we get $c_1(E(-2)) = -4$, and our notion gives back usual k -instantons. However, when i_X is odd, $c_1(E)$ is also odd, and we speak thus of *odd instantons*.

A survey of results on instantons and more generally on moduli spaces of stable sheaves on Fano threefolds will be given at the beginning of Chapter 3, including some sketches of related topics such as the map of periods and derived categories. In fact, one of the main classical tools to analyze the moduli space $\mathrm{MI}_{\mathbb{P}^3}(k)$ is provided by monads, via Beilinson's theorem, cf. [26]. We will overview the way to set up this tool, relying on the structure of the derived category of X , for the other Fano threefolds X with trivial intermediate Jacobian arriving to a parametrization of $\mathrm{MI}_{\mathbb{P}^3}(k)$ as geometric quotient of a space of self-dual monads.

Even when the intermediate Jacobian $J(X)$ of our threefold X is not trivial, the situation is quite well understood at least when X is rational. In this case, $J(X)$ is the Jacobian of a curve C of positive genus, related to X by Kuznetsov's homological projective duality, cf. [217]. In fact the curve C in this case provides the non-trivial component of the derived category $\mathbf{D}^b(X)$, in the sense that this category is orthogonal to an exceptional sequence of maximal length. This subcategory conjecturally captures important features of a given variety, as for instance a smooth cubic fourfold should be rational if and only if this category is equivalent to the derived category of a K3 surface (Kuznetsov's conjecture, [211]); while a cubic threefold is determined by such subcategory (categorical Torelli of [35]). Homological projective duality gives a very efficient tool to control this subcategory in terms of the dual variety, the only drawback being the little number of examples available today, essentially projective bundles and two Grassmannians, $\mathbb{G}(2, 6)$ and $\mathbb{G}(2, 7)$. We hope that at least one more construction could come from the Cayley plane via the desingularization of the Cartan cubic, with applications to cubic sevenfolds, [137, 182]. The point we would like to stress here is however focused on the use of this

setup in the study of moduli of vector bundles, replacing exceptional sequences and monads by semiorthogonal decompositions and Fourier-Mukai functors.

We look at this situation in detail in a specific case, namely that of *prime Fano threefolds of genus 10*. Such a threefold X is obtained as double hyperplane section of a variety Σ of dimension 5, homogeneous under the exceptional Lie group G_2 . The dual variety of Σ is a sextic hypersurface in $\mathbb{P}^{13}$, and the pencil of hyperplanes defining X cuts this sextic at 6 points. Taking the associated double cover of $\mathbb{P}^1$, we get the homologically dual curve C which in this case has genus 2.

We will rely on the method developed in [48, 50] to describe this curve as a moduli space of bundles of rank 3 on X (Theorem II.1), whereby refining a result of Kapustka and Ranestad, [198]. This will allow us to provide a description of the Hilbert schemes of lines, conics, and rational cubics contained in X in terms of divisor of class 3Θ in $J(C)$ (for lines), or $J(C)$ itself (for conics), or, for cubics, as a projective bundle $\mathbb{P}(\mathcal{V})$, where $\mathcal{V}$ is a particular stable vector bundle on C , of rank 3, and with trivial determinant. This will allow us to parametrize odd k -instanton on X as simple bundles on C , with rank and degree k with at least $k - 1$ independent morphisms from $\mathcal{V}^*$ (plus two slightly more technical conditions).

Moreover, we will see in Theorem II.7 that the bundle $\mathcal{V}$ lies in a special divisor of the moduli space $M_C(3)$ of stable bundles of rank 3 with trivial determinant on C , called the *Coble-Dolgachev sextic*. Indeed, the so-called theta map provides a $2 : 1$ morphism $M_C(3) \rightarrow \mathbb{P}^8$, which is ramified along a sextic hypersurface, dual to the Coble cubic in $\mathbb{P}^8$ (cf. [255]). We conjecture that the general fibre of the period map of Fano threefolds of genus 10 is an open dense subset of the Coble-Dolgachev sextic. In other words, the choice of $\mathcal{V}$ should allow us to recover X , although for the moment we have not been able to check this. This is closely related to conjectures of Mukai (on realisation of X as a Brill-Noether locus) and of Kuznetsov (on the relation between X and the intersection of two quadrics in $\mathbb{P}^5$ in terms of instantons and derived categories).

The three main chapters of this work contain a short discussion of open problems that seem interesting to me. Most results cited here are cited without proof. Besides some exceptions, detailed arguments are only given in case they have not appeared so far (at least to my knowledge).

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Acknowledgements

I am deeply indebted with Giorgio Ottaviani for teaching me his way of doing “concrete” algebraic geometry, at the time of my Ph. D. in Florence. I remember with pleasure the feeling, a mixture of confusion and enthusiasm

(usually with a greater proportion of the second one), after a long discussion in his office. I may well say that almost all the techniques I feel comfortable with today, I learned from him.

I am also grateful to Robin Hartshorne, for the encouragement and support he offered some years later, especially concerning my work on ACM bundles on cubic surfaces. I am not sure that I would have continued to do research at that time if I did not receive his letter.

A special thank goes to Olivier Debarre, Edoardo Sernesi and Frank-Olaf Schreyer, who kindly accepted to serve as referees of this text, and even more so in view of my personal admiration of their work. I am indebted to them for suggesting several improvements to this write-up.

My colleagues and true friends Vincent Florens and Jean Vallès of *équipe d'algèbre et géométrie de Pau*, together with Jacky Cresson, also deserve my greatest gratitude. It has been a pure pleasure to work together these years (and hopefully it will continue to be so), while trying to strengthen the tradition in pure mathematics in Pau, in spite of our little number. We could have certainly done better, but not in a more pleasant atmosphere of free discussion on any topic that one can possibly imagine (including, occasionally, mathematics). Life in Pau's department would surely have been unhappy, however, if several other friends like Isabelle Greff, Charles Pierre, Christian Paroissin, did not help make it so nice.

On a "south western" level, I would like to thank the persons that organized with me our "PTT Pau-Tarbes-Toulouse seminar", namely Marcello Bernardara, Thomas Dedieu, Benoît Bertrand, Jean Vallès, together with all the recurrent or sporadic participants of it, for the good time we shared in those occasions, and also for patiently listening to the many lectures I delivered there in the last years. In fact, I first got acquainted with Coble cubics and related material at a special two-days session organized in Toulouse by Thomas Dedieu, in the framework of PTT, in 2012, with nice lectures of Arnaud Beauville, Michele Bolognesi, Thomas Dedieu, Angela Ortega, and Christian Pauly.

Several other friends and collaborators should be credited for continuous support, invitations and discussions. For this, and for everything they taught me, I am particularly grateful to Laurent Manivel, Emilia Mezzetti, Lucia Fania, Enrique Artal, José Ignacio Cogolludo, Laura Costa and Rosa Maria Miró Roig. I also wish to thank Gianfranco Casnati, Francesco Malaspina, Ada Boralevi, Chiara Brambilla, Alessandro Chiodo, Luca Chiantini, Christian Peskine, Frédéric Han, Johannes Huisman, Jan Nagel, Michele Bolognesi, Olivier Serman, Roland Abuaf, Alessandra Sarti, Laurent Evain, Luis Solà Conde and many others that I might have forgotten for discussions and advices. Special thanks

to Takuro Abe, Daniel Matei and Masahiko Yoshinaga for the time they spent teaching to me the basics of arrangements.

The deepest thank of all, anyway, is reserved to my family, and I guess they know why.

CHAPTER 0

Preliminaries and background

Our point of view on algebraic varieties is based on vector bundles, or more generally on sheaves, together with a collection of related notions, such as characteristic classes, moduli spaces, deformation theory, derived categories, and so forth.

We give here a reminder of some of these notions, essentially aimed at fixing terminology and notation. We refer mainly to [162] and [178].

I. Notation and conventions

By k we will always denote a base field, in principle an arbitrary one. However, we will mainly deal with algebraically closed field of characteristic zero (in practice, with $\mathbb{C}$).

I.1. Polynomial rings. Let V be a vector space over k . Given an integer $d \geq 0$, we will denote by $S^d V$ the d -th symmetric power of V , i.e., the quotient of $V^{\otimes d}$ by the subspace generated by $u_1 \otimes \cdots \otimes u_d - u_{\sigma 1} \otimes \cdots \otimes u_{\sigma d}$ for all $\sigma \in \mathfrak{S}_d$ and all $u_1, \dots, u_d \in V$. One has the multiplication map $S^d V \otimes S^e V \rightarrow S^{d+e} V$, and the comultiplication $S^d V \rightarrow \bigoplus_e S^e V \otimes S^{d-e} V$, which is induced by the diagonal inclusion $V \rightarrow V \oplus V$.

For a positive integer n , we will denote by S the polynomial ring $k[x_0, \dots, x_n]$, and by S_d the homogeneous piece of degree d of S . If V has dimension $n+1$, then S_d can be identified with $S^d V$ and S is the symmetric algebra $\bigoplus_{d \geq 0} S^d V$. We write $\mathfrak{m}$ for the maximal ideal $(x_0, \dots, x_n)$ of S .

Given a multiindex i , we will write $\mathbf{x}^i$ for the monomial $\prod_j \mathbf{x}_j^{i_j}$, so in this situation $\mathbf{x}$ is implicitly written as indexed variable. In this framework, $\binom{i}{k}$ is defined as $\prod_j \binom{i_j}{k_j}$. We also denote by ∂_i the derivation in S with respect to the variable $\mathbf{x}_i$.

Given a polynomial f in S , or more generally and ideal I of S , we denote the zero-locus of f , or of I , in $\mathbb{A}^n$ by $\mathbb{V}(f)$. If f , or I , are homogeneous, $\mathbb{V}(f)$ will be thought of as a subvariety of the projective space.

I.2. Projective spaces and Grassmannians. Let V be a vector space of dimension $n+1$ over k . We will write $\mathbb{P}(V)$ for the projective space of 1-dimensional quotients of V . We will set $\mathbb{P}^n = \mathbb{P}_k^n = \mathbb{P}(k^{n+1})$. The set $\check{\mathbb{P}}^n$ of

hyperplanes of $\mathbb{P}^n$ is called the dual projective space. This is the space of 1-dimensional vector subspaces of V , and is identified with $\mathbb{P}(V^*)$, we also write $\check{\mathbb{P}}(V) = \mathbb{P}(V^*)$. Accordingly, $\Gamma(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(1))$ is naturally identified with V while $\Gamma(\check{\mathbb{P}}(V), \mathcal{O}_{\check{\mathbb{P}}(V)}(1)) \simeq V^*$.

Likewise, we will write $\mathbb{G}(V, k)$ for the Grassmannian of k -dimensional quotient spaces of V , and $\mathbb{G}(k, V)$ for the k -dimensional subspaces of V , so that $\mathbb{G}(k, V)$ is identified with $\mathbb{G}(V^*, k)$.

Also, there are natural vector bundles $\mathcal{U}$ and $\mathcal{Q}$ on $\mathbb{G}(V, k)$, called *tautological bundles*, whose fibres over a point Λ of $\mathbb{G}(V, k)$ corresponding to a quotient space Λ of V are given by $\mathcal{Q}_\Lambda = \Lambda$ and $\mathcal{U}_\Lambda = \ker(V \rightarrow \Lambda)$. So we have $\mathrm{rk}(\mathcal{Q}) = k$, $\mathrm{rk}(\mathcal{U}) = n + 1 - k$, and an exact sequence:

$$(I.1) \quad 0 \rightarrow \mathcal{U} \rightarrow V \otimes \mathcal{O}_{\mathbb{G}(V, k)} \rightarrow \mathcal{Q} \rightarrow 0.$$

Beware that, when dealing with $\mathbb{G}(k, V)$, the tautological bundles will have different conventions, namely we will have $\mathrm{rk}(\mathcal{U}) = k$, $\mathrm{rk}(\mathcal{Q}) = n + 1 - k$.

Given integers $0 < n_1 < \dots < n_s < n$, we will occasionally use the notation $\mathbf{F}(n_1, \dots, n_s)$ for the flag manifold of subspaces $\Lambda_1 \subset \dots \subset \Lambda_s$ of V having dimension $\dim(\Lambda_i) = n_i$.

I apologize with the reader for this “double notation” issue, which is due to the contrast between the more familiar notion of projective space (with subspaces), and the better sheaf-theoretic behaviour of the dual notion (with quotient spaces). In a minute, we will see set up the notation for the relative case too.

I.3. Varieties. Let us fix a base field $\mathbf{k}$. A *variety* X over $\mathbf{k}$ will mean a separated scheme of finite type over $\mathbf{k}$. A *point* of a variety, of a scheme X will usually mean a closed point of X , although in two or three occasions we will use S -valued points, S being an arbitrary $\mathbf{k}$ -scheme, but this will be pointed out explicitly. A *manifold* will mean a smooth variety. We will write X_{sm} for the set of smooth points of a reduced variety X .

Given a subscheme Y of a variety X , we will denote by $\mathcal{I}_{Y/X}$ the ideal sheaf of Y in X , and by $\mathcal{N}_{Y/X}$ the normal sheaf of Y in X . Given a coherent sheaf E on X , we will sometimes write E_Y for $E|_Y = E \otimes \mathcal{O}_Y$.

If s is a global section of a coherent sheaf E on X , the scheme-theoretic zero-locus of s will also be denoted by $\mathbb{V}(s)$.

Given a variety X , we will use the cohomology groups $H^i(X, E)$ attached to a coherent sheaf E on X , which are finitely-dimensional $\mathbf{k}$ -vector spaces, and we will write $h^i(X, E)$ for $\dim H^i(X, E)$. For a pair of coherent sheaves E, F on X , we will write $\chi(E, F) = \sum (-1)^i \mathrm{ext}_X^i(E, F)$ (where $\mathrm{ext}_X^i(E, F) = \dim_{\mathbf{k}} \mathrm{Ext}_X^i(E, F)$) and $\chi(E) = \chi(\mathcal{O}_X, E)$.

For a pair of varieties X and Y , we will write π_X and π_Y for the projections from $X \times Y$ onto X and Y . If E and F are coherent sheaves defined respectively on X and Y , we write $E \boxtimes F$ for $\pi_X^*(E) \otimes \pi_Y^*(F)$.

I.4. Divisors, polarized varieties. A *polarized variety* will mean a pair consisting of a variety X and the linear equivalence class of an ample Cartier divisor h over X . Given the class h , we will write $\mathcal{O}_X(h)$ for the associated line bundle. In case there is a morphism $f : Y \rightarrow X$, we will write by abuse of notation $\mathcal{O}_Y(h)$ for $\mathcal{O}_Y(f^*(h))$.

A Weil divisor D on a (geometrically) integral variety X has *simple (or strict) normal crossings* if, locally in the Zariski topology, in the algebraic closure of $\mathbf{k}$, all the irreducible components of D are smooth and intersect transversely. If, in the algebraic closure of $\mathbf{k}$, all the components of D intersect transversely locally in the étale topology, then D has *normal crossings*.

II. Basic material on coherent sheaves

We will need several notions concerning coherent sheaves on algebraic varieties, which are a geometric counterpart or finitely generated modules over commutative rings. We refer to [162] for the basics on coherent sheaves, vector bundles (i.e., locally free sheaves), sheaf cohomology, and so forth.

II.1. Semistable sheaves and moduli spaces. Let X be an m -dimensional connected manifold over a field $\mathbf{k}$.

II.1.1. Chern classes. The Chern classes $c_k(F)$ are defined for any coherent sheaf F on X . According to the specific situation, we will consider them as elements of the Chow ring $\mathrm{CH}^*(X)$ (see [146]), or, when $\mathbf{k} = \mathbb{C}$ as elements of $H^{k,k}(X)$ (see for instance [69]). In the sequel, the Chern classes will be written as integers as soon as the corresponding ambient space has dimension 1 and the choice of a generator is made. All these conventions should be clear from the context.

The Chern polynomial of a coherent sheaf F on X is defined as $c_F(t) = 1 + c_1(F)t + \dots + c_m(F)t^m$. Let Z be an integral subscheme of X , of codimension $p \geq 1$, denote by $[Z]$ its fundamental class in $H^{p,p}(X)$ (or in $\mathrm{CH}^p(X)$). We recall that when a sheaf T is supported at Z , and has rank r at a generic point of Z , then we have $c_k(T) = 0$ for $1 \leq k \leq p-1$ and:

$$c_p(T) = (-1)^{p-1} r [Z].$$

If Z is not integral, a similar formula holds by taking the sum over all integral components of minimal codimension appearing in the support of T , weighted by their multiplicity.

II.1.2. *Torsion-free, reflexive sheaves.* Let X be an integral, locally factorial variety over $\mathbf{k}$. Given a coherent sheaf F on X , we write $\mathrm{rk}(F)$ for the rank of F at a general point of X . We denote by $F^* = \mathcal{H}om_X(F, \mathcal{O}_X)$ the dual of F . Recall that a coherent sheaf F on X is *reflexive* if the natural map $F \rightarrow F^{**}$ of F to its double dual is an isomorphism. Anyway F^{**} is called the *reflexive hull* of F , the map $F \rightarrow F^{**}$ is injective if F is torsion-free, and the support of its cokernel is has codimension at least 2 in X . Moreover, any locally free sheaf is reflexive, and any reflexive sheaf is torsion-free (recall that a coherent sheaf E on X is *torsion-free* if, for all points x of X , and all $0 \neq f \in \mathcal{O}_{X,x}$ the multiplication by $f : E_x \rightarrow E_x$ is injective). It is true that a coherent sheaf F on X is reflexive if and only if it can be included into a locally free sheaf E with E/F torsion-free, see [165, Proposition 1.1]. Moreover, by [165, Proposition 1.9], any reflexive rank-1 sheaf is invertible (because X is integral and locally factorial).

Finally, we will use a straightforward generalization of [165, Proposition 2.6] which implies that the third Chern class $c_3(F)$ of a rank 2 reflexive sheaf F on a smooth projective threefold satisfies $c_3(F) \geq 0$, with equality attained iff F is locally free.

II.1.3. *Hilbert polynomial.* Let (X, h_X) be a polarized variety of dimension m . Given a coherent sheaf F on X , we usually simplify $F(th_X)$ to $F(t)$. We denote by $p(F, t)$ the Hilbert polynomial of F namely $\chi(F(t))$. If F is torsion-free of generic rank $r \neq 0$, the dominant term of this polynomial is $\mathrm{rk}(F) \deg(X)/m!$. We write $p(F, t)$ for the *reduced Hilbert polynomial*:

$$p(F, t) = \frac{\chi(F(t))}{r}.$$

Given polynomials $p, q \in \mathbb{Q}[t]$, we write $p(t) \succeq q(t)$ if $p(t) \geq q(t)$ for $t \gg 0$ and $p(t) \succ q(t)$ if $p(t) > q(t)$ for $t \gg 0$. We let $\mathrm{Hilb}_p(X)$ be the Hilbert scheme of subschemes Y of X having Hilbert polynomial $\chi(\mathcal{O}_Y(t))$ equal to $p(t)$. Given integers d, g , we let $\mathcal{H}_g^d(X)$ be the union of components of $\mathrm{Hilb}_{dt+1-g}(X)$ containing locally Cohen-Macaulay curves $Y \subset X$ (i.e., curves Y with no isolated or embedded components) having degree d and arithmetic genus g .

II.1.4. *Summary on semistable vector bundles and sheaves.* We will deal with semistable sheaves in characteristic zero only, even though many constructions can be carry out in greater generality (cf. [222]). However we set $\mathrm{char}(\mathbf{k}) = 0$, and assume $\mathbf{k}$ algebraically closed.

Let X be a connected m -dimensional manifold over $\mathbf{k}$, and let h_X be an ample divisor class on X . A torsion-free coherent sheaf F on X is h_X -*semistable* in the sense of Gieseker-Maruyama if, for any coherent subsheaf E of F , with $\mathrm{rk}(E) < \mathrm{rk}(F)$, one has the inequality of reduced Hilbert polynomials:

$$p(F, t) \succeq p(E, t).$$

The words “in the sense of Gieseker-Maruyama” will be tacitly omitted. The sheaf F is called *stable* if for all E as above we have $p(F, t) \succ p(E, t)$. A semistable sheaf is called *polystable* if it is the direct sum of stable sheaves having the same reduced Hilbert polynomial.

The *slope* of a torsion-free sheaf $F \neq 0$, sometimes denoted by $\mu(F)$ is the rational number defined as $\deg(F)/\mathrm{rk}(F)$, where $\deg(F) = h_X^{m-1} \cdot c_1(F)$. The *normalized twist* F_{norm} of F is set to be the unique sheaf $F(t)$ whose slope is in $(-1, 0]$. In general, we say that a coherent sheaf F on X has a certain property *up to a twist* if there is t such that $F(t)$ has that property.

A torsion-free coherent sheaf F is semistable in the sense of Mumford-Takemoto, or *slope-semistable* if the slope of any coherent subsheaf E with $\mathrm{rk}(E) < \mathrm{rk}(F)$, is at most the slope of F . The sheaf F is called *slope-stable* if we require strict inequality. We define the *discriminant* of F as:

$$\Delta(F) = 2rc_2(F) - (r-1)c_1(F)^2.$$

Bogomolov’s inequality, see for instance [178, Theorem 3.4.1], states that if F is μ -semistable, then we have:

$$\Delta(F) \cdot h_X^{n-2} \geq 0.$$

Another useful tool is Hoppe’s criterion, see [172, Lemma 2.6], or [5, Theorem 1.2]. It says that, if the line bundle h_X is very ample and generates $\mathrm{Pic}(X)$, and F is a vector bundle on X of rank r , we have:

if $H^0(X, (\wedge^p F)_{\mathrm{norm}}) = 0$, $\forall 0 < p < r$, then the bundle F is slope-stable.

A basic property of semistable sheaves is that their tensor product remains semistable, see for instance [178, Theorem 3.1.4].

II.1.5. Moduli spaces. Let again $\mathbf{k}$ be algebraically closed of characteristic zero. We introduce here some notation concerning moduli spaces. Recall that two semistable sheaves are *S-equivalent* if the direct sum of all successive quotients associated with their Jordan-Hölder filtrations are isomorphic. We denote by $M_X(r, c_1, \dots, c_m)$ the moduli space of *S*-equivalence classes of rank r torsion-free semistable sheaves on X with Chern classes $c_1, \dots, c_m$, considered as elements of the intersection ring. We will drop the values of the classes c_k from k_0 on when they are zero from k_0 on. The class in $M_X(r, c_1, \dots, c_m)$ of a given sheaf F will be denoted again by F .

The moduli space $M_X(r, c_1, \dots, c_m)$ is a projective separated $\mathbf{k}$ -scheme of finite type; however not much more is known about it in general. The Zariski tangent space of this space at the point corresponding to a stable sheaf F is naturally identified with $\mathrm{Ext}_X^1(F, F)$. The obstructions at F lie in $\mathrm{Ext}_X^2(F, F)$, so if this space is zero then $M_X(r, c_1, \dots, c_m)$ is smooth at F .

We and refer to the book [178] for more details on these notions.

II.2. Cohomology and derived categories. Let us introduce some notation and recall some notions concerning some of the cohomological tools that we will use.

II.2.1. *Cohomology of sheaves and modules.* Let (X, h_X) be a polarized variety, and F be a coherent sheaf on X . The following notation is standard:

$$H_*^i(X, F) = \bigoplus_{t \in \mathbb{Z}} H^i(X, F(t)), \quad \Gamma_*(X, F) = H_*^0(X, F).$$

Here, $\Gamma_*(X, \mathcal{O}_X)$ is a graded k -algebra, and the $H_*^i(X, F)$ are modules over it. By Serre's vanishing, $H_*^i(X, F)$ is zero in sufficiently high degree for $i > 0$. Also, F is said to have *natural cohomology* if, for all $t \in \mathbb{Z}$, there is at most one value of i such that $H^i(X, F(t)) \neq 0$.

Let now $X \subset \mathbb{P}^n$ be an embedded variety. Then we write I_X for the homogeneous ideal of X in S , and S_X for the graded quotient k -algebra S/I_X . This is a subalgebra of $\Gamma_*(X, \mathcal{O}_X)$. If F is a coherent sheaf on X then $H_*^i(X, F)$ is a graded S_X -module, whose component in degree j is $H^i(X, F(j))$.

Further, for a given S -module F , we write $H_m^i(F)$ for the local cohomology of F with respect to m . The S -module $H_m^0(F)$ is defined as:

$$H_m^0(F) = \{\nu \in F \mid m^k \cdot \nu = 0, \text{ for some } k \geq 0\},$$

and $H_m^i(-)$ is the right derived functor of order i of the functor $H_m^0(-)$. The dimension and depth of F are, respectively, the maximum and the minimum i such that $H_m^i(F) \neq 0$.

Given a coherent sheaf F on X , set $F = \Gamma_*(X, F)$. This is an S_X -module whose sheafification is F . On the other hand, if F is a module over S_X , if we let F be its sheafification, then there is a long exact sequence:

$$0 \rightarrow H_m^0(F) \rightarrow F \rightarrow \Gamma_*(X, F) \rightarrow H_m^1(F) \rightarrow 0,$$

while for $i > 0$ there are isomorphisms:

$$H_m^{i+1}(F) \simeq H_*^i(X, F).$$

We refer for instance to [119, Appendix 4] and [292].

II.2.2. *Derived categories.* Let X be a smooth projective variety over k . We will use the derived category $\mathbf{D}^b(X)$ of bounded complexes of coherent sheaves on X . We refer to [149, 177] for a detailed account of this triangulated category, and to [69] for a very nice survey on its basic features.

An object $\mathcal{E}$ of $\mathbf{D}^b(X)$ is a bounded complex of coherent sheaves on X :

$$\dots \rightarrow \mathcal{E}^{-1} \rightarrow \mathcal{E}^0 \rightarrow \mathcal{E}^1 \rightarrow \dots$$

As usual, we write $[j]$ for the j -th shift to the left in the derived category, so $\mathcal{E}[j]^i = \mathcal{E}^{i+j}$. The coherent sheaves $\mathcal{H}^i(\mathcal{E})$ are defined as cohomology in degree i of the complex $\mathcal{E}$. We will denote by $H^i(X, \mathcal{E})$ the hypercohomology in

degree i of $\mathcal{E}$, associated to the functor of global sections of $\mathcal{E}$. This is identified with $\mathrm{Hom}_{\mathbf{D}^b(X)}(\mathcal{O}_X, \mathcal{E}[i])$. Similarly, given two complexes $\mathcal{E}$ and $\mathcal{F}$ of $\mathbf{D}^b(X)$, the hypercohomology $\mathrm{Ext}_X^i(\mathcal{E}, \mathcal{F}[i])$ is identified with $\mathrm{Hom}_{\mathbf{D}^b(X)}(\mathcal{E}, \mathcal{F}[i])$.

An object $\mathcal{E}$ of $\mathbf{D}^b(X)$ is *exceptional* if $\mathrm{Ext}_X^i(\mathcal{E}, \mathcal{E})$ is 0 when $i \neq 0$ and k when $i = 0$ (this last condition says that $\mathcal{E}$ is *simple*). A sequence $(\mathcal{E}_1, \dots, \mathcal{E}_s)$ of objects of $\mathbf{D}^b(X)$ is an *exceptional collection* if all the $\mathcal{E}_i$ are exceptional, and $\mathrm{Ext}_X^i(\mathcal{E}_j, \mathcal{E}_k) = 0$ for all $1 \leq k < j \leq s$ and all i . Given a set S of objects of $\mathbf{D}^b(X)$, we denote by $\langle S \rangle$ the smallest full triangulated subcategory of $\mathbf{D}^b(X)$ containing S . A sequence $(\mathcal{E}_1, \dots, \mathcal{E}_s)$ of objects of $\mathbf{D}^b(X)$ is a *full exceptional collection* if it is an exceptional collection, and $\langle \mathcal{E}_1, \dots, \mathcal{E}_s \rangle = \mathbf{D}^b(X)$.

Given a full subcategory $\mathcal{A}$ of $\mathbf{D}^b(X)$, we write $\mathcal{A}^\perp$ for the *right orthogonal* of $\mathcal{A}$, namely the full triangulated subcategory of objects $\mathcal{F}$ of $\mathbf{D}^b(X)$ such that $\mathrm{Hom}_X(\mathcal{E}, \mathcal{F}) = 0$ for all objects $\mathcal{E}$ of $\mathcal{A}$. Similarly one defines the *left orthogonal* ${}^\perp\mathcal{A}$. A subcategory $\mathcal{A}$ of $\mathbf{D}^b(X)$ is called *left* or *right admissible* if the inclusion $i_{\mathcal{A}} : \mathcal{A} \hookrightarrow \mathbf{D}^b(X)$ has a left or right adjoint, which will be denoted as usual by $i_{\mathcal{A}}^*$ and $i_{\mathcal{A}}^!$. Also, $\mathcal{A}$ is called *admissible* if it is so in both ways. Assuming $\mathcal{A}$ admissible, we have $\mathbf{D}^b(X) = \langle \mathcal{A}, {}^\perp\mathcal{A} \rangle = \langle \mathcal{A}^\perp, \mathcal{A} \rangle$, $\mathcal{A}^\perp$ is left admissible and ${}^\perp\mathcal{A}$ is right admissible. In this situation, the left and right mutations through $\mathcal{A}$ are defined respectively as:

$$L_{\mathcal{A}} = i_{\mathcal{A}^\perp} i_{\mathcal{A}}^* \quad \text{and} \quad R_{\mathcal{A}} = i_{{}^\perp\mathcal{A}} i_{{}^\perp\mathcal{A}}^!.$$

We refer to [40, 150] for more details.

If $\mathcal{A}$ is generated by an exceptional object A , and B is an object of $\mathbf{D}^b(X)$, the left and right mutations of B through A are defined, respectively, by the triangles:

$$\begin{aligned} L_A B[-1] &\rightarrow \mathrm{Hom}(A, B) \otimes A \rightarrow B \rightarrow L_A B, \\ R_A B &\rightarrow A \rightarrow \mathrm{Hom}(A, B)^* \otimes B \rightarrow R_A B[1]. \end{aligned}$$

We will make use a couple of times of Grothendieck duality, we refer to [177, Theorem 3.34] for a statement sufficient for our purposes.

III. Projective and Grassmann bundles and cokernel sheaves

Let W be a quasi-projective variety over k , and let $\mathcal{F}$ be a coherent sheaf over W . For any integer k , there is a variety $\mathbb{G}(\mathcal{F}, k)$, parametrizing k -dimensional quotient modules of $\mathcal{F}$, see for instance [178, Example 2.2.3]. This variety, called the *k-th Grassmann bundle of $\mathcal{F}$* , is a special case of Grothendieck's Quot-scheme, and as such it is universal for families of k -quotients of $\mathcal{F}$ and is equipped with a projective morphism $\pi : \mathbb{G}(\mathcal{F}, k) \rightarrow W$.

As a functor of points, $\mathbb{G}(\mathcal{F}, k)$ is described as follows. Take a k -scheme S and a morphism $f : S \rightarrow W$, and write $\mathcal{F}_S = f^* \mathcal{F}$. Then, the corresponding S -point $[\lambda]$ of $\mathbb{G}(\mathcal{F}, k)$ is given by the set of equivalence classes of surjective

maps $\lambda : \mathcal{F}_S \rightarrow \mathcal{V}$, with $\mathcal{V}$ locally free of rank k on S , where $\lambda' : \mathcal{F}_S \rightarrow \mathcal{V}'$ is equivalent to λ if there is an isomorphism $\varphi : \mathcal{V} \simeq \mathcal{V}'$ such that $\lambda' = \varphi \circ \lambda$. We refer again to [178] for related material on S -valued points of a $\mathbf{k}$ -variety. A closed point (or a $\mathbf{k}$ -point) of $\mathbb{G}(\mathcal{F}, k)$, lying over a point $x \in W$, is a $\mathbf{k}$ -vector space Λ of dimension k , equipped with a surjective linear map $\lambda : \mathcal{F}_x \rightarrow \Lambda$. On the Grassmann bundle, there is a natural rank- k quotient bundle $\mathcal{Q}_{\mathcal{F}}$ of $\pi^*(\mathcal{F})$, and we sometimes denote by $\mathcal{U}_{\mathcal{F}}$ the kernel of the canonical projection, so that we have:

$$0 \rightarrow \mathcal{U}_{\mathcal{F}} \rightarrow \pi^*(\mathcal{F}) \rightarrow \mathcal{Q}_{\mathcal{F}} \rightarrow 0.$$

The subscript $\mathcal{F}$ will be frequently omitted. Of course, in case W is a point, then $\mathcal{F}$ is just a $\mathbf{k}$ -vector space V , so $\mathbb{G}(\mathcal{F}, k) \simeq \mathbb{G}(V, k)$ and the above sequence becomes (I.1). We have $\pi_*(S^t \mathcal{Q}) \simeq S^t \mathcal{F}$, for all $t \geq 0$, where S^t denotes the t -th symmetric power of $\mathcal{Q}$.

The most important special case is that of the projective $\mathbb{P}(\mathcal{F})$ arising as $\mathbb{G}(\mathcal{F}, 1)$, in which case one writes $\mathcal{Q}_{\mathcal{F}}$ is a line bundle, usually denoted by $\mathcal{O}_{\mathcal{F}}(1)$, namely the Grothendieck tautological line bundle. In this case $\pi_*(\mathcal{O}_{\mathcal{F}}(t)) \simeq S^t \mathcal{F}$, for all $t \geq 0$. Some important instances of this arise when $\mathcal{F}$ is a vector bundle or rank r on W , in which case π is a Zariski-locally trivial fibration over W whose fibres are projective spaces of dimension $r - 1$; and when $\mathcal{F} \simeq \mathcal{I}_{Z/W} \otimes \mathcal{L}$, where Z is a subvariety of W and $\mathcal{L}$ is a line bundle on W , in which case $\mathbb{P}(\mathcal{F})$ is the blow-up of W along Z .

III.1. Grassmann bundles and morphisms of vector bundles. Let now W be an integral variety over $\mathbf{k}$, and let $\mathcal{E}$ and $\mathcal{F}$ be coherent sheaves on W . Set $u = \text{rk}(\mathcal{E})$ and $v = \text{rk}(\mathcal{F})$ and let $\phi : \mathcal{E} \rightarrow \mathcal{F}$ be a morphism of coherent sheaves. Set $\mathcal{C}_{\phi} = \text{coker}(\phi)$. For any $1 \leq k \leq v$, we consider the Grassmann bundle $\mathbb{G}(\mathcal{F}, k)$. There is a natural isomorphism:

$$\text{Hom}_W(\mathcal{E}, \mathcal{F}) \simeq \text{Hom}_{\mathbb{G}(\mathcal{F}, k)}(\pi^*(\mathcal{E}), \mathcal{Q}).$$

Let us write s_{ϕ} for the image of ϕ under the above isomorphism, and consider its vanishing locus $Y_{\phi} = \mathbb{V}(s_{\phi})$.

Lemma III.1. *There is an isomorphism $Y_{\phi} \simeq \mathbb{G}(\mathcal{C}_{\phi}, k)$.*

PROOF. By Yoneda's lemma, we are authorized to prove that the functors of points induced by Y_{ϕ} and $\mathbb{G}(\mathcal{C}_{\phi}, k)$ are isomorphic. Set $\mathcal{C} = \mathcal{C}_{\phi}$. The surjection $g : \mathcal{F} \rightarrow \mathcal{C}$ induces a closed embedding $g^* : \mathbb{G}(\mathcal{C}, k) \hookrightarrow \mathbb{G}(\mathcal{F}, k)$, which is given explicitly as follows. For any $\mathbf{k}$ -scheme S , an S -valued point $[\lambda]$ of $\mathbb{G}(\mathcal{C}, k)$ is given by a morphism $f : S \rightarrow W$ and the class of a quotient $\lambda : \mathcal{C}_S \rightarrow \mathcal{V}$ where $\mathcal{V}$ is locally free of rank k on S . So, with $[\lambda]$ we associate the class $[\mu]$ of $\mu = \lambda \circ g : \mathcal{F}_S \rightarrow \mathcal{V}$. Clearly, $\mu \circ \phi_S = 0$. In fact, it is also clear that the condition $\mu \circ \phi_S = 0$ defines $\mathbb{G}(\mathcal{C}, k)$ as a subscheme of $\mathbb{G}(\mathcal{F}, k)$, for a quotient μ of $\mathcal{F}_S$ factors through $\mathcal{C}_S$ if $\mu \circ \phi_S = 0$.

It remains to see that the condition $\mu \circ \phi_S = 0$ is also equivalent the vanishing of s_ϕ at $[\mu]$. Of course $\mu \circ \phi_S = 0$ is equivalent to ask that, for all $a \in \mathcal{E}_S$, one has $\mu(\phi_S(a)) = 0$. Now we identify $\mathcal{E}_S$ and $\pi^*(\mathcal{E}_S)$ via π_* , and use the identification $\mathcal{Q}_S \simeq \mathcal{V}$ given by μ and the structure formula in $\mathcal{Q}_S$:

$$s_\phi(b)(\mu) = \mu\phi(\pi_*(b)), \quad \text{for all } b \in \pi^*(\mathcal{E}_S).$$

This says that $[\mu]$ is an S -point of $\mathbb{G}(\mathcal{C}, k)$ if and only if $s_\phi(b)$ vanishes at $[\mu]$ for all $b \in \mathcal{E}_S$, i.e., if and only if $[\mu]$ lies in $\mathbb{V}(s_\phi)$. Summing up we have proved the isomorphism $\mathbb{V}(s_\phi) \simeq \mathbb{G}(\mathcal{C}, k)$. $\square$

In particular, $\mathbb{P}(\mathcal{C})$ is the subvariety of $\mathbb{P}(\mathcal{F})$ defined as the zero-locus of the global section s_ϕ of $\pi^*(\mathcal{E}^*) \otimes \mathcal{O}_{\mathcal{F}}(1)$ naturally given by ϕ .

III.2. Grassmann bundles and degeneracy loci. Let again W be a connected manifold, and assume $\mathcal{E}$ and $\mathcal{F}$ locally free, again of rank, respectively, u and v , with $v \leq u$. Let $1 \leq r \leq v$ be an integer, and consider the r -th degeneracy locus $D_r(\phi)$ of ϕ , defined set-theoretically by:

$$D_r(\phi) = \{x \in W \mid \text{rk}(\phi_x) \leq r\}.$$

Of course, $D_k(\phi)$ is naturally a subvariety of W , defined by the minors of order $r + 1$ of ϕ , locally on a trivializing cover of $\mathcal{E}$ and $\mathcal{F}$.

Lemma III.2. *Set $k = v - r$. Assume that, for x general in any component of $D_r(\phi)$, we have $\text{rk}(\phi_x) = r$. Then $D_r(\phi)$ is birational to $\mathbb{V}(s_\phi)$.*

PROOF. Set $\mathcal{C}$ for the cokernel sheaf $\mathcal{C}_\phi = \text{coker}(\phi)$, and recall that $\mathbb{V}(s_\phi) \simeq \mathbb{G}(\mathcal{C}, k)$. Note that $\pi(\mathbb{V}(s_\phi)) \subset D_r(\phi)$ (at least set theoretically). Indeed, if $x \in W$ is a closed point lying in the image of π , then a point in $\pi^{-1}(\{x\})$ is given by a k -dimensional vector space Λ and a quotient map $\mathcal{C}_x \rightarrow \Lambda$, i.e. a quotient $\mu : \mathcal{F}_x \rightarrow \Lambda$ with $\mu \circ \phi_x = 0$. For such quotient to exist at all, ϕ has must have corank at least k at x , i.e. ϕ_x has rank at most r , hence x lies in the degeneracy locus $D_r(\phi)$. Working with S -valued points on a trivializing cover of $\mathcal{E}$ and $\mathcal{F}$, we see analogously that π maps $\mathbb{V}(s_\phi)$ to $D_r(\phi)$ as schemes.

Under our assumption, the locus $U = D_r(\phi) \setminus D_{r-1}(\phi)$ is open and dense in $D_r(\phi)$. Also, over the open subset U , the sheaf $\mathcal{C}$ is locally free of rank k , and therefore $\mathbb{G}(\mathcal{C}_U, k) \simeq U$, which says that $U \simeq \mathbb{V}(s_\phi) \times_{D_r(\phi)} U$, so that $D_r(\phi)$ is birational to $\mathbb{V}(s_\phi)$. $\square$

CHAPTER 1

Logarithmic vector fields along a divisor

In this chapter, I will give an introduction to the sheaves of logarithmic vector fields and logarithmic 1-forms, mainly for divisors of the projective space, with a focus on freeness issues and Torelli problems.

I will describe these sheaves under various points of view, sketch the proof of two theorems extracted from [138] and [139], show one result essentially from the Ph. D. thesis of E. Angelini, and mention many well-known theorems and constructions, few of which will be proposed with a proof, variably similar to the original one.

I. Logarithmic derivations and syzygies of the Jacobian ideal

Let $D \subset \mathbb{P}^n$ be a hypersurface over a field k , defined by a homogeneous polynomial $f \in \mathcal{S}$ of degree d . The singularities of D are controlled by the *Jacobian ideal* J_D given by the partial derivatives of f .

$$J_D = (\partial_0 f, \dots, \partial_n f).$$

The generators of J_D are thus given by the gradient of f :

$$\nabla f : \mathcal{S}^{n+1} \xrightarrow{(\partial_0 f, \dots, \partial_n f)} J_D(d-1).$$

Definition I.1. The *graded module of logarithmic derivations* $\text{Der}_0(-\log D)$ of D is defined as $\ker(\nabla f)$.

$$0 \rightarrow \text{Der}_S(-\log D)_0 \rightarrow \mathcal{S}^{n+1} \xrightarrow{(\partial_0 f, \dots, \partial_n f)} J_D(d-1) \rightarrow 0.$$

Formally, once fixed a k -vector space V of dimension $n+1$, so that $\mathbb{P}^n = \mathbb{P}(V)$, we have $V^* = \langle \partial_0, \dots, \partial_n \rangle$. The derivation map $\mathcal{S}_d \otimes V^* \rightarrow \mathcal{S}_{d-1}$ is induced by the comultiplication $\mathcal{S}_d \rightarrow V \otimes \mathcal{S}_{d-1}$. In characteristic zero, the module $\text{Der}_S(-\log D)_0$ is as a direct summand of the *reduced module of logarithmic derivations*, $\text{Der}_S(-\log D)$, which we now define (hence the subscript 0). Let Der_S be the free $\mathcal{S}$ -module of polynomial first-order differential operators of order on $\mathcal{S}$, so:

$$\text{Der}_S = \partial_0 \mathcal{S} \oplus \dots \oplus \partial_n \mathcal{S}.$$

An element $\theta \in \mathcal{S}$ is a derivation of the form $g_0 \partial_0 + \dots + g_n \partial_n$. We define the graded $\mathcal{S}$ -module of *affine logarithmic derivations*:

$$\text{Der}_S(-\log D) = \{\theta \in \text{Der}_S \mid \theta(f) \in (f)\}.$$

Consider the quotient ring $\mathcal{S}(D) = \mathcal{S}/(f)$, and set J_D^D for the *polar ideal* of D , i.e., the image in $\mathcal{S}(D)$ of J_D . Then we have the defining exact sequence:

$$(I.1) \quad 0 \rightarrow \mathrm{Der}_{\mathcal{S}}(-\log D) \rightarrow \mathcal{S}^{n+1} \xrightarrow{\nabla f} J_D^D(d-1) \rightarrow 0.$$

If $\mathrm{char}(\mathbf{k})$ does not divide $\deg(f)$, then $J_D^D \simeq J_D/\mathcal{S}(-d)$ because of the Euler relation $\sum \mathbf{x}_i \partial_i f = \deg(f)f$, and the Euler derivation gives a splitting:

$$\mathrm{Der}_{\mathcal{S}}(-\log D) \simeq \mathrm{Der}_{\mathcal{S}}(-\log D)_0 \oplus \mathcal{S}(-1).$$

Both the modules $\mathrm{Der}_{\mathcal{S}}(-\log D)$ and $\mathrm{Der}_{\mathcal{S}}(-\log D)_0$ are called, in the literature, *modules of logarithmic derivations*, although we added the adjective “affine” to the first of them. Anyway, the context should always make clear of which of them one is speaking.

Occasionally, we will prefer speaking of logarithmic derivations for non-homogeneous polynomials, i.e., for hypersurfaces of an affine space $\mathbb{A}^n$. In this case, given a hypersurface $D = \mathbb{V}(f) \subset \mathbb{A}^n$ we write $D_h = \mathbb{V}(f_h) \subset \mathbb{P}^n$, where f_h is the homogenization of f . Else, one could study the affine hypersurface given by a homogeneous polynomial, i.e. the affine cone $\hat{D}$ over the projective hypersurface D . For $D \subset \mathbb{A}^n$, we also have the cone $cD = \hat{D}_h \subset \mathbb{A}^{n+1}$. This terminology is quite frequent in the literature devoted to hyperplane arrangements; the cones cD are in this case called *central* arrangements.

I.1. Logarithmic vector fields and differentials. The definition proposed above of the graded module of logarithmic derivations has a natural sheaf-theoretic counterpart: Deligne-Saito’s sheaf of vector fields with logarithmic poles along a reduced hypersurface D . Dually, we have a sheaf of 1-forms with logarithmic poles along D . This sheaf, together the whole complex of higher order forms with logarithmic poles, was extensively used for the study of Hodge theory of quasiprojective manifolds, as a refinement of the Grothendieck de Rham complex of forms with arbitrary meromorphic poles along D , [155]. It was defined by Deligne for divisors with normal crossing, cf. [97], and generalized by Saito to arbitrary reduced divisors, cf. [277]. Let us review briefly the construction here.

I.1.1. Logarithmic vector fields. Let $D \subset \mathbb{P}^n$ be a reduced hypersurface. The sheaf of logarithmic vector fields along D is the sheafification:

$$\mathcal{T}_{\mathbb{P}^n}(-\log D) = \mathrm{Der}_{\mathcal{S}}(-\log D)_0^\sim.$$

of $\mathrm{Der}_{\mathcal{S}}(-\log D)_0$ as a subsheaf of the tangent bundle $\mathcal{T}_{\mathbb{P}^n}$. We denote by $\mathcal{J}_D$ the Jacobian ideal sheaf, so that our defining exact sequence of $\mathcal{T}_{\mathbb{P}^n}(-\log D)$ reads:

$$0 \rightarrow \mathcal{T}_{\mathbb{P}^n}(-\log D) \rightarrow \mathcal{O}_{\mathbb{P}^n}^{n+1} \xrightarrow{\nabla f} \mathcal{J}_D(d-1) \rightarrow 0.$$

One can work more generally over a smooth connected variety W over k and a reduced subvariety $X \subset W$. Then, the *equisingular normal sheaf* $\mathcal{N}'_{X/W}$, cf. [287] is defined by the exact sequence: D

$$(I.2) \quad 0 \rightarrow \mathcal{T}_X \rightarrow \mathcal{T}_W|_X \rightarrow \mathcal{N}'_{X/W} \rightarrow 0.$$

Then we have an exact commutative diagram:

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ & & \mathcal{I}_{X/W} \otimes \mathcal{T}_W = \mathcal{I}_{X/W} \otimes \mathcal{T}_W & & & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{T}_W \langle X \rangle & \longrightarrow & \mathcal{T}_W & \longrightarrow & \mathcal{N}'_{X/W} \rightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & \mathcal{T}_X & \longrightarrow & \mathcal{T}_W|_X & \longrightarrow & \mathcal{N}'_{X/W} \rightarrow 0 \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

Where the sheaf $\mathcal{T}_W \langle X \rangle$ of logarithmic differentials (general version), is defined by the sequence.

If D is a reduced hypersurface of W , the sheaf $\mathcal{T}_W \langle D \rangle$ is given, in terms of local sections, on a open subset $U \subset W$ by taking an equation $f_U \in \Gamma(U, \mathcal{O}_W)$ locally defining $D \cap U$ and considering:

$$\{\theta \in \Gamma(U, \mathcal{T}_W) \mid \theta(f_U) \in (f_U)\}.$$

If $X = D$ is a hypersurface of degree d in $W = \mathbb{P}^n$, the equisingular normal sheaf $\mathcal{N}'_{D/\mathbb{P}^n}$ is precisely the sheafified polar ideal $\mathcal{J}_D^D(d)$. So the two modules of logarithmic derivations $\mathcal{T}_{\mathbb{P}^n} \langle D \rangle$ and $\mathcal{T}_{\mathbb{P}^n}(-\log D)$ are related just by a shift by one in degree:

$$(I.3) \quad \mathcal{T}_{\mathbb{P}^n} \langle D \rangle \simeq \mathcal{T}_{\mathbb{P}^n}(-\log D) \otimes \mathcal{O}_{\mathbb{P}^n}(1).$$

One defines the sheaf of logarithmic 1-forms $\Omega_W(\log D)$ as:

$$\Omega_W(\log D) = \mathcal{T}_X \langle D \rangle^*.$$

Example I.2 ($\text{char}(k) \neq 2$). Let $D = \mathbb{V}(f) \subset \mathbb{P}^n$ is a quadric hypersurface of rank $m+1$, and L be the kernel of f . We consider the projection $\mathbb{P}^n \dashrightarrow \mathbb{P}^m$. Let $\tilde{\mathbb{P}}^n$ be the blow-up of $\mathbb{P}(L)$, so that this projection factors through the natural morphisms $\pi_L : \tilde{\mathbb{P}}^n \rightarrow \mathbb{P}^m$ and $\sigma_L : \tilde{\mathbb{P}}^n \rightarrow \mathbb{P}^n$. Then $\mathcal{T}_{\mathbb{P}^n}(-\log D) \simeq \mathcal{O}_{\mathbb{P}^n}^{n-m} \oplus \sigma_{L*}(\pi_L^*(\Omega_{\mathbb{P}^m}(1)))$. In particular, the sheaf $\mathcal{T}_{\mathbb{P}^n}(-\log D)$ is locally free if and only if D is smooth, in which case we get an isomorphism $\mathcal{T}_{\mathbb{P}^n}(-\log D) \simeq \Omega_{\mathbb{P}^n}(1)$.

I.1.2. Logarithmic 1-forms. Here we work over $\mathbb{C}$, in order to sketch the original definition of $\Omega_W(\log D)$, which goes as follows. Let W be a connected complex manifold, and D be a reduced divisor on W . Then, for a given open subset U of W , if s_U is an equation locally defining $D \cap U$, the local sections

of $\Omega_W(\log D)$ are the meromorphic 1-forms ω such that both $f\omega$ and $f d\omega$ are holomorphic. If D has simple normal crossings and W has dimension m , given a point x of W , we can choose local coordinates $(z_1, \dots, z_n)$ in a neighborhood U of x such that the irreducible components $D_1, \dots, D_k$ of $D \cap U$ passing through x are defined by $D_i = \mathbb{V}(z_i)$ so $D \cap U = \mathbb{V}(z_1 \cdots z_k)$. In this case $\Omega_W(\log D)$ restricts to U as:

$$\Omega_W(\log D)|_U \simeq \left\langle \frac{dz_1}{z_1}, \dots, \frac{dz_k}{z_k} \right\rangle \otimes \mathcal{O}_U \oplus \langle dz_{k+1}, \dots, dz_m \rangle \otimes \mathcal{O}_U.$$

The relation $d \log z_i = \frac{dz_i}{z_i}$ should thus explain the name “logarithmic forms”.

