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Transport branché et structures fractales

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Introduction

L'objet de cette thèse est l'étude du transport branché, de problèmes variationnels qui y sont liés et des structures fractales qui peuvent y apparaître. Avant d'introduire le transport branché proprement dit et d'en faire le panorama, commençons par observer que c'est une branche du transport optimal, dont on donne un très bref aperçu.

Transport optimal

Le problème du transport optimal remonte à 1781, lorsque Gaspard Monge publie son "Mémoire sur la théorie des déblais et des remblais" [Mon81]. Une répartition initiale de masse (il s'agissait de terre dans la formulation originale) étant donnée, et une nouvelle répartition étant souhaitée, il s'agit pour chaque particule se trouvant en x , de la déplacer vers une destination $T(x)$ de façon à obtenir cette nouvelle répartition, et ce de la manière la plus économique possible. Dans le modèle de Monge, le coût de transport d'une particule est proportionnel à sa masse multipliée par la distance parcourue, et le but est de minimiser la somme de toutes ces quantités. Dans un langage moderne, le problème se formalise de la manière suivante : étant données deux mesures de même masse μ et ν , disons de probabilité, sur un compact $K \subseteq \mathbb{R}^d$, trouver une fonction $T : K \rightarrow K$ solution du problème

$$\min_T \left\{ \int_K |T(x) - x| d\mu(x) : T_{\#}\mu = \nu \right\}, \quad (\text{M})$$

où $T_{\#}\mu$ est la mesure image de μ par T , c'est-à-dire $T_{\#}\mu(B) = \mu(T^{-1}(B))$ pour tout borélien B , ou encore $\int \phi \circ T d\mu = \int \phi d\nu$ pour tout $\phi \in \mathcal{C}(K)$. Dans ce cadre, on suppose que toute la masse $\mu(x)$ se trouvant en x est envoyée en un même point $T(x)$, sans qu'elle ne se sépare. Posé sous cette forme, c'est un problème dont le coût est certes convexe, mais dont la contrainte est fortement non convexe. En effet, si μ et ν ont pour densité f et g , par changement de variable la contrainte $T_{\#}\mu = \nu$ se reformule

$$f = |\det DT| \cdot g \circ T, \quad (0.0.1)$$

en admettant que T soit régulière et injective. Ainsi, la contrainte ne passe pas à la limite pour des topologies faibles, et l'existence de solutions¹ est restée inconnue jusque très récemment.

¹Question que les mathématiciens ne se posaient pas à l'époque de Monge.

Une généralisation au problème de Monge a été posée par Kantorovich en 1942 [Kan04b; Kan04a], où l'inconnue n'est plus une fonction mais un *plan de transport* $\pi \in \mathbf{\Pi}(\mu, \nu)$, soit une mesure sur l'espace produit $\pi \in \text{Prob}(\mathbb{R}^d \times \mathbb{R}^d)$ telle que $(p_1)_\# \pi = \mu$, $(p_2)_\# \pi = \nu$. Dans cette formulation, on autorise désormais les particules qui se trouvaient au départ en x à rejoindre différentes destinations, le mouvement étant décrit par π : $d\pi(x, y)$ représente simplement la quantité de particules transportée de x vers y . Le problème étendu, appelé *problème de Monge-Kantorovich*, s'exprime ainsi :

$$\min_{\pi} \left\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} |y - x| d\pi(x, y) : \pi \in \mathbf{\Pi}(\mu, \nu) \right\}. \quad (\text{MK})$$

Il est possible de voir ce nouveau problème comme une extension (par relaxation) du problème de Monge. En particulier, à chaque fonction T est associé canoniquement un *plan de transport* $\pi_T \in \mathbf{\Pi}(\mu, \nu)$ de sorte que les coûts de π_T et de T coïncident : π_T est la mesure concentrée sur le graphe de T , telle que la mesure d'une portion du graphe corresponde à la mesure, selon μ , de sa projection sur l'axe des abscisses : en d'autres termes $\pi_T = (\text{Id}, T)_\# \mu$. L'intérêt du problème de Monge-Kantorovich est qu'il s'agit d'un problème de programmation linéaire : c'est un problème linéaire sous contraintes convexes. On obtient de manière quasi-immédiate la compacité et la continuité suffisantes pour conclure à l'existence d'un minimiseur, et toute la théorie de dualité convexe s'applique, fournissant de nombreux outils pour étudier le problème.

Néanmoins, cela ne prouve pas l'existence d'une solution au problème de Monge (M), les solutions du problème de Monge-Kantorovich (MK) n'étant pas nécessairement concentrées sur le graphe d'une fonction. Une stratégie pour résoudre le problème originel de Monge a été proposée en 1979 par Sudakov [Sud79], puis mise en oeuvre avec succès 20 ans plus tard par Ambrosio dans [Amb03]; d'autres preuves ont été établies dans [EG99; TW01; CFM02]. Par ailleurs, de nombreuses variantes au problème de Monge-Kantorovich ont été étudiées, en particulier des généralisations de la forme :

$$\min_{\pi} \left\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} c(x, y) d\pi(x, y) : \pi \in \mathbf{\Pi}(\mu, \nu) \right\}, \quad (\text{TO})$$

où $c : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}_+$ est une fonction continue. Une attention particulière a été portée aux cas $c_p(x, y) = |y - x|^p$, p un exposant convexe ($p \geq 1$) ou concave ($p \leq 1$). Un cas très naturel est le cas $p = 2$ (pour ses liens avec la physique), pour lequel Brenier a démontré en 1991 l'existence et l'unicité de la solution, et que celle-ci est concentrée sur un graphe. Plus précisément :

Théorème de Brenier ([Bre91]). Soit K un compact de \mathbb{R}^d et $\mu, \nu \in \text{Prob}(K)$ telles que $\mu \ll \mathcal{L}^d$. Alors il existe un unique plan de transport optimal pour le problème

$$\min_{\pi} \left\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} |y - x|^2 d\pi(x, y) : \pi \in \mathbf{\Pi}(\mu, \nu) \right\},$$

et il est donné par le gradient d'une fonction convexe : $\pi = (\text{Id}, \nabla\phi)_\# \mu$ où ϕ est convexe.

Une vaste théorie a été élaborée à la suite des travaux pionniers de Brenier [Bre91], reliant le transport optimal à de nombreux domaines : géométrie dans les espaces métrique, EDPs d'évolution, traitement d'images, inégalités fonctionnelles et géométriques etc. Un vaste panorama de la théorie et ses principales applications pourra être trouvé dans les ouvrages [San15], [Vil03; Vil09], [AGS08].

Du transport optimal au transport branché

Dans le problème du transport optimal “classique”, le coût pour chaque particule ne dépend que de sa position de départ et de sa position d’arrivée, et non de sa trajectoire. C’est un implicite qui vient du problème posé au départ par Monge : le coût de transport d’une particule de masse m est proportionnel à $m \times \ell$ où ℓ est la distance parcourue, or cette distance est minimale le long d’une géodésique, c’est-à-dire d’un segment dans \mathbb{R}^d , de sorte que si le transport est optimal, les particules p doivent se déplacer en ligne droite et le coût total s’exprime bien comme $\sum_p m_p |y_p - x_p|$. En réalité, dès lors que le coût de transport c d’une particule ne dépend que de sa masse m et de sa propre trajectoire γ , le problème peut se réduire à un coût ne dépendant que de m et du couple (x, y) .

