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UNIVERSITÉ DE BOURGOGNE  
Institut de Mathématiques de Bourgogne



UNIVERSITÉ DE SCIENCE DE HO CHI MINH VILLE  
Faculté de Mathématiques - Informatique



# THÈSE

présentée par

**NGUYEN Le Hoang Anh**

en vue d'obtenir le titre de

DOCTEUR

DE L'UNIVERSITÉ DE BOURGOGNE

ET DE L'UNIVERSITÉ DE SCIENCE DE HO CHI MINH VILLE

Discipline: MATHÉMATIQUES

## QUELQUES THÈMES EN L'ANALYSE VARIATIONNELLE ET OPTIMISATION

23 Février 2014

Directeurs de thèse

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# Résumé

Dans cette thèse, j'étudie d'abord la théorie des  $\Gamma$ -limites. En dehors de quelques propriétés fondamentales des  $\Gamma$ -limites, les expressions de  $\Gamma$ -limites séquentielles généralisant des résultats de Greco sont présentées. En outre, ces limites nous donnent aussi une idée d'une classification unifiée de la tangence et la différentiation généralisée. Ensuite, je développe une approche des théories de la différentiation généralisée. Cela permet de traiter plusieurs dérivées généralisées des multi-applications définies directement dans l'espace primal, tels que des ensembles variationnels, des ensembles radiaux, des dérivées radiales, des dérivées de Studniarski. Finalement, j'étudie les règles de calcul de ces dérivées et les applications liées aux conditions d'optimalité et à l'analyse de sensibilité.

# Abstract

In this thesis, we first study the theory of  $\Gamma$ -limits. Besides some basic properties of  $\Gamma$ -limits, expressions of sequential  $\Gamma$ -limits generalizing classical results of Greco are presented. These limits also give us a clue to a unified classification of derivatives and tangent cones. Next, we develop an approach to generalized differentiation theory. This allows us to deal with several generalized derivatives of set-valued maps defined directly in primal spaces, such as variational sets, radial sets, radial derivatives, Studniarski derivatives. Finally, we study calculus rules of these derivatives and applications related to optimality conditions and sensitivity analysis.

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# Contents

<b>Abstract</b>	<b>ii</b>
<b>Acknowledgements</b>	<b>iii</b>
<b>Preface</b>	<b>viii</b>
<b>1 Motivations</b>	<b>1</b>
1.1 $\Gamma$ -limits . . . . .	1
1.2 Sensitivity analysis . . . . .	2
1.3 Optimality conditions . . . . .	3
1.4 Calculus rules and applications . . . . .	4
<b>2 Preliminaries</b>	<b>5</b>
2.1 Some definitions in set theory . . . . .	5
2.2 Some definitions in set-valued analysis . . . . .	6
<b>3 The theory of <math>\Gamma</math>-limits</b>	<b>13</b>
3.1 Introduction . . . . .	13
3.2 $\Gamma$ -limits in two variables . . . . .	15
3.3 $\Gamma$ -limits valued in completely distributive lattices . . . . .	20
3.3.1 Limitoids . . . . .	20
3.3.2 Representation theorem . . . . .	22
3.4 Sequential forms of $\Gamma$ -limits for extended-real-valued functions . . . . .	24
3.4.1 Two variables . . . . .	26
3.4.2 Three variables . . . . .	31
3.4.3 More than three variables . . . . .	34

3.5	Applications . . . . .	39
3.5.1	Generalized derivatives . . . . .	39
3.5.2	Tangent cones . . . . .	41
<b>4</b>	<b>Variational sets and applications to sensitivity analysis for vector optimization problems</b>	<b>45</b>
4.1	Introduction . . . . .	45
4.2	Variational sets of set-valued maps . . . . .	46
4.2.1	Definitions . . . . .	46
4.2.2	Relationships between variational sets of $F$ and those of its profile map	50
4.3	Variational sets of perturbation maps . . . . .	56
4.4	Sensitivity analysis for vector optimization problems . . . . .	62
<b>5</b>	<b>Radial sets, radial derivatives and applications to optimality conditions for vector optimization problems</b>	<b>71</b>
5.1	Introduction . . . . .	71
5.2	Radial sets and radial derivatives . . . . .	73
5.2.1	Definitions and properties . . . . .	73
5.2.2	Sum rule and chain rule . . . . .	79
5.3	Optimality conditions . . . . .	86
5.4	Applications in some particular problems . . . . .	92
<b>6</b>	<b>Calculus rules and applications of Studniarski derivatives to sensitivity and implicit function theorems</b>	<b>97</b>
6.1	Introduction . . . . .	97
6.2	The Studniarski derivative . . . . .	98
6.3	Calculus rules . . . . .	103
6.4	Applications . . . . .	114
6.4.1	Studniarski derivatives of solution maps to inclusions . . . . .	114
6.4.2	Implicit multifunction theorems . . . . .	115
	<b>Conclusions</b>	<b>119</b>
	<b>Further works</b>	<b>121</b>



*Contents*

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<b>Publications</b>	<b>123</b>
<b>Bibliography</b>	<b>124</b>
<b>Index</b>	<b>131</b>

# Preface

Variational analysis is related to a broad spectrum of mathematical theories that have grown in connection with the study of problems of optimization and variational convergence.

To my knowledge, many concepts of convergence for sequences of functions have been introduced in mathematical analysis. These concepts are designed to approach the limit of sequences of variational problems and are called variational convergence. Introduced by De Giorgi in the early 1970s,  $\Gamma$ -convergence plays an important role among notions of convergences for variational problems. Moreover, many applications of this concept have been developed in other fields of variational analysis such as calculus of variations and differential equations.

Recently, nonsmoothness has become one of the most characteristic features of modern variational analysis. In fact, many fundamental objects frequently appearing in the frame work of variational analysis (e.g., the distance function, value functions in optimization and control problems, maximum and minimum functions, solution maps to perturbed constraint and variational systems, etc.) are inevitably of nonsmooth and/or set-valued structures. This requires the development of new forms of analysis that involve generalized differentiation.

The analysis above motivates us to study some topics on  $\Gamma$ -limits, generalized differentiation of set-valued maps and their applications.

# Chapter 1

## Motivations

### 1.1 $\Gamma$ -limits

Several last decades have seen an increasing interest for variational convergences and for their applications to different fields, like approximation of variational problems and nonsmooth analysis, see [22, 32, 113, 120, 131, 133, 150]. Among variational convergences, definitions of  $\Gamma$ -convergence, introduced in [48] by Ennio De Giorgi and Franzoni in 1975, have become commonly-recognized notions (see [37] of Dal Maso for more detail introduction). Under suitable conditions,  $\Gamma$ -convergence implies stability of extremal points, while some other convergences, such as pointwise convergence, do not. Moreover, almost all other variational convergences can be easily expressed in the language of  $\Gamma$ -convergence. As explained in [16, 57, 169], this concept plays a fundamental role in optimization theory, decision theory, homogenization problems, phase transitions, singular perturbations, the theory of integral functionals, algorithmic procedures, and in many others.

In 1983 Greco introduced in [82] a concept of limitoid and noticed that all the  $\Gamma$ -limits are special limitoids. Each limitoid defines its support, which is a family of subsets of the domain of the limitoid, which in turn determines this limitoid. Besides, Greco presented in [82, 84] a representation theorem for which each relationship of limitoids corresponds a relationship in set theory. This theorem enabled a calculus of supports and was instrumental in the discovery of a limitation of equivalence between  $\Gamma$ -limits and sequential  $\Gamma$ -limits, see [83].

Recently, a lot of research has been carried in the realm of tangency and differentiation and their applications, see [2, 4, 8, 14, 15, 50, 77, 101, 109, 134]. We propose a unified approach to approximating tangency cones and generalized derivatives based on the theory of  $\Gamma$ -limits. This means that most of them can be expressed in terms of  $\Gamma$ -limits.

The analysis above motivates us to study the theory of  $\Gamma$ -limits.

## 1.2 Sensitivity analysis

Stability and sensitivity analyses are of great importance for optimization from both the theoretical and practical view points. As usual, stability is understood as a qualitative analysis, which concerns mainly studies of various continuity (or semicontinuity) properties of solution maps and optimal-value maps. Sensitivity means a quantitative analysis, which can be expressed in terms of various derivatives of the mentioned maps. For sensitivity results in nonlinear programming using classical derivatives, we can see the book [64] of Fiacco. However, practical optimization problems are often nonsmooth. To cope with this crucial difficulty, most of approaches to studies of optimality conditions and sensitivity analysis are based on generalized derivatives.

Nowadays, set-valued maps (also known as multimaps or multifunctions) are involved frequently in optimization-related models. In particular, for vector optimization, both perturbation and solution maps are set-valued. One of the most important derivatives of a multimap is the contingent derivative. In [107–109, 153, 154, 162, 163], behaviors of perturbation maps for vector optimization were investigated quantitatively by making use of contingent derivatives. Results on higher-order sensitivity analysis were studied in [158, 167], applying kinds of contingent derivatives. To the best of our knowledge, no other kinds of generalized derivatives have been used in contributions to this topic, while so many notions of generalized differentiability have been introduced and applied effectively in investigations of optimality conditions, see books [11] of Aubin and Frankowska, [129, 130] of Mordukhovich, and [147] of Rockafellar and Wets.

We mention in more detail only several recent papers on generalized derivatives of set-valued maps and optimality conditions. Radial epiderivatives were used to get optimality conditions for nonconvex vector optimization in [66] by Flores-Bazan and for set-valued optimization in [102] by Kasimbeyli. Variants of higher-order radial derivatives for establishing higher-order conditions were proposed by Anh et al. in [2, 4, 8]. The higher-order lower Hadamard directional derivative was the tool for set-valued vector optimization presented by Ginchev in [71, 72]. Higher-order variational sets of a multimap were proposed in [105, 106] by Khanh and Tuan in dealing with optimality conditions for set-valued optimization.

We expect that many generalized derivatives, besides the contingent ones, can be employed effectively in sensitivity analysis. Thus, we choose variational sets for higher-order consider-

ations of perturbation maps, since some advantages of this generalized differentiability were shown in [7, 105, 106], e.g., almost no assumptions are required for variational sets to exist (to be nonempty); direct calculating of these sets is simply a computation of a set limit; extensions to higher orders are direct; they are bigger than corresponding sets of most derivatives (this property is decisively advantageous in establishing necessary optimality conditions by separation techniques), etc. Moreover, Anh et al. established calculus rules for variational sets in [7] to ensure the applicability of variational sets.

### **1.3 Optimality conditions**

Various problems encountered in the areas of engineering, sciences, management science, economics and other fields are based on the fundamental idea of mathematical formulation. Optimization is an essential tool for the formulation of many such problems expressed in the form of minimization/maximization of a function under certain constraints like inequalities, equalities, and/or abstract constraints. It is thus rightly considered a science of selecting the best of the many possible decisions in a complex real-life environment.

All initial theories of optimization theory were developed with differentiability assumptions of functions involved. Meanwhile, efforts were made to shed the differentiability hypothesis, there by leading to the development of nonsmooth analysis as a subject in itself. This added a new chapter to optimization theory, known as nonsmooth optimization. Optimality conditions in nonsmooth problems have been attracting increasing efforts of mathematicians around the world for half a century. For systematic expositions about this topic, including practical applications, see books [11] of Aubin and Frankowska, [29] of Clarke, [92] of Jahn, [129, 130] of Mordukhovich, [142] of Penot, [146] of Rockafellar and Wets, and [149] of Schirotzek. A significant number of generalized derivatives have been introduced to replace the Fréchet and Gâteaux derivatives which do not exist for studying optimality conditions in nonsmooth optimization.

One can roughly separate the wide range of methods for nonsmooth problems into two groups : the primal space and the dual space approaches. The primal space approach has been more developed, since it exhibits a clear geometry, originated from the famous works of Fermat and Lagrange. Most derivatives in this stream are based on kinds of tangency/linear approximations. Among tangent cones, contingent cone plays special roles, both in direct uses as derivatives/linear approximations and in combination with other ideas to provide kinds of generalized derivatives (contingent epiderivatives by Jahn and Rauh in [96], contingent variations

by Frankowska and Quincampoix in [68], variational sets by Khanh et al. in [7, 105, 106], generalized (adjacent) epiderivatives by Li et al. in [27, 166, 168], etc).

Similarly as for generalized derivatives defined based on kinds of tangent cones, the radial derivative was introduced by Taa in [160]. Coupling the idea of tangency and epigraphs, like other epiderivatives, radial epiderivatives were defined and applied to investigating optimality conditions in [65–67] by Flores-Bazan and in [102] by Kasimbeyli. To include more information in optimality conditions, higher-order derivatives should be defined.

The discussion above motivates us to define a kind of higher-order radial derivatives and use them to obtain higher-order optimality conditions for set-valued vector optimization.

## 1.4 Calculus rules and applications

The investigation of optimality conditions for nonsmooth optimization problems has implied many kinds of generalized derivatives (introduced in above subsections). However, to the best of our knowledge, there are few research on their calculus rules. We mention in more detail some recent papers on generalized derivatives of set-valued maps and their calculus rules. In [94], some calculus rules for contingent epiderivatives of set-valued maps were given by Jahn and Khan. In [116], Li et al. obtained some calculus rules for intermediate derivative-like multifunctions. Similar ideas had also been utilized for the calculus rules for contingent derivatives of set-valued maps and for generalized derivatives of single-valued nonconvex functions in [11, 164, 165]. Anh et al. developed elements of calculus of higher-order variational sets for set-valued mappings in [7].

In [156], Studniarski introduced another way to get higher-order derivatives (do not depend on lower orders) for extended-real-valued functions, known as Studniarski derivatives, and obtained necessary and sufficient conditions for strict minimizers of order greater than 2 for optimization problems with vector-valued maps as constraints and objectives. Recently, these derivatives have been extended to set-valued maps and applied to optimality conditions for set-valued optimization problems in [1, 117, 159]. However, there are no results on their calculus rules.

The analysis above motivates us to study on calculus rules of Studniarski derivatives and their applications.

# Chapter 2

## Preliminaries

### 2.1 Some definitions in set theory

**Definition 2.1.1.** ([23, 24]) Let  $S$  be a subset of a topological sapce  $X$ .

- (i) A family  $\mathcal{F}$  of subsets of  $S$  is called a *non-degenerate family* on  $S$  if  $\emptyset \notin \mathcal{F}$ .
- (ii) A non-degenerate family  $\mathcal{F}$  on  $S$  is called a *semi-filter* if

$$G \supseteq F \in \mathcal{F} \implies G \in \mathcal{F}.$$

- (iii) A semi-filter  $\mathcal{F}$  on  $S$  is called a *filter* if

$$F_0, F_1 \in \mathcal{F} \implies F_0 \cap F_1 \in \mathcal{F}.$$

The *set of filters* and the *set of semi-filters* on  $S$  are denoted by  $\mathbb{F}(S)$  and  $\mathbb{SF}(S)$ , respectively. If  $\mathcal{A}, \mathcal{B}$  are two families, then  $\mathcal{B}$  is called *finer* than  $\mathcal{A}$  (denoted by  $\mathcal{A} \leq \mathcal{B}$ ) if for each  $A \in \mathcal{A}$  there exists  $B \in \mathcal{B}$  such that  $B \subseteq A$ . We say that  $\mathcal{A}$  and  $\mathcal{B}$  are *equivalent* ( $\mathcal{A} \approx \mathcal{B}$ ) if  $\mathcal{A} \leq \mathcal{B}$  and  $\mathcal{B} \leq \mathcal{A}$ . A subfamily  $\mathcal{B}$  of a non-degenerate family  $\mathcal{F}$  is said a *base* of  $\mathcal{F}$  (or  $\mathcal{B}$  generates  $\mathcal{F}$ ) if  $\mathcal{F} \leq \mathcal{B}$ . We say that  $\mathcal{A}$  and  $\mathcal{B}$  *mesh* (denoted by  $\mathcal{A} \# \mathcal{B}$ ) if  $A \cap B \neq \emptyset$  for every  $A \in \mathcal{A}$  and  $B \in \mathcal{B}$ .

The *grill* of a family  $\mathcal{A}$  on  $S$ , denoted by  $\mathcal{A}^\#$ , is defined by

$$\mathcal{A}^\# := \{A \subseteq S : \forall_{F \in \mathcal{A}} A \cap F \neq \emptyset\}.$$

Therefore  $\mathcal{A} \# \mathcal{B}$  is equivalent to  $\mathcal{A} \subseteq \mathcal{B}^\#$  and to  $\mathcal{B} \subseteq \mathcal{A}^\#$ .

If  $\mathcal{F}$  is a filter, then  $\mathcal{F} \subseteq \mathcal{F}^\#$ . In  $\mathbb{SF}(S)$ , the operation of grills is an *involution*, i.e., the following equalities hold (see [55])

$$\mathcal{A}^{\#\#} = \mathcal{A}, \quad \left(\bigcup_i \mathcal{A}_i\right)^\# = \bigcap_i (\mathcal{A}_i^\#), \quad \left(\bigcap_i \mathcal{A}_i\right)^\# = \bigcup_i (\mathcal{A}_i^\#). \quad (2.1)$$

Semi-filters, filters, and grills were thoroughly studied in [53] by Dolecki.

**Definition 2.1.2.** ([18]) (i) A set  $S$  with a binary relation ( $\leq$ ) satisfying three properties : reflexivity, antisymmetry, and transitivity is called an *ordered set*  $S$  (also called a *poset*).

(ii) Let  $S$  be a subset of a poset  $P$ . An element  $a \in P$  is called an upper bound (or lower bound) of  $S$  if  $a \geq s$  ( $a \leq s$ , respectively) for all  $s \in S$ .

(iii) An upper bound  $a$  (lower bound, respectively) of a subset  $S$  is called the *least upper bound* (or the *greatest lower bound*) of  $S$ , denoted by  $\sup S$  or  $\bigvee S$  ( $\inf S$  or  $\bigwedge S$ , respectively) if  $a \leq b$  ( $a \geq b$ , respectively) for all  $b$  be another upper bound (lower bound, respectively) of  $S$ .

**Definition 2.1.3.** ([18, 82]) (i) A poset  $L$  is called a *lattice* if each couple of its elements has a least upper bound or “join” denoted by  $x \vee y$ , and a greatest lower bound or “meet” denoted by  $x \wedge y$ .

(ii) A lattice  $L$  is called *complete* if each of its subsets  $S$  has a greatest lower bound and a least upper bound in  $L$ .

(iii) A complete lattice  $L$  is called *completely distributive* if

$$\begin{aligned} \text{(a)} \quad \bigvee_{j \in J} \bigwedge_{i \in A_j} f(j, i) &= \bigwedge_{\varphi \in \prod_{j \in J} A_j} \bigvee_{j \in J} f(j, \varphi(j)), \\ \text{(b)} \quad \bigwedge_{j \in J} \bigvee_{i \in A_j} f(j, i) &= \bigvee_{\varphi \in \prod_{j \in J} A_j} \bigwedge_{j \in J} f(j, \varphi(j)), \end{aligned}$$

for each non-empty family  $\{A_j\}_{j \in J}$  of non-empty sets and for each function  $f$  defined on  $\{(j, i) \in J \times I : i \in A_j\}$  with values in  $L$ , and  $\prod_{j \in J} A_j := \{\varphi \in (\bigcup_{j \in J} A_j)^J : \forall_{j \in J} \varphi(j) \in A_j\}$ , where  $(\bigcup_{j \in J} A_j)^J$  denotes the set of functions from  $J$  into  $\bigcup_{j \in J} A_j$ .

(iv) A non-empty subset  $S$  of a lattice  $L$  is called a *sublattice* if for every pair of elements  $a, b$  in  $S$  both  $a \wedge b$  and  $a \vee b$  are in  $S$ .

(v) A sublattice  $S$  of a complete lattice  $L$  is called *closed* if for every non-empty subset  $A$  of  $S$  both  $\bigwedge A$  and  $\bigvee A$  are in  $L$ .

## 2.2 Some definitions in set-valued analysis

Let  $X, Y$  be *vector spaces*,  $C$  be a non-empty cone in  $Y$ , and  $A \subseteq Y$ . We denote sets of positive integers, of real numbers, and of non-negative real numbers by  $\mathbb{N}$ ,  $\mathbb{R}$ , and  $\mathbb{R}_+$ , respectively. We often use the following notations

$$\text{cone} A := \{\lambda a : \lambda \geq 0, a \in A\}, \quad \text{cone}_+ A := \{\lambda a : \lambda > 0, a \in A\},$$



$$C^* := \{y^* \in Y^* : \langle y^*, c \rangle \geq 0, \forall c \in C\}, \quad C^{+i} := \{y^* \in Y^* : \langle y^*, c \rangle > 0, \forall c \in C \setminus \{0\}\}.$$

A subset  $B$  of a cone  $C$  is called a *base* of  $C$  if and only if  $C = \text{cone } B$  and  $0 \notin \text{cl } B$ .

For a *set-valued map*  $F : X \rightarrow 2^Y$ ,  $F + C$  is called the *profile map* of  $F$  with respect to  $C$  defined by  $(F + C)(x) := F(x) + C$ . The domain, graph, epigraph and hypograph of  $F$  are denoted by  $\text{dom } F$ ,  $\text{gr } F$ ,  $\text{epi } F$ , and  $\text{hypo } F$ , respectively, and defined by

$$\text{dom } F := \{x \in X : F(x) \neq \emptyset\}, \quad \text{gr } F := \{(x, y) \in X \times Y : y \in F(x)\},$$

$$\text{epi } F := \text{gr}(F + C), \quad \text{hypo } F := \text{gr}(F - C).$$

A subset  $M \subseteq X \times Y$  can be considered as a set-valued map  $M$  from  $X$  into  $Y$ , called a *relation* from  $X$  into  $Y$ . The image of a singleton  $\{x\}$  by  $Mx$  is denoted by  $Mx := \{y \in Y : (x, y) \in M\}$ , and of a subset  $S$  of  $X$  is denoted by  $MS := \bigcup_{x \in S} Mx$ . The preimage of a subset  $K$  of  $Y$  by  $M$  is denoted by  $M^{-1}K := \{x : Mx \cap K \neq \emptyset\}$ .

**Definition 2.2.1.** Let  $C$  be a convex cone,  $F : X \rightarrow 2^Y$  and  $(x_0, y_0) \in \text{gr } F$ .

(i)  $F$  is called a *convex map* on a convex set  $S \subseteq X$  if, for all  $\lambda \in [0, 1]$  and  $x_1, x_2 \in S$ ,

$$(1 - \lambda)F(x_1) + \lambda F(x_2) \subseteq F((1 - \lambda)x_1 + \lambda x_2).$$

(ii)  $F$  is called a *C-convex map* on a convex set  $S$  if, for all  $\lambda \in [0, 1]$  and  $x_1, x_2 \in S$ ,

$$(1 - \lambda)F(x_1) + \lambda F(x_2) \subseteq F((1 - \lambda)x_1 + \lambda x_2) + C.$$

**Definition 2.2.2.** Let  $F : X \rightarrow 2^Y$  and  $(x_0, y_0) \in \text{gr } F$ .

(i)  $F$  is called a *lower semicontinuous* map at  $(x_0, y_0)$  if for each  $V \in \mathcal{N}(y_0)$  there is a neighborhood  $U \in \mathcal{N}(x_0)$  such that  $V \cap F(x) \neq \emptyset$  for each  $x \in U$ .

(ii) Suppose that  $X, Y$  are normed spaces. The map  $F$  is called a *m-th order locally pseudo-Hölder calm* map at  $x_0$  for  $y_0 \in F(x_0)$  if  $\exists \lambda > 0, \exists U \in \mathcal{N}(x_0), \exists V \in \mathcal{N}(y_0), \forall x \in U$ ,

$$(F(x) \cap V) \subseteq \{y_0\} + \lambda \|x - x_0\|^m B_Y,$$

where  $B_Y$  stands for the closed unit ball in  $Y$ .

For  $m = 1$ , the word ‘‘Hölder’’ is replaced by ‘‘Lipschitz’’. If  $V = Y$ , then ‘‘locally pseudo-Hölder calm’’ becomes ‘‘locally Hölder calm’’.

**Example 2.2.3.** (i) For  $F : \mathbb{R} \rightarrow \mathbb{R}$  defined by  $F(x) = \{y : -x^2 \leq y \leq x^2\}$  and  $(x_0, y_0) = (0, 0)$ ,  $F$  is the second order locally pseudo-Hölder calm map at  $x_0$  for  $y_0$ .

(ii) Let  $F : \mathbb{R} \rightarrow \mathbb{R}$  be defined by

$$F(x) = \begin{cases} \{0, 1/x\}, & \text{if } x \neq 0, \\ \{0, (1/n)_{n \in \mathbb{N}}\}, & \text{if } x = 0, \end{cases}$$

and  $(x_0, y_0) = (0, 0)$ . Then, for all  $m \geq 1$ ,  $F$  is not  $m$ -th order locally pseudo-Hölder calm at  $x_0$  for  $y_0$ .

Observe that if  $F$  is  $m$ -th order locally (pseudo-)Hölder calm at  $x_0$  for  $y_0$ , it is also  $n$ -th order locally (pseudo-)Hölder calm at  $x_0$  for  $y_0$  for all  $m > n$ . However, the converse may not hold. The following example shows the case.

**Example 2.2.4.** Let  $F : \mathbb{R} \rightarrow \mathbb{R}$  be defined by

$$F(x) = \begin{cases} x^2 \sin(1/x), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases}$$

and  $(x_0, y_0) = (0, 0)$ . Obviously,  $F$  is second order locally Hölder calm  $x_0$  for  $y_0$ , but  $F$  is not third order locally Hölder calm at  $x_0$  for  $y_0$ .

In the rest of this section, we introduce some definitions in vector optimization. Let  $C \subseteq Y$ , we consider the following relation  $\leq_C$  in  $Y$ , for  $y_1, y_2 \in Y$ ,

$$y_1 \leq_C y_2 \iff y_2 - y_1 \in C.$$

Recall that a cone  $K$  in  $Y$  is called *pointed* if  $K \cap -K \subseteq \{0\}$ .

**Proposition 2.2.5.** *If  $C$  is a cone, then  $\leq_C$  is*

- (i) *reflexive if and only if  $0 \in C$ ,*
- (ii) *antisymmetric if and only if  $C$  is pointed,*
- (iii) *transitive if and only if  $C$  is convex.*

*Proof.* (i) Suppose that  $\leq_C$  is reflexive, then  $y \leq_C y$  for all  $y \in Y$ . This means  $0 = y - y \in C$ . Conversely, since  $0 \in C$ ,  $y - y \in C$  for all  $y \in Y$ . Thus,  $y \leq_C y$ .

(ii) Suppose that  $\leq_C$  is antisymmetric. If  $C \cap -C$  is empty, we are done. Assume that  $y \in C \cap -C$ , then  $0 \leq_C y$ ,  $y \leq_C 0$ . This implies  $y = 0$ . Conversely, let  $y_1, y_2 \in Y$  such that  $y_1 \leq_C y_2$  and  $y_2 \leq_C y_1$ . Then,  $y_2 - y_1 \in C \cap -C$ . Since  $C$  is pointed,  $y_2 = y_1$ .

(iii) Suppose that  $\leq_C$  is transitive. Let  $y_1, y_2 \in C$  and  $\lambda \in (0, 1)$ . Since  $C$  is cone,  $\lambda y_1 \in C$  and  $(1 - \lambda)y_2 \in C$ . It follows from  $\lambda y_1 \in C$  that  $0 \leq_C \lambda y_1$ . Similarly,  $-(-(1 - \lambda)y_2) = (1 - \lambda)y_2 \in C$  means  $-(1 - \lambda)y_2 \leq_C 0$ . This implies  $-(1 - \lambda)y_2 \leq_C \lambda y_1$ . Thus,  $\lambda y_1 + (1 - \lambda)y_2 \in C$ .

Conversely, let  $y_1, y_2, y_3 \in Y$  such that  $y_1 \leq_C y_2$  and  $y_2 \leq_C y_3$ . It means that  $y_2 - y_1 \in C$  and  $y_3 - y_2 \in C$ . Since  $C$  is cone,  $\frac{1}{2}(y_2 - y_1) \in C$  and  $\frac{1}{2}(y_3 - y_2) \in C$ . It follows from the convexity of  $C$  that  $\frac{1}{2}(y_3 - y_2) + \frac{1}{2}(y_2 - y_1) = \frac{1}{2}(y_3 - y_1) \in C$ . Thus,  $y_1 \leq_C y_3$ . □

A relation  $\leq_C$  satisfying three properties in the proposition above is called an order (or order structure) in  $Y$ . Proposition 2.2.5 gives us conditions for which a cone  $C$  generates an order in  $Y$ .

We now recall some conditions on  $C$ , introduced in [28] by Choquet, to ensure that  $(Y, \leq_C)$  is a lattice. Recall that in  $\mathbb{R}^n$ , a  $n$ -simplex is the convex hull of  $n + 1$  (affinely) independent points.

**Proposition 2.2.6.** ([28]) *Suppose that  $C$  is a convex cone in  $\mathbb{R}^n$ . Then  $(Y, \leq_C)$  is a lattice if and only if there exists a base of  $C$  which is a  $(n - 1)$ -simplex in  $\mathbb{R}^{n-1}$ .*

*Proof.* It follows from Proposition 28.3 in [28]. □

By the proposition above,  $(\mathbb{R}^2, \leq_C)$  is a lattice if and only if  $C$  has a base which is a line segment. In  $\mathbb{R}^3$ , the base of  $C$  must be triangle to ensure that  $(\mathbb{R}^3, \leq_C)$  is a lattice.

Let  $C$  be a convex cone in  $Y$ . A main concept in vector optimization is Pareto efficiency.  $A \subseteq Y$ , recall that  $a_0$  is a *Pareto efficient point* of  $A$  with respect to  $C$  if

$$(A - a_0) \cap (-C \setminus l(C)) = \emptyset, \tag{2.2}$$

where  $l(C) := C \cap -C$ . We denote the *set of all Pareto efficient points* of  $A$  by  $\text{Min}_{C \setminus l(C)} A$ .

If, additionally,  $C$  is closed and pointed, then (2.2) becomes  $(A - a_0) \cap (-C \setminus \{0\}) = \emptyset$  and is denoted by  $a_0 \in \text{Min}_{C \setminus \{0\}} A$ .

Next, we are concerned also with the other concepts of efficiency as follows.

**Definition 2.2.7.** ([88]) Let  $A \subseteq Y$ .

(i) Supposing  $\text{int} C \neq \emptyset^1$ ,  $a_0 \in A$  is a *weak efficient point* of  $A$  with respect to  $C$  if  $(A - a_0) \cap -\text{int} C = \emptyset$ .

(ii)  $a_0 \in A$  is a *strong efficient point* of  $A$  with respect to  $C$  if  $A - a_0 \subseteq C$ .

---

<sup>1</sup> $\text{int} C$  denotes the interior of  $C$ .

(iii) Supposing  $C^{+i} \neq \emptyset$ ,  $a_0 \in A$  is a *positive-proper efficient point* of  $A$  with respect to  $C$  if there exists  $\varphi \in C^{+i}$  such that  $\varphi(a) \geq \varphi(a_0)$  for all  $a \in A$ .

(iv)  $a_0 \in A$  is a *Geoffrion-proper efficient point* of  $A$  with respect to  $C$  if  $a_0$  is a Pareto efficient point of  $A$  and there exists a constant  $M > 0$  such that, whenever there is  $\lambda \in C^*$  with norm one and  $\lambda(a_0 - a) > 0$  for some  $a \in A$ , one can find  $\mu \in C^*$  with norm one such that  $\langle \lambda, a_0 - a \rangle \leq M \langle \mu, a - a_0 \rangle$ .

(v)  $a_0 \in A$  is a *Henig-proper efficient point* of  $A$  with respect to  $C$  if there exists a pointed convex cone  $K$  with  $C \setminus \{0\} \subseteq \text{int} K$  such that  $(A - a_0) \cap (-\text{int} K) = \emptyset$ .

(vi) Supposing  $C$  has a convex base  $B$ ,  $a_0 \in A$  is a *strong Henig-proper efficient point* of  $A$  with respect to  $C$  if there is  $\varepsilon > 0$  such that  $\text{cl cone}(A - a_0) \cap (-\text{cl cone}(B + \varepsilon B_Y)) = \{0\}^2$ .

Note that Geoffrion originally defined the properness notion in (iv) for  $\mathbb{R}^n$  with the ordering cone  $\mathbb{R}_+^n$ . The above general definition of Geoffrion properness is taken from [103].

To unify the notation of these above efficiency (with Pareto efficiency), we introduce the following definition. Let  $Q \subseteq Y$  be a nonempty cone, different from  $Y$ , unless otherwise specified.

**Definition 2.2.8.** ([88]) We say that  $a_0$  is a  *$Q$ -efficient point* of  $A$  if

$$(A - a_0) \cap -Q = \emptyset.$$

We define the set of  $Q$ -efficient points by  $\text{Min}_Q A$ .

Recall that a cone in  $Y$  is said to be a *dilating cone* (or a dilation) of  $C$ , or dilating  $C$  if it contains  $C \setminus \{0\}$ . Let  $B$  be, as before, a convex base of  $C$ . Setting  $\delta := \inf\{\|b\| : b \in B\} > 0$ , for  $\varepsilon \in (0, \delta)$ , we associate to  $C$  a pointed convex cone  $C_\varepsilon(B) := \text{cone}(B + \varepsilon B_Y)$ . For  $\varepsilon > 0$ , we also associate to  $C$  another cone  $C(\varepsilon) := \{y \in Y : d_C(y) < \varepsilon d_{-C}(y)\}$ .

Any kind of efficiency in Definition 2.2.7 is in fact a  $Q$ -efficient point with  $Q$  being appropriately chosen as follows.

**Proposition 2.2.9.** ([88]) (i) *Supposing  $\text{int} C \neq \emptyset$ ,  $a_0$  is a weak efficient point of  $A$  with respect to  $C$  if and only if  $a_0 \in \text{Min}_Q A$  with  $Q = \text{int} C$ .*

(ii)  *$a_0$  is a strong efficient point of  $A$  with respect to  $C$  if and only if  $a_0 \in \text{Min}_Q A$  with  $Q = Y \setminus (-C)$ .*

(iii) *Supposing  $C^{+i} \neq \emptyset$ ,  $a_0$  is a positive-proper efficient point of  $A$  with respect to  $C$  if and only if  $a_0 \in \text{Min}_Q A$  with  $Q = \{y \in Y : \varphi(y) > 0\}$  (denoted by  $Q = \{\varphi > 0\}$ ),  $\varphi$  being some functional in  $C^{+i}$ .*

---

<sup>2</sup>Let  $E$  be a set, then  $\text{cl} E$  denotes the closure of  $E$ .

(iv)  $a_0$  is a Geoffrion-proper efficient point of  $A$  with respect to  $C$  if and only if  $a_0 \in \text{Min}_Q A$  with  $Q = C(\varepsilon)$  for some  $\varepsilon > 0$ .

(v)  $a_0$  is a Henig-proper efficient point of  $A$  with respect to  $C$  if and only if  $a_0 \in \text{Min}_Q A$  with  $Q$  being pointed open convex, and dilating  $C$ .

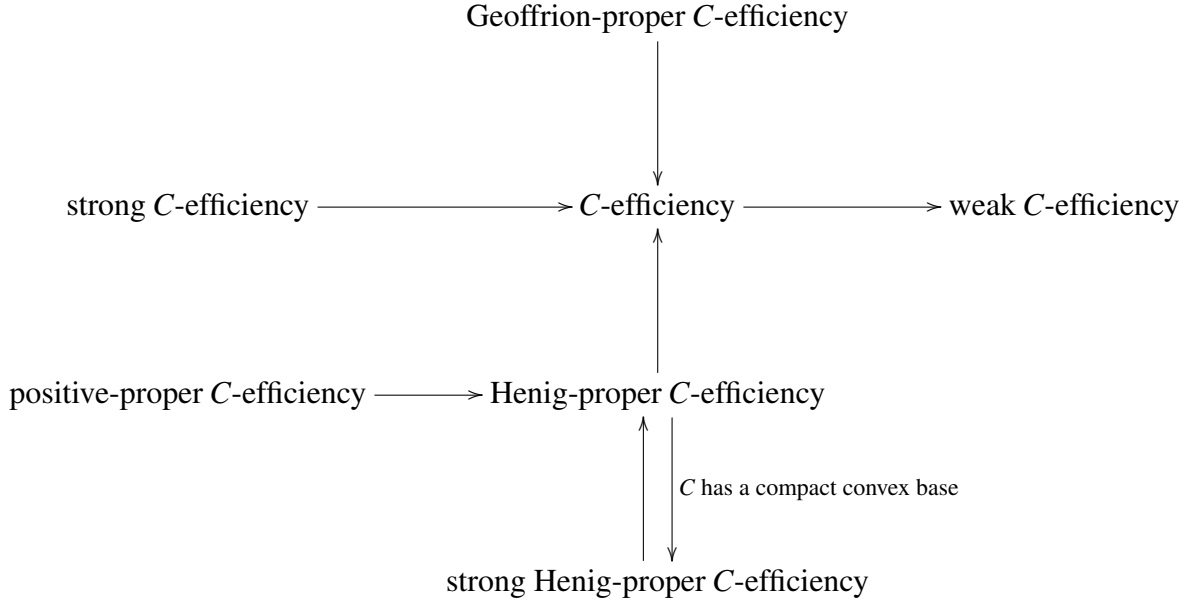
(vi) Supposing  $C$  has a convex base  $B$ ,  $a_0$  is a strong Henig-proper efficient point of  $A$  with respect to  $C$  if and only if  $a_0 \in \text{Min}_Q A$  with  $Q = \text{int}C_\varepsilon(B)$ ,  $\varepsilon$  satisfying  $0 < \varepsilon < \delta$ .

The above proposition gives us a unified way to denote sets of efficient points by the following table

Sets of	Notations
$C$ -efficiency	$\text{Min}_{C \setminus \{0\}}$
weak $C$ -efficiency	$\text{Min}_{\text{int}C}$
strong $C$ -efficiency	$\text{Min}_{Y \setminus (-C)}$
positive-proper $C$ -efficiency	$\bigcup_{\varphi \in C^+} \text{Min}_{\{\varphi > 0\}}$
Geoffrion-proper $C$ -efficiency	$\bigcup_{\varepsilon > 0} \text{Min}_{C(\varepsilon)}$
Henig-proper $C$ -efficiency	$\text{Min}_Q$ where $Q$ is pointed open convex, and dilating $C$
strong Henig-proper $C$ -efficiency	$\text{Min}_{\text{int}C_\varepsilon(B)}$ $\varepsilon$ satisfying $0 < \varepsilon < \delta$ , where $\delta := \inf\{\ b\  : b \in B\}$

For relations of the above properness concepts and also other kinds of efficiency see, e.g.,

[87, 88, 103, 104, 125]. Some of them are collected in the diagram below as examples, see [88].



Let us observe that

**Proposition 2.2.10.** *Suppose that  $Q$  is any cone given in Proposition 2.2.9. Then*

$$Q + C \subseteq Q.$$

*Proof.* It is easy to prove the assertion, when  $Q = \text{int}C$ ,  $Q = Y \setminus (-C)$ ,  $Q = \{y \in Y : \varphi(y) > 0\}$  for  $\varphi \in C^{+i}$ , or  $Q$  is a pointed open convex cone dilating  $C$ .

Now let  $Q = C(\varepsilon)$  for some  $\varepsilon > 0$ ,  $y \in Q$  and  $c \in C$ . We show that  $y + c \in Q$ . It is easy to see that  $d_C(y + c) \leq d_C(y)$  and  $d_{-C}(y) \leq d_{-C}(y + c)$ . Because  $y \in Q$ , we have  $d_C(y) < \varepsilon d_{-C}(y)$ . Thus,  $d_C(y + c) < \varepsilon d_{-C}(y + c)$  and hence  $y + c \in Q$ .

For  $Q = \text{int}C_\varepsilon(B)$ , it is easy to see that  $C \subseteq Q$  for any  $\varepsilon$  satisfying  $0 < \varepsilon < \delta$ . So,  $Q + C \subseteq Q + Q \subseteq Q$ .  $\square$

# Chapter 3

## The theory of $\Gamma$ -limits

### 3.1 Introduction

$\Gamma$ -convergence were introduced by Ennio De Giorgi in a series of papers published between 1975 and 1985. In the same years, De Giorgi developed the theoretical framework of  $\Gamma$ -convergence and explored multifarious applications of this tool. We now give a brief on the development of  $\Gamma$ -convergence in this period.

In 1975, a formal definition of  $\Gamma$ -convergence for a sequence of functions on a topological vector space appeared in [48] by De Giorgi and Franzoni. It included the old notion of  $G$ -convergence (introduced in [155] by Spagnolo for elliptic operators) as a particular case, and provided a unified framework for the study of many asymptotic problems in the calculus of variations.

In 1977, De Giorgi defined in [38] the so called multiple  $\Gamma$ -limits, i.e.,  $\Gamma$ -limits for functions depending on more than one variable. These notions have been a starting point for applications of  $\Gamma$ -convergence to the study of asymptotic behaviour of saddle points in min-max problems and of solutions to optimal control problems.

In 1981, De Giorgi formulated in [40, 41] the theory of  $\Gamma$ -limits in a very general abstract setting and also explored a possibility of extending these notions to complete lattices. This project was accomplished in [43] by De Giorgi and Buttazzo in the same year. The paper also contains some general guide-lines for the applications of  $\Gamma$ -convergence to the study of limits of solutions of ordinary and partial differential equations, including also optimal control problems.

Other applications of  $\Gamma$ -convergence was considered in [39, 45] by De Giorgi et al. in 1981. These papers deal with the asymptotic behaviour of the solutions to minimum problems for the Dirichlet integral with unilateral obstacles. In [44], De Giorgi and Dal Maso gave an account

of main results on  $\Gamma$ -convergence and of its most significant applications to the calculus of variations.

In 1983, De Giorgi proposed in [42] several notions of convergence for measures defined on the space of lower semicontinuous functions, and formulated some problems whose solutions would be useful to identify the most suitable notion of convergence for the study of  $\Gamma$ -limits of random functionals. This notion of convergence was pointed out and studied in detail by De Giorgi et al. in [46,47].

In 1983 in [82] Greco introduced limitoids and showed that all the  $\Gamma$ -limits are special limitoids. In a series of papers published between 1983 and 1985, he developed many applications of this tool in the general theory of limits. The most important result regarding limitoids, presented in [82,84], is the representation theorem for which each relationship of limitoids becomes a relationship of their supports in set theory. In 1984, by applying this theorem, Greco stated in [83] important results on sequential forms of De Giorgi's  $\Gamma$ -limits via a decomposition of their supports in the setting of completely distributive lattice. These results simplify calculation of complicated  $\Gamma$ -limits. This enabled him to find many errors in the literature.

In this chapter, we first introduce definitions and some basic properties of  $\Gamma$ -limits. Greco's results on sequential forms of  $\Gamma$ -limits are also recalled. Finally, we give some applications of  $\Gamma$ -limits to derivatives and tangent cones.

Consider  $n$  sets  $S_1, \dots, S_n$  and a function  $f$  from  $S_1 \times \dots \times S_n$  into  $\overline{\mathbb{R}}$ . Given non-degenerate families  $\mathcal{A}_1, \dots, \mathcal{A}_n$  on  $S_1, \dots, S_n$ , respectively, and  $\alpha_1, \dots, \alpha_n \in \{+, -\}$ .

**Definition 3.1.1.** ([38]) Let

$$\Gamma(\mathcal{A}_1^{\alpha_1}, \dots, \mathcal{A}_n^{\alpha_n}) \lim f := \text{ext}_{A_n \in \mathcal{A}_n}^{-\alpha_n} \dots \text{ext}_{A_1 \in \mathcal{A}_1}^{-\alpha_1} \text{ext}_{x_1 \in A_1}^{\alpha_1} \dots \text{ext}_{x_n \in A_n}^{\alpha_n} f(x_1, \dots, x_n),$$

where  $\text{ext}^+ = \sup$  and  $\text{ext}^- = \inf$ .

The expression above, called a  $\Gamma$ -limit of  $f$ , is a (possibly infinite) number. It is obvious that

$$\Gamma(\mathcal{A}_1^{\alpha_1}, \dots, \mathcal{A}_n^{\alpha_n}) \lim f = -\Gamma(\mathcal{A}_1^{-\alpha_1}, \dots, \mathcal{A}_n^{-\alpha_n}) \lim (-f). \quad (3.1)$$

Given topologies  $\tau_1, \dots, \tau_n$  on  $S_1, \dots, S_n$ , we write

$$(\Gamma(\tau_1^{\alpha_1}, \dots, \tau_n^{\alpha_n}) \lim f)(x_1, \dots, x_n) := \Gamma(\mathcal{N}_{\tau_1}(x_1)^{\alpha_1}, \dots, \mathcal{N}_{\tau_n}(x_n)^{\alpha_n}) \lim f^1. \quad (3.2)$$

Notice that  $\Gamma(\tau_1^{\alpha_1}, \dots, \tau_n^{\alpha_n}) \lim f$  is a function from  $S_1 \times \dots \times S_n$  into  $\overline{\mathbb{R}}$ .

---

<sup>1</sup>If  $(X, \tau)$  is a topological space, then  $\mathcal{N}_\tau(x)$  stands for the set of all neighborhoods of  $x$ .



**Proposition 3.1.2.** ([50]) (i) If  $\mathcal{A}_k \leq \mathcal{B}_k$ , then

$$\Gamma(\dots, \mathcal{A}_k^-, \dots) \lim f \leq \Gamma(\dots, \mathcal{B}_k^-, \dots) \lim f,$$

$$\Gamma(\dots, \mathcal{A}_k^+, \dots) \lim f \geq \Gamma(\dots, \mathcal{B}_k^+, \dots) \lim f.$$

(ii) Suppose that  $\mathcal{A}_i$ ,  $i = 1, \dots, n$ , are filters. Then

$$\Gamma(\dots, \mathcal{A}_k^-, \dots) \lim f \leq \Gamma(\dots, \mathcal{A}_k^+, \dots) \lim f,$$

$$\Gamma(\dots, \mathcal{A}_k^+, \mathcal{A}_{k+1}^-, \dots) \lim f \leq \Gamma(\dots, \mathcal{A}_{k+1}^-, \mathcal{A}_k^+, \dots) \lim f.$$

It is a simple observation that “sup” and “inf” operations are examples of  $\Gamma$ -limits:

$$\inf_B f(x) = \Gamma(N_\iota(B)^-) f, \quad \sup_B f(x) = \Gamma(N_\iota(B)^+) f,$$

where  $\iota$  stands for the *discrete topology*,  $N_\iota(B)$  is the filter of all supersets of the set  $B$ . If  $B$  is the whole space, we may also use the *chaotic topology*  $o$ .

## 3.2 $\Gamma$ -limits in two variables

Let  $f : I \times X \rightarrow \overline{\mathbb{R}}$  defined by  $f(i, x) := f_i(x)$ , where  $\{f_i\}_{i \in I}$  is a family of functions from  $X$  into  $\overline{\mathbb{R}}$  and filtered by a filter  $\mathcal{F}$  on  $I$ . Thus, results on  $\Gamma$ -limits of  $f$  implies those on limits of  $\{f_i\}_{i \in I}$ .

From Definition 3.1.1, we get for  $x \in X$ ,

$$\begin{aligned} (\Gamma(\mathcal{F}^+; \tau^-) \lim f)(x) &= \Gamma(\mathcal{F}^+, \mathcal{N}_\tau(x)^-) \lim f \\ &= \sup_{U \in \mathcal{N}_\tau(x)} \inf_{F \in \mathcal{F}} \sup_{i \in F} \inf_{y \in U} f(i, y) \\ &= \sup_{U \in \mathcal{N}_\tau(x)} \limsup_{\mathcal{F}} \inf_{y \in U} f_i(y), \\ &= \sup_{U \in \mathcal{N}_\tau(x)} \Gamma(\mathcal{F}^+) \inf_{y \in U} f_i(y), \end{aligned}$$

$$\begin{aligned} (\Gamma(\mathcal{F}^-; \tau^-) \lim f)(x) &= \Gamma(\mathcal{F}^-, \mathcal{N}_\tau(x)^-) \lim f \\ &= \sup_{U \in \mathcal{N}_\tau(x)} \sup_{F \in \mathcal{F}} \inf_{i \in F} \inf_{y \in U} f(i, y) \\ &= \sup_{U \in \mathcal{N}_\tau(x)} \liminf_{\mathcal{F}} \inf_{y \in U} f_i(y), \\ &= \sup_{U \in \mathcal{N}_\tau(x)} \Gamma(\mathcal{F}^-) \inf_{y \in U} f_i(y), \end{aligned}$$

$$\begin{aligned}
 (\Gamma(\mathcal{F}^+; \tau^+) \lim f)(x) &= \Gamma(\mathcal{F}^+, \mathcal{N}_\tau(x)^+) \lim f \\
 &= \inf_{U \in \mathcal{N}_\tau(x)} \inf_{F \in \mathcal{F}} \sup_{i \in F} \sup_{y \in U} f(i, y) \\
 &= \inf_{U \in \mathcal{N}_\tau(x)} \limsup_{\mathcal{F}} \sup_{y \in U} f_i(y), \\
 &= \inf_{U \in \mathcal{N}_\tau(x)} \Gamma(\mathcal{F}^+) \sup_{y \in U} f_i(y), \\
 (\Gamma(\mathcal{F}^-; \tau^+) \lim f)(x) &= \Gamma(\mathcal{F}^-, \mathcal{N}_\tau(x)^+) \lim f \\
 &= \inf_{U \in \mathcal{N}_\tau(x)} \sup_{F \in \mathcal{F}} \inf_{i \in F} \sup_{y \in U} f(i, y) \\
 &= \inf_{U \in \mathcal{N}_\tau(x)} \liminf_{\mathcal{F}} \sup_{y \in U} f_i(y), \\
 &= \inf_{U \in \mathcal{N}_\tau(x)} \Gamma(\mathcal{F}^-) \sup_{y \in U} f_i(y).
 \end{aligned}$$

Based on these limits above, we can show that some well-known limits are special cases of  $\Gamma$ -limits as follows.

**Remark 3.2.1.** (i) If functions  $f_i(x)$  are independent of  $x$ , i.e., for every  $i$  there exists a constant  $a_i \in \overline{\mathbb{R}}$  such that  $f_i(x) = a_i$  for every  $x \in X$ , then

$$\begin{aligned}
 (\Gamma(\mathcal{F}^+; \tau^-) \lim f)(x) &= (\Gamma(\mathcal{F}^+; \tau^+) \lim f)(x) = \limsup_{\mathcal{F}} a_i, \\
 (\Gamma(\mathcal{F}^-; \tau^-) \lim f)(x) &= (\Gamma(\mathcal{F}^-; \tau^+) \lim f)(x) = \liminf_{\mathcal{F}} a_i.
 \end{aligned}$$

(ii) If functions  $f_i(x)$  are independent of  $i$ , i.e., there exists  $g : X \rightarrow \overline{\mathbb{R}}$  such that  $f_i(x) = g(x)$  for every  $x \in X, i \in I$ , then

$$\begin{aligned}
 (\Gamma(\mathcal{F}^-; \tau^+) \lim f)(x) &= (\Gamma(\mathcal{F}^+; \tau^+) \lim f)(x) = \limsup_{y \rightarrow_\tau x} g(y), \\
 (\Gamma(\mathcal{F}^-; \tau^-) \lim f)(x) &= (\Gamma(\mathcal{F}^+; \tau^-) \lim f)(x) = \liminf_{y \rightarrow_\tau x} g(y).
 \end{aligned}$$

In [36],  $\Gamma(\mathcal{F}^+; \tau^-) \lim f$  and  $\Gamma(\mathcal{F}^-; \tau^-) \lim f$  are called, by Dal Maso, the  $\Gamma$ -upper limit and the  $\Gamma$ -lower limit of the family  $f_i$  and are denoted by  $\Gamma\text{-}\limsup_{\mathcal{F}}^\tau f_i$  and  $\Gamma\text{-}\liminf_{\mathcal{F}}^\tau f_i$ , respectively. If there exists a function  $f_0$  such that for all  $x \in X$ ,

$$(\Gamma(\mathcal{F}^+; \tau^-) \lim f)(x) \leq f_0(x) \leq (\Gamma(\mathcal{F}^-; \tau^-) \lim f)(x),$$

then we say that  $\{f_i\}$   $\Gamma$ -converges to  $f_0$  or  $f_0$  is a  $\Gamma$ -limit of  $\{f_i\}$ .