In this case, it is clear that $\Omega_W(\log D)$ is locally free of rank m , and taking residues we obtain a surjection onto $\bigoplus_{i=1, \dots, k} \mathcal{O}_{D_i}$, whose kernel are just holomorphic 1-forms. Globally, this gives the residue exact sequence:

$$0 \rightarrow \Omega_W \rightarrow \Omega_W(\log D) \rightarrow \bigoplus_{j=1, \dots, \ell} \mathcal{O}_{D_j} \rightarrow 0,$$

where $D_1, \dots, D_\ell$ are the irreducible components of D .

I.2. Dolgachev’s logarithmic forms and residue sequence. Assume k algebraically closed of characteristic zero. Let W be a connected manifold over k , and let D be a reduced hypersurface of W . There is a second type of logarithmic sheaf associated to $D \subset W$, namely Dolgachev’s sheaf, see [105] that we denote by $\tilde{\Omega}_W(\log D)$. This sheaf has worse local properties than $\Omega_W(\log D)$, in particular it is rarely locally free. On the other hand, $\Omega_W(\log D)$ is recovered as double dual of the refined sheaf $\tilde{\Omega}_W(\log D)$. Something might be lost however in the process of taking reflexive hull, and for this reason $\tilde{\Omega}_W(\log D)$ carries richer information on D .

Dolgachev’s definition of $\tilde{\Omega}_W(\log D)$ goes as follow. We consider an embedded resolution of singularities of D , namely a proper birational map:

$$\mu : \tilde{W} \rightarrow W,$$

such that the strict transform $\tilde{D}$ is smooth. Then $\mu^*(D) = \tilde{D} + F$, where F is supported the exceptional locus of μ . The adjoint ideal is defined as $\mu_*(\omega_{\tilde{W}/W}(-F))$, and the conductor ideal sheaf $\mathfrak{c}_D$ is the image of the adjoint ideal in $\mathcal{O}_D$. This ideal contains $\mathcal{J}_D^D$, and there is a natural chain of morphisms:

$$\Omega_W(\log D) \rightarrow \mathcal{E}xt_W^1(\mathcal{J}_D^D(D), \mathcal{O}_W) \rightarrow \mathcal{E}xt_W^2(\mathfrak{c}_D / \mathcal{J}_D^D(D), \mathcal{O}_W).$$

The sheaf $\tilde{\Omega}_W(\log D)$ is the kernel of this composition. One has:

$$\tilde{\Omega}_W(\log D)^{**} \simeq \Omega_W(\log D).$$

Dolgachev proves that $\tilde{\Omega}_W(\log(D))$ is locally free if D has simple normal crossings, and that it agrees with $\Omega_W(\log(D))$ if D has normal crossings in codimension ≤ 2 . However, what will be most relevant to us is that $\tilde{\Omega}_W(\log D)$

always fits into a functorial *residue exact sequence*:

$$0 \rightarrow \Omega_W \rightarrow \tilde{\Omega}_W(\log D) \rightarrow v_*(\mathcal{O}_{\tilde{D}}) \rightarrow 0,$$

where $v : \tilde{D} \rightarrow D$ is the restriction of μ to $\tilde{D}$. If D has simple normal crossings, then $\tilde{D}$ is the disjoint union of the irreducible components $D_1, \dots, D_\ell$ of D , so $v_*(\mathcal{O}_{\tilde{D}}) \simeq \bigoplus_{j=1, \dots, \ell} \mathcal{O}_{D_j}$ and the two residue exact sequences we have written are the same thing.

I.3. Logarithmic vector fields and deformations. Here we follow [287]. Assume k algebraically closed. The sheaf of logarithmic vector fields can be seen in connection with the general framework of deformations of closed embeddings. Indeed, let W be a smooth connected manifold over k .

Take a reduced subvariety X of W , and let j be the embedding. We can consider the functor of infinitesimal locally trivial Def_j deformations of X , [287, Section 3.4.4]. This functor takes a local Artinian k -algebra A with residue field k to the set of deformations $\text{Def}_j(A)$ of j over $\text{Spec}(A)$. These are isomorphism classes of diagrams of the following form:

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{X} \\ \downarrow j & & \downarrow J \\ W & \longrightarrow & W \times \text{Spec}(A) \\ \downarrow & & \downarrow \psi \\ \text{Spec}(k) & \xrightarrow{a_0} & \text{Spec}(A) \end{array}$$

where J and ψJ are flat, and the vertical morphisms on the left column are induced by the ones on the right column by pull-back via a_0 . Locally trivial here means that J induces locally a trivial deformation. Roughly speaking, we put j into a slightly larger family of maps J parametrized by $\text{Spec}(A)$, which specializes to j at the point a_0 .

The deformation theory of j is controlled by the sheaf of logarithmic derivations in the sense that, as it turns out, the functor Def_j has a formal semiuniversal deformation space, whose tangent space is $H^1(W, \mathcal{T}_W \langle X \rangle)$, and whose obstruction space is $H^2(W, \mathcal{T}_W \langle X \rangle)$. By (I.3) we get $H^1(\mathbb{P}^n, \mathcal{T}_{\mathbb{P}^n}(-\log D)(1))$ as tangent space to those deformations of the embedding of D in $\mathbb{P}^n$, along which the polar ideal of D is deformed flatly (equisingular deformations). The space $H^2(\mathbb{P}^n, \mathcal{T}_{\mathbb{P}^n}(-\log D)(1))$ contains the local obstructions to such deformations.

I.4. Logarithmic vector fields and duality. Here k is an arbitrary field. As a reference for this part the reader may consult [299]. We start with an integral variety X embedded in the dual projective space $\mathbb{P}(V^*) = \check{\mathbb{P}}^n = \check{\mathbb{P}}^n$. Write (I.2) and twist by $\mathcal{O}_X(-1)$. This reads:

$$0 \rightarrow \mathcal{T}_X(-1) \rightarrow \mathcal{T}_{\check{\mathbb{P}}^n}(-1)|_X \rightarrow \mathcal{N}'_{X/\check{\mathbb{P}}^n}(-1) \rightarrow 0.$$

We consider the variety $\mathbf{F}_X = \mathbb{P}(\mathcal{N}'_{X/\check{\mathbb{P}}^n}(-1))$. There is a chain of surjections:

$$V \otimes \mathcal{O}_{\check{\mathbb{P}}^n} \rightarrow \mathcal{T}_{\check{\mathbb{P}}^n}(-1) \rightarrow \mathcal{T}_{\check{\mathbb{P}}^n}(-1)|_X \rightarrow \mathcal{N}'_{X/\check{\mathbb{P}}^n}(-1)$$

Therefore, $\mathbf{F}_X$ sits in $\mathbf{F}(1, n) \simeq \mathbb{P}(\mathcal{T}_{\check{\mathbb{P}}^n}(-1)) \subset \check{\mathbb{P}}^n \times \mathbb{P}^n$. In fact, the variety $\mathbf{F}_X$ is the Zariski closure in $\check{\mathbb{P}}^n \times \mathbb{P}^n$ of:

$$\mathbf{F}_X^\circ = \{(x, y) \in X_{\text{sm}} \times \mathbb{P}^n \mid H_y \text{ is tangent to } X \text{ at } x\}.$$

Let $D \subset \mathbb{P}^n$ be the dual variety of X , i.e. the variety of tangent hyperplanes to X . The integral variety D is the image of $\mathbf{F}_X$ by the second projection $\pi_{\mathbb{P}^n}$. We have thus the diagram:

$$(I.4) \quad \begin{array}{ccc} & \mathbf{F}_X & \\ \pi_{\check{\mathbb{P}}^n} \swarrow & & \searrow \pi_{\mathbb{P}^n} \\ \check{\mathbb{P}}^n \supset X & & D \subset \mathbb{P}^n \end{array}$$

The reflexivity theorem asserts that the dual of D is X . This is based on the identification:

$$\mathbf{F}_X = \mathbb{P}(\mathcal{N}'_{X/\check{\mathbb{P}}^n}(-1)) \simeq \mathbb{P}(\mathcal{N}'_{D/\mathbb{P}^n}(-1)) = \mathbf{F}_D.$$

Since $\pi_{\mathbb{P}^n}$ and $\pi_{\check{\mathbb{P}}^n}$ are the morphism associated, respectively, with the tautological line bundles $\mathcal{O}_{\mathcal{N}'_{X/\check{\mathbb{P}}^n}}(1)$ and $\mathcal{O}_{\mathcal{N}'_{D/\mathbb{P}^n}}(1)$ on $\mathbf{F}_X \simeq \mathbf{F}_D$, we obtain:

$$(I.5) \quad \pi_{\check{\mathbb{P}}^n*}(\pi_{\mathbb{P}^n}^*(\mathcal{O}_X(1))) \simeq \mathcal{N}'_{X/\check{\mathbb{P}}^n}(-1), \quad \pi_{\mathbb{P}^n*}(\pi_{\check{\mathbb{P}}^n}^*(\mathcal{O}_X(1))) \simeq \mathcal{N}'_{D/\mathbb{P}^n}(-1).$$

Assume now that X is positive-dimensional, embedded by the complete linear system associated with $\mathcal{O}_X(1)$, so that $H^0(X, \mathcal{O}_X(1))$ is identified with V^* . Denote by $\widehat{\mathcal{T}}_X$ the affine tangent bundle of X , fitting into the canonical extension:

$$(I.6) \quad 0 \rightarrow \mathcal{O}_X \rightarrow \widehat{\mathcal{T}}_X \rightarrow \mathcal{T}_X \rightarrow 0.$$

Then we have the natural exact sequence:

$$0 \rightarrow \widehat{\mathcal{T}}_X(-1) \rightarrow V \otimes \mathcal{O}_X \rightarrow \mathcal{N}'_{X/\check{\mathbb{P}}^n}(-1) \rightarrow 0.$$

We rewrite (I.4) in the extended form:

$$\begin{array}{ccccc} & \mathbf{F}_X & \hookrightarrow & X \times \mathbb{P}^n & \\ \pi_X \swarrow & & \searrow \pi_D & & \searrow \pi_{\mathbb{P}^n} \\ X & & D & \hookrightarrow & \mathbb{P}^n \end{array}$$

Now, the exact sequence (I.6) says that $\mathbf{F}_X$ is cut in $X \times \mathbb{P}^n$ linearly on the fibres of $\pi_{\mathbb{P}^n}$ by a (co)section of the pull-back of $\widehat{\mathcal{T}}_X(-1)$. Since such cosection vanishes in the expected codimension, we may write the Koszul resolution:

$$\cdots \rightarrow \wedge^2 \widehat{\mathcal{T}}_X(-2) \boxtimes \mathcal{O}_{\mathbb{P}^n}(-2) \rightarrow \widehat{\mathcal{T}}_X(-1) \boxtimes \mathcal{O}_{\mathbb{P}^n}(-1) \rightarrow \mathcal{O}_{X \times \mathbb{P}^n} \rightarrow \mathcal{O}_{\mathbf{F}_X} \rightarrow 0,$$

where the image of the map with target into $\mathcal{O}_{X \times \mathbb{P}^n}$ is of course $\mathcal{I}_{\mathbf{F}_X/X \times \mathbb{P}^n}$.

Assume now that D is a hypersurface of $\mathbb{P}^n$, and let $D = \mathbb{V}(f)$. We tensor the Koszul complex above with $\pi_X^*(\mathcal{O}_X(1))$ and take direct image via $\pi_{\mathbb{P}^n}$. Using (I.5), since V^* is identified with $H^0(X, \mathcal{O}_X(1))$, we get an exact sequence:

$$0 \rightarrow \pi_{\mathbb{P}^n}(\mathcal{I}_{\mathbb{F}_X/X \times \mathbb{P}^n}(1, 0)) \rightarrow V^* \otimes \mathcal{O}_{\mathbb{P}^n} \rightarrow \mathcal{N}'_{D/\mathbb{P}^n}$$

The map $V^* \otimes \mathcal{O}_{\mathbb{P}^n} \rightarrow \mathcal{N}'_{D/\mathbb{P}^n}$ is identified with $\nabla(f)$, and it is therefore surjective (although perhaps not on global sections). After sheafifying (I.1), we have thus proved:

Lemma I.3. *Let X be a reduced variety of positive dimension embedded in $\mathbb{P}^n$ by the complete linear series $|\mathcal{O}_X(1)|$, and assume that the dual of X is a hypersurface D of $\mathbb{P}^n$. Then we have:*

$$\mathrm{Der}_{\mathcal{S}}(-\log D)^\sim \simeq \pi_{\mathbb{P}^n}(\mathcal{I}_{\mathbb{F}_X/X \times \mathbb{P}^n}(1, 0)).$$

In view of this lemma, we have a strategy to calculate $\mathrm{Der}_{\mathcal{S}}(-\log D)$ based on Weyman's method (cf. [317]) that computes a graded free resolution of the $\mathcal{S}$ -module $J_D^D(\deg f - 1)$ by a complex $F^\bullet = (F^0 \leftarrow F^1 \leftarrow \dots)$ whose term F^i is:

$$(I.7) \quad F^i = \bigoplus_{j \geq 0} H^j(X, \bigwedge^{i+j} \widehat{\mathcal{T}}_X(1 - i - j)) \otimes \mathcal{S}(-i - j).$$

I.5. Logarithmic vector fields, Jacobian ring and primitive cohomology. For this part we follow [288], and we work over $\mathbb{C}$. The *Jacobian ring* of a reduced hypersurface $D \subset \mathbb{P}^n$ defined by a homogeneous polynomial $f \in \mathcal{S}_d$ is $R_D = \mathcal{S}/J_D$. In case D is smooth, R_D is an Artinian Gorenstein ring, with socle degree $(n+1)(d-2)$. We have an isomorphism of graded $\mathcal{S}$ -modules:

$$R_D \simeq H_*^1(\mathbb{P}^n, \mathcal{T}_{\mathbb{P}^n}(-\log D))(1-d).$$

As shown by Griffiths in [154], this ring carries information on the primitive cohomology in middle dimension of D and on the period map of D (see also [314]).

On the other hand, if D is singular, the degree-0 local cohomology of R is better behaved. Indeed there is a canonical isomorphism:

$$H_m^0(R_D)(d-1) \simeq H_*^1(\mathbb{P}^n, \mathcal{T}_{\mathbb{P}^n}(-\log D)).$$

The result of Griffiths has been rephrased in [288] to hypersurfaces $D \subset \mathbb{P}^n$ of degree d with simple normal crossings in the following sense. Let $D_1, \dots, D_s$ be the irreducible components of D . Then, we have:

$$\bigoplus_{i=1, \dots, s} H^{n-1,0}(D_i) \simeq H_*^1(\mathbb{P}^n, \mathcal{T}_{\mathbb{P}^n}(-\log D))_{-n}.$$

II. Free hypersurfaces

Let k be a base field. A hypersurface in $\mathbb{P}^n$ over k is free if its module of logarithmic derivations is free. This notion turns out to be very interesting, as free hypersurfaces tend to be quite special and to often exhibit unique properties. Still, it is not quite clear how to decide whether a hypersurface is free relying on partial information such as local singularities or other combinatorial data, or topological invariants of the complement of the hypersurface.

I will give here an account of this notion and of some of the problems related to it, with emphasis to the case of hyperplane arrangements. These will be described in a little proportion of their many aspects, mainly a sketch of their invariants (Poincaré polynomial, Orlik-Solomon algebra) and a brief discussion of free arrangements, where I will mention some results from my paper with J. Vallès [139].

Definition II.1. Let $D \subset \mathbb{P}^n$ be a reduced hypersurface. Then D is said to be *free* if $\text{Der}_S(-\log D)_0$ is a free graded S -module. In this case, D is said to be *free with exponents* $(a_1, \dots, a_n)$ if:

$$\text{Der}_S(-\log D)_0 \simeq S(-a_1) \oplus \cdots \oplus S(-a_n).$$

If $\text{Der}_S(-\log D) \simeq S(-a_0) \oplus \cdots \oplus S(-a_n)$ it is also common to say that D is free with exponents $(a_0, \dots, a_n)$. Note that the number of exponents typically allows to understand which module is stated to be free. Sometimes one writes the exponents as $(a_1^{r_1}, \dots, a_s^{r_s})$ where $a_i^{r_i}$ means a_i repeated r_i times.

For instance, any hypersurface D is free if $n = 1$. The exponent in this case is $\deg(D) - 1$.

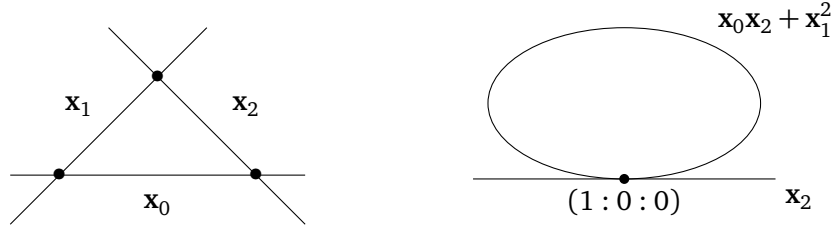
Example II.2 (Boolean arrangement). Set $f = x_0 x_1 \cdots x_n$, so that $D = \mathbb{V}(f)$ is the union of the coordinate hyperplanes. Then D is free with exponents $(1, \dots, 1)$. Indeed, we have the obvious resolution:

$$0 \rightarrow S(-1)^n \hookrightarrow S^{n+1} \xrightarrow{(x_1 \cdots x_n, \dots, x_0 \cdots x_{n-1})} J_D(n) \rightarrow 0$$

where the syzygy matrix is given (e.g. for $n = 4$) by:

$$\begin{pmatrix} x_0 & 0 & 0 & 0 \\ 0 & x_1 & 0 & 0 \\ 0 & 0 & x_2 & 0 \\ 0 & 0 & 0 & x_3 \\ -x_4 & -x_4 & -x_4 & -x_4 \end{pmatrix}$$

Example II.3. Another free divisors of $\mathbb{P}^2$ is the union of a smooth conic and a tangent line. We can see that this example and the previous one give, up to projective equivalence, the only free divisors of $\mathbb{P}^2$ with exponents $(1, 1)$.

FIGURE 1. Free divisors with exponents $(1, 1)$ in $\mathbb{P}^2$.

Indeed, such a divisor must have degree 3. But we can compute, as an exercise, the resolution of $\mathcal{T}_{\mathbb{P}^2}(-\log D)$ of one divisor D for each projective equivalence class of divisors of degree 3.

If D is the union of three lines meeting at one point, then $\mathcal{T} \simeq \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-2)$. If D is the union of a conic and a secant line, or if D is a cuspidal cubic, then we have a sheafified minimal graded free resolution of $\mathcal{T}_{\mathbb{P}^2}(-\log D)$:

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^2}(-3) \rightarrow \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-2)^2 \rightarrow \mathcal{T}_{\mathbb{P}^2}(-\log D) \rightarrow 0.$$

If D is a nodal cubic, the resolution of $\mathcal{T}_{\mathbb{P}^2}(-\log D)$ reads:

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^2}(-2)^2 \rightarrow \mathcal{O}_{\mathbb{P}^2}(-1)^4 \rightarrow \mathcal{T}_{\mathbb{P}^2}(-\log D) \rightarrow 0.$$

If D is a smooth cubic, then we have a resolution:

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^2}(-4) \rightarrow \mathcal{O}_{\mathbb{P}^2}(-2)^3 \rightarrow \mathcal{T}_{\mathbb{P}^2}(-\log D) \rightarrow 0.$$

Lemma II.4. *The hypersurface D is free with exponents $(a_1, \dots, a_n)$ if and only if the blow-up of the Jacobian ideal is a complete intersection in $\mathbb{P}^n \times \check{\mathbb{P}}^n$ of n divisors $A_1, \dots, A_n$ with A_i of bidegree $(a_i, 1)$.*

PROOF. We use the setting of Lemma III.1. We have the defining exact sequence of the sheaf of logarithmic derivations:

$$0 \rightarrow \mathcal{T}_{\mathbb{P}^n}(-\log D) \rightarrow V^* \otimes \mathcal{O}_{\mathbb{P}^n} \xrightarrow{\nabla f} \mathcal{J}_D(d-1) \rightarrow 0.$$

The surjection $V^* \otimes \mathcal{O}_{\mathbb{P}^n} \rightarrow \mathcal{J}_D(d-1)$ induces a closed embedding $\mathbb{P}(\mathcal{J}_D) \subset \mathbb{P}^n \times \check{\mathbb{P}}^n$, because the trivial bundle $V^* \otimes \mathcal{O}_{\mathbb{P}^n}$ is naturally identified with $\mathbb{P}^n \times \check{\mathbb{P}}^n$. By the previous exact sequence, the pull-back of $\mathcal{T}_{\mathbb{P}^n}(-\log D)$ to $\mathbb{P}^n \times \check{\mathbb{P}}^n$ cuts $\mathbb{P}(\mathcal{J}_D(d-1)) \simeq \mathbb{P}(\mathcal{J}_D)$, linearly along the fibres of the projection $\pi_{\check{\mathbb{P}}^n}$, i.e., we have an exact sequence:

$$\mathcal{T}_{\mathbb{P}^n}(-\log D) \boxtimes \mathcal{O}_{\check{\mathbb{P}}^n}(-1) \rightarrow \mathcal{O}_{\mathbb{P}^n \times \check{\mathbb{P}}^n} \rightarrow \mathcal{O}_{\mathbb{P}(\mathcal{J}_D)} \rightarrow 0.$$

In other words, $\mathbb{P}(\mathcal{J}_D)$ is obtained as the vanishing locus of a global cosection of $\mathcal{T}_{\mathbb{P}^n}(-\log D) \boxtimes \mathcal{O}_{\check{\mathbb{P}}^n}(-1)$, taking place in expected codimension, indeed $\text{codim}(\mathbb{P}(\mathcal{J}_D)) = \text{rk}(\mathcal{T}_{\mathbb{P}^n}(-\log D)) = n$. Then, $\mathcal{T}_{\mathbb{P}^n}(-\log D) \boxtimes \mathcal{O}_{\check{\mathbb{P}}^n}(-1)$ splits as the direct sum $\oplus_i \mathcal{O}_{\mathbb{P}^n \times \check{\mathbb{P}}^n}(-a_i, -1)$ if and only if $\mathbb{P}(\mathcal{J}_D)$ is the complete intersection of divisors of bidegrees $(a_1, 1), \dots, (a_n, 1)$, which proves our claim. $\square$

The next lemma says that freeness is an open property. Let D be a family of hypersurfaces parametrized by the points s of an integral base scheme S , and write $D_s \subset \mathbb{P}^n$ for the hypersurface corresponding to $s \in S$.

Lemma II.5. *Assume that D_{s_0} is free with exponents $(a_1, \dots, a_n)$, for some $s_0 \in S$, and that $c_i(\mathcal{T}_{\mathbb{P}^n}(-\log D_s)) = c_i(\mathcal{T}_{\mathbb{P}^n}(-\log D_{s_0}))$ for all s in S . Then there is a Zariski-open dense subset U of S such that D_{s_0} is free with exponents $(a_1, \dots, a_n)$, for all $s \in U$.*

PROOF. Set $\mathcal{T}_s = \mathcal{T}_{\mathbb{P}^n}(-\log D_s)$. Since $\mathcal{T}_s$ is torsion-free, we have that D_s is free if and only if $H^k(\mathbb{P}^n, \mathcal{T}_s(t)) = 0$ for all $t \in \mathbb{Z}$ and $0 < k < n$, cf. for instance [37] for this generalization of Horrocks splitting criterion [173] (or use [166, Lemma 1.1] to reduce to the locally free case).

The assumption that the Chern classes of $\mathcal{T}_s$ are constant on s guarantees that the coherent sheaf $\mathcal{T}_{\mathbb{P}^n \times S}(-\log(D \times S))$ is flat over S . Therefore, by semi-continuity of cohomology, since $H^k(\mathbb{P}^n, \mathcal{T}_{s_0}(t)) = 0$ for all $t \in \mathbb{Z}$ and $0 < k < n$, the same vanishing takes place for all s in a non-empty Zariski-open neighbourhood U of s_0 , so that D_s is free for all s in U . Since S is irreducible, U is dense in S . Finally, we have $\sum_{i=0, \dots, n} c_i h^i = \prod_{j=1, \dots, n} 1 - a_j h$, so the exponents are determined by the Chern classes. This says that D_s is free with exponents $(a_1, \dots, a_n)$ for all s in U . $\square$

II.1. Free hyperplane arrangements. For this part, we refer to [251], see also [281] for a nice survey of this topic. Here the base field is arbitrary, except for some considerations on the topology of the complement of hyperplane arrangement divisors, where we will work over $\mathbb{C}$.

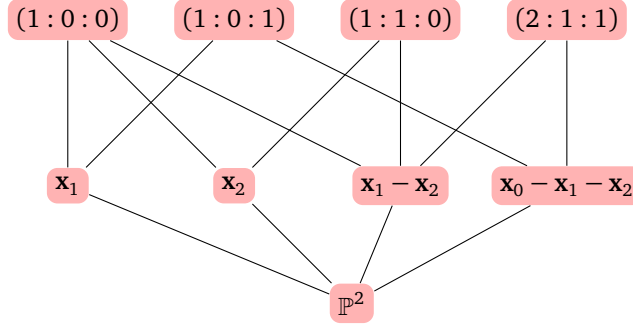
A hyperplane arrangement in $\mathbb{P}^n$ is a collection $\mathcal{A} = (H_1, \dots, H_\ell)$ of distinct hyperplanes H_i of $\mathbb{P}^n$. In other words all the irreducible components of our hypersurface D have degree 1, and we speak of D as a *hyperplane arrangement divisor*, written as $D_{\mathcal{A}} = H_1 \cup \dots \cup H_\ell$. In this situation, it is natural to introduce some combinatorial invariants of $D = D_{\mathcal{A}}$, and we will see that these invariants actually capture many deep features of $\mathcal{A}$, although some of them, and in particular freeness, only conjecturally.

II.1.1. Intersection lattice and combinatorial type. We define thus the *intersection lattice*, or *Hasse diagram* of $\mathcal{A}$, usually denoted by $L_{\mathcal{A}}$ as follows:

$$L_{\mathcal{A}} = \{H_{i_1} \cap \dots \cap H_{i_s} \neq \emptyset \mid i_1 \leq \dots \leq i_s, s = 0, \dots, \ell\}.$$

That is to say, $L_{\mathcal{A}}$ is the set of all non-empty intersections of elements of $\mathcal{A}$. This set is partially ordered by *reverse inclusion*, and is equipped with a rank function given by codimension, that makes it a geometric lattice. The *combinatorial type* of $\mathcal{A}$ is the isomorphism class of this lattice.

It is thus natural to ask which invariants of $\mathcal{A}$ are combinatorial, i.e., depend only on the isomorphism class of $L_{\mathcal{A}}$, and which are not.

FIGURE 2. The Hasse diagram of $x_1x_2(x_1 - x_2)(x_0 - x_1 - x_2) \subset \mathbb{P}^2$.

II.1.2. *Poincaré polynomial.* The first combinatorial invariant is the Poincaré polynomial. Its definition goes as follows. First one defines the Möbius function $\mu : L_{\mathcal{A}} \rightarrow \mathbb{Z}$ recursively, starting with $\mu(\mathbb{P}^n) = 1$ and setting $\mu(X)$ as the sum of $-\mu(Y)$ for all Y strictly containing X . Then, the *Poincaré polynomial* $\pi(\mathcal{A}, t)$ of $\mathcal{A}$ is defined as:

$$\pi(\mathcal{A}, t) = \sum_{X \in L_{\mathcal{A}}} \mu(X)(-t)^{\text{codim} X}.$$

We also write $\pi(\mathcal{A}, t) = \sum_i b_i(\mathcal{A})t^i$, for some integers b_i . For instance, the arrangement of Figure II.1.1 has $\pi(\mathcal{A}, t) = 1 + 4t + 5t^2$.

The Poincaré polynomial has the following nice geometric interpretation:

- i) If $\mathbf{k} = \mathbb{R}$, and $\mathcal{A}$ is an essential affine arrangement (not all hyperplanes are parallel to a single direction), then $|\pi(\mathcal{A}, 1)|$ is the number of connected components of $\mathbb{R}^n \setminus D_{\mathcal{A}}$, while $|\pi(\mathcal{A}, -1)|$ is the number of bounded components, cf. [319].
- ii) If $\mathbf{k} = \mathbb{C}$, and $\mathcal{A}$ is a central arrangement, then the coefficients of $\pi(\mathcal{A}, t)$ are the Betti numbers of $\mathbb{C}^n \setminus D_{\mathcal{A}}$, cf. [249].
- iii) If $\mathbf{k}$ is the finite field $\mathbb{F}_q$ with q elements, then $\pi(\mathcal{A}, q)$ is the number of points in $\mathbb{F}_q^n \setminus D_{\mathcal{A}}$, cf. [251, Theorem 2.69].

II.1.3. *Orlik-Solomon algebra.* One of the main combinatorial invariants of a complex hyperplane arrangement is the cohomology algebra of the complement $M_{\mathcal{A}}$ of $\mathcal{A}$ in $\mathbb{P}^n$. To better fit the classical notation, we take the affine cone over $\mathcal{A}$, which is a *central* arrangement in $\mathbb{C}^{n+1}$, and we denote by $H_1, \dots, H_m$ the corresponding vector hyperplanes. We thus consider the vector space E generated by m vectors $e_1, \dots, e_m$, and the exterior algebra $\wedge E$. Given a subset $B = i_1, \dots, i_k \subset \{1, \dots, m\}$, we set $e_B = e_{i_1} \wedge \dots \wedge e_{i_k}$. We also set ∂ for the standard differential of $\wedge E$ namely $\partial e_B = \sum_j (-1)^j e_{B \setminus \{i_j\}}$. We then define the ideal $I_{\mathcal{A}}$ of $\wedge E$ generated by non-transverse intersections: set

$$I_{\mathcal{A}} = (\partial e_B \mid B \subset \{1, \dots, m\}, \text{codim}(\cap_{j \in B} H_j) < |B|).$$

The algebra $\wedge E/I_{\mathcal{A}}$ is called the Orlik-Solomon algebra. It was proved in [249] that this algebra is isomorphic to the cohomology algebra $H^*(M_{\mathcal{A}}, \mathbb{Z})$, equipped with the cup product.

II.1.4. *Logarithmic forms and cohomology algebra.* Let $\mathcal{A}$ be a complex hyperplane arrangement in $\mathbb{P}^n$, and denote by $\hat{\mathcal{A}} = (H_1, \dots, H_m)$ the collection of affine cones over the hyperplanes of $\mathcal{A}$, so that $\hat{\mathcal{A}}$ is a central arrangement in $\mathbb{C}^{n+1}$. Let f_i be a linear form defining H_i as $\ker(f_i)$. Consider the meromorphic 1-forms:

$$\omega_k = \frac{1}{2\pi i} \frac{df_k}{f_k}.$$

It was proved by Brieskorn in [55], answering a question of Arnold [10] that the ring of differential forms generated by the cohomology classes of $(\omega_1, \dots, \omega_m)$ is isomorphic to the cohomology ring $H^*(M_{\mathcal{A}}, \mathbb{C})$.

For free (affine) arrangements, there is a surprising and beautiful relation between the exponents and the Poincaré polynomial. Indeed, in [296] Terao proved that, if $\mathcal{A}$ is a free central affine arrangement with exponents $(a_1, \dots, a_n)$ then:

$$\pi(\mathcal{A}, t) = \prod_i (1 + a_i t).$$

Of course, much more is known concerning the geometry of the complement $\mathbb{C}^n \setminus D_{\mathcal{A}}$, and on its partial dependency on $L_{\mathcal{A}}$. Just to mention a few results in this framework, let us recall that the topology of the complement of a complexified real arrangements is computed by the Salvetti complex, [278], generalized in [39] to arbitrary arrangements (a more efficient complex for line arrangements was given in [140]). However combinatorics are not enough to determine the topology of the complement, as there are combinatorially equivalent arrangements whose complements have different fundamental group, cf. [276]. In a different spirit, the diffeomorphism type of $\mathbb{C}^n \setminus D_{\mathcal{A}}$ is also quite well understood, see e.g. [192].

In a different direction, by a theorem of Deligne, [98], if $\mathcal{A}$ is the set of complexified reflecting hyperplanes of a Coxeter group W , then $\mathbb{C}^n \setminus D_{\mathcal{A}}$ is an Eilenberg-Mac Lane space, and the same property holds for the quotient X_W of this space by W . The space X_W is homeomorphic to the discriminant locus in the parameter space of the versal deformation of the corresponding rational singularity, [54]. The fundamental group of X_W is an Artin-Tits group (for instance, the braid group in n strands if $\mathcal{A}$ is the braid arrangement and $W = \mathfrak{S}_n$). A presentation for these groups was first given by Brieskorn, [53]. Also, it was originally conjectured by Saito that free arrangements are Eilenberg-Mac Lane spaces, but this implication is not always true, cf. [142] for a discussion.

II.1.5. *Examples and families of free arrangements.* Some basic examples of free arrangements are the following.

Example II.6 (Braid arrangement). Let $\mathcal{A}$ be the central arrangement in $\mathbb{k}^{n+1}$ defined by the Vandermonde determinant:

$$f(\mathbf{x}) = \prod_{0 \leq i < j \leq n} x_i - x_j.$$

Then $\mathcal{A}$ is free with exponents $(0, 1, \dots, n)$. This can be checked easily for instance using Saito's criterion, cf [251] indeed an explicit basis of derivations is known in this case. The appearance of 0 among the exponents corresponds to the fact that the set Z of points in dual space given by $\mathcal{A}$ is contained in a hyperplane, namely the sum of the coordinates of these points is zero. In fact, the exponent 0^r appears if and only if Z is contained in a linear subspace of codimension r .

Example II.7 (Finite field arrangement). Let $\mathcal{A}$ be the central arrangements given by all hyperplanes in $\mathbb{F}_q^{n+1}$. Then $\mathcal{A}$ is free with exponents $(1, q, q^2, \dots, q^n)$.

We gave these arrangements in the affine space because the exponents are “nicer”, however they can be defined of course also in projective space.

As a method to analyze the sheaf $\mathcal{T}_{\mathbb{P}^n}(-\log D_{\mathcal{A}})$ associated with an arrangement $\mathcal{A}$, one may try to remove hyperplanes one by one (*deletion*) and see how the sheaf changes, or to start from the empty set and add arrangements one by one (*addition*). Removing a hyperplane $H_0 \simeq \mathbb{P}^{n-1}$ from $\mathcal{A}$ creates a triple $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ where $\mathcal{A}'$ is $\mathcal{A} \setminus \{H_0\}$ and $\mathcal{A}''$ is the arrangement in $\mathbb{P}^{n-1}$ obtained intersecting the hyperplane of $\mathcal{A}'$ with H_0 . Freeness in this case can be controlled, up to a certain extent, via the *addition-deletion theorem*. It asserts that any two of the following three statements implies the third:

- i) $\mathcal{A}$ is free with exponents $(a_1, \dots, a_{n-1}, a_n)$;
- ii) $\mathcal{A}'$ is free with exponents $(a_1, \dots, a_{n-1}, a_{n-1})$;
- iii) $\mathcal{A}''$ is free with exponents $(a_1, \dots, a_{n-1})$.

In the definition of $\mathcal{A}''$, we could have chosen to take hyperplanes with multiplicity, by attaching to any hyperplane $H \cap H_0$ the number of hyperplanes H' of $\mathcal{A}$ such that $H \cap H_0 = H' \cap H_0$. This leads to Ziegler's multiarrangements, [320]. A natural definition of weighted derivation module is attached to multiarrangements, and the study of this module is already interesting for multiarrangements of $\mathbb{P}^1$, [316] as it controls the splitting type of $\mathcal{T}_{\mathbb{P}^2}(-\log D_{\mathcal{A}})$ on the lines of $\mathcal{A}$ (this is in fact the one of the main results of [318]). Multiarrangements have been extensively studied, and addition-deletion theorem is known also in this setting, see [3]; cf. also [2] for the characteristic polynomial of a multiarrangement. However, we will not pursue the analysis of multiarrangements further here.

We turn instead to families of free hyperplane arrangements of $\mathbb{P}^n$. One such family is that of *inductively free arrangements*. They are the smallest set

of arrangements IF containing the empty arrangement and all arrangements $\mathcal{A}$ that admit a hyperplane H_0 of such that, looking at the associated triple $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$, we get $\mathcal{A}'$ and $\mathcal{A}''$ in IF , with the exponents of $\mathcal{A}''$ contained in those of $\mathcal{A}'$. The braid arrangement for instance is inductively free.

There are more interesting classes of free arrangements, such as recursively free, supersolvable (cf. [251, Pages 121-122]) and reflection arrangements (see [251, Chapter 6]), but we omit this analysis for lack of space.

Finally, the following is among my favourite examples of free arrangements.

Example II.8 (Hesse arrangements). The Hesse arrangement arises as follows. Consider a smooth complex plane cubic curve $C \subset \mathbb{P}^2$, and its 9 inflection points $(x_1, \dots, x_9) \in \mathbb{P}^2$. Then, for any $i \neq j$, it turns out that any line passing through x_i and x_j passes through a third point x_k with $i \neq j$. There are 12 such lines.

- i) The *Hesse arrangement* consists of the union of these 12 lines. It is free with exponents $(4, 7)$.
- ii) The *dual Hesse arrangement* is the collection of 9 inflection lines. It free with exponents $(4, 4)$.
- iii) The union of these two sets of 9 and 12 lines respectively forms another free arrangement, this time with exponents $(7, 13)$.

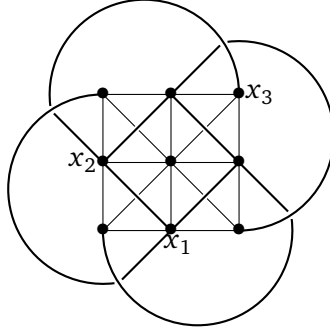


FIGURE 3. Hesse arrangement.

II.1.6. Free arrangements and duality. For hyperplane arrangements, Lemma I.3 takes a slightly different form, as we pointed out in [139]. Indeed, let us consider the dual projective space $\check{\mathbb{P}}^n$. To a finite (reduced) set of ℓ points Z in $\check{\mathbb{P}}^n$ corresponds the arrangement $\mathcal{A}_Z = (H_z \mid z \in Z)$, where we denote by H_z the hyperplane in $\mathbb{P}^n$ corresponding to $z \in \check{\mathbb{P}}^n$. Write D_Z for $D_{\mathcal{A}_Z}$.

Then, we consider the incidence variety $\mathbf{F}(1, n) \simeq \mathbb{P}(\mathcal{T}_{\mathbb{P}^n}(-1)) \subset \mathbb{P}^n \times \check{\mathbb{P}}^n$, so $\mathbf{F}(1, n)$ is the set of pairs (x, y) , with $x \in \mathbb{P}^n$ and $y \in \check{\mathbb{P}}^n$, such that x lies in

H_Y . We have then the diagram:

$$\begin{array}{ccc} & \mathbf{F}(1, n) & \\ \pi_{\mathbb{P}^n} \swarrow & & \searrow \pi_{\check{\mathbb{P}}^n} \\ \mathbb{P}^n & & \check{\mathbb{P}}^n \end{array}$$

Let us denote by $\pi_{\mathbb{P}^n}$ and $\pi_{\check{\mathbb{P}}^n}$ also the projections from $\mathbf{F}(1, n)$ onto $\mathbb{P}^n$ and $\check{\mathbb{P}}^n$.

Now, the first main result of [139] is that there is a natural isomorphism:

$$\mathcal{T}_{\mathbb{P}^n}(-\log D_Z) \simeq \pi_{\mathbb{P}^n*}(\pi_{\check{\mathbb{P}}^n}^*(\mathcal{I}_{Z/\check{\mathbb{P}}^n}(1))).$$

This point, rooted in [307, 308], is actually the key of our method.

The next results of [139] are mainly devoted to line arrangements. As for Chern classes of a line arrangement $\mathcal{A} = \mathcal{A}_Z$ in $\mathbb{P}^2$, denoting by $b_{\mathcal{A}, t}$ the number of points of multiplicity t in $D_{\mathcal{A}}$, we have the simple relations (cf. also [283]):

$$(II.1) \quad \sum_{j \geq 2} \binom{j}{2} b_{\mathcal{A}, j} = \binom{\ell}{2},$$

$$(II.2) \quad \sum_{j \geq 2} \binom{j}{2} b_{\mathcal{A}, j+1} = \binom{\ell-1}{2} - c_2(\mathcal{T}_{\mathbb{P}^n}(-\log D_{\mathcal{A}})).$$

For instance we proved the following result.

Theorem II.9. *Let $k \geq 1$, $r \geq 0$ be integers, set $\ell = 2k + r + 1$, and consider a line arrangement $\mathcal{A}$ of ℓ lines with a point of multiplicity h with $k \leq h \leq k + r + 1$. Then $\mathcal{A}$ is free with exponents $(k, k + r)$ if and only if $c_2(\mathcal{T}_{\mathbb{P}^2}(-\log D_{\mathcal{A}})) = k(k + r)$.*

Here is a sketch of the proof. One direction is obvious. The other is indeed related to duality, for the point of multiplicity h becomes a line L in the dual space $\check{\mathbb{P}}^2$ containing h points of the set of ℓ points Z dual to $\mathcal{A}$. Setting $Z' = Z \setminus L$, we get an exact sequence:

$$0 \rightarrow \mathcal{I}_{Z'} \rightarrow \mathcal{I}_Z(1) \rightarrow \mathcal{O}_L(1 - h) \rightarrow 0.$$

Applying to this sequence our functor $\pi_{\mathbb{P}^n*} \circ \pi_{\check{\mathbb{P}}^n}^*$, we thus obtain:

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^2}(h - \ell) \rightarrow \mathcal{T}_{\mathbb{P}^2}(-\log D_{\mathcal{A}}) \rightarrow \mathcal{I}_{\Gamma}(1 - h) \rightarrow 0,$$

for some finite-length subscheme $\Gamma \subset \mathbb{P}^2$. It turns out by computing Chern classes that Γ has positive length. However, an easy lemma on vector bundles on $\mathbb{P}^2$ shows that, under the assumption $c_2(\mathcal{T}_{\mathbb{P}^2}(-\log D_{\mathcal{A}})) = k(k + r)$, $\mathcal{A}$ is not free if and only if:

$$H^0(\mathbb{P}^2, \mathcal{T}_{\mathbb{P}^2}(-\log D_{\mathcal{A}}) \otimes \mathcal{O}_{\mathbb{P}^2}(k - 1)) \neq 0.$$

So in this case we would have $H^0(\mathbb{P}^2, \mathcal{I}_{\Gamma}(k - h)) = 0$, which contradicts Γ being non-empty since $k \leq h$.

II.1.7. Terao's conjecture. One main issue in the theory of arrangements is to what extent the sheaf $\mathcal{T}_{\mathbb{P}^n}(-\log D_{\mathcal{A}})$ depends on the combinatorial type

of $\mathcal{A}$, defined as the isomorphism type of the lattice $L_{\mathcal{A}}$. A very important conjecture of Terao (reported in [251]) asserts that if $\mathcal{A}$ and $\mathcal{A}'$ have the same combinatorial type, and $\mathcal{A}$ is free with exponents $(a_1, \dots, a_n)$, then $\mathcal{A}'$ is also free with exponents $(a_1, \dots, a_n)$.

Terao's conjecture is known to hold for several classes of arrangements, most notably:

- i) inductively free arrangements;
- ii) fibre type arrangements, reflection arrangements (cf. [251]), Shi and Catalan arrangements (which are generalized Weyl arrangements cf. Section V.1);
- iii) arrangements of up to 12 lines;
- iv) line arrangements with exponents (a, b) such that $a \leq 5$ (cf. [139, Theorem 6]);
- v) line arrangements admitting a line containing double and triple points only.

For line arrangements, the main point is that, even though the Chern classes of $\mathcal{T}_{\mathbb{P}^n}(-\log D_Z)$ are easily determined by combinatorics, its generic splitting is only presumed to do so. However a proof is lacking at the moment, even though it would entail a very significant progress in all the theory of arrangements.

We contribute to this with [139, Theorem 4]. This asserts that, writing the splitting of $\mathcal{T}_{\mathbb{P}^n}(-\log D_Z)$ on a general line H in $\mathbb{P}^2$ as $\mathcal{O}_H(-a) \oplus \mathcal{O}_H(-b)$, with $a \leq b$, the number a is equal to the smallest integer d such that, for a general point $y \in \mathbb{P}^2$, there exists a curve of degree $d + 1$ in $\mathbb{P}^2$, containing Z , with a point of multiplicity d at y .

II.2. Free hypersurfaces with components of higher degree. Let us give here a very brief outline of free hypersurfaces besides the case of hyperplane arrangements.

II.2.1. Buchsbaum problem. One interesting problem, usually attributed to Buchsbaum (cf. [289, Introduction]), and originally formulated in terms of depth of the Jacobian ideal, is whether one can produce free irreducible hypersurfaces $D \subset \mathbb{P}^n$.

This question has been recently addressed by Simis and Tohaneanu for plane curves of any degree, [290]. These curves are defined by polynomials of the form:

$$f(\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2) = \mathbf{x}_2^{d-1} \mathbf{x}_0 + a_1 \mathbf{x}_1^d + a_2 \mathbf{x}_1^2 \mathbf{x}_2^{d-2} + a_3 \mathbf{x}_1 \mathbf{x}_2^{d-1} + a_4 \mathbf{x}_2^d,$$

with $a_1, a_2 \neq 0$, and are free with exponents $(2, d - 3)$. However, I don't have a geometric intuition on these examples, except perhaps if they are related to Conjecture V.4. Also, I do not know which sequences of integers are exponents

of free irreducible hypersurfaces in $\mathbb{P}^n$, for any $n \geq 2$. To my knowledge, the first example of free irreducible plane curve was given in [289].

We are going to see in a minute that some discriminants of forms in several variables are also free.

II.2.2. Discriminants. This section is almost entirely contained in [148]. Let k be a field, $m, d \geq 1$ be integers. Consider the polynomial ring S in m variables over k , the space S_d of homogeneous forms of degree d in $m + 1$ variables, and let $\check{\mathbb{P}}^N = \check{\mathbb{P}}(S_d)$. The discriminant hypersurface D of $\check{\mathbb{P}}^N$ is the dual of the Veronese embedding V_d of $\mathbb{P}^m = \mathbb{P}(V)$ in $\mathbb{P}^N = \mathbb{P}(S^d V) = \mathbb{P}(S_d)$. In this case we get the following result (certainly well-known, cf. for instance [94] for the case $m = 1$):

Proposition II.10. *The discriminant hypersurface D is free if and only (m, d) is one the following pairs:*

- i) $(1, d)$, in which case D has exponents $(1^4, 2^{d-3})$;
- ii) $(2, 3)$, with exponents $(1^9, 3)$;
- iii) $(2, 4)$, with exponents $(1^9, 3^6)$;
- iv) $(3, 3)$, with exponents $(1^{16}, 4^4)$.

PROOF. We use Lemma I.3 and Weyman's complex $F^\bullet$ whose terms are given by formula (I.7). In this case, X is $\mathbb{P}^m$ embedded by the complete linear system $|\mathcal{O}_{\mathbb{P}^m}(d)|$ so the affine tangent bundle $\widehat{\mathcal{T}}_{\mathbb{P}^m}(-1)$ is just the trivial bundle $V^* \otimes \mathcal{O}_{\mathbb{P}^m}$. The condition that D is free amounts to ask that $F^\bullet$ is concentrated in degrees 0 and 1 only, and that F^0 is $V \otimes S$. Indeed, for each t , $\mathcal{O}_{\mathbb{P}^m}(t)$ has cohomology in degree 0 or m only, so no cancellation takes place in $F^\bullet$.

This in turn implies that $d \geq 3$, for otherwise we have the non-trivial term $H^0(\mathbb{P}^m, \wedge^2 V^* \otimes \mathcal{O}_{\mathbb{P}^m}(2-d)) \otimes S(-2)$ in F^2 . Moreover, looking at F^0 our condition imposes, for $m > 0$, the vanishing:

$$H^m(\mathbb{P}^m, \wedge^m V^* \otimes \mathcal{O}_{\mathbb{P}^m}(m - (m-1)d)) = 0.$$

By Serre duality, this cannot happen for any $m \geq 2$ in the range $d \geq 2 + \frac{3}{m-1}$, which leaves out precisely the cases of (m, d) of our list. Conversely, for these cases a direct computation of F^{-1} gives the desired answer. $\square$

Clearly this setup is invariant under the action of $GL(V)$, and one might wish to know the representations of $GL(V)$ occurring in the summands of $\mathcal{T}_{\check{\mathbb{P}}^N}(-\log D)$ in the cases of the above list. These are:

- i) for $(1, d)$, $\mathcal{T}_{\check{\mathbb{P}}^N}(-\log D) \simeq \text{End}(V) \otimes S(-1) \oplus S^{d-4} V^* \otimes S(-2)$;
- ii) for $(2, 3)$, $\mathcal{T}_{\check{\mathbb{P}}^N}(-\log D) \simeq \text{End}(V) \otimes S(-1) \oplus S(-3)$;
- iii) for $(2, 4)$, $\mathcal{T}_{\check{\mathbb{P}}^N}(-\log D) \simeq \text{End}(V) \otimes S(-1) \oplus S^2 V^* \otimes S(-3)$;
- iv) for $(3, 3)$, $\mathcal{T}_{\check{\mathbb{P}}^N}(-\log D) \simeq \text{End}(V) \otimes S(-1) \oplus V^* \otimes S(-4)$.

One can compute this way the resolution of $\mathcal{T}_{\check{\mathbb{P}}^N}(-\log D)$ for many more classical invariant hypersurfaces D . Freeness of such hypersurfaces is quite

rare, however the resolution tends to be much shorter than for random hypersurfaces.