Évidemment, cet axiome n’est pas satisfaisant pour modéliser nombre de situations où le coût de transport de chaque particule peut dépendre du mouvement des autres particules, par exemple en fonction de la quantité de particules empruntant le même chemin. Dans le cas d’une foule voulant évacuer une pièce, chaque personne désire rejoindre une issue tout en évitant de se rentrer dedans : les fortes concentrations sont coûteuses voire proscrites, ce qui se traduit par une pénalisation ou une contrainte sur la densité de transport. C’est le domaine de ce qu’on appelle le *transport congestionné* [BCS10; CJS08]. À l’inverse, si plusieurs personnes veulent rejoindre une ville B à partir d’une ville A par la route, il leur sera moins coûteux de covoiturer pour mutualiser les frais d’essence et d’autoroute, auquel cas la concentration est favorisée : on entre le domaine du *transport branché*. Dans ce modèle, les particules peuvent se grouper et le coût d’un paquet de particules de masse totale m et voyageant conjointement est $H(m) \times \ell$ où ℓ est la distance parcourue. Si $H(m) = m$, H est additive, i.e. $H(a + b) = H(a) + H(b)$, et le coût des particules ensemble est égal au coût des particules considérées séparément : on retrouve en effet le problème de Monge. En revanche, si H est strictement sous-additive, i.e. $H(a + b) < H(a) + H(b)$ si $a, b > 0$, alors il revient moins cher de se déplacer ensemble.

Ce type de coût apparaît en réalité de manière très naturelle, dès lors qu’existe un phénomène “d’économie d’échelle” : transporter ou construire quelque chose de taille m coûte davantage lorsque m augmente mais proportionnellement moins, rapporté à la taille m . C’est le cas par exemple du transport routier, où un certain coût doit être payé pour construire les routes, mais il ne coûte pas beaucoup plus cher d’augmenter le trafic et ou d’élargir le réseau. De manière générale, ce phénomène apparaît lorsqu’une partie du coût réside dans le coût de construction d’une certaine structure. Cette économie d’échelle se traduit par le fait que $H(m)/m$ est décroissante en m , ce qui implique que H est sous-additive. Ces conditions sont vérifiées en particulier pour toute fonction H croissante, concave et telle que $H(0) = 0$, par exemple $x \mapsto x^\alpha$, $\alpha \in [0, 1]$.

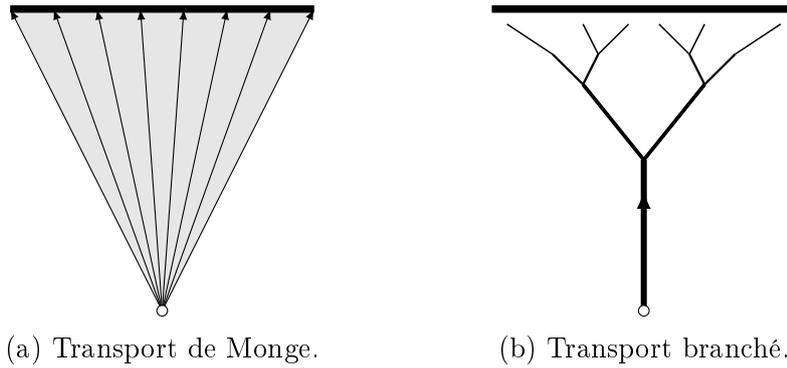


Figure 1: Transport d'un point vers un segment.

Dans le cas du transport de Monge, si l'on veut transporter une masse située en un point vers une distribution uniformément répartie sur un segment, le transport optimal consiste à faire voyager toutes les particules en ligne droite, le long d'une infinité (non dénombrable) de rayons de transport (voire Figure 1a); si l'on considère le transport routier, ce n'est évidemment par la manière la plus économique de procéder ! Dans un modèle de transport branché où le déplacement conjoint est moins coûteux, il est préférable de faire voyager les particules ensemble le plus longtemps possible puis de les faire se séparer afin qu'elles rejoignent leurs différentes destinations, d'où l'apparition de points de branchements et d'un réseau de transport comme l'illustre la Figure 1b.

De nombreuses structures naturelles, comme les réseaux racinaires, les branches des arbres, les réseaux fluviaux, le système de vaisseaux sanguins dans le poumon etc. partagent des traits similaires, de même que de nombreuses structures construites par l'homme : réseaux routiers, gazoducs, oléoducs, systèmes d'irrigation, réseaux électriques, réseaux d'assainissement etc. Une explication peut tenir au fait que toutes ces structures partagent un principe commun : elles doivent irriguer une certaine distribution de masse à partir d'une source (ponctuelle ou étendue) tout en optimisant un coût favorisant les fortes concentrations (par économie d'échelle dans les réseaux artificiels, par soucis de préservation dans les réseaux naturels ; par exemple en biologie, les gros vaisseaux sanguins sont préférables car moins fragiles). Un objectif de la théorie du transport branché est notamment de mettre en évidence des propriétés qualitatives générales de ces structures à l'aide d'un modèle commun.

Le transport branché et ses modèles

Le principal axiome que l'on supposera est le suivant : déplacer une masse m sur une distance ℓ a un coût proportionnel à $m^\alpha \ell$ où α est un exposant dit concave : $\alpha \in [0, 1]$. Le cas plus général de la H -masse (introduit dans [DH03] suite aux travaux [Whi99]), où H est croissante, sous-additive et $H(0) = 0$ a commencé à être étudié récemment [BW15; Col+17], mais nous nous concentrerons dans cette thèse sur le cas $H(m) = m^\alpha$. Le premier chapitre de la thèse définit précisément les principaux modèles du transport branché en fournissant certaines preuves – on l'espère – simplifiées. Commençons par le modèle le plus simple : le modèle discret, qui date des travaux de Gilbert [Gil67] en 1967, dont l'objet était l'optimisation des réseaux de communications.

On se donne deux ensembles finis de points $(x_i)_{i \leq N}, (y_j)_{j \leq M}$ dans \mathbb{R}^d , pondérés par des masses $(a_i)_i, (b_j)_j$ et on suppose qu'ils ont même masse total, disons unitaire : $\sum_i a_i = \sum_j b_j = 1$. On peut les voir comme deux mesures de probabilité finiment atomiques $\mu = \sum_{i \leq N} a_i \delta_{x_i}$ et $\nu = \sum_{j \leq M} a_j \delta_{y_j}$. Le but est de connecter ces deux mesures via un graphe pondéré orienté G qui consiste en un triplet $G = (V, E, \Theta = (\theta_e)_{e \in E})$ où V est l'ensemble des sommets, E celui des arêtes (orientées) et θ_e représente pour chaque arête e un poids associé. Un tel graphe connecte μ et ν s'il vérifie les lois de Kirchhoff : en chaque sommet $v \in V$,

$$\text{masse rentrant en } v = \text{masse sortant en } v,$$

les poids a_i étant considérés comme rentrant en x_i , les b_j comme sortant en y_j . À chaque graphe est associé un coût qu'on appelle *énergie de Gilbert* :

$$\mathbf{E}^\alpha(G) = \sum_{e \in E} \theta_e^\alpha |e|,$$

où $|e|$ désigne la longueur de l'arête e . Le problème de transport branché discret consiste à trouver un minimiseur de cette énergie parmi tous les graphes envoyant μ sur ν , autrement dit

$$\min \left\{ \mathbf{E}^\alpha(G) \doteq \sum_{e \in E} \theta_e^\alpha |e| : \mu \xrightarrow{G} \nu \right\}. \quad (\text{DI}_\alpha)$$

Par la méthode directe du calcul des variations, en faisant quelques réductions sur une suite minimisante, il n'est pas difficile de démontrer l'existence d'une solution ; une preuve est donnée dans le premier chapitre. Que peut-on alors dire sur les minimiseurs ? D'abord, qu'ils ne sont pas nécessairement uniques, comme le montre la Figure 2, où le problème présente des symétries mais admet plusieurs solutions qui ne sont pas symétriques.