The following examples show that, in general,  $\Gamma$ -convergence and pointwise convergence are independent.

**Example 3.2.2.** ([36]) Let  $X = \mathbb{R}$  (with a usual topology  $\nu$  on  $\mathbb{R}$ ) and  $\{f_n\}$  be defined by

(i)  $f_n(x) = \sin(nx)$ . Then,  $\{f_n\}$   $\Gamma$ -converges to the constant function  $f = -1$ , whereas  $\{f_n\}$  does not converge pointwise in  $\mathbb{R}$ .

$$(ii) f_n(x) = \begin{cases} nxe^{-2n^2x^2}, & \text{if } n \text{ is even,} \\ 2nxe^{-2n^2x^2}, & \text{if } n \text{ is odd.} \end{cases}$$

By calculating,  $\{f_n\}$  converges pointwise to 0, but  $\{f_n\}$  does not  $\Gamma$ -converge since

$$(\Gamma(\mathcal{I}^-; \nu^-) \lim f)(x) = \begin{cases} -e^{-1/2}, & \text{if } x = 0, \\ 0, & \text{if } x \neq 0, \end{cases}$$

$$(\Gamma(\mathcal{I}^+; \nu^-) \lim f)(x) = \begin{cases} -\frac{1}{2}e^{-1/2}, & \text{if } x = 0, \\ 0, & \text{if } x \neq 0, \end{cases}$$

where  $f(n, x) := f_n(x)$  and  $\mathcal{I} := \{I_m\}_{m \in \mathbb{N}}$  with  $I_m := \{n \in \mathbb{N} : n \geq m\}$ .

We now compare the notion of  $\Gamma$ -limits with some classical notions of convergence.

**Definition 3.2.3.** A family  $\{f_i\}_{i \in I}$  is said to be *continuously convergent* to a function  $g : X \rightarrow \overline{\mathbb{R}}$  if for every  $x \in X$  and for every neighborhood  $V$  of  $g(x)$ , there exist  $F \in \mathcal{F}$  and  $U \in \mathcal{N}_\tau(x)$  such that  $f_i(y) \in V$  for every  $i \in F$  and for every  $y \in U$ .

It follows immediately from the definitions that  $\{f_i\}$  is continuously convergent to  $g$  if and only if

$$\Gamma(\mathcal{F}^+; \tau^+) \lim f \leq g \leq \Gamma(\mathcal{F}^-; \tau^-) \lim f.$$

**Definition 3.2.4.** ([111]) Let  $\{A_i\}_{i \in I}$  be a family of subsets in  $(X, \tau)$  filtered by  $\mathcal{F}$ .

(i) The *K-upper limit* of  $\{A_i\}_{i \in I}$  is defined by

$$\text{Limsup}_{\mathcal{F}}^\tau A_i = \bigcap_{F \in \mathcal{F}} \text{cl}_\tau \bigcup_{i \in F} A_i.$$

(ii) The *K-lower limit* of  $\{A_i\}_{i \in I}$  is defined by

$$\text{Liminf}_{\mathcal{F}}^\tau A_i = \bigcap_{F \in \mathcal{F}^\#} \text{cl}_\tau \bigcup_{i \in F} A_i.$$

(iii) If there exists a subset  $A$  in  $X$  such that

$$\text{Limsup}_{\mathcal{F}}^\tau A_i \subseteq A \subseteq \text{Liminf}_{\mathcal{F}}^\tau A_i,$$

then we say that  $\{A_i\}$  *K-converges* to  $A$ .

It follows from the above definition that  $x \in \text{Limsup}_{\mathcal{F}}^{\tau} A_i$  if and only if for every  $U \in \mathcal{N}_{\tau}(x)$  every  $F \in \mathcal{F}$ , there is  $i \in F$  such that  $U \cap A_i \neq \emptyset$ ; due to the duality of filters and their grills, if for every  $U \in \mathcal{N}_{\tau}(x)$  there is  $H \in \mathcal{F}^{\#}$  such that  $U \cap A_i \neq \emptyset$  for each  $i \in H$ .

A point  $x \in \text{Liminf}_{\mathcal{F}}^{\tau} A_i$  if and only if for every  $U \in \mathcal{N}_{\tau}(x)$  and every  $H \in \mathcal{F}^{\#}$ , there is  $i \in H$  such that  $U \cap A_i \neq \emptyset$ . Dually, if for every  $U \in \mathcal{N}_{\tau}(x)$  there is  $F \in \mathcal{F}$  such that for each  $i \in F$ ,  $U \cap A_i \neq \emptyset$ .

When  $X$  is equipped with the discrete topology  $\iota$ , the discussed limits become set-theoretical, that is

$$\text{Limsup}_{\mathcal{F}}^{\iota} A_i = \bigcap_{F \in \mathcal{F}} \bigcup_{i \in F} A_i,$$

and

$$\text{Liminf}_{\mathcal{F}}^{\iota} A_i = \bigcap_{H \in \mathcal{F}^{\#}} \bigcup_{i \in H} A_i = \bigcup_{F \in \mathcal{F}} \bigcap_{i \in F} A_i.$$

**Remark 3.2.5.** It is generally admitted among those who study optimization, that modern definition of limits of sets by Painlevé and Kuratowski (see [110, 172, 173]). However, Peano already introduced them in 1887. Indeed, in [135], Peano defined the lower limit of a family, indexed by the reals, of a subsets  $A_{\lambda}$  of an affine Euclidean space  $A$  by

$$\text{Liminf}_{\lambda \rightarrow +\infty} A_{\lambda} := \{y \in X : \lim_{\lambda \rightarrow +\infty} d(y, A_{\lambda}) = 0\}.$$

In [137], he also defined the upper limits of  $\{A_{\lambda}\}$

$$\text{Limsup}_{\lambda \rightarrow +\infty} A_{\lambda} := \{y \in X : \liminf_{\lambda \rightarrow +\infty} d(y, A_{\lambda}) = 0\}$$

that he also expresses as

$$\text{Limsup}_{n \rightarrow +\infty} A_{\lambda} = \bigcap_{n \in \mathbb{N}} \text{cl} \bigcup_{k \geq n} A_k.$$

In 1948, Kuratowski, by his work (see [111]), has definitely propagated the concept of limits of variable sets and established the use of them in mathematics, that are called today upper and lower Kuratowski limits.

Recall that, for every  $A \subseteq X$ , the *characteristic function* of  $A$  is defined by

$$\chi_A(x) := \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

Let  $\Omega \subseteq (S_1, \tau_1) \times \dots \times (S_n, \tau_n)$ . We define

$$\chi_{G(\tau_1^{\alpha_1}, \dots, \tau_n^{\alpha_n})\Omega} := \Gamma(\tau_1^{\alpha_1}, \dots, \tau_n^{\alpha_n}) \lim \chi_\Omega.$$

Then  $G(\tau_1^{\alpha_1}, \dots, \tau_n^{\alpha_n})\Omega$  is called  $G$ -limit of  $\Omega$ , see [38].

The following proposition shows that  $K$ -limits of a family of subsets can be expressed in terms of  $G$ -limits.

**Proposition 3.2.6.** ([36]) *Let  $\{A_i\}_{i \in I}$  be a family of subsets of  $X$ . Then*

$$G(\mathcal{F}^-; \tau^+) \mathbb{A} = \text{Liminf}_{\mathcal{F}}^{\tau} A_i, \quad G(\mathcal{F}^+; \tau^+) \mathbb{A} = \text{Limsup}_{\mathcal{F}}^{\tau} A_i,$$

where  $\mathbb{A}$  is a relation from  $I$  into  $X$ , i.e.  $\mathbb{A} \subseteq I \times X$ , defined by  $\mathbb{A}(i) := A_i$ .

*In particular,  $\{A_i\}$   $K$ -converges to  $A$  if and only if  $\{\chi_{A_i}\}$   $\Gamma$ -converges to  $\chi_A$ .*

*Proof.* We prove only the first equality, the other one being analogous. Since  $\chi_{G(\mathcal{F}^-; \tau^+) \mathbb{A}} = \Gamma(\mathcal{F}^-; \tau^+) \lim \chi_{\mathbb{A}}$  takes only the values 0 and 1, it is enough to show that

$$(\Gamma(\mathcal{F}^-; \tau^+) \lim \chi_{\mathbb{A}})(x) = 1 \iff x \in \text{Liminf}_{\mathcal{F}}^{\tau} A_i. \quad (3.3)$$

By (3.2),  $(\Gamma(\mathcal{F}^-; \tau^+) \lim \chi_{\mathbb{A}})(x) = 1$  if and only if

$$\inf_{U \in \mathcal{N}_\tau(x)} \sup_{F \in \mathcal{F}} \inf_{i \in F} \sup_{x' \in U} \chi_{\mathbb{A}}(i, x') = 1.$$

It means that for every  $U \in \mathcal{N}_\tau(x)$  there is  $F \in \mathcal{F}$  such that for each  $i \in F$ , there is  $x' \in U$  satisfying  $x' \in A_i$ , i.e.,  $U \cap A_i \neq \emptyset$ . Thus,  $x \in \text{Liminf}_{\mathcal{F}}^{\tau} A_i$ . This prove (3.3) and concludes the proof of the proposition.  $\square$

The next result shows a connection between  $\Gamma$ -convergence of functions and  $K$ -convergence of their epigraphs or hypographs.

**Proposition 3.2.7.** ([36]) *Let  $\{f_i\}_{i \in I}$  be a family of extended-real-valued functions, and let*

$$f^- := \Gamma(\mathcal{F}^-; \tau^-) \lim f, \quad f^+ := \Gamma(\mathcal{F}^+; \tau^-) \lim f.$$

*Then*

$$\begin{aligned} \text{epi}(f^-) &= \text{Limsup}_{\mathcal{F}}^{\tau} \text{epi}(f_i), & \text{epi}(f^+) &= \text{Liminf}_{\mathcal{F}}^{\tau} \text{epi}(f_i), \\ \text{hypo}(f^-) &= \text{Liminf}_{\mathcal{F}}^{\tau} \text{hypo}(f_i), & \text{hypo}(f^+) &= \text{Limsup}_{\mathcal{F}}^{\tau} \text{hypo}(f_i). \end{aligned}$$

*In particular,  $\{f_i\}$   $\Gamma$ -converges to  $f$  if and only if  $\{\text{epi}(f_i)\}$  (or  $\{\text{hypo}(f_i)\}$ )  $K$ -converges to  $\text{epi}(f)$  ( $\{\text{hypo}(f)\}$ ), respectively.*

*Proof.* By the similarity, we prove only the first equality. A point  $(x, t) \in X \times \mathbb{R}$  belongs to  $\text{epi}(f^-)$  if and only if  $f^-(x) \leq t$ . By the definition of  $f^-$ , this happens if and only if for every  $\varepsilon > 0$ , and for every  $U \in \mathcal{N}_\tau(x)$  we have

$$\liminf_{\mathcal{F}} \inf_{y \in U} f_i(y) < t + \varepsilon,$$

and this is equivalent to say that for every  $\varepsilon > 0, U \in \mathcal{N}_\tau(x), F \in \mathcal{F}$  there exists  $i \in F$  such that  $\inf_{y \in U} f_i(y) < t + \varepsilon$ . Since this inequality is equivalent to

$$\text{epi}(f_i) \cap (U \times (t - \varepsilon, t + \varepsilon)) \neq \emptyset,$$

and the sets of the form  $U \times (t - \varepsilon, t + \varepsilon)$ , with  $U \in \mathcal{N}_\tau(x)$  and  $\varepsilon > 0$ , are a base for the neighborhood systems of  $(x, t)$  in  $X \times \mathbb{R}$ , we have proved that  $(x, t) \in \text{epi}(f^-)$  if and only if  $(x, t) \in \text{Limsup}_{\mathcal{F}}^\tau \text{epi}(f_i)$ .  $\square$

### 3.3 $\Gamma$ -limits valued in completely distributive lattices

This section presents some results related to  $\Gamma$ -limits given by Greco in [82, 83]. More precisely, he defined functionals called limitoids and proved that  $\Gamma$ -limits are special limitoids. Then he proved representation theorem showing that for which each relationship of limitoids corresponds a relationship in set theory.

#### 3.3.1 Limitoids

Let  $S$  be a set with at least two elements and  $L$  be a complete lattice with the minimum element  $0_L$  and the maximum element  $1_L$  ( $0_L \neq 1_L$ ).

**Definition 3.3.1.** ([82]) A function  $T : L^S \rightarrow L$  is said  $L$ -limitoid in  $S$  (or *limitoid*, in short) if for every  $f, g \in L^S$  and for each *complete homomorphism*  $\varphi$  of  $L$  in  $L$ ,

- (i)  $g \leq f \implies T(g) \leq T(f)$ ,
- (ii)  $T(\varphi \circ g) = \varphi(T(g))$ ,
- (iii)  $T(g) \in \overline{g(S)}^L$ ,

where  $\overline{g(S)}^L$  is the smallest closed sublattice of  $L$  containing  $g(S)$ , and  $L^S$  denotes the set of functions from  $S$  into  $L$ .

We recall that a *complete homomorphism*  $\varphi : L \rightarrow L'$  between two complete lattices is a function verifying two equalities  $\varphi(\bigvee A) = \bigvee \varphi(A)$  and  $\varphi(\bigwedge A) = \bigwedge \varphi(A)$  for each non-empty subset  $A$  of  $L$ , see [82].

Simple examples of limitoids are limit inferior and limit superior. Let  $f$  be a function from  $S$  into  $L$  and  $\mathcal{A}$  be a non-degenerate family of subsets of  $S$ , the *limit inferior* and *limit superior* of  $f$  along  $\mathcal{A}$  are defined, respectively,

$$\begin{aligned} \liminf_{\mathcal{A}} f &= \bigvee_{A \in \mathcal{A}} \bigwedge_{x \in A} f(x) \left( = \sup_{A \in \mathcal{A}} \inf_{x \in A} f(x) \right), \\ \limsup_{\mathcal{A}} f &= \bigwedge_{A \in \mathcal{A}} \bigvee_{x \in A} f(x) \left( = \inf_{A \in \mathcal{A}} \sup_{x \in A} f(x) \right). \end{aligned}$$

The  $\Gamma$ -limit introduced in Definition 3.1.1 is another example of the limitoid.

It is evident that the limit inferior, limit superior and  $\Gamma$ -limit do not change if we use equivalent families. Since  $\mathcal{A}^{\#\#} = \mathcal{A}$  for each family  $\mathcal{A}$ , we have

$$\liminf_{\mathcal{A}} f = \liminf_{\mathcal{A}^{\#\#}} f, \quad \limsup_{\mathcal{A}} f = \limsup_{\mathcal{A}^{\#\#}} f.$$

The following result characterises of a completely distributive lattice  $L$ .

**Proposition 3.3.2.** ([82]) *A complete lattice  $L$  is completely distributive if and only if*

$$\liminf_{\mathcal{A}} f = \limsup_{\mathcal{A}^{\#}} f, \tag{3.4}$$

for each non-degenerate family  $\mathcal{A}$  of subsets of  $S$  and for each function  $f$  from  $S$  into  $L$ .

*Proof.* It follows from Proposition D.3 in [82]. □

**Definition 3.3.3.** ([83]) The *support* of a limitoid  $T$  in  $S$ , denoted by  $\mathbf{st}(T)$ , is the family of sets defined by

$$\mathbf{st}(T) := \{A \subseteq S : T(\chi_A^L) = 1_L\},$$

where  $\chi_A^L : S \rightarrow L$  is to  $1_L$  on  $A$  and to  $0_L$  on  $S \setminus A$ .

A support of a limitoid  $T$  in  $S$  is a semi-filter, i.e.,  $\mathbf{st}(T) \in \mathbb{SF}(S)$ . Recall that,  $\mathbb{SF}(S)$  is a completely distributive lattice with respect to inclusion, see [83], with its operations  $\bigwedge$  and  $\bigvee$  be the intersection and the union of sets, respectively.

The support of  $\Gamma$ -limit in Definition 3.1.1 is indicated with  $(\mathcal{A}_1^{\alpha_1}, \dots, \mathcal{A}_n^{\alpha_n})$ . In [83], Greco proved recursively that

$$\begin{aligned} (\mathcal{A}^-) &\approx \mathcal{A}^{\#\#}, & (\mathcal{A}^+) &\approx \mathcal{A}^\#, \\ (\mathcal{A}_1^{\alpha_1}, \dots, \mathcal{A}_{n-1}^{\alpha_{n-1}}, \mathcal{A}_n^-) &\approx (\mathcal{A}_1^{\alpha_1}, \dots, \mathcal{A}_{n-1}^{\alpha_{n-1}}) \times \mathcal{A}_n, \\ (\mathcal{A}_1^{\alpha_1}, \dots, \mathcal{A}_{n-1}^{\alpha_{n-1}}, \mathcal{A}_n^+) &\approx ((\mathcal{A}_1^{\alpha_1}, \dots, \mathcal{A}_{n-1}^{\alpha_{n-1}})^\# \times \mathcal{A}_n)^\#, \end{aligned}$$

where  $\mathcal{A} \times \mathcal{B}$  indicates the family generated by  $\{A \times B : A \in \mathcal{A}, B \in \mathcal{B}\}$ . Some special cases are

$$\begin{aligned} (\mathcal{A}^-, \mathcal{B}^-) &\approx \mathcal{A} \times \mathcal{B}, & (\mathcal{A}^+, \mathcal{B}^-) &\approx \mathcal{A}^\# \times \mathcal{B}, \\ (\mathcal{A}^+, \mathcal{B}^+) &\approx (\mathcal{A} \times \mathcal{B})^\#, & (\mathcal{A}^-, \mathcal{B}^+) &\approx (\mathcal{A}^\# \times \mathcal{B})^\#, \\ (\mathcal{A}^+, \mathcal{B}^-, \mathcal{C}^-) &\approx \mathcal{A}^\# \times \mathcal{B} \times \mathcal{C}, & (\mathcal{A}^-, \mathcal{B}^+, \mathcal{C}^-) &\approx (\mathcal{A}^\# \times \mathcal{B})^\# \times \mathcal{C}. \end{aligned}$$

Then we get the following property

$$(\mathcal{A}_1^{-\alpha_1}, \dots, \mathcal{A}_n^{-\alpha_n}) \approx (\mathcal{A}_1^{\alpha_1}, \dots, \mathcal{A}_n^{\alpha_n})^\#. \quad (3.5)$$

### 3.3.2 Representation theorem

The representation theorem of Greco showed that each limitoid valued in a completely distributive lattice is a limit inferior.

**Theorem 3.3.4.** ([82]) (The representation of limitoids) *Let  $L$  be a completely distributive lattice and  $T$  be a limitoid in  $S$ . Then, for each  $f \in L^S$ ,*

$$T(f) = \liminf_{\mathbf{st}(T)} f, \quad (3.6)$$

where  $\mathbf{st}(T)$  is the support of  $T$ .

*Proof.* First, we check that for each  $f \in L^S$ ,

$$\liminf_{\mathbf{st}(T)} f \leq T(f) \leq \limsup_{(\mathbf{st}(T))^\#} f, \quad (3.7)$$

where  $(\mathbf{st}(T))^\# := \{A \subseteq S : A \cap F \neq \emptyset, \forall F \in \mathbf{st}(T)\}$ . For each  $A \in \mathbf{st}(T)$ , we put  $g := \chi_A^L \wedge (\bigwedge f(A))$ . Since  $L$  is completely distributive, the function  $\varphi(x) := x \wedge (\bigwedge f(A))$  is a complete homomorphism of  $L$  in  $L$ . So, from condition (ii) in Definition 3.3.1 and the definition of carriers, we get  $T(g) = \bigwedge f(A)$ . Since  $g \leq f$ , it follows from condition (i) in Definition 3.3.1 that  $\bigwedge f(A) \leq T(f)$ . Therefore,

$$\liminf_{\mathbf{st}(T)} f \leq T(f).$$



On the other hand, let  $A \in (\mathbf{st}(T))^\#$  and  $g := \chi_{S \setminus A}^L \vee (\vee f(A))$ . From the definition of  $(\mathbf{st}(T))^\#$ , we have  $(S \setminus A) \notin \mathbf{st}(T)$ , so  $T(\chi_{S \setminus A}^L) = 0_L$ . Since the function  $\varphi$  defined by  $\varphi(x) := x \vee (\vee f(A))$  is a complete homomorphism of  $L$  in  $L$ , so  $T(g) = \vee f(A)$ . Finally, since  $g \geq f$ , we obtain  $\vee f(A) \geq T(f)$ , which implies

$$T(f) \leq \limsup_{(\mathbf{st}(T))^\#} f.$$

Thus, we have proved (3.7). (3.6) is obtained by Proposition 3.3.2.  $\square$

It follows from (3.6) that

$$\Gamma(\mathcal{A}_1^{\alpha_1}, \dots, \mathcal{A}_n^{\alpha_n}) \lim f = \liminf_{(\mathcal{A}_1^{\alpha_1}, \dots, \mathcal{A}_n^{\alpha_n})} f = \sup_{A \in (\mathcal{A}_1^{\alpha_1}, \dots, \mathcal{A}_n^{\alpha_n})} \inf_{(x_1, \dots, x_n) \in A} f(x_1, \dots, x_n).$$

By virtue of (3.4) and (3.5), we get

$$\Gamma(\mathcal{A}_1^{-\alpha_1}, \dots, \mathcal{A}_n^{-\alpha_n}) \lim f = \liminf_{(\mathcal{A}_1^{\alpha_1}, \dots, \mathcal{A}_n^{\alpha_n})^\#} f = \inf_{A \in (\mathcal{A}_1^{\alpha_1}, \dots, \mathcal{A}_n^{\alpha_n})} \sup_{(x_1, \dots, x_n) \in A} f(x_1, \dots, x_n).$$

The representation of limitoids allows us to describe the structure of lattice  $\mathbf{Lim}(S, L)$  of limitoids in  $S$  valued in  $L$ , where  $L$  is a completely distributive lattice. The set  $\mathbf{Lim}(S, L)$  of limitoids in  $S$  valued in  $L$  is the complete lattice with respect to the order defined by  $T \leq T'$  if and only if  $T(g) \leq T'(g)$  for each  $g \in L^S$ . In  $\mathbf{Lim}(S, L)$ , the limitoids  $\vee_i T_i, \wedge_i T_i$  are defined by for each  $g \in L^S$ ,

$$\left( \bigvee_i T_i \right) (g) = \bigvee_i (T_i(g)), \quad \left( \bigwedge_i T_i \right) (g) = \bigwedge_i (T_i(g)).$$

Furthermore, the function  $\mathbf{st} : \mathbf{Lim}(S, L) \rightarrow \mathbb{SF}(S)$  is a complete homomorphism of  $\mathbf{Lim}(S, L)$  on  $\mathbb{SF}(S)$ , see [82], since for each semi-filter  $\mathcal{A}$  in  $S$ ,

$$\mathbf{st} \left( \bigwedge_i T_i \right) = \bigcap_i \mathbf{st}(T_i), \quad \mathbf{st} \left( \bigvee_i T_i \right) = \bigcup_i \mathbf{st}(T_i), \quad \mathbf{st}(\liminf_{\mathcal{A}}) = \mathcal{A}.$$

On the other hand, if  $L$  is completely distributive, then two limitoids in  $S$  with the same supports are equal (by the representation of limitoids). Therefore

**Theorem 3.3.5.** ([82]) (The structure of lattice of limitoids) *If  $L$  is a completely distributive lattice, then the function that associates each limitoid in  $S$  to its support is a complete isomorphism, i.e., a bijective complete homomorphism, from  $\mathbf{Lim}(S, L)$  into  $\mathbb{SF}(S)$ .*

The function  $\liminf : \mathbb{SF}(S) \rightarrow \mathbf{Lim}(S, L)$  that associates each semi-filter  $\mathcal{A}$  in  $S$  to the limit inferior with respect to  $\mathcal{A}$  is the inverse isomorphism. Therefore, for a completely distributive lattice  $L$ , we have

$$\liminf_{\bigcap_i \mathcal{A}_i} f = \bigwedge_i \liminf_{\mathcal{A}_i} f, \tag{3.8}$$

$$\liminf_{\bigcup_i \mathcal{A}_i} f = \bigvee_i \liminf_{\mathcal{A}_i} f, \quad (3.9)$$

for each  $f \in L^S$  and  $\{\mathcal{A}_i\}_i \subseteq \mathbb{SF}(S)$ .

These analyses above mean that with a completely distributive lattice  $L$ , each theorem in  $\mathbf{Lim}(S, L)$  becomes a theorem in set theory in  $\mathbb{SF}(S)$ , and vice versa.

### 3.4 Sequential forms of $\Gamma$ -limits for extended-real-valued functions

In this section, we extend Greco's results to more general filters related to sequentiality, like Fréchet, strongly Fréchet, and productively Fréchet filters.

Let  $\mathcal{F}$  be a filter on  $X$ . We recall that, see [53]

- $\mathcal{F}$  is called a *principal filter* if there exists a nonempty subset  $A$  of  $X$  such that  $\mathcal{F} = \{B \subseteq X : A \subseteq B\}$ . The set of principal filters on  $X$  is denoted by  $\mathbb{F}_0(X)$ .

- $\mathcal{F}$  is called a *sequential filter* if there exists a sequence  $\{x_n\}_n$  in  $X$  such that the family  $\{\{x_n : n \geq m\} : m \in \mathbb{N}\}$  is a base of  $\mathcal{F}$ . Then, we denote  $\mathcal{F} \approx \{x_n\}_n$ . The set of sequential filters on  $X$  is denoted by  $\mathbb{F}_{\text{seq}}(X)$ .

- $\mathcal{F}$  is called a *countably based filter* if it admits a countable base. The set of countably based filters on  $X$  is denoted by  $\mathbb{F}_1(X)$ .

Principal filters and sequential filters are special cases of countably based filters. If  $\mathcal{F} \approx \{x_n\}_n$ , then (see [83])

$$\liminf_{\mathcal{F}} f = \liminf_{n \rightarrow +\infty} f(x_n), \quad \liminf_{\mathcal{F}^\#} f = \limsup_{n \rightarrow +\infty} f(x_n).$$

To facilitate for results in the sequel, we denote  $\text{Seq}(\mathcal{F}) := \{\mathcal{E} \in \mathbb{F}_{\text{seq}}(X) : \mathcal{E} \geq \mathcal{F}\}$ .

**Definition 3.4.1.** ([62]) A topological space  $X$  is called

(i) *Fréchet* if whenever  $A \subseteq X$ , and  $x \in \text{cl}A$ , there exists a sequence  $\{x_n\}_n$  on  $A$  such that  $x = \lim_{n \rightarrow +\infty} x_n$ .

(ii) *strongly Fréchet* if for each a decreasing sequence  $\{A_n\}_n$  of subsets of  $X$  and  $x \in \bigcap_n \text{cl}(A_n)$ , there exists a sequence  $\{x_n\}_n$  such that  $x_n \in A_n$  and  $x = \lim_{n \rightarrow +\infty} x_n$ .

(iii) *first countable* if for all  $x \in X$ ,  $\mathcal{N}(x)$  is a countably based filter.

Definitions of Fréchet and strongly Fréchet spaces can be rephrased in terms of filters as follows, see [52, 100].

• A space  $X$  is Fréchet if and only if for all  $x \in X$ ,  $\mathcal{N}(x)$  is a Fréchet filter on  $X$  in the following sense: a filter  $\mathcal{F}$  is *Fréchet* if

$$\forall \mathcal{G} \in \mathbb{F}_0(X) : \mathcal{G} \# \mathcal{F} \implies \exists \mathcal{H} \in \mathbb{F}_{\text{seq}}(X) : \mathcal{H} \geq \mathcal{F} \vee \mathcal{G}, \quad (3.10)$$

where  $\mathcal{F} \vee \mathcal{G} := \{F \cap G : F \in \mathcal{F}, G \in \mathcal{G}\}$  is the supremum of  $\mathcal{F}$  and  $\mathcal{G}$ .

• A space  $X$  is strongly Fréchet if and only if for all  $x \in X$ ,  $\mathcal{N}(x)$  is a strongly Fréchet filter on  $X$  in the following sense: a filter  $\mathcal{F}$  is *strongly Fréchet* if

$$\forall \mathcal{G} \in \mathbb{F}_1(X) : \mathcal{G} \# \mathcal{F} \implies \exists \mathcal{H} \in \mathbb{F}_{\text{seq}}(X) : \mathcal{H} \geq \mathcal{F} \vee \mathcal{G}. \quad (3.11)$$

Based on definitions above, Jordan and Mynard introduced a productively Fréchet space by using a new filter, called productively Fréchet filter, as follows

**Definition 3.4.2.** ([100]) A space  $X$  is productively Fréchet if and only if for all  $x \in X$ ,  $\mathcal{N}(x)$  is a productively Fréchet filter on  $X$  in the following sense: a filter  $\mathcal{F}$  is *productively Fréchet* if

$$\forall \mathcal{G} \in \mathbb{F}_{\text{sF}}(X) : \mathcal{G} \# \mathcal{F} \implies \exists \mathcal{H} \in \mathbb{F}_1(X) : \mathcal{H} \geq \mathcal{F} \vee \mathcal{G},$$

where  $\mathbb{F}_{\text{sF}}(X)$  denotes the set of strongly Fréchet filters on  $X$ .

In [100], Jordan and Mynard showed that

first countable filter  $\implies$  productively Fréchet filter  $\implies$  strongly Fréchet filter  $\implies$  Fréchet filter,

i.e.,

first countable space  $\implies$  productively Fréchet space  $\implies$  strongly Fréchet space  $\implies$  Fréchet space.

**Proposition 3.4.3.** ([100]) A filter  $\mathcal{F}$  is productively Fréchet if and only if  $\mathcal{F} \times \mathcal{G}$  is a Fréchet filter (equivalently a strongly Fréchet filter) for every strongly Fréchet filter  $\mathcal{G}$ .

*Proof.* It follows from Theorem 9 in [100]. □

**Remark 3.4.4.** ([53]) (i) For every semi-filter  $\mathcal{F}$ , we have

$$H \notin \mathcal{F}^\# \iff H^c \in \mathcal{F}, \quad (3.12)$$

where  $H^c$  denotes the complement of  $H$ . In fact, by definition,  $H \notin \mathcal{F}^\#$  whenever there is  $F \in \mathcal{F}$  such that  $H \cap F = \emptyset$ , equivalently  $F \subseteq H^c$ , that is,  $H^c \in \mathcal{F}$  since  $\mathcal{F}$  is filter.

(ii) If  $\mathcal{F}$  is a filter on a set  $X$ ,  $\mathcal{G}$  is a filter on a set  $Y$ , and  $\mathcal{H}$  is a filter on  $X \times Y$ , we denote by  $\mathcal{H} \mathcal{F}$  the filter on  $Y$  generated by the sets

$$HF = \{y : \exists x \in F, (x, y) \in H\},$$

for  $H \in \mathcal{H}$  and  $F \in \mathcal{F}$ , and by  $\mathcal{H}^{-1} \mathcal{G}$  the filter on  $X$  generated by the sets

$$H^{-1}G = \{x : \exists y \in G, (x, y) \in H\},$$

for  $H \in \mathcal{H}$  and  $G \in \mathcal{G}$ . Notice that

$$\mathcal{H} \# (\mathcal{F} \times \mathcal{G}) \iff (\mathcal{H} \mathcal{F}) \# \mathcal{G} \iff \mathcal{F} \# (\mathcal{H}^{-1} \mathcal{G}).$$

We now recall some definitions introduced in [83] by Greco as follows

**Definition 3.4.5.** ([83]) Let  $\mathcal{N}$  be a sequential filter associated with a sequence  $\{n\}_n$ , and let  $\mathcal{A}_1, \dots, \mathcal{A}_k$  be filters.

$$(\mathcal{N}^{\alpha_0}, \mathcal{A}_1^{\alpha_1}, \dots, \mathcal{A}_k^{\alpha_k})_{\text{seq}} := \text{ext}_{\{x_n^1\}_n \in \text{Seq}(\mathcal{A}_1)}^{\alpha_1} \dots \text{ext}_{\{x_n^k\}_n \in \text{Seq}(\mathcal{A}_k)}^{\alpha_k} \text{ext}_{m \in \mathbb{N}}^{-\alpha_0} \text{ext}_{n \geq m}^{\alpha_0} \{(n, x_n^1, \dots, x_n^k)\}_n,$$

where  $\alpha_0, \alpha_1, \dots, \alpha_k$  are signs of  $+$ ,  $-$  and  $\text{ext}^- = \bigcap, \text{ext}^+ = \bigcup$ .

From Definition 3.4.5, the sequential form of the  $\Gamma$ -limit is defined as follows

**Definition 3.4.6.** ([83]) Let  $\mathcal{N}$  be the sequential filter associated with a sequence  $\{n\}_n$ ,  $\mathcal{A}_1, \dots, \mathcal{A}_k$  be filters in  $S_1, \dots, S_k$ , respectively, and  $f : \mathbb{N} \times S_1 \times \dots \times S_k \rightarrow \overline{\mathbb{R}}$ . The *sequential  $\Gamma$ -limit* is defined by

$$\Gamma_{\text{seq}}(\mathcal{N}^{\alpha_0}, \mathcal{A}_1^{\alpha_1}, \dots, \mathcal{A}_k^{\alpha_k}) \lim f := \text{ext}_{\{x_n^1\}_n \in \text{Seq}(\mathcal{A}_1)}^{\alpha_1} \dots \text{ext}_{\{x_n^k\}_n \in \text{Seq}(\mathcal{A}_k)}^{\alpha_k} \text{ext}_{m \in \mathbb{N}}^{-\alpha_0} \text{ext}_{n \geq m}^{\alpha_0} f(n, x_n^1, \dots, x_n^k),$$

where  $\text{ext}^- = \inf, \text{ext}^+ = \sup$ ,  $\alpha_0, \alpha_1, \dots, \alpha_k$  are signs of  $+$ ,  $-$ .

The  $\Gamma_{\text{seq}}$ -limit is a limitoid and its support is the family  $(\mathcal{N}^{\alpha_0}, \mathcal{A}_1^{\alpha_1}, \dots, \mathcal{A}_k^{\alpha_k})_{\text{seq}}$ .

### 3.4.1 Two variables

**Proposition 3.4.7.** *Suppose that  $\mathcal{F}$  is a Fréchet filter. Then*

$$\mathcal{F} = \bigcap_{\mathcal{E} \in \text{Seq}(\mathcal{F})} \mathcal{E} = \bigcap_{\mathcal{E} \in \text{Seq}(\mathcal{F})} \mathcal{E}^{\#}, \quad (3.13)$$

$$\mathcal{F}^{\#} = \bigcup_{\mathcal{E} \in \text{Seq}(\mathcal{F})} \mathcal{E} = \bigcup_{\mathcal{E} \in \text{Seq}(\mathcal{F})} \mathcal{E}^{\#}. \quad (3.14)$$

*Proof.* Since (3.13) implies (3.14) (by (2.1)), we prove only (3.13). Let  $F \in \mathcal{F}$  and an arbitrary  $\mathcal{E} \in \text{Seq}(\mathcal{F})$ . Since  $\mathcal{E} \geq \mathcal{F}$ , there exists  $E \in \mathcal{E}$  such that  $E \subseteq F$ , so  $F \in \mathcal{E}$ . This implies  $F \in \bigcap_{\mathcal{E} \in \text{Seq}(\mathcal{F})} \mathcal{E}$ . Thus,  $\mathcal{F} \subseteq \bigcap_{\mathcal{E} \in \text{Seq}(\mathcal{F})} \mathcal{E}$ . Since  $\mathcal{E} \subseteq \mathcal{E}^\#$ , we get

$$\mathcal{F} \subseteq \bigcap_{\mathcal{E} \in \text{Seq}(\mathcal{F})} \mathcal{E} \subseteq \bigcap_{\mathcal{E} \in \text{Seq}(\mathcal{F})} \mathcal{E}^\#.$$

We now prove  $\bigcap_{\mathcal{E} \in \text{Seq}(\mathcal{F})} \mathcal{E}^\# \subseteq \mathcal{F}$ . It follows from the definition of Fréchet filters that

$$\forall A \in \mathcal{F}^\# \implies \exists \mathcal{E} \in \text{Seq}(\mathcal{F}) : A \in \mathcal{E}. \quad (3.15)$$

Suppose that  $H \notin \mathcal{F}$ . This implies  $H^c \in \mathcal{F}^\#$ . From (3.15), there is  $\mathcal{E} \in \text{Seq}(\mathcal{F})$  such that  $H^c \in \mathcal{E}$ , i.e.,  $H \notin \mathcal{E}^\#$ . Thus,  $H \notin \bigcap_{\mathcal{E} \in \text{Seq}(\mathcal{F})} \mathcal{E}^\#$ .  $\square$

**Proposition 3.4.8.** *Let  $\mathcal{F}, \mathcal{G}$  be filters.*

(i) *Suppose that  $\mathcal{F}$  is Fréchet. Then*

$$(\mathcal{F}^+, \mathcal{G}^-) = \bigcup_{\mathcal{E} \in \text{Seq}(\mathcal{F})} (\mathcal{E}^-, \mathcal{G}^-) = \bigcup_{\mathcal{E} \in \text{Seq}(\mathcal{F})} (\mathcal{E}^+, \mathcal{G}^-), \quad (3.16)$$

$$(\mathcal{F}^-, \mathcal{G}^+) = \bigcap_{\mathcal{E} \in \text{Seq}(\mathcal{F})} (\mathcal{E}^+, \mathcal{G}^+) = \bigcap_{\mathcal{E} \in \text{Seq}(\mathcal{F})} (\mathcal{E}^-, \mathcal{G}^+). \quad (3.17)$$

(ii) *Suppose that  $\mathcal{F}$  is strongly Fréchet and  $\mathcal{G}$  is productively Fréchet (or vice versa). Then*

$$(\mathcal{F}^-, \mathcal{G}^-) = \bigcap_{\mathcal{E} \in \text{Seq}(\mathcal{F})} (\mathcal{E}^-, \mathcal{G}^-) = \bigcap_{\mathcal{E} \in \text{Seq}(\mathcal{F})} (\mathcal{E}^+, \mathcal{G}^-), \quad (3.18)$$

$$(\mathcal{F}^+, \mathcal{G}^+) = \bigcup_{\mathcal{E} \in \text{Seq}(\mathcal{F})} (\mathcal{E}^+, \mathcal{G}^+) = \bigcup_{\mathcal{E} \in \text{Seq}(\mathcal{F})} (\mathcal{E}^-, \mathcal{G}^+). \quad (3.19)$$

*Proof.* (i) Since (3.16) implies (3.17), we prove only (3.16). Because  $(\mathcal{E}^-, \mathcal{G}^-) \approx \mathcal{E} \times \mathcal{G}$ ,  $(\mathcal{E}^+, \mathcal{G}^-) \approx \mathcal{E}^\# \times \mathcal{G}$  and  $\mathcal{E} \subseteq \mathcal{E}^\#$ , we get

$$\bigcup_{\mathcal{E} \in \text{Seq}(\mathcal{F})} (\mathcal{E}^-, \mathcal{G}^-) \subseteq \bigcup_{\mathcal{E} \in \text{Seq}(\mathcal{F})} (\mathcal{E}^+, \mathcal{G}^-).$$

It follows from (3.14) that

$$(\mathcal{F}^+, \mathcal{G}^-) \approx \mathcal{F}^\# \times \mathcal{G} = \left( \bigcup_{\mathcal{E} \in \text{Seq}(\mathcal{F})} \mathcal{E} \right) \times \mathcal{G} \subseteq \bigcup_{\mathcal{E} \in \text{Seq}(\mathcal{F})} (\mathcal{E} \times \mathcal{G}).$$

Thus,

$$(\mathcal{F}^+, \mathcal{G}^-) \subseteq \bigcup_{\mathcal{E} \in \text{Seq}(\mathcal{F})} (\mathcal{E}^-, \mathcal{G}^-) \subseteq \bigcup_{\mathcal{E} \in \text{Seq}(\mathcal{F})} (\mathcal{E}^+, \mathcal{G}^-).$$

Let  $A \in \bigcup_{\mathcal{E} \in \text{Seq}(\mathcal{F})} (\mathcal{E}^+, \mathcal{G}^-)$ , then there exists  $\mathcal{E} \in \text{Seq}(\mathcal{F})$  such that  $A \in \mathcal{E}^\# \times \mathcal{G}$ . It means that there exist  $A_1 \in \mathcal{E}^\#$  and  $A_2 \in \mathcal{G}$  such that  $A \supseteq A_1 \times A_2$ . Suppose that  $A \notin (\mathcal{F}^+, \mathcal{G}^-) \approx \mathcal{F}^\# \times \mathcal{G}$ . It follows from  $A_2 \in \mathcal{G}$  that  $A_1 \notin \mathcal{F}^\#$ . By (3.12), we get  $A_1^c \in \mathcal{F}$ . Because  $\mathcal{E} \geq \mathcal{F}$ , we get  $A_1^c \in \mathcal{E}$ , which contradicts to  $A_1 \in \mathcal{E}^\#$ .

(ii) Similarly, we prove only (3.18). It follows from Proposition 3.4.3 that  $\mathcal{F} \times \mathcal{G}$  is a Fréchet filter. We can see that

$$\begin{aligned} (\mathcal{F}^-, \mathcal{G}^-) &\approx \mathcal{F} \times \mathcal{G} \\ &= \left( \bigcap_{\mathcal{E} \in \text{Seq}(\mathcal{F})} \mathcal{E} \right) \times \mathcal{G} \text{ (by (3.13))} \\ &\subseteq \bigcap_{\mathcal{E} \in \text{Seq}(\mathcal{F})} (\mathcal{E} \times \mathcal{G}) \\ &\subseteq \bigcap_{\mathcal{E} \in \text{Seq}(\mathcal{F})} (\mathcal{E}^\# \times \mathcal{G}) \\ &= \bigcap_{\mathcal{E} \in \text{Seq}(\mathcal{F})} (\mathcal{E}^\# \times \left( \bigcap_{\mathcal{B} \in \text{Seq}(\mathcal{G})} \mathcal{B} \right)) \\ &\subseteq \bigcap_{\mathcal{E} \in \text{Seq}(\mathcal{F})} \bigcap_{\mathcal{B} \in \text{Seq}(\mathcal{G})} (\mathcal{E}^\# \times \mathcal{B}) \\ &\subseteq \bigcap_{\mathcal{E} \in \text{Seq}(\mathcal{F})} \bigcap_{\mathcal{B} \in \text{Seq}(\mathcal{G})} (\mathcal{E} \times \mathcal{B})^\# \\ &\subseteq \bigcap_{\mathcal{D} \in \text{Seq}(\mathcal{F} \times \mathcal{G})} \mathcal{D}^\# \\ &= \mathcal{F} \times \mathcal{G} \text{ (by (3.13)).} \end{aligned}$$

□

**Remark 3.4.9.** In [100] Jordan and Mynard showed that if  $\mathcal{F}$  is not a strongly Fréchet filter (or not a productively Fréchet filter), then there exists a countably based filter  $\mathcal{G}$  (a strongly Fréchet filter  $\mathcal{G}$ , respectively) such that  $\mathcal{F} \times \mathcal{G}$  is not a Fréchet filter. On the other hand, by Proposition 3.4.3,  $\bigcap_{\mathcal{E} \in \text{Seq}(\mathcal{F})} (\mathcal{E}^-, \mathcal{G}^-)$  is a Fréchet filter. Thus

$$(\mathcal{F}^-, \mathcal{G}^-) \neq \bigcap_{\mathcal{E} \in \text{Seq}(\mathcal{F})} (\mathcal{E}^-, \mathcal{G}^-).$$

**Proposition 3.4.10.** Let  $\mathcal{E} \approx \{x_n\}_n$ .

(i) If  $\mathcal{G}$  is strongly Fréchet filter, then

$$(\mathcal{E}^-, \mathcal{G}^-) = \bigcap_{\{y_n\}_n \geq \mathcal{G}} \{(x_n, y_n)\}_n, \quad (\mathcal{E}^+, \mathcal{G}^+) = \bigcup_{\{y_n\}_n \geq \mathcal{G}} \{(x_n, y_n)\}_n^\#.$$

(ii) If  $\mathcal{G}$  is countably based filter, then

$$(\mathcal{E}^+, \mathcal{G}^-) = \bigcap_{\{y_n\}_n \geq \mathcal{G}} \{(x_n, y_n)\}_n^\#, \quad (\mathcal{E}^-, \mathcal{G}^+) = \bigcup_{\{y_n\}_n \geq \mathcal{G}} \{(x_n, y_n)\}_n.$$

*Proof.* (i) We prove only  $(\mathcal{E}^-, \mathcal{G}^-) = \bigcap_{\{y_n\}_n \in \text{Seq}(\mathcal{G})} \{(x_n, y_n)\}_n$  since this formula implies the other one by properties of grills. It follows from Proposition 3.4.3 that  $\mathcal{E} \times \mathcal{G}$  is Fréchet filter. We can see that

$$\begin{aligned} (\mathcal{E}^-, \mathcal{G}^-) &\approx \mathcal{E} \times \mathcal{G} \\ &= \mathcal{E} \times \left( \bigcap_{\mathcal{B} \in \text{Seq}(\mathcal{G})} \mathcal{B} \right) \text{ (by (3.13))} \\ &\subseteq \bigcap_{\mathcal{B} \in \text{Seq}(\mathcal{G})} (\mathcal{E} \times \mathcal{B}) \\ &= \bigcap_{\mathcal{B} \in \text{Seq}(\mathcal{G})} \left( \left( \bigcap_{\mathcal{D} \in \text{Seq}(\mathcal{E})} \mathcal{D} \right) \times \mathcal{B} \right) \text{ (by (3.13))} \\ &\subseteq \bigcap_{\mathcal{B} \in \text{Seq}(\mathcal{G})} \bigcap_{\mathcal{D} \in \text{Seq}(\mathcal{E})} (\mathcal{D} \times \mathcal{B}) \\ &\subseteq \bigcap_{\mathcal{L} \in \text{Seq}(\mathcal{E} \times \mathcal{G})} \mathcal{L} \\ &= \mathcal{E} \times \mathcal{G} \text{ (by (3.13)).} \end{aligned}$$

Because  $\mathcal{E} \approx \{x_n\}_n$ , we get

$$(\mathcal{E}^-, \mathcal{G}^-) = \bigcap_{\{y_n\}_n \geq \mathcal{G}} \{(x_n, y_n)\}_n.$$

(ii) We prove only  $(\mathcal{E}^-, \mathcal{G}^+) = \bigcup_{\{y_n\}_n \geq \mathcal{G}} \{(x_n, y_n)\}_n$ . Let  $\mathcal{G} \approx \{G_n\}_n$ , where  $\{G_n\}_n$  is a decreasing sequence of subsets, and  $A \in (\mathcal{E}^-, \mathcal{G}^+) \approx (\mathcal{E}^\# \times \mathcal{G})^\#$ . In other words, for each  $n$  and  $H \in \mathcal{E}^\#$ ,

$$(H \times G_n) \cap A \neq \emptyset \iff AG_n \cap H \neq \emptyset, \quad (3.20)$$

that is,  $AG_n \in \mathcal{E}^{\#\#} = \mathcal{E}$  for each  $n$ . This means that for every  $n$  there is  $k_n$  such that  $\{x_k : k \geq k_n\} \subseteq AG_n$ , hence there exists  $\{y_k^n : k \geq k_n\} \subseteq G_n$  with

$$\{(x_k, y_k^n) : k \geq k_n\} \subseteq A.$$

Using induction, we can get a strictly increasing sequence  $\{k_n\}_n$  with this property. Let

$$y_k := y_k^n \text{ if } k_n \leq k < k_{n+1}.$$

Then  $\{y_k\}_k \geq \mathcal{G}$  and  $\{(x_k, y_k)\} \subseteq A$  for each  $k \geq k_1$ .

Conversely, let  $A \in \bigcup_{\{y_n\}_{n \geq \mathcal{G}}} \{(x_n, y_n)\}_n$ , then there is  $D \approx \{y_k\}_k \geq \mathcal{G}$  such that  $(x_k, y_k) \subseteq A$ . We can check that (3.20) holds, then  $A \in (\mathcal{E}^-, \mathcal{G}^+)$ . □

**Theorem 3.4.11.** *Let  $f : \mathbb{N} \times X \rightarrow \overline{\mathbb{R}}$ .*

(i) *If  $\mathcal{G}$  is a strongly Fréchet filter on  $X$ , then*

$$\Gamma(\mathcal{N}^-, \mathcal{G}^-) = \Gamma_{\text{seq}}(\mathcal{N}^-, \mathcal{G}^-), \quad \Gamma(\mathcal{N}^+, \mathcal{G}^+) = \Gamma_{\text{seq}}(\mathcal{N}^+, \mathcal{G}^+).$$

(ii) *If  $\mathcal{G}$  is a countably based filter on  $X$ , then*

$$\Gamma(\mathcal{N}^-, \mathcal{G}^+) = \Gamma_{\text{seq}}(\mathcal{N}^-, \mathcal{G}^+), \quad \Gamma(\mathcal{N}^+, \mathcal{G}^-) = \Gamma_{\text{seq}}(\mathcal{N}^+, \mathcal{G}^-).$$

*Proof.* We check only for  $\Gamma(\mathcal{N}^+, \mathcal{G}^+) = \Gamma_{\text{seq}}(\mathcal{N}^+, \mathcal{G}^+)$ . The other cases are analogous. Let  $\mathcal{E} = \mathcal{N}$ , from (3.4.10), we get

$$(\mathcal{N}^+, \mathcal{G}^+) = \bigcup_{\{y_n\}_{n \in \text{Seq}(\mathcal{G})}} \{(n, y_n)\}_n^\#.$$

It follows from (3.6) and (3.9) that for all  $f : \mathbb{N} \times X \rightarrow \overline{\mathbb{R}}$ ,

$$\begin{aligned} \Gamma(\mathcal{N}^+, \mathcal{G}^+) \lim f &= \liminf_{(\mathcal{N}^+, \mathcal{G}^+)} f \\ &= \sup_{\{y_n\}_{n \in \text{Seq}(\mathcal{G})}} \liminf_{\{(n, y_n)\}_n^\#} f \\ &= \sup_{\{y_n\}_{n \in \text{Seq}(\mathcal{G})}} \limsup_{n \rightarrow +\infty} f(n, x_n) \\ &= \Gamma_{\text{seq}}(\mathcal{N}^+, \mathcal{G}^+) \lim f. \end{aligned}$$

□

The following example shows that the strongly Fréchetness of  $\mathcal{G}$  in Theorem 3.4.11(i) is necessary.

**Example 3.4.12.** Let a sequential fan  $S_{\mathbb{N}} := \{\bigcup_{n \in \mathbb{N}} X_n\} \cup \{x_\infty\}$ , where  $X_n := \{x_{n,k} : k \in \mathbb{N}\}$  and  $X_n \cap X_m = \emptyset$  for all  $n \neq m$ , be equipped with a topology defined as follows

- each point  $x_{n,k}$  is isolated;
- a basic open neighborhood of  $x_\infty$  in the form

$$O_f(x_\infty) := \{x_\infty\} \cup \{x_{n,k} : k \geq f(n)\},$$



for each function  $f \in \mathbb{N}^{\mathbb{N}}$ , where  $\mathbb{N}^{\mathbb{N}} := \{f : \mathbb{N} \rightarrow \mathbb{N}\}$ .

First, we prove that  $S_{\mathbb{N}}$  is not strongly Fréchet, i.e., there exists  $\{A_n\}$  be a decreasing sequence of subsets in  $S_{\mathbb{N}}$  such that  $x_{\infty} \in \bigcap_n \text{cl}(A_n)$ , but there is no  $x_n \in A_n$  with  $x_{\infty} = \lim_{n \rightarrow \infty} x_n$ .