II.2.3. Conic-line arrangements. Let us work over $\mathbb{C}$ for the following discussion. Freeness has been investigated for conic-line arrangements by Schenck and Tohaneanu in [282]. These are hypersurfaces in $\mathbb{P}^2$ whose irreducible components are lines or smooth conics. Let us recall that, according to [87], the cohomology ring of the complement of a (reduced) divisor D in $\mathbb{P}^2$ consisting of a union of rational curves is determined by the number of components of D and the number of analytic branches through the singular points of D . One defines also in this case the intersection poset of the components of D and attaches to the vertices corresponding to singular points $x \in D$ the Milnor number:

$$\mu_x(D) = \dim_{\mathbb{C}}(\mathbb{C}\{x, y\} / (\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y})),$$

where in a suitable local chart x is the point $(0, 0) \in \mathbb{C}^2$ and D is locally defined as $\mathbb{V}(g)$. One also defines the Tjurina in the same setting as:

$$\tau_x(D) = \dim_{\mathbb{C}}(\mathbb{C}\{x, y\} / (g, \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y})).$$

It turns out that these numbers may differ, i.e., it may happen that $\tau_x(D) < \mu_x(D)$. They are equal if and only if the singularity of D at x is *quasi-homogeneous*, which is to say that under a convenient holomorphic change of variables the defining polynomial $g(x, y)$ of D at 0 becomes homogeneous when x and y are raised to suitable rational exponents. However the contribution at a singular point x of D to the length of the Jacobian ideal of D (and hence to the logarithmic tangent sheaf) is provided by the Tjurina number $\tau_x(D)$. It turns out, cf. [282], that freeness of conic-line arrangements is not purely combinatorial, for there are such arrangements D and D' with the same combinatorial type, such that D is free and D' is not. However the reason is that freeness of D' is spoiled by a jump of Tjurina number, so that D' is not even numerically free, i.e. $\mathcal{T}_{\mathbb{P}^2}(-\log D')$ does not have the Chern classes of a decomposable bundle. So one should ask that D and D' have the same intersection graph with the same Tjurina numbers at each point to obtain a “non-cheating” extension of Terao’s conjecture. Counterexamples to this refined conjecture are not known to me.

III. Polar map

Let $D \subset \mathbb{P}^n$ be a hypersurface, defined by a homogeneous polynomial $f \in S$ of degree d . The linear system $\langle \partial_0 f, \dots, \partial_n f \rangle$ defines the polar map:

$$\mathcal{P}_D : \mathbb{P}^n \dashrightarrow \check{\mathbb{P}}^n$$

Definition III.1. The *polar degree* of D is the degree of the rational map $\mathcal{P}_D$.

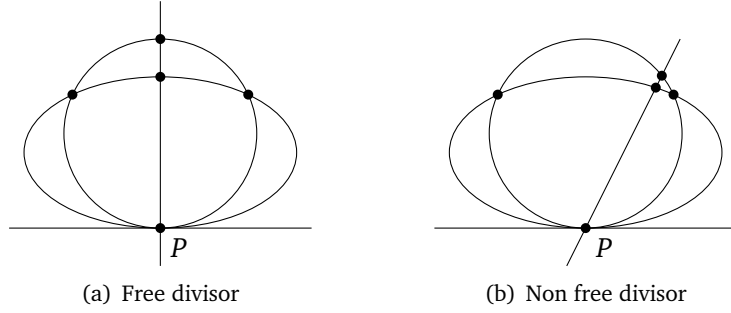


FIGURE 4. Free and non-free conic-line arrangements

Blowing up the Jacobian ideal, we get:

$$\begin{array}{ccc}
 & \mathbb{P}(\mathcal{J}_D) & \\
 \swarrow & & \searrow \\
 \mathbb{P}^n & \dashrightarrow & \check{\mathbb{P}}^n \\
 & \mathcal{P}_D &
 \end{array}$$

III.1. Polar map and logarithmic derivations. There is a visible connection between the features of the polar map associated with D and those of the logarithmic sheaf $\mathcal{T}_{\mathbb{P}^n}(-\log D)$. One instance of this is the following well-known lemma.

Lemma III.2. *If $\mathcal{P}_D$ is generically finite, then:*

$$\deg(\mathcal{P}_D) = (-1)^n c_n(\mathcal{T}_{\mathbb{P}^n}(-\log D)).$$

PROOF. If the degree of $\mathcal{P}_D$ is finite, it equals the length of a subscheme Z of $\mathbb{P}(\mathcal{J}_D) \subset \mathbb{P}^n \times \check{\mathbb{P}}^n$ obtained as general fibre of $\mathcal{P}_D$ over a point, say y , of $\check{\mathbb{P}}^n$. In other words, if we let $(s_1, \dots, s_n)$ be independent linear forms vanishing at y , the subscheme Z is the intersection of $\mathbb{P}(\mathcal{J}_D)$ with the span of $(s_1, \dots, s_n)$, i.e., $Z = \mathbb{P}(\mathcal{J}_D) \cap \mathbb{P}\langle s_1, \dots, s_n \rangle \subset \mathbb{P}^n \times \check{\mathbb{P}}^n$. Therefore, the image Z' of Z in $\mathbb{P}^n$ (which has the same length as Z) is obtained as common zero locus of the sections $\tilde{s}_1, \dots, \tilde{s}_n$ of $\mathcal{J}_D(d-1)$, where $\tilde{s}_i$ is obtained from s_i by the natural isomorphism:

$$H^0(\mathbb{P}(\mathcal{J}_D(d-1)), \mathcal{O}_{\mathcal{J}_D(d-1)}(1)) \simeq H^0(\mathbb{P}^n, \mathcal{J}_D(d-1)).$$

We write thus the exact diagram:

$$\begin{array}{ccccc}
 & \langle s_1, \dots, s_n \rangle \otimes \mathcal{O}_{\mathbb{P}^n} & & & \\
 & \downarrow & & & \\
 \mathcal{T}_{\mathbb{P}^n}(-\log D) & \longrightarrow & V^* \otimes \mathcal{O}_{\mathbb{P}^n} & \longrightarrow & \mathcal{J}_D(d-1) \\
 \downarrow & & \downarrow & & \downarrow \\
 \mathcal{J}_{Z'/\mathbb{P}^n} & \longrightarrow & \mathcal{O}_{\mathbb{P}^n} & \longrightarrow & \mathcal{O}_{Z'}(d-1) \\
 \downarrow & & \downarrow & & \downarrow \\
 0 & & 0 & & 0
 \end{array}$$

Then the 0-dimensional subscheme $Z' \subset \mathbb{P}^n$ appears as zero locus of the co-section $\tilde{s} : \mathcal{T}_{\mathbb{P}^n}(-\log D) \rightarrow \mathcal{O}_{\mathbb{P}^n}$. Hence its length is just $c_n((\mathcal{T}_{\mathbb{P}^n}(-\log D))^*) = (-1)^n c_n(\mathcal{T}_{\mathbb{P}^n}(-\log D))$. $\square$

III.2. Homaloidal polynomials. Let $D \subset \mathbb{P}^n$ be a reduced hypersurface. A particularly interesting, and historically well-studied case is when D has polar degree 1, which is to say that $\mathcal{P}_D$ is birational. In this case D is said to be *homaloidal*. In fact, as far as the polar degree is concerned and we are working over the complex numbers, we might even work with non-reduced D , since the polar degree of D and of its reduced locus are the same, cf. [102]. See [144] for a different approach. An algebraic proof of this fact is not known to me except for line arrangements, [59]. Anyway, we carry out this discussion over $\mathbb{C}$, except for the last proposition.

Irreducible homaloidal hypersurfaces of arbitrarily large degree in $\mathbb{P}^n$ for any $n \geq 3$ were produced in [82]. We refer to this paper for a more detailed account, including many interesting references and a historical perspective, on homaloidal polynomials and on the related notion of homogeneous polynomials f with vanishing Hessian, i.e. such that the determinant $h(f)$ of the Hesse matrix $(\partial f / \partial x_i \partial x_j)$ is identically zero.

An important notion in this setting is that of *prehomogeneous vector space*, cf. [279], i.e. a complex vector space V equipped with a representation $G \rightarrow \mathrm{GL}(V)$ of an algebraic group G , possessing an open G -orbit whose complement in V is a hypersurface $\mathbb{V}(f)$. The function given by polynomial f , which we may assume to be square-free, is then G -equivariant, up the choice of a suitable character of G . The space V is said to be *regular* if $h(f) \neq 0$, and f is called a *relative invariant* of V . Relative invariants of regular prehomogeneous vector spaces for G reductive are classified, cf. [279].

The point of recalling these notions in this context is the fact that relative invariants of regular prehomogeneous vector spaces are homaloidal polynomials, cf. [116, 128]; see [210]. In this sense, homaloidal divisors D of degree 3 correspond precisely to Severi varieties (cf. also [78]), in the sense that D is the dual (or the secant) variety of a Severi variety. These are the $V_2 \subset \mathbb{P}^5$ (and the dual is the determinant of symmetric matrices of size 3); $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$ (and the dual is the determinant of matrices of size 3); $\mathbb{G}(2, 6) \subset \mathbb{P}^{14}$ (and the dual is the Pfaffian of skew-symmetric matrices of size 6); $\mathbb{O}\mathbb{P}^2 \subset \mathbb{P}^{26}$ (and the dual is the Cartan cubic).

A conjecture of Dimca-Papadima, recently proved by J. Huh in [174], is that if D is homaloidal with isolated singularities, then D is a smooth quadric, or the union of 3 lines not passing through a point in $\mathbb{P}^2$, or the union of a conic and a tangent line again in $\mathbb{P}^2$. Huh's proof is of topological nature, I don't know of any algebraic proof. For curves, this was already a result of Dolgachev, [104], of which we give an alternative proof here.

Proposition III.3. *Let k be an algebraically closed field, and let D be a reduced homaloidal curve in $\mathbb{P}^2$. Then D is a smooth conic, or a union of 3 lines not passing through a point, or the union of a conic and a tangent line.*

PROOF. We have seen in Lemma III.2 that, if D is a reduced homaloidal curve of degree d , then $\mathcal{T} = \mathcal{T}_{\mathbb{P}^2}(-\log D)$ has $c_1(\mathcal{T}) = 1 - d$ and $c_2(\mathcal{T}) = 1$.

We look first at the range $d \geq 4$, in which case we have to look for a contradiction. The argument is more easily dealt with if we divide into cases according to whether d is even or odd. So we assume that d is even (the case when d is odd is completely analogous and we omit it).

So assume $d = 2a$ and $a \geq 2$, hence $c_1(\mathcal{T}) = 1 - 2a$. We compute $\chi(\mathcal{T}(a - 1)) = a(a - 1) > 0$, so $h^0(\mathbb{P}^2, \mathcal{T}(a - 1)) = 0$ implies $h^2(\mathbb{P}^2, \mathcal{T}(a - 1)) \neq 0$. But by Serre duality, since $\mathcal{T}^* \simeq \mathcal{T}(2a - 1)$, this value equals $h^0(\mathbb{P}^2, \mathcal{T}(a - 3))$ which vanishes if $h^0(\mathbb{P}^2, \mathcal{T}(a - 1)) = 0$. So $h^0(\mathbb{P}^2, \mathcal{T}(a - 1)) \neq 0$.

Choose now the greatest integer b such that $h^0(\mathbb{P}^2, \mathcal{T}(a - 1 - b)) \neq 0$. We have just proved that $b \geq 0$. Also, since $\mathcal{T}$ is a subbundle of $\mathcal{O}_{\mathbb{P}^2}^3$, we obviously get $b \leq a - 1$. In fact we rather obtain $b \leq a - 2$ for if $b = a - 1$, composing the inclusion of $\mathcal{O}_{\mathbb{P}^2}$ into $\mathcal{T}$ with the injection of $\mathcal{T}$ into $\mathcal{O}_{\mathbb{P}^2}^3$ we see $\mathcal{O}_{\mathbb{P}^2}$ splits off as a direct summand of $\mathcal{T}$, so $c_2(\mathcal{T}) = 0$, which is not the case.

Now, it is very easy to show that $c_2(\mathcal{T}(a - 1 - b)) \geq 0$ for a non-zero global section of $\mathcal{T}(a - 1 - b)$ vanishes on a subscheme of $\mathbb{P}^2$ having precisely this length. Now we compute this length, and find $-a^2 + b^2 + a + b + 1$, which is easily seen to be negative for $0 \leq b \leq a - 2$. A contradiction!

The range $d \geq 3$ is easily studied with a case-by-case analysis. If $d = 1$, we know that $\mathcal{T} \simeq \mathcal{O}_{\mathbb{P}^2}^2$. In case $d = 2$, we know that $\mathcal{T} \simeq \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$ if D is the union of two distinct lines and $\mathcal{T} \simeq \Omega_{\mathbb{P}^2}(1)$ if D is a smooth conic. If $d = 3$, the statement is clear after computing Chern classes in Example II.3. $\square$

Interesting questions concern hypersurfaces of low polar degree. For instance, J. Huh proposed a conjecturally complete list of complex projective hypersurfaces with isolated singularities of polar degree 2, which should be:

- i) a normal cubic surface containing one line or two lines, or three lines and three binodes;
- ii) two smooth conics meeting at a single point, with or without their common tangent;
- iii) a smooth conic, a tangent line, and a line passing through the tangency point;
- iv) a smooth conic and two tangent lines;
- v) three lines passing through a point x and a line disjoint from x ;
- vi) a cuspidal cubic, and its tangent at the cusp;
- vii) a cuspidal cubic and its tangent at the flex;
- viii) a cuspidal cubic;

ix) a smooth conic and a secant line.

One may think that, once fixed k , there should be no hypersurface of degree d in $\mathbb{P}^n$ of polar degree k with isolated singularities, if d and n are large enough.

IV. Torelli problem for hypersurfaces

Generally speaking, a Torelli problem consists in asking whether a variety X can be reconstructed from a particular invariant, typically related to the Hodge structure of X .

For instance, a smooth complex projective curve is determined up to isomorphism by its Jacobian $J(C)$, polarized by Riemann's Theta divisor. This is the content of the original Torelli theorem, [302]. We refer to [81] for a very nice account of this result, with a proof and a historical perspective. The theorem holds, however, over any field, cf. for instance Serre's appendix of [225] - one has to assume C geometrically integral. Torelli theorems also exist for more general curves (e.g. stable curves, cf. [70]), invariants of different kind (cf. the global Torelli theorem [153]) or other manifolds such as K3 surfaces [263], smooth cubic threefolds [84], and generic hypersurfaces of degree d in $\mathbb{P}^n$ for $n \geq 3$, except when $n = d = 3$, or $d \mid n + 1$ or $d = 4$, $n \equiv 1$ modulo 4, or $d = 6$, $n \equiv 2$ modulo 6, [108].

Let us formulate the precise Torelli problem that we are going to be interested in, for reduced hypersurface of projective space, over a field k .

Problem IV.1. *Let D and D' be reduced projective hypersurfaces of $\mathbb{P}^n$. Assume $\mathcal{T}_{\mathbb{P}^n}(-\log D) \simeq \mathcal{T}_{\mathbb{P}^n}(-\log D')$. Then, do we have $D = D'$?*

If the answer is positive, we will say that D is *Torelli*. We have said that $\tilde{\Omega}_{\mathbb{P}^n}(\log D)$ is a refinement of the dual of $\mathcal{T}_{\mathbb{P}^n}(-\log D)$, so another Torelli problem arises for Dolgachev's sheaf. This apparently innocent modification of the problem turns out to give a totally different answer, as we shall see.

Problem IV.2. *Let D and D' be reduced projective hypersurfaces of $\mathbb{P}^n$. Assume $\tilde{\Omega}_{\mathbb{P}^n}(\log D) \simeq \tilde{\Omega}_{\mathbb{P}^n}(\log D')$. Then, do we have $D = D'$?*

Also in this case we will say that D is *Torelli* if it satisfies the above property: the problem under consideration should be clear from the context. Should it not be so, we will refer to the first property as *strong Torelli*. A tightly related question is, however, the following:

Problem IV.3. *Assume the reduced projective hypersurface D is not Torelli. Then, how to describe the set of hypersurfaces D' such that $\tilde{\Omega}_{\mathbb{P}^n}(\log D) \simeq \tilde{\Omega}_{\mathbb{P}^n}(\log D')$?*

Let us mention some of the Torelli theorems available in the literature directly related to our problem. Ueda-Yoshinaga proved in [304] that a smooth complex hypersurface $\mathbb{V}(f) \subset \mathbb{P}^n$ is Torelli if and only if f is not of

Thom-Sebastiani type, i.e. it cannot be split non-trivially as $f(\mathbf{x}_0, \dots, \mathbf{x}_n) = f_1(\mathbf{x}_0, \dots, \mathbf{x}_k) + f_2(\mathbf{x}_{k+1}, \dots, \mathbf{x}_n)$. This reproves the theorem of [303], namely that a smooth plane cubic over $\mathbb{C}$ is Torelli if and only if its J invariant is non-zero.

We would like to give here an overview of these two problems in some interesting situations, namely hyperplane arrangements (with normal crossings or not), and “generic” arrangements of hypersurfaces with components of arbitrary degree.

IV.1. Torelli theorems for hyperplane arrangement. Let us first look at the case when all the irreducible components of D have degree 1, in other words, when D is a *hyperplane arrangement divisor*.

The results of this section are originally formulated over $\mathbb{C}$, but hold in fact for any field k .

IV.1.1. Generic hyperplane arrangements. Let D be a hyperplane arrangement divisor with normal crossings, so D corresponds to an arrangement $\mathcal{A} = (H_1, \dots, H_\ell)$, such that any k distinct hyperplanes among the H_i ’s meet along a $\mathbb{P}^{n-k}$. Then, D is said to be a *generic arrangement*.

In this case, Problems IV.1 and IV.2 are the same thing. Indeed, in view of [105], we know that $\tilde{\Omega}_{\mathbb{P}^n}(\log(D_{\mathcal{A}}))$ is locally free if and only if $D_{\mathcal{A}}$ is generic, and that it agrees with $\tilde{\Omega}_{\mathbb{P}^n}(\log(D_{\mathcal{A}}))$ if $D_{\mathcal{A}}$ has normal crossings in codimension ≤ 2 .

For generic arrangements, the answer to Problems IV.1 and IV.2 is the main result of [107]. It is proved there that, if $\ell \geq 2n + 3$, then $\mathcal{A}$ is Torelli if and only if $\mathcal{A}$ does not osculate a rational normal curve. The result was extended to the range $\ell \geq n + 2$ in [306].

However this result only covers generic arrangement. On the other hand, typically the most interesting arrangements are quite far from being generic.

For instance, the Hesse arrangements $\mathcal{A}$ that we mentioned in Example II.8 is free with exponents $(4, 7)$, for whatever smooth cubic curve C we may start with. Then, of course we cannot reconstruct our C , nor the 12 lines of our arrangement, from $\mathcal{T}_{\mathbb{P}^2}(-\log D_{\mathcal{A}}) \simeq \mathcal{O}_{\mathbb{P}^2}(-4) \oplus \mathcal{O}_{\mathbb{P}^2}(-7)$. Anyway, we will be able to recover the 12 lines from $\tilde{\Omega}_{\mathbb{P}^2}(\log D_{\mathcal{A}})$. This also explains the big difference between Problems IV.1 and IV.2. However, one should be aware that, by [315], $\mathcal{A}$ can be recovered from $\mathcal{J}_{D_{\mathcal{A}}}$. This last result is closely related to [108, Proposition 1.1], where it is proved that any hypersurface D in $\mathbb{P}^n$ is recovered from $\mathcal{J}_D$ up to projective equivalence for $n \geq 3$.

IV.1.2. Arbitrary hyperplane arrangements. According to the observations contained in the previous paragraph, for an arbitrary hyperplane arrangement we study Problem IV.2 rather than IV.1. This part is a sketch of the results of [138].

As in Section II.1.6, we use our approach based on projective duality. So again we write $H = H_z$ for the hyperplane of $\mathbb{P}^n$ given by the point z of $\check{\mathbb{P}}^n$, and $\mathcal{A} = \mathcal{A}_Z$ for the hyperplane arrangement corresponding to a finite set Z of points of $\check{\mathbb{P}}^n$ (i.e. a reduced subscheme of finite length of $\check{\mathbb{P}}^n$). Of course, if $\mathcal{A}$ consists of ℓ hyperplanes, then Z has length ℓ . We write D_Z for $D_{\mathcal{A}_Z}$. We say that $Z \subset \check{\mathbb{P}}^n$ is *Torelli* according to whether D_Z is Torelli or not. This will depend on whether Z is contained in a certain type of varieties, that we call *Kronecker-Weierstrass varieties*.

Definition IV.4. Let $s \geq 0$ and $(d, n_1, \dots, n_s)$ be a string of $s+1$ positive integers such that $n = d + n_1 + \dots + n_s$. Then $Y \subset \check{\mathbb{P}}^n$ is a *Kronecker-Weierstrass (KW) variety of type $(d; s)$* if $Y = C \cup L_1 \cup \dots \cup L_s \subset \check{\mathbb{P}}^n$, where the L_i 's are linear subspaces of dimension n_i and C is a smooth rational curve of degree d (called the *curve part* of Y) spanning a linear space L of dimension d such that:

- i) for all i , $L \cap L_i$ is a single point which lies in C ;
- ii) the span of $L \cup L_1 \cup \dots \cup L_s$ is all of $\check{\mathbb{P}}^n$.

If $d = 0$ and $s \geq 2$ a KW variety of type $(0; s)$ is defined as $Y = L_1 \cup \dots \cup L_s \subset \check{\mathbb{P}}^n$ spanning $\check{\mathbb{P}}^n$ where the L_i 's are linear subspaces of dimension n_i with $n = n_1 + \dots + n_s$ and all the linear spaces L_i meet only at a point y , which is called the *distinguished point* of Y .



FIGURE 5. KW varieties of types $(n; 0)$ and $(d; 2)$.

Some examples of KW are the following.

- i) A rational normal curve is a KW variety of type $(n; 0)$.
- ii) A union of two lines in $\mathbb{P}^2$ is a KW variety in three ways, two of them of type $(1; 1)$, and one of type $(0; 2)$.

Remark IV.5. We will use the previous definition in a minute to assert that a finite-length subscheme Z of $\check{\mathbb{P}}^n$, whose reduced structure is contained in no hyperplane, is not Torelli if and only if Z lies in a KW variety (whose distinguished point lies away from Z in case $d = 0$).

With a little abuse of terminology, we state the result here even for Z non-reduced. In fact, in the outline of the proof, we will give a more general definition of $\tilde{\Omega}_{\mathbb{P}^n}(\log D_Z)$, which is meaningful even when Z is not reduced.

Also, since Z is non-degenerate, in Definition IV.4, still assuming $n = d + n_1 + \dots + n_s$, condition (ii) can be omitted, and (i) can be replaced by the requirement that $L \cap L_i$ have non-empty intersection along C .

Theorem IV.6. *Let $Z \subset \check{\mathbb{P}}^n$ be a finite-length, set-theoretically non-degenerate subscheme. Then Z fails to be Torelli if and only if Z is contained in a KW variety $Y \subset \check{\mathbb{P}}^n$ of type $(d; s)$ such that either $d > 0$, $s \geq 0$, or $d = 0$, $s \geq 2$, and the distinguished point of Y does not lie in Z .*

A key ingredient is the notion of *unstable hyperplane*, which we now define.

Definition IV.7. Given $y \in \check{\mathbb{P}}^n$, H_y is an *unstable hyperplane* for $\tilde{\Omega}_{\mathbb{P}^n}(\log D_Z)$ if:

$$H^{n-1}(H_y, \tilde{\Omega}_{\mathbb{P}^n}(\log D_Z)|_{H_y}(-n)) \neq 0.$$

Let us now sketch the proof of the above theorem.

Step 1. *Give an alternative definition of $\tilde{\Omega}_{\mathbb{P}^n}(\log D_Z)$ via integral functors.*

This is a derived version of the construction of Section II.1.6, and goes as follows. Again, consider the projective bundle $\mathbb{P}(\mathcal{T}_{\mathbb{P}^n}(-1))$, and the diagram:

$$\begin{array}{ccc} & \mathbf{F}(1, n) & \\ \pi_{\mathbb{P}^n} \swarrow & & \searrow \pi_{\check{\mathbb{P}}^n} \\ \mathbb{P}^n & & \check{\mathbb{P}}^n \end{array}$$

Then, take the ideal sheaf $\mathcal{I}_{Z/\check{\mathbb{P}}^n}(1)$, and the derived direct image:

$$R\pi_{\mathbb{P}^n*}(\pi_{\mathbb{P}^n}^*(\mathcal{I}_{Z/\check{\mathbb{P}}^n}(1))).$$

This is a 2-term complex that is easily seen to be quasi-isomorphic, at least when Z is (scheme-theoretically) non-degenerate, to a complex of the form:

$$\mathcal{O}_{\mathbb{P}^n}^{\ell-1}(-1) \rightarrow \mathcal{O}_{\mathbb{P}^n}^{\ell-n-1}.$$

Note that this makes sense even when Z is non-reduced, and that Z must have length $\ell \geq n+1$ in order to be non-degenerate.

Finally, apply $R\mathcal{H}om_{\mathbb{P}^n}(-, \mathcal{O}_{\mathbb{P}^n}(-1))$, to the direct image complex above, and show that the result is a *pure sheaf*, which is in fact isomorphic to $\tilde{\Omega}_{\mathbb{P}^n}(\log D_Z)$ (if Z is reduced, otherwise we get the announced generalization of this sheaf for non-reduced subschemes). This sheaf is thus a *Steiner sheaf*, as already proved by Dolgachev, i.e. we can see it as cokernel of the transpose of the above matrix:

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^n}^{\ell-n-1}(-1) \rightarrow \mathcal{O}_{\mathbb{P}^n}^{\ell-1} \rightarrow \tilde{\Omega}_{\mathbb{P}^n}(\log D_Z) \rightarrow 0.$$

Step 2. *Give a condition on $y \in \check{\mathbb{P}}^n \setminus Z$ and $\mathcal{I}_{Z/\check{\mathbb{P}}^n}$ equivalent to H_y being an unstable hyperplane for $\tilde{\Omega}_{\mathbb{P}^n}(\log D_Z)$.*

Given $y \in \check{\mathbb{P}}^n$, consider n independent linear forms $(s_1, \dots, s_n)$ vanishing at y , and define the torsion-free sheaf $\mathcal{S}_y$:

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^n} \xrightarrow{(s_1, \dots, s_n)} \mathcal{O}_{\mathbb{P}^n}^n(1) \rightarrow \mathcal{S}_y \rightarrow 0.$$

Now, in view of the alternative definition we have given of $\tilde{\Omega}_{\mathbb{P}^n}(\log D_Z)$, we argue using the properties of integral functors under consideration that H_y is

unstable for $\tilde{\Omega}_{\mathbb{P}^n}(\log D_Z)$ if and only if:

$$H^0(\check{\mathbb{P}}^n, \mathcal{S}_y \otimes \mathcal{I}_Z) \neq 0.$$

Step 3. Use the previous step to show that, if $y \in \check{\mathbb{P}}^n \setminus Z$ is unstable for $\tilde{\Omega}_{\mathbb{P}^n}(\log D_Z)$, then Z is contained in some KW variety.

To achieve this, we note that a non-zero element s of $H^0(\check{\mathbb{P}}^n, \mathcal{S}_y \otimes \mathcal{I}_Z)$ lifts to $\tilde{s}$ as in the diagram:

$$\begin{array}{ccccccc} & & & & \mathcal{O}_{\check{\mathbb{P}}^n} & & \\ & & & \nearrow \tilde{s} & \downarrow s & & \\ 0 & \longrightarrow & \mathcal{O}_{\check{\mathbb{P}}^n} & \xrightarrow{(s_1, \dots, s_n)} & \mathcal{O}_{\check{\mathbb{P}}^n}^n(1) & \longrightarrow & \mathcal{S}_y \longrightarrow 0. \end{array}$$

Now $\tilde{s}$ is given by $(t_1, \dots, t_n)$, where the t_i 's are linear forms and the row $(t_1, \dots, t_n)$ is not proportional to $(s_1, \dots, s_n)$. Then s vanishes on Z only if Z is contained in the locus Y cut by the 2×2 minors of the matrix:

$$M = \begin{pmatrix} t_1 & \cdots & t_n \\ s_1 & \cdots & s_n \end{pmatrix},$$

Note that Y is not all of $\check{\mathbb{P}}^n$, because the two rows of M are not proportional.

Now, one uses the approach of Lemma III.1 of Chapter 0 to show that $\mathbb{P}(\text{coker}(M))$ is isomorphic to $\mathbb{P}(\text{coker}(N))$, where N is the matrix:

$$N : \mathcal{O}_{\mathbb{P}^1}(-1)^n \rightarrow \mathcal{O}_{\mathbb{P}^1}^{n+1},$$

defined using the rows of M to index the variables of N , and the variables of M to index the rows of N . The degeneracy locus Y is the image in $\check{\mathbb{P}}^n$ of $\mathbb{P}(\text{coker}(M)) \simeq \mathbb{P}(\text{coker}(N))$, and $\text{coker}(N)$ is the direct sum of a line bundle of degree $d \geq 0$ on $\mathbb{P}^1$ (which gives the curve part of Y if $d > 0$) and of torsion sheaves at points of $\mathbb{P}^1$ (which give the linear part of Y). In fact, we will see in detail how to work with matrices of linear forms in Chapter 2, cf. in particular Section IV.4.2.

The final point is to check that, since Z is set-theoretically non-degenerate, Y is reduced, so it satisfies precisely our definition of KW variety.

To conclude the proof, one argues that, as remarked by Dolgachev, all points of Z give unstable hyperplanes for $\tilde{\Omega}_{\mathbb{P}^n}(\log D_Z)$. If there were points y not in Z , giving unstable hyperplanes, then Z would be contained in a KW variety Y (suppose, for simplicity, with a curve part). Hence if Z is not contained in any such variety from the beginning, it means that the points of Z are precisely the only unstable hyperplanes of $\tilde{\Omega}_{\mathbb{P}^n}(\log D_Z)$, and Z is reconstructed this way, so that Torelli holds. It Y has no curve part one has to pay a little more attention, but this is really a detail and we skip it here.

IV.2. Generic Torelli theorem for hypersurfaces. Let us now go back to our Torelli problem for hypersurfaces having irreducible components of higher

degree. First of all, we formulate the problem a bit more explicitly, in a framework involving projective duality.

Let us fix $n \geq 2$, an integer $s \geq 1$, and a degree vector $\mathbf{d}$, i.e. $\mathbf{d} = (d_1, \dots, d_s)$ with $1 \leq d_1 < \dots < d_s$. For any integer d , we consider the projective space $\check{\mathbb{P}}(\mathcal{S}_d)$ as the parameter space of hypersurfaces of degree d . Of course, the space $\check{\mathbb{P}}(\mathcal{S}_1)$ is just the dual projective space $\check{\mathbb{P}}^n$, and in this sense will generalize the approach of the previous section. So, given a degree vector $\mathbf{d}$ we let:

$$\check{\mathbb{P}}(\mathcal{S}_{\mathbf{d}}) = \coprod_{i=1, \dots, s} \check{\mathbb{P}}(\mathcal{S}_{d_i}).$$

Set $N_i = \binom{n+d_i}{n} - 1$. Clearly, we have $\check{\mathbb{P}}(\mathcal{S}_{\mathbf{d}}) \simeq \coprod_{i=1, \dots, s} \check{\mathbb{P}}^{N_i}$.

Let $\ell = (\ell_1, \dots, \ell_s)$ be a sequence of integers. A reduced hypersurface D having for all i , a number ℓ_i of irreducible components of degree d_i corresponds uniquely to a collection Z_i of ℓ_i points of $\check{\mathbb{P}}(\mathcal{S}_{d_i})$. The degree of D is $\sum_i \ell_i d_i$. Set $\ell = \sum_i \ell_i$. The collection of ℓ points $Z = \cup_i Z_i$ lives in the space $\check{\mathbb{P}}(\mathcal{S}_{\mathbf{d}})$, and we denote by D_Z the hypersurface in $\mathbb{P}^n$ corresponding to Z . The hypersurface D_Z consists of the union of components $\cup_{z \in Z} D_z$.

When D consists of many sufficiently general hypersurfaces in $\check{\mathbb{P}}^n$, the Torelli problem has a positive answer. This is what we call generic Torelli theorem. This result has been obtained in collaboration with Elena Angelini, and is in great proportion contained in her thesis, [7] and in the preprint [6]. However, the result has never appeared so far, so we present it with a full proof.

So let $Z = \cup_i Z_i$ and $Z' = \cup_i Z'_i$ be finite sets of points in $\check{\mathbb{P}}(\mathcal{S}_{\mathbf{d}})$, with Z_i and Z'_i in $\check{\mathbb{P}}(\mathcal{S}_{d_i})$. Consider the hypersurfaces $D = D_Z$ and $D' = D_{Z'}$ of $\mathbb{P}^n$ associated with Z and Z' . The following result says that, under some generality assumptions, $\Omega_{\mathbb{P}^n}(\log D)$ and $\Omega_{\mathbb{P}^n}(\log D')$ are not the same, unless D and D' are the same.

Theorem IV.8. *Fix notations as above, and assume D and D' simple normal crossings. Suppose that, for all i , Z_i is in general linear position and does not lie in a rational normal curve of degree N_i .*

Then, whenever $\Omega_{\mathbb{P}^n}(\log D) \simeq \Omega_{\mathbb{P}^n}(\log D')$, we must have $Z = Z'$.

PROOF. Since D is a divisor with simple normal crossings, $\Omega_{\mathbb{P}^n}(\log D)$ is locally free, and we have the residue exact sequence:

$$0 \rightarrow \Omega_{\mathbb{P}^n} \rightarrow \Omega_{\mathbb{P}^n}(\log D) \rightarrow \bigoplus_{z \in Z} \mathcal{O}_{D_z} \rightarrow 0.$$

From the above sequence we deduce that, for all $z \in Z$ we have:

$$(IV.1) \quad \text{Hom}_{\mathbb{P}^n}(\Omega_{\mathbb{P}^n}(\log D), \mathcal{O}_{D_z}) \neq 0.$$

In analogy with Definition IV.7 for unstable hyperplanes, given $y \in \check{\mathbb{P}}(\mathcal{S}_e)$, we say that the corresponding hypersurface D_y is an *unstable hypersurface* for

$\Omega_{\mathbb{P}^n}(\log D)$ if:

$$(IV.2) \quad \text{Hom}_{\mathbb{P}^n}(\Omega_{\mathbb{P}^n}(\log D), \mathcal{O}_{D_y}) \neq 0.$$

So (IV.1) says that the hypersurfaces of Z are unstable for $\Omega_{\mathbb{P}^n}(\log D)$. Note that, by Serre duality, since $\Omega_{\mathbb{P}^n}(\log D)$ is locally free, this condition boils down to Definition IV.7 in case $y \in \check{\mathbb{P}}(\mathcal{S}_1) = \check{\mathbb{P}}^n$.

Likewise, the hypersurfaces of Z' are unstable for $\Omega_{\mathbb{P}^n}(\log D')$, i.e., we also have $\text{Hom}_{\mathbb{P}^n}(\Omega_{\mathbb{P}^n}(\log D'), \mathcal{O}_{D_z}) \neq 0$ for all $z \in Z'$. Therefore, by the assumption $\Omega_{\mathbb{P}^n}(\log D) \simeq \Omega_{\mathbb{P}^n}(\log D')$, we get (IV.2) for all $y \in Z'$.

What we want to show is that, under our assumptions, $\Omega_{\mathbb{P}^n}(\log D)$ has no unstable hypersurfaces other than Z . In other words, to conclude $Z = Z'$, we have to show that, if $z \notin Z$, then:

$$(IV.3) \quad \text{Hom}_{\mathbb{P}^n}(\Omega_{\mathbb{P}^n}(\log D), \mathcal{O}_{D_z}) = 0.$$

We divide the proof of this fact into several steps.

Step 1. *Set up of multiple Veronese product.* Let us set $N_i = N_{d_i}$, and consider the product space $\mathbb{P}^{N_d} = \prod_{i=1}^s \mathbb{P}^{N_i}$. This space contains the diagonal Veronese image V_d of the projective space $\mathbb{P}^n$, embedded in $\mathbb{P}^{N_i}$ by the monomials of degree d_i in $n+1$ variables.

$$\mathbb{P}^n \rightarrow V_d \hookrightarrow \mathbb{P}^{N_d}.$$

We have the obvious exact sequence:

$$(IV.4) \quad 0 \rightarrow \mathcal{I}_{V_d/\mathbb{P}^N} \rightarrow \mathcal{O}_{\mathbb{P}^N} \rightarrow \mathcal{O}_{V_d} \rightarrow 0.$$

Each of the sets of points $Z_i \subset \check{\mathbb{P}}(\mathcal{S}_{d_i}) = \check{\mathbb{P}}^{N_i}$ gives an arrangement $\mathcal{A}_i = \mathcal{A}_{Z_i}$ of hyperplanes in $\mathbb{P}^{N_i}$, and the collection of components of D_i is given as $\mathcal{A}_i \cap V_d$. In detail, for any z belonging to some $Z_i \subset Z$, we have a hyperplane in $\mathbb{P}^{N_i}$, and thus its pull-back $H_z \subset \mathbb{P}^{N_d}$, and $D_z = H_z \cap V_d$. Denote by $\pi_i : \mathbb{P}^{N_d} \rightarrow \mathbb{P}^{N_i}$ be the i -th projection, and $h_i = c_1(\pi_i^*(\mathcal{O}_{\mathbb{P}^{N_i}}(1)))$. We get an arrangement $\mathcal{A} = \cup_i \mathcal{A}_i$ of hypersurfaces of $\mathbb{P}^{N_d}$, where $\mathcal{A}_i$ is an irreducible divisor of class h_i . Given $z \in \check{\mathbb{P}}^{N_i}$, we have:

$$(IV.5) \quad 0 \rightarrow \mathcal{O}_{\mathbb{P}^{N_d}}(-h_i) \rightarrow \mathcal{O}_{\mathbb{P}^d} \rightarrow \mathcal{O}_{H_z} \rightarrow 0.$$

Step 2. *Get restricted unstable hyperplanes from unstable hypersurfaces.* Since each component of D is smooth, the manifold V_d intersects transversely the hyperplanes of $\mathcal{A}_i$, so that by [105, Proposition 2.11] we get an exact sequence:

$$(IV.6) \quad 0 \rightarrow \mathcal{N}_{V_d}^* \rightarrow \Omega_{\mathbb{P}^{N_d}}(\log \mathcal{A})|_{V_d} \rightarrow \Omega_{\mathbb{P}^n}(\log D) \rightarrow 0.$$

By contradiction with (IV.3), let $z \notin Z$ satisfy (IV.2). By the previous sequence, it follows that $\text{Hom}_{\mathbb{P}^n}(\Omega_{\mathbb{P}^{N_d}}(\log \mathcal{A})|_{V_d}, \mathcal{O}_{D_z}) \neq 0$, i.e.:

$$(IV.7) \quad H^0(V_d, \Omega_{\mathbb{P}^{N_d}}(\log \mathcal{A})^*|_{V_d \cap H_z}) \neq 0.$$

We say that H_z is a *restricted unstable hyperplane* for $\Omega_{\mathbb{P}^{N_d}}(\log \mathcal{A})$ when the above non-vanishing condition holds. We have thus proved that unstable hypersurfaces for $\Omega_{\mathbb{P}^n}(\log D)$ give rise to restricted unstable hyperplanes for the sheaf $\Omega_{\mathbb{P}^{N_d}}(\log \mathcal{A})$.

Step 3. Prove that, for all $z \in \check{\mathbb{P}}^{N_s}$, we have the vanishing:

$$(IV.8) \quad H^1(\mathbb{P}^{N_d}, \mathcal{J}_{V_d/\mathbb{P}^{N_d}} \otimes \Omega_{\mathbb{P}^{N_d}}(\log \mathcal{A})^*|_{H_z}) = 0.$$

This is the most technical part. First of all, since $\mathbb{P}^{N_d}$ is a product and each subarrangement $\mathcal{A}_i$ of $\mathcal{A}$ is a pull-back via the projection map π_i , we get a splitting:

$$\Omega_{\mathbb{P}^{N_d}}(\log \mathcal{A})^* \simeq \bigoplus_{i=1, \dots, s} \pi_i^*(\Omega_{\mathbb{P}^{N_i}}(\log \mathcal{A}_i)^*).$$

Let us look at the single summands $\Omega_{\mathbb{P}^{N_i}}(\log \mathcal{A}_i)^*$. Since $Z_i \subset \check{\mathbb{P}}^{N_i}$ is general linear position and is not contained in a rational normal curve of degree N_i , we have that Z_i is non-degenerate, so [107, Theorem 3.5] implies that $\Omega_{\mathbb{P}^{N_i}}(\log \mathcal{A}_i)^*$ is a vector bundle fitting into an exact sequence of the form:

$$0 \rightarrow \Omega_{\mathbb{P}^{N_i}}(\log \mathcal{A}_i)^* \rightarrow \mathcal{O}_{\mathbb{P}^{N_i}}^{\ell_i-1} \rightarrow \mathcal{O}_{\mathbb{P}^{N_i}}(1)^{\ell_i-N_i-1} \rightarrow 0.$$

Summing over all i we get:

$$0 \rightarrow \Omega_{\mathbb{P}^{N_d}}(\log \mathcal{A})^* \rightarrow \mathcal{O}_{\mathbb{P}^{N_d}}^{\ell-s} \rightarrow \bigoplus_{i=1, \dots, s} \mathcal{O}_{\mathbb{P}^{N_d}}(h_i)^{\ell_i-N_i-1} \rightarrow 0.$$

Now we tensor this sequence with $\mathcal{J}_{V_d/\mathbb{P}^{N_d}}$ and we restrict to H_z , i.e., we tensor with $\mathcal{J}_{V_d \cap H_z/H_z}$. To prove (IV.8), it suffices thus to prove:

$$(IV.9) \quad H^{1-k}(H_z, \mathcal{J}_{V_d \cap H_z/H_z}(kh_i)), \quad \text{for } k = 0, 1, \text{ for all } i = 1, \dots, s \text{ if } z \in \check{\mathbb{P}}^{N_s}.$$

To accomplish this, we put together (IV.5) and (IV.4) to form the following exact commutative diagram.

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & \mathcal{J}_{V_d/\mathbb{P}^{N_d}}(h_i - h_j) & \rightarrow & \mathcal{O}_{\mathbb{P}^{N_d}}(h_i - h_j) & \rightarrow & \mathcal{O}_{\mathbb{P}^n}(d_i - d_j) \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & \mathcal{J}_{V_d/\mathbb{P}^{N_d}}(h_i) & \rightarrow & \mathcal{O}_{\mathbb{P}^{N_d}}(h_i) & \rightarrow & \mathcal{O}_{\mathbb{P}^n}(d_i) \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & \mathcal{J}_{V_d \cap H_z/H_z}(h_i) & \rightarrow & \mathcal{O}_{H_z}(h_i) & \rightarrow & \mathcal{O}_{V_d \cap H_z}(d_i) \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

Clearly, the map $H^0(\mathbb{P}^{N_d}, \mathcal{O}_{\mathbb{P}^{N_d}}(h_i)) \rightarrow H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d_i))$ is an isomorphism, so we get the vanishing:

$$(IV.10) \quad H^0(\mathbb{P}^{N_d}, \mathcal{J}_{V_d/\mathbb{P}^{N_d}}(h_i)) = H^1(\mathbb{P}^{N_d}, \mathcal{J}_{V_d/\mathbb{P}^{N_d}}(h_i)) = 0.$$

Assume $j = s$, i.e., $z \in \check{\mathbb{P}}^{N_s}$, so that $d_s \geq d_i$ for all $i = 1, \dots, s$. In this case, looking at the first row of the above diagram, we see that:

$$(IV.11) \quad H^1(\mathbb{P}^{N_d}, \mathcal{I}_{V_d/\mathbb{P}^{N_d}}(h_i - h_j)) = 0, \quad \text{for } j = s.$$

which is obvious for $i = j$, and follows from $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d_i - d_j)) = 0$ for $i \neq j$ since in any case $H^1(\mathbb{P}^{N_d}, \mathcal{O}_{\mathbb{P}^{N_d}}(h_i - h_j)) = 0$.

Putting together (IV.10) and (IV.11), we obtain, by the leftmost column of the above diagram:

$$H^0(H_z, \mathcal{I}_{V_d \cap H_z/H_z}(h_i)) = 0, \quad \text{for all } i = 1, \dots, s, \text{ for all } z \in \check{\mathbb{P}}^{N_s}.$$

Moreover, since $V_d \cap H_z$ is connected, we get the vanishing:

$$H^1(H_z, \mathcal{I}_{V_d \cap H_z/H_z}) = 0.$$

We have finally proved (IV.9), so the proof of this step is finished.

Step 4. Deduce that $Z_s = Z'_s$. Tensoring by $\Omega_{\mathbb{P}^{N_d}}(\log \mathcal{A})^*|_{H_z}$ the sequence (IV.4), by the previous steps (i.e., using (IV.7) and (IV.8)), we get, for any $z \in \check{\mathbb{P}}^{N_s}$:

$$H^0(\mathbb{P}^{N_d}, \Omega_{\mathbb{P}^{N_d}}(\log \mathcal{A})^*|_{H_z}) \neq 0.$$

In other words, the hyperplanes corresponding to $z \in \check{\mathbb{P}}^{N_s}$, i.e., to hypersurfaces of maximal degree, are unstable for $\Omega_{\mathbb{P}^{N_d}}(\log \mathcal{A})$. Namely, we have lifted the restricted unstable hyperplanes to true unstable hyperplanes, at least in maximal degree.

Now we use the previously mentioned results of [107, 306] which assert that, as soon as Z_s is not contained in a rational normal curve of degree N_s , the unstable hyperplanes of $\Omega_{\mathbb{P}^{N_s}}(\log \mathcal{A}_s)$ are exactly the ones of $\mathcal{A}_s$. This means that the unstable hypersurfaces of degree s of $\Omega_{\mathbb{P}^{N_d}}(\log \mathcal{A})$ are precisely the points of Z_s . We conclude that $Z_s = Z'_s$.

Step 5. Use reduction to conclude $Z = Z'$. So far we have shown $Z_s = Z'_s$. Set:

$$Z^s = \bigcup_{i=1, \dots, s-1} Z_i, \quad \mathcal{A}^s = \mathcal{A}_{Z^s}.$$

We go back to the residue exact sequence, this time for $\Omega_{\mathbb{P}^{N_d}}(\log \mathcal{A})$, that reads:

$$0 \rightarrow \Omega_{\mathbb{P}^{N_d}} \rightarrow \Omega_{\mathbb{P}^{N_d}}(\log \mathcal{A}) \rightarrow \bigoplus_{z \in Z} \mathcal{O}_{H_z} \rightarrow 0.$$

We now *reduce* $\Omega_{\mathbb{P}^N_d}(\log \mathcal{A})$ via the hyperplanes of Z_s , i.e., we consider the projection onto $\bigoplus_{z \in Z_s} \mathcal{O}_{H_z}$ and take the kernel $\mathcal{F}$, thus getting the diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \rightarrow & \Omega_{\mathbb{P}^N_d} & \longrightarrow & \mathcal{F} & \longrightarrow & \bigoplus_{z \in Z^s} \mathcal{O}_{H_z} \rightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow \\
 0 & \rightarrow & \Omega_{\mathbb{P}^N_d} & \rightarrow & \Omega_{\mathbb{P}^N_d}(\log \mathcal{A}) & \rightarrow & \bigoplus_{z \in Z} \mathcal{O}_{H_z} \rightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & \bigoplus_{z \in Z_s} \mathcal{O}_{H_z} & = & \bigoplus_{z \in Z_s} \mathcal{O}_{H_z} \rightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
 \end{array}$$

We want to prove that $\mathcal{F} \simeq \Omega_{\mathbb{P}^N_d}(\log \mathcal{A}^s)$. To show this, first note that $\mathcal{F}$ is locally free, as one sees applying $\mathcal{H}om_{\mathbb{P}^N_d}(-, \mathcal{O}_{\mathbb{P}^N_d})$ to the central column of the diagram and checking that $\mathcal{E}xt_{\mathbb{P}^N_d}^k(\mathcal{F}, \mathcal{O}_{\mathbb{P}^N_d}) = 0$ for all $k > 0$, which in turn easily follows from the fact that $\Omega_{\mathbb{P}^N_d}(\log \mathcal{A})$ is locally free.

Further, we know that $\mathcal{F}$ is given as an extension associated with:

$$\xi \in \bigoplus_{z \in Z^s} \text{Ext}_{\mathbb{P}^N_d}^1(\mathcal{O}_{H_z}, \Omega_{\mathbb{P}^N_d}).$$

Write ξ_z for the component of ξ along $\text{Ext}_{\mathbb{P}^N_d}^1(\mathcal{O}_{H_z}, \Omega_{\mathbb{P}^N_d})$. By the residue exact sequence, $\Omega_{\mathbb{P}^N_d}(\log \mathcal{A}^s)$ is given by another element, say ξ' , in the same space. All the components ξ_z and ξ'_z must be non-zero, for otherwise the associated extensions would not give locally free sheaves. Moreover, if $z \in \check{\mathbb{P}}(\mathcal{S}_{d_j})$, we have $R\mathcal{H}om_{\mathbb{P}^N_d}(\mathcal{O}_{H_z}, \mathcal{O}_{\mathbb{P}^N_d}) \simeq \mathcal{O}_{H_z}(h_j)[-1]$, which easily implies:

$$\begin{aligned}
 \text{Ext}_{\mathbb{P}^N_d}^1(\mathcal{O}_{H_z}, \Omega_{\mathbb{P}^N_d}) &\simeq H^0(\mathbb{P}^N_d, \Omega_{\mathbb{P}^N_d}(h_j)|_{H_z}) \simeq \\
 &\simeq \bigoplus_{i \neq j} H^0(\mathbb{P}^{N_i}, \Omega_{\mathbb{P}^{N_i}}) \oplus H^0(\mathbb{P}^{N_j}, \Omega_{\mathbb{P}^{N_j}}(h_j)|_{H_z}) \simeq k,
 \end{aligned}$$

where we have used $\Omega_{\mathbb{P}^N_d} \simeq \bigoplus_{i=1, \dots, s} \pi_i^*(\Omega_{\mathbb{P}^{N_i}})$. Therefore, since $\xi_z \neq 0$ and $\xi'_z \neq 0$ for all $z \in Z^s$, the extension ξ is taken to ξ' by the automorphism $(\xi'_z/\xi_z)_{z \in Z^s}$ of $\bigoplus_{z \in Z^s} \mathcal{O}_{H_z}$, and we deduce $\mathcal{F} \simeq \Omega_{\mathbb{P}^N_d}(\log \mathcal{A}^s)$.

