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et soutenue publiquement par

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10 Mars 2015

## Micromagnétisme des films minces

Thèse dirigée par Rejeb HADIJI

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# Résumé

Les matériaux ferromagnétiques possèdent la propriété de devenir magnétiques, c'est à dire de s'aimanter, lorsqu'ils sont en présence d'un champ magnétique et de conserver une partie de leur magnétisation lorsque le champ est supprimé. C'est pour cette raison, ces matériaux sont devenus d'usage dans des nombreuses applications industrielles. Le modèle mathématique du micromagnétisme a été introduit par W.F. Brown (voir [11]) pour décrire le comportement de l'aimantation dans les matériaux ferromagnétiques depuis les années 40.

Pour étudier ce phénomène, on le transforme en un système l'étude de ces équations donnent les informations physiques attendus dans des espaces appropriés. Dans cette thèse on est intéressé à des structures minces de films ferromagnétiques. En pratique, une structure mince est un objet tridimensionnel ayant une ou deux directions prépondérantes comme par exemple une plaque, une barre ou un fil. Nous étudions le comportement de l'énergie quand l'épaisseur du film tend vers zéro.

Dans le premier travail, nous généralisons un résultat dû à Gioia et James à des dimensions supérieures à 4. Plus précisément, on considère un domaine mince borné ferromagnétique dans  $\mathbb{R}^n$ , le but est d'étudier les comportements asymptotiques de l'énergie libre du domaine mince ferromagnétique.

Dans le deuxième travail, on s'intéresse à une approche dynamique de problème micromagnétisme. On étudie le comportement asymptotique des solutions des équations Landau Lifshitz dans un multi-structure mince ferromagnétique composée de deux films minces orthogonaux d'épaisseur respectif  $h^a$  et  $h^b$ . On distingue différents régimes: lorsque  $\lim_{\frac{h^a}{h^b}} \in ]0, \infty[$ . On identifie le problème limite et on montre que ce dernier est couplé par une condition de jonction sur l'axe vertical  $x_2$ , pour tout  $x_2 \in ]-\frac{1}{2}, \frac{1}{2}[$ . La troisième partie est liée à ce dernier travail, nous complétons l'étude précédente lorsque  $\lim_{\frac{h^a}{h^b}} = 0$  et  $+\infty$  (voir [2]).

Ensuite dans la quatrième chapitre, on a étudié des phénomènes de micromagnétisme dans un multi-structure mince: il s'agit d'un ouvert connexe de  $\mathbb{R}^3$  composé de deux parties ayant un angle  $\theta \in ]0, \pi[$ , le but est d'étudier les comportements asymptotiques de l'énergie libre dans ce domaine lorsque l'épaisseur tend vers zéro. Il s'agit d'un problème non convexe et non local. On identifie le problème limite, et on montre que l'aimantation  $m^{(h)}$  converge vers une fonc-

tion  $\mu = (\mu^a, \mu^b)$  qui minimise ce dernier. Le problème limite obtenue est local, couplé par une condition de jonction  $\mu^a(x_2, 0) = \mu^b(0, x_2)$  pour  $x_2 \in ]-\frac{1}{2}, \frac{1}{2}[$ .

MOTS CLÉS : ferromagnétisme, micromagnétisme, film mince, multi-structure, jonction, analyse asymptotique, équation Landau Lifschitz, énergie d'échange, énergie de Zeeman, énergie d'anisotropie, énergie de démagnétisante.

# Abstrat

The ferromagnetic materials possess the property of becoming magnetic when placed in a magnetic field and conserve a part of this magnetism when the field is removed. For this reason, these materials have become usual in many industrial applications. The model micromagnetism was introduced by W. F. Brown in the 40s (see [11]) to describe the behavior of the magnetization from ferromagnetic materials.

To study thin physical phenomenon, we transform it into a system of partial differential equation. The results obtained from the study of these equation give the physical information in this space. In this thesis, we are interested in thin ferromagnetic structures. In practice, a thin structure is a three-dimensional object having one or two direction preponderant. For example a plate, a bar and a wire. We study the asymptotic behavior, as the thickness of the film tends to zero.

In the first work, We generalize a result due to Gioia and James to dimensions superior to 4. More precisely, we consider a thin bounded ferromagnetic domain in  $\mathbb{R}^n$ . The goal is to study the asymptotic behavior, where the free energy of this thin ferromagnetic domain.

In the second work, we are interested in a dynamic approach to the problem micromagnetism. We study the asymptotic behavior of solution of equations Landau-Lifshitz in a thin multi-structure ferromagnetic, composed of two thin orthogonal films a thickness  $h_n^a$  and  $h_n^b$  respectively. We distinguish different regimes depending on the limit  $\lim \frac{h_n^a}{h_n^b} \in ]0, +\infty[$ . We identify the limit problem and we prove that it is coupled by a junction condition on the vertical axis  $x_2$ , for all  $x_2 \in ]-\frac{1}{2}, \frac{1}{2}[$ .

The third part is linked to the second part. We complete the previous work by studying the cases when  $\lim \frac{h_n^a}{h_n^b} = 0$  and  $+\infty$  (see. [2]).

Then in the fourth chapter, we study the micromagnetism phenomenon in a thin multi-structure domain. This domain is an open connected in  $\mathbb{R}^3$  composed of two parts forming an angle  $\theta \in ]0, \pi[$  between them. The goal is to study the asymptotic behavior of the free energy in this domain when the thickness tends to zero, this is non-convex and non-local problem. We identify the limit problem, and we prove that the magnetization  $m^{(h)}$  converges to  $\mu = (\mu^a, \mu^b)$ , which minimizes our limit problem. the obtained limit problem is local, and coupled by the junction condition  $\mu^a(x_2, 0) = \mu^b(0, x_2)$  pour  $x_2 \in ]-\frac{1}{2}, \frac{1}{2}[$ .

KEYWORDS: Ferromagnetism, micromagnetism, thin film, multi-structure, junction, asymptotic analysis, Landau-Lifschitz equation, energy exchange, zee-man energy, anisotropy energy, demagnetizing energy.

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*Je dédie cette thèse à Diaa  
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# Introduction

## 0.1 Propriétés des matériaux ferromagnétiques

Les matériaux ferromagnétiques ont la particularité de présenter une aimantation non nulle en l'absence de toute excitation extérieure, et ceci sera une de leurs importantes caractéristiques. Grâce à cette propriété les matériaux ferromagnétiques sont devenus d'usage dans de nombreux secteurs d'activités industrielles, par exemples (télécommunication, protection de radar, stockage informatique,...).

Cette théorie a été développée par W.F. Brown dans les années 40 (voir [11]), a pour but d'identifier les principaux phénomènes intervenant dans la configuration de l'aimantation au sein d'un échantillon et leur associer une énergie. Les positions d'équilibre correspondent aux minima de l'énergie totale. Il est également possible d'utiliser un modèle dynamique, décrivant l'aimantation au cours du temps, en utilisant un système introduit par L.D. Landau et E.M. Lifschitz en 1935 (voir [50]).

## 0.2 Propriétés de base de l'aimantation

Dans la théorie classique du micromagnétisme, un matériau ferromagnétique est caractérisé par une aimantation spontanée représentée par un moment magnétique  $m$ :

$$m : \Omega \subset \mathbb{R}^3 \longrightarrow S^2$$

où  $\Omega$  est un ouvert borné de  $\mathbb{R}^3$ , il représente l'échantillon dans lequel le matériau est confiné,  $S^2$  est la sphère unité de  $\mathbb{R}^3$ , le corps est toujours localement magnétisé à une aimantation saturée  $|m(x)| = m_s(T)$ , où  $T$  est la température locale. Soit  $T_c$  la température de Curie,  $m_s > 0$  si  $T \geq T_c$ , et  $m_s = 0$  sinon. En principe, l'aimantation  $m$  n'est pas définie dans l'espace entier  $\mathbb{R}^3$ , mais uniquement dans l'échantillon  $\Omega$  du matériau ferromagnétique. Dans toute la suite, on considérera  $\tilde{m}$  comme un champ défini sur tout  $\mathbb{R}^3$ ,  $\tilde{m}$  prolonge  $m$  par 0 à l'extérieur de  $\Omega$ , ce qui permet de donner un sens aux diverses équations. Dans cette thèse, le moment magnétique  $\tilde{m}$  et le champ magnétique  $H$  sont liés par l'une des équations de Maxwell (sans charge, ni courant)

$$B = H + \tilde{m}, \quad \text{dans } \mathbb{R}^3$$

où  $B$  et  $H(\tilde{m})$  sont définis sur tout  $\mathbb{R}^3$ , avec  $H(\tilde{m}) \equiv -D\zeta$  ( $\zeta$  est le potentiel magnétique).

### 0.3 Energie libre du modèle classique en micro-magnétisme

Les moments magnétiques observés sont des minimiseurs (locaux) de l'énergie ferromagnétique

$$(0.3.1) \quad E(m) = \int_{\Omega} \left( \alpha |Dm|^2 + \varphi(m) + \frac{1}{2} D\zeta m - 2fm \right) dx$$

nous allons expliquer chaque terme.

#### 0.3.1 L'énergie d'échange

L'énergie d'échange est due à l'existence d'une force responsable à aligner les spins voisins des deux atomes. Cette contribution est locale, elle dépend des propriétés microscopiques du matériaux, cette énergie s'écrit:

$$E_e(m) = \int_{\Omega} \alpha |Dm|^2 dx,$$

où  $\alpha$  est le coefficient d'échange.

#### 0.3.2 L'énergie d'anisotropie

L'anisotropie magnéto-cristalline tient compte des effets d'anisotropie dus à la structure cristalline du matériau. On en tient compte grâce à une fonction paire et continue

$$\varphi : S^2 \longrightarrow \mathbb{R}^+$$

Nous observons principalement deux cas

- *L'anisotropie uniaxiale*: cette anisotropie se trouve dans les cristaux hexagonaux (*Co*). L'expression de cette énergie est :

$$\varphi(m) = -K_1 m_z^2 + K_2 m_z^4$$

où  $m_z$  est la composante selon l'axe  $z$  du vecteur  $m$ ,  $K_1$  et  $K_2$  sont les coefficients d'anisotropie dépendant de la température, les valeurs de ces coefficients sont tirées de l'expérience (dans les plupart de cas même  $K_2$  est négligeable) par exemple dans le cas du matériel fer on a  $K_1 = 4,81 \times 10^6 J.m^{-3}$  et  $K_2 = 1,2 \times 10^4$ .

- *L'anisotropie cubique*: Dans le cas des cristaux cubiques (*Fe, Ni*), les axes  $x$ ,  $y$  et  $z$  étant choisis le long des axes du cristal, l'expression de l'énergie est la suivante:

$$\varphi(m) = K_1 (m_x^2 m_y^2 + m_y^2 m_z^2 + m_z^2 m_x^2) + K_2 m_x^2 m_y^2 m_z^2$$

où  $K_1$  et  $K_2$  sont les coefficients d'anisotropie dépendant de la température et  $m_x$ ,  $m_y$  et  $m_z$  les composantes du vecteur  $m$ . À température ambiante, voici les valeurs calculées pour certains matériaux ferromagnétique (voir [21])

Matériaux	<i>Fe(cc)</i>	<i>Ni(cc)</i>	<i>Ni<sub>80</sub>Fe<sub>20</sub></i>
$K_1(J.m^{-3})$	$4,81 \times 10^6$	$-5,48 \times 10^5$	0
$K_2(J.m^{-3})$	$1,2 \times 10^4$	$-2,47 \times 10^5$	$-1,5 \times 10^3$

### 0.3.3 L'énergie magnétostatique

Le troisième terme de (0.3.1) représente l'énergie magnétostatique en présence d'un champ magnétique qu'elle produit par elle même. En partant des équations de Maxwell sans charge ni courant, on trouve que le champ magnétique est déterminé à partir de l'aimantation comme solution du problème suivant:

$$(0.3.2) \quad \begin{cases} \operatorname{div}(H(\tilde{m}) + \tilde{m}) = 0, & \text{dans } \mathbb{R}^3, \\ \operatorname{rot}H(\tilde{m}) = 0 & \text{dans } \mathbb{R}^3. \end{cases}$$

Par conséquent, on obtient

$$E_{mag}(m) = \frac{1}{2} \int_{\Omega} D\zeta m dx = \frac{1}{2} \int_{\mathbb{R}^3} |D\zeta|^2 dx.$$

Remarquons que l'énergie magnétostatique exprime une interaction non locale.

### 0.3.4 L'énergie de Zeeman ou extérieure

L'énergie de Zeeman, ou l'énergie du champ externe, est l'énergie d'un corps aimanté dans un champ magnétique externe, elle est donnée par:

$$E_{ext}(m) = -2 \int_{\Omega} f m dx,$$

où  $f : \Omega \rightarrow \mathbb{R}^3$  est le champ extérieur.

## 0.4 $nD-pD$ Réduction dimensionnelle des structures micromagnétiques

### 0.4.1 Présentation du problème

Ce travail est publié dans Recherche di Matematica (2014). Dans ce travail je généralise un résultat dû à Gioia et James (voir [39]). Ces auteurs ont étudié le comportement asymptotique d'un film mince ferromagnétique de  $\mathbb{R}^3$ , d'épaisseur  $h$ ,  $h$  est un paramètre positif qui tend vers zéro. Ce chapitre généralise leur



résultat, précisément je détermine via une analyse asymptotique l'énergie libre d'un domaine ferromagnétique  $p$ -dimensionnel dans  $\mathbb{R}^n$ ,  $1 \leq p < n$ .

On considère un domaine mince dans  $\mathbb{R}^n$ , défini par  $\Omega_h = \Theta \times ]-h, h[$ ,  $h^{[n-p} \subset \mathbb{R}^n$ , où  $h$  est un paramètre qui tend vers zéro,  $\Theta$  est un ouvert borné de  $\mathbb{R}^p$ . Le but de ce chapitre est d'étudier le comportement asymptotique, quand  $h$  tend vers zéro, du problème suivant:

$$(0.4.1) \quad \frac{1}{h^{n-p}} \int_{\Omega_h} \left( |Dm|^2 + \varphi(m) + \frac{1}{2} Dzm - 2F_h m \right) dx, m \in H^1(\Omega_h, S^{n-1})$$

avec

$$(0.4.2) \quad \left\{ \begin{array}{l} z \in \left\{ u \in L^1_{loc}(\mathbb{R}^n) : u \in L^2(B), Du \in (L^2(\mathbb{R}^n))^n, \int_B u dx = 0 \right\}, \\ \int_{\mathbb{R}^n} Dz D\zeta dx = \int_{\mathbb{R}^n} \tilde{m} D\zeta dx, \quad \forall \zeta \in \mathcal{U}, \end{array} \right.$$

où  $B$  est un ouvert contenant  $\overline{\Omega_h}$ ,  $\varphi : S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\} \rightarrow [0, +\infty[$  est une fonction paire continue,  $F_h \in L^2(\Omega_h, \mathbb{R}^n)$ , et  $m = 0$  à l'extérieur de  $\Omega_h$  et vérifie les propriétés de base de l'aimantation (voir paragraphe 0.2). La fonction  $z : \mathbb{R}^n \rightarrow \mathbb{R}$  est le potentiel magnétique, il est lié à l'aimantation par (0.4.2). D'après Visintin (voir [57]), le problème (0.4.1) admet au moins une solution. Afin de travailler sur un domaine fixe, on utilise le changement d'échelle suivant

$$(x', x'') \in \Omega \rightarrow (x', hx'') \in \Omega_h,$$

où  $(x', x'')$  est un point de  $\mathbb{R}^n$ , avec  $x' = (x_1, \dots, x_p)$  et  $x'' = (x_{p+1}, \dots, x_n)$ . Posons

$$f_h(x) = F_h(x', hx''), \quad x \text{ p.p. dans } \Omega,$$

on va supposer que

$$(0.4.3) \quad f_h \rightharpoonup f \quad \text{faiblement dans } L^2(\Omega, \mathbb{R}^n)$$

quand  $h$  tends vers zéro.

## 0.4.2 Les principaux résultats

En étudiant les comportements asymptotiques de l'énergie libre on trouve que nos résultats principaux dépendent de  $n - p \in \mathbb{N}^*$ .

Soit

$$(0.4.4) \quad M = \{\mu \in H^1(\Omega, S^{n-1}) : \mu \text{ est indépendant de } x''\} \simeq H^1(\Theta, S^{n-1}).$$

Dans le cas  $n - p \geq 2$ , on y établit le résultat suivant:

**Théorème 0.4.1.** Soit  $m^{(h)}$  est une solution minimisante de (0.4.1) et  $\zeta^{(h)}$  est l'unique solution de (0.4.2) qui correspond à  $m^{(h)}$ . Alors, il existe une sous-suite notée aussi  $(m^{(h)})_h$  et une fonction  $\hat{\mu} \in M$  dépendant de la sous-suite, tels que

$$\begin{aligned} m^{(h)} &\longrightarrow \hat{\mu} \quad \text{fortement dans } H^1(\Omega, S^{n-1}), \\ \frac{1}{h} D_{x''} m^{(h)} &\longrightarrow 0 \quad \text{fortement dans } L^2(\Omega, S^{n-1}), \end{aligned}$$

et

$$\begin{cases} D_{x'} \zeta^{(h)} \longrightarrow 0 & \text{fortement dans } L^2(\mathbb{R}^n), \\ \frac{1}{h} D_{x''} \zeta^{(h)} \longrightarrow \hat{\xi} & \text{fortement dans } L^2(\mathbb{R}^n), \end{cases}$$

où  $\hat{\mu}$  est une solution de problème suivant:

$$E_0(\hat{\mu}) = \min \{E_0(\mu), \mu \in M\},$$

avec

$$\hat{\xi}(x) = \begin{cases} \sum_{i=p+1}^n \hat{\mu}_i(x') D_{x''} P_i(x'') & \text{p.p. dans } \Theta \times \mathbb{R}^{n-p}, \\ 0 & \text{p.p. dans } \mathbb{R}^p \setminus \Theta \times \mathbb{R}^{n-p}, \end{cases}$$

de plus on a

$$\lim_{h \rightarrow 0} E_h^{(h)}(m^{(h)}) = E_0(\hat{\mu}).$$

Avec

$$(0.4.5) \quad \begin{aligned} E_0 : \mu \in M &\longrightarrow \int_{\Omega} (\alpha |D_{x'} \mu|^2 + \varphi(\mu) - 2f\mu) dx + \\ &\frac{1}{2} \left( \sum_{i=p+1}^n \int_{\mathbb{R}^{n-p}} (|D_{x''} P_i|^2 \int_{\Theta} |\mu_i|^2 dx') dx'' \right) + \\ &\frac{1}{2} \left( \sum_{i,j \neq p+1}^n \int_{\mathbb{R}^{n-p}} (D_{x''} P_i D_{x''} P_j \int_{\Theta} \mu_i \mu_j dx') dx'' \right). \end{aligned}$$

où  $\hat{\mu}_i$ ,  $i \in \{p+1, \dots, n\}$  est le  $i^{\text{ème}}$  composante de  $\hat{\mu}$ , et  $P_i$ ,  $i \in \{p+1, \dots, n\}$  est l'unique solution du problème suivant:

$$(0.4.6) \quad \begin{cases} P_i \in BL_0(\mathbb{R}^{n-p}), \\ \int_{\mathbb{R}^{n-p}} D_{x''} P_i D_{x''} \phi dx'' = \int_{]-1,1[^{n-p}} D_{x_i} \phi dx'', \quad \forall \phi \in BL_0(\mathbb{R}^{n-p}), \end{cases}$$

où  $BL_0(\mathbb{R}^{n-p})$  est l'espace Beppo-Levi dans  $\mathbb{R}^{n-p}$ , voir [34] pour plus de détail en dimension 3.

La preuve du théorème répose sur la proposition suivante:

**Proposition 0.4.1.** *On suppose que  $n \geq 3$ ,  $p \geq 1$  et  $n - p \geq 2$ . Soit  $(m_h)_h \subset L^2(\Omega, \mathbb{R}^n)$ . Supposons qu'il existe  $\mu \in L^2(\Omega, \mathbb{R}^n)$  qui ne dépend pas de  $x''$  tel que*

$$m_h \longrightarrow \mu \quad \text{fortement dans } L^2(\Omega, \mathbb{R}^n),$$

quand  $h$  tends vers zéro. Soit  $\zeta_h$  est l'unique solution du problème (0.4.2) qui correspond à  $m_h$ . Alors, on a

$$\begin{cases} D_{x'} \zeta_h \longrightarrow 0 & \text{fortement dans } (L^2(\mathbb{R}^n))^p, \\ \frac{1}{h} D_{x''} \zeta_h \longrightarrow \xi & \text{fortement dans } (L^2(\mathbb{R}^n))^{n-p}, \end{cases}$$

quand  $h$  tends vers zéro, où

$$\xi(x) = \begin{cases} \sum_{i=p+1}^n \mu_i(x') D_{x''} P_i(x''), & \text{p.p. dans } \Theta \times \mathbb{R}^{n-p}, \\ 0, & \text{p.p. dans } (\mathbb{R}^p \setminus \Theta) \times \mathbb{R}^{n-p}, \end{cases}$$

avec  $P_i$ ,  $i \in \{p+1, \dots, n\}$ , l'unique solution de (0.4.6). De plus,

$$\begin{aligned} \lim_{h \rightarrow 0} E_h^{mag}(m_h) &= \frac{1}{2} \sum_{i=p+1}^n \int_{\mathbb{R}^{n-p}} |D_{x''} P_i|^2 dx'' \int_{\Theta} |\mu_i|^2 dx' + \\ &\frac{1}{2} \sum_{\substack{i, j = p+1 \\ i \neq j}}^n \int_{\mathbb{R}^{n-p}} D_{x''} P_i D_{x''} P_j dx'' \int_{\Theta} \mu_i \mu_j dx'. \end{aligned}$$

Si  $n - p = 1$  on prouve le résultat suivant

**Théorème 0.4.2.** *Soit  $m^{(h)}$  est une solution minimisante de (0.4.1) et  $\zeta^{(h)}$  est l'unique solution de (0.4.2) qui correspond à  $m^{(h)}$ . Alors, il existe une sous-suite de  $m^{(h)}$  notée aussi  $(m^{(h)})_h$  et  $\hat{\mu} \in M$ , dépendant de la sous-suite tels que, pour tout  $j = 1, \dots, n - 1$*

$$\begin{aligned} m^{(h)} &\longrightarrow \hat{\mu} \quad \text{fortement dans } H^1(\Omega, S^{n-1}), \\ \frac{1}{h} D_{x_n} m^{(h)} &\longrightarrow 0 \quad \text{fortement dans } L^2(\Omega, S^{n-1}), \end{aligned}$$

et

$$\begin{cases} D_{x_j} \zeta^{(h)} \longrightarrow 0 & \text{fortement dans } L^2(\mathbb{R}^n), \\ \frac{1}{h} D_{x_n} \zeta^{(h)} \longrightarrow \tilde{\mu}_n & \text{fortement dans } L^2(\mathbb{R}^n), \end{cases}$$

où  $\tilde{\mu}_n$  est le prolongement par zéro de  $\hat{\mu}_n$  sur  $\mathbb{R}^n$ , et  $\hat{\mu}$  est une solution du problème suivant:

$$E_1(\hat{\mu}) = \min \{E_1(\mu), \mu \in M\}.$$

De plus, on a

$$\lim_{h \rightarrow 0} E_h^{(h)}(m^{(h)}) = E_1(\hat{\mu}).$$

Où

$$E_1 : \mu \in M \longrightarrow \int_{\Omega} (\alpha |D_{x'} \mu|^2 + \varphi(\mu) - 2f\mu) dx + \frac{1}{2} \int_{\Omega} |\mu_n|^2 dx.$$

Comme précédemment pour démontrer ce théorème on commence par prouver le résultat de convergence de l'énergie magnétostatique.

## 0.5 Problème en présence de poids

Je suis en train de généraliser ces résultats le même problème en présence de poids  $\alpha^{(h)}(x)$ ,  $\beta^{(h)}(x)$  and  $\gamma^{(h)}(x)$  dépendant de  $h$

## 0.6 Problem with weight

Dans ce travail, je reprends le problème précédente en prsence de poids. Plus précisément, on étudie les comportements asymptotiques du problème suivant:

$$(0.6.1) \quad \min \left\{ \frac{1}{h^{n-p}} \int_{\Omega_h} \left( \underline{\alpha}_h(x) |D\underline{m}|^2 + \varphi(\underline{m}) + \frac{1}{2} \underline{\beta}_h(x) D\underline{\zeta} \underline{m} - 2F_h \underline{m} \right) dx, \underline{m} \in H^1(\Omega_h, S^{n-1}) \right\}$$

à condition que

$$(0.6.2) \quad \operatorname{div} \left( -\underline{\gamma}_h(x) D\underline{\zeta} + \underline{\beta}_h(x) \underline{m} \right) = 0 \quad \text{dans } \mathbb{R}^n,$$

où

$$(0.6.3) \quad \underline{\alpha}_h \in L^\infty(\Omega_h), \quad \underline{\beta}_h \in L^\infty(\Omega_h), \quad \underline{\gamma}_h \in L^\infty(\mathbb{R}^n),$$

$$(0.6.4) \quad 0 \leq \underline{\beta}_h(x), \quad x \text{ p.p. dans } \Omega_h, \quad \forall h,$$

il existe une constante  $c_1 > 0$  tels que

$$(0.6.5) \quad c_1 \leq \underline{\alpha}_h(x), \quad x \text{ p.p. dans } \Omega_h, \quad \forall h,$$

et

$$(0.6.6) \quad c_1 \leq \underline{\gamma}_h(x), \quad x \text{ p.p. dans } \mathbb{R}^n, \quad \forall h.$$

Afin de travailler sur un domaine fixe, posons

$$\alpha_h(x) = \underline{\alpha}_h(x', hx''), \quad \beta_h(x) = \underline{\beta}_h(x', hx''), \quad x \text{ p.p. dans } \Omega,$$

$$\gamma_h(x) = \underline{\gamma}_h(x', hx''), \quad x \text{ p.p. dans } \mathbb{R}^n,$$

$$f_h(x) = F_h(x', hx''), \quad x \text{ p.p. dans } \Omega,$$

et supposons que

$$(0.6.7) \quad \begin{cases} \alpha_h \rightarrow \alpha & \text{fortement dans } L^\infty(\Omega), \\ \beta_h \rightarrow \beta & \text{fortement dans } L^\infty(\Omega), \\ \gamma_h \rightarrow \gamma & \text{fortement dans } L^\infty(\mathbb{R}^n), \end{cases}$$

De même, notre résultat principal dépend de  $n - p$ . Précisément, si  $n - p \geq 2$ , je prouve que

**Theorem 0.6.1.** *Soit  $n \geq 3$ ,  $p \geq 1$  et  $n - p \geq 2$ . Pour tout  $h$ , soit  $m_h$  est une solution minimisante de (0.6.1) et soit  $\zeta_h$  est l'unique solution de (0.6.2) qui correspond à  $m_h$ . Supposons (0.6.3)÷(0.6.7). Alors, il existe une sous-suite notée aussi  $\{m_h\}_h$ , et  $\hat{\mu} \in M$ , dépendant de la sous-suite, tels que*

$$\begin{aligned} m_h &\longrightarrow \hat{\mu} && \text{fortement dans } H^1(\Omega, S^{n-1}), \\ \frac{1}{h} D_{x''} m_h &\longrightarrow 0 && \text{fortement dans } (L^2(\Omega))^{n-p}, \end{aligned}$$

et

$$\begin{cases} D_{x'} \zeta_h \longrightarrow 0 & \text{fortement dans } (L^2(\mathbb{R}^n))^p, \\ \frac{1}{h} D_{x''} \zeta_h \longrightarrow \hat{\xi} & \text{fortement dans } (L^2(\mathbb{R}^n))^{n-p}, \end{cases}$$

où  $\hat{\mu}$  est une solution de problème suivant

$$E_0(\hat{\mu}) = \min \{E_0(\mu), \quad \mu \in M\}$$

avec

$$\hat{\xi}(x) = \begin{cases} \sum_{i=p+1}^n \hat{\mu}_i(x') D_{x''} P_i(x''), & p.p. \text{ dans } \Theta \times \mathbb{R}^{n-p}, \\ 0, & p.p. \text{ dans } (\mathbb{R}^p \setminus \Theta) \times \mathbb{R}^{n-p}, \end{cases}$$

où  $P_i$ ,  $i \in \{p+1, \dots, n\}$ , est l'unique solution de (1.2.1). De plus

$$\lim_{h \rightarrow 0} E_h(m_h) = E_0(\hat{\mu}).$$

Avec

$$(0.6.8) \quad \begin{aligned} E_0 : \mu \in M &\longrightarrow \int_{\Omega} (\alpha |D_{x'} \mu|^2 + \varphi(\mu) - 2f\mu) dx + \\ &\frac{1}{2} \sum_{i=p+1}^n \int_{\mathbb{R}^{n-p}} \left( |D_{x''} P_i|^2 \int_{\Theta} \gamma |\mu_i|^2 dx' \right) dx'' + \\ &\frac{1}{2} \sum_{\substack{i, j = p+1 \\ i \neq j}}^n \int_{\mathbb{R}^{n-p}} \left( D_{x''} P_i D_{x''} P_j \int_{\Theta} \gamma \mu_i \mu_j dx' \right) dx''. \end{aligned}$$

Si  $n - p = 1$ , on prouve le résultat suivant

**Théorème 0.6.1.** *Soit  $n \geq 2$  et  $n - p = 1$ . Pour tout  $h$ , soit  $m_h$  est une solution minimisante de (0.6.1) et  $\zeta_h$  est l'unique solution de (0.6.2) correspond à  $m_h$ . Supposons (0.6.3)÷(0.6.7). Alors, il existe une sous-suite notée aussi  $\{m_h\}_h$ , et  $\hat{\mu} \in M$ , dépendant de la sous-suite, tels que*

$$\begin{aligned} m_h &\longrightarrow \hat{\mu} \quad \text{fortement dans } H^1(\Omega, S^{n-1}), \\ \frac{1}{h} D_{x_n} m_h &\longrightarrow 0 \quad \text{fortement dans } L^2(\Omega), \end{aligned}$$

et, pour  $j = 1, \dots, n - 1$ ,

$$\begin{cases} D_{x_j} \zeta_h \longrightarrow 0 & \text{fortement dans } L^2(\mathbb{R}^n), \\ \frac{1}{h} D_{x_n} \zeta_h \longrightarrow \frac{\beta}{\gamma} \tilde{\mu}_n & \text{fortement dans } L^2(\mathbb{R}^n), \end{cases}$$

où  $\hat{\mu}$  est une solution de problème suivant:

$$E_1(\hat{\mu}) = \min \{E_1(\mu), \quad \mu \in M\}$$

et  $\tilde{\mu}_n$  est le prolongement par zero à l'extérieur de  $\Omega$ . De plus on a

$$\lim_{h \rightarrow 0} E_h(m_h) = E_1(\hat{\mu}).$$

Avec

$$E_1 : \mu \in M \longrightarrow \int_{\Omega} (\alpha |D_{x'} \mu|^2 + \varphi(\mu) - 2f\mu) dx + \frac{1}{2} \int_{\Omega} \frac{\beta^2}{\gamma} |\mu_n|^2 dx.$$

## 0.7 Jonction des films minces ferromagnétiques en micromagnétisme

Le deuxième et le troisième travail est en collaboration avec L. Faella sont consacrés à étudier le comportement asymptotique d'un modèle quasi-stationnaire

ferromagnétique, composé de deux films minces perpendiculaires, et qui se joint par une condition de jonction sur  $(h_n^a x_1, x_2, 0)$ . D'un point de vue physique, ce modèle décrit les propagations d'ondes électromagnétiques dans un milieu ferromagnétique. Plus précisément, pour  $n \in \mathbb{N}$ , on considère  $\Omega_n = \Omega_n^a \cup \Omega_n^b$ , avec

$$\Omega_n^a = \left( \left[ -\frac{h_n^a}{2}, \frac{h_n^a}{2} \right] \times \left[ -\frac{1}{2}, \frac{1}{2} \right] \times ]0, 1[ \right) \quad \Omega_n^b = \left( \left[ -\frac{1}{2}, \frac{1}{2} \right]^2 \times ]-h_n^b, 0[ \right),$$

où  $\{h_n^a\}_{n \in \mathbb{N}}, \{h_n^b\}_{n \in \mathbb{N}} \subset ]0, 1[$  tel que

$$(0.7.1) \quad \lim_n h_n^a = 0 = \lim_n h_n^b, \quad \lim_n \frac{h_n^b}{h_n^a} = q \in [0, +\infty].$$

D'après les propriétés de base de l'aimantation le moment magnétique  $M$  induit un champ magnétique  $H(M)$  donnée par

$$(0.7.2) \quad \begin{cases} H(M) \in (L^2(\mathbb{R}^3))^3 \\ \operatorname{curl} H(M) = 0 \\ \operatorname{div}(H(M) + \overline{M}) = 0 \end{cases} \quad \begin{array}{l} \text{dans } D'(\mathbb{R}^3) \\ \text{dans } D'(\mathbb{R}^3) \end{array}.$$

Le but de ce travail est d'étudier le comportement asymptotique, quand  $n$  diverge, du problème quasi-stationnaire suivant

$$(0.7.3) \quad \begin{cases} \frac{\partial M}{\partial t} + M \wedge \frac{\partial M}{\partial t} = 2M \wedge (-\Delta M + DU_M) & \text{in } ]0, T[ \times \Omega_n \\ \text{avec (0.7.2)} \end{cases}$$

Dans ces chapitres l'aimantation  $M$  dépend du temps  $t$ , soit  $M_0$  l'aimantation au temps  $t = 0$ . Pour  $M_0(x) \in H^1(\Omega_n)$ ,  $|M_0(X)| = 1$  p.p et  $\operatorname{div}(M_0) + M_0 = 0$  sous ces hypothèses initiales, problème (0.7.3) admet au moins une solution faible  $M \in L^\infty(0, T; H^1(\Omega_n))$  de telle sorte que  $\frac{\partial M}{\partial t} \in L^2(0, T; H^1(\Omega_n))$  (voir [14] et [57]). En étudiant le comportement asymptotique du problème (0.7.3), on remarque que notre résultat dépend de  $q \in [0, +\infty]$ . Dans le chapitre 2 on traite le cas  $q \in ]0, +\infty[$ , nous obtenons deux problèmes  $2D$  couplés par un condition de jonction sur  $] -\frac{1}{2}, \frac{1}{2}[$ . Soit

$$\mathcal{M} = \left\{ \begin{array}{l} \psi = (\psi^a, \psi^b) \in H^1(\Omega^a, S^2) \times H^1(\Omega^b, S^2) : \psi^a \text{ ne dépend pas de } x_1, \\ \psi^b \text{ ne dépend pas de } x_3, \psi^a(0, x_2, 0) = \psi^b(0, x_2, 0), \text{ pour } x_2 \text{ p.p. dans } ]-\frac{1}{2}, \frac{1}{2}[ \end{array} \right\}$$

Dans ce chapitre on y établit le résultat suivant

**Théorème 0.7.1.** *Soit  $\underline{m}_n = (\underline{m}_n^a, \underline{m}_n^b)$  est une solution du notre problème*

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(0.7.3) avec ces hypothèses initiales:

$$\begin{aligned} \underline{\mu}_0 &= (\underline{\mu}_0^a, \underline{\mu}_0^b) \in \mathcal{M}, \\ \underline{m}_{n_0}^a &\rightarrow \underline{\mu}_0^a \text{ fortement dans } H^1(\Omega^a, S^2), \\ \underline{m}_{n_0}^b &\rightarrow \underline{\mu}_0^b \text{ fortement dans } H^1(\Omega^b, S^2), \\ \frac{1}{h_n^a} D_{x_1} \underline{m}_{n_0}^a &\rightarrow 0 \text{ fortement dans } L^2(\Omega^a, \mathbb{R}^3), \\ \frac{1}{h_n^b} D_{x_3} \underline{m}_{n_0}^b &\rightarrow 0 \text{ fortement dans } L^2(\Omega^b, \mathbb{R}^3). \end{aligned}$$

Notons  $u_n = (u_n^a, u_n^b)$  l'unique solution du Problème (0.7.2) qui correspond à  $\underline{m}_n$ . Alors, il existe une sous-suite  $(m_n)_n$  et  $\underline{\mu} = (\underline{\mu}^a, \underline{\mu}^b) = \left( (\underline{\mu}_1^a, \underline{\mu}_2^a, \underline{\mu}_3^a), (\underline{\mu}_1^b, \underline{\mu}_2^b, \underline{\mu}_3^b) \right) \in L^\infty(0, T; \mathcal{M})$ , dépend de la sous-suite de telle sorte que

$$\underline{m}_n \rightharpoonup \underline{\mu} \text{ faible } * \text{ dans } L^\infty(0, T; \mathcal{M}; S^2)$$

$$\underline{m}_n \rightarrow \underline{\mu} \text{ fortement dans } L^2(0, T; L^2(\Omega^a) \times L^2(\Omega^b)) \text{ et p. p. dans } [0, T] \times (\Omega^a \cup \Omega^b)$$

où  $\underline{\mu}$  est la solution du problème suivant:

$$\left\{ \begin{array}{l} \underline{\mu}(t=0) = \underline{\mu}_0 = (\underline{\mu}_0^a, \underline{\mu}_0^b) \in \mathcal{M} \\ \underline{\mu} = (\underline{\mu}^a, \underline{\mu}^b) \in L^\infty(0, T; \mathcal{M}) \\ \forall t \in [0, T] \quad |\underline{\mu}| = 1 \text{ pour p.p. } x \in \Omega^a \cup \Omega^b \\ \frac{\partial \underline{\mu}}{\partial t} \in L^2\left(0, T; L^2\left(]-\frac{1}{2}, \frac{1}{2}[ \times ]0, 1[, S^2\right) \times L^2\left(]-\frac{1}{2}, \frac{1}{2}[^2, S^2\right)\right) \\ \forall \varphi \in D(0, T) \text{ et } g = (g^a, g^b) \in \mathcal{M} \\ \int_0^T \int_{]-\frac{1}{2}, \frac{1}{2}[ \times ]0, 1[} \left( \frac{\partial \underline{\mu}^a}{\partial t} + \underline{\mu}^a \wedge \frac{\partial \underline{\mu}^a}{\partial t} \right) \varphi g^a dx dt + \int_0^T \int_{]-\frac{1}{2}, \frac{1}{2}[^2} \left( \frac{\partial \underline{\mu}^b}{\partial t} + \underline{\mu}^b \wedge \frac{\partial \underline{\mu}^b}{\partial t} \right) \varphi g^b dx dt = \\ -2 \int_0^T \int_{]-\frac{1}{2}, \frac{1}{2}[ \times ]0, 1[} \sum_{i=2}^3 \underline{\mu}^a \wedge \frac{\partial \underline{\mu}^a}{\partial x_i} \frac{\partial g^a}{\partial x_i} \varphi - 2q \int_0^T \int_{]-\frac{1}{2}, \frac{1}{2}[^2} \sum_{i=1}^2 \underline{\mu}^b \wedge \frac{\partial \underline{\mu}^b}{\partial x_i} \frac{\partial g^b}{\partial x_i} \varphi \\ -2 \int_0^T \int_{]-\frac{1}{2}, \frac{1}{2}[ \times ]0, 1[} \underline{\mu}^a \wedge (\underline{\mu}^a, e_1) e_1 g^a \varphi - 2q \int_0^T \int_{]-\frac{1}{2}, \frac{1}{2}[^2} \underline{\mu}^b \wedge (\underline{\mu}^b, e_3) e_3 g^b \varphi. \\ \forall t \in [0, T] \\ E_q(\underline{\mu}(t, \cdot)) + \int_0^t \left\| \frac{\partial \underline{\mu}^a}{\partial t} \right\|_{L^2(\Omega^a)}^2 dx dt + q \int_0^t \left\| \frac{\partial \underline{\mu}^b}{\partial t} \right\|_{L^2(\Omega^b)}^2 dx dt \leq E_q(\underline{\mu}(0, \cdot)) \end{array} \right.$$

où

$$E_q(\underline{\mu}(t, \cdot)) = \int_{]-\frac{1}{2}, \frac{1}{2}[ \times ]0, 1[} |D\underline{\mu}^a|^2 + q \int_{]-\frac{1}{2}, \frac{1}{2}[^2} |D\underline{\mu}^b|^2 + \frac{1}{2} \int_{]-\frac{1}{2}, \frac{1}{2}[ \times ]0, 1[} |\underline{\mu}_1^a|^2 dx_2 dx_3 + \frac{1}{2} q \int_{]-\frac{1}{2}, \frac{1}{2}[^2} |\underline{\mu}_3^b|^2 dx_1 dx_2.$$



Dans le troisième travail, on complète l'étude précédente dans les cas  $q = 0$  et  $q = +\infty$ , la structure se comporte comme un seul film. Plus précisément, lorsque  $q = 0$  (c.à.d  $h_n^b \ll h_n^a$ ) nous prouvons que le problème limite se réduit à un film mince verticale  $2D$  et on perd la condition de jonction. Pareillement, pour  $q = +\infty$  (c.à.d  $h_n^a \ll h_n^b$ ) nous prouvons que le problème limite se réduit à un film mince horizontale  $2D$  et on perd aussi la condition de jonction (voir Chapitre 2).