Setting  $A_n := \{X_m : m \geq n\}$ . It is easy to check that  $A_{n+1} \subseteq A_n$  and  $x_{\infty} \in \bigcap_n \text{cl}(A_n)$ . For each  $n$ , choose any finite set  $F_n$  of  $A_n$  and denote  $F := \bigcup_n F_n$ . We define a function  $h \in \mathbb{N}^{\mathbb{N}}$  by

$$h(n) := \begin{cases} 1 + \max\{k : x_{n,k} \in F\}, & \text{if } F \cap X_n \neq \emptyset, \\ 1, & \text{otherwise.} \end{cases}$$

Then,  $O_h(x_{\infty}) \cap F = \emptyset$ , i.e.,  $x_{\infty} \notin \text{cl}F$ . Since  $F_n$  is arbitrary for all  $n$ , there is no  $x_n \in A_n$  with  $x_{\infty} = \lim_{n \rightarrow +\infty} x_n$ .

Let  $\mathbf{g} : \mathbb{N} \times S_{\mathbb{N}} \rightarrow [0, 1]$  be defined by

$$\mathbf{g}(n, x) = \begin{cases} \frac{1}{\sqrt[n]{m}}, & \text{if } x \in X_m, \\ 1, & \text{otherwise.} \end{cases}$$

By calculating, we get

$$\Gamma(\mathcal{N}^-, \mathcal{N}_{\tau}(x_{\infty})^-) \lim \mathbf{g} = 0.$$

Let  $\{x^m\}$  be a sequence converging to  $x_{\infty}$ . If  $x^m = x_{\infty}$  for infinitely many  $m \in \mathbb{N}$ , we get  $\lim_{m \rightarrow +\infty} \mathbf{g}(m, x^m) = 1$ . We now consider  $x^m \neq x_{\infty}$  for all  $m \in \mathbb{N}$ . Then, there exists  $n_0 \in \mathbb{N}$  such that  $\{x^m\} \subseteq \bigcup_{n=1}^{n_0} X_n$ . Indeed, if not, for all  $n \in \mathbb{N}$ , there exists  $m_n$  such that  $x^{m_n} \notin \bigcup_{l=1}^n X_l$ . This implies that  $x^{m_n} \in A_{n+1}$ , where  $\{A_n\}$  is the decreasing sequence of subsets defined above. By the previous analysis,  $\{x^{m_n}\}_n$  does not converge to  $x_{\infty}$ , which is a contradiction. Thus, each sequence  $\{x^m\}$  ( $\neq \{x_{\infty}\}$ ) converging to  $x_{\infty}$  is in type of  $\{x_{p,k} : \exists n_0 \in \mathbb{N}, p \in [1, n_0], k \rightarrow +\infty\}$ . By calculating, we get

$$\liminf_{m \rightarrow +\infty} \mathbf{g}(m, x^m) = \liminf_{k \rightarrow +\infty} \mathbf{g}(k, x_{p,k}) = 1.$$

This implies that

$$\Gamma_{\text{seq}}(\mathcal{N}^-, \mathcal{N}_{\tau}(x_{\infty})^-) \lim \mathbf{g} = 1.$$

Thus

$$\Gamma(\mathcal{N}^-, \mathcal{N}_{\tau}(x_{\infty})^-) \lim \mathbf{g} < \Gamma_{\text{seq}}(\mathcal{N}^-, \mathcal{N}_{\tau}(x_{\infty})^-) \lim \mathbf{g}.$$

### 3.4.2 Three variables

**Proposition 3.4.13.** *Let  $\mathcal{F}, \mathcal{G}$  be filters.*

(i) If  $\mathcal{F}$  is strongly Fréchet and  $\mathcal{G}$  is productively Fréchet (or vice versa), then

$$(\mathcal{F}^-, \mathcal{G}^-) = \bigcap_{\{x_n\}_n \geq \mathcal{F}} \bigcap_{\{y_n\}_n \geq \mathcal{G}} \{(x_n, y_n)\}_n = \bigcap_{\{x_n\}_n \geq \mathcal{F}} \bigcap_{\{y_n\}_n \geq \mathcal{G}} \{(x_n, y_n)\}_n^\#$$

(ii) If  $\mathcal{F}$  is Fréchet and  $\mathcal{G}$  is countably based, then

$$(\mathcal{F}^+, \mathcal{G}^-) = \bigcup_{\{x_n\}_n \geq \mathcal{F}} \bigcap_{\{y_n\}_n \geq \mathcal{G}} \{(x_n, y_n)\}_n = \bigcup_{\{x_n\}_n \geq \mathcal{F}} \bigcap_{\{y_n\}_n \geq \mathcal{G}} \{(x_n, y_n)\}_n^\#$$

*Proof.* (i) It follows from Proposition 3.4.3 that  $\mathcal{F} \times \mathcal{G}$  is a Fréchet filter. We can see that

$$\begin{aligned} (\mathcal{F}^-, \mathcal{G}^-) &= \bigcap_{\{x_n\}_n \geq \mathcal{F}} (\{x_n\}_n^-, \mathcal{G}^-) \text{ (by Proposition 3.4.8)} \\ &= \bigcap_{\{x_n\}_n \geq \mathcal{F}} \bigcap_{\{y_n\}_n \geq \mathcal{G}} \{(x_n, y_n)\}_n \text{ (by Proposition 3.4.10)} \\ &\subseteq \bigcap_{\{x_n\}_n \geq \mathcal{F}} \bigcap_{\{y_n\}_n \geq \mathcal{G}} \{(x_n, y_n)\}_n^\# \\ &\subseteq \bigcap_{\mathcal{L} \in \text{Seq}(\mathcal{F} \times \mathcal{G})} \mathcal{L}^\# \\ &= \mathcal{F} \times \mathcal{G} \text{ (by (3.13)).} \end{aligned}$$

(ii) We can see that

$$\begin{aligned} (\mathcal{F}^+, \mathcal{G}^-) &= \bigcup_{\{x_n\}_n \geq \mathcal{F}} (\{x_n\}_n^-, \mathcal{G}^-) \text{ (by Proposition 3.4.8)} \\ &= \bigcup_{\{x_n\}_n \geq \mathcal{F}} \bigcap_{\{y_n\}_n \geq \mathcal{G}} \{(x_n, y_n)\}_n \text{ (by Proposition 3.4.10).} \end{aligned}$$

Besides,

$$\begin{aligned} (\mathcal{F}^+, \mathcal{G}^-) &= \bigcup_{\{x_n\}_n \geq \mathcal{F}} (\{x_n\}_n^+, \mathcal{G}^-) \text{ (by Proposition 3.4.8)} \\ &= \bigcup_{\{x_n\}_n \geq \mathcal{F}} \bigcap_{\{y_n\}_n \geq \mathcal{G}} \{(x_n, y_n)\}_n^\# \text{ (by Proposition 3.4.10).} \end{aligned}$$

□

**Proposition 3.4.14.** Let  $\mathcal{E} \approx \{x_n\}_n$ , and  $\mathcal{G}, \mathcal{H}$  are filters.

(i) Suppose that  $\mathcal{G}$  is strongly Fréchet and  $\mathcal{H}$  is productively Fréchet (or vice versa). Then

$$(\mathcal{E}^-, \mathcal{G}^-, \mathcal{H}^-) = \bigcap_{\{y_n\}_n \geq \mathcal{G}} \bigcap_{\{z_n\}_n \geq \mathcal{H}} \{(x_n, y_n, z_n)\}_n$$

(ii) Suppose that  $\mathcal{G}$  is countably based and  $\mathcal{H}$  is strongly Fréchet. Then

$$(\mathcal{E}^-, \mathcal{G}^+, \mathcal{H}^-) = \bigcup_{\{y_n\}_n \geq \mathcal{G}} \bigcap_{\{z_n\}_n \geq \mathcal{H}} \{(x_n, y_n, z_n)\}_n$$

(iii) Suppose that  $\mathcal{G}, \mathcal{H}$  are countably based. Then

$$(\mathcal{E}^+, \mathcal{G}^-, \mathcal{H}^-) = \bigcap_{\{y_n\}_n \geq \mathcal{G}} \bigcap_{\{z_n\}_n \geq \mathcal{H}} \{(x_n, y_n, z_n)\}_n^\#$$

(iv) Suppose that  $\mathcal{G}$  is strongly Fréchet and  $\mathcal{H}$  is countably based. Then

$$(\mathcal{E}^+, \mathcal{G}^+, \mathcal{H}^-) = \bigcup_{\{y_n\}_n \geq \mathcal{G}} \bigcap_{\{z_n\}_n \geq \mathcal{H}} \{(x_n, y_n, z_n)\}_n^\#$$

*Proof.* We only prove (iii). The other ones are analogous. It follows from Proposition 3.4.3 that  $\mathcal{E} \times \mathcal{G} \times \mathcal{H}$  is a Fréchet filter. We have

$$\begin{aligned} (\mathcal{E}^+, \mathcal{G}^-, \mathcal{H}^-) &\approx \mathcal{E}^\# \times \mathcal{G} \times \mathcal{H} \\ &= \left( \bigcap_{\{y_n\}_n \geq \mathcal{G}} \{(x_n, y_n)\}_n^\# \right) \times \mathcal{H} \text{ (by Proposition 3.4.10)} \\ &\subseteq \bigcap_{\{y_n\}_n \geq \mathcal{G}} (\{(x_n, y_n)\}_n^\# \times \mathcal{H}) \\ &= \bigcap_{\{y_n\}_n \geq \mathcal{G}} \bigcap_{\{z_n\}_n \geq \mathcal{H}} \{(x_n, y_n, z_n)\}_n^\# \text{ (by Proposition 3.4.10)} \\ &\subseteq \bigcap_{\mathcal{B} \in \text{Seq}(\mathcal{E} \times \mathcal{G} \times \mathcal{H})} \mathcal{B}^\# \\ &= \mathcal{E} \times \mathcal{G} \times \mathcal{H} \text{ (by (3.13))} \\ &\subseteq \mathcal{E}^\# \times \mathcal{G} \times \mathcal{H}. \end{aligned}$$

□

**Corollary 3.4.15.** Let  $\mathcal{E} \approx \{x_n\}_n$ , and  $\mathcal{G}, \mathcal{H}$  are filters.

(i) Suppose that  $\mathcal{G}$  is strongly Fréchet and  $\mathcal{H}$  is productively Fréchet (or vice versa). Then

$$(\mathcal{E}^+, \mathcal{G}^+, \mathcal{H}^+) = \bigcup_{\{y_n\}_n \geq \mathcal{G}} \bigcup_{\{z_n\}_n \geq \mathcal{H}} \{(x_n, y_n, z_n)\}_n^\#$$

(ii) Suppose that  $\mathcal{G}$  is countably based and  $\mathcal{H}$  is strongly Fréchet. Then

$$(\mathcal{E}^+, \mathcal{G}^-, \mathcal{H}^+) = \bigcap_{\{y_n\}_n \geq \mathcal{G}} \bigcup_{\{z_n\}_n \geq \mathcal{H}} \{(x_n, y_n, z_n)\}_n^\#$$

(iii) Suppose that  $\mathcal{G}, \mathcal{H}$  are countably based. Then

$$(\mathcal{E}^-, \mathcal{G}^+, \mathcal{H}^+) = \bigcup_{\{y_n\}_n \geq \mathcal{G}} \bigcup_{\{z_n\}_n \geq \mathcal{H}} \{(x_n, y_n, z_n)\}_n$$

(iv) Suppose that  $\mathcal{G}$  is strongly Fréchet and  $\mathcal{H}$  is countably based. Then

$$(\mathcal{E}^-, \mathcal{G}^-, \mathcal{H}^+) = \bigcap_{\{y_n\}_n \geq \mathcal{G}} \bigcup_{\{z_n\}_n \geq \mathcal{H}} \{(x_n, y_n, z_n)\}_n$$

*Proof.* It follows from Proposition 3.4.14. □

**Theorem 3.4.16.** *Let  $\mathcal{G}, \mathcal{H}$  be filters on  $X, Y$ , respectively, and  $f : \mathbb{N} \times X \times Y \rightarrow \overline{\mathbb{R}}$ .*

(i) *Suppose that  $\mathcal{G}$  is strongly Fréchet and  $\mathcal{H}$  is productively Fréchet (or vice versa). Then*

$$\Gamma(\mathcal{N}^-, \mathcal{G}^-, \mathcal{H}^-) = \Gamma_{\text{seq}}(\mathcal{N}^-, \mathcal{G}^-, \mathcal{H}^-),$$

$$\Gamma(\mathcal{N}^+, \mathcal{G}^+, \mathcal{H}^+) = \Gamma_{\text{seq}}(\mathcal{N}^+, \mathcal{G}^+, \mathcal{H}^+).$$

(ii) *Suppose that  $\mathcal{G}$  is countably based and  $\mathcal{H}$  is strongly Fréchet. Then*

$$\Gamma(\mathcal{N}^-, \mathcal{G}^+, \mathcal{H}^-) = \Gamma_{\text{seq}}(\mathcal{N}^-, \mathcal{G}^+, \mathcal{H}^-),$$

$$\Gamma(\mathcal{N}^+, \mathcal{G}^-, \mathcal{H}^+) = \Gamma_{\text{seq}}(\mathcal{N}^+, \mathcal{G}^-, \mathcal{H}^+).$$

(iii) *Suppose that  $\mathcal{G}, \mathcal{H}$  are countably based. Then*

$$\Gamma(\mathcal{N}^+, \mathcal{G}^-, \mathcal{H}^-) = \Gamma_{\text{seq}}(\mathcal{N}^+, \mathcal{G}^-, \mathcal{H}^-),$$

$$\Gamma(\mathcal{N}^-, \mathcal{G}^+, \mathcal{H}^+) = \Gamma_{\text{seq}}(\mathcal{N}^-, \mathcal{G}^+, \mathcal{H}^+).$$

(iv) *Suppose that  $\mathcal{G}$  is strongly Fréchet and  $\mathcal{H}$  is countably based. Then*

$$\Gamma(\mathcal{N}^+, \mathcal{G}^+, \mathcal{H}^-) = \Gamma_{\text{seq}}(\mathcal{N}^+, \mathcal{G}^+, \mathcal{H}^-).$$

$$\Gamma(\mathcal{N}^-, \mathcal{G}^-, \mathcal{H}^+) = \Gamma_{\text{seq}}(\mathcal{N}^-, \mathcal{G}^-, \mathcal{H}^+).$$

*Proof.* Based on Proposition 3.4.14 and Corollary 3.4.15, the proof is similar to that of Theorem 3.4.11. □

### 3.4.3 More than three variables

Let  $T$  and  $T_i$ ,  $i \in I$ , be  $\overline{\mathbb{R}}$ -limitoids in  $S$ , and  $\mathbf{st}(T) = \bigcap_{i \in I} \mathbf{st}(T_i)$ . It follows from (3.6) and (3.8) that for all  $f : S \rightarrow \overline{\mathbb{R}}$ ,

$$T(f) = \inf_{i \in I} T_i(f). \tag{3.21}$$

The following lemma give us a condition for which “inf” in (3.21) can be attained.

**Lemma 3.4.17.** *The following properties are equivalent*

(i) *for each  $f : S \rightarrow \overline{\mathbb{R}}$ ,*

$$T(f) = \min_{i \in I} T_i(f),$$

(ii) *for each countably based filter  $\mathcal{F}$ ,*

$$(\mathbf{st}(T)^-, \mathcal{F}^-) = \bigcap_{i \in I} (\mathbf{st}(T_i)^-, \mathcal{F}^-).$$

*Proof.* Suppose that (ii) holds. By setting  $\mathcal{F} := \{S\}$ , we get  $\mathbf{st}(T) = \bigcap_{i \in I} \mathbf{st}(T_i)$ . This implies

$$T(f) = \inf_{i \in I} T_i(f).$$

Let  $\mathcal{H} \approx \{\{r : r < r_n\} : n \in \mathbb{N}\}$ , where  $\{r_n\}_n$  is a strictly decreasing sequence converging to  $T(f)$  such that  $T(f) < r_n$  for all  $n$ . It follows from (3.6) that

$$T(f) = \liminf_{\mathbf{st}(T)} f = \sup_{A \in \mathbf{st}(T)} \inf_{x \in A} f < r_n,$$

that is,  $\{x \in S : f(x) < r_n\} \in \mathbf{st}(T)^\#$  for all  $n$ , that is,  $(f^{-1}(\mathcal{H}))^\#(\mathbf{st}(T))$  or else  $\text{gr}(f) \in (\mathbf{st}(T) \times \mathcal{H})^\#$ . By (ii), there is  $i \in I$  such that  $\text{gr}(f) \in (\mathbf{st}(T_i) \times \mathcal{H})^\#$ , equivalently for all  $n$ ,  $T_i(f) = \liminf_{\mathbf{st}(T_i)} f < r_n$ . Then,

$$T_i(f) \leq \lim_{n \rightarrow +\infty} r_n = T(f),$$

which implies (i).

Conversely, suppose that (i) holds and  $\mathcal{F} \approx \{F_n : n \in \mathbb{N}\}$ , where  $\{F_n\}_n$  is a decreasing sequence of subsets. We prove that

$$(\mathbf{st}(T)^-, \mathcal{F}^-)^\# = \bigcup_{i \in I} (\mathbf{st}(T_i)^-, \mathcal{F}^-)^\#.$$

Let  $H \in (\mathbf{st}(T)^-, \mathcal{F}^-)^\# \approx (\mathbf{st}(T) \times \mathcal{F})^\#$ . By Remark 3.4.4(ii),  $H \in (\mathbf{st}(T) \times \mathcal{F})^\#$  if and only if  $\mathbf{st}(T)^\#(H^{-1}\mathcal{F})$ , that is,  $H^{-1}F_n \in (\mathbf{st}(T))^\#$  for all  $n$ . Let

$$f_H(x) := \begin{cases} \inf \{\frac{1}{n} : x \in H^{-1}F_n\}, & \text{if } \{\frac{1}{n} : x \in H^{-1}F_n\} \neq \emptyset, \\ 1, & \text{otherwise.} \end{cases}$$

Then if  $n > \frac{1}{r}$ , that is,  $r > \frac{1}{n}$ , and  $x \in H^{-1}F_n$ , then  $f_H(x) < r$ , so that  $H^{-1}F_n \subseteq \{x : f_H(x) < r\}$ .

It follows from

$$T(f_H) = \liminf_{\mathbf{st}(T)} f = \sup_{A \in \mathbf{st}(T)} \inf_{x \in A} f = \inf_{B \in (\mathbf{st}(T))^\#} \sup_{x \in B} f$$

that  $T(f_H) < r$  for each  $r > 0$ . By (i), there exists  $i \in I$  such that  $T_i(f_H) = 0$ , i.e., for each  $r > 0$ ,

$$T_i(f_H) = \liminf_{\mathbf{st}(T_i)} f_H = \sup_{A \in \mathbf{st}(T_i)} \inf_{x \in A} f_H < r,$$

equivalently,  $\{x : f_H(x) < r\} \in \mathbf{st}(T_i)^\#$  for all  $r > 0$ . Therefore, if  $\frac{1}{n} < r < \frac{1}{n-1}$ , then

$$H^{-1}F_n \subseteq \{x : f_H(x) < r\} \subseteq H^{-1}F_{n-1},$$

so that  $H^{-1}F_{n-1} \in \mathbf{st}(T_i)^\#$  for each  $n > 1$ , that is  $H \in (\mathbf{st}(T_i) \times \mathcal{F})^\#$ .

□

**Proposition 3.4.18.** *Suppose that  $\mathcal{F}$  is a strongly Fréchet filter,  $\mathcal{G}, \mathcal{H}$  are countably based filters. Then*

$$(\mathcal{F}^-, \mathcal{G}^+, \mathcal{H}^-) = \bigcap_{\mathcal{E} \in \text{Seq}(\mathcal{F})} (\mathcal{E}^-, \mathcal{G}^+, \mathcal{H}^-). \quad (3.22)$$

However, there exist countably based filters  $\mathcal{F}, \mathcal{G}, \mathcal{H}$  such that

$$(\mathcal{F}^-, \mathcal{G}^+, \mathcal{H}^-) \neq \bigcap_{\mathcal{E} \in \text{Seq}(\mathcal{F})} (\mathcal{E}^+, \mathcal{G}^+, \mathcal{H}^-).$$

*Proof.* It follows from Lemma 3.4.17 that (3.22) will be proved if for each extended-real-valued function  $f$ ,

$$\Gamma(\mathcal{F}^-, \mathcal{G}^+) \lim f = \min_{\mathcal{E} \in \text{Seq}(\mathcal{F})} \Gamma(\mathcal{E}^-, \mathcal{G}^+) \lim f. \quad (3.23)$$

It follows from (3.17) that  $(\mathcal{F}^-, \mathcal{G}^+) = \bigcap_{\mathcal{E} \in \text{Seq}(\mathcal{F})} (\mathcal{E}^-, \mathcal{G}^+)$ . This implies

$$\Gamma(\mathcal{F}^-, \mathcal{G}^+) \lim f = \inf_{\mathcal{E} \in \text{Seq}(\mathcal{F})} \Gamma(\mathcal{E}^-, \mathcal{G}^+) \lim f.$$

Setting  $b := \inf_{\mathcal{E} \in \text{Seq}(\mathcal{F})} \Gamma(\mathcal{E}^-, \mathcal{G}^+) \lim f$ . Let  $\{\mathcal{E}_n\}_n$  be a sequence of sequential filters such that  $\mathcal{E}_n \geq \mathcal{F}$  for all  $n$  and  $b = \inf_n \Gamma(\mathcal{E}_n^-, \mathcal{G}^+) \lim f$ . Let  $\{r_n\}_n$  be a strictly decreasing sequence converging to  $b$  and  $\Gamma(\mathcal{E}_n^-, \mathcal{G}^+) \lim f < r_n$  for all  $n$ . By the definition of  $\Gamma$ -limits, for all  $n$ ,

$$\inf_{G \in \mathcal{G}} \Gamma(\mathcal{E}_n^-) \sup_{y \in G} f(x, y) < r_n.$$

It means that for every  $n$ , there exists  $G_n \in \mathcal{G}$  such that

$$\Gamma(\mathcal{E}_n^-) \sup_{y \in G_n} f(x, y) < r_n,$$

that is,

$$\left\{ x : \sup_{y \in G_n} f(x, y) < r_n \right\} \in \mathcal{E}_n^\#.$$

Hence

$$\left\{ x : \sup_{y \in G_n} f(x, y) < r_n \right\}_n \subseteq \mathcal{F}^\#.$$

Since  $\mathcal{F}$  is a strongly Fréchet filter, there is a sequential  $\mathcal{E}_0 \approx \{x_n\}_n \geq \mathcal{F}$  such that

$$\sup_{y \in G_n} f(x_n, y) < r_n.$$

This implies

$$\inf_{G_n \in \mathcal{G}} \Gamma(\mathcal{E}_0^-, \mathcal{G}^+) \sup_{y \in G_n} f(x, y) < r_n.$$

Thus,  $\Gamma(\mathcal{E}_0^-, \mathcal{G}^+) \lim f \leq b = \bigcap_{\mathcal{E} \in \text{Seq}(\mathcal{F})} \Gamma(\mathcal{E}^-, \mathcal{G}^+) \lim f$ , which implies (3.23).

For the second part of (iii), we consider the following example. □

**Example 3.4.19.** ([83]) Let  $\mathcal{F} = \mathcal{G} = \mathcal{H} = \mathcal{N}(0)$ , where  $\mathcal{N}(0)$  is a filter of neighborhoods of 0 on  $S = [0, 1]$ , and  $g : [0, 1] \times [0, 1] \rightarrow \overline{\mathbb{R}}$  be defined by

$$g(x, y) := \begin{cases} 2^{-m}, & \text{if } \exists n, m \in \mathbb{N} \text{ such that } x = 2^{-n}(1 - 2^{-m}), 0 \leq y \leq 2^{-m}, \\ 1, & \text{otherwise.} \end{cases}$$

Setting  $\mathcal{E}_m := \{2^{-n}(1 - 2^{-m})\}_n$ . It is evident that  $\Gamma(\mathcal{E}_m^+, \mathcal{N}(0)^+) \lim g = 2^{-m}$ . By virtue of (3.8) and Proposition 3.4.18(ii), we get

$$0 = \inf_{\mathcal{E} \geq \mathcal{N}(0)} \Gamma(\mathcal{E}^+, \mathcal{N}(0)^+) \lim g = \Gamma(\mathcal{N}(0)^-, \mathcal{N}(0)^+) \lim g.$$

Recall that  $\text{hypo}(g) \in (\mathcal{N}(0)^-, \mathcal{N}(0)^+, \mathcal{N}(0)^-)$  if and only if  $\Gamma(\mathcal{N}(0)^-, \mathcal{N}(0)^+) \lim g > 0$ , see Lemma 3.4 in [83]. This implies  $\text{hypo}(g) \notin (\mathcal{N}(0)^-, \mathcal{N}(0)^+, \mathcal{N}(0)^-)$ . Let  $\mathcal{E}$  be an sequential filter associated with a sequence  $\{x_n\} \subseteq [0, 1]$  converging to 0. If there exists  $m$  such that  $\{x_n\}_n \cap \{2^{-n}(1 - 2^{-m})\}_n$  is infinite, we have  $\Gamma(\mathcal{E}^+, \mathcal{N}(0)^+) \lim g \geq 2^{-m}$ ; otherwise,  $\Gamma(\mathcal{E}^+, \mathcal{N}(0)^+) \lim g = 1$ . Then,  $\text{hypo}(g) \in (\mathcal{E}^+, \mathcal{N}(0)^+, \mathcal{N}(0)^-)$  for each sequential filter  $\mathcal{E}$  finer than  $\mathcal{N}(0)$ . In conclusion, we demonstrate that  $\text{hypo}(g) \notin (\mathcal{N}(0)^-, \mathcal{N}(0)^+, \mathcal{N}(0)^-)$  and  $\text{hypo}(g) \in \bigcap_{\mathcal{E} \geq \mathcal{N}(0)} (\mathcal{E}^+, \mathcal{N}(0)^+, \mathcal{N}(0)^-)$ .

**Proposition 3.4.20.** *Suppose that  $\mathcal{F}$  is a strongly Fréchet filter, and  $\mathcal{G}, \mathcal{H}$  are countably based filters. Then*

$$(\mathcal{F}^-, \mathcal{G}^+, \mathcal{H}^-) = \bigcap_{\{u_n\}_n \geq \mathcal{F}} \bigcup_{\{y_n\}_n \geq \mathcal{G}} \bigcap_{\{z_n\}_n \geq \mathcal{H}} \{(u_n, y_n, z_n)\}_n.$$

However, there exist countably based filters  $\mathcal{F}, \mathcal{G}, \mathcal{H}$  such that

$$(\mathcal{F}^-, \mathcal{G}^+, \mathcal{H}^-) \neq \bigcap_{\{u_n\}_n \geq \mathcal{F}} \bigcup_{\{y_n\}_n \geq \mathcal{G}} \bigcap_{\{z_n\}_n \geq \mathcal{H}} \{(u_n, y_n, z_n)\}_n^\#.$$

*Proof.* It follows from Propositions 3.4.14(ii) and 3.4.18. □

**Proposition 3.4.21.** *Let  $\mathcal{E} \approx \{x_n\}_n$ ,  $\mathcal{F}$  be a strongly Fréchet filter,  $\mathcal{G}, \mathcal{H}$  be countably based filters. Then*

$$(\mathcal{E}^-, \mathcal{F}^-, \mathcal{G}^+, \mathcal{H}^-) = \bigcap_{\{u_n\}_n \geq \mathcal{F}} \bigcup_{\{y_n\}_n \geq \mathcal{G}} \bigcap_{\{z_n\}_n \geq \mathcal{H}} \{(x_n, u_n, y_n, z_n)\}_n.$$

However, there exist countably based filters  $\mathcal{F}, \mathcal{G}, \mathcal{H}$  such that

$$(\mathcal{E}^+, \mathcal{F}^-, \mathcal{G}^+, \mathcal{H}^-) \neq \bigcap_{\{u_n\}_n \geq \mathcal{F}} \bigcup_{\{y_n\}_n \geq \mathcal{G}} \bigcap_{\{z_n\}_n \geq \mathcal{H}} \{(x_n, u_n, y_n, z_n)\}_n^\#.$$

*Proof.* It is implied from Propositions 3.4.10 and 3.4.20. □

The results above define the possibility to express filters. They are mainstay for De Giorgi's  $\Gamma$ -limits, for example, if  $\mathcal{F}, \mathcal{G}, \mathcal{H}, \mathcal{C}$  are countably based filters then the following extensions of previous results are not true in general

$$(\mathcal{C}^+, \mathcal{F}^-, \mathcal{G}^+, \mathcal{H}^-) = \bigcup_{\{w_n\}_n \geq \mathcal{C}} \bigcap_{\{u_n\}_n \geq \mathcal{F}} \bigcup_{\{y_n\}_n \geq \mathcal{G}} \bigcap_{\{z_n\}_n \geq \mathcal{H}} \{(w_n, u_n, y_n, z_n)\}_n$$

and

$$(\mathcal{C}^+, \mathcal{F}^-, \mathcal{G}^+, \mathcal{H}^-) = \bigcup_{\{w_n\}_n \geq \mathcal{C}} \bigcap_{\{u_n\}_n \geq \mathcal{F}} \bigcup_{\{y_n\}_n \geq \mathcal{G}} \bigcap_{\{z_n\}_n \geq \mathcal{H}} \{(w_n, u_n, y_n, z_n)\}_n^\#.$$

**Theorem 3.4.22.** *Let  $\mathcal{F}$  be a strongly Fréchet filter on  $X$ , and  $\mathcal{G}, \mathcal{H}$  be countably based filters on  $Y, Z$ , respectively. Then, for every  $f : \mathbb{N} \times X \times Y \times Z \rightarrow \overline{\mathbb{R}}$ ,*

$$\Gamma(\mathcal{N}^-, \mathcal{F}^-, \mathcal{G}^+, \mathcal{H}^-) \lim f = \Gamma_{\text{seq}}(\mathcal{N}^-, \mathcal{F}^-, \mathcal{G}^+, \mathcal{H}^-) \lim f.$$

However, there exist countably based filters  $\mathcal{F}, \mathcal{G}, \mathcal{H}$  and an extended-real-valued function  $f$  such that

$$\Gamma(\mathcal{N}^+, \mathcal{F}^-, \mathcal{G}^+, \mathcal{H}^-) \lim f \neq \Gamma_{\text{seq}}(\mathcal{N}^+, \mathcal{F}^-, \mathcal{G}^+, \mathcal{H}^-) \lim f.$$

*Proof.* It follows from Theorem 3.3.5, Proposition 3.4.21 with  $\mathcal{E} = \mathcal{N}$ . □

**Corollary 3.4.23.** *Let  $\mathcal{F}$  be a strongly Fréchet filter on  $X$ , and  $\mathcal{G}, \mathcal{H}$  be countably based filters on  $Y, Z$ , respectively. Then, for every  $f : \mathbb{N} \times X \times Y \times Z \rightarrow \overline{\mathbb{R}}$ ,*

$$\Gamma(\mathcal{N}^+, \mathcal{F}^+, \mathcal{G}^-, \mathcal{H}^+) \lim f = \Gamma_{\text{seq}}(\mathcal{N}^+, \mathcal{F}^+, \mathcal{G}^-, \mathcal{H}^+) \lim f.$$

However, there exist countably based filters  $\mathcal{F}, \mathcal{G}, \mathcal{H}$  and an extended-real-valued function  $f$  such that

$$\Gamma(\mathcal{N}^-, \mathcal{F}^+, \mathcal{G}^-, \mathcal{H}^+) \lim f \neq \Gamma_{\text{seq}}(\mathcal{N}^-, \mathcal{F}^+, \mathcal{G}^-, \mathcal{H}^+) \lim f.$$



*Proof.* It follows from Theorem 3.4.22. □

From the theorem and corollary above, we see that for  $k \geq 3$ , regardless of the sign  $\alpha_0$  there are countably based filters  $\mathcal{F}_1, \dots, \mathcal{F}_k$  and  $\alpha_1, \dots, \alpha_k$  such that

$$\Gamma(\mathcal{N}^{\alpha_0}, \mathcal{F}_1^{\alpha_1}, \mathcal{F}_2^{\alpha_2}, \dots, \mathcal{F}_k^{\alpha_k})\text{lim} \neq \Gamma_{\text{seq}}(\mathcal{N}^{\alpha_0}, \mathcal{F}_1^{\alpha_1}, \mathcal{F}_2^{\alpha_2}, \dots, \mathcal{F}_k^{\alpha_k})\text{lim}.$$

## 3.5 Applications

In this section, applications of  $\Gamma$ -limits to generalized derivatives and tangency are given.

### 3.5.1 Generalized derivatives

It is generally admitted that definition of differentiability of functions in Euclidean spaces was introduced in [69] by Fréchet in 1911. However, the definition of derivative at a point  $x$  of a real-valued function defined on a subset of Euclidean space was already given in [135] by Peano in 1887 and generalized in 1908 in [138] to function valued in Euclidean space.

Let  $A$  be a subset in  $\mathbb{R}^m$  and  $x$  be an accumulation point of  $A$ . A function  $f : A \rightarrow \mathbb{R}^n$  is said to be differentiable at  $x$  if there exists a linear map  $L : \mathbb{R}^m \rightarrow \mathbb{R}^n$  such that

$$\lim_{y \rightarrow x} \frac{f(y) - f(x) - L(y-x)}{\|y-x\|} = 0. \quad (3.24)$$

It should be stressed that Peano's definition appeared in a rigorous modern form (3.24), that is used nowadays in contrast to the standard language of mathematical definition in that epoch, was usually informal and often vague. Even if, in giving this definition, Peano referred to the concepts of Grassmann in [81] and of Jacobi in [91], those however were more rudimentary. As  $A$  is not the whole of Euclidean space, in general the linear operator  $L$  in (3.24) is not unique; if it is, it is called the derivative of  $f$  at  $x$  and is denoted by  $Df(x)$ .

$Df(x)$  is called nowadays the Fréchet derivative of  $f$  at  $x$ , although Fréchet gave its informal (geometric) definition only in [69] in 1911. Fréchet was apparently unaware of Peano's definition, because one month later, he published in [70] another note, acknowledging contributions of some authors, such as Stolz, Pierpoint and Young, but not that of Peano.

In 1892, Peano introduced the strict differentiability of  $f$  at  $x$ , see [136], that is, if (3.24) is strengthened to

$$\lim_{y, u \rightarrow x, y \neq u} \frac{f(y) - f(u) - L(y-u)}{\|y-u\|} = 0.$$

He also noticed that strict differentiability amounts to continuous differentiability. This definition is frequently referred to Leach in [114], where it is called strong differentiability, and to Bourbaki in [21].

Not only recently, a prominent role that nonsmooth analysis plays in connection with optimization theory is widely recognized, especially since the latter has natural mechanisms that generate nonsmoothness: duality theory, sensitivity and stability analysis, decomposition techniques, etc. Therefore, theories of generalized differentiability have been started. One of its important applications is the topic of optimality conditions for nonsmooth and nonconvex problems. References [19, 129, 130, 147] are recent books that contain systematic expositions and references on generalized differentiation and their applications to optimization-related problems, including optimality conditions. Also, [89, 141] are also detailed treatments on the issues.

Although a whole spectrum of denitions of differentiability can be given in analytical and/or geometrical ways, we can observe that using of kinds of directional derivatives is often a first step for a differential construction; see e.g. [89, 97, 147] for often-met notions of directional derivatives. Therefore, applying directional derivatives is a simple way to deal with optimization-related problems in general and optimality conditions in particular, see [71, 73–76, 157].

In this subsection, by using  $\Gamma$ -limits, we introduce a unified notation of derivatives. Let  $\mathcal{N}_+(0) := \mathcal{N}(0) \cap (0, +\infty)$  be a filter on  $(0, +\infty)$ ,  $\tau$  be a topology on  $X$ , and  $f : X \rightarrow \overline{\mathbb{R}}$ .

Let  $\vartheta_f$  be the supremum of  $\tau$  and of the coarsest topology in  $X$  for which  $f$  is continuous. An unified notation of derivatives of  $f$  at  $x_0$  is defined by

$$D(\mathcal{N}_+(0)^{\alpha_1}; \vartheta_f^{\alpha_2}, \tau^{\alpha_3})f(x_0)(h) := \left( \Gamma(\mathcal{N}_+(0)^{\alpha_1}; \vartheta_f^{\alpha_2}, \tau^{\alpha_3}) \lim \frac{f(x+tu) - f(x)}{t} \right) (x_0, h), \quad (3.25)$$

with  $\alpha_1 \in \{+, -\}$ ,  $\alpha_2, \alpha_3 \in \{+, -, *\}$  where  $\alpha_2 = *$  (or  $\alpha_3 = *$ ) means that  $x$  ( $u$ , respectively) is fixed and equal to  $x_0$  ( $h$ , respectively).

The formula (3.25) can be abbreviated as follows

$$D^{(\alpha_1, \alpha_2, \alpha_3)}f(x_0)(h) = \Gamma((t \rightarrow 0^+)^{\alpha_1}, (x \rightarrow x_0)^{\alpha_2}, (u \rightarrow h)^{\alpha_3}) \lim \frac{f(x+tu) - f(x)}{t}.$$

By using (3.25), we get the following definition.

**Definition 3.5.1.** Let  $f : (X, \tau) \rightarrow \overline{\mathbb{R}}$  and  $x_0 \in X$ .

(i) The *upper directional derivative* of  $f$  at  $x_0$  in direction  $h \in X$  is

$$D^{(+, *, *)}f(x_0)(h) = \Gamma((t \rightarrow 0^+)^+) \lim \frac{f(x_0+th) - f(x_0)}{t}.$$

(ii) The *lower directional derivative* of  $f$  at  $x_0$  in direction  $h \in X$  is

$$D^{(-,*,*)}f(x_0)(h) = \Gamma((t \rightarrow 0^+)^-) \lim \frac{f(x_0 + th) - f(x_0)}{t}.$$

(iii) The *upper tangent derivative* of  $f$  at  $x_0$  in direction  $h \in X$  is

$$D^{(+,*,+)}f(x_0)(h) = \Gamma((t \rightarrow 0^+)^+, (u \rightarrow h)^+) \lim \frac{f(x_0 + tu) - f(x_0)}{t}.$$

(iv) The *lower tangent derivative* of  $f$  at  $x_0$  in direction  $h \in X$  is

$$D^{(-,*,^-)}f(x_0)(h) = \Gamma((t \rightarrow 0^+)^-, (u \rightarrow h)^-) \lim \frac{f(x_0 + tu) - f(x_0)}{t}.$$

(v) The *upper paratangent derivative* of  $f$  at  $x_0$  in direction  $h \in X$  is

$$D^{(+,+,+)}f(x_0)(h) = \Gamma((t \rightarrow 0^+)^+, (x \rightarrow x_0)^+, (u \rightarrow h)^+) \lim \frac{f(x + tu) - f(x)}{t}.$$

(vi) The *lower paratangent derivative* of  $f$  at  $x_0$  in direction  $h \in X$  is

$$D^{(-,-,-)}f(x_0)(h) = \Gamma((t \rightarrow 0^+)^-, (x \rightarrow x_0)^-, (u \rightarrow h)^-) \lim \frac{f(x + tu) - f(x)}{t}.$$

In short, the results above can be expressed as follows

Unifying terminology	Notation	Traditional terminology
upper directional derivative	$D^{(+,*,*)}f(x_0)(h)$	upper Dini derivative
lower directional derivative	$D^{(-,*,*)}f(x_0)(h)$	lower Dini derivative
upper tangent derivative	$D^{(+,*,+)}f(x_0)(h)$	upper Hadamard derivative
lower tangent derivative	$D^{(-,*,^-)}f(x_0)(h)$	lower Hadamard derivative
upper paratangent derivative	$D^{(+,+,+)}f(x_0)(h)$	paratangent derivative
lower paratangent derivative	$D^{(-,-,-)}f(x_0)(h)$	Clarke derivative

### 3.5.2 Tangent cones

In this subsection, we recall some well-known tangent cones in optimization theory and express them in terms of  $\Gamma$ -limits.

Recently, various types of tangent cones have been studied in the literature. Their definitions depend on variants of the limiting process. The most known contribution to the investigation of these concepts is due to Bouligand in 1932, see [20]. One can find a mention about other

contributors in papers of Guareschi [85, 86], Saks [148], Severi [151, 152], Federer [63] and Whitney [170]. We can say that, for tangent cones, main references are Bouligand in optimization theory, Federer in geometric measure theory and calculus of variations, and Whitney in differential geometry.

However, by [54], we discover that tangent cones were already known by Peano at the end of 19th century. Indeed, in 1887, Peano gave in [135] a metric definition of tangent straight line and tangent plane, then reaches, in a natural way, a unifying notion as follows

$$\text{tang}(A, x) := x + \liminf_{\lambda \rightarrow +\infty} \lambda(A - x).$$

Later, in 1908, he introduced in [138] another types of tangent cone, namely

$$\text{Tang}(A, x) := x + \limsup_{\lambda \rightarrow +\infty} \lambda(A - x).$$

To distinguish two above notions, we shall call the first lower tangent cone and the second upper tangent cone. As usual, after abstract investigation of a notion, Peano considered significant special cases and calculated the upper tangent cone in several basic figures (closed ball, curves and surfaces parametrized in a regular way).

Let  $S$  be a subset of  $X$ . The *homothety* of  $S$ , see [50, 139, 145], is the set-valued map from  $(0, +\infty) \times X$  into  $X$  defined by

$$\mathcal{H}_S(t, x) := \frac{1}{t}(S - x). \quad (3.26)$$

$\mathcal{H}_S$  can be considered as a relation in  $(0, +\infty) \times X \times X$ . If  $x_0$  is fixed, (3.26) is called the homothety of  $S$  at  $x_0$  and is denoted by  $\mathcal{H}_{S, x_0}$ .

Let  $\tau, \theta$  be topologies on  $X$ . Based on the above homothety, we now give a unified notation of cones as follows

$$v \in T_S(\mathcal{N}_+(0)^{\alpha_1}; \theta_S^{\alpha_2}, \tau^{\alpha_3})(x_0) \iff (\Gamma(\mathcal{N}_+(0)^{\alpha_1}; \theta_S^{\alpha_2}, \tau^{\alpha_3}) \lim \chi_{(\mathcal{H}_S)})(x_0, v) = 1, \quad (3.27)$$

with  $\theta_S$  be a topology induced on  $S$  by  $\theta$ ,  $\alpha_1, \alpha_3 \in \{+, -\}$ ,  $\alpha_2 \in \{+, -, *\}$ , where  $\alpha_2 = *$  means that  $x$  is fixed and equal to  $x_0$ .

The formula (3.27) can be abbreviated as follows

$$v \in T_S^{(\alpha_1, \alpha_2, \alpha_3)}(x_0) \iff \Gamma((t \rightarrow 0^+)^{\alpha_1}, (x \rightarrow x_0)^{\alpha_2}, (u \rightarrow v)^{\alpha_3}) \lim \chi_{(\mathcal{H}_S)} = 1.$$

By (3.27), we introduce the following definition.

**Definition 3.5.2.** Let  $S \subseteq X$  and  $x_0 \in \text{cl}S$ .

(i) The *upper tangent cone* of  $S$  at  $x_0$  is defined by

$$v \in T_S^{(+,*,+)}(x_0) \iff \Gamma((t \rightarrow 0^+)^+, (u \rightarrow v)^+) \lim \chi_{(\mathcal{H}_{S,x_0})} = 1.$$

(ii) The *lower tangent cone* of  $S$  at  $x_0$  is defined by

$$v \in T_S^{(-,*,+)}(x_0) \iff \Gamma((t \rightarrow 0^+)^-, (u \rightarrow v)^+) \lim \chi_{(\mathcal{H}_{S,x_0})} = 1.$$

(iii) The *upper paratangent cone* of  $S$  at  $x_0$  is defined by

$$v \in T_S^{(+,+,+)}(x_0) \iff \Gamma((t \rightarrow 0^+)^+, (x \rightarrow x_0)^+, (u \rightarrow v)^+) \lim \chi_{(\mathcal{H}_S)} = 1.$$

(iv) The *lower paratangent cone* of  $S$  at  $x_0$  is defined by

$$v \in T_S^{(-,-,+)}(x_0) \iff \Gamma((t \rightarrow 0^+)^-, (x \rightarrow x_0)^-, (u \rightarrow v)^+) \lim \chi_{(\mathcal{H}_S)} = 1.$$

**Proposition 3.5.3.** (i)  $v \in T_S^{(+,*,+)}(x_0)$  if and only if for every  $Q \in \mathcal{N}_\tau(v)$ ,  $t > 0$ , there are  $t' \leq t$  and  $v' \in Q$  such that  $x_0 + t'v' \in S$ .

(ii)  $v \in T_S^{(-,*,+)}(x_0)$  if and only if for every  $Q \in \mathcal{N}_\tau(v)$  there is  $t > 0$  such that for all  $t' \leq t$  there is  $v' \in Q$  satisfying  $x_0 + t'v' \in S$ .

(iii)  $v \in T_S^{(+,+,+)}(x_0)$  if and only if for every  $Q \in \mathcal{N}_\tau(v)$ ,  $W \in \mathcal{N}_{\theta_S}(x_0)$ ,  $t > 0$  there are  $v' \in Q$ ,  $x' \in W$ ,  $t' > 0$  such that  $x' + t'v' \in S$ .

(iv)  $v \in T_S^{(-,-,+)}(x_0)$  if and only if for every  $Q \in \mathcal{N}_\tau(v)$  there are  $W \in \mathcal{N}_{\theta_S}(x_0)$  and  $t > 0$  such that for all  $t' \leq t$ ,  $x' \in W$  there is  $v' \in Q$  satisfying  $x' + t'v' \in S$ .

*Proof.* By the similarity, we prove only (iv). It follows from (3.27) that  $v \in T_S^{(-,-,+)}(x_0)$  if and only if  $(\Gamma(\mathcal{N}_+(0)^-; \theta_S^-, \tau^+) \lim \chi_{(\mathcal{H}_S)})(x_0, v) = 1$ , i.e.,

$$\inf_{Q \in \mathcal{N}_\tau(v)} \sup_{W \in \mathcal{N}_{\theta_S}(x_0)} \sup_{t > 0} \inf_{t' \leq t} \inf_{x' \in W} \sup_{v' \in Q} \chi_{(\mathcal{H}_S)}(t', x', v') = 1.$$

It means that for every  $Q \in \mathcal{N}_\tau(v)$  there are  $W \in \mathcal{N}_{\theta_S}(x_0)$  and  $t > 0$  such that for all  $t' \leq t$ ,  $x' \in W$  there is  $v' \in Q$  satisfying  $x' + t'v' \in S$ . □

**Remark 3.5.4.** When  $X$  is a normed space. We get the sequential form of these tangent cones as follows

(i)  $T_S^{(+,*,+)}(x_0) = \{v \in X : \exists t_n \rightarrow 0^+, \exists v_n \rightarrow v, x_0 + t_n v_n \in S\}$ .

(ii)  $T_S^{(-,*,+)}(x_0) = \{v \in X : \forall t_n \rightarrow 0^+, \exists v_n \rightarrow v, x_0 + t_n v_n \in S\}$ .

(iii)  $T_S^{(+,+,+)}(x_0) = \{v \in X : \exists t_n \rightarrow 0^+, \exists x_n \in S : x_n \rightarrow x_0, \exists v_n \rightarrow v, x_n + t_n v_n \in S\}$ .

(iv)  $T_S^{(-,-,+)}(x_0) = \{v \in X : \forall t_n \rightarrow 0^+, \forall x_n \in S : x_n \rightarrow x_0, \exists v_n \rightarrow v, x_n + t_n v_n \in S\}$ .

These cones, also called the contingent cone, the adjacent cone, the paratangent cone, and the Clarke cone, respectively, were thoroughly studied in [11–13, 17, 50, 60, 61, 77–79, 161] in detail.

These results above give us a unified way to denote cones as follows

Unifying terminology	Notation	Traditional terminology
upper paratangent cone	$T_S^{(+,+,+)}(x_0)$	paratangent cone
upper tangent cone	$T_S^{(+,*,+)}(x_0)$	contingent cone
lower tangent cone	$T_S^{(-,*,+)}(x_0)$	adjacent cone
lower paratangent cone	$T_S^{(-,-,+)}(x_0)$	Clarke cone

The tangent cones above play an important role in the study of various mathematical problems, including optimization, viability theory, and control theory. Moreover, once one has a concept of tangent cone, one can construct a corresponding derivative of a set-valued map. Some of them are presented in the table below as examples, see book [11] of Aubin and Frankowska. Let  $F : X \rightarrow 2^Y$  and  $(x_0, y_0) \in \text{gr}F$ .

Traditional terminology	Unifying notation	Definition
paratangent derivative	$DF^{(+,+,+)}(x_0, y_0)$	$\text{gr}(DF^{(+,+,+)}(x_0, y_0)) = T_{\text{gr}F}^{(+,+,+)}(x_0, y_0)$
contingent derivative	$DF^{(+,*,+)}(x_0, y_0)$	$\text{gr}(DF^{(+,*,+)}(x_0, y_0)) = T_{\text{gr}F}^{(+,*,+)}(x_0, y_0)$
adjacent derivative	$DF^{(-,*,+)}(x_0, y_0)$	$\text{gr}(DF^{(-,*,+)}(x_0, y_0)) = T_{\text{gr}F}^{(-,*,+)}(x_0, y_0)$
circatangent derivative	$DF^{(-,-,+)}(x_0, y_0)$	$\text{gr}(DF^{(-,-,+)}(x_0, y_0)) = T_{\text{gr}F}^{(-,-,+)}(x_0, y_0)$

Inspired by them, many kinds of generalized derivatives have been defined and applied to optimization. Some of them, e.g., variational sets, radial sets, radial derivatives, and Studniarski derivatives, will be introduced in Chapters 4, 5, and 6. They can also be expressed in terms of  $\Gamma$ -limits.

# Chapter 4

## Variational sets and applications to sensitivity analysis for vector optimization problems

### 4.1 Introduction

First-order derivatives (of various types, classical or generalized) of a map are used to approximate a given map to simplify a problem under consideration. To have better approximations, higher-order derivatives are applied. For generalized derivatives and their applications in variational analysis, see books [19] of Bonnans and Shapiro, [129, 130] of Mordukhovich, [147] of Rockafellar and Wets, and long papers [89] of Ioffe, and [141] of Penot. Examining the existing optimality conditions, we can observe that the key argument is included in a separation of suitable sets. To explain the idea, let us take the well-known scheme of Dubovitskii-Milyutin in [58] for first-order optimality conditions in single-valued scalar optimization problems : the intersection of the cone of decrease directions of the objective function and the cone of feasible directions defined by constraints must be empty at a local minimizer. Here, the cone of decrease directions is defined by a kind of derivatives. For other theories of optimality conditions, especially of higher-order conditions, we may have separations of sets, not cones. An important point for a necessary optimality condition of this type is that the larger the separated sets, the stronger the result. This was a motivation for Khanh and Tuan in [105, 106] to propose variational sets replacing derivatives so that they are bigger than sets defined by known derivatives and can be used in the mentioned separation. Some advantages of this generalized differentiability were shown by Khanh et al. in [7, 105, 106], e.g., almost no assumptions are required for variational sets to exist; extensions to higher orders are direct; they are bigger than corresponding sets of most

derivatives, which implies advantageous results in establishing necessary optimality conditions by separation techniques, etc.

In this chapter, we will present our results, published in [3], on variational sets in sensitivity analysis. We study properties of perturbation maps, in terms of higher-order variational sets. Regarding solutions of vector optimization, we restrict ourselves to basic notions of (Pareto) efficient points and weak efficient points. Correspondingly, our concern is to deal with perturbation maps and weak perturbation maps. We employ variational sets in both assumptions and conclusions. We also show cases where our results can be employed but some existing results cannot. Examples are provided to ensure the essentialness of each imposed assumption.