Restricting the central column of the previous diagram to V_d , and using again (IV.6), we get the following exact sequence, called the *reduction sequence*:

$$0 \rightarrow \Omega_{\mathbb{P}^n}(\log(D^s)) \rightarrow \Omega_{\mathbb{P}^n}(\log D) \rightarrow \bigoplus_{z \in Z_s} \mathcal{O}_{D_z} \rightarrow 0.$$

For any $i < s$, applying $\text{Hom}_{\mathbb{P}^n}(-, \mathcal{O}_{D_w})$ to this sequence, for all $w \in \check{\mathbb{P}}^{N_i}$, we easily see that $\Omega_{\mathbb{P}^n}(\log(D^s))$ and $\Omega_{\mathbb{P}^n}(\log D)$ share the same unstable hyper-surfaces of degree smaller than s . Therefore, repeating the previous steps for $\Omega_{\mathbb{P}^n}(\log(D^s))$ instead of $\Omega_{\mathbb{P}^n}(\log D)$, we conclude that $Z_{s-1} = Z'_{s-1}$. Iterating this procedure, we finally obtain $Z = Z'$.

□

It should be noted that this theorem is very far from being sharp, in the sense that Z might very well be Torelli even if it does not satisfy the condition of the theorem. However, it says that Torelli holds for generic arrangements of many hypersurfaces.

Corollary IV.9. *Let $Z = \cup Z_i$ be a finite set in $\check{\mathbb{P}}(\mathcal{S}_d)$, with $Z_i \subset \check{\mathbb{P}}(\mathcal{S}_{d_i})$ of length ℓ_i . If $\ell_i \geq N_i + 4$ for all i and each Z_i is sufficiently general, then Z is Torelli.*

PROOF. In the range $\ell_i \geq N_i + 4$, there is no rational normal curve in $\mathbb{P}^{N_i}$ through a general set of ℓ_i points of $\mathbb{P}^{N_i}$. All the conditions of Theorem IV.8 are open, so the proof is finished. □

V. Open questions

We already mentioned some open problems and conjectures related to logarithmic sheaves, in particular the general Torelli problems, Terao's conjecture on the combinatorial nature of freeness for hyperplane arrangements, Buchsbaum problem of producing free irreducible hypersurfaces, and Huh's conjectural classification of hypersurfaces of polar degree 2. Let us review briefly some more speculations.

V.1. Generalized Weyl arrangements. A very interesting class of arrangements comes from root systems.

Let V be a Euclidean vector space, equipped with the scalar product $(-, -)$. Given $0 \neq \alpha \in V$, write s_α for the reflection along the hyperplane $H_\alpha = \alpha^\perp$. A (reduced) *crystallographic root system* Φ is a finite set of non-zero vectors of V , such that:

- i) the set Φ spans V ;
- ii) for all $\alpha \in \Phi$, $t\alpha$ lies in Φ if and only if $t = \pm 1$;
- iii) for all $\alpha \in \Phi$, the set Φ is stable for s_α ;
- iv) for any $\alpha, \beta \in \Phi$, the value $\langle \beta, \alpha \rangle = \frac{(\alpha, \beta)}{(\alpha, \alpha)}$ is an integer.

A root system is irreducible if it does not arise as product of two root systems.

The classification of irreducible root systems, or equivalently of simple Lie algebras over $\mathbb{C}$, relies on beautiful work of Killing, Cartan, Dynkin. These root systems are in bijection with Dynkin diagrams of types $A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2$. We refer to [45, 175].

Choosing a hyperplane of V skew to Φ results in dividing Φ into the positive and negative parts $\Phi_\pm$. The *simple roots* Δ of Φ are an integral basis of Φ . The *Weyl group* of Φ is the subgroup of $\text{GL}_n(\mathbb{R})$ generated by the reflections s_α , for $\alpha \in \Phi$. Taking the product of the s_α , for all $\alpha \in \Delta$, we obtain a *Coxeter element* of W , and its order is called the *Coxeter number* of Φ , usually denoted by h .

The eigenvalues of a Coxeter element acting on V are powers of $e^{2i\pi/h}$, say $a_1, \dots, a_n$, which are called the *exponents* of Φ . In fact, $a_1 = 1$, and $a_n = h - 1$.

The *Weyl arrangement* is the affine arrangement of hyperplanes given by H_α for $\alpha \in \Phi_+$. These arrangements are free, and their exponents are given by the so-called dual partition of the height distribution. We refer to [1] for an arrangement-theoretic approach. The first proof not based on classification of root system was given in [209].

Given the root system Φ , one can further construct the arrangements of Shi and Catalan, as follows. First, we define the translated hyperplanes $H_{\alpha,k} = \{x \in V \mid \alpha(x) = k\}$. Then, we denote, for a given pair of integers $a \leq b \in \mathbb{Z}$:

$$\mathcal{A}_\Phi^{[a,b]} = (H_{\alpha,k} \mid \alpha \in \Phi^+, k \in \mathbb{Z}, a \leq k \leq b).$$

An important result of Yoshinaga (cf. [318]) characterizes the free arrangements arising this way. These are the cones (cf. [251, Section 1.15] or Section I) $c\mathcal{A}_\Phi^{[-k,k]}$ (Catalan) and $c\mathcal{A}_\Phi^{[1-k,k]}$ (Shi) over $\mathcal{A}_\Phi^{[-k,k]}$ and $\mathcal{A}_\Phi^{[1-k,k]}$.

Besides the cases giving rise to free arrangements, not much is known. However, combining a guess of Yoshinaga (private communication) with some experimental evidence, we propose the following.

Conjecture V.1. *Let h be the Coxeter number of Φ , $-1 \leq a \leq b$ two integers, $(-1, -1) \neq (a, b) \neq (-1, 0)$.*

- i) *The logarithmic sheaf associated with $c\mathcal{A}_\Phi^{[-a,b]}$ and its dual, twisted with $\mathcal{O}_{\mathbb{P}^n}(-(a+b+1)h)$, have the same graded Betti numbers;*
- ii) *The logarithmic sheaf associated with $c\mathcal{A}_\Phi^{[-a-1,b+1]}$, twisted with $\mathcal{O}_{\mathbb{P}^n}(h)$, also has the same graded Betti numbers.*
- iii) *Let Φ be a root system of rank n (for instance A_n). Then the projective dimension of the logarithmic sheaf associated with $c\mathcal{A}_\Phi^{[-a-1,b+1]}$ is the minimum among $n-1$ and $b-a-1$.*
- iv) *The logarithmic sheaf associated with $c\mathcal{A}_{A_n}^{[-a-1,b+1]}$ has a linear resolution.*

In fact, the best result would be to explicitly compute these Betti numbers; the conjectural values of these numbers are some binomial coefficients at least for root systems of type A_n . Together with T. Abe and J. Vallès, I recently obtained a proof of all conjectures for the root system A_2 . The expectation of writing explicitly a resolution for any Φ , a, b is less optimistic.

Example V.2. This example has been worked out with Macaulay2, [151]. The reader is redirected to the file <http://web.univ-pau.fr/~dfaenzi1/code/coxeter-arrangements-examples.m2> for the code of this computation. More experimental material on logarithmic derivations of generalized Coxeter arrangements of type $\mathcal{A}_\Phi^{[a,b]}$ for root systems of type A_2, A_3, A_4, B_2 , can be found in this file.

Let Φ be the root system of type A_4 , so the exponents are $(1, 2, 3, 4)$ and the Coxeter number is 5. Consider $\mathcal{A} = \check{\mathbb{P}}c\mathcal{A}_\Phi^{[1,1]}$, which consists of 11 hyperplanes in $\mathbb{P}^4$.

It turns out that $\mathcal{T}_{\mathbb{P}^n}(-\log \mathcal{A})$ and its dual twisted by $\mathcal{O}_{\mathbb{P}^4}(-5)$, are Steiner sheaves defined as cokernel of matrices say M and N of the form:

$$\mathcal{O}_{\mathbb{P}^4}(-5)^4 \rightarrow \mathcal{O}_{\mathbb{P}^4}(-4)^{10}.$$

However, N gives rise to 11 unstable hyperplanes, for in fact N is just the matrix presenting $\tilde{\Omega}_{\mathbb{P}^n}(\log \mathcal{A})$, tensored with $\mathcal{O}_{\mathbb{P}^n}(-4)$.

On the other hand M gives only 6 unstable hyperplanes! So the two sheaves under considerations are not isomorphic. However, they share the same Betti numbers, and they both fail to be locally free (in this case, along 20 lines).

V.2. Free divisors associated with fibrations. The following idea is based on some computations and discussions done this year with Enrique Artal, José Ignacio Cogolludo and Jean Vallès, aiming at the construction of free plane curves. The statement of the conjectures is due to Enrique Artal.

Conjecture V.3. *Let $D \subset \mathbb{P}^2$ be a curve such that there exists a locally trivial fibration $\varphi : \mathbb{P}^2 \setminus D \rightarrow Y$, where Y is a quasi-projective curve. Then D is free.*

This conjecture is true, even in higher dimension, for hyperplane arrangements. Such arrangements are called *fibre-type* according to terminology of Falk and Randell, [141]. Being fibre-type is in fact a combinatorial property, equivalent to being *supersolvable*, cf. [297], and supersolvable arrangements are free.

In fact, even more should be true, namely this conjecture generalizes to the orbifold setting in the following sense. Let $\varphi : U \rightarrow \Delta_n$ be a holomorphic map where U is a complex surface and Δ is an open disk in $\mathbb{C}$ such that the origin is an orbifold point of index n . Then φ is *partially locally trivial* if its pull-back by $t \mapsto t^{nm}$, for some m , is locally trivial.

Conjecture V.4. *Let $D \subset \mathbb{P}^2$ be a curve such that there exists a partially locally trivial fibration map $\varphi : \mathbb{P}^2 \setminus D \rightarrow X^{orb}$, where X^{orb} is a quasi-projective rational orbicurve. Then D is free.*

For instance, the Hesse arrangement $\mathcal{A}$ naturally arises when looking at the singular fibres of the pencil generated by a smooth complex cubic in $\mathbb{P}^2$ and its Hessian. Removing these fibres we obtain a locally trivial fibration $\mathbb{P}^2 \setminus D_{\mathcal{A}} \rightarrow \mathbb{P}^1 \setminus x_1, \dots, x_4$. Adding an arbitrary number of smooth cubics in this pencil to the Hesse arrangement still gives a free arrangement. The point is that this phenomenon should be general.

CHAPTER 2

Cohen-Macaulay bundles

This chapter is devoted to maximal Cohen-Macaulay modules and arithmetically Cohen-Macaulay bundles, a notion that ties together tightly algebraic geometry and commutative algebra. I will first give a survey of their main features, focusing on existence and classification problems. Doing so, I will give a couple of alternative proofs of well-known results; these are sketched in some detail for they look a bit simpler (for me!) than the original ones. Then, I will describe ACM bundles in detail in a special situation, showing that a certain surface in $\mathbb{P}^5$ admits only families of dimension at most 1 of such (indecomposable) bundles. This part has never appeared before, so it will be presented with a “full” proof.

I. ACM varieties and bundles

We first define ACM varieties, which are projective varieties whose graded coordinate ring has a minimal graded free resolution over $\mathcal{S}$ which is as short as it can be, namely of length equal to the codimension of the variety. We will then define in similar fashion ACM sheaves and MCM modules. After that, we will say some words on a class of particularly nice bundles, namely Ulrich bundles, characterized by the fact that the minimal graded free resolution is not only short but also linear.

I.1. ACM varieties. Let k be a field, and $m \geq 0$ and $n \geq 1$ be integers. Consider a subvariety $X \subset \mathbb{P}^n$ over k , embedded by the complete linear series defined by a divisor class h , and denote by I_X the homogeneous saturated ideal of X . We have defined the graded algebra $\mathcal{S}_X = \mathcal{S}/I_X$, and the ideal sheaf $\mathcal{I}_X/\mathbb{P}^n$.

Definition I.1. The variety X is *ACM* (for *arithmetically Cohen-Macaulay*) if $\mathcal{S}_X$ is a Cohen-Macaulay ring, i.e., $\text{depth}(\mathcal{S}_X) = m + 1$.

Equivalently, X is ACM if the projective dimension of $\mathcal{S}_X$ as an $\mathcal{S}$ -module equals $\text{codim}(X, \mathbb{P}^n) = n - m$, which is to say that the ideal I_X has a free resolution over $\mathcal{S}$ of the form:

$$0 \rightarrow F_{n-m} \rightarrow \cdots \rightarrow F_1 \rightarrow I_X \rightarrow 0.$$

This is also equivalent to the following vanishing conditions:

$$H_*^1(\mathbb{P}^n, \mathcal{I}_X/\mathbb{P}^n) = 0; \quad H_*^i(X, \mathcal{O}_X) = 0, \quad \text{for } 0 < i < m.$$

Of course this notion depends on the divisor class h on X .

Example I.2. The following examples are classical.

- i) Complete intersections in $\mathbb{P}^n$ are ACM.
- ii) Determinantal varieties are ACM, cf. for instance [234]. The resolution of S_X as an S -module is called the Eagon-Northcott complex.

The following example is also well-known. Take a reductive algebraic group G over an algebraically closed field k of characteristic zero and a projective manifold X homogeneous under the action of G (for a related result in positive characteristic see [274]).

Lemma I.3. *An equivariant embedding of X is ACM.*

PROOF. Let P be the parabolic subgroup of G given by the stabilizer of a point of X . An equivariant embedding of X is defined by a lined bundle $\mathcal{L}_\lambda$ associated with a dominant weight λ of G . If P is given by the choice of simple roots $\alpha_1, \dots, \alpha_s$ of G , then $\lambda = a_1 \lambda_1 + \dots + a_s \lambda_s$, where the λ_i 's are the corresponding fundamental weights and a_i 's are positive integers. So, denoting by V_λ the G -module given by the weight λ , we have $\mathbb{P}^n = \mathbb{P}(V_\lambda)$.

It is clear that $H_*^i(X, \mathcal{O}_X)$ for $0 < i < m$ (see e.g. [317]), so we should check $H_*^1(\mathbb{P}^n, \mathcal{O}_X) = 0$. This holds if the restriction map $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(t)) \rightarrow H^0(X, \mathcal{L}_{t\lambda})$ is surjective for any integer $t \geq 0$. In turn, this is clear since this map is the canonical projection:

$$H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(t)) \simeq S^t V_\lambda \rightarrow V_{t\lambda} \simeq H^0(X, \mathcal{L}_{t\lambda}).$$

□

I.2. ACM sheaves. Let us now proceed to define arithmetically Cohen-Macaulay sheaves. Again k is a field.

Definition I.4. Let $X \subset \mathbb{P}^n$ be an ACM subvariety of dimension $m > 0$ and let E be a coherent sheaf on a X . Then E is an *ACM sheaf* if E is locally maximal CM, and if E has no intermediate cohomology:

$$H_*^i(X, E) = 0, \quad \text{for all } 0 < i < m.$$

A sheaf E is ACM if $\Gamma_*(X, E)$ is a maximal Cohen-Macaulay (MCM) S -module, i.e., its depth is $m + 1$. If E is ACM, then $\Gamma_*(X, E)$ has a minimal graded free resolution of the form:

$$(I.1) \quad 0 \rightarrow F_{n-m} \rightarrow \dots \rightarrow F_0 \rightarrow \Gamma_*(X, E) \rightarrow 0.$$

The condition of being locally maximal CM amounts to ask that the depth of any localization of E equals the dimension of the corresponding local ring. It can be replaced by $H^0(X, E(t)) = 0$ for $t \ll 0$, for in this case $\Gamma_*(X, E)$ is finitely generated over S_X and $H_*^i(X, E) = 0$ for $0 < i < m$ makes it into an MCM

module. The fact that E is locally maximal CM is also equivalent to $H_*^i(X, E)$ being a finite dimensional vector space for all $0 < i < m$. If X is smooth, locally maximal CM sheaves are precisely locally free sheaves.

Example I.5. Of course, a vector bundle on a curve is an ACM sheaf. The next well-known examples will be fundamental for us.

- i) A line bundle on a projective space is an ACM sheaf. It is important to keep in mind the opposite implication, the already mentioned theorem of Horrocks [173], which asserts that an ACM indecomposable sheaf on $\mathbb{P}^n$, up to twist, is just $\mathcal{O}_{\mathbb{P}^n}$.
- ii) Spinor bundles on smooth quadrics are ACM sheaves. There are two non-isomorphic spinor bundles on a smooth quadric of even dimension, or just one of them in odd dimension, cf. [62, 257, 293].

This time too, something very precise can be said in the opposite direction. Indeed, an important result of Knörrer, [206] says that an indecomposable ACM sheaf on a smooth quadric hypersurface is, up to a twist, the structure sheaf or a spinor bundle. We refer to [256, 258] for the discussion of splitting criteria on quadrics, and to [196, 197] for spinor bundles in connection with full exceptional collections on quadrics.

- iii) Determinantal varieties support various ACM sheaves. The graded free resolution as $\mathbf{S}$ -modules of the associated MCM modules is given by the Buchsbaum-Rim complexes cf. for instance [119, 317].

Once again, in some cases one can go in the other direction: this works well for instance for projective hypersurfaces. For a modern account of this procedure, going back at least to [100, 103], we refer for e.g. to [28, 270]. Indeed, once given an ACM sheaf E over an integral hypersurface X embedded in $\mathbb{P}^n$ by ι , we have a minimal graded free resolution:

$$0 \rightarrow F_1 \xrightarrow{M} F_0 \rightarrow \Gamma_*(X, E) \rightarrow 0,$$

and its sheafification:

$$(I.2) \quad 0 \rightarrow F_1 \xrightarrow{M} F_0 \rightarrow \iota_*(E) \rightarrow 0,$$

where the F_i 's are split bundles. The determinant of the square matrix M above is a power of the defining equation of X , and this power is just the generic rank of E . We will see below a slightly more detailed survey on various kinds of determinantal representations of hypersurfaces.

I.3. Ulrich sheaves. Let again $X \subset \mathbb{P}^n$ be an m -dimensional ACM subvariety, $m > 0$, and E be a coherent sheaf on X . We say that E is *initialized* if $H^0(X, E) \neq 0$ and $H^0(X, E(-1)) = 0$, i.e. if $\Gamma_*(X, E)$ is zero in negative degrees and non-zero in positive degrees. Of course we have to specify with respect to which line bundle is a sheaf initialized, should it be unclear from the context. Any coherent sheaf E with no 0-dimensional torsion part has an *initialized*

twist, i.e. there is a unique integer t such that $E(t)$ is initialized. In particular an ACM sheaf has a normalized twist if $m > 0$.

Given an ACM sheaf E of rank r on X , its initialized twist $E(t_0)$ satisfies the inequality (cf. [305]):

$$(I.3) \quad h^0(X, E(t_0)) \leq r \deg(X).$$

Of course the degree of X here is the degree with respect to h , i.e. h^m .

Definition I.6. We say that E is *Ulrich* if equality is attained in (I.3).

Ulrich bundles have been studied intensively, and are related to many features of X . In first place, in view of [125], they are related to linear resolutions (and to the theory of Chow forms), which is to say that E is Ulrich if and only if all differentials in (I.1) are matrices of linear forms. In this case, the rank of the F_i 's is particularly easy to determine as some prescribed multiples of binomial coefficients. Ulrich bundles are characterized by an even more extreme cohomology vanishing than just intermediate cohomology, namely:

$$H^i(X, E(-i)) = H^j(X, E(-j-1)) = 0, \quad \text{for all } i > 0 \text{ and } j < m.$$

Second, Ulrich bundles are connected to Boij-Söderberg theory, cf. [122–124]. Indeed, by [285], the existence of a single Ulrich sheaf on a given integral projective variety X entails equality between the Boij-Söderberg cones of coherent sheaves on X and of a projective space of dimension $\dim(X)$.

Here is a (very partial) survey of varieties that admit Ulrich bundles.

- i) Any projective curve, over any field, admits an Ulrich sheaf, by [122, Corollary 4.5].
- ii) Hypersurfaces admit Ulrich bundles, [52, 170]. In general the rank of these bundles is exponential in the number of variables.
- iii) Veronese varieties admit Ulrich bundles. See [125, Section 5] for a proof relying on representation theory and on Borel-Bott-Weil's. See [122] for a very simple proof valid over arbitrary fields.
- iv) Many K3 surface (cf. [8]) and all smooth quartic surfaces (cf. [88]) admit Ulrich bundles of rank 2.

Ulrich bundles are known to exist also on determinantal varieties, on Grassmannians (work in progress of L. Costa and R. M. Miró-Roig), Segre-Veronese products, etc. The class of varieties admitting Ulrich bundles is closed under Segre products and transverse intersection. However a general existence result is far from being clear.

II. CM type of varieties

By analogy with the case of representations of quivers, keeping in mind the subdivision of quivers into finite, tame, and wild representation types (see

[36, 147] for a classification of quivers of the former two types related to simply laced Dynkin diagrams, and [193] for an analysis of the latter type), a notion of finite, tame, and wild representation type for ACM varieties has been proposed recently, cf. [72, 111].

Definition II.1. Let $X \subset \mathbb{P}^n$ be an integral ACM variety of positive dimension. Then X is said to be of *finite representation type*, or of *finite CM type* if it supports only finitely many isomorphism classes of indecomposable ACM sheaves, up to twist.

If X is not of this type, it is said to be of *tame representation type* or of *tame CM type* if, for each rank r , the indecomposable ACM sheaves of rank r form a finite number of families of dimension at most one, up to twist and isomorphism, and not all such families are zero-dimensional.

Finally if, up to twist and isomorphism, there are m -dimensional families of indecomposable ACM sheaves, for arbitrarily large m , then X is said to be of *wild representation type*, or of *wild CM type*.

In these definitions it makes sense to restrict, in the singular case, to locally free sheaves ACM sheaves, in which case one speaks of VB (vector bundle) type of a variety.

The simplest framework to test these notions is given by curves. A smooth projective curve in X of genus g embedded in $\mathbb{P}^n$ is either of finite, tame, or wild CM type according to whether g is 0, 1 or ≥ 2 .

- i) For $g = 0$ (i.e., for $X = \mathbb{P}^1$) of course we have finite CM type, as any vector bundle on $\mathbb{P}^1$ splits as a direct sum of line bundles. This works over any field.
- ii) For $g = 1$, X is of tame CM type. Indeed, a famous result of Atiyah (cf. [14, Theorem 7]) stipulates that, for any given pair of integers (r, d) , the set of isomorphism classes of indecomposable vector bundles on X having rank r and degree d is identified with X itself. Atiyah's theorem holds over any algebraically closed field.
- iii) For $g \geq 2$, X is of wild CM type. Indeed, once fixed (r, d) as above, the moduli space of stable bundles on X having rank r and degree d has dimension $r^2(g - 1) - 1$, so there are arbitrarily large families of non-isomorphic bundles of rank r on X .

These definitions can be naturally given also in the framework of local or graded rings, replacing ACM sheaves with MCM modules (or even in a more general setting: non-commutative etc). In the local analytic setting, over an algebraically closed field k of characteristic different from 2, a remarkable fact is that simple hypersurface singularities are precisely the local rings over k of finite CM type, see again [63, 206].

For surfaces, simple singularities are exactly the rational double points, indexed by the simply laced Dynkin diagrams, and the characterization of finite CM rings among surface singularities goes back to [13, 18, 127, 169], and works even in characteristic 2. In this case (say if $k = \mathbb{C}$) there is a bijection between CM modules over the coordinate ring of a rational double point and the irreducible representations of the associated subgroup of $\mathrm{SL}_2(\mathbb{C})$, called the McKay correspondence.

Minimally elliptic surface singularities are of tame CM type, cf. [101, 112, 195]. A detailed study of several classes of non-isolated surface singularities, some of which are of tame CM type, has been carried out recently, see [67].

Going back to curves, in the singular case Drozd and Greuel proved in [110, 111] that nodal projective curves are VB finite, tame or wild, according to their arithmetic genus being 0, 1, or ≥ 2 .

The finite-tame-wild trichotomy, however, does not take place in general for ACM sheaves over singular varieties. Indeed, [63, §4], quadric cones over a point have an infinite discrete set of ACM sheaves. It is unclear whether the trichotomy holds in the class of smooth projective ACM varieties.

II.1. Varieties of finite CM type. Let us now briefly describe the easiest situation, namely when our variety has finite CM type. It turns out that these varieties are classified, as we shall recall in a minute. We will then sketch a method to classify ACM sheaves on these varieties.

II.1.1. Classification of varieties of finite CM type. We have already encountered three classes of varieties of CM finite type, namely projective spaces, quadrics, and rational curves. In fact, these exhaust all three infinite series of ACM varieties of finite CM type, at least over an algebraically closed field of characteristic zero. A theorem of Eisenbud and Herzog gives the complete classification of such varieties, [121]. They are the following:

- i) three or fewer distinct points;
- ii) projective spaces;
- iii) smooth quadric hypersurfaces;
- iv) rational normal curves;
- v) the Veronese surface in $\mathbb{P}^5$;
- vi) a smooth cubic scroll in $\mathbb{P}^4$.

The theorem follows from [152] for $\dim(X) = 0$. For positive dimension, a key ingredient is the case of hypersurfaces, treated in [63]. Besides hypersurfaces, with the help of a theorem of Auslander [17] controlling the singularities of X , one reduces to the classification of varieties of minimal degree of Del Pezzo and Bertini (i.e., integral non-degenerate varieties attaining equality in the universal bound $\deg(X) \geq 1 + \mathrm{codim}(X)$, cf. [120] for an account of this notion). In turn CM-finite varieties of this kind are classified in [19].

II.1.2. *Classification of ACM bundles on varieties of CM finite type.* As we have just mentioned, the classification of ACM bundles on the Veronese surface in $\mathbb{P}^5$ is well-known, see [19, 291]. We give here an elementary proof based on Beilinson's theorem. It is valid over any field.

Proposition II.2. *Let E be an ACM indecomposable bundle on $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$. Then, up to twist by $\mathcal{O}_{\mathbb{P}^2}(2t)$ for a certain $t \in \mathbb{Z}$, the bundle E is isomorphic to one of the three bundles $\mathcal{O}_{\mathbb{P}^2}$, $\mathcal{O}_{\mathbb{P}^2}(1)$, $\Omega_{\mathbb{P}^2}(1)$.*

We use the theorem of Beilinson, [34]. We refer for instance to [177, Chapter 8] for a detailed description of this fundamental result. A nice description of it with several applications is also given in [4, 260].

This theorem states that, for a given bundle E on $\mathbb{P}^n$, there is a complex of coherent sheaves F , whose cohomology is E :

$$\dots \rightarrow F^{-1} \xrightarrow{d_0} F^0 \xrightarrow{d_1} F^1 \rightarrow \dots,$$

with $\mathcal{H}^0(F) \simeq E$ and $\mathcal{H}^i(F) = 0$ for $i \neq 0$. In other words F is acyclic except in degree 0, and the 0-th cohomology of F is E . For any j , the term F^j is:

$$(II.1) \quad F^j = \bigoplus_k H^{k+j}(\mathbb{P}^n, E(-k)) \otimes \Omega_{\mathbb{P}^n}^k(k).$$

The theorem of Beilinson also says that F is *minimal*, which means that all maps $\Omega_{\mathbb{P}^n}^k(k) \rightarrow \Omega_{\mathbb{P}^n}^k(k)$ induced by the differential of F is *zero*. This terminology comes from minimal graded free resolutions of modules over a polynomial ring. Indeed, such resolution is *minimal* if and only if any map of degree 0 extracted from the differentials vanishes, and this condition amounts to ask that there is no resolution with terms of smaller rank of the same module.

PROOF. Let E be an ACM indecomposable bundle on $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$. The complex F of Beilinson then reads:

$$\begin{aligned} & \mathcal{O}_{\mathbb{P}^2}(-1)^{h^2(\mathbb{P}^2, E(-2))} \\ & \rightarrow \Omega_{\mathbb{P}^2}(1)^{h^0(\mathbb{P}^2, E(-1))} \xrightarrow{d_0} \Omega_{\mathbb{P}^2}(1)^{h^1(\mathbb{P}^2, E(-1))} \xrightarrow{d_1} \Omega_{\mathbb{P}^2}(1)^{h^2(\mathbb{P}^2, E(-1))} \rightarrow \\ & \quad \oplus \\ & \quad \mathcal{O}_{\mathbb{P}^2}^{h^0(\mathbb{P}^2, E)} \end{aligned}$$

Since the complex is minimal, the central term $\Omega_{\mathbb{P}^2}(1)^{h^1(\mathbb{P}^2, E(-1))}$ is sent by d_1 to $0 \in F^1$. Likewise, the image of d_0 has intersection zero with $\Omega_{\mathbb{P}^2}(1)^{h^1(\mathbb{P}^2, E(-1))}$. Therefore, $\Omega_{\mathbb{P}^2}(1)^{h^1(\mathbb{P}^2, E(-1))}$ is a direct summand of E . It follows, since E is indecomposable, that $E \simeq \Omega_{\mathbb{P}^2}(1)$, or $H^1(\mathbb{P}^2, E(-1)) = 0$.

In the second case, we look at $E(2)$. By the same argument, $E(2) \simeq \Omega_{\mathbb{P}^2}(1)$, or $H^1(\mathbb{P}^2, E(1)) = 0$. Continuing this way, we deduce that, if $E(2t)$ is not isomorphic to $\Omega_{\mathbb{P}^2}(1)$ for any $t \in \mathbb{Z}$, then $H^1(\mathbb{P}^2, E(2t+1)) = 0$ for all $t \in \mathbb{Z}$.

But in this case E is ACM on $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$, so it splits by Horrocks's criterion. It is thus isomorphic to $\mathcal{O}_{\mathbb{P}^2}$ or $\mathcal{O}_{\mathbb{P}^2}(1)$ up to a twist by $\mathcal{O}_{\mathbb{P}^2}(2t)$. $\square$

II.2. Some varieties of wild CM type. The class of varieties which are known to be of wild representation type is quite large.

- i) Hypersurfaces of degree $d \geq 4$ in $\mathbb{P}^n$ with $n \geq 2$ are of wild CM type according to [92, 113]. The same happens for complete intersections in $\mathbb{P}^n$ of codimension ≥ 3 , having one defining polynomial of degree ≥ 3 .
- ii) Segre varieties are of wild CM type except for the well-known cases of finite CM type appearing in the list of Section II.1.1, according to [90]. We will see in a while that Segre-Veronese varieties have an interesting behaviour.
- iii) Smooth rational ACM surfaces in $\mathbb{P}^4$ other than the cubic scroll are of wild CM type, cf. [236]. Del Pezzo surfaces are CM wild, [267], see also [89].
- iv) The third Veronese embedding of any variety is of wild CM type, [237].
- v) Given integers $s > 0$ and $0 > a_1 \geq \dots \geq a_s$, the rational normal scroll $S(a_1, \dots, a_s)$ is the variety $\mathbb{P}(\oplus_i \mathcal{O}_{\mathbb{P}^1}(a_i))$ embedded in projective space by the tautological ample line bundle. Rational normal scrolls are almost always of wild CM type according to [235]. Indeed, this happens unless $s = 1$, or $s = 2$, and $a_1 + a_2 \leq 4$. It should be noted that, in [235], it is claimed that even $S(2, 2)$ and $S(1, 3)$ should be of wild CM type, while we will show in the next section that $S(2, 2)$ actually leads to a variety of tame CM type. The same should happen to $S(1, 3)$ (work in progress). However, the argument of [235] fails for these cases only.

III. Existence and classification problems for ACM bundles

We have mentioned in Section I the problem of existence of nontrivial ACM and Ulrich bundles, and in the previous section the question of measuring how many, or how large families of such bundles exist on a given variety. We would like to go here a bit further in this analysis, showing some known results on existence of ACM and Ulrich with special properties (small rank, symmetry, etc), and on their classification in some bounded region (rigid bundles, or again bundles of small rank), with a view towards related topics in real and convex algebraic geometry.

III.1. Hypersurfaces. We have mentioned the relation between ACM bundles and determinantal representations. Let us go into a bit more detailed account of this phenomenon.

III.1.1. Symmetric and linear determinants. Let us work over an algebraically closed field k of characteristic other than 2 for this subsection. Given an irreducible polynomial $f \in S_d$, it is clear by (I.2) an ACM line bundle E on $D = \mathbb{V}(f) \subset \mathbb{P}^n$ gives an expression of f as $\det(M)$. We call this a *determinantal*

representation of f , or of D , and we say that is *nontrivial* if the associated bundle is not a direct sum $\bigoplus \mathcal{O}_X(a_i)$ (which is to say that M is not just a diagonal matrix with f on the diagonal). The entries of M are linear forms if and only if E is Ulrich, in which case we speak of a *linear determinantal representation* of f . If f is not irreducible, one usually treats each component at a time.

While any plane curve $D \subset \mathbb{P}^2$ has a linear determinantal representation (we refer again e.g. to [28]), not every surface of degree $d \geq 4$ in $\mathbb{P}^3$ has determinantal representations (linear or not), nor does any smooth D in $\mathbb{P}^n$ for $n \geq 4$, as it is clear from Grothendieck-Lefschetz.

A smooth cubic surface D , instead, has 72 non-equivalent linear determinantal representations, in natural correspondence with double-sixers (cf. [106, Chapter 9]), and 27 determinantal representations with quadratic and linear forms (plus with their transpose), which are, as the reader will have undoubtedly imagined, in natural bijection with lines contained in D .

One can specify even more M , by requiring that it is symmetric. This amounts to ask that E be equipped with a symmetric duality $E \simeq E^*(t)$, for some $t \in \mathbb{Z}$. A symmetric determinantal hypersurface $\mathbb{V}(\det(M))$ in $\mathbb{P}^n$ is necessarily singular if $n \geq 3$. In fact singular determinantal surfaces, together with their double covers, have been deeply studied in [75], and play an important role (in a slightly generalized form) while seeking surfaces with many nodes, cf. [77].

On the other hand, any plane curve D in $\mathbb{P}^2$ has a linear symmetric determinantal representation. Indeed, D can be taken to be integral, and one just needs to find, on the normalization $\tilde{D}$ of D , a theta characteristic E (i.e. $E^{\otimes 2} \simeq \omega_{\tilde{D}}$), with $H^0(\tilde{D}, E) = 0$. But the existence of such E is guaranteed by Riemann's singularity theorem (cf. again [28] and references therein).

III.1.2. *Determinantal representations and LMIs.* Determinantal representations of polynomials play an important role in control theory and semidefinite programming via the notion of *linear matrix inequality* (LMI). Let us review this briefly here, referring to [311] for an excellent survey.

Fix $\mathbf{k} = \mathbb{R}$ and $\mathbf{S} = \mathbb{R}[\mathbf{x}_1, \dots, \mathbf{x}_n]$. Take M to be a symmetric matrix of size ℓ whose entries are affine linear forms in the variables $\mathbf{x}_1, \dots, \mathbf{x}_n$. Then the *spectrahedron* associated with M is the set:

$$\mathcal{S}_M = \{\mathbf{x} \in \mathbb{R}^n \mid M(\mathbf{x}) \succeq 0\},$$

where $M(\mathbf{x}) \succeq 0$ means that the evaluation of M at $\mathbf{x}$ is positive semidefinite. It is clear this locus is convex. If it has non-empty interior, then by conveniently reducing the size of M , we may assume that $M(\mathbf{x}) \succ 0$ for all $\mathbf{x}$ in the interior of $\mathcal{S}_M$. A convex set $C \subset \mathbb{R}^n$ (with non-empty interior) is said to admit an *LMI representation* if $C = \mathcal{S}_M$ for some M as above.

A polynomial $f \in \mathcal{S}$ is *real-zero* with respect to $\mathbf{x} \in \mathbb{R}^n$ if, for any $\mathbf{y} \in \mathbb{R}^n$, the following polynomial in t has only real solutions:

$$f(\mathbf{x} + t\mathbf{y}).$$

Given $0 \neq f \in \mathcal{S}$ and $\mathbf{x} \in \mathbb{R}^n$, we set $\mathcal{J}_{\mathbf{x}}(f)$ for the connected component containing $\mathbf{x}$ of $\{\mathbf{y} \in \mathbb{R}^n \mid f(\mathbf{y}) > 0\}$. A set $C \subset \mathbb{R}^n$ is called *algebraic interior* if $C = \mathcal{J}_{\mathbf{x}}(f)$ for some f as above.

It turns out that, if a convex set $C \subset \mathbb{R}^n$ with interior $I \neq \emptyset$ has an LMI representation M , then it is an algebraic interior with respect to a real-zero polynomial f which is positive at a point of I . Further, $\det(M) = f(\mathbf{x})g(\mathbf{x})$, for some polynomial g positive on I .

A beautiful result of Helton-Vinnikov (see [167]) asserts that, when $n = 2$, given any real-zero polynomial f , the algebraic interior $\mathcal{J}_{\mathbf{x}}(f)$ has an LMI representation, of size $\ell = \deg(f)$, and with $M(\mathbf{x}) = \mathbb{I}_{\ell}$. The result is rooted in [310], where a condition on existence of LMI representations was given in terms of nesting of the ovals of the real algebraic curve $\mathbb{V}(f)$. In turn, this analysis is a refinement of the fact, that we have already observed, that curves in complex projective plane have a linear determinantal representation: indeed positivity and reality issues will now forbid some choices of our theta characteristic E .

The construction has been carried out in an algorithmic form in [265]. For plane quartics, an explicit LMI representation can be constructed starting from bitangents, cf. [264]. For rational curves a very explicit description of this is given in [168]. For elliptic curves, see [280].

III.1.3. Rank two bundles and Pfaffian hypersurfaces. We mentioned symmetric determinants already, in connection with theta characteristics and convex geometry. Skew-symmetric matrices M of even size, on the other hand, are related to ACM bundles of rank 2, which is readily understood if we think that, on a point of $\mathbb{V}(\det(M))$, the matrix M will generally have corank 2, hence determine a rank-2 kernel (and cokernel). In this case the relevant invariant is the *Pfaffian* $\text{Pf}(M)$ of M , that satisfies $\text{Pf}(M)^2 = \det(M)$, so we speak of *Pfaffian and linear Pfaffian representations* of a hypersurface $D \subset \mathbb{P}^n$.

It is immediate to see that any plane curve has a linear Pfaffian representation, and one can even try to parametrize all such representations for a given smooth projective plane curve D . A detailed analysis of this parametrization in terms of the moduli space $M_D(2, \omega_D)$ has been carried out in [65, 66].

For surfaces, it is known that a *general* surface D of degree d in $\mathbb{P}^3$ has a linear Pfaffian representation if and only if $d \leq 15$, [28, Appendix]. The same result holds if we replace linear Pfaffian with “almost” linear Pfaffian, namely we allow one row and column of M to have higher degree, [131]. These two results are tightly related since, in both cases, for any rank-2 ACM bundle E on D giving such presentation, the vanishing locus of a general global

section of E vanishes on a codimension-2 subscheme of D , whose resolution as a codimension-3 subscheme of $\mathbb{P}^n$ is linear, cf. [60].

For surfaces of low degree d , the situation is analyzed in detail in [79, 132]. It turns out that, for any d , we have bounds $3 - d \leq c_1(E) \leq d - 1$ and the implications:

$$\begin{aligned} c_1(E) = 3 - d &\iff c_2(E) = 1, \\ c_1(E) = 4 - d &\iff c_2(E) = 2, \\ c_1(E) = d - 2 &\implies c_2(E) = \frac{d(d-1)(d-2)}{3}, \\ c_1(E) = d - 1 &\implies c_2(E) = \frac{d(d-1)(2d-1)}{6}. \end{aligned}$$

Ulrich bundles here correspond to extremal values $c_1(E) = d - 1$, $c_2(E) = \frac{d(d-1)(2d-1)}{6}$. For $d \leq 5$, any intermediate value of c_2 is actually attained.

If X is a smooth cubic surface $X \subset \mathbb{P}^3$, the classification is a bit more intricate. Indeed, the rich structure of $\text{Pic}(X)$ allows for existence of more classes ACM bundles E on X , and for some of these classes $c_1(E)$ is not a multiple of h . As it turns out, these are precisely the bundles that do not extend to a general cubic threefold containing X . The classification of all these cases is the content of [132].

For hypersurfaces of dimension $n - 1 \geq 3$, the situation is settled by the work of [238, 239] (cf. also [272]), together with [80, 227]. It turns out that, for $n = 4$ and $d \geq 6$, a general hypersurface X in $\mathbb{P}^4$ does not support ACM bundles of rank 2, so X has no Pfaffian representations. These bundles are classified for $d \leq 5$. For $n \geq 5$, the same non-existence phenomenon takes place for $d \geq 3$. For $n \geq 6$, only singular hypersurfaces can support such bundles (cf. [203]).

III.1.4. Higher rank ACM bundles on cubics. Stable Ulrich bundles of arbitrary rank have been found on cubic surfaces in [72]. The construction goes through a higher-rank analogue of the Hartshorne-Serre construction, in the spirit of [312]. It is shown in [73] that their moduli space of stable bundles, even when c_1 is not a multiple of h , is smooth and irreducible, as soon as it is non-empty. It is also proved in *loc. cit.* that stable Ulrich bundles exist of any rank exist on a general cubic threefold.

A more sophisticated approach was developed in [221] to study ACM bundles on cubic threefolds and cubic fourfolds containing a plane H , relying on the quadric bundle structure of the cubic X . In this setting, one considers the quadric bundle structure induced by the projection $X \dashrightarrow \mathbb{P}^2$ from a line in X (when $n = 4$), or from H (when $n = 5$), and looks at the induced sheaf of (even parts of) Clifford algebras $\mathcal{B}$ over $\mathbb{P}^2$. A semiorthogonal summand of the derived category $\mathbf{D}^b(X)$ of X has been described by Kuznetsov's in [218] as the

derived category of $\mathcal{B}$ -modules over $\mathbb{P}^2$, denoted by $\mathbf{D}^b(\mathbb{P}^2, \mathcal{B})$. In [221] this category is used to construct stable Ulrich bundles, and, when $n = 5$, to recover the K3 surface associated with X as a moduli space of Ulrich bundles on X .

III.2. Fano threefolds. A Fano manifold, for us, is a smooth connected complex projective variety X of dimension m , such that the anticanonical line bundle ω_X^* is ample. The interest in studying ACM bundles on Fano threefolds is perhaps rooted in the following remark: let i_X be the index of X , i.e. the largest integer such that $\omega_X \simeq \mathcal{O}_X(-i_X h)$, for some (ample) divisor class h on X . By a famous result of Kobayashi-Ochiai [207] (see also [208]), one has $1 \leq i_X \leq m + 1$, and if $i_X = m + 1$ then $X \simeq \mathbb{P}^m$, while if $i_X = m$ then X is a quadric hypersurface (cf. also [194] for the extension of this result to positive characteristic).

Since in the cases $i_X = m + 1$ and $i_X = m$ ACM bundles are well understood, one may hope for similar results in slightly lower index, at least for $m \leq 3$ for these varieties are then classified (cf. [188]). However, it soon appeared that even when $m = 2$ and $i_X = 1$ (del Pezzo surfaces) one should expect the situation to be much more complicated (cf. the already mentioned case of cubic surfaces).

In spite of this, for rank 2 some nice classification results are available. Indeed, ACM bundles of rank 2 on del Pezzo threefolds (i.e. $i_X = 2$) of Picard number 1 were classified in [11]. For higher Picard number, we tackled the problem in [74]. The main difficulty is to control the zero locus of a general section of the initialized twist of such a bundle, for in this case it might happen, in principle, that it vanishes along a divisor. It turns out, however, that this phenomenon happens only for decomposable bundles.

The situation for threefolds of index 1 and Picard number 1 is discussed in [49], where also some moduli spaces are computed (see also [12] for threefolds of maximal genus).

III.3. Classification of rigid ACM bundles on Veronese varieties. Here, we rephrase in terms of vector bundles a couple of very interesting results of Iyama and Yoshino, cf. [189, Theorem 1.2 and Theorem 1.3], on the classification of rigid indecomposable MCM modules over two Veronese embeddings in $\mathbb{P}^9$ given, respectively, by plane cubics and space quadrics. The theorem presented below is not exactly stated in the same way as the result just mentioned, however it is strictly equivalent to it. The proof of Iyama and Yoshino relies on techniques of cluster tilting.

There is also at least another proof, that makes use of Orlov's singularity category, cf. [200]. "At least", since [201] seems to contain yet another argument. In spite of the title of [200], these three papers go far beyond the scope of the next theorem.

The simple-minded proof presented here is essentially a refinement of Proposition II.2. I assume $k = \mathbb{C}$, because a couple of results are used where this assumption is made, namely [109, 161]. However, everything seems to work smoothly if k is algebraically closed or finite.

III.3.1. *ACM bundles on the third Veronese surface.* Consider $\mathbb{P}^2$, and the polarization h given by plane cubics, which is to say that h is associated with $\mathcal{O}_{\mathbb{P}^2}(3)$, so that the linear system $|\mathcal{O}_{\mathbb{P}^2}(h)|$ embeds $\mathbb{P}^2$ as the *third Veronese surface* in $\mathbb{P}^9$. A coherent sheaf E on $\mathbb{P}^2$ is ACM with respect to h if and only if E is locally free and :

$$(III.1) \quad H^1(\mathbb{P}^2, E(3t)) = 0, \quad \text{for all } t \in \mathbb{Z}.$$

We are going to classify simple ACM sheaves E on the third Veronese surface, with a special attention to the case of *rigid* bundles, namely when $\text{Ext}_{\mathbb{P}^2}^1(E, E) = 0$. To do this, we define the *Fibonacci numbers*, starting from an integer ℓ and setting:

$$a_{\ell,k} = \frac{(\ell + \sqrt{\ell^2 - 4})^k - (\ell - \sqrt{\ell^2 - 4})^k}{2^k \sqrt{\ell^2 - 4}}.$$

Equivalently, $a_{\ell,k}$ is defined by the relations:

$$a_{\ell,0} = 0, \quad a_{\ell,1} = 1, \quad a_{\ell,k+1} = \ell a_{\ell,k} - a_{\ell,k-1}.$$

For instance $(a_{3,k})_k$ is given by the odd values of the Fibonacci sequence:

k	0	1	2	3	4	5	6
$a_{3,k}$	0	1	3	8	21	55	144

Theorem III.1. *Let E be a bundle on $\mathbb{P}^2$ satisfying (III.1).*

- i) *If E is simple, then there are integers $a, b \geq 0$ such that, up to a twist by $\mathcal{O}_{\mathbb{P}^2}(s)$, E or E^* have a resolution of the form:*

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^2}(-2)^b \rightarrow \mathcal{O}_{\mathbb{P}^2}(-1)^a \rightarrow E \rightarrow 0.$$

- ii) *If E is rigid, then there is $k \geq 1$ such that, up to tensoring with $\mathcal{O}_{\mathbb{P}^2}(s)$, E or E^* have a resolution of the form:*

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^2}(-2)^{a_{3,k-1}} \rightarrow \mathcal{O}_{\mathbb{P}^2}(-1)^{a_{3,k}} \rightarrow E \rightarrow 0.$$

- iii) *Conversely for any $k \geq 1$, there is a unique indecomposable bundle E having resolution of the form:*

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^2}(-2)^{a_{3,k-1}} \xrightarrow{M} \mathcal{O}_{\mathbb{P}^2}(-1)^{a_{3,k}} \rightarrow E \rightarrow 0,$$

and both E and E^ are ACM and exceptional.*

Remark III.2. The rank of the bundle E_k is given the Fibonacci number between $a_{3,k-1}$ and $a_{3,k}$. Also, E_{2k} (respectively, E_{2k+1}) is the k -th sheafified

syzygy occurring in the resolution of $\mathcal{O}_{\mathbb{P}^2}(1)$ (respectively, of $\mathcal{O}_{\mathbb{P}^2}(2)$) over the Veronese ring, twisted by $\mathcal{O}_{\mathbb{P}^2}(3(k-1))$.

It should be noted that, in [189, Theorem 1.2 and Theorem 1.3], the ACM bundle E on the given Veronese variety is assumed to have a rigid module of global sections. This implies, respectively, $\text{Ext}_{\mathbb{P}^2}^1(E, E(3t)) = 0$, or $\text{Ext}_{\mathbb{P}^3}^1(E, E(2t)) = 0$, for all $t \in \mathbb{Z}$. A priori, this is a stronger requirement than just $\text{Ext}_{\mathbb{P}^n}^1(E, E) = 0$. However, our proof shows that the two conditions are actually equivalent for ACM bundles.

Also in (i), we actually need the weaker assumption that E has no endomorphism factoring through $\mathcal{O}_{\mathbb{P}^2}(3t)$ for any $t \in \mathbb{Z}$.

PROOF. Let us first prove (i). So let E be a simple vector bundle on $\mathbb{P}^2$ satisfying (III.1). Let E_0 be the initialized twist of E with respect to $\mathcal{O}_{\mathbb{P}^2}(1)$ and set $\alpha_{i,j} = h^i(\mathbb{P}^2, E_0(-j))$. Of course we have $\alpha_{0,j} = 0$ if and only if $j > 0$. The Beilinson complex F associated with E_0 , see (II.1) reads:

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^2}(-1)^{\alpha_{1,2}} \xrightarrow{d_0} \begin{array}{c} \mathcal{O}_{\mathbb{P}^2}(-1)^{\alpha_{1,2}} \\ \oplus \\ \Omega_{\mathbb{P}^2}(1)^{\alpha_{1,1}} \\ \oplus \\ \mathcal{O}_{\mathbb{P}^2}^{\alpha_{0,0}} \end{array} \xrightarrow{d_1} \begin{array}{c} \Omega_{\mathbb{P}^2}(1)^{\alpha_{2,1}} \\ \oplus \\ \mathcal{O}_{\mathbb{P}^2}^{\alpha_{1,0}} \end{array} \xrightarrow{d_2} \mathcal{O}_{\mathbb{P}^2}^{\alpha_{2,0}} \rightarrow 0.$$

The term consisting of three summands in the above complex sits in degree 0 (we call it “middle term”), and the cohomology of this complex is E_0 . By condition (III.1), at least one of the $\alpha_{1,j}$ is zero, for $j = 0, 1, 2$.

If $\alpha_{1,2} = 0$, then $d_0 = 0$. By minimality of the Beilinson complex the restriction of d_1 to the summand $\mathcal{O}_{\mathbb{P}^2}^{\alpha_{0,0}}$ of the middle term is also zero. Therefore $\mathcal{O}_{\mathbb{P}^2}^{\alpha_{0,0}}$ is a direct summand of E_0 , so $E_0 \simeq \mathcal{O}_{\mathbb{P}^2}$ by indecomposability of E .

If $\alpha_{1,1} = 0$, then the non-zero component of d_0 is just a map $\mathcal{O}_{\mathbb{P}^2}(-1)^{\alpha_{1,2}} \rightarrow \mathcal{O}_{\mathbb{P}^2}^{\alpha_{0,0}}$, and a direct summand of E_0 is the cokernel of this map. By indecomposability of E , in this case $E_0(-1)$ has a resolution of the desired form with $a = \alpha_{0,0}$ and $b = \alpha_{1,2}$.