## 0.8 Analyse asymptotique de deux films minces obliques ferromagnétiques

Ce travail est écrit en collaboration avec R. Hadiji. Un multi-domaine mince est une structure de deux domaines minces qui sont liés par une condition de jonction, et qui sont très fins. Gaudiello et Hadiji ont beaucoup étudié sur ce type de multi-domaine (voir [31], [33],[34]).

### 0.8.1 Mise en équations du problème

Dans ce chapitre on considère une multi-structure composé de deux films minces ferromagnétiques, formé de deux films minces ayant un angle  $\theta \in ]0, \pi[$  et sont liés par une condition de jonction sur l'axe  $(h_n^a x_1, x_2, 0)$ ,  $\forall (x_1, x_2) \in ]-\frac{1}{2}, \frac{1}{2}[ \times ]-\frac{1}{2}, \frac{1}{2}[$ . Plus précisément, on considère  $\Omega_n^{\theta_0} = \Omega_n^{a,\theta_0} \cup \Omega_n^b$ , avec  $n \in \mathbb{N}$ ,  $\Omega_n^b = ]-\frac{1}{2}, \frac{1}{2}[ \times ]-\frac{1}{2}, \frac{1}{2}[$  et  $h_n^b, 0[$  et

$$\Omega_n^{a,\theta_0} = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 : |x_1 - \cot \theta_0 x_3| < \frac{h_n^a}{2}, (x_2, x_3) \in ]-\frac{1}{2}, \frac{1}{2}[ \times [0, \sin \theta_0[ \right\},$$

où  $h_n^a$  et  $h_n^b$  sont les épaisseurs de  $\Omega_n^{a,\theta_0}$  et  $\Omega_n^b$  respectivement, tels que

$$(0.8.1) \quad \begin{cases} \lim_n h_n^a = 0 = \lim_n h_n^b, \\ \lim_n \frac{h_n^b}{h_n^a} = q \in [0, +\infty]. \end{cases}$$

Le but de ce chapitre est d'étudier le comportement asymptotique quand  $n$  diverge du problème suivant:

$$(0.8.2) \quad \min \left\{ \int_{\Omega_n^{\theta_0}} (\alpha |D\underline{m}|^2 + \varphi(\underline{m}) + \frac{1}{2} D\underline{\zeta} \underline{m} - 2F_n \underline{m}) dx, \underline{m} \in H^1(\Omega_n^{\theta_0}, S^2) \right\}$$

avec

$$(0.8.3) \quad \begin{cases} z \in \left\{ u \in L^1_{loc}(\mathbb{R}^3) : u \in L^2(B), Du \in (L^2(\mathbb{R}^3))^3, \int_B u dx = 0 \right\}, \\ \int_{\mathbb{R}^3} Dz D\underline{\zeta} dx = \int_{\Omega} m D\underline{\zeta} dx, \quad \forall \underline{\zeta} \in \mathcal{U}, \end{cases}$$

où  $B$  est un ouvert contenant  $\overline{\Omega_n^\theta}$ . Le problème (0.8.2) admet au moins une solution minimisante voir [57]. Afin de travailler sur un domaine fixe, on utilise le changement de variable suivant:

$$\left\{ \begin{array}{l} (x_1, x_2, x_3) \in \Omega^{a, \theta_0} = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 : |x_1 - \cot \theta_0 x_3| < \frac{1}{2}, (x_2, x_3) \in ] -\frac{1}{2}, \frac{1}{2}[ \times ] 0, \sin \theta_0[ \right\} \\ \qquad \qquad \qquad \longrightarrow (h_n^a x_1 + \frac{\cos \theta_0}{\sin \theta_0} (1 - h_n^a) x_3, x_2, x_3) \in \text{int}(\Omega_n^{a, \theta_0}), \\ (x_1, x_2, x_3) \in \Omega^b = ] -\frac{1}{2}, \frac{1}{2}[^2 \times ] -1, 0[ \longrightarrow (x_1, x_2, h_n^b x_3) \in \Omega_n^b. \end{array} \right.$$

Cette fois l'estimation de l'énergie libre du système est établie d'une part grâce au résultat de convergence de l'énergie magnétostatique et d'autre part en utilisant le résultat de densité suivant.

Soit

$$\mathcal{M}_{reg} = \left\{ \mu = (\mu^a, \mu^b) \in C^1(\overline{\Omega^{a, \theta_0}}, S^2) \times C(\overline{\Omega^b}, S^2) : \right.$$

$$\mu^a \text{ ne dépend pas de } x_1, \mu^b \text{ ne dépend pas de } x_3,$$

$$\mu^b / ] -\frac{1}{2}, 0[ \times ] -\frac{1}{2}, \frac{1}{2}[ \times ] -1, 0[ \in C^1([-\frac{1}{2}, 0] \times [-\frac{1}{2}, \frac{1}{2}] \times [-1, 0], S^2),$$

$$\mu^b / ] 0, \frac{1}{2}[ \times ] -\frac{1}{2}, \frac{1}{2}[ \times ] -1, 0[ \in C^1([0, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}] \times [-1, 0], S^2),$$

$$\left. \mu^a(x_2, 0) = \mu^b(0, x_2) \text{ pour } x_2 \in ] -\frac{1}{2}, \frac{1}{2}[ \right\},$$

et

$$\mathcal{M} = \left\{ \mu = (\mu^a, \mu^b) \in H^1(\Omega^{a, \theta_0}, S^2) \times H^1(\Omega^b, S^2) : \right.$$

$$\mu^a \text{ ne dépend pas } x_1, \mu^b \text{ ne dépend pas } x_3,$$

$$\left. \mu^a(x_2, 0) = \mu^b(0, x_2), \text{ pour } x_2 \text{ p.p. dans } ] -\frac{1}{2}, \frac{1}{2}[ \right\}$$

**Proposition 0.8.1.**  $\mathcal{M}_{reg}$  dense dans  $M$ .

## 0.8.2 Les principaux résultats

En étudiant le comportement asymptotique de l'énergie libre on trouve que nos résultats dépendent de  $\lim_n \frac{h_n^b}{h_n^a} = q \in [0, +\infty]$ .

Plus précisément, dans le cas  $q \in ]0, +\infty[$  on obtient le résultat suivant:

**Théorème 0.8.1.** *Supposons (0.8.1) avec  $q \in ]0, +\infty[$ . Pour tout  $n \in \mathbb{N}$ , soit  $m_n = (m_n^a, m_n^b)$  est une solution minimisante de (0.8.2) et  $\zeta_n = (\zeta_n^a, \zeta_n^b)$  est l'unique solution de (0.8.3) qui correspond à  $m_n$ , soit  $\xi_n^{(1)}(x_2, x_3) = \int_{-\frac{1}{2}}^{\frac{1}{2}} m_n^a(x_1 + \cot \theta_0 x_3, x_2, x_3) dx_1$ . Alors, il existe une sous-suite  $(m_{n_i})_{i \in \mathbb{N}}$  et  $(\hat{\mu}^a, \hat{\mu}^b, \hat{\xi}^a) \in \mathcal{M} \times \mathcal{F}$  dépendant de la sous-suite choisie, tel que*

$$\left\{ \begin{array}{l} m_{n_i}^a \rightarrow \hat{\mu}^a \text{ fortement dans } H^1(\Omega^{a, \theta_0}, S^2), m_{n_i}^b \rightarrow \hat{\mu}^b \text{ fortement dans } H^1(\Omega^b, S^2), \\ \frac{1}{h_{n_i}^a}(m_{n_i}^a - \xi_{n_i}^{(1)}) \rightharpoonup \hat{\xi}^a \text{ faiblement dans } \mathcal{F}, \end{array} \right.$$

$$\left\{ \begin{array}{l} \frac{1}{h_n^a} D_{x_1} m_n^a \rightarrow D_{x_1} \hat{\xi}^a \text{ fortement dans } L^2(\Omega^{a, \theta_0}, \mathbb{R}^3), \\ \frac{1}{h_n^b} D_{x_3} m_n^b \rightarrow 0 \text{ fortement dans } L^2(\Omega^b, \mathbb{R}^3), \end{array} \right.$$

et

$$\left\{ \begin{array}{l} \frac{1}{h_{n_i}^a} D_{x_1} \zeta_{n_i}^a \rightarrow \sin^2 \theta_0 \tilde{\mu}_1^a - \sin \theta_0 \cos \theta_0 \tilde{\mu}_3^a, \quad D_{x_2} \zeta_{n_i}^a \rightarrow 0, \quad D_{x_3} \zeta_{n_i}^a \rightarrow 0 \text{ fort. dans } L^2(\mathbb{R}_+^3), \\ D_{x_1} \zeta_{n_i}^b \rightarrow 0, \quad D_{x_2} \zeta_{n_i}^b \rightarrow 0, \quad \frac{1}{h_{n_i}^b} D_{x_3} \zeta_{n_i}^b \rightarrow \tilde{\mu}_3^b \text{ fort. dans } L^2(\mathbb{R}_-^3), \end{array} \right.$$

quand  $i$  et  $n$  divergent, où  $\tilde{\mu}_1^a, \tilde{\mu}_3^a$  et  $\tilde{\mu}_3^b$  sont les prolongements par zéro de  $\hat{\mu}_1^a, \hat{\mu}_3^a$  et  $\hat{\mu}_3^b$  dans  $\mathbb{R}^3$  respectivement, et  $(\hat{\mu}, \hat{\xi}^a)$  est une solution de problème suivant

$$E_q(\hat{\mu}, \hat{\xi}^a) = \min \{E_q(\mu, \xi^a) : (\mu, \xi^a) \in \mathcal{M} \times \mathcal{F}\}.$$

De plus, on a

$$\lim_n E_n(m_n) = E_q(\hat{\mu}^a, \hat{\xi}^a),$$

où

$$E_q : (\mu^a, \mu^b, \xi^a) \in \mathcal{M} \times \mathcal{F} \longrightarrow$$

$$\int_{\Omega^{a, \theta_0}} \left( \alpha |(D_{x_1} \xi^a, D_{x_2} \mu^a, D_{x_3} \mu^a - \cot \theta_0 D_{x_1} \xi^a)|^2 + \varphi(\mu^a) + \frac{1}{2} |\sin \theta_0 \mu_1^a - \cos \theta_0 \mu_3^a|^2 \right) dx -$$

$$2 \int_{\Omega^{a, \theta_0}} f^a(x_1, x_2, x_3) \mu^a dx - 2q \int_{\Omega^b} f^b(x_1, x_2, x_3) \mu^b dx +$$

$$q \int_{]-\frac{1}{2}, \frac{1}{2}[^2} \left( \alpha |(D_{x_1} \mu^b, D_{x_2} \mu^b)|^2 + \varphi(\mu^b) + \frac{1}{2} |\mu_3^b|^2 \right) dx_1 dx_2,$$

avec

$$\mathcal{F} = \{f \in L^2(\Omega^{a, \theta_0}) : D_{x_1} f \in L^2(\Omega^{a, \theta_0})\}.$$

Pour  $q = 0$  (i.e.  $h_n^b \ll h_n^a$ ), le problème se réduit à un problème en  $2D$  (un film mince vertical), et on perd la condition de jonction. De manière analogue, si  $q = +\infty$  (i.e.  $h_n^a \ll h_n^b$ ), le problème se réduit à un problème en  $2D$  (un film mince horizontal), et on perd ainsi la condition de jonction.



# Chapter 1

## $nD - pD$ Dimensional reduction of micromagnetic structures

S. Soueid.  $nD - pD$  Dimensional reduction of micromagnetic structures, accepté pour publication Ricerchie Mat. Doi: 10.1007/s11587-014-0186-8.

**Abstract.** Starting from a  $nD$ ,  $n \geq 2$ , non-convex and nonlocal micromagnetic energy, we determine, via an asymptotic analysis, the free energy of a  $pD$  ferromagnetic domain,  $1 \leq p < n$ .

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**Keywords:** micromagnetics, variational problem, dimensional reduction.

### 1.1 Introduction

This paper is devoted to generalize a result due to G. Gioia and R. D. James in [39]. By starting from the classical micromagnetic energy for a bounded domain in  $\mathbb{R}^n$ ,  $n \geq 2$  (cf. L. D. Landau and E. M. Lifshitz in [50]), we determine via an asymptotic analysis, the free energy of a ferromagnetic  $p$ -dimensional domain in  $\mathbb{R}^n$ ,  $1 \leq p < n$ .

Magnetic thin-film elements are used in many applications: inductive thin films heads, magnetic recording, megneto-resistive sensors, thin films memories, etc. (see [46]).

We consider a thin domain  $\Omega_h = \Theta \times ]-h, h[ \subset \mathbb{R}^n$  with small thickness  $h$ , where  $h$  is a positive parameter tending to zero, and with cross-section  $\Theta$ , where  $\Theta$  is a smooth bounded open subset of  $\mathbb{R}^p$ . The aim of this paper is to study the asymptotic behavior, as  $h$  vanishes, of the following problem

$$(1.1.1) \quad \min \left\{ \frac{1}{h^{n-p}} \int_{\Omega_h} \left( |D\underline{m}|^2 + \varphi(\underline{m}) + \frac{1}{2} D\underline{\zeta} \underline{m} - 2F_h \underline{m} \right) dx, \quad \underline{m} \in H^1(\Omega_h, S^{n-1}) \right\}$$

subjected to

$$(1.1.2) \quad \operatorname{div}(-D\underline{\zeta} + \underline{m}) = 0 \quad \text{in } \mathbb{R}^n,$$

where  $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ ,  $\varphi : S^{n-1} \rightarrow [0, +\infty[$  is a continuous even function and  $F_h \in L^2(\Omega_h, \mathbb{R}^n)$ . In (1.1.2) it is understood  $\underline{m} = 0$  in  $\mathbb{R}^n \setminus \Omega_h$ .

In the classical theory of micromagnetics, if  $n = 3$ ,  $\underline{m} : \Omega_h \rightarrow \mathbb{R}^3$  denotes the magnetization and the body is always locally magnetized to a saturation magnetization  $|\underline{m}(x)| = m_s(T) > 0$  unless the local temperature  $T$  is greater or equal to Curie temperature depending on the body, in the latter case  $m_s(T) = 0$ . This model was proposed by Brown in [11]. We suppose that the temperature is constant and lower than Curie temperature and, without loss of generality, we assume that  $|\underline{m}| = 1$ , i.e.  $\underline{m}(x) \in S^{n-1}$  a.e. in  $\Omega_h$ . The function  $\underline{\zeta} : \mathbb{R}^3 \rightarrow \mathbb{R}$  denotes the magnetic field potential. The magnetic field potential and the magnetization  $\underline{m}$  are connected by equation (1.1.2). The energy in (1.1.1) consists of several contributions. The exchange energy  $\int_{\Omega_h} |D\underline{m}|^2 dx$  penalizes the spatial variation of  $\underline{m}$ , driving the body to have large regions of uniform magnetization separated by thin transition layers. The magnetostatic energy  $\int_{\Omega_h} D\underline{\zeta} \underline{m} dx = \int_{\mathbb{R}^n} |D\underline{\zeta}|^2 dx$  favors  $\operatorname{div} \underline{m} = 0$  in  $\Omega_h$  and  $\underline{m} \cdot \nu = 0$  on  $\partial\Omega_h$ , where  $\nu$  denotes the exterior unit normal to  $\partial\Omega_h$ . The anisotropy energy  $\int_{\Omega_h} \varphi(\underline{m}) dx$  models the existence of preferred directions of magnetization. The external (Zeemann) energy  $\int_{\Omega_h} F_h \underline{m} dx$  favors magnetization parallel to an externally applied field.

In the sequel, following [17], we reformulate our problem on a fixed domain  $\Omega = \Theta \times ]-1, 1[^{n-p}$ . Precisely, for describing the limit problem, we set

$$(1.1.3) \quad f_h(x) = F_h(x', hx''), \quad x \text{ a.e. in } \Omega,$$

where  $x = (x', x'')$  denotes the generic point of  $\mathbb{R}^n$ , with  $x' = (x_1, \dots, x_p)$  and  $x'' = (x_{p+1}, \dots, x_n)$ , and we assume

$$(1.1.4) \quad f_h \rightharpoonup f \quad \text{weakly in } L^2(\Omega, \mathbb{R}^n),$$

as  $h$  vanishes.

Our main results depend on  $n - p$ . Precisely, if  $n - p \geq 2$ , we prove that the limit problem is given by

$$(1.1.5) \quad \min \left\{ \int_{\Omega} (|D_{x'} \mu|^2 + \varphi(\mu) - 2f(x)\mu) dx + \frac{1}{2} \sum_{i=p+1}^n \int_{\mathbb{R}^{n-p}} |D_{x''} P_i|^2 dx'' \int_{\Theta} |\mu_i|^2 dx' + \right. \\ \left. \frac{1}{2} \sum_{\substack{i,j=p+1 \\ i \neq j}}^n \int_{\mathbb{R}^{n-p}} D_{x''} P_i D_{x''} P_j dx'' \int_{\Theta} \mu_i \mu_j dx' \right\}, \quad \mu \in H^1(\Theta, S^{n-1})$$

where  $\mu_i$ ,  $i \in \{p+1, \dots, n\}$ , is the  $i^{\text{th}}$  component of  $\mu$  and  $P_i$ ,  $i \in \{p+1, \dots, n\}$ , is the unique solution of the following problem

$$\begin{cases} P_i \in BL_0(\mathbb{R}^{n-p}), \\ \int_{\mathbb{R}^{n-p}} D_{x''} P_i D_{x''} \phi dx'' = \int_{]-1,1[^{n-p}} D_{x_i} \phi dx'', \quad \forall \phi \in BL_0(\mathbb{R}^{n-p}), \end{cases}$$

$BL_0(\mathbb{R}^{n-p})$  denoting the Beppo-Levi space on  $\mathbb{R}^{n-p}$  (cf. Section 1.2) and  $(x_{p+1}, \dots, x_n)$  the generic point of  $\mathbb{R}^{n-p}$ .

**Remark 1.1.1.** *If  $\varphi = 0$  and  $f = 0$ , the solutions of (1.1.5) are given by constant fields  $(c_1, \dots, c_p, 0, \dots, 0)$  with  $\sum_{i=1}^p c_i^2 = 1$ . This last claim follows immediately from the fact that*

$$(1.1.6) \quad \sum_{i=p+1}^n \int_{\mathbb{R}^{n-p}} |D_{x''} P_i|^2 dx'' \int_{\Theta} |\mu_i|^2 dx' + \sum_{\substack{i,j=p+1 \\ i \neq j}}^n \int_{\mathbb{R}^{n-p}} D_{x''} P_i D_{x''} P_j dx'' \int_{\Theta} \mu_i \mu_j dx' = \int_{\Theta \times \mathbb{R}^{n-p}} \left| \sum_{i=p+1}^n \mu_i D_{x''} P_i \right|^2 dx$$

and that  $D_{x''} P_i$ ,  $i = p+1, \dots, n$ , are linearly independent.

In the case  $n-p=1$ , we prove that the limit problem is given by

$$(1.1.7) \quad \min \left\{ \int_{\Omega} \left( |D_{x'} \mu|^2 + \varphi(\mu) + \frac{1}{2} |\mu_n|^2 - 2f\mu \right) dx, \quad \mu \in H^1(\Theta, S^{n-1}) \right\},$$

$\mu_n$  denoting the last component of  $\mu$ .

**Remark 1.1.2.** *If  $\varphi = 0$  and  $f = 0$ , the solutions of (1.1.7) are given by constant fields  $(c_1, \dots, c_{n-1}, 0)$  with  $\sum_{i=1}^{n-1} c_i^2 = 1$ .*

**Remark 1.1.3.** *We point out that the limit problem remains non-convex, but it becomes local and it depends on  $n-p$ . Roughly speaking, the limit problem behaves as a ferromagnetic thin film if  $n-p=1$ , while it behaves as a thin wire if  $n-p \geq 2$ .*

Several results regarding the study of a single ferromagnetic thin film via dimensional reduction appear in literature. In [39] the authors proved that the limit energy becomes local. In [19] and [49], the authors considered also others regimes. The time dependent case was treated in [5] and in [12]. In [41] and [42], the authors studied a micromagnetic thin film with degenerate exchange



energy. Other models of micromagnetic thin films were proposed in [2], [3] and [52]. For related problems, some interesting regularity theorems were obtained in [43]. A ferroelectric thin film was studied in [35].

For ferromagnetic thin wires we refer to [15], [53] and [55].

In what concerns the study of ferromagnetic multistructures, we refer to [33] for two joined thin films, to [34] for two joined thin wires and for a thin wire in junction with a thin film. We explicitly remark that formula (1.1.5) was obtained in [34] in the case  $n = 3$  and  $p = 1$ . For junction 1D-2D we refer to [31] and [36]. For junction 1D-1D we refer to [32].

## 1.2 Some preliminary results

Let

$$BL(\mathbb{R}^{n-p}) = \left\{ u \in \mathcal{D}'(\mathbb{R}^{n-p}) : Du \in (L^2(\mathbb{R}^{n-p}))^{n-p} \right\}.$$

$BL_0(\mathbb{R}^{n-p}) = BL(\mathbb{R}^{n-p})/\mathbb{R}$  endowed with the inner product

$$(u, v)_{BL} = \int_{\mathbb{R}^{n-p}} Du Dv dy.$$

is a Hilbert space (see [18]). Consequently, for every  $i$  in  $\{p+1, \dots, n\}$  and for every  $c = (c_{p+1}, \dots, c_n) \in \mathbb{R}^{n-p}$ , the following problems

$$(1.2.1) \quad \begin{cases} P_i \in BL_0(\mathbb{R}^{n-p}), \\ \int_{\mathbb{R}^{n-p}} D_{x''} P_i D_{x''} \phi dx'' = \int_{]-1,1[^{n-p}} D_{x_i} \phi dx'', \quad \forall \phi \in BL_0(\mathbb{R}^{n-p}), \end{cases}$$

$$(1.2.2) \quad \begin{cases} P_c \in BL_0(\mathbb{R}^{n-p}), \\ \int_{\mathbb{R}^{n-p}} D_{x''} P_c D_{x''} \phi dx'' = \int_{]-1,1[^{n-p}} c D_{x''} \phi dx'', \quad \forall \phi \in BL_0(\mathbb{R}^{n-p}), \end{cases}$$

admit a unique solution. Moreover, it is evident that

$$(1.2.3) \quad P_c = \sum_{i=p+1}^n c_i P_i.$$

By using a result of Kryloff (cf. [54], Th. XV, page 181) one has

**Proposition 1.2.1.** *If  $n - p \geq 3$ , then*

$$BL(\mathbb{R}^{n-p}) = \left\{ u \in L_{loc}^{\frac{2(n-p)}{(n-p)-2}}(\mathbb{R}^{n-p}) : Du \in (L^2(\mathbb{R}^{n-p}))^{n-p} \right\}.$$



admits a unique solution  $\underline{z}$  and it is characterized as the unique minimiser of the following problem

$$(1.3.2) \quad \min \left\{ \frac{1}{2} \int_{\mathbb{R}^n} |D\underline{\zeta} - \underline{m}|^2 dx : \underline{\zeta} \in \mathcal{U} \right\},$$

where it is understood that  $\underline{m} = 0$  in  $\mathbb{R}^n \setminus \Omega_h$ . Moreover,  $\underline{z}$  belong to  $H^1(\mathbb{R}^n)$  up to additive constant (see. [48]).

Let us consider the following problem

$$(1.3.3) \quad \min \left\{ \frac{1}{h^{n-p}} \int_{\Omega_h} \left( |D\underline{m}|^2 + \varphi(\underline{m}) + \frac{1}{2} D\underline{z} \underline{m} - 2F_h \underline{m} \right) dx : \underline{m} \in H^1(\Omega_h, S^{n-1}) \right\},$$

where  $\underline{m}$  and  $\underline{z}$  are linked by (1.3.1), and  $F_h \in L^2(\Omega_h)$ . This problem has at least one solution (compare [57]). In general, the solution of (1.3.3) is not unique. In fact, if  $\underline{m}_h$  is a solution of (1.3.3) with  $F_h = 0$ , then also  $-\underline{m}_h$  is a solution.

The aim of this paper is to study the asymptotic behavior of problem (1.3.3), as  $h$  vanishes.

## 1.4 The rescaled problem

In this section, we reformulate problem (1.3.3) on a fixed domain, using the following rescaling

$$(x', x'') \in \Omega \rightarrow (x', hx'') \in \Omega_h,$$

where  $x' = (x_1, \dots, x_p)$ ,  $x'' = (x_{p+1}, \dots, x_n)$ , and denoting with  $f_h$  the function defined in (1.1.3). Namely,  $\mathcal{U}$  defined in previous section is rescaled in the following one

$$(1.4.1) \quad \mathcal{U}_h = \left\{ \zeta \in L^1_{loc}(\mathbb{R}^n) : \zeta \in L^2(B_h), D\zeta \in (L^2(\mathbb{R}^n))^n, \int_{B_h} \zeta dx = 0 \right\},$$

where  $B_h = ]-a, a[ \times ]-\frac{2}{h}, \frac{2}{h}[^{n-p}$ . Then, for every  $m \in L^2(\Omega, \mathbb{R}^n)$ , the rescaling of equation (1.3.1)

$$(1.4.2) \quad \begin{cases} z \in \mathcal{U}_h, \\ \int_{\mathbb{R}^n} (D_{x'} z, \frac{1}{h} D_{x''} z) (D_{x'} \zeta, \frac{1}{h} D_{x''} \zeta) dx = \int_{\Omega} m (D_{x'} \zeta, \frac{1}{h} D_{x''} \zeta) dx, \quad \forall \zeta \in \mathcal{U}_h, \end{cases}$$

admits a unique solution  $z$ , which is also the unique solution of the following problem

$$(1.4.3) \quad \min \{ j_h(\zeta) : \zeta \in \mathcal{U}_h \},$$

where

$$(1.4.4) \quad j_h : \zeta \in \mathcal{U}_h \mapsto \frac{1}{2} \int_{\mathbb{R}^n} \left| (D_{x'} \zeta, \frac{1}{h} D_{x''} \zeta) - m \right|^2 dx,$$

understanding  $m = 0$  in  $\mathbb{R}^n \setminus \Omega$ . Remark that  $z$  belong to  $H^1(\mathbb{R}^n)$  up to additive constant.

Let

$$E_h : m \in H^1(\Omega, S^{n-1}) \mapsto \int_{\Omega} \left( |(D_{x'} m, \frac{1}{h} D_{x''} m)|^2 + \varphi(m) + \frac{1}{2} (D_{x'} z, \frac{1}{h} D_{x''} z) m - 2f_h m \right) dx,$$

where  $m$  and  $z$  are linked by (1.4.2). Then, the function defined by  $\underline{m}_h(x', hx'')$ , with  $\underline{m}_h$  solution of (1.3.3), solves the following problem

$$(1.4.5) \quad \min \left\{ E_h(m) : m \in H^1(\Omega, S^{n-1}) \right\}.$$

The goal of this paper becomes to study the asymptotic behavior, as  $h$  vanishes, of problem (1.4.5).

By setting

$$(1.4.6) \quad E_h^{mag}(m) = \frac{1}{2} \int_{\mathbb{R}^n} \left| (D_{x'} z, \frac{1}{h} D_{x''} z) \right|^2 dx,$$

from (1.4.2) it follows that

$$(1.4.7) \quad E_h(m) = \int_{\Omega} \left( \left| (D_{x'} m, \frac{1}{h} D_{x''} m) \right|^2 + \varphi(m) - 2f_h m \right) dx + E_h^{mag}(m), \quad \forall m \in H^1(\Omega, S^{n-1}).$$

## 1.5 The main results

Let

$$(1.5.1) \quad M = \left\{ \mu \in H^1(\Omega, S^{n-1}) : \mu \text{ is independent of } x'' \right\} \simeq H^1(\Theta, S^{n-1}).$$

For  $n - p \geq 2$ , let

$$(1.5.2) \quad \begin{aligned} E_0 : \mu \in M &\longrightarrow \int_{\Omega} (|D_{x'} \mu|^2 + \varphi(\mu) - 2f\mu) dx + \\ &\frac{1}{2} \sum_{i=p+1}^n \int_{\mathbb{R}^{n-p}} |D_{x''} P_i|^2 dx'' \int_{\Theta} |\mu_i|^2 dx' + \\ &\frac{1}{2} \sum_{\substack{i,j=p+1 \\ i \neq j}}^n \int_{\mathbb{R}^{n-p}} D_{x''} P_i D_{x''} P_j dx'' \int_{\Theta} \mu_i \mu_j dx'. \end{aligned}$$

The following theorem contains our first main result.

**Theorem 1.5.1.** *Let  $n \geq 3$ ,  $p \geq 1$  and  $n - p \geq 2$ . For every  $h$ , let  $m_h$  be a solution of (1.4.5) and let  $\zeta_h$  be the unique solution of (1.4.3) corresponding to  $m_h$ . Assume (1.1.4). Then, there exist a subsequence of  $\{m_h\}_h$ , still denoted by  $\{m_h\}_h$ , a subsequence of  $\{\zeta_h\}_h$ , still denoted by  $\{\zeta_h\}_h$ , and  $\hat{\mu} \in M$ , depending on the selected subsequence, such that*

$$(1.5.3) \quad m_h \longrightarrow \hat{\mu} \quad \text{strongly in } H^1(\Omega, S^{n-1}),$$

$$(1.5.4) \quad \frac{1}{h} D_{x''} m_h \longrightarrow 0 \quad \text{strongly in } (L^2(\Omega))^{n-p},$$

and

$$(1.5.5) \quad \begin{cases} D_{x'} \zeta_h \longrightarrow 0 & \text{strongly in } (L^2(\mathbb{R}^n))^p, \\ \frac{1}{h} D_{x''} \zeta_h \longrightarrow \hat{\xi} & \text{strongly in } (L^2(\mathbb{R}^n))^{n-p}, \end{cases}$$

as  $h$  vanishes, where  $\hat{\mu}$  solves

$$(1.5.6) \quad E_0(\hat{\mu}) = \min \{E_0(\mu), \quad \mu \in M\}$$

and

$$(1.5.7) \quad \hat{\xi}(x) = \begin{cases} \sum_{i=p+1}^n \hat{\mu}_i(x') D_{x''} P_i(x''), & \text{a.e. in } \Theta \times \mathbb{R}^{n-p}, \\ 0, & \text{a.e. in } (\mathbb{R}^p \setminus \Theta) \times \mathbb{R}^{n-p}, \end{cases}$$

with  $P_i$ ,  $i \in \{p+1, \dots, n\}$ , the unique solution of (1.2.1). Moreover, the convergence of the energies hold true for the whole sequence, i.e.

$$(1.5.8) \quad \lim_{h \rightarrow 0} E_h(m_h) = E_0(\hat{\mu}).$$

In the case  $n - p = 1$ , let

$$(1.5.9) \quad E_1 : \mu \in M \longrightarrow \int_{\Omega} (|D_{x'} \mu|^2 + \varphi(\mu) - 2f\mu + \frac{1}{2} |\mu_n|^2) dx.$$

The following theorem contains our second main result.

**Theorem 1.5.2.** *Let  $n \geq 2$  and  $n - p = 1$ . For every  $h$ , let  $m_h$  be a solution of (1.4.5) and let  $\zeta_h$  be the unique solution of (1.4.3) corresponding to  $m_h$ . Assume (1.1.4). Then, there exist a subsequence of  $\{m_h\}_h$ , still denoted by  $\{m_h\}_h$ , a subsequence of  $\{\zeta_h\}_h$ , still denoted by  $\{\zeta_h\}_h$ , and  $\hat{\mu} \in M$ , depending on the selected subsequence, such that*

$$(1.5.10) \quad m_h \longrightarrow \hat{\mu} \quad \text{strongly in } H^1(\Omega, S^{n-1}),$$

$$(1.5.11) \quad \frac{1}{h} D_{x_n} m_h \longrightarrow 0 \quad \text{strongly in } L^2(\Omega),$$

and

$$(1.5.12) \quad \begin{cases} D_{x_j} \zeta_h \longrightarrow 0 & \text{strongly in } L^2(\mathbb{R}^n), \quad \forall j = 1, \dots, n-1, \\ \frac{1}{h} D_{x_n} \zeta_h \longrightarrow \tilde{\mu}_n & \text{strongly in } L^2(\mathbb{R}^n), \end{cases}$$

where  $\hat{\mu}$  solves

$$(1.5.13) \quad E_1(\hat{\mu}) = \min \{E_1(\mu), \quad \mu \in M\}$$

and  $\tilde{\mu}_n$  denote the zero-extension of  $\hat{\mu}_n$  to  $\mathbb{R}^n$ . Moreover, the convergence of the energies hold true for the whole sequence, i.e.

$$(1.5.14) \quad \lim_{h \rightarrow 0} E_h(m_h) = E_1(\hat{\mu}).$$

**Remark 1.5.3.** In previous theorems, in the cases where the limit is zero the convergences hold true for the whole sequence.

## 1.6 The proof in the case $n - p \geq 2$

We begin by proving a general convergence result for the magnetostatic energy.

**Proposition 1.6.1.** Let  $n \geq 3$ ,  $p \geq 1$  and  $n - p \geq 2$ . Let  $\{m_h\}_h \subset L^2(\Omega, \mathbb{R}^n)$ . Assume that there exists  $\mu \in L^2(\Omega, \mathbb{R}^n)$  independent of  $x''$  such that

$$(1.6.1) \quad m_h \longrightarrow \mu \quad \text{strongly in } L^2(\Omega, \mathbb{R}^n),$$

as  $h$  vanishes. Moreover, let  $\zeta_h$  be the unique solution of (1.4.3) corresponding to  $m_h$ . Let  $E_h^{mag}$  be defined by (1.4.6). Then, it results

$$(1.6.2) \quad \begin{cases} D_{x'} \zeta_h \longrightarrow 0 & \text{strongly in } (L^2(\mathbb{R}^n))^p, \\ \frac{1}{h} D_{x''} \zeta_h \longrightarrow \xi & \text{strongly in } (L^2(\mathbb{R}^n))^{n-p}, \end{cases}$$

as  $h$  vanishes, where

$$(1.6.3) \quad \xi(x) = \begin{cases} \sum_{i=p+1}^n \mu_i(x') D_{x''} P_i(x''), & \text{a.e. in } \Theta \times \mathbb{R}^{n-p}, \\ 0, & \text{a.e. in } (\mathbb{R}^p \setminus \Theta) \times \mathbb{R}^{n-p}, \end{cases}$$

with  $P_i$ ,  $i \in \{p+1, \dots, n\}$ , the unique solution of (1.2.1). Furthermore,

$$(1.6.4) \quad \lim_{h \rightarrow 0} E_h^{mag}(m_h) = \frac{1}{2} \sum_{i=p+1}^n \int_{\mathbb{R}^{n-p}} |D_{x''} P_i|^2 dx'' \int_{\Theta} |\mu_i|^2 dx' + \frac{1}{2} \sum_{\substack{i, j = p+1 \\ i \neq j}}^n \int_{\mathbb{R}^{n-p}} D_{x''} P_i D_{x''} P_j dx'' \int_{\Theta} \mu_i \mu_j dx'.$$

*Proof.* The proof is developed in several steps and it is inspired by [34]. In the sequel,  $c$  denotes any positive constant independent of  $h$ .

By choosing  $\zeta = 0$  as test function in (1.4.3) corresponding to  $m_h$  and using the fact that  $|m_h| = 1$ , one derives

$$(1.6.5) \quad \begin{cases} \|D_{x'} \zeta_h\|_{(L^2(\mathbb{R}^n))^p}^2 \leq c, \\ \left\| \frac{1}{h} D_{x''} \zeta_h \right\|_{(L^2(\mathbb{R}^n))^{n-p}}^2 \leq c, \end{cases} \quad \forall h.$$

Since  $\zeta_h \in H^1(\mathbb{R}^n)$  up to an additive constant and  $n \geq 3$ , the Sobolev-Gagliardo-Nirenberg inequality and (1.6.5) provide

$$(1.6.6) \quad \|\zeta_h + c_h\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \leq c, \quad \forall h,$$

for a sequence  $\{c_h\}_h \subset \mathbb{R}$ . Estimates (1.6.5) and (1.6.6) ensure the existence of a function  $\zeta \in L^{\frac{2n}{n-2}}(\mathbb{R}^n)$ , with  $D\zeta \in (L^2(\mathbb{R}^n))^n$  and  $\zeta$  independent of  $x''$ , and a function  $\xi = (\xi_{p+1}, \dots, \xi_n) \in (L^2(\mathbb{R}^n))^{n-p}$  such that on extraction of a suitable subsequence (not relabelled)

$$(1.6.7) \quad \begin{cases} \zeta_h + c_h \rightharpoonup \zeta & \text{weakly in } L^{\frac{2n}{n-2}}(\mathbb{R}^n), \\ D\zeta_h \rightharpoonup D\zeta & \text{weakly in } (L^2(\mathbb{R}^n))^n, \end{cases}$$

$$(1.6.8) \quad \frac{1}{h} D_{x''} \zeta_h \rightharpoonup \xi \quad \text{weakly in } (L^2(\mathbb{R}^n))^{n-p},$$

as  $h$  vanishes. Moreover, the fact that  $\zeta$  is independent of  $x''$  and  $D\zeta \in (L^2(\mathbb{R}^n))^n$  provide

$$\begin{cases} +\infty > \int_{\mathbb{R}^n} |D\zeta|^2 dx \geq \int_{]a,b[^{n-p}} \left( \int_{\mathbb{R}^p} |D_{x'} \zeta|^2 dx' \right) dx'' = (b-a)^{n-p} \int_{\mathbb{R}^p} |D_{x'} \zeta|^2 dx', \\ \forall a, b \in \mathbb{R} \text{ with } a < b, \end{cases}$$

which gives that  $\zeta$  is independent of  $x'$ , too. Then,  $\zeta = 0$ , since zero is the only constant in  $L^{\frac{2n}{n-2}}(\mathbb{R}^n)$ . Consequently, from (1.6.7) it follows that

$$(1.6.9) \quad D\zeta_h \rightharpoonup 0 \quad \text{weakly in } (L^2(\mathbb{R}^n))^n,$$

as  $h$  vanishes, and this convergence holds true for all the sequence.

The next step is devoted to identify  $\xi$ . To this aim, starting from

$$D_{x_j} \left( \frac{1}{h} D_{x_i} \zeta_h \right) = D_{x_i} \left( \frac{1}{h} D_{x_j} \zeta_h \right), \quad \text{in } D'(\mathbb{R}^n), \quad \forall i, j = p+1, \dots, n,$$

and using (1.6.8), we derive

$$(1.6.10) \quad \int_{\mathbb{R}^n} \xi_i D_{x_j} \varphi dx = \int_{\mathbb{R}^n} \xi_j D_{x_i} \varphi dx, \quad \forall \varphi \in H^1(\mathbb{R}^n), \quad \forall i, j = p+1, \dots, n.$$

If we choose  $\varphi(x) = \phi(x'')\chi(x')$  in (1.6.10), with  $\phi \in H^1(\mathbb{R}^{n-p})$  and  $\chi \in C_0^\infty(\mathbb{R}^p)$ , recalling that  $H^1(\mathbb{R}^{n-p})$  is separable, we easily obtain

$$(1.6.11) \quad \left\{ \begin{array}{l} \text{for } x' \text{ a.e in } \mathbb{R}^p, \\ \int_{\mathbb{R}^{n-p}} \xi_i(x', x'') D_{x_j} \phi(x'') dx'' = \int_{\mathbb{R}^{n-p}} \xi_j(x', x'') D_{x_i} \phi(x'') dx'', \quad \forall \phi \in H^1(\mathbb{R}^{n-p}). \end{array} \right.$$

Consequently, by virtue of Lemma 1.2.2,

$$(1.6.12) \quad \text{for } x' \text{ a.e. in } \mathbb{R}^p, \quad \exists! P(x', \cdot) \in BL_0(\mathbb{R}^{n-p}) : \xi(x', \cdot) = D_{x''} P(x', \cdot), \text{ a.e in } \mathbb{R}^{n-p}.$$

Now, choosing  $\zeta = -\frac{h}{|B_h|} \int_{B_h} \varphi dx + h\varphi$ , with  $\varphi \in C_0^\infty(\mathbb{R}^n)$ , as test function in (1.4.2) when  $m = m_h$ , we have

$$(1.6.13) \quad \int_{\mathbb{R}^n} (D_{x'} \zeta_h, \frac{1}{h} D_{x''} \zeta_h) (h D_{x'} \varphi, D_{x''} \varphi) dx = \int_{\Omega} m_h (h D_{x'} \varphi, D_{x''} \varphi) dx, \quad \forall \varphi \in C_0^\infty(\mathbb{R}^n).$$

Then, passing to the limit in (1.6.13) and using (1.6.1), (1.6.5) and (1.6.8), we obtain

$$\int_{\mathbb{R}^n} (\xi_{p+1}, \dots, \xi_n) (D_{x_{p+1}} \varphi, \dots, D_{x_n} \varphi) = \int_{\Omega} (\mu_{p+1}, \dots, \mu_n) (D_{x_{p+1}} \varphi, \dots, D_{x_n} \varphi) dx, \quad \forall \varphi \in C_0^\infty(\mathbb{R}^n).$$

Consequently, using (1.6.12) and arguing as in the proof of (1.6.11), it follows that for  $x'$  a.e. in  $\mathbb{R}^p \setminus \Theta$

$$\left\{ \begin{array}{l} P(x', \cdot) \in BL_0(\mathbb{R}^{n-p}), \\ \int_{\mathbb{R}^{n-p}} (D_{x_{p+1}} P(x), \dots, D_{x_n} P(x)) (D_{x_{p+1}} \phi(x''), \dots, D_{x_n} \phi(x'')) dx'' = 0, \quad \forall \phi \in BL_0(\mathbb{R}^{n-p}); \end{array} \right.$$

while, for  $x'$  a.e. in  $\Theta$ ,

$$(1.6.14) \quad \left\{ \begin{array}{l} P(x', \cdot) \in BL_0(\mathbb{R}^{n-p}), \\ \int_{\mathbb{R}^{n-p}} (D_{x_{p+1}} P(x), \dots, D_{x_n} P(x)) (D_{x_{p+1}} \phi(x''), \dots, D_{x_n} \phi(x'')) dx'' = \\ (\mu_{p+1}(x'), \dots, \mu_n(x')) \int_{]-1, 1[^{n-p}} (D_{x_{p+1}} \phi(x''), \dots, D_{x_n} \phi(x'')) dx'', \quad \forall \phi \in BL_0(\mathbb{R}^{n-p}). \end{array} \right.$$



Then, by virtue of (1.2.3), it results that, for  $x'$  a.e in  $\mathbb{R}^p$ ,

$$(1.6.15) \quad P(x', x'') = \begin{cases} 0 & \text{a.e. in } \mathbb{R}^{n-p}, \quad \text{if } x' \in \mathbb{R}^p \setminus \Theta, \\ \sum_{i=p+1}^n \mu_i(x') P_i(x''), & \text{a.e. in } \mathbb{R}^{n-p}, \quad \text{if } x' \in \Theta, \end{cases}$$

with  $P_i$ ,  $i \in \{p+1, \dots, n\}$ , the unique solution of (1.2.1).

