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Wilfredo Salazar

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Normandie Université

THESE

Pour obtenir le diplôme de doctorat

Spécialité Mathématiques

Préparée au sein de l'Institut National des Sciences Appliquées de Rouen

**Contribution aux équations aux dérivées partielles non linéaires et non locales
et application au trafic routier**

Présentée et soutenue par
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Thèse soutenue publiquement le 7 octobre 2016
devant le jury composé de

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Résumé : Cette thèse porte sur la modélisation, l'analyse et l'analyse numérique des équations aux dérivées partielles non-linéaires et non-locales avec des applications au trafic routier. Le trafic routier peut être modélisé à des différentes échelles. En particulier, on peut considérer l'échelle microscopique qui décrit la dynamique de chaque véhicule individuellement et l'échelle macroscopique qui voit le trafic comme un fluide et qui décrit le trafic en utilisant des quantités macroscopiques comme la densité des véhicules et la vitesse moyenne. Dans cette thèse, en utilisant la théorie des solutions de viscosité, on fait le passage entre les modèles microscopiques et les modèles macroscopiques. L'intérêt de ce passage est que les modèles microscopiques sont plus intuitifs et faciles à manipuler pour simuler des situations particulières (bifurcations, feux tricolores,...) mais ils ne sont pas adaptés à des grosses simulations (pour simuler le trafic dans toute une ville par exemple). Au contraire, les modèles macroscopiques sont moins évidents à modifier (pour simuler une situation particulière) mais ils peuvent être utilisés pour des simulations à grande échelle. L'idée est donc de trouver le modèle macroscopique équivalent à un modèle microscopique qui décrit un scénario précis (une jonction, une bifurcation, des différents types de conducteurs, une zone scolaire,...). La première partie de cette thèse contient un résultat d'homogénéisation et d'homogénéisation numérique pour un modèle microscopique avec différents types de conducteurs. Dans une seconde partie, on obtient des résultats d'homogénéisation et d'homogénéisation numérique pour des modèles microscopiques contenant une perturbation locale (ralentisseur, zone scolaire,...). Finalement, on présente un résultat d'homogénéisation dans le cadre d'une bifurcation.

Abstract: This work deals with the modelling, analysis and numerical analysis of non-linear and non-local partial differential equations and their application to traffic flow. Traffic can be simulated at different scales. Mainly, we have the microscopic scale which describes the dynamics of each of the vehicles individually and the macroscopic scale which describes the traffic as a fluid using macroscopic quantities such as the density of vehicles and the average speed. In this PhD thesis, using the theory of viscosity solutions, we derive macroscopic models from microscopic models. The interest of these results is that microscopic models are very intuitive and easy to manipulate to describe a particular situation (bifurcation, a traffic light,...), however, they are not adapted for big simulations (to simulate the traffic in an entire city for example). Conversely, macroscopic models are less easy to modify (to simulate a particular situation) but they can be used for big simulations. The idea is then to find the macroscopic model equivalent to a microscopic model describing a particular scenario (a junction, a bifurcation, different types of drivers, a school zone,...). The first part of this work contains an homogenization result and a numerical homogenization result for a microscopic model with different types of drivers. The second part contains an homogenization and numerical homogenization result for microscopic models with a local perturbation (a moderator, a school zone,...). Finally, we present an homogenization result for a bifurcation.

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- *Numerical homogenization of a second order discrete model for traffic flow*. Computers & Mathematics with Applications, 71(1):29–45, 2016.

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- (avec N. Forcadel et M. Zaydan) *Homogenization of second order discrete model with local perturbation and application to traffic flow*. <hal-01311363>, 2016.
- *Numerical specified homogenization of a discrete model with a local perturbation and application to traffic flow*. <hal-01302943>, 2016.
- (avec N. Forcadel) *Homogenization of a discrete model for a bifurcation and application to traffic flow*. <hal-01332787>, 2015.

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Introduction générale

Cette thèse porte sur l'analyse et l'analyse numérique d'équations aux dérivées partielles non-locales avec des applications en particulier en trafic routier. La modélisation du trafic routier est particulièrement importante et permet de simuler comment le trafic réagirait à un changement dans les infrastructures ou encore à optimiser le flux du trafic. En effet, il existe des exemples où un changement dans l'infrastructure des routes n'a pas contribué à l'amélioration du trafic. Par exemple en Allemagne à Stuttgart en 1969, après un investissement dans un nouveau réseau routier la situation du trafic routier ne s'est pas améliorée jusqu'à la fermeture d'une des nouvelles routes (voir [Knö69]). Ceci est connu comme le paradoxe de Braess. Durant les dernières années, beaucoup de travaux concernant la modélisation et simulation du trafic routier ont donc été réalisés.

Le trafic peut être modélisé à des différentes échelles : l'échelle microscopique (qui décrit la dynamique de chaque véhicule), l'échelle macroscopique (qui décrit des quantités macroscopiques comme la densité des véhicules, la vitesse moyenne,...) ou encore l'échelle mesoscopique (qui utilise la densité des véhicules et la vitesse moyenne des véhicules mais qui a également accès aux dynamiques de tous les véhicules). Simuler le trafic à chacune de ces échelles a ses propres avantages et désavantages. Dans ce travail on se concentre sur les échelles microscopiques et macroscopiques que nous détaillons maintenant.

Les modèles microscopiques sont très précis et intuitifs puisqu'ils décrivent comment chaque voiture réagit à une situation donnée. Par exemple, c'est facile de modéliser comment un véhicule réagit à la présence d'un ralentisseur, d'un feu ou de tout autre phénomène microscopique. On peut également prendre en compte le fait que tous les conducteurs n'ont pas le même comportement (quelques uns sont plus "agressif", autres prennent plus de temps à réagir, les limitations peuvent être différentes entre les voitures et les camions,...). Cependant, si l'on souhaite simuler le trafic à l'échelle d'une ville, on aurait besoin de prendre en compte tous les véhicules et toutes leurs interactions ce qui serait extrêmement coûteux du point de vue calcul informatique.

Dans cette situation, il est plus judicieux de considérer un modèle macroscopique qui modélise le trafic grâce à des quantités macroscopiques comme la densité de véhicules et la vitesse moyenne des véhicules. La contre partie est qu'il est beaucoup plus difficile de modéliser des phénomènes microscopiques comme la présence d'un feu tricolore par exemple. Cela vient également du fait que en général, les modèles macroscopiques sont

basés sur des hypothèse difficilement vérifiables.

Par conséquent, il est très intéressant de pouvoir justifier les modèles macroscopiques grâce à des changements d'échelle dans les modèles microscopiques et voir comment les modèles macroscopiques peuvent garder en mémoire des phénomènes microscopiques (feu tricolore, ralentisseur, zone scolaire, différents types de conducteurs,...). Le problème de dériver des modèles macroscopiques à partir de modèles microscopiques a déjà été étudié pour des modèles microscopiques du type "follow-the-leader" (par exemple dans [AKRM02, DFR15, Hel98, LLK01]). Ces résultats ont été obtenus dans des cadres périodiques et pour des modèles assez simples. Dans cette thèse, nous considérons des modèles plus compliqués (et plus réalistes) et nous traitons également des problèmes avec des modèles non-périodiques.

D'un point de vue mathématique, les modèles microscopiques peuvent être représentés par des équations aux dérivées partielles non-linéaires et non-locales (car la dynamique d'un véhicule dépend de la distance à celui de devant). Le bon cadre pour résoudre ces problèmes est la théorie des solutions de viscosité introduite par Crandall et Lions [CL81, CL83] (on renvoie au User's guide de Crandall, Ishii et Lions [CIL92] et au livre de Barles [Bar94] pour une excellente introduction à cette théorie). En ce qui concerne les modèles macroscopiques, ils peuvent être vu comme des équations de Hamilton-Jacobi posées sur des réseaux et nous renvoyons à Imbert et Monneau [IM14] pour une introduction à cette théorie.

Dans les sections qui suivent, nous présentons rapidement des résultats existant pour le trafic routier, et on donne quelques modèles microscopiques et macroscopiques que l'on peut trouver dans la littérature. On montre rapidement comment les modèles classiques sont liés. On présente également les résultats mathématiques qu'on a utilisé dans cette thèse. Enfin, on présente les résultats principaux obtenus dans cette thèse.

1 Différents types de modèles pour le trafic routier

1.1 Modèles microscopiques pour le trafic routier

Dans cette section, on récapitule quelques modèles microscopiques classiques pour le trafic routier. A l'échelle microscopique, on considère que les véhicules sont sur une seule route et on utilise les notations suivantes : $U_j(t)$ est la position du j -ème véhicule au temps t et l_j est la longueur de la voiture (comme illustré dans la Figure 1.1).

Les modèles du premier ordre et second ordre du type "follow-the-leader" (car-following models) décrivent respectivement la vitesse et l'accélération d'une voiture en fonction de la distance à la voiture qui se trouve devant. Le problème de simuler correctement le trafic a été largement étudié au cours des dernières années, notamment dans [BHN⁺95, BT10, CHM58, Edi61, GHR61, GCM35, New61, Pip53], où on peut trouver des modèles du premier ou second ordre. Pour une introduction plus détaillé sur les modèles du type "follow-the-leader" on renvoie à [BM99, HB01].

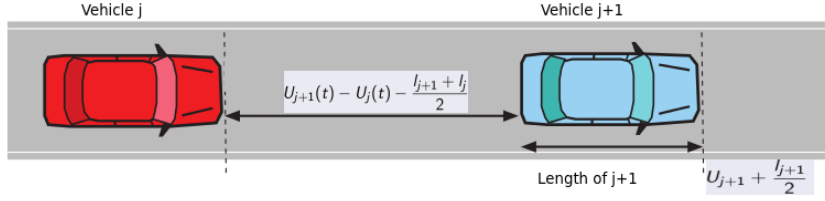


Figure 1.1: Représentation schématique des notations.

Modèles classiques du premier ordre

Un modèle microscopique du premier ordre décrit la vitesse de chacune des voitures. Ceci le rend peu réaliste dans le sens où il n'y a pas de temps de réaction (c'est à dire qu'un véhicule va adapter immédiatement sa vitesse en fonction de la distance à la voiture de devant, sans prendre en compte le temps de réaction du conducteur où le délai de freinage ou d'accélération). Ces modèles sont de la forme suivante, pour tout $j \in \mathbb{Z}$ et $t > 0$,

$$\dot{U}_j(t) = V_j(U_{j+1}(t) - U_j(t)), \quad (1.1)$$

où \dot{U}_j est la vitesse de la j -ème voiture. La fonction V_j est une fonction de vitesse optimale (OVF) dont les propriétés sont listées plus loin. On observe que ce modèle ne prend pas en compte la longueur des voitures, ceci vient du fait que la longueur peut être absorbée dans la fonction V_j . On renvoie à [BT10] pour des exemples de fonctions de vitesse optimale. Les hypothèses classiques sur la fonction V_j est qu'elle soit continue, croissante, bornée supérieurement et qu'il existe un $h_0 \geq 0$ tel que $V_j(h) = 0$ si $h \leq h_0$ (h_0 représente une distance de sécurité). La Figure 1.2 donne une représentation schématique d'une fonction de vitesse optimale. Dans la suite on considérera souvent la cas où $V_j = V$ pour tout $j \in \mathbb{Z}$, c'est à dire que tous les véhicules ont la même dynamique.

Modèles classiques du second ordre

On présente maintenant un modèle du second ordre (Modèle de Bando [BHN⁺95]) qui décrit l'accélération de chacune des voitures. Ces modèles sont plus réaliste puisqu'ils prennent en compte un temps de réaction. On considère le modèle suivant, pour tout $j \in \mathbb{Z}$ et $t > 0$,

$$\ddot{U}_j(t) = a_j(V_j(U_{j+1}(t) - U_j(t)) - \dot{U}_j), \quad (1.2)$$

où, \ddot{U}_j est l'accélération de la j -ème voiture. Le coefficient a_j est la sensibilité du conducteur numéro j et V_j est encore une fois une fonction de vitesse optimale (OVF). Le modèle cherche à faire coïncider la vitesse réelle des voitures avec la vitesse idéale donnée par la fonction de vitesse optimale. Dans les articles [BHN⁺95, BT10] on peut trouver différents types de fonctions de vitesse optimale.

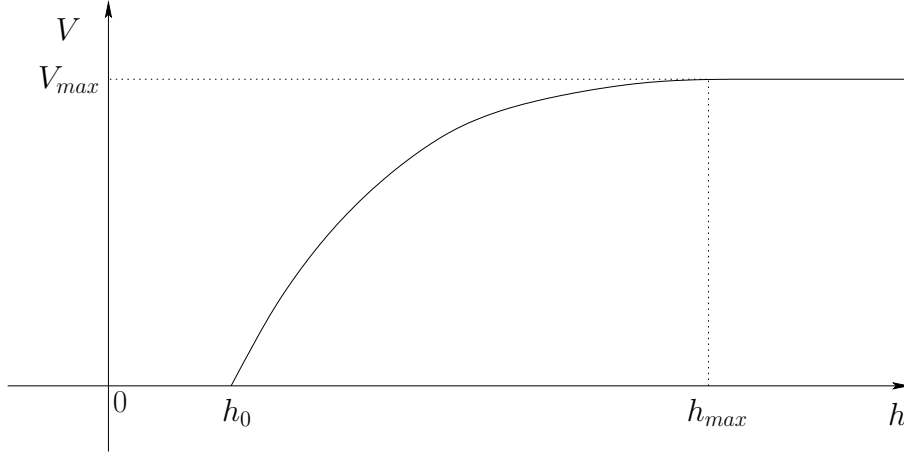


Figure 1.2: Représentation schématique d'une fonction de vitesse optimale.

Autres modèles microscopiques du second ordre

On présente maintenant rapidement d'autres modèles microscopiques du second ordre qui sont toujours du type "follow-the-leader". Néanmoins, dans ces modèles l'accélération de chacune des voitures ne dépend pas uniquement de la distance à la voiture qui se trouve devant mais aussi de la différence des vitesses entre les véhicules.

En particulier, on voudrait mentionner le modèle introduit par Aw, Rascle et Materne [AKRM02], inspiré par celui introduit dans [GHR61]. En utilisant les mêmes notations que précédemment, on considère pour tout $t > 0$, et tout $i \in \mathbb{Z}$,

$$\ddot{U}_i = C \frac{\dot{U}_{i+1} - \dot{U}_i}{(U_{i+1} - U_i)^{1+\gamma}} + \frac{A}{T_r} \left(V \left(\frac{U_{i+1} - U_i}{l} \right) - \dot{U}_i \right), \quad (1.3)$$

avec $A > 0$, $C > 0$, $\gamma > 0$, $l > 0$ et le temps de relaxation $T_r > 0$. Les constantes C , A et γ sont à choisir selon la situation que l'on veut modéliser (trafic congestionné ou fluide) et l représente la longueur d'une voiture. On peut voir que si $C = 0$, $l = 1$ et que l'on considère A/T_r la sensibilité des conducteurs, on retrouve le modèle classique de Bando (1.2).

On voudrait aussi mentionner l'intelligent-driver-model (IDM) développé par Treiber, Hennecke et Helbing [THH00]. L'idée de ce modèle est de pouvoir prendre en compte les différents états du trafic (congestionné ou fluide) dans un seul modèle. Pour mieux voir ce que représentent chacun des termes, on introduit $v_i = \dot{U}_i$ et on considère pour tout $t > 0$ et tout $i \in \mathbb{Z}$,

$$\dot{v}_i = a \left[1 - \left(\frac{v_i}{v_0} \right)^\delta - \left(\frac{s^*(v_i, \Delta v_i)}{s_i} \right)^2 \right], \quad (1.4)$$

1. Différents types de modèles pour le trafic routier

avec $\Delta v_i = v_i - v_{i+1}$ et s^* "l'inter-distance idéale" définie par

$$s^*(v, \Delta v) = h_0 + vT + \frac{v\Delta v}{2\sqrt{ab}}. \quad (1.5)$$

Dans le tableau suivant on résume la signification de chacun des paramètres.

| Paramètre | signification | valeur classique |
|-----------|--|---------------------|
| v_0 | vitesse idéale | 130km/h |
| a | accélération maximale | 1.4m/s ² |
| h_0 | distance de sécurité (distance minimale) | 2.0m |
| T | temps de sécurité (temps de réaction) | 1.5s |
| b | décélération idéale | 2.0m/s ² |

Dans ce modèle, $v_{free}(v) = a[1 - (v/v_0)^\delta]$ est le terme d'accélération libre qui est dominant quand le trafic est fluide. Le terme $-a(s^*/s)^2$ est le terme de décélération quand le trafic est congestionné, c'est le terme dominant quand la voiture i est trop proche de la voiture $i+1$. On observe que la décélération dépend du rapport entre l'inter-distance idéale et l'inter-distance réelle. L'IDM est plus réaliste et plus complexe que le modèle de Bando dans le sens où il prend en compte la différence entre les vitesses et qu'il peut correctement simuler le trafic congestionné. L'IDM reste un modèle pour une seule route, mais dans l'article de Kesting, Treiber et Helbing [KTH10], les auteurs ont modifié l'IDM pour prendre en compte (de manière plus réaliste) des changements de voies. Ceci permettant de simuler le trafic avec plusieurs voies et avec des voitures pouvant changer de voie.

1.2 Modèle macroscopique classique : le modèle LWR

Concernant les modèles macroscopiques, le plus connu est le modèle LWR (Lighthill-Whitham-Richards) qui a été introduit dans [LW55, Ric56]. Il est donné par l'équation aux dérivées partielles suivante :

$$\partial_t \rho + \partial_y f(\rho) = 0, \quad (1.6)$$

où $\rho(t, y)$ est la densité des véhicules au point $y \in \mathbb{R}$ (un point physique de la route) au temps $t \in (0, +\infty)$, et $v(\rho)$ est la vitesse moyenne des véhicules. On appelle $f(\rho) = \rho v(\rho)$ le flux du trafic, et c'est aussi le diagramme fondamental qui caractérise le modèle. On peut remarquer que (1.6) utilise des coordonnées Eulériennes (y est un point sur la route). Cependant, Wagner a montré dans [Wag87] (pour les équations de dynamiques des gaz) que le problème (1.6) est équivalent à

$$\partial_t s - \partial_x v^*(s) = 0, \quad (1.7)$$

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où $s(t, x) = 1/\rho$ est l'espace entre les véhicules, x (une variable continue) représente le véhicule d'indice x et $v^*(s) = v(1/s)$. On peut voir que l'équation (1.7) utilise des coordonnées Lagrangiennes. De plus, si on note $u^0(t, x)$ la position du véhicule x , on a que (1.7) est équivalent (voir [LLC08]) à

$$\partial_t u^0(t, x) = v^* \left(\partial_x u^0 \right), \quad (1.8)$$

avec $s(t, x) = \partial_x u^0(t, x)$. Cela implique que les modèles de type LWR peuvent être vus comme des équations de Hamilton-Jacobi du type (1.8). C'est essentiellement cette formulation que nous utiliserons pour les modèles macroscopiques apparaissant dans cette thèse.

Il existent différents types de diagrammes fondamentaux qui ont été introduit par différents auteurs. Dans le tableau suivant on en liste quelques uns.

| Auteur | Diagramme fondamental |
|-----------------------------|--|
| [CHM58, Chandler(1958)] | $f(\rho) = f_{max} \left(1 - \frac{\rho}{\rho_{max}} \right)$ |
| [Gre59, Greenberg(1959)] | $f(\rho) = \rho \cdot V_{critique} \cdot \ln \left(\frac{\rho}{\rho_{max}} \right)$ |
| [GCM35, Greenshields(1935)] | $f(\rho) = \rho \cdot V_{max} \cdot \left(1 - \frac{\rho}{\rho_{max}} \right)$ |
| [Edi61, Edie(1960)] | $f(\rho) = \rho \cdot V_{max} \cdot \exp -\frac{\rho}{\rho_{max}}$ |
| [May90, May(1990)] | $f(\rho) = \rho \cdot V_{max} \cdot \exp -\frac{1}{a} \cdot \frac{\rho}{\rho_{max}}$ |

(1.9)

1.3 Autres modèles macroscopiques

Il est important de mentionner qu'il existe d'autres modèles macroscopiques. En effet, le modèle LWR présente plusieurs inconvénients le principal étant qu'il peut développer des discontinuités en temps finie. Ceci vient du fait que le modèle permet à la vitesse d'un véhicule de changer instantanément quand il traverse un choc (entre deux régions régulières). C'est pour cette raison que les modèles d'ordre supérieur ont été développés.

On peut citer par exemple le modèle Payne-Whitham (PW) [Pay71, Whi74]. Le modèle LWR suppose que la vitesse moyenne v dépend uniquement de la densité des véhicules ρ ce qui peut ne pas être valide dans certains cas. Pour corriger ceci, le modèle PW contient une équation sur la vitesse moyenne :

$$\begin{cases} \rho_t + (\rho v)_x = 0 \\ v_t + v v_x + \frac{1}{\rho} (A_e(\rho))_x = \frac{1}{\tau} (v_e(\rho) - v), \end{cases}$$

où $v_e(\rho)$ est la valeur d'équilibre pour la vitesse, $(A_e(\rho))_x/\rho$ est le terme d'anticipation et $(v_e(\rho) - v)/\tau$ s'appelle le terme de relaxation. En 1995 Daganzo [Dag95] a mis en évidence quelques inconvénients du modèle PW. En particulier, dans certaines situations les véhicules pouvait avoir des vitesses négatives.

Le modèle Aw-Rascle [AR00] a ensuite été proposé pour surmonter quelques remarques de Daganzo. Le modèle est donné par le système suivant

$$\begin{cases} \rho_t + (\rho v)_x = 0 \\ (v + p(\rho))_t + v(v + p(\rho))_x = 0, \end{cases} \quad (1.10)$$

avec p la "pression" qui est une fonction croissante de la densité. Ce modèle corrige les inconvénients du modèle PW. Pour plus d'information sur les modèles macroscopiques, on renvoie au livre de Piccoli et Garavello [GP06].

1.4 Lien entre les modèles microscopiques et macroscopiques

Le problème de dériver des modèles macroscopiques à partir de modèles microscopiques a déjà été étudié pour des modèles du type "follow-the-leader". Notamment dans [DFR15, Hel98, LLK01], les auteurs, en utilisant la mesure empirique de la position des véhicules, obtiennent une loi de conservation scalaire (modèle LWR). Dans le papier de Aw *et al.* [AKRM02], les auteurs, en utilisant une formulation équivalente de (1.3), ont été capables de dériver (dans le cas où $A = 0$), un système équivalent au modèle Aw-Rascle (1.10). Dans le cas général, ils arrivent à montrer que la discrétisation standard du modèle Aw-Rascle (en coordonnées Lagrangiennes) est équivalent à (1.3) discrétisé en temps.

La plus part des résultats d'homogénéisation pour le trafic routier ont été obtenus pour une seule route, et il n'y a pas beaucoup de résultats dans les réseaux. Cependant, récemment Cristiani et Sahu ont présenté dans [CS15] un modèle du premier ordre dans un réseau et montrent le lien avec un modèle multi-path (voir [BBC14, BC14]). En fait, ils considèrent pour chaque chemin possible une population de véhicules. Leur résultat d'homogénéisation est alors fait dans un cadre très général, mais en supposant la convergence de la mesure empirique (pour chacune des populations) ce qui leur permet de montrer que la limite satisfait un modèle multi-path.

Dans le cadre de cette thèse, pour dériver les modèles macroscopiques, on utilise une version de la primitive de la mesure empirique de la position des véhicules comme introduit dans [FIM09a, FIM09b] pour la dynamique des dislocations. De plus, les résultats concernant le passage du microscopique au macroscopique dans cette section et dans le reste de cette thèse utilisent la théorie des solutions de viscosité. On renvoie au User's guide de Crandall, Ishii et Lions [CIL92] et au livre de Barles [Bar94] pour une introduction aux solutions de viscosité et à [Ish92, IK91, Len88] et leurs références pour des résultats concernant les solutions de viscosité dans les systèmes faiblement couplés.

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On commence par un résultat très simple où $(U_j(\cdot))_{j \in \mathbb{Z}}$ est une solution classique du modèle du premier ordre (1.1) ou du modèle classique du second ordre (1.2) avec $V_j = V$ et $a_j = a$ pour tout $j \in \mathbb{Z}$. Pour commencer, on introduit une fonction avec un changement d'échelle,

$$\begin{cases} u^\varepsilon(t, x) = \varepsilon U \left[\frac{x}{\varepsilon} \right] \left(\frac{t}{\varepsilon} \right), \\ u^\varepsilon(0, x) = u_0(x) \quad \forall \varepsilon > 0, \end{cases} \quad (1.11)$$

où x est une variable continue qui représente l'indice de chacune des voitures et $[\cdot]$ est la partie entière inférieure. La condition initiale u_0 est simplement une fonction régulière telle que pour tout $j \in \mathbb{Z}$ et pour tout $\varepsilon > 0$

$$\frac{1}{\varepsilon} u_0(j\varepsilon) = U_j(0).$$

On suppose que l'OVF et le coefficient a satisfont les hypothèses suivantes :

(A1) (Régularité)

$$\begin{cases} V \text{ est continue et positive.} \\ V \text{ est lipschitzienne et on note } L \text{ sa constante de Lipschitz.} \end{cases}$$

(A2) (Monotonie)

$$\begin{cases} V \text{ est croissante.} \\ a \geq 2L. \end{cases}$$

(A3) (Borne supérieure)

$$\lim_{h \rightarrow +\infty} V(h) < +\infty. \quad (1.12)$$

On note $V_{max} = \|V\|_\infty$ et $h_0 = 2V_{max}/a$.

(A4) (Borne inférieure)

$$V(h) = 0 \quad \text{pour tout } h \leq 2h_0.$$

On a alors le résultat standard suivant dont la preuve peut se trouver, entre autres, dans [Cos14, FIM09a].

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Theorem 1.1 (Homogénéisation des systèmes classiques). *Supposons que (A1)-(A4) soient vérifiées et que la condition initiale u_0 ait été choisie correctement. On suppose également que $V_j = V$ et $a_j = a$ pour tout $j \in \mathbb{Z}$. Alors la fonction u^ε définie dans (1.11) converge localement uniformément sur $(0, +\infty) \times \mathbb{R}$ quand ε tend vers 0 vers l'unique solution de viscosité u^0 de l'équation aux dérivées partielles suivante,*

$$\begin{cases} \partial_t u^0 = V(\partial_x u^0) & \forall (t, x) \in [0, T] \times \mathbb{R}, \\ u^0(0, x) = u_0(x) & \forall x \in \mathbb{R}. \end{cases} \quad (1.13)$$

Remark 1.2. *En utilisant le résultat présenté dans la Section 1.2, concernant les modèles macroscopiques, on a que (1.13) est équivalent au modèle macroscopique LWR avec le diagramme fondamental $f(\rho) = \rho \cdot V(1/\rho)$. Aussi, on remarque que la fonction u^0 est en fait la primitive de la densité des véhicules en coordonnées Lagrangiennes.*

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Comme évoqué précédemment, la bonne notion pour résoudre les équations étudiées dans cette thèse est la théorie de solutions de viscosité, introduite par Crandall et Lions [CL81]. Il s'agit de solutions faibles pour des équations aux dérivées partielles de type Hamilton-Jacobi. On renvoie encore une fois au User's guide de Crandall, Ishii et Lions [CIL92], et au livre de Barles [Bar94] pour une excellente introduction à cette théorie.

Une façon de voir que la notion de solutions de viscosité est bien adaptée est de considérer la fonction u définie par (1.11), avec $\varepsilon = 1$ et $(U_j)_j$ solution du modèle du premier ordre (1.1). La fonction u est une fonction discontinue et il faut donc définir une notion de solution faible. On peut alors montrer que cette fonction est une solution de viscosité (voir définition plus bas) de l'EDP (équation de Hamilton-Jacobi) suivante, qui est non-locale et non-linéaire,

$$\frac{\partial u}{\partial t}(t, x) = V(u(t, x+1) - u(t, x)) \quad \text{pour tout } (t, x) \in (0, +\infty) \times \mathbb{R}. \quad (2.1)$$

Ceci nous permet d'injecter tout le système d'EDOs (1.1) dans une seule EDP. Pour définir une solution de viscosité de (2.1), on introduit les enveloppes semi-continues supérieurement et semi-continue inférieurement de u :

$$u^*(t, x) = \limsup_{\substack{y \rightarrow x \\ s \rightarrow t}} u(s, y) \quad \text{et} \quad u_*(t, x) = \liminf_{\substack{y \rightarrow x \\ s \rightarrow t}} u(s, y).$$

Definition 2.1. *Soit $\Omega = (0, +\infty) \times \mathbb{R}$. Une fonction semi-continue supérieurement (resp. inférieurement) $u : \Omega \rightarrow \mathbb{R}$ est une sous-solution (resp. une sur-solution) de (2.1) sur Ω ,*

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si pour toute fonction test $\varphi \in C^1(\Omega)$ et tout $(t, x) \in \Omega$ tel que $u - \varphi$ atteint un maximum (resp. un minimum) en (t, x) , on a

$$\varphi_t(t, x) \leq V(u(t, x + 1) - u(t, x)) \quad \left(\text{resp. } \varphi_t \geq V(u(t, x + 1) - u(t, x)) \right).$$

Une fonction u est une solution de (2.1), si u^* est une sous-solution de (2.1) et u_* est une sur-solution de 2.1.

La notion de solutions de viscosité est très puissante. Elle permet de définir une notion de solution avec très peu de régularité et permet également d'obtenir des résultats très utiles.

- Résultats de stabilité (voir [Bar13, CL81, CL83, Lio82]) : ils permettent de passer à la limite à l'intérieur de l'équation, en particulier, si on a une suite de sous-solutions (ou de sur-solutions) on peut passer à la limite (sous des conditions très faibles) et obtenir une sous-solution (ou une sur-solution).
- Principe de comparaison (voir [Bar13, CL81, CL83, Lio82]) : il permet de comparer des sous-solutions et des sur-solutions et est souvent utilisé pour obtenir l'unicité de la solution de viscosité du problème considéré. C'est l'argument principal de la théorie.
- Résultats d'existence via la méthode de Perron : H. Ishii a généralisé la méthode de Perron dans [Ish87] pour les équations de Hamilton-Jacobi. Il s'agit d'une méthode qui permet de construire une solution (a priori discontinue) d'une équation de Hamilton-Jacobi avec des hypothèses très faibles. Ensuite en utilisant le principe de comparaison on peut obtenir l'unicité (et aussi la continuité) de la solution.
- Homogénéisation : depuis les papiers de Lions, Papanicolaou et Varadhan [LPV88] puis d'Evans [Eva89, Eva92], la théorie de l'homogénéisation pour les équations de Hamilton-Jacobi a connu de grands développements.

Dans ce qui suit, nous présentons des résultats concernant les solutions de viscosité posées sur des réseaux et sur l'homogénéisation spécifique.

2.1 Résultats sur les solutions de viscosité dans les réseaux

Dans les exemples précédents, les véhicules étaient sur la droite réelle. La réalité est bien sûr beaucoup plus compliquée et on peut imaginer des bifurcations, des jonctions, ou même tout un réseau. On peut donc voir que travailler dans un réseau devient nécessaire pour traiter des problèmes liés au trafic routier.

Récemment, la théorie des équations de Hamilton-Jacobi dans les réseaux a connu de grands progrès. Les premiers résultats ont été obtenus par Schieborn [Sch06] pour des

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équations eikonales dans les réseaux. Des travaux plus récents ont ensuite été effectués par Achdou, Camilli, Cutri et Tchou [ACCT13], Imbert, Monneau et Zidani [IMZ13], et Schieborn et Camilli [SC13]. Dans ces trois papiers, les hamiltoniens sont convexes et les auteurs utilisent un approche du type contrôle optimal (en particulier pour montrer le principe de comparaison). On renvoie à [CM13], pour une comparaison entre les différentes notions de solutions de viscosité proposées dans [ACCT13], [IMZ13] et [SC13]. De même, on renvoie au travail récent de Camilli, Marchi et Schieborn [CMS13] pour les équations eikonales dans les réseaux.

Imbert et Monneau [IM14] ont ensuite proposé une approche complètement EDP pour des équations de Hamilton-Jacobi quasi-convexe dans les réseaux. Cette approche repose sur la construction d'une fonction "sommet" qui permet de dédoubler les variables correctement près des jonctions. Nous présentons ci-dessous les éléments essentiels de cette approche qui sera utilisée pour les modèles macroscopiques apparaissant dans cette thèse.

Ils considèrent le cadre suivant pour une jonction. Ils étudient un nombre fini de branches, pour $i = 1, \dots, N$, chacune des branches R_i est isométrique à $[0, +\infty)$ et

$$\bar{R} = \bigcup_{i=1, \dots, N} R_i \quad \text{avec } R_i \cap R_j = \{0\} \text{ pour } i \neq j.$$

Toutes les branches sont collées à l'origine (le nœud). On note $\partial_x u(x)$ la dérivée spatiale de u en $x \in R_i$ qui est définie par

$$\partial_x u(x) = \begin{cases} \partial_i u(x) & \text{si } x \in R_i^* := R_i \setminus \{0\} \\ (\partial_1 u(0), \dots, \partial_N u(0)) & \text{si } x = 0. \end{cases}$$

Les auteurs considèrent alors l'équation de Hamilton-Jacobi suivante posée dans le réseau \bar{R}

$$\begin{cases} u_t + H_i(u_x) = 0 & \text{pour } t \in (0, +\infty) \text{ et } x \in R_i^* \\ u_t + F(u_x) = 0 & \text{pour } t \in (0, +\infty) \text{ et } x = 0 \\ u(0, x) = u_0(x). \end{cases} \quad (2.2)$$

Les hypothèses sur les hamiltoniens H_i , pour $i = 1, \dots, N$, sont les suivantes

Hypothèse (FH)

(Continuité) $H_i \in C(\mathbb{R})$.

(Convexité des ensembles de niveaux) Pour chaque $i = 1, \dots, N$, il existe un nombre réel $p_i^0 \in \mathbb{R}$ tel que

$$\begin{cases} H_i \text{ est décroissante sur } (-\infty, p_i^0] \\ H_i \text{ est croissante sur } [p_i^0, +\infty). \end{cases} \quad (2.3)$$

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(Coercivité) $\lim_{|p| \rightarrow +\infty} H_i(p) = +\infty$.

Concernant la condition de jonction $F : \mathbb{R}^N \rightarrow \mathbb{R}$, on suppose qu'il s'agit d'une fonction continue et décroissante par rapport à toutes ses variables.

Afin de pouvoir introduire une bonne définition de solution de viscosité pour (2.2), les auteurs introduisent une classe particulière de fonction test.

Classe de fonction test pour (2.2). Pour $T > 0$, soit $R_T = (0, T) \times \bar{R}$. On définit une classe de fonction test sur R_T par

$$C^1(R_T) = \{\varphi \in C(R_T), \text{ la restriction de } \varphi \text{ à } (0, T) \times R_i^* \text{ est } C^1 \text{ pour } i = 1, \dots, N\}.$$

Definition 2.2 (Solution de viscosité pour (2.2)). *Une fonction semi-continue supérieurement (resp. inférieurement) $u : [0, T) \times \bar{R} \rightarrow \mathbb{R}$ est une sous-solution (resp. une sur-solution) de (2.2) sur R_T si pour toute fonction test $\varphi \in C^1(R_T)$ et tout point $(t, x) \in R_T$ tel que $u - \varphi$ atteint un maximum local (resp. un minimum local) en $(t, x) \in R_T$, on a*

$$\begin{aligned} \varphi_t(t, x) + H_i(\varphi_x(t, x)) &\leq 0 \text{ (resp. } \geq) && \text{si } x \in R_i^* \\ \varphi_t(t, x) + F(\varphi_x(t, x)) &\leq 0 \text{ (resp. } \geq) && \text{si } x = 0. \end{aligned}$$

-Une fonction u est une solution de (2.2) sur R_T si u^* est une sous-solution de (2.2) et si u_* est une sur-solution de (2.2) sur R_T .

-Une fonction u est une solution de (2.2) sur $[0, T) \times \bar{R}$ si elle est solution sur R_T et si $u^*(0, x) \leq u_0(x)$ et $u_*(0, x) \geq u_0(x)$ pour tout $x \in \bar{R}$.

La définition précédente peut être vue comme la définition standard de solution de viscosité. Néanmoins, pour avoir un résultat de stabilité pour (2.2), Imbert et Monneau ont introduit la notion de solution de viscosité "relaxée".

Definition 2.3 (Solution de viscosité relaxée pour (2.2)). *Une fonction semi-continue supérieurement (resp. inférieurement) $u : [0, T) \times \bar{R} \rightarrow \mathbb{R}$ est une sous-solution relaxée (resp. une sur-solution relaxée) de (2.2) sur R_T si pour toute fonction test $\varphi \in C^1(R_T)$ et tout point $(t, x) \in R_T$ tel que $u - \varphi$ atteint un maximum local (resp. un minimum local) en $(t, x) \in R_T$, on a*

-si $x \in R_i^*$

$$\varphi_t(t, x) + H_i(\varphi_x(t, x)) \leq 0 \text{ (resp. } \geq)$$

-si $x = 0$,

$$\begin{aligned} \text{soit } \varphi_t(t, x) + H_i(\varphi_x(t, x)) &\leq 0 \text{ (resp. } \geq) && \text{avec } i \in \{1, \dots, N\} \\ \text{ou } \varphi_t(t, x) + F(\varphi_x(t, x)) &\leq 0 \text{ (resp. } \geq). \end{aligned}$$

Une fonction u est une solution relaxée de (2.2) sur R_T si u^* est une sous-solution relaxée de (2.2) et u_* est une sur-solution relaxée de (2.2) sur R_T .

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Les auteurs de [IM14] ont également montré qu'une condition de jonction F générale pouvait être remplacée par une condition de jonction F_A de la forme suivante, pour tout $p = (p_1, \dots, p_N)$

$$F_A(p) = \max \left(A, \max_i H_i^-(p_i) \right),$$

avec

$$H_i^-(q) = \begin{cases} H_i(q) & \text{si } q \leq p_i^0, \\ H_i(p_i^0) & \text{si } q \geq p_i^0. \end{cases}$$

Parmi les résultats de [IM14], les résultats suivants sont les plus importants dans le cadre de cette thèse :

- Pour une condition initiale u_0 uniformément continue, (2.2) admet une solution relaxée.
- En utilisant Définition 2.3, il y a stabilité des sous-solutions (resp. des sur-solutions) par passage au supremum (resp. au infimum).
- Les conditions de jonction de la forme F_A sont toujours satisfaites au sens classique de viscosité (Définition 2.2). C'est à dire que une sous-solution (resp. sur-solution) relaxée est une sous-solution (resp. sur-solution) classique dans le cas où la condition de jonction est donné par F_A .
- La condition de jonction générale F peut être réduite à une condition de jonction de la forme F_A . En particulier, sous l'hypothèse (FH) il existe une constante $A_F \in \mathbb{R}$ telle que n'importe quelle sous-solution (resp. sur-solution) relaxée de (2.2) est une sous-solution (resp. sur-solution) de

$$\begin{cases} u_t + H_i(u_x) = 0 & \text{pour } t \in (0, +\infty) \text{ et } x \in R_i^* \\ u_t + F_A(u_x) = 0 & \text{pour } t \in (0, +\infty) \text{ et } x = 0, \end{cases} \quad (2.4)$$

avec $A = A_F$. On appelle une sous-solution de (2.4) une sous-solution "A-flux limited" (on appelle de manière similaire les sur-solutions et les solutions de (2.4)).

- Il existe un principe de comparaison pour (2.4) avec la condition initiale $u(0, x) = u_0(x)$.

Pour finir, on voudrait rappeler un résultat essentiel de [IM14] et qui sera utilisé en particulier dans la preuve d'homogénéisation au point de jonction. Il s'agit d'une définition équivalente à la définition de sous et sur-solution au point de jonction. On considère l'équation de Hamilton-Jacobi suivante sur $\bar{R} \setminus \{0\}$,

$$u_t + H_i(u_x) = 0 \quad (t, x) \in (0, +\infty) \times R_i^* \text{ pour } i = 1, \dots, N. \quad (2.5)$$

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Theorem 2.4 (Définition équivalente pour les sous/sur-solutions (Théorème 2.11 dans [IM14])). *Supposons que pour $i = 1, \dots, N$, H_i satisfait l'hypothèse (HF) et on considère $A \in [A_0, +\infty)$ avec $A_0 = \max_i \min_{p \in \mathbb{R}} H_i(p)$. Soient $p_i^A \in \mathbb{R}$ des solutions arbitraires, pour $i = 1, \dots, N$, de*

$$H_i(p_i^A) = H_i^+(p_i^A) = A \quad (2.6)$$

on fixe $\phi_0(x)$ n'importe quelle fonction indépendante du temps telle que

$$\partial_i \phi_0(0) = p_i^A. \quad (2.7)$$

Étant donné une fonction $u : [0, T) \times \mathbb{R} \rightarrow \mathbb{R}$, les propriétés suivantes sont vraies.

1. Si u est une sous-solution semi-continue supérieurement de (2.5) satisfaisant

$$u(t, 0) = \limsup_{(s, y) \rightarrow (t, 0), y \in \mathbb{R}_i^*} u(s, y) \quad (2.8)$$

alors u est une sous-solution " A_0 -flux limited".

2. Étant donné $A > A_0$ et $t_0 \in (0, T)$, si u est une sous-solution semi-continue supérieurement de (2.5) satisfaisant (2.8) et si pour n'importe quelle fonction test φ touchant u par au-dessus en $(t_0, 0)$ avec

$$\varphi(t, x) = g(t) + \phi_0(x) \quad (2.9)$$

pour un certain $g \in C^1(0, +\infty)$, on a

$$\varphi_t + F_A(\varphi_x) \leq 0 \text{ en } (t_0, 0), \quad (2.10)$$

alors u est une sous-solution " A -flux limited" en $(t_0, 0)$.

3. Étant donné $t_0 \in (0, T)$, si u est une sur-solution semi-continue inférieurement de (2.5) et si pour toute fonction test φ touchant u par en-dessous en $(t_0, 0)$ avec φ satisfaisant (2.9), on a

$$\varphi_t + F_A(\varphi_x) \geq 0 \text{ en } (t_0, 0), \quad (2.11)$$

alors u est une sur-solution " A -flux limited" en $(t_0, 0)$.

2.2 Résultats existant sur l'homogénéisation

Comme évoqué précédemment, depuis le papier novateur de Lions, Papanicolau et Varadhan [LPV88], les problèmes d'homogénéisation pour les équations de Hamilton-Jacobi ont été énormément étudiés. Nous listons ici quelques travaux particulièrement importants pour cette thèse.

Tout d'abord, une grande avancée a été faite dans le travail de Evans [Eva89] où il a introduit la méthode de la fonction test perturbée qui permet de faire les passages à la limite dans les problèmes d'homogénéisation. Concernant l'homogénéisation pour des équations non-locales, on renvoie au travail de Imbert, Monneau et Rouy [IMR08] ainsi qu'aux articles de Forcadel, Imbert et Monneau [FIM09a, FIM09b, FIM12] qui permettent d'homogénéiser des systèmes de particules en interaction (décrits par des EDO) en injectant le système d'EDOs dans une seule EDP (ou dans un système d'EDPs faiblement couplées). Ces résultats ont été particulièrement importants pour cette thèse car pour obtenir nos résultats d'homogénéisation pour les problèmes de trafic routier (typiquement des EDOs) nous avons appliqué cette approche qui consiste à injecter le système d'EDOs dans une seule EDP (ou système d'EDPs) puis à homogénéiser cette EDP. Concernant l'homogénéisation pour les systèmes, on renvoie également à Camilli, Ley et Loreti [CLL10].

Les travaux précédents ont été faits dans un cadre périodique. De nombreux travaux ont également été faits dans le cadre stochastique (par exemple [AC15, AS12a, AS12b, AS13, AT14, ATY15, ATY16]) mais dans cette thèse nous n'aborderons pas ce cadre.

Plus récemment la théorie de l'homogénéisation spécifiée est apparue. Au contraire de l'homogénéisation classique qui va moyenniser les phénomènes et donc "oublier" les petits défauts, le but de l'homogénéisation spécifiée est de décrire ces défauts de manière précise. Du point de vue trafic routier, le cas le plus (qui sera décrit plus loin) est le cas d'une perturbation localisée par exemple à l'origine dans le modèle microscopique (par exemple un ralentisseur, une zone scolaire,...). A l'échelle macroscopique on veut garder en mémoire cette perturbation et la décrire comme une condition de jonction. Dans ce type de problème, la principale difficulté est la construction des correcteurs à la jonction. Dans ce cadre, on mentionne les travaux récents de Achdou et Tchou [AT15], de Galise, Imbert et Monneau [GIM15] ainsi que les cours de Lions au "Collège de France" [Lio14]. L'idée principale derrière ces travaux est que les correcteurs à la jonction doivent avoir des pentes particulières à l'infini. L'idée est alors de construire des correcteurs dans un domaine tronqué avec des "bonnes" conditions aux bords et ensuite de passer à la limite sur la taille du domaine.

3 Résultats principaux de cette thèse

Comme évoqué précédemment, il est important de faire le lien entre les modèles microscopiques et les modèles macroscopiques. Ce lien a souvent été étudié dans des cas très

simples. On présente maintenant les résultats principaux de la thèse. Ces résultats seront présentés en détail dans les chapitres suivants.

3.1 Homogénéisation d'un modèle discret du second ordre avec $n_0 \in \mathbb{N}$ types de conducteurs

On commence par introduire les résultats de [1, 5] qui sont détaillés dans les Chapitres 1 et 2.

Modèle général du second ordre avec $n_0 \in \mathbb{N}^*$ types de conducteur

Le premier résultat de cette thèse est une généralisation du Théorème 1.1 au cas plus réaliste où l'on considère différents types de conducteurs et différents types de véhicules. Plus précisément, on considère $n_0 \in \mathbb{N}$ types de conducteurs et on rappelle que la dynamique est définie, pour tout $j \in \mathbb{Z}$ et $t \geq 0$, par

$$\ddot{U}_j(t) = a_j(V_j(U_{j+1}(t) - U_j(t)) - \dot{U}_j), \quad (3.1)$$

où U_j dénote la position du j -ème véhicule, \dot{U}_j sa vitesse et \ddot{U}_j son accélération. Les coefficients a_j et les fonctions de vitesse optimale V_j dépendent du conducteur. On fait les hypothèses suivantes sur les coefficients a_j et les fonctions V_j .

(B1) (Regularité) Pour tout $j \in \{1, \dots, n_0\}$,

$$\begin{cases} V_j \text{ est continue et positive.} \\ V_j \text{ est lipschitzienne et on note } L_j \text{ sa constante de Lipschitz.} \end{cases}$$

On note $L = \max_{j \in \{1, \dots, n_0\}} L_j$.

(B2) (Monotonie) Pour tout $j \in \{1, \dots, n_0\}$,

$$\begin{cases} V_j \text{ est croissante.} \\ a_j \geq 4L. \end{cases}$$

(B3) (Borne supérieure) Pour tout $j \in \{1, \dots, n_0\}$,

$$\lim_{h \rightarrow +\infty} V_j(h) < +\infty. \quad (3.2)$$

On note $V_{max} = \max_j (\|V_j\|_\infty)$, $h_0 = V_{max}/\alpha$ et $\alpha = \min_{j \in \{1, \dots, n_0\}} (a_j)/2$.

(B4) (Borne inférieure) Pour tout $j \in \{1, \dots, n_0\}$,

$$V_j(h) = 0 \quad \text{pour tout } h \leq 2h_0.$$

Remark 3.1. *Comme précédemment, les hypothèses (B1)-(B4) sont classiques pour le modèle de Bando (voir par exemple [BHN⁺95, BT10]). Concernant le modèle du première ordre (1.1), on pourrait aussi le modifier pour prendre en compte différents types de conducteurs en modifiant les fonctions de vitesse optimale. Mais comme précisé plus haut, les modèle du second ordre sont plus réalistes. C'est pour cela que l'on considère uniquement ici le modèle du second ordre.*

Aux hypothèses précédentes on ajoute une condition de périodicité (nécessaire pour réussir à faire le passage du modèle microscopique au modèle macroscopique).

(B5) (Périodicité sur le type de conducteur) Pour tout $j \in \mathbb{Z}$,

$$a_{j+n_0} = a_j \quad \text{et} \quad V_{j+n_0} = V_j.$$

Homogénéisation du modèle du second ordre généralisé

Pour passer au modèle macroscopique, la technique consiste à introduire une fonction avec des variables continues : $(u_j)_{j \in \mathbb{Z}}$, définie par

$$u_j(t, x) = U_{j+\lfloor x \rfloor n_0}(t) \quad \text{pour tout } (t, x) \in [0, +\infty) \times \mathbb{R}, \quad (3.3)$$

où $\lfloor \cdot \rfloor$ est la partie entière inférieure. On fait un changement d'échelle en introduisant un petit paramètre $\varepsilon > 0$, et la fonction $(u_j^\varepsilon)_{j \in \mathbb{Z}}$ définie par

$$u_j^\varepsilon(t, x) = \varepsilon u_j \left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon} \right). \quad (3.4)$$

En fait, quand ε tend vers 0, le nombre de véhicules par unité de longueur tend vers l'infini et le résultat principal de ce travail est que toutes les fonctions u_j^ε convergent localement uniformément sur $(0, +\infty) \times \mathbb{R}$ quand ε tend vers 0, vers la solution du problème suivant,

$$\begin{cases} u_t^0(t, x) = \bar{F}(u_x^0(t, x)) & \text{pour } (t, x) \in (0, +\infty) \times \mathbb{R}, \\ u^0(0, x) = u_0(x) & \text{pour tout } x \in \mathbb{R}, \end{cases} \quad (3.5)$$

où \bar{F} reste à être déterminée et la fonction u_0 est une fonction lipschitzienne telle que pour $\varepsilon > 0$,

$$\frac{1}{\varepsilon} u_0 \left(\frac{j\varepsilon}{n_0} \right) = U_j(0).$$

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Theorem 3.2 (Homogénéisation d'un système avec n_0 types of conducteurs (Théorème 1.3 dans [1])). *Supposons que (B1)-(B5) soient vérifiées. Considérons la fonction $(u_j^\varepsilon)_{j \in \mathbb{Z}}$ définie par (3.4). Alors, il existe une fonction continue $\bar{F} : \mathbb{R} \rightarrow \mathbb{R}$ telle que pour tout $j \in \mathbb{Z}$, les fonctions u_j^ε convergent localement uniformément sur $(0, +\infty) \times \mathbb{R}$ quand ε tend vers 0 vers l'unique solution de viscosité u^0 de (3.5).*

Remark 3.3. *On a vu précédemment que (3.5) est équivalent à un modèle macroscopique LWR avec un diagramme fondamental donné par $f(\rho) = \rho \cdot \bar{F}(1/\rho)$ et avec u_x^0 qui est l'inverse de la densité des voitures.*

Propriétés qualitatives de l'hamiltonien effectif \bar{F}

Theorem 3.4 (Propriétés qualitatives de \bar{F} (Théorème 1.9 dans [1])). *Supposons (B1)-(B5). Alors, pour tout $p \in (0, +\infty)$, les propriétés suivantes sont vérifiées :*

(i) **(Borne inférieure)** *si $p \leq 2h_0n_0$, on a*

$$\bar{F}(p) = 0.$$

(ii) **(Borne supérieure)**

$$\lim_{p \rightarrow +\infty} \bar{F}(p) = \min_{j \in \{1, \dots, n_0\}} (\|V_j\|_\infty).$$

(iii) **(Monotonie)** *\bar{F} est croissante.*

Injection du système d'EDOs dans un système d'EDPs

Pour que le résultat d'homogénéisation du Théorème 3.2 soit complet, il faut déterminer l'hamiltonien effectif \bar{F} . On explique maintenant comment on le construit. Pour commencer, il faut injecter le système (3.1) dans un système d'EDPs. On utilise l'idée de [FIM09a, FIM12]. On introduit pour tout $j \in \mathbb{Z}$

$$\Xi_j(t) = U_j(t) + \frac{1}{\alpha} \dot{U}_j(t) \quad \text{où} \quad \alpha = \frac{1}{2} \min_{j \in \{1, \dots, n_0\}} (a_j). \quad (3.6)$$

On obtient alors le système d'EDOs suivant : pour tout $j \in \mathbb{Z}$ et $t \in (0, +\infty)$,

$$\begin{cases} \dot{U}_j(t) = \alpha(\Xi_j(t) - U_j(t)) \\ \dot{\Xi}_j(t) = (a_j - \alpha)(U_j(t) - \Xi_j(t)) + \frac{a_j}{\alpha} V_j(U_{j+1}(t) - U_j(t)). \end{cases} \quad (3.7)$$

On injecte ensuite le système d'EDOs dans un système d'EDPs, en introduisant les fonctions

$$(u, \xi) = ((u_j(t, x))_{j \in \mathbb{Z}}, (\xi_j(t, x))_{j \in \mathbb{Z}}),$$

définies par

$$u_j(t, x) = U_{j+[x]n_0}(t) \quad \text{et} \quad \xi_j(t, x) = \Xi_{j+[x]n_0}(t),$$

où $[\cdot]$ est la partie entière inférieure. Par exemple, dans le cas où $n_0 = 1$, on a

$$u(t, x) = U_{[x]}(t) \quad \text{et} \quad \xi(t, x) = \Xi_{[x]}(t),$$

où x est une variable continue qui représente l'indice de chacune des voitures. Dans le cas où $n_0 > 1$, l'entier j nous permet de prendre en compte la périodicité des coefficients a_j et des fonctions V_j .

La fonction (u, ξ) satisfait alors le système d'équations suivant, pour tout $(t, x) \in (0, +\infty) \times \mathbb{R}$ et pour tout $j \in \mathbb{Z}$,

$$\left\{ \begin{array}{l} \frac{\partial u_j}{\partial t}(t, x) = \alpha(\xi_j(t, x) - u_j(t, x)) \\ \frac{\partial \xi_j}{\partial t}(t, x) = (a_j - \alpha)(u_j(t, x) - \xi_j(t, x)) + \frac{a_j}{\alpha} V_j(u_{j+1}(t, x) - u_j(t, x)) \\ u_{j+n_0}(t, x) = u_j(t, x + 1) \\ \xi_{j+n_0}(t, x) = \xi_j(t, x + 1). \end{array} \right. \quad (3.8)$$

On introduit ensuite le changement d'échelle suivant

$$u_j^\varepsilon(t, x) = \varepsilon u_j\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right) \quad \text{et} \quad \xi_j^\varepsilon(t, x) = \varepsilon \xi_j\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right). \quad (3.9)$$

Quand ε tend vers 0, le nombre de véhicules par unité de longueur tend vers l'infini. On peut alors montrer que la fonction $(u^\varepsilon, \xi^\varepsilon) = ((u_j^\varepsilon(t, x))_{j \in \mathbb{Z}}, (\xi_j^\varepsilon(t, x))_{j \in \mathbb{Z}})$ satisfait alors le problème de Cauchy suivant, pour tout $(t, x) \in (0, +\infty) \times \mathbb{R}$,

$$\left\{ \begin{array}{l} \frac{\partial u_j^\varepsilon}{\partial t}(t, x) = \alpha \frac{\xi_j^\varepsilon(t, x) - u_j^\varepsilon(t, x)}{\varepsilon} \\ \frac{\partial \xi_j^\varepsilon}{\partial t}(t, x) = (a_j - \alpha) \frac{u_j^\varepsilon(t, x) - \xi_j^\varepsilon(t, x)}{\varepsilon} + \frac{a_j}{\alpha} V_j \left(\frac{u_{j+1}^\varepsilon(t, x) - u_j^\varepsilon(t, x)}{\varepsilon} \right) \\ u_{j+n_0}^\varepsilon(t, x) = u_j^\varepsilon(t, x + \varepsilon) \\ \xi_{j+n_0}^\varepsilon(t, x) = \xi_j^\varepsilon(t, x + \varepsilon), \end{array} \right. \quad (3.10)$$

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avec comme condition initiale,

$$u_j^\varepsilon(0, x) = u_0\left(x + \frac{j\varepsilon}{n_0}\right) \quad \text{et} \quad \xi_j^\varepsilon(0, x) = \xi_0^\varepsilon\left(x + \frac{j\varepsilon}{n_0}\right). \quad (3.11)$$

Remark 3.5. *Les fonctions de la condition initiale ont été artificiellement introduite, mais simplement il s'agit de fonctions régulières telles que pour tout $\varepsilon > 0$, on a pour tout $j \in \mathbb{Z}$,*

$$\frac{1}{\varepsilon}u_0\left(\frac{j\varepsilon}{n_0}\right) = U_j(0) \quad \text{et} \quad \frac{1}{\varepsilon}\xi_0^\varepsilon\left(\frac{j\varepsilon}{n_0}\right) = U_j(0) + \frac{1}{\alpha}\dot{U}_j(0).$$

On suppose que la condition initiale satisfait l'hypothèse suivante

(B0) (Borne de gradient) Il existe $k_0, K_0 > 0$ telles que

$$\begin{cases} 0 < k_0 \leq (u_0)_x \leq K_0 \\ 0 < k_0 \leq (\xi_0^\varepsilon)_x \leq K_0. \end{cases}$$

On suppose également que

$$0 \leq \alpha(\xi_0^\varepsilon(x) - u_0(x)) \leq \min\left(V_{max}\varepsilon, \alpha \cdot \frac{u_0\left(x + \frac{\varepsilon}{n_0}\right) - u_0(x)}{2}\right) \quad \text{pour tout } x \in \mathbb{R}.$$

Remark 3.6. *La condition (B0) implique qu'initialement, les véhicules ont assez d'espace entre eux. Concernant la dernière égalité dans (B0), formellement, on peut voir que si on utilise (3.6), on a*

$$\xi_j(t, x) = u_j(t, x) + \frac{1}{\alpha} \frac{\partial u}{\partial t}(t, x).$$

Ce qui veut dire que, après le changement d'échelle, en utilisant (3.9), on a

$$\xi_j^\varepsilon(t, x) = u^\varepsilon(t, x) + \frac{\varepsilon}{\alpha} \frac{\partial u^\varepsilon}{\partial t}(t, x).$$

Par conséquent, la condition initiale doit satisfaire

$$\xi_0^\varepsilon\left(x + \frac{j\varepsilon}{n_0}\right) = u_0\left(x + \frac{j\varepsilon}{n_0}\right) + \frac{\varepsilon}{\alpha} \frac{\partial u_j^\varepsilon}{\partial t}(0, x).$$

Tout d'abord, cela implique que la condition initiale ξ_0^ε dépend de ε . La borne sur $\alpha(\xi_0^\varepsilon(x) - u_0(x))$ peut s'expliquer dans le sens où, initialement, dans le cas où les véhicules

ont assez d'espace entre eux ($k_0 \geq 2h_0n_0$), alors la vitesse initiale de chacun des véhicules doit être inférieure à V_{max} , c'est à dire

$$\alpha (\xi_0^\varepsilon(x) - u_0(x)) \leq V_{max}\varepsilon.$$

Dans le cas où initialement les véhicules ont moins d'espace entre eux, ($k_0 \leq 2h_0n_0$), la vitesse initiale de chacun des véhicules doit être bornée de telle manière à garantir la préservation de l'ordre des véhicules ($u_j \leq u_{j+1}$ et $\xi_j \leq \xi_{j+1}$), ce qui se traduit par

$$\alpha (\xi_0^\varepsilon(x) - u_0(x)) \leq V_{max} \cdot \frac{u_0\left(x + \frac{\varepsilon}{n_0}\right) - u_0(x)}{2h_0} = \alpha \cdot \frac{u_0\left(x + \frac{\varepsilon}{n_0}\right) - u_0(x)}{2}.$$

En combinant les deux dernières inégalités, on retrouve la dernière inégalité de (B0).

Détermination de l'hamiltonien effectif \bar{F}

Pour déterminer l'hamiltonien effectif, on utilise la notion de fonction enveloppe (comme dans [FIM12]) pour le système (3.8). La fonction \bar{F} est déterminée implicitement avec le problème suivant : pour tout $p \in \mathbb{R}^+$, trouver l'unique réel $\lambda = \bar{F}(p)$ tel qu'il existe une famille de couples de fonctions enveloppes $((h_j)_j, (g_j)_j)$ telle que $(u_j(t, x), \xi_j(t, x)) = (h_j(\lambda t + px), g_j(\lambda t + px))$ soit une solution de (3.8) sur $\mathbb{R} \times \mathbb{R}$. Plus précisément, on a la définition suivante.

Definition 3.7 (Fonctions enveloppes pour un système avec n_0 types de conducteurs). Soient $(V_j)_j$ et $(a_j)_j$ telles que (B1)-(B5) soient satisfaites, $p \in \mathbb{R}^+$, et un réel $\lambda \in \mathbb{R}$, on dit que la famille de fonctions $((h_j)_{j \in \mathbb{Z}}, (g_j)_{j \in \mathbb{Z}})$ est une fonction enveloppe pour (3.8) si elle satisfait pour tout $j \in \{1, \dots, n_0\}$ et pour tout $z \in \mathbb{R}$,

$$\left\{ \begin{array}{l} \lambda = \alpha(g_j - h_j) \\ h_{j+n_0}(z) = h_j(z + p) \\ h_{j+1}(z) \geq h_j(z) \\ h_j(z) = z + h_j(0) \end{array} \right. \left\{ \begin{array}{l} \lambda = (a_j - \alpha)(h_j - g_j) + \frac{a_j}{\alpha}V_j(h_{j+1} - h_j) \\ g_{j+n_0}(z) = g_j(z + p) \\ g_{j+1}(z) \geq g_j(z) \\ g_j(z) = z + g_j(0) \end{array} \right. \quad (3.12)$$

Remark 3.8. La notion de fonction enveloppe est différente de celle présentée dans [FIM12]. Ceci vient du fait que notre système (3.8) est invariant par addition de constantes alors que celui considéré dans [FIM12] est seulement invariant par addition de constantes entières. Ceci nous a permis de montrer que $h'_j = 1$ et $g'_j = 1$ et donc d'obtenir la forme particulière des h_j et des g_j .

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Theorem 3.9 (Hamiltonien effectif et fonction enveloppe (Théorème 1.7 dans [1])). *Supposons que (A1)-(A5) soient vérifiées et soit $p \in (0, +\infty)$. Alors il existe un unique réel λ pour lequel il existe une fonction enveloppe $((h_j)_j, (g_j)_j)$ satisfaisant (3.12). De plus, le réel $\lambda = \bar{F}(p)$, vu comme une fonction de p , est continue sur $(0, +\infty)$.*

Remark 3.10. *Par un simple calcul on trouve*

$$\bar{F}(p) = V_j(h_{j+1}(0) - h_j(0)) \quad \forall j \in \mathbb{Z}. \quad (3.13)$$

Calcul numérique de l'hamiltonien effectif

Nous présentons maintenant des résultats concernant le calcul numérique du \bar{F} . En utilisant la Remarque 3.10, dans le cas où l'on a un seul type de conducteur (modèle classique du second ordre) on a $\bar{F} = V$ (avec $h_j(z) = z + j \cdot p$). Mais si $n_0 > 1$, on n'a pas de formule explicite et l'on est obligé d'utiliser des méthodes numériques pour déterminer la fonction \bar{F} . Concernant les méthodes numériques pour l'approximation des hamiltoniens effectifs, on renvoie, entre autres, aux papiers de Capuzzo-Dolcetta et Ishii [CDI01] et Camilli, Capuzzo-Dolcetta et Gomes [CCDG08] et dans le cadre des équations non-locales, on renvoie aux papiers de Ghorbel, Hoch et Monneau [GHM08] et Cacace, Chambolle et Monneau [CCM12].

Notre résultat d'homogénéisation numérique est basé sur le résultat suivant.

Proposition 3.11 (Solution particulière des solutions de (3.8) et approximation de $\bar{F}(p)$ (Proposition 1.2 dans [5])). *Supposons (B1)-(B5) et soit $p > 0$. Soit $((u_j)_j, (\xi_j)_j)$ la solution de (3.8) avec une condition initiale telle que $(u_0)_x = (\xi_0)_x = p$. Alors $((u_j)_j, (\xi_j)_j)$ satisfait*

$$u_j(t, x) = px + u_j(t, 0) \quad \text{et} \quad \xi_j(t, x) = px + \xi_j(t, 0), \quad (3.14)$$

et il existe une constante C , qui dépend uniquement de p, V_{max}, α et n_0 telle que pour tout $T \in (0, +\infty)$,

$$\left| \frac{u_1(T, 0)}{T} - \bar{F}(p) \right| \leq \frac{C}{T}. \quad (3.15)$$

Remark 3.12. *Cette proposition nous donne une manière d'approximer la fonction \bar{F} en un seul point $p \in \mathbb{R}$. Pour obtenir \bar{F} sur un intervalle, l'idée est alors de discrétiser l'intervalle et pour chaque point de discrétisation p , d'approximer $((u_j)_j, (\xi_j)_j)$, la solution de (3.8) avec une condition initiale telle que $(u_0)_x = (\xi_0)_x = p$.*

Schémas numériques pour (3.8). On présente ensuite les schémas pour (3.8) utilisés dans [5]. On note Δt le pas en temps et $t_n = n\Delta t$. Le but est de calculer une approximation de la solution de (3.8) avec la condition initiale $(px + pj/n, px + pj/n)$. En utilisant la forme particulière des solutions donnée par la Proposition 3.11, on est seulement intéressé par l'approximation numérique de $u_j(t_n, 0)$ et de $\xi_j(t_n, 0)$ que l'on notera respectivement u_j^n et ξ_j^n . On propose un schéma explicite et un schéma implicite pour calculer u_j^n et ξ_j^n .

Schéma explicite aux différences finies. On considère le schéma numérique suivant, pour tout $j \in \mathbb{Z}$ et pour tout $n \in \mathbb{N}$,

$$\left\{ \begin{array}{l} \frac{u_j^{n+1} - u_j^n}{\Delta t} = \alpha (\xi_j^n - u_j^n) \\ \frac{\xi_j^{n+1} - \xi_j^n}{\Delta t} = (a_j - \alpha) (u_j^n - \xi_j^n) + \frac{a_j}{\alpha} V_j (u_{j+1}^n - u_j^n) \\ u_{n_0+j}^n = u_j^n + p \\ \xi_{n_0+j}^n = \xi_j^n + p, \end{array} \right. \quad (3.16)$$

avec la condition initiale

$$u_j^0 = \frac{pj}{n_0} \quad \text{et} \quad \xi_j^0 = \frac{pj}{n_0}. \quad (3.17)$$

Schéma implicite aux différences finies. On considère également le schéma numérique suivant, pour tout $j \in \mathbb{Z}$, et pour tout $n \in \mathbb{N}$,

$$\left\{ \begin{array}{l} \frac{u_j^{n+1} - u_j^n}{\Delta t} = \alpha (\xi_j^{n+1} - u_j^{n+1}) \\ \frac{\xi_j^{n+1} - \xi_j^n}{\Delta t} = (a_j - \alpha) (u_j^{n+1} - \xi_j^{n+1}) + \frac{a_j}{\alpha} V_j (u_{j+1}^{n+1} - u_j^{n+1}) \\ u_{j+n_0}^n = u_j^n + p \\ \xi_{j+n_0}^n = \xi_j^n + p \end{array} \right. \quad (3.18)$$

avec la condition initiale donnée par (3.17).

Remark 3.13. Grâce à la forme particulière de la solution de (3.8) avec la condition initiale $(px + pj/n, px + pj/n)$, l'approximation de $u_j(t, x)$ et de $\xi_j(t, x)$ est donnée respectivement par

$$u_j^n + px \quad \text{et} \quad \xi_j^n + px. \quad (3.19)$$

Estimation numérique de $\bar{F}(p)$, $p > 0$. Étant donné $n_T \in \mathbb{N}$, on définit $T = n_T \Delta t$ et on introduit notre approximation de l'hamiltonien effectif,

$$\lambda^T = \frac{u_1^{n_T}}{T}. \quad (3.20)$$

Theorem 3.14 (Estimation numérique de l'hamiltonien effectif (Théorème 1.5 dans [5])). Soit $p > 0$, et soit λ^T l'approximation numérique de $\bar{F}(p)$ donné par (3.20). Si u_j^n a été calculé avec le schéma explicite (resp. avec le schéma implicite), on suppose également que

$$\Delta t \leq \frac{1}{\max_{j \in \{1, \dots, n_0\}} (a_j)} \quad \left(\text{resp.} \quad \Delta t \leq \frac{1}{2\alpha} \left(1 + \min_{j \in \{1, \dots, n_0\}} \sqrt{a_j + \frac{4\alpha}{a_j}} \right) \right).$$

Alors il existe deux constantes $C_1 > 0$ et $C_2 > 0$, telles que,

$$|\lambda^T - \bar{F}(p)| \leq \frac{C_1}{T} + C_2(\Delta t + \sqrt{\Delta t}). \quad (3.21)$$

3.2 Homogénéisation d'un modèle contenant une perturbation locale

On présente maintenant les résultats de [2], [6] et [4] qui sont présentés dans les Chapitres 3, 4 et 5. Dans le premier papier, on obtient un résultat d'homogénéisation pour un modèle microscopique du première ordre contenant une perturbation locale. Dans le deuxième papier, on présente un résultat d'homogénéisation numérique qui vient compléter celui de [2]. Dans le dernier papier, on obtient un résultat d'homogénéisation pour un modèle microscopique du second ordre qui contient une perturbation locale.

L'idée de nos travaux est de considérer à l'échelle microscopique une perturbation locale qui ralentit les véhicules. On veut comprendre comment cette perturbation influence la dynamique du trafic à l'échelle macroscopique. On considère une perturbation située autour de l'origine (voir la Figure 3.3) avec un rayon d'influence $r \in (0, +\infty)$.

A l'échelle macroscopique, il est naturel d'obtenir une équation de Hamilton-Jacobi avec une condition de jonction en 0 (vu que la taille de la perturbation diminue avec le

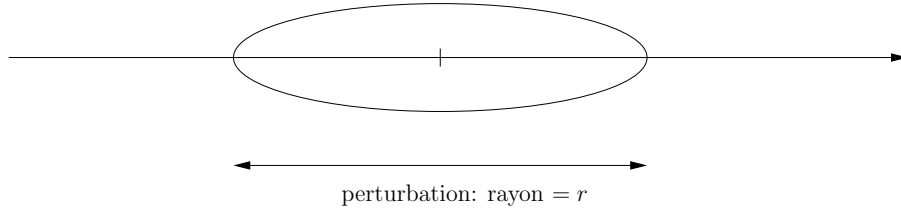


Figure 3.3: Représentation schématique d'une perturbation locale de rayon r à l'échelle microscopique.



Figure 3.4: Représentation schématique de la condition de jonction à l'échelle macroscopique.

changement d'échelle) et un limiteur de flux effectif (voir la Figure 3.4). La difficulté est alors de construire ce limiteur de flux effectif.

Récemment, la théorie des équations de Hamilton-Jacobi avec des conditions de jonction, ou plus généralement dans les réseaux, a connu de grands développements, particulièrement à partir des travaux de Achdou, Camilli, Cutri, et Tchou [ACCT13] et de Imbert, Monneau et Zidani [IMZ13]. On renvoie également au travail de Imbert et Monneau [IM14] dans lequel, comme évoqué précédemment, les auteurs donnent une définition (de solution de viscosité) adaptée à des problèmes munis de conditions de jonction ainsi que des résultats de comparaison et de stabilité.

Dans cette section, on utilise, en particulier, l'idée développée par Achdou et Tchou dans [AT15], puis par Galise, Imbert et Monneau dans [GIM15], et par Lions dans son cours au "Collège de France" [Lio14], qui consiste à construire des correcteurs dans des domaines tronqués.

Dans toute cette section, on considère uniquement le cas où $V_j = V$ et $a_j = a$ pour tout $j \in \mathbb{Z}$.

Modèle général du premier ordre

Dans cette section on introduit rapidement les résultats de [2] et qui sont présentés en détail dans le Chapitre 3. On considère un modèle microscopique de la forme

$$\dot{U}_j(t) = V(U_{j+1}(t) - U_j(t)) \cdot \phi(U_j(t)), \quad (3.22)$$

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où la fonction $\phi : \mathbb{R} \rightarrow [0, 1]$ simule la présence d'une perturbation locale autour de l'origine qui ralentit les véhicules. On note r le rayon d'influence de cette perturbation.

Le but de ce modèle est de simuler la présence d'une perturbation locale autour de l'origine, par exemple un ralentisseur, un accident sur le bord de la route (les conducteurs ralentissent pour voir l'accident), une zone scolaire,... L'originalité de ce modèle est que l'on considère une perturbation locale et non pas périodique.

On fait les hypothèses suivantes sur V et ϕ :

Hypothèse (C)

- (C1) $V : \mathbb{R} \rightarrow \mathbb{R}$ est lipschitzienne, positive.
- (C2) V est croissante sur \mathbb{R} .
- (C3) Il existe $h_0 \in (0, +\infty)$ tel que pour tout $h \leq h_0$, $V(h) = 0$.
- (C4) Il existe $h_{max} \in (h_0, +\infty)$ tel que pour tout $h \geq h_{max}$, $V(h) = V(h_{max}) =: V_{max}$.
- (C5) Il existe un réel $p_0 \in [-1/h_0, 0)$ tel que la fonction $p \mapsto pV(-1/p)$ soit décroissante sur $[-1/h_0, p_0)$ et croissante sur $[p_0, 0)$.
- (C6) La fonction $\phi : \mathbb{R} \rightarrow [0, 1]$ est lipschitzienne et $\phi(x) = 1$ pour $|x| \geq r$.

Remark 3.15. *Les hypothèses (C1)-(C2)-(C3)-(C5) sont équivalentes aux hypothèses (A1)-(A2)-(A3)-(A4), on a ajouté l'hypothèse (C4) pour travailler avec V' avec un support compact. Mais en modifiant légèrement les fonctions de vitesse optimal classiques, on peut construire des fonctions qui satisfont hypothèse (C). Par exemple, la fonction de Greenshield :*

$$V(h) = \begin{cases} 0 & \text{for } h \leq h_0, \\ V_{max} \left(1 - \left(\frac{h_0}{h} \right)^n \right) & \text{for } h_0 < h \leq h_{max}, \\ V_{max} \left(1 - \left(\frac{h_0}{h_{max}} \right)^n \right) & \text{for } h_{max} < h, \end{cases}$$

avec $n \in \mathbb{N} \setminus \{0\}$. On remarque également que la condition (C4) est pertinente d'un point de vue de la modélisation et signifie que si la voiture qui est devant est très loin (par exemple si on ne peut plus la voir) alors on souhaite rouler à la vitesse maximale V_{max} . Voir la Figure 1.2 pour une représentation schématique d'une fonction de vitesse optimale satisfaisant l'hypothèse (C).

Injection du système d'EDOs dans une seule EDP

Remark 3.16 (Première approche : formulation lagrangienne). *Une première idée pour traiter le problème d'homogénéisation est de faire comme précédemment (comme dans [1]), et de considérer la fonction*

$$v : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}, \quad v(t, y) = U_{\lfloor y \rfloor}(t).$$

Cette fonction satisfait pour tout $(t, y) \in [0, T] \times \mathbb{R}$

$$\begin{cases} v_t(t, y) = V(u(t, y+1) - u(t, y)) \cdot \phi(v(t, y)), \\ v(0, y) = v_0(y). \end{cases} \quad (3.23)$$

La difficulté avec cette formulation est que la fonction ϕ est évaluée en $v(t, y)$ et non pas sur un point physique de la route et formellement, la jonction sera localisée en $v = 0$. Dans ce cas, la notion de jonction n'est pas bien définie et c'est pour cela que nous avons décidé d'utiliser une autre formulation (dans laquelle la perturbation sera évaluée sur un point physique de la route) en utilisant la même technique que dans le papier de Forcadel, Imbert et Monneau [FIM09b]. C'est à dire en utilisant la primitive de la mesure empirique des positions des véhicules. De cette manière, on peut utiliser les résultats de Imbert et Monneau [IM14] pour les hamiltoniens quasi-convexes avec des conditions de jonction.

On veut étudier le trafic quand le nombre de véhicules par unité de longueur tend vers l'infini. Pour cela, on introduit la fonction de "distribution cumulative", ρ^ε , définie par

$$\rho^\varepsilon(t, y) = -\varepsilon \left(\sum_{i \geq 0} H(y - \varepsilon U_i(t/\varepsilon)) + \sum_{i < 0} (-1 + H(y - \varepsilon U_i(t/\varepsilon))) \right),$$

avec

$$H(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases}$$

Sous l'hypothèse (C), on peut montrer que la fonction ρ^ε est une solution (éventuellement discontinue) de l'EDP non-locale suivante,

$$u_t^\varepsilon + M^\varepsilon \left[\frac{u^\varepsilon(t, \cdot)}{\varepsilon} \right] (x) \cdot \phi \left(\frac{x}{\varepsilon} \right) \cdot |u_x^\varepsilon| = 0 \quad \text{sur } (0, +\infty) \times \mathbb{R} \quad (3.24)$$

où M^ε est un opérateur non-local défini par

$$M^\varepsilon[U](x) = \int_{-\infty}^{+\infty} J(z) E(U(x + \varepsilon z) - U(x)) dz - \frac{3}{2} V_{max}$$

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et

$$E(z) = \begin{cases} 0 & \text{si } z \geq 0 \\ 1/2 & \text{si } -1 \leq z < 0 \\ 3/2 & \text{si } z < -1, \end{cases} \quad \text{et } J = V' \text{ sur } \mathbb{R}.$$

À (3.24), on ajoute la condition initiale

$$u^\varepsilon(0, x) = u_0(x) \quad \text{sur } \mathbb{R}. \quad (3.25)$$

On suppose aussi que cette condition initiale satisfait l'hypothèse suivante :

(C0) (Borne de gradient) La fonction u_0 est Lipschitz et satisfait

$$-k_0 \leq (u_0)_x \leq 0. \quad (3.26)$$

Remark 3.17. *Cette condition assure qu'initialement, les véhicules ont au moins une distance h_0 entre eux et comme on travaille avec un modèle du premier ordre, cette distance de sécurité est conservée. De plus, on choisira u_0 une fonction régulière telle que pour tout $\varepsilon > 0$,*

$$|\rho^\varepsilon(0, x) - u_0(x)| \leq f(\varepsilon),$$

avec $f(\varepsilon) \rightarrow 0$ quand ε tend vers 0.

Homogénéisation spécifiée du modèle du premier ordre

On définit $k_0 = 1/h_0$ et $\bar{H} : \mathbb{R} \rightarrow \mathbb{R}$, par

$$\bar{H}(p) = \begin{cases} -p - k_0 & \text{pour } p < -k_0, \\ -V\left(\frac{-1}{p}\right)|p| & \text{pour } -k_0 \leq p \leq 0, \\ p & \text{pour } p > 0. \end{cases}$$

On remarque que la fonction \bar{H} est continue, coercive $\left(\lim_{|p| \rightarrow +\infty} \bar{H}(p) = +\infty\right)$ et grâce à (C5), il existe un unique point $p_0 \in [-k_0, 0]$ tel que

$$\begin{cases} \bar{H} \text{ est strictement décroissante sur } (-\infty, p_0), \\ \bar{H} \text{ est strictement croissante sur } (p_0, +\infty). \end{cases}$$

On note

$$H_0 = \min_{p \in \mathbb{R}} \bar{H}(p) = \bar{H}(p_0).$$

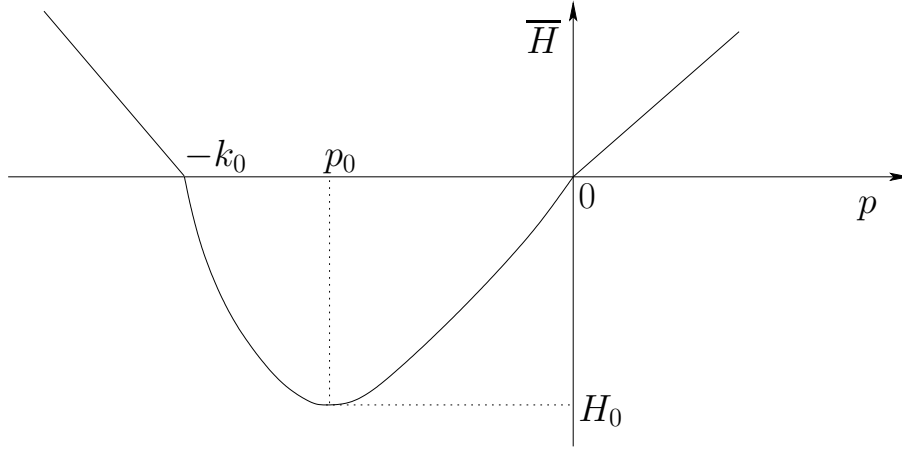


Figure 3.5: Représentation schématique de \bar{H} .

La Figure 3.5 contient une représentation schématique de la fonction \bar{H} .

Le résultat principal concernant l'homogénéisation d'un modèle avec perturbation local est que la solution de viscosité de (3.24)-(3.25) converge localement uniformément dans $(0, +\infty) \times \mathbb{R}$ quand ε tend vers 0 vers l'unique solution de viscosité du problème suivant

$$\begin{cases} u_t^0 + \bar{H}(u_x^0) = 0 & \text{pour } (t, x) \in (0, +\infty) \times (-\infty, 0) \\ u_t^0 + \bar{H}(u_x^0) = 0 & \text{pour } (t, x) \in (0, +\infty) \times (0, +\infty) \\ u_t^0 + F_{\bar{A}}^-(u_x^0(t, 0^-), u_x^0(t, 0^+)) = 0 & \text{for } (t, x) \in (0, +\infty) \times \{0\} \\ u^0(0, x) = u_0(x) & \text{pour } x \in \mathbb{R}, \end{cases} \quad (3.27)$$

où \bar{A} est à déterminer et $F_{\bar{A}}$ est définie par

$$F_{\bar{A}}(p_-, p_+) = \max(\bar{A}, \bar{H}^+(p_-), \bar{H}^-(p_+)),$$

avec

$$\bar{H}^-(p) = \begin{cases} \bar{H}(p) & \text{si } p \leq p_0, \\ \bar{H}(p_0) & \text{si } p \geq p_0, \end{cases} \quad \text{et} \quad \bar{H}^+(p) = \begin{cases} \bar{H}(p_0) & \text{si } p \leq p_0, \\ \bar{H}(p) & \text{si } p \geq p_0. \end{cases} \quad (3.28)$$

Theorem 3.18 (Condition de jonction par homogénéisation : application au trafic routier (Théorème 2.5 dans [2])). *Supposons que (C) est satisfaite et que, au temps initial, on a pour tout $i \in \mathbb{Z}$,*

$$U_i(0) \leq U_{i+1}(0) - h_0.$$

On suppose également qu'il existe une constant $R > 0$ telle que, pour tout $i \in \mathbb{Z}$, si $|U_i(0)| \geq R$

$$U_{i+1}(0) - U_i(0) = h,$$

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avec $h \geq h_0$. On définit alors la fonction u_0 (satisfaisant (A0)), par $u_0(x) = -x/h$ pour tout $x \in \mathbb{R}$.

Alors il existe $\bar{A} \in [H_0, 0]$ tel que la fonction ρ^ε définie par (3.24) converge vers l'unique solution u^0 de (3.27). De plus, on a que u^0 satisfait $-k_0 \leq u_x^0 \leq 0$.

Remark 3.19. On remarque que (3.27) est équivalent (en dérivant en espace) à un modèle LWR (voir [LW55, Ric56]) avec une condition limitant le flux à l'origine. En fait, le diagramme fondamental est $\rho V(1/\rho)$ et u_x^0 correspond à la densité des véhicules.

Propriétés qualitatives du limiteur de flux effectif

Proposition 3.20 (Propriétés qualitatives du limiteur de flux (Proposition 2.12 dans [2])). *Supposons que (C) est vérifiée. On a les propriétés qualitatives suivantes sur le limiteur de flux effectif :*

(i) (Monotonie du limiteur de flux). Soient $\phi_1, \phi_2 : \mathbb{R} \rightarrow [0, 1]$ deux fonctions satisfaisant (C6). Soient \bar{A}_1 et \bar{A}_2 les limiteurs de flux correspondants donnés par le Théorème 3.18. Si, pour tout $x \in \mathbb{R}$, on a

$$\phi_1(x) \leq \phi_2(x)$$

alors

$$\bar{A}_1 \geq \bar{A}_2.$$

(ii) (Interruption du flux) Soit ϕ une fonction satisfaisant (C6). Si $\phi = 0$ dans un intervalle ouvert, alors on a

$$\bar{A} = 0.$$

Détermination du limiteur de flux effectif \bar{A}

Pour compléter le résultat de la section précédente, il est important d'expliquer comment est déterminé le limiteur de flux effectif \bar{A} .

Comme toujours, le limiteur de flux effectif est déterminé de manière implicite comme étant l'unique constante λ telle qu'il existe une solution w à l'équation de Hamilton-Jacobi suivante

$$M[w](x) \cdot \phi(x) \cdot |w_x| = \lambda \quad \text{pour } x \in \mathbb{R}, \tag{3.29}$$

3. Résultats principaux de cette thèse

avec les bonnes pentes à l'infini. Plus précisément, si on définit w^ε par $w^\varepsilon(x) = \varepsilon w(x/\varepsilon)$, on impose que w^ε converge quand ε tend vers 0 vers une fonction W de la forme

$$W(x) = p^- x 1_{\{x < 0\}} + p^+ x 1_{\{x \geq 0\}},$$

avec p^- et p^+ satisfaisant

$$\overline{H}(p^-) = \overline{H}^-(p^-) = \overline{A} \quad \text{et} \quad \overline{H}(p^+) = \overline{H}^+(p^+) = \overline{A}.$$

Pour construire le couple (\overline{A}, w) , l'idée introduite par Achdou, Tchou [AT15] et Galise, Imbert et Monneau [GIM15], est de considérer un problème dans la cellule sur un domaine tronqué puis de passer à la limite quand la taille du domaine tend vers l'infini. Pour prédire les bonnes pentes à l'infini, il faut donc imposer de bonnes conditions aux bords. Dans notre cas, l'équation (3.29) étant non-locale, il est difficile d'imposer ces conditions aux bords. C'est pour cette raison que nous introduisons un nouvel hamiltonien qui sera égal à l'opérateur non-local dans l'intervalle $[-R, R]$ et qui sera remplacé par un hamiltonien local en dehors de $[-R-1, R+1]$ afin de pouvoir imposer des conditions aux limites. Plus précisément, pour $l \in (r, +\infty)$, $r \ll l$ et $r \leq R \ll l$, on considère le problème suivant : trouver $\lambda_{l,R}$, tel qu'il existe une solution $w^{l,R}$ de

$$\begin{cases} G_R(x, [w^{l,R}], w_x^{l,R}) = \lambda_{l,R} & \text{si } x \in (-l, l) \\ \overline{H}^-(w_x^{l,R}) = \lambda_{l,R} & \text{si } x = -l \\ \overline{H}^+(w_x^{l,R}) = \lambda_{l,R} & \text{si } x = l, \end{cases} \quad (3.30)$$

avec

$$G_R(x, [U], q) = \psi_R(x) \phi(x) \cdot M[U](x) \cdot |q| + (1 - \psi_R(x)) \cdot \overline{H}(q), \quad (3.31)$$

et $\psi_R \in C^\infty$, $\psi_R : \mathbb{R} \rightarrow [0, 1]$, telle que

$$\psi_R \equiv \begin{cases} 1 & \text{sur } [-R, R] \\ 0 & \text{dehors de } [-R-1, R+1], \end{cases} \quad \text{et} \quad \psi_R(x) < 1 \quad \forall x \notin [-R, R]. \quad (3.32)$$

Le limiteur de flux effectif \overline{A} est alors construit comme la limite quand l tend vers $+\infty$ puis R tend vers $+\infty$ de $\lambda_{l,R}$. La fonction $w^{l,R}$ permet de construire la fonction w avec les bonnes pentes à l'infini.

On a également le résultat suivant qui caractérise le limiteur de flux.

Theorem 3.21 (Limiteur de flux effectif (Théorème 2.10 dans [2])). *Supposons que l'hypothèse (C) soit satisfaite. On définit l'ensemble de fonctions suivant*

$$\mathcal{S} = \left\{ w \text{ t.q. } \exists \text{ une fonction lipschitzienne } m \quad \text{et une constante } C \geq 0 \text{ t.q.} \right. \\ \left. \|w - m\|_{L^\infty(\mathbb{R})} \leq C \right\}.$$

Alors on a

$$\overline{A} = \inf \{ \lambda \in \mathbb{R} : \exists w \in \mathcal{S} \text{ solution de (3.29)} \}.$$

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Calcul numérique du limiteur de flux \bar{A}

Comme on l'a dit précédemment, le limiteur de flux \bar{A} est la limite de $\lambda_{l,R}$ quand l puis R tendent vers l'infini. On cherche alors à avoir une approximation de $\lambda_{l,R}$. Pour cela, on suit la construction faite dans [2] et on considère le problème tronqué dans la cellule approchée, pour tout $\delta > 0$

$$\begin{cases} \delta v^\delta + \psi_R(x)M[v^\delta](x) \cdot \phi(x) \cdot |v_x^\delta| + (1 - \psi_R(x))\bar{H}(v_x^\delta) = 0 & \text{pour } x \in (-l, l) \\ \delta v^\delta + \bar{H}^-(v_x^\delta) = 0 & \text{pour } x = -l \\ \delta v^\delta + \bar{H}^+(v_x^\delta) = 0 & \text{pour } x = l. \end{cases} \quad (3.33)$$

Il a été montré dans [2, Preuve de la Proposition 6.4] que $-\delta v^\delta(0) \rightarrow \lambda_{l,R}$ quand $\delta \rightarrow 0$. On cherche maintenant à obtenir une approximation numérique de v^δ (solution de (3.33)).

Schéma numérique pour (3.33). Le schéma numérique que nous utilisons pour l'opérateur non-local a été inspiré par celui de [CCM12, For08] et le schéma numérique pour l'opérateur local a été inspiré par celui de [CLM15].

On considère une grille uniforme de l'intervalle $[-l, l]$ avec $2n + 1$ points, $n \in \mathbb{N} \setminus \{0\}$, et on note le pas de discrétisation $\Delta x = l/n$. Pour tout $i \in \{1, \dots, n\}$, on note $x_i = \Delta x \cdot i$ les nœuds de la grille. En particulier, on a $x_0 = 0$, $x_{-n} = -l$ et $x_n = l$.

Pour toute fonction discrète $v : \{1, \dots, n\} \rightarrow \mathbb{R}$, on note $v_\#$ son extension sur \mathbb{R} continue par morceaux et définie par

$$v_\#(x) = \sum_{i=1}^n v_i \cdot \chi_{Q_i}(x) \quad (3.34)$$

avec

$$Q_i = \begin{cases} [-l, -l + \Delta x/2) & \text{si } i = -n, \\ [x_i - \Delta x/2, x_i + \Delta x/2) & \text{si } i \in \{-n + 1, \dots, n - 1\} \\ [l - \Delta x/2, l] & \text{si } i = n. \end{cases} \quad (3.35)$$

Discrétisation de l'opérateur non-local. Pour toute fonction discrète $v : \{1, \dots, n\} \rightarrow \mathbb{R}$, on définit un opérateur discret non-local

$$M_i^d[v] := M[v_\#](x_i) \quad \text{et} \quad \tilde{M}_i^d[v] := \tilde{M}[v_\#](x_i). \quad (3.36)$$

Discrétisation du gradient. On considère les différences finies standard forward et backward du premier ordre :

$$D^+v(x_i) = \frac{v(x_{i+1}) - v(x_i)}{\Delta x} \quad \text{et} \quad D^-v(x_i) = \frac{v(x_i) - v(x_{i-1})}{\Delta x}. \quad (3.37)$$

Enfin, on définit $Dv_i = (D^-v(x_i), D^+v(x_i))$.

Comme dans [CCM12], on utilise une discrétisation monotone de Osher et Sethian [OS88] pour le gradient. Soit $S = (p, q) \in \mathbb{R}^2$. On définit une fonction qu'on utilise par la suite pour la discrétisation du gradient

$$G^+(S) = \left(\max(p, 0)^2 + \min(q, 0)^2 \right)^{1/2}.$$

Discrétisation de l'opérateur local. En ce qui concerne l'opérateur $\overline{H}(\cdot)$, comme dans [CLM15] on considère la discrétisation suivante

$$\overline{H}_d(Dv_i) = \max \left\{ \overline{H}^+(D^-v_i), \overline{H}^-(D^+v_i) \right\}, \quad (3.38)$$

où \overline{H}^- et \overline{H}^+ ont été définies dans (3.28).

Finalement, on introduit pour toute fonction discrète $v : \{1, \dots, n\} \rightarrow \mathbb{R}$,

$$R_i[v] := \mathcal{R}_i([v], Dv_i) = \psi_R(x_i) \cdot M_i^d[v] \cdot \phi(x_i) \cdot G^+(Dv_i) + (1 - \psi_R(x_i)) \cdot \overline{H}_d(Dv_i).$$

On définit similairement \tilde{R} et $\tilde{\mathbb{R}}$ en remplaçant M^d par \tilde{M}^d .

Pour résumer, la version discrète de (3.33) est

$$\begin{cases} \delta v_i + R_i[v] = 0 & \text{pour } i \in \{-n + 1, \dots, n - 1\} \\ \delta v_i + \overline{H}^-(D^+v_i) = 0 & \text{pour } i = -n \\ \delta v_i + \overline{H}^+(D^-v_i) = 0 & \text{pour } i = n. \end{cases} \quad (3.39)$$

Theorem 3.22 (Convergence (Théorème 3.11 dans [6])). *En utilisant les mêmes notations que dans (3.34)-(3.35). Soit $(v_i^{\Delta x})_{i \in \{-n, \dots, n\}}$ une solution de (3.39) (voir la définition exacte dans le Chapitre 4), alors la fonction $v_{\#}^{\Delta x}$ (définie en (3.34)-(3.35)) converge localement uniformément quand $\Delta x \rightarrow 0$ vers l'unique solution (continue) de viscosité de (3.33).*

Homogénéisation spécifiée d'un modèle du second ordre contenant une perturbation

On présente maintenant les résultats de [4]. Les résultats pour le modèle du second ordre sont similaires à ceux présentés juste avant pour le modèle du premier ordre. L'intérêt de travailler avec un modèle du second ordre est qu'il est plus réaliste vu qu'il prend en compte le temps de réaction des conducteurs. Cependant, le fait de travailler avec un modèle du second ordre ajoute beaucoup de difficultés techniques. En effet, on a une combinaison entre les difficultés provenant du fait d'avoir un modèle contenant une perturbation locale mais aussi du fait qu'il s'agit d'un modèle du second ordre.

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Au départ on considère le modèle classique de Bando (1.2) et on le modifie afin de pouvoir simuler la présence d'une perturbation locale. On obtient le système suivant pour tout $t > 0$ et $j \in \mathbb{Z}$,

$$\ddot{U}_j(t) = a \left(V(U_{j+1}(t) - U_j(t)) \cdot \phi(U_j(t)) - \dot{U}_j(t) \right), \quad (3.40)$$

où U_j est la position de la j -ème voiture, \dot{U}_j sa vitesse et \ddot{U}_j son accélération. Comme précédemment, la fonction ϕ simule la présence d'une perturbation locale autour de l'origine et on note r son rayon d'influence. Dans ce modèle, a et V représentent respectivement la sensibilité des conducteurs et la fonction de vitesse optimale. On suppose que V et ϕ satisfont l'hypothèse (C) et que l'on a également l'hypothèse suivante :

(D) (Monotonie) $a \geq 4 \|V'\|_\infty \|\phi'\|_\infty + \|\phi'\|_\infty \|V\|_\infty.$

Remark 3.23 (Remarque sur (D)). *L'hypothèse (D) implique que pour tout $(b, x) \in \mathbb{R}^2$, la fonction $f : z \rightarrow \frac{a}{2}z - 2V(b+z)\phi(x-z)$ est croissante. Ceci est en fait, équivalent à l'hypothèse (B2) adaptée au fait que l'on a une perturbation. En effet, si l'on considère $\phi \equiv 1$, on retrouve exactement (B2).*

Pour obtenir un résultat d'homogénéisation pour (3.40) on est obligé de combiner les techniques utilisées pour le modèle (3.1) et le modèle (3.22). C'est à dire qu'il faut transformer (3.40) en un modèle du premier ordre à deux variables et ensuite utiliser deux fonctions de densité cumulatives. Plus précisément, on introduit pour tout $j \in \mathbb{Z}$,

$$\Xi_j(t) = U_j(t) + \frac{1}{\alpha} \dot{U}_j(t) \quad \text{avec} \quad \alpha = \frac{a}{2}.$$

Le système suivant est équivalent à (3.40) : pour tout $j \in \mathbb{Z}$, pour tout $t \in (0, +\infty)$,

$$\begin{cases} \dot{U}_j(t) = \alpha (\Xi_j(t) - U_j(t)) \\ \dot{\Xi}_j(t) = \alpha (U_j(t) - \Xi_j(t)) + 2V(U_{j+1}(t) - U_j(t)) \cdot \phi(U_j(t)). \end{cases} \quad (3.41)$$

On peut voir qu'il s'agit de la même idée que dans [FIM09a, FIM12, 1]. Ensuite, on introduit deux fonctions cumulatives de distribution (comme dans [FIM09b, 2]) : ρ^ε et σ^ε définies par

$$\rho^\varepsilon(t, y) = -\varepsilon \left(\sum_{i \geq 0} H(y - \varepsilon U_i(t/\varepsilon)) + \sum_{i < 0} (-1 + H(y - \varepsilon U_i(t/\varepsilon))) \right), \quad (3.42)$$

et

$$\sigma^\varepsilon(t, y) = -\varepsilon \left(\sum_{i \geq 0} H(y - \varepsilon \Xi_i(t/\varepsilon)) + \sum_{i < 0} (-1 + H(y - \varepsilon \Xi_i(t/\varepsilon))) \right). \quad (3.43)$$

On peut montrer que $(\rho^\varepsilon, \sigma^\varepsilon)$ est une solution discontinue d'un système d'EDPs non-locales (voir Chapitre 5 pour plus de détails). On a alors le résultat suivant.

Theorem 3.24 (Condition de jonction par homogénéisation : application au trafic routier (Théorème 3.3 dans [4])). *Supposons que (C) et (D) sont vérifiées et qu'au temps initial $(U_i(0), \Xi_i(0))_i$ satisfait pour tout $i \in \mathbb{Z}$*

$$0 \leq \Xi_i(0) - U_i(0) \leq \frac{V_{max}}{\alpha}, \quad U_{i+1}(0) - \Xi_i(0) \geq h_0, \quad \text{et} \quad U_{i+1}(0) - U_i(0) \leq h_{max}.$$

De plus, on suppose qu'il existe une constante $R > 0$ telle que, pour tout $i \in \mathbb{Z}$, si $|U_i(0)| \geq R$,

$$U_{i+1}(0) - U_i(0) = h$$

et si $|\Xi_i(0)| \geq R$

$$\Xi_{i+1}(0) - \Xi_i(0) = h,$$

avec $h_{max} \geq h \geq h_0$. On définit $u_0(x) = -x/h$. Alors il existe une unique constante $\bar{A} \in [H_0, 0]$ telle que les fonctions ρ^ε et σ^ε définies respectivement par (3.42) et (3.43) convergent localement uniformément vers l'unique solution de viscosité u^0 de (3.27).

3.3 Homogénéisation d'un modèle microscopique d'une bifurcation

On présente maintenant les résultats de [3] qui seront détaillés dans le Chapitre 6. Dans ce papier, on a réussi à obtenir un résultat d'homogénéisation pour un modèle microscopique du premier ordre modélisant une bifurcation simple.

Modèle microscopique du premier ordre pour une bifurcation

La Figure 3.6 donne une représentation schématique du modèle microscopique que l'on considère.

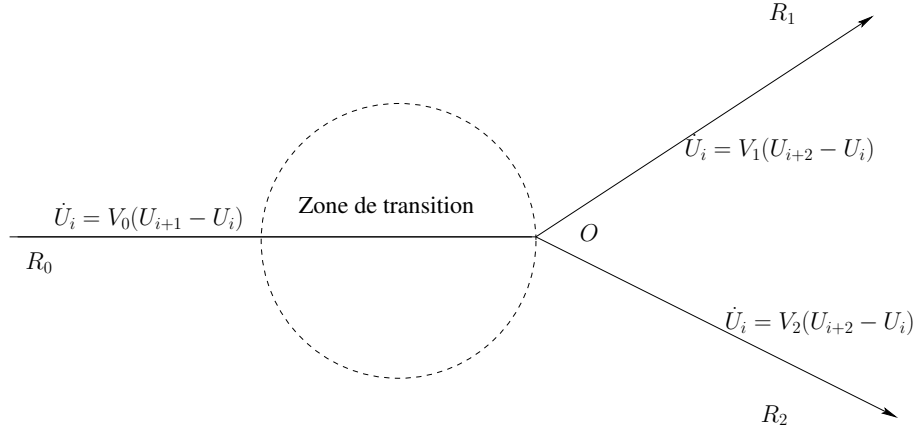


Figure 3.6: Représentation schématique simplifiée du modèle microscopique.

Plus précisément, on considère une voie entrante R_0 qui se sépare en deux voies sortantes R_1 et R_2 . On note toujours $U_i(t)$ la position du i -ème véhicule, et l'on suppose que les véhicules avec des indices impairs vont dans R_1 et que ceux avec des indices pairs vont vers R_2 . Pour finir, on suppose que sur chaque voie R_i la vitesse de chacune des voitures est donnée par une fonction de vitesse optimal V_i . A l'échelle macroscopique on s'attend à obtenir une équation de Hamilton-Jacobi dans chacune des voies et une condition de jonction à l'origine (voir la Figure 3.7, avec u_x^0 qui sera liée à la densité des véhicules et les hamiltoniens effectifs \bar{H}_i qui sont définies plus bas).

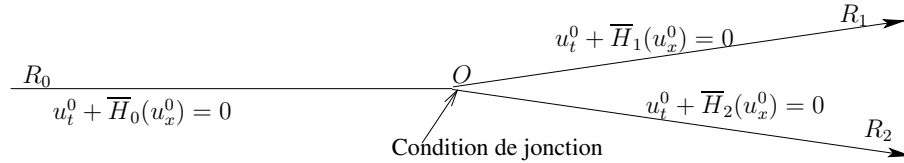


Figure 3.7: Représentation schématique du modèle macroscopique.

Avant la bifurcation (i.e pour $U_i(t) \ll 0$) on suppose que la vitesse de chacune des voitures est donnée par

$$\dot{U}_i(t) = V_0 (U_{i+1}(t) - U_i(t)),$$

et après la bifurcation (i.e pour $U_i(t) \gg 0$) on suppose que

$$\dot{U}_i(t) = \begin{cases} V_1 (U_{i+2}(t) - U_i(t)) & \text{si } i \text{ est impair} \\ V_2 (U_{i+2}(t) - U_i(t)) & \text{si } i \text{ est pair.} \end{cases}$$

Pour faire le passage de la vitesse V_0 à V_i , $i = 1$ ou 2 , on introduit une fonction de transition

$\bar{\phi}$ et on considère le système suivant, pour tout $i \in \mathbb{Z}$ et pour tout $t > 0$:

$$\dot{U}_i(t) = \begin{cases} \bar{\phi}\left(U_i(t), V_0(U_{i+1}(t) - U_i(t)), V_1(U_{i+2}(t) - U_i(t))\right) & \text{si } i \text{ est impair} \\ \bar{\phi}\left(U_i(t), V_0(U_{i+1}(t) - U_i(t)), V_2(U_{i+2}(t) - U_i(t))\right) & \text{si } i \text{ est pair.} \end{cases} \quad (3.44)$$

La fonction $\bar{\phi}$ permet de faire la transition près de la bifurcation et est définie par

$$\bar{\phi}(x, a, b) = \begin{cases} a & \text{si } x < -h_0 - h_{max} - 1, \\ (h_0 + h_{max})(\min(a, b) - a) & \text{si } -1 \leq x + h_0 + h_{max} < 0, \\ -|x|(\min(a, b) - a) + \min(a, b) & \text{if } -h_0 - h_{max} \leq x < -h_0, \\ \min(a, b) & \text{if } -h_0 - h_{max} \leq x < -h_0, \\ \frac{|x|}{h_0} \min(a, b) + \left(1 - \frac{|x|}{h_0}\right) b & \text{si } -h_0 \leq x \leq 0, \\ b & \text{si } x > 0, \end{cases}$$

avec $h_{max} > h_0 > 0$. La Figure 3.8 donne une représentation schématique du modèle (3.44).

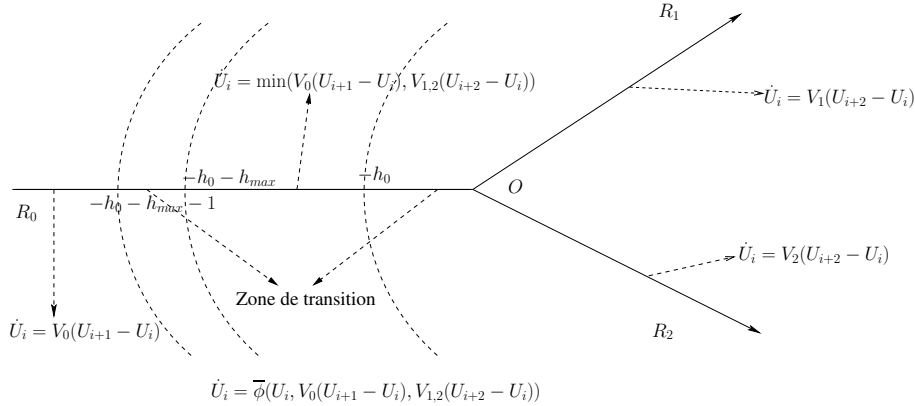


Figure 3.8: Représentation schématique du modèle microscopique.

Dans le modèle (3.44), les véhicules avec des indices impairs vont dans R_1 et les autres dans R_2 . Puisqu'on travaille avec un modèle du premier ordre, sur R_0 la vitesse de chacun des véhicules dépend de la distance $U_{i+1} - U_i$, mais sur R_1 et R_2 elle dépend de $U_{i+2} - U_i$. De plus, on définit

$$\bar{R} = R_0 \cup R_1 \cup R_2 \quad \text{et} \quad R_i \cap R_j = \{0\} \quad \text{pour } i \neq j.$$

On remarque que physiquement les véhicules sont sur \bar{R} , par contre le modèle (3.44) ne fait pas la différence quant à la localisation des véhicules sur \bar{R} . Ceci vient du fait que l'on

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peut identifier $R_0 \cup R_1$ et $R_0 \cup R_2$ à la ligne réelle. Par exemple, les véhicules avec des indices impairs vont bouger dans $R_0 \cup R_1$.

On appelle, zone de transition, l'intervalle $[-h_0 - h_{max} - 1, 0]$, où les voitures vont changer d'une vitesse à l'autre. Concernant la fonction de vitesse optimale V_i , pour $i = 0, 1, 2$, et $\bar{\phi}$, on suppose qu'elles satisfont l'hypothèse (C) mais au lieu de (C5) et (C6) qu'elles satisfont les hypothèses suivantes :

- (C5') Les fonctions $p \mapsto pV_0(-1/p)$ et $p \mapsto pV_j(-2/p)$, pour $j = 1, 2$, sont strictement convexes respectivement sur $[-1/h_0, 0)$ et sur $[-2/h_0, 0)$.
- (C6') La fonction $\bar{\phi} : \mathbb{R}^3 \rightarrow \mathbb{R}$ est lipschitzienne par rapport à toutes ses variables. On note $\|\phi'\|_\infty$ sa constante de Lipschitz. Pour tout $(x, a, b) \in \mathbb{R}$, les fonctions $\bar{\phi}(x, \cdot, b)$ et $\bar{\phi}(x, a, \cdot)$ sont croissantes.

Pour simplifier les notations, on note (C') l'hypothèse (C) mais avec (C5') et (C6') au lieu de (C5) et (C6).

Remark 3.25 (Zone de transition). *Les fonctions de vitesse optimale V_i décrivent les dynamiques de chacune des voitures dans chacune des voies. Le rôle de la fonction de transition (et de la zone de transition) est de faire un passage continu d'une dynamique à l'autre. On peut voir qu'étant donnée la forme de la fonction de transition $\bar{\phi}$, si initialement les voitures ont suffisamment d'espace entre elles, il y aura toujours au moins une distance h_0 entre deux voitures consécutives, ce qui veut dire que dans le modèle (3.44) il y a un distance de sécurité h_0 qui évite toute collision.*

Injection du système d'EDOs dans un système d'EDPs

Comme précédemment, on injecte le système d'EDOs dans un système d'EDPs. Pour faire cela, on sépare les véhicules en deux groupes, ceux qui vont en R_1 et ceux qui vont en R_2 et on introduit deux fonctions, la première étant la fonction de "distribution cumulative" pour les voitures avec des indices impairs,

$$\rho_1^\varepsilon(t, x) = -2\varepsilon \cdot \left(\sum_{i[2]=1, i \geq 0} H(x - \varepsilon U_i(t/\varepsilon)) + \sum_{i[2]=1, i < 0} (-1 + H(x - \varepsilon U_i(t/\varepsilon))) \right)$$

et la seconde étant la fonction de "distribution cumulative" des voitures avec des indices pairs,

$$\rho_2^\varepsilon(t, x) = -2\varepsilon \cdot \left(\sum_{i[2]=0, i \geq 0} H(x - \varepsilon U_i(t/\varepsilon)) + \sum_{i[2]=0, i < 0} (-1 + H(x - \varepsilon U_i(t/\varepsilon))) \right) - \varepsilon.$$

Ici et dans le reste de cette thèse, $i[2]$ correspond au reste de la division euclidienne de i par 2 (soit 0 ou 1). On remarque que ces fonctions ont été définies sur \mathbb{R} et non pas sur \bar{R} ,

3. Résultats principaux de cette thèse

ceci vient encore une fois du fait que l'on peut identifier $R_0 \cup R_1$ et $R_0 \cup R_2$ à \mathbb{R} . De plus, le coefficient 2 dans les fonctions précédentes a été ajouté artificiellement pour simplifier les calculs.

Sous l'hypothèse (C'), la fonction $(\rho_1^\varepsilon, \rho_2^\varepsilon)$ est une solution (de viscosité discontinue) de l'équation non-locale et non-linéaire suivante, pour $(t, x) \in (0, +\infty) \times \mathbb{R}$,

$$\begin{cases} u_t^\varepsilon + \phi\left(\frac{x}{\varepsilon}, N_0^\varepsilon\left(\frac{u^\varepsilon}{\varepsilon}, \left[\frac{\xi^\varepsilon(t, \cdot)}{\varepsilon}\right]\right)(x), M_1^\varepsilon\left[\frac{u^\varepsilon(t, \cdot)}{\varepsilon}\right](x)\right) \cdot |u_x^\varepsilon| = 0 \\ \xi_t^\varepsilon + \phi\left(\frac{x}{\varepsilon}, N_0^\varepsilon\left(\frac{\xi^\varepsilon}{\varepsilon}, \left[\frac{u^\varepsilon(t, \cdot)}{\varepsilon}\right]\right)(x), M_2^\varepsilon\left[\frac{\xi^\varepsilon(t, \cdot)}{\varepsilon}\right](x)\right) \cdot |\xi_x^\varepsilon| = 0, \end{cases} \quad (3.45)$$

où N_0^ε et M_i^ε pour $i = 1, 2$, sont des opérateurs non-locaux définis par

$$N_0^\varepsilon(U, [\Xi])(x) = \int_{\mathbb{R}} J_0(z) F(\Xi(x + \varepsilon z) - U(x)) dz - \frac{3}{2} V_{max}^0,$$

et

$$M_i^\varepsilon[U](x) = \int_{\mathbb{R}} J_i(z) E(U(x + \varepsilon z) - U(x)) dz - \frac{3}{2} V_{max}^i,$$

avec $J_i = V_i'$ pour $i = 0, 1, 2$,

$$F(z) = \begin{cases} 0 & \text{si } z \geq 1, \\ 1/2 & \text{si } -1 \leq z < 1, \\ 3/2 & \text{si } z < -1, \end{cases} \quad \text{et } E(z) = \begin{cases} 0 & \text{si } z \geq 0, \\ 1/2 & \text{si } -2 \leq z < 0, \\ 3/2 & \text{si } z < -2. \end{cases}$$

Pour le problème (3.45), on considère la condition initiale

$$\begin{cases} u^\varepsilon(0, x) = u_0(x) \\ \xi^\varepsilon(0, x) = \xi_0(x) \end{cases} \quad \text{pour tout } x \in \mathbb{R},$$

et on fait l'hypothèse suivante

(E0) (Initial condition). Pour tout $x \leq 0$,

$$u_0(x) = \xi_0(x).$$

De plus, on suppose que pour tout $x \in \mathbb{R}$

$$-2k_0 = -\frac{2}{h_0} \leq (u_0)_x \leq 0 \quad \text{et} \quad -2k_0 \leq (\xi_0)_x \leq 0.$$

Résultat de convergence pour une bifurcation simple

Avant de donner le résultat principal de [3], on introduit trois fonctions, \bar{H}_0 , \bar{H}_1 , et \bar{H}_2 que l'on utilise pour présenter notre résultat d'homogénéisation. Il s'agit des hamiltoniens effectifs dans chacune des voies. On définit $k_0 = 1/h_0$, et $\bar{H}_0 : \mathbb{R} \rightarrow \mathbb{R}$, par

$$\bar{H}_0(p) = \begin{cases} -p - 2k_0 & \text{pour } p < -2k_0, \\ -V_0 \left(\frac{-1}{p} \right) \cdot |p| & \text{pour } -2k_0 \leq p \leq 0, \\ p & \text{pour } p > 0. \end{cases}$$

On introduit également, pour $i = 1, 2$, $\bar{H}_i : \mathbb{R} \rightarrow \mathbb{R}$, définies par

$$\bar{H}_i(p) = \begin{cases} -p - 2k_0 & \text{pour } p < -2k_0, \\ -V_i \left(\frac{-2}{p} \right) \cdot |p| & \text{pour } -2k_0 \leq p \leq 0, \\ p & \text{pour } p > 0. \end{cases}$$

Pour $i = 0, 1, 2$, on peut remarquer que la fonction \bar{H}_i est continue, coercive

$\left(\lim_{|p| \rightarrow +\infty} \bar{H}_i(p) = +\infty \right)$ et grâce à (C5'), il existe un unique point $p_i \in [-2k_0, 0]$ tel que

$$\begin{cases} \bar{H}_i \text{ est décroissante sur } (-\infty, p_i), \\ \bar{H}_i \text{ est strictement croissante sur } (p_i, +\infty). \end{cases}$$

On note

$$H_0 = \max_{i \in \{0,1,2\}} \min_{p \in \mathbb{R}} \bar{H}_i(p).$$

La Figure 3.9 donne une représentation schématique de \bar{H}_0 , \bar{H}_1 , et \bar{H}_2 .

On veut un résultat d'homogénéisation quand le nombre de véhicules par unité de longueur tend vers l'infini, c'est à dire que l'on veut connaître le comportement de $(\rho_1^\varepsilon, \rho_2^\varepsilon)$ quand ε tend vers 0. Tout d'abord, on peut voir que le rayon de la zone de transition tend vers 0, et donc à l'échelle macroscopique on obtiendra une condition de jonction en 0.

On introduit, pour une fonction lisse $u : [0, T] \times \bar{R} \rightarrow \mathbb{R}$, $\partial_i u(t, x)$ qui correspond à la dérivée spatiale de u en $x \in R_i$, pour $i = 0, 1, 2$, et on définit

$$u_x(t, x) := \begin{cases} \partial_i u(t, x) & \text{si } x \in R_i^* := R_i \setminus \{0\}, \\ (\partial_0 u(t, 0), \partial_1 u(t, 0), \partial_2 u(t, 0)) & \text{si } x = 0. \end{cases}$$

Finalement, on introduit pour tout $x, y \in \bar{R}$ la distance $d(x, y)$ sur \bar{R} ,

$$d(x, y) = \begin{cases} |x - y| & \text{si } x, y \text{ se trouvent dans la même voie,} \\ |x| + |y| & \text{sinon.} \end{cases}$$

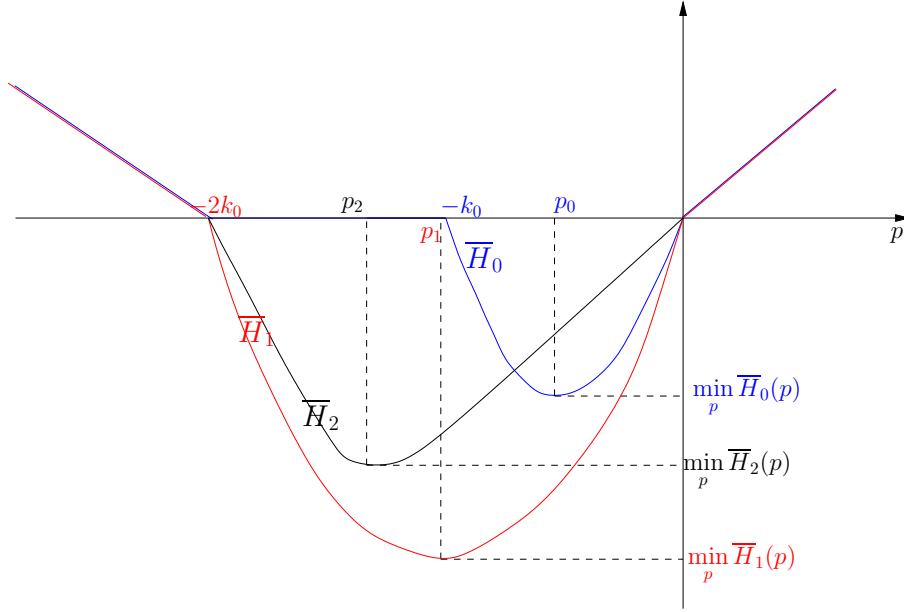


Figure 3.9: Représentation schématique des hamiltoniens effectifs

Theorem 3.26 (Théorème 2.5 dans [3]). *Supposons (C') vraie, et qu'au temps initial pour tout $i \in \mathbb{Z}$, il y a au moins une distance h_0 entre deux véhicules consécutifs. On suppose également qu'il existe une constante $R > 0$ telle que, pour tout entier $i \in \mathbb{Z}$, si $U_i(0) \geq R$*

$$U_{i+1}(0) - U_i(0) = \begin{cases} h_1 & \text{si } i[2] = 1 \\ h_2 & \text{si } i[2] = 0 \end{cases}$$

et si $U_i(0) \leq -R$

$$U_{i+1}(0) - U_i(0) = h,$$

avec $h, h_1, h_2 \geq h_0$. On définit deux fonctions u_0 et ξ_0 (satisfaisant (E0)) par

$$u_0(x) = \begin{cases} -x/h & \text{si } x \leq 0 \\ -2x/h_1 & \text{si } x > 0 \end{cases} \quad \text{et} \quad \xi_0(x) = \begin{cases} -x/h & \text{si } x \leq 0 \\ -2x/h_2 & \text{si } x > 0 \end{cases} \quad \text{pour tout } x \in \mathbb{R}.$$

Alors il existe une constante $\bar{A} \in [H_0, 0]$ telle que la fonction

$$\tilde{\rho}^\varepsilon(t, y) = \begin{cases} \rho_1^\varepsilon(t, -d(0, y)) & \text{pour } (t, y) \in (0, +\infty) \times R_0, \\ \rho_1^\varepsilon(t, d(0, y)) & \text{pour } (t, y) \in (0, +\infty) \times R_1^*, \\ \rho_2^\varepsilon(t, d(0, y)) & \text{pour } (t, y) \in (0, +\infty) \times R_2^*, \end{cases}$$

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converge localement uniformément vers u^0 , l'unique solution de viscosité de

$$\begin{cases} u_t^0 + \bar{H}_0(u_x^0) = 0 & \text{pour } (t, x) \in [0, +\infty) \times R_0^* \\ u_t^0 + \bar{H}_1(u_x^0) = 0 & \text{pour } (t, x) \in [0, +\infty) \times R_1^* \\ u_t^0 + \bar{H}_2(u_x^0) = 0 & \text{pour } (t, x) \in [0, +\infty) \times R_2^* \\ u_t^0 + F_{\bar{A}}(\partial_0 u^0(t, 0), \partial_1 u^0(t, 0), \partial_2 u^0(t, 0)) = 0 & \text{pour } (t, x) \in [0, +\infty) \times \{0\} \\ u^0(0, x) = \begin{cases} u_0(d(0, x)) & \text{pour } x \in R_0 \cup R_1, \\ \xi_0(d(0, x)) & \text{pour } x \in R_2^*, \end{cases} \end{cases}$$

où \bar{A} est une constante qui reste à déterminer et où $F_{\bar{A}}$ est définie par

$$F_{\bar{A}}(p_0, p_1, p_2) = \max(\bar{A}, \bar{H}_0^+(p_0), \bar{H}_1^-(p_1), \bar{H}_2^-(p_2)),$$

avec, pour $i = 0, 1, 2$,

$$\bar{H}_i^-(p) = \begin{cases} \bar{H}_i(p) & \text{si } p \leq p_i, \\ \bar{H}_i(p_i) & \text{si } p \geq p_i, \end{cases} \quad \text{et} \quad \bar{H}_i^+(p) = \begin{cases} \bar{H}_i(p_i) & \text{si } p \leq p_i, \\ \bar{H}_i(p) & \text{si } p \geq p_i. \end{cases}$$

De plus, on a pour tout $(t, x) \in [0, +\infty) \times \bar{R}$, $-2k_0 \leq u_x^0 \leq 0$ avec k_0 définie dans (E0).