Let us look at the case $\alpha_{1,0} = 0$. Note that the restriction of d_1 to $\Omega_{\mathbb{P}^2}(1)^{\alpha_{1,1}} \oplus \mathcal{O}_{\mathbb{P}^2}^{\alpha_{0,0}}$ is zero, which implies that a direct summand of E_0 (hence all of E_0 by indecomposability) has the resolution:

$$(III.2) \quad 0 \rightarrow \mathcal{O}_{\mathbb{P}^2}(-1)^{\alpha_{1,2}} \xrightarrow{d_0} \Omega_{\mathbb{P}^2}(1)^{\alpha_{1,1}} \oplus \mathcal{O}_{\mathbb{P}^2}^{\alpha_{0,0}} \rightarrow E_0 \rightarrow 0$$

and we have $\alpha_{2,j} = 0$ for $j = 0, 1, 2$. Now, we compute $\chi(E_0(-3)) = 3\alpha_{1,2} - 3\alpha_{1,1} + \alpha_{0,0}$, so:

$$h^0(\mathbb{P}^2, E_0^*) = h^2(\mathbb{P}^2, E_0(-3)) = 3\alpha_{1,2} - 3\alpha_{1,1} + \alpha_{0,0}.$$

If this value is positive, then there is a non-trivial morphism $g : E_0 \rightarrow \mathcal{O}_{\mathbb{P}^2}$, and since $\alpha_{0,0} \neq 0$ there also exists $0 \neq f : \mathcal{O}_{\mathbb{P}^2} \rightarrow E_0$. If E_0 is not isomorphic to $\mathcal{O}_{\mathbb{P}^2}$, then $f \circ g$ is not a multiple of the identity, so that E_0 is not simple.

Hence we may assume $3\alpha_{1,2} - 3\alpha_{1,1} + \alpha_{0,0}$, in other words $\alpha_{0,3} = 0$. Therefore, the Beilinson complex associated with $E_0(-1)$ gives a resolution:

$$0 \rightarrow E_0(-1) \rightarrow \Omega_{\mathbb{P}^2}(1)^{\alpha_{1,2}} \rightarrow \mathcal{O}_{\mathbb{P}^2}^{\alpha_{1,1}} \rightarrow 0.$$

It is classical to convert this resolution into the form we want. Namely we consider the diagram:

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & E_0(-1) & \longrightarrow & \Omega_{\mathbb{P}^2}(1)^{\alpha_{1,2}} & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}^{\alpha_{1,1}} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}^{3\alpha_{1,2}-\alpha_{1,1}} & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}^{3\alpha_{1,2}} & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}^{\alpha_{1,1}} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \\ & & \mathcal{O}_{\mathbb{P}^2}(1)^{\alpha_{1,2}} & = & \mathcal{O}_{\mathbb{P}^2}(1)^{\alpha_{1,2}} & & \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

From the leftmost column, it follows that E_0^* has a resolution of the desired form, with $a = 3\alpha_{1,2} - \alpha_{1,1}$ and $b = \alpha_{1,2}$. Claim (i) is thus proved.

Let us now prove (ii). By [109, Corollaire 7], if E is rigid then E is a direct sum of exceptional bundles (this result seems to be true over perfect fields). Therefore we can apply (i), so that E or E^* have a resolution by a matrix of linear forms.

Then, we apply a result of Kac [193] (that works if k is algebraically closed or finite), see also [46, 47] (over $\mathbb{C}$). Indeed, an indecomposable vector bundle E with a resolution as in (i) is rigid if and only if (a, b) is a real Schur root of the Kronecker quiver with two vertices and 3 arrows pointing in the same direction. This in turn is equivalent to ask that a and b are two consecutive Fibonacci numbers of the form $a_{3,k}$ and $a_{3,k-1}$.

In this case, there is a unique such bundle, up to isomorphism, let us call it E_k . It turns out that E_k is an exceptional bundle, and that a general matrix of linear forms $\mathcal{O}_{\mathbb{P}^2}(-2)^{a_{3,k-1}} \rightarrow \mathcal{O}_{\mathbb{P}^2}(-1)^{a_{3,k}}$ defines E_k , cf. [46]. This implies that E_k has natural cohomology (cf. [171, Corollaire 3.1], this holds if k is algebraically closed). So $H^1(\mathbb{P}^2, E_k(3t)) = 0$ for all $t \in \mathbb{Z}$ because:

$$\chi(E_k(3t)) = \frac{3t}{2}(3t(a_{3,k} - a_{3,k-1}) + a_{3,k} + a_{3,k-1}) \geq 0,$$

for all $t \in \mathbb{Z}$. This inequality, in turn, follows from the elementary fact that $a_{3,k} + a_{3,k-1} \leq 3(a_{3,k} - a_{3,k-1})$, easily proved by induction on k .

The fact that E^* is also ACM is obvious by Serre duality. $\square$

III.3.2. *ACM bundles on the second Veronese threefold.* The techniques we have just seen apply to the embedding of $\mathbb{P}^3$ in $\mathbb{P}^9$ by $|\mathcal{O}_{\mathbb{P}^3}(2)|$. This time, an ACM sheaf is a vector bundle E on $\mathbb{P}^3$ with:

$$(III.3) \quad H^1(\mathbb{P}^3, E(2t)) = H^2(\mathbb{P}^3, E(2t)) = 0, \quad \text{for all } t \in \mathbb{Z}.$$

Theorem III.3. *Let E be an indecomposable bundle on $\mathbb{P}^3$ satisfying (III.3).*

i) *If E is simple, then there are $a, b \geq 0$ such that, up to a twist by $\mathcal{O}_{\mathbb{P}^3}(s)$, E or E^* have a resolution of the form:*

$$0 \rightarrow \Omega_{\mathbb{P}^3}^2(1)^b \rightarrow \mathcal{O}_{\mathbb{P}^3}(-1)^a \rightarrow E \rightarrow 0.$$

ii) *If E is rigid, then there is $k \geq 1$ such that, up to a twist by $\mathcal{O}_{\mathbb{P}^3}(s)$, E or E^* have a resolution of the form:*

$$0 \rightarrow \Omega_{\mathbb{P}^3}^2(1)^{a_{6,k-1}} \rightarrow \mathcal{O}_{\mathbb{P}^3}(-1)^{a_{6,k-1}} \rightarrow E \rightarrow 0.$$

iii) *For any $k \geq 1$, there is a unique indecomposable bundle E having resolution:*

$$0 \rightarrow \Omega_{\mathbb{P}^3}^2(1)^{a_{6,k-1}} \rightarrow \mathcal{O}_{\mathbb{P}^3}(-1)^{a_{6,k}} \rightarrow E \rightarrow 0,$$

and both E and E^ are ACM and exceptional.*

PROOF. The proof goes in the same way as in the previous theorem. Namely we consider the initialized twist E_0 of E and we set $\alpha_{i,j} = h^i(\mathbb{P}^3, E_0(-j))$. If (III.3) gives $\alpha_{1,1} = \alpha_{2,1} = 0$, then $E_0(-1)$ has the desired resolution. On the other hand, if (III.3) tells $\alpha_{1,0} = \alpha_{2,0} = \alpha_{1,2} = \alpha_{2,2} = 0$, then we are left with a resolution of the form:

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-1)^{\alpha_{1,3}} \xrightarrow{d_0} \Omega_{\mathbb{P}^3}(1)^{\alpha_{1,1}} \oplus \mathcal{O}_{\mathbb{P}^3}^{\alpha_{0,0}} \rightarrow E_0 \rightarrow 0.$$

This time we also have $\alpha_{0,4} = 0$, and $\alpha_{1,4} = \alpha_{2,4} = 0$ again by (III.3), and simplicity of E gives $\alpha_{3,4} = h^0(\mathbb{P}^3, E_0^*) = 0$. So $E_0(-1)$ has a resolution of the form:

$$0 \rightarrow E_0(-1) \rightarrow \Omega_{\mathbb{P}^2}^2(2)^{\alpha_{1,3}} \rightarrow \mathcal{O}_{\mathbb{P}^3}^{\alpha_{1,1}} \rightarrow 0.$$

Then, using the same trick as in the proof of the previous theorem, we see that E_0^* has the desired resolution, with $a = 6\alpha_{1,3} - \alpha_{1,1}$ and $b = \alpha_{1,3}$.

This proves the first statement. The rest follows again by [46, 47, 193], where Drezet's theorem has to be replaced by [161]. $\square$

IV. A smooth projective surface of tame CM type

Here I will show that a smooth projective surface obtained as product of a line and a conic is of tame representation type. This result has been announced in [136]. However, the complete proof has not yet appeared, and for this reason I present it here in some detail.

IV.1. Segre-Veronese varieties. Given an integer s and a sequence $\mathbf{n} = (n_1, \dots, n_s)$ of s integers, we write $\mathbb{P}^{\mathbf{n}}$ for $\mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_s}$. Given a sequence $\mathbf{d} = (d_1, \dots, d_s)$ of s integers, we write $\mathcal{O}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{d}) = \mathcal{O}_{\mathbb{P}^{\mathbf{n}}}(d_1, \dots, d_s)$ for the line bundle on $\mathbb{P}^{\mathbf{n}}$ obtained by taking the tensor product of all the line bundles obtained as the pull-back to $\mathbb{P}^{\mathbf{n}}$ of $\mathcal{O}_{\mathbb{P}^{n_i}}(d_i)$ under the projection to the i -th factor $\mathbb{P}^{n_i}$, when i varies from 1 to s . If all the integers d_i are positive, we denote by $V_{\mathbf{n}}(\mathbf{d})$ the variety $\mathbb{P}^{\mathbf{n}}$ polarized by the divisor class $h = c_1(\mathcal{O}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{d}))$.

It can be shown, although we will not do it here, that $V_{\mathbf{n}}(\mathbf{d})$ is of wild representation type except in the following cases:

- (1) $s = 1$ and $\mathbf{d} = (1)$ (projective spaces);
- (2) $s = 1$ and $\mathbf{n} = (1)$ (rational normal curves);
- (3) $s = 1$ and $\mathbf{n} = \mathbf{d} = (2)$ (second Veronese surface);
- (4) $s = 2$ and $\mathbf{n} = \mathbf{d} = (1, 1)$ (quadric surface);
- (5) $s = 2$ and $\mathbf{n} = (1, 1)$ and $\mathbf{d} = (1, 2)$ (product of line and conic).

We have said that the first four cases are of finite CM type. In most cases, we have even learned which are the finitely many ACM bundles supported by these varieties.

The goal of this section is to prove that the last example is, instead, of tame CM type. This surface can also be seen as the scroll $S(2, 2)$, cf. (v) at Section II.2. Although this is actually work in progress, let us mention that the other quartic scroll $S(1, 3)$ in $\mathbb{P}^5$ also seems to be of tame CM type, even if structure of ACM bundles on it is slightly more complicated than on $S(2, 2)$.

We also mentioned the fact that, besides the elliptic curve, no other smooth projective variety of tame CM type is known. This, together with the analysis of some other cases (homogeneous spaces, K3 surfaces, etc) leads to formulate the following.

Conjecture IV.1. *Let $X \subset \mathbb{P}^n$ be a smooth projective positive-dimensional ACM variety of tame CM type. Then X is an elliptic curve or a quartic scroll in $\mathbb{P}^5$.*

Let us now go back to our theorem. So let U be a 2-dimensional $\mathbf{k}$ -vector space, and let $\mathbb{P}^1 = \mathbb{P}(U)$ so that $U^* = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$. Let $X = \mathbb{P}^1 \times \mathbb{P}^1$, $h = c_1(\mathcal{O}_X(1, 2))$ and consider $X = V_{1,1}(1, 2) = (X, h)$. As already mentioned, the polarized variety X sits in $\mathbb{P}^5$ as an ACM submanifold of degree 4, namely the rational quartic scroll $S(2, 2)$.

Theorem IV.2. *Let $(X, h) = V_{1,1}(1, 2)$.*

- (i) *Up to a twist, any indecomposable ACM bundle on X is isomorphic either to $\mathcal{O}_X$, or $\mathcal{O}_X(-1, 0)$ or $\mathcal{O}_X(-1, -1)$, or to an Ulrich bundle E_{ξ} fitting into:*

$$0 \rightarrow \mathcal{O}_X(0, -1)^a \rightarrow E_{\xi} \rightarrow \mathcal{O}_X(-1, 1)^b \rightarrow 0,$$

for some integers a, b with $|a - b| \leq 1$ and a class $\xi \in U^ \otimes \mathbf{k}^b \otimes \mathbf{k}^a$.*

- (ii) For any (a, b) with $a = b \pm 1$, there exists a unique indecomposable bundle of the form E_ξ as above, and moreover this bundle is exceptional.
- (iii) For any $a = b$, the isomorphism classes of indecomposable bundles of the form E_ξ , form a 1-dimensional rational family.

In particular, (X, h) is of tame representation type.

IV.2. ACM line bundles. Let us first classify ACM line bundles on X .

Lemma IV.3. *Let $\mathcal{L}$ be an initialized ACM line bundle on X . Then $\mathcal{L}$ is isomorphic to $\mathcal{O}_X$, or to $\mathcal{O}_X(0, 1)$ or $\mathcal{O}_X(0, 2)$ or to one of the two Ulrich line bundles $\mathcal{O}_X(1, 1)$ and $\mathcal{O}_X(0, 3)$.*

PROOF. Of course, any line bundle $\mathcal{L}$ on X is isomorphic to $\mathcal{O}_X(a, b)$ for some $a, b \in \mathbb{Z}$, and assuming $\mathcal{L}$ to be initialized amounts to assume that $a = 0$ and $b \geq 0$, or $a \geq 0$ and $0 \leq b \leq 1$.

In the first case $a = 0$, we have $H^1(X, \mathcal{L}(-2, -4)) \neq 0$ as soon as $b \geq 4$, so $\mathcal{L}$ ACM implies $0 \leq b \leq 3$. On the other hand $\mathcal{O}_X$, $\mathcal{O}_X(0, 1)$, $\mathcal{O}_X(0, 2)$ and $\mathcal{O}_X(0, 3)$ are immediately seen to be ACM on X , with moreover $\mathcal{O}_X(0, 3)$ is clearly Ulrich.

In the second case $b = 0$ implies $H^1(X, \mathcal{L}(-1, -2)) \neq 0$ if $a \geq 1$, while $b = 1$ implies $H^1(X, \mathcal{L}(-2, -4)) \neq 0$ if $a \geq 2$. This leaves $\mathcal{L} \simeq \mathcal{O}_X(1, 1)$ as the only case, and $\mathcal{L}$ is easily proved to be an Ulrich bundle on X . $\square$

IV.3. Computing resolutions of ACM bundles. Let π be the projection of $X = \mathbb{P}^1 \times \mathbb{P}^1$ onto the second factor, and recall that, by a result of Orlov [252]:

$$(IV.1) \quad \mathbf{D}^b(X) = \langle \pi^* \mathbf{D}^b(\mathbb{P}^1) \otimes \mathcal{O}_X(-1, 0), \pi^* \mathbf{D}^b(\mathbb{P}^1) \rangle.$$

In turn, we have $\mathbf{D}^b(\mathbb{P}^1) = \langle \mathcal{O}_{\mathbb{P}^1}(t-1), \mathcal{O}_{\mathbb{P}^1}(t) \rangle$, for any $t \in \mathbb{Z}$ by Beilinson's theorem, see for instance [177].

IV.3.1. The unbalanced exceptional collection. We define here two full exceptional collections in $\mathbf{D}^b(X)$, dual to each other, adapted to the study of ACM bundles on X . We denote:

$$\mathcal{E}_3 = \mathcal{O}_X, \quad \mathcal{E}_2 = \mathcal{O}_X(0, -1), \quad \mathcal{E}_1 = \mathcal{O}_X(-1, -1), \quad \mathcal{E}_0 = \mathcal{O}_X(-1, -2).$$

Then $(\mathcal{E}_3, \dots, \mathcal{E}_0)$ forms a strongly exceptional collection. We call it the *unbalanced exceptional collection*. In view of (IV.1), we have the semiorthogonal decomposition:

$$\mathbf{D}^b(X) = \langle \mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3 \rangle.$$

According to [150], the objects $\mathcal{F}_i$ of the dual collection $(\mathcal{F}_0, \dots, \mathcal{F}_3)$ are defined for all i by $\mathcal{F}_i = L_{\mathcal{E}_3^*} \cdots L_{\mathcal{E}_{i-1}^*} \mathcal{E}_i^*[-1]$. An easy computation gives:

$$\mathcal{F}_3 \simeq \mathcal{O}_X, \quad \mathcal{F}_2 \simeq \mathcal{O}_X(0, -1), \quad \mathcal{F}_1 \simeq \mathcal{O}_X(-1, 1)[-1], \quad \mathcal{F}_0 \simeq \mathcal{O}_X(-1, 0)[-1].$$

The Stokes matrix of the dual collection is:

$$(\chi(\mathcal{F}_i, \mathcal{F}_j))_{i,j} = \begin{pmatrix} 1 & 2 & 0 & -2 \\ & 1 & 2 & 0 \\ & & 1 & 2 \\ & & & 1 \end{pmatrix},$$

and in fact for all i we have:

$$\mathrm{Hom}_{\mathbf{D}^b(X)}(\mathcal{F}_1, \mathcal{F}_3[i]) = \mathrm{Hom}_{\mathbf{D}^b(X)}(\mathcal{F}_0, \mathcal{F}_2[i]) = 0.$$

IV.3.2. *Computing resolutions using the unbalanced collection.* Let $\Gamma \subset X \times \mathbb{P}^1$ be the graph of the projection $\pi : X \rightarrow \mathbb{P}^1$ onto the second factor. Note that $\mathcal{O}_\Gamma$ is given by:

$$(IV.2) \quad 0 \rightarrow \mathcal{O}_{X \times \mathbb{P}^1}(0, -1, -1) \rightarrow \mathcal{O}_{X \times \mathbb{P}^1} \rightarrow \mathcal{O}_\Gamma \rightarrow 0.$$

Denote by φ and ψ the projections of $X \times \mathbb{P}^1$ onto X and onto $\mathbb{P}^1$ respectively. Given a line bundle $\mathcal{L} = \mathcal{O}_{X \times \mathbb{P}^1}(a, b, c)$, we define the functor $\Psi_{\mathcal{L}} = R\varphi_*(\psi^*(-) \otimes \mathcal{L} \otimes \mathcal{O}_\Gamma)$. By computing $\Psi_{\mathcal{L}}$ on $\mathcal{O}_{\mathbb{P}^1}$ and $\mathcal{O}_{\mathbb{P}^1}(-1)$, since $\mathbf{D}^b(\mathbb{P}^1) = \langle \mathcal{O}_{\mathbb{P}^1}(-1), \mathcal{O}_{\mathbb{P}^1} \rangle$ we easily see that $\Psi_{\mathcal{L}}$ is the composition of $\pi^* : \mathbf{D}^b(\mathbb{P}^1) \rightarrow \mathbf{D}^b(X)$ and of the tensor product with $\mathcal{O}_X(a, b+c)$. In order to define π^* as $\Psi_{\mathcal{L}}$, we can thus choose $\mathcal{L} = \mathcal{O}_{X \times \mathbb{P}^1}(0, 1, -1)$. Similarly, the functor $\pi^*(-) \otimes \mathcal{O}_X(-1, 0)$ is given as $\Psi_{\mathcal{L}'}$ with $\mathcal{L}' = \mathcal{O}_{X \times \mathbb{P}^1}(-1, 0, 0)$. We have chosen here $\mathcal{L}$ and $\mathcal{L}'$ in order to adapt them to the unbalanced exceptional collection.

The derived dual of $\mathcal{O}_\Gamma$ in $\mathbf{D}^b(X \times \mathbb{P}^1)$ is $\mathcal{O}_\Gamma(0, 1, 1)[-1]$. Therefore, the right adjoint functor of $\Psi = \Psi_{\mathcal{L}}$ is:

$$\begin{aligned} \Psi^! &= R\psi_*(\varphi^*(-) \otimes \mathcal{O}_\Gamma(0, 1, 1)[-1] \otimes \mathcal{L}^* \otimes \psi^*(\omega_{\mathbb{P}^1}[1])) \simeq \\ &\simeq R\psi_*(\varphi^*(-) \otimes \mathcal{O}_\Gamma). \end{aligned}$$

Next, the factor $\pi^*(\mathbf{D}^b(\mathbb{P}^1)) \otimes \mathcal{O}_X(-1, 0)$ of $\mathbf{D}^b(X)$, is the image of the functor $\Theta = \Psi_{\mathcal{L}'}$. The left adjoint functor of Θ is:

$$\begin{aligned} \Theta^* &= R\psi_*(\varphi^*(-) \otimes \mathcal{O}_\Gamma(0, 1, 1)[-1] \otimes (\mathcal{L}')^* \otimes \varphi^*(\omega_X[2])) \simeq \\ &\simeq R\psi_*(\varphi^*(-) \otimes \mathcal{O}_\Gamma(-1, -1, 1)[1]). \end{aligned}$$

Given a coherent sheaf E on X , we have a functorial distinguished triangle:

$$(IV.3) \quad \Psi\Psi^!E \xrightarrow{\gamma} E \xrightarrow{\delta} \Theta\Theta^*E.$$

Using (IV.2), we see that the complex $\Psi^!E$, that lies in $\mathbf{D}^b(\mathbb{P}^1)$, fits into a functorial distinguished triangle:

$$(IV.4) \quad H^\bullet(X, E(0, -1)) \otimes \mathcal{O}_{\mathbb{P}^1}(-1) \rightarrow H^\bullet(X, E) \otimes \mathcal{O}_{\mathbb{P}^1} \rightarrow \Psi^!E.$$

Since $\Psi \simeq \pi^*$, we have that $\Psi\Psi^!E$ fits into a functorial distinguished triangle:

$$H^\bullet(X, E(0, -1)) \otimes \mathcal{O}_X(0, -1) \xrightarrow{\alpha} H^\bullet(X, E) \otimes \mathcal{O}_X \rightarrow \Psi\Psi^!E,$$

i.e., $\Psi\Psi^!E$ is the cone of α . Similarly, $\Theta\Theta^*E[-1]$ is the cone of a map β :

$$(IV.5) \quad H^\bullet(X, E(-1, -2)) \otimes \mathcal{O}_X(-1, 0) \xrightarrow{\beta} H^\bullet(X, E(-1, -1)) \otimes \mathcal{O}_X(-1, 1).$$

IV.3.3. *Splitting of ACM bundles.* Let us prove the first part of Theorem IV.2.

Proposition IV.4. *Let E be an indecomposable ACM bundle on (X, h) . Then, up to a twist, either E is isomorphic to $\mathcal{O}_X$, or $\mathcal{O}_X(-1, 0)$ or $\mathcal{O}_X(-1, -1)$, either there are integers a, b such that E fits into:*

$$0 \rightarrow \mathcal{O}_X(0, -1)^a \rightarrow E \rightarrow \mathcal{O}_X(-1, 1)^b \rightarrow 0.$$

PROOF. Set $a = h^1(X, E(0, -1))$ and $b = h^1(X, E(-1, -1))$.

Let us look at the cohomology of $\Psi\Psi^!E$. Since $H^1(X, E) = 0$, we have exact sequences:

$$(IV.6) \quad \cdots \xrightarrow{f_2} H^0(X, E) \otimes \mathcal{O}_X \xrightarrow{f_1} \mathcal{H}^0(\Psi\Psi^!E) \xrightarrow{f_0} \mathcal{O}_X(0, -1)^a \rightarrow 0,$$

$$(IV.7) \quad 0 \rightarrow \mathcal{H}^1(\Psi\Psi^!E) \rightarrow H^2(X, E(0, -1)) \otimes \mathcal{O}_X(0, -1) \rightarrow \cdots.$$

Looking at the cohomology of $\Theta\Theta^*E$, since $H^1(X, E(-1, -2)) = 0$, we get:

$$0 \rightarrow \mathcal{O}_X(-1, 1)^b \xrightarrow{g_0} \mathcal{H}^0(\Theta\Theta^*E) \xrightarrow{g_1} H^2(X, E(-1, -2)) \otimes \mathcal{O}_X(-1, 0) \xrightarrow{g_2} \cdots$$

and also:

$$(IV.8) \quad \cdots \rightarrow H^0(X, E(-1, -1)) \otimes \mathcal{O}_X(-1, 1) \rightarrow \mathcal{H}^{-1}(\Theta\Theta^*E) \rightarrow 0.$$

With a slight abuse of notation, we still denote by δ and γ the maps obtained by taking cohomology of (IV.3), and we have a long exact sequence:

$$(IV.9) \quad 0 \rightarrow \mathcal{H}^{-1}(\Theta\Theta^*E) \rightarrow \mathcal{H}^0(\Psi\Psi^!E) \xrightarrow{\gamma} E \xrightarrow{\delta} \mathcal{H}^0(\Theta\Theta^*E) \rightarrow \mathcal{H}^1(\Psi\Psi^!E) \rightarrow 0.$$

Claim IV.5. *Let I be the image of f_1 and J be the image of g_1 . Then:*

$$\mathcal{H}^0(\Psi\Psi^!E) = I \oplus \mathcal{O}_X(0, -1)^a, \quad \mathcal{H}^0(\Theta\Theta^*E) = J \oplus \mathcal{O}_X(-1, 1)^b.$$

PROOF OF CLAIM IV.5. We use (IV.6) to look more closely at $\mathcal{H}^0(\Psi\Psi^!E)$, and we have to prove that f_0 splits. To see this, we can write the cohomology of (IV.4):

$$\cdots \xrightarrow{h_2} H^0(X, E) \otimes \mathcal{O}_{\mathbb{P}^1} \xrightarrow{h_1} \mathcal{H}^0(\Psi^!E) \xrightarrow{h_0} \mathcal{O}_{\mathbb{P}^1}(-1)^a \rightarrow 0.$$

It suffices to check that h_0 splits since applying Ψ to the above sequence we get (IV.6). To check that h_0 splits, we let M be the image of h_1 and we prove that $\text{Ext}_{\mathbb{P}^1}^1(\mathcal{O}_{\mathbb{P}^1}(-1), M) = 0$. Note that this extension space is isomorphic to $H^1(\mathbb{P}^1, M(1))$, and that since M is dominated by $H^0(X, E) \otimes \mathcal{O}_{\mathbb{P}^1}$ we have a surjection:

$$H^0(X, E) \otimes H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) \twoheadrightarrow H^1(\mathbb{P}^1, M(1))$$

We deduce that $H^1(\mathbb{P}^1, M(1)) = 0$ so h_0 splits. In a similar way one proves that g_0 splits. $\square$

Now observe that the restriction of γ to the summand $\mathcal{O}_X(0, -1)^a$ of $\mathcal{H}^0(\Psi\Psi^!E)$ is injective, since $\ker(\gamma) \simeq \mathcal{H}^{-1}(\Theta\Theta^*E)$ is dominated by the bundle $H^0(X, E(-1, -1)) \otimes \mathcal{O}_X(-1, 1)$ in view of (IV.8), and there are no nontrivial maps $\mathcal{O}_X(-1, 1) \rightarrow \mathcal{O}_X(0, -1)$. Similarly, δ is surjective onto $\mathcal{O}_X(-1, 1)^b$. We can define thus the sheaves P and Q by the exact sequences:

$$(IV.10) \quad 0 \rightarrow Q \rightarrow J \rightarrow \mathcal{H}^1(\Psi\Psi^!E) \rightarrow 0,$$

$$(IV.11) \quad 0 \rightarrow \mathcal{H}^{-1}(\Theta\Theta^*E) \rightarrow I \rightarrow P \rightarrow 0,$$

and (IV.9) becomes:

$$0 \rightarrow P \oplus \mathcal{O}_X(0, -1)^a \xrightarrow{\gamma} E \xrightarrow{\delta} Q \oplus \mathcal{O}_X(-1, 1)^b \rightarrow 0.$$

Claim IV.6. We have $\text{Ext}_X^1(Q, \mathcal{O}_X(0, -1)) = \text{Ext}_X^1(\mathcal{O}_X(-1, 1), P) = 0$.

PROOF OF CLAIM IV.6. We show only $\text{Ext}_X^1(Q, \mathcal{O}_X(0, -1)) = 0$, the other vanishing being analogous. Note that this extension space is dual to $\text{Ext}_X^1(\mathcal{O}_X(0, -1), Q(-2, -2))$ which in turn is isomorphic to $H^1(X, Q(-2, -1))$. We have an exact sequence:

$$H^0(X, \mathcal{H}^1(\Psi\Psi^!E) \otimes \mathcal{O}_X(-2, -1)) \rightarrow H^1(X, Q(-2, -1)) \rightarrow H^1(X, J(-2, -1)),$$

and we show that the outer terms of this sequence vanish. To check that the leftmost term is zero, we use (IV.7) twisted by $\mathcal{O}_X(-2, -1)$. Taking global sections gives the result. To show that the rightmost term vanishes, we recall that J is the image of g_1 , we let K be the image of g_2 , and we use the exact sequences (obtained since $\Theta\Theta^*E[-1]$ is the cone of (IV.5)):

$$(IV.12) \quad 0 \rightarrow J \xrightarrow{g_1} H^2(X, E(-1, -2)) \otimes \mathcal{O}_X(-1, 0) \rightarrow K \rightarrow 0,$$

$$(IV.13) \quad 0 \rightarrow K \rightarrow H^2(X, E(-1, -1)) \otimes \mathcal{O}_X(-1, 1) \rightarrow \dots$$

Twisting these sequences with $\mathcal{O}_X(-2, -1)$ and taking cohomology, we get $H^1(X, J(-2, -1)) = 0$ since both $H^1(X, \mathcal{O}_X(-3, -1))$ and $H^0(X, \mathcal{O}_X(-3, 0))$ vanish. $\square$

Now, in view of the above claim, we deduce that E is the direct sum of E' and E'' , where E' fits into an exact sequence:

$$0 \rightarrow \mathcal{O}_X(0, -1)^a \rightarrow E' \rightarrow \mathcal{O}_X(-1, 1)^b \rightarrow 0,$$

and E'' fits into:

$$0 \rightarrow P \rightarrow E'' \rightarrow Q \rightarrow 0,$$

and satisfies:

$$H^1(X, E''(0, -1)) = H^1(X, E''(-1, -1)) = 0.$$

Letting t vary in $\mathbb{Z}$, and calculating for each twist $E(t, 2t)$ the spaces $H^1(X, E(t, 2t - 1))$ and $H^1(X, E(t - 1, 2t - 1))$, we use the process just described to split off from E finitely many extension bundles, until we reduce E to an ACM bundle E_0 such that:

$$(IV.14) \quad H^1(X, E_0(t, 2t - 1)) = 0, \quad H^1(X, E_0(t - 1, 2t - 1)) = 0, \quad \text{for all } t \in \mathbb{Z}.$$

The proof of the proposition will be completed by the following lemma, in combination with Lemma IV.3. $\square$

Lemma IV.7. *An ACM bundle E_0 satisfying (IV.14) splits as a direct sum of line bundles.*

PROOF. We set $E = E_0$, and borrow the notation from the proof of the above proposition, where we assume $a = b = 0$. We can assume that E is initialized, hence $H^0(X, E(-1, -2)) = 0$. We get:

$$(IV.15) \quad \begin{aligned} \mathcal{H}^{-2}(\Theta\Theta^*E) &= \mathcal{H}^{-1}(\Psi\Psi^!E) = 0, \\ \mathcal{H}^{-1}(\Theta\Theta^*E) &\simeq H^0(X, E(-1, -1)) \otimes \mathcal{O}_X(-1, 1). \end{aligned}$$

Rewriting (IV.6), since $a = 0$ we get:

$$(IV.16) \quad 0 \rightarrow H^0(X, E(0, -1)) \otimes \mathcal{O}_X(0, -1) \xrightarrow{f_2} H^0(X, E) \otimes \mathcal{O}_X \xrightarrow{f_1} I \rightarrow 0.$$

Claim IV.8. *Given P and Q as above and $E = E_0$ satisfying (IV.14), we have:*

$$(IV.17) \quad \text{Ext}_X^1(Q, P) = 0.$$

PROOF OF CLAIM IV.8. To show this, we apply $\text{Hom}_X(Q, -)$ to (IV.11), obtaining:

$$\text{Ext}_X^1(Q, I) \rightarrow \text{Ext}_X^1(Q, P) \rightarrow \text{Ext}_X^2(Q, \mathcal{H}^{-1}(\Theta\Theta^*E)).$$

We want to show that the outer terms of this exact sequence vanish. For the leftmost term, using (IV.16), we are reduced to show:

$$(IV.18) \quad \text{Ext}_X^1(Q, \mathcal{O}_X) \simeq H^1(X, Q(-2, -2))^* = 0,$$

$$(IV.19) \quad \text{Ext}_X^2(Q, \mathcal{O}_X(0, -1)) \simeq H^0(X, Q(-2, -1))^* = 0,$$

where the isomorphisms are given by Serre duality. For the rightmost term, making use (IV.15) we need to prove:

$$(IV.20) \quad \text{Ext}_X^2(Q, \mathcal{O}_X(-1, 1)) \simeq H^0(X, Q(-1, -3))^* = 0.$$

In order to prove these vanishing results, by (IV.10), it suffices to show:

$$(IV.21)$$

$$H^0(X, J(-2, -1)) = H^0(X, J(-1, -3)) = 0, \quad \text{for (IV.19), (IV.20),}$$

$$(IV.22)$$

$$H^1(X, J(-2, -2)) = H^0(X, \mathcal{H}^1(\Psi\Psi^!E) \otimes \mathcal{O}_X(-2, -2)) = 0, \quad \text{for (IV.18).}$$

Now, (IV.21) easily follows from taking global sections of (IV.12), twisted by $\mathcal{O}_X(-2, -1)$, or by $\mathcal{O}_X(-1, -3)$. On the other hand, the first vanishing required for (IV.22) follows from (IV.12) since $H^1(X, \mathcal{O}_X(-3, -2)) = 0$ and $H^0(X, K(-2, -2)) = 0$, which in turn is easily derived from (IV.13). The second vanishing appearing in (IV.22) follows taking global sections of (IV.7), twisted by $\mathcal{O}_X(-2, -2)$. $\square$

Let us proceed to finish the proof of our lemma. In view of (IV.17), we have that E is the direct sum of P and Q . To conclude, it remains to prove that P is a direct sum of line bundles (by splitting off P from E and using induction on the rank, the proof of our lemma will be settled). To check this we note that P is locally free (it is a direct summand of E) so I is torsion-free, hence by (IV.16) we have:

$$I \simeq \bigoplus_{i=1, \dots, r} \mathcal{O}_X(0, a_i),$$

for some integers $r \geq 1$ and $a_i \geq 0$. Using (IV.15), the exact sequence (IV.11) becomes:

$$0 \rightarrow H^0(X, E(-1, -1)) \otimes \mathcal{O}_X(-1, 1) \rightarrow \bigoplus_{i=1, \dots, r} \mathcal{O}_X(0, a_i) \rightarrow P \rightarrow 0.$$

Twisting this sequence by $\mathcal{O}_X(-1, -3)$, since $H^k(X, \mathcal{O}_X(-1, a_i - 3)) = 0$ for any a_i and any k , we get:

$$H^1(X, P(-1, -3)) \simeq H^0(X, E(-1, -1)).$$

On the other hand, this space must be zero since P is a direct summand of E , and $E = E_0$ satisfies (IV.14). We deduce that: $P \simeq \bigoplus_{i=1, \dots, r} \mathcal{O}_X(0, a_i)$. $\square$

Remark IV.9. Lemma IV.7 can be proved also by using the notion of regularity developed in [20, 21].

IV.4. Kronecker-Weierstrass canonical form for extension bundles. Let A, B be k -vector spaces, set $a = \dim(A)$, $b = \dim(B)$. We keep also in mind our 2-dimensional k -vector space U . The aim of this section is to classify the bundles E fitting into:

$$0 \rightarrow B \otimes \mathcal{O}_X(0, -1) \rightarrow E \rightarrow A^* \otimes \mathcal{O}_X(-1, 1) \rightarrow 0.$$

IV.4.1. Parametrizing rank-2 extension bundles. Let us now construct the basic family of extension bundles. First of all, note that:

$$\text{Ext}_X^1(\mathcal{O}_X(-1, 1), \mathcal{O}_X(0, -1)) \simeq H^1(X, \mathcal{O}_X(1, -2)) \simeq U^*.$$

Given $a \in U^*$, we have an extension bundle E_a of the form:

$$(IV.23) \quad 0 \rightarrow \mathcal{O}_X(0, -1) \rightarrow E_a \rightarrow \mathcal{O}_X(-1, 1) \rightarrow 0.$$

If $a \neq 0$ then E_a depends on the class a of a in the projective line $\mathbf{P} = \mathbb{P}(U^*)$ so we also write $E_a = E_{\mathbf{a}}$. There is a universal extension bundle $\mathcal{U}$ over $X \times \mathbf{P}$ parametrizing the bundles of the form E_a , in such a way that for all $\mathbf{a} \in \mathbf{P}$ we have $\mathcal{U}|_{X \times \{\mathbf{a}\}} \simeq E_{\mathbf{a}}$.

Call $\sigma : X \times \mathbf{P} \rightarrow \mathbf{P}$ and $\tau : X \times \mathbf{P} \rightarrow X$ the projections and consider the functor $\Phi : \mathbf{D}^b(X) \rightarrow \mathbf{D}^b(\mathbf{P})$ defined as $\Phi = R\tau_*(\sigma^*(-) \otimes \mathcal{U})$. The right adjoint functor of Φ is $\Phi^! = R\sigma_*(\tau^*(-) \otimes \mathcal{U}^* \otimes \sigma^*(\omega_{\mathbf{P}})[1])$.

Lemma IV.10. *The universal extension $\mathcal{U}$ fits into:*

$$(IV.24) \quad 0 \rightarrow \mathcal{O}_{X \times \mathbf{P}}(0, -1, 0) \rightarrow \mathcal{U} \rightarrow \mathcal{O}_{X \times \mathbf{P}}(-1, 1, -1) \rightarrow 0.$$

Furthermore Φ is fully faithful and we have $\Phi(\mathbf{D}^b(\mathbf{P})) = \langle \mathcal{F}_2, \mathcal{F}_1 \rangle$, and:

$$(IV.25) \quad \Phi(\mathcal{O}_{\mathbb{P}^1}) \simeq \mathcal{O}_X(0, -1), \quad \Phi(\mathcal{O}_{\mathbb{P}^1}(-1)) \simeq \mathcal{O}_X(-1, 1)[-1], \quad E_{\mathbf{a}} \simeq \Phi(\mathcal{O}_{\mathbf{a}}).$$

Finally, we have:

$$(IV.26) \quad \Phi^!(\mathcal{O}_X(0, -1)) \simeq \mathcal{O}_{\mathbb{P}^1}, \quad \Phi^!(\mathcal{O}_X(-1, 1)) \simeq \mathcal{O}_{\mathbb{P}^1}(-1)[1].$$

PROOF. For the first claim, we know that there are line bundles $\mathcal{L}$ and $\mathcal{L}'$ on $\mathbf{P}$ such that $\mathcal{U}$ fits into:

$$0 \rightarrow \tau^*(\mathcal{O}_X(0, -1)) \otimes \sigma^*(\mathcal{L}') \rightarrow \mathcal{U} \rightarrow \tau^*(\mathcal{O}_X(-1, 1)) \otimes \sigma^*(\mathcal{L}) \rightarrow 0.$$

Further, $\mathcal{U}$ is determined up to a twist by a line bundle on $\mathbf{P}$, so we can assume $\mathcal{L}' \simeq \mathcal{O}_{\mathbf{P}}$, and say $\mathcal{L} \simeq \mathcal{O}_{\mathbf{P}}(t)$. The extension corresponding to $\mathcal{U}$ is clearly non-zero and $\mathrm{PGL}_2(k)$ -invariant, and lies in:

$$\mathrm{Ext}_{X \times \mathbf{P}}^1(\mathcal{O}_{X \times \mathbf{P}}(-1, 1, t), \mathcal{O}_{X \times \mathbf{P}}(0, -1, 0)) \simeq U^* \otimes S^{-t} U,$$

where we take $S^{-t} U = 0$ for positive t . So $t = -1$, since only in this case $U^* \otimes S^{-t} U$ contains a non-zero invariant element.

For the second claim, first of all the isomorphisms (IV.25) are clear, for it suffices to compute $\Phi(\mathcal{O}_{\mathbb{P}^1})$ and $\Phi(\mathcal{O}_{\mathbb{P}^1}(-1))$ making use of (IV.24), and to observe that $\Phi(\mathcal{O}_{\mathbf{a}})$ is just $\mathcal{U}|_{X \times \{\mathbf{a}\}} \simeq E_{\mathbf{a}}$.

Note that (IV.25) amounts to say that $\Phi(\mathcal{O}_{\mathbf{P}}) \simeq \mathcal{F}_2$ and $\Phi(\mathcal{O}_{\mathbf{P}}(-1)) \simeq \mathcal{F}_1$. Observe now that:

$$\mathrm{Hom}_{\mathbf{D}^b(\mathbf{P})}(\mathcal{O}_{\mathbb{P}^1}(-1), \mathcal{O}_{\mathbb{P}^1}) \simeq \mathrm{Hom}_{\mathbf{D}^b(X)}(\mathcal{F}_1, \mathcal{F}_2) \simeq U^*.$$

Since $\mathbf{D}^b(\mathbf{P}) = \langle \mathcal{O}_{\mathbb{P}^1}(-1), \mathcal{O}_{\mathbb{P}^1} \rangle$, this implies that Φ is fully faithful, and that $\Phi(\mathbf{D}^b(\mathbf{P})) = \langle \mathcal{F}_2, \mathcal{F}_1 \rangle$. Finally, the isomorphisms (IV.26) are clear since $\Phi^! \circ \Phi$ is the identity functor. $\square$

Note that the isomorphisms (IV.26) can also be derived directly by computing $\Phi^!$ making use of the exact sequence:

$$0 \rightarrow \mathcal{O}_{X \times \mathbf{P}}(1, -1, -1) \rightarrow \mathcal{U}^* \otimes \sigma^*(\omega_{\mathbf{P}}) \rightarrow \mathcal{O}_{X \times \mathbf{P}}(0, 1, -2) \rightarrow 0.$$

IV.4.2. *Kronecker-Weierstrass theory for matrix pencils.* Consider the vector space $A \otimes B \otimes U^*$, and remark that:

$$\mathrm{Hom}_{\mathbb{P}^1}(A^* \otimes \mathcal{O}_{\mathbb{P}^1}(-1), B \otimes \mathcal{O}_{\mathbb{P}^1}) \simeq A \otimes B \otimes U^*.$$

Therefore, an element ξ of $A \otimes B \otimes U^*$ corresponds to a matrix M_ξ of linear forms on $\mathbb{P}^1$, or a matrix pencil:

$$M_\xi : A^* \otimes \mathcal{O}_{\mathbb{P}^1}(-1) \rightarrow B \otimes \mathcal{O}_{\mathbb{P}^1}.$$

The pencil $M = M_\xi$ can be classified according to its Kronecker-Weierstrass canonical form. Let us sketch this here, and refer to [68, Chapter 19.1] for proofs.

First of all, the classification takes place up to linear coordinate change on A and B , and one writes $M \simeq M' \boxplus M''$ if M is equivalent to a block matrix having M' and M'' on the diagonal and zero off the diagonal. Fixing variables x, y on $\mathbb{P}^1$, and given positive integers u, v , one defines:

$$\mathfrak{C}_u = \begin{pmatrix} x & & & & \\ y & x & & & \\ & y & \ddots & & \\ & & \ddots & x & \\ & & & & y \end{pmatrix}, \quad \mathfrak{B}_v = \begin{pmatrix} x & y & & & \\ & x & y & & \\ & & \ddots & \ddots & \\ & & & x & y \end{pmatrix},$$

where $\mathfrak{C}_u$ has size $(u+1) \times u$ and $\mathfrak{B}_v$ has size $v \times (v+1)$. Also, given $a \in \mathbf{k}$ and a positive integer n one defines:

$$J_{a,n} = \begin{pmatrix} a & 1 & & & \\ & \ddots & \ddots & & \\ & & a & 1 & \\ & & & & a \end{pmatrix} \in \mathbf{k}^{n \times n}, \quad \text{and:} \quad \mathfrak{J}_{a,n} = xI_n + yJ_{a,n}.$$

The next proposition is obtained combining [68, Theorem 19.2 and 19.3], with the caveat that, up to changing basis in $\mathbb{P}^1$, we can assume that a matrix pencil M has no *infinite elementary divisors*, i.e., the morphism $M_\xi : A^* \otimes \mathcal{O}_{\mathbb{P}^1}(-1) \rightarrow B \otimes \mathcal{O}_{\mathbb{P}^1}$ has constant rank around $\infty = (0 : 1) \in \mathbf{P}$.

Lemma IV.11. *Up to possibly changing basis in $\mathbb{P}^1$, any matrix pencil M is equivalent to:*

$$\mathfrak{C}_{u_1} \boxplus \cdots \boxplus \mathfrak{C}_{u_r} \boxplus \mathfrak{B}_{v_1} \boxplus \cdots \boxplus \mathfrak{B}_{v_s} \boxplus \mathfrak{J}_{n_1, a_1} \boxplus \cdots \boxplus \mathfrak{J}_{n_t, a_t} \boxplus \mathfrak{Z}_{a_0, b_0},$$

for some integers r, s, t, a_0, b_0 and u_i, v_j, n_k , and some $a_1, \dots, a_t \in \mathbf{k}$, where $\mathfrak{Z}_{a_0, b_0}$ is the zero pencil of size $a_0 \times b_0$.

We say that a matrix pencil is *irreducible* if only one summand appears in the above decomposition, and this summand is not $\mathfrak{Z}_{a_0, b_0}$ with $(a_0, b_0) \neq (1, 0)$ or $(0, 1)$.

We have the following straightforward isomorphisms:

$$\mathrm{coker}(\mathfrak{C}_u) \cong \mathcal{O}_P(u), \quad \ker(\mathfrak{B}_v) \cong \mathcal{O}_P(-v-1), \quad \mathrm{coker}(\mathfrak{J}_{n,a}) \cong \mathcal{O}_{na},$$

where $\mathcal{O}_{na}$ is the skyscraper sheaf over the point a with multiplicity n . Allowing a to vary in P instead of k only amounts to authorizing infinite elementary divisors too.

IV.4.3. *From extension bundles to matrix pencils.* We consider the space:

$$\mathrm{Ext}_X^1(A^* \otimes \mathcal{O}_X(-1, 1), B \otimes \mathcal{O}_X(0, -1)) \simeq A \otimes B \otimes U^*.$$

The extension associated to an element ξ of $A \otimes B \otimes U^*$ gives a bundle that we denote by E_ξ , fitting into:

$$(IV.27) \quad 0 \rightarrow B \otimes \mathcal{O}_X(0, -1) \rightarrow E_\xi \rightarrow A^* \otimes \mathcal{O}_X(-1, 1) \rightarrow 0.$$

In other words, ξ lies in $\mathrm{Hom}_{\mathbf{D}^b(X)}(A^* \otimes \mathcal{F}_1, B \otimes \mathcal{F}_2)$ and E_ξ is the cone of the morphism ξ .

Applying $\Phi^!$ to the exact sequence defining E_ξ , and using Lemma IV.10, we find a matrix pencil M_ξ . Equivalently, $\Phi^!$ transforms the morphism $\xi : A^* \otimes \mathcal{F}_1 \rightarrow B \otimes \mathcal{F}_2$ into the morphism $M_\xi : A^* \otimes \mathcal{O}_P(-1) \rightarrow B \otimes \mathcal{O}_P$. Likewise, applying Φ to a matrix pencil M_ξ and using again Lemma IV.10, we find an exact sequence like (IV.27). Clearly, these are mutually inverse equivalences.

Lemma IV.12. *Set $E = E_\xi$, let $M = M_\xi$ be the associated matrix pencil and set $\mathcal{F} = \Phi^!(E)$. Then $E \simeq \Phi(\mathcal{F})$, and, for all i , we have:*

$$(IV.28) \quad \mathrm{Ext}_X^i(E, E) \simeq \mathrm{Ext}_P^i(\mathcal{F}, \mathcal{F}).$$

Further, E is a semistable Ulrich bundle, which is indecomposable if and only if M is irreducible, and this happens if and only if $\mathcal{F} \simeq \mathcal{O}_P(t)$ for some t up to a shift, or $\mathcal{F} \simeq \mathcal{O}_{na}$ for some $n \geq 1$ and $a \in P$.

PROOF. The isomorphism $E \simeq \Phi(\mathcal{F})$ is clear by Lemma IV.10, since E lies in $\langle \mathcal{F}_2, \mathcal{F}_1 \rangle$, and the restriction of $\Phi^!$ to $\langle \mathcal{F}_2, \mathcal{F}_1 \rangle$ is the inverse functor of $\Phi : \mathbf{D}^b(P) \rightarrow \mathbf{D}^b(X)$, whose image is $\langle \mathcal{F}_2, \mathcal{F}_1 \rangle$. Since Φ is fully faithful, we immediately get (IV.28).

Next, note that E is an ACM bundle, since the exact sequence (IV.27) immediately gives $H_*^1(X, E) = 0$, and moreover $E(1, 2)$ is initialized and in fact an Ulrich bundle, since again (IV.27) gives $H^0(X, E) = 0$ and $h^0(X, E(1, 2)) = 4(a + b) = \deg(X) \mathrm{rk}(E)$.

Furthermore, E is semistable, since it is an extension of line bundles having the same Hilbert polynomial.

Let us now look at indecomposability of E . If $M = M' \boxplus M''$ then $A = A' \oplus A''$ and $B = B' \oplus B''$ and there are $\xi' \in A' \otimes B' \otimes U^*$ and $\xi'' \in A'' \otimes B'' \otimes U^*$ such that $M' = M_{\xi'}$ and $M'' = M_{\xi''}$. It is clear that $E \simeq E_{\xi'} \oplus E_{\xi''}$. This proves that M is irreducible if E is indecomposable.

Conversely, assume that M is irreducible. Then M is equivalent to $\mathfrak{C}_u$ for some $u \geq 1$, or $\mathfrak{B}_v$ for some $v \geq 1$, or $\mathfrak{J}_{n,a}$ for some $n \geq 1$ and $a \in \mathbf{P}$, or to $\mathfrak{J}_{1,0}$ or to $\mathfrak{J}_{0,1}$. Let us analyze each case.