Finally, since Tonelli's Theorem ensures that  $\xi$  and  $\sum_{i=p+1}^n \mu_i D_{x''} P_i$  belong to  $(L^2(\mathbb{R}^n))^{n-p} \subset (L^1_{loc}(\mathbb{R}^n))^{n-p}$ , using Fubini's Theorem with (1.6.12) and (1.6.15), it results that

$$\begin{aligned} \int_{\mathbb{R}^n} \xi \varphi dx &= \int_{\mathbb{R}^p} \left( \int_{\mathbb{R}^{n-p}} D_{x''} P(x', x'') \varphi dx'' \right) dx' \\ &= \int_{\Theta \times \mathbb{R}^{n-p}} \sum_{i=p+1}^n \mu_i(x') D_{x''} P_i(x'') \varphi dx, \quad \forall \varphi \in C_0^\infty(\mathbb{R}^n), \end{aligned}$$

from which, it follows that

$$(1.6.16) \quad \xi(x) = \begin{cases} \sum_{i=p+1}^n \mu_i(x') D_{x''} P_i(x''), & \text{a.e. in } \Theta \times \mathbb{R}^{n-p}, \\ 0, & \text{a.e. in } (\mathbb{R}^p \setminus \Theta) \times \mathbb{R}^{n-p}, \end{cases}$$

with  $P_i$ ,  $i \in \{p+1, \dots, n\}$ , the unique solution of (1.2.1). Consequently, convergence (1.6.8) holds true for the whole sequence.

The last step is devoted to obtain convergence (1.6.4) and to prove that convergences in (1.6.8) and (1.6.9) are strong. By passing to the limit in (1.4.6) with  $m = m_h$ , and using (1.4.2), (1.6.1), (1.6.8), (1.6.9), (1.6.16) and equation

(1.6.14) with test function  $\sum_{i=p+1}^n \mu_i(x') P_i(x'')$ , we obtain

$$\begin{aligned}
\lim_{h \rightarrow 0} E_h^{mag}(m_h) &= \lim_{h \rightarrow 0} \frac{1}{2} \int_{\mathbb{R}^n} |(D_{x'} \zeta_h, \frac{1}{h} D_{x''} \zeta_h)|^2 dx \\
&= \lim_{h \rightarrow 0} \frac{1}{2} \int_{\Omega} m_h (D_{x'} \zeta_h, \frac{1}{h} D_{x''} \zeta_h) dx \\
&= \frac{1}{2} \int_{\Omega} (\mu_{p+1}, \dots, \mu_n) \sum_{i=p+1}^n \mu_i D_{x''} P_i dx \\
&= \frac{1}{2} \int_{\Theta \times \mathbb{R}^{n-p}} \left| \sum_{i=p+1}^n \mu_i D_{x''} P_i \right|^2 dx \\
&= \frac{1}{2} \sum_{i=p+1}^n \int_{\mathbb{R}^{n-p}} |D_{x''} P_i|^2 dx'' \int_{\Theta} |\mu_i|^2 dx' \\
&\quad + \frac{1}{2} \sum_{\substack{i, j = p+1 \\ i \neq j}}^n \int_{\mathbb{R}^{n-p}} D_{x''} P_i D_{x''} P_j dx'' \int_{\Theta} \mu_i \mu_j dx'.
\end{aligned}$$

Finally, combining this result with (1.6.8) and (1.6.9), we deduce (1.6.2).  $\square$

### 1.6.1 Proof of Theorem 1.5.1

In the sequel,  $c$  denotes any positive constant independent of  $h$ .

Since  $m_h$  is a minimizer of  $E_h$ , choosing  $m = e$  (constant unit vector) as test function in (1.4.5) and taking into account decomposition (1.4.7), it follows that

$$(1.6.17) \quad E_h(m_h) \leq E_h(e) \leq |\Omega| \varphi(e) + 2|\Omega|^{\frac{1}{2}} \|f_h\|_{L^2(\Omega)} + E_h^{mag}(e), \quad \forall h.$$

On the other side, taking into account equation (1.4.2) with  $m = e$ , it results

$$(1.6.18) \quad E_h^{mag}(e) \leq c, \quad \forall h.$$

Then, combining (1.6.17) with (1.6.18) and (1.1.4), one obtains

$$E_h(m_h) \leq c, \quad \forall h.$$

Consequently, by virtue of (1.1.4) and of the fact that  $m_h = 1$ , one derives

$$(1.6.19) \quad \begin{cases} \|D_{x'} m_h\|_{(L^2(\Omega))^p}^2 \leq c, \\ \|\frac{1}{h} D_{x''} m_h\|_{(L^2(\Omega))^{n-p}}^2 \leq c, \end{cases} \quad \forall h.$$

Thus, taking into account that  $|m_h| = 1$ , there exists a subsequence of  $\{m_h\}_h$ , still denoted by  $\{m_h\}_h$ ,  $\hat{\mu} \in M$  and  $g \in (L^2(\Omega, \mathbb{R}^n))^{n-p}$ , depending on the selected subsequence, such that

$$(1.6.20) \quad m_h \rightharpoonup \hat{\mu} \quad \text{weakly in } H^1(\Omega, \mathbb{R}^n),$$

$$(1.6.21) \quad \frac{1}{h} D_{x''} m_h \rightharpoonup g \quad \text{weakly in } (L^2(\Omega, \mathbb{R}^n))^{n-p}.$$

as  $h$  vanishes. Consequently, by virtue of Proposition 1.6.1, convergences in (1.5.5) and identification (1.5.7) hold true. Moreover, it results that

$$(1.6.22) \quad \begin{aligned} \lim_{h \rightarrow 0} E_h^{mag}(m_h) &= \frac{1}{2} \sum_{i=p+1}^n \int_{\mathbb{R}^{n-p}} |D_{x''} P_i|^2 dx'' \int_{\Theta} |\hat{\mu}_i|^2 dx' + \\ &\frac{1}{2} \sum_{\substack{i,j=p+1 \\ i \neq j}}^n \int_{\mathbb{R}^{n-p}} D_{x''} P_i D_{x''} P_j dx'' \int_{\Theta} \gamma \hat{\mu}_i \hat{\mu}_j dx'. \end{aligned}$$

Now, the goal is to identify  $\hat{\mu}$  and  $g$ , to prove convergence (1.5.8) and to obtain strong convergences in (1.6.20) and (1.6.21). To this aim, using a l.s.c. argument, (1.6.20), (1.6.21) and Proposition 1.6.1, we obtain that

$$(1.6.23) \quad \int_{\Omega} |g|^2 dx + E_0(\hat{\mu}) \leq \liminf_h E_h(m_h) \leq \limsup_h E_h(m_h) \leq \lim_h E_h(\mu) = E_0(\mu), \quad \forall \mu \in \mathcal{M}.$$

Consequently,  $g = 0$ ,  $\hat{\mu}$  solve (1.5.6) and the convergence (1.5.8) holds true. Finally, combining (1.5.8) with (1.6.20) and (1.6.21), we obtain that convergences in (1.6.20) and (1.6.21) are strong.

## 1.7 Case $n - p = 1$

In this case,  $x' = (x_1, \dots, x_n)$  and  $x'' = x_n$ .

### 1.7.1 A convergence result for the magnetostatic energy

**Proposition 1.7.1.** *Let  $n \geq 2$  and  $n - p = 1$ . Let  $\{m_h\}_h \in L^2(\Omega, \mathbb{R}^n)$ . Assume that there exists  $\mu \in L^2(\Omega, \mathbb{R}^n)$ , independent of  $x_n$ , such that*

$$(1.7.1) \quad m_h \longrightarrow \mu \quad \text{strongly in } L^2(\Omega, \mathbb{R}^n),$$

as  $h$  vanishes. Moreover, let  $\zeta_h$  be the unique solution of (1.4.3) corresponding to  $m_h$ . Let  $E_h^{mag}$  be defined by (1.4.6). Then, it results

$$(1.7.2) \quad \begin{cases} D_{x'} \zeta_h \longrightarrow 0 & \text{strongly in } (L^2(\mathbb{R}^n))^{n-1}, \\ \frac{1}{h} D_{x_n} \zeta_h \longrightarrow \tilde{\mu}_n & \text{strongly in } L^2(\mathbb{R}^n), \end{cases}$$

as  $h$  vanishes, where  $\tilde{\mu}_n$  denotes the zero-extension of  $\mu_n$  to  $\mathbb{R}^n$ . Moreover,

$$(1.7.3) \quad \lim_{h \rightarrow 0} E_h^{mag}(m_h) = \frac{1}{2} \int_{\Omega} |\mu_n|^2 dx.$$

*Proof.* By arguing as in the first part of the proof of Proposition 1.6.1, one obtains

$$(1.7.4) \quad D\zeta_h \rightharpoonup 0 \quad \text{weakly in } (L^2(\mathbb{R}^n))^n,$$

and the existence of a function  $\xi \in L^2(\mathbb{R}^n)$  such that on extraction of a suitable subsequence (not relabelled)

$$(1.7.5) \quad \frac{1}{h} D_{x_n} \zeta_h \rightharpoonup \xi \quad \text{weakly in } L^2(\mathbb{R}^n).$$

Now, choosing  $\zeta = -\frac{h}{|B_h|} \int_{B_h} \varphi dx + h\varphi$ , with  $\varphi \in C_0^\infty(\mathbb{R}^n)$ , as test function in (1.4.2) when  $m = m_h$ , and passing to the limit as  $h$  vanishes, arguing as in the proof of Proposition 1.6.1 we obtain

$$\int_{\mathbb{R}^n} \xi D_{x_n} \varphi dx = \int_{\Omega} \mu_n D_{x_n} \varphi dx, \quad \forall \varphi \in C_0^\infty(\mathbb{R}^n).$$

This relation provides that, for  $x'$  a.e. in  $\mathbb{R}^{n-1}$ , function  $\xi(x', \cdot) - \tilde{\mu}_n(x', \cdot)$  is constant in  $\mathbb{R}$ , where  $\tilde{\mu}_n$  denotes the zero extension of  $\mu_n$  to  $\mathbb{R}^n$ . Moreover, since  $\xi(x', \cdot) - \tilde{\mu}_n(x', \cdot) \in L^2(\mathbb{R})$  for  $x'$  a.e. in  $\mathbb{R}^{n-1}$ , one concludes

$$(1.7.6) \quad \xi(x', \cdot) = \tilde{\mu}_n(x', \cdot) \quad \text{a.e. in } \mathbb{R}.$$

Then, combining (1.7.5) and (1.7.6), one has

$$(1.7.7) \quad \frac{1}{h} D_{x_n} \zeta_h \rightharpoonup \tilde{\mu}_n \quad \text{weakly in } L^2(\mathbb{R}^n),$$

By passing to the limit in (1.4.6) with  $m = m_h$ , and using (1.7.4) and (1.7.7) one obtains

$$(1.7.8) \quad \begin{aligned} \lim_h E_h^{mag}(m_h) &= \lim_h \left( \frac{1}{2} \int_{\mathbb{R}^n} |(D_{x'} \zeta_h, \frac{1}{h} D_{x_n} \zeta_h)|^2 dx \right) \\ &= \lim_h \left( \frac{1}{2} \int_{\mathbb{R}^n} (D_{x'} \zeta_h, \frac{1}{h} D_{x_n} \zeta_h) m_h dx \right) \\ &= \frac{1}{2} \int_{\Omega} |\mu_n|^2 dx. \end{aligned}$$

So convergence (1.7.3) is proved. Finally, from (1.7.4), (1.7.7) and (1.7.8) one deduces (1.7.2).  $\square$

By using Proposition 1.7.1 and arguing as in the proof of Theorem 1.5.1, one proves Theorem 1.5.2.



## Chapter 2

# Junction of quasi-stationary ferromagnetic thin films

U. De Maio, L. Faella, S. Soueid. Quasi-stationary ferromagnetic thin films in degenerated cases, à soumettre.

**Abstract.** In this paper we study the asymptotic behavior of the solutions of time dependent micromagnetism problem in a multi-domain consisting of two joined ferromagnetic thin films. We distinguish different regimes depending on the limit of the ratio between the small thickness of the two films.

**Keywords:** Micromagnetics, variational problem, thin films, Landau-Lifschitz equation.

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### 2.1 Introduction

Ferromagnetic behavior, in particular the presence of spontaneous magnetization even in the absence of an applied magnetic field, can be examined by the theory started by Weiss in 1907 and perfected by Landau and Lifshitz in 1935 (see [50] and for a modern analysis see [11]). It is proposed that, under suitable conditions, in particular when the temperature is below a critical point (the so called Curie's temperature, characteristic of the material), a ferromagnetic body breaks up into uniformly magnetized region (Weiss domains) separated by thin transition layers (Bloch wall), even in the absence of any applied magnetic field. Then, the phenomena can be described by a magnetization field  $M$ , defined on the domain  $\Omega$  in which the material is confined, which on a microscopic scale has a fixed modulus  $|M| = m = \text{const}$  and variable orientation, because of the presence of a strong molecular field. So, the system can be studied through the functional representing its magnetic energy. This energy consists in several terms: the so called exchange energy, which contains the space derivative of  $M$  and is peculiar to ferromagnetic behavior, a term corresponding to magnetic

anisotropy, and another one depending on the magnetic field  $H$ , which is related to  $M$  via the equations of magnetostatic.

More precisely, assume the body is homogeneous and has a uniform temperature. Then the magnetic induction  $B$ , the magnetic field  $H$  and the magnetization  $M$  are connected by: the relation  $B = H(M) + \bar{M}$  where  $\bar{M}$  is the extension by zero of  $M$  outside  $\Omega$ ; the static Maxwell equation and the magnetostatic equation (Faraday law):

$$(2.1.1) \quad \begin{cases} \nabla \times H(M) = 0, \\ \operatorname{div} (H(M) + \bar{M}) = \operatorname{div} (B) = 0 . \end{cases}$$

So the steady state configuration of  $M$  corresponds to a minimum of the following functional  $E$  representing the magnetic energy:

$$(2.1.2) \quad \begin{cases} E = \int_{\Omega} (|DM|^2 + |H(M)|^2 + \varphi(M)) dx \\ \text{subject to conditions (2.1.1)} . \end{cases}$$

where  $E$  is obtained by summing up the exchange energy  $E^{exc} = \int_{\Omega} |DM|^2 dx$ , the magnetostatic energy  $E^{mag} = \int_{\Omega} |H(M)|^2 dx$ , which is related to  $M$  via equation of magnetostatic, and the anisotropy energy  $E^{an} = \int_{\Omega} \varphi(M) dx$ .

Existence of the minimizers of  $E$  is proved in [57]; here the author showed that the total magnetic energy as a functional of  $M$  is convex, coercive and lower-semicontinuous in Sobolev space  $H^1$ , hence the corresponding minimization problem has at least one solution. Of course this is not unique, in general, because of the non-convexity of the constraint  $|M| = m$ . Regularity results about these minimizers are proved in [13], [45] and [57].

Gioia and James in [39] study the asymptotic behavior arbitrary-shaped very thin films of small thickness. They analyze "rescaled energies". So they prove that the thickness of the film imports an artificial anisotropy which disfavours out of plane magnetization. Moreover the limiting energy is completely local, that is to say the magnetostatic equation which contains the magnetization  $m$  in the original problem, disappears from the limiting one.

Problems of dimension reduction in magnetostatic were treated by several authors. A pioneering work is the paper of Stoner and Wohlfarth (1948). A rigorous treatment in this case was given by De Simone [19]. Carbou treated the case of thin films again in [12]. Other regimes are considered in [19] and [20] in the case of the films.

In [33] and [34] Gaudiello and Hadiji studies the behavior of minimizers of problem 2.1.2 in a multidomain. More precisely, in [33] Gaudiello and Hadiji,

study the free energy of two joined ferromagnetic thin films distinguishing different regimes depending on the limit of the ratio between the small thickness of the two films. In what concerns the study of a ferroelectric materials see also [35]. See [36], [37], [31] for junction 3D-1D, and [32] for junction 1D-1D.

When the body is isotropic, one can assume  $\phi(M) = 0$ . In this case the quasy-stationary model situation is governed by Landau-Lifshitz's equation (see [50], [14])

$$(2.1.3) \quad \frac{\partial M}{\partial t} + M \wedge \frac{\partial M}{\partial t} = 2M \wedge (\Delta M + H(M))$$

subject to conditions (2.1.1).

The existence result for this problem is proved, in a more general case, in [57] (see Theorem 2) and in [14] (see Section 3 and Section 5). We observe (see [14] and [57]) that the corresponding configuration satisfies the following energy estimate:

$$E(t) \leq E(0), \quad \forall t \in [0, T].$$

Then Carbou in [12], as Gioia and James [39] in the stationary case, studies the limit behavior of the isotropic ferromagnetic films when the thicknesses goes to zero, in the quasy stationary case. Other similar problems are studied by Ammari et al. [5].

In this work we study, as Gaudiello and Hadji in the stationary case (see [33]), the limit behavior of a system governed by Landau-Lifshitz equation, in a isotropic ferromagnetic multi-domain consisting of two joined thin films when the thicknesses goes to zero.

More precisely, for every  $n \in N$ , let

$$(2.1.4) \quad \Omega_n^a = \left( \left] -\frac{h_n^a}{2}, \frac{h_n^a}{2} \left[ \times \left] -\frac{1}{2}, \frac{1}{2} \left[ \times [0, 1[ \right), \quad \Omega_n^b = \left( \left] -\frac{1}{2}, \frac{1}{2} \left[ \times \left] -h_n^b, 0 \left[ \right), \right. \\ \Omega_n = \Omega_n^a \cup \Omega_n^b$$

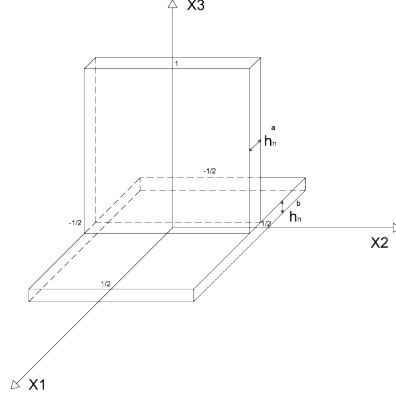
be a 3D ferromagnetic multidomain consisting of two orthogonal joined films, as in Figure 2.1, with small thicknesses  $\{h_n^a\}_{n \in N}, \{h_n^b\}_{n \in N} \subset ]0, 1[$  such that

$$(2.1.5) \quad \lim_n h_n^a = 0 = \lim_n h_n^b, \quad \lim_n \frac{h_n^b}{h_n^a} = q \in [0, +\infty].$$

(For instance, such structure appears as a component of a rotor of a permanent magnetic synchronous micro-machine, see Irudayarai and Emadi [47]).

Then, for any fixed  $T > 0$ , the aim of our paper is to study the asymptotic behavior, as  $h_n^a \rightarrow 0$  and  $h_n^b \rightarrow 0$ , of the following problem:



Figure 2.1:  $\Omega_n$ 

$$(2.1.6) \quad \begin{cases} \frac{\partial M_n}{\partial t} + M_n \wedge \frac{\partial M_n}{\partial t} = 2M_n \wedge (\Delta M_n + H(M_n)) & \text{in } \Omega_n \times ]0, T[, \\ M_n(0, x) = M_{0_n}(x) & \text{in } \Omega_n, \\ \text{subject to conditions (2.1.1).} \end{cases}$$

where  $M_{0_n}(x) \in H^1(\Omega_n)$ ,  $|M_{0_n}(x)| = 1$  a. e. in  $\Omega_n$  and  $\operatorname{div}(H(M_{0_n}) + \overline{M}_{0_n}) = 0$  in  $\mathbb{R}^3$ . After having reformulated our problem in a fixed domain  $\Omega = \Omega^a \cup \Omega^b$ , with  $\Omega^a = ]-\frac{1}{2}, \frac{1}{2}[^2 \times ]0, 1[$  and  $\Omega^b = ]-\frac{1}{2}, \frac{1}{2}[^2 \times ]-1, 0[$ , through appropriate rescalings of the kind proposed by Ciarlet and Destuynder [17], using the main ideas of  $\Gamma$ -convergence method introduced by De Giorgi [40], we derive the limit problem which depends on the limit  $\lim_n \frac{h_n^b}{h_n^a} = q \in [0, +\infty]$ , that is the ratio between the thickness  $h_n^b$  and  $h_n^a$ .

If  $q \in ]0, +\infty[$  (i.e.  $h_n^b \simeq h_n^a$ ), we show the solutions of rescaled equation converge, in some sense, to the solutions of a limit problem "morally" defined only on two perpendicular sections.

Indeed, for any fixed  $T > 0$ , this problem is described by a coupled of

magnetic field  $\mu^a$  and  $\mu^b$  acting on a couple of perpendicular surface, i. e.

$$(2.1.7) \quad \left\{ \begin{array}{l} \frac{\partial \mu^a}{\partial t} + \mu^a \wedge \frac{\partial \mu^a}{\partial t} = 2\mu^a \wedge (\Delta \mu^a - (\mu^a, e_1) e_1) \quad \text{in } ]0, T[ \times ]-\frac{1}{2}, \frac{1}{2}[ \times ]0, 1[ \\ q \frac{\partial \mu^b}{\partial t} + q \left( \mu^b \wedge \frac{\partial \mu^b}{\partial t} \right) = 2q\mu^b \wedge (\Delta \mu^b - (\mu^b, e_3) e_3) \quad \text{in } ]0, T[ \times ]-\frac{1}{2}, \frac{1}{2}[^2, \\ \mu^a(0, x) = \mu_0^a, \quad \text{in } ]-\frac{1}{2}, \frac{1}{2}[ \times ]0, 1[ \\ \mu^b(0, x) = \mu_0^b \quad \text{in } ]-\frac{1}{2}, \frac{1}{2}[^2 \\ \mu^a(\cdot, x_2, 0) = \mu^b(\cdot, 0, x_2), \quad \text{for } x_2 \text{ a.e. in } ]-\frac{1}{2}, \frac{1}{2}[ \\ |\mu^a| = 1 \quad \text{for a.e. } x \text{ in } \Omega^a, \quad |\mu^b| = 1 \quad \text{for a.e. } x \text{ in } \Omega^b. \end{array} \right.$$

We explicitly observe that the couplage between the two 2D problems is given by the junction condition  $\mu^a(x_2, 0) = \mu^b(0, x_2)$ , for  $x_2$  a.e. in  $]-\frac{1}{2}, \frac{1}{2}[$ . Moreover  $\mu_0^a$  and  $\mu_0^b$  in (2.1.7) are limits in some sense of the "rescaled" initial data  $M_{0_n}$ . We note that this problem is completely local. The cases  $q = 0$  and  $q = +\infty$  were studied in [24]. Here we proved that if  $q = 0$  (i.e.  $h_n^b \ll h_n^a$ ) the limit problem reduces to a 2D local problem in a vertical thin film losing the junction condition (see [24] Theorem 3.1). Analogously, if  $q = +\infty$  (i.e.  $h_n^a \ll h_n^b$ ) the limit problem reduces to a 2D local problem in an horizontal thin film (see [24] Theorem 3.2).

The paper is organized as follows. Section 2 is devoted to the statement of the problem. In Section 3, after some preliminaries, we give the main results. Section 4 consists of three steps. The first step is devoted to obtain some compactness properties which also provides some uniform a priori estimates for the rescaled magnetization  $M_n$ , in the second one we get the convergence results for the rescaled magnetostatic energies. The main result, Theorem 2.3.1, is proved, using appropriate test functions, in the third step. In particular here we consider a couple of test functions satisfying transmission conditions between the two films. Moreover the estimate for the limit energy is also proved.

## 2.2 Statement of the problem

### 2.2.1 Preliminary notations and weak formulation of (2.1.6)

Let  $x = (x_1, x_2, x_3)$  denote the generic point of  $\mathbb{R}^3$ . If  $a, b, c \in \mathbb{R}^3$ , then  $(a | b | c)$  denotes the  $3 \times 3$  real matrix having  $a^T$  as first column  $b^T$  as second column and  $c^T$  as third column. In according with this notation if  $v : A \subset \mathbb{R}^3 \rightarrow \mathbb{R}^3$  then  $Dv$  denotes the  $3 \times 3$  real matrix  $(D_{x_1}v | D_{x_2}v | D_{x_3}v)$ , where  $D_{x_i}v \in \mathbb{R}^3$ ,  $i = 1, 2, 3$ , stands for the derivative of  $v$  with respect to  $x_i$ . Let  $S^2$  be the unit

sphere of  $\mathbb{R}^3$ .

Let  $\Omega^a = ]-\frac{1}{2}, \frac{1}{2}[ \times ]-\frac{1}{2}, \frac{1}{2}[ \times ]0, 1[$ ,  $\Omega^b = ]-\frac{1}{2}, \frac{1}{2}[^2 \times ]-1, 0[$  and  $\Omega = \Omega^a \cup \Omega^b$ .

Let  $B$  be a rectangle containing  $\bar{\Omega}$ , for instance let  $B = ]-1, 1[^2 \times ]-2, 2[$ .

Let us consider the space

$$(2.2.1) \quad \mathcal{U} = \left\{ U \in L^1_{loc}(\mathbb{R}^3) : U \in L^2(B), DU \in (L^2(\mathbb{R}^3))^3, \int_B U dx = 0 \right\}.$$

It is easy to prove that  $\mathcal{U}$  is contained in  $L^2_{loc}(\mathbb{R}^3)$  and it is an Hilbert space with the inner product  $(U, V) = \int_{\mathbb{R}^3} DUDV dx + \int_B UV dx$ . Moreover, from Poincaré -

Wirtinger inequality it follows that a norm on  $\mathcal{U}$  equivalent to  $(U, U)^{\frac{1}{2}}$  is given by  $\left( \int_{\mathbb{R}^3} |DU|^2 dx \right)^{\frac{1}{2}}$ .

To reformulate conditions (2.1.1), related to the equation (2.1.3), as usual let us introduce  $U_M$ , the scalar magnetostatic potential, which satisfies the equation  $div(-D_x U_M + \bar{M}) = 0$ , where  $\bar{M}$  denotes the zero extension of  $M$  outside  $\Omega_n$ . Posing  $H = -DU_M$ , obviously, we obtain  $\nabla \times H(M) = 0$  and then conditions (2.1.1).

Let  $M \in L^2(\Omega_n, \mathbb{R}^3)$  then the following problem

$$(2.2.2) \quad \begin{cases} U_M \in \mathcal{U} \\ \int_{\mathbb{R}^3} DU_M DU = \int_{\Omega_n} M DU dx \quad \forall U \in \mathcal{U}, \end{cases}$$

admits a unique solution  $U_M \in \mathcal{U}$ . This solution is characterized as the unique minimizer of the following problem:

$$(2.2.3) \quad \min \left\{ \frac{1}{2} \int_{\mathbb{R}^3} |DU - \bar{M}|^2 dx : U \in \mathcal{U} \right\},$$

where as usual  $\bar{M}$  denotes the zero extension of  $M$  in  $\mathbb{R}^3 \setminus \Omega_n$ . Moreover  $U_M \in H^1(\mathbb{R}^3)$  up to an additive constant, see [48].

Then a weak formulation of the Landau-Lifschitz equation (2.1.6) in our case is the following.

Fixed  $M_{0,n} \in H^1(\Omega_n, S^2)$ ,  $U_{0,n} \in \mathcal{U}$  being the corresponding solution of problem (2.2.2), find  $M_n$  which satisfies

$$(2.2.4) \quad \left\{ \begin{array}{l} M_n \in L^\infty(0, T; H^1(\Omega_n, S^2)) \\ \frac{\partial M_n}{\partial t} \in L^2(0, T; L^2(\Omega_n, \mathbb{R}^3)) \\ \forall \phi \in D([0, T] \times \overline{\Omega_n}, \mathbb{R}^3) \\ \int_0^T \int_{\Omega_n} \left( \frac{\partial M_n}{\partial t} + M_n \wedge \frac{\partial M_n}{\partial t} \right) \cdot \phi \, dx \, dt = -2 \int_0^T \int_{\Omega_n} \sum_{i=1}^3 M_n \wedge D_{x_i} M_n \cdot (D_{x_i} \phi) - 2 \int_0^T \int_{\Omega_n} M_n \wedge DU_{M_n} \cdot \phi, \\ M_n(0, x) = M_{0_n}. \end{array} \right.$$

where  $U_{M_n}$  and  $M_n$  are linked by (2.2.2)

In [14] and [57] this problem has been solved. Moreover, it is proved the following energy estimate:

$$(2.2.5) \quad E(t) + \int_0^t \left\| \frac{\partial M_n}{\partial t} \right\|_{(L^2(\Omega_n))^3}^2 \, ds \leq E(0), \quad \forall t \in [0, T]$$

where

$$E(t) = \int_{\Omega_n} |DM_n(t, x)|^2 \, dx + \int_{\mathbb{R}^3} |DU_{M_n}(t, x)|^2 \, dx,$$

is the magnetic energy.

**Remark 2.2.1.** *By a density argument (for instance see [28]), problem (2.2.4) it is equivalent to that obtained choosing test functions  $\phi = \chi g$  with  $\chi \in D([0, T])$  and  $g \in H^1(\Omega_n, \mathbb{R}^3)$ .*

## 2.2.2 The rescaled problem

We can reformulate problem (2.2.4) on a fixed domain through the following rescalings

$$\left\{ \begin{array}{l} (x_1, x_2, x_3) \in \Omega^a = ]-\frac{1}{2}, \frac{1}{2}[ \times ]-\frac{1}{2}, \frac{1}{2}[ \times ]0, 1[ \rightarrow (h_n^a x_1, x_2, x_3) \in \text{Int}(\Omega_n^a) \\ (x_1, x_2, x_3) \in \Omega^b = ]-\frac{1}{2}, \frac{1}{2}[ \times ]-\frac{1}{2}, \frac{1}{2}[ \times ]-1, 0[ \rightarrow (x_1, x_2, h_n^b x_3) \in \Omega_n^b \end{array} \right.$$

where  $\text{Int}(\Omega_n^a)$  denotes the interior of  $\Omega_n^a$ .

For every  $n \in \mathbb{N}$ , the space  $\mathcal{U}$ , defined in (2.2.1), is rescaled in the following one:

$$(2.2.6) \quad \begin{aligned} \mathcal{U}_n = \{ v = (v^a, v^b) \in L^1_{loc}(\overline{\mathbb{R}^3_+}) \times L^1_{loc}(\overline{\mathbb{R}^3_-}) : (v^a_{|_{B_n^a}}, v^b_{|_{B_n^b}}) \in L^2(B_n^a) \times L^2(B_n^b), \\ (Dv^a, Dv^b) \in (L^2(\mathbb{R}^3_+))^3 \times (L^2(\mathbb{R}^3_-))^3, \int_{B_n^a} v^a \, dx + \frac{h_n^b}{h_n^a} \int_{B_n^b} v^b \, dx = 0, \\ v^a(x_1, x_2, 0) = v^b(h_n^a x_1, x_2, 0), \text{ for } (x_1, x_2) \text{ a.e. in } \mathbb{R}^2 \}, \end{aligned}$$

where  $\mathbb{R}_+^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > 0\}$ ,  $\mathbb{R}_-^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 < 0\}$ ,  
 $B_n^a = ]-\frac{1}{h_n^a}, \frac{1}{h_n^a}[ \times ]-1, 1[ \times ]0, 2[$  and  $B_n^b = ]-1, 1[^2 \times ]-\frac{2}{h_n^b}, 0[$ .

For  $\underline{m} \in L^2(\Omega, \mathbb{R}^3)$ , the following equation  
(2.2.7)

$$\begin{cases} u_{\underline{m}} = (u_{\underline{m}}^a, u_{\underline{m}}^b) \in \mathcal{U}_n, \\ \int_{\mathbb{R}_+^3} \left( \frac{1}{h_n^a} D_{x_1} u_{\underline{m}}^a, D_{x_2} u_{\underline{m}}^a, D_{x_3} u_{\underline{m}}^a \right) \cdot \left( \frac{1}{h_n^a} D_{x_1} u^a, D_{x_2} u^a, D_{x_3} u^a \right) dx + \\ \frac{h_n^b}{h_n^a} \int_{\mathbb{R}_-^3} \left( D_{x_1} u_{\underline{m}}^b, D_{x_2} u_{\underline{m}}^b, \frac{1}{h_n^b} D_{x_3} u_{\underline{m}}^b \right) \cdot \left( D_{x_1} u^b, D_{x_2} u^b, \frac{1}{h_n^b} D_{x_3} u^b \right) dx = \\ \int_{\Omega_a} \left( \frac{1}{h_n^a} D_{x_1} u^a, D_{x_2} u^a, D_{x_3} u^a \right) \cdot \underline{m} dx + \\ \frac{h_n^b}{h_n^a} \int_{\Omega_b} \left( D_{x_1} u^b, D_{x_2} u^b, \frac{1}{h_n^b} D_{x_3} u^b \right) \cdot \underline{m} dx, \quad \forall u = (u^a, u^b) \in \mathcal{U}_n, \end{cases}$$

which rescales equation (2.2.2), admits a unique solution.

Its solution,  $u_{\underline{m}} = (u_{\underline{m}}^a, u_{\underline{m}}^b) \in \mathcal{U}_n$  is characterized as the unique minimizer of the problem

$$(2.2.8) \quad \min \left\{ \frac{1}{2} \int_{\mathbb{R}_+^3} \left| \left( \frac{1}{h_n^a} D_{x_1} u^a, D_{x_2} u^a, D_{x_3} u^a \right) - \underline{m} \right|^2 dx \right. \\ \left. + \frac{1}{2} \frac{h_n^b}{h_n^a} \int_{\mathbb{R}_-^3} \left| \left( \frac{1}{h_n^a} D_{x_1} u^b, D_{x_2} u^b, D_{x_3} u^b \right) - \underline{m} \right|^2 dx : u \in \mathcal{U}_n \right\},$$

understanding  $\underline{m} = 0$  in  $\mathbb{R}^3 \setminus \Omega$ .

For every  $n \in \mathbb{N}$ , let us consider the following space

$$(2.2.9) \quad \mathcal{W}_n = \{ \underline{m} = (\underline{m}^a, \underline{m}^b) \in H^1(\Omega^a, \mathbb{R}^3) \times H^1(\Omega^b, \mathbb{R}^3) : \\ \underline{m}^a(x_1, x_2, 0) = \underline{m}^b(h_n^a x_1, x_2, 0), \text{ for } (x_1, x_2) \text{ a.e. in } ]-\frac{1}{2}, \frac{1}{2}[^2 \}.$$

For simplicity of notation, let us introduce the space

$$(2.2.10) \quad \mathcal{M}_n = \mathcal{W}_n \cap (H^1(\Omega^a, S^2) \times H^1(\Omega^b, S^2)).$$

which explicitly takes into account the condition  $|m| = 1$ . Now, it is possible to study problem (2.2.4) on a fixed domain.

Let  $\underline{m}_{0_n} = (\underline{m}_{0_n}^a, \underline{m}_{0_n}^b) \in \mathcal{M}_n$ , then there exists at least a weak solution of problem (2.2.4) renormalized by the two thickness  $h_n^a$  and  $h_n^b$ . This solution

satisfies the following problem:

$$(2.2.11) \quad \left\{ \begin{array}{l} \underline{m}_n = (\underline{m}_n^a, \underline{m}_n^b) \in L^\infty(0, T; \mathcal{M}_n) \\ \frac{\partial \underline{m}_n}{\partial t} = \left( \frac{\partial \underline{m}_n^a}{\partial t}, \frac{\partial \underline{m}_n^b}{\partial t} \right) \in L^2(0, T; L^2(\Omega^a, \mathbb{R}^3)) \times L^2(0, T; L^2(\Omega^b, \mathbb{R}^3)), \\ \forall \chi \in D(0, T) \text{ and } g = (g^a, g^b) \in \mathcal{W}_n \\ \int_0^T \int_{\Omega^a} \left( \frac{\partial \underline{m}_n^a}{\partial t} + \underline{m}_n^a \wedge \frac{\partial \underline{m}_n^a}{\partial t} \right) \chi g^a dx dt + \frac{h_n^b}{h_n^a} \int_0^T \int_{\Omega^b} \left( \frac{\partial \underline{m}_n^b}{\partial t} + \underline{m}_n^b \wedge \frac{\partial \underline{m}_n^b}{\partial t} \right) \chi g^b dx dt = \\ -2 \int_0^T \int_{\Omega^a} \left[ \underline{m}_n^a \wedge \frac{1}{h_n^a} D_{x_1} \underline{m}_n^a \cdot \frac{1}{h_n^a} (D_{x_1} g^a) \chi + \sum_{i=2}^3 \underline{m}_n^a \wedge D_{x_i} \underline{m}_n^a \cdot (D_{x_i} g^a) \chi \right] dx dt \\ -2 \int_0^T \int_{\Omega^a} \underline{m}_n^a \wedge \left( \frac{1}{h_n^a} D_{x_1} u_{\underline{m},n}^a, D_{x_2} u_{\underline{m},n}^a, D_{x_3} u_{\underline{m},n}^a \right) \chi g^a dx dt \\ -2 \frac{h_n^b}{h_n^a} \int_0^T \int_{\Omega^b} \left[ \underline{m}_n^b \wedge \frac{1}{h_n^b} D_{x_3} \underline{m}_n^b \cdot \frac{1}{h_n^b} (D_{x_3} g^b) \chi + \sum_{i=1}^2 \underline{m}_n^b \wedge D_{x_i} \underline{m}_n^b \cdot (D_{x_i} g^b) \chi \right] dx dt \\ -2 \int_0^T \int_{\Omega^b} \underline{m}_n^b \wedge \left( D_{x_1} u_{\underline{m},n}^b, D_{x_2} u_{\underline{m},n}^b, \frac{1}{h_n^b} D_{x_3} u_{\underline{m},n}^b \right) \chi g^b dx dt, \\ \underline{m}_n(0, x) = \underline{m}_{0_n} = (\underline{m}_{0_n}^a, \underline{m}_{0_n}^b). \end{array} \right.$$

(Let us point out that Problem (2.2.11) rescales Problem (2.2.4) and that  $u_{\underline{m}}$  and  $\underline{m}$  are linked by (2.2.7).)