## 4.2 Variational sets of set-valued maps

In this section, we introduce the concept of variational sets of set-valued maps and establish some results on the relationship between variational sets of  $F$  and its profile map.

### 4.2.1 Definitions

Let  $X$  and  $Y$  be normed spaces,  $C$  be a pointed closed convex cone in  $Y$ . To approximate multivalued map  $F : X \rightarrow 2^Y$  at  $(x_0, y_0) \in \text{gr}F$ , we recall two types of higher-order variational sets as follows.

**Definition 4.2.1.** ([105, 106]) Let  $v_1, \dots, v_{m-1} \in Y$ .

(i) The first, second, and higher-order *variational sets of type 1* are the following

$$V^1(F, x_0, y_0) := \text{Limsup}_{x \xrightarrow{F} x_0, t \rightarrow 0^+} \frac{1}{t}(F(x) - y_0),$$

$$V^2(F, x_0, y_0, v_1) := \text{Limsup}_{x \xrightarrow{F} x_0, t \rightarrow 0^+} \frac{1}{t^2}(F(x) - y_0 - tv_1),$$

$$V^m(F, x_0, y_0, v_1, \dots, v_{m-1}) := \text{Limsup}_{x \xrightarrow{F} x_0, t \rightarrow 0^+} \frac{1}{t^m}(F(x) - y_0 - tv_1 - \dots - t^{m-1}v_{m-1}),$$

where  $x \xrightarrow{F} x_0$  means that  $x \in \text{dom}F$  and  $x \rightarrow x_0$ .

(ii) The first, second, and higher-order *variational sets of type 2* are the following

$$W^1(F, x_0, y_0) := \text{Limsup}_{x \xrightarrow{F} x_0, t \rightarrow 0^+} \text{cone}_+(F(x) - y_0),$$



$$W^2(F, x_0, y_0, v_1) := \operatorname{Limsup}_{x \xrightarrow{F} x_0, t \rightarrow 0^+} \frac{1}{t} (\operatorname{cone}_+(F(x) - y_0) - v_1),$$

$$W^m(F, x_0, y_0, v_1, \dots, v_{m-1}) := \operatorname{Limsup}_{x \xrightarrow{F} x_0, t \rightarrow 0^+} \frac{1}{t^{m-1}} (\operatorname{cone}_+(F(x) - y_0) - v_1 - \dots - t^{m-2}v_{m-1}).$$

(iii) If the upper limits in (i) are equal to the lower ones, then these limits are called the first, second, and higher-order *proto-variational sets* of type 1 of  $F$  at  $(x_0, y_0)$ . Similar terminology is defined for type 2.

By using equivalent formulations for the upper limit of a set-valued map (see [11]), i.e.,

$$\operatorname{Limsup}_{x \xrightarrow{F} x_0} F(x) = \{y \in Y : \exists x_n \in \operatorname{dom} F : x_n \rightarrow x_0, \exists y_n \in F(x_n), y_n \rightarrow y\},$$

we easily obtain the following formulae of two types of variational sets.

**Proposition 4.2.2.** ([7]) (Equivalent formulations of  $V^m$ )  $V^m(F, x_0, y_0, v_1, \dots, v_{m-1})$  is equal to all of the following sets

- (i)  $\{y \in Y : \liminf_{x \xrightarrow{F} x_0, t \rightarrow 0^+} \frac{1}{t^m} d(y_0 + tv_1 + \dots + t^{m-1}v_{m-1} + t^m y, F(x)) = 0\}$ ,
- (ii)  $\{y \in Y : \exists t_n \rightarrow 0^+, \exists x_n \xrightarrow{F} x_0, \exists r(t_n^m) = 0(t_n^m), \forall n, y_0 + t_n v_1 + \dots + t_n^{m-1} v_{m-1} + t_n^m y + r(t_n^m) \in F(x_n)\}$ ,
- (iii)  $\{y \in Y : \exists t_n \rightarrow 0^+, \exists x_n \xrightarrow{F} x_0, \exists v_n \rightarrow y, \forall n, y_0 + t_n v_1 + \dots + t_n^{m-1} v_{m-1} + t_n^m v_n \in F(x_n)\}$ ,
- (iv)  $\{y \in Y : \exists t_n \rightarrow 0^+, \exists x_n \xrightarrow{F} x_0, \exists y_n \in F(x_n), \lim_{n \rightarrow \infty} \frac{1}{t_n^m} (y_n - y_0 - t_n v_1 - \dots - t_n^{m-1} v_{m-1}) = y\}$ ,
- (v)  $\bigcap_{\varepsilon > 0} \bigcap_{\substack{\alpha > 0 \\ \beta > 0}} \bigcup_{\substack{0 < t \leq \alpha \\ \|x - x_0\| \leq \beta}} \left( \frac{1}{t^m} (F(x) - y_0 - tv_1 - \dots - t^{m-1} v_{m-1}) + \varepsilon B_Y \right)$ ,
- (vi)  $\bigcap_{\substack{\alpha > 0 \\ \beta > 0}} \operatorname{cl} \bigcup_{\substack{0 < t \leq \alpha \\ \|x - x_0\| \leq \beta}} \frac{1}{t^m} (F(x) - y_0 - tv_1 - \dots - t^{m-1} v_{m-1})$ .

**Proposition 4.2.3.** ([7]) (Equivalent formulations of  $W^m$ )  $W^m(F, x_0, y_0, v_1, \dots, v_{m-1})$  has the following equivalent expressions

- (i)  $\{y \in Y : \liminf_{x \xrightarrow{F} x_0, t \rightarrow 0^+} \frac{1}{t^{m-1}} d(v_1 + \dots + t^{m-2} v_{m-1} + t^{m-1} y, \operatorname{cone}_+(F(x) - y_0)) = 0\}$ ,

§4. Variational sets and applications to sensitivity analysis for vector optimization problems

- (ii)  $\{y \in Y : \exists t_n \rightarrow 0^+, \exists x_n \xrightarrow{F} x_0, \exists r(t_n^{m-1}) = 0(t_n^{m-1}), \forall n, v_1 + \dots + t^{m-2}v_{m-1} + t_n^{m-1}y + r(t_n^{m-1}) \in \text{cone}_+(F(x_n) - y_0)\}$ ,
- (iii)  $\{y \in Y : \exists t_n \rightarrow 0^+, \exists x_n \xrightarrow{F} x_0, \exists v_n \rightarrow y, \forall n, v_1 + \dots + t^{m-2}v_{m-1} + t_n^{m-1}v_n \in \text{cone}_+(F(x_n) - y_0)\}$ ,
- (iv)  $\{y \in Y : \exists t_n \rightarrow 0^+, \exists x_n \xrightarrow{F} x_0, \exists y_n \in \text{cone}_+(F(x_n) - y_0), \lim_{n \rightarrow \infty} \frac{1}{t_n^{m-1}}(y_n - v_1 - \dots - t_n^{m-2}v_{m-1}) = y\}$ ,
- (v)  $\bigcap_{\varepsilon > 0} \bigcap_{\substack{\alpha > 0 \\ \beta > 0}} \bigcup_{\substack{0 < t \leq \alpha \\ \|x - x_0\| \leq \beta}} \left( \frac{1}{t^{m-1}}(\text{cone}_+(F(x) - y_0) - v_1 - \dots - t^{m-2}v_{m-1}) + \varepsilon B_Y \right)$ ,
- (vi)  $\bigcap_{\substack{\alpha > 0 \\ \beta > 0}} \text{cl} \bigcup_{\substack{0 < t \leq \alpha \\ \|x - x_0\| \leq \beta}} \frac{1}{t^{m-1}}(\text{cone}_+(F(x) - y_0) - v_1 - \dots - t^{m-2}v_{m-1})$ .

**Remark 4.2.4.** For all  $m \geq 1$ , we have

- (i)  $V^m(F, x_0, y_0, v_1, \dots, v_{m-1}) \subseteq W^m(F, x_0, y_0, v_1, \dots, v_{m-1})$ .
- (ii)  $V^m(F, x_0, y_0, 0, \dots, 0) = V^1(F, x_0, y_0)$ ,  $W^m(F, x_0, y_0, 0, \dots, 0) = W^1(F, x_0, y_0)$ .
- (iii) If  $v_1 \notin V^1(F, x_0, y_0)$  then  $V^2(F, x_0, y_0, v_1) = \emptyset$ . If one of the conditions  $v_1 \in V^1(F, x_0, y_0)$ ,  $\dots, v_{m-1} \in V^{m-1}(F, x_0, y_0, v_1, \dots, v_{m-2})$  is violated, then  $V^m(F, x_0, y_0, v_1, \dots, v_{m-1}) = \emptyset$ . The variational sets of type 2 have the same property.
- (iv) Variational sets can be expressed in terms of  $\Gamma$ -limits as follows

$$\begin{aligned}
 & y \in V^m(F, x_0, y_0, v_1, \dots, v_{m-1}) \\
 \iff & \inf_{Q \in \mathcal{N}(y)} \inf_{W \in \mathcal{N}(x_0, \text{dom} F)} \inf_{t > 0} \sup_{0 < t' < t} \sup_{x' \in W} \sup_{y' \in Q} \chi_{\text{gr}(\mathcal{L}_{F, y_0, v_1, \dots, v_{m-1}})}(t', x', y') = 1 \\
 \iff & \Gamma(\mathcal{N}_+(0)^+, \mathcal{N}(x_0, \text{dom} F)^+, \mathcal{N}(y)^+) \lim \chi_{\text{gr}(\mathcal{L}_{F, y_0, v_1, \dots, v_{m-1}})} = 1,
 \end{aligned}$$

where  $\mathcal{N}(x_0, \text{dom} F) := \mathcal{N}(x_0) \cap \text{dom} F$ , and  $\mathcal{L}_{F, y_0, v_1, \dots, v_{m-1}} : (0, +\infty) \times X \rightarrow 2^Y$  is defined by

$$\begin{aligned}
 & \mathcal{L}_{F, y_0, v_1, \dots, v_{m-1}}(t', x') := \frac{1}{t'^m}(F(x') - y_0 - t'v_1 - \dots - t'^{m-1}v_{m-1}). \\
 & y \in W^m(F, x_0, y_0, v_1, \dots, v_{m-1}) \\
 \iff & \inf_{Q \in \mathcal{N}(y)} \inf_{W \in \mathcal{N}(x_0, \text{dom} F)} \inf_{t > 0} \sup_{0 < t' < t} \sup_{x' \in W} \sup_{y' \in Q} \chi_{\text{gr}(\mathcal{H}_{F, y_0, v_1, \dots, v_{m-1}})}(t', x', y') = 1 \\
 \iff & \Gamma(\mathcal{N}_+(0)^+, \mathcal{N}(x_0, \text{dom} F)^+, \mathcal{N}(y)^+) \lim \chi_{\text{gr}(\mathcal{H}_{F, y_0, v_1, \dots, v_{m-1}})} = 1,
 \end{aligned}$$

where  $\mathcal{H}_{F,y_0,v_1,\dots,v_{m-1}} : (0, +\infty) \times X \rightarrow 2^Y$  is defined by

$$\mathcal{H}_{F,y_0,v_1,\dots,v_{m-1}}(t', x') := \frac{1}{t'^m} (\text{cone}_+(F(x') - y_0) - t'v_1 - \dots - t'^{m-1}v_{m-1}).$$

The inclusion in Remark 4.2.4(i) may be a strict inclusion or an equality as shown by the following examples.

**Example 4.2.5.** Let  $X = \mathbb{R}$ ,  $Y = \mathbb{R}^2$  and, for  $n = 1, 2, \dots$ ,

$$F(x) = \begin{cases} \{(0, 0)\}, & \text{if } x = 0, \\ \{(-n, n)\}, & \text{if } x = \frac{1}{n}, \\ \{(\frac{1}{n}, 0)\}, & \text{if } x = \ln(1 + \frac{1}{n}), \\ \{(1, \frac{1}{n^2})\}, & \text{if } x = \sin(\frac{1}{n}), \\ \emptyset, & \text{otherwise.} \end{cases}$$

Then, for  $(x_0, y_0) = (0, (0, 0)) \in \text{gr}F$  and  $v_1 = (1, 0) \in Y$ , one has

$$\begin{aligned} V^1(F, x_0, y_0) &= \{(y_1, 0) \in Y : y_1 \geq 0\}, \\ V^2(F, x_0, y_0, v_1) &= \{(y_1, 0) \in Y : y_1 \in \mathbb{R}\}, \\ W^1(F, x_0, y_0) &= \{(y_1, 0) \in Y : y_1 \geq 0\} \cup \{(-y_1, y_1) \in Y : y_1 \geq 0\}, \\ W^2(F, x_0, y_0, v_1) &= \{(y_1, y_2) \in Y : y_2 \geq 0\}. \end{aligned}$$

**Example 4.2.6.** Let  $X = \mathbb{R}$ ,  $Y = \mathbb{R}^2$  and, for  $n = 1, 2, \dots$ ,

$$F(x) = \begin{cases} \{(0, 0)\}, & \text{if } x = 0, \\ \{(-n, n)\}, & \text{if } x = \frac{1}{n}, \\ \{(\frac{1}{n}, 0)\}, & \text{if } x = \ln(1 + \frac{1}{n}), \\ \emptyset, & \text{otherwise.} \end{cases}$$

Then, for  $(x_0, y_0) = (0, (0, 0)) \in \text{gr}F$  and  $v_1 = (1, 0) \in Y$ , one has

$$\begin{aligned} V^1(F, x_0, y_0) &= \{(y_1, 0) \in Y : y_1 \geq 0\}, \\ W^1(F, x_0, y_0) &= \{(y_1, 0) \in Y : y_1 \geq 0\} \cup \{(-y_1, y_1) \in Y : y_1 \geq 0\}, \\ V^2(F, x_0, y_0, v_1) &= W^2(F, x_0, y_0, v_1) = \{(y_1, 0) \in Y : y_1 \in \mathbb{R}\}. \end{aligned}$$

**Remark 4.2.7.** Recall that the *higher-order contingent derivative* of  $F$  at  $(x_0, y_0)$  (relative to  $(u_1, v_1), \dots, (u_{m-1}, v_{m-1})$ ) is the map  $D^m F(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1}) : X \rightarrow 2^Y$  (see [11]) defined by

$$D^m F(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1})(u) := \operatorname{Limsup}_{u' \rightarrow u, t \rightarrow 0^+} \frac{1}{t^m} (F(x_0 + tu_1 + \dots + t^{m-1}u_{m-1} + t^m u') - y_0 - tv_1 - \dots - t^{m-1}v_{m-1}).$$

We can say roughly that the contingent derivative is a directional variant of variational set  $V^m$ . Similarly, most of generalized derivatives (e.g., the (upper) Dini derivative, Hadamard derivative, adjacent derivative, etc) are also based on directional rates, while for the variational sets we allow the flexibility  $x_n \xrightarrow{F} x_0$ . That is why these sets are big

$$D^m F(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1})X \subseteq V^m(F, x_0, y_0, v_1, \dots, v_{m-1}) \subseteq W^m(F, x_0, y_0, v_1, \dots, v_{m-1}).$$

More comparisons between variational sets with well-known derivatives were stated in Proposition 4.1 in [105] by Khanh and Tuan.

## 4.2.2 Relationships between variational sets of $F$ and those of its profile map

The first simple result about a relation between variational sets of the two maps  $F$  and its profile map is as follows.

**Proposition 4.2.8.** (i)  $V^m(F, x_0, y_0, v_1, \dots, v_{m-1}) + C \subseteq V^m(F + C, x_0, y_0, v_1, \dots, v_{m-1})$ ,  
(ii)  $W^m(F, x_0, y_0, v_1, \dots, v_{m-1}) + C \subseteq W^m(F + C, x_0, y_0, v_1, \dots, v_{m-1})$ .

*Proof.* By the similarity, we present only a proof for (ii). Let  $y \in W^m(F, x_0, y_0, v_1, \dots, v_{m-1}) + C$ , i.e., there exist  $v \in W^m(F, x_0, y_0, v_1, \dots, v_{m-1})$  and  $c \in C$  such that  $y = v + c$ . Then, there are  $t_n \rightarrow 0^+$ ,  $x_n \xrightarrow{F} x_0$  and  $v_n \rightarrow v$  such that

$$h_n(v_1 + \dots + t_n^{m-2}v_{m-1} + t_n^{m-1}(v_n + c)) \in F(x_n) + C - y_0.$$

So,  $v + c \in W^m(F + C, x_0, y_0, v_1, \dots, v_{m-1})$ . □

The inclusions opposite to those in Proposition 4.2.8 may not hold as the following example shows.

**Example 4.2.9.** Let  $X = \mathbb{R}$ ,  $Y = \mathbb{R}^2$ ,  $C = \mathbb{R}_+^2$ ,  $(x_0, y_0) = (0, (0, 0))$ , and

$$F(x) = \begin{cases} \{(0, 0)\}, & \text{if } x = 0, \\ \{(-1, -1)\}, & \text{if } x \neq 0. \end{cases}$$

Then, we have  $V^1(F, x_0, y_0) = \{(0, 0)\}$  and  $V^1(F + C, x_0, y_0) = \mathbb{R}^2$ . Thus,  $V^1(F + C, x_0, y_0) \not\subseteq V^1(F, x_0, y_0) + C$ . Let  $v_1 = (0, 1) \in V^1(F + C, x_0, y_0)$ . Then,  $V^2(F + C, x_0, y_0, v_1) \neq \emptyset$  and  $V^2(F, x_0, y_0, v_1) = \emptyset$ . Consequently,

$$V^2(F + C, x_0, y_0, v_1) \not\subseteq V^2(F, x_0, y_0, v_1) + C.$$

For variational sets of type 2, one has  $W^1(F, x_0, y_0) + C = \mathbb{R}^2 = W^1(F + C, x_0, y_0)$  and  $v_1 \in W^1(F + C, x_0, y_0)$ . But,  $W^2(F + C, x_0, y_0, v_1) = \mathbb{R}^2$  and  $W^2(F, x_0, y_0, v_1) = \emptyset$ . Hence,  $W^2(F + C, x_0, y_0, v_1) \not\subseteq W^2(F, x_0, y_0, v_1) + C$ .

**Proposition 4.2.10.** Suppose  $C$  have a compact base. Then

- (i)  $\text{Min}_{C \setminus \{0\}} V^m(F + C, x_0, y_0, v_1, \dots, v_{m-1}) \subseteq V^m(F, x_0, y_0, v_1, \dots, v_{m-1})$ ,
- (ii)  $\text{Min}_{C \setminus \{0\}} W^m(F + C, x_0, y_0, v_1, \dots, v_{m-1}) \subseteq W^m(F, x_0, y_0, v_1, \dots, v_{m-1})$ .

*Proof.* We prove only (i). Let  $v \in \text{Min}_{C \setminus \{0\}} V^m(F + C, x_0, y_0, v_1, \dots, v_{m-1})$ . Then, there exist  $t_n \rightarrow 0^+$ ,  $x_n \xrightarrow{F} x_0$ ,  $y_n \in F(x_n)$ , and  $c_n \in C$  such that, for all  $n$ ,  $\bar{v}_n := t_n^{-m}(y_n + c_n - t_n v_1 - \dots - t_n^{m-1} v_{m-1}) \rightarrow v$ . Then,

$$y_0 + t_n v_1 + \dots + t_n^{m-1} v_{m-1} + t_n^m \bar{v}_n - c_n \in F(x_n). \quad (4.1)$$

We claim that  $c_n/t_n^m \rightarrow 0$  (for a subsequence). For a compact base  $Q$  of  $C$ , one has  $c_n = \alpha_n b_n$  for some  $\alpha_n \geq 0$ ,  $b_n \in Q$ . If  $\alpha_n = 0$  for infinitely many  $n \in \mathbb{N}$ , we are done. Hence, let  $\alpha_n > 0$  for all  $n$ , we may assume that  $b_n \rightarrow b \in Q$ . Then,  $c_n/t_n^m = \alpha_n b_n/t_n^m \rightarrow 0$  if and only if  $\alpha_n/t_n^m \rightarrow 0^+$ . Suppose that  $\alpha_n/t_n^m$  does not converge to 0. Then, nothing is lost by assuming that  $\alpha_n/t_n^m \geq \varepsilon$  for some  $\varepsilon > 0$ . Let  $\bar{c}_n := (\varepsilon t_n^m / \alpha_n) c_n$ . Then,  $\bar{c}_n - c_n \in -C$  and

$$y_0 + t_n v_1 + \dots + t_n^{m-1} v_{m-1} + t_n^m \bar{v}_n - \bar{c}_n \in F(x_n) + C.$$

Since  $\bar{c}_n/t_n^m \rightarrow \varepsilon b \neq 0$ , one has  $\bar{v}_n - t_n^{-m} \bar{c}_n \rightarrow v - \varepsilon b$ , and hence  $v - \varepsilon b \in V^m(F + C, x_0, y_0, v_1, \dots, v_{m-1})$ . Thus,

$$-\varepsilon b \in (V^m(F + C, x_0, y_0, v_1, \dots, v_{m-1}) - v) \cap (-C \setminus \{0\}),$$

contradicting the efficiency of  $v$ . Therefore,  $c_n/t_n^m \rightarrow 0$ . It follows from (4.1) that

$$y_0 + t_n v_1 + \cdots + t_n^{m-1} v_{m-1} + t_n^m \left( \frac{c_n}{v_n} - \frac{c_n}{t_n^m} \right) \in F(x_n),$$

and  $\bar{v}_n - t_n^{-m} c_n \rightarrow v$ . So,  $v \in V^m(F, x_0, y_0, v_1, \dots, v_{m-1})$ .  $\square$

For weak efficiency, we do not have a similar result, as indicated by the following example.

**Example 4.2.11.** Let  $X = \mathbb{R}$ ,  $Y = \mathbb{R}^2$ ,  $C = \mathbb{R}_+^2$ ,  $(x_0, y_0) = (0, (0, 0))$ , and

$$F(x) = \begin{cases} \{(0, 0)\}, & \text{if } x = 0, \\ \{(0, -1)\}, & \text{if } x = \frac{1}{n}, \\ \{(\frac{1}{n}, \frac{-1}{n})\}, & \text{if } x = \sin \frac{1}{n} \text{ for } n \in \mathbb{N}, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Then, we have

$$V^1(F, x_0, y_0) = \{(x, y) \in Y : y = -x, x \geq 0\},$$

$$W^1(F, x_0, y_0) = \{(0, y) \in Y : y \leq 0\} \cup \{(x, y) \in Y : y = -x, x \geq 0\},$$

$$V^1(F + C, x_0, y_0) = W^1(F + C, x_0, y_0) = \mathbb{R}_+ \times \mathbb{R}.$$

Consequently,  $\text{Min}_{\text{int}C} V^1(F + C, x_0, y_0) = \text{Min}_{\text{int}C} W^1(F + C, x_0, y_0) = \{0\} \times \mathbb{R}$ . Therefore,

$$\text{Min}_{\text{int}C} V^1(F + C, x_0, y_0) \not\subseteq V^1(F, x_0, y_0), \text{Min}_{\text{int}C} W^1(F + C, x_0, y_0) \not\subseteq W^1(F, x_0, y_0).$$

If  $\text{int}C \neq \emptyset$ , for weak efficiency, we have the following analogous properties.

**Proposition 4.2.12.** Suppose  $\widehat{C} \subseteq \text{int}C \cup \{0\}$  be a closed convex cone with a compact base. Then

$$(i) \text{Min}_{\text{int}C} V^m(F + \widehat{C}, x_0, y_0, v_1, \dots, v_{m-1}) \subseteq V^m(F, x_0, y_0, v_1, \dots, v_{m-1}),$$

$$(ii) \text{Min}_{\text{int}C} W^m(F + \widehat{C}, x_0, y_0, v_1, \dots, v_{m-1}) \subseteq W^m(F, x_0, y_0, v_1, \dots, v_{m-1}).$$

*Proof.* We prove only (ii). Since  $\widehat{C} \subseteq \text{int}C \cup \{0\}$ , any  $v \in \text{Min}_{\text{int}C} W^m(F + \widehat{C}, x_0, y_0, v_1, \dots, v_{m-1})$  satisfies

$$v \in W^m(F + \widehat{C}, x_0, y_0, v_1, \dots, v_{m-1}) \cap \text{Min}_{\widehat{C} \setminus \{0\}} W^m(F + \widehat{C}, x_0, y_0, v_1, \dots, v_{m-1}), \quad (4.2)$$

where  $\text{Min}_{\widehat{C} \setminus \{0\}} W^m(F + \widehat{C}, x_0, y_0, v_1, \dots, v_{m-1})$  is the set of efficient points of  $W^m(F + \widehat{C}, x_0, y_0, v_1, \dots, v_{m-1})$  with respect to the cone  $\widehat{C}$ . Hence, there exist  $t_n \rightarrow 0^+$ ,  $x_n \xrightarrow{F} x_0$ ,  $v_n \rightarrow v$ ,  $c_n \in \widehat{C}$ , and  $h_n > 0$  such that, for all  $n$ ,

$$\frac{v_1 + \dots + t_n^{m-2} v_{m-1} + t_n^{m-1} v_n}{h_n} - c_n \in F(x_n) - y_0. \quad (4.3)$$

For a compact base  $\widehat{Q}$  of  $\widehat{C}$ , there exist  $\alpha_n \geq 0$  and  $q_n \in \widehat{Q}$  such that  $c_n = \alpha_n q_n$ . We may assume that  $q_n \rightarrow q \in \widehat{Q}$ . We claim that  $h_n \alpha_n / t_n^{m-1} \rightarrow 0^+$  (for a subsequence). This is true if  $\alpha_n = 0$  for infinitely many  $n \in \mathbb{N}$ . Now, suppose to the contrary that  $\alpha_n > 0$ , and  $h_n \alpha_n / t_n^{m-1}$  does not converge to 0. Then, we may assume that  $h_n \alpha_n / t_n^{m-1} \geq \varepsilon$  for some  $\varepsilon > 0$ . Let  $\bar{c}_n := (\varepsilon t_n^{m-1} / h_n \alpha_n) c_n \in \widehat{C}$ . Then, we have  $c_n - \bar{c}_n \in \widehat{C}$ . By (4.3), we obtain

$$\frac{v_1 + \dots + t_n^{m-2} v_{m-1} + t_n^{m-1} v_n}{h_n} - \bar{c}_n \in F(x_n) + \widehat{C} - y_0.$$

As  $h_n \bar{c}_n / t_n^{m-1} \rightarrow \varepsilon q \neq 0$ , this implies that  $v - \varepsilon q \in W^m(F + \widehat{C}, x_0, y_0, v_1, \dots, v_{m-1})$ . Therefore,

$$-\varepsilon q \in (W^m(F + \widehat{C}, x_0, y_0, v_1, \dots, v_{m-1}) - v) \cap (-\widehat{C} \setminus \{0\}),$$

which contradicts (4.2). Hence,  $h_n \alpha_n / t_n^{m-1} \rightarrow 0^+$  and  $v_n - t_n^{-(m-1)} h_n c_n \rightarrow v$ . It follows from (4.3) that  $v \in W^m(F, x_0, y_0, v_1, \dots, v_{m-1})$ .  $\square$

To get the equalities in Proposition 4.2.8, we need the following new notions.

**Definition 4.2.13.** Let  $(x_0, y_0) \in \text{gr}F$ ,  $v_1, \dots, v_{m-1} \in Y$ , and  $m \in \mathbb{N}$ . The  $m$ -th order *singular variational set* of type 1 (type 2, respectively) of  $F$  at  $(x_0, y_0)$  is defined by

$$V^{\infty(m)}(F, x_0, y_0, v_1, \dots, v_{m-1}) := \left\{ y \in Y : \exists x_n \xrightarrow{F} x_0, \exists t_n \rightarrow 0^+, \exists \lambda_n \rightarrow 0^+, \right. \\ \left. \exists y_n \in \frac{F(x_n) - y_0 - t_n v_1 - \dots - t_n^{m-1} v_{m-1}}{t_n^m}, \lambda_n y_n \rightarrow y \right\}$$

$$(W^{\infty(m)}(F, x_0, y_0, v_1, \dots, v_{m-1})) := \left\{ y \in Y : \exists x_n \xrightarrow{F} x_0, \exists t_n \rightarrow 0^+, \exists \lambda_n \rightarrow 0^+, \right. \\ \left. \exists y_n \in \frac{\text{cone}_+(F(x_n) - y_0) - v_1 - \dots - t_n^{m-2} v_{m-1}}{t_n^{m-1}}, \lambda_n y_n \rightarrow y \right\}.$$

**Definition 4.2.14.** Let  $A \subseteq Y$ .

- (i)  $A$  is said to have the *domination property* if and only if  $A \subseteq \text{Min}_{\widehat{C} \setminus \{0\}} A + \widehat{C}$ .
- (ii) When  $\text{int}C \neq \emptyset$ , we say that  $A$  has the *weak domination property* with respect to  $\widehat{C}$  if and only if  $A \subseteq \text{Min}_{\text{int}C} A + \widehat{C}$ , where  $\widehat{C} \subseteq \text{int}C \cup \{0\}$  is a closed convex cone.

**Proposition 4.2.15.** *Let  $C$  have a compact base.*

(i) *Let either of the following conditions hold:*

(i<sub>1</sub>)  $V^m(F + C, x_0, y_0, v_1, \dots, v_{m-1})$  *has the domination property,*

(i<sub>2</sub>)  $V^{\infty(m)}(F, x_0, y_0, v_1, \dots, v_{m-1}) \cap (-C) = \{0\}$ .

*Then*

$$V^m(F + C, x_0, y_0, v_1, \dots, v_{m-1}) = V^m(F, x_0, y_0, v_1, \dots, v_{m-1}) + C, \quad (4.4)$$

$$\text{Min}_{C \setminus \{0\}} V^m(F + C, x_0, y_0, v_1, \dots, v_{m-1}) = \text{Min}_{C \setminus \{0\}} V^m(F, x_0, y_0, v_1, \dots, v_{m-1}). \quad (4.5)$$

(ii) *Let either of the following two conditions hold:*

(ii<sub>1</sub>)  $W^m(F + C, x_0, y_0, v_1, \dots, v_{m-1})$  *has the domination property,*

(ii<sub>2</sub>)  $W^{\infty(m)}(F, x_0, y_0, v_1, \dots, v_{m-1}) \cap (-C) = \{0\}$ .

*Then*

$$W^m(F + C, x_0, y_0, v_1, \dots, v_{m-1}) = W^m(F, x_0, y_0, v_1, \dots, v_{m-1}) + C,$$

$$\text{Min}_{C \setminus \{0\}} W^m(F + C, x_0, y_0, v_1, \dots, v_{m-1}) = \text{Min}_{C \setminus \{0\}} W^m(F, x_0, y_0, v_1, \dots, v_{m-1}).$$

*Proof.* We prove only (i). First, we check (4.4). By Proposition 4.2.8(i), we need simply to verify that

$$V^m(F + C, x_0, y_0, v_1, \dots, v_{m-1}) \subseteq V^m(F, x_0, y_0, v_1, \dots, v_{m-1}) + C.$$

If (i<sub>1</sub>) holds, then  $V^m(F + C, x_0, y_0, v_1, \dots, v_{m-1}) \subseteq \text{Min}_{C \setminus \{0\}} V^m(F + C, x_0, y_0, v_1, \dots, v_{m-1}) + C$ .

Hence, (4.4) is satisfied since we have (by Proposition 4.2.10)

$$\text{Min}_{C \setminus \{0\}} V^m(F + C, x_0, y_0, v_1, \dots, v_{m-1}) + C \subseteq V^m(F, x_0, y_0, v_1, \dots, v_{m-1}) + C.$$

If (i<sub>2</sub>) holds and  $v \in V^m(F + C, x_0, y_0, v_1, \dots, v_{m-1})$ , then there exist  $t_n \rightarrow 0^+$ ,  $x_n \xrightarrow{F} x_0$ ,  $y_n \in F(x_n)$ , and  $c_n \in C$  such that  $t_n^{-m}(y_n + c_n - y_0 - t_n v_1 - \dots - t_n^{m-1} v_{m-1}) \rightarrow v$ . If one has  $n_0$  such that  $c_n = 0$  for all  $n \geq n_0$ , then  $v \in V^m(F, x_0, y_0, v_1, \dots, v_{m-1})$ . If there is a subsequence, denoted again by  $\{c_n\}$  with  $c_n \neq 0$ , we claim that  $\{\|c_n\|/t_n^m\}$  be bounded. Indeed, otherwise we may assume that  $\|c_n\|/t_n^m \rightarrow \infty$  and  $c_n/\|c_n\| \rightarrow \bar{c} \in C \setminus \{0\}$ . Setting

$$v_n := \frac{y_n + c_n - y_0 - t_n v_1 - \dots - t_n^{m-1} v_{m-1}}{t_n^m}, \quad \lambda_n := \frac{t_n^m}{\|c_n\|},$$

we get

$$\lambda_n \frac{y_n - y_0 - t_n v_1 - \dots - t_n^{m-1} v_{m-1} - t_n^m v_n}{t_n^m} \rightarrow -\bar{c} \in -C \setminus \{0\}.$$



As  $\lambda_n \rightarrow 0^+$ , this means  $-\bar{c} \in V^{\infty(m)}(F, x_0, y_0, v_1, \dots, v_{m-1}) \cap -C \setminus \{0\}$ , contradicting (i<sub>2</sub>). So,  $\{\|c_n\|/t_n^m\}$  is bounded and  $\|c_n\|/t_n^m \rightarrow a \geq 0$ . With  $\bar{v}_n := \|c_n\|^{-1}(y_n - y_0 - t_n v_1 - \dots - t_n^{m-1} v_{m-1} - t_n^m v_n)$ , one has

$$y_0 + t_n v_1 + \dots + t_n^{m-1} v_{m-1} + t_n^m \left( v_n + \frac{\|c_n\| \bar{v}_n}{t_n^m} \right) = y_n \in F(x_n).$$

It easy to see that  $v_n + t_n^{-m} \|c_n\| \bar{v}_n \rightarrow v - a\bar{c}$ . Thus,  $v - a\bar{c} \in V^m(F, x_0, y_0, v_1, \dots, v_{m-1})$  and (4.4) is satisfied. (4.5) is implied directly from (4.4).  $\square$

The following example shows that conditions in Proposition 4.2.15 are essential.

**Example 4.2.16.** Let  $X = \mathbb{R}$ ,  $Y = \mathbb{R}^2$ ,  $C = \mathbb{R}_+^2$ ,  $x_0 = 0$ ,  $y_0 = (0, 0)$ , and

$$F(x) = \begin{cases} \{(0, 0)\}, & \text{if } x \leq 0, \\ \{(0, 0), (-1, -1)\}, & \text{if } x > 0. \end{cases}$$

Then,

$$(F + C)(x) = \begin{cases} \mathbb{R}_+^2, & \text{if } x \leq 0, \\ \{(y_1, y_2) : y_1 \geq -1, y_2 \geq -1\}, & \text{if } x > 0. \end{cases}$$

By calculating, we get  $V^1(F, x_0, y_0) = \{(0, 0)\}$  and  $V^1(F + C, x_0, y_0) = \mathbb{R}^2$ . So the conclusions in Proposition 4.2.15(i) does not hold. The reason is that both conditions in Proposition 4.2.15(i) are not satisfied.

Obviously,  $V^1(F + C, x_0, y_0)$  does not have the domination property. Next, we show that  $V^{\infty(1)}(F, x_0, y_0) \cap (-C) \neq \{(0, 0)\}$ . Indeed, by chosing  $x_n = \frac{1}{n} \rightarrow x_0$ ,  $t_n = \frac{1}{n} \rightarrow 0^+$ ,  $\lambda_n = \frac{1}{n} \rightarrow 0^+$  and  $y_n = (-n, -n)$ , it is easy to check that

$$y_n \in \frac{F(x_n) - y_0}{t_n} \text{ and } \lambda_n y_n \rightarrow (-1, -1).$$

Thus,  $(-1, -1) \in V^{\infty(1)}(F, x_0, y_0) \cap (-C)$ .

The following result for weak efficiency can be proved similarly as Proposition 4.2.15.

**Proposition 4.2.17.** Let  $\widehat{C} \subseteq \text{int}C \cup \{0\}$  be a closed convex cone with a compact base.

(i) Impose either of the following two conditions:

(i<sub>1</sub>)  $V^m(F + \widehat{C}, x_0, y_0, v_1, \dots, v_{m-1})$  has the weak domination property with respect to  $\widehat{C}$ ,

$$(i_2) V^{\infty(m)}(F, x_0, y_0, v_1, \dots, v_{m-1}) \cap (-\widehat{C}) = \{0\}.$$

Then

$$\text{Min}_{\text{int}C} V^m(F, x_0, y_0, v_1, \dots, v_{m-1}) = \text{Min}_{\text{int}C} V^m(F + \widehat{C}, x_0, y_0, v_1, \dots, v_{m-1}).$$

(ii) Let either of the following conditions hold:

(ii<sub>1</sub>)  $W^m(F + \widehat{C}, x_0, y_0, v_1, \dots, v_{m-1})$  has the weak domination property with respect to  $\widehat{C}$ ,

$$(ii_2) W^{\infty(m)}(F, x_0, y_0, v_1, \dots, v_{m-1}) \cap (-\widehat{C}) = \{0\}.$$

Then

$$\text{Min}_{\text{int}C} W^m(F, x_0, y_0, v_1, \dots, v_{m-1}) = \text{Min}_{\text{int}C} W^m(F + \widehat{C}, x_0, y_0, v_1, \dots, v_{m-1}).$$

### 4.3 Variational sets of perturbation maps

In this section, we apply results of subsection 4.2.2 to set-valued optimization. Let  $U$  be a normed space of perturbation parameters,  $Y$  be an objective (normed) space ordered by a pointed closed convex cone  $C$  and  $F : U \rightarrow 2^Y$ . One aims at finding the set of efficient points or the set of weak efficient points of  $F(u)$  for a given parameter value  $u$ . Hence, we define set-valued maps  $G$  and  $S$  from  $U$  to  $Y$  by, for  $u \in U$ ,

$$G(u) := \text{Min}_{C \setminus \{0\}} F(u), \quad S(u) := \text{Min}_{\text{int}C} F(u).$$

As it is well-known,  $G$  and  $S$  are called the *perturbation map* and *weak perturbation map*, respectively. The purpose of this section is to investigate relationships between variational sets of  $F$  and that of  $G$  and  $S$ , including relations between the set of efficient points or the set of weak efficient points of these variational sets.

A map  $F$  is said to have the *domination property* around  $u_0$  if and only if there exists a neighborhood  $V$  of  $u_0$  such that  $F(u)$  has the domination property for all  $u \in V$ . The map  $F$  is said to have the *weak domination property* around  $u_0$  with respect to  $\widehat{C}$  if and only if there exists a neighborhood  $V$  of  $u_0$  such that  $F(u)$  has the weak domination property with respect to  $\widehat{C}$  for all  $u \in V$ , where  $\widehat{C} \subseteq \text{int}C \cup \{0\}$  is a closed convex cone.

**Remark 4.3.1.** (i) Suppose  $y_0 \in G(u_0)$  and  $F$  have the domination property around  $u_0$ . Then

$$V^m(G + C, u_0, y_0, v_1, \dots, v_{m-1}) = V^m(F + C, u_0, y_0, v_1, \dots, v_{m-1}),$$

$$W^m(G + C, u_0, y_0, v_1, \dots, v_{m-1}) = W^m(F + C, u_0, y_0, v_1, \dots, v_{m-1}).$$

(ii) Suppose  $y_0 \in S(u_0)$  and  $F$  have the weak domination property around  $u_0$  with respect to  $\widehat{C}$ . Then

$$V^m(S + \widehat{C}, u_0, y_0, v_1, \dots, v_{m-1}) = V^m(F + \widehat{C}, u_0, y_0, v_1, \dots, v_{m-1}),$$

$$W^m(S + \widehat{C}, u_0, y_0, v_1, \dots, v_{m-1}) = W^m(F + \widehat{C}, u_0, y_0, v_1, \dots, v_{m-1}).$$

The first result on efficiency is as follows.

**Theorem 4.3.2.** *Let  $(u_0, y_0) \in \text{gr } G$  and  $v_1, \dots, v_{m-1} \in Y$ . Suppose  $F$  have the domination property around  $u_0$  and  $C$  have a compact base.*

(i) *Assume further either of the following two conditions:*

(i<sub>1</sub>)  $V^m(F + C, u_0, y_0, v_1, \dots, v_{m-1})$  *has the domination property,*

(i<sub>2</sub>)  $V^{\infty(m)}(F, u_0, y_0, v_1, \dots, v_{m-1}) \cap (-C) = \{0\}$ .

*Then*

$$\text{Min}_{C \setminus \{0\}} V^m(F, u_0, y_0, v_1, \dots, v_{m-1}) = \text{Min}_{C \setminus \{0\}} V^m(G, u_0, y_0, v_1, \dots, v_{m-1}).$$

(ii) *Impose either of the following conditions:*

(ii<sub>1</sub>)  $W^m(F + C, u_0, y_0, v_1, \dots, v_{m-1})$  *has the domination property,*

(ii<sub>2</sub>)  $W^{\infty(m)}(F, u_0, y_0, v_1, \dots, v_{m-1}) \cap (-C) = \{0\}$ .

*Then*

$$\text{Min}_{C \setminus \{0\}} W^m(F, u_0, y_0, v_1, \dots, v_{m-1}) = \text{Min}_{C \setminus \{0\}} W^m(G, u_0, y_0, v_1, \dots, v_{m-1}).$$

*Proof.* We prove only assertion (i). Remark 4.3.1(i) yields that  $V^m(G + C, u_0, y_0, v_1, \dots, v_{m-1})$  also has the domination property. Because either (i<sub>1</sub>) or (i<sub>2</sub>) holds, from Proposition 4.2.15 we get

$$\begin{aligned} \text{Min}_{C \setminus \{0\}} V^m(F, u_0, y_0, v_1, \dots, v_{m-1}) &= \text{Min}_{C \setminus \{0\}} V^m(F + C, u_0, y_0, v_1, \dots, v_{m-1}) \\ &= \text{Min}_{C \setminus \{0\}} V^m(G + C, u_0, y_0, v_1, \dots, v_{m-1}) = \text{Min}_{C \setminus \{0\}} V^m(G, u_0, y_0, v_1, \dots, v_{m-1}). \end{aligned}$$

□

The following example illustrates Theorem 4.3.2.

**Example 4.3.3.** Let  $U = \mathbb{R}$ ,  $Y = \mathbb{R}^2$ ,  $C = \mathbb{R}_+^2$ ,  $u_0 = 0$ ,  $y_0 = (0, 0)$ , and  $F(u) = \{(y_1, y_2) \in Y : y_1 = u, y_2 \geq |y_1|\}$  for  $u \in U$ . Then,  $G(u) = \{(y_1, y_2) \in Y : y_1 = u, y_2 = |y_1|\}$ . Let  $v_i = (-1, 1)$  for  $i = 1, \dots, m-1$ . Direct calculations give

$$\begin{aligned} V^m(F, u_0, y_0, v_1, \dots, v_{m-1}) &= W^m(F, u_0, y_0, v_1, \dots, v_{m-1}) \\ &= \begin{cases} \{(y_1, y_2) \in Y : y_2 \geq |y_1|\}, & \text{if } m = 1, \\ \{(y_1, y_2) \in Y : y_1 + y_2 \geq 0\}, & \text{if } m > 1. \end{cases} \end{aligned}$$

$$\begin{aligned} V^m(G, u_0, y_0, v_1, \dots, v_{m-1}) &= W^m(G, u_0, y_0, v_1, \dots, v_{m-1}) \\ &= \begin{cases} \{(y_1, y_2) \in Y : y_2 = |y_1|\}, & \text{if } m = 1, \\ \{(y_1, y_2) \in Y : y_1 + y_2 = 0\}, & \text{if } m > 1. \end{cases} \end{aligned}$$

We can check that assumptions of Theorem 4.3.2 are satisfied for all  $m$ . Direct checking yields

$$\begin{aligned} \text{Min}_{C \setminus \{0\}} V^m(F, u_0, y_0, v_1, \dots, v_{m-1}) &= \text{Min}_{C \setminus \{0\}} V^m(G, u_0, y_0, v_1, \dots, v_{m-1}) \\ &= \text{Min}_{C \setminus \{0\}} W^m(G, u_0, y_0, v_1, \dots, v_{m-1}) \\ &= \begin{cases} \{(y_1, y_2) \in Y : y_1 \leq 0, y_2 = |y_1|\}, & \text{if } m = 1, \\ \{(y_1, y_2) \in Y : y_1 + y_2 = 0\}, & \text{if } m > 1. \end{cases} \end{aligned}$$

Similarly, by Remark 4.3.1(ii) and Proposition 4.2.17, we have the following for weak efficiency.

**Theorem 4.3.4.** Let  $(u_0, y_0) \in \text{gr}S$  and  $v_1, \dots, v_{m-1} \in Y$ . Suppose  $F$  have the weak domination property around  $u_0$  with respect to  $\widehat{C}$ , where  $\widehat{C} \subseteq \text{int}C \cup \{0\}$  is a closed convex cone having a compact base.

(i) Let either of the following two conditions hold:

(i<sub>1</sub>)  $V^m(F + \widehat{C}, u_0, y_0, v_1, \dots, v_{m-1})$  has the weak domination property with respect to  $\widehat{C}$ ,

(i<sub>2</sub>)  $V^{\infty(m)}(F, u_0, y_0, v_1, \dots, v_{m-1}) \cap (-\widehat{C}) = \{0\}$ .

Then

$$\text{Min}_{\text{int}C} V^m(F, u_0, y_0, v_1, \dots, v_{m-1}) = \text{Min}_{\text{int}C} V^m(S, u_0, y_0, v_1, \dots, v_{m-1}).$$

(ii) Impose one of the following two conditions:

(ii<sub>1</sub>)  $W^m(F + \widehat{C}, u_0, y_0, v_1, \dots, v_{m-1})$  has the weak domination property with respect to  $\widehat{C}$ ,

$$(ii_2) W^{\infty(m)}(F, u_0, y_0, v_1, \dots, v_{m-1}) \cap (-\widehat{C}) = \{0\}.$$

Then

$$\text{Min}_{\text{int}C} W^m(F, u_0, y_0, v_1, \dots, v_{m-1}) = \text{Min}_{\text{int}C} W^m(S, u_0, y_0, v_1, \dots, v_{m-1}).$$

**Example 4.3.5.** Let  $U = \mathbb{R}$ ,  $Y = \mathbb{R}^2$ ,  $C = \mathbb{R}_+^2$ ,  $u_0 = 0$ ,  $y_0 = (0, 0)$ , and

$$F(u) = \begin{cases} \{(y_1, y_2) \in Y : y_1 = u, y_2 \geq -y_1\}, & \text{if } u \leq 0, \\ \{(y_1, y_2) \in Y : 0 \leq y_1 \leq u, y_2 \geq 0\}, & \text{if } u > 0. \end{cases}$$

Then,

$$S(u) = \begin{cases} \{(y_1, y_2) \in Y : y_1 = u, y_2 \geq -y_1\}, & \text{if } u \leq 0, \\ \{(y_1, y_2) \in Y : 0 \leq y_1 \leq u, y_2 = 0\}, & \text{if } u > 0. \end{cases}$$

Let  $v_i = (1, 0)$  for  $i = 1, \dots, m-1$ . Direct computations yield that

$$V^m(F, u_0, y_0, v_1, \dots, v_{m-1}) = \begin{cases} \mathbb{R}_+^2 \cup \{(y_1, y_2) \in Y : y_1 \leq 0, y_2 \geq -y_1\}, & \text{if } m = 1, \\ \mathbb{R} \times \mathbb{R}_+, & \text{if } m > 1, \end{cases}$$

and

$$V^m(S, u_0, y_0, v_1, \dots, v_{m-1}) = \begin{cases} \{(y_1, y_2) \in Y : y_1 \leq 0, y_2 \geq -y_1\} \cup (\mathbb{R}_+ \times \{0\}), & \text{if } m = 1, \\ \mathbb{R} \times \{0\}, & \text{if } m > 1. \end{cases}$$

For each of  $F$  and  $S$ , variational sets of two types coincide for all  $m \geq 1$ . We can check that assumptions of Theorem 4.3.4 are fulfilled for all  $m$  (for an arbitrary closed convex cone  $\widehat{C}$  such that  $\widehat{C} \in \text{int} \mathbb{R}_+^2 \cup \{(0, 0)\}$ ). Direct verifying gives

$$\begin{aligned} \text{Min}_{\text{int}C} V^m(F, u_0, y_0, v_1, \dots, v_{m-1}) &= \text{Min}_{\text{int}C} V^m(S, u_0, y_0, v_1, \dots, v_{m-1}) \\ &= \text{Min}_{\text{int}C} W^m(S, u_0, y_0, v_1, \dots, v_{m-1}) \\ &= \begin{cases} \{(y_1, y_2) : y_1 \leq 0, y_2 = -y_1\} \cup (\mathbb{R}_+ \times \{0\}), & \text{if } m = 1, \\ \mathbb{R} \times \{0\}, & \text{if } m > 1. \end{cases} \end{aligned}$$

Note that the set of (Pareto) efficient points is much smaller than that of weak efficient points

$$G(u) = \begin{cases} \{(y_1, y_2) \in Y : y_1 = u, y_2 = -y_1\}, & \text{if } u \leq 0, \\ \{(0, 0)\}, & \text{if } u > 0. \end{cases}$$

For  $v_i = (1, 0)$ ,  $i = 1, \dots, m-1$ , we have  $V^1(G, u_0, y_0) = W^m(G, u_0, y_0) = \{(y_1, y_2) \in Y : y_1 \leq 0, y_2 = -y_1\}$ , and they are empty for  $m > 1$ . We can check that assumptions of Theorem 4.3.2 are satisfied for  $m = 1$  and

$$\begin{aligned} \text{Min}_{C \setminus \{0\}} V^1(F, u_0, y_0) &= \text{Min}_{C \setminus \{0\}} V^1(G, u_0, y_0) \\ &= \text{Min}_{C \setminus \{0\}} W^1(G, u_0, y_0) \\ &= \{(y_1, y_2) \in Y : y_1 \leq 0, y_2 = -y_1\}. \end{aligned}$$

In the following case, Theorems 4.3.2 and 4.3.4 can be used, but some recent existing results cannot.

**Example 4.3.6.** Let  $U = \mathbb{R}$ ,  $Y = \mathbb{R}^2$ ,  $C = \mathbb{R}_+^2$ ,  $u_0 = 0$ ,  $y_0 = (0, 0)$ , and

$$F(u) = \begin{cases} \{(0, 0)\}, & \text{if } u = 0, \\ \left\{ (0, 0); \left(\frac{1}{n^3}, \frac{-1}{n^3}\right); \left(\frac{-1}{n^3}, \frac{1}{n^3}\right) \right\}, & \text{if } u = \frac{1}{n} \text{ for } n \in \mathbb{N}, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Then,  $S(u) = G(u) = F(u)$ . For  $v_1 = (1, -1)$ ,  $v_2 = (-1, 1)$ . Calculations give

$$\begin{aligned} V^1(F, u_0, y_0) &= W^1(F, u_0, y_0) = \{(y_1, y_2) \in Y : y_1 + y_2 = 0\}, \\ V^2(F, u_0, y_0, v_1) &= W^2(F, u_0, y_0, v_1) = \{(y_1, y_2) \in Y : y_1 + y_2 = 0\}, \\ V^3(F, u_0, y_0, v_1, v_2) &= W^3(F, u_0, y_0, v_1, v_2) = \{(y_1, y_2) \in Y : y_1 + y_2 = 0\}. \end{aligned}$$

We can check that assumptions of Theorems 4.3.2 and 4.3.4 are satisfied. Calculating the lower Studniarski derivative of  $F$  at  $(u_0, y_0)$  (see [158] for the definition), we have  $\underline{d}^m F(u_0, y_0)(u)$  is empty for all  $u \in \mathbb{R}$ . Hence, Theorems 4.1-4.3 and Corollaries 4.1-4.3 of [158] cannot be in use.

Since  $D^2 F(u_0, y_0, u_1, v_1)(u) = \emptyset$ , for all  $u \in \mathbb{R}$ , Theorems 4.3, 4.7, and 4.10 of [167] in terms of second-order contingent derivatives cannot be applied either.