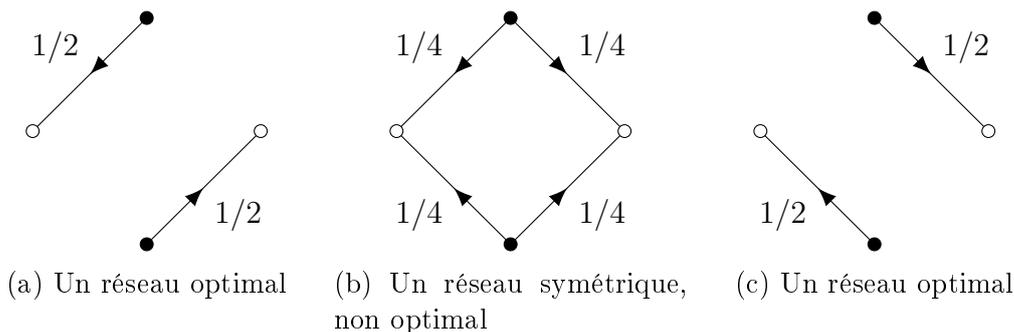


Figure 2: Situation sans unicité ($1/2 < \alpha < 1$).

La principale propriété qualitative des minimiseurs est que ce sont tous des arbres (lorsque $\alpha < 1$). Plus précisément, le fait que $H(x) = x^\alpha$ satisfasse $H(0) = 0$ et H croissante empêche la présence de circuits (boucles qui suivent l'orientation des arêtes), et sa stricte concavité empêche la présence de cycles (boucles non orientées). Prenons l'exemple d'une source s et de deux puits p_1, p_2 . Dans ce cas, trois topologies sont a priori possible : en V, en Y ou en L. Lorsque les deux puits sont à même distance de

la source, la forme en L n'est jamais optimale, et lorsque l'angle $\sphericalangle(p_1 - s, p_2 - s)$ n'est pas trop grand, la forme en Y est préférable, comme l'illustre la Figure 3.

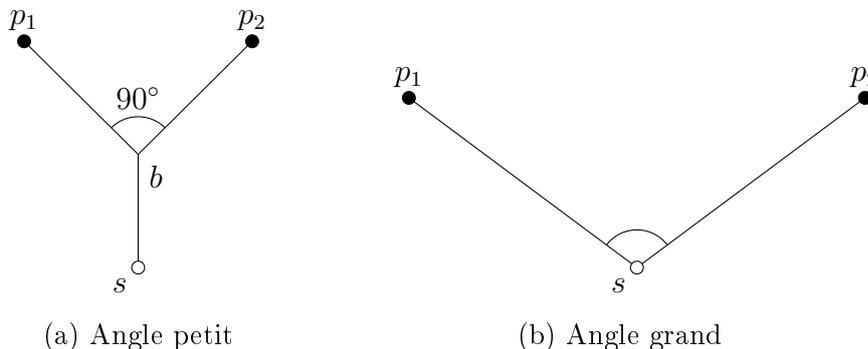


Figure 3: Graphes optimaux ($\alpha = 1/2$).

Des extensions continues au modèle discret ont été proposées depuis les années 2000, permettant de formuler le problème du transport branché, ou *d'irrigation*, pour des mesures $\mu, \nu \in \text{Prob}(K)$ quelconques et non nécessairement atomiques. Des modèles dynamiques existent, mais on s'est intéressé dans cette thèse uniquement aux deux principaux modèles statiques (dont chacun existe dans moult déclinaisons en fonction du formalisme choisi) : le *modèle eulérien* introduit par Xia dans [Xia03], et le *modèle lagrangien* développé par Bernot, Caselles, Morel et Morel, Maddalena, Solimini dans [BCM09; MSM03]. Commençons par le modèle eulérien, plus facile à introduire à partir du problème discret, et qui est formulé en termes de mesures vectorielles². Partant du modèle discret, on considère deux mesures finiment atomiques μ et ν et G un graphe pondéré orienté. On peut construire une mesure vectorielle v_G à partir de G :

$$v_G = \sum_{e \in E(G)} \theta_e \hat{e} \mathcal{H}^1 \llcorner e,$$

où $\hat{e} \in \mathbb{S}^{d-1}$ est la direction du segment e . On constate que la contrainte $G : \mu \rightarrow \nu$ et son énergie de Gilbert peuvent s'exprimer entièrement en fonction de v_G : G envoie μ sur ν si et seulement si $\nabla \cdot v_G = \mu - \nu$ au sens des distributions sur \mathbb{R}^d , et $\mathbf{E}^\alpha(G)$ se réécrit

$$\mathbf{E}^\alpha(G) \doteq \sum_e \theta_e^\alpha |e| = \int_K |v_G(x)|^\alpha d\mathcal{H}^1(x).$$

Ceci motive l'introduction du problème d'irrigation dans sa forme eulérienne, où l'inconnue est mesure vectorielle v qu'on appelle *flot d'irrigation* :

$$\min \{ \mathbf{M}^\alpha(v) : \nabla \cdot v = \mu - \nu \}, \tag{EI_\alpha}$$

où la α -masse \mathbf{M}_α est définie comme

$$\mathbf{M}^\alpha(v) = \begin{cases} \int_K \left| \frac{dv}{d\mathcal{H}^1}(x) \right|^\alpha d\mathcal{H}^1(x) & \text{si } v \text{ est rectifiable,} \\ +\infty & \text{sinon.} \end{cases}$$

²En théorie géométrique de la mesure, on le formule plutôt en termes de courants.

On dit que v est rectifiable si c'est une mesure vectorielle de la forme $v = \theta \tau \mathcal{H}^1 \llcorner E$ où θ est une fonction positive appelée multiplicité, E est un ensemble 1-rectifiable³, $\tau(x)$ un vecteur unitaire tangent à E en x . Des propriétés de coercivité et de semicontinuité peuvent être démontrées pour établir l'existence de minimiseurs. Notons toutefois que la topologie considérée doit être plus forte que la convergence faible des mesures pour v , puisqu'on peut approcher faiblement un champs de vecteur lisse par une mesure rectifiable de α -masse bornée, de sorte que \mathbf{M}^α n'est pas semicontinue inférieurement pour cette topologie.