Moreover the following energy estimate holds true

$$(2.2.12) \quad E_n(t) + \int_0^t \left\| \frac{\partial \underline{m}_n^a}{\partial t} \right\|_{(L^2(\Omega^a))^3}^2 ds + \frac{h_n^b}{h_n^a} \int_0^t \left\| \frac{\partial \underline{m}_n^b}{\partial t} \right\|_{(L^2(\Omega^b))^3}^2 ds \leq E_n(0), \forall t \in [0, T]$$

where

$$(2.2.13) \quad \begin{aligned} E_n(t) &= \int_{\Omega^a} \left| \left( \frac{1}{h_n^a} D_{x_1} \underline{m}_n^a, D_{x_2} \underline{m}_n^a, D_{x_3} \underline{m}_n^a \right) \right|^2 dx + \frac{1}{2} \int_{\mathbb{R}_+^3} \left| \left( \frac{1}{h_n^a} D_{x_1} u_{\underline{m},n}^a, D_{x_2} u_{\underline{m},n}^a, D_{x_3} u_{\underline{m},n}^a \right) \right|^2 dx \\ &+ \frac{h_n^b}{h_n^a} \int_{\Omega^b} \left| \left( D_{x_1} \underline{m}_n^b, D_{x_2} \underline{m}_n^b, \frac{1}{h_n^b} D_{x_3} \underline{m}_n^b \right) \right|^2 dx + \frac{1}{2} \frac{h_n^b}{h_n^a} \int_{\mathbb{R}_-^3} \left| \left( D_{x_1} u_{\underline{m},n}^b, D_{x_2} u_{\underline{m},n}^b, \frac{1}{h_n^b} D_{x_3} u_{\underline{m},n}^b \right) \right|^2 dx. \end{aligned}$$

Setting for every  $n \in \mathbb{N}$  and for every  $t \in [0, T]$

(2.2.14)

$$E_n^{mag}(t) = \frac{1}{2} \int_{\mathbb{R}_+^3} \left| \left( \frac{1}{h_n^a} D_{x_1} u_{\underline{m},n}^a, D_{x_2} u_{\underline{m},n}^a, D_{x_3} u_{\underline{m},n}^a \right) \right|^2 dx + \frac{1}{2} \frac{h_n^b}{h_n^a} \int_{\mathbb{R}_-^3} \left| \left( D_{x_1} u_{\underline{m},n}^b, D_{x_2} u_{\underline{m},n}^b, \frac{1}{h_n^b} D_{x_3} u_{\underline{m},n}^b \right) \right|^2 dx$$

by virtue (3.2.8),  $E_n(t)$  can be rewritten as

$$(2.2.15) \quad \begin{aligned} E_n(t) &= \int_{\Omega^a} \left| \left( \frac{1}{h_n^a} D_{x_1} \underline{m}_n^a, D_{x_2} \underline{m}_n^a, D_{x_3} \underline{m}_n^a \right) \right|^2 dx \\ &+ \frac{h_n^b}{h_n^a} \int_{\Omega^b} \left| \left( D_{x_1} \underline{m}_n^b, D_{x_2} \underline{m}_n^b, \frac{1}{h_n^b} D_{x_3} \underline{m}_n^b \right) \right|^2 dx \\ &+ E_n^{mag}(t). \end{aligned}$$

At this point we can observe that, for every  $t \in [0, T]$ , the function defined by

$$M_n(t, h_n^a x_1, x_2, x_3) \quad \text{for a.e. in } \Omega^a \quad M_n(t, x_1, x_2, h_n^b x_3) \quad \text{for a.e. in } \Omega^b$$

with  $M_n$  solution of the Problem (2.2.4), is a solution of Problem (2.2.11) with the following initial data:

$$(2.2.16) \quad \begin{cases} \underline{m}_{0,n}^a(x_1, x_2, x_3) = M_{0,n}(h_n^a x_1, x_2, x_3) \text{ for a.e. in } \Omega^a \\ \underline{m}_{0,n}^b(x_1, x_2, x_3) = M_{0,n}(x_1, x_2, h_n^b x_3) \text{ for a.e. in } \Omega^b. \end{cases}$$

### 2.3 The main results

We state now the main results of this paper, which describe the asymptotic behavior of Problem (2.2.11).

Let us introduce the following spaces

$$(2.3.1) \quad \begin{aligned} \mathcal{W} &= \left\{ \begin{array}{l} \psi = (\psi^a, \psi^b) \in H^1(\Omega^a, \mathbb{R}^3) \times H^1(\Omega^b, \mathbb{R}^3) : \psi^a \text{ is independent of } x_1, \\ \psi^b \text{ is independent of } x_3, \psi^a(0, x_2, 0) = \psi^b(0, x_2, 0), \text{ for } x_2 \text{ a.e. in } ]-\frac{1}{2}, \frac{1}{2}[ \end{array} \right\} \\ &\simeq \left\{ \begin{array}{l} \psi = (\psi^a, \psi^b) \in H^1\left(]-\frac{1}{2}, \frac{1}{2}[ \times ]0, 1[, \mathbb{R}^3\right) \times H^1\left(]-\frac{1}{2}, \frac{1}{2}[^2, \mathbb{R}^3\right) : \\ \psi^a(x_2, 0) = \psi^b(0, x_2), \text{ for } x_2 \text{ a.e. in } ]-\frac{1}{2}, \frac{1}{2}[ \end{array} \right\}, \end{aligned}$$

and, for uniformity of notation

$$(2.3.2) \quad \begin{aligned} \mathcal{M} &= \mathcal{W} \cap \left\{ H^1(\Omega^a, S^2) \times H^1(\Omega^b, S^2) \right\} \\ &\simeq \mathcal{W} \cap \left\{ H^1\left(]-\frac{1}{2}, \frac{1}{2}[ \times ]0, 1[, S^2\right) \times H^1\left(]-\frac{1}{2}, \frac{1}{2}[^2, S^2\right) \right\}, \end{aligned}$$

To enunciate our theorem, we need some hypothesis on the initial data. At this aim, let us suppose that

$$(2.3.3) \quad \left\{ \begin{array}{l} \underline{m}_{0_n} = (\underline{m}_{0_n}^a, \underline{m}_{0_n}^b) \in \mathcal{M}_n, \quad \underline{\mu}_0 = (\underline{\mu}_0^a, \underline{\mu}_0^b) \in \mathcal{M}, \\ \underline{m}_{0_n}^a \rightarrow \underline{\mu}_0^a \text{ strongly in } H^1(\Omega^a, S^2), \quad \underline{m}_{0_n}^b \rightarrow \underline{\mu}_0^b \text{ strongly in } H^1(\Omega^b, S^2), \\ \frac{1}{h_n^a} D_{x_1} \underline{m}_{0_n}^a \rightarrow 0 \text{ strongly in } L^2(\Omega^a, \mathbb{R}^3), \quad \frac{1}{h_n^b} D_{x_3} \underline{m}_{0_n}^b \rightarrow 0 \text{ strongly in } L^2(\Omega^b, \mathbb{R}^3). \end{array} \right.$$

For example, let  $\mu_0^a(y, z) \in H^1\left(]-\frac{1}{2}, \frac{1}{2}[ \times ]0, 1[, S^2\right)$  and  $\mu_0^b(x, y) \in H^1\left(]-\frac{1}{2}, \frac{1}{2}[^2, S^2\right)$  such that there exist  $\alpha < 0$  and  $\beta > 0$  such that  $\mu_0^a(y, 0) = \mu_0^b(x, y) \forall \alpha < x < \beta$ ,  $\forall y \in ]-\frac{1}{2}, \frac{1}{2}[$ . If  $m_n \in L^\infty(0, T; \mathcal{M}_n)$  is a weak solution of Problem (2.2.11) then assumptions (2.3.3) are satisfied with initial data

$$m_n^0(x, y, z) = \begin{cases} \mu_0^a(y, z) & \text{in } ]-\frac{1}{2}, \frac{1}{2}[ \times ]0, 1[ \\ \mu_0^b(x, y) & \text{in } ]-\frac{1}{2}, \frac{1}{2}[^2, \end{cases}$$

**Theorem 2.3.1.** *Let  $\mathcal{M}$  be defined by (2.3.2). Assume (2.1.5) with  $q \in ]0, +\infty[$ . For every  $n \in \mathbb{N}$ , let  $\underline{m}_n = (\underline{m}_n^a, \underline{m}_n^b)$  be the solution of Problem (2.2.11) with assumptions (2.3.3) on the initial data. Then, there exists an increasing sequence of positive integer numbers  $\{n_i\}_{i \in \mathbb{N}}$ , still denoted by  $\{n\}$ , and  $\underline{\mu} = (\underline{\mu}^a, \underline{\mu}^b) = \left( (\underline{\mu}_1^a, \underline{\mu}_2^a, \underline{\mu}_3^a), (\underline{\mu}_1^b, \underline{\mu}_2^b, \underline{\mu}_3^b) \right) \in L^\infty(0, T; \mathcal{M})$ , depending on the selected subsequence, such that*

$$(2.3.4) \quad \left\{ \begin{array}{l} \underline{m}_n^a \rightharpoonup \underline{\mu}^a \text{ weakly } * \text{ in } L^\infty(0, T; H^1(\Omega^a, \mathbb{R}^3)) \\ \underline{m}_n^b \rightharpoonup \underline{\mu}^b \text{ weakly } * \text{ in } L^\infty(0, T; H^1(\Omega^b, \mathbb{R}^3)) \\ \underline{m}_n^a \rightarrow \underline{\mu}^a \text{ strongly in } L^2(0, T; L^2(\Omega^a, \mathbb{R}^3)) \text{ and a. e. in } ]0, T[ \times \Omega^a \\ \underline{m}_n^b \rightarrow \underline{\mu}^b \text{ strongly in } L^2(0, T; L^2(\Omega^b, \mathbb{R}^3)) \text{ and a. e. in } ]0, T[ \times \Omega^b \end{array} \right.$$



where  $\underline{\mu}$  is the solution of the following problem:

$$(2.3.5) \quad \left\{ \begin{array}{l} \underline{\mu}(0, x) = \underline{\mu}_0 = (\underline{\mu}_0^a, \underline{\mu}_0^b) \in \mathcal{M} \\ \underline{\mu} = (\underline{\mu}^a, \underline{\mu}^b) \in L^\infty(0, T; \mathcal{M}) \\ \frac{\partial \underline{\mu}}{\partial t} \in L^2(0, T; L^2(\Omega^a, \mathbb{R}^3) \times L^2(\Omega^b, \mathbb{R}^3)) \\ \forall \varphi \in D(0, T) \text{ and } g = (g^a, g^b) \in \mathcal{W} \\ \int_0^T \int_{]-\frac{1}{2}, \frac{1}{2}[ \times ]0, 1[} \left( \frac{\partial \underline{\mu}^a}{\partial t} + \underline{\mu}^a \wedge \frac{\partial \underline{\mu}^a}{\partial t} \right) \varphi g^a dx dt + \int_0^T \int_{]-\frac{1}{2}, \frac{1}{2}[^2} \left( \frac{\partial \underline{\mu}^b}{\partial t} + \underline{\mu}^b \wedge \frac{\partial \underline{\mu}^b}{\partial t} \right) \varphi g^b dx dt = \\ -2 \int_0^T \int_{]-\frac{1}{2}, \frac{1}{2}[ \times ]0, 1[} \sum_{i=2}^3 \underline{\mu}^a \wedge \frac{\partial \underline{\mu}^a}{\partial x_i} \frac{\partial g^a}{\partial x_i} \varphi - 2q \int_0^T \int_{]-\frac{1}{2}, \frac{1}{2}[^2} \sum_{i=1}^2 \underline{\mu}^b \wedge \frac{\partial \underline{\mu}^b}{\partial x_i} \frac{\partial g^b}{\partial x_i} \varphi \\ -2 \int_0^T \int_{]-\frac{1}{2}, \frac{1}{2}[ \times ]0, 1[} \underline{\mu}^a \wedge (\underline{\mu}^a, e_1) e_1 g^a \varphi - 2q \int_0^T \int_{]-\frac{1}{2}, \frac{1}{2}[^2} \underline{\mu}^b \wedge (\underline{\mu}^b, e_3) e_3 g^b \varphi. \end{array} \right.$$

Moreover, the following energy estimate holds:

$$(2.3.6) \quad E_q(t) + \int_0^t \left\| \frac{\partial \underline{\mu}^a}{\partial t} \right\|_{(L^2(]-\frac{1}{2}, \frac{1}{2}[ \times ]0, 1[))^3}^2 ds + q \int_0^t \left\| \frac{\partial \underline{\mu}^b}{\partial t} \right\|_{(L^2(]-\frac{1}{2}, \frac{1}{2}[^2))^3}^2 ds \leq E_q(0), \quad \forall t \in [0, T]$$

where

$$(2.3.7) \quad E_q(t) = \int_{]-\frac{1}{2}, \frac{1}{2}[ \times ]0, 1[} |D \underline{\mu}^a|^2 + q \int_{]-\frac{1}{2}, \frac{1}{2}[^2} |D \underline{\mu}^b|^2 + \frac{1}{2} \int_{]-\frac{1}{2}, \frac{1}{2}[ \times ]0, 1[} |\underline{\mu}_1^a|^2 dx_2 dx_3 + \frac{1}{2} q \int_{]-\frac{1}{2}, \frac{1}{2}[^2} |\underline{\mu}_3^b|^2 dx_1 dx_2.$$

We remark, again, that the solution  $\underline{\mu} = (\underline{\mu}^a, \underline{\mu}^b) \in L^\infty(0, T; \mathcal{M})$  of the limit Problem (2.3.5) is such that  $\underline{\mu}^a$  and  $\underline{\mu}^b$  does not depend on  $x_1$  and  $x_3$  respectively.

## 2.4 Proof of main result

The proof is developed in several steps.

### Step 1: A priori estimates

Let us introduce the following compactness results:

**Theorem 2.4.1.** *Let  $\{\underline{m}_n\}_{n \in \mathbb{N}}$  be a sequence such that*

$$(2.4.1) \quad \begin{cases} i) & \underline{m}_n = (\underline{m}_n^a, \underline{m}_n^b) \in L^\infty(0, T; \mathcal{M}_n), \\ ii) & \|D_{x_1} \underline{m}_n^a\|_{L^\infty(0, T; L^2(\Omega^a))} \leq Ch_n^a, \quad \|D_{x_2} \underline{m}_n^a\|_{L^\infty(0, T; L^2(\Omega^a))} \leq C, \quad \|D_{x_3} \underline{m}_n^a\|_{L^\infty(0, T; L^2(\Omega^a))} \leq C, \\ iii) & \|D_{x_1} \underline{m}_n^b\|_{L^\infty(0, T; L^2(\Omega^b))} \leq C, \quad \|D_{x_2} \underline{m}_n^b\|_{L^\infty(0, T; L^2(\Omega^b))} \leq C, \quad \|D_{x_3} \underline{m}_n^b\|_{L^\infty(0, T; L^2(\Omega^b))} \leq Ch_n^b, \\ iv) & \left\| \frac{\partial \underline{m}_n^a}{\partial t} \right\|_{L^2(0, T; L^2(\Omega^a))} \leq C, \quad \left\| \frac{\partial \underline{m}_n^b}{\partial t} \right\|_{L^2(0, T; L^2(\Omega^b))} \leq C, \end{cases}$$

for every  $n \in \mathbb{N}$ , where  $C$  is a constant independent on  $n$ . Then there exists an increasing sequence of positive integer numbers  $\{n_i\}_{i \in \mathbb{N}}$ , still denoted by  $\{n\}$ , and  $\underline{\mu} = (\underline{\mu}^a, \underline{\mu}^b) = \left( (\underline{\mu}_1^a, \underline{\mu}_2^a, \underline{\mu}_3^a), (\underline{\mu}_1^b, \underline{\mu}_2^b, \underline{\mu}_3^b) \right) \in L^\infty(0, T; \mathcal{M})$ , depending on the selected subsequence, such that

$$(2.4.2) \quad \begin{cases} i) & \underline{m}_n^a \rightharpoonup \underline{\mu}^a \text{ weakly } * \text{ in } L^\infty(0, T; H^1(\Omega^a, \mathbb{R}^3)), \\ & \text{strongly } L^2(0, T; L^2(\Omega^a, \mathbb{R}^3)) \text{ and a. e. in } ]0, T[ \times \Omega^a, \\ ii) & \underline{m}_n^b \rightharpoonup \underline{\mu}^b \text{ weakly } * \text{ in } L^\infty(0, T; H^1(\Omega^b, \mathbb{R}^3)), \\ & \text{strongly } L^2(0, T; L^2(\Omega^b)) \text{ and a. e. in } ]0, T[ \times \Omega^b, \\ iii) & \frac{\partial \underline{m}_n^a}{\partial t} \rightharpoonup \frac{\partial \underline{\mu}^a}{\partial t} \text{ weakly in } L^2(0, T; L^2(\Omega^a, \mathbb{R}^3)), \\ iv) & \frac{\partial \underline{m}_n^b}{\partial t} \rightharpoonup \frac{\partial \underline{\mu}^b}{\partial t} \text{ weakly in } L^2(0, T; L^2(\Omega^b, \mathbb{R}^3)). \end{cases}$$

*Proof.* From (2.4.1), it follows that there exist  $\underline{\mu}^a \in L^\infty(0, T; H^1(\Omega^a))$ ,  $\underline{\mu}^b \in L^\infty(0, T; H^1(\Omega^b))$ ,  $\frac{\partial \underline{\mu}^a}{\partial t} \in L^2(0, T; L^2(\Omega^a))$  and  $\frac{\partial \underline{\mu}^b}{\partial t} \in L^2(0, T; L^2(\Omega^b))$  such that convergences (2.4.2) hold true. Moreover, since  $\underline{m}_n^a \rightarrow \underline{\mu}^a$  and  $\underline{m}_n^b \rightarrow \underline{\mu}^b$  almost everywhere we have that  $|\underline{\mu}^a| = 1$  a.e. in  $(0, T) \times \Omega^a$  and  $|\underline{\mu}^b| = 1$  a.e. in  $(0, T) \times \Omega^b$ . In order to prove that the limit function  $\underline{\mu} = (\underline{\mu}^a, \underline{\mu}^b)$  belongs to  $L^\infty(0, T; \mathcal{M})$  we need to prove that

$$(2.4.3) \quad \underline{\mu}^a(t, x_2, 0) = \underline{\mu}^b(t, 0, x_2) \text{ a.e. in } ]0, T[ \times \left] -\frac{1}{2}, \frac{1}{2} \right[.$$

The proof of (2.4.3) will be developed in several steps. To this aim, we have

$$(2.4.4) \quad \lim_i \int_{[0,T] \times ]-\frac{1}{2}, \frac{1}{2}[^2} (\underline{m}_{n_i}^a(t, x_1, x_2, 0) - \underline{\mu}^a(t, x_2, 0)) \varphi(x_2) \chi(t) dx_1 dx_2 dt = 0,$$

$$\forall \varphi \in C_0^\infty(]-\frac{1}{2}, \frac{1}{2}[), \forall \chi \in C_0^\infty([0, T]).$$

Set

$$\rho_i(x_3) = \int_{[0,T] \times ]-\frac{1}{2}, \frac{1}{2}[^2} (|D_{x_1} \underline{m}_{n_i}^b(t, x_1, x_2, x_3)|^2 + |D_{x_2} \underline{m}_{n_i}^b(t, x_1, x_2, x_3)|^2 + |\underline{m}_{n_i}^b(t, x_1, x_2, x_3)|^2) dx_1 dx_2 dt \\ \text{a.e. in } ]-1, 0[.$$

From Fatou's Lemma, (2.4.1) *iii*) and the fact that  $|m_{n_i}^b| = 1$  a.e. in  $\Omega^b$ , it follows that

$$\int_{-1}^0 \liminf_i \rho_i(x_3) dx_3 \leq \liminf_i \int_{-1}^0 \rho_i(x_3) dx_3 < +\infty$$

Consequently, there exists two constants  $c \in ]0, +\infty[$  and  $\bar{x}_3 \in ]-1, 0[$ , and a subsequence  $(\rho_{i_k})_{k \in \mathbb{N}}$  such that

$$\rho_{i_k}(\bar{x}_3) < c, \forall k \in \mathbb{N}.$$

Then, by (2.4.2) *ii*), it results that

$$(2.4.5) \quad \underline{m}_{n_{i_k}}^b(\cdot, \cdot, \cdot, \bar{x}_3) \rightharpoonup \underline{\mu}^b \quad * \text{ weak in } L^\infty([0, T], H^1(]-\frac{1}{2}, \frac{1}{2}[^2, \mathbb{R}^3)),$$

as  $k$  diverges.

The second step is devoted to prove that

$$(2.4.6) \quad \lim_i \int_{[0,T] \times ]-\frac{1}{2}, \frac{1}{2}[^2} (\underline{\mu}_{n_{i_k}}^b(t, h_{n_{i_k}}^a(x_1, x_2, 0)) - \underline{\mu}^b(t, 0, x_2)) \varphi(x_2) \chi(t) dx_1 dx_2 dt = 0,$$

$$\forall \varphi \in C_0^\infty(]-\frac{1}{2}, \frac{1}{2}[), \forall \chi \in C_0^\infty([0, T]).$$

Moreover, for  $\varphi \in C_0^\infty(]-\frac{1}{2}, \frac{1}{2}[)$  and  $\chi \in C_0^\infty([0, T])$ , one has

$$(2.4.7) \quad \int_{[0,T] \times ]-\frac{1}{2}, \frac{1}{2}[^2} (\underline{m}_{n_{i_k}}^b(t, h_{n_{i_k}}^a(x_1, x_2, 0)) - \underline{\mu}^b(t, 0, x_2)) \varphi(x_2) \chi(t) dx_1 dx_2 dt = \\ \int_{[0,T] \times ]-\frac{1}{2}, \frac{1}{2}[^2} (\underline{m}_{n_{i_k}}^b(t, h_{n_{i_k}}^a(x_1, x_2, 0)) - \underline{m}_{n_{i_k}}^b(t, h_{n_{i_k}}^a(x_1, x_2, \bar{x}_3))) \varphi(x_2) \chi(t) dx_1 dx_2 dt + \\ \int_{[0,T] \times ]-\frac{1}{2}, \frac{1}{2}[^2} (\underline{m}_{n_{i_k}}^b(t, h_{n_{i_k}}^a(x_1, x_2, \bar{x}_3)) - \underline{m}_{n_{i_k}}^b(t, 0, x_2, \bar{x}_3)) \varphi(x_2) \chi(t) dx_1 dx_2 dt + \\ \int_{[0,T] \times ]-\frac{1}{2}, \frac{1}{2}[^2} (\underline{m}_{n_{i_k}}^b(t, 0, x_2, \bar{x}_3) - \underline{\mu}^b(t, 0, x_2)) \varphi(x_2) \chi(t) dx_1 dx_2 dt, \quad \forall k \in \mathbb{N}.$$

Now will pass to the limit, as  $k$  diverges, in each term of this decomposition. By (2.4.1) *iii*) and (2.1.5) with  $q \neq +\infty$ , there exists  $C \in ]0, +\infty[$  such that

$$\begin{aligned}
 (2.4.8) \quad & \limsup_k \left| \int_{[0,T] \times ]-\frac{1}{2}, \frac{1}{2}[^2} \left( \underline{m}_{n_{i_k}}^b(t, h_{n_{i_k}}^a(x_1, x_2, 0)) - \underline{m}_{n_{i_k}}^b(t, h_{n_{i_k}}^a(x_1, x_2, \bar{x}_3)) \right) \varphi(x_2) \chi(t) dx_1 dx_2 dt \right| = \\
 & \limsup_k \left| \int_{[0,T] \times ]-\frac{1}{2}, \frac{1}{2}[^2} \left( \int_{\bar{x}_3}^0 D_{x_1} \underline{m}_{n_{i_k}}^b(t, h_{n_{i_k}}^a(x_1, x_2, s)) ds \right) \varphi(x_2) \chi(t) dx_1 dx_2 dt \right| \leq \\
 & \|\varphi\|_{L^\infty(]-\frac{1}{2}, \frac{1}{2}[)} \|\chi\|_{L^\infty([0,T])} T^{\frac{1}{2}} |\Omega^b|^{\frac{1}{2}} \limsup_k \left( \int_{[0,T] \times \Omega^b} |D_{x_3} \underline{m}_{n_{i_k}}^b(t, h_{n_{i_k}}^a(x_1, x_2, x_3))|^2 dx dt \right)^{\frac{1}{2}} \leq \\
 & \|\varphi\|_{L^\infty(]-\frac{1}{2}, \frac{1}{2}[)} \|\chi\|_{L^\infty([0,T])} T^{\frac{1}{2}} |\Omega^b|^{\frac{1}{2}} \limsup_k \left( \frac{1}{h_{n_{i_k}}^a} \|D_{x_3} \underline{m}_{n_{i_k}}^b\|_{L^2([0,T] \times \Omega^b)} \right) \leq \\
 & \|\varphi\|_{L^\infty(]-\frac{1}{2}, \frac{1}{2}[)} \|\chi\|_{L^\infty([0,T])} T^{\frac{1}{2}} |\Omega^b|^{\frac{1}{2}} C \limsup_k \frac{h_{n_{i_k}}^b}{\sqrt{h_{n_{i_k}}^a}} = 0
 \end{aligned}$$

On the other side, from (2.4.5), there exists  $C \in ]0, +\infty[$  such that

$$\begin{aligned}
 (2.4.9) \quad & \limsup_k \left| \int_{[0,T] \times ]-\frac{1}{2}, \frac{1}{2}[^2} \left( \underline{m}_{n_{i_k}}^b(t, h_{n_{i_k}}^a(x_1, x_2, \bar{x}_3)) - \underline{m}_{n_{i_k}}^b(t, 0, x_2, \bar{x}_3) \right) \varphi(x_2) \chi(t) dx_1 dx_2 dt \right| = \\
 & \limsup_k \left| \int_{[0,T] \times ]-\frac{1}{2}, \frac{1}{2}[^2} \left( \int_0^{h_{n_{i_k}}^a x_1} D_s \underline{m}_{n_{i_k}}^b(t, s, x_2, \bar{x}_3) ds \right) \varphi(x_2) \chi(t) dx_1 dx_2 dt \right| \leq \\
 & \frac{1}{2} \|\varphi\|_{L^\infty(]-\frac{1}{2}, \frac{1}{2}[)} \|\chi\|_{L^\infty([0,T])} \limsup_k \int_{[0,T] \times ]-\frac{1}{2}, \frac{1}{2}[} \left( \int_0^{\frac{h_{n_{i_k}}^a}{2}} |D_s \underline{m}_{n_{i_k}}^b(t, s, x_2, \bar{x}_3)| ds \right) dx_2 dt + \\
 & \frac{1}{2} \|\varphi\|_{L^\infty(]-\frac{1}{2}, \frac{1}{2}[)} \|\chi\|_{L^\infty([0,T])} \limsup_k \int_{[0,T] \times ]-\frac{1}{2}, \frac{1}{2}[} \left( \int_{-\frac{h_{n_{i_k}}^a}{2}}^0 |D_s \underline{m}_{n_{i_k}}^b(t, s, x_2, \bar{x}_3)| ds \right) dx_2 dt \leq \\
 & \frac{1}{2} \|\varphi\|_{L^\infty(]-\frac{1}{2}, \frac{1}{2}[)} \|\chi\|_{L^\infty([0,T])} T^{\frac{1}{2}} \limsup_k \left( \frac{h_{n_{i_k}}^a}{2} \int_{[0,T] \times ]-\frac{1}{2}, \frac{1}{2}[^2} |D_{x_1} \underline{m}_{n_{i_k}}^b(t, x_1, x_2, \bar{x}_3)|^2 dx_1 dx_2 dt \right)^{\frac{1}{2}} \leq \\
 & \frac{C}{\sqrt{2}} \|\varphi\|_{L^\infty(]-\frac{1}{2}, \frac{1}{2}[)} \|\chi\|_{L^\infty([0,T])} T^{\frac{1}{2}} \lim_k \sqrt{h_{n_{i_k}}^a} = 0
 \end{aligned}$$

In virtue of (2.4.5) it follows that

$$\underline{m}_{n_{i_k}}^b(\cdot, \cdot, \cdot, \bar{x}_3) \rightarrow \underline{\mu}^b \quad \text{strongly in } L^2([0, T], L^2(]-\frac{1}{2}, \frac{1}{2}[^2, \mathbb{R}^3)) \text{ and a.e.}$$

Consequently, we obtain that

(2.4.10)

$$\lim_k \int_{[0,T] \times ]-\frac{1}{2}, \frac{1}{2}[^2} \left( \underline{m}_{n_{i_k}}^b(t, 0, x_2, \bar{x}_3) - \underline{\mu}^b(t, 0, x_2) \right) \varphi(t) \chi(x_2) dx_1 dx_2 dt =$$

$$\lim_k \int_{[0,T] \times ]-\frac{1}{2}, \frac{1}{2}[} \left( \underline{m}_{n_{i_k}}^b(t, 0, x_2, \bar{x}_3) - \underline{\mu}^b(t, 0, x_2) \right) \varphi(t) \chi(x_2) dx_2 dt = 0$$

$$\forall \varphi \in C_0^\infty[0, T] \quad \forall \chi \in C_0^\infty(]-\frac{1}{2}, \frac{1}{2}[)$$

Then, by passing to the limit in (2.4.7), as  $k$  diverges, and taking into account (2.4.8) – (2.4.10), one obtains (2.4.6).

Taking into account the following junction condition

$$(2.4.11) \quad \begin{aligned} & \int_{[0,T] \times ]-\frac{1}{2}, \frac{1}{2}[^2} \underline{m}_{n_{i_k}}^a(t, x_1, x_2, 0) \varphi(t) \chi(x_2) dx_1 dx_2 dt = \\ & \int_{[0,T] \times ]-\frac{1}{2}, \frac{1}{2}[^2} \underline{m}_{n_{i_k}}^b(t, h_{n_{i_k}}^a x_1, x_2, 0) \varphi(t) \chi(x_2) dx_1 dx_2 dt \end{aligned}$$

$$\forall k \in \mathbb{N}, \forall \varphi \in C_0^\infty[0, T] \quad \forall \chi \in C_0^\infty(]-\frac{1}{2}, \frac{1}{2}[),$$

By using (2.4.4) and (2.4.6) to pass to the limit in (2.4.11) we get (2.4.3).  $\square$

Let us obtain a priori estimates for the sequence of the solutions of the Problem (2.2.11).

**Proposition 2.4.2.** *Assume (2.1.5) with  $q \in [0, +\infty[$ . For every  $n \in \mathbb{N}$ , let  $\underline{m}_n = (\underline{m}_n^a, \underline{m}_n^b)$  be the solution of Problem (2.2.11). Then it results that*

$$(2.4.12) \quad \exists c \in ]0, +\infty[ : E_n(\underline{m}_n(t, \cdot)) \leq c, \quad \forall t \in [0, T] \quad \forall n \in \mathbb{N}.$$

*Proof.* Taking into account (2.2.14) and (2.2.15), we have

$$(2.4.13) \quad \begin{aligned} E_n(\underline{m}_n(0, \cdot)) &= \int_{\Omega^a} \left| \left( \frac{1}{h_n^a} D_{x_1} \underline{m}_{0,n}^a, D_{x_2} \underline{m}_{0,n}^a, D_{x_3} \underline{m}_{0,n}^a \right) \right|^2 dx + \\ &+ \frac{h_n^b}{h_n^a} \int_{\Omega^b} \left| \left( D_{x_1} \underline{m}_{0,n}^b, D_{x_2} \underline{m}_{0,n}^b, \frac{1}{h_n^b} D_{x_3} \underline{m}_{0,n}^b \right) \right|^2 dx + E_n^{mag}(\underline{m}_n(0, \cdot)) \end{aligned}$$

since  $\underline{m}_{0,n} = (\underline{m}_{0,n}^a, \underline{m}_{0,n}^b)$  is such that  $\underline{m}_{0,n}^a$  does not depend on  $x_1$  and  $\underline{m}_{0,n}^b$  does not depend on  $x_3$ .

By choosing  $\underline{m}_{0,n} = ((0, 1, 0), (0, 1, 0))$  as test function in (2.2.11) for  $t = 0$ , it follows that:

$$E_n(\underline{m}_n(0, \cdot)) \leq E_{n,((0,1,0),(0,1,0))}^{mag}(\underline{m}_n(0, \cdot)) \quad \forall n \in \mathbb{N}.$$

Consequently it remains to prove that

$$\forall t \in [0, T] \quad \exists c \in ]0, +\infty[ : E_{n,((0,1,0),(0,1,0))}^{mag}(\underline{m}_n(0, \cdot)) \leq c, \quad \forall n \in \mathbb{N}.$$

By choosing  $u = \left( u_{n,((0,1,0),(0,1,0))}^a, u_{n,((0,1,0),(0,1,0))}^b \right)$  as test function in (2.2.8) and using Cauchy-Schwarz inequality, one derives that

$$2E_{n,((0,1,0),(0,1,0))}^{mag}(\underline{m}_n(0, \cdot)) \leq \left( 1 + \left( \frac{h_n^b}{h_n^a} \right)^{\frac{1}{2}} \right) \left( 2E_{n,((0,1,0),(0,1,0))}^{mag}(\underline{m}_n(0, \cdot)) \right)^{\frac{1}{2}} \quad \forall n \in \mathbb{N},$$

which gives (2.4.12) since  $q \neq +\infty$ .  $\square$

The following results are an immediate consequence of Theorem 2.4.1, Proposition 2.4.2.

**Corollary 2.4.3.** *Let  $\mathcal{M}$  be defined by (2.3.2). Assume (2.1.5) with  $q \in ]0, +\infty[$ . For every  $n \in \mathbb{N}$ , let  $\underline{m}_n = (\underline{m}_n^a, \underline{m}_n^b)$  be the solution of problem (2.2.11).*

*Then, there exist an increasing sequence of positive integer numbers  $\{n_i\}_{i \in \mathbb{N}}$ , still denoted by  $\{n\}$ , and  $\underline{\mu} = (\underline{\mu}^a, \underline{\mu}^b) = \left( (\mu_1^a, \mu_2^a, \mu_3^a), (\mu_1^b, \mu_2^b, \mu_3^b) \right) \in L^\infty(0, T; \mathcal{M})$ , depending on the selected subsequence, such that:*

$$(2.4.14) \quad \left\{ \begin{array}{l} \underline{m}_n^a \rightharpoonup \underline{\mu}^a \text{ weakly } * \text{ in } L^\infty(0, T; H^1(\Omega^a, S^2)), \text{ strongly in } L^2(0, T; L^2(\Omega^a, S^2)) \text{ and a. e.}, \\ \underline{m}_n^b \rightharpoonup \underline{\mu}^b \text{ weakly } * \text{ in } L^\infty(0, T; H^1(\Omega^b, S^2)), \text{ strongly in } L^2(0, T; L^2(\Omega^b, S^2)) \text{ and a. e.}, \\ \frac{\partial \underline{m}_n^a}{\partial t} \rightharpoonup \frac{\partial \underline{\mu}^a}{\partial t} \text{ weakly in } L^2(0, T; L^2(\Omega^a, \mathbb{R}^3)), \\ \frac{\partial \underline{m}_n^b}{\partial t} \rightharpoonup \frac{\partial \underline{\mu}^b}{\partial t} \text{ weakly in } L^2(0, T; L^2(\Omega^b, \mathbb{R}^3)) \end{array} \right.$$

### Step 2: A convergence result for the magnetostatic energy, limit identification

Let us identify the limit function for the magnetostatic energy and its potential.

**Proposition 2.4.4.** *Assume (2.1.5) with  $q \in ]0, +\infty[$ . Let  $\{\underline{m}_n = (\underline{m}_n^a, \underline{m}_n^b)\}_{n \in \mathbb{N}} \in L^\infty(0, T; \mathcal{M}_n)$ , and  $\underline{\mu} = (\underline{\mu}^a, \underline{\mu}^b) = \left( (\mu_1^a, \mu_2^a, \mu_3^a), (\mu_1^b, \mu_2^b, \mu_3^b) \right) \in L^\infty(0, T; \mathcal{M})$  be such that*

$$(2.4.15) \quad \begin{aligned} (\underline{m}_n^a, \underline{m}_n^b) &\rightharpoonup (\underline{\mu}^a, \underline{\mu}^b) \quad \text{weakly } * \text{ in } L^\infty(0, T; H^1(\Omega^a, \mathbb{R}^3)) \times L^\infty(0, T; H^1(\Omega^b, \mathbb{R}^3)) \\ (\underline{m}_n^a, \underline{m}_n^b) &\rightarrow (\underline{\mu}^a, \underline{\mu}^b) \quad \text{strongly in } L^2(0, T; L^2(\Omega^a, \mathbb{R}^3)) \times L^2(0, T; L^2(\Omega^b, \mathbb{R}^3)) \end{aligned}$$

as  $n$  diverges. Let  $u_n = (u_n^a, u_n^b)$  be the unique solution of (2.2.7) corresponding to  $\underline{m}_n$ . Then it result that

$$(2.4.16) \quad \begin{cases} \frac{1}{h_n^a} D_{x_1} u_n^a \rightharpoonup \mu_1^a & D_{x_2} u_n^a \rightharpoonup 0 & D_{x_3} u_n^a \rightharpoonup 0 & \text{weakly } * \text{ in } L^\infty(0, T; L^2(\mathbb{R}_+^3)) \\ D_{x_1} u_n^b \rightharpoonup 0 & D_{x_2} u_n^b \rightharpoonup 0 & \frac{1}{h_n^b} D_{x_3} u_n^b \rightharpoonup \mu_3^b & \text{weakly } * \text{ in } L^\infty(0, T; L^2(\mathbb{R}_-^3)) \end{cases}$$

as  $n$  diverges, and for  $t \in [0, T]$

$$(2.4.17) \quad \lim_n E_n^{mag}(\underline{m}_n(t, \cdot)) = \frac{1}{2} \left( \int_{\Omega^a} |\mu_1^a|^2 dx + q \int_{\Omega^b} |\mu_3^b|^2 dx \right).$$

*Proof.* For a given  $u_{m,n}$  be a solution of (2.2.7), in particular, that potential takes a larger value for  $u_{m,n} = (0, 0)$ , it results that

$$(2.4.18) \quad \begin{aligned} & \exists c \in ]0, +\infty[ : \int_{\mathbb{R}_+^3} \left| \left( \frac{1}{h_n^a} D_{x_1} u_n^a, D_{x_2} u_n^a, D_{x_3} u_n^a \right) - m_n^a \right|^2 dx + \\ & \frac{h_n^b}{h_n^a} \int_{\mathbb{R}_-^3} \left| \left( D_{x_1} u_n^b, D_{x_2} u_n^b, \frac{1}{h_n^b} D_{x_3} u_n^b \right) - m_n^b \right|^2 dx \leq c, \forall n \in \mathbb{N}. \end{aligned}$$

Consequently, by applying the triangle inequality and recalling that  $|m_n| = 1$  and by taking into account (2.1.5) with  $q \neq +\infty$ , and for all  $t \in [0, T]$ , we obtains

$$(2.4.19) \quad \begin{cases} \exists c \in ]0, +\infty[, \\ \left\| \frac{1}{h_n^a} D_{x_1} u_n^a \right\|_{L^2(\mathbb{R}_+^3)} \leq c, \left\| D_{x_2} u_n^a \right\|_{L^2(\mathbb{R}_+^3)} \leq c, \left\| D_{x_3} u_n^a \right\|_{L^2(\mathbb{R}_+^3)} \leq c, \\ \left\| D_{x_1} u_n^b \right\|_{L^2(\mathbb{R}_-^3)} \leq c, \left\| D_{x_2} u_n^b \right\|_{L^2(\mathbb{R}_-^3)} \leq c, \left\| \frac{1}{h_n^b} D_{x_3} u_n^b \right\|_{L^2(\mathbb{R}_-^3)} \leq c, \end{cases}$$

From the Sobolev-Gagliardo-Niremberg inequality and (2.4.19), we obtains

$$(2.4.20) \quad \exists c \in ]0, +\infty[ : \|u_{m,n}\|_{L^6(\mathbb{R}^3)} \leq c, \quad \forall n \in \mathbb{N}, \forall t \geq 0.$$

Moreover, estimates (2.4.18) and (2.4.19) guarantee, for all  $t \in [0, T]$ , the existence of a function  $u = (u^a, u^b) \in L^6(\mathbb{R}_+^3) \times L^6(\mathbb{R}_-^3)$  and  $Du = (Du^a, Du^b) \in (L^2(\mathbb{R}_+^3))^3 \times (L^2(\mathbb{R}_-^3))^3$ ,  $u^a$  is independent of  $x_1$  and  $u^b$  is independent of  $x_3$  such that

$$(2.4.21) \quad \begin{cases} u_n^a \rightharpoonup u^a & \text{weak in } L^6(\mathbb{R}_+^3), & Du_n^a \rightharpoonup Du^a & \text{weak in } (L^2(\mathbb{R}_+^3))^3, \\ u_n^b \rightharpoonup u^b & \text{weak in } L^6(\mathbb{R}_-^3), & Du_n^b \rightharpoonup Du^b & \text{weak in } (L^2(\mathbb{R}_-^3))^3, \end{cases}$$

as  $n$  diverges. Moreover, for all  $t \in [0, T]$  and the fact that  $u^a$  is independent of

$x_1$  and  $Du^a \in (L^2(\mathbb{R}^3))^3$ , we have

$$+\infty > \int_{\mathbb{R}_+^3} |Du^a(t, x)|^2 dx \geq \int_{]a, b[} \left( \int_{\mathbb{R}_+^2} (|D_{x_2} u^a(t, x_2, x_3)|^2 + |D_{x_3} u^a(t, x_2, x_3)|^2) d(x_2, x_3) \right) dx_1 =$$

$$(b-a) \int_{\mathbb{R}_+^2} (|D_{x_2} u^a(t, x_2, x_3)|^2 + |D_{x_3} u^a(t, x_2, x_3)|^2) d(x_2, x_3), \forall a, b \in \mathbb{R} \text{ where } a < b.$$

