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# The free wreath product of a compact quantum group by a quantum automorphism group

Lorenzo Pittau

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UNIVERSITÉ DE CERGY - PONTOISE

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École Doctorale Économie, Management, Mathématiques et Physique  
ED n°405

**Le produit en couronne libre  
d'un groupe quantique compact  
par  
un groupe quantique d'automorphismes**

**THÈSE DE DOCTORAT**

Spécialité : Mathématiques

présentée par

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To my parents



# **Le produit en couronne libre d'un groupe quantique compact par un groupe quantique d'automorphismes**

## Résumé

Dans cette thèse on définit et étudie le produit en couronne libre d'un groupe quantique compact par un groupe quantique d'automorphismes, en généralisant la notion de produit en couronne libre par le groupe quantique symétrique introduite par Bichon.

Notre recherche est divisée en deux parties. Dans la première, on définit le produit en couronne libre d'un groupe discret par un groupe quantique d'automorphismes. Ensuite, on montre comment décrire les entrelaceurs de ce nouveau objet à l'aide de partitions non-croisées et décorées; à partir de cela et grâce à un résultat de Lemeux, on déduit les représentations irréductibles et les règles de fusion. Ensuite, on prouve des propriétés des algèbres d'opérateurs associées à ce groupe quantique compact, comme la simplicité de la  $C^*$ -algèbre réduite et la propriété d'Haagerup de l'algèbre de von Neumann.

La deuxième partie est une généralisation de la première. D'abord, on définit la notion de produit en couronne libre d'un groupe quantique compact par un groupe quantique d'automorphismes. Après, on généralise la description des espaces des entrelaceurs donnée dans le cas discret et, en adaptant un résultat d'équivalence monoïdale de Lemeux et Tarrago, on trouve les représentations irréductibles et les règles de fusion. Ensuite, on montre des propriétés de stabilité de l'opération de produit en couronne libre. En particulier, on prouve sous quelles conditions deux produits en couronne libres sont monoïdalement équivalents ou ont le semi-anneau de fusion isomorphe. Enfin, on démontre certaines propriétés algébriques et analytiques du groupe quantique duale et des algèbres d'opérateurs associées à un produit en couronne. Comme dernier résultat, on prouve que le produit en couronne de deux groupes quantiques d'automorphismes est isomorphe à un quotient d'un particulier groupe quantique d'automorphismes.



# The free wreath product of a compact quantum group by a quantum automorphism group

## Abstract

In this thesis, we define and study the free wreath product of a compact quantum group by a quantum automorphism group and, in this way, we generalize the previous notion of free wreath product by the quantum symmetric group introduced by Bichon.

Our investigation is divided into two parts. In the first, we define the free wreath product of a discrete group by a quantum automorphism group. We show how to describe its intertwiners by making use of decorated noncrossing partitions and from this, thanks to a result of Lemeux, we deduce the irreducible representations and the fusion rules. Then, we prove some properties of the operator algebras associated to this compact quantum group, such as the simplicity of the reduced  $C^*$ -algebra and the Haagerup property of the von Neumann algebra.

The second part is a generalization of the first one. We start by defining the notion of free wreath product of a compact quantum group by a quantum automorphism group. We generalize the description of the spaces of the intertwiners obtained in the discrete case and, by adapting a monoidal equivalence result of Lemeux and Tarrago, we find the irreducible representations and the fusion rules. Then, we prove some stability properties of the free wreath product operation. In particular, we find under which conditions two free wreath products are monoidally equivalent or have isomorphic fusion semirings. We also establish some analytic and algebraic properties of the dual quantum group and of the operator algebras associated to a free wreath product. As a last result, we prove that the free wreath product of two quantum automorphism groups can be seen as the quotient of a suitable quantum automorphism group.



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# Introduction

The general research area of this thesis is the theory of quantum groups so we begin with a brief presentation of this notion. The term quantum group is not associated to a unique definition, but it is used to denote a multitude of similar objects. Anyway, the underlying idea common to all these objects is to extend the notion of group to the framework of the noncommutative geometry. There are two main approaches to this subject: the first one is completely algebraic while the second one is more analytic. In the algebraic approach, the first important results come from Drinfeld and Jimbo in [Dri86, Dri87, Jim85]. By making use of a parameter  $q$ , they deformed the universal enveloping algebra of some Lie algebras and gave to these objects a Hopf algebra structure. This kind of quantum groups and their representation theory have been widely studied and investigated. The main results achieved within this setting can be found in the books [Kas95, CP94].

Now, we focus on the second approach, from which the theories used in this thesis were obtained. It has a more analytic flavour and has been developed in the context of operator algebras. Pontryagin proved in [Pon34] that it is possible to give a structure of locally compact abelian group to the set of the characters of a locally compact abelian group  $G$ , the so-called dual of Pontryagin. Moreover, he showed that  $G$  is naturally isomorphic to its bidual. This construction is no longer valid if the hypothesis of commutativity is dropped, therefore the problem at the origin of this point of view is the construction of a generalisation to the non-abelian case of the Pontryagin duality. A first significant contribution in this direction was given in [Tan38] by Tannaka. By observing that the characters of an abelian group correspond to its irreducible representations, he defined the dual of a

compact group  $G$  as the category of the finite dimensional unitary representations of  $G$  endowed with the operations of direct sum and tensor product. The work of Tannaka was further developed by Krein (see [Kre49b, Kre49a, Kre50]) and all these results are known as Tannaka-Krein duality. Other possible notions of duality were introduced and generalized in the following, but all these theories did not extend to every locally compact group. The first general answer was given by Vainerman and Kac in [VK73, VK74] and by Enock and Schwartz in [ES73, ES75] (see also [ES92]). The object at the center of their theory is called Kac algebra; it is a von Neumann algebra endowed with a special structure. As in the case of the Pontryagin duality, it is possible to define a dual Kac algebra and every Kac algebra is isomorphic to its bidual.

In this context Woronowicz presented his theory centred on the notion of compact matrix pseudogroup [Wor87, Wor91]. These objects, defined in the  $C^*$ -algebraic framework, give rise to the theory of compact quantum groups which is at the base of this thesis. In [Wor87] Woronowicz defined a compact matrix pseudogroup as a unital  $C^*$ -algebra with an additional structure (it is also a Hopf algebra) and which satisfies certain properties. The main example for this new theory is  $SU_q(2)$ , a quantum deformation of the classic  $SU(2)$ . In [Wor98], while presenting a sort of revision and simplification of his own theory, he extended this notion to the slightly more general one of compact quantum group. Every commutative compact quantum group is isomorphic to the  $C^*$ -algebras of continuous functions on a compact group, so a general (noncommutative) compact quantum group should be imagined as the  $C^*$ -algebra of continuous functions on an abstract compact object. For this reason, all the properties and the statements about compact quantum groups actually refer to the corresponding concrete Hopf algebras. The main point of strength of this theory is the existence and the uniqueness of a Haar state, the analogue of the Haar measure of a classic compact group. Another crucial result due to Woronowicz is a quantum version of the Tannaka-Krein duality. By using the results of Tannaka and Krein as a starting point, Woronowicz was able to link every compact quantum group to its representation theory. In particular, he proved that a compact quantum group can be completely reconstructed by knowing its ir-

reducible representations and the fusion rules between them, i.e. the rules allowing to decompose the tensor product of two irreducible representations as direct sum of irreducible representations (see [Wor88]). All these data permit to determine the representation category of a compact quantum group, whose objects are the finite dimensional unitary representations endowed with the operation of direct sum and tensor product. In the C\*-algebraic framework, a more general approach to the problem, allowing to deal with the non-compact case and to include some examples of quantum groups excluded from Woronowicz's theory, was proposed by Baaaj and Skandalis in [BS93] with the notion of multiplicative unitary. In particular, the multiplicative unitary associated to a compact quantum group satisfying certain basic properties can be seen as a huge source of information, as it allows to entirely reconstruct the group itself and its dual. This object, however, turned out to be quite difficult to use in the general context of quantum groups. The approach which is nowadays considered as the most general and comprehensive was introduced by Kustermans and Vaes in [KV99] and [KV00]. Their theory which can be developed in both the frameworks of C\*-algebras and of von Neumann algebras, provides and studies the notion of locally compact quantum group. With respect to the previous and less general theories, its main specificity is the assumption of the existence of left and right Haar weights.

The first Woronowicz compact quantum group which has been studied is  $SU_q(2)$ , for  $q \in [-1, 1]$ ,  $q \neq 0$ . Later on, the quantum versions of the classic groups  $O_n$  and  $U_n$  were introduced by Wang in [Wan93, Wan98] and by Wang and Van Daele in [VDW96]; they are denoted  $O_n^+$  and  $U_n^+$  respectively. The quantum analogue of  $S_n$ , denoted  $S_n^+$ , was defined by Wang in [Wan98]. They are the noncommutative versions of the spaces of the continuous functions  $C(O_n)$ ,  $C(U_n)$  and  $C(S_n)$ . These spaces can be seen as the C\*-algebras generated by the matrix coordinates  $u_{ij}$  of a matrix  $u$  of order  $n$  subject to some relations. The presentations of these commutative C\*-algebras can be chosen in order to include the commutativity relations  $u_{ij}u_{kl} = u_{kl}u_{ij}$  between the generators. The quantum versions are then obtained by removing these commutativity relations. The matrix  $u$  is a representation of the group and it is called fundamental because it allows to reconstruct all the

other ones. The operation of "liberation" of the generators from some of the relations explains and justifies the terminology of free quantum groups used to refer to these objects. A first significant step in their investigation was done by Banica in [Ban96, Ban97, Ban99] with the description of their representation theory. He also proved the presence (or absence) of some properties of the associated algebras such as simplicity and amenability.

This way of proceeding can be adopted as a sort of natural scheme when analysing a specific compact quantum group. Indeed, in view of the Tannaka-Krein duality, the representation category is a crucial and primordial object to consider in the analysis. To this purpose, it is possible to use different techniques depending on the group considered; for the needs of our thesis, we want to stress the combinatorial approach adopted for example in [Ban99, Ban02, BS09]. In all these cases the spaces of intertwiners between tensor products of the fundamental representation have been described by using Temperley-Lieb diagrams or possibly coloured (noncrossing) partitions. Moreover, in [BS09] the term easy quantum group was introduced to denote a family of compact quantum groups whose spaces of intertwiners can be described by means of noncrossing partitions. These descriptions have subsequently allowed to deduce the irreducible representations and the fusion rules.

As said, another aspect of the study of a compact quantum group concerns the analytic properties of the associated algebras, such as the universal or reduced  $C^*$ -algebra and the von Neumann algebra. In many cases, the knowledge of the representation category and, in particular, of the fusion rules has been fundamental to prove these properties. As an example, now we recall some results obtained in this context. The simplicity of the reduced  $C^*$ -algebras of  $U_n^+$  and  $S_n^+$  was proved by Banica in [Ban97] and by Brannan in [Bra13] respectively. The Haagerup property of the von Neumann algebras associated to  $O_n^+$ ,  $U_n^+$  and  $S_n^+$  was established by Brannan in [Bra12, Bra13]. Vaes and Vergnioux proved in [VV07] the exactness of  $C_r(O_n^+)$  and the fullness of  $L^\infty(O_n^+)$ . The weak amenability of this von Neumann algebra is due to Freslon [Fre13]. In [Fim10], Fima analysed the property T and proved that free quantum groups do not have this property. De Commer, Fres-

lon and Yamashita in [DCFY14] demonstrated that the discrete duals of  $O_n^+$  and  $U_n^+$  have the central ACPAP (almost completely positive approximation property) which implies the CCAP (completely contractive approximation property). They also proved that the associated von Neumann algebras do not have any Cartan subalgebra.

From these basic compact quantum groups it is possible to construct many other examples. They can be obtained as a generalization of these groups or by making use of different kinds of product operations. The families  $O_F^+$  and  $U_F^+$ , for example, are obtained by modifying the usual relations of the free quantum orthogonal and unitary groups with a matrix  $F$ . As a generalisation of the quantum symmetric group  $S_n^+$  and of the classic notion of automorphism group Wang in [Wan98] defined the quantum automorphism group. Always Wang in [Wan95] introduced the notion of free product of two compact quantum groups and reconstructed its representation theory from the representations of the factors. Another kind of product was introduced by Bichon in [Bic04]; it is the free wreath product of a compact quantum group by the quantum symmetric group.

In this thesis we want to extend this product to the free wreath product of a compact quantum group by a quantum automorphism group. Therefore, we focus now more specifically on the notions and on the results at the basis of our work.

It is well known what the automorphism group of a given space is. As a basic example, it is always useful to think to  $S_n$  as the universal group acting on a set of  $n$  points or equivalently to the group of automorphisms on a set of  $n$  point. It is then natural to look for a quantum analogue of this notion. As already said, the answer was given by Wang, who introduced in [Wan98] the notion of quantum automorphism group of a quantum finite measured space. He proved that, in general, it is not possible to define the notion of quantum automorphism group of a finite dimensional  $C^*$ -algebra  $B$ , but, if  $B$  is endowed with a state  $\psi$ , the category of compact quantum groups acting on  $B$  and leaving the state  $\psi$  invariant admits a universal object, called quantum automorphism group and denoted  $\mathbb{G}^{aut}(B, \psi)$ . Similarly to the classical case, a basic example here is given by  $S_n^+$ , the quantum symmetric group. In effect,  $C(S_n^+)$  is exactly  $C(\mathbb{G}^{aut}(\mathbb{C}^n, \text{tr}))$ . The representation

theory of  $\mathbb{G}^{aut}(B, \psi)$  was first studied by Banica in [Ban99, Ban02]. He showed that, if  $\psi$  is a  $\delta$ -form and  $\dim(B) \geq 4$ , the irreducible representations and the fusion rules are the same as  $SO(3)$ . In order to obtain these results, he analysed the spaces of intertwiners between tensor products of the fundamental representation and showed that there is a bijection between a linear basis of these spaces and the Temperley-Lieb diagrams. Later on, in [BS09] Banica and Speicher proved that, in the case of  $C(S_N^+)$ , it was possible to describe the intertwiners in a simpler way by making use of noncrossing partitions. Always in [BS09] this combinatorial interpretation in terms of noncrossing partitions was extended to  $O_N^+$ ,  $B_N^+$  and  $H_N^+$  (respectively the orthogonal, bistochastic and hyperoctahedral groups).

The other important notion to consider is that of free wreath product by the quantum symmetric group. In the classical case, the wreath product of a group  $G$  by  $S_n$ , denoted  $G \wr S_n$  is defined thanks to the natural action of  $S_n$  on a set of  $n$  copies of  $G$ . Bichon in [Bic04] introduced the quantum version  $\mathbb{G} \wr_* S_n^+$  by using an action of  $S_n^+$  on  $n$  copies of  $\mathbb{G}$ . In [Bic04] a first easy case was investigated more in detail and its representation theory was described: the free wreath product  $\widehat{\mathbb{Z}}_2 \wr S_n^+$ . A more general analysis of these products was done in three successive steps. In [BV09], Banica and Vergnioux showed that the quantum reflection group  $H_N^{s+}$  is isomorphic to  $\widehat{\mathbb{Z}}_s \wr_* S_n^+$  and found its irreducible representations and fusion rules in the case  $n \geq 4$ . Later on, Lemeux in [Lem14] extended this result to the free wreath product  $\widehat{\Gamma} \wr_* S_n^+$ , where  $\Gamma$  is a discrete group and  $n \geq 4$ . An even more general result was finally presented by Lemeux and Tarrago in [LT14], where they considered the case of the free wreath product of a compact matrix quantum group of Kac type  $\mathbb{G}$  by  $S_n^+$  and found its representation category by using an argument of monoidal equivalence.

In this last article, it is also possible to find many results concerning the properties of the operator algebras associated to a free wreath product. More precisely, by using a result from [DCFY14], it has been proved that, if  $\mathbb{G}$  has the Haagerup property, also the von Neumann algebra  $L^\infty(\mathbb{G} \wr_* S_n^+)$  has this property. Moreover, the reduced C\*-algebra  $C_r(\mathbb{G} \wr_* S_n^+)$  is exact if  $C_r(\mathbb{G})$  is exact. In [Lem14], Lemeux proved the simplicity and uniqueness of the trace for the reduced C\*-algebra in

the discrete case. His argument, based on the so-called Powers method and on the simplicity and uniqueness of the trace of  $C_r(S_n^+)$  ( $n \geq 8$ ) demonstrated by Brannan in [Bra13], was extended by Wahl in [Wah14] to the general case of a matrix quantum group of Kac type.

The definition of the free wreath product by  $S_n^+$  can be interpreted also from a more geometric point of view; indeed, in analogy with the classical case, it can be used to describe the quantum symmetry group of  $n$  copies of a finite graph in term of the symmetry group of the graph and of  $S_n^+$ . It is well known that the automorphism group of a finite graph  $X$  with  $n$  vertices can be seen as a suitable quotient of  $S_n$ . This object which is also called group of the symmetries of  $X$ , is usually denoted  $G(X)$ . The geometric interpretation of the classic notion of wreath product between symmetry groups is given by formulas such as  $G(X * Y) \cong G(X) \wr G(Y)$ , for a suitable notion of product  $*$  and for graphs satisfying certain conditions. For example, when dealing with the so-called lexicographic product, a characterisation of the graphs satisfying such a relation was given by Sabidussi in [Sab59].

The quantum analogue of  $G(X)$  was introduced by Bichon. In [Bic03], he defined the notion of quantum automorphism group of a finite graph  $X$  as a subgroup of  $S_n^+$ . Such a subgroup which is usually denoted  $G^+(X)$ , is obtained by adding to  $C(S_n^+)$  the relations corresponding to the commutativity between its fundamental representation and the adjacency matrix of  $X$ . Then, as in the classic case, the notion of free wreath product can be geometrically interpreted by formulas such as  $G^+(X * Y) \cong G^+(X) \wr_* G^+(Y)$ . The investigation was started by Bichon in [Bic04] and a first significant result was given some years later by Banica and Bichon in [BB07]. They proved the validity of the formula when  $X * Y$  is a coloured lexicographic product. In [Cha15], Chassaniol considered the lexicographic (non coloured) product and proved that Sabidussi's characterisation of the graphs verifying such a relation can be extended the quantum case.

In this thesis, we aim at providing a further generalization of these results, by taking into account the free wreath product of a compact quantum group by a general quantum automorphism group.

The structure of the thesis will reflect the successive phases of development

of the project; first, we will define and investigate the free wreath product of a discrete group by a quantum automorphism group and, only in a second time, we will deal with the general case. The definition of the free wreath product by a quantum automorphism group is not an immediate generalisation of Bichon's definition, but it is inspired by Example 2.5 in [Bic04] where it is proved that, for every discrete group  $\Gamma$  and  $n \in \mathbb{N}^*$ , the universal  $C^*$ -algebra generated by the coefficients  $a_{ij}(g)$ ,  $1 \leq i, j \leq n$ ,  $g \in \Gamma$  of the matrices  $a(g)$  and with relations  $a_{ij}(g)^* = a_{ij}(g^{-1})$

$$a_{ij}(g)a_{ik}(h) = \delta_{jk}a_{ij}(gh) \quad a_{ij}(g)a_{kj}(h) = \delta_{ik}a_{ij}(gh) \quad \sum_{j=1}^n a_{ij}(e) = 1 = \sum_{i=1}^n a_{ij}(e)$$

can be endowed with a compact quantum group structure and it is isomorphic to the free wreath product  $\widehat{\Gamma} \wr_* S_n^+$ . For our purposes, it is interesting to observe that these relations are equivalent to the following ones, for all  $g, h \in \Gamma$ :

$$a(g) \text{ is unitary} \quad m \in \text{Hom}(a(g) \otimes a(h), a(gh)) \quad \eta \in \text{Hom}(1, a(1))$$

where  $m$  and  $\eta$  are the multiplication and the unity of  $\mathbb{C}^n$  respectively. This characterisation is particularly significant because, in [Ban99], Banica proved that the quantum automorphism group of a  $n$ -dimensional  $C^*$ -algebra  $B$  endowed with a state  $\psi$  can be defined as the universal  $C^*$ -algebra generated by the coefficients  $u_{ij}$ ,  $1 \leq i, j \leq n$  of a matrix  $u$  with relations such that

$$u \text{ is unitary} \quad m \in \text{Hom}(u^{\otimes 2}, u) \quad \eta \in \text{Hom}(1, u)$$

where  $m$  and  $\eta$  are the multiplication and the unity of  $B$  respectively. With this informations in mind, after recalling that  $S_n^+ = \mathbb{G}^{\text{aut}}(\mathbb{C}^n, \text{tr})$ , we can give the following definition.

**Definition.** Let  $\Gamma$  be a discrete group and  $B$  a finite dimensional  $C^*$ -algebra endowed with a faithful state  $\psi$ . Let  $C^*(\Gamma) *_w C(\mathbb{G}^{\text{aut}}(B, \psi))$  be the universal unital  $C^*$ -algebra with generators  $a(g) \in \mathcal{L}(B) \otimes C^*(\Gamma) *_w C(\mathbb{G}^{\text{aut}}(B, \psi))$ ,  $g \in \Gamma$  and relations such that, for every  $g, h \in \Gamma$ :

$$a(g) \text{ is unitary} \quad m \in \text{Hom}(a(g) \otimes a(h), a(gh)) \quad \eta \in \text{Hom}(1, a(e))$$

The C\*-algebra  $C^*(\Gamma) *_w C(\mathbb{G}^{aut}(B, \psi))$  endowed with a suitable comultiplication map is a compact quantum group. It is called the free wreath product of  $\widehat{\Gamma}$  by  $\mathbb{G}^{aut}(B, \psi)$  and will be denoted  $\widehat{\Gamma} \wr_* \mathbb{G}^{aut}(B, \psi)$  or  $H_{(B, \psi)}^+(\widehat{\Gamma})$ .

The first step in order to investigate this object is to revise the representation theory of  $\mathbb{G}^{aut}(B, \psi)$ , when  $\dim(B) \geq 4$  and  $\psi$  is a  $\delta$ -form. In particular, we will present a new description of its spaces of intertwiners which makes use of noncrossing partitions instead of Temperley-Lieb diagrams. This alternative presentation can be seen as a generalisation of the description introduced in [BS09] for  $S_n^+$ ; in our case the computation of the intertwiner associated to every partition is more complicated because of the structure of  $\mathbb{G}^{aut}(B, \psi)$ . By relying on this new graphical interpretation, we can describe the intertwiners of  $\widehat{\Gamma} \wr_* \mathbb{G}^{aut}(B, \psi)$ , for  $\dim(B) \geq 4$  and  $\psi$   $\delta$ -form, by using noncrossing partitions decorated with the elements of the discrete group  $\Gamma$ . This result will allow to prove that the irreducible representations can be indexed by the elements of the monoid of the words written with the elements of  $\Gamma$  and to compute the fusion rules, by generalising the argument used by Lemeux in [Lem14].

The knowledge of the representation category will be crucial for the analysis of the properties of the reduced and of the von Neumann algebras in the particular case of a  $\delta$ -trace  $\psi$ . More precisely, it is proved that  $C_r(H_{(B, \psi)}^+(\widehat{\Gamma}))$  is simple and has a unique trace, when  $\dim(B) \geq 8$ . Moreover, if the group  $\Gamma$  is finite, we show that the von Neumann algebra  $L^\infty(H_{(B, \psi)}^+(\widehat{\Gamma}))$ ,  $\dim(B) \geq 4$  has the Haagerup property.

All these results are also presented in a more general form by dropping the  $\delta$  condition on the state  $\psi$ , because, in this case,  $\widehat{\Gamma} \wr_* \mathbb{G}^{aut}(B, \psi)$  can be decomposed as the free product of smaller free wreath products  $\widehat{\Gamma} \wr_* \mathbb{G}^{aut}(B_i, \psi_i)$ , where  $\psi_i$  is a  $\delta_i$ -form.

In the last part of the thesis, we will deal with the more general case of the free wreath product of a compact quantum group by a quantum automorphism group. First of all, we need to extend the definition given in the discrete case. We have

**Definition.** Let  $\mathbb{G}$  be a compact quantum group and for each  $\alpha \in \text{Irr}(\mathbb{G})$  let  $H_\alpha$  be a space for the representation. Let  $B$  be a finite dimensional C\*-algebra endowed

with a faithful state  $\psi$ . Let  $C(\mathbb{G}) *_w C(\mathbb{G}^{aut}(B, \psi))$  be the universal unital  $C^*$ -algebra with generators  $a(\alpha) \in \mathcal{L}(B \otimes H_\alpha) \otimes C(\mathbb{G}) *_w C(\mathbb{G}^{aut}(B, \psi))$  and relations such that:

- $a(\alpha)$  is unitary for every  $\alpha \in \text{Irr}(\mathbb{G})$
- $\forall \alpha, \beta, \gamma \in \text{Irr}(\mathbb{G}), \forall S \in \text{Hom}(\alpha \otimes \beta, \gamma)$

$$\widetilde{m \otimes S} := (m \otimes S) \circ \Sigma_{23} \in \text{Hom}(a(\alpha) \otimes a(\beta), a(\gamma))$$

where  $\Sigma_{23} : B \otimes H_\alpha \otimes B \otimes H_\beta \longrightarrow B^{\otimes 2} \otimes (H_\alpha \otimes H_\beta), x_1 \otimes x_2 \otimes x_3 \otimes x_4 \mapsto x_1 \otimes x_3 \otimes x_2 \otimes x_4$  is the unitary map that exchanges the legs 2 and 3 in the tensor product.

- $\eta \in \text{Hom}(1, a(1_{\mathbb{G}}))$ , where 1 is the unity of  $C(\mathbb{G}) *_w C(\mathbb{G}^{aut}(B, \psi))$  and  $1_{\mathbb{G}}$  denotes the trivial representations of  $\mathbb{G}$

The  $C^*$ -algebra  $C(\mathbb{G}) *_w C(\mathbb{G}^{aut}(B, \psi))$  endowed with a suitable comultiplication map is a compact quantum group. It is called the free wreath product of  $\mathbb{G}$  by  $\mathbb{G}^{aut}(B, \psi)$  and will be denoted  $\mathbb{G} \wr_* \mathbb{G}^{aut}(B, \psi)$  or  $H_{(B, \psi)}^+(\mathbb{G})$ .

By relying on the same scheme as in the discrete case, we will show how to describe some spaces of intertwiners by means of noncrossing partitions decorated with morphisms of  $\mathbb{G}$ . By generalizing a monoidal equivalence argument used in [LT14], it is then possible to find the irreducible representations and the fusion rules. In this case, the irreducible representations will be indexed by the elements of the monoid of the words whose letters are the irreducible representations of  $\mathbb{G}$ . From the description of the intertwining spaces we will also deduce that the monoidal equivalence is preserved by the free wreath product operation.

**Theorem.** *Let  $\mathbb{G}_1$  and  $\mathbb{G}_2$  be two compact quantum group monoidally equivalent. Let  $B, B'$  be two finite dimensional  $C^*$ -algebras of dimension at least 4 endowed with the  $\delta$ -form  $\psi$  and the  $\delta'$ -form  $\psi'$  respectively. Suppose that the associated quantum automorphism groups  $\mathbb{G}^{aut}(B, \psi)$  and  $\mathbb{G}^{aut}(B', \psi')$  are monoidally equivalent. Then*

$$H_{(B, \psi)}^+(\mathbb{G}_1) \simeq_{\text{mon}} H_{(B', \psi')}^+(\mathbb{G}_2)$$

A stability result is also proved when looking at the fusion semiring.

**Theorem.** *Let  $\mathbb{G}_1$  and  $\mathbb{G}_2$  be two compact quantum groups. Suppose that there exists an isomorphism  $\phi : R^+(\mathbb{G}_1) \longrightarrow R^+(\mathbb{G}_2)$  of their fusion semirings and that  $\phi$  restricted to  $\text{Irr}(\mathbb{G}_1)$  is a bijection of  $\text{Irr}(\mathbb{G}_1)$  onto  $\text{Irr}(\mathbb{G}_2)$ . Let  $B, B'$  be two finite dimensional  $C^*$ -algebras of dimension at least 4 endowed with the  $\delta$ -form  $\psi$  and the  $\delta'$ -form  $\psi'$  respectively. Then, the fusion semirings remain isomorphic when passing to the free wreath product by a quantum automorphism group*

$$R^+(H_{(B,\psi)}^+(\mathbb{G}_1)) \cong R^+(H_{(B',\psi')}^+(\mathbb{G}_2))$$

*and the isomorphism is still a bijection between the spaces of the irreducible representations.*

Finally, we will analyse some properties of the dual quantum group and of the associated operator algebras. More in detail, by using some results from [DCFY14], it will be proved that the dual of  $H_{(B,\psi)}^+(\mathbb{G})$  has the central ACPAP if  $\widehat{\mathbb{G}}$  has the central ACPAP; it follows that in this case the corresponding von Neumann algebra has the Haagerup property. Similarly, a result from [VV07] allows us to show that the exactness of  $\widehat{\mathbb{G}}$  implies the exactness of the dual of  $H_{(B,\psi)}^+(\mathbb{G})$ . Furthermore, if  $\psi$  is a  $\delta$ -trace and  $\dim(B) \geq 8$ , we can generalize an argument of Lemeux [Lem14], based on a result of Brannan [Bra13] and on the Powers method adapted by Banica [Ban97], in order to show the simplicity and uniqueness of the trace for  $C_r(H_{(B,\psi)}^+(\mathbb{G}))$ .

As in the discrete case, all these properties as well as the representation theory are extended to the case of a general faithful state  $\psi$  thanks to a suitable free product decomposition.

As a last result, we take into account the free wreath product of two quantum automorphism groups. We know that, under some assumptions, the free wreath product between the groups of quantum symmetries of two graphs is isomorphic to the quantum symmetric group of a suitable graph. Therefore, our aim is to find an analogous result in the framework of quantum automorphism groups. In particular, we will show that the free wreath product  $\mathbb{G}^{\text{aut}}(B', \psi') \wr_* \mathbb{G}^{\text{aut}}(B, \psi)$  is isomorphic to a suitable quotient of  $\mathbb{G}^{\text{aut}}(B \otimes B', \psi \otimes \psi')$ .



# Chapter 1

## Preliminaries

### 1.1 Compact quantum groups

In this section we recall some basic notions about the theory of compact quantum groups introduced by S. L. Woronowicz. The main references are the original articles of Woronowicz [Wor87, Wor88, Wor91, Wor98]. The same results can be found also in [Tim08, MVD98, NT13].

First of all, we fix the following notation.

*Notation 1.* The symbol  $\otimes$  will be used to denote the tensor product of Hilbert spaces, the minimal tensor product of C\*-algebras or the tensor product of von Neumann algebras, depending on the context.

**Definition 1.1.1.** A Woronowicz compact quantum group  $\mathbb{G}$  is a pair  $(C(\mathbb{G}), \Delta)$  where  $C(\mathbb{G})$  is a unital C\*-algebra and  $\Delta : C(\mathbb{G}) \longrightarrow C(\mathbb{G}) \otimes C(\mathbb{G})$  a \*-homomorphism such that

- $(\text{id} \otimes \Delta)\Delta = (\Delta \otimes \text{id})\Delta$  (coassociativity)
- $\overline{\Delta(C(\mathbb{G}))(C(\mathbb{G}) \otimes 1)} = C(\mathbb{G}) \otimes C(\mathbb{G})$  and  $\overline{\Delta(C(\mathbb{G}))(1 \otimes C(\mathbb{G}))} = C(\mathbb{G}) \otimes C(\mathbb{G})$  (cancellation law)

Woronowicz showed that this definition is equivalent to the following one.

**Definition 1.1.2.** A Woronowicz compact quantum group  $\mathbb{G}$  is a pair  $(C(\mathbb{G}), \Delta)$  where  $C(\mathbb{G})$  is a unital  $C^*$ -algebra and  $\Delta : C(\mathbb{G}) \longrightarrow C(\mathbb{G}) \otimes C(\mathbb{G})$  a  $*$ -homomorphism together with a family of unitary matrices  $(u^\alpha)_{\alpha \in I}$ ,  $u^\alpha \in M_{d_\alpha}(C(\mathbb{G}))$  such that:

- the  $*$ -subalgebra generated by the entries  $u_{ij}^\alpha$  of the matrices  $u^\alpha$  is dense in  $C(\mathbb{G})$
- for all  $\alpha \in I$  and  $1 \leq i, j \leq d_\alpha$  we have  $\Delta(u_{ij}^\alpha) = \sum_{k=1}^{d_\alpha} u_{ik}^\alpha \otimes u_{kj}^\alpha$
- for all  $\alpha \in I$  the transposed matrix  $(u^\alpha)^t$  is invertible

In what follows, we will essentially make use of this second version of the definition.

The first examples of compact quantum groups arise when considering a compact group and a discrete group.

**Example 1.1.3.** Let  $G$  be a compact group and consider the algebra  $C(G) = \{f : G \longrightarrow \mathbb{C}, f \text{ continuous}\}$  endowed with the comultiplication map  $\Delta : C(G) \longrightarrow C(G) \otimes C(G)$ ,  $\Delta(f)(x, y) = f(xy)$  which is correctly defined because of the isomorphism  $C(G) \otimes C(G) \cong C(G \times G)$ . Then,  $(C(G), \Delta)$  is a commutative compact quantum group. Furthermore, thanks to the Gelfand theorem, all the commutative compact quantum groups are of this type.

*Notation 2.* This example explains and justifies the notation  $\mathbb{G} = (C(\mathbb{G}), \Delta)$  which we will use to denote a compact quantum group. In the general case of a noncommutative  $C^*$ -algebra  $C(\mathbb{G})$ , the Gelfand theorem is no longer valid and  $C(\mathbb{G})$  is not isomorphic to the  $C^*$ -algebra of the continuous functions on a compact space. However, we keep the same notation.

**Example 1.1.4.** Let  $\Gamma$  be a discrete group and consider its reduced group  $C^*$ -algebra  $C_r^*(\Gamma)$  which is the  $C^*$ -algebra generated by the image of the left regular representation  $\lambda : \Gamma \longrightarrow \mathcal{L}(\ell^2(\Gamma)), g \mapsto \lambda_g$  of  $\Gamma$ . Let  $\Delta : C_r^*(\Gamma) \longrightarrow C_r^*(\Gamma) \otimes C_r^*(\Gamma)$  be the comultiplication such that  $\Delta(\lambda_g) = \lambda_g \otimes \lambda_g$ . Then  $(C_r^*(\Gamma), \Delta)$  is a compact quantum group. In particular, it is cocommutative, i.e.  $\Delta^{op} := \sigma \circ \Delta = \Delta$ , where  $\sigma$  is the flip operation. There is also the universal version of this compact quantum

group. It is the completion of the group algebra  $\mathbb{C}[\Gamma]$  with respect to the universal norm and will be denoted  $\widehat{\Gamma} = (C^*(\Gamma), \Delta)$ . Furthermore, every cocommutative compact quantum groups is included between the reduced group C\*-algebra and the full group C\*-algebra of a suitable discrete group.

One of the main properties of compact quantum groups is the existence of a Haar state.

**Theorem 1.1.5.** *Let  $(C(\mathbb{G}), \Delta)$  be a compact quantum group. Then, there exists a unique state  $h \in C(\mathbb{G})^*$  such that*

$$(h \otimes id_{C(\mathbb{G})})\Delta(\cdot) = h(\cdot) \text{ and } (id_{C(\mathbb{G})} \otimes h)\Delta(\cdot) = h(\cdot)$$

We will say that a compact quantum group is of Kac type if its Haar state is a trace.

*Remark 1.1.* This property generalizes the existence of a left and right invariant Haar measure  $\mu$  for any compact group  $G$  which is expressed by the following equality:

$$\int_G f(hg)d\mu(g) = \int_G f(g)d\mu(g) = \int_G f(gh)d\mu(g)$$

for any  $f \in C(G)$  and  $h \in G$ .

We can now introduce the representation theory of compact quantum groups which can be considered as a generalisation of the Peter-Weyl theory of the compact groups.

**Definition 1.1.6.** A representation of the compact quantum group  $(C(\mathbb{G}), \Delta)$  on a Hilbert space  $H$  is an element  $u \in M(C(\mathbb{G}) \otimes K(H))$  such that

$$(\Delta \otimes id)(u) = u_{(13)}u_{(23)} \tag{1.1}$$

where  $u_{(13)}$  and  $u_{(23)}$  are defined according to the leg numbering notation.

A representation is said to be unitary if the multiplier  $u$  is unitary as well. In what follows, the Hilbert space of a given representation  $u$  will be denoted  $H_u$ . If the representation is finite dimensional, then there exists  $n \in \mathbb{N}$  such that  $u \in$

$C(\mathbb{G}) \otimes M_n(\mathbb{C})$ . Therefore, in this case, the representation  $u$  can be interpreted as a matrix of order  $n$  with coefficients in  $C(\mathbb{G})$  and the condition 1.1 is equivalent to

$$\Delta(u_{ij}) = \sum_{k=1}^n u_{ik} \otimes u_{kj} \quad (1.2)$$

*Remark 1.2.* Relation 1.2 allows us to give an interpretation of Definition 1.1.2. The generating family  $u_\alpha$  is indeed a family of finite dimensional unitary representations because of the unitarity condition and the definition of  $\Delta$ . The compact quantum group is then defined by means of its own representation theory.

*Remark 1.3.* We observe that, in the case of a compact quantum group  $\mathbb{G} = (C(G), \Delta)$  obtained from a concrete compact group  $G$ , the representations of  $\mathbb{G}$  just introduced are exactly the usual representations of the compact group  $G$ . Let  $\pi : G \rightarrow \mathcal{L}(H)$  be a strongly continuous unitary representation of the compact group  $G$  on the Hilbert space  $H$ . This implies that  $\pi$  is still a continuous map when  $\mathcal{L}(H)$  is considered with the strict operator topology and identified as  $M(K(H))$ , so  $\pi \in M(C(G) \otimes K(H))$ . Then, the elements in  $M(C(G) \otimes C(G) \otimes K(H)) \cong M(C(G \times G) \otimes K(H))$  can be identified as strictly continuous functions on  $G \times G$  with values on  $\mathcal{L}(H)$ . In particular, this implies that

$$\pi_{(13)}(p, q) = \pi(p) \quad \text{and} \quad \pi_{(23)}(p, q) = \pi(q)$$

Finally, by recalling the definition of comultiplication introduced for  $C(G)$  in Example 1.1.3, we have  $(\Delta \otimes \text{id}_{\mathcal{L}(H)})(\pi)(p, q) = \pi(pq)$ . Therefore, in the compact case, equation 1.1 is equivalent to the usual condition  $\pi(pq) = \pi(p)\pi(q)$ .

**Definition 1.1.7.** Let  $\mathbb{G}$  be a compact quantum group and  $u$  a representation such that  $C(\mathbb{G})$  is the closure of the linear span of the coefficients of  $u$ . Then, the pair  $(\mathbb{G}, u)$  is called compact matrix quantum group (or compact matrix pseudogroup) and we will refer to  $u$  as the fundamental representation of  $\mathbb{G}$ .

**Definition 1.1.8.** Let  $u$  and  $v$  be two representations of a compact quantum group  $\mathbb{G}$  on the Hilbert spaces  $H_u$  and  $H_v$ . An intertwiner between  $u$  and  $v$  is a linear map  $T \in \mathcal{L}(H_u, H_v)$  such that

$$v(1 \otimes T) = (1 \otimes T)u$$

The space of intertwiners will be denoted  $\text{Hom}(u, v)$ . A representation  $u$  is said to be irreducible if  $\text{Hom}(u, u) = \mathbb{C}\text{id}$ .

Moreover, in the finite dimensional case, the two representations can be seen as matrices  $u \in M_{n_u}(C(\mathbb{G}))$  and  $v \in M_{n_v}(C(\mathbb{G}))$ . Therefore, an intertwiner is a map  $T \in M_{n_v, n_u}(\mathbb{C})$ .

**Definition 1.1.9.** Two representations  $u$  and  $v$  are said to be equivalent and we will write  $u \sim v$  if there exists an invertible intertwiner  $T \in \text{Hom}(u, v)$ . If  $T$  is also unitary, they are said to be unitarily equivalent.

*Notation 3.* The element  $1 \otimes 1 \in C(\mathbb{G}) \otimes \mathbb{C}$  is the trivial representation of the compact quantum group  $\mathbb{G}$  and it will be denoted  $1_{\mathbb{G}}$ .

*Notation 4.* Let  $\mathbb{G}$  be a compact quantum group. We will denote  $\text{Rep}(\mathbb{G})$  the set of the classes of equivalence of the finite dimensional representations of  $\mathbb{G}$ . Similarly,  $\text{Irr}(\mathbb{G})$  will be the set of the irreducible representations of  $\mathbb{G}$ , up to equivalence.

Now, we can define the fundamental operations between representations.

**Definition 1.1.10.** Let  $u$  and  $v$  be two representation of a compact quantum group  $\mathbb{G} = (C(\mathbb{G}), \Delta)$  on the Hilbert spaces  $H_u$  and  $H_v$ . We define the following operations:

- the direct sum of  $u$  and  $v$ , denoted  $u \oplus v$ , is the element of  $M(C(\mathbb{G}) \otimes K(H_u \oplus H_v))$  obtained as the diagonal sum of the two representations
- the tensor product of  $u$  and  $v$  is the element of  $M(C(\mathbb{G}) \otimes K(H_u \otimes H_v))$  defined by  $u \otimes v := u_{(12)}v_{(13)}$

The following fundamental result of Woronowicz will allow us to deal only with finite dimensional representations.

**Theorem 1.1.11.** *Every irreducible representation of a compact quantum group is finite dimensional and equivalent to a unitary one. Furthermore, every representation can be decomposed as a direct sum of irreducible representations.*

*Remark 1.4.* For every representation  $r \in \text{Rep}(\mathbb{G})$ , there is a family of irreducible representations  $\alpha_i \in \text{Irr}(\mathbb{G})$  such that:

$$r = \bigoplus_i \alpha_i$$

If a representation  $\alpha \in \text{Irr}(\mathbb{G})$  is in the decomposition of  $r \in \text{Rep}(\mathbb{G})$  we will write  $\alpha \subset r$ . The existence of a subrepresentation  $\alpha$  is equivalent to the existence of an isometric intertwiner  $T \in \text{Hom}(\alpha, r)$ . The dimension of  $\text{Hom}(\alpha, r)$  is the multiplicity of  $\alpha$  in  $r$ , i.e. the number of times that  $\alpha$  is present in the sum.

The formulas describing the decomposition into irreducible representations of the tensor product of two irreducible representations are called fusion rules.

Woronowicz showed that we can define the notion of conjugate representation as follows.

**Definition 1.1.12.** Let  $u$  be a finite dimensional unitary representation on the Hilbert space  $H$ . Let  $j : \mathcal{L}(H) \longrightarrow \mathcal{L}(\bar{H})$  be the application sending an operator in its dual. Then  $u^c = (\text{id} \otimes j)(u^{-1}) \in C(\mathbb{G}) \otimes \mathcal{L}(\bar{H})$  is a representation of  $\mathbb{G}$ , called the contragredient representation.

By choosing a basis of  $H$ , we can think to the representation  $u$  as a matrix  $(u_{ij})_{ij}$ . In this case  $u^c = (u_{ij}^*)_{ij}$  with respect to the dual basis.

The representation  $u^c$  can be non unitary, but we know from Theorem 1.1.11 that it is equivalent to a unitary one. The conjugate representation of  $u$ , denoted  $\bar{u}$ , is the unique, up to equivalence, unitary version of  $u^c$ .

Moreover, if  $u$  is a unitary irreducible representation, then  $\bar{u}$  is, up to equivalence, the unique unitary irreducible representation such that  $u \otimes \bar{u}$  and  $\bar{u} \otimes u$  contain at least a copy of the trivial representation  $1_{\mathbb{G}}$ .

We have this characterisation of the property of being Kac.

**Proposition 1.1.13.** *The contragredient representations of all the unitary representations of a compact quantum group  $\mathbb{G}$  are unitary if and only if  $\mathbb{G}$  is of Kac type.*

**Definition 1.1.14.** The fusion semiring of a compact quantum group  $\mathbb{G}$ , denoted  $(R^+(\mathbb{G}), \oplus, \otimes, \bar{\phantom{x}})$ , is the set of equivalence classes of finite dimensional representations endowed with the operations of direct sum, tensor product and conjugate. We will say that two fusion semirings  $R_1^+$  and  $R_2^+$  are isomorphic if there is a bijection  $\phi : R_1^+ \longrightarrow R_2^+$  which is compatible with the three operations of the semirings.

Now, we recall a more specific result concerning the conjugate representation which will be used in what follows.

**Proposition 1.1.15.** *Let  $\alpha \in \text{Rep}(\mathbb{G})$  and  $H_\alpha$  be a space for the representation, let  $\bar{\alpha}$  be the conjugate representation and  $H_{\bar{\alpha}}$  the associated Hilbert space. Let  $S \in \text{Hom}(\alpha \otimes \bar{\alpha}, 1_{\mathbb{G}})$  and  $S' \in \text{Hom}(\bar{\alpha} \otimes \alpha, 1_{\mathbb{G}})$  be two non-trivial morphisms. Then, there exist a basis  $(e_i^\alpha)$  of  $H_\alpha$ , a basis  $(e_i^{\bar{\alpha}})$  of  $H_{\bar{\alpha}}$  and a family  $(\lambda_i^\alpha)$  of positive scalars such that, up to a scalar coefficient,*

$$S(\xi) = \sum_{i=1}^n \lambda_{i,\alpha} \langle \xi, e_i^\alpha \otimes e_i^{\bar{\alpha}} \rangle \quad \text{and} \quad S'(\eta) = \sum_{i=1}^n \lambda_{i,\alpha}^{-1} \langle \eta, e_i^{\bar{\alpha}} \otimes e_i^\alpha \rangle$$

*Proof.* Let  $E \in \text{Hom}(1, \alpha \otimes \bar{\alpha})$  be a non zero invariant vector and view it as an element in  $H_\alpha \otimes H_{\bar{\alpha}}$ . Define  $J_E : H_\alpha \longrightarrow H_{\bar{\alpha}}$  as the invertible antilinear application which satisfies  $\langle J_E \xi, \eta \rangle_{H_{\bar{\alpha}}} = \langle E, \xi \otimes \eta \rangle_{H_\alpha \otimes H_{\bar{\alpha}}}$  for all  $\xi \in H_\alpha$  and  $\eta \in H_{\bar{\alpha}}$ . Let  $Q_E = J_E J_E^* \in \mathcal{L}(H_\alpha)$ . We observe that the map  $E \mapsto J_E$  is linear, therefore  $Q_{\lambda E} = |\lambda|^2 Q_E$  for all  $\lambda \in \mathbb{C} \setminus \{0\}$  and for all  $E \in \text{Hom}(1, \alpha \otimes \bar{\alpha}) \setminus \{0\}$ . Moreover, we have that  $\text{Tr}(Q_{\lambda E}) = |\lambda|^2 \text{Tr}(Q_E)$  and  $\text{Tr}(Q_{\lambda E}^{-1}) = |\lambda|^2 \text{Tr}(Q_E^{-1})$ . It follows that by replacing  $E$  by  $\lambda E$ , for  $\lambda = \sqrt[4]{\text{Tr}(Q_E^{-1})/\text{Tr}(Q_E)}$ , we have that  $\text{Tr}(Q_E) = \text{Tr}(Q_E^{-1})$ . Moreover, for this particular choice of  $E$ , we can find a unique non zero vector  $\bar{E} \in \text{Hom}(1, \bar{\alpha} \otimes \alpha)$  such that  $(E^* \otimes 1)(1 \otimes \bar{E}) = 1$ . It follows that,  $J_{\bar{E}} = J_E^{-1}$  and  $Q_{\bar{E}} = (J_E J_E^*)^{-1}$ . In this proof, we will always suppose that the invariant vectors  $E$  and  $\bar{E}$  are normalized in this way (this means also that they are unique up to  $\mathbb{S}^1$ ) and we will use the following notations:  $E_\alpha = E$ ,  $E_{\bar{\alpha}} = \bar{E}$ ,  $J_\alpha = J_E$ ,  $J_{\bar{\alpha}} = J_{\bar{E}}$  and  $Q_\alpha = Q_E$ . Then, we can assume  $\|E_\alpha\| = \|E_{\bar{\alpha}}\|$  and  $J_{\bar{\alpha}} = J_\alpha^{-1}$ .