Passons maintenant au modèle lagrangien, plus riche au sens où on suit, dans ce modèle, la trajectoire de chaque particule plutôt que de considérer uniquement leur champ de vitesse de ces particules, comme c'était le cas dans le modèle eulérien. Sans entrer dans les détails, un mouvement de particules est modélisé par une mesure η sur un ensemble Γ de courbes paramétrées (pour des raisons techniques, elles seront définies sur $[0, +\infty]$ et 1-Lipschitziennes), par exemple une mesure de probabilité $\eta \in \text{Prob}(\Gamma)$, de sorte que $d\eta(\gamma)$ représente la quantité de particules voyageant suivant la trajectoire de γ . On appelle η un *plan d'irrigation*. La contrainte imposant à η d'envoyer μ sur ν se formalise par les conditions $(e_0)_\# \eta = \mu, (e_\infty)_\# \eta = \nu$ où $e_0(\gamma) = \gamma(0)$ et $e_\infty(\gamma) = \lim_{t \rightarrow \infty} \gamma(t)$, ce qu'on note $\mu \xrightarrow{\eta} \nu$. En chaque point x , la quantité de masse $\theta_\eta(x)$ visitant x , appelée *multiplicité*, est définie comme

$$\theta_\eta(x) = \eta(\gamma \in \Gamma : x \in \gamma).$$

On note $\int_\gamma \theta_\eta(x)^{\alpha-1} dx := \int_0^{+\infty} \theta_\eta(\gamma(t))^{\alpha-1} |\dot{\gamma}(t)| dt$ l'intégrale de $\theta_\eta^{\alpha-1}$ le long de γ , et on définit le α -coût de η comme la somme pour toutes les courbes γ de cette quantité, soit :

$$\mathbf{I}^\alpha(\eta) = \int_\Gamma \int_\gamma \theta_\eta(x)^{\alpha-1} dx d\eta(\gamma).$$

Cette expression, a priori barbare, admet toutefois une expression très simple sous certaines hypothèses raisonnables sur η :

$$\mathbf{I}^\alpha(\eta) = \begin{cases} \int_K \theta_\eta(x)^\alpha d\mathcal{H}^1(x) & \text{si } \eta \text{ est rectifiable,} \\ +\infty & \text{sinon,} \end{cases} \quad (0.0.2)$$

qui n'est pas sans ressemblance avec la α -masse. L'introduction de la première vient du fait qu'aucune contrainte de rectifiabilité n'y apparaît a priori et qu'il est ainsi assez aisé de démontrer sa semicontinuité inférieure. Une preuve simple des propriétés de semicontinuité, compacité, coercivité menant à l'existence d'optimiseurs, et ne recourant pas à l'utilisation de paramétrisations des plans d'irrigation comme c'est souvent le cas dans la littérature existant, est donnée dans le Chapitre 1. Le problème d'irrigation lagrangien s'écrit alors

$$\min \left\{ \mathbf{I}^\alpha(\eta) \doteq \int_\Gamma \int_\gamma \theta_\eta(x)^{\alpha-1} dx d\eta(\gamma) : \mu \xrightarrow{\eta} \nu \right\}. \quad (\text{LI}_\alpha)$$

Les modèles eulérien et lagrangien sont équivalents, au sens que les minima des deux problèmes d'irrigation correspondants sont les mêmes, et qu'il est possible de

³Une réunion dénombrables de courbes lipschitziennes et d'un ensemble de mesure \mathcal{H}^1 nulle.

construire un minimiseur d'un problème à partir d'un minimiseur de l'autre. Ce fait était connu depuis un certain temps mais non démontré en toute rigueur et en totalité. Le Chapitre 2 est consacré à la preuve de cette équivalence. Un ingrédient essentiel est la preuve de la formule d'énergie (0.0.2), qui résulte d'une simple application du théorème de Fubini-Tonelli, dont l'utilisation licite n'a pas toujours été clairement justifiée, puisque les mesures doivent être σ -finies pour appliquer le théorème. C'est la raison pour laquelle on démontre au préalable la rectifiabilité des plans d'irrigation de coût fini. Écrivons formellement le calcul:

$$\begin{aligned} \mathbf{I}^\alpha(\eta) &\doteq \int_\Gamma \int_\gamma \theta_\eta(x)^{\alpha-1} dx d\eta(\gamma) = \int_\Gamma \int_K \mathbf{1}_{x \in \gamma} \theta_\eta(x)^{\alpha-1} d\mathcal{H}^1(x) d\eta(\gamma) \\ &= \int_K \theta_\eta(x)^{\alpha-1} \eta(\{\gamma : x \in \gamma\}) d\mathcal{H}^1(x) \\ &= \int_K \theta_\eta(x)^\alpha d\mathcal{H}^1(x). \end{aligned}$$

La deuxième égalité résulte d'un changement de variable (ou la co-aire), pour peu que les courbes soient injective, la suivante par une interversion de l'ordre d'intégration, et la dernière par définition de la multiplicité $\theta_\eta(x)$. Cette forme est très proche de la α -masse, de sorte que la comparaison des énergies sera aisée, si tant est qu'on puisse construire un flot d'irrigation de coût comparable à partir d'un plan d'irrigation et réciproquement. Du cas lagrangien au cas eulérien, les choses sont plus faciles. À partir de η on construit un flot v_η en superposant les flots associés à chaque courbe, en posant pour toute fonction $\psi \in \mathcal{C}(K)^d$

$$\langle v_\eta, \psi \rangle = \int_\Gamma \int_\gamma \psi(x) \cdot dx d\eta(\gamma),$$

où $\int_\gamma \psi(x) \cdot dx := \int_0^{+\infty} \psi(\gamma(t)) \cdot \dot{\gamma}(t) dt$. Un flot v étant donné, la construction d'un plan d'irrigation induisant v s'obtient par une décomposition de Smirnov [Smi93] obtenue dans le cadre des mesures vectorielles à divergence mesurée par une approche à la Dacorogna-Moser dans [San14], lorsque v est sans cycle. On aboutit au théorème principal du Chapitre 2:

Theorem 0.0.1 (Équivalence des modèles). *Si $\alpha < 1$ and $\mu, \nu \in \text{Prob}(K)$, le problème eulérien (\mathbf{EI}_α) et le problème lagrangien (\mathbf{LI}_α) sont équivalents au sens où :*

(i) *les minima sont les mêmes*

$$\min_{\eta: \mu \rightarrow \nu} \mathbf{I}^\alpha(\eta) = \min_{v: \mu \rightarrow \nu} \mathbf{M}^\alpha(v),$$

(ii) *si v est optimal, il peut être représenté par un plan d'irrigation $v = v_\eta$ où $\eta : \mu \rightarrow \nu$ est optimal,*

(iii) *if η est optimal, alors v_η est optimal.*

Un bref aperçu de la théorie

Donnons un aperçu rapide de la théorie, en restant assez vague sur le formalisme, certains résultats s'exprimant plus simplement dans un modèle ou dans un autre.

Angles minimaux Reprenons le cas d'une source x_0 de masse m_0 irriguant deux puits x_1, x_2 à même distance de la source, de masses m_1, m_2 telles que $m_0 = m_1 + m_2$. Considérons un graphe G_x de structure en Y, de point de branchement situé en x . Si G_x est d'énergie minimale, x doit être un point critique de la fonction

$$x \mapsto \mathbf{E}^\alpha(G_x) = m_0^\alpha |x_0 - x| + m_1^\alpha |x_1 - x| + m_2^\alpha |x_2 - x|,$$

ce qui s'écrit

$$m_0^\alpha n_0 + m_1^\alpha n_1 + m_2^\alpha n_2 = 0 \quad \text{où} \quad n_i = \frac{x_i - x}{|x_i - x|}.$$

En posant $k_1 = \frac{m_1}{m_0}, k_2 = \frac{m_2}{m_0}$, on obtient les relations sur les angles:

$$\cos \theta_1 = \frac{k_1^{2\alpha} + 1 - k_2^{2\alpha}}{2k_1^\alpha} \tag{0.0.3}$$

$$\cos \theta_2 = \frac{k_2^{2\alpha} + 1 - k_1^{2\alpha}}{2k_2^\alpha} \tag{0.0.4}$$

$$\cos(\theta_1 + \theta_2) = \frac{1 - k_2^{2\alpha} - k_1^{2\alpha}}{2k_1^\alpha k_2^\alpha}. \tag{0.0.5}$$