In particular,  $(b-a)$  can be any arbitrarily large positive number, hence for all  $t$  in  $[0, T]$ ,  $u^a$  is independent of  $x_2$  and  $x_3$ , then  $u^a(t, \cdot) = 0$  in  $L^6(\mathbb{R}_+^3)$ . Similarly, for all  $t \in [0, T]$  we obtains  $u^b(t, \cdot) = 0$  in  $L^6(\mathbb{R}_-^3)$ . we concludes that

$$(2.4.22) \quad \begin{cases} Du_n^a \rightharpoonup 0 \text{ weak* in } L^\infty(0, T; (L^2(\mathbb{R}_+^3))^3), \\ Du_n^b \rightharpoonup 0 \text{ weak* in } L^\infty(0, T; (L^2(\mathbb{R}_-^3))^3), \end{cases}$$

and that there exist  $\xi^a \in L^2(\mathbb{R}_+^3)$  and  $\xi^b \in L^2(\mathbb{R}_-^3)$  such that

$$(2.4.23) \quad \begin{cases} \frac{1}{h_n^a} D_{x_1} u_n^a \rightharpoonup \xi^a \text{ weak* in } L^\infty(0, T; (L^2(\mathbb{R}_+^3))), \\ \frac{1}{h_n^b} D_{x_3} u_n^b \rightharpoonup \xi^b \text{ weak* in } L^\infty(0, T; (L^2(\mathbb{R}_-^3))), \end{cases}$$

as  $n$  diverges. The following step is devoted to identify  $\xi^a$  and  $\xi^b$ . Let us fix  $t \in [0, T]$ . In equation (2.2.7), choose

$$(2.4.24) \quad \begin{cases} m = m_n, \\ u^a = c_n^a + \varphi \quad \text{where } \varphi \in C_0^\infty(\mathbb{R}_+^3), \\ u^b = c_n^a, \end{cases}$$

where  $c_n^a = -(|B_n^a| + \frac{h_n^b}{h_n^a} |B_n^b|)^{-1} \int_{B_n^a} \varphi dx$ . After having multiplied this equation by  $h_n^a$ , we have:

$$\int_{\mathbb{R}_+^3} \left( \frac{1}{h_n^a} D_{x_1} u_n^a, D_{x_2} u_n^a, D_{x_3} u_n^a \right) (D_{x_1} \varphi, h_n^a D_{x_2} \varphi, h_n^a D_{x_3} \varphi) dx =$$

$$\int_{\Omega^a} (D_{x_1} \varphi, h_n^a D_{x_2} \varphi, h_n^a D_{x_3} \varphi) m_n^a dx, \forall \varphi \in C_0^\infty(\mathbb{R}_+^3)$$

Then passing to the limit as  $n$  diverges, we obtains

$$(2.4.25) \quad \int_{\mathbb{R}_+^3} \xi^a D_{x_1} \varphi dx = \int_{\mathbb{R}_+^3} \tilde{\mu}_1^a D_{x_1} \varphi dx$$

where  $\tilde{\mu}_1^a$  denotes the zero extension of  $\mu_1^a$  on  $\mathbb{R}^3/\Omega^a$ . This proves that the function  $\xi^a(t, \cdot) - \tilde{\mu}_1^a(t, \cdot)$  is constant with respect to  $x_1$ . Consequently, since



$\xi^a(t, \cdot) - \tilde{\mu}_1^a(t, \cdot) \in L^2(\mathbb{R}^3)$ , it results that

$$(2.4.26) \quad \xi^a(t, x) = \begin{cases} \mu_1^a(t, x_2, x_3) & \text{a.e. in } [0, T] \times \Omega^a, \\ 0 & \text{also.} \end{cases}$$

Similarly, now, in equation (2.2.7), we choose

$$\begin{cases} m = m_n, \\ u^b = c_n^b + \varphi & \text{where } \varphi \in C_0^\infty([0, T], C_0^\infty(\mathbb{R}_-^3)), \\ u^a = c_n^b, \end{cases}$$

where  $c_n^b = -(|B_n^a| + \frac{h_n^b}{h_n^a}|B_n^b|)^{-1} \frac{h_n^b}{h_n^a} \int_{B_n^b} \varphi dx$ , one obtains

$$(2.4.27) \quad \xi^b(t, x) = \begin{cases} \mu_3^b(t, x_1, x_2) & \text{a.e. in } [0, T] \times \Omega^b, \\ 0 & \text{also.} \end{cases}$$

The last step is devoted to prove the convergence of the magnetostatic energies. To this aim, remark that from (2.4.23), and for  $t$  a.e. in  $[0, T]$ ,  $\forall T \geq 0$  we have

$$(2.4.28) \quad \begin{cases} \frac{1}{h_n^a} D_{x_1} u_n^a \rightharpoonup \mu_1^a & \text{weak in } L^2(\mathbb{R}_+^3), \\ \frac{1}{h_n^b} D_{x_3} u_n^b \rightharpoonup \mu_3^b & \text{weak in } L^2(\mathbb{R}_-^3), \end{cases}$$

Then by passing to the limit in (2.2.14), and by taking into account (2.4.26), (2.4.27) and (2.4.28) we obtains

(2.4.29)

$$\begin{aligned} \lim_n E_n^{mag}(\underline{m}_n(t, \cdot)) &= \lim_n \left[ \frac{1}{2} \int_{\mathbb{R}_+^3} \left| \left( \frac{1}{h_n^a} D_{x_1} u_n^a, D_{x_2} u_n^a, D_{x_3} u_n^a \right) \right|^2 dx + \right. \\ &\quad \left. \frac{1}{2} \frac{h_n^b}{h_n^a} \int_{\mathbb{R}_-^3} \left| \left( D_{x_1} u_n^b, D_{x_2} u_n^b, \frac{1}{h_n^b} D_{x_3} u_n^b \right) \right|^2 dx \right] = \\ &= \lim_n \left[ \frac{1}{2} \int_{\Omega^a} \left( \left( \frac{1}{h_n^a} D_{x_1} u_n^a, D_{x_2} u_n^a, D_{x_3} u_n^a \right) m_n^a \right) dx + \right. \\ &\quad \left. \frac{1}{2} \frac{h_n^b}{h_n^a} \int_{\Omega^b} \left( \left( D_{x_1} u_n^b, D_{x_2} u_n^b, \frac{1}{h_n^b} D_{x_3} u_n^b \right) m_n^b \right) dx \right] = \\ &= \frac{1}{2} \left( \int_{\Omega^a} |\mu_1^a|^2 dx + q \int_{\Omega^b} |\mu_3^b|^2 dx \right). \end{aligned}$$

□

Let

$$\mathcal{M}_{reg} = \left\{ \begin{array}{l} \psi = (\psi^a, \psi^b) \in C^1 \left( ]-\frac{1}{2}, \frac{1}{2}[ \times [0, 1], \mathbb{R}^3 \right) \times C \left( \left[-\frac{1}{2}, \frac{1}{2}\right]^2, \mathbb{R}^3 \right) : \\ \psi^b|_{[-\frac{1}{2}, 0] \times [-\frac{1}{2}, \frac{1}{2}]} \in C^1 \left( ]-\frac{1}{2}, 0[ \times \left[-\frac{1}{2}, \frac{1}{2}\right], \mathbb{R}^3 \right), \\ \psi^b|_{[0, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]} \in C^1 \left( \left[0, \frac{1}{2}\right] \times \left[-\frac{1}{2}, \frac{1}{2}\right], \mathbb{R}^3 \right), \\ \psi^a(x_2, 0) = \psi^b(0, x_2), \text{ for } x_2 \text{ a.e. in } ]-\frac{1}{2}, \frac{1}{2}[. \end{array} \right\}$$

**Remark 2.4.5.**  $\mathcal{M}_{reg} \subset W^{1,\infty} \left( ]-\frac{1}{2}, \frac{1}{2}[ \times [0, 1], \mathbb{R}^3 \right) \times W^{1,\infty} \left( \left[-\frac{1}{2}, \frac{1}{2}\right]^2, \mathbb{R}^3 \right)$ . Moreover  $\mathcal{M}_{reg}$  is dense in  $\mathcal{M}$ , as proved in Proposition 5.5 in [33], but, in this case with more simple consideration.

At first we recall a result which is proved in Proposition 5.4 in [33].

**Lemma 2.4.6.** Let us consider a function  $\psi = (\psi^a, \psi^b) \in \mathcal{M}_{reg}$ . Then there exists a sequence  $\{g_n\}_{n \in \mathbb{N}} \subset H^1(\Omega^a, \mathbb{R}^3)$  such that

$$(2.4.30) \quad \left\{ \begin{array}{l} g_n \rightarrow \psi^a \text{ strongly in } L^2(\Omega^a, \mathbb{R}^3) \text{ as } n \rightarrow +\infty \\ \left( \frac{1}{h_n^a} D_{x_1} g_n | D_{x_2} g_n | D_{x_3} g_n \right) \rightarrow (0 | D_{x_2} \psi^a | D_{x_3} \psi^a) \text{ strongly in } L^2(\Omega^a, \mathbb{R}^9) \text{ as } n \rightarrow +\infty \\ g_n(x_1, x_2, 0) = \psi^b(h_n^a x_1, x_2) \text{ for } x_2 \in ]-\frac{1}{2}, \frac{1}{2}[^2, \quad \forall n \in \mathbb{N}. \end{array} \right.$$

### Step 3: Proof of Theorem 2.3.1

Now we are able to prove Theorem 2.3.1.

#### a) Identification of $\underline{\mu}$ as solution of Problem (2.3.5)

Let choose  $(g_n, \psi^b)$ , the coupled function given in (2.4.30), as test function in (2.2.11)

$$(2.4.31) \quad \begin{aligned} & \int_0^T \int_{\Omega^a} \left( \frac{\partial \underline{m}_n^a}{\partial t} + \underline{m}_n^a \wedge \frac{\partial \underline{m}_n^a}{\partial t} \right) \chi g_n dx dt + \frac{h_n^b}{h_n^a} \int_0^T \int_{\Omega^b} \left( \frac{\partial \underline{m}_n^b}{\partial t} + \underline{m}_n^b \wedge \frac{\partial \underline{m}_n^b}{\partial t} \right) \chi \psi^b dx dt = \\ & -2 \int_0^T \int_{\Omega^a} \left[ \underline{m}_n^a \wedge \frac{1}{h_n^a} D_{x_1} \underline{m}_n^a \cdot \frac{1}{h_n^a} (D_{x_1} g_n) \chi + \sum_{i=2}^3 \underline{m}_n^a \wedge D_{x_i} \underline{m}_n^a \cdot (D_{x_i} g_n) \chi \right] dx dt \\ & -2 \int_0^T \int_{\Omega^a} \underline{m}_n^a \wedge \left( \frac{1}{h_n^a} D_{x_1} u_n^a, D_{x_2} u_n^a, D_{x_3} u_n^a \right) \chi g_n dx dt \\ & -2 \frac{h_n^b}{h_n^a} \int_0^T \int_{\Omega^b} \left[ \underline{m}_n^b \wedge \frac{1}{h_n^b} D_{x_3} \underline{m}_n^b \cdot \frac{1}{h_n^b} (D_{x_3} \psi^b) \chi + \sum_{i=1}^2 \underline{m}_n^b \wedge D_{x_i} \underline{m}_n^b \cdot (D_{x_i} \psi^b) \chi \right] dx dt \\ & -2 \int_0^T \int_{\Omega^b} \underline{m}_n^b \wedge \left( D_{x_1} u_n^b, D_{x_2} u_n^b, \frac{1}{h_n^b} D_{x_3} u_n^b \right) \chi \psi^b dx dt. \\ & \forall \chi \in D(0, T) \end{aligned}$$

Since  $\underline{m}_n$  converges to  $\underline{\mu}$  almost everywhere in  $[0, T] \times \Omega$ , we can deduce that  $|\underline{\mu}| \equiv 1 \forall t \in [0, T]$ . Passing to the limit as  $n$  diverges, we obtain that there exists  $\underline{\mu} = (\underline{\mu}^a, \underline{\mu}^b) \in L^\infty(0, T; \mathcal{M})$  which satisfies:

$$(2.4.32) \quad \begin{aligned} & \int_0^T \int_{]-\frac{1}{2}, \frac{1}{2}[ \times ]0, 1[} \left( \frac{\partial \underline{\mu}^a}{\partial t} + \underline{\mu}^a \wedge \frac{\partial \underline{\mu}^a}{\partial t} \right) \chi \psi^a dx dt + q \int_0^T \int_{]-\frac{1}{2}, \frac{1}{2}[^2} \left( \frac{\partial \underline{\mu}^b}{\partial t} + \underline{\mu}^b \wedge \frac{\partial \underline{\mu}^b}{\partial t} \right) \chi \psi^b dx dt = \\ & -2 \int_0^T \int_{]-\frac{1}{2}, \frac{1}{2}[ \times ]0, 1[} \sum_{i=2}^3 \underline{\mu}^a \wedge \frac{\partial \underline{\mu}^a}{\partial x_i} \frac{\partial \psi^a}{\partial x_i} \chi - 2q \int_0^T \int_{]-\frac{1}{2}, \frac{1}{2}[^2} \underline{\mu}^b \wedge \sum_{i=1}^2 \frac{\partial \underline{\mu}^b}{\partial x_i} \frac{\partial \psi^b}{\partial x_i} \chi + \\ & -2 \int_0^T \int_{]-\frac{1}{2}, \frac{1}{2}[ \times ]0, 1[} \underline{\mu}^a \wedge (\underline{\mu}_1^a, 0, 0) \chi \psi^a - 2q \int_0^T \int_{]-\frac{1}{2}, \frac{1}{2}[^2} \underline{\mu}^b \wedge (0, 0, \underline{\mu}_3^b) \chi \psi^b \\ & \forall \chi \in D(0, T), \quad \forall (\psi^a, \psi^b) \in \mathcal{M}_{reg}. \end{aligned}$$

Let us observe that  $(\psi^a, \psi^b)$  can be any arbitrarily element of  $\mathcal{M}_{reg}$ . Being  $\mathcal{M}_{reg}$  dense in  $\mathcal{M}$ , we obtain that (2.4.32) holds true with  $(\psi^a, \psi^b) \in \mathcal{M}$ . Consequently, we derive the weak formulation Problem (2.3.5)

$$(2.4.33) \quad \left\{ \begin{array}{l} \frac{\partial \underline{\mu}^a}{\partial t} + \underline{\mu}^a \wedge \frac{\partial \underline{\mu}^a}{\partial t} = 2\underline{\mu}^a \wedge (\Delta \underline{\mu}^a - (\underline{\mu}^a, e_1) e_1) \quad \text{in } ]0, T[ \times ]-\frac{1}{2}, \frac{1}{2}[ \times ]0, 1[ \\ q \frac{\partial \underline{\mu}^b}{\partial t} + q \left( \underline{\mu}^b \wedge \frac{\partial \underline{\mu}^b}{\partial t} \right) = 2q \underline{\mu}^b \wedge (\Delta \underline{\mu}^b - (\underline{\mu}^b, e_3) e_3) \quad \text{in } ]0, T[ \times ]-\frac{1}{2}, \frac{1}{2}[^2, \\ \underline{\mu}^a(0, x) = \mu_0^a, \quad \text{in } ]-\frac{1}{2}, \frac{1}{2}[ \times ]0, 1[ \\ \underline{\mu}^b(0, x) = \mu_0^b \quad \text{in } ]-\frac{1}{2}, \frac{1}{2}[^2 \\ \underline{\mu}^a(\cdot, x_2, 0) = \underline{\mu}^b(\cdot, 0, x_2), \quad \text{for } x_2 \text{ a.e. in } ]-\frac{1}{2}, \frac{1}{2}[, \\ \underline{\mu}^a(\cdot, x) \in S^2, \underline{\mu}^b(\cdot, x) \in S^2. \end{array} \right.$$

**Remark 2.4.7.** *Let us observe that the integrand of the limit does not depend on  $x_1$  on  $\Omega^a$  and does not depend on  $x_3$  on  $\Omega^b$ .*

### b) Energy estimates

It remains to prove that  $\underline{\mu}$  satisfies the energy inequality.

By lower semicontinuity of the norm for the weak topology, we obtain

$$E_q(\underline{\mu}(t, \cdot)) + \int_0^t \left\| \frac{\partial \underline{\mu}^a}{\partial t} \right\|_{L^2(\Omega^a)}^2 dx dt + q \int_0^t \left\| \frac{\partial \underline{\mu}^b}{\partial t} \right\|_{L^2(\Omega^b)}^2 \leq \liminf E_n(\underline{m}_n(0, \cdot))$$

with

$$E_q(\underline{\mu}(t, \cdot)) = \int_{]-\frac{1}{2}, \frac{1}{2}[ \times ]0, 1[} |D\underline{\mu}^a|^2 + q \int_{]-\frac{1}{2}, \frac{1}{2}[^2} |D\underline{\mu}^b|^2 + \frac{1}{2} \int_{]-\frac{1}{2}, \frac{1}{2}[ \times ]0, 1[} |\underline{\mu}_1^a|^2 dx_2 dx_3 + \frac{1}{2} q \int_{]-\frac{1}{2}, \frac{1}{2}[^2} |\underline{\mu}_3^b|^2 dx_1 dx_2.$$

Let us compute  $E_n(\underline{m}_n(0, \cdot))$ . We have

$$(2.4.34) \quad \begin{aligned} E_n(\underline{m}_n(0, \cdot)) &= \int_{\Omega^a} \left| \left( \frac{1}{h_n^a} D_{x_1} \underline{m}_{0_n}^a, D_{x_2} \underline{m}_{0_n}^a, D_{x_3} \underline{m}_{0_n}^a \right) \right|^2 dx + \frac{1}{2} \int_{\mathbb{R}_+^3} \left| \left( \frac{1}{h_n^a} D_{x_1} u_{0_n}^a, D_{x_2} u_{0_n}^a, D_{x_3} u_{0_n}^a \right) \right|^2 dx + \\ &+ \frac{h_n^b}{h_n^a} \int_{\Omega^b} \left| \left( D_{x_1} \underline{m}_{0_n}^b, D_{x_2} \underline{m}_{0_n}^b, \frac{1}{h_n^b} D_{x_3} \underline{m}_{0_n}^b \right) \right|^2 dx + \frac{1}{2} \frac{h_n^b}{h_n^a} \int_{\mathbb{R}_-^3} \left| \left( D_{x_1} u_{0_n}^b, D_{x_2} u_{0_n}^b, \frac{1}{h_n^b} D_{x_3} u_{0_n}^b \right) \right|^2 dx. \end{aligned}$$

Under the hypotheses (2.3.3), by using the stationary result given in Theorem 4.1 in [33], we have that

$$(2.4.35) \quad \begin{aligned} \frac{1}{h_n^a} D_{x_1} \underline{u}_{0_n}^a &\rightarrow \underline{\mu}_{1_0}^a \quad D_{x_2} \underline{u}_{0_n}^a \rightarrow 0 \quad D_{x_3} \underline{u}_{0_n}^a \rightarrow 0 \text{ strongly in } L^2(\mathbb{R}_+^3) \\ D_{x_1} \underline{u}_{0_n}^a &\rightarrow 0 \quad D_{x_2} \underline{u}_{0_n}^a \rightarrow 0 \quad \frac{1}{h_n^b} D_{x_3} \underline{u}_{0_n}^b \rightarrow \underline{\mu}_{3_0}^b \text{ strongly in } L^2(\mathbb{R}_-^3). \end{aligned}$$

Passing to the limit in the (2.4.34) by (2.3.3) and (2.4.35) we get

$$(2.4.36) \quad \begin{aligned} \lim E_n(\underline{m}_n(0, \cdot)) &= \int_{\Omega^a} |(D_{x_2}^a \mu_0^a, D_{x_3} \mu_0^a)|^2 dx + q \int_{\Omega^b} |(D_{x_1}^b \mu_0^b, D_{x_2} \mu_0^b)|^2 dx \\ &+ \frac{1}{2} \int_{\mathbb{R}_+^3} |\underline{\mu}_{1_0}^a|^2 dx + \frac{1}{2} q \int_{\mathbb{R}_-^3} |\underline{\mu}_{3_0}^b|^2 = E_q(\underline{\mu}(0, \cdot)) \end{aligned}$$

which concludes the proof.



## Chapter 3

# Quasi-stationary ferromagnetic thin films in degenerated cases

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**Abstract.** In this paper we study the asymptotic behavior of a quasy-stationary ferromagnetic problem in a multi-domain consisting of two joined thin films. It is possible to distinguish different regimes depending on the limit  $q$  of the ratio between the small thickness of the two films. Here the case  $q = 0$  and  $q = +\infty$  are analyzed.

**Keywords:** Micromagnetics, thin films, Landau-Lifschitz equation.  
**AMS** 78A25, 74K35, 78M35.

### 3.1 Introduction

For every  $n \in N$ , let  $\Omega_n = \Omega_n^a \cup \Omega_n^b$ , where

$$\Omega_n^a = \left( \left[ -\frac{h_n^a}{2}, \frac{h_n^a}{2} \right] \times \left[ -\frac{1}{2}, \frac{1}{2} \right] \times [0, 1[ \right) \text{ and } \Omega_n^b = \left( \left[ -\frac{1}{2}, \frac{1}{2} \right]^2 \times ]-h_n^b, 0[ \right),$$

be a 3D ferromagnetic multidomain consisting of two orthogonal joined thin films with small thicknesses  $h_n^a$  and  $h_n^b \in ]0, 1[$ , respectively, such that

$$(3.1.1) \quad \lim_n h_n^a = 0 = \lim_n h_n^b, \quad \lim_n \frac{h_n^b}{h_n^a} = q \in \{0, +\infty\}.$$

Let  $B = ]-1, 1[^2 \times ]-2, 2[$  and set

$$(3.1.2) \quad \mathcal{U} = \left\{ U \in L^1_{loc}(\mathbb{R}^3) : U \in L^2(B), DU \in (L^2(\mathbb{R}^3))^3, \int_B U dx = 0 \right\}.$$

For  $M \in L^2(\Omega_n, \mathbb{R}^3)$ , let  $U_M$  be the unique solution of the following problem

$$(3.1.3) \quad \begin{cases} U_M \in \mathcal{U}, \\ \int_{\mathbb{R}^3} DU_M DU = \int_{\Omega_n} MDU dx, \quad \forall U \in \mathcal{U}. \end{cases}$$

For  $T > 0$  and  $M_{0,n} \in H^1(\Omega_n, S^2)$ , the following problem

$$(3.1.4) \quad \begin{cases} M_n \in L^\infty(0, T; H^1(\Omega_n, S^2)), \quad \frac{\partial M_n}{\partial t} \in L^2(0, T; L^2(\Omega_n, \mathbb{R}^3)), \\ \forall \phi \in D(]0, T[ \times \overline{\Omega_n}, \mathbb{R}^3) \quad \int_0^T \int_{\Omega_n} \left( \frac{\partial M_n}{\partial t} + M_n \wedge \frac{\partial M_n}{\partial t} \right) \cdot \phi dx dt = \\ -2 \int_0^T \int_{\Omega_n} \sum_{i=1}^3 M_n \wedge D_{x_i} M_n \cdot D_{x_i} \phi - 2 \int_0^T \int_{\Omega_n} M_n \wedge DU_{M_n} \cdot \phi, \\ M_n(0, x) = M_{0,n}. \end{cases}$$

admits solution (see [15]), where  $\wedge$  denotes the cross product in  $\mathbb{R}^3$ ,  $\cdot$  denotes the inner product in  $\mathbb{R}^3$  and  $S^2$  denotes the unit sphere of  $\mathbb{R}^3$ . Moreover (see [15] again),  $M_n$  satisfies

$$(3.1.5) \quad \mathcal{E}(t) + \int_0^t \left\| \frac{\partial M_n(s, \cdot)}{\partial t} \right\|_{(L^2(\Omega_n))^3}^2 ds \leq \mathcal{E}(0), \quad \forall t \in [0, T],$$

where

$$\mathcal{E}(t) = \int_{\Omega_n} |DM_n(t, x)|^2 dx + \int_{\mathbb{R}^3} |DU_{M_n}(t, x)|^2 dx.$$

By a density argument (for instance see [28]), problem (3.1.4) it is equivalent to that obtained choosing test functions  $\phi = \chi g$  with  $\chi \in D(]0, T[)$  and  $g \in H^1(\Omega_n, \mathbb{R}^3)$ .

After the reformulation of our problem in a fixed domain  $\Omega = \Omega^a \cup \Omega^b$ , with

$$\Omega^a = \left] -\frac{1}{2}, \frac{1}{2} \right[ \times ]0, 1[ \quad \text{and} \quad \Omega^b = \left] -\frac{1}{2}, \frac{1}{2} \right[ \times ]-1, 0[,$$

through rescalings of the kind proposed in [54] and appropriate convergence assumptions on the initial data, we study the asymptotic behavior, as  $n$  diverges, of problem (3.1.4) under assumption (3.1.1). If  $q = 0$  (i.e.  $h_n^b \ll h_n^a$ ), we prove

that the limit problem reduces to a 2D local problem in a vertical thin film losing the junction condition (see Theorem 3.3.1). Analogously, if  $q = +\infty$  (i.e.  $h_n^a \ll h_n^b$ ), the limit problem reduces to a 2D local problem in an horizontal thin film (see Theorem 3.3.2).

In the case  $q \in ]0, +\infty[$  (i.e.  $h_n^b \simeq h_n^a$ ), studied in [2], the problems on the two thin films are linked by a junction condition.

In Section 2, we rescale the problem. In Section 3, we state the main results. In Section 4, we sketch the proofs for the case  $q = 0$ . We omit the proofs for the case  $q = +\infty$ , since they are quite similar to previous ones.

This paper is inspired by [12] and [33], where the authors study a quasy-stationary ferromagnetic thin film and the junction of stationary ferromagnetic thin films, respectively.

About ferromagnetic theory see [11], [29], [50], [57] and the references therein. About ferroelectric thin structure see [1], [2], [3], [5], [6], [15], [16], [19], [39], [41], [48], [49], [52], [53], [55], [56] and the references therein. About ferroelectric thin multi-structure see [34] and the references therein. For related problems in thin structure see [9], [32], [31], [35] and [36], in composites with imperfect interface see [26, 27]. For recent papers on joined structure see also [4], [22], [23], [25], [30] and [38].

## 3.2 The rescaled problem

We reformulate problem (3.1.4) on a fixed domain through the following rescalings

$$(3.2.1) \quad \begin{cases} (x_1, x_2, x_3) \in \Omega^a \rightarrow (h_n^a x_1, x_2, x_3) \in \text{Int}(\Omega_n^a), \\ (x_1, x_2, x_3) \in \Omega^b \rightarrow (x_1, x_2, h_n^b x_3) \in \Omega_n^b, \end{cases}$$

where  $\text{Int}(\Omega_n^a)$  denotes the interior of  $\Omega_n^a$ .

For every  $n \in \mathbb{N}$ , space  $\mathcal{U}$  defined in (3.1.2) is rescaled in the following one

$$\begin{aligned} \mathcal{U}_n = \{v = (v^a, v^b) \in L^1_{loc}(\overline{\mathbb{R}_+^3}) \times L^1_{loc}(\overline{\mathbb{R}_-^3}) : (v^a|_{B_n^a}, v^b|_{B_n^b}) \in L^2(B_n^a) \times L^2(B_n^b), \\ (Dv^a, Dv^b) \in (L^2(\mathbb{R}_+^3))^3 \times (L^2(\mathbb{R}_-^3))^3, \int_{B_n^a} v^a dx + \frac{h_n^b}{h_n^a} \int_{B_n^b} v^b dx = 0, \\ v^a(x_1, x_2, 0) = v^b(h_n^a x_1, x_2, 0), \text{ for } (x_1, x_2) \text{ a.e. in } \mathbb{R}^2\}, \end{aligned}$$

where  $\mathbb{R}_+^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > 0\}$ ,  $\mathbb{R}_-^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 < 0\}$ ,  $B_n^a = ]-\frac{1}{h_n^a}, \frac{1}{h_n^a}[ \times ]-1, 1[ \times ]0, 2[$  and  $B_n^b = ]-1, 1[^2 \times ]-\frac{2}{h_n^b}, 0[$ .



For  $\underline{m} = (\underline{m}^a, \underline{m}^b) \in L^2(\Omega^a, \mathbb{R}^3) \times L^2(\Omega^b, \mathbb{R}^3)$ , the following equation

$$(3.2.2) \quad \begin{cases} \underline{u}_{\underline{m}} = (u_{\underline{m}}^a, u_{\underline{m}}^b) \in \mathcal{U}_n, \\ \int_{\mathbb{R}_+^3} \left( \frac{1}{h_n^a} D_{x_1} u_{\underline{m}}^a, D_{x_2} u_{\underline{m}}^a, D_{x_3} u_{\underline{m}}^a \right) \left( \frac{1}{h_n^a} D_{x_1} u^a, D_{x_2} u^a, D_{x_3} u^a \right) dx + \\ \frac{h_n^b}{h_n^a} \int_{\mathbb{R}_-^3} \left( D_{x_1} u_{\underline{m}}^b, D_{x_2} u_{\underline{m}}^b, \frac{1}{h_n^b} D_{x_3} u_{\underline{m}}^b \right) \left( D_{x_1} u^b, D_{x_2} u^b, \frac{1}{h_n^b} D_{x_3} u^b \right) dx = \\ \int_{\Omega_a} \left( \frac{1}{h_n^a} D_{x_1} u^a, D_{x_2} u^a, D_{x_3} u^a \right) \underline{m}^a dx + \\ \frac{h_n^b}{h_n^a} \int_{\Omega_b} \left( D_{x_1} u^b, D_{x_2} u^b, \frac{1}{h_n^b} D_{x_3} u^b \right) \underline{m}^b dx, \quad \forall u = (u^a, u^b) \in \mathcal{U}_n, \end{cases}$$

which rescales equation (3.1.3), admits a unique solution. Moreover, its solution  $\underline{u}_{\underline{m}} = (u_{\underline{m}}^a, u_{\underline{m}}^b) \in \mathcal{U}_n$  is characterized as the unique minimizer of the following problem

$$(3.2.3) \quad \min \left\{ \frac{1}{2} \int_{\mathbb{R}_+^3} \left| \left( \frac{1}{h_n^a} D_{x_1} u^a, D_{x_2} u^a, D_{x_3} u^a \right) - \underline{m}^a \right|^2 dx \right. \\ \left. + \frac{1}{2} \frac{h_n^b}{h_n^a} \int_{\mathbb{R}_-^3} \left| \left( \frac{1}{h_n^a} D_{x_1} u^b, D_{x_2} u^b, D_{x_3} u^b \right) - \underline{m}^b \right|^2 dx : u \in \mathcal{U}_n \right\},$$

understanding  $\underline{m}^a = 0$  in  $\mathbb{R}_+^3 \setminus \Omega^a$  and  $\underline{m}^b = 0$  in  $\mathbb{R}_-^3 \setminus \Omega^b$ .

For every  $n \in \mathbb{N}$ , let us consider the following spaces

$$(3.2.4) \quad \mathcal{W}_n = \{ \underline{m} = (\underline{m}^a, \underline{m}^b) \in H^1(\Omega^a) \times H^1(\Omega^b) : \\ \underline{m}^a(x_1, x_2, 0) = \underline{m}^b(h_n^a x_1, x_2, 0), \text{ for } (x_1, x_2) \text{ a.e. in } ]-\frac{1}{2}, \frac{1}{2}[ \}$$

and

$$(3.2.5) \quad \mathcal{M}_n = \mathcal{W}_n \cap (H^1(\Omega^a, S^2) \times H^1(\Omega^b, S^2)).$$

Let  $\underline{m}_{0_n} = (\underline{m}_{0_n}^a, \underline{m}_{0_n}^b) \in \mathcal{M}_n$  and  $\underline{m}_n = (\underline{m}_n^a, \underline{m}_n^b) \in L^\infty(0, T; \mathcal{M}_n)$  be the rescaling through (3.2.1) of the initial data  $M_{0_n}$  and of a solution  $M_n$  of problem

(3.1.4). Then it results

(3.2.6)

$$\left\{ \begin{array}{l} \underline{m}_n = (\underline{m}_n^a, \underline{m}_n^b) \in L^\infty(0, T; \mathcal{M}_n), \\ \frac{\partial \underline{m}_n}{\partial t} = \left( \frac{\partial \underline{m}_n^a}{\partial t}, \frac{\partial \underline{m}_n^b}{\partial t} \right) \in L^2(0, T; L^2(\Omega^a, \mathbb{R}^3)) \times L^2(0, T; L^2(\Omega^b, \mathbb{R}^3)), \\ \forall \chi \in D(]0, T[) \text{ and } \forall g = (g^a, g^b) \in \mathcal{W}_n \\ \int_0^T \int_{\Omega^a} \left( \frac{\partial \underline{m}_n^a}{\partial t} + \underline{m}_n^a \wedge \frac{\partial \underline{m}_n^a}{\partial t} \right) \chi g^a dx dt + \frac{h_n^b}{h_n^a} \int_0^T \int_{\Omega^b} \left( \frac{\partial \underline{m}_n^b}{\partial t} + \underline{m}_n^b \wedge \frac{\partial \underline{m}_n^b}{\partial t} \right) \chi g^b dx dt = \\ -2 \int_0^T \int_{\Omega^a} \left[ \underline{m}_n^a \wedge \frac{1}{h_n^a} D_{x_1} \underline{m}_n^a \cdot \frac{1}{h_n^a} (D_{x_1} g^a) \chi + \sum_{i=2}^3 \underline{m}_n^a \wedge D_{x_i} \underline{m}_n^a \cdot (D_{x_i} g^a) \chi \right] dx dt \\ -2 \int_0^T \int_{\Omega^a} \underline{m}_n^a \wedge \left( \frac{1}{h_n^a} D_{x_1} u_{\underline{m}_n^a}^a, D_{x_2} u_{\underline{m}_n^a}^a, D_{x_3} u_{\underline{m}_n^a}^a \right) \chi g^a dx dt \\ -2 \frac{h_n^b}{h_n^a} \int_0^T \int_{\Omega^b} \left[ \underline{m}_n^b \wedge \frac{1}{h_n^b} D_{x_3} \underline{m}_n^b \cdot \frac{1}{h_n^b} (D_{x_3} g^b) \chi + \sum_{i=1}^2 \underline{m}_n^b \wedge D_{x_i} \underline{m}_n^b \cdot (D_{x_i} g^b) \chi \right] dx dt \\ -2 \frac{h_n^b}{h_n^a} \int_0^T \int_{\Omega^b} \underline{m}_n^b \wedge \left( D_{x_1} u_{\underline{m}_n^b}^b, D_{x_2} u_{\underline{m}_n^b}^b, \frac{1}{h_n^b} D_{x_3} u_{\underline{m}_n^b}^b \right) \chi g^b dx dt, \\ \underline{m}_n(0, x) = \underline{m}_{0_n}, \end{array} \right.$$

and

(3.2.7)

$$E_n(t) + \int_0^t \left\| \frac{\partial \underline{m}_n^a}{\partial t} \right\|_{(L^2(\Omega^a))^3}^2 ds + \frac{h_n^b}{h_n^a} \int_0^t \left\| \frac{\partial \underline{m}_n^b}{\partial t} \right\|_{(L^2(\Omega^b))^3}^2 ds \leq E_n(0), \forall t \in [0, T],$$

where

$$(3.2.8) \quad \begin{aligned} E_n(t) &= \int_{\Omega^a} \left| \left( \frac{1}{h_n^a} D_{x_1} \underline{m}_n^a, D_{x_2} \underline{m}_n^a, D_{x_3} \underline{m}_n^a \right) \right|^2 dx \\ &+ \frac{1}{2} \int_{\mathbb{R}_+^3} \left| \left( \frac{1}{h_n^a} D_{x_1} u_{\underline{m}_n^a}^a, D_{x_2} u_{\underline{m}_n^a}^a, D_{x_3} u_{\underline{m}_n^a}^a \right) \right|^2 dx \\ &+ \frac{h_n^b}{h_n^a} \int_{\Omega^b} \left| \left( D_{x_1} \underline{m}_n^b, D_{x_2} \underline{m}_n^b, \frac{1}{h_n^b} D_{x_3} \underline{m}_n^b \right) \right|^2 dx \\ &+ \frac{1}{2} \frac{h_n^b}{h_n^a} \int_{\mathbb{R}_-^3} \left| \left( D_{x_1} u_{\underline{m}_n^b}^b, D_{x_2} u_{\underline{m}_n^b}^b, \frac{1}{h_n^b} D_{x_3} u_{\underline{m}_n^b}^b \right) \right|^2 dx. \end{aligned}$$

In the sequel we set

$$(3.2.9) \quad \begin{aligned} E_n^{mag}(t) &= \frac{1}{2} \int_{\mathbb{R}_+^3} \left| \left( \frac{1}{h_n^a} D_{x_1} u_{\underline{m}_n}^a, D_{x_2} u_{\underline{m}_n}^a, D_{x_3} u_{\underline{m}_n}^a \right) \right|^2 dx \\ &+ \frac{1}{2} \frac{h_n^b}{h_n^a} \int_{\mathbb{R}^3} \left| \left( D_{x_1} u_{\underline{m}_n}^b, D_{x_2} u_{\underline{m}_n}^b, \frac{1}{h_n^b} D_{x_3} u_{\underline{m}_n}^b \right) \right|^2 dx, \quad \forall t \in [0, T]. \end{aligned}$$

### 3.3 The main results

#### 3.3.1 Case $q = 0$

Let us introduce the following space

$$(3.3.1) \quad \mathcal{M}_0 = \{ \psi^a \in H^1(\Omega^a, S^2) : \psi^a \text{ is indep. of } x_1 \}.$$

**Theorem 3.3.1.** *Let  $\mathcal{M}_0$  be defined by (3.3.1). Assume (3.1.1) with  $q = 0$ . For every  $n \in \mathbb{N}$ , let  $\underline{m}_n = (\underline{m}_n^a, \underline{m}_n^b)$  be a solution of Problem (3.2.6) with the following assumptions on the initial data*

$$(3.3.2) \quad \left\{ \begin{array}{l} \underline{m}_{0_n} = (\underline{m}_{0_n}^a, \underline{m}_{0_n}^b) \in \mathcal{M}_n, \quad \underline{\mu}_0^a \in \mathcal{M}_0, \\ \underline{m}_{0_n}^a \rightarrow \underline{\mu}_0^a \text{ strongly in } H^1(\Omega^a, S^2), \\ \left( \frac{h_n^b}{h_n^a} \right)^{\frac{1}{2}} \underline{m}_{0_n}^b \rightarrow 0 \text{ strongly in } H^1(\Omega^b, \mathbb{R}^3), \\ \frac{1}{h_n^a} D_{x_1} \underline{m}_{0_n}^a \rightarrow 0 \text{ strongly in } L^2(\Omega^a, \mathbb{R}^3), \\ \frac{1}{h_n^b} \left( \frac{h_n^b}{h_n^a} \right)^{\frac{1}{2}} D_{x_3} \underline{m}_{0_n}^b \rightarrow 0 \text{ strongly in } L^2(\Omega^b, \mathbb{R}^3). \end{array} \right.$$

Then, there exist an increasing sequence of positive integer numbers, still denoted by  $\{n\}$ , and  $\underline{\mu}^a = (\underline{\mu}_1^a, \underline{\mu}_2^a, \underline{\mu}_3^a) \in L^\infty(0, T; \mathcal{M}_0)$ , depending on the selected subsequence, such that

$$\left\{ \begin{array}{l} \underline{m}_n^a \rightharpoonup \underline{\mu}^a \text{ weakly } * \text{ in } L^\infty(0, T; H^1(\Omega^a, S^2)), \\ \left( \frac{h_n^b}{h_n^a} \right)^{\frac{1}{2}} \underline{m}_n^b \rightharpoonup 0 \text{ weakly } * \text{ in } L^\infty(0, T; H^1(\Omega^b, \mathbb{R}^3)), \\ \underline{m}_n^a \rightarrow \underline{\mu}^a \text{ strongly in } L^2(0, T; L^2(\Omega^a, S^2)) \text{ and a. e. in } ]0, T[ \times \Omega^a, \\ \left( \frac{h_n^b}{h_n^a} \right)^{\frac{1}{2}} \underline{m}_n^b \rightarrow 0 \text{ strongly in } L^2(0, T; L^2(\Omega^b, \mathbb{R}^3)) \text{ and a. e. in } ]0, T[ \times \Omega^b, \end{array} \right.$$

where  $\underline{\mu}^a$  is the solution of the following problem

$$(3.3.3) \quad \left\{ \begin{array}{l} \underline{\mu}^a \in L^\infty(0, T; \mathcal{M}_0), \quad \frac{\partial \underline{\mu}^a}{\partial t} \in L^2(0, T; L^2(\text{]}-\frac{1}{2}, \frac{1}{2}[ \times ]0, 1[, \mathbb{R}^3)), \\ \forall \chi \in D(0, T), \quad \forall g^a \in H^1(\Omega^a, \mathbb{R}^3), \\ \int_0^T \int_{\text{]}-\frac{1}{2}, \frac{1}{2}[ \times ]0, 1[} \left( \frac{\partial \underline{\mu}^a}{\partial t} + \underline{\mu}^a \wedge \frac{\partial \underline{\mu}^a}{\partial t} \right) \chi g^a dx dt = \\ -2 \int_0^T \int_{\text{]}-\frac{1}{2}, \frac{1}{2}[ \times ]0, 1[} \sum_{i=2}^3 \underline{\mu}^a \wedge D_{x_i} \underline{\mu}^a D_{x_i} g^a \chi dx dt \\ -2 \int_0^T \int_{\text{]}-\frac{1}{2}, \frac{1}{2}[ \times ]0, 1[} \underline{\mu}^a \wedge (\underline{\mu}_1^a, 0, 0) \chi g^a dx dt, \\ \underline{\mu}^a(0, \cdot) = \underline{\mu}_0^a \in \mathcal{M}_0. \end{array} \right.$$