Now, the maps  $S$  and  $S'$ , up to a scalar coefficient, can be written as  $S(\xi) = \langle \xi, E_\alpha \rangle$  for all  $\xi \in H_\alpha \otimes H_{\bar{\alpha}}$  and  $S'(\eta) = \langle \eta, E_{\bar{\alpha}} \rangle$  for all  $\eta \in H_{\bar{\alpha}} \otimes H_\alpha$ . We observe that  $Q_\alpha$  is a self-adjoint positive operator and, in light of it, it is diagonalizable. Let  $(e_i^\alpha)$  be

an orthonormal basis of  $H_\alpha$  of eigenvectors and  $\lambda_{i,\alpha}$  the corresponding eigenvalues such that  $Q_\alpha e_i^\alpha = \lambda_{i,\alpha} e_i^\alpha$ . Choose  $e_i^{\bar{\alpha}} := J_\alpha e_i^\alpha$  as basis of  $H_{\bar{\alpha}}$ .

Then, we have  $\langle E_\alpha, e_i^\alpha \otimes e_j^{\bar{\alpha}} \rangle = \langle J_\alpha e_i^\alpha, e_j^{\bar{\alpha}} \rangle = \langle J_\alpha e_i^\alpha, J_\alpha e_j^\alpha \rangle = \langle J_\alpha^* J_\alpha e_i^\alpha, e_j^\alpha \rangle = \langle Q_\alpha e_i^\alpha, e_j^\alpha \rangle = \lambda_{i,\alpha} \langle e_i^\alpha, e_j^\alpha \rangle = \delta_{ij} \lambda_{i,\alpha}$  so  $E_\alpha = \sum_{i=1}^n \lambda_{i,\alpha} e_i^\alpha \otimes e_i^{\bar{\alpha}}$

It follows that  $S(\xi) = \langle \xi, E_\alpha \rangle = \sum_{i=1}^n \lambda_{i,\alpha} \langle \xi, e_i^\alpha \otimes e_i^{\bar{\alpha}} \rangle$ .

In order to prove the second relation, we observe that  $Q_{\bar{\alpha}} = J_\alpha^* J_\alpha = (J_\alpha J_\alpha^*)^{-1} = J_\alpha Q_\alpha^{-1} J_\alpha^{-1}$  and so  $Q_{\bar{\alpha}} e_i^{\bar{\alpha}} = \lambda_{i,\alpha}^{-1} e_i^{\bar{\alpha}}$ . Then, as in the previous case, we can prove that  $S'(\eta) = \sum_{i=1}^n \lambda_{i,\alpha}^{-1} \langle \eta, e_i^{\bar{\alpha}} \otimes e_i^\alpha \rangle$ .  $\square$

*Remark 1.5.* A compact quantum group  $\mathbb{G}$  is of Kac type if and only if the antilinear map  $J_\alpha$  is anti-unitary for all  $\alpha \in \text{Irr}(\mathbb{G})$ . In particular, this means that  $J_\alpha^* = J_\alpha^{-1}$ , therefore  $Q_\alpha = \text{id}$  and  $\lambda_{i,\alpha} = 1$  for all  $i$ .

Now, we can give some important examples of the representations of a compact quantum group. We start by describing the representation theory of  $(C_r^*(\Gamma), \Delta)$  and we introduce the GNS construction afterwards.

**Example 1.1.16.** Let  $\Gamma$  be a discrete group and consider the compact quantum group  $C_r^*(\Gamma)$  introduced in Example 1.1.4. The generators  $\lambda_g \in \mathcal{L}(\ell^2(\Gamma))$  are given by  $\lambda_g(\delta_h) = \delta_{gh}$ , where the  $\delta_g \in \ell^2(\Gamma)$  are defined by  $\delta_g(r) = \delta_{g,r}$  and form a basis of  $\ell^2(\Gamma)$ . The  $\lambda_g$  are all the irreducible representations of the compact quantum group. Moreover, they are all of dimension one and the fusion rules are simply given by  $\lambda_g \otimes \lambda_h = \lambda_{gh}$ . This example will be important in what follows and the representation  $\lambda_g, g \in \Gamma$  will be denoted  $g$  for simplicity.

We are now ready to introduce the GNS construction associated to a compact quantum group  $\mathbb{G} = (C(\mathbb{G}), \Delta)$  with Haar state  $h$ . Consider the scalar product  $\langle x, y \rangle := h(x^*y)$  induced by  $h$  on  $C(\mathbb{G})$  and define  $L^2(\mathbb{G})$  to be the completion of  $C(\mathbb{G})$  with respect to the norm induced by this scalar product, possibly passing to a suitable quotient to make the scalar product non-degenerate. Let  $\Lambda : C(\mathbb{G}) \longrightarrow L^2(\mathbb{G})$  be the quotient map. Then

$$\pi_h : C(\mathbb{G}) \longrightarrow \mathcal{L}(L^2(\mathbb{G})), \quad \pi_h(x)\Lambda(y) = \Lambda(xy)$$

is a representation of  $C(\mathbb{G})$  with cyclic vector  $\xi_0 = \Lambda(1)$  and associated state  $h$ , i.e we have  $h(x) = \langle \xi_0, \pi_h(x)\xi_0 \rangle$  for all  $x \in C(\mathbb{G})$ . In what follows, the GNS construction will be characterised by the triple  $(L^2(\mathbb{G}), \pi_h, \xi_0)$ .

**Definition 1.1.17.** The reduced C\*-algebra associated to a compact quantum group  $\mathbb{G}$  is the image of  $C(\mathbb{G})$  through the GNS representation  $\pi_h$ . We have

$$C_r(\mathbb{G}) = \pi_h(C(\mathbb{G}))$$

It is easy to prove that  $(\pi_h \otimes \pi_h)\Delta$  can be factorized through  $\pi_h$ , therefore there exists a map  $\Delta_r : C_r(\mathbb{G}) \longrightarrow C_r(\mathbb{G}) \otimes C_r(\mathbb{G})$  such that

$$\Delta_r \circ \pi_h = (\pi_h \otimes \pi_h)\Delta$$

It follows that the pair  $(C_r(\mathbb{G}), \Delta_r)$  has a natural structure of compact quantum group inherited from  $\mathbb{G}$ . The Haar state  $h_r$  is similarly defined by factorizing  $h$  through  $\pi_h$ ; it satisfies the relation  $h_r \circ \pi_r = h$  and is always faithful.

For a given compact quantum group  $\mathbb{G}$ , we will denote  $\text{Pol}(\mathbb{G})$  the subspace of  $C(\mathbb{G})$  generated by all the coefficients  $(\text{id} \otimes \omega)(u)$ ,  $\omega \in \mathcal{L}(H_u)^*$  of any finite dimensional representation  $u$  on the Hilbert space  $H_u$ . It is naturally endowed with a \*-algebra structure inherited from  $C(\mathbb{G})$ , but it is possible to obtain a Hopf-\*algebra structure by defining the right coalgebra operations and an antipode. Let  $u$  be a finite dimensional unitary representation; we know that  $u \in M_n(C(\mathbb{G}))$ , so its coefficients can be chosen to be exactly the entries  $u_{ij}$  of the matrix. Thanks to Theorem 1.1.11, it is enough to define the following maps only on these coefficients. The comultiplication is  $\Delta|_{\text{Pol}(\mathbb{G})}$ , i.e. the restriction of the map which gives the compact quantum group structure, so in particular  $\Delta(u_{ij}) = \sum_{k=1}^n u_{ik} \otimes u_{kj}$ . The counit is  $\varepsilon(u_{ij}) = \delta_{ij}$  and the antipode is  $S(u_{ij}) = u_{ij}^*$ .

We have the following two important results due to Woronowicz.

**Theorem 1.1.18.** *The Hopf-\*algebra  $\text{Pol}(\mathbb{G})$  is dense in  $C(\mathbb{G})$ .*

**Proposition 1.1.19.** *The compact quantum group  $\mathbb{G}$  is of Kac type if and only if the antipode satisfies the relation  $S^2 = \text{id}_{C(\mathbb{G})}$ .*

A compact quantum group admits not only a reduced version, but also a maximal version.

**Definition 1.1.20.** Let  $\mathbb{G}$  be a compact quantum group. Let  $C_{\max}(\mathbb{G})$  be the enveloping  $C^*$ -algebra of  $\text{Pol}(\mathbb{G})$ . The comultiplication of  $\mathbb{G}$  can be extended to a comultiplication  $\Delta_{\max}$  on  $C_{\max}(\mathbb{G})$  by universality. Then, the pair  $(C_{\max}(\mathbb{G}), \Delta_{\max})$  is a compact quantum group called the maximal version of  $\mathbb{G}$ . A compact quantum group such that its underlying  $C^*$ -algebra is maximal is said to be full.

By making use of the GNS construction, we can also introduce the regular representation of a compact quantum group  $\mathbb{G}$ . Woronowicz proved that there exists a unitary operator  $u$  on  $L^2(\mathbb{G}) \otimes L^2(\mathbb{G})$  such that, for all  $a \in C(\mathbb{G})$  and  $\eta \in L^2(\mathbb{G})$ , we have

$$u^*(\eta \otimes \pi_h(a)\xi_0) = (\pi_h \otimes \pi_h)\Delta(a)(\eta \otimes \xi_0)$$

The operator  $u$  is a representation of  $\mathbb{G}$  and it is called left regular representation, because, in the case of a compact group,  $u$  is the classic left regular representation. Therefore,  $u$  is also an element of  $M(C_r(\mathbb{G}) \otimes K(L^2(\mathbb{G})))$  and, since it satisfies the pentagonal equation

$$u_{(12)}u_{(13)}u_{(23)} = u_{(23)}u_{(12)}$$

it is a multiplicative unitary, as defined by Baaj and Skandalis in [BS93]. In the same way, the right regular representation is the unitary operator  $v$  such that, for all  $a \in C(\mathbb{G})$  and  $\eta \in L^2(\mathbb{G})$ , we have

$$v(\eta \otimes \pi_h(a)\xi_0) = (\pi_h \otimes \pi_h)\Delta^{op}(a)(\eta \otimes \xi_0)$$

In this case as well  $v \in M(C_r(\mathbb{G}) \otimes K(L^2(\mathbb{G})))$  and it is a multiplicative unitary. We recall this important result of Woronowicz.

**Proposition 1.1.21.** *Every irreducible unitary representation is contained in the (left or right) regular representation with multiplicity equal to its dimension.*

The multiplicative unitary of a compact quantum group contains all the information necessary to reconstruct the group itself. By using the operator  $u$ , we have

that

$$C_r(\mathbb{G}) = \overline{\langle (\text{id} \otimes \omega)u \mid \omega \in \mathcal{L}(L^2(\mathbb{G}))_* \rangle} \quad \text{and} \quad \Delta(a) = u^*(1 \otimes a)u$$

where  $\mathcal{L}(L^2(\mathbb{G}))_*$  denotes the predual which is the space of the normal linear functionals on  $\mathcal{L}(L^2(\mathbb{G}))$ . Furthermore, thanks to some particular properties of the multiplicative unitary corresponding to the regular representation, we can define a second quantum group which is the dual quantum group. It is not a compact quantum group, but a discrete quantum group. We can think of the groups of this type as being the dual of a compact one. It will be denoted  $\widehat{\mathbb{G}} = (C_0(\widehat{\mathbb{G}}), \widehat{\Delta})$ , where

$$C_0(\widehat{\mathbb{G}}) = \overline{\langle (\omega \otimes \text{id})u \mid \omega \in \mathcal{L}(L^2(\mathbb{G}))_* \rangle}$$

and

$$\widehat{\Delta} : C_0(\widehat{\mathbb{G}}) \longrightarrow M(C_0(\widehat{\mathbb{G}}) \otimes C_0(\widehat{\mathbb{G}})), \quad \widehat{\Delta}(a) = \sigma u(a \otimes 1)u^* \sigma$$

The map  $\sigma$  is the flip.

The dual quantum group  $\widehat{\mathbb{G}}$  of a compact quantum group  $\mathbb{G}$  can also be defined by using the irreducible representations of  $\mathbb{G}$ . We have that

$$C_0(\widehat{\mathbb{G}}) \cong \bigoplus_{\alpha \in \text{Irr}(\mathbb{G})} \mathcal{L}(H_\alpha)$$

while the comultiplication  $\widehat{\Delta}$  is such that, for  $x \in \mathcal{L}(H_\alpha)$ ,  $\alpha, \beta, \gamma \in \text{Irr}(\mathbb{G})$  and  $T \in \text{Hom}(\alpha, \beta \otimes \gamma)$ , we have  $\widehat{\Delta}(x)T = Tx$ .

Finally, the von Neumann algebra of the dual quantum group  $\widehat{\mathbb{G}}$  is

$$l^\infty(\widehat{\mathbb{G}}) \cong \prod_{\alpha \in \text{Irr}(\mathbb{G})} \mathcal{L}(H_\alpha)$$

Now, we recall some definitions and results of the category theory. They will be immediately used in order to introduce the quantum version of the Tannaka-Krein duality, presented by Woronowicz in [Wor88]. More details and precisions can be found in [NT13].

**Definition 1.1.22.** Let  $\mathcal{C}$  be a category. We will say that  $\mathcal{C}$  is a  $C^*$ -category if the following conditions hold

- a) for all  $U, V, W \in \text{Ob}(\mathcal{C})$  the class of morphisms  $\text{Hom}(U, V)$  is a Banach space and  $\text{Hom}(V, W) \times \text{Hom}(U, V) \longrightarrow \text{Hom}(U, V)$ ,  $(R, S) \mapsto RS$  is a bilinear map such that  $\|RS\| \leq \|R\| \|S\|$
- b)  $\mathcal{C}$  is endowed with an antilinear contravariant functor  $*$  :  $\mathcal{C} \longrightarrow \mathcal{C}$  which is the identity on the objects (i.e. if  $T \in \text{Hom}(U, V)$ , then  $T^* \in \text{Hom}(V, U)$ ) and such that, for any  $T \in \text{Hom}(U, V)$ ,  $T^{**} = T$  and  $\|T^*T\| = \|T\|^2$ . Moreover,  $\text{End}(U)$  is a  $C^*$ -algebra for any  $U \in \text{Ob}(\mathcal{C})$  and  $T^*T \in \text{End}(U)$  is a positive element for any  $T \in \text{Hom}(U, V)$ .

We will say that  $\mathcal{C}$  is a monoidal  $C^*$ -category if, in addition, there is a bilinear bifunctor  $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ ,  $(U, V) \mapsto U \otimes V$ , a unit object  $1_{\mathcal{C}}$  and, for any  $U, V, W \in \text{Ob}(\mathcal{C})$ , natural unitary isomorphisms  $\alpha_{U,V,W} : (U \otimes V) \otimes W \longrightarrow U \otimes (V \otimes W)$ ,  $\lambda_U : 1_{\mathcal{C}} \otimes U \longrightarrow U$  and  $\rho_U : U \otimes 1_{\mathcal{C}} \longrightarrow U$  such that

1.  $\alpha_{U,V,W \otimes X} \circ \alpha_{U \otimes V, W, X} = (\text{id}_U \otimes \alpha_{V,W,X}) \circ \alpha_{U,V \otimes W, X} \circ (\alpha_{U,V,W} \otimes \text{id}_X)$  in the space  $\text{Hom}(((U \otimes V) \otimes W) \otimes X, U \otimes (V \otimes (W \otimes X)))$
2.  $\rho_U \otimes \text{id}_V = (\text{id}_U \otimes \lambda_V) \circ \alpha_{U,1_{\mathcal{C}},V}$  in  $\text{Hom}((U \otimes 1_{\mathcal{C}}) \otimes V, U \otimes V)$
3.  $\lambda_{1_{\mathcal{C}}} = \rho_{1_{\mathcal{C}}}$
4.  $(S \otimes T)^* = S^* \otimes T^*$  for any  $S, T$  morphisms

We will also suppose that the following conditions are verified:

5. for any  $U, V \in \text{Ob}(\mathcal{C})$  there exist  $W \in \text{Ob}(\mathcal{C})$  and two isometries  $u \in \text{Hom}(U, W)$ ,  $v \in \text{Hom}(V, W)$  such that  $uu^* + vv^* = \text{id}_W$
6. for any  $U \in \text{Ob}(\mathcal{C})$  and projection  $p \in \text{End}(U)$  there exist  $V \in \text{Ob}(\mathcal{C})$  and an isometry  $q \in \text{Hom}(V, U)$  such that  $qq^* = p$
7.  $\text{End}(1_{\mathcal{C}}) = \mathbb{C}\text{id}_{1_{\mathcal{C}}}$
8. the class  $\text{Ob}(\mathcal{C})$  is a set

A category is said to be strict if  $(U \otimes V) \otimes W = U \otimes (V \otimes W)$ ,  $1_{\mathcal{C}} \otimes U = U \otimes 1_{\mathcal{C}} = U$  and the isomorphisms  $\alpha$ ,  $\lambda$  and  $\rho$  are the identity morphisms. A useful result from [ML98] is that every monoidal category can be strictified. Therefore, in what follows, these categories will be supposed to be strict.

**Definition 1.1.23.** Let  $\mathcal{C}$  be a monoidal  $C^*$ -category and  $A \subset Ob(\mathcal{C})$ . We say that the subset  $A$  generates the category  $\mathcal{C}$  if, for any  $V \in Ob(\mathcal{C})$ , there exists a finite family of morphisms  $p_i \in Hom(U_i, V)$ , where  $U_i$  is a tensor product of elements of  $A$ , such that  $\sum_i p_i p_i^* = id_V$

**Definition 1.1.24.** Let  $\mathcal{C}$  and  $\mathcal{D}$  be two monoidal  $C^*$ -categories. A functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  is called a tensor functor if it is linear on the morphisms and there exist isomorphisms  $F_0 : 1_{\mathcal{D}} \rightarrow F(1_{\mathcal{C}})$  and  $F_2 : F(U) \otimes F(V) \rightarrow F(U \otimes V)$  such that

1.  $F(\alpha_{U,V,W}) \circ F_2 \circ (F_2 \otimes id_{F(W)}) = F_2 \circ (id_{F(U)} \otimes F_2) \circ \alpha_{F(U),F(V),F(W)}$  in the space  $Hom((F(U) \otimes F(V)) \otimes F(W), F(U \otimes (V \otimes W)))$
2.  $F(\lambda_U) \circ F_2 = \lambda_{F(U)} \circ (F_0 \otimes id_{F(U)})$  in  $Hom(F(1_{\mathcal{C}}) \otimes F(U), F(U))$
3.  $F(\rho_U) \circ F_2 = \rho_{F(U)} \circ (id_{F(U)} \otimes F_0)$  in  $Hom(F(U) \otimes F(1_{\mathcal{C}}), F(U))$

A tensor functor is said to be unitary if  $F(T)^* = F(T^*)$  for any morphism  $T$  and  $F_0, F_2$  are unitary.

The conditions 5 and 6 in Definition 1.1.22 tell us that a monoidal  $C^*$ -category is required to be complete with respect to direct sums and subobjects. Anyway, a category  $\mathcal{C}$  which verifies all the properties of a (strict) monoidal  $C^*$ -category with the exception of 5 and 6 can be completed to a category  $\tilde{\mathcal{C}}$  satisfying all these properties. The category  $\tilde{\mathcal{C}}$  is the so called Karoubi envelope (or Cauchy completion) of the additive completion. More details on the existence and on the construction of  $\tilde{\mathcal{C}}$  can be found in [NT13]. We refer to [Bor94a, Bor94b] for a more general analysis of the notion of completion of a category.

**Definition 1.1.25.** Let  $F, G$  be two tensor functors  $\mathcal{C} \rightarrow \mathcal{D}$ . A natural isomorphism  $p : F \rightarrow G$  is called monoidal if  $F_2 \circ p = (p \otimes p) \circ G_2$  in  $Hom(F(U) \otimes F(V), G(U \otimes V))$  and  $G_0 = F_0 \circ p$  in  $Hom(1_{\mathcal{D}}, G(1_{\mathcal{C}}))$ .

**Definition 1.1.26.** We say that two monoidal  $C^*$ -categories  $\mathcal{C}$  and  $\mathcal{D}$  are monoidally equivalent if there exist two tensor functors  $F : \mathcal{C} \rightarrow \mathcal{D}$  and  $G : \mathcal{D} \rightarrow \mathcal{C}$  such that  $FG \cong \text{id}$  and  $GF \cong \text{id}$  are monoidal isomorphisms. If  $F$ ,  $G$  and the two isomorphisms can be chosen to be unitary then  $\mathcal{C}$  and  $\mathcal{D}$  are unitarily monoidally isomorphic.

**Definition 1.1.27.** A monoidal  $C^*$ -category  $\mathcal{C}$  is rigid if for any  $U \in \text{Ob}(\mathcal{C})$  there exist  $\bar{U} \in \text{Ob}(\mathcal{C})$  and two morphisms  $R \in \text{Hom}(1, \bar{U} \otimes U)$ ,  $\bar{R} \in \text{Hom}(1, U \otimes \bar{U})$  such that

$$(\bar{R}^* \otimes \text{id}_U)(\text{id}_U \otimes R) = \text{id}_U \quad (R^* \otimes \text{id}_{\bar{U}})(\text{id}_{\bar{U}} \otimes \bar{R}) = \text{id}_{\bar{U}}$$

The object  $\bar{U}$  is called the conjugate of  $U$  and the conditions satisfied by  $R, \bar{R}$  are the conjugate equations.

The notion of conjugate is at the base of the following theorem, known as Frobenius reciprocity.

**Theorem 1.1.28.** *Let  $U$  be an object of a monoidal  $C^*$ -category with conjugate  $\bar{U}$  and let  $R, \bar{R}$  be the morphisms solving the conjugate equations. Then, the linear application  $\text{Hom}(U \otimes V, W) \rightarrow \text{Hom}(V, \bar{U} \otimes W)$  given by  $T \mapsto (\text{id}_{\bar{U}} \otimes T)(R \otimes \text{id}_V)$  is an isomorphism. Similarly, we have  $\text{Hom}(V \otimes U, W) \cong \text{Hom}(V, W \otimes \bar{U})$*

Now, we introduce some examples which will be particularly important in what follows. Let us fix a set of Hilbert spaces. Then, the category  $\text{Hilb}_f$ , whose objects are chosen to be the spaces of this set, can be endowed with a structure of rigid monoidal  $C^*$ -category. Another category which admits such a structure and which will play a prominent role in this thesis is  $\text{Rep}(\mathbb{G})$ , the category of the finite dimensional unitary representations of a compact quantum group  $\mathbb{G}$ . We will always suppose that the Hilbert spaces of the representations are objects of  $\text{Hilb}_f$ .

**Definition 1.1.29.** A tensor functor  $F : \mathcal{C} \rightarrow \text{Hilb}_f$  which is injective on the morphisms and exact is called a fiber functor. A monoidal  $C^*$ -category endowed with a fiber functor is said to be concrete.

An important example of fiber functor is the functor which associates to every representation of  $\text{Rep}(\mathbb{G})$  its underlying Hilbert space.

We can now state the quantum version of the Tannaka-Krein duality, presented by Woronowicz in [Wor88].

**Theorem 1.1.30.** *Let  $\mathcal{C}$  be a rigid concrete monoidal  $C^*$ -category generated by a family of objects  $(v_i)_{i \in I}$  together with their conjugates. Then, there exists, up to isomorphism, a unique full compact quantum group  $\mathbb{G} = (C(\mathbb{G}), \Delta)$ , whose  $C^*$ -algebra  $C(\mathbb{G})$  is generated by a family of finite dimensional unitary representations  $u_i$ ,  $i \in I$  (with  $v_i$  and  $u_i$  indexed by the same  $I$ ) and such that, if  $\mathbb{G}_1 = (C(\mathbb{G}_1), \Delta_1)$  is a full compact quantum group such that*

- $C(\mathbb{G}_1)$  is generated by the coefficients of a family of finite dimensional unitary representations  $w_i$ ,  $i \in I$
- for all  $(i_1, \dots, i_k) \in I^k$  and  $(j_1, \dots, j_l) \in I^l$ , we have

$$\text{Hom}(v_{i_1} \otimes \dots \otimes v_{i_k}, v_{j_1} \otimes \dots \otimes v_{j_l}) \subseteq \text{Hom}(w_{i_1} \otimes \dots \otimes w_{i_k}, w_{j_1} \otimes \dots \otimes w_{j_l})$$

then, there exists a surjective  $*$ -homomorphism  $\phi : C(\mathbb{G}) \longrightarrow C(\mathbb{G}_1)$  such that  $(\text{id} \otimes \phi)(u_i) = w_i$ .

To be precise, this theorem was proved by Woronowicz in the case of a compact matrix quantum group, i.e when  $|I| = 1$ . However, the proof can be generalized and the result is true also in this more general case.

This theorem is particularly important as it allows us to reconstruct a compact quantum group, up to isomorphism, by starting from its representation category.

We say that two compact quantum groups are monoidally equivalent if their representation categories are unitarily monoidally equivalent. In what follows, however, the monoidal equivalence results will not be proved by referring to the general definition above, but to this more explicit equivalent definition (see [BDRV06]).

**Definition 1.1.31.** Let  $\mathbb{G}_1$  and  $\mathbb{G}_2$  be two compact quantum groups. They are monoidally equivalent (written  $\mathbb{G}_1 \simeq_{\text{mon}} \mathbb{G}_2$ ) if there exists a bijection  $\phi :$

$\text{Irr}(\mathbb{G}_1) \longrightarrow \text{Irr}(\mathbb{G}_2)$ ,  $\phi(1_{\mathbb{G}_1}) = 1_{\mathbb{G}_2}$  such that, for any  $k, l \in \mathbb{N}$  and for any  $\alpha_i, \beta_j \in \text{Irr}(\mathbb{G})$ ,  $1 \leq i \leq k$ ,  $1 \leq j \leq l$ , there is an isomorphism

$$\phi : \text{Hom}_{\mathbb{G}_1}(\alpha_1 \otimes \dots \otimes \alpha_k; \beta_1 \otimes \dots \otimes \beta_l) \longrightarrow \text{Hom}_{\mathbb{G}_2}(\phi(\alpha_1) \otimes \dots \otimes \phi(\alpha_k); \phi(\beta_1) \otimes \dots \otimes \phi(\beta_l))$$

such that:

- i)  $\phi(\text{id}) = \text{id}$
- ii)  $\phi(F \otimes G) = \phi(F) \otimes \phi(G)$
- iii)  $\phi(F^*) = \phi(F)^*$
- iv)  $\phi(FG) = \phi(F)\phi(G)$  for  $F, G$  composable morphisms

The proof of a monoidal equivalence between two compact quantum groups can be simplified by making use of the following proposition.

**Proposition 1.1.32.** *Let  $\mathcal{C}, \mathcal{D}$  be two monoidal rigid  $C^*$ -categories, possibly non complete with respect to direct sums and subobjects. Let  $\tilde{\mathcal{C}}, \tilde{\mathcal{D}}$  be their completions. If  $\psi : \mathcal{C} \longrightarrow \mathcal{D}$  is a unital monoidal equivalence between the two categories  $\mathcal{C}$  and  $\mathcal{D}$ , then there exists a unital monoidal equivalence  $\tilde{\psi} : \tilde{\mathcal{C}} \longrightarrow \tilde{\mathcal{D}}$  which extends  $\psi$ .*

This is a standard result in category theory, we refer to [Bor94a, Bor94b] for the proof and for further details.

## 1.2 Free compact quantum groups

In this section, we present some important families of free compact quantum groups. For the first examples, the basic idea is to build a quantum noncommutative version of the spaces of the continuous functions on the classic groups  $U_n, O_n$  and  $S_n$ . The  $C^*$ -algebras  $C(U_n), C(O_n)$  and  $C(S_n)$  are commutative so we want to liberate them from this condition in order to find a noncommutative quantum analogue.

We start by defining the free unitary quantum group, introduced by Wang and Van Daele ([Wan93, VDW96]).

**Definition 1.2.1.** Let  $F \in GL_n(\mathbb{C})$ ,  $n \geq 2$ . Consider the following universal unital  $C^*$ -algebra

$$A_u(F) = \langle (v_{ij})_{ij=1, \dots, n} | v \text{ and } F\bar{v}F^{-1} \text{ are unitaries} \rangle$$

The  $C^*$ -algebra  $A_u(F)$  endowed with the comultiplication such that

$$\Delta(v_{ij}) = \sum_{k=1}^n v_{ik} \otimes v_{kj}$$

is a compact quantum group. It is called free unitary quantum group and will be denoted  $U^+(F)$ . In particular, when  $F = I_n$ ,  $A_u(I_n)$  is exactly the noncommutative version of  $C(U_n)$ . This can be considered as the basic free unitary quantum group and will be denoted  $U_n^+ = (C(U_n^+), \Delta)$ .

The irreducible representations and the fusion rules of the free unitary groups  $A_u(F)$ ,  $F \in GL_n(\mathbb{C})$  were calculated by Banica in [Ban97].

*Notation 5.* Let  $\mathbb{N} * \mathbb{N}$  be the free product between two copies of the monoid  $\mathbb{N}$  with  $a$  and  $b$  as generators; denote by  $e$  the neutral element. Define an anti-multiplicative operation of involution by  $\bar{a} = b$ ,  $\bar{b} = a$  and  $\bar{e} = e$ .

**Theorem 1.2.2.** *The equivalence classes of irreducible representations of  $U^+(F) = (A_u(F), \Delta)$  can be indexed by the elements of  $\mathbb{N} * \mathbb{N}$  and will be denoted  $v_x$ ,  $x \in \mathbb{N} * \mathbb{N}$ . In particular we have that  $v_e = 1_{U^+(F)}$ ,  $v_a = v$  and  $v_b = \bar{v}$ . The adjoint representation is given by  $\bar{v}_x = v_{\bar{x}}$  and the fusion rules are*

$$v_x \otimes v_y = \sum_{x=rt, y=\bar{t}s} v_{rs}$$

To Wang and van Daele is due also the notion of free orthogonal quantum group.

**Definition 1.2.3.** Let  $F \in GL_n(\mathbb{C})$ ,  $n \geq 2$  such that  $F\bar{F} = cI$ ,  $c \in \mathbb{R}$ . Consider the following universal unital  $C^*$ -algebra

$$A_o(F) = \langle (u_{ij})_{ij=1, \dots, n} | u = F\bar{u}F^{-1}, u \text{ unitary} \rangle$$

The  $C^*$ -algebra  $A_o(F)$  endowed with the comultiplication such that

$$\Delta(u_{ij}) = \sum_{k=1}^n u_{ik} \otimes u_{kj}$$

is a compact quantum group. It is called free orthogonal quantum group and will be denoted  $O^+(F)$ . As in the unitary case, when  $F = I_n$ ,  $A_o(I_n)$  is the noncommutative version of  $C(O_n)$ . This can be considered as the basic free unitary quantum group and will be denoted  $O_n^+ = (C(O_n^+), \Delta)$ . Then, the  $C^*$ -algebra  $C(O_n^+)$  can be seen as being generated by the coefficients, supposed self-adjoint, of a unitary matrix.

Also in this case, the representation theory was calculated by Banica (see [Ban96]).

**Theorem 1.2.4.** *The irreducible non-equivalent representations of the free orthogonal quantum group  $O^+(F) = (A_o(F), \Delta)$  can be indexed by the elements of  $\mathbb{N}$  and will be denoted  $u_k$ ,  $k \in \mathbb{N}$ . In particular, we have that  $u_0 = 1_{O^+(F)}$  and  $u_1 = u$ . All the irreducible representations are self-adjoint, i.e.  $\bar{u}_k = u_k$  for all  $k \in \mathbb{N}$ . The fusion rules are*

$$u_k \otimes u_l = u_{|k-l|} \oplus u_{|k-l|+2} \oplus \dots \oplus u_{k+l-2} \oplus u_{k+l} = \bigoplus_{t=0}^{\min(k,l)} u_{k+l-2t}$$

*Remark 1.6.* We observe that these are the same fusion rules as  $SU(2)$ . Moreover, we can notice that, if  $F = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ , we have  $A_o(F) = C(SU(2))$ .

Banica in [Ban97] proved also the following proposition. Broadly speaking, he showed that,  $U^+(F)$  can be seen as the free complexification of  $O^+(F)$ , if  $F\bar{F}$  is a multiple of the identity.

**Proposition 1.2.5.** *Let  $F \in GL_n(\mathbb{C})$ ,  $n \geq 2$  such that  $F\bar{F} = cI$ ,  $c \in \mathbb{R}$ . Then there is an embedding  $C_r(A_u(F)) \hookrightarrow C(\mathbb{S}^1) *_{red} C_r(A_o(F))$  given by  $v_{ij} \mapsto zu_{ij}$  where  $z$  is the generator of  $C(\mathbb{S}^1)$ .*

Wang in [Wan98] defined another important family of compact quantum groups by introducing the notion of quantum automorphism group of a finite dimensional  $C^*$ -algebra. This notion can be seen as a generalisation of the definition of quantum symmetric group. We begin by introducing this last one.

**Definition 1.2.6.** Let  $n \in \mathbb{N}^*$ . Consider the following universal unital  $C^*$ -algebra

$$C(S_n^+) = \langle (u_{ij})_{i,j=1,\dots,n} \mid u_{ij} = u_{ij}^2 = u_{ij}^* \ \forall i, j \text{ and } \sum_{i=1}^n u_{ij_0} = 1 = \sum_{j=1}^n u_{i_0j} \ \forall i_0, j_0 \rangle$$

A matrix  $u$ , whose coefficients are projections and sum up to 1 on the lines and on the columns, is said to be magic. Then,  $C(S_n^+)$  is generated by the coefficients of a magic (unitary) matrix. By endowing  $C(S_n^+)$  with the comultiplication such that

$$\Delta(u_{ij}) = \sum_{k=1}^n u_{ik} \otimes u_{kj}$$

we get the quantum symmetric group, denoted  $S_n^+ = (C(S_n^+), \Delta)$ .

*Remark 1.7.* If  $n = 1, 2, 3$ ,  $C(S_n^+)$  is always commutative and equal to  $C(S_n)$ , while, if  $n \geq 4$ , the two  $C^*$ -algebras are different.

The representation theory of  $S_n^+$  was computed by Banica in the more general framework of the quantum automorphism groups (we will give a unique description afterwards). We start with some definitions in order to explain the construction of a quantum automorphism group. Since we will widely rely on this object in what follows, we give more details than usual.

**Definition 1.2.7.** Let  $B$  be a finite dimensional  $C^*$ -algebra endowed with a state  $\psi$ . Let  $(C(\mathbb{G}), \Delta)$  be a full compact quantum group (i.e.  $C(\mathbb{G}) = C_{\max}(\mathbb{G})$ ) and let  $\varepsilon : C(\mathbb{G}) \rightarrow \mathbb{C}$  be the counit of  $C(\mathbb{G})$ . An action of  $\mathbb{G}$  on  $B$  is a  $*$ -homomorphism  $\alpha : B \rightarrow B \otimes C(\mathbb{G})$  such that

$$(\text{id}_B \otimes \Delta)\alpha = (\alpha \otimes \text{id}_{C(\mathbb{G})})\alpha \quad \text{and} \quad (\text{id}_B \otimes \varepsilon)\alpha = \text{id}_B \quad (1.3)$$

Moreover, if

$$(\psi \otimes \text{id}_{C(\mathbb{G})})\alpha = \psi(\cdot)1_{C(\mathbb{G})}$$

the action is said to be  $\psi$ -invariant.

In what follows, when considering an action on  $(B, \psi)$ , we will always assume that the  $\psi$ -invariance condition is satisfied. The next proposition shows a link between the actions and the representations of a compact quantum group.

**Proposition 1.2.8.** *By using the previous notations, let  $\{b_1, \dots, b_n\}$  be an orthonormal basis of the  $C^*$ -algebra  $B$ . Denote  $m : B \otimes B \rightarrow B$  its multiplication map and  $\eta : \mathbb{C} \rightarrow B$  its unity. Consider the linear map*

$$\alpha : B \rightarrow B \otimes C(\mathbb{G})$$

$$\alpha(b_i) = \sum_{j=1}^n b_j \otimes u_{ji}$$

It is natural to associate to this map  $\alpha$  the matrix  $u \in M_n(C(\mathbb{G}))$  given by  $u = (u_{ij})_{ij}$ . Then, the map  $\alpha$  satisfies the relations 1.3 if and only if  $u$  is a representation of  $\mathbb{G}$ .

The remaining properties which make  $\alpha$  a  $\psi$ -invariant action, i.e. the  $*$ -morphism conditions and the  $\psi$ -invariance, can be similarly translated in relations concerning  $u$ .

**Proposition 1.2.9.** *Let  $u \in M_n(C(\mathbb{G}))$  be the matrix associated to a linear map  $\alpha$  satisfying equations 1.3. Then:*

1.  $\alpha$  is multiplicative if and only if  $m \in \text{Hom}(u^{\otimes 2}, u)$
2.  $\alpha$  is unital if and only if  $\eta \in \text{Hom}(1, u)$
3.  $\alpha$  is  $\psi$ -invariant if and only if  $\eta \in \text{Hom}(1, u^*)$

Furthermore, if 1, 2 and 3 are satisfied we have:

4.  $\alpha$  is involutive if and only if  $u$  is unitary.

This proposition explains and justifies the next definition.

**Definition 1.2.10.** Let  $B$  be a  $n$ -dimensional  $C^*$ -algebra with multiplication  $m : B \otimes B \rightarrow B$  and unity  $\eta : \mathbb{C} \rightarrow B$ . Let  $\psi$  be a state on  $B$ . Let  $C(\mathbb{G}^{\text{aut}}(B, \psi))$  be the universal unital  $C^*$ -algebra generated by the coefficients of an element  $u \in \mathcal{L}(B) \otimes C(\mathbb{G}^{\text{aut}}(B, \psi))$  which satisfies the following relations

- $u$  is unitary
- $m \in \text{Hom}(u^{\otimes 2}, u)$
- $\eta \in \text{Hom}(1, u)$

The C\*-algebra  $C(\mathbb{G}^{aut}(B, \psi))$ , endowed with the unique comultiplication such that

$$(\text{id}_B \otimes \Delta)(u) = u_{(12)}u_{(13)}$$

is a compact quantum group. It is called quantum automorphism group of the C\*-algebra  $(B, \psi)$  and denoted  $\mathbb{G}^{aut}(B, \psi)$ .

The quantum automorphism group  $\mathbb{G}^{aut}(B, \psi)$  is the universal object in the category of the compact quantum groups acting on  $B$  and leaving the state  $\psi$  invariant.

*Remark 1.8.* The choice of a state  $\psi$  allows us to define a Hilbert space structure on the C\*-algebra  $B$  and to have a notion of adjoint. It follows that the condition asking for  $u$  to be unitary depends on the state  $\psi$  chosen.

*Remark 1.9.* By choosing an orthonormal basis for  $B$ , it is possible to transform the three defining conditions of a quantum automorphism group in a set of relations between the coefficients of  $u$ . The relations depend on the basis, but the quantum automorphism group generated is of course independent from this choice (see [Ban99]).

*Remark 1.10.* If we choose  $B = \mathbb{C}^n$  endowed with the canonical trace  $\text{tr}$ , the associated quantum automorphism group  $\mathbb{G}^{aut}(\mathbb{C}^n, \text{tr})$  is exactly the quantum symmetric group  $S_n^+$ . This observation, linked to Remark 1.7, implies that, if  $\dim(B) \leq 3$  the quantum automorphism group would be  $C(S_n)$ . Because of this, in what follows we will always suppose  $\dim(B) \geq 4$ , in order to get a non-degenerate situation.

As previously said, the investigation of the representation theory was done by Banica in the case of particular states  $\psi$ .

**Definition 1.2.11.** Let  $B$  be a  $n$ -dimensional C\*-algebra as in definition 1.2.10 and  $\delta > 0$ . A faithful state  $\psi : B \rightarrow \mathbb{C}$  is a  $\delta$ -form if the multiplication map of  $B$  and its adjoint with respect to the inner product induced by  $\psi$  (i.e.  $\langle x, y \rangle = \psi(y^*x)$ ) satisfy  $mm^* = \delta \cdot \text{id}_B$ .

If such a  $\psi$  is also a trace, it is called a tracial  $\delta$ -form or a  $\delta$ -trace.

*Remark 1.11.* The convention which we adopted in the definition of a  $\delta$ -form is slightly different from the standard one. Usually, the condition which a state  $\psi$  has to satisfy in order to be a  $\delta$ -form is  $mm^* = \delta^2 \cdot \text{id}_B$ . However, some computations and some results of this thesis lead us to prefer the use of the condition without the square.

**Theorem 1.2.12.** *Let  $B$  be a  $n$ -dimensional  $C^*$ -algebra,  $n \geq 4$ , endowed with a  $\delta$ -form  $\psi$ . Then, the classes of equivalence of irreducible representations of  $\mathbb{G}^{\text{aut}}(B, \psi)$  can be indexed by  $\mathbb{N}$  and will be denoted  $u_k$ ,  $k \in \mathbb{N}$ . In particular, we have that  $u_0 = 1_{\mathbb{G}^{\text{aut}}(B, \psi)}$  and  $u_1 = u$ . All the irreducible representations are self-adjoint, i.e.  $\bar{u}_k = u_k$  for all  $k \in \mathbb{N}$ . The fusion rules are*

$$u_k \otimes u_l = u_{|k-l|} \oplus u_{|k-l|+1} \oplus \dots \oplus u_{k+l-1} \oplus u_{k+l} = \bigoplus_{t=0}^{2 \min(k,l)} u_{k+l-t}$$

We observe that these are the same fusion rules as  $SO(3)$  and that the fusion semiring depends neither on the dimension or structure of  $B$  nor on the  $\delta$  of  $\psi$ .

The last family of compact quantum groups which we will describe has been introduced by Bichon in [Bic04]. Its elements are the free wreath products of a compact quantum group by a quantum permutation group.

**Definition 1.2.13.** Let  $\mathbb{G} = (C(\mathbb{G}), \Delta_{C(\mathbb{G})})$  be a compact quantum group and  $n \in \mathbb{N}$ ,  $n \geq 4$ . Consider the  $C^*$ -algebra free product  $C(\mathbb{G})^{*n}$  and let  $\nu_i : C(\mathbb{G}) \rightarrow C(\mathbb{G})^{*n}$ ,  $i \in \{1, \dots, n\}$  be the  $*$ -homomorphism sending the elements of  $C(\mathbb{G})$  in its  $i$ -th copy. Define  $C(\mathbb{G}) *_w C(S_n^+)$  to be the quotient of the free product  $C^*$ -algebra  $C(\mathbb{G})^{*n} *_w C(S_n^+)$  by the two-sided ideal generated by

$$\nu_k(a)u_{ki} - u_{ki}\nu_k(a) \quad \text{for } 1 \leq i, k \leq n \text{ and } a \in C(\mathbb{G})$$

where  $u = (u_{ij})_{ij}$  is the magic matrix which generates  $C(S_n^+)$ .

The  $C^*$ -algebra  $C(\mathbb{G}) *_w C(S_n^+)$  is the free wreath product of  $\mathbb{G}$  by  $S_n^+$ . Endowed with the unique comultiplication such that

$$\Delta(u_{ij}) = \sum_{k=1}^n u_{ik} \otimes u_{kj} \quad \text{and} \quad \Delta(\nu_i(a)) = \sum_{k=1}^n (\nu_i \otimes \nu_k(\Delta_{C(\mathbb{G})}(a)))(u_{ik} \otimes 1)$$

it becomes a compact quantum group, denoted  $\mathbb{G} \wr_* S_n^+$ .

The representation theory of these groups was first investigated by Banica and Vergnioux in [BV09], when  $\mathbb{G} = \widehat{\mathbb{Z}/s\mathbb{Z}}$  or  $\mathbb{G} = \widehat{\mathbb{Z}}$ . In this case, the free wreath product is the so-called quantum reflection group. It admits also this alternative construction.

**Definition 1.2.14.** Let  $s \geq 2$ ,  $n \in \mathbb{N}^*$ . Consider the universal unital C\*-algebra

$$C(H_n^{s+}) = \langle u = (u_{ij})_{i,j=1,\dots,n} \mid u, u^t \text{ are unitary,} \\ u_{ij}u_{ij}^* \text{ is a projection, } u_{ij}^s = u_{ij}u_{ij}^* = u_{ij}^*u_{ij} \rangle$$

endowed with the comultiplication such that  $\Delta(u_{ij}) = \sum_{k=1}^n u_{ik} \otimes u_{kj}$ .

Then  $H_n^{s+} = (C(H_n^{s+}), \Delta) = \widehat{\mathbb{Z}/s\mathbb{Z}} \lambda_* S_n^+$  is a quantum reflection group.

When  $s = 1$ , i.e.  $\mathbb{G} = \widehat{\mathbb{Z}}$ , we consider the C\*-algebra  $C(H_n^{\infty+})$ , obtained by removing the condition  $u_{ij}^s = u_{ij}u_{ij}^*$  in the previous case. The comultiplication remains unchanged and we get the quantum reflection group  $H_n^{\infty+}$ .

The results in [BV09] were first generalized by Lemeux in [Lem14], where he considered the free wreath product of the dual of a discrete group  $\Gamma$  by  $S_n^+$ . This compact quantum group is denoted  $H_n^+(\widehat{\Gamma})$ . A further generalisation was successively given by Lemeux and Tarrago in [LT14], where the general case of the free wreath product of a compact matrix quantum group of Kac type by  $S_n^+$  was analysed.

Now, we describe the representation theory in the more general case.

**Definition 1.2.15.** Let  $\mathbb{G}$  be a compact matrix quantum group of Kac type and consider the monoid  $M = \langle \text{Irr}(\mathbb{G}) \rangle$  composed by the words in the alphabet of the irreducible representations of  $\mathbb{G}$ . Define the following operations:

- involution  $\overline{(\alpha_1, \dots, \alpha_k)} = (\bar{\alpha}_k, \dots, \bar{\alpha}_1)$
- concatenation  $(\alpha_1, \dots, \alpha_k), (\beta_1, \dots, \beta_l) = (\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_l)$
- fusion of two non-empty words:  $(\alpha_1, \dots, \alpha_k) \cdot (\beta_1, \dots, \beta_l)$  is the multiset composed by the words  $(\alpha_1, \dots, \alpha_{k-1}, \gamma, \beta_2, \dots, \beta_l)$  for all the possible  $\gamma \subset \alpha_k \otimes \beta_1$ ; the multiplicity of each word is given by  $\dim(\text{Hom}(\gamma, \alpha_k \otimes \beta_1))$ , i.e. by the multiplicity of the representation  $\gamma$  in the tensor product  $\alpha_k \otimes \beta_1$

**Theorem 1.2.16.** *The classes of irreducible non-equivalent representations can be indexed by the elements of the monoid  $M$  and denoted  $r_x$ ,  $x \in M$ . The involution is given by  $\bar{r}_x = r_{\bar{x}}$  and the fusion rules are:*

$$r_x \otimes r_y = \sum_{\substack{x=u,t \\ y=\bar{t},v}} r_{u,v} \oplus \sum_{\substack{x=u,t \ y=\bar{t},v \\ u \neq \emptyset, v \neq \emptyset \\ w \in u.v}} r_w$$

# Chapter 2

## The free wreath product

In this chapter, we will take into account the compact quantum group obtained as the free wreath product by a quantum automorphism group.