Remark 3.27 (Lien avec les modèles macroscopiques). *On peut voir que notre résultat d'homogénéisation ne donne pas directement les dynamiques des véhicules. En effet, sur R_0 , u_x^0 est la densité des véhicules mais sur R_1 et R_2 c'est deux fois la densité des véhicules. Donc l'intégrale de la densité des véhicules sur \bar{R} est donné par*

$$\tilde{\rho}^0(t, x) = \begin{cases} u^0(t, x) & \text{pour } x \in R_0, \\ u^0(t, x)/2 & \text{pour } x \in R_1^* \cup R_2^*. \end{cases}$$

Cependant, on ne peut pas expliciter la dynamique de $\tilde{\rho}^0$ à cause de sa définition à l'origine.

Chapitre 1

Homogenization of second order discrete model and application to traffic flow

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Abstract

The goal of this paper is to derive traffic flow macroscopic models from microscopic models. At the microscopic scales, we consider a Bando model, of the type following the leader, i.e. the acceleration of each vehicle depends on the distance to the vehicle in front of it. We take into account the possibility that each driver can have different characteristics such as sensitivity to other drivers or optimal velocities. After rescaling, we prove that the solution of this system of ODEs converges to the solution of a macroscopic homogenized Hamilton-Jacobi equation which can be seen as a LWR (Lighthill-Whitham-Richards) model.

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Keywords: Traffic flow, macroscopic model, microscopic model, homogenization, viscosity solution.

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1 Introduction

The goal of this paper is to obtain an homogenization result for a traffic flow model. More precisely, we are interested in a discrete model (of type "following the leader") which describes the dynamics of vehicles on a straight road. The microscopic model we consider was introduced by Bando *et al.* [BHN⁺95] and is an optimal velocity model. The goal is then to describe the collective behaviour of the vehicles (in term of the density of vehicles) as the number of vehicles per unit length goes to infinity. We will see in particular that this problem can be seen as an homogenization result. Let us mention that the theory of homogenization for periodic Hamilton-Jacobi equations has known an important development since the pioneer works of Lions, Papanicolaou, Varadhan [LPV88] and Evans [Eva89]. We would like in particular to mention [CLL10] which is concerned with the homogenization of system and [FIM09a, FIM09b, FIM12, IMR08] for the homogenization of non-local equations (or systems).

1.1 General model with n_0 types of drivers

We begin by recalling the model introduced in [BHN⁺95]. We consider that we have $n_0 \in \mathbb{N}$ types of drivers (or vehicles), and we consider the following optimal velocity model, for all $j \in \mathbb{Z}$ and $t \geq 0$,

$$\ddot{U}_j(t) = a_j(V_j(U_{j+1}(t) - U_j(t)) - \dot{U}_j), \quad (1.1)$$

where U_j denotes the position of j -th vehicle, \dot{U}_j is its velocity and \ddot{U}_j its acceleration. The coefficients a_j are the sensitivities of the drivers and V_j are called optimal velocity functions (OVF) and depend on the driver.

To simplify the study and in order to be able to get homogenization, we impose the following periodic conditions

$$a_{j+n_0} = a_j \quad \text{and} \quad V_{j+n_0} = V_j \quad \text{for all } j \in \mathbb{Z}.$$

The model we consider has some similarities with the one studied in [FIM12] (in a different context). The main difference here is that the a_j can depend on j which was a crucial arguments in the proof of [FIM12], in particular to get the order of the particles. Nevertheless, we will use the same strategy and we introduce for all $j \in \mathbb{Z}$

$$\Xi_j(t) = U_j(t) + \frac{1}{\alpha} \dot{U}_j(t) \quad \text{where} \quad \alpha = \frac{1}{2} \min_{j \in \{1, \dots, n_0\}} (a_j). \quad (1.2)$$

We then obtain the following system of ODEs: for all $j \in \mathbb{Z}$ and $t \in (0, +\infty)$,

$$\begin{cases} \dot{U}_j(t) = \alpha(\Xi_j(t) - U_j(t)) \\ \dot{\Xi}_j(t) = (a_j - \alpha)(U_j(t) - \Xi_j(t)) + \frac{a_j}{\alpha} V_j(U_{j+1}(t) - U_j(t)). \end{cases} \quad (1.3)$$

Let us now give the assumptions on the functions V_j and the coefficients a_j :

(A1) (Regularity) For all $j \in \{1, \dots, n_0\}$,

$$\begin{cases} V_j \text{ is continuous and non-negative.} \\ V_j \text{ is Lipschitz continuous and we denote by } L_j \text{ its Lipschitz constant.} \end{cases}$$

We denote by $L = \max_{j \in \{1, \dots, n_0\}} L_j$.

(A2) (Monotonicity) For all $j \in \{1, \dots, n_0\}$,

$$\begin{cases} V_j \text{ is non-decreasing.} \\ a_j \geq 4L. \end{cases}$$

(A3) (Upper bound) For all $j \in \{1, \dots, n_0\}$,

$$\lim_{h \rightarrow +\infty} V_j(h) < +\infty. \tag{1.4}$$

We denote by $V_{max} = \max_j (\|V_j\|_\infty)$ and $h_0 = V_{max}/\alpha$.

(A4) (Lower bound) For all $j \in \{1, \dots, n_0\}$,

$$V_j(h) = 0 \quad \text{for all } h \leq 2h_0.$$

Remark 1.1. *Conditions (A1)-(A3) are classical for the Bando model (see for example [BHN⁺95], [BT10]). Assumption (A4) appears for example in [BT10]. We note that the second condition in (A2) appears in [BHN⁺95] to get stability. Here, it allows us to show that $U_j \mapsto (a_j - \alpha)U_j + \frac{a_j}{\alpha}V_j(U_{j+1} - U_j)$ is non decreasing, and so the system (1.3) is monotone.*

(A5) (Periodicity of the type of drivers) For all $j \in \mathbb{Z}$,

$$a_{j+n_0} = a_j \quad \text{and} \quad V_{j+n_0} = V_j.$$

1.2 General system of PDE with n_0 types of drivers

As in [FIM09a, FIM12], we inject the system of (ODE) in a system of (PDE) by considering the functions

$$(u, \xi) = ((u_j(t, x))_{j \in \mathbb{Z}}, (\xi_j(t, x))_{j \in \mathbb{Z}}),$$

defined by

$$u_j(t, x) = U_{j+\lfloor x \rfloor n_0}(t) \quad \text{and} \quad \xi_j(t, x) = \Xi_{j+\lfloor x \rfloor n_0}(t),$$

where $\lfloor \cdot \rfloor$ denotes the floor integer part. For instance in the case $n_0 = 1$, we have

$$u(t, x) = U_{\lfloor x \rfloor}(t) \quad \text{and} \quad \xi(t, x) = \Xi_{\lfloor x \rfloor}(t),$$

where x is a continuous variable representing the index of each vehicle. In the case $n_0 > 1$, the integer j allows us to take into account the periodicity of the coefficients a_j and the functions V_j .

The function (u, ξ) satisfies the following system of equations (see Proposition 6.1), for all $(t, x) \in (0, +\infty) \times \mathbb{R}$ and for all $j \in \mathbb{Z}$,

$$\left\{ \begin{array}{l} \frac{\partial u_j}{\partial t}(t, x) = \alpha(\xi_j(t, x) - u_j(t, x)) \\ \frac{\partial \xi_j}{\partial t}(t, x) = (a_j - \alpha)(u_j(t, x) - \xi_j(t, x)) + \frac{a_j}{\alpha} V_j(u_{j+1}(t, x) - u_j(t, x)) \\ u_{j+n_0}(t, x) = u_j(t, x + 1) \\ \xi_{j+n_0}(t, x) = \xi_j(t, x + 1). \end{array} \right. \quad (1.5)$$

However, we are more interested in the rescaled system, defined by

$$u_j^\varepsilon(t, x) = \varepsilon u_j\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right) \quad \text{and} \quad \xi_j^\varepsilon(t, x) = \varepsilon \xi_j\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right). \quad (1.6)$$

As ε goes to 0, the number of vehicles per length unit goes to infinity. The function $(u^\varepsilon, \xi^\varepsilon) = ((u_j^\varepsilon(t, x))_{j \in \mathbb{Z}}, (\xi_j^\varepsilon(t, x))_{j \in \mathbb{Z}})$ satisfy the following Cauchy problem, for all $(t, x) \in (0, +\infty) \times \mathbb{R}$,

$$\left\{ \begin{array}{l} \frac{\partial u_j^\varepsilon}{\partial t}(t, x) = \alpha \frac{\xi_j^\varepsilon(t, x) - u_j^\varepsilon(t, x)}{\varepsilon} \\ \frac{\partial \xi_j^\varepsilon}{\partial t}(t, x) = (a_j - \alpha) \frac{u_j^\varepsilon(t, x) - \xi_j^\varepsilon(t, x)}{\varepsilon} + \frac{a_j}{\alpha} V_j \left(\frac{u_{j+1}^\varepsilon(t, x) - u_j^\varepsilon(t, x)}{\varepsilon} \right) \\ u_{j+n_0}^\varepsilon(t, x) = u_j^\varepsilon(t, x + \varepsilon) \\ \xi_{j+n_0}^\varepsilon(t, x) = \xi_j^\varepsilon(t, x + \varepsilon), \end{array} \right. \quad (1.7)$$

completed with the initial condition

$$u_j^\varepsilon(0, x) = u_0\left(x + \frac{j\varepsilon}{n_0}\right) \quad \text{and} \quad \xi_j^\varepsilon(0, x) = \xi_0^\varepsilon\left(x + \frac{j\varepsilon}{n_0}\right). \quad (1.8)$$

We assume that the initial condition satisfies the following assumption,

(A0) (Gradient bound) There exist $k_0, K_0 > 0$ such that

$$\begin{cases} 0 < k_0 \leq (u_0)_x \leq K_0 \\ 0 < k_0 \leq (\xi_0^\varepsilon)_x \leq K_0. \end{cases}$$

We also assume that

$$0 \leq \alpha(\xi_0^\varepsilon(x) - u_0(x)) \leq \min\left(V_{max}\varepsilon, \alpha \cdot \frac{u_0\left(x + \frac{\varepsilon}{n_0}\right) - u_0(x)}{2}\right) \quad \text{for all } x \in \mathbb{R}.$$

Remark 1.2. Condition (A0) implies that initially the vehicles have enough space between them. Concerning the last inequality in (A0), formally we can see, using (1.2), that we have

$$\xi_j(t, x) = u_j(t, x) + \frac{1}{\alpha} \frac{\partial u}{\partial t}(t, x),$$

which means that after rescaling, using (1.6), we have

$$\xi_j^\varepsilon(t, x) = u^\varepsilon(t, x) + \frac{\varepsilon}{\alpha} \frac{\partial u^\varepsilon}{\partial t}(t, x).$$

Therefore, the initial condition must satisfy

$$\xi_0^\varepsilon\left(x + \frac{j\varepsilon}{n_0}\right) = u_0\left(x + \frac{j\varepsilon}{n_0}\right) + \frac{\varepsilon}{\alpha} \frac{\partial u_j^\varepsilon}{\partial t}(0, x).$$

First, this implies that the initial condition ξ_0^ε depends on ε . The bound on $\alpha(\xi_0^\varepsilon(x) - u_0(x))$ can be explained as follows, in the case where initially the vehicles have enough space between them ($k_0 \geq 2h_0n_0$), the initial velocity of each vehicle must be less than V_{max} , so we have

$$\alpha(\xi_0^\varepsilon(x) - u_0(x)) \leq V_{max}\varepsilon.$$

Ch.1. Homogenization of a second order discrete model

In the case where vehicles have less space between them ($k_0 \leq 2h_0n_0$), the initial velocity of each vehicle must be bounded in such a way that the vehicles remain ordered ($u_j \leq u_{j+1}$ and $\xi_j \leq \xi_{j+1}$) (see Theorem 2.7), so we have

$$\alpha (\xi_0^\varepsilon(x) - u_0(x)) \leq V_{max} \cdot \frac{u_0\left(x + \frac{\varepsilon}{n_0}\right) - u_0(x)}{2h_0} = \alpha \cdot \frac{u_0\left(x + \frac{\varepsilon}{n_0}\right) - u_0(x)}{2}.$$

Combining the two previous inequalities we obtain the last inequality in (A0).

The main purpose of this article is to prove that the viscosity solution of (1.7)-(1.8) converges uniformly on compact subsets of $(0, +\infty) \times \mathbb{R}$ as ε goes to 0, to the unique solution of the following problem

$$\begin{cases} u_t^0(t, x) = \bar{F}(u_x^0(t, x)) & \text{for } (t, x) \in (0, +\infty) \times \mathbb{R}, \\ u^0(0, x) = u_0(x) & \text{for } x \in \mathbb{R}, \end{cases} \quad (1.9)$$

where \bar{F} has to be determined.

Theorem 1.3 (Homogenization of systems with n_0 types of drivers). *Assume that (A1)-(A5) holds and that the initial datum u_0, ξ_0^ε satisfy (A0). Consider the solution $((u_j^\varepsilon)_{j \in \mathbb{Z}}, (\xi_j^\varepsilon)_{j \in \mathbb{Z}})$ of (1.7)-(1.8). Then, there exists a continuous function $\bar{F} : \mathbb{R} \rightarrow \mathbb{R}$ such that, for all $j \in \mathbb{Z}$, the functions u_j^ε and ξ_j^ε converge uniformly on compact subsets of $(0, +\infty) \times \mathbb{R}$ as ε goes to 0 to the unique viscosity solution u^0 of (1.9).*

Theorem 1.4 (Homogenization of systems with n_0 types of sensitivities and one OVF). *Assume that (A1)-(A5) are satisfied and that the initial datum u_0, ξ_0^ε satisfy (A0). We also assume that $V_j = V$ for all $j \in \mathbb{N}$. Let us consider the solution $((u_j^\varepsilon)_{j \in \mathbb{Z}}, (\xi_j^\varepsilon)_{j \in \mathbb{Z}})$ of (1.7)-(1.8). Then the effective Hamiltonian \bar{F} is given by*

$$\bar{F}(p) = V \left(\frac{p}{n_0} \right) \quad \text{for all } p \in \mathbb{R}^+. \quad (1.10)$$

1.3 Hull functions

We recall the notion of hull function (presented as in [FIM12]) for the system (1.5) which is necessary for the proof of Theorem 1.3. It will allow us in particular to define the effective Hamiltonian \bar{F} . We look for functions $((h_j)_j, (g_j)_j)$ such that $(u_j(t, x), \xi_j(t, x)) = (h_j(\lambda t + px), g_j(\lambda t + px))$ is a solution of (1.5) on $\mathbb{R} \times \mathbb{R}$.

Definition 1.5 (Hull function for system with n_0 types of drivers). *Given $(V_j)_j$ and $(a_j)_j$ satisfying (A1)-(A5), $p \in \mathbb{R}^+$, and a real number $\lambda \in \mathbb{R}$, we say that a family of functions*

$((h_j)_{j \in \mathbb{Z}}, (g_j)_{j \in \mathbb{Z}})$ is a hull function for (1.5) if it satisfies for all $j \in \{1, \dots, n_0\}$ and for all $z \in \mathbb{R}$,

$$\left\{ \begin{array}{l} \lambda = \alpha(g_j - h_j) \\ h_{j+n_0}(z) = h_j(z + p) \\ h_{j+1}(z) \geq h_j(z) \\ h_j(z) = z + h_j(0) \end{array} \right. \left\{ \begin{array}{l} \lambda = (a_j - \alpha)(h_j - g_j) + \frac{a_j}{\alpha} V_j(h_{j+1} - h_j) \\ g_{j+n_0}(z) = g_j(z + p) \\ g_{j+1}(z) \geq g_j(z) \\ g_j(z) = z + g_j(0) \end{array} \right. \quad (1.11)$$

Remark 1.6. *The notion of hull functions is a little bit different from the one presented in [FIM12]. This comes from the fact that our system is invariant by addition of constant while the one considered in [FIM12] was invariant by addition of integer constant only. This allows us to show that $h'_j = 1$ and $g'_j = 1$ and so to get the special form for h_j and g_j .*

Theorem 1.7 (Effective Hamiltonian and hull functions). *Assume (A1)-(A5) and let $p \in (0, +\infty)$. Then there exists a unique real λ for which there exists a hull function $((h_j)_j, (g_j)_j)$ satisfying (1.11). Moreover the real $\lambda = \bar{F}(p)$, seen as a function of p , is continuous in $(0, +\infty)$.*

Remark 1.8. *A simple computation gives us that*

$$\bar{F}(p) = V_j(h_{j+1}(0) - h_j(0)) \quad \forall j \in \mathbb{Z}. \quad (1.12)$$

1.4 Qualitative properties of the effective Hamiltonian

We have the following results concerning \bar{F} , and concerning the homogenized Hamilton-Jacobi equation (1.9).

Theorem 1.9 (Qualitative properties of \bar{F}). *Assume (A1)-(A5). For any $p \in (0, +\infty)$, let $\bar{F}(p)$ denote the effective Hamiltonian given by Theorem 1.7. Then we have the following properties*

(i) **(Lower boundary)** *if $p \leq 2h_0n_0$, we have*

$$\bar{F}(p) = 0.$$

(ii) **(Upper boundary)**

$$\lim_{p \rightarrow +\infty} \bar{F}(p) = \min_{j \in \{1, \dots, n_0\}} (\|V_j\|_\infty).$$

(iii) (Monotonicity) \bar{F} is non-decreasing.

Remark 1.10. For example, an effective Hamiltonian can be of the form:

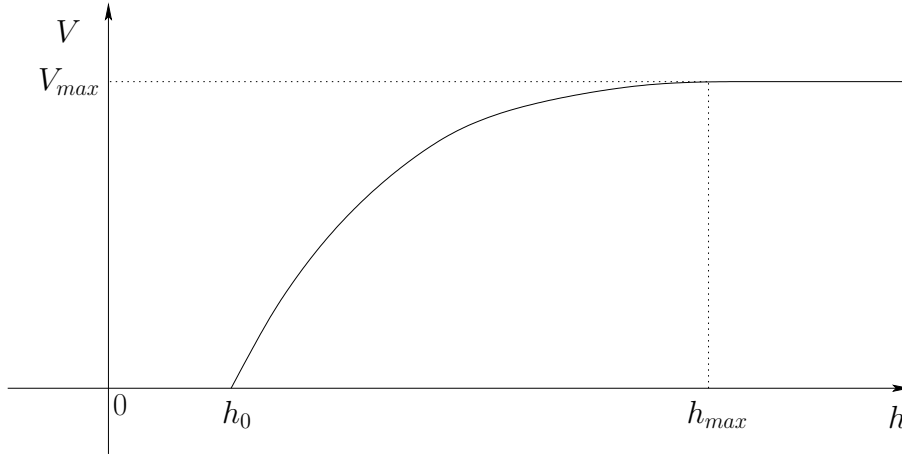


Figure 1.1: Schematic representation of the effective Hamiltonian.

Link with macroscopic models. In the literature we can find different types of macroscopic models. But we will focus on the first order model LWR (Lighthill-Whitham-Richards) (for more information on the LWR model see [LW55, Ric56] or for instance [GP06]), which is defined by

$$\partial_t \rho + \partial_y(\rho v(\rho)) = 0, \tag{1.13}$$

where $\rho(t, y)$ is the density of vehicles at the point $y \in \mathbb{R}$ (physical point on the road) at time $t \in (0, +\infty)$, and $v(\rho)$ is the average speed of vehicles. We call $f(\rho) = \rho v(\rho)$ the traffic flux. It can be remarked that (1.13) uses Eulerian coordinates (y is physical point on the road). However, it was proven by Wagner in [Wag87] (for equations of gas dynamics) that the problem (1.13) is equivalent to

$$\partial_t s - \partial_x v^*(s) = 0, \tag{1.14}$$

where $s(t, x) = 1/\rho$ is the spacing between the vehicles, x stands for the vehicle x (seen as a continuous variable) and $v^*(s) = v(1/s)$. We can see that equation (1.14) uses Lagrangian coordinates. Moreover, if we denote by $u^0(t, x)$ the position of the x vehicle, we have that (1.14) is equivalent (see [LLC08]) to

$$\partial_t u^0(t, x) = v^*(\partial_x u^0), \tag{1.15}$$

with $s(t, x) = \partial_x u^0(t, x)$. From this we can see that equation (1.9) is equivalent to a macroscopic model of traffic flow of the LWR type, with

$$v(\rho) := \bar{F} \left(\frac{1}{\rho} \right).$$

Using Theorem 1.9, we can see that the flux of the macroscopic model, $f(\rho) = \rho v(\rho)$, satisfies some of the properties presented in [GP06]:

1. f is a continuous function.
2. $f(0) = f(\rho_{max}) = 0$, with $\rho_{max} = \frac{n_0}{h_0}$.

The problem of deriving macroscopic models from microscopic models has already been studied for models of the type following the leader. We refer for example to [AKRM02, DFR15, Hel98, LLK01] where the authors rescaled the empirical measure and obtain a scalar conservation law (LWR model). In these papers at the microscopic scale the authors consider only one type of driver (a single optimal velocity function).

The originality of our work is that we work with the primitive of the empirical measure and at the macroscopic scale we obtain a Hamilton-Jacobi equation which is equivalent to the primitive of the LWR model. This approach gives different techniques of proof than those used when considering the empirical measure. This method allows us to treat more complex problems. For instance, to consider at the microscopic scale, a model that takes into account n_0 different types of drivers (considering the coefficients a_j and the optimal velocity functions V_j for $j \in \{1, \dots, n_0\}$). Another example is the case of microscopic models with a singular perturbation (see [2]). Finally, simpler cases, for instance when we have a single optimal velocity function, are easily treated. In fact in that case we can explicitly determine the effective Hamiltonian (see Theorem 1.4).

1.5 Organisation of the article

In Section 2 we give some results concerning viscosity solutions for systems. In Section 3, we prove Theorem 1.3 assuming Theorem 1.7. In Section 4 we give the results concerning the existence of the hull functions.

2 Viscosity Solutions

This section is devoted to the definition and to useful results for viscosity solutions for systems like (1.5). The reader is referred to the user's guide of Crandall, Ishii, Lions [CIL92] and the book of Barles [Bar94] for an introduction to viscosity solutions and to [Ish92, IK91, Len88] and references therein for results concerning viscosity solutions for weakly coupled systems.

2.1 Definitions

We consider for $0 < T \leq +\infty$ the following Cauchy problem, for $j \in \mathbb{Z}$, $t > 0$ and $x \in \mathbb{R}$,

$$\left\{ \begin{array}{l} \frac{\partial u_j}{\partial t}(t, x) = \alpha(\xi_j(t, x) - u_j(t, x)) \\ \frac{\partial \xi_j}{\partial t}(t, x) = (a_j - \alpha)(u_j(t, x) - \xi_j(t, x)) + \frac{a_j}{\alpha} V_j(u_{j+1}(t, x) - u_j(t, x)) \\ u_{j+n_0}(t, x) = u_j(t, x + 1) \\ \xi_{j+n_0}(t, x) = \xi_j(t, x + 1), \end{array} \right. \quad (2.1)$$

with the initial condition

$$u_j(0, x) = u_0 \left(x + \frac{j}{n_0} \right) \quad \text{and} \quad \xi_j(0, x) = \xi_0 \left(x + \frac{j}{n_0} \right). \quad (2.2)$$

We recall the definition of the upper and lower semi-continuous envelopes, u^* and u_* , of a locally bounded function u ,

$$u^*(t, x) = \limsup_{(\tau, y) \rightarrow (t, x)} u(\tau, y) \quad \text{and} \quad u_*(t, x) = \liminf_{(\tau, y) \rightarrow (t, x)} u(\tau, y).$$

Definition 2.1 (Viscosity Solutions). *Let $T > 0$, $u_0 : \mathbb{R} \rightarrow \mathbb{R}$ and $\xi_0 : \mathbb{R} \rightarrow \mathbb{R}$ satisfying (A0). For all j , let $u_j : \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}$ and $\xi_j : \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}$ be upper semi-continuous (resp. lower semi-continuous) locally bounded functions. We set $\Omega = (0, T) \times \mathbb{R}$. Let us consider that $((u_j)_j, (\xi_j)_j)$ satisfies*

$$\forall j \in \mathbb{Z}, \forall (t, x) \in \Omega, \quad u_{j+n_0}(t, x) = u_j(t, x + 1) \quad \text{and} \quad \xi_{j+n_0}(t, x) = \xi_j(t, x + 1).$$

-A function $((u_j)_j, (\xi_j)_j)$ is a sub-solution (resp. a super-solution) of (2.1) on Ω if for all $(t, x) \in \Omega$ and for any test function $\varphi \in C^1(\Omega)$ such that $u_j - \varphi$ attains a local maximum (resp. a local minimum) at the point (t, x) , we have

$$\varphi_t(t, x) \leq \alpha(\xi_j(t, x) - u_j(t, x)) \quad (\text{resp. } \geq), \quad (2.3)$$

and for all $(t, x) \in \Omega$, and any test function $\varphi \in C^1(\Omega)$ such that $\xi_j - \varphi$ attains a local maximum (resp. a local minimum) at the point (t, x) , we have

$$\varphi_t(t, x) \leq (a_j - \alpha)(u_j(t, x) - \xi_j(t, x)) + \frac{a_j}{\alpha} V_j(u_{j+1}(t, x) - u_j(t, x)) \quad (\text{resp. } \geq). \quad (2.4)$$

-A function $((u_j)_j, (\xi_j)_j)$ is a sub-solution (resp. a super-solution) of (2.1)-(2.2) if $((u_j)_j, (\xi_j)_j)$ is a sub-solution (resp. a super solution) of (2.1) on Ω and if it satisfies moreover for all $x \in \mathbb{R}$, $j \in \{1, \dots, n_0\}$,

$$u_j(0, x) \leq u_0 \left(x + \frac{j}{n_0} \right) \quad (\text{resp. } \geq) \quad \text{and} \quad \xi_j(0, x) \leq \xi_0 \left(x + \frac{j}{n_0} \right) \quad (\text{resp. } \geq).$$

-A function $((u_j)_j, (\xi_j)_j)$ is a viscosity solution of (2.1) (resp. of (2.1)-(2.2)) if $((u_j^*)_j, (\xi_j^*)_j)$ is a sub-solution and $((u_j)_*, (\xi_j)_*)$ is a super solution of (2.1) (resp. of (2.1)-(2.2)).

2.2 Results for viscosity solutions of (2.1)

Proposition 2.2 (Comparison Principle). *Assume (A0) and (A1)-(A5). Let (u_j, ξ_j) and (v_j, ζ_j) be respectively a sub-solution and a super-solution of (2.1)-(2.2). We also assume that there is a constant $K > 0$ such that for all $j \in \{1, \dots, n_0\}$ and for all $(t, x) \in [0; T] \times \mathbb{R}$, we have*

$$\begin{aligned} u_j(t, x) &\leq u_j(0, x) + K(1 + t), & \xi_j(t, x) &\leq \xi_j(0, x) + K(1 + t) \\ -v_j(t, x) &\leq -v_j(0, x) + K(1 + t), & -\zeta_j(t, x) &\leq -\zeta_j(0, x) + K(1 + t). \end{aligned} \tag{2.5}$$

If

$$u_j(0, x) \leq v_j(0, x) \quad \text{and} \quad \xi_j(0, x) \leq \zeta_j(0, x) \quad \text{for all } x \in \mathbb{R}, j \in \mathbb{Z},$$

then

$$u_j(t, x) \leq v_j(t, x) \quad \text{and} \quad \xi_j(t, x) \leq \zeta_j(t, x) \quad \text{for all } x \in \mathbb{R}, j \in \mathbb{Z}, t \in [0; T].$$

Proof of Proposition 2.2.

This proof is similar to the one of Proposition 2.2 in [FIM12], because the system (2.1) is monotone as the one studied in [FIM12] thanks to assumption (A2). Indeed, for systems of the following form, for $j \in \{1, \dots, n_0\}$,

$$\begin{aligned} \frac{\partial u_j}{\partial t} &= f_j(\xi_1, \dots, \xi_{n_0}, u_1, \dots, u_{n_0}), \\ \frac{\partial \xi_j}{\partial t} &= g_j(\xi_1, \dots, \xi_{n_0}, u_1, \dots, u_{n_0}), \end{aligned}$$

we say that the system is monotone in the sense of Ishii and Koike [IK91], if for all $i, j \in \{1, \dots, n_0\}$, $i \neq j$,

$$\frac{\partial f_j}{\partial \xi_i} \geq 0, \quad \frac{\partial f_j}{\partial u_i} \geq 0, \quad \frac{\partial g_j}{\partial \xi_i} \geq 0 \quad \text{and} \quad \frac{\partial g_j}{\partial u_i} \geq 0. \tag{2.6}$$

System (2.1) satisfies the previous definition, thanks to (A2). For the readers convenience, we give the details of the proof of Proposition 2.2.

Thanks to the periodicity in (2.1), we only need to prove that

$$M = \sup_{(0; T) \times \mathbb{R}} \max_{j \in \{1, \dots, n_0\}} \max(u_j(t, x) - v_j(t, x), \xi_j(t, x) - \zeta_j(t, x)) \leq 0.$$

We argue by contradiction and assume that $M > 0$.

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Step 1: the test functions. We introduce, for $\varepsilon, \gamma > 0$ and $\eta > 0$ small parameters, the functions

$$\varphi(t, x, y, j) = u_j(t, x) - v_j(t, y) - \frac{(x - y)^2}{\varepsilon^2} - \frac{\eta}{T - t} - \gamma x^2$$

and

$$\phi(t, x, y, j) = \xi_j(t, x) - \zeta_j(t, x) - \frac{(x - y)^2}{\varepsilon^2} - \frac{\eta}{T - t} - \gamma x^2.$$

First, we notice that these functions satisfy

$$\varphi(t, x, y, j) \leq 2K(1 + T) + K_0|x - y| - \frac{(x - y)^2}{\varepsilon^2} - \gamma x^2$$

and

$$\phi(t, x, y, j) \leq 2K(1 + T) + K_0|x - y| - \frac{(x - y)^2}{\varepsilon^2} - \gamma x^2,$$

where we used (2.5) and (A0). This implies that

$$\lim_{|x|, |y| \rightarrow +\infty} \varphi = \lim_{|x|, |y| \rightarrow +\infty} \phi = -\infty.$$

Therefore, given that φ and ϕ are upper semi-continuous functions, we have that the function $\psi = \max(\varphi, \phi)$ reaches a maximum at some finite point that we denote by $(\bar{t}, \bar{x}, \bar{y}, \bar{j}) \in [0, T] \times \mathbb{R} \times \mathbb{R} \times \{1, \dots, n_0\}$. We define $M_{\varepsilon, \eta, \gamma} = \psi(\bar{t}, \bar{x}, \bar{y}, \bar{j})$.

Classically we have the following lemma.

Lemma 2.3. *For η and γ small enough, we have the following properties.*

(i) $0 < \frac{M}{2} \leq M_{\varepsilon, \eta, \gamma}$.

(ii) $\gamma|\bar{x}| \rightarrow 0$ as $\gamma \rightarrow 0$.

(iii) $|\bar{x} - \bar{y}| \rightarrow 0$ as $\varepsilon \rightarrow 0$.

Step 2: $\bar{t} > 0$. By contradiction, let us assume that ψ reaches its maximum for $\bar{t} = 0$. Let us for instance assume that $\psi(\bar{t}, \bar{x}, \bar{y}, \bar{j}) = \varphi(\bar{t}, \bar{x}, \bar{y}, \bar{j})$. In this case we have

$$0 < \frac{M}{2} \leq M_{\varepsilon, \eta, \gamma} \leq u_0 \left(\bar{x} + \frac{\bar{j}}{n_0} \right) - u_0 \left(\bar{y} + \frac{\bar{j}}{n_0} \right) \leq K_0|\bar{x} - \bar{y}|,$$

where we have used assumption (A0) for the last inequality. Using the fact that $|\bar{x} - \bar{y}| \rightarrow 0$ as $\varepsilon \rightarrow 0$, we get a contradiction for ε small enough. Similarly, we get a contradiction if we assume that $\psi(\bar{t}, \bar{x}, \bar{y}, \bar{j}) = \phi(\bar{t}, \bar{x}, \bar{y}, \bar{j})$.

Step 3: viscosity inequalities in the case $\psi(\bar{t}, \bar{x}, \bar{y}, \bar{j}) = \varphi(\bar{t}, \bar{x}, \bar{y}, \bar{j})$. By duplication of the time variable and passing to the limit we have that there exist two real numbers $a, b \in \mathbb{R}$, such that

$$a - b = \frac{\eta}{(T - \bar{t})^2}, \quad a \leq \alpha \left(u_{\bar{j}}(\bar{t}, \bar{x}) - \xi_{\bar{j}}(\bar{t}, \bar{x}) \right), \quad \text{and} \quad b \geq \alpha \left(v_{\bar{j}}(\bar{t}, \bar{y}) - \zeta_{\bar{j}}(\bar{t}, \bar{y}) \right).$$

Combining these inequalities, we obtain

$$\frac{\eta}{(T - \bar{t})^2} \leq \alpha(u_{\bar{j}}(\bar{t}, \bar{x}) - v_{\bar{j}}(\bar{t}, \bar{y}) - (\xi_{\bar{j}}(\bar{t}, \bar{x}) - \zeta_{\bar{j}}(\bar{t}, \bar{y}))) \leq 0,$$

where we have used the fact that $\varphi(\bar{t}, \bar{x}, \bar{y}, \bar{j}) \geq \phi(\bar{t}, \bar{x}, \bar{y}, \bar{j})$ for the last inequality. This gives us a contradiction from the fact that $\eta > 0$.

Step 4: viscosity inequalities in the case $\psi(\bar{t}, \bar{x}, \bar{y}, \bar{j}) = \phi(\bar{t}, \bar{x}, \bar{y}, \bar{j})$. Like in the previous step, by duplication of the time variable and passing to the limit we have that there exist two real numbers $a, b \in \mathbb{R}$, such that

$$a - b = \frac{\eta}{(T - \bar{t})^2} \tag{2.7}$$

and

$$\begin{cases} a \leq (a_{\bar{j}} - \alpha) \left(u_{\bar{j}}(\bar{t}, \bar{x}) - \xi_{\bar{j}}(\bar{t}, \bar{x}) \right) + \frac{a_{\bar{j}}}{\alpha} V_{\bar{j}} \left(u_{\bar{j}+1}(\bar{t}, \bar{x}) - u_{\bar{j}}(\bar{t}, \bar{x}) \right) \\ b \geq (a_{\bar{j}} - \alpha) \left(v_{\bar{j}}(\bar{t}, \bar{y}) - \zeta_{\bar{j}}(\bar{t}, \bar{y}) \right) + \frac{a_{\bar{j}}}{\alpha} V_{\bar{j}} \left(v_{\bar{j}+1}(\bar{t}, \bar{y}) - v_{\bar{j}}(\bar{t}, \bar{y}) \right). \end{cases} \tag{2.8}$$

Using the fact that $\psi(\bar{t}, \bar{x}, \bar{y}, \bar{j}) = \phi(\bar{t}, \bar{x}, \bar{y}, \bar{j})$, we obtain the following properties.

- For all $k \in \{1, \dots, n_0\}$

$$u_k(\bar{t}, \bar{x}) \leq v_k(\bar{t}, \bar{y}) + \xi_{\bar{j}}(\bar{t}, \bar{x}) - \zeta_{\bar{j}}(\bar{t}, \bar{y}). \tag{2.9}$$

- $\forall k \notin \{1, \dots, n_0\}$, there is an $l_k \in \mathbb{Z}$ and a $\tilde{k} \in \{1, \dots, n_0\}$ such that $k = \tilde{k} + l_k n_0$,

$$\begin{aligned} u_k(\bar{t}, \bar{x}) - v_k(\bar{t}, \bar{y}) &= u_{\tilde{k}}(\bar{t}, \bar{x} + l_k) - v_{\tilde{k}}(\bar{t}, \bar{y} + l_k) \leq \xi_{\bar{j}}(\bar{t}, \bar{x}) - \zeta_{\bar{j}}(\bar{t}, \bar{y}) - \gamma \bar{x}^2 \\ &\quad + \gamma(l_k + \bar{x})^2 \\ &\leq \xi_{\bar{j}}(\bar{t}, \bar{x}) - \zeta_{\bar{j}}(\bar{t}, \bar{y}) + o_\gamma(1), \end{aligned} \tag{2.10}$$

where we used the periodicity in (2.1) and the fact that $\varphi(\bar{t}, \bar{x} + l_k, \bar{y} + l_k, \tilde{k}) \leq \phi(\bar{t}, \bar{x}, \bar{y}, \tilde{j})$ for the first inequality and Lemma 2.3 for the second inequality.

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Combining the inequalities from (2.7) and (2.8), we get

$$\begin{aligned}
\frac{\eta}{T^2} &\leq (a_{\bar{j}} - \alpha) \left(u_{\bar{j}}(\bar{t}, \bar{x}) - v_{\bar{j}}(\bar{t}, \bar{y}) - (\xi_{\bar{j}}(\bar{t}, \bar{x}) - \zeta_{\bar{j}}(\bar{t}, \bar{y})) \right) \\
&\quad + \frac{a_{\bar{j}}}{\alpha} V_{\bar{j}} \left(u_{\bar{j}+1}(\bar{t}, \bar{x}) - u_{\bar{j}}(\bar{t}, \bar{x}) \right) - \frac{a_{\bar{j}}}{\alpha} V_{\bar{j}} \left(v_{\bar{j}+1}(\bar{t}, \bar{y}) - v_{\bar{j}}(\bar{t}, \bar{y}) \right) \\
&\leq \frac{a_{\bar{j}}}{\alpha} V_{\bar{j}} \left(u_{\bar{j}+1}(\bar{t}, \bar{x}) - v_{\bar{j}}(\bar{t}, \bar{y}) - (\xi_{\bar{j}}(\bar{t}, \bar{x}) - \zeta_{\bar{j}}(\bar{t}, \bar{y})) \right) - \frac{a_{\bar{j}}}{\alpha} V_{\bar{j}} \left(v_{\bar{j}+1}(\bar{t}, \bar{y}) - v_{\bar{j}}(\bar{t}, \bar{y}) \right) \\
&\leq \frac{a_{\bar{j}}}{\alpha} \left(V_{\bar{j}} \left(v_{\bar{j}+1}(\bar{t}, \bar{y}) - v_{\bar{j}}(\bar{t}, \bar{y}) + o_\gamma(1) \right) - V_{\bar{j}} \left(v_{\bar{j}+1}(\bar{t}, \bar{y}) - v_{\bar{j}}(\bar{t}, \bar{y}) \right) \right) \\
&\leq \frac{a_{\bar{j}}}{\alpha} L \cdot o_\gamma(1) = o_\gamma(1),
\end{aligned}$$

where we have used the monotonicity property (A2) (see Remark 1.1), (2.9), and the fact that $\bar{j} \in \{1, \dots, n_0\}$ for the second inequality, the fact that the function $V_{\bar{j}}$ is non-decreasing (see assumption (A2)) and (2.10) for the third inequality and the fact that the function $V_{\bar{j}}$ is Lipschitz continuous (see assumption (A1)) for the last inequality.

From the previous inequalities we get a contradictory for γ small enough. This ends the proof of Proposition 2.2. \square

We now give a comparison principle on bounded sets, to do this, we define, for a given point $(t_0, x_0) \in (0, T) \times \mathbb{R}$ and for all $r, R > 0$, the set

$$\mathcal{Q}_{r,R} = (t_0 - r, t_0 + r) \times (y_0 - R, y_0 + R).$$

Proposition 2.4 (Comparison principle on bounded sets). *Assume (A1)-(A5). Let $((u_j)_j, (\xi_j)_j)$ (resp. $((v_j)_j, (\zeta_j)_j)$) be a sub-solution (resp. a super-solution) of (2.1) on the open set $\mathcal{Q}_{r,R} \subset (0, T) \times \mathbb{R}$. Assume also that for all $j \in \{1, \dots, n_0\}$,*

$$u_j \leq v_j \quad \text{and} \quad \xi_j \leq \zeta_j \quad \text{on} \quad \bar{\mathcal{Q}}_{r,R+1} \setminus \mathcal{Q}_{r,R},$$

then

$$u_j \leq v_j \quad \text{and} \quad \xi_j \leq \zeta_j \quad \text{on} \quad \mathcal{Q}_{r,R} \text{ for } j \in \{1, \dots, n_0\}.$$

We now turn to the existence of a solution for equation (2.1). To do this we will use the following lemma.

Lemma 2.5 (Existence of Barriers). *Assume (A0) and (A1)-(A3). There exist a constant $K_1 > 0$ such that*

$$((u_j^+(t, x))_j, (\xi_j^+(t, x))_j) = \left(\left(u_0 \left(x + \frac{j}{n_0} \right) + K_1 t \right)_j, \left(\xi_0 \left(x + \frac{j}{n_0} \right) + K_1 t \right)_j \right),$$

and

$$((u_j^-(t, x))_j, (\xi_j^-(t, x))_j) = \left(\left(u_0 \left(x + \frac{j}{n_0} \right) - K_1 t \right)_j, \left(\xi_0 \left(x + \frac{j}{n_0} \right) - K_1 t \right)_j \right),$$

are respectively super and sub-solutions of (2.1)-(2.2) for all $T > 0$. Moreover, the constant K_1 can be chosen to be

$$K_1 = \max_{j \in \{1, \dots, n_0\}} (a_j) \cdot \frac{V_{max}}{\alpha}.$$

Proof. Let us prove that $((u_j^+(t, x))_j, (\xi_j^+(t, x))_j)$ is a super-solution of (2.1)-(2.2).

First, using (A0) with $\varepsilon = 1$ we have for all $j \in \{1, \dots, n_0\}$,

$$\alpha(\xi_j^+(t, x) - u_j^+(t, x)) = \alpha \left(\xi_0 \left(x + \frac{j}{n_0} \right) - u_0 \left(x + \frac{j}{n_0} \right) \right) \leq V_{max} \leq K_1,$$

and

$$(a_j - \alpha)(u_j^+(t, x) - \xi_j^+(t, x)) + \frac{a_j}{\alpha} V_j (u_{j+1}^+(t, x) - u_j^+(t, x)) \leq \frac{\max(a_j)}{\alpha} V_{max} \leq K_1, \quad (2.11)$$

where we have used (A0) and the fact that for all j , $\|V_j\|_\infty \leq V_{max}$. □

By applying Perron's method, joint to the comparison principle, we get the following result.

Theorem 2.6 (Existence and uniqueness of viscosity solutions for (2.1)). *Assume (A0) and (A1)-(A5). Then there exists a unique solution $((u_j)_j, (\xi_j)_j)$ of (2.1)-(2.2). Moreover, the functions u_j, ξ_j are continuous for all $j \in \mathbb{Z}$.*

We now prove that the cars remain in ordered during the evolution.

Theorem 2.7 (Ordering of the cars). *Assume (A0) and (A1)-(A5). Let $((u_j)_j, (\xi_j)_j)$ be a solution of (2.1). Then u_j and ξ_j are non-decreasing with respect to j .*

In order to do the proof of this theorem we need the following lemma.

Lemma 2.8 (Bound on time-derivative). *Assume (A1)-(A5). Let $((u_j)_j, (\xi_j)_j)$ be a solution of (1.5), with initial condition $((u_j(0, x))_j, (\xi_j(0, x))_j)$ satisfying (A0). Then for all $(t, x) \in [0, T] \times \mathbb{R}$ and for all $j \in \mathbb{Z}$,*

$$0 \leq \xi_j(t, x) - u_j(t, x) \leq \frac{V_{max}}{\alpha}.$$

Proof. Since $\xi_j - u_j$ is periodic in j it is sufficient to do this proof for $j \in \{1, \dots, n_0\}$. We will only do the proof for the upper bound since the proof for the lower bound is similar.

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Step 1: test function. We introduce

$$M = \sup_{(t,x) \in [0,T] \times \mathbb{R}} \max_{j \in \{1, \dots, n_0\}} \left(\xi_j(t, x) - u_j(t, x) - \frac{V_{max}}{\alpha} \right).$$

We want to prove that $M \leq 0$. To do this, we argue by contradiction and we assume that $M > 0$. We define the following function, with $\varepsilon, \eta, \gamma > 0$ small parameters,

$$\varphi(t, x, y, j) = \xi_j(t, x) - u_j(t, y) - \frac{|x - y|^2}{\varepsilon^2} - \frac{\eta}{T - t} - \frac{V_{max}}{\alpha} - \gamma|x|^2.$$

We can see that the function $\varphi(t, x, y, j)$ reaches a maximum at a finite point $(\bar{t}, \bar{x}, \bar{y}, \bar{j})$ thanks to the existence of barriers (Lemma 2.5). By classical arguments, we have,

$$\begin{cases} M_{\varepsilon, \eta, \gamma} = \varphi(\bar{t}, \bar{x}, \bar{y}, \bar{j}) \geq \frac{M}{2} \text{ for } \gamma \text{ and } \eta \text{ small enough.} \\ |\bar{x} - \bar{y}| \rightarrow 0 \text{ as } \varepsilon \rightarrow 0. \\ \gamma|\bar{x}| \rightarrow 0 \text{ as } \alpha \rightarrow 0. \end{cases} \quad (2.12)$$

Step 2: $\bar{t} > 0$ for ε small enough. By contradiction, let us assume $\bar{t} = 0$. Then using the fact that $M_{\varepsilon, \eta, \gamma} \geq M/2 > 0$, we get

$$\begin{aligned} 0 < M_{\varepsilon, \eta, \gamma} &\leq \xi_{\bar{j}}(0, \bar{x}) - u_{\bar{j}}(0, \bar{y}) - \frac{|\bar{x} - \bar{y}|^2}{\varepsilon^2} - \frac{\eta}{T} - \gamma|\bar{x}|^2 - \frac{V_{max}}{\alpha} \\ &\Rightarrow \frac{\eta}{T} < u_{\bar{j}}(0, \bar{x}) - u_{\bar{j}}(0, \bar{y}) + \xi_{\bar{j}}(0, \bar{x}) - u_{\bar{j}}(0, \bar{x}) - \frac{V_{max}}{\alpha} \\ &\Rightarrow \frac{\eta}{T} < K_0|\bar{x} - \bar{y}| + \frac{V_{max}}{\alpha} - \frac{V_{max}}{\alpha} \leq K_0|\bar{x} - \bar{y}|, \end{aligned} \quad (2.13)$$

where we have used for the third line assumption (A0). This is a contradiction for ε small enough.

Step 3: viscosity inequalities. We classically do a duplication of the time variable and passing to the limit, we get that there are real numbers $a, b \in \mathbb{R}$ such that

$$a - b = \frac{\eta}{(T - \bar{t})^2},$$

and

$$\begin{aligned} a &\leq (a_{\bar{j}} - \alpha)(u_{\bar{j}}(\bar{t}, \bar{x}) - \xi_{\bar{j}}(\bar{t}, \bar{x})) + \frac{a_{\bar{j}}}{\alpha} V_{\bar{j}} \left(u_{\bar{j}+1}(\bar{t}, \bar{x}) - u_{\bar{j}}(\bar{t}, \bar{x}) \right) \\ b &\geq \alpha \left(\xi_{\bar{j}}(\bar{t}, \bar{y}) - u_{\bar{j}}(\bar{t}, \bar{y}) \right). \end{aligned} \quad (2.14)$$

Step 4: passing to the limit. Subtracting the two previous inequalities we obtain

$$\begin{aligned} \frac{\eta}{T^2} &\leq (a_{\bar{j}} - \alpha) \left(u_{\bar{j}}(\bar{t}, \bar{x}) - \xi_{\bar{j}}(\bar{t}, \bar{x}) \right) + \alpha \left(u_{\bar{j}}(\bar{t}, \bar{y}) - \xi_{\bar{j}}(\bar{t}, \bar{y}) \right) \\ &\quad + \frac{a_{\bar{j}}}{\alpha} V_{\bar{j}} \left(u_{\bar{j}+1}(\bar{t}, \bar{x}) - u_{\bar{j}}(\bar{t}, \bar{x}) \right) \\ &\leq \alpha \left(u_{\bar{j}}(\bar{t}, \bar{y}) - u_{\bar{j}}(\bar{t}, \bar{x}) + \xi_{\bar{j}}(\bar{t}, \bar{x}) - \xi_{\bar{j}}(\bar{t}, \bar{y}) \right) + a_{\bar{j}} \left(u_{\bar{j}}(\bar{t}, \bar{x}) - \xi_{\bar{j}}(\bar{t}, \bar{x}) \right) + a_{\bar{j}} \frac{V_{max}}{\alpha}. \end{aligned} \quad (2.15)$$

Using that $a_{\bar{j}} \frac{V_{max}}{\alpha} \leq a_{\bar{j}} (\xi_{\bar{j}}(\bar{t}, \bar{x}) - u_{\bar{j}}(\bar{t}, \bar{y}))$, we get that

$$\frac{\eta}{T^2} \leq \alpha \left(u_{\bar{j}}(\bar{t}, \bar{y}) - u_{\bar{j}}(\bar{t}, \bar{x}) + \xi_{\bar{j}}(\bar{t}, \bar{x}) - \xi_{\bar{j}}(\bar{t}, \bar{y}) \right) + a_{\bar{j}} \left(u_{\bar{j}}(\bar{t}, \bar{y}) - u_{\bar{j}}(\bar{t}, \bar{x}) \right).$$

Sending $\varepsilon \rightarrow 0$, we get $\frac{\eta}{T^2} \leq 0$, which is a contradiction. □

We now turn to the proof of Theorem 2.7.

Proof of Theorem 2.7.

Case 1: $k_0 \geq 2h_0 n_0$. In this case we have for all $j \in \mathbb{Z}$ and $y \in \mathbb{R}$,

$$\frac{2V_{max}}{\alpha} = 2h_0 \leq u_0 \left(y + \frac{j+1}{n_0} \right) - u_0 \left(y + \frac{j}{n_0} \right)$$

Now, we would like to prove that $u_j(t, y) < u_{j+1}(t, y) - h_0$. We argue by contradiction, let us assume that there exists a time

$$t^* = \inf \{ t, \text{ s.t. } \exists i \in \mathbb{Z}, y \in \mathbb{R} \text{ s.t. } u_i(t, y) = u_{i+1}(t, y) - h_0 \}.$$

Let us consider $y \in \mathbb{R}$ and $i \in \mathbb{Z}$ such that $u_i(t^*, y) = u_{i+1}(t^*, y) - h_0$. By continuity, there exists a time $t_0 \in [0, t^*)$ such that

$$u_i(t_0, y) = u_{i+1}(t_0, y) - 2h_0 \quad \text{and} \quad u_{i+1}(t, y) - u_i(t, y) \in [h_0, 2h_0] \quad \text{for all } t \in [t_0, t^*].$$

Let us now see the equation satisfied by (u_i, ξ_i) for $t \in [t_0, t^*]$, using that $V_i(u_{i+1}(t, y) - u_i(t, y)) = 0$,

$$\begin{cases} (u_i)_t(t, y) = \alpha(\xi_i(t, y) - u_i(t, y)) \\ (\xi_i)_t(t, y) = (a_i - \alpha)(u_i(t, y) - \xi_i(t, y)) \\ u_i(t_0, y) = u_{i+1}(t_0, y) - 2h_0 \\ \xi_i(t_0, y) \leq u_{i+1}(t_0, y) - 2h_0 + \frac{V_{max}}{\alpha}. \end{cases} \quad (2.16)$$

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The last inequality is justified by Lemma 2.8. We now construct a super-solution for this system by considering

$$\begin{cases} \bar{u}_i(t, y) = \frac{V_{max}}{a_i}(1 - e^{-a_i(t-t_0)}) + u_{i+1}(t_0, y) - 2h_0 \\ \bar{\xi}_i(t, y) = \bar{u}_i(t, y) + \frac{1}{\alpha}(\bar{u}_i)_t(t, y), \end{cases}$$

with the initial condition

$$\begin{cases} \bar{u}_i(t_0, y) = u_{i+1}(t_0, y) - 2h_0 \\ \bar{\xi}_i(t_0, y) = u_{i+1}(t_0, y) - 2h_0 + \frac{V_{max}}{\alpha}. \end{cases}$$

Since $t_0 < t^*$, we have

$$\begin{aligned} \bar{u}_i(t^*, y) &< \frac{V_{max}}{\alpha} - h_0 + u_{i+1}(t_0, y) - h_0 \\ &= u_{i+1}(t_0, y) - h_0 \\ &\leq u_{i+1}(t^*, y) - h_0, \end{aligned} \tag{2.17}$$

where we used for the first line the fact that $1 - e^{a_i(t^*-t_0)} < 1$ and that $\alpha \leq a_i$, for the second line the fact that $h_0 = V_{max}/\alpha$, and for the third line the fact that $t^* > t_0$ and Lemma 2.8 which implies that the u_j are non-decreasing in t . Using the comparison principle for (2.16) yields

$$u_i(t^*, y) \leq \bar{u}_i(t^*, y) < u_{i+1}(t^*, y) - h_0. \tag{2.18}$$

This is a contradiction with the definition of t^* . Therefore, we have for all $j \in \{1, \dots, n_0\}$, for all $(t, x) \in [0, T] \times \mathbb{R}$, that

$$u_j(t, x) < u_{j+1}(t, x) - h_0.$$

Now from Lemma 2.8 we know that for all $(t, x) \in (0, +\infty) \times \mathbb{R}$ and for all $j \in \mathbb{Z}$,

$$\begin{cases} V_{max} \geq \alpha(\xi_j(t, x) - u_j(t, x)) \geq 0 \\ V_{max} \geq \alpha(\xi_{j+1}(t, x) - u_{j+1}(t, x)) \geq 0, \end{cases}$$

from which we can easily deduce that for all $j \in \mathbb{Z}$, for all $(t, x) \in [0, T] \times \mathbb{R}$, we have

$$\xi_{j+1}(t, x) \geq \xi_j(t, x).$$

Case 2: $0 < k_0 \leq 2h_0n_0$. We use the following lemma:

Lemma 2.9. *Assume (A1)-(A5), let $i \in \mathbb{Z}$, $y \in \mathbb{R}$, and let $2h_0 - k_0/n_0 \geq \delta > 0$, also let us assume that*

$$u_i(0, y) = u_{i+1}(0, y) - (2h_0 - \delta) \quad \text{and} \quad 0 \leq \xi_i(0, y) - u_i(0, y) \leq \frac{2h_0 - \delta}{2}, \quad (2.19)$$

then we have

$$u_i(t, y) \leq u_{i+1}(t, y) \quad \text{and} \quad \xi_i(t, y) \leq \xi_{i+1}(t, y),$$

for all time $t \in [0, T]$ such that

$$u_{i+1}(t, y) - u_i(t, y) \leq 2h_0.$$

Proof. Let us denote by \hat{t} , the time

$$\hat{t} = \inf\{t \text{ s.t. } u_{i+1}(t, y) - u_i(t, y) = 2h_0\},$$

then for all $t \in [0, \hat{t}]$, $u_{i+1}(t, y) - u_i(t, y) \leq 2h_0$, and (u_i, ξ_i) is solution to the following system, for $t \in [0, \hat{t}]$

$$\begin{cases} (u_i)_t = \alpha(\xi_i - u_i) \\ (\xi_i)_t = (a_i - \alpha)(u_i - \xi_i) \\ u_i(0, y) = u_{i+1}(0, y) - (2h_0 - \delta) \\ \xi_i(0, y) \leq u_{i+1}(0, y) - 2h_0 + \delta + \frac{2h_0 - \delta}{2}, \end{cases}$$

We can construct a super-solution of this system by considering

$$\begin{cases} \bar{u}_i(t, y) = \alpha \frac{2h_0 - \delta}{2a_i} (1 - e^{-a_i t}) + u_{i+1}(0, y) - 2h_0 + \delta \\ \bar{\xi}_i(t, y) = \bar{u}_i(t, y) + \frac{1}{\alpha} (\bar{u}_i)_t(t, y). \end{cases} \quad (2.20)$$

Therefore, for all $t \in [0, \hat{t}]$, we have

$$\begin{aligned} u_{i+1}(t, y) - u_i(t, y) &\geq u_{i+1}(0, y) - u_i(t, y) \\ &\geq u_{i+1}(0, y) - \bar{u}_i(t, y) \\ &\geq 2h_0 - \delta - \alpha \frac{2h_0 - \delta}{2a_i} \\ &\geq 2h_0 - \delta - \frac{2h_0 - \delta}{2} \\ &\geq \frac{2h_0 - \delta}{2} \geq 0, \end{aligned} \quad (2.21)$$

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where we have used for the first line the fact that u_{i+1} is non-decreasing in time (see Lemma 2.8), for the second line a comparison between u_i and \bar{u}_i , for the third line the definition of $\bar{u}_i(t, y)$ and the fact that $1 - e^{-a_i t} \leq 1$, and for the fourth line the fact that $\alpha \leq a_i$. Similarly, we have

$$\begin{aligned} \xi_{i+1}(t, y) - \xi_i(t, y) &\geq u_{i+1}(t, y) - \bar{\xi}_i(t, y) \\ &\geq u_{i+1}(t, y) - \bar{u}_i(t, y) - \frac{1}{\alpha}(\bar{u}_i)_t(t, y) \\ &\geq \frac{2h_0 - \delta}{2} - \frac{2h_0 - \delta}{2} \geq 0, \end{aligned} \tag{2.22}$$

where we have used for the first line Lemma 2.8 and the fact that $\bar{\xi}_i \geq \xi_i$, for the second line the definition of $\bar{\xi}$. □

Using this lemma, we deduce that in the case where there exist $i \in \mathbb{Z}$ and $y \in \mathbb{R}$ such that

$$\frac{u_{i+1}(0, y) - u_i(0, y)}{2} \leq \frac{V_{max}}{\alpha} = h_0, \tag{2.23}$$

we have

$$u_i(t, y) \leq u_{i+1}(t, y) \quad \text{and} \quad \xi_i(t, y) \leq \xi_{i+1}(t, y),$$

for all time $t \in [0, t^*]$, where t^* is defined by

$$t^* = \inf\{t, u_{i+1}(t, y) - u_i(0, y) > 2h_0\}.$$

We can then use the same proof as in Case 1 to deduce that

$$u_{i+1}(t, y) \geq u_i(t, y) + h_0 \quad \text{and} \quad \xi_{i+1}(t, y) \geq \xi_i(t, y) \quad \text{for all } t \geq t^*.$$

□

3 Convergence

This section contains the proof of the main homogenization result (Theorem 1.3). This proof relies on the existence of hull functions and some properties of the effective Hamiltonian.

We recall two lemmas necessary for the proof of Theorem 1.3. We begin by a first result which is a direct consequence of Perron's method and Lemma 2.5 (with a rescaling in ε).

Lemma 3.1 (Barriers uniform in ε). *Assume (A1)-(A5) and (A0). Then there is a constant $C > 0$, such that for all $\varepsilon > 0$, the solution $((u_j^\varepsilon)_j, (\xi_j^\varepsilon)_j)$ of (1.7)-(1.8) satisfies for all $t > 0$ and $x \in \mathbb{R}$,*

$$\left| u_j^\varepsilon(t, x) - u_0 \left(x + \frac{j\varepsilon}{n_0} \right) \right| \leq Ct \quad \text{and} \quad \left| \xi_j^\varepsilon(t, x) - \xi_0 \left(x + \frac{j\varepsilon}{n_0} \right) \right| \leq Ct.$$

Lemma 3.2 (ε -bounds on the gradient). *Assume (A1)-(A5) and (A0). Then, the solution $((u_j^\varepsilon)_j, (\xi_j^\varepsilon)_j)$ of (1.7)-(1.8) satisfies for all $t > 0$, $x \in \mathbb{R}$, $z \geq 0$ and $j \in \mathbb{Z}$,*

$$zk_0 \leq u_j^\varepsilon(t, x+z) - u_j^\varepsilon(t, x) \leq zK_0, \tag{3.1}$$

and

$$zk_0 \leq \xi_j^\varepsilon(t, x+z) - \xi_j^\varepsilon(t, x) \leq zK_0. \tag{3.2}$$

Proof. We prove the lower bound (the proof for the upper bound is similar). Using assumption (A0), we get that for all $j \in \mathbb{Z}$, for all $x \in \mathbb{R}$, $z \geq 0$,

$$u_j^\varepsilon(0, x+z) = u_0 \left(x+z + \frac{j\varepsilon}{n_0} \right) \geq u_0 \left(x + \frac{j\varepsilon}{n_0} \right) + zk_0 \geq u_j^\varepsilon(0, x) + zk_0, \tag{3.3}$$

and

$$\xi_j^\varepsilon(0, x+z) \geq \xi_j^\varepsilon(0, x) + zk_0.$$

From the form of system (1.7), we know that the equation is invariant by addition of constants to the solutions. For this reason the solution associated to the initial data $((u_j^\varepsilon(0, x) + zk_0)_j, (\xi_j^\varepsilon(0, x) + zk_0)_j)$ is $((u_j^\varepsilon(t, x) + zk_0)_j, (\xi_j^\varepsilon(t, x) + zk_0)_j)$. We can also see that the equation is invariant by space translations. Therefore the solution associated to the initial data $((u_j(0, x+z))_j, (\xi_j^\varepsilon(0, x+z))_j)$ is $((u_j(t, x+z))_j, (\xi_j^\varepsilon(t, x+z))_j)$. Finally, from (3.3) and from the comparison principle (Proposition 2.2), we get

$$u_j^\varepsilon(t, x+z) \geq u_j(t, x) + zk_0 \quad \text{and} \quad \xi_j^\varepsilon(t, x+z) \geq \xi_j^\varepsilon(t, x) + zk_0.$$

□

Proof of Theorem 1.3.

Since, for all $j \in \mathbb{Z}$, $u_{j+n_0}^\varepsilon(t, x) = u_j^\varepsilon(t, x + \varepsilon)$ and $\xi_{j+n_0}^\varepsilon(t, x) = \xi_j(t, x + \varepsilon)$ it is sufficient to do this proof for $j \in \{1, \dots, n_0\}$.

We introduce for all $j \in \{1, \dots, n_0\}$,

$$\bar{u}_j(t, x) = \limsup_{\varepsilon \rightarrow 0, (t', x') \rightarrow (t, x)} u_j^\varepsilon(t', x') \quad \text{and} \quad \bar{\xi}_j(t, x) = \limsup_{\varepsilon \rightarrow 0, (t', x') \rightarrow (t, x)} \xi_j^\varepsilon(t', x'),$$

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$$\underline{u}_j(t, x) = \liminf_{\varepsilon \rightarrow 0, (t', x') \rightarrow (t, x)} u_j^\varepsilon(t', x') \quad \text{and} \quad \underline{\xi}_j(t, x) = \liminf_{\varepsilon \rightarrow 0, (t', x') \rightarrow (t, x)} \xi_j^\varepsilon(t', x').$$

Thanks to Lemma 3.1 we know that this functions are well defined. We also introduce

$$\bar{v} = \max_{j \in \{1, \dots, n_0\}} \max(\bar{u}_j, \bar{\xi}_j) \quad \text{and} \quad \underline{v} = \min_{j \in \{1, \dots, n_0\}} \min(\underline{u}_j, \underline{\xi}_j). \quad (3.4)$$

Using the two previous lemmas we get that the function $w = \bar{v}, \underline{v}$, satisfies for all $t > 0$ and $x, x' \in \mathbb{R}$, $x \leq x'$,

$$\begin{aligned} |w(t, x) - u_0(x)| &\leq Ct, \\ k_0|x - x'| &\leq w(t, x') - w(t, x) \leq K_0|x - x'|. \end{aligned} \quad (3.5)$$

We want to prove that \bar{v} is a sub-solution of (1.9) and that \underline{v} is a super-solution of (1.9). Indeed, in this case, the comparison principle will imply that $\bar{v} \leq \underline{v}$. But by construction $\underline{v} \leq \bar{v}$, hence $\underline{v} = \bar{v} = u^0$, the unique solution of (1.9). This implies that for all $j \in \{1, \dots, n_0\}$, $\bar{u}_j = \underline{u}_j = \bar{\xi}_j = \underline{\xi}_j = u^0$ and so u_j^ε and ξ_j^ε converge locally uniformly to u^0 .

To prove that \bar{v} is a sub-solution of (1.9), we argue by contradiction, we assume that there is a point $(\bar{t}, \bar{x}) \in \mathbb{R}^+ \times \mathbb{R}$ and a test function $\phi \in C^1$ such that

$$\left\{ \begin{array}{ll} \bar{v}(\bar{t}, \bar{x}) = \phi(\bar{t}, \bar{x}) & \\ \bar{v} \leq \phi & \text{on } Q_{r, 2r}(\bar{t}, \bar{x}) \quad \text{with } r > 0 \\ \bar{v} \leq \phi - 2\eta & \text{on } Q_{r, 2r}(\bar{t}, \bar{x}) \setminus Q_{r, r}(\bar{t}, \bar{x}) \quad \text{with } \eta > 0 \\ \phi_t(\bar{t}, \bar{x}) = \bar{F}(\phi_x(\bar{t}, \bar{x})) + \theta, & \text{with } \theta > 0. \end{array} \right. \quad (3.6)$$

We define $p = \phi_x(\bar{t}, \bar{x})$ that according to (3.5) satisfies

$$0 < k_0 \leq p \leq K_0.$$

Using Theorem 1.7, we define the hull functions $((h_j)_j, (g_j)_j)$ associated to p such that

$$\lambda = \bar{F}(p)$$

We now apply the perturbed test function method introduced by Evans [Eva89] in terms here of hull functions instead of correctors. Let us consider the following perturbed test functions for $j \in \{1, \dots, n_0\}$,

$$\phi_j^\varepsilon(t, x) = \varepsilon h_j \left(\frac{\phi(t, x)}{\varepsilon} \right) = \phi(t, x) + \varepsilon h_j(0)$$

and

$$\psi_j^\varepsilon(t, x) = \varepsilon g_j \left(\frac{\phi(t, x)}{\varepsilon} \right) = \phi(t, x) + \varepsilon g_j(0).$$

We define the family of test functions $(\phi_j^\varepsilon)_{j \in \mathbb{Z}}$ and $(\psi_j^\varepsilon)_{j \in \mathbb{Z}}$ by using the relation

$$\phi_{j+kn_0}^\varepsilon(t, x) = \phi_j^\varepsilon(t, x + \varepsilon k) \quad \text{and} \quad \psi_{j+kn_0}^\varepsilon(t, x) = \psi_j^\varepsilon(t, x + \varepsilon k).$$

We first want to prove that $((\phi_j^\varepsilon)_j, (\psi_j^\varepsilon)_j)$ is a super-solution of (1.7) in a neighbourhood of (\bar{t}, \bar{x}) .

To do this, we simply check the equations satisfied by the perturbed test functions, we denote by $z = \frac{\phi(t, x)}{\varepsilon}$ to simplify the notations. For $j \in \{1, \dots, n_0\}$, we have

$$\begin{aligned} (\phi_j^\varepsilon)_t(t, x) &= \phi_t(t, x) + \alpha(g_j(z) - h_j(z)) - \alpha(g_j(z) - h_j(z)) \\ &= \frac{\alpha}{\varepsilon}(\psi_j^\varepsilon(t, x) - \phi_j^\varepsilon(t, x)) + (\phi_t(t, x) - \lambda) \\ &= \frac{\alpha}{\varepsilon}(\psi_j^\varepsilon(t, x) - \phi_j^\varepsilon(t, x)) + \left(\phi_t(t, x) - \phi_t(\bar{t}, \bar{x}) + \theta \right) \\ &\geq \frac{\alpha}{\varepsilon}(\psi_j^\varepsilon(t, x) - \phi_j^\varepsilon(t, x)), \end{aligned}$$

where we have used the equations satisfied by the hull functions for the second line, the definition of λ for the third line. For the fourth line we have used the fact that for $r > 0$ small enough, we have $\left(\phi_t(t, x) - \phi_t(\bar{t}, \bar{x}) + \frac{\theta}{2} \right) \geq 0$, because $\theta > 0$ and ϕ is C^1 . Similarly, we have

$$\begin{aligned} (\psi_j^\varepsilon)_t(t, x) &= \phi_t(t, x) \\ &= (a_j - \alpha)(h_j(z) - g_j(z)) + \frac{a_j}{\alpha} V_j(h_{j+1}(z) - h_j(z)) - \lambda + \phi_t(t, x) \\ &\geq \frac{(a_j - \alpha)}{\varepsilon}(\phi_j^\varepsilon(t, x) - \psi_j^\varepsilon(t, x)) + \frac{a_j}{\alpha} V_j \left(\frac{\phi_{j+1}^\varepsilon(t, x) - \phi_j^\varepsilon(t, x)}{\varepsilon} \right) \\ &\quad + \left(\phi_t(t, x) - \phi_t(\bar{t}, \bar{x}) + \frac{\theta}{2} \right) \\ &\quad + \frac{\theta}{2} + \frac{a_j}{\alpha} \left[V_j(h_{j+1}(z) - h_j(z)) - V_j \left(\frac{\phi_{j+1}^\varepsilon(t, x) - \phi_j^\varepsilon(t, x)}{\varepsilon} \right) \right]. \end{aligned}$$

It is then enough to prove that

$$\frac{\theta}{2} + \frac{a_j}{\alpha} \left[V_j(h_{j+1}(z) - h_j(z)) - V_j \left(\frac{\phi_{j+1}^\varepsilon(t, x) - \phi_j^\varepsilon(t, x)}{\varepsilon} \right) \right] \geq 0.$$

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If $j + 1 \in \{1, \dots, n_0\}$, then,

$$h_j(z) = \frac{\phi_j^\varepsilon(t, x)}{\varepsilon} \quad \text{and} \quad h_{j+1}(z) = \frac{\phi_{j+1}^\varepsilon(t, x)}{\varepsilon},$$

and the result is trivial.

If $j + 1 \notin \{1, \dots, n_0\}$ then $j = n_0$ and we have

$$h_{j+1}(z) = h_{1+n_0}(z) = h_1(z + p) = \frac{\phi(t, x)}{\varepsilon} + p + h_1(0),$$

$$\begin{aligned} \phi_{j+1}^\varepsilon(t, x) &= \phi_{n_0+1}^\varepsilon(t, x) = \phi_1^\varepsilon(t, x + \varepsilon) = \phi(t, x + \varepsilon) + \varepsilon h_1(0). \\ &= \phi(t, x) + \varepsilon p + \varepsilon h_1(0) + o_\varepsilon(\varepsilon). \end{aligned}$$

This implies that

$$\frac{\phi_{j+1}^\varepsilon(t, x)}{\varepsilon} = \frac{\phi(t, x)}{\varepsilon} + p + h_1(0) + o_\varepsilon(1) = h_{j+1}(z) + o_\varepsilon(1).$$

This allows us to see that for $r > 0$ small enough, we get

$$\frac{\theta}{2} + \frac{a_j}{\alpha} \left[V_j(h_{j+1}(z) - h_j(z)) - V_j \left(\frac{\phi_{j+1}^\varepsilon(t, x) - \phi_j^\varepsilon(t, x)}{\varepsilon} \right) \right] \geq 0.$$

Getting a contradiction. By definition: $\phi_j^\varepsilon \rightarrow \phi$ and $\psi_j^\varepsilon \rightarrow \phi$ as $\varepsilon \rightarrow 0$. Moreover, $\bar{u}_j \leq \bar{v} \leq \phi - 2\eta$ on $Q_{r,2r}(\bar{t}, \bar{x}) \setminus Q_{r,r}(\bar{t}, \bar{x})$ therefore, for ε small enough

$$u_j^\varepsilon \leq \phi_j^\varepsilon - \eta \quad \text{on} \quad Q_{r,2r}(\bar{t}, \bar{x}) \setminus Q_{r,r}(\bar{t}, \bar{x}).$$

Similarly, we have

$$\xi_j^\varepsilon \leq \psi_j^\varepsilon - \eta \quad \text{on} \quad Q_{r,2r}(\bar{t}, \bar{x}) \setminus Q_{r,r}(\bar{t}, \bar{x}).$$

Using the comparison principle on bounded sets for (1.7), we get

$$u_j^\varepsilon \leq \phi_j^\varepsilon - \eta \quad \text{and} \quad \xi_j^\varepsilon \leq \psi_j^\varepsilon - \eta \quad \text{on} \quad Q_{r,r}(\bar{t}, \bar{x}). \quad (3.7)$$

Passing to the limit as $\varepsilon \rightarrow 0$, we get $\bar{v} \leq \phi - \eta$ on $Q_{r,r}(\bar{t}, \bar{x})$ and this contradicts the fact that $\bar{v}(\bar{t}, \bar{x}) = \phi(\bar{t}, \bar{x})$.

Therefore \bar{v} is a sub-solution of (1.9) on $(0, +\infty) \times \mathbb{R}$. Similarly, \underline{v} is a super-solution of the same equation. Therefore, $\underline{v} = \bar{v} = u^0$ and u_j and ξ_j converge locally uniformly to u^0 for $j \in \{1, \dots, n_0\}$.

□

4 Ergodicity and construction of hull functions

In this section, we construct the hull functions for (2.1). The construction follows the one of [FIM12] but we use here the fact that the system is invariant by addition of constants. This allows us to get the particular form of the hull functions.

4.1 Ergodicity

Proposition 4.1 (Particular form of the solution of (2.1)). *Assume (A1)-(A5) and let $p > 0$. Let $((u_j)_j, (\xi_j)_j)$ be the solution of (2.1) with $u_0(y) = \xi_0(y) = py$. Then $((u_j)_j, (\xi_j)_j)$ satisfies*

$$u_j(t, y) = py + u_j(t, 0) \quad \text{and} \quad \xi_j(t, y) = py + \xi_j(t, 0). \quad (4.1)$$

Proof of Proposition 4.1. Using that equation (2.1) is invariant by space translations, invariant by addition of constants, and the fact that for all $y, z \in \mathbb{R}, j \in \mathbb{Z}$

$$u_0\left(y + z + \frac{j}{n_0}\right) - pz = u_0\left(y + \frac{j}{n_0}\right) \quad \text{and} \quad \xi_0\left(y + z + \frac{j}{n_0}\right) - pz = \xi_0\left(y + \frac{j}{n_0}\right),$$

we deduce, by the comparison principle, that

$$u_j(t, y + z) - pz = u_j(t, y) \quad \text{and} \quad \xi_j(t, y + z) - pz = \xi_j(t, y).$$

Taking $y = 0$, we deduce the result. □

Proposition 4.2 (Ergodicity). *Assume (A1)-(A5), let $((u_j)_j, (\xi_j)_j)$ be a solution of (2.1) with initial data $u_0(y) = \xi_0(y) = py$ for some $p > 0$. Then there exists a constant $\lambda \in \mathbb{R}$ such that, for all $(t, y) \in [0; +\infty) \times \mathbb{R}, j \in \{1, \dots, n_0\}$,*

$$|u_j(t, 0) - \lambda t| \leq C_1 \quad \text{and} \quad |\xi_j(t, 0) - \lambda t| \leq C_1, \quad (4.2)$$

and

$$|\lambda| \leq K_1, \quad (4.3)$$

with

$$C_1 = 4p + 2 \frac{V_{max}}{\alpha}, \quad (4.4)$$

and K_1 defined in Lemma 2.5.

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The proof of Proposition 4.2 is done in different steps, it uses the following classical lemma from ergodic theory (see for instance [Kat95]).

Lemma 4.3. *Consider $\Lambda : \mathbb{R}^+ \rightarrow \mathbb{R}$ a continuous function which is sub-additive, meaning that: for all $t, s \geq 0$,*

$$\Lambda(t + s) \leq \Lambda(t) + \Lambda(s).$$

Then $\frac{\Lambda(t)}{t}$ has a limit l as $t \rightarrow +\infty$ and

$$l = \inf_{t > 0} \frac{\Lambda(t)}{t}. \quad (4.5)$$

Proof of Proposition 4.2.

The main idea of the proof is to control the time oscillations. To do this we will use the following continuous functions for all $T > 0$,

$$\lambda_+^u(T) = \sup_{j \in \{1, \dots, n_0\}} \sup_{t \geq 0} \frac{u_j(t + T, 0) - u_j(t, 0)}{T},$$

$$\lambda_-^u(T) = \inf_{j \in \{1, \dots, n_0\}} \inf_{t \geq 0} \frac{u_j(t + T, 0) - u_j(t, 0)}{T},$$

and

$$\lambda_+^\xi(T) = \sup_{j \in \{1, \dots, n_0\}} \sup_{t \geq 0} \frac{\xi_j(t + T, 0) - \xi_j(t, 0)}{T},$$

$$\lambda_-^\xi(T) = \inf_{j \in \{1, \dots, n_0\}} \inf_{t \geq 0} \frac{\xi_j(t + T, 0) - \xi_j(t, 0)}{T}.$$

We also introduce

$$\lambda_+(T) = \sup(\lambda_+^u(T), \lambda_+^\xi(T)) \quad \text{and} \quad \lambda_-(T) = \inf(\lambda_-^u(T), \lambda_-^\xi(T)).$$

To get the result, it suffices to prove that $\lambda_+(T)$ and $\lambda_-(T)$ have a common limit λ as $T \rightarrow +\infty$ such that $|\lambda_\pm - \lambda| \leq \frac{C_1}{T}$. To do this we would like to apply Lemma 4.3. Because of their definitions, we know that $T \mapsto T\lambda_+^u(T)$ and $T \mapsto T\lambda_+^\xi(T)$ are sub-additive, in the same way $T \mapsto -T\lambda_-^u(T)$ and $T \mapsto -T\lambda_-^\xi(T)$ are also sub-additive. Therefore if $\lambda_\pm^u(T)$ and $\lambda_\pm^\xi(T)$ are finite, we will get the convergence, and we will only have to prove that they have the same limit.

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Step 1: $\lambda_+(T)$ and $\lambda_-(T)$ converge as T goes to $+\infty$. We want to use Lemma 4.3. Since $T \mapsto T\lambda_+^u$, $T \mapsto T\lambda_+^\xi$, $T \mapsto -T\lambda_-^u$ and $T \mapsto -T\lambda_-^\xi$ are sub-additive, we deduce that $T \mapsto T\lambda_+$ and $T \mapsto -T\lambda_-$ are sub-additive. It just remains to show that λ_+ and λ_- are bounded (to get a finite limit). Using Lemma 2.8, we get that for all $j \in \{1, \dots, n_0\}$ and for all $t, T > 0$, we have

$$-K_1 \leq 0 \leq \frac{u_j(t+T, 0) - u_j(t, 0)}{T} \leq V_{max} \leq K_1 \quad (4.6)$$

and

$$-K_1 \leq -\max(a_j - \alpha) \frac{V_{max}}{\alpha} \leq \frac{\xi_j(t+T, 0) - \xi_j(t, 0)}{T} \leq \frac{V_{max}}{\alpha} \max_{j \in \{1, \dots, n_0\}} (a_j) \leq K_1, \quad (4.7)$$

where we have used the equation satisfied by ξ_j and the facts that

$$-a_j \frac{V_{max}}{\alpha} \leq -(a_j - \alpha) \frac{V_{max}}{\alpha} \leq (a_j - \alpha)(u_j(t, x) - \xi_j(t, x)) \leq 0,$$

and

$$0 \leq V_j(u_{j+1}(t, x) - u_j(t, x)) \leq V_{max}.$$

Step 2: Control on the time oscillations We now prove that λ_+ and λ_- have the same limit. More precisely, we will prove that

$$|\lambda_+(T) - \lambda_-(T)| \leq \frac{C_1}{T},$$

with C_1 defined in Proposition 4.2.

By definition of $\lambda_\pm(T)$, for all $\varepsilon > 0$, there exists τ^\pm and $v^\pm \in \{u_1, \dots, u_n, \xi_1, \dots, \xi_n\}$ such that

$$\left| \lambda_\pm(T) - \frac{v^\pm(\tau^\pm + T, 0) - v^\pm(\tau^\pm, 0)}{T} \right| \leq \varepsilon$$

Let us set for all $j \in \{1, \dots, n_0\}$,

$$\Delta_j^u = u_j(\tau^+, 0) - u_j(\tau^-, 0), \quad \Delta_j^\xi = \xi_j(\tau^+, 0) - \xi_j(\tau^-, 0)$$

and

$$\Delta = \sup_{j \in \{1, \dots, n_0\}} \sup(\Delta_j^u, \Delta_j^\xi).$$

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Using Proposition 4.1, we have

$$u_j(\tau^+, y) = u_j(\tau^-, y) + u_j(\tau^+, 0) - u_j(\tau^-, 0) \leq u_j(\tau^-, y) + \Delta$$

and

$$\xi_j(\tau^+, y) = \xi_j(\tau^-, y) + \xi_j(\tau^+, 0) - \xi_j(\tau^-, 0) \leq \xi_j(\tau^-, y) + \Delta.$$

Using the comparison principle we get

$$u_j(\tau^+ + T, y) \leq u_j(\tau^- + T, y) + \Delta \quad (4.8)$$

and

$$\xi_j(\tau^+ + T, y) \leq \xi_j(\tau^- + T, y) + \Delta. \quad (4.9)$$

Now we would like to estimate Δ . Let us assume that the maximum in Δ is reached for the index \bar{j} . We then have for all $j \in \{1, \dots, n_0\}$,

$$\begin{aligned} \Delta &\leq u_{\bar{j}}(\tau^+, 0) - u_{\bar{j}}(\tau^-, 0) + \frac{V_{max}}{\alpha} \\ &\leq u_{j+n_0}(\tau^+, 0) - u_{j-n_0}(\tau^-, 0) + \frac{V_{max}}{\alpha} \\ &\leq u_j(\tau^+, 1) - u_j(\tau^-, -1) + \frac{V_{max}}{\alpha} \\ &\leq u_j(\tau^+, 0) - u_j(\tau^-, 0) + \frac{V_{max}}{\alpha} + 2p, \end{aligned} \quad (4.10)$$

where we have used for the first line Lemma 2.8 (to compare $u_{\bar{j}}$ and $\xi_{\bar{j}}$), for the second line, the fact that $(u_j)_j$ is non-decreasing in j , for the third line the periodicity of the function u_j , and for the last line we have used Proposition 4.1. Similarly we have

$$\Delta \leq \xi_j(\tau^+, 0) - \xi_j(\tau^-, 0) + \frac{V_{max}}{\alpha} + 2p.$$

We now inject this results in (4.8) and (4.9), with $y = 0$, to obtain

$$u_j(\tau^+ + T, 0) - u_j(\tau^+, 0) \leq u_j(\tau^- + T, 0) - u_j(\tau^-, 0) + \frac{V_{max}}{\alpha} + 2p,$$

and

$$\xi_j(\tau^+ + T, 0) - \xi_j(\tau^+, 0) \leq \xi_j(\tau^- + T, 0) - \xi_j(\tau^-, 0) + \frac{V_{max}}{\alpha} + 2p.$$

Using this two results we get that

$$v^+(\tau^+ + T, 0) - v^+(\tau^+, 0) \leq v^-(\tau^- + T, 0) - v^-(\tau^-, 0) + 2\frac{V_{max}}{\alpha} + 4p,$$

where the possible comparison between ξ_j and u_j adds an additional V_{max}/α , and the possible comparison between u_j and u_k adds an additional $2p$. This implies that

$$T\lambda_+(T) \leq T\lambda_-(T) + 2\varepsilon T + C_1.$$

Since this is true for all $\varepsilon > 0$, we get

$$|\lambda_+(T) - \lambda_-(T)| \leq \frac{C_1}{T}. \tag{4.11}$$

Step 3: Conclusion. From the previous step we know that λ_{\pm} have the same limit. Let us denote it by λ , and by Lemma 4.3 we have, for all $T > 0$,

$$\lambda_-(T) \leq \lambda \leq \lambda_+(T).$$

Using (4.11), we deduce that

$$|\lambda_{\pm} - \lambda| \leq \frac{C_1}{T}. \tag{4.12}$$

□

4.2 Construction of hull functions

We now would like to prove the existence of time-space global solutions of (2.1).

Proposition 4.4. *Let $p > 0$ and assume (A1)-(A5). Then, there exist some constants $((u_j^\infty(0, 0))_j, (\xi_j^\infty(0, 0))_j)$ and a real number $\lambda \in \mathbb{R}$ such that for all $(\tau, y) \in \mathbb{R}^2$,*

$$\left((u_j^\infty(\tau, y))_j, (\xi_j^\infty(\tau, y))_j \right) = \left((py + \lambda\tau + u_j^\infty(0, 0))_j, (py + \lambda\tau + \xi_j^\infty(0, 0))_j \right),$$

is a solution of (2.1). These constants satisfy, for all $j \in \mathbb{Z}$,

$$u_{j+1}^\infty(0, 0) \geq u_j^\infty(0, 0) \quad \text{and} \quad \xi_{j+1}^\infty(0, 0) \geq \xi_j^\infty(0, 0). \tag{4.13}$$

The interest of this result is that if we consider for all $z \in \mathbb{R}$,

$$\begin{cases} h_j(z) = z + u_j^\infty(0, 0) & \text{if } j \in \{1, \dots, n_0\} \\ h_{j+n_0}(z) = h_j(z + p), \end{cases} \quad \begin{cases} g_j(z) = z + \xi_j^\infty(0, 0) & \text{if } j \in \{1, \dots, n_0\} \\ g_{j+n_0}(z) = g_j(z + p), \end{cases} \tag{4.14}$$

then we have directly the following result.

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Corollary 4.5. (*Existence of hull functions*). Assume (A1)-(A5), then there exists $\lambda \in \mathbb{R}$ such that there exist a hull function $((h_j)_j, (g_j)_j)$ defined as in Definition 1.5.

Proof of Proposition 4.4.

Construction of a solution. In this step, we use the functions $((u_j)_j, (\xi_j)_j)$ solution of (2.1) with $u_0(y) = \xi_0(y) = py$. For $m \in \mathbb{R}$, we consider

$$u_j^m(t, 0) = u_j(t + m, 0) - \lambda m \quad \text{and} \quad \xi_j^m(t, 0) = \xi_j(t + m, 0) - \lambda m.$$

Since the equation is invariant by addition of constants and by time-translations, we deduce that

$$\left((u_j^m(t, y) = py + u_j^m(t, 0))_j, (\xi_j^m(t, y) = py + u_j^m(t, 0))_j \right), \quad (4.15)$$

is a solution of (2.1). Moreover, u_j^m is Lipschitz continuous in time thanks to Lemma 2.8 and as a consequence ξ_j^m is also Lipschitz continuous in time. Therefore we can use Ascoli Theorem to deduce that there is a sub-sequence of $((u_j^m)_j, (\xi_j^m)_j)$ converging uniformly on compact sets to a Lipschitz continuous function $((u_j^\infty)_j, (\xi_j^\infty)_j)$ which satisfies, for all $k \in \mathbb{R}$,

$$\begin{cases} u_j^\infty(t + k, 0) = u_j^\infty(t, 0) + \lambda k \\ u_j^\infty(t, y) = py + u_j^\infty(t, 0) \\ u_{j+1}^\infty \geq u_j^\infty \end{cases} \quad \begin{cases} \xi_j^\infty(t + k, 0) = \xi_j^\infty(t, 0) + \lambda k \\ \xi_j^\infty(t, y) = py + \xi_j^\infty(t, 0) \\ \xi_{j+1}^\infty \geq \xi_j^\infty. \end{cases} \quad (4.16)$$

However, since $k \in \mathbb{R}$, we deduce that

$$u_j^\infty(t, 0) = u_j^\infty(0, 0) + \lambda t \quad \text{and} \quad \xi_j^\infty(t, 0) = \xi_j^\infty(0, 0) + \lambda t, \quad (4.17)$$

which implies the result. □

Proof of Theorem 1.7.

Uniqueness of λ . Given some $p \in (0, +\infty)$, let us assume that there exist $\lambda_1, \lambda_2 \in \mathbb{R}$ with their corresponding hull functions $((h_j^1)_j, (g_j^1)_j), ((h_j^2)_j, (g_j^2)_j)$. Then we define for $i = 1, 2$ and $j \in \{1, \dots, n_0\}$,

$$u_j^i(t, y) = h_j^i(\lambda_i t + py) \quad \text{and} \quad \xi_j^i(t, y) = g_j^i(\lambda_i t + py),$$

4. Ergodicity and construction of hull functions

solution of (2.1). Let us denote by $C = \max_{j \in \{1, \dots, n_0\}} \max_{i \in \{1, 2\}} (|h_j^i(0)|, |g_j^i(0)|)$, then we have

$$u_j^1(0, y) \leq u_j^2(0, y) + 2C \quad \text{and} \quad \xi_j^1(0, y) \leq \xi_j^2(0, y) + 2C.$$

Using the comparison principle we get for all $(t, x) \in (0, +\infty) \times \mathbb{R}$,

$$u_j^1(t, y) \leq u_j^2(t, y) + 2C \quad \text{and} \quad \xi_j^1(t, y) \leq \xi_j^2(t, y) + 2C.$$

Now we set $y = 0$ to deduce that for all $t \in (0, +\infty)$,

$$h_j^1(\lambda_1 t) \leq h_j^2(\lambda_2 t) + 2C \quad \text{and} \quad g_j^1(\lambda_1 t) \leq g_j^2(\lambda_2 t) + 2C,$$

which implies that

$$\lambda_1 t \leq \lambda_2 t + 4C.$$

Because this is true for all $t \in (0, +\infty)$, we deduce that

$$\lambda_1 \leq \lambda_2.$$

The reverse inequality is obtained by exchanging $((h_j^1)_j, (g_j^1)_j)$ and $((h_j^2)_j, (g_j^2)_j)$, which proves that $\lambda_1 = \lambda_2$ and therefore the uniqueness of $\lambda = \bar{F}(p)$.

Continuity of the map $p \mapsto \bar{F}(p)$. This proof is similar to the one in [FIM12] so we skip it. □

Proof of Theorem 1.4 . It suffices to remark that

$$h_j(z) = z + h_j(0) \quad \text{and} \quad g_j(z) = z + g_j(0).$$

with

$$h_j(0) = \frac{pj}{n_0} \quad \text{and} \quad g_j(0) = \frac{pj}{n_0} + \frac{1}{\alpha} V \left(\frac{p}{n_0} \right), \tag{4.18}$$

is a solution of (1.11) with

$$\lambda = \bar{F}(p) = V \left(\frac{p}{n_0} \right).$$

□

5 Qualitative properties of the effective Hamiltonian

Proof of Theorem 1.9.

Step 1: proof of the lower bound . If we have $p \leq 2h_0n_0$, then we can see that

$$\left((h_j(z))_j, (g_j(z))_j \right) = \left(\left(z + \frac{pj}{n_0} \right)_j, \left(z + \frac{pj}{n_0} \right)_j \right),$$

is a hull function for $\lambda = 0$. In fact we have

$$\lambda = \alpha(g_j(z) - h_j(z)) = 0,$$

and

$$\begin{aligned} \lambda &= (a_j - \alpha)(h_j(z) - g_j(z)) + \frac{a_j}{\alpha} V_j \left(\frac{p}{n_0} \right) \\ &= 0, \end{aligned}$$

where we have used assumption (A4). Now by uniqueness of the effective Hamiltonian we have that $\bar{F}(p) = 0$.

Step 2: proof of the upper bound. A simple computation gives

$$\bar{F}(p) = V_j(h_{j+1}(0) - h_j(0)), \quad \text{for all } j \in \mathbb{Z}$$

However, we also know that $h_{j+1}(0) \geq h_j(0)$ and that $h_{j+n_0}(0) = h_j(0) + p$ this means that there exists a $i \in \{j, \dots, j + n_0\}$ such that $h_{i+1}(0) - h_i(0) \geq p/n_0$. Therefore, using the fact that V_i is non-decreasing, we have

$$V_i \left(\frac{p}{n_0} \right) \leq \bar{F}(p) \leq \min_{j \in \{1, \dots, n_0\}} (\|V_j\|_\infty). \quad (5.1)$$

Passing to the limit as p goes to $+\infty$, we get the desired result.

Step 3: proof of the monotonicity. Let $p_1, p_2 \in (0, +\infty)$, and let $\lambda_1 = \bar{F}(p_1), \lambda_2 = \bar{F}(p_2)$ be their respective effective Hamiltonians, each associated to the hull functions $((h_j^1)_j, (g_j^1)_j)$ and $((h_j^2)_j, (g_j^2)_j)$. We assume that $p_2 > p_1$. Therefore, we have

$$h_{n_0}^1(0) - h_0^1(0) = p_1 < p_2 = h_{n_0}^2(0) - h_0^2(0).$$

From this, we can deduce that there exists an integer $k \in \{0, \dots, n_0 - 1\}$ such that

$$h_{k+1}^1(0) - h_k^1(0) < h_{k+1}^2(0) - h_k^2(0).$$

Now, using the monotonicity of V_k , we get,

$$V_k \left(h_{k+1}^1(0) - h_k^1(0) \right) \leq V_k \left(h_{k+1}^2(0) - h_k^2(0) \right),$$

which implies that $\lambda_1 \leq \lambda_2$. Therefore the function \bar{F} is non decreasing. □

6 Appendix

Proposition 6.1. *Let $((U_j)_j, (\Xi_j)_j)$ be a classical solution of (1.3), then the function $((u_j)_j, (\xi_j)_j)$ defined by*

$$u_j(t, x) = U_{j+\lfloor x \rfloor n_0}(t) \quad \text{and} \quad \xi_j(t, x) = \Xi_{j+\lfloor x \rfloor n_0}(t)$$

is a viscosity solution of the following system

$$\left\{ \begin{array}{l} \frac{\partial u_j}{\partial t}(t, x) = \alpha(\xi_j(t, x) - u_j(t, x)) \\ \frac{\partial \xi_j}{\partial t}(t, x) = (a_j - \alpha)(u_j(t, x) - \xi_j(t, x)) + \frac{a_j}{\alpha} V_j(u_{j+1}(t, x) - u_j(t, x)) \\ u_{j+n_0}(t, x) = u_j(t, x + 1) \\ \xi_{j+n_0}(t, x) = \xi_j(t, x + 1). \end{array} \right. \quad (6.1)$$

Proof. First, the periodicity condition, can easily be checked, in fact we have

$$u_{j+n_0}(t, x) = U_{j+n_0+\lfloor x \rfloor n_0}(t) = U_{j+\lfloor x+1 \rfloor n_0}(t) = u_j(t, x + 1). \quad (6.2)$$

We have a similar result for ξ_j .

We now prove that the function $((u_j)_j, (\xi_j)_j)$ satisfies the equations in (6.1) in the viscosity sense (see Definition 2.1). Let φ be a test function such that $u_j^* \leq \varphi$ and $u_j^*(t_0, x_0) = \varphi(t_0, x_0)$, by definition the functions u_j are upper semi-continuous, so we have

$$U_{j+\lfloor x_0 \rfloor n_0}(t) \leq \varphi(t, x_0) \quad \text{and} \quad \varphi(t_0, x_0) = U_{j+\lfloor x_0 \rfloor n_0}(t_0).$$

Given that $((U_j)_j, (\Xi_j)_j)$ is a classical solution of (1.3), it is also a viscosity solution of (1.3) and we obtain

$$\frac{\partial \varphi}{\partial t}(t_0, x_0) \leq \alpha \left(\Xi_{j+\lfloor x_0 \rfloor n_0}(t_0) - U_{j+\lfloor x_0 \rfloor n_0}(t_0) \right) = \alpha \left(\xi_j^*(t_0, x_0) - u_j^*(t_0, x_0) \right). \quad (6.3)$$

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We have a similar result for ξ_j^* , therefore $((u_j)_j, (\xi_j)_j)$ is a sub-solution of (6.1). The proof for super-solutions is similar and we skip it.

□

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Chapitre 2

Numerical homogenization of a second order discrete model for traffic flow

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Abstract

The goal of this paper is to obtain a numerical approximation of the effective Hamiltonian for a system of PDEs deriving from a second order discrete model for traffic flow. We will propose an explicit and an implicit discretization for this effective Hamiltonian and we will provide the corresponding error estimates of Crandall-Lions type. Finally, we will also present some numerical simulations.

AMS Classification: 90B20, 65L12, 49L25.

Keywords: Traffic flow, macroscopic model, explicit scheme, implicit scheme, effective Hamiltonian, viscosity solution.

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1 Introduction

The problem of correctly simulating traffic flow has a great interest since it can be used to see how traffic would react to a change in the infrastructure of the road (to understand if it is interesting to place a traffic light, if a bridge would help the traffic flow, how would a moderator affect the traffic...). Traffic flow can be simulated at different scales: the microscopic scale (which describes the dynamics of all the vehicles), the macroscopic scale (which describes macroscopic quantities such as the vehicle density, the average speed,...) and the mesoscopic scale (it uses the vehicle density and the average speed of the vehicles but still keeps track of the dynamics of all the vehicles). In the present work we only focus on the microscopic and the macroscopic scale.

The microscopic models are very precise and intuitive since they describe how each vehicle reacts to a situation. For instance it is easy to simulate how a vehicle would react to the presence of a moderator (at some point the vehicle notices the moderators and slows down, passes the moderator and then gradually increases its speed). However, to use microscopic models at a large scale would be computationally very costly. For instance if we would like to simulate the traffic in an entire city, we would need to consider all the vehicles.

The macroscopic models are more adapted to simulate traffic at a large scale, since they do not consider all the car-to-car interactions. However, they are often based on assumption that are hard to verify and also they are not very easy to modify. How does the density of vehicle reacts to the presence of a moderator? An example of a macroscopic model would be the LWR (Lighthill-Whitham-Richards) model introduced in [LW55, Ric56] and inspired by fluid dynamics.

We can see that it is interesting to pass from a microscopic model to a macroscopic model, it can help to rigorously derive a macroscopic model from a model with solid assumptions and which is adapted to the situation we want to simulate (traffic light, moderator, bifurcation,...).

In this paper, we are interested in the numerical homogenization of a system of PDE that derives from a second order discrete model for traffic flow. The system studied in this paper was introduced in [1], and derives from a microscopic model of "follow the leader" type, that was introduced by Bando *et al* [BHN⁺95].

In [1], the homogenization of the system was obtained and as it turns out, the homogenized system gives a macroscopic model for traffic flow which is defined by the so-called effective Hamiltonian. The main difficulty to use this result in practice is that the effective Hamiltonian cannot be explicitly computed in general. Therefore, the numerical computation of this effective Hamiltonian is very important and this is the main focus of the present paper. We will present an explicit and an implicit discretization of this effective Hamiltonian and we will give the corresponding error estimates.

1.1 General model with n_0 types of drivers

For the readers convenience we detail the microscopic model that we use in this paper. We consider a model with $n_0 \in \mathbb{N}$ types of drivers, defined, for all $j \in \mathbb{Z}$ and for all $t \in (0, +\infty)$, by

$$\ddot{U}_j = a_j \left(V_j [U_{j+1} - U_j] - \dot{U}_j \right), \tag{1.1}$$

where $U_j(t)$ denotes the position of the vehicle j , $\dot{U}_j(t)$ its velocity and $\ddot{U}_j(t)$ its acceleration at a time $t \in (0, +\infty)$. The coefficients a_j represent the drivers sensitivity (affects how quickly a driver adapts to a change in the road) and the functions V_j are the optimal velocity functions (OVFs) of the drivers. Like in [1], in order to simplify the scenario and to be able to obtain an homogenization result, we impose the following periodic conditions,

$$a_{j+n_0} = a_j \quad \text{and} \quad V_{j+n_0} = V_j \quad \text{for all } j \in \mathbb{Z}.$$

Proceeding as in [FIM12], we introduce a well chosen artificial variable in order to work with a first order system, we consider for all $j \in \mathbb{Z}$ and for all $t \in [0, +\infty)$,

$$\Xi_j(t) = U_j(t) + \frac{1}{\alpha} \dot{U}_j(t) \quad \text{where } \alpha = \frac{1}{2} \min_{i \in \{1, \dots, n_0\}} (a_i). \tag{1.2}$$

We obtain the following first order system of ODEs (which is equivalent to (1.1)), for all $j \in \mathbb{Z}$ and for all $t \in (0, +\infty)$,

$$\begin{cases} \dot{U}_j(t) = \alpha(\Xi_j(t) - U_j(t)) \\ \dot{\Xi}_j(t) = (a_j - \alpha)(U_j(t) - \Xi_j(t)) + \frac{a_j}{\alpha} V_j [U_{j+1}(t) - U_j(t)]. \end{cases} \tag{1.3}$$

Like in [1], we consider the following assumptions concerning the coefficients a_j and the functions V_j :

(A1) (Regularity) For all $j \in \{1, \dots, n_0\}$,

- $\left\{ \begin{array}{l} V_j \text{ is non-negative.} \\ V_j \text{ is Lipschitz continuous and we denote by } L_j \text{ its Lipschitz constant.} \end{array} \right.$

We denote by $L = \max_{j \in \{1, \dots, n_0\}} L_j$.

(A2) (Monotonicity) For all $j \in \{1, \dots, n_0\}$,

$$\begin{cases} V_j \text{ is non-decreasing,} \\ a_j \geq 4L. \end{cases}$$

(A3) (Upper boundary) For all $j \in \{1, \dots, n_0\}$,

$$\lim_{h \rightarrow +\infty} V_j[h] < +\infty.$$

(1.4)

We define $V_{max} = \max_j (\|V_j\|_\infty)$ and $h_0 \geq V_{max}/\alpha$.

(A4) (Lower boundary) For all $j \in \{1, \dots, n_0\}$,

$$V_j[h] = 0 \quad \text{for all } h \leq 2h_0.$$

(A5) (Periodicity of the type of drivers) For all $j \in \mathbb{Z}$,

$$a_{j+n_0} = a_j \quad \text{and} \quad V_{j+n_0} = V_j.$$

1.2 General continuous system with n_0 types of drivers

In order to obtain an homogenization result, it is necessary to inject the system of ODEs (1.3) into a system of PDEs (see [FIM09a, FIM12, 1]). This is done by considering the functions

$$(u, \xi) = ((u_j)_{j \in \mathbb{Z}}, (\xi_j)_{j \in \mathbb{Z}}), \quad (1.5)$$

defined by

$$u_j(t, x) = U_{j+[x]n_0}(t) \quad \text{and} \quad \xi_j(t, x) = \Xi_{j+[x]n_0}(t) \quad \text{for all } (t, x) \in (0, +\infty) \times \mathbb{R},$$

where $[\cdot]$ denotes the floor integer part.

Moreover, the function (u, ξ) satisfies (see [1, Proposition A.1]) the following system of PDEs, for all $(t, x) \in (0, +\infty) \times \mathbb{R}$ and for all $j \in \mathbb{Z}$,

$$\begin{cases} \frac{\partial u_j}{\partial t}(t, x) = \alpha(\xi_j(t, x) - u_j(t, x)) \\ \frac{\partial \xi_j}{\partial t}(t, x) = (a_j - \alpha)(u_j(t, x) - \xi_j(t, x)) + \frac{a_j}{\alpha} V_j [u_{j+1}(t, x) - u_j(t, x)] \\ u_{j+n_0}(t, x) = u_j(t, x + 1) \\ \xi_{j+n_0}(t, x) = \xi_j(t, x + 1). \end{cases} \quad (1.6)$$

We complete the previous system with the initial condition

$$u_j(0, x) = u_0 \left(x + \frac{j}{n_0} \right) \quad \text{and} \quad \xi_j(0, x) = \xi_0 \left(x + \frac{j}{n_0} \right). \quad (1.7)$$

Remark 1.1. *The initial condition functions are artificially introduced, but simply they are regular functions such that, we have for all $j \in \mathbb{Z}$,*

$$u_0 \left(\frac{j}{n_0} \right) = U_j(0) \quad \text{and} \quad \xi_0 \left(\frac{j}{n_0} \right) = U_j(0) + \frac{1}{\alpha} \dot{U}_j(0).$$

If we denote by $((u_j)_j, (\xi_j)_j)$ the solution of (1.6)-(1.7), then in [1] it was proven that for all $j \in \mathbb{Z}$, the rescaled functions u_j^ε and ξ_j^ε defined by

$$u_j^\varepsilon(t, x) = \varepsilon u_j \left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon} \right) \quad \text{and} \quad \xi_j^\varepsilon(t, x) = \varepsilon \xi_j \left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon} \right), \quad (1.8)$$

converge uniformly on compact subsets of $(0, +\infty) \times \mathbb{R}$ as ε goes to 0, to the unique solution of

$$\begin{cases} u_t^0(t, x) = \bar{F}(u_x^0(t, x)) & \text{for } (t, x) \in (0, +\infty) \times \mathbb{R}, \\ u^0(0, x) = u_0(x) & \text{for } x \in \mathbb{R}, \end{cases} \quad (1.9)$$

where \bar{F} is the effective Hamiltonian to be determined.

As it turns out, the solution of (1.6) with the following initial condition, for all $x \in \mathbb{R}$, and for all $j \in \mathbb{Z}$,

$$u_j(0, x) = u_0 \left(x + \frac{j}{n_0} \right) = \xi_j(0, x) = \xi_0 \left(x + \frac{j}{n_0} \right) = p \cdot \left(x + \frac{j}{n_0} \right), \quad (1.10)$$

can be used to compute the effective Hamiltonian that we denote by $\lambda = \bar{F}(p)$. Here and in the rest of the paper \cdot symbolizes the product of two real numbers. We have the following result (see [1, Proposition 4.1]).

Proposition 1.2 (Particular form of the solution of (1.6) and approximation of λ). *Assume (A1)-(A5) and let $p > 0$. Let $((u_j)_j, (\xi_j)_j)$ be the solution of (1.6) with an initial condition such that $(u_0)_x = (\xi_0)_x = p$. Then $((u_j)_j, (\xi_j)_j)$ satisfies*

$$u_j(t, x) = px + u_j(t, 0) \quad \text{and} \quad \xi_j(t, x) = px + \xi_j(t, 0), \quad (1.11)$$

and there exists a constant C , depending only on p, V_{max}, α and n_0 such that for all $T \in (0, +\infty)$,

$$\left| \frac{u_1(T, 0)}{T} - \lambda \right| \leq \frac{C}{T}. \quad (1.12)$$

Remark 1.3. *The well chosen initial conditions (1.10) simply translates the fact that the vehicles are initially uniformly distributed along the real line. This initial condition is used in [FIM09a, FIM12, 1] in order to obtain the homogenization results.*

1.3 Numerical schemes

We denote by Δt the time step and by $t_n = n\Delta t$. As explained before, the goal is to compute the solution of (1.6) with initial condition (1.10). Using the particular form of this solution given by Proposition 1.2, we are only interested in the numerical approximation of $u_j(t_n, 0)$ and $\xi_j(t_n, 0)$ that we will denote respectively by u_j^n and ξ_j^n . We propose an explicit and an implicit discretization of u_j^n and ξ_j^n .

Explicit finite difference scheme

We consider the following numerical scheme, for all $j \in \mathbb{Z}$ and for all $n \in \mathbb{N}$,

$$\left\{ \begin{array}{l} \frac{u_j^{n+1} - u_j^n}{\Delta t} = \alpha (\xi_j^n - u_j^n) \\ \frac{\xi_j^{n+1} - \xi_j^n}{\Delta t} = (a_j - \alpha) (u_j^n - \xi_j^n) + \frac{a_j}{\alpha} V_j [u_{j+1}^n - u_j^n] \\ u_{n_0+j}^n = u_j^n + p \\ \xi_{n_0+j}^n = \xi_j^n + p, \end{array} \right. \quad (1.13)$$

with the initial condition

$$u_j^0 = \frac{pj}{n_0} \quad \text{and} \quad \xi_j^0 = \frac{pj}{n_0}. \quad (1.14)$$

Implicit finite difference scheme

We also consider the following numerical scheme, for all $j \in \mathbb{Z}$, and for all $n \in \mathbb{N}$,

$$\left\{ \begin{array}{l} \frac{u_j^{n+1} - u_j^n}{\Delta t} = \alpha (\xi_j^{n+1} - u_j^{n+1}) \\ \frac{\xi_j^{n+1} - \xi_j^n}{\Delta t} = (a_j - \alpha) (u_j^{n+1} - \xi_j^{n+1}) + \frac{a_j}{\alpha} V_j [u_{j+1}^{n+1} - u_j^{n+1}] \\ u_{j+n_0}^n = u_j^n + p \\ \xi_{j+n_0}^n = \xi_j^n + p \end{array} \right. \quad (1.15)$$

with initial condition given by (1.14).

Remark 1.4. Thanks to the particular form of the solution of (1.6) with the initial condition (1.10), the approximations of $u_j(t, x)$ and $\xi_j(t, x)$ are given respectively by

$$u_j^n + px \quad \text{and} \quad \xi_j^n + px. \tag{1.16}$$

1.4 Numerical estimate of the effective Hamiltonian

Given $n_T \in \mathbb{N}$, we define $T = n_T \Delta t$ and we introduce our approximation for the effective Hamiltonian,

$$\lambda^T = \frac{u_1^{n_T}}{T}. \tag{1.17}$$

Theorem 1.5 (Numerical estimate of the effective Hamiltonian). *Let $p > 0$, and let λ^T be the numerical approximation of λ given by (1.17). If u_j^n is computed with the explicit scheme (resp. by the implicit scheme), we will assume that*

$$\Delta t \leq \frac{1}{\max_{j \in \{1, \dots, n_0\}} (a_j)} \left(\text{resp. } \Delta t < \frac{1}{2\alpha} \left(1 + \min_{j \in \{1, \dots, n_0\}} \sqrt{a_j + \frac{4\alpha}{a_j}} \right) \right).$$

Then there exist two constants C_1 and C_2 , such that,

$$|\lambda^T - \lambda| \leq \frac{C_1}{T} + C_2(\Delta t + \sqrt{\Delta t}). \tag{1.18}$$

Corollary 1.6. *Let $p > 0$, and let λ^T be the numerical approximation of λ given by (1.17). If*

$$\Delta t \leq \frac{1}{T^2},$$

then we have,

$$|\lambda^T - \lambda| = O\left(\frac{1}{T}\right). \tag{1.19}$$

1.5 Organisation of the article

In Section 2, we give some definitions and results for viscosity solutions of the continuous problem (1.6). In Section 3, we give an error estimate between the continuous solution of (1.6) and its numerical approximation. In Section 4, we use the results of Section 3 to do the proof of Theorem 1.5. Finally, Section 5 is devoted to numerical simulations. Different examples are provided and we also present a numerical study of the error for the effective Hamiltonian.

2 Viscosity solutions

In this section we present the definition of viscosity solutions for the system (1.6). We refer to the user's guide of Crandall, Ishii, Lions [CIL92] and the book of Barles [Bar94] for a good introduction to viscosity solutions. We also refer to [Ish92, IK91, Len88] and references therein for results concerning solutions for systems of weakly coupled partial differential equations.

The proof of the results concerning the continuous problem can be founded in [FIM12] and [1].

2.1 Definitions

Definition 2.1 (Viscosity Solutions). *Let $T > 0$, $u_0 : \mathbb{R} \rightarrow \mathbb{R}$ and $\xi_0 : \mathbb{R} \rightarrow \mathbb{R}$ defined by (1.10). For all j , let $u_j : \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}$ and $\xi_j : \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}$ be upper semi-continuous (resp. lower semi-continuous) locally bounded functions. We set $\Omega = (0, T) \times \mathbb{R}$. Let us consider that $((u_j)_j, (\xi_j)_j)$ satisfies*

$$\forall j \in \mathbb{Z}, \forall (t, x) \in \Omega, u_{j+n_0}(t, x) = u_j(t, x + 1) \quad \text{and} \quad \xi_{j+n_0}(t, x) = \xi_j(t, x + 1).$$

-A function $((u_j)_j, (\xi_j)_j)$ is a sub-solution (resp. a super-solution) of (1.6) on Ω if for all $(t, x) \in \Omega$ and for any test function $\varphi \in C^1(\Omega)$ such that $u_j - \varphi$ attains a local maximum (resp. a local minimum) at the point (t, x) , we have

$$\varphi_t(t, x) \leq \alpha(\xi_j(t, x) - u_j(t, x)) \quad (\text{resp. } \geq), \tag{2.1}$$

and for all $(t, x) \in \Omega$, and any test function $\varphi \in C^1(\Omega)$ such that $\xi_j - \varphi$ attains a local maximum (resp. a local minimum) at the point (t, x) , we have

$$\varphi_t(t, x) \leq (a_j - \alpha)(u_j(t, x) - \xi_j(t, x)) + \frac{a_j}{\alpha} V_j [u_{j+1}(t, x) - u_j(t, x)] \quad (\text{resp. } \geq) \tag{2.2}$$

-A function $((u_j)_j, (\xi_j)_j)$ is a sub-solution (resp. a super-solution) of (1.6)-(1.10) if $((u_j)_j, (\xi_j)_j)$ is a sub-solution (resp. a super solution) of (1.6) on Ω and if it satisfies moreover for all $x \in \mathbb{R}$, $j \in \{1, \dots, n_0\}$,

$$u_j(0, x) \leq u_0 \left(x + \frac{j}{n_0} \right) \quad (\text{resp. } \geq) \quad \text{and} \quad \xi_j(0, x) \leq \xi_0 \left(x + \frac{j}{n_0} \right) \quad (\text{resp. } \geq).$$

-A function $((u_j)_j, (\xi_j)_j)$ is a viscosity solution of (1.6) (resp. of (1.6)-(1.10)) on Ω if $((u_j^)_j, (\xi_j^*)_j)$ is a sub-solution and $((u_j)_j, (\xi_j)_j)$ is a super solution of (1.6) (resp. of (1.6)-(1.10)).*

2.2 Results for viscosity solutions

We now recall some of the results we find in [1] for the continuous problem (1.6), that will help us in the rest of the paper. We also give a result for the discrete solutions.

Theorem 2.2 (Existence and uniqueness of viscosity solutions for (1.6)). *Assume (A1)-(A5). Then there exists a unique continuous solution $((u_j)_j, (\xi_j)_j)$ of (1.6)-(1.10). Moreover, the solution satisfies*

$$p \cdot \left(x + \frac{j}{n_0} \right) - K_1 t \leq u_j(t, x), \xi_j(t, x) \leq p \cdot \left(x + \frac{j}{n_0} \right) + K_1 t, \quad (2.3)$$

with

$$K_1 = \max_{j \in \{1, \dots, n_0\}} (a_j) \cdot \frac{V_{max}}{\alpha}. \quad (2.4)$$

Proposition 2.3 (Solutions of the numerical scheme (1.15)). *Given $((u_j^0)_j, (\xi_j^0)_j)$ an initial condition, if we have*

$$\Delta t < \frac{1}{2\alpha} \left(1 + \min_{j \in \{1, \dots, n_0\}} \sqrt{a_j + \frac{4\alpha}{a_j}} \right), \quad (2.5)$$

then for any $n \in \mathbb{N}$, there exists a unique $((u_j^n)_j, (\xi_j^n)_j)$ provided by the numerical scheme (1.15).

Proof. We do the proof by induction, and we want to prove that for $((u_j^n)_j, (\xi_j^n)_j)$ there exists a unique $((u_j^{n+1})_j, (\xi_j^{n+1})_j)$.

To do this, we use an equivalent formulation of (1.15), we have for all $j \in \{1, \dots, n_0 - 1\}$,

$$\begin{aligned} \xi_j^{n+1} &= \frac{1}{\Delta t(a_j - \alpha)} \left(\xi_j^n + \Delta t(a_j - \alpha)u_j^{n+1} + \frac{\Delta t a_j}{\alpha} V_j [u_{j+1}^{n+1} - u_j^{n+1}] \right) \\ u_j^{n+1} &= \frac{1}{1 + \alpha \Delta t} \left(u_j^n + \Delta t \alpha \xi_j^{n+1} \right) \\ &= \frac{1 + \Delta t(a_j - \alpha)}{1 + a_j \Delta t} u_j^n + \frac{\alpha \Delta t}{1 + a_j \Delta t} \xi_j^n + \frac{a_j \Delta t^2}{1 + a_j \Delta t} V_j [u_{j+1}^{n+1} - u_j^{n+1}] =: G_j(u^{n+1}), \end{aligned}$$

and

$$\begin{aligned} u_{n_0}^{n+1} &= \frac{1 + \Delta t(a_{n_0} - \alpha)}{1 + a_{n_0} \Delta t} u_{n_0}^n + \frac{\alpha \Delta t}{1 + a_{n_0} \Delta t} \xi_{n_0}^n + \frac{a_{n_0} \Delta t^2}{1 + a_{n_0} \Delta t} V_{n_0} [u_1^{n+1} - u_{n_0}^{n+1} + p] \\ &= G_{n_0}(u^{n+1}). \end{aligned}$$

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We want to prove that there exists a unique solution to $u = G(u)$, with $G = (G_j)_{j=1,\dots,n_0}$. We want to use the fixed point theorem and it can easily be seen that

$$\frac{\partial G_j}{\partial u_j^{n+1}} = -\frac{a_j \Delta t^2}{1 + a_j \Delta t} V'_j [u_{j+1}^{n+1} - u_j^{n+1}], \quad \frac{\partial G_j}{\partial u_{j+1}^{n+1}} = \frac{a_j \Delta t^2}{1 + a_j \Delta t} V'_j [u_{j+1}^{n+1} - u_j^{n+1}],$$

$$\frac{\partial G_{n_0}}{\partial u_{n_0}^{n+1}} = -\frac{a_{n_0} \Delta t^2}{1 + a_{n_0} \Delta t} V'_{n_0} [u_1^{n+1} - u_{n_0}^{n+1} + p],$$

and

$$\frac{\partial G_{n_0}}{\partial u_1^{n+1}} = \frac{a_{n_0} \Delta t^2}{1 + a_{n_0} \Delta t} V'_{n_0} [u_1^{n+1} - u_{n_0}^{n+1} + p].$$

Therefore, we have

$$\|DG\|_1 \leq \max_{j \in \{1, \dots, n_0\}} \left(\frac{2a_j \Delta t^2}{1 + a_j \Delta t} \|V'\|_\infty \right) \leq \max_{j \in \{1, \dots, n_0\}} \left(\frac{a_j \Delta t^2}{1 + a_j \Delta t} \right) \cdot \alpha.$$

Using (2.5), we have that for all $j \in \{1, \dots, n_0\}$,

$$\frac{a_j \Delta t^2}{1 + a_j \Delta t} < 1. \quad (2.6)$$

Therefore G is a contraction mapping on \mathbb{R}^{n_0} and by the fixed point theorem, there exists a unique fixed point u^{n+1} such that $u^{n+1} = G(u^{n+1})$. By uniqueness of u^{n+1} we have uniqueness of ξ^{n+1} . □

Proposition 2.4 (Discrete barriers). *Assume (A1)-(A5). Then*

$$\left((u_j^{+,n})_j, (\xi_j^{+,n})_j \right) = \left(\left(\frac{pj}{n_0} + K_1 n \Delta t \right)_j, \left(\frac{pj}{n_0} + K_1 n \Delta t \right)_j \right),$$

and

$$\left((u_j^{-,n})_j, (\xi_j^{-,n})_j \right) = \left(\left(\frac{pj}{n_0} - K_1 n \Delta t \right)_j, \left(\frac{pj}{n_0} - K_1 n \Delta t \right)_j \right),$$

are respectively super- and sub-solution of (1.13) and of (1.15) for all $T > 0$, with K_1 defined as in (2.4).