First, let us look at $\mathfrak{C}_u$. In this case we have $\mathcal{F} \simeq \text{coker}(\mathfrak{C}_u) \simeq \mathcal{O}_{\mathbf{P}}(u)$ (with $u \geq 1$). Using (IV.28) we get that E is an exceptional bundle, hence E is simple, and a fortiori indecomposable. For the case $\mathfrak{B}_v$, we have $\mathcal{F} \simeq \ker(\mathfrak{B}_v)[1] \simeq \mathcal{O}_{\mathbf{P}}(-v-1)[1]$ (with $v \geq 1$). Again $\mathcal{F}$ is exceptional, hence so is E by (IV.28), so E is indecomposable too. The same argument works for $\mathfrak{J}_{1,0}$ in which case we get $\mathcal{F} \simeq \mathcal{O}_{\mathbf{P}}(-1)[1]$ and $E \simeq \mathcal{O}_X(-1, 1)$, and for $\mathfrak{J}_{0,1}$, that gives $\mathcal{F} \simeq \mathcal{O}_{\mathbf{P}}$ and $E \simeq \mathcal{O}_X(0, 1)$.

Let us look at the case when M is of the form $\mathfrak{J}_{n,a}$, so that $\mathcal{F} \simeq \mathcal{O}_{na}$. Note that $\mathcal{F}$ is filtered by the sheaves $\mathcal{G}_m = \mathcal{O}_{ma}$, for $1 \leq m \leq n$ and $\mathcal{G}_m/\mathcal{G}_{m-1} \simeq \mathcal{G}_1$. This induces a filtration of E by the sheaves $E_m = \Phi(\mathcal{G}_m)$ having quotients $E_m/E_{m-1} \simeq E_a$ (recall (IV.25)). Note that E_a is a simple bundle with $\text{Ext}_X^1(E_a, E_a) \simeq \mathbf{k}$ by (IV.28), and that (IV.23) gives its Jordan-Hölder filtration. Therefore, any indecomposable summand of E must have a Jordan-Hölder filtration with quotients of the form $\mathcal{O}_X(0, -1)$ or $\mathcal{O}_X(-1, 1)$, i.e., it must be of the form $E_{\xi'}$ for some $\xi' \in A' \otimes B' \otimes U^*$, with $A = A' \oplus A''$ and $B = B' \oplus B''$. This would induce a decomposition of M into $M' \boxplus M''$, a contradiction.

The proof of the lemma is now finished. $\square$

IV.4.4. *Classification of extension bundles.* The next result proves the remaining claims of Theorem IV.2.

Proposition IV.13. *Let ξ be an element of $A \otimes B \otimes U^*$ and set $E = E_{\xi}$. Then E is a semistable Ulrich bundle.*

Further, if E is indecomposable, then $\text{Ext}_X^2(E, E) = 0$, $|a - b| \leq 1$ and:

- (i) *if $a = b \pm 1$ then E is exceptional;*
- (ii) *if $a = b$ then E varies in a 1-dimensional family.*

PROOF. We have already proved that E is a semistable Ulrich bundle. Consider another element ξ' of $A \otimes B \otimes U^*$, define $E' = E_{\xi'}$, and assume that E_{ξ} is also indecomposable. Setting $\mathcal{F} = \Phi^1(E)$ and $\mathcal{F}' = \Phi^1(E')$. Using that Φ is fully faithful and $E \simeq \Phi(\mathcal{F})$ and $E' \simeq \Phi(\mathcal{F}')$ we strengthen (IV.28) to:

$$\text{Ext}_X^i(E, E') \simeq \text{Ext}_{\mathbf{P}}^i(\mathcal{F}, \mathcal{F}'),$$

for all i . Therefore, using the analysis of Lemma IV.12, we get $\text{Ext}_X^2(E, E') = 0$ by dimension reasons unless $\mathcal{F} \simeq \mathcal{O}_{\mathbf{P}}(-v-1)$ for some $v \geq 1$, in which case this Ext^2 space is isomorphic to $H^1(\mathbf{P}, \mathcal{F}'(v+1))$. This space is zero unless $\mathcal{F}' \simeq \mathcal{O}_{\mathbf{P}}(-v'-1)$, for some $v' \geq 3$ and in this case this H^1 space is dual to $H^0(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(v'-v-2))$. This shows $\text{Ext}_X^2(E, E) = 0$ in case E is indecomposable.

By the classification of Lemma IV.11, together with the correspondence established by Lemma IV.12, we see that, if E is indecomposable, then the associated matrix pencil is of the form $\mathfrak{C}_u$ for some $u \geq 1$ or $\mathfrak{B}_v$ for some $v \geq 1$, or $\mathfrak{J}_{1,0}$ or $\mathfrak{J}_{0,1}$ (and in all these cases $|a - b| = 1$) or of the form $\mathfrak{J}_{a,n}$ for some $a \in \mathbf{P}$ and $n \geq 1$, (in which case $a = b$). In case $|a - b| = 1$, we have seen in the case-by-case analysis of Lemma IV.12 that the associated bundle E is exceptional. In case $a = b$, again in the proof of Lemma IV.12 we have seen that the bundle E is given as $\Phi(\mathcal{O}_{na})$, and the deformation space of E is exactly that of $\mathcal{O}_{na}$, i.e., the motions of a in the projective line $\mathbf{P}$ itself. $\square$

V. Open questions

We have sketched already several problems and conjectures throughout the chapter. Let us outline a few more here.

V.1. ACM and Ulrich bundles. A challenging problem, appearing as main conjecture in [125], is to determine whether any given variety X admits Ulrich bundles at all.

It seems also difficult to determine what would be the minimal rank of such bundle, and more generally to determine the minimal rank of an ACM bundle, not containing line bundles as direct summands. In fact an important conjecture stated in [63] asserts that, if X is a smooth hypersurface in $\mathbb{P}^n$, this rank should be at least $2^{\lfloor \frac{n}{2} \rfloor - 1}$. The bound would be sharp, for smooth quadric hypersurfaces support spinor bundles, which have precisely this rank.

We propose the following “generic” analogue.

Conjecture V.1. *Let X be a sufficiently general hypersurface in $\mathbb{P}^n$ of degree $d \gg 0$, and E an ACM non-split vector bundle on X . Then, the rank of E is at least 2^{n-2} .*

A first step would be to prove that, for $n = 4$, a threefold hypersurface of degree $d \gg 0$ supports no ACM bundles of rank 3 (the case of rank 2 being settled, as already mentioned, in [239]). Anyway, the bundle is sharp also in this case, for the ACM bundle associated with a point contained in X has rank precisely 2^{n-2} and is, in general, not split. This conjecture is somehow related to the rank conjecture of Buchsbaum-Eisenbud-Horrocks, see [60, Page 453] and [164, Problem 24].

Another interesting question is whether smooth cubic hypersurfaces in $\mathbb{P}^n$, for $n \geq 3$, are always of wild representation type (as for higher degree this is settled, as we have said, cf. [92]).

V.2. Families of determinantal varieties. We mentioned already several problems concerning determinantal varieties or more generally degeneracy loci of morphisms of vector bundles $\phi : E \rightarrow F$ on a connected projective manifold W . A general issue concerning these varieties is the question of when

one can parametrize completely their infinitesimal families by morphisms in a neighbourhood of ϕ , or if our degeneracy locus X_ϕ can deform (flatly) to a non-determinantal variety.

So let $u = \text{rk}(\mathcal{E})$, $v = \text{rk}(\mathcal{F})$, assume $u \geq v$ and choose an integer $0 \leq r \leq v - 1$. Assume that for a general choice of ϕ the variety $D_r(\phi)$ is of the expected dimension and that its singularities also lie in the expected dimension (this can be ensured, at least in characteristic zero, by asking $\mathcal{E}^* \otimes \mathcal{F}$ to be globally generated).

Denote by $\mathcal{Y}$ a smooth variety representing an open dense subset of the quotient $\text{Hom}_W(\mathcal{E}, \mathcal{F}) / \text{Aut}(\mathcal{E}) \times \text{Aut}(\mathcal{F})$ (since this group is in general not reductive, we are contented with “some” open set, which exists by [275]). Write $\mathcal{H}$ for the union of components of the Hilbert scheme of subschemes of W having the same Hilbert polynomial as X_ϕ , for general ϕ . Let $\rho : \mathcal{Y} \dashrightarrow \mathcal{H}$ be the rational map sending $[\phi]$ to $D_r(\phi)$.

Problem V2. *For which choices of $\mathcal{E}$, $\mathcal{F}$, r and W is ρ birational?*

This problem is rooted in early work of Ellingsrud [126], who studied the case $W = \mathbb{P}^n$ with $n \geq 3$ and $\mathcal{E}$, $\mathcal{F}$ split bundles, and $r = v - 1 = u - 2$. Ellingsrud’s result, relying on Hilbert-Burch’s theorem (cf. [119, Chapter 20.4]), says that ACM subvarieties X of codimension 2 in $\mathbb{P}^n$, with $n \geq 3$ fill in a smooth open subset of the Hilbert scheme, which is covered by connected subsets determined by the Betti numbers of X . The answer is also affirmative when $\mathcal{E}$ and $\mathcal{F}$ are split bundles (with a certain positivity condition), $W = \mathbb{P}^n$, $r = v - 1$ and $n + u \geq v + 3$ (which amounts to $\dim(X_\phi) \geq 2$), as proved in [134] (this was conjectured in [204, 205]). When $W = \mathbb{P}^n$, $\mathcal{F}$ is trivial, $\mathcal{E} \simeq \mathcal{T}_{\mathbb{P}^n}(-2)$ and $r = v - 1$, the answer is also affirmative for a wide range of choices of n, v (cf. [27, 135, 294] for the precise range). However answers for more general choices of r are lacking.

V3. Generalized Lax conjecture. We go back to LMI representations of Section III.1.2, and we borrow notation from there. The result of Helton-Vinnikov originally answered a question of P. Lax. A generalization to higher dimension would sound as follows.

Conjecture V3. *Let $f \in S$ be a real-zero polynomial and $\mathbf{x} \in \mathbb{R}^n$ with $f(\mathbf{x}) = 1$. Then there is a real-zero polynomial g , with $\mathcal{J}_{\mathbf{x}}(f) \subset \mathcal{J}_{\mathbf{x}}(g)$, such that $f g$ has an LMI representation M of size $\ell = \deg(f g)$ with $M(\mathbf{x}) = \mathbb{I}_\ell$.*

The relevant theory of determinantal representations of singular hypersurfaces has been developed in [202]. It should be noted that the naive generalization of Helton-Vinnikov’s result, namely by taking $g = 1$, fails by easy dimension counts (cf. for instance [266]), and even taking g to be a power of f will not be enough as shown in [51].

Another approach in this direction is to study in more detail some particular situations, such as Itenberg-Degtyarev's [96] construction of a linear symmetric real quartic surface whose 10 nodes all lie in the innermost oval. Similar constructions for singular real plane curves are unknown, even for rational curves.

CHAPTER 3

Odd instantons on Fano threefolds

In this chapter I will describe a class of vector bundles defined on certain threefolds, that may serve as generalization of instanton bundles on projective 3-space. The overview appearing in the next chapter is inspired mostly on [48, 50, 133]. My personal interest for this subject goes back my thesis, and was first stimulated by a series of lectures on the cubic threefold delivered in Florence by Fabio Bardelli, many years ago. I should also mention that some of the topics covered here are studied as well in the independent work of Kuznetsov, cf. [220].

The main focus will be on Fano threefolds of genus 10. The material appearing in Section II devoted to this class of manifolds has never appeared before. Accordingly, the new results come with (hopefully) full proofs.

Throughout the chapter the base field will be $\mathbb{C}$.

I. Introduction to even and odd instantons

Instanton bundle have been widely studied by several authors, starting from the foundational papers [15, 16], relating instantons on $\mathbb{P}^3$ satisfying a reality condition to self-dual Yang-Mills $\mathrm{Sp}(1)$ -connections over the 4-dimensional sphere via the Penrose-Ward twistor transform. In terms of algebraic geometry, an *instanton bundle* on $\mathbb{P}^3$ is a slope-stable vector bundle E of rank 2 with trivial determinant and $H^1(\mathbb{P}^3, E(-2)) = 0$. If $c_2(E) = k$, we say that E is a k -instanton, or an *instanton of charge k* and we denote the *moduli space of k -instantons* by $\mathrm{MI}_{\mathbb{P}^3}(k)$. This space is a subscheme of the Maruyama's moduli space of stable sheaves, which is a quasi-projective algebraic variety. We refer to [178] for an exhaustive treatment. The space of instantons is defined by open conditions in this space, namely a cohomology vanishing and slope-stability.

Alternatively a k -instanton can be defined as the cohomology of a self-dual *monad* (see [26]), i.e., a complex of coherent sheaves with cohomology in degree zero only, of the following form:

$$\mathcal{O}_{\mathbb{P}^3}(-1)^k \xrightarrow{JA^t} \mathcal{O}_{\mathbb{P}^3}^{2k+2} \xrightarrow{A} \mathcal{O}_{\mathbb{P}^3}(1)^k,$$

where J is a fixed skew-symmetric duality of $\mathbb{C}^{2k+2}$. So, the abstract concept of *moduli space* of instantons of charge k is translated in the following concrete

terms: the space of matrices A of size $(2k+2) \times k$, whose entries are linear forms, such that A is surjective at every point and satisfies $AJA^t = 0$, up to coordinate change in $\mathbb{C}^{2k+2} \times \mathbb{C}^k$.

The history of the study of $\text{MI}_{\mathbb{P}^3}(k)$ is quite long; however a culminating point has been reached recently, for a number of long-standing conjectures concerning nice properties of this space have now been established. Indeed, smoothness of $\text{MI}_{\mathbb{P}^3}(k)$ has been shown for all k in [190, 191] with methods of hyperkähler geometry. An algebraic proof is lacking at the moment. Irreducibility of $\text{MI}_{\mathbb{P}^3}(k)$ for odd k has also been proved recently, see [300]. Rationality of $\text{MI}_{\mathbb{P}^3}(k)$, conditional to irreducibility, has been announced as well, see [231]. In contrast to this, the number of irreducible components of $\text{M}_{\mathbb{P}^3}(2, c_1, c_2)$ tends to infinity as c_2 increases, see [115]. The same holds for the moduli space of stable rank-2 bundles over any smooth polarized threefold having c_2 of growing degree, see [22].

It is perhaps worth mentioning here that, for instanton bundles which are general enough (in the main component), several more properties are known: for instance that they have natural cohomology [166], their minimal free resolution [271], their jumping lines [57], their restriction to planes [58].

This is even more pertinent here, for this good knowledge of general instantons has been exploited recently in quite different directions, for instance in Boij-Söderberg theory, cf. [122], and in the construction of matrices of linear forms of constant rank, as we did in [44].

Let us now try to move to other 3-dimensional manifolds. A class of varieties that share a certain similarity with $\mathbb{P}^3$ is that of Fano threefolds X of Picard number 1, say $\text{Pic}(X)$ is generated by the ample divisor class h_X . The even cohomology ring of X looks as follows: $H^{2,2}(X)$ and $H^{3,3}(X)$ are one-dimensional, generated respectively by the classes ℓ_X and p_X of a line and a point contained in X , with the relation $h_X^2 = \deg(X)\ell_X$, where $\deg(X) = h_X^3$ (Chern classes will be denoted as integers from now on, with obvious meaning). A more subtle invariant of X comes from $H^3(X)$. Indeed the non-trivial Hodge theory of X is encoded by the intermediate Jacobian $J(X)$. This is an abelian variety, whose structure of complex torus is defined as $H^{2,1}(X)^*/H_3(X, \mathbb{Z})$, where $H_3(X, \mathbb{Z})$ is viewed as a lattice in $H^{2,1}(X)^*$ via a higher-dimensional analogue of the Abel-Jacobi map. Also, we have $\omega_X \simeq \mathcal{O}_X(-i_X h_X)$ for some integer $1 \leq i_X \leq 4$, that we called the *index* of X (cf. Section III.2 of Chapter 2), another basic invariant of X .

It is worth recalling immediately that Fano threefolds are completely classified. For Picard number 1 there are 17 deformation families, see for instance [188, Chapter 12.2]. If $i_X = 4$ then $X = \mathbb{P}^3$, and if $i_X = 3$, then X is a smooth quadric. On the other hand, in case $i_X = 2$ one speaks of Del Pezzo threefolds of Picard number 1: they come in 5 deformation families, one for

each degree $1 \leq \deg(X) \leq 5$. For $i_X = 1$, one speaks of *prime Fano threefolds* (beware that terminology is not uniform in the literature!) of genus g , namely g is the genus of the *canonical curve* obtained as general linear section of codimension 2 in X . One has $2g - 2 = \deg(X)$, and there are 10 deformation families of these varieties, one for each g in $[2, 12] \setminus \{11\}$.

So much for our class of threefolds; let us now describe our class of bundles. Denote by q and ϵ be the quotient and the remainder of the division of i_X by 2. Let E be a stable bundle of rank 2 on X with $c_1(E) = -\epsilon$. Then, $F = E(-q)$ satisfies $F \simeq F^* \otimes \omega_X$. We say that E is a *k-instanton on X* if $c_2(E) = k$, and $H^1(X, F) = 0$. In this case, it is easy to check that $H^k(X, F) = 0$ for all k , so that F is right-orthogonal to $\mathcal{O}_X$, in the sense of semiorthogonal decompositions [42]. When $X = \mathbb{P}^3$, this notion gives back the classical *k*-instantons. Sometimes, when $\epsilon = 1$, to emphasize the fact that E has odd determinant we say that E is an *odd instanton*. Anyway, we denote by $\text{MI}_X(k)$ the space of *k*-instantons on X , or instantons of charge k .

These bundles occur, sometimes unexpectedly, and especially for low charge, in several aspects of the study of Fano threefolds, and have attracted considerable attention. Yet, some of their properties are still unclear. For instance, we don't know if, at least when X is general in its moduli space, the space $\text{MI}_X(k)$ is smooth and/or irreducible. But more elementary questions are also open, for instance given F in $\text{MI}_X(k)$, we don't know if there is a line $L \subset X$ with *ordinary splitting*, i.e. $F|_L \simeq \mathcal{O}_L \oplus \mathcal{O}_L(\epsilon)$. See Section III for more open questions. We give here a quick overview of some of the main results and questions concerning these bundles.

I.1. Existence of instantons. Let $k \geq 1$. It is not difficult to show that $\text{MI}_X(k)$, and in fact all of $\text{M}_X(2, \epsilon, k)$, are empty for $i_X = 2, 3$, $k = 1$, and for $i_X = 1$ and $2k < g + 2$. In the second case, the lowest value of k such that $\text{M}_X(2, 1, k)$ is not empty is $m_g = \lceil \frac{g+2}{2} \rceil$. We call this the *minimal charge*. The value $m_g + 1$ is also important for us, we will call it *almost minimal charge* (see below for an overview of these cases).

Above these bounds, combining the results of [48, 133], we are ensured that $\text{MI}_X(k)$ is not empty, at least for threefolds with a mild generality assumption, namely the existence of a generically reduced component in the Hilbert scheme $\text{Hilb}_{t+1}(X)$ (cf. Section II.1.3 of Chapter 0 for notation) of lines in case $i_X = 1$: a condition equivalent to the existence of an *ordinary line* $L \subset X$, i.e., such that $\mathcal{N}_L \simeq \mathcal{O}_L \oplus \mathcal{O}_L(-1)$. We call these threefolds *ordinary*, in contrast with *exotic* threefolds where this Hilbert scheme is nowhere reduced.

One should be aware that, in principle, a Fano threefold can be ordinary and exotic at the same time, if $\text{Hilb}_{t+1}(X)$ has two or more components, one generically reduced and the other not. However, we need ordinary, and non-exotic implies ordinary. By the way, if $g \geq 9$, in view of [156, 269] we know

that X is non-exotic unless it is isomorphic to Mukai-Umemura's threefold (cf. [244]), which is the union of three orbits for $\mathrm{SL}_2(\mathbb{C})$ acting on binary forms of degree 12, in which case $\mathrm{Hilb}_{t+1}(X)$ is a double conic. In fact, the only other known examples of exotic prime Fano threefolds are those containing a cone, for instance the Fermat quartic threefold in $\mathbb{P}^4$ ($g = 3$), then $\mathrm{Hilb}_{t+1}(X)$ is a curve with 40 irreducible components, each of multiplicity 2 so this threefold is not ordinary (see [295]). Also, in case $i_X = 1$, one has to assume that X is not hyperelliptic, which means that $|\mathcal{O}_X(1)|$ is very ample. This excludes the case $g = 2$ and a proper closed subset of the moduli space of Fano threefolds of genus 3.

Proposition I.1. *Let X be a smooth Fano threefold of index i_X , and assume X ordinary and non-hyperelliptic of genus g in case $i_X = 1$. Then, the moduli space $\mathrm{MI}_X(k)$ admits a generically smooth component of dimension δ_X :*

i_X	4	3	2	1
δ	$8k - 3$	$6k - 6$	$4k - 3$	$2k - g - 2$

The proof of this fact works by induction on k . As basis of the recursion, one has to study $\mathrm{MI}_X(k)$ for minimal k . This is a bit easier in case $i_X \geq 2$, but requires a case-by-case analysis for $i_X = 1$. In fact, the whole $\mathrm{M}_X(2, \epsilon, k)$ can be described in detail for minimal k ; this has been done in a series of papers. We refer to [240] for g even and ≥ 6 (cf. also [158] for $g = 6$, [159, 160] for $g = 8$ and [130, 212, 284] for $g = 12$). One may look at [38, 49, 227] for $g = 3$ and to [49, 228]. We refer to [184, 214], for $g = 7$, and to [49, 186] for $g = 9$. This list, though very long, certainly forgets some contributions to this problem, and we apologize for this. The final outcome in this case is the following (see [49, Theorem 3.2]).

Proposition I.2. *Let X be a smooth non-hyperelliptic Fano threefold of index 1 and genus g . Then any sheaf F lying in $\mathrm{M}_X(2, 1, m_g)$ is locally free and ACM, is globally generated if $g \geq 4$, and there is a line $L \subset X$ where F has generic splitting. Further, $\mathrm{M}_X(2, 1, m_g)$ can be described as follows:*

- i) the curve $\mathrm{Hilb}_{t+1}(X)$ parametrizing lines contained in X if $g = 3$;
- ii) a length-2 scheme if $g = 4$, reduced iff X is contained in a smooth quadric;
- iii) a double cover of the discriminant septic curve if $g = 5$;
- iv) a single smooth point if $g = 6, 8, 10, 12$;
- v) a smooth non-tetragonal curve of genus 7 if $g = 7$;
- vi) a smooth plane quartic if $g = 9$.

Moreover, assuming X ordinary for $g = 3$, or X contained in a smooth quadric for $g = 4$, then there is a sheaf F in $\mathrm{M}_X(2, 1, m_g)$ with $\mathrm{Ext}_X^2(F, F) = 0$.

To run the induction argument, we add a line. This, in turn, is done in two steps. First, one takes a sheaf F in $\mathrm{MI}_X(k)$ with $\mathrm{Ext}_X^2(F, F) = 0$, and a line $L \subset X$

such that F has generic splitting on L (these data exist by induction hypothesis, except in the case $g = 4$ and X contained in no smooth quadric, but this case can be worked out separately). Then, we obtain a modification of F along L as the sheaf F' defined as kernel of a surjection $F \rightarrow \mathcal{O}_L$. Unfortunately, this sheaf is not locally free, in fact it fails to be reflexive since its double dual is just F . However, the sheaf F' turns out to be stable, and to lie in $M_X(2, \epsilon, k + 1)$.

Second, one checks that F' is unobstructed, has generic splitting on any line L' near L in $\text{Hilb}_{t+1}(X)$, and still satisfies $H^1(X, F'(-q)) = 0$. It turns out that F' can be deformed to a locally free sheaf with the same cohomological vanishing. This deformed sheaf now lies in $MI_X(k + 1)$, and the induction process can continue.

I.2. Instantons with small charge, Jacobians, periods. Although there is apparently no precise explanation for this, it seems that moduli space of bundles with minimal invariants on X capture some of the key features of X , such as intermediate Jacobian, periods, Hilbert scheme of curves of low degree, and so forth.

I.2.1. Small charge for index 2. To have a first impression of this phenomenon, let us look at the case $i_X = 2$ (i.e. Del Pezzo threefolds of Picard number 1 and degree $d = h_X^3$), we have:

- i) if $d = 4$ (i.e., X is the complete intersection of two quadrics in $\mathbb{P}^5$), the space $M_X(2, 0, 2)$ is a smooth curve of genus 2. This result (cf. [42, 242]) is quite paradigmatic of the phenomena we will encounter in the sequel, namely the modular relation between a threefold X and a curve, dual to X in some sense. It is a counterpart to another classical fact: the description of X as moduli space of rank-2 bundles on C with fixed odd determinant, cf. [99].
- ii) if $d = 3$ (i.e., X is a cubic threefold), the space $M_X(2, 0, 2)$ is the blow-up of $J(X)$ along a translate of the Fano surface of lines in X , cf. [29, 114, 229]. This beautiful result, closely related to foundational material of [84, 301], will turn out to have many “cousins” in different situations, see below.

Cubic threefolds X are related also to many other interesting objects. We mentioned that the structure of conic bundle on X obtained via the projection $X \dashrightarrow \mathbb{P}^2$ from line contained in X has been used to study higher rank bundles (which are even ACM) in terms of sheaves on a non-commutative $\mathbb{P}^2$, cf. [218, 221]. We will also see that cubic threefolds X are naturally related to threefolds of genus 8, via the Palatini quartic.

- iii) if $d = 2$ (i.e., X is a double cover of $\mathbb{P}^3$ ramified along a quartic surface), in view of [230] we know that $M_X(2, 0, 3)$ maps $84 : 1$ onto a theta divisor in $J(X)$ via the Abel-Jacobi map: this again is related to more classical work [83, 313].

I.2.2. *One-dimensional homological dual: genus 7 and 9.* Let us now turn to Fano threefolds of index 1 (and still of Picard number 1). In this case, an interesting behaviour appears for (almost) minimal charge, i.e., when $k = m_g$ or $k = m_g + 1$. Indeed, the moduli space $\mathrm{MI}_X(2, 1, m_g + 1)$ can be studied in detail, and is often related to nice geometric properties of X . I have studied particularly the case when X has a 1-dimensional homological dual (e.g. $g = 7, 9$), since in this case vector bundles on X can be studied in terms of a smooth curve, a priori a much simpler object.

For Fano threefolds of genus 7, minimal charge corresponds to $m_7 = 5$. The moduli space $\mathrm{MI}_X(5) = \mathrm{M}_X(2, 1, 5)$ is a smooth projective non-tetragonal curve C of genus 7, which is the homological dual of X , in fact the first example of such duality, according to [214]. This moduli space and the one of almost minimal charge $\mathrm{M}_X(2, 1, 6)$ are studied in detail in [48, 184, 185]. It turns out that $\mathrm{M}_X(2, 1, 6)$ is related to the singular locus of Riemann's Θ divisor in $\mathrm{Pic}^6(C)$, namely it is isomorphic to the locus of line bundles of degree 6 with 2 independent global sections. Instantons of higher charge are related to higher-rank Brill-Noether loci in C .

For genus 9, we have $m_9 = 6$, and $\mathrm{MI}_X(6) = \mathrm{M}_X(2, 1, 6)$ also in this case is a smooth projective curve C which is the homological dual of X , cf. [216]. On the other hand, the space $\mathrm{MI}_X(2, 1, 7)$ of sheaves of almost minimal charge is the blow-up of $\mathrm{Pic}^2(C)$ along $\mathrm{Hilb}_{t+1}(X)$. Much of the geometry here is controlled by a rank-2 bundle $\mathcal{V}$ on C , having the property that $\mathrm{Hilb}_{2t+1}(X) \simeq \mathbb{P}(\mathcal{V})$, and X can be defined as a certain Brill-Noether locus of stable rank-2 bundles on C that have enough sections when twisted with $\mathcal{V}$. We refer to [49, 179, 186, 242]. The moduli of $\mathcal{V}$ in this case give the periods of X .

I.2.3. *Genus 8.* We describe a bit this case since, besides being particularly beautiful, it provides a conjectural model for other constructions (cf. Section III).

A smooth prime Fano threefold X of genus 8 is birational to a smooth cubic threefold Y in $\mathbb{P}^4$. This correspondence can be explicitly described via an instanton bundle on Y , according to [213]. Indeed, X is the complete intersection of $\mathbb{G}(2, V)$, where $V \simeq \mathbb{C}^6$, with a $\mathbb{P}^9$ in $\check{\mathbb{P}}(\wedge^2 V)$. Write $\mathcal{U}_X$ for the universal rank-2 subbundle on X . The associated choice of 5 independent hyperplane sections gives a 5-dimensional subspace $W \subset \wedge^2 V^*$, so that X lives in $\check{\mathbb{P}}(W^\perp)$, where $W^\perp$ is the kernel of $\wedge^2 V \rightarrow W^*$. Over $\mathbb{P}^4 = \check{\mathbb{P}}(W)$, we obtain a morphism $V \rightarrow V^* \otimes \mathcal{H}^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(1))$. This gives rise to the following datum:

$$0 \rightarrow V \otimes \mathcal{O}_{\mathbb{P}^4}(-1) \rightarrow V^* \otimes \mathcal{O}_{\mathbb{P}^4} \rightarrow E(1) \rightarrow 0,$$

where the middle map, given by our morphism $V \rightarrow V^* \otimes W^*$, is skew symmetric and injective. The Pfaffian of this map is a cubic form that defines Y in $\mathbb{P}^4$, and its cokernel $E(1)$ is a vector bundle of rank 2 on Y , in fact E lies in $\mathrm{MI}_Y(2)$.

On the other hand, one could consider $\mathbb{P}^5 = \check{\mathbb{P}}(V)$, and recall that $H^0(\mathbb{P}^5, \Omega_{\mathbb{P}^5}(2)) \simeq \wedge^2 V^*$. Then we would interpret $W \hookrightarrow \wedge^2 V^*$ as a map:

$$W \otimes \mathcal{O}_{\mathbb{P}^5} \rightarrow \Omega_{\mathbb{P}^5}(2).$$

The degeneracy locus of this map is a quartic in $\mathbb{P}^5$, called the Palatini quartic Z . It turns out that $\mathbb{P}(\mathcal{U}_X^*)$ and $\mathbb{P}(E(1))$ are both birational to Z , by the relatively ample linear system, that naturally maps to $\mathbb{P}^5$ since $H^0(Y, E(1)) \simeq V^* \simeq H^0(X, \mathcal{U}_X^*)$. The Palatini quartic Z is singular along a curve C , which is the image of the universal line over $X \times \text{Hilb}_{t+1}(X)$. Let us mention that X can be recovered from the curve C , together with its embedding in $\mathbb{P}^5$, according to [145]. Let us also mention that the Hilbert scheme of the degeneracy loci of maps $\mathcal{O}_{\mathbb{P}(V)}^m \rightarrow \Omega_{\mathbb{P}^n}(2)$ have been studied in [135], and more recently by F. Tanturri in his thesis, [294]. See also [134] for the Hilbert scheme of determinantal varieties in this spirit, related to the framework of Problem V.2 of Chapter 2.

This correspondence goes further. Indeed, the “non-trivial components” ${}^\perp \langle \mathcal{O}_X, \mathcal{U}_X^* \rangle \subset \mathbf{D}^b(X)$ and $\langle \mathcal{O}_Y, \mathcal{O}_Y(1) \rangle^\perp \subset \mathbf{D}^b(Y)$ are equivalent by [213], see also [215]. This subcategory also determines Y by [35].

In terms of moduli spaces, we have $m_8 = 5$, and we know by [180] that $M_X(2, 1, 8)$ is isomorphic to the Hilbert scheme of lines in Y .

1.2.4. Periods. The period map $\wp$ associated with a smooth Fano threefold X with fixed index and degree its intermediate Jacobian $J(X)$ as a principally polarized abelian variety. This map can be viewed in families (at least stack-theoretically), from the “moduli space” of Fano threefolds with fixed index and degree to a moduli space of polarized abelian varieties. Closely related to this is the Torelli-type problem, i.e., the question whether $\wp$ is injective. For instance, the Torelli-type result that a smooth cubic threefold Y is determined by $(J(Y), \Theta)$ goes back to Clemens-Griffiths, [84], a more precise result being due to Mumford [245], cf. also [71] for a converse statement. Of course this is also related to the discussion of Chapter 1, Section IV.

However it turns out that $\wp$ need not always be injective, as shown for instance by Fano threefolds of index 1 and genus 12, that have trivial intermediate Jacobian, despite having 6-dimensional moduli. Its fibres however, for given index and degree, are of particular interest since they describe non-isomorphic threefolds with the same Hodge theory.

In spite of this, sometimes one can reconstruct X from some other datum, for instance for $i_X = 1$ and $g = 8$ or $g = 12$, X is recovered from the Hilbert scheme $\text{Hilb}_{t+1}(X)$ parametrizing lines in X , together with a theta-characteristic on it (we mentioned already [145] for $g = 8$, and we refer to [241, 243] for $g = 12$); or one could look at categorical invariants cf. [35]. We will see more on this later on.

Going back to the period map $\wp$, the link with vector bundles appears in a quite interesting fashion, precisely in view of the previous subsection. Indeed, for Fano threefolds of index 1 and genus 14, the fibres of the period map (i.e. families of Fano threefolds of genus 8 birational to the same cubic threefold Y) are birational to $J(Y)$, which is in turn birational to $M_Y(2, 0, 2)$ as we have seen.

And there are more examples. For genus 6 (so $m_6 = 4$), it is proved in [95] that the general fibres of $\wp$ have at least two connected components, one of them is $\text{Hilb}_{2t+1}(X)$, and the other is the space of almost minimal charge $M_X(2, 1, 5)$.

Periods for Fano threefolds of genus 10 are still to be determined. We set a possible starting point for this question in the next section.

I.3. Instantons on Fano threefolds with trivial Hodge theory. Next, we focus on the case when the variety X satisfies $H^3(X) = 0$, i.e. the intermediate Jacobian of X is trivial. This holds if the derived category $\mathbf{D}^b(X)$ of coherent sheaves on X admits a full strongly exceptional collection, and a case-by-case analysis shows that in fact the two conditions are equivalent. Indeed, there are only 4 classes of such varieties, one for each index, namely:

- i) the projective space $\mathbb{P}^3$, for $i_X = 4$;
- ii) a quadric hypersurface in $\mathbb{P}^4$, for $i_X = 3$;
- iii) a linear section $X = \mathbb{P}^6 \cap \mathbb{G}(2, \mathbb{C}^5) \subset \mathbb{P}^9$, with $H_X^3 = 5$, for $i_X = 2$;
- iv) a prime Fano threefold $X \subset \mathbb{P}^{13}$ of genus 12, in case $i_X = 1$.

In all these case, there are vector bundles $\mathcal{E}_i$ on X such that:

$$\mathbf{D}^b(X) = \langle \mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3 \rangle,$$

and the $\mathcal{E}_i$'s can be chosen in such a way that:

$$(I.1) \quad \mathcal{E}_0 \simeq \mathcal{O}_X(-q - \epsilon), \quad \mathcal{E}_3^*(-\epsilon) \simeq \mathcal{E}_1, \quad \mathcal{E}_2^*(-\epsilon) \simeq \mathcal{E}_2.$$

Set $U = \text{Hom}_X(\mathcal{E}_2, \mathcal{E}_3)$, and note that $U \simeq \text{Hom}_X(\mathcal{E}_1, \mathcal{E}_2)$. To obtain a *monadic description* of $\text{MI}_X(k)$, given an integer k , fix vector spaces I and W , and an isomorphism $D : W \rightarrow W^*$ with $D^t = (-1)^{\epsilon+1}D$ (an $(\epsilon + 1)$ -symmetric duality). According to the values of i_X and k , we need to choose the dimension of I and W as follows:

i_X	k	$\dim(I)$	$\dim(W)$
4	$k \geq 1$	k	$2k + 2$
3	$k \geq 2$	$k - 1$	k
2	$k \geq 2$	k	$4k + 2$
1	$k \geq 8$	$k - 7$	$3k - 20$

The lower bounds for k appear in order to ensure non-emptiness of $\text{MI}_X(k)$. Let us write $G(W, D)$ for the symplectic group $\text{Sp}(W, D)$, or for the orthogonal group $\text{O}(W, D)$ depending on whether $\epsilon = 0$ or 1, so that $\eta \in G(W, D)$ operates

on W and satisfies $\eta^t D \eta = D$. We look at an element A of $I \otimes W \otimes U$ as a map:

$$A : W^* \otimes \mathcal{E}_2 \rightarrow I \otimes \mathcal{E}_3,$$

and, under the dualities (I.1), we can consider:

$$DA^t : I^* \otimes \mathcal{E}_1 \rightarrow W^* \otimes \mathcal{E}_2.$$

We define the subvariety $\mathcal{Q}_{X,k}$ of $I \otimes W \otimes U$ by:

$$\mathcal{Q}_{X,k} = \{A \in I \otimes W \otimes U \mid ADA^t = 0\},$$

and its open piece:

$$\mathcal{Q}_{X,k}^\circ = \{A \in \mathcal{Q}_{X,k} \mid A : W^* \otimes \mathcal{E}_2 \rightarrow I \otimes \mathcal{E}_3 \text{ is surjective}\}.$$

We also define the group:

$$G_k = \mathrm{GL}(I) \times \mathrm{G}(W, D),$$

acting on $I \otimes W \otimes U$ on the left by $(\xi, \eta).A = (\xi A \eta^t)$. The group G_k acts on the variety $\mathcal{Q}_{X,k}$, since, for all $A \in \mathcal{Q}_{X,k}$, we have $\xi A \eta^t D \eta A^t \xi^t = \xi A D A^t \xi^t = 0$. Clearly, G_k acts also on $\mathcal{Q}_{X,k}^\circ$. Then, $\mathrm{MI}_X(k)$ is described by the following result.

Proposition I.3. *Let X be a smooth Fano threefold of Picard number 1 and $H^3(X) = 0$. Let I, W, D and the $\mathcal{E}_i$'s be as above. Then a k -instanton E on X is the cohomology of a monad of the form:*

$$I^* \otimes \mathcal{E}_1 \xrightarrow{DA^t} W^* \otimes \mathcal{E}_2 \xrightarrow{A} I \otimes \mathcal{E}_3,$$

and conversely the cohomology of such a monad is a k -instanton. The moduli space $\mathrm{MI}_X(k)$ is isomorphic to the geometric quotient:

$$\mathcal{Q}_{X,k}^\circ / G_k.$$

More specific results can be given for each threefold. When X is a quadric threefold, it turns out that $\mathrm{MI}_X(k)$ is affine, in analogy with the same result, valid on $\mathbb{P}^n$, proved in [91]. A detailed analysis of $\mathrm{M}_X(2, c_1, k)$ for low k is carried out in [117, 261]. Also, the behaviour of stability with respect to restriction to hyperplane sections is developed in [85].

When $H^3(X) \neq 0$, there are not enough exceptional objects on X to generate $\mathbf{D}^b(X)$. However, as we already pointed out, one can rely on a semiorthogonal decomposition of $\mathbf{D}^b(X)$ containing a subcategory equivalent to $\mathbf{D}^b(C)$, where C is a curve whose Jacobian is isomorphic to the intermediate Jacobian of X , cf. [216]. This allows to describe instantons in terms of vector bundles over C , as it is done in [48, 50] (for threefolds of index 1 and genus 7, 9) and [133, 220] (for threefolds of index 2 and degree 4).

In the next section I will show how to extend these methods to genus 10.

I.4. Summary of basic formulas for Fano threefolds. Let X be a smooth Fano threefold of Picard number 1 and index 1, i.e. such that $\omega_X \simeq \mathcal{O}_X(-h_X)$, with h_X ample and $\text{Pic}(X) \simeq \langle h_X \rangle$. The genus g of X , i.e., the genus a curve obtained as vanishing locus of a general pencil of hyperplane sections, satisfies $\deg(X) = 2g - 2$, where $\deg(X) = \deg(h_X) = h_X^3$.

Given a smooth projective curve $D \subset X$ of degree d and genus p_a , we have:

$$c_1(\mathcal{O}_D) = 0, \quad c_2(\mathcal{O}_D) = -d, \quad c_3(\mathcal{O}_D) = 2 - 2p_a - d.$$

Applying the theorem of Riemann-Roch to a coherent sheaf F on X , of (generic) rank r and with Chern classes c_1, c_2, c_3 , we obtain the following formulas:

$$\begin{aligned} \chi(F) &= r + \frac{11+g}{6}c_1 + \frac{g-1}{2}c_1^2 - \frac{1}{2}c_2 + \frac{g-1}{3}c_1^3 - \frac{1}{2}c_1c_2 + \frac{1}{2}c_3, \\ \chi(F, F) &= r^2 - \frac{1}{2}\Delta(F). \end{aligned}$$

Recall that if $T \neq 0$ is a torsion sheaf supported in codimension $p > 0$, then $c_k(T) = 0$ for $k < p$, while $(-1)^{p-1}c_p(T)$ is the class of the scheme-theoretic support of T in $\mathbb{H}^{p,p}(X)$ (see e.g. [146]). Moreover since $\chi(T(t))$ is positive for $t \gg 0$, looking at the dominant term of $\chi(T(t))$, we see that $(-1)^{p-1}c_p(T) > 0$.

A hyperplane section of S is a K3 surface, namely S is a simply connected surface with $\omega_S \simeq \mathcal{O}_S$. If E is a stable sheaf of rank r on S , with $c_i(E) = c_i \in \mathbb{H}^{i,i}(S)$. Then (see [178, Chapter 6]), the dimension of $M_S(r, c_1, c_2)$ at E is:

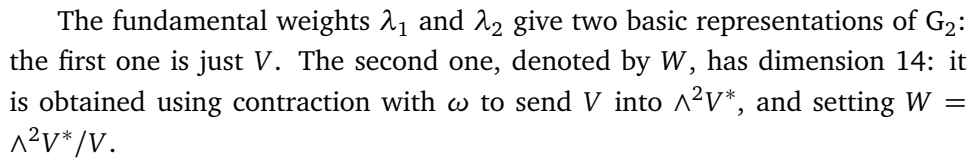
$$(I.2) \quad 2rc_2 - (r-1)c_1^2 - 2(r^2 - 1)$$

II. Fano threefolds of genus 10

In this section, I will outline a study of odd instantons on a Fano threefold of genus 10, pointing out a link with the theory of binary sextics. Indeed, given such threefold, the homologically dual genus-2 curve C is equipped with a stable vector bundle of rank 3 that lies in the Coble-Dolgachev sextic in $\mathbb{P}^8$ via the theta map that sends $\text{SU}_C(3)$ to $\mathbb{P}^8$. This bundle, obtained as direct image of the universal bundle on X via the Fourier-Mukai kernel on $X \times C$, plays an important role in many features of X , such as Hilbert scheme of curves, instanton bundles, and (hopefully) fibres of the period map and the Torelli problem.

II.1. Basic features of Fano threefolds of genus 10. A smooth Fano threefold X of genus 10 is a double hyperplane section of a 5-dimensional manifold Σ , homogeneous under the complex Lie group G_2 , naturally embedded in $\mathbb{P}^{13}$. The projective dual of Σ is a sextic hypersurface in $\check{\mathbb{P}}^{13}$, and we let Γ be the double cover of $\check{\mathbb{P}}^{13}$ ramified along this sextic. The line in $\check{\mathbb{P}}^{13}$ dual to the pencil of hyperplanes containing X cuts this sextic at six points, and the associated double cover of $\mathbb{P}^1$ is a smooth projective curve $C \subset \Gamma$ of genus 2.

II.1.1. *The G_2 -manifold and the 5-dimensional quadric.* We consider the simple complex Lie group G_2 , see [45]. This group can be obtained by fixing a 7-dimensional vector space V over the field $\mathbb{C}$ of complex numbers, and a general alternating 3-form $\omega \in \wedge^3 V^*$. Then, G_2 is the closed subgroup of $SL(V)$ of linear transformations preserving ω . The group G_2 has two simple positive roots α_1 and α_2 .





Q_5



Σ

$$0 \rightarrow \mathcal{U} \rightarrow V \otimes \mathcal{O}_{\mathbb{G}(2,V)} \rightarrow \mathcal{Q} \rightarrow 0.$$

These two G_2 -homogeneous varieties are connected by the complete G_2 -flag G_2/B , where B is a Borel subgroup of G_2 . This manifold has Picard number 2 and appears as $\mathbb{P}(\mathcal{U}^*)$, or equivalently as $\mathbb{P}(\mathcal{C})$, where $\mathcal{C}$ is the so-called Cayley bundle on Q_5 , cf. for instance [259]. This is a stable G_2 -homogeneous

bundle of rank 2 with $c_1(\mathcal{C}) = 3$ and $H^0(Q_5, \mathcal{C}) \simeq W$. We have thus a diagram:

$$\begin{array}{ccc} & G_2/B & \\ p \swarrow & & \searrow q \\ Q_5 & & \Sigma \end{array}$$

and natural isomorphisms:

$$q_*(p^*(\mathcal{O}_{Q_5}(1))) \simeq \mathcal{U}_\Sigma^*, \quad p_*(q^*(\mathcal{O}_\Sigma(1))) \simeq \mathcal{C}.$$

Another way to see $\mathcal{C}$ is the following. We consider $\mathbb{P}^6 = \check{\mathbb{P}}(V)$ and identify $H^0(\mathbb{P}^6, \Omega^2(3)) \simeq \wedge^3 V^*$ with the space of skew-symmetric morphisms $\mathcal{T}_{\mathbb{P}^6}(-1) \rightarrow \Omega_{\mathbb{P}^6}(2)$. Then ω gives an exact sequence:

$$0 \rightarrow \mathcal{T}_{\mathbb{P}^6}(-1) \xrightarrow{\omega} \Omega_{\mathbb{P}^6}(2) \rightarrow \mathcal{C} \rightarrow 0.$$

II.1.2. The G_2 -manifold and Fano threefolds of genus 10. A smooth Fano threefold X of genus 10 is cut in Σ by a pencil of hyperplane sections. We have $\omega_X \simeq \mathcal{O}_X(-1)$. Let S be a smooth hyperplane section surface of X . Then S is a K3 surface of sectional genus 10. Also, choosing S generally in the system of surfaces through $x \in X$, we get that S has Picard number 1. We have

$$(II.1) \quad 0 \rightarrow \mathcal{O}_X(-1) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_S \rightarrow 0.$$

Taking a further smooth hyperplane section, we get a curve of genus 10, embedded by its canonical system. As a side remark, we recall that this curve is not general in the moduli space of curves of genus 10, in fact curves arising this way fill a divisor of the moduli space, that turns out to be quite interesting for other purposes, cf. [143].

Let us continue now the description of X . The tautological exact sequence restricted to X becomes:

$$(II.2) \quad 0 \rightarrow \mathcal{U}_X \rightarrow V \otimes \mathcal{O}_X \rightarrow \mathcal{Q}_X \rightarrow 0.$$

Applying the theorem of Borel-Bott-Weil (cf. for instance [317]) to the homogeneous bundles $\wedge^p \mathcal{U}_\Sigma(t)$ and $\wedge^p \mathcal{Q}_\Sigma(t)$, making use of the Koszul complex defining X and S in Σ , together with Hoppe's criterion, one easily sees that the bundles $\mathcal{U}_X$ and $\mathcal{Q}_X$ are stable, and that, and that $\mathcal{U}_S$ and $\mathcal{Q}_S$ are stable too as soon as S has Picard number 1.

It is not hard to see (cf. [48, Lemma 3.1]) that a semistable sheaf F of rank 2 with $c_1(F) = 1$ must satisfy $c_2(F) \geq 6$, and in fact we must have $F \simeq \mathcal{U}_X^*$ as soon as $c_2(F) = 6$, so $MI_X(2, 1, 6)$ consists of a single point which is also reduced cf. [49, Theorem 3.2].

II.1.3. Semiorthogonal decomposition and bundles of rank 3. An analogue of Beilinson's theorem on $\mathbb{P}^3$, suitable for our threefold X , is provided by Kuznetsov's homological projective duality approach, see in particular [216, Section 8]. The outcome is as follows. First, $\mathcal{O}_X$ and $\mathcal{U}_X^*$ are exceptional and

$\mathcal{U}_X^*$ is left-orthogonal to $\mathcal{O}_X$, i.e. $H^i(X, \mathcal{U}_X) = 0$ for all i . Second, there exists a vector bundle $\mathcal{F}$ of rank 3 over $X \times C$, giving rise to the integral functor:

$$\Phi = \Phi_{\mathcal{F}} : \mathbf{D}^b(C) \rightarrow \mathbf{D}^b(X), \quad \Phi(E) = R\pi_{X*}(\pi_C^*(E) \otimes \mathcal{F}),$$

and to its right and left adjoints $\Phi^!$ and Φ^* :

$$\Phi^!(E) = R\pi_{C*}(\pi_X^*(E) \otimes \mathcal{F}^*[1]) \otimes \omega_C, \quad \Phi^*(E) = R\pi_{C*}(\pi_X^*(E(-1)) \otimes \mathcal{F}^*[3]).$$

Third, the functor Φ provided by $\mathcal{F}$ is fully faithful, and gives the semiorthogonal decomposition:

$$\langle \mathcal{O}_X, \mathcal{U}_X^*, \Phi(\mathbf{D}^b(C)) \rangle.$$

This means several things. One of them is the following: denote by $\mathcal{F}_y$ (or $\mathcal{F}_x$) the restriction of $\mathcal{F}$ to $X \simeq X \times \{y\}$ for $y \in C$ (or to $C \simeq \{x\} \times C$ for $x \in X$). Then $H^i(X, \mathcal{F}_y^*) = H^i(X, \mathcal{U}_X^* \otimes \mathcal{F}_y^*) = 0$ for all i . Also, $\mathcal{F}_y$ is simple and $\text{Ext}_X^i(\mathcal{F}_y, \mathcal{F}_{y'}) = 0$ for $y \neq y'$ and all i . The main consequence is that, for any coherent sheaf E on X , we have a distinguished triangle:

$$(II.3) \quad \Phi\Phi^!(E) \rightarrow E \rightarrow \Psi\Psi^*(E),$$

where Ψ is the embedding of $\langle \mathcal{O}_X, \mathcal{U}_X^* \rangle$ into $\mathbf{D}^b(X)$.

Fourth, $\Psi\Psi^*(E)$ is a minimal complex whose k -th term is:

$$(II.4) \quad \Psi\Psi^*(E)^k = H^{3+k}(X, E(-1)) \otimes \mathcal{O}_X \oplus H^{2+k}(X, \mathcal{U}_X \otimes E(-1)) \otimes \mathcal{U}_X^*.$$

By *minimal* complex here we mean that any endomorphism of $\mathcal{O}_X$ or of $\mathcal{U}_X^*$ induced by the differential actually vanishes.