In the first section, we introduce some notations and recall some known results, in the second we revise the basic theory of noncrossing partitions and add some particular notion which will be crucial later. The third one is dedicated to the quantum automorphism group; in particular we show how to describe its intertwining spaces by using noncrossing partitions instead of Temperley-Lieb diagrams. This new description is fundamental for the study of the free wreath product.

The remaining two sections are entirely devoted to the analysis of the free wreath product and, in particular, the second can be seen as a generalization of the first one. This inclusion reflects two successive phases of this project: first, I defined and studied the free wreath product of a discrete group by a quantum automorphism group and, only in a second time, I considered the more general case obtained by replacing the discrete group by a compact quantum group.

### 2.1 Preliminaries

We recall that every finite dimensional  $C^*$ -algebra  $B$  is isomorphic to a multi-matrix  $C^*$ -algebra so in what follows we will consider the decomposition

$$B = \bigoplus_{\alpha=1}^c M_{n_\alpha}(\mathbb{C})$$

Let  $\mathcal{B} = \{(e_{ij}^\alpha)_{i,j=1,\dots,n_\alpha}, \alpha = 1, \dots, c\}$  be a basis of matrix units and define on  $B$  the standard operations of:

- multiplication  $m : B \otimes B \longrightarrow B$ ,  $m(e_{ij}^\alpha \otimes e_{kl}^\beta) = \delta_{jk} \delta_{\alpha\beta} e_{il}^\alpha$
- unity  $\eta : \mathbb{C} \longrightarrow B$ ,  $\eta(1) = \sum_{\alpha=1}^c \sum_{i=1}^{n_\alpha} e_{ii}^\alpha$

Moreover, each finite dimensional  $C^*$ -algebra  $B$  can be endowed with a Hilbert space structure by considering the scalar product  $\langle x, y \rangle_\psi := \psi(y^*x)$  induced by a faithful state  $\psi$  on  $B$ .

Now, we recall some particularly important definitions and results about the structure of  $\psi$  and the  $\delta$ -form condition. A faithful state  $\psi : B \longrightarrow \mathbb{C}$  is a  $\delta$ -form if  $mm^* = \delta \cdot \text{id}_B$ , where  $\delta > 0$  and  $m^*$  is the adjoint with respect to  $\langle \cdot, \cdot \rangle_\psi$ .

If  $\psi : M_n(\mathbb{C}) \longrightarrow \mathbb{C}$  there exists  $Q \in M_n(\mathbb{C})$ ,  $Q > 0$ ,  $\text{Tr}(Q) = 1$  such that  $\psi = \text{Tr}(Q \cdot)$ . Moreover, we notice that every such  $\psi$  is a  $\delta$ -form, with  $\delta = \text{Tr}(Q^{-1})$ . More generally, if  $B = \bigoplus_{\alpha=1}^c M_{n_\alpha}(\mathbb{C})$ , then every faithful state  $\psi : B \longrightarrow \mathbb{C}$  is of the form  $\psi = \bigoplus_{\alpha=1}^c \text{Tr}(Q_\alpha \cdot)$  for a suitable family  $Q_\alpha \in M_{n_\alpha}(\mathbb{C})$ ,  $Q_\alpha > 0$ ,  $\sum_\alpha \text{Tr}(Q_\alpha) = 1$ . In this case,  $\psi$  is a  $\delta$ -form if  $\text{Tr}(Q_\alpha^{-1}) = \delta$  for all  $\alpha$ .

It is well known that every positive complex matrix is diagonalizable. It follows that the matrices  $Q_\alpha$  are always similar to diagonal matrices with positive real eigenvalues. In what follows, when considering the basis  $\mathcal{B}$  of a finite dimensional  $C^*$ -algebra  $B$  endowed with a faithful state  $\psi$ , we will always suppose to choose the matrix units  $e_{ij}^\alpha$  with respect to a basis which diagonalizes  $Q_\alpha$ . We will denote  $Q_{i,\alpha}$  the eigenvalue in position  $(i, i)$  of the matrix  $Q_\alpha$  written with respect to this fixed diagonal basis. We observe that  $\psi(e_{ij}^\alpha) = \text{Tr}(Q_\alpha e_{ij}^\alpha) = \delta_{ij} Q_{i,\alpha}$ . The basis  $\mathcal{B}$  is then always orthogonal with respect to the scalar product induced by  $\psi$ . By normalizing  $\mathcal{B}$  we obtain the following orthonormal basis

$$\mathcal{B}' = \{b_{ij}^\alpha | b_{ij}^\alpha = \psi(e_{jj}^\alpha)^{-\frac{1}{2}} e_{ij}^\alpha = Q_{j,\alpha}^{-\frac{1}{2}} e_{ij}^\alpha, i, j = 1, \dots, n_\alpha, \alpha = 1, \dots, c\}$$

which will be widely used in this thesis.

## 2.2 Noncrossing partitions

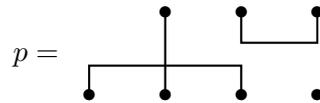
Noncrossing partitions have a crucial role in the description of the spaces of intertwiners of quantum automorphism groups and free wreath products. Now, we recall the basic definitions and define the three fundamental operations between these diagrams.

**Definition 2.2.1.** Let  $k, l \in \mathbb{N}$ . Let  $p = P_1 \sqcup P_2 \sqcup \dots \sqcup P_t$  be a partition of the set  $I_{k+l} = \{1, \dots, k+l\}$ . The subsets  $P_i$ ,  $i = 1, \dots, t$  are called the blocks of the partition. The partition  $p$  is said to be a noncrossing partition if, for every possible choice of elements  $r_1 < r_2 < r_3 < r_4$ ,  $r_j \in I_{k+l}$  such that  $r_1$  and  $r_3$  belong to the same block, then  $r_2$  and  $r_4$  belong to different blocks. As we fixed  $k$  and  $l$ , such a noncrossing partition  $p$  can be represented by a diagram with  $k$  upper points and  $l$  lower points constructed as follows:

- consider two horizontal imaginary lines and draw  $k$  points on the upper one and  $l$  points on the lower one
- number the  $k$  upper points from 1 to  $k$  and from the left to the right
- number the  $l$  lower points from  $k+1$  to  $k+l$  and from the right to the left
- connect to each other the points in a same block of the partition by drawing strings in the part of the plane between the two imaginary lines

From the non crossing condition, it follows that the strings which connect points of different blocks can be drawn in such a way that they do not intersect. We denote  $NC(k, l)$  the set of noncrossing partitions between  $k$  upper points and  $l$  lower points. The total number of blocks of  $p \in NC(k, l)$  is denoted  $b(p)$ .

**Example 2.2.2.** The following diagram represents a noncrossing partition  $p \in NC(3, 4)$  with  $b(p) = 3$ .



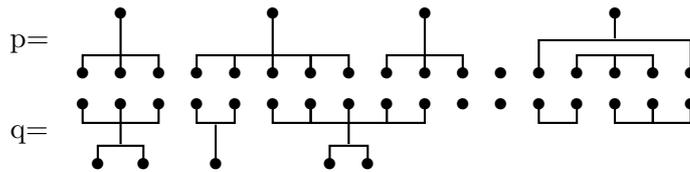
**Definition 2.2.3.** Let  $p \in NC(k, l), q \in NC(v, w)$ . We define the following diagram operations:

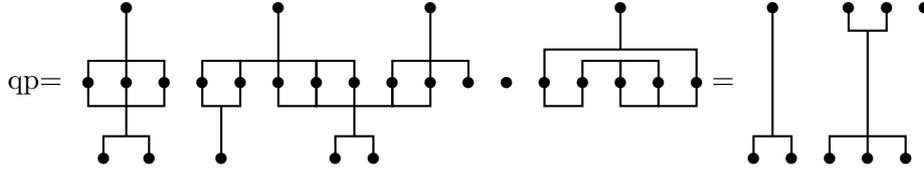
1. the tensor product  $p \otimes q$  is the diagram in  $NC(k + v, l + w)$  obtained by horizontal concatenation of the diagrams  $p$  and  $q$
2. if  $l = v$  it is possible to define the composition  $qp$  as the diagram in  $NC(k, w)$  obtained by identifying the lower points of  $p$  with the upper points of  $q$  and by removing all the blocks which have possibly appeared and which contain neither one of the upper points of  $p$  nor one of the lower points of  $q$ ; such operation, when it is defined, is associative
3. the adjoint  $p^*$  is the diagram in  $NC(l, k)$  obtained by reflecting the diagram  $p$  with respect to an horizontal line between the two rows of points

*Notation 6.* When multiplying two noncrossing partitions  $p \in NC(k, l), q \in NC(l, w)$  we get a unique partition  $qp \in NC(k, w)$  but, as observed, there can be some blocks composed only of lower points of  $p$ /upper points of  $q$  which are removed. We refer to them as central blocks and their number is denoted  $cb(p, q)$ . Furthermore, the vertical concatenation can produce some (closed) cycles which will not appear in the final noncrossing partition either. Intuitively, they are the rectangles which are obtained when two or more central points are connected both in the upper and in the lower noncrossing partition (see example below). In a more formal way, the number of cycles is denoted  $cy(p, q)$  and defined as

$$cy(p, q) = l + b(qp) + cb(p, q) - b(p) - b(q)$$

**Example 2.2.4.** In order to clarify the multiplication operation and the concepts of block, central block as well as cycle, we can think of  $p \in NC(4, 17)$  and  $q \in NC(17, 5)$  in the following example:





We suddenly have  $b(p) = 6$ ,  $b(q) = 7$ ,  $b(qp) = 3$  and  $cb(p, q) = 1$ .  
 Then, the number of cycles is  $cy(p, q) = 17 + 3 + 1 - 6 - 7 = 8$ .

In the following proposition we introduce a simple but useful relation concerning the number of cycles obtained by multiplying three noncrossing partitions.

**Proposition 2.2.5.** *Let  $p \in NC(k, l)$ ,  $r \in NC(l, m)$  and  $s \in NC(m, v)$ . Then the following relation holds:*

$$cy(p, sr) = cy(p, r) + cy(rp, s) - cy(r, s) \tag{2.1}$$

*Proof.* By making use of the – just introduced – definition of a cycle and of the associativity of the composition, the relation (2.1) reduces to  $cb(p, sr) = cb(p, r) + cb(rp, s) - cb(r, s)$ . In order to complete the proof, it is then enough to observe that  $cb(p, sr) = cb(p, r)$  because the number of central blocks obtained by concatenating  $p$  and  $sr$  does not depend on the noncrossing partition  $s$ ; in the same way,  $cb(rp, s) = cb(r, s)$ . □

### 2.2.1 Intertwining spaces

In this subsection, we recall first how noncrossing partitions can be used to describe the intertwining spaces of the quantum symmetric group  $S_n^+ = \mathbb{G}^{aut}(\mathbb{C}^n, \text{tr})$  and, later on, we introduce a particular kind of partitions, called Temperley-Lieb diagrams.

We start by showing how to associate a linear map  $T_p \in \mathcal{L}((\mathbb{C}^n)^{\otimes k}, (\mathbb{C}^n)^{\otimes l})$  to every  $p \in NC(k, l)$ . All these results come from [BS09].

**Definition 2.2.6.** Let  $p \in NC(k, l)$  and suppose to decorate the  $k$  upper points with the multi-index  $i = (i_1, \dots, i_k)$ ; similarly, decorate the  $l$  lower points with the multi-index  $j = (j_1, \dots, j_l)$ . Then, we define the following coefficient:

$$\delta_p(i, j) = \begin{cases} 1 & \text{if every string of } p \text{ joins equal indices} \\ 0 & \text{if at least a string of } p \text{ joins different indices} \end{cases}$$

We can now describe the linear map

**Definition 2.2.7.** To every  $p \in NC(k, l)$  we associate the map  $T_p : (\mathbb{C}^n)^{\otimes k} \longrightarrow (\mathbb{C}^n)^{\otimes l}$ :

$$T_p(e_{i_1} \otimes \dots \otimes e_{i_k}) = \sum_j \delta_p(i, j)(e_{j_1} \otimes \dots \otimes e_{j_l})$$

This association is well defined in light of this important compatibility result.

**Proposition 2.2.8.** Let  $p \in NC(l, k), q \in NC(v, w)$ . We have:

1.  $T_{p \otimes q} = T_p \otimes T_q$
2.  $T_p^* = T_{p^*}$
3. if  $k = v$  then  $T_{qp} = n^{-cb(p,q)} T_q T_p$

From this the description of the intertwining spaces follows.

**Theorem 2.2.9.** Let  $n \in \mathbb{N}, n \geq 4$  and consider the quantum symmetric group  $S_n^+$  with fundamental representation  $u$ . Then for all  $k, l \in \mathbb{N}$

$$\text{Hom}(u^{\otimes k}, u^{\otimes l}) = \text{span}\{T_p | p \in NC(k, l)\}$$

Furthermore, the maps associated to distinct noncrossing partitions in  $NC(k, l)$  are linearly independent.

We recall now the notion of Temperley-Lieb diagram.

**Definition 2.2.10.** Let  $k, l \in \mathbb{N}$ . We will denote  $TL(2k, 2l)$  the set of noncrossing partitions between  $2k$  upper points and  $2l$  lower points such that the cardinality of each block is 2. All the notations introduced for noncrossing partitions can be also used for these diagrams.

These diagrams have been used by Banica in [Ban02] to reconstruct the intertwining spaces of a quantum automorphism group. More precisely, he shows that it is possible to reconstruct all the Temperley-Lieb diagrams by starting from two basic diagrams. These diagrams can be thought as corresponding to the generating morphisms  $m$  and  $\eta$  of Definition 1.2.10.

## 2.3 The quantum automorphism group $\mathbb{G}^{aut}(B, \psi)$

In this section, we take into account the quantum automorphism group  $\mathbb{G}^{aut}(B, \psi)$ . Thanks to some remarks on the  $\delta$ -form  $\psi$ , we reduce the study of  $\mathbb{G}^{aut}(B, \psi)$  to some particular cases. This will allow us to introduce a new description of its intertwining spaces which makes use of noncrossing partitions instead of Temperley-Lieb diagrams and which is more explicit, i.e. to every noncrossing partition will correspond a morphism. This different approach will be widely used when studying the representation theory of a free wreath product by a quantum automorphism group.

An introduction to the construction and to the structure of the quantum automorphism group can be found in the first chapter, here we only recall its definition (see [Ban99, Wan98]).

**Definition 2.3.1.** Let  $B$  be a  $n$ -dimensional  $C^*$ -algebra endowed with a state  $\psi$ . Fix a basis of  $B$ , orthonormal with respect to the scalar product induced by  $\psi$ , and identify  $B \cong \mathbb{C}^n$  as Hilbert spaces. Consider the following universal unital  $C^*$ -algebra:

$$C(\mathbb{G}^{aut}(B, \psi)) = \langle (u_{ij})_{i,j=1,\dots,n} \mid u = (u_{ij}) \text{ unitary}, m \in Hom(u^{\otimes 2}, u), \eta \in Hom(1, u) \rangle$$

Then  $C(\mathbb{G}^{aut}(B, \psi))$  endowed with the comultiplication  $\Delta$  such that  $\Delta(u_{ij}) = \sum_{k=1}^n u_{ik} \otimes u_{kj}$  is a compact quantum group. It is called quantum automorphism group of the  $C^*$ -algebra  $(B, \psi)$  and denoted  $\mathbb{G}^{aut}(B, \psi)$ .

As pointed out in Remark 1.9, the definition of  $\mathbb{G}^{aut}(B, \psi)$  does not depend on the choice of the orthonormal basis of  $B$ , up to isomorphism. In this thesis, however, the choice of a good basis of  $B$  is of fundamental importance in order to prove many results and to simplify the computations. We will always use the orthonormal basis  $\mathcal{B}'$  introduced in Section 2.1.

### 2.3.1 New description of the intertwining spaces

The representation theory of  $\mathbb{G}^{aut}(B, \psi)$  is well known from [Ban02], but, in order to generalize it to the free wreath product, we need a description of the

intertwining spaces in terms of noncrossing partition. For this reason, the goal is to generalise the construction recalled in Section 2.2.1 which is valid in the special case of the quantum symmetric group  $\mathbb{G}^{aut}(\mathbb{C}^n, \text{tr})$ .

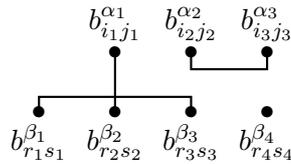
The idea is to assign a linear map to every noncrossing partition. In order to define such a map we will make use of the following notation which generalizes the classical one.

*Notation 7.* Consider a diagram  $p \in NC(k, l)$  and associate to every point an element of the basis  $\mathcal{B}'$  of  $B$ . Let  $(b_{i_1 j_1}^{\alpha_1}, \dots, b_{i_k j_k}^{\alpha_k})$  be the ordered set of elements associated to the upper points and  $(b_{r_1 s_1}^{\beta_1}, \dots, b_{r_l s_l}^{\beta_l})$  the elements associated to the lower points. Let  $ij$  and  $\alpha$  be the multi-index notation for the indices of these matrices, in particular  $ij = ((i_1, j_1), \dots, (i_k, j_k))$  and  $\alpha = (\alpha_1, \dots, \alpha_k)$ ; in a similar way, we define  $rs$  and  $\beta$ .

Denote  $b_v, v = 1, \dots, m$  the different blocks of  $p$  and let  $b_v^\uparrow$  ( $b_v^\downarrow$ ) be the ordered product of the matrix units associated to the upper (lower) points of the block  $b_v$ . Such a product is conventionally the identity matrix, if there are no upper (lower) points in the block. Define

$$\delta_p^{\alpha, \beta}(ij, rs) := \prod_{v=1}^m \psi((b_v^\downarrow)^* b_v^\uparrow) \quad (2.2)$$

**Example 2.3.2.** Consider the following noncrossing partition  $p$  in which we associated an element of the basis to every point.



In this case the coefficient just introduced is

$$\delta_p^{\alpha, \beta}(ij, rs) = \psi((b_{r_1 s_1}^{\beta_1} b_{r_2 s_2}^{\beta_2} b_{r_3 s_3}^{\beta_3})^* b_{i_1 j_1}^{\alpha_1}) \psi(b_{i_2 j_2}^{\alpha_2} b_{i_3 j_3}^{\alpha_3}) \psi((b_{r_4 s_4}^{\beta_4})^*)$$

*Remark 2.1.* It is possible to give a more concrete interpretation of the coefficient  $\delta_p^{\alpha, \beta}(ij, rs)$ . First of all, we notice that it can be non zero only if the indices  $\alpha_x, \beta_y$  are equal in the points of a same block. In this case, it will be effectively non zero if the following condition is satisfied for each block. Let  $((i_{c_1}, j_{c_1}), \dots, (i_{c_t}, j_{c_t}))$  and

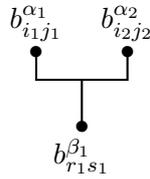
$((r_{d_1}, s_{d_1}), \dots, (r_{d_w}, s_{d_w}))$  be the pairs of indices of the matrix units associated, for a fixed block, to the upper points and to the lower points respectively. Then, the second index of each pair must be equal to the first of the following one, assuming that:

- the first index of the first of the upper points is equal to the first index of the first of the lower points ( $i_{c_1} = r_{d_1}$ )
- the second index of the last of the upper points is equal to the second index of the last of the lower points ( $j_{c_t} = s_{d_w}$ )

**Definition 2.3.3.** We associate to every element  $p \in NC(k, l)$  the linear map  $T_p : B^{\otimes k} \longrightarrow B^{\otimes l}$  which is defined by:

$$T_p(b_{i_1 j_1}^{\alpha_1} \otimes \dots \otimes b_{i_k j_k}^{\alpha_k}) = \sum_{r, s, \beta} \delta_p^{\alpha, \beta}(i, j, r, s) b_{r_1 s_1}^{\beta_1} \otimes \dots \otimes b_{r_l s_l}^{\beta_l}$$

**Example 2.3.4.** The diagram  $p$  which is associated to the multiplication map  $m$  (writing explicitly  $b_{i_1, j_1}^{\alpha_1}, b_{i_2, j_2}^{\alpha_2}$  on the upper points and  $b_{r_1, s_1}^{\beta_1}$  on the lower point) is:



Here, by applying the definition

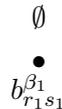
$$\begin{aligned} \delta_p^{\alpha, \beta}((i_1, j_1, i_2, j_2), (r_1, s_1)) &= \psi((\psi(e_{s_1 s_1}^{\beta_1})^{-\frac{1}{2}} e_{r_1 s_1}^{\beta_1}) * \psi(e_{j_1 j_1}^{\alpha_1})^{-\frac{1}{2}} e_{i_1 j_1}^{\alpha_1} \psi(e_{j_2 j_2}^{\alpha_2})^{-\frac{1}{2}} e_{i_2 j_2}^{\alpha_2}) \\ &= \psi(e_{s_1 s_1}^{\beta_1})^{-\frac{1}{2}} \psi(e_{j_1 j_1}^{\alpha_1})^{-\frac{1}{2}} \psi(e_{j_2 j_2}^{\alpha_2})^{-\frac{1}{2}} \psi(e_{s_1 r_1}^{\beta_1} e_{i_1 j_1}^{\alpha_1} e_{i_2 j_2}^{\alpha_2}) \\ &= \delta_{\beta_1, \alpha_1} \delta_{\alpha_1, \alpha_2} \delta_{r_1, i_1} \delta_{j_1, i_2} \delta_{s_1, j_2} \psi(e_{j_1 j_1}^{\alpha_1})^{-\frac{1}{2}} \end{aligned}$$

so the associated map  $T_p : B^{\otimes 2} \longrightarrow B$  is given by

$$T_p(b_{i_1 j_1}^{\alpha_1} \otimes b_{i_2 j_2}^{\alpha_2}) = \delta_{\alpha_1, \alpha_2} \delta_{j_1, i_2} \psi(e_{j_1 j_1}^{\alpha_1})^{-\frac{1}{2}} b_{i_1 j_2}^{\alpha_1}$$

which is the multiplication  $m$ .

The diagram  $q$  associated to the unity map  $\eta$  is:



By applying the definition we have

$$\delta_p^{\beta_1}(\emptyset, (r_1, s_1)) = \psi((\psi(e_{s_1 s_1}^{\beta_1})^{-\frac{1}{2}} e_{r_1, s_1}^{\beta_1})^*) = \delta_{r_1, s_1} \psi(e_{s_1 s_1}^{\beta_1})^{\frac{1}{2}}$$

therefore  $T_q : \mathbb{C} \longrightarrow B$  is given by  $T_q(1) = \sum_{r_1, \beta_1} \psi(e_{r_1 r_1}^{\beta_1})^{\frac{1}{2}} b_{r_1, r_1}^{\beta_1}$  which is exactly the unity map  $\eta$ .

As usual, the diagram of the identity map (with two basis elements) is:

$$\begin{array}{c} b_{i_1 j_1}^{\alpha_1} \\ \bullet \\ \downarrow \\ b_{r_1 s_1}^{\beta_1} \end{array}$$

We can now state an important result of compatibility between the standard operations of tensor product, composition and adjoint of linear maps and the same operations between diagrams introduced in Definition 2.2.3 (see [BS09, Proposition 1.9] for the case of  $\mathbb{C}^n$  with the canonical trace).

**Proposition 2.3.5.** *Let  $p \in NC(l, k)$  and  $q \in NC(v, w)$ . We have:*

1.  $T_{p \otimes q} = T_p \otimes T_q$
2.  $T_p^* = T_{p^*}$
3. if  $k = v$  then  $T_{qp} = \delta^{-cy(p,q)} T_q T_p$

*Proof.* Even if the core of the proof is essentially the same as [BS09], it is necessary to pay much more attention to the computations in order to prove that this is the correct way to associate a morphism to a noncrossing partition; in particular, it is crucial that  $\psi$  is a  $\delta$ -form.

The relation 1 is clear because  $\delta_p^{\alpha, \beta}(ij, rs) \delta_q^{\alpha', \beta'}(IJ, RS) = \delta_{p \otimes q}^{\alpha \alpha', \beta \beta'}(ijIJ, rsRS)$ .

The relation 2 follows from  $\delta_p^{\alpha, \beta}(ij, rs) = \delta_{p^*}^{\beta, \alpha}(rs, ij)$  which is true because we have  $\psi((b_{ij}^\alpha)^*) = \psi(b_{ij}^\alpha)$  (we want to emphasize that the matrix associated to  $\psi$  has real eigenvalues).

The relation 3 is less obvious and it is not possible to prove it just by looking at the definition of the coefficients, because its validity strongly depends on the geometric

structure of the noncrossing partitions too. In such a case, the relation to prove is

$$\sum_{\beta} \sum_{r,s=1}^{n_{\beta}} \delta_p^{\alpha,\beta}(ij, rs) \delta_q^{\beta,\gamma}(rs, RS) = \delta^{cy(p,q)} \delta_{qp}^{\alpha,\gamma}(ij, RS) \quad (2.3)$$

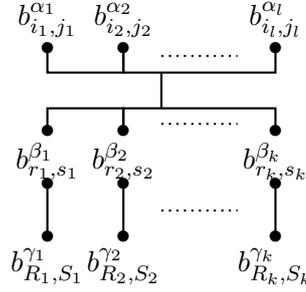
We remind that any noncrossing partition is obtained by using compositions, tensor products and adjoints of the basic morphisms  $m, \eta$  and  $\text{id}_B$ . In order to prove the composition formula between the maps associated to two noncrossing partitions  $p$  and  $q$ , we can think of decomposing  $q$  in the composition of a sequence of noncrossing partitions corresponding to elementary maps of type  $\text{id}_B^{\otimes u} \otimes f \otimes \text{id}_B^{\otimes v}$  where  $u, v \in \mathbb{N}$  and  $f = \text{id}_B, m, m^*, \eta, \eta^*$ .

If we suppose that relation 3 holds for the composition of a general noncrossing partition  $p$  with this kind of maps and let  $q = q_s \dots q_2 q_1$  be such a decomposition of  $q$ , then we have

$$\begin{aligned} T_{qp} &= T_{q_s \dots q_2 q_1 p} \\ &= \delta^{-cy(q_{s-1} \dots q_1 p, q_s)} T_{q_s} T_{q_{s-1} \dots q_1 p} \\ &= \delta^{-cy(q_{s-1} \dots q_1 p, q_s) - cy(q_{s-2} \dots q_1 p, q_{s-1}) - \dots - cy(p, q_1)} T_{q_s} T_{q_{s-1}} \dots T_{q_1} T_p \\ &= \delta^{-cy(q_{s-1} \dots q_1 p, q_s) - cy(q_{s-2} \dots q_1 p, q_{s-1}) - \dots - cy(p, q_1) + cy(q_1, q_2)} T_{q_s} \dots T_{q_2 q_1} T_p \\ &= \delta^{-cy(q_{s-1} \dots q_1 p, q_s) - cy(q_{s-2} \dots q_1 p, q_{s-1}) - \dots - cy(p, q_1) + cy(q_1, q_2) + \dots + cy(q_{s-1} \dots q_1, q_s)} T_{q_s \dots q_2 q_1} T_p \\ &= \delta^{-cy(p, q_s \dots q_1)} T_{q_s \dots q_1} T_p \\ &= \delta^{-cy(p, q)} T_q T_p \end{aligned}$$

where the second to last equality follows by applying  $s$  times Proposition 2.2.5.

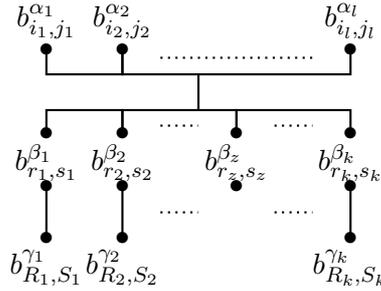
Now we only need to show relation (2.3) when composing the map associated to  $p$  with an elementary map. Furthermore, we can consider that the noncrossing partition  $p$  has only one block (or possibly two when  $f = m$ ): it is possible to reconstruct the multi-block case by using a tensor product argument or by generalizing the following proof in an obvious way. Let us start now the computations in the different cases. Let  $p \in NC(l, k)$  be a one block noncrossing partition. The first case we take into account is the composition of  $T_p$  with  $T_q = \text{id}_B^{\otimes k}$ . The corresponding diagram is the following:



In this case relation (2.3) is satisfied, indeed

$$\begin{aligned}
& \sum_{\beta} \sum_{r,s=1}^{n_{\beta}} \delta_p^{\alpha,\beta}(ij, rs) \delta_q^{\beta,\gamma}(rs, RS) = \\
& \sum_{\beta} \sum_{r,s=1}^{n_{\beta}} \psi((b_{r_1, s_1}^{\beta_1} \dots b_{r_k, s_k}^{\beta_k})^* (b_{i_1, j_1}^{\alpha_1} \dots b_{i_l, j_l}^{\alpha_l})) \prod_{t=1}^k \psi((b_{R_t, S_t}^{\gamma_t})^* b_{r_t, s_t}^{\beta_t}) = \\
& \sum_{\beta} \sum_{r,s=1}^{n_{\beta}} \psi((b_{r_1, s_1}^{\beta_1} \dots b_{r_k, s_k}^{\beta_k})^* (b_{i_1, j_1}^{\alpha_1} \dots b_{i_l, j_l}^{\alpha_l})) \prod_{t=1}^k \delta_{\gamma_t \beta_t} \delta_{R_t r_t} \delta_{S_t s_t} = \\
& \psi((b_{R_1, S_1}^{\gamma_1} \dots b_{R_k, S_k}^{\gamma_k})^* (b_{i_1, j_1}^{\alpha_1} \dots b_{i_l, j_l}^{\alpha_l})) = \\
& \delta_{qp}^{\alpha,\gamma}(ij, RS)
\end{aligned}$$

A second possible case is the composition of  $T_p$  with  $T_q = \text{id}_B^{\otimes u} \otimes \eta^* \otimes \text{id}_B^{\otimes v}$ . Here there are two different situations which deserve to be considered. If  $T_p = \eta$ , a simple computation shows that  $\eta^* \eta = \text{id}_{\mathbb{C}}$  (because  $\psi$  is unital) and the relation (2.3) is satisfied. For all the other possible  $T_p$  the general diagram is the following:

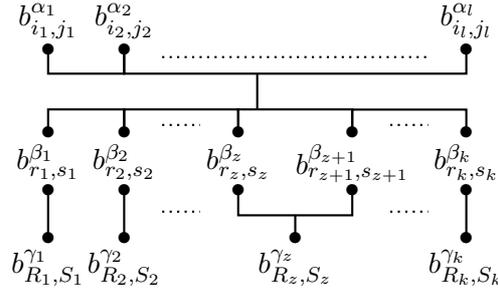


With respect to the previous case in the lower noncrossing partition  $q$ , one of the identity maps was replaced by  $\eta^*$  (in this case, with a little abuse of notation, we removed  $b_{R_z, S_z}^{\gamma_z}$  but did not reassign the index  $z$ ).

The relation (2.3) is verified because

$$\begin{aligned}
 & \sum_{\beta} \sum_{r,s=1}^{n_{\beta}} \delta_p^{\alpha,\beta}(ij, rs) \delta_q^{\beta,\gamma}(rs, RS) = \\
 & \sum_{\beta} \sum_{r,s=1}^{n_{\beta}} \psi((b_{r_1,s_1}^{\beta_1} \dots b_{r_k,s_k}^{\beta_k})^* (b_{i_1,j_1}^{\alpha_1} \dots b_{i_l,j_l}^{\alpha_l})) \psi(b_{r_z,s_z}^{\beta_z}) \prod_{t=1,t \neq z}^k \psi((b_{R_t,S_t}^{\gamma_t})^* b_{r_t,s_t}^{\beta_t}) = \\
 & \sum_{\beta} \sum_{r,s=1}^{n_{\beta}} \psi((b_{r_1,s_1}^{\beta_1} \dots b_{r_k,s_k}^{\beta_k})^* (b_{i_1,j_1}^{\alpha_1} \dots b_{i_l,j_l}^{\alpha_l})) \delta_{r_z s_z} Q_{s_z, \beta_z}^{\frac{1}{2}} \prod_{t=1,t \neq z}^k (\delta_{\gamma_t \beta_t} \delta_{R_t r_t} \delta_{S_t s_t}) = \\
 & \sum_{\beta} \sum_{r_z=1}^{n_{\beta}} \psi((b_{R_1,S_1}^{\gamma_1} \dots e_{r_z,s_z}^{\beta_z} \dots b_{R_k,S_k}^{\gamma_k})^* (b_{i_1,j_1}^{\alpha_1} \dots b_{i_l,j_l}^{\alpha_l})) = \\
 & \psi((b_{R_1,S_1}^{\gamma_1} \dots b_{R_{z-1},S_{z-1}}^{\gamma_{z-1}} b_{R_{z+1},S_{z+1}}^{\gamma_{z+1}} \dots b_{R_k,S_k}^{\gamma_k})^* (b_{i_1,j_1}^{\alpha_1} \dots b_{i_l,j_l}^{\alpha_l})) = \\
 & \delta_{qp}^{\alpha,\gamma}(ij, RS)
 \end{aligned}$$

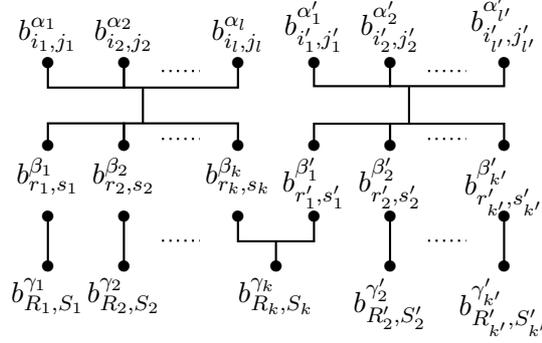
The third case we analyse is the composition of  $T_p$  with  $T_q = \text{id}_B^{\otimes u} \otimes m \otimes \text{id}_B^{\otimes v}$ . There are two different situations which deserve to be considered: the two upper points of  $m$  can be connected either to one block or to two different blocks of the noncrossing partition  $p$ . We observe that in the first sub-case a cycle appears and the diagram is:



As in the previous case one of the points of the lower line was removed and its index (in this case  $z + 1$ ) was not reassigned, for the sake of the clarity of the notation. The relation (2.3) is still verified and the factor  $\delta$  implied by the cycle appears. We have

$$\begin{aligned}
 & \sum_{\beta} \sum_{r,s=1}^{n_{\beta}} \delta_p^{\alpha,\beta}(ij, rs) \delta_q^{\beta,\gamma}(rs, RS) = \\
 & \sum_{\beta} \sum_{r,s=1}^{n_{\beta}} \psi((b_{r_1,s_1}^{\beta_1} \dots b_{r_k,s_k}^{\beta_k})^* (b_{i_1,j_1}^{\alpha_1} \dots b_{i_l,j_l}^{\alpha_l})) \psi((b_{R_z,S_z}^{\gamma_z})^* b_{r_z,s_z}^{\beta_z} b_{r_{z+1},s_{z+1}}^{\beta_{z+1}}) \\
 & \prod_{t=1,t \neq z,z+1}^k \psi((b_{R_t,S_t}^{\gamma_t})^* b_{r_t,s_t}^{\beta_t}) = \\
 & \sum_{\beta} \sum_{r,s=1}^{n_{\beta}} \psi((b_{r_1,s_1}^{\beta_1} \dots b_{r_k,s_k}^{\beta_k})^* (b_{i_1,j_1}^{\alpha_1} \dots b_{i_l,j_l}^{\alpha_l})) \delta_{\beta_z \gamma_z} \delta_{\beta_{z+1} \gamma_z} \delta_{r_z R_z} \delta_{s_z r_{z+1}} \delta_{s_{z+1} S_z} Q_{s_z, \beta_z}^{-\frac{1}{2}} \\
 & \prod_{t=1,t \neq z,z+1}^k (\delta_{\gamma_t \beta_t} \delta_{R_t r_t} \delta_{S_t s_t}) = \\
 & \sum_{s_z=1}^{n_{\gamma_z}} Q_{s_z, \gamma_z}^{-1} \psi((b_{R_1,S_1}^{\gamma_1} \dots e_{R_z,S_z}^{\gamma_z} b_{R_z,S_z}^{\gamma_z} b_{R_{z+2},S_{z+2}}^{\gamma_{z+2}} \dots b_{R_k,S_k}^{\gamma_k})^* (b_{i_1,j_1}^{\alpha_1} \dots b_{i_l,j_l}^{\alpha_l})) = \\
 & \delta \cdot \psi((b_{R_1,S_1}^{\gamma_1} \dots b_{R_z,S_z}^{\gamma_z} b_{R_{z+2},S_{z+2}}^{\gamma_{z+2}} \dots b_{R_k,S_k}^{\gamma_k})^* (b_{i_1,j_1}^{\alpha_1} \dots b_{i_l,j_l}^{\alpha_l})) = \\
 & \delta \cdot \delta_{qp}^{\alpha,\gamma}(ij, RS)
 \end{aligned}$$

The diagram of the sub-case with two blocks (with  $p \in NC(l + l', k + k')$ ) follows:



Relation (2.3) is still verified:

$$\begin{aligned}
& \sum_{\beta, \beta'} \sum_{r, s=1}^{n_\beta} \sum_{r', s'=1}^{n_{\beta'}} \delta_p^{\alpha\alpha', \beta\beta'} (i' j j', r r' s s') \delta_q^{\beta\beta', \gamma\gamma'} (r r' s s', R R' S S') = \\
& \sum_{\beta, \beta'} \sum_{r, s=1}^{n_\beta} \sum_{r', s'=1}^{n_{\beta'}} \psi((b_{r_1, s_1}^{\beta_1} \dots b_{r_k, s_k}^{\beta_k})^* (b_{i_1, j_1}^{\alpha_1} \dots b_{i_l, j_l}^{\alpha_l})) \psi((b_{r'_1, s'_1}^{\beta'_1} \dots b_{r'_{k'}, s'_{k'}}^{\beta'_{k'}})^* (b_{i'_1, j'_1}^{\alpha'_1} \dots b_{i'_{l'}, j'_{l'}}^{\alpha'_{l'}})) \\
& \quad \prod_{t=1}^{k-1} \psi((b_{R_t, S_t}^{\gamma_t})^* b_{r_t, s_t}^{\beta_t}) \psi((b_{R_k, S_k}^{\gamma_k})^* b_{r_k, s_k}^{\beta_k} b_{r'_1, s'_1}^{\beta'_1}) \prod_{t=2}^{k'} \psi((b_{R'_t, S'_t}^{\gamma'_t})^* b_{r'_t, s'_t}^{\beta'_t}) = \\
& \sum_{\beta, \beta'} \sum_{r, s=1}^{n_\beta} \sum_{r', s'=1}^{n_{\beta'}} \psi((b_{r_1, s_1}^{\beta_1} \dots b_{r_k, s_k}^{\beta_k})^* (b_{i_1, j_1}^{\alpha_1} \dots b_{i_l, j_l}^{\alpha_l})) \psi((b_{r'_1, s'_1}^{\beta'_1} \dots b_{r'_{k'}, s'_{k'}}^{\beta'_{k'}})^* (b_{i'_1, j'_1}^{\alpha'_1} \dots b_{i'_{l'}, j'_{l'}}^{\alpha'_{l'}})) \\
& \quad \prod_{t=1}^{k-1} (\delta_{\gamma_t \beta_t} \delta_{R_t r_t} \delta_{S_t s_t}) \delta_{\beta_k \gamma_k} \delta_{\beta'_1 \gamma_k} \delta_{r_k R_k} \delta_{s_k r'_1} \delta_{s'_1 S_k} Q_{s_k, \beta_k}^{-\frac{1}{2}} \prod_{t=2}^{k'} (\delta_{\gamma'_t \beta'_t} \delta_{R'_t r'_t} \delta_{S'_t s'_t}) = \\
& \sum_{s_k=1}^{n_{\gamma_k}} Q_{s_k, \gamma_k}^{-\frac{1}{2}} \psi((b_{R_1, S_1}^{\gamma_1} \dots b_{R_k, S_k}^{\gamma_k})^* (b_{i_1, j_1}^{\alpha_1} \dots b_{i_l, j_l}^{\alpha_l})) \psi((b_{s_k, S_k}^{\gamma_k} b_{R'_2, S'_2}^{\gamma'_2} \dots b_{R'_{k'}, S'_{k'}}^{\gamma'_{k'}})^* (b_{i'_1, j'_1}^{\alpha'_1} \dots b_{i'_{l'}, j'_{l'}}^{\alpha'_{l'}})) = \\
& Q_{j_l, \gamma_k}^{-\frac{1}{2}} \psi((b_{R_1, S_1}^{\gamma_1} \dots b_{R_k, j_l}^{\gamma_k})^* (b_{i_1, j_1}^{\alpha_1} \dots b_{i_l, j_l}^{\alpha_l})) \psi((b_{j_l, S_k}^{\gamma_k} b_{R'_2, S'_2}^{\gamma'_2} \dots b_{R'_{k'}, S'_{k'}}^{\gamma'_{k'}})^* (b_{i'_1, j'_1}^{\alpha'_1} \dots b_{i'_{l'}, j'_{l'}}^{\alpha'_{l'}})) = \\
& Q_{j_l, \gamma_k}^{-1} \psi((b_{R_1, S_1}^{\gamma_1} \dots e_{R_k, j_l}^{\gamma_k})^* (b_{i_1, j_1}^{\alpha_1} \dots b_{i_l, j_l}^{\alpha_l})) \psi((b_{j_l, S_k}^{\gamma_k} b_{R'_2, S'_2}^{\gamma'_2} \dots b_{R'_{k'}, S'_{k'}}^{\gamma'_{k'}})^* (b_{i'_1, j'_1}^{\alpha'_1} \dots b_{i'_{l'}, j'_{l'}}^{\alpha'_{l'}})) = \\
& \psi((b_{R_1, S_1}^{\gamma_1} \dots b_{R_k, S_k}^{\gamma_k} b_{R'_2, S'_2}^{\gamma'_2} \dots b_{R'_{k'}, S'_{k'}}^{\gamma'_{k'}})^* (b_{i_1, j_1}^{\alpha_1} \dots b_{i_l, j_l}^{\alpha_l} b_{i'_1, j'_1}^{\alpha'_1} \dots b_{i'_{l'}, j'_{l'}}^{\alpha'_{l'}})) = \\
& \delta_{qp}^{\alpha\alpha', \gamma\gamma'} (i' j j', R R' S S')
\end{aligned}$$

where the second to last equality is obtained by simplifying  $Q_{j_l, \gamma_k}^{-1}$  with the value given by the first  $\psi$  and by inserting all the  $\delta$  conditions and coefficients of its argument in the argument of the second  $\psi$ . This is essentially due to the fact that the index  $j_l$  is in both arguments.

With similar computations, it is finally possible to prove that the formula still holds in the remaining cases of  $\eta$  (trivial case) and  $m^*$ .  $\square$

*Remark 2.2.* It is interesting to observe that, with respect to the classical composition formula of the maps associated to two noncrossing partitions, in this more general case the correction factor depends on the number of cycles which appear

instead of the number of central blocks (see Proposition 2.3.5(3) and compare with Proposition 1.9(2) in [BS09]). This is due to a different choice made during the analysis of the two cases. In the classical case of the quantum symmetric group  $S_n^+ = \mathbb{G}^{aut}(\mathbb{C}^n, tr)$ , the  $\delta$ -form considered is the usual trace  $tr$  which is not unital but is a 1-form, while in our general case of  $\mathbb{G}^{aut}(B, \psi)$  the  $\delta$ -form  $\psi$  is always unital (by definition of a state). Now the correction factor  $n^{-cb(p,q)}$  present in the classical formula can actually be seen as  $tr(1)^{-cb(p,q)}$  so it disappears when dealing with a unital  $\delta$ -form while, on the other side, it is obvious that the dependency from the cycles can be ignored if  $\delta = 1$ . Of course, for a given  $\delta$ -form, it is not possible, in general, to find a normalization constant such that, in addition,  $\delta = 1$ .

For some of the following results it is necessary that the scalar coefficient which can appear when composing depends on the number of central blocks as in the standard case, so we need to state a modified version of Proposition 2.3.5. In this case, we consider the quantum automorphism group  $\mathbb{G}^{aut}(B, \tilde{\psi})$  where  $\tilde{\psi} := \delta\psi$  and  $\psi$  is, as usual, a  $\delta$ -form on  $B$ . It is clear that  $\mathbb{G}^{aut}(B, \tilde{\psi}) = \mathbb{G}^{aut}(B, \psi)$ . We observe that  $\tilde{\psi}$  is a 1-form but it is in general not unital. In this case we have the following compatibility result.

**Proposition 2.3.6.** *Consider the quantum automorphism group  $\mathbb{G}^{aut}(B, \tilde{\psi})$  where  $\tilde{\psi}$  is a 1-form (in general non-unital). Let  $p \in NC(l, k), q \in NC(v, w)$ . We have:*

1.  $T_{p \otimes q} = T_p \otimes T_q$
2.  $T_p^* = T_{p^*}$
3. if  $k = v$  then  $T_{qp} = \tilde{\psi}(1)^{-cb(p,q)} T_q T_p$

*Proof.* The proof is based exactly on the same techniques of the previous one and the same analysis applies, therefore we will only point out the changes due to the slightly different hypothesis ( $\tilde{\psi}$  instead of  $\psi$ ). The first two relations are clear. In order to prove the compatibility with respect to the multiplication, we observe that, in this case, the relation to prove is

$$\sum_{\beta} \sum_{r,s=1}^{n_{\beta}} \delta_p^{\alpha,\beta}(ij, rs) \delta_q^{\beta,\gamma}(rs, RS) = \tilde{\psi}(1)^{cb(p,q)} \delta_{qp}^{\alpha,\gamma}(ij, RS) \quad (2.4)$$

As for the previous proposition, the proof can now be reduced to some elementary compositions: by making use of the same notations, we have that  $T_{qp} = \tilde{\psi}(1)^{-cb(p,q)} T_q T_p$ . This follows by recalling that Proposition 2.2.5 is true also when considering the number of closed blocks instead of the cycles (see the proof of the proposition itself).

In order to verify that the relation 2.4 holds in the basic cases, we proceed exactly as before. The only differences are in the computations where cycles or central blocks are concerned. In the case of a basic composition where a cycle appears, we observe that the relation 2.4 is verified because we have  $\delta = 1$ . The only case where a central block appears is that of  $T_p = \eta$  and  $T_q = \eta^*$ . We have

$$\begin{array}{c}
 \emptyset \\
 \bullet \\
 b_{r,s}^\beta \\
 \bullet \\
 \emptyset
 \end{array}$$

$$\begin{aligned}
 \sum_{\beta} \sum_{r,s=1}^{n_{\beta}} \delta_p^{\emptyset,\beta}(\emptyset, rs) \delta_q^{\beta,\emptyset}(rs, \emptyset) &= \sum_{\beta} \sum_{r,s=1}^{n_{\beta}} \tilde{\psi}((b_{rs}^{\beta})^*) \tilde{\psi}(b_{rs}^{\beta}) \\
 &= \sum_{\beta} \sum_{r=1}^{n_{\beta}} Q_{r,\beta} \delta_{rs} \\
 &= \tilde{\psi}(1) \delta_{qp}^{\emptyset,\emptyset}(\emptyset, \emptyset)
 \end{aligned}$$

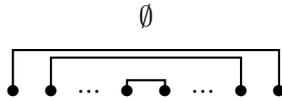
where the  $\emptyset$  symbol means that there are no indices.