Si x_1 et x_2 sont équidistants de x_0 et $m_1 = m_2$, alors $\theta_1 = \theta_2 =: \theta$ et x doit être équidistant de x_1 et x_2 . De plus $2\theta = \arccos(2^{2\alpha-1} - 1) =: \theta_\alpha$ et $\theta = \arccos(2^{\alpha-1})$. Ainsi si $\theta \geq \theta_\alpha$, l'optimum ne peut être une structure en Y. En notant $\ell = |x_1 - x_0| = |x_2 - x_0|$, le coût de la structure en V est

$$c_V = \ell m^\alpha 2^{1-\alpha},$$

tandis que lorsque $\theta < \theta_\alpha$ la structure en Y (avec l'angle θ_α au branchement) coûte, après un calcul élémentaire :

$$c_Y = \ell m^\alpha 2^{1-\alpha} \cos((\theta_\alpha - \theta)/2) < c_V,$$

et c'est le minimiseur.

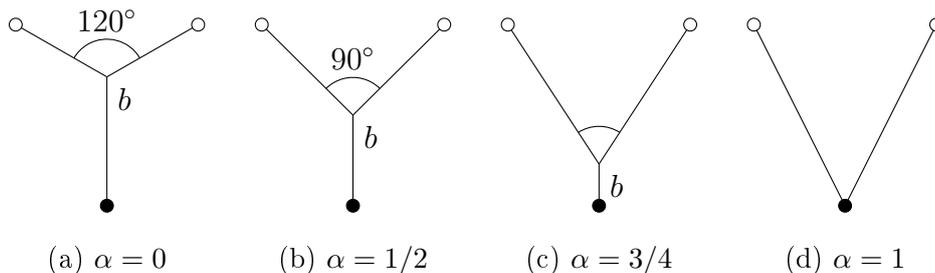


Figure 4: Graphes optimaux dans le cas d'une source et deux puits de même masse.

De la même manière, lorsque x_1, x_2, m_1, m_2 sont quelconques, on peut vérifier que la structure en Y est préférable dès lors que $\cos(\theta) < \frac{1 - k_2^{2\alpha} - k_1^{2\alpha}}{2k_1^\alpha k_2^\alpha}$, c'est à dire

$$\theta > \arccos\left(\frac{1 - k_2^{2\alpha} - k_1^{2\alpha}}{2k_1^\alpha k_2^\alpha}\right) \geq \arccos(2^{2\alpha-1} - 1) =: \theta_\alpha.$$

Ainsi, si x est un point de branchement, c'est-à-dire qu'il y a au moins 2 arêtes entrantes ou 2 arêtes sortantes en x , notées e_1, e_2 , par optimalité, l'angle entre e_1 et e_2 doit être supérieur ou égal à θ_α .

Régularité du réseau Intéressons-nous maintenant au cas continu. Commençons par dire qu'il est possible de montrer, en un certain sens, que les optimiseurs ne possèdent pas de cycles, comme dans le modèle discret. Dans le modèle lagrangien, par exemple, on définit le réseau d'un plan d'irrigation η comme l'ensemble des points par lesquels circulent des particules :

$$N_\eta = \{x : \theta_\eta(x) > 0\}.$$

Des propriétés de régularité du réseau ont été démontrées par Xia [Xia04; Xia11] dans le modèle eulérien et [BCM08] dans le modèle lagrangien. Loin des mesures μ et ν , à "l'intérieur du réseau", on retrouve la structure d'un graphe.

Theorem 0.0.2 (Xia [Xia04]). *Si v est un flot optimal entre μ et ν et B est une boule à distance positive de $\text{supp } \mu \cup \text{supp } \nu$ alors v restreint à B est un graphe discret.*

Passons maintenant à la régularité au "bord" du réseau.

Theorem 0.0.3 ([BCM08; Xia11]). *Si x est un point du réseau, le nombre de branches qui en partent est borné par une quantité $Q = Q(d, \alpha)$.*

Dans le cas d'une simple source $\mu = \delta_0$, sous des hypothèses de régularité sur ν , on peut en dire davantage.

Theorem 0.0.4 ([MS10]). *Si γ est une courbe "du réseau", alors elle est à courbure localement bornée. De plus, elle admet des demi-tangentes en chaque point, et une tangente en chaque point qui n'est pas un point de branchement.*

Dans [MS10], les auteurs montrent de plus que γ vérifie une certaine EDO elliptique.

Irrigabilité et distance d'irrigation Les mesures μ et ν étant fixées, il n'est pas garanti qu'elles puissent être connectées avec un α -coût fini. Néanmoins, la relation " μ et ν peuvent être connectées à α -coût fini" est une relation d'équivalence, et tout couple de mesures peut être connecté si toute mesure peut se connecter à une masse de Dirac, auquel cas la mesure μ est dite α -irrigable. Un raisonnement dyadique a permis à Xia de démontrer un critère d'irrigabilité.

Theorem 0.0.5 (Irrigabilité [Xia03]). *Si $\alpha > 1 - \frac{1}{d}$, alors toute mesure μ supportée sur un compact de \mathbb{R}^d est α -irrigable.*

Cette condition est nécessaire au sens où la mesure uniforme sur un cube n'est pas α -irrigable si $\alpha = 1 - \frac{1}{d}$ et $d > 1$.

Si $\alpha > 1 - \frac{1}{d}$, alors pour toute paire de mesures $\mu, \nu \in \text{Prob}(K)$ où K est un compact fixé de \mathbb{R}^d , la quantité

$$d_\alpha(\mu, \nu) = \min\{\mathbf{I}^\alpha(\eta) : \mu \xrightarrow{\eta} \nu\}$$

est finie, et c'est une distance sur $\text{Prob}(K)$ qui métrise la convergence faible- \star des mesures, appelée *distance d'irrigation*. C'est un analogue aux distances de Wasserstein en transport optimal classique. On peut étendre naturellement d_α aux mesures positives de même masse, et le coût en transport branché étant de la forme $m^\alpha \ell$, on peut démontrer que

$$d_\alpha(\mu, \nu) \leq C m^\alpha \text{diam}(\text{supp } \mu \cup \text{supp } \nu),$$

où $m = \|\mu\| = \|\nu\|$ et $C = C(\alpha, d)$. Cette distance sera utilisée à maintes reprises dans la thèse.

Fonction paysage Dans le cas d'une seule source $\mu = \delta_0$, on peut définir une *fonction paysage* z_η associée à un plan d'irrigation optimale $\eta : \delta_0 \rightarrow \nu$, qui est devenue un outil fondamental en transport branché. Elle a été introduite par Santambrogio dans [San07], et est inspirée de la fonction du même nom utilisée par les géophysiciens (voir [RR01]). Elle est définie comme

$$z_\eta(x) = \int_\gamma \theta_\eta(x)^{\alpha-1} dx$$

si γ est une "courbe du réseau" (en un sens à définir) de 0 à x , et $z_\eta(x) = +\infty$ s'il n'existe pas de telle courbe. Évidemment, il faut vérifier que cette définition est cohérente et ne dépend pas du choix de γ . Un intérêt de la fonction paysage est par exemple le fait qu'elle joue le rôle de variation première pour la fonctionnelle

$$\mathbf{X}^\alpha(\nu) = d_\alpha(\delta_0, \nu)$$

au sens où

$$\mathbf{X}^\alpha(\nu') \leq \mathbf{X}^\alpha(\nu) + \alpha \int z_\eta d(\nu' - \nu)$$

pour une certaine classe de compétiteurs ν' . La fonction paysage est donc essentielle dans l'étude de problèmes faisant intervenir la distance d'irrigation (au départ d'une source), ce qui est l'objet du Chapitre 3. Par ailleurs, des propriétés de régularité Hölder peuvent être établies sous des hypothèses de régularité pour ν , une propriété utilisée pour démontrer la courbure localement bornée des branches, la bornitude uniforme des longueurs des branches, et des comportements fractals des structures optimales.