In the next section, we are going to relate this construction with the moduli spaces $M_X(3, 2, 17, 12)$ and $M_X(3, 1, 9, 2)$, which will turn out to be both isomorphic to C , by proving that $\mathcal{F}$ is in fact a universal bundle for the moduli space $M_X(3, 2, 17, 12)$. Note that the correspondence $E \mapsto E^*(1)$ identifies the open pieces consisting of locally free sheaves in $M_X(3, 2, 17, 12)$ and $M_X(3, 1, 9, 2)$. We will see in a minute that these open pieces coincide in fact with the whole moduli space.

II.2. Rank-3 bundles and homological projective duality. The main goal of this section is to prove the following result, that serves as basis of our analysis of Fano threefolds of genus 10. The ideas to tackle this kind of result appear in [48, 49].

Theorem II.1. *The curve C is identified with $M_X(3, 2, 17, 12)$ and $\mathcal{F}$ is a universal bundle for this moduli space. For any $y \in C$, the corresponding sheaf $\mathcal{F}_y$ is locally free, $\mathcal{F}_{\iota y}^*(1)$ is globally generated, and we have the exact sequences:*

$$(II.5) \quad 0 \rightarrow \mathcal{F}_y(-1) \rightarrow \mathcal{O}_X^6 \rightarrow \mathcal{F}_{\iota y}^*(1) \rightarrow 0,$$

$$(II.6) \quad 0 \rightarrow \mathcal{F}_{\iota y}^*(1) \rightarrow (\mathcal{U}_X^*)^3 \rightarrow \mathcal{F}_y \rightarrow 0.$$

PROOF. Our first task will be to show, for any F in $M_X(3, 1, 9, 2)$:

$$(II.7) \quad H^k(X, F(-1)) = 0,$$

for all k . To do this, we first check $H^2(X, F) = 0$. Note that this space is dual to $\text{Ext}_X^1(F, \mathcal{O}_X(-1))$, so a non-zero element in it would give a non-split extension:

$$(II.8) \quad 0 \rightarrow \mathcal{O}_X(-1) \rightarrow \tilde{F} \rightarrow F \rightarrow 0.$$

One checks that $\tilde{F}$ is slope-semistable. Indeed, note that $c_1(\tilde{F}) = 0$, consider a stable destabilizing quotient Q , and let K be the kernel of $\tilde{F} \rightarrow Q$. Set $K' = K \cap \mathcal{O}_X(-1) \subset \tilde{F}$ and $K'' = K/K'$. Note that $K' = \mathcal{O}_X(-1)$ or $K' = 0$, because $\mathcal{O}_X(-1)/K$ sits in the torsion-free sheaf Q . If $\text{rk}(Q) = 1$, then $c_1(Q) \leq -1$ would entail, by stability of F , a non-zero map $\mathcal{O}_X(-1) \rightarrow Q$, hence $c_1(Q) = -1$ and in fact $Q \simeq \mathcal{O}_X(-1)$ so (II.8) would split, which is absurd. If $\text{rk}(Q) \geq 2$, then, both choosing $K' = 0$ or $K' = \mathcal{O}_X(-1)$, we get that K'' destabilizes F , which is again absurd. We have proved that $\tilde{F}$ is slope-semistable. But this contradicts Bogomolov's inequality. This shows $H^2(X, F) = 0$.

Next, we let S be a general hyperplane section surface of X , and we observe that $F|_S$ is stable. To check this, first note that F^* is also a slope-stable sheaf, for a destabilizing quotient of F^* would destabilize F^{**} hence also F since the cokernel of $F \hookrightarrow F^{**}$ has trivial c_1 (i.e., a torsion-free sheaf and its reflexive hull only differ in codimension 2). Then, we observe that $\chi(F) = 6$ by Riemann-Roch, hence there is a non-zero global section $s : \mathcal{O}_X \rightarrow F$ because $H^2(X, F) = 0$. By stability of F , we have $H^0(X, F(-1)) = 0$, so the transpose of s gives an exact sequence:

$$0 \rightarrow F^S \rightarrow F^* \rightarrow \mathcal{I}_{D/X} \rightarrow 0,$$

for some $D \subset X$ of codimension 2 and some rank-2 torsion-free sheaf F^S , which is clearly slope-stable. Restricting this sequence to S , we get that $F^S|_S$ is stable by Maruyama's theorem (see [232]), so that $F^*|_S$ is also stable, hence $F|_S$ as well.

To continue the proof of (II.7), we work again on S and show that $H^1(S, F|_S) = 0$. Note that this group is dual to $\text{Ext}_S^1(F|_S, \mathcal{O}_S)$, hence a non-zero element of this space would give a non-split extension of the form:

$$0 \rightarrow \mathcal{O}_S \rightarrow \widetilde{F|_S} \rightarrow F|_S \rightarrow 0.$$

Now, with the same argument used above for $\tilde{F}$, we can check that the sheaf $\widetilde{F|_S}$ is slope-stable. Indeed, a destabilizing quotient Q cannot have rank 1 and $c_1(Q) \leq 0$ for otherwise we would easily get a splitting of the above sequence; nor Q can have higher rank because this would give a destabilizing subsheaf of $F|_S$. But the dimension count (I.2) gives a negative number: a contradiction. We have proved $H^1(S, F|_S) = 0$. We can now use:

$$0 \rightarrow F(-1) \rightarrow F \rightarrow F|_S \rightarrow 0$$

and stability of F to deduce (II.7). Indeed, $H^0(X, F(-1)) = 0$ is clear, $H^3(X, F(-1))$ is dual to $\text{Hom}_X(F, \mathcal{O}_X)$ which vanishes by stability, so the vanishing of $H^1(S, F|_S)$ and $H^2(X, F)$, combined with Riemann-Roch, give (II.7) for all k .

Now, we use [198] and [216]. It is proved in [198] that C is indeed a component of $M_X(3, 1, 9, 2)$, and that, denoting by F_y the sheaf over X corresponding to $y \in C$, we have an exact sequence:

$$(II.9) \quad 0 \rightarrow F_{\iota_y}^* \rightarrow \mathcal{O}_X^6 \rightarrow F_y \rightarrow 0.$$

On the other hand, following [216, Page 525] we see that the vector bundle $\mathcal{F}$ of rank 3 over $X \times C$ associated with Φ is built in such a way that, for any $y \in C$, we have an exact sequence:

$$(II.10) \quad 0 \rightarrow \mathcal{O}_X \rightarrow (\mathcal{U}_X^*)^2 \rightarrow \mathcal{F}_y \rightarrow \omega_D \rightarrow 0,$$

where $D \subset X$ is a curve of degree 3 and arithmetic genus 0. By [48, Lemma 3.2], the curve D is locally Cohen-Macaulay (we call such curve a *twisted cubic*). It is then clear that $H^0(X, \mathcal{F}_y(-1)) = 0$ and $\text{Hom}_X(\mathcal{F}_y, \mathcal{O}_X) = 0$, which says that $\mathcal{F}_y$ is stable by Hoppe's criterion. A Chern class computation now shows that $\mathcal{F}_y$ lies in $M_X(3, 2, 17, 12)$. In fact, up to twisting $\mathcal{F}$ with a line bundle on C , one can deduce the Chern classes in $H^*(X \times C)$ of $\mathcal{F}$ by Grothendieck-Riemann-Roch using the relations:

$$\Phi^!(\mathcal{O}_X) = \Phi^!(\mathcal{U}_X^*) = 0, \quad \Phi^!(\mathcal{F}_y) \simeq \mathcal{O}_y.$$

Denoting by p_C the cohomology class of a point in C , the result reads:

$$c_1(\mathcal{F}) = 2h_X + 2p_C, \quad c_2(\mathcal{F}) = 27\ell_X + 3h_X p_C + \eta, \quad c_3(\mathcal{F}) = 7p_X + 22\ell_X p_C,$$

where η lies in $H_3(X, \mathbb{Z}) \otimes H_1(C, \mathbb{Z})$ and satisfies $\eta^2 = 4$.

Let now F be any sheaf in $M_X(3, 1, 9, 2)$. By Serre duality, since (II.7) holds for all k , we have $\text{Ext}_X^k(F, \mathcal{O}_X) = 0$ for all k . Then, applying $\text{Hom}_X(F, -)$ to (II.9), we get:

$$H^{3-i}(X, F_y^* \otimes F(-1))^* \simeq \text{Ext}_X^i(F, F_y) \simeq \text{Ext}_X^{i+1}(F, F_{\iota_y}^*) \simeq H_X^{2-i}(F_{\iota_y} \otimes F(-1))^*,$$

where we also used Serre duality. This space vanishes for $i = 3$ and $i = 2$ by stability. Also, we have $\chi(F, F_y) = 0$. On the other hand, if $\text{Hom}_X(F, F_y) \neq 0$, then $F \simeq F_y$. So $\text{Ext}_X^i(F, F_y) = 0$ for all i if $F \not\simeq F_y$.

Setting $F = \mathcal{F}_{y'}^*(1)$ and letting y' vary in C , we get $\Phi^!(F_y^*) = 0$ unless $F_y^* \simeq \mathcal{F}_{y'}^*(-1)$ for some $y' \in C$, and in this case clearly $y = y'$. But, using Grothendieck-Riemann-Roch, one sees that the Chern character of $\Phi^!(F_y^*)$ is non-zero, so $\Phi^!(F_y^*) \neq 0$, hence $F_y^* \simeq \mathcal{F}_y^*(-1)$. We have now identified the component C of [198] with Kuznetsov's construction of C as homological dual of X . The same argument now shows that any F in $M_X(3, 1, 9, 2)$ is of the form

$\mathcal{F}_y^*(1)$ for some $y \in C$, so:

$$C \simeq M_X(3, 1, 9, 2) \simeq M_X(3, 2, 17, 12),$$

and (II.5) is just (II.9). At this point, [198] tells that any F in $M_X(3, 1, 9, 2)$ is a globally generated vector bundle, with $h^0(X, F) = 6$, $H^i(X, F) = 0$ for $i > 0$, and $H^i(X, F^*) = 0$ for all i .

It remains to prove (II.6). We will first show the vanishing:

$$(II.11) \quad H^i(X, \mathcal{U}_X \otimes \mathcal{F}_y) = 0,$$

for all $\mathcal{F}_y$ in C , and all $i > 0$, which will give $h^0(X, \mathcal{U}_X \otimes \mathcal{F}_y) = 3$ by Riemann-Roch. First note that $h^i(X, \mathcal{U}_X \otimes \mathcal{F}_y) = h^{3-i}(X, \mathcal{U}_X^* \otimes \mathcal{F}_y^*) = 0$ by Serre duality, since $\mathcal{U}_X^*(-1) \simeq \mathcal{U}_X$. Stability gives this vanishing for $i = 3$. Also, tensoring (II.2) with $\mathcal{F}_y^*$ then gives (II.11) for $i = 2$ since $H^i(X, \mathcal{F}_y^*) = 0$ for all i and $H^0(X, \mathcal{Q}_X \otimes \mathcal{F}_y^*) = 0$ by stability.

It follows that $H^0(X, \mathcal{U}_X \otimes \mathcal{F}_y)$ has dimension 3 at least: let us consider a non-zero element s of this space as a map $\mathcal{F}_y^*(1) \rightarrow \mathcal{U}_X^*$. By stability of these two bundles, the image of s must have slope $1/2$, so $\ker(s)$ is a reflexive sheaf of rank 1 so that $c_1(\ker(s)) = 0$ tells $\ker(s) \simeq \mathcal{O}_X$. Computing Chern classes and dualizing, we see that (the transpose of) s gives rise to the exact sequence:

$$(II.12) \quad 0 \rightarrow \mathcal{U}_X^* \xrightarrow{s} \mathcal{F}_y \rightarrow \mathcal{I}_{D/X}(1) \rightarrow 0,$$

where $D \subset X$ is a twisted cubic. What we have to prove now is that, for any twisted cubic $D \subset X$, we have:

$$(II.13) \quad h^j(X, \mathcal{U}_X \otimes \mathcal{I}_{D/X}) = \text{ext}_X^{3-j}(\mathcal{I}_{D/X}, \mathcal{U}_X) = \delta_{2,j},$$

$$(II.14) \quad h^k(X, \mathcal{U}_X^* \otimes \mathcal{I}_{D/X}) = 2\delta_{0,k},$$

where $\delta_{i,j}$ is the Kronecker symbol. Note that the relation between h^j and ext^{3-j} in (II.13) is given by Serre duality, so by stability the equality with $\delta_{2,j}$ is also clear for $j = 0, 3$. By Riemann-Roch, (II.13) will be proved if we show it for $j = 1$. Further, observe that $H^j(X, \mathcal{I}_{D/X}) = 0$ for all j by [48, Lemma 3.2]. Then, tensoring (II.2) by $\mathcal{I}_{D/X}$, we reduce to prove $H^0(X, \mathcal{Q}_X \otimes \mathcal{I}_{D/X}) = 0$. But this is given by the following claim: *a non-zero global section of Q vanishes nowhere, or on a line contained in X , or on a point of X .*

In turn, to prove the claim, we let s be such a section. Using Borel-Bott-Weil, it is easy to see that s lifts to a global section $\tilde{s}$ of $\mathcal{Q}_{\mathbb{G}(2,V)}$ on $\mathbb{G}(2, V)$, which vanishes on a $\mathbb{P}^5$. Also, $\mathcal{Q}_{\mathbb{G}(2,V)}^*(1)$ restricts to $\mathbb{P}^5$ as $\Omega_{\mathbb{P}^5}(2)$. Then, ω corresponds to a global section of s_ω of $\Omega_{\mathbb{P}^5}(2)$, and as such vanishes nowhere, or on a $\mathbb{P}^3$, or on a line. Then, $\mathbb{V}(s)$ is the intersection of $\mathbb{V}(s_\omega)$ with the linear span of X , and as such is $\emptyset$, or a linear space. But since $\text{Pic}(X)$ is generated by the hyperplane section of X , this linear space can be at most of dimension 1.

So far we have proved (II.13). To prove (II.14), we first dualize (II.2) and tensor with $\mathcal{I}_{D/X}$. Then, we observe that $H^3(X, \mathcal{Q}_X \otimes \mathcal{I}_{D/X}) = 0$ holds by stability, so (II.14) holds for $k = 2, 3$. By Riemann-Roch, it remains to show (II.14) for $k = 1$. We consider again the hyperplane section S , and we intersect D with S to obtain a subscheme $Z \subset S$ of length 3. We have thus:

$$0 \rightarrow \mathcal{I}_{D/X} \rightarrow \mathcal{I}_{D/X}(1) \rightarrow \mathcal{I}_{Z/S}(1) \rightarrow 0,$$

and we tensor this sequence with $\mathcal{U}_X$. In view of (II.13), to prove it suffices now to show $h^1(S, \mathcal{U}_S^* \otimes \mathcal{I}_{Z/S}) = 1$. By Serre duality, if this value was greater than 1, we would have a non-split extension:

$$0 \rightarrow \mathcal{U}_S^2 \rightarrow E \rightarrow \mathcal{I}_{Z/S} \rightarrow 0,$$

for a torsion-free sheaf E of slope $-2/5$ on S . With the same method as above, one shows that E is stable, while the parameter count (I.2) of Chapter 0, applied to E , gives a negative number. This is a contradiction, and the proof of (II.14) is thus finished. Applying $\text{Hom}_X(\mathcal{U}^*, -)$ to (II.12), since $\mathcal{U}^*$ is exceptional, using (II.14) we get (II.11).

We can now finish the proof of (II.6). First of all, let us show that the evaluation map $e_y : (\mathcal{U}_X^*)^3 \rightarrow \mathcal{F}_y$ is surjective for all $y \in C$. Dualizing (II.10) we get an exact sequence:

$$0 \rightarrow \mathcal{F}_y^*(1) \rightarrow (\mathcal{U}_X^*)^2 \rightarrow \mathcal{I}_{D/X}(1) \rightarrow 0,$$

for some twisted cubic $D \subset X$. Also, we have proved in (II.13) that $\text{ext}_X^1(\mathcal{I}_{D/X}(1), \mathcal{U}_X^*) = 1$. We claim that this corresponds to an extension:

$$0 \rightarrow \mathcal{U}_X^* \rightarrow \mathcal{F}_{y'} \rightarrow \mathcal{I}_{D/X}(1) \rightarrow 0,$$

for some $y' \in C$. Indeed, the extension sheaf is easily seen to have the correct Chern classes and to be stable. Therefore, the maps the surjection $(\mathcal{U}_X^*)^2 \rightarrow \mathcal{I}_{D/X}(1)$ and the section $\mathcal{U}_X^* \rightarrow \mathcal{F}_{y'}$ patch together to give an exact sequence:

$$0 \rightarrow \mathcal{F}_y^*(1) \rightarrow (\mathcal{U}_X^*)^3 \xrightarrow{e_{y'}} \mathcal{F}_{y'} \rightarrow 0.$$

Dualizing and tensoring with $\mathcal{O}_X(1)$ we get that e_y is surjective. Finally, we must have $y' = \iota y$ since the previous sequence is an element of $\text{Ext}_X^1(\mathcal{F}_{y'}, \mathcal{F}_y^*(1)) \simeq \text{Ext}_X^2(F_y, F_y^*)^*$ (with the previous convention $\mathcal{F}_y \simeq F_y^*(1)$) and we have seen that this space vanishes for $y' \neq \iota y$. $\square$

II.3. Curves of low degree. The study of curves of low degree in X is quite essential for us. Indeed, curves of degree up to 3 are tightly related to some structural objects on X , such as the homologically dual curve C , its Picard variety, the Coble cubic, the theta divisors in $\text{Pic}^1(C)$ and so forth. We will start this analysis here, with conics and rational cubics, and end it a bit further on with lines.

II.3.1. *Conics.* It is well-known that the Hilbert scheme of conics contained in X is isomorphic to the Jacobian of C . This has been proved at least two times: in [198, Corollary 1.4] and [181, Proposition 3]. We provide a third proof, of homological flavour.

Proposition II.2. *The map $D \mapsto \Phi^!(\mathcal{I}_{D/X}(1))[-1]$ defines an isomorphism:*

$$\mathrm{Hilb}_{2t+1}(X) \simeq \mathrm{Pic}^0(C).$$

PROOF. Note that in this case $\mathcal{H}_0^2(X) = \mathrm{Hilb}_{2t+1}(X)$ (cf. Section II.1.3 of Chapter 0 for notation) by [48, Lemma 3.2]. First of all we prove that, given a conic $D \subset X$ (i.e., a subvariety of X of dimension 1 whose Hilbert polynomial in t is $2t + 1$), we have $\mathcal{I}_{D/X}(1) \simeq \Phi^!(\mathcal{I}_{D/X}(1))$. By [48, Lemma 3.2], we have $H^i(X, \mathcal{I}_{D/X}) = 0$ for all k . It is easy to see, using stability and Serre duality, that $H^3(X, \mathcal{Q}_X \otimes \mathcal{I}_{D/X}) = 0$. By the same reason, we get that $H^k(X, \mathcal{U}_X \otimes \mathcal{I}_{D/X})$ vanishes for $k = 0$ and $k = 3$. Hence, tensoring (II.2) with $\mathcal{I}_{D/X}$ we obtain $H^2(X, \mathcal{Q}_X \otimes \mathcal{I}_{D/X}) = 0$. Moreover, by the claim emphasized in the proof of Theorem II.1, we see that $H^0(X, \mathcal{Q}_X \otimes \mathcal{I}_{D/X}) = 0$. By Riemann-Roch, we also get $H^1(X, \mathcal{Q}_X \otimes \mathcal{I}_{D/X}) = 0$. It follows from (II.3) that $\mathcal{I}_{D/X}(1) \simeq \Phi^!(\mathcal{I}_{D/X}(1))$.

Now let us check that $\Phi^!(\mathcal{I}_{D/X}(1))[-1]$ is a line bundle of degree 0. By the vanishing we have just proved, tensoring (II.5) with $\mathcal{I}_{D/X}$ and (II.6) with $\mathcal{I}_{D/X}(-1)$, we get:

$$H^i(X, \mathcal{F}_y^* \otimes \mathcal{I}_{D/X}(1)) \simeq H^{i+2}(X, \mathcal{F}_y^* \otimes \mathcal{I}_{D/X}).$$

These groups obviously vanish for $i \geq 2$, and also for $i = 1$, as one easily checks again using stability and Serre duality. By Riemann-Roch, we get $h^0(X, \mathcal{F}_y^* \otimes \mathcal{I}_{D/X}(1)) = 1$, so $\Phi^!(\mathcal{I}_{D/X}(1))[-1]$ is a locally free sheaf N_D of rank 1. By Grothendieck-Riemann-Roch, we see that $\deg(N_D) = 0$.

We have thus a morphism from $\mathrm{Hilb}_{2t+1}(X)$ to $\mathrm{Pic}^0(C)$ defined by $D \mapsto N_D$, and this morphism is clearly an embedding for $\mathcal{I}_{D/X}(1) \simeq \Phi(N_D)[1]$. But $\mathrm{Pic}^0(C)$ is an irreducible surface and any component of $\mathrm{Hilb}_{2t+1}(X)$ has dimension ≥ 2 by [187], so $\mathrm{Hilb}_{2t+1}(X) \simeq \mathrm{Pic}^0(C)$. $\square$

II.3.2. *The first anti-autoequivalence.* We now define a useful anti-autoequivalence τ of the subcategory $\langle \mathcal{O}_X(1) \rangle^\perp$ of $\mathbf{D}^b(X)$. Given E in $\mathbf{D}^b(X)$, we set:

$$\tau(E) = \mathcal{O}_X(1) \otimes_{\mathcal{O}_X} (\mathbf{R}\mathcal{H}om_X(E(-1), \mathcal{O}_X)))[-2].$$

Lemma II.3. *The functor τ is an involutive anti-autoequivalence of $\langle \mathcal{O}_X(1) \rangle^\perp$ that fixes $\Phi(\mathbf{D}^b(C))$, and for E in $\mathbf{D}^b(C)$ we have:*

$$\tau(\Phi(E)) \simeq \Phi(\mathbf{R}\mathcal{H}om_C(\iota^*E, \mathcal{O}_C)).$$

PROOF. The anti-equivalence $\mathbf{R}\mathcal{H}om_X(-, \mathcal{O}_X(1))$ sends $\langle \mathcal{O}_X(1) \rangle^\perp$ onto ${}^\perp\langle \mathcal{O}_X \rangle$, while $L_{\mathcal{O}_X}$ is an equivalence of ${}^\perp\langle \mathcal{O}_X \rangle$ onto $\langle \mathcal{O}_X \rangle^\perp$ by [41]. Tensoring

with $\mathcal{O}_X(1)$ (and shifting), we get an anti-autoequivalence of $\langle \mathcal{O}_X(1) \rangle^\perp$. To see that τ^2 is the identity, we note that for any E in $\mathbf{D}^b(X)$ there is an exact triangle:

$$\mathbf{R}\mathrm{Hom}_X(E(-1), \mathcal{O}_X) \otimes \mathcal{O}_X(1) \rightarrow \mathbf{R}\mathcal{H}om_X(E(-1), \mathcal{O}_X(1)) \rightarrow \tau(E)[2]$$

Tensoring with $\mathcal{O}_X(-1)$ and dualizing, we get:

$$\mathbf{R}\mathcal{H}om_X(\tau(E), \mathcal{O}_X(1))[-2] \rightarrow E(-1) \rightarrow \mathbf{R}\mathrm{Hom}_X(E(-1), \mathcal{O}_X)^* \otimes \mathcal{O}_X.$$

Of course, one has $L_{\mathcal{O}_X}(\mathcal{O}_X) = 0$, and assuming that E lies in $\langle \mathcal{O}_X(1) \rangle^\perp$ we have $L_{\mathcal{O}_X}(E(-1)) \simeq E(-1)$. So, applying $L_{\mathcal{O}_X}$ to the previous triangle and tensoring with $\mathcal{O}_X(1)$ we see that $\tau^2(E) \simeq E$.

Next, since we have a parametrization of $\Phi(\mathbf{D}^b(C))$ by the points of C , to check that τ fixes $\Phi(\mathbf{D}^b(C))$ it suffices to see that $\tau(\mathcal{F}_y)$ lies in $\Phi(\mathbf{D}^b(C))$. But τ is the composition of $(-\otimes \mathcal{O}_X(-1))$, $\mathbf{R}\mathcal{H}om_C(-, \mathcal{O}_X)$, $L_{\mathcal{O}_X}$, and again $-\otimes \mathcal{O}_X(1)$, so it transforms $\mathcal{F}_y$ into $\mathcal{F}_{\iota y}[-1]$ via:

$$\mathcal{F}_y \mapsto \mathcal{F}_y(-1) \mapsto \mathcal{F}_y^*(1) \mapsto \mathcal{F}_{\iota y}(-1)[-1] \mapsto \mathcal{F}_{\iota y}[-1],$$

where we have used (II.6) to apply $L_{\mathcal{O}_X}[-2]$. To see the induced anti-autoequivalence τ_C on $\mathbf{D}^b(C)$, we recall from [43, Theorem 3.1] that any antiequivalence of $\mathbf{D}^b(C)$ is the composition of $\mathbf{R}\mathcal{H}om_C(-, \mathcal{O}_C)$ and of an automorphism of C , the operation of taking tensor product with a line bundle on C , and a shift in the triangulated category. Rewriting the previous display as $\tau(\Phi(\mathcal{O}_y)) \simeq \Phi(\mathcal{O}_{\iota y}[-1])$, and using the fact that $\mathbf{R}\mathcal{H}om_C(\mathcal{O}_y, \mathcal{O}_C) \simeq \mathcal{O}_y[-1]$, we get that τ_C is $N \otimes \iota^* \mathbf{R}\mathcal{H}om_C(-, \mathcal{O}_C)$, for some line bundle N on C .

It remains to check that N is $\mathcal{O}_C$. To get this, consider a conic D contained in X . Note that $\tau(\mathcal{I}_{D/X}(1)) \simeq \mathcal{I}_{D/X}(1)[-2]$, indeed $\omega_D \simeq \mathcal{O}_D(-1)$ gives $\mathbf{R}\mathcal{H}om_X(\mathcal{O}_{D/X}, \mathcal{O}_X) \simeq \mathcal{O}_D[-2]$, which easily implies $L_{\mathcal{O}_X}(\mathbf{R}\mathcal{H}om_X(\mathcal{I}_{D/X}, \mathcal{O}_X)) \simeq \mathcal{I}_{D/X}$. By the previous proposition, $N_D = \Phi^!(\mathcal{I}_{D/X}(1))[-1]$ is a line bundle of degree zero. Also, we have $\tau(\Phi(N_D[1])) \simeq \Phi(N_D)[-1]$, hence $\tau_C(N_D[1]) \simeq N_D[-1]$, i.e., $\iota^*(N_D^*) \otimes N \simeq N_D$. Therefore, $N \simeq \mathcal{O}_C$ and we are done. $\square$

II.3.3. Twisted cubics. One of the main characters here is a rank-3 vector bundle on C , obtained as direct image of $\mathcal{U}_X^*$. This bundle is directly related to twisted cubics contained in X , but also to the other important actor in this story: the Coble cubic.

Definition II.4. Set $\mathcal{V} = \Phi^*(\mathcal{U}_X^*)^*$.

We have seen in Theorem II.1 that, for any $y \in C$, $H^i(X, \mathcal{F}_y^* \otimes \mathcal{U}_X^*) = 0$ for $i \neq 3$, and that $H^3(X, \mathcal{F}_y^* \otimes \mathcal{U}_X^*)$ has constant dimension 3. So, $\mathcal{V}$ is a vector bundle of rank 3 on C . By Grothendieck duality one can easily check:

$$\mathcal{V} \simeq \pi_{C*}(\mathcal{F} \otimes \pi_X^*(\mathcal{U}_X)).$$

Also, by Grothendieck-Riemann-Roch we see that $\mathcal{V}$ has degree zero.

Lemma II.5. *We have $\mathcal{V}^* \simeq \iota^* \mathcal{V}$, and:*

$$\mathcal{H}^0(\Phi(\mathcal{V}^*)) \simeq \mathcal{U}_X^*, \quad \mathcal{H}^1(\Phi(\mathcal{V}^*)) \simeq \mathcal{Q}_X^*(1).$$

PROOF. Set $\mathcal{A} = \Phi(\mathbf{D}^b(C))$. Consider the decomposition $\mathbf{D}^b(X) = \langle \mathcal{O}_X, \mathcal{U}_X^*, \mathcal{A} \rangle$ and apply $L_{\mathcal{O}_X}$ to $\mathcal{U}_X^*$ in order to mutate it to $\langle \mathcal{Q}_X^*, \mathcal{O}_X, \mathcal{A} \rangle$. In this setting (cf. [40]) we have $\mathcal{Q}_X^*(1)[-3] \simeq R_{\mathcal{A}} R_{\mathcal{O}_X} \mathcal{Q}_X^* \simeq R_{\mathcal{A}} \mathcal{U}_X^*[-1]$, because right-mutating a generator E of $\mathbf{D}^b(X)$ all around a semiorthogonal decomposition we must get $E \otimes \omega_X^*[\dim X]$. Hence $\mathcal{Q}_X^*(1)[-2] \simeq R_{\mathcal{A}} \mathcal{U}_X^*$. But we have an exact mutation triangle:

$$R_{\mathcal{A}} \mathcal{U}_X^* \rightarrow \mathcal{U}_X^* \rightarrow \Phi(\Phi^*(\mathcal{U}_X^*)),$$

Note that this triangle reads:

$$\mathcal{Q}_X^*(1)[-2] \rightarrow \mathcal{U}_X^* \rightarrow \Phi(\mathcal{V}^*).$$

Taking cohomology we obtain the displayed isomorphisms.

To see that $\mathcal{V}^* \simeq \iota^* \mathcal{V}$, we apply the anti-autoequivalence τ to the above mutation triangle. Using (II.2), we easily see that this reads:

$$\mathcal{Q}_X^*(1)[-2] \rightarrow \mathcal{U}_X^* \rightarrow \tau(\Phi(\mathcal{V}^*)).$$

But $\text{Hom}_X(\mathcal{Q}_X^*(1)[-2], \mathcal{U}_X^*) \simeq \text{Ext}_X^1(\mathcal{U}_X^*, \mathcal{Q}_X^*)^*$ by Serre duality, and this group is easily seen to be one-dimensional. Therefore, the two previous triangles are isomorphic, and $\tau(\Phi(\mathcal{V}^*)) \simeq \Phi(\mathcal{V}^*)$. This implies $\mathcal{V}^* \simeq \iota^*(\mathcal{V})$ by Lemma II.3. $\square$

Proposition II.6. *There is an isomorphism $\text{Hilb}_{3t+1}(X) \simeq \mathbb{P}(\mathcal{V}^*)$. For any $x \in X$, there is a triple cover of C parametrizing twisted cubics in X through x .*

PROOF. Actually, we have almost completed the proof of the first statement already in Theorem II.1. Again, we have $\mathcal{H}_0^3(X) = \text{Hilb}_{3t+1}(X)$ by [48, Lemma 3.2]. We have shown in Theorem II.1, that, given a twisted cubic $D \subset X$, we have $H^i(X, \mathcal{I}_{D/X}) = 0$ for all i and that $h^j(X, \mathcal{U}_X \otimes \mathcal{I}_{D/X}) = \delta_{2,j}$ so that $\Psi(\Phi^*(\mathcal{I}_{D/X}(1)))$ is $\mathcal{U}_X^*[1]$. Also, we have obtained the exact sequence (II.12), so that applying $\Phi^!$ we see that $\mathcal{I}_{D/X}(1)$ gives a point $y \in C$ via $\Phi^!(\mathcal{I}_{D/X}(1)) \simeq \mathcal{O}_y$. Also, again by (II.12) the curve $D \subset X$ gives a non-zero section $s \in H^0(X, \mathcal{U}_X \otimes \mathcal{F}_y) \simeq \mathcal{V}_y$. The transpose of s is then a point of $\mathbb{P}(\mathcal{V}_y^*)$. This defines a morphism $\text{Hilb}_{3t+1}(X) \rightarrow \mathbb{P}(\mathcal{V}^*)$.

The inverse morphism is defined again in the proof of Theorem II.1. Indeed, given a point $y \in C$ and a non-zero global section $s \in H^0(X, \mathcal{U}_X \otimes \mathcal{F}_y)$ we have said that the cokernel of the corresponding map $\mathcal{U}_X^* \rightarrow \mathcal{F}_y$ is $\mathcal{I}_{D/X}(1)$ for some D in $\text{Hilb}_{3t+1}(X)$. Clearly these maps are mutually inverse, so our first statement is proved.

Let us now globalize (II.6). We have a natural map $\pi_C^*(\mathcal{V}) \rightarrow \pi_X^*(\mathcal{U}_X) \otimes \mathcal{F}$ and hence a map $e : \mathcal{U}_X^* \boxtimes \mathcal{V} \rightarrow \mathcal{F}$ on the product $X \times C$, which is surjective

by (II.6) as it restricts to e_y for any $y \in C$. For simplicity, we denote by ι also the product $\text{id}_X \times \iota$. Of course, $\ker(e)$ restricts to $\mathcal{F}_{\iota y}^*(1)$ by (II.6), so $\ker(e) \simeq \iota^* \mathcal{F}^*(h_X + \ell)$ for some divisor ℓ in C . We have thus an exact sequence:

$$0 \rightarrow \iota^* \mathcal{F}^*(h_X + \ell) \rightarrow \mathcal{U}_X^* \boxtimes \mathcal{V} \rightarrow \mathcal{F} \rightarrow 0.$$

Taking its dual, applying ι^* and tensoring with $\mathcal{O}_{X \times C}(h_X + \ell)$, we get:

$$0 \rightarrow \iota^* \mathcal{F}^*(h_X + \ell) \rightarrow \mathcal{U}_X^* \boxtimes \iota^* \mathcal{V}^*(\ell) \rightarrow \mathcal{F} \rightarrow 0.$$

In order to lift the identity of $\mathcal{F}$ to a morphism between the middle terms of these sequences, we prove:

$$\text{Ext}_{X \times C}^1(\mathcal{U}_X^* \boxtimes \mathcal{V}, \iota^* \mathcal{F}^*(h_X + \ell)) \simeq H^1(X \times C, \mathcal{U}_X \boxtimes \mathcal{V}^* \otimes \iota^* \mathcal{F}^*(h_X + \ell)) = 0.$$

By the Leray spectral sequence, this is clear since $R\pi_C(\mathcal{U}_X \boxtimes \mathcal{V}^* \otimes \iota^* \mathcal{F}^*(h_X + \ell)) = 0$, which in turn follows from the fact that, for any $y \in C$, we have $H^i(X, \mathcal{F}_{\iota y}^* \otimes \mathcal{U}_X^*) = 0$ for all i , which is equivalent to saying that $\mathcal{U}_X^*$ is in the right-orthogonal to $\Phi(\mathbf{D}^b(C))$ in $\mathbf{D}^b(X)$.

We have now a non-zero morphism $f : \mathcal{U}_X^* \boxtimes \mathcal{V} \rightarrow \mathcal{U}_X^* \boxtimes \iota^* \mathcal{V}^*(\ell)$. In fact, f is an isomorphism, as it can be easily seen along a fibre of $y \in C$ by stability of the involved bundles. This induces an isomorphism $\mathcal{V}(-\ell) \simeq \iota^* \mathcal{V}^*$. Then, by Lemma II.5, we get $\ell \equiv 0$.

Finally, let us define the coherent sheaf $\mathcal{W} = \pi_{C*}(\iota^* \mathcal{F}^*(h_X))$ on C . Since we proved $h^k(X, \mathcal{F}_y^*(1)) = 6\delta_{k,0}$, we have that $\mathcal{W}$ is a vector bundle of rank 6 on C , and by Grothendieck-Riemann-Roch we see that it has degree -6 . Note that, by definition, $\iota^* \mathcal{W} \otimes \omega_C \simeq \Phi^!(\mathcal{O}_X(1))[-1]$. Observe also that the natural map $f : \pi_C^*(\mathcal{W}) \rightarrow \iota^* \mathcal{F}^*(h_X)$ is surjective by (II.5) so that $\ker(f) \simeq \mathcal{F}(-h_X - m)$ for some divisor m of C , and it turns out that $\mathcal{O}_C(m)$ lies in $\text{Pic}^2(C)$. We would like to show that (II.6) and (II.5) globalize in fact to an exact diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{F}(-h_X - h_C) & \longrightarrow & \mathcal{O}_X \boxtimes \mathcal{W} & \longrightarrow & \mathcal{U}_X^* \boxtimes \mathcal{V} \longrightarrow \mathcal{F} \longrightarrow 0. \\ & & & & \searrow & & \nearrow \\ & & & & & \iota^* \mathcal{F}^*(h_X) & \end{array}$$

The extension corresponding to the rightmost half of this diagram lies in:

$$H^1(X \times C, \mathcal{F}^* \otimes \iota^* \mathcal{F}^*(h_X)).$$

Now, $H^k(X, \mathcal{F}_y^*)$ for all k and all $y \in C$, so $R\pi_{C*}(\mathcal{O}_X \boxtimes \mathcal{W} \otimes \mathcal{F}^*) = 0$ hence:

$$R^i \pi_{C*}(\mathcal{F}^* \otimes \iota^* \mathcal{F}^*(h_X)) \simeq R^{i+1} \pi_{C*}(\mathcal{F}^* \otimes \mathcal{F}(-h_X - m)).$$

Using Grothendieck duality, we see that this sheaf is isomorphic to:

$$\mathcal{O}_C(-m) \otimes (R^{2-i} \pi_{C*}(\mathcal{F}^* \otimes \mathcal{F}))^* \simeq \begin{cases} \mathcal{O}_C(-m) & \text{for } i=2, \\ \omega_C(-m) & \text{for } i=1, \\ 0 & \text{otherwise.} \end{cases}$$

By the Leray spectral sequence, for our extension to be non-split we must then have $H^0(C, \omega_C(-m)) \neq 0$, i.e., $h_C = m$, and we have obtained our diagram.

Consider now the universal curve $\mathcal{C}$ over $X \times \text{Hilb}_{3t+1}(X)$. To describe it, we consider the tautological relatively ample line bundle $\mathcal{O}_{\mathcal{V}^*}(1)$ over $\mathbb{P}(\mathcal{V}^*)$ and we let $\pi : \mathbb{P}(\mathcal{V}^*) \rightarrow C$ be the natural projection. To simplify notation we denote by π also the projection $X \times \mathbb{P}(\mathcal{V}^*) \rightarrow X \times C$. We have a natural map $\mathcal{O}_{\mathcal{V}^*}(-1) \rightarrow \pi^*(\mathcal{V})$, and hence a morphism $\mathcal{U}_X^* \boxtimes \mathcal{O}_{\mathcal{V}^*}(-1) \rightarrow \pi^*\mathcal{F}$. It is now clear that, chosen any $y \in C$ and any $0 \neq s \in H^0(X, \mathcal{U}_X \otimes \mathcal{F}_y)$, this map restricts to natural map appearing in (II.12), whose cokernel is $\mathcal{I}_{D/X}(1)$ where D is the twisted cubic associated with s . This says that the ideal of the universal curve $\mathcal{C}$ in $X \times \text{Hilb}_{3t+1}(X)$ has the following resolution:

$$0 \rightarrow \mathcal{U}_X^* \boxtimes \mathcal{O}_{\mathcal{V}^*}(-1) \rightarrow \pi^*\mathcal{F} \rightarrow \mathcal{I}_{\mathcal{C}} \otimes N \rightarrow 0,$$

for some line bundle N over $X \times \text{Hilb}_{3t+1}(X)$. Restricting to a point of X and to a point of C , we get 3 points as degeneracy locus of a 2×3 matrix of linear forms, that are the 3 fibres of our desired cover. $\square$

II.4. Fano threefolds of genus 10 and Coble cubic. Here, we carry out a construction relating a smooth Fano threefold X of genus 10 to the Coble cubic associated to the genus-2 curve, homologically dual to X .

II.4.1. Introduction to theta map and Coble cubic. Let us first give an account of the Coble cubic, and related material on moduli spaces of bundles on curves, particularly in genus 2. We will mainly follow [32, 255] to introduce the construction of the theta map, going back to [246]. So we start with a smooth complex projective curve C of genus $g \geq 2$, and consider the abelian variety $\text{Pic}^{g-1}(C)$ parametrizing line bundles of degree $g-1$ on C . This contains Riemann's theta divisor Θ , defined (set-theoretically) by:

$$\Theta = \{N \in \text{Pic}^{g-1}(C) \mid H^0(C, L) \neq 0\}.$$

This is an ample Cartier divisor, and the linear system $|r\Theta|$ has dimension $r^g - 1$.

Consider now the moduli space $M_C(r, d)$ of semistable vector bundles of rank r and degree d on C (sometimes denoted by $U_C(r, d)$). Taking determinant gives a fibration $M_C(r, d) \rightarrow \text{Pic}^d(C)$ whose fibre over a line bundle N is the moduli space $M_C(r, N)$ of semistable bundles E of rank r with $\wedge^r E \simeq N$. When $N = \mathcal{O}_C$, we write $M_C(r) = M_C(r, \mathcal{O}_C)$ (or sometimes $SU_C(r)$). The closed points in $M_C(r)$ are isomorphism classes of vector bundles with trivial determinant, that are direct sums of stable vector bundles of degree zero.

This space is an integral, normal, unirational projective variety of dimension $(r^2 - 1)(g - 1)$, with Gorenstein singularities. The singular locus of $M_C(r)$ is given by decomposable bundles, except for $g = r = 2$, in which case

$M_C(2) \simeq \mathbb{P}^3$. The Picard number of $M_C(r)$ is 1, and it is generated by the so-called *determinant line bundle* $\mathcal{L}$ given as follows: choose $N \in \text{Pic}^{g-1}(C)$, and consider:

$$\Delta_N = \{E \in M_C(r) \mid H^0(C, E \otimes N) \neq 0\}.$$

This is a Cartier divisor in $|\mathcal{L}|$, and does not depend on N . Also, $\omega_{M_C(r)}^* \simeq \mathcal{L}^{\otimes 2}$.

Viceversa, for any E in $M_C(r)$ we let:

$$\theta(E) = \{N \in \text{Pic}^{g-1}(C) \mid H^0(C, E \otimes N) \neq 0\}.$$

It turns out that either $\theta(E) = \text{Pic}^{g-1}(C)$, or $\theta(E)$ is a divisor of class $r\Theta$ (in which case one says that E has a *theta*), cf. [33], see also [32]. This way we define a rational map:

$$\theta : M_C(r) \dashrightarrow |r\Theta|.$$

By [33], there is a natural way to identify $|r\Theta|$ and $|\mathcal{L}|^*$.

The map θ has been widely investigated. Here are some of its properties, and some questions on about it.

- i) The base locus can be non-empty, the first examples being due to Raynaud [273]. More examples are given in [9, 268].
- ii) For rank 2, if C is not hyperelliptic, θ is an embedding, see [99, 309]. Also, θ is everywhere defined for $r = 2$, and for $r = 3$ in case C is generic, or has genus 2 or 3 [31, 273]).
- iii) For $g \geq 3$, is it true that θ is generically 2 : 1 if C is hyperelliptic, and generically 1 : 1 if C is not hyperelliptic? Cf. [32, Speculation 6.1].
- iv) Is it true that θ is everywhere defined if $r = 3$? Cf. [32, Conjecture 6.2].

Let us now turn to the case $g = 2$, the most relevant for us. In this case, some numerical coincidences make the situation even more interesting. This time, θ is generically finite (cf. [31]), and in fact an isomorphism for $r = 2$ (cf. [246, 247]), but not always a morphism for $r \geq 4$ (cf. [273]). Raynaud's bundle has rank 4 and slope $1/4$, and gives 16 base points for θ . This bundle also appears in the analysis of instanton bundles over the complete intersection of 2 quadrics in $\mathbb{P}^5$, cf. [133, 220].

For $r = 3$, according to [224], θ is a 2 : 1 cover, whose branch locus is, by [255] (see also [248]), a sextic hypersurface, in fact the projective dual of the Coble cubic in $\mathbb{P}^8$. This was originally conjectured by Dolgachev (as reported in [224]); we call this sextic the *Coble-Dolgachev sextic* in $\mathbb{P}^8$.

This cubic mentioned above first arose in Coble's paper [86]. He claimed (and Barth proved in [25]) that the image of the Jacobian $J(C)$ in $\mathbb{P}^8$ by the linear system 3Θ is the singular locus of a cubic, whose partial derivatives generate the ideal of $J(C)$. We refer to [176, Remark 5.3.1] for a nice treatment of this cubic. More material on $M_C(3)$ and Coble's cubic can be found in [30, 233]. For more on the local structure of $M_C(3)$ in this case, see [286].

II.4.2. *Fano threefold of genus 10 and Coble-Dolgachev sextic.* Now let us go back to our situation, so X is a smooth Fano threefold of genus 10, and C is the homological projective dual of X , which is a smooth projective curve of genus 2. This curve is equipped with the rank-3 bundle $\mathcal{V}^*$, obtained as direct image of $\mathcal{U}_X^*$ from the universal bundle coming from the interpretation of C as moduli space of rank-3 bundles on X . The result is the following.

Theorem II.7. *The sheaf $\mathcal{V}^*$ is a stable bundle of rank 3 with trivial determinant, satisfying $\iota^*\mathcal{V} \simeq \mathcal{V}^*$, so that its image via θ lies in the Coble-Dolgachev sextic.*

PROOF. We have proved $\iota^*\mathcal{V} \simeq \mathcal{V}^*$ in Lemma II.5. The fact that a stable bundle with trivial determinant is mapped via θ to a point in the dual of the Coble cubic is proved in [255].

Let us check that $\mathcal{V}$ is stable. It suffices to prove that, given any line bundle N on C of degree 0, there cannot be a non-trivial map $\mathcal{V}^* \rightarrow N$. Indeed, in this case $\mathcal{V}$ contains no line bundle N' of non-negative degree (i.e., $\mathcal{V}$ cannot be destabilized by quotient sheaves of rank 2), since any such N' contains a line bundle N of degree 0, so that $\mathcal{V}^*$ maps non-trivially to N^* . But then $\mathcal{V}$ is not destabilized by quotient sheaves of rank 1 either, since any such sheaf is contained into a line bundle N of degree zero, so $\mathcal{V}^*$ would map non-trivially to $\iota^*(N)$.

Now we show that, given a line bundle N in $\text{Pic}^0(C)$, we have $\text{Hom}_C(\mathcal{V}^*, N) = 0$. Observe that, since $\Phi^*(\mathcal{U}_X^*) \simeq \mathcal{V}^*$ and Φ^* is left-adjoint to Φ , this space is isomorphic to $\text{Hom}_X(\mathcal{U}_X^*, \Phi(N))$. Recall by Proposition II.2 that there exists a conic $D \subset X$ such that $N \simeq \Phi^!(\mathcal{I}_{D/X}(1))[-1]$. We have also seen that $\Phi(N)[1] \simeq \mathcal{I}_{D/X}(1)$. We obtain:

$$\text{Hom}_C(\mathcal{V}^*, N) \simeq \text{Hom}_X(\mathcal{U}_X^*, \mathcal{I}_{D/X}(1)[-1]) = 0.$$

To check that $\mathcal{V}$ has trivial determinant, first recall that $\deg(\mathcal{V}) = 0$. Then, we use again the construction of Kuznetsov. More precisely, we recall that the variety Γ , that turns out to be a double cover of $\mathbb{P}^6$ ramified along a sextic hypersurface, is constructed in [216, Section 5] as a certain moduli space of representations of the Kronecker quiver

$$Q_V : \quad \bullet \xrightarrow{V} \bullet$$

This quiver is the oriented graph with 2 vertices, called the source and the target, and 7 arrows pointing from the source to the target (the notation Q_V is supposed to remind the reader that $\dim(V) = 7$ so the 7 arrows are indexed by a basis of V). The moduli space in question is that of representations of Q_V with dimension vector $(6, 3)$, semistable with respect to the character $\vartheta : \text{GL}_6 \times \text{GL}_3 \rightarrow \mathbb{C}^*$ of weight vector $(-1, 2)$, i.e. $\vartheta(g_1, g_2) = \det(g_2)^2 / \det(g_1)$ (see for instance [56] for an overview of representations of quivers and their moduli spaces). Since $\text{Pic}(\Gamma)$ is generated by the hyperplane section of

$\mathbb{P}^6$, it suffices to show that $\mathcal{V}$ and $\mathcal{W}$ lift to Γ to prove $\det(\mathcal{V}) \simeq \mathcal{O}_C$ and $\det(\mathcal{W}) \simeq \mathcal{O}_C(-3h_C)$. So, to complete the argument, we can show, for each $y \in C$, the ϑ -semistability of the representation of Q_V provided by the diagram of Proposition (II.6) via the identification $H^0(X, \mathcal{U}_X^*) = V^*$. This way, the diagram itself will be a pull-back to $X \times C$ of the diagram at [216, Page 524].

To achieve this, given $y \in C$, we regard the map $\mathcal{O}_X^6 \rightarrow (\mathcal{U}_X^*)^3$ corresponding to y as a representation of Q_V , and we consider a destabilizing subrepresentation ρ of it, of dimension vector (a, b) hence $2b - a \leq 0$ with $(0, 0) \neq (a, b) \neq (6, 3)$. Denote by F' , F and F'' the kernel, image and cokernel of the map $\mathcal{O}_X^a \rightarrow (\mathcal{U}_X^*)^b$ corresponding to ρ . We have a commutative exact diagram:

$$\begin{array}{ccccccccc}
 0 & \rightarrow & \mathcal{F}_y(-1) & \rightarrow & \mathcal{O}_X^6 & \xrightarrow{\quad} & (\mathcal{U}_X^*)^3 & \rightarrow & \mathcal{F}_y & \rightarrow & 0 \\
 & & \uparrow & & \uparrow & \searrow & \uparrow & & \uparrow & & \\
 & & & & & \mathcal{F}_{\iota y}^*(1) & & & & & \\
 0 & \rightarrow & F' & \rightarrow & \mathcal{O}_X^a & \xrightarrow{\quad} & (\mathcal{U}_X^*)^b & \rightarrow & F'' & \rightarrow & 0 \\
 & & & & \searrow & \nearrow & & & & & \\
 & & & & & F & & & & &
 \end{array}$$

The slope of $\mathcal{F}_{\iota y}^*(1)$ is $1/3$, so $c_1(F) \leq 0$ or F has slope $1/3$.

In the first case, since $c_1(F') \leq 0$ and $c_1(F') = -c_1(F)$ we get $c_1(F) = c_1(F') = 0$ which implies $F' = 0$ since $F' \subset \mathcal{F}_y(-1)$. Therefore $F' \simeq \mathcal{O}_X^a$ hence F'' has slope $\frac{b}{2b-a}$ and this value is thus $\leq 2/3$. But this is compatible with $2b \leq 0$ only if $(a, b) = (0, 0)$: contradiction!