Proof. Let us prove that $\left((u_j^{+,n})_j, (\xi_j^{+,n})_j \right)$ is a super-solution for (1.13) (the proof for (1.15) is the same so we skip it).

$$\frac{u_j^{+,n+1} - u_j^{+,n}}{\Delta t} = K_1 \geq \alpha (\xi_j^{+,n} - u_j^{+,n}) = 0,$$

and

$$\begin{aligned} (a_j - \alpha) (u_j^{+,n} - \xi_j^{+,n}) + \frac{a_j}{\alpha} V_j [u_{j+1}^{+,n} - u_j^{+,n}] &\leq \max_{j \in \{1, \dots, n_0\}} (a_j) \frac{V_{max}}{\alpha} \\ &\leq K_1 = \frac{\xi_j^{+,n+1} - \xi_j^{+,n}}{\Delta t}. \end{aligned}$$

The proof for $\left((u_j^{-,n})_j, (\xi_j^{-,n})_j \right)$ is similar, so we skip it. □

The following lemma is applied later in the paper in order to use the viscosity inequalities of (1.6) at time T .

Lemma 2.5 (Viscosity inequality at time T). *Let $((u_j)_j, (\xi_j)_j)$ be a continuous sub-solution of (1.6) and let $T > 0$. For every test function $\varphi \in C^1((0, +\infty) \times \mathbb{R})$ such that*

$$\max(u_j - \varphi) = u_j(T, x_0) - \varphi(T, x_0) \quad \text{for some } x_0 \in \mathbb{R},$$

the following viscosity inequality holds:

$$\varphi_t(T, x_0) \leq \alpha (\xi_j(T, x_0) - u_j(T, x_0)).$$

Similarly, for every test function $\varphi \in C^1((0, +\infty) \times \mathbb{R})$ such that

$$\max(\xi_j - \varphi) = \xi_j(T, x_0) - \varphi(T, x_0) \quad \text{for some } x_0 \in \mathbb{R},$$

the following viscosity inequality holds:

$$\varphi_t(T, x_0) \leq (a_j - \alpha) (u_j(T, x_0) - \xi_j(T, x_0)) + \frac{a_j}{\alpha} V [u_{j+1}(T, x_0) - u_j(T, x_0)].$$

Proof. The proof of this lemma is similar to the proof of Lemma 4.4 in [CCM12], so we skip it. □

3 Crandall-Lions type error estimates for (1.6)

In this section, we prove an error estimate of Crandall-Lions type between the viscosity solution for the continuous problem (1.6)-(1.10) and the discrete solutions of the schemes (1.13)-(1.14) and (1.15)-(1.14) namely, Theorem 1.5. The following proofs use the method introduced by Crandall and Lions in [CL84] and adapted in [CCM12].

We define for any $(u_j^n)_n$ the following piecewise constant function

$$u_j^\sharp(t) = \sum_{n \in \{1, \dots, n_T\}} u_j^n \cdot \chi_{[t_n, t_{n+1})}(t).$$

This function is simply a piecewise constant extension in time to $[0, T]$ of $(u_j^n)_n$. We will also use

$$q^{\Delta t}(u^\sharp) = \sup_{j \in \{1, \dots, n_0\}} \sup_{|t-s| \geq \Delta t, t, s \in [0, 1]} \left(\frac{|u_j^\sharp(t) - u_j^\sharp(s)|}{|t-s|} \right). \quad (3.1)$$

It should be noticed that since $t, s \in [0, 1]$ in (3.1), $q^{\Delta t}(u^\sharp)$ does not depend on T . We also introduce

$$q^{\Delta t}(u^\sharp, v^\sharp) = \sup(q^{\Delta t}(u^\sharp), q^{\Delta t}(v^\sharp)).$$

3.1 Error estimate using the explicit scheme

Since we are working with an explicit scheme, we give a sufficient condition on the time-step to make the numerical scheme monotone.

Proposition 3.1 (Monotonicity of the explicit numerical scheme (1.13)). *If we have*

$$\Delta t \leq \frac{1}{\max_{j \in \{1, \dots, n_0\}} (a_j)}, \quad (3.2)$$

then the numerical scheme (1.13) is monotone.

Proof. We have

$$\begin{cases} u_j^{n+1} = (1 - \alpha \Delta t) u_j^n + \alpha \xi_j^n \Delta t \\ \xi_j^{n+1} = (1 - (a_j - \alpha) \Delta t) \xi_j^n + (a_j - \alpha) u_j^n \Delta t + \Delta t \frac{a_j}{\alpha} V_j [u_{j+1}^n - u_j^n], \end{cases}$$

therefore the monotonicity condition is satisfied if

$$\Delta t \leq \min \left(\frac{1}{\alpha}, \frac{1}{\max_{j \in \{1, \dots, n_0\}} (a_j - \alpha)} \right). \quad (3.3)$$

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It can easily be seen that (3.2) implies (3.3). □

Lemma 3.2 (Bound on discrete-time derivative). *Assume (A1)-(A5). Let $((u_j^n)_j, (\xi_j^n)_j)$ be a solution of (1.13), with initial data (u_j^0, ξ_j^0) as defined in (1.14). If condition (3.2) is fulfilled, then for all $j \in \mathbb{Z}$ and for all $n \in \mathbb{N}$,*

$$0 \leq \frac{u_j^{n+1} - u_j^n}{\Delta t} \leq V_{max}, \quad (3.4)$$

and

$$-\frac{V_{max}}{\alpha} \max_{j \in \{1, \dots, n_0\}} (a_j) \leq \frac{\xi_j^{n+1} - \xi_j^n}{\Delta t} \leq \frac{V_{max}}{\alpha} \max_{j \in \{1, \dots, n_0\}} (a_j). \quad (3.5)$$

Proof of Lemma 3.2.

Step 1: first inequality. We do the first part of the proof by induction on n , let us notice that (3.4) is equivalent to

$$0 \leq \xi_j^n - u_j^n \leq \frac{V_{max}}{\alpha}, \quad (3.6)$$

for all $n \in \mathbb{N}$. Let $j \in \mathbb{Z}$, thanks to the initial condition (1.14), we have

$$0 \leq \xi_j^0 - u_j^0 \leq \frac{V_{max}}{\alpha}.$$

Let us assume that

$$0 \leq \xi_j^n - u_j^n \leq \frac{V_{max}}{\alpha}. \quad (3.7)$$

Using the definition of $((u_j^n)_j, (\xi_j^n)_j)$ as a solution of (1.13), we have

$$\begin{aligned} \xi_j^{n+1} - u_j^{n+1} &\leq \xi_j^n - u_j^n + \Delta t \left(a_j (u_j^n - \xi_j^n) + \frac{a_j}{\alpha} V_j [u_{j+1}^n - u_j^n] \right) \\ &\leq (1 - a_j \Delta t) (\xi_j^n - u_j^n) + \frac{a_j}{\alpha} V_{max} \Delta t \\ &\leq (1 - a_j \Delta t) \frac{V_{max}}{\alpha} + \frac{a_j}{\alpha} V_{max} \Delta t \\ &\leq \frac{V_{max}}{\alpha}, \end{aligned} \quad (3.8)$$

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where we have used assumption (A3) for the second line and (3.7) and (3.2) for the third line. In the same way we have

$$\begin{aligned}\xi_j^{n+1} - u_j^{n+1} &\geq \xi_j^n - u_j^n + \Delta t \left(a_j (u_j^n - \xi_j^n) + \frac{a_j}{\alpha} V_j [u_{j+1}^n - u_j^n] \right) \\ &\geq (1 - \Delta t a_j) (\xi_j^n - u_j^n) + 0 \\ &\geq 0,\end{aligned}$$

where we have used assumption (A1) for the second line, and we have used (3.2) and (3.7) for the last line. This ends the proof of (3.6).

Step 2: second inequality. Let $j \in \mathbb{Z}$ and $n \in \mathbb{N}$, then we have

$$\begin{aligned}\xi_j^{n+1} - \xi_j^n &\leq \Delta t \left[(a_j - \alpha) (u_j^n - \xi_j^n) + \frac{a_j}{\alpha} V_j (u_{j+1}^n - u_j^n) \right] \\ &\leq \frac{a_j}{\alpha} V_{max} \Delta t \\ &\leq \Delta t \frac{V_{max}}{\alpha} \max_{j \in \{1, \dots, n_0\}} (a_j),\end{aligned}$$

where we have used assumption (A3) and (3.6) for the second line and (3.6) (upper inequality) for the third line. In the same, way we have

$$\begin{aligned}\xi_j^{n+1} - \xi_j^n &\geq \Delta t \left((a_j - \alpha) (u_j^n - \xi_j^n) + \frac{a_j}{\alpha} V_j [u_{j+1}^n - u_j^n] \right) \\ &\geq a_j \Delta t (u_j^n - \xi_j^n) \\ &\geq -\Delta t \frac{V_{max}}{\alpha} \max_{j \in \{1, \dots, n_0\}} (a_j),\end{aligned} \tag{3.9}$$

where we have used assumption (A3) and (3.6). □

Theorem 3.3 (Error estimate for the explicit scheme). *Assume (A1)-(A5). Let $T > 0$ and let $((u_j)_j, (\xi_j)_j)$ be a solution of the continuous problem (1.6) with initial data $((u_0(x + j/n_0))_j, (\xi_0(x + j/n_0))_j)$ such that $(u_0)_x = (\xi_0)_x = p$. Let $((v_j)_j, (\zeta_j)_j)$ be a solution of (1.13) with initial data (v_j^0, ζ_j^0) and with Δt such that (3.2) is satisfied. Then there exist constants $K_2, K_3 > 0$ such that*

$$\begin{aligned}\sup_{t \in [0, T]} \max_{j \in \mathbb{Z}} \max(|v_j^\sharp(t) - u_j(t, 0)|, |\zeta_j^\sharp(t) - \xi_j(t, 0)|) &\leq \mu_0 + (K_2 + K_3) \Delta t \\ &\quad + (T + 1) \cdot (K_2 + 1) \sqrt{\Delta t} \\ &\quad + T(K_2 + K_3) \Delta t,\end{aligned}$$

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with

$$\begin{aligned} K_2 &= \max \left(2K_1, 4K_1 q^{\Delta t} (v^\sharp, \zeta^\sharp), \right), \\ \mu_0 &= \max_{j \in \{1, \dots, n_0\}} \max \left(|v_j^\sharp(0) - u_j(0, 0)|, |\zeta_j^\sharp(0) - \xi_j(0, 0)| \right), \\ K_3 &= \max_{j \in \{1, \dots, n_0\}} \max \left((a_j + \alpha), (a_j - \alpha) \left(\frac{\alpha + a_j}{\alpha} \right) + \frac{2L}{\alpha} a_j \right) V_{max}. \end{aligned}$$

and K_1 defined in (2.4).

Proof. To do this proof, we will only prove that

$$\begin{aligned} \sup_{t \in [0, T]} \max_{j \in \mathbb{Z}} \max(v_j^\sharp(t) - u_j(t, 0), \zeta_j^\sharp(t) - \xi_j(t, 0)) &\leq \mu_0 + (K_2 + K_3)\Delta t \\ &\quad + (T + 1) \cdot (K_2 + 1) \sqrt{\Delta t} \\ &\quad + T(K_2 + K_3)\Delta t, \end{aligned}$$

given that the other inequality is obtain similarly by exchanging the roles of $((u_j)_j, (\xi_j)_j)$ and $((v_j^\sharp)_j, (\zeta_j^\sharp)_j)$. The proof of this theorem is inspired by the one of Crandall-Lions [CL84]. We do the proof for $j \in \{1, \dots, n_0\}$ and we obtain the result for $j \in \mathbb{Z}$ by periodicity of the solutions. In this proof we will consider $T \leq 1$, and we will prove that

$$\begin{aligned} \sup_{t \in [0, T]} \max_{j \in \mathbb{Z}} \max(v_j^\sharp(t) - u_j(t, 0), \zeta_j^\sharp(t) - \xi_j(t, 0)) &\leq \mu_0 + (K_2 + 1)\sqrt{T\Delta t} \\ &\quad + (K_2 + K_3)\Delta t. \end{aligned} \tag{3.10}$$

In fact if $T \in (0, +\infty)$ and we have the result for $T \leq 1$, (3.10), then

$$\begin{aligned} &\sup_{t \in [0, T]} \max_{j \in \mathbb{Z}} \max(v_j^\sharp(t) - u_j(t, 0), \zeta_j^\sharp(t) - \xi_j(t, 0)) \\ &\leq \sup_{t \in [0, T - [T]]} \max_{j \in \mathbb{Z}} \max(v_j^\sharp - u_j, \zeta_j^\sharp - \xi_j) + [T] \cdot (K_2 + 1)\sqrt{\Delta t} + [T] \cdot (K_2 + K_3)\Delta t \\ &\leq \mu_0 + \sqrt{T - [T]} \cdot (K_2 + 1)\sqrt{\Delta t} + [T] \cdot (K_2 + 1)\sqrt{\Delta t} + (1 + [T]) \cdot (K_2 + K_3)\Delta t \\ &\leq \mu_0 + (T + 1) \left((K_2 + 1)\sqrt{\Delta t} + (K_2 + K_3)\Delta t \right). \end{aligned}$$

where we have used the fact that the result takes into account the error at the initial time.

We now introduce two test functions, for all $j \in \{1, \dots, n_0\}$

$$\begin{cases} \varphi(t, s, j) = v_j^\sharp(t) - u_j(s, 0) - \frac{(t-s)^2}{2\nu} - \eta s \\ \phi(t, s, j) = \zeta_j^\sharp(t) - \xi_j(s, 0) - \frac{(t-s)^2}{2\nu} - \eta s. \end{cases}$$

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We can see that the function $\psi = \max(\varphi, \phi)$ reaches a maximum at a finite point $(\bar{t}, \bar{s}, \bar{j}) \in [0, T] \times [0, T] \times \{1, \dots, n_0\}$. We define

$$M_{\nu, \eta} = \max_{(t, s, j) \in [0, T] \times [0, T] \times \{1, \dots, n_0\}} \psi(t, s, j).$$

Step 1: estimate of the maximum point of ψ . We want to prove that

$$|\bar{t} - \bar{s}| \leq 4\nu q^{\Delta t} (v^\#, \zeta^\#) + 2\Delta t. \quad (3.11)$$

To do this, we denote $\hat{s} = \bar{n}\Delta t$, with $\bar{n} \in \mathbb{N}$, such that $|\bar{s} - \hat{s}| \leq \Delta t$. We can assume that $|\bar{t} - \bar{s}| \geq 2\Delta t$ (otherwise the result is trivial). Then, using the fact that $\psi(\hat{s}, \bar{s}, \bar{j}) \leq \psi(\bar{t}, \bar{s}, \bar{j})$ (let us assume that for instance $\psi = \varphi$, we have the same result in the other case), we have

$$v_j^\#(\hat{s}) - \frac{(\hat{s} - \bar{s})^2}{2\nu} \leq v_j^\#(\bar{t}) - \frac{(\bar{t} - \bar{s})^2}{2\nu},$$

This implies that

$$\begin{aligned} \frac{(\bar{t} - \bar{s})^2}{2\nu} &\leq q^{\Delta t} (v^\#, \zeta^\#) |\bar{t} - \hat{s}| + \frac{\Delta t^2}{2\nu} \\ &\leq q^{\Delta t} (v^\#, \zeta^\#) (|\bar{t} - \bar{s}| + \Delta t) + \frac{|\bar{t} - \bar{s}|^2}{8\nu} \\ \Rightarrow \frac{3|\bar{t} - \bar{s}|^2}{8\nu} &\leq \frac{3}{2} q^{\Delta t} (v^\#, \zeta^\#) |\bar{t} - \bar{s}|, \\ \Rightarrow |\bar{t} - \bar{s}| &\leq 4\nu q^{\Delta t} (v^\#, \zeta^\#), \end{aligned}$$

where we used $\Delta t \leq |\bar{t} - \bar{s}|/2$ for the second and third line.

Now we would like to prove that for η big enough, we will have $\bar{t} = 0$ or $\bar{s} = 0$. We argue by contradiction and we assume that $\bar{t} > 0$ and $\bar{s} > 0$.

Step 2: case $\psi(\bar{t}, \bar{s}, \bar{j}) = \varphi(\bar{t}, \bar{s}, \bar{j})$.

Step 2.1: continuous viscosity inequality. Let us consider the following test function

$$w(s, 0) = v_j^\#(\bar{t}) - \frac{(\bar{t} - s)^2}{2\nu} - \eta s,$$

then $u_{\bar{j}} - w$ reaches a minimum at $(\bar{s}, 0)$ and since $u_{\bar{j}}$ is a super-solution, we get

$$q_\nu - \eta \geq \alpha(\xi_{\bar{j}}(\bar{s}, 0) - u_{\bar{j}}(\bar{s}, 0)),$$

with $q_\nu = (\bar{t} - \bar{s})/\nu$.

3.Crandall-Lions type error estimates for (1.6)

Step 2.2: discrete viscosity inequality. Let $\bar{n} \in \mathbb{N}$ such that $\bar{t} = \bar{n}\Delta t$. Let us consider for all $t \geq \bar{t} - \Delta t$,

$$g(t) = \frac{(t - \bar{s})^2}{2\nu}.$$

We now use the fact that $\varphi(t, \bar{s}, \bar{j}) \leq \varphi(\bar{t}, \bar{s}, \bar{j})$, which allows us to see that

$$v_j^\sharp(t) - g(t) \leq v_j^\sharp(\bar{t}) - g(\bar{t})$$

$$\Leftrightarrow g(\bar{t}) - g(t) \leq v_j^\sharp(\bar{t}) - v_j^\sharp(t).$$

Now we use the fact that v_j^\sharp is a piecewise constant function and choosing $t := \bar{t} - \Delta t$,

$$\begin{aligned} \frac{g(\bar{t}) - g(\bar{t} - \Delta t)}{\Delta t} &\leq \frac{v_j^\sharp(\bar{n}\Delta t) - v_j^\sharp((\bar{n} - 1)\Delta t)}{\Delta t} \\ \frac{(\bar{t} - \bar{s})^2}{2\nu\Delta t} - \frac{(\bar{t} - \bar{s} - \Delta t)^2}{2\nu\Delta t} &\leq \frac{v_j^{\bar{n}} - v_j^{\bar{n}-1}}{\Delta t} \\ \frac{(\bar{t} - \bar{s})}{\nu} - \frac{\Delta t}{2\nu} &\leq \alpha \left(\zeta_j^{\bar{n}-1} - v_j^{\bar{n}-1} \right) \\ &\leq \alpha \left(\zeta_j^{\bar{n}} + \max_{j \in \{1, \dots, n_0\}} (a_j) \frac{V_{max}}{\alpha} \Delta t - v_j^{\bar{n}} + V_{max} \Delta t \right) \\ \Rightarrow q_\nu - \frac{\Delta t}{2\nu} &\leq \alpha \left(\zeta_j^{\bar{n}} - v_j^{\bar{n}} \right) + K_3 \Delta t, \end{aligned} \tag{3.12}$$

where we have used for the third line the fact that (v_j, ζ_j) is a sub-solution of (1.13), for the fourth line we have used Lemma 3.2, and for the fifth line we have used the definition of K_3 .

Step 2.3: subtracting the viscosity inequalities. We obtain directly

$$\begin{aligned} \eta &\leq \frac{\Delta t}{2\nu} + K_3 \Delta t + \alpha \left(\zeta_j^\sharp(\bar{t}) - \xi_j(\bar{s}, 0) - \left(v_j^\sharp(\bar{t}) - u_j(\bar{s}, 0) \right) \right) \\ &\leq \frac{\Delta t}{2\nu} + K_3 \Delta t, \end{aligned} \tag{3.13}$$

where we used the fact that $\varphi(\bar{t}, \bar{s}, \bar{j}) \geq \phi(\bar{t}, \bar{s}, \bar{j})$. We can see that if we define $\bar{\eta} := \frac{\Delta t}{2\nu} + K_3 \Delta t$ and we choose $\eta > \bar{\eta}$, we get a contradiction.

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Step 3: case $\psi(\bar{t}, \bar{s}, \bar{j}) = \phi(\bar{t}, \bar{s}, \bar{j})$. In this case, we proceed as before. When we subtract the two viscosity inequalities, we then obtain

$$\begin{aligned} \eta \leq & \frac{\Delta t}{2\nu} + K_3 \Delta t + (a_{\bar{j}} - \alpha) \left(v_{\bar{j}}^{\sharp}(\bar{t}) - u_{\bar{j}}(\bar{s}, 0) - (\zeta_{\bar{j}}^{\sharp}(\bar{t}) - \xi_{\bar{j}}(\bar{s}, 0)) \right) \\ & + \frac{a_{\bar{j}}}{\alpha} \left(V_{\bar{j}} \left[v_{\bar{j}+1}^{\sharp}(\bar{t}) - v_{\bar{j}}^{\sharp}(\bar{t}) \right] - V_{\bar{j}} \left[u_{\bar{j}+1}(\bar{s}, 0) - u_{\bar{j}}(\bar{s}, 0) \right] \right). \end{aligned} \quad (3.14)$$

We can see that, in the case $j \in \{1, \dots, n_0\}$, using $\phi(\bar{t}, \bar{s}, \bar{j}) \geq \varphi(\bar{t}, \bar{s}, j)$ we get

$$v_{\bar{j}}^{\sharp}(\bar{t}) - u_{\bar{j}}(\bar{s}, 0) \leq \zeta_{\bar{j}}^{\sharp}(\bar{t}) - \xi_{\bar{j}}(\bar{s}, 0). \quad (3.15)$$

In the case $j = n_0 + 1$, using the periodicity conditions in the systems (1.6) and (1.13) with the particular form of the solution (see Proposition 1.2), we get

$$\begin{aligned} v_{n_0+1}^{\sharp}(\bar{t}) - u_{n_0+1}(\bar{s}, 0) &= v_1^{\sharp}(\bar{t}) + p - u_1(\bar{s}, 0) - p \\ &\leq \zeta_1^{\sharp}(\bar{t}) - \xi_1(\bar{s}, 0). \end{aligned} \quad (3.16)$$

Therefore, for all $j \in \{1, \dots, n_0 + 1\}$,

$$v_{\bar{j}}^{\sharp}(\bar{t}) \leq u_{\bar{j}}(\bar{s}, 0) + \zeta_{\bar{j}}^{\sharp}(\bar{t}) - \xi_{\bar{j}}(\bar{s}, 0). \quad (3.17)$$

Now using (3.17) on (3.14) we get

$$\begin{aligned} \eta \leq & \frac{\Delta t}{2\nu} + K_3 \Delta t + (a_{\bar{j}} - \alpha) \left(v_{\bar{j}}^{\sharp}(\bar{t}) - u_{\bar{j}}(\bar{s}, 0) - (\zeta_{\bar{j}}^{\sharp}(\bar{t}) - \xi_{\bar{j}}(\bar{s}, 0)) \right) \\ & + \frac{a_{\bar{j}}}{\alpha} \left(V_{\bar{j}} \left[v_{\bar{j}+1}^{\sharp}(\bar{t}) - v_{\bar{j}}^{\sharp}(\bar{t}) \right] - V_{\bar{j}} \left[u_{\bar{j}+1}(\bar{s}, 0) - u_{\bar{j}}(\bar{s}, 0) \right] \right) \\ \leq & \frac{\Delta t}{2\nu} + K_3 \Delta t \\ & + \frac{a_{\bar{j}}}{\alpha} \left(V_{\bar{j}} \left[v_{\bar{j}+1}^{\sharp}(\bar{t}) - \left(u_{\bar{j}}(\bar{s}, 0) + \zeta_{\bar{j}}^{\sharp}(\bar{t}) - \xi_{\bar{j}}(\bar{s}, 0) \right) \right] - V_{\bar{j}} \left[u_{\bar{j}+1}(\bar{s}, 0) - u_{\bar{j}}(\bar{s}, 0) \right] \right) \\ \leq & \frac{\Delta t}{2\nu} + K_3 \Delta t, \end{aligned}$$

where we have used the monotonicity assumption (A2) and (3.17) with $j = \bar{j}$ for the second inequality and the monotonicity assumption (A2) and (3.17) with $j = \bar{j} + 1$ for the last inequality. Therefore, we obtain

$$\eta \leq \frac{\Delta t}{2\nu} + K_3 \Delta t.$$

As before, it suffices to choose $\eta > \bar{\eta}$ to obtain a contradiction. Therefore we have $\bar{t} = 0$ or $\bar{s} = 0$.

3.Crandall-Lions type error estimates for (1.6)

Step 4: bound on the error. Let us assume that $\psi(\bar{t}, \bar{s}, \bar{j}) = \varphi(\bar{t}, \bar{s}, \bar{j})$, (the other case being similar).

Step 4.1: case $\bar{t} = 0$. We get

$$\begin{aligned}
 M_{\nu, \eta} &= v_j^\sharp(0) - u_{\bar{j}}(\bar{s}, 0) - \frac{\bar{s}^2}{2\nu} - \eta\bar{s} \\
 &\leq v_j^\sharp(0) - u_{\bar{j}}(\bar{s}, 0) \\
 &\leq \mu_0 + K_1\bar{s} \\
 &\leq \mu_0 + K_1|\bar{s} - \bar{t}| \\
 &\leq \mu_0 + K_1(4\nu q^{\Delta t}(v^\sharp, \zeta^\sharp) + 2\Delta t),
 \end{aligned}$$

where we have used for the third line Theorem 2.2, for the fourth line, the definition of the initial data (1.14). Then, for the last line we have used inequality (3.11).

Step 4.2: case $\bar{s} = 0$. Similarly, we get

$$\begin{aligned}
 M_{\nu, \eta} &\leq v_j^\sharp(\bar{t}) - u_{\bar{j}}(0, 0) \\
 &\leq K_1\bar{t} + v_j^\sharp(0) - u_{\bar{j}}(0, 0) \\
 &\leq \mu_0 + K_1(4\nu q^{\Delta t}(v^\sharp, \zeta^\sharp) + 2\Delta t).
 \end{aligned}$$

Step 4.3: conclusion. We can see that we have the same upper bound in the two cases, therefore

$$M_{\nu, \eta} \leq \max(2K_1, 4K_1q^{\Delta t}(v^\sharp, \zeta^\sharp)).(\nu + \Delta t) + \mu_0 =: M. \quad (3.18)$$

Now we can see that for all $j \in \mathbb{Z}$ and for all $t \in [0, T]$ (we recall that we have chosen $T \leq 1$),

$$\begin{aligned}
 v_j^\sharp(t) - u_j(t, 0) - \eta t &\leq M_{\nu, \eta} \leq M, \\
 \Rightarrow v_j^\sharp(t) - u_j(t, 0) &\leq M + T\eta^*,
 \end{aligned}$$

with $\eta^* = K_3\Delta t + \Delta t/\nu = K_3\Delta t + \sqrt{\Delta t}/\sqrt{T}$, choosing $\nu = \sqrt{T \cdot \Delta t}$. We have a similar result for $\zeta_j^\sharp - \xi_j$, therefore we have

$$M \leq \max(2K_1, 4K_1q^{\Delta t}(v^\sharp, \zeta^\sharp)).(\sqrt{T\Delta t} + \Delta t) + \mu_0.$$

Finally, we have

$$\begin{aligned}
 \sup_{t \in [0, 1]} \max_{j \in \mathbb{Z}} \max(v_j^\sharp(t) - u_j(t, 0), \zeta_j^\sharp(t) - \xi_j(t, 0)) &\leq (K_2 + 1) \cdot \sqrt{T\Delta t} \\
 &\quad + (K_2 + K_3T) \cdot \Delta t + \mu_0 \\
 &\leq (K_2 + 1) \cdot \sqrt{T\Delta t} + (K_2 + K_3) \cdot \Delta t \\
 &\quad + \mu_0,
 \end{aligned}$$

where we have used the fact that $T \leq 1$ and with

$$K_2 = \max(2K_1, 4K_1q^{\Delta t}(v^\sharp, \zeta^\sharp)).$$

□

3.2 Error estimate using an implicit scheme

Theorem 3.4 (Error estimate for the implicit scheme). *Assume (A1)-(A5). Let $T > 0$ and let $((u_j)_j, (\xi_j)_j)$ be a solution of the continuous problem (1.6) with initial data $((u_0(x + j/n_0))_j, (\xi_0(x + j/n_0))_j)$. Let $((v_j)_j, (\zeta_j)_j)$ be a solution of (1.15) with initial data (v_j^0, ζ_j^0) and with Δt such that (2.5) is satisfied. Then there exist a constant $K_2 > 0$ such that*

$$\begin{aligned} \sup_{t \in [0, T]} \max_{j \in \mathbb{Z}} \max(|v_j^\sharp(t) - u_j(t, 0)|, |\zeta_j^\sharp(t) - \xi_j(t, 0)|) \leq & \mu_0 + K_2 \Delta t (T + 1) \\ & + (T + 1) \cdot (K_2 + 1) \sqrt{\Delta t}, \end{aligned}$$

with

$$\begin{aligned} K_2 &= \max\left(2K_1, 4K_1q^{\Delta t}(v^\sharp, \zeta^\sharp)\right), \\ \mu_0 &= \max_{j \in \{1, \dots, n_0\}} \max\left(v_j^\sharp(0) - u_j(0, 0), \zeta_j^\sharp(0) - \xi_j(0, 0)\right). \end{aligned}$$

Proof. The proof of this theorem is similar to the one of Theorem 3.3, and even simpler. The difference is that when obtaining the discrete inequality (like in (3.12)) there is no need to use Lemma 3.2, we have the result directly using the definition of the numerical scheme (1.15). That is the reason why for this theorem there is no constant K_3 . For the readers convenience, we give the details of the proof of Theorem 3.4.

We only prove that

$$\sup_{t \in [0, T]} \max_{j \in \mathbb{Z}} \max(v_j^\sharp(t, 0) - u_j(t, 0), \zeta_j^\sharp(t, 0) - \xi_j(t, 0)) \leq \mu_0 + (T + 1) \left(K_2 \Delta t + (K_2 + 1) \sqrt{\Delta t}\right)$$

given that the other inequality is obtain similarly by exchanging the roles of $((u_j)_j, (\xi_j)_j)$ and $((v_j^\sharp)_j, (\zeta_j^\sharp)_j)$. The proof of this theorem is inspired by the one of Crandall-Lions [CL84]. We do the proof for $j \in \{1, \dots, n_0\}$ and we obtain the result for $j \in \mathbb{Z}$ by periodicity of the solutions. As in the proof of Theorem 3.3, we consider only $T \leq 1$. We introduce two test functions, for all $j \in \{1, \dots, n_0\}$,

$$\begin{cases} \varphi(t, s, j) = v_j^\sharp(t) - u_j(s, 0) - \frac{(t-s)^2}{2\nu} - \eta s \\ \phi(t, s, j) = \zeta_j^\sharp(t) - \xi_j(s, 0) - \frac{(t-s)^2}{2\nu} - \eta s. \end{cases}$$

3.Crandall-Lions type error estimates for (1.6)

We can see that the function $\psi = \max(\varphi, \phi)$ reaches a maximum at a finite point $(\bar{t}, \bar{s}, \bar{j}) \in [0, T] \times [0, T] \times \{1, \dots, n_0\}$. We define

$$M_{\nu, \eta} = \max_{(t, s, j) \in [0, T] \times [0, T] \times \{1, \dots, n_0\}} \psi(t, s, j).$$

Step 1: estimate of the maximum point of ψ . Like before, we have

$$|\bar{t} - \bar{s}| \leq 4\nu q^{\Delta t} (v^{\sharp}, \zeta^{\sharp}) + 2\Delta t.$$

Now we would like to prove that for η big enough, we will have $\bar{t} = 0$ or $\bar{s} = 0$. We argue by contradiction and we assume that $\bar{t} > 0$ and $\bar{s} > 0$.

Step 2: case $\psi(\bar{t}, \bar{s}, \bar{j}) = \varphi(\bar{t}, \bar{s}, \bar{j})$.

Step 2.1: continuous viscosity inequality. Let us consider the following test function

$$w(s, 0) = v_j^{\sharp}(\bar{t}) - \frac{(\bar{t} - s)^2}{2\nu} - \eta s,$$

then $u_{\bar{j}} - w$ reaches a minimum at $(\bar{s}, 0)$ and since $u_{\bar{j}}$ is a super-solution, we get

$$q_{\nu} - \eta \geq \alpha(\xi_{\bar{j}}(\bar{s}, 0) - u_{\bar{j}}(\bar{s}, 0)),$$

with $q_{\nu} = (\bar{t} - \bar{s})/\nu$.

Step 2.2: discrete viscosity inequality. Let $\bar{n} \in \mathbb{N}$ be such that $\bar{t} = \bar{n}\Delta t$. Let us consider for all $t \geq \bar{t} - \Delta t$,

$$g(t) = \frac{(t - \bar{s})^2}{2\nu}.$$

We now use the fact that $\varphi(t, \bar{s}, \bar{j}) \leq \varphi(\bar{t}, \bar{s}, \bar{j})$, which allows us to see that

$$v_j^{\sharp}(t) - g(t) \leq v_j^{\sharp}(\bar{t}) - g(\bar{t})$$

$$\Leftrightarrow g(\bar{t}) - g(t) \leq v_j^{\sharp}(\bar{t}) - v_j^{\sharp}(t).$$

Now we use the fact that v_j^{\sharp} is a piecewise constant function and choosing $t := \bar{t} - \Delta t$,

$$\begin{aligned} \frac{g(\bar{t}) - g(\bar{t} - \Delta t)}{\Delta t} &\leq \frac{v_j^{\sharp}(\bar{n}\Delta t) - v_j^{\sharp}((\bar{n} - 1)\Delta t)}{\Delta t} \\ \frac{(\bar{t} - \bar{s})^2}{2\nu\Delta t} - \frac{(\bar{t} - \bar{s} - \Delta t)^2}{2\nu\Delta t} &\leq \frac{v_j^{\bar{n}} - v_j^{\bar{n}-1}}{\Delta t} \\ \Rightarrow q_{\nu} - \frac{\Delta t}{2\nu} &\leq \alpha \left(\zeta_j^{\bar{n}} - v_j^{\bar{n}} \right), \end{aligned} \tag{3.19}$$

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For the third line we simply used the fact that (v_j, ζ_j) is a solution of (1.15).

Step 2.3: subtracting the viscosity inequalities. We obtain directly

$$\begin{aligned} \eta &\leq \frac{\Delta t}{2\nu} + \alpha \left(\zeta_j^\sharp(\bar{t}) - \xi_j(\bar{s}, 0) - \left(v_j^\sharp(\bar{t}) - u_j(\bar{s}, 0) \right) \right) \\ &\leq \frac{\Delta t}{2\nu}, \end{aligned} \tag{3.20}$$

where we used the fact that $\varphi(\bar{t}, \bar{s}, \bar{j}) \geq \phi(\bar{t}, \bar{s}, \bar{j})$. We can see that if we define $\bar{\eta} := \frac{\Delta t}{2\nu}$ and we choose $\eta > \bar{\eta}$, we get a contradiction.

Step 3: case $\psi(\bar{t}, \bar{s}, \bar{j}) = \phi(\bar{t}, \bar{s}, \bar{j})$. In this case, we proceed as before. When we subtract the two viscosity inequalities, we then obtain

$$\begin{aligned} \eta &\leq \frac{\Delta t}{2\nu} + (a_{\bar{j}} - \alpha) \left(v_j^\sharp(\bar{t}) - u_j(\bar{s}, 0) - (\zeta_j^\sharp(\bar{t}) - \xi_j(\bar{s}, 0)) \right) \\ &\quad + \frac{a_{\bar{j}}}{\alpha} \left(V_{\bar{j}} \left[v_{\bar{j}+1}^\sharp(\bar{t}) - v_j^\sharp(\bar{t}) \right] - V_{\bar{j}} \left[u_{\bar{j}+1}(\bar{s}, 0) - u_j(\bar{s}, 0) \right] \right). \end{aligned}$$

We can see that, in the case $j \in \{1, \dots, n_0\}$, using $\phi(\bar{t}, \bar{s}, j) \geq \varphi(\bar{t}, \bar{s}, j)$ we get

$$v_j^\sharp(\bar{t}) - u_j(\bar{s}, 0) \leq \zeta_j^\sharp(\bar{t}) - \xi_j(\bar{s}, 0). \tag{3.21}$$

In the case $\bar{j} = n_0$, using the periodicity conditions in the systems (1.6) and (1.13) with the particular form of the solution (see Proposition 1.2), we get

$$\begin{aligned} v_{n_0+1}^\sharp(\bar{t}) - u_{n_0+1}(\bar{s}, 0) &= v_1^\sharp(\bar{t}) + p - u_1(\bar{s}, 0) - p \\ &\leq \zeta_j^\sharp(\bar{t}) - \xi_j(\bar{s}, 0). \end{aligned} \tag{3.22}$$

Therefore, for all $j \in \{1, \dots, n_0 + 1\}$,

$$v_j^\sharp(\bar{t}) \leq u_j(\bar{s}, 0) + \zeta_j^\sharp(\bar{t}) - \xi_j(\bar{s}, 0)$$

Now using monotonicity assumption (A2) (like in the proof of Theorem 3.3) we get

$$\eta \leq \frac{\Delta t}{2\nu}.$$

As before, it suffices to choose $\eta > \bar{\eta}$ to obtain a contradiction. Therefore we have $\bar{t} = 0$ or $\bar{s} = 0$.

3. Crandall-Lions type error estimates for (1.6)

Step 4: bound on the error. Let us assume that $\psi(\bar{t}, \bar{s}, \bar{j}) = \varphi(\bar{t}, \bar{s}, \bar{j})$, (the other case being similar we skip it).

Step 4.1: case $\bar{t} = 0$. We get

$$\begin{aligned}
 M_{\nu, \eta} &= v_j^\sharp(0) - u_{\bar{j}}(\bar{s}, 0) - \frac{\bar{s}^2}{2\nu} - \eta\bar{s} \\
 &\leq v_j^\sharp(0) - u_{\bar{j}}(\bar{s}, 0) \\
 &\leq \mu_0 + K_1\bar{s} \\
 &\leq \mu_0 + K_1|\bar{s} - \bar{t}| \\
 &\leq \mu_0 + K_1(4\nu q^{\Delta t}(v^\sharp, \zeta^\sharp) + 2\Delta t),
 \end{aligned}$$

where we used for the third line Theorem 2.2, for the fourth line, the definition of the initial data (1.14). Then, for the last line we used inequality (3.11).

Step 4.2: case $\bar{s} = 0$. Similarly, we get

$$\begin{aligned}
 M_{\nu, \eta} &\leq v_j^\sharp(\bar{t}) - u_{\bar{j}}(0, 0) \\
 &\leq K_1\bar{t} + v_j^\sharp(0) - u_{\bar{j}}(0, 0) \\
 &\leq \mu_0 + K_1(4\nu q^{\Delta t}(v^\sharp, \zeta^\sharp) + 2\Delta t).
 \end{aligned}$$

Step 4.3: conclusion. We can see that we have the same upper bound in the two cases, therefore

$$M_{\nu, \eta} \leq \max(2K_1, 4K_1q^{\Delta t}(v^\sharp, \zeta^\sharp)).(\nu + \Delta t) + \mu_0 =: M. \quad (3.23)$$

Now we can see that for all $j \in \mathbb{Z}$ and for all $t \in [0, T]$ (we recall that we have chosen $T \leq 1$),

$$\begin{aligned}
 v_j^\sharp(t) - u_j(t, 0) - \eta t &\leq M_{\nu, \eta} \leq M, \\
 \Rightarrow v_j^\sharp(t) - u_j(t, 0) &\leq M + T\eta^*,
 \end{aligned}$$

with $\eta^* = \Delta t/\nu = \sqrt{\Delta t}/\sqrt{T}$, choosing $\nu = \sqrt{T\Delta t}$. We have a similar result for $\zeta_j^\sharp - \xi_j$, therefore we have

$$M \leq \max(2K_1, 4K_1q^{\Delta t}(v^\sharp, \zeta^\sharp)).(\sqrt{T\Delta t} + \Delta t) + \mu_0.$$

Finally, we have

$$\sup_{t \in [0, 1]} \max_{j \in \mathbb{Z}} \sup(v_j^\sharp(t) - u_j(t, 0), \zeta_j^\sharp(t) - \xi_j(t, 0)) \leq (K_2 + 1)\sqrt{T\Delta t} + K_2\Delta t + \mu_0, \quad (3.24)$$

with

$$K_2 = \max(2K_1, 4K_1q^{\Delta t}(v^\sharp, \zeta^\sharp)).$$

This ends the proof of Theorem 3.4. □

4 Estimate on the effective Hamiltonian for a discrete traffic flow model

This section is devoted to the proof of Theorem 1.5.

Proof of Theorem 1.5. Let us consider $((u_j)_j, (\xi_j)_j)$ the solution to (1.6)-(1.10), and let us now consider a numerical approximation of $u_1(T, 0)$, given by (1.13) or (1.15), that we denote by u_1^{nT} . From Theorem 3.3 and Theorem 3.4, we have

$$\left| \frac{u_1^{nT} - u_1(T, 0)}{T} \right| \leq \max(K_2 + 1, K_2 + K_3)(\Delta t + \sqrt{\Delta t}) \left(1 + \frac{1}{T} \right).$$

Therefore, we have the following estimate

$$\begin{aligned} |\lambda^T - \lambda| &= \left| \frac{u_1^{nT}}{T} - \frac{u_1(T, 0)}{T} + \frac{u_1(T, 0)}{T} - \lambda \right| \\ &\leq \frac{C_1}{T} + C_2(\Delta t + \sqrt{\Delta t}), \end{aligned}$$

where we have also used Proposition 1.2. □

5 Numerical Simulations

As seen in [1], the Cauchy problem (1.9) is equivalent to a LWR macroscopic model (see [LW55, Ric56] or [GP06] for more information on macroscopic models) of the form

$$\partial_t \rho + \partial_y (\rho v(\rho)) = 0, \tag{5.1}$$

where $\rho(t, y)$ is the density of vehicles at time t at the physical point y (point on the road) and $v(\rho) := \bar{F}\left(\frac{1}{\rho}\right)$ is the average speed of the vehicles. The equivalence between (1.9) and (5.1), was done in [LLC08] and [Wag87].

Since the interest of the homogenisation of (1.6) is to pass to a macroscopic model, we will give our numerical results in the form $v(\rho) := \bar{F}\left(\frac{1}{\rho}\right)$. Moreover, according to Theorem 1.4 in [1], and in the case there is only one optimal velocity function, meaning $V \equiv V_j$, for all $j \in \{1, \dots, n_0\}$, we have that the effective Hamiltonian is given by

$$\bar{F}(p) = V \left[\frac{p}{n_0} \right]. \quad (5.2)$$

We recall that to simulate traffic flow at a macroscopic scale, it is necessary to use a numerical scheme for the equivalent LWR model. We recommend [Har83, Roe81, Swe84] for numerical scheme for conservation laws like the LWR model.

5.1 Setting of the computation

To obtain a numerical estimation of $\lambda = \bar{F}(p)$, for $p \in [0, +\infty)$, we have to fix T , and solve (1.6) using one of our two finite difference schemes, with the initial condition (1.14) for $p \in [0, +\infty)$, to obtain u_1^{nT} , then our estimate is given by $\lambda^T = u_1^{nT}/T$. To obtain the function $\bar{F}\left(\frac{1}{\rho}\right)$, we simply repeat the same process for different values of $\rho \in (0, \rho_0]$. We denote by $\Delta\rho$ our step in space for the interval $(0, \rho_0]$.

5.2 First case: one type of driver

Let us begin with a simple case, we consider that we have only one type of driver, $n_0 = 1$, therefore, we have only one optimal velocity function V and one driver sensitivity a . We consider the model based on the Greenshields optimal velocity function,

$$V[h] = \begin{cases} 0 & \text{if } h \leq h_0, \\ V_{max} \left(1 - \left(\frac{h_0}{h} \right)^n \right) & \text{if } h > h_0, \end{cases} \quad (5.3)$$

with $n \in \mathbb{N}^*$. For the values of the different parameters we take similar values to the ones in [BT10, Table 2], for the Lincoln tunnel,

$$\begin{cases} V_{max} = 58.86 \text{ km/h}, \\ h_0 = 9.64 \text{ m}, \\ a = \frac{4nV_{max}}{h_0} = 20.35 \text{ s}^{-1}, \\ n = 3, \end{cases} \quad (5.4)$$

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we also fix $\rho_0 = 180$ *vehicles/km*, $\Delta\rho = 0.35$ *vehicles/km* and $T = 2000$ *s*. Using (5.2) and the definition of $v(\rho)$ in (5.1), we get that

$$v(\rho) = \begin{cases} 0 & \text{if } \rho \geq \rho_{max}, \\ V_{max} \left(1 - \left(\frac{\rho}{\rho_{max}}\right)^n\right) & \text{if } \rho < \rho_{max}, \end{cases} \quad (5.5)$$

with $\rho_{max} = \frac{1}{h_0}$. For the numerical tests, we choose, for the explicit scheme $\Delta t = 1/\max_{j \in \{1, \dots, n_0\}}(a_j)$ and for the implicit scheme $\Delta t = 1/\alpha$ (which, given the definition of α , satisfies (2.5) because $a > 1$) to get the average speed ($v(\rho)$) on Figure 5.1.

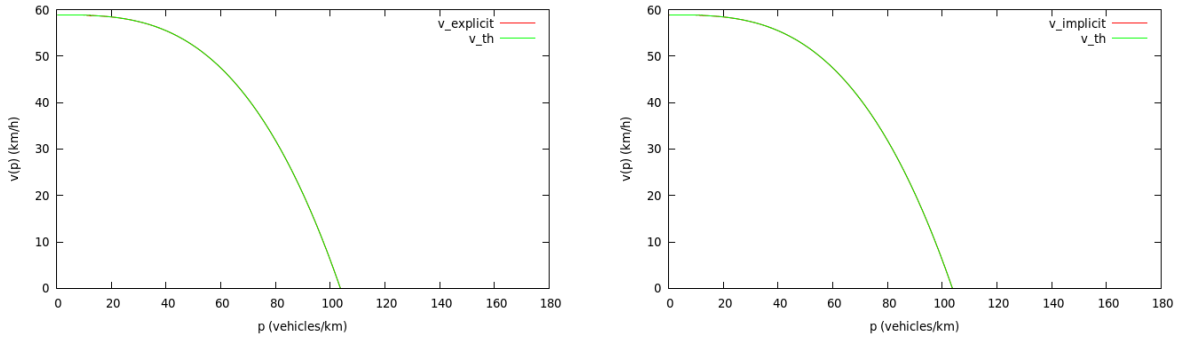


Figure 5.1: Average speed using the explicit scheme (left) and the implicit scheme (right). We consider only one type of driver, $n_0 = 1$. In red we have the numerical approximation and in green the theoretical value.

Using the fact that we know the explicit form of $v(\rho)$ (see (5.5)), we can analyse the relative error of our numerical estimation against T . To do this, we take different values for T from the interval $\left[\frac{T_{final}}{20}, T_{final}\right]$, with $T_{final} = 200$ *s* and $\Delta t = 1/T^2$ for both numerical scheme. We choose the same time step for both schemes, in order to numerically verify Corollary 1.6. In fact, we expect to have a linear relationship between $1/T$ and the approximation error to the average speed.

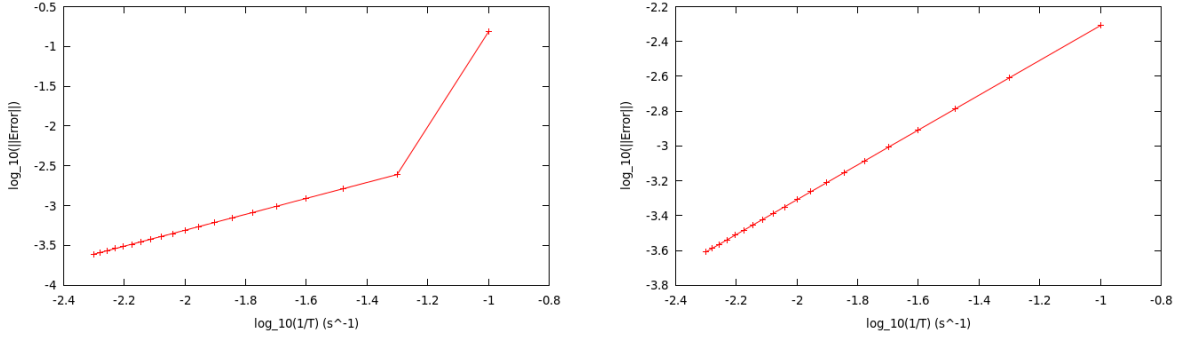


Figure 5.2: Relative error for the explicit scheme (left) and the implicit scheme (right) against $1/T$.

If we denote by $\tilde{\lambda}_i$ the approximation of $v(\rho_i) = \bar{F}\left(\frac{1}{\rho_i}\right)$ with

$$\rho_i = i\Delta\rho \text{ with } i \in \left\{1, \dots, \left\lceil \frac{\rho_0}{n_0\Delta\rho} \right\rceil =: n_{max}\right\}, \tag{5.6}$$

then we define the error plotted in Figure 5.2 by

$$\text{Error} = \frac{\max_{i \in \{1, \dots, n_{max}\}} |\tilde{\lambda}_i - v(\rho_i)|}{\max_{i \in \{1, \dots, n_{max}\}} |v(\rho_i)|}. \tag{5.7}$$

From Figure 5.2 we can see that the error decreases linearly with $\frac{1}{T}$, which numerically verifies Corollary 1.6 in this particular case. Notice that for the explicit scheme, the error increases greatly for the last value of $1/T$, this is due to the fact that for such a point the monotonicity condition (3.2) is no longer satisfied.

5.3 Second case: one optimal velocity function

Let us consider the same optimal velocity function as before, but with $n_0 = 10$ different sensitivities, we choose the a_j coefficients at random in the interval $[a, (1 + \theta)a]$, with a defined as before.

Here and in the rest of the paper, the parameter $\theta \in [0, 1]$ is there to simply define the amplitude of the intervals in which we take the values of the coefficients appearing in the models. For this example, we consider the following numerical values for the parameters of the model:

$$\begin{cases} V_{max} = 58.86 \text{ km/h}, \\ h_0 = 9.64 \text{ m}, \\ a_j \in [a, (1 + \theta)a], \quad \text{with } a = 20.35 \text{ s}^{-1}, \quad j \in \{1, \dots, n_0\}, \\ n = 3. \end{cases} \tag{5.8}$$

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Let us choose for example $\theta = 0.99$. We fix $\rho_{max} = 180$ vehicles/km, $\Delta\rho = 0.35$ and $T = 20000$ s. It is easily verified that with these values we satisfy all of our assumptions. We also choose for the time steps $1/\max_{j \in \{1, \dots, n_0\}}(a_j)$ for the explicit scheme and $1/\alpha$ for the implicit scheme (like before, given that $a_j > 1$ for all $j \in \{1, \dots, n_0\}$, condition (2.5) is satisfied). In Figure 5.3, we can see the average speed that we obtain in this particular case.

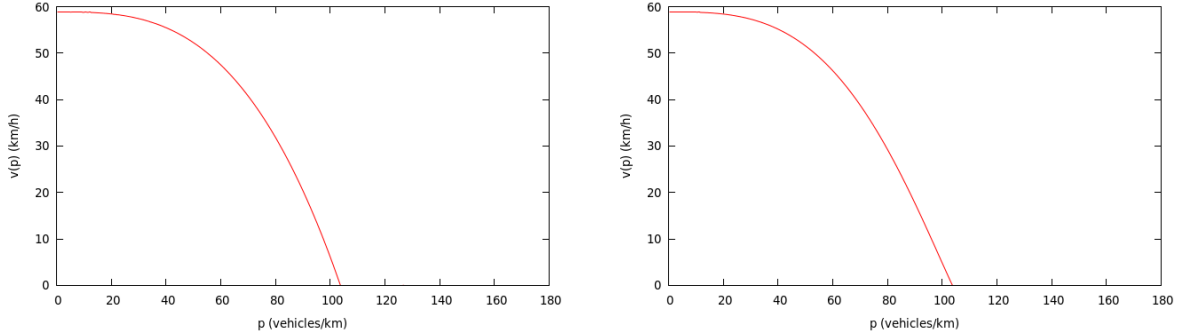


Figure 5.3: Average speed using the explicit scheme (left) and the implicit scheme (right). We consider only optimal velocity function and $n_0 = 10$ different driver sensitivities.

Remark 5.1. We can see that we have the same results in the first two cases, but we know that we should have $v(\rho') = V[1/(n_0\rho')]$, the problem is that ρ' is in (vehicles/ n_0)/km, so we denote by $\rho = n_0\rho'$ the density in vehicles/km. That is the reason why we have the same results in both cases once we pass to the same units. Moreover, this numerical test allows us to numerically verify Theorem 1.4 in [1], that gave us Formula (5.2), meaning that in the case we have only one optimal velocity function, we will have the same effective Hamiltonian, independently of the number of different driver sensitivities. Also it should be remarked that once we start having different values for the coefficients it becomes necessary to use larger values for T , and the use of the implicit scheme becomes more and more interesting to reduce the computation time.

5.4 Third case: n_0 type of drivers

Let us now consider a generalized Newell-based model (see [GP06] for other models), that uses the following optimal velocity functions, for $j \in \{1, \dots, n_0\}$,

$$V_j[h] = \begin{cases} 0, & h \leq h_0 \\ V_{max}^j \left(1 - \exp\left(-\left(\frac{h-h_0}{b_j}\right)^n\right) \right), & h > h_0. \end{cases} \quad (5.9)$$

For the different parameters we have chosen values that are close to the ones in [BT10, Table 2]. We set $h_0 = 6.50 m$, and the rest of the parameters we have chosen at random in the following intervals, for all $j \in \{1, \dots, n_0\}$,

$$\left\{ \begin{array}{l} b_j \in [(1 - \theta)b, b], \text{ with } b = 0.013 \text{ km}, \\ V_{max}^j \in [(1 - \theta)V_{max}, V_{max}], \text{ with } V_{max} = 54.11 \text{ km/h}, \\ a_j \in \left[\frac{4V_{max}}{\min_{j \in \{1, \dots, n_0\}}(b_j)}, (1 + \theta) \frac{4V_{max}}{\min_{j \in \{1, \dots, n_0\}}(b_j)} \right], \end{array} \right. \quad (5.10)$$

with $\theta \in [0, 1)$.

Numerical test. We choose $\theta = 0.55$, $n_0 = 10$, $T = 2.10^6$, and for the time step we choose, for the implicit scheme $1/\alpha$ and $1/\max_{i \in \{1, \dots, n_0\}}(a_i)$ for the explicit scheme. In this example we have $\min_{j \in \{1, \dots, n_0\}}(\|V_j\|_\infty) = 26.50 \text{ km/h}$.

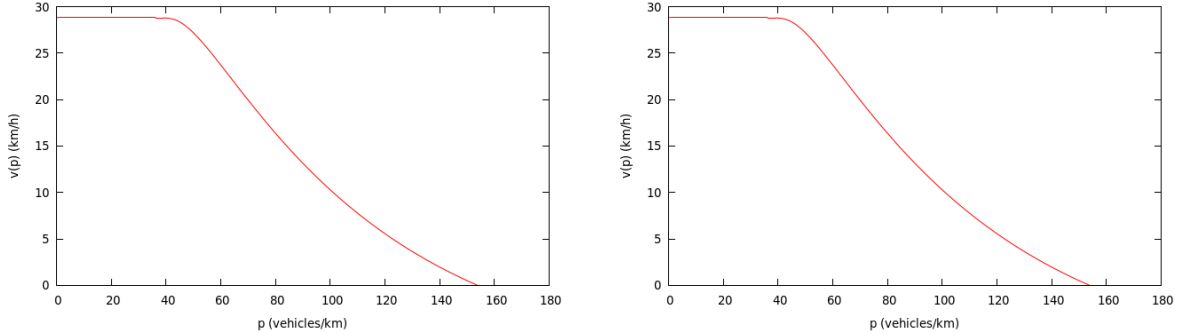


Figure 5.4: Average speed using the explicit scheme (left) and the implicit scheme (right). Using the Newell optimal velocity function and considering $n_0 = 10$ different types of drivers.

Remark 5.2. We can see that the effective Hamiltonian that we obtain in Figure 5.4 satisfies all the qualitative properties that were proven in [1]: $v(\rho) = \bar{F}\left(\frac{1}{\rho}\right)$ is non-increasing, continuous and

$$\lim_{\rho \rightarrow 0} \bar{F}\left(\frac{1}{\rho}\right) = \min_{j \in \{1, \dots, n_0\}}(\|V_j\|_\infty). \quad (5.11)$$

Let us take $n_0 = 1$ (one type of driver with the values of the parameters given by (5.10) for $\theta = 0$), in order to compare the deformation. We can see the results in Figure 5.5. At

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the beginning of Section 5, we introduce the LWR model. In general such models are often defined by their flux $f(\rho) = \rho \cdot v(\rho)$, also called the fundamental diagram. In Figure 5.6, we present one last graphic where we can see the difference in the fundamental diagrams when we consider $n_0 = 1$ type of driver (with $\theta = 0$) and when we consider $n_0 = 10$ types of drivers (with $\theta = 0.55$).

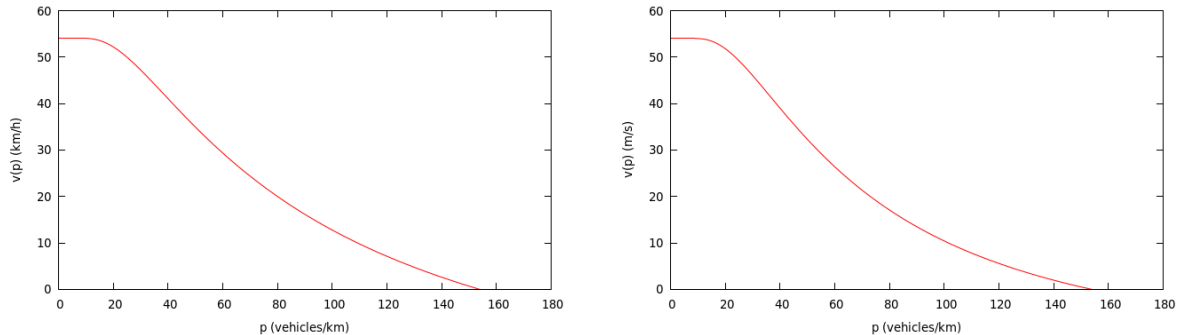


Figure 5.5: Average speed using the explicit scheme (left) and the implicit scheme (right). Considering $n_0 = 1$ and $\theta = 0$, we only consider one type of driver.

Remark 5.3. *Unlike for the cases where we only had one optimal velocity function, we can see that the average speed function that we obtain in Figure 5.4 is different from the one in Figure 5.5. However, the form of the average speed remains the same and we also can see that $v(\rho) = 0$ for all $\rho \geq \rho_{max} \simeq 153$ vehicles/km which is another qualitative property of the effective Hamiltonian that was proven in [1].*

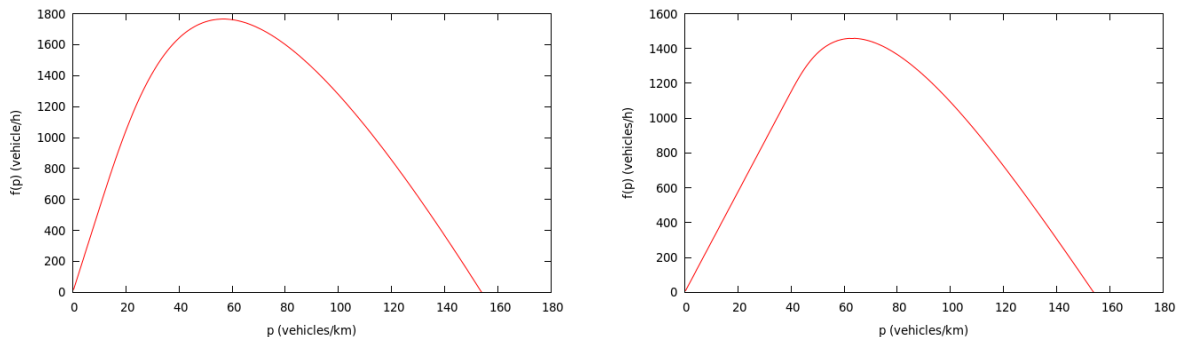


Figure 5.6: Flux of vehicles obtained using the implicit scheme and $n_0 = 1$ (with $\theta = 0$) (left) and the flux of vehicles obtained using the implicit scheme with $n_0 = 10$ (with $\theta = 0.55$) (and $n_0 = 10$) (right).

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Chapitre 3

Specified homogenization of a discrete model with a local perturbation and application to traffic flow

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Abstract

In this paper, we focus on deriving traffic flow macroscopic models from microscopic models containing a local perturbation. At the microscopic scale, we consider a first order model of the form "follow the leader" i.e. the velocity of each vehicle depends on the distance to the vehicle in front of it. We consider a local perturbation located at the origin that slows down the vehicles. At the macroscopic scale, we obtain an explicit Hamilton-Jacobi equation left and right of the origin and a junction condition at the origin (in the sense of [IM14]) which keeps the memory of the local perturbation. As it turns out, the macroscopic model is equivalent to a LWR model, with a flux limiting condition at the junction. Finally, we also present qualitative properties concerning the flux limiter at the junction.

AMS Classification: 35D40, 90B20, 35B27, 35F20, 45K05.

Keywords: specified homogenization, Hamilton-Jacobi equations, integro-differential operators, Slepčev formulation, viscosity solutions, traffic flow, microscopic models, macroscopic models.

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1 Introduction

The goal of this paper is to derive a macroscopic model for traffic flow problems from a microscopic model. The idea is to rescale the microscopic model, which describes the dynamics of each vehicle individually, in order to get a macroscopic model which describes the dynamics of density of vehicles.

The problem of deriving macroscopic models from microscopic ones has already been studied for models of the type following the leader (i.e. the velocity or the acceleration of each vehicle depends only on the distance to the vehicle in front of it). We refer for example to [AKRM02, DFR15, Hel98, LLK01] where the authors rescaled the empirical measure and obtained a scalar conservation law (LWR model [LW55, Ric56]). Recently, another approach has been introduced in [FIM09b] (see also [FIM09a, FIM12, 1]) where the authors work on the primitive of the empirical measure and, at the limit, obtain a Hamilton-Jacobi equation which is the primitive of the LWR model.

The originality of our work is that we assume that there is a local perturbation that slows down the vehicles and we want to understand how this local perturbation influences the macroscopic dynamics. If the local perturbation is located around zero, at the macroscopic scale it is natural to get an Hamilton-Jacobi equation with a junction condition at zero and an effective flux limiter, the difficulty being to construct this effective flux limiter.

Recently, the theory of Hamilton-Jacobi equations with junction or more generally on networks has known important developments in particular since the works of Achdou, Camilli, Cutri, and Tchou [ACCT13] and Imbert, Monneau, and Zidani [IMZ13]. In this direction, we would like to mention the recent work of Imbert and Monneau [IM14] in which they give a suitable definition of (viscosity) solutions at the junction which allows to prove comparison principle, stability and so on.

In this paper, we will use the ideas developed in [FIM09b] in order to pass from microscopic models to macroscopic ones. In particular, we will show that this problem can be seen as an homogenization result. The difficulty here is that, due to the local perturbation, we are not in a periodic setting and so the construction of suitable correctors is more complicated. In particular, we will use the idea developed by Achdou and Tchou in [AT15], by Galise, Imbert, and Monneau in [GIM15], and in the lectures of Lions at the "College de France" [Lio14], which consists in constructing correctors on truncated domains.

2 Main results

2.1 General model: first order model with a single perturbation

In this paper, we are interested in a first order microscopic model of the form

$$\dot{U}_j(t) = V(U_{j+1}(t) - U_j(t)) \cdot \phi(U_j(t)), \quad (2.1)$$

where $U_j : [0, +\infty) \rightarrow \mathbb{R}$ denotes the position of the j -th vehicle and \dot{U}_j is its velocity. The function $\phi : \mathbb{R} \rightarrow [0, 1]$ simulates the presence of a local perturbation around the origin. We denote by r the radius of influence of the perturbation.

The function V is called the optimal velocity function and we make the following assumptions on V and ϕ :

Assumption (A)

- (A1) $V : \mathbb{R} \rightarrow \mathbb{R}^+$ is Lipschitz continuous, non-negative.
- (A2) V is non-decreasing on \mathbb{R} .
- (A3) There exists a $h_0 \in (0, +\infty)$ such that for all $h \leq h_0$, $V(h) = 0$.
- (A4) There exists $h_{max} \in (h_0, +\infty)$ such that for all $h \geq h_{max}$, $V(h) = V(h_{max}) =: V_{max}$.
- (A5) There exists a real $p_0 \in [-1/h_0, 0)$ such that the function $p \mapsto pV(-1/p)$ is decreasing on $[-1/h_0, p_0)$ and increasing on $[p_0, 0)$.
- (A6) The function $\phi : \mathbb{R} \rightarrow [0, 1]$ is Lipschitz continuous and $\phi(x) = 1$ for $|x| \geq r$.

Remark 2.1. *Assumptions (A1)-(A2)-(A3)-(A5) are satisfied by several classical optimal velocity functions, we have added assumption (A4) to work with V' with a bounded support. But by modifying slightly the classical optimal velocity functions, we obtain a function that satisfies all the assumptions. For instance, in the case of the Greenshields based models [GCM35](see also [BT10]):*

$$V(h) = \begin{cases} 0 & \text{for } h \leq h_0, \\ V_{max} \left(1 - \left(\frac{h_0}{h} \right)^n \right) & \text{for } h_0 < h \leq h_{max}, \\ V_{max} \left(1 - \left(\frac{h_0}{h_{max}} \right)^n \right) & \text{for } h > h_{max}, \end{cases}$$

with $n \in \mathbb{N} \setminus \{0\}$. Another optimal velocity function, based on the Newell model [New61](see also [Edi61]), is given by:

$$V(h) = \begin{cases} 0 & \text{for } h \leq h_0, \\ V_{max} \left(1 - \exp \left(- \left(\frac{h - h_0}{b} \right)^n \right) \right) & \text{for } h_0 < h \leq h_{max}, \\ V_{max} \left(1 - \exp \left(- \left(\frac{h_{max} - h_0}{b} \right)^n \right) \right) & \text{for } h > h_{max}, \end{cases}$$

with $n \in \mathbb{N} \setminus \{0\}$ and $b \in [0, +\infty)$. See Figure 2.1 for a schematic representation of an optimal velocity function satisfying assumption (A).

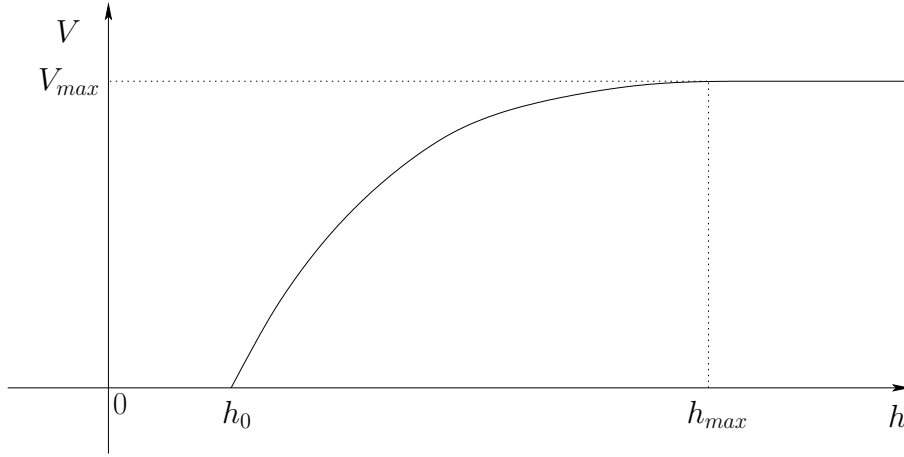


Figure 2.1: Schematic representation of the optimal velocity function V .

2.2 Injecting the system of ODEs into a single PDE

In this paper, we will study the traffic flow when the number of vehicles per unit length tends to infinity by introducing the rescaled "cumulative distribution function" of vehicles, ρ^ε , defined by

$$\rho^\varepsilon(t, y) = -\varepsilon \left(\sum_{i \geq 0} H(y - \varepsilon U_i(t/\varepsilon)) + \sum_{i < 0} (-1 + H(y - \varepsilon U_i(t/\varepsilon))) \right), \quad (2.2)$$

with

$$H(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases} \quad (2.3)$$

Under assumption (A), the function ρ^ε satisfies in the viscosity sense (see Definition 3.1 and Theorem 8.1 for the proof of this result) the following non-local equation

$$u_t^\varepsilon + M^\varepsilon \left[\frac{u^\varepsilon(t, \cdot)}{\varepsilon} \right] (x) \cdot \phi \left(\frac{x}{\varepsilon} \right) \cdot |u_x^\varepsilon| = 0 \quad \text{on } (0, +\infty) \times \mathbb{R}, \quad (2.4)$$

where M^ε is a non-local operator defined by

$$M^\varepsilon[U](x) = \int_{-\infty}^{+\infty} J(z) E(U(x + \varepsilon z) - U(x)) dz - \frac{3}{2} V_{max} \quad (2.5)$$

and with

$$E(z) = \begin{cases} 0 & \text{if } z \geq 0 \\ 1/2 & \text{if } -1 \leq z < 0 \\ 3/2 & \text{if } z < -1, \end{cases} \quad \text{and } J = V' \text{ on } \mathbb{R}. \quad (2.6)$$

In the rest of this paper, we couple equation (2.4) with the following initial condition

$$u^\varepsilon(0, x) = u_0(x) \quad \text{on } \mathbb{R}. \tag{2.7}$$

We also assume that the initial condition satisfies the following assumption:

(A0) (Gradient bound) The function u_0 is Lipschitz continuous and satisfies

$$-k_0 \leq (u_0)_x \leq 0. \tag{2.8}$$

Remark 2.2. *This condition ensures that initially the vehicles have a security distance between them and since we are working with a first order model, this security distance will be preserved. We choose u_0 a regular function such that for all ε ,*

$$|\rho^\varepsilon(0, x) - u_0(x)| \leq f(\varepsilon),$$

with $f(\varepsilon) \rightarrow 0$ as ε goes to 0.

Remark 2.3 (Lagrangian formulation). *Another way to treat this problem is to consider a Lagrangian formulation, like in [1], considering the function,*

$$v : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}, \quad v(t, y) = U_{\lfloor y \rfloor}(t).$$

This function satisfies for all $(t, y) \in [0, T] \times \mathbb{R}$

$$\begin{cases} v_t(t, y) = V(u(t, y+1) - u(t, y)) \cdot \phi(v(t, y)), \\ v(0, y) = v_0(y). \end{cases} \tag{2.9}$$

The difficulty with this formulation is that the function ϕ is evaluated at $v(t, y)$ and not at a physical point of the road. The notion of junction in this case is not well defined and this is why we use the formulation (2.4) (where the perturbation function is evaluated at a point of the road) instead of (2.9). This will allow us to use the results of Imbert and Monneau [IM14] concerning quasi-convex Hamiltonians with a junction condition.

2.3 Convergence result

We define $k_0 = 1/h_0$ and $\bar{H} : \mathbb{R} \rightarrow \mathbb{R}$, by

$$\bar{H}(p) = \begin{cases} -p - k_0 & \text{for } p < -k_0, \\ -V\left(\frac{-1}{p}\right) |p| & \text{for } -k_0 \leq p \leq 0, \\ p & \text{for } p > 0. \end{cases} \tag{2.10}$$

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Note that such a \bar{H} is continuous, coercive $\left(\lim_{|p| \rightarrow +\infty} \bar{H}(p) = +\infty\right)$ and because of (A5), there exists a unique point $p_0 \in [-k_0, 0]$ such that

$$\begin{cases} \bar{H} \text{ is decreasing on } (-\infty, p_0), \\ \bar{H} \text{ is increasing on } (p_0, +\infty). \end{cases} \quad (2.11)$$

We denote by

$$H_0 = \min_{p \in \mathbb{R}} \bar{H}(p) = \bar{H}(p_0) \quad (2.12)$$

and we refer to Figure 2.2 for an schematic representation of \bar{H} .

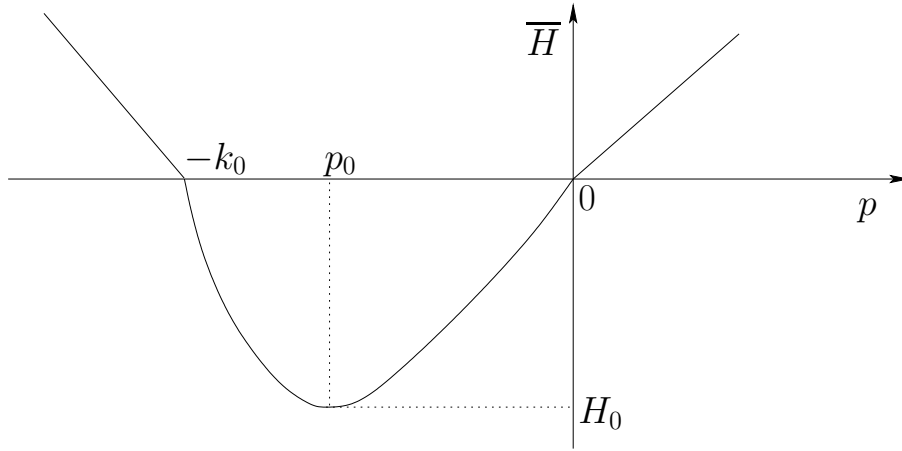


Figure 2.2: Schematic representation of \bar{H} .

The main purpose of this article is to prove that the viscosity solution of (2.4)-(2.7) converges uniformly on compact subsets of $(0, +\infty) \times \mathbb{R}$ as ε goes to 0 to the unique viscosity solution of the following problem

$$\begin{cases} u_t^0 + \bar{H}(u_x^0) = 0 & \text{for } (t, x) \in (0, +\infty) \times (-\infty, 0) \\ u_t^0 + \bar{H}(u_x^0) = 0 & \text{for } (t, x) \in (0, +\infty) \times (0, +\infty) \\ u_t^0 + F_{\bar{A}}(u_x^0(t, 0^-), u_x^0(t, 0^+)) = 0 & \text{for } (t, x) \in (0, +\infty) \times \{0\} \\ u^0(0, x) = u_0(x) & \text{for } x \in \mathbb{R}, \end{cases} \quad (2.13)$$

where \bar{A} has to be determined and $F_{\bar{A}}$ is defined by

$$F_{\bar{A}}(p_-, p_+) = \max(\bar{A}, \bar{H}^+(p_-), \bar{H}^-(p_+)), \quad (2.14)$$

with

$$\bar{H}^-(p) = \begin{cases} \bar{H}(p) & \text{if } p \leq p_0, \\ \bar{H}(p_0) & \text{if } p \geq p_0, \end{cases} \quad \text{and} \quad \bar{H}^+(p) = \begin{cases} \bar{H}(p_0) & \text{if } p \leq p_0, \\ \bar{H}(p) & \text{if } p \geq p_0. \end{cases} \quad (2.15)$$

The following theorems are the main results of this paper, and their proof are postponed. The proofs of Theorem 2.4 and Theorem 2.8 are done in Section 5 and the proof of Theorem 2.5 is done in Section 8.

Theorem 2.4 (Junction condition by homogenisation). *Assume (A) and (A0). For $\varepsilon > 0$, let u^ε be the solution of (2.4)-(2.7). Then there exists $\bar{A} \in [H_0, 0]$ such that u^ε converges locally uniformly to the unique viscosity solution u^0 of (2.13) (in the sense of Definition 3.6).*

Theorem 2.5 (Junction condition by homogenisation: application to traffic flow). *Assume (A) and that at the initial time, we have, for all $i \in \mathbb{Z}$,*

$$U_i(0) \leq U_{i+1}(0) - h_0. \tag{2.16}$$

We also assume that there exists a constant $R > 0$ such that, for all $i \in \mathbb{Z}$, if $|U_i(0)| \geq R$

$$U_{i+1}(0) - U_i(0) = h,$$

with $h \geq h_0$. We define the function u_0 (satisfying (A0)) by $u_0(x) = -x/h$ for all $x \in \mathbb{R}$. Then there exists $\bar{A} \in [H_0, 0]$ such that the function ρ^ε defined by (2.2) converges towards the unique solution u^0 of (2.13).

Remark 2.6. *Condition (2.16) means that the initial condition is well-prepared.*

Remark 2.7. *We notice that in the case of traffic flow, (2.13) is equivalent (deriving in space) to a LWR model (see [LW55, Ric56]) with a flux limiting condition at the origin. In fact, the fundamental diagram of the model is $pV(1/p)$ and u_x^0 corresponds to the density of vehicles.*

The following theorem ensures that when we use (2.13) we only evaluate the function \bar{H} in $[-k_0, 0]$.

Theorem 2.8. *Assume (A0)-(A). Let u^0 be the unique solution of (2.13), then we have for all $(t, x) \in [0, T] \times \mathbb{R}$,*

$$-k_0 \leq u_x^0 \leq 0,$$

with k_0 defined in (A0).

Remark 2.9 (Extension of the effective Hamiltonian). *This theorem implies in particular that in the case of traffic flow, the effective Hamiltonian only needs to be computed for $p \in [-k_0, 0]$. However, for the construction of the correctors it is necessary to work with a coercive Hamiltonian in \mathbb{R} that is why we extend the function \bar{H} in (2.10).*

2.4 Effective Hamiltonian and effective flux-limiter

We define the non-local operator M_p by

$$M_p[U](x) = \int_{-\infty}^{+\infty} J(z)E(U(x+z) - U(x) + p \cdot z) dz - \frac{3}{2}V_{max}. \quad (2.17)$$

We then have the following result

Proposition 2.10 (Homogenization left and right of the perturbation). *Assume (A). Then for every $p \in [-k_0, 0]$, there exists a unique $\lambda \in \mathbb{R}$, such that there exists a bounded solution v of*

$$\begin{cases} M_p[v](x) \cdot |v_x + p| = \lambda, & x \in \mathbb{R}, \\ v \text{ is } \mathbb{Z}\text{-periodic}, \end{cases} \quad (2.18)$$

with M_p defined in (2.17). Moreover, for $p \in [-k_0, 0]$, we have $\lambda = \overline{H}(p)$.

Proof. To prove this proposition, it is only necessary to notice that $v = 0$ is an obvious solution of (2.18) with $\lambda = \overline{H}(p)$. The uniqueness of λ is classical (see for instance [FIM09a, Proof of Proposition 4.6]) so we skip it. □

To construct the effective flux-limiter \overline{A} , we consider the following cell problem: find $\lambda \in \mathbb{R}$ such that there exists a solution w of the following Hamilton-Jacobi equation

$$M[w](x) \cdot \phi(x) \cdot |w_x| = \lambda \quad \text{for } x \in \mathbb{R}. \quad (2.19)$$

More precisely, we have the following result, which proof is postponed until Section 6.

Theorem 2.11 (Effective flux limiter). *Assume (A). We define the following set of functions*

$$\mathcal{S} = \{w \text{ s.t. } \exists \text{ a Lipschitz continuous function } m \text{ and } C \geq 0 \text{ such that } \|w - m\|_{L^\infty(\mathbb{R})} \leq C\}.$$

Then we have

$$\overline{A} = \inf \{ \lambda \in \mathbb{R} : \exists w \in \mathcal{S} \text{ solution of (2.19)} \}.$$

Remark 2.12. *This theorem allows us to characterize and give uniqueness to the flux limiter that we present in Section 4 which construction is presented in Section 6.*

2.5 Qualitative properties of the effective flux limiter

We have the following qualitative properties on the effective flux limiter \bar{A} , the proof of this result is postponed until Section 7.

Proposition 2.13 (Qualitative properties of the flux limiter). *Assume (A). We have the following qualitative properties on the flux limiter.*

(i) (Monotonicity of the flux-limiter). *Let $\phi_1, \phi_2 : \mathbb{R} \rightarrow [0, 1]$ be two functions satisfying (A6). Let \bar{A}_1 and \bar{A}_2 be their respective flux limiters given by Theorem 2.4. If, for all $x \in \mathbb{R}$, we have*

$$\phi_1(x) \leq \phi_2(x),$$

then

$$\bar{A}_1 \geq \bar{A}_2.$$

(ii) (Flux interruption) *Let ϕ be a function satisfying (A6). If $\phi = 0$ on an open interval, then we have*

$$\bar{A} = 0.$$

2.6 Notations

We recall the definition of the non-local operators that we used in this paper,

$$M[U](x) = \int_{-\infty}^{+\infty} J(z)E(U(x+z) - U(x)) dz - \frac{3}{2}V_{max}, \tag{2.20}$$

$$M_p[U](x) = \int_{-\infty}^{+\infty} J(z)E(U(x+z) - U(x) + p \cdot z) dz - \frac{3}{2}V_{max}. \tag{2.21}$$

To each operator M , we associate the operator \tilde{M} which is defined in the same way except that the function E is replaced by the function \tilde{E} , defined by

$$\tilde{E}(z) = \begin{cases} 0 & \text{if } z > 0 \\ 1/2 & \text{if } -1 < z \leq 0 \\ 3/2 & \text{if } z \leq -1. \end{cases} \tag{2.22}$$

Remark 2.14. *Using the fact that E and V are bounded, we get that for every function U and every $x \in \mathbb{R}$, we have*

$$-M_0 = -\frac{3}{2}V_{max} \leq M[U](x) \leq 0. \quad (2.23)$$

We also use the following notations for the upper and lower semi-continuous envelopes of a locally bounded function u :

$$u^*(t, x) = \limsup_{s \rightarrow t, y \rightarrow x} u(s, y) \quad \text{and} \quad u_*(t, x) = \liminf_{s \rightarrow t, y \rightarrow x} u(s, y).$$

2.7 Organization of the article

Section 3 contains the definition of the viscosity solutions for the problems we consider in the entire article and it also contains some results for those problems. In Section 4 we present some results on the correctors at the junction (Theorem 4.1) that will be used in Section 5 to prove Theorem 2.4. Section 6 contains the proof of Theorem 4.1. In Section 7 we give the proof of the qualitative properties of the flux-limiter. Finally, Section 8 details the link between the system of ODEs (2.1) and the PDE (2.4) (with $\varepsilon = 1$).

3 Viscosity solutions for (2.4) and (2.13)

3.1 Definitions

In order to give a general definition for all the non-local problems we consider, we will give the definition for the following equation, with $p \in \mathbb{R}$, for all $(t, x) \in (0, +\infty) \times \mathbb{R}$,

$$\begin{cases} u_t + \psi(x) \cdot M_p[u(t, \cdot)](x) \cdot \phi(x) \cdot |p + u_x| + (1 - \psi(x)) \cdot \overline{H}(u_x) = 0 \\ u(0, x) = u_0(x), \end{cases} \quad (3.1)$$

with $\psi : \mathbb{R} \rightarrow [0, 1]$ a Lipschitz continuous function.

Definition 3.1 (Viscosity solutions for (3.1)). *Let $T > 0$. An upper semi-continuous function (resp. lower semi-continuous) $u : [0, +\infty) \times \mathbb{R} \rightarrow \mathbb{R}$ is a viscosity sub-solution (resp. super-solution) of (3.1) on $[0, T] \times \mathbb{R}$, if $u(0, x) \leq u_0(x)$ (resp. $u(0, x) \geq u_0(x)$) and for all $(t, x) \in (0, T) \times \mathbb{R}$ and for all $\varphi \in C^2([0, T] \times \mathbb{R})$ such that $u - \varphi$ reaches a maximum (resp. a minimum) at the point (t, x) , we have*

$$\begin{aligned} & \varphi_t(t, x) + \psi(x) \cdot \phi(x) \cdot M_p[u(t, \cdot)](x) \cdot |p + \varphi_x(t, x)| + (1 - \psi(x)) \overline{H}(\varphi_x(t, x)) \leq 0 \\ & \left(\text{resp.} \quad \begin{aligned} & \varphi_t(t, x) + \psi(x) \cdot \phi(x) \cdot \tilde{M}_p[u(t, \cdot)](x) \cdot |p + \varphi_x(t, x)| \\ & + (1 - \psi(x)) \overline{H}(\varphi_x(t, x)) \geq 0 \end{aligned} \right). \end{aligned}$$

3. Viscosity solutions for (2.4) and (2.13)

We say that a function u is a viscosity solution of (3.1) if u^* and u_* are respectively a sub-solution and a super-solution of (3.1).

Remark 3.2. We use this definition in order to have a stability result for the non-local term. We refer to [DLKS04, Sle03] for such kind of definition and to [FIM09b, Proposition 4.2]. For the readers convenience we give the stability result and its proof.

Proposition 3.3 (Stability of the solutions of (3.1)). *Let $(u_n)_n$ be a sequence of uniformly bounded upper semi-continuous functions (resp. lower semi-continuous functions) and let \bar{u} denote $\limsup^* u_n$ (resp. $\underline{u} = \liminf_* u_n$). Let $(t_n, x_n, p_n) \rightarrow (t_0, x_0, p)$ in \mathbb{R}^3 be such that $u_n(t_n, x_n) \rightarrow \bar{u}(t_0, x_0)$ (resp. $u_n(t_n, x_n) \rightarrow \underline{u}(t_0, x_0)$). Then*

$$\limsup_{n \rightarrow +\infty}^* M_{p_n}[u_n(t_n, \cdot)](x_n) \geq M_p[\bar{u}(t_0, \cdot)](x_0) \quad (3.2)$$

$$\left(\text{resp. } \liminf_{n \rightarrow +\infty} \tilde{M}_{p_n}[u_n(t_n, \cdot)](x_n) \leq \tilde{M}_p[\underline{u}(t_0, \cdot)](x_0) \right). \quad (3.3)$$

In order to prove Proposition 3.3, we use the following lemma which proof can be found in [Sle03].

Lemma 3.4. *Let $(f_n)_n$ be a sequence of measurable functions on \mathbb{R} , and consider*

$$\bar{f} = \limsup^* f_n$$

and

$$\underline{f} = \liminf_* f_n. \quad (3.4)$$

Let $(a_n)_n$ be a sequence of \mathbb{R} converging to zero. Then

$$\mathcal{L}(\{f_n \geq a_n\} \setminus \{\bar{f} \geq 0\}) \rightarrow 0 \quad \text{as } n \rightarrow +\infty$$

and

$$\mathcal{L}(\{\underline{f} > 0\} \setminus \{f_n > a_n\}) \rightarrow 0 \quad \text{as } n \rightarrow +\infty,$$

where $\mathcal{L}(A)$ denotes the Lesbegue measure of measurable set A .

Proof of Proposition 3.3. We just do the proof for \bar{u} . Let $\varepsilon > 0$. Using (2.6), we have that

$$E(\beta) = \frac{1}{2} 1_{\{\beta \in [-1, 0)\}} + \frac{3}{2} 1_{\{\beta < -1\}} = \frac{1}{2} 1_{\{\beta < 0\}} + 1_{\{\beta < -1\}}.$$

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We get that

$$\begin{aligned}
& \int_{\mathbb{R}} J(z) E(u_n(t_n, x_n + z) - u_n(t_n, x_n) + p_n \cdot z) dz \\
& \quad - \int_{\mathbb{R}} J(z) E(\bar{u}(t_0, x_0 + z) - \bar{u}(t_0, x_0) + p \cdot z) dz \\
= & \int_{\mathbb{R}} J(z) \left\{ 1_{\{u_n(t_n, x_n + z) - u_n(t_n, x_n) + p_n \cdot z < -1\}} - 1_{\{\bar{u}(t_0, x_0 + z) - \bar{u}(t_0, x_0) + p \cdot z < -1\}} \right\} dz \quad (3.5) \\
& \quad + \int_{\mathbb{R}} \frac{1}{2} J(z) \left\{ 1_{\{u_n(t_n, x_n + z) - u_n(t_n, x_n) + p_n \cdot z < 0\}} - 1_{\{\bar{u}(t_0, x_0 + z) - \bar{u}(t_0, x_0) + p \cdot z < 0\}} \right\} dz
\end{aligned}$$

Using Lemma 3.4, we have for n big enough,

$$\begin{aligned}
& \int_{\mathbb{R}} J(z) \left\{ 1_{\{\bar{u}(t_0, x_0 + z) - \bar{u}(t_0, x_0) + p \cdot z \geq -1\}} - 1_{\{u_n(t_n, x_n + z) - u_n(t_n, x_n) + p_n \cdot z \geq -1\}} \right\} dz \\
& \quad \geq - \int_{\mathbb{R}} J(z) 1_{\{A_n(z) \setminus A(z)\}} \geq -\frac{\varepsilon}{2}, \\
& \int_{\mathbb{R}} \frac{1}{2} J(z) \left\{ 1_{\{\bar{u}(t_0, x_0 + z) - \bar{u}(t_0, x_0) + p \cdot z \geq 0\}} - 1_{\{u_n(t_n, x_n + z) - u_n(t_n, x_n) + p_n \cdot z \geq 0\}} \right\} dz \quad (3.6) \\
& \quad \geq -\frac{1}{2} \int_{\mathbb{R}} J(z) 1_{\{B_n(z) \setminus B(z)\}} \geq -\frac{\varepsilon}{2},
\end{aligned}$$

with

$$\begin{cases} A_n(z) = & \{u_n(t_n, x_n + z) - u_n(t_n, x_n) + p_n \cdot z \geq -1\} \\ & \cup \{\bar{u}(t_0, x_0 + z) - \bar{u}(t_0, x_0) + p \cdot z \geq -1\} \\ A(z) = & \{\bar{u}(t_0, x_0 + z) - \bar{u}(t_0, x_0) + p \cdot z \geq -1\} \end{cases}$$

and

$$\begin{cases} B_n(z) = & \{u_n(t_n, x_n + z) - u_n(t_n, x_n) + p_n \cdot z \geq 0\} \\ & \cup \{\bar{u}(t_0, x_0 + z) - \bar{u}(t_0, x_0) + p \cdot z \geq 0\} \\ B(z) = & \{\bar{u}(t_0, x_0 + z) - \bar{u}(t_0, x_0) + p \cdot z \geq 0\}. \end{cases}$$

Using (3.5) and (3.6), we deduce that

$$M_{p_n}[u_n(t_n, \cdot)](x_n) \geq M_p[\bar{u}(t_0, \cdot)](x_0) - \varepsilon, \quad (3.7)$$

for n big enough. This implies (3.2). □

Definition 3.5 (Class of test functions for (2.13)). *We denote by $J_\infty := (0, +\infty) \times \mathbb{R}$, $J_\infty^+ := (0, +\infty) \times [0, +\infty)$ and $J_\infty^- := (0, \infty) \times (-\infty, 0]$. We define a class of test functions on J_∞ by*

$$C^1(J_\infty) = \left\{ \varphi \in C(J_\infty), \text{ the restriction of } \varphi \text{ to } J_\infty^+ \text{ and to } J_\infty^- \text{ is } C^1 \right\}.$$

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Definition 3.6 (Viscosity solutions for (2.13)). *Let \bar{H} be given by (2.10) and $\bar{A} \in \mathbb{R}$. An upper semi-continuous (resp. lower semi-continuous) function $u : [0, +\infty) \times \mathbb{R} \rightarrow \mathbb{R}$ is a viscosity sub-solution (resp. super-solution) of (2.13) if $u(0, x) \leq u_0(x)$ (resp. $u(0, x) \geq u_0(x)$) and for all $(t, x) \in J_\infty$ and for all $\varphi \in \mathcal{C}^1(J_\infty)$ such that*

$$u \leq \varphi \text{ (resp. } u \geq \varphi) \text{ in a neighbourhood of } (t, x) \in J_\infty \text{ and } u(t, x) = \varphi(t, x),$$

we have

$$\begin{aligned} \varphi_t(t, x) + \bar{H}(\varphi_x(t, x)) &\leq 0 \quad (\text{resp. } \geq 0) && \text{if } x \neq 0, \\ \varphi_t(t, x) + F_{\bar{A}}(\varphi_x(t, 0^-), \varphi_x(t, 0^+)) &\leq 0 \quad (\text{resp. } \geq 0) && \text{if } x = 0. \end{aligned}$$

We say that a function u is a viscosity solution of (2.13) if u^* and u_* are respectively a sub-solution and a super-solution of (2.13). We refer to this solution as \bar{A} -flux limited solution.

3.2 Results for viscosity solutions of (3.1)

Proposition 3.7 (Comparison principle for (3.1)). *Let $T > 0$. Assume (A0) and (A). Let u be a sub-solution of (3.1) and v be a super-solution of (3.1). Let us also assume that there exists a constant $K > 0$ such that for all $(t, x) \in [0, T] \times \mathbb{R}$,*

$$u(t, x) \leq u_0(x) + Kt \quad \text{and} \quad -v(t, x) \leq -u_0(x) + Kt. \quad (3.8)$$

Then we have $u(t, x) \leq v(t, x)$ for all $(t, x) \in [0, T] \times \mathbb{R}$.

Proof. The only difficulty in proving the comparison principle comes from the non-local term, but in our case the proof is similar to the proof of [FIM09b, Theorem 4.4]. For the readers convenience we give the details of the proof of Proposition 3.7.

Let us introduce

$$M = \sup_{(t,x) \in [0,T] \times \mathbb{R}} \{u(t, x) - v(t, x)\}.$$

We want to prove that $M \leq 0$, we argue by contradiction and assume that $M > 0$.

Step 1: the test function. Let us introduce the following test function,

$$\varphi(t, x, y) = u(t, x) - v(t, y) + p(x - y) - \frac{\eta}{T - t} - e^{Bt} \frac{(x - y)^2}{2\varepsilon} - \alpha x^2,$$

where η, ε , and α are small, strictly positive parameters, and B is a constant to be chosen later. We can notice that we have

$$\lim_{|x|, |y| \rightarrow +\infty} \varphi(t, x, y) = -\infty.$$

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In fact, using (3.8) and (A0) we have

$$\begin{aligned}\varphi(t, x, y) &\leq u_0(x) - u_0(y) + p(x - y) + 2KT - \alpha x^2 - e^{Bt} \frac{(x - y)^2}{2\varepsilon} \\ &\leq (k_0 + |p|)|x - y| + 2KT - \alpha x^2 - e^{Bt} \frac{(x - y)^2}{2\varepsilon}.\end{aligned}$$

Using the fact that our test function is upper semi-continuous we can see that it reaches a maximum at a finite point that we denote by $(\bar{t}, \bar{x}, \bar{y}) \in [0, T] \times \mathbb{R} \times \mathbb{R}$. We also have the following result.

Lemma 3.8. *For α and η small enough, we have*

- $M_{\eta, \varepsilon, \alpha} := \varphi(\bar{t}, \bar{x}, \bar{y}) \geq M/2 > 0$.
- $|\bar{x} - \bar{y}| \rightarrow 0$ as $\varepsilon \rightarrow 0$.
- $\alpha|\bar{x}| \rightarrow 0$ as $\alpha \rightarrow 0$.

Proof of Lemma 3.8. By definition of M , for all $\theta > 0$, $\exists(t_\theta, x_\theta) \in [0, T] \times \mathbb{R}$ such that

$$\begin{aligned}M - \theta &\leq u(t_\theta, x_\theta) - v(t_\theta, x_\theta) \\ \Rightarrow M - \theta - \frac{\eta}{T - t_\theta} - \alpha x_\theta^2 &\leq u(t_\theta, x_\theta) - v(t_\theta, x_\theta) - \frac{\eta}{T - t_\theta} - \alpha x_\theta^2,\end{aligned}$$

choosing η and α small enough we have $0 < M/2 \leq M_{\eta, \varepsilon, \alpha}$. Using this result we can see that we have

$$\alpha \bar{x}^2 + e^{B\bar{t}} \frac{(\bar{x} - \bar{y})^2}{2\varepsilon} \leq u(\bar{t}, \bar{x}) - v(\bar{t}, \bar{y}) + p(x - y) \leq (k_0 + |p|)|\bar{x} - \bar{y}| + 2KT,$$

where we have used 3.8 and (A0), this inequality allows us to directly deduce the rest of the lemma. □

Step 2: case $\bar{t} = 0$. In this particular case, using Lemma 3.8 we have

$$\begin{aligned}0 < \varphi(0, \bar{x}, \bar{y}) &= u_0(\bar{x}) - u_0(\bar{y}) + p(x - y) - \frac{\eta}{T} - \frac{(\bar{x} - \bar{y})^2}{2\varepsilon} - \alpha \bar{x}^2, \\ \Rightarrow \frac{\eta}{T} &\leq (k_0 + |p|)|\bar{x} - \bar{y}|,\end{aligned}$$

using the fact that $|\bar{x} - \bar{y}| \rightarrow 0$ as $\varepsilon \rightarrow 0$, we get a contradiction.

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Step 3: case $\bar{t} > 0$. By duplication of the time variable and passing to the limit, we have that there exist two real numbers $a, b \in \mathbb{R}$ such that

$$a - b = \frac{\eta}{(T - \bar{t})^2} + B e^{B\bar{t}} \frac{(\bar{x} - \bar{y})^2}{2\varepsilon}$$

and

$$\begin{aligned} a &\leq -M_p[u(\bar{t}, \cdot)](\bar{x}) \cdot \phi(\bar{x}) \cdot |e^{B\bar{t}} p_\varepsilon + 2\alpha\bar{x}| - \psi(\bar{x}) \bar{H}(e^{B\bar{t}} p_\varepsilon + 2\alpha\bar{x}) \\ b &\geq -\tilde{M}_p[v(\bar{t}, \cdot)](\bar{y}) \cdot \phi(\bar{y}) \cdot |e^{B\bar{t}} p_\varepsilon| - \psi(\bar{y}) \bar{H}(e^{B\bar{t}} p_\varepsilon), \end{aligned}$$

with $p_\varepsilon = (\bar{x} - \bar{y})/\varepsilon$. Combining these inequalities we obtain

$$\begin{aligned} B e^{B\bar{t}} \frac{(\bar{x} - \bar{y})^2}{2\varepsilon} + \frac{\eta}{(T - \bar{t})^2} &\leq \tilde{M}_p[v(\bar{t}, \cdot)](\bar{y}) \cdot \phi(\bar{y}) \cdot |e^{B\bar{t}} p_\varepsilon| \\ &\quad - M_p[u(\bar{t}, \cdot)](\bar{x}) \cdot \phi(\bar{x}) \cdot |e^{B\bar{t}} p_\varepsilon + 2\alpha\bar{x}| \\ &\quad + \psi(\bar{y}) \bar{H}(e^{B\bar{t}} p_\varepsilon) - \psi(\bar{x}) \bar{H}(e^{B\bar{t}} p_\varepsilon + 2\alpha\bar{x}) \\ &\leq M_0 |p_\varepsilon| \cdot \|\phi'\|_\infty \cdot |\bar{x} - \bar{y}| e^{B\bar{t}} \\ &\quad + \tilde{M}_p[v(\bar{t}, \cdot)](\bar{y}) \cdot \phi(\bar{x}) \cdot |e^{B\bar{t}} p_\varepsilon| + o_\alpha(1) \\ &\quad - M_p[u(\bar{t}, \cdot)](\bar{x}) \cdot \phi(\bar{x}) \cdot |e^{B\bar{t}} p_\varepsilon| \\ &\quad + \psi(\bar{x}) \left(\bar{H}(e^{B\bar{t}} p_\varepsilon) - \bar{H}(e^{B\bar{t}} p_\varepsilon + 2\alpha\bar{x}) \right) \\ &\quad + \|\psi'\|_\infty |\bar{x} - \bar{y}| \bar{H}(e^{B\bar{t}} p_\varepsilon) \\ &\leq M_0 |p_\varepsilon| \cdot \|\phi'\|_\infty \cdot |\bar{x} - \bar{y}| e^{B\bar{t}} \\ &\quad + \tilde{M}_p[v(\bar{t}, \cdot)](\bar{y}) \cdot \phi(\bar{x}) \cdot |e^{B\bar{t}} p_\varepsilon| + o_\alpha(1) \\ &\quad - M_p[u(\bar{t}, \cdot)](\bar{x}) \cdot \phi(\bar{x}) \cdot e^{B\bar{t}} |p_\varepsilon| \\ &\quad + \|\psi'\|_\infty |\bar{x} - \bar{y}| \bar{H}(e^{B\bar{t}} p_\varepsilon), \end{aligned} \tag{3.9}$$

where we have used (2.23), the fact that ϕ is Lipschitz continuous, Lemma 3.8, and the fact that ψ is Lipschitz continuous for the second inequality. Moreover, we have used Lemma 3.8 and the fact that \bar{H} is Lipschitz continuous for the last inequality.

As in [FIM09b], we define

$$\mathcal{A} := \left\{ z : \tilde{E} \left(v(\bar{t}, z) - v(\bar{t}, \bar{y}) + p(z - \bar{y}) \right) \leq E \left(u(\bar{t}, z) - u(\bar{t}, \bar{x}) + p(z - \bar{x}) \right) \right\}.$$

However, we know that $M_{\eta, \varepsilon, \alpha}$ is reached at $(\bar{t}, \bar{x}, \bar{y})$, we have that for all $z \in \mathbb{R}$, $\varphi(\bar{t}, z, z) \leq \varphi(\bar{t}, \bar{x}, \bar{y})$ meaning that we have

$$\begin{aligned} u(\bar{t}, z) - u(\bar{t}, \bar{x}) + p(z - \bar{x}) &\leq v(\bar{t}, z) - v(\bar{t}, \bar{y}) + p(z - \bar{y}) \\ &\quad + \left(\alpha z^2 - e^{B\bar{t}} \frac{(\bar{x} - \bar{y})^2}{2\varepsilon} - \alpha \bar{x}^2 \right). \end{aligned} \tag{3.10}$$

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This implies that

$$\mathcal{A}^c \subset \{|z| \geq R_{\varepsilon, \alpha}\}, \quad \text{with } R_{\varepsilon, \alpha}^2 = \frac{1}{\alpha} \left(e^{B\bar{t}} \frac{(\bar{x} - \bar{y})^2}{2\varepsilon} + \alpha \bar{x}^2 \right).$$

We distinguish two cases.

Case 1: there exists a constant $C_\varepsilon > 0$, such that for any α small enough we have

$$\frac{|\bar{x} - \bar{y}|}{2\varepsilon} \geq C_\varepsilon. \quad (3.11)$$

In this case, we have

$$\{|z - \bar{y}| \geq R_{\varepsilon, \alpha}\} \subset \{|z| \geq \bar{R}_{\varepsilon, \alpha}\},$$

with $\bar{R}_{\varepsilon, \alpha} = -|\bar{y}| + R_{\varepsilon, \alpha} \rightarrow +\infty$ as $\alpha \rightarrow 0$ (see Da Lio *et al.* Lemma 2.5 in [DLFM08]). This implies that

$$\begin{aligned} \tilde{M}_p [v(\bar{t}, \cdot)] (\bar{y}) &= \int_{\mathbb{R}} J(z - \bar{y}) \tilde{E} \left(v(\bar{t}, z) - v(\bar{t}, \bar{y}) + p(z - \bar{y}) \right) dz - \frac{3}{2} V_{max} \\ &\leq \int_{\mathbb{R}} J(z - \bar{y}) E \left(u(\bar{t}, z) - v(\bar{t}, \bar{x}) + p(z - \bar{x}) \right) dz - \frac{3}{2} V_{max} + o_\alpha(1). \end{aligned}$$

Using this and the fact that by definition $\forall p \in \mathbb{R} \quad |\bar{H}(p)| \leq V_{max}|p|$, (3.9) becomes

$$\begin{aligned} \frac{\eta}{T^2} + B e^{B\bar{t}} \frac{(\bar{x} - \bar{y})^2}{2\varepsilon} &\leq (2M_0 \|\phi'\|_\infty + 2V_{max} \|\psi'\|_\infty) \frac{(\bar{x} - \bar{y})^2}{2\varepsilon} e^{B\bar{t}} + o_\alpha(1) \\ &\quad + e^{B\bar{t}} |p_\varepsilon| \phi(\bar{x}) \cdot \int_{\mathbb{R}} J(z - \bar{y}) \cdot E \left(u(\bar{t}, z) - u(\bar{t}, \bar{x}) + p(z - x) \right) dz \\ &\quad - e^{B\bar{t}} |p_\varepsilon| \phi(\bar{x}) \int_{\mathbb{R}} J(z - \bar{x}) \cdot E \left(u(\bar{t}, z) - u(\bar{t}, \bar{x}) + p(z - x) \right) dz \\ &\leq \left(2M_0 \|D\phi\|_\infty + 3\|\phi\|_\infty \cdot \|DJ\|_{L^1(\mathbb{R})} \right) \frac{(\bar{x} - \bar{y})^2}{2\varepsilon} e^{B\bar{t}} + o_\alpha(1) \\ &\quad + 2V_{max} \|\psi'\|_\infty \frac{(\bar{x} - \bar{y})^2}{2\varepsilon} e^{B\bar{t}} \end{aligned}$$

Choosing

$$B = 2M_0 \|\phi'\|_\infty + 3\|\phi\|_\infty \cdot \|DJ\|_{L^1(\mathbb{R})} + 2V_{max} \|\psi'\|_\infty,$$

we get a contradiction for α small enough.

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Case 2: there exists a subsequence α_n , such that

$$\frac{|\bar{x} - \bar{y}|}{2\varepsilon} \rightarrow 0 \quad \text{as } n \rightarrow +\infty. \quad (3.12)$$

In this case, (3.9), gives us a contradiction, choosing $B = 2M_0\|\phi'\|_\infty + 2V_{max}\|\psi'\|_\infty$ and passing to the limit as $n \rightarrow +\infty$. □

We now give a comparison principle on bounded sets, to do this, we define for a given point $(t_0, x_0) \in (0, T) \times \mathbb{R}$ and for $\bar{r}, \bar{R} > 0$, the set

$$\mathcal{Q}_{\bar{r}, \bar{R}}(t_0, x_0) = (t_0 - \bar{r}, t_0 + \bar{r}) \times (x_0 - \bar{R}, x_0 + \bar{R}).$$

Theorem 3.9 (Comparison principle on bounded sets for (3.1)). *Assume (A). Let u be a sub-solution of (3.1) and let v be a super-solution of (3.1) on the open set $\mathcal{Q}_{\bar{r}, \bar{R}} \subset (0, T) \times \mathbb{R}$. We assume that u (resp. v) is upper semi-continuous (resp. lower semi-continuous) on $\overline{\mathcal{Q}_{\bar{r}, \bar{R}}}$. Also assume that*

$$u \leq v \quad \text{outside } \mathcal{Q}_{\bar{r}, \bar{R}},$$

then

$$u \leq v \quad \text{on } \mathcal{Q}_{\bar{r}, \bar{R}}.$$

Proof. The proof of this theorem is similar to the one of Proposition 3.7, so we skip it. □

Lemma 3.10 (Existence of barriers for (3.1)). *Assume (A0) and (A). There exists a constant $K_1 > 0$ such that*

$$u^+(t, x) = K_1 t + u_0(x) \quad \text{and} \quad u^-(t, x) = u_0(x),$$

are respectively super and sub-solutions of (3.1).

Proof. We define $K_1 = M_0 \cdot (|p| + k_0) + |H_0|$. Let us prove that u^+ is a super-solution of (3.1). Using assumption (A0) and the form of the non-local operator and of \bar{H} , we have

$$\begin{aligned} \phi(x)\psi(x)M_p[u_0](x) \cdot |p + (u_0)_x| + (1 - \psi(x)) \cdot \bar{H}((u_0)_x) &\geq -M_0 \cdot |p + (u_0)_x| + H_0 \\ &\geq -M_0(|p| + k_0) - |H_0| \\ &= -K_1, \end{aligned}$$

where we used (2.23) and (2.12). The proof for u^- is simpler, it uses (2.23) and (2.12),

$$\phi(x)\psi(x)M_p[u_0](x) \cdot |p + (u_0)_x| + (1 - \psi(x)) \cdot \bar{H}((u_0)_x) \leq 0.$$

□

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Applying Perron's method (see [IMR08, Proof of Theorem 6], [AT96] or [Imb05] to see how to apply Perron's method for problems with non-local terms), joint to the comparison principle, we obtain the following result.

Theorem 3.11 (Existence and uniqueness of viscosity solutions for (3.1)). *Assume (A0) and (A). Then, there exists a unique solution u of (3.1). Moreover, the function u is continuous and there exists a constant K_1 such that*

$$u_0(x) \leq u(t, x) \leq u_0(x) + K_1 t.$$

3.3 Results for viscosity solutions of (2.13)

Now we recall an equivalent definition (see [IM14, Theorem 2.5]) for sub and super solution at the junction. We will also consider the following problem,

$$u_t + \bar{H}(u_x) = 0 \quad \text{for } t \in (0, T) \text{ and } x \in \mathbb{R} \setminus \{0\}. \quad (3.13)$$

Theorem 3.12 (Equivalent definition for sub/super-solutions). *Let \bar{H} given by (2.10) and consider $A \in [H_0, +\infty)$ with H_0 defined in (2.12). Given arbitrary solutions $p_{\pm}^A \in \mathbb{R}$ of*

$$\bar{H}(p_+^A) = \bar{H}^+(p_+^A) = A = \bar{H}^-(p_-^A) = \bar{H}(p_-^A), \quad (3.14)$$

let us fix any time independent test function $\phi^0(x)$ satisfying

$$\phi_x^0(0^{\pm}) = p_{\pm}^A.$$

Given a function $u : (0, T) \times \mathbb{R} \rightarrow \mathbb{R}$, the following properties hold true.

i) If u is an upper semi-continuous sub-solution of (3.13) satisfying

$$u(t, 0) = \limsup_{(s,y) \rightarrow (t,0), y \in \mathbb{R} \setminus \{0\}} u(s, y), \quad (3.15)$$

then u is a H_0 -flux limited sub-solution.

ii) Given $A > H_0$ and $t_0 \in (0, T)$, if u is an upper semi-continuous sub-solution of (3.13), satisfying (3.15), and if for any test function φ touching u from above at $(t_0, 0)$ with

$$\varphi(t, x) = \psi(t) + \phi^0(x), \quad (3.16)$$

for some $\psi \in C^2(0, +\infty)$, we have

$$\varphi_t + F_A(\varphi_x) \leq 0 \quad \text{at } (t_0, 0),$$

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then u is a A -flux limited sub-solution at $(t_0, 0)$.

iii) Given $t_0 \in (0, T)$, if u is a lower semi-continuous super-solution of (3.13) and if for any test function φ satisfying (3.16) touching u from above at $(t_0, 0)$ we have

$$\varphi_t + F_A(\varphi_x) \geq 0 \quad \text{at } (t_0, 0),$$

then u is a A -flux limited super-solution at $(t_0, 0)$.

Proof. The proof of Theorem 3.12 can be founded in [IM14, Theorem 2.5]. □

3.4 Control of the oscillations for (2.4)-(2.7)

Theorem 3.13 (Control of the oscillations). *Let $T > 0$. Assume (A0)-(A) and let u be a solution of (2.4)-(2.7), with $\varepsilon = 1$. Then there exists a constants $C_1 > 0$ such that for all $x, y \in \mathbb{R}$, $x \geq y$ and for all $t, s \in [0, T]$, $t \geq s$, we have*

$$0 \leq u(t, x) - u(s, x) \leq C_1(t - s) \quad \text{and} \quad -k_0(x - y) - 1 \leq u(t, x) - u(t, y) \leq 0, \quad (3.17)$$

with k_0 defined in (2.8).

Proof. In this proof we used the barriers given by Lemma 3.10 (with $p = 0$ and $\psi \equiv 1$), which means that the solution u of (2.4)-(2.7) with $\varepsilon = 1$ satisfies for all $(t, x) \in [0, +\infty) \times \mathbb{R}$,

$$0 \leq u(t, x) - u_0(x) \leq M_0 k_0 t. \quad (3.18)$$

In the rest of the proof we will use the following notation:

$$\Omega = \left\{ (t, x, y) \in [0, T) \times \mathbb{R}^2 \text{ s.t. } x \geq y \right\}.$$

Proof of the bound on the time derivative. For all $h \geq 0$, we have

$$u(0, x) \leq u(h, x) \leq M_0 k_0 h + u(0, x).$$

Using the fact that equation (2.4) is invariant by addition of constants to the solution and by translations in time, we deduce by the comparison principle that, for all $(t, x) \in [0, +\infty) \times \mathbb{R}$, we have

$$u(t, x) \leq u(t + h, x) \leq M_0 k_0 h + u(t, x).$$

We deduce the result by choosing $C_1 = M_0 k_0$.

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Proof of the upper inequality for the control of the space oscillations. We introduce,

$$M = \sup_{(t,x,y) \in \Omega} \{u(t,x) - u(t,y)\}.$$

We want to prove that $M \leq 0$. We argue by contradiction and assume that $M > 0$.

Step 1: the test function. For $\eta, \alpha > 0$, small parameters, we define

$$\varphi(t,x,y) = u(t,x) - u(t,y) - \frac{\eta}{T-t} - \alpha x^2 - \alpha y^2.$$

Using (3.18), we have that

$$\varphi(t,x,y) \leq u_0(x) - u_0(y) + 2M_0 k_0 T - \alpha(x^2 + y^2) \leq -\alpha(x^2 + y^2) + 2M_0 k_0 T,$$

where we used assumption (A0) for the second inequality. Therefore we have

$$\lim_{|x|,|y| \rightarrow +\infty} \varphi(t,x,y) = -\infty.$$

Since φ is upper-semi continuous, it reaches a maximum at a point that we denote by $(\bar{t}, \bar{x}, \bar{y}) \in \Omega$. Classically we have for η and α small enough,

$$\begin{cases} 0 < \frac{M}{2} \leq \varphi(\bar{t}, \bar{x}, \bar{y}), \\ \alpha|\bar{x}|, \alpha|\bar{y}| \rightarrow 0 \text{ as } \alpha \rightarrow 0. \end{cases}$$

Step 2: $\bar{t} > 0$ and $\bar{x} > \bar{y}$. By contradiction, assume first that $\bar{t} = 0$. Then we have

$$\frac{\eta}{T} < u_0(\bar{x}) - u_0(\bar{y}) \leq 0,$$

where we used that u_0 is non-increasing, and we get a contradiction. The fact that $\bar{x} > \bar{y}$, comes directly from the fact that $\varphi(\bar{t}, \bar{x}, \bar{y}) > 0$.

Step 3: viscosity inequalities. By doing a duplication of the time variable and passing to the limit we get that

$$\begin{aligned} \frac{\eta}{(T-\bar{t})^2} &\leq \tilde{M}[u(\bar{t}, \cdot)](\bar{y}) \cdot |2\alpha\bar{y}| \cdot \phi(\bar{y}) - M[u(t, \cdot)](\bar{x}) \cdot \phi(\bar{x}) \cdot |2\alpha\bar{x}| \\ &\leq 2M_0 \cdot \alpha(|\bar{x}| + |\bar{y}|), \end{aligned}$$

passing to the limit as α goes to 0, we obtain a contradiction.

Proof of the lower inequality for the control of the space oscillations Let us introduce,

$$M = \sup_{(t,x,y) \in \Omega} \{u(t,y) - u(t,x) - 1 - k_0(x-y)\}.$$

We want to prove that $M \leq 0$. We argue by contradiction and assume that $M > 0$.

Step 1: the test function. For $\alpha, \eta > 0$, small parameters we consider the function

$$\varphi(t,x,y) = u(t,y) - u(t,x) - 1 - k_0(x-y) - \alpha(x^2 + y^2) - \frac{\eta}{T-t}.$$

We have that

$$\begin{aligned} \varphi(t,x,y) &\leq u_0(y) - u_0(x) - \alpha(x^2 + y^2) + 2M_0k_0T - k_0(x-y) - 1 \\ &\leq -\alpha(x^2 + y^2) + 2M_0k_0T. \end{aligned}$$

Therefore, we have

$$\lim_{|x|,|y| \rightarrow +\infty} \varphi(t,x,y) = -\infty.$$

Using the fact that φ is upper-semi continuous we deduce that φ reaches a maximum at a finite point that we denote $(\bar{t}, \bar{x}, \bar{y}) \in \Omega$. Classically we have for η and α small enough,

$$\begin{cases} 0 < \frac{M}{2} \leq \varphi(\bar{t}, \bar{x}, \bar{y}), \\ \alpha|\bar{x}|, \alpha|\bar{y}| \rightarrow 0 \text{ as } \alpha \rightarrow 0. \end{cases}$$

Step 2: $\bar{t} > 0$ and $\bar{x} > \bar{y}$. By contradiction, assume that $\bar{t} = 0$. Using the fact that $\varphi(\bar{t}, \bar{x}, \bar{y}) > 0$ and (A0), we have

$$\frac{\eta}{T} < u(0, \bar{y}) - u(0, \bar{x}) - k_0(\bar{x} - \bar{y}) - 1 \leq -1,$$

which is a contradiction. Hence $\bar{t} > 0$. Using that $\varphi(\bar{t}, \bar{x}, \bar{y}) > 0$, we also deduce that $\bar{x} > \bar{y}$.

Step 3: viscosity inequalities. By duplicating the time variable and passing to the limit we have that there exists two real numbers a, b , such that $(a, -k_0 + 2\alpha\bar{y}) \in \overline{\mathcal{D}}^+ u(\bar{t}, \bar{y})$, $(b, -k_0 + 2\alpha\bar{x}) \in \overline{\mathcal{D}}^- u(\bar{t}, \bar{x})$ and

$$a - b = \frac{\eta}{(T - \bar{t})^2}. \tag{3.19}$$

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Using that u is a sub-solution of (2.4)-(2.7) (with $\varepsilon = 1$), we get

$$a + M[u(\bar{t}, \cdot)](\bar{y}) \cdot \phi(\bar{y}) \cdot | -k_0 + 2\alpha\bar{y}| \leq 0. \quad (3.20)$$

We claim that

$$M[u(\bar{t}, \cdot)](\bar{y}) = \int_{\mathbb{R}} J(z) E(u(\bar{t}, \bar{y} + z) - u(\bar{t}, \bar{y})) dz - \frac{3}{2} V_{max} = 0.$$

Indeed, let $z \in (h_0, h_{max}]$. If $\bar{y} + z \geq \bar{x}$, using that u is non-increasing in space, we get

$$u(\bar{t}, \bar{y} + z) - u(\bar{t}, \bar{y}) \leq u(\bar{t}, \bar{x}) - u(\bar{t}, \bar{y}) \leq -k_0(\bar{x} - \bar{y}) - 1 < -1.$$

If $\bar{y} + z < \bar{x}$, using the fact that $\varphi(\bar{t}, \bar{x}, \bar{y} + z) \leq \varphi(\bar{t}, \bar{x}, \bar{y})$ we obtain

$$u(\bar{t}, \bar{y} + z) - u(\bar{t}, \bar{y}) \leq -k_0 z < -1.$$

This implies that we have for all $z \in (h_0, h_{max}]$,

$$E(u(\bar{t}, \bar{y} + z) - u(\bar{t}, \bar{y})) = \frac{3}{2}.$$

Injecting this in the non-local term, we deduce the claim.