Also in this case the new relation is verified. □

*Remark 2.3.* Proposition 2.3.5 allows us to define the concrete monoidal  $C^*$ -category of noncrossing partitions. It will be denoted  $\mathcal{NC}$  and:

- $Ob(\mathcal{NC}) = \mathbb{N}$
- $Hom(k, l) = span\{T_p | p \in NC(k, l)\}$

It is endowed with a canonical fiber functor  $\mathcal{NC} \rightarrow \text{Hilb}_f$  which sends every  $n \in Ob(\mathcal{NC})$  to the Hilbert space  $B^{\otimes n}$ . Moreover, in this category every object is equal to its conjugate, because for every  $k \in \mathbb{N}$  the map associated to the following diagram of  $NC(0, 2k)$  satisfies the conjugate condition (see Definition 1.1.27).



Therefore  $\mathcal{NC}$  is also rigid.

We now reformulate a result from [Ban99, Ban02, BS09] about the description of the intertwining spaces of the category of representations of  $\mathbb{G}^{aut}(B, \psi)$ . The main difference is that the morphisms are associated to noncrossing partitions instead of Temperley-Lieb diagrams, as in [Ban02], where the  $\delta$ -form case is taken into account.

**Theorem 2.3.7.** *Let  $B$  be a  $n$ -dimensional  $C^*$ -algebra,  $n \geq 4$  and consider the quantum automorphism group  $\mathbb{G}^{aut}(B, \psi)$  with fundamental representation  $u$ . Then for all  $k, l \in \mathbb{N}$*

$$\text{Hom}(u^{\otimes k}, u^{\otimes l}) = \text{span}\{T_p | p \in NC(k, l)\}$$

*Furthermore, the maps associated to distinct noncrossing partitions in  $NC(k, l)$  are linearly independent.*

*Proof.* For the first inclusion ( $\supseteq$ ) it is enough to observe that all noncrossing partitions can be obtained from the basic ones (diagrams of multiplication, unity and identity) by using the operations of Definition 2.2.3 (this is true because the theorem has already been proved for  $(B, \psi) = (C^n, \text{tr})$ ). The inclusion follows because the maps associated to these basic diagrams are intertwiners.

For the second inclusion ( $\subseteq$ ) we apply the Tannaka-Krein duality to the concrete rigid monoidal  $C^*$ -category  $\mathcal{NC}$ . This implies that there exists a compact quantum group  $\mathbb{G} = (C(\mathbb{G}), \Delta)$  with fundamental representation  $v$  and such that  $\text{Hom}(v^{\otimes k}, v^{\otimes l}) = \text{span}\{T_p | p \in NC(k, l)\}$ . Because of the universality of the Tannaka-Krein construction, from the inclusion already proved it follows that there is a surjective map  $\phi : C(\mathbb{G}) \longrightarrow \mathbb{G}^{aut}(B, \psi)$  such that  $(id_B \otimes \phi)(v) = u$ . In order to complete the proof we have to show that the map is an isomorphism. This follows from the universality of the quantum automorphism group construction after observing that the matrix  $v$  is unitary and verifying the two conditions  $m \in \text{Hom}(v^{\otimes 2}, v)$  and  $\eta \in \text{Hom}(1, v)$  because  $m$  and  $\eta$  correspond to two noncrossing partitions.

The independence of the maps follows from a dimension count, as observed in [BS09], because the dimensions of the intertwining spaces computed in [Ban99] are still true in this case (see also [Ban02]).  $\square$

We recall now a proposition attributed to Brannan in [DCFY14] in order to generalize the representation theory to the case of a state  $\psi$ :

**Proposition 2.3.8.** *Let  $(B, \psi)$  be a finite dimensional  $C^*$ -algebra equipped with a state. Let  $B = \bigoplus_{i=1}^k B_i$  be the coarsest direct sum decomposition into  $C^*$ -algebras such that, for each  $i$ , the normalization  $\psi_i$  of  $\psi|_{B_i}$  is a  $\delta_i$ -form for a suitable  $\delta_i$ . Then,  $\mathbb{G}^{aut}(B, \psi)$  is isomorphic to the free product  $\hat{*}_{i=1}^k \mathbb{G}^{aut}(B_i, \psi_i)$ .*

*Remark 2.4.* A suitable decomposition of  $B$  always exists. If  $B = \bigoplus_{\alpha=1}^c M_{n_\alpha}$  is the standard multimatrix decomposition, first we observe that the restriction of  $\psi$  to every summand (after normalization) is a  $\delta$ -form for a suitable  $\delta$ . Then the summands  $B_i$  of the decomposition in Proposition 2.3.8 are given by the direct sum of the  $M_{n_\alpha}$  with a common  $\delta$ . The isomorphism is proved by using the universal properties.

The representation theory of a free product has been described by Wang in [Wan95] and it is completely determined by the representation theory of the factors. In particular, the non trivial irreducible representations are the alternating tensor product of the non trivial irreducible representations of the factors.

## 2.4 The free wreath product $\hat{\Gamma} \wr_* \mathbb{G}^{aut}(B, \psi)$

In this section, we define the free wreath product of a discrete group by a quantum automorphism group. We will then describe its representation theory and some properties of its reduced  $C^*$ -algebra and von Neumann algebra.

Bichon in [Bic04] introduced the notion of free wreath product by a quantum permutation group, so the first goal is to generalize his definition in order to consider the free wreath product by a quantum automorphism group. Our definition is not an immediate generalisation of his first definition (see Definition 1.2.13), where the free wreath product is seen as a quotient of a particular free product, but we generalise an alternative definition given in the case of the product of a discrete group by  $S_N^+$ . One of the great advantages of this approach is that it simplifies the description of the intertwining spaces.

### 2.4.1 Definition

First of all, we recall the equivalent definition given by Bichon in the particular case of the free wreath product  $\widehat{\Gamma} \wr_* S_n^+$ .

**Definition 2.4.1.** Let  $\Gamma$  be a discrete group and  $n \in \mathbb{N}^*$ . Let  $A_n(\Gamma)$  be the universal unital  $C^*$ -algebra generated by the coefficients of the matrices  $a(g) = (a_{ij}(g))_{i,j=1,\dots,n}$ ,  $g \in \Gamma$  with the following relations:

$$a_{ij}(g)a_{ik}(h) = \delta_{jk}a_{ij}(gh) \quad a_{ij}(g)a_{kj}(h) = \delta_{ik}a_{ij}(gh)$$

$$\sum_{k=1}^n a_{ik}(e) = 1 \quad \sum_{k=1}^n a_{kj}(e) = 1 \quad a_{ij}(g)^* = a_{ij}(g^{-1})$$

for all  $1 \leq i, j, k \leq n$  and  $g \in \Gamma$ .

Then  $A_n(\Gamma)$  endowed with the comultiplication  $\Delta : A_n(\Gamma) \longrightarrow A_n(\Gamma) \otimes A_n(\Gamma)$  such that

$$\Delta(a_{ij}(g)) = \sum_{k=1}^n a_{ik}(g) \otimes a_{kj}(g)$$

is a compact quantum group isomorphic to  $\widehat{\Gamma} \wr_* S_n^+$ .

*Remark 2.5.* Let  $m$  and  $\eta$  be the multiplication and the unity map of  $\mathbb{C}^n$  respectively. The  $C^*$ -algebra  $A_n(\Gamma)$  admits this equivalent presentation:

$$A_n(\Gamma) = \langle a(g) = (a_{ij}(g))_{i,j=1,\dots,n}, g \in \Gamma \mid a(g) \text{ unitary}, \\ m \in \text{Hom}(a(g) \otimes a(h), a(gh)), \eta \in \text{Hom}(1, a(e)) \rangle$$

The three conditions required here (the unitarity and  $m, \eta$  morphisms) correspond to equations concerning the coefficients of the matrices  $a(g)$ . Let us check that these equations are exactly the relations of the definition above. The basis of  $\mathbb{C}^n$  used in the following computations is  $(e_i)_i$ , the canonical one. We will denote  $e_{ij}$  the matrix units of  $M_n(\mathbb{C})$  with respect to the canonical basis of  $\mathbb{C}^n$ . We start by considering the condition  $m \in \text{Hom}(a(g) \otimes a(h), a(gh))$  which is equivalent to  $(m \otimes 1)a(g)_{(13)}a(h)_{(23)} = a(gh)(m \otimes 1)$ . Let us compute the left and right-hand sides of the equality on the element  $e_p \otimes e_q \otimes 1 \in \mathbb{C}^n \otimes \mathbb{C}^n \otimes A_n(\Gamma)$ . We have

$$\begin{aligned}
& (m \otimes 1)a(g)_{(13)}a(h)_{(23)}(e_p \otimes e_q \otimes 1) = \\
& (m \otimes 1) \sum_{i,j,k,l} e_{ij} \otimes e_{kl} \otimes a_{ij}(g)a_{kl}(h)(e_p \otimes e_q \otimes 1) = \\
& (m \otimes 1) \sum_{i,k} e_i \otimes e_k \otimes a_{ip}(g)a_{kq}(h) = \\
& \sum_i e_i \otimes a_{ip}(g)a_{iq}(h)
\end{aligned}$$

and

$$a(gh)(m \otimes 1)(e_p \otimes e_q \otimes 1) = (\sum_{i,j} e_{ij} \otimes a_{ij}(gh))(\delta_{pq}e_p \otimes 1) = \sum_i e_i \otimes \delta_{pq}a_{ip}(gh)$$

Therefore, the condition on the multiplication is equivalent to the following family of relations  $a_{ij}(g)a_{ik}(h) = \delta_{jk}a_{ij}(gh)$ . The family of relations with the indices inverted can be obtained from the condition  $m^* \in \text{Hom}(a(gh), a(g) \otimes a(h))$  with a similar computation (the condition on  $m^*$  holds because the  $a(g)$  are unitary and can not be deduced uniquely from the condition on  $m$ ). Let us compute on the element  $1_{\mathbb{C}} \otimes 1$  the relations corresponding to  $\eta \in \text{Hom}(1, a(e))$  which is equivalent to  $(\eta \otimes 1) = a(e)(\eta \otimes 1)$ . We have  $\eta(1) \otimes 1 = \sum_i e_i \otimes 1$  and  $(\sum_{i,j} e_{ij} \otimes a_{ij}(e))(\sum_k e_k \otimes 1) = \sum_i e_i \otimes (\sum_k a_{ik}(e))$ . In this case we obtain the relations  $\sum_{k=1}^n a_{ik}(e) = 1$  and, as before, the relations with the indices inverted are equivalent to  $\eta^* \in \text{Hom}(a(e), 1)$ . A simple computation proves that the condition  $a(g)$  unitary is equivalent to  $\sum_j a_{ij}(g)a_{kj}(g)^* = \delta_{ik}$ . By multiplying by  $a_{it}(g^{-1})$ , we have  $\sum_j a_{it}(g^{-1})a_{ij}(g)a_{kj}(g)^* = \delta_{ik}a_{it}(g^{-1})$ . By using the relations obtained from  $m$  we find  $a_{it}(e)a_{kt}(g)^* = \delta_{ik}a_{it}(g^{-1})$ . Finally, by summing over  $i$  and by applying the relations obtained from  $\eta$  we get the desired relation  $a_{kt}(g)^* = a_{kt}(g^{-1})$ .

Then, the origin of the following, more general, definition is clear.

**Definition 2.4.2.** Let  $\Gamma$  be a discrete group and consider the quantum automorphism group  $\mathbb{G}^{aut}(B, \psi)$ , where  $\psi$  is a state on  $B$ . Let  $C^*(\Gamma) *_w C(\mathbb{G}^{aut}(B, \psi))$  be the universal unital C\*-algebra with generators  $a(g) \in \mathcal{L}(B) \otimes C^*(\Gamma) *_w C(\mathbb{G}^{aut}(B, \psi))$ ,  $g \in \Gamma$  and relations such that:

- $a(g)$  is unitary for every  $g \in \Gamma$
- $m \in \text{Hom}(a(g) \otimes a(h), a(gh))$  for every  $g, h \in \Gamma$
- $\eta \in \text{Hom}(1, a(e))$

Such a universal C\*-algebra can be endowed with a compact quantum group structure, but as far as this construction is concerned, we need to go deeper into the generators  $a(g)$ .

*Notation 8.* Consider the matrices  $a = (a_{ij,\alpha}^{kl,\beta})$  and  $b = (b_{ij,\alpha}^{kl,\beta})$  with coefficients in a C\*-algebra where  $1 \leq \alpha, \beta \leq c$ ,  $1 \leq i, j \leq n_\alpha$ ,  $1 \leq k, l \leq n_\beta$ .

Their multiplication is defined by:

$$(ab)_{ij,\alpha}^{kl,\beta} = \sum_{\gamma=1}^c \sum_{r,s=1}^{n_\gamma} a_{rs,\gamma}^{kl,\beta} b_{ij,\alpha}^{rs,\gamma}$$

The adjoint matrix is:

$$a^* = ((a_{kl,\beta}^{ij,\alpha})^*)_{ij,\alpha}^{kl,\beta}$$

*Remark 2.6.* As the definition of the quantum automorphism group  $\mathbb{G}^{aut}(B, \psi)$  does not depend on the choice of an orthonormal basis of  $B$ , also the definition of the universal C\*-algebra  $C^*(\Gamma) *_w C(\mathbb{G}^{aut}(B, \psi))$  does not depend on such a choice. When it will be necessary to fix a basis of  $B$ , we will always use  $\mathcal{B}'$ , as this will allow us to consider as diagonal the matrices  $Q_\alpha$  associated to the state  $\psi$ .

*Remark 2.7.* Let us fix  $\mathcal{B}'$  as basis of the C\*-algebra  $B$ . Then, the generators of the C\*-algebra  $C^*(\Gamma) *_w C(\mathbb{G}^{aut}(B, \psi))$  can be seen as matrices of type  $a(g) = (a_{ij,\alpha}^{kl,\beta}(g))$ ,  $1 \leq \alpha, \beta \leq c$ ,  $1 \leq i, j \leq n_\alpha$ ,  $1 \leq k, l \leq n_\beta$ ,  $g \in \Gamma$ . By using the conventions introduced in Notation 8, we can change the three conditions of Definition 2.4.2 into the following relations:

$$\begin{aligned} \sum_{l=1}^{n_\gamma} Q_{l,\gamma}^{-\frac{1}{2}} a_{ik,\alpha}^{rl,\gamma}(g) a_{pj,\beta}^{ls,\gamma}(h) &= \delta_{\alpha\beta} \delta_{kp} Q_{k,\alpha}^{-\frac{1}{2}} a_{ij,\alpha}^{rs,\gamma}(gh) \\ \sum_{k=1}^{n_\alpha} Q_{k,\alpha}^{-\frac{1}{2}} a_{ik,\alpha}^{rp,\beta}(g) a_{kj,\alpha}^{qs,\gamma}(h) &= \delta_{\beta\gamma} \delta_{pq} Q_{p,\beta}^{-\frac{1}{2}} a_{ij,\alpha}^{rs,\beta}(gh) \\ \sum_{\alpha=1}^c \sum_{j=1}^{n_\alpha} Q_{j,\alpha}^{\frac{1}{2}} a_{jj,\alpha}^{kl,\beta}(e) &= \delta_{kl} Q_{l,\beta}^{\frac{1}{2}} & \sum_{\beta=1}^c \sum_{k=1}^{n_\beta} Q_{k,\beta}^{\frac{1}{2}} a_{kk,\beta}^{ij,\alpha}(e) &= \delta_{ij} Q_{i,\alpha}^{\frac{1}{2}} \\ (a_{ij,\alpha}^{kl,\beta}(g))^* &= \left(\frac{Q_{l,\beta}}{Q_{j,\alpha}}\right)^{\frac{1}{2}} \left(\frac{Q_{k,\beta}}{Q_{i,\alpha}}\right)^{-\frac{1}{2}} a_{ji,\alpha}^{lk,\beta}(g^{-1}) \end{aligned} \quad (2.5)$$

**Proposition 2.4.3.** *There exists a unique  $*$ -homomorphism*

$$\Delta : C^*(\Gamma) *_w C(\mathbb{G}^{aut}(B, \psi)) \longrightarrow C^*(\Gamma) *_w C(\mathbb{G}^{aut}(B, \psi)) \otimes C^*(\Gamma) *_w C(\mathbb{G}^{aut}(B, \psi))$$

such that, for any  $g \in \Gamma$

$$(id \otimes \Delta)(a(g)) = a(g)_{(12)}a(g)_{(13)}$$

Moreover,  $\Delta$  is a comultiplication and the pair  $(C^*(\Gamma) *_w C(\mathbb{G}^{aut}(B, \psi)), \Delta)$  is a compact quantum group which is called the free wreath product of  $\widehat{\Gamma}$  by  $\mathbb{G}^{aut}(B, \psi)$  and will be denoted  $\widehat{\Gamma} \lambda_* \mathbb{G}^{aut}(B, \psi)$  or  $H_{(B, \psi)}^+(\widehat{\Gamma})$ .

*Proof.* In order to prove the existence of  $\Delta$ , we have to check that the images of the generators  $a(g)$  satisfy the same relations. It is clear that  $a(g)_{(12)}a(g)_{(13)}$  is unitary. When considering the condition on the multiplication, we have

$$\begin{aligned} (m \otimes 1^{\otimes 2})(a(g)_{(12)}a(g)_{(13)} \otimes a(h)_{(12)}a(h)_{(23)}) &= \\ (m \otimes 1^{\otimes 2})(a(g)_{(13)}a(g)_{(14)}a(h)_{(23)}a(h)_{(24)}) &= \\ (m \otimes 1^{\otimes 2})(a(g)_{(13)}a(h)_{(23)})(a(g)_{(14)}a(h)_{(24)}) &= \\ (a(gh)_{(12)}a(gh)_{(13)})(m \otimes 1^{\otimes 2}) \end{aligned}$$

The condition on the unity map is simply

$$a(e)_{(12)}a(e)_{(13)}(\eta \otimes 1^{\otimes 2}) = \eta \otimes 1^{\otimes 2}$$

Therefore, by the universality of the free wreath product construction, the existence of the map  $\Delta$  is proved. The uniqueness is an immediate consequence of the fact that the image of all the generators is fixed.

Now, we have to verify that the defining properties of a compact quantum group are satisfied. In the preliminaries of the thesis we gave two equivalent definitions and for this proof we will consider the second one. We observe that the matrices  $a(g)$  are unitary and, by construction, their entries generate a dense  $*$ -subalgebra of  $C^*(\Gamma) *_w C(\mathbb{G}^{aut}(B, \psi))$ . We just proved the existence of a suitable comultiplication  $\Delta$ . What is left is to prove that the transposed matrices  $(a(g))^t$  are invertible. As in Remark 2.13, the basis of  $B$  which will be used for the computations is  $\mathcal{B}'$ . For every  $(a(g))^t$  the inverse is given by  $b(g) = ((\frac{Q_{l,\beta}}{Q_{j,\alpha}})^{-\frac{1}{2}}(\frac{Q_{k,\beta}}{Q_{i,\alpha}})^{\frac{1}{2}}a_{ji,\alpha}^{lk,\beta}(g^{-1}))_{ij,\alpha}^{kl,\beta}$ . Indeed,

we have that

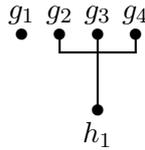
$$\begin{aligned} (b(g)(a(g))^t)_{rs,\gamma}^{kl,\beta} &= \sum_{\alpha=1}^c \sum_{i,j=1}^{n_\alpha} \left(\frac{Q_{i,\beta}}{Q_{j,\alpha}}\right)^{-\frac{1}{2}} \left(\frac{Q_{k,\beta}}{Q_{i,\alpha}}\right)^{\frac{1}{2}} a_{ji,\alpha}^{lk,\beta}(g^{-1}) a(g)_{ij,\alpha}^{rs,\gamma} \\ &= \delta_{\beta\gamma} \delta_{kr} \sum_{\alpha=1}^c \sum_{j=1}^{n_\alpha} \left(\frac{Q_{l,\beta}}{Q_{j,\alpha}}\right)^{-\frac{1}{2}} a_{jj,\alpha}^{ls,\beta}(e) \\ &= \delta_{\beta,\gamma} \delta_{kr} \delta_{ls} \end{aligned}$$

so  $b(g)(a(g))^t = \text{Id}$ . In the same way it is possible to prove that  $(a(g))^t b(g) = \text{Id}$ . It follows that  $a(g)$  is invertible and  $H_{(B,\psi)}^+(\widehat{\Gamma})$  is a compact quantum group.  $\square$

### 2.4.2 Spaces of intertwiners

The first step now is to study the representation theory of  $H_{(B,\psi)}^+(\widehat{\Gamma})$  in the case of a  $\delta$ -form  $\psi$ . Such a result will then be extended to the case of a state  $\psi$  with a result analogous to Proposition 2.3.8.

*Notation 9.* We denote  $NC_{\widehat{\Gamma}}(g_1, \dots, g_k; h_1, \dots, h_l)$  the set of diagrams in  $NC(k, l)$  where the  $k$  upper points are decorated by some  $g_i \in \Gamma$  and the  $l$  lower points by elements  $h_j \in \Gamma$  such that, in every block, the product of the upper elements is equal to the product of the lower elements (with the convention that, if the block connects only upper or only lower points, the product must be the unit of  $\Gamma$ ). For example



is in  $NC_{\widehat{\Gamma}}(g_1, g_2, g_3, g_4; h_1)$  if  $g_1 = e$ ,  $g_2 g_3 g_4 = h_1$ .

The operations between noncrossing partitions introduced in Definition 2.2.3 as well as the compatibility results of Propositions 2.3.5 and 2.3.6 naturally extend to decorated diagrams.

**Proposition 2.4.4.** *Let  $p \in NC_{\widehat{\Gamma}}(g_1, \dots, g_k; h_1, \dots, h_l)$ ,  $q \in NC_{\widehat{\Gamma}}(g'_1, \dots, g'_v; h'_1, \dots, h'_w)$ .*

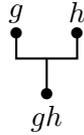
*We have:*

1.  $T_{p \otimes q} = T_p \otimes T_q$  where  $p \otimes q \in NC_{\widehat{\Gamma}}(g_1, \dots, g_k, g'_1, \dots, g'_v; h_1, \dots, h_l, h'_1, \dots, h'_w)$  is obtained by horizontal concatenation

2.  $T_p^* = T_{p^*}$  where  $p^* \in NC_{\widehat{\Gamma}}(h_1, \dots, h_l; g_1, \dots, g_k)$  is obtained by reflecting  $p$  with respect to an horizontal line between the two rows of points
3. if  $(h_1, \dots, h_l) = (g'_1, \dots, g'_v)$  there are two possible cases:
  - a. if  $\psi$  is a (unital)  $\delta$ -form, then  $T_{qp} = \delta^{-cy(p,q)} T_q T_p$  where  $qp \in NC_{\widehat{\Gamma}}(g_1, \dots, g_k; h'_1, \dots, h'_w)$  is obtained by vertical concatenation
  - b. if  $\tilde{\psi}$  is a (possibly non unital) 1-form  $\delta\psi$ , then  $T_{qp} = \tilde{\psi}(1)^{-cb(p,q)} T_q T_p$  where  $qp \in NC_{\widehat{\Gamma}}(g_1, \dots, g_k; h'_1, \dots, h'_w)$  is obtained by vertical concatenation

*Proof.* The proof is essentially the same as Proposition 2.3.5, we have only to observe that the operations between noncrossing partitions are well defined with respect to the decoration of the diagrams, i.e. the operations of tensor product, adjoint and composition always produce diagrams with an admissible decoration.  $\square$

**Example 2.4.5.** The fundamental maps  $m$ ,  $\eta$  and  $\text{id}_B$  can be represented by making use of decorated noncrossing partitions. Their diagrams are the same diagrams introduced in Example 2.3.4 with all the admissible decorations. In particular, for all  $g, h \in \Gamma$ , the multiplication  $m$ , corresponds to the following noncrossing partition of  $NC_{\widehat{\Gamma}}(g, h; gh)$



while the unity  $\eta$  and the identity  $\text{id}_B$  correspond respectively to the following decorated diagrams in  $NC_{\widehat{\Gamma}}(\emptyset; e)$  and in  $NC_{\widehat{\Gamma}}(g; g)$



**Theorem 2.4.6.** Let  $\Gamma$  be a discrete group and  $(B, \psi)$  be a finite dimensional  $C^*$ -algebra with a  $\delta$ -form  $\psi$  and  $\dim(B) \geq 4$ . The spaces of intertwiners of  $\widehat{\Gamma} \wr \lambda_*$

$\mathbb{G}^{aut}(B, \psi)$  is spanned by the linear maps associated to some decorated noncrossing partitions. In particular for any  $g_i, h_j \in \Gamma$  we have

$$\text{Hom}\left(\bigotimes_{i=1}^k a(g_i), \bigotimes_{j=1}^l a(h_j)\right) = \text{span}\{T_p | p \in NC_{\widehat{\Gamma}}(g_1, \dots, g_k; h_1, \dots, h_l)\}$$

with the convention that, if  $k = 0$ ,  $\bigotimes_{i=1}^k a(g_i) = 1_{H_{(B, \psi)}^+(\widehat{\Gamma})}$  and the space of the noncrossing partitions is  $NC_{\widehat{\Gamma}}(\emptyset; h_1, \dots, h_l)$ , i.e. it does not have upper points. Similarly, if  $l = 0$ .

*Proof.* We prove this result by showing the double inclusion. The first inclusion we take into account is the one of the right space in the left one ( $\supseteq$ ). It is well known that all noncrossing partitions can be built by using the operations of tensor product, composition and adjoint on the noncrossing partitions corresponding to the maps of multiplication, unity and identity. This fact can be easily generalized to the context of the noncrossing partitions decorated with the elements of  $\Gamma$ . Let  $p \in NC_{\widehat{\Gamma}}(g_1, \dots, g_k; h_1, \dots, h_l)$  be a decorated noncrossing partition. Its decomposition in terms of the decorated noncrossing partitions corresponding to  $m$ ,  $\eta$  and  $\text{id}$  is simply obtained by considering the usual decomposition in terms of (non decorated) noncrossing partition and by observing that it is always possible to decorate all these partitions in an admissible way. Now, the linear maps corresponding to the decorated noncrossing partitions of the decomposition are intertwiners of  $H_{(B, \psi)}^+(\widehat{\Gamma})$  by definition of free wreath product. It follows that  $T_p \in \text{Hom}(\bigotimes_{i=1}^k a(g_i), \bigotimes_{j=1}^l a(h_j))$ .

For the second inclusion ( $\subseteq$ ), we observe that, similarly to the proof of Theorem 2.3.7, the noncrossing partitions decorated with the elements of  $\Gamma$  form a concrete rigid monoidal C\*-category  $\mathcal{NC}_{\widehat{\Gamma}}$ , whose objects are the finite sequences  $(g_1, \dots, g_k)$ ,  $g_i \in \Gamma$  and whose spaces of morphisms are  $\text{Hom}((g_1, \dots, g_k), (h_1, \dots, h_l)) = \text{span}\{T_p | p \in NC_{\widehat{\Gamma}}(g_1, \dots, g_k; h_1, \dots, h_l)\}$ . Therefore, by the Tannaka-Krein duality, there exists a compact quantum group  $\mathbb{G} = (C(\mathbb{G}), \Delta)$ , such that  $C(\mathbb{G})$  is generated by the coefficients of a family of finite dimensional unitary representations  $a(g_i)'$  and  $\text{Hom}(\bigotimes_{i=1}^k a(g_i)', \bigotimes_{j=1}^l a(h_j)') = \text{span}\{T_p | p \in NC_{\widehat{\Gamma}}(g_1, \dots, g_k; h_1, \dots, h_l)\}$ . Moreover, the inclusion showed in the first part of the proof, together with the

universality of the Tannaka-Krein construction, imply that there is a surjective map  $\phi : C(\mathbb{G}) \longrightarrow H_{(B,\psi)}^+(\widehat{\Gamma})$  such that  $(\text{id} \otimes \phi)(a(g)') = a(g)$ , for all  $g \in \Gamma$ . In order to complete the proof we have to show that the map is an isomorphism. We observe that the representations  $a(g)'$  are such that  $m \in \text{Hom}(a(g)' \otimes a(h)', a(gh)')$  and  $\eta \in \text{Hom}(1, a(e)')$  because these maps correspond to well decorated noncrossing partitions. Therefore, because of the universality of the free wreath product construction we have the inverse morphism and the proof is complete.  $\square$

### 2.4.3 Irreducible representations and fusion rules

As in [Lem14, Cor 2.21] we can immediately deduce a result about basic representations. The proof is identical.

**Proposition 2.4.7.** *The basic representations  $a(g)$ ,  $g \in \Gamma$  of  $H_{(B,\psi)}^+(\widehat{\Gamma})$  are irreducible and pairwise non-equivalent if  $g \neq e$ ; the remaining representation is  $a(e) = 1 \oplus \omega(e)$ , where  $\omega(e)$  is irreducible and non-equivalent to any  $a(g)$ ,  $g \neq e$ .*

**Definition 2.4.8.** Let  $\Gamma$  be a discrete group and  $M = \langle \Gamma \rangle$  be the monoid of the words written by using the elements of  $\Gamma$  as letters. We define the following operations:

- involution:  $\overline{(g_1, \dots, g_k)} = (g_k^{-1}, \dots, g_1^{-1})$
- concatenation:  $(g_1, \dots, g_k), (h_1, \dots, h_l) = (g_1, \dots, g_k, h_1, \dots, h_l)$
- fusion:  $(g_1, \dots, g_k).(h_1, \dots, h_l) = (g_1, \dots, g_k h_1, \dots, h_l)$

We can now state the main theorem, generalizing Theorem 2.25 in [Lem14] to the case of  $H_{(B,\psi)}^+(\widehat{\Gamma})$ . The proof is the same of [Lem14], because it only relies on the fact that the intertwining spaces can be described by making use of noncrossing partitions. As it was proved in Theorem 2.4.6 this is possible also in this more general context.

**Theorem 2.4.9.** *The irreducible representations of  $H_{(B,\psi)}^+(\widehat{\Gamma})$  are indexed by the words of  $M$  and denoted  $\omega(x)$ ,  $x \in M$  with involution  $\bar{\omega}(x) = \omega(\bar{x})$ . In particular*

for  $g \in \Gamma$  we have  $\omega(g) = a(g) \ominus \delta_{g,e}1$ .

The fusion rules are:

$$\omega(x) \otimes \omega(y) = \sum_{\substack{x=u,t \\ y=t,v}} \omega(u, v) \oplus \sum_{\substack{x=u,t \\ y=t,v \\ u \neq \emptyset, v \neq \emptyset}} \omega(u.v)$$

As in [Lem14] the same representations can be indexed in a different way.

**Proposition 2.4.10.** *Let  $L$  be the monoid generated by an element  $a$  together with a family of elements  $z_g$ ,  $g \in \Gamma$  which satisfy the same relations of the corresponding elements of  $\Gamma$ . The neutral element of  $L$  is  $z_e$  and it is identified with  $a^0$ . Let  $S$  be the submonoid of  $L$  generated by the elements  $az_g a$ ,  $g \in \Gamma$ . Then, there is a bijection between  $S$  and the irreducible representations of  $H_{(B, \psi)}^+(\widehat{\Gamma})$ .*

The following proposition allows us to extend these results to the case of a state  $\psi$ .

**Proposition 2.4.11.** *Let  $B = \bigoplus_{\alpha=1}^c M_{n_\alpha}(\mathbb{C})$  be a finite dimensional  $C^*$ -algebra with a state  $\psi = \bigoplus_{\alpha=1}^c \text{Tr}(Q_\alpha \cdot)$  on it. The state  $\psi$  restricted to every summand  $M_{n_\alpha}(\mathbb{C})$  (and normalized) is a  $\delta$ -form with  $\delta = \text{Tr}(Q_\alpha^{-1})$ . Consider the decomposition  $B = \bigoplus_{i=1}^d B_i$  where every  $B_i$  is the direct sum of all the  $M_{n_\alpha}(\mathbb{C})$  such that  $\text{Tr}(Q_\alpha^{-1})$  is a constant value denoted  $\delta_i$ . Let  $\psi_i$  be the state on  $B_i$  obtained by normalizing  $\psi|_{B_i}$ . Then*

$$\widehat{\Gamma} \wr_* \mathbb{G}^{aut}(B, \psi) \cong \widehat{*}_{i=1}^d (\widehat{\Gamma} \wr_* \mathbb{G}^{aut}(B_i, \psi_i))$$

is a  $*$ -isomorphism which intertwines the comultiplications.

*Proof.* The proof consists in the explicit construction of the isomorphism. We fix the notations  $M = C(H_{(B, \psi)}^+(\widehat{\Gamma}))$  and  $N_i = C(H_{(B_i, \psi_i)}^+(\widehat{\Gamma}))$  for  $1 \leq i \leq d$ . Let  $a(g) \in \mathcal{L}(B) \otimes M$ ,  $g \in \Gamma$  be the family of generators of  $M$  and let  $a(g)_i \in \mathcal{L}(B_i) \otimes N_i$ ,  $g \in \Gamma$  be the family of generators of  $N_i$ , for  $1 \leq i \leq d$ . Let  $m, \eta$  be the multiplication and the unity of  $B$  and let  $m_i, \eta_i$  be the multiplication and the unity of  $B_i$ . Moreover, let  $\nu_i : B_i \rightarrow B$  be a family of isometries such that  $\nu_i \nu_i^*$  are pairwise orthogonal projections and  $\sum_i \nu_i \nu_i^* = \text{id}_B$ . Define the element  $v(g) \in \mathcal{L}(B) \otimes \widehat{*}_{i=1}^d N_i$  by

$$v(g) = \sum_i (\nu_i \otimes 1) a(g)_i (\nu_i^* \otimes 1)$$

We claim that there exists a unital  $*$ -homomorphism  $\Psi : M \longrightarrow *_{i=1}^d N_i$  such that  $(\text{id}_B \otimes \Psi)a(g) = v(g)$ . By the universality of the free wreath product construction it is enough to verify that

1.  $v(g)$  is unitary
2.  $m \in \text{Hom}(v(g) \otimes v(h), v(gh))$
3.  $\eta \in \text{Hom}(1, v(e))$

Let us prove (1). Since the  $\nu_i \nu_i^*$  are pairwise orthogonal we have  $\nu_i^* \nu_k = 0$  if  $i \neq k$  and  $\nu_i^* \nu_i = \text{id}_{B_i}$ . It follows that

$$\begin{aligned} v(g)v(g)^* &= \sum_{i,k} (\nu_i \otimes 1) a(g)_i (\nu_i^* \otimes 1) (\nu_k \otimes 1) a(g)_k^* (\nu_k^* \otimes 1) \\ &= \sum_i (\nu_i \otimes 1) a(g)_i a(g)_i^* (\nu_i^* \otimes 1) \\ &= \text{id}_B \otimes 1 \end{aligned}$$

Similarly,  $v(g)^*v(g) = \text{id}_B \otimes 1$ .

Let us prove (2). Observe that  $\nu_j^* m(\nu_i \otimes \nu_k) = \delta_{ik} \delta_{ij} m$  and that  $\sum_i \nu_i m_i (\nu_i^* \otimes \nu_i^*) = m$ . Then

$$\begin{aligned} (m \otimes 1)v(g) \otimes v(h) &= \\ (m \otimes 1) \sum_{i,k} (\nu_i \otimes \nu_k \otimes 1) (a(g)_i \otimes a(h)_k) (\nu_i^* \otimes \nu_k^* \otimes 1) &= \\ \sum_{i,k} (m(\nu_i \otimes \nu_k) \otimes 1) (a(g)_i \otimes a(h)_k) (\nu_i^* \otimes \nu_k^* \otimes 1) &= \\ \sum_{i,j,k} (\nu_j \otimes 1) (\nu_j^* m(\nu_i \otimes \nu_k) \otimes 1) (a(g)_i \otimes a(h)_k) (\nu_i^* \otimes \nu_k^* \otimes 1) &= \\ \sum_i (\nu_i \otimes 1) (m_i \otimes 1) (a(g)_i \otimes a(h)_i) (\nu_i^* \otimes \nu_i^* \otimes 1) &= \\ \sum_i (\nu_i \otimes 1) a(gh)_i (m_i (\nu_i^* \otimes \nu_i^*) \otimes 1) &= \\ \sum_i (\nu_i \otimes 1) a(gh)_i (\nu_i^* \otimes 1) (\sum_k \nu_k m_k (\nu_k^* \otimes \nu_k^*) \otimes 1) &= \\ v(gh)(m \otimes 1) \end{aligned}$$

Let us prove (3). Observe that  $\nu_i^* \eta = \eta_i$  and  $\sum_i \nu_i \eta_i = \eta$ . We have

$$\begin{aligned} v(e)(\eta \otimes 1) &= \sum_i (\nu_i \otimes 1) a(e)_i (\nu_i^* \otimes 1) (\eta \otimes 1) \\ &= \sum_i (\nu_i \otimes 1) a(e)_i (\eta_i \otimes 1) \\ &= \sum_i (\nu_i \otimes 1) (\eta_i \otimes 1) \\ &= \eta \otimes 1 \end{aligned}$$

A simple verification allows us to show that this homomorphism intertwines the comultiplications. This ends the first part of the proof.

In order to construct the inverse homomorphism we need some preliminary results.

We claim that, for all  $i$ ,  $\nu_i \nu_i^* \in \text{Hom}(a(g), a(g))$ . Consider the morphism  $m \in \text{Hom}(a(g) \otimes a(e), a(g))$  and observe that

$$mm^* = \sum_{i=1}^d \delta_i \cdot \nu_i \nu_i^* \in \text{Hom}(a(g), a(g))$$

For a suitable constant  $K$ , we have

$$\nu_i \nu_i^* = K \prod_{\substack{k=1 \\ k \neq i}}^d (\delta_k \text{id}_B - \sum_l \delta_l \nu_l \nu_l^*)$$

This implies that  $\nu_i \nu_i^* \in \text{Hom}(a(g), a(g))$ .

Now, for all  $1 \leq i \leq d$  define the element  $v(g)_i \in \mathcal{L}(B_i) \otimes M$  by

$$v(g)_i = (\nu_i^* \otimes 1)a(g)(\nu_i \otimes 1)$$

We claim that, for all  $i$ , there exists a unital  $*$ -homomorphism  $\Phi_i : N_i \longrightarrow M$  such that  $(\text{id}_{B_i} \otimes \Phi_i)a(g)_i = v(g)_i$ . By the universality of the  $C^*$ -algebra  $N_i$  it is enough to verify that

1.  $v(g)_i$  is unitary
2.  $m_i \in \text{Hom}(v(g)_i \otimes v(h)_i, v(gh)_i)$
3.  $\eta_i \in \text{Hom}(1, v(e)_i)$

Let us prove (1). We have

$$\begin{aligned} v(g)_i v(g)_i^* &= (\nu_i^* \otimes 1)a(g)(\nu_i \otimes 1)(\nu_i^* \otimes 1)a(g)^*(\nu_i \otimes 1) \\ &= (\nu_i^* \otimes 1)a(g)(\nu_i \nu_i^* \otimes 1)a(g)^*(\nu_i \otimes 1) \\ &= (\nu_i^* \otimes 1)(\nu_i \nu_i^* \otimes 1)a(g)a(g)^*(\nu_i \otimes 1) \\ &= \text{id}_{B_i} \otimes 1 \end{aligned}$$

Similarly,  $v(g)_i^* v(g)_i = \text{id}_{B_i} \otimes 1$ .

Let us prove (2). Recall that  $m_i = \nu_i^* m(\nu_i \otimes \nu_i)$ , then  $\nu_i m_i(\nu_i^* \otimes \nu_i^*) = (\nu_i \nu_i^*)m(\nu_i \nu_i^* \otimes \nu_i \nu_i^*) \in \text{Hom}(a(g)_i \otimes a(h)_i, a(gh)_i)$ . Hence

$$\begin{aligned} (m_i \otimes 1)v(g)_i \otimes v(h)_i &= \\ (m_i \otimes 1)(\nu_i^* \otimes \nu_i^* \otimes 1)(a(g) \otimes a(h))(\nu_i \otimes \nu_i \otimes 1) &= \\ (m_i(\nu_i^* \otimes \nu_i^*) \otimes 1)(a(g) \otimes a(h))(\nu_i \otimes \nu_i \otimes 1) &= \\ (\nu_i^* \otimes 1)(\nu_i m_i(\nu_i^* \otimes \nu_i^*) \otimes 1)(a(g) \otimes a(h))(\nu_i \otimes \nu_i \otimes 1) &= \\ (\nu_i^* \otimes 1)a(gh)(\nu_i m_i(\nu_i^* \otimes \nu_i^*) \otimes 1)(\nu_i \otimes \nu_i \otimes 1) &= \end{aligned}$$

$$\begin{aligned}
& (\nu_i^* \otimes 1)a(gh)(\nu_i m_i (\nu_i^* \nu_i \otimes \nu_i^* \nu_i) \otimes 1) = \\
& (\nu_i^* \otimes 1)a(gh)(\nu_i \otimes 1)(m_i \otimes 1) = \\
& v(gh)_i(m_i \otimes 1)
\end{aligned}$$

Let us prove (3). Observe that  $\nu_i \eta_i = (\nu_i \nu_i^*) \eta \in \text{Hom}(1, a(e))$ . Then

$$\begin{aligned}
v(e)_i(\eta_i \otimes 1) &= (\nu_i^* \otimes 1)a(e)(\nu_i \otimes 1)(\eta_i \otimes 1) \\
&= (\nu_i^* \nu_i \otimes 1)(\eta_i \otimes 1) \\
&= (\eta_i \otimes 1)
\end{aligned}$$

This completes the proof of the existence of the morphism  $\Phi_i : N_i \longrightarrow M$ , for all  $i$ . Then, because of the universality of the free product construction, there exists a unital  $*$ -homomorphism  $\Phi : *_{i=1}^d N_i \longrightarrow M$  such that  $(\text{id}_{B_i} \otimes \Phi)a(g)_i = v(g)_i$  and it is easy to verify that this morphism intertwines the comultiplications. Finally, a simple computation allows us to prove that  $\Psi$  and  $\Phi$  are inverse to each other and this ends the proof.  $\square$

The non trivial irreducible representations of  $\widehat{\Gamma} \lambda_* \mathbb{G}^{aut}(B, \psi)$  are then given by an alternating tensor product of non trivial irreducible representations of the factors  $\widehat{\Gamma} \lambda_* \mathbb{G}^{aut}(B_i, \psi_i)$  (see [Wan95]).

We conclude this section with a remark about the spectral measure on a subalgebra of  $C^*(\Gamma) *_w C(\mathbb{G}^{aut}(B, \psi))$  ( $\psi$   $\delta$ -form) which will be useful in the following section.

*Remark 2.8.* The description of the intertwining spaces in term of noncrossing partitions allows us to give a result concerning the Haar measure of some particular elements. It is well known that the character of the fundamental representation of the quantum symmetric group follows the free Poisson law (see e.g. [BC07]). We observe that a similar result is still valid in the case of the free wreath product  $H_{(B, \psi)}^+(\widehat{\Gamma})$ , when  $\psi$  is a  $\delta$ -form. Let  $\chi(a(e)) := (\text{Tr} \otimes \text{id})(a(e))$  be the character of the representation  $a(e)$ . It follows immediately from relation (2.5) that  $\chi(a(e))$  is self-adjoint. Therefore, in order to find its spectral measure, it is enough to compute the moments  $h(\chi(a(e))^k)$ . By denoting  $p_k$  the orthogonal projection onto the fixed points space  $\text{Hom}(1, a(e)^{\otimes k})$  and thanks to some classic results of Woronowicz (see [Wor88]) we have  $h(\chi(a(e))^k) = h((\text{Tr} \otimes \text{id})(a(e))^k) = \text{Tr}((\text{id} \otimes h)(a(e)^{\otimes k})) =$

$\text{Tr}(p_k) = \dim(\text{Hom}(1, a(e)^{\otimes k})) = \#NC(0, k) = C_k$  where  $C_k$  are the Catalan numbers. They are exactly the moments of the free Poisson law of parameter 1 so this is the spectral measure of  $\chi(a(e))$ .

### 2.4.4 Algebraic and analytic properties

#### Simplicity and uniqueness of the trace for the reduced algebra

We prove that, under certain conditions, the reduced C\*-algebra  $C_r(H_{(B,\psi)}^+(\widehat{\Gamma}))$  is simple and has a unique trace.

*Remark 2.9.* The free product decomposition given in Proposition 2.4.11 implies that the Haar measure of  $\widehat{\Gamma} \wr_* \mathbb{G}^{aut}(B, \psi)$  is the free product of the Haar measures of its factors, by using a well known result of Wang (see [Wan95]). It follows that the decomposition is still true at the level of the reduced C\* and von Neumann algebras so the following isomorphism holds:

$$(C_r(\widehat{\Gamma} \wr_* \mathbb{G}^{aut}(B, \psi)), h) \cong \underset{\text{red}_{i=1}}{*}^k (C_r(\widehat{\Gamma} \wr_* \mathbb{G}^{aut}(B_i, \psi_i)), h_i)$$

$$(L^\infty(\widehat{\Gamma} \wr_* \mathbb{G}^{aut}(B, \psi)), h) \cong \underset{i=1}{*}^k (L^\infty(\widehat{\Gamma} \wr_* \mathbb{G}^{aut}(B_i, \psi_i)), h_i)$$

where  $h$  and  $h_i$  are the Haar states on the respective C\*-algebras.

**Proposition 2.4.12.** *Let  $(B, \psi)$  be a finite dimensional C\*-algebra endowed with a trace  $\psi$ . Let  $\Gamma$  be a discrete group,  $|\Gamma| \geq 4$ . Consider the free product decomposition of the reduced C\*-algebra  $C_r(H_{(B,\psi)}^+(\widehat{\Gamma}))$  given in Remark 2.9. If there is either only one factor (i.e.  $\psi$  is a  $\delta$ -trace) and  $\dim(B) \geq 8$  or there are two or more factors with  $\dim(B_i) \geq 4$  for all  $i$ , then  $C_r(H_{(B,\psi)}^+(\widehat{\Gamma}))$  is simple with a unique trace given by the free product of the Haar measures.*

*Proof.* If there are two or more factors, the result follows from a proposition of Avitzour (see [Avi82, Section 3]). The latter states that, given two C\*-algebras  $A$  and  $A'$  endowed with tracial Haar states  $h_A$  and  $h_{A'}$ , the reduced free product C\*-algebra  $A \underset{\text{red}}{*} A'$  is simple with unique trace if there exist two unitary elements of  $\text{Ker}(h_A)$  which are orthogonal with respect to the scalar product induced by  $h_A$

and a unitary element in  $\text{Ker}(h_{A'})$ . In order to show that, in our case, these elements exist we use a result from [DHR97, Proposition 4.1 (i)] according to which, if a  $C^*$ -algebra  $A$  endowed with a normalized trace  $\tau$  admits an abelian sub- $C^*$ -algebra  $F$  so that the spectral measure corresponding to  $\tau|_F$  is diffuse, then there is a unitary element  $u \in A$  such that  $\tau(u^n) = 0$  for each  $n \in \mathbb{Z}, n \neq 0$ . We aim to apply this proposition to every factor of the decomposition in order to satisfy the Avitzour's condition. But this follows from Remark 2.8 where we observed that, when considering the generator  $a(e)$  of an indecomposable free wreath product  $\widehat{\Gamma} \wr_* \mathbb{G}^{aut}(B, \psi)$ ,  $\psi$   $\delta$ -trace, the spectral measure associated to its character  $\chi(a(e))$  is the free Poisson law of parameter 1 which is diffuse. The simplicity and uniqueness of the trace in the multifactor case are then proved.