Fractalité en transport branché

En regardant les réseaux optimaux calculés numériquement dans [OS11] (dans les cas non-atomiques), or aux réseaux fluviaux naturels et leurs bassins d'irrigation, on est tenté de les décrire comme fractals. Des travaux venant de la géophysique et l'hydrologie vont dans ce sens [RR01]. En fait, même si le réseau sous-jacent N_η a une infinité de branchements, il s'agit toujours d'un ensemble 1-rectifiable, il n'est donc pas clair en quel sens la fractalité apparaît. Le terme fractals est usuellement associé ou bien à des propriétés d'autosimilarité d'objets non lisses, ou à la présence d'une dimension non entière. Un premier résultat rigoureux en transport branché en tombant d'une certaine manière dans la première catégorie a été prouvé par Brancolini et Solimini dans [BS14] : pour des mesures suffisamment diffuses (par exemple la mesure de Lebesgue sur un ensemble régulier), le nombre de branches de longueur $\sim \varepsilon$ partant d'une branche de longueur ℓ est de l'ordre de ℓ/ε . Cela peut être vu comme une propriété d'auto-similarité car d'une certaine façon la longueur total est préservée en regardant les sous-branches à toute échelle.

Et qu'en est-il pour l'autre versant de la fractalité, c'est-à-dire de la présence de dimension non entière (ou fractale) ? Plusieurs conjectures ont été émises quant aux ensembles pouvant présenter une dimension fractales ; ce peut être la frontière des bassins d'irrigation (une conjecture de J.-M. Morel), ou bien certains ensembles de niveaux de la fonction paysage introduite précédemment. On s'est intéressé avec F. Santambrogio et Q. Xia à un autre candidat potentiel relié au transport branché : le

bord de ce qu'on pourrait appeler les "boules unité" en transport branché. Ce sont les ensembles de volume 1 les plus proches de l'origine 0 au sens de l'irrigation. C'est l'objet du Chapitre 3. Pour donner un sens rigoureux à cela, tous les objets sont vus comme des mesures, 0 devenant δ_0 et un ensemble borélien E comme $\mathcal{L} \llcorner E$, ce qui donne lieu lorsque $\alpha > 1 - 1/d$ au problème d'optimisation de forme

$$\min \{d_\alpha(\delta_0, \mathcal{L} \llcorner E) \doteq \mathbf{X}^\alpha(\mathcal{L} \llcorner E) : \text{Vol}(E) = 1\}. \quad (0.0.6)$$

Le problème d'optimisation de forme est traité en considérant un problème relaxé, remplaçant E par des mesures à densité encadrée par 0 et 1, comme suit

$$\min \{\mathbf{X}^\alpha(\nu) : 0 \leq \nu \leq 1, \nu \in \text{Prob}(\mathbb{R}^d)\}.$$

La fonction paysage est essentielle dans l'étude de ce problème, les minimiseurs (dont l'existence se démontre par la méthode directe en calcul des variations) étant de la forme

$$E = \{x : z(x) \leq z^*\}$$

où z est une fonction paysage associé à un plan d'irrigation optimal $\eta : \delta_0 \rightarrow \mathcal{L} \llcorner E$ et $z^* < \infty$. Notre étude a permis de démontrer des propriétés topologiques élémentaires sur E (compacité, connexité, propriétés sur l'intérieur) et la régularité Hölder de la fonction paysage z , bien qu'on ne connaisse pas la régularité de E (et qu'on conjecture l'inverse). De cette façon, on a pu obtenir une borne supérieure sur la dimension de Minkowski du bord

Theorem 0.0.6. *Soit E solution de (0.0.6). On a*

$$\overline{\dim}_M(\partial E) \leq d - \beta,$$

où $\beta = 1 + d\alpha - d$ est l'exposant de Hölder de la fonction paysage.

On constate que l'exposant β varie de 0 à 1 lorsque α va de $1 - 1/d$ à 1, or on s'attend à ce que le réseau devienne de plus en plus diffus lorsque $\alpha \rightarrow 1$ pour retomber sur le problème de Monge. Or on sait que l'ensemble le plus proche de 0 pour la distance de Wasserstein W_1 est évidemment la boule, dont le bord est lisse et de dimension $d - 1$. On fait donc la conjecture suivante :

Conjecture 0.0.7. *Si E est solution de (0.0.6), alors*

$$\dim_H(\partial E) = \dim_M(\partial E) = d - \beta.$$

Démontrer ce résultat revient à obtenir la minoration $\underline{\dim}_H \partial E \geq d - \beta$, mais nos stratégies n'ont pas encore permis de l'établir, laissant pour l'heure la question de la fractalité ouverte. Une résolution numérique du problème a été entreprise, basée sur le modèle de type Modica-Mortola proposé par Oudet et Santambrogio [OS11]. La fonctionnelle du transport branché, dans sa version eulérienne, est approchée au sens de la Γ -convergence par

$$\mathbf{M}_\varepsilon^\alpha(v) = \varepsilon^{-\gamma_1} \int |v(x)|^\sigma dx + \varepsilon^{\gamma_2} \int |\nabla v(x)|^2 dx,$$

où $\sigma, \gamma_1, \gamma_2$ sont bien choisis et $\varepsilon \rightarrow 0$. Il s'agit donc de minimiser cette fonctionnelle sous contrainte inégalité sur la divergence de v pour notre problème d'optimisation de forme : $0 \leq \nabla \cdot \delta^\varepsilon - v_\varepsilon \leq 1$ où δ^ε est une certaine approximation de δ_0 . Dans le problème à source et destination fixées, on imposait une contrainte égalité sur la divergence, dont la projection pouvait se faire très rapidement numériquement (par simple résolution d'un Laplacien). Dans notre approche, où la projection est plus compliquée, une méthode de descente de type quasi-Newton proximale a été utilisée. Des expériences numériques ont été menées, dont on donne un échantillon en Figure 5.