**Proposition 4.3.7.** Let  $(u_0, y_0) \in \text{gr } S$  and  $v_1, \dots, v_{m-1} \in Y$ . Suppose  $F$  have a proto-variational set of order  $m$  of type 1 at  $(u_0, y_0)$ . Then

$$V^m(S, u_0, y_0, v_1, \dots, v_{m-1}) \subseteq \text{Min}_{\text{int } C} V^m(F, u_0, y_0, v_1, \dots, v_{m-1}).$$

*Proof.* Let  $y \in V^m(S, u_0, y_0, v_1, \dots, v_{m-1})$ , i.e., there exist  $t_n \rightarrow 0^+$ ,  $u_n \xrightarrow{S} u_0$ , and  $y_n \rightarrow y$  such that

$$y_0 + t_n v_1 + \dots + t_n^m y_n \in S(u_n) \subseteq F(u_n), \quad (4.6)$$

so  $y \in V^m(F, u_0, y_0, v_1, \dots, v_{m-1})$ . Suppose  $y \notin \text{Min}_{\text{int}C} V^m(F, u_0, y_0, v_1, \dots, v_{m-1})$ , i.e., there exists some  $y'$  in  $V^m(F, u_0, y_0, v_1, \dots, v_{m-1})$  such that  $y - y' \in \text{int}C$ . For the above sequences  $t_n$  and  $u_n$ , there exists  $y'_n \rightarrow y'$  such that  $y_0 + t_n v_1 + \dots + t_n^m y'_n \in F(u_n)$ , and  $y_n - y'_n \in \text{int}C$  for large  $n$ . Consequently,

$$(y_0 + t_n v_1 + \dots + t_n^m y_n) - (y_0 + t_n v_1 + \dots + t_n^m y'_n) = t_n^m (y_n - y'_n) \in \text{int}C,$$

i.e.,  $y_0 + t_n v_1 + \dots + t_n^m y_n \notin \text{Min}_{\text{int}C} F(u_n) = S(u_n)$ , which contradicts to (4.6). □

Unfortunately, the similar result is not true for  $W^m$ , as indicated by the next example.

**Example 4.3.8.** Let  $U = \mathbb{R}$ ,  $Y = \mathbb{R}^2$ ,  $C = \mathbb{R}_+^2$ , and

$$F(u) = \begin{cases} (\{0\} \times \mathbb{R}) \cup (\mathbb{R} \times \{0\}) \cup \{(x, y) \in Y : x^2 + y^2 = 1\}, & \text{if } u = 0, \\ \emptyset, & \text{if } u \neq 0. \end{cases}$$

Then,

$$S(u) = \begin{cases} ((-\infty, -1) \times \{0\}) \cup (\{0\} \times (-\infty, -1)) \cup \{(x, y) : x^2 + y^2 = 1, x \leq 0, y \leq 0\}, & \text{if } u = 0, \\ \emptyset, & \text{if } u \neq 0. \end{cases}$$

The map  $F$  has a proto-variational set at  $(0, (-1, 0))$  and  $W^1(F, 0, (-1, 0)) = (\mathbb{R}_+ \times \mathbb{R}) \cup (\mathbb{R}_- \times \{0\})$ . However, we have

$$\text{Min}_{\text{int}C} W^1(F, 0, (-1, 0)) = (\{0\} \times \mathbb{R}_-) \cup (\mathbb{R}_- \times \{0\}),$$

$$W^1(S, 0, (-1, 0)) = (\mathbb{R}_- \times \{0\}) \cup \{(x, y) \in Y : y \leq -x, x \geq 0\},$$

and hence  $W^1(S, 0, (-1, 0)) \not\subseteq \text{Min}_{\text{int}C} W^1(F, 0, (-1, 0))$ .

Can we get a similar result when “ $G$ ” and “ $\text{Min}_{C \setminus \{0\}}$ ” replacing “ $S$ ” and “ $\text{Min}_{\text{int}C}$ ” in Proposition 4.3.7? The following example gives a negative answer.

**Example 4.3.9.** Let  $U = \mathbb{R}$ ,  $Y = \mathbb{R}^2$ ,  $C = \mathbb{R}_+^2$ , and

$$F(u) = \begin{cases} \{(x, y) \in Y : x > 0, y < -x\} \cup \{(x, y) \in Y : y = -x\}, & \text{if } u = 0, \\ \emptyset, & \text{if } u \neq 0. \end{cases}$$

Then,  $G(u) \equiv \text{Min}_{C \setminus \{0\}} F(u)$  is defined by  $G(0) = \{(x, y) \in Y : y = -x, x \leq 0\}$  and  $G(u) = \emptyset$  for any  $u \neq 0$ . We see that  $F$  has the following proto-variational set

$$V^1(F, 0, (0, 0)) = \{(x, y) \in Y : x \geq 0, y < -x\} \cup \{(x, y) \in Y : y = -x\}.$$

Since  $\text{Min}_{C \setminus \{0\}} V^1(F, 0, (0, 0)) = \{(x, y) \in Y : y = -x, x < 0\}$  and  $V^1(G, 0, (0, 0)) = \{(x, y) \in Y : y = -x, x \leq 0\}$ , one has

$$V^1(G, 0, (0, 0)) \not\subseteq \text{Min}_{C \setminus \{0\}} V^1(F, 0, (0, 0)).$$

**Theorem 4.3.10.** *Let  $(u_0, y_0) \in \text{gr} S$ ,  $v_1, \dots, v_{m-1} \in Y$ , and  $\widehat{C}$  be a closed convex cone contained in  $\text{int} C \cup \{0\}$  and have a compact base. Suppose the following conditions be satisfied:*

(i) *either of the following holds*

(i<sub>1</sub>)  $V^m(F + \widehat{C}, u_0, y_0, v_1, \dots, v_{m-1})$  *has the weak domination property with respect to  $\widehat{C}$ ,*

$$(i_2) V^{\infty(m)}(F, u_0, y_0, v_1, \dots, v_{m-1}) \cap (-\widehat{C}) = \{0\},$$

(ii)  $F$  *has the weak domination property around  $u_0$  with respect to  $\widehat{C}$ ,*

(iii)  $F$  *has a proto-variational set of order  $m$  of type 1 at  $(u_0, y_0)$ .*

*Then*

$$V^m(S, u_0, y_0, v_1, \dots, v_{m-1}) = \text{Min}_{\text{int} C} V^m(F, u_0, y_0, v_1, \dots, v_{m-1}).$$

*Proof.* Obviously, by Proposition 4.3.7, we need to prove only that

$$V^m(S, u_0, y_0, v_1, \dots, v_{m-1}) \supseteq \text{Min}_{\text{int} C} V^m(F, u_0, y_0, v_1, \dots, v_{m-1}).$$

Propositions 4.2.12, 4.2.17 and Remark 4.3.1(iii) together imply that

$$\begin{aligned} \text{Min}_{\text{int} C} V^m(F, u_0, y_0, v_1, \dots, v_{m-1}) &= \text{Min}_{\text{int} C} V^m(F + \widehat{C}, u_0, y_0, v_1, \dots, v_{m-1}) \\ &= \text{Min}_{\text{int} C} V^m(S + \widehat{C}, u_0, y_0, v_1, \dots, v_{m-1}) \subseteq V^m(S, u_0, y_0, v_1, \dots, v_{m-1}). \end{aligned}$$

□

## 4.4 Sensitivity analysis for vector optimization problems

In this section, we consider the following two constrained vector optimization problems, where both the objective map and the constraint set depend on a perturbation parameter,

$$\text{Min}_{C \setminus \{0\}} F(x, u), \quad \text{subject to } x \in X(u), \quad (4.7)$$



$$\text{Min}_{\text{int}C} F(x, u), \text{ subject to } x \in X(u). \quad (4.8)$$

Here, as before,  $U, W, Y$  are normed spaces,  $C$  is a pointed closed convex ordering cone in  $Y$ ,  $F$  is a set-valued objective map from  $W \times U$  to  $Y$ , and  $X$  is a set-valued map from  $U$  to  $W$ . We define a set-valued map  $H$  from  $U$  to  $Y$  by

$$H(u) := F(X(u), u) = \{y \in Y : y \in F(x, u), x \in X(u)\}.$$

So,  $H(u)$  is the parameterized feasible set in the objective space. In problems (4.7) and (4.8), we aim to obtain efficient points and weak efficient points of  $H(u)$ , respectively. Solution sets in  $Y$  of problems (4.7) and (4.8) are denoted by  $\text{Min}_{C \setminus \{0\}} H(u)$  and  $\text{Min}_{\text{int}C} H(u)$ , respectively. Like in Section 4.3, we define

$$G(u) := \text{Min}_{C \setminus \{0\}} H(u), \quad S(u) := \text{Min}_{\text{int}C} H(u).$$

We need the following new definition.

**Definition 4.4.1.** Let  $W, U, Y$  be normed spaces,  $F : W \times U \rightarrow 2^Y$ ,  $((x_0, u_0), y_0) \in \text{gr} F$ ,  $x \in W$ , and  $(w_i, v_i) \in W \times Y$  for  $i = 1, \dots, m-1$ .

(i) The  $m$ -th order *upper (lower, respectively) variation* of  $F$  at  $((x_0, u_0), y_0)$  with respect to  $x$  is

$$\begin{aligned} V_q^m(F, (x_0[x], u_0), y_0, w_1, v_1, \dots, w_{m-1}, v_{m-1}) &:= \{v \in Y : \exists t_n \rightarrow 0^+, \exists h_n \rightarrow 0^+, \exists x_n \rightarrow x, \exists u_n \rightarrow u_0, \\ &\quad \exists v_n \rightarrow v, \forall n, y_0 + h_n v_1 + \dots + h_n^{m-1} v_{m-1} + h_n^m v_n \in F(x_0 + t_n w_1 + \dots + t_n^{m-1} w_{m-1} + t_n^m x_n, u_n)\} \\ \underline{V}_q^m(F, (x_0[x], u_0), y_0, w_1, v_1, \dots, w_{m-1}, v_{m-1}) &:= \{v \in Y : \forall t_n \rightarrow 0^+, \forall x_n \rightarrow x, \forall u_n \rightarrow u_0, \exists v_n \rightarrow v, \forall n, \\ &\quad y_0 + t_n v_1 + \dots + t_n^{m-1} v_{m-1} + t_n^m v_n \in F(x_0 + t_n w_1 + \dots + t_n^{m-1} w_{m-1} + t_n^m x_n, u_n)\}. \end{aligned}$$

(ii)  $F$  is said to have a  $m$ -th order *proto variation* of  $F$  at  $((x_0, u_0), y_0)$  if and only if, for all  $x$ ,

$$V_q^m(F, (x_0[x], u_0), y_0, w_1, v_1, \dots, w_{m-1}, v_{m-1}) = \underline{V}_q^m(F, (x_0[x], u_0), y_0, w_1, v_1, \dots, w_{m-1}, v_{m-1}).$$

We recall that a map  $M : X \rightarrow 2^Y$  is said to be *calm* around  $x_0 \in \text{dom} M$  if and only if there exist a neighborhood  $V$  of  $x_0$  and  $L > 0$  such that  $\forall x \in V$ ,

$$M(x) \subseteq M(x_0) + L\|x - x_0\|B_Y.$$

We now investigate connections of a proto variation of  $F$  and a variational set of  $X$  to the corresponding variational set of  $H$ .

**Proposition 4.4.2.** *Let  $u_0 \in U$ ,  $x_0 \in X(u_0)$ , and  $y_0 \in F(x_0, u_0)$ . If  $F$  has a  $m$ -th order proto variation at  $((x_0, u_0), y_0)$ , then*

$$\bigcup_{x \in V^m(X, u_0, x_0, w_1, \dots, w_{m-1})} V_q^m(F, (x_0[x], u_0), y_0, w_1, v_1, \dots, w_{m-1}, v_{m-1}) \subseteq V^m(H, u_0, y_0, v_1, \dots, v_{m-1}). \quad (4.9)$$

Moreover, if  $W$  is finite dimensional,  $\tilde{X}(u, y) := \{x \in \mathbb{R}^n : x \in X(u), y \in F(x, u)\}$  is calm around  $(u_0, y_0)$ ,  $\tilde{X}(u_0, y_0) = \{x_0\}$ , and  $V_q^1(\tilde{X}, (u_0, y_0[0]), x_0) = \{0\}$ , then the inclusion opposite to (4.9) is valid.

*Proof.* Let  $x \in V^m(X, u_0, x_0, w_1, \dots, w_{m-1})$  such that there exists  $v$  satisfying

$$v \in V_q^m(F, (x_0[x], u_0), y_0, w_1, v_1, \dots, w_{m-1}, v_{m-1}).$$

Since  $x \in V^m(X, u_0, x_0, w_1, \dots, w_{m-1})$ , there exist  $t_n \rightarrow 0^+$ ,  $u_n \rightarrow u_0$ ,  $x_n \xrightarrow{X} x$  such that, for all  $n$ ,

$$x_0 + t_n w_1 + \dots + t_n^{m-1} w_{m-1} + t_n^m x_n \in X(u_n).$$

Then,

$$F(x_0 + t_n w_1 + \dots + t_n^{m-1} w_{m-1} + t_n^m x_n, u_n) \subseteq H(u_n). \quad (4.10)$$

Because  $v \in V_q^m(F, (x_0[x], u_0), y_0, w_1, v_1, \dots, w_{m-1}, v_{m-1})$ , with the above  $t_n, u_n, x_n$ , there exists  $y_n \in F(x_0 + t_n w_1 + \dots + t_n^{m-1} w_{m-1} + t_n^m x_n, u_n)$  such that  $t_n^{-m}(y_n - y_0 - t_n v_1 - \dots - t_n^{m-1} v_{m-1}) \rightarrow v$ . So, we have

$$\begin{aligned} y_0 + t_n v_1 + \dots + t_n^{m-1} v_{m-1} + t_n^m \left( \frac{y_n - y_0 - t_n v_1 - \dots - t_n^{m-1} v_{m-1}}{t_n^m} \right) &= y_n \in \\ &\in F(x_0 + t_n w_1 + \dots + t_n^{m-1} w_{m-1} + t_n^m x_n, u_n). \end{aligned}$$

It follows from (4.10) that  $v \in V^m(H, u_0, y_0, v_1, \dots, v_{m-1})$ .

Next, we prove the inclusion reverse to (4.9). Let  $v \in V^m(H, u_0, y_0, v_1, \dots, v_{m-1})$ , i.e., there exist  $t_n \rightarrow 0^+$ ,  $u_n \rightarrow u_0$ , and  $v_n \xrightarrow{H} v$  such that  $y_0 + t_n v_1 + \dots + t_n^{m-1} v_{m-1} + t_n^m v_n \in H(u_n)$  for all  $n$ . Then, there exists  $x_n \in X(u_n)$  such that  $y_0 + t_n v_1 + \dots + t_n^{m-1} v_{m-1} + t_n^m v_n \in F(x_n, u_n)$ . Hence,  $x_n \in \tilde{X}(u_n, y_0 + t_n v_1 + \dots + t_n^{m-1} v_{m-1} + t_n^m v_n)$ . The calmness of  $\tilde{X}$  yields  $M > 0$  such that

$$\|x_n - x_0\| \leq M \| (u_n, y_0 + t_n v_1 + \dots + t_n^{m-1} v_{m-1} + t_n^m v_n) - (u_0, y_0) \|.$$

Then,  $x_n \rightarrow x_0$  and hence  $(x_n - x_0 - t_n w_1 - \dots - t_n^{m-1} w_{m-1}) \rightarrow 0$ . We claim that  $\{t_n^{-m}(x_n - x_0 - t_n w_1 - \dots - t_n^{m-1} w_{m-1})\}$  is bounded. Indeed, we have

$$x_0 + \|x_n - x_0\| \frac{(x_n - x_0)}{\|x_n - x_0\|} = x_n \in \tilde{X}(u_n, y_0 + t_n v_1 + \dots + t_n^{m-1} v_{m-1} + t_n^m v_n). \quad (4.11)$$

We may assume that  $a_n := (x_n - x_0)/\|x_n - x_0\| \rightarrow a$  with norm one. Setting  $r_n^m := \|x_n - x_0 - t_n w_1 - \dots - t_n^{m-1} w_{m-1}\|$ , we have  $r_n^m \rightarrow 0^+$  and

$$\begin{aligned} y_0 + t_n v_1 + \dots + t_n^{m-1} v_{m-1} + t_n^m v_n &= y_0 + r_n \frac{t_n}{r_n} v_1 + \dots + r_n^{m-1} \frac{t_n^{m-1}}{r_n^{m-1}} v_{m-1} + r_n^m \frac{t_n^m}{r_n^m} v_n \\ &= y_0 + r_n \left( \frac{t_n}{r_n} v_1 + \dots + r_n^{m-2} \frac{t_n^{m-1}}{r_n^{m-1}} v_{m-1} + r_n^{m-1} \frac{t_n^m}{r_n^m} v_n \right) =: y_0 + r_n q_n. \end{aligned}$$

For  $h_n := \|x_n - x_0\|$ , (4.11) is written equivalently as  $x_0 + h_n a_n \in \tilde{X}(u_n, y_0 + r_n q_n)$ . If  $t_n^m / r_n^m \rightarrow 0^+$ , then  $q_n \rightarrow 0$  and  $a \in V_q^1(\tilde{X}, (u_0, y_0[0]), x_0)$ , which is impossible. Thus,  $\{t_n^{-m}(x_n - x_0 - t_n w_1 - \dots - t_n^{m-1} w_{m-1})\}$  is bounded and

$$\bar{x}_n := \frac{1}{t_n^m} (x_n - x_0 - t_n w_1 - \dots - t_n^{m-1} w_{m-1})$$

converges to some  $\bar{x} \in \mathbb{R}^n$ . Since  $x_0 + t_n w_1 + \dots + t_n^{m-1} w_{m-1} + t_n^m \bar{x}_n \in X(u_n)$ , one has

$$y_0 + t_n v_1 + \dots + t_n^{m-1} v_{m-1} + t_n^m v_n \in F(x_0 + t_n w_1 + \dots + t_n^{m-1} w_{m-1} + t_n^m \bar{x}_n, u_n).$$

Therefore,  $\bar{x} \in V^m(X, u_0, x_0, w_1, \dots, w_{m-1})$  and  $v \in V_q^m(F, (x_0[\bar{x}], u_0), y_0, w_1, v_1, \dots, w_{m-1}, v_{m-1})$ .  $\square$

The following four examples ensure the essentialness of each assumption of Proposition 4.4.2.

**Example 4.4.3.** ( $\tilde{X}(u_0, y_0) = \{x_0\}$  is needed)

Let  $U = W = Y = \mathbb{R}$ ,  $F(x, u) = \{x(x-1)\}$ ,  $u_0 = 0$ ,  $x_0 = 1$ ,  $y_0 = 0 \in F(x_0, u_0)$ , and

$$X(u) = \begin{cases} \{x \in W : 0 \leq x \leq 1\}, & \text{if } u = 0, \\ \{x \in W : -u \leq x \leq 1\}, & \text{if } u = \frac{1}{n}, n \in \mathbb{N}, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Then,

$$\tilde{X}(u, y) = \begin{cases} \left\{ \frac{1 - \sqrt{1+4y}}{2}, \frac{1 + \sqrt{1+4y}}{2} \right\}, & \text{if } u \in \{0\} \cup \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}, \frac{-1}{4} \leq y \leq 0, \\ \left\{ \frac{1 - \sqrt{1+4y}}{2} \right\}, & \text{if } u \in \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}, 0 < y \leq u(u+1), \\ \emptyset, & \text{otherwise,} \end{cases}$$

and

$$H(u) = \begin{cases} \{y \in Y : \frac{-1}{4} \leq y \leq 0\}, & \text{if } u = 0, \\ \{y \in Y : \frac{-1}{4} \leq y \leq u(u+1)\}, & \text{if } u = \frac{1}{n}, n \in \mathbb{N}, \\ \emptyset, & \text{otherwise.} \end{cases}$$

The map  $\tilde{X}$  is clearly calm around  $(u_0, y_0)$  and we can obtain by direct calculations that

$$V_q^1(\tilde{X}, (u_0, y_0[0]), x_0) = \{0\}, V^1(X, u_0, x_0) = -\mathbb{R}_+,$$

$$V_q^1(F, (x_0[x], u_0), y_0) = \{x\}, V^1(H, u_0, y_0) = \mathbb{R}.$$

So,

$$\bigcup_{x \in V^1(X, u_0, x_0)} V_q^1(F, (x_0[x], u_0), y_0) = -\mathbb{R}_+.$$

Thus, since  $\tilde{X}(u_0, y_0) = \{0, 1\} \neq \{x_0\}$ , we have

$$V^1(H, u_0, y_0) \not\subseteq \bigcup_{x \in V^1(X, u_0, x_0)} V_q^1(F, (x_0[x], u_0), y_0).$$

**Example 4.4.4.** (the calmness around  $(u_0, y_0)$  cannot be dropped)

Let  $U = W = Y = \mathbb{R}$ ,  $F(x, u) = \{x(x-1)\}$ ,  $u_0 = 0$ ,  $x_0 = 1$ ,  $y_0 = 0 \in F(x_0, u_0)$ , and

$$X(u) = \begin{cases} \{x \in W : 0 < x \leq 1\}, & \text{if } u = 0, \\ \{x \in W : -u < x \leq 1\}, & \text{if } u = \frac{1}{n}, n \in \mathbb{N}, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Then,

$$\tilde{X}(u, y) = \begin{cases} \{1\}, & \text{if } u \in \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}, y = 0, \\ \left\{ \frac{1 - \sqrt{1+4y}}{2}, \frac{1 + \sqrt{1+4y}}{2} \right\}, & \text{if } u \in \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}, \frac{-1}{4} \leq y < 0, \\ \left\{ \frac{1 - \sqrt{1+4y}}{2} \right\}, & \text{if } u \in \{\frac{1}{n} : n \in \mathbb{N}\}, 0 < y < u(u+1), \\ \emptyset, & \text{otherwise,} \end{cases}$$

and  $H(u)$  is as in Example 4.4.3 with only “ $y \leq u(u+1)$ ” replaced by the strict inequality.

Hence,

$$\tilde{X}(u_0, y_0) = \{x_0\}, V_q^1(\tilde{X}, (u_0, y_0[0]), x_0) = \{0\},$$

$$V^1(X, u_0, x_0) = -\mathbb{R}_+, V_q^1(F, (x_0[x], u_0), y_0) = \{x\}, V^1(H, u_0, y_0) = \mathbb{R}.$$

Consequently, because  $\tilde{X}$  is not calm around  $(u_0, y_0)$ , we really have

$$\bigcup_{x \in V^1(X, u_0, x_0)} V_q^1(F, (x_0[x], u_0), y_0) = -\mathbb{R}_+,$$

$$V^1(H, u_0, y_0) \not\subseteq \bigcup_{x \in V^1(X, u_0, x_0)} V_q^1(F, (x_0[x], u_0), y_0).$$

**Example 4.4.5.**  $(V_q(\tilde{X}, (u_0, y_0[0]), x_0) = \{0\})$  is essential

Let  $U = Y = \mathbb{R}$ ,  $W = \mathbb{R}^2$ ,  $X(u) = \{x \in W : x_1 = u, x_2 = 0\}$ ,  $F(x, u) = \{x_1^2(x_1 - 1)\}$ ,  $u_0 = 0$ ,  $x_0 = (0, 0)$ , and  $y_0 = f(x_0, u_0) = 0$ . Then,

$$\tilde{X}(u, y) = \begin{cases} \{(u, 0)\}, & \text{if } u \in \mathbb{R}, y = u^2(u - 1), \\ \emptyset, & \text{otherwise,} \end{cases}$$

and  $H(u) = \{u^2(u - 1)\}$ . Hence,  $\tilde{X}(u_0, y_0) = \{x_0\}$  and  $\tilde{X}$  is calm around  $(u_0, y_0)$ . Direct calculations give  $V^1(X, u_0, x_0) = \mathbb{R} \times \{0\}$ ,  $V_q^1(F, (x_0[x], u_0), y_0) = \{0\}$ . Therefore,

$$\bigcup_{x \in V^1(X, u_0, x_0)} V_q^1(F, (x_0[x], u_0), y_0) = \{0\}.$$

By taking  $t_n = \frac{1}{n}$ ,  $u_n = \frac{1}{\sqrt{n}}$ ,  $x_n = (u_n, 0) \in X(u_n)$ ,  $v_n = \frac{1}{\sqrt{n}} - 1 \rightarrow -1$ , we can check that  $y_0 + t_n v_n \in H(u_n)$ . Thus,  $-1 \in V^1(H, u_0, y_0)$ . Consequently,

$$V^1(H, u_0, y_0) \not\subseteq \bigcup_{x \in V^1(X, u_0, x_0)} V_q^1(F, (x_0[x], u_0), y_0).$$

To see the cause, let  $t_n = \frac{1}{n}$ ,  $u_n = \frac{1}{n}$ ,  $y_n = \frac{1}{n}(\frac{1}{n} - 1) \rightarrow 0$ ,  $x_n = (1, 0)$  to have that  $x_0 + t_n x_n \in \tilde{X}(u_n, y_0 + t_n y_n)$ , and so  $(1, 0) \in V_q^1(\tilde{X}, (u_0, y_0[0]), x_0)$ .

**Example 4.4.6.** ( $W$  needs be finite dimensional)

Let  $U = Y = \mathbb{R}$ , and  $W = l_1$ , the space of all real sequences  $x = (x^i)_{i \in \mathbb{N}}$  with  $\sum_{i=1}^{\infty} |x^i| < \infty$ . Let

$$X(u) = \begin{cases} \{0\}, & \text{if } u = 0, \\ \{x = (x^i)_{i \in \mathbb{N}} \in W : x^i = u \text{ if } i = n; x^i = 0 \text{ if } i \neq n\}, & \text{if } u = \frac{1}{n}, n \in \mathbb{N}, \\ \emptyset, & \text{otherwise,} \end{cases}$$

$F(x, u) = \{\|x\|(|x| - 1)\}$ ,  $u_0 = 0$ ,  $x_0 = 0 \in X(u_0)$ , and  $y_0 = 0 \in F(x_0, u_0)$ . Then,

$$\tilde{X}(u, y) = \begin{cases} \{0\}, & \text{if } u = 0, \\ \{(x^i)_{i \in \mathbb{N}} \in W : x^i = u \text{ if } i = n; x^i = 0 \text{ if } i \neq n\}, & \text{if } u = \frac{1}{n}, n \in \mathbb{N}, y = |u|(|u| - 1), \\ \emptyset, & \text{otherwise,} \end{cases}$$

and

$$H(u) = \begin{cases} \{0\}, & \text{if } u = 0, \\ \{|u|(|u| - 1)\}, & \text{if } u = \frac{1}{n}, n \in \mathbb{N}, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Hence,  $\tilde{X}$  is calm around  $(u_0, y_0)$ . We can compute directly that

$$\tilde{X}(u_0, y_0) = \{x_0\}, V_q^1(\tilde{X}, (u_0, y_0[0]), x_0) = \{0\},$$

$$V_q^1(F, (x_0[x], u_0), y_0) = \{-\|x\|\}, V^1(X, u_0, x_0) = \{0\}.$$

Therefore,

$$\bigcup_{x \in V^1(X, u_0, x_0)} V_q^1(F, (x_0[x], u_0), y_0) = \{0\}.$$

By taking  $t_n = \frac{1}{n}$ ,  $u_n = \frac{1}{n}$ ,  $x_n = (x_n^i)_{i \in \mathbb{N}} \in X(u_n)$  satisfying  $x_n^i = u_n$  if  $i = n$  and  $x_n^i = 0$  if  $i \neq n$ , and  $v_n = \frac{1}{n} - 1 \rightarrow -1$ , we can check that  $y_0 + t_n v_n \in H(u_n)$ . Hence,  $-1 \in V^1(H, u_0, y_0)$ . Thus,

$$V^1(H, u_0, y_0) \not\subseteq \bigcup_{x \in V^1(X, u_0, x_0)} V_q^1(F, (x_0[x], u_0), y_0).$$

Finally, invoking to Proposition 4.4.2 and results of Section 4.3, we easily establish relations between the set of efficient points and the set of weak efficient points of the mentioned variational sets stated in the following theorems.

**Theorem 4.4.7.** *Let  $(u_0, y_0) \in \text{gr}G$ ,  $x_0 \in X(u_0)$ ,  $y_0 \in F(x_0, u_0)$ ,  $W$  be finite dimensional, and  $C$  have a compact base. Suppose that*

(i)  $H$  has the domination property around  $u_0$ ,

(ii) either of the following two conditions holds:

(ii<sub>1</sub>)  $V^m(H + C, u_0, y_0, v_1, \dots, v_{m-1})$  has the domination property,

(ii<sub>2</sub>)  $V^{\infty(m)}(H, x_0, y_0, v_1, \dots, v_{m-1}) \cap (-C) = \{0\}$ ,

(iii)  $F$  has a  $m$ -th order proto variation at  $((x_0, u_0), y_0)$ ,

(iv)  $\tilde{X}$  is calm around  $(u_0, y_0)$ ,

(v)  $\tilde{X}(u_0, y_0) = \{x_0\}$  and  $V_q^1(\tilde{X}, (u_0, y_0[0]), x_0) = \{0\}$ .

Then

$$\text{Min}_{C \setminus \{0\}} \left( \bigcup_{x \in V^m(X, u_0, x_0, w_1, \dots, w_{m-1})} V_q^m(F, (x_0[x], u_0), y_0, w_1, v_1, \dots, w_{m-1}, v_{m-1}) \right) = \\ \text{Min}_{C \setminus \{0\}} V^m(G, x_0, y_0, v_1, \dots, v_{m-1}).$$

*Proof.* This follows from Theorem 4.3.2 and Proposition 4.4.2.  $\square$

**Theorem 4.4.8.** Let  $(u_0, y_0) \in \text{gr} S$ ,  $x_0 \in X(u_0)$ ,  $y_0 \in F(x_0, u_0)$ ,  $W$  be finite dimensional, and  $\hat{C}$  be a closed convex cone contained in  $\text{int} C \cup \{0\}$ , and have a compact base. Suppose that

(i)  $Y$  has the weak domination property around  $u_0$  with respect to  $\hat{C}$ ,

(ii) either of the following two conditions is satisfied:

(ii<sub>1</sub>)  $V^m(H + \hat{C}, u_0, y_0, v_1, \dots, v_{m-1})$  has the weak domination property with respect to  $\hat{C}$ ,

(ii<sub>2</sub>)  $V^{\infty(m)}(H, x_0, y_0, v_1, \dots, v_{m-1}) \cap (-\hat{C}) = \{0\}$ ,

(iii)  $F$  has a  $m$ -th order proto variation at  $((x_0, u_0), y_0)$ ,

(iv)  $\tilde{X}$  is calm around  $((u_0, y_0), x_0)$ ,

(v)  $\tilde{X}(u_0, y_0) = \{x_0\}$  and  $V_q^1(\tilde{X}, (u_0, y_0[0]), x_0) = \{0\}$ .

Then

$$\text{Min}_{\text{int} C} \left( \bigcup_{x \in V^m(X, u_0, x_0, w_1, \dots, w_{m-1})} V_q^m(F, (x_0[x], u_0), y_0, w_1, v_1, \dots, w_{m-1}, v_{m-1}) \right) = \\ \text{Min}_{\text{int} C} V^m(S, x_0, y_0, v_1, \dots, v_{m-1}).$$

*Proof.* Theorem 4.3.4(i) and Proposition 4.4.2 together imply this theorem.  $\square$

**Theorem 4.4.9.** Let  $(u_0, y_0) \in \text{gr} S$ , the assumptions of Theorem 4.4.8 be satisfied, and  $H$  have a proto-variational set of order  $m$  of type I at  $(u_0, y_0)$ . Then

$$V^m(S, u_0, y_0, v_1, \dots, v_{m-1}) = \\ \text{Min}_{\text{int} C} \left( \bigcup_{x \in V^m(X, u_0, x_0, w_1, \dots, w_{m-1})} V_q^m(F, (x_0[x], u_0), y_0, w_1, v_1, \dots, w_{m-1}, v_{m-1}) \right).$$

*Proof.* Applying Theorem 4.3.10 and Proposition 4.4.2, we are done.  $\square$

**Remark 4.4.10.** Though there have been several contributions to analysis of perturbation map  $G$  and weak perturbation map  $S$  for unconstrained feasible map  $F$  (defined in Section 4.3), we see only Tanino in [162] dealing with this topic for a map  $F$  in a set-constrained smooth single-valued problem. That paper was limited to first-order results in terms of gradients of  $F$ . The present section is the first attempt of higher-order considerations of  $F$  for a set-constrained nonsmooth multivalued problem. The extension has been performed in several aspects. Furthermore, we have extended successfully almost directly Theorem 4.1 of Tanino in [162]. However, a drawback here is that the results are technically complicated. We hope that, excluding inevitable complexity, e.g., with higher-order derivatives (at least because of long expressions) and a high level of nonsmoothness, improvements can be obtained in future. In this section, we restrict ourselves to making sure that the relatively complicated assumptions imposed in the results cannot be avoided by showing (in examples) their essentialness.



# Chapter 5

## Radial sets, radial derivatives and applications to optimality conditions for vector optimization problems

### 5.1 Introduction

On optimality conditions for nonsmooth problems, to meet the increasing diversity of practical situations, a broad spectrum of generalized derivatives has been developed to replace the Fréchet and Gâteaux derivatives. Each of them is suitable for several models, and none is universal. Note that the wide range of methods in nonsmooth optimization can be roughly separated into the primal and the dual space approaches. Almost notions of generalized derivatives in the primal space approach are based on corresponding tangency concepts and hence carry only local information. In other words, such derivatives are local linear approximations of a considered map.

Until now, only few concepts of such derivatives have been extended to orders greater than two, which are naturally understood to be inevitable for higher-order optimality conditions, such as contingent, adjacent and Clarke derivatives (see [26, 59, 67, 80, 87, 95, 99, 103, 115, 118, 119, 125], variational sets (see [7, 105, 106]), etc. (The derivatives constructed in the dual space approach are hardly extended to orders greater than two.) The radial derivative, introduced by Taa in [160], is in the first approach, but encompasses the idea of a conical hull, and hence contains global information of a map as its conical hull, and is closed and can be used to obtain optimality conditions for global solutions without convexity assumptions.

We recall that, for a subset  $S$  of a normed space  $X$  and  $x_0 \in \text{cl}S$ , the radial cone of  $S$  at  $x_0$  is

defined by Taa in [160] as follows

$$R_S(x_0) := \{u \in X : \exists t_n > 0, \exists u_n \rightarrow u, \forall n, x_0 + t_n u_n \in S\}.$$

For  $F : X \rightarrow 2^Y$ ,  $X, Y$  be normed spaces,  $(x_0, y_0) \in \text{gr}F$  and  $u \in X$ , the first-order radial derivative of  $F$  at  $(x_0, y_0)$  is defined in [160] by  $\text{gr}(D_R F(x_0, y_0)) = R_{\text{gr}F}(x_0, y_0)$ . A kind of higher-order radial derivatives was proposed by Anh et al. in [8] as follows. The  $m$ -th order outer radial derivative of  $F$  at  $(x_0, y_0) \in \text{gr}F$  is

$$\overline{D}_R^m F(x_0, y_0)(u) := \{v \in Y : \exists t_n > 0, \exists (u_n, v_n) \rightarrow (u, v), \forall n, y_0 + t_n^m v_n \in F(x_0 + t_n u_n)\}, \quad (5.1)$$

the  $m$ -th order inner radial derivative of  $F$  at  $(x_0, y_0) \in \text{gr}F$  is

$$\overline{D}_R^{\flat(m)} F(x_0, y_0)(u) := \{v \in Y : \forall t_n > 0, \exists (u_n, v_n) \rightarrow (u, v), \forall n, y_0 + t_n^m v_n \in F(x_0 + t_n u_n)\}.$$

Observe that the graph of the above higher-order radial derivatives is not a higher-order tangent set of the graph of the map, because the rates of change of the points under consideration in  $X$  and  $Y$  are different ( $t_n$  and  $t_n^m$ ). The graph of many other higher-order derivatives is such a corresponding graph. For instance, for  $F : X \rightarrow 2^Y$ ,  $(x_0, y_0) \in \text{gr}F$  and  $(u_i, v_i) \in X \times Y$ ,  $i = 1, \dots, m-1$ , recall that the  $m$ -th order contingent derivative of  $F$  at  $(x_0, y_0)$  with respect to  $(u_1, v_1), \dots, (u_{m-1}, v_{m-1})$  is

$$D^m F(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1})(u) := \{v \in Y : \exists t_n \rightarrow 0^+, \exists (u_n, v_n) \rightarrow (u, v), \\ \forall n, y_0 + t_n v_1 + \dots + t_n^m v_n \in F(x_0 + t_n u_1 + \dots + t_n^m u_n)\}.$$

Its graph is just the  $m$ -th order contingent set of the graph of  $F$ . In some sense, this property has a better geometry and is more natural.

The discussion above motivates the aim of this chapter: to define another kind of higher-order radial derivatives based on (higher-order) radial sets and use it to obtain higher-order optimality conditions for set-valued vector optimization. It turns out that, in general this kind of radial derivatives is incomparable with our previous definitions in [8], but it provides a tool for establishing new optimality conditions, which also sharpen or improve a number of the existing results in the literature. Note further that the obtained optimality conditions have global characters and do not need any convexity assumptions. The content of this chapter is also our research published in [2, 4].

## 5.2 Radial sets and radial derivatives

Definitions of higher-order radial sets and corresponding derivatives are introduced in this section, followed by their properties and basic calculus rules like those for a sum or composition of mappings.

### 5.2.1 Definitions and properties

**Definition 5.2.1.** Let  $X$  be a normed space,  $x_0 \in S \subseteq X$ , and  $u_1, \dots, u_{m-1} \in X$  with  $m \geq 1$ .

(i) The  $m$ -th order *upper radial set* of  $S$  at  $x_0$  with respect to  $u_1, \dots, u_{m-1}$  is defined as

$$T_S^{r(m)}(x_0, u_1, \dots, u_{m-1}) := \{y \in X : \exists t_n > 0, \exists y_n \rightarrow y, \forall n, x_0 + t_n u_1 + \dots + t_n^{m-1} u_{m-1} + t_n^m y_n \in S\}.$$

(ii) The  $m$ -th order *lower radial set* of  $S$  at  $x_0$  with respect to  $u_1, \dots, u_{m-1}$  is

$$T_S^{rb(m)}(x_0, u_1, \dots, u_{m-1}) := \{y \in X : \forall t_n > 0, \exists y_n \rightarrow y, \forall n, x_0 + t_n u_1 + \dots + t_n^{m-1} u_{m-1} + t_n^m y_n \in S\}.$$

**Definition 5.2.2.** Let  $X, Y$  be normed spaces,  $F : X \rightarrow 2^Y$ ,  $(x_0, y_0) \in \text{gr} F$ , and  $(u_i, v_i) \in X \times Y$ ,  $i = 1, \dots, m-1$  with  $m \geq 1$ .

(i) The  $m$ -th order *upper radial derivative* of  $F$  at  $(x_0, y_0)$  with respect to  $(u_1, v_1), \dots, (u_{m-1}, v_{m-1})$  is the multimap  $D_R^m F(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1}) : X \rightarrow 2^Y$  whose graph is

$$\text{gr} D_R^m F(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1}) := T_{\text{gr} F}^{r(m)}(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1}).$$

(ii) The  $m$ -th order *lower radial derivative* of  $F$  at  $(x_0, y_0)$  with respect to  $(u_1, v_1), \dots, (u_{m-1}, v_{m-1})$  is the multimap  $D_R^{b(m)} F(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1}) : X \rightarrow 2^Y$  with

$$\text{gr} D_R^{b(m)} F(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1}) := T_{\text{gr} F}^{rb(m)}(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1}).$$

We easily obtain the following formulae, for  $x \in X$ ,

$$D_R^m F(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1})(x) = \{v \in Y : \exists t_n > 0, \exists x_n \rightarrow x, \exists v_n \rightarrow v, \forall n,$$

$$= y_0 + t_n v_1 + \dots + t_n^{m-1} v_{m-1} + t_n^m v_n \in F(y_0 + t_n u_1 + \dots + t_n^{m-1} u_{m-1} + t_n^m x_n)\}$$

$$D_R^{b(m)} F(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1})(x) = \{v \in Y : \forall t_n > 0, \exists x_n \rightarrow x, \exists v_n \rightarrow v, \forall n,$$

$$y_0 + t_n v_1 + \dots + t_n^{m-1} v_{m-1} + t_n^m v_n \in F(y_0 + t_n u_1 + \dots + t_n^{m-1} u_{m-1} + t_n^m x_n)\}.$$

**Remark 5.2.3.** (i) It follows that

$$D_R^m F(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1})(X) \subseteq T_{F(X)}^{r(m)}(y_0, v_1, \dots, v_{m-1}).$$

(ii) Radial sets are especial cases of  $\Gamma$ -limits. Indeed,

$$y \in T_S^{r(m)}(x_0, u_1, \dots, u_{m-1}) \iff \inf_{Q \in \mathcal{N}(y)} \sup_{t > 0} \sup_{y' \in Q} \chi_{\text{gr}(\mathcal{H}_{S, x_0, u_1, \dots, u_{m-1}})}(t, y') = 1$$

$$\iff \Gamma(\mathcal{R}^+, \mathcal{N}(y)^+) \lim \chi_{\text{gr}(\mathcal{H}_{S, x_0, u_1, \dots, u_{m-1}})} = 1,$$

$$y \in T_S^{rb(m)}(x_0, u_1, \dots, u_{m-1}) \iff \inf_{Q \in \mathcal{N}(y)} \inf_{t > 0} \sup_{y' \in Q} \chi_{\text{gr}(\mathcal{H}_{S, x_0, u_1, \dots, u_{m-1}})}(t, y') = 1$$

$$\iff \Gamma(\mathcal{R}^-, \mathcal{N}(y)^+) \lim \chi_{\text{gr}(\mathcal{H}_{S, x_0, u_1, \dots, u_{m-1}})} = 1,$$

where  $\mathcal{R} := \{(0, +\infty)\}$  be a filter on  $(0, +\infty)$ , and  $\mathcal{H}_{S, x_0, u_1, \dots, u_{m-1}} : (0, +\infty) \rightarrow 2^X$  is defined by

$$\mathcal{H}_{S, x_0, u_1, \dots, u_{m-1}}(t) := \frac{1}{t^m} (S - x_0 - tu_1 - \dots - t^{m-1}u_{m-1}).$$

(iii) Radial derivatives can be expressed in terms of  $\Gamma$ -limits as follows

$$y \in D_R^m F(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1})(x)$$

$$\iff \inf_{Q \in \mathcal{N}(y)} \inf_{W \in \mathcal{N}(x)} \sup_{t > 0} \sup_{x' \in W} \sup_{y' \in Q} \chi_{\text{gr}(\mathcal{H}_{F, (x_0, y_0), (u_1, v_1), \dots, (u_{m-1}, v_{m-1})})}(t, x', y') = 1$$

$$\iff \Gamma(\mathcal{R}^+, \mathcal{N}(x)^+, \mathcal{N}(y)^+) \lim \chi_{\text{gr}(\mathcal{H}_{F, (x_0, y_0), (u_1, v_1), \dots, (u_{m-1}, v_{m-1})})} = 1,$$

$$y \in D_R^{b(m)} F(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1})(x)$$

$$\iff \inf_{Q \in \mathcal{N}(y)} \inf_{W \in \mathcal{N}(x)} \inf_{t > 0} \sup_{x' \in W} \sup_{y' \in Q} \chi_{\text{gr}(\mathcal{H}_{F, (x_0, y_0), (u_1, v_1), \dots, (u_{m-1}, v_{m-1})})}(t, x', y') = 1$$

$$\iff \Gamma(\mathcal{R}^-, \mathcal{N}(x)^+, \mathcal{N}(y)^+) \lim \chi_{\text{gr}(\mathcal{H}_{F, (x_0, y_0), (u_1, v_1), \dots, (u_{m-1}, v_{m-1})})} = 1,$$

where  $\mathcal{H}_{F, (x_0, y_0), (u_1, v_1), \dots, (u_{m-1}, v_{m-1})} : (0, +\infty) \times X \rightarrow 2^Y$  is defined by

$$\mathcal{H}_{F, (x_0, y_0), (u_1, v_1), \dots, (u_{m-1}, v_{m-1})}(t, x') := \frac{1}{t^m} (F(x_0 + tu_1 + \dots + t^{m-1}u_{m-1} + t^m x') - y_0 - tv_1 - \dots - t^{m-1}v_{m-1}).$$

(iv) Definitions 5.2.1, 5.2.2 correspond to the following known definition of the contingent objects

$$T_S^m(x_0, u_1, \dots, u_{m-1}) := \{y \in X : \exists t_n \rightarrow 0, \exists y_n \rightarrow y, \forall n, x_0 + t_n u_1 + \dots + t_n^{m-1} u_{m-1} + t_n^m y_n \in S\},$$

$$\text{gr} D^m F(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1}) := T_{\text{gr} F}^m(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1}).$$

Observe another fact, which makes the radial set and derivative different from the contingent set and derivative, and hence also from other tangency notions and derivatives in variational analysis. Let us explain this difference only between  $T_S^{r(2)}(x_0, u)$  and  $T_S^2(x_0, u)$  for simplicity. It is known that if  $u \notin T_S(x_0)$ , then  $T_S^2(x_0, u) = \emptyset$ . But, the simple example with  $X = \mathbb{R}^2$ ,  $S = \{0_{\mathbb{R}^2}, e_1\}$ ,  $x_0 = 0_{\mathbb{R}^2}$ ,  $e_1 = (1, 0)$ ,  $e_2 = (0, 1)$  shows that this is not valid for radial cones:  $u = e_1 - e_2 \notin T_S^r(x_0)$ , but  $e_2 \in T_S^{r(2)}(x_0, u)$ , i.e., the last cone is nonempty.

(v) The objects defined in Definitions 5.2.1, 5.2.2 are called upper and lower radial sets and derivatives, respectively. However, though being applied to establish necessary optimality conditions similarly as the upper radial concept is, this lower radial object yields weaker results, and it is not convenient for dealing with sufficient optimality conditions. Therefore, in this chapter we develop only the upper radial concepts and thus omit the term “upper”.

In the following example, we compute a radial derivative and a contingent derivative in an infinite dimensional case.

**Example 5.2.4.** Let  $X = \mathbb{R}$  and  $Y = l^2$ , the Hilbert space of the numerical sequences  $x = (x_i)_{i \in \mathbb{N}}$  with  $\sum_{i=1}^{\infty} x_i^2$  being convergent. By  $(e_i)_{i \in \mathbb{N}}$  we denote standard unit basis of  $l^2$ . Consider  $F : X \rightarrow 2^Y$  defined by

$$F(x) := \begin{cases} \left\{ \frac{1}{n}(-e_1 + 2e_n) \right\}, & \text{if } x = \frac{1}{n}, \\ \{y = (y_i)_{i=1}^{\infty} \in l^2 : y_1 \geq 0, y_1^2 \geq \sum_{i=2}^{\infty} y_i^2\}, & \text{if } x = n, \\ \{0\}, & \text{otherwise.} \end{cases}$$

It is easy to see that  $C := \{y = (y_i)_{i=1}^{\infty} \in l^2 : y_1 \geq 0, y_1^2 \geq \sum_{i=2}^{\infty} y_i^2\}$  is a closed, convex, and pointed cone. For  $(x_0, y_0) = (0, 0)$ , we compute  $T_{\text{gr}F}^{r(1)}(x_0, y_0)$ . If  $(u, v) \in T_{\text{gr}F}^{r(1)}(x_0, y_0)$ , by definition, there exist  $t_n > 0$  and  $(u_n, v_n) \rightarrow (u, v)$  such that

$$t_n v_n \in F(t_n u_n). \quad (5.2)$$

If  $t_n u_n \notin \{1/n, n\}$ , then from (5.2) a direct computation gives  $v = 0$ . Now assume that  $t_n u_n \in \{1/n, n\}$ . Consider the first case with  $t_n u_n = 1/n$ . From (5.2), one has

$$t_n v_n = \frac{1}{n}(-e_1 + 2e_n). \quad (5.3)$$

We have two subcases. If  $u = 0$ , then  $u_n = 1/(nt_n) \rightarrow 0$ , and hence (5.3) implies that  $v_n = u_n(-e_1 + 2e_n) \rightarrow 0$ , i.e.,  $v = 0$ . In the second subcase with  $u > 0$ , we claim that there is no  $v$  such

that  $(u, v) \in T_{\text{gr}F}^{r(1)}(x_0, y_0)$ . Suppose there exists such a  $(u, v)$  (with  $u > 0$ ). Then, from (5.3), the sequence  $(nt_n)^{-1}(-e_1 + 2e_n) = u_n(-e_1 + 2e_n)$  converges. Hence, as  $u_n e_1 \rightarrow u e_1$ , the sequence  $d_n := 2u_n e_n$  is also convergent. Suppose that  $d_n$  converges to  $d$ . By a direct computation, one has

$$\|2u_n e_n - d\|^2 = (2u_n)^2 + \|d\|^2 + 2\langle 2u_n e_n, -d \rangle \rightarrow 0.$$

Therefore,  $4(u)^2 + \|d\|^2 = 0$  (since  $\{e_n\}$  converges weakly to 0,  $\langle e_n, d \rangle \rightarrow 0$ ), which is a contradiction.

Now consider the second case with  $t_n u_n = n$ . From (5.2) we get  $t_n v_n \in C$ . Thus,  $v \in C$  since  $C$  is a closed cone. It follows from  $u_n = n/t_n$  that  $u \geq 0$ . Consequently, we have proved that

$$T_{\text{gr}F}^{r(1)}(x_0, y_0) \subseteq ([0, +\infty) \times C) \cup ((-\infty, 0) \times \{0\}).$$

We now show the reverse inclusion. Let  $(u, v) \in ([0, +\infty) \times C) \cup ((-\infty, 0) \times \{0\})$ . We prove that there exist  $t_n > 0$ ,  $u_n \rightarrow u$ , and  $v_n \rightarrow v$  such that (5.2) holds for all  $n$ . Indeed, depending on  $u$  and  $v$ , such  $t_n$ ,  $u_n$ , and  $v_n$  can be chosen as follows.

- For  $(0, v)$  such that  $v \in C$ , we take  $t_n = n^2$ ,  $u_n = 1/n$ ,  $v_n \equiv v$ .
- For  $(u, v) \in (0, +\infty) \times C$ , we take  $t_n = n/u$ ,  $u_n \equiv u$ ,  $v_n \equiv v$ .
- For  $(u, v) \in (-\infty, 0) \times \{0\}$ , we take  $t_n = n/|u|$ ,  $u_n \equiv u$ ,  $v_n \equiv 0$ .

So,

$$T_{\text{gr}F}^{r(1)}(x_0, y_0) = ([0, +\infty) \times C) \cup ((-\infty, 0) \times \{0\}).$$

Therefore,

$$D_R^1 F(x_0, y_0)(u) = \begin{cases} C, & \text{if } u \geq 0, \\ \{0\}, & \text{if } u < 0. \end{cases}$$

By a similar way, with simpler calculations, we get  $T_{\text{gr}F}^1(x_0, y_0) = \mathbb{R} \times \{0\}$ , and hence for all  $u \in \mathbb{R}$ ,  $D^1 F(x_0, y_0)(u) = \{0\}$ .

The next example highlights detailed differences between (5.1) and the radial derivative introduced in Definition 5.2.2(i).

**Example 5.2.5.** Let  $X = Y = \mathbb{R}$  and  $F(x) = \{x^2\}$  and  $(x_0, y_0) = (0, 0)$ . Direct calculations yield

$$\overline{D}_R^1 F(x_0, y_0)(x) = D_R^1 F(x_0, y_0)(x) = \mathbb{R}_+.$$

Without any information, we have  $\overline{D}_R^2 F(x_0, y_0)(x) = x^2$ . Now let  $(u_1, v_1) = (0, 0)$  be given, then  $D_R^2 F(x_0, y_0, u_1, v_1)(x) = \mathbb{R}_+$ . For another given direction  $(u_1, v_1) = (1, 0)$ ,  $D_R^2 F(x_0, y_0, u_1, v_1)(x) = \{1 + a^2 x^2 + 2ax : a \geq 0\}$ .