Finally, the fact that $u_t \geq 0$ implies that $a, b \geq 0$. Therefore, inequality (3.20) implies

$$a = 0.$$

Finally, using (3.19), we obtain

$$\frac{\eta}{T^2} \leq 0,$$

which is a contradiction. This ends the proof. □

4 Correctors for the junction

The key ingredient to prove the convergence result is to construct correctors for the junction. The main result of this section is the existence of appropriate correctors. The proof of this theorem is presented in Section 6. Given $\bar{A} \in \mathbb{R}$, $\bar{A} \geq H_0$, we introduce two real numbers $\bar{p}_+, \bar{p}_- \in \mathbb{R}$, such that

$$\bar{H}(\bar{p}_+) = \bar{H}^+(\bar{p}_+) = \bar{H}(\bar{p}_-) = \bar{H}^-(\bar{p}_-) = \bar{A}. \quad (4.1)$$

Due to the form of \bar{H} (see (2.10)) this two real numbers exist and are unique.

Theorem 4.1 (Existence of a global corrector for the junction). *Assume (A).*

i) (General properties) There exists a constant $\bar{A} \in [H_0, 0]$ such that there exists a solution w of (2.19) with $\lambda = \bar{A}$ and such that there exists a constant C and a globally Lipschitz continuous function m such that for all $x \in \mathbb{R}$,

$$|w(x) - m(x)| \leq C. \quad (4.2)$$

ii) (Bound from below at infinity) If $\bar{A} > H_0$, then there exists a γ_0 such that for every $\gamma \in (0, \gamma_0)$, we have

$$\begin{cases} w(x+h) - w(x) \geq (\bar{p}_+ - \gamma)h - C & \text{for } x \geq r \text{ and } h \geq 0, \\ w(x-h) - w(x) \geq (-\bar{p}_- - \gamma)h - C & \text{for } x \leq -r \text{ and } h \geq 0. \end{cases} \quad (4.3)$$

iii) (Rescaling w) For $\varepsilon > 0$, we set

$$w^\varepsilon(x) = \varepsilon w\left(\frac{x}{\varepsilon}\right),$$

then (along a subsequence $\varepsilon_n \rightarrow 0$) we have that w^ε converges locally uniformly towards a function $W = W(x)$ which satisfies

$$\begin{cases} |W(x) - W(y)| \leq C|x - y| & \text{for all } x, y \in \mathbb{R}, \\ \bar{H}(W_x) = \bar{A} & \text{for all } x \in \mathbb{R} \setminus \{0\}, \end{cases} \quad (4.4)$$

In particular, we have (with $W(0) = 0$)

$$W(x) = \bar{p}_+ x 1_{\{x>0\}} + \bar{p}_- x 1_{\{x<0\}}. \quad (4.5)$$

5 Proof of convergence

This section contains the proof of the main homogenization result (Theorem 2.4). This proof relies on the existences of correctors (Proposition 2.10 and Theorem 4.1).

We begin with two useful lemmas for the proof of Theorem 2.4. The first result is a direct consequence of Perron's method and Lemma 3.10.

Lemma 5.1 (Barriers uniform in ε). *Assume (A0) and (A). There exists a constant $C > 0$ (depending only on M_0 and k_0) such that for all $t > 0$ and $x \in \mathbb{R}$,*

$$|u^\varepsilon(t, x) - u_0(x)| \leq Ct.$$

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The following lemma is a direct result of Theorem 3.13.

Lemma 5.2 (Uniform gradient bound). *Assume (A0) and (A). Then the solution u^ε of (2.4)-(2.7) satisfies for all $t > 0$, for all $x, y \in \mathbb{R}$, $x \geq y$,*

$$-k_0(x - y) - \varepsilon \leq u^\varepsilon(t, x) - u^\varepsilon(t, y) \leq 0. \quad (5.1)$$

Before passing to the proof of Theorem 2.4, let us show how it allows us to prove Theorem 2.8.

Proof of Theorem 2.8. We want to prove that for all $t \in [0, +\infty)$ and for all $x, y \in \mathbb{R}$, $x \geq y$,

$$-k_0(x - y) \leq u^0(t, x) - u^0(t, y) \leq 0. \quad (5.2)$$

Using Lemma 5.2, we have that the solution u^ε of (2.4)-(2.7), satisfies for all $(t, x, y) \in [0, +\infty) \times \mathbb{R} \times \mathbb{R}$, with $x \geq y$,

$$-k_0(x - y) - \varepsilon \leq u^\varepsilon(t, x) - u^\varepsilon(t, y) \leq 0.$$

Now using Theorem 2.4, passing to the limit as $\varepsilon \rightarrow 0$, we obtain the result. □

We now turn to the proof of Theorem 2.4.

Proof of Theorem 2.4. We introduce

$$\bar{u}(t, x) = \limsup_{\varepsilon \rightarrow 0}^* u^\varepsilon \quad \text{and} \quad \underline{u}(t, x) = \liminf_{\varepsilon \rightarrow 0}^* u^\varepsilon. \quad (5.3)$$

Thanks to Lemma 5.1, we know that these functions are well defined. We want to prove that \bar{u} and \underline{u} are respectively a sub-solution and a super-solution of (2.13). In this case, the comparison principle will imply that $\bar{u} \leq \underline{u}$. But, by construction, we have $\underline{u} \leq \bar{u}$, hence we will get $\underline{u} = \bar{u} = u^0$, the unique solution of (2.13).

Let us prove that \bar{u} is a sub-solution of (2.13) (the proof for \underline{u} is similar and we skip it). We argue by contradiction and assume that there exist a test function $\varphi \in \mathcal{C}^1(J_\infty)$ (in the sense of Definition 3.5), and a point $(\bar{t}, \bar{x}) \in (0, +\infty) \times \mathbb{R}$ such that

$$\begin{cases} \bar{u}(\bar{t}, \bar{x}) = \varphi(\bar{t}, \bar{x}) \\ \bar{u} \leq \varphi & \text{on } \mathcal{Q}_{\bar{r}, \bar{r}}(\bar{t}, \bar{x}) & \text{with } \bar{r} > 0 \\ \bar{u} \leq \varphi - 2\eta & \text{outside } \mathcal{Q}_{\bar{r}, \bar{r}}(\bar{t}, \bar{x}) & \text{with } \eta > 0 \\ \varphi_t(\bar{t}, \bar{x}) + \bar{H}(\bar{x}, \varphi_x(\bar{t}, \bar{x})) = \theta & \text{with } \theta > 0, \end{cases} \quad (5.4)$$

where

$$\bar{H}(\bar{x}, \varphi_x(\bar{t}, \bar{x})) := \begin{cases} \bar{H}(\varphi_x(\bar{t}, \bar{x})) & \text{if } \bar{x} \neq 0, \\ \bar{F}_A(\varphi_x(\bar{t}, 0^-), \varphi_x(\bar{t}, 0^+)) & \text{if } \bar{x} = 0. \end{cases}$$

Given Lemma 5.2 and (5.3), we can assume (up to changing φ at infinity) that for ε small enough, we have

$$u^\varepsilon \leq \varphi - \eta \quad \text{outside } \mathcal{Q}_{\bar{r}, \bar{r}}(\bar{t}, \bar{x}).$$

Using the previous lemmas we get that the function \bar{u} satisfies for all $t > 0$ and $x, y \in \mathbb{R}$, $x \geq y$,

$$\begin{aligned} |\bar{u}(t, x) - u_0(x)| &\leq Ct, \\ -k_0(x - y) &\leq \bar{u}(t, x) - \bar{u}(t, y) \leq 0. \end{aligned} \tag{5.5}$$

First case: $\bar{x} \neq 0$. We only consider $\bar{x} > 0$, since the other case ($\bar{x} < 0$) is treated in the same way. We define $p = \varphi_x(\bar{t}, \bar{x})$ that according to (5.5) satisfies

$$-k_0 \leq p \leq 0.$$

We choose \bar{r} small enough so that $\bar{x} - 2\bar{r} > 0$. Let us prove that the test function φ satisfies in the viscosity sense, the inequality

$$\varphi_t + \tilde{M}^\varepsilon \left[\frac{\varphi}{\varepsilon}(t, \cdot) \right] (x) \cdot \phi \left(\frac{x}{\varepsilon} \right) \cdot |\varphi_x| \geq \frac{\theta}{2} \quad \text{for } (t, x) \in \mathcal{Q}_{\bar{r}, \bar{r}}(\bar{t}, \bar{x}). \tag{5.6}$$

Let us notice that for ε small enough we have

$$\phi \left(\frac{x}{\varepsilon} \right) = 1 \quad \text{for all } (t, x) \in \mathcal{Q}_{\bar{r}, \bar{r}}(\bar{t}, \bar{x}).$$

For all $(t, x) \in \mathcal{Q}_{\bar{r}, \bar{r}}(\bar{t}, \bar{x})$, we have for \bar{r} small enough

$$\begin{aligned} \varphi_t(t, x) + \tilde{M}^\varepsilon \left[\frac{\varphi}{\varepsilon}(t, \cdot) \right] (x) \cdot |\varphi_x| &= \varphi_t(\bar{t}, \bar{x}) + o_{\bar{r}}(1) + \tilde{M}^\varepsilon \left[\frac{\varphi}{\varepsilon}(t, \cdot) \right] (x) \cdot |\varphi_x| \\ &= \theta + o_{\bar{r}}(1) + \tilde{M}^\varepsilon \left[\frac{\varphi}{\varepsilon}(t, \cdot) \right] (x) \cdot |p| - \bar{H}(p) \\ &=: \Delta, \end{aligned} \tag{5.7}$$

where we have used (5.4). We recall that for $-k_0 \leq p \leq 0$,

$$\bar{H}(p) = M_p0|p|.$$

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Moreover, for all $z \in [h_0, h_{max}]$, and for ε and \bar{r} small enough we have that

$$\begin{aligned} \frac{\varphi(t, x + \varepsilon z) - \varphi(t, x)}{\varepsilon} &= z\varphi_x(t, y) + \varepsilon z^2\varphi_{xx}(t, \xi(x, x + \varepsilon z)) \\ &\leq pz + o_{\bar{r}}(1) + c\varepsilon, \end{aligned}$$

where we have used the fact that $\varphi \in \mathcal{C}^2$ and that $z \in [h_0, h_{max}]$. Now using the fact that \tilde{E} is decreasing we have

$$\tilde{E}(pz + c\varepsilon + o_{\bar{r}}(1)) \leq \tilde{E}\left(\frac{\varphi(t, x + \varepsilon z) - \varphi(t, x)}{\varepsilon}\right).$$

Using this result and replacing the non-local operators in (5.7) by their definition (see 2.17), we obtain

$$\begin{aligned} \Delta &\geq \theta + o_{\bar{r}}(1) + |p| \int_{h_0}^{h_{max}} J(z)\tilde{E}(pz + c\varepsilon + o_{\bar{r}}(1))dz \\ &\quad - |p| \int_{h_0}^{h_{max}} J(z)\tilde{E}(pz)dz. \end{aligned} \tag{5.8}$$

We can see that if we have $p = 0$, we obtain directly our result. However, if $-k_0 \leq p < 0$,

$$\begin{aligned} \int_{\mathbb{R}} J(z)\tilde{E}(pz + c\varepsilon + o_{\bar{r}}(1))dz &= -V\left(\frac{-1 - c\varepsilon + o_{\bar{r}}(1)}{p}\right) - \frac{1}{2}V\left(-\frac{c\varepsilon + o_{\bar{r}}(1)}{p}\right) + \frac{3}{2}V_{max}, \\ \int_{\mathbb{R}} J(z)\tilde{E}(pz)dz &= -V\left(\frac{-1}{p}\right) + \frac{3}{2}V_{max}. \end{aligned} \tag{5.9}$$

Injecting (5.9) in (5.8) and choosing ε and \bar{r} , we obtain

$$\begin{aligned} \Delta &\geq \theta + o_{\bar{r}}(1) + |p| \cdot \left[-V\left(\frac{-1 - c\varepsilon + o_{\bar{r}}(1)}{p}\right) + V\left(\frac{-1}{p}\right) \right] \\ &\geq \theta + o_{\bar{r}}(1) - \|V'\|_{\infty} \cdot (c\varepsilon + o_{\bar{r}}(1)) \\ &\geq \frac{\theta}{2}, \end{aligned}$$

where we have used assumption (A1) for the second line.

Getting a contradiction. By definition, we have for ε small enough,

$$u^\varepsilon \leq \varphi - \eta \quad \text{outside } \mathcal{Q}_{\bar{r}, \bar{r}}(\bar{t}, \bar{x}).$$

Using the comparison principle on bounded subsets for (2.4)-(2.7), we get

$$u^\varepsilon \leq \varphi - \eta \quad \text{on } \mathcal{Q}_{\bar{r}, \bar{r}}(\bar{t}, \bar{x}).$$

Passing to the limit as $\varepsilon \rightarrow 0$, we get $\bar{u} \leq \varphi - \eta$ on $\mathcal{Q}_{\bar{r}, \bar{r}}(\bar{t}, \bar{x})$ and this contradicts the fact that $\bar{u}(\bar{t}, \bar{x}) = \varphi(\bar{t}, \bar{x})$.

Second case: $\bar{x} = 0$. Using Theorem 3.12, we may assume that the test function has the following form

$$\varphi(t, x) = g(t) + \bar{p}_- x 1_{\{x < 0\}} + \bar{p}_+ x 1_{\{x > 0\}} \quad \text{on } \mathcal{Q}_{2\bar{r}, 2\bar{r}}(\bar{t}, 0), \quad (5.10)$$

where g is a C^1 function defined in $(0, +\infty)$. The last line in condition (5.4) becomes

$$g'(t) + F_{\bar{A}}(\bar{p}_-, \bar{p}_+) = g'(t) + \bar{A} = \theta \quad \text{at } (\bar{t}, 0). \quad (5.11)$$

Let us consider w the solution of (2.19) provided by Theorem 4.1, and let us denote

$$\varphi^\varepsilon(t, x) = \begin{cases} g(t) + w^\varepsilon(x) & \text{on } \mathcal{Q}_{2\bar{r}, 2\bar{r}}(\bar{t}, 0), \\ \varphi(t, x) & \text{outside } \mathcal{Q}_{2\bar{r}, 2\bar{r}}(\bar{t}, 0). \end{cases} \quad (5.12)$$

We would like to prove that this function satisfies in the viscosity sense, for \bar{r} and ε small enough,

$$\varphi^\varepsilon(t, x) + \tilde{M}^\varepsilon \left[\frac{\varphi^\varepsilon}{\varepsilon}(t, \cdot) \right] (x) \cdot \phi \left(\frac{x}{\varepsilon} \right) \cdot |\varphi^\varepsilon| \geq \frac{\theta}{2} \quad \text{on } \mathcal{Q}_{\bar{r}, \bar{r}}(\bar{t}, 0).$$

Let h be a test function touching φ^ε from below at $(t_1, x_1) \in \mathcal{Q}_{\bar{r}, \bar{r}}(\bar{t}, 0)$, so we have

$$w \left(\frac{x_1}{\varepsilon} \right) = \frac{1}{\varepsilon} (h(t_1, x_1) - g(t_1)),$$

and

$$w(y) \geq \frac{1}{\varepsilon} (h(t_1, \varepsilon y) - g(t_1)),$$

for y in a neighbourhood of $\frac{x_1}{\varepsilon}$. Since w does not depend on time, we have

$$h_t(t_1, x_1) = g'(t_1).$$

Therefore, we have

$$h_t(t_1, x_1) - g'(t_1) + \tilde{M} [w] \left(\frac{x_1}{\varepsilon} \right) \cdot \phi \left(\frac{x_1}{\varepsilon} \right) \cdot |h_x(t_1, x_1)| \geq \bar{A}.$$

This implies that (using (5.11) and taking \bar{r} small enough)

$$h_t(t_1, x_1) + \tilde{M} [w] \left(\frac{x_1}{\varepsilon} \right) \cdot \phi \left(\frac{x_1}{\varepsilon} \right) \cdot |h_x(t_1, x_1)| \geq \bar{A} + g'(t_1) \geq \frac{\theta}{2}.$$

Now for ε small enough such that $\varepsilon h_{max} \leq \bar{r}$, we deduce using the fact that \tilde{M} is a non-local operator with a bounded support that we have

$$\tilde{M} [w] \left(\frac{x_1}{\varepsilon} \right) = \tilde{M}^\varepsilon \left[\frac{\varphi^\varepsilon(t_1, \cdot)}{\varepsilon} \right] (x_1).$$

This implies that we have

$$h_t(t_1, x_1) + \tilde{M}^\varepsilon \left[\frac{\varphi^\varepsilon(t_1, \cdot)}{\varepsilon} \right] (x_1) \cdot \phi \left(\frac{x_1}{\varepsilon} \right) \cdot |h_x(t_1, x_1)| \geq \frac{\theta}{2}.$$

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Getting the contradiction. We have that for ε small enough

$$u^\varepsilon + \eta \leq \varphi = g(t) + \bar{p}_- x 1_{\{x < 0\}} + \bar{p}_+ x 1_{\{x > 0\}} \quad \text{on } \mathcal{Q}_{2\bar{r}, 2\bar{r}}(\bar{t}, 0) \setminus \mathcal{Q}_{\bar{r}, \bar{r}}(\bar{t}, 0).$$

Using the fact that $w^\varepsilon \rightarrow W$, and using (4.5), we have for ε small enough

$$u^\varepsilon + \frac{\eta}{2} \leq \varphi^\varepsilon \quad \text{on } \mathcal{Q}_{2\bar{r}, 2\bar{r}}(\bar{t}, 0) \setminus \mathcal{Q}_{\bar{r}, \bar{r}}(\bar{t}, 0).$$

Combining this with (5.12), we get that

$$u^\varepsilon + \frac{\eta}{2} \leq \varphi^\varepsilon \quad \text{outside } \mathcal{Q}_{\bar{r}, \bar{r}}(\bar{t}, 0),$$

By the comparison principle on bounded subsets the previous inequality holds in $\mathcal{Q}_{\bar{r}, \bar{r}}(\bar{t}, 0)$. Passing to the limit as $\varepsilon \rightarrow 0$ and evaluating the inequality in $(\bar{t}, 0)$, we obtain

$$\bar{u}(\bar{t}, 0) + \frac{\eta}{2} \leq \varphi(\bar{t}, 0) = \bar{u}(\bar{t}, 0),$$

which is a contradiction. □

6 Truncated cell problems

This section contains the proof of Theorem 4.1. To do this, we will construct correctors on truncated domains and then pass to the limit as the size of the domain goes to infinity. This idea comes from [AT15] and [GIM15]. For $l \in (r, +\infty)$, $r \ll l$ and $r \leq R \ll l$, we want to find $\lambda_{l,R}$, such that there exists a solution $w^{l,R}$ of

$$\begin{cases} G_R(x, [w^{l,R}], w_x^{l,R}) = \lambda_{l,R} & \text{if } x \in (-l, l) \\ \bar{H}^-(w_x^{l,R}) = \lambda_{l,R} & \text{if } x = -l \\ \bar{H}^+(w_x^{l,R}) = \lambda_{l,R} & \text{if } x = l, \end{cases} \quad (6.1)$$

with

$$G_R(x, [U], q) = \psi_R(x) \phi(x) \cdot M[U](x) \cdot |q| + (1 - \psi_R(x)) \cdot \bar{H}(q), \quad (6.2)$$

and $\psi_R \in C^\infty$, $\psi_R : \mathbb{R} \rightarrow [0, 1]$, with

$$\psi_R \equiv \begin{cases} 1 & \text{on } [-R, R] \\ 0 & \text{outside } [-R - 1, R + 1], \end{cases} \quad \text{and } \psi_R(x) < 1 \quad \forall x \notin [-R, R]. \quad (6.3)$$

To G_R , we associate \tilde{G}_R which is defined in the same way but the operator M is replaced by \tilde{M} .

Remark 6.1. *The operator G_R is used to have a local operator near the boundary and then to well define the boundary conditions.*

6.1 Comparison principle for a truncated problem

Proposition 6.2 (Comparison principle on truncated domains). *Let us consider the following problem for $r < l_1 < l_2$ and $\lambda \in \mathbb{R}$, with and $l_2 \gg R$.*

$$\begin{cases} \tilde{G}_R(x, [v], v_x) \geq \lambda & \text{for } x \in (l_1, l_2) \\ \overline{H}^+(v_x) \geq \lambda & \text{for } x = l_2, \end{cases} \quad (6.4)$$

and for $\varepsilon_0 > 0$

$$\begin{cases} G_R(x, [u], u_x) \leq \lambda - \varepsilon_0 & \text{for } x \in (l_1, l_2) \\ \overline{H}^+(u_x) \leq \lambda - \varepsilon_0 & \text{for } x = l_2, \end{cases} \quad (6.5)$$

Then if $u(l_1) \leq v(l_1)$ we have $u \leq v$ in $[l_1, l_2]$.

Proof. The only difficulty in proving this result is the comparison at the boundary $\{l_2\}$. However, for x close to l_2 , the function G_R is actually the effective Hamiltonian \overline{H} . Therefore, we can proceed as in the proof of [GIM15, Proposition 4.1] and so we skip the proof. \square

Remark 6.3. *We have a similar result for $l_1 < l_2 < -r$ and if for all $x \in [l_2, l_2 + h_{max}]$, $u(x) \leq v(x)$ and the following conditions are imposed at $x = l_1$:*

$$\begin{cases} \overline{H}^-(v_x) \geq \lambda & \text{for } x = l_1, \\ \overline{H}^-(u_x) \leq \lambda - \varepsilon_0 & \text{for } x = l_1. \end{cases}$$

6.2 Existence of correctors on a truncated domain

Proposition 6.4 (Existence of correctors on truncated domains). *There exists a unique $\lambda_{l,R} \in \mathbb{R}$ such that there exists a solutions $w^{l,R}$ of (6.1). Moreover, there exists a constant C (depending only on k_0), and a Lipschitz continuous function $m^{l,R}$, such that*

$$\begin{cases} H_0 \leq \lambda_{l,R} \leq 0, \\ |m^{l,R}(x) - m^{l,R}(y)| \leq C|x - y| & \text{for } x, y \in [-l, l], \\ |w^{l,R}(x) - m^{l,R}(x)| \leq C & \text{for } x \in [-l, l], \end{cases} \quad (6.6)$$

with $H_0 = \min \overline{H}$.

Proof. Given that G_R does not depend explicitly on the time variable, we will classically consider the approximated problem

$$\begin{cases} \delta v^\delta + \psi_R(x)M[v^\delta](x) \cdot \phi(x) \cdot |v_x^\delta| + (1 - \psi_R(x))\overline{H}(v_x^\delta) = 0 & \text{for } x \in (-l, l) \\ \delta v^\delta + \overline{H}^-(v_x^\delta) = 0 & \text{for } x = -l \\ \delta v^\delta + \overline{H}^+(v_x^\delta) = 0 & \text{for } x = l \end{cases} \quad (6.7)$$

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Step 1: construction of barriers. Using that 0 and $\delta^{-1}C_0$ are respectively sub and super-solution of (6.7) with $C_0 = |H_0|$, and that we have a comparison principle, we deduce that there exists a continuous viscosity solution, v^δ of (6.7) which satisfies

$$0 \leq v^\delta \leq \frac{C_0}{\delta}. \quad (6.8)$$

Step 2: control of the space oscillations of v^δ .

Lemma 6.5. *The function v^δ satisfies for all $x, y \in [-l, l]$, $x \geq y$,*

$$-k_0(x - y) - 1 \leq v^\delta(x) - v^\delta(y) \leq 0,$$

with k_0 defined in (A0).

Proof of Lemma 6.5. In the rest of the proof we will use the following notation,

$$\Omega = \{(x, y) \in [-l, l]^2 \text{ such that } x \geq y\}.$$

Step 2.1: proof of the upper inequality. We want to prove that

$$M = \sup_{(x,y) \in \Omega} \{v^\delta(x) - v^\delta(y)\} \leq 0.$$

We argue by contradiction and assume that $M > 0$. We can see that M is reached for a finite point that we denote by $(\bar{x}, \bar{y}) \in \Omega$. Given that $M > 0$, we deduce that $\bar{x} \neq \bar{y}$. Therefore, we can use the viscosity inequalities for (6.7).

-If $(\bar{x}, \bar{y}) \in (-l, l)$, we have

$$\begin{aligned} \delta v^\delta(\bar{x}) + G_R(\bar{x}, [v^\delta], 0) &\leq 0 \\ \delta v^\delta(\bar{y}) + G_R(\bar{y}, [v^\delta], 0) &\geq 0, \end{aligned}$$

combining these two inequalities with the fact that $G_R(x, [U], 0) = 0$, we obtain

$$\delta M \leq 0.$$

-If $\bar{x} = l$ and $\bar{y} \in (-l, l)$, similarly we obtain

$$\delta M \leq 0,$$

where we have used the fact that $\bar{H}^+(0) = 0$.

-If $\bar{x} \in (-l, l)$ and $\bar{y} = -l$, we obtain

$$\delta M \leq H_0 \leq 0,$$

where we used the fact that $\bar{H}^-(0) = H_0$.

-If $\bar{x} = l$ and $\bar{y} = -l$, we obtain

$$\delta M \leq H_0 \leq 0.$$

For every value of \bar{x} and \bar{y} we obtain a contradiction, therefore we have $M \leq 0$.

Step 2.2: proof of the lower inequality. We want to prove that

$$M = \sup_{(x,y) \in \Omega} \left\{ v^\delta(y) - v^\delta(x) - k_0(x - y) - 1 \right\} \leq 0.$$

We argue by contradiction and assume that $M > 0$. We can see that M is reached for a finite point that we denote by (\bar{x}, \bar{y}) . Since $M > 0$, we deduce that $\bar{x} \neq \bar{y}$. Therefore, we can use the viscosity inequalities for (6.7).

Case 1: $\bar{y} \in (-l, l)$. If $\bar{y} \in (-l, l)$, we have

$$\delta v^\delta(\bar{y}) + \psi_R(\bar{y})M[v^\delta](\bar{y}) \cdot \phi(\bar{y}) \cdot |-k_0| + (1 - \psi_R(\bar{y}))\bar{H}(-k_0) \leq 0. \quad (6.9)$$

We claim that $M[v^\delta](\bar{y}) = 0$.

Indeed, for all $z > h_0$, if $\bar{x} > \bar{y} + z$ using the fact that the maximum is reached for (\bar{x}, \bar{y}) , we deduce that

$$v^\delta(\bar{y} + z) - v^\delta(\bar{x}) - k_0(\bar{x} - \bar{y} - z) - 1 \leq v^\delta(\bar{y}) - v^\delta(\bar{x}) - k_0(\bar{x} - \bar{y}) - 1$$

which implies that

$$v^\delta(\bar{y} + z) - v^\delta(\bar{y}) \leq -k_0z < -1.$$

On the contrary, if $\bar{x} \leq \bar{y} + z$, using the fact that v^δ is non-increasing in space, we have

$$v^\delta(\bar{y} + z) - v^\delta(\bar{y}) \leq v^\delta(\bar{x}) - v^\delta(\bar{y}) \leq -k_0(\bar{x} - \bar{y}) - 1 < -1.$$

We can therefore, conclude that for all $z \in (h_0, +\infty)$, $E(v^\delta(\bar{y} + z) - v^\delta(\bar{y})) = -\frac{3}{2}$ and so we get $M[v^\delta](\bar{y}) = 0$. Using also that $\bar{H}(-k_0) = 0$, equation (6.9) becomes

$$\delta v^\delta(\bar{y}) \leq 0.$$

However, using the fact that $v^\delta \geq 0$ (see (6.8)), we get

$$\delta M \leq \delta v^\delta(\bar{y}) - \delta v^\delta(\bar{x}) \leq 0,$$

which is a contradiction.

Case 2: $\bar{y} = -l$. In this situation, the viscosity inequality becomes

$$\delta v^\delta(\bar{y}) + \bar{H}^-(-k_0) \leq 0.$$

Using the fact that $\bar{H}^-(-k_0) = \bar{H}(-k_0) = 0$, we obtain

$$\delta v^\delta(\bar{y}) \leq 0,$$

and as in the previous case, we obtain a contradiction. This ends the proof of the lemma. \square

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Step 3: construction of a Lipschitz estimate.

Lemma 6.6. *There exists a Lipschitz continuous function m^δ , such that there exists a constant C , (independent of l, R and δ) such that*

$$\begin{cases} |m^\delta(x) - m^\delta(y)| \leq C|x - y| & \text{for all } x, y \in [-l, l], \\ |v^\delta(x) - m^\delta(x)| \leq C & \text{for all } x \in [-l, l]. \end{cases} \quad (6.10)$$

Proof of Lemma 6.6. Let us define m^δ as an affine function in each interval of the form $[ih_0, (i+1)h_0]$, with $i \in \mathbb{Z}$, such that

$$m^\delta(ih_0) = v^\delta(ih_0) \quad \text{and} \quad m^\delta((i+1)h_0) = v^\delta((i+1)h_0).$$

Since m^δ, v^δ are non-increasing and $|v^\delta((i+1)h_0) - v^\delta(ih_0)| \leq k_0h_0 + 1 = 2$, we deduce that $\forall x \in [ih_0, (i+1)h_0]$,

$$-2 \leq v^\delta((i+1)h_0) - m^\delta(ih_0) \leq v^\delta(x) - m^\delta(x) \leq v^\delta(ih_0) - m^\delta((i+1)h_0) \leq 2,$$

and for all $x, y \in [-l, l]$,

$$|m^\delta(x) - m^\delta(y)| \leq 2k_0|x - y|.$$

□

Step 4: passing to the limit as δ goes to 0. Using (6.8) and (6.10), we deduce that there exists $\delta_n \rightarrow 0$ such that

$$\begin{aligned} \delta_n v^{\delta_n}(0) &\rightarrow -\lambda_{l,R} & \text{as } n \rightarrow +\infty, \\ m^{\delta_n} - m^{\delta_n}(0) &\rightarrow m^{l,R} & \text{as } n \rightarrow +\infty, \end{aligned}$$

the second convergence being locally uniform. Let us consider,

$$\overline{w}^{l,R}(t, x) = \limsup_{\delta_n \rightarrow 0}^* (v^{\delta_n} - v^{\delta_n}(0)) \quad \text{and} \quad \underline{w}^{l,R} = \liminf_{\delta_n \rightarrow 0} (v^{\delta_n} - v^{\delta_n}(0)).$$

Therefore, we have that $\lambda_{l,R}, m^{l,R}, \overline{w}^{l,R}$ and $\underline{w}^{l,R}$ satisfy

$$\begin{aligned} H_0 &\leq \lambda_{l,R} \leq 0, \\ |\overline{w}^{l,R} - m^{l,R}| &\leq C, \\ |\underline{w}^{l,R} - m^{l,R}| &\leq C, \\ |m_x^{l,R}| &\leq C. \end{aligned} \quad (6.11)$$

By stability of the solutions we have that $\overline{w}^{l,R} - 2C$ and $\underline{w}^{l,R}$ are respectively a sub-solution and a super-solution of (6.1) and

$$\overline{w}^{l,R} - 2C \leq \underline{w}^{l,R}.$$

By Perron's method we can construct a solution $w^{l,R}$ of (6.1) and thanks to (6.8) and (6.11), $m^{l,R}$, $\lambda_{l,R}$ and $w^{l,R}$ satisfy (6.6).

The uniqueness of $\lambda_{l,R}$ is classical so we skip it. This ends the proof of Proposition 6.4. □

Proposition 6.7 (First definition of the flux limiter). *The following limits exist (up to a subsequence)*

$$\begin{cases} \bar{A}_R = \lim_{l \rightarrow +\infty} \lambda_{l,R} \\ \bar{A} = \lim_{R \rightarrow +\infty} \bar{A}_R. \end{cases} \quad (6.12)$$

Moreover, we have

$$H_0 \leq \bar{A}_R, \bar{A} \leq 0.$$

Proof. This results comes from the fact that we have the following bound on $\lambda_{l,R}$ which is independent of l and R (see Proposition 6.4),

$$H_0 \leq \lambda_{l,R} \leq 0.$$

□

Remark 6.8. *This proposition does not ensure the uniqueness of the flux limiter \bar{A} . However, since we know that such a limit exists, we can obtain the converge result. The uniqueness of \bar{A} is given in Theorem 2.11.*

Proposition 6.9 (Control of the slopes on a truncated domain). *Assume that l and R are big enough. Let $w^{l,R}$ be the solution of (6.1) given by Proposition 6.4. We also assume that up to a sub-sequence $\bar{A} = \lim_{R \rightarrow +\infty} \lim_{l \rightarrow +\infty} \lambda_{l,R} > H_0$. Then there exists a $\gamma_0 > 0$ such that for all $\gamma \in (0, \gamma_0)$, there exists a constant C (independent of l and R) such that for all $x \geq r$ and $h \geq 0$*

$$w^{l,R}(x+h) - w^{l,R}(x) \geq (\bar{p}_+ - \gamma)h - C. \quad (6.13)$$

Similarly, for all $x \leq -r$ and $h \geq 0$,

$$w^{l,R}(x-h) - w^{l,R}(x) \geq (-\bar{p}_- - \gamma)h - C. \quad (6.14)$$

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Proof. We only prove (6.13) since the proof for (6.14) is similar. For $\mu > 0$ small enough, we denote by p_+^μ the real number such that

$$\overline{H}(p_+^\mu) = \overline{H}^+(p_+^\mu) = \lambda_{l,R} - \mu.$$

Using that

$$H_0 < \lambda_{l,R} \leq 0,$$

we deduce that p_+^μ exists, is unique and satisfies $-k_0 \leq p_+^\mu \leq 0$ for μ small enough.

Let us now consider the function $w^+ = p_+^\mu x$ that satisfies

$$\overline{H}(w_x^+) = \lambda_{l,R} - \mu \quad \text{for } x \in \mathbb{R}.$$

We also have

$$\begin{aligned} M[w^+](x) &= \int_{\mathbb{R}} J(z) E(p_+^\mu(x+z) - p_+^\mu x) dz - \frac{3}{2} V_{max} \\ &= \int_0^{\frac{-1}{p_+^\mu}} \frac{1}{2} J(z) dz + \int_{\frac{-1}{p_+^\mu}}^{+\infty} \frac{3}{2} J(z) dz - \frac{3}{2} V_{max} \\ &= -V \left(\frac{-1}{p_+^\mu} \right). \end{aligned}$$

For all $x \in (r, l)$, using that $\phi(x) = 1$, we deduce that

$$M[w^+](x) \cdot \phi(x) \cdot |w_x^+| = -V \left(\frac{-1}{p_+^\mu} \right) \cdot |p_+^\mu| = \overline{H}(p_+^\mu) = \lambda_{l,R} - \mu,$$

and so the restriction of w^+ to $(r, l]$ satisfies

$$\begin{cases} G_R(x, [w^+], w_x^+) = \lambda_{l,R} - \mu & \text{for } x \in (r, l) \\ \overline{H}^+(w_x^+) = \lambda_{l,R} - \mu & \text{for } x = l. \end{cases}$$

Let us denote by $g = w^{l,R} - w^{l,R}(x_0)$ and $u = w^+ - w^+(x_0) - 2C$, for some $x_0 \in (r, l)$ and C defined as in Proposition 6.4. Then we have

$$g(x_0) = 0 \geq -2C = u(x_0).$$

Using that g is a solution of (6.4) (with $\varepsilon_0 = \mu$) and u is a solution of (6.5) joint to the comparison principle (Proposition 6.2) we get that

$$w^{l,R}(x) - w^{l,R}(x_0) = g(x) \geq u(x) = p_+^\mu(x - x_0) - 2C.$$

This implies that for all $h \geq 0$ and for all $x \in (r, l)$,

$$w^{l,R}(x+h) - w^{l,R}(x) \geq p_+^\mu h - 2C.$$

Finally, if we choose $\gamma_0 < |p_0 - \bar{p}_+|$ (with p_0 defined in (2.12)), then

$$\bar{H}(\bar{p}_+ - \gamma) = \bar{H}^+(\bar{p}_+ - \gamma),$$

and we can choose $\mu > 0$ such that

$$p_+^\mu = \bar{p}_+ - \gamma.$$

This implies inequality (6.13). □

Proof of Theorem 4.1. The proof is performed two steps.

Step 1: proof of i) and ii). The goal is to pass to the limit as $l \rightarrow +\infty$ and then as $R \rightarrow +\infty$. Using Proposition 6.4, there exists $l_n \rightarrow +\infty$, such that

$$m^{l_n,R} - m^{l_n,R}(0) \rightarrow m^R \quad \text{as } n \rightarrow +\infty,$$

the convergence being locally uniform. We also define

$$\begin{aligned} \bar{w}^R(x) &= \limsup_{l_n \rightarrow +\infty}^* (w^{l_n,R} - w^{l_n,R}(0)), \\ \underline{w}^R(x) &= \liminf_{l_n \rightarrow +\infty_*} (w^{l_n,R} - w^{l_n,R}(0)). \end{aligned}$$

Thanks to (6.6), we know that \bar{w}^R and \underline{w}^R are finite and satisfy

$$m^R - C \leq \underline{w}^R \leq \bar{w}^R \leq m^R + C.$$

By stability of viscosity solutions, $\bar{w}^R - 2C$ and \underline{w}^R are respectively a sub and a super-solution of

$$G_R(x, [w^R], w_x^R) = \bar{A}_R \quad \text{for } x \in \mathbb{R} \tag{6.15}$$

Therefore, using Perron's method, we can construct a solution w^R of (6.15) with m^R, \bar{A}^R and w^R satisfying

$$\begin{cases} |m^R(x) - m^R(y)| \leq C|x - y| & \text{for all } x, y \in \mathbb{R}, \\ |w^R(x) - m^R(x)| \leq C & \text{for } x \in \mathbb{R} \times \mathbb{R}, \\ H_0 \leq \bar{A}_R \leq 0. \end{cases} \tag{6.16}$$

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Using Proposition 6.9, if $\bar{A} > H_0$, we know that there exists a γ_0 and a constant C , such that for all $\gamma \in (0, \gamma_0)$,

$$\begin{cases} w^R(x+h) - w^R(x) \geq (\bar{p}_+ - \gamma)h - C & \text{for all } x \geq r, h \geq 0, \\ w^R(x-h) - w^R(x) \geq (-\bar{p}_- - \gamma)h - C & \text{for all } x \leq -r, h \geq 0. \end{cases} \quad (6.17)$$

We now pass to the limit as $R \rightarrow +\infty$. We consider (up to some subsequence)

$$\begin{cases} \bar{w}(x) = \limsup_{R \rightarrow +\infty}^* (w^R - w^R(0)), \\ \underline{w}(x) = \liminf_{R \rightarrow +\infty} (w^R - w^R(0)), \\ \bar{A} = \lim_{R \rightarrow +\infty} \bar{A}_R, \\ m = \lim_{R \rightarrow +\infty} (m^R - m^R(0)). \end{cases}$$

The last convergence being locally uniform. Thanks to (6.16), we know that \bar{w} and \underline{w} are finite and satisfy

$$m - C \leq \underline{w} \leq \bar{w} \leq m + C.$$

By stability of viscosity solutions, $\bar{w} - 2C$ and \underline{w} are respectively a sub and a super-solution of (2.19) with $\lambda = \bar{A}$. Using Perron's method, we can then construct a solution w of (2.19) with $\lambda = \bar{A}$ that satisfies (4.2) and (4.3).

Step 2: proof of iii). We are now interested in the rescaled function $w^\varepsilon(x) = \varepsilon w\left(\frac{x}{\varepsilon}\right)$.

Using (4.3), we have that

$$w^\varepsilon(x) = \varepsilon m \left(\frac{x}{\varepsilon}\right) + O(\varepsilon).$$

Therefore, we can find a sequence $\varepsilon_n \rightarrow 0$, such that

$$w^{\varepsilon_n} \rightarrow W \quad \text{locally uniformly as } n \rightarrow +\infty,$$

with $W(0) = 0$. Like in [IM14], arguing as in the proof of convergence away from the junction point, we have that W satisfies

$$\bar{H}(W_x) = \bar{A} \quad \text{for } x \neq 0.$$

For all $\gamma \in (0, \gamma_0)$, we have that if $\bar{A} > H_0$ and $x > 0$,

$$W_x \geq \bar{p}_+ - \gamma,$$

where we have used (4.3). Therefore we get

$$W_x = \bar{p}_+ \quad \text{for } x > 0,$$

this result remains valid even if $\bar{A} = H_0$ (in this particular case $W_x = p_0$). Similarly, we get

$$W_x = \bar{p}_- \quad \text{for } x < 0.$$

which implies (4.4) and (4.5). This ends the proof of Theorem 4.1. □

6.3 Proof of Theorem 2.11

Proof of Theorem 2.11. Up to a sub-sequence, we assume that $\bar{A} = \lim_{R \rightarrow +\infty} \lim_{l \rightarrow +\infty} \lambda_{l,R}$. We want to prove that $\bar{A} = \inf E$, where

$$E = \{\lambda \in [h_0, 0] : \exists w \in \mathcal{S} \text{ solution of (2.19)}\},$$

with

$$\mathcal{S} = \{w \text{ s.t. } \exists \text{ a Lipschitz continuous function } m \quad \text{and } C \geq 0 \text{ such that } \|w - m\|_{L^\infty(\mathbb{R})} \leq C\}.$$

We argue by contradiction and assume that there exists a $\lambda < \bar{A}$ and a function $w^\lambda \in \mathcal{S}$ solution of (2.19). We assume that $w^\lambda(0) = 0$ (if we are not in this situation, we do a translation since we have $w^\lambda - w^\lambda(0) \in \mathcal{S}$). Arguing as in the proof of Theorem 4.1, we deduce that the function

$$w_\lambda^\varepsilon(x) = \varepsilon w^\lambda\left(\frac{x}{\varepsilon}\right)$$

has a limit W^λ (with $W^\lambda(0) = 0$) which satisfies

$$\bar{H}(W_x^\lambda) = \lambda \quad \text{for } x > 0,$$

which means that for all $x > 0$,

$$W_x^\lambda \leq p_+^\lambda < \bar{p}_+ \quad \text{with } \bar{H}(p_+^\lambda) = \bar{H}^+(p_+^\lambda) = \lambda. \tag{6.18}$$

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Similarly we have for all $x < 0$,

$$W_x^\lambda \geq p_-^\lambda > \bar{p}_- \quad \text{with } \bar{H}(p_-^\lambda) = \bar{H}^-(p_-^\lambda) = \lambda. \quad (6.19)$$

These inequalities imply that for all $\gamma > 0$, there exists a constant $\tilde{C}_\gamma > 0$ such that

$$w^\lambda(x) \leq \begin{cases} (p_+^\lambda + \gamma)x + \tilde{C}_\gamma & \text{for } x > 0, \\ (p_-^\lambda - \gamma)x + \tilde{C}_\gamma & \text{for } x < 0, \end{cases} \quad (6.20)$$

In fact, if w^λ does not satisfies (6.20), we cannot have (6.18) and (6.19). Using Theorem 4.1, we get

$$w^\lambda < w \quad \text{for } |x| \geq \tilde{R}$$

if γ is small enough and \tilde{R} big enough. This implies that there exists a constant $C_{\tilde{R}} > 0$ such that for all $x \in \mathbb{R}$, we have

$$w^\lambda(x) < w(x) + C_{\tilde{R}}.$$

Let us now introduce, $u(t, x) = w(x) + C_{\tilde{R}} - \bar{A}t$ and $u_\lambda(t, x) = w^\lambda(x) - \lambda t$ both solutions of (2.4) with $\varepsilon = 1$ and $u_\lambda(0, x) \leq u(0, x)$. Therefore, the comparison principle implies

$$w^\lambda(x) - \lambda t \leq w(x) + C_{\tilde{R}} - \bar{A}t$$

Dividing by t and passing to the limit as t goes to infinity, we get

$$\bar{A} \leq \lambda,$$

which is a contradiction. □

7 Qualitative properties of the flux limiter

This section is devoted to the proof of Proposition 2.13.

Proof of Proposition 2.13. We perform the proof of each item separately.

Proof of (i). In order to establish the monotonicity, we have to consider the approximated truncated cell problem (6.7). Let us consider v_1^δ and v_2^δ viscosity solutions of (6.7), respectively for ϕ_1 and ϕ_2 , with $0 \leq \phi_1 \leq \phi_2$. First, using the fact that the non-local operator is negative, we have

$$G_R^2(x, [U], q) \leq G_R^1(x, [U], q),$$

with

$$G_R^i(x, [U], q) = \phi_i(x) \cdot M[U](x) \cdot \psi_R(x) \cdot |q| + (1 - \psi_R(x))\overline{H}(q), \quad \text{for } i = 1, 2.$$

Therefore, we have

$$0 = \delta v_1^\delta + G_R^1(x, [v_1^\delta], (v_1^\delta)_x) \geq \delta v_1^\delta + G_R^2(x, [v_1^\delta], (v_1^\delta)_x),$$

meaning that v_1^δ is a sub-solution of (6.7) with ϕ_2 . The comparison principle and (6.8) imply that

$$0 \leq \delta v_1^\delta \leq \delta v_2^\delta \leq |H_0|.$$

Passing to the limit as $\delta \rightarrow 0$, we obtain

$$0 \geq \lambda_{l,R}^1 \geq \lambda_{l,R}^2 \geq H_0.$$

Passing to the limit as $l, R \rightarrow +\infty$, we get the result.

Proof of (ii). If $\phi = 0$ on an open interval, then using [AHFM13, Lemme B.1], we can use the definition of a viscosity solution of (6.7) at a point where $\phi = 0$ and therefore, we have

$$\overline{A} = 0.$$

□

8 Link between the system of ODEs and the PDE

This section is devoted to the proof of Theorem 2.5, which is a direct application of our convergence result, Theorem 2.4.

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Theorem 8.1. For $\varepsilon = 1$, the cumulative distribution function ρ defined by (2.2) is a discontinuous viscosity solution of

$$\rho_t + M[\rho(t, \cdot)](x) \cdot \phi(x) \cdot |\rho_x| = 0 \quad \text{for } (t, x) \in [0, +\infty) \times \mathbb{R}. \quad (8.1)$$

Conversely, if u is a bounded and continuous viscosity solution of (8.1) satisfying for some time $T > 0$, and for all $t \in (0, T)$

$$u(t, x) \text{ is decreasing in } x,$$

then the points $U_j(t)$, defined by $u(t, U_j(t)) = -(j+1)$ for $j \in \mathbb{Z}$, satisfy the system (2.1) on $(0, T)$.

Before giving the proof of Theorem 8.1, let us do the proof of Theorem 2.5.

Proof of Theorem 2.5. We recall that in Theorem 2.5, we have $u_0(x) = -x/h$, with $h \geq h_0$. First, we would like to prove that for all $\varepsilon > 0$, we have

$$|\rho^\varepsilon(0, x) - u_0(x)| \leq f(\varepsilon) \quad \text{for all } x \in \mathbb{R}, \quad (8.2)$$

with $f(\varepsilon) \rightarrow 0$ as ε goes to 0. To do this, we define a piece-wise affine function v satisfying

$$\rho^1(0, x) = v(x) \quad \text{for } x = U_i(0), \text{ for all } i \in \mathbb{Z}.$$

Given that for all $U_{i+1}(0) - U_i(0) \geq h_0$, we notice that v is k_0 -Lipschitz continuous and by definition of $\rho^1(0, x)$, we have

$$|\rho^1(0, x) - v(x)| \leq 1 \quad \text{for all } x \in \mathbb{R}.$$

Let us consider the integer $i_0 \in \mathbb{N}$ defined by

$$i_0 = \sup \{i \in \mathbb{Z}, \text{ s.t. } U_i(0) \leq -R\}.$$

Using the assumption that for all $i \in \mathbb{Z}$ such that $U_i(0) \leq -R$ we have $U_{i+1}(0) - U_i(0) = h$, we deduce that for all $x \leq U_{i_0}(0)$

$$v(x) = -\frac{x}{h} + \frac{U_{i_0}(0)}{h} + \rho^1(0, U_{i_0}(0)) = -\frac{x}{h} + \frac{U_{i_0}(0)}{h} - i_0 - 1.$$

Let us now consider the integer $i_1 \in \mathbb{N}$ defined by

$$i_1 = \inf \{i \in \mathbb{Z}, \text{ s.t. } U_i(0) \geq R\}.$$

Now using the assumption that for all $i \in \mathbb{Z}$ such that $U_i(0) \geq R$ we have $U_{i+1}(0) - U_i(0) = h$, we deduce that for all $x \geq U_{i_1}(0)$

$$v(x) = -\frac{x}{h} + \frac{U_{i_1}(0)}{h} + \rho^1(0, U_{i_1}(0)) = -\frac{x}{h} + \frac{U_{i_1}(0)}{h} - i_1 - 1.$$

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Moreover, we recall that for all $\varepsilon > 0$, we have $\rho^\varepsilon(0, x) = \varepsilon\rho^1(0, x/\varepsilon)$, this implies that for all $x \notin [\varepsilon U_{i_0}(0), \varepsilon U_{i_1}(0)]$,

$$\begin{aligned} |\rho^\varepsilon(0, x) - u_0(x)| &\leq \left| \rho^\varepsilon(0, x) - \varepsilon v\left(\frac{x}{\varepsilon}\right) \right| + \left| \varepsilon v\left(\frac{x}{\varepsilon}\right) - u_0(x) \right| \\ &\leq \varepsilon + \varepsilon \max \left(\left| \frac{U_{i_1}(0)}{h} - i_1 - 1 \right|, \left| \frac{U_{i_0}(0)}{h} - i_0 - 1 \right| \right). \end{aligned} \quad (8.3)$$

Similarly, we have for all $x \in [\varepsilon U_{i_0}(0), \varepsilon U_{i_1}(0)]$,

$$\begin{aligned} |\rho^\varepsilon(0, x) - u_0(x)| &\leq \left| \rho^\varepsilon(0, x) - \varepsilon v\left(\frac{x}{\varepsilon}\right) \right| + \left| \varepsilon v\left(\frac{x}{\varepsilon}\right) - \varepsilon u_0\left(\frac{x}{\varepsilon}\right) \right| \\ &\leq \varepsilon + \varepsilon \max_{y \in [U_{i_0}(0), U_{i_1}(0)]} (|v(y) - u_0(y)|), \end{aligned} \quad (8.4)$$

where we have used the fact that $\varepsilon u_0(x/\varepsilon) = u_0(x)$. Combining (8.3) and (8.4) and choosing

$$\begin{aligned} f(\varepsilon) = \varepsilon + \varepsilon \max \left(\left| \frac{U_{i_0}(0)}{h} - i_0 - 1 \right|, \right. \\ \left. \max_{y \in [U_{i_0}(0), U_{i_1}(0)]} (|v(y) - u_0(y)|), \left| \frac{U_{i_1}(0)}{h} - i_1 - 1 \right| \right) \end{aligned}$$

we deduce (8.2). Notice also that thanks to (8.2), we have

$$|(\rho^\varepsilon)^*(0, x) - u_0(x)| \leq f(\varepsilon) + \varepsilon. \quad (8.5)$$

Therefore, we have

$$u_0(x) - f(\varepsilon) \leq \rho^\varepsilon(0, x) \leq (\rho^\varepsilon)^*(0, x) \leq u_0(x) + f(\varepsilon) + \varepsilon.$$

Using the fact that ρ^ε is a viscosity solution of (2.4) and the comparison principle (Proposition 3.7) we deduce that (with u^ε the continuous solution of (2.4) associated to the initial condition $u_0(x) = -x/h$)

$$u^\varepsilon(t, x) - f(\varepsilon) \leq \rho^\varepsilon(t, x) \leq (\rho^\varepsilon)^*(t, x) \leq u^\varepsilon(t, x) + f(\varepsilon) + \varepsilon,$$

where we have used the fact that (2.4) is invariant by addition of constants to the solutions. Passing to the limit as $\varepsilon \rightarrow 0$ and using Theorem 2.4 we get that $\rho^\varepsilon \rightarrow u^0$, which ends the proof of Theorem 2.5. □

Proof of Theorem 8.1. Theorem 8.1 is a consequence of the following lemma.

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Lemma 8.2 (Link between the velocities). *Assume (A). Let $((U_j)_j)$ be the solution of (2.1) with*

$$U_{j+1}(0) - U_j(0) > h_0. \quad (8.6)$$

Then we have

$$\dot{U}_j(t) = -M[u(t, \cdot)](U_j(t)) \cdot \phi(U_j(t)), \quad (8.7)$$

where E and J are defined in (2.6) and $u(t, x)$ is a continuous function such that

$$\begin{cases} u(t, x) = \rho_*(t, x) = \rho(t, x) \text{ for } x = U_j(t), j \in \mathbb{Z}, \\ u \text{ is decreasing in } x, \end{cases} \quad (8.8)$$

with ρ defined in (2.2) (with $\varepsilon = 1$).

Proof. We drop the time dependence to simplify the presentation. Let $j \in \mathbb{Z}$. Using the fact that $u(U_j) = -(j + 1)$ and (8.8), we have for all $z \in [0, +\infty)$,

$$\begin{cases} 0 \geq u(U_j + z) - u(U_j) > u(U_{j+1}) - u(U_j) = -1 & \text{if } z \in [0, U_{j+1} - U_j) \\ -1 \geq u(U_j + z) - u(U_j) & \text{if } z \in [U_{j+1} - U_j, +\infty). \end{cases}$$

Given that u is continuous, this implies that

$$M[u](U_j) = \int_0^{U_{j+1}-U_j} \frac{1}{2} J(z) dz + \int_{U_{j+1}-U_j}^{+\infty} \frac{3}{2} J(z) dz - \frac{3}{2} V_{max} = -V (U_{j+1} - U_j).$$

Combining this result with (2.1), we obtain (8.7). □

Noticing that because of (8.8), we have for $x = U_j(t)$, $j \in \mathbb{Z}$,

$$\tilde{M}[\rho_*(t, \cdot)](x) = \tilde{M}[u(t, \cdot)](x) = M[u(t, \cdot)](x),$$

and using Lemma 8.2, and Definition 3.1, we can see that ρ_* is a discontinuous viscosity super-solution of (8.1). We obtain a similar result for ρ^* , therefore, ρ is a discontinuous viscosity solution of (8.1).

We prove the converse. For the readers convenience we recall Proposition 4.8 from [FIM09b] that we will use later. The proof of this proposition remains almost the same in our case the only difference being the definition of the functions E and \tilde{E} .

Lemma 8.3. *Assume that $\theta : \mathbb{R} \rightarrow \mathbb{R}$ is a non-decreasing and upper semi-continuous (resp. lower semi-continuous). Assume also that*

$$\theta(v) - v \text{ is } 1\text{-periodic in } v.$$

Assume that $\varepsilon = 1$ in (2.4). Consider also a sub-solution (resp. a super-solution) u of (2.4). Then $\theta(u)$ is also a sub-solution (resp. a super-solution) of (2.4).

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Using Lemma 8.3 we can conclude that $\rho_* = \lceil u \rceil$ (resp. $\rho^* = \lfloor u \rfloor$) is a viscosity super-solution (resp. sub-solution) of

$$\partial_t \rho - \tilde{c}(t, x) \partial_x \rho = 0 \quad \text{with } \tilde{c}(t, x) = M[u(t, \cdot)](x) \cdot \phi(x) = \tilde{M}[u(t, \cdot)](x) \cdot \phi(x).$$

Using the fact that u is decreasing in space, we define

$$U_i(t) = \inf\{x, u(t, x) \leq -(i+1)\} = (u(t, \cdot))^{-1}(-i-1)$$

and we consider the functions $t \mapsto U_i(t)$. They are continuous because u is decreasing in x and is continuous in (t, x) .

We now prove that the functions U_i are viscosity solutions of (2.1). Let φ be a test function such that $\varphi(t) \leq U_i(t)$ and $\varphi(t_0) = U_i(t_0)$. Let us now define $\hat{\varphi}(t, x) = -(i+1) + \varphi(t) - x$. It satisfies

$$\hat{\varphi}(t_0, U_i(t_0)) = \rho_*(t_0, U_i(t_0)),$$

and

$$\hat{\varphi}(t, x) \leq \rho_*(t, x) \quad \text{for } U_i(t) - 1 < x < U_{i+1}(t).$$

This implies that

$$\begin{aligned} \varphi_t(t_0) + \tilde{c}(t_0, U_i(t_0)) &\geq 0 \\ \Leftrightarrow \varphi_t(t_0) &\geq -\tilde{c}(t_0, U_i(t_0)) = -\bar{c}_i(t_0) = V(U_{i+1}(t_0) - U_i(t_0)) \cdot \phi(U_i(t_0)). \end{aligned}$$

This proves that U_i are viscosity super-solutions of (2.1). The proof for sub-solutions is similar and we skip it. Moreover, since \bar{c}_i is continuous, we deduce that $U_i \in C^1$ and it is therefore a classical solution of (2.1). □

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Chapitre 4

Numerical specified homogenization of a discrete model with a local perturbation

W. Salazar¹

Abstract

The goal of this work is to present a numerical homogenization of a non-local PDE deriving from a first order discrete model for traffic flow that simulates the presence of a local perturbation. In a previous work, we have shown that the solution of the discrete microscopic model converges to the (unique) solution of a Hamilton-Jacobi equation posed on a network and with a junction condition (it can be seen as a flux limiter that keeps the memory of the local perturbation). The goal of this paper is to provide a numerical scheme able to provide an approximation of this flux-limiter. We show the convergence of this scheme and we provide some numerical results.

AMS Classification: 35D40, 90B20, 35B27, 35F20, 45K05, 65M06.

Keywords: numerical specified homogenization, Hamilton-Jacobi equations, viscosity solutions, traffic flow, microscopic models, macroscopic models, convergence of numerical scheme.

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1 Introduction

The problem of simulating traffic flow is very important, particularly because it allows us to know how the traffic would react to a change in the infrastructure of the road. Traffic flow can be simulated at different scales: the microscopic scale (which describes the dynamics of all the vehicles), the macroscopic scale (which describes macroscopic quantities such as the vehicle density, the average speed,...) and the mesoscopic scale (between the microscopic and the macroscopic scale). In this paper we only consider the microscopic and the macroscopic scales.

In this paper, we are interested in the numerical homogenization of a non-local PDE that derives from a microscopic first order model for traffic flow that simulates the presence of a local perturbation that does not depend on time (for instance a school zone, a moderator,...). The PDE was introduced in [2] and derives from a first order traffic flow model of the type "follow-the-leader".

In [2], the homogenization of the PDE was obtained and it turns out that the homogenized system is defined by a function call the effective Hamiltonian and by a constant call the flux limiter. The effective Hamiltonian describes the dynamics of the traffic flow and the flux-limiter will defined how the local perturbation affects the macroscopic (homogenized) model. In fact, the effective Hamiltonian has been explicitly determined, however the flux-limiter constant is only implicitly determined. In this paper, we provide a numerical scheme for the computation of an approximation of the flux-limiter.

1.1 General model: first order model with a local perturbation

For the readers convenience, we detail the microscopic model from which derives the PDE we study later in this paper. We consider the following model where all the vehicles are considered as points placed in the real line, for all $t > 0$,

$$\dot{U}_j(t) = V(U_{j+1}(t) - U_j(t)) \cdot \phi(U_j(t)), \quad (1.1)$$

where $U_j : [0, +\infty) \rightarrow \mathbb{R}$ denotes the position of the j -th vehicle and \dot{U}_j is its velocity. The function $\phi : \mathbb{R} \rightarrow [0, 1]$ simulates the presence of a local perturbation around the origin. We denote by r the radius of influence of the perturbation.

The function V is called the optimal velocity function and we make the following assumptions on V and ϕ :

Assumption (A)

- (A1) $V : \mathbb{R} \rightarrow \mathbb{R}^+$ is Lipschitz continuous, non-negative.
- (A2) V is non-decreasing on \mathbb{R} .

- (A3) There exists $h_0 \in (0, +\infty)$ such that for all $h \leq h_0$, $V(h) = 0$.
- (A4) There exists $h_{max} \in (h_0, +\infty)$ such that for all $h \geq h_{max}$, $V(h) = V(h_{max}) =: V_{max}$.
- (A5) There exists a real number $p_0 \in [-1/h_0, 0)$ such that the function $p \mapsto pV(-1/p)$ is decreasing on $[-1/h_0, p_0)$ and increasing on $[p_0, 0)$.
- (A6) The function $\phi : \mathbb{R} \rightarrow [0, 1]$ is Lipschitz continuous and $\phi(x) = 1$ for $|x| \geq r$.

Remark 1.1. *Assumptions (A1)-(A2)-(A3)-(A5) are satisfied by several classical optimal velocity functions, we have added assumption (A4) to work with V' with a bounded support. But by modifying slightly the classical optimal velocity functions, we obtain a function that satisfies all the assumptions. For instance, in the case of the Greenshields based models [GCM35](see also [BT10, Edi61, New61, GP06] for other classical optimal velocity functions):*

$$V(h) = \begin{cases} 0 & \text{for } h \leq h_0, \\ V_{max} \left(1 - \left(\frac{h_0}{h} \right)^2 \right) & \text{for } h_0 < h \leq h_{max}, \\ V_{max} \left(1 - \left(\frac{h_0}{h_{max}} \right)^2 \right) & \text{for } h > h_{max}. \end{cases}$$

In Figure 1.1 we give a schematic representation of an optimal velocity function satisfying assumption (A).

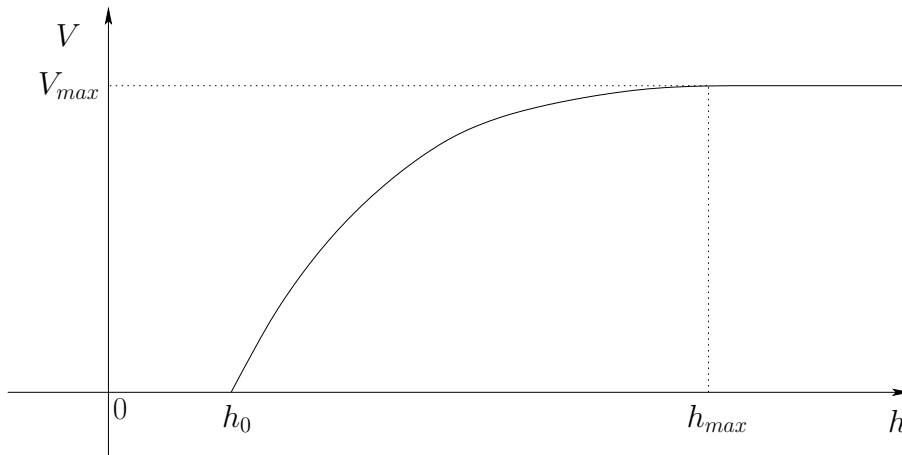


Figure 1.1: Schematic representation of the optimal velocity function V .

1.2 Injecting the system of ODEs into a single PDE

In order to obtain an homogenization result, the authors borrowed the idea from [FIM09b] and injected the system of ODE (1.1) into a single PDE. To do this, in [2], it was introduced the following "cumulative distribution function" of vehicles ρ^ε :

$$\rho^\varepsilon(t, y) = -\varepsilon \left(\sum_{i \geq 0} H(y - \varepsilon U_i(t/\varepsilon)) + \sum_{i < 0} (-1 + H(y - \varepsilon U_i(t/\varepsilon))) \right), \quad (1.2)$$

with

$$H(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases} \quad (1.3)$$

It was proven that under assumption (A) the function ρ^ε satisfies in the viscosity sense the following non-local equation

$$u_t^\varepsilon + M^\varepsilon \left[\frac{u^\varepsilon(t, \cdot)}{\varepsilon} \right] (x) \cdot \phi \left(\frac{x}{\varepsilon} \right) \cdot |u_x^\varepsilon| = 0 \quad \text{on } (0, +\infty) \times \mathbb{R}, \quad (1.4)$$

where M^ε is a non-local operator defined by

$$M^\varepsilon[U](x) = \int_{-\infty}^{+\infty} J(z) E(U(x + \varepsilon z) - U(x)) dz - \frac{3}{2} V_{max} \quad (1.5)$$

with

$$E(z) = \begin{cases} 0 & \text{if } z \geq 0 \\ 1/2 & \text{if } -1 \leq z < 0 \\ 3/2 & \text{if } z < -1, \end{cases} \quad \text{and } J = V' \text{ on } \mathbb{R}. \quad (1.6)$$

1.3 Convergence result

We define $k_0 = 1/h_0$ and $\bar{H} : \mathbb{R} \rightarrow \mathbb{R}$, by

$$\bar{H}(p) = \begin{cases} -p - k_0 & \text{for } p < -k_0, \\ -V \left(\frac{-1}{p} \right) |p| & \text{for } -k_0 \leq p \leq 0, \\ p & \text{for } p > 0. \end{cases} \quad (1.7)$$

Note that such a \bar{H} is continuous, coercive $\left(\lim_{|p| \rightarrow +\infty} \bar{H}(p) = +\infty \right)$ and because of (A5), there exists a unique point $p_0 \in [-k_0, 0]$ such that

$$\begin{cases} \bar{H} \text{ is decreasing on } (-\infty, p_0), \\ \bar{H} \text{ is increasing on } (p_0, +\infty). \end{cases} \quad (1.8)$$

We denote by

$$H_0 = \min_{p \in \mathbb{R}} \bar{H}(p) = \bar{H}(p_0) \tag{1.9}$$

and we refer to Figure 1.2 for an schematic representation of \bar{H} .

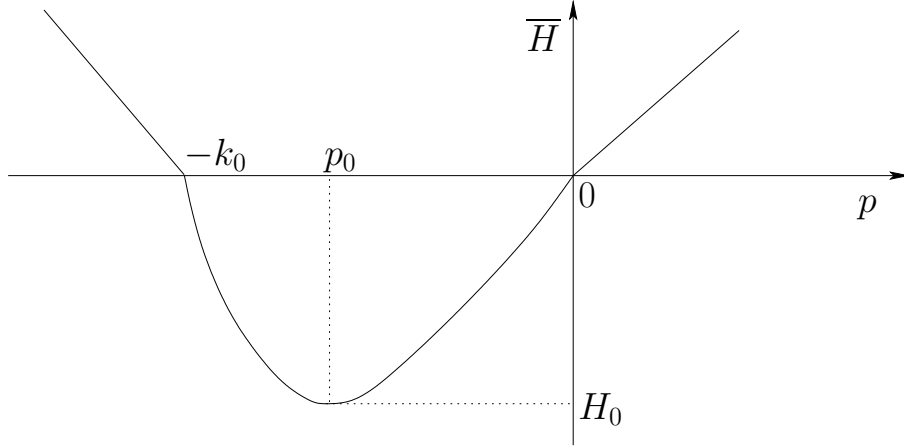


Figure 1.2: Schematic representation of \bar{H} .

From [2], we have the following homogenization result.

Theorem 1.2 (Junction condition by homogenisation). *Assume (A) and that at the initial time, we have, for all $i \in \mathbb{Z}$,*

$$U_i(0) \leq U_{i+1}(0) - h_0.$$

We also assume that there exists a constant $R > 0$ such that, for all $i \in \mathbb{Z}$, if $|U_i(0)| \geq R$

$$U_{i+1}(0) - U_i(0) = h,$$

with $h \geq h_0$. We define the function u_0 (satisfying (A0)) by $u_0(x) = -x/h$ for all $x \in \mathbb{R}$. Then there exists $\bar{A} \in [H_0, 0]$ such that the function ρ^ε defined by (1.2) converges uniformly on compact subsets of $(0, +\infty) \times \mathbb{R}$ as ε goes to 0 towards the unique solution u^0 of

$$\begin{cases} u_t^0 + \bar{H}(u_x^0) = 0 & \text{for } (t, x) \in (0, +\infty) \times (-\infty, 0) \\ u_t^0 + \bar{H}(u_x^0) = 0 & \text{for } (t, x) \in (0, +\infty) \times (0, +\infty) \\ u_t^0 + F_{\bar{A}}(u_x^0(t, 0^-), u_x^0(t, 0^+)) = 0 & \text{for } (t, x) \in (0, +\infty) \times \{0\} \\ u^0(0, x) = u_0(x) & \text{for } x \in \mathbb{R}, \end{cases} \tag{1.10}$$

where \bar{A} has to be determined and $F_{\bar{A}}$ is defined by

$$F_{\bar{A}}(p_-, p_+) = \max(\bar{A}, \bar{H}^+(p_-), \bar{H}^-(p_+)), \tag{1.11}$$

with

$$\overline{H}^-(p) = \begin{cases} \overline{H}(p) & \text{if } p \leq p_0, \\ \overline{H}(p_0) & \text{if } p \geq p_0, \end{cases} \quad \text{and} \quad \overline{H}^+(p) = \begin{cases} \overline{H}(p_0) & \text{if } p \leq p_0, \\ \overline{H}(p) & \text{if } p \geq p_0. \end{cases} \quad (1.12)$$

Moreover, u^0 satisfies $-k_0 \leq u_x^0 \leq 0$.

Remark 1.3. We notice that in the case of traffic flow, (1.10) is equivalent (deriving in space) to a LWR model (see [LW55, Ric56]) with a flux limiting condition at the origin. In fact, the fundamental diagram of the model is $pV(1/p)$ and u_x^0 corresponds to the density of vehicles.

2 Construction of the flux-limiter

As we can see from Theorem 1.2, in order to have a complete homogenization result we only need to determine the flux limiter \overline{A} . In the rest of the paper, we focus on obtaining a numerical approximation of the flux limiter. First we give a few characterizations of \overline{A} that can be founded in [2].

In fact, the flux limiter is the unique constant λ such that there exists a solution w of the following Hamilton-Jacobi equation

$$M[w](x) \cdot \phi(x) \cdot |w_x| = \lambda \quad \text{for } x \in \mathbb{R}, \quad (2.1)$$

with the right slopes at infinity. We refer to [GIM15] and [2] for a more detailed explanation on these slopes.

In order to construct the corrector for the junction w and \overline{A} , in [2] the authors used the idea from [GIM15], [AT15] and from the lectures of Lions at "Collège de France" [Lio14], which is to construct the correctors in a truncated domain with good boundary conditions and then to expand the domain.

The effective flux limiter is then obtained as the limit as $l \rightarrow +\infty$, and then $R \rightarrow +\infty$ of $\lambda_{l,R}$ which is the unique constant for which the following truncated cell problem admits a solution (see [2, Proposition 6.4]): for $l \in (r, +\infty)$, $r \ll l$ and $r \leq R \ll l$, find $\lambda_{l,R}$, such that there exists a solution $w^{l,R}$ of

$$\begin{cases} G_R(x, [w^{l,R}], w_x^{l,R}) = \lambda_{l,R} & \text{if } x \in (-l, l) \\ \overline{H}^-(w_x^{l,R}) = \lambda_{l,R} & \text{if } x = -l \\ \overline{H}^+(w_x^{l,R}) = \lambda_{l,R} & \text{if } x = l, \end{cases} \quad (2.2)$$

with

$$G_R(x, [U], q) = \psi_R(x)\phi(x) \cdot M[U](x) \cdot |q| + (1 - \psi_R(x)) \cdot \overline{H}(q), \quad (2.3)$$

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and $\psi_R \in C^\infty$, $\psi_R : \mathbb{R} \rightarrow [0, 1]$, with

$$\psi_R \equiv \begin{cases} 1 & \text{on } [-R, R] \\ 0 & \text{outside } [-R - 10, R + 10], \end{cases} \quad \text{and} \quad \psi_R(x) < 1 \quad \forall x \notin [-R, R]. \quad (2.4)$$

Remark 2.1. *The operator G_R is used to have a local operator near the boundary and then to well define the boundary conditions.*

To an approximation of $\lambda_{l,R}$ we follow the construction done in [2] and consider the approximated truncated cell problem, for all $\delta > 0$.

$$\begin{cases} \delta v^\delta + \psi_R(x)M[v^\delta](x) \cdot \phi(x) \cdot |v_x^\delta| + (1 - \psi_R(x))\overline{H}(v_x^\delta) = 0 & \text{for } x \in (-l, l) \\ \delta v^\delta + \overline{H}^-(v_x^\delta) = 0 & \text{for } x = -l \\ \delta v^\delta + \overline{H}^+(v_x^\delta) = 0 & \text{for } x = l. \end{cases} \quad (2.5)$$

It is proven in [2] that $-\delta v^\delta(0) \rightarrow \lambda_{l,R}$ as $\delta \rightarrow 0$.

2.1 Organization of the paper

In Section 3 we introduce the definition of a viscosity solution for (2.5), we also present some classical results like the comparison principle and the stability. In the same section, we present a numerical scheme for (2.5), and we announce our main result which is a convergence result for the scheme. In Section 4, we provide the proof of the convergence result. In Section 5, we prove some properties regarding our numerical scheme that we use in practice to obtain a numerical approximation of the solution of (2.5). Finally, Section 6 contains some numerical simulations for different types of perturbations and we numerical verify some of the qualitative properties of \overline{A} that were proven in [2].

3 Viscosity solutions for the approximated cell problem

In this section, we study (2.5). In order to simplify the notations, we drop the index δ in v^δ . We also present the definition of viscosity solutions for (2.5). We refer to the User's guide of Crandall, Ishii, Lions [CIL92] and the book of Barles [Bar94] for a good introduction to viscosity solutions.

3.1 Viscosity solution for the continuous approximated cell problem

For $l \in (r, +\infty)$, $r \ll l$ and $r \leq R \ll l$, we consider the following problem

$$\begin{cases} \delta v + \psi_R(x)M[v](x) \cdot \phi(x) \cdot |v_x| + (1 - \psi_R(x))\overline{H}(v_x) = 0 & \text{for } x \in (-l, l) \\ \delta v + \overline{H}^-(v_x) = 0 & \text{for } x = -l \\ \delta v + \overline{H}^+(v_x) = 0 & \text{for } x = l, \end{cases} \quad (3.1)$$

with

$$M[U](x) = \int_{h_0}^{h_{max}} J(z)E(U(x+z) - U(x)) dz - \frac{3}{2}V_{max} \quad (3.2)$$

with

$$E(z) = \begin{cases} 0 & \text{if } z \geq 0 \\ 1/2 & \text{if } -1 \leq z < 0 \\ 3/2 & \text{if } z < -1, \end{cases} \quad \text{and } J = V' \text{ on } \mathbb{R}.$$

Similarly, we define

$$\tilde{M}[U](x) = \int_{h_0}^{h_{max}} J(z)\tilde{E}(U(x+z) - U(x)) dz - \frac{3}{2}V_{max} \quad (3.3)$$

with

$$\tilde{E}(z) = \begin{cases} 0 & \text{if } z > 0 \\ 1/2 & \text{if } -1 < z \leq 0 \\ 3/2 & \text{if } z \leq -1. \end{cases} \quad (3.4)$$

In order to simplify the notations, we introduce the following function

$$F(x, [v], v_x) = \begin{cases} \psi_R(x)M[v](x) \cdot \phi(x) \cdot |v_x| + (1 - \psi_R(x))\overline{H}(v_x) & \text{if } x \in (-l, l), \\ \overline{H}^-(v_x) & \text{if } x = -l, \\ \overline{H}^+(v_x) & \text{if } x = l. \end{cases}$$

Similarly, we define \tilde{F} by replacing M by \tilde{M} .

In this paper, we work with viscosity solutions, and the boundary conditions of (3.1) are interpreted in the classical sense of viscosity solutions with Neumann boundary conditions. That is why we introduce

$$\mathcal{I}(x, [v], v_x) = \begin{cases} \psi_R(x)M[v](x) \cdot \phi(x) \cdot |v_x| + (1 - \psi_R(x))\overline{H}(v_x) & \text{if } x \in (-l, l), \\ \min(\overline{H}(v_x), \overline{H}^-(v_x)) & \text{if } x = -l, \\ \min(\overline{H}(v_x), \overline{H}^+(v_x)) & \text{if } x = l, \end{cases}$$

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and

$$\tilde{\mathcal{I}}(x, [v], v_x) = \begin{cases} \psi_R(x)\tilde{M}[v](x) \cdot \phi(x) \cdot |v_x| + (1 - \psi_R(x))\bar{H}(v_x) & \text{if } x \in (-l, l), \\ \max(\bar{H}(v_x), \bar{H}^-(v_x)) & \text{if } x = -l, \\ \max(\bar{H}(v_x), \bar{H}^+(v_x)) & \text{if } x = l. \end{cases}$$

Definition 3.1 (Viscosity solution for the continuous approximated cell problem). *An upper semi-continuous function (resp. lower semi-continuous) $v : [-l, l] \rightarrow \mathbb{R}$ is a viscosity sub-solution (resp. a super-solution) of (3.1) in $[-l, l]$, if for all $x \in [-l, l]$ and all $\varphi \in C^1([-l, l])$ such that $u - \varphi$ reaches a maximum (resp. a minimum) at the point x , we have*

$$\delta v(x) + \mathcal{I}(x, [v], \varphi') \leq 0 \quad (\text{resp. } \delta v(x) + \tilde{\mathcal{I}}(x, [v], \varphi') \geq 0).$$

We say that a function v is a solution of (3.1) if v^* and v_* are respectively a sub and a super-solution of (3.1).

Remark 3.2. *Like in [2], we use this type of definition in order to have a stability result for the non-local term. We refer to [DLFM08, Sle03] for such kind of definition.*

We now give a slightly stronger result than a stability result for the sub and super-solutions of (3.1) that will be used to prove the consistency of the numerical scheme we present later in this paper.

Proposition 3.3 (Stability for (3.1)). *Let $(u_m)_m$ be a sequence of measurable functions and let \bar{u} denote $\limsup^* u_m$ (resp. $\underline{u} = \liminf_* u_m$). Let $(x_m, p_m) \rightarrow (x_0, p)$ in $[-l, l] \times \mathbb{R}$ be such that $u_m(x_m) \rightarrow \bar{u}(x_0)$ (resp. $u_m(x_m) \rightarrow \underline{u}(x_0)$). Then*

$$\liminf_{m \rightarrow +\infty} F(x_m, [u_m], p_m) \geq \mathcal{I}(x_0, [\bar{u}], p) \tag{3.5}$$

$$\left(\text{resp. } \limsup_{m \rightarrow +\infty} \tilde{F}(x_m, [u_m], p_m) \leq \tilde{\mathcal{I}}(x_0, [\underline{u}], p) \right). \tag{3.6}$$

In order to prove Proposition 3.3, we use the following lemma which proof can be found in [Sle03].

Lemma 3.4. *Let $(f_m)_m$ be a sequence of measurable functions on \mathbb{R} , and consider*

$$\bar{f} = \limsup^* f_m$$

and

$$\underline{f} = \liminf_* f_m. \tag{3.7}$$

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Let $(a_m)_m$ be a sequence of \mathbb{R} converging to zero. Then

$$\mathcal{L}(\{f_m \geq a_m\} \setminus \{\bar{f} \geq 0\}) \rightarrow 0 \quad \text{as } m \rightarrow +\infty$$

and

$$\mathcal{L}(\{\underline{f} > 0\} \setminus \{f_m > a_m\}) \rightarrow 0 \quad \text{as } m \rightarrow +\infty,$$

where $\mathcal{L}(A)$ denotes the Lesbegue measure of measurable set A .

Proof of Proposition 3.3. We just do the proof for \bar{u} . We distinguish two cases.

Case 1: $x_0 \in (-l, l)$. From the definition of F , we can see that the only problem we have is the non-local operator. We claim that for m big enough we have

$$M[u_m](x_m) \geq M[\bar{u}](x_0) - \varepsilon_m \quad \text{with } \lim_{m \rightarrow +\infty} \varepsilon_m = 0. \quad (3.8)$$

Using (1.6), we have that

$$E(\beta) = \frac{1}{2}1_{\{\beta \in [-1, 0)\}} + \frac{3}{2}1_{\{\beta < -1\}} = \frac{1}{2}1_{\{\beta < 0\}} + 1_{\{\beta < -1\}}.$$

We get that

$$\begin{aligned} & \int_{\mathbb{R}} J(z) E(u_m(x_m + z) - u_m(x_m)) dz - \int_{\mathbb{R}} J(z) E(\bar{u}(x_0 + z) - \bar{u}(x_0)) dz \\ &= \int_{\mathbb{R}} J(z) \left\{ 1_{\{u_m(x_m+z) - u_m(x_m) < -1\}} - 1_{\{\bar{u}(x_0+z) - \bar{u}(x_0) < -1\}} \right\} dz \\ & \quad + \int_{\mathbb{R}} \frac{1}{2} J(z) \left\{ 1_{\{u_m(x_m+z) - u_m(x_m) < 0\}} - 1_{\{\bar{u}(x_0+z) - \bar{u}(x_0) < 0\}} \right\} dz \end{aligned} \quad (3.9)$$

Using Lemma 3.4, we have for n big enough,

$$\begin{aligned} & \int_{\mathbb{R}} J(z) \left\{ 1_{\{\bar{u}(x_0+z) - \bar{u}(x_0) \geq -1\}} - 1_{\{u_m(x_m+z) - u_m(x_m) \geq -1\}} \right\} dz \\ & \geq - \int_{\mathbb{R}} J(z) 1_{\{A_m(z) \setminus A(z)\}} \geq -\frac{\varepsilon_m}{2}, \\ & \int_{\mathbb{R}} \frac{1}{2} J(z) \left\{ 1_{\{\bar{u}(x_0+z) - \bar{u}(x_0) \geq 0\}} - 1_{\{u_m(x_m+z) - u_m(x_m) \geq 0\}} \right\} dz \\ & \geq -\frac{1}{2} \int_{\mathbb{R}} J(z) 1_{\{B_m(z) \setminus B(z)\}} \geq -\frac{\varepsilon_m}{2}, \end{aligned} \quad (3.10)$$

with

$$\begin{cases} A_m(z) = \{u_m(x_m + z) - u_m(x_m) \geq -1\} \cup \{\bar{u}(x_0 + z) - \bar{u}(x_0) \geq -1\}, \\ A(z) = \{\bar{u}(x_0 + z) - \bar{u}(x_0) \geq -1\} \end{cases}$$

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and

$$\begin{cases} B_m(z) = \{u_m(x_m + z) - u_m(x_m) \geq 0\} \cup \{\bar{u}(x_0 + z) - \bar{u}(x_0) \geq 0\}, \\ B(z) = \{\bar{u}(x_0 + z) - \bar{u}(x_0) \geq 0\}. \end{cases}$$

Using (3.9) and (3.10), we prove (3.8).

Given that $x_0 \in (-l, l)$, for m big enough we have $x_m \in (-l, l)$. Using the definition of F and (3.8), we have

$$\begin{aligned} & \psi_R(x_m)M[u_m](x_m) \cdot \phi(x_m) \cdot |p_m| + (1 - \psi_R(x_m))\bar{H}(p_m) \\ & \geq \psi_R(x_m)M[\bar{u}](x_0) \cdot \phi(x_m) \cdot |p_m| + (1 - \psi_R(x_m))\bar{H}(p_m) - \varepsilon_m \psi_R(x_m) \cdot \phi(x_m) \cdot |p_m|. \end{aligned}$$

Using the fact that the terms on the right are continuous, we pass to the limit as m goes to infinity to obtain (3.5).

Case 2: $x_0 = -l$ or $x_0 = l$. In this case, using Definition 3.1 and the continuity of \bar{H} , we obtain (3.5). This ends the proof of Proposition 3.3. □

Theorem 3.5 (Comparison principle). *Let u and v be respectively a sub and a super-solution of (3.1) then we have for all $x \in [-l, l]$,*

$$u(x) \leq v(x).$$

Proof. The proof of this Theorem comes from the comparison principle [FIM09b, Theorem 4.4] for the non-local term. The only remaining difficulty is proving this result at the boundaries. However, for x close to l_2 , the function G_R is actually the effective Hamiltonian \bar{H} . Therefore, we can proceed as in the proof of [GIM15, Proposition 4.1] in which the authors consider the boundaries as a network composed of a single lane and a junction point then they use the results from [IM14] and so we skip the rest of the proof. Notice that Definition 3.1 is equivalent at the boundaries to the definition of relaxed viscosity solution [IM14, Definition 2.1] in the case of a single lane with a junction point. □

Theorem 3.6 (Existence and uniqueness). *Assume (A). There exists a unique solution v^δ of (3.1). Moreover, this solution is continuous and we have that for all $x \in [-l, l]$*

$$0 \leq v^\delta(x) \leq \frac{C_0}{\delta}, \quad \text{with } C_0 = \left| \min_{p \in \mathbb{R}} \bar{H}(p) \right|.$$

Proof. To prove this theorem we only need to notice that 0 and C_0/δ are respectively a sub-solution and a super-solution of (3.1). Then combining Perron's method with the comparison principle (Theorem 3.5) we get the result we wanted. We refer to [IMR08, Proof of Theorem 6] and to [AT96, Imb05] to see how to apply Perron's method for problems with non-local terms. □

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Finally, we have the following result from [2, Proof of Proposition 6.4], which justifies considering (3.1) and looking for a numerical approximation of v^δ .

Proposition 3.7. *We have the following limit as δ goes to 0 (up to a subsequence)*

$$\lim_{\delta \rightarrow 0} -\delta v^\delta(0) = \lambda_{l,R}. \quad (3.11)$$

3.2 Numerical scheme for (3.1)

The numerical scheme we use was inspired by the one from [CCM12, For08] for the non-local operator and by [CLM15] for the local operator. We consider a uniform grid of the interval $[-l, l]$ with $2n+1$ points, $n \in \mathbb{N} \setminus \{0\}$, and we denote by $\Delta x = l/n$ the discretization step. For all $i \in \{1, \dots, n\}$, we denote by $x_i = \Delta x \cdot i$ the nodes of the grid. In particular we have that $x_0 = 0$, $x_{-n} = -l$ and $x_n = l$.

For every discrete function $v : \{1, \dots, n\} \rightarrow \mathbb{R}$, we denote by v_\sharp its piecewise constant extension to \mathbb{R} , defined by

$$v_\sharp(x) = \sum_{i=1}^n v_i \cdot \chi_{Q_i}(x) \quad \text{with} \quad Q_i = \begin{cases} [-l, -l + \Delta x/2) & \text{if } i = -n, \\ [x_i - \Delta x/2, x_i + \Delta x/2) & \text{if } i \neq -n \text{ and } i \neq n \\ [l - \Delta x/2, l] & \text{if } i = n. \end{cases} \quad (3.12)$$

Discretization of the non-local operator For all discrete function $v : \{1, \dots, n\} \rightarrow \mathbb{R}$, we define the following discrete non-local operators

$$M_i^d[v] := M[v_\sharp](x_i) \quad \text{and} \quad \tilde{M}_i^d[v] := \tilde{M}[v_\sharp](x_i). \quad (3.13)$$

Discretization of the gradient We consider the standard forward and backward first order differences:

$$D^+v(x_i) = \frac{v(x_{i+1}) - v(x_i)}{\Delta x} \quad \text{and} \quad D^-v(x_i) = \frac{v(x_i) - v(x_{i-1})}{\Delta x}. \quad (3.14)$$

Finally, we consider $Dv_i = (D^-v(x_i), D^+v(x_i))$.

Similarly to [CCM12], we consider the following Osher, Sethian [OS88] upwind discretization of the modulus of the gradient. Let $S = (p, q) \in \mathbb{R}^2$. We define the following function, that we will use for the discretization of the gradient

$$G^+(S) = \left(\max(p, 0)^2 + \min(q, 0)^2 \right)^{1/2}.$$

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Discretization of the local operator Concerning the local operator $\overline{H}(\cdot)$, like in [CLM15] we consider the following discretization

$$\overline{H}_d(Dv_i) = \max \left\{ \overline{H}^+(D^-v_i), \overline{H}^-(D^+v_i) \right\}, \quad (3.15)$$

where \overline{H}^- and \overline{H}^+ were defined in (1.12).

Finally, we introduce for any discrete function $v : \{1, \dots, n\} \rightarrow \mathbb{R}$,

$$R_i[v] := \mathcal{R}_i([v], Dv_i) = \psi_R(x_i) \cdot M_i^d[v] \cdot \phi(x_i) \cdot G^+(Dv_i) + (1 - \psi_R(x_i)) \cdot \overline{H}_d(Dv_i). \quad (3.16)$$

Similarly, we define \tilde{R} and $\tilde{\mathcal{R}}$ by replacing M^d with \tilde{M}^d .

To summarize, in the rest of the paper, for all discrete function $v : \{1, \dots, n\} \rightarrow \mathbb{R}$, we consider the following numerical scheme.

$$\begin{cases} \delta v_i + \mathcal{R}_i([v], Dv_i) = 0 & \text{for } i \in \{-n+1, \dots, n-1\} \\ \delta v_i + \overline{H}^-(D^+v_i) = 0 & \text{for } i = -n \\ \delta v_i + \overline{H}^+(D^-v_i) = 0 & \text{for } i = n. \end{cases} \quad (3.17)$$

Remark 3.8 (Notation for the discretization of the non-local operator). *Since the function J inside the non-local operator is of bounded support, we introduce the following notations, which are the discrete equivalents of h_0 and h_{max} ,*

$$j_0 = \max \{j \in \{1, \dots, n\} \text{ s.t. } x_j - \Delta x/2 < h_0\} \quad (3.18)$$

and

$$j_{max} = \min \{j \in \{1, \dots, n\} \text{ s.t. } x_j + \Delta x/2 > h_{max}\}. \quad (3.19)$$

3.3 Viscosity solution for the numerical scheme for the approximated cell problem

To simplify the notations, we introduce the following notation

$$F_i([v], Dv_i) = \begin{cases} \mathcal{R}_i([v], Dv_i) & \text{if } x_i \in (-l, l), \\ \overline{H}^-(D^+v_i) & \text{if } x_i = -l, \\ \overline{H}^+(D^-v_i) & \text{if } x_i = l. \end{cases} \quad (3.20)$$

Similarly, we define \tilde{F} by replacing \mathcal{R} with $\tilde{\mathcal{R}}$. We introduce the following definition of viscosity solution for (3.17).

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Definition 3.9 (Viscosity solution for the approximated cell problem scheme). *Let us consider a function $v : \{-n, \dots, n\} \rightarrow \mathbb{R}$. We say that v is a sub-solution (resp. a super-solution) of (3.17) if for all $i \in \{-n, \dots, n\}$ we have*

$$\delta v_i + F_i([v], Dv_i) \leq 0 \quad (\text{resp. } \delta v_i + \tilde{F}_i([v], Dv_i) \geq 0).$$

Then we say that v is a solution of (3.17) if and only if it is a sub and a super-solution.

Remark 3.10. *The notion of discrete viscosity solutions is necessary here because of the discontinuity inside the non-local operator. In fact, we could not work with a regularise version of E because we do not have a stability result with respect to E . Moreover, in Section 4 and Section 5 we can see the interest in working with such a definition. The proof of existence of solutions for (3.17), will be postponed until Section 5.*

The main result of this paper is the following convergence result which proof is postponed until Section 4.

Theorem 3.11 (Convergence). *Using the same notations as in (3.12). Let $(v_i^{\Delta x})_{i \in \{-n, \dots, n\}}$ be a solution of (3.17), then the function $v_{\frac{\Delta x}{4}}$ (defined as in (3.12)) converges locally uniformly as $\Delta x \rightarrow 0$ to the unique continuous viscosity solution of (3.1).*

Remark 3.12 (Condition on the discretization step). *In the rest of the paper, we consider that the integer n is big enough (Δx is small enough) so that $j_0 > 1$. In fact, given the standard values of $h_0 (> 2$ meters), this is not a very restrictive condition but it helps to simplify the computations (regarding the monotonicity of the scheme) since for any $i \in \{1, \dots, n\}$ the term v_{i+1} appears only on the gradient in $F_i([v], Dv_i)$.*

Remark 3.13 (Bounds on the non-local operator). *Given the definition of the non-local operators M and \tilde{M} , we have that for any function $U : \mathbb{R} \rightarrow \mathbb{R}$ and any $x \in \mathbb{R}$,*

$$-\frac{3}{2}V_{max} \leq M[U](x) \leq \tilde{M}[U](x) \leq 0.$$

Given the definition of M^d and \tilde{M}^d , these inequalities hold for the discrete non-local operators.

Moreover, we have the following properties regarding the numerical scheme (3.17).

Lemma 3.14 (Monotonicity of \tilde{F} and F). *Assume (A). Let v, w be two discrete functions such that*

$$v_j \leq w_j \quad \text{for all } j \in \mathbb{Z}, \tag{3.21}$$

also assume that there exists an index $i \in \mathbb{Z}$ such that $v_i = w_i$, then we have

$$F_i([v], Dv_i) \geq F_i([w], Dw_i) \quad \text{and} \quad \tilde{F}_i([v], Dv_i) \geq \tilde{F}_i([w], Dw_i).$$

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Proof. We present the proof for \tilde{F}_i and we skip it for F_i since the proof is similar. Let us begin by proving the monotonicity for the non-local term, first we want to prove that $\tilde{M}_i^d[v] \geq \tilde{M}_i^d[w]$. In fact, using the notations from Remark 3.8, we have

$$\begin{aligned}
\tilde{M}^d[v_{\#}](x_i) &= \int_{h_0}^{h_{max}} J(z) E^*(v_{\#}(x_i + z) - v_{\#}(x_i)) dz - \frac{3}{2} V_{max} \\
&= \sum_{j=j_0}^{j_{max}} \int_{x_j - \Delta x/2}^{x_i + \Delta x/2} J(z) E^*(v_{\#}(x_i + z) - v_{\#}(x_j)) dz - \frac{3}{2} V_{max} \\
&= \sum_{j=j_0}^{j_{max}} \int_{\mathcal{Q}_j} J(z) dz \{E^*(v_{i+j} - v_i)\} - \frac{3}{2} V_{max} \\
&\geq \sum_{j=j_0}^{j_{max}} \int_{\mathcal{Q}_j} J(z) dz \{E^*(w_{i+j} - w_i)\} - \frac{3}{2} V_{max} = \tilde{M}_i^d[w],
\end{aligned}$$

where we have used for the last line the fact that \tilde{E} is non-increasing, with (3.21) and the fact that J is non-negative.

Moreover, using (3.21), we have that

$$D^+ v_i \leq D^+ w_i \quad \text{and} \quad D^- v_i \geq D^- w_i.$$

This implies in particular that

$$\max(D^+ v_i, 0)^2 \leq \max(D^+ w_i, 0)^2 \quad \text{and} \quad \min(D^- v_i, 0)^2 \leq \min(D^- w_i, 0)^2.$$

Combining the previous inequalities we have that $G^+(Dv_i) \leq G^+(Dw_i)$. We recall that \tilde{M}^d is non-positive (Remark 3.13) and therefore, we have

$$\tilde{M}_i^d[v] \cdot G^+(Dv_i) \geq \tilde{M}_i^d[v] \cdot G^+(Dw_i) \geq \tilde{M}_i^d[w] \cdot G^+(Dw_i). \quad (3.22)$$

Let us now prove the monotonicity for the local term, using the fact that \bar{H}^+ is non-decreasing and that \bar{H}^- is non-increasing, we have that

$$\bar{H}^+(D^- v_i) \geq \bar{H}^+(D^- w_i) \quad \text{and} \quad \bar{H}^-(D^+ v_i) \geq \bar{H}^-(D^+ w_i),$$

this implies in particular that

$$\begin{aligned}
\bar{H}_d(Dv_i) &= \max(\bar{H}^-(D^+ v_i), \bar{H}^+(D^- v_i)) \\
&\geq \max(\bar{H}^-(D^+ w_i), \bar{H}^+(D^- w_i)) = \bar{H}_d(Dw_i).
\end{aligned} \quad (3.23)$$

Combining (3.22) and (3.23), we get $\tilde{F}_i([v], Dv_i) \geq \tilde{F}_i([w], Dw_i)$.

□

4 Convergence of the numerical scheme for the approximated cell problem

This section is devoted to the proof of Theorem 3.11 which is an adaptation of the proof of convergence from [BS91] to a non-local PDE. Before passing to the proof of Theorem 3.11, we give some preliminary results, concerning the discrete barriers of the solutions of (3.17).

Lemma 4.1 (Existence of discrete barriers for (3.1)). *Assume (A). The discrete functions defined by*

$$v_i^- = 0 \quad \text{and} \quad v_i^+ = \frac{C_0}{\delta} \quad \text{for all } i \in \{1, \dots, n\}$$

with $C_0 = |H_0| = |\min_{p \in \mathbb{R}} \overline{H}(p)|$ are respectively a sub and a super-solution of (3.1).

Proof. We only prove that v^+ is a super-solution, since the sub-solution case is similar and even simpler. Using the form of (3.1), we have that for all $i \in \{1, \dots, n\}$

$$\delta v_i^+ + \tilde{F}_i([v^+], (0, 0)) \geq \delta v_i^+ + H_0 = |H_0| + H_0 = 0.$$

Therefore, v^+ is a super-solution of (3.1). □

Proposition 4.2 (Discrete barriers). *Let $u_i^- = 0$ and $v_i^+ = C_0/\delta$ for all $i \in \{1, \dots, n\}$ with $C_0 = |H_0|$. Then every solution v of (3.17) satisfies*

$$u^- \leq v \leq v^+.$$

Proof. Let us begin by proving that $v - v^+ \leq 0$. We introduce $M = \max_{i \in \{1, \dots, n\}} \{v_i - v_i^+\}$, we assume the maximum is reached for an index $i_0 \in \{1, \dots, n\}$. Therefore, we have

$$v_{i_0} - M = v_{i_0}^+ \quad \text{and} \quad \tilde{v}_i := v_i - M \leq v_i^+ \quad \text{for all } i \in \{1, \dots, n\}. \quad (4.1)$$

We notice that $F_i([v], Dv_i)$ is invariant by addition of constant to v and therefore, $F_i([\tilde{v}], D\tilde{v}_i) = F_i([v], Dv_i)$. Moreover, using Lemme 3.14 and (4.1), we have

$$F_{i_0}([v], Dv_{i_0}) = F_{i_0}([\tilde{v}], D\tilde{v}_{i_0}) \geq F_{i_0}([v^+], Dv_{i_0}^+). \quad (4.2)$$

Using the fact that v is a solution of (3.17) and in particular a sub-solution, we have

$$0 \geq \delta v_{i_0} + F_{i_0}([v], Dv_{i_0}) \geq \delta v_{i_0}^+ + \delta M + F_{i_0}([v^+], Dv_{i_0}^+).$$

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In particular, replacing $v_{i_0}^+ = C_0/\delta$ we have

$$\delta M \leq -C_0 - F_{i_0}([v^+], Dv_{i_0}^+) = \begin{cases} 0 & \text{if } i_0 = -n \\ -C_0 & \text{if } i_0 \in \{-n+1, \dots, n\}, \end{cases}$$

where we have used for the equality the definition of F and the fact that $Dv_{i_0}^+ = (0, 0)$ (in this case the only term that is not equal to 0 is $\overline{H}^-(0) = H_0 = -C_0$). Therefore, $M \leq 0$. The proof that $v^- - v \leq 0$ is similar to the previous one and we skip it. □

Proof of Theorem 3.11. We introduce

$$\bar{v}(x) = \limsup_{\substack{y \rightarrow x \\ \Delta x \rightarrow 0}} v_{\#}^{\Delta x}(y) \quad \text{and} \quad \underline{v}(x) = \liminf_{\substack{y \rightarrow x \\ \Delta x \rightarrow 0}} v_{\#}^{\Delta x}(y).$$

Here we have added the superscript Δx in order for the proof to be clearer. Like in (3.12) the function $v_{\#}^{\Delta x}$ is the piecewise extension of a discrete function $v^{\Delta x}$ which is a solution of (3.17) (see Definition 3.9).

We want to prove that \bar{v} and \underline{v} are respectively a sub and a super-solution of (3.1). In fact if that is true, the comparison principle for (3.1) implies that $\bar{v} \leq \underline{v}$ on $[-l, l]$. However by construction we have that $\underline{v} \leq \bar{v}$, which will imply that $\underline{v} = \bar{v} = v^\delta$ the unique continuous solution of (3.1), this implies the local uniform convergence of $v_{\#}^{\Delta x}$.

Let us now prove that \bar{v} is a sub-solution of (3.1). We only do the proof in the sub-solution case, since the super-solution case is very similar and we skip it. We argue by contradiction and assume that there exists a function $\varphi \in C^1([-l, l])$ and a point $\bar{x} \in [-l, l]$ such that $\bar{v} - \varphi$ reaches a strict local maximum at \bar{x} and that we have

$$\delta \bar{v}(\bar{x}) + \mathcal{I}(\bar{x}, [\bar{v}], \varphi_x(\bar{x})) = \theta > 0.$$

Moreover, without any loss of generality, we assume that $\bar{u}(\bar{x}) = \varphi(\bar{x})$ and that $\varphi \geq 2 \sup_{\Delta x} \|v_{\#}^{\Delta x}\|_\infty$ outside the ball $B(\bar{x}, r)$, where $r > 0$ is such that

$$\bar{u}(x) - \varphi(x) \leq 0 = \bar{u}(\bar{x}) - \varphi(\bar{x}) \quad \text{in } B(\bar{x}, r). \tag{4.3}$$

Then there exists sequences $\Delta x_m \in [0, +\infty)$ and $y_m \in [-l, l]$, such that as $m \rightarrow +\infty$

$$\Delta x_m \rightarrow 0, \quad y_m \rightarrow \bar{x}, \quad v_{\#}^{\Delta x_m}(y_m) \rightarrow \bar{v}(\bar{x}), \quad \text{and} \\ y_m \text{ is a global maximum point of } v_{\#}^{\Delta x_m}(\cdot) - \varphi(\cdot).$$

We denote by $\xi_m = v_{\#}^{\Delta x_m}(y_m) - \varphi(y_m)$, and we have that $\xi_m \rightarrow 0$ as $m \rightarrow +\infty$. Moreover, we have for all $x \in [-l, l]$

$$v_{\#}^{\Delta x_m}(x) \leq \varphi(x) + \xi_m \quad \text{and} \quad v_{\#}^{\Delta x_m}(y_m) = \varphi(y_m) + \xi_m.$$

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We denote by x_{i_m} the point in the grid such that $y_m \in \mathcal{Q}_{i_m}$, therefore we have that $|x_{i_m} - y_m| \leq \Delta x_m$ and

$$\begin{aligned} 0 &\geq \delta v_{i_m}^{\Delta x_m} + F_{i_m} \left(\left[v^{\Delta x_m} \right], Dv_{i_m}^{\Delta x_m} \right) \\ &\geq \delta v_{i_m}^{\Delta x_m} + F_{i_m} \left(\left[v^{\Delta x_m} \right], D\varphi(y_m) \right) \\ &\geq \delta v_{\sharp}^{\Delta x_m}(y_m) + F \left(x_{i_m}, \left[v_{\sharp}^{\Delta x_m} \right], \varphi'(x_0) + o(\Delta x_m) \right), \end{aligned}$$

where we have used the fact that $v^{\Delta x_m}$ is a sub-solution of (3.17) for the first line. For the second line, we have considered

$$D\varphi(y_m) = \left(\frac{\varphi(y_m) - \varphi(y_m - \Delta x_m)}{\Delta x_m}, \frac{\varphi(y_m + \Delta x_m) - \varphi(y_m)}{\Delta x_m} \right)$$

and have used the monotonicity of the discrete operator F_{i_m} (Lemma 3.14). Finally, for the third line, we have used the fact that $\varphi \in C^1$ and the definition of F_{i_m} . We now pass to the limit in the previous inequality

$$\begin{aligned} 0 &\geq \liminf_{m \rightarrow +\infty} \left(\delta v_{\sharp}^{\Delta x_m}(i_m) + F(x_{i_m}, [v_{\sharp}^{\Delta x_m}], \varphi'(x_0) + o(\Delta x_m)) \right) \\ &\geq \delta \bar{v}(\bar{x}) + \mathcal{I}(\bar{x}, [\bar{v}], \varphi'(\bar{x})), \end{aligned}$$

where we have used Proposition 3.3 with $u_m = v_{\sharp}^{\Delta x_m}$, $x_m := x_{i_m}$, $p_m := \varphi'(\bar{x}) + o(\Delta x_m)$ and notice that $v_{\sharp}^{\Delta x_m}(x_{i_m}) = v_{\sharp}^{\Delta x_m}(y_m)$. The last inequality provides us with a contradiction which ends the proof of Theorem 3.11. □

5 Discrete approximated cell problem

The following results are similar to the ones in [CCM12], for the reader convenience, we recall them and adapt them to our problem. The idea is to obtain some results on the numerical scheme that will help us to construct a solution to (3.17).

5.1 Comparisons for the numerical scheme

In this section, we prove the existence of solutions for the numerical scheme (3.17). We use the following notations, for every $s \in \mathbb{R}$ and every discrete function v , we define

$$D_{v_i}^+(s) = \frac{v_{i+1} - s}{\Delta x}, \quad D_{v_i}^-(s) = \frac{s - v_{i-1}}{\Delta x}, \quad D_{v_i}(s) = \left(D_{v_i}^-(s), D_{v_i}^+(s) \right).$$

For every $i \in \{1, \dots, n\}$, we set

$$M_i^d[v](s) = \int_{h_0}^{h_{max}} J(z) E(v_{\sharp}(x_i + z) - s) dz - \frac{3}{2} V_{max}$$

$$\tilde{M}_i^d[v](s) = \int_{h_0}^{h_{max}} J(z) \tilde{E}(v_{\sharp}(x_i + z) - s) dz - \frac{3}{2} V_{max}$$

and

$$\mathcal{R}_i[v](s) = \psi_R(x_i) \cdot \phi(x_i) \cdot M_i^d[v](s) \cdot G^+(Dv_i(s)) + (1 - \psi_R(x_i)) \cdot \bar{H}_d(Dv_i(s)).$$

We define similarly $\tilde{\mathcal{R}}_i$ by replacing M_i^d with \tilde{M}_i^d . Finally, we define for all $i \in \{1, \dots, n\}$,

$$F_i[v](s) = \begin{cases} \mathcal{R}_i[v](s) & \text{if } i \in \{-n+1, \dots, n-1\} \\ \bar{H}^-(Dv_i^+(s)) & \text{if } i = -n \\ \bar{H}^+(Dv_i^-(s)) & \text{if } i = n. \end{cases}$$

Similarly we define \tilde{F}_i by replacing \mathcal{R}_i by $\tilde{\mathcal{R}}_i$.

Remark 5.1 (Monotonicity of the numerical scheme). *By the definition of F_i and \tilde{F}_i and by Lemma 3.14, we have for all $i \in \{1, \dots, n\}$,*

- (i) $F_i[v](s)$ and $\tilde{F}_i[v](s)$ are non-decreasing with respect to s .
- (ii) $F_i[v](s)$ and $\tilde{F}_i[v](s)$ are non-increasing with respect to v .
- (iii) $F_i[v](v_i) = F_i([v], Dv_i)$ and $\tilde{F}_i[v](v_i) = \tilde{F}_i([v], Dv_i)$.
- (iv) $\tilde{F}_i[v](s) \leq F_i[v](t)$ for all $s < t$.

Lemma 5.2 (Comparison). *Let v be a discrete function and let $i \in \{1, \dots, n\}$. There exists a unique $s_i \in \mathbb{R}$ such that*

$$-\tilde{F}_i[v](s_i) \leq \delta s_i \leq -F_i[v](s_i). \tag{5.1}$$

Moreover, let w be a discrete function. Then the following implications hold:

- (i) $\delta w_i + F_i[w](w_i) \leq 0$ and $w \leq v \Rightarrow w_i \leq s_i$.
- (ii) $\delta w_i + F_i[w](w_i) > 0$ and $w \geq v \Rightarrow w_i > s_i$.
- (iii) $\delta w_i + \tilde{F}_i[w](w_i) \geq 0$ and $w \geq v \Rightarrow w_i \geq s_i$.

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(iv) $\delta w_i + \tilde{F}_i[w](w_i) < 0$ and $w \leq v \Rightarrow w_i < s_i$

Proof. The existence of a $s_i \in \mathbb{R}$ satisfying (5.1) comes from the fact that $-F_i[v](s)$ and $-\tilde{F}_i[v](s)$ are non-increasing in s and that δs is strictly increasing in s . As for the uniqueness of s_i , let us argue by contradiction and assume there exists s_i^1 and s_i^2 with $s_i^1 < s_i^2$ such that

$$-\tilde{F}_i[v](s_i^1) \leq \delta s_i^1 \leq -F_i[v](s_i^1) \quad \text{and} \quad -\tilde{F}_i[v](s_i^2) \leq \delta s_i^2 \leq -F_i[v](s_i^2).$$

However, using (iv) from Remark 5.1, we have that $-F_i[v](s_i^2) \leq -\tilde{F}_i[v](s_i^1)$, using this and combining the previous inequalities we obtain that

$$F_i[v](s_i^1) - \tilde{F}_i[v](s_i^2) \leq \delta(s_i^2 - s_i^1) \leq \tilde{F}_i[v](s_i^1) - F_i[v](s_i^2) \leq 0. \quad (5.2)$$

This gives us that $s_i^2 - s_i^1 \leq 0$ which is a contradiction.

We now prove the implication (i), again we argue by contradiction and assume that $w_i > s_i$. Then using (5.1), we have

$$\delta w_i > \delta s_i \geq -\tilde{F}_i[v](s_i) \geq -\tilde{F}_i[w](s_i) \geq -F_i[w](w_i) \geq \delta w_i,$$

where we have used for the third and fourth inequality respectively (ii) and (iv) from Remark 5.1. The previous inequality gives us the desired contradiction.

We now prove implication (ii). We argue by contradiction and assume that $w_i \leq s_i$, using (5.1), we have

$$\delta w_i \leq \delta s_i \leq -F_i[v](s_i) \leq -F_i[w](s_i) \leq -F_i[w](w_i) < \delta w_i,$$

where we have used for the third and fourth inequality respectively (ii) and (i) from the Remark 5.1. The previous inequality gives us the desired contradiction. The proofs of implications (iii) and (iv) are similar and we skip them. □

5.2 Construction of minimal and maximal solutions

This section is devoted to the proof of existence and to the construction of minimal and maximal solutions. In order to prove the existence of discrete solution for (3.17), we will provide a constructive method that will also provide us with the minimal and maximal solutions.

Proposition 5.3 (Definition of the map Φ). *There exists a map $\Phi : \mathbb{R}^{2n+1} \rightarrow \mathbb{R}^{2n+1}$ satisfying the following properties*

a) Let u^- be a sub-solution of (3.17), i.e.,

$$\delta u_i^- + F_i[u^-](u_i^-) \leq 0 \quad \text{for all } i \in \{1, \dots, n\}.$$

Then

- i) $u^- \leq \Phi[u^-]$ (with $u^- = \Phi[u^-]$ if and only if u^- is a solution).
- ii) $\Phi[u^-]$ is a sub-solution of (3.17).

b) Let u^+ be a super-solution of (3.17), i.e.,

$$\delta u_i^- + \tilde{F}_i[u^-](u_i^-) \geq 0 \quad \text{for all } i \in \{1, \dots, n\}.$$

Then

- i) $u^+ \geq \Phi[u^+]$ (with $u^+ = \Phi[u^+]$ if and only if u^+ is a solution).
- ii) $\Phi[u^+]$ is a super-solution of (3.17).

Proof. Let us prove the result for the sub-solutions (a).

Using Lemma 5.2, for every $i \in \{1, \dots, n\}$ there exists a unique $s_i^- \in \mathbb{R}$ such that

$$-\tilde{F}_i[u^-](s_i^-) \leq \delta s_i^- \leq -F_i[u^-](s_i^-) \quad \text{and} \quad u_i^- \leq s_i^-,$$

the second inequality comes from (i) in Lemma 5.2 considering $w = u^-$. Using the real numbers s_i^- we construct the map Φ :

$$\Phi[u^-]_i = s_i^- \quad \text{for all } i \in \{1, \dots, n\}.$$

By construction we have $u^- \leq \Phi[u^-]$, with the equality if and only if u^- is a solution of (3.17) (which proves (i) in (a)). Let us now prove that $\Phi[u^-]$ is a sub-solution of (3.17). For all $i \in \{1, \dots, n\}$, by the definition of s_i^- , we have

$$0 \geq \delta s_i^- + F_i[u^-](s_i^-) \geq \delta s_i^- + F_i[\Phi[u^-]](s_i^-) = \delta \Phi[u^-]_i + F_i[\Phi[u^-]](\Phi[u^-]_i),$$

where we have used for the second inequality the fact that $F_i[\cdot](s)$ is non-increasing and that $u^- \leq \Phi[u^-]$. Therefore, $\Phi[u^-]$ is a sub-solution of (3.17).

In the case of the super-solutions, to prove (b), we define the map Φ in the same way. Using Lemma 5.2, for every $i \in \{1, \dots, n\}$ there exists a unique $s_i^+ \in \mathbb{R}$ such that

$$-\tilde{F}_i[u^+](s_i^+) \leq \delta s_i^+ \leq -F_i[u^+](s_i^+) \quad \text{and} \quad u_i^+ \geq s_i^+.$$

Like before, the map Φ is constructed using the real numbers s_i^+ :

$$\Phi[u^+]_i = s_i^+ \quad \text{for all } i \in \{1, \dots, n\}.$$

Proceeding like before we can prove the rest of (b). We skip the rest of the proof and this ends the proof of Proposition 5.3. □

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Proposition 5.4 (Partial comparison principle). *Let u^- and u^+ be respectively a sub and a super-solution of (3.17) such that $u^- \leq u^+$ then $\Phi[u^-] \leq \Phi[u^+]$.*

Proof. We argue by contradiction and assume that there exists a $i_0 \in \{1, \dots, n\}$ such that

$$\Phi[u^-]_{i_0} = s_{i_0}^- > s_{i_0}^+ = \Phi[u^+]_{i_0}.$$

By definition of $s_{i_0}^+$, we have

$$0 \leq \delta s_{i_0}^+ + \tilde{F}_{i_0}[u^+](s_{i_0}^+) < \delta s_{i_0}^- + \tilde{F}_{i_0}[u^+](s_{i_0}^+) \leq \delta s_{i_0}^- + F_{i_0}[u^+](s_{i_0}^-) \leq \delta s_{i_0}^- + F_{i_0}[u^-](s_{i_0}^-), \quad (5.3)$$

where we have used the fact that $s_{i_0}^- > s_{i_0}^+$ for the second inequality, and (i) and (ii) from Remark 5.1 for the third and fourth inequalities respectively. This inequality gives us the desired contradiction because of the definition of $s_{i_0}^-$. \square

Proposition 5.5 (Construction of solutions). *Let u^- and u^+ be respectively a sub and a super-solution of (3.17) such that $u^- \leq u^+$. We consider for every $k \in \mathbb{N}$,*

$$u^{-,k+1} = \Phi[u^{-,k}], \quad \text{with } u^{-,0} = u^-,$$

and

$$v^{+,k+1} = \Phi[v^{+,k}], \quad \text{with } v^{+,0} = u^+.$$

There exist two discrete functions u and v such that $u^{-,k} \rightarrow u$ and $v^{+,k} \rightarrow v$ as $k \rightarrow +\infty$. Moreover u and v are two solutions of (3.17). We define

$$\Psi[u^-] := u \quad \text{and} \quad \Psi[v^+] := v.$$

Then we have

$$u^- \leq \Psi[u^-] \leq \Psi[u^+] \leq u^+,$$

and $\Psi[u] = u$ if and only if u is a solution of (3.17).

Proof. Using Proposition 5.4 we get the following inequalities

$$u^- \leq \Phi[u^{-,k}] \leq \Phi[u^{-,k+1}] \leq \dots \leq \Phi[v^{+,k+1}] \leq \Phi[v^{+,k}] \leq v^+.$$

Therefore, the sequence $(u^{-,k})_k$ is non-decreasing and bounded from above by v^+ , and the sequence $(v^{+,k})_k$ is non-increasing and bounded from below by u^- . Passing to the limit as k goes to infinity in the previous inequalities, we obtain

$$u^- \leq \lim_{k \rightarrow +\infty} u^{-,k} =: u =: \Psi[u^-] \leq \Psi[v^+] := v := \lim_{k \rightarrow +\infty} v^{+,k} \leq v^+.$$

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Let us now prove that u is a solution of (3.17) (The proof for v is similar and we skip it). By definition of the sequence $(u^{-,k})_k$, we have for all $k \in \mathbb{N}$ and for all $i \in \{1, \dots, n\}$,

$$-\tilde{F}_i[u^{-,k}](u_i^{-,k+1}) \leq \delta u_i^{-,k+1} \leq -F_i[u^{-,k}](u_i^{-,k+1}). \quad (5.4)$$

We recall that E and \tilde{E} are respectively lower and upper semi-continuous, which implies that R and \tilde{R} are also lower and upper semi-continuous and in particular that $-R$ and $-\tilde{R}$ are respectively upper and lower semi-continuous. Adding this to the continuity of the discrete gradient and of the functions \bar{H} , \bar{H}^+ , and \bar{H}^- and passing to the limit as k goes to $+\infty$ in (5.4) implies

$$-\tilde{F}_i[u](u_i) \leq \delta u_i \leq -F_i[u](u_i),$$

which means that u is a solution of (3.17). Finally, the fact that $\Psi[u] = u$ if and only if u is a solution comes from the properties of Φ (see Proposition 5.3). □

Proposition 5.6 (Extremal solutions in the interval (u^-, v^+)). *Let u^- and v^+ be respectively a sub and super-solution of (3.17) such that $u^- \leq v^+$. Let $\Psi[u^-]$ and $\Psi[v^+]$ be the two solutions provided by Proposition 5.5. Then every solution v of (3.17) such that $u^- \leq v \leq v^+$ satisfies*

$$u^- \leq \Psi[u^-] \leq v \leq \Psi[v^+] \leq v^+.$$

Proof. Considering v as a super-solution of (3.17), and using Proposition 5.5, we get

$$u^- \leq \Psi[u^-] \leq \Psi[v] = v.$$

Similarly, considering v as a sub-solution of (3.17), using Proposition 5.5, we have

$$v = \Psi[v] \leq \Psi[v^+] \leq v^+.$$

□

Combining Lemma 4.1, Proposition 4.2, and Proposition 5.6 we obtain the following result.

Corollary 5.7. *Let $u_i^- = 0$ and $v_i^+ = C_0/\delta$ for all $i \in \{1, \dots, n\}$. Let $\Psi[u^-]$ and $\Psi[v^+]$ be the solutions provided by Proposition 5.5, then every solution v of (3.17) satisfies*

$$\Psi[u^-] \leq v \leq \Psi[v^+].$$

6 Numerical simulations

This section contains the application of the results from the previous sections. First we provide the algorithm used to obtain a numerical approximation of the flux-limiter and then we provide some numerical tests.

6.1 The algorithm

The following algorithm is inspired by the one in [CCM12], and by the results from Section 5. The idea of the algorithm is to build the extremal solutions from Corollary 5.7, to build the biggest sub-solution and the smallest super-solution. Therefore obtaining an interval that contains all the solutions of (3.17) and therefore obtaining an approximation of the solution of (3.1).

We introduce two parameters ε_d and ε_c respectively a tolerance to quit the dichotomy process updating the sub and super-solutions (numerical equivalent of Proposition 5.5) and a tolerance for the convergence of the numerical scheme.