In the second case, when  $\psi$  is a  $\delta$ -trace and there is not a free product decomposition, the proof is a generalisation of the proof presented in [Lem14, Theorem 3.5] and relies on the simplicity of  $C_r(\mathbb{G}^{aut}(B, \psi))$  for a  $\delta$ -trace proved in [Bra13]. We will give only a sketch of the arguments.

Let  $E$  be the subset of the monoid  $M$  containing only words composed by using the neutral element of  $\Gamma$ , denoted  $e$ . Let  $A$  be the subspace of  $C_r(H_{(B,\psi)}^+(\widehat{\Gamma}))$  generated by the coefficients of the irreducible representations associated to the words of  $E$  and  $A'$  its closure in  $C_r(H_{(B,\psi)}^+(\widehat{\Gamma}))$ . We have that  $A' \cong C_r(\mathbb{G}^{aut}(B, \psi))$ . Furthermore, there exists a unique conditional expectation  $P : C_r(H_{(B,\psi)}^+(\widehat{\Gamma})) \longrightarrow A'$  such that  $h = h_{A'} \circ P$ , where  $h$  is the Haar state of  $C_r(H_{(B,\psi)}^+(\widehat{\Gamma}))$  and  $h_{A'} := h|_{A'}$ .

Now, let  $I \subseteq C_r(H_{(B,\psi)}^+(\widehat{\Gamma}))$  be an ideal; we want to prove that it is trivial or equal to  $C_r(H_{(B,\psi)}^+(\widehat{\Gamma}))$ . Consider the ideal  $P(I) \subseteq A'$ ; the simplicity of  $C_r(\mathbb{G}^{aut}(B, \psi))$  implies that  $P(I)$  is either trivial or  $A'$ .

If  $P(I) = \{0\}$  then  $I \subseteq \ker(P) \subseteq \ker(h)$ . Suppose  $x \in I$ , then  $x^*x \in I$ , so  $h(x^*x) = 0$ .  $h$  being faithful we get immediately  $x = 0$  so in this case  $I = \{0\}$ .

If  $P(I) = A'$  the proof is more complicated, but the core idea is that, by making use of the Powers method, adapted by Banica in [Ban97], if  $x \in I$  it is possible to build a sequence  $(b_i)_i$  in  $C_r(H_{(B,\psi)}^+(\widehat{\Gamma}))$  such that  $\|1 - \sum_i b_i x b_i^*\|_r < 1$ . This implies that the element  $\sum_i b_i x b_i^* \in I$  and it is invertible.

In order to prove the uniqueness of the trace we introduce the space  $D$  generated

by the coefficients of the irreducible representations associated to words in  $E^c$ , i.e. which contain at least a letter different from  $e$ . Let  $D'$  be its closure and notice that  $C_r(H_{(B,\psi)}^+(\widehat{\Gamma})) = A' \oplus D'$ . The first step of the proof consists in showing that any faithful normal trace on  $C_r(H_{(B,\psi)}^+(\widehat{\Gamma}))$  coincide with the Haar state when restricted to  $D'$ . The uniqueness of the trace on  $C_r(\mathbb{G}^{aut}(B, \psi)) \cong A'$  implies the equality also when restricting to  $A'$  so the proof is finished.  $\square$

### Haagerup property

The aim of this paragraph is to prove that, under some hypothesis, the von Neumann algebra  $L^\infty(H_{(B,\psi)}^+(\widehat{\Gamma}))$  has the Haagerup property. Firstly, we recall the basic definition.

**Definition 2.4.13.** Let  $\mathbb{G}$  be a compact quantum group with Haar state  $h$ . We say that  $L^\infty(\mathbb{G})$  has the Haagerup property if there exists a net  $\{(\varphi_x)\}_{x \in A}$  of normal unital completely positive  $h$ -preserving maps on  $L^\infty(\mathbb{G})$  such that the extension to  $L^2(\mathbb{G})$  is a compact operator and converges pointwise to the identity in  $L^2$ -norm.

Now, we can prove the following result.

**Proposition 2.4.14.** *Let  $\Gamma$  be a finite group and  $(B, \psi)$  a finite dimensional  $C^*$ -algebra ( $\dim(B) \geq 4$ ) with a  $\delta$ -trace. Then  $L^\infty(H_{(B,\psi)}^+(\widehat{\Gamma}))$  has the Haagerup approximation property.*

This proposition extends Theorem 3.12 in [Lem14] and the proof uses the same arguments.

In what follows  $\pi : C(H_{(B,\psi)}^+(\widehat{\Gamma})) \longrightarrow C(\mathbb{G}^{aut}(B, \psi))$  will be the canonical map given by  $(\text{id} \otimes \pi)(a(g)) = u \forall g \in \Gamma$ , where  $u$  is the fundamental representation of  $\mathbb{G}^{aut}(B, \psi)$ . Consider the Gelfand isomorphism

$$\omega : C^*(\chi_\alpha, \alpha \in \text{Irr}(\mathbb{G}^{aut}(B, \psi))) \longrightarrow C(\text{Spec}(\chi))$$

where  $\chi$  is the character of the fundamental representation of  $\mathbb{G}^{aut}(B, \psi)$  (i.e.  $\chi = (\text{Tr} \otimes \text{id})(u)$ ) and  $\chi_\alpha$  is the character of the irreducible representation  $\alpha$ . From [Bra13, Proposition 4.8] we know that that  $[0, \dim(B)] \subseteq \text{Spec}(\chi) \subset \mathbb{R}$ .

We recall that the irreducible representations of a compact quantum group allow to decompose the Hilbert space associated with the GNS construction. In our case, for  $S = \text{Irr}(H_{(B,\psi)}^+(\widehat{\Gamma}))$ , we have  $L^2(H_{(B,\psi)}^+(\widehat{\Gamma})) = \bigoplus_{\alpha \in S} L_\alpha^2(H_{(B,\psi)}^+(\widehat{\Gamma}))$ , where  $L_\alpha^2(H_{(B,\psi)}^+(\widehat{\Gamma}))$  is the space generated by the coefficients of the irreducible representation  $\alpha$ . Let  $P_\alpha : L^2(H_{(B,\psi)}^+(\widehat{\Gamma})) \longrightarrow L_\alpha^2(H_{(B,\psi)}^+(\widehat{\Gamma}))$  be the orthogonal projection on the summand.

Finally, we can state this crucial lemma from [Bra13].

**Lemma 2.4.15.** *Let  $I = [0, 4]$  if  $\dim(B) = 4$  and  $I = [4, \dim(B)]$  if  $\dim(B) \geq 5$ . The net  $(T_{\varphi_x})_{x \in I}$  is made up of normal, unital, completely positive  $h$ -preserving maps on  $L^\infty(H_{(B,\psi)}^+(\widehat{\Gamma}))$  defined by:*

$$T_{\varphi_x} = \sum_{\alpha \in S} \frac{\varphi_x(\chi_{\bar{\alpha}})}{d_\alpha} P_\alpha$$

where  $\varphi_x = ev_x \circ \omega \circ \pi$  is a state on  $C^*(\chi_\alpha | \alpha \in \text{Irr}(H_{(B,\psi)}^+(\widehat{\Gamma})))$ ,  $ev_x$  is the evaluation function in the point  $x$  and  $d_\alpha$  is the dimension of the irreducible representation  $\alpha$ .

*Proof of Proposition 2.4.14.* In order to prove that  $L^\infty(H_{(B,\psi)}^+(\widehat{\Gamma}))$  has the Haagerup approximation property, it is necessary to show that the  $L^2$  extension of the net  $(T_{\varphi_x})_{x \in I}$  is a compact operator which converges pointwise to the identity.

Being each  $T_{\varphi_x}$  the direct sum of projections over finite dimensional spaces, the proof of the compactness reduces to show that the net of the coefficients vanishes at infinity, i.e. that  $\frac{\varphi_x(\chi_{\bar{\alpha}})}{d_\alpha} \rightarrow 0$  as  $\alpha \rightarrow \infty$ . This can be proved exactly as in [Lem15, Propositions 3.3 and 3.4] in the case of  $\widehat{\mathbb{Z}}_s \wr S_N^+$  because the proof only relies on the fusion rules which are the same.

Following [Lem15, Proposition 3.5], we also have the pointwise convergence.  $\square$

Since the Haagerup property is stable under tracial free products (see [Boc93, Proposition 3.9]) and by using Remark 2.9 we have the generalisation to the case of a trace.

**Proposition 2.4.16.** *Let  $(B, \psi)$  be a finite dimensional  $C^*$ -algebra endowed with a trace  $\psi$  and  $\Gamma$  be a finite group. Consider the free product decomposition of the reduced von Neumann algebra  $L^\infty(\widehat{\Gamma} \wr \mathbb{G}^{\text{aut}}(B, \psi))$  given in Remark 2.9. If for each  $i$ ,  $\dim(B_i) \geq 4$  then  $L^\infty(H_{(B,\psi)}^+(\widehat{\Gamma}))$  has the Haagerup property.*

## 2.5 The free wreath product $\mathbb{G} \wr_* \mathbb{G}^{aut}(B, \psi)$

This section is a generalisation of the results obtained in the previous one. As previously, our first aim is to correctly define the object that we will take into account: the free wreath product of a compact quantum group by a quantum automorphism group. The definition is based on the same idea already used in the case of a discrete group, but it needs to be adapted to this new context. In this more general situation too, we will describe the spaces of intertwiners by means of specially decorated noncrossing partitions. This will be fundamental in order to prove a monoidal equivalence result, from which the fusion rules and some other properties will be deduced.

### 2.5.1 Definition

**Definition 2.5.1.** Let  $\mathbb{G}$  be a compact quantum group and, for each  $\alpha \in \text{Irr}(\mathbb{G})$ , let  $H_\alpha$  be a space for the representation. Consider the quantum automorphism group  $\mathbb{G}^{aut}(B, \psi)$ , where  $\psi$  is a faithful state on a finite dimensional C\*-algebra  $B$ . Let  $C(\mathbb{G}) *_w C(\mathbb{G}^{aut}(B, \psi))$  be the universal unital C\*-algebra with generators  $a(\alpha) \in \mathcal{L}(B \otimes H_\alpha) \otimes C(\mathbb{G}) *_w C(\mathbb{G}^{aut}(B, \psi))$  and relations such that:

- $a(\alpha)$  is unitary for any  $\alpha \in \text{Irr}(\mathbb{G})$
- $\forall \alpha, \beta, \gamma \in \text{Irr}(\mathbb{G}), \forall S \in \text{Hom}(\alpha \otimes \beta, \gamma)$

$$\widetilde{m \otimes S} := (m \otimes S) \circ \Sigma_{23} \in \text{Hom}(a(\alpha) \otimes a(\beta), a(\gamma))$$

where  $\Sigma_{23} : B \otimes H_\alpha \otimes B \otimes H_\beta \longrightarrow B^{\otimes 2} \otimes (H_\alpha \otimes H_\beta), x_1 \otimes x_2 \otimes x_3 \otimes x_4 \mapsto x_1 \otimes x_3 \otimes x_2 \otimes x_4$  is the unitary map that exchanges the legs 2 and 3 in the tensor product.

- $\eta \in \text{Hom}(1, a(1_{\mathbb{G}}))$ , where 1 is the unity of  $C(\mathbb{G}) *_w C(\mathbb{G}^{aut}(B, \psi))$  and  $1_{\mathbb{G}}$  denote the trivial representations of  $\mathbb{G}$

*Remark 2.10.* In order to seem more coherent with the second relation, the third one can be rewritten as  $\eta \otimes S \in \text{Hom}(1, a(1_{\mathbb{G}}))$ , where  $S = \text{id}_{1_{\mathbb{G}}} : \mathbb{C} \longrightarrow \mathbb{C}$ . In this

case, there is no need to reorder the spaces with a map of type  $\Sigma$ . Moreover, being the map  $S$  a morphism of one dimensional representations, it is clear that the two conditions are exactly the same. This remark also shows the link to the definition given in the simpler case  $\mathbb{G} = \widehat{\Gamma}$ . In this case, all the irreducible representations of  $\widehat{\Gamma}$  are one dimensional; therefore, the morphisms  $S$  can be ignored, since they are scalar multiples of  $\text{id}_{\mathbb{C}}$ . Then, this second definition is a generalization of the first one.

*Remark 2.11.* The definition of  $C(\mathbb{G}) *_w C(\mathbb{G}^{\text{aut}}(B, \psi))$  does not depend on the choice of the basis of  $B = \bigoplus_{T=1}^c M_{n_T}(\mathbb{C})$  and of the spaces  $H_\alpha$ ,  $\dim(H_\alpha) = d_\alpha$  for every  $\alpha \in \text{Irr}(\mathbb{G})$ . We observe also that, by choosing a basis of  $B$  and of the  $H_\alpha$ , the generators  $a(\alpha)$  of this C\*-algebra can be seen as matrices with 8 indices. More precisely

$$a(\alpha) = \sum_{R,Z=1}^c \sum_{i,j=1}^{n_R} \sum_{k,l=1}^{n_Z} \sum_{p,q=1}^{d_\alpha} e_{ij,R}^{kl,Z} \otimes e_{pq} \otimes a_{ij,R}^{kl,Z}(\alpha_{pq}) \quad (2.6)$$

where the  $e_{ij,R}^{kl,Z}$  and the  $e_{pq}$  are the matrix units with respect to the chosen basis.

In order to correctly deal with these objects during the following computations, we fix this notation.

*Notation 10.* Consider the matrices  $a = (a_{ij,R,pq}^{kl,Z})$  and  $b = (b_{ij,R,pq}^{kl,Z})$  with coefficients in a C\*-algebra where  $1 \leq R, Z \leq c$ ,  $1 \leq i, j \leq n_R$ ,  $1 \leq k, l \leq n_Z$ ,  $1 \leq p, q \leq N$ . Their multiplication is defined by:

$$(ab)_{ij,R,pq}^{kl,Z} = \sum_{T=1}^c \sum_{r,s=1}^{n_T} \sum_{t=1}^N a_{rs,T,pt}^{kl,Z} b_{ij,R,tq}^{rs,T}$$

The transpose matrix is:

$$a^t = (a_{kl,Z,qp}^{ij,R})_{ij,R,pq}^{kl,Z}$$

The adjoint matrix is:

$$a^* = ((a_{kl,Z,qp}^{ij,R})^*)_{ij,R,pq}^{kl,Z}$$

**Proposition 2.5.2.** *There exists a unique \*-homomorphism*

$$\Delta : C(\mathbb{G}) *_w C(\mathbb{G}^{\text{aut}}(B, \psi)) \longrightarrow C(\mathbb{G}) *_w C(\mathbb{G}^{\text{aut}}(B, \psi)) \otimes C(\mathbb{G}) *_w C(\mathbb{G}^{\text{aut}}(B, \psi))$$

such that, for any  $\alpha \in \text{Irr}(\mathbb{G})$

$$(id \otimes \Delta)(a(\alpha)) = a(\alpha)_{(12)}a(\alpha)_{(13)}$$

Moreover,  $\Delta$  is a comultiplication of  $C(\mathbb{G}) *_w C(\mathbb{G}^{aut}(B, \psi))$  and the pair  $(C(\mathbb{G}) *_w C(\mathbb{G}^{aut}(B, \psi)), \Delta)$  is a compact quantum group. It is called the free wreath product of  $\mathbb{G}$  by  $\mathbb{G}^{aut}(B, \psi)$  and will be denoted  $\mathbb{G} \lambda_* \mathbb{G}^{aut}(B, \psi)$  or  $H_{(B, \psi)}^+(\mathbb{G})$ .

*Proof.* In order to verify that  $\Delta$  exists, we have to check that the images of the generators satisfy the same relations of the generators. For this verification, we need to think to the  $a(\alpha)$  as three legs objects, as in formula (2.6). Therefore, the condition on  $\Delta$  can be rewritten as  $(id \otimes \Delta)(a(\alpha)) = a(\alpha)_{(123)}a(\alpha)_{(124)}$ . It is easy to check that  $a(\alpha)_{(123)}a(\alpha)_{(124)}$  is unitary. Now, we verify the relations concerning the multiplication map. We have

$$\begin{aligned} & (m \otimes S \otimes 1^{\otimes 2})\Sigma_{23}(a(\alpha)_{(123)}a(\alpha)_{(124)} \otimes a(\beta)_{(123)}a(\beta)_{(124)}) = \\ & (m \otimes S \otimes 1^{\otimes 2})\Sigma_{23}(a(\alpha)_{(125)}a(\alpha)_{(126)}a(\beta)_{(345)}a(\beta)_{(346)}) = \\ & (m \otimes S \otimes 1^{\otimes 2})\Sigma_{23}(a(\alpha)_{(125)}a(\beta)_{(345)})(a(\alpha)_{(126)}a(\beta)_{(346)}) = \\ & (a(\gamma)_{(123)}a(\gamma)_{(124)})(m \otimes S \otimes 1^{\otimes 2})\Sigma_{23} \end{aligned}$$

When considering the condition on the unity map we have trivially that

$$a(1_{\mathbb{G}})_{(12)}a(1_{\mathbb{G}})_{(13)}(\eta \otimes 1^{\otimes 2}) = \eta \otimes 1^{\otimes 2}$$

Then, the  $*$ -homomorphism  $\Delta$  exists by the universality of the free wreath product construction. The uniqueness is an immediate consequence of the fact that the image of all the generators is fixed.

The proof of the compact quantum group structure consists in verifying the conditions of Definition 1.1.2. It is clear that the matrices  $a(\alpha)$  are unitary and we just proved that the comultiplication  $\Delta$  exists. What is left is to show that the  $a(\alpha)^t$  are invertible. Let us fix  $\mathcal{B}'$  as basis of  $B$ . Let  $H_\alpha$ ,  $\dim(H_\alpha) = d_\alpha$  be the space of the representation  $\alpha \in \text{Irr}(\mathbb{G})$  with the basis introduced in Proposition 1.1.15. Then, as explained in Remark 2.11, the generators of the free wreath product can be seen as matrices. The inverse of the transposed of  $a(\alpha) = (a_{ij,R}^{kl,Z}(\alpha_{pr}))$ , is

$$b(\alpha) = \left( \left( \frac{Q_{k,Z}Q_{j,R}}{Q_{i,R}Q_{l,Z}} \right)^{\frac{1}{2}} \frac{\lambda_{p,\alpha}}{\lambda_{r,\alpha}} a_{ji,R}^{lk,Z}(\bar{\alpha}_{pr}) \right)_{ij,R,pr}^{kl,Z}$$

where the coefficients of type  $Q_{i,R}$  are the eigenvalues of the matrices associated to the state  $\psi$  and the coefficients of type  $\lambda_{p,\alpha}$  were introduced in Proposition 1.1.15

to describe a morphism  $S \in \text{Hom}(\alpha \otimes \bar{\alpha}, 1_{\mathbb{G}})$  (such a morphism exists by definition of conjugate representation).

In order to show that this is really the inverse, we need to compute explicitly some of the relations introduced in the definition of the free wreath product. Let us start by considering the condition  $\widetilde{m \otimes S} \in \text{Hom}(a(\alpha) \otimes a(\bar{\alpha}), a(1_{\mathbb{G}}))$ . According to Proposition 1.1.15, the morphism  $S$  can be assumed to be of the form  $S(\xi) = \sum_{i=1}^{d_{\alpha}} \lambda_{i,\alpha} \langle \xi, e_i^{\alpha} \otimes e_i^{\bar{\alpha}} \rangle$ , where  $(e_i^{\alpha})_i$  and  $(e_i^{\bar{\alpha}})_i$  denote the well chosen basis of  $H_{\alpha}$  and  $H_{\bar{\alpha}}$  respectively. Then, the condition asking for  $\widetilde{m \otimes S}$  to be a morphism is equivalent to the following family of relations

$$\sum_{l=1}^{n_T} \sum_{r=1}^{d_{\alpha}} \lambda_{r,\alpha} Q_{l,T}^{-\frac{1}{2}} a_{ik,R}^{dl,T}(\alpha_{rt}) a_{pj,Z}^{ls,T}(\bar{\alpha}_{rq}) = \delta_{RZ} \delta_{kp} \delta_{tq} Q_{k,R}^{-\frac{1}{2}} \lambda_{t,\alpha} a_{ij,R}^{ds,T}(1_{\mathbb{G}}) \quad (2.7)$$

while the condition  $\eta^* \in \text{Hom}(a(1_{\mathbb{G}}), 1)$  corresponds to

$$\sum_{Z=1}^c \sum_{k=1}^{n_Z} Q_{k,Z}^{\frac{1}{2}} a_{ij,R}^{kk,Z}(1_{\mathbb{G}}) = \delta_{ij} Q_{i,R}^{\frac{1}{2}} \quad (2.8)$$

It follows that  $b(\alpha)$  is a right inverse, indeed

$$\begin{aligned} (a(\alpha)^t b(\alpha))_{ts,T,pq}^{kl,Z} &= \sum_{R=1}^c \sum_{i,j=1}^{n_R} \sum_{r=1}^{d_{\alpha}} \left( \frac{Q_{s,T} Q_{i,R}}{Q_{j,R} Q_{t,T}} \right)^{\frac{1}{2}} \frac{\lambda_{r,\alpha}}{\lambda_{q,\alpha}} a_{kl,Z}^{ij,R}(\alpha_{rp}) a_{st,T}^{j_i,R}(\bar{\alpha}_{rq}) \\ &= \sum_{R=1}^c \sum_{i=1}^{n_R} \delta_{ZT} \delta_{ls} \delta_{pq} a_{kt,T}^{ii,R}(1_{\mathbb{G}}) \left( \frac{Q_{i,R}}{Q_{t,T}} \right)^{\frac{1}{2}} \\ &= \delta_{ZT} \delta_{ls} \delta_{pq} \delta_{kt} \end{aligned}$$

where the second equality is given by 2.7 and the third by 2.8.

Similarly, the explicit relations corresponding to  $\widetilde{m^* \otimes S^*} \in \text{Hom}(a(1_{\mathbb{G}}), a(\alpha) \otimes a(\bar{\alpha}))$ , where  $S^*(1) = \sum_{i=1}^{d_{\alpha}} \lambda_{i,\alpha} e_i^{\alpha} \otimes e_i^{\bar{\alpha}}$  and to  $\eta \in \text{Hom}(1, a(1_{\mathbb{G}}))$  allow to show that  $b(\alpha)$  is also a left inverse, therefore it is the inverse. The compact quantum group structure is proved.  $\square$

*Remark 2.12.* The definition of the compact quantum group  $H_{(B,\psi)}^+(\mathbb{G})$  implies that every irreducible representation can be obtained as a sub-representation of a suitable tensor product of the basic representations  $a(\alpha)$ ,  $\alpha \in \text{Irr}(\mathbb{G})$ .

*Remark 2.13.* It is possible to transform the three conditions of Definition 2.5.1 into explicit relations between the coefficients of the  $a(\alpha)$ . Of course, this implies the choice of a basis for all the vector spaces concerned, i.e. the  $C^*$ -algebra  $B$  and

the spaces  $H_\alpha$ ,  $\dim(H_\alpha) = d_\alpha$ . As in the proof of Proposition 2.5.2, we choose  $\mathcal{B}'$  as basis of  $B$  and, for each  $H_\alpha$ ,  $\alpha \in \text{Irr}(\mathbb{G})$ , we use the basis constructed in Proposition 1.1.15. Even if the definition of the free wreath product does not depend on the choice of these basis, the following relations do. Therefore, a different choice, in general, leads to different relations. In the proof of Proposition 2.5.2 we already calculated the relations corresponding to  $\widetilde{m \otimes S} \in \text{Hom}(a(\alpha) \otimes a(\bar{\alpha}), a(1_{\mathbb{G}}))$ . This simplified formula, fundamental for specific computations, does not have a direct analogue in the general case, because it relies on the possibility to diagonalise the morphism  $S$ .

In order to give a general formulation we need to introduce some notations. Let  $\alpha, \beta, \gamma \in \text{Irr}(\mathbb{G})$ , let  $S \in \text{Hom}(\alpha \otimes \beta, \gamma)$  and denote the corresponding matrix by  $S = (\varphi_z^{rs})_{z,(r,s)}$  where  $1 \leq z \leq d_\gamma$ ,  $1 \leq r \leq d_\alpha$ ,  $1 \leq s \leq d_\beta$ . Similarly, let  $S' \in \text{Hom}(\gamma, \alpha \otimes \beta)$  and denote its matrix by  $S' = (\varphi_{rs}^z)_{(r,s),z}$  where  $1 \leq z \leq d_\gamma$ ,  $1 \leq r \leq d_\alpha$ ,  $1 \leq s \leq d_\beta$ .

The family of relations corresponding to  $\widetilde{m \otimes S} \in \text{Hom}(a(\alpha) \otimes a(\beta), a(\gamma))$  is then

$$\sum_{t=1}^{n_Z} \sum_{r=1}^{d_\alpha} \sum_{s=1}^{d_\beta} \varphi_z^{rs} Q_{t,Z}^{-\frac{1}{2}} a_{iv,R}^{kt,Z}(\alpha_{rp}) a_{wj,T}^{tl,Z}(\beta_{sq}) = \delta_{RT} \delta_{vw} Q_{v,R}^{-\frac{1}{2}} \sum_{y=1}^{d_\gamma} \varphi_y^{pq} a_{ij,R}^{kl,Z}(\gamma_{zy}) \quad (2.9)$$

while the relations associated to  $\widetilde{m^* \otimes S'} \in \text{Hom}(a(\gamma), a(\alpha) \otimes a(\beta))$  are

$$\sum_{v=1}^{n_R} \sum_{p=1}^{d_\alpha} \sum_{q=1}^{d_\beta} \varphi_{pq}^y Q_{v,R}^{-\frac{1}{2}} a_{iv,R}^{kt,Z}(\alpha_{rp}) a_{vj,R}^{wl,T}(\beta_{sq}) = \delta_{ZT} \delta_{tw} Q_{t,Z}^{-\frac{1}{2}} \sum_{z=1}^{d_\gamma} \varphi_{rs}^z a_{ij,R}^{kl,Z}(\gamma_{zy}) \quad (2.10)$$

The relations corresponding to the intertwiners  $\eta \in \text{Hom}(1_{H_{(B,\psi)}^+(\mathbb{G})}, a(1_{\mathbb{G}}))$  and  $\eta^* \in \text{Hom}(a(1_{\mathbb{G}}), 1_{H_{(B,\psi)}^+(\mathbb{G})})$  are respectively

$$\sum_{R=1}^c \sum_{i=1}^{n_R} Q_{i,R}^{\frac{1}{2}} a_{ii,R}^{kl,Z}(1_{\mathbb{G}}) = \delta_{kl} Q_{k,Z}^{\frac{1}{2}} \quad \sum_{Z=1}^c \sum_{k=1}^{n_Z} Q_{k,Z}^{\frac{1}{2}} a_{ij,R}^{kk,Z}(1_{\mathbb{G}}) = \delta_{ij} Q_{i,R}^{\frac{1}{2}} \quad (2.11)$$

The relations concerning the adjoints are not explicitly stated in the definition of the free wreath product (they can be easily deduced). We decided to include them in this list because, in this way, the last condition which asks for the  $a(\alpha)$ ,  $\alpha \in \text{Irr}(\mathbb{G})$  to be unitary, can simply be replaced by the following definition of the involution operation

$$a_{ij,R}^{kl,Z}(\alpha_{pr})^* = \left( \frac{Q_{i,R} Q_{l,Z}}{Q_{k,Z} Q_{j,R}} \right)^{\frac{1}{2}} \frac{\lambda_{r,\alpha}}{\lambda_{p,\alpha}} a_{ji,R}^{lk,Z}(\bar{\alpha}_{pr}) \quad (2.12)$$

where the coefficients of type  $\lambda_{p,\alpha}$  are the scalars describing a morphism  $S \in \text{Hom}(\alpha \otimes \bar{\alpha}, 1_{\mathbb{G}})$  as explained in Proposition 1.1.15. The relations 2.9, 2.10 and 2.11 can be calculated directly from the definition of intertwiner. In order to show that the last relation is the good definition of involution (i.e. that it can be deduced from Definition 2.5.1) and that, together with the other relations, generates the free wreath product, we need to combine different relations and results. Firstly, it is necessary to observe that Proposition 1.1.15 implies that  $\lambda_{p,\bar{\alpha}} = (\lambda_{p,\alpha})^{-1}$  as well as to use the relations 2.11. Moreover, we need to use relation 2.7 which is a special case of relation 2.9, and the relation corresponding to  $\widetilde{m^* \otimes S^*} \in \text{Hom}(a(1_{\mathbb{G}}), a(\alpha) \otimes a(\bar{\alpha}))$ , where  $S^*(1) = \sum_{i=1}^{d_\alpha} \lambda_{i,\alpha} e_i^\alpha \otimes e_i^{\bar{\alpha}}$ . Of course, this last one is a particular case of relation 2.10 and it is given by

$$\sum_{v=1}^{n_R} \sum_{p=1}^{d_\alpha} \lambda_{p,\alpha} Q_{v,R}^{-\frac{1}{2}} a_{iv,R}^{kt,Z}(\alpha_{rp}) a_{vj,R}^{wl,T}(\bar{\alpha}_{sp}) = \delta_{ZT} \delta_{tw} \delta_{rs} Q_{t,Z}^{-\frac{1}{2}} \lambda_{r,\alpha} a_{ij,R}^{kl,Z}(1_{\mathbb{G}}) \quad (2.13)$$

### 2.5.2 Spaces of intertwiners

In this section we want to generalize some previous results and some results from [LT14], in order to describe the spaces of intertwiners of the free wreath product  $H_{(B,\psi)}^+(\mathbb{G})$  by means of decorated noncrossing partitions.

**Definition 2.5.3.** Let  $p \in NC(k, l)$ . Suppose to decorate the  $k$  upper points of  $p$  with the representations of the tuple  $\alpha = (\alpha_1, \dots, \alpha_k) \in \text{Rep}(\mathbb{G})^k$  and the  $l$  lower points with the representations of the tuple  $\beta = (\beta_1, \dots, \beta_l) \in \text{Rep}(\mathbb{G})^l$ . Denote by  $b_v$ ,  $v = 1, \dots, m$  the different blocks of  $p$ , let  $U_v$  ( $L_v$ ) be the upper (lower) points of the block  $b_v$  and let  $\alpha_{U_v}$  ( $\beta_{L_v}$ ) be the tensor product of the representations which decorate the upper (lower) points of  $b_v$ . Similarly, let  $H_{U_v}$  ( $H_{L_v}$ ) be the tensor product of the Hilbert spaces associated to these representations.

If for every block  $b_v$  there is at least a non-zero morphism  $S_v \in \text{Hom}(\alpha(U_v), \beta(L_v))$ , then  $p$  is considered well decorated. If a block  $b_v$  does not have lower points, the map  $S_v$  must exist in  $\text{Hom}(\alpha(U_v), 1_{\mathbb{G}})$  so the upper points can be thought as being connected to an imaginary lower point decorated by the trivial representation; similarly, if there are no upper points.

We will denote  $NC_{\mathbb{G}}(\alpha_1, \dots, \alpha_k; \beta_1, \dots, \beta_l)$  the set of these well decorated noncrossing partitions.

*Notation 11.* Let  $p \in NC_{\mathbb{G}}(\alpha_1, \dots, \alpha_k; \beta_1, \dots, \beta_l)$  and denote by  $H_{\gamma}$  the Hilbert space of a representation  $\gamma \in \text{Rep}(\mathbb{G})$ . For each block of  $p$  choose a morphism  $S_v$ . Let  $S := \bigotimes_{v=1}^m S_v$  be the morphism obtained by means of the tensor product operation.

$$S : \bigotimes_{v=1}^m H_{U_v} \longrightarrow \bigotimes_{v=1}^m H_{L_v}$$

Then, it is quite natural to consider the map

$$T_p \otimes S : B^{\otimes k} \otimes \bigotimes_{v=1}^m H_{U_v} \longrightarrow B^{\otimes l} \otimes \bigotimes_{v=1}^m H_{L_v}$$

but, as we can observe, a morphism  $f \in \text{Hom}(a(\alpha_1) \otimes \dots \otimes a(\alpha_k), a(\beta_1) \otimes \dots \otimes a(\beta_l))$  is a map  $f \in \mathcal{L}(\bigotimes_{i=1}^k (B \otimes H_{\alpha_i}), \bigotimes_{j=1}^l (B \otimes H_{\beta_j}))$  so it is necessary to correctly put in order the spaces.

*Notation 12.* By making use of the previous notations, we define  $s_{p,U} : \bigotimes_{i=1}^k (B \otimes H_{\alpha_i}) \longrightarrow B^{\otimes k} \otimes \bigotimes_{v=1}^m H_{U_v}$  and  $s_{p,L} : \bigotimes_{j=1}^l (B \otimes H_{\beta_j}) \longrightarrow B^{\otimes l} \otimes \bigotimes_{v=1}^m H_{L_v}$  as the applications which reorder the spaces associated to the upper and lower points of  $p$  respectively.

**Definition 2.5.4.** The map in  $\mathcal{L}(\bigotimes_{i=1}^k (B \otimes H_{\alpha_i}), \bigotimes_{j=1}^l (B \otimes H_{\beta_j}))$  associated to a decorated noncrossing partition  $p \in NC_{\mathbb{G}}(\alpha_1, \dots, \alpha_k; \beta_1, \dots, \beta_l)$  endowed with a morphism  $S \in \bigotimes_{v=1}^m \text{Hom}(\alpha_{U_v}, \beta_{L_v})$  is

$$T_{p,S} := s_{p,L}^{-1} \circ (T_p \otimes S) \circ s_{p,U}$$

From Proposition 2.3.5 this compatibility result easily follows.

**Proposition 2.5.5.** *Let  $p \in NC_{\mathbb{G}}(\alpha_1, \dots, \alpha_k; \beta_1, \dots, \beta_l)$  be a decorated noncrossing partition endowed with the morphism  $S \in \bigotimes_{v=1}^m \text{Hom}(\alpha_{U_v}, \beta_{L_v})$ . Similarly, let  $q \in NC_{\mathbb{G}}(\alpha'_1, \dots, \alpha'_{k'}; \beta'_1, \dots, \beta'_{l'})$  be a decorated noncrossing partition endowed with the morphism  $S' \in \bigotimes_{v=1}^{m'} \text{Hom}(\alpha'_{U_v}, \beta'_{L_v})$ .*

*Then:*

1.  $T_{p \otimes q, S \otimes S'} = T_{p, S} \otimes T_{q, S'}$
2.  $T_{p, S}^* = T_{p^*, S^*}$
3. if  $l = k'$  and  $\beta_i = \alpha'_i$  for all  $i = 1, \dots, k'$  there are two possibilities:
  - a. if  $\psi$  is a (unital)  $\delta$ -form, then

$$T_{qp, S'S} = \delta^{-cy(p,q)} T_{q, S'} T_{p, S}$$

- b. if  $\tilde{\psi}$  is a (possibly non unital) 1-form, then

$$T_{qp, S'S} = \tilde{\psi}(1)^{-cb(p,q)} T_{q, S'} T_{p, S}$$

*Proof.* The first relation follows from

$$\begin{aligned} T_{p, S} \otimes T_{q, S'} &= \\ (s_{p, L}^{-1} \circ (T_p \otimes S) \circ s_{p, U}) \otimes (s_{q, L}^{-1} \circ (T_q \otimes S') \circ s_{q, U}) &= \\ (s_{p, L}^{-1} \otimes s_{q, L}^{-1}) \circ (T_p \otimes S \otimes T_q \otimes S') \circ (s_{p, U} \otimes s_{q, U}) &= \\ (s_{p, L}^{-1} \otimes s_{q, L}^{-1}) (\text{id} \otimes \sigma_1^{-1} \otimes \text{id}) (\text{id} \otimes \sigma_1 \otimes \text{id}) (T_p \otimes S \otimes T_q \otimes S') (\text{id} \otimes \sigma_2^{-1} \otimes \text{id}) \\ & \qquad \qquad \qquad (\text{id} \otimes \sigma_2 \otimes \text{id}) (s_{p, U} \otimes s_{q, U}) = \\ s_{p \otimes q, L}^{-1} \circ (T_{p \otimes q} \otimes S \otimes S') \circ s_{p \otimes q, U} &= \\ T_{p \otimes q, S \otimes S'} & \end{aligned}$$

where  $\sigma_1$  and  $\sigma_2$  are maps which reorder the spaces as necessary. In particular,  $\sigma_1 : \bigotimes_{i=1}^l H_{\beta_i} \otimes B^{\otimes l'} \longrightarrow B^{\otimes l'} \otimes \bigotimes_{i=1}^l H_{\beta_i}$  and  $\sigma_2 : \bigotimes_{i=1}^k H_{\alpha_i} \otimes B^{\otimes k'} \longrightarrow B^{\otimes k'} \otimes \bigotimes_{i=1}^k H_{\alpha_i}$ .

For the second relation we observe that

$$\begin{aligned} T_{p, S}^* &= (s_{p, L}^{-1} \circ (T_p \otimes S) \circ s_{p, U})^* \\ &= s_{p, U}^{-1} \circ (T_p^* \otimes S^*) \circ s_{p, L} \\ &= s_{p^*, L}^{-1} \circ (T_{p^*} \otimes S^*) \circ s_{p^*, U} \\ &= T_{p^*, S^*} \end{aligned}$$

The compatibility with the multiplication (case 3a) follows from

$$\begin{aligned} T_{q, S'} T_{p, S} &= (s_{q, L}^{-1} \circ (T_q \otimes S') \circ s_{q, U}) \circ (s_{p, L}^{-1} \circ (T_p \otimes S) \circ s_{p, U}) \\ &= s_{q, L}^{-1} \circ (T_q \otimes S') \circ (T_p \otimes S) \circ s_{p, U} \\ &= \delta^{cy(p,q)} (s_{qp, L}^{-1} \circ (T_{qp} \otimes S'S) \circ s_{qp, U}) \\ &= \delta^{cy(p,q)} T_{qp, S'S} \end{aligned}$$

The proof of the case 3b is identical but based on the results of Proposition 2.3.6.  $\square$

The following lemma is a sort of linearity result concerning these morphisms and will be important in the proof of the next theorem.

**Lemma 2.5.6.** *Let  $p \in NC_{\mathbb{G}}(\alpha_1, \dots, \alpha_k; \beta_1, \dots, \beta_l)$  be a decorated noncrossing partition which can be endowed with the morphisms  $S, S' \in \bigotimes_{v=1}^m \text{Hom}(\alpha_{U_v}, \beta_{L_v})$ . Let  $\lambda, \mu \in \mathbb{C}$ . Then*

$$\lambda T_{p,S} + \mu T_{p,S'} = T_{p,\lambda S + \mu S'}$$

*Proof.* By applying the definition and by using the linearity of the different maps we have:

$$\begin{aligned} \lambda T_{p,S} + \mu T_{p,S'} &= \lambda(s_{p,L}^{-1} \circ (T_p \otimes S) \circ s_{p,U}) + \mu(s_{p,L}^{-1} \circ (T_p \otimes S') \circ s_{p,U}) \\ &= s_{p,L}^{-1} \circ (T_p \otimes \lambda S) \circ s_{p,U} + s_{p,L}^{-1} \circ (T_p \otimes \mu S') \circ s_{p,U} \\ &= s_{p,L}^{-1} \circ ((T_p \otimes \lambda S) \circ s_{p,U} + (T_p \otimes \mu S') \circ s_{p,U}) \\ &= s_{p,L}^{-1} \circ ((T_p \otimes \lambda S + T_p \otimes \mu S') \circ s_{p,U}) \\ &= s_{p,L}^{-1} \circ ((T_p \otimes \lambda S + \mu S') \circ s_{p,U}) \\ &= T_{p,\lambda S + \mu S'} \end{aligned}$$

$\square$

**Theorem 2.5.7.** *Let  $B$  be a  $n$ -dimensional  $C^*$ -algebra ( $n \geq 4$ ) endowed with a  $\delta$ -form  $\psi$  and  $\mathbb{G}$  a compact quantum group. Consider the free wreath product  $H_{(B,\psi)}^+(\mathbb{G})$  with basic representations  $a(\alpha)$ , where  $\alpha \in \text{Irr}(\mathbb{G})$ . Then, for all  $k, l \in \mathbb{N}$*

$$\begin{aligned} \text{Hom}(\bigotimes_{i=1}^k a(\alpha_i), \bigotimes_{j=1}^l a(\beta_j)) = \\ \text{span}\{T_{p,S} \mid p \in NC_{\mathbb{G}}(\alpha_1, \dots, \alpha_k; \beta_1, \dots, \beta_l), S \in \bigotimes_{v=1}^m \text{Hom}(\alpha_{U_v}, \beta_{L_v})\} \end{aligned}$$

with the convention that, if  $k = 0$ ,  $\bigotimes_{i=1}^k a(\alpha_i) = 1_{H_{(B,\psi)}^+(\mathbb{G})}$  and the space of the noncrossing partitions is  $NC_{\mathbb{G}}(\emptyset; \bigotimes_{j=1}^l a(\beta_j))$ , i.e. it does not have upper points. Similarly, if  $l = 0$ .

Moreover, its dimension is given by  $\sum_{p \in NC_{\mathbb{G}}(\alpha_1, \dots, \alpha_k; \beta_1, \dots, \beta_l)} \prod_{v=1}^m \dim \text{Hom}(\alpha_{U_v}, \beta_{L_v})$ .

*Proof.* In order to prove the inclusion  $\supseteq$ , we have to show that every linear map  $T_{p,S}$  obtained from a decorated noncrossing partition  $p$  endowed with a suitable

morphism  $S$  is an intertwiner of  $H_{(B,\psi)}^+(\mathbb{G})$ . In particular, we will prove that every  $T_{p,S}$  can be decomposed as a linear combination of tensor products, compositions and adjoints of the basic morphisms  $\widetilde{m \otimes S}$ ,  $\eta$  and  $\text{id}$ . From Theorem 2.3.7, we know that such a decomposition at the level of the noncrossing partitions exists. The more difficult point here is to decorate every block of the decomposition with irreducible representations and to associate the right morphisms such that, if we compose all the diagrams, we obtain the original map.

By making use of the Frobenius reciprocity (Theorem 1.1.28), we have that

$$\text{Hom}\left(\bigotimes_{i=1}^k a(\alpha_i), \bigotimes_{j=1}^l a(\beta_j)\right) \cong \text{Hom}\left(1, a(\beta_1) \otimes \dots \otimes a(\beta_l) \otimes a(\bar{\alpha}_k) \otimes \dots \otimes a(\bar{\alpha}_1)\right)$$

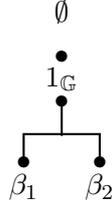
Moreover, from the previous results it follows that the noncrossing partitions decorated with the elements of  $\text{Irr}(\mathbb{G})$  form a monoidal rigid  $C^*$ -category, denoted  $\mathcal{NC}_{\mathbb{G}}$ . The objects are all the finite sequences  $(\alpha_1, \dots, \alpha_k)$  for all  $\alpha_i \in \text{Irr}(\mathbb{G})$  and  $k \in \mathbb{N}$  (plus the empty word  $\emptyset$ ) and the spaces of the morphisms are given by  $\text{Hom}((\alpha_1, \dots, \alpha_k), (\beta_1, \dots, \beta_l)) = \text{span}\{T_{p,S} \mid p \in NC_{\mathbb{G}}(\alpha_1, \dots, \alpha_k; \beta_1, \dots, \beta_l), S \in \bigotimes_{v=1}^m \text{Hom}(\alpha_{U_v}, \beta_{L_v})\}$ . By applying the Frobenius reciprocity in this case we get

$$\text{Hom}((\alpha_1, \dots, \alpha_k), (\beta_1, \dots, \beta_l)) \cong \text{Hom}(\emptyset, (\beta_1, \dots, \beta_l, \bar{\alpha}_k, \dots, \bar{\alpha}_1))$$

It follows that it is enough to prove the inclusion for  $k = 0$ . Moreover, we can restrict ourselves to prove the result in the case of a one-block non crossing partition, because the map associated to any decorated noncrossing partition in  $NC_{\mathbb{G}}(\emptyset, (\beta_1, \dots, \beta_l))$  can easily be obtained through compositions and tensor products of the maps associated to one-block noncrossing partitions and of the identity map. Let  $p \in NC_{\mathbb{G}}(\emptyset; \beta_1, \dots, \beta_l)$ ,  $b(p) = 1$  be a decorated noncrossing partition endowed with the morphism  $S$  and consider the map  $T_{p,S}$ . The condition  $b(p) = 1$  implies that  $S \in \text{Hom}(1_{\mathbb{G}}, \bigotimes_{j=1}^l \beta_j)$  and the diagram we have to consider is like this:

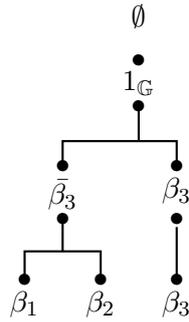
$$p_l = \begin{array}{c} \emptyset \\ \hline \bullet \quad \bullet \quad \dots \quad \bullet \\ \beta_1 \quad \beta_2 \quad \dots \quad \beta_l \end{array}$$

We will prove the result by induction. If  $l = 0$  the result is trivial, if  $l = 1$  we obtain the map  $\eta$  which is in  $\text{Hom}(1, a(1_{\mathbb{G}}))$  by definition. If  $l = 2$ ,  $S \in \text{Hom}(1_{\mathbb{G}}, \beta_1 \otimes \beta_2)$  and the diagram can be decomposed as follows



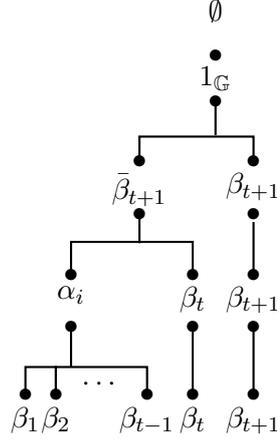
Therefore  $T_{p_2, S} = \Sigma_{23}(m^* \otimes S)\eta \in \text{Hom}(1, a(\beta_1) \otimes a(\beta_2))$  because it can be seen as the composition of two intertwiners. Moreover, we observe that this situation is possible only if  $\beta_2 = \bar{\beta}_1$ .

If  $l = 3$  and the morphism associated to the noncrossing partition is  $S \in \text{Hom}(1_{\mathbb{G}}, \beta_1 \otimes \beta_2 \otimes \beta_3)$ , we have the following decomposition.



In order to complete the description of the decomposition, we need to associate a morphism to every noncrossing partition. The morphism of the noncrossing partition corresponding to  $\eta$  is clearly  $\text{id}_{1_{\mathbb{G}}}$ , while the morphism of the lower block corresponding to the identity is  $\text{id}_{H_{\beta_3}}$ . In order to define the remaining morphisms we recall the notation introduced in Definition 1.1.27 to denote the invariant vectors. Let  $R \in \text{Hom}(1, \bar{\beta}_3 \otimes \beta_3)$  and  $\bar{R} \in \text{Hom}(1, \beta_3 \otimes \bar{\beta}_3)$  be the morphisms satisfying the conjugate equations. Then, the morphisms associated to the two blocks corresponding to  $m^*$  are  $R \in \text{Hom}(1, \bar{\beta}_3 \otimes \beta_3)$  and  $(\text{id}_{H_{\beta_1} \otimes H_{\beta_2}} \otimes \bar{R}^*)(S \otimes \text{id}_{H_{\bar{\beta}_3}}) \in \text{Hom}(\bar{\beta}_3, \beta_1 \otimes \beta_2)$  respectively. An easy computation allows us to verify that  $S = [(\text{id}_{H_{\beta_1} \otimes H_{\beta_2}} \otimes \bar{R}^*)(S \otimes \text{id}_{H_{\bar{\beta}_3}}) \otimes \text{id}_{H_{\beta_3}}]R$ . This means that  $T_{p_3, S}$  can be decomposed in term of some of the basic morphisms introduced in the definition of the free wreath product and therefore is in  $\text{Hom}(1, \beta_1 \otimes \beta_2 \otimes \beta_3)$ .