**Definition 5.2.6.** Let  $F : X \rightarrow 2^Y$ ,  $(x_0, y_0) \in \text{gr } F$  and  $(u_i, v_i) \in X \times Y$ ,  $i = 1, \dots, m-1$  with  $m \geq 1$ .

(i) If  $T_{F(X)}^{r(m)}(y_0, v_1, \dots, v_{m-1}) = T_{F(X)}^{rb(m)}(y_0, v_1, \dots, v_{m-1})$ , then this set is called a  $m$ -th order proto-radial set of  $F(X)$  at  $y_0$  with respect to  $v_1, \dots, v_{m-1}$ .

(ii) If  $D_R^m F(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1})(x) = \{v \in Y : \forall t_n > 0, \forall x_n \rightarrow x, \exists v_n \rightarrow v : y_0 + t_n v_1 + \dots + t_n^{m-1} v_{m-1} + t_n^m v_n \in F(y_0 + t_n u_1 + \dots + t_n^{m-1} u_{m-1} + t_n^m x_n), \forall n\}$ , for any  $x \in \text{dom } D_R^m F(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1})$ , then this derivative is called a  $m$ -th order radial semiderivative of  $F$  at  $(x_0, y_0)$  with respect to  $(u_1, v_1), \dots, (u_{m-1}, v_{m-1})$ .

Note that, following strictly Definition 5.2.2, we would define that  $D_R^m F(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1})$  is a  $m$ -th order radial semiderivative if

$$\text{gr } D_R^m F(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1}) = T_{\text{gr } F}^{rb(m)}(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1}).$$

But, this last condition is equivalent to

$$\begin{aligned} D_R^m F(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1})(x) &= \{v \in Y : \forall t_n > 0, \exists x_n \rightarrow x, \exists v_n \rightarrow v, \forall n, y_0 + t_n v_1 + \dots \\ &\quad + t_n^{m-1} v_{m-1} + t_n^m v_n \in F(x_0 + t_n u_1 + \dots + t_n^{m-1} u_{m-1} + t_n^m x_n)\}, \end{aligned}$$

which is weaker than Definition 5.2.6(ii). (This weaker condition was used to define proto-contingent derivatives in many papers in the literature.) Definition 5.2.6 is restrictive. However, it may be satisfied as shown in the following.

**Example 5.2.7.** Let  $X = Y = \mathbb{R}$ ,  $F : X \rightarrow 2^Y$  be defined by  $F(x) = \{y \in Y : y \geq x\}$ , and  $(x_0, y_0) = (0, 0)$ . Then, direct calculations yield  $T_{F(X)}^{r(1)}(y_0) = T_{F(X)}^{rb(1)}(y_0) = \mathbb{R}$  and

$$\begin{aligned} D_R^1 F(x_0, y_0)(x) &= \{v \in Y : \forall t_n > 0, \forall x_n \rightarrow x, \exists v_n \rightarrow v, \forall n, y_0 + t_n v_n \in F(x_0 + t_n x_n)\} \\ &= \{v \in Y : v \geq x\}. \end{aligned}$$

So,  $T_{F(X)}^{r(1)}(y_0)$  is a first-order proto-radial set of  $F(X)$  at  $y_0$ , and  $D_R^1 F(x_0, y_0)$  is a first-order radial semiderivative of  $F$  at  $(x_0, y_0)$ .

For examples of second-orders, with any  $v_1 \in \mathbb{R}$ , direct computations show that

$$T_{F(X)}^{r(2)}(y_0, v_1) = T_{F(X)}^{rb(2)}(y_0, v_1) = \mathbb{R}.$$

Thus,  $T_{F(X)}^{r(2)}(y_0, v_1)$  is a second-order proto-radial set of  $F(X)$  at  $y_0$  with respect to  $v_1 \in \mathbb{R}$ .

Passing to derivatives, let  $(u_1, v_1) = (1, 1)$ . Direct computations indicate that both  $D_R^2 F(x_0, y_0, u_1, v_1)(x)$  and the set on the right-hand side of the equality in Definition 5.2.6(ii) are equal to

$\{v \in Y : v \geq x\}$ . Thus,  $F$  has a second-order radial semiderivative at  $(x_0, y_0)$  with respect to  $(1, 1)$ . However, for  $(u_1, v_1) = (0, -1)$ , the derivatives are “worse”:

$$D_R^2 F(x_0, y_0, u_1, v_1)(x) = \{v \in Y : v \geq x\}$$

and the other mentioned set is empty. So,  $F$  does not have a second-order radial semiderivative at  $(x_0, y_0)$  with respect to  $(0, -1)$ .

**Proposition 5.2.8.** *If  $S$  is convex, then so is  $T_S^{rb(m)}(x_0, u_1, \dots, u_{m-1})$ .*

*Proof.* If  $T_S^{rb(m)}(x_0, u_1, \dots, u_{m-1}) = \emptyset$  or is a singleton, the result holds trivially. Now assume that there are  $v_1, v_2 \in T_S^{rb(m)}(x_0, u_1, \dots, u_{m-1})$  and  $\lambda \in (0, 1)$ . It follows from the definition that, for any  $t_n > 0$ , there exist sequences  $v_n^1$  and  $v_n^2$  such that  $v_n^1 \rightarrow v^1$ ,  $v_n^2 \rightarrow v^2$  and for  $i = 1, 2$ ,

$$x_0 + t_n u_1 + \dots + t_n^{m-1} u_{m-1} + t_n^m v_n^i \in S.$$

From the convexity of  $S$ , we have

$$x_0 + t_n u_1 + \dots + t_n^{m-1} u_{m-1} + t_n^m (\lambda v_n^1 + (1 - \lambda) v_n^2) \in S.$$

Thus,  $\lambda v_1 + (1 - \lambda) v_2 \in T_S^{rb(m)}(x_0, u_1, \dots, u_{m-1})$  and the proof is complete.  $\square$

**Proposition 5.2.9.** *If  $S$  is convex and  $u_1, \dots, u_{m-1} \in S$ , then*

$$\begin{aligned} T_S^{rb(m)}(x_0, u_1 - x_0, \dots, u_{m-1} - x_0) &\subseteq T_S^{b(m)}(x_0, u_1 - x_0, \dots, u_{m-1} - x_0) = \\ &= T_S^m(x_0, u_1 - x_0, \dots, u_{m-1} - x_0) = T_S^{r(m)}(x_0, u_1 - x_0, \dots, u_{m-1} - x_0) \end{aligned}$$

*Proof.* From the definitions, we have

$$T_S^{rb(m)}(x_0, u_1 - x_0, \dots, u_{m-1} - x_0) \subseteq T_S^{b(m)}(x_0, u_1 - x_0, \dots, u_{m-1} - x_0)$$

and

$$T_S^{r(m)}(x_0, u_1 - x_0, \dots, u_{m-1} - x_0) = \text{cl} \left( \bigcup_{t>0} \frac{S - x_0 - t(u_1 - x_0) - \dots - t^{m-1}(u_{m-1} - x_0)}{t^m} \right).$$

Since  $S$  is convex, Proposition 3.1 of [118] says that the right side of the last equality is equal to  $T_S^m(x_0, u_1 - x_0, \dots, u_{m-1} - x_0) = T_S^{b(m)}(x_0, u_1 - x_0, \dots, u_{m-1} - x_0)$  and we are done  $\square$



The inclusion in Proposition 5.2.9 may be strict as for  $S = [0, 1]$  and  $x_0 = 0$ , since we have  $T_S^{rb(1)}(x_0) = \{0\}$  and  $T_S^{b(1)}(x_0) = T_S^1(x_0) = T_S^{r(1)}(x_0) = \mathbb{R}_+$ .

It is known (Corollary 3.1 of [118]) that if  $S$  is convex and  $u_1, \dots, u_{m-1} \in S$ , then  $T_S^{b(m)}(x_0, u_1 - x_0, \dots, u_{m-1} - x_0)$  is convex. Therefore, Proposition 5.2.9 implies the following.

**Corollary 5.2.10.** *If  $S$  is convex and  $u_1, \dots, u_{m-1} \in S$ , then  $T_S^{r(m)}(x_0, u_1 - x_0, \dots, u_{m-1} - x_0)$  is convex.*

**Proposition 5.2.11.** *Let  $S = \text{dom } F$  and  $(x_0, y_0) \in \text{gr } F$ . Then, for all  $x \in S$ ,*

- (i)  $F(x) - y_0 \subseteq D_R^1 F(x_0, y_0)(x - x_0)$ ,
- (ii)  $F(x) - y_0 \subseteq T_{F(S)}^{r(1)}(y_0)$ .

*Proof.* Let  $x \in S$ ,  $y \in F(x) - y_0$ , then  $y_0 + y \in F(x)$ . Therefore, there exist  $t_n = 1, y_n = y$  and  $x_n = x - x_0$ , for all  $n$ , such that  $y_0 + t_n^m y_n \in F(x_0 + t_n x_n)$  for all  $n$ . Hence,  $y \in D_R^1 F(x_0, y_0)(x - x_0)$ .

(ii) Let  $x \in S$ ,  $y \in F(x) - y_0$ , then  $y_0 + y \in F(x)$ . Therefore, there exist  $t_n = 1, y_n = y$  and  $x_n = x - x_0$ ,  $\forall n$  such that  $y_0 + t_n^m y_n \in F(x_0 + t_n x_n) \subseteq F(S)$ . So,  $y \in T_{F(S)}^{r(1)}(y_0)$ .  $\square$

Note that these assertions say, in particular, that radial sets and derivatives possess global properties without any (relaxed) convexity assumption. To make this clear, recall that  $F : X \rightarrow 2^Y$  is termed pseudoconvex at  $(x_0, y_0) \in \text{gr } F$  if  $\text{epi } F - (x_0, y_0) \subseteq T_{\text{epi } F}(x_0, y_0)$ . Furthermore, if  $F$  is pseudoconvex at  $(x_0, y_0)$ , then,  $\forall x \in \text{dom } F$ ,  $F(x) - y_0 \subseteq V^1(F_+, x_0, y_0)$  (see Proposition 2.1 of Khanh and Tuan in [105]). Roughly speaking, that is why, in the sufficient condition (see Theorem 5.3.7), no convexity assumption is needed.

## 5.2.2 Sum rule and chain rule

**Proposition 5.2.12.** (Sum rule) *Let  $F_i : X \rightarrow 2^Y$ ,  $x_0 \in \Omega := \text{dom } F_1 \cap \text{dom } F_2$ ,  $y_i \in F_i(x_0)$  for  $i = 1, 2$ .*

(i) *If either  $F_1(\Omega)$  or  $F_2(\Omega)$  has a  $m$ -th order proto-radial set at  $y_1$  with respect to  $v_{1,1}, \dots, v_{1,m-1}$  or at  $y_2$  with respect to  $v_{2,1}, \dots, v_{2,m-1}$ , respectively, then*

$$\begin{aligned} & T_{F_1(\Omega)}^{r(m)}(y_1, v_{1,1}, \dots, v_{1,m-1}) + T_{F_2(\Omega)}^{r(m)}(y_2, v_{2,1}, \dots, v_{2,m-1}) \\ & \subseteq T_{(F_1+F_2)(\Omega)}^{r(m)}(y_1 + y_2, v_{1,1} + v_{2,1}, \dots, v_{1,m-1} + v_{2,m-1}). \end{aligned}$$

(ii) If either  $F_1$  or  $F_2$  has a  $m$ -th order radial semiderivative at  $(x_0, y_1)$  with respect to  $(u_1, v_{1,1}), \dots, (u_{m-1}, v_{1,m-1})$  or at  $(x_0, y_2)$  with respect to  $(u_1, v_{2,1}), \dots, (u_{m-1}, v_{2,m-1})$ , respectively, then for any  $u \in X$ ,

$$\begin{aligned} & D_R^m F_1(x_0, y_1, u_1, v_{1,1}, \dots, u_{m-1}, v_{1,m-1})(u) + D_R^m F_2(x_0, y_2, u_1, v_{2,1}, \dots, u_{m-1}, v_{2,m-1})(u) \\ & \subseteq D_R^m (F_1 + F_2)(x_0, y_1 + y_2, u_1, v_{1,1} + v_{2,1}, \dots, u_{m-1}, v_{1,m-1} + v_{2,m-1})(u). \end{aligned}$$

*Proof.* (i) Let  $u_i \in T_{F_i(\Omega)}^{r(m)}(y_i, v_{i,1}, \dots, v_{i,m-1})$  for  $i = 1, 2$ . Then, there exist  $t_n > 0$  and  $u_n^1 \rightarrow u_1$  such that

$$y_1 + t_n v_{1,1} + \dots + t_n^{m-1} v_{1,m-1} + t_n^m u_n^1 \in F_1(\Omega).$$

Suppose that  $F_2(\Omega)$  has a  $m$ -th order proto-radial set at  $y_2$ . Then, with  $t_n$  above, there exists  $u_n^2 \rightarrow u_2$  such that

$$y_2 + t_n v_{2,1} + \dots + t_n^{m-1} v_{2,m-1} + t_n^m u_n^2 \in F_2(\Omega).$$

Thus,

$$(y_1 + y_2) + t_n(v_{1,1} + v_{2,1}) + \dots + t_n^{m-1}(v_{1,m-1} + v_{2,m-1}) + t_n^m(u_n^1 + u_n^2) \in (F_1 + F_2)(\Omega).$$

Hence,  $u_1 + u_2 \in T_{(F_1 + F_2)(\Omega)}^{r(m)}(y_1 + y_2, v_{1,1} + v_{2,1}, \dots, v_{1,m-1} + v_{2,m-1})$ .

(ii) If  $u$  does not belong to the intersection of  $\text{dom } D_R^m F_1(x_0, y_1, u_1, v_{1,1}, \dots, u_{m-1}, v_{1,m-1})$  and  $\text{dom } D_R^m F_2(x_0, y_1, u_1, v_{2,1}, \dots, u_{m-1}, v_{2,m-1})$ , the conclusion is trivial. Suppose that  $u \in \text{dom } D_R^m F_i(x_0, y_i, u_1, v_{i,1}, \dots, u_{m-1}, v_{i,m-1})$  for  $i = 1, 2$  and let  $v_i \in D_R^m F_i(x_0, y_i, u_1, v_{i,1}, \dots, u_{m-1}, v_{i,m-1})(u)$  for  $i = 1, 2$ . Then, there exist  $t_n > 0$  and  $(u_n, v_n^1) \rightarrow (u, v_1)$  such that

$$y_1 + t_n v_{1,1} + \dots + t_n^{m-1} v_{1,m-1} + t_n^m v_n^1 \in F_1(x_0 + t_n u_1 + \dots + t_n^{m-1} u_{m-1} + t_n^m u_n).$$

For  $v_2$ , suppose that  $F_2$  has a  $m$ -th order radial semiderivative at  $(x_0, y_2)$ , with  $t_n, u_n$  above, there exists  $v_n^2 \rightarrow v_2$  such that

$$y_2 + t_n v_{2,1} + \dots + t_n^{m-1} v_{2,m-1} + t_n^m v_n^2 \in F_2(x_0 + t_n u_1 + \dots + t_n^{m-1} u_{m-1} + t_n^m u_n).$$

Thus,

$$\begin{aligned} & (y_1 + y_2) + t_n(v_{1,1} + v_{2,1}) + \dots + t_n^{m-1}(v_{1,m-1} + v_{2,m-1}) + t_n^m(v_n^1 + v_n^2) \\ & \in (F_1 + F_2)(x_0 + t_n u_1 + \dots + t_n^{m-1} u_{m-1} + t_n^m u_n). \end{aligned}$$

Hence,  $v_1 + v_2 \in D_R^m (F_1 + F_2)(x_0, y_1 + y_2, u_1, v_{1,1} + v_{2,1}, \dots, u_{m-1}, v_{1,m-1} + v_{2,m-1})(u)$ .  $\square$

The following example shows that the assumption about proto-radial sets in Proposition 5.2.12 cannot be dropped.

**Example 5.2.13.** Let  $X = Y = \mathbb{R}$ ,  $C = \mathbb{R}_+$ , and  $F_1, F_2 : X \rightarrow 2^Y$  be given by

$$F_1(x) = \begin{cases} \{1\}, & \text{if } x = \frac{1}{n}, n \in \mathbb{N}, \\ \{0\}, & \text{if } x = 0, \end{cases}$$

$$F_2(x) = \begin{cases} \{0\}, & \text{if } x = \frac{1}{n}, n \in \mathbb{N}, \\ \{1\}, & \text{if } x = 0. \end{cases}$$

It is easy to see that  $F_1$  and  $F_2$  have neither first order proto-radial set (of  $F_1(\Omega)$  and  $F_2(\Omega)$ ) nor first order radial semiderivative at  $(0,0)$  and  $(0,1)$ , respectively. We have

$$T_{F_1(\Omega)}^{r(1)}(0) = \mathbb{R}_+, \quad T_{F_2(\Omega)}^{r(1)}(1) = \mathbb{R}_-,$$

where  $\Omega := \text{dom } F_1 = \text{dom } F_2 = \left\{0, \frac{1}{n}\right\}_{n \in \mathbb{N}}$ , and

$$D_R^1 F_1(0,0)(0) = \mathbb{R}_+, \quad D_R^1 F_2(0,1)(0) = \mathbb{R}_-.$$

On the other hand,

$$(F_1 + F_2)(x) = \begin{cases} \{1\}, & \text{if } x = \frac{1}{n}, n \in \mathbb{N}, \\ \{1\}, & \text{if } x = 0. \end{cases}$$

Direct calculations yield

$$T_{(F_1+F_2)(\Omega)}^{r(1)}(1) = \{0\}, \quad D_R^1(F_1 + F_2)(0,1)(0) = \{0\}.$$

Thus,

$$T_{F_1(\Omega)}^{r(1)}(0) + T_{F_2(\Omega)}^{r(1)}(1) \not\subseteq T_{(F_1+F_2)(\Omega)}^{r(1)}(1)$$

and

$$D_R^1 F_1(0,0)(0) + D_R^1 F_2(0,1)(0) \not\subseteq D_R^1(F_1 + F_2)(0,1)(0).$$

**Proposition 5.2.14.** (Chain rule) Let  $G : X \rightarrow 2^Y$ ,  $F : Y \rightarrow 2^Z$  with  $\text{Im } G \subseteq \text{dom } F$ ,  $(x_0, y_0) \in \text{gr } G$ ,  $(y_0, z_0) \in \text{gr } F$  and  $(u_1, v_1, w_1), \dots, (u_{m-1}, v_{m-1}, w_{m-1}) \in X \times Y \times Z$ . Suppose that  $F$  has a  $m$ -th order radial semiderivative at  $(y_0, z_0)$  with respect to  $(v_1, w_1), \dots, (v_{m-1}, w_{m-1})$ . Then

- (i)  $D_R^m F(y_0, z_0, v_1, w_1, \dots, v_{m-1}, w_{m-1})[T_{G(X)}^{r(m)}(y_0, v_1, \dots, v_{m-1})] \subseteq T_{(F \circ G)(X)}^{r(m)}(z_0, w_1, \dots, w_{m-1})$ ,
- (ii)  $D_R^m F(y_0, z_0, v_1, w_1, \dots, v_{m-1}, w_{m-1})[D_R^m G(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1})(X)] \subseteq T_{(F \circ G)(X)}^{r(m)}(z_0, w_1, \dots, w_{m-1})$ .

*Proof.* (i) Let  $z \in D_R^m F(y_0, z_0, v_1, w_1, \dots, v_{m-1}, w_{m-1})[T_{G(X)}^{r(m)}(y_0, v_1, \dots, v_{m-1})]$ . There exists  $v \in T_{G(X)}^{r(m)}(y_0, v_1, \dots, v_{m-1})$  such that  $z \in D_R^m F(y_0, z_0, v_1, w_1, \dots, v_{m-1}, w_{m-1})(v)$ . Hence, there exist  $t_n > 0$  and  $v_n \rightarrow v$  such that

$$y_0 + t_n v_1 + \dots + t_n^{m-1} v_{m-1} + t_n^m v_n \in G(X).$$

Because  $F$  has a  $m$ -th order radial semiderivative of  $F$  at  $(y_0, z_0)$  with respect to  $(v_1, w_1), \dots, (v_{m-1}, w_{m-1})$ , there exists  $z_n \rightarrow z$  such that

$$z_0 + t_n w_1 + \dots + t_n^{m-1} w_{m-1} + t_n^m z_n \in F(y_0 + t_n v_1 + \dots + t_n^{m-1} v_{m-1} + t_n^m v_n) \subseteq F(G(X)).$$

Thus,  $z \in T_{(F \circ G)(X)}^{r(m)}(z_0, w_1, \dots, w_{m-1})$ .

(ii) This follows from (i) and Remark 5.2.3(i).  $\square$

These rules will be applied in the sequel since they are simple (at least their formulations are). However, being a proto-radial set or radial semiderivative is a restrictive condition. Hence, we develop now another sum rule and another chain rule for possible better applications. For a sum  $M + N$  of two multimaps  $M, N : X \rightarrow 2^Y$ , we express it as a composition as follows. Define  $G : X \rightarrow 2^{X \times Y}$  and  $F : X \times Y \rightarrow 2^Y$  by, for the identity map  $I$  on  $X$ ,

$$G = I \times M \quad \text{and} \quad F(x, y) = y + N(x). \quad (5.4)$$

Then, clearly  $M + N = F \circ G$ . So, we will apply a chain rule. The chain rule given in Proposition 5.2.14, though simple and relatively direct, is not suitable for dealing with this composition  $F \circ G$ , since the intermediate space ( $Y$  there and  $X \times Y$  here) is little involved. We develop another chain rule as follows. Let general multimaps  $G : X \rightarrow 2^Y$  and  $F : Y \rightarrow 2^Z$  be considered, where  $X, Y, Z$  be normed spaces. The so-called resultant multimap  $C : X \times Z \rightarrow 2^Y$  is defined by  $C(x, z) := G(x) \cap F^{-1}(z)$ . Then,  $\text{dom } C = \text{gr}(F \circ G)$ .

We can obtain a general chain rule suitable for dealing with a sum expressed as a composition as above, and without assumption about radial semiderivatives, as follows.

**Proposition 5.2.15.** *Let  $(x_0, z_0) \in \text{gr}(F \circ G)$ ,  $y_0 \in C(x_0, z_0)$ , and  $(u_i, v_i, w_i) \in X \times Y \times Z$ .*

(i) *If, for all  $w \in Z$ , one has*

$$\begin{aligned} & T_{G(X)}^{r(m)}(y_0, v_1, \dots, v_{m-1}) \cap D_R^m F^{-1}(z_0, y_0, w_1, v_1, \dots, w_{m-1}, v_{m-1})(w) \\ & \subseteq D_R^m C_X(z_0, y_0, w_1, v_1, \dots, w_{m-1}, v_{m-1})(w), \end{aligned} \quad (5.5)$$

where  $C_X : Z \rightarrow 2^Y$  is defined by  $C_X(z) := C(X, z)$ , then

$$D_R^m F(y_0, z_0, v_1, w_1, \dots, v_{m-1}, w_{m-1})[T_{G(X)}^{r(m)}(y_0, v_1, \dots, v_{m-1})] \subseteq T_{(F \circ G)(X)}^{r(m)}(z_0, w_1, \dots, w_{m-1}).$$

(ii) If, for all  $(u, w) \in X \times Z$ , one has

$$\begin{aligned} & D_R^m G(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1})(u) \cap D_R^m F^{-1}(z_0, y_0, w_1, v_1, \dots, w_{m-1}, v_{m-1})(w) \\ & \subseteq D_R^m C((x_0, z_0), y_0, (u_1, w_1), v_1, \dots, (u_{m-1}, w_{m-1}), v_{m-1})(u, w), \end{aligned} \quad (5.6)$$

then

$$\begin{aligned} & D_R^m F(y_0, z_0, v_1, w_1, \dots, v_{m-1}, w_{m-1})[D_R^m G(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1})(u)] \\ & \subseteq D_R^m (F \circ G)(x_0, z_0, u_1, w_1, \dots, u_{m-1}, w_{m-1})(u). \end{aligned}$$

*Proof.* By the similarity, we prove only (i). Let  $w \in D_R^m F(y_0, z_0, v_1, w_1, \dots, v_{m-1}, w_{m-1})[T_{G(X)}^{r(m)}(y_0, v_1, \dots, v_{m-1})]$ , i.e., there exists some  $y \in T_{G(X)}^{r(m)}(y_0, v_1, \dots, v_{m-1})$  such that  $y \in D_R^m F^{-1}(z_0, y_0, w_1, v_1, \dots, w_{m-1}, v_{m-1})(w)$ . Then, (5.5) ensures that  $y \in D_R^m C_X(y_0, z_0, w_1, v_1, \dots, w_{m-1}, v_{m-1})(w)$ . This means the existence of  $t_n > 0$  and  $(y_n, w_n) \rightarrow (y, w)$  such that, for all  $n \in \mathbb{N}$ ,

$$y_0 + t_n v_1 + \dots + t_n^{m-1} v_{m-1} + t_n^m y_n \in C(X, z_0 + t_n w_1 + \dots + t_n^{m-1} w_{m-1} + t_n^m w_n).$$

From the definition of the map  $C$ , we get  $z_0 + t_n w_1 + \dots + t_n^{m-1} w_{m-1} + t_n^m w_n \in (F \circ G)(X)$ . So,  $w \in T_{(F \circ G)(X)}^{r(m)}(z_0, w_1, \dots, w_{m-1})$ .  $\square$

We now show the essentialness of the assumption (5.6) in Proposition 5.2.15 (it is similar for (5.5)) by the following.

**Example 5.2.16.** Let  $X = Y = Z = \mathbb{R}$ ,  $G : X \rightarrow 2^Y$  and  $F : Y \rightarrow 2^Z$  be defined by

$$G(x) = \begin{cases} \{1, 2\}, & \text{if } x = 1, \\ \{0\}, & \text{if } x = 0, \end{cases} \quad F(y) = \begin{cases} \{0\}, & \text{if } y = 1, \\ \{1\}, & \text{if } y = 0. \end{cases}$$

Then,

$$\begin{aligned} (F \circ G)(x) &= \begin{cases} \{0\}, & \text{if } x = 1, \\ \{1\}, & \text{if } x = 0, \end{cases} \quad F^{-1}(z) = \begin{cases} \{0\}, & \text{if } z = 1, \\ \{1\}, & \text{if } z = 0, \end{cases} \\ C(x, z) &= G(x) \cap F^{-1}(z) = \begin{cases} \{1\}, & \text{if } (x, z) = (1, 0), \\ \{0\}, & \text{if } (x, z) = (0, 1). \end{cases} \end{aligned}$$

Let  $(x_0, z_0) = (0, 1)$  and  $y_0 = 0 \in C(x_0, z_0)$ . Direct calculations give

$$D_R^1 G(x_0, y_0)(1/2) = \{1/2, 1\}, \quad D_R^1 F(y_0, z_0)(1) = \{-1\},$$

$$D_R^1 F(y_0, z_0)(1/2) = \{-1/2\}, \quad D_R^1 (F \circ G)(x_0, z_0)(1/2) = \{-1/2\}.$$

So, the conclusion of Proposition 5.2.15(ii) does not hold. The reason is the violence of (5.6): for  $(u, w) = (1/2, -1)$ ,  $D_R^1 F^{-1}(z_0, y_0)(-1) = \{1\}$ ,  $D_R^1 C((x_0, z_0), y_0)(u, w) = \emptyset$ , and hence

$$D_R^1 G(x_0, y_0)(u) \cap D_R^1 F^{-1}(z_0, y_0)(w) \not\subseteq D_R^1 C((x_0, z_0), y_0)(u, w).$$

Now we apply the preceding composition rule to establish a sum rule for  $M, N : X \rightarrow 2^Y$ . For this purpose, we use  $G : X \rightarrow 2^{X \times Y}$  and  $F : X \times Y \rightarrow 2^Y$  defined in (5.4). For  $(x, z) \in X \times Y$ , we set  $H(x, z) := M(x) \cap (z - N(x))$ . Then, the resultant multimap  $C : X \times Y \rightarrow 2^{X \times Y}$  associated to these  $F$  and  $G$  is  $C(x, z) = \{x\} \times H(x, z)$ .

**Proposition 5.2.17.** *Let  $(x_0, z_0) \in \text{gr}(M + N)$ ,  $y_0 \in H(x_0, z_0)$  and  $(u_i, v_i, w_i) \in X \times Y \times Y$ .*

(i) *If, for all  $w \in Y$ , one has*

$$\begin{aligned} & T_{M(X)}^{r(m)}(y_0, v_1, \dots, v_{m-1}) \cap [w - T_{N(X)}^{r(m)}(z_0 - y_0, w_1, \dots, w_{m-1})] \\ & \subseteq D_R^m H_X(z_0, y_0, v_1 + w_1, v_1, \dots, v_{m-1} + w_{m-1}, v_{m-1})(w), \end{aligned} \quad (5.7)$$

where  $H_X : Y \rightarrow 2^Y$  is defined by  $H_X(y) := H(X, y)$ , then

$$\begin{aligned} & T_{M(X)}^{r(m)}(y_0, v_1, \dots, v_{m-1}) + T_{N(X)}^{r(m)}(z_0 - y_0, w_1, \dots, w_{m-1}) \\ & \subseteq T_{(M+N)(X)}^{r(m)}(z_0, v_1 + w_1, \dots, v_{m-1} + w_{m-1}). \end{aligned}$$

(ii) *If, for all  $(u, w) \in X \times Y$ , one has*

$$\begin{aligned} & D_R^m M(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1})(u) \cap [w - D_R^m N(x_0, z_0 - y_0, u_1, w_1, \dots, u_{m-1}, w_{m-1})(u)] \\ & \subseteq D_R^m H((x_0, z_0), y_0, (u_1, v_1 + w_1), v_1, \dots, (u_{m-1}, v_{m-1} + w_{m-1}), v_{m-1})(u, w), \end{aligned} \quad (5.8)$$

then

$$\begin{aligned} & D_R^m M(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1})(u) + D_R^m N(x_0, z_0 - y_0, u_1, w_1, \dots, u_{m-1}, w_{m-1})(u) \\ & \subseteq D_R^m (M + N)(x_0, z_0, u_1, v_1 + w_1, \dots, u_{m-1}, v_{m-1} + w_{m-1})(u). \end{aligned}$$

*Proof.* By the similarity, we prove only (ii). Let  $w \in D_R^m M(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1})(u) + D_R^m N(x_0, z_0 - y_0, u_1, w_1, \dots, u_{m-1}, w_{m-1})(u)$ , i.e., there exists  $y \in D_R^m M(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1})(u)$  such that  $y \in w - D_R^m N(x_0, z_0 - y_0, u_1, w_1, \dots, u_{m-1}, w_{m-1})(u)$ . Then, (5.8) ensures that  $y \in D_R^m H((x_0, z_0), y_0, (u_1, v_1 + w_1), v_1, \dots, (u_{m-1}, v_{m-1} + w_{m-1}), v_{m-1})(u, w)$ . This means the existence of  $t_n > 0$  and  $(u_n, y_n, w_n) \rightarrow (u, y, w)$  such that

$$y_0 + t_n v_1 + \dots + t_n^{m-1} v_{m-1} + t_n^m y_n \in$$

$$H(x_0 + t_n u_1 + \dots + t_n^{m-1} u_{m-1} + t_n^m u_n, z_0 + t_n(v_1 + w_1) + \dots + t_n^{m-1}(v_{m-1} + w_{m-1}) + t_n^m w_n).$$

From the definition of  $H$ , we get

$$z_0 + t_n(v_1 + w_1) + \dots + t_n^{m-1}(v_{m-1} + w_{m-1}) + t_n^m w_n \in (M + N)(x_0 + t_n u_1 + \dots + t_n^{m-1} u_{m-1} + t_n^m u_n).$$

So,  $w \in D_R^m (M + N)(x_0, z_0, u_1, v_1 + w_1, \dots, u_{m-1}, v_{m-1} + w_{m-1})(u)$ .  $\square$

The following example shows that assumptions (5.7) and (5.8) cannot be dispensed and are not difficult to check.

**Example 5.2.18.** Let  $X = Y = \mathbb{R}$  and  $M, N : X \rightarrow 2^Y$  be given by

$$M(x) = \begin{cases} \{1\}, & \text{if } x = \frac{1}{n}, n \in \mathbb{N}, \\ \{0\}, & \text{if } x = 0, \end{cases} \quad N(x) = \begin{cases} \{0\}, & \text{if } x = \frac{1}{n}, n \in \mathbb{N}, \\ \{1\}, & \text{if } x = 0. \end{cases}$$

Then,

$$H(x, z) = M(x) \cap (z - N(x)) = \begin{cases} \{0\}, & \text{if } (x, z) = (0, 1), \\ \{1\}, & \text{if } (x, z) = \left(\frac{1}{n}, 1\right), n \in \mathbb{N}, \\ \emptyset, & \text{otherwise,} \end{cases}$$

$$(M + N)(x) = \begin{cases} \{1\}, & \text{if } x = \frac{1}{n}, n \in \mathbb{N}, \\ \{1\}, & \text{if } x = 0. \end{cases}$$

Choose  $x_0 = 0, z_0 = 1, y_0 = 0 \in \text{cl}H(x_0, z_0)$  and  $u = w = 0$ . Then, one can easily show

$$T_{M(X)}^{r(1)}(y_0) = \mathbb{R}_+, T_{N(X)}^{r(1)}(z_0 - y_0) = \mathbb{R}_-, D_R^1 H_X(z_0, y_0)(w) = \{0\},$$

$$D_R^1 M(x_0, y_0)(u) = \mathbb{R}_+, D_R^1 N(x_0, z_0 - y_0)(u) = \mathbb{R}_-, D_R^1 H((x_0, z_0), y_0)(u, w) = \{0\}.$$

Thus, (5.7) and (5.8) are violated:

$$T_{M(X)}^{r(1)}(y_0) \cap [w - T_{N(X)}^{r(1)}(z_0 - y_0)] \not\subseteq D_R^1 H_X(z_0, y_0)(w),$$

$$D_R^1 M(x_0, y_0)(u) \cap [w - D_R^1 N(x_0, z_0 - y_0)(u)] \not\subseteq D_R^1 H((x_0, z_0), y_0)(u, w).$$

Direct computations show that conclusions of Proposition 5.2.17 do not hold:

$$T_{M(X)}^{r(1)}(y_0) + T_{N(X)}^{r(1)}(z_0 - y_0) \not\subseteq T_{(M+N)(X)}^{r(1)}(z_0),$$

$$D_R^1 M(x_0, y_0)(u) + D_R^1 N(x_0, z_0 - y_0)(u) \not\subseteq D_R^1 (M + N)(x_0, z_0)(u),$$

since  $T_{(M+N)(X)}^{r(1)}(z_0) = \{0\}$  and  $D_R^1 (M + N)(x_0, z_0)(u) = \{0\}$ .

### 5.3 Optimality conditions

Let  $X, Y$  and  $Z$  be normed spaces,  $C \subseteq Y$  and  $D \subseteq Z$  be pointed closed convex cones, not the entire space,  $S \subseteq X$  nonempty, and  $F : S \rightarrow 2^Y$ ,  $G : S \rightarrow 2^Z$ . Our problem is

$$(P) \quad \text{Min}_Q F(x), \text{ s.t. } x \in S, G(x) \cap -D \neq \emptyset.$$

We denote  $A := \{x \in S : G(x) \cap -D \neq \emptyset\}$  (the feasible set).

In this section, both necessary and sufficient optimality conditions for the mentioned efficient solutions of the problem (P) are established. As  $Q$ -efficiency (see Definition 2.2.7) includes many other kinds of solutions as particular cases (see Proposition 2.2.9), we first prove necessary optimality conditions of this notion.

**Proposition 5.3.1.** *Let  $(x_0, y_0) \in \text{gr}F$  be a  $Q$ -efficient solution of (P),  $(u_i, v_i, w_i) \in X \times (-C) \times (-D)$ ,  $i = 1, \dots, m-1$ , and  $z_0 \in G(x_0) \cap -D$ . Suppose that the open cone  $Q$  satisfies  $Q + C \subseteq Q$ . Then, the following separations hold*

$$T_{(F,G)_+(S)}^{r(m)}(y_0, z_0, (v_1, w_1), \dots, (v_{m-1}, w_{m-1})) \cap (-Q \times -\text{int}D) = \emptyset, \quad (5.9)$$

$$D_R^m (F, G)_+(x_0, y_0, z_0, (u_1, v_1, w_1), \dots, (u_{m-1}, v_{m-1}, w_{m-1}))(X) \cap (-Q \times -\text{int}D) = \emptyset. \quad (5.10)$$

*Proof.* Suppose (5.9) does not hold. Then, there exists  $(y, z)$  such that

$$(y, z) \in T_{(F,G)_+(S)}^{r(m)}(y_0, z_0, (v_1, w_1), \dots, (v_{m-1}, w_{m-1})), \quad (5.11)$$

$$(y, z) \in (-Q \times -\text{int}D). \quad (5.12)$$



It follows from (5.11) and the definition of  $m$ -th order radial sets that there exist sequences  $t_n > 0$ ,  $x_n \in S$ , and  $(y_n, z_n) \in (F, G)(x_n) + C \times D$  such that

$$\frac{(y_n, z_n) - (y_0, z_0) - t_n(v_1, w_1) - \dots - t_n^{m-1}(v_{m-1}, w_{m-1})}{t_n^m} \rightarrow (y, z). \quad (5.13)$$

From (5.12) and (5.13), one has, for large  $n$ ,

$$y_n - y_0 - t_n v_1 - \dots - t_n^{m-1} v_{m-1} \in -Q, \quad z_n \in -\text{int}D. \quad (5.14)$$

As  $v_1, \dots, v_{m-1} \in -C$ , (5.14) implies that, for large  $n$ ,

$$y_n - y_0 \in -Q. \quad (5.15)$$

Because  $z_n \in G(x_n) + D$ , there exist  $\bar{z}_n \in G(x_n)$  and  $d_n \in D$  such that  $z_n = \bar{z}_n + d_n$ . (5.14) implies also that  $\bar{z}_n \in G(x_n) \cap (-D)$  for large  $n$ , and then  $x_n \in A$ . Because  $y_n \in F(x_n) + C$ , there exist  $\bar{y}_n \in F(x_n)$  and  $c_n \in C$  such that  $y_n = \bar{y}_n + c_n$ . Then, (5.15) implies that  $\bar{y}_n - y_0 \in -Q$  for large  $n$ . Therefore,  $\bar{y}_n - y_0 \in (F(A) - y_0) \cap (-Q)$ , which contradicts the  $Q$ -efficiency of  $(x_0, y_0)$ . Thus, (5.9) holds. (5.10) follows from (5.9) and the evident fact that

$$D_R^m F(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1})(X) \subseteq T_{F(X)}^{r(m)}(y_0, v_1, \dots, v_{m-1}).$$

□

Propositions 5.3.1 and 2.2.9 together yield the following result.

**Theorem 5.3.2.** *Let  $(x_0, y_0) \in \text{gr}F$ ,  $(u_i, v_i, w_i) \in X \times (-C) \times (-D)$ ,  $i = 1, \dots, m-1$ , and  $z_0 \in G(x_0) \cap -D$ . Then, (5.9) and (5.10) hold in each of the following cases*

- (i)  $(x_0, y_0)$  is a weak efficient solution of  $(P)$  and  $Q = \text{int}C$ ,
- (ii)  $(x_0, y_0)$  is a strong efficient solution of  $(P)$  and  $Q = Y \setminus -C$ ,
- (iii)  $(x_0, y_0)$  is a positive-proper efficient solution of  $(P)$  and  $Q = \{y : \varphi(y) > 0\}$  for some functional  $\varphi \in C^{+i}$ ,
- (iv)  $(x_0, y_0)$  is a Geoffrion-proper efficient solution of  $(P)$  and  $Q = C(\varepsilon)$  for  $\varepsilon > 0$ ,
- (v)  $(x_0, y_0)$  is a Henig-proper efficient solution of  $(P)$  and  $Q = K$  for some pointed open convex cone  $K$  dilating  $C$ ,
- (vi)  $(x_0, y_0)$  is a strong Henig-proper efficient solution of  $(P)$  and  $Q = \text{int}C_\varepsilon(B)$  for  $\varepsilon$  satisfying  $0 < \varepsilon < \delta$ .

The next example illustrates Theorem 5.3.2(vi).

**Example 5.3.3.** Let  $X = Y = Z = \mathbb{R}$ ,  $S = X$ ,  $C = D = \mathbb{R}_+$ ,  $G(x) \equiv \mathbb{R}$ , and  $F(x) = \{y \in Y : y \geq |x|\}$ . Choose the base  $B = \{1\}$ . Then,  $\delta = 1$  and  $C_\varepsilon(B) = \mathbb{R}_+$  for all  $\varepsilon \in (0, \delta)$ . Let  $(x_0, y_0) = (0, 0)$  and  $z_0 = 0$ . It is easy to see that  $(x_0, y_0)$  is a strong Henig-proper efficient solution. For any  $(v_1, w_1) \in -(C \times D)$ , direct computations give

$$T_{(F,G)_+(S)}^{r(1)}(y_0, z_0) = \mathbb{R}_+ \times \mathbb{R}, \quad T_{(F,G)_+(S)}^{r(2)}(y_0, z_0, v_1, w_1) = \mathbb{R}_+ \times \mathbb{R},$$

and hence

$$T_{(F,G)_+(S)}^{r(1)}(y_0, z_0) \cap -\text{int}(C_\varepsilon(B) \times D) = \emptyset, \quad T_{(F,G)_+(S)}^{r(2)}(y_0, z_0, v_1, w_1) \cap -\text{int}(C_\varepsilon(B) \times D) = \emptyset,$$

i.e., the necessary optimality condition of Theorem 5.3.2(vi) holds.

We now compare Theorem 5.3.2 with some known results. Because variational sets (see Section 4.2), played the role of generalized derivatives (see [105, 106]), are bigger than the image of  $X$  through most of kinds of generalized derivatives (see Remark 2.1 and Proposition 4.1 in [105]), optimality conditions (obtained by separating sets as usual) in terms of these variational sets are strong. So, we will compare our results with those using variational sets. However, in general, these sets are incomparable with radial sets as follows

**Example 5.3.4.** Suppose that  $X = \mathbb{R}$ ,  $Y = \mathbb{R}^2$ ,  $(x_0, y_0) = (0, (0, 0))$ , and  $F : X \rightarrow 2^Y$  be defined by

$$F(x) = \begin{cases} \{(0, 0)\}, & \text{if } x = 0, \\ \{(1, 0)\}, & \text{otherwise.} \end{cases}$$

Then,

$$W^1(F, x_0, y_0) = T_{F(X)}^{r(1)}(x_0, y_0) = \{(y_1, y_2) \in \mathbb{R}_+^2 : y_2 = 0\}.$$

Let  $v_1 = (1, 0)$ . Then,

$$W^2(F, x_0, y_0, v_1) = \{(y_1, y_2) \in \mathbb{R}^2 : y_2 = 0\},$$

$$T_{F(X)}^{r(2)}(y_0, v_1) = \{(y_1, y_2) \in \mathbb{R}^2 : y_1 = y_2^2 + y_2, y_2 \leq 0\}.$$

Hence, the latter may be more advantageous in cases as ensured by the following.

**Example 5.3.5.** Let  $X = Y = Z = \mathbb{R}$ ,  $S = \{0, 1\}$ ,  $C = D = \mathbb{R}_+$ ,  $(x_0, y_0) = (0, 0)$ ,  $G$  and  $F$  be

$$G(x) = F(x) = \begin{cases} \{0\}, & \text{if } x = 0, \\ \{-1\}, & \text{if } x = 1, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Choose the base  $B = \{1\}$ . Then,  $\delta = 1$  and  $C_\varepsilon(B) = \mathbb{R}_+$  for all  $\varepsilon \in (0, \delta)$ . Let  $z_0 = 0$ . Let us try to use optimality conditions given in [106] by Khanh and Tuan in terms of variational sets to eliminate  $(x_0, y_0)$  as a candidate for a strong Henig-proper efficient solution. We can compute directly that  $V^1((F, G)_{+, x_0, y_0, z_0}) = \mathbb{R}_+ \times \mathbb{R}_+$ . Let  $(v_1, w_1) \in V^1((F, G)_{+, x_0, y_0, z_0}) \cap -\text{bd}(C_\varepsilon(B) \times D(z_0)), \dots, (v_{m-1}, w_{m-1}) \in V^{m-1}((F, G)_{+, x_0, y_0, z_0, v_1, w_1, \dots, v_{m-2}, w_{m-2}}) \cap -\text{bd}(C_\varepsilon(B) \times D(z_0))$ , for  $m \geq 2$ , where  $\text{bd} A$  means the boundary of  $A$ . It is easy to check that  $(v_1, w_1) = \dots = (v_{m-1}, w_{m-1}) = (0, 0)$  and

$$V^m((F, G)_{+, x_0, y_0, z_0, v_1, w_1, \dots, v_{m-1}, w_{m-1}}) = \mathbb{R}_+ \times \mathbb{R}_+.$$

Thus, for all  $m \geq 1$ , we get

$$V^m((F, G)_{+, x_0, y_0, z_0, v_1, w_1, \dots, v_{m-1}, w_{m-1}}) \cap -\text{int}(C_\varepsilon(B) \times D(z_0)) = \emptyset.$$

For variational sets of type 2, by direct calculating we get

$$W^1((F, G)_{+, x_0, y_0, z_0}) = \mathbb{R}_+ \times \mathbb{R}_+,$$

$$W^1((F, G)_{+, x_0, y_0, z_0}) \cap -\text{int}(C_\varepsilon(B) \times D) = \emptyset.$$

Let  $(v_1, w_1) \in W^1((F, G)_{+, x_0, y_0, z_0}) \cap -\text{bd}(C_\varepsilon(B) \times D)$ ,  $(v_2, w_2) \in W^2((F, G)_{+, x_0, y_0, z_0, v_1, w_1}) \cap -\text{bd}(C_\varepsilon(B) \times D(w_1)), \dots, (v_{m-1}, w_{m-1}) \in W^{m-1}((F, G)_{+, x_0, y_0, z_0, v_1, w_1, \dots, v_{m-2}, w_{m-2}}) \cap -\text{bd}(C_\varepsilon(B) \times D(w_1))$ ,  $m \geq 3$ . We have  $(v_1, w_1) = \dots = (v_{m-1}, w_{m-1}) = (0, 0)$  and

$$W^m((F, G)_{+, x_0, y_0, z_0, v_1, w_1, \dots, v_{m-1}, w_{m-1}}) = \mathbb{R}_+ \times \mathbb{R}_+.$$

Thus, for all  $m \geq 2$ , we get

$$W^m((F, G)_{+, x_0, y_0, z_0, v_1, w_1, \dots, v_{m-1}, w_{m-1}}) \cap -\text{int}(C_\varepsilon(B) \times D(w_1)) = \emptyset.$$

So, Theorems 3.4 and 3.5 of Khanh and Tuan in [106] say nothing about  $(x_0, y_0)$  being strong Henig-proper efficient or not. By virtue of Remark 3.3 in [106] and Proposition 2.2 in [106], we can see that Theorems 4.1, 4.2, 5.1, and 5.2 of Li and Chen in [115], Theorem 3.1, Proposition 3.1 of Gong et al. in [80] and Theorem 1 of Liu and Gong in [119] cannot be in use to reject

$(x_0, y_0)$  either. On the other hand, since  $T_{(F,G)_+(S)}^{r(1)}(y_0, z_0) = \mathbb{R} \times \mathbb{R}$ , Theorem 5.3.2(vi) rejects the candidate  $(x_0, y_0)$ .

Moreover, Theorem 3.2(i) of Khanh and Tuan in [105] says nothing about  $(x_0, y_0)$  being weak efficiency for (P). By Proposition 4.1 in [105], we can see that Theorem 7 of Jahn and Rauh in [96], Theorem 5 of Chen and Jahn in [26], Theorem 4.1 of Corley in [33], Proposition 3.1 of Taa in [160], Theorem 2.7(a) of Jahn and Khan in [93], Theorem 2 of Crepsi et al. in [34], Theorem 4.1 of Crepsi et al. in [35] and Theorem 3.1 of Jahn et al. in [95] cannot be in use to reject  $(x_0, y_0)$  either. On the other hand, by using Theorem 5.3.2(i),  $(x_0, y_0)$  is not a weak efficiency for (P).

Finally we discuss sufficient conditions for the mentioned efficient solutions of problem (P).

**Proposition 5.3.6.** *Let  $(x_0, y_0) \in \text{gr}F$  and  $x_0 \in A$ , the feasible set. Suppose that there exists  $z_0 \in G(x_0) \cap (-D)$  such that, for  $(u_i, v_i, w_i) \in X \times (-C) \times (-D)$ ,  $i = 1, \dots, m-1$ , and  $x \in S$ , either of the following separations holds*

$$T_{(F,G)_+(S)}^{r(m)}((y_0, z_0), (v_1, w_1), \dots, (v_{m-1}, w_{m-1})) \cap -(Q \times D(z_0)) = \emptyset, \quad (5.16)$$

$$D_R^m(F, G)_+(x_0, y_0, z_0, u_1, v_1, w_1, \dots, u_{m-1}, v_{m-1}, w_{m-1})(x - x_0) \cap -(Q \times D(z_0)) = \emptyset. \quad (5.17)$$

*Then,  $(x_0, y_0)$  is a  $Q$ -efficient solution of (P), for any non-empty open cone  $Q$ .*

*Proof.* By the similarity, we prove only (5.16). Note that (5.16) is required to be satisfied also for  $v_i = 0 \in -C$  and  $w_i = 0 \in -D$ ,  $i = 1, \dots, m-1$ . Therefore,  $T_{(F,G)_+(S)}^{r(1)}(y_0, z_0) \cap -(Q \times D(z_0)) = \emptyset$ . It follows from Proposition 5.2.11 that  $(y - y_0, z - z_0) \in T_{(F,G)_+(S)}^{r(1)}(y_0, z_0)$  for all  $y \in F(S)$ ,  $z \in G(S)$ . Then,

$$(F, G)(S) - (y_0, z_0) \cap -(Q \times D(z_0)) = \emptyset.$$

Suppose the existence of  $x \in A$  and  $y \in F(x)$  such that  $y - y_0 \in -Q$ . Then, there exists  $z \in G(x) \cap -D$  such that  $(y, z) - (y_0, z_0) \in -(Q \times D(z_0))$ , a contradiction.  $\square$

From Propositions 5.3.6 and 2.2.9, we obtain immediately the following result.

**Theorem 5.3.7.** *Let  $(x_0, y_0) \in \text{gr}F$  and  $x_0 \in A$ . Suppose that there exists  $z_0 \in G(x_0) \cap (-D)$  such that, for  $(u_i, v_i, w_i) \in X \times (-C) \times (-D)$ ,  $i = 1, \dots, m-1$ , and  $x \in S$ , either of (5.16) or (5.17) holds. Then, one has the following assertions*

- (i)  $(x_0, y_0)$  is a weak efficient solution of (P) and  $Q = \text{int}C$ ,

- (ii)  $(x_0, y_0)$  is a strong efficient solution of  $(P)$  and  $Q = Y \setminus -C$ ,
- (iii)  $(x_0, y_0)$  is a positive-proper efficient solution of  $(P)$  and  $Q = \{y : \varphi(y) > 0\}$  for some functional  $\varphi \in C^{+i}$ ,
- (iv)  $(x_0, y_0)$  is a Geoffrion-proper efficient solution of  $(P)$  and  $Q = C(\varepsilon)$  for  $\varepsilon > 0$ ,
- (v)  $(x_0, y_0)$  is a Henig-proper efficient solution of  $(P)$  and  $Q = K$  for some pointed open convex cone  $K$  dilating  $C$ ,
- (vi)  $(x_0, y_0)$  is a strong Henig-proper efficient solution of  $(P)$  and  $Q = \text{int}C_\varepsilon(B)$  for  $\varepsilon$  satisfying  $0 < \varepsilon < \delta$ .