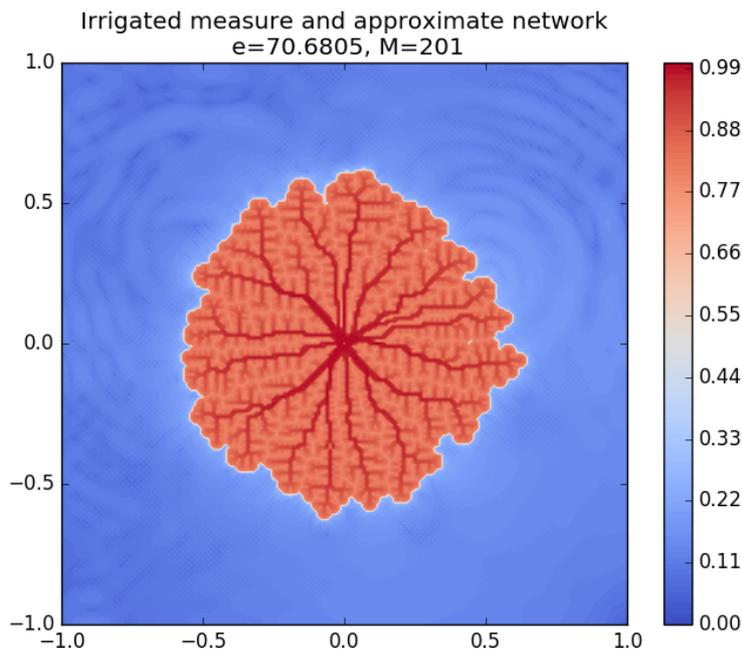


Figure 5: Mesure irriguée et réseau, $\alpha = 0.55$

Problèmes variationnels en transport branché

Pour modéliser certaines phénomènes, on peut être amené à considérer des problèmes du type

$$\min \{ \mathbf{X}^\alpha(\nu) + \mathcal{F}(\nu) \},$$

le terme $\mathbf{X}^\alpha(\nu)$ représentant le coût de construction d'un réseau couvrant ν ou un coût d'acheminement (d'un bien, d'un fluide etc.) vers ν , tandis que $\mathcal{F}(\nu)$ est un coût associé à la distribution irriguée. Une fonctionnelle \mathcal{F} convexe, comme

$$\mathcal{F}_2(\nu) = \begin{cases} \int_{\mathbb{R}^d} \frac{f(x)^2}{2} dx & \text{si } \nu = f \mathcal{L} \text{ et } \int_{\mathbb{R}^d} f = 1, \\ +\infty & \text{sinon,} \end{cases}$$

comme envisagé dans [San07], favorisera une répartition diffuse et pourrait schématiquement modéliser une feuille ou le feuillage dont le but est de maximiser la surface

à même de capter la lumière, ou la répartition d'une densité de population voulant se répartir le plus proche du centre-ville (par un réseau) mais en évitant de fortes densités. D'autres problèmes variationnels faisant intervenir \mathbf{X}^α sont proposés dans [MSM03].

Le problème d'optimisation de forme défini précédemment entre dans ce cadre, avec

$$\mathcal{F}(\nu) = \begin{cases} 0 & \text{si } \nu \in \text{Prob}(\mathbb{R}^d) \text{ et } 0 \leq \nu \leq 1, \\ +\infty & \text{sinon,} \end{cases}$$

de même que le problème d'évolution de forme envisagé dans le Chapitre 5, où on considère une suite de fonctionnelles $\mathcal{F} = \mathcal{F}_k^\tau$ avec

$$\mathcal{F}_k(\nu) = \int_{\mathbb{R}^d} \frac{d_s(x, E_k)}{\tau} d\nu - \lambda \int_{\mathbb{R}^d} d\nu,$$

où $\lambda > 0$ est fixé et $d_s(\cdot, E_k)$ représente la distance signée à l'ensemble E_k .

Un outil essentiel pour l'étude de tels problèmes est la fonction paysage z . Par exemple, dans le cas $\mathcal{F} = \mathcal{F}_2$, des résultats descriptifs et de régularité en fonction de z devraient pouvoir se démontrer. En particulier, l'équation d'Euler-Lagrange devrait être de la forme

$$\alpha z + f = \text{cst} \quad \text{sur } \{f > 0\},$$

de sorte que f s'exprime en fonction de s . De plus, comme $z(x) \geq |x|$ on obtient une borne sur le support de f . Enfin, si z est β -Hölder, ce qui est le cas lorsque ν est assez régulière – qu'il faudrait donc démontrer au préalable – alors f l'est également.

Néanmoins, ce type d'étude ne peut être effectué que pour la fonctionnelle $\mathbf{X}^\alpha = d_\alpha(\delta_0, \nu)$, pour laquelle on sait définir la fonction paysage, or on aimerait pouvoir remplacer δ_0 par plusieurs sources, éventuelle une mesure diffuse μ . Le Chapitre 4 s'intéresse donc à définir la fonction paysage dans le cas de sources multiples, et montrer des résultats de régularité Hölder tels que ceux connus dans le cas d'une simple source. Pour l'instant, on arrive à généraliser la définition et les résultats de régularités pour un plan d'irrigation optimal $\eta : \mu \rightarrow \nu$ qui possède un système fini de racines : il existe N points x_1, \dots, x_N tels que η -presque toute courbe passe par l'un de ces points. Cette condition est en particulier garantie lorsque $d(\text{supp } \mu, \text{supp } \nu) > 0$. Ceci est un premier pas vers une définition générale de la fonction paysage pour μ et ν quelconques. La régularité de la fonction paysage devrait nécessiter en revanche certaines hypothèses sur le réseau ou les mesures à connecter (séparation des mesures, réseau admettant un nombre fini de racine, par exemple) qui ne sont pas réalisées en général.

La thèse s'achève par un dernier chapitre dédié à des perspectives pouvant prolonger les travaux présentés dans les précédents chapitres.

General theory

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The aim of this first chapter is threefold. First of all, we want to give a brief but precise description of what we consider to be the main models of branched transport. We are going to define the discrete model, which is the oldest one as it was introduced in the 60's by Gilbert in [Gil67], though it was not termed branched transport at that time. After that, we will go on with the two main continuous and static models: the *Lagrangian model*, which uses the measures on path formalism, and the *Eulerian model*, which uses the language of 1-dimensional currents or vector measures. Our second purpose is to make an emphasis on the Lagrangian model, which is the most used in the next chapters, and which we want to rephrase using measures on paths, which we call *irrigation plans*, rather than “parameterized” traffic plans used in the works of Bernot, Caselles, Morel [BCM09], or patterns defined by Morel, Maddalena, Solimini [MSM03]. With this formalism we are able to give simple proofs of the elementary properties of minimizers, and to provide a rigorous basis to establish the equivalence between the two continuous models, which is the object of Chapter 2. The final purpose of this chapter is to set all the definitions and state all the results to be used in the rest of the thesis, which will consequently be almost self-contained. Most of the results of this chapter were already proved or could be assembled from the existing literature, but we give proofs of some of them when we feel we were able to provide a simpler proof or when the adaptation of the existing proofs to our irrigation plans framework is not straightforward.

1.1 The discrete model

Let us start with the discrete model, which is the most concrete and easiest to grasp. We consider two atomic probability measures μ, ν on \mathbb{R}^d :

$$\mu = \sum_{i=1}^n a_i \delta_{x_i}, \quad \nu = \sum_{j=1}^m b_j \delta_{y_j},$$

and we want to connect them in the cheapest possible way. In this discrete case, the connection will be achieved through a graph, and the cost associated to each graph will be the so-called Gilbert energy [Gil67].

Irrigation graph An irrigation graph G is a weighted oriented graph, that is a triple $G = (V, E, \Theta = (\theta_e)_{e \in E})$, where V is a set of vertices, E a set of oriented edges between vertices and θ_e is a nonnegative weight associated to each edge e . We say that G sends μ to ν if

- (i) it satisfies Kirchhoff's Law, that is at each vertex $v \in V$,

$$\text{incoming mass at } v = \text{outgoing mass at } v,$$

- (ii) μ is considered as incoming and ν as outgoing: if $v = x_i$ then a_i counts as incoming and if $v = y_j$ then b_j counts as outgoing.