Now, we are ready for the inductive step. Let us suppose the inclusion true for a  $l = t \geq 3$  and prove it for  $t+1$ . As usual, let  $S \in \text{Hom}(1, \otimes_{j=1}^{t+1} \beta_j)$  be the morphism associated to the noncrossing partition  $p_{t+1}$ . The decomposition which we need to consider in this case is



where  $\alpha_i \subset \bar{\beta}_{t+1} \otimes \bar{\beta}_t$ .

Now, we have to assign a suitable morphism to every noncrossing partition of the decomposition. As in the case  $l = 3$ , we associate the identity map to the diagrams corresponding to  $\eta$  and to  $\text{id}$ . The morphisms on the other blocks are less obvious and we need to introduce some notations. Let us denote  $R_t \in \text{Hom}(1, \bar{\beta}_t \otimes \beta_t)$ ,  $\bar{R}_t \in \text{Hom}(1, \beta_t \otimes \bar{\beta}_t)$ ,  $R_{t+1} \in \text{Hom}(1, \bar{\beta}_{t+1} \otimes \beta_{t+1})$  and  $\bar{R}_{t+1} \in \text{Hom}(1, \beta_{t+1} \otimes \bar{\beta}_{t+1})$  two pair of invariant vectors satisfying the conjugate equations. For every  $\alpha_i \subset \bar{\beta}_{t+1} \otimes \bar{\beta}_t$ , we know that there is an isometry  $r_i \in \text{Hom}(\alpha_i, \bar{\beta}_{t+1} \otimes \bar{\beta}_t)$  such that  $r_i r_i^* \in \text{End}(\bar{\beta}_{t+1} \otimes \bar{\beta}_t)$  is a projection and  $\sum_i r_i r_i^* = \text{id}_{H_{\bar{\beta}_{t+1}} \otimes H_{\bar{\beta}_t}}$ . There are still three morphisms to assign; they will be denoted  $S_{1,i}$ ,  $S_{2,i}$  and  $S_{3,i}$ . From the top to the bottom, they are the following ones. The morphism  $S_{1,i} \in \text{Hom}(1_G, \bar{\beta}_{t+1} \otimes \beta_{t+1})$  is  $R_{t+1}$ . The morphism  $S_{2,i} \in \text{Hom}(\bar{\beta}_{t+1}, \alpha_i \otimes \beta_t)$  is  $(t_i^* \otimes \text{id}_{H_{\beta_t}})(\text{id}_{H_{\bar{\beta}_{t+1}}} \otimes R_t)$ . Finally, the morphism  $S_{3,i} \in \text{Hom}(\alpha_i, \otimes_{j=1}^{t-i} \beta_j)$  is

$$(\text{id}_{H_{\beta_1} \otimes \dots \otimes H_{\beta_{t-1}}} \otimes \bar{R}_t^*(\text{id}_{H_{\beta_t}} \otimes \bar{R}_{t+1}^* \otimes \text{id}_{H_{\bar{\beta}_t}}))(S \otimes \text{id}_{H_{\bar{\beta}_{t+1}} \otimes H_{\bar{\beta}_t}})t_i$$

These are all morphisms because they are obtained through the operations of tensor product, composition and adjoint from known morphisms. Moreover, by making use of the Frobenius reciprocity and of the inductive hypothesis, we know

that the linear map in  $\mathcal{L}(B \otimes H_{\alpha_i}, \otimes_{j=1}^{t-1} (B \otimes H_{\beta_j}))$  associated to the noncrossing partition endowed with the morphism  $S_{3,i}$  is in  $\text{Hom}(a(\alpha_i), \otimes_{j=1}^{t-1} a(\beta_j))$ . The fact that the linear maps corresponding to the other decorated noncrossing partitions are intertwiners follows from the definition of free wreath product. An easy computation allows us to verify that  $\sum_i (S_{3,i} \otimes \text{id}_{H_{\beta_t} \otimes H_{\beta_{t+1}}})(S_{2,i} \otimes \text{id}_{H_{\beta_{t+1}}})S_{1,i} = S$ , therefore, by making use of Lemma 2.5.6 and of Proposition 2.5.5, we have that  $T_{p_{t+1}, S} \in \text{Hom}(1, \otimes_{j=1}^{t+1} a(\beta_j))$  as it is possible to write  $T_{p_{t+1}, S}$  as a linear combination of compositions, adjoints and tensor products of intertwiners.

For the second inclusion ( $\subseteq$ ), we apply the Tannaka-Krein duality to the concrete rigid monoidal  $C^*$ -category  $\mathcal{N}\mathcal{C}_{\mathbb{G}}$ . Then, there exists a compact quantum group  $\mathbb{G} = (C(\mathbb{G}), \Delta)$  such that  $C(\mathbb{G})$  is generated by the coefficients of a family of finite dimensional unitary representations  $a(\alpha_i)'$  and  $\text{Hom}(\otimes_{i=1}^k a(\alpha_i)', \otimes_{j=1}^l a(\beta_j)') = \text{span}\{T_p | p \in NC_{\widehat{\Gamma}}(\alpha_1, \dots, \alpha_k; \beta_1, \dots, \beta_l)\}$ . Moreover, because of the universality of the Tannaka-Krein construction, from the inclusion already proved we can deduce that there is a surjective map  $\phi : C(\mathbb{G}) \longrightarrow H_{(B, \psi)}^+(\mathbb{G})$  such that  $(\text{id} \otimes \phi)(a(\alpha)') = a(\alpha)$ , for all  $\alpha \in \text{Irr}(\mathbb{G})$ . In order to complete the proof, we have to show that this map is an isomorphism. The existence of the inverse morphism follows from the universality of the free wreath product construction. It is enough to observe that the unitary representations  $a(\alpha)'$  are such that  $\widetilde{m \otimes S} \in \text{Hom}(a(\alpha)' \otimes a(\beta)', a(\gamma)')$  for any  $S \in \text{Hom}(\alpha \otimes \beta, \gamma)$  and  $\eta \in \text{Hom}(1, a(1_{\mathbb{G}})')$  because the maps  $\widetilde{m \otimes S}$  and  $\eta$  correspond to well decorated noncrossing partitions.

The dimension formula follows by recalling that the maps  $T_p$  associated to distinct noncrossing partitions in  $NC(k, l)$  are linearly independent (see Theorem 2.3.7).  $\square$

*Remark 2.14.* The description of the space of intertwiners does not depend on considering a unital  $\delta$ -form  $\psi$  or the associated non-unital 1-form  $\tilde{\psi}$ . Instead, in what follows, it will be necessary to be in the  $\tilde{\psi}$  case, in order to prove a monoidal equivalence result for the free wreath product.

*Remark 2.15.* As in the case of a discrete group, we can compute the Haar measure of some particular elements. Consider the free wreath product  $H_{(B, \psi)}^+(\mathbb{G})$ , where  $\psi$  is a  $\delta$ -form, and let  $\chi(a(1_{\mathbb{G}})) := (\text{Tr} \otimes \text{id})(a(1_{\mathbb{G}}))$  be the character of the representa-

tion  $a(1_{\mathbb{G}})$ . We easily observe that  $\chi(a(1_{\mathbb{G}}))$  is self-adjoint. Now, we want to compute the moments  $h(\chi(a(1_{\mathbb{G}}))^k)$ ; let  $p_k$  be the orthogonal projection onto the fixed points space  $Hom(1, a(1_{\mathbb{G}})^{\otimes k})$ . Thanks to some classic results of Woronowicz (see [Wor88]) we have  $h(\chi(a(1_{\mathbb{G}}))^k) = h((\text{Tr} \otimes \text{id})(a(1_{\mathbb{G}}))^k) = \text{Tr}((\text{id} \otimes h)(a(1_{\mathbb{G}})^{\otimes k})) = \text{Tr}(p_k) = \dim(Hom(1, a(1_{\mathbb{G}})^{\otimes k})) = \#NC(0, k) = C_k$  where  $C_k$  are the Catalan numbers. They are the moments of the free Poisson law of parameter 1 which is then the spectral measure of  $\chi(a(1_{\mathbb{G}}))$ .

### 2.5.3 Monoidal equivalence

In this part, we will prove a monoidal equivalence result for the free wreath product. An analogous result has been proved in [LT14, Theorem 5.11], in the particular case of the free wreath product of a compact matrix quantum group of Kac type by the quantum symmetric group. We will show that it can be extended to this more general context. The monoidal equivalence will allow us to reconstruct the representation theory of  $H_{(B,\psi)}^+(\mathbb{G})$  and to prove some properties of the operator algebras associated to the free wreath product. We recall the fundamental definition.

**Definition 2.5.8.** Let  $\mathbb{G}_1$  and  $\mathbb{G}_2$  be two compact quantum groups. They are monoidally equivalent (written  $\mathbb{G}_1 \simeq_{mon} \mathbb{G}_2$ ) if there exists a bijection  $\phi : \text{Irr}(\mathbb{G}_1) \longrightarrow \text{Irr}(\mathbb{G}_2)$ ,  $\phi(1_{\mathbb{G}_1}) = 1_{\mathbb{G}_2}$  such that, for any  $k, l \in \mathbb{N}$  and for any  $\alpha_i, \beta_j \in \text{Irr}(\mathbb{G})$ ,  $1 \leq i \leq k$ ,  $1 \leq j \leq l$ , there is an isomorphism

$$\phi : \text{Hom}_{\mathbb{G}_1}(\alpha_1 \otimes \dots \otimes \alpha_k; \beta_1 \otimes \dots \otimes \beta_l) \longrightarrow \text{Hom}_{\mathbb{G}_2}(\phi(\alpha_1) \otimes \dots \otimes \phi(\alpha_k); \phi(\beta_1) \otimes \dots \otimes \phi(\beta_l))$$

such that:

- i)  $\phi(\text{id}) = \text{id}$
- ii)  $\phi(F \otimes G) = \phi(F) \otimes \phi(G)$
- iii)  $\phi(F^*) = \phi(F)^*$
- iv)  $\phi(FG) = \phi(F)\phi(G)$  for  $F, G$  composable morphisms

The following theorem generalises the monoidal equivalence result proved in [LT14, Theorem 5.11], where the case of the free wreath product of a compact matrix quantum group of Kac type by the quantum symmetric group was considered.

**Theorem 2.5.9.** *Let  $(B, \tilde{\psi})$  be a finite dimensional  $C^*$ -algebra,  $\dim(B) \geq 4$ , endowed with a possibly non-unital 1-form  $\tilde{\psi}$ . Let  $0 < q \leq 1$  be such that  $q + q^{-1} = \sqrt{\tilde{\psi}(1)}$ . Let  $\mathbb{G}$  be a compact quantum group and consider the free wreath product  $\mathbb{G} \wr_* \mathbb{G}^{aut}(B, \tilde{\psi})$ . Let  $\mathbb{H}$  be the compact quantum subgroup of  $\mathbb{G} \hat{*} SU_q(2)$  given by*

$$C(\mathbb{H}) = \langle b_{ij}ab_{kl} \mid 1 \leq i, j, k, l \leq 2, a \in C(\mathbb{G}) \rangle \subset C(\mathbb{G}) * C(SU_q(2))$$

$$\Delta_{\mathbb{H}} := \Delta_{\mathbb{G} \hat{*} SU_q(2)|_{\mathbb{H}}}$$

More precisely

$$\Delta(b_{ij}ab_{kl}) = \sum_{r,s,v} b_{ir}a_{(1)}b_{ks} \otimes b_{rj}a_{(2)}b_{sl} \in C(\mathbb{H}) \otimes C(\mathbb{H})$$

where  $b = (b_{ij})_{ij}$  is the generating matrix of  $SU_q(2)$  and  $\Delta_{\mathbb{G}}(a) = \sum a_{(1)} \otimes a_{(2)}$ .

Then

$$\mathbb{G} \wr_* \mathbb{G}^{aut}(B, \tilde{\psi}) \simeq_{mon} \mathbb{H}$$

*Proof.* By doing some minor changes and remarks, the proof presented in [LT14] works also in this more general case, so we will only give a sketch of the arguments, pointing out the critical passages in the adaptation to this context. The first remark is about the existence of a  $q \in (0, 1]$  such that  $q + q^{-1} = \sqrt{\tilde{\psi}(1)}$ . An easy computation shows that  $q + q^{-1} : (0, 1] \rightarrow [2, \infty)$  is a bijection, therefore the monoidal equivalence makes sense only if  $\tilde{\psi}(1) \geq 4$ . We recall that  $\tilde{\psi} : B \rightarrow \mathbb{C}$  is a 1-form so it can be rewritten as  $\tilde{\psi}(\cdot) = \sum_{\lambda} \text{Tr}(Q_{\lambda} \cdot)$  for a suitable family of positive diagonal matrices  $Q_{\lambda}$  such that  $\text{Tr}(Q_{\lambda}^{-1}) = 1$  for every  $\lambda$ . With this in mind, the condition  $\tilde{\psi}(1) \geq 4$  is a consequence of the following arithmetic lemma.

**Lemma 2.5.10.** *Let  $(x_i)_{i=1, \dots, n}, x_i > 0$  and  $n \geq 2$  be a family of positive real numbers such that  $\sum_{i=1}^n x_i \leq 1$ . Then  $\sum_{i=1}^n x_i^{-1} \geq 4$ .*

This lemma can simply be proved by recalling the inequality between the harmonic mean and the arithmetic mean.

The proof of the monoidal equivalence is based on the construction of an explicit isomorphism between the intertwining spaces. In this first phase, we will take into account intertwiners between the tensor products of the representations which generate the compact quantum groups; only in a second time, this isomorphism will be extended to intertwiners between tensor products of all the irreducible representations. Consider the family of representations of  $\mathbb{H}$  given by  $s(\alpha) := b \otimes \alpha \otimes b$ ,  $\alpha \in \text{Irr}(\mathbb{G})$ . By using the description of the irreducible representations of a free product given by Wang in [Wan95], we have that  $\{s(\alpha) | \alpha \in \text{Irr}(\mathbb{G}), \alpha \neq 1_{\mathbb{G}}\} \subset \text{Irr}(\mathbb{H})$ . Moreover, we observe that, for any  $\beta \in \text{Irr}(\mathbb{H})$ , there exists a finite family  $(\alpha_i)_{i=1, \dots, k}$ ,  $\alpha_i \in \text{Irr}(\mathbb{G})$  such that  $\beta \subset \bigotimes_{i=1}^k s(\alpha_i)$  because the coefficients of the representations  $s(\alpha)$  are dense in  $C(\mathbb{H})$ . Let us denote  $\phi$  the map such that  $\phi(s(\alpha)) := a(\alpha)$ ,  $\phi(1_{\mathbb{H}}) = 1_{H^+_{(B, \psi)}(\mathbb{G})}$ . Then, it is possible to define an isomorphism

$$\phi : \text{Hom}\left(\bigotimes_{i=1}^k s(\alpha_i), \bigotimes_{j=1}^l s(\beta_j)\right) \longrightarrow \text{Hom}\left(\bigotimes_{i=1}^k a(\alpha_i), \bigotimes_{j=1}^l a(\beta_j)\right)$$

which satisfies the properties of a monoidal equivalence. The core idea in order to define this map is to find a good description of the two spaces of intertwiners. The spaces on the right have been described in terms of decorated noncrossing partitions in Theorem 2.5.7.

The intertwiners of  $\mathbb{H}$  can be described by means of semi-decorated noncrossing partitions in  $NC(3k, 3l)$  such that, when numbering each line of points from the left to the right, the points with a number equal to 0 or 2 modulo 3 form a Temperley-Lieb diagram in  $TL(2k, 2l)$  and the remaining points form a decorated noncrossing partition in  $NC((\alpha_1, \dots, \alpha_k), (\beta_1, \dots, \beta_l))$  endowed with a morphism  $S$ . This presentation can be proved by recalling that the intertwiners of  $SU_q(2)$  can be described in terms of Temperley-Lieb diagrams (such that the coefficient  $q + q^{-1}$  is introduced for every central block removed during the composition operation) and by knowing the description of the intertwiners of a free product of two compact quantum groups in terms of the intertwiners of the factors (see Proposition 2.15 in [Lem14] and observe that the result is true for every compact quantum groups, even if it is stated only for compact matrix quantum groups).

Now, in order to describe the map  $\phi$ , we recall that there is an isomorphism

$$\rho : TL_x(2k, 2l) \longrightarrow NC_{x^2}(k, l), \quad x \in \mathbb{R}^+$$

which satisfies all the compatibility properties of Definition 2.5.8 (but at the level of the diagrams). The subscripts  $x$  and  $x^2$  mean that when composing two diagrams, the final diagram is multiplied by a coefficient  $x$  or  $x^2$  for every central block appeared. More precisely,  $\rho$  acts by associating to each Temperley-Lieb diagram the noncrossing partition obtained by identifying the pairs of consecutive points and by multiplying this partition by a suitable coefficient (which depends on the diagram). This coefficient is crucial to assure the compatibility with the multiplication given by  $\rho(t_2 t_1) = \rho(t_2) \circ \rho(t_1)$  for  $t_1, t_2$  composable Temperley-Lieb diagrams.

The map  $\phi$  is then defined by sending every special diagram described above to the noncrossing partition obtained after applying the map  $\rho$  to the Temperley-Lieb diagram and decorating the points with the  $\alpha_i, \beta_j$ . Finally, this noncrossing partition is endowed with the map  $S$  (actually, some twist operations can be necessary, but for simplicity we keep the same notation). In this case, the subscript  $x$ , introduced in the definition of  $\rho$ , has to be chosen equal to  $q + q^{-1}$ , the coefficient corresponding to central blocks for  $SU_q(2)$ .

Then, every central block appeared when composing the associated noncrossing partitions, will correspond to  $(q + q^{-1})^2$  and this factor is by hypothesis equal to  $\tilde{\psi}(1)$ , the coefficient corresponding to central blocks for  $H_{(B, \tilde{\psi})}^+(\mathbb{G})$ . Thanks to this choice, it is possible to verify that  $\phi$  is a well defined isomorphism and satisfies all the properties of Definition 2.5.8.

We observe that, in order to use the isomorphism  $\rho$ , it has been crucial the dependence on the number of central blocks (instead of on the number of cycles) of the coefficient possibly appeared when composing two noncrossing partitions. This explains the use of  $\tilde{\psi}$  instead of  $\psi$ .

In order to complete the proof, it is enough to observe that the map  $\phi$  is an equivalence between the categories containing the tensor products of the generating representations of the two compact quantum groups and that there is a correspondence between these generators. By applying Proposition 1.1.32, we can extend  $\phi$

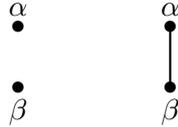
to an equivalence  $\tilde{\phi}$  between the completions of the two categories with respect to direct sums and sub-objects. The map  $\tilde{\phi}$  is in particular a bijection between the irreducible representations and the monoidal equivalence is proved.  $\square$

### 2.5.4 Irreducible representations and fusion rules

We find the irreducible representations and the fusion rules of the free wreath product  $\mathbb{G} \wr_* \mathbb{G}^{aut}(B, \psi)$ . The next result follows from Theorem 2.5.7 and is a generalisation of [LT14, Cor 3.9].

**Proposition 2.5.11.** *Let  $B$  be a finite dimensional  $C^*$ -algebra,  $\dim(B) \geq 4$ , endowed with a  $\delta$ -form  $\psi$ . Let  $\mathbb{G}$  be a compact quantum group. The basic representations  $a(\alpha)$ ,  $\alpha \in \text{Irr}(\mathbb{G})$  of  $H_{(B, \psi)}^+(\mathbb{G})$  are irreducible and pairwise non-equivalent if  $\alpha \not\simeq 1_{\mathbb{G}}$ . The representation  $a(1_{\mathbb{G}})$  can be decomposed as  $1_{H_{(B, \psi)}^+(\mathbb{G})} \oplus r_{1_{\mathbb{G}}}$ , where  $r_{1_{\mathbb{G}}}$  is irreducible and non-equivalent to any  $a(\alpha)$ ,  $\alpha \not\simeq 1_{\mathbb{G}}$ .*

*Proof.* As in [LT14], this proposition is proved looking at the dimension of the space  $\text{Hom}(a(\alpha), a(\beta))$  for  $\alpha, \beta \in \text{Irr}(\mathbb{G})$ . There are only two possible noncrossing partitions to decorate.



If  $\alpha \simeq \beta \not\simeq 1_{\mathbb{G}}$ , only the second diagram is admissible, indeed, in this case, we have that  $\dim \text{Hom}(\alpha, 1_{\mathbb{G}}) = 0$  and  $\dim \text{Hom}(\alpha, \alpha) = 1$ , so  $\dim \text{Hom}(a(\alpha), a(\alpha)) = 1$  and the irreducibility of the  $a(\alpha)$  is proved. If  $\alpha \not\simeq \beta$ , it is clear that  $\dim \text{Hom}(\alpha, \beta) = 0$  and  $\dim \text{Hom}(a(\alpha), a(\beta)) = 0$ . This proves the non-equivalence.

If  $\alpha \simeq \beta \simeq 1_{\mathbb{G}}$ , both diagrams are admissible because  $\dim \text{Hom}(1_{\mathbb{G}}, 1_{\mathbb{G}}) = 1$ . It follows that  $\dim \text{Hom}(a(1_{\mathbb{G}}), a(1_{\mathbb{G}})) = 2$ , so the linear independence of the intertwiners associated to distinct noncrossing partitions together with the remark that  $\dim \text{Hom}(1_{H_{(B, \psi)}^+(\mathbb{G})}, a(1_{\mathbb{G}})) = 1$  (the multiples of  $\eta$ ) give the decomposition.  $\square$

Now, we can describe the fusion semiring of  $H_{(B, \psi)}^+(\mathbb{G})$ . As in the discrete case, the irreducible representations will be indexed by the elements of a monoid.

**Definition 2.5.12.** Let  $M$  be the monoid whose elements are the words written by using the irreducible representations of  $\mathbb{G}$  as letters. We define the following operations:

- involution:  $\overline{(\alpha_1, \dots, \alpha_k)} = (\overline{\alpha_k}, \dots, \overline{\alpha_1})$
- concatenation:  $(\alpha_1, \dots, \alpha_k), (\beta_1, \dots, \beta_l) = (\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_l)$
- fusion of two non-empty words:  $(\alpha_1, \dots, \alpha_k) \cdot (\beta_1, \dots, \beta_l)$  is the multiset composed by the words  $(\alpha_1, \dots, \alpha_{k-1}, \gamma, \beta_2, \dots, \beta_l)$  for all the possible  $\gamma \subset \alpha_k \otimes \beta_1$ ; the multiplicity of each word is given by  $\dim(\text{Hom}(\gamma, \alpha_k \otimes \beta_1))$ , i.e. by the multiplicity of the representation  $\gamma$  in the tensor product  $\alpha_k \otimes \beta_1$ .

**Theorem 2.5.13.** *Let  $B$  be a finite dimensional  $C^*$ -algebra,  $\dim(B) \geq 4$ , endowed with a  $\delta$ -form  $\psi$ . Let  $\mathbb{G}$  be a compact quantum group. The classes of irreducible non-equivalent representations of  $H_{(B, \psi)}^+(\mathbb{G})$  can be indexed by the elements of the monoid  $M$  and denoted  $r_x$ ,  $x \in M$ . The involution is given by  $\overline{r_x} = r_{\overline{x}}$  and the fusion rules are:*

$$r_x \otimes r_y = \sum_{\substack{x=u, t \\ y=\overline{t}, v}} r_{u, v} \oplus \sum_{\substack{x=u, t \quad y=\overline{t}, v \\ u \neq \emptyset, v \neq \emptyset \\ w \in u \cdot v}} r_w$$

Because of the monoidal equivalence proved in Theorem 2.5.9, it is enough to verify that these are the fusion rules of  $\mathbb{H}$ ; this proof can be found in [LT14].

*Remark 2.16.* If  $\mathbb{G}$  is a matrix quantum group with fundamental representation  $u$ , it is possible to find a fundamental representation also for the free wreath product  $H_{(B, \psi)}^+(\mathbb{G})$ . Let  $u = \bigoplus_{i=1}^t \alpha_i$  be the decomposition of  $u$  in terms of the irreducible representations  $\alpha_i \in \text{Irr}(\mathbb{G})$ . Then,  $\bigoplus_{i=1}^t a(\alpha_i)$  is a representation of  $H_{(B, \psi)}^+(\mathbb{G})$  which can be considered as fundamental because its coefficients generates a  $C^*$ -algebra dense in  $H_{(B, \psi)}^+(\mathbb{G})$ . Indeed, for every  $r \in \text{Irr}(\mathbb{G})$ , there exists  $k \in \mathbb{N}$  such that  $r \subseteq u^{\otimes k}$ ; this implies that  $r \subseteq \bigotimes_{j=1}^k \alpha_{i_j}$ . Therefore, by making use of the fusion rules just found, we deduce that  $a(r) \subseteq \bigotimes_{j=1}^k a(\alpha_{i_j}) \subseteq (\bigoplus_{i=1}^t a(\alpha_i))^{\otimes k}$ . This proves that every generator  $a(r)$  of the free wreath product is included in a suitable tensor product of copies of the fundamental representation.

The description of the fusion semiring just given can be generalized to the case of a state  $\psi$  which is not a  $\delta$ -form, thanks to the following proposition. This result will be widely used also in what follows, in order to prove some algebraic and analytical properties in a more general framework.

**Proposition 2.5.14.** *Let  $B = \bigoplus_{T=1}^c M_{n_T}(\mathbb{C})$  be a finite dimensional  $C^*$ -algebra and  $\psi = \bigoplus_{T=1}^c \text{Tr}(Q_T \cdot)$  a state on  $B$ . Consider the decomposition  $B = \bigoplus_{i=1}^d B_i$  obtained by summing up all the matrix spaces  $M_{n_T}(\mathbb{C})$  with a common value of  $\text{Tr}(Q_T^{-1})$  in a unique summand  $B_i$  and let  $\delta_i$  be the value of such a trace. Let  $\psi_i$  be the normalized version of  $\psi|_{B_i}$ . Then*

$$\mathbb{G} \wr_* \mathbb{G}^{\text{aut}}(B, \psi) \cong \hat{*}_{i=1}^d \mathbb{G} \wr_* \mathbb{G}^{\text{aut}}(B_i, \psi_i)$$

is a  $*$ -isomorphism intertwining the comultiplications.

*Proof.* The proof consists in the explicit construction of the isomorphism. We fix the notations  $M = C(H_{(B,\psi)}^+(\mathbb{G}))$  and  $N_i = C(H_{(B_i,\psi_i)}^+(\mathbb{G}))$  for  $1 \leq i \leq d$ . Let  $a(\alpha) \in \mathcal{L}(B \otimes H_\alpha) \otimes M$ ,  $\alpha \in \text{Irr}(\mathbb{G})$  be the family of generators of  $M$  and let  $a(\alpha)_i \in \mathcal{L}(B_i \otimes H_\alpha) \otimes N_i$ ,  $\alpha \in \text{Irr}(\mathbb{G})$  be the family of generators of  $N_i$ , for  $1 \leq i \leq d$ . Let  $m, \eta$  be the multiplication and the unity of  $B$  and let  $m_i, \eta_i$  be the multiplication and the unity of  $B_i$ . Moreover, let  $\nu_i : B_i \rightarrow B$  be a family of isometries such that  $\nu_i \nu_i^*$  are pairwise orthogonal projections and  $\sum_i \nu_i \nu_i^* = \text{id}_B$ . Define the element  $v(\alpha) \in \mathcal{L}(B \otimes H_\alpha) \otimes \hat{*}_{i=1}^d N_i$  by

$$v(\alpha) = \sum_i (\nu_i \otimes \text{id}_{H_\alpha} \otimes 1) a(\alpha)_i (\nu_i^* \otimes \text{id}_{H_\alpha} \otimes 1)$$

We claim that there exists a unital  $*$ -homomorphism  $\Psi : M \rightarrow \hat{*}_{i=1}^d N_i$  such that  $(\text{id}_{B \otimes H_\alpha} \otimes \Psi) a(\alpha) = v(\alpha)$ . By the universality of the free wreath product construction it is enough to verify that

1.  $v(\alpha)$  is unitary
2.  $(m \otimes S) \Sigma_{23} \in \text{Hom}(v(\alpha) \otimes v(\beta), v(\gamma))$  for any  $\alpha, \beta, \gamma \in \text{Irr}(\mathbb{G})$  and  $S \in \text{Hom}(\alpha \otimes \beta, \gamma)$
3.  $\eta \in \text{Hom}(1, v(1))$

Let us prove (1). Since the  $\nu_i \nu_i^*$  are pairwise orthogonal we have  $\nu_i^* \nu_k = 0$  if  $i \neq k$  and  $\nu_i^* \nu_i = \text{id}_{B_i}$ . It follows that

$$\begin{aligned} v(\alpha)v(\alpha)^* &= \sum_{i,k} (\nu_i \otimes \text{id} \otimes 1) a(\alpha)_i (\nu_i^* \otimes \text{id} \otimes 1) (\nu_k \otimes \text{id} \otimes 1) a(\alpha)_k^* (\nu_k^* \otimes \text{id} \otimes 1) \\ &= \sum_i (\nu_i \otimes \text{id} \otimes 1) a(\alpha)_i a(\alpha)_i^* (\nu_i^* \otimes \text{id} \otimes 1) \\ &= \text{id}_B \otimes \text{id} \otimes 1 \end{aligned}$$

Similarly,  $v(\alpha)^* v(\alpha) = \text{id}_B \otimes \text{id} \otimes 1$ .

Let us prove (2). Observe that  $\nu_j^* m(\nu_i \otimes \nu_k) = \delta_{ik} \delta_{ij} m$  and that  $\sum_i \nu_i m_i (\nu_i^* \otimes \nu_i^*) = m$ . Then

$$\begin{aligned} ((m \otimes S) \Sigma_{23} \otimes 1) v(\alpha) \otimes v(\beta) &= \\ ((m \otimes S) \Sigma_{23} \otimes 1) \sum_{i,k} (\nu_i \otimes \text{id} \otimes \nu_k \otimes \text{id} \otimes 1) (a(\alpha)_i \otimes a(\beta)_k) (\nu_i^* \otimes \text{id} \otimes \nu_k^* \otimes \text{id} \otimes 1) &= \\ \sum_{i,k} ((m(\nu_i \otimes \nu_k) \otimes S) \Sigma_{23} \otimes 1) (a(\alpha)_i \otimes a(\beta)_k) (\nu_i^* \otimes \text{id} \otimes \nu_k^* \otimes \text{id} \otimes 1) &= \\ \sum_{i,j,k} (\nu_j \otimes \text{id} \otimes 1) ((\nu_j^* m(\nu_i \otimes \nu_k) \otimes S) \Sigma_{23} \otimes 1) (a(\alpha)_i \otimes a(\beta)_k) (\nu_i^* \otimes \text{id} \otimes \nu_k^* \otimes \text{id} \otimes 1) &= \\ \sum_i (\nu_i \otimes \text{id} \otimes 1) ((m_i \otimes S) \Sigma_{23} \otimes 1) (a(\alpha)_i \otimes a(\beta)_i) (\nu_i^* \otimes \text{id} \otimes \nu_i^* \otimes \text{id} \otimes 1) &= \\ \sum_i (\nu_i \otimes \text{id} \otimes 1) a(\gamma)_i ((m_i (\nu_i^* \otimes \nu_i^*) \otimes S) \Sigma_{23} \otimes 1) &= \\ \sum_i (\nu_i \otimes \text{id} \otimes 1) a(\gamma)_i (\nu_i^* \otimes \text{id} \otimes 1) ((\sum_k \nu_k m_k (\nu_k^* \otimes \nu_k^*) \otimes S) \Sigma_{23} \otimes 1) &= \\ v(\gamma) ((m \otimes S) \Sigma_{23} \otimes 1) \end{aligned}$$

Let us prove (3). Observe that  $\nu_i^* \eta = \eta_i$  and  $\sum_i \nu_i \eta_i = \eta$ . We have

$$\begin{aligned} v(1)(\eta \otimes 1) &= \sum_i (\nu_i \otimes 1) a(1)_i (\nu_i^* \otimes 1) (\eta \otimes 1) \\ &= \sum_i (\nu_i \otimes 1) a(1)_i (\eta_i \otimes 1) \\ &= \sum_i (\nu_i \otimes 1) (\eta_i \otimes 1) \\ &= \eta \otimes 1 \end{aligned}$$

A simple verification allows us to show that this homomorphism intertwines the comultiplications. This ends the first part of the proof.

In order to construct the inverse homomorphism we need some preliminary results.

We claim that, for all  $i$ ,  $\nu_i \nu_i^* \otimes \text{id}_{H_\alpha} \in \text{Hom}(a(\alpha), a(\alpha))$ . Consider the morphism  $\widetilde{m \otimes S} \in \text{Hom}(a(\alpha) \otimes a(1_{\mathbb{G}}), a(\alpha))$ , where  $S \in \text{Hom}(\alpha \otimes 1_{\mathbb{G}}, \alpha)$  is the identity morphism and observe that

$$\widetilde{(m \otimes S)} \circ \widetilde{(m \otimes S)}^* = m m^* \otimes \text{id}_{H_\alpha} = \sum_{i=1}^d \delta_i \cdot \nu_i \nu_i^* \otimes \text{id}_{H_\alpha} \in \text{Hom}(a(\alpha), a(\alpha))$$

For a suitable constant  $K$ , we have

$$\nu_i \nu_i^* \otimes \text{id} = K \prod_{\substack{k=1 \\ k \neq i}}^d (\delta_k \text{id}_B - \sum_l \delta_l \nu_l \nu_l^*) \otimes \text{id}$$

This implies that  $\nu_i \nu_i^* \otimes \text{id} \in \text{Hom}(a(\alpha), a(\alpha))$ .

Now, for all  $1 \leq i \leq d$  define the element  $v(\alpha)_i \in \mathcal{L}(B_i \otimes H_\alpha) \otimes M$  by

$$v(\alpha)_i = (\nu_i^* \otimes \text{id}_{H_\alpha} \otimes 1)a(\alpha)(\nu_i \otimes \text{id}_{H_\alpha} \otimes 1)$$

We claim that, for all  $i$ , there exists a unital  $*$ -homomorphism  $\Phi_i : N_i \longrightarrow M$  such that  $(\text{id}_{B_i \otimes H_\alpha} \otimes \Phi_i)a(\alpha)_i = v(\alpha)_i$ . By the universality of the  $C^*$ -algebra  $N_i$  it is enough to verify that

1.  $v(\alpha)_i$  is unitary
2.  $(m_i \otimes S)\Sigma_{23} \in \text{Hom}(v(\alpha)_i \otimes v(\beta)_i, v(\gamma)_i)$  for any  $\alpha, \beta, \gamma \in \text{Irr}(\mathbb{G})$  and  $S \in \text{Hom}(\alpha \otimes \beta, \gamma)$
3.  $\eta_i \in \text{Hom}(1, v(1)_i)$

Let us prove (1). We have

$$\begin{aligned} v(\alpha)_i v(\alpha)_i^* &= (\nu_i^* \otimes \text{id} \otimes 1)a(\alpha)(\nu_i \otimes \text{id} \otimes 1)(\nu_i^* \otimes \text{id} \otimes 1)a(\alpha)^*(\nu_i \otimes \text{id} \otimes 1) \\ &= (\nu_i^* \otimes \text{id} \otimes 1)a(\alpha)(\nu_i \nu_i^* \otimes \text{id} \otimes 1)a(\alpha)^*(\nu_i \otimes \text{id} \otimes 1) \\ &= (\nu_i^* \otimes \text{id} \otimes 1)(\nu_i \nu_i^* \otimes \text{id} \otimes 1)a(\alpha)a(\alpha)^*(\nu_i \otimes \text{id} \otimes 1) \\ &= \text{id}_{B_i} \otimes \text{id} \otimes 1 \end{aligned}$$

Similarly,  $v(\alpha)_i^* v(\alpha)_i = \text{id}_{B_i} \otimes \text{id} \otimes 1$ .

Let us prove (2). Recall that  $m_i = \nu_i^* m(\nu_i \otimes \nu_i)$ , then  $(\nu_i m_i (\nu_i^* \otimes \nu_i^*) \otimes S)\Sigma_{23} = (\nu_i \nu_i^* \otimes \text{id})(m \otimes S)\Sigma_{23}(\nu_i \nu_i^* \otimes \text{id} \otimes \nu_i \nu_i^* \otimes \text{id}) \in \text{Hom}(a(\alpha)_i \otimes a(\beta)_i, a(\gamma)_i)$ . Hence

$$\begin{aligned} ((m_i \otimes S)\Sigma_{23} \otimes 1)v(\alpha)_i \otimes v(\beta)_i &= \\ ((m_i \otimes S)\Sigma_{23} \otimes 1)(\nu_i^* \otimes \text{id} \otimes \nu_i^* \otimes \text{id} \otimes 1)(a(\alpha) \otimes a(\beta))(\nu_i \otimes \text{id} \otimes \nu_i \otimes \text{id} \otimes 1) &= \\ ((m_i (\nu_i^* \otimes \nu_i^*) \otimes S)\Sigma_{23} \otimes 1)(a(\alpha) \otimes a(\beta))(\nu_i \otimes \text{id} \otimes \nu_i \otimes \text{id} \otimes 1) &= \\ (\nu_i^* \otimes \text{id} \otimes 1)((\nu_i m_i (\nu_i^* \otimes \nu_i^*) \otimes S)\Sigma_{23} \otimes 1)(a(\alpha) \otimes a(\beta))(\nu_i \otimes \text{id} \otimes \nu_i \otimes \text{id} \otimes 1) &= \\ (\nu_i^* \otimes \text{id} \otimes 1)a(\gamma)((\nu_i m_i (\nu_i^* \otimes \nu_i^*) \otimes S)\Sigma_{23} \otimes 1)(\nu_i \otimes \text{id} \otimes \nu_i \otimes \text{id} \otimes 1) &= \\ (\nu_i^* \otimes \text{id} \otimes 1)a(\gamma)((\nu_i m_i (\nu_i^* \nu_i \otimes \nu_i^* \nu_i) \otimes S)\Sigma_{23} \otimes 1) &= \\ (\nu_i^* \otimes \text{id} \otimes 1)a(\gamma)(\nu_i \otimes \text{id} \otimes 1)((m_i \otimes S)\Sigma_{23} \otimes 1) &= \\ v(\gamma)_i((m_i \otimes S)\Sigma_{23} \otimes 1) \end{aligned}$$

Let us prove (3). Observe that  $\nu_i \eta_i = (\nu_i \nu_i^*)\eta \in \text{Hom}(1, a(1))$ . Then

$$\begin{aligned} v(1)_i(\eta_i \otimes 1) &= (\nu_i^* \otimes 1)a(1)(\nu_i \otimes 1)(\eta_i \otimes 1) \\ &= (\nu_i^* \nu_i \otimes 1)(\eta_i \otimes 1) \\ &= (\eta_i \otimes 1) \end{aligned}$$

This completes the proof of the existence of the morphism  $\Phi_i : N_i \longrightarrow M$ , for all  $i$ . Then, because of the universality of the free product construction, there exists a unital  $*$ -homomorphism  $\Phi : \ast_{i=1}^d N_i \longrightarrow M$  such that  $(\text{id}_{B_i \otimes H_\alpha} \otimes \Phi)a(\alpha)_i = v(\alpha)_i$  and it is easy to verify that this morphism intertwines the comultiplications. Finally, a simple computation allows us to prove that  $\Psi$  and  $\Phi$  are inverse to each other and this ends the proof.  $\square$

The results in [Wan95] together with this proposition allow us to describe the irreducible representations and fusion rules of the free wreath product  $\mathbb{G} \wr_* \mathbb{G}^{aut}(B, \psi)$ , for a generic state  $\psi$ .

### 2.5.5 Stability properties of the free wreath product

We present some stability results concerning the operation of free wreath product. More precisely, we prove that the free wreath product preserves the relation of monoidal equivalence and we find under which conditions two free wreath products have isomorphic fusion semiring.

We start by recalling a result from [DRVV10] about the monoidal equivalence of quantum automorphism groups.

**Theorem 2.5.15.** *Consider the quantum automorphism groups  $(C(\mathbb{G}^{aut}(B, \psi)), u)$  and  $(C(\mathbb{G}^{aut}(B', \psi')), u')$ , where  $\psi$  and  $\psi'$  are a  $\delta$ -form and a  $\delta'$ -form respectively. Then they are monoidally equivalent if and only if  $\delta = \delta'$*

All the details of the proof can be found in [DRVV10]. Anyway, we want to give a simpler demonstration of the first "if" because we will use the same technique to prove some results concerning the free wreath product.

*Proof.* We have to construct the maps and to verify the properties of Definition 2.5.8. First of all, let  $\phi$  be the bijection satisfying  $\phi(u) = u'$  and  $\phi(1) = 1$ , where  $u$  and  $u'$  are the fundamental representations. We use the same notation to denote, for every  $k, l \in \mathbb{N}$ , the map  $\phi(T_p) = T'_p$ , where  $T_p$  is the morphism in  $\text{Hom}_{\mathbb{G}^{aut}(B, \psi)}(u^{\otimes k}, u^{\otimes l})$  associated to a noncrossing partition  $p \in NC(k, l)$  and  $T'_p$  is the morphism in  $\text{Hom}_{\mathbb{G}^{aut}(B', \psi')}(u'^{\otimes k}, u'^{\otimes l})$  associated to the same noncrossing

partition  $p$ . As the  $T_p$  and the  $T'_p$  are a basis of the respective spaces of intertwiners,  $\phi$  can be extended by linearity to an isomorphism

$$\phi : \text{Hom}_{\mathbb{G}^{aut}(B, \psi)}(u^{\otimes k}, u^{\otimes l}) \longrightarrow \text{Hom}_{\mathbb{G}^{aut}(B', \psi')}(u'^{\otimes k}, u'^{\otimes l})$$

It is clear that  $\phi(\text{id}) = \text{id}$  because  $\Phi(T_{|\dots|}) = T'_{|\dots|}$ , where  $|\dots|$  is the noncrossing partition in  $NC(k, k)$  which connects each of the  $k$  upper points to the respective lower point.

The second property which is required is the compatibility with the tensor product:  $\Phi(P \otimes Q) = \Phi(P) \otimes \Phi(Q)$  for all  $P, Q$  morphisms. For  $P = T_p$  and  $Q = T_q$  we have  $\Phi(T_p \otimes T_q) = \Phi(T_{p \otimes q}) = T'_{p \otimes q} = T'_p \otimes T'_q = \Phi(T_p) \otimes \Phi(T_q)$ . The result holds for all the pairs  $P, Q$  of morphisms by linearity of  $\phi$ .

The third property is the compatibility with respect to the adjoint:  $\Phi(P^*) = \Phi(P)^*$  for all morphisms  $P$ . If  $P = T_p$  we have  $\Phi(T_p^*) = \Phi(T_{p^*}) = T'_{p^*} = T'_p{}^* = \Phi(T_p)^*$ . The result holds for all morphisms  $P$  by linearity of  $\phi$ .

The compatibility with the composition is a little more subtle and it is the part of the proof where the hypothesis  $\delta = \delta'$  is used. We want to prove that  $\phi(S \circ R) = \phi(S) \circ \phi(R)$  for all  $R, S$  composable morphisms. Suppose  $S = T_p$  and  $R = T_q$ , then we have  $\phi(T_p \circ T_q) = \phi(\delta^{cy(p,q)} T_{pq}) = \delta^{cy(p,q)} T'_{pq} = T'_p \circ T'_q = \phi(T_p) \circ \phi(T_q)$ , where the second to last equality is true only because of the assumption  $\delta = \delta'$ . The result holds for all the pairs of composable morphisms by linearity of  $\phi$ .

In order to complete the proof, we observe that  $\phi$  is an equivalence between the categories

$$\mathcal{C} = \{(u^{\otimes k}, k \in \mathbb{N}), (\text{Hom}(u^{\otimes k}, u^{\otimes l}), k, l \in \mathbb{N})\}$$

and

$$\mathcal{D} = \{(u'^{\otimes k}, k \in \mathbb{N}), (\text{Hom}(u'^{\otimes k}, u'^{\otimes l}), k, l \in \mathbb{N})\}$$

Then, by applying Proposition 1.1.32, it is possible to extend  $\phi$  to an equivalence  $\tilde{\phi}$  between the completion of the two categories with respect to direct sums and sub-objects.

Moreover, the map  $\tilde{\phi}$  can be restricted to a bijection

$$\tilde{\phi}|_{\text{Irr}(\mathbb{G}^{aut}(B, \psi))} : \text{Irr}(\mathbb{G}^{aut}(B, \psi)) \longrightarrow \text{Irr}(\mathbb{G}^{aut}(B', \psi'))$$

and the monoidal equivalence is proved. □

**Theorem 2.5.16.** *Let  $\mathbb{G}_1$  and  $\mathbb{G}_2$  be two monoidally equivalent compact quantum groups. Let  $B, B'$  be two finite dimensional  $C^*$ -algebras of dimension at least 4 endowed with the  $\delta$ -form  $\psi$  and the  $\delta'$ -form  $\psi'$  respectively. Suppose that the associated quantum automorphism groups  $\mathbb{G}^{aut}(B, \psi)$  and  $\mathbb{G}^{aut}(B', \psi')$  are monoidally equivalent. Then*

$$H_{(B, \psi)}^+(\mathbb{G}_1) \simeq_{mon} H_{(B', \psi')}^+(\mathbb{G}_2)$$

*The monoidal equivalence is preserved by the free wreath product by a quantum automorphism group.*

*Proof.* Let  $\phi : \text{Irr}(\mathbb{G}_1) \rightarrow \text{Irr}(\mathbb{G}_2)$  be the map which establishes the monoidal equivalence between  $\mathbb{G}_1$  and  $\mathbb{G}_2$ . The proof is divided into two parts: firstly, we define the map  $\Phi$  (which satisfies the properties of the monoidal equivalence) on the basic representations which generate the two free wreath products and later on we will observe that we can extend  $\Phi$  to all the irreducible representations. Let us denote by  $a(\alpha), \alpha \in \text{Irr}(\mathbb{G}_1)$  the basic representations of  $H_{(B, \psi)}^+(\mathbb{G}_1)$ . Then, we define the following bijection:

$$\Phi(a(\alpha)) = a(\phi(\alpha)) \quad \Phi(1) = 1$$

We use the same notation to define, for every  $\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_l \in \text{Irr}(\mathbb{G}_1)$ , the map

$$\Phi(T_{p,S}) = T'_{p,\phi(S)}$$

where  $T_{p,S}$  is the morphism in  $\text{Hom}_{H_{(B, \psi)}^+(\mathbb{G}_1)}(a(\alpha_1) \otimes \dots \otimes a(\alpha_k), a(\beta_1) \otimes \dots \otimes a(\beta_l))$  associated to a noncrossing partition  $p \in NC_{\mathbb{G}_1}(\alpha_1, \dots, \alpha_k; \beta_1, \dots, \beta_l)$  decorated with an intertwiner  $S$  and  $T'_{p,\phi(S)}$  is the morphism in  $\text{Hom}_{H_{(B', \psi')}^+(\mathbb{G}_2)}(a(\phi(\alpha_1)) \otimes \dots \otimes a(\phi(\alpha_k)), a(\phi(\beta_1)) \otimes \dots \otimes a(\phi(\beta_l)))$  associated to the same noncrossing partition  $p$ , decorated with the intertwiner  $\phi(S)$ . Now, we recall that the morphisms of type  $T_{p,S}$  and  $T'_{p,\phi(S)}$  span the respective spaces of intertwiners, the maps  $T_p$  ( $T'_p$ ) associated to distinct noncrossing partitions are linearly independent,  $\phi$  is an isomorphism and we have Lemma 2.5.6. The map  $\Phi$  can then be extended by linearity

to the following isomorphism:

$$\Phi : \text{Hom}_{H_{(B,\psi)}^+(\mathbb{G}_1)}(a(\alpha_1) \otimes \dots \otimes a(\alpha_k), a(\beta_1) \otimes \dots \otimes a(\beta_l)) \longrightarrow \\ \text{Hom}_{H_{(B',\psi')}^+(\mathbb{G}_2)}(a(\phi(\alpha_1)) \otimes \dots \otimes a(\phi(\alpha_k)), a(\phi(\beta_1)) \otimes \dots \otimes a(\phi(\beta_l)))$$

Moreover, the properties required by Definition 2.5.8 are verified.