Moreover, the following energy estimates hold

(3.3.4)

$$E_0(t) + \int_0^t \left\| \frac{\partial \underline{\mu}^a}{\partial t} \right\|_{(L^2(\text{]}-\frac{1}{2}, \frac{1}{2}[ \times ]0, 1[))}^2 dx_2 dx_3 dt \leq E_0(0), \quad \forall t \in [0, T],$$

where

$$(3.3.5) \quad E_0(t) = \int_{\text{]}-\frac{1}{2}, \frac{1}{2}[ \times ]0, 1[} |D \underline{\mu}^a|^2 dx_2 dx_3 + \frac{1}{2} \int_{\text{]}-\frac{1}{2}, \frac{1}{2}[ \times ]0, 1[} |\underline{\mu}_1^a|^2 dx_2 dx_3.$$

### 3.3.2 Case $q = +\infty$

Let us introduce the following space

$$(3.3.6) \quad \mathcal{M}_\infty = \{ \psi^b \in H^1(\Omega^b, S^2) : \psi^b \text{ is independent of } x_3 \}.$$

**Theorem 3.3.2.** *Let  $\mathcal{M}_\infty$  be defined by (3.3.6). Assume (3.1.1) with  $q = +\infty$ . For every  $n \in \mathbb{N}$ , let  $\underline{m}_n = (\underline{m}_n^a, \underline{m}_n^b)$  be a solution of Problem (3.2.6) with the following assumptions on the initial data*

$$\left\{ \begin{array}{l} \underline{m}_{0_n} = (\underline{m}_{0_n}^a, \underline{m}_{0_n}^b) \in \mathcal{M}_n, \quad \underline{\mu}_0^b \in \mathcal{M}_\infty, \\ \left( \frac{h_n^a}{h_n^b} \right)^{\frac{1}{2}} \underline{m}_{0_n}^a \rightarrow 0 \text{ strongly in } H^1(\Omega^a, \mathbb{R}^3), \\ \underline{m}_{0_n}^b \rightarrow \underline{\mu}_0^b \text{ strongly in } H^1(\Omega^b, S^2), \\ \frac{1}{h_n^a} \left( \frac{h_n^a}{h_n^b} \right)^{\frac{1}{2}} D_{x_1} \underline{m}_{0_n}^a \rightarrow 0 \text{ strongly in } L^2(\Omega^a, \mathbb{R}^3), \\ \frac{1}{h_n^b} D_{x_3} \underline{m}_{0_n}^b \rightarrow 0 \text{ strongly in } L^2(\Omega^b, \mathbb{R}^3). \end{array} \right.$$

Then, there exist an increasing sequence of positive integer numbers, still denoted by  $\{n\}$ , and  $\underline{\mu}^b = (\underline{\mu}_1^b, \underline{\mu}_2^b, \underline{\mu}_3^b) \in L^\infty(0, T; \mathcal{M}_\infty)$ , depending on the selected subsequence, such that

$$\left\{ \begin{array}{l} \left( \frac{h_n^a}{h_n^b} \right)^{\frac{1}{2}} \underline{m}_n^a \rightharpoonup 0 \text{ weakly } * \text{ in } L^\infty(0, T; H^1(\Omega^a, \mathbb{R}^3)), \\ \underline{m}_n^b \rightharpoonup \underline{\mu}^b \text{ weakly } * \text{ in } L^\infty(0, T; H^1(\Omega^b, S^2)), \\ \left( \frac{h_n^a}{h_n^b} \right)^{\frac{1}{2}} \underline{m}_n^a \rightarrow 0 \text{ strongly in } L^2(0, T; L^2(\Omega^a, \mathbb{R}^3)) \text{ and a. e. in } ]0, T[ \times \Omega^a, \\ \underline{m}_n^b \rightarrow \underline{\mu}^b \text{ strongly in } L^2(0, T; L^2(\Omega^b, S^2)) \text{ and a. e. in } ]0, T[ \times \Omega^b, \end{array} \right.$$

where  $\underline{\mu}^b$  is the solution of the following problem

$$\left\{ \begin{array}{l} \underline{\mu}^b \in L^\infty(0, T; \mathcal{M}_\infty), \quad \frac{\partial \underline{\mu}^b}{\partial t} \in L^2\left(0, T; L^2\left(]-\frac{1}{2}, \frac{1}{2}[^2, \mathbb{R}^3\right)\right), \quad \underline{\mu}^b(0, \cdot) = \underline{\mu}_0^b \in \mathcal{M}_\infty, \\ \forall \chi \in D(0, T), \quad \forall g^b \in H^1(\Omega^b, \mathbb{R}^3), \quad \int_0^T \int_{]-\frac{1}{2}, \frac{1}{2}[^2} \left( \frac{\partial \underline{\mu}^b}{\partial t} + \underline{\mu}^b \wedge \frac{\partial \underline{\mu}^b}{\partial t} \right) \chi g^b dx_1 dx_2 dt = \\ -2 \int_0^T \int_{]-\frac{1}{2}, \frac{1}{2}[^2} \sum_{i=1}^2 \underline{\mu}^b \wedge D_{x_i} \underline{\mu}^b D_{x_i} g^b \chi dx_1 dx_2 dt - 2 \int_0^T \int_{]-\frac{1}{2}, \frac{1}{2}[^2} \underline{\mu}^b \wedge (0, 0, \underline{\mu}_3^b) g^b \chi dx_1 dx_2 dt. \end{array} \right.$$

Moreover the following energy estimates hold

$$E_\infty(t) + \int_0^t \left\| \frac{\partial \underline{\mu}^b}{\partial t} \right\|_{(L^2(]-\frac{1}{2}, \frac{1}{2}[^2))^3}^2 dx_1 dx_2 dt \leq E_\infty(0), \quad \forall t \in [0, T],$$

where

$$E_\infty(t) = \int_{]-\frac{1}{2}, \frac{1}{2}[^2} |D \underline{\mu}^b|^2 dx_1 dx_2 + \frac{1}{2} \int_{]-\frac{1}{2}, \frac{1}{2}[^2} |\underline{\mu}_3^b|^2 dx_1 dx_2.$$

### 3.4 The proof in the case $q = 0$

#### 3.4.1 A priori estimates

Let us recall the following result which can be proved as in [33].

**Proposition 3.4.1.** *Assume (3.1.1) with  $q = 0$  and (3.3.2). For every  $n \in \mathbb{N}$ , let  $\underline{m}_n = (\underline{m}_n^a, \underline{m}_n^b)$  be a solution of Problem (3.2.6). Then, it results that*

$$(3.4.1) \quad \exists c \in ]0, +\infty[ : E_n(0) \leq c, \quad \forall n \in \mathbb{N}.$$

**Remark 3.4.2.** Similarly, in the case  $q = +\infty$ , one can prove that

$$(3.4.2) \quad \exists c \in ]0, +\infty[ : \frac{h_n^a}{h_n^b} E_n(0) \leq c, \quad \forall n \in \mathbb{N}.$$

The following result is an immediate consequence of Proposition 3.4.1.

**Corollary 3.4.3.** Let  $\mathcal{M}_0$  be defined by (3.3.1). Assume (3.1.1) with  $q = 0$  and (3.3.2). For every  $n \in \mathbb{N}$ , let  $\underline{m}_n = (\underline{m}_n^a, \underline{m}_n^b)$  be a solution of Problem (3.2.6). Then, there exist an increasing sequence of positive integer numbers, still denoted by  $\{n\}$ , and  $\underline{\mu}^a = (\underline{\mu}_1^a, \underline{\mu}_2^a, \underline{\mu}_3^a) \in L^\infty(0, T; \mathcal{M}_0)$ , depending on the selected subsequence, such that

$$(3.4.3) \quad \left\{ \begin{array}{l} \underline{m}_n^a \rightharpoonup \underline{\mu}^a \text{ weakly } * \text{ in } L^\infty(0, T; H^1(\Omega^a, S^2)), \\ \hspace{10em} \text{strongly in } L^2(]0, T[ \times \Omega^a, S^2) \text{ and a. e. in } ]0, T[ \times \Omega^a, \\ \left( \frac{h_n^b}{h_n^a} \right)^{\frac{1}{2}} \underline{m}_n^b \rightharpoonup 0 \text{ weakly } * \text{ in } L^\infty(0, T; H^1(\Omega^b, \mathbb{R}^3)), \\ \hspace{10em} \text{strongly in } L^2(]0, T[ \times \Omega^b, \mathbb{R}^3) \text{ and a. e. in } ]0, T[ \times \Omega^b, \\ \frac{\partial \underline{m}_n^a}{\partial t} \rightharpoonup \frac{\partial \underline{\mu}^a}{\partial t} \text{ weakly in } L^2(]0, T[ \times \Omega^a, \mathbb{R}^3), \\ \left( \frac{h_n^b}{h_n^a} \right)^{\frac{1}{2}} \frac{\partial \underline{m}_n^b}{\partial t} \rightharpoonup 0 \text{ weakly in } L^2(]0, T[ \times \Omega^b, \mathbb{R}^3). \end{array} \right.$$

### 3.4.2 A convergence result for the magnetostatic energy

Let us identify the limit magnetostatic energy.

**Proposition 3.4.4.** Assume (3.1.1) with  $q = 0$ . For every  $n \in \mathbb{N}$ , let  $\underline{m}_n = (\underline{m}_n^a, \underline{m}_n^b) \in L^\infty(0, T; L^2(\Omega^a, \mathbb{R}^3)) \times L^\infty(0, T; L^2(\Omega^b, \mathbb{R}^3))$  and  $\mu^a = (\mu_1^a, \mu_2^a, \mu_3^a) \in L^\infty(0, T; L^2(\Omega^a, \mathbb{R}^3))$  be such that

$$(3.4.4) \quad \left( \underline{m}_n^a, \left( \frac{h_n^b}{h_n^a} \right)^{\frac{1}{2}} \underline{m}_n^b \right) \rightarrow (\mu^a, 0) \\ \text{strongly in } L^2(]0, T[ \times \Omega^a, \mathbb{R}^3) \times L^2(]0, T[ \times \Omega^b, \mathbb{R}^3),$$

as  $n$  diverges. Let  $u_n = (u_n^a, u_n^b)$  be the unique solution of (3.2.2) corresponding to  $\underline{m}_n$ . Then it result that

$$(3.4.5) \quad \left\{ \begin{array}{l} \frac{1}{h_n^a} D_{x_1} u_n^a \rightharpoonup \mu_1^a, \quad D_{x_2} u_n^a \rightharpoonup 0, \quad D_{x_3} u_n^a \rightharpoonup 0 \\ \hspace{10em} \text{weakly } * \text{ in } L^\infty(0, T; L^2(\mathbb{R}_+^3)), \\ \left( \frac{h_n^b}{h_n^a} \right)^{\frac{1}{2}} D_{x_1} u_n^b \rightharpoonup 0, \quad \left( \frac{h_n^b}{h_n^a} \right)^{\frac{1}{2}} D_{x_2} u_n^b \rightharpoonup 0, \quad \frac{1}{h_n^b} \left( \frac{h_n^b}{h_n^a} \right)^{\frac{1}{2}} D_{x_3} u_n^b \rightharpoonup 0 \\ \hspace{10em} \text{weakly } * \text{ in } L^\infty(0, T; L^2(\mathbb{R}_-^3)), \end{array} \right.$$

as  $n$  diverges, and for  $t \in [0, T]$

$$(3.4.6) \quad \lim_n E_n^{mag}(t) = \frac{1}{2} \int_{\Omega^a} |\mu_1^a|^2 dx.$$

*Proof.* Let us choose  $u = (0, 0)$  as test function in problem 3.2.3. Then, arguing as in Proposition 3.2 in [34], there exists a constant  $c$  (independent of  $t$  and  $n$ ) such that

$$(3.4.7) \quad \begin{cases} \|\frac{1}{h_n^a} D_{x_1} u_n^a\|_{L^2(\mathbb{R}_+^3)} \leq c, \|D_{x_2} u_n^a\|_{L^2(\mathbb{R}_+^3)} \leq c, \|D_{x_3} u_n^a\|_{L^2(\mathbb{R}_+^3)} \leq c, \\ \|D_{x_1} u_n^b\|_{L^2(\mathbb{R}_-^3)} \leq c, \|D_{x_2} u_n^b\|_{L^2(\mathbb{R}_-^3)} \leq c, \|\frac{1}{h_n^b} D_{x_3} u_n^b\|_{L^2(\mathbb{R}_-^3)} \leq c, \end{cases}$$

and

$$(3.4.8) \quad \begin{cases} Du_n^a \rightarrow 0 \text{ strongly in } (L^2(\mathbb{R}_+^3))^3, \\ \frac{1}{h_n^a} D_{x_1} u_n^a \rightarrow \mu_1^a \text{ strongly in } L^2(\mathbb{R}_+^3), \\ \left(\frac{h_n^b}{h_n^a}\right)^{\frac{1}{2}} Du_n^b \rightarrow 0 \text{ strongly in } (L^2(\mathbb{R}_-^3))^3, \\ \frac{1}{h_n^b} \left(\frac{h_n^b}{h_n^a}\right)^{\frac{1}{2}} D_{x_3} u_n^b \rightarrow 0 \text{ strongly in } L^2(\mathbb{R}_-^3), \end{cases}$$

as  $n$  diverges, for  $t \in ]0, T[$ , where it is understood that  $\mu_1^a = 0$  in  $\mathbb{R}_+^3 \setminus \Omega^a$ . Limits in (3.4.5) are obtained combining (3.4.7) with (3.4.8).

Finally, passing to the limit in (3.2.9) and taking into account (2.4.22), we obtain

$$(3.4.9) \quad \begin{aligned} \lim_n E_n^{mag}(t) &= \lim_n \left[ \frac{1}{2} \int_{\mathbb{R}_+^3} \left| \left( \frac{1}{h_n^a} D_{x_1} u_n^a, D_{x_2} u_n^a, D_{x_3} u_n^a \right) \right|^2 dx \right. \\ &\quad \left. + \frac{1}{2} \frac{h_n^b}{h_n^a} \int_{\mathbb{R}_-^3} \left| \left( D_{x_1} u_n^b, D_{x_2} u_n^b, \frac{1}{h_n^b} D_{x_3} u_n^b \right) \right|^2 dx \right] = \frac{1}{2} \int_{\Omega^a} |\mu_1^a|^2 dx. \end{aligned}$$

□

### 3.4.3 Proof of theorem 3.3.1

Let us choose  $g = (\psi^a, \psi^a(x_2, 0))$  as test function in (3.2.6), with  $\psi^a \in C^\infty([-\frac{1}{2}, \frac{1}{2}] \times [0, 1], \mathbb{R}^3)$ . Then it results

$$\begin{aligned}
 & \int_0^T \int_{\Omega^a} \left( \frac{\partial \underline{m}_n^a}{\partial t} + \underline{m}_n^a \wedge \frac{\partial \underline{m}_n^a}{\partial t} \right) \chi \psi^a dx dt \\
 & + \int_0^T \int_{\Omega^b} \left( \frac{h_n^b}{h_n^a} \frac{\partial \underline{m}_n^b}{\partial t} + \left( \frac{h_n^b}{h_n^a} \right)^{\frac{1}{2}} \underline{m}_n^b \wedge \left( \frac{h_n^b}{h_n^a} \right)^{\frac{1}{2}} \frac{\partial \underline{m}_n^b}{\partial t} \right) \chi \psi^a(x_2, 0) dx dt = \\
 & -2 \int_0^T \int_{\Omega^a} \sum_{i=2}^3 \underline{m}_n^a \wedge D_{x_i} \underline{m}_n^a \cdot (D_{x_i} \psi^a) \chi dx dt \\
 & -2 \int_0^T \int_{\Omega^a} \underline{m}_n^a \wedge \left( \frac{1}{h_n^a} D_{x_1} u_n^a, D_{x_2} u_n^a, D_{x_3} u_n^a \right) \chi \psi^a dx dt \\
 & -2 \int_0^T \int_{\Omega^b} \left( \frac{h_n^b}{h_n^a} \right)^{\frac{1}{2}} \underline{m}_n^b \wedge \left( \frac{h_n^b}{h_n^a} \right)^{\frac{1}{2}} D_{x_2} \underline{m}_n^b \cdot (D_{x_2} \psi^a(x_2, 0)) \chi dx dt \\
 & -2 \int_0^T \int_{\Omega^b} \left( \frac{h_n^b}{h_n^a} \right)^{\frac{1}{2}} \underline{m}_n^b \wedge \left( \left( \frac{h_n^b}{h_n^a} \right)^{\frac{1}{2}} D_{x_1} u_n^b, \left( \frac{h_n^b}{h_n^a} \right)^{\frac{1}{2}} D_{x_2} u_n^b, \frac{1}{h_n^b} \left( \frac{h_n^b}{h_n^a} \right)^{\frac{1}{2}} D_{x_3} u_n^b \right) \chi \psi^a(x_2, 0) dx dt, \\
 & \forall \chi \in D(0, T), \forall \psi^a \in C^\infty([-\frac{1}{2}, \frac{1}{2}] \times [0, 1], \mathbb{R}^3).
 \end{aligned}$$

Then passing to the limit in this equation, as  $n$  diverges, and using (3.4.3) and (3.4.5), we obtain that  $\underline{\mu}^a \in L^\infty(0, T; \mathcal{M}_0)$  satisfies

$$\left\{ \begin{array}{l} \int_0^T \int_{[-\frac{1}{2}, \frac{1}{2}] \times [0, 1]} \left( \frac{\partial \underline{\mu}^a}{\partial t} + \underline{\mu}^a \wedge \frac{\partial \underline{\mu}^a}{\partial t} \right) \chi \psi^a dx dt = \\ -2 \int_0^T \int_{[-\frac{1}{2}, \frac{1}{2}] \times [0, 1]} \sum_{i=2}^3 \underline{\mu}^a \wedge D_{x_i} \underline{\mu}^a D_{x_i} \psi^a \chi - 2 \int_0^T \int_{[-\frac{1}{2}, \frac{1}{2}] \times [0, 1]} \underline{\mu}^a \wedge (\underline{\mu}_1^a, 0, 0) \chi \psi^a, \\ \forall \chi \in D(0, T), \forall \psi^a \in C^\infty([-\frac{1}{2}, \frac{1}{2}] \times [0, 1], \mathbb{R}^3). \end{array} \right.$$

Since  $C^\infty([-\frac{1}{2}, \frac{1}{2}] \times [0, 1], \mathbb{R}^3)$  is dense in  $H^1(\Omega^a, \mathbb{R}^3)$ , it follows that  $\underline{\mu}^a$  solves system in (3.3.3).

The initial condition can be obtained in a classical way. Estimate (3.3.4) can be obtained as in [12], taking into account that assumption (3.3.2) and the



stationary result given in Theorem 4.1 in [33] provide

$$\begin{aligned} \frac{1}{h_n^a} D_{x_1} \underline{u}_{0_n}^a &\rightarrow \underline{\mu}_{0_1}^a, \quad D_{x_2} \underline{u}_{0_n}^a \rightarrow 0, \quad D_{x_3} \underline{u}_{0_n}^a \rightarrow 0, \text{ strongly in } L^2(\mathbb{R}_+^3), \\ \left(\frac{h_n^b}{h_n^a}\right)^{\frac{1}{2}} D_{x_1} \underline{u}_{0_n}^b &\rightarrow 0, \quad \left(\frac{h_n^b}{h_n^a}\right)^{\frac{1}{2}} D_{x_2} \underline{u}_{0_n}^b \rightarrow 0, \quad \frac{1}{h_n^b} \left(\frac{h_n^b}{h_n^a}\right)^{\frac{1}{2}} D_{x_3} \underline{u}_{0_n}^b \rightarrow 0 \text{ strongly in } L^2(\mathbb{R}_-^3), \end{aligned}$$

where  $\underline{\mu}_{0_1}^a$  is the first component of  $\underline{\mu}_0^a$ .

## Chapter 4

# Asymptotic analysis for two joined thin slanting Ferromagnetic films

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**Abstract.** Starting from a 3D non-convex and nonlocal micromagnetic energy for ferromagnetic materials, we determine, via an asymptotic analysis, the free energy of two joined ferromagnetic thin films. Precisely, how the limit problem depend on limit of the ratio between the volumes of the two joined films.

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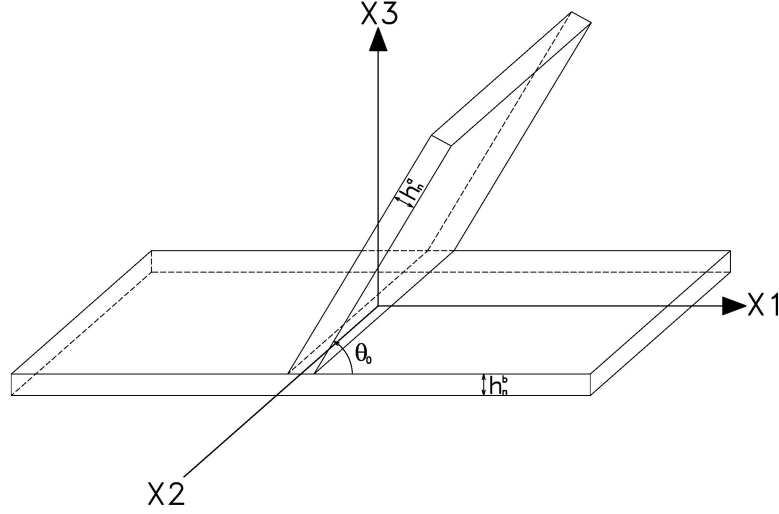
**Keywords:** micromagnetics, variational problem, thin film, junctions.

### 4.1 Introduction

In this paper, starting from the classical 3D micromagnetic energy (cf. L.D. Landau and E.M. Lifschitz [50]), we determine via an asymptotic analysis, the free energy of a multi-structure ferromagnetic, composed of two joined films, forming an angle  $\theta_0 \in ]0, \pi[$  at the junction point  $(h_n^a x_1, x_2, 0)$  (see Figure 4.1). Precisely, let  $\Omega_n^{\theta_0} = \Omega_n^{a, \theta_0} \cup \Omega_n^b$ , with  $n \in \mathbb{N}$ ,  $\Omega_n^b = ]-\frac{1}{2}, \frac{1}{2}[ \times ]-h_n^b, 0[$  and  $\Omega_n^{a, \theta_0}$  is a bounded domain in  $\mathbb{R}^3$ , so that form an angle  $\theta_0$  with the domain  $\Omega_n^b$  such that, for all  $n \in \mathbb{N}$

$$\Omega_n^{a, \theta_0} = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 : |x_1 - \cot \theta_0 x_3| < \frac{h_n^a}{2}, (x_2, x_3) \in ]-\frac{1}{2}, \frac{1}{2}[ \times [0, \sin \theta_0] \right\},$$

where  $h_n^a$  and  $h_n^b$  are the small thickness of  $\Omega_n^{a, \theta_0}$  and  $\Omega_n^b$  respectively, such that

Figure 4.1:  $\Omega_n^{a,\theta_0}$ 

$$(4.1.1) \quad \begin{cases} \lim_n h_n^a = 0 = \lim_n h_n^b, \\ \lim_n \frac{h_n^b}{h_n^a} = q \in [0, +\infty], \end{cases}$$

In general, this type of thin films, it is used in apparatus for the storage of numerical information, such as the hard disks, magnetoresistive sensors, inductive thin films heads in magnetic, etc. The aim of this paper, is to study the asymptotic behavior as  $n$  diverges of the following problem:

$$(4.1.2) \quad \min \left\{ \int_{\Omega_n^{\theta_0}} (\alpha |D\underline{m}|^2 + \varphi(\underline{m}) + \frac{1}{2} D\underline{\zeta} \underline{m} - 2F_n \underline{m}) dx, \underline{m} \in H^1(\Omega_n^{\theta_0}, S^2) \right\}$$

subject to,

$$(4.1.3) \quad D(-D\underline{\zeta} + \underline{m}) = 0 \quad \text{in } \mathbb{R}^3$$

where  $\alpha$  is the exchange constant and let  $\varphi : S^2 \rightarrow [0, +\infty[$  is a continuous and even function,  $F_n \in L^2(\Omega_n^{\theta_0}, \mathbb{R}^3)$ , and  $S^2$  denotes the unit sphere of  $\mathbb{R}^3$ . In classical theory of micromagnetics,  $\underline{m} : \Omega_n^{\theta_0} \rightarrow \mathbb{R}^3$  denotes the magnetization and the body is always locally magnetized to a saturation magnetization  $|\underline{m}(x)| = m_s(T) > 0$  unless the local temperature  $T$  is greater or equal to Curie

temperature depending on the body, in the latter case  $\underline{m}(T) = 0$ . It is understood that  $\underline{m}(x) = 0$  in  $\mathbb{R}^3 \setminus \Omega_n^{\theta_0}$ . This model was proposed by Brown in [11]. We suppose that the temperature is a constant and lower than Curie temperature and, without loss of generality, we assume that  $|\underline{m}| = 1$ , i.e.  $\underline{m}(x) \in S^2$  a.e in  $\Omega_n^{\theta_0}$ . The function  $\zeta : \mathbb{R}^3 \rightarrow \mathbb{R}$  denotes the magnetic field potential. The magnetic field potential and the magnetization  $\underline{m}$  are connected by equation 4.1.3. Afin to reformulate the problem on a fixed domain  $\Omega^{\theta_0} = \Omega^{a,\theta_0} \cup \Omega^b$ , where  $\Omega^b = ]-\frac{1}{2}, \frac{1}{2}[^2 \times ]-1, 0[$  and  $\Omega^{a,\theta_0}$  is a the rescaled domain of  $\Omega_n^{a,\theta_0}$ , we derive the limit problem which depends on the limit  $\lim_n \frac{h_n^b}{h_n^a} = q \in [0, +\infty[$ .

Precisely, if  $q \in ]0, +\infty[$ , we prove that the limit problem is given by

$$(4.1.4) \quad \min \left\{ \int_{\Omega^{a,\theta_0}} \left( \alpha |(D_{x_1} \xi^a, D_{x_2} \mu^a, D_{x_3} \mu^a - \cot \theta_0 D_{x_1} \xi^a)|^2 + \varphi(\mu^a) + \frac{1}{2} |\sin \theta_0 \mu_1^a - \cos \theta_0 \mu_3^a|^2 \right) dx \right. \\ \left. - 2 \int_{\Omega^{a,\theta_0}} f^a(x_1, x_2, x_3) \mu^a dx - 2q \int_{\Omega^b} f^b(x_1, x_2, x_3) \mu^b dx + \right. \\ \left. q \int_{]-\frac{1}{2}, \frac{1}{2}[^2} \left( \alpha |(D_{x_1} \mu^b, D_{x_2} \mu^b)|^2 + \varphi(\mu^b) + \frac{1}{2} |\mu_3^b|^2 \right) dx_1 dx_2, \right. \\ \left. (\mu^a, \mu^b, \xi^a) \in H^1(\Omega^{a,\theta_0}, S^2) \times H^1(\Omega^b, S^2) \times \mathcal{F}, \right. \\ \left. \mu^a(x_2, 0) = \mu^b(0, x_2) \text{ in } ]-\frac{1}{2}, \frac{1}{2}[ \right\},$$

where  $\mu_1^a, \mu_3^a$  is the first and the third component of  $\mu^a$  respectively,  $\mu_3^b$  is the third component of  $\mu^b$ ,  $\mathcal{F} = \{g \in L^2(\Omega^{a,\theta_0}) : D_{x_1} g \in L^2(\Omega^{a,\theta_0})\}$ , and  $f^a$  and  $f^b$  are the  $L^2$ -weak limits of the rescaled exterior fields in  $\Omega^{a,\theta_0}$  and  $\Omega^b$  respectively, we obtain two problem coupled by the junction condition  $\mu^a(x_2, 0) = \mu^b(0, x_2)$  in  $]-\frac{1}{2}, \frac{1}{2}[$ .

**Remark 4.1.1.** • *It is easy to see that if  $\varphi = 0$  and  $f = 0$ , then the limit problem is zero and is attained by  $((0, \pm 1, 0), (0, \pm 1, 0))$ .*

- *If  $\varphi \neq 0$  and  $f = 0$ , there exist several type of anisotropy. In the uniaxial anisotropy, for wich only one direction  $u \in S^2$  is privileged, and we have:*

$$\varphi(m) = 1 - (m \cdot u)^2.$$

*In this case, we suppose that  $u \neq (0, \pm 1, 0)$ , it is easy to see that*

$$0 = \varphi(\pm u) < \varphi(m), \quad \text{for all } m \neq \pm u.$$

*In this case, we can show easily here that the minimizer of (4.1.4) does not attained by  $\pm u$ . Therefore, this limit is not parallel to the junction axis.*

In the second case, for  $q = 0$  the limit problem reduces to a single domain  $\Omega^{a,\theta_0}$ . Precisely, we prove that the limit problem is

$$\min \left\{ \int_{\Omega^{a,\theta_0}} (\alpha |(D_{x_1}\xi^a, D_{x_2}\mu^a, D_{x_3}\mu^a - \cot\theta_0 D_{x_1}\xi^a)|^2 + \varphi(\mu^a) + \frac{1}{2} |\sin\theta_0\mu_1^a - \cos\theta_0\mu_3^a|^2) dx \right. \\ \left. - 2 \int_{\Omega^{a,\theta_0}} f^a \mu^a dx, (\mu^a, \xi^a) \in H^1(\Omega^{a,\theta_0}, S^2) \times \mathcal{F} \right\}.$$

Similarly in the last case, for  $q = +\infty$  the limit problem reduces to a single domain  $\Omega^b$ . Precisely, we prove that the limit problem is

$$\min \left\{ \int_{] -\frac{1}{2}, \frac{1}{2} ]^2} \left( \alpha |(D_{x_1}\mu^b, D_{x_2}\mu^b)|^2 + \varphi(\mu^b) + \frac{1}{2} |\mu_3^b|^2 \right) dx_1 dx_2 \right. \\ \left. - 2 \int_{\Omega^b} f^b(x_1, x_2, x_3) \mu^b dx, \mu^b \in H^1(\Omega^b, S^2) \right\}.$$

The proof of these results are developed in several steps. If  $q \in ]0, +\infty[$ , we begin by proving a general convergence result for the magnetostatic energy. In the next step, we prove a weak convergence of the magnetization in  $H^1(\Omega^{a,\theta_0}, S^2) \times H^1(\Omega^b, S^2)$ , verifying the junction condition.

Our study generalizes the cases studied by Gaudiello and Hadji (cf. [31],[32],[33]), they studied the junction of ferromagnetic bodies. In [33], they considered two orthogonal ferromagnetic thin films and they proved that the limit magnetization are coupled when the volumes of the two thin films vanish with the same rate. In [31] and [32], they developed an asymptotic analysis of minimizing maps with values in  $S^2$  for the energy  $\int_{\Omega_n} (|DM|^2 - 2F_n M) dx$ , neglecting the term with non-local magnetostatic energy which characterizes the actual paper.

In what concerns the study of a single ferromagnetic thin film, several results regarding the study of a single ferromagnetic thin film are presented by Gioia and James in [39]. They considered a ferromagnetic thin film ( $\Omega_h$  of thickness  $h$ ), and studied the behavior of a thin film, with a external magnetic field equal to zero, when  $h \rightarrow 0$ , the energy of this film depending on the magnetization  $\bar{m}$ , Gioia and James proved that the magnetization minimizing the energy converge when  $h \rightarrow 0$ , to a magnetization minimizing the limit energy, and this limit energy is local. This result was extended by Leone and Alicandro in [1] to the case of non-convex exchange energy. In [41] and [42], the authors studied a micromagnetic thin film with degenerate exchange energy. For related problems, some interesting regularity theorems were obtained in [43]. The time dependent case was treated in [12] and in [14]. In general, magnetic thin-film elements are used in many applications: inductive thin films heads, magnetic recording, megneto-resistive sensors, thin films memories, etc. About this subject we refer to Hubert and Schafer in [46]. For recent papers on joined structure see also [38] and [56]

### 4.1.1 The setting of the problem

Let  $B$  be a open bounded containing  $\overline{\Omega_n^{\theta_0}}$  for every  $n \in \mathbb{N}$ , and set

$$(4.1.5) \quad \mathcal{U} = \left\{ U \in L^1_{loc}(\mathbb{R}^3) : U \in L^2(B), DU \in (L^2(\mathbb{R}^3))^3, \int_B U dx = 0 \right\}.$$

Remark that  $\mathcal{U}$  is contained in  $L^2_{loc}(\mathbb{R}^3)$  and it is a Hilbert space with the inner product:

$$(U, V) = \int_{\mathbb{R}^3} DUDV dx + \int_B UV dx.$$

From the Poincaré-Wirtinger inequality it follows that a norm on  $\mathcal{U}$  is given by

$$(U, U)^{\frac{1}{2}} = \left( \int_{\mathbb{R}^3} |DU|^2 dx \right)^{\frac{1}{2}}.$$

Then, for  $\underline{m} \in L^2(\Omega_n^{\theta_0}, \mathbb{R}^3)$ , Lax-Milgram Theorem ensures that the following problem

$$(4.1.6) \quad \begin{cases} \underline{\zeta}_{n, \underline{m}} \in \mathcal{U}, \\ \int_{\mathbb{R}^3} D\underline{\zeta}_{n, \underline{m}} D\underline{\zeta} dx = \int_{\Omega_n^{\theta_0}} \underline{m} D\underline{\zeta} dx = 0, \quad \forall \underline{\zeta} \in \mathcal{U}, \end{cases}$$

admits a unique solution  $\underline{\zeta}_{n, \underline{m}}$  in  $\mathcal{U}$  and it is characterized as the unique solution of the following problem:

$$(4.1.7) \quad \min \left\{ \frac{1}{2} \int_{\mathbb{R}^3} |D\underline{\zeta} - \underline{m}|^2 dx : \underline{\zeta} \in \mathcal{U} \right\},$$

where it is understood that  $\underline{m} = 0$  in  $\mathbb{R}^3 \setminus \Omega_n^{\theta_0}$ . Moreover,  $\underline{\zeta}_{n, \underline{m}}$  belong to  $H^1(\mathbb{R}^3)$  up to additive constant (see. [48]) Let us consider the following problem

$$(4.1.8) \quad \min \left\{ \int_{\Omega_n^{\theta_0}} \left( \alpha |D\underline{m}|^2 + \varphi(\underline{m}) + \frac{1}{2} \underline{\zeta}_{n, \underline{m}} \underline{m} - 2F_n \underline{m} \right) dx : \underline{m} \in H^1(\Omega_n^{\theta_0}, S^2) \right\},$$

has at least one solution(cf. [57]). The aim of this paper is to study the asymptotic behavior of problem (4.1.8), as  $n$  diverges.

### 4.1.2 The rescaled problem

In this section, we reformulate problem (4.1.8) on a fixed domain, using the following rescaling

$$\left\{ \begin{array}{l} (x_1, x_2, x_3) \in \Omega^{a, \theta_0} = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 : |x_1 - \cot \theta_0 x_3| < \frac{1}{2}, (x_2, x_3) \in ] -\frac{1}{2}, \frac{1}{2}[ \times ] 0, \sin \theta_0[ \right\} \\ \qquad \qquad \qquad \longrightarrow (h_n^a x_1 + \frac{\cos \theta_0}{\sin \theta_0} (1 - h_n^a) x_3, x_2, x_3) \in \text{int}(\Omega_n^{a, \theta_0}), \\ (x_1, x_2, x_3) \in \Omega^b = ] -\frac{1}{2}, \frac{1}{2}[^2 \times ] -1, 0[ \longrightarrow (x_1, x_2, h_n^b x_3) \in \Omega_n^b, \end{array} \right.$$

In fact, for every  $n \in \mathbb{N}$  set

$$m : x \in \Omega^{\theta_0} = \Omega^{a,\theta_0} \cup \Omega^b \mapsto m(x) = \begin{cases} m^a(x) = \underline{m}(h_n^a x_1 + \frac{\cos \theta_0}{\sin \theta_0}(1 - h_n^a)x_3, x_2, x_3) & \text{for } x \text{ a.e. in } \Omega^{a,\theta_0}, \\ m^b(x) = \underline{m}(x_1, x_2, h_n^b x_3) & \text{for } x \text{ a.e. in } \Omega^b \end{cases}$$

Let as set  $\mathbb{R}_+^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > 0\}$ ,  $\mathbb{R}_-^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 < 0\}$ .