In the second case, $c_1(F') = -1$ implies that $a = 6$ for otherwise F' would destabilize $\mathcal{F}_{\iota y}^*(1)$. Then the slope of F'' is $\frac{b-1}{2b-3}$. Imposing to this value to be at most $2/3$, we get $b = 3$, so $(a, b) = (6, 3)$, again a contradiction. $\square$

II.5. Instanton bundles. Here is our description of the moduli space of (odd) instanton bundles on Fano threefolds of genus 10.

Theorem II.8. *For any $k \geq 1$, the space $\text{MI}_X(k+6)$ is isomorphic to the moduli space of simple bundles E of rank and degree k on C with $E \simeq \omega_C \otimes \iota^* E^*$, and:*

$$(II.15) \quad h^0(C, \mathcal{V} \otimes E) = h^1(C, \mathcal{V} \otimes E) = k - 1, \quad H^1(C, \mathcal{F}_x \otimes E) = 0, \quad \forall x \in X.$$

Any F in $\text{MI}_X(k+6)$ fits into a functorial exact sequence:

$$(II.16) \quad 0 \rightarrow (\mathcal{U}_X^*)^{k-1} \rightarrow \Phi(\Phi^!(F)) \rightarrow F \rightarrow 0.$$

PROOF. Let us first show that, for any F in $\text{MI}_X(k+6)$, we have the extension (II.16). We do this using (II.3) in the explicit formulation given by (II.4). Then we note that, for any F in $\text{M}_X(2, 1, k+6)$ satisfying $H^1(X, F(-1)) = 0$ we obtain $H^i(X, F(-1)) = 0$ for all i , or equivalently $\text{Ext}_X^j(F, \mathcal{O}_X)$ for all j (this is proved in [48, Lemma 5.1]). So the component along $\mathcal{O}_X$ of $\Psi(\Psi^*(F))$ is zero

Let us now compute $H^i(X, \mathcal{Q}_X \otimes F(-1))$. For $i = 0$ this space vanishes by stability. For $i = 3$, this space is dual to $\text{Hom}_X(F, \mathcal{Q}_X^*)$, which is zero by stability. To obtain the vanishing for $i = 2$ we apply $\text{Hom}_X(F, -)$ to the dual of (II.2) and use $\text{Ext}_X^i(F, \mathcal{O}_X) = 0$ for all i to reduce to $\text{Hom}_X(F, \mathcal{U}_X^*) = 0$. In turn, this space is zero since it is isomorphic to $\text{Hom}_X(\mathcal{U}_X, F^*) = 0$, any map $\mathcal{U}_X \rightarrow F^*$ should be an isomorphism, while we have $c_2(F^*) > c_2(\mathcal{U}_X) = 6$ (we have used here that F is locally free). By Riemann-Roch we have now $h^1(X, \mathcal{Q}_X \otimes F(-1)) = k - 1$. We have obtained (II.16), and $\Phi(\Phi^!(F))$ is a vector bundle of rank $2k$, left-orthogonal to $\mathcal{U}_X^*$.

Let us now show that, for any sheaf F in $M_X(2, 1, k + 6)$ with $H^1(X, F(-1)) = 0$, and any $y \in C$, $\phi^!(F)$ is a vector bundle of rank and degree k on C . To do this, we have to prove $H^i(X, \mathcal{F}_y^* \otimes F) = 0$ for $i \neq 1$. For $i = 0$ and $i = 3$, again this is clear by stability. For $i = 2$, we apply ι^* to (II.5) and we tensor with $F(-1)$. Via Serre duality, we reduce thus the vanishing to $\text{Hom}_X(F(-1), \mathcal{F}_y^*) = 0$, which is clear by stability. Riemann-Roch now implies that $h^1(X, \mathcal{F}_y^* \otimes F) = k$, so $E = \Phi^!(F)$ is a vector bundle of rank k on C . By Grothendieck-Riemann-Roch E has degree k .

The next step is to show that our bundle $E = \Phi^!(F)$ has the required properties. First, since we have proved that $\Phi(E)$ is concentrated in degree 0, by base change we get that $H^1(C, \mathcal{F}_x \otimes E) = 0$ for all $x \in X$.

Then, to finish the proof of (II.15), by Riemann-Roch it suffices to prove $h^0(C, \mathcal{V} \otimes E) = k - 1$. In turn, this is equivalent to $h^0(X, \mathcal{U}_X \otimes \Phi(E)) = k - 1$ because Φ^* is left adjoint to Φ and $\mathcal{V}^* \simeq \Phi^*(\mathcal{U}_X^*)$. Tensoring (II.16) with $\mathcal{U}_X$, since $\mathcal{U}_X$ is exceptional, we just have to check $H^0(X, \mathcal{U}_X \otimes F) = 0$. But this is clear since there is no non-zero map $\mathcal{U}_X^* \rightarrow F$, as we already pointed out.

We still haven't checked that E is simple and satisfies $E^* \otimes \omega_C \simeq \iota^*(E)$. To check the first property, we apply $\text{Hom}_X(-, F)$ to (II.16) and again we use $\text{Hom}_X(\mathcal{U}_X^*, F) = 0$ to check that $\text{Hom}_X(\Phi(E), F) \simeq \mathbb{C}$. But $\Phi^!$ is right-adjoint to Φ , so $\text{Hom}_X(E, E) \simeq \mathbb{C}$ and E is simple.

In order to show $E^* \otimes \omega_C \simeq \iota^*(E)$, we define a second anti-autoequivalence on a $\Phi(\mathbf{D}^b(C))$. This time we set, for any G in $\mathbf{D}^b(X)$,

$$\sigma(G) = R_{\mathcal{U}_X^*} \mathcal{R}\mathcal{H}om_X(G, \mathcal{O}_X(1)).$$

To check that σ operates on $\Phi(\mathbf{D}^b(C))$, we let $y \in C$ and we compute $\sigma(\mathcal{F}_y)$. Clearly $\mathcal{R}\mathcal{H}om_X(\mathcal{F}_y, \mathcal{O}_X(1))$ is just $\mathcal{F}_y^*(1)$. Using (II.6), we see that $R_{\mathcal{U}_X^*}(cF_y^*(1)) \simeq \mathcal{F}_{\iota_y}[-1]$. Then σ is an anti-autoequivalence of $\Phi(\mathbf{D}^b(C))$. Again, this says that σ is induced by an anti-autoequivalence σ_C of $\mathbf{D}^b(C)$ of the form $\sigma_C = N \otimes \iota^* \mathcal{R}\mathcal{H}om_C(-, \mathcal{O}_C)$, for some line bundle N on C .

We will see in a minute that $N \simeq \omega_C$. Prior to this, let us check the property that will be useful to us, namely that for any F in $\text{MI}_X(k + 6)$, once set $E =$

$\Phi^!(F)$, we have:

$$\sigma(\Phi(E)) \simeq \Phi(E).$$

Indeed, first of all note that (II.16) says that $\Phi(E) \simeq R_{\mathcal{U}_X^*}(F)$. Then, applying $R\mathcal{H}om_X(-, \mathcal{O}_X(1))$ to (II.16), and using $F \simeq R\mathcal{H}om_X(F, \mathcal{O}_X(1))$, we get an exact sequence:

$$0 \rightarrow F \rightarrow R\mathcal{H}om_X(\Phi(E), \mathcal{O}_X(1)) \rightarrow (\mathcal{U}_X^*)^{k-1} \rightarrow 0.$$

Applying $R_{\mathcal{U}_X^*}$ we get that σ fixes $\Phi(E)$ since $R_{\mathcal{U}_X^*}\mathcal{U}_X^* = 0$ and $\Phi(E) \simeq R_{\mathcal{U}_X^*}(F)$.

Let us now go ahead to check $N \simeq \omega_C$. Let F lie in $\text{MI}_X(2, 1, 7)$. We have said that $E = \Phi^!(F)$ is a line bundle of degree 1 on C , and that $\sigma(\Phi(E)) \simeq \Phi(E)$ so that $E \simeq \sigma_C(E)$ hence $E \simeq N \otimes \iota^*E^*$. Then, N is a line bundle of degree 2 which is invariant by ι and as such must be ω_C . In conclusion we have proved that, for any F in $\text{MI}_X(k+6)$, once set $E = \Phi^!(F)$, there is an isomorphism $E \simeq \omega_C \otimes \iota^*E^*$.

To define the inverse map of $\Phi^!$ from our moduli space of bundles on C to $\text{MI}_X(k+6)$, we consider a vector bundle E over C and let $G = \Phi(E)$. By the assumption $H^1(C, \mathcal{F}_X \otimes E) = 0$ for all $x \in X$, we identify G with a vector bundle on X which is left-orthogonal to $\mathcal{O}_X$ and $\mathcal{U}_X^*$. Note that $\sigma(G) \simeq G$ because $E \simeq \omega_C \otimes \iota^*E^*$. This means $G \simeq R_{\mathcal{U}_X^*}(G^*(1))$. Writing down the mutation exact triangle and taking cohomology we get a long exact sequence:

$$0 \rightarrow H^1(X, \mathcal{U}_X \otimes G)^* \otimes \mathcal{U}_X^* \rightarrow G \rightarrow G^*(1) \rightarrow H^0(X, \mathcal{U}_X \otimes G)^* \otimes \mathcal{U}_X^* \rightarrow 0,$$

together with its dual, tensored with $\mathcal{O}_X(1)$:

$$(II.17) \quad 0 \rightarrow H^0(X, \mathcal{U}_X \otimes G) \otimes \mathcal{U}_X^* \rightarrow G \rightarrow G^*(1) \rightarrow H^1(X, \mathcal{U}_X \otimes G) \otimes \mathcal{U}_X^* \rightarrow 0.$$

Note that, for all i :

$$\begin{aligned} H^i(X, \mathcal{U}_X \otimes G) &\simeq \text{Ext}_X^i(\mathcal{U}_X^*, \Phi(E)) \simeq \text{Ext}_C^i(\Phi^*(\mathcal{U}_X^*), E) \simeq H^i(C, \mathcal{V} \otimes E). \\ H^i(C, \mathcal{V} \otimes E) &\simeq H^{1-i}(C, \mathcal{V}^* \otimes E^* \otimes \omega_C)^* \simeq H^{1-i}(C, \iota^*(\mathcal{V} \otimes E))^*. \end{aligned}$$

Let F be the image of the middle map in (II.17). In view of the previous isomorphisms, $h^i(X, \mathcal{U}_X \otimes G) = k-1$ for $i = 0, 1$, and the Chern classes of G are computed by Grothendieck-Riemann-Roch, so that F is a vector bundle of rank 2 on X with $c_1(F) = 1$ and $c_2(F) = k$. Moreover, applying $\text{Hom}_X(-, \mathcal{O}_X)$ we see that $\text{Ext}_X^i(F, \mathcal{O}_X) = 0$ for all i so that F is stable and actually lies in $\text{MI}_X(k+6)$.

Clearly, applying $\Phi^!$ to the exact sequence defining F we see that $E \simeq \Phi^!(G) \simeq \Phi^!(F)$. Also, the left half of the exact sequence defining F is just (II.16), so that the instanton associated with $\Phi^!(F)$ is F . So our maps are mutually inverse. $\square$

II.6. Lines and Theta divisors. The Hilbert scheme $\text{Hilb}_{t+1}(X)$ of lines contained in a smooth Fano threefold X of index 1, sometimes called the *Fano*

variety of lines of X , has also been very well studied, see [188, Theorem 4.2.7] and, references therein for an account on this. In the case of genus 10 that we are interested in here, this curve has arithmetic genus 10, and is smooth and connected for X general enough.

Here, we show that $\text{Hilb}_{t+1}(X)$ is isomorphic to $\theta(\mathcal{V})$. We have said that $\theta(\mathcal{V})$ is a divisor of class 3Θ in $\text{Pic}^1(C)$, cf. [273]. This agrees with the well-known fact that $\text{Hilb}_{t+1}(X)$ has dimension 1, see the results of Iskovskih in [187].

Theorem II.9. *The map $L \mapsto \Phi^1(\mathcal{I}_{L/X})$ defines an isomorphism:*

$$\text{Hilb}_{t+1}(X) \simeq \theta(\mathcal{V}) = \{E \in \text{Pic}^1(C) \mid H^0(C, \mathcal{V} \otimes E) \neq 0\}.$$

Moreover, $M_X(2, 1, 7) \simeq \text{Pic}^1(C)$, and $\text{MI}_X(7)$ is the complement of $\theta(\mathcal{V})$.

PROOF. As a preliminary step, given a line $L \subset X$, we compute a resolution of $\mathcal{O}_L$ in terms of our semiorthogonal decomposition $\mathbf{D}^b(X) = \langle \mathcal{O}_X, \mathcal{U}_X^*, \Phi(\mathbf{D}^b(C)) \rangle$. In other words, we write a resolution of $\mathcal{O}_L$ via (II.3) using the expression (II.4).

First, we have $H^i(X, \mathcal{O}_L(-1)) = 0$ for all i . Second, note that $\mathcal{Q}_L$ is a globally generated bundle of rank 5 and degree 1, and as such it splits as $\mathcal{O}_L^4 \oplus \mathcal{O}_L(1)$. Then $h^i(X, \mathcal{Q}_X \otimes \mathcal{O}_L(-1)) = \delta_{i,3}$. We have thus computed $\Psi(\Psi^*(\mathcal{O}_L))$, and we deduce that:

$$\mathcal{H}^{-1}(\Phi(\Phi^1(\mathcal{O}_L))) \simeq \mathcal{U}_X^*, \quad \mathcal{H}^0(\Phi(\Phi^1(\mathcal{O}_L))) \simeq \mathcal{O}_L.$$

Our task is now to show that our map $\text{Hilb}_{t+1}(X) \rightarrow \text{Pic}^1(C)$ is well-defined. Remark that $\Phi^1(\mathcal{O}_L) \simeq \Phi^1(\mathcal{I}_{L/X})[1]$ since $\Phi^1(\mathcal{O}_X) = 0$. We have to check that $\Phi^1(\mathcal{O}_L)[-1]$ is a line bundle on C . For any $y \in C$, consider $(\mathcal{F}_y^*(1))|_L$. Again, this is a globally generated vector bundle of rank 3 and degree 1, so it must split as $\mathcal{O}_L^2 \oplus \mathcal{O}_L(1)$. So, $\mathcal{F}_y^*|_L \simeq \mathcal{O}_L^2(-1) \oplus \mathcal{O}_L$, hence $h^i(X, \mathcal{F}_y^* \otimes \mathcal{O}_L) = \delta_{i,0}$. This says that $N_L = \Phi^1(\mathcal{O}_L)[-1] \simeq \Phi^1(\mathcal{O}_L)$ is a line bundle on C , and Grothendieck-Riemann-Roch shows that N_L has degree 1.

Next, we show that N_L belongs to $\theta(\mathcal{V})$. We have to prove that $\text{Hom}_C(\mathcal{V}^*, N_L)$ is 1-dimensional. This follows from the isomorphisms:

$$\text{Hom}_C(\mathcal{V}^*, N_L) \simeq \text{Hom}_X(\Phi(\mathcal{V}^*), \mathcal{O}_L[-1]) \simeq \text{Hom}_X(\mathcal{H}^1(\Phi(\mathcal{V}^*)), \mathcal{O}_L),$$

where the last isomorphism is given by a degenerate hypercohomology spectral sequence. In fact, the last space in the previous display is isomorphic to $H^0(X, \mathcal{Q}_X(-1) \otimes \mathcal{O}_L)$, and we have seen that this space is 1-dimensional.

So far we have shown that $L \mapsto \Phi^1(\mathcal{I}_{L/X})$ defines a morphism $\text{Hilb}_{t+1}(X) \rightarrow \theta(\mathcal{V})$, which is clearly an embedding since for any line L in X we have $\mathcal{H}^0(\Phi(\Phi^1(\mathcal{O}_L))) \simeq \mathcal{O}_L$.

We pass now to the description of $M_X(2, 1, 7)$. We resume the setting of Theorem II.8 in the case of bundles with $c_1 = 1$ and *almost minimal charge*

namely with $c_2 = 7$, but this time not only for $\text{MI}_X(7)$ but for the full moduli space $\text{M}_X(2, 1, 7)$. By [48, Proposition 3.4], we have that any sheaf F in $\text{M}_X(2, 1, 7)$ satisfies $H^1(X, F(-1)) = 0$ hence again $H^i(X, F(-1)) = 0$ for all i , so the component along $\mathcal{O}_X$ of $\Psi(\Psi^*(F))$ is still zero. In other words, the complement of $\text{MI}_X(7)$ in $\text{M}_X(2, 1, 7)$ consists of the sheaves that are not locally free. Moreover, again by [48, Proposition 3.4], F is either locally free (i.e., it lies in $\text{MI}_X(7)$) or fits into:

$$(II.18) \quad 0 \rightarrow F \rightarrow \mathcal{U}_X^* \rightarrow \mathcal{O}_L \rightarrow 0,$$

for some line $L \subset X$. In the latter case we have $E \simeq \Phi^!(\mathcal{O}_L)[-1]$. In any case, $\Phi^!$ maps $\text{M}_X(2, 1, 7)$ to $\text{Pic}^1(C)$.

Let us look at the component along $\mathcal{U}_X^*$ of $\Psi(\Psi^*(F))$ for non-locally free F . Again we have $H^i(X, \mathcal{Q}_X \otimes F(-1)) = 0$ for $i = 0, 3$ and we compute as in the proof of Theorem II.8:

$$H^2(X, \mathcal{Q}_X \otimes F(-1)) \simeq \text{Hom}_X(F, \mathcal{U}_X^*)^*.$$

This space is non-zero if and only if it is 1-dimensional, and this happens precisely F is not locally free: we checked one implication in Theorem II.8, and the converse is clear by (II.18). Also, by Riemann-Roch we have:

$$h^2(X, \mathcal{Q}_X \otimes F(-1)) = h^1(X, \mathcal{Q}_X \otimes F(-1)).$$

So, if F in $\text{M}_X(2, 1, 7)$ is not locally free we get a long exact sequence:

$$0 \rightarrow \mathcal{U}_X^* \rightarrow \mathcal{H}^0(\Phi(E)) \rightarrow F \rightarrow \mathcal{U}_X^* \rightarrow \mathcal{H}^1(\Phi(E)) \rightarrow 0.$$

Comparing with (II.18), we get:

$$(II.19) \quad \mathcal{H}^0(\Phi(E)) \simeq \mathcal{U}_X^*, \quad \mathcal{H}^1(\Phi(E)) \simeq \mathcal{O}_L.$$

Moreover, recall the hypercohomology spectral sequence:

$$\text{Ext}_C^{i+j}(\mathcal{V}^*, \Phi^!(F)) \simeq \text{Ext}_X^{i+j}(\Phi(\mathcal{V}^*), F) \Leftarrow \text{Ext}_X^i(\mathcal{H}^{-j}\Phi(\mathcal{V}^*), F),$$

Hence, since $H^i(X, \mathcal{U}_X \otimes F) = 0$ for all i , from Lemma II.5, we get:

$$h^2(X, \mathcal{Q}_X \otimes F(-1)) = h^0(C, \mathcal{V} \otimes E).$$

Therefore, $h^0(C, \mathcal{V} \otimes E)$ is either zero or one, and the second case takes place if and only if F fails to be locally free.

Summing up, we have proved that $\Phi^!$ defines a morphism $\text{MI}_X(2, 1, 7) \rightarrow \text{Pic}^1(C)$. The restriction to $\text{MI}_X(7)$ of this morphism is an embedding since $F \simeq \Phi(E)$ for any $F \in \text{MI}_X(7)$. Also, the image of $\text{MI}_X(7)$ actually lies in the complement of $\theta(\mathcal{V})$ since $H^0(C, \mathcal{V} \otimes E) = 0$ when F lies in $\text{MI}_X(7)$ (equivalently when F is locally free), while $h^0(C, \mathcal{V} \otimes E) = 1$ if F lies in the complement of $\text{MI}_X(7)$.

In fact, any irreducible component of $\text{MI}_X(7)$ has dimension at least 2 by the usual dimension estimate, so $\text{MI}_X(7)$ is open and dense in $\text{Pic}^1(C)$. Since $\text{M}_X(2, 1, 7)$ is projective and $\text{Pic}^1(C)$ is irreducible, we conclude that the map $\text{M}_X(2, 1, 7) \rightarrow \text{Pic}^1(C)$ is also surjective.

To continue the proof we show that, given E in $\theta(\mathcal{V})$, there is a line $L \subset X$ such that $\mathcal{H}^1(\Phi(E)) \simeq \mathcal{O}_L$. This will provide an inverse to our map $\text{Hilb}_{t+1}(X) \rightarrow \theta(\mathcal{V})$. Since our morphism $\text{M}_X(2, 1, 7) \rightarrow \text{Pic}^1(C)$ is surjective, there is a non-locally free sheaf $F \in \text{M}_X(2, 1, 7)$ such that $\Phi^1(F) \simeq E$. But we have just proved that $\mathcal{H}^1(\Phi(E)) \simeq \mathcal{O}_L$ in (II.19). This shows $\text{Hilb}_{t+1}(X) \simeq \theta(\mathcal{V})$.

Finally, by (II.19) the complex $\Phi(E)$ is given by an element $s \in \text{Hom}_X(\mathcal{O}_L[-2], \mathcal{U}^*) \simeq H^1(X, \mathcal{U}_X \otimes \mathcal{O}_L(-1))$, which is non-zero. Indeed, otherwise $\Phi(E) \simeq \mathcal{U}_X^* \oplus \mathcal{O}_L[-1]$, in which case $\text{Hom}_X(\Phi(E), \Phi(E))$ contains at least $\mathbb{C}^2$, while we know $\text{Hom}_X(\Phi(E), \Phi(E)) \simeq \text{Hom}_C(E, E) \simeq \mathbb{C}$. Also, note that $h^1(X, \mathcal{U} \otimes \mathcal{O}_L(-1)) = 1$ so s is obtained (up to multiplication by a non-zero scalar) by applying $R\mathcal{H}om_X(-\mathcal{O}_X(1))$ to the natural projection $\mathcal{U}_X^* \rightarrow \mathcal{O}_L$, whose kernel is F . This says that the functor $E \mapsto R\mathcal{H}om_X(\Phi(E), \mathcal{O}_X(1))$ provides an inverse to our morphism $\text{M}_X(2, 1, 7) \rightarrow \text{Pic}^1(C)$. The proof of the theorem is now finished. $\square$

Corollary II.10. *There is an isomorphism of $\theta(\mathcal{V})$ and the locus:*

$$\delta(\mathcal{V}) = \{E \in \text{M}_C(2, 1) \mid H^0(C, \mathcal{V} \otimes E) \neq 0\}.$$

The resulting isomorphism $\text{Hilb}_{t+1}(X) \rightarrow \delta(E)$ is realized by $L \mapsto \Phi^1(\mathcal{J}_{L/X}(1))[-1]$.

PROOF. First of all we note that, given $N \in \theta(E)$, we have in fact $h^0(C, \mathcal{V} \otimes N) = 1$. Indeed, for any such N there is a line $L \subset X$ such that $N \simeq N_L = \Phi^1(\mathcal{O}_L)[-1]$, and we have said that $H^0(C, \mathcal{V} \otimes N_L) = 1$. Therefore, for any $N \in \theta(\mathcal{V})$, we have a canonical evaluation map $e : \mathcal{V}^* \rightarrow N$, and we observe that e is surjective. Indeed, otherwise we would have a quotient bundle of $\mathcal{V}^*$, of rank 1 and degree ≤ 0 , contradicting stability of $\mathcal{V}$.

Next, we consider the bundle $\ker(e)$, fitting into:

$$(II.20) \quad 0 \rightarrow \ker(e) \rightarrow \mathcal{V}^* \rightarrow N \rightarrow 0.$$

We note that $\ker(e)$ is a stable bundle of rank 2 and degree 1. Indeed, a destabilizing line bundle contained in $\ker(e)$ should have degree ≥ 0 , hence it would also destabilize $\mathcal{V}^*$. We can consider thus the bundle $E_N = \iota^*(\ker(e))^*$. We also denote E_N by E_L when $N \simeq \phi^1(\mathcal{O}_L)[-1]$. Now, since $\mathcal{V}^* \simeq \iota^*\mathcal{V}$, we have $H^0(C, \mathcal{V} \otimes E_N) \neq 0$ so that E_N lies in $\delta(\mathcal{V})$. This defines a morphism $\theta(\mathcal{V}) \rightarrow \delta(\mathcal{V})$. But in fact this construction is clearly reversible, since for any $E \in \delta(\mathcal{V})$, and any non-zero map $s : \mathcal{V}^* \rightarrow E$, stability of $\mathcal{V}$ and E easily implies that s is surjective, so that $\iota^*(\ker(s))^*$ lies in $\theta(\mathcal{V})$.

The required isomorphism is thus established. Let us now see that the composition of this isomorphism with the one of Theorem II.9 is given by $L \mapsto \Phi^!(\mathcal{I}_{L/X}(1))[-1]$. To this purpose, given a line $L \subset X$, we compute a resolution of $\mathcal{I}_{L/X}(1)$ in terms of (II.3) using (II.4). We have $H^i(X, \mathcal{I}_{L/X}) = 0$ for all i , and $H^3(X, \mathcal{Q}_X \otimes \mathcal{I}_{L/X}) = 0$ by stability of $\mathcal{Q}_X$ and Serre duality, and similarly we see that $H^3(X, \mathcal{U}_X \otimes \mathcal{I}_{L/X}) = 0$. Tensoring (II.2) with $\mathcal{I}_{L/X}$ we get $H^2(X, \mathcal{Q}_X \otimes \mathcal{I}_{L/X}) = 0$. By the claim emphasized in the proof of Theorem II.1, we see that $h^0(X, \mathcal{Q}_X \otimes \mathcal{I}_{L/X}) \leq 1$, since if there were two independent global sections of $\mathcal{Q}_X$ vanishing on L then X would contain a plane, which is not the case as we know from $\text{Pic}(X) \simeq \langle h_X \rangle$. By Riemann-Roch, we get $H^1(X, \mathcal{Q}_X \otimes \mathcal{I}_{L/X}) = 0$, and in fact $h^0(X, \mathcal{Q}_X \otimes \mathcal{I}_{L/X}) = 1$. Therefore, setting $G_L = \Phi^!(\mathcal{I}_{L/X}(1))[-1]$, we have a distinguished triangle:

$$(II.21) \quad \mathcal{U}_X^*[1] \rightarrow \Phi(G_L)[1] \rightarrow \mathcal{I}_{L/X}(1),$$

where the middle term is a vector bundle of rank 3, which is easily seen to be simple.

Moreover, it is clear that $\mathcal{I}_{L/X}(1)$ and $\mathcal{O}_L$ belong to $\langle \mathcal{O}_X(1) \rangle^\perp$. Also, we compute $\tau(\mathcal{I}_{L/X}(1)) \simeq \mathcal{O}_L[-3]$ since from $R\mathcal{H}om_X(\mathcal{O}_L, \mathcal{O}_X) \simeq \mathcal{O}_L(-1)[-2]$ we get the exact triangle:

$$\mathcal{O}_L(-1)[-2] \rightarrow \mathcal{O}_X \rightarrow R\mathcal{H}om_X(\mathcal{I}_{L/X}, \mathcal{O}_X),$$

which gives the result once we apply $L_{\mathcal{O}_X}$ and tensor with $\mathcal{O}_X(1)[-2]$. Therefore, applying τ to (II.21) and using $L_{\mathcal{O}_X}(\mathcal{U}_X^*) \simeq \mathcal{Q}_X^*(1)[1]$, we get the exact triangle:

$$(II.22) \quad \mathcal{O}_L[-1] \rightarrow \tau(\Phi(G_L))[1] \rightarrow \mathcal{Q}_X^*(1)$$

We now look back to (II.20), apply Φ to it, and set $K = \Phi(\ker(e))$. Since the map $\mathcal{H}^0(\Phi(N)) \rightarrow \mathcal{U}_X^*$ is an isomorphism, we have $\mathcal{H}^0(K) = 0$ so that K is concentrated in degree 1. We have then the distinguished triangle:

$$(II.23) \quad \mathcal{O}_L[-1] \rightarrow K[1] \rightarrow \mathcal{Q}_X^*(1).$$

Now, both triangles (II.22) and (II.23) are given by elements of $\text{Hom}_X(\mathcal{Q}_X^*(1), \mathcal{O}_L) \simeq H^0(L, \mathcal{Q}(-1)|_L) \simeq \mathbb{C}$ (recall that $\mathcal{Q}_L \simeq \mathcal{O}_L^4 \oplus \mathcal{O}_L(1)$). Both these elements are non-zero. Indeed, otherwise the middle terms $K[1]$ or $\tau(\Phi(G_L))[1]$ would be of the form $\mathcal{O}_L[-1] \oplus \mathcal{Q}_X^*(1)$, and hence their endomorphism spaces would contain at least a $\mathbb{C}^2$, while we know that both $K[1]$ and $\tau(\Phi(G_L))[1]$ are simple, i.e. their endomorphism spaces are one-dimensional. From the fact that $\text{Hom}_X(\mathcal{Q}_X^*(1), \mathcal{O}_L)$ is also one-dimensional, we now deduce that $K \simeq \tau(\Phi(G_L))$. This implies that $G_L \simeq \iota^*(\ker(e))^*$. We conclude that $\Phi^!(\mathcal{I}_{L/X}(1))[1] \simeq \iota^*(\ker(e))^*$, which is what we wanted. $\square$

II.7. Instantons of charge 8. Here we study one space of instanton bundles, still for low charge, but this time above the maximum by 2. This kind of study, in the spirit of the early works on instanton bundles such as [24, 157, 199], tries to give a detailed description of manageable moduli space. The approach is based on the study of bundles over the homologically dual curve C .

Proposition II.11. *The space $\text{MI}_X(8)$ is a smooth fourfold in $\text{M}_C(2, 2)$.*

PROOF. Let us check that, for any F in $\text{MI}_X(8)$, the rank-2 bundle $E = \Phi^!(F)$ is stable. We have seen that E is simple, and satisfies $H^0(C, \mathcal{V} \otimes E) \neq 0$. It suffices to show that E cannot be destabilized by a quotient bundle N of rank 1 and degree 1. By contradiction, consider such a line bundle N , let $f : E \rightarrow N$ be a surjection, and set $K = \ker(f)$. Note that K and N both lie in $\text{Pic}^1(C)$, and write the exact sequence:

$$0 \rightarrow K \rightarrow E \rightarrow N \rightarrow 0.$$

Observe now that any map $\mathcal{V}^* \rightarrow E$ is zero when composed with f . Indeed, otherwise N would lie in $\theta(E)$ by Theorem II.9, so that $\mathcal{H}^0(\Phi(f))$ would be a non-zero map $\Phi(E) \rightarrow \mathcal{U}_X^*$. But $\text{Hom}_X(\Phi(E), \mathcal{U}_X^*) \simeq \text{Hom}_C(E, \Phi^!(\mathcal{U}_X^*)) = 0$. Then, once set $F' = \Phi(N)$, we have that F' lies in $\text{MI}_X(7)$. Moreover, any map $\mathcal{V}^* \rightarrow E$ factors through K , so that $K \simeq \Phi^!(\mathcal{O}_L)[-1]$, for some line $L \subset X$, again by Theorem II.9. In this case, applying Φ to the previous display and taking cohomology, we get a long exact sequence:

$$0 \rightarrow \mathcal{U}_X^* \rightarrow \Phi(E) \rightarrow F' \rightarrow \mathcal{O}_L \rightarrow 0,$$

where the image of the middle map is F . It follows that F is not locally free, precisely along L , and that $F^{**} \simeq F'$. However, this contradicts the assumption that F lies in $\text{MI}_X(8)$.

It remains to check that, if F lies in $\text{MI}_X(8)$, then E is a smooth point of the divisor defined by $H^0(C, \mathcal{V} \otimes E) \neq 0$. Consider then any non-zero element s of $H^0(C, \mathcal{V} \otimes E)$. This gives a map $s : \mathcal{V}^* \rightarrow E$. Note that, since $\mathcal{V}^*$ and E are stable bundles of slope 0 and 1, the image I of s cannot be a line bundle, so $\text{rk}(I) = 2$. We have now two possibilities.

i) s is surjective. In this case, we apply $\text{Hom}_C(-, \mathcal{V}^*)$ to the sequence:

$$0 \rightarrow \ker(s) \rightarrow \mathcal{V}^* \rightarrow E \rightarrow 0.$$

We get a long exact sequence, where π_E is the Petri map:

$$\cdots \rightarrow \text{Ext}_C^1(E, E) \xrightarrow{\pi_E} \text{Ext}_C^1(\mathcal{V}^*, E) \rightarrow \text{Ext}_C^1(\ker(s), E) \rightarrow 0.$$

Hence, the kernel of π_E is naturally identified with the fibre of the cotangent sheaf at F of $\text{MI}_X(8)$. But the rightmost term is dual to

$H^0(C, E^*(h_C) \otimes \ker(s)^*)$, which vanishes by stability. So $\ker(\pi_E)$ has dimension 4, and $\text{MI}_X(8)$ is smooth at F .

- ii) s is not surjective. In this case, the image of s has rank 2 and degree 1, and we immediately see that $\text{Im}(s)$ must be stable and to lie in fact in $\delta(\mathcal{V})$, cf. Corollary II.10. Therefore, we have $\text{Im}(s) \simeq \Phi^!(\mathcal{I}_{L/X}(1))[-1]$ for some line $L \subset X$. We write:

$$0 \rightarrow \text{Im}(s) \rightarrow E \rightarrow \mathcal{O}_y \rightarrow 0,$$

for some $y \in C$. Applying Φ to this sequence, by II.10 we get:

$$0 \rightarrow F \rightarrow \mathcal{F}_y \rightarrow \mathcal{I}_{L/X}(1) \rightarrow 0.$$

Applying $\text{Hom}_X(-, F)$ to this sequence, we get:

$$\rightarrow H^2(X, \mathcal{F}_y^* \otimes F) \rightarrow \text{Ext}_X^2(F, F) \rightarrow \text{Ext}_X^3(\mathcal{I}_{L/X}(1), F) \rightarrow$$

The leftmost space vanishes: indeed we have seen that $H^i(X, \mathcal{F}_y^* \otimes F) = 0$ for $i \neq 1$; the rightmost one does too, since it is dual to $\text{Hom}_X(F, \mathcal{I}_{L/X})$ which vanishes by stability. Then $\text{MI}_X(8)$ is smooth at F .

□

II.8. Bundles of rank 3 with canonical determinant. We propose here a result that allows to think of X as a moduli space of stable bundles of rank 3 with canonical determinant over C , by means of the universal bundle $\mathcal{F}$. We show that all bundles corresponding to points of X have a fixed number of sections when twisted with $\mathcal{V}$, which makes X into some Brill-Noether locus over C .

Theorem II.12. *The threefold X is a subvariety of $\text{M}_C(3, \omega_C)$, and any bundle E corresponding to a point of X satisfies $h^0(C, \mathcal{V}^* \otimes E) = 2$.*

PROOF. We have the vector bundle $\mathcal{F}$ over $X \times C$, and we would like to check that, for any $x \in X$, the rank-3 bundle $\mathcal{F}_x$ over C is stable, has canonical determinant, and satisfies $h^0(C, \mathcal{V}^* \otimes \mathcal{F}_x) = 2$.

To start the proof, we show three preliminary facts. The first one is that, for any point x in X , we have:

$$(II.24) \quad 0 \rightarrow \mathcal{O}_X \rightarrow (\mathcal{U}_X^*)^5 \rightarrow \mathcal{H}^0(\Phi(\mathcal{F}_x^* \otimes \omega_C)) \rightarrow 0,$$

and $\mathcal{H}^1(\Phi(\mathcal{F}_x^* \otimes \omega_C)) \simeq \mathcal{O}_x$. To see this, note that by definition $\Phi^!(\mathcal{O}_x) \simeq \mathcal{F}_y^* \otimes \omega_C[1]$. Further, computing a resolution of $\mathcal{O}_x$ via (II.3) we get $h^i(X, \mathcal{O}_x(-1)) = \delta_{0,i}$ and $h^i(X, \mathcal{Q}_x(-1)) = 5\delta_{0,i}$. Then, writing (II.3) for $\mathcal{O}_x$ via II.4 and taking cohomology we get our claim.

The second fact we prove is that the map $D \mapsto \Phi^!(\mathcal{O}_D)$ is an isomorphism of $\text{Hilb}_{2t+1}(X)$ onto $\text{Pic}^2(C)$, which is the composition of the isomorphism of Proposition II.2 with the isomorphism $\text{Pic}^0(C) \rightarrow \text{Pic}^2(C)$ defined by

$N \mapsto \omega_C \otimes \iota^* N^*$. Indeed, let $D \subset X$ be a conic, and set $N = \Phi^!(\mathcal{I}_{D/X}(1))[-1]$. Since $H^i(X, \mathcal{I}_{D/X}) = 0$ for all i , we have, tensoring with $\mathcal{I}_{D/X}$ the leftmost part of the diagram of Proposition II.6, we get:

$$N \simeq R\pi_{C*}(\pi_X^*(\mathcal{I}_{D/X}(-1)) \otimes \iota^* \mathcal{F})[1].$$

We deduce, since $H^i(X, \mathcal{F}_y(-1)) = 0$ for all i and all $y \in C$, we have, by the exact sequence defining $\mathcal{I}_{D/X}$ as kernel of $\mathcal{O}_X \rightarrow \mathcal{O}_D$:

$$N \simeq R\pi_{C*}(\pi_X^*(\mathcal{O}_D(-1)) \otimes \iota^* \mathcal{F}).$$

Using Grothendieck duality, since $R\mathcal{H}om_X(\mathcal{O}_D, \mathcal{O}_X) \simeq \mathcal{O}_D[-2]$, we obtain thus:

$$N^* \simeq R\pi_{C*}(\pi_X^*(\mathcal{O}_D) \otimes \iota^* \mathcal{F}^*)[1].$$

This sheaf is $\iota^* \Phi^!(\mathcal{O}_D) \otimes \omega_C^*$, which implies our claim.

The third thing we do is to compute a resolution of $\mathcal{O}_D$ in the sense of (II.3). We have $h^i(X, \mathcal{O}_D(-1)) = \delta_{i,1}$. Of course, the vanishing $H^i(X, \mathcal{Q}_D(-1)) = 0$ takes place for $i = 2, 3$. We would like to show $H^0(X, \mathcal{Q}_D(-1)) = 0$. Note that it is enough to check $H^1(X, \mathcal{Q}_X \otimes \mathcal{I}_{D/X}(-1)) = 0$. To do this, we consider a general hyperplane section S of X (so in particular we assume $\text{Pic}(S) \simeq \langle h_S \rangle$), and we let $Z = D \cap S$, so that Z is a subscheme of length 2 of S . We tensor with $\mathcal{Q}_X \otimes \mathcal{I}_{D/X}$ the exact sequence (II.1). Since we already proved $H^i(X, \mathcal{Q} \otimes \mathcal{I}_{D/X}) = 0$ for all i in Proposition II.2, we are reduced to prove $H^0(S, \mathcal{Q}_S \otimes \mathcal{I}_{Z/S}) = 0$. But the claim emphasized in the proof of Theorem II.1 can be rephrased on S , asserting that a global section of $\mathcal{Q}_S$ vanishes nowhere, or on a reduced point of S . Indeed, this vanishing locus is obtained as intersection of a linear subspace of Σ with the span of S , and this cannot have positive dimension since S contains no line by assumption. The consequence is that no non-zero section of $\mathcal{Q}_S$ vanishes on Z , so $H^0(S, \mathcal{Q}_S \otimes \mathcal{I}_{Z/S}) = 0$. Summing up, $H^i(X, \mathcal{Q}_X \otimes \mathcal{I}_{D/X}(-1)) = 0$ for $i \neq 1$, and by Riemann-Roch we get $h^1(X, \mathcal{Q}_D(-1)) = 3$.

The conclusion is that, given D in $\text{Hilb}_{2t+1}(X)$, we have:

$$(II.25) \quad 0 \rightarrow \mathcal{O}_X \rightarrow (\mathcal{U}_X^*)^3 \rightarrow \Phi(\Phi^!(\mathcal{O}_D)) \rightarrow \mathcal{O}_D \rightarrow 0.$$

Let us now check that $\mathcal{F}_X$ is stable. First we prove that, given any line bundle of degree 0 on C , there are no non-trivial morphisms from $\mathcal{F}_X$ to it. We write $N^* \otimes \omega_C$ such a line bundle. Note that the maps $\mathcal{F}_X \rightarrow N^* \otimes \omega_C$ are transpose of maps $N \rightarrow \mathcal{F}_X^* \otimes \omega_C \simeq \Phi^!(\mathcal{O}_X)[-1]$. Since N has degree 2, by the previous step there is a conic $D \subset X$ such that $\Phi^!(\mathcal{O}_D) \simeq N$, so that in particular $\Phi(N) \simeq \Phi(\Phi^!(\mathcal{O}_D))$. We have thus:

$$\text{Hom}_C(\mathcal{F}, N^* \otimes \omega_C) \simeq \text{Hom}_C(N, \Phi^!(\mathcal{O}_X)[-1]) \simeq \text{Hom}_X(\Phi(N), \mathcal{O}_X[-1]),$$

which is zero because $\Phi(N)$ is concentrated in degree zero by (II.25).

Next we show that, given any line bundle of degree 1 on C , there are no non-trivial morphisms from it to $\mathcal{F}_x$. We write this line bundle $M^* \otimes \omega_C$ for some M in $\text{Pic}^1(C)$, and again given a morphism $M^* \otimes \omega_C \rightarrow \mathcal{F}_x$ we transpose to $\mathcal{F}_x^* \otimes \omega_C \rightarrow M$. By Theorem II.9 there is F in $M_X(2, 1, 7)$ such that $M \simeq \Phi^!(F)$, so that:

$$\text{Hom}_C(\mathcal{F}_x^* \otimes \omega_C, M) \simeq \text{Hom}_X(\Phi(\mathcal{F}_x^* \otimes \omega_C), F).$$

So this space is zero, by a spectral sequence argument, if we prove:

$$\text{Ext}_X^i(\mathcal{H}^i(\Phi(\mathcal{F}_x^* \otimes \omega_C)), F) = 0.$$

By the first step of this proof, we have to deal with $i = 0, 1$. As for $i = 0$, we apply $\text{Hom}_X(-, F)$ to the exact sequence appearing in (II.24), and we conclude because $\text{Hom}_X(\mathcal{U}_x^*, F) = 0$, since any non-zero element in this space would give $\mathcal{U}^* \simeq F$. On the other hand, for $i = 1$ we recall that $\mathcal{H}^1(\Phi(\mathcal{F}_x^* \otimes \omega_C)) \simeq \mathcal{O}_x$. So the vanishing is clear by Serre duality if F is locally free. But F is not, applying $\text{Hom}_X(\mathcal{O}_x, -)$ to (II.18), we easily see that $\text{Ext}_X^1(\mathcal{O}_x, F) = 0$ since $\text{Ext}_X^1(\mathcal{O}_x, \mathcal{U}_x^*) = 0$ (obvious by Serre duality since $\mathcal{U}_x$ is locally free) and $\text{Hom}_X(\mathcal{O}_x, \mathcal{O}_L) = 0$ (obvious since $\mathcal{O}_x$ is torsion over L).

This shows that $\mathcal{F}_x$ is stable. To see its determinant, recall that $\wedge^3 \mathcal{F}_y$ is a line bundle of degree 2 (cf. proof of Theorem II.1). However, by the diagram of Proposition II.6, we have, for any $x \in X$ an exact sequence of the form:

$$0 \rightarrow \iota^* \mathcal{F}_x^* \rightarrow \mathcal{V}^2 \rightarrow \mathcal{F}_x \rightarrow 0,$$

and $\wedge^3 \mathcal{V} \simeq \mathcal{O}_C$. Then $\wedge^3 \mathcal{F}_y$ is invariant for ι , hence it is isomorphic to ω_C .

It remains to check that $h^0(C, \mathcal{V}^* \otimes \mathcal{F}_x) = 2$. This is obtained by the following isomorphisms:

$$\begin{aligned} H^0(C, \mathcal{F}_x \otimes \mathcal{V}^*)^* &\simeq \text{Hom}_C(\mathcal{V}^*, \mathcal{F}_x^* \otimes \omega_C[1]) \simeq \\ &\simeq \text{Hom}_C(\mathcal{V}^*, \Phi^!(\mathcal{O}_x)) \simeq \\ &\simeq \text{Hom}_X(\Phi(\mathcal{V}^*), \mathcal{O}_x) \simeq \\ &\simeq \text{Hom}_X(\mathcal{H}^0(\Phi(\mathcal{V}^*)), \mathcal{O}_x) \simeq \\ &\simeq \text{Hom}_X(\mathcal{U}_x^*, \mathcal{O}_x) \simeq \mathbb{C}^2. \end{aligned}$$

□

III. Open questions

I give here a brief account of some of the questions related to the material above, that seem important to me. Some of them are explicitly formulated as conjectures in the literature; others are the subject of work currently in progress. I will be rather sloppy here on the language concerning moduli spaces (actually I do not know to what extent these spaces are defined as schemes).

III.1. Properties of the moduli space of instantons. Perhaps the most important questions on the moduli space $\mathrm{MI}_X(k)$ concern its smoothness, and irreducibility. It might be natural to conjecture that these properties hold when X is general in its moduli space. In some cases these properties do hold, in particular for low values of k . For instance, this is the case for $i_X = 3$ (i.e. X is a quadric threefold) and $k = 2, 3$ (see [261]), for most Del Pezzo threefolds (i.e. $i_X = 2$) when $k = 2$, (see [129, 229, 230]), and for many prime Fano threefolds (i.e. $i_X = 1$) when $\frac{g_X}{2} + 1 \leq k \leq \frac{g_X+1}{2} + 2$. Some papers where these cases are studied in detail are [48–50, 180, 183, 185].

However, it should be clear that these properties do not necessarily hold when X is not general in its moduli space. For instance, for $i_X = 1$ and $g_X = 5$ (i.e. X is the intersection of 3 quadrics in $\mathbb{P}^6$), the moduli space $\mathrm{MI}_X(4)$ is isomorphic to a double cover of the discriminant septic, as proved in [49]. For special X , this septic can be singular and can have many irreducible components. Examples of threefolds X with $i_X = 1$ and $g_X = 7$ such that $\mathrm{MI}_X(6)$ is singular are given in [48]. Still $\mathrm{MI}_X(6)$ is always connected in this case. Finally, A. Langer outlined an argument based on [223] that suggests that $\mathrm{MI}_X(k)$ cannot be smooth and irreducible for all k when X is a smooth quadric threefold.

III.2. A conjecture of Kuznetsov. This is taken from [219]. Let $\mathcal{Y}_d$ be the moduli space of smooth Fano threefolds of index 2, Picard number 1, and degree d , and $\mathcal{X}_g$ be the moduli space of smooth Fano threefold of index 1, Picard number 1, and genus g . It turns out that, for any element X in $\mathcal{X}_{2d+2}$, there is a semiorthogonal decomposition:

$$\mathbf{D}^b(X) = \langle \mathcal{A}_X, \mathcal{E}_X, \mathcal{O}_X \rangle,$$

where $\mathcal{E}_X$ is an exceptional bundle of rank 2, and $\mathcal{A}_X$ is a certain admissible triangulated subcategory of $\mathbf{D}^b(X)$. There is also a semiorthogonal decomposition, for any Y in $\mathcal{Y}_d$:

$$\mathbf{D}^b(Y) = \langle \mathcal{B}_Y, \mathcal{O}_Y, \mathcal{O}_Y(1) \rangle,$$

with $\mathcal{B}_Y$ admissible in $\mathbf{D}^b(X)$. Let us also write $\mathcal{I}_{\mathcal{X}_g}(k)$ for the moduli space of pairs (F, X) , where X lies in $\mathcal{X}_g$ and E lies in $\mathrm{MI}_X(k)$, and similarly for $\mathcal{I}_{\mathcal{Y}_d}(k)$.

Kuznetsov's conjectures [219, Conjectures 3.7 and 4.12] look as follows.

- i) There should be a correspondence $\mathcal{Z} \subset \mathcal{Y}_d \times \mathcal{X}_{2d+2}$, dominant on each factor, such that for any pair (Y, X) of $\mathcal{Z}$ one has $\mathcal{B}_Y \simeq \mathcal{A}_X$.
- ii) For all $d \in [1, 5]$ there is some k for which there is an isomorphism $\mathcal{I}_{\mathcal{Y}_d}(k) \rightarrow \mathcal{I}_{\mathcal{X}_{2d+2}}(2k + d + 1)$, whose graph is $\mathcal{Z}$, such that, if (X, F) corresponds to (Y, E) , then $H^0(X, F^*) \simeq H^0(Y, E(1))$, and there is a birational map $\mathbb{P}(E) \rightarrow \mathbb{P}(F)$ commuting with this isomorphism.

The first part is true for $d = 3, 4, 5$, see again [219]. The proof is a case-by-case analysis: could one imagine a uniform proof? The second part, to my

knowledge, has been established only for $d = 3$, cf. [213]. The previous section might serve as basis to look at the case $d = 4$.

III.3. A conjecture of Mukai. The question whether a Fano threefold X of genus 10 can be expressed as a Brill-Noether locus of stable bundle over a curve of genus 2 goes back to Mukai. However, Theorem II.12 does not fully answer the question, that would be: given a smooth projective curve C of genus 2 with hyperelliptic involution ι , and a stable vector bundle $\mathcal{V}$ in $M_C(3)$ with $\iota^*\mathcal{V} \simeq \mathcal{V}^*$, is it true that the subvariety X of $M_C(3, \omega_C)$, given by bundles E having $h^0(C, \mathcal{V}^* \otimes E) = 2$ is a smooth Fano threefold of genus 10 (or at least does this happen generically)? If so, Theorem II.12 says that any such threefolds arise this way.

Closely related to this is the problem of periods of Fano threefolds of genus 10. It is natural to expect that, given a smooth projective curve C of genus 2, an open dense subset of the Coble-Dolgachev sextic is the fibre of the period map from $\mathcal{X}_{10}$ to the moduli space of principally polarized abelian varieties of genus 2.

III.4. Instantons and the non-commutative plane. We already quoted a nice result, again by Kuznetsov, that gives a semiorthogonal decomposition of an intersection of quadrics, in terms of modules over a sheaves of Clifford algebras, see [218]. A Fano threefold of genus 5 is the complete intersection of 3 quadrics in $\mathbb{P}^6$. So, for genus 5, the space $MI_X(k)$ can be probably described satisfactorily in terms of monads over a non-commutative $\mathbb{P}^2$: this seems to deserve a closer look.

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