The first condition  $\Phi(\text{id}) = \text{id}$  is clear because  $\Phi(T_{|\dots|,\text{id}}) = T'_{|\dots|,\phi(\text{id})}$  and  $\phi(\text{id}) = \text{id}$ . The second property which is required is the compatibility with the tensor product:  $\Phi(P \otimes Q) = \Phi(P) \otimes \Phi(Q)$  for all  $P, Q$  morphisms. If  $P = T_{p,S}$  and  $Q = T_{q,R}$  we have  $\Phi(T_{p,S} \otimes T_{q,R}) = \Phi(T_{p \otimes q, S \otimes R}) = T'_{p \otimes q, \phi(S \otimes R)} = T'_{p \otimes q, \phi(S) \otimes \phi(R)} = T'_{p, \phi(S)} \otimes T'_{q, \phi(R)} = \Phi(T_{p,S}) \otimes \Phi(T_{q,R})$ . The results holds for all the pairs  $P, Q$  of morphisms by linearity of  $\Phi$ .

The third property is the compatibility with respect to the adjoint:  $\Phi(P^*) = \Phi(P)^*$  for all morphisms  $P$ . If  $P = T_{p,S}$  we have  $\Phi(T_{p,S}^*) = \Phi(T_{p^*, S^*}) = T'_{p^*, \phi(S^*)} = T'_{p^*, \phi(S)^*} = T'_{p, \phi(S)}^* = \Phi(T_{p,S})^*$ . The results holds for all the morphisms  $P$  by linearity of  $\Phi$ .

The last condition is the compatibility with respect to the composition:  $\Phi(P \circ Q) = \Phi(P) \circ \Phi(Q)$  for all composable morphisms  $P, Q$ . We observe that, because of Theorem 2.5.15, we have  $\delta = \delta'$ . Now, suppose that  $P = T_{p,S}$  and  $Q = T_{q,R}$ . We have  $\Phi(T_{p,S} \circ T_{q,R}) = \Phi(\delta^{cy(p,q)} T_{qp,RS}) = \delta^{cy(p,q)} T'_{qp, \phi(RS)} = \delta^{cy(p,q)} T'_{qp, \phi(R) \circ \phi(S)} = T'_{p, \phi(R)} \circ T'_{q, \phi(S)} = \Phi(T_{p,R}) \circ \Phi(T_{q,S})$ . The results holds for all the pairs  $P, Q$  of composable morphisms by linearity of  $\Phi$ .

As in the proof of Theorem 2.5.15, we can observe that  $\Phi$  is an equivalence between the categories

$$\mathcal{C} = \left\{ \left( \bigotimes_{i=1}^k a(\alpha_i), \alpha_i \in \text{Irr}(\mathbb{G}_1), k \in \mathbb{N} \right), \left( \text{Hom} \left( \bigotimes_{i=1}^k a(\alpha_i), \bigotimes_{j=1}^l a(\beta_j) \right), \alpha_i, \beta_j \in \text{Irr}(\mathbb{G}_1) \right) \right\}$$

and

$$\mathcal{D} = \left\{ \left( \bigotimes_{i=1}^k a(\alpha_i), \alpha_i \in \text{Irr}(\mathbb{G}_2), k \in \mathbb{N} \right), \left( \text{Hom} \left( \bigotimes_{i=1}^k a(\alpha_i), \bigotimes_{j=1}^l a(\beta_j) \right), \alpha_i, \beta_j \in \text{Irr}(\mathbb{G}_2) \right) \right\}$$

Then, by applying Proposition 1.1.32, it is possible to extend  $\Phi$  to an equivalence  $\tilde{\Phi}$  between the completion of the two categories with respect to the direct sums

and the sub-objects.

Moreover, the representations generating the two free wreath products are in correspondence so we can restrict  $\tilde{\Phi}$  to the bijection

$$\tilde{\Phi}|_{\text{Irr}(H_{(B,\psi)}^+(\mathbb{G}_1))} : \text{Irr}(H_{(B,\psi)}^+(\mathbb{G}_1)) \longrightarrow \text{Irr}(H_{(B,\psi)}^+(\mathbb{G}_2))$$

and the monoidal equivalence is proved. □

**Theorem 2.5.17.** *Let  $\mathbb{G}_1$  and  $\mathbb{G}_2$  be two compact quantum groups. Suppose that there exists an isomorphism  $\phi : R^+(\mathbb{G}_1) \longrightarrow R^+(\mathbb{G}_2)$  of their fusion semirings and that  $\phi$  restricted to  $\text{Irr}(\mathbb{G}_1)$  is a bijection of  $\text{Irr}(\mathbb{G}_1)$  onto  $\text{Irr}(\mathbb{G}_2)$ . Let  $B, B'$  be two finite dimensional  $C^*$ -algebras of dimension at least 4 endowed with the  $\delta$ -form  $\psi$  and the  $\delta'$ -form  $\psi'$  respectively. Then, the fusion semirings remain isomorphic when passing to the free wreath product by a quantum automorphism group*

$$R^+(H_{(B,\psi)}^+(\mathbb{G}_1)) \cong R^+(H_{(B',\psi')}^+(\mathbb{G}_2))$$

and the isomorphism is still a bijection between the spaces of the irreducible representations.

*Proof.* We recall that the irreducible representations of the free wreath product  $H_{(B,\psi)}^+(\mathbb{G}_1)$  can be indexed by the elements of the monoid  $M$ , i.e. by the words written using as letters the irreducible representations of  $\mathbb{G}_1$ . The monoid is endowed with the three operations of involution, concatenation and fusion introduced in Definition 2.5.12. We will denote  $r_x, x \in M$  the irreducible representations. Let  $\Phi$  be the map given by  $\Phi(r_x) = r_{\phi(x)}$ , where for every word  $x = (\alpha_1, \dots, \alpha_k) \in M$ , we define  $\phi(x) := (\phi(\alpha_1), \dots, \phi(\alpha_k))$ . We observe that the map  $\Phi$  can be extended by additivity to

$$\Phi : R^+(H_{(B,\psi)}^+(\mathbb{G}_1)) \longrightarrow R^+(H_{(B',\psi')}^+(\mathbb{G}_2))$$

Moreover,  $\Phi$  is an isomorphism because  $\phi$  is a bijection and  $\text{Irr}(\mathbb{G})$  is a basis of  $R^+(\mathbb{G})$ . Then, the proof reduces to show that, for all  $x, y \in M$ , we have

$$\Phi(r_x \otimes r_y) = \Phi(r_x) \otimes \Phi(r_y) \tag{2.14}$$

$$\Phi(\overline{r_x}) = \overline{\Phi(r_x)} \tag{2.15}$$

For this verification it is necessary to state a preliminary result which assures the compatibility between the map  $\phi$  and the operations of the monoid  $M$ : more precisely, we need to prove that, for all  $u, v \in M$ ,  $\phi(u, v) = \phi(u), \phi(v)$ ,  $\phi(\bar{u}) = \overline{\phi(u)}$  and  $\phi(u.v) = \phi(u).\phi(v)$ . To this aim, let  $u = (\alpha_1, \dots, \alpha_k)$  and  $v = (\beta_1, \dots, \beta_l)$ . It follows that  $\phi(u, v) = \phi((\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_l)) = (\phi(\alpha_1), \dots, \phi(\alpha_k), \phi(\beta_1), \dots, \phi(\beta_l)) = \phi(u), \phi(v)$ .

Similarly  $\phi(\bar{u}) = \phi((\bar{\alpha}_k, \dots, \bar{\alpha}_1)) = (\phi(\bar{\alpha}_k), \dots, \phi(\bar{\alpha}_1)) = (\overline{\phi(\alpha_k)}, \dots, \overline{\phi(\alpha_1)}) = \overline{\phi(u)}$ .

For the third relation we have

$$\begin{aligned} \phi(u.v) &= \phi(\bigoplus_{\gamma \subset \alpha_k \otimes \beta_1} (\alpha_1, \dots, \alpha_{k-1}, \gamma, \beta_2, \dots, \beta_l)) \\ &= \bigoplus_{\phi(\gamma) \subset \phi(\alpha_k \otimes \beta_1)} (\phi(\alpha_1), \dots, \phi(\alpha_{k-1}), \phi(\gamma), \phi(\beta_2), \dots, \phi(\beta_l)) \\ &= \bigoplus_{\phi(\gamma) \subset \phi(\alpha_k) \otimes \phi(\beta_1)} (\phi(\alpha_1), \dots, \phi(\alpha_{k-1}), \phi(\gamma), \phi(\beta_2), \dots, \phi(\beta_l)) \\ &= \phi((\alpha_1, \dots, \alpha_k)).\phi((\beta_1, \dots, \beta_l)) \end{aligned}$$

Now, we are ready to prove equation 2.14. We have

$$\begin{aligned} \Phi(r_x \otimes r_y) &= \Phi(\sum_{\substack{x=u,t \\ y=\bar{t},v}} r_{u,v} \oplus \sum_{\substack{x=u,t \\ y=\bar{t},v \\ u \neq \emptyset, v \neq \emptyset}} r_{u,v}) \\ &= \sum_{\substack{x=u,t \\ y=\bar{t},v}} \Phi(r_{u,v}) \oplus \sum_{\substack{x=u,t \\ y=\bar{t},v \\ u \neq \emptyset, v \neq \emptyset}} \Phi(r_{u,v}) \\ &= \sum_{\substack{\phi(x)=\phi(u,t) \\ \phi(y)=\phi(\bar{t},v)}} r_{\phi(u,v)} \oplus \sum_{\substack{\phi(x)=\phi(u,t) \\ \phi(y)=\phi(\bar{t},v) \\ \phi(u) \neq \emptyset, \phi(v) \neq \emptyset}} r_{\phi(u,v)} \\ &= \sum_{\substack{\phi(x)=\phi(u), \phi(t) \\ \phi(y)=\phi(\bar{t}), \phi(v)}} r_{\phi(u), \phi(v)} \oplus \sum_{\substack{\phi(x)=\phi(u), \phi(t) \\ \phi(y)=\phi(\bar{t}), \phi(v) \\ \phi(u) \neq \emptyset, \phi(v) \neq \emptyset}} r_{\phi(u), \phi(v)} \\ &= r_{\phi(x)} \otimes r_{\phi(y)} \\ &= \Phi(r_x) \otimes \Phi(r_y) \end{aligned}$$

The relation 2.15 is clear because  $\Phi(\overline{r_x}) = \Phi(r_{\bar{x}}) = r_{\phi(\bar{x})} = r_{\overline{\phi(x)}} = \overline{r_{\phi(x)}} = \overline{\Phi(r_x)}$ .  $\square$

### 2.5.6 Algebraic and analytic properties

The monoidal equivalence result proved in Theorem 2.5.9 allows us to prove some properties of the reduced C\*-algebra and of the von Neumann C\*-algebra associated to a free wreath product. Before taking into account these properties,

we state some preliminary results which will be useful in what follows. First of all, we observe that an analogue of Proposition 2.5.14 holds when dealing with the associated operator algebras.

*Remark 2.17.* By using the free wreath product decomposition introduced in Proposition 2.5.14 and a classic result of Wang [Wan95], we observe that the Haar measure of  $H_{(B,\psi)}^+(\mathbb{G})$  is the free product of the Haar measures of its factors. Hence, the following isomorphisms hold:

$$(C_r(H_{(B,\psi)}^+(\mathbb{G})), h) \cong \underset{\text{red } i=1}{*}^k (C_r(H_{(B_i,\psi_i)}^+(\mathbb{G})), h_i)$$

$$(L^\infty(H_{(B,\psi)}^+(\mathbb{G})), h) \cong \underset{i=1}{*}^k (L^\infty(H_{(B_i,\psi_i)}^+(\mathbb{G})), h_i)$$

where  $h$  and  $h_i$  are the Haar states on the respective C\*-algebras.

We observe also that the free wreath product of compact quantum groups of Kac type is still of Kac type.

**Proposition 2.5.18.** *If  $\mathbb{G}$  is a compact quantum group of Kac type and  $\psi$  is a  $\delta$ -trace, then the free wreath product  $H_{(B,\psi)}^+(\mathbb{G})$  is of Kac type.*

*Proof.* First of all, we recall that  $\mathbb{G}^{aut}(B, \psi)$  is of Kac type, if  $\psi$  is a  $\delta$ -trace (see [Wan98]). Therefore, the proposition can be read as a sort of stability of the free wreath product with respect to the property of being Kac.

In order to show that  $H_{(B,\psi)}^+(\mathbb{G})$  is of Kac type, we will prove that the antipode is involutory. According to Woronowicz, the antipode is given by the map  $S : H_{(B,\psi)}^+(\mathbb{G}) \longrightarrow H_{(B,\psi)}^+(\mathbb{G}), a_{ij,R}^{kl,Z}(\alpha_{pq}) \mapsto a_{kl,Z}^{ij,R}(\alpha_{qp})^*$ . By applying the definition, we have that

$$\begin{aligned} S^2(a_{ij,R}^{kl,Z}(\alpha_{pq})) &= S(a_{kl,Z}^{ij,R}(\alpha_{qp})^*) \\ &= S\left(\left(\frac{Q_{k,Z}Q_{i,R}}{Q_{i,R}Q_{l,Z}}\right)^{\frac{1}{2}} \frac{\lambda_{p,\alpha}}{\lambda_{q,\alpha}} a_{lk,Z}^{ji,R}(\bar{\alpha}_{qp})\right) \\ &= \left(\frac{Q_{k,Z}Q_{i,R}}{Q_{i,R}Q_{l,Z}}\right)^{\frac{1}{2}} \frac{\lambda_{p,\alpha}}{\lambda_{q,\alpha}} a_{ji,R}^{lk,Z}(\bar{\alpha}_{pq})^* \\ &= \frac{Q_{k,Z}Q_{j,R}}{Q_{i,R}Q_{l,Z}} \frac{\lambda_{p,\alpha}}{\lambda_{q,\alpha}} \frac{\lambda_{q,\bar{\alpha}}}{\lambda_{p,\bar{\alpha}}} a_{ij,R}^{kl,Z}(\alpha_{pq}) \end{aligned}$$

Therefore, in order to prove that the free wreath product is Kac, we have to show that  $\frac{Q_{k,Z}Q_{j,R}}{Q_{i,R}Q_{l,Z}} \frac{\lambda_{p,\alpha}}{\lambda_{q,\alpha}} \frac{\lambda_{q,\bar{\alpha}}}{\lambda_{p,\bar{\alpha}}} = 1$ .

Firstly, we observe that  $\psi(e_{ij,R}e_{kl,Z}) = \delta_{RZ}\delta_{jk}\psi(e_{il,R}) = \delta_{RZ}\delta_{jk}\delta_{il}Q_{i,R}$  and  $\psi(e_{kl,Z}e_{ij,R}) =$

$\delta_{RZ}\delta_{il}\psi(e_{kj,R}) = \delta_{RZ}\delta_{jk}\delta_{il}Q_{k,R}$ . If  $\psi$  is a  $\delta$ -trace, the resulting values must be equal. This implies that, for every  $R$  and every  $i, j$ , we have  $Q_{i,R} = Q_{j,R}$ , i.e. that the matrices  $Q_R$  characterizing  $\psi$  are multiples of the identity. In particular, this means that  $\frac{Q_{k,Z}Q_{i,R}}{Q_{i,R}Q_{i,Z}} = 1$ .

In order to complete the proof, it is enough to observe that  $\lambda_{i,\alpha} = 1$  for all  $i, \alpha$  because  $\mathbb{G}$  is of Kac type (see Remark 1.5).  $\square$

We can now prove some approximation properties by making use of some results in [DCFY14]. First of all, we recall some definitions.

**Definition 2.5.19.** Let  $(C(\mathbb{G}), \Delta)$  be a full compact quantum group. The space of the functionals  $C(\mathbb{G})^*$ , endowed with the multiplication  $(\phi_1\phi_2)(x) = (\phi_1 \otimes \phi_2)\Delta(x)$  for  $\phi_1, \phi_2 \in C(\mathbb{G})^*$ ,  $x \in C(\mathbb{G})$  and the involution  $\phi^*(x) = \overline{\phi(S(x^*))}$  for  $\phi \in C(\mathbb{G})^*$ ,  $x \in C(\mathbb{G})$  is a unital associative  $*$ -algebra.

A functional in  $C(\mathbb{G})^*$  is said to be central if it commutes with all the other functionals.

If  $\phi \in C(\mathbb{G})^*$  is central, the map  $T_\phi : C(\mathbb{G}) \longrightarrow C(\mathbb{G})$  given by  $T_\phi = (\phi \otimes \text{id})\Delta$  is called the central multiplier associated to  $\phi$ .

**Definition 2.5.20** ([DCFY14]). A discrete quantum group  $\widehat{\mathbb{G}}$  is said to have the central almost completely positive approximation property (central ACPAP) if there is a net of central functionals  $(\varphi_\lambda)_{\lambda \in I}$  on  $C(\mathbb{G})$  such that:

- the operator  $T_{\varphi_\lambda} = (\varphi_\lambda \otimes \text{id}) \circ \Delta$  induces a unital completely positive map on  $C_r(\mathbb{G})$  for every  $\lambda \in I$
- the operator  $T_{\varphi_\lambda}$  is approximated in the cb-norm by finitely supported central multipliers for every  $\lambda \in I$
- $\lim_{\lambda \in I} \varphi_\lambda(\chi_\alpha) \dim(\alpha)^{-1} = 1$  for every  $\alpha \in \text{Irr}(\mathbb{G})$

*Remark 2.18.* The central ACPAP is preserved under monoidal equivalence. Moreover, the central ACPAP is equivalent to the ACPAP (where the centrality condition is not required), if  $\mathbb{G}$  is of Kac type.

**Proposition 2.5.21.** *Let  $B$  be a finite dimensional  $C^*$ -algebra endowed with a state  $\psi$ . Let  $\widehat{\mathbb{G}}$  be a discrete quantum group with the central ACPAP and consider the free wreath product  $H_{(B,\psi)}^+(\mathbb{G})$ . Suppose that, in the free product decomposition  $\widehat{*}_{i=1}^k H_{(B_i,\psi_i)}^+(\mathbb{G})$ , we have  $\dim(B_i) \geq 4$  for all  $i$ . Then, the dual of  $H_{(B,\psi)}^+(\mathbb{G})$  has the central ACPAP.*

*Proof.* From [DCFY14, Proposition 24] we know that the central ACPAP is preserved by the operation of free product so it is enough to prove the result when  $\psi$  is a  $\delta$ -form. In this case, the free wreath product is monoidally equivalent to a quantum subgroup of  $\mathbb{G} \widehat{*} SU_q(2)$ ,  $q \in (0, 1]$  by Theorem 2.5.9. Now, the dual of  $SU_q(2)$ ,  $q \in [-1, 1]$ ,  $q \neq 0$  has the central ACPAP ([DCFY14, Theorem 25]) so the free product has the central ACPAP. This property is also preserved when passing to quantum subgroups ([DCFY14, Lemma 23]) so the proof is complete.  $\square$

The Haagerup property is implied by the central ACPAP, therefore we have the following corollary.

**Corollary 2.5.22.** *Consider the assumptions and the notations of Proposition 2.5.21. Then, the von Neumann algebra  $L^\infty(H_{(B,\psi)}^+(\mathbb{G}))$  has the Haagerup property and the  $W^*$ -CCAP.*

**Example 2.5.23.** From [DCFY14], we know that the dual of  $SU_q(2)$  with  $q \in [-1, 1]$ ,  $q \neq 0$  as well as the duals of the free unitary and orthogonal quantum groups  $U^+(F)$  and  $O^+(F)$  with  $\dim(F) \geq 2$ , have the central ACPAP. It follows that, for any finite dimensional  $C^*$ -algebra  $B$ ,  $\dim(B) \geq 4$  endowed with a  $\delta$ -form  $\psi$ , the von Neumann  $C^*$ -algebras  $L^\infty(H_{(B,\psi)}^+(U^+(F)))$ ,  $L^\infty(H_{(B,\psi)}^+(O^+(F)))$  and  $L^\infty(H_{(B,\psi)}^+(SU_q(2)))$  have the Haagerup property.

By means of similar stability results it is possible to prove the exactness. The definition of exactness is given in terms of the dual of a compact quantum group. This definition is, however, equivalent to a second one, a sort of characterisation which we will adopt as definition in this context.

**Definition 2.5.24.** A discrete quantum group  $\widehat{\mathbb{G}}$  is said to be exact if and only if its reduced  $C^*$ -algebra  $C_r(\mathbb{G})$  is exact, i.e. if the tensor product operation  $C_r(\mathbb{G}) \otimes_{\min} \cdot$  sends short exact sequences in short exact sequences.

**Proposition 2.5.25.** *Let  $B$  be a finite dimensional  $C^*$ -algebra endowed with a state  $\psi$ . Let  $\widehat{\mathbb{G}}$  be an exact discrete quantum group and consider the reduced  $C^*$ -algebra  $C_r(H_{(B,\psi)}^+(\mathbb{G}))$  with free product decomposition  $\ast_{red_i=1}^k C_r(H_{(B_i,\psi_i)}^+(\mathbb{G}))$ . If  $\dim(B_i) \geq 4$  for all  $i$ , then the dual of  $H_{(B,\psi)}^+(\mathbb{G})$  is exact.*

*Proof.* According to Definition 2.5.24, the proof consists in showing that the reduced  $C^*$ -algebra  $C_r(H_{(B,\psi)}^+(\mathbb{G}))$  is exact. In [Dyk04] it is proved that the exactness is preserved by reduced free products, so it is enough to show the result for the factors  $C_r(\mathbb{G} \wr_* \mathbb{G}^{aut}(B_i, \psi_i))$ . Now, we use that exactness is conserved under monoidal equivalence (see [VV07]), so we only need to prove the exactness of  $C_r(\mathbb{H})$ . This is a subalgebra of a free product whose factors are exact:  $\widehat{\mathbb{G}}$  is exact by hypothesis and  $\widehat{SU_q(2)}$  is exact as a consequence of its amenability. Exactness passes to subalgebras and free products so  $C_r(\mathbb{H})$  is exact.  $\square$

**Example 2.5.26.** Let  $B$  a finite dimensional  $C^*$ -algebra,  $\dim(B) \geq 4$ , endowed with a  $\delta$ -form  $\psi$ . It is well known that, if  $G$  is a compact group, the reduced  $C^*$ -algebra of the commutative quantum group  $\mathbb{G} = (C(G), \Delta)$  is exact. Then, from the proposition above it follows that the dual of  $H_{(B,\psi)}^+(\mathbb{G})$  is exact.

Moreover,  $SU_q(2)$ ,  $q \in (-1, 1)$ ,  $q \neq 0$  is exact, so the dual of  $H_{(B,\psi)}^+(SU_q(2))$  is exact. In [VV07] it is proved that also  $O^+(F)$ , for  $\dim(F) \geq 3$  is exact; the exactness of the dual of  $H_{(B,\psi)}^+(O^+(F))$  follows.

Now, we prove, under some ulterior hypothesis, the simplicity of the reduced  $C^*$ -algebra and the uniqueness of the trace.

**Proposition 2.5.27.** *Let  $B$  be a finite dimensional  $C^*$ -algebra endowed with a trace  $\psi$ . Let  $\mathbb{G}$  be a compact quantum group of Kac type. Consider the reduced  $C^*$ -algebra  $C_r(H_{(B,\psi)}^+(\mathbb{G}))$  and its free product decomposition  $\ast_{red_i=1}^k C_r(H_{(B_i,\psi_i)}^+(\mathbb{G}))$ . If there is either only one factor (i.e.  $\psi$  is a  $\delta$ -trace) and  $\dim(B) \geq 8$  or there are two or more factors with  $\dim(B_i) \geq 4$  for all  $i$ , then  $C_r(H_{(B,\psi)}^+(\mathbb{G}))$  is simple with unique trace.*

*Proof.* The techniques and the results introduced for the proof of the analogous result in the case of a discrete group (see Proposition 2.4.12) can be applied also

here. In particular, if there are two or more factors we can still use a proposition of Avitzour (see [Avi82, Section 3]). It states that, given two C\*-algebras  $A$  and  $A'$  endowed with tracial Haar states  $h_A$  and  $h_{A'}$ , the reduced free product C\*-algebra  $A \underset{\text{red}}{*} A'$  is simple with unique trace if there exist two unitary elements of  $\text{Ker}(h_A)$  orthogonal with respect to the scalar product induced by  $h_A$  and a unitary element in  $\text{Ker}(h_{A'})$ . In order to show that, in our case, these elements exist we use a result from [DHR97, Proposition 4.1 (i)] according to which, if a C\*-algebra  $A$  endowed with a normalized trace  $\tau$  admits an abelian sub-C\*-algebra  $F$  so that the spectral measure corresponding to  $\tau|_F$  is diffuse, then there is a unitary element  $u \in A$  such that  $\tau(u^n) = 0$  for each  $n \in \mathbb{Z}, n \neq 0$ . Thanks to Remark 2.15, we know that the spectral measure of the character of the generator  $a(1_{\mathbb{G}})$  of an indecomposable free wreath product  $\mathbb{G} \wr_* \mathbb{G}^{aut}(B, \psi)$ ,  $\psi$   $\delta$ -trace, is the free Poisson law of parameter 1 which is diffuse. Then, we can find some elements which satisfy the Avitzour's condition and the proof of the simplicity and of the uniqueness of the trace in the multifactor case is then completed.

In the second case, when  $\psi$  is a  $\delta$ -trace and there is not a free product decomposition, the proof is still a generalisation of the proof presented in [Lem14, Theorem 3.5]. Moreover, by making use of a trick introduced in [Wah14], where the proof is extended to the case of  $\mathbb{G} \wr_* S_n^+$ , it is possible to remove the assumption of Lemeux on the minimum number of irreducible representations of  $\mathbb{G}$ .  $\square$

### 2.5.7 The free wreath product of two quantum automorphism groups

We recall that  $G(X)$ , the group of symmetries of a graph  $X$  with  $n$  vertices, can be seen as a quotient of the symmetric group  $S_n$ . Moreover, when dealing with the usual notion of wreath product, we can give a sort of geometric interpretation thanks to formulas such as

$$G(X * Y) \cong G(X) \wr G(Y)$$

for a suitable notion of product  $*$  and only for graphs satisfying certain conditions. The definition of free wreath product by a quantum permutation group given by

Bichon allows us to find a quantum analogue of these results. More precisely, if we denote by  $G^+(X)$  the group of the quantum symmetries of a finite graph  $X$ , we have formulas such as

$$G^+(X * Y) \cong G^+(X) \wr_* G^+(Y)$$

for a suitable notion of  $*$  and some assumptions on the graphs (see [Bic04, BB07, Cha15]).

In this paper, we introduced the notion of free wreath product by a quantum automorphism group, generalising the previous one. Therefore, our aim is to find a sort of analogue of these formulas in this more general context. We observe that, while the quantum group  $S_n^+$  can be seen as the symmetry group of a graph composed by  $n$  vertices and no edges, such an interpretation is not possible for a quantum automorphism group. We will think of  $\mathbb{G}^{aut}(B, \psi)$  as being the group of the quantum symmetries of a finite quantum measured space.

Now, we prove some general results which will be fundamental for the proof of such a formula. First of all, we introduce some notations.

Let  $B$  be a finite dimensional  $C^*$ -algebra endowed with a  $\delta$ -form  $\psi$ . Let  $m$  and  $\eta$  be the multiplication and unity on  $B$  respectively. Consider the free wreath product  $H_{(B, \psi)}^+(\mathbb{G})$  of a compact quantum group  $\mathbb{G}$  by the quantum automorphism group  $\mathbb{G}^{aut}(B, \psi)$ . Choose a complete set of irreducible representations  $u_\alpha \in \mathcal{L}(H_\alpha) \otimes C(\mathbb{G})$ ,  $\alpha \in \text{Irr}(\mathbb{G})$ . We recall that  $C(H_{(B, \psi)}^+(\mathbb{G}))$  is generated by the coefficients of a family of unitary representations  $a(\alpha)$ ,  $\alpha \in \text{Irr}(\mathbb{G})$  such that  $\eta \in \text{Hom}(1, a(1))$  and, for all  $\alpha, \beta, \gamma \in \text{Irr}(\mathbb{G})$  and all  $S \in \text{Hom}(u_\alpha \otimes u_\beta, u_\gamma)$ , we have a morphism  $a(S) := \widetilde{m \otimes S} = (m \otimes S) \circ \Sigma_{23} \in \text{Hom}(a(\alpha) \otimes a(\beta), a(\gamma))$ .

For any finite dimensional representation  $u \in \mathcal{L}(H_u) \otimes C(\mathbb{G})$  we define an element  $a(u) \in \mathcal{L}(B \otimes H_u) \otimes C(H_{(B, \psi)}^+(\mathbb{G}))$  in the following way. For all  $\alpha \in \text{Irr}(\mathbb{G})$  such that  $\alpha \subset u$ , we choose a family of isometries  $S_{\alpha, k} \in \mathcal{L}(H_\alpha, H_u)$  such that  $S_{\alpha, k} \in \text{Hom}(u_\alpha, u)$ ,  $1 \leq k \leq \dim(\text{Hom}(u_\alpha, u))$  and  $S_{\alpha, k} S_{\alpha, k}^*$  are pairwise orthogonal projection with  $\sum_{\alpha, k} S_{\alpha, k} S_{\alpha, k}^* = \text{id}_{H_u}$ . Hence,  $u = \sum_{\alpha, k} (S_{\alpha, k} \otimes 1) u_\alpha (S_{\alpha, k}^* \otimes 1)$ . Define

$$a(u) = \sum_{\alpha, k} (\text{id}_B \otimes S_{\alpha, k} \otimes 1) a(\alpha) (\text{id}_B \otimes S_{\alpha, k}^* \otimes 1) \in \mathcal{L}(B \otimes H_u) \otimes C(H_{(B, \psi)}^+(\mathbb{G}))$$

Observe that the definition is coherent with the original  $a(\alpha)$ . For every finite dimensional unitary representations of  $u, v$  and  $w$  of  $\mathbb{G}$  and every morphism  $S \in \text{Hom}(u \otimes v, w)$  define the linear map  $a(S) : B \otimes H_u \otimes B \otimes H_v \rightarrow B \otimes H_w$  by  $a(S) = (m_B \otimes S)\Sigma_{23}$ . Observe that  $a(S)$  is coherent with the original definition, when  $u, v$  and  $w$  are in  $\text{Irr}(\mathbb{G})$ .

**Proposition 2.5.28.** *For all finite dimensional unitary representation  $u, v$  of  $\mathbb{G}$  and all  $S \in \text{Hom}(u, v)$  the following holds.*

0. *The definition of  $a(u)$  does not depend on the isometries  $S_{\alpha,k}$  chosen*
  1.  *$a(u)$  is a unitary representation of  $H_{(B,\psi)}^+(\mathbb{G})$  on  $B \otimes H_u$ .*
  2.  *$\text{id}_B \otimes S \in \text{Hom}(a(u), a(v))$ .*
  3. *If  $u \simeq v$  then  $a(u) \simeq a(v)$ .*
1. *For all finite dimensional unitary representations  $w$  of  $\mathbb{G}$  and all  $S \in \text{Hom}(u \otimes v, w)$ ,  $a(S) \in \text{Hom}(a(u) \otimes a(v), a(w))$ . Moreover,  $a(S)a(S)^* = \delta \text{id}_B \otimes SS^*$ . In particular, if  $S^*$  is isometric then  $\delta^{-\frac{1}{2}}a(S)^*$  is isometric.*

*Proof.* Let us prove (0). Let  $T_{\beta,l} \in \mathcal{L}(H_\beta, H_u)$  be another family of isometries such that  $T_{\beta,l} \in \text{Hom}(u_\beta, u)$ ,  $1 \leq l \leq \dim(\text{Hom}(u_\beta, u))$  and  $T_{\beta,l}T_{\beta,l}^*$  are pairwise orthogonal projections with  $\sum_{\beta,l} T_{\beta,l}T_{\beta,l}^* = \text{id}_{H_u}$ . Observe that  $T_{\beta,l}^*S_{\alpha,k} \in \text{Hom}(u_\alpha, u_\beta)$ . Therefore, there exists  $\lambda_{kl}^\beta \in \mathbb{C}$  such that  $T_{\beta,l}^*S_{\alpha,k} = \delta_{\alpha,\beta}\lambda_{kl}^\beta \text{id}_{H_{u_\beta}}$ . Also note that  $\sum_k \lambda_{kl}^\beta S_{\beta,k}^* = \sum_k T_{\beta,l}^*S_{\beta,k}S_{\beta,k}^* = \sum_{\alpha,k} T_{\beta,l}^*S_{\alpha,k}S_{\alpha,k}^*$  since  $T_{\beta,l}^*S_{\alpha,k} = 0$  for  $\alpha \neq \beta$ . Hence,  $\sum_k \lambda_{kl}^\beta S_{\beta,k}^* = T_{\beta,l}^* \left( \sum_{\alpha,k} S_{\alpha,k}S_{\alpha,k}^* \right) = T_{\beta,l}^*$ . It follows that

$$\begin{aligned}
 & \sum_{\alpha,k} (\text{id}_B \otimes S_{\alpha,k} \otimes 1) a(\alpha) (\text{id}_B \otimes S_{\alpha,k}^* \otimes 1) = \\
 & \sum_{\alpha,\beta,k,l} (\text{id}_B \otimes T_{\beta,l} T_{\beta,l}^* S_{\alpha,k} \otimes 1) a(\alpha) (\text{id}_B \otimes S_{\alpha,k}^* \otimes 1) = \\
 & \sum_{\alpha,\beta,k,l} (\text{id}_B \otimes T_{\beta,l} \delta_{\alpha,\beta} \lambda_{kl}^\beta \otimes 1) a(\alpha) (\text{id}_B \otimes S_{\alpha,k}^* \otimes 1) = \\
 & \sum_{\beta,k,l} (\text{id}_B \otimes T_{\beta,l} \otimes 1) a(\beta) (\text{id}_B \otimes \lambda_{kl}^\beta S_{\beta,k}^* \otimes 1) = \\
 & \sum_{\beta,l} (\text{id}_B \otimes T_{\beta,l} \otimes 1) a(\beta) (\text{id}_B \otimes \left( \sum_k \lambda_{kl}^\beta S_{\beta,k}^* \right) \otimes 1) = \\
 & \sum_{\beta,l} (\text{id}_B \otimes T_{\beta,l} \otimes 1) a(\beta) (\text{id}_B \otimes T_{\beta,l}^* \otimes 1)
 \end{aligned}$$

(1) is obvious and (3) follows from (2). Let us prove (2). Write  $T_{\beta,l} \in \mathcal{L}(H_\beta, H_v)$  the chosen isometries such that  $T_{\beta,l} \in \text{Hom}(u_\beta, v)$ ,  $1 \leq l \leq \dim(\text{Hom}(u_\beta, v))$  and

$T_{\beta,l}T_{\beta,l}^*$  are pairwise orthogonal projections with  $\sum_{\beta,l} T_{\beta,l}T_{\beta,l}^* = \text{id}_{H_v}$ . Observe that  $T_{\beta,l}^*SS_{\alpha,k} \in \text{Hom}(u_\alpha, u_\beta)$ . Therefore, there exists  $\lambda_{kl}^\beta \in \mathbb{C}$  such that  $T_{\beta,l}^*SS_{\alpha,k} = \delta_{\alpha,\beta}\lambda_{kl}^\beta$ . Also note that  $\sum_k \lambda_{kl}^\beta S_{\beta,k}^* = \sum_k T_{\beta,l}^*SS_{\beta,k}S_{\beta,k}^* = \sum_{\alpha,k} T_{\beta,l}^*SS_{\alpha,k}S_{\alpha,k}^*$  since  $T_{\beta,l}^*SS_{\alpha,k} = 0$  for  $\alpha \neq \beta$ . Hence,  $\sum_k \lambda_{kl}^\beta S_{\beta,k}^* = T_{\beta,l}^*S \left( \sum_{\alpha,k} S_{\alpha,k}S_{\alpha,k}^* \right) = T_{\beta,l}^*S$ . It follows that

$$\begin{aligned}
(\text{id}_B \otimes S \otimes 1)a(u) &= \sum_{\alpha,k} (\text{id}_B \otimes SS_{\alpha,k} \otimes 1)a(\alpha)(\text{id}_B \otimes S_{\alpha,k}^* \otimes 1) \\
&= \sum_{\alpha,\beta,k,l} (\text{id}_B \otimes T_{\beta,l}T_{\beta,l}^*SS_{\alpha,k} \otimes 1)a(\alpha)(\text{id}_B \otimes S_{\alpha,k}^* \otimes 1) \\
&= \sum_{\alpha,\beta,k,l} (\text{id}_B \otimes T_{\beta,l}\delta_{\alpha,\beta}\lambda_{kl}^\beta \otimes 1)a(\alpha)(\text{id}_B \otimes S_{\alpha,k}^* \otimes 1) \\
&= \sum_{\beta,k,l} (\text{id}_B \otimes T_{\beta,l} \otimes 1)a(\beta)(\text{id}_B \otimes \lambda_{kl}^\beta S_{\beta,k}^* \otimes 1) \\
&= \sum_{\beta,l} (\text{id}_B \otimes T_{\beta,l} \otimes 1)a(\beta)(\text{id}_B \otimes \left( \sum_k \lambda_{kl}^\beta S_{\beta,k}^* \right) \otimes 1) \\
&= a(v)(\text{id}_B \otimes S \otimes 1).
\end{aligned}$$

Let us prove (4). Consider the decompositions  $u = \sum_{\alpha,k} (U_{\alpha,k} \otimes 1)u_\alpha(U_{\alpha,k}^* \otimes 1)$ ,  $v = \sum_{\beta,l} (V_{\beta,l} \otimes 1)u_\beta(V_{\beta,l}^* \otimes 1)$  and  $w = \sum_{\gamma,j} (W_{\gamma,j} \otimes 1)u_\gamma(W_{\gamma,j}^* \otimes 1)$ . Then, with  $A = C(H_{(B,\psi)}^+(\mathbb{G}))$ ,

$$\begin{aligned}
a(u)_{13}a(v)_{23} &= \sum_{\alpha,\beta,k,l} (\text{id}_B \otimes U_{\alpha,k} \otimes \text{id}_{B \otimes H_v} \otimes 1_A)a(\alpha)_{13} \\
&\quad (\text{id}_B \otimes U_{\alpha,k}^* \otimes \text{id}_B \otimes V_{\beta,l} \otimes 1_A)a(\beta)_{23}(\text{id}_{B \otimes H_u} \otimes \text{id}_B \otimes V_{\beta,l}^* \otimes 1_A)
\end{aligned}$$

We have  $a(S)(\text{id}_B \otimes U_{\alpha,k} \otimes \text{id}_{B \otimes H_v}) = m_B \otimes (S \circ (U_{\alpha,k} \otimes \text{id}_{H_v})) \circ \Sigma_{23}$  and, by using  $\text{id}_{H_v} = \sum V_{\beta,l}V_{\beta,l}^*$  and  $\text{id}_{H_w} = \sum W_{\gamma,j}W_{\gamma,j}^*$ , we find

$$S \circ (U_{\alpha,k} \otimes \text{id}_{H_v}) = \sum W_{\gamma,j} (W_{\gamma,j}^* \circ S \circ (U_{\alpha,k} \otimes V_{\beta,l})) V_{\beta,l}^*$$

Hence,

$$\begin{aligned}
&a(S)(\text{id}_B \otimes U_{\alpha,k} \otimes \text{id}_{B \otimes H_v}) = \\
&\sum_{\beta,\gamma,l,j} \left[ m_B \otimes \left( W_{\gamma,j} (W_{\gamma,j}^* \circ S \circ (U_{\alpha,k} \otimes V_{\beta,l})) V_{\beta,l}^* \right) \right] \circ \Sigma_{23} = \\
&\sum (\text{id}_B \otimes W_{\gamma,j}) \left[ m_B \otimes \left( W_{\gamma,j}^* \circ S \circ U_{\alpha,k} \otimes V_{\beta,l} \right) \right] (\text{id}_B \otimes \text{id}_B \otimes \text{id}_{H_\alpha} \otimes V_{\beta,l}^*) \circ \Sigma_{23} = \\
&\sum (\text{id}_B \otimes W_{\gamma,j}) \left[ m_B \otimes \left( W_{\gamma,j}^* \circ S \circ U_{\alpha,k} \otimes V_{\beta,l} \right) \right] \Sigma_{23} (\text{id}_B \otimes \text{id}_{H_\alpha} \otimes \text{id}_B \otimes V_{\beta,l}^*) = \\
&\sum_{\beta,\gamma,l,j} (\text{id}_B \otimes W_{\gamma,j}) a(W_{\gamma,j}^* \circ S \circ U_{\alpha,k} \otimes V_{\beta,l}) (\text{id}_B \otimes \text{id}_{H_\alpha} \otimes \text{id}_B \otimes V_{\beta,l}^*),
\end{aligned}$$

where  $W_{\gamma,j}^* \circ S \circ U_{\alpha,k} \otimes V_{\beta,l} \in \text{Hom}(u_\alpha \otimes u_\beta, u_\gamma)$ . Hence,  $a(W_{\gamma,j}^* \circ S \circ U_{\alpha,k} \otimes V_{\beta,l}) \in \text{Hom}(a(\alpha) \otimes a(\beta), a(\gamma))$  and, by using 3, we find  $T_{\alpha,\beta,\gamma,k,l,j} = (\text{id}_B \otimes W_{\gamma,j}^*)a(W_{\gamma,j}^* \circ S \circ U_{\alpha,k} \otimes V_{\beta,l}) \in \text{Hom}(a(\alpha) \otimes a(\beta), a(w))$ . Hence, we find that  $(a(S) \otimes 1_A)(a(u)_{13}a(v)_{23})$  is equal to:

$$\sum_{\alpha,\beta,\beta',\gamma,k,l,l',j} T_{\alpha,\beta',\gamma,k,l',j} (\text{id}_B \otimes \text{id}_{H_\alpha} \otimes \text{id}_B \otimes V_{\beta',l'}^*) \otimes 1_A a(\alpha)_{13} \\ (\text{id}_B \otimes U_{\alpha,k}^* \otimes \text{id}_B \otimes V_{\beta,l} \otimes 1_A) a(\beta)_{23} (\text{id}_{B \otimes H_u} \otimes \text{id}_B \otimes V_{\beta,l}^* \otimes 1_A)$$

Since  $(\text{id}_B \otimes \text{id}_{H_\alpha} \otimes \text{id}_B \otimes V_{\beta',l'}^* \otimes 1_A) a(\alpha)_{13} (\text{id}_B \otimes U_{\alpha,k}^* \otimes \text{id}_B \otimes V_{\beta,l} \otimes 1_A)$  is equal to

$$a(\alpha)_{13} (\text{id}_B \otimes U_{\alpha,k}^* \otimes \text{id}_B \otimes V_{\beta',l'}^* \otimes 1_A) = \delta_{\beta,\beta'} \delta_{l,l'} a(\alpha)_{13} (\text{id}_B \otimes U_{\alpha,k}^* \otimes \text{id}_B \otimes \text{id}_{H_\beta} \otimes 1_A),$$

we find that  $(a(S) \otimes 1_A)(a(u)_{13}a(v)_{23})$  is equal to:

$$\sum_{\alpha,\beta,\gamma,k,l,j} (T_{\alpha,\beta,\gamma,k,l,j} \otimes 1_A) a(\alpha)_{13} (\text{id}_B \otimes U_{\alpha,k}^* \otimes \text{id}_{B \otimes H_\beta} \otimes 1_A) a(\beta)_{23} (\text{id}_{B \otimes H_u} \otimes \text{id}_B \otimes V_{\beta,l}^* \otimes 1_A) \\ = \sum_{\alpha,\beta,\gamma,k,l,j} (T_{\alpha,\beta,\gamma,k,l,j} \otimes 1_A) a(\alpha)_{13} a(\beta)_{23} (\text{id}_B \otimes U_{\alpha,k}^* \otimes \text{id}_B \otimes V_{\beta,l}^* \otimes 1_A) \\ = \sum_{\alpha,\beta,\gamma,k,l,j} a(w) (T_{\alpha,\beta,\gamma,k,l,j} \otimes 1_A) (\text{id}_B \otimes U_{\alpha,k}^* \otimes \text{id}_B \otimes V_{\beta,l}^* \otimes 1_A).$$

Hence, it suffices to check that  $a(S) = \sum_{\alpha,\beta,\gamma,k,l,j} T_{\alpha,\beta,\gamma,k,l,j} \circ (\text{id}_B \otimes U_{\alpha,k}^* \otimes \text{id}_B \otimes V_{\beta,l}^*)$ . This follows from the equation  $a(S)(\text{id}_B \otimes U_{\alpha,k} \otimes \text{id}_{B \otimes H_v}) = \sum_{\beta,\gamma,l,j} T_{\alpha,\beta,\gamma,k,l,j} (\text{id}_B \otimes \text{id}_{H_\alpha} \otimes \text{id}_B \otimes V_{\beta,l}^*)$  for all  $\alpha, k$  and the fact that  $\text{id}_{H_\alpha} = \sum_{\alpha,k} U_{\alpha,k} U_{\alpha,k}^*$ . Finally, we have

$$a(S)a(S)^* = (m_B \otimes S) \Sigma_{23} \Sigma_{23}^* (m_B^* \otimes S^*) = (m_B m_B^* \otimes S S^*) = \delta \text{id}_B \otimes S S^*$$

□

*Remark 2.19.* If we apply assertion 4 of Proposition 2.5.28 with  $w = u \otimes v$  and  $S = \text{id}_{H_u \otimes H_v}$ , we get a morphism  $S_{u,v} \in \text{Hom}(a(u) \otimes a(v), a(u \otimes v))$ . Hence,  $T_{u,v} = \delta^{-\frac{1}{2}} S_{u,v}^* \in \text{Hom}(a(u \otimes v), a(u) \otimes a(v))$  is isometric so we always have  $a(u \otimes v) \subset a(u) \otimes a(v)$ .

**Theorem 2.5.29.** *Let  $B, B'$  be two finite dimensional  $C^*$ -algebras,  $\dim B, B' \geq 4$ , endowed with a  $\delta$ -form  $\psi$  and a  $\delta'$ -form  $\psi'$  respectively. Consider the quantum automorphism group  $\mathbb{G}^{aut}(B \otimes B', \psi \otimes \psi')$  and let  $U$  be its fundamental representation. Then, we have the following isomorphism of  $C^*$ -algebras.*

$$C(\mathbb{G}^{aut}(B \otimes B', \psi \otimes \psi'))/I \cong C(\mathbb{G}^{aut}(B', \psi')) *_w C(\mathbb{G}^{aut}(B, \psi))$$

where  $I \subset C(\mathbb{G}^{aut}(B \otimes B', \psi \otimes \psi'))$  is the closed two-sided  $*$ -ideal generated by the relations corresponding to the condition  $\text{id}_B \otimes \eta_{B'} \eta_{B'}^* \in \text{End}(U)$ .