- 1) Initialization: for $i = -n, \dots, n$,

$$u_i^- = 0 \quad \text{and} \quad v_i^+ = \frac{C_0}{\delta}.$$

- 2) Initialize dichotomy intervals: for $i = -n, \dots, n$,

$$s_{left,i}^- = u_i^- \quad \text{and} \quad s_{right,i}^- = u_i^- + 0, 1k_i^-$$

with k_i^- the first integer such that

$$\delta s_{right,i}^- + F_i[u^-](s_{right,i}^-) > 0,$$

and

$$s_{left,i}^+ = v_i^+ - 0, 1k_i^+ \quad \text{and} \quad s_{right,i}^+ = v_i^+$$

with k_i^+ the first integer such that

$$\delta s_{left,i}^+ + \tilde{F}_i[v^+](s_{left,i}^+) < 0.$$

- 3) Dichotomy process: for $i = -n, \dots, n$ optimize respectively in $s_i^- \in [s_{left,i}^-, s_{right,i}^-]$ and $s_i^+ \in [s_{left,i}^+, s_{right,i}^+]$, the inequalities

$$\delta s_i^- + F_i[u^-](s_i^-) \leq 0 \quad \text{and} \quad \delta s_i^+ + \tilde{F}_i[v^+](s_i^+) \geq 0$$

until $s_{right,i}^- - s_{left,i}^- < \varepsilon_d$ and $s_{right,i}^+ - s_{left,i}^+ < \varepsilon_d$.

If $\|u^- - s_{left,i}^-\|_\infty \leq \varepsilon_c$ and $\|v^+ - s_{left,i}^+\|_\infty \leq \varepsilon_c$ go to Step 4 else swap $u^- \leftrightarrow s_{left,i}^-$ and $v^+ \leftrightarrow s_{right,i}^+$ and go to Step 2.

- 4) The interval $[u^-, v^+]$ contains all the solutions of the numerical scheme (3.17) and therefore gives an approximation of the solution of (3.1). In particular, the value of \bar{A} is approximated by the interval $[-\delta v_0^+, -\delta u_0^-]$.

Remark 6.1. *Notice that this algorithm can be extremely costly computationally. However, we can easily apply parallel programming to steps 2) and 3) to accelerate the process.*

6.2 Setting of the computation

We consider an uniform grid of the interval $[-l, l]$ with $2n + 1$ points, $n \in \mathbb{N} \setminus \{0\}$, and we denote by $\Delta x = l/n$ the discretization step. For all $i \in \{1, \dots, n\}$, we denote by $x_i = \Delta x \cdot i$ the nodes of the grid.

For all the numerical computations, we consider for equation (3.1), the following values for the different parameters.

$$l = 200, \quad R = 100, \quad \varepsilon_c = \varepsilon_d = 0.001, \quad \text{and } \delta = 0.001. \quad (6.1)$$

For all the numerical computations, regarding the values for the discretization, we consider

$$n = 400 \quad \text{and} \quad \Delta x = 0.5. \quad (6.2)$$

For the computation of the discrete non-local operator, we recall the following result, using the notations from Remark 3.8 and from (3.12),

$$\begin{aligned} M[v_{\#}](x_i) &= \int_{h_0}^{h_{max}} J(z)E(v_{\#}(x_i + z) - v_{\#}(x_i)) dz - \frac{3}{2}V_{max} \\ &= \sum_{j=j_0}^{j_{max}} \int_{x_j - \Delta x/2}^{x_j + \Delta x/2} J(z)E(v_{\#}(x_i + z) - v_{\#}(x_j)) dz - \frac{3}{2}V_{max} \\ &= \sum_{j=j_0}^{j_{max}} \int_{x_j - \Delta x/2}^{x_j + \Delta x/2} J(z)E(v_{i+j} - v_i) dz - \frac{3}{2}V_{max} \\ &= \sum_{j=j_0}^{j_{max}} E(v_{i+j} - v_i) \int_{x_j - \Delta x/2}^{x_j + \Delta x/2} J(z) dz - \frac{3}{2}V_{max} \\ &= \sum_{j=j_0}^{j_{max}} E(v_{i+j} - v_i) J_j - \frac{3}{2}V_{max}, \end{aligned}$$

with $J_j = V(x_j + \Delta x/2) - V(x_j - \Delta x/2)$.

6.3 Qualitative properties of \bar{A}

Before passing to the numerical tests, we recall a final result from [2] regarding the qualitative properties of \bar{A} , which we will numerically verify in the next section.

Proposition 6.2 (Qualitative properties of the flux limiter). *Assume (A). We have the following qualitative properties on the flux limiter:*

(i) (Monotonicity of the flux-limiter). *Let $\phi_1, \phi_2 : \mathbb{R} \rightarrow [0, 1]$ be two functions satisfying (A6). Let \bar{A}_1 and \bar{A}_2 be their respective flux limiters. If, for all $x \in \mathbb{R}$, we have*

$$\phi_1(x) \leq \phi_2(x),$$

then

$$\bar{A}_1 \geq \bar{A}_2.$$

(ii) (Flux interruption) *Let ϕ be a function satisfying (A6). If $\phi = 0$ on an open interval, then we have*

$$\bar{A} = 0.$$

6.4 Numerical tests

Influence of ϕ_0

First, we would like to numerically verify (i) from Proposition 6.2 and to see the influence of $\phi_0 = \min_x \phi(x)$. We consider a Greenshields optimal velocity function,

$$V(h) = \begin{cases} 0 & \text{if } h \leq h_0, \\ V_{max} \left(1 - \left(\frac{h_0}{h} \right)^2 \right) & \text{if } h > h_0, \\ V_{max} \left(1 - \left(\frac{h_0}{h_{max}} \right)^2 \right) & \text{if } h > h_{max}. \end{cases} \quad (6.3)$$

For the perturbation, we consider a function $\phi : \mathbb{R} \rightarrow [0, 1]$ defined by

$$\phi(x) = \begin{cases} 1 & \text{if } |x| \geq r \\ \phi_0 & \text{if } |x| \leq r/8 \\ -\frac{8x(1-\phi_0)}{7r} + \frac{8\phi_0-1}{7} & \text{if } x \in (-r, -r/8) \\ \frac{8x(1-\phi_0)}{7r} + \frac{8\phi_0-1}{7} & \text{if } x \in (r/8, r). \end{cases} \quad (6.4)$$

For the values of the different parameters for the optimal velocity function, we take

$$\begin{cases} V_{max} = 58 \text{ km/h}, \\ h_0 = 2 \text{ m}, \\ h_{max} = 25 \text{ m}. \end{cases} \quad (6.5)$$

For the local perturbation, we consider the radius of the perturbation $r = 45 \text{ m}$. In Figure 6.3, we have an example of the local perturbation for two different values of ϕ_0 .

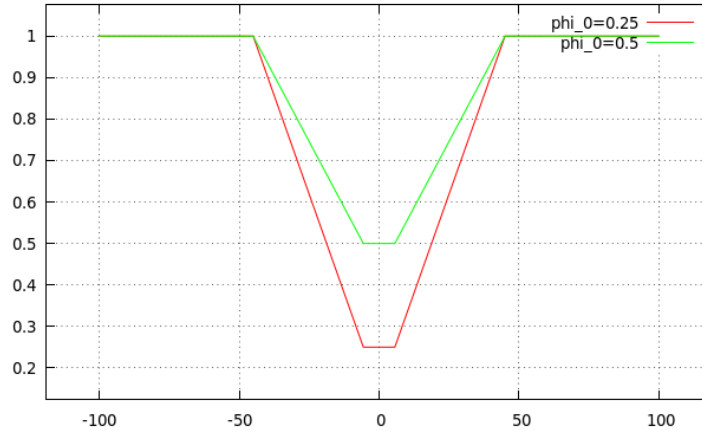


Figure 6.3: Example of ϕ with $r = 45 \text{ m}$ and $\phi_0 = 0, 25$ (red) and $\phi_0 = 0, 5$ (green).

Notice that given the definition (6.4), we have that if we consider two functions ϕ^1 and ϕ^2 with their respective minima $\phi_0^1 < \phi_0^2$ then for all $x \in \mathbb{R}$, we have $\phi^1(x) \leq \phi^2(x)$.

In order to see the influence of $\phi_0 \in [0, 1]$ on \bar{A} , we discretize the interval $[0, 1] \ni \phi_0$ in 21 points (a step of 0.05) and we compute our estimate of \bar{A} for each of those ϕ_0 . For each ϕ_0 , we plot two points, since we have an interval that approximates \bar{A} .

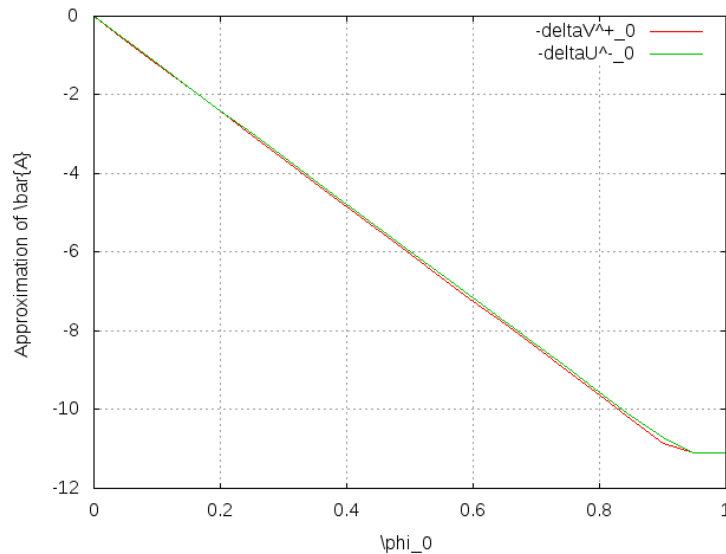


Figure 6.4: Approximation of \bar{A} plotted versus different values of ϕ_0 .

From Figure 6.4, we notice that the approximation is decreasing with ϕ_0 , which numerically confirms (i) of Proposition (6.2) and notice that for $\phi_0 = 0$, we have that \bar{A} is close to 0 which numerically confirms (ii) of (6.2).

In the case $\phi_0 = 1$, the model is actually equivalent to a model without a perturbation. Therefore, we should not have a flux-limiting condition. Given the definition (1.11) of $F_{\bar{A}}$ this can only happen if $\bar{A} = \min_p \bar{H}(p)$. In our computational setting we have $H_0 \simeq -11.16$ and we have an approximation of -11.11 which is not very far and which also validates our numerical approach.

Influence of the radius of influence of the perturbation

We consider the same optimal velocity function as before and the same perturbation and we make the radius of influence of the perturbation vary in the interval $[25, 75]$. Figure 6.5 contains the approximation of \bar{A} for $\phi_0 = 0, 25$ and for different values of $r \in [25, 75]$.

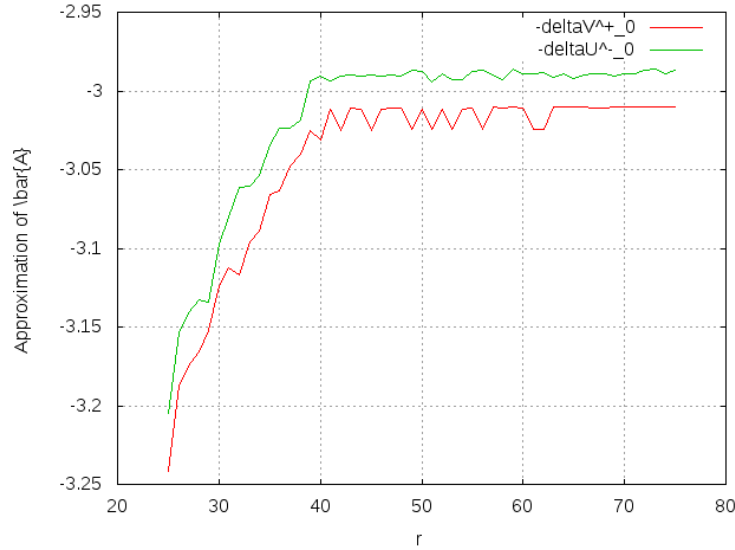


Figure 6.5: Approximation of \bar{A} plotted versus different values of r and $\phi_0 = 0.25$.

From Figure 6.5, we can see that in this case the approximation of \bar{A} increases with the radius of the perturbation. However, for $r \geq 40m$ the approximation remains the same which could imply that for r big enough, the radius of the perturbation does not influence the value of \bar{A} . Moreover, for smaller values of r , we can see that \bar{A} is smaller (meaning that the flux is less limited) which is logical, since for a radius $r = 0$, we expect to have $\bar{A} = H_0$ (in which case there is no perturbation).

Two different perturbations

Now we would like to take into account two different perturbations and see how our approximation of \bar{A} changes with each perturbation. We consider the same perturbation as before and we introduce the following perturbation

$$\tilde{\phi}(x) = \begin{cases} 1 & \text{if } |x| \geq r \\ \frac{(1 - \phi_0)x^2}{r^2} + \phi_0 & \text{if } |x| \leq r. \end{cases} \quad (6.6)$$

We consider the same radius of influence $r = 45 m$. In Figure 6.6, we plotted the two perturbations ϕ and $\tilde{\phi}$ with $\phi_0 = 0, 25$.

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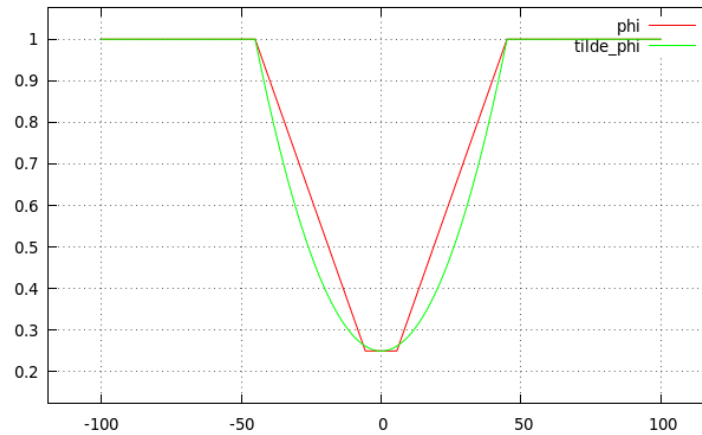


Figure 6.6: The functions ϕ and $\tilde{\phi}$, with $r = 45 m$ and $\phi_0 = 0, 25$.

Like before, we make ϕ_0 vary inside the interval $[0, 1]$. In Figure 6.7, we compare the upper and lower bound of the approximation of \bar{A} for different values of ϕ_0 .

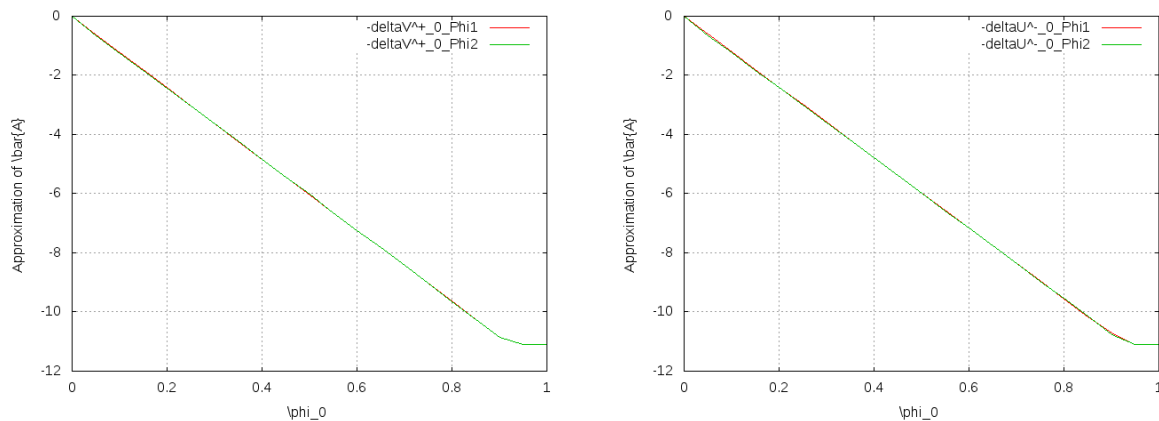


Figure 6.7: Lower bound for the approximation of \bar{A} for ϕ and $\tilde{\phi}$ versus different values of ϕ_0 (left) upper bound for the approximation of \bar{A} for ϕ and $\tilde{\phi}$ versus different values of ϕ_0 (right).

From Figure 6.7, we first notice that the approximation of \bar{A} for $\tilde{\phi}$ numerically verifies Proposition 6.2. Moreover, we notice that the values for both perturbations are very similar. This actually could imply that the form of the perturbation does not influence the result but it is only ϕ_0 that determines the value of the flux limiter. In Figure 6.8, we plot the absolute difference between the two approximations and we notice the difference is very small.

6.Numerical simulations

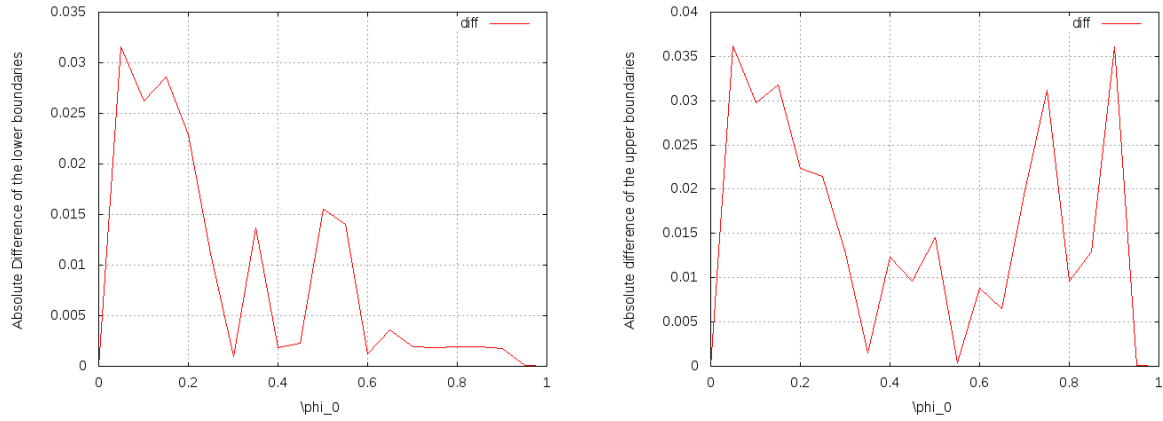


Figure 6.8: Absolute difference of the lower bound for the approximation of \bar{A} for ϕ and $\tilde{\phi}$ versus different values of ϕ_0 (left). Absolute difference of the upper bound for the approximation of \bar{A} for ϕ and $\tilde{\phi}$ versus different values of ϕ_0 (right).

Influence of δ

We consider the optimal velocity function (6.3) with (6.5) and the perturbation (6.4), with $r = 45 \text{ m}$ and $\phi_0 = 0, 25$. To see the influence of δ on the approximation of \bar{A} , we fix l and R to the values of (6.1) and we make δ vary in $[0.001, 0.1]$ with a step of 0.001.

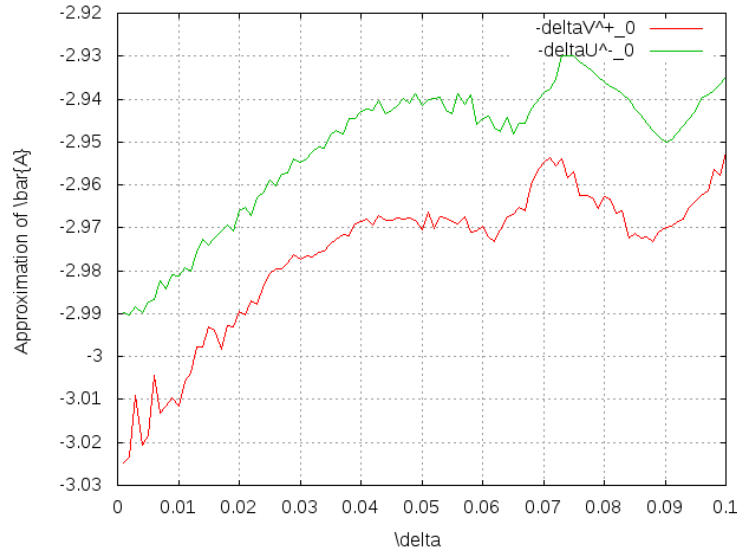


Figure 6.9: Approximation of \bar{A} versus different values of δ .

From Figure 6.9, we first we notice that there is a lot of oscillations on the behaviour

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of the estimates of \bar{A} . However, the upper and lower bound remain close to each other. Given that the difference between the estimates is small, we can assume that considering $\delta = 0.001$ gives a good enough approximation of the flux-limiter.

Influence of R (transition between the non-local and local operators)

We consider the same optimal velocity function and perturbation as before, with the same parameters. To see the influence of R on the approximation of \bar{A} , we fix l and δ to the values of (6.1) and we make R vary in $[80, 150]$ with a step of 1.

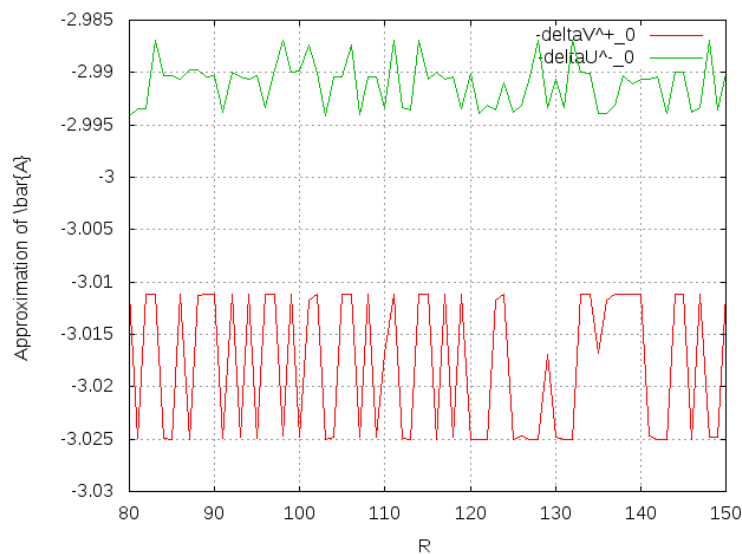


Figure 6.10: Approximation of \bar{A} versus different values of R .

From Figure 6.10, we notice a lot of oscillation on the behaviour of the estimates for \bar{A} when we make R vary. However, we notice that for $R > 80$ the difference between the upper and lower estimate is very small (less than 0.4). This would suggest that considering $R > 80$ is enough for a good approximation of \bar{A} .

Influence of l

We consider the same optimal velocity function and perturbation as before, with the same parameters. To see the influence of l on the approximation of \bar{A} , we fix R and δ to the values of (6.1) and we make l vary in $[180, 300]$ with a step of 1.

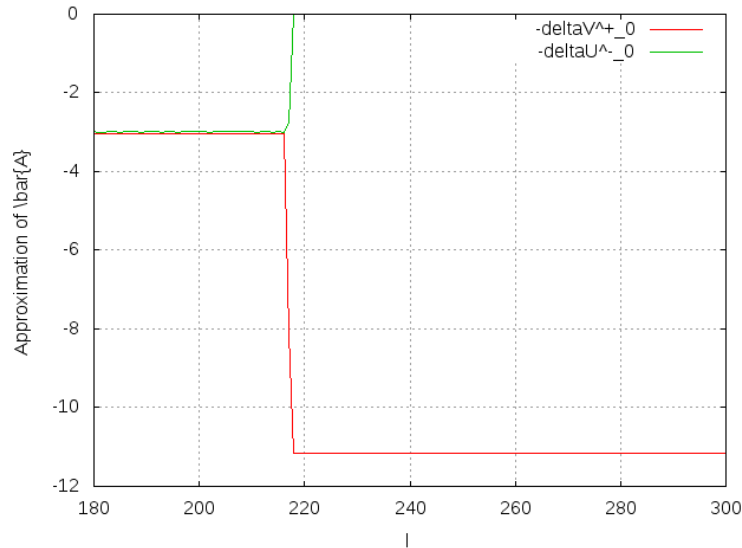


Figure 6.11: Approximation of \bar{A} versus different values of l .

From Figure 6.11 we remark that for $l \in [180, 220]$ the approximation of \bar{A} remains almost the same. The behaviour of the approximation for bigger values of l can be explain by the fact that first δ should go to 0 before passing to the limit as l goes to infinity. Therefore, there is a compromise to be made between l and δ . However, for $\delta = 0.001$ taking $l = 200$ seems to give a reasonable approximation.

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Chapitre 5

Homogenization of second order discrete model with local perturbation and application to traffic flow

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Abstract

The goal of this paper is to derive a traffic flow macroscopic model from a second order microscopic model with a local perturbation. At the microscopic scale, we consider a Bando model of the type following the leader, i.e the acceleration of each vehicle depends on the distance of the vehicle in front of it. We consider also a local perturbation like an accident at the roadside that slows down the vehicles. After rescaling, we prove that the "cumulative distribution functions" of the vehicles converges towards the solution of a macroscopic homogenized Hamilton-Jacobi equation with a flux limiting condition at junction which can be seen as a LWR (Lighthill-Whitham-Richards) model.

AMS Classification: 35D40, 90B20, 35B27, 35F20, 45K05.

Keywords: specified homogenization, Hamilton-Jacobi equations, integro-differential operators, Slepčev formulation, viscosity solutions, traffic flow, microscopic models, macroscopic models.

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1 Introduction

The modelling and simulation of traffic flow is a challenging task in particular in order to design infrastructure. Indeed, there are some examples in which the construction of a new infrastructure did not improve the traffic. For example, in Stuttgart, Germany, after investments into the road network in 1969, the traffic situation did not improve until a section of newly build road was closed for traffic again (see [Knö69]). This is known as the Braess' paradox. In the past years, a lot of work has been done concerning the modelling and simulation of traffic flows problems.

Traffic flow can be modelled at different scales depending on the level of details one wants to observe: the microscopic scale (describes the dynamics of each of the vehicles), the macroscopic scale (describes the dynamics of the density of vehicles) and the mesoscopic scale (describes the dynamics of the density of vehicles but the car-to-car interactions are not lost).

Microscopic models are considered more justifiable because the behaviour of every single vehicle can be described with high precision whereas macroscopic models are based on assumptions which are less verifiable. Another way to justify macroscopic models is to derive them from microscopic models by rescaling arguments.

The problem of deriving macroscopic models from microscopic ones has already been studied for models of the type following the leader (i.e. the velocity or the acceleration of each vehicle depends only on the distance to the vehicle in front of it). We refer for example to [AKRM02, DFR15, Hel98, HP10, LLK01] where the authors rescaled the empirical measure and obtained a scalar conservation law (LWR model). In particular, passing from microscopic to macroscopic model for second-order models was instead investigated in [AKRM02, Gre01], where the Aw-Rascle model is derived as the limit of a second order follow-the-leader model.

In this paper we establish a connection between a car-following model and a fluid-dynamic model. This result is a generalization of the results of [2] to a second order microscopic model. We consider a second order microscopic model of *follow-the-leader* type with a local perturbation. In such model, the whole traffic flow is determined by the dynamics of the very first vehicle (the *leader*). We will establish a connection between this second order discrete model and a macroscopic model equivalent to a LWR model. The idea is to rescale the microscopic model, which describes the dynamics of each vehicle individually, in order to get a macroscopic model which describes the dynamics of density of vehicles.

The model we study here is similar to the one considered in [1], but in our work, as in [2], we assume that there is a local perturbation (located at the origin for example) that slows down the vehicles and we want to understand how this local perturbation influences the macroscopic dynamics. Due to this perturbation, it is natural to get an Hamilton-Jacobi equation with a junction condition at the origin and an effective flux limiter. Further,

our result is stronger than the one in [2] because our microscopic model is a second order model which is more realistic than the first order model considered in the last paper. From a mathematical point of view the fact of considering a second order model presents many technical difficulties. First, we need to consider a system of two non-local PDEs instead of a single equation [FIM12, 1]. Moreover, the two functions that we consider have to satisfy certain properties that derive from the physical characteristics of the microscopic model and those properties need to be proven for the system of non-local PDEs which is more complicated in the case of a second order model than in the case of a first order model.

Paper organization. The paper is organized as follows. In Section 2, we present the microscopic model for which we will present an homogenization result. In Section 3, we inject the system of ODEs into a system of PDEs and we present our main results. Section 3.3 is dedicated to the definition of the non-local operators which appear in the PDEs given in Section 3. In Section 4, we introduce the notion of viscosity solutions for the considered problems and give stability, existence and uniqueness results. In Sections 5 and 6 we present the correctors necessary for the proof of convergence which is located in Section 7. Section 8 contains the proof of existence of correctors for the junction, where we use the idea developed in [AT15, GIM15] and in the lectures of Lions at the "College de France" [Lio14], which consists in constructing correctors on truncated domains. In Section 9 we show the link between the system of ODEs and the system of PDEs which proof is in Appendix 11. Finally in Appendix 10 we analyse the properties of the microscopic model.

2 A first main result

In this paper, we are interested in a second order microscopic model that can simulate the presence of a local perturbation. In order to do that, we considered a modified version of the model introduced by Bando *et al* in [BHN⁺95]. More precisely, we consider a "follow-the-leader" model of the following form

$$\ddot{U}_j(t) = a \left(V(U_{j+1}(t) - U_j(t)) \cdot \phi(U_j(t)) - \dot{U}_j(t) \right), \tag{2.1}$$

where U_j denotes the position of the j -th vehicle, \dot{U}_j its velocity and \ddot{U}_j its acceleration. The function ϕ simulates the presence of a local perturbation located at the origin and we denote by r its radius of influence. In this model, $a \in \mathbb{R}$ and $V : \mathbb{R} \rightarrow \mathbb{R}$ represent respectively the drivers sensitivity and the optimal velocity function. We make the following assumptions on V , ϕ and on the coefficient a .

Assumption (A)

- (A1) $V : \mathbb{R} \rightarrow \mathbb{R}^+$ is Lipschitz continuous, non-negative.
- (A2) V is non-decreasing on \mathbb{R} .

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- (A3) There exists $h_0 \in (0, +\infty)$ such that for all $h \leq h_0$, $V(h) = 0$.
- (A4) There exists $h_{max} \in (h_0, +\infty)$ such that for all $h \geq h_{max}$, $V(h) = V(h_{max}) =: V_{max}$.
- (A5) The function $p \mapsto pV(-1/p)$ is strictly convex on $[-1/h_0, 0)$.
- (A6) The function $\phi : \mathbb{R} \rightarrow (0, 1]$ is Lipschitz continuous and $\phi(x) = 1$ for $|x| \geq r$. We denote by $\phi_0 = \min_{x \in [-r, r]} \phi(x) > 0$.
- (A7)(Monotonicity). $a \geq 4 \|V'\|_\infty \|\phi\|_\infty + 4 \|\phi'\|_\infty \|V\|_\infty$.

Remark 2.1 (Remark on (A6)). *In the case $\phi = 0$ on an open interval (therefore $\phi_0 = 0$) all the vehicles left of the perturbation would come to a full stop. This case lacks any interest and therefore we can assume that $\phi_0 > 0$.*

Remark 2.2 (Remark on (A7)). *Assumption (A7) yields that for all $(b, x) \in \mathbb{R}^2$, the function*

$$f : z \mapsto \frac{a}{2}z - 2V(b+z)\phi(x-z)$$

is non-decreasing. This result is particularly important later in the paper because it implies that the systems we consider later in this work are monotone in the sense of Ishii and Koike [IK91], which will imply the uniqueness of the solution we consider.

As we said in the introduction, in order to obtain an homogenization result for (3.1), we will inject the system of ODEs into a system of PDEs. To do so, we proceed as in [FIM09b, 2] by introducing the rescaled "cumulative distribution function", which is the primitive of the rescaled empirical measure, defined by,

$$\rho^\varepsilon(t, y) = -\varepsilon \left(\sum_{i \geq 0} H(y - \varepsilon U_i(t/\varepsilon)) + \sum_{i < 0} (-1 + H(y - \varepsilon U_i(t/\varepsilon))) \right) \quad (2.2)$$

with

$$H(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases} \quad (2.3)$$

The macroscopic model

We define $\bar{H} : \mathbb{R} \rightarrow \mathbb{R}$, by

$$\bar{H}(p) = \begin{cases} -p - k_0 & \text{for } p < -k_0, \\ -V\left(\frac{-1}{p}\right) \cdot |p| & \text{for } -k_0 \leq p < 0, \\ p & \text{for } p \geq 0. \end{cases} \quad (2.4)$$

Note that such \bar{H} is continuous, coercive and because of (A5), there exists a unique point $p_0 \in [-k_0, 0]$ such that

$$\begin{cases} \bar{H} & \text{is decreasing on } (-\infty, p_0) \\ \bar{H} & \text{is increasing on } (p_0, +\infty), \end{cases} \quad (2.5)$$

and we denote by

$$H_0 := \bar{H}(p_0) = \min_{p \in \mathbb{R}} \bar{H}(p) < 0. \quad (2.6)$$

We want to show that the rescaled "cumulative distribution function" converges to the solution of the following macroscopic model.

$$\begin{cases} u_t^0 + \bar{H}(u_x^0) = 0 & \text{for } (t, x) \in (0, +\infty) \times (-\infty, 0) \\ u_t^0 + \bar{H}(u_x^0) = 0 & \text{for } (t, x) \in (0, +\infty) \times (0, +\infty) \\ u_t^0 + F_{\bar{A}}(u_x^0(t, 0^-), u_x^0(t, 0^+)) = 0 & \text{for } (t, x) \in (0, +\infty) \times \{0\} \\ u^0(0, x) = u_0(x) & \text{for } x \in \mathbb{R}. \end{cases} \quad (2.7)$$

where \bar{A} has to be determined and $F_{\bar{A}}$ is defined by

$$F_{\bar{A}}(p_-, p_+) = \max(\bar{A}, \bar{H}^+(p_-), \bar{H}^-(p_+)), \quad (2.8)$$

with

$$\bar{H}^-(p) = \begin{cases} \bar{H}(p) & \text{if } p \leq p_0 \\ \bar{H}(p_0) & \text{if } p \geq p_0 \end{cases} \quad \text{and} \quad \bar{H}^+(p) = \begin{cases} \bar{H}(p_0) & \text{if } p \leq p_0 \\ \bar{H}(p) & \text{if } p \geq p_0. \end{cases} \quad (2.9)$$

The initial condition u_0 is a function that satisfies

$$-k_0 \leq (u_0)_x \leq 0 \quad \text{and for all } \varepsilon > 0 \quad |\rho^\varepsilon(0, x) - u_0(x)| \leq f(\varepsilon), \quad (2.10)$$

with $f(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. According to [IM14], for all $\bar{A} \in \mathbb{R}$, there exists a unique solution u^0 of (2.7).

Remark 2.3. *We notice that in the case of traffic flow, (2.7) is equivalent (deriving in space) to a LWR model (see [LW55, Ric56]) with a flux limiting condition at the origin. In fact, the fundamental diagram of the model is $pV(1/p)$ and u_x^0 corresponds to the density of vehicles.*

Passage from a microscopic to a macroscopic model

The main result of this paper is the following convergence result.

Theorem 2.4 (Passage from a microscopic to a macroscopic model). *Assume (A). There exists a unique $\bar{A} \in [H_0, 0]$ such that the function ρ^ε defined by (2.2) converges locally uniformly towards the unique solution of (2.7).*

3 Main results

3.1 Injecting the system of ODEs into a system of PDEs

In the rest of the paper, we will work with an equivalent formulation of (2.1). We borrow the idea from [FIM09a, FIM12, 1] and consider for all $j \in \mathbb{Z}$,

$$\Xi_j(t) = U_j(t) + \frac{1}{\alpha} \dot{U}_j(t) \quad \text{with} \quad \alpha = \frac{a}{2}.$$

Using this new function, we obtain the following system of ODEs equivalent to (2.1) for all $j \in \mathbb{Z}$, for all $t \in (0, +\infty)$,

$$\begin{cases} \dot{U}_j(t) = \alpha (\Xi_j(t) - U_j(t)) \\ \dot{\Xi}_j(t) = \alpha (U_j(t) - \Xi_j(t)) + 2V (U_{j+1}(t) - U_j(t)) \cdot \phi (U_j(t)). \end{cases} \quad (3.1)$$

In Appendix 10, we give some properties of system 3.1, such as maximal velocities of the vehicles and minimal and maximal distance between two consecutive vehicles.

We now introduce the "cumulative distribution function" for $(\Xi_j)_j$, defined by

$$\sigma^\varepsilon(t, y) = -\varepsilon \left(\sum_{i \geq 0} H(y - \varepsilon \Xi_i(t/\varepsilon)) + \sum_{i < 0} (-1 + H(y - \varepsilon \Xi_i(t/\varepsilon))) \right). \quad (3.2)$$

Under assumption (A), $(\rho^\varepsilon, \sigma^\varepsilon)$ is a discontinuous viscosity solution (see Theorem 3.3) of the following non-local equation, for all $(t, x) \in (0, +\infty) \times \mathbb{R}$,

$$\begin{cases} u_t^\varepsilon + M^\varepsilon \left(\frac{u^\varepsilon}{\varepsilon}(t, x), \left[\frac{\xi^\varepsilon}{\varepsilon}(t, \cdot) \right] \right) (x) \cdot |u_x^\varepsilon| = 0 \\ \xi_t^\varepsilon + L^\varepsilon \left(\frac{x}{\varepsilon}, \frac{\xi^\varepsilon}{\varepsilon}(t, x), \left[\frac{u^\varepsilon}{\varepsilon}(t, \cdot) \right] \right) (x) \cdot |\xi_x^\varepsilon| = 0. \end{cases} \quad (3.3)$$

The definition of M^ε and L^ε is postponed to the next section. We submit equation (3.3) to the following initial condition. For all $x \in \mathbb{R}$,

$$\begin{cases} u^\varepsilon(0, x) = u_0(x) \\ \xi^\varepsilon(0, x) = \xi_0^\varepsilon(x). \end{cases} \quad (3.4)$$

We also assume that the initial condition satisfies the following assumption.

(A0) (Gradient bound) Let $k_0 = 1/h_0$. The functions u_0 and ξ_0^ε are Lipschitz continuous functions, such that

$$-k_0 \leq (u_0)_x \leq 0 \tag{3.5}$$

$$-k_0 \leq (\xi_0^\varepsilon)_x \leq 0, \tag{3.6}$$

and

$$0 \leq \xi_0^\varepsilon(x) - u_0(x) \leq \varepsilon. \tag{3.7}$$

Remark 3.1. *The initial conditions u_0 and ξ_0^ε are "regular" functions such that for all $\varepsilon > 0$ we have*

$$|\rho^\varepsilon(0, x) - u_0(x)| \leq f(\varepsilon) \quad \text{and} \quad |\sigma^\varepsilon(0, x) - \xi_0^\varepsilon(x)| \leq g(\varepsilon), \tag{3.8}$$

with $f(\varepsilon), g(\varepsilon) \rightarrow 0$ as ε goes to 0. For $\varepsilon = 1$, the conditions on the gradients translate the fact that at the initial time there is at least h_0 meters between two consecutive vehicles. In the rest of the paper we are interested in the behaviour of ρ^ε and σ^ε as ε goes to 0. This in fact translates to studying the behaviour of the traffic as the number of vehicles per unit length goes to infinity. For $\varepsilon = 1$ condition (3.7) translate the fact that at initial time the velocity of the vehicles must be bounded so the ordering of the vehicles is kept.

The fact that ξ_0^ε depends on ε comes from the rescaling. In fact, given that σ^ε is the "cumulative distribution function" of $(\Xi_j)_j$ which are defined using the velocity of the vehicles, an ε appears multiplying the velocity when rescaling (see [1, Remark 1.2]). Therefore, ξ_0^ε tends to u_0 as ε goes to zero. Finally, to simplify the notations, we denote by $\xi_0 = \xi_0^\varepsilon$ for $\varepsilon = 1$.

3.2 Convergence result

Theorem 2.4 is a consequence of the following theorems. The proof of Theorem 3.2 is postponed until Section 7 and the proof of Theorem 3.3 is postponed until Section 9.

Theorem 3.2 (Junction condition by homogenization). *Assume (A) and (A0). For $\varepsilon > 0$, let $(u^\varepsilon, \xi^\varepsilon)$ be the solution of (3.3)-(3.4). Then there exists $\bar{A} \in [H_0, 0]$ such that u^ε and ξ^ε converge locally uniformly to the unique viscosity solution u^0 of (2.7).*

Theorem 3.3 (Junction condition by homogenization: application to traffic flow). *Assume (A) and that at initial time $(U_i(0), \Xi_i(0))_i$ satisfies*

$$0 \leq \Xi_i(0) - U_i(0) \leq \frac{V_{max}}{\alpha}, \quad U_{i+1}(0) - \Xi_i(0) \geq h_0, \quad \text{and} \quad U_{i+1}(0) - U_i(0) \leq h_{max}.$$

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We also assume that there exists a constant $R > 0$ such that, for all $i \in \mathbb{Z}$, if $|U_i(0)| \geq R$

$$U_{i+1}(0) - U_i(0) = h \quad (3.9)$$

and if $|\Xi_i(0)| \geq R$

$$\Xi_{i+1}(0) - \Xi_i(0) = h, \quad (3.10)$$

with $h \in [h_0, h_{max}]$. We define two function u_0 and ξ_0^ε (satisfying (A0)) by $u_0(x) = \xi_0^\varepsilon(x) = -x/h$ for all $x \in \mathbb{R}$. Then there exists a unique $\bar{A} \in [H_0, 0]$ such that the functions ρ^ε and σ^ε defined by (2.2) and (3.2) converge locally uniformly towards the unique solution u^0 of (2.7).

Remark 3.4. Conditions (3.9) and (3.10) mean that the initial condition is well-prepared.

The following theorem ensures that when we use (2.7) we only evaluate the function \bar{H} in the interval $[-k_0, 0]$. The proof of Theorem 3.5 is postponed until Section 7.

Theorem 3.5 (Gradient bound). *Assume (A0)-(A). Let u^0 be the unique solution of (2.7), then we have for all $(t, x) \in [0, T] \times \mathbb{R}$,*

$$-k_0 \leq u_x^0 \leq 0,$$

with k_0 defined in (A0).

3.3 Definition of the non-local operators

In this section, we clarify equation (3.3). We will give the definition of M and L , and then the definition of M^ε and L^ε . To do this, we first introduce the following functions.

$$E(z) = \begin{cases} -\alpha & \text{if } z \geq 0 \\ 0 & \text{if } z < 0, \end{cases} \quad F(z) = \begin{cases} 1 & \text{if } z < 0 \\ 0 & \text{if } z \geq 0, \end{cases} \quad I(z) = \begin{cases} 1 & \text{if } z \geq -1 \\ 0 & \text{if } z < -1, \end{cases}$$

$$\tilde{E}(z) = \begin{cases} -\alpha & \text{if } z > 0 \\ 0 & \text{if } z \leq 0, \end{cases} \quad \tilde{F}(z) = \begin{cases} 1 & \text{if } z \leq 0 \\ 0 & \text{if } z > 0, \end{cases} \quad \text{and} \quad \tilde{I}(z) = \begin{cases} 1 & \text{if } z > -1 \\ 0 & \text{if } z \leq -1. \end{cases}$$

For $x, p \in \mathbb{R}$, we then define the following non-local operators

$$M_p(U(x), [\Sigma]) (x) = \int_0^D E(\Sigma(x+z) - U(x) + pz) dz,$$

$$K_p(\Sigma(x), [U]) (x) = \int_0^D F(U(x-z) - \Sigma(x) - pz) dz,$$

$$N_p(\Sigma(x), [U]) (x) = \int_0^D I(U(x+z) - \Sigma(x) + pz) dz,$$

with $D = h_{max} + 3V_{max}/(2\alpha) + 2r/\phi_0$ (see Appendixes 10 and 11 for more details on where the constant D comes from). We can now define L^p . For $x, y \in \mathbb{R}$,

$$\begin{aligned} L_p(y, \Sigma(x), [U])(x) &= \alpha K_p(\Sigma(x), [U])(x) \\ &\quad - 2V \left(N_p(\Sigma(x), [U])(x) + K_p(\Sigma(x), [U])(x) \right) \cdot \phi(y - K_p(\Sigma(x), [U])(x)). \end{aligned} \quad (3.11)$$

In the same way, we define \tilde{M}_p, \tilde{K}_p and \tilde{N}_p by replacing E, F and I respectively by \tilde{E}, \tilde{F} and \tilde{I} . Similarly,

$$\begin{aligned} \tilde{L}_p(y, \Sigma(x), [U])(x) &= \alpha \tilde{K}_p(\Sigma(x), [U])(x) \\ &\quad - 2V \left(\tilde{N}_p(\Sigma(x), [U])(x) + \tilde{K}_p(\Sigma(x), [U])(x) \right) \cdot \phi(y - \tilde{K}_p(\Sigma(x), [U])(x)). \end{aligned} \quad (3.12)$$

For $p = 0$, we define

$$M(U(x), [\Sigma])(x) := M_0(U(x), [\Sigma]) = \int_0^D E(\Sigma(x+z) - U(x)) dz, \quad (3.13)$$

$$K(\Sigma(x), [U])(x) := K_0(\Sigma(x), [U])(x) = \int_0^D F(U(x-z) - \Sigma(x)) dz, \quad (3.14)$$

$$N(\Sigma(x), [U])(x) := N_0(\Sigma(x), [U])(x) = \int_0^D I(U(x+z) - \Sigma(x)) dz, \quad (3.15)$$

and

$$\begin{aligned} L(y, \Sigma(x), [U])(x) &= \alpha K(\Sigma(x), [U])(x) \\ &\quad - 2V \left(N(\Sigma(x), [U])(x) + K(\Sigma(x), [U])(x) \right) \cdot \phi(y - K(\Sigma(x), [U])(x)). \end{aligned} \quad (3.16)$$

Remark 3.6 (Remarks on the non-local operators). *First let us notice that the domain of integration in the non-local operators is bounded by a constant $D := h_{max} + 3V_{max}/(2\alpha) + 2r/\phi_0$, this comes from the fact that the velocities of the vehicles as well as the distance between two consecutive vehicles from model 3.1 are bounded (see Appendix 10). In particular, there exists a constant $M_0 > 0$ (independent of p), such that we have the following bounds on the non-local operators,*

$$\begin{aligned} -M_0 &\leq -\alpha D \leq M_p(U(x), [\Sigma])(x) \leq 0, \\ M_0 &\geq D \geq K_p(\Sigma(x), [U])(x) \geq 0, \\ M_0 &\geq D \geq N_p(\Sigma(x), [U])(x) \geq 0, \\ M_0 &\geq \alpha D \geq L_p(y, \Sigma(x), [U])(x) \geq -2V_{max} \geq -M_0, \end{aligned}$$

with $M_0 = \max(2V_{max}, \alpha D, D)$.

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Finally, we would like to point out that given the fact that the function V is non-decreasing (assumption (A2)) and that the function $F \geq 0$ and therefore $K(\Sigma, [U])(x) \geq 0$, we have

$$L(y, \Sigma(x), [U])(x) \geq -2V\left(N(\Sigma(x), [U])(x)\right). \quad (3.17)$$

Finally, we introduce for $\varepsilon > 0$,

$$M^\varepsilon(U(x), [\Sigma])(x) = \int_0^D E(\Sigma(x + \varepsilon z) - U(x)) dz, \quad (3.18)$$

$$K^\varepsilon(\Sigma(x), [U])(x) = \int_0^D F(U(x - \varepsilon z) - \Sigma(x)) dz, \quad (3.19)$$

$$N^\varepsilon(\Sigma(x), [U])(x) = \int_0^D I(U(x + \varepsilon z) - \Sigma(x)) dz, \quad (3.20)$$

$$(3.21)$$

and

$$\begin{aligned} L^\varepsilon(y, \Sigma(x), [U])(x) &= \alpha K^\varepsilon(\Sigma(x), [U])(x) \\ &\quad - 2V\left(N^\varepsilon(\Sigma(x), [U])(x) + K^\varepsilon(\Sigma(x), [U])(x)\right) \cdot \phi(y - K^\varepsilon(\Sigma(x), [U])(x)). \end{aligned} \quad (3.22)$$

The bounds provided by Remark 3.6 remain valid for the non-local operators depending on $\varepsilon > 0$.

Remark 3.7 (Lagrangian formulation). *Another way to treat this problem is to consider a Lagrangian formulation, like in [1], considering the functions,*

$$u(t, y) = U_{[y]}(t) \quad \text{and} \quad \xi(t, y) = \Sigma_{[y]}(t).$$

The couple (u, ξ) satisfies for all $(t, y) \in [0, T] \times \mathbb{R}$

$$\begin{cases} u_t(t, y) = \alpha(\xi(t, y) - u(t, y)) \\ \xi_t(t, y) = \alpha(u(t, y) - \xi(t, y)) + 2V(u(t, y + 1) - u(t, y)) \cdot \phi(u(t, y)) \\ u(0, y) = u_0(y) \\ \xi(0, y) = \xi_0^\varepsilon(y). \end{cases}$$

We note that the system we obtain is much more simple. Nevertheless, the difficulty with this formulation is that the function ϕ is evaluated at $u(t, y)$ and not at a physical point of the road. At the macroscopic scale, we then expect to get a junction condition located at $u = 0$. The notion of junction in this case is not well defined and this is why we use the formulation (3.3) (where the perturbation function is evaluated at a point of the road). This will allow us to use the results of Imbert and Monneau [IM14] concerning quasi-convex Hamiltonians with a junction condition.

4 Viscosity Solutions

This section is devoted to the definition and useful results for viscosity solutions of the problems considered in this paper. The reader is referred to the user's guide of Crandall, Ishii, Lions [CIL92] and the book of Barles [Bar94] for an introduction to viscosity solutions. In order to give a general definition, we will give the definition of viscosity solutions for the following equation, with $p \in \mathbb{R}$, and for all $(t, x) \in (0, +\infty) \times \mathbb{R}$,

$$\begin{cases} u_t + \psi(x) \cdot M_p(u(t, x), [\xi(t, \cdot)])(x) \cdot |p + u_x| + (1 - \psi(x)) \cdot \bar{H}(u_x) = 0 \\ \xi_t + \psi(x) \cdot L_p(x, \xi(t, x), [u(t, \cdot)])(x) \cdot |p + \xi_x| + (1 - \psi(x)) \cdot \bar{H}(\xi_x) = 0 \\ u(0, x) = u_0(x) \\ \xi(0, x) = \xi_0(x), \end{cases} \quad (4.1)$$

with $\psi : \mathbb{R} \rightarrow [0, 1]$ a Lipschitz continuous function. We also use the following notations for the upper and lower semi-continuous envelopes of a locally bounded function u :

$$u^*(t, x) = \limsup_{s \rightarrow t, y \rightarrow x} u(s, y) \quad \text{and} \quad u_*(t, x) = \liminf_{s \rightarrow t, y \rightarrow x} u(s, y).$$

4.1 Definitions

Definition 4.1 (Viscosity solutions for (4.1)). *Let $T > 0$. Let $u : \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}$ and $\xi : \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}$ be upper semi-continuous (resp. lower semi-continuous) functions. We say that (u, ξ) is a viscosity sub-solution (resp. super-solution) of (4.1) on $[0, T] \times \mathbb{R}$ if $u(0, x) \leq u_0(x)$ and $\xi(0, x) \leq \xi_0(x)$ (resp. $u(0, x) \geq u_0(x)$ and $\xi(0, x) \geq \xi_0(x)$) and for all $(t, x) \in (0, T) \times \mathbb{R}$, and for any test function $\varphi \in C^1((0, T) \times \mathbb{R})$ such that $u - \varphi$ attains a local maximum (resp. a local minimum) at the point (t, x) , we have*

$$\begin{aligned} & \varphi_t + \psi(x) \cdot M_p(u(t, x), [\xi(t, \cdot)])(x) \cdot |p + \varphi_x| + (1 - \psi(x)) \cdot \bar{H}(\varphi_x) \leq 0, \\ (\text{resp. } & \varphi_t + \psi(x) \cdot \tilde{M}_p(u(t, x), [\xi(t, \cdot)])(x) \cdot |p + \varphi_x| + (1 - \psi(x)) \cdot \bar{H}(\varphi_x) \geq 0), \end{aligned}$$

and for all $(t, x) \in (0, T) \times \mathbb{R}$ and any test function $\varphi \in C^1((0, T) \times \mathbb{R})$ such that $\xi - \varphi$ attains a local maximum (resp. a local minimum) at the point (t, x) , we have

$$\begin{aligned} & \varphi_t + \psi(x) \cdot L_p(x, \xi(t, x), [u(t, \cdot)])(x) \cdot |p + \varphi_x| + (1 - \psi(x)) \cdot \bar{H}(\varphi_x) \leq 0, \\ (\text{resp. } & \varphi_t + \psi(x) \cdot \tilde{L}_p(x, \xi(t, x), [u(t, \cdot)])(x) \cdot |p + \varphi_x| + (1 - \psi(x)) \cdot \bar{H}(\varphi_x) \geq 0). \end{aligned}$$

We say that (u, ξ) is a viscosity solution of (4.1) if (u^*, ξ^*) and (u_*, ξ_*) are respectively a sub-solution and a super-solution of (4.1).

Remark 4.2. *We use this definition in order to have a stability result for the non-local terms. We refer to [DLKS04, Sle03] for such kind of definition. For the readers convenience, we give the stability result and its proof.*

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Proposition 4.3 (Stability of solution of (4.1)). *Let (u_n, ξ_n) be a sequence of uniformly bounded upper semi-continuous functions (resp. lower semi-continuous).*

Let $(\bar{u}, \bar{\xi}) = (\limsup^ u_n, \limsup^* \xi_n)$ (resp. $(\underline{u}, \underline{\xi}) = (\liminf_* u_n, \liminf_* \xi_n)$) and let $(t_n, x_n, p_n) \rightarrow (t_0, x_0, p)$ in $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ be such that*

$$\begin{aligned} (u_n(t_n, x_n), \xi_n(t_n, x_n)) &\rightarrow (\bar{u}(t_0, x_0), \bar{\xi}(t_0, x_0)), \\ (\text{resp. } (u_n(t_n, x_n), \xi_n(t_n, x_n)) &\rightarrow (\underline{u}(t_0, x_0), \underline{\xi}(t_0, x_0))), \end{aligned}$$

then

$$\limsup_{n \rightarrow +\infty}^* M_{p_n}(u_n(t_n, x_n), [\xi_n(t_n, \cdot)])(x_n) \geq M_p(\bar{u}(t_0, x_0), [\bar{\xi}(t_0, \cdot)])(x_0), \quad (4.2)$$

$$\left(\text{resp. } \liminf_{n \rightarrow +\infty}^* \tilde{M}_{p_n}(u_n(t_n, x_n), [\xi_n(t_n, \cdot)])(x_n) \leq \tilde{M}_p(\bar{u}(t_0, x_0), [\bar{\xi}(t_0, \cdot)])(x_0) \right),$$

and

$$\limsup_{n \rightarrow +\infty}^* L_{p_n}(x_n, \xi_n(t_n, x_n), [u_n(t_n, \cdot)])(x_n) \geq L_p(x_0, \bar{\xi}(t_0, x_0), [\bar{u}(t_0, \cdot)])(x_0), \quad (4.3)$$

$$\left(\text{resp. } \liminf_{n \rightarrow +\infty}^* \tilde{L}_{p_n}(x_n, \xi_n(t_n, x_n), [u_n(t_n, \cdot)])(x_n) \leq \tilde{L}_p(x_0, \bar{\xi}(t_0, x_0), [\bar{u}(t_0, \cdot)])(x_0) \right).$$

In order to do the proof of Proposition 4.3, we use the following lemma which proof is given in [Sle03].

Lemma 4.4. *Let (f_n) be a sequence of measurable functions of \mathbb{R} and consider*

$$\bar{f} = \limsup^* f_n$$

and

$$\underline{f} = \liminf_* f_n.$$

Let $(a_n)_n$ be a sequence of \mathbb{R} converging to zero as n goes to infinity. Then

$$\mathcal{L}(\{f_n \geq a_n\} \setminus \{\bar{f} \geq 0\}) \rightarrow 0 \quad \text{as } n \rightarrow +\infty$$

and

$$\mathcal{L}(\{\underline{f} > 0\} \setminus \{f_n > a_n\}) \rightarrow 0 \quad \text{as } n \rightarrow +\infty,$$

where $\mathcal{L}(A)$ denotes the lebesgue measure of the measurable set A .

Proof of proposition 4.3. We begin by proving (4.2). Let $\varepsilon > 0$, then by definition

$$\begin{aligned}
 & M_{p_n}(u_n(t_n, x_n), [\xi(t_n, \cdot)])(x_n) - M_p(\bar{u}(t_0, x_0), [\bar{\xi}(t_0, \cdot)])(x_0) \\
 &= \int_0^D E(\xi_n(t_n, x_n + z) - u_n(t_n, x_n) + p_n z) dz - \int_0^D E(\bar{\xi}(t_0, x_0 + z) - \bar{u}(t_0, x_0) + p z) dz \\
 &= \alpha \int_0^D \left(\mathbf{1}_{\{\bar{\xi}(t_0, x_0 + z) - \bar{u}(t_0, x_0) + p z \geq 0\}} - \mathbf{1}_{\{\xi_n(t_n, x_n + z) - u_n(t_n, x_n) + p_n z \geq 0\}} \right) dz \\
 &\geq -\alpha \int_0^D \mathbf{1}_{\{\{f_n \geq 0\} \cup \{\bar{f} \geq 0\} \setminus \{\bar{f} \geq 0\}\}} dz
 \end{aligned}$$

with

$$f_n(z) = \xi_n(t_n, x_n + z) - u_n(t_n, x_n) + p_n z \quad \text{and} \quad \bar{f}(z) = \bar{\xi}(t_0, x_0 + z) - \bar{u}(t_0, x_0) + p z.$$

Using Lemma 4.4, for n big we have that

$$-\alpha \int_0^D \mathbf{1}_{\{\{f_n \geq 0\} \cup \{\bar{f} \geq 0\} \setminus \{\bar{f} \geq 0\}\}} dz \geq -\varepsilon. \tag{4.4}$$

Using (4.4), we deduce that

$$M_{p_n}(u_n(t_n, x_n), [\xi(t_n, \cdot)])(x_n) \geq M_p(\bar{u}(t_0, x_0), [\bar{\xi}(t_0, \cdot)])(x_0) - \varepsilon,$$

which implies (4.2).

We turn now to the proof of (4.3). We recall that the definition of L is given in (3.16). To simplify the notations, we introduce

$$\begin{aligned}
 L_{p_n} &= L_{p_n}(x_n, \xi_n(t_n, x_n), [u_n(t_n, \cdot)])(x_n) & L_p &= L_p(x_0, \bar{\xi}(t_0, x_0), [\bar{u}(t_0, \cdot)])(x_0), \\
 K_{p_n} &= K_{p_n}(\xi_n(t_n, x_n), [u_n(t_n, \cdot)])(x_n) & K_p &= K_p(\bar{\xi}(t_0, x_0), [\bar{u}(t_0, \cdot)])(x_0), \\
 N_{p_n} &= N_{p_n}(\xi_n(t_n, x_n), [u_n(t_n, \cdot)])(x_n) & N_p &= N_p(\bar{\xi}(t_0, x_0), [\bar{u}(t_0, \cdot)])(x_0).
 \end{aligned}$$

Proceeding like before, we have for n big enough

$$K_{p_n} \geq K_p - \varepsilon, \tag{4.5}$$

$$N_{p_n} \leq N_p + \varepsilon, \tag{4.6}$$

and

$$|x_n - x_0| \leq \varepsilon. \tag{4.7}$$

Therefore, we have that

$$\begin{aligned}
 L_{p_n} &= \alpha K_{p_n} - 2V(N_{p_n} + K_{p_n})\phi(x_n - K_{p_n}) \\
 &\geq \alpha(K_p - \varepsilon) - 2V(N_{p_n} + K_p - \varepsilon)\phi(x_n - K_p + \varepsilon) \\
 &\geq \alpha(K_p - \varepsilon) - 2V(N_p + \varepsilon + K_p - \varepsilon)\phi(x_n - K_p + \varepsilon) \\
 &\geq \alpha K_p - 2V(N_p + K_p)\phi(x_0 - K_p) - \alpha\varepsilon - 4V_{max}\|\phi'\|_\infty\varepsilon \\
 &= L_p - \alpha\varepsilon - 4V_{max}\|\phi'\|_\infty\varepsilon,
 \end{aligned}$$

where we have used Remark 2.2 combined with (4.5) for the first inequality, (4.6) and the fact that V is non-decreasing for the second inequality. Finally, for the third inequality, we have used (4.7) and the fact that ϕ is Lipschitz continuous.

The previous inequality is true for all $\varepsilon > 0$, for n big enough and implies (4.3). This ends the proof of Proposition 4.3. □

4.2 Viscosity solutions for (2.7)

The theory of viscosity solutions for Hamilton-Jacobi equations on networks was recently treated in several papers. We give here some results for viscosity solutions of (2.7) that will be used in the rest of paper and we refer to [IM14] for the general theory and for the proofs.

Definition 4.5 (Class of test function for (2.7)). *We denote by $J_\infty := (0, +\infty) \times \mathbb{R}$, $J_\infty^+ := (0, +\infty) \times (0, +\infty)$ and $J_\infty^- := (0, +\infty) \times (-\infty, 0)$, we define a class of test function on J_∞ by*

$$C^1(J_\infty) = \{\varphi \in C(J_\infty), \text{ the restriction of } \varphi \text{ to } J_\infty^+ \text{ and to } J_\infty^- \text{ are } C^1\}.$$

Definition 4.6 (Viscosity solution for (2.7)). *An upper semi-continuous (resp. lower semi-continuous) function $u : [0, +\infty) \times \mathbb{R} \rightarrow \mathbb{R}$ is a viscosity sub-solution (resp. super-solution) of (2.7) if $u(0, x) \leq u_0(x)$ (resp. $u(0, x) \geq u_0(x)$) and for all $(t, x) \in J_\infty$ and for all $\varphi \in C^1(J_\infty)$ such that $u \leq \varphi$ (resp. $u \geq \varphi$) in a neighbourhood of $(t, x) \in J_\infty$ and $u(t, x) = \varphi(t, x)$, we have*

$$\begin{aligned}
 \varphi_t(t, x) + \overline{H}(\varphi_x(t, x)) &\leq 0 \quad (\text{resp. } \geq 0) && \text{if } x \neq 0 \\
 \varphi_t(t, x) + F_A(\varphi_x(t, 0^-), \varphi_x(t, 0^+)) &\leq 0 \quad (\text{resp. } \geq 0) && \text{if } x = 0.
 \end{aligned}$$

We say that a function u is a viscosity solution of (2.7) if u^ and u_* are respectively a sub-solution and a super-solution of (2.7). We refer to this solution as A -flux-limited solution.*

Now we recall an equivalent definition (Theorem 2.5 in [IM14]) for sub and super solution at the junction. We will also consider the following problem,

$$u_t + \overline{H}(u_x) = 0 \quad \text{for } t \in (0, T) \text{ and } x \in \mathbb{R} \setminus \{0\}. \tag{4.8}$$

Theorem 4.7 (Equivalent definition for sub/super-solutions). *Let \bar{H} given by (2.4) and consider $A \in [H_0, +\infty)$ with H_0 defined in (2.6). Given arbitrary solutions $p_{\pm}^A \in \mathbb{R}$ of*

$$\bar{H}(p_+^A) = \bar{H}^+(p_+^A) = A = \bar{H}^-(p_-^A) = \bar{H}(p_-^A), \quad (4.9)$$

let us fix any time independent test function $\phi^0(x)$ satisfying

$$\phi_x^0(0^\pm) = p_{\pm}^A.$$

Given a function $u : (0, T) \times \mathbb{R} \rightarrow \mathbb{R}$, the following properties hold true.

1. If u is an upper semi-continuous sub-solution of (4.8) satisfying

$$u(t, 0) = \limsup_{(t,y) \rightarrow (t,0), y \in J_t^*} u(s, y), \quad (4.10)$$

then u is a H_0 -flux limited sub-solution.

2. Given $A > H_0$ and $t_0 \in (0, T)$, if u is an upper semi-continuous sub-solution of (4.8) satisfying (4.10) and if for any test function φ touching u from above at $(t_0, 0)$ with

$$\varphi(t, x) = \psi(t) + \phi^0(x), \quad (4.11)$$

for some $\psi \in C^1(0, +\infty)$, we have

$$\varphi_t + F_A(\varphi_x(t_0, 0^-), \varphi_x(t_0, 0^+)) \leq 0 \quad \text{at } (t_0, 0),$$

then u is a A -flux limited sub-solution at $(t_0, 0)$.

3. Given $t_0 \in (0, T)$, if u is a lower semi-continuous super-solution of (4.8) and if for any test function φ satisfying (4.11) touching u from above at $(t_0, 0)$ we have

$$\varphi_t + F_A(\varphi_x(t_0, 0^-), \varphi_x(t_0, 0^+)) \geq 0 \quad \text{at } (t_0, 0),$$

then u is a A -flux limited super-solution at $(t_0, 0)$.

4.3 Existence and uniqueness of viscosity solution for (4.1) with $p = 0$

We recall that for $p = 0$, our equation is

$$\begin{cases} u_t + M(u(t, x), [\xi(t, \cdot)])(x) \cdot |u_x| = 0 & \text{for } (t, x) \in (0, +\infty) \times \mathbb{R}, \\ \xi_t + L(x, \xi(t, x), [u(t, \cdot)])(x) \cdot |\xi_x| = 0 & \text{for } (t, x) \in (0, +\infty) \times \mathbb{R}, \\ u(0, x) = u_0(x) & \text{for } x \in \mathbb{R}, \\ \xi(0, x) = \xi_0(x) & \text{for } x \in \mathbb{R}. \end{cases} \quad (4.12)$$

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Lemma 4.8 (Existence of barriers for (4.12)). *Assume (A) and (A0). There exists a constant $K_1 > 0$ such that*

$$(u^-, \xi^-) = (u_0 - K_1 t, \xi_0 - K_1 t) \quad \text{and} \quad (u^+, \xi^+) = (u_0 + K_1 t, \xi_0 + K_1 t) \quad (4.13)$$

are respectively sub-solution and super-solution of (4.12).

Proof. We define $K_1 = M_0 k_0$. Let us prove that (u^+, ξ^+) is a super-solution of (4.12). In fact, we have that

$$u_t^+ + \tilde{M}(u^+(t, x), [\xi^+(t, \cdot)])(x) |u_x^+| \geq K_1 - M_0 k_0 = 0,$$

where we have used Remark 3.6 for the second inequality. Similarly, using that $\tilde{K} \geq 0$ and $K_1 \geq 2\|V\|_\infty k_0$, we have that

$$\xi_t^+ + \tilde{L}(x, \xi^+(t, x), [u^+(t, \cdot)])(x) |\xi_t^+| \geq 0.$$

The proof that (u^-, ξ^-) is a sub-solution is similar and we skip it. □

Proposition 4.9 (Comparison principle). *Let $T > 0$. Assume (A)-(A0). Let (u, ξ) and (v, ζ) be respectively a sub-solution and a super-solution of (4.12). We also assume that there exists a constant $C > 0$ such that for all $(t, x) \in [0, T] \times \mathbb{R}$, we have*

$$u_0(x) - Ct \leq u(t, x) \leq u_0(x) + Ct, \quad \xi_0(x) - Ct \leq \xi(t, x) \leq \xi_0(x) + Ct \quad (4.14)$$

and

$$-u_0(x) - Ct \leq -v(t, x) \leq -u_0(x) + Ct, \quad -\xi_0(x) - Ct \leq -\zeta(t, x) \leq -\xi_0(x) + Ct \quad (4.15)$$

If

$$u(0, x) \leq v(0, x) \quad \text{and} \quad \xi(0, x) \leq \zeta(0, x) \quad \text{for all } x \in \mathbb{R},$$

then

$$u(t, x) \leq v(t, x) \quad \text{and} \quad \xi(t, x) \leq \zeta(t, x) \quad \text{for all } x \in \mathbb{R}, t \in [0, T].$$

Proof. Let us introduce

$$\overline{M} = \sup_{t \in [0, T], x \in \mathbb{R}} \max(u(t, x) - v(t, x), \xi(t, x) - \zeta(t, x)).$$

We want to prove that $\overline{M} \leq 0$. We argue by contradiction by assuming that $\overline{M} > 0$.

Step 1: test functions. We introduce the following test functions

$$\varphi(t, x, y) = u(t, x) - v(t, y) - \frac{\eta}{T-t} - e^{At} \left(\frac{(x-y)^2}{2\varepsilon} + \gamma \frac{x^2}{2} \right)$$

and

$$\psi(t, x, y) = \xi(t, x) - \zeta(t, y) - \frac{\eta}{T-t} - e^{At} \left(\frac{(x-y)^2}{2\varepsilon} + \gamma \frac{x^2}{2} \right),$$

with η, γ small parameters, and A a constant to be chosen later. We denote by $\Phi(t, x, y) = \max(\varphi(t, x, y), \psi(t, x, y))$. Using (4.14) and (4.15) we have that

$$\begin{aligned} \varphi(t, x, y) &\leq u_0(x) - u_0(y) + 2CT - \frac{\eta}{T-t} - e^{At} \left(\frac{(x-y)^2}{2\varepsilon} + \gamma \frac{x^2}{2} \right) \\ &\leq 2CT + k_0|x-y| - \frac{\eta}{T-t} - e^{At} \left(\frac{(x-y)^2}{2\varepsilon} + \gamma \frac{x^2}{2} \right). \end{aligned}$$

We have a similar result for ψ which yields that

$$\lim_{|x|, |y| \rightarrow +\infty} \Phi = -\infty.$$

Using the fact that our test functions are upper semi continuous, we can see that Φ reaches a maximum at some finite point that we denote by $(\bar{t}, \bar{x}, \bar{y}) \in [0, T) \times \mathbb{R} \times \mathbb{R}$. Classically we have for η and γ small enough,

$$\begin{cases} M_{\eta, \varepsilon, \gamma} = \Phi(\bar{t}, \bar{x}, \bar{y}) \geq \frac{\overline{M}}{2} > 0, \\ |\bar{x} - \bar{y}| \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0, \\ \gamma|\bar{x}| \rightarrow 0 \quad \text{as } \gamma \rightarrow 0. \end{cases}$$

Step 2: $\bar{t} > 0$ for ε small enough. By contradiction, let us assume that Φ reaches its maximum for $\bar{t} = 0$. Let us for instance assume that $\Phi(\bar{t}, \bar{x}, \bar{y}) = \varphi(\bar{t}, \bar{x}, \bar{y})$. In this case, we have

$$0 < \frac{\overline{M}}{2} \leq u(0, \bar{x}) - v(0, \bar{y}) - \frac{\eta}{T} \leq k_0|\bar{x} - \bar{y}| - \frac{\eta}{T}.$$

Therefore, $\frac{\eta}{T} < k_0|\bar{x} - \bar{y}|$ and for ε small enough we get a contradiction. In the same way, we get a contradiction if we assume that $\Phi(\bar{t}, \bar{x}, \bar{y}) = \psi(\bar{t}, \bar{x}, \bar{y})$.

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Step 3: viscosity inequalities in the case $\Phi(\bar{t}, \bar{x}, \bar{y}) = \varphi(\bar{t}, \bar{x}, \bar{y})$. By duplication of the time variable and passing to the limit we have that there exist two real numbers $a, b \in \mathbb{R}$ such that

$$a - b = \frac{\eta}{(T - \bar{t})^2} + Ae^{A\bar{t}} \left(\frac{(\bar{x} - \bar{y})^2}{2\varepsilon} + \gamma \frac{\bar{x}^2}{2} \right) \quad (4.16)$$

$$a + M(u(\bar{t}, \bar{x}), [\xi(\bar{t}, \cdot)])(\bar{x}) |e^{A\bar{t}}(p_\varepsilon + \gamma\bar{x})| \leq 0 \quad (4.17)$$

$$b + \tilde{M}(v(\bar{t}, \bar{y}), [\zeta(\bar{t}, \cdot)])(\bar{y}) |e^{A\bar{t}}p_\varepsilon| \geq 0 \quad (4.18)$$

with $p_\varepsilon = \frac{\bar{x} - \bar{y}}{\varepsilon}$. Combining (4.16), (4.17) and (4.18), we obtain

$$\begin{aligned} \frac{\eta}{(T - \bar{t})^2} + Ae^{A\bar{t}} \left(\frac{(\bar{x} - \bar{y})^2}{2\varepsilon} + \gamma \frac{\bar{x}^2}{2} \right) &\leq |e^{A\bar{t}}p_\varepsilon| \tilde{M}(v(\bar{t}, \bar{y}), [\zeta(\bar{t}, \cdot)])(\bar{y}) \\ &\quad - |e^{A\bar{t}}p_\varepsilon| M(u(\bar{t}, \bar{x}), [\xi(\bar{t}, \cdot)])(\bar{x}) + o_\gamma, \end{aligned} \quad (4.19)$$

where we have used the fact that $M(u(\bar{t}, \bar{x}), [\xi(\bar{t}, \cdot)])(\bar{x})$ is finite according to Remark 3.6.

We distinguish two cases.

Case 1: there exists a subsequence γ_n such that

$$\frac{|\bar{x} - \bar{y}|}{\varepsilon} \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

In this case, taking γ going to zero in (4.19) yields a contradiction.

Case 2: there exists a constant $C_\varepsilon > 0$ such that for any γ small enough we have,

$$\frac{|\bar{x} - \bar{y}|}{\varepsilon} \geq C_\varepsilon.$$

Changing variables in (4.19) we can write

$$\begin{aligned} \frac{\eta}{(T - \bar{t})^2} + Ae^{A\bar{t}} \frac{(\bar{x} - \bar{y})^2}{2\varepsilon} &\leq |e^{A\bar{t}}p_\varepsilon| \int_{\bar{y}}^{D+\bar{y}} \tilde{E}(\zeta(\bar{t}, z) - v(\bar{t}, \bar{y})) dz \\ &\quad - |e^{A\bar{t}}p_\varepsilon| \int_{\bar{x}}^{D+\bar{x}} E(\xi(\bar{t}, z) - u(\bar{t}, \bar{x})) dz + o_\gamma(1) \\ &\leq |e^{A\bar{t}}p_\varepsilon| \int_{\bar{y}}^{D+\bar{y}} \tilde{E}(\zeta(\bar{t}, z) - v(\bar{t}, \bar{y})) - E(\xi(\bar{t}, z) - u(\bar{t}, \bar{x})) dz \\ &\quad + |e^{A\bar{t}}p_\varepsilon| \left| \int_{\bar{y}}^{\bar{x}} E(\xi(\bar{t}, z) - u(\bar{t}, \bar{x})) dz \right| \\ &\quad + |e^{A\bar{t}}p_\varepsilon| \left| \int_{D+\bar{x}}^{D+\bar{y}} E(\xi(\bar{t}, z) - u(\bar{t}, \bar{x})) dz \right| + o_\gamma(1). \end{aligned} \quad (4.20)$$

We define

$$\mathcal{A} = \left\{ z \in \mathbb{R} : \tilde{E}(\zeta(\bar{t}, z) - v(\bar{t}, \bar{y})) \leq E(\xi(\bar{t}, z) - u(\bar{t}, \bar{x})) \right\}.$$

The inequality $\varphi(\bar{t}, \bar{x}, \bar{y}) \geq \psi(\bar{t}, z, z)$ yields

$$\zeta(\bar{t}, z) - v(\bar{t}, \bar{y}) \geq \xi(\bar{t}, z) - u(\bar{t}, \bar{x}) + e^{A\bar{t}} \left(\frac{(\bar{x} - \bar{y})^2}{2\varepsilon} + \gamma \frac{\bar{x}^2}{2} - \gamma \frac{z^2}{2} \right).$$

This implies that

$$\mathcal{A}^c \subset \{|z| \geq R_{\varepsilon, \gamma}\} \quad \text{with } R_{\varepsilon, \gamma}^2 = \frac{2}{\gamma} \left(\frac{(\bar{x} - \bar{y})^2}{2\varepsilon} + \gamma \frac{\bar{x}^2}{2} \right).$$

Moreover $\tilde{R}_{\varepsilon, \gamma} = R_{\varepsilon, \gamma} - |\bar{y}| \rightarrow +\infty$ as $\gamma \rightarrow 0$ (see Da Lio *et al.* in [DLFM08, Lemma 2.5]). This implies that

$$\begin{aligned} \int_{\bar{y}}^{D+\bar{y}} \tilde{E}(\zeta(\bar{t}, z) - v(\bar{t}, \bar{y})) dz &= \int_{[\bar{y}, D+\bar{y}] \cap \mathcal{A}} \tilde{E}(\zeta(\bar{t}, z) - v(\bar{t}, \bar{y})) dz \\ &\quad + \int_{[\bar{y}, D+\bar{y}] \cap \mathcal{A}^c} \tilde{E}(\zeta(\bar{t}, z) - v(\bar{t}, \bar{y})) dz. \end{aligned}$$

However, from Remark 3.6, we have that for γ small enough

$$\begin{aligned} 0 &\leq \int_{[\bar{y}, D+\bar{y}] \cap \mathcal{A}^c} -\tilde{E}(\zeta(\bar{t}, z) - v(\bar{t}, \bar{y})) dz = \int_{[\bar{y}, D+\bar{y}] \cap \{|z| \geq R_{\varepsilon, \gamma}\}} -\tilde{E}(\zeta(\bar{t}, z) - v(\bar{t}, \bar{y})) dz \\ &= \int_{[0, D] \cap \{|z+\bar{y}| \geq R_{\varepsilon, \gamma}\}} -\tilde{E}(\zeta(\bar{t}, z + \bar{y}) - v(\bar{t}, \bar{y})) dz \\ &\leq \int_{[0, D] \cap \{|z| \geq \tilde{R}_{\varepsilon, \gamma}\}} -\tilde{E}(\zeta(\bar{t}, z + \bar{y}) - v(\bar{t}, \bar{y})) dz \\ &= 0. \end{aligned}$$

We deduce that for γ small enough,

$$\int_{\bar{y}}^{D+\bar{y}} \tilde{E}(\zeta(\bar{t}, z) - v(\bar{t}, \bar{y})) dz \leq \int_{\bar{y}}^{D+\bar{y}} E(\xi(\bar{t}, z) - u(\bar{t}, \bar{x})) dz.$$

Then for γ small enough (4.20) implies

$$\begin{aligned} \frac{\eta}{(T - \bar{t})^2} + Ae^{A\bar{t}} \frac{(\bar{x} - \bar{y})^2}{2\varepsilon} &\leq \left| e^{A\bar{t}} p_\varepsilon \right| \left| \int_{\bar{y}}^{\bar{x}} E(\xi(\bar{t}, z) - u(\bar{t}, \bar{x})) dz + \int_{D+\bar{x}}^{D+\bar{y}} E(\xi(\bar{t}, z) - u(\bar{t}, \bar{x})) dz \right| \\ &\quad + o_\gamma \\ &\leq 2\alpha e^{A\bar{t}} \frac{(\bar{x} - \bar{y})^2}{\varepsilon} + o_\gamma. \end{aligned}$$

Choosing $A = 4\alpha$, we get a contradiction.

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Step 4: viscosity inequalities in the case $\Phi(\bar{t}, \bar{x}, \bar{y}) = \psi(\bar{t}, \bar{x}, \bar{y})$. By duplication of the time variable and passing to the limit, we have that there exist two real numbers $a, b \in \mathbb{R}$ such that

$$a - b = \frac{\eta}{(T - \bar{t})^2} + Ae^{A\bar{t}} \left(\frac{(\bar{x} - \bar{y})^2}{2\varepsilon} + \gamma \frac{\bar{x}^2}{2} \right) \quad (4.21)$$

$$a + L(\bar{x}, \xi(\bar{t}, \bar{x}), [u(\bar{t}, \cdot)])(\bar{x}) \cdot |e^{A\bar{t}}(p_\varepsilon + \gamma\bar{x})| \leq 0 \quad (4.22)$$

$$b + \tilde{L}(\bar{y}, \xi(\bar{t}, \bar{y}), [u(\bar{t}, \cdot)])(\bar{y}) \cdot |e^{A\bar{t}}p_\varepsilon| \geq 0 \quad (4.23)$$

with $p_\varepsilon = \frac{\bar{x} - \bar{y}}{\varepsilon}$. Combining (4.21), (4.22) and (4.23), we obtain that

$$\begin{aligned} \frac{\eta}{(T - \bar{t})^2} + Ae^{A\bar{t}} \frac{(\bar{x} - \bar{y})^2}{2\varepsilon} &\leq |e^{A\bar{t}}p_\varepsilon| \left(\tilde{L}(\bar{y}, \xi(\bar{t}, \bar{y}), [u(\bar{t}, \cdot)])(\bar{y}) - L(\bar{x}, \xi(\bar{t}, \bar{x}), [u(\bar{t}, \cdot)])(\bar{x}) \right) \\ &\quad + o_\gamma. \end{aligned} \quad (4.24)$$

We recall that we defined L and \tilde{L} using K and V (see (3.11) and (3.12)). Therefore, we can see that the right part of inequality (4.24) is finite (using Remark 3.6). We distinguish two cases.

Case 1: there exists a subsequence γ_n such that

$$\frac{|\bar{x} - \bar{y}|}{\varepsilon} \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

In this case, taking γ to zero in (4.24) yields a contradiction.

Case 2: there exists a constant $C_\varepsilon > 0$, such that for any γ small enough we have

$$\frac{|\bar{x} - \bar{y}|}{\varepsilon} \geq C_\varepsilon.$$

To simplify, we introduce

$$\begin{aligned} L &= L(\bar{x}, \xi(\bar{t}, \bar{x}), [u(\bar{t}, \cdot)])(\bar{x}) & \tilde{L} &= \tilde{L}(\bar{y}, \zeta(\bar{t}, \bar{y}), [v(\bar{t}, \cdot)])(\bar{y}), \\ K &= K(\xi(\bar{t}, \bar{x}), [u(\bar{t}, \cdot)])(\bar{x}) & \tilde{K} &= \tilde{K}(\zeta(\bar{t}, \bar{y}), [v(\bar{t}, \cdot)])(\bar{y}), \\ N &= N(\xi(\bar{t}, \bar{x}), [u(\bar{t}, \cdot)])(\bar{x}) & \tilde{N} &= \tilde{N}(\zeta(\bar{t}, \bar{y}), [v(\bar{t}, \cdot)])(\bar{y}). \end{aligned}$$

As above, we can prove

$$\tilde{K} - K \leq |\bar{x} - \bar{y}| \quad \text{and} \quad N - \tilde{N} \leq |\bar{x} - \bar{y}|.$$

We have that

$$\begin{aligned}
 \tilde{L} - L &= \alpha \tilde{K} - 2V(\tilde{N} + \tilde{K}) \phi(\bar{y} - \tilde{K}) - L \\
 &\leq \alpha(K + |\bar{x} - \bar{y}|) - 2V(\tilde{N} + K + |\bar{x} - \bar{y}|) \phi(\bar{y} - K - |\bar{x} - \bar{y}|) - L \\
 &\leq \alpha(K + |\bar{x} - \bar{y}|) - 2V(N + K) \phi(\bar{y} - K - |\bar{x} - \bar{y}|) - L \\
 &\leq \alpha|\bar{x} - \bar{y}| + 2V(N + K) (\phi(\bar{x} - K) - \phi(\bar{y} - K - |\bar{x} - \bar{y}|)) \\
 &\leq \alpha|\bar{x} - \bar{y}| + 2\|V\|_\infty \|\phi'\|_\infty |\bar{x} - \bar{y}|, \tag{4.25}
 \end{aligned}$$

where we have used for the first inequality the monotonicity (see Remark 2.2). The monotonicity of V yields the second inequality. The third and the final inequalities come from the definition of L and the fact that ϕ is a Lipschitz function and that V is bounded. Finally, combining (4.25) with (4.24), we obtain

$$\frac{\eta}{(T - \bar{t})^2} + Ae^{A\bar{t}} \frac{(\bar{x} - \bar{y})^2}{2\varepsilon} \leq e^{A\bar{t}} \frac{(\bar{x} - \bar{y})^2}{\varepsilon} (\alpha + 2\|V\|_\infty \|\phi'\|_\infty) + o_\gamma(1). \tag{4.26}$$

Taking $A = 2(\alpha + 2\|V\|_\infty \|\phi'\|_\infty)$, we get a contradiction in (4.26). The proof of Proposition 4.9 is now complete. □

We now give a comparison principle on bounded sets, to do this, we define for a given point $(t_0, x_0) \in (0, T) \times \mathbb{R}$ and for $\bar{r}, \bar{R} > 0$, the set

$$\mathcal{Q}_{\bar{r}, \bar{R}}(t_0, x_0) = (t_0 - \bar{r}, t_0 + \bar{r}) \times (x_0 - \bar{R}, x_0 + \bar{R}).$$

Proposition 4.10 (Comparison principle on bounded sets for (4.12)). *Assume (A). Let (u, ξ) be a sub-solution of (4.12) and let (v, ζ) be a super-solution of (4.12) on the open set $\mathcal{Q}_{\bar{r}, \bar{R}} \subset (0, T) \times \mathbb{R}$. We assume that u and ξ (resp. v and ζ) are upper semi-continuous (resp. lower semi-continuous) on $\overline{\mathcal{Q}_{\bar{r}, \bar{R}}}$. Also assume that*

$$u \leq v \quad \text{and} \quad \xi \leq \zeta \quad \text{outside } \mathcal{Q}_{\bar{r}, \bar{R}},$$

then

$$u \leq v \quad \text{and} \quad \xi \leq \zeta \quad \text{on } \mathcal{Q}_{\bar{r}, \bar{R}}.$$

Applying Perron's method (see [IMR08, Proof of Theorem 6], [AT96] or [Imb05] to see how to apply Perron's method for problems with non-local terms), joint to the comparison principle, we obtain the following result.

Theorem 4.11 (Existence and uniqueness of viscosity solutions for (4.12)). *Assume (A0) and (A). Then, there exists a unique solution (u, ξ) of (4.12). Moreover, the functions u and ξ are continuous and there exists a constant $K_1 > 0$ such that*

$$u_0(x) - K_1 t \leq u(t, x) \leq u_0(x) + K_1 t \quad \text{and} \quad \xi_0(x) - K_1 t \leq \xi(t, x) \leq \xi_0(x) + K_1 t. \tag{4.27}$$

4.4 Control of the oscillations for (4.12)

We now present a theorem that provides a control on the oscillations in space of the solution of (4.12). This is a very important theorem, first because it will allow us to prove Theorem 3.5 and also because it presents some of the arguments we use later to build the correctors at the junction.

Theorem 4.12 (Control of the space oscillations). *Let $T > 0$. Assume (A0)-(A) and let (u, ξ) be the solution of (4.12) provided by Theorem 4.11. Then for all $x, y \in \mathbb{R}, x \geq y$ and for all $t \in [0, T]$, we have*

$$-k_0(x - y) - 1 \leq u(t, x) - u(t, y) \leq 0 \quad (4.28)$$

and

$$-k_0(x - y) - 1 \leq \xi(t, x) - \xi(t, y) \leq 0, \quad (4.29)$$

with k_0 defined in (A0).

Proof. We use the following notation,

$$\Omega = \{(t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R} \quad \text{s.t. } x \geq y\}.$$

Proof of the upper bound. We introduce

$$\bar{M} = \sup_{(t,x,y) \in \Omega} \max(u(t, x) - u(t, y), \xi(t, x) - \xi(t, y)).$$

We want to prove that $\bar{M} \leq 0$. We argue by contradiction and assume that $\bar{M} > 0$.

Step 1: the test functions. For $\eta, \gamma > 0$ small parameters, we define

$$\varphi(t, x, y) = u(t, x) - u(t, y) - \frac{\eta}{T-t} - \gamma x^2 - \gamma y^2$$

and

$$\psi(t, x, y) = \xi(t, x) - \xi(t, y) - \frac{\eta}{T-t} - \gamma x^2 - \gamma y^2.$$

We denote by $\Phi(t, x, y) = \max(\varphi(t, x, y), \psi(t, x, y))$. For $x \geq y$, using (4.27) and (A0) we have

$$\varphi(t, x, y) \leq u_0(x) - u_0(y) + 2K_1T - \frac{\eta}{T-t} - \gamma x^2 - \gamma y^2 \leq 2K_1T - \gamma x^2 - \gamma y^2$$

$$\psi(t, x, y) \leq \xi_0(x) - \xi_0(y) + 2K_1T - \frac{\eta}{T-t} - \gamma x^2 - \gamma y^2 \leq 2K_1T - \gamma x^2 - \gamma y^2.$$

Therefore, we deduce

$$\lim_{|x|, |y| \rightarrow +\infty} \Phi(t, x, y) = -\infty.$$

Since φ, ψ are upper semi continuous, Φ reaches a maximum on Ω at a point that we denote by $(\bar{t}, \bar{x}, \bar{y})$. Classically we have for η and γ small enough

$$\begin{cases} 0 < \frac{\bar{M}}{2} \leq \Phi(\bar{t}, \bar{x}, \bar{y}), \\ \gamma|\bar{x}|, \gamma|\bar{y}| \rightarrow 0 \quad \text{as } \gamma \rightarrow 0. \end{cases}$$

Step 2: $\bar{t} > 0$ and $\bar{x} > \bar{y}$. By contradiction, assume first that $\bar{t} = 0$. For instance, we assume that $\Phi(\bar{t}, \bar{x}, \bar{y}) = \varphi(\bar{t}, \bar{x}, \bar{y})$. In this case, we have that

$$\frac{\eta}{T} \leq u_0(\bar{x}) - u_0(\bar{y}) \leq 0,$$

where we have used the fact that u_0 is non increasing, and we get a contradiction. In the same way, we get a contradiction if $\Phi(\bar{t}, \bar{x}, \bar{y}) = \psi(\bar{t}, \bar{x}, \bar{y})$. The fact that $\bar{x} > \bar{y}$ is obtained directly using that $\Phi(\bar{t}, \bar{x}, \bar{y}) > 0$.

Step 3: viscosity inequalities in the case $\Phi(\bar{t}, \bar{x}, \bar{y}) = \varphi(\bar{t}, \bar{x}, \bar{y})$. By duplication of the time variable and passing to the limit we get that

$$\frac{\eta}{T^2} \leq \frac{\eta}{(T-\bar{t})^2} \leq -M(u(\bar{t}, \bar{x}), [\xi(\bar{t}, \cdot)])(\bar{x}) \cdot |2\gamma\bar{x}|, \quad (4.30)$$

where we have used the fact that $\tilde{M}(u(\bar{t}, \bar{y}), [\xi(\bar{t}, \cdot)])(\bar{y}) \leq 0$. Using Remark 3.6, we have that $-M(u(\bar{t}, \bar{x}), [\xi(\bar{t}, \cdot)])(\bar{x})$ is bounded. Taking γ to zero, we get a contradiction in (4.30).

Step 4: viscosity inequalities in the case $\Phi(\bar{t}, \bar{x}, \bar{y}) = \psi(\bar{t}, \bar{x}, \bar{y})$. By duplication of the time variable and passing to the limit we get that

$$\begin{aligned} \frac{\eta}{(T-\bar{t})^2} &\leq \tilde{L}(\bar{y}, \xi(\bar{t}, \bar{y}), [u(\bar{t}, \cdot)])(\bar{y})|2\gamma\bar{y}| - L(\bar{x}, \xi(\bar{t}, \bar{x}), [u(\bar{t}, \cdot)])(\bar{x})|2\gamma\bar{x}| \\ &\leq 2M_0 (|\gamma\bar{x}| + |\gamma\bar{y}|) \end{aligned}$$

where we have used the bounds on L and \tilde{L} (see Remark 3.6). Taking γ to zero, we get a contradiction.

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Proof of the lower bound. In order to prove our result, we will use the following lemma which proof is postponed.

Lemma 4.13. *For all $(t, x) \in [0, T] \times \mathbb{R}$, we have*

$$0 \leq \xi(t, x) - u(t, x) \leq 1. \quad (4.31)$$

Now we would like to prove that for all $\varepsilon > 0$,

$$\bar{M} = \sup_{(t,x,y) \in \Omega} \{ \xi(t, y) - u(t, x) - (k_0 + \varepsilon)(x - y) - 1 \} \leq 0. \quad (4.32)$$

In fact, if (4.32) is true, then taking ε to 0 and using (4.31) we directly obtain the lower inequalities in (4.28) and (4.29). We argue by contradiction and assume that $\bar{M} > 0$.

Step 1: the test function. For $\eta, \gamma > 0$ small parameters, we define

$$\varphi(t, x, y) = \xi(t, y) - u(t, x) - (k_0 + \varepsilon)(x - y) - 1 - \frac{\eta}{T - t} - \gamma x^2.$$

Using (A0) and (4.27), we obtain that

$$\begin{aligned} \varphi(t, x, y) &\leq \xi_0(y) - u_0(x) + 2K_1T - (k_0 + \varepsilon)(x - y) - 1 - \frac{\eta}{T - t} - \gamma x^2 \\ &\leq 1 + k_0(x - y) + 2K_1T - (k_0 + \varepsilon)(x - y) - 1 - \frac{\eta}{T - t} - \gamma x^2 \\ &\leq 2K_1T - \gamma x^2 - \varepsilon(x - y). \end{aligned}$$

Therefore, we have that for $(t, x, y) \in \Omega$

$$\lim_{|x|, |y| \rightarrow +\infty} \varphi(t, x, y) = -\infty.$$

Since φ is upper semi continuous, φ reaches a maximum on Ω at a point that we denote by $(\bar{t}, \bar{x}, \bar{y})$. Classically we have for η and γ small enough

$$\begin{cases} 0 < \frac{\bar{M}}{2} \leq \varphi(\bar{t}, \bar{x}, \bar{y}), \\ \gamma |\bar{x}| \rightarrow 0 \quad \text{as } \gamma \rightarrow 0. \end{cases}$$

Step 2: $\bar{t} > 0$ and $\bar{x} > \bar{y}$. By contradiction, assume first that $\bar{t} = 0$. Using (A0), we get a contradiction writing that

$$\frac{\eta}{T} < \xi_0(\bar{y}) - u_0(\bar{x}) - (k_0 + \varepsilon)(\bar{x} - \bar{y}) - 1 \leq 1 + k_0(\bar{x} - \bar{y}) - (k_0 + \varepsilon)(\bar{x} - \bar{y}) - 1 \leq 0.$$

If we assume that $\bar{x} = \bar{y}$ then, using the fact that $\varphi(\bar{t}, \bar{x}, \bar{y}) > 0$, we get that

$$0 < \xi(\bar{t}, \bar{x}) - u(\bar{t}, \bar{x}) - 1 - \frac{\eta}{T - \bar{t}} \leq 1 - 1 - \frac{\eta}{T} = -\frac{\eta}{T}.$$

This inequality yields a contradiction.

Step 3: viscosity inequalities. By duplication of the time variable and passing to the limit we get that

$$\begin{aligned} \frac{\eta}{(T - \bar{t})^2} &\leq \tilde{M}(u(t, \bar{x}), [\xi(t, \cdot)])(\bar{x}) \cdot |2\gamma\bar{x} + k_0 + \varepsilon| - L(\bar{y}, \xi(\bar{t}, \bar{y}), [u(\bar{t}, \cdot)])(\bar{y}) \cdot |k_0 + \varepsilon| \\ &\leq -L(\bar{y}, \xi(\bar{t}, \bar{y}), [u(\bar{t}, \cdot)])(\bar{y}) \cdot |k_0 + \varepsilon|, \end{aligned}$$

where we have used the fact that $\tilde{M}(u(\bar{t}, \bar{x}), [\xi(\bar{t}, \cdot)])(\bar{x}) \leq 0$. We replace L by its definition (3.16) and using (3.17), we have

$$\frac{\eta}{(T - \bar{t})^2} \leq 2V \left(N(\xi(\bar{t}, \bar{y}), [u(\bar{t}, \cdot)])(\bar{y}) \right) |k_0 + \varepsilon|. \quad (4.33)$$

Now we want to prove that $N(\xi(\bar{t}, \bar{y}), [u(\bar{t}, \cdot)])(\bar{y}) \leq h_0$. Indeed, if it is true, we will get a contradiction in (4.33) because $V(h) = 0 \forall h \leq h_0$. Let then $z > h_0$.

If $\bar{y} + z \geq \bar{x}$, then using that $u(\bar{t}, \cdot)$ is non increasing, we get that

$$u(\bar{t}, \bar{y} + z) - \xi(\bar{t}, \bar{y}) \leq u(\bar{t}, \bar{x}) - \xi(\bar{t}, \bar{y}) < -k_0(\bar{x} - \bar{y}) - 1 < -1.$$

If $\bar{y} + z < \bar{x}$, using the fact that $\varphi(\bar{t}, \bar{x}, \bar{y} + z) \leq \varphi(\bar{t}, \bar{x}, \bar{y})$, we obtain

$$\xi(\bar{t}, \bar{y} + z) - \xi(\bar{t}, \bar{y}) < -k_0 z \leq -1.$$

Using Lemma 4.13, we get that $u(\bar{t}, \bar{y} + z) - \xi(\bar{t}, \bar{y}) < -1$. We deduce that $I(u(\bar{t}, \bar{y} + z) - \xi(\bar{t}, \bar{y})) = 0$ for $z \geq h_0$ and so $N(\xi(\bar{t}, \bar{y}), [u(\bar{t}, \cdot)])(\bar{y}) \leq h_0$. □

We now turn to the proof of Lemma 4.13.

Proof of Lemma 4.13. The proof is divided into several steps.

Step 1: proof of the lower bound. We introduce

$$\bar{M} = \sup_{(t,x) \in [0,T] \times \mathbb{R}} \{u(t, x) - \xi(t, x)\}.$$

We want to prove that $\bar{M} \leq 0$ and argue by contradiction by assuming that $\bar{M} > 0$.

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Step 1.1: the test function. Let η, γ be small parameters, and A a constant to be chosen later. We introduce

$$\varphi(t, x, y) = u(t, x) - \xi(t, y) - \frac{\eta}{T-t} - e^{At} \frac{(x-y)^2}{2\varepsilon} - \gamma x^2.$$

Classically, φ reaches a maximum on $[0, T] \times \mathbb{R} \times \mathbb{R}$ at $(\bar{t}, \bar{x}, \bar{y})$ and we have for η, γ small enough,

$$\begin{cases} 0 < \frac{\overline{M}}{2} \leq \varphi(\bar{t}, \bar{x}, \bar{y}), \\ \gamma \bar{x} \rightarrow 0 \quad \text{as } \gamma \rightarrow 0, \\ |\bar{x} - \bar{y}| \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0. \end{cases}$$

Step 1.2: $\bar{t} > 0$ for ε small enough. We assume by contradiction that $\bar{t} = 0$. We have that

$$0 < u_0(\bar{x}) - \xi_0(\bar{y}) - \frac{\eta}{T} \leq k_0 |\bar{x} - \bar{y}| - \frac{\eta}{T}.$$

Taking ε small enough, we get a contradiction.

Step 1.3: viscosity inequalities. By duplication of the time variable and passing to the limit, we get that

$$\begin{aligned} \frac{\eta}{(T-\bar{t})^2} + Ae^{A\bar{t}} \frac{(\bar{x}-\bar{y})^2}{2\varepsilon} &\leq \left| e^{A\bar{t}} \frac{\bar{x}-\bar{y}}{\varepsilon} \right| \left(\tilde{L}(\bar{y}, \xi(\bar{t}, \bar{y}), [u(\bar{t}, \cdot)])(\bar{y}) - M(u(\bar{t}, \bar{x}), [\xi(\bar{t}, \cdot)])(\bar{x}) \right) \\ &\quad + o_\gamma(1) \\ &\leq \left| e^{A\bar{t}} \frac{\bar{x}-\bar{y}}{\varepsilon} \right| \left(\alpha \tilde{K}(\xi(\bar{t}, \bar{y}), [u(\bar{t}, \cdot)])(\bar{y}) - M(u(\bar{t}, \bar{x}), [\xi(\bar{t}, \cdot)])(\bar{x}) \right) \\ &\quad + o_\gamma(1), \end{aligned} \tag{4.34}$$

where we have used the fact that $V \geq 0$. We claim that

$$-M(u(\bar{t}, \bar{x}), [\xi(\bar{t}, \cdot)])(\bar{x}) \leq \alpha |\bar{x} - \bar{y}| \quad \text{and} \quad \tilde{K}(\xi(\bar{t}, \bar{y}), [u(\bar{t}, \cdot)])(\bar{y}) \leq |\bar{x} - \bar{y}|. \tag{4.35}$$

Indeed, for $z > |\bar{x} - \bar{y}|$, using that $\xi(\bar{t}, \cdot)$ is non increasing and that $\varphi(\bar{t}, \bar{x}, \bar{y}) > 0$, we have that

$$\xi(\bar{t}, \bar{x} + z) - u(\bar{t}, \bar{x}) \leq \xi(\bar{t}, \bar{y}) - u(\bar{t}, \bar{x}) < 0.$$

Therefore, using the definition of E we obtain that (for ε small enough such that $|\bar{x} - \bar{y}| \leq D$)

$$-M(u(\bar{t}, \bar{x}), [\xi(\bar{t}, \cdot)])(\bar{x}) = - \int_0^{|\bar{x}-\bar{y}|} E(\xi(\bar{t}, \bar{x} + z) - u(\bar{t}, \bar{x})) dz \leq \alpha |\bar{x} - \bar{y}|.$$

Similarly, using the fact that $u(\bar{t}, \cdot)$ is non increasing, for all $z > |\bar{x} - \bar{y}|$, we have that

$$u(\bar{t}, \bar{y} - z) - \xi(\bar{t}, \bar{y}) \geq u(\bar{t}, \bar{x}) - \xi(\bar{t}, \bar{y}) > 0.$$

Therefore,

$$\tilde{K}(\xi(\bar{t}, \bar{y}), [u(\bar{t}, \cdot)])(\bar{y}) = \int_0^{|\bar{x} - \bar{y}|} \tilde{F}(u(\bar{t}, \bar{y} - z) - \xi(\bar{t}, \bar{y})) dz \leq |\bar{x} - \bar{y}|.$$

This ends the proof of (4.35). Injecting (4.35) into (4.34), we get that

$$\frac{\eta}{(T - \bar{t})^2} + Ae^{A\bar{t}} \frac{(\bar{x} - \bar{y})^2}{2\varepsilon} \leq 2\alpha e^{A\bar{t}} \frac{(\bar{x} - \bar{y})^2}{\varepsilon} + o_\gamma(1).$$

Taking $A = 4\alpha$, we get a contradiction for γ small enough.

Step 2: proof of the upper bound. We introduce

$$\bar{M} = \sup_{(t,x) \in [0,T] \times \mathbb{R}} \{ \xi(t, x) - u(t, x) - 1 \}.$$

We want to prove that $\bar{M} \leq 0$. We argue by contradiction and assume that $\bar{M} > 0$.

Let η, γ be small parameters. We consider

$$\varphi(t, x, y) = \xi(t, x) - u(t, y) - 1 - \frac{\eta}{T - t} - \frac{(x - y)^2}{2\varepsilon} - \gamma x^2.$$

Classically, φ reaches a maximum on $[0, T] \times \mathbb{R} \times \mathbb{R}$ at $(\bar{t}, \bar{x}, \bar{y})$ and we have the following result for η and γ small enough

$$\begin{cases} 0 < \frac{\bar{M}}{2} \leq \varphi(\bar{t}, \bar{x}, \bar{y}), \\ |\gamma \bar{x}| \rightarrow 0 \quad \text{as } \gamma \rightarrow 0, \\ |\bar{x} - \bar{y}| \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0. \end{cases} \quad (4.36)$$

As in the previous Step 1.2, we get that $\bar{t} > 0$.

By duplication of the time variable and passing to the limit we then get that

$$\begin{aligned} \frac{\eta}{(T - \bar{t})^2} &\leq \left(\tilde{M}(u(\bar{t}, \bar{y}), [\xi(\bar{t}, \cdot)])(\bar{y}) - L(\bar{x}, \xi(\bar{t}, \bar{x}), [u(\bar{t}, \cdot)])(\bar{x}) \right) \left| \frac{\bar{x} - \bar{y}}{\varepsilon} \right| + o_\gamma(1) \\ &\leq 2V \left(N(\xi(\bar{t}, \bar{x}), [u(\bar{t}, \cdot)])(\bar{x}) \right) \left| \frac{\bar{x} - \bar{y}}{\varepsilon} \right| + o_\gamma(1), \end{aligned} \quad (4.37)$$

where we have used the fact that $\tilde{M} \leq 0$ and (3.17). We want to prove that $N(\xi(\bar{t}, \bar{x}), [u(\bar{t}, \cdot)])(\bar{x}) \leq h_0$. In fact for all $z \geq h_0$, we have that $\bar{x} + z > \bar{y}$ for ε small enough, so using that $\varphi(\bar{t}, \bar{x}, \bar{y}) > 0$ we get that

$$u(\bar{t}, \bar{x} + z) - \xi(\bar{t}, \bar{x}) \leq u(\bar{t}, \bar{y}) - \xi(\bar{t}, \bar{x}) < -1.$$

We deduce that $N(\xi(\bar{t}, \bar{x}), [u(\bar{t}, \cdot)])(\bar{x}) = \int_0^{h_0} I(u(\bar{t}, \bar{x} + z) - \xi(\bar{t}, \bar{x})) dz \leq h_0$. Using that $V(h) = 0$ for $h \leq h_0$, we get a contradiction in (4.37) for γ small enough. \square

5 Effective Hamiltonian and effective flux-limiter

In this section we provide a justification for the definition of the effective Hamiltonian \bar{H} provided in (2.4), we use the following proposition.

Proposition 5.1. (*Homogenization left and right of the perturbation*). Assume (A). Then for every $p \in [-k_0, 0]$, there exists a unique $\lambda \in \mathbb{R}$, such that there exists a bounded solution (w, χ) of

$$\begin{cases} M_p(w(x), [\chi])(x) |p + w_x| = \lambda \\ \left(\alpha K_p(\chi(x), [w])(x) - 2V \left(N_p(\chi(x), [w])(x) + K_p(\chi(x), [w])(x) \right) \right) |p + \chi_x| = \lambda \end{cases} \quad x \in \mathbb{R} \quad (5.1)$$

Moreover, for $p \in [-k_0, 0]$, we have $\lambda = \bar{H}(p) = -V \left(\frac{-1}{p} \right) |p|$.

Proof. We claim that $(w, \chi) = \left(0, -\frac{p}{\alpha} V \left(\frac{-1}{p} \right) \right)$ is a solution of (5.1) for $\lambda = -|p|V \left(\frac{-1}{p} \right)$.

-If $p = 0$, the result is obvious.

-If $p \in [-k_0, 0)$, since $\frac{-p}{\alpha} V \left(\frac{-1}{p} \right) + pz \geq 0$ if and only if $z \in [0, V(-1/p)/\alpha]$, then we have

$$M_p(w(x), [\chi])(x) = \int_0^D E \left(-\frac{p}{\alpha} V \left(\frac{-1}{p} \right) + pz \right) dz = -V \left(\frac{-1}{p} \right), \quad (5.2)$$

we recall that $D = h_{max} + 3V_{max}/(2\alpha) + 2r/\phi_0$. Similarly, for all $z > 0$, we have $\frac{p}{\alpha} V \left(\frac{-1}{p} \right) - pz < 0$ if and only if $z < V(-1/p)/\alpha$, then

$$K_p(\chi(x), [w])(x) = \int_0^D F \left(\frac{p}{\alpha} V \left(\frac{-1}{p} \right) - pz \right) dz = \frac{1}{\alpha} V \left(\frac{-1}{p} \right). \quad (5.3)$$

Finally, by definition we have that

$$N_p(\chi(x), [w])(x) = \int_0^D I\left(\frac{p}{\alpha}V\left(\frac{-1}{p}\right) + pz\right) dz.$$

First, notice that thanks to assumption (A7), for all $p \in [-k_0, 0)$, we have $\frac{1}{\alpha}V\left(\frac{-1}{p}\right) + \frac{1}{p} < 0$. Moreover, $\frac{p}{\alpha}V\left(\frac{-1}{p}\right) + pz > -1$ for $z < \frac{-1}{\alpha}V\left(\frac{-1}{p}\right) - \frac{1}{p}$. We distinguish two cases.

Case 1: $\frac{-1}{\alpha}V\left(\frac{-1}{p}\right) - \frac{1}{p} \leq D$. In this case, we have

$$N_p(\chi(x), [w])(x) = \frac{-1}{\alpha}V\left(\frac{-1}{p}\right) - \frac{1}{p}$$

and

$$N_p(\chi(x), [w])(x) + K_p(\chi(x), [w])(x) = -\frac{1}{p}. \quad (5.4)$$

Finally, using (5.2), (5.3), and (5.4), we obtain our desired result.

Case 2: $\frac{-1}{\alpha}V\left(\frac{-1}{p}\right) - \frac{1}{p} > D$. In particular we have $-1/p \geq h_{max}$. Therefore, we have

$$N_p(\chi(x), [w])(x) = D \quad \text{and} \quad K_p(\chi(x), [w])(x) = \frac{V_{max}}{\alpha},$$

this implies that

$$N_p(\chi(x), [w])(x) + K_p(\chi(x), [w])(x) = D + \frac{V_{max}}{\alpha} > h_{max}.$$

Combining this result to (5.3), we obtain

$$\begin{aligned} \left(\alpha K_p(\chi(x), [w])(x) - 2V\left(N_p(\chi(x), [w])(x) + K_p(\chi(x), [w])(x)\right)\right) |p| &= -V_{max}|p| \\ &= -V\left(\frac{-1}{p}\right) \cdot |p|. \end{aligned} \quad (5.5)$$

Using (5.2) and (5.5), we obtain our desired result. The uniqueness of λ is classical and we skip it. The proof is now complete. □

6 Correctors for the junction

In order to obtain an homogenization result, we need to find the effective flux-limiter. That is why we consider the following cell problem: find $\lambda \in \mathbb{R}$ such that there exists a solution (w, χ) of the following Hamilton-Jacobi equation, for $x \in \mathbb{R}$,

$$\begin{cases} M(w(x), [\chi(\cdot)]) \cdot |w_x| = \lambda \\ L(x, \chi(x), [w(\cdot)])(x) \cdot |\chi_x| = \lambda. \end{cases} \quad (6.1)$$

In this section we present a result of existence of correctors for the junction, which will be used for the proof of Theorem 3.2. We use the following notation: given $\bar{A} \in \mathbb{R}$, $\bar{A} \geq H_0$, we define two real numbers \bar{p}_- and \bar{p}_+ defined by

$$\bar{H}(\bar{p}_+) = \bar{H}^+(\bar{p}_+) = \bar{H}(\bar{p}_-) = \bar{H}^-(\bar{p}_-) = \bar{A}. \quad (6.2)$$

Given the form of \bar{H} , there exists only one couple of real numbers satisfying (6.2).

Theorem 6.1 (Existence of global corrector for the junction). *Assume (A).*

i) (General properties) There exists a constant $\bar{A} \in [H_0, 0]$ such that there exists a solution (w, χ) of (6.1) with $\lambda = \bar{A}$ and such that there exists a constant $C > 0$ and a globally Lipschitz continuous function m such that for all $x \in \mathbb{R}$,

$$|w(x) - m(x)| \leq C \quad \text{and} \quad |\chi(x) - m(x)| \leq C. \quad (6.3)$$

ii) (Bound from below at infinity) If $\bar{A} > H_0$, then there exists a $\gamma_0 > 0$ such that for every $\gamma \in (0, \gamma_0)$, we have for all $x \geq r + V_{max}/\alpha$ and $h \geq 0$,

$$\begin{aligned} w(x+h) - w(x) &\geq (\bar{p}_+ - \gamma)h, \\ \chi(x+h) - \chi(x) &\geq (\bar{p}_+ - \gamma)h \end{aligned} \quad (6.4)$$

and for $x \leq -r - V_{max}/\alpha$ and $h \geq 0$,

$$\begin{aligned} w(x-h) - w(x) &\geq (-\bar{p}_- - \gamma)h, \\ \chi(x-h) - \chi(x) &\geq (-\bar{p}_- - \gamma)h. \end{aligned} \quad (6.5)$$

iii) (Rescaling) For $\varepsilon > 0$, we set

$$w^\varepsilon(x) = \varepsilon w\left(\frac{x}{\varepsilon}\right) \quad \text{and} \quad \chi^\varepsilon(x) = \varepsilon \chi\left(\frac{x}{\varepsilon}\right),$$

then (up to a sub-sequence $\varepsilon_n \rightarrow 0$) we have that w^ε and χ^ε converge locally uniformly towards a function W which satisfies

$$\begin{cases} |W(x) - W(y)| \leq C|x - y| & \text{for all } x, y \in \mathbb{R}, \\ \bar{H}(W_x) = \bar{A} & \text{for all } x \neq 0. \end{cases} \quad (6.6)$$

In particular, we have (with $W(0) = 0$),

$$W(x) = \bar{p}_+ x 1_{\{x>0\}} + \bar{p}_- x 1_{\{x<0\}}. \quad (6.7)$$

The proof of this theorem is postponed until Section 8.

7 Proof of convergence

This section is devoted to the proof of Theorem 3.2 which relies on the existence of correctors provided by Proposition 5.1 and Theorem 6.1. We will use the following lemmas, the first one being a direct consequence of Theorem 4.11.

Lemma 7.1. (*Barriers uniform in ε*). *Assume (A0) and (A). There exist a constant $K_1 > 0$ such that for all $t \geq 0$ and $x \in \mathbb{R}$, we have*

$$|u^\varepsilon(t, x) - u_0(x)| \leq K_1 t \quad \text{and} \quad |\xi^\varepsilon(t, x) - \xi_0(x)| \leq K_1 t \quad (7.1)$$

The following lemma is a direct consequence of Theorem 4.12.

Lemma 7.2. (*Uniform gradient bound*). *Assume (A0) and (A). Then the solution $(u^\varepsilon, \xi^\varepsilon)$ of (3.3) satisfies for all $t \geq 0$, for all $x, y \in \mathbb{R}$, $x \geq y$,*

$$\begin{aligned} -k_0(x - y) - \varepsilon &\leq u^\varepsilon(t, x) - u^\varepsilon(t, y) \leq 0, \\ -k_0(x - y) - \varepsilon &\leq \xi^\varepsilon(t, x) - \xi^\varepsilon(t, y) \leq 0. \end{aligned} \quad (7.2)$$

Before passing to the proof of Theorem 3.2, let us mention that Theorem 3.5 is a direct consequence of this result joint to Theorem 3.2.

We now turn to the proof of Theorem 3.2.

Proof of Theorem 3.2. We introduce

$$\begin{aligned} \bar{u}(t, x) &= \limsup_{\varepsilon \rightarrow 0}^* u^\varepsilon & \bar{\xi}(t, x) &= \limsup_{\varepsilon \rightarrow 0}^* \xi^\varepsilon, \\ \underline{u}(t, x) &= \liminf_{\varepsilon \rightarrow 0}^* u^\varepsilon & \underline{\xi}(t, x) &= \liminf_{\varepsilon \rightarrow 0}^* \xi^\varepsilon, \end{aligned}$$

and

$$\bar{v} = \max(\bar{u}, \bar{\xi}) \quad \underline{v} = \min(\underline{u}, \underline{\xi}).$$

We want to prove that \bar{v} is a sub-solution of (2.7) and that \underline{v} is a super-solution of (2.7). Indeed, in this case, the comparison principle will imply that $\bar{v} \leq \underline{v}$. But by construction $\underline{v} \leq \bar{v}$, hence $\bar{v} = \underline{v} = u^0$, the unique solution of (2.7). This implies that $\bar{u} = \underline{u} = \bar{\xi} = \underline{\xi} = u^0$ and so u^ε and ξ^ε converge locally uniformly to u^0 .

To prove that \bar{v} is a sub-solution of (2.7), we argue by contradiction and assume that there is a point $(\bar{t}, \bar{x}) \in (0, +\infty) \times \mathbb{R}$ and a test function $\varphi \in C^1(J_\infty)$ such that

$$\begin{cases} \bar{v}(\bar{t}, \bar{x}) = \varphi(\bar{t}, \bar{x}), \\ \bar{v} \leq \varphi \quad \text{on} \quad Q_{\bar{r}, \bar{r}}(\bar{t}, \bar{x}) & \text{with } \bar{r} > 0, \\ \bar{v} \leq \varphi - 2\eta \quad \text{outside} \quad Q_{\bar{r}, \bar{r}}(\bar{t}, \bar{x}) & \text{with } \eta > 0, \\ \varphi_t(\bar{t}, \bar{x}) + \bar{H}(\bar{x}, \varphi_x(\bar{t}, \bar{x})) = \theta > 0, \end{cases} \quad (7.3)$$

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where

$$\bar{H}(\bar{x}, \varphi_x(\bar{t}, \bar{x})) = \begin{cases} \bar{H}(\varphi_x(\bar{t}, \bar{x})) & \text{if } \bar{x} \neq 0 \\ F_{\bar{A}}(\varphi_x(\bar{t}, 0^-), \varphi_x(\bar{t}, 0^+)) & \text{if } \bar{x} = 0. \end{cases}$$

We can assume that for ε small enough (up to changing φ at infinity), we have

$$u^\varepsilon, \xi^\varepsilon \leq \varphi - \eta \quad \text{outside } Q_{\bar{r}, \bar{r}}(\bar{t}, \bar{x}). \quad (7.4)$$

Using Lemmas 7.1 and 7.2 we get that the functions \bar{u} and $\bar{\xi}$ satisfy for all $t > 0$,

$$|\bar{u}(t, x) - u_0(x)| \leq K_1 t \quad \text{and} \quad |\bar{\xi}(t, x) - \xi_0(x)| \leq K_1 t \quad \text{for all } x \in \mathbb{R}, \quad (7.5)$$

and for $x \geq y$,

$$-k_0(x - y) \leq \bar{u}(t, x) - \bar{u}(t, y) \leq 0 \quad \text{and} \quad -k_0(x - y) \leq \bar{\xi}(t, x) - \bar{\xi}(t, y) \leq 0. \quad (7.6)$$

We distinguish two cases.

Case 1: $\bar{x} \neq 0$. We only consider the case $\bar{x} > 0$, since the other case ($\bar{x} < 0$) is treated in the same way. We define $p = \varphi_x(\bar{t}, \bar{x})$, that according to (7.6), satisfies

$$-k_0 \leq p \leq 0 \quad (7.7)$$

We choose \bar{r} small enough so that $\bar{x} - 2\bar{r} > 0$. We introduce

$$\psi^\varepsilon(t, x) = \varphi(t, x) - \varepsilon \frac{p}{\alpha} V\left(\frac{-1}{p}\right).$$

We have the following lemma.

Lemma 7.3. $(\varphi, \psi^\varepsilon)$ satisfies, in the viscosity sense, the inequality

$$\begin{cases} \varphi_t + \tilde{M}^\varepsilon\left(\frac{\varphi}{\varepsilon}(t, x), \left[\frac{\psi^\varepsilon}{\varepsilon}(t, \cdot)\right]\right)(x) \cdot |\varphi_x| \geq \frac{\theta}{2} \\ \psi_t^\varepsilon + \tilde{L}^\varepsilon\left(\frac{x}{\varepsilon}, \frac{\psi^\varepsilon}{\varepsilon}(t, x), \left[\frac{\varphi}{\varepsilon}(t, \cdot)\right]\right)(x) \cdot |\psi_x^\varepsilon| \geq \frac{\theta}{2} \end{cases} \quad \text{on } Q_{\bar{r}, \bar{r}}(\bar{t}, \bar{x}). \quad (7.8)$$

Proof of Lemma 7.3. For all $(t, x) \in Q_{\bar{r}, \bar{r}}(\bar{t}, \bar{x})$, we have for \bar{r} small enough

$$\begin{aligned}
 \varphi_t(t, x) + \tilde{M}^\varepsilon \left(\frac{\varphi}{\varepsilon}(t, x), \left[\frac{\psi^\varepsilon}{\varepsilon}(t, \cdot) \right] \right) (x) \cdot |\varphi_x(t, x)| &= \varphi_t(\bar{t}, \bar{x}) + o_{\bar{r}}(1) \\
 &+ \tilde{M}^\varepsilon \left(\frac{\varphi}{\varepsilon}(t, x), \left[\frac{\psi^\varepsilon}{\varepsilon}(t, \cdot) \right] \right) (x) \cdot |p| \\
 &= \theta + o_{\bar{r}}(1) \\
 &+ \tilde{M}^\varepsilon \left(\frac{\varphi}{\varepsilon}(t, x), \left[\frac{\psi^\varepsilon}{\varepsilon}(t, \cdot) \right] \right) (x) \cdot |p| \\
 &- \bar{H}(p) \\
 &=: \Delta
 \end{aligned}$$

where we have used for the first equality the regularity of the test function φ and the fact that the non-local operator \tilde{M}^ε is bounded (see Remark 3.6) and (7.3) for the second equality.