For every  $n \in \mathbb{N}$ , let  $B_n^a$  and  $B_n^b$  is a open bounded in  $\mathbb{R}^3$  containing  $\overline{\Omega}^{a,\theta_0}$  and  $\overline{\Omega}^b$  respectively, let

$$(4.1.9) \quad \mathcal{U}_n = \left\{ \begin{aligned} &u = (u^a, u^b) \in L_{loc}^1(\overline{\mathbb{R}}_+^3) \times L_{loc}^1(\overline{\mathbb{R}}_-^3) : (u^a|_{B_n^a}, u^b|_{B_n^b}) \in L^2(B_n^a) \times L^2(B_n^b), \\ &(Du^a, Du^b) \in (L^2(\mathbb{R}_+^3))^3 \times (L^2(\mathbb{R}_-^3))^3, \quad \int_{B_n^a} u^a dx + \frac{h_n^b}{h_n^a} \int_{B_n^b} u^b dx = 0, \\ &u^a(x_1, x_2, 0) = u^b(h_n^a x_1, x_2, 0), \text{ for } (x_1, x_2) \text{ a.e. in } \mathbb{R}^2 \end{aligned} \right\}.$$

We state problem (4.1.6) in the fixed domain  $\Omega^{\theta_0}$ , for every  $m = (m^a, m^b) \in L^2(\Omega^{a,\theta_0}, \mathbb{R}^3) \times L^2(\Omega^b, \mathbb{R}^3)$ , the following equation:

$$(4.1.10) \quad \left\{ \begin{aligned} &\zeta_{n,m} = (\zeta_{n,m}^a, \zeta_{n,m}^b) \in \mathcal{U}_n \\ &\int_{\mathbb{R}_+^3} \left( \frac{1}{h_n^a} D_{x_1} \zeta_{n,m}^a, D_{x_2} \zeta_{n,m}^a, D_{x_3} \zeta_{n,m}^a + \frac{\cos \theta_0}{\sin \theta_0} \left(1 - \frac{1}{h_n^a}\right) D_{x_1} \zeta_{n,m}^a \right) \\ &\quad \left( \frac{1}{h_n^a} D_{x_1} \zeta^a, D_{x_2} \zeta^a, D_{x_3} \zeta^a + \frac{\cos \theta_0}{\sin \theta_0} \left(1 - \frac{1}{h_n^a}\right) D_{x_1} \zeta^a \right) dx + \\ &\quad \frac{h_n^b}{h_n^a} \int_{\mathbb{R}_-^3} \left( D_{x_1} \zeta_{n,m}^b, D_{x_2} \zeta_{n,m}^b, \frac{1}{h_n^b} D_{x_3} \zeta_{n,m}^b \right) \left( D_{x_1} \zeta^b, D_{x_2} \zeta^b, \frac{1}{h_n^b} D_{x_3} \zeta^b \right) dx = \\ &\int_{\Omega^{a,\theta_0}} \left( \frac{1}{h_n^a} D_{x_1} \zeta^a, D_{x_2} \zeta^a, D_{x_3} \zeta^a + \frac{\cos \theta_0}{\sin \theta_0} \left(1 - \frac{1}{h_n^a}\right) D_{x_1} \zeta^a \right) m^a dx + \\ &\quad \frac{h_n^b}{h_n^a} \int_{\Omega^b} \left( D_{x_1} \zeta^b, D_{x_2} \zeta^b, \frac{1}{h_n^b} D_{x_3} \zeta^b \right) m^b dx, \quad \zeta = (\zeta^a, \zeta^b) \in \mathcal{U}_n, \end{aligned} \right.$$

admits a unique solution  $\zeta_{n,m} = (\zeta_{n,m}^a, \zeta_{n,m}^b) \in \mathcal{U}_n$ , and this solution is characterized as the unique solution of the following problem:

$$(4.1.11) \quad j_n(\zeta_{n,m}) = \min \{j_n(\zeta) : \zeta \in \mathcal{U}_n\},$$

where

$$(4.1.12) \quad j_n : \zeta = (\zeta^a, \zeta^b) \in \mathcal{U}_n \longrightarrow \frac{1}{2} \int_{\mathbb{R}_+^3} \left| \left( \frac{1}{h_n^a} D_{x_1} \zeta^a, D_{x_2} \zeta^a, D_{x_3} \zeta^a + \frac{\cos \theta_0}{\sin \theta_0} \left(1 - \frac{1}{h_n^a}\right) D_{x_1} \zeta^a \right) - m^a \right|^2 dx \\ + \frac{1}{2} \frac{h_n^b}{h_n^a} \int_{\mathbb{R}_-^3} \left| \left( D_{x_1} \zeta^b, D_{x_2} \zeta^b, \frac{1}{h_n^b} D_{x_3} \zeta^b \right) - m^b \right|^2 dx.$$

Remark that,  $\zeta_{n,m} = (\zeta_{n,m}^a, \zeta_{n,m}^b)$  belongs to  $H^1(\mathbb{R}_+^3) \times H^1(\mathbb{R}_-^3)$  up to additive constant. For every  $n \in \mathbb{N}$ , let

(4.1.13)

$$\mathcal{M}_n = \left\{ m = (m^a, m^b) \in H^1(\Omega^{a,\theta_0}, S^2) \times H^1(\Omega^b, S^2) : \right.$$

$$\left. m^a(x_1, x_2, 0) = m^b(h_n^a x_1, x_2, 0), \text{ for } (x_1, x_2) \text{ a.e. in } ] -\frac{1}{2}, \frac{1}{2}[^2 \right\},$$

$$f_n : x \in \Omega^{a,\theta_0} \cup \Omega^b \longmapsto f_n(x) = \begin{cases} f_n^a(x) = F_n(h_n^a x_1 + \frac{\cos \theta_0}{\sin \theta_0} (1 - h_n^a) x_3, x_2, x_3) & \text{for } x \text{ a.e. in } \Omega^{a,\theta_0}, \\ f_n^b(x) = F_n(x_1, x_2, h_n^b x_3) & \text{for } x \text{ a.e. in } \Omega^b, \end{cases}$$

let

(4.1.14)

$$E_n : m = (m^a, m^b) \in \mathcal{M}_n \longrightarrow$$

$$\int_{\Omega^{a,\theta_0}} \left( \alpha \left| \left( \frac{1}{h_n^a} D_{x_1} m^a, D_{x_2} m^a, D_{x_3} m^a + \frac{\cos \theta_0}{\sin \theta_0} \left(1 - \frac{1}{h_n^a}\right) D_{x_1} m^a \right) \right|^2 + \varphi(m^a) - 2f_n^a m^a \right) dx +$$

$$\frac{h_n^b}{h_n^a} \int_{\Omega^b} \left( \alpha \left| \left( D_{x_1} m^b, D_{x_2} m^b, \frac{1}{h_n^b} D_{x_3} m^b \right) \right|^2 + \varphi(m^b) - 2f_n^b m^b \right) dx +$$

$$\frac{1}{2} \int_{\Omega^{a,\theta_0}} \left( \frac{1}{h_n^a} D_{x_1} \zeta_{n,m}^a, D_{x_2} \zeta_{n,m}^a, D_{x_3} \zeta_{n,m}^a + \frac{\cos \theta_0}{\sin \theta_0} \left(1 - \frac{1}{h_n^a}\right) D_{x_1} \zeta_{n,m}^a \right) m^a dx +$$

$$\frac{1}{2} \frac{h_n^b}{h_n^a} \int_{\Omega^b} \left( D_{x_1} \zeta_{n,m}^b, D_{x_2} \zeta_{n,m}^b, \frac{1}{h_n^b} D_{x_3} \zeta_{n,m}^b \right) m^b dx.$$

Then the function defined by

$$\underline{m}(h_n^a x_1 + \frac{\cos \theta_0}{\sin \theta_0} (1 - h_n^a) x_3, x_2, x_3) \text{ for } x \text{ a.e. in } \Omega^{a,\theta_0}, \quad \underline{m}(x_1, x_2, h_n^b x_3) \text{ for } x \text{ a.e. in } \Omega^b,$$

with  $\underline{m}$  solution of problem (4.1.8), as a minimizer of the following problem:

$$(4.1.15) \quad \min \{ E_n(m) : m \in \mathcal{M}_n \}.$$



Remark that, setting for every  $n \in \mathbb{N}$

$$\begin{aligned}
(4.1.16) \quad E_n^{mag} : m = (m^a, m^b) &\in L^2(\Omega^{a,\theta_0}, \mathbb{R}^3) \times L^2(\Omega^b, \mathbb{R}^3) \longrightarrow \\
&\frac{1}{2} \int_{\Omega^{a,\theta_0}} \left( \frac{1}{h_n^a} D_{x_1} \zeta_{n,m}^a, D_{x_2} \zeta_{n,m}^a, D_{x_3} \zeta_{n,m}^a + \frac{\cos \theta_0}{\sin \theta_0} \left(1 - \frac{1}{h_n^a}\right) D_{x_1} \zeta_{n,m}^a \right) m^a dx + \\
&\frac{1}{2} \frac{h_n^b}{h_n^a} \int_{\Omega^b} \left( D_{x_1} \zeta_{n,m}^b, D_{x_2} \zeta_{n,m}^b, \frac{1}{h_n^b} D_{x_3} \zeta_{n,m}^b \right) m^b dx = \\
&\frac{1}{2} \int_{\mathbb{R}_+^3} \left| \left( \frac{1}{h_n^a} D_{x_1} \zeta_{n,m}^a, D_{x_2} \zeta_{n,m}^a, D_{x_3} \zeta_{n,m}^a + \frac{\cos \theta_0}{\sin \theta_0} \left(1 - \frac{1}{h_n^a}\right) D_{x_1} \zeta_{n,m}^a \right) \right|^2 dx + \\
&\frac{1}{2} \frac{h_n^b}{h_n^a} \int_{\mathbb{R}_-^3} \left| \left( D_{x_1} \zeta_{n,m}^b, D_{x_2} \zeta_{n,m}^b, \frac{1}{h_n^b} D_{x_3} \zeta_{n,m}^b \right) \right|^2 dx,
\end{aligned}$$

from (4.1.10), it follows that

$$\begin{aligned}
(4.1.17) \quad E_n(m) &= \int_{\Omega^{a,\theta_0}} \left( \alpha \left| \left( \frac{1}{h_n^a} D_{x_1} m^a, D_{x_2} m^a, D_{x_3} m^a + \frac{\cos \theta_0}{\sin \theta_0} \left(1 - \frac{1}{h_n^a}\right) D_{x_1} m^a \right) \right|^2 + \varphi(m^a) - 2f_n^a m^a \right) dx \\
&+ \frac{h_n^b}{h_n^a} \int_{\Omega^b} \left( \alpha \left| \left( D_{x_1} m^b, D_{x_2} m^b, \frac{1}{h_n^b} D_{x_3} m^b \right) \right|^2 + \varphi(m^b) - 2f_n^b m^b \right) dx + E_n^{mag}(m), \\
\forall m = (m^a, m^b) &\in \mathcal{M}_n, \quad \forall n \in \mathbb{N}.
\end{aligned}$$

Actually, the goal of this paper is to study the asymptotic behavior, as  $n$  diverges, of problem (4.1.15). To this aim, Assume that

$$(4.1.18) \quad f_n^a \rightharpoonup f^a \text{ weakly in } L^2(\Omega^{a,\theta_0}, \mathbb{R}^3), \quad f_n^b \rightharpoonup f^b \text{ weakly in } L^2(\Omega^b, \mathbb{R}^3).$$

### 4.1.3 preliminary

Let us define

$$(4.1.19) \quad \mathcal{F} = \{g \in L^2(\Omega^{a,\theta_0}) : D_{x_1} g \in L^2(\Omega^{a,\theta_0})\},$$

remark that  $\mathcal{F}$  it is a Hilbert space with the norm

$$(u, u)_{\mathcal{F}} = \left( \int_{\Omega^{a,\theta_0}} |u|^2 dx \right)^{\frac{1}{2}} + \left( \int_{\Omega^{a,\theta_0}} |D_{x_1} u|^2 dx \right)^{\frac{1}{2}},$$

Using standard arguments (see. [10]), we have  $C^\infty(\overline{\Omega^{a,\theta_0}})$  is dense in  $\mathcal{F}$ .

## 4.2 The main results

Let

$$\begin{aligned}
 \mathcal{M} &= \left\{ \mu = (\mu^a, \mu^b) \in H^1(\Omega^{a,\theta_0}, S^2) \times H^1(\Omega^b, S^2) : \right. \\
 &\mu^a \text{ is independent of } x_1, \mu^b \text{ is independent of } x_3, \\
 &\left. \mu^a(x_2, 0) = \mu^b(0, x_2), \text{ for } x_2 \text{ a.e. in } ] - \frac{1}{2}, \frac{1}{2}[ \right\} \\
 (4.2.1) \quad &\simeq \left\{ \mu = (\mu^a, \mu^b) \in H^1(] - \frac{1}{2}, \frac{1}{2}[ \times ]0, \sin \theta_0[, S^2) \times H^1(] - \frac{1}{2}, \frac{1}{2}[^2, S^2) : \right. \\
 &\mu^a \text{ is independent of } x_1, \mu^b \text{ is independent of } x_3, \\
 &\left. \mu^a(x_2, 0) = \mu^b(0, x_2), \text{ for } x_2 \text{ a.e. in } ] - \frac{1}{2}, \frac{1}{2}[ \right\}
 \end{aligned}$$

and for  $q \in ]0, \infty[$ , let

$$\begin{aligned}
 (4.2.2) \quad E_q &: (\mu^a, \mu^b, \xi^a) \in \mathcal{M} \times \mathcal{F} \longrightarrow \\
 &\int_{\Omega^{a,\theta_0}} \left( \alpha |(D_{x_1} \xi^a, D_{x_2} \mu^a, D_{x_3} \mu^a - \cot \theta_0 D_{x_1} \xi^a)|^2 + \varphi(\mu^a) + \frac{1}{2} |\sin \theta_0 \mu_1^a - \cos \theta_0 \mu_3^a|^2 \right) dx - \\
 &2 \int_{\Omega^{a,\theta_0}} f^a(x_1, x_2, x_3) \mu^a dx - 2q \int_{\Omega^b} f^b(x_1, x_2, x_3) \mu^b dx + \\
 &q \int_{] - \frac{1}{2}, \frac{1}{2}[^2} \left( \alpha |(D_{x_1} \mu^b, D_{x_2} \mu^b)|^2 + \varphi(\mu^b) + \frac{1}{2} |\mu_3^b|^2 \right) dx_1 dx_2,
 \end{aligned}$$

Our first main result is

**Theorem 4.2.1.** *Assume (4.1.1) with  $q \in ]0, +\infty[$  and (4.1.18). For every  $n \in \mathbb{N}$ , let  $m_n = (m_n^a, m_n^b)$  be a solution of (4.1.15) and  $\zeta_n = (\zeta_n^a, \zeta_n^b)$  be the unique solution of (4.1.11) corresponding to  $m_n$ , let  $\xi_n^{(1)}(x_2, x_3) = \int_{-\frac{1}{2}}^{\frac{1}{2}} m_n^a(x_1 + \cot \theta_0 x_3, x_2, x_3) dx_1$ . Then, there exist a subsequence  $(m_{n_i})_{i \in \mathbb{N}}$  and  $(\hat{\mu}^a, \hat{\mu}^b, \hat{\xi}^a) \in \mathcal{M} \times \mathcal{F}$ , depending on the selected subsequence, such that*

$$(4.2.3) \quad \begin{cases} m_{n_i}^a \rightarrow \hat{\mu}^a \text{ strongly in } H^1(\Omega^{a,\theta_0}, S^2), m_{n_i}^b \rightarrow \hat{\mu}^b \text{ strongly in } H^1(\Omega^b, S^2), \\ \frac{1}{h_{n_i}^a} (m_{n_i}^a - \xi_{n_i}^{(1)}) \rightharpoonup \hat{\xi}^a \text{ weakly in } \mathcal{F}, \end{cases}$$

$$(4.2.4) \quad \begin{cases} \frac{1}{h_{n_i}^a} D_{x_1} m_{n_i}^a \rightarrow D_{x_1} \hat{\xi}^a \text{ strongly in } L^2(\Omega^{a, \theta_0}, \mathbb{R}^3), \\ \frac{1}{h_{n_i}^b} D_{x_3} m_{n_i}^b \rightarrow 0 \text{ strongly in } L^2(\Omega^b, \mathbb{R}^3), \end{cases}$$

and

$$(4.2.5) \quad \begin{cases} \frac{1}{h_{n_i}^a} D_{x_1} \zeta_{n_i}^a \rightarrow \sin^2 \theta_0 \tilde{\mu}_1^a - \sin \theta_0 \cos \theta_0 \tilde{\mu}_3^a, & D_{x_2} \zeta_{n_i}^a \rightarrow 0, & D_{x_3} \zeta_{n_i}^a \rightarrow 0 & \text{strongly in } L^2(\mathbb{R}_+^3), \\ D_{x_1} \zeta_{n_i}^b \rightarrow 0, & D_{x_2} \zeta_{n_i}^b \rightarrow 0, & \frac{1}{h_{n_i}^b} D_{x_3} \zeta_{n_i}^b \rightarrow \tilde{\mu}_3^b & \text{strongly in } L^2(\mathbb{R}_-^3), \end{cases}$$

as  $i$  and  $n$  diverges, where  $\tilde{\mu}_1^a$ ,  $\tilde{\mu}_3^a$  and  $\tilde{\mu}_3^b$  denote the zero extension of  $\hat{\mu}_1^a$ ,  $\hat{\mu}_3^a$  and  $\hat{\mu}_3^b$  on  $\mathbb{R}^3$  respectively, and  $(\hat{\mu}, \hat{\xi}^a)$  is a solution of the following problem

$$(4.2.6) \quad E_q(\hat{\mu}, \hat{\xi}^a) = \min \{E_q(\mu, \xi^a) : (\mu, \xi^a) \in \mathcal{M} \times \mathcal{F}\},$$

where  $\mathcal{M}$  and  $E_q$  be defined (4.2.1) and (4.2.2), respectively. Moreover, it results that

$$(4.2.7) \quad \lim_n E_n(m_n) = E_q(\hat{\mu}^a, \hat{\xi}^a).$$

If  $q = 0$ , let

$$(4.2.8) \quad \mathcal{M}_0 = \{\mu^a \in H^1(\Omega^{a, \theta_0}, S^2) : \mu^a \text{ is independent of } x_1\}$$

and

$$(4.2.9) \quad E_0 : (\mu^a, \xi^a) \in \mathcal{M}_0 \times \mathcal{F} \longrightarrow$$

$$\int_{\Omega^{a, \theta_0}} \left( \alpha |(D_{x_1} \xi^a, D_{x_2} \mu^a, D_{x_3} \mu^a - \cot \theta_0 D_{x_1} \xi^a)|^2 + \varphi(\mu^a) + \frac{1}{2} |\sin \theta_0 \mu_1^a - \cos \theta_0 \mu_3^a|^2 \right) dx \\ - 2 \int_{\Omega^{a, \theta_0}} f^a(x_1, x_2, x_3) \mu^a dx,$$

then, it results that

**Theorem 4.2.2.** *Assume (4.1.1) with  $q = 0$  and (4.1.18). For every  $n \in \mathbb{N}$ , let  $m_n = (m_n^a, m_n^b)$  be a solution of (4.1.15) and  $\zeta_n = (\zeta_n^a, \zeta_n^b)$  be the unique solution of (4.1.11) corresponding to  $m_n$ , let  $\xi_n^{(1)}(x_2, x_3) = \int_{-\frac{1}{2}}^{\frac{1}{2}} m_n^a(x_1 +$*

$\cot \theta_0 x_3, x_2, x_3) dx_1$ . Then, there exist a subsequence  $(m_{n_i})_{i \in \mathbb{N}}$  and  $(\hat{\mu}^a, \hat{\mu}^b, \hat{\xi}^a) \in \mathcal{M} \times \mathcal{F}$ , depending on the selected subsequence, such that

$$(4.2.10) \quad \begin{cases} m_{n_i}^a \rightarrow \hat{\mu}^a \text{ strongly in } H^1(\Omega^{a, \theta_0}, S^2), m_{n_i}^b \rightarrow \hat{\mu}^b \text{ strongly in } H^1(\Omega^b, S^2), \\ \frac{1}{h_{n_i}^a} (m_{n_i}^a - \xi_{n_i}^{(1)}) \rightharpoonup \hat{\xi}^a \text{ weakly in } \mathcal{F}, \end{cases}$$

$$(4.2.11) \quad \begin{cases} \frac{1}{h_{n_i}^a} D_{x_1} m_{n_i}^a \rightarrow D_{x_1} \hat{\xi}^a \text{ strongly in } L^2(\Omega^{a, \theta_0}, \mathbb{R}^3), \\ \frac{1}{h_{n_i}^b} \left( \frac{h_{n_i}^b}{h_{n_i}^a} \right)^{\frac{1}{2}} D_{x_3} m_{n_i}^b \rightarrow 0 \text{ strongly in } L^2(\Omega^b, \mathbb{R}^3), \end{cases}$$

and

$$(4.2.12) \quad \begin{cases} \frac{1}{h_{n_i}^a} D_{x_1} \zeta_{n_i}^a \rightarrow \sin^2 \theta_0 \tilde{\mu}_1^a - \sin \theta_0 \cos \theta_0 \tilde{\mu}_3^a, \quad D_{x_2} \zeta_{n_i}^a \rightarrow 0, \quad D_{x_3} \zeta_{n_i}^a \rightarrow 0 \text{ strongly in } L^2(\mathbb{R}_-^3), \\ \left( \frac{h_{n_i}^b}{h_{n_i}^a} \right)^{\frac{1}{2}} D_{x_1} \zeta_{n_i}^b \rightarrow 0, \quad \left( \frac{h_{n_i}^b}{h_{n_i}^a} \right)^{\frac{1}{2}} D_{x_2} \zeta_{n_i}^b \rightarrow 0, \quad \frac{1}{h_{n_i}^b} \left( \frac{h_{n_i}^b}{h_{n_i}^a} \right)^{\frac{1}{2}} D_{x_3} \zeta_{n_i}^b \rightarrow 0 \text{ strongly in } L^2(\mathbb{R}_+^3), \end{cases}$$

as  $i$  and  $n$  diverges, where  $\tilde{\mu}_1^a$  and  $\tilde{\mu}_3^a$  denote the zero extension of  $\hat{\mu}_1^a$  and  $\hat{\mu}_3^a$  on  $\mathbb{R}^3$  respectively, and  $(\hat{\mu}^a, \hat{\xi}^a)$  is a solution of the following problem:

$$(4.2.13) \quad E_0(\hat{\mu}^a, \hat{\xi}^a) = \min\{E_0(\mu^a, \xi^a) : (\mu^a, \xi^a) \in \mathcal{M}_0 \times \mathcal{F}\}.$$

Moreover, it results that

$$(4.2.14) \quad \lim_n E_n(m_n) = E_0(\hat{\mu}^a, \hat{\xi}^a).$$

In the case  $q = +\infty$ , let

$$(4.2.15) \quad \mathcal{M}_\infty = \{\mu^b \in H^1(\Omega^b, S^2) : \mu^b \text{ is independent of } x_3\} \simeq H^1\left(\left]-\frac{1}{2}, \frac{1}{2}\right]^2, S^2\right)$$

and

$$(4.2.16) \quad E_\infty : \mu = \mu^b \in \mathcal{M}_\infty \rightarrow$$

$$\int_{\left]-\frac{1}{2}, \frac{1}{2}\right]^2} \left( \alpha |(D_{x_1} \mu^b, D_{x_2} \mu^b)|^2 + \varphi(\mu^b) + \frac{1}{2} |\mu_3^b|^2 \right) dx_1 dx_2 - 2 \int_{\Omega^b} f^b(x_1, x_2, x_3) \mu^b dx,$$

then, it results that:

**Theorem 4.2.3.** *Assume (4.1.1) with  $q = +\infty$  and (4.1.18). For every  $n \in \mathbb{N}$ , let  $m_n = (m_n^a, m_n^b)$  be a solution of (4.1.15) and  $\zeta_n = (\zeta_n^a, \zeta_n^b)$  be the unique solution of (4.1.11) corresponding to  $m_n$ . Then, there exist a subsequence  $(m_{n_i})_{i \in \mathbb{N}}$  and  $\hat{\mu}^b \in \mathcal{M}_\infty$ , depending on the selected subsequence, such that*

$$(4.2.17) \quad \begin{cases} \left( \frac{h_{n_i}^a}{h_{n_i}^b} \right)^{\frac{1}{2}} m_{n_i}^a \rightarrow 0 & \text{strongly in } H^1(\Omega^{a, \theta_0}, S^2), \\ m_{n_i}^b \rightarrow \hat{\mu}^b & \text{strongly in } H^1(\Omega^b, S^2), \end{cases}$$

$$(4.2.18) \quad \begin{cases} \frac{1}{h_{n_i}^a} \left( \frac{h_{n_i}^a}{h_{n_i}^b} \right)^{\frac{1}{2}} D_{x_1} m_{n_i}^a \rightarrow 0, & \text{strongly in } L^2(\Omega^{a, \theta_0}, \mathbb{R}^3), \\ \frac{1}{h_{n_i}^b} D_{x_3} m_{n_i}^b \rightarrow 0 & \text{strongly in } L^2(\Omega^b, \mathbb{R}^3), \end{cases}$$

and

$$(4.2.19) \quad \begin{cases} \frac{1}{h_{n_i}^a} \left( \frac{h_{n_i}^a}{h_{n_i}^b} \right)^{\frac{1}{2}} D_{x_1} \zeta_{n_i}^a \rightarrow 0, & \left( \frac{h_{n_i}^a}{h_{n_i}^b} \right)^{\frac{1}{2}} D_{x_2} \zeta_{n_i}^a \rightarrow 0, & \left( \frac{h_{n_i}^a}{h_{n_i}^b} \right)^{\frac{1}{2}} D_{x_3} \zeta_{n_i}^a \rightarrow 0 & \text{strongly in } L^2(\mathbb{R}_+^3), \\ D_{x_1} \zeta_{n_i}^b \rightarrow 0, & D_{x_2} \zeta_{n_i}^b \rightarrow 0, & \frac{1}{h_{n_i}^b} D_{x_3} \zeta_{n_i}^b \rightarrow \hat{\mu}_3^b & \text{strongly in } L^2(\mathbb{R}_-^3), \end{cases}$$

as  $i$  and  $n$  diverges, where  $\tilde{\mu}_3^b$  denote the zero extension of  $\hat{\mu}_3^b$  on  $\mathbb{R}^3$ , and  $\hat{\mu}^b$  is a solution of the following problem:

$$(4.2.20) \quad E_\infty(\hat{\mu}^b) = \min\{E_\infty(\mu^b) : \mu^b \in \mathcal{M}_\infty\}.$$

Moreover, it results that

$$(4.2.21) \quad \lim_n \left( \frac{h_n^a}{h_n^b} E_n(m_n) \right) = E_\infty(\hat{\mu}^b)$$

### 4.3 If $q \in ]0, +\infty[$

The proof of Theorem 4.2.1 well developed in several steps. We begin by proving a general convergence result for the magnetostatic energy.

#### 4.3.1 A convergence result for the magnetostatic energy

**Proposition 4.3.1.** *Assume (4.1.1), with  $q \in ]0, +\infty[$ . Let  $\{m_n = (m_n^a, m_n^b)\}_{n \in \mathbb{N}} \subset L^2(\Omega^{a, \theta_0}, \mathbb{R}^3) \times L^2(\Omega^b, \mathbb{R}^3)$  and  $\mu = (\mu^a, \mu^b) = ((\mu_1^a, \mu_2^a, \mu_3^a), (\mu_1^b, \mu_2^b, \mu_3^b)) \in L^2(\Omega^{a, \theta_0}, \mathbb{R}^3) \times L^2(\Omega^b, \mathbb{R}^3)$  such that  $\mu^a$  and  $\mu^b$  be a independent of  $x_1$  and  $x_3$  respectively, and*

$$(4.3.1) \quad (m_n^a, m_n^b) \longrightarrow (\mu^a, \mu^b) \quad \text{strongly in } L^2(\Omega^{a, \theta_0}, \mathbb{R}^3) \times L^2(\Omega^b, \mathbb{R}^3)$$

as  $n$  diverges. Let  $\zeta_n = (\zeta_n^a, \zeta_n^b)$  be the unique solution of (4.1.11) corresponding to  $m_n$ . Then, it results that

$$(4.3.2) \quad \begin{cases} \frac{1}{h_n^a} D_{x_1} \zeta_n^a \rightarrow \sin^2 \theta_0 \mu_1^a - \sin \theta_0 \cos \theta_0 \mu_3^a, & D_{x_2} \zeta_n^a \rightarrow 0, & D_{x_3} \zeta_n^a \rightarrow 0 & \text{strongly in } L^2(\mathbb{R}_+^3), \\ D_{x_1} \zeta_n^b \rightarrow 0, & D_{x_2} \zeta_n^b \rightarrow 0, & \frac{1}{h_n^b} D_{x_3} \zeta_n^b \rightarrow \mu_3^b & \text{strongly in } L^2(\mathbb{R}_-^3), \end{cases}$$

as  $n$  diverges, and

$$(4.3.3) \quad \lim_n E_n^{mag}(m_n) = \frac{1}{2} \left( \int_{\Omega^{a, \theta_0}} |\sin \theta_0 \mu_1^a - \cos \theta_0 \mu_3^a|^2 dx_2 dx_3 + q \int_{]-\frac{1}{2}, \frac{1}{2}[^2} |\mu_3^b|^2 dx_1 dx_2 \right).$$

*Proof.* For a given  $\zeta_n$  is a solution of the problem (4.1.11), and by choosing  $\zeta = (0, 0)$  as test function in (4.1.11) corresponding to  $m_n$ , it results that

$$(4.3.4) \quad \begin{aligned} \exists c \in ]0, +\infty[ : \int_{\mathbb{R}_+^3} \left| \left( \frac{1}{h_n^a} D_{x_1} \zeta_n^a, D_{x_2} \zeta_n^a, D_{x_3} \zeta_n^a + \frac{\cos \theta_0}{\sin \theta_0} \left(1 - \frac{1}{h_n^a}\right) D_{x_1} \zeta_n^a \right) - m_n^a \right|^2 dx + \\ \frac{h_n^b}{h_n^a} \int_{\mathbb{R}_-^3} \left| \left( D_{x_1} \zeta_n^b, D_{x_2} \zeta_n^b, \frac{1}{h_n^b} D_{x_3} \zeta_n^b \right) - m_n^b \right|^2 dx \leq c, \forall n \in \mathbb{N}. \end{aligned}$$

Applying the triangle inequality and by taking into account the fact that  $|m_n| = 1$  and (4.1.1), we prove the existence of a positive constant  $c \in ]0, +\infty[$ , such that

$$(4.3.5) \quad \begin{cases} \exists c \in ]0, +\infty[, \\ \left\| \frac{1}{h_n^a} D_{x_1} \zeta_n^a \right\|_{L^2(\mathbb{R}_+^3)} \leq c, \left\| D_{x_2} \zeta_n^a \right\|_{L^2(\mathbb{R}_+^3)} \leq c, \left\| D_{x_3} \zeta_n^a \right\|_{L^2(\mathbb{R}_+^3)} \leq c, \\ \left\| D_{x_1} \zeta_n^b \right\|_{L^2(\mathbb{R}_-^3)} \leq c, \left\| D_{x_2} \zeta_n^b \right\|_{L^2(\mathbb{R}_-^3)} \leq c, \left\| \frac{1}{h_n^b} D_{x_3} \zeta_n^b \right\|_{L^2(\mathbb{R}_-^3)} \leq c, \end{cases}$$

From the Sobolev-Gagliardo-Nirenberg inequality and (3.5), we obtain

$$(4.3.6) \quad \exists c \in ]0, +\infty[ : \|\zeta_n\|_{L^6(\mathbb{R}^3)} \leq c, \forall n \in \mathbb{N}.$$

Moreover, the estimates (4.3.5) and (4.3.6) guarantee the existence of a function  $\zeta = (\zeta^a, \zeta^b) \in L^6(\mathbb{R}_+^3) \times L^6(\mathbb{R}_-^3)$  and  $D\zeta = (D\zeta^a, D\zeta^b) \in (L^2(\mathbb{R}_+^3))^3 \times (L^2(\mathbb{R}_-^3))^3$ , where  $\zeta^a$  and  $\zeta^b$  is independent of  $x_1$  and  $x_3$  respectively, such that

$$(4.3.7) \quad \begin{cases} \zeta_n^a \rightharpoonup \zeta^a & \text{weakly in } L^6(\mathbb{R}_+^3), & D\zeta_n^a \rightharpoonup D\zeta^a & \text{weakly in } (L^2(\mathbb{R}_+^3))^3, \\ \zeta_n^b \rightharpoonup \zeta^b & \text{weakly in } L^6(\mathbb{R}_-^3), & D\zeta_n^b \rightharpoonup D\zeta^b & \text{weakly in } (L^2(\mathbb{R}_-^3))^3, \end{cases}$$

as  $n$  diverges. Moreover, the fact that  $\zeta^a$  is independent of  $x_1$  and  $D\zeta^a \in$

$(L^2(\mathbb{R}_+^3))^3$ , gives that

$$+\infty > \int_{\mathbb{R}_+^3} |D\zeta^a|^2 dx \geq \int_{|a,b|} \left( \int_{\mathbb{R}_+^2} (|D_{x_2}\zeta^a|^2 + |D_{x_3}\zeta^a|^2) d(x_2, x_3) \right) dx_1 =$$

$$(b-a) \int_{\mathbb{R}_+^2} (|D_{x_2}\zeta^a|^2 + |D_{x_3}\zeta^a|^2) d(x_2, x_3), \forall a, b \in \mathbb{R} \text{ where } a < b$$

which gives that  $\zeta^a$  is independent of  $x_2$  and  $x_3$ , then  $\zeta^a = 0$  in  $L^6(\mathbb{R}_+^3)$ . Similarly, we obtain  $\zeta^b = 0$  in  $L^6(\mathbb{R}_-^3)$ . It results that

$$(4.3.8) \quad \begin{cases} D\zeta_n^a \rightharpoonup 0 & \text{weakly in } (L^2(\mathbb{R}_+^3))^3, \\ D\zeta_n^b \rightharpoonup 0 & \text{weakly in } (L^2(\mathbb{R}_-^3))^3, \end{cases}$$

as  $n$  diverges. Moreover, (4.3.5) guarantee the existence of  $\rho^a \in L^2(\mathbb{R}_+^3)$  and  $\rho^b \in L^2(\mathbb{R}_-^3)$  such that

$$(4.3.9) \quad \begin{cases} \frac{1}{h_n^a} D_{x_1}\zeta_n^a \rightharpoonup \rho^a & \text{weakly in } L^2(\mathbb{R}_+^3), \\ \frac{1}{h_n^b} D_{x_3}\zeta_n^b \rightharpoonup \rho^b & \text{weakly in } L^2(\mathbb{R}_-^3), \end{cases}$$

as  $n$  diverges. The next step is devoted to identify  $\rho^a$  and  $\rho^b$ . In this aim, now in equation (4.1.10), choose

$$(4.3.10) \quad \begin{cases} m = m_n, \\ \zeta^a = c_n^a + \varphi & \text{where } \varphi \in C_0^\infty(\mathbb{R}_+^3), \\ \zeta^b = c_n^b, \end{cases}$$

with  $c_n^a = -(|B_n^a| + \frac{h_n^b}{h_n^a}|B_n^b|)^{-1} \int_{B_n^a} \varphi dx$ .

After having multiplied this equation by  $h_n^a$ , we obtain

$$\int_{\mathbb{R}_+^3} \left( \frac{1}{h_n^a} D_{x_1}\zeta_n^a, D_{x_2}\zeta_n^a, D_{x_3}\zeta_n^a + \frac{\cos\theta_0}{\sin\theta_0} \left(1 - \frac{1}{h_n^a}\right) D_{x_1}\zeta_n^a \right) \left( D_{x_1}\varphi, h_n^a D_{x_2}\varphi, h_n^a D_{x_3}\varphi + \frac{\cos\theta_0}{\sin\theta_0} (h_n^a - 1) D_{x_1}\varphi \right) dx$$

$$= \int_{\Omega^{a,\theta_0}} \left( D_{x_1}\varphi, h_n^a D_{x_2}\varphi, h_n^a D_{x_3}\varphi + \frac{\cos\theta_0}{\sin\theta_0} (h_n^a - 1) D_{x_1}\varphi \right) m_n^a dx, \quad \forall \varphi \in C_0^\infty(\mathbb{R}_+^3),$$

by passing to the limit as  $n$  diverges and by taking into account (4.3.8) and (4.3.9) we obtain

$$(4.3.11) \quad \int_{\mathbb{R}_+^3} \frac{1}{\sin^2\theta_0} \rho^a D_{x_1}\varphi dx = \int_{\mathbb{R}_+^3} \left( \tilde{\mu}_1^a - \frac{\cos\theta_0}{\sin\theta_0} \tilde{\mu}_3^a \right) D_{x_1}\varphi dx,$$

where  $\tilde{\mu}_1^a$  and  $\tilde{\mu}_3^a$  denotes the zero extension of  $\mu_1^a$  and  $\mu_3^a$  on  $\mathbb{R}_+^3 \setminus \Omega^{a,\theta_0}$  respectively, this proves that for  $(x_2, x_3)$  a.e. in  $\mathbb{R}_+^2$ , the function  $\frac{1}{\sin^2 \theta_0} \rho^a(\cdot, x_2, x_3) - \left( \tilde{\mu}_1^a(\cdot, x_2, x_3) - \frac{\cos \theta_0}{\sin \theta_0} \tilde{\mu}_3^a(\cdot, x_2, x_3) \right)$  is constant in  $\mathbb{R}$ , and the fact that  $\frac{1}{\sin^2 \theta_0} \rho^a(\cdot, x_2, x_3) - \left( \tilde{\mu}_1^a(\cdot, x_2, x_3) - \frac{\cos \theta_0}{\sin \theta_0} \tilde{\mu}_3^a(\cdot, x_2, x_3) \right) \in L^2(\mathbb{R})$ , we concludes

$$\rho^a(\cdot, x_2, x_3) = \sin^2 \theta_0 \tilde{\mu}_1^a(\cdot, x_2, x_3) - \sin \theta_0 \cos \theta_0 \tilde{\mu}_3^a(\cdot, x_2, x_3) \quad \text{a.e. in } \mathbb{R}$$

Then

$$(4.3.12) \quad \rho^a(x) = \begin{cases} \sin^2 \theta_0 \mu_1^a - \sin \theta_0 \cos \theta_0 \mu_3^a & \text{a.e. in } \Omega^{a,\theta_0}, \\ 0 & \text{a.e. in } \mathbb{R}^3 \setminus \Omega^{a,\theta_0}, \end{cases}$$

similarly, now, in equation (4.1.10), choose

$$\begin{cases} m = m_n, \\ \zeta^b = c_n^b + \varphi \quad \text{where } \varphi \in C_0^\infty(\mathbb{R}_-^3), \\ \zeta^a = c_n^b, \end{cases}$$

with  $c_n^b = -(|B_n^a| + \frac{h_n^b}{h_n^a} |B_n^b|)^{-1} \frac{h_n^b}{h_n^a} \int_{B_n^b} \varphi dx$ , we obtain

$$(4.3.13) \quad \rho^b(x) = \begin{cases} \mu_3^b(x_2, x_3) & \text{a.e. in } \Omega^b, \\ 0 & \text{a.e. in } \mathbb{R}^3 \setminus \Omega^b, \end{cases}$$

The last step is devoted to prove that convergences in (4.3.8) and (4.3.9) are strong, and we have convergence (4.3.3). Let  $w_n = D_{x_3} \zeta_n^a + \frac{\cos \theta_0}{\sin \theta_0} (1 - \frac{1}{h_n^a}) D_{x_1} \zeta_n^a$ , from (4.3.8) and (4.3.9) we have

$$w_n \rightharpoonup -\frac{\cos \theta_0}{\sin \theta_0} \rho^a \quad \text{weakly in } L^2(\mathbb{R}_+^3).$$

By passing to the limit in (4.1.16) with  $m = m_n$  and using (4.3.1), (4.3.8) and (4.3.9) we obtain the convergence of the energie:



(4.3.14)

$$\begin{aligned}
\lim_n E_n^{mag}(m_n) &= \lim_n \left[ \frac{1}{2} \int_{\mathbb{R}_+^3} \left| \left( \frac{1}{h_n^a} D_{x_1} \zeta_n^a, D_{x_2} \zeta_n^a, D_{x_3} \zeta_n^a + \frac{\cos \theta_0}{\sin \theta_0} \left(1 - \frac{1}{h_n^a}\right) D_{x_1} \zeta_n^a \right) \right|^2 dx + \right. \\
&\quad \left. \frac{1}{2} \frac{h_n^b}{h_n^a} \int_{\mathbb{R}_-^3} \left| \left( D_{x_1} \zeta_n^b, D_{x_2} \zeta_n^b, \frac{1}{h_n^b} D_{x_3} \zeta_n^b \right) \right|^2 dx \right] = \\
&\quad \lim_n \left[ \frac{1}{2} \int_{\mathbb{R}_+^3} \left( \left| \frac{1}{h_n^a} D_{x_1} \zeta_n^a \right|^2 + |D_{x_2} \zeta_n^a|^2 + |w_n|^2 \right) dx \right. \\
&\quad \left. + \frac{1}{2} \frac{h_n^b}{h_n^a} \int_{\mathbb{R}_-^3} \left| \left( D_{x_1} \zeta_n^b, D_{x_2} \zeta_n^b, \frac{1}{h_n^b} D_{x_3} \zeta_n^b \right) \right|^2 dx \right] = \\
&\quad \lim_n \left[ \frac{1}{2} \int_{\Omega^{a, \theta_0}} \left( \left( \frac{1}{h_n^a} D_{x_1} \zeta_n^a, D_{x_2} \zeta_n^a, D_{x_3} \zeta_n^a + \frac{\cos \theta_0}{\sin \theta_0} \left(1 - \frac{1}{h_n^a}\right) D_{x_1} \zeta_n^a \right) m_n^a \right) dx + \right. \\
&\quad \left. \frac{1}{2} \frac{h_n^b}{h_n^a} \int_{\Omega^b} \left( \left( D_{x_1} \zeta_n^b, D_{x_2} \zeta_n^b, \frac{1}{h_n^b} D_{x_3} \zeta_n^b \right) m_n^b \right) dx \right] = \\
&\quad \frac{1}{2} \left( \int_{\Omega^{a, \theta_0}} |\sin \theta_0 \mu_1^a - \cos \theta_0 \mu_3^a|^2 dx + q \int_{\Omega^b} |\mu_3^b|^2 dx_1 dx_2 \right)
\end{aligned}$$

By multiplying the last lign of (4.3.14) by  $\sin^2 \theta_0 + \cos^2 \theta_0 = 1$ , we obtain

(4.3.15)

$$\begin{aligned}
\lim_n E_n^{mag}(m_n) &= \frac{1}{2} \left( \int_{\Omega^{a, \theta_0}} \left( |\sin^2 \theta_0 \mu_1^a - \cos \theta_0 \sin \theta_0 \mu_3^a|^2 + |\sin \theta_0 \cos \theta_0 \mu_1^a - \cos^2 \theta_0 \mu_3^a|^2 \right) dx \right. \\
&\quad \left. + q \int_{\Omega^b} |\mu_3^b|^2 dx \right) = \frac{1}{2} \left( \int_{\Omega^{a, \theta_0}} \left( |\rho^a|^2 + \left| \frac{\cos \theta_0}{\sin \theta_0} \rho^a \right|^2 \right) dx + q \int_{\Omega^b} |\mu_3^b|^2 dx \right)
\end{aligned}$$

and by combining this result with (4.3.8), (4.3.9), (4.3.12) and (4.3.13), we deduce the convergences (4.3.2).  $\square$

**Proposition 4.3.2.** *Assume (4.1.1), with  $q \in ]0, +\infty[$ . For every  $n \in \mathbb{N}$ , let  $m_n = (m_n^a, m_n^b)$  be a solution of (4.1.15). Then there exist  $c \in ]0, +\infty[$ , such that:*

$$E_n(m_n) \leq c, \quad \forall n \in \mathbb{N},$$

*Proof.* Let  $m = ((0, 1, 0), (0, 1, 0))$  as the test function in (4.1.17), it follows that

$$\begin{aligned} E_n(m_n) &\leq \varphi((0, 1, 0)) + 2\|f_n^a\|_{(L^2(\Omega^{a,\theta_0}))^3} + \frac{h_n^b}{h_n^a} \left( \varphi((0, 1, 0)) + 2\|f_n^b\|_{(L^2(\Omega^b))^3} \right) \\ &\quad + E_n^{mag} \left( ((0, 1, 0), (0, 1, 0)) \right), \forall n \in \mathbb{N} \end{aligned}$$