*Proof.* We fix the notations  $M = C(\mathbb{G}^{aut}(B \otimes B', \psi \otimes \psi'))$  and  $N = C(\mathbb{G}^{aut}(B', \psi')) *_w C(\mathbb{G}^{aut}(B, \psi))$ . Let  $u \in \mathcal{L}(B') \otimes C(\mathbb{G}^{aut}(B', \psi'))$  be the fundamental representation of  $\mathbb{G}^{aut}(B', \psi')$ . Choose a complete set of representative of irreducible representations  $u_n \in \mathcal{L}(H_n) \otimes C(\mathbb{G}^{aut}(B', \psi'))$  with  $u_0 = 1$  and  $u \simeq u_0 \oplus u_1$ . Define the unitary representation  $v \in \mathcal{L}(B \otimes B') \otimes N$  of  $\mathbb{G}^{aut}(B', \psi') \wr_* \mathbb{G}^{aut}(B, \psi)$  by

$$v = a(u) = (\text{id}_B \otimes \eta_{B'} \otimes 1_N) a(u_0) (\text{id}_B \otimes \eta_{B'}^* \otimes 1_N) + (\text{id}_B \otimes S_1 \otimes 1_N) a(u_1) (\text{id}_B \otimes S_1^* \otimes 1_N)$$

where  $S_1 \in \text{Hom}(u_1, u)$  is the unique isometry, up to  $\mathbb{S}^1$ , such that  $\eta_{B'} \eta_{B'}^*$  and  $S_1 S_1^*$  are two orthogonal projections such that  $\eta_{B'} \eta_{B'}^* + S_1 S_1^* = \text{id}_{B'}$ .

We claim that there exists a unital  $*$ -homomorphism  $\Psi : M \rightarrow N$  such that  $(\text{id} \otimes \Psi)(U) = v$ . By the universal property of the  $C^*$ -algebra  $M$ , it suffices to check the following conditions.

1.  $\eta_{B \otimes B'} \in \text{Hom}(1, v)$ .
2.  $m_{B \otimes B'} \in \text{Hom}(v^{\otimes 2}, v)$ .

Let us prove (1). Since  $\eta_{B'} \eta_{B'}^*$  and  $S_1 S_1^*$  are orthogonal we have  $S_1^* \eta_{B'} = 0$ ; it follows that  $(\text{id}_B \otimes S_1^*) \eta_{B \otimes B'} = 0$ . Hence,

$$v(\eta_{B \otimes B'} \otimes 1_N) = (\text{id}_B \otimes \eta_{B'} \otimes 1_N) a(u_0) ((\text{id}_B \otimes \eta_{B'}^*) \circ \eta_{B \otimes B'}) \otimes 1_N$$

Since  $(\text{id}_B \otimes \eta_{B'}^*) \circ \eta_{B \otimes B'} = \eta_B \in \text{Hom}(1, a(u_0))$  and  $(\text{id}_B \otimes \eta_{B'}) \circ \eta_B = \eta_{B \otimes B'}$  we find

$$v(\eta_{B \otimes B'} \otimes 1_N) = (\text{id}_B \otimes \eta_{B'} \otimes 1_N) a(u_0) (\eta_B \otimes 1_N) = ((\text{id}_B \otimes \eta_{B'}) \circ \eta_B) \otimes 1_N = \eta_{B \otimes B'} \otimes 1_N$$

This proves (1). Let us prove (2). Observe that  $m_{B \otimes B'} = (m_B \otimes m_{B'})\Sigma_{23} = a(m_{B'})$ , where  $m_{B'} \in \text{Hom}(u \otimes u, u)$ . It follows from Proposition 2.5.28 that  $m_{B \otimes B'} \in \text{Hom}(v \otimes v, v)$ .

Let  $\pi : M \rightarrow M/I$  be the canonical quotient map. If we apply assertion (2) of Proposition 2.5.28 with  $\eta_{B'}\eta_{B'}^* \in \text{End}(u)$ , we find that  $\text{id}_B \otimes \eta_{B'}\eta_{B'}^* \in \text{End}(v)$ . It follows that the map  $\Psi$  can be factorized through  $M/I$ . This means that there exists a unique map  $\tilde{\Psi} : M/I \rightarrow N$  such that  $v = (\text{id}_{B \otimes B'} \otimes \Psi)(U) = (\text{id}_{B \otimes B'} \otimes \tilde{\Psi})(\text{id}_{B \otimes B'} \otimes \pi)(U)$ .

In order to construct a morphism in the opposite direction we need to define some linear maps and to introduce some notations. All these definitions are given, unless otherwise stated, for  $k \in \mathbb{N}, k \geq 1$ .

Let  $\Sigma_k$  be the unitary map  $\Sigma_k : (B \otimes B')^{\otimes k} \rightarrow B^{\otimes k} \otimes B'^{\otimes k}$ ,  $\otimes_{i=1}^k (b_i \otimes b'_i) \mapsto \otimes_{i=1}^k b_i \otimes \otimes_{i=1}^k b'_i$ , where  $b_i \in B$  and  $b'_i \in B'$ . We observe that, with this new notation  $\Sigma_2 = \Sigma_{23}$ .

Let  $m_B^{(k)} : B^{\otimes k} \rightarrow B$  be the map which multiplies  $k$  elements of  $B$ ; we set  $m_B^{(1)} = \text{id}_B$  by convention. We observe that this map is unique and well defined by the associativity of the multiplication. In particular, we have  $m_B^{(2)} = m_B$ . We claim that, for any  $k \geq 2$ ,  $m_B^{(k)}(m_B^{(k)})^* = \delta^{k-1}\text{id}_B$ . The proof is by induction. If  $k = 1$ , it is trivially true; if  $k = 2$ , it is clear that  $m_B m_B^* = \delta \text{id}_B$ . Let us suppose the result true for  $k = l$ , i.e.  $m_B^{(l)}(m_B^{(l)})^* = \delta^{l-1}\text{id}_B$ . We have that  $m_B^{(l+1)} = m_B^{(l)}(m_B \otimes \text{id}_B^{\otimes l-1})$  by associativity. Hence,  $m_B^{(l+1)}(m_B^{(l+1)})^* = m_B^{(l)}(m_B m_B^* \otimes \text{id}_B^{\otimes l-1})(m_B^{(l)})^* = \delta m_B^{(l)}(m_B^{(l)})^* = \delta^l \text{id}_B$ . Then, the equality is true for  $k = l + 1$ . This completes the proof.

Define the map  $T_k = (m_B^{(k)} \otimes \text{id}_{B'}^{\otimes k})\Sigma_k \in \mathcal{L}((B \otimes B')^{\otimes k}, B \otimes B'^{\otimes k})$ .

Let  $S_k \in \text{Hom}(u_k, u^{\otimes k})$  be the unique isometry, up to  $\mathbb{S}^1$ , and define the isometry  $Q_k = \delta^{-\frac{k-1}{2}} T_k^* \circ (\text{id}_B \otimes S_k) \in \mathcal{L}(B \otimes H_k, (B \otimes B')^{\otimes k})$  for  $k \geq 2$ . For  $k = 0$ , we define the isometry  $Q_0 = \text{id}_B \otimes \eta_{B'} \in \text{Hom}(a(u_0), v)$ .

Finally, for  $k \geq 1$ , consider the elements  $A_k = (Q_k^* \otimes 1_M)U^{\otimes k}(Q_k \otimes 1_M) \in \mathcal{L}(B \otimes H_k) \otimes M$  and, for  $k = 0$ ,  $A_0 = (Q_0^* \otimes 1_M)U(Q_0 \otimes 1_M) \in \mathcal{L}(B) \otimes M$ . Denote by  $\tilde{A}_k = (\text{id} \otimes \pi)(A_k) \in \mathcal{L}(B \otimes H_k) \otimes M/I$  the projections of the  $A_k, k \in \mathbb{N}$  on the quotient. Let  $V = (\text{id} \otimes \pi)(U) \in \mathcal{L}(B \otimes B') \otimes M/I$  be the projection of the fundamental representation  $U$ .

We claim that there exists a unital  $*$ -homomorphism  $\Phi : N \longrightarrow M/I$  such that  $(\text{id} \otimes \Phi)(a(u_k)) = \tilde{A}_k$  for all  $k \in \mathbb{N}$ . By the universal property of the  $C^*$ -algebra  $N$  it suffices to check the following.

1.  $A_0(\eta_B \otimes 1_M) = \eta_B \otimes 1_M$ .
2.  $\tilde{A}_k \in \mathcal{L}(B \otimes H_k) \otimes M/I$  is unitary for all  $k \geq 0$ .
3. For all  $k, l, t \in \mathbb{N}$  and  $R \in \text{Hom}(u_k \otimes u_l, u_t)$ ,  $(m_B \otimes R)\Sigma_{23} \in \text{Hom}(\tilde{A}_k \otimes \tilde{A}_l, \tilde{A}_t)$ .

Let us prove (1). Since  $\eta_B \otimes \eta_{B'} = \eta_{B \otimes B'} \in \text{Hom}(1, U)$ , we have

$$\begin{aligned} A_0(\eta_B \otimes 1_M) &= (\text{id}_B \otimes \eta_{B'}^* \otimes 1_M)U(\text{id}_B \otimes \eta_{B'} \otimes 1_M)(\eta_B \otimes 1_M) \\ &= (\text{id}_B \otimes \eta_{B'}^* \otimes 1_M)U(\eta_B \otimes \eta_{B'} \otimes 1_M) \\ &= \eta_B \otimes \eta_{B'}^* \eta_{B'} \otimes 1_M \\ &= \eta_B \otimes 1_M. \end{aligned}$$

Let us prove (2). We have  $A_k A_k^* = (Q_k^* \otimes 1_M)U^{\otimes k}(Q_k Q_k^* \otimes 1_M)(U^{\otimes k})^*(Q_k \otimes 1_M)$ . We have  $Q_0 Q_0^* = \text{id}_B \otimes \eta_{B'} \eta_{B'}^*$  and  $Q_1 Q_1^* = \text{id}_B \otimes S_1 S_1^* = \text{id}_B \otimes (\text{id}_{B'} - \eta_{B'} \eta_{B'}^*) = \text{id}_{B \otimes B'} - Q_0 Q_0^*$ . By definition of  $I$ , we have

$$V(Q_0 Q_0^* \otimes 1_{M/I}) = (Q_0 Q_0^* \otimes 1_{M/I})V,$$

and

$$\begin{aligned} V(Q_1 Q_1^* \otimes 1_{M/I}) &= V(1 - Q_0 Q_0^* \otimes 1_{M/I}) \\ &= (1 - Q_0 Q_0^* \otimes 1_{M/I})V \\ &= (Q_1 Q_1^* \otimes 1_{M/I})V. \end{aligned}$$

It follows that, for  $k = 0, 1$ ,

$$\begin{aligned} \tilde{A}_k \tilde{A}_k^* &= (Q_k^* \otimes 1_{M/I})V(Q_k Q_k^* \otimes 1_{M/I})V^*(Q_k \otimes 1_{M/I}) \\ &= (Q_k^* Q_k Q_k^* \otimes 1_{M/I})VV^*(Q_k \otimes 1_{M/I}) \\ &= (Q_k^* Q_k Q_k^* Q_k \otimes 1_{M/I}) \\ &= \text{id}_{B \otimes H_k} \otimes 1_{M/I}. \end{aligned}$$

The proof of  $\tilde{A}_k^* \tilde{A}_k = \text{id}_{B \otimes H_k} \otimes 1_{M/I}$  when  $k = 0, 1$  is the same.

In order to prove (2) when  $k \geq 2$  and to check (3) we introduce the following lemma.

**Lemma 2.5.30.** *For all  $k, l \in \mathbb{N}$ ,  $k, l \geq 1$  and  $T \in \mathcal{L}(B'^{\otimes k}, B'^{\otimes l})$  we define the map*

$$\phi_{k,l}(T) = \Sigma_l^*((m_B^{(l)})^* m_B^{(k)} \otimes T) \Sigma_k \in \mathcal{L}((B \otimes B')^{\otimes k}, (B \otimes B')^{\otimes l})$$

We fix the notation  $\phi_k := \phi_{k,k}(\text{id}_{B'}^{\otimes k})$ . Then, we have

(i).  $\phi_k \in \text{End}(V^{\otimes k})$

(ii). For all  $T \in \mathcal{L}(B'^{\otimes k}, B'^{\otimes l})$  and  $S \in \mathcal{L}(B'^{\otimes l}, B'^{\otimes t})$  we have:

- $\phi_{l,t}(S) \phi_{k,l}(T) = \delta^{l-1} \phi_{k,t}(S \circ T)$
- $\phi_{k,l}(T)^* = \phi_{l,k}(T^*)$

(iii).  $\phi_{k,k-1}(\text{id}_{B'}^{\otimes s} \otimes m_{B'} \otimes \text{id}_{B'}^{\otimes s'}) = \phi_{k-1} \circ (\text{id}_{B \otimes B'}^{\otimes s} \otimes m_{B \otimes B'} \otimes \text{id}_{B \otimes B'}^{\otimes s'}) \in \text{Hom}(V^{\otimes k}, V^{\otimes k-1})$   
for all  $k \geq 2$  and all  $s, s' \geq 0$  such that  $s + s' + 2 = k$

(iv). •  $\phi_{k,k-1}(\text{id}_{B'}^{\otimes s} \otimes \eta_{B'}^* \otimes \text{id}_{B'}^{\otimes s'}) = \phi_{k-1} \circ (\text{id}_{B \otimes B'}^{\otimes s-1} \otimes (m_B \otimes \text{id}_{B'} \otimes \eta_{B'}^*) \Sigma_{23} \otimes \text{id}_{B \otimes B'}^{\otimes s'}) \in \text{Hom}(V^{\otimes k}, V^{\otimes k-1})$  for all  $k \geq 2$  and all  $s \geq 1, s' \geq 0$  such that  $s + s' + 1 = k$

- $\phi_{k,k-1}(\eta_{B'}^* \otimes \text{id}_{B'}^{\otimes k-1}) = \phi_{k-1} \circ ((m_B \otimes \eta_{B'}^* \otimes \text{id}_{B'}) \Sigma_{23} \otimes \text{id}_{B \otimes B'}^{\otimes k-2}) \in \text{Hom}(V^{\otimes k}, V^{\otimes k-1})$   
for all  $k \geq 2$

(v).  $\phi_{k,l}(T) \in \text{Hom}(V^{\otimes k}, V^{\otimes l})$  for all  $T \in \text{Hom}(u^{\otimes k}, u^{\otimes l})$

*Proof.* (i). If  $k = 1$ ,  $\phi_1 = \text{id}_{B \otimes B'} \in \text{End}(V)$ .

When  $k \geq 2$  we prove the result by induction on  $k$ . If  $k = 2$ , we have  $\phi_2 = \Sigma_{23}(m_B^* m_B \otimes \text{id}_{B'}^{\otimes 2}) \Sigma_{23}$ . We want to prove that  $\phi_2 \in \text{End}(V^{\otimes 2})$ .

Let  $L = \Sigma_3^*((m_B^{(3)})^* m_B \otimes \text{id}_{B'} \otimes \eta_{B'} \otimes \text{id}_{B'}) \Sigma_2$ . We claim that

$$L = (\text{id}_{B \otimes B'} \otimes m_{B \otimes B'} \otimes \text{id}_{B \otimes B'}) (\Sigma_2^*(m_B^* \otimes \text{id}_{B'} \otimes \eta_{B'}) \otimes \Sigma_2^*(m_B^* \otimes \eta_{B'} \otimes \text{id}_{B'})) \quad (2.16)$$

Let us evaluate the two maps on the element  $b_1 \otimes b'_1 \otimes b_2 \otimes b'_2 \in (B \otimes B')^{\otimes 2}$ . In the first case we have

$$\begin{aligned} & \Sigma_3^*((m_B^{(3)})^*m_B \otimes \text{id}_{B'} \otimes \eta_{B'} \otimes \text{id}_{B'})\Sigma_2(b_1 \otimes b'_1 \otimes b_2 \otimes b'_2) = \\ & \Sigma_3^*((m_B^{(3)})^*m_B(b_1 \otimes b_2) \otimes b'_1 \otimes 1_{B'} \otimes b'_2) = \\ & \Sigma b_{(1)}^{12} \otimes b'_1 \otimes b_{(2)}^{12} \otimes 1_{B'} \otimes b_{(3)}^{12} \otimes b'_2 \end{aligned}$$

where we used the notation  $(m_B^{(3)})^*m_B(b_1 \otimes b_2) = \Sigma b_{(1)}^{12} \otimes b_{(2)}^{12} \otimes b_{(3)}^{12}$ .

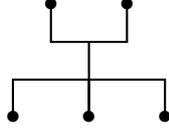
In the second case we have

$$\begin{aligned} & (\text{id}_{B \otimes B'} \otimes m_{B \otimes B'} \otimes \text{id}_{B \otimes B'}) (\Sigma_2^*(m_B^* \otimes \text{id}_{B'} \otimes \eta_{B'}) \otimes \Sigma_2^*(m_B^* \otimes \eta_{B'} \otimes \text{id}_{B'})) \\ & \quad (b_1 \otimes b'_1 \otimes b_2 \otimes b'_2) = \\ & (\text{id}_{B \otimes B'} \otimes m_{B \otimes B'} \otimes \text{id}_{B \otimes B'}) (\Sigma_2^*(m_B^*(b_1) \otimes b'_1 \otimes 1_{B'}) \otimes \Sigma_2^*(m_B^*(b_2) \otimes 1_{B'} \otimes b'_2)) = \\ & (\text{id}_{B \otimes B'} \otimes m_{B \otimes B'} \otimes \text{id}_{B \otimes B'}) (\Sigma b_{(1)}^1 \otimes b'_1 \otimes b_{(2)}^1 \otimes 1_{B'} \otimes b_{(1)}^2 \otimes 1_{B'} \otimes b_{(2)}^2 \otimes b'_2) = \\ & (\Sigma b_{(1)}^1 \otimes b'_1 \otimes m_B(b_{(2)}^1 \otimes b_{(1)}^2) \otimes 1_{B'} \otimes b_{(2)}^2 \otimes b'_2) \end{aligned}$$

where we used the notation  $m_B^*(b_i) = \Sigma b_{(1)}^i \otimes b_{(2)}^i$  for  $i = 1, 2$ .

We observe that the two elements of  $(B \otimes B')^{\otimes 3}$  which we found are equal if and only if  $(m_B^{(3)})^*m_B = (\text{id}_B \otimes m_B \otimes \text{id}_B)(m_B^* \otimes m_B^*)$ . This can be verified by drawing the noncrossing partitions associated to the different maps and by using the compatibility with respect to the multiplication proved in Proposition 2.3.5.

In both cases, the noncrossing partition obtained after the composition is



It follows that relation 2.16 is verified. Now, let  $T = (m_B \otimes \eta_{B'}^* \otimes \text{id}_{B'})\Sigma_{23}$ . Observe that

$$m_{B'} \circ (\eta_{B'} \eta_{B'}^* \otimes \text{id}_{B'}) = \eta_{B'}^* \otimes \text{id}_{B'}$$

Hence,

$$\begin{aligned} T &= [m_B \otimes (m_{B'} \circ (\eta_{B'} \eta_{B'}^* \otimes \text{id}_{B'}))] \Sigma_{23} \\ &= [(m_B \otimes m_{B'}) \circ (\text{id}_B \otimes \text{id}_B \otimes \eta_{B'} \eta_{B'}^* \otimes \text{id}_{B'})] \Sigma_{23} \\ &= (m_B \otimes m_{B'}) \Sigma_{23} (\text{id}_B \otimes \eta_{B'} \eta_{B'}^* \otimes \text{id}_B \otimes \text{id}_{B'}) \\ &= m_{B \otimes B'} \circ ((\text{id}_B \otimes \eta_{B'} \eta_{B'}^*) \otimes \text{id}_{B \otimes B'}) \end{aligned}$$

By definition of  $I$  we have  $\text{id}_B \otimes \eta_{B'} \eta_{B'}^* \in \text{End}(V)$  and since  $m_{B \otimes B'} \in \text{Hom}(U^{\otimes 2}, U)$  we deduce that  $T \in \text{End}(V^{\otimes 2}, V)$ .

Similarly, let  $Z = (m_B \otimes \text{id}_{B'} \otimes \eta_{B'}^*)\Sigma_{23}$ . By using that  $\text{id}_{B'} \otimes \eta_{B'}^* = m_{B'} \circ (\text{id}_{B'} \otimes \eta_{B'} \eta_{B'}^*)$ , we find  $Z = m_{B \otimes B'} \circ (\text{id}_{B \otimes B'} \otimes (\text{id}_B \otimes \eta_{B'} \eta_{B'}^*))$ . It follows that  $Z \in \text{Hom}(V^{\otimes 2}, V)$ .

We observe that  $L = (\text{id}_{B \otimes B'} \otimes m_{B \otimes B'} \otimes \text{id}_{B \otimes B'})(Z^* \otimes T^*)$ ; it follows that  $L \in \text{Hom}(V^{\otimes 2}, V^{\otimes 3})$ .

In order to complete the proof of the case  $k = 2$ , it is enough to observe that

$$\begin{aligned} L^*L &= \Sigma_2^*(m_B^*m_B^{(3)} \otimes \text{id}_{B'} \otimes \eta_{B'}^* \otimes \text{id}_{B'}) \Sigma_3 \Sigma_3^*((m_B^{(3)})^*m_B \otimes \text{id}_{B'} \otimes \eta_{B'}^* \otimes \text{id}_{B'}) \Sigma_2 \\ &= \Sigma_2^*(m_B^*m_B^{(3)}(m_B^{(3)})^*m_B \otimes \text{id}_{B'}^{\otimes 2}) \Sigma_2 \\ &= \delta^2 \phi_2 \end{aligned}$$

where we used that  $m_B^{(3)}(m_B^{(3)})^* = \delta^2 \text{id}_B$ . It follows that  $\phi_2 \in \text{End}(V^{\otimes 2})$ .

Now, let us prove that, if  $\phi_{k-1} \in \text{End}(V^{\otimes k-1})$ , then  $\phi_k \in \text{End}(V^{\otimes k})$ . We claim that, for any  $k \geq 2$ , the following holds

$$\phi_k = (\text{id}_{B \otimes B'}^{\otimes k-2} \otimes \phi_2)(\phi_{k-1} \otimes \text{id}_{B \otimes B'}) \quad (2.17)$$

Let us evaluate this map on the general element  $\bigotimes_{i=1}^k (b_i \otimes b'_i) \in (B \otimes B')^{\otimes k}$ . We have

$$\begin{aligned} &(\text{id}_{B \otimes B'}^{\otimes k-2} \otimes \phi_2)(\phi_{k-1} \otimes \text{id}_{B \otimes B'}) (\bigotimes_{i=1}^k (b_i \otimes b'_i)) = \\ &(\text{id}_{B \otimes B'}^{\otimes k-2} \otimes \phi_2)[(\Sigma_{k-1}^*((m_B^{(k-1)})^*m_B^{(k-1)} \otimes \text{id}_{B'}^{\otimes k-1}) \Sigma_{k-1}) \otimes \text{id}_{B \otimes B'}] (\bigotimes_{i=1}^k (b_i \otimes b'_i)) = \\ &(\text{id}_{B \otimes B'}^{\otimes k-2} \otimes \Sigma_2^*(m_B^*m_B \otimes \text{id}_{B'}^{\otimes 2}) \Sigma_2)(\Sigma_{k-1}^*(\Sigma \bigotimes_{i=1}^{k-1} b_{(i)}^{C_1} \otimes \bigotimes_{i=1}^{k-1} b'_i) \otimes b_k \otimes b'_k) = \\ &\Sigma \bigotimes_{i=1}^{k-2} (b_{(i)}^{C_1} \otimes b'_i) \otimes \Sigma_2^*(m_B^*m_B(b_{(k-1)}^{C_1} \otimes b_k) \otimes b'_{k-1} \otimes b'_k) = \\ &\Sigma \bigotimes_{i=1}^{k-2} (b_{(i)}^{C_1} \otimes b'_i) \otimes b_{(1)}^{C_2} \otimes b'_{k-1} \otimes b_{(2)}^{C_2} \otimes b'_k \end{aligned}$$

where we used the notations  $(m_B^{(k-1)})^*m_B^{(k-1)}(\bigotimes_{i=1}^{k-1} b_i) = \Sigma \bigotimes_{i=1}^{k-1} b_{(i)}^{C_1}$  and  $m_B^*m_B(b_{(k-1)}^{C_1} \otimes b_k) = \Sigma b_{(1)}^{C_2} \otimes b_{(2)}^{C_2}$

When we evaluate  $\phi_k$  according to its definition, we get

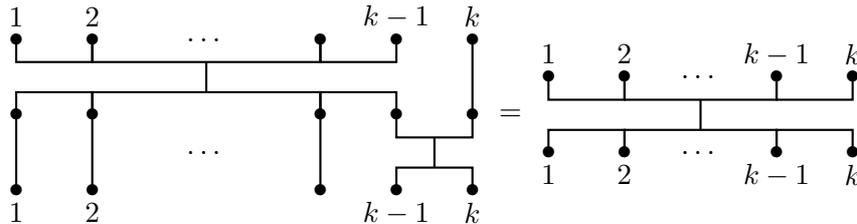
$$\Sigma_k^*((m_B^{(k)})^*m_B^{(k)} \otimes \text{id}_{B'}^{\otimes k}) \Sigma_k(\bigotimes_{i=1}^k (b_i \otimes b'_i)) = \Sigma \bigotimes_{i=1}^k (b_{(i)}^{C_3} \otimes b'_i)$$

where we used the notation  $(m_B^{(k)})^*m_B^{(k)}(\bigotimes_{i=1}^k b_i) = \Sigma \bigotimes_{i=1}^k b_{(i)}^{C_3}$ .

We observe that the two elements obtained are equal if and only if

$$(m_B^{(k)})^*m_B^{(k)} = (\text{id}_B^{\otimes k-2} \otimes m_B^*m_B)((m_B^{(k-1)})^*m_B^{(k-1)} \otimes \text{id}_B)$$

This formula can be verified by considering the noncrossing partitions associated to the different maps and by applying Proposition 2.3.5. We have



This completes the proof of relation 2.17. In order to complete the induction it is enough to use the inductive hypothesis. The map  $\phi_k$  can be obtained through tensor products and compositions of the intertwiners  $\text{id}_{B \otimes B'}$ ,  $\phi_2$ ,  $\phi_{k-1}$ ; it follows that  $\phi_k \in \text{End}(V^{\otimes k})$ .

(ii). The composition formula can be checked as follows

$$\begin{aligned} \phi_{l,t}(S)\phi_{k,l}(T) &= \Sigma_t^*((m_B^{(t)})^* m_B^{(l)} \otimes S) \Sigma_l \Sigma_l^*((m_B^{(l)})^* m_B^{(k)} \otimes T) \Sigma_k \\ &= \Sigma_t^*((m_B^{(t)})^* m_B^{(l)} (m_B^{(l)})^* m_B^{(k)} \otimes ST) \Sigma_k \\ &= \delta^{l-1} \Sigma_t^*((m_B^{(t)})^* m_B^{(k)} \otimes ST) \Sigma_k \\ &= \delta^{l-1} \phi_{k,t}(S \circ T) \end{aligned}$$

where we used that  $m_B^{(l)} (m_B^{(l)})^* = \delta^{l-1} \text{id}_B$ .

Now, we prove the compatibility with respect to the adjoint operation. We have

$$\begin{aligned} \phi_{k,l}(T)^* &= (\Sigma_l^*((m_B^{(l)})^* m_B^{(k)} \otimes T) \Sigma_k)^* \\ &= \Sigma_k^*((m_B^{(k)})^* m_B^{(l)} \otimes T^*) \Sigma_l \\ &= \phi_{l,k}(T^*) \end{aligned}$$

(iii). We have

$$\begin{aligned} \phi_{k-1} \circ (\text{id}_{B \otimes B'}^{\otimes s} \otimes m_{B \otimes B'} \otimes \text{id}_{B \otimes B'}^{\otimes s'}) &= \\ \Sigma_{k-1}^*((m_B^{(k-1)})^* m_B^{(k-1)} \otimes \text{id}_{B'}^{\otimes k-1}) \Sigma_{k-1} (\text{id}_{B \otimes B'}^{\otimes s} \otimes m_{B \otimes B'} \otimes \text{id}_{B \otimes B'}^{\otimes s'}) &= \\ \Sigma_{k-1}^*((m_B^{(k-1)})^* m_B^{(k-1)} \otimes \text{id}_{B'}^{\otimes k-1}) (\text{id}_B^{\otimes s} \otimes m_B \otimes \text{id}_{B'}^{\otimes s'} \otimes \text{id}_B^{\otimes s} \otimes m_{B'} \otimes \text{id}_{B'}^{\otimes s'}) \Sigma_k &= \\ \Sigma_{k-1}^*((m_B^{(k-1)})^* m_B^{(k)} \otimes \text{id}_{B'}^{\otimes s} \otimes m_{B'} \otimes \text{id}_{B'}^{\otimes s'}) \Sigma_k &= \\ \phi_{k,k-1} (\text{id}_{B'}^{\otimes s} \otimes m_{B'} \otimes \text{id}_{B'}^{\otimes s'}) & \end{aligned}$$

where the second equality follows from

$$\begin{aligned} \Sigma_{k-1} (\text{id}_{B \otimes B'}^{\otimes s} \otimes m_{B \otimes B'} \otimes \text{id}_{B \otimes B'}^{\otimes s'}) (\otimes_{i=1}^k (b_i \otimes b'_i)) &= \\ \Sigma_{k-1} (\otimes_{i=1}^s (b_i \otimes b'_i) \otimes m_B (b_{s+1} \otimes b_{s+2}) \otimes m_{B'} (b'_{s+1} \otimes b'_{s+2}) \otimes \otimes_{i=s+3}^k (b_i \otimes b'_i)) &= \\ \otimes_{i=1}^s b_i \otimes m_B (b_{s+1} \otimes b_{s+2}) \otimes \otimes_{i=s+3}^k b_i \otimes \otimes_{i=1}^s b'_i \otimes m_{B'} (b'_{s+1} \otimes b'_{s+2}) \otimes \otimes_{i=s+3}^k b'_i &= \\ (\text{id}_B^{\otimes s} \otimes m_B \otimes \text{id}_B^{\otimes s'} \otimes \text{id}_{B'}^{\otimes s} \otimes m_{B'} \otimes \text{id}_{B'}^{\otimes s'}) \Sigma_k (\otimes_{i=1}^k (b_i \otimes b'_i)) & \end{aligned}$$

and for the third equality we observe that  $m_B^{(k-1)} (\text{id}_B^{\otimes s} \otimes m_B \otimes \text{id}_B^{\otimes s'}) = m_B^{(k)}$  because the multiplication is associative.

Now,  $m_{B \otimes B'} \in \text{Hom}(U^{\otimes 2}, U)$  by definition and  $\phi_{k-1} \in \text{End}(V^{\otimes k-1})$  by the assertion

(i) of this Lemma. It follows that  $\phi_{k,k-1} (\text{id}_{B'}^{\otimes s} \otimes m_{B'} \otimes \text{id}_{B'}^{\otimes s'}) \in \text{Hom}(V^{\otimes k}, V^{\otimes k-1})$

(iv). We have

$$\begin{aligned}
 & \phi_{k-1} \circ (\text{id}_{B \otimes B'}^{\otimes s-1} \otimes (m_B \otimes \text{id}_{B'} \otimes \eta_{B'}^*) \Sigma_{23} \otimes \text{id}_{B \otimes B'}^{\otimes s'}) = \\
 & \Sigma_{k-1}^* ((m_B^{(k-1)})^* m_B^{(k-1)} \otimes \text{id}_{B'}^{\otimes k-1}) \Sigma_{k-1} (\text{id}_{B \otimes B'}^{\otimes s-1} \otimes (m_B \otimes \text{id}_{B'} \otimes \eta_{B'}^*) \Sigma_{23} \otimes \text{id}_{B \otimes B'}^{\otimes s'}) = \\
 & \Sigma_{k-1}^* ((m_B^{(k-1)})^* m_B^{(k-1)} \otimes \text{id}_{B'}^{\otimes k-1}) (\text{id}_B^{\otimes s-1} \otimes m_B \otimes \text{id}_{B'}^{\otimes s'} \otimes \text{id}_{B'}^{\otimes s} \otimes \eta_{B'}^* \otimes \text{id}_{B'}^{\otimes s'}) \Sigma_k = \\
 & \Sigma_{k-1}^* ((m_B^{(k-1)})^* m_B^{(k)} \otimes \text{id}_{B'}^{\otimes s} \otimes \eta_{B'}^* \otimes \text{id}_{B'}^{\otimes s'}) \Sigma_k = \\
 & \phi_{k,k-1} (\text{id}_{B'}^{\otimes s} \otimes \eta_{B'}^* \otimes \text{id}_{B'}^{\otimes s'})
 \end{aligned}$$

As in the proof of assertion (iii) the second equality can be checked by evaluating the two maps on an element of  $(B \otimes B')^{\otimes k}$  and the third follows from the associativity of  $m_B$ .

If  $s = 0$  and  $s' = k - 1$  the formula is slightly different but the computations are analogous. We have

$$\begin{aligned}
 & \phi_{k-1} \circ ((m_B \otimes \eta_{B'}^* \otimes \text{id}_{B'}) \Sigma_{23} \otimes \text{id}_{B \otimes B'}^{\otimes k-2}) = \\
 & \Sigma_{k-1}^* ((m_B^{(k-1)})^* m_B^{(k-1)} \otimes \text{id}_{B'}^{\otimes k-1}) \Sigma_{k-1} ((m_B \otimes \eta_{B'}^* \otimes \text{id}_{B'}) \Sigma_{23} \otimes \text{id}_{B \otimes B'}^{\otimes k-2}) = \\
 & \Sigma_{k-1}^* ((m_B^{(k-1)})^* m_B^{(k-1)} \otimes \text{id}_{B'}^{\otimes k-1}) (m_B \otimes \text{id}_B^{\otimes k-2} \otimes \eta_{B'}^* \otimes \text{id}_{B'}^{\otimes k-1}) \Sigma_k = \\
 & \Sigma_{k-1}^* ((m_B^{(k-1)})^* m_B^{(k)} \otimes \eta_{B'}^* \otimes \text{id}_{B'}^{\otimes k-1}) \Sigma_k = \\
 & \phi_{k,k-1} (\eta_{B'}^* \otimes \text{id}_{B'}^{\otimes k-1})
 \end{aligned}$$

Since in the proof of assertion (i) we showed that  $(m_B \otimes \text{id}_{B'} \otimes \eta_{B'}^*) \Sigma_{23}$  and  $(m_B \otimes \eta_{B'}^* \otimes \text{id}_{B'}) \Sigma_{23}$  are in  $\text{Hom}(V^{\otimes 2}, V)$  and  $\phi_{k-1} \in \text{End}(V^{\otimes k-1})$  by (i), we have that  $\phi_{k,k-1} (\text{id}_{B'}^{\otimes s} \otimes \eta_{B'}^* \otimes \text{id}_{B'}^{\otimes s'}) \in \text{Hom}(V^{\otimes k}, V^{\otimes k-1})$ .

(v). Thanks to Theorem 2.3.7 we know that the morphisms associated to the noncrossing partitions in  $NC(k, l)$  form a linear basis of  $\text{Hom}(u^{\otimes k}, u^{\otimes l})$ . Moreover, every morphism of such a basis can be seen as the composition of the morphisms  $\text{id}_{B'}^{\otimes s} \otimes m_{B'} \otimes \text{id}_{B'}^{\otimes s'} \in \text{Hom}(u^{\otimes s+s'+2}, u^{\otimes s+s'+1})$  and  $\text{id}_{B'}^{\otimes s} \otimes \eta_{B'} \otimes \text{id}_{B'}^{\otimes s'} \in \text{Hom}(u^{\otimes s+s'}, u^{\otimes s+s'+1})$  and of their adjoints. This fact, together with the assertions (ii), (iii) and (iv) of Lemma 2.5.30, implies that  $\phi_{k,l}(T) \in \text{Hom}(V^{\otimes k}, V^{\otimes l})$  for all  $T \in \text{Hom}(u^{\otimes k}, u^{\otimes l})$ .  $\square$

Now, we go back to the proof of (2), when  $k \geq 2$ . We have that,

$$\begin{aligned}
 Q_k Q_k^* &= \delta^{-(k-1)} T_k^* \circ (\text{id}_B \otimes S_k S_k^*) \circ T_k \\
 &= \delta^{-(k-1)} \Sigma_k^* ((m_B^{(k)})^* m_B^{(k)} \otimes S_k S_k^*) \Sigma_k \\
 &= \delta^{-(k-1)} \phi_{k,k} (S_k S_k^*)
 \end{aligned}$$

Since  $S_k S_k^* \in \text{End}(u^{\otimes k})$ ; it follows that  $Q_k Q_k^* \in \text{End}(V^{\otimes k})$  by Lemma 2.5.30 (v).

It is now easy to verify that  $\tilde{A}_k$  is unitary.

Let us check the last condition. We have to prove that, for all  $k, l, t \in \mathbb{N}$  and  $R \in \text{Hom}(u_k \otimes u_l, u_t)$ ,  $(m_B \otimes R)\Sigma_{23} \in \text{Hom}(\tilde{A}_k \otimes \tilde{A}_l, \tilde{A}_t)$ . Since the  $Q_s$  are isometries and  $Q_s Q_s^* \in \text{End}(V^{\otimes s})$ , we have the following sequence of equivalent conditions.

$$\begin{aligned} & ((m_B \otimes R)\Sigma_{23} \otimes 1_{M/I})(Q_k^* \otimes Q_l^* \otimes 1_{M/I})V^{\otimes k+l}(Q_k \otimes Q_l \otimes 1_{M/I}) = \\ & \quad (Q_t^* \otimes 1_{M/I})V^{\otimes t}(Q_t \otimes 1_{M/I})((m_B \otimes R)\Sigma_{23} \otimes 1_{M/I}) \\ & (Q_t \otimes 1_{M/I})((m_B \otimes R)\Sigma_{23} \otimes 1_{M/I})(Q_k^* \otimes Q_l^* \otimes 1_{M/I})V^{\otimes k+l}(Q_k Q_k^* \otimes Q_l Q_l^* \otimes 1_{M/I}) = \\ & \quad (Q_t Q_t^* \otimes 1_{M/I})V^{\otimes t}(Q_t \otimes 1_{M/I})((m_B \otimes R)\Sigma_{23} \otimes 1_{M/I})(Q_k^* \otimes Q_l^* \otimes 1_{M/I}) \\ & (Q_t \otimes 1_{M/I})((m_B \otimes R)\Sigma_{23} \otimes 1_{M/I})(Q_k^* Q_k Q_k^* \otimes Q_l^* Q_l Q_l^* \otimes 1_{M/I})V^{\otimes k+l} = \\ & \quad V^{\otimes t}(Q_t Q_t^* Q_t \otimes 1_{M/I})((m_B \otimes R)\Sigma_{23} \otimes 1_{M/I})(Q_k^* \otimes Q_l^* \otimes 1_{M/I}) \\ & (Q_t \otimes 1_{M/I})((m_B \otimes R)\Sigma_{23} \otimes 1_{M/I})(Q_k^* \otimes Q_l^* \otimes 1_{M/I})V^{\otimes k+l} = \\ & \quad V^{\otimes t}(Q_t \otimes 1_{M/I})((m_B \otimes R)\Sigma_{23} \otimes 1_{M/I})(Q_k^* \otimes Q_l^* \otimes 1_{M/I}) \end{aligned}$$

Then, the original condition is equivalent to

$$Q_t(m_B \otimes R)\Sigma_{23}(Q_k^* \otimes Q_l^*) \in \text{Hom}(V^{\otimes k+l}, V^{\otimes t})$$

Now, if we replace every  $Q_i$  with its definition and we fix  $K = \delta^{-\frac{k+l+t-3}{2}}$ , we get

$$\begin{aligned} & Q_t(m_B \otimes R)\Sigma_{23}(Q_k^* \otimes Q_l^*) = \\ & K\Sigma_t^*(m_B^{(t)*} \otimes \text{id}_{B'}^{\otimes t})(\text{id}_B \otimes S_t)(m_B \otimes R)\Sigma_{23} \\ & \quad [(\text{id}_B \otimes S_k^*)(m_B^{(k)} \otimes \text{id}_{B'}^{\otimes k})\Sigma_k \otimes (\text{id}_B \otimes S_l^*)(m_B^{(l)} \otimes \text{id}_{B'}^{\otimes l})\Sigma_l] = \\ & K\Sigma_t^*(m_B^{(t)*} \otimes S_t)(m_B \otimes R)\Sigma_{23}(m_B^{(k)} \otimes m_B^{(l)} \otimes S_k^* \otimes S_l^*)(\Sigma_k \otimes \Sigma_l) = \\ & K\Sigma_t^*(m_B^{(t)*} \otimes S_t)(m_B \otimes R)(m_B^{(k)} \otimes m_B^{(l)} \otimes S_k^* \otimes S_l^*)\widetilde{\Sigma}_{23}(\Sigma_k \otimes \Sigma_l) = \\ & K\Sigma_t^*(m_B^{(t)*} m_B^{(k+l)} \otimes S_t R(S_k^* \otimes S_l^*))\widetilde{\Sigma}_{23}(\Sigma_k \otimes \Sigma_l) = \\ & K\Sigma_t^*(m_B^{(t)*} m_B^{(k+l)} \otimes S_t R(S_k^* \otimes S_l^*))\Sigma_{k+l} = \\ & K\phi_{k+l,t}(S_t R(S_k^* \otimes S_l^*)) \end{aligned}$$

where  $\widetilde{\Sigma}_{23} : B^{\otimes k} \otimes B'^{\otimes k} \otimes B^{\otimes l} \otimes B'^{\otimes l} \longrightarrow B^{\otimes k+l} \otimes B'^{\otimes k+l}$  is the map that exchanges  $B'^{\otimes k}$  and  $B^{\otimes l}$ . It is easy to check that  $\widetilde{\Sigma}_{23}(\Sigma_k \otimes \Sigma_l) = \Sigma_{k+l}$ . In the third equality we used that  $m_B(m_B^{(k)} \otimes m_B^{(l)}) = m_B^{(k+l)}$ ; this is due to the associativity of the multiplication. Since  $S_t, S_k, S_l$  and  $R$  are intertwiners of  $\mathbb{G}^{\text{aut}}(B', \psi')$ , we have that  $S_t R(S_k^* \otimes S_l^*) \in \text{Hom}(u^{\otimes k} \otimes u^{\otimes l}, u^{\otimes t})$ . We can then apply Lemma 2.5.30 (v) and find that  $\phi_{k+l,t}(S_t R(S_k^* \otimes S_l^*)) \in \text{Hom}(V^{\otimes k+l}, V^{\otimes t})$ . This completes the proof of

(3).

What is left is to prove that the morphisms  $\tilde{\Psi}$  and  $\Phi$  are inverse to each other. We have

$$\begin{aligned}
 (\text{id} \otimes \Phi \tilde{\Psi})(V) &= (\text{id} \otimes \Phi)(\text{id} \otimes \Psi)(U) \\
 &= (\text{id} \otimes \Phi)(v) \\
 &= (Q_0 \otimes 1_{M/I})\tilde{A}_0(Q_0^* \otimes 1_{M/I}) + (Q_1 \otimes 1_{M/I})\tilde{A}_1(Q_1^* \otimes 1_{M/I}) \\
 &= (Q_0 Q_0^* \otimes 1_M)V(Q_0 Q_0^* \otimes 1_M) + (Q_1 Q_1^* \otimes 1_M)V(Q_1 Q_1^* \otimes 1_M) \\
 &= ((Q_0 Q_0^* + Q_1 Q_1^*) \otimes 1_M)V((Q_0 Q_0^* + Q_1 Q_1^*) \otimes 1_M) \\
 &= V
 \end{aligned}$$

since, for  $s = 0, 1$ , the  $Q_s$  are isometries such that  $Q_s Q_s^* \in \text{End}(V)$  and  $Q_0 Q_0^* + Q_1 Q_1^* = \text{id}_{B \otimes B'}$ . Similarly

$$\begin{aligned}
 (\text{id} \otimes \tilde{\Psi})(\text{id} \otimes \Phi)(a(u_k)) &= (\text{id} \otimes \tilde{\Psi})(\tilde{A}_k) \\
 &= (\text{id} \otimes \tilde{\Psi})(\text{id} \otimes \pi)(A_k) \\
 &= (\text{id} \otimes \Psi)(A_k) \\
 &= (Q_k^* \otimes 1_N)(\text{id} \otimes \Psi)(U^{\otimes k})(Q_k \otimes 1_N) \\
 &= (Q_k^* \otimes 1_N)v^{\otimes k}(Q_k \otimes 1_N) \\
 &= a(u_k)
 \end{aligned}$$

The last equality requires particular attention. It is verified if and only if  $Q_k \in \text{Hom}(a(u_k), a(u)^{\otimes k})$ , therefore, in order to complete the proof, we have to check that the map  $Q_k$  defined during the proof is in  $\text{Hom}(a(u_k), a(u)^{\otimes k})$ . If  $k = 0, 1$ , it is clear. In the general case, for  $k \geq 2$ , we recall that

$$Q_k^* = \delta^{-\frac{k-1}{2}}(\text{id}_B \otimes S_k) \circ T_k = \delta^{-\frac{k-1}{2}}(\text{id}_B \otimes S_k)(m_B^{(k)} \otimes \text{id}_{B'}^{\otimes k})\Sigma_k$$

We claim that

$$\begin{aligned}
 Q_k^* &= (\text{id}_B \otimes S_k) \circ \delta'^{-\frac{1}{2}}(m_B \otimes \text{id}_{B'}^{\otimes k})\Sigma_{23} \circ (\text{id}_{B \otimes B'} \otimes \delta'^{-\frac{1}{2}}(m_B \otimes \text{id}_{B'}^{\otimes k-1})\Sigma_{23}) \circ \dots \\
 &\quad \dots \circ (\text{id}_{B \otimes B'}^{\otimes k-2} \otimes \delta'^{-\frac{1}{2}}(m_B \otimes \text{id}_{B'}^{\otimes 2})\Sigma_{23})
 \end{aligned}$$

This can be easily verified by evaluating the two formulations of  $Q_k^*$  on a general element of  $(B \otimes B')^{\otimes k}$ . The equality depends essentially on the associativity of the multiplication. Moreover, we observe that  $\delta'^{-\frac{1}{2}}(m_B \otimes \text{id}_{B'}^{\otimes k})\Sigma_{23} \in \text{Hom}(a(u) \otimes a(u^{\otimes k-1}), a(u^{\otimes k}))$  by Proposition 2.5.28 (4). Therefore, the linear map  $Q_k^*$  can be obtained as composition and tensor product of morphisms. It follows that  $Q_k^* \in \text{Hom}(a(u)^{\otimes k}, a(u_k))$  and  $Q_k \in \text{Hom}(a(u_k), a(u)^{\otimes k})$ .

The two morphisms  $\Phi$  and  $\tilde{\Psi}$  are then inverse to each other and the isomorphism

is proved. □

*Remark 2.20.* We observe that this theorem is coherent with the previous results of Banica and Bichon. In [BB07], they investigated the free wreath product of two quantum permutation groups and, in the particular case of two quantum symmetric groups, they proved that

$$C(S_{mn}^+)/I \cong C(S_m^+) *_w C(S_n^+)$$

where  $I \subset C(S_{mn}^+)$  is the closed two-sided  $*$ -ideal generated by the relations corresponding to the condition  $\text{id}_{\mathbb{C}^n} \otimes \eta_{\mathbb{C}^m} \eta_{\mathbb{C}^m}^* \in \text{End}(U)$  and  $U$  is the fundamental representation of  $S_{mn}^+$ .

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