If $p = 0$, we obtain directly our result. We then assume that $p \in [-k_0, 0)$. For all $D \geq z \geq 0$, and for ε and \bar{r} small enough we have that

$$\frac{\psi^\varepsilon(t, x + \varepsilon.z) - \varphi(t, x)}{\varepsilon} \leq pz - \frac{p}{\alpha} V \left(\frac{-1}{p} \right) + o_{\bar{r}}(1) + o_\varepsilon(1),$$

where we have used the fact that $\varphi \in C^1$. Now using the fact that \tilde{E} is non increasing, we have

$$\tilde{E} \left(\frac{\psi^\varepsilon(t, x + \varepsilon.z) - \varphi(t, x)}{\varepsilon} \right) \geq \tilde{E} \left(pz - \frac{p}{\alpha} V \left(\frac{-1}{p} \right) + o_{\bar{r}}(1) + o_\varepsilon(1) \right). \quad (7.9)$$

Moreover, we have that

$$pz - \frac{p}{\alpha} V \left(\frac{-1}{p} \right) + o_{\bar{r}}(1) + o_\varepsilon(1) \geq 0 \quad \text{iff} \quad z \leq \frac{1}{\alpha} V \left(\frac{-1}{p} \right) + o_{\bar{r}}(1) + o_\varepsilon(1).$$

We deduce that

$$\begin{aligned}
 \tilde{M}^\varepsilon \left(\frac{\varphi}{\varepsilon}(t, x), \left[\frac{\psi^\varepsilon}{\varepsilon}(t, \cdot) \right] \right) (x) &\geq \int_0^D \tilde{E} \left(pz - \frac{p}{\alpha} V \left(\frac{-1}{p} \right) + o_{\bar{r}}(1) + o_\varepsilon(1) \right) dz \\
 &\geq -V \left(\frac{-1}{p} \right) + o_{\bar{r}}(1) + o_\varepsilon(1).
 \end{aligned} \quad (7.10)$$

Using (7.9),(7.10) and the definition of \bar{H} , we have for \bar{r} and ε small enough,

$$\Delta \geq \theta + o_{\bar{r}}(1) - V \left(\frac{-1}{p} \right) |p| + o_{\bar{r}}(1) + o_\varepsilon(1) + V \left(\frac{-1}{p} \right) |p| = \theta + o_{\bar{r}}(1) + o_\varepsilon(1) \geq \frac{\theta}{2}.$$

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We now prove the second inequality in (7.8). Let us notice that for ε small enough, using the fact that the non-local operator \tilde{K}^ε is bounded (see Remark 3.6) and the definition of ϕ , we have that

$$\phi\left(\frac{x}{\varepsilon} - \tilde{K}^\varepsilon\left(\frac{\psi^\varepsilon}{\varepsilon}(t, x), \left[\frac{\varphi}{\varepsilon}(t, \cdot)\right]\right)(x)\right) = 1 \quad \text{for all } (t, x) \in Q_{\bar{r}, \bar{r}}(\bar{t}, \bar{x}).$$

For all $(t, x) \in Q_{\bar{r}, \bar{r}}(\bar{t}, \bar{x})$, we have for \bar{r} small enough

$$\begin{aligned} \psi_t^\varepsilon(t, x) + \tilde{L}^\varepsilon\left(\frac{x}{\varepsilon}, \frac{\psi^\varepsilon}{\varepsilon}(t, x), \left[\frac{\varphi}{\varepsilon}(t, \cdot)\right]\right)(x) |\psi_x^\varepsilon(t, x)| &= \varphi_t(t, x) \\ &+ \tilde{L}^\varepsilon\left(\frac{x}{\varepsilon}, \frac{\psi^\varepsilon}{\varepsilon}(t, x), \left[\frac{\varphi}{\varepsilon}(t, \cdot)\right]\right)(x) |\varphi_x(t, x)| \\ &= \theta + o_{\bar{r}}(1) \\ &+ \tilde{L}^\varepsilon\left(\frac{x}{\varepsilon}, \frac{\psi^\varepsilon}{\varepsilon}(t, x), \left[\frac{\varphi}{\varepsilon}(t, \cdot)\right]\right)(x) |p| \\ &- \bar{H}(p) \\ &=: \Delta' \end{aligned}$$

If $p = 0$, we obtain directly our result. We then assume that $p \in [-k_0, 0)$. For all $D \geq z \geq 0$, and for ε and \bar{r} small enough we have that

$$\frac{\varphi(t, x - \varepsilon z) - \psi^\varepsilon(t, x)}{\varepsilon} \leq -pz + \frac{p}{\alpha} V\left(\frac{-1}{p}\right) + o_{\bar{r}}(1) + o_\varepsilon(1).$$

Now, using the fact that \tilde{F} is non increasing, we have that

$$\int_0^D \tilde{F}\left(-pz + \frac{p}{\alpha} V\left(\frac{-1}{p}\right) + o_{\bar{r}}(1) + o_\varepsilon(1)\right) dz \leq \tilde{K}^\varepsilon\left(\frac{\psi^\varepsilon}{\varepsilon}(t, x), \left[\frac{\varphi}{\varepsilon}(t, \cdot)\right]\right)(x)$$

which yields that

$$\frac{1}{\alpha} V\left(\frac{-1}{p}\right) + o_{\bar{r}}(1) + o_\varepsilon(1) \leq \tilde{K}^\varepsilon\left(\frac{\psi^\varepsilon}{\varepsilon}(t, x), \left[\frac{\varphi}{\varepsilon}(t, \cdot)\right]\right)(x). \quad (7.11)$$

We now compute $\tilde{N}^\varepsilon\left(\frac{\psi^\varepsilon}{\varepsilon}(t, x), \left[\frac{\varphi}{\varepsilon}(t, \cdot)\right]\right)(x)$. As above, and using the fact that \tilde{I} is non decreasing, we have

$$\tilde{N}^\varepsilon\left(\frac{\psi^\varepsilon}{\varepsilon}(t, x), \left[\frac{\varphi}{\varepsilon}(t, \cdot)\right]\right)(x) \leq \int_0^D \tilde{I}\left(pz + \frac{p}{\alpha} V\left(\frac{-1}{p}\right) + o_{\bar{r}}(1) + o_\varepsilon(1)\right) dz. \quad (7.12)$$

We notice that thanks to assumption (A7), for all $p \in [-k_0, 0)$ we have $\frac{1}{p} + \frac{1}{\alpha}V\left(\frac{-1}{p}\right) < 0$. Using that $pz + \frac{p}{\alpha}V\left(\frac{-1}{p}\right) + o_{\bar{r}}(1) + o_{\varepsilon}(1) > -1$ if and only if $z < -\frac{1}{p} - \frac{1}{\alpha}V\left(\frac{-1}{p}\right) + o_{\bar{r}}(1) + o_{\varepsilon}(1)$, we have distinguish two cases.

First case: $-\frac{1}{p} - \frac{1}{\alpha}V\left(\frac{-1}{p}\right) + o_{\bar{r}}(1) + o_{\varepsilon}(1) \leq D$. In this case,

$$\begin{aligned} \tilde{N}^{\varepsilon}\left(\frac{\psi^{\varepsilon}}{\varepsilon}(t, x), \left[\frac{\varphi}{\varepsilon}(t, \cdot)\right]\right)(x) &\leq \int_0^D \tilde{I}\left(pz + \frac{p}{\alpha}V\left(\frac{-1}{p}\right) + o_{\bar{r}}(1) + o_{\varepsilon}(1)\right) dz \\ &\leq -\frac{1}{p} - \frac{1}{\alpha}V\left(\frac{-1}{p}\right) + o_{\bar{r}}(1) + o_{\varepsilon}(1). \end{aligned} \quad (7.13)$$

Then,

$$\begin{aligned} \Delta' &\geq \theta + o_{\bar{r}}(1) + \tilde{L}^{\varepsilon}\left(\frac{x}{\varepsilon}, \frac{\psi^{\varepsilon}}{\varepsilon}(t, x), \left[\frac{\varphi}{\varepsilon}(t, \cdot)\right]\right)(x)|p| - \bar{H}(p) \\ &\geq \theta + o_{\bar{r}}(1) + \alpha\tilde{K}^{\varepsilon}\left(\frac{\psi^{\varepsilon}}{\varepsilon}(t, x), \left[\frac{\varphi}{\varepsilon}(t, \cdot)\right]\right)(x) \\ &\quad - 2V\left(\tilde{N}^{\varepsilon}\left(\frac{\psi^{\varepsilon}}{\varepsilon}(t, x), \left[\frac{\varphi}{\varepsilon}(t, \cdot)\right]\right)(x) + \tilde{K}^{\varepsilon}\left(\frac{\psi^{\varepsilon}}{\varepsilon}(t, x), \left[\frac{\varphi}{\varepsilon}(t, \cdot)\right]\right)(x) + V\left(\frac{-1}{p}\right)|p|\right) \\ &\geq \theta + o_{\bar{r}}(1) + V\left(\frac{-1}{p}\right) + o_{\bar{r}}(1) + o_{\varepsilon}(1) \\ &\quad - 2V\left(\tilde{N}^{\varepsilon}\left(\frac{\psi^{\varepsilon}}{\varepsilon}(t, x), \left[\frac{\varphi}{\varepsilon}(t, \cdot)\right]\right)(x) + \frac{1}{\alpha}V\left(\frac{-1}{p}\right) + o_{\bar{r}}(1) + o_{\varepsilon}(1) + V\left(\frac{-1}{p}\right)|p|\right) \\ &\geq \theta + o_{\bar{r}}(1) + V\left(\frac{-1}{p}\right)|p| + o_{\bar{r}}(1) + o_{\varepsilon}(1) - 2V\left(\frac{-1}{p} + o_{\bar{r}}(1) + o_{\varepsilon}(1)\right)|p| + V\left(\frac{-1}{p}\right)|p| \\ &\geq \theta + o_{\bar{r}}(1) + o_{\varepsilon}(1) \geq \frac{\theta}{2}, \end{aligned}$$

where we have used the definition of \tilde{L}^{ε} for the second inequality, (7.11) combined with assumption (A7) (see Remark 2.2) for the third inequality, (7.13) combined with the fact that V is non-decreasing for the fourth inequality and the fact V is a Lipschitz continuous function for the last inequality.

Second case: $-\frac{1}{p} - \frac{1}{\alpha}V\left(\frac{-1}{p}\right) + o_{\bar{r}}(1) + o_{\varepsilon}(1) > D$. In particular, by definition of D , we have $-1/p \geq h_{max}$ for ε and \bar{r} small enough. Then using (7.11) and the definition

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of \tilde{N}^ε , we obtain

$$\tilde{N}^\varepsilon \left(\frac{\psi^\varepsilon}{\varepsilon}(t, x), \left[\frac{\varphi}{\varepsilon}(t, \cdot) \right] \right) (x) \leq D$$

and

$$\frac{V_{max}}{\alpha} + o_{\bar{r}}(1) + o_\varepsilon(1) \leq \tilde{K}^\varepsilon \left(\frac{\psi^\varepsilon}{\varepsilon}(t, x), \left[\frac{\varphi}{\varepsilon}(t, \cdot) \right] \right) (x).$$

Using assumption (A7) (see Remark 2.2) and the previous inequalities, we get, using the definition of \tilde{L}^ε , that

$$\begin{aligned} \tilde{L}^\varepsilon \left(\frac{x}{\varepsilon}, \frac{\psi^\varepsilon}{\varepsilon}(t, x), \left[\frac{\varphi}{\varepsilon}(t, \cdot) \right] \right) (x) &= \alpha \tilde{K}^\varepsilon \left(\frac{\psi^\varepsilon}{\varepsilon}(t, x), \left[\frac{\varphi}{\varepsilon}(t, \cdot) \right] \right) (x) \\ &\quad - 2V \left(\tilde{N}^\varepsilon \left(\frac{\psi^\varepsilon}{\varepsilon}(t, x), \left[\frac{\varphi}{\varepsilon}(t, \cdot) \right] \right) (x) + \tilde{K}^\varepsilon \left(\frac{\psi^\varepsilon}{\varepsilon}(t, x), \left[\frac{\varphi}{\varepsilon}(t, \cdot) \right] \right) (x) \right) \\ &\geq V_{max} + o_{\bar{r}}(1) + o_\varepsilon(1) - 2V \left(D + \frac{V_{max}}{\alpha} + o_{\bar{r}}(1) + o_\varepsilon(1) \right) \\ &\geq -V_{max} + o_{\bar{r}}(1) + o_\varepsilon(1). \end{aligned}$$

Therefore, we have

$$\begin{aligned} \Delta' &\geq \theta + o_{\bar{r}}(1) - V_{max}|p| + o_{\bar{r}}(1) + o_\varepsilon(1) + V \left(\frac{-1}{p} \right) |p| \\ &\geq \theta + o_{\bar{r}}(1) + o_\varepsilon(1) \\ &\geq \frac{\theta}{2}, \end{aligned}$$

where we have used assumption (A4) ($V(h) = V_{max} \forall h \geq h_{max}$) and that $-1/p \geq h_{max}$. This ends the proof of Lemma 7.3. □

Getting a contradiction. Using (7.4), we have for ε small enough,

$$u^\varepsilon \leq \varphi - \eta \quad \text{and} \quad \xi^\varepsilon \leq \psi^\varepsilon - \eta \quad \text{outside } Q_{\bar{r}, \bar{r}}(\bar{t}, \bar{x}).$$

Using the comparison principle on bounded subsets for (3.3), we get

$$u^\varepsilon \leq \varphi - \eta \quad \text{and} \quad \xi^\varepsilon \leq \psi^\varepsilon - \eta \quad \text{on } Q_{\bar{r}, \bar{r}}(\bar{t}, \bar{x}).$$

Passing to the limit as $\varepsilon \rightarrow 0$, we get $\bar{u} \leq \varphi - \eta$ and $\bar{\xi} \leq \varphi - \eta$ on $Q_{\bar{r}, \bar{r}}(\bar{t}, \bar{x})$ and this contradicts the fact that $\bar{v}(\bar{t}, \bar{x}) = \max(\bar{u}(\bar{t}, \bar{x}), \bar{\xi}(\bar{t}, \bar{x})) = \varphi(\bar{t}, \bar{x})$.

Case 2: $\bar{x} = 0$. Using Theorem 4.11, we may assume that the test function has the following form

$$\varphi(t, x) = g(t) + \bar{p}_- x 1_{\{x < 0\}} + \bar{p}_+ x 1_{\{x > 0\}} \quad \text{on } Q_{2\bar{r}, 2\bar{r}}(\bar{t}, 0), \quad (7.14)$$

where g is a C^1 function defined on $(0, +\infty)$. The last line in condition (7.3) then becomes

$$g'(\bar{t}) + F_{\bar{A}}(\bar{p}_-, \bar{p}_+) = g'(\bar{t}) + \bar{A} = \theta.$$

Let us consider (w, ζ) the solution of (6.1) provided by Theorem 6.1. We define

$$\varphi^\varepsilon(t, x) = \begin{cases} g(t) + w^\varepsilon(x) & \text{on } Q_{2\bar{r}, 2\bar{r}}(\bar{t}, 0) \\ \varphi(t, x) & \text{outside } Q_{2\bar{r}, 2\bar{r}}(\bar{t}, 0), \end{cases} \quad (7.15)$$

$$\psi^\varepsilon(t, x) = \begin{cases} g(t) + \zeta^\varepsilon(x) & \text{on } Q_{2\bar{r}, 2\bar{r}}(\bar{t}, 0) \\ \varphi(t, x) & \text{outside } Q_{2\bar{r}, 2\bar{r}}(\bar{t}, 0). \end{cases} \quad (7.16)$$

We have the following lemma,

Lemma 7.4. $(\varphi^\varepsilon, \psi^\varepsilon)$ satisfies in the viscosity sense, for \bar{r} and ε small enough on $Q_{\bar{r}, \bar{r}}(\bar{t}, 0)$,

$$\begin{cases} \varphi_t^\varepsilon + \tilde{M}^\varepsilon \left(\frac{\varphi^\varepsilon}{\varepsilon}(t, x), \left[\frac{\psi^\varepsilon}{\varepsilon}(t, \cdot) \right] \right) (x) \cdot |\varphi_x^\varepsilon| \geq \frac{\theta}{2} \\ \psi_t^\varepsilon + \tilde{L}^\varepsilon \left(\frac{x}{\varepsilon}, \frac{\psi^\varepsilon}{\varepsilon}(t, x), \left[\frac{\varphi^\varepsilon}{\varepsilon}(t, \cdot) \right] \right) (x) \cdot |\psi_x^\varepsilon| \geq \frac{\theta}{2}. \end{cases} \quad (7.17)$$

Proof of Lemma 7.4. Let h be a test function touching φ^ε from below at $(t_1, x_1) \in Q_{\bar{r}, \bar{r}}(\bar{t}, 0)$, so we have

$$w \left(\frac{x_1}{\varepsilon} \right) = \frac{1}{\varepsilon} (h(t_1, x_1) - g(t_1))$$

and

$$w(y) \geq \frac{1}{\varepsilon} (h(t_1, \varepsilon y) - g(t_1)),$$

for y in a neighbourhood of $\frac{x_1}{\varepsilon}$. Since w does not depend on time, we have that

$$h_t(t_1, x_1) = g'(t_1).$$

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Using that (w, ζ) is a solution of (6.1), we then deduce that

$$h_t(t_1, x_1) - g'(t_1) + \tilde{M} \left(w \left(\frac{x_1}{\varepsilon} \right), [\zeta] \right) \left(\frac{x_1}{\varepsilon} \right) \cdot |h_x(t_1, x_1)| \geq \bar{A},$$

which implies

$$h_t(t_1, x_1) + \tilde{M} \left(w \left(\frac{x_1}{\varepsilon} \right), [\zeta] \right) \left(\frac{x_1}{\varepsilon} \right) \cdot |h_x(t_1, x_1)| \geq \bar{A} + g'(t_1) \geq \frac{\theta}{2},$$

i.e.

$$h_t(t_1, x_1) + \tilde{M}^\varepsilon \left(\frac{\varphi^\varepsilon}{\varepsilon}(t_1, x_1), \left[\frac{\psi^\varepsilon}{\varepsilon}(t_1, \cdot) \right] \right) (x_1) \cdot |h_x(t_1, x_1)| \geq \frac{\theta}{2}. \quad (7.18)$$

Let f be a test function touching ψ^ε from below at $(t_2, x_2) \in Q_{\bar{r}, \bar{r}}(\bar{t}, 0)$. We have

$$\zeta \left(\frac{x_2}{\varepsilon} \right) = \frac{1}{\varepsilon} (f(t_2, x_2) - g(t_2))$$

and

$$\zeta(y) \geq \frac{1}{\varepsilon} (f(t_2, \varepsilon y) - g(t_2))$$

for y in a neighbourhood of $\frac{x_2}{\varepsilon}$. Since ζ does not depend on time, we have that

$$f_t(t_2, x_2) = g'(t_2).$$

Therefore, using that (w, ζ) is a solution of (6.1), we get

$$f_t(t_2, x_2) - g'(t_2) + \tilde{L} \left(\frac{x_2}{\varepsilon}, \zeta \left(\frac{x_2}{\varepsilon} \right), [w] \right) \left(\frac{x_2}{\varepsilon} \right) \cdot |f_x(t_2, x_2)| \geq \bar{A},$$

which implies

$$f_t(t_2, x_2) + \tilde{L} \left(\frac{x_2}{\varepsilon}, \zeta \left(\frac{x_2}{\varepsilon} \right), [w] \right) \left(\frac{x_2}{\varepsilon} \right) \cdot |f_x(t_2, x_2)| \geq \bar{A} + g_t(t_2) \geq \frac{\theta}{2}.$$

Now for ε small enough such that $\varepsilon D \leq \bar{r}$, we deduce from the previous inequality and using the fact that we consider non-local operators with bounded support, that we have

$$f_t(t_2, x_2) + \tilde{L}^\varepsilon \left(\frac{x_2}{\varepsilon}, \frac{\psi^\varepsilon}{\varepsilon}(t_2, x_2), \left[\frac{\varphi^\varepsilon}{\varepsilon}(t_2, \cdot) \right] \right) (x_2) \cdot |f_x(t_2, x_2)| \geq \frac{\theta}{2}.$$

□

8. Proof of the existence of correctors at the junction

Getting the contradiction. We have that for ε small enough

$$\begin{aligned} u^\varepsilon + \eta &\leq \varphi = g(t) + \bar{p}_- x 1_{\{x < 0\}} + \bar{p}_+ x 1_{\{x > 0\}} && \text{on } Q_{2\bar{r}, 2\bar{r}}(\bar{t}, 0) \setminus Q_{\bar{r}, \bar{r}}(\bar{t}, 0) \\ \xi^\varepsilon + \eta &\leq \varphi = g(t) + \bar{p}_- x 1_{\{x < 0\}} + \bar{p}_+ x 1_{\{x > 0\}} && \text{on } Q_{2\bar{r}, 2\bar{r}}(\bar{t}, 0) \setminus Q_{\bar{r}, \bar{r}}(\bar{t}, 0). \end{aligned}$$

Using the fact that $w^\varepsilon, \zeta^\varepsilon \rightarrow W$ with $W(x) = \bar{p}_- x 1_{\{x < 0\}} + \bar{p}_+ x 1_{\{x > 0\}}$ (see Theorem 6.1), we deduce that for ε small enough, we have

$$u^\varepsilon + \frac{\eta}{2} \leq \varphi^\varepsilon \quad \text{and} \quad \xi^\varepsilon + \frac{\eta}{2} \leq \varphi^\varepsilon \quad \text{on } Q_{2\bar{r}, 2\bar{r}}(\bar{t}, 0) \setminus Q_{\bar{r}, \bar{r}}(\bar{t}, 0).$$

Combining this with (7.15) and (7.16), we get that

$$u^\varepsilon + \frac{\eta}{2} \leq \varphi^\varepsilon \quad \text{and} \quad \xi^\varepsilon + \frac{\eta}{2} \leq \varphi^\varepsilon \quad \text{outside } Q_{\bar{r}, \bar{r}}(\bar{t}, 0).$$

By the comparison principle on bounded subsets the previous inequality holds in $Q_{\bar{r}, \bar{r}}(\bar{t}, 0)$. Passing to the limit as $\varepsilon \rightarrow 0$ and evaluating the inequality in $(\bar{t}, 0)$, we obtain

$$\bar{u}(\bar{t}, 0) + \frac{\eta}{2} \leq \varphi(\bar{t}, 0) \quad \text{and} \quad \bar{\xi}(\bar{t}, 0) + \frac{\eta}{2} \leq \varphi(\bar{t}, 0)$$

which is a contradiction with the fact that $\bar{v}(\bar{t}, 0) = \max(\bar{u}(\bar{t}, 0), \bar{\xi}(\bar{t}, 0)) = \varphi(\bar{t}, 0)$. □

8 Proof of the existence of correctors at the junction

This section contains the proof of Theorem 6.1. We proceed as in [2, GIM15] and we will construct correctors on a truncated domain and then pass to the limit as the size of the domain goes to infinity.

For $l \in (r, +\infty)$, $r \ll l$ and $r \leq R \ll l$ we want to find $\lambda_{l,R} \in \mathbb{R}$ such that there exists a solution $(w^{l,R}, \chi^{l,R})$ of

$$\left\{ \begin{array}{ll} \left\{ \begin{array}{l} G_R^1(x, w^{l,R}(x), [\chi^{l,R}], w_x^{l,R}) = \lambda_{l,R} \\ G_R^2(x, \chi^{l,R}(x), [w^{l,R}], \chi_x^{l,R}) = \lambda_{l,R} \end{array} \right. & \text{if } x \in (-l, l) \\ \left\{ \begin{array}{l} \bar{H}^+(w_x^{l,R}) = \lambda_{l,R} \\ \bar{H}^+(\chi_x^{l,R}) = \lambda_{l,R} \end{array} \right. & \text{if } x = l \\ \left\{ \begin{array}{l} \bar{H}^-(w_x^{l,R}) = \lambda_{l,R} \\ \bar{H}^-(\chi_x^{l,R}) = \lambda_{l,R} \end{array} \right. & \text{if } x = -l \end{array} \right. \quad (8.1)$$

with

$$G_R^1(x, w(x), [\chi], q) = \psi_R(x)M(w(x), [\chi])(x)|q| + (1 - \psi_R(x))\bar{H}(q), \quad (8.2)$$

$$G_R^2(x, \chi(x), [w], q) = \psi_R(x)L(x, \chi(x), [w])(x)|q| + (1 - \psi_R(x))\bar{H}(q). \quad (8.3)$$

Moreover, $\psi_R \in C^\infty$, $\psi_R : \mathbb{R} \rightarrow [0, 1]$, with

$$\psi_R \equiv \begin{cases} 1 & \text{on } [-R, R] \\ 0 & \text{on } (-\infty, -R - 1] \cup [R + 1, +\infty), \end{cases} \quad \text{and } \psi_R(x) < 1 \ \forall x \notin [-R, R]. \quad (8.4)$$

As in the previous sections, to $G_R^{1,2}$ we associate $\tilde{G}_R^{1,2}$ which is defined in the same way but we replace the non-local operators M and L respectively by \tilde{M} and \tilde{L} .

8.1 Comparison principle for a truncated problem

Proposition 8.1 (Comparison principle on a truncated domain). *Let us consider the following problem for $r < l_1 < l_2$ and $\lambda \in \mathbb{R}$, with $l_2 \gg R$.*

$$\left\{ \begin{array}{l} \left\{ \begin{array}{l} \tilde{G}_R^1(x, u(x), [\xi], u_x) \geq \lambda \\ \tilde{G}_R^2(x, \xi(x), [u], \xi_x) \geq \lambda \end{array} \right. \quad \text{if } x \in (l_1, l_2) \\ \left\{ \begin{array}{l} \bar{H}^+(u_x) \geq \lambda \\ \bar{H}^+(\xi_x) \geq \lambda \end{array} \right. \quad \text{if } x = l_2 \end{array} \right. \quad (8.5)$$

and for $\varepsilon_0 > 0$,

$$\left\{ \begin{array}{l} \left\{ \begin{array}{l} G_R^1(x, v(x), [\zeta], v_x) \leq \lambda - \varepsilon_0 \\ G_R^2(x, \zeta(x), [v], \zeta_x) \leq \lambda - \varepsilon_0 \end{array} \right. \quad \text{if } x \in (l_1, l_2) \\ \left\{ \begin{array}{l} \bar{H}^+(v_x) \leq \lambda - \varepsilon_0 \\ \bar{H}^+(\zeta_x) \leq \lambda - \varepsilon_0 \end{array} \right. \quad \text{if } x = l_2 \end{array} \right. \quad (8.6)$$

Then if $v(l_1) \leq u(l_1)$ and $\zeta(l_1) \leq \xi(l_1)$, we have $v \leq u$ and $\zeta \leq \xi$ in $[l_1, l_2]$.

Proof. Like in [2], the only new difficulty to prove this proposition is the comparison at l_2 . But since near l_2 , the system decouples itself, we can proceed as in [GIM15, Proposition 4.1].

□

Remark 8.2. *We have a similar result if we exchange the boundary conditions, that is to say for $l_1 < l_2 < -r$ and if for all $x \in [l_2, l_2 + D]$, $v(x) \leq u(x)$ and $\zeta(x) \leq \xi(x)$, and the following conditions are imposed at $x = l_1$,*

$$\left\{ \begin{array}{l} \left\{ \begin{array}{l} \bar{H}^-(u_x) \geq \lambda \\ \bar{H}^-(\xi_x) \geq \lambda \end{array} \right. \quad \text{if } x = l_1 \\ \left\{ \begin{array}{l} \bar{H}^-(v_x) \leq \lambda - \varepsilon_0 \\ \bar{H}^-(\zeta_x) \leq \lambda - \varepsilon_0 \end{array} \right. \quad \text{if } x = l_1. \end{array} \right.$$

8.2 Existence of correctors on a truncated domain

Proposition 8.3 (Existence of correctors on a truncated domain). *There exists a constant $\lambda_{l,R} \in \mathbb{R}$ such that there exists a solution $(w^{l,R}, \chi^{l,R})$ of (8.1) for which there exists a constant C (depending only on k_0) and a Lipschitz continuous function $m^{l,R}$, such that*

$$\begin{cases} H_0 \leq \lambda_{l,R} \leq 0, \\ |w^{l,R}(x) - m^{l,R}(x)| \leq C & \text{for all } x \in [-l, l], \\ |\chi^{l,R}(x) - m^{l,R}(x)| \leq C & \text{for all } x \in [-l, l], \\ |m^{l,R}(x) - m^{l,R}(y)| \leq C|x - y| & \text{for all } x, y \in [-l, l], \\ |w^{l,R}(x) - \chi^{l,R}(x)| \leq C & \text{for all } x \in [-l, l], \end{cases} \quad (8.7)$$

with H_0 defined in (2.6).

Proof. Classically, we consider the approximated truncated cell problem,

$$\begin{cases} \begin{cases} \delta v^\delta + G_R^1(x, v^\delta(x), [\zeta^\delta], v_x^\delta) = 0 \\ \delta \zeta^\delta + G_R^2(x, \zeta^\delta(x), [v^\delta], \zeta_x^\delta) = 0 \end{cases} & \text{if } x \in (-l, l) \\ \begin{cases} \delta v^\delta + \overline{H}^+(v_x^\delta) = 0 \\ \delta \zeta^\delta + \overline{H}^+(\zeta_x^\delta) = 0 \end{cases} & \text{if } x = l \\ \begin{cases} \delta v^\delta + \overline{H}^-(v_x^\delta) = 0 \\ \delta \zeta^\delta + \overline{H}^-(\zeta_x^\delta) = 0 \end{cases} & \text{if } x = -l. \end{cases} \quad (8.8)$$

Step 1: construction of barriers. Using that $(0, 0)$ and $(C_0/\delta, C_0/\delta)$ are respectively obvious sub and super-solution of (8.8), with $C_0 = |\min_{p \in \mathbb{R}} \overline{H}_0(p)| = -H_0$ and that we have a comparison principle, we deduce that there exists a continuous viscosity solution (v^δ, ζ^δ) of (8.8) which satisfies

$$0 \leq v^\delta \leq \frac{C_0}{\delta} \quad \text{and} \quad 0 \leq \zeta^\delta \leq \frac{C_0}{\delta}. \quad (8.9)$$

Step 2: control of the oscillations of v^δ and ζ^δ .

Lemma 8.4. *The functions v^δ and ζ^δ satisfy for all $x, y \in [-l, l]$, $x \geq y$,*

$$-k_0(x - y) - 1 \leq v^\delta(x) - v^\delta(y) \leq 0 \quad \text{and} \quad -k_0(x - y) - 1 \leq \zeta^\delta(x) - \zeta^\delta(y) \leq 0. \quad (8.10)$$

Proof of Lemma 8.4. In the rest of the proof we will use the following notation

$$\Omega = \{(x, y) \in [-l, l]^2 \text{ s.t. } x \geq y\}.$$

Proof of the upper inequality. We want to prove that

$$\overline{M} = \sup_{(x,y) \in \Omega} \max \left(v^\delta(x) - v^\delta(y), \zeta^\delta(x) - \zeta^\delta(y) \right) \leq 0. \quad (8.11)$$

We argue by contradiction and assume that $\overline{M} > 0$. Since v^δ and ζ^δ are continuous and x, y belong to a compact, \overline{M} is reached for a finite point that we denote by $(\bar{x}, \bar{y}) \in \Omega$. Given that $\overline{M} > 0$, we deduce that $\bar{x} \neq \bar{y}$. Therefore, we can use the viscosity inequalities for (8.8).

Let us for instance assume that $\overline{M} = v^\delta(\bar{x}) - v^\delta(\bar{y})$, the other case is similar so we skip it. We distinguish 3 cases:

-If $(\bar{x}, \bar{y}) \in (-l, l)$, we have

$$\begin{aligned} \delta v^\delta(\bar{x}) + G_R^1(\bar{x}, v^\delta(\bar{x}), [\zeta^\delta], 0) &\leq 0 \\ \delta v^\delta(\bar{y}) + \tilde{G}_R^1(\bar{y}, v^\delta(\bar{y}), [\zeta^\delta], 0) &\geq 0. \end{aligned}$$

Combining these inequalities with the fact that $G_R^i(x, U, [\Xi], 0) = 0$ for $i = 1, 2$, we obtain

$$\delta \overline{M} \leq 0.$$

-If $\bar{x} = l$ and $\bar{y} \in [-l, l)$, we obtain similarly

$$\delta \overline{M} \leq 0, \quad (8.12)$$

using the fact that $\overline{H}^+(0) = 0$.

-If $\bar{x} \in (-l, l]$ and $\bar{y} = -l$, we obtain

$$\delta \overline{M} \leq H_0 < 0,$$

where we have used the fact that $\overline{H}^-(0) = H_0 < 0$.

For every value of \bar{x}, \bar{y} we obtain a contradiction, therefore $\overline{M} \leq 0$.

Proof of the lower inequalities. In order to proof these inequalities, we will use the following lemma which proof is postponed.

Lemma 8.5. *For all $x \in [-l, l]$, we have*

$$0 \leq \zeta^\delta(x) - v^\delta(x) \leq 1. \quad (8.13)$$

In order to prove (8.10), using Lemma 8.5 it is sufficient to prove that

$$\overline{M} = \sup_{(x,y) \in \Omega} \left(\zeta^\delta(y) - v^\delta(x) - k_0(x - y) - 1 \right) \leq 0. \quad (8.14)$$

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We argue by contradiction and assume that $\overline{M} > 0$. Since Ω is compact and v^δ and ζ^δ are continuous, \overline{M} is reached for a finite point that we denote by $(\bar{x}, \bar{y}) \in \Omega$. Since $\overline{M} > 0$, we deduce that $\bar{x} > \bar{y}$ (thanks to Lemma 8.5). Therefore, we can use the viscosity inequalities for (8.8). We distinguish 4 cases:

-If $\bar{x}, \bar{y} \in (-l, l)$, we obtain

$$\begin{aligned} \delta\zeta^\delta(\bar{y}) + G_R^2(\bar{y}, \zeta^\delta(\bar{y}), [v^\delta], -k_0) &\leq 0 \\ \delta v^\delta(\bar{x}) + \tilde{G}_R^1(\bar{x}, v^\delta(\bar{x}), [\zeta^\delta], -k_0) &\geq 0, \end{aligned}$$

combining these inequalities and using the definition of \overline{M} , we obtain

$$\delta\overline{M} \leq \delta\zeta^\delta(\bar{y}) - \delta v^\delta(\bar{x}) \leq \begin{aligned} &\tilde{G}_R^1(\bar{x}, v^\delta(\bar{x}), [\zeta^\delta], -k_0) \\ &- G_R^2(\bar{y}, \zeta^\delta(\bar{y}), [v^\delta], -k_0). \end{aligned} \quad (8.15)$$

Since the non-local operator \tilde{M} is negative and that $\overline{H}(-k_0) = 0$ we deduce that

$$\tilde{G}_R^1(\bar{x}, v^\delta(\bar{x}), [\zeta^\delta], -k_0) \leq 0.$$

We now claim that $G_R^2(\bar{y}, \zeta^\delta(\bar{y}), [v^\delta], -k_0) \geq 0$. Using $\overline{H}(-k_0) = 0$ and (3.17), we get that

$$\begin{aligned} G_R^2(\bar{y}, \zeta^\delta(\bar{y}), [v^\delta], -k_0) &= L(\bar{y}, \zeta^\delta(\bar{y}), [v^\delta(\cdot)])(\bar{y}) \cdot k_0 \psi_R(\bar{y}) \\ &\geq -2k_0 V(N(\zeta^\delta(\bar{y}), [v^\delta(\cdot)])(\bar{y})). \end{aligned} \quad (8.16)$$

Let us now prove that $N(\zeta^\delta(\bar{y}), [v^\delta(\cdot)])(\bar{y}) \leq h_0$. In fact, it is sufficient to prove that for all $z \in (h_0, D]$, we have

$$v^\delta(\bar{y} + z) - \zeta^\delta(\bar{y}) < -1. \quad (8.17)$$

First, if $z \geq \bar{x} - \bar{y}$, using the fact that v^δ is non increasing and that $\overline{M} > 0$, we obtain

$$v^\delta(\bar{y} + z) - \zeta^\delta(\bar{y}) \leq v^\delta(\bar{x}) - \zeta^\delta(\bar{y}) \leq -k_0(\bar{x} - \bar{y}) - 1 < -1.$$

Second, in the case $z < \bar{x} - \bar{y}$, using the fact that

$$\zeta^\delta(\bar{y} + z) - v^\delta(\bar{x}) - k_0(\bar{x} - \bar{y} - z) - 1 \leq \zeta^\delta(\bar{y}) - v^\delta(\bar{x}) - k_0(\bar{x} - \bar{y}) - 1,$$

and using Lemma 8.5 we deduce that

$$v^\delta(\bar{y} + z) - \zeta^\delta(\bar{y}) \leq -k_0 z < -1. \quad (8.18)$$

This implies that $N(\zeta^\delta(\bar{y}), [v^\delta(\cdot)])(\bar{y}) \leq h_0$. Using assumption (A3) ($V(h) = 0$ if $h \leq h_0$) and injecting this result in (8.16) we get that $G_R^2(\bar{y}, \zeta^\delta(\bar{y}), [v^\delta], -k_0) \geq 0$. Using (8.15) we then get a contradiction.

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-If $\bar{x} \in (-l, l)$ and $\bar{y} = -l$, we obtain

$$\begin{aligned}\delta\zeta^\delta(\bar{y}) + \overline{H}^-(-k_0) &\leq 0 \\ \delta v^\delta(\bar{x}) + \tilde{G}_R^1(\bar{x}, v^\delta(\bar{x}), [\zeta^\delta], -k_0) &\geq 0.\end{aligned}$$

Using the fact that $\overline{H}^-(-k_0) = 0$ and that $\tilde{G}_R^1(\bar{x}, v^\delta(\bar{x}), [\zeta^\delta], -k_0) \leq 0$ we obtain $\delta\overline{M} \leq 0$.

-If $\bar{x} = l$ and $\bar{y} \in (-l, l)$, we obtain

$$\begin{aligned}\delta\zeta^\delta(\bar{y}) + G_R^2(\bar{y}, \zeta^\delta(\bar{y}), [v^\delta], -k_0) &\leq 0 \\ \delta v^\delta(\bar{x}) + \overline{H}^+(-k_0) &\geq 0,\end{aligned}$$

using that $G_R^2(\bar{y}, \zeta^\delta(\bar{y}), [v^\delta], -k_0) \geq 0$ (see the first case) , and the fact that $\overline{H}^+(-k_0) < 0$, we directly obtain $\delta\overline{M} \leq 0$.

-If $\bar{x} = l$ and $\bar{y} = -l$, we obtain

$$\begin{aligned}\delta\zeta^\delta(\bar{y}) + \overline{H}^-(-k_0) &\leq 0 \\ \delta v^\delta(\bar{x}) + \overline{H}^+(-k_0) &\geq 0,\end{aligned}$$

and so, we get $\delta\overline{M} \leq 0$.

For every value of $\bar{x}, \bar{y} \in [-l, l]$ we get a contradiction, therefore we have $\overline{M} \leq 0$. This ends the proof of Lemma 8.4. □

Step 3: construction of a Lipschitz estimate. We want to construct a Lipschitz continuous function m^δ , such that there exists a constant $C > 0$ (independent of l and R) such that

$$\left\{ \begin{array}{ll} |v^\delta(x) - m^\delta(x)| \leq C & \text{for all } x \in [-l, l], \\ |\zeta^\delta(x) - m^\delta(x)| \leq C & \text{for all } x \in [-l, l], \\ |m^\delta(x) - m^\delta(y)| \leq C|x - y| & \text{for all } x, y \in [-l, l]. \end{array} \right. \quad (8.19)$$

We define m^δ as an affine function in each interval of the form $[ih_0, (i+1)h_0]$, with $i \in \mathbb{Z}$, such that

$$m^\delta(ih_0) = v^\delta(ih_0) \quad \text{and} \quad m^\delta((i+1)h_0) = v^\delta((i+1)h_0).$$

Since m^δ and v^δ are non-increasing, and $|v^\delta((i+1)h_0) - v^\delta(ih_0)| \leq k_0h_0 + 1 = 2$, we deduce that for all $x \in [ih_0, (i+1)h_0]$,

$$-2 \leq v^\delta((i+1)h_0) - m^\delta(ih_0) \leq v^\delta(x) - m^\delta(x) \leq v^\delta(ih_0) - m^\delta((i+1)h_0) \leq 2, \quad (8.20)$$

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and for all $x, y \in [-l, l]$,

$$|m^\delta(x) - m^\delta(y)| \leq 2k_0|x - y|.$$

Now using Lemma 8.5, we have

$$|\zeta^\delta(x) - m^\delta(x)| \leq 3.$$

Choosing $C = \max(2k_0, 3)$, we obtain (8.19).

Step 4: passing to the limit as δ goes to 0. Using (8.9), Lemma 8.5 and (8.19), we deduce that there exists a subsequence $\delta_n \rightarrow 0$ such that

$$\begin{aligned} \delta_n v^{\delta_n}(0) &\rightarrow -\lambda_{l,R} && \text{as } n \rightarrow +\infty, \\ \delta_n \zeta^{\delta_n}(0) &\rightarrow -\lambda_{l,R} && \text{as } n \rightarrow +\infty, \\ m^{\delta_n} - m^{\delta_n}(0) &\rightarrow m^{l,R} && \text{as } n \rightarrow +\infty. \end{aligned}$$

The last convergence being locally uniform. Let us consider,

$$\bar{w}^{l,R} = \limsup_{\delta_n \rightarrow 0}^* (v^{\delta_n} - v^{\delta_n}(0)) \quad \text{and} \quad \underline{w}^{l,R} = \liminf_{\delta_n \rightarrow 0} (v^{\delta_n} - v^{\delta_n}(0))$$

and

$$\bar{\chi}^{l,R} = \limsup_{\delta_n \rightarrow 0}^* (\zeta^{\delta_n} - \zeta^{\delta_n}(0)) \quad \text{and} \quad \underline{\chi}^{l,R} = \liminf_{\delta_n \rightarrow 0} (\zeta^{\delta_n} - \zeta^{\delta_n}(0)).$$

Therefore, we have that $\lambda_{l,R}$, $\bar{w}^{l,R}$, $\underline{w}^{l,R}$, $\bar{\chi}^{l,R}$, $\underline{\chi}^{l,R}$ and $m^{l,R}$ satisfy

$$\begin{aligned} H_0 &\leq \lambda_{l,R} \leq 0, \\ |\bar{w}^{l,R} - m^{l,R}| &\leq C, \quad |\underline{w}^{l,R} - m^{l,R}| \leq C, \\ |\bar{\chi}^{l,R} - m^{l,R}| &\leq C, \quad |\underline{\chi}^{l,R} - m^{l,R}| \leq C, \\ &|m_x^{l,R}| \leq C, \end{aligned} \tag{8.21}$$

and thanks to Lemma 8.5, we have

$$|\underline{\chi}^{l,R} - \bar{w}^{l,R}|, |\bar{\chi}^{l,R} - \underline{w}^{l,R}| \leq 1. \tag{8.22}$$

By stability of viscosity solutions, we have that $(\bar{w}^{l,R} - 2C, \bar{\chi}^{l,R} - 2C)$ and $(\underline{w}^{l,R}, \underline{\chi}^{l,R})$ are respectively a sub-solution and a super-solution of (8.1), and

$$\bar{w}^{l,R} - 2C \leq \underline{w}^{l,R} \quad \text{and} \quad \bar{\chi}^{l,R} - 2C \leq \underline{\chi}^{l,R}.$$

By Perron's method, we can construct a solution $(w^{l,R}, \chi^{l,R})$ of (8.1) and thanks to (8.21) and (8.22), $m^{l,R}$, $w^{l,R}$, $\chi^{l,R}$ and $\lambda_{l,R}$ satisfy (8.7).

The uniqueness of $\lambda_{l,R}$ is classical so we skip it. This ends the proof of Proposition 8.3. \square

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Proof of Lemma 8.5. We separate the proof in two parts. This proof uses the vertex test function of the work of Imbert and Monneau [IM14, Theorem 3.2] to treat the comparison between v^δ and ζ^δ near $-l$ and l . In fact, we consider that we have a network composed of a single branch with two nodes (one in $-l$ and the other in l). Near $-l$ we consider an outgoing branch and near l we consider an incoming branch.

Step 1: proof of $v^\delta(x) - \zeta^\delta(x) \leq 0$ for all $x \in [-l, l]$. We want to prove that

$$\bar{M} = \sup_{x \in [-l, l]} (v^\delta(x) - \zeta^\delta(x)) \leq 0.$$

We argue by contradiction and assume that $\bar{M} > 0$. Given that v^δ and ζ^δ are continuous, \bar{M} is reached at a finite point that we denote by $\bar{x} \in [-l, l]$. We distinguish 3 cases according to the position of \bar{x} in the interval $[-l, l]$.

Case 1: $\bar{x} \in (-l, l)$. We define for ε a small parameter,

$$\varphi(x, y) = v^\delta(x) - \zeta^\delta(y) - \frac{(x - y)^2}{2\varepsilon} - \frac{1}{2} \left((x - \bar{x})^2 + (y - \bar{x})^2 \right).$$

Since $[-l, l]$ is compact and v^δ and ζ^δ are continuous functions, the function φ reaches a maximum at a finite point that we denote by $(x_\varepsilon, y_\varepsilon) \in [-l, l]$. If we denote $M_\varepsilon = \varphi(x_\varepsilon, y_\varepsilon)$, by classical arguments, we have that

$$\lim_{\varepsilon \rightarrow 0} M_\varepsilon = M, \quad \lim_{\varepsilon \rightarrow 0} |x_\varepsilon - y_\varepsilon| = 0, \quad \text{and} \quad (x_\varepsilon, y_\varepsilon) \rightarrow (\bar{x}, \bar{x}) \text{ as } \varepsilon \text{ goes to } 0. \quad (8.23)$$

We can also prove that

$$\frac{(x_\varepsilon - y_\varepsilon)^2}{\varepsilon} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0. \quad (8.24)$$

Furthermore, for ε small enough we have $x_\varepsilon, y_\varepsilon \in (-l, l)$, and using the viscosity inequalities we obtain

$$\begin{aligned} \delta v^\delta(x_\varepsilon) + G_R^1(x_\varepsilon, v^\delta(x_\varepsilon), [\zeta^\delta], p_\varepsilon + (x_\varepsilon - \bar{x})) &\leq 0 \\ \delta \zeta^\delta(y_\varepsilon) + \tilde{G}_R^2(y_\varepsilon, \zeta^\delta(y_\varepsilon), [v^\delta], p_\varepsilon - (y_\varepsilon - \bar{x})) &\geq 0, \end{aligned}$$

with $p_\varepsilon = (x_\varepsilon - y_\varepsilon)/\varepsilon$. Combining these inequalities and using the definition of \bar{M} , we

obtain that

$$\begin{aligned}
\delta\bar{M} &\leq \tilde{G}_R^2\left(y_\varepsilon, \zeta^\delta(y_\varepsilon), [v^\delta], p_\varepsilon - (y_\varepsilon - \bar{x})\right) - G_R^1\left(x_\varepsilon, v^\delta(x_\varepsilon), [\zeta^\delta], p_\varepsilon + (x_\varepsilon - \bar{x})\right) \\
&\leq (\psi_R(x_\varepsilon) - \psi_R(y_\varepsilon))\bar{H}(p_\varepsilon) + \|\psi_R\|_\infty \|\bar{H}'\|_\infty (|y_\varepsilon - \bar{x}| + |x_\varepsilon - \bar{x}|) \\
&\quad + \psi_R(y_\varepsilon)\alpha\tilde{K}(\zeta^\delta(y_\varepsilon), [v^\delta(\cdot)])(y_\varepsilon)\cdot|p_\varepsilon - y_\varepsilon + \bar{x}| \\
&\quad - \psi_R(x_\varepsilon)M(v^\delta(x_\varepsilon), [\zeta^\delta(\cdot)])(x_\varepsilon)\cdot|p_\varepsilon + x_\varepsilon - \bar{x}| \\
&\leq (\psi_R(x_\varepsilon) - \psi_R(y_\varepsilon))\bar{H}(p_\varepsilon) + \|\psi_R\|_\infty \|\bar{H}'\|_\infty (|y_\varepsilon - \bar{x}| + |x_\varepsilon - \bar{x}|) \\
&\quad + \psi_R(y_\varepsilon)\alpha\tilde{K}(\zeta^\delta(y_\varepsilon), [v^\delta(\cdot)])(y_\varepsilon)\cdot|p_\varepsilon| - \psi_R(x_\varepsilon)M(v^\delta(x_\varepsilon), [\zeta^\delta(\cdot)])(x_\varepsilon)\cdot|p_\varepsilon| \\
&\quad + (\alpha M_0|y_\varepsilon - \bar{x}| + M_0|x_\varepsilon - \bar{x}|) \\
&\leq (\psi_R(x_\varepsilon) - \psi_R(y_\varepsilon))\bar{H}(p_\varepsilon) + o_\varepsilon(1) \\
&\quad + \psi_R(y_\varepsilon)\alpha\tilde{K}(\zeta^\delta(y_\varepsilon), [v^\delta(\cdot)])(y_\varepsilon)\cdot|p_\varepsilon| - \psi_R(x_\varepsilon)M(v^\delta(x_\varepsilon), [\zeta^\delta(\cdot)])(x_\varepsilon)\cdot|p_\varepsilon|
\end{aligned} \tag{8.25}$$

where we have replaced G_R^1 and \tilde{G}_R^2 by their definitions, used the fact that by definition \bar{H} is a Lipschitz function and that that $V \geq 0$ for the second inequality, used Remark 3.6 for the third inequality and (8.23) for the last inequality.

We will compute the right part of the inequality in different steps.

1-Concerning the local operator.

$$\begin{aligned}
\left|(\psi_R(x_\varepsilon) - \psi_R(y_\varepsilon))\bar{H}(p_\varepsilon)\right| &\leq \|D\psi_R\|_\infty|x_\varepsilon - y_\varepsilon|\bar{H}(p_\varepsilon)| \\
&\leq \|D\psi_R\|_\infty V_{max} \frac{(x_\varepsilon - y_\varepsilon)^2}{\varepsilon} \\
&= o_\varepsilon(1)
\end{aligned} \tag{8.26}$$

where we have used the regularity of ψ_R for the first inequality, used the fact that by definition of \bar{H} , we have $|\bar{H}| \leq V_{max}|p|$ for the second inequality and used (8.24) for the last inequality.

2-Concerning the non-local operator M . We claim that

$M(v^\delta(x_\varepsilon), [\zeta^\delta(\cdot)])(x_\varepsilon) \leq |x_\varepsilon - y_\varepsilon|$. To prove this, it suffices to prove that for all $z > |x_\varepsilon - y_\varepsilon|$

$$\zeta^\delta(x_\varepsilon + z) - v^\delta(x_\varepsilon) < 0.$$

Using the fact that ζ^δ is decreasing, that $x_\varepsilon + z \geq y_\varepsilon$ and that $M_\varepsilon > 0$, we obtain

$$\zeta^\delta(x_\varepsilon + z) - v^\delta(x_\varepsilon) \leq \zeta^\delta(y_\varepsilon) - v^\delta(x_\varepsilon) < 0.$$

Therefore we have

$$\begin{aligned}
-\psi_R(x_\varepsilon)M(v^\delta(x_\varepsilon), [\zeta^\delta])(x_\varepsilon) &= -\psi_R(x_\varepsilon) \int_0^{|x_\varepsilon - y_\varepsilon|} E(\zeta^\delta(x_\varepsilon + z) - v^\delta(x_\varepsilon))dz \\
&\leq \alpha|x_\varepsilon - y_\varepsilon|.
\end{aligned} \tag{8.27}$$

In particular, this implies that

$$\left|\psi_R(x_\varepsilon)M(v^\delta(x_\varepsilon), [\zeta^\delta])(x_\varepsilon)\right| |p_\varepsilon| \leq \alpha \frac{(x_\varepsilon - y_\varepsilon)^2}{\varepsilon} = o_\varepsilon(1). \tag{8.28}$$

3-Concerning the non-local operator \tilde{K} . We claim that $|\tilde{K}(\zeta\delta(y_\varepsilon), [v^\delta(\cdot)])(y_\varepsilon)| \leq |x_\varepsilon - y_\varepsilon|$. As before, it suffices to prove that for all $z > |x_\varepsilon - y_\varepsilon|$

$$v^\delta(y_\varepsilon - z) - \zeta^\delta(y_\varepsilon) > 0.$$

Using the fact that v^δ is decreasing, that $x_\varepsilon \geq y_\varepsilon - z$ and that $M_\varepsilon > 0$, we obtain

$$v^\delta(y_\varepsilon - z) - \zeta^\delta(y_\varepsilon) \geq v^\delta(x_\varepsilon) - \zeta^\delta(y_\varepsilon) > 0.$$

Therefore we have

$$\left| \psi_R(y_\varepsilon) \tilde{K}(\zeta^\delta(y_\varepsilon), [v^\delta]) (y_\varepsilon) \right| \leq |x_\varepsilon - y_\varepsilon|. \quad (8.29)$$

Injecting (8.26), (8.27), and (8.29) into (8.25), we obtain $\delta\bar{M} \leq o_\varepsilon(1)$ and we get a contradiction for ε small enough.

Case 2: $\bar{x} = l$. In this case, we use the vertex test function introduced by Imbert and Monneau. We refer to [IM14] for a detailed description of the vertex test function, but for the readers convenience we recall the properties that we used to complete this proof. The vertex test function G^γ is associated to the single Hamiltonian \bar{H} . We fix $\gamma = \delta M/2$. It satisfies the following properties.

1. (Regularity)

$$G^\gamma \in C([-l, l]^2) \quad \begin{cases} G^\gamma(x, \cdot) \in C^1([-l, l]) & \text{for all } x \in [-l, l] \\ G^\gamma(\cdot, y) \in C^1([-l, l]) & \text{for all } y \in [-l, l]. \end{cases} \quad (8.30)$$

2. (Bound from below) $G^\gamma \geq 0 = G(0, 0)$.

3. (Super-linearity) There exists $g : [0, +\infty) \rightarrow \mathbb{R}$ non-decreasing and such that for all $(x, y) \in [-l, l]^2$

$$g(|x - y|) \leq G^\gamma(x, y) \quad \text{and} \quad \lim_{a \rightarrow +\infty} \frac{g(a)}{a} = +\infty.$$

4. (Compatibility condition on the gradient)

$$\bar{H}(y, -G_y^\gamma(x, y)) - \bar{H}(x, G_x^\gamma(x, y)) \leq \gamma, \quad (8.31)$$

with for all $x \in [-l, l]$ and $p \in \mathbb{R}$,

$$\bar{H}(x, p) = \begin{cases} \bar{H}(p) & \text{if } x \in [-l, l) \\ \bar{H}^+(p) & \text{if } x = l. \end{cases} \quad (8.32)$$

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We introduce the following test function, for $\varepsilon > 0$ a small parameter,

$$\varphi(x, y) = v^\delta(x) - \zeta^\delta(y) - \varepsilon G^\gamma \left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon} \right) - \frac{1}{2} \left((x - \bar{x})^2 + (y - \bar{y})^2 \right).$$

which like before reaches a maximum at a finite point $(x_\varepsilon, y_\varepsilon) \in [-l, l]$ and (8.23) remains true.

Using the viscosity equations, we have that

$$\begin{cases} \delta v^\delta(x_\varepsilon) + \bar{H} \left(x_\varepsilon, G_x^\gamma \left(\frac{x_\varepsilon}{\varepsilon}, \frac{y_\varepsilon}{\varepsilon} \right) + (x_\varepsilon - \bar{x}) \right) \leq 0 \\ \delta \zeta^\delta(y_\varepsilon) + \bar{H} \left(y_\varepsilon, -G_y^\gamma \left(\frac{x_\varepsilon}{\varepsilon}, \frac{y_\varepsilon}{\varepsilon} - (y_\varepsilon - \bar{y}) \right) \right) \geq 0. \end{cases}$$

Using the definition of \bar{M} and combining the previous inequalities, we get that

$$\begin{aligned} \delta \bar{M} &\leq \bar{H} \left(y_\varepsilon, -G_y^\gamma \left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon} - (y_\varepsilon - \bar{y}) \right) \right) - \bar{H} \left(x_\varepsilon, G_x^\gamma \left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon} \right) + (x_\varepsilon - \bar{x}) \right) \\ &\leq \bar{H} \left(y_\varepsilon, -G_y^\gamma \left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon} \right) \right) - \bar{H} \left(x_\varepsilon, G_x^\gamma \left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon} \right) \right) + o_\varepsilon(1), \end{aligned}$$

where we have used (8.23) combined with the fact that both \bar{H} and \bar{H}^+ are Lipschitz continuous for the second inequality. Using the compatibility condition on the gradient of the vertex test function (8.31) we obtain

$$\delta \bar{M} \leq \gamma + o_\varepsilon(1),$$

and given that $\gamma = \delta \bar{M} / 2$, we get a contradiction for ε small enough.

Case 3: $\bar{x} = -l$. This case is exactly like the previous one with the exception that the vertex test function must be adapted to treat the junction at $-l$. In particular, (8.32) is replaced by

$$\bar{H}(x, p) = \begin{cases} \bar{H}(p) & \text{if } x \in (-l, l] \\ \bar{H}^-(p) & \text{if } x = -l. \end{cases}$$

We skip the rest of the computation for this case.

In conclusion, we have $\bar{M} \leq 0$ and for all $x \in [-l, l]$, $0 \leq \zeta^\delta(x) - v^\delta(x)$.

Step 2: proof of $\zeta^\delta(x) - v^\delta(x) \leq 1$. We want to prove that

$$\bar{M} = \sup_{x \in [-l, l]} \left(\zeta^\delta(x) - v^\delta(x) - 1 \right) \leq 0.$$

We argue by contradiction and assume that $\bar{M} > 0$. Give that v^δ and ζ^δ are continuous, \bar{M} is reached at a finite point that we denote by $\bar{x} \in [-l, l]$. We distinguish 2 cases according to the position of \bar{x} in the interval $[-l, l]$.

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Case 1: $\bar{x} \in (-l, l)$. We define for ε a small parameter,

$$\varphi(x, y) = v^\delta(x) - \zeta^\delta(y) - 1 - \frac{(x - y)^2}{2\varepsilon} - \frac{1}{2} \left((x - \bar{x})^2 + (y - \bar{x})^2 \right).$$

Using the same arguments as before, the test function reaches a maximum at a finite point that we denote by $(x_\varepsilon, y_\varepsilon) \in [-l, l]$. If we denote $M_\varepsilon = \varphi(x_\varepsilon, y_\varepsilon)$ (8.23) and (8.24) remain valid.

For ε small enough we have $x_\varepsilon, y_\varepsilon \in (-l, l)$, and using the viscosity inequalities we get that

$$\begin{aligned} \delta\zeta^\delta(x_\varepsilon) + G_R^2(x_\varepsilon, \zeta^\delta(x_\varepsilon), [v^\delta], p_\varepsilon) &\leq 0 \\ \delta v^\delta(y_\varepsilon) + \tilde{G}_R^1(y_\varepsilon, v^\delta(y_\varepsilon), [\zeta^\delta], p_\varepsilon) &\geq 0, \end{aligned}$$

with $p_\varepsilon = (x_\varepsilon - y_\varepsilon)/\varepsilon$. Combining these inequalities and using the definition of \bar{M} , we obtain

$$\begin{aligned} \delta\bar{M} &\leq \tilde{G}_R^1(y_\varepsilon, v^\delta(y_\varepsilon), [\zeta^\delta], p_\varepsilon) - G_R^2(x_\varepsilon, \zeta^\delta(x_\varepsilon), [v^\delta], p_\varepsilon) \\ &\leq (\psi_R(x_\varepsilon) - \psi_R(y_\varepsilon))\bar{H}(p_\varepsilon) + 2\psi_R(x_\varepsilon)V\left(N(\zeta^\delta(x_\varepsilon), [v^\delta(\cdot)])(x_\varepsilon)\right) \cdot |p_\varepsilon|, \end{aligned} \quad (8.33)$$

where we have replaced G_R^2 and \tilde{G}_R^1 by their definition and used (3.17) and that $\tilde{M} \leq 0$. We will compute the right part of (8.33) in different steps.

1-Concerning the local operator. Like before, we have

$$\left| (\psi_R(x_\varepsilon) - \psi_R(y_\varepsilon))\bar{H}(p_\varepsilon) \right| \leq \|D\psi_R\|_\infty |x_\varepsilon - y_\varepsilon| |\bar{H}(p_\varepsilon)| = o_\varepsilon(1). \quad (8.34)$$

2-Concerning the non-local operator N . We claim that

$$N(\zeta^\delta(x_\varepsilon), [v^\delta(\cdot)])(x_\varepsilon) \leq h_0.$$

To prove this, it suffices to prove that for all $z \geq h_0$, we have

$$v^\delta(x_\varepsilon + z) - \zeta^\delta(x_\varepsilon) < -1.$$

Since $|x_\varepsilon - y_\varepsilon| \rightarrow 0$ as ε goes to 0, we have for all $z \geq h_0$ and ε small enough that $x_\varepsilon + z \geq y_\varepsilon$. Therefore, we get

$$v^\delta(x_\varepsilon + z) - \zeta^\delta(x_\varepsilon) \leq v^\delta(y_\varepsilon) - \zeta^\delta(x_\varepsilon) < -1,$$

where we have used the fact that v^δ is decreasing for the first inequality and the fact that $M_\varepsilon > 0$ for the second inequality. This implies that

$$V\left(N(\zeta^\delta(x_\varepsilon), [v^\delta(\cdot)])(x_\varepsilon)\right) \leq V(h_0) = 0. \quad (8.35)$$

8.Proof of the existence of correctors at the junction

Injecting (8.34) and (8.35) in (8.33), we obtain $\delta\bar{M} \leq o_\varepsilon(1)$, and we get a contradiction for ε small enough.

Case 2: $\bar{x} = l$ of $\bar{x} = -l$. Proceeding like in the previous step we obtain directly a contradiction by using the properties of the vertex test function.

This ends the proof of Lemma 8.5. □

Proposition 8.6 (First definition of the flux limiter). *The following limits exists (up to some sub-sequence),*

$$\begin{cases} \bar{A} = \lim_{R \rightarrow +\infty} \bar{A}_R, \\ \bar{A}_R = \lim_{l \rightarrow +\infty} \lambda_{R,l}. \end{cases}$$

Moreover, we have

$$H_0 \leq \bar{A}, \quad \bar{A}_R \leq 0. \tag{8.36}$$

Proof. This proposition is a direct consequence of (8.7). □

Proposition 8.7 (Control of the slopes on a truncated domain). *Assume that l and R are big enough. Let $(w^{l,R}, \chi^{l,R})$ be the solution of (8.1) given by Proposition 8.3. We also assume up to a sub-sequence, that $\bar{A} = \lim_{R \rightarrow +\infty} \lim_{l \rightarrow +\infty} \lambda_{l,R} > H_0$. Then there exists $\gamma_0 > 0$ and a constant $C > 0$ (independent of l and R) such that for all $\gamma \in (0, \gamma_0)$ and for all $x \geq r + D$, $h \geq 0$ we have*

$$w^{l,R}(x+h) - w^{l,R}(x) \geq (\bar{p}_+ - \gamma)h - C \tag{8.37}$$

and

$$\chi^{l,R}(x+h) - \chi^{l,R}(x) \geq (\bar{p}_+ - \gamma)h - C. \tag{8.38}$$

Similarly, for all $x \leq -r - D$ and $h \geq 0$,

$$w^{l,R}(x-h) - w^{l,R}(x) \geq (-\bar{p}_- - \gamma)h - C \tag{8.39}$$

and

$$\chi^{l,R}(x-h) - \chi^{l,R}(x) \geq (-\bar{p}_- - \gamma)h - C. \tag{8.40}$$

Proof. We only do the proof of (8.37)-(8.38), since the proof of (8.39)-(8.40) is similar and we skip it. For $\mu > 0$, small enough, we denote by p_μ^+ the real number defined by

$$\bar{H}(p_\mu^+) = \bar{H}^+(p_\mu^+) = \lambda_{l,R} - \mu. \tag{8.41}$$

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Using that

$$H_0 < \lambda_{l,R} \leq 0,$$

we deduce that p_μ^+ exists for μ small enough and $p_\mu^+ \in [-k_0, 0)$.

Let us now consider

$$\begin{cases} w^+ = p_\mu^+ x, \\ \chi^+ = p_\mu^+ x - \frac{p_\mu^+}{\alpha} V\left(\frac{-1}{p_\mu^+}\right), \end{cases}$$

that satisfy

$$\overline{H}(w_x^+) = \overline{H}^+(w_x^+) = \overline{H}(\chi_x^+) = \overline{H}^+(\chi_x^+) = \lambda_{l,R} - \mu \quad \text{for } x \in \mathbb{R}. \quad (8.42)$$

Let us consider $(w, \chi) = \left(0, -\frac{p_\mu^+}{\alpha} V\left(\frac{-1}{p_\mu^+}\right)\right)$ the correctors provided by Proposition 5.1 for $p = p_\mu^+$. Given the definition of w^+ and χ^+ , we get

$$M(w^+(x), [\chi^+])(x) = M_{p_\mu^+}(w(x), [\chi])(x), \quad K(\chi^+(x), [w^+])(x) = K_{p_\mu^+}(\chi(x), [w])(x),$$

and

$$N(\chi^+(x), [w^+])(x) = N_{p_\mu^+}(\chi(x), [w])(x).$$

In particular this implies that

$$M(w^+(x), [\chi^+])(x) = -V\left(\frac{-1}{p_\mu^+}\right)$$

and

$$\alpha K(\chi^+(x), [w^+])(x) - 2V\left(K(\chi^+(x), [w^+])(x) + N(\chi^+(x), [w^+])(x)\right) = -V\left(\frac{-1}{p_\mu^+}\right).$$

Finally, given that the non-local operator K is bounded by D (see Remark 3.6), we have for all $x \in (r + D, l]$

$$\phi\left(x - K(\chi^+(x), [w^+])(x)\right) = 1.$$

Combining the previous results, we can see that the restriction of (w^+, χ^+) to $(r + D, l]$ satisfies

$$\begin{cases} \begin{cases} G_R^1(x, w^+(x), [\chi^+], w_x^+) = \overline{H}(p_\mu^+) = \lambda_{l,R} - \mu \\ G_R^2(x, \chi^+(x), [w^+], \chi_x^+) = \overline{H}(p_\mu^+) = \lambda_{l,R} - \mu \end{cases} & \text{if } x \in (r + D, l) \\ \begin{cases} \overline{H}^+(w_x^+) = \lambda_{l,R} - \mu \\ \overline{H}^+(\chi_x^+) = \lambda_{l,R} - \mu \end{cases} & \text{if } x = l \end{cases} \quad (8.43)$$

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Let us introduce, for some $x_0 \in (r + D, l]$,

$$\begin{cases} g = w^{l,R} - w^{l,R}(x_0) \\ h = \chi^{l,R} - w^{l,R}(x_0), \end{cases} \quad \text{and} \quad \begin{cases} u = w^+ - w^+(x_0) - C - \frac{k_0}{\alpha} V_{max} \\ v = \chi^+ - w^+(x_0) - C - \frac{k_0}{\alpha} V_{max}, \end{cases} \quad (8.44)$$

with $C > 0$ the constant provided by Proposition 8.3. Then we have

$$g(x_0) = 0 \geq -C - \frac{k_0}{\alpha} V_{max} = u(x_0)$$

and

$$h(x_0) = \chi^{l,R}(x_0) - w^{l,R}(x_0) \geq -C \geq -C - \frac{k_0}{\alpha} V_{max} - \frac{p_\mu^+}{\alpha} V \left(\frac{-1}{p} \right) = v(x_0),$$

where we have used the fact that $p_\mu^+ \in [-k_0, 0)$ and $\|V\|_\infty \leq V_{max}$. Using that (g, h) is a solution of (8.5) and (u, v) is a solution of (8.6) (with $\varepsilon_0 = \mu$), joint to the comparison principle (Proposition 8.1), up to changing the value of the constant C , we get that

$$\begin{cases} w^{l,R}(x) - w^{l,R}(x_0) \geq p_\mu^+(x - x_0) - C \\ \chi^{l,R}(x) - \chi^{l,R}(x_0) \geq p_\mu^+(x - x_0) - C. \end{cases}$$

This implies that for all $h \geq 0$, and for all $x \in (r + D, l)$,

$$\begin{cases} w^{l,R}(x + h) - w^{l,R}(x) \geq p_\mu^+ h - C \\ \chi^{l,R}(x + h) - \chi^{l,R}(x) \geq p_\mu^+ h - C. \end{cases}$$

Finally, if we choose $\gamma_0 < |p_0 - \bar{p}_+|$, then we have

$$\bar{H}(\bar{p}_+ - \gamma) = \bar{H}^+(\bar{p}_+ - \gamma).$$

Choosing $\mu > 0$ such that

$$p_\mu^+ = \bar{p}_+ - \gamma.$$

we obtain (8.37)-(8.38). □

Proof of Theorem 6.1. The proof is performed in two steps.

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Step 1: proof of i) and ii) We want to pass to the limit as $l \rightarrow +\infty$ and then as $R \rightarrow +\infty$ on the solution of (8.1) given by Proposition 8.3. Using (8.3), there exists $l_n \rightarrow +\infty$, such that

$$m^{l_n, R} - m^{l_n, R}(0) \rightarrow m^R \quad \text{as } n \rightarrow +\infty,$$

the convergence being locally uniform. We also define

$$\bar{w}^R(x) = \limsup_{n \rightarrow +\infty}^* (w^{l_n, R} - w^{l_n, R}(0)), \quad \underline{w}^R(x) = \liminf_{n \rightarrow +\infty} (w^{l_n, R} - w^{l_n, R}(0)),$$

and

$$\bar{\chi}^R(x) = \limsup_{n \rightarrow +\infty}^* (\chi^{l_n, R} - \chi^{l_n, R}(0)), \quad \underline{\chi}^R(x) = \liminf_{n \rightarrow +\infty} (\chi^{l_n, R} - \chi^{l_n, R}(0))$$

Thanks to (8.3), we know that these limits are finite and satisfy

$$m^R - C \leq \underline{w}^R \leq \bar{w}^R \leq m^R + C. \quad \text{and} \quad m^R - C \leq \underline{\chi}^R \leq \bar{\chi}^R \leq m^R + C.$$

By stability of viscosity solutions $(\bar{w}^R - 2C, \bar{\chi}^R - 2C)$ and $(\underline{w}^R, \underline{\chi}^R)$ are respectively a sub-solution and a super-solution of

$$\begin{cases} G_R^1(x, w^R(x), [\chi^R], w_x^R) = \bar{A}_R \\ G_R^2(x, \chi^R(x), [w^R], \chi_x^R) = \bar{A}_R. \end{cases} \quad (8.45)$$

Therefore, using Perron's method, we can construct a solution (w^R, χ^R) of (8.45), with m^R, \bar{A}_R, w^R and χ^R satisfying

$$\begin{cases} |m^R(x) - m^R(y)| \leq C|x - y| & \text{for all } x, y \in \mathbb{R}, \\ |w^R(x) - m^R(x)| \leq C, |\chi^R(x) - m^R(x)| \leq C & \text{for all } x \in \mathbb{R}, \\ |w^R(x) - \chi^R(x)| \leq C & \text{for all } x \in \mathbb{R}, H_0 \leq \bar{A}_R \leq 0. \end{cases} \quad (8.46)$$

Using Proposition 8.7, if $\bar{A} > H_0$, we know that there exists a $\gamma_0 > 0$ and a constant $C > 0$ such that for all $\gamma \in (0, \gamma_0)$, for all $x \geq r + D$, and $h \geq 0$,

$$w^R(x + h) - w^R(x) \geq (\bar{p}_+ - \gamma)h - C \quad \text{and} \quad \chi^R(x + h) - \chi^R(x) \geq (\bar{p}_+ - \gamma)h - C.$$

Similarly, for all $x \leq -r - D$ and $h \geq 0$,

$$w^R(x - h) - w^R(x) \geq (-\bar{p}_- - \gamma)h - C \quad \text{and} \quad \chi^R(x - h) - \chi^R(x) \geq (-\bar{p}_- - \gamma)h - C.$$

Proceeding like before, we pass to the limit as $R \rightarrow +\infty$ in order to build a solution (w, χ) of (6.1) with $\lambda = \bar{A}$ that satisfies (6.3), (6.4) and (6.5).

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Step 2: proof of iii). Let us now consider the rescaled functions $w^\varepsilon = \varepsilon w(x/\varepsilon)$ and $\chi^\varepsilon(x) = \varepsilon \chi(x/\varepsilon)$. Using (6.3), we have that

$$w^\varepsilon(x) = \varepsilon m\left(\frac{x}{\varepsilon}\right) + O(\varepsilon) \quad \text{and} \quad \chi^\varepsilon(x) = \varepsilon m\left(\frac{x}{\varepsilon}\right) + O(\varepsilon). \quad (8.47)$$

Therefore, there exists a subsequence $\varepsilon_n \rightarrow 0$ as $n \rightarrow +\infty$, such that

$$w^{\varepsilon_n}, \chi^{\varepsilon_n} \rightarrow W \quad \text{locally uniformly as } n \rightarrow +\infty, \quad (8.48)$$

with $W(0) = 0$. Proceeding as in the proof of convergence (Section 7), away from the junction point, we have that W satisfies

$$\overline{H}(W_x) = \overline{A} \quad \text{for } x \neq 0.$$

This proves (6.6). Let us now prove (6.7).

For $x < 0$, we have for all $\gamma \in (0, \gamma_0)$, if $\overline{A} > H_0$,

$$W_x \leq \overline{p}_- + \gamma,$$

where we have used (6.5). Therefore, we have $W_x = \overline{p}_-$ for $x < 0$, this equality remains valid if $\overline{A} = H_0$ (indeed, if $\overline{A} = H_0$, we have $\overline{p}_+ = \overline{p}_- = p_0 = W_x$).

For $x > 0$, we have for all $\gamma \in (0, \gamma_0)$, if $\overline{A} > H_0$,

$$W_x \geq \overline{p}_+ - \gamma,$$

where we have used (6.4). Therefore, we have that $W_x = \overline{p}_+$ for $x > 0$, this result is still valid if $\overline{A} = H_0$.

Combining these results, we obtain (6.7). □

Theorem 8.8 (Effective flux limiter). *Assume (A). We define the following set of functions,*

$$\mathcal{S} = \{(v, \zeta) \text{ s.t. } \exists \text{ a Lipschitz continuous function } m \text{ (with } m(0)=0) \text{ and constant } C > 0 \text{ s.t. } \|v - m\|_\infty, \|\zeta - m\|_\infty \leq C\}.$$

Then we have

$$\overline{A} = \inf\{\lambda \in [H_0, 0] : \exists (v, \zeta) \in \mathcal{S} \text{ solution of (6.1)}\}. \quad (8.49)$$

Proof of Theorem 8.8. Up to a sub-sequence, let $\overline{A} = \lim_{R \rightarrow +\infty} \lim_{l \rightarrow +\infty} \lambda_{l,R}$. We want to prove that $\overline{A} = \inf E$, with

$$E = \{\lambda \in [H_0, 0] : \exists (v, \zeta) \in \mathcal{S} \text{ solution of (6.1)}\}.$$

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We argue by contradiction and assume that there exists $\lambda \in E$ such that $\lambda < \bar{A}$. We denote by $(v^\lambda, \zeta^\lambda)$ a solution of (6.1) associated to λ . Arguing as in the proof of Theorem 6.1, Step 2, we deduce that the functions

$$v_\lambda^\varepsilon(x) = \varepsilon v^\lambda\left(\frac{x}{\varepsilon}\right) \quad \text{and} \quad \zeta_\lambda^\varepsilon(x) = \varepsilon \zeta^\lambda\left(\frac{x}{\varepsilon}\right) \quad (8.50)$$

have a limit W^λ (with $W^\lambda(0) = 0$) which satisfies

$$\bar{H}(W_x^\lambda) = \lambda \quad \text{for } x \neq 0.$$

This means that for all $x > 0$, we have

$$W_x^\lambda \leq p_+^\lambda < \bar{p}_+ \quad \text{with} \quad \bar{H}(p_+^\lambda) = \bar{H}^+(p_+^\lambda) = \lambda. \quad (8.51)$$

Similarly, for all $x < 0$, we have

$$W_x^\lambda \geq p_-^\lambda > \bar{p}_- \quad \text{with} \quad \bar{H}(p_-^\lambda) = \bar{H}^-(p_-^\lambda) = \lambda. \quad (8.52)$$

These inequalities imply that for all $\gamma > 0$, there exists a constant \tilde{C}_γ such that

$$v^\lambda(x), \zeta^\lambda(x) \leq \begin{cases} (p_+^\lambda + \gamma)x + \tilde{C}_\gamma & \text{for } x > 0, \\ (p_-^\lambda - \gamma)x + \tilde{C}_\gamma & \text{for } x < 0. \end{cases} \quad (8.53)$$

Using Theorem 6.1 (ii), we have for γ small enough,

$$v^\lambda \leq w \quad \text{and} \quad \zeta^\lambda \leq \chi \quad \text{for } |x| \geq \tilde{R}.$$

This implies that there exists a constant $C_{\tilde{R}}$ such that for all $x \in \mathbb{R}$, we have

$$v^\lambda(x) < w(x) + C_{\tilde{R}} \quad \text{and} \quad \zeta^\lambda(x) < \chi(x) + C_{\tilde{R}}.$$

Let us now introduce two functions (u, ξ) and (u^λ, ξ^λ) , defined by

$$\begin{cases} u(t, x) = w(x) + C_{\tilde{R}} - \bar{A}t, \\ \xi(t, x) = \chi(x) + C_{\tilde{R}} - \bar{A}t, \end{cases} \quad \text{and} \quad \begin{cases} u^\lambda(t, x) = v^\lambda(x) - \lambda t, \\ \xi^\lambda(t, x) = \zeta^\lambda(x) - \lambda t. \end{cases}$$

Both functions are solutions of (3.3) (with $\varepsilon = 1$) and

$$u^\lambda(0, x) \leq u(0, x) \quad \text{and} \quad \xi^\lambda(0, x) \leq \xi(0, x).$$

Using the comparison principle (Proposition 4.9), we obtain

$$v^\lambda(x) - \lambda t \leq w(x) - \bar{A}t + C_{\tilde{R}}.$$

Passing to the limit as t goes to infinity, we get $\bar{A} \leq \lambda$, which is a contradiction. □

9 Link between the system of ODEs and the PDE

This section is devoted to the proof of Theorem 3.3, which is a direct application of our convergence result, Theorem 3.2 joint to the following result.

Theorem 9.1. *For $\varepsilon = 1$, (ρ, σ) defined by (2.2) and (3.2) is a discontinuous viscosity solution of the following equation*

$$\begin{cases} \rho_t + M(\rho(t, x), [\sigma(t, \cdot)])(x) \cdot |\rho_x| = 0 \\ \sigma_t + L(x, \sigma(t, x), [\rho(t, \cdot)])(x) \cdot |\sigma_x| = 0 \end{cases} \quad \text{for } (t, x) \in (0, +\infty) \times \mathbb{R}. \quad (9.1)$$

Conversely, if u and ξ are bounded and continuous functions such that (u, ξ) is a solution of (9.1) satisfying for some time $T > 0$, and for all $t \in (0, T)$

$$\begin{aligned} u(t, x) &\text{ is non-increasing in } x, \\ \xi(t, x) &\text{ is non-increasing in } x, \end{aligned} \quad (9.2)$$

then the points $(U_j(t), \Xi_j(t))$ defined by $u(t, U_j(t)) = -(j + 1)$ and $\xi(t, \Xi_j(t)) = -(j + 1)$ for $j \in \mathbb{Z}$, satisfy the system (3.1) on $(0, T)$.

The proof of Theorem 9.1 is given in Appendix 11. Let us use Theorem 9.1 to do the proof of Theorem 3.3.

Proof of Theorem 3.3. We recall that in Theorem 3.3 we have $u_0(x) = \xi_0^\varepsilon(x) = -x/h$. Let us begin by proving that for all $x \in \mathbb{R}$ and for all $\varepsilon > 0$, we have

$$|\rho^\varepsilon(0, x) - u_0(x)| \leq f(\varepsilon) \quad \text{and} \quad |\sigma^\varepsilon(0, x) - \xi_0^\varepsilon(x)| \leq g(\varepsilon), \quad (9.3)$$

with $f(\varepsilon), g(\varepsilon) \rightarrow 0$ as ε goes to 0. Let us define a piece-wise affine function v satisfying

$$\rho^1(0, x) = v(x) \quad \text{for } x = U_i(0), \text{ for all } i \in \mathbb{Z}.$$

Given that for all $U_{i+1}(0) - U_i(0) \geq h_0$, we notice that v is k_0 -Lipschitz continuous and by definition of $\rho^1(0, x)$, we have

$$|\rho^1(0, x) - v(x)| \leq 1 \quad \text{for all } x \in \mathbb{R}.$$

Let us consider the integer $i_0 \in \mathbb{N}$ defined by

$$i_0 = \sup \{i \in \mathbb{Z}, \text{ s.t. } U_i(0) \leq -R\}.$$

Using the assumption that for all $i \in \mathbb{Z}$ such that $U_i(0) \leq -R$ we have $U_{i+1}(0) - U_i(0) = h$, we deduce that for all $x \leq U_{i_0}(0)$

$$v(x) = -\frac{x}{h} + \frac{U_{i_0}(0)}{h} + \rho^1(0, U_{i_0}(0)) = -\frac{x}{h} + \frac{U_{i_0}(0)}{h} - i_0 - 1.$$

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Let us now consider the integer $i_1 \in \mathbb{N}$ defined by

$$i_1 = \inf \{i \in \mathbb{Z}, \text{ s.t. } U_i(0) \geq R\}.$$

Now using the assumption that for all $i \in \mathbb{Z}$ such that $U_i(0) \geq R$ we have $U_{i+1}(0) - U_i(0) = h$, we deduce that for all $x \geq U_{i_1}(0)$

$$v(x) = -\frac{x}{h} + \frac{U_{i_1}(0)}{h} + \rho^1(0, U_{i_1}(0)) = -\frac{x}{h} + \frac{U_{i_1}(0)}{h} - i_1 - 1.$$

Moreover, we recall that for all $\varepsilon > 0$, we have $\rho^\varepsilon(0, x) = \varepsilon \rho^1(0, x/\varepsilon)$, this implies that for all $x \notin [\varepsilon U_{i_0}(0), \varepsilon U_{i_1}(0)]$,

$$\begin{aligned} |\rho^\varepsilon(0, x) - u_0(x)| &\leq \left| \rho^\varepsilon(0, x) - \varepsilon v\left(\frac{x}{\varepsilon}\right) \right| + \left| \varepsilon v\left(\frac{x}{\varepsilon}\right) - u_0(x) \right| \\ &\leq \varepsilon + \varepsilon \max \left(\left| \frac{U_{i_1}(0)}{h} - i_1 - 1 \right|, \left| \frac{U_{i_0}(0)}{h} - i_0 - 1 \right| \right). \end{aligned} \quad (9.4)$$

Similarly, we have for all $x \in [\varepsilon U_{i_0}(0), \varepsilon U_{i_1}(0)]$,

$$\begin{aligned} |\rho^\varepsilon(0, x) - u_0(x)| &\leq \left| \rho^\varepsilon(0, x) - \varepsilon v\left(\frac{x}{\varepsilon}\right) \right| + \left| \varepsilon v\left(\frac{x}{\varepsilon}\right) - \varepsilon u_0\left(\frac{x}{\varepsilon}\right) \right| \\ &\leq \varepsilon + \varepsilon \max_{y \in [U_{i_0}(0), U_{i_1}(0)]} (|v(y) - u_0(y)|), \end{aligned} \quad (9.5)$$

where we have used the fact that $\varepsilon u_0(x/\varepsilon) = u_0(x)$. Combining (9.4) and (9.5) and choosing

$$\begin{aligned} f(\varepsilon) &= \varepsilon + \varepsilon \max \left(\left| \frac{U_{i_0}(0)}{h} - i_0 - 1 \right|, \right. \\ &\quad \left. \max_{y \in [U_{i_0}(0), U_{i_1}(0)]} (|v(y) - u_0(y)|), \left| \frac{U_{i_1}(0)}{h} - i_1 - 1 \right| \right) \end{aligned}$$

we deduce the first inequality in (9.3) and proceeding in the same way we obtain the second inequality. Using (9.3), we deduce that for all $x \in \mathbb{R}$ and for all $\varepsilon > 0$, we have

$$|(\rho^\varepsilon)^*(0, x) - u_0(x)| \leq f(\varepsilon) + \varepsilon \quad \text{and} \quad |(\sigma^\varepsilon)^*(0, x) - \xi_0^\varepsilon(x)| \leq g(\varepsilon) + \varepsilon. \quad (9.6)$$

Combining (9.3) and (9.6), we get

$$\begin{cases} u_0(x) - \max(f(\varepsilon), g(\varepsilon)) \leq \rho^\varepsilon(0, x) \leq (\rho^\varepsilon)^*(0, x) \leq u_0(x) + \max(f(\varepsilon), g(\varepsilon)) + \varepsilon \\ \xi_0^\varepsilon(x) - \max(f(\varepsilon), g(\varepsilon)) \leq \sigma^\varepsilon(0, x) \leq (\sigma^\varepsilon)^*(0, x) \leq \xi_0^\varepsilon(x) + \max(f(\varepsilon), g(\varepsilon)) + \varepsilon. \end{cases}$$

Using the fact that $(\rho^\varepsilon, \sigma^\varepsilon)$ is a viscosity solution of (3.3) and the comparison principle (Proposition 4.9) we deduce that (with $(u^\varepsilon, \xi^\varepsilon)$ the continuous solution of (3.3) associated to the initial condition $u_0(x) = \xi_0^\varepsilon(x) = -x/h$)

$$\begin{cases} u^\varepsilon(t, x) - \max(f(\varepsilon), g(\varepsilon)) \leq \rho^\varepsilon(t, x) \leq (\rho^\varepsilon)^*(t, x) \leq u^\varepsilon(t, x) + \max(f(\varepsilon), g(\varepsilon)) + \varepsilon \\ \xi^\varepsilon(t, x) - \max(f(\varepsilon), g(\varepsilon)) \leq \sigma^\varepsilon(t, x) \leq (\sigma^\varepsilon)^*(t, x) \leq \xi^\varepsilon(t, x) + \max(f(\varepsilon), g(\varepsilon)) + \varepsilon, \end{cases}$$

where we have used the fact that (3.3) is invariant by addition of constants to the solutions. Passing to the limit as $\varepsilon \rightarrow 0$ and using Theorem 3.2 we get that $\rho^\varepsilon, \sigma^\varepsilon \rightarrow u^0$ (the unique solution of (2.7) with (u_0, ξ_0^ε) for initial condition), which ends the proof of Theorem 3.3. \square

10 Appendix: analysis of system (3.1)

In this section we present some properties of the solution $(U_i, \Xi_i)_{i \in \mathbb{Z}}$ of

$$\begin{cases} \dot{U}_j(t) = \alpha (\Xi_j(t) - U_j(t)) \\ \dot{\Xi}_j(t) = \alpha (U_j(t) - \Xi_j(t)) + 2V (U_{j+1}(t) - U_j(t)) \cdot \phi(U_j(t)). \end{cases} \quad (10.1)$$

We couple system (10.1) with an initial condition $(U_i(0), \Xi_i(0))_i$ that satisfy the following assumption.

(A0') (Initial conditions for (10.1)). For all $i \in \mathbb{Z}$,

$$0 \leq \Xi_i(0) - U_i(0) \leq \frac{V_{max}}{\alpha}, \quad U_{i+1}(0) - \Xi_i(0) \geq h_0, \quad \text{and} \quad U_{i+1}(0) - U_i(0) \leq h_{max}. \quad (10.2)$$

Proposition 10.1 (Bounds on the velocities of the vehicles). *Assume (A) and (A0'), then the solution $(U_i, \Xi_i)_i$ of (10.1) satisfies for all $i \in \mathbb{Z}$*

$$0 \leq \Xi_i(t) - U_i(t) \leq \frac{V_{max}}{\alpha} \quad \text{for all } t > 0. \quad (10.3)$$

Proof. Let us consider the equation satisfied by $\Xi_i - U_i$,

$$\begin{cases} \frac{d(\Xi_i - U_i)}{dt} = -2\alpha (\Xi_i - U_i) + 2V (U_{j+1} - U_j) \cdot \phi(U_j) \quad \text{for all } t > 0, \\ 0 \leq \Xi_i(0) - U_i(0) \leq \frac{V_{max}}{\alpha}. \end{cases}$$

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Step 1: proof of the upper bound in (10.3). Using assumptions (A1), (A4), and (A6), we notice that $\Xi_i - U_i$ is a sub-solution of

$$\begin{cases} \dot{z} = -2\alpha z + 2V_{max}, \\ z(0) = \frac{V_{max}}{\alpha}. \end{cases} \quad (10.4)$$

By comparison, we have

$$\Xi_i(t) - U_i(t) \leq z(t) = \frac{V_{max}}{\alpha} \quad \text{for all } t \geq 0.$$

Step 2: proof of the lower bound in (10.3). Using assumptions (A1), (A3), and (A6), we notice that $\Xi_i - U_i$ is a super-solution of

$$\begin{cases} \dot{z} = -2\alpha z, \\ z(0) = 0. \end{cases} \quad (10.5)$$

By comparison, we have

$$\Xi_i(t) - U_i(t) \geq z(t) = 0 \quad \text{for all } t \geq 0.$$

This ends the proof of Proposition 10.1. □

Proposition 10.2 (Conservation of the order in (10.1)). *Assume (A) and (A0'), then the solution $(U_i, \Xi_i)_i$ of (10.1) satisfies for all $i \in \mathbb{Z}$,*

$$U_{i+1}(t) - \Xi_i(t) \geq h_0 \quad \text{for all } t > 0. \quad (10.6)$$

In particular, using Proposition 10.1, this result implies that

$$U_{i+1}(t) - U_i(t) \geq h_0 \quad \text{and} \quad \Xi_{i+1}(t) - \Xi_i(t) \geq h_0 \quad \text{for all } t > 0. \quad (10.7)$$

Proof. We will prove that for all $\delta > 0$ small, we have

$$U_{i+1}(t) - \Xi_i(t) \geq h_0 - \delta \quad \text{for all } t > 0.$$

Then passing to the limit as δ goes to 0 we will obtain (10.6).

Let $\delta > 0$, we argue by contradiction and assume there exists a time

$$t^* = \inf\{t, \text{ s.t. } \exists j \in \mathbb{Z} \text{ s.t. } U_{j+1}(t) - \Xi_j(t) = h_0 - \delta\}.$$

10. Appendix: analysis of system (3.1)

Let us consider $j \in \mathbb{Z}$ such that $U_{j+1}(t^*) - \Xi_j(t^*) = h_0 - \delta$. By continuity, there exists a time $t_0 \in [0, t^*)$ such that

$$U_{j+1}(t_0) - \Xi_j(t_0) = h_0 \quad \text{and} \quad U_{j+1}(t) - \Xi_j(t) \in [h_0 - \delta, h_0] \quad \text{for all } t \in [t_0, t^*].$$

Using Proposition 10.1, in particular that $U_j \leq \Xi_j$, and assumption (A7) combined with Remark 2.2, we have that

$$\alpha(U_j - \Xi_j) + 2V(U_{j+1} - U_j) \cdot \phi(U_j) \leq 2V(U_{j+1} - \Xi_j) \cdot \phi(\Xi_j) \leq 2V(h_0) \cdot \phi(\Xi_j) = 0 \quad (10.8)$$

This implies that (U_j, Ξ_j) satisfies for all $t \in [t_0, t^*]$,

$$\begin{cases} \dot{U}_j = \alpha(\Xi_j - U_j) \\ \dot{\Xi}_j \leq 0, \end{cases} \quad \text{with} \quad \begin{cases} U_j(t_0) \leq \Xi_j(t_0) \\ \Xi_j(t_0) = U_{j+1}(t_0) - h_0. \end{cases}$$

Therefore, we have for all $t \in [t_0, t^*]$

$$\Xi_j(t) \leq U_{j+1}(t_0) - h_0.$$

Using again Proposition 10.1, in particular that the functions $(U_i)_i$ are non-decreasing in time, we obtain that

$$\Xi_j(t^*) \leq U_{j+1}(t^*) - h_0,$$

which is a contradiction. This ends the proof of Proposition 10.2. □

Proposition 10.3 (Maximal distance between two vehicles). *Assume (A) and (A0'), then the solution $(U_i, \Xi_i)_i$ of (10.1) satisfies for all $i \in \mathbb{Z}$,*

$$U_{i+1}(t) - U_i(t) \leq h_{max} + \frac{3V_{max}}{2\alpha} + \frac{2r}{\phi_0} \quad \text{for all } t > 0. \quad (10.9)$$

In particular, using Proposition 10.1, we have that for all $i \in \mathbb{Z}$,

$$U_{i+1}(t) - \Xi_i(t) \leq h_{max} + \frac{3V_{max}}{2\alpha} + \frac{2r}{\phi_0} \quad \text{for all } t > 0. \quad (10.10)$$

Proof. We will prove that for all $\delta > 0$ small, we have for all $i \in \mathbb{Z}$,

$$U_{i+1}(t) - U_i(t) \leq h_{max} + \frac{3V_{max}}{2\alpha} + \frac{2r}{\phi_0} + \delta \quad \text{for all } t > 0. \quad (10.11)$$

Passing to the limit in the previous inequality as δ goes to 0, we will obtain (10.9).

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Let $\delta > 0$, we argue by contradiction and assume there exists a time

$$t^* = \inf \left\{ t \text{ s.t. } \exists j \in \mathbb{Z} \text{ s.t. } U_{j+1}(t) - U_j(t) > h_{max} + \frac{2r}{\phi_0} + \frac{3V_{max}}{2\alpha} + \delta \right\}.$$

Let us consider $j \in \mathbb{Z}$ such that $U_{j+1}(t^*) - U_j(t^*) = h_{max} + \frac{2r}{\phi_0} + \frac{3V_{max}}{2\alpha} + \delta$. By continuity and (A0'), there exists a time $t_0 \in [0, t^*)$ such that

$$U_{j+1}(t_0) - U_j(t_0) = h_{max} \quad (10.12)$$

and

$$U_{j+1}(t) - U_j(t) \in \left[h_{max}, h_{max} + \frac{2r}{\phi_0} + \frac{3V_{max}}{2\alpha} + \delta \right] \quad \text{for all } t \in [t_0, t^*].$$

We distinguish three cases.

Case 1: $U_j(t_0) \in [-r, r]$. The couple (U_j, Ξ_j) satisfy for all $t \in [t_0, t^*]$

$$\begin{cases} \dot{U}_j = \alpha(\Xi_j - U_j) \\ \dot{\Xi}_j = \alpha(U_j - \Xi_j) + 2V_{max} \cdot \phi(U_j), \end{cases} \quad \text{with} \quad \begin{cases} U_j(t_0) = U_{j+1}(t_0) - h_{max} \\ 0 \leq \Xi_j(t_0) - U_j(t_0) \leq \frac{V_{max}}{\alpha}. \end{cases} \quad (10.13)$$

In order to compare the distance $U_{j+1} - U_j$ when U_j is inside the perturbation, we consider the worst case scenario where the vehicle j advances at a speed $V_{max}\phi_0$ and $j+1$ advances at a speed V_{max} , until $U_j \geq r$ (meaning that the vehicle j is outside the perturbation). To be more exact, we notice that the couple (U_j, Ξ_j) is a super-solution of the following system

$$\begin{cases} \dot{v} = \alpha(\zeta - v) \\ \dot{\zeta} = \alpha(v - \zeta) + 2V_{max}\phi_0, \end{cases} \quad \text{with} \quad \begin{cases} v(t_0) = U_{j+1}(t_0) - h_{max} \\ \zeta(t_0) = v(t_0). \end{cases} \quad (10.14)$$

Computing the solution of (10.14) we get

$$\begin{cases} v(t) = \frac{V_{max}\phi_0}{2\alpha} e^{-2\alpha(t-t_0)} - \frac{V_{max}\phi_0}{2\alpha} + V_{max}\phi_0(t-t_0) + v(t_0) \\ \zeta(t) = -\frac{V_{max}\phi_0}{2\alpha} e^{-2\alpha(t-t_0)} + \frac{V_{max}\phi_0}{2\alpha} + V_{max}\phi_0(t-t_0) + v(t_0) \end{cases} \quad (10.15)$$

By comparison, we obtain that

$$U_j(t) \geq v(t) = \frac{V_{max}\phi_0}{2\alpha} e^{-2\alpha(t-t_0)} - \frac{V_{max}\phi_0}{2\alpha} + V_{max}\phi_0(t-t_0) + v(t_0). \quad (10.16)$$

Let $\hat{t} = \frac{1}{V_{\max}\phi_0} \left(\frac{V_{\max}\phi_0}{2\alpha} + r - U_j(t_0) \right) + t_0$. Using (10.16), we have that $U_j(\hat{t}) \geq r$. We now prove that $\hat{t} < t^*$. In fact, for all $t \in [t_0, \hat{t}]$, we have

$$\begin{aligned} U_{j+1}(t) - U_j(t) &\leq U_{j+1}(\hat{t}) - U_j(t_0) \leq V_{\max}(\hat{t} - t_0) + U_{j+1}(t_0) - U_j(t_0) \\ &= V_{\max} \left(\frac{1}{V_{\max}\phi_0} \left(\frac{V_{\max}\phi_0}{2\alpha} + r - U_j(t_0) \right) \right) + U_{j+1}(t_0) - U_j(t_0) \\ &\leq \frac{V_{\max}}{2\alpha} + \left(\frac{r - U_j(t_0)}{\phi_0} \right) + h_{\max} \\ &\leq \frac{V_{\max}}{2\alpha} + \frac{2r}{\phi_0} + h_{\max}, \end{aligned}$$

where we have used Proposition 10.1 for the first line. From the previous inequality and the definition of t^* , we deduce that $\hat{t} < t^*$.

The couple (U_j, Ξ_j) satisfies for all $t \in [\hat{t}, t^*]$,

$$\begin{cases} \dot{U}_j = \alpha(\Xi_j - U_j) \\ \dot{\Xi}_j = \alpha(U_j - \Xi_j) + 2V_{\max}, \end{cases} \quad (10.17)$$

with

$$\begin{cases} h_{\max} \leq U_{j+1}(\hat{t}) - U_j(\hat{t}) \leq h_{\max} + \frac{2r}{\phi_0} + \frac{V_{\max}}{2\alpha} \\ 0 \leq \Xi_j(\hat{t}) - U_j(\hat{t}) \leq \frac{V_{\max}}{\alpha}. \end{cases} \quad (10.18)$$

We can easily compute the explicit form of the solution of (10.18),

$$U_j(t) = \left(\frac{V_{\max}}{\alpha} - \Xi_j(\hat{t}) + U_j(\hat{t}) \right) \frac{e^{-2\alpha(t-\hat{t})}}{2} - \frac{V_{\max}}{2\alpha} + V_{\max}(t - \hat{t}) + \frac{1}{2} \left(\Xi_j(\hat{t}) + U_j(\hat{t}) \right)$$

and

$$\Xi_j(t) = \left(\Xi_j(\hat{t}) - U_j(\hat{t}) - \frac{V_{\max}}{\alpha} \right) \frac{e^{-2\alpha(t-\hat{t})}}{2} + \frac{V_{\max}}{2\alpha} + V_{\max}(t - \hat{t}) + \frac{1}{2} \left(\Xi_j(\hat{t}) + U_j(\hat{t}) \right).$$

Using Proposition 10.1, for all $t \in [\hat{t}, t^*]$, we have that

$$U_{j+1}(t) \leq V_{\max}(t - \hat{t}) + U_{j+1}(\hat{t}). \quad (10.19)$$

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Therefore, combining the previous results, we have for all $t \in [\hat{t}, t^*]$

$$\begin{aligned}
 U_{j+1}(t) - U_j(t) &\leq V_{max}(t - \hat{t}) + U_{j+1}(\hat{t}) - V_{max}(t - \hat{t}) - \frac{1}{2} (\Xi_j(\hat{t}) + U_j(\hat{t})) \\
 &\quad - \left(\frac{V_{max}}{\alpha} - \Xi_j(\hat{t}) + U_j(\hat{t}) \right) \frac{e^{-2\alpha(t-\hat{t})}}{2} + \frac{V_{max}}{2\alpha} \\
 &\leq U_{j+1}(\hat{t}) - \frac{1}{2} (\Xi_j(\hat{t}) + U_j(\hat{t})) + \frac{V_{max}}{2\alpha} \\
 &\leq U_{j+1}(\hat{t}) - U_j(\hat{t}) + \frac{V_{max}}{2\alpha} \\
 &\leq h_{max} + \frac{2r}{\phi_0} + \frac{V_{max}}{\alpha},
 \end{aligned}$$

where we have used Proposition 10.1 for the second and third inequality and we have used (10.18) for the last inequality. The previous inequality remains valid for $t = t^*$ which gives us a contradiction.

Case 2: $U_j(t_0) > r$. In this case, the couple (U_j, Ξ_j) satisfies system (10.17) for all $t \in (t_0, t^*]$, with the following initial conditions

$$\begin{cases} U_j(t_0) = U_{j+1}(t_0) - h_{max} \\ 0 \leq \Xi_j(t_0) - U_j(t_0) \leq \frac{V_{max}}{\alpha}. \end{cases} \quad (10.20)$$

As above, the explicit solution of (10.17)-(10.20) has the following form,

$$\begin{aligned}
 U_j(t) &= \left(\frac{V_{max}}{\alpha} - \Xi_j(t_0) + U_j(t_0) \right) \frac{e^{-2\alpha(t-t_0)}}{2} - \frac{V_{max}}{2\alpha} \\
 &\quad + V_{max}(t - t_0) + \frac{1}{2} (\Xi_j(t_0) + U_j(t_0))
 \end{aligned}$$

and

$$\begin{aligned}
 \Xi_j(t) &= \left(\Xi_j(t_0) - U_j(t_0) - \frac{V_{max}}{\alpha} \right) \frac{e^{-2\alpha(t-t_0)}}{2} \\
 &\quad + \frac{V_{max}}{2\alpha} + V_{max}(t - t_0) + \frac{1}{2} (\Xi_j(t_0) + U_j(t_0)).
 \end{aligned}$$

Arguing as above, we will obtain $U_{j+1}(t^*) - U_j(t^*) \leq h_{max} + \frac{V_{max}}{2\alpha}$ which is a contradiction.

Case 3: $U_j(t_0) < -r$. We treat this case in 3 steps.

Step 1: left of the perturbation. We denote by

$$\hat{t} = \inf \{t \geq t_0 \text{ s.t. } U_j(t) = -r\}.$$

For all $t \in [t_0, \hat{t}]$, the couple (U_j, Ξ_j) satisfies (10.17)-(10.20) and therefore has the same form as the one presented in Case 2. In particular, for all $t \in [t_0, \hat{t}]$, we have

$$U_{j+1}(t) - U_j(t) \leq h_{max} + \frac{V_{max}}{2\alpha}. \quad (10.21)$$

This implies that $\hat{t} < t^*$.

Step 2: inside the perturbation. In the interval $[\hat{t}, t^*]$, the couple (U_j, Ξ_j) satisfies (10.13) with the following initial condition

$$\begin{cases} U_{j+1}(\hat{t}) - U_j(\hat{t}) \leq h_{max} + \frac{V_{max}}{2\alpha} \\ 0 \leq \Xi_j(\hat{t}) - U_j(\hat{t}) \leq \frac{V_{max}}{\alpha}. \end{cases}$$

The couple (U_j, Ξ_j) is a super-solution of

$$\begin{cases} \dot{v} = \alpha(\zeta - v) \\ \dot{\zeta} = \alpha(v - \zeta) + 2V_{max}\phi_0, \end{cases} \quad \text{with} \quad \begin{cases} v(\hat{t}) = U_{j+1}(\hat{t}) + h_{max} + \frac{V_{max}}{2\alpha} \\ \zeta(\hat{t}) = v(\hat{t}). \end{cases} \quad (10.22)$$

Computing the solution of (10.22), and by comparison, for all $t \in [\hat{t}, t^*]$, we have

$$U_j(t) \geq \frac{V_{max}\phi_0}{2\alpha} e^{-2\alpha(t-\hat{t})} - \frac{V_{max}\phi_0}{2\alpha} + V_{max}\phi_0(t - \hat{t}) + v(\hat{t}).$$

Let $\tilde{t} = \frac{1}{V_{max}\phi_0} \left(\frac{V_{max}\phi_0}{2\alpha} + r - U_j(\hat{t}) \right) + \hat{t}$. Using (10.16), we have that $U_j(\tilde{t}) \geq r$. We now prove that $\tilde{t} < t^*$. We recall that $U_j(\hat{t}) = -r$. In fact, for all $t \in [\hat{t}, \tilde{t}]$, we have

$$\begin{aligned} U_{j+1}(t) - U_j(t) &\leq U_{j+1}(\tilde{t}) - U_j(\hat{t}) \leq V_{max}(\tilde{t} - \hat{t}) + U_{j+1}(\hat{t}) - U_j(\hat{t}) \\ &= V_{max} \left(\frac{1}{V_{max}\phi_0} \left(\frac{V_{max}\phi_0}{2\alpha} + r - U_j(\hat{t}) \right) \right) + U_{j+1}(\hat{t}) - U_j(\hat{t}) \\ &\leq \frac{V_{max}}{\alpha} + \frac{2r}{\phi_0} + h_{max}, \end{aligned}$$

where we have used Proposition 10.1 for the first line. From the previous inequality and the definition of t^* , we deduce that $\tilde{t} < t^*$.

Step 3: right of the perturbation. In the interval $[\tilde{t}, t^*]$, the couple (U_j, Ξ_j) satisfies (10.17), with the following initial condition

$$U_{j+1}(\tilde{t}) - U_j(\tilde{t}) \leq \frac{V_{max}}{\alpha} + \frac{2r}{\phi_0} + h_{max}.$$

Proceeding like before, we can prove that for all $t \in [\tilde{t}, t^*]$, we have

$$U_{j+1}(t) - U_j(t) \leq \frac{3V_{max}}{2\alpha} + \frac{2r}{\phi_0} + h_{max},$$

which gives us a contradiction for $t = t^*$. This ends the proof of Proposition 10.3. \square

11 Appendix: proof of Theorem 9.1

Before we give the proof of Theorem 9.1, we need the following result.

Lemma 11.1 (Link between the velocities). *Assume (A). Let $((U_j)_j, (\Xi_j)_j)$ be the solution of (3.1) with an initial condition $(U_j(0), \Xi_j(0))_j$ satisfying (A0'). Then we have*

$$\dot{U}_j(t) = -M(u(t, U_j(t)), [\xi(t, \cdot)])(U_j(t)) \quad (11.1)$$

and

$$\dot{\Xi}_j(t) = -L(\Xi_j(t), \xi(t, \Xi_j(t)), [u(t, \cdot)])(\Xi_j(t)), \quad (11.2)$$

where u and ξ are continuous functions such that

$$\begin{cases} u(t, x) = \rho_*(t, x) = \rho(t, x) \text{ for } x = U_j(t), j \in \mathbb{Z}, \\ u \text{ is decreasing in } x, \end{cases} \quad (11.3)$$

$$\begin{cases} \xi(t, x) = \sigma_*(t, x) = \sigma(t, x) \text{ for } x = \Xi_j(t), j \in \mathbb{Z}, \\ \xi \text{ is decreasing in } x, \end{cases} \quad (11.4)$$

where ρ and σ are defined respectively in (2.2) and (3.2) (with $\varepsilon = 1$).

Proof. We drop the time dependence to simplify the presentation. Let $j \in \mathbb{Z}$. We recall that we chose $D = h_{max} + 3V_{max}/(2\alpha) + 2r/\phi_0$. Using the fact that $u(t, U_j(t)) = -(j+1)$ and (11.3), we have for all $z \in [0, +\infty)$,

$$\begin{cases} \xi(U_j + z) - u(U_j) > \xi(\Xi_j) - u(U_j) = 0 & \text{if } z \in [0, \Xi_j - U_j) \\ \xi(U_j + z) - u(U_j) \leq 0 & \text{if } z \in [\Xi_j - U_j, +\infty). \end{cases}$$

Using Proposition 10.1, in particular that $\Xi_j - U_j \leq D$, we have

$$\begin{aligned} M(u(t, U_j(t)), [\xi(t, \cdot)])(U_j(t)) &= \int_0^D E(\xi(U_j + z) - u(U_j)) dz \\ &= \int_0^{\Xi_j - U_j} E(\xi(U_j + z) - u(U_j)) dz + \int_{\Xi_j - U_j}^D E(\xi(U_j + z) - u(U_j)) dz \\ &= -\alpha(\Xi_j - U_j). \end{aligned}$$

Combining this result with (3.1), we obtain (11.1). We now turn to the proof of (11.2). We begin by computing $K(\xi(t, \Xi_j(t)), [u(t, \cdot)])(\Xi_j(t))$.

Using the fact that $\xi(t, \Xi_j(t)) = -(j+1)$ and (11.4), we have for all $z \in [0, +\infty)$,

$$\begin{cases} u(\Xi_j - z) - \xi(\Xi_j) < u(U_j) - \xi(\Xi_j) = 0 & \text{if } z \in [0, \Xi_j - U_j) \\ u(\Xi_j - z) - \xi(\Xi_j) \geq 0 & \text{if } z \in [\Xi_j - U_j, +\infty). \end{cases}$$

Thanks to Proposition 10.1, this implies that

$$K(\xi(t, \Xi_j(t)), [u(t, \cdot)])(\Xi_j(t)) = \int_0^{\Xi_j - U_j} F(u(\Xi_j - z) - \xi(\Xi_j)) dz = \Xi_j - U_j.$$

We now turn to the computation of $N(\xi(t, \Xi_j(t)), [u(t, \cdot)])(\Xi_j(t))$. We recall that thanks to Proposition 10.2, we have $U_{j+1} - \Xi_j \geq h_0$. In particular, we have that

$$\begin{cases} u(\Xi_j + z) - \xi(\Xi_j) > u(U_{j+1}) - \xi(\Xi_j) = -1 & \text{if } z \in [0, U_{j+1} - \Xi_j) \\ u(\Xi_j + z) - \xi(\Xi_j) \leq -1 & \text{if } z \in [U_{j+1} - \Xi_j, +\infty). \end{cases}$$

Once more thanks to Proposition 10.3, we obtain

$$N(\xi(t, \Xi_j(t)), [u(t, \cdot)])(\Xi_j(t)) = \int_0^{U_{j+1} - \Xi_j} I(u(\Xi_j + z) - \xi(\Xi_j)) dz = U_{j+1} - \Xi_j.$$

Combining the previous results with (3.16) and (3.1), we obtain (11.2). □

Proof of Theorem 9.1. We remark that thanks to (11.3) and (11.4), we have for $x = U_j(t)$ and $y = \Xi_j(t)$, $j \in \mathbb{Z}$,

$$\tilde{M}(\rho_*(t, x), [\sigma_*(t, \cdot)])(x) = \tilde{M}(u(t, x), [\xi(t, \cdot)])(x) \geq M(u(t, x), [\xi(t, \cdot)])(x),$$

and

$$\tilde{L}(y, \sigma_*(t, y), [\rho_*(t, \cdot)])(y) = \tilde{L}(y, \xi_*(t, y), [u(t, \cdot)])(y) \geq L(y, \xi_*(t, y), [u(t, \cdot)])(y).$$

Using Lemma 11.1, and Definition 4.1, we can see that (ρ_*, σ_*) is a discontinuous viscosity super-solution of (9.1). We obtain a similar result for (ρ^*, σ^*) , therefore, (ρ, σ) is a discontinuous viscosity solution of (9.1).

We prove the converse. For the readers convenience we recall from [FIM09b, Proposition 4.8] that we will use later.

Ch.5. Specified homogenization of a second order discrete model

Lemma 11.2. *Assume that $\theta : \mathbb{R} \rightarrow \mathbb{R}$ is a non-decreasing and upper semi-continuous (resp. lower semi-continuous). Assume also that*

$$\theta(v) - v \text{ is } 1\text{-periodic in } v.$$

Assume that $\varepsilon = 1$ in (3.3). Consider also a sub-solution (resp. a super-solution) (u, ξ) of (3.3). Then $(\theta(u), \theta(\xi))$ is also a sub-solution (resp. a super-solution) of (3.3).

Using Lemma 11.2 we can conclude that $(\rho_*, \sigma_*) = (\lceil u \rceil, \lceil \xi \rceil)$ (resp. $(\rho^*, \sigma^*) = (\lfloor u \rfloor, \lfloor \xi \rfloor)$) is a viscosity super-solution (resp. sub-solution) of

$$\begin{cases} \partial_t \rho - \tilde{c}(t, x) \partial_x \rho = 0 \\ \partial_t \sigma - \tilde{d}(t, x) \partial_x \sigma = 0, \end{cases}$$

with

$$\begin{cases} \tilde{c}(t, x) = M(u(t, x), [\xi(t, \cdot)])(x) = \tilde{M}(u(t, x), [\xi(t, \cdot)])(x) \\ \tilde{d}(t, x) = L(x, \xi(t, x), [u(t, \cdot)])(x) = \tilde{L}(x, \xi(t, x), [u(t, \cdot)])(x). \end{cases}$$

Using the fact that u and ξ are decreasing in space, we define

$$\begin{cases} U_i(t) = \inf\{x, u(t, x) \leq -(i+1)\} = (u(t, \cdot))^{-1}(-i-1) \\ \Xi_i(t) = \inf\{x, \xi(t, x) \leq -(i+1)\} = (\xi(t, \cdot))^{-1}(-i-1), \end{cases}$$

and we consider the functions $t \mapsto U_i(t)$ and $t \mapsto \Xi_i(t)$. They are continuous because u and ξ are decreasing in x and are continuous in (t, x) .

We now prove that the functions (U_i, Ξ_i) are viscosity solutions of (3.1). Let φ be a test function such that $\varphi(t) \leq U_i(t)$ and $\varphi(t_0) = U_i(t_0)$. Let us now define $\hat{\varphi}(t, x) = -(i+1) + \varphi(t) - x$. It satisfies

$$\hat{\varphi}(t_0, U_i(t_0)) = \rho_*(t_0, U_i(t_0)),$$

and

$$\hat{\varphi}(t, x) \leq \rho_*(t, x) \quad \text{for } U_i(t) - 1 < x < U_{i+1}(t).$$

This implies that

$$\begin{aligned} \varphi_t(t_0) + \tilde{c}(t_0, U_i(t_0)) &\geq 0 \\ \Leftrightarrow \varphi_t(t_0) &\geq -\tilde{c}(t_0, U_i(t_0)) = -\bar{c}_i(t_0) = \alpha(\Xi_i(t_0) - U_i(t_0)). \end{aligned}$$

Let ψ be a test function such that $\psi(t) \leq \Xi_i(t)$ and $\psi(t_1) = \Xi_i(t_1)$. Let us now define $\hat{\psi}(t, x) = -(i+1) + \psi(t) - x$. It satisfies

$$\hat{\psi}(t_1, \Xi_i(t_1)) = \sigma_*(t_1, \Xi_i(t_1)),$$

and

$$\hat{\psi}(t, x) \leq \sigma_*(t, x) \quad \text{for } \Xi_i(t) - 1 < x < \Xi_{i+1}(t).$$

This implies that

$$\begin{aligned} \psi_t(t_1) + \tilde{d}(t_1, \Xi_i(t_1)) \geq 0 &\Leftrightarrow \psi_t(t_1) \geq -\tilde{d}(t_1, \Xi_i(t_1)) = -\bar{d}_i(t_1) \\ &= \alpha(U_i(t_1) - \Xi_i(t_1)) + 2V(U_{i+1}(t_1) - U_i(t_1)) \phi(U_i(t_1)) \end{aligned}$$

This proves that (U_i, Ξ_i) are viscosity super-solutions of (3.1). The proof for sub-solutions is similar and we skip it. Moreover, since \bar{c}_i and \bar{d}_i are continuous, we deduce that $U_i, \Xi_i \in C^1$ and therefore (U_i, Ξ_i) are classical solution of (3.1). □

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Chapitre 6

Homogenization of a discrete model for a bifurcation and application to traffic flow

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Abstract

The goal of this paper is to derive a macroscopic traffic flow model, for a simple bifurcation, from a microscopic model. At the microscopic scale, we consider a first order model of the form "follow the leader" i.e. the velocity of each vehicle depends on the distance to the vehicle in front of it. We consider the case of a very simple bifurcation in which one road separates into two and one vehicle over two goes to the right and the other goes to the left. At the bifurcation, we then have to add a phase of transition because the vehicle in front will change. Moreover, we assume that the velocity on each of the roads can be different. At the macroscopic scale, we obtain an explicit Hamilton-Jacobi equation on each road and a junction condition (in the sense of [IM14]) located at the bifurcation. From this case of a simple bifurcation, we then extend to more general scenarios. For instance, the case of a different distribution of the vehicles at the bifurcation or even to consider more than two outgoing roads. For these extensions we only present the results and explain how to adapt the proofs from the case of a simple bifurcation.

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1 Introduction

Traffic flow can be characterized at different scales. The first one is the microscopic scale in which we describe the dynamics of each vehicles individually. Another one is the macroscopic scale that describes the collective dynamics of the vehicles with macroscopic quantities such as the density of vehicles and the average speed. The link between microscopic and macroscopic models has been extensively studied specially in the case of a single road. However, there are not many results concerning the homogenization of microscopic traffic flow models in networks.

In the case of a single road, we refer to [AKRM02, DFR15, Hel98, LLK01] where the authors rescaled the empirical measure in order to obtain a Lighthill-Whitham-Richards (LWR) model (see [LW55, Ric56]) at the macroscopic scale. Other works have been done by rescaling the primitive of the empirical measure in order to obtain a Hamilton-Jacobi equation (which is the primitive of a LWR model) at the macroscopic scale. Among those works we refer the readers to the papers [FIM09a, FIM09b, FIM12, 1], the last one being directly applied to traffic flow.

Moreover, the recent works [AT15, GIM15], and the ideas in the lectures of Lions at the "College de France" [Lio14], concerning specified homogenization enabled us to consider a traffic flow model with a local perturbation and to deduce a macroscopic model with a junction condition (see [2]). We would also like to refer to the paper of Colombo and Goatin [AKR11], where the authors present an homogenization result from a LWR model with a discontinuous flow in space to a LWR model with a flow limiting condition at a single point.