We denote the set of irrigation graphs by $\mathbf{IG}(\mathbb{R}^d)$ and by $\mathbf{IG}(\mu, \nu)$ the set of those sending μ to ν . For short, we will write $G : \mu \rightarrow \nu$ or $\mu \xrightarrow{G} \nu$ to say that G sends μ to ν .

Gilbert energy For $\alpha \in [0, 1]$, we associate to every irrigation graph G its Gilbert energy (see [Gil67; GP68]):

$$\mathbf{E}^\alpha(G) = \sum_{e \in E} \theta_e^\alpha |e|,$$

where $|e|$ denotes the length of e . Notice that the cost is proportional to $m^\alpha \times \ell$: if one wants to send a mass m at distance ℓ along the graph then one should pay $m^\alpha \ell$.

The discrete irrigation problem The goal is to minimize Gilbert energy among all graphs which send μ to ν , which reads:

$$\min \left\{ \mathbf{E}^\alpha(G) \doteq \sum_{e \in E} \theta_e^\alpha |e| : \mu \xrightarrow{G} \nu \right\}. \quad (\text{DI}_\alpha)$$

Since the map $x \mapsto x^\alpha$ is subadditive, i.e.

$$(x + y)^\alpha \leq x^\alpha + y^\alpha, \quad \forall x, y \geq 0,$$

it is less expensive for the mass to travel together as much as possible. This is responsible for the observed branching structures in optimizers in the strictly subadditive case ($\alpha < 1$). Let us look at the two extreme cases.

- $\alpha = 0$

The mass does not appear in the cost, leaving only the total length of the graph: it is called *Steiner problem* (see [GP68]),

- $\alpha = 1$

The cost is linear in the mass, which corresponds the classical *Monge problem* in optimal transport (see [San15] for a general reference on optimal transport).

Consequently, one may view the irrigation problem as an interpolation between two well-known problems: Steiner and Monge's problems.

Before sketching a proof of existence, we will need some terminology. If e is an oriented edge, $\hat{e} \in \mathbb{S}^{d-1}$ denotes its orientation, \bar{e} the same edge with opposite orientation, $|e|$ its length, and e^-, e^+ its initial and final extremity respectively.

Definition 1.1.1. Given $p = (e_1, \dots, e_k)$ a sequence of oriented edges, we say that p is

- a *circuit* if $e_i^+ = e_{i+1}^-$ for every $i = 1, \dots, k$,
- a *cycle* if $\tilde{p} = (\tilde{e}_1, \dots, \tilde{e}_k)$ is a circuit, where $\tilde{e}_i \in \{e_i, \bar{e}_i\}$,

with the convention $e_{k+1} = e_1$ and $\tilde{e}_{k+1} = \tilde{e}_1$.

Theorem 1.1.2. *The discrete irrigation problem (DI_α) admits solutions. Moreover, if G is a solution, then*

- (i) *it has no (non-trivial) inner vertex of degree 2,*
- (ii) *it lies in the convex envelope of the sources and targets,*
- (iii) *it has no circuit,*
- (iv) *it has no cycle if $\alpha < 1$.*

Proof. Let us make some remarks concerning general graphs $G : \mu \rightarrow \nu$.

Support. The measures μ, ν are concentrated on the convex envelope C of the atoms, which is a compact convex set, and one may project any graph $G : \mu \rightarrow \nu$ on C getting another graph \tilde{G} still sending μ to ν , but with reduced cost $\mathbf{E}^\alpha(\tilde{G}) \leq \mathbf{E}^\alpha(G)$, since the projection is 1-Lipschitz. Notice that the inequality is strict if G does not lie entirely in C .

Circuits and cycles. If G has a circuit $p = (e_1, \dots, e_k)$, then one may remove it from the graph by removing a mass $\epsilon = \min_i \theta_{e_i}$ to the edges e_i (and then removing edges with 0 multiplicity), thus getting a new graph \tilde{G} . Since p was a circuit of G , \tilde{G} still satisfies Kirchhoff's Law and sends μ to ν . The fact that $x \mapsto x^\alpha$ is increasing and that we have for sure removed at least one edge guarantees that $\mathbf{E}^\alpha(\tilde{G}) < \mathbf{E}^\alpha(G)$. Hence one may remove circuits iteratively while strictly reducing the cost.

Now, if G has a cycle $p = (e_1, \dots, e_k)$, choose an orientation τ . For $|t| < \epsilon = \min_i \theta_{e_i}$, add t to the weights of the edges of orientation τ (we write $e_i \in \tau$) and remove t to those of opposite orientation, resulting in a graph G_t . At each vertex v , we have added and removed the same amount of mass flowing in, thus Kirchhoff's Law is still satisfied so that $G_t : \mu \rightarrow \nu$. Now let us look at the cost:

$$\mathbf{E}^\alpha(G_t) - \mathbf{E}^\alpha(G) = \sum_{i:e_i \in \tau} |e_i|((\theta_{e_i} + t)^\alpha - \theta_{e_i}^\alpha) + \sum_{i:e_i \notin \tau} |e_i|((\theta_{e_i} - t)^\alpha - \theta_{e_i}^\alpha) =: f(t).$$

If $\alpha < 1$ then $x \mapsto x^\alpha$ is strictly concave, and so is f , hence $f(t) < f(0) + tf'(0) = tf'(0)$. Now choose t with a sign opposite to $f'(0)$ so that $\mathbf{E}^\alpha(G_t) < \mathbf{E}^\alpha(G)$. One may repeat the operation until there is no more cycle while strictly reducing the cost.

Degree 2 vertices. One may remove vertices v which are neither a source nor a target and which have degree 2 (i.e. they have exactly two adjacent edges), replacing the two adjacent edges by a single one. Here again we reduce the cost $\mathbf{E}^\alpha(\tilde{G}) \leq \mathbf{E}^\alpha(G)$ and preserve Kirchhoff's Law with $\tilde{G} : \mu \rightarrow \nu$. Notice that the inequality is strict if there are degree 2 vertex which are not trivial (i.e. such that the adjacent edges are not colinear).

Existence of minimizers In the case $\alpha = 1$, it is easy to build a minimizer: construct G as a superposition of straight paths going from each source x_i to each destination y_j with mass $a_i b_j$. Now we place ourselves in the case $\alpha < 1$ and we consider a minimizing sequence $(G_n)_{n \in \mathbb{N}}$. Thanks to the previous paragraphs, we may assume that the graphs G_n enjoy all the properties (i)–(iv). We may get a uniform bound on the number of edges and vertices of the graphs G_n . Denote by $f(L)$ the supremal number of vertices of graphs connecting L sources and targets, termed *leaves*, and satisfying (i) and (iv). Let us show that $f(L) \leq 2(L - 1)$ by induction. For $L = 2$ it is clear. Now suppose that it is true for $k < L$ and take such a graph G with L leaves. Consider a simple non-oriented path p joining two leaves x, y with a maximal number of vertices. Set \hat{x} the next vertex after x along this path and \hat{e} the corresponding edge. Since \hat{x} has degree at least 3, it is connected to a point which does not belong to p . We claim that the connected component \mathcal{C} of \hat{x} in $G \setminus p$ consists only of edges joining \hat{x} and a leaf. If it was not the case, one could either find a simple path $\hat{p} \in G \setminus p$ connecting \hat{x} to another vertex u which has at least two edges, or an edge $e \notin p$ between \hat{x} and a

point u