In the other hand, choosing  $(\zeta^a, \zeta^b)$  the potentiel corresponding to  $((0, 1, 0), (0, 1, 0))$  as test function in (4.1.16) and by using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} 2E_n^{mag}((0, 1, 0), (0, 1, 0)) &\leq \left( \int_{\mathbb{R}_+^3} \left| \left( \frac{1}{h_n^a} D_{x_1} \zeta^a, D_{x_2} \zeta^a, D_{x_3} \zeta^a + \frac{\cos \theta_0}{\sin \theta_0} \left(1 - \frac{1}{h_n^a}\right) D_{x_1} \zeta^a \right) \right|^2 dx \right)^{\frac{1}{2}} \\ &\quad + \frac{h_n^a}{h_n^b} \left( \int_{\mathbb{R}_-^3} \left| \left( D_{x_1} \zeta^b, D_{x_2} \zeta^b, \frac{1}{h_n^b} D_{x_3} \zeta^b \right) \right|^2 dx \right)^{\frac{1}{2}} \\ &\leq \left( 2E_n^{mag}((0, 1, 0), (0, 1, 0)) \right)^{\frac{1}{2}} + \left( \frac{h_n^a}{h_n^b} \right)^{\frac{1}{2}} \left( 2E_n^{mag}((0, 1, 0), (0, 1, 0)) \right)^{\frac{1}{2}}, \forall n \in \mathbb{N}. \end{aligned}$$

Then, there exist a constant  $c \in ]0, +\infty[$ , such that:

$$E_n(m_n) \leq c, \quad \forall n \in \mathbb{N}.$$

□

**Corollary 4.3.3.** *Assume (4.1.1) with  $q \in ]0, +\infty[$  and (4.1.18). For every  $n \in \mathbb{N}$  let  $m_n = (m_n^a, m_n^b)$  is a solution of (4.1.15). Then, there exist  $c \in ]0, +\infty[$ , such that:*

$$\|D_{x_1} m_n^a\|_{(L^2(\Omega^{a,\theta_0}))^3} \leq ch_n^a, \quad \|D_{x_2} m_n^a\|_{(L^2(\Omega^{a,\theta_0}))^3} \leq c, \quad \|D_{x_3} m_n^a\|_{(L^2(\Omega^{a,\theta_0}))^3} \leq c, \quad \forall n \in \mathbb{N},$$

$$\|D_{x_1} m_n^b\|_{(L^2(\Omega^b))^3} \leq c, \quad \|D_{x_2} m_n^b\|_{(L^2(\Omega^b))^3} \leq c, \quad \|D_{x_3} m_n^b\|_{(L^2(\Omega^b))^3} \leq ch_n^b, \quad \forall n \in \mathbb{N}.$$

**Proposition 4.3.4.** *Let  $m_n = (m_n^a, m_n^b)$  is a solution of (4.1.15),  $\mathcal{M}$  defined in (4.2.1), and let  $\xi_{n_i}^{(1)}(x_2, x_3) = \int_{-\frac{1}{2}}^{\frac{1}{2}} m_{n_i}^a(x_1 + \cot \theta_0 x_3, x_2, x_3) dx_1$ . Then, there exist a subsequence  $(m_{n_i})_{i \in \mathbb{N}}$  and  $(\hat{\mu}^a, \hat{\mu}^b, \hat{\xi}^a) \in \mathcal{M} \times \mathcal{F}$ , such that*

$$(4.3.16) \quad \begin{cases} m_{n_i}^a \rightharpoonup \hat{\mu}^a & \text{weakly in } H^1(\Omega^{a,\theta_0}, \mathbb{R}^3), \quad m_{n_i}^b \rightharpoonup \hat{\mu}^b & \text{weakly in } H^1(\Omega^b, \mathbb{R}^3), \\ \frac{1}{h_{n_i}^a} (m_{n_i}^a - \xi_{n_i}^{(1)}) \rightharpoonup \hat{\xi}^a & \text{weakly in } \mathcal{F}, \quad \frac{1}{h_{n_i}^a} D_{x_1} m_{n_i}^a \longrightarrow D_{x_1} \hat{\xi}^a & \text{weakly in } L^2(\Omega^{a,\theta_0}, \mathbb{R}^3), \end{cases}$$

*Proof.* By taking into account  $|m_n| = 1$  a.e. in  $\Omega^{\theta_0}$ , Corollary 4.3.3 and  $q \in ]0, +\infty[$ , there exist a subsequence  $(m_{n_i})_{i \in \mathbb{N}}$ ,  $\hat{\mu}^a \in H^1(\Omega^{a, \theta_0}, S^2)$  is independent of  $x_1$  and  $\hat{\mu}^b \in H^1(\Omega^b, S^2)$  is independent of  $x_3$ , such that

$$(4.3.17) \quad \begin{cases} m_{n_i}^a \rightharpoonup \hat{\mu}^a & \text{weakly in } H^1(\Omega^{a, \theta_0}, \mathbb{R}^3), \\ m_{n_i}^b \rightharpoonup \hat{\mu}^b & \text{weakly in } H^1(\Omega^b, \mathbb{R}^3). \end{cases}$$

To prove the last convergence in (4.3.16), for every  $n \in \mathbb{N}$  we set

$$\xi_n^{(1)}(x_2, x_3) = \int_{\cot \theta_0 x_3 - \frac{1}{2}}^{\cot \theta_0 x_3 + \frac{1}{2}} m_n^a(x_1, x_2, x_3) dx_1,$$

by virtue of the Poincaré-Wirtinger inequality, there exist a constant  $c \in ]0, +\infty[$  such that

$$\begin{aligned} & \left\| \frac{1}{h_n^a} \left( m_n^a(x_1, x_2, x_3) - \xi_n^{(1)}(x_2, x_3) \right) \right\|_{L^2(\cot \theta_0 x_3 - \frac{1}{2}, \cot \theta_0 x_3 + \frac{1}{2}, dx_1)} \leq \\ & c \left\| \frac{1}{h_n^a} D_{x_1} m_n^a(x_1, x_2, x_3) \right\|_{L^2(\cot \theta_0 x_3 - \frac{1}{2}, \cot \theta_0 x_3 + \frac{1}{2}, dx_1)}, \quad \forall n \in \mathbb{N}, \end{aligned}$$

it gives that

$$\begin{aligned} & \left\| \frac{1}{h_n^a} \left( m_n^a(x_1, x_2, x_3) - \xi_n^{(1)}(x_2, x_3) \right) \right\|_{H^1(\cot \theta_0 x_3 - \frac{1}{2}, \cot \theta_0 x_3 + \frac{1}{2}, dx_1)} \leq \\ & (1 + c) \left\| \frac{1}{h_n^a} D_{x_1} m_n^a(x_1, x_2, x_3) \right\|_{L^2(\cot \theta_0 x_3 - \frac{1}{2}, \cot \theta_0 x_3 + \frac{1}{2}, dx_1)}, \quad \forall n \in \mathbb{N}, \end{aligned}$$

for  $(x_2, x_3)$  a.e. in  $] -\frac{1}{2}, \frac{1}{2}[ \times ]0, \sin \theta_0[$ . Thus integrating this inequality over  $(x_2, x_3) \in ] -\frac{1}{2}, \frac{1}{2}[ \times ]0, \sin \theta_0[$ , the first estimate in Corollary 4.3.3 ensures the existence of a subsequence of  $\{n_i\}_{i \in \mathbb{N}}$  and a function  $\hat{\xi}^a \in \mathcal{F}$  such that

$$\frac{1}{h_{n_i}^a} \left( m_{n_i}^a - \xi_{n_i}^{(1)} \right) \rightharpoonup \hat{\xi}^a \quad \text{weakly in } \mathcal{F},$$

consequently,

$$\frac{1}{h_{n_i}^a} D_{x_1} m_{n_i}^a \rightharpoonup D_{x_1} \hat{\xi}^a \quad \text{weakly in } L^2(\Omega^{a, \theta_0}, \mathbb{R}^3).$$

For asserting that  $(\hat{\mu}^a, \hat{\mu}^b) \in \mathcal{M}$ , it remains to prove that

$$(4.3.18) \quad \hat{\mu}^a(x_2, 0) = \hat{\mu}^b(0, x_2) \quad \text{for } x_2 \text{ a.e. in } \left] -\frac{1}{2}, \frac{1}{2} \right[.$$

The proof of (4.3.18) will be performed in several steps. At first remark that

$$(4.3.19) \quad \lim_i \int_{]-\frac{1}{2}, \frac{1}{2}[^2} \left( m_{n_i}^a(x_1, x_2, 0) - \hat{\mu}^a(x_2, 0) \right) \varphi(x_2) dx_1 dx_2 = 0 \quad \forall \varphi \in C_0^\infty \left( \left] -\frac{1}{2}, \frac{1}{2} \right[ \right).$$

Now, for every  $i \in \mathbb{N}$ , set

$$\rho_i(x_3) = \int_{]-\frac{1}{2}, \frac{1}{2}[^2} \left( |D_{x_1} m_{n_i}^b(x_1, x_2, x_3)|^2 + |D_{x_2} m_{n_i}^b(x_1, x_2, x_3)|^2 + |m_{n_i}^b(x_1, x_2, x_3)|^2 \right) dx_1 dx_2$$

From Fatou's Lemma and the fact that  $|m_{n_i}^b| = 1$  a.e. in  $\Omega^b$ , it follows that

$$\int_{]-1, 0[} \liminf_i \rho_i(x_3) dx_3 \leq \liminf_i \int_{]-1, 0[} \rho_i(x_3) dx_3 < +\infty$$

Consequently, there exist two constants  $c \in ]0, +\infty[$  and  $\bar{x}_3 \in ]-1, 0[$ , and a subsequence  $(\rho_{i_k})_{k \in \mathbb{N}}$  such that

$$\rho_{i_k}(\bar{x}_3) < c, \forall k \in \mathbb{N}$$

It results that

$$(4.3.20) \quad m_{n_{i_k}}^b(\cdot, \cdot, \bar{x}_3) \rightharpoonup \hat{\mu}^b \quad \text{weakly in } H^1\left(]-\frac{1}{2}, \frac{1}{2}[^2, \mathbb{R}^3\right),$$

as  $k$  diverges.

The second step is devoted to prove that

$$(4.3.21) \quad \lim_k \int_{]-\frac{1}{2}, \frac{1}{2}[^2} \left( m_{n_{i_k}}^b(h_{n_{i_k}}^a x_1, x_2, 0) - \hat{\mu}^b(0, x_2) \right) \varphi(x_2) dx_1 dx_2 = 0, \quad \forall \varphi \in C_0^\infty\left(]-\frac{1}{2}, \frac{1}{2}[ \right).$$

Moreover, for  $\varphi \in C_0^\infty\left(]-\frac{1}{2}, \frac{1}{2}[ \right)$  we have

$$(4.3.22) \quad \begin{aligned} & \int_{]-\frac{1}{2}, \frac{1}{2}[^2} \left( m_{n_{i_k}}^b(h_{n_{i_k}}^a x_1, x_2, 0) - \hat{\mu}^b(0, x_2) \right) \varphi(x_2) dx_1 dx_2 = \\ & \int_{]-\frac{1}{2}, \frac{1}{2}[^2} \left( m_{n_{i_k}}^b(h_{n_{i_k}}^a x_1, x_2, 0) - m_{n_{i_k}}^b(h_{n_{i_k}}^a x_1, x_2, \bar{x}_3) \right) \varphi(x_2) dx_1 dx_2 + \\ & \int_{]-\frac{1}{2}, \frac{1}{2}[^2} \left( m_{n_{i_k}}^b(h_{n_{i_k}}^a x_1, x_2, \bar{x}_3) - m_{n_{i_k}}^b(0, x_2, \bar{x}_3) \right) \varphi(x_2) dx_1 dx_2 + \\ & \int_{]-\frac{1}{2}, \frac{1}{2}[^2} \left( m_{n_{i_k}}^b(0, x_2, \bar{x}_3) - \hat{\mu}^b(0, x_2) \right) \varphi(x_2) dx_1 dx_2 \quad \forall k \in \mathbb{N} \end{aligned}$$

and we will pass to the limit, as  $k$  diverges, in each term of this decomposition,

there exist  $c \in ]0, +\infty[$  such that

$$\begin{aligned}
(4.3.23) \quad & \limsup_k \left| \int_{]-\frac{1}{2}, \frac{1}{2}[^2} \left( m_{n_{i_k}}^b(h_{n_{i_k}}^a x_1, x_2, 0) - m_{n_{i_k}}^b(h_{n_{i_k}}^a x_1, x_2, \bar{x}_3) \right) \varphi(x_2) dx_1 dx_2 \right| = \\
& \limsup_k \left| \int_{]-\frac{1}{2}, \frac{1}{2}[^2} \left( \int_0^{\bar{x}_3} D_t m_{n_{i_k}}^b(h_{n_{i_k}}^a x_1, x_2, t) dt \right) \varphi(x_2) dx_1 dx_2 \right| \leq \\
& \|\varphi\|_{L^\infty(]-\frac{1}{2}, \frac{1}{2}[)} |\Omega^b|^{\frac{1}{2}} \limsup_k \left( \int_{\Omega^b} |D_{x_3} m_{n_{i_k}}^b(h_{n_{i_k}}^a x_1, x_2, x_3)|^2 dx \right)^{\frac{1}{2}} \leq \\
& \|\varphi\|_{L^\infty(]-\frac{1}{2}, \frac{1}{2}[)} |\Omega^b|^{\frac{1}{2}} \limsup_k \left( \frac{1}{h_{n_{i_k}}^a} \|D_{x_3} m_{n_{i_k}}^b\|_{L^2(\Omega^b)} \right)^{\frac{1}{2}} \leq \\
& c \|\varphi\|_{L^\infty(]-\frac{1}{2}, \frac{1}{2}[)} |\Omega^b|^{\frac{1}{2}} \limsup_k \frac{h_{n_{i_k}}^b}{\sqrt{h_{n_{i_k}}^a}} = 0
\end{aligned}$$

In the other hand, there exist  $c \in ]0, +\infty[$  such that

$$\begin{aligned}
(4.3.24) \quad & \limsup_k \left| \int_{]-\frac{1}{2}, \frac{1}{2}[^2} \left( m_{n_{i_k}}^b(h_{n_{i_k}}^a x_1, x_2, \bar{x}_3) - m_{n_{i_k}}^b(0, x_2, \bar{x}_3) \right) \varphi(x_2) dx_1 dx_2 \right| = \\
& \limsup_k \left| \int_{]-\frac{1}{2}, \frac{1}{2}[^2} \left( \int_0^{h_{n_{i_k}}^a} D_t m_{n_{i_k}}^b(t, x_2, \bar{x}_3) dt \right) \varphi(x_2) dx_1 dx_2 \right| \leq \\
& \frac{1}{2} \|\varphi\|_{L^\infty(]-\frac{1}{2}, \frac{1}{2}[)} \limsup_k \int_{]-\frac{1}{2}, \frac{1}{2}[} \left( \int_0^{\frac{h_{n_{i_k}}^a}{2}} |D_t m_{n_{i_k}}^b(t, x_2, \bar{x}_3)| dt \right) dx_2 + \\
& \frac{1}{2} \|\varphi\|_{L^\infty(]-\frac{1}{2}, \frac{1}{2}[)} \limsup_k \int_{]-\frac{1}{2}, \frac{1}{2}[} \left( \int_{-\frac{h_{n_{i_k}}^a}{2}}^0 |D_t m_{n_{i_k}}^b(t, x_2, \bar{x}_3)| dt \right) dx_2 \leq \\
& \frac{1}{2} \|\varphi\|_{L^\infty(]-\frac{1}{2}, \frac{1}{2}[)} \limsup_k \left( h_{n_{i_k}}^a \int_{]-\frac{1}{2}, \frac{1}{2}[^2} |D_{x_1} m_{n_{i_k}}^b(x_1, x_2, \bar{x}_3)|^2 dx_1 dx_2 \right)^{\frac{1}{2}} \leq \\
& \frac{c}{2} \|\varphi\|_{L^\infty(]-\frac{1}{2}, \frac{1}{2}[)} \lim_k \sqrt{h_{n_{i_k}}^a} = 0
\end{aligned}$$

From (4.3.20), it follows that

$$\begin{aligned}
(4.3.25) \quad & \lim_k \int_{]-\frac{1}{2}, \frac{1}{2}[^2} \left( m_{n_{i_k}}^b(0, x_2, \bar{x}_3) - \hat{\mu}^b(0, x_2) \right) \varphi(x_2) dx_1 dx_2 = \\
& \lim_k \int_{]-\frac{1}{2}, \frac{1}{2}[} \left( m_{n_{i_k}}^b(0, x_2, \bar{x}_3) - \hat{\mu}^b(0, x_2) \right) \varphi(x_2) dx_2 = 0
\end{aligned}$$

Then, by passing to the limit in (4.3.22), as  $k$  diverges, and by taking into account (4.3.23)-(4.3.25) we obtain (4.3.21). Finally, the junction condition (4.3.18), is obtained by passing to the limit, as  $k$  diverges in

$$\begin{aligned}
(4.3.26) \quad & \int_{]-\frac{1}{2}, \frac{1}{2}[^2} m_{n_{i_k}}^a(x_1, x_2, 0) \varphi(x_2) dx_1 dx_2 = \int_{]-\frac{1}{2}, \frac{1}{2}[^2} m_{n_{i_k}}^b(h_{n_{i_k}}^a x_1, x_2, 0) \varphi(x_2) dx_1 dx_2 \\
& \forall k \in \mathbb{N}, \forall \varphi \in C_0^\infty\left(]-\frac{1}{2}, \frac{1}{2}[ \right),
\end{aligned}$$

and by using (4.3.19) and (4.3.21).  $\square$

## 4.4 A density result

Let

(4.4.1)

$$\mathcal{M}_{reg} = \left\{ \mu = (\mu^a, \mu^b) \in C^1(\overline{\Omega^{a,\theta_0}}, S^2) \times C(\overline{\Omega^b}, S^2) : \right.$$

$\mu^a$  is independent of  $x_1$ ,  $\mu^b$  is independent of  $x_3$ ,

$$\mu^b /_{[-\frac{1}{2}, 0] \times [-\frac{1}{2}, \frac{1}{2}] \times [-1, 0]} \in C^1([-\frac{1}{2}, 0] \times [-\frac{1}{2}, \frac{1}{2}] \times [-1, 0], S^2),$$

$$\mu^b /_{[0, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}] \times [-1, 0]} \in C^1([0, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}] \times [-1, 0], S^2),$$

$$\left. \mu^a(x_2, 0) = \mu^b(0, x_2) \text{ for } x_2 \in ] -\frac{1}{2}, \frac{1}{2}[ \right\}$$

$$\simeq \left\{ \mu = (\mu^a, \mu^b) \in C^1([-\frac{1}{2}, \frac{1}{2}] \times [0, \sin \theta_0], S^2) \times C([-\frac{1}{2}, \frac{1}{2}]^2, S^2) : \right.$$

$\mu^a$  is independent of  $x_1$ ,  $\mu^b$  is independent of  $x_3$ ,

$$\mu^b /_{[-\frac{1}{2}, 0] \times [-\frac{1}{2}, \frac{1}{2}]} \in C^1([-\frac{1}{2}, 0] \times [-\frac{1}{2}, \frac{1}{2}], S^2), \mu^b /_{[0, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]} \in C^1([0, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}], S^2),$$

$$\left. \mu^a(x_2, 0) = \mu^b(0, x_2) \text{ for } x_2 \in ] -\frac{1}{2}, \frac{1}{2}[ \right\}$$

Remark that,  $\mathcal{M}_{reg} \subset W^{1,\infty}(\Omega^{a,\theta_0}, S^2) \times W^{1,\infty}(\Omega^b, S^2)$ . The proof is based on the combination of an approximation result proved in [7] with splitting techniques introduced in [51] and a projection technique from  $\mathbb{R}^3$  into  $S^2$  as in [44].

**Proposition 4.4.1.** *Let  $\mathcal{M}_{reg}$  and  $\mathcal{M}$  be the space defined in (4.4.1) and (4.2.1), respectively. Then  $\mathcal{M}_{reg}$  is dense in  $\mathcal{M}$ .*

*Proof.* this proof is very similar to the proof of proposition 5.5 in [33]. So omit it.  $\square$

### 4.4.1 Proof of theorem 4.2.1

*Proof.* From Proposition 4.3.4, there exist of subsequence  $\{m_{n_i}\}_{i \in \mathbb{N}}$  of  $m_n$ ,  $(\hat{\mu}^a, \hat{\mu}^b, \hat{\xi}^a) \in \mathcal{M} \times \mathcal{F}$  and from Corollary 4.3.3, there exist  $\eta^b \in L^2(\Omega^b, \mathbb{R}^3)$

such that

$$(4.4.2) \quad \begin{cases} m_{n_i}^a \rightharpoonup \hat{\mu}^a & \text{weakly in } H^1(\Omega^{a,\theta_0}, \mathbb{R}^3), \quad m_{n_i}^b \rightharpoonup \hat{\mu}^b & \text{weakly in } H^1(\Omega^b, \mathbb{R}^3), \\ \frac{1}{h_{n_i}^a}(m_{n_i}^a - \xi_{n_i}^{(1)}) \rightharpoonup \hat{\xi}^a & \text{weakly in } \mathcal{F}, \\ \frac{1}{h_{n_i}^a}D_{x_1}m_{n_i}^a \rightharpoonup D_{x_1}\hat{\xi}^a & \text{weakly in } L^2(\Omega^{a,\theta_0}, \mathbb{R}^3), \quad \frac{1}{h_{n_i}^b}D_{x_3}m_{n_i}^b \rightharpoonup \eta^b & \text{weakly in } L^2(\Omega^b, \mathbb{R}^3), \end{cases}$$

as  $i$  diverges. Consequently, by virtue of Proposition 4.3.1 and convergences (4.2.5) hold true, it results that

$$(4.4.3) \quad \lim_i E_{n_i}^{mag}(m_{n_i}) = \frac{1}{2} \left( \int_{\Omega^{a,\theta_0}} |\sin \theta_0 \hat{\mu}_1^a - \cos \theta_0 \hat{\mu}_3^a|^2 dx + q \int_{] -\frac{1}{2}, \frac{1}{2} [^2} |\mu_3^b|^2 dx_1 dx_2 \right)$$

Let  $(\mu^a, \mu^b, \xi^a) \in \mathcal{M}_{reg} \times C^\infty(\overline{\Omega^{a,\theta_0}})$ . We shall build a sequence  $\{g_n\}_{n \in \mathbb{N}} \subset H^1(\Omega^{a,\theta_0}, \mathbb{R}^3)$ , such that

$$(4.4.4) \quad \begin{cases} g_n \rightharpoonup \mu^a & \text{weakly in } H^1(\Omega^{a,\theta_0}, \mathbb{R}^3), \\ \left( \frac{1}{h_n^a}D_{x_1}g_n, D_{x_2}g_n, D_{x_3}g_n + \frac{\cos \theta_0}{\sin \theta_0} \left(1 - \frac{1}{h_n^a}\right)D_{x_1}g_n \right) \longrightarrow \\ \quad (D_{x_1}\xi^a, D_{x_2}\mu^a, D_{x_3}\mu^a - \cot \theta_0 D_{x_1}\xi^a) & \text{strongly in } L^2(\Omega^{a,\theta_0}, \mathbb{R}^3), \\ g_n(x_1, x_2, 0) = \mu^b(h_n^a x_1, x_2), & \text{for } (x_1, x_2) \in \left] -\frac{1}{2}, \frac{1}{2} \right[ ^2, \end{cases}$$

Let  $W_{n,1}^a = \left\{ x \in \mathbb{R}^3 : (x_2, x_3) \in \left] -\frac{1}{2}, \frac{1}{2} \right[ \times ] h_n^a \sin \theta_0, \sin \theta_0 [ , |x_1 - \cot \theta_0 x_3| < \frac{1}{2} \right\}$

and  $W_{n,2}^a = \left\{ x \in \mathbb{R}^3 : (x_2, x_3) \in \left] -\frac{1}{2}, \frac{1}{2} \right[ \times [0, h_n^a \sin \theta_0], |x_1 - \cot \theta_0 x_3| < \frac{1}{2} \right\}$ ,

where  $\Omega^{a,\theta_0} = W_{n,1}^a \cup W_{n,2}^a$ . To this aim, for every  $n \in \mathbb{N}$ , since for  $h_n^a$  small  $\|\mu^a + h_n^a \xi^a\|_\infty \neq 0$ , we set

$$(4.4.5) \quad g_n(x) = \begin{cases} \frac{\mu^a(x_2, x_3) + h_n^a \xi^a(x_1, x_2, x_3)}{\|\mu^a + h_n^a \xi^a\|_\infty}, & \text{if } x \in W_{n,1}^a, \\ \left( \frac{\mu^a(x_2, h_n^a \sin \theta_0) + h_n^a \xi^a(x_1, x_2, h_n^a \sin \theta_0)}{\|\mu^a + h_n^a \xi^a\|_\infty} \right) \frac{x_3}{h_n^a \sin \theta_0} + \mu^b(h_n^a x_1, x_2) \frac{h_n^a \sin \theta_0 - x_3}{h_n^a \sin \theta_0}, & \text{if } x \in W_{n,2}^a. \end{cases}$$



Remark that  $\{g_n\}_{n \in \mathbb{N}} \subset H^1(\Omega^{a, \theta_0}, \mathbb{R}^3)$ , and the last line of (4.4.4) is verified. Moreover, by using the properties of  $\mathcal{M}_{reg}$  and there exist  $c \in ]0, +\infty[$ , we have

$$\begin{aligned} \lim_n \int_{W_{n,2}^a} \left| \frac{1}{h_n^a} D_{x_1} g_n(x) \right|^2 dx &= \lim_n \int_{W_{n,2}^a} \left| D_{x_1} \mu^b(h_n^a x_1, x_2) \frac{h_n^a \sin \theta_0 - x_3}{h_n^a \sin \theta_0} + \frac{D_{x_1} \xi^a(x_1, x_2, h_n^a \sin \theta_0)}{\|\mu^a + h_n^a \xi^a\|_\infty} \frac{x_3}{h_n^a \sin \theta_0} \right|^2 dx \\ &\leq 2 \lim_n \int_{W_{n,2}^a} \left| D_{x_1} \mu^b(h_n^a x_1, x_2) \frac{h_n^a \sin \theta_0 - x_3}{h_n^a \sin \theta_0} \right|^2 dx \\ &\quad + 2 \lim_n \int_0^{h_n^a \sin \theta_0} \int_{\cot \theta_0 x_3 - \frac{1}{2}}^{\cot \theta_0 x_3 + \frac{1}{2}} \int_{\frac{1}{2}}^{\frac{1}{2}} \left| \frac{D_{x_1} \xi^a(x_1, x_2, h_n^a \sin \theta_0)}{\|\mu^a + h_n^a \xi^a\|_\infty} \frac{x_3}{h_n^a \sin \theta_0} \right|^2 dx \\ &\leq c \left( \|\mu^b\|_{W^{1,\infty}(\] - \frac{1}{2}, \frac{1}{2} [^2, S^2))}^2 \lim_n h_n^a + \|\xi^a\|_{W^{1,\infty}(\Omega^{a, \theta_0}, \mathbb{R}^3)}^2 \lim_n \frac{h_n^a}{\|\mu^a + h_n^a \xi^a\|_\infty^2} \right) = 0 \end{aligned}$$

$$\begin{aligned} \lim_n \int_{W_{n,2}^a} \left| D_{x_2} g_n(x) \right|^2 dx &= \lim_n \int_{W_{n,2}^a} \left| \frac{D_{x_2} \mu^a(x_2, h_n^a \sin \theta_0)}{\|\mu^a + h_n^a \xi^a\|_\infty} \frac{x_3}{h_n^a \sin \theta_0} + \frac{D_{x_2} \xi^a(x_1, x_2, h_n^a \sin \theta_0)}{\|\mu^a + h_n^a \xi^a\|_\infty} \frac{x_3}{\sin \theta_0} + \right. \\ &\quad \left. D_{x_2} \mu^b(h_n^a x_1, x_2) \frac{h_n^a \sin \theta_0 - x_3}{h_n^a \sin \theta_0} \right|^2 dx \\ &\leq c \left( \left( \|\mu^a\|_{W^{1,\infty}(\Omega^{a, \theta_0}, S^2)}^2 + \|\mu^b\|_{W^{1,\infty}(\] - \frac{1}{2}, \frac{1}{2} [^2, S^2))}^2 \right) \lim_n \frac{h_n^a}{\|\mu^a + h_n^a \xi^a\|_\infty^2} + \|\xi^a\|_{W^{1,\infty}(\Omega^{a, \theta_0}, \mathbb{R}^3)}^2 \lim_n h_n^a \right) = 0 \end{aligned}$$

and

$$\begin{aligned} \lim_n \int_{W_{n,2}^a} \left| D_{x_3} g_n(x) \right|^2 dx &= \lim_n \int_{W_{n,2}^a} \frac{1}{h_n^a \sin^2 \theta_0} \left| \frac{\mu^a(x_2, h_n^a \sin \theta_0) - \mu^a(x_2, 0)}{\|\mu^a + h_n^a \xi^a\|_\infty} + \frac{h_n^a \xi^a(x_1, x_2, h_n^a \sin \theta_0)}{\|\mu^a + h_n^a \xi^a\|_\infty} + \mu^b(0, x_2) - \mu^b(h_n^a x_1, x_2) \right|^2 dx \\ &\leq c \left( \left( \|\mu^a\|_{W^{1,\infty}(\Omega^{a, \theta_0}, S^2)}^2 + \|\xi^a\|_{L^\infty(\Omega^{a, \theta_0}, \mathbb{R}^3)}^2 \right) \lim_n \frac{h_n^a}{\|\mu^a + h_n^a \xi^a\|_\infty^2} + \|\mu^b\|_{W^{1,\infty}(\] - \frac{1}{2}, \frac{1}{2} [^2, S^2)}^2 \lim_n h_n^a \right) = 0, \end{aligned}$$

that is

$$(4.4.6) \quad \lim_n \left\| \left( \frac{1}{h_n^a} D_{x_1} g_n, D_{x_2} g_n, D_{x_3} g_n + \frac{\cos \theta_0}{\sin \theta_0} \left( 1 - \frac{1}{h_n^a} \right) D_{x_1} g_n \right) \right\|_{L^2(W_{n,2}^a, \mathbb{R}^9)} = 0,$$

In the other hand,

$$\lim_n \int_{W_{n,1}^a} \left| \frac{1}{h_n^a} D_{x_1} g_n(x) - D_{x_1} \xi^a(x_1, x_2, x_3) \right|^2 dx = \lim_n \int_{W_{n,1}^a} \left| \frac{D_{x_1} \xi^a(x_1, x_2, x_3)}{\|\mu^a + h_n^a \xi^a\|_\infty} - D_{x_1} \xi^a(x_1, x_2, x_3) \right|^2 dx = 0$$

$$\begin{aligned} & \lim_n \int_{W_{n,1}^a} \left| D_{x_2} g_n(x) - D_{x_2} \mu^a(x_2, x_3) \right|^2 dx = \lim_n \int_{W_{n,1}^a} \left| \frac{D_{x_2} \mu^a(x_2, x_3)}{\|\mu^a + h_n^a \xi^a\|_\infty} + h_n^a \frac{D_{x_2} \xi^a(x_1, x_2, x_3)}{\|\mu^a + h_n^a \xi^a\|_\infty} - D_{x_2} \mu^a(x_2, x_3) \right|^2 dx \\ & \leq 2 \lim_n \int_{W_{n,1}^a} \frac{(h_n^a)^2}{\|\mu^a + h_n^a \xi^a\|_\infty^2} \left| D_{x_2} \xi^a(x_1, x_2, x_3) \right|^2 dx + 2 \lim_n \int_{W_{n,1}^a} \left| \frac{D_{x_2} \mu^a(x_2, x_3)}{\|\mu^a + h_n^a \xi^a\|_\infty} - D_{x_2} \mu^a(x_2, x_3) \right|^2 dx = 0 \end{aligned}$$

and

$$\begin{aligned} & \lim_n \int_{W_{n,1}^a} \left| D_{x_3} g_n(x) - D_{x_3} \mu^a(x_2, x_3) \right|^2 dx = \lim_n \int_{W_{n,1}^a} \left| \frac{D_{x_3} \mu^a(x_2, x_3)}{\|\mu^a + h_n^a \xi^a\|_\infty} + h_n^a \frac{D_{x_3} \xi^a(x_1, x_2, x_3)}{\|\mu^a + h_n^a \xi^a\|_\infty} - D_{x_3} \mu^a(x_2, x_3) \right|^2 dx \\ & \leq 2 \lim_n \int_{W_{n,1}^a} \frac{(h_n^a)^2}{\|\mu^a + h_n^a \xi^a\|_\infty^2} \left| D_{x_3} \xi^a(x_1, x_2, x_3) \right|^2 dx + 2 \lim_n \int_{W_{n,1}^a} \left| \frac{D_{x_3} \mu^a(x_2, x_3)}{\|\mu^a + h_n^a \xi^a\|_\infty} - D_{x_3} \mu^a(x_2, x_3) \right|^2 dx = 0 \end{aligned}$$

Consequently, convergence (4.4.4) holds true. Remark that,  $|g_n(x)| \leq 1$  in  $\Omega^{a,\theta_0}$ . To normalize  $g_n$ , it suffices to project on  $S^2$ . To this aim, let  $y \in B_{\frac{1}{2}}(0) = \{y \in \mathbb{R}^3 : |y| \leq \frac{1}{2}\}$ , project  $x \in \mathbb{R}^3 \setminus \{y\}$  on  $S^2$  by the intersection of  $S^2$  with the half line  $y + t(x - y) : t \in ]0, +\infty[$ , introduce the projection

$$\pi_y : x \in \mathbb{R}^3 \setminus \{y\} \mapsto y + \frac{-y(x - y) + \sqrt{(y(x - y))^2 + |x - y|^2(1 - |y|^2)}}{|x - y|^2} (x - y) \in S^2$$

It is easy to see that

$$\pi_y(x) = x, \quad \forall x \in S^2$$

Moreover, it results that (see. [44])

$$(4.4.7) \quad \exists c \in ]0, +\infty[ : |D\pi_y(x)|^2 \leq \frac{c}{|x - y|^2}, \quad \forall y \in B_{\frac{1}{2}}(0), \quad \forall x \in \mathbb{R}^3 \setminus \{y\}$$

with  $c$  independent on  $y \in B_{\frac{1}{2}}(0) \setminus G$ , let

$$G_n = \left\{ y \in B_{\frac{1}{2}}(0) : \exists x \in W_{n,2}^a, \text{ with } g_n(x) = y, \text{ rank}(Dg_n(x)) < 3 \right\},$$

and set

$$G = \cup_{n \in \mathbb{N}} G_n.$$

By Sard's lemma,  $meas(G) = 0$ . Consequently, for every  $n \in \mathbb{N}$ , and for every  $y \in B_{\frac{1}{2}}(0) \setminus G$ , the function  $\pi_y \circ g_n$  is well defined and, by virtue of (4.4.7) there exist a constant  $c > 0$  such that

$$\begin{aligned} & \int_{B_{\frac{1}{2}}(0) \setminus G} \left( \int_{W_2^a} |D(\pi_y \circ g_n(x))|^2 dx \right) dy = \int_{B_{\frac{1}{2}}(0) \setminus G} \int_{W_2^a} |D\pi_y(g_n(x))|^2 |Dg_n(x)|^2 dx dy \\ & \leq \int_{B_{\frac{1}{2}}(0) \setminus G} c \left( \int_{W_2^a} \frac{|Dg_n(x)|^2}{|g_n(x) - y|^2} dx \right) dy \\ & \leq c \int_{B_{\frac{3}{2}}(0)} \frac{1}{|z|^2} dz \left( \int_{W_2^a} |Dg_n(x)|^2 dx \right), \quad \forall n \in \mathbb{N}, \end{aligned}$$

Consequently by using the Fatou Lemma and by taking into account (4.4.6) and the fact that  $\int_{B_{\frac{3}{2}}(0)} \frac{1}{|z|^2} dz < \infty$ , there exist a subsequence of  $\{n\}$ , still denoted by  $\{n\}$ , and  $\bar{y} \in B_{\frac{1}{2}}(0) \setminus G$  such that

$$(4.4.8) \quad \lim_n \|D(\pi_{\bar{y}} \circ g_n(x))\|_{L^2(W_2^a, \mathbb{R}^3)} = 0.$$

Consequently, we have  $\{(v_n, \mu^b)\}_{n \in \mathbb{N}} = \{(\pi_{\bar{y}} \circ g_n, \mu^b)\}_{n \in \mathbb{N}} \subset \mathcal{M}_{reg}$ , Remark that the first line in (4.4.4) and Proposition 4.3.1 provide that

$$(4.4.9) \quad \lim_n E_n^{mag}(v_n, \mu^b) = \frac{1}{2} \left( \int_{\Omega^{a, \theta_0}} |\sin \theta_0 \mu_1^a - \cos \theta_0 \mu_3^a|^2 dx + q \int_{\Omega^b} |\mu_3^b|^2 dx_1 dx_2 \right)$$

Finally, choosing  $(v_n, \mu^b)$  as test function in (4.1.15) and taking into account (4.1.17), one has that

$$(4.4.10) \quad \begin{aligned} & \lim_n \left( \int_{\Omega^{a, \theta_0}} \left( \alpha \left| \left( \frac{1}{h_n^a} D_{x_1} v_n, D_{x_2} v_n, D_{x_3} v_n + \frac{\cos \theta_0}{\sin \theta_0} \left( 1 - \frac{1}{h_n^a} \right) D_{x_1} v_n \right) \right|^2 + \varphi(v_n) - 2f_n^a v_n \right) dx \right. \\ & \left. + \frac{h_n^b}{h_n^a} \int_{\Omega^b} \left( \alpha |(D_{x_1} \mu^b, D_{x_2} \mu^b, 0)|^2 + \varphi(\mu^b) - 2f_n^b \mu^b \right) dx + E_n^{mag}(v_n, \mu^b) \right) = \\ & \int_{\Omega^{a, \theta_0}} \left( \alpha |(D_{x_1} \xi^a, D_{x_2} \mu^a, D_{x_3} \mu^a - \cot \theta_0 D_{x_1} \xi^a)|^2 + \varphi(\mu^a) + \frac{1}{2} |\sin \theta_0 \mu_1^a - \cos \theta_0 \mu_3^a|^2 \right) dx \\ & - 2 \int_{\Omega^{a, \theta_0}} f^a(x_1, x_2, x_3) \mu^a dx - 2q \int_{\Omega^b} f^b(x_1, x_2, x_3) \mu^b dx + \\ & q \int_{] -\frac{1}{2}, \frac{1}{2} [^2} \left( \alpha |(D_{x_1} \mu^b, D_{x_2} \mu^b)|^2 + \varphi(\mu^b) + \frac{1}{2} |\mu_3^b|^2 \right) dx_1 dx_2 = E_q(\mu^a, \mu^b), \end{aligned}$$

The last step is devoted to prove that convergences in (4.4.2) are strong. To this aim, by using a l.s.c. argument, with  $q \in ]0, +\infty[$  and from (4.4.2), it follows that

$$(4.4.11) \quad q \int_{\Omega^b} \alpha |\eta^b|^2 + E_q(\hat{\mu}^a, \hat{\mu}^b, \hat{\xi}^a) \leq \liminf_i E_{n_i}(m_{n_i}) \leq \limsup_i E_{n_i}(m_{n_i}) \leq \limsup_i E_{n_i}(v_{n_i}, \mu^b) = E_q(\mu^a, \mu^b, \xi^a)$$

Since  $(\mu^a, \mu^b, \xi^a)$  can be any arbitrarily element of  $\mathcal{M}_{reg} \times C^\infty(\overline{\Omega^{a, \theta_0}})$ , by virtue of the density result stated in Proposition 4.4.1 and the fact that  $C^\infty(\overline{\Omega^{a, \theta_0}})$  dense in  $\mathcal{F}$ , this inequality also holds true with  $(\mu^a, \mu^b, \xi^a) = (\hat{\mu}^a, \hat{\mu}^b, \hat{\xi}^a)$ . Consequently, it results that  $\eta^b = 0$ , and by combining this results and (4.4.2), we

concludes the convergences of (4.2.3), (4.2.4) and (4.2.7). Moreover (4.4.2) and (4.4.11) provide that  $\hat{\mu}$  is a solution of (4.2.6).  $\square$

**Remark 4.4.2.** *In the other cases, the structure behaves like a single thin film. Precisely, when  $q = 0$  (i.e.  $h_n^b \ll h_n^a$ ), the limit problem reduces to a 2D problem in the vertical thin film losing the junction condition. Analogously, when  $q = +\infty$  (i.e.  $h_n^b \gg h_n^a$ ), the limit problem reduces to a 2D problem in the horizontal thin film. For shorting the paper, we omit the proofs of these results.*



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