Concerning homogenization results on networks, we would like to mention the recent work of Cristiani and Sahu [CS15] where the authors present a first order microscopic model on a network and show the link to a multi-path model (see [BBC14, BC14]). In fact they consider for each possible path a different population of vehicles. Their homogenization result is set in a very general network, however, they assume the convergence of the empirical measure (of each population) and they prove that the limit satisfies a multi-path model.

The present work focuses on obtaining a macroscopic model from a microscopic model for traffic flow in the case of a simple bifurcation. The schematic representation of the microscopic model is given in Figure 1.1.

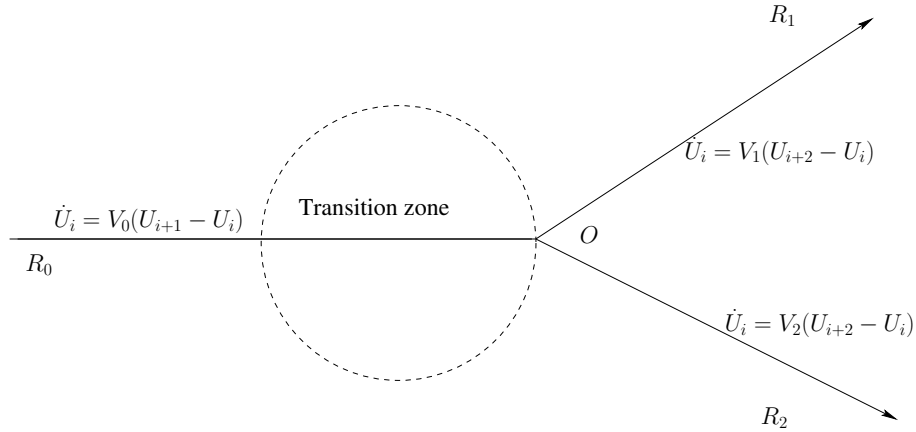


Figure 1.1: Simplified schematic representation of the microscopic model.

More precisely, we consider one incoming road R_0 which separates into two outgoing roads R_1 and R_2 . We denote by $U_i(t)$ the position of the i th vehicle, and we assume that the vehicles with an odd index go to R_1 while the ones with an even index go to R_2 . Finally, we assume that on each road R_i the velocity of each vehicle is given by a function V_i . In order to obtain our result we will proceed as in [FIM09a, FIM12, 2, 1] and rescale the microscopic model which describes the dynamics of each vehicle, in order to get a macroscopic model that describes the dynamics of the density of vehicles. At the macroscopic scale we will obtain a Hamilton-Jacobi equation on each branch and a junction condition at the origin (see Figure 1.2, where u_x^0 is related to the density of vehicles (see below) and the effective Hamiltonians \bar{H}_i are defined in the next section.

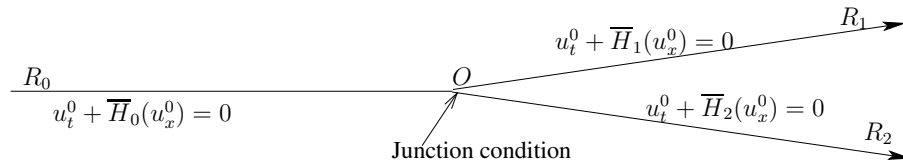


Figure 1.2: Schematic representation of the macroscopic (homogenized) model.

Finally, we give some extensions from the case of a simple bifurcation. We present the results in the case of more than two outgoing roads, and in the case we have a more general (but still periodic) distribution of the vehicles on each road. For the extensions, we do not give the details of the proofs, we only give some tips on how to adapt the proofs already presented in this paper.

In this paper we will use the recent developments on Hamilton-Jacobi equations on networks, particularly the paper of Imbert and Monneau [IM14] which gives a suitable definition of viscosity solutions at the junction.

1.1 General first order microscopic model for a junction

In this paper we are interested in a first order microscopic model for a simple bifurcation located at the origin, where we consider that the vehicles with odd indexes go to the left and vehicles with even indexes go to the right. We denote by U_i the position of the i th vehicle and \dot{U}_i its velocity. For $i = 0, 1, 2$ we call V_i an optimal velocity functions. Before the bifurcation (i.e for $U_i(t) \ll 0$) we assume that the velocity of each vehicle is given by

$$\dot{U}_i(t) = V_0 (U_{i+1}(t) - U_i(t)),$$

while after the bifurcation (i.e for $U_i(t) > 0$) we assume that

$$\dot{U}_i(t) = \begin{cases} V_1 (U_{i+2}(t) - U_i(t)) & \text{if } i \text{ is odd} \\ V_2 (U_{i+2}(t) - U_i(t)) & \text{if } i \text{ is even.} \end{cases}$$

In order to pass from the velocity V_0 to V_i , $i = 1$ or 2 , we introduce a transition function $\bar{\phi}$ and we consider the following system for all $t > 0$:

$$\dot{U}_i(t) = \begin{cases} \bar{\phi} \left(U_i(t), V_0 (U_{i+1}(t) - U_i(t)), V_1 (U_{i+2}(t) - U_i(t)) \right) & \text{if } i \text{ is odd} \\ \bar{\phi} \left(U_i(t), V_0 (U_{i+1}(t) - U_i(t)), V_2 (U_{i+2}(t) - U_i(t)) \right) & \text{if } i \text{ is even.} \end{cases} \quad (1.1)$$

The function $\bar{\phi}$ allows a transition near the junction, and is defined by

$$\bar{\phi}(x, a, b) = \begin{cases} a & \text{if } x < -h_0 - h_{max} - 1, \\ (h_0 + h_{max})(\min(a, b) - a) & \text{if } -1 \leq x + h_0 + h_{max} < 0, \\ -|x|(\min(a, b) - a) + \min(a, b) & \text{if } -h_0 - h_{max} \leq x < -h_0, \\ \frac{|x|}{h_0} \min(a, b) + \left(1 - \frac{|x|}{h_0}\right) b & \text{if } -h_0 \leq x \leq 0, \\ b & \text{if } x > 0, \end{cases} \quad (1.2)$$

with $h_{max} > h_0 > 0$. Figure 1.3 represents model (1.1).

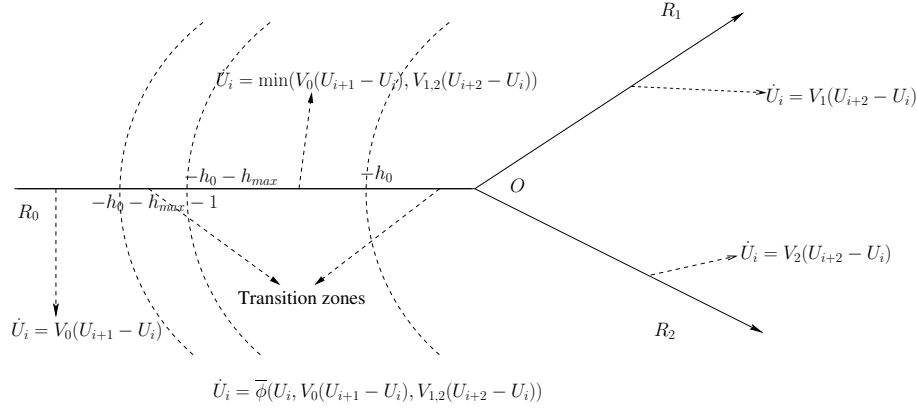


Figure 1.3: Schematic representation of the microscopic model.

In model (1.1), the vehicles with an odd index go to the road R_1 and the others go to the road R_2 . Since we work with a first order model, on R_0 the velocity of each vehicle depends on the distance $U_{i+1} - U_i$, but on R_1 and R_2 it depends on $U_{i+2} - U_i$.

We will call the transition zone, the interval $[-h_0 - h_{max}, 0]$, where the vehicles will change from one model to the other. Concerning the optimal velocity functions V_i , for $i = 0, 1, 2$, and $\bar{\phi}$, we do the following assumptions.

Assumption (A)

- (A1) $V_i : \mathbb{R} \rightarrow \mathbb{R}^+$ is Lipschitz continuous, non-negative.
- (A2) V_i is non-decreasing on \mathbb{R} .
- (A3) There exists $h_0 \in (0, +\infty)$ such that for all $h \leq h_0$, $V_i(h) = 0$.
- (A4) There exists $h_{max} \in (h_0, +\infty)$ such that for all $h \geq h_{max}$, $V_i(h) = V_i(h_{max}) =: V_{max}^i$.
- (A5) The functions $p \mapsto pV_0(-1/p)$ and $p \mapsto pV_j(-2/p)$, for $j=1,2$, are strictly convex respectively on $[-1/h_0, 0)$ and on $[-2/h_0, 0)$.
- (A6) The function $\bar{\phi} : \mathbb{R}^3 \rightarrow \mathbb{R}$ is Lipschitz continuous with respect to each variable. For all $(x, a, b) \in \mathbb{R}$, the functions $\bar{\phi}(x, \cdot, b)$ and $\bar{\phi}(x, a, \cdot)$ are non-decreasing. We denote by $\|\bar{\phi}'\|_\infty$, the smallest constant such that for all $(x, a, b), (x', a', b') \in \mathbb{R}^3$, we have

$$|\bar{\phi}(x, a, b) - \bar{\phi}(x', a', b')| \leq \|\bar{\phi}'\|_\infty (|x - x'| + |a - a'| + |b - b'|).$$

Remark 1.1 (An extra perturbation). *In order to make the microscopic model more realistic, it is possible to modify it and add a local perturbation around the junction point (by multiplying the velocity with a certain function like in [2]), that would decrease the speed of the vehicles near the origin. However, in the rest of the paper the function $\bar{\phi}$ is already treated like a local perturbation; adding an extra one would just complicate the notations without adding a mathematical interest (see [2] for how to treat a local perturbation in a microscopic model).*

Remark 1.2 (The transition zone). *The optimal velocity functions V_i describe the dynamics of the vehicles on each branch. The role of the transition zone is to do a continuous transition from one dynamic to the next one. Notice that given the form of the transition (1.2), if initially the vehicles have enough space between them, there will always be at least a distance h_0 between two vehicles. Meaning that in model (1.1) there is always a safety distance h_0 that avoids any collisions.*

2 Main results

Like in [FIM09b, 2], we inject the system of ODE into a system of PDE. To do this, we separate the vehicles into two groups, those going into R_1 and those going into R_2 .

2.1 Injecting the system of ODEs into a system of PDEs

We introduce two functions, the first one is the rescaled "cumulative distribution function" of vehicles with an odd index

$$\rho_1^\varepsilon(t, x) = -2\varepsilon \cdot \left(\sum_{i[2]=1, i \geq 0} H(x - \varepsilon U_i(t/\varepsilon)) + \sum_{i[2]=1, i < 0} (-1 + H(x - \varepsilon U_i(t/\varepsilon))) \right) \quad (2.1)$$

and the second one is the rescaled "cumulative distribution function" of vehicles with an even index

$$\rho_2^\varepsilon(t, x) = -2\varepsilon \cdot \left(\sum_{i[2]=0, i \geq 0} H(x - \varepsilon U_i(t/\varepsilon)) + \sum_{i[2]=0, i < 0} (-1 + H(x - \varepsilon U_i(t/\varepsilon))) \right) - \varepsilon, \quad (2.2)$$

with

$$H(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases} \quad (2.3)$$

Here and in the rest of this paper $i[2]$ denotes the rest of the euclidean division of i by 2 (either 0 or 1). Under assumption (A), the function $(\rho_1^\varepsilon, \rho_2^\varepsilon)$ is a (discontinuous viscosity) solution (see Theorem 7.1) of the following non-local equation, for $(t, x) \in (0, +\infty) \times \mathbb{R}$,

$$\begin{cases} u_t^\varepsilon + \phi\left(\frac{x}{\varepsilon}, N_0^\varepsilon\left(\frac{u^\varepsilon}{\varepsilon}, \left[\frac{\xi^\varepsilon(t, \cdot)}{\varepsilon}\right]\right)(x), M_1^\varepsilon\left[\frac{u^\varepsilon(t, \cdot)}{\varepsilon}\right](x)\right) \cdot |u_x^\varepsilon| = 0 \\ \xi_t^\varepsilon + \phi\left(\frac{x}{\varepsilon}, N_0^\varepsilon\left(\frac{\xi^\varepsilon}{\varepsilon}, \left[\frac{u^\varepsilon(t, \cdot)}{\varepsilon}\right]\right)(x), M_2^\varepsilon\left[\frac{\xi^\varepsilon(t, \cdot)}{\varepsilon}\right](x)\right) \cdot |\xi_x^\varepsilon| = 0, \end{cases} \quad (2.4)$$

where N_0^ε and M_i^ε for $i = 1, 2$, are non-local operators defined by

$$N_0^\varepsilon(U, [\Xi])(x) = \int_{\mathbb{R}} J_0(z)F(\Xi(x + \varepsilon z) - U(x))dz - \frac{3}{2}V_{max}, \quad (2.5)$$

and

$$M_i^\varepsilon[U](x) = \int_{\mathbb{R}} J_i(z)E(U(x + \varepsilon z) - U(x))dz - \frac{3}{2}V_{max}, \quad (2.6)$$

with $J_i = V_i'$ for $i = 0, 1, 2$ and

$$F(z) = \begin{cases} 0 & \text{if } z \geq 1, \\ 1/2 & \text{if } -1 \leq z < 1, \\ 3/2 & \text{if } z < -1, \end{cases} \quad \text{and } E(z) = \begin{cases} 0 & \text{if } z \geq 0, \\ 1/2 & \text{if } -2 \leq z < 0, \\ 3/2 & \text{if } z < -2. \end{cases} \quad (2.7)$$

Finally, the function $\phi : \mathbb{R}^3 \rightarrow \mathbb{R}$, is defined by $\phi(x, a, b) = -\bar{\phi}(x, -a, -b)$ for all $(x, a, b) \in \mathbb{R}^3$. In particular it has the following form

$$\phi(x, a, b) = \begin{cases} a & \text{if } x < -h_0 - h_{max} - 1, \\ (h_0 + h_{max})(\max(a, b) - a) & \text{if } -1 \leq x + h_0 + h_{max} < 0, \\ -|x|(\max(a, b) - a) + \max(a, b) & \text{if } -h_0 - h_{max} \leq x < -h_0, \\ \max(a, b) & \text{if } -h_0 \leq x < 0, \\ \frac{|x|}{h_0} \max(a, b) + \left(1 - \frac{|x|}{h_0}\right) b & \text{if } -h_0 \leq x \leq 0, \\ b & \text{if } x > 0. \end{cases} \quad (2.8)$$

Remark 2.1 (Choice of the "cumulative distribution functions"). *Contrary to [2], it is impossible to work with the integral of the empirical measure of the positions of all the vehicles. Indeed, near the bifurcation, the vehicles stop being in order (some go to R_1 and others to R_2), which makes it impossible to recover the distance $U_{i+1} - U_i$. To overcome this difficulty, we consider a modified version of the integral of the empirical measure of the two types of vehicles (the ones going on R_1 and the ones going to R_2). This modification*

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simply allows us to obtain the index of the vehicles more easily: for $\varepsilon = 1$, if $i[2] = 1$ then $\rho_1^\varepsilon(t, U_i(t)) = -(i + 1)$ and if $i[2] = 0$ then $\rho_2^\varepsilon(t, U_i(t)) = -(i + 1)$.

In particular, this implies (see Section 7) that if for instance $i[2] = 1$ and $\varepsilon = 1$, we have

$$\begin{cases} N_0^\varepsilon(\rho_1^\varepsilon(t, \cdot), [\rho_2^\varepsilon]) (U_i(t)) = -V_0 (U_{i+1}(t) - U_i(t)) \\ M_1^\varepsilon[\rho_1^\varepsilon(t, \cdot)](U_i(t)) = -V_1 (U_{i+1}(t) - U_i(t)). \end{cases}$$

We obtain a similar result if $i[2] = 0$. This results helps us inject the system of ODEs into a system of PDEs.

The new transition function ϕ comes from the fact that with the non-local operators we recover the opposite of the velocity of the vehicles.

Remark 2.2. Given the definition of the cumulative distribution function ρ_1^ε (resp. ρ_2^ε) we expect that $(\rho_1)_x$ (resp. $(\rho_2)_x$) (the gradient of the limit of ρ_1^ε (resp. ρ_2^ε) as ε goes to 0) is going to be the density of vehicles on R_0 and twice the density on R_1 (resp. R_2).

We complete (2.4) with the following initial conditions

$$\begin{cases} u^\varepsilon(0, x) = u_0(x) & \text{for } x \in \mathbb{R}, \\ \xi^\varepsilon(0, x) = \xi_0(x) & \text{for } x \in \mathbb{R}, \end{cases} \quad (2.9)$$

and we make the following assumptions:

(A0) (Initial condition). For all $x \leq 0$,

$$u_0(x) = \xi_0(x).$$

Moreover, we assume for all $x \in \mathbb{R}$

$$-2k_0 = -\frac{2}{h_0} \leq (u_0)_x \leq 0 \quad \text{and} \quad -2k_0 \leq (\xi_0)_x \leq 0.$$

2.2 Convergence result

The effective Hamiltonians

Here, we introduce three Hamiltonians, \overline{H}_0 , \overline{H}_1 , and \overline{H}_2 that we will use in the rest of the paper. They are the effective Hamiltonians on each of the branches R_0 , R_1 , and R_2 . We define $k_0 = 1/h_0$ and $\overline{H}_0 : \mathbb{R} \rightarrow \mathbb{R}$ by

$$\overline{H}_0(p) = \begin{cases} -p - 2k_0 & \text{for } p < -2k_0, \\ -V_0 \left(\frac{-1}{p} \right) \cdot |p| & \text{for } -2k_0 \leq p \leq 0, \\ p & \text{for } p > 0. \end{cases} \quad (2.10)$$

We also define, for $i = 1, 2$, $\bar{H}_i : \mathbb{R} \rightarrow \mathbb{R}$ by

$$\bar{H}_i(p) = \begin{cases} -p - 2k_0 & \text{for } p < -2k_0, \\ -V_i \left(\frac{-2}{p} \right) \cdot |p| & \text{for } -2k_0 \leq p \leq 0, \\ p & \text{for } p > 0. \end{cases} \quad (2.11)$$

For $i = 0, 1, 2$, let us notice that such \bar{H}_i is continuous, coercive $\left(\lim_{|p| \rightarrow +\infty} \bar{H}_i(p) = +\infty \right)$ and because of (A5), there exists a unique point $p_i \in [-2k_0, 0]$ such that

$$\begin{cases} \bar{H}_i \text{ is non-increasing on } (-\infty, p_i), \\ \bar{H}_i \text{ is increasing on } (p_i, +\infty), \end{cases} \quad (2.12)$$

We denote by

$$H_0 = \max_{i \in \{0, 1, 2\}} \min_{p \in \mathbb{R}} \bar{H}_i(p). \quad (2.13)$$

Figure 2.4 gives a schematic representation of \bar{H}_0 , \bar{H}_1 , and \bar{H}_2 .

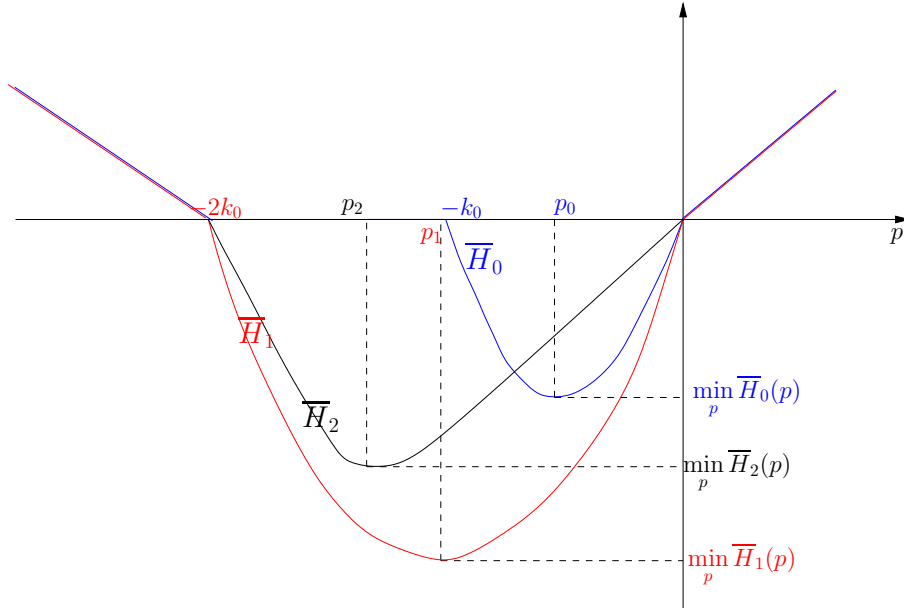


Figure 2.4: Schematic representation of the effective Hamiltonians.

The macroscopic model

The main objective of this article is to obtain an homogenisation result when the number of vehicles per unit length goes to infinity, that is to say what is the behavior of $(\rho_1^\varepsilon, \rho_2^\varepsilon)$ as ε goes to 0. First we can notice that the radius of the transition zone will go to 0, therefore at the macroscopic scale we will obtain a junction condition in 0 in the sense of Imbert and Monneau [IM14]. We consider one incoming road R_0 (isometric to $(-\infty, 0]$) and two outgoing roads R_1 and R_2 (isometric to $[0, +\infty)$) and all the branches are glued at the origin. Moreover, we define

$$\bar{R} = R_0 \cup R_1 \cup R_2 \quad \text{and} \quad R_i \cap R_j = \{0\} \text{ for } i \neq j.$$

In order to give a more physical interpretation, we consider that the branches R_0, R_1 and R_2 are generated respectively by the vectors \vec{e}_0, \vec{e}_1 and \vec{e}_2 , such that

$$R_0 = (-\infty, 0] \cdot \vec{e}_0 \quad \text{and} \quad R_{1,2} = [0, +\infty) \cdot \vec{e}_{1,2}.$$

For a smooth function $u : [0, T] \times \bar{R} \rightarrow \mathbb{R}$, we denote by $\partial_i u(t, x)$ the spatial derivative of u at $x \in R_i$, for $i = 0, 1, 2$ and we define

$$u_x(t, x) := \begin{cases} \partial_i u(t, x) & \text{if } x \in R_i^* := R_i \setminus \{0\}, \\ (\partial_0 u(t, 0), \partial_1 u(t, 0), \partial_2 u(t, 0)) & \text{if } x = 0. \end{cases} \quad (2.14)$$

Finally, we introduce for all $x, y \in \bar{R}$ the distance $d(x, y)$ on \bar{R} ,

$$d(x, y) = \begin{cases} |x - y| & \text{if } x, y \text{ belong to the same branch,} \\ |x| + |y| & \text{if } x, y \text{ belong to different branches.} \end{cases} \quad (2.15)$$

The main result of this article is to prove that the function \tilde{u}^ε defined by

$$\tilde{u}^\varepsilon(t, x) = \begin{cases} u^\varepsilon(t, -d(0, x)) & \text{for } (t, x) \in (0, +\infty) \times R_0, \\ u^\varepsilon(t, d(0, x)) & \text{for } (t, x) \in (0, +\infty) \times R_1^*, \\ \xi^\varepsilon(t, d(0, x)) & \text{for } (t, x) \in (0, +\infty) \times R_2^*, \end{cases} \quad (2.16)$$

converges locally uniformly on $(0, +\infty) \times \bar{R}$ as ε goes to 0 to the unique viscosity solution of the following problem

$$\begin{cases} u_t^0 + \bar{H}_0(u_x^0) = 0 & \text{for } (t, x) \in [0, +\infty) \times R_0^*, \\ u_t^0 + \bar{H}_1(u_x^0) = 0 & \text{for } (t, x) \in [0, +\infty) \times R_1^*, \\ u_t^0 + \bar{H}_2(u_x^0) = 0 & \text{for } (t, x) \in [0, +\infty) \times R_2^*, \\ u_t^0 + F_{\bar{A}}(\partial_0 u^0(t, 0), \partial_1 u^0(t, 0), \partial_2 u^0(t, 0)) = 0 & \text{for } (t, x) \in [0, +\infty) \times \{0\}, \\ u^0(0, x) = \bar{u}_0(x) = \begin{cases} u_0(d(0, x)) & \text{for } x \in R_0 \cup R_1, \\ \xi_0(d(0, x)) & \text{for } x \in R_2^*, \end{cases} \end{cases} \quad (2.17)$$

where \bar{A} is a constant to be determined and $F_{\bar{A}}$ is defined by

$$F_{\bar{A}}(p_0, p_1, p_2) = \max(\bar{A}, \bar{H}_0^+(p_0), \bar{H}_1^-(p_1), \bar{H}_2^-(p_2)), \quad (2.18)$$

and for $i = 0, 1, 2$, we define

$$\bar{H}_i^-(p) = \begin{cases} \bar{H}_i(p) & \text{if } p \leq p_i, \\ \bar{H}_i(p_i) & \text{if } p \geq p_i, \end{cases} \quad \text{and} \quad \bar{H}_i^+(p) = \begin{cases} \bar{H}_i(p_i) & \text{if } p \leq p_i, \\ \bar{H}_i(p) & \text{if } p \geq p_i. \end{cases} \quad (2.19)$$

Remark 2.3. In (2.16), we can replace u^ε by ξ^ε for $x \in R_0$ and we obtain the same homogenisation result. The coefficient 2 that appears in \bar{H}_i for $i = 1, 2$ and not in \bar{H}_0 comes from the fact that we look one vehicle over two on R_i and therefore the density of the vehicles is divided by 2.

The homogenization result

Theorem 2.4 (Junction condition by homogenization). *Assume (A0) and (A). For $\varepsilon > 0$, let $(u^\varepsilon, \xi^\varepsilon)$ be the solution of (2.4). Then there exists a unique constant $\bar{A} \in [H_0, 0]$ such that the function \tilde{u}^ε defined by (2.16) converges locally uniformly to the unique viscosity solution u^0 of (2.17).*

The previous result will allow us to get the following homogenization result for the vehicles.

Theorem 2.5. *Assume (A) and that at initial time we have, for all $i \in \mathbb{Z}$, if the vehicles i and $i + 1$ are both in R_0 ,*

$$U_i(0) \leq U_{i+1}(0) - h_0, \quad (2.20)$$

and if not

$$U_i(0) \leq U_{i+2}(0) - h_0. \quad (2.21)$$

We also assume that there exists a constant $R > 0$ such that for all $i \in \mathbb{Z}$, if $U_i(0) \geq R$

$$U_{i+1}(0) - U_i(0) = \begin{cases} h_1 & \text{if } i[2] = 1 \\ h_2 & \text{if } i[2] = 0 \end{cases} \quad (2.22)$$

and if $U_i(0) \leq -R$

$$U_{i+1}(0) - U_i(0) = h, \quad (2.23)$$

with $h, h_1, h_2 \geq h_0$. We define two functions u_0 and ξ_0 (satisfying (A0)) by

$$u_0(x) = \begin{cases} -x/h & \text{if } x \leq 0 \\ -2x/h_1 & \text{if } x > 0 \end{cases} \quad \text{and} \quad \xi_0(x) = \begin{cases} -x/h & \text{if } x \leq 0 \\ -2x/h_2 & \text{if } x > 0 \end{cases} \quad \text{for all } x \in \mathbb{R}.$$

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Then there exists a unique constant $\bar{A} \in [H_0, 0]$ such that the function

$$\tilde{\rho}^\varepsilon(t, y) = \begin{cases} \rho_1^\varepsilon(t, -d(0, y)) & \text{for } (t, y) \in (0, +\infty) \times R_0, \\ \rho_1^\varepsilon(t, d(0, y)) & \text{for } (t, y) \in (0, +\infty) \times R_1^*, \\ \rho_2^\varepsilon(t, d(0, y)) & \text{for } (t, y) \in (0, +\infty) \times R_2^*, \end{cases} \quad (2.24)$$

converges locally uniformly to the unique solution u^0 of (2.17).

Remark 2.6. Conditions (2.22) and (2.23) mean that the initial condition is well-prepared.

The goal of the following theorem is to show that the effective Hamiltonians in (2.17) are only evaluated for values in $[-2k_0, 0]$. However, it is convenient to work with the extended Hamiltonians presented in (2.10)-(2.11) because it is necessary to have coercive Hamiltonians, in order to apply the results developed by Imbert and Monneau in [IM14].

Theorem 2.7. Assume (A0)-(A). Let u^0 be the unique solution of (2.17), then we have for all $(t, x) \in [0, +\infty) \times \bar{R}$,

$$-2k_0 \leq u_x^0 \leq 0,$$

with k_0 defined in (A0).

To construct the effective flux limiter \bar{A} , we consider the following cell problem: find $\lambda \in \mathbb{R}$ such that there exists a solution (v, ζ) of the following Hamilton-Jacobi equation

$$\begin{cases} \phi(x, N_0(v, [\zeta])(x), M_1[v](x)) \cdot |v_x| = \lambda, & \text{for } x \in \mathbb{R}, \\ \phi(x, N_0(\zeta, [v])(x), M_2[\zeta](x)) \cdot |\zeta_x| = \lambda, & \text{for } x \in \mathbb{R}. \end{cases} \quad (2.25)$$

Theorem 2.8 (Effective flux limiter). Assume (A). We define the following set of functions,

$$\mathcal{S} = \{(v, \zeta) \text{ s.t. } \exists \text{ two Lipschitz continuous functions } m_1 \text{ and } m_2 \text{ s.t.} \\ m_1(0) = m_2(0) = 0 \text{ and a constant } C > 0 \text{ s.t. } \|v - m_1\|_\infty, \|\zeta - m_2\|_\infty \leq C\}.$$

Then \bar{A} is given by

$$\bar{A} = \inf\{\lambda \in [H_0, 0] : \exists (v, \zeta) \in \mathcal{S} \text{ solution of (2.25)}\}. \quad (2.26)$$

Link with macroscopic models

Notice that the homogenization of (2.4), does not directly give the dynamics of the density of vehicles. In fact, in R_0 , u^0 is the primitive of the density of vehicles, but in R_1 and R_2 it is twice the primitive of the density of vehicles. Therefore, the integral of the density of vehicles in \bar{R} is given by

$$\tilde{\rho}^0(t, x) = \begin{cases} u^0(t, x) & \text{for } x \in R_0, \\ u^0(t, x)/2 & \text{for } x \in R_1^* \cup R_2^*. \end{cases} \quad (2.27)$$

However, we cannot explicit the dynamics of $\tilde{\rho}^0$ because of its definition at the origin.

2.3 Notations and organization of the paper

We recall the definition of the non-local operators that we use in this paper,

$$N_0(U, [\Xi])(x) = \int_{-\infty}^{+\infty} J_0(z)F(\Xi(x+z) - U(x))dz - \frac{3}{2}V_{max}^0, \quad (2.28)$$

and for $i = 1, 2$,

$$M_i[U](x) = \int_{-\infty}^{+\infty} J_i(z)E(U(x+z) - U(x))dz - \frac{3}{2}V_{max}^i. \quad (2.29)$$

To each operator N and M , we associate the operators \tilde{N} and \tilde{M} , which are defined in the same way except that the functions F and E are replaced by the functions \tilde{F} and \tilde{E} defined by

$$\tilde{F}(z) = \begin{cases} 0 & \text{if } z > 1, \\ 1/2 & \text{if } -1 < z \leq 1, \\ 3/2 & \text{if } z \leq -1, \end{cases} \quad \text{and} \quad \tilde{E}(z) = \begin{cases} 0 & \text{if } z > 0, \\ 1/2 & \text{if } -2 < z \leq 0, \\ 3/2 & \text{if } z \leq -2. \end{cases} \quad (2.30)$$

Remark 2.9. *Using the fact that E , F and V are bounded, there exists a constant M_0 such that for every U , Ξ and every $x \in \mathbb{R}$, we have for $i = 1, 2$,*

$$\begin{cases} -M_0 \leq -\frac{3}{2}V_{max}^0 \leq N_0(U, [\Xi])(x) \leq 0, \\ -M_0 \leq -\frac{3}{2}V_{max}^i \leq M_i[U](x) \leq 0, \end{cases}$$

with

$$M_0 = \max_{i \in \{0,1,2\}} \frac{3}{2}V_{max}^i.$$

Remark 2.10 (Choice of the transition function). *In some computation it is useful to specify the explicit dependence of the function $\phi(x, a, b)$ on $\max(a, b)$. That is why we use the following notation,*

$$\tilde{\phi}(x, a, \max(a, b), b) = \phi(x, a, b) \quad \text{for all } (x, a, b) \in \mathbb{R}^3. \quad (2.31)$$

Finally, for a given point $(t_0, x_0) \in (0, +\infty) \times \mathbb{R}$, we define for $\bar{r}, \hat{r} > 0$, the set

$$\mathcal{P}_{\bar{r}, \hat{r}}(t_0, x_0) = (t_0 - \bar{r}, t_0 + \bar{r}) \times (x_0 - \hat{r}, x_0 + \hat{r}) \quad (2.32)$$

and

$$B(x_0, \bar{r}) = \{x \in \mathbb{R}, \text{ s.t. } |x - x_0| \leq \bar{r}\}.$$

Also for a given point $(t_0, y_0) \in (0, +\infty) \times \overline{R}$, we define for $\bar{r}, \hat{r} > 0$, the set

$$\mathcal{Q}_{\bar{r}, \hat{r}}(t_0, y_0) = (t_0 - \bar{r}, t_0 + \bar{r}) \times \mathcal{B}_{\hat{r}}(y_0),$$

with

$$\mathcal{B}_{\hat{r}}(y_0) = \{y \in \overline{R} \text{ s.t. } d(y, y_0) < \hat{r}\}.$$

We denote by $C > 0$ a generic constant that may vary from one line to the next.

Organization of the paper. In Section 3, we introduce the definition of viscosity solutions for the considered problems and give stability, existence, and uniqueness results. Section 4 contains the results concerning the effective Hamiltonians and the correctors for the junction (Theorem 4.3) that are necessary for the proof of convergence (Theorem 2.4) which proof is located in Section 5. In Section 6, we present the proof of Theorem 4.3. In Section 7, we show the link between the ODEs and the system of PDEs. Finally, in Section 8, from the case of a simple bifurcation, we extend to more general scenarios.

3 Viscosity solutions

In this section we give the definition of viscosity solution of the equations we treat in this paper. We refer the reader to the user's guide of Crandall, Ishii, Lions [CIL92] and the book of Barles [Bar94] for an introduction to viscosity solutions. We also refer the reader to [Ish92, IK91, Len88] for results concerning viscosity solutions for weakly coupled systems and to [IM14] for viscosity solutions on networks.

3.1 Definitions

In order to give a more general definition for all the non-local equations we will consider in this paper, we give the definition of a viscosity solution of the following equation, with $p \in \mathbb{R}$,

$$\begin{cases} u_t + G_p^1(x, u(t, x), [\xi(t, \cdot)], [u(t, \cdot)], u_x) = 0 \\ \xi_t + G_p^2(x, \xi(t, x), [u(t, \cdot)], [\xi(t, \cdot)], \xi_x) = 0 \end{cases} \quad \text{for all } (t, x) \in (0, +\infty) \times \mathbb{R}, \quad (3.1)$$

with, for $i = 1, 2$,

$$\begin{aligned} G_p^i(x, U, [\Xi], [U], q) = & (1 - \psi^+(x))\overline{H}_i(q) + (1 - \psi^-(x))\overline{H}_0(q) \\ & + \psi^+(x)\psi^-(x)\phi\left(x, N_p^0(U, [\Xi])(x), M_p^i[U](x)\right) |q + p|, \end{aligned}$$

and $\psi^\pm : \mathbb{R} \rightarrow [0, 1]$ two Lipschitz continuous functions such that $\text{supp}(1 - \psi^-) \cap \text{supp}(1 - \psi^+) = \emptyset$. To G_p^i we associate \tilde{G}_p^i defined in the same way but with the non-local operators \tilde{N}_p^0 and \tilde{M}_p^i . For $p \in [-2k_0, 0]$, the non-local operators N_p^0 and M_p^i are defined by

$$N_p^0(U, [\Xi])(x) = \int_{-\infty}^{+\infty} J_0(z) F(\Xi(x+z) - U(x) + p \cdot z) dz - \frac{3}{2} V_{max}^0, \quad (3.2)$$

and for $i = 1, 2$

$$M_p^i[U](x) = \int_{-\infty}^{+\infty} J_i(z) E(U(x+z) - U(x) + p \cdot z) dz - \frac{3}{2} V_{max}^i. \quad (3.3)$$

We also consider the following initial condition

$$\begin{cases} u(0, x) = u_0(x) \\ \xi(0, x) = \xi_0(x), \end{cases} \quad (3.4)$$

that satisfies (A0).

We recall the definition of the upper and lower semi-continuous envelopes, u^* and u_* , of a locally bounded function u , for all $(t, x) \in [0, +\infty) \times \mathbb{R}$,

$$u^*(t, x) = \limsup_{(s,y) \rightarrow (t,x)} u(s, y) \quad \text{and} \quad u_*(t, x) = \liminf_{(s,y) \rightarrow (t,x)} u(s, y). \quad (3.5)$$

We can now give the definition of viscosity solutions for (3.1).

Definition 3.1 (Definition of viscosity solutions for (3.1)). *Let $T > 0$, $u_0 : \mathbb{R} \rightarrow \mathbb{R}$, $\xi_0 : \mathbb{R} \rightarrow \mathbb{R}$ satisfying (A0). Let $u : [0, +\infty) \times \mathbb{R} \rightarrow \mathbb{R}$, $\xi : [0, +\infty) \times \mathbb{R} \rightarrow \mathbb{R}$ upper-semi continuous (resp. lower-semi continuous) locally bounded functions. We set $\Omega = (0, T) \times \mathbb{R}$. A function (u, ξ) is a sub-solution (resp. a super-solution) of (3.1) if for all $(t, x) \in \Omega$ and for any test function $\varphi \in C^1(\Omega)$ such that $u - \varphi$ attains a local maximum (resp. local minimum) at the point $(t, x) \in \Omega$, we have*

$$\varphi_t(t, x) + G_p^1(x, u(t, x), [\xi(t, \cdot)], [u(t, \cdot)], \varphi_x) \leq 0$$

$$\left(\text{resp. } \varphi_t(t, x) + \tilde{G}_p^1(x, u(t, x), [\xi(t, \cdot)], [u(t, \cdot)], \varphi_x) \geq 0 \right),$$

and if for all $(t, x) \in \Omega$ and for any test function φ such that $\xi - \varphi$ attains a local maximum (resp. local minimum) at (t, x) , we have

$$\varphi_t(t, x) + G_p^2(x, \xi(t, x), [u(t, \cdot)], [\xi(t, \cdot)], \varphi_x) \leq 0$$

$$\left(\text{resp. } \varphi_t(t, x) + \tilde{G}_p^2(x, \xi(t, x), [u(t, \cdot)], [\xi(t, \cdot)], \varphi_x) \geq 0 \right).$$

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-A function (u, ξ) is a sub-solution (resp. super-solution) of (3.1)-(3.4) if (u, ξ) is a sub-solution (resp. super-solution) of (3.1) on Ω and satisfies

$$u(0, x) \leq u_0(x) \text{ (resp. } \geq) \quad \text{and} \quad \xi(0, x) \leq \xi_0(x) \text{ (resp. } \geq). \quad (3.6)$$

-A function (u, ξ) is a viscosity solution of (3.1) (resp. of (3.1)-(3.4)) if (u^*, ξ^*) is a sub-solution and (u_*, ξ_*) is a super-solution of (3.1) (resp. of (3.1)-(3.4)).

Remark 3.2. We use this definition in order to have a stability result for the non-local operators (see [DLKS04, Sle03] for similar definitions). We refer to [FIM09b, Proposition 4.2] for the corresponding stability result.

Definition 3.3 (Class of test functions for (2.17)). We denote $R^\infty := (0, +\infty) \times \bar{R}$ and $R_i^\infty := (0, +\infty) \times R_i^*$ for $i = 0, 1, 2$. We define a class of test functions on R^∞ by

$$C^2(R^\infty) = \{\varphi \in C(R^\infty), \text{ the restriction of } \varphi \text{ to } R_i^\infty, i = 0, 1, 2 \text{ is } C^2\}.$$

Definition 3.4 (Definition of viscosity solutions for (2.17)). Let \bar{H}_0 and \bar{H}_i , for $i = 1, 2$ be given respectively by (2.10) and (2.11). Let $\bar{A} \in \mathbb{R}$. An upper semi-continuous (resp. lower semi-continuous) function $u : [0, +\infty) \times \bar{R} \rightarrow \mathbb{R}$ is a viscosity sub-solution (resp. super-solution) of (2.17) if $u(0, x) \leq \bar{u}_0(x)$ (resp. $u(0, x) \geq \bar{u}_0(x)$) and for all $(t, x) \in R^\infty$ and for all $\varphi \in C^2(R^\infty)$ such that

$$u \leq \varphi \text{ (resp. } u \geq \varphi) \text{ in a neighbourhood of } (t, x) \in R^\infty \text{ and } u(t, x) = \varphi(t, x),$$

we have

$$\begin{cases} \varphi_t(t, x) + \bar{H}_0(\varphi_x(t, x)) \leq 0 \text{ (resp. } \geq 0) & \text{if } x \in R_0^*, \\ \varphi_t(t, x) + \bar{H}_1(\varphi_x(t, x)) \leq 0 \text{ (resp. } \geq 0) & \text{if } x \in R_1^*, \\ \varphi_t(t, x) + \bar{H}_2(\varphi_x(t, x)) \leq 0 \text{ (resp. } \geq 0) & \text{if } x \in R_2^*, \\ \varphi_t(t, x) + F_{\bar{A}}(\varphi_x(t, x)) \leq 0 \text{ (resp. } \geq 0) & \text{if } x = 0. \end{cases}$$

We say that u is a viscosity solution of (2.17) if u^* and u_* are respectively a sub-solution and a super-solution of (2.17). We refer to this solution as \bar{A} -flux limited solution.

Thanks to the work of Imbert and Monneau [IM14], we have the following result which gives an equivalent definition of viscosity solutions for (2.17). We use this equivalent definition in the proof of Theorem 2.4 in Section 5.

Theorem 3.5 (Equivalent definition for sub/super-solutions). Let \bar{H}_0 and \bar{H}_i for $i = 1, 2$ be given by (2.10)-(2.11) and consider $A \in [H_0, +\infty)$ with $H_0 = \max_{i \in \{0, 1, 2\}} \min_{p \in \mathbb{R}} \bar{H}_i(p)$. Given arbitrary solutions $p_0 \in \mathbb{R}$ and $p_i^A \in \mathbb{R}$ for $i = 1, 2$ of

$$\overline{H}_0(p_0^A) = \overline{H}^-(p_0^A) = A \quad \text{and} \quad \overline{H}_i^+(p_i^A) = \overline{H}_i(p_i^A) = A, \quad (3.7)$$

let us fix any time independent test function $\phi^0(x)$ satisfying, for $i = 0, 1, 2$,

$$\partial_i \phi^0(0) = p_i^A.$$

Given a function $u : (0, T) \times \overline{R} \rightarrow \mathbb{R}$, the following properties hold true.

i) If u is an upper semi-continuous sub-solution of (2.17) with $A = H_0$, for $x \neq 0$, satisfying

$$u(t, 0) = \limsup_{(s, y) \rightarrow (t, 0), y \in R_i^*} u(s, y), \quad (3.8)$$

then u is a H_0 -flux limited sub-solution.

ii) Given $A > H_0$ and $t_0 \in (0, T)$, if u is an upper semi-continuous sub-solution of (2.17) for $x \neq 0$, satisfying (3.8), and if for any test function φ touching u from above at $(t_0, 0)$ with

$$\varphi(t, x) = \psi(t) + \phi^0(x), \quad (3.9)$$

for some $\psi \in C^2(0, +\infty)$, we have

$$\varphi_t + F_A(\varphi_x) \leq 0 \quad \text{at } (t_0, 0),$$

then u is a A -flux limited sub-solution at $(t_0, 0)$.

iii) Given $t_0 \in (0, T)$, if u is a lower semi-continuous super-solution of (2.17) for $x \neq 0$ and if for any test function φ satisfying (3.9) touching u from above at $(t_0, 0)$ we have

$$\varphi_t + F_A(\varphi_x) \geq 0 \quad \text{at } (t_0, 0),$$

then u is a A -flux limited super-solution at $(t_0, 0)$.

3.2 Results for viscosity solutions of (2.4)

Lemma 3.6 (Existence of barriers for (3.1)). *Assume (A0) and (A). There exists a constant $K_1 > 0$ such that the functions (u^+, ξ^+) and (u^-, ξ^-) defined by*

$$(u^+(t, x), \xi^+(t, x)) = (K_1 t + u_0(x), K_1 t + \xi_0(x))$$

and

$$(u^-(t, x), \xi^-(t, x)) = (u_0(x), \xi_0(x))$$

are respectively a super and sub-solution of (3.1).

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Proof. We define $K_1 = M_0 \cdot (|p| + 2k_0) + 2 \max_{i=0,1,2} \left(\max_{q \in [-2k_0, 0]} |\overline{H}_i(q)| \right)$, where M_0 is defined in Remark 2.9. Let us prove that (u^+, ξ^+) is a super-solution of (3.1). Using assumption (A0) and the form of the non-local operators and of \overline{H}_i , $i = 0, 1, 2$ we have

$$\begin{aligned} \tilde{G}_p^1(x, u^+(t, x), [\xi^+(t, \cdot)], [u^+(t, \cdot)], (u_0)_x) &\geq -M_0 \cdot |p + (u_0)_x| \\ &\quad - 2 \max_{i=0,1,2} \left(\max_{q \in [-2k_0, 0]} |\overline{H}_i(q)| \right) \\ &\geq -M_0(|p| + 2k_0) \\ &\quad - 2 \max_{i=0,1,2} \left(\max_{q \in [-2k_0, 0]} |\overline{H}_i(q)| \right) \\ &= -K_1, \end{aligned}$$

where we have used Remark (2.9). Similarly, we have

$$\begin{aligned} \tilde{G}_p^2(x, \xi^+(t, x), [u^+(t, \cdot)], [\xi^+(t, \cdot)], (\xi_0)_x) &\geq -M_0(|p| + 2k_0) \\ &\quad - 2 \max_{i=0,1,2} \left(\max_{x \in [-2k_0, 0]} |\overline{H}_i(x)| \right) \\ &= -K_1. \end{aligned} \tag{3.10}$$

The proof for the sub-solution is similar using that the non-local operators and $\overline{H}_i((u_0)_x)$ for $i = 0, 1, 2$ are non-positive. This ends the proof of Lemma 3.6. \square

Proposition 3.7 (Existence and uniqueness for (3.1)). *Assume (A). Let (u, ξ) (resp. (v, ζ)) be a sub-solution (resp. a super-solution) of (3.1)-(3.4). We also assume that there exists a constant $K > 0$ such that for all $(t, x) \in [0, T] \times \mathbb{R}$, we have*

$$\begin{aligned} u(t, x) &\leq u_0(x) + K(1+t), & \xi(t, x) &\leq \xi_0(x) + K(1+t) \\ -v(t, x) &\leq -u_0(x) + K(1+t), & \zeta(t, x) &\leq -\xi_0(x) + K(1+t). \end{aligned} \tag{3.11}$$

If

$$u(0, x) \leq v(0, x) \quad \text{and} \quad \xi(0, x) \leq \zeta(0, x) \quad \text{for all } x \in \mathbb{R},$$

then

$$u(t, x) \leq v(t, x) \quad \text{and} \quad \xi(t, x) \leq \zeta(t, x) \quad \text{for all } (t, x) \in [0, T] \times \mathbb{R}.$$

In particular, the previous result combined with the Lemma 3.6 imply that there exists a unique solution (u, ξ) of (3.1)-(3.4). Moreover, the functions u and ξ are continuous and there exists a constant $K_1 > 0$ such that for all $(t, x) \in [0, +\infty) \times \mathbb{R}$

$$0 \leq u(t, x) - u_0(x) \leq K_1 t \quad \text{and} \quad 0 \leq \xi(t, x) - \xi_0(x) \leq K_1 t.$$

Proof. The first part of this result (comparison principle) is classical and uses the monotonicity properties of G (see for example [DLFM08, FIM09b] for a similar result) but for the readers convenience we give the proof in Appendix 9.

To prove the rest of Proposition 3.7, we apply Perron’s method (see [IMR08, Proof of Theorem 6], [AT96, Imb05] to see how to apply Perron’s method for problems with non-local terms), joint to the comparison principle. □

We now give a comparison principle on bounded sets, we use the notations from (2.32).

Proposition 3.8 (Comparison principle on bounded sets for (3.1)). *Assume (A). Let (u, ξ) be a sub-solution of (3.1) and let (v, ζ) be a super-solution of (3.1) on the open set $\mathcal{P}_{\bar{r}, \hat{r}} \subset (0, T) \times \mathbb{R}$. We assume that u and ξ (resp. v and ζ) are upper semi-continuous (resp. lower semi-continuous) on $\overline{\mathcal{P}}_{\bar{r}, \hat{r}}$. Also assume that*

$$u \leq v \quad \text{and} \quad \xi \leq \zeta \quad \text{outside } \mathcal{P}_{\bar{r}, \hat{r}},$$

then

$$u \leq v \quad \text{and} \quad \xi \leq \zeta \quad \text{on } \mathcal{P}_{\bar{r}, \hat{r}}.$$

We now give a result on the control of the oscillations for the solution of (2.4) (with $\varepsilon = 1$). This result will be used in particular to prove Theorem 2.7.

Theorem 3.9 (Control of the oscillations). *Let $T > 0$. Assume (A0)-(A) and let (u, ξ) be a solution of (2.4)-(2.9), with $\varepsilon = 1$. Then there exists a constant $C_1 > 0$ such that for all $x, y \in \mathbb{R}$, $x \geq y$ and for all $t, s \in [0, T]$, $t \geq s$, we have*

$$\begin{aligned} 0 \leq u(t, x) - u(s, x) &\leq C_1(t - s), & \text{and} & & -K_0(x - y) - 2 \leq u(t, x) - u(t, y) \leq 0, \\ 0 \leq \xi(t, x) - \xi(s, x) &\leq C_1(t - s) & & & -K_0(x - y) - 2 \leq \xi(t, x) - \xi(t, y) \leq 0, \end{aligned} \tag{3.12}$$

with $K_0 := 2k_0$.

Proof. Using the barriers constructed in Lemma 3.6 (with $p = 0$, $\psi^+ \equiv 1$ and $\psi^- \equiv 1$) we deduce that (u, ξ) satisfies for all $(t, x) \in [0, +\infty) \times \mathbb{R}$,

$$0 \leq u(t, x) - u_0(x) \leq M_0 K_0 t \quad \text{and} \quad 0 \leq \xi(t, x) - \xi_0(x) \leq M_0 K_0 t. \tag{3.13}$$

In the rest of the proof we use the following notation

$$\tilde{\Omega} = \left\{ (t, x, y) \in [0, T) \times \mathbb{R}^2 \text{ s.t. } x \geq y \right\}.$$

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Step 1: proof of the bound of the time derivative. For all $h \geq 0$, we have

$$u(0, x) \leq u(h, x) \leq M_0 K_0 h + u(0, x) \quad \text{and} \quad \xi(0, x) \leq \xi(h, x) \leq M_0 K_0 h + \xi(0, x).$$

Using the fact that (2.4) is invariant by addition of constant to the solution and by translation in time, we deduce by the comparison principle that, for all $(t, x) \in [0, +\infty) \times \mathbb{R}$, we have

$$u(t, x) \leq u(t + h, x) \leq M_0 K_0 h + u(t, x)$$

and

$$\xi(t, x) \leq \xi(t + h, x) \leq M_0 K_0 h + \xi(t, x).$$

We deduce the result by choosing $C_1 = M_0 K_0$.

Step 2: proof of the upper inequality for the control of the space oscillations.

We introduce

$$M = \sup_{(t,x,y) \in \tilde{\Omega}} \max(u(t, x) - u(t, y), \xi(t, x) - \xi(t, y)). \quad (3.14)$$

We want to prove that $M \leq 0$. We argue by contradiction and assume that $M > 0$.

Step 2.1: the test functions. For $\eta, \alpha > 0$, small parameters, we define

$$\begin{cases} \varphi_1(t, x, y) = u(t, x) - u(t, y) - \frac{\eta}{T-t} - \alpha x^2 - \alpha y^2, \\ \varphi_2(t, x, y) = \xi(t, x) - \xi(t, y) - \frac{\eta}{T-t} - \alpha x^2 - \alpha y^2. \end{cases}$$

Using (3.13), we have that

$$\lim_{|x|, |y| \rightarrow \infty} \varphi_i(t, x, y) = -\infty.$$

Since φ_1, φ_2 are upper semi-continuous, the function $\psi = \max(\varphi_1, \varphi_2)$ reaches a maximum at a finite point that we denote by $(\bar{t}, \bar{x}, \bar{y}) \in \tilde{\Omega}$. Classically, we have for η and α small enough,

$$\begin{cases} 0 < \frac{M}{2} \leq \psi(\bar{t}, \bar{x}, \bar{y}), \\ \alpha |\bar{x}|, \alpha |\bar{y}| \rightarrow 0 \text{ as } \alpha \rightarrow 0. \end{cases}$$

Step 2.2: $\bar{t} > 0$ and $\bar{x} > \bar{y}$. By contradiction, assume first that $\bar{t} = 0$. Then we have,

$$\frac{\eta}{T} \leq u_0(\bar{x}) - u_0(\bar{y}) \leq 0 \quad \text{or} \quad \frac{\eta}{T} \leq \xi_0(\bar{x}) - \xi_0(\bar{y}) \leq 0,$$

where we have used (A0), and we get a contradiction. The fact that $\bar{x} > \bar{y}$ comes directly from the fact that $\psi(\bar{t}, \bar{x}, \bar{y}) > 0$.

Step 2.3: viscosity inequalities. By doing a duplication of the time variable and passing to the limit we get, if $\psi(\bar{t}, \bar{x}, \bar{y}) = \varphi_1(\bar{t}, \bar{x}, \bar{y})$,

$$\frac{\eta}{(T - \bar{t})^2} \leq -\phi\left(\bar{x}, N_0(u(\bar{t}, \cdot), [\xi(\bar{t}, \cdot)])(\bar{x}), M_1[u(\bar{t}, \cdot)](x)\right) \cdot |2\alpha\bar{x}| \leq 2M_0|\alpha\bar{x}|$$

or, if $\psi(\bar{t}, \bar{x}, \bar{y}) = \varphi_2(\bar{t}, \bar{x}, \bar{y})$,

$$\frac{\eta}{(T - \bar{t})^2} \leq -\phi\left(\bar{x}, N_0(\xi(\bar{t}, \cdot), [u(\bar{t}, \cdot)])(\bar{x}), M_2[\xi(\bar{t}, \cdot)](x)\right) \cdot |2\alpha\bar{x}| \leq 2M_0|\alpha\bar{x}|,$$

where we have used the fact that the non-local operators are negative. This is a contradiction for α small enough.

Step 3: proof of the lower inequality for the control of the space oscillations.

In order to do this part of the proof, we use the following lemma, which proof is postponed.

Lemma 3.10. *Assume (A0)-(A). Let $T > 0$, then the solution (u, ξ) of (2.4)-(2.9) (with $\varepsilon = 1$) satisfies*

$$|\xi(t, x) - u(t, x)| \leq 1 \quad \text{for } (t, x) \in [0, T] \times (-\infty, -h_0]. \quad (3.15)$$

We now introduce,

$$M = \sup_{(t,x,y) \in \tilde{\Omega}} \max(u(t, y) - u(t, x) - K_0(x - y) - 2)$$

and we want to prove that $M \leq 0$. We argue by contradiction and assume that $M > 0$.

Step 3.1: the test functions. For η and α small parameters, we introduce

$$\varphi(t, x, y) = u(t, y) - u(t, x) - K_0(x - y) - 2 - \frac{\eta}{T - t} - \alpha x^2$$

Using (3.13), we deduce that the function φ reaches a maximum at a finite point that we denote by $(\bar{t}, \bar{x}, \bar{y}) \in \tilde{\Omega}$ and that for η and α small enough, we have

$$\begin{cases} 0 < \frac{M}{2} \leq \psi(\bar{t}, \bar{x}, \bar{y}), \\ \alpha|\bar{x}| \rightarrow 0 \text{ as } \alpha \rightarrow 0. \end{cases} \quad (3.16)$$

Moreover, as in Step 2.2 we get that $\bar{t} > 0$ and $\bar{x} > \bar{y}$.

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Step 3.2: getting a contradiction. By doubling the time variable and passing to the limit, we get

$$\begin{aligned} \frac{\eta}{(T-\bar{t})^2} &\leq \phi\left(\bar{x}, \tilde{N}_0(u(\bar{t}, \cdot), [\xi(\bar{t}, \cdot)])(\bar{x}), \tilde{M}_1[u(\bar{t}, \cdot)](\bar{x})\right) \cdot |-K_0 - 2\alpha\bar{x}| \\ &\quad - \phi\left(\bar{y}, N_0(u(\bar{t}, \cdot), [\xi(\bar{t}, \cdot)])(\bar{y}), M_1[u(\bar{t}, \cdot)](\bar{y})\right) \cdot |K_0| \\ &\leq -\phi\left(\bar{y}, N_0(u(\bar{t}, \cdot), [\xi(\bar{t}, \cdot)])(\bar{y}), M_1[u(\bar{t}, \cdot)](\bar{y})\right) \cdot K_0, \end{aligned} \quad (3.17)$$

where we have used the fact that the non-local operators are non-positive.

Let us now prove that

$$M_1[u(\bar{t}, \cdot)](\bar{y}) = \int_{h_0}^{h_{max}} J_1(z) E(u(\bar{t}, \bar{y} + z) - u(\bar{t}, \bar{y})) dz - \frac{3}{2} V_{max}^1 = 0. \quad (3.18)$$

It is sufficient to prove that for all $z \in (h_0, h_{max}]$, we have

$$u(\bar{t}, \bar{y} + z) - u(\bar{t}, \bar{y}) < -2.$$

Let $z \in (h_0, h_{max}]$, if $\bar{y} + z < \bar{x}$, using the fact that $\varphi(\bar{t}, \bar{x}, \bar{y} + z) \leq \varphi(\bar{t}, \bar{x}, \bar{y})$, we observe that

$$u(\bar{t}, \bar{y} + z) - u(\bar{t}, \bar{y}) \leq -K_0 z < -2.$$

If $\bar{y} + z \geq \bar{x}$, using the fact that u is non-increasing in space and that $\varphi(\bar{t}, \bar{x}, \bar{y}) > 0$, we obtain

$$u(\bar{t}, \bar{y} + z) - u(\bar{t}, \bar{y}) \leq u(\bar{t}, \bar{x}) - u(\bar{t}, \bar{y}) < -K_0 z - 2 < -2.$$

We recall that (Remark 2.10) $\phi(x, a, b) = \tilde{\phi}(x, a, \max(a, b), b)$. In particular, given that the non-local operators are non-positive (Remark 2.9), we have

$$\max\left(N_0(u(\bar{t}, \cdot), [\xi(\bar{t}, \cdot)])(\bar{y}), M_1[u(\bar{t}, \cdot)](\bar{y})\right) = 0. \quad (3.19)$$

Therefore, given the definition of ϕ in (2.8), the right-hand side term in (3.17) is equal to zero unless $\bar{y} \leq -h_0 - h_{max}$. Let us now prove that if $\bar{y} \leq -h_0 - h_{max}$, we have $N_0(u(\bar{t}, \cdot), [\xi(\bar{t}, \cdot)])(\bar{y}) = 0$, which will directly give us a contradiction from (3.17).

We claim that

$$N_0(u(\bar{t}, \cdot), [\xi(\bar{t}, \cdot)])(\bar{y}) = \int_{h_0}^{h_{max}} J_0(z) F(\xi(\bar{t}, \bar{y} + z) - u(\bar{t}, \bar{y})) dz - \frac{3}{2} V_{max}^0 = 0. \quad (3.20)$$

In fact it is sufficient to prove that for all $z \in (h_0, h_{max}]$, we have

$$\xi(\bar{t}, \bar{y} + z) - u(\bar{t}, \bar{y}) < -1.$$

Let $z \in (h_0, h_{max}]$, if $\bar{y} + z < \bar{x}$, we obtain

$$\begin{aligned} \xi(\bar{t}, \bar{y} + z) - u(\bar{t}, \bar{y}) &\leq u(\bar{t}, \bar{y} + z) - u(\bar{t}, \bar{x}) + 1 \\ &\leq 1 - K_0 z \\ &< -1, \end{aligned}$$

where we have used Lemma 3.10 for the first line (since $\bar{y} + z \leq \bar{y} + h_{max} \leq -h_0$), the fact that $\varphi(\bar{t}, \bar{x}, \bar{y} + z) \leq \varphi(\bar{t}, \bar{x}, \bar{y})$ for the second line and the fact that $K_0 z > 2$ for the third line.

If $\bar{y} + z \geq \bar{x}$, we then have

$$\begin{aligned} \xi(\bar{t}, \bar{y} + z) - u(\bar{t}, \bar{y}) &\leq u(\bar{t}, \bar{y} + z) - u(\bar{t}, \bar{y}) + 1 \\ &\leq u(\bar{t}, \bar{x}) - u(\bar{t}, \bar{y}) + 1 \\ &\leq 1 - K_0(\bar{x} - \bar{y}) - 2 < -1, \end{aligned}$$

where we have used Lemma 3.10 for the first line, the fact that u is non-increasing in space for the second line, and the fact that $\varphi(\bar{t}, \bar{x}, \bar{y}) > 0$ for the third line. Injecting (3.18), (3.19), and (3.20) in (3.17), we obtain $\eta/T^2 \leq 0$, which is a contradiction.

The proof of the lower bound on the control of the space oscillations of ξ is done similarly and we skip it. This ends the proof of Theorem 3.9. □

Proof of Lemma 3.10. We only prove that $\xi(t, x) - u(t, x) \leq 1$ for $x \leq -h_0$ and $t \in [0, T)$, the proof for the second part of the result is similar so we skip it. We will prove this result under a different form using the technique introduced by Ishii and Lions in [IL90] for the proof of local gradient estimates. Let $z \in (-\infty, -h_0)$ and $\delta > 0$ a small parameter be such that $z \leq -h_0 - \delta$. We introduce

$$\Delta = [0, T) \times B(z, \delta).$$

We will prove that for a constant $L := (4K_1 T + 8k_0 \delta)/\delta^2$, we have

$$\xi(t, x) - u(t, x) \leq 1 + L(x - z)^2 \quad \text{for all } (t, x) \in \Delta. \tag{3.21}$$

In fact, if we take $x = z$ in (3.21), we get

$$\xi(t, z) - u(t, z) \leq 1 \quad \forall (t, z) \in [0, T) \times (-\infty, -h_0).$$

Using the continuity of the solution (u, ξ) , we can pass to the limit in the previous inequality as z goes to $-h_0$ and we obtain the complete result. In order to prove (3.21), we introduce

$$M = \sup_{(t,x) \in \Delta} \left(\xi(t, x) - u(t, x) - 1 - L(x - z)^2 \right) \tag{3.22}$$

and we want to prove that $M \leq 0$. We argue by contradiction and assume that $M > 0$.

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Step 1: the test function. For η and ε small parameters, we define

$$\varphi(t, x, y) = \xi(t, x) - u(t, y) - 1 - \frac{L}{2}(x - z)^2 - \frac{L}{2}(y - z)^2 - \frac{(x - y)^2}{2\varepsilon} - \frac{\eta}{T - t}.$$

As in the previous proof, we have that φ reaches a maximum at a finite point that we denote by

$$(\bar{t}, \bar{x}, \bar{y}) \in [0, T) \times B(z, \delta) \times B(z, \delta).$$

By classical arguments, we have that

$$\begin{cases} 0 < \frac{M}{2} \leq \varphi(\bar{t}, \bar{x}, \bar{y}), \\ |\bar{x} - \bar{y}| \rightarrow 0 \text{ as } \varepsilon \rightarrow 0. \end{cases}$$

Step 2: $\bar{t} > 0$ for ε small enough. By contradiction assume that $\bar{t} = 0$, then we have

$$\frac{\eta}{T} < \xi_0(x) - u_0(x) - 1 + 2k_0|x - y| \leq o_\varepsilon(1),$$

where we have used assumption (A0) and we get a contradiction for ε small enough.

Step 3: $|\bar{x} - z| < \delta$ and $|\bar{y} - z| < \delta$. By contradiction assume that either $|\bar{x} - z| = \delta$ or $|\bar{y} - z| = \delta$, in which case, we have

$$\begin{aligned} 0 &< \varphi(\bar{t}, \bar{x}, \bar{y}) = \xi(\bar{t}, \bar{x}) - u(\bar{t}, \bar{y}) - 1 - \frac{L}{2}(\bar{x} - z)^2 - \frac{L}{2}(\bar{y} - z)^2 - \frac{(\bar{x} - \bar{y})^2}{2\varepsilon} \\ &\leq 2K_1T + \xi_0(\bar{x}) - \xi_0(\bar{y}) + \xi_0(\bar{y}) - u_0(\bar{y}) - 1 - \frac{L}{2}(\bar{x} - z)^2 - \frac{L}{2}(\bar{y} - z)^2 - \frac{(\bar{x} - \bar{y})^2}{2\varepsilon} \quad (3.23) \\ &\leq 2K_1T + 2k_0|\bar{x} - \bar{y}| - \frac{(\bar{x} - \bar{y})^2}{2\varepsilon} - \frac{L}{2}\delta^2 \\ &\leq 2K_1T + 4k_0\delta - \frac{L}{2}\delta^2 \leq 0, \end{aligned}$$

where we have used the barriers from Lemma 3.6 for the second line, assumption (A0) for the third line, and for the fourth line the fact that $L = (4K_1T + 8k_0\delta)/\delta^2$. This gives us a contradiction.

Step 3: viscosity inequalities. By doubling the time variable and passing to the limit, we obtain that there exists two real numbers $a, b \in \mathbb{R}$ such that

$$a - b = \frac{\eta}{(T - \bar{t})^2},$$

and

$$\begin{aligned} a + \phi \left(\bar{x}, N_0(\xi(\bar{t}, \cdot), [u(\bar{t}, \cdot)])(\bar{x}), M_2[\xi(\bar{t}, \cdot)](\bar{x}) \right) \cdot |p_\varepsilon + L(\bar{x} - z)| &\leq 0, \\ b + \phi \left(\bar{y}, \tilde{N}_0(u(\bar{t}, \cdot), [\xi(\bar{t}, \cdot)])(\bar{y}), \tilde{M}_1(\bar{t}, \cdot)](\bar{y}) \right) \cdot |p_\varepsilon - L(\bar{y} - z)| &\geq 0, \end{aligned}$$

with $p_\varepsilon = (\bar{x} - \bar{y})/\varepsilon$. Combining these inequalities we obtain

$$\frac{\eta}{T^2} \leq -\phi \left(\bar{x}, N_0(\xi(\bar{t}, \cdot), [u(\bar{t}, \cdot)])(\bar{x}), M_2[\xi(\bar{t}, \cdot)](\bar{x}) \right) \cdot |p_\varepsilon + L(\bar{x} - z)|. \quad (3.24)$$

Notice that because $\bar{x} < z + \delta \leq -h_0$, and because of the form of ϕ (Remark 2.10), the right-hand side term in (3.24) is only a combination of the non-local term $N_0(\xi(\bar{t}, \cdot), [u(\bar{t}, \cdot)])(\bar{x})$ and of $\max(N_0(\xi(\bar{t}, \cdot), [u(\bar{t}, \cdot)])(\bar{x}), M_2[\xi(\bar{t}, \cdot)](\bar{x}))$.

Let us prove that $N_0(\xi(\bar{t}, \cdot), [u(\bar{t}, \cdot)])(\bar{x}) = 0$, which, given that the non-local operators are non-positive, will imply that the entire right-hand side term in (3.24) is equal to zero.

As a matter of fact, using that $|\bar{x} - \bar{y}| \rightarrow 0$ as ε goes to 0, for all $h \in [h_0, h_{max}]$, we can assume that $\bar{x} + h > \bar{y}$. Using the fact that u is decreasing in space we deduce that

$$u(\bar{t}, \bar{x} + h) - \xi(\bar{t}, \bar{x}) \leq u(\bar{t}, \bar{y}) - \xi(\bar{t}, \bar{x}) < -1, \quad (3.25)$$

where we have used the fact that $\varphi(\bar{t}, \bar{x}, \bar{y}) > 0$. This implies that $N_0(\xi(\bar{t}, \cdot), [u(\bar{t}, \cdot)])(\bar{x}) = 0$. This gives us a contradiction and ends the proof of Lemma 3.10. □

4 Effective Hamiltonians and correctors for the junction

This section is devoted to the construction of correctors at the junction and far from the junction.

4.1 Correctors far from the junction

The following propositions explicit the construction of the effective Hamiltonians and of the correctors far from the junction.

Proposition 4.1 (Homogenization on R_0). *Assume (A). Then for $p \in [-2k_0, 0]$, there exists a unique $\lambda \in \mathbb{R}$, such that there exists a bounded solution (v, ζ) of*

$$\begin{cases} N_p^0(v, [\zeta])(x) \cdot |v_x + p| = \lambda, \\ N_p^0(\zeta, [v])(x) \cdot |\zeta_x + p| = \lambda, \\ v \text{ and } \zeta \text{ are } \mathbb{Z}\text{-periodic.} \end{cases} \quad (4.1)$$

Moreover, for $p \in [-2k_0, 0]$, we have $\lambda = \bar{H}_0(p) = -V_0 \left(\frac{-1}{p} \right) |p|$.

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Proposition 4.2 (Homogenization on R_1 and R_2). *Assume (A). Then for $i = 1, 2$ and for $p \in [-2k_0, 0]$, there exists a unique $\lambda \in \mathbb{R}$, such that there exists a bounded solution v of*

$$\begin{cases} M_p^i[v](x) \cdot |v_x + p| = \lambda, \\ v \text{ is } \mathbb{Z} - \text{periodic}, \end{cases} \quad (4.2)$$

with M_p^i defined in (3.3). Moreover, for $p \in [-2k_0, 0]$ we have $\lambda = \overline{H}_i(p) = -V_i \left(\frac{-2}{p} \right) |p|$.

To prove the previous propositions, it is only necessary to notice that $v = \zeta \equiv 0$ are obvious solutions to each problem.

4.2 Correctors at the junction

Like in [2, GIM15] in order to prove the convergence result we need to construct correctors for the junction. We now present the existence result of the appropriate correctors. The proof of this result is postponed to Section 6. Given $\overline{A} \in \mathbb{R}$, $\overline{A} \geq H_0$, we introduce the real numbers $\overline{p}_+^1, \overline{p}_+^2 \in \mathbb{R}$ such that

$$\overline{H}_1(\overline{p}_+^1) = \overline{H}_1^+(\overline{p}_+^1) = \overline{A} \quad \text{and} \quad \overline{H}_2(\overline{p}_+^2) = \overline{H}_2^+(\overline{p}_+^2) = \overline{A} \quad (4.3)$$

thanks to the form of \overline{H}_1 and of \overline{H}_2 , there is only one couple of real numbers satisfying (4.3). We also introduce two real number \hat{p}_-^0 and \overline{p}_-^0 defined by

$$\begin{cases} \hat{p}_-^0 = \min\{p, p \in E_0\} \\ \overline{p}_-^0 = \max\{p, p \in E_0\} \end{cases} \quad \text{with} \quad E_0 = \{p \in [-2k_0, 0], \overline{H}_0^-(p) = \overline{H}_0(p) = \overline{A}\}. \quad (4.4)$$

Note that if $\overline{A} \neq 0$, then $\hat{p}_-^0 = \overline{p}_-^0$.

Theorem 4.3 (Existence of global corrector for the junction). *Assume (A).*

i) (General properties) There exists a constant $\overline{A} \in [H_0, 0]$ such that there exists a solution (w, χ) of (2.25) with $\lambda = \overline{A}$ and such that there exists a constant $C > 0$ and two globally Lipschitz continuous functions m_1 and m_2 with $m_1(0) = m_2(0) = 0$ such that, for all $x \in \mathbb{R}$,

$$|w(x) - m_1(x)| \leq C \quad \text{and} \quad |\chi(x) - m_2(x)| \leq C, \quad (4.5)$$

and for all $x \leq h_0$,

$$|w(x) - \chi(x)| \leq C. \quad (4.6)$$

4. Effective Hamiltonians and correctors for the junction

ii) (Bound from below at infinity) There exist two constants $\gamma_0 > 0$ and $C > 0$ such that for every $\gamma \in (0, \gamma_0)$, we have for $x > 0$ and $h \geq 0$,

$$w(x+h) - w(x) \geq (\bar{p}_+^1 - \gamma)h - C \quad \chi(x+h) - \chi(x) \geq (\bar{p}_+^2 - \gamma)h - C \quad (4.7)$$

and for $x \leq -h_0 - h_{max} - 1$ and $h \geq 0$

$$w(x-h) - w(x) \geq (-\bar{p}_-^0 - \gamma)h - C \quad \chi(x-h) - \chi(x) \geq (-\bar{p}_-^0 - \gamma)h - C. \quad (4.8)$$

The first inequality in (4.7) is valid only if $\bar{A} > \min \bar{H}_1$ and the second inequality is valid only if $\bar{A} > \min \bar{H}_2$. Similarly, the inequalities in (4.8) are valid only if $\bar{A} > \min \bar{H}_0$.

iii) (Rescaling) For $\varepsilon > 0$, we set

$$w^\varepsilon(x) = \varepsilon w\left(\frac{x}{\varepsilon}\right) \quad \text{and} \quad \chi^\varepsilon(x) = \varepsilon \chi\left(\frac{x}{\varepsilon}\right),$$

then (up to a sub-sequence $\varepsilon_n \rightarrow 0$) we have that w^ε and χ^ε converge locally uniformly respectively towards W and X which satisfy respectively

$$\begin{cases} |W(x) - W(y)| \leq C|x - y| & \text{for all } x, y \in \mathbb{R}, \\ \bar{H}_1(W_x) = \bar{A} & \text{for all } x > 0, \\ \bar{H}_0(W_x) = \bar{A} & \text{for all } x < 0. \end{cases} \quad (4.9)$$

and

$$\begin{cases} |X(x) - X(y)| \leq C|x - y| & \text{for all } x, y \in \mathbb{R}, \\ \bar{H}_2(X_x) = \bar{A} & \text{for all } x > 0, \\ \bar{H}_0(X_x) = \bar{A} & \text{for all } x < 0. \end{cases} \quad (4.10)$$

In particular, we have (with $W(0) = 0 = X(0)$),

$$\bar{p}_+^1 x 1_{\{x>0\}} + \bar{p}_-^0 x 1_{\{x<0\}} \leq W(x) \leq \bar{p}_+^1 x 1_{\{x>0\}} + \hat{p}_-^0 x 1_{\{x<0\}} \quad (4.11)$$

and

$$\bar{p}_+^2 x 1_{\{x>0\}} + \bar{p}_-^0 x 1_{\{x<0\}} \leq X(x) \leq \bar{p}_+^2 x 1_{\{x>0\}} + \hat{p}_-^0 x 1_{\{x<0\}}. \quad (4.12)$$

5 Proof of convergence

This section contains the proof of Theorem 2.4, which relies on the existence of correctors provided by Propositions 4.1, 4.2 and Theorem 4.3.

We will use the following lemmas for the proof of Theorem 2.4, the first one is a direct consequence of Perron's method and Lemma 3.6 and the second one is a direct consequence of Theorem 3.9.

Lemma 5.1 (Barriers uniform in ε). *Assume (A0) and (A). Then there exists a constant $C > 0$, such that for all $\varepsilon > 0$, the solution $(u^\varepsilon, \xi^\varepsilon)$ of (2.4)-(2.9) satisfies for all $t > 0$ and $x \in \mathbb{R}$,*

$$|u^\varepsilon(t, x) - u_0(x)| \leq Ct \quad \text{and} \quad |\xi^\varepsilon(t, x) - \xi_0(x)| \leq Ct.$$

Lemma 5.2 (Control of the space oscillations). *Assume (A0) and (A). Then the solution $(u^\varepsilon, \xi^\varepsilon)$ of (2.4)-(2.9) satisfies for all $t > 0$, $x, y \in \mathbb{R}$, $x \geq y$,*

$$\begin{cases} -2k_0(x - y) - 2\varepsilon \leq u^\varepsilon(t, x) - u^\varepsilon(t, y) \leq 0, \\ -2k_0(x - y) - 2\varepsilon \leq \xi^\varepsilon(t, x) - \xi^\varepsilon(t, y) \leq 0. \end{cases} \quad (5.1)$$

Before doing the proof of Theorem 2.4, let us show how it will allow us to prove Theorem 2.7.

Proof of Theorem 2.7. We want to prove that for all $t \in [0, +\infty)$, $x, y \in R_0 \cup R_1$, $x \geq y$ (or for all $t \in [0, +\infty)$, $x, y \in R_0 \cup R_2$, $x \geq y$)

$$-2k_0 \cdot d(x, y) \leq u^0(t, x) - u^0(t, y) \leq 0.$$

Using Lemma 5.2, we have that \tilde{u}^ε , defined by (2.16), satisfies for all $t \in [0, +\infty)$, $x, y \in R_0 \cup R_1$, $x \geq y$ (or for all $t \in [0, +\infty)$, $x, y \in R_0 \cup R_2$, $x \geq y$),

$$-2k_0 \cdot d(x, y) - 2\varepsilon \leq \tilde{u}^\varepsilon(t, x) - \tilde{u}^\varepsilon(t, y) \leq 0.$$

Now using Theorem 2.4, and passing to the limit as $\varepsilon \rightarrow 0$, we obtain the result. □

Proof of Theorem 2.4. Let us introduce for $(t, x) \in (0, T) \times \mathbb{R}$,

$$\bar{u}(t, x) = \limsup_{\varepsilon \rightarrow 0}^* u^\varepsilon(t, x) \quad \text{and} \quad \bar{\xi}(t, x) = \limsup_{\varepsilon \rightarrow 0}^* \xi^\varepsilon(t, x), \quad (5.2)$$

$$\underline{u}(t, x) = \liminf_{\varepsilon \rightarrow 0} u^\varepsilon(t, x) \quad \text{and} \quad \underline{\xi}(t, x) = \liminf_{\varepsilon \rightarrow 0} \xi^\varepsilon(t, x). \quad (5.3)$$

Thanks to Lemma 5.1 these functions are well defined. We also introduce

$$\bar{v}(t, y) = \begin{cases} \max(\bar{u}(t, -d(0, y)), \bar{\xi}(t, -d(0, y))) & \text{for } (t, y) \in (0, T) \times R_0, \\ \bar{u}(t, d(0, y)) & \text{for } (t, y) \in (0, T) \times R_1^*, \\ \bar{\xi}(t, d(0, y)) & \text{for } (t, y) \in (0, T) \times R_2^*, \end{cases}$$

and

$$\underline{v}(t, y) = \begin{cases} \min(\underline{u}(t, -d(0, y)), \underline{\xi}(t, -d(0, y))) & \text{for } (t, y) \in (0, T) \times R_0, \\ \underline{u}(t, d(0, y)) & \text{for } (t, y) \in (0, T) \times R_1^*, \\ \underline{\xi}(t, d(0, y)) & \text{for } (t, y) \in (0, T) \times R_2^*. \end{cases}$$

We want to prove that \bar{v} and \underline{v} are respectively a sub-solution and a super-solution of (2.17). In this case, the comparison principle will imply that $\bar{v} \leq \underline{v}$, but by construction we have $\underline{v} \leq \bar{v}$, hence we will get $\underline{v} = \bar{v} = u^0$, the unique solution of (2.17).

Let us prove that \bar{v} is a sub-solution of (2.17) (the proof for \underline{v} is similar and we skip it). We argue by contradiction and assume that there exists a test function $\varphi \in C^2((0, +\infty) \times \bar{R})$ (in the sense of Definition 3.3), and a point $(\bar{t}, \bar{y}) \in (0, +\infty) \times \bar{R}$ such that

$$\begin{cases} \bar{v}(\bar{t}, \bar{y}) = \varphi(\bar{t}, \bar{y}) \\ \bar{v} \leq \varphi & \text{on } \mathcal{Q}_{\bar{r}, \bar{r}}(\bar{t}, \bar{y}) & \text{with } \bar{r} > 0 \\ \bar{v} \leq \varphi - 2\eta & \text{outside } \mathcal{Q}_{\bar{r}, \bar{r}}(\bar{t}, \bar{y}) & \text{with } \eta > 0 \\ \varphi_t + \bar{H}(\bar{y}, \varphi_y(\bar{t}, \bar{y})) = \theta & \text{with } \theta > 0, \end{cases} \quad (5.4)$$

where

$$\bar{H}(\bar{y}, \varphi_y(\bar{t}, \bar{y})) = \begin{cases} \bar{H}_0(\partial_0 \varphi(\bar{t}, \bar{y})) & \text{if } \bar{y} \in R_0^*, \\ \bar{H}_1(\partial_1 \varphi(\bar{t}, \bar{y})) & \text{if } \bar{y} \in R_1^*, \\ \bar{H}_2(\partial_2 \varphi(\bar{t}, \bar{y})) & \text{if } \bar{y} \in R_2^*, \\ F_{\bar{A}}(\partial_0 \varphi(\bar{t}, 0), \partial_1 \varphi(\bar{t}, 0), \partial_2 \varphi(\bar{t}, 0)) & \text{if } \bar{y} = 0. \end{cases}$$

We denote by $\bar{x} = \text{sign}(\bar{y})d(0, \bar{y})$, with $\text{sign} : \bar{R} \rightarrow \mathbb{R}$,

$$\text{sign}(y) = \begin{cases} 1 & \text{if } y \in R_1^* \cup R_2^*, \\ -1 & \text{if } y \in R_0^*, \\ 0 & \text{if } y = 0. \end{cases}$$

Given Lemma 5.2 and (5.2)-(5.3), we can assume (up to changing φ at infinity) that for ε small enough, we have

$$\begin{aligned} u^\varepsilon(\tau, \text{sign}(y)d(0, y)) &\leq \varphi(\tau, y) - \eta \quad \text{for } (\tau, y) \in \left(\mathcal{Q}_{\bar{r}, \bar{r}}(\bar{t}, \bar{y})\right)^c \cap (R_0 \cup R_1), \\ \xi^\varepsilon(\tau, \text{sign}(y)d(0, y)) &\leq \varphi(\tau, y) - \eta \quad \text{for } (\tau, y) \in \left(\mathcal{Q}_{\bar{r}, \bar{r}}(\bar{t}, \bar{y})\right)^c \cap (R_0 \cup R_2). \end{aligned} \quad (5.5)$$

Using the previous lemmas we get that the function \bar{v} satisfies for all $t > 0$, $x, y \in R_0^*$ (or $x, y \in R_1^*$ or $x, y \in R_2^*$), such that $\text{sign}(x)d(0, x) \geq \text{sign}(y)d(0, y)$,

$$-2k_0 \cdot d(x, y) \leq \bar{v}(t, x) - \bar{v}(t, y) \leq 0. \quad (5.6)$$

Ch.6. Homogenization of a discrete model for a bifurcation

First case: $\bar{y} \neq 0$. We only consider $\bar{y} \in R_0^*$, since the other cases ($\bar{y} \in R_1^*$ and $\bar{y} \in R_2^*$) can be treated in the same way. We define $p = \partial_0 \varphi(\bar{t}, \bar{y})$ which, according to (5.6) satisfies

$$-2k_0 \leq p \leq 0. \quad (5.7)$$

We choose, $\bar{r} > 0$ small enough so that $\bar{x} + \bar{r} < 0$. Let us prove that the test function $\tilde{\varphi}$ defined by

$$\tilde{\varphi}(t, x) = \varphi(t, x\bar{e}_0) \quad \text{for } x < 0,$$

(notice that $\tilde{\varphi}_x(\bar{t}, \bar{x}) = \partial_0 \varphi(\bar{t}, \bar{y})$ and $\tilde{\varphi}_t(\bar{t}, \bar{x}) = \varphi_t(\bar{t}, \bar{y})$) satisfies in the viscosity sense, the following inequality for $(t, x) \in \mathcal{P}_{\bar{r}, \bar{r}}(\bar{t}, \bar{x})$,

$$\tilde{\varphi}_t + \phi \left(\frac{x}{\varepsilon}, \tilde{N}_0^\varepsilon \left(\frac{\tilde{\varphi}(t, \cdot)}{\varepsilon}, \left[\frac{\tilde{\varphi}(t, \cdot)}{\varepsilon} \right] \right) (x), \tilde{M}_1^\varepsilon \left[\frac{\tilde{\varphi}(t, \cdot)}{\varepsilon} \right] (x) \right) \cdot |\tilde{\varphi}_x| \geq \frac{\theta}{2}. \quad (5.8)$$

Let us notice that for ε small enough we have for all $(t, x) \in \mathcal{P}_{\bar{r}, \bar{r}}(\bar{t}, \bar{x})$, for $k = 1, 2$,

$$\phi \left(\frac{x}{\varepsilon}, \tilde{N}_0^\varepsilon \left(\frac{\tilde{\varphi}(t, \cdot)}{\varepsilon}, \left[\frac{\tilde{\varphi}(t, \cdot)}{\varepsilon} \right] \right) (x), \tilde{M}_k^\varepsilon \left[\frac{\tilde{\varphi}(t, \cdot)}{\varepsilon} \right] (x) \right) = \tilde{N}_0^\varepsilon \left(\frac{\tilde{\varphi}(t, \cdot)}{\varepsilon}, \left[\frac{\tilde{\varphi}(t, \cdot)}{\varepsilon} \right] \right) (x). \quad (5.9)$$

For all $(t, x) \in \mathcal{P}_{\bar{r}, \bar{r}}(\bar{t}, \bar{x})$, we have for \bar{r} small enough

$$\begin{aligned} \tilde{\varphi}_t(t, x) \\ + \tilde{N}_0^\varepsilon \left(\frac{\tilde{\varphi}(t, \cdot)}{\varepsilon}, \left[\frac{\tilde{\varphi}(t, \cdot)}{\varepsilon} \right] \right) (x) \cdot |\tilde{\varphi}_x(t, x)| &= \tilde{\varphi}_t(\bar{t}, \bar{x}) + o_{\bar{r}}(1) \\ &+ \tilde{N}_0^\varepsilon \left(\frac{\tilde{\varphi}(t, \cdot)}{\varepsilon}, \left[\frac{\tilde{\varphi}(t, \cdot)}{\varepsilon} \right] \right) (x) \cdot |p| \\ &= \theta - \bar{H}_0(p) + o_{\bar{r}}(1) \\ &+ \tilde{N}_0^\varepsilon \left(\frac{\tilde{\varphi}(t, \cdot)}{\varepsilon}, \left[\frac{\tilde{\varphi}(t, \cdot)}{\varepsilon} \right] \right) (x) \cdot |p| \\ &=: \Delta, \end{aligned} \quad (5.10)$$

where we have used (5.4) and the definition of p . We recall that for $-2k_0 \leq p \leq 0$, we have

$$\bar{H}_0(p) = N_p^0(0, [0])(0) \cdot |p| = \tilde{N}_p^0(0, [0])(0) \cdot |p|.$$

Moreover, for all $z \in [h_0, h_{max}]$, and for ε and \bar{r} small enough we have that

$$\begin{aligned} \frac{\tilde{\varphi}(t, x + \varepsilon z) - \tilde{\varphi}(t, x)}{\varepsilon} &= z\tilde{\varphi}_x(t, x) + O(\varepsilon) \\ &\leq pz + o_{\bar{r}}(1) + c\varepsilon, \end{aligned}$$

with $c > 0$ a fixed constant. Now using the fact that \tilde{F} is decreasing we have

$$\tilde{F}(pz + c\varepsilon + o_{\bar{r}}(1)) \leq \tilde{F}\left(\frac{\tilde{\varphi}(t, x + \varepsilon z) - \tilde{\varphi}(t, x)}{\varepsilon}\right).$$

Using this result and replacing the non-local operators in (5.10) by their definition we obtain

$$\begin{aligned} \Delta &\geq \theta + o_{\bar{r}} + |p| \int_{h_0}^{h_{max}} J_0(z) \tilde{F}(pz + c\varepsilon + o_{\bar{r}}(1)) dz \\ &\quad - |p| \int_{h_0}^{h_{max}} J_0(z) \tilde{F}(pz) dz. \end{aligned} \tag{5.11}$$

We can see that if we have $p = 0$, we obtain directly our result. However, if $-2k_0 \leq p < 0$,

$$\begin{aligned} \int_{\mathbb{R}} J_0(z) \tilde{F}(pz + c\varepsilon + o_{\bar{r}}(1)) dz &= -V_0 \left(\frac{-1 - c\varepsilon - o_{\bar{r}}(1)}{p} \right) \\ &\quad - \frac{1}{2} V_0 \left(-\frac{-1 + c\varepsilon + o_{\bar{r}}(1)}{p} \right) + \frac{3}{2} V_{max}^0, \\ \int_{\mathbb{R}} J_0(z) \tilde{F}(pz) dz &= -V_0 \left(\frac{-1}{p} \right) + \frac{3}{2} V_{max}^0. \end{aligned} \tag{5.12}$$

Injecting (5.12) in (5.11) and choosing ε and \bar{r} small enough, we obtain

$$\begin{aligned} \Delta &\geq \theta + o_{\bar{r}}(1) + |p| \cdot \left[-V_0 \left(\frac{-1 - c\varepsilon + o_{\bar{r}}(1)}{p} \right) + V_0 \left(\frac{-1}{p} \right) \right] \\ &\geq \theta + o_{\bar{r}}(1) - \|V_0'\|_{\infty} \cdot (c\varepsilon + o_{\bar{r}}(1)) \\ &\geq \frac{\theta}{2}, \end{aligned} \tag{5.13}$$

where we have used assumption (A1) for the second line.

Getting the contradiction. By definition, for ε small enough and using (5.5), we have

$$u^\varepsilon \leq \tilde{\varphi} - \eta \quad \text{and} \quad \xi^\varepsilon \leq \tilde{\varphi} - \eta \quad \text{outside } \mathcal{P}_{\bar{r}, \bar{r}}(\bar{t}, \bar{x}).$$

Using the comparison principle on bounded sets for (2.4), we get

$$u^\varepsilon \leq \tilde{\varphi} - \eta \quad \text{and} \quad \xi^\varepsilon \leq \tilde{\varphi} - \eta \quad \text{on } \mathcal{P}_{\bar{r}, \bar{r}}(\bar{t}, \bar{x}).$$

Passing to the limit as $\varepsilon \rightarrow 0$, this implies $\bar{v} \leq \varphi - \eta$ on $\mathcal{Q}_{\bar{r}, \bar{r}}(\bar{t}, \bar{y})$ and this contradicts the fact that $\bar{v}(\bar{t}, \bar{y}) = \varphi(\bar{t}, \bar{y})$.

Remark 5.3. *In the case $\bar{y} \in R_1^*$ or $\bar{y} \in R_2^*$, (2.4) is completely decoupled for ε small enough in $\mathcal{P}_{\bar{r}, \bar{r}}(\bar{t}, \bar{x})$ and therefore we can consider each line of (2.4) as independent.*

Ch.6. Homogenization of a discrete model for a bifurcation

Second case: $\bar{y} = 0$. Thanks to (5.6) (which implies (3.8)), we may use Theorem 3.5 and assume that the test function has the following form

$$\varphi(t, y) = g(t) + \bar{p}_-^0 y 1_{\{y \in R_0^*\}} + \bar{p}_+^1 y 1_{\{y \in R_1^*\}} + \bar{p}_+^2 y 1_{\{y \in R_2^*\}} \quad \text{on } \mathcal{Q}_{2\bar{r}, 2\bar{r}}(\bar{t}, 0), \quad (5.14)$$

where g is a C^1 function defined in $(0, +\infty)$. The last line in condition (5.4) becomes

$$g'(t) + F_{\bar{A}}(\bar{p}_-, \bar{p}_+, \bar{p}_+) = g'(t) + \bar{A} = \theta \quad \text{at } (\bar{t}, 0). \quad (5.15)$$

Let us consider (w, χ) the solution of (2.25) provided by Theorem 4.3, and let us denote

$$\varphi^\varepsilon(t, x) = \begin{cases} g(t) + \varepsilon w\left(\frac{x}{\varepsilon}\right) & \text{on } \mathcal{P}_{2\bar{r}, 2\bar{r}}(\bar{t}, 0), \\ \varphi\left(t, x\bar{e}_0 1_{\{x \leq 0\}} + x\bar{e}_1 1_{\{x \geq 0\}}\right) & \text{outside } \mathcal{P}_{2\bar{r}, 2\bar{r}}(\bar{t}, 0), \end{cases} \quad (5.16)$$

and

$$\psi^\varepsilon(t, x) = \begin{cases} g(t) + \varepsilon \chi\left(\frac{x}{\varepsilon}\right) & \text{on } \mathcal{P}_{2\bar{r}, 2\bar{r}}(\bar{t}, 0), \\ \varphi\left(t, x\bar{e}_0 1_{\{x \leq 0\}} + x\bar{e}_2 1_{\{x \geq 0\}}\right) & \text{outside } \mathcal{P}_{2\bar{r}, 2\bar{r}}(\bar{t}, 0). \end{cases} \quad (5.17)$$

We claim that these functions satisfy in the viscosity sense, for \bar{r} and ε small enough, on $\mathcal{P}_{\bar{r}, \bar{r}}(\bar{t}, 0)$,

$$\begin{cases} \varphi_t^\varepsilon(t, x) + \phi\left(\frac{x}{\varepsilon}, \tilde{N}_0^\varepsilon\left(\frac{\varphi^\varepsilon(t, \cdot)}{\varepsilon}, \left[\frac{\psi^\varepsilon(t, \cdot)}{\varepsilon}\right]\right), \tilde{M}_1^\varepsilon\left[\frac{\varphi^\varepsilon}{\varepsilon}(t, \cdot)\right](x)\right) \cdot |\varphi_x^\varepsilon| \geq \frac{\theta}{2}, \\ \psi_t^\varepsilon(t, x) + \phi\left(\frac{x}{\varepsilon}, \tilde{N}_0^\varepsilon\left(\frac{\psi^\varepsilon(t, \cdot)}{\varepsilon}, \left[\frac{\varphi^\varepsilon(t, \cdot)}{\varepsilon}\right]\right), \tilde{M}_2^\varepsilon\left[\frac{\psi^\varepsilon}{\varepsilon}(t, \cdot)\right](x)\right) \cdot |\psi_x^\varepsilon| \geq \frac{\theta}{2}, \end{cases}$$

We only prove that φ^ε is a super-solution (the case for ψ^ε is similar and we skip it), let h be a test function touching φ^ε from below at $(t_1, x_1) \in \mathcal{P}_{\bar{r}, \bar{r}}(\bar{t}, 0)$, so we have

$$w\left(\frac{x_1}{\varepsilon}\right) = \frac{1}{\varepsilon} (h(t_1, x_1) - g(t_1)),$$

and

$$w(y) \geq \frac{1}{\varepsilon} (h(t_1, \varepsilon y) - g(t_1)),$$

for y in a neighbourhood of $\frac{x_1}{\varepsilon}$. Since w does not depend on time, we have

$$h_t(t_1, x_1) = g'(t_1).$$

Therefore, we have

$$h_t(t_1, x_1) - g'(t_1) + \phi\left(\frac{x_1}{\varepsilon}, \tilde{N}_0(w, [\chi])\left(\frac{x_1}{\varepsilon}\right), \tilde{M}_1[w]\left(\frac{x_1}{\varepsilon}\right)\right) \cdot |h_x(t_1, x_1)| \geq \bar{A}.$$

This implies that (using (5.15) and taking \bar{r} small enough)

$$h_t(t_1, x_1) + \phi\left(\frac{x_1}{\varepsilon}, \tilde{N}_0(w, [\chi])\left(\frac{x_1}{\varepsilon}\right), \tilde{M}_1[w]\left(\frac{x_1}{\varepsilon}\right)\right) \cdot |h_x(t_1, x_1)| \geq \bar{A} + g'(t_1) \geq \frac{\theta}{2}.$$

Now for ε small enough such that $\varepsilon h_{max} \leq \bar{r}$, we deduce from the previous inequality and using the fact that we consider non-local operators with bounded support, that we have

$$h_t(t_1, x_1) + \phi\left(\frac{x_1}{\varepsilon}, \tilde{N}_0^\varepsilon\left(\frac{\varphi^\varepsilon(t_1, \cdot)}{\varepsilon}, \left[\frac{\psi^\varepsilon(t_1, \cdot)}{\varepsilon}\right]\right)(x_1), \tilde{M}_1^\varepsilon\left[\frac{\varphi^\varepsilon(t_1, \cdot)}{\varepsilon}\right](x_1)\right) \cdot |h_x(t_1, x_1)| \geq \frac{\theta}{2}.$$

Getting the contradiction. Using (5.5) and (5.14), we have for ε small enough

$$\begin{cases} u^\varepsilon + \eta \leq g(t) + \bar{p}_-^0 x 1_{\{x < 0\}} + \bar{p}_+^1 x 1_{\{x > 0\}} \\ \xi^\varepsilon + \eta \leq g(t) + \bar{p}_-^0 x 1_{\{x < 0\}} + \bar{p}_+^2 x 1_{\{x > 0\}}, \end{cases} \quad \text{on } \mathcal{P}_{2\bar{r}, 2\bar{r}}(\bar{t}, 0) \setminus \mathcal{P}_{\bar{r}, \bar{r}}(\bar{t}, 0),$$

Using the fact that $w^\varepsilon \rightarrow W$ and $\zeta^\varepsilon \rightarrow X$, and using (4.11), we have for ε small enough

$$\begin{cases} u^\varepsilon + \frac{\eta}{2} \leq \varphi^\varepsilon \\ \xi^\varepsilon + \frac{\eta}{2} \leq \psi^\varepsilon, \end{cases} \quad \text{on } \mathcal{P}_{2\bar{r}, 2\bar{r}}(\bar{t}, 0) \setminus \mathcal{P}_{\bar{r}, \bar{r}}(\bar{t}, 0).$$

Combining this result with (5.16)-(5.17), we get

$$\begin{cases} u^\varepsilon + \frac{\eta}{2} \leq \varphi^\varepsilon \\ \xi^\varepsilon + \frac{\eta}{2} \leq \psi^\varepsilon, \end{cases} \quad \text{outside } \mathcal{P}_{\bar{r}, \bar{r}}(\bar{t}, 0).$$

By the comparison principle on bounded subsets the previous inequality holds in $\mathcal{P}_{\bar{r}, \bar{r}}(\bar{t}, 0)$.

Passing to the limit as $\varepsilon \rightarrow 0$, we obtain

$$\bar{v} + \frac{\eta}{2} \leq \varphi \quad \text{on } \mathcal{Q}_{\bar{r}, \bar{r}}(\bar{t}, 0).$$

Evaluating this inequality in $(\bar{t}, 0)$, we obtain a contradiction.

Remark 5.4. *In order to prove that \underline{v} is a super-solution, the test function in the case $\bar{y} = 0$, needs to be*

$$\varphi(t, y) = g(t) + \hat{p}_-^0 y 1_{\{y \in R_0^*\}} + \bar{p}_+^1 y 1_{\{y \in R_1^*\}} + \bar{p}_+^2 y 1_{\{y \in R_2^*\}} \quad \text{on } \mathcal{Q}_{2\bar{r}, 2\bar{r}}(\bar{t}, 0). \quad (5.18)$$

This ends the proof of Theorem 2.4. □

6 Truncated cell problem

This section contains the proof of Theorem 4.3. We proceed as in [2, GIM15] and we will construct correctors on a truncated domain and then pass to the limit as the size of the domain goes to infinity. For $l \in (h_0 + h_{max} + 1, +\infty)$, $h_0 \ll l$ and $h_0 + h_{max} + 1 \leq R \ll l$, we want to find $\lambda_{l,R} \in \mathbb{R}$ such that there exists a solution $(w^{l,R}, \chi^{l,R})$ of

$$\left\{ \begin{array}{l} \left\{ \begin{array}{l} G_R^1(x, w^{l,R}(x), [\chi^{l,R}], [w^{l,R}], w_x^{l,R}) = \lambda_{l,R} \\ G_R^2(x, \chi^{l,R}(x), [w^{l,R}], [\chi^{l,R}], \chi_x^{l,R}) = \lambda_{l,R} \end{array} \right. & \text{if } x \in (-l, l) \\ \left\{ \begin{array}{l} \overline{H}_1^+(w_x^{l,R}) = \lambda_{l,R} \\ \overline{H}_2^+(\chi_x^{l,R}) = \lambda_{l,R} \end{array} \right. & \text{if } x = l \\ \left\{ \begin{array}{l} \overline{H}_0^-(w_x^{l,R}) = \lambda_{l,R} \\ \overline{H}_0^-(\chi_x^{l,R}) = \lambda_{l,R} \end{array} \right. & \text{if } x = -l \end{array} \right. \quad (6.1)$$

with, for $j = 1, 2$,

$$G_R^j(x, w(x), [\chi], [w], q) = \psi_R^-(x)\psi_R^+(x)\phi(x, N_0(w, [\chi])(x), M_j[w](x))|q| + (1 - \psi_R^+(x))\overline{H}_j(q) + (1 - \psi_R^-(x))\overline{H}_0(q), \quad (6.2)$$

and $\psi_R^+, \psi_R^- \in C^\infty$, $\psi_R^\pm : \mathbb{R} \rightarrow [0, 1]$, with $\psi_R^-(x) = \psi_R^+(-x)$ and

$$\psi_R^+ \equiv \begin{cases} 1 & \text{on } (-\infty, R] \\ 0 & \text{on } [R+1, +\infty), \end{cases} \quad \text{and } \psi_R(x) < 1 \quad \forall x \notin (-\infty, R]. \quad (6.3)$$

To G_R^j we associate \tilde{G}_R^j which is defined in the same way by replacing the non-local operators N_0 and M_j respectively by \tilde{N}_0 and \tilde{M}_j .

6.1 Comparison principle for a truncated problem

Proposition 6.1 (Comparison principle on a truncated domain). *Let us consider the following problem for $0 < l_1 < l_2$ and $\lambda \in \mathbb{R}$, with $l_2 \gg R$.*

$$\left\{ \begin{array}{l} \left\{ \begin{array}{l} \tilde{G}_R^1(x, u(x), [\xi], [u], u_x) \geq \lambda \\ \tilde{G}_R^2(x, \xi(x), [u], [\xi], \xi_x) \geq \lambda \end{array} \right. & \text{if } x \in (l_1, l_2) \\ \left\{ \begin{array}{l} \overline{H}_1^+(u_x) \geq \lambda \\ \overline{H}_2^+(\xi_x) \geq \lambda \end{array} \right. & \text{if } x = l_2 \end{array} \right. \quad (6.4)$$

and for $\varepsilon_0 > 0$,

$$\left\{ \begin{array}{l} \left\{ \begin{array}{l} G_R^1(x, v(x), [\zeta], [v], v_x) \leq \lambda - \varepsilon_0 \\ G_R^2(x, \zeta(x), [v], [\zeta], \zeta_x) \leq \lambda - \varepsilon_0 \end{array} \right. & \text{if } x \in (l_1, l_2) \\ \left\{ \begin{array}{l} \overline{H}_1^+(v_x) \leq \lambda - \varepsilon_0 \\ \overline{H}_2^+(\zeta_x) \leq \lambda - \varepsilon_0 \end{array} \right. & \text{if } x = l_2 \end{array} \right. \quad (6.5)$$

Then if $v(l_1) \leq u(l_1)$ and $\zeta(l_1) \leq \xi(l_1)$, we have $v \leq u$ and $\zeta \leq \xi$ in $[l_1, l_2]$.

Proof. Like in [2], the only new difficulty to prove this proposition is the comparison at l_2 . But since near l_2 , the system decouples itself, we can proceed as in [GIM15, Proposition 4.1]. □

Remark 6.2. We have a similar result if we exchange the boundary conditions, that is to say for $l_1 < l_2 < -h_0 - h_{max} - 1$, and if for all $x \in [l_2, l_2 + h_{max}]$ $v(x) \leq u(x)$ and $\zeta(x) \leq \xi(x)$ and the following conditions are imposed at $x = l_1$,

$$\left\{ \begin{array}{l} \left\{ \begin{array}{l} \overline{H}_0^-(u_x) \geq \lambda \\ \overline{H}_0^-(\xi_x) \geq \lambda \end{array} \right. \quad \text{if } x = l_1 \\ \left\{ \begin{array}{l} \overline{H}_0^-(v_x) \leq \lambda - \varepsilon_0 \\ \overline{H}_0^-(\zeta_x) \leq \lambda - \varepsilon_0 \end{array} \right. \quad \text{if } x = l_1. \end{array} \right. \quad (6.6)$$

6.2 Existence of correctors on a truncated domain

Proposition 6.3 (Existence of correctors on a truncated domain). *There exists a constant $\lambda_{l,R} \in \mathbb{R}$ such that there exists a solution $(w^{l,R}, \chi^{l,R})$ of (6.1). Moreover, there exists a constant C (depending only on k_0) and two Lipschitz continuous functions $m_1^{l,R}$ and $m_2^{l,R}$, such that*

$$\left\{ \begin{array}{l} H_0 = \max_{i \in \{0,1,2\}} \min_{p \in \mathbb{R}} \overline{H}_i(p) \leq \lambda_{l,R} \leq 0, \\ |w^{l,R}(x) - m_1^{l,R}(x)| \leq C \quad \text{for all } x \in [-l, l], \\ |\chi^{l,R}(x) - m_2^{l,R}(x)| \leq C \quad \text{for all } x \in [-l, l], \\ |m_i^{l,R}(x) - m_i^{l,R}(y)| \leq C|x - y| \quad \text{for all } x, y \in [-l, l], \quad \text{for } i = 1, 2 \\ |w^{l,R}(x) - \chi^{l,R}(x)| \leq C \quad \text{for all } x \in [-l, h_0]. \end{array} \right. \quad (6.7)$$

Proof. We consider the approximated truncated cell problem,

$$\left\{ \begin{array}{l} \left\{ \begin{array}{l} \delta v^\delta + G_R^1(x, v^\delta(x), [\zeta^\delta], [v^\delta], v_x^\delta) = 0 \\ \delta \zeta^\delta + G_R^2(x, \zeta^\delta(x), [v^\delta], [\zeta^\delta], \zeta_x^\delta) = 0 \end{array} \right. \quad \text{if } x \in (-l, l) \\ \left\{ \begin{array}{l} \delta v^\delta + \overline{H}_1^+(v_x^\delta) = 0 \\ \delta \zeta^\delta + \overline{H}_2^+(\zeta_x^\delta) = 0 \end{array} \right. \quad \text{if } x = l \\ \left\{ \begin{array}{l} \delta v^\delta + \overline{H}_0^-(v_x^\delta) = 0 \\ \delta \zeta^\delta + \overline{H}_0^-(\zeta_x^\delta) = 0 \end{array} \right. \quad \text{if } x = -l \end{array} \right. \quad (6.8)$$

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Step 1: existence of a solution. Using that $(0, 0)$ and $(C_0/\delta, C_0/\delta)$ are respectively sub and super-solution of (6.8), with $C_0 = |\min_{p \in \mathbb{R}} \bar{H}_0(p)|$ and that we have a comparison principle, we deduce that there exists a continuous viscosity solution (v^δ, ζ^δ) of (6.8) which satisfies

$$0 \leq v^\delta \leq \frac{C_0}{\delta} \quad \text{and} \quad 0 \leq \zeta^\delta \leq \frac{C_0}{\delta}. \quad (6.9)$$

Step 2: control of the oscillations of v^δ and ζ^δ .

Lemma 6.4. *The functions v^δ and ζ^δ satisfy for all $x, y \in [-l, l]$, $x \geq y$,*

$$-K_0(x - y) - 2 \leq v^\delta(x) - v^\delta(y) \leq 0 \quad \text{and} \quad -K_0(x - y) - 2 \leq \zeta^\delta(x) - \zeta^\delta(y) \leq 0, \quad (6.10)$$

with $K_0 = 2k_0$.

Proof of Lemma 6.4. In the rest of the proof we will use the following notation

$$\hat{\Omega} = \{(x, y) \in [-l, l]^2 \text{ s.t. } x \geq y\}.$$

Proof of the upper inequality. We want to prove that

$$M = \sup_{(x, y) \in \hat{\Omega}} \max(v^\delta(x) - v^\delta(y), \zeta^\delta(x) - \zeta^\delta(y)) \leq 0. \quad (6.11)$$

We argue by contradiction and assume that $M > 0$. Since $\hat{\Omega}$ is compact and v^δ and ζ^δ are continuous, we can see that M is reached for a finite point that we denote by $(\bar{x}, \bar{y}) \in \hat{\Omega}$. Given that $M > 0$, we deduce that $\bar{x} \neq \bar{y}$. Therefore, we can use the viscosity inequalities for (6.8).

Let us assume for instance that $M = v^\delta(\bar{x}) - v^\delta(\bar{y})$ (the other case is similar so we skip it).

We distinguish 3 cases.

-If $(\bar{x}, \bar{y}) \in (-l, l)$, we have

$$\begin{aligned} \delta v^\delta(\bar{x}) + G_R^1(\bar{x}, v^\delta(\bar{x}), [\zeta^\delta], [v^\delta], 0) &\leq 0 \\ \delta v^\delta(\bar{y}) + \tilde{G}_R^1(\bar{y}, v^\delta(\bar{y}), [\zeta^\delta], [v^\delta], 0) &\geq 0 \end{aligned}$$

combining these inequalities with the fact that $G(x, U, [\Xi], [U], 0) = 0$, we obtain

$$\delta M \leq 0.$$

-If $\bar{x} = l$ and $\bar{y} \in (-l, l)$, we obtain similarly

$$\delta M \leq 0, \quad (6.12)$$

using the fact that $\overline{H}_1^+(0) = 0$.

-If $\bar{x} \in (-l, l]$ and $\bar{y} = -l$, we obtain

$$\delta M \leq \overline{H}_0^-(0) \leq 0.$$

For every value of \bar{x}, \bar{y} we obtain a contradiction, therefore $M \leq 0$.

Proof of the lower inequality. In order to proof this inequalities, we will use the following lemma whose proof is postponed.

Lemma 6.5. *For all $x \in [-l, -h_0]$, we have*

$$|\zeta^\delta(x) - v^\delta(x)| \leq 1. \tag{6.13}$$

We want to prove that

$$M = \sup_{(x,y) \in \hat{\Omega}} \max \left(v^\delta(y) - v^\delta(x) - K_0(x - y) - 2, \zeta^\delta(y) - \zeta^\delta(x) - K_0(x - y) - 2 \right) \leq 0.$$

We argue by contradiction and assume that $M > 0$. We can see that M is reached for a finite point that we denote by $(\bar{x}, \bar{y}) \in \hat{\Omega}$. Since $M > 0$, we deduce that $\bar{x} > \bar{y}$. Therefore, we can use the viscosity inequalities from (6.8).

Let us for instance assume that $M = v^\delta(\bar{y}) - v^\delta(\bar{x}) - K_0(\bar{x} - \bar{y}) - 2$ (the other case is similar and we skip it).

We distinguish 4 cases.

-If $\bar{x}, \bar{y} \in (-l, l)$, we obtain

$$\begin{aligned} \delta v^\delta(\bar{y}) + G_R^1(\bar{y}, v^\delta(\bar{y}), [\zeta^\delta], [v^\delta], -K_0) &\leq 0 \\ \delta v^\delta(\bar{x}) + \tilde{G}_R^1(\bar{x}, v^\delta(\bar{x}), [\zeta^\delta], [v^\delta], -K_0) &\geq 0, \end{aligned}$$

combining these inequalities and using the definition of M , we obtain

$$\delta M \leq \tilde{G}_R^1(\bar{x}, v^\delta(\bar{x}), [\zeta^\delta], [v^\delta], -K_0) - G_R^1(\bar{y}, v^\delta(\bar{y}), [\zeta^\delta], [v^\delta], -K_0). \tag{6.14}$$

We recall that the non-local operators are non-positive (see Remark 2.9) and that $\overline{H}_i(-K_0) = 0$ for $i = 0, 1, 2$. This implies that

$$\delta M \leq -G_R^1(\bar{y}, v^\delta(\bar{y}), [\zeta^\delta], [v^\delta], -K_0) = \psi_R^-(\bar{y})\psi_R^+(\bar{y})\phi\left(\bar{y}, N_0(v^\delta, [\zeta^\delta])(\bar{y}), M_1[v^\delta](\bar{y})\right) \cdot K_0. \tag{6.15}$$

To treat (6.15), we will compute $M_1[v^\delta](\bar{y})$ and $N_0(v^\delta, [\zeta^\delta])(\bar{y})$.

Let us prove that $M_1[v^\delta](\bar{y}) = 0$. In fact, it is sufficient to prove that for all $z \in (h_0, h_{max}]$, we have

$$v^\delta(\bar{y} + z) - v^\delta(\bar{y}) < -2.$$

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Let $z \in (h_0, h_{max}]$, if $\bar{y} + z \leq \bar{x}$, then using the fact that

$$v^\delta(\bar{y} + z) - v^\delta(\bar{x}) - K_0(\bar{x} - \bar{y} - z) - 2 \leq v^\delta(\bar{y}) - v^\delta(\bar{x}) - K_0(\bar{x} - \bar{y}) - 2,$$

we deduce that

$$v^\delta(\bar{y} + z) - v^\delta(\bar{y}) \leq -K_0 z < -2.$$

If $\bar{y} + z > \bar{x}$, using the fact that v^δ is non-increasing and that $M > 0$, we obtain

$$v^\delta(\bar{y} + z) - v^\delta(\bar{y}) \leq v^\delta(\bar{x}) - v^\delta(\bar{y}) \leq -K_0(\bar{x} - \bar{y}) - 2 < -2.$$

Using that $N_0(v^\delta, [\zeta^\delta])(\bar{y}) \leq 0$, and that $M_1[v^\delta](\bar{y}) = 0$, we get

$$\max \left(N_0(v^\delta, [\zeta^\delta])(\bar{y}), M_1[v^\delta](\bar{y}) \right) = 0.$$

We can therefore assume that $-R - 1 \leq \bar{y} \leq -h_0 - h_{max}$ (if not we get a contradiction in (6.15)).

Let us now prove that for all $-R - 1 \leq \bar{y} \leq -h_0 - h_{max}$, $N_0(v^\delta, [\zeta^\delta])(\bar{y}) = 0$. In fact, it suffices to prove that for all $z \in (h_0, h_{max}]$, we have

$$\zeta^\delta(\bar{y} + z) - v^\delta(\bar{y}) < -1.$$

Let $z \in (h_0, h_{max}]$. If $\bar{y} + z \geq \bar{x}$, using the fact that ζ^δ is non-increasing and Lemma 6.5 (since $\bar{y} + z \leq \bar{y} + h_{max} \leq -h_0$), we obtain

$$\begin{aligned} \zeta^\delta(\bar{y} + z) - v^\delta(\bar{y}) &\leq v^\delta(\bar{y} + z) - v^\delta(\bar{y}) + 1 \\ &\leq v^\delta(\bar{x}) - v^\delta(\bar{y}) + 1 \\ &\leq -K_0(\bar{x} - \bar{y}) - 1 < -1, \end{aligned}$$

where we used the fact that $M > 0$ for the third line. If $\bar{y} + z < \bar{x}$, using Lemma 6.5 and that

$$v^\delta(\bar{y} + z) - v^\delta(\bar{x}) - K_0(\bar{x} - \bar{y} - z) - 2 \leq v^\delta(\bar{y}) - v^\delta(\bar{x}) - K_0(\bar{x} - \bar{y}) - 2,$$

we obtain

$$\begin{aligned} \zeta^\delta(\bar{y} + z) - v^\delta(\bar{y}) &\leq v^\delta(\bar{y} + z) - v^\delta(\bar{y}) + 1 \\ &\leq 1 - K_0 z < -1. \end{aligned}$$

Injecting the previous results into (6.15), we get $\delta M \leq 0$, which is a contradiction.

-If $\bar{x} \in (-l, l)$ and $\bar{y} = -l$, we obtain

$$\begin{aligned} \delta v^\delta(\bar{y}) + \overline{H_0}^-(-K_0) &\leq 0 \\ \delta v^\delta(\bar{x}) &\geq \delta v^\delta(\bar{x}) + \tilde{G}_R^1(\bar{x}, v^\delta(\bar{x}), [\zeta^\delta], [v^\delta], -K_0) \geq 0. \end{aligned}$$

Using the fact that $\overline{H}_0^-(-K_0) = 0$ we obtain $\delta M \leq 0$, which is a contradiction.

-If $\bar{x} = l$ and $\bar{y} \in (-l, l)$, we obtain

$$\begin{aligned} \delta v^\delta(\bar{y}) + G_R^1(\bar{y}, v^\delta(\bar{y}), [\zeta^\delta], [v^\delta], -K_0) &\leq 0 \\ \delta v^\delta(\bar{x}) + \overline{H}_1^+(-K_0) &\geq 0, \end{aligned}$$

using the result of the first case, and the fact that $\overline{H}_1^+(-K_0) < 0$, we directly obtain $\delta M \leq 0$, which is a contradiction.

-If $\bar{x} = l$ and $\bar{y} = -l$, we obtain

$$\begin{aligned} \delta v^\delta(\bar{y}) + \overline{H}_0^-(-K_0) &\leq 0 \\ \delta v^\delta(\bar{x}) + \overline{H}_1^+(-K_0) &\geq 0, \end{aligned}$$

just like before, we obtain $\delta M \leq 0$, which is a contradiction.

For every value of $\bar{x}, \bar{y} \in [-l, l]$ we obtain a contradiction, therefore we have $M \leq 0$. This ends the proof of Lemma 6.4. □

Step 3: construction of the Lipschitz continuous function.

Lemma 6.6. *There exists two Lipschitz continuous functions m_1^δ and m_2^δ , such that there exists a constant $C > 0$ (independent of l and R) such that*

$$\left\{ \begin{array}{ll} |v^\delta(x) - m_1^\delta(x)| \leq C & \text{for all } x \in [-l, l], \\ |\zeta^\delta(x) - m_2^\delta(x)| \leq C & \text{for all } x \in [-l, l], \\ |m_i^\delta(x) - m_i^\delta(y)| \leq C|x - y| & \text{for all } x, y \in [-l, l], \text{ for } i = 1, 2. \end{array} \right. \quad (6.16)$$

Moreover, for all $x \in [-l, h_0]$, we have

$$|v^\delta(x) - \zeta^\delta(x)| \leq C. \quad (6.17)$$

Proof. We only do the construction of m_1^δ , since the construction of m_2^δ is similar and we skip it. We define m_1^δ as an affine function in each interval of the form $[ih_0, (i+1)h_0]$, with $i \in \mathbb{Z}$, such that

$$m_1^\delta(ih_0) = v^\delta(ih_0) \quad \text{and} \quad m_1^\delta((i+1)h_0) = v^\delta((i+1)h_0).$$

Since m_1^δ and v^δ are non-increasing, and $|v^\delta((i+1)h_0) - v^\delta(ih_0)| \leq K_0 h_0 + 2 = 4$, we deduce that for all $x \in [ih_0, (i+1)h_0]$,

$$-4 \leq v^\delta((i+1)h_0) - m_1^\delta(ih_0) \leq v^\delta(x) - m_1^\delta(x) \leq v^\delta(ih_0) - m_1^\delta((i+1)h_0) \leq 4, \quad (6.18)$$

and for all $x, y \in [-l, l]$,

$$|m_1^\delta(x) - m_1^\delta(y)| \leq 2K_0|x - y|. \quad (6.19)$$

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We build m_2^δ the same way but using ζ^δ instead of v^δ .