In the following example, Theorems 5.3.7(vi) works, while several existing results do not.

**Example 5.3.8.** Let  $X = Y = Z = \mathbb{R}$ ,  $C = D = \mathbb{R}_+$ ,  $G(x) \equiv \{0\}$ , and

$$F(x) = \begin{cases} \{0\}, & \text{if } x = 0, \\ \left\{ \frac{1}{n^2} \right\}, & \text{if } x = n, n \in \mathbb{N}, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Choose the base  $B = \{1\}$ . Then,  $\delta = 1$  and  $C_\varepsilon(B) = \mathbb{R}_+$  for all  $\varepsilon \in (0, \delta)$ . Let  $(x_0, y_0) = (0, 0)$  and  $z_0 = 0$ . It is easy to see that  $A = \mathbb{N} \cup \{0\}$ . Then,  $T_{(F,G)_+(A)}^{r(1)}(y_0, z_0) = \mathbb{R}_+ \times \mathbb{R}_+$ . It follows from Theorem 5.3.7(vi) that  $(x_0, y_0)$  is a strong Henig-proper efficient solution. It is easy to see that  $\text{dom}F = \mathbb{N} \cup \{0\}$  is not convex and  $F$  is not pseudoconvex at  $(x_0, y_0)$ . So, Theorem 3.6 of Khanh and Tuan in [106], Theorems 5.3, 5.4 of Li and Chen in [115], and Theorem 2 of Liu and Gong in [119] cannot be applied.

Moreover, it is easy to see that  $(x_0, y_0)$  is also a weak efficient solution. But, Theorem 8 of Jahn and Rauh in [96], Theorem 6 of Chen and Jahn in [26] and Theorem 3.3 of Khanh and Tuan in [105] cannot be applied.

A natural question now arises: can we replace  $D$  by  $D(z_0)$  in the necessary condition given by Theorem 5.3.2 to obtain a smaller gap with the sufficient one expressed by Theorem 5.3.7? Unfortunately, a negative answer is supplied by the following example.

**Example 5.3.9.** Suppose that  $X = Y = Z = \mathbb{R}$ ,  $S = X$ ,  $C = D = \mathbb{R}_+$ , and  $F : X \rightarrow 2^Y$ ,  $G : X \rightarrow 2^Z$  are given by

$$F(x) = \begin{cases} \{y : y \geq x^2\}, & \text{if } x \in [-1, 1], \\ \{-1\}, & \text{if } x \notin [-1, 1], \end{cases}$$

$$G(x) = \{z \in Z : z = x^2 - 1\}.$$

We see that  $(x_0, y_0) = (0, 0)$  is a weak efficient pair of (P). Take  $z_0 = -1 \in G(x_0) \cap (-D)$ . We have  $T_{(F,G)_+(S)}^{r(1)}(y_0, z_0) = \mathbb{R} \times \mathbb{R}_+$  and hence

$$T_{(F,G)_+(S)}^{r(1)}(y_0, z_0) \cap -\text{int}(C \times D) = \emptyset.$$

On the other hand,  $D(z_0) = \mathbb{R}$ . Thus,

$$T_{(F,G)_+(S)}^{r(1)}(y_0, z_0) \cap -\text{int}(C \times D(z_0)) \neq \emptyset.$$

## 5.4 Applications in some particular problems

In this section, we apply calculus rules to establish necessary conditions for some kinds of efficient solutions of several particular optimization problems. We first prove a simple characterization of this notion.

**Proposition 5.4.1.** *Let  $X, Y$  and  $Q$  as before,  $F : X \rightarrow 2^Y$ ,  $(x_0, y_0) \in \text{gr}F$  and  $(u_i, v_i) \in X \times (-C)$ ,  $i = 1, \dots, m-1$ . Suppose that the open cone  $Q$  satisfies  $Q + C \subseteq Q$ . Then,  $y_0$  is a  $Q$ -efficient point of  $F(X)$  if and only if one of the following separations holds*

$$T_{F_+(X)}^{r(m)}(y_0, v_1, \dots, v_{m-1}) \cap (-Q) = \emptyset, \quad (5.18)$$

$$D_R^m F_+(x_0, y_0, (u_1, v_1), \dots, (u_{m-1}, v_{m-1}))(X) \cap (-Q) = \emptyset. \quad (5.19)$$

*Proof.* It follows from Propositions 5.3.1 and 5.3.6. □

Let  $F : X \rightarrow 2^Y$  and  $G : X \rightarrow 2^X$ . Consider

$$(P_1) \quad \text{Min}_Q F(x') \text{ s.t. } x \in X \text{ and } x' \in G(x).$$

This problem can be restated as the unconstrained problem:  $\text{Min}_Q (F \circ G)(x)$ . Recall that  $(x_0, y_0)$  is called a  $Q$ -efficient solution if  $y_0 \in (F \circ G)(x_0)$  and  $((F \circ G)(X) - y_0) \cap (-Q) = \emptyset$ .

**Proposition 5.4.2.** *Assume for (P<sub>1</sub>) that  $\text{Im}G \subseteq \text{dom}F$ ,  $(x_0, z_0) \in \text{gr}G$ ,  $(z_0, y_0) \in \text{gr}F$ , and  $(u_1, v_1, w_1), \dots, (u_{m-1}, v_{m-1}, w_{m-1}) \in X \times X \times (-C)$ . Suppose that an open cone  $Q$  satisfies  $Q + C \subseteq Q$  and  $(x_0, y_0)$  is a  $Q$ -efficient solution of (P<sub>1</sub>).*

(i) If either  $F_+$  has a  $m$ -th order radial semiderivative at  $(z_0, y_0)$  with respect to  $(v_1, w_1), \dots, (v_{m-1}, w_{m-1})$  or (5.5) holds for  $F_+$  and  $G$ , then

$$D_R^m F_+(z_0, y_0, v_1, w_1, \dots, v_{m-1}, w_{m-1})[T_{G(X)}^{r(m)}(z_0, v_1, \dots, v_{m-1})] \cap (-Q) = \emptyset.$$

(ii) If either  $F_+$  has a  $m$ -th order radial semiderivative at  $(z_0, y_0)$  with respect to  $(v_1, w_1), \dots, (v_{m-1}, w_{m-1})$  or (5.6) holds for  $F_+$  and  $G$ , then

$$D_R^m F_+(z_0, y_0, v_1, w_1, \dots, v_{m-1}, w_{m-1})[D_R^m G(x_0, z_0, u_1, v_1, \dots, u_{m-1}, v_{m-1})(X)] \cap (-Q) = \emptyset.$$

*Proof.* We prove (i). By Proposition 5.4.1,  $T_{(F \circ G)_+(X)}^{r(m)}(y_0, w_1, \dots, w_{m-1}) \cap (-Q) = \emptyset$ . Propositions 5.2.14(i) and 5.2.15(i) say that

$$D_R^m F_+(z_0, y_0, v_1, w_1, \dots, v_{m-1}, w_{m-1})[T_{G(X)}^{r(m)}(z_0, v_1, \dots, v_{m-1})] \subseteq T_{(F \circ G)_+(X)}^{r(m)}(y_0, w_1, \dots, w_{m-1}).$$

□

From Propositions 5.4.2 and 2.2.9, we obtain immediately the following result for  $(P_1)$ .

**Theorem 5.4.3.** Assume for  $(P_1)$  that  $\text{Im } G \subseteq \text{dom } F$ ,  $(x_0, z_0) \in \text{gr } G$ ,  $(z_0, y_0) \in \text{gr } F$ , and  $(u_1, v_1, w_1), \dots, (u_{m-1}, v_{m-1}, w_{m-1}) \in X \times X \times (-C)$ . Then, assertions (i) and (ii) in Proposition 5.4.2 hold in each of the following cases

- (i)  $(x_0, y_0)$  is a weak efficient solution of  $(P_1)$  and  $Q = \text{int } C$ ,
- (ii)  $(x_0, y_0)$  is a strong efficient solution of  $(P_1)$  and  $Q = Y \setminus -C$ ,
- (iii)  $(x_0, y_0)$  is a positive-proper efficient solution of  $(P_1)$  and  $Q = \{y : \varphi(y) > 0\}$  for some functional  $\varphi \in C^{+i}$ ,
- (iv)  $(x_0, y_0)$  is a Geoffrion-proper efficient solution of  $(P_1)$  and  $Q = C(\varepsilon)$  for  $\varepsilon > 0$ ,
- (v)  $(x_0, y_0)$  is a Henig-proper efficient solution of  $(P_1)$  and  $Q = K$  for some pointed open convex cone  $K$  dilating  $C$ ,
- (vi)  $(x_0, y_0)$  is a strong Henig-proper efficient solution of  $(P_1)$  and  $Q = \text{int } C_\varepsilon(B)$  for  $\varepsilon$  satisfying  $0 < \varepsilon < \delta$ .

To compare with a result of Jahn and Khan in [94], we recall the definition of *contingent epiderivatives*. For a multimap  $F$  between normed spaces  $X$  and  $Y$ ,  $Y$  being partially ordered by a pointed convex cone  $C$  and a point  $(x, y) \in \text{gr } F$ , a single-valued map  $EDF(x, y) : X \rightarrow Y$  satisfying  $\text{epi}(EDF(x, y)) = T_{\text{epi}F}(x, y) \equiv T_{\text{gr}F_+}(x, y)$  is said to be the contingent epiderivative of  $F$  at  $(x, y)$ .

**Example 5.4.4.** Let  $X = Y = \mathbb{R}$ ,  $C = \mathbb{R}_+$ ,  $G(x) = \{-|x|\}$ , and  $F$  be defined by

$$F(x) = \begin{cases} \mathbb{R}_-, & \text{if } x \leq 0, \\ \emptyset, & \text{if } x > 0. \end{cases}$$

Since  $G$  is single-valued we can try to make use of Proposition 5.2 of [94]. By a direct computation we have  $T_{\text{epi}F}(G(0), 0) = \mathbb{R}^2$ . So, the contingent epiderivative  $EDF(G(0), 0)(h)$  does not exist for any  $h \in X$ . Hence, the necessary condition in the mentioned Proposition 5.2 says nothing about the candidate point  $(0, 0)$  for weak efficiency. However,  $F_+$  has the first order radial semiderivative at  $(G(0), 0)$  and  $T_{G(X)}^{r(1)}(G(0)) = \mathbb{R}_-$ . Furthermore,  $D_R^m F_+(G(0), 0)[T_{G(X)}^{r(1)}(G(0))] = \mathbb{R}$ , which meets  $-\text{int}C$ , and hence Theorem 5.4.3(i) above rejects this candidate.

Our sum rule can be applied directly to the following problem

$$(P_2) \quad \text{Min}_Q F(x) \text{ s.t. } g(x) \leq 0,$$

where  $X, Y$  are as for problem  $(P_1)$ ,  $F : X \rightarrow 2^Y$  and  $g : X \rightarrow Y$ . Denote  $A := \{x \in X : g(x) \leq 0\}$  (the feasible set). Define  $G : X \rightarrow 2^Y$  by  $G(x) := \{0\}$  if  $x \in A$  and  $G(x) := \{g(x)\}$  otherwise. Consider the following unconstrained set-valued optimization problem, for arbitrary  $s > 0$ ,

$$(P_3) \quad \text{Min}_Q (F + sG)(x).$$

In the particular case, when  $Y = \mathbb{R}$  and  $F$  is single-valued,  $(P_3)$  is used to approximate  $(P_2)$  in penalty methods (see [147]). We will apply our calculus rules for radial sets to get the following necessary condition for a  $Q$ -minimal solution of  $(P_3)$ .

**Proposition 5.4.5.** *Let  $y_0 \in F(x_0)$ ,  $x_0 \in \Omega = \text{dom}F \cap \text{dom}G$ , and  $(u_1, v_{1,1}), \dots, (u_{m-1}, v_{i,m-1}) \in X \times (-C)$  for  $i = 1, 2$ . Suppose that an open cone  $Q$  satisfies  $Q + C \subseteq Q$  and  $(x_0, y_0)$  is a  $Q$ -efficient solution of  $(P_3)$ . Then*

(i) *if either  $F_+(\Omega)$  (or  $sG_+(\Omega)$ ) has a  $m$ -th order proto-radial set at  $y_0$  with respect to  $v_{1,1}, \dots, v_{1,m-1}$  (at 0 with respect to  $v_{2,1}, \dots, v_{2,m-1}$ , respectively) or (5.7) holds for  $F_+$  and  $sG_+$ , then*

$$(T_{F_+(\Omega)}^{r(m)}(y_0, v_{1,1}, \dots, v_{1,m-1}) + sT_{G_+(\Omega)}^{r(m)}(0, v_{2,1}/s, \dots, v_{2,m-1}/s)) \cap (-Q) = \emptyset.$$

(ii) *if either  $F_+$  (or  $sG_+$ ) has a  $m$ -th order radial semiderivative at  $(x_0, y_0)$  with respect to  $(u_1, v_{1,1}), \dots, (u_{m-1}, v_{1,m-1})$  (at  $(x_0, 0)$  with respect to  $(u_1, v_{2,1}), \dots, (u_{m-1}, v_{2,m-1})$ , respectively) or (5.8) holds for  $F_+$  and  $sG_+$ , then, for any  $u \in X$ ,*

$$(D_R^m F_+(x_0, y_0, u_1, v_{1,1}, \dots, u_{m-1}, v_{1,m-1})(u) + sD_R^m G_+(x_0, 0, u_1, v_{2,1}/s, \dots, u_{m-1}, v_{2,m-1}/s)(u)) \cap (-Q) = \emptyset.$$



*Proof.* We prove (i). It follows from Propositions 5.2.12(i) and 5.2.17(i) that

$$\begin{aligned} T_{F_+(\Omega)}^{r(m)}(y_0, v_{1,1}, \dots, v_{1,m-1}) + T_{sG_+(\Omega)}^{r(m)}(0, v_{2,1}, \dots, v_{2,m-1}) &\subseteq \\ T_{(F+sG)_+(\Omega)}^{r(m)}(y_0, v_{1,1} + v_{2,1}, \dots, v_{1,m-1} + v_{2,m-1}). \end{aligned}$$

It is easy to see that

$$\begin{aligned} T_{sG_+(\Omega)}^{r(m)}(0, v_{2,1}, \dots, v_{2,m-1}) &= sT_{G_+(\Omega)}^{r(m)}(0, v_{2,1}/s, \dots, v_{2,m-1}/s), \\ T_{F_+(\Omega)}^{r(m)}(y_0, v_{1,1}, \dots, v_{1,m-1}) + sT_{G_+(\Omega)}^{r(m)}(0, v_{2,1}/s, \dots, v_{2,m-1}/s) \\ &\subseteq T_{(F+sG)_+(\Omega)}^{r(m)}(y_0, v_{1,1} + v_{2,1}, \dots, v_{1,m-1} + v_{2,m-1}). \end{aligned}$$

By Proposition 5.4.1, one gets

$$T_{(F+sG)_+(\Omega)}^{r(m)}(y_0, v_{1,1} + v_{2,1}, \dots, v_{1,m-1} + v_{2,m-1}) \cap (-Q) = \emptyset,$$

and hence the proof is complete.  $\square$

From Propositions 5.4.5 and 2.2.9, we obtain immediately the following statement for  $(P_3)$ .

**Theorem 5.4.6.** *Let  $y_0 \in F(x_0)$ ,  $x_0 \in \Omega = \text{dom}F \cap \text{dom}G$ , and  $(u_1, v_{i,1}), \dots, (u_{m-1}, v_{i,m-1}) \in X \times (-C)$  for  $i = 1, 2$ . Then, assertions (i) and (ii) in Proposition 5.4.5 hold in each of the following cases*

- (i)  $(x_0, y_0)$  is a weak efficient solution of  $(P_3)$  and  $Q = \text{int}C$ ,
- (ii)  $(x_0, y_0)$  is a strong efficient solution of  $(P_3)$  and  $Q = Y \setminus -C$ ,
- (iii)  $(x_0, y_0)$  is a positive-proper efficient solution of  $(P_3)$  and  $Q = \{y : \varphi(y) > 0\}$  for some functional  $\varphi \in C^{+i}$ ,
- (iv)  $(x_0, y_0)$  is a Geoffrion-proper efficient solution of  $(P_3)$  and  $Q = C(\varepsilon)$  for  $\varepsilon > 0$ ,
- (v)  $(x_0, y_0)$  is a Henig-proper efficient solution of  $(P_3)$  and  $Q = K$  for some pointed open convex cone  $K$  dilating  $C$ ,
- (vi)  $(x_0, y_0)$  is a strong Henig-proper efficient solution of  $(P_3)$  and  $Q = \text{int}C_\varepsilon(B)$  for  $\varepsilon$  satisfying  $0 < \varepsilon < \delta$ .

The next example illustrates a case, Theorem 5.4.6 is more advantageous than earlier existing results.

**Example 5.4.7.** Let  $X = Y = \mathbb{R}$ ,  $C = \mathbb{R}_+$ ,  $g(x) = x^4 - 2x^3$ , and  $F(x) = \mathbb{R}_-$  for all  $x \in X$ . Then,  $S = [0, 2]$  and  $G(x) = \max\{0, g(x)\}$ . Furthermore, since  $T_{\text{epi}F}(0, 0) = \mathbb{R}^2$  and  $T_{\text{epi}G}(0, 0) = \{(x, y) : y \geq 0\}$ , the contingent epiderivative  $EDF(0, 0)(h)$  does not exist for any  $h \in X$  and Proposition 5.1 of Jahn and Khan in [94] cannot be applied. But we have proto-radial sets  $T_{F_+(X)}^{r(1)}(0) = \mathbb{R}$  and  $T_{G_+(X)}^{r(1)}(0) = \mathbb{R}_+$ . So,

$$(T_{F_+(X)}^{r(1)}(0) + sT_{G_+(X)}^{r(1)}(0)) \cap (-\text{int}C) \neq \emptyset.$$

By Theorem 5.4.6(i),  $(x_0, y_0)$  is not a weak efficient solution of  $(P_3)$ . This fact can be checked directly too.

# Chapter 6

## Calculus rules and applications of Studniarski derivatives to sensitivity and implicit function theorems

### 6.1 Introduction

In set-valued analysis, one of the most popular and useful higher-order derivatives is the contingent derivative introduced in [10] by Aubin in 1981. However, the set  $D^m F(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1})(u)$  (its definition is given in Remark 4.2.7 in Chapter 4) is non-empty only if  $v_1 \in DF(x_0, y_0)(u_1)$ ,  $\dots$ ,  $v_{m-1} \in D^{m-1} F(x_0, y_0, u_1, v_1, \dots, u_{m-2}, v_{m-2})(u_{m-1})$ . In applications, even the need of having these  $m - 1$  points may lead to inconvenience. In 1986, the following modification was proposed in [156] by Studniarski, without the “intermediate” orders in definition and hence without the need of these  $m - 1$  points,

$$D^m F(x_0, y_0)(x) := \{v \in Y : \exists t_n \rightarrow 0^+, \exists (x_n, v_n) \rightarrow (x, v), \forall n, y_0 + t_n^m v_n \in F(x_0 + t_n x_n)\}.$$

We can write the following two equivalent formulations as follows

$$D^m F(x_0, y_0)(u) = \operatorname{Limsup}_{(t, u') \rightarrow (0^+, u)} \frac{F(x_0 + t u') - y_0}{t^m},$$

and, by setting  $(x_n, y_n) := (x_0 + t_n u_n, y_0 + t_n^m v_n)$ ,  $\gamma_n = t_n^{-1}$ ,

$$D^m F(x_0, y_0)(u) = \{v \in Y : \exists \gamma_n > 0, \exists (x_n, y_n) \in \operatorname{gr} F : (x_n, y_n) \rightarrow (x_0, y_0), \\ (\gamma_n(x_n - x_0), \gamma_n^m(y_n - y_0)) \rightarrow (u, v)\}.$$

This object is called, by several authors, the Studniarski derivative. In nonsmooth optimization, it was applied in obtaining optimality conditions, e.g., in [97, 99, 117, 124, 144, 156, 159] and in discussing sensitivity analysis in [158] by Sun and Li.

The idea of omitting “intermediate” orders in defining higher-order derivatives was continued in [8, 49]. Namely, several notions of higher-order derivatives were developed, combining the idea of extending to higher-order the radial derivative proposed by Taa in [160] (for the first-order) with this omitting. In that way, global (not local as with the above two derivatives) higher-order optimality conditions were established for nonconvex optimization. (The main technical change in the above definitions is replacing  $\exists t_n \rightarrow 0^+$  by  $\exists t_n > 0$ .) The possibility for global consideration is good for optimality conditions, but for some other topics like sensitivity analysis or implicit function theorems, this may be inconvenient.

Thus, in this chapter, we return to the Studniarski derivative. Namely, we are concerned with two topics. First we develop calculus rules for this derivative, observing that these rules have not been studied, but a kind of derivatives is significant only if it enjoys enough calculus rules. Later, we use the Studniarski derivative just to the two mentioned topics of sensitivity analysis and implicit function theorems to ensure that in this paper we can investigate what is difficult for the derivatives considered in [8]. The content of this chapter is also our results in [6].

## 6.2 The Studniarski derivative

Let  $X, Y$  be normed spaces,  $F : X \rightarrow 2^Y$ ,  $(x_0, y_0) \in \text{gr}F$ ,  $u \in X$ , and  $m \geq 1$ .

**Definition 6.2.1.** The  $m$ -th order Studniarski derivative of  $F$  at  $(x_0, y_0)$  is defined by

$$D^m F(x_0, y_0)(x) := \text{Limsup}_{(t, x') \rightarrow (0^+, x)} \frac{F(x_0 + tx') - y_0}{t^m},$$

or, equivalently,

$$D^m F(x_0, y_0)(x) = \{v \in Y : \exists t_n \rightarrow 0^+, \exists (x_n, v_n) \rightarrow (x, v), \forall n, y_0 + t_n^m v_n \in F(x_0 + t_n x_n)\}.$$

The  $m$ -th order Studniarski derivative can be expressed as the  $\Gamma$ -limit as follows

$$\begin{aligned} v \in D^m F(x_0, y_0)(x) &\iff \inf_{Q \in \mathcal{N}(v)} \inf_{W \in \mathcal{N}(x)} \inf_{t > 0} \sup_{0 < t' < t} \sup_{x' \in W} \sup_{v' \in Q} \chi_{\text{gr}(\mathcal{L}_{F, (x_0, y_0)}}(t', x', v') = 1 \\ &\iff \Gamma(\mathcal{N}_+(0)^+, \mathcal{N}(x)^+, \mathcal{N}(v)^+) \chi_{\text{gr}(\mathcal{L}_{F, (x_0, y_0)}} = 1, \end{aligned}$$

where  $\mathcal{L}_{F,(x_0,y_0)} : (0, +\infty) \times X \rightarrow 2^Y$  is defined by

$$\mathcal{L}_{F,(x_0,y_0)}(t', x') := \frac{1}{t'^m} (F(x_0 + t'x') - y_0).$$

If the upper limit in Definition 6.2.1 is a full limit, i.e., the upper limit coincides with the lower limit for all  $u$ , then the map  $F$  is called to have a  $m$ -th order *proto-Studniarski derivative* at  $(x_0, y_0)$ .

**Example 6.2.2.** Let  $X = Y = \mathbb{R}$  and  $F_n : X \rightarrow 2^Y$ ,  $n \in \mathbb{N}$ , be defined by, for all  $x \in X$ ,  $F_n(x) = \{y \in Y : y \geq x^n\}$ . By calculating, we can find the  $m$ -th order Studniarski derivatives of  $F_n$  at  $(x_0, y_0) = (0, 0)$  as follows

If  $m = n$ , then for all  $u \in X$ ,  $D^m F_n(x_0, y_0)(u) = \{y \in Y : y \geq u^n\}$ .

If  $m < n$ , then for all  $u \in X$ ,  $D^m F_n(x_0, y_0)(u) = \mathbb{R}_+$ .

If  $m > n$ , then

$$D^m F_n(x_0, y_0)(u) = \begin{cases} \mathbb{R}, & \text{if } n = 2k - 1 \text{ (} k = 1, 2, \dots \text{) and } u \leq 0, \\ \mathbb{R}_+, & \text{if } n = 2k \text{ (} k = 1, 2, \dots \text{) and } u = 0, \\ \emptyset, & \text{otherwise.} \end{cases}$$

In the following example, we compute the Studniarski derivative of a map into an infinite dimensional case.

**Example 6.2.3.** Let  $X = \mathbb{R}$  and  $Y = l^2$ , the Hilbert space of the numerical sequences  $x = (x_i)_{i \in \mathbb{N}}$  with  $\sum_{i=1}^{\infty} x_i^2$  being convergent. By  $(e_i)_{i \in \mathbb{N}}$  we denote standard unit basis of  $l^2$ . Let  $f : X \rightarrow Y$  be defined by

$$f(x) := \begin{cases} \frac{1}{n}(-e_1 + 2e_n), & \text{if } x = \frac{1}{n}, \\ 0, & \text{otherwise} \end{cases}$$

and  $(x_0, y_0) = (0, 0)$ . We find the higher-order Studniarski derivatives of  $f$  at  $(x_0, y_0)$ . It follows from Definition 2.1 that  $v \in D^m f(x_0, y_0)(u)$  means the existence of  $t_k \rightarrow 0^+$ ,  $u_k \rightarrow u$ , and  $v_k \rightarrow v$  such that

$$y_0 + t_k^m v_k \in f(x_0 + t_k u_k). \quad (6.1)$$

For all  $u \in X$ , we can choose  $t_k \rightarrow 0^+$ ,  $u_k \rightarrow u$  such that  $t_k u_k \neq 1/k$ . So, for all  $u \in X$ ,

$$\{0\} \subseteq D^m f(x_0, y_0)(u).$$

We now prove that, for each  $v \in Y \setminus \{0\}$ ,  $v \notin D^m f(x_0, y_0)(u)$  for all  $u \in X$ . Suppose, on the contrary, there exist  $u \in U$  and  $v \in Y \setminus \{0\}$  such that  $v \in D^m f(x_0, y_0)(u)$ , i.e., there are  $t_k \rightarrow 0^+$ ,  $u_k \rightarrow u$ ,  $v_k \rightarrow v$  such that (6.1) holds. If  $t_k u_k \neq 1/k$  for infinitely many  $k \in \mathbb{N}$ , we get a contradiction easily. Hence, assume that  $t_k u_k = 1/k$ . Then (6.1) becomes  $v_k = \frac{1}{k \cdot t_k^m} (-e_1 + 2e_k)$ . If  $1/(k \cdot t_k^m) \rightarrow +\infty$ , we get a contradiction to the convergence of the sequence  $(-e_1 + 2e_k)/(k \cdot t_k^m)$ . Suppose  $1/(k \cdot t_k^m) \rightarrow a \geq 0$ . As  $e_1/(k \cdot t_k^m) \rightarrow a e_1$ , the sequence  $e_k/(k \cdot t_k^m)$  converges to some  $c$ , i.e.,

$$\left\| \frac{2}{k \cdot t_k^m} e_k - c \right\|^2 \rightarrow 0,$$

that is,

$$\left\| \frac{2}{k \cdot t_k^m} e_k - c \right\|^2 = \left( \frac{2}{k \cdot t_k^m} \right)^2 + \|c\|^2 + 2 \left\langle \frac{2}{k \cdot t_k^m} e_k, -c \right\rangle \rightarrow 0. \quad (6.2)$$

Since  $(e_k)$  converges to 0 with respect to the weak topology, then  $\langle e_k, -c \rangle \rightarrow 0$ . From (6.2), we get  $4a^2 + \|c\|^2 = 0$ . If  $a = 0$ , then  $c = v (\neq \emptyset)$  since  $(-e_1 + 2e_k)/(k \cdot t_k^m) \rightarrow v$ . If  $a > 0$ , then  $4a^2 + \|c\|^2 \neq 0$ . Therefore, we always have a contradiction. Thus, for all  $u \in X$ ,  $D^m f(x_0, y_0)(u) = \{0\}$ .

We now present a condition for a  $m$ -th order Studniarski derivative to be nonempty.

**Proposition 6.2.4.** *Let  $\dim Y < +\infty$ ,  $(x_0, y_0) \in \text{gr } F$ , and  $x_0 \in \text{int}(\text{dom } F)$ . Suppose that*

- (i)  *$F$  is lower semicontinuous at  $(x_0, y_0)$ ,*
- (ii)  *$F$  is  $m$ -th order locally pseudo-Hölder calm at  $x_0$  for  $y_0$ .*

*Then,  $D^m F(x_0, y_0)(x) \neq \emptyset$  for all  $x \in X$ .*

*Proof.* For  $x = 0$ , this is trivial because we always have  $0 \in D^m F(x_0, y_0)(0)$ . By assumption (ii), there exist  $\lambda > 0$ ,  $U_1 \in \mathcal{N}(x_0)$  and  $V \in \mathcal{N}(y_0)$  such that  $\forall x' \in U_1$ ,

$$(F(x') \cap V) \subseteq \{y_0\} + \lambda \|x' - x_0\|^m B_Y.$$

By assumption (i), with  $V$  above, there exists  $U_2 \in \mathcal{N}(x_0)$  such that  $\forall \hat{x} \in U_2$ ,  $V \cap F(\hat{x}) \neq \emptyset$ . It follows from  $x_0 \in \text{int}(\text{dom } F)$  that there exists  $U_3 \in \mathcal{N}(x_0)$  such that  $U_3 \subseteq \text{dom } F$ . Setting  $\hat{U} = U_1 \cap U_2 \cap U_3$ , we get  $\hat{U} \in \mathcal{N}(x_0)$ . Let an arbitrary  $x \in X \setminus \{0\}$  and  $t_n \rightarrow 0^+$ . Because  $x_0 + t_n x \rightarrow x_0$ , we get  $x_0 + t_n x \in \hat{U}$  for large  $n$ . Hence, there exists  $y_n \in F(x_0 + t_n x) \cap V$  such that

$$t_n^{-m} \|y_n - y_0\| \leq \lambda \|x\|^m.$$

So,  $t_n^{-m}(y_n - y_0)$  is a bounded sequence and hence has a convergent subsequence. By Definition 6.2.1, the limit of this subsequence is an element of the set  $D^m F(x_0, y_0)(x)$ .  $\square$

**Example 6.2.5.** (assumption (ii) is essential) Let  $F : \mathbb{R} \rightarrow 2^{\mathbb{R}}$  be defined by

$$F(x) = \begin{cases} \{x^{1/3}\}, & \text{if } 0 \leq x \leq 1, \\ \{x\}, & \text{if } x > 1, \\ \{-x\}, & \text{if } -1 \leq x < 0, \\ \{-x^{1/3}\}, & \text{if } x < -1. \end{cases}$$

Direct computations yield that  $D^m F(0,0)(1) = \emptyset$  for all  $m \geq 1$ . Here,  $F$  is lower semicontinuous at  $(0,0)$ , but the  $m$ -th order locally pseudo-Hölder calmness fails.

**Example 6.2.6.** (assumption (i) cannot be dropped) Let  $F : \mathbb{R} \rightarrow 2^{\mathbb{R}}$  be defined by

$$F(x) = \begin{cases} \{1\}, & \text{if } x = 0, \\ \{y : y \leq x\}, & \text{if } x \neq 0. \end{cases}$$

Then, assumption (ii) is satisfied at  $(0,1)$ . Direct calculations give that  $D^m F(0,1)(1) = \emptyset$  for all  $m \geq 1$ . The cause is that  $F$  is not lower semicontinuous at  $(0,1)$ , since  $F$  is  $m$ th-order locally pseudo-Hölder calm at 0 for 1. Indeed, pick  $\lambda = 1, U = \{x \in \mathbb{R} : -1/2 < x < 1/2\}, V = \{y \in \mathbb{R} : 1/2 < y < 3/2\}$ . Then,  $F(x) = \{y \in \mathbb{R} : y \leq x\} \subset (-\infty, 1/2)$  for all  $x \in U \setminus \{0\}$ . Therefore,  $F(x) \cap V = \emptyset$  for all  $x \in U \setminus \{0\}$ , and

$$F(0) \cap V = \{1\} \subset \{y_0\} + \|x\|^m B_Y$$

for all  $m \geq 1$ .

A map  $F$  is said to have a *strict Studniarski derivative* at  $(x_0, y_0) \in \text{gr}F$  if

$$D^m F(x_0, y_0)(u) = \{v \in Y : \forall t_n \rightarrow 0^+, \exists (u_n, v_n) \rightarrow (u, v), \forall n, y_0 + t_n^m v_n \in F(x_0 + t_n u_n)\}.$$

**Proposition 6.2.7.** Let  $F : X \rightarrow 2^Y, (x_0, y_0) \in \text{gr}F$ , and  $F$  be a convex map and have a strict Studniarski derivative at  $(x_0, y_0)$ . Then,  $D^m F(x_0, y_0)$  is convex.

*Proof.* Let  $x^1, x^2 \in X$  and  $y^i \in D^m F(x_0, y_0)(x^i), i = 1, 2$ , i.e., for any  $t_n \rightarrow 0^+$ , there exists  $(x_n^i, y_n^i) \rightarrow (x^i, y^i)$  such that, for all  $n, y_n^i \in t_n^{-m}(F(x_0 + t_n x_n^i) - y_0)$ . Since  $F$  is convex, for all  $\lambda \in [0, 1]$ ,

$$\lambda \left( \frac{F(x_0 + t_n x_n^1) - y_0}{t_n^m} \right) + (1 - \lambda) \left( \frac{F(x_0 + t_n x_n^2) - y_0}{t_n^m} \right) \subseteq \frac{F(\lambda(x_0 + t_n x_n^1) + (1 - \lambda)(x_0 + t_n x_n^2)) - y_0}{t_n^m}.$$

Therefore,

$$\lambda y_n^1 + (1 - \lambda)y_n^2 \in \frac{F(x_0 + t_n(\lambda x_n^1 + (1 - \lambda)x_n^2)) - y_0}{t_n^m}.$$

Hence,  $\lambda y^1 + (1 - \lambda)y^2 \in D^m F(x_0, y_0)(\lambda x^1 + (1 - \lambda)x^2)$ .  $\square$

The next statement is a relation between the Studniarski derivative of  $F$  and that of the profile map.

**Proposition 6.2.8.** *Let  $F : X \rightarrow 2^Y$ , and  $(x_0, y_0) \in \text{gr } F$ . Then, for all  $x \in X$ ,*

$$D^m F(x_0, y_0)(x) + C \subseteq D^m(F + C)(x_0, y_0)(x). \quad (6.3)$$

If  $\dim Y < +\infty$  and  $F$  is  $m$ -th order locally Hölder calm at  $x_0$  for  $y_0$ , then (6.3) becomes an equality.

*Proof.* Let  $w \in D^m F(x_0, y_0)(x) + C$ , i.e., there exists  $v \in D^m F(x_0, y_0)(x)$  and  $c \in C$  such that  $w = v + c$ . We then have sequences  $t_n \rightarrow 0^+$ ,  $x_n \rightarrow x$ , and  $v_n \rightarrow v$  such that, for all  $n$ ,

$$y_0 + t_n^m(v_n + c) \in F(x_0 + t_n x_n) + t_n^m c \subseteq F(x_0 + t_n x_n) + C.$$

So,  $v + c \in D^m(F + C)(x_0, y_0)(x)$ .

Let  $w \in D^m(F + C)(x_0, y_0)(x)$ , i.e., there exist  $t_n \rightarrow 0^+$ ,  $x_n \rightarrow x$ ,  $w_n \rightarrow w$  such that  $y_0 + t_n^m w_n \in F(x_0 + t_n x_n) + C$ . Then, there exist  $y_n \in F(x_0 + t_n x_n)$  and  $c_n \in C$  satisfying

$$w_n = t_n^{-m}(y_n - y_0) + t_n^{-m}c_n. \quad (6.4)$$

Because  $F$  is  $m$ -th order locally Hölder calm at  $x_0$  for  $y_0$ , there exists  $\lambda > 0$  such that, for large  $n$ ,

$$y_n \in F(x_0 + t_n x_n) \subseteq \{y_0\} + \lambda \|t_n x_n\|^m B_Y.$$

So,

$$t_n^{-m} \|y_n - y_0\| \leq \lambda \|x_n\|^m.$$

Since  $\dim Y < +\infty$ ,  $t_n^{-m}(y_n - y_0)$  (using a subsequence, if necessary) converges to some  $v$  and  $v \in D^m F(x_0, y_0)(x)$ . From (6.4), the sequence  $c_n/t_n^m$  converges to some  $c \in C$  and  $w = v + c$ . Thus  $w \in D^m F(x_0, y_0)(x) + C$ .  $\square$



### 6.3 Calculus rules

**Proposition 6.3.1.** (Sum rule) *Let  $F_1, F_2 : X \rightarrow 2^Y$ ,  $x_0 \in \text{dom} F_1 \cap \text{dom} F_2$ ,  $y_i \in F(x_i)$  ( $i=1,2$ ) and  $u \in X$ . Suppose either  $F_1$  or  $F_2$  has a  $m$ -th order proto-Studniarski derivative at  $(x_0, y_1)$  or  $(x_0, y_2)$ , respectively. Then*

$$D^m F_1(x_0, y_1)(u) + D^m F_2(x_0, y_2)(u) \subseteq D^m(F_1 + F_2)(x_0, y_1 + y_2)(u). \quad (6.5)$$

*If, additionally,  $\dim Y < +\infty$  and either  $F_1$  or  $F_2$  is  $m$ -th order locally Hölder calm at  $x_0$  for  $y_1$  or  $y_2$ , respectively, then (6.5) becomes an equality.*

*Proof.* Consider  $v^i \in D^m F_i(x_0, y_i)(u)$ . For  $v^1$ , there exist  $t_n \rightarrow 0^+$ ,  $u_n \rightarrow u$ , and  $v_n^1 \rightarrow v^1$  such that, for all  $n$ ,  $y_1 + t_n^m v_n^1 \in F_1(x_0 + t_n u_n)$ . For  $v^2$ , supposing that  $F_2$  has the  $m$ -th order proto-Studniarski derivative at  $(x_0, y_2)$ , with  $t_n, u_n$  above, there exists  $v_n^2 \rightarrow v^2$  such that  $y_2 + t_n^m v_n^2 \in F_2(x_0 + t_n u_n)$ . Hence,  $y_1 + y_2 + t_n^m(v_n^1 + v_n^2) \in (F_1 + F_2)(x_0 + t_n u_n)$  and  $v^1 + v^2 \in D^m(F_1 + F_2)(x_0, y_1 + y_2)(u)$ .

Let  $v \in D^m(F_1 + F_2)(x_0, y_1 + y_2)(u)$ , i.e., there exist  $t_n \rightarrow 0^+$ ,  $u_n \rightarrow u$ , and  $v_n \rightarrow v$  such that

$$y_1 + y_2 + t_n^m v_n \in (F_1 + F_2)(x_0 + t_n u_n) = F_1(x_0 + t_n u_n) + F_2(x_0 + t_n u_n).$$

This means that there exist  $y_n^i \in F_i(x_0 + t_n u_n)$ ,  $i = 1, 2$ , such that

$$v_n = t_n^{-m}(y_n^1 - y_1) + t_n^{-m}(y_n^2 - y_2). \quad (6.6)$$

Suppose  $F_1$  is  $m$ -th order locally Hölder calm at  $x_0$  for  $y_1$ , i.e., there exists  $L > 0$  such that, for large  $n$ ,

$$y_n^1 \in F_1(x_0 + t_n^m u_n) \subseteq \{y_1\} + L \|t_n u_n\|^m B_Y.$$

Because  $\dim Y < +\infty$ ,  $t_n^{-m}(y_n^1 - y_1)$  (using a subsequence, if necessary) converges to some  $v^1$  and hence  $v^1 \in D^m F_1(x_0, y_1)(u)$ . From (6.6), the sequence  $t_n^{-m}(y_n^2 - y_2)$  also converges to some  $v^2$  such that  $v^2 = v - v^1$ , and  $v^2 \in D^m F_2(x_0, y_2)(u)$ . Thus,  $v \in D^m F_1(x_0, y_1)(u) + D^m F_2(x_0, y_2)(u)$ .  $\square$

**Proposition 6.3.2.** (Chain rule) *Let  $F : X \rightarrow 2^Y$ ,  $G : Y \rightarrow 2^Z$ ,  $(x_0, y_0) \in \text{gr} F$ ,  $(y_0, z_0) \in \text{gr} G$ , and  $\text{Im} F \subseteq \text{dom} G$ .*

(i) *Suppose  $G$  has a  $m$ -th order proto-Studniarski derivative at  $(y_0, z_0)$ . Then, for all  $u \in X$ ,*

$$D^m G(y_0, z_0)(D^1 F(x_0, y_0)(u)) \subseteq D^m(G \circ F)(x_0, z_0)(u). \quad (6.7)$$

If, additionally,  $\dim Y < +\infty$  and  $F$  is locally Lipschitz calm at  $x_0$  for  $y_0$ , then (6.7) becomes an equality.

(ii) Suppose  $G$  has a first-order proto-Studniarski derivative at  $(y_0, z_0)$ . Then, for all  $u \in X$ ,

$$D^1 G(y_0, z_0)(D^m F(x_0, y_0)(u)) \subseteq D^m(G \circ F)(x_0, z_0)(u). \quad (6.8)$$

If, additionally,  $\dim Y < +\infty$  and  $F$  is  $m$ -th order locally Hölder calm at  $x_0$  for  $y_0$ , then (6.8) becomes an equality.

*Proof.* By the similarity, we prove only (i). Let  $w \in D^m G(y_0, z_0)(D^1 F(x_0, y_0)(u))$ , i.e., there exists  $v \in D^1 F(x_0, y_0)(u)$  such that  $w \in D^m G(y_0, z_0)(v)$ . There exist  $t_n \rightarrow 0^+$ ,  $u_n \rightarrow u$ , and  $v_n \rightarrow v$  such that  $y_0 + t_n v_n \in F(x_0 + t_n u_n)$ . With  $t_n, v_n$  above, we have  $w_n \rightarrow w$  such that  $z_0 + t_n^m w_n \in G(y_0 + t_n v_n)$ . So,  $z_0 + t_n^m w_n \in G(F(x_0 + t_n u_n))$ . Thus,  $w \in D^m(G \circ F)(x_0, z_0)(u)$ .

Let  $w \in D^m(G \circ F)(x_0, z_0)(u)$ , i.e., there exists  $t_n \rightarrow 0^+$ ,  $u_n \rightarrow u$ , and  $w_n \rightarrow w$  such that  $z_0 + t_n^m w_n \in G(F(x_0 + t_n u_n))$ . Then, there exists  $y_n \in F(x_0 + t_n u_n)$  such that  $z_0 + t_n^m w_n \in G(y_n)$ . Due to the local Lipschitz calmness of  $F$  and the finiteness of  $\dim Y$ , the sequence  $v_n := t_n^{-1}(y_n - y_0)$ , or a subsequence, converges to some  $v$  and  $v \in D^1 F_1(x_0, y_0)(u)$ . This implies that  $z_0 + t_n^m w_n \in G(y_0 + t_n v_n)$  and hence  $w \in D^m G(y_0, z_0)(v)$ .  $\square$

We next discuss calculus rules for the following operations.

**Definition 6.3.3.** (i) For  $F_1, F_2 : X \rightarrow 2^{\mathbb{R}^k}$ ,  $\mathbb{R}^k$  being an Euclidean space, the product of  $F_1$  and  $F_2$  is the set-valued map  $\langle F_1, F_2 \rangle : X \rightarrow 2^{\mathbb{R}^k}$  defined by  $\langle F_1, F_2 \rangle(x) := \{\langle y_1, y_2 \rangle : y_1 \in F_1(x), y_2 \in F_2(x)\}$ .

(ii) For  $F_1, F_2 : X \rightarrow 2^{\mathbb{R}}$ , the quotient of  $F_1$  and  $F_2$  is the set-valued map  $F_1/F_2 : X \rightarrow 2^{\mathbb{R}}$  defined by  $(F_1/F_2)(x) := \{y_1/y_2 : y_1 \in F_1(x), y_2 \in F_2(x), y_2 \neq 0\}$ .

**Proposition 6.3.4.** (Product rule) Let  $F_1, F_2 : X \rightarrow 2^{\mathbb{R}^k}$ ,  $x_0 \in \text{dom } F_1 \cap \text{dom } F_2$ ,  $y_i \in F_1(x_0)$  ( $i=1,2$ ). Suppose either  $F_1$  or  $F_2$  has a  $m$ -th order proto-Studniarski derivative at  $(x_0, y_1)$  or  $(x_0, y_2)$ , respectively. Then, for all  $u \in X$ ,

$$\langle y_2, D^m F_1(x_0, y_1)(u) \rangle + \langle y_1, D^m F_2(x_0, y_2)(u) \rangle \subseteq D^m(\langle F_1, F_2 \rangle)(x_0, \langle y_1, y_2 \rangle)(u). \quad (6.9)$$

If, additionally,  $F_i$  are  $m$ -th order locally Hölder calm at  $x_0$  for  $y_i$ ,  $i = 1, 2$ , then (6.9) becomes an equality.

*Proof.* Consider  $v^i \in D^m F_i(x_0, y_i)(u)$ . There exist  $t_n \rightarrow 0^+$ ,  $u_n \rightarrow u$ , and  $v_n^1 \rightarrow v^1$  such that  $y_1 + t_n^m v_n^1 \in F_1(x_0 + t_n u_n)$ . Supposing  $F_2$  has the  $m$ -th order proto-Studniarski derivative at  $(x_0, y_2)$ ,

with  $t_n, u_n$  above, there exists  $v_n^2 \rightarrow v^2$  such that  $y_2 + t_n^m v_n^2 \in F_2(x_0 + t_n u_n)$ . We have

$$\langle y_1 + t_n^m v_n^1, y_2 + t_n^m v_n^2 \rangle = \langle y_1, y_2 \rangle + t_n^m (\langle y_1, v_n^2 \rangle + \langle y_2, v_n^1 \rangle + t_n^m \langle v_n^1, v_n^2 \rangle),$$

and

$$\langle y_1 + t_n^m v_n^1, y_2 + t_n^m v_n^2 \rangle \in \langle F_1, F_2 \rangle(x_0 + t_n u_n).$$

This implies that  $\langle y_1, v^2 \rangle + \langle y_2, v^1 \rangle \in D^m(\langle F_1, F_2 \rangle)(x_0, \langle y_1, y_2 \rangle)(u)$ .

Let  $v \in D^m(\langle F_1, F_2 \rangle)(x_0, \langle y_1, y_2 \rangle)(u)$ , i.e., there exist  $t_n \rightarrow 0^+$ ,  $u_n \rightarrow u$ , and  $v_n \rightarrow v$  such that  $\langle y_1, y_2 \rangle + t_n^m v_n \in \langle F_1, F_2 \rangle(x_0 + t_n u_n)$ . Then, there exist  $y_n^i \in F_i(x_0 + t_n u_n)$  such that  $\langle y_1, y_2 \rangle + t_n^m v_n = \langle y_n^1, y_n^2 \rangle$ . We have

$$\langle y_n^1, y_n^2 \rangle = \langle y_n^1 - y_1 + y_1, y_n^2 - y_2 + y_2 \rangle = \langle y_n^1 - y_1, y_n^2 - y_2 \rangle + \langle y_n^1 - y_1, y_2 \rangle + \langle y_n^2 - y_2, y_1 \rangle + \langle y_1, y_2 \rangle.$$

This implies that

$$v_n = \left\langle \frac{y_n^1 - y_1}{t_n^m}, y_2 \right\rangle + \left\langle \frac{y_n^2 - y_2}{t_n^m}, y_1 \right\rangle + t_n^m \left\langle \frac{y_n^1 - y_1}{t_n^m}, \frac{y_n^2 - y_2}{t_n^m} \right\rangle. \quad (6.10)$$

Because  $F_i$  is  $m$ -th order locally Hölder calm at  $x_0$  for  $y_i$ , there exists  $L_i > 0$  such that, for large  $n$ ,

$$y_n^i \in F_i(x_0 + t_n^m u_n) \subseteq \{y_i\} + L_i \|t_n u_n\|^m B_Y.$$

This implies that there are two subsequence (the subscripts of the second one are taken among those of the first), denoted by the same notations  $t_n^{-m}(y_n^i - y_i)$ , converging to some  $v^i \in \mathbb{R}^k$  and  $v^i \in D^m F_i(x_0, y_i)(u)$ ,  $i = 1, 2$ . Thus, from (6.10),  $v \in \langle D^m F_1(x_0, y_1)(u), y_2 \rangle + \langle D^m F_2(x_0, y_2)(u), y_1 \rangle$ .  $\square$

**Proposition 6.3.5.** (Quotient rule) *Let  $F_1, F_2 : X \rightarrow 2^{\mathbb{R}}$ ,  $x_0 \in \text{dom} F_1 \cap \text{dom} F_2$ , and  $y_i \in F_i(x_0)$  ( $i=1,2$ ) with  $y_2 \neq 0$ . Suppose either  $F_1$  or  $F_2$  has a  $m$ -th order proto-Studniarski derivative at  $(x_0, y_1)$  or  $(x_0, y_2)$ , respectively. Then, for all  $u \in X$ ,*

$$\frac{1}{y_2} (y_2 D^m F_1(x_0, y_1)(u) - y_1 D^m F_2(x_0, y_2)(u)) \subseteq D^m((F_1/F_2)(x_0, y_1/y_2)(u)). \quad (6.11)$$

*If, in addition,  $F_2$  is  $m$ -th order locally Hölder calm at  $x_0$  for  $y_2$ , then (6.11) becomes an equality.*

*Proof.* Consider  $v^i \in D^m F_i(x_0, y_i)(u)$ . There exist  $t_n \rightarrow 0^+$ ,  $u_n \rightarrow u$ , and  $v_n^1 \rightarrow v^1$  such that  $y_1 + t_n^m v_n^1 \in F_1(x_0 + t_n u_n)$ . Supposing  $F_2$  has the  $m$ -th order proto-Studniarski derivative at  $(x_0, y_2)$ , with  $t_n, u_n$  above, there exists  $v_n^2 \rightarrow v^2$  such that  $y_2 + t_n^m v_n^2 \in F_2(x_0 + t_n u_n)$ . We have

$$\frac{y_1 + t_n^m v_n^1}{y_2 + t_n^m v_n^2} = \frac{y_1}{y_2} + t_n^m \left( \frac{y_2 v_n^1 - y_1 v_n^2}{y_2^2 + t_n^m v_n^2 y_2} \right) \in (F_1/F_2)(x_0 + t_n u_n).$$

This implies that  $y_2^{-2}(y_2v^1 - y_1v^2) \in D^m((F_1/F_2)(x_0, y_1/y_2))(u)$ .