Moreover, let $x \in [-l, h_0]$, first if $x \leq -h_0$, using Lemma 6.5, we have that

$$|v^\delta(x) - \zeta^\delta(x)| \leq 1.$$

If $x \in [-h_0, h_0]$, we have

$$\begin{aligned} |v^\delta(x) - \zeta^\delta(x)| &\leq |v^\delta(x) - m_1(x)| + |\zeta^\delta(x) - m_2(x)| + |m_1(x) - m_2(x)| \\ &\leq 8 + |m_1(x) - m_1(-h_0)| + |m_2(-h_0) - m_2(x)| \\ &\quad + |m_1(-h_0) - m_2(-h_0)| \\ &\leq 8 + 1 + 4K_0|x + h_0| \\ &\leq 9 + 8K_0h_0 = 25, \end{aligned}$$

where we have used for the third inequality the fact that

$$|m_1(-h_0) - m_2(-h_0)| = |v^\delta(-h_0) - \zeta^\delta(-h_0)| \leq 1$$

and for the last inequality, the fact that $x \in [-h_0, h_0]$.

Choosing $C = \max(2K_0, 25)$ we obtain (6.16) and (6.17). This ends the proof of Lemma 6.6. □

Step 4: passing to the limit as δ goes to 0. Using (6.9), Lemma 6.6 and (6.16), we deduce that there exists a subsequence $\delta_n \rightarrow 0$ such that

$$\begin{aligned} \delta_n v^{\delta_n}(0) &\rightarrow -\lambda_{l,R} && \text{as } n \rightarrow +\infty, \\ \delta_n \zeta^{\delta_n}(0) &\rightarrow -\lambda_{l,R} && \text{as } n \rightarrow +\infty, \\ m_i^{\delta_n} - m_i^{\delta_n}(0) &\rightarrow m_i^{l,R} && \text{as } n \rightarrow +\infty \quad \text{for } i = 1, 2. \end{aligned}$$

The last convergence being locally uniform. Let us consider,

$$\overline{w}^{l,R} = \limsup_{\delta_n \rightarrow 0}^* (v^{\delta_n} - v^{\delta_n}(0)) \quad \text{and} \quad \underline{w}^{l,R} = \liminf_{\delta_n \rightarrow 0} (v^{\delta_n} - v^{\delta_n}(0))$$

and

$$\overline{\chi}^{l,R} = \limsup_{\delta_n \rightarrow 0}^* (\zeta^{\delta_n} - v^{\delta_n}(0)) \quad \text{and} \quad \underline{\chi}^{l,R} = \liminf_{\delta_n \rightarrow 0} (\zeta^{\delta_n} - v^{\delta_n}(0)).$$

Therefore, we have that $\lambda_{l,R}$, $\overline{w}^{l,R}$, $\underline{w}^{l,R}$, $\overline{\chi}^{l,R}$, $\underline{\chi}^{l,R}$ and $m_i^{l,R}$, satisfy for $i = 1, 2$,

$$\begin{aligned} \min_{p \in \mathbb{R}} \overline{H}_0(p) &\leq \lambda_{l,R} \leq 0, \\ |\overline{w}^{l,R} - m_1^{l,R}| &\leq C, \quad |\underline{w}^{l,R} - m_1^{l,R}| \leq C, \\ |\overline{\chi}^{l,R} - m_2^{l,R}| &\leq C, \quad |\underline{\chi}^{l,R} - m_2^{l,R}| \leq C, \\ &|\partial_x m_i^{l,R}| \leq C, \end{aligned} \tag{6.20}$$

and for $x \leq h_0$,

$$|\bar{\chi}^{l,R} - \underline{w}^{l,R}| \leq C, \quad |\underline{\chi}^{l,R} - \bar{w}^{l,R}| \leq C. \quad (6.21)$$

By stability of viscosity solutions, we have that $(\bar{w}^{l,R} - 2C, \bar{\chi}^{l,R} - 2C)$ and $(\underline{w}^{l,R}, \underline{\chi}^{l,R})$ are respectively a sub-solution and a super-solution of (6.1), and

$$\bar{w}^{l,R} - 2C \leq \underline{w}^{l,R} \quad \text{and} \quad \bar{\chi}^{l,R} - 2C \leq \underline{\chi}^{l,R}.$$

By Perron's method, we can construct a solution $(w^{l,R}, \chi^{l,R})$ of (6.1) and thanks to (6.20)-(6.21), $m^{l,R}$, $w^{l,R}$, $\chi^{l,R}$ and $\lambda_{l,R}$ satisfy (6.7) in which the first inequality is replaced by

$$\min_{p \in \mathbb{R}} \bar{H}_0(p) \leq \lambda_{l,R} \leq 0.$$

The first inequality of (6.7) comes from the fact that near $\{-l\}$ and $\{l\}$, (6.1) contains only local operators \bar{H}_i for $i = 0, 1, 2$. Indeed, using [AHFM13, Lemme B.1], we can touch from above, near $\{-l\}$ or $\{l\}$, the functions $(w^{l,R}, \chi^{l,R})$ with a regular test function φ . This implies that $\bar{H}_i(\varphi_x) \leq \lambda_{l,R}$ for $i = 0, 1$ or 2 which gives

$$\max_{i \in \{0,1,2\}} \min_{p \in \mathbb{R}} \bar{H}_i(p) \leq \lambda_{l,R}. \quad (6.22)$$

The uniqueness of $\lambda_{l,R}$ is classical so we skip it. This ends the proof of Proposition 6.3. \square

Proof of Lemma 6.5. This proof uses similar arguments to ones used in the proof of Theorem 3.9. We only prove that for all $x \in [-l, -h_0]$ we have $\zeta^\delta(x) - v^\delta(x) \leq 1$, the proof for the second part of the result is similar and we skip it. First, let us prove the inequality on the interval $[-R, -h_0]$.

Step 1: $x \in [-R, -h_0]$. Take $z \in (-R, -h_0)$ and let $\alpha > 0$ be some small parameter such that $z \in [-R + \alpha, -h_0 - \alpha]$. We denote by

$$\Delta = \{x \in [-l, l] \text{ s.t. } |z - x| \leq \alpha\}.$$

We want to prove that for $L := 4C_0/(\delta\alpha^2)$, we have

$$\zeta^\delta(x) - v^\delta(x) \leq 1 + L(x - z)^2 \quad \text{for all } x \in \Delta. \quad (6.23)$$

In fact, if we take $x = z$ in the previous inequality, we obtain

$$\zeta^\delta(x) - v^\delta(x) \leq 1 \quad \text{for all } z \in (-R, -h_0)$$

Using the continuity of v^δ and ζ^δ , we deduce the result in $[-R, -h_0]$.

In order to prove (6.23), we introduce

$$M = \sup_{x \in \Delta} \left(\zeta^\delta(x) - v^\delta(x) - 1 - L(x - z)^2 \right). \quad (6.24)$$

We argue by contradiction and assume that $M > 0$.

Step 1.1: the test function. We define for ε a small parameter, the function

$$\varphi(x, y) = \zeta^\delta(x) - v^\delta(y) - 1 - \frac{L}{2}(x - z)^2 - \frac{L}{2}(y - z)^2 - \frac{(x - y)^2}{2\varepsilon}.$$

The function φ reaches a maximum at a finite point (\bar{x}, \bar{y}) in the domain

$$\{(x, y) \in [-l, l]^2 \text{ s.t. } |z - x| \leq \alpha \text{ and } |z - y| \leq \alpha\}. \quad (6.25)$$

By classical arguments, we deduce that

$$\begin{cases} 0 < M \leq M_\varepsilon \\ |\bar{x} - \bar{y}| \rightarrow 0 \text{ as } \varepsilon \rightarrow 0 \\ \frac{(\bar{x} - \bar{y})^2}{2\varepsilon} \rightarrow 0 \text{ as } \varepsilon \rightarrow 0. \end{cases} \quad (6.26)$$

Step 1.2: $|\bar{x} - z| < \alpha$ **and** $|\bar{y} - z| < \alpha$. By contradiction, assume that $|\bar{x} - z| = \alpha$ or $|\bar{y} - z| = \alpha$, in which case, we have

$$\begin{aligned} 0 < \varphi(\bar{x}, \bar{y}) &= \zeta^\delta(\bar{x}) - v^\delta(\bar{y}) - 1 - \frac{L}{2}(\bar{x} - z)^2 - \frac{L}{2}(\bar{y} - z)^2 - \frac{(\bar{x} - \bar{y})^2}{2\varepsilon} \\ &\leq 2\frac{C_0}{\delta} - \frac{L}{2}\alpha^2 = 0, \end{aligned} \quad (6.27)$$

where we have used (6.9) and the fact that $L = 4C_0/(\delta\alpha^2)$ for the second line, which gives us a contradiction.

Step 1.3: viscosity inequalities. Using the viscosity inequalities we obtain

$$\begin{aligned} \delta\zeta^\delta(\bar{x}) + G_R^2(\bar{x}, \zeta^\delta(\bar{x}), [v^\delta], [\zeta^\delta], L(\bar{x} - z) + p_\varepsilon) &\leq 0 \\ \delta v^\delta(\bar{y}) + \tilde{G}_R^1(\bar{y}, v^\delta(\bar{y}), [\zeta^\delta], [v^\delta], -L(\bar{y} - z) + p_\varepsilon) &\geq 0, \end{aligned}$$

with $p_\varepsilon = (\bar{x} - \bar{y})/\varepsilon$. Combining these inequalities and using the definition of M , we obtain

$$\begin{aligned} \delta M &\leq \tilde{G}_R^1(\bar{y}, v^\delta(\bar{y}), [\zeta^\delta], [v^\delta], -L(\bar{y} - z) + p_\varepsilon) - G_R^2(\bar{x}, \zeta^\delta(\bar{x}), [v^\delta], [\zeta^\delta], L(\bar{x} - z) + p_\varepsilon) \\ &\leq \phi(\bar{y}, \tilde{N}_0(v^\delta, [\zeta^\delta])(\bar{y}), \tilde{M}_1[v^\delta](\bar{y})) \cdot |-L(\bar{y} - z) + p_\varepsilon| \\ &\quad - \phi(\bar{x}, N_0(\zeta^\delta, [v^\delta])(\bar{x}), M_2[\zeta^\delta](\bar{x})) \cdot |L(\bar{x} - z) + p_\varepsilon| \\ &\leq -\phi(\bar{x}, N_0(\zeta^\delta, [v^\delta])(\bar{x}), M_2[\zeta^\delta](\bar{x})) \cdot |L(\bar{x} - z) + p_\varepsilon|, \end{aligned} \quad (6.28)$$

where we have used the fact that the non-local operators are non-positive (Remark 2.9) and that $\bar{x}, \bar{y} \in [-R, -h_0]$ which implies that $\psi_R^\pm(\bar{y}) = \psi_R^\pm(\bar{x}) = 1$.

Let us prove that $N_0(\zeta^\delta, [v^\delta])(\bar{x}) = 0$. It is sufficient to prove that for all $h \in [h_0, h_{max}]$, we have

$$v^\delta(\bar{x} + h) - \zeta^\delta(\bar{x}) < -1.$$

In fact, using (6.26), for ε small enough, we can assume that for all $h \in [h_0, h_{max}]$, we have $\bar{x} + h > \bar{y}$. Using the fact that v^δ is non-increasing and that $M > 0$, we have

$$v^\delta(\bar{x} + h) - \zeta^\delta(\bar{x}) \leq v^\delta(\bar{y}) - \zeta^\delta(\bar{x}) < -1.$$

Using that the non-local operators are non-positive, we have that

$$\max(N_0(\zeta^\delta, [v^\delta])(\bar{x}), M_2[\zeta^\delta](\bar{x})) = 0.$$

Given the form of the transition function (see Remark 2.10), (6.28) becomes

$$\delta M \leq -\tilde{\phi}(\bar{x}, 0, 0, M_2[\zeta^\delta](\bar{x})) \cdot |L(\bar{x} - z) + p_\varepsilon| = 0. \tag{6.29}$$

This gives us a contradiction and therefore $M \leq 0$.

Step 2: $x \in [-l, -R]$. We introduce

$$M = \sup_{x \in [-l, -R]} (\zeta^\delta(x) - v^\delta(x) - 1). \tag{6.30}$$

We want to prove that $M \leq 0$ and we argue by contradiction and by assuming that $M > 0$. Given that ζ^δ and v^δ are continuous, M is reached for a point $\bar{x} \in [-l, -R]$. First, let us notice that thanks to Step 1, M cannot be reached for $\bar{x} = -R$. We distinguish 2 cases.

Case 1: $\bar{x} \in (-l, -R)$. We introduce

$$\varphi(x, y) = \zeta^\delta(x) - v^\delta(x) - 1 - \frac{(x - y)^2}{2\varepsilon} - \frac{1}{2} \left((x - \bar{x})^2 + (y - \bar{x})^2 \right),$$

which reaches a maximum at a point $(x_\varepsilon, y_\varepsilon) \in [-l, -R]^2$. We denote by $M_\varepsilon = \varphi(x_\varepsilon, y_\varepsilon)$. By classical arguments, we can prove that

$$\lim_{\varepsilon \rightarrow 0} M_\varepsilon = M, \quad \lim_{\varepsilon \rightarrow 0} \frac{(x_\varepsilon - y_\varepsilon)^2}{2\varepsilon} = 0, \quad \text{and} \quad (x_\varepsilon, y_\varepsilon) \rightarrow (\bar{x}, \bar{x}) \text{ as } \varepsilon \text{ goes to } 0. \tag{6.31}$$

Therefore, for ε small enough, we can assume that $x_\varepsilon, y_\varepsilon \in (-l, -R)$. Using the viscosity inequalities, we obtain

$$\begin{aligned} \delta \zeta^\delta(x_\varepsilon) + G_R^2(x_\varepsilon, \zeta^\delta(x_\varepsilon), [v^\delta], [\zeta^\delta], (x_\varepsilon - \bar{x}) + p_\varepsilon) &\leq 0 \\ \delta v^\delta(y_\varepsilon) + \tilde{G}_R^1(y_\varepsilon, v^\delta(y_\varepsilon), [\zeta^\delta], [v^\delta], -(y_\varepsilon - \bar{x}) + p_\varepsilon) &\geq 0, \end{aligned}$$

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with $p_\varepsilon = (x_\varepsilon - y_\varepsilon)/\varepsilon$. Combining these inequalities and using the definition of M , we obtain

$$\begin{aligned}
\delta M &\leq \tilde{G}_R^1(y_\varepsilon, v^\delta(y_\varepsilon), [\zeta^\delta], [v^\delta], -(y_\varepsilon - \bar{x}) + p_\varepsilon) \\
&\quad - G_R^2(x_\varepsilon, \zeta^\delta(\bar{x}), [v^\delta], [\zeta^\delta], (x_\varepsilon - \bar{x}) + p_\varepsilon) \\
&\leq (1 - \psi_R^-(y_\varepsilon))\overline{H}_0(-(y_\varepsilon - \bar{x}) + p_\varepsilon) - (1 - \psi_R^-(x_\varepsilon))\overline{H}_0((x_\varepsilon - \bar{x}) + p_\varepsilon) \\
&\quad + \psi_R^-(y_\varepsilon)\phi\left(y_\varepsilon, \tilde{N}_0(v^\delta, [\zeta^\delta])(y_\varepsilon), \tilde{M}_1^\delta[v^\delta](y_\varepsilon)\right) \cdot |-(y_\varepsilon - \bar{x}) + p_\varepsilon| \\
&\quad - \psi_R^-(x_\varepsilon)\phi\left(x_\varepsilon, N_0(\zeta^\delta, [v^\delta])(x_\varepsilon), M_2[\zeta^\delta](x_\varepsilon)\right) \cdot |(x_\varepsilon - \bar{x}) + p_\varepsilon| \\
&\leq (1 - \psi_R^-(y_\varepsilon))\overline{H}_0(-(y_\varepsilon - \bar{x}) + p_\varepsilon) - (1 - \psi_R^-(x_\varepsilon))\overline{H}_0((x_\varepsilon - \bar{x}) + p_\varepsilon) \\
&\quad - \psi_R^-(x_\varepsilon)N_0(\zeta^\delta, [v^\delta])(x_\varepsilon) \cdot |(x_\varepsilon - \bar{x}) + p_\varepsilon|,
\end{aligned} \tag{6.32}$$

where we have used that $\phi(x_\varepsilon, a, b) = a$ for $x_\varepsilon \leq -h_0 - h_{max} - 1$.

Using that $N_0(\zeta^\delta, [v^\delta])(x_\varepsilon) = 0$ (see Step 1), we get

$$\delta M \leq (1 - \psi_R^-(y_\varepsilon))\overline{H}_0(-(y_\varepsilon - \bar{x}) + p_\varepsilon) - (1 - \psi_R^-(x_\varepsilon))\overline{H}_0(x_\varepsilon - \bar{x} + p_\varepsilon). \tag{6.33}$$

This implies that

$$\begin{aligned}
\delta M &\leq (1 - \psi_R^-(y_\varepsilon))\left(\overline{H}_0(-(y_\varepsilon - \bar{x}) + p_\varepsilon) - \overline{H}_0((x_\varepsilon - \bar{x}) + p_\varepsilon)\right) \\
&\quad + \|(\psi_R^-)'\|_\infty |x_\varepsilon - y_\varepsilon| \left|\overline{H}_0((x_\varepsilon - \bar{x}) + p_\varepsilon)\right| \\
&\leq \|\overline{H}'_0\|_\infty (|x_\varepsilon - \bar{x}| + |y_\varepsilon - \bar{x}|) + \|(\psi_R^-)'\|_\infty |x_\varepsilon - y_\varepsilon| \cdot V_{max}^0 |x_\varepsilon - \bar{x}| + p_\varepsilon| \\
&\leq o_\varepsilon(1)
\end{aligned} \tag{6.34}$$

where we have used the regularity of ψ_R^- for the first inequality, the form of \overline{H}_0 (the fact that $|\overline{H}_0(p)| \leq V_{max}^0 |p|$, see (2.10)), and (6.31) for the last inequality.

Injecting (6.34) into (6.33), we obtain that

$$\delta M \leq o_\varepsilon(1),$$

which gives a contradiction for ε small enough.

Case 2: $\bar{x} = -l$. We introduce

$$\varphi(x, y) = \zeta^\delta(x) - v^\delta(x) - 1 - \varepsilon G^\gamma\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) - \frac{1}{2}\left((x - \bar{x})^2 + (y - \bar{x})^2\right),$$

where G^γ is the vertex test function introduced by Imbert and Monneau in [IM14, Theorem 3.2]. We fix $\gamma = \delta M/2$. We refer to [IM14] for a detailed description on the vertex test function and on how to build it, but for the readers convenience we recall the properties we use in this proof.

1. (Regularity)

$$G^\gamma \in C([-l, l]^2) \quad \begin{cases} G^\gamma(x, \cdot) \in C^1([-l, l]) & \text{for all } x \in [-l, l] \\ G^\gamma(\cdot, y) \in C^1([-l, l]) & \text{for all } y \in [-l, l]. \end{cases} \quad (6.35)$$

2. (Bound from below) $G^\gamma \geq 0 = G(0, 0)$.

3. (Super-linearity) There exists $g : [0, +\infty) \rightarrow \mathbb{R}$ non-decreasing and such that for all $(x, y) \in [-l, l]^2$

$$g(|x - y|) \leq G^\gamma(x, y) \quad \text{and} \quad \lim_{a \rightarrow +\infty} \frac{g(a)}{a} = +\infty.$$

4. (Compatibility condition on the gradient)

$$\overline{H}_0(y, -G_y^\gamma(x, y)) - \overline{H}_0(x, G_x^\gamma(x, y)) \leq \gamma, \quad (6.36)$$

with, for all $x \in [-l, l]$ and $p \in \mathbb{R}$,

$$\overline{H}_0(x, p) = \begin{cases} \overline{H}_0^-(p) & \text{if } x = -l \\ \overline{H}_0(p) & \text{if } x \in (-l, l] \end{cases} \quad (6.37)$$

Like in the previous case, the test function φ reaches a maximum at a point $(x_\varepsilon, y_\varepsilon)$ and (6.31) remains valid with $\bar{x} = -l$. Proceeding like before, we obtain (6.32) but for ε small enough the only terms that remain are the local ones. Therefore, we have

$$\begin{aligned} \delta M &\leq \overline{H}_0\left(y_\varepsilon, -(y_\varepsilon - \bar{x}) - G_y^\gamma\left(\frac{x_\varepsilon}{\varepsilon}, \frac{y_\varepsilon}{\varepsilon}\right)\right) - \overline{H}_0\left(x_\varepsilon, x_\varepsilon - \bar{x} + G_x^\gamma\left(\frac{x_\varepsilon}{\varepsilon}, \frac{y_\varepsilon}{\varepsilon}\right)\right) \\ &\leq \overline{H}_0\left(y_\varepsilon, -G_y^\gamma\left(\frac{x_\varepsilon}{\varepsilon}, \frac{y_\varepsilon}{\varepsilon}\right)\right) - \overline{H}_0\left(x_\varepsilon, G_x^\gamma\left(\frac{x_\varepsilon}{\varepsilon}, \frac{y_\varepsilon}{\varepsilon}\right)\right) + \|\overline{H}'_0\|_\infty (|x_\varepsilon + l| + |y_\varepsilon + l|) \\ &\leq \gamma + o_\varepsilon(1). \end{aligned} \quad (6.38)$$

This implies that $\delta M/2 \leq o_\varepsilon(1)$ and we obtain a contradiction for ε small enough. \square

Proposition 6.7 (First definition of the flux limiter). *The following limits exist (up to some sub-sequence)*

$$\begin{cases} \overline{A}_R = \lim_{l \rightarrow +\infty} \lambda_{l,R}, \\ \overline{A} = \lim_{R \rightarrow +\infty} \overline{A}_R. \end{cases} \quad (6.39)$$

Moreover, we have

$$H_0 \leq \overline{A}, \overline{A}_R \leq 0.$$

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Proof. This proof is a direct consequence of (6.7). □

Proposition 6.8 (Control of the slopes on a truncated domain). *Assume that l and R are large enough. Let $(w^{l,R}, \chi^{l,R})$ be the solution of (6.1) given by Proposition 6.3.*

If we assume up to a sub-sequence that $\bar{A} = \lim_{R \rightarrow +\infty} \lim_{l \rightarrow +\infty} \lambda_{l,R} > \min \bar{H}_1$. Then there exist $\gamma_0 > 0$ and a constant $C > 0$ (independent of l and R) such that for all $\gamma \in (0, \gamma_0)$ and for all $x \geq 0$ and $h \geq 0$

$$w^{l,R}(x+h) - w^{l,R}(x) \geq (\bar{p}_+^1 - \gamma)h - C \quad (6.40)$$

Similarly, if we assume that $\bar{A} > \min \bar{H}_2$ then we have for all $x \geq 0$ and $h \geq 0$

$$\chi^{l,R}(x+h) - \chi^{l,R}(x) \geq (\bar{p}_+^2 - \gamma)h - C. \quad (6.41)$$

Finally, if we assume that $\bar{A} > \min \bar{H}_0$, we have for all $x \leq -h_0 - h_{max} - 1$ and $h \geq 0$,

$$w^{l,R}(x-h) - w^{l,R}(x) \geq (-\bar{p}_-^0 - \gamma)h - C \quad (6.42)$$

and

$$\chi^{l,R}(x-h) - \chi^{l,R}(x) \geq (-\bar{p}_-^0 - \gamma)h - C. \quad (6.43)$$

Proof. We do the proof of Proposition 6.8 in two steps.

Step 1: proof of (6.40)-(6.41). We do the proof only for $w^{l,R}$, since the truncated cell problem (6.1) decouples itself for $x > 0$. The proof for $\chi^{l,R}$ is similar and we skip it. For $\mu > 0$, small enough, we denote by p_μ^1 the real numbers defined by

$$\bar{H}_1(p_\mu^1) = \bar{H}_1^+(p_\mu^1) = \lambda_{l,R} - \mu. \quad (6.44)$$

Using that

$$\min \bar{H}_1 < \lambda_{l,R} \leq 0,$$

we deduce that p_μ^1 exists for μ small enough and $p_\mu^1 \in [-2k_0, 0)$.

Let us now consider $w^+ = p_\mu^1 x$ that satisfies

$$\bar{H}_1(w_x^+) = \lambda_{l,R} - \mu \quad \text{for } x \in \mathbb{R}. \quad (6.45)$$

We also have

$$\begin{aligned} M_1[w^+](x) &= \int_{\mathbb{R}} J_1(z) E(p_\mu^1(x+z) - p_\mu^1 x) dz - \frac{3}{2} V_{max}^1 \\ &= \int_0^{\frac{-2}{p_\mu^1}} \frac{1}{2} J_1(z) dz + \int_{\frac{-2}{p_\mu^1}}^{+\infty} \frac{3}{2} J_1(z) dz - \frac{3}{2} V_{max}^1 \\ &= -V_1 \left(\frac{-2}{p_\mu^1} \right). \end{aligned}$$

For all $x \in (0, l)$, we have

$$M_1[w^+](x) \cdot |w_x^+| = -V_1 \left(\frac{-2}{p_\mu^1} \right) \cdot |p_\mu^1| = \overline{H}_1(p_\mu^+) = \lambda_{l,R} - \mu. \quad (6.46)$$

Using (6.46), we can see that the restriction of w^+ to $(0, l]$ satisfies

$$\begin{cases} G_R^1(x, w^+, [\chi^+], [w^+], w_x^+) = \lambda_{l,R} - \mu & \text{if } x \in (0, l) \\ \overline{H}_1^+(w_x^+) = \lambda_{l,R} - \mu & \text{if } x = l, \end{cases}$$

notice that since $x > 0$ there is no actual dependence on χ^+ on the previous equation.

Let us introduce, for some $x_0 \in (0, l]$,

$$g = w^{l,R} - w^{l,R}(x_0) \quad \text{and} \quad u = w^+ - w^+(x_0) \quad (6.47)$$

Then we have

$$g(x_0) = 0 \geq u(x_0). \quad (6.48)$$

Using that g is a solution of (6.4) (in $(0, l]$ the solutions are invariant by addition of constants) and u is a solution of (6.5) (with $\varepsilon_0 = \mu$), joint to the comparison principle (Proposition 6.1), we get that

$$w^{l,R}(x) - w^{l,R}(x_0) \geq p_\mu^1(x - x_0)$$

This implies that for all $h \geq 0$, and for all $x \in (0, l)$,

$$w^{l,R}(x+h) - w^{l,R}(x) \geq p_\mu^1 h$$

Finally, if we choose $\gamma_0 < |p_1 - \overline{p}_+^1|$, then we have

$$\overline{H}_1(\overline{p}_+^1 - \gamma) = \overline{H}_1^+(\overline{p}_+^1 - \gamma).$$

Choosing $\mu > 0$ such that

$$p_\mu^1 = \overline{p}_+^1 - \gamma.$$

we obtain (6.40).

Step 2: proof of (6.42) and (6.43). The arguments are similar to the previous ones but for the readers convenience, we detail where the two proofs differ. For $\mu > 0$, small enough, we denote by p_μ^0 a solution of

$$\overline{H}_0(p_\mu^0) = \overline{H}_0^-(p_\mu^0) = \lambda_{l,R} - \mu$$

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Moreover, since $\min \bar{H}_0 < \lambda_{l,R}$, for μ small enough, we have $p_\mu^0 \in [-2k_0, 0)$. We introduce $w^- = p_\mu^0 x = \chi^-$. Like before, we can see that the restriction of w^- to $[-l, -h_0 - h_{max} - 1)$ satisfies

$$\begin{cases} G_R^1(x, w^-, [\chi^-], [w^-], w_x^-) = \lambda_{l,R} - \mu & \text{if } x \in (-l, -h_0 - h_{max} - 1) \\ \bar{H}_0^-(w_x^-) = \lambda_{l,R} - \mu & \text{if } x = -l. \end{cases}$$

Similarly, the restriction of χ^- to $[-l, -h_{max} - h_0 - 1)$ satisfies

$$\begin{cases} G_R^2(x, \chi^-, [w^-], [\chi^-], \chi_x^-) = \lambda_{l,R} - \mu & \text{if } x \in (-l, -h_0 - h_{max} - 1) \\ \bar{H}_0^-(\chi_x^-) = \lambda_{l,R} - \mu & \text{if } x = -l. \end{cases}$$

Let us introduce, for some $x_0 \in [-l, -h_0 - h_{max} - 1)$,

$$\begin{cases} g = w^{l,R} - w^{l,R}(x_0) \\ h = \chi^{l,R} - w^{l,R}(x_0), \end{cases} \quad \text{and} \quad \begin{cases} u = w^- - w^-(x_0) - \tilde{C} \\ v = \chi^- - \chi^-(x_0) - \tilde{C}, \end{cases} \quad (6.49)$$

with $\tilde{C} = Ch_{max} + 3C$ where $C > 0$ is the constant provided by Proposition 6.3. Then, using Proposition 6.3, we have for all $x \in [x_0, x_0 + h_{max}]$

$$\begin{aligned} g(x) = w^{l,R}(x) - w^{l,R}(x_0) &\geq -C|x - x_0| - 2C \geq -Ch_{max} - 2C \geq -\tilde{C} \\ &\geq -\tilde{C} + p_\mu^0(x - x_0) = u(x) \end{aligned}$$

and

$$\begin{aligned} h(x) = \chi^{l,R}(x) - w^{l,R}(x_0) &= \chi^{l,R}(x) - \chi^{l,R}(x_0) + \chi^{l,R}(x_0) - w^{l,R}(x_0) \\ &\geq -Ch_{max} - 2C - C \geq -\tilde{C} \geq -\tilde{C} + p_\mu^0(x - x_0) = v(x). \end{aligned}$$

Using the comparison principle (Proposition 6.1) with the boundary conditions (6.6) (see Remark 6.2), we get that

$$\begin{cases} w^{l,R}(x) - w^{l,R}(x_0) \geq p_\mu^0(x - x_0) - C \\ \chi^{l,R}(x) - w^{l,R}(x_0) \geq p_\mu^0(x - x_0) - C. \end{cases}$$

Using Proposition 6.3, we have that $|\chi^{l,R}(x_0) - w^{l,R}(x_0)| \leq C$. Therefore, up to changing the constant $\tilde{C} > 0$, we have

$$\begin{cases} w^{l,R}(x) - w^{l,R}(x_0) \geq p_\mu^0(x - x_0) - \tilde{C} \\ \chi^{l,R}(x) - \chi^{l,R}(x_0) \geq p_\mu^0(x - x_0) - \tilde{C}. \end{cases}$$

This implies that for all $h \geq 0$, and for all $x \in (-l, -h_0 - h_{max} - 1)$,

$$\begin{cases} w^{l,R}(x - h) - w^{l,R}(x) \geq -p_\mu^0 h - \tilde{C} \\ \chi^{l,R}(x - h) - \chi^{l,R}(x) \geq -p_\mu^0 h - \tilde{C}. \end{cases}$$

Finally, if we choose $\gamma_0 < |p_0 - \bar{p}_-^0|$, then we have

$$\bar{H}_0(\bar{p}_-^0 + \gamma) = \bar{H}_0^-(\bar{p}_-^0 + \gamma).$$

Choosing $\mu > 0$ such that

$$p_\mu^0 = \bar{p}_+^0 + \gamma.$$

we obtain (6.42) and (6.43) . □

Proof of Theorem 4.3. The proof is performed in two steps.

Step 1: proof of i) and ii). We want to pass to the limit as $l \rightarrow +\infty$ and then as $R \rightarrow +\infty$ for the solution of (6.1) given by Proposition 6.3. Using (6.3), there exists $l_n \rightarrow +\infty$, such that for $i = 1, 2$, we have

$$m_i^{l_n, R} - m_i^{l_n, R}(0) \rightarrow m_i^R \quad \text{as } n \rightarrow +\infty,$$

the convergence being locally uniform. We also define

$$\begin{aligned} \bar{w}^R(x) &= \limsup_{n \rightarrow +\infty}^* (w^{l_n, R} - w^{l_n, R}(0)), & \bar{\chi}^R(x) &= \limsup_{n \rightarrow +\infty}^* (\chi^{l_n, R} - w^{l_n, R}(0)), \\ \underline{w}^R(x) &= \liminf_{n \rightarrow +\infty} (w^{l_n, R} - w^{l_n, R}(0)), & \underline{\chi}^R(x) &= \liminf_{n \rightarrow +\infty} (\chi^{l_n, R} - w^{l_n, R}(0)). \end{aligned} \quad \text{and}$$

Thanks to (6.3), we know that these limits are finite and satisfy

$$m_1^R - C \leq \underline{w}^R \leq \bar{w}^R \leq m_1^R + C. \quad \text{and} \quad m_2^R - C \leq \underline{\chi}^R \leq \bar{\chi}^R \leq m_2^R + C.$$

By stability of viscosity solutions $(\bar{w}^R - 2C, \bar{\chi}^R - 2C)$ and $(\underline{w}^R, \underline{\chi}^R)$ are respectively a sub-solution and a super-solution of

$$\begin{cases} G_R^1(x, w^R(x), [\chi^R], [w^R], w_x^R) = \bar{A}_R \\ G_R^2(x, \chi^R(x), [w^R], [\chi^R], \chi_x^R) = \bar{A}_R \end{cases} \quad (6.50)$$

Therefore, using Perron's method, we can construct a solution (w^R, χ^R) of (6.50), with m^R, \bar{A}_R, w^R and χ^R satisfying

$$\begin{cases} |m_i^R(x) - m_i^R(y)| \leq C|x - y| & \text{for all } x, y \in \mathbb{R}, \text{ for } i = 1, 2, \\ |w^R(x) - m_1^R(x)| \leq C, |\chi^R(x) - m_2^R(x)| \leq C & \text{for all } x \in \mathbb{R}, \\ |w^R(x) - \chi^R(x)| \leq C & \text{for all } x \leq h_0, \\ H_0 \leq \bar{A}_R \leq 0. \end{cases} \quad (6.51)$$

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Using Proposition 6.8, if $\bar{A} > \bar{H}_1$ (resp. $\bar{A} > \min \bar{H}_2$), we know that there exists γ_0 and a constant $C > 0$, such that for all $\gamma \in (0, \gamma_0)$, for all $x \geq 0$, and $h \geq 0$,

$$w^R(x+h) - w^R(x) \geq (\bar{p}_+^1 - \gamma)h - C \quad \left(\text{resp. } \chi^R(x+h) - \chi^R(x) \geq (\bar{p}_+^1 - \gamma)h - C \right) \quad (6.52)$$

Similarly, if we assume $\bar{A} > \min \bar{H}_0$, then we have similarly, for all $x \leq -h_0 - h_{max} - 1$ and $h \geq 0$,

$$w^R(x-h) - w^R(x) \geq (-\bar{p}_-^0 - \gamma)h - C \quad \text{and} \quad \chi^R(x-h) - \chi^R(x) \geq (-\bar{p}_+^0 - \gamma)h - C \quad (6.53)$$

We now pass to the limit as $R \rightarrow +\infty$. We consider (up to a sub-sequence), for $i = 1, 2$,

$$\left\{ \begin{array}{l} \bar{w}(x) = \limsup_{R \rightarrow +\infty}^* (w^R - w^R(0)), \quad \underline{w}(x) = \liminf_{R \rightarrow +\infty} (w^R - w^R(0)), \\ \bar{\chi}(x) = \limsup_{R \rightarrow +\infty}^* (\chi^R - w^R(0)), \quad \underline{\chi}(x) = \liminf_{R \rightarrow +\infty} (\chi^R - w^R(0)), \\ \bar{A} = \lim_{R \rightarrow +\infty} \bar{A}_R, \quad m_i = \lim_{R \rightarrow +\infty} (m_i^R - m_i^R(0)). \end{array} \right.$$

The last convergence being locally uniform. Thanks to (6.51), we know that the previous limits are finite and that

$$m_1 - C \leq \underline{w} \leq \bar{w} \leq m_1 + C. \quad \text{and} \quad m_2 - C \leq \underline{\chi} \leq \bar{\chi} \leq m_2 + C. \quad (6.54)$$

By stability of viscosity solutions $(\bar{w} - 2C, \bar{\chi} - 2C)$ and $(\underline{w}, \underline{\chi})$ are respectively sub-solution and super-solution of (2.25) with $\lambda = \bar{A}$. Using Perron's method we can construct a solution (w, χ) of (2.25) with $\lambda = \bar{A}$ that satisfies (4.5), (4.6), (4.7) and (4.8).

Step 2: proof of iii). Let us now consider the rescaled functions $w^\varepsilon = \varepsilon w(x/\varepsilon)$ and $\chi^\varepsilon(x) = \varepsilon \chi(x/\varepsilon)$. Using (4.5), we have that

$$w^\varepsilon(x) = \varepsilon m_1 \left(\frac{x}{\varepsilon} \right) + O(\varepsilon) \quad \text{and} \quad \chi^\varepsilon(x) = \varepsilon m_2 \left(\frac{x}{\varepsilon} \right) + O(\varepsilon). \quad (6.55)$$

Therefore, there exists a subsequence $\varepsilon_n \rightarrow 0$ as $n \rightarrow +\infty$, such that

$$w^{\varepsilon_n} \rightarrow W \quad \text{and} \quad \chi^{\varepsilon_n} \rightarrow X \quad \text{locally uniformly as } n \rightarrow +\infty, \quad (6.56)$$

with $W(0) = X(0) = 0$. Arguing as in the proof of convergence away from the junction point, we have that (W, X) satisfies

$$\left\{ \begin{array}{l} \left\{ \begin{array}{l} \bar{H}_0(W_x) = \bar{A} \\ \bar{H}_0(X_x) = \bar{A} \end{array} \right. \quad \text{for } x < 0, \\ \left\{ \begin{array}{l} \bar{H}_1(W_x) = \bar{A} \\ \bar{H}_2(X_x) = \bar{A} \end{array} \right. \quad \text{for } x > 0. \end{array} \right.$$

This proves (4.9). Let us now prove (4.11).

For $x < 0$, we have for all $\gamma \in (0, \gamma_0)$, if $\bar{A} > \min \bar{H}_0$,

$$W_x \leq \bar{p}_-^0 + \gamma \quad \text{and} \quad X_x \leq \bar{p}_-^0 + \gamma,$$

where we have used (4.8). Therefore, we have

$$\hat{p}_-^0 \leq W_x, X_x \leq \bar{p}_-^0, \tag{6.57}$$

this inequality remains valid if $\bar{A} = \min \bar{H}_0$ (in which case, given the form of \bar{H}_0 , we have $W_x = X_x = \hat{p}_-^0 = \bar{p}_-^0 = p_0$).

For $x > 0$, we have for all $\gamma \in (0, \gamma_0)$, if $\bar{A} > \min \bar{H}_1$ (resp. $\bar{A} > \min \bar{H}_2$),

$$W_x \geq \bar{p}_+^1 - \gamma \quad (\text{resp. } X_x \geq \bar{p}_+^2 - \gamma),$$

where we have used (4.7). Therefore given the form of \bar{H}_1 (resp. of \bar{H}_2), we have that $W_x = \bar{p}_+^1$ (resp. $X_x = \bar{p}_+^2$) for $x > 0$, this result is still valid if $\bar{A} = \min \bar{H}_1$ (resp. $\bar{A} = \bar{H}_2$).

Combining these results, we obtain (4.11). □

Proof of Theorem 2.8. Up to a sub-sequence, we assume that $\bar{A} = \lim_{R \rightarrow +\infty} \lim_{l \rightarrow +\infty} \lambda_{l,R}$. We want to prove that $\bar{A} = \inf E$, with

$$E = \{\lambda \in [H_0, 0] : \exists (v, \zeta) \in \mathcal{S} \text{ solution of (2.25)}\},$$

and

$$\mathcal{S} = \{(v, \zeta) \text{ s.t. } \exists \text{ two Lipschitz continuous functions } m_1 \text{ and } m_2 \text{ s.t. } \\ m_1(0) = m_2(0) = 0 \text{ and a constant } C > 0 \text{ s.t. } \|v - m_1\|_\infty, \|\zeta - m_2\|_\infty \leq C\}.$$

We argue by contradiction and assume that there exists $\lambda \in E$ such that $\lambda < \bar{A}$, associated with $(v^\lambda, \zeta^\lambda)$ solution of (2.25). Arguing as in the proof of Theorem 4.3, we deduce that the functions

$$v_\lambda^\varepsilon(x) = \varepsilon v^\lambda\left(\frac{x}{\varepsilon}\right) \quad \text{and} \quad \zeta_\lambda^\varepsilon(x) = \varepsilon \zeta^\lambda\left(\frac{x}{\varepsilon}\right) \tag{6.58}$$

have as limits, as ε goes to 0, W^λ and X^λ (with $W^\lambda(0) = X^\lambda(0) = 0$) which satisfies

$$\left\{ \begin{array}{l} \left\{ \begin{array}{l} \bar{H}_0(W_x^\lambda) = \lambda \\ \bar{H}_0(X_x^\lambda) = \lambda \end{array} \right. \text{ for } x < 0, \\ \left\{ \begin{array}{l} \bar{H}_1(W_x^\lambda) = \lambda \\ \bar{H}_2(X_x^\lambda) = \lambda \end{array} \right. \text{ for } x > 0. \end{array} \right.$$

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This means that for all $x > 0$, we have

$$W_x^\lambda \leq p_+^{\lambda,1} < \bar{p}_+^1 \quad \text{with} \quad \bar{H}_1(p_+^{\lambda,1}) = \bar{H}_1^+(p_+^{\lambda,1}) = \lambda, \quad (6.59)$$

and

$$X_x^\lambda \leq p_+^{\lambda,2} < \bar{p}_+^2 \quad \text{with} \quad \bar{H}_2(p_+^{\lambda,2}) = \bar{H}_2^+(p_+^{\lambda,2}) = \lambda. \quad (6.60)$$

Similarly, for all $x < 0$, we have

$$W_x^\lambda, X_x^\lambda \geq p_-^\lambda > \bar{p}_-^0 \quad \text{with} \quad p_-^\lambda = \min\{p \in [-2k_0, 0] : \bar{H}_0(p) = \bar{H}_0^-(p) = \lambda\}. \quad (6.61)$$

These inequalities imply that for all $\gamma > 0$, there exists a constant \tilde{C}_γ such that

$$v^\lambda(x) \leq \begin{cases} (p_+^{\lambda,1} + \gamma)x + \tilde{C}_\gamma & \text{for } x > 0, \\ (p_-^\lambda - \gamma)x + \tilde{C}_\gamma & \text{for } x < 0. \end{cases} \quad (6.62)$$

and

$$\zeta^\lambda(x) \leq \begin{cases} (p_+^{\lambda,2} + \gamma)x + \tilde{C}_\gamma & \text{for } x > 0, \\ (p_-^\lambda - \gamma)x + \tilde{C}_\gamma & \text{for } x < 0. \end{cases} \quad (6.63)$$

In fact if $(v^\lambda, \zeta^\lambda)$ does not satisfy (6.62) and (6.63), we cannot have (6.59), (6.60), and (6.61). Using Theorem 4.3, we have for γ small enough and for $\tilde{R} > 0$ big enough,

$$v^\lambda \leq v \quad \text{and} \quad \zeta^\lambda < \zeta \quad \text{for } |x| \geq \tilde{R}.$$

This implies that there exists a constant $C_{\tilde{R}}$ such that for all $x \in \mathbb{R}$, we have

$$v^\lambda(x) < v(x) + C_{\tilde{R}} \quad \text{and} \quad \zeta^\lambda(x) < \zeta(x) + C_{\tilde{R}}.$$

Let us now introduce two functions (u, ξ) and (u^λ, ξ^λ) , defined by

$$\begin{cases} u(t, x) = v(x) + C_{\tilde{R}} - \bar{A}t, \\ \xi(t, x) = \zeta(x) + C_{\tilde{R}} - \bar{A}t, \end{cases} \quad \text{and} \quad \begin{cases} u^\lambda(t, x) = v^\lambda(x) - \lambda t, \\ \xi^\lambda(t, x) = \zeta^\lambda(x) - \lambda t. \end{cases}$$

Both functions are solutions of (2.4) (with $\varepsilon = 1$) and

$$u^\lambda(0, x) \leq u(0, x) \quad \text{and} \quad \xi^\lambda(0, x) \leq \xi(0, x).$$

Using the comparison principle (Proposition 3.7), we obtain

$$v^\lambda(x) - \lambda t \leq v(x) - \bar{A}t + C_{\tilde{R}}.$$

Dividing by t and passing to the limit as t goes to infinity, we get $\bar{A} \leq \lambda$, which is a contradiction. □

7 Link between the system of ODEs and the system of PDEs

Theorem 7.1. *For $\varepsilon = 1$, the cumulative distribution function (ρ_1, ρ_2) defined by (2.1)-(2.2), is a discontinuous viscosity solution, for $(t, x) \in [0, +\infty) \times \mathbb{R}$, of*

$$\begin{cases} (\rho_1)_t + \phi(x, N_0(\rho_1, [\rho_2(t, \cdot)])(x), M_1[\rho_1(t, \cdot)](x)) \cdot |\partial_x \rho_1| = 0 \\ (\rho_2)_t + \phi(x, N_0(\rho_2, [\rho_1(t, \cdot)])(x), M_2[\rho_2(t, \cdot)](x)) \cdot |\partial_x \rho_2| = 0. \end{cases} \quad (7.1)$$

Conversely, if (u, ξ) is a bounded and continuous viscosity solution of (7.1) satisfying for some $T > 0$, and for all $t \in (0, T)$,

$$u(t, x) \text{ and } \xi(t, x) \text{ are decreasing in } x,$$

then the points $U_i(t)$, defined by $u(t, U_i(t)) = -(i + 1)$ for $i \in \mathbb{Z}$ if i odd, and defined by $\xi(t, U_i(t)) = -(i + 1)$ for $i \in \mathbb{Z}$ if i even, satisfy the system (1.1) on $(0, T)$.

Before the proof of Theorem 7.1, let us do the proof of Theorem 2.5.

Proof of Theorem 2.5. We recall that in Theorem 2.5, we have

$$u_0(x) = \begin{cases} -x/h & \text{if } x \leq 0 \\ -2x/h_1 & \text{if } x > 0 \end{cases} \quad \text{and} \quad \xi_0(x) = \begin{cases} -x/h & \text{if } x \leq 0 \\ -2x/h_2 & \text{if } x > 0 \end{cases} \quad \text{for all } x \in \mathbb{R}.$$

First, we would like to prove that for all $\varepsilon > 0$, we have

$$|\rho_1^\varepsilon(0, x) - u_0(x)| \leq f(\varepsilon) \quad \text{for all } x \in \mathbb{R} \quad (7.2)$$

and

$$|\rho_1^\varepsilon(0, x) - \xi_0(x)| \leq g(\varepsilon) \quad \text{for all } x \in \mathbb{R}, \quad (7.3)$$

with $f(\varepsilon), g(\varepsilon) \rightarrow 0$ as ε goes to 0. Let us begin by proving (7.2). To do this, we consider a piece-wise affine function v such that

$$\rho_1^1(0, x) = v(x) \quad \text{for } x = U_i(0), \text{ for all } i \in \mathbb{Z} \text{ such that } i[2] = 1. \quad (7.4)$$

First, given that for all $i \in \mathbb{Z}$, $U_{i+1}(0) - U_i(0) \geq h_0$, we notice that v is $2k_0$ -Lipschitz continuous and by definition of ρ_1^1 , we have

$$|\rho_1^1(0, x) - v(x)| \leq 2 \quad \text{for all } x \in \mathbb{R}. \quad (7.5)$$

Let us consider the integer $i_0 \in \mathbb{N}$ defined by

$$i_0 = \sup \{i \in \mathbb{Z}, \text{ s.t. } i[2] = 1, U_i(0) \leq -R\}.$$

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Using the assumption that for all $i \in \mathbb{Z}$ such that $U_i(0) \leq -R$ we have $U_{i+1}(0) - U_i(0) = h$, we deduce that for all $x \leq U_{i_0}(0)$

$$v(x) = -\frac{x}{h} + \frac{U_{i_0}(0)}{h} + \rho_1^1(0, U_{i_0}(0)) = -\frac{x}{h} + \frac{U_{i_0}(0)}{h} - i_0 - 1.$$

Let us now consider the integer $i_1 \in \mathbb{N}$ defined by

$$i_1 = \inf \{i \in \mathbb{Z}, \text{ s.t. } i[2] = 1, U_i(0) \geq R\}.$$

Now using the assumption that for all $i \in \mathbb{Z}$ such that $i[2] = 1$ and $U_i(0) \geq R$ we have $U_{i+2}(0) - U_i(0) = h_1$, we deduce that for all $x \geq U_{i_1}(0)$

$$v(x) = -\frac{2x}{h_1} + \frac{2U_{i_1}(0)}{h_1} + \rho_1^1(0, U_{i_1}(0)) = -\frac{2x}{h_1} + \frac{2U_{i_1}(0)}{h_1} - (i_1 + 1).$$

Moreover, we recall that for all $\varepsilon > 0$, we have $\rho_1^\varepsilon(0, x) = \varepsilon \rho_1^1(0, x/\varepsilon)$, this implies that for all $x \notin [\varepsilon U_{i_0}(0), \varepsilon U_{i_1}(0)]$,

$$\begin{aligned} |\rho_1^\varepsilon(0, x) - u_0(x)| &\leq \left| \rho_1^\varepsilon(0, x) - \varepsilon v\left(\frac{x}{\varepsilon}\right) \right| + \left| \varepsilon v\left(\frac{x}{\varepsilon}\right) - u_0(x) \right| \\ &\leq 2\varepsilon + \varepsilon \max \left(\left| \frac{2U_{i_1}(0)}{h_1} - i_1 - 1 \right|, \left| \frac{U_{i_0}(0)}{h} - i_0 - 1 \right| \right). \end{aligned} \quad (7.6)$$

Similarly, we have for all $x \in [\varepsilon U_{i_0}(0), \varepsilon U_{i_1}(0)]$,

$$\begin{aligned} |\rho_1^\varepsilon(0, x) - u_0(x)| &\leq \left| \rho_1^\varepsilon(0, x) - \varepsilon v\left(\frac{x}{\varepsilon}\right) \right| + \left| \varepsilon v\left(\frac{x}{\varepsilon}\right) - \varepsilon u_0\left(\frac{x}{\varepsilon}\right) \right| \\ &\leq 2\varepsilon + \varepsilon \max_{y \in [U_{i_0}(0), U_{i_1}(0)]} (|v(y) - u_0(y)|), \end{aligned} \quad (7.7)$$

where we have used the fact that $\varepsilon u_0(x/\varepsilon) = u_0(x)$. Combining (7.6) and (7.7) and choosing

$$\begin{aligned} f(\varepsilon) &= 2\varepsilon + \varepsilon \max \left(\left| \frac{2U_{i_1}(0)}{h_1} - i_1 - 1 \right|, \right. \\ &\quad \left. \max_{y \in [U_{i_0}(0), U_{i_1}(0)]} (|v(y) - u_0(y)|), \left| \frac{U_{i_0}(0)}{h} - i_0 - 1 \right| \right) \end{aligned}$$

we deduce (7.2). Similarly, we can construct $g(\varepsilon)$ such that (7.3) is satisfied. Notice also that thanks to (7.2) and (7.3), we have

$$|(\rho_1^\varepsilon)^*(0, x) - u_0(x)| \leq f(\varepsilon) + \varepsilon \quad \text{and} \quad |(\rho_2^\varepsilon)^*(0, x) - \xi_0(x)| \leq g(\varepsilon) + \varepsilon. \quad (7.8)$$

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Therefore, we have

$$\begin{cases} u_0(x) - \max(f(\varepsilon), g(\varepsilon)) \leq \rho_1^\varepsilon(0, x) \leq (\rho_1^\varepsilon)^*(0, x) \leq u_0(x) + \max(f(\varepsilon), g(\varepsilon)) + \varepsilon. \\ \xi_0(x) - \max(f(\varepsilon), g(\varepsilon)) \leq \rho_2^\varepsilon(0, x) \leq (\rho_2^\varepsilon)^*(0, x) \leq \xi_0(x) + \max(f(\varepsilon), g(\varepsilon)) + \varepsilon. \end{cases}$$

Using the fact that $(\rho_1^\varepsilon, \rho_2^\varepsilon)$ is a viscosity solution of (2.4) and the comparison principle (Proposition 3.7) we deduce that (with $(u^\varepsilon, \xi^\varepsilon)$ a continuous solution of (2.4) associated to the initial condition (u_0, ξ_0))

$$\begin{cases} u^\varepsilon(t, x) - \max(f(\varepsilon), g(\varepsilon)) \leq \rho_1^\varepsilon(t, x) \leq (\rho_1^\varepsilon)^*(t, x) \leq u^\varepsilon(t, x) + \max(f(\varepsilon), g(\varepsilon)) + \varepsilon. \\ \xi^\varepsilon(t, x) - \max(f(\varepsilon), g(\varepsilon)) \leq \rho_2^\varepsilon(t, x) \leq (\rho_2^\varepsilon)^*(t, x) \leq \xi^\varepsilon(t, x) + \max(f(\varepsilon), g(\varepsilon)) + \varepsilon. \end{cases}$$

Passing to the limit as $\varepsilon \rightarrow 0$ and using Theorem 2.4 we get that

$$\tilde{\rho}^\varepsilon(t, x) = \begin{cases} \rho_1^\varepsilon(t, -d(0, x)) & \text{for } (t, x) \in (0, +\infty) \times R_0, \\ \rho_1^\varepsilon(t, d(0, x)) & \text{for } (t, x) \in (0, +\infty) \times R_1^*, \\ \rho_2^\varepsilon(t, d(0, x)) & \text{for } (t, x) \in (0, +\infty) \times R_2^*, \end{cases} \quad (7.9)$$

converges locally uniformly to u^0 (unique solution of (2.17)), which ends the proof of Theorem 2.5. □

Proof of Theorem 7.1. Theorem 7.1 is a consequence of the following lemma.

Lemma 7.2 (Link between the velocities). *Assume (A). Let $((U_i)_i)$ be the solution of (1.1) with*

$$\begin{cases} U_i(0) \leq U_{i+1}(0) - h_0 & \text{if } U_i(0), U_{i+1}(0) \in R_0 \\ U_i(0) \leq U_{i+2}(0) - h_0 & \text{otherwise.} \end{cases} \quad (7.10)$$

Then we have for all $i \in \mathbb{Z}$, such that $i[2] = 1$,

$$\dot{U}_i(t) = -\phi(U_i(t), N_0(u(t, \cdot), [\xi(t, \cdot)])(U_i(t)), M_1[u(t, \cdot)](U_i(t))), \quad (7.11)$$

and for all $i \in \mathbb{Z}$ such that $i[2] = 0$,

$$\dot{U}_i(t) = -\phi(U_i(t), N_0(\xi(t, \cdot), [u(t, \cdot)])(U_i(t)), M_2[\xi(t, \cdot)](U_i(t))), \quad (7.12)$$

where E and F are defined in (2.7), $J_i = V_i'$ for $i = 0, 1, 2$, and u and ξ are continuous functions, decreasing in x , such that

$$\begin{cases} u(t, x) = \rho_1(t, x) = (\rho_1)_*(t, x) & \text{for } x = U_i(t), \text{ for all } i \in \mathbb{Z}, \text{ such that } i[2] = 1, \\ \xi(t, x) = \rho_2(t, x) = (\rho_2)_*(t, x) & \text{for } x = U_i(t), \text{ for all } i \in \mathbb{Z}, \text{ such that } i[2] = 0. \end{cases} \quad (7.13)$$

with (ρ_1, ρ_2) defined in (2.1)-(2.2) (with $\varepsilon = 1$).

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Proof. We drop the time dependence to simplify the presentation. We only do the proof in the case $i \in \mathbb{Z}$, $i[2] = 0$, the other case being identical.

In the case $U_i \leq 0$, all the vehicles remain in order, meaning that $\xi(U_i) = -(i+1)$ and $-(i+2) = u(U_{i+1}) < u(U_i) < u(U_{i-1}) = -i$, using this and (7.13) we have for all $z \in [0, +\infty)$,

$$\begin{cases} -1 < u(U_i + z) - \xi(U_i) < 1 & \text{if } z \in [0, U_{i+1} - U_i), \\ u(U_i + z) - \xi(U_i) \leq -1 & \text{if } z \in [U_{i+1} - U_i, +\infty). \end{cases}$$

Given that u and ξ are continuous, this implies that

$$N_0(\xi, [u])(U_i) = \int_0^{U_{i+1}-U_i} \frac{1}{2} J_0(z) dz + \int_{U_{i+1}-U_i}^{+\infty} \frac{3}{2} J_0(z) dz - \frac{3}{2} V_{max}^0 = -V_0(U_{i+1} - U_i). \quad (7.14)$$

Now using (7.13), and the fact that $\xi(U_i) = -(i+1)$, we obtain

$$\begin{cases} -2 < \xi(U_i + z) - \xi(U_i) \leq 0 & \text{if } z \in [0, U_{i+2} - U_i), \\ \xi(U_i + z) - \xi(U_i) \leq -2 & \text{if } z \in [U_{i+2} - U_i, +\infty). \end{cases}$$

Again, given that ξ is continuous, this implies that

$$M_2[\xi](U_i) = \int_0^{U_{i+2}-U_i} \frac{1}{2} J_2(z) dz + \int_{U_{i+2}-U_i}^{+\infty} \frac{3}{2} J_2(z) dz - \frac{3}{2} V_{max}^2 = -V_2(U_{i+2} - U_i). \quad (7.15)$$

Combining (7.14) and (7.15), we get (7.12). □

Thanks to (7.13), we have for $x = U_i(t)$, for all $i \in \mathbb{Z}$, $i[2] = 1$,

$$\begin{aligned} \tilde{N}_0((\rho_1)_*(t, \cdot), [(\rho_2)_*(t, \cdot)])(x) &= \tilde{N}_0(u(t, \cdot), [\xi(t, \cdot)])(x) = N_0(u(t, \cdot), [\xi(t, \cdot)])(x), \\ \tilde{M}_1[(\rho_1)_*(t, \cdot)](x) &= \tilde{M}_1[u(t, \cdot)](x) = M_1[u(t, \cdot)](x). \end{aligned}$$

Similarly, we have for all $x = U_i(t)$, for all $i \in \mathbb{Z}$, $i[2] = 0$,

$$\begin{aligned} \tilde{N}_0((\rho_2)_*(t, \cdot), [(\rho_1)_*(t, \cdot)])(x) &= \tilde{N}_0(\xi(t, \cdot), [u(t, \cdot)])(x) = N_0(\xi(t, \cdot), [u(t, \cdot)])(x), \\ \tilde{M}_2[(\rho_2)_*(t, \cdot)](x) &= \tilde{M}_2[\xi(t, \cdot)](x) = M_2[\xi(t, \cdot)](x) \end{aligned}$$

and using Lemma 7.2, and Definition 3.1, we can see that $((\rho_1)_*, (\rho_2)_*)$ is a discontinuous viscosity super-solution of (7.1). We obtain a similar result for $((\rho_1)^*, (\rho_2)^*)$ therefore, (ρ_1, ρ_2) is a discontinuous viscosity solution of (7.1).

We prove the converse. Using from [FIM09b, Proposition 11], we can conclude that $((\rho_1)_* = [u], (\rho_2)_* = [\xi])$ (resp. $(\rho_1^* = [u], \rho_2^* = [\xi])$) is a viscosity super-solution (resp. sub-solution) of

$$\begin{cases} (\rho_1)_t - \tilde{c}_1(t, x)(\rho_1)_x = 0 & \text{with } \tilde{c}_1(t, x) = \phi(x, N_0(u(t, \cdot), [\xi(t, \cdot)])(x), M_1[u(t, \cdot)](x)), \\ (\rho_2)_t - \tilde{c}_2(t, x)(\rho_2)_x = 0 & \text{with } \tilde{c}_2(t, x) = \phi(x, N_0(\xi(t, \cdot), [u(t, \cdot)])(x), M_2[\xi(t, \cdot)](x)). \end{cases} \quad (7.16)$$

Using the fact that u and ξ are decreasing, we define for all $i \in \mathbb{Z}$, $i[2] = 1$,

$$U_i(t) = \inf \{x, u(t, x) \leq -(i + 1)\} = (u(t, \cdot))^{-1}(-i - 1), \tag{7.17}$$

and for all $i \in \mathbb{Z}$, $i[2] = 0$,

$$U_i(t) = \inf \{x, \xi(t, x) \leq -(i + 1)\} = (\xi(t, \cdot))^{-1}(-i - 1), \tag{7.18}$$

and we consider the functions $t \mapsto U_i(t)$. They are continuous because u and ξ are decreasing in x and continuous in (t, x) .

We now prove that the functions U_i are viscosity solutions of (1.1). Let φ be a test function such that $\varphi(t) \leq U_i(t)$ and $\varphi(t_0) = U_i(t_0)$. Let us now define $\hat{\varphi}(t, x) = -(i + 1) + \varphi(t) - x$. Let us for instance consider $i \in \mathbb{Z}$, such that $i[2] = 0$ then we have

$$\hat{\varphi}(t_0, U_i(t_0)) = (\rho_2)_*(t_0, U_i(t_0))$$

and

$$\hat{\varphi}(t, x) \leq (\rho_2)_*(t, x) \quad \text{for } U_i(t) - 1 < x < U_{i+2}(t).$$

This implies that

$$\begin{aligned} & \varphi_t(t_0) + \tilde{c}_2(t_0, U_i(t_0)) \geq 0 \\ & \Leftrightarrow \varphi_t(t_0) \geq -\phi(U_i(t_0), -V_0(U_{i+1}(t_0) - U_i(t_0)), -V_2(U_{i+2}(t_0) - U_i(t_0))) \\ & \Leftrightarrow \varphi_t(t_0) \geq \bar{\phi}(U_i(t_0), V_0(U_{i+1}(t_0) - U_i(t_0)), V_2(U_{i+2}(t_0) - U_i(t_0))). \end{aligned}$$

We obtain a similar result in the case $i[2] = 1$. This proves that U_i are viscosity super-solutions of (1.1). The proof for sub-solutions is similar and we skip it. Moreover, $\tilde{c}_i(\cdot, U_i(\cdot))$, for $i = 1, 2$, is continuous, we deduce that $U_i \in C^1$, and is therefore a classical solution of (1.1). □

8 Extensions

In this section, we will introduce some extensions of model (1.1) for which an homogenization result is possible by using the same arguments as the ones used in the case presented in this paper. However, since the models we introduce in this section are more complex, many technical difficulties appear. Particularly, we will no longer be working with a system of two equations but with a more general system. We will not go into details of the proofs for each of the models, however we give some guidelines for any reader that would like to do the proofs in detail.

8.1 One incoming road, n outgoing roads

General model

Let us begin by considering a model where we have one incoming road R_0 and $n \in \mathbb{N} \setminus \{0\}$ outgoing roads that we denote by R_k , for $k = 1, \dots, n$. We consider a simple periodic setting and assume that the vehicle $i \in \mathbb{Z}$ such that $i[n] = k - 1$ goes into R_k , where $i[n]$ denotes the rest of the euclidean division of i by n (therefore $i[n] \in \{0, \dots, n - 1\}$). We consider the following model, if $i[n] = k - 1$, for all $t > 0$,

$$\dot{U}_i(t) = \bar{\phi}(U_i(t), V_0(U_{i+1}(t) - U_i(t)), V_k(U_{i+n}(t) - U_i(t))), \quad (8.1)$$

where U_i denotes the position of the i th vehicle and \dot{U}_i its velocity. For $i = 0, \dots, n$, V_i is an optimal velocity function. The function $\bar{\phi}$ is the same as before and is defined in (1.2).

In Figure 8.5, we have an schematic representation of model (8.1).

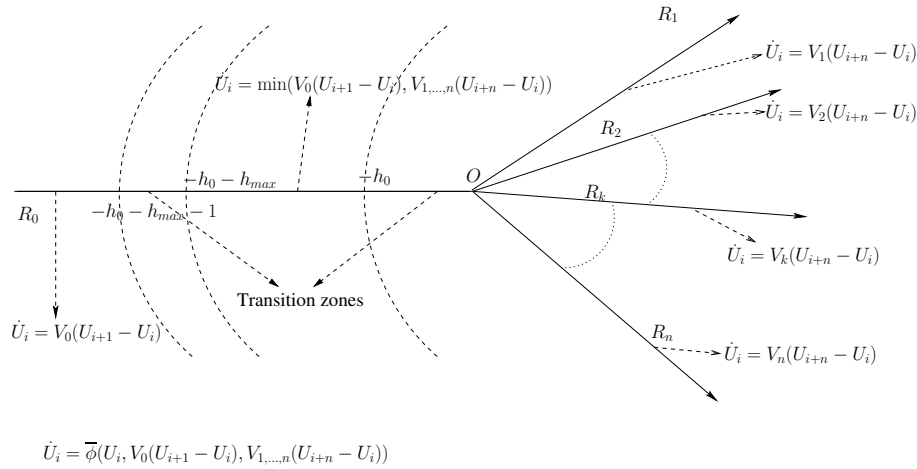


Figure 8.5: Schematic representation of the microscopic model.

We assume that the optimal velocities satisfy assumption (A). However, we need to change assumption (A5), to take into account the fact that we have n possible exits.

- (A5') The function $p \mapsto pV_0(-1/p)$ and $p \mapsto pV_k(-n/p)$ for $k = 1, \dots, n$ are strictly convex respectively on $[-1/h_0, 0)$ and on $[-n/h_0, 0)$.

To simplify we call (A') assumption (A) with (A5') instead of (A5).

Injecting the system of ODEs into a system of PDEs

The technique remains the same as before and we inject the system of ODE (8.1) into a system of PDEs by considering n "cumulative distribution functions". For $j = 0, \dots, n-1$,

$$\begin{aligned} \rho_j^\varepsilon(t, x) = & -n\varepsilon \cdot \left(\sum_{i[n]=j, i \geq 0} H\left(x - \varepsilon U_i\left(\frac{t}{\varepsilon}\right)\right) + \sum_{i[n]=j, i < 0} \left(-1 + H\left(x - \varepsilon U_i\left(\frac{t}{\varepsilon}\right)\right)\right) \right) \\ & + \varepsilon(n-1-j), \end{aligned} \tag{8.2}$$

with H the heaviside function defined in (2.3).

Remark 8.1. We choose this type of "cumulative distribution function", to simplify the computations, because if $i[n] = j$, then we have $\rho_j(t, U_i(t)) = -(i+1)$.

Under assumption (A') we can prove that $(\rho_j)_{j=0, \dots, n-1}$ is a (discontinuous viscosity) solution of the following non-local system of PDEs, for $(t, x) \in (0, +\infty) \times \mathbb{R}$,

$$\begin{cases} \partial_t u_j^\varepsilon + \phi\left(\frac{x}{\varepsilon}, N_0^\varepsilon\left(\frac{u_j^\varepsilon}{\varepsilon}, \left[\frac{u_{j+1}^\varepsilon(t, \cdot)}{\varepsilon}\right]\right)(x), M_{j+1}^\varepsilon\left[\frac{u_j^\varepsilon(t, \cdot)}{\varepsilon}\right](x)\right) \cdot |\partial_x u_j^\varepsilon| = 0 \\ \text{for } j = 0, \dots, n-2 \\ \partial_t u_{n-1}^\varepsilon + \phi\left(\frac{x}{\varepsilon}, N_0^\varepsilon\left(\frac{u_{n-1}^\varepsilon}{\varepsilon}, \left[\frac{u_0^\varepsilon(t, \cdot)}{\varepsilon}\right]\right)(x), M_n^\varepsilon\left[\frac{u_{n-1}^\varepsilon(t, \cdot)}{\varepsilon}\right](x)\right) \cdot |\partial_x u_{n-1}^\varepsilon| = 0, \end{cases} \tag{8.3}$$

where N_0^ε and M_j^ε for $j = 1, \dots, n$ are non-local operators defined by

$$N_0^\varepsilon(U, [\Xi])(x) = \int_{\mathbb{R}} J_0(z) F(\Xi(x + \varepsilon z) - U(x)) dz - \frac{3}{2} V_{max}^0, \tag{8.4}$$

and

$$M_j^\varepsilon[U](x) = \int_{\mathbb{R}} J_j(z) E(U(x + \varepsilon z) - U(x)) dz - \frac{3}{2} V_{max}^j, \tag{8.5}$$

with $J_j = V_j'$ for $j = 0, 1, \dots, n$,

$$F(z) = \begin{cases} 0 & \text{if } z \geq n-1, \\ 1/2 & \text{if } -1 \leq z < n-1, \\ 3/2 & \text{if } z < -1, \end{cases} \quad \text{and} \quad E(z) = \begin{cases} 0 & \text{if } z \geq 0, \\ 1/2 & \text{if } -n \leq z < 0, \\ 3/2 & \text{if } z < -n. \end{cases} \tag{8.6}$$

Finally, the function ϕ is the same as the one in (2.8).

We will consider the following initial conditions,

$$u_j^\varepsilon(0, x) = u_{j,0}(x) \quad \text{for } x \in \mathbb{R} \text{ and for } j = 0, \dots, n-1, \tag{8.7}$$

and we make the following assumptions.

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(A0') (Initial condition). For all $x \leq 0$, and all $j = 1, \dots, n-1$,

$$u_{0,0}(x) = u_{j,0}(x).$$

Moreover, we assume that for all $x \in \mathbb{R}$

$$-nk_0 = -\frac{n}{h_0} \leq (u_{j,0})_x \leq 0 \quad \text{for all } j = 0, \dots, n-1.$$

The effective Hamiltonians

Like in the case of a simple bifurcation, we will have an effective Hamiltonian on each road. We introduce \bar{H}_k , for $k = 0, \dots, n$ which are the effective Hamiltonian on each of the roads R_k for $k = 0, \dots, n$. We define $k_0 = 1/h_0$ and $\bar{H}_0 : \mathbb{R} \rightarrow \mathbb{R}$ by

$$\bar{H}_0(p) = \begin{cases} -p - nk_0 & \text{for } p < -nk_0, \\ -V_0 \left(\frac{-1}{p} \right) \cdot |p| & \text{for } -nk_0 \leq p \leq 0, \\ p & \text{for } p > 0. \end{cases} \quad (8.8)$$

We also define, for $k = 1, \dots, n$, $\bar{H}_k : \mathbb{R} \rightarrow \mathbb{R}$ by

$$\bar{H}_k(p) = \begin{cases} -p - nk_0 & \text{for } p < -nk_0, \\ -V_k \left(\frac{-n}{p} \right) \cdot |p| & \text{for } -nk_0 \leq p \leq 0, \\ p & \text{for } p > 0. \end{cases} \quad (8.9)$$

For $k = 0, \dots, n$, let us notice that \bar{H}_k is continuous, coercive $\left(\lim_{|p| \rightarrow +\infty} \bar{H}_k(p) = +\infty \right)$ and because of (A5'), there exists a unique point $p_k \in [-nk_0, 0]$ such that

$$\begin{cases} \bar{H}_k \text{ is non-increasing on } (-\infty, p_k), \\ \bar{H}_k \text{ is increasing on } (p_k, +\infty), \end{cases} \quad (8.10)$$

We denote by

$$H_0 = \max_{k \in \{0, \dots, n\}} \min_{p \in \mathbb{R}} \bar{H}_k(p). \quad (8.11)$$

Convergence result

Let $(u_j^\varepsilon)_j$ be the solution of (8.3)-(8.7). It is possible to prove that the function defined by

$$\tilde{u}^\varepsilon(t, x) = \begin{cases} u_0^\varepsilon(t, -d(0, x)) & \text{for } (t, x) \in (0, +\infty) \times R_0, \\ u_{k-1}^\varepsilon(t, d(0, x)) & \text{for } (t, x) \in (0, +\infty) \times R_k^*, \quad k \in \{1, \dots, n\}, \end{cases} \quad (8.12)$$

converges locally uniformly on compact subsets of $(0, +\infty) \times \bar{R}$ as ε goes to 0 to the unique viscosity solution of the following problem

$$\begin{cases} u_t^0 + \bar{H}_0(u_x^0) = 0 & \text{for } (t, x) \in [0, +\infty) \times R_0^*, \\ u_t^0 + \bar{H}_k(u_x^0) = 0 & \text{for } (t, x) \in [0, +\infty) \times R_k^*, \\ u_t^0 + F_{\bar{A}}(\partial_0 u^0(t, 0), \dots, \partial_n u^0(t, 0)) = 0 & \text{for } (t, x) \in [0, +\infty) \times \{0\}, \\ u^0(0, x) = \bar{u}_0(x) = \begin{cases} u_{0,0}(-d(0, x)) & \text{for } x \in R_0, \\ u_{k-1,0}(d(0, x)) & \text{for } x \in R_k \end{cases} \end{cases} \quad (8.13)$$

where \bar{A} is a constant to be determined and $F_{\bar{A}}$ is defined by

$$F_{\bar{A}}(p_0, \dots, p_n) = \max(\bar{A}, \bar{H}_0^+(p_0), \max_{k=1, \dots, n} \bar{H}_k^-(p_k)), \quad (8.14)$$

where for $k = 0, \dots, n$,

$$\bar{H}_k^-(p) = \begin{cases} \bar{H}_k(p) & \text{if } p \leq p_k \\ \bar{H}_k(p_k) & \text{if } p \geq p_k \end{cases} \quad \text{and} \quad \bar{H}_k^+(p) = \begin{cases} \bar{H}_k(p_k) & \text{if } p \leq p_k \\ \bar{H}_k(p) & \text{if } p \geq p_k. \end{cases} \quad (8.15)$$

Theorem 8.2 (Junction condition by homogenization). *Assume (A0') and (A'). For $\varepsilon > 0$, let $(u_j^\varepsilon)_j$ be the solution of (8.3)-(8.7). Then there exists $\bar{A} \in [H_0, 0]$ such that the function defined by (8.12) converges locally uniformly to the unique viscosity solution u^0 of (8.13).*

Theorem 8.3. *Assume (A') and that at initial time we have, for all $i \in \mathbb{Z}$, if the vehicle $i + 1$ is in R_0 ,*

$$U_i(0) \leq U_{i+1}(0) - h_0, \quad (8.16)$$

and if not

$$U_i(0) \leq U_{i+n}(0) - h_0. \quad (8.17)$$

We also assume that there exists a constant $R > 0$ such that, for all $i \in \mathbb{Z}$, if $U_i(0) \geq R$

$$U_{i+n}(0) - U_i(0) = h_j \quad \text{if } i[n] = j - 1$$

and if $U_i(0) \leq -R$

$$U_{i+1}(0) - U_i(0) = h,$$

with $h \geq h_0$ and for $j \in \{1, \dots, n\}$, $h_j \geq h_0$. We define the functions $(u_{j,0})_j$ (satisfying (A0')) by

$$u_{j-1,0}(x) = \begin{cases} \frac{-x}{h} & \text{if } x \leq 0 \\ \frac{-nx}{h_j} & \text{if } x > 0 \end{cases} \quad \text{for all } x \in \mathbb{R} \text{ for } j = 1, \dots, n.$$

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Then there exists a constant $\bar{A} \in [H_0, 0]$ such that the function

$$\tilde{\rho}^\varepsilon(t, y) = \begin{cases} \rho_0^\varepsilon(t, -d(0, y)) & \text{for } (t, y) \in (0, +\infty) \times R_0, \\ \rho_{k-1}^\varepsilon(t, d(0, y)) & \text{for } (t, y) \in (0, +\infty) \times R_k^*, \quad k \in \{1, \dots, n\}, \end{cases} \quad (8.18)$$

converges locally uniformly to u^0 the unique solution of (8.13).

Theorem 8.4. Assume $(A0')$ - (A') . Let u^0 be the unique solution of (8.13), then we have for all $(t, x) \in [0, +\infty) \times \bar{R}$,

$$-nk_0 \leq u_x^0 \leq 0,$$

with k_0 defined in $(A0')$.

Remark 8.5. First, notice that, at the macroscopic scale, we obtain a similar result like the one from the case of a simple bifurcation (and of course if $n = 2$ we find the same result).

In the introduction we mentioned that the main difficulty to obtain an homogenization result was to build the correctors at the junction since we are in a non-periodic setting. However, notice that in the proof of Proposition 6.3 (in the core of the proof of Theorem 4.3 which gives the existence of correctors at the junction) the key element is that we are able to control the oscillations in space of the solutions in the truncated domain for the approximated cell problem (Lemma 6.4). Notice also that the arguments used in that lemma are actually similar to the ones used to prove Theorem 3.9. That is why we will give the equivalent theorem in the case of n outgoing roads and then give some guidelines on how to prove it.

Theorem 8.6 (Control of the oscillations). Let $T > 0$. Assume $(A0')$ - (A') and let $(u_j)_j$ be a solution of (8.3)-(8.7), with $\varepsilon = 1$. Then there exists a constant $C_1 > 0$ such that for all $x, y \in \mathbb{R}$, $x \geq y$ and for all $t, s \in [0, T]$, $t \geq s$, we have

$$\begin{aligned} 0 \leq u(t, x) - u(s, x) &\leq C_1(t - s), & \text{and} & & -K_0(x - y) - n \leq u(t, x) - u(t, y) \leq 0, \\ 0 \leq \xi(t, x) - \xi(s, x) &\leq C_1(t - s) & & & -K_0(x - y) - n \leq \xi(t, x) - \xi(t, y) \leq 0, \end{aligned} \quad (8.19)$$

with $K_0 := nk_0$.

Sketch of the proof of Theorem 8.6. We only give the ideas necessary to prove Theorem 8.6, and the order in which the proof must be done.

1. Prove the control of the oscillations in time using the barriers and the fact that the solutions are invariant by additions of constants and by translations in time.
2. Like in Theorem 3.9, prove that the functions u_j are non-increasing in space.

3. Now we need a comparison between the functions $(u_j)_j$ solution of (8.3) (with $\varepsilon = 1$) for all $x \leq -h_0$. We want to prove that for all $x \leq -h_0$, all $t \in [0, T]$ and for $j = 0, \dots, n - 2$,

$$-1 \leq u_{j+1}(t, x) - u_j(t, x) \leq n - 1 \quad \text{and} \quad -1 \leq u_0(t, x) - u_{n-1}(t, x) \leq n - 1 \quad (8.20)$$

To prove (8.20), we need to proceed in the following order.

- (a) Using a localisation argument, like in the proof of Lemma 3.10, we prove that for all $x \leq -h_0$, all $t \in [0, T]$ and for $j = 0, \dots, n - 2$,

$$-1 \leq u_{j+1}(t, x) - u_j(t, x) \quad \text{and} \quad -1 \leq u_0(t, x) - u_{n-1}(t, x).$$

- (b) Using the previous result, we deduce that for all $x \leq -h_0$, all $t \in [0, T]$ and for $j = 0, \dots, n - 2$

$$u_{j+1}(x) - u_j(x) \leq n - 1 \quad \text{and} \quad u_0(x) - u_{n-1}(x) \leq n - 1.$$

4. Using the previous results, proceeding like in the proof of Theorem 3.9 prove the lower bounds on the control of the space oscillations.

□

8.2 A more general distribution of vehicles

Let us consider the case we have one incoming road R_0 and two outgoing roads R_1 and R_2 like in the case treated in detail in the previous sections. However, instead of considering one vehicle going to the left and one to the right, we consider a more general distribution of the vehicles (but still a periodic distribution). More precisely, let $n \geq 2$ and assume that the vehicle $i \in \mathbb{Z}$ such that $i[n] = 0$ goes into R_1 and the rest of the vehicles go into R_2 (one vehicle goes left and $n - 1$ go right). We then consider the following model for all $i \in \mathbb{Z}$ and all $t > 0$,

-if $i[n] = 0$,

$$\dot{U}_i(t) = \bar{\phi} \left(U_i(t), V_0(U_{i+1}(t) - U_i(t)), V_1(U_{i+n}(t) - U_i(t)) \right), \quad (8.21)$$

-if $i[n] = k \in \{1, \dots, n - 2\}$,

$$\dot{U}_i(t) = \bar{\phi} \left(U_i(t), V_0(U_{i+1}(t) - U_i(t)), V_2(U_{i+1}(t) - U_i(t)) \right), \quad (8.22)$$

-if $i[n] = n - 1$,

$$\dot{U}_i(t) = \bar{\phi} \left(U_i(t), V_0(U_{i+1}(t) - U_i(t)), V_2(U_{i+2}(t) - U_i(t)) \right), \quad (8.23)$$

where U_i denotes the position of the i -th vehicle and \dot{U}_i its velocity.

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For $i = 0, 1, 2$, V_i is an optimal velocity function. The function $\bar{\phi}$ is the same as before (see (1.2)).

In Figure 8.6, we have a schematic representation of model (8.21)-(8.22)-(8.23).

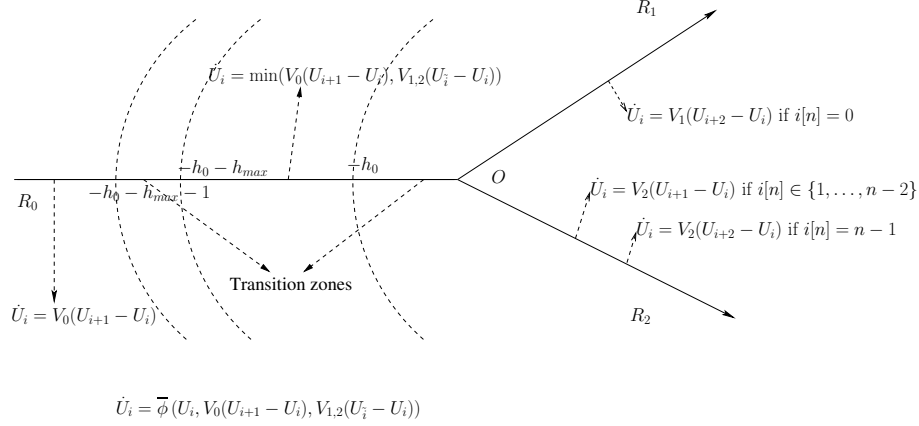


Figure 8.6: Schematic representation of the microscopic model (8.21)-(8.22)-(8.23). The index \tilde{i} denotes the index of the vehicle that will be in front of the vehicle i after the bifurcation.

We assume that the optimal velocities satisfy assumption (A), however, we need to change assumption (A5), to take into account the fact that we have a different distribution of vehicles.

- (A5'') The function $p \mapsto pV_0(-1/p)$, $p \mapsto pV_1(-n/p)$ and $p \mapsto pV_2(-n/(p(n-1)))$ are strictly convex respectively on $[-1/h_0, 0)$, on $[-n/h_0, 0)$, and on $[-nk_0/(n-1), 0)$.

To simplify we call (A'') assumption (A) with (A5'') instead of (A5).

As before, for $j = 0, \dots, n-1$, we define the following "cumulative distribution functions"

$$\rho_j^\varepsilon(t, x) = -n\varepsilon \cdot \left(\sum_{i[n]=j, i \geq 0} H \left(x - \varepsilon U_i \left(\frac{t}{\varepsilon} \right) \right) + \sum_{i[n]=j, i < 0} \left(-1 + H \left(x - \varepsilon U_i \left(\frac{t}{\varepsilon} \right) \right) \right) \right) + \varepsilon(n-1-j). \quad (8.24)$$

Under assumption (A'') we can prove that $(\rho_j)_{j=0, \dots, n-1}$ is a (discontinuous viscosity)

solution of the following non-local system of PDEs, for $(t, x) \in (0, +\infty) \times \mathbb{R}$,

$$\begin{cases} \partial_t u_0^\varepsilon + \phi\left(\frac{x}{\varepsilon}, N_0^\varepsilon\left(\frac{u_0^\varepsilon}{\varepsilon}, \left[\frac{u_1^\varepsilon(t, \cdot)}{\varepsilon}\right]\right)(x), M_1^\varepsilon\left[\frac{u_0^\varepsilon(t, \cdot)}{\varepsilon}\right](x)\right) \cdot |\partial_x u_0^\varepsilon| = 0 \\ \partial_t u_j^\varepsilon + \phi\left(\frac{x}{\varepsilon}, N_0^\varepsilon\left(\frac{u_j^\varepsilon}{\varepsilon}, \left[\frac{u_{j+1}^\varepsilon(t, \cdot)}{\varepsilon}\right]\right)(x), N_2^\varepsilon\left(\frac{u_j^\varepsilon}{\varepsilon}, \left[\frac{u_{j+1}^\varepsilon(t, \cdot)}{\varepsilon}\right]\right)(x)\right) \cdot |\partial_x u_j^\varepsilon| = 0 \\ \text{for } j = 1, \dots, n-2 \\ \partial_t u_{n-1}^\varepsilon + \phi\left(\frac{x}{\varepsilon}, N_0^\varepsilon\left(\frac{u_{n-1}^\varepsilon}{\varepsilon}, \left[\frac{u_0^\varepsilon(t, \cdot)}{\varepsilon}\right]\right)(x), L_2^\varepsilon\left(\frac{u_{n-1}^\varepsilon}{\varepsilon}, \left[\frac{u_1^\varepsilon(t, \cdot)}{\varepsilon}\right]\right)(x)\right) \cdot |\partial_x u_{n-1}^\varepsilon| = 0 \end{cases} \quad (8.25)$$

where N_0^ε and M_1^ε are defined respectively in (8.4) and (8.5). The non-local operator N_2^ε is defined like N_0^ε but with J_2 and V_{max}^2 instead of J_0 and V_{max}^0 (in order to recover the velocity V_2 on R_2). Finally, the non-local operator L_2^ε is defined by

$$L_2^\varepsilon(U, [\Xi])(x) = \int_{\mathbb{R}} J_2(z) G(\Xi(x + \varepsilon z) - U(x)) dz - \frac{3}{2} V_{max}^2, \quad (8.26)$$

with $J_2 = V_2'$ and

$$G(z) = \begin{cases} 0 & \text{if } z \geq n-2, \\ 1/2 & \text{if } -2 \leq z < n-2, \\ 3/2 & \text{if } z < -2. \end{cases} \quad (8.27)$$

The function ϕ is the same as the one in (2.8).

The effective Hamiltonians

Like in the previous scenarios, we have an effective Hamiltonian on each road. The effective Hamiltonians $\bar{H}_0 : \mathbb{R} \rightarrow \mathbb{R}$ and $\bar{H}_1 : \mathbb{R} \rightarrow \mathbb{R}$ are defined respectively in (8.8) and (8.9) while \bar{H}_2 is defined by (with $k_0 = 1/h_0$)

$$\bar{H}_2(p) = \begin{cases} -p - nk_0 & \text{for } p < -nk_0 \\ -V_2\left(\frac{-n}{p(n-1)}\right) & \text{for } -nk_0 \leq p \leq 0, \\ p & \text{for } p > 0. \end{cases} \quad (8.28)$$

For $k = 0, 1, 2$, let us notice that such \bar{H}_k is continuous, coercive $\left(\lim_{|p| \rightarrow +\infty} \bar{H}_k(p) = +\infty\right)$ and because of (A5''), there exists a unique point $p_k \in [-nk_0, 0]$ such that

$$\begin{cases} \bar{H}_k \text{ is non-increasing on } (-\infty, p_k), \\ \bar{H}_k \text{ is increasing on } (p_k, +\infty), \end{cases} \quad (8.29)$$

We denote by

$$H_0 = \max_{k \in \{0, 1, 2\}} \min_{p \in \mathbb{R}} \bar{H}_k(p).$$

Convergence result

Theorem 8.7. *Assume (A'') and that at initial time between two consecutive vehicles there is atleast a distance of h_0 between them. We also assume that there exists a constant $R > 0$ such that for all $i \in \mathbb{Z}$, if $U_i(0) \geq R$*

$$U_{i+n}(0) - U_i(0) = h_1 \quad \text{if } i[n] = 0,$$

$$\begin{cases} U_{i+1}(0) - U_i(0) = h_2 & \text{if } i[n] \neq 0, n-1 \\ U_{i+2}(0) - U_i(0) = h_2 & \text{if } i[n] = n-1, \end{cases}$$

and if $U_i(0) \leq -R$

$$U_{i+1}(0) - U_i(0) = h,$$

with $h, h_1, h_2 \geq h_0$. We define the functions $u_{0,0}$ and $u_{1,0}$ (satisfying (A0')) by

$$u_{k-1,0}(x) = \begin{cases} \frac{-x}{h} & \text{if } x \leq 0 \\ \frac{-nx}{h_k} & \text{if } x > 0 \end{cases} \quad \text{for all } x \in \mathbb{R} \text{ and } k = 1 \text{ or } 2.$$

Then there exists a constant $\bar{A} \in [H_0, 0]$ such that, for any $k \in \{0, \dots, n-1\}$ and any $j \in \{1, \dots, n-1\}$, the function

$$\tilde{\rho}^\varepsilon(t, y) = \begin{cases} \rho_k^\varepsilon(t, -d(0, y)) & \text{for } (t, y) \in (0, +\infty) \times R_0, \\ \rho_0^\varepsilon(t, d(0, y)) & \text{for } (t, y) \in (0, +\infty) \times R_1^*, \\ \rho_j^\varepsilon(t, d(0, y)) & \text{for } (t, y) \in (0, +\infty) \times R_2^*, \end{cases} \quad (8.30)$$

converges locally uniformly to u^0 the unique solution of

$$\begin{cases} u_t^0 + \bar{H}_0(u_x^0) = 0 & \text{for } (t, x) \in [0, +\infty) \times R_0^*, \\ u_t^0 + \bar{H}_1(u_x^0) = 0 & \text{for } (t, x) \in [0, +\infty) \times R_1^*, \\ u_t^0 + \bar{H}_2(u_x^0) = 0 & \text{for } (t, x) \in [0, +\infty) \times R_2^*, \\ u_t^0 + F_{\bar{A}}^-(\partial_0 u^0(t, 0), \partial_1 u^0(t, 0), \partial_2 u^0(t, 0)) = 0 & \text{for } (t, x) \in [0, +\infty) \times \{0\}, \\ u^0(0, x) = \bar{u}_0(x) = \begin{cases} u_{0,0}(-d(0, x)) & \text{for } x \in R_0, \\ u_{0,0}(d(0, x)) & \text{for } x \in R_1 \\ u_{1,0}(d(0, x)) & \text{for } x \in R_2 \end{cases} \end{cases} \quad (8.31)$$

where \bar{A} is a constant to be determined and $F_{\bar{A}}^-$ is defined by

$$F_{\bar{A}}^-(p_0, p_1, p_2) = \max(\bar{A}, \bar{H}_0^+(p_0), \bar{H}_1^-(p_1), \bar{H}_2^-(p_2)), \quad (8.32)$$

and for $k = 0, 1, 2$, we define

$$\overline{H}_k^-(p) = \begin{cases} \overline{H}_k(p) & \text{if } p \leq p_k, \\ \overline{H}_k(p_k) & \text{if } p \geq p_k, \end{cases} \quad \text{and} \quad \overline{H}_k^+(p) = \begin{cases} \overline{H}_k(p_k) & \text{if } p \leq p_k, \\ \overline{H}_k(p) & \text{if } p \geq p_k. \end{cases}$$

Moreover, the unique solution u^0 of (8.31) satisfies

$$-nk_0 \leq u_x^0 \leq 0,$$

with $k_0 = 1/h_0$.

Remark 8.8. Let us notice that if we choose $n = 2$, we recover the same result as the one from the case of a simple bifurcation (one vehicle goes left, the other goes right).

Like in the previous extension, we will give the equivalent of Theorem 3.9 in the case of the more general distribution of vehicles. Contrary to the case of n outgoing roads, there is a slight difference, in this case, when building the correctors away from the junction in R_2 . We will explain how to correctly choose them and how to obtain the effective Hamiltonian \overline{H}_2 .

Theorem 8.9 (Control of the oscillations). *Let $T > 0$. Assume that (A'') and (A0') are valid and let $(u_j)_j$ be a solution of (8.25), with $\varepsilon = 1$. Then there exists a constant $C_1 > 0$ such that for all $x, y \in \mathbb{R}$, $x \geq y$ and for all $t, s \in [0, T]$, $t \geq s$, we have*

$$\begin{aligned} 0 \leq u(t, x) - u(s, x) &\leq C_1(t - s), & \text{and} & & -K_0(x - y) - n \leq u(t, x) - u(t, y) \leq 0, \\ 0 \leq \xi(t, x) - \xi(s, x) &\leq C_1(t - s) & & & -K_0(x - y) - n \leq \xi(t, x) - \xi(t, y) \leq 0, \end{aligned} \quad (8.33)$$

with $K_0 := nk_0$.

Sketch of the proof of Theorem 8.9. We only give the ideas necessary to prove Theorem 8.9, and the order in which the proof must be done.

1. Prove the control of the oscillations in time using the barriers and the fact that the solutions are invariant by additions of constants and by translations in time.
2. Like in Theorem 3.9, prove that the functions u_j are non-increasing in space.
3. Now we need a comparison between the functions $(u_j)_j$ solution of (8.25) (with $\varepsilon = 1$). We want to prove that for all $x \in \mathbb{R}$, all $t \in [0, T]$ and for $j = 1, \dots, n - 2$, we have

$$-1 \leq u_{j+1}(t, x) - u_j(t, x) \leq n - 1. \quad (8.34)$$

We also want to prove that for all $x \leq -h_0$ and all $t \in [0, T]$,

$$-1 \leq u_1(t, x) - u_0(t, x) \leq n - 1 \quad \text{and} \quad -1 \leq u_0(t, x) - u_{n-1}(t, x) \leq n - 1. \quad (8.35)$$

Finally, we want to prove that for all $x \geq -h_0$ and all $t \in [0, T]$, we have

$$-2 \leq u_1(t, x) - u_{n-1}(t, x) \leq n - 2. \quad (8.36)$$

To prove (8.36)-(8.35)-(8.34), we need to proceed in the following order.

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(a) First, prove that for $j = 1, \dots, n - 2$

$$-1 \leq u_{j+1}(t, x) - u_j(t, x) \quad \text{for all } x \in \mathbb{R}, \text{ all } t \in [0, T]. \quad (8.37)$$

To do this, we argue classically by contradiction and assume that the supremum of $-1 - u_{j+1}(t, x) + u_j(t, x)$ over $[0, T] \times \mathbb{R}$ is strictly positive.

(b) Using a localisation term, like in the proof of Lemma 3.10, prove that

$$-1 \leq u_1(t, x) - u_0(t, x) \quad \text{for all } x \leq -h_0 \text{ and all } t \in [0, T].$$

(c) Similarly, prove that

$$-1 \leq u_0(t, x) - u_{n-1}(t, x) \quad \text{for all } x \leq -h_0 \text{ and all } t \in [0, T].$$

(d) Using the previous results, deduce that for all $x \leq -h_0$, all $t \in [0, T]$, and for $j = 0, \dots, n - 2$, we have

$$u_{j+1}(t, x) - u_j(t, x) \leq n - 1 \quad \text{and} \quad u_0(t, x) - u_{n-1}(t, x) \leq n - 1.$$

(e) Using a localisation term, prove that for all $x \geq -h_0$ and all $t \in [0, T]$, we have

$$-2 \leq u_1(t, x) - u_{n-1}(t, x).$$

(f) Using the previous result and (8.37), deduce that for all $x \geq -h_0$, all $t \in [0, T]$ and for $j = 1, \dots, n - 2$, we have

$$u_{j+1}(t, x) - u_j(t, x) \leq n - 1 \quad \text{and} \quad u_1(t, x) - u_{n-1}(t, x) \leq n - 2.$$

4. Using the previous results, proceeding like in the proof of Theorem 3.9 prove the lower bounds on the control of the space oscillations.

□

Proposition 8.10 (Homogenization on R_2). *Assume (A''). Then for $p \in [-nk_0, 0]$, there exists a unique $\lambda \in \mathbb{R}$, such that there exists a bounded solution $(v_j)_{j=1, \dots, n-1}$ of*

$$\begin{cases} N_p^2(v_j, [v_{j+1}])(x) \cdot |\partial_x v_j + p| = \lambda & \text{for } j=1, \dots, n-2, \\ L_p^2(v_{n-1}, [v_1])(x) \cdot |\partial_x v_{n-1} + p| = \lambda, \\ v_j \text{ for } j = 1, \dots, n-1 \text{ is } \mathbb{Z} - \text{periodic}, \end{cases}$$

with

$$N_p^2(U, [\Xi])(x) = \int_{-\infty}^{+\infty} J_2(z) F(\Xi(x+z) - U(x) + pz) dz - \frac{3}{2} V_{max}^2$$

and

$$L_p^2(U, [\Xi])(x) = \int_{-\infty}^{+\infty} J_2(z) G(\Xi(x+z) - U(x) + pz) dz - \frac{3}{2} V_{max}^2.$$

Moreover, for $p \in [-nk_0, 0]$ we have $\lambda = \bar{H}_2(p) = -V_2 \left(\frac{-n}{p(n-1)} \right) |p|$.

Proof. Contrary to the construction of the correctors on R_0 and R_1 , we cannot consider the correctors equal to zero. However, we can see that choosing for $j = 1, \dots, n - 1$,

$$v_j \equiv \frac{j - 1}{n - 1},$$

we obtain for $j = 1, \dots, n - 2$

$$L_p^2(v_{n-1}, [v_1])(x) = N_p^2(v_j, [v_{j+1}])(x) = -V_2 \left(\frac{-n}{p(n - 1)} \right). \quad (8.38)$$

This ends the proof of Proposition (8.10). □

Remark 8.11. *From the previous extensions, we can imagine even more complex scenarios. For instance having a bifurcation with n outgoing roads, with a very general (but still periodic) distribution of vehicles. The technique remains the same for that type of problem but one has to consider a great number of "cumulative distribution functions".*

9 Appendix

Proof of comparison principle in Proposition 3.7. We introduce

$$M = \sup_{(t,x) \in (0,T) \times \mathbb{R}} \max(u(t, x) - v(t, x), \xi(t, x) - \zeta(t, x)),$$

we want to prove that $M \leq 0$. We argue by contradiction and assume that $M > 0$.

Step 1: the test functions. For η, α and ε small parameters and $A > 0$ a constant to be chosen later, we define

$$\begin{cases} \varphi_1(t, x, y) = u(t, x) - v(t, y) + p(x - y) - \frac{(x - y)^2}{2\varepsilon} e^{At} - \frac{\eta}{T - t} - \alpha x^2 \\ \varphi_2(t, x, y) = \xi(t, x) - \zeta(t, y) + p(x - y) - \frac{(x - y)^2}{2\varepsilon} e^{At} - \frac{\eta}{T - t} - \alpha x^2. \end{cases}$$

Using (3.11) and (A0), we deduce that

$$\begin{aligned} \varphi_1(t, x, y) &\leq u_0(x) - u_0(y) + p(x - y) + 2K(1 + T) - \frac{(x - y)^2}{2\varepsilon} e^{At} - \alpha x^2 \\ &\leq (2k_0 + |p|)|x - y| + 2K(1 + T) - \frac{(x - y)^2}{2\varepsilon} e^{At} - \alpha x^2. \end{aligned}$$

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We have the same upper bound on φ_2 . Therefore, for $i = 1, 2$, we have

$$\lim_{|x|, |y| \rightarrow +\infty} \varphi_i(t, x, y) = -\infty.$$

Since φ_1 and φ_2 are upper semi-continuous, the function $\psi = \max(\varphi_1, \varphi_2)$ reaches a maximum at a finite point that we denote by $(\bar{t}, \bar{x}, \bar{y}) \in [0, T) \times \mathbb{R} \times \mathbb{R}$. Classically, we have for α and η small enough,

$$\begin{cases} 0 < \frac{M}{2} \leq \psi(\bar{t}, \bar{x}, \bar{y}), \\ \alpha|\bar{x}| \rightarrow 0 \text{ as } \alpha \rightarrow 0, \\ |\bar{x} - \bar{y}| \rightarrow 0 \text{ as } \varepsilon \rightarrow 0. \end{cases} \quad (9.1)$$

Step 2: $\bar{t} > 0$. By contradiction, assume $\bar{t} = 0$. Then we have (if for instance $\psi(\bar{t}, \bar{x}, \bar{y}) = \varphi_1(\bar{t}, \bar{x}, \bar{y})$, we have a similar result in the other case),

$$\frac{\eta}{T} < u(0, \bar{x}) - v(0, \bar{y}) + p(\bar{x} - \bar{y}) \leq u_0(\bar{x}) - u_0(\bar{y}) + p(\bar{x} - \bar{y}) \leq (2k_0 + |p|)|\bar{x} - \bar{y}|.$$

Using that $|\bar{x} - \bar{y}| \rightarrow 0$ as $\varepsilon \rightarrow 0$, we obtain a contradiction for ε small enough.

Step 3: viscosity inequalities. We only consider the case $\psi(\bar{t}, \bar{x}, \bar{y}) = \varphi_1(\bar{t}, \bar{x}, \bar{y})$, since the other case is similar and so we skip it. By doubling the time variable and passing to the limit, there exists two real numbers $a, b \in \mathbb{R}$ such that

$$a - b = \frac{\eta}{(T - \bar{t})^2} + Ae^{A\bar{t}} \frac{(\bar{x} - \bar{y})^2}{2\varepsilon},$$

and

$$\begin{aligned} a + G_p^1(\bar{x}, u(\bar{t}, \bar{x}), [\xi(\bar{t}, \cdot)], [u(\bar{t}, \cdot)], p_\varepsilon e^{A\bar{t}} + 2\alpha\bar{x} - p) &\leq 0 \\ b + \tilde{G}_p^1(\bar{y}, v(\bar{t}, \bar{y}), [\zeta(\bar{t}, \cdot)], [v(\bar{t}, \cdot)], p_\varepsilon e^{A\bar{t}} - p) &\geq 0, \end{aligned} \quad (9.2)$$

with $p_\varepsilon = (\bar{x} - \bar{y})/\varepsilon$.

Combining these inequalities we obtain

$$\begin{aligned}
a - b &\leq \tilde{G}_p^1(\bar{y}, v(\bar{t}, \bar{y}), [\zeta(\bar{t}, \cdot)], [v(\bar{t}, \cdot)], p_\varepsilon e^{A\bar{t}} - p) \\
&\quad - G_p^1(\bar{x}, u(\bar{t}, \bar{x}), [\xi(\bar{t}, \cdot)], [u(\bar{t}, \cdot)], p_\varepsilon e^{A\bar{t}} + 2\alpha\bar{x} - p) \\
&\leq \tilde{G}_p^1(\bar{y}, v(\bar{t}, \bar{y}), [\zeta(\bar{t}, \cdot)], [v(\bar{t}, \cdot)], p_\varepsilon e^{A\bar{t}} - p) \\
&\quad - G_p^1(\bar{x}, u(\bar{t}, \bar{x}), [\xi(\bar{t}, \cdot)], [u(\bar{t}, \cdot)], p_\varepsilon e^{A\bar{t}} - p) + o_\alpha(1) \\
&\leq (\psi^+(\bar{x}) - \psi^+(\bar{y}))\bar{H}_1(p_\varepsilon e^{A\bar{t}} - p) + (\psi^-(\bar{x}) - \psi^-(\bar{y}))\bar{H}_0(p_\varepsilon e^{A\bar{t}} - p) \\
&\quad + \psi^+(\bar{y})\psi^-(\bar{y})\phi(\bar{y}, \tilde{N}_p^0(v(\bar{t}, \cdot), [\zeta(\bar{t}, \cdot)])(\bar{y}), \tilde{M}_p^1[v(\bar{t}, \cdot)](\bar{y})) |p_\varepsilon e^{A\bar{t}}| \\
&\quad - \psi^+(\bar{x})\psi^-(\bar{x})\phi(\bar{x}, N_p^0(v(\bar{t}, \cdot), [\zeta(\bar{t}, \cdot)])(\bar{x}), M_p^1[v(\bar{t}, \cdot)](\bar{x})) |p_\varepsilon e^{A\bar{t}}| + o_\alpha(1) \quad (9.3) \\
&\leq (\psi^+(\bar{x}) - \psi^+(\bar{y}))\bar{H}_1(p_\varepsilon e^{A\bar{t}} - p) + (\psi^-(\bar{x}) - \psi^-(\bar{y}))\bar{H}_0(p_\varepsilon e^{A\bar{t}} - p) \\
&\quad + \psi^+(\bar{y})\psi^-(\bar{y})\phi(\bar{y}, \tilde{N}_p^0(v(\bar{t}, \cdot), [\zeta(\bar{t}, \cdot)])(\bar{y}), \tilde{M}_p^1[v(\bar{t}, \cdot)](\bar{y})) |p_\varepsilon e^{A\bar{t}}| \\
&\quad - \psi^+(\bar{y})\psi^-(\bar{y})\phi(\bar{y}, N_p^0(v(\bar{t}, \cdot), [\zeta(\bar{t}, \cdot)])(\bar{x}), M_p^1[v(\bar{t}, \cdot)](\bar{x})) |p_\varepsilon e^{A\bar{t}}| \\
&\quad + M_0 \|\phi'\|_\infty \|D(\psi^+\psi^-)\|_\infty \frac{(\bar{x} - \bar{y})^2}{\varepsilon} e^{A\bar{t}} + o_\alpha(1) \\
&= \mathcal{I}_1 + \mathcal{I}_2 + M_0 \|\phi'\|_\infty \|D(\psi^+\psi^-)\|_\infty \frac{(\bar{x} - \bar{y})^2}{\varepsilon} e^{A\bar{t}} + o_\alpha(1),
\end{aligned}$$

with

$$\mathcal{I}_1 = (\psi^+(\bar{x}) - \psi^+(\bar{y}))\bar{H}_1(p_\varepsilon e^{A\bar{t}} - p) + (\psi^-(\bar{x}) - \psi^-(\bar{y}))\bar{H}_0(p_\varepsilon e^{A\bar{t}} - p)$$

and

$$\begin{aligned}
\mathcal{I}_2 &= \psi^+(\bar{y})\psi^-(\bar{y})\phi(\bar{y}, \tilde{N}_p(v(\bar{t}, \cdot), [\zeta(\bar{t}, \cdot)])(\bar{y}), \tilde{M}_p[v(\bar{t}, \cdot)](\bar{y})) |p_\varepsilon e^{A\bar{t}}| \\
&\quad - \psi^+(\bar{y})\psi^-(\bar{y})\phi(\bar{y}, N_p(v(\bar{t}, \cdot), [\zeta(\bar{t}, \cdot)])(\bar{x}), M_p[v(\bar{t}, \cdot)](\bar{x})) |p_\varepsilon e^{A\bar{t}}|.
\end{aligned}$$

Step 3.1: bound on \mathcal{I}_1 . Using (2.10), we have that

$$\begin{aligned}
(\psi^-(\bar{x}) - \psi^-(\bar{y}))\bar{H}_0(p_\varepsilon e^{A\bar{t}} - p) &\leq \|D\psi^-\|_\infty |\bar{x} - \bar{y}| |\bar{H}_0(p_\varepsilon e^{A\bar{t}} - p)| \\
&\leq \|D\psi^-\|_\infty |\bar{x} - \bar{y}| V_{max}^0 |p_\varepsilon e^{A\bar{t}} - p| \\
&\leq \|D\psi^-\|_\infty V_{max}^0 \left(\frac{(\bar{x} - \bar{y})^2}{\varepsilon} e^{A\bar{t}} + |\bar{x} - \bar{y}| \cdot |p| \right) \\
&\leq 2V_{max}^0 \|D\psi^-\|_\infty \frac{(\bar{x} - \bar{y})^2}{2\varepsilon} e^{A\bar{t}} + o_\varepsilon(1).
\end{aligned}$$

Similarly, we get a similar bound on \bar{H}_1 and so,

$$\mathcal{I}_1 \leq 2 \left(V_{max}^1 \|D\psi^+\|_\infty + V_{max}^0 \|D\psi^-\|_\infty \right) \frac{(\bar{x} - \bar{y})^2}{2\varepsilon} e^{A\bar{t}} + o_\varepsilon(1). \quad (9.4)$$

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Step 3.2: bound on \mathcal{I}_2 . Like in [DLFM08] and [FIM09b], we distinguish two cases.

CASE 1: there exists a subsequence α_n , such that,

$$\frac{|\bar{x} - \bar{y}|}{2\varepsilon} \rightarrow 0 \text{ as } n \rightarrow +\infty.$$

In this case, we choose

$$A = 2M_0 \|\phi'\|_\infty \|D(\psi^+ \psi^-)\|_\infty + 2 \left(V_{max}^1 \|D\psi^+\|_\infty + V_{max}^0 \|D\psi^-\|_\infty \right).$$

Using the fact that the non-local operators are bounded and choosing ε small enough and then passing to the limit as n goes to infinity (note that $p_\varepsilon \rightarrow 0$), we get a contradiction from (9.3).

CASE 2: there exists a constant \tilde{C}_ε such that for any $\alpha > 0$, small enough, we have

$$\frac{|\bar{x} - \bar{y}|}{2\varepsilon} \geq \tilde{C}_\varepsilon.$$

Let us now begin by comparing the non-local operators \tilde{N}_p^0 and N_p^0 , we introduce

$$\mathcal{A} = \{z / \tilde{F}(\zeta(\bar{t}, z) - v(\bar{t}, \bar{y}) + p(z - \bar{y})) \leq F(\xi(\bar{t}, z) - u(\bar{t}, \bar{x}) + p(z - \bar{x}))\}.$$

Using that $\varphi_2(\bar{t}, z, z) \leq \varphi_1(\bar{t}, \bar{x}, \bar{y})$, we obtain that

$$\begin{aligned} \xi(\bar{t}, z) - u(\bar{t}, \bar{x}) + p(z - \bar{x}) &\leq \zeta(\bar{t}, z) - v(\bar{t}, \bar{y}) + p(z - \bar{y}) \\ &\quad - \left(e^{A\bar{t}} \frac{(\bar{x} - \bar{y})^2}{2\varepsilon} + \alpha \bar{x}^2 - \alpha z^2 \right), \end{aligned}$$

therefore, we have

$$\mathcal{A}^c \subset \left\{ z / |z| > R_{\varepsilon, \alpha} := \left(\frac{e^{A\bar{t}} (\bar{x} - \bar{y})^2}{\alpha} + \bar{x}^2 \right)^{1/2} \right\}.$$

We also have that

$$\{|z + \bar{x}| \geq R_{\varepsilon, \alpha}\} \subset \{|z| \geq \tilde{R}_{\varepsilon, \alpha} := R_{\varepsilon, \alpha} - |\bar{x}|\},$$

and as in [DLFM08], $\tilde{R}_{\varepsilon, \alpha} \rightarrow +\infty$ as α goes to 0. Given that J_0 has a bounded support, this implies that for α small enough, we have

$$\begin{aligned} &\int_{\mathbb{R}} J_0(z - \bar{x}) F(\xi(\bar{t}, z) - u(\bar{t}, \bar{x}) + p(z - \bar{x})) dz \\ &\geq \int_{\mathbb{R}} J_0(z - \bar{x}) \tilde{F}(\zeta(\bar{t}, z) - v(\bar{t}, \bar{y}) + p(z - \bar{y})) dz. \end{aligned}$$

Using this result, we can see that

$$\tilde{N}_p^0(v(\bar{t}, \cdot), [\zeta(\bar{t}, \cdot)])(\bar{y}) \leq N_p^0(u(\bar{t}, \cdot), [\xi(\bar{t}, \cdot)])(\bar{x}) + \|DJ_0\|_{L^1(\mathbb{R})}|\bar{x} - \bar{y}|. \quad (9.5)$$

In order to compare the non-local operators \tilde{M}_p^1 and M_p^1 , we introduce

$$\mathcal{B} = \{z / \tilde{E}(v(\bar{t}, z) - v(\bar{t}, \bar{y}) + p(z - \bar{y})) \leq E(u(\bar{t}, z) - u(\bar{t}, \bar{x}) + p(z - \bar{x}))\}.$$

Given that $\varphi_1(\bar{t}, z, z) \leq \varphi_1(\bar{t}, \bar{x}, \bar{y})$, we get that

$$\begin{aligned} u(\bar{t}, z) - u(\bar{t}, \bar{x}) + p(z - \bar{x}) &\leq v(\bar{t}, z) - v(\bar{t}, \bar{y}) + p(z - \bar{y}) \\ &\quad - \left(e^{A\bar{t}} \frac{(\bar{x} - \bar{y})^2}{2\varepsilon} + \alpha\bar{x}^2 - \alpha z^2 \right), \end{aligned}$$

therefore, we have

$$\mathcal{B}^c \subset \{z / |z| > R_{\varepsilon, \alpha}\}.$$

Proceeding as before, we obtain for α small enough

$$\begin{aligned} &\int_{\mathbb{R}} J_1(z - \bar{x}) E(u(\bar{t}, z) - u(\bar{t}, \bar{x}) + p(z - \bar{x})) dz \\ &\geq \int_{\mathbb{R}} J_1(z - \bar{y}) \tilde{E}(v(\bar{t}, z) - v(\bar{t}, \bar{y}) + p(z - \bar{y})) dz. \end{aligned}$$

With this result, we deduce that

$$\tilde{M}_p^1[v(\bar{t}, \cdot)](\bar{y}) \leq M_p^1[v(\bar{t}, \cdot)](\bar{x}) + \|DJ_1\|_{L^1(\mathbb{R})}|\bar{x} - \bar{y}| \quad (9.6)$$

$$. \quad (9.7)$$

Injecting (9.4), (9.5) and (9.6), in (9.3), we obtain

$$\begin{aligned} a - b &\leq \left(2M_0 \|\phi'\|_{\infty} \|D(\psi^+ \psi^-)\|_{\infty} + 2 \left(V_{max}^1 \|D\psi^+\|_{\infty} + V_{max}^0 \|D\psi^-\|_{\infty} \right) \frac{(\bar{x} - \bar{y})^2}{2\varepsilon} e^{A\bar{t}} \right. \\ &\quad \left. + o_{\alpha}(1) + \psi^+(\bar{y}) \psi^-(\bar{y}) \phi(\bar{y}, \tilde{N}_p^0(v(\bar{t}, \cdot), [\zeta(\bar{t}, \cdot)])(\bar{y}), \tilde{M}_p^1[v(\bar{t}, \cdot)](\bar{y})) \right) |p_{\varepsilon} e^{A\bar{t}}| \\ &\quad - \psi^+(\bar{y}) \psi^-(\bar{y}) \phi(\bar{y}, N_p^0(v(\bar{t}, \cdot), [\zeta(\bar{t}, \cdot)])(\bar{x}), M_p^1[v(\bar{t}, \cdot)](\bar{x})) \left) |p_{\varepsilon} e^{A\bar{t}}| \right. \\ &\leq \left(2M_0 \|\phi'\|_{\infty} \|D(\psi^+ \psi^-)\|_{\infty} + 2 \left(V_{max}^1 \|D\psi^+\|_{\infty} + V_{max}^0 \|D\psi^-\|_{\infty} \right) \frac{(\bar{x} - \bar{y})^2}{2\varepsilon} e^{A\bar{t}} \right. \\ &\quad \left. + o_{\alpha}(1) + 2\|\psi^+ \phi^-\|_{\infty} \|\phi'\|_{\infty} \left(\|DJ_1\|_{L^1(\mathbb{R})} + \|DJ_0\|_{L^1(\mathbb{R})} \right) \frac{(\bar{x} - \bar{y})^2}{2\varepsilon} e^{A\bar{t}}, \right. \end{aligned}$$

where we have used assumption (A6) for the last inequality. Finally, choosing correctly A , we obtain a contradiction for α and ε small enough. This ends the proof of Proposition 3.7. \square

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Conclusion et perspectives

Dans cette thèse, nous avons obtenu des résultats nouveaux concernant les modèles macroscopiques pour le trafic routier, en utilisant des techniques d'analyse et d'analyse numérique. En particulier, nous avons obtenu des justifications rigoureuses de modèles macroscopiques connus (équivalent au modèle LWR) dans le cas d'une seule route. On a également obtenu des modèles macroscopiques équivalents pour des phénomènes microscopiques plus compliqués : différents types de conducteurs, des perturbations locales et des bifurcations. Il reste désormais plusieurs problèmes ouverts que je vais détailler ci-dessous.

Homogénéisation stochastique

Dans la plupart des travaux de cette thèse, il y a une hypothèse de périodicité, par exemple dans [1] on considère que l'on a des blocs de n_0 types de véhicules, dans [3] on considère une répartition uniforme des véhicules dans les voies sortantes de la bifurcation. Une possibilité pour surmonter ce manque de réalisme est de considérer des modélisations stochastiques. Par exemple, cela permettrait d'avoir des conducteurs avec différentes caractéristiques sans la condition de périodicité. De même, on pourrait considérer dans le cas d'une bifurcation avec une voie entrante et deux sortantes qu'en moyenne une voiture sur deux part à gauche et le reste à droite, ceci généraliserait le résultat de [3].

Homogénéisation dans les réseaux

Dans cette thèse, on a réussi à obtenir un résultat d'homogénéisation pour une bifurcation. Par contre, il reste encore beaucoup de situations à traiter surtout du point de vue des réseaux pour le trafic routier. On peut chercher à déterminer le modèle macroscopique équivalent à une jonction (deux voies entrantes et une sortante), à un rond-point...

D'autre part, on pourrait imaginer, à l'échelle microscopique, un modèle avec deux perturbations locales (voir la Figure 9.7).

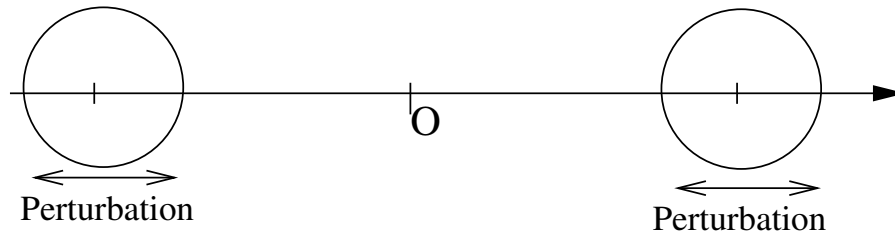


Figure 9.7: Représentation schématique de deux perturbations à l'échelle microscopique.

En utilisant nos techniques d'homogénéisation on retrouve un modèle macroscopique avec une condition de limitation de flux à l'origine (voir la Figure 9.8).



Figure 9.8: Représentation schématique du modèle macroscopique en utilisant la méthode classique.

Mais si les perturbations sont assez éloignées, on pourrait imaginer que le modèle macroscopique à considérer devrait être une équation de Hamilton-Jacobi sur chacune des sections avec deux conditions de jonctions aux centres des perturbations (voir la Figure 9.9).

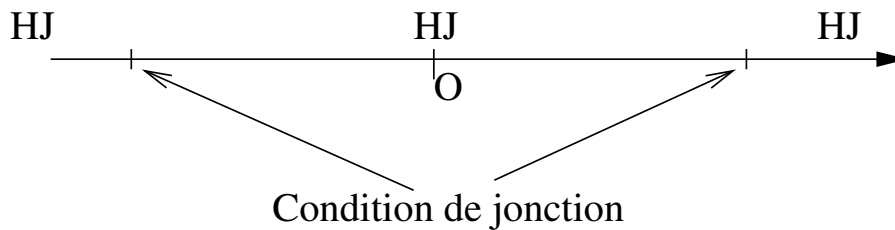


Figure 9.9: Représentation schématique du modèle macroscopique attendu.

La difficulté est alors de savoir comment il est possible d'obtenir ce modèle macroscopique et comment on peut conserver séparément l'information de ces deux perturbations.

Homogénéisation de modèles avec accélération bornée

Dans les modèles macroscopiques que nous avons présenté dans cette thèse, l'accélération des véhicules n'est pas bornée. Par contre, il existe des modèles macroscopiques construits

pour surmonter ce défaut. Il s'agit des modèles à accélération bornée, étudiés par Leclercq [Lec02, Lec07] et Lebacque [Leb02, Leb03]. Ces modèles couplent l'équation de transport du modèle LWR avec une condition qui borne l'accélération des véhicules. On peut alors chercher à rigoureusement obtenir ce modèle macroscopique. Il faudrait donc trouver le modèle microscopique adéquat et utiliser la bonne technique d'homogénéisation pour garder la condition d'accélération bornée à l'échelle macroscopique.

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