Let  $v \in D^m(F_1/F_2)(x_0, (y_1/y_2))(u)$ , i.e., there exist  $t_n \rightarrow 0^+$ ,  $u_n \rightarrow u$ , and  $v_n \rightarrow v$  such that  $(y_1/y_2) + t_n^m v_n \in (F_1/F_2)(x_0 + t_n u_n)$ . So, there exist  $y_n^i \in F_i(x_0 + t_n u_n)$  such that  $(y_1/y_2) + t_n^m v_n = y_n^1/y_n^2$ . We get

$$\frac{y_n^1}{y_n^2} = \frac{y_1}{y_2} + \frac{y_2(y_n^1 - y_1) - y_1(y_n^2 - y_2)}{y_2^2 + y_2(y_n^2 - y_2)},$$

and hence

$$v_n = \frac{y_2(y_n^1 - y_1)/t_n^m - y_1(y_n^2 - y_2)/t_n^m}{y_2^2 + t_n^m \cdot y_2(y_n^2 - y_2)/t_n^m}. \quad (6.12)$$

Because  $F_2$  is  $m$ -th order locally Hölder calm at  $x_0$  for  $y_2$ ,  $(y_n^2 - y_2)/t_n^m$  converges to some  $v^2$  with  $v^2 \in D^m F_2(x_0, y_2)(u)$ . From (6.12), the sequence  $(y_n^1 - y_1)/t_n^m$  also converges to  $v^1$  such that  $v^1 \in D^m F_1(x_0, y_1)(u)$  and  $v = y_2^{-2}(y_2v^1 - y_1v^2)$ . Thus,  $v \in y_2^{-2}(y_2 D^m F_1(x_0, y_1)(u) - y_1 D^m F_2(x_0, y_2)(u))$ .  $\square$

**Corollary 6.3.6.** (Reciprocal rule) *Let  $F : X \rightarrow 2^{\mathbb{R}}$ ,  $y_0 \in F(x_0)$  with  $y_0 \neq 0$ . Then, for all  $u \in X$ ,*

$$-y_0^{-2}(D^m F(x_0, y_0)(u) \subseteq D^m(1/F)(x_0, 1/y_0)(u)). \quad (6.13)$$

*If, in addition,  $F$  is  $m$ -th order locally Hölder calm at  $x_0$  for  $y_0$ , then (6.13) becomes an equality.*

In the rest of this section, we discuss other sum and chain rules, which may be more useful in some cases (see, e.g., Section 6.4). Let  $X, Y, Z$  be normed spaces. To investigate the sum  $M + N$  of multifunctions  $M, N : X \rightarrow 2^Y$ , we express  $M + N$  as a composition as follows. Define  $F : X \rightarrow 2^{X \times Y}$  and  $G : X \times Y \rightarrow 2^Y$  by, for  $I$  being the identity map on  $X$  and  $(x, y) \in X \times Y$ ,

$$F = I \times M \quad \text{and} \quad G(x, y) = y + N(x). \quad (6.14)$$

Then, clearly  $M + N = G \circ F$ .

First, we develop a chain rule. Let general multimaps  $F : X \rightarrow 2^Y$  and  $G : Y \rightarrow 2^Z$  be considered. The so-called resultant set-valued map  $C : X \times Z \rightarrow 2^Y$  is defined by

$$C(x, z) := F(x) \cap G^{-1}(z).$$

Then,  $\text{dom} C = \text{gr}(G \circ F)$ . We need the following compactness properties.

**Definition 6.3.7.** Let  $H : X \rightarrow 2^Y$  be a set-valued map.

(i)  $H$  is said to be *compact*, see [140], at  $x \in \text{cl}(\text{dom} H)$  if any sequence  $y_n \in H(x_n)$  satisfying  $x_n \rightarrow x$  has a convergent subsequence.

(ii)  $H$  is said to be *closed* at  $x$  if  $(\text{cl} H)(x) = H(x)$ , where  $\text{cl} H$  is the closure map of  $H$  defined by  $\text{gr}(\text{cl} H) = \text{cl}(\text{gr} H)$ .

Note that when  $H$  is compact at  $x$ , the image  $H(x)$  still may be not closed. Simply think of  $H : \mathbb{R} \rightarrow 2^{\mathbb{R}}$  equal to  $(0, 1)$  if  $x = 0$ , and to  $\{0\}$  if  $x \neq 0$ . Then,  $H$  is compact at 0, but  $H(0) = (0, 1)$  is not closed.

We define other kinds of  $m$ -th order Studniarski derivatives of  $G \circ F$  with respect to variable  $y$  as follows.

**Definition 6.3.8.** Let  $((x, z), y) \in \text{gr}C$ .

(i) The  $m$ -th order  $y$ -Studniarski derivative of  $G \circ F$  at  $((x, z), y)$  is defined as, for  $u \in X$ ,

$$D^m(G \circ_y F)(x, z)(u) := \{w \in Z : \exists t_n \rightarrow 0^+, \exists (u_n, y_n, w_n) \rightarrow (u, y, w), \forall n, y_n \in C(x + t_n u_n, z + t_n^m w_n)\}.$$

(ii) For an integer  $k$ , the  $m$ -th order pseudo-Studniarski derivative of the map  $C$  at  $(x, z)$  with respect to  $k$  is defined as, for  $(u, w) \in X \times Z$ ,

$$D_p^{m(k)}C((x, z), y)(u, w) := \{\bar{y} \in Y : \exists t_n \rightarrow 0^+, \exists (u_n, \bar{y}_n, w_n) \rightarrow (u, \bar{y}, w), \forall n, y + t_n^k \bar{y}_n \in C(x + t_n u_n, z + t_n^m w_n)\}.$$

If  $k = m$ , the set in Definition 6.3.8(ii) is denoted shortly by  $D_p^m C((x, z), y)(u, w)$ . One has a relationship between  $D^m(G \circ_y F)(x, z)(u)$  and  $D^m(G \circ F)(x, z)(u)$  in the following statement.

**Proposition 6.3.9.** Let  $(x, z) \in \text{gr}(G \circ F)$  and  $u \in X$ .

(i) For  $y \in C(x, z)$ , one has

$$D^m(G \circ_y F)(x, z)(u) \subseteq D^m(G \circ F)(x, z)(u).$$

(ii) If  $C$  is compact and closed at  $(x, z)$ , then

$$\bigcup_{y \in C(x, z)} D^m(G \circ_y F)(x, z)(u) = D^m(G \circ F)(x, z)(u).$$

*Proof.* (i) This follows immediately from the definitions.

(ii) “ $\subseteq$ ” follows from (i). For “ $\supseteq$ ”, let  $w \in D^m(G \circ F)(x, z)(u)$ , i.e., there exist sequences  $t_n \rightarrow 0^+$  and  $(u_n, w_n) \rightarrow (u, w)$  such that  $z + t_n^m w_n \in (G \circ F)(x + t_n u_n)$ . So, there exists  $y_n \in Y$  with  $y_n \in C(x + t_n u_n, z + t_n^m w_n)$ . Since  $C$  is compact at  $(x, z)$ ,  $y_n$  (or a subsequence) has a limit  $y$ . Since  $(x + t_n u_n, z + t_n^m w_n, y_n) \rightarrow (x, z, y)$ , one has  $y \in (\text{cl}C)(x, z)$ . It follows from the closedness of  $C$  at  $(x, z)$  that  $y \in C(x, z)$ . □

The first chain rule for  $G \circ F$  using these Studniarski derivatives is

**Proposition 6.3.10.** Let  $(x, z) \in \text{gr}(G \circ F)$  and  $y \in C(x, z)$ . Suppose, for all  $(u, w) \in X \times Z$ ,

$$D^m F(x, y)(u) \cap (D^1 G(y, z))^{-1}(w) \subseteq D_p^m C((x, z), y)(u, w). \quad (6.15)$$

Then

$$D^1 G(y, z)[D^m F(x, y)(u)] \subseteq D^m(G \circ_y F)(x, z)(u).$$

*Proof.* Let  $v \in D^1 G(y, z)[D^m F(x, y)(u)]$ , i.e., there exists  $\bar{y} \in D^m F(x, y)(u)$  such that  $\bar{y} \in (D^1 G(y, z))^{-1}(v)$ . Then, (6.15) ensures that  $\bar{y} \in D_p^m C((x, z), y)(u, v)$ . This means the existence of  $t_n \rightarrow 0^+$  and  $(u_n, \bar{y}_n, v_n) \rightarrow (u, \bar{y}, v)$  such that  $y + t_n^m \bar{y}_n \in C(x + t_n u_n, z + t_n^m v_n)$ . We have  $y_n := y + t_n^m \bar{y}_n \in C(x + t_n u_n, z + t_n^m v_n)$ . So,  $v \in D^m(G \circ_y F)(x, z)(u)$  and we are done.  $\square$

**Proposition 6.3.11.** Let  $(x, z) \in \text{gr}(G \circ F)$  and  $y \in C(x, z)$ . Suppose, for all  $(u, w) \in X \times Z$ ,

$$D^1 F(x, y)(u) \cap (D^m G(y, z))^{-1}(w) \subseteq D_p^{m(1)} C((x, z), y)(u, w). \quad (6.16)$$

Then

$$D^m G(y, z)[D^1 F(x, y)(u)] \subseteq D^m(G \circ_y F)(x, z)(u).$$

*Proof.* The proof is similar to that of Proposition 6.3.10.  $\square$

Note that, when  $m = 1$ , we have  $(D^1 G(y, z))^{-1} = D^1 G^{-1}(z, y)$ , however this is not true for  $m \geq 2$  as shown in the following example.

**Example 6.3.12.** Let  $F : \mathbb{R} \rightarrow \mathbb{R}$  be defined by  $F(x) = x^2$ . Then,

$$F^{-1}(y) = \begin{cases} \{-\sqrt{y}, \sqrt{y}\}, & \text{if } y \geq 0, \\ \emptyset, & \text{if } y < 0. \end{cases}$$

Direct computations yield that  $D^1 F(0, 0)(u) = \{0\}$  for all  $u \in \mathbb{R}$ , which implies that  $(D^1 F(0, 0))^{-1}(0) = \mathbb{R}$  and  $(D^1 F(0, 0))^{-1}(v) = \emptyset$  for  $v \neq 0$ . It is easy to check that  $D^1 F^{-1}(0, 0)$  coincides with  $(D^1 F(0, 0))^{-1}$ .

For  $m = 2$ ,  $D^2 F(0, 0)(u) = \{u^2\}$  for all  $u \in \mathbb{R}$ , which implies

$$(D^2 F(0, 0))^{-1}(y) = \begin{cases} \{-\sqrt{y}, \sqrt{y}\}, & \text{if } y \geq 0, \\ \emptyset, & \text{if } y < 0. \end{cases}$$

However,

$$D^2F^{-1}(0,0)(v) = \begin{cases} \mathbb{R}, & \text{if } v = 0, \\ \emptyset, & \text{if } v \neq 0. \end{cases}$$

To get a chain rule for Studniarski derivatives in the form of equalities, we first prove the inclusions reverse to those in Propositions 6.3.10(i), 6.3.11(i) under additional assumptions as follows.

**Proposition 6.3.13.** *Let  $y \in C(x, z)$  and  $Y$  be finite dimensional.*

(i) *If*

$$D_p^m C((x, z), y)(0, 0) = \{0\}, \quad (6.17)$$

*then*

$$D^m(G \circ_y F)(x, z)(u) \subseteq D^1 G(y, z)[D^m F(x, y)(u)].$$

(ii) *If*

$$D_p^{m(1)} C((x, z), y)(0, 0) = \{0\}, \quad (6.18)$$

*then*

$$D^m(G \circ_y F)(x, z)(u) \subseteq D^m G(y, z)[D^1 F(x, y)(u)].$$

*Proof.* By the similarity, we prove only (i). Let  $w \in D^m(G \circ_y F)(x, z)(u)$ , i.e., there exist  $t_n \rightarrow 0^+$  and  $(u_n, y_n, w_n) \rightarrow (u, y, w)$  such that  $y_n \in C(x + t_n u_n, z + t_n^m w_n)$ . If  $y_k = y$  for infinitely many  $k \in \mathbb{N}$ , one has  $0 \in D^m F(x, y)(u)$ ,  $w \in D^1 G(y, z)(0)$  and we are done. Thus, suppose  $y_n \neq y$  for all  $n$  and, for  $s_n := \|y_n - y\|^{1/m}$ , the sequence  $v_n := s_n^{-m}(y_n - y)$  or some subsequence has a limit  $v$  of norm one. If  $t_n/s_n \rightarrow 0$ , since

$$y + s_n^m v_n = y_n \in C\left(x + s_n \left(\frac{t_n u_n}{s_n}\right), z + s_n^m \left(\frac{t_n^m w_n}{s_n^m}\right)\right),$$

one sees that  $v \in D_p^m C((x, z), y)(0, 0)$ , contradicting (6.17). Consequently,  $t_n^{-1} s_n$  has a bounded subsequence and one may assume that  $t_n^{-1} s_n$  tends to  $q \in R_+$ . So,

$$y + t_n^m (s_n^m v_n / t_n^m) = y_n \in C(x + t_n u_n, z + t_n^m w_n)$$

and then one gets  $q^m v \in D_p^m C((x, z), y)(u, w)$ . It follows from the definition of  $D_p^m C((x, z), y)(u, w)$  that  $q^m v \in D^m F(x, y)(u)$  and  $w \in D^1 G(y, z)(q^m v)$ .  $\square$

Combining Propositions 6.3.9 - 6.3.13, we arrive at the following chain rule.

**Proposition 6.3.14.** *Suppose  $Y$  is finite dimensional and  $(x, z) \in \text{gr}(G \circ F)$  is such that  $C$  is compact and closed at  $(x, z)$ .*

(i) *Assume that (6.17) holds for every  $y \in C(x, z)$ . Then*

$$D^m(G \circ F)(x, z)(u) \subseteq \bigcup_{y \in C(x, z)} D^1 G(y, z)[D^m F(x, y)(u)]. \quad (6.19)$$

*If, additionally, (6.15) holds for every  $y \in C(x, z)$ , then (6.19) is an equality.*

(ii) *Assume that (6.18) holds for every  $y \in C(x, z)$ . Then*

$$D^m(G \circ F)(x, z)(u) \subseteq \bigcup_{y \in C(x, z)} D^m G(y, z)[D^1 F(x, y)(u)]. \quad (6.20)$$

*If, additionally, (6.16) holds for every  $y \in C(x, z)$ , then (6.20) is an equality.*

Now we apply the preceding chain rules to establish sum rules for  $M, N : X \rightarrow 2^Y$ . For this purpose we use  $F : X \rightarrow 2^{X \times Y}$  and  $G : X \times Y \rightarrow 2^Y$  defined in (6.14). For  $(x, z) \in X \times Y$ , set

$$S(x, z) := M(x) \cap (z - N(x)).$$

Then, the resultant map  $C : X \times Y \rightarrow 2^{X \times Y}$  associated to these  $F$  and  $G$  is

$$C(x, z) = \{x\} \times S(x, z).$$

Given  $((x, z), y) \in \text{gr} S$ , the  $m$ -th order  $y$ -Studniarski derivative of  $M + N$  at  $(x, z)$  is defined as, for  $u \in X$ ,

$$D^m(M +_y N)(x, z)(u) := \{w \in Y : \exists t_n \rightarrow 0^+, \exists (u_n, y_n, w_n) \rightarrow (u, y, w), \forall n, y_n \in S(x + t_n u_n, z + t_n^m w_n)\}.$$

Observe that

$$D^m(M +_y N)(x, z)(u) = D^m(G \circ_y F)(x, z)(u). \quad (6.21)$$

One has a relationship between  $D^m(M +_y N)(x, z)(u)$  and  $D^m(M + N)(x, z)(u)$  as noted in the next statement.

**Proposition 6.3.15.** *Let  $(x, z) \in \text{gr}(M + N)$  and  $y \in S(x, z)$ .*

(i)  $D^m(M +_y N)(x, z)(u) \subseteq D^m(M + N)(x, z)(u)$ .

(ii) *If  $S$  is compact and closed at  $(x, z)$ , then*

$$\bigcup_{y \in S(x, z)} D^m(M +_y N)(x, z)(u) = D^m(M + N)(x, z)(u).$$



*Proof.* (i) This is an immediate consequence of the definitions.

(ii) When  $S$  is compact and closed at  $(x, z)$ ,  $C$  is compact and closed at  $(x, z)$ . Hence, the equality in Proposition 6.3.9(ii) holds. In view of (6.21), this relation implies the required equality.  $\square$

For higher-order sum rules, we have

**Proposition 6.3.16.** *Let  $(x, z) \in \text{gr}(M + N)$  and  $y \in S(x, z)$ .*

(i) *Suppose, for all  $(u, v) \in X \times Y$ ,*

$$D^m M(x, y)(u) \cap [v - D^m N(x, z - y)(u)] \subseteq D_p^m S((x, z), y)(u, v). \quad (6.22)$$

*Then*

$$D^m M(x, y)(u) + D^m N(x, z - y)(u) \subseteq D^m(M +_y N)(x, z)(u).$$

(ii) *If (6.22) holds for all  $y \in S(x, z)$ , then*

$$\bigcup_{y \in S(x, z)} (D^m M(x, y)(u) + D^m N(x, z - y)(u)) \subseteq D^m(M + N)(x, z)(u).$$

*Proof.* (i) Let  $w \in D^m M(x, y)(u) + D^m N(x, z - y)(u)$ , i.e., there exists  $\bar{y} \in D^m M(x, y)(u)$  such that  $\bar{y} \in w - D^m N(x, z - y)(u)$ . Hence, (6.22) ensures that  $\bar{y} \in D_p^m S((x, z), y)(u, w)$ . Therefore, there exist  $t_n \rightarrow 0^+$  and  $(u_n, \bar{y}_n, w_n) \rightarrow (u, \bar{y}, w)$  such that  $y + t_n^m \bar{y}_n \in S(x + t_n u_n, z + t_n^m w_n)$ . Setting  $y_n = y + t_n^m \bar{y}_n$ , we have  $y_n \in S(x + t_n u_n, z + t_n^m w_n)$ . Consequently,  $w \in D^m(M +_y N)(x, z)(u)$ .

(ii) This follows from (i) and Proposition 6.3.15(i).  $\square$

We can impose an additional condition to get equalities in the above sum rules as follows.

**Proposition 6.3.17.** *Let  $Y$  be finite dimensional and  $(x, z) \in \text{gr}(M + N)$ .*

(i) *Suppose, for  $y \in S(x, z)$ ,*

$$D_p^m S((x, z), y)(0, 0) = \{0\}. \quad (6.23)$$

*Then*

$$D^m(M +_y N)(x, z)(u) \subseteq D^m M(x, y)(u) + D^m N(x, z - y)(u).$$

(ii) *If  $S$  is compact and closed at  $(x, z)$  and (6.23) holds for every  $y \in S(x, z)$ , then one has*

$$D^m(M + N)(x, z)(u) \subseteq \bigcup_{y \in S(x, z)} (D^m M(x, y)(u) + D^m N(x, z - y)(u)). \quad (6.24)$$

*If, additionally, (6.22) holds for every  $y \in S(x, z)$ , then (6.24) becomes an equality.*

*Proof.* (i) Let  $w \in D^m(M +_y N)(x, z)(u)$ , i.e., there exist  $t_n \rightarrow 0^+$  and  $(u_n, y_n, w_n) \rightarrow (u, y, w)$  such that  $y_n \in S(x + t_n u_n, z + t_n^m w_n)$ . If  $y_k = y$  for infinitely many  $k \in \mathbb{N}$  one has  $0 \in D^m M(x, y)(u)$  and  $w \in D^m N(x, z - y)(u)$ , and we are done. Thus, suppose  $y_n \neq y$  for all  $n$  and, for  $s_n := \|y_n - y\|^{1/m}$ , the sequence  $v_n := s_n^{-m}(y_n - y)$  converges to  $v$  of norm one. If  $t_n/s_n \rightarrow 0$ , since

$$y + s_n^m v_n = y_n \in S\left(x + s_n \left(\frac{t_n u_n}{s_n}\right), z + s_n^m \left(\frac{t_n^m w_n}{s_n^m}\right)\right),$$

one sees that  $v \in D_p^m S((x, z), y)(0, 0)$ , contradicting (6.23). Consequently,  $s_n/t_n$  has a bounded subsequence and we may assume that  $s_n/t_n$  tends to  $q \in R_+$ . So,

$$y + t_n^m \left(\frac{s_n^m}{t_n^m} v_n\right) = y_n \in S(x + t_n u_n, z + t_n^m w_n)$$

and then  $q^m v \in D_p^m S((x, z), y)(u, w)$ . It follows from the definition of  $D_p^m S((x, z), y)(u, w)$  that  $q^m v \in D^m M(x, y)(u)$  and  $w - q^m v \in D^m N(x, z - y)(u)$ .

(ii) This follows from (i) and Propositions 6.3.15 and 6.3.16.  $\square$

Next, we define two other  $m$ -th order Studniarski derivatives, which are slight modifications of those in the above definitions and suitable for applications to variational inequalities in Section 6.4. Let  $P$  be also a normed space,  $F : P \times X \rightarrow 2^Y$  and  $N : P \times X \rightarrow 2^Y$ . Let  $\widehat{S} : P \times X \times Y \rightarrow 2^Y$  be given by

$$\widehat{S}(p, x, y) := F(p, x) \cap (y - N(p, x)).$$

**Definition 6.3.18.** Given  $y_0 \in \widehat{S}(p, x, y)$  and  $(u, v) \in P \times X$ , we define

$$D^m(F +_{y_0} N)((p, x), y)(u, v) := \{w \in Y : \exists t_n \rightarrow 0^+, \exists (u_n, v_n, y_n, w_n) \rightarrow (u, v, y_0, w),$$

$$y_n \in \widehat{S}(p + t_n u_n, x + t_n^m v_n, y + t_n^m w_n)\},$$

and

$$D_p^m \widehat{S}((p, x, y), y_0)(u, v, s) := \{w \in Y : \exists t_n \rightarrow 0^+, \exists (u_n, v_n, s_n, w_n) \rightarrow (u, v, s, w),$$

$$y_0 + t_n^m w_n \in \widehat{S}(p + t_n u_n, x + t_n^m v_n, y + t_n^m s_n)\}.$$

**Proposition 6.3.19.** Let  $Y$  be finite dimensional and  $((p, x), y) \in \text{gr}(F + N)$ .

(i) Suppose, for  $y_0 \in \widehat{S}(p, x, y)$ ,

$$D_p^m \widehat{S}((p, x, y), y_0)(0, 0, 0) = \{0\}. \quad (6.25)$$

Then

$$D^m(F +_{y_0} N)((p, x), y)(u, v) \subseteq D_p^m F((p, x), y_0)(u, v) + D_p^m N((p, x), y - y_0)(u, v).$$

(ii) If  $\widehat{S}$  is compact and closed at  $(p, x, y)$  and (6.25) holds for every  $y_0 \in \widehat{S}(p, x, y)$ , then one has

$$D_p^m(F + N)((p, x), y)(u, v) \subseteq \bigcup_{y_0 \in \widehat{S}(p, x, y)} (D_p^m F((p, x), y_0)(u, v) + D_p^m N((p, x), y - y_0)(u, v)).$$

*Proof.* (i) Let  $w \in D^m(F +_{y_0} N)((p, x), y)(u, v)$ , i.e., there exist  $t_n \rightarrow 0^+$  and  $(u_n, v_n, y_n, w_n) \rightarrow (u, v, y_0, w)$  such that  $y_n \in \widehat{S}(p + t_n u_n, x + t_n^m v_n, y + t_n^m w_n)$ . If  $y_k = y_0$  for infinitely many  $k \in \mathbb{N}$ , one has  $0 \in D_p^m F((p, x), y_0)(u, v)$  and  $w \in D_p^m N((p, x), y - y_0)(u, v)$ , and we are done. Now suppose  $y_n \neq y_0$  for all  $n$  and, for  $s_n := \|y_n - y_0\|^{1/m}$ , the sequence  $l_n := s_n^{-m}(y_n - y_0)$  converges to  $l$  of norm one. If  $t_n/s_n \rightarrow 0$ , since

$$y_0 + s_n^m l_n = y_n \in \widehat{S}\left(p + s_n \frac{t_n u_n}{s_n}, x + s_n \left(\frac{t_n^m v_n}{s_n}\right), y + s_n^m \left(\frac{t_n^m w_n}{s_n^m}\right)\right),$$

one sees that  $l \in D_p^m \widehat{S}((p, x, y), y_0)(0, 0, 0)$ , contradicting (6.25). Consequently, one may assume that  $s_n/t_n$  tends to  $q \in \mathbb{R}_+$ . So,

$$y_0 + t_n^m \left(\frac{s_n^m}{t_n^m} l_n\right) = y_n \in \widehat{S}(p + t_n u_n, x + t_n^m v_n, y + t_n^m w_n)$$

and thus  $q^m l \in D_p^m \widehat{S}((p, x, y), y_0)(u, v, w)$ . By the definition of  $D_p^m \widehat{S}((p, x, y), y_0)(u, v, w)$ , one has  $q^m l \in D_p^m F((p, x), y_0)(u, v)$  and  $w - q^m l \in D_p^m N((p, x), y - y_0)(u, v)$ .

(ii) We need to prove that, if  $\widehat{S}$  is compact and closed at  $(p, x, y)$ , then

$$D_p^m(F + N)((p, x), y)(u, v) = \bigcup_{y_0 \in \widehat{S}(p, x, y)} D^m(F +_{y_0} N)((p, x), y)(u, v).$$

The containment “ $\supseteq$ ” follows from definitions. For “ $\subseteq$ ”, let  $w \in D_p^m(F + N)((p, x), y)(u, v)$ . There exist  $t_n \rightarrow 0^+$  and  $(u_n, v_n, w_n) \rightarrow (u, v, w)$  such that  $y + t_n^m w_n \in F(p + t_n u_n, x + t_n^m v_n) + N(p + t_n u_n, x + t_n^m v_n)$ . Then, one can find  $y_n \in F(p + t_n u_n, x + t_n^m v_n)$  such that  $y + t_n^m w_n - y_n \in N(p + t_n u_n, x + t_n^m v_n)$ . Therefore,  $y_n \in \widehat{S}(p + t_n u_n, x + t_n^m v_n, y + t_n^m w_n)$ . Since  $\widehat{S}$  is compact at  $(p, x, y)$ , one may assume that  $y_n$  converges to  $y_0$ . As  $(p + t_n u_n, x + t_n^m v_n, y + t_n^m w_n) \rightarrow (p, x, y)$ , one has  $y_0 \in (\text{cl } \widehat{S})(p, x, y)$ . It follows from the closedness of  $\widehat{S}$  that  $y_0 \in \widehat{S}(p, x, y)$ .  $\square$

## 6.4 Applications

### 6.4.1 Studniarski derivatives of solution maps to inclusions

Let  $M : P \times X \rightarrow 2^Z$  be a set-valued map between normed spaces. Then, the map  $S$  defined by

$$S(p) := \{x \in X : 0 \in M(p, x)\}, \quad (6.26)$$

is said to be the solution map of the *parametrized inclusion*  $0 \in M(p, x)$ .

**Theorem 6.4.1.** *For a solution map  $S$  defined by (6.26) and  $\bar{x} \in S(\bar{p})$ , we have, for  $p \in P$ ,*

$$D^m S(\bar{p}, \bar{x})(p) \subseteq \{x \in X : 0 \in D_p^m M((\bar{p}, \bar{x}), 0)(p, x)\}.$$

*Proof.* Let  $(p, x) \in \text{gr } D^m S(\bar{p}, \bar{x})$ , i.e., there exist sequences  $p_n \rightarrow p$ ,  $x_n \rightarrow x$ , and  $t_n \rightarrow 0^+$  such that  $\bar{x} + t_n^m x_n \in S(\bar{p} + t_n p_n)$ . This implies that  $0$  is an element of the set  $M(\bar{p} + t_n p_n, \bar{x} + t_n^m x_n)$ . Hence, for  $z_n = 0$ , the inclusion  $0 + t_n^m z_n \in M(\bar{p} + t_n p_n, \bar{x} + t_n^m x_n)$  holds, i.e.,  $0 \in D_p^m M((\bar{p}, \bar{x}), 0)(p, x)$ .  $\square$

In parameterized optimization, we frequently meet  $M$  of the form

$$M(p, x) = F(p, x) + N(p, x), \quad (6.27)$$

where  $F : P \times X \rightarrow 2^Z$  and  $N : P \times X \rightarrow 2^Z$ . Let  $\widehat{S} : P \times X \times Z \rightarrow 2^Z$  be defined by

$$\widehat{S}(p, x, z) := F(p, x) \cap (z - N(p, x)).$$

The following theorem gives an approximation of the  $m$ -th order Studniarski derivative of  $S$  when  $M$  is defined by (6.27).

**Theorem 6.4.2.** *For the solution map  $S(p) = \{x \in X : 0 \in F(p, x) + N(p, x)\}$  and  $\bar{x} \in S(\bar{p})$  with  $Z$  being finite dimensional, suppose either of the following conditions hold*

(i)  $\widehat{S}$  is compact and closed at  $(\bar{p}, \bar{x}, 0)$  and  $D_p^m \widehat{S}((\bar{p}, \bar{x}, 0), y)(0, 0, 0) = \{0\}$  for all  $y \in \widehat{S}(\bar{p}, \bar{x}, 0)$ ;

(ii) there exists  $y \in \widehat{S}(\bar{p}, \bar{x}, 0)$  such that either  $F$  or  $N$  is  $m$ -th order locally Hölder calm at  $(\bar{p}, \bar{x})$  for  $y$  or  $-y$ , respectively.

Then

$$D^m S(\bar{p}, \bar{x})(p) \subseteq \{x \in X : 0 \in \bigcup_{y \in \widehat{S}(\bar{p}, \bar{x}, 0)} (D_p^m F((\bar{p}, \bar{x}), y)(p, x) + D_p^m N((\bar{p}, \bar{x}), 0 - y)(p, x))\}.$$

*Proof.* We first prove that

$$D_p^m M((\bar{p}, \bar{x}), 0)(p, x) \subseteq \bigcup_{y \in \widehat{S}(\bar{p}, \bar{x}, 0)} (D_p^m F((\bar{p}, \bar{x}), y)(p, x) + D_p^m N((\bar{p}, \bar{x}), 0 - y)(p, x)).$$

If (i) holds, the above inclusion is followed by Proposition 6.3.19. For the case (ii) and  $y \in \widehat{S}(\bar{p}, \bar{x}, 0)$ , we see that  $y \in F(\bar{p}, \bar{x})$  and  $-y \in N(\bar{p}, \bar{x})$ . Let  $v \in D_p^m M((\bar{p}, \bar{x}), 0)(p, x)$ , i.e., there exist  $t_n \rightarrow 0^+$ ,  $(p_n, x_n) \rightarrow (p, x)$ , and  $v_n \rightarrow v$  such that

$$0 + t_n^m v_n \in M(\bar{p} + t_n p_n, \bar{x} + t_n^m x_n) = F(\bar{p} + t_n p_n, \bar{x} + t_n^m x_n) + N(\bar{p} + t_n p_n, \bar{x} + t_n^m x_n).$$

Then, there exist  $y_n^1 \in F(\bar{p} + t_n p_n, \bar{x} + t_n^m x_n)$  and  $y_n^2 \in N(\bar{p} + t_n p_n, \bar{x} + t_n^m x_n)$  such that

$$v_n = t_n^{-m} (y_n^1 - y) + t_n^{-m} (y_n^2 - (-y)). \quad (6.28)$$

Suppose  $F$  is  $m$ -th order locally Hölder calm at  $(\bar{p}, \bar{x})$  for  $y$ . Then, there exists  $L > 0$  such that for large  $n$ ,

$$y_n^1 \in F(\bar{p} + t_n p_n, \bar{x} + t_n^m x_n) \subseteq \{y\} + L \|(t_n p_n, t_n^m x_n)\|^m B_Z.$$

Because  $\dim Z < +\infty$ ,  $t_n^{-m} (y_n^1 - y)$ , or a subsequence, converges to some  $v^1 \in Z$  and so  $v^1 \in D_p^m F((\bar{p}, \bar{x}), y)(p, x)$ . From (6.28), the sequence  $t_n^{-m} (y_n^2 - (-y))$  also converges to some  $v^2$  such that  $v^2 = v - v^1$ , and  $v^2 \in D_p^m N((\bar{p}, \bar{x}), -y)(p, x)$ . Thus,

$$v \in D_p^m F((\bar{p}, \bar{x}), y)(p, x) + D_p^m N((\bar{p}, \bar{x}), -y)(p, x).$$

Now applying Theorem 6.4.1 completes the proof.  $\square$

## 6.4.2 Implicit multifunction theorems

Let  $M : P \times X \rightarrow Z$  and  $S(p) := \{x \in X : M(p, x) = 0\}$ , be the set of solutions to the *parametrized equation*  $M(x, p) = 0$ . We impose the condition

$$(*) \begin{cases} \exists \bar{x} \in X \text{ such that } M(0, \bar{x}) = 0 \text{ and} \\ M_p \text{ is continuous in a neighborhood } (U, V) \in \mathcal{N}(0) \times \mathcal{N}(\bar{x}), \end{cases}$$

where  $M_p$  denotes the partial Fréchet derivative with respect to  $p$ . Let  $H = V \cap M(0, \cdot)^{-1}$ , i.e.,

$$H(z) = \{x \in V : M(0, x) = z\}.$$

Under the hypotheses of usual implicit function theorems for  $M \in C^1$ ,  $S$  and  $H$  are single-valued and smooth (with derivatives  $DS$ ,  $DH$ ), and there holds

$$DS(0) = -DH(0)M_p(0, \bar{x}) = -M_x(0, \bar{x})^{-1}M_p(0, \bar{x}).$$

Now we are interested in a similar formula of the  $m$ -th order Studniarski derivative  $D^m S(0, \bar{x})(\cdot)$  of the map  $S$  under the assumption (\*). For  $(p, x)$  near  $(0, \bar{x})$ , we consider the map

$$r(p, x) := M(p, x) - M(0, x) - M_p(0, \bar{x})p.$$

By the mean-value theorem, one obtains

$$r(p, x) = \int_0^1 [M_p(\theta p, x) - M_p(0, \bar{x})]pd\theta,$$

where

$$\alpha(p, x, \theta) := \|M_p(\theta p, x) - M_p(0, \bar{x})\|$$

can be estimated (uniformly for  $0 < \theta < 1$ ) by

$$\alpha(p, x, \theta) \leq 0(p, x) \text{ with } 0(p, x) \rightarrow 0^+ \text{ as } x \rightarrow \bar{x} \text{ and } \|p\| \rightarrow 0^+.$$

Due to  $\|r(p, x)\| \leq 0(p, x)\|p\|$ , one easily sees that  $\|p\|^{-1}\|r(p, x)\| \rightarrow 0^+$  as  $x \rightarrow \bar{x}$  and  $\|p\| \rightarrow 0^+$ , and

$$r(p(t), x(t)) = o_2(t) \text{ if } x(t) \rightarrow \bar{x} \text{ and } p(t) = tq + o_1(t) \text{ with some } q \in P,$$

where  $o_k(t)$  means that  $\frac{\|o_k(t)\|}{t} \rightarrow 0^+$  as  $t \rightarrow 0^+$ .

For  $(p, x)$  near  $(0, \bar{x})$ , we have

$$M(p, x) = 0 \text{ if and only if } M(0, x) = -M_p(0, \bar{x})p - r(p, x),$$

i.e.,

$$x \in S(p) \text{ if and only if } x \in H(-M_p(0, \bar{x})p - r(p, x)).$$

Let  $\widehat{M} : P \times X \rightarrow Z$  be defined by

$$\widehat{M}(p, x) := -M_p(0, \bar{x})(p) - r(p, x).$$

Then,  $x \in S(p)$  if and only if  $x \in H(\widehat{M}(p, x))$ . Set  $C(p, x) := \widehat{M}(p, x) \cap H^{-1}(x)$ . It is easy to see that  $C(0, \bar{x}) = \{0\}$ .

The following result is a slight modification of that in Proposition 6.3.13(ii).

**Lemma 6.4.3.** *Let  $Z$  be finite dimensional and either of the following conditions hold*

- (i)  $\widehat{M}$  is locally Lipschitz calm at  $(0, \bar{x})$  for 0;
- (ii)  $C$  is compact and closed at  $(0, \bar{x})$  and

$$D_p^{m(1)}C((0, \bar{x}), 0)(0, 0) = \{0\}. \quad (6.29)$$

*Then,  $x \in D^mS(0, \bar{x})(q)$  implies that  $x \in D^mH(0, \bar{x})[D_p^{m(1)}\widehat{M}((0, \bar{x}), 0)(q, x)]$ .*

*Proof.* Let  $x \in D^mS(0, \bar{x})(q)$ , i.e., there exist  $t_n \rightarrow 0^+$ ,  $q_n \rightarrow q$ , and  $x_n \rightarrow x$  such that  $\bar{x} + t_n^m x_n \in S(0 + t_n q_n)$ . This implies that

$$\bar{x} + t_n^m x_n \in H(\widehat{M}(0 + t_n q_n, \bar{x} + t_n^m x_n)). \quad (6.30)$$

Then, there exists  $y_n \in \widehat{M}(0 + t_n q_n, \bar{x} + t_n^m x_n)$  such that  $\bar{x} + t_n^m x_n \in H(y_n)$ . Suppose (i) hold. Because  $\widehat{M}$  is locally Lipschitz calm at  $(0, \bar{x})$  for 0, there exists  $L > 0$  such that, for large  $n$ ,

$$y_n \in \widehat{M}(0 + t_n q_n, \bar{x} + t_n^m x_n) \subseteq \{0\} + L\|(t_n q_n, t_n^m x_n)\|B_Z.$$

Since  $\dim Z < +\infty$ ,  $v_n := (t_n)^{-1}(y_n - 0)$  converges to some  $v \in Z$ . So,  $v \in D_p^{m(1)}\widehat{M}((0, \bar{x}), 0)(q, x)$ . It implies that  $\bar{x} + t_n^m x_n \in H(0 + t_n v_n)$ . Thus,  $x \in D^mH(0, \bar{x})(v)$ .

If assumption (ii) holds, it follows from (6.30) that there exists

$$y_n \in \widehat{M}(0 + t_n q_n, \bar{x} + t_n^m x_n) \cap H^{-1}(\bar{x} + t_n^m x_n) = C(0 + t_n q_n, \bar{x} + t_n^m x_n).$$

Since  $C$  is compact at  $(0, \bar{x})$ ,  $y_n$  (or a subsequence) has a limit  $y$ . Since  $(0 + t_n q_n, \bar{x} + t_n^m x_n, y_n) \rightarrow (0, \bar{x}, y)$ , one has  $y \in (\text{cl } C)(0, \bar{x})$ . It follows from the closedness of  $C$  at  $(0, \bar{x})$  that  $y \in C(0, \bar{x}) = 0$ .

If  $y_k = 0$  for infinitely many  $k \in \mathbb{N}$ , one has  $0 \in D_p^{m(1)}\widehat{M}((0, \bar{x}), 0)(q, w)$  and  $w \in D^mH(0, \bar{x})(0)$ , and we are done. Thus, one may suppose, for  $s_n := \|y_n\|$ , the sequence  $v_n := y_n/s_n$  has a limit  $v$  of norm one. If  $t_n/s_n \rightarrow 0$ , since

$$0 + s_n v_n = y_n \in C\left(0 + s_n\left(\frac{t_n q_n}{s_n}, \bar{x} + s_n^m\left(\frac{t_n^m x_n}{s_n^m}\right)\right),\right),$$

one sees that  $v \in D_p^{m(1)}C((0, \bar{x}), 0)(0, 0)$ , contradicting (6.29). Consequently, one may assume that  $s_n/t_n$  converges to  $q \in R_+$ . So,

$$0 + t_n\left(\frac{s_n}{t_n}v_n\right) = y_n \in C(0 + t_n q_n, \bar{x} + t_n^m x_n)$$

and thus  $qv \in D_p^{m(1)}C((0, \bar{x}), 0)(q, x)$ . It follows from the definition of  $D_p^{m(1)}C((0, \bar{x}), 0)(q, x)$  that  $qv \in D_p^{m(1)}\widehat{M}((0, \bar{x}), 0)(q, x)$  and  $x \in D^mH(y, z)(qv)$ . □

**Lemma 6.4.4.** *Let  $Z$  be finite dimensional, the assumptions of Lemma 6.4.3 be satisfied and*

$$D_p^{m(1)}\widehat{M}((0,\bar{x}),0)(q,x) \cap (D^m H(0,\bar{x}))^{-1}(x) \subseteq D_p^{m(1)}C((0,\bar{x}),0)(q,x). \quad (6.31)$$

*Then,  $x \in D^m S(0,\bar{x})(q)$  if and only if  $x \in D^m H(0,\bar{x})[D_p^{m(1)}\widehat{M}((0,\bar{x}),0)(q,x)]$ .*

*Proof.* By Lemma 6.4.3, we need to prove that  $x \in D^m H(0,\bar{x})[D_p^{m(1)}\widehat{M}((0,\bar{x}),0)(q,x)]$  implies  $x \in D^m S(0,\bar{x})(q)$ .  $x \in D^m H(0,\bar{x})[D_p^{m(1)}\widehat{M}((0,\bar{x}),0)(q,x)]$  means the existence of  $v \in D_p^{m(1)}\widehat{M}((0,\bar{x}),0)(q,x) \cap (D^m H(0,\bar{x}))^{-1}(x)$ . Then, (6.31) ensures that  $v \in D_p^{m(1)}C((0,\bar{x}),0)(q,x)$ . This means the existence of  $t_n \rightarrow 0^+$  and  $(q_n, x_n, v_n) \rightarrow (q, x, v)$  such that

$$0 + t_n v_n \in C(0 + t_n q_n, \bar{x} + t_n^m x_n).$$

From the definition of the map  $C$ , we get  $0 + t_n v_n \in \widehat{M}(0 + t_n q_n, \bar{x} + t_n^m x_n)$  and  $\bar{x} + t_n^m x_n \in H(0 + t_n v_n)$ , which imply that  $\bar{x} + t_n^m x_n \in H(\widehat{M}(0 + t_n q_n, \bar{x} + t_n^m x_n))$ . Thus, we have  $\bar{x} + t_n^m x_n \in S(0 + t_n q_n)$  and  $x \in D^m S(0,\bar{x})(q)$ . □

**Theorem 6.4.5.** *Impose the assumptions of Lemma 6.4.3. Then,*

$$D^m S(0,\bar{x})(q) \subseteq D^m H(0,\bar{x})[-M_p(0,\bar{x})(q)]. \quad (6.32)$$

*If, additionally, (6.31) holds, then (6.32) becomes an equality.*

*Proof.* By Lemmas 6.4.3 and 6.4.4, we need to prove that  $D_p^{m(1)}\widehat{M}((0,\bar{x}),0)(q,x) = -M_p(0,\bar{x})(q)$ . Let  $v \in D_p^{m(1)}\widehat{M}((0,\bar{x}),0)(q,x)$ . There exist  $t_n \rightarrow 0^+$  and  $(q_n, x_n, v_n) \rightarrow (q, x, v)$  such that

$$0 + t_n v_n = \widehat{M}(0 + t_n q_n, \bar{x} + t_n^m x_n) = -M_p(0,\bar{x})(0 + t_n q_n) - r(0 + t_n q_n, \bar{x} + t_n^m x_n).$$

Therefore,

$$v_n = -M_p(0,\bar{x})(q_n) - t_n^{-1} r(t_n q_n, \bar{x} + t_n^m x_n) \rightarrow -M_p(0,\bar{x})(q).$$

Thus,  $v = -M_p(0,\bar{x})(q)$  and we are done. □



# Conclusions

In this thesis, we have presented results related to some topics of variational analysis. First, we have stated definitions and basic properties of  $\Gamma$ -limits. Then, several generalized results on sequential forms of  $\Gamma$ -limits have been given. We have also got important applications of  $\Gamma$ -limits to tangency and generalized differentiation theory. It turns out that most of generalized derivatives can be expressed in terms of  $\Gamma$ -limits. Finally, we have introduced some kinds of generalized derivatives and their applications. In detail,

- We have discussed higher-order analysis for quantitative properties of perturbation maps of nonsmooth vector optimization in terms of variational sets, a kind of generalized derivatives which is suitable for a high level of nonsmoothness and relatively easy to compute. We have established relations between variational sets of a perturbation map and weak perturbation map or the efficiency/weak efficiency of these sets and the corresponding ones of the feasible-set map to the objective space. These results have been applied to sensitivity analysis for set-constrained vector optimization. As some results look complicated, we have tried to confirm the essentialness of each imposed assumption, as well as to illustrate advantages of our results by a number of examples, which indicate also that computing variational sets is not a hard work.

- Realizing advantages in some aspects of radial sets and derivatives, we have aimed to establish both necessary and sufficient higher-order conditions in terms of radial sets and derivatives for various kinds of efficiency concepts in set-valued vector optimization. We have chosen the  $Q$ -efficiency defined in [88] to unify these concepts. Thus, we have first discussed optimality conditions for  $Q$ -efficiency and then rephrase the results for the other kinds of solutions. Besides, we have also discussed properties and basic calculus rules of radial sets and derivatives like those for a sum or composition of maps. Furthermore, direct applications of these rules in proving optimality conditions for some particular problems have been given.

- Some calculus rules for Studniarski derivatives have been given to ensure that it can be used in practice. Most of the usual rules, from the sum and chain rules to various operations in

analysis, have been investigated. It turns out that Studniarski derivatives possesses many fundamental and comprehensive calculus rules. Although this construction is not comparable with objects in the dual approach like Mordukhovich's coderivatives (see books [129, 130] and papers [89, 90, 128]) in enjoying rich calculus, it may be better in dealing with higher-order properties. We have paid attention also on relations between the established calculus rules and applications of some rules to get others. As such applications we have provided a direct employment of sum rules to establishing an explicit formula for the Studniarski derivative of the solution map to a parametrized inclusion in terms of Studniarski derivatives of the data. Furthermore, chain rules have been also used to get implicit multifunction theorems.

## Further works

1) Studying on stability theories with respect to solution sets under perturbations of the data has been of great interest in variational convergence. Some papers presented the convergence of solution sets and efficient sets of perturbed problems in general convergence spaces, by using some kinds of convergence, such as adherence and persistence, see [51, 56]. For vector problems, the notion of continuous convergence of the sequence of perturbed vector valued objective functions has usually been used, see [51, 56, 122, 146, 169, 171]. Extending this convergence, in [132, 133], by using  $\Gamma$ -limits, Oppezzi and Rossi introduced the notion of  $\Gamma$ -convergence for sequences of vector valued functions and investigated stability results for convex vector optimization problems. Some recent papers dealing with this convergence are [9, 112, 113, 120, 123]. For possible developments of Chapter 3, we think that, by using  $\Gamma$ -limits, ones can get stability results to vector programming problem, where the objective function holds some conditions on convexity as the inequality constraints, while the equality constraints are linear.

A well established practice to solve a vector optimization problem is through an associated scalar optimization problem. There are various approaches, see [121, 126, 127, 143], to scalarize a vector optimization and obtain a complete characterization for several types of solution sets of the vector optimization in terms of solution sets of the scalarized problem. Thus, we can establish the convergence of solution sets of scalarized problems by using  $\Gamma$ -limits.

2) For possible developments of Chapter 4, we think that, besides going deeper in relations for perturbation and weak perturbation maps in terms of variational sets for set-constrained problems, one can consider the most important case of optimization problems with constraints defined by inequalities and equalities. Furthermore, sensitivity analysis in terms of generalized derivatives other than contingent ones and variational sets is a promising research direction. In fact, first results of this kind have been obtained very recently by Diem et al. in [49], using the so-called contingent-radial derivatives. This generalized derivative is defined by a combination of the ideas of contingent derivatives and radial ones (for the latter derivative see recent works

of Anh and Khanh in [2, 5]).

3) Note that separation theorems are a basic tool for proving optimality conditions and duality statements. With the development of nonconvex optimization theory, there has come into existence a need for nonconvex separation theorems, see [25]. So, for possible developments of Chapter 5, we can derive optimality conditions of several kinds of efficient solutions for set-valued optimization problems by using a nonconvex separation function given by Certh and Weidner in [25]. Then, we can use these results to derive duality results for vector optimization when the objectives and the constraints are nonconvex. Besides, motivated by [14, 30, 31], we can obtain some calculations of higher-order radial sets in certain functional spaces.

# Publications

- 1) N.L.H. Anh, P.Q. Khanh, and L.T. Tung, *Variational sets : calculus and applications to nonsmooth vector optimization*, Nonlinear Anal. TMA **74** (2011), 2358 - 2379.
- 2) N.L.H. Anh, P.Q. Khanh, and L.T. Tung, *Higher-order radial derivatives and optimality conditions in nonsmooth vector optimization*, Nonlinear Anal. TMA **74** (2011), 7365 - 7379.
- 3)\* N.L.H. Anh and P.Q. Khanh, *Higher-order optimality conditions in set-valued optimization using radial sets and radial derivatives*, J. Global Optim. **56** (2013), 519 - 536.
- 4)\* N.L.H. Anh and P.Q. Khanh, *Variational sets of perturbation maps and applications to sensitivity analysis for constrained vector optimization*, J. Optim. Theory Appl. **158** (2013), 363 - 384.
- 5)\* N.L.H. Anh and P.Q. Khanh, *Higher-order optimality conditions for proper efficiency in nonsmooth vector optimization using radial sets and radial derivatives*, J. Global Optim., DOI10.1007/s10898-013-0077-7 (2013), Online First.
- 6) N.L.H. Anh, *Higher-order optimality conditions in set-valued optimization using Studniarski derivatives and applications to duality*, Positivity, DOI10.1007/s11117-013-0254-4 (2013), Online First.
- 7) N.L.H. Anh and P.Q. Khanh, *Higher-order radial epiderivatives and optimality conditions in nonsmooth vector optimization*, submitted to Math. Meth. Oper. Res.
- 8)\* N.L.H. Anh and P.Q. Khanh, *Calculus and applications of Studniarski derivatives to sensitivity and implicit function theorems*, Control and Cybernetics, Accepted.

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\*: The contents of these papers are presented in the thesis.

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# Index

- Base of
  - a non-degenerate family, 5
  - a cone, 7
- Bound
  - greatest lower, 6
  - least upper, 6
- Complete homomorphism, 20, 21
- Cone
  - adjacent, 44
  - Clarke, 44
  - contingent, 44
  - lower paratangent, 43
  - paratangent, 44
  - upper paratangent, 43
  - upper tangent, 43
  - dilating, 10
  - pointed, 8
- Contingent epiderivative, 93
- Convergence
  - $\Gamma$ , 16
  - continuous, 17
  - $K$ , 17
- Derivative
  - $\gamma$ -Studniarski, 107
  - Clarke, 41
  - lower Dini, 41
  - lower directional, 41
  - lower Hadamard, 41
  - lower paratangent, 41
  - lower tangent, 41
  - paratangent, 41
  - upper Dini, 41
  - upper directional, 40
  - upper Hadamard, 41
  - upper paratangent, 41
  - upper tangent, 41
  - higher-order contingent, 50
  - lower radial, 73
  - proto-Studniarski, 99
  - pseudo-Studniarski, 107
  - strict Studniarski, 101
  - Studniarski, 98
  - upper radial, 73
- Domination property of a map, 56
  - weak, 56
- Domination property of a set, 53
  - weak, 53
- Efficient point
  - $Q$ , 10
  - positive-proper, 10
  - strong, 9
  - weak, 9

- Geoffrion-proper, 10
- Henig-proper, 10
- Pareto, 9
- strong Henig-proper, 10
- Filter, 5
  - Fréchet, 25
  - countably based, 24
  - principal, 24
  - productively Fréchet, 25
  - sequential, 24
  - strongly Fréchet, 25
- Finer, 5
- Function
  - characteristic, 18
- Grill, 5
- Homothety, 42
- Involution, 5
- Lattice, 6
  - complete, 6
  - completely distributive, 6
- Limit
  - $\Gamma$ , 14
  - $\Gamma$ -lower, 16
  - $\Gamma$ -upper, 16
  - $G$ , 19
  - $K$ -lower, 17
  - $K$ -upper, 17
  - sequential  $\Gamma$ , 26
  - inferior, 21
  - superior, 21
- Limitoid, 20
  - support of, 21
- Mesh, 5
- Non-degenerate family, 5
- Ordered set, 6
- Parametrized
  - equation, 115
  - inclusion, 114
- Perturbation map, 56
  - weak, 56
- Poset, 6
- Radial semiderivative, 77
- Radial set
  - lower, 73
  - proto, 77
  - upper, 73
- Relation, 7
- Semi-filter, 5
- Set of
  - filters, 5
  - Pareto efficient points, 9
  - semi-filters, 5
- Set-valued map, 7
  - $C$ -convex, 7
  - profile, 7
  - calm, 63
  - closed, 106
  - compact, 106
  - convex, 7
  - locally pseudo-Hölder calm, 7

lower semicontinuous, 7

Simplex, 9

Space

first countable, 24

Fréchet, 24

vector, 6

strongly Fréchet, 24

Sublattice, 6

closed, 6

Topology

chaotic, 15

discrete, 15

Variation

lower, 63

proto, 63

upper, 63

Variational set

of type 1, 46

of type 2, 46

proto, 47

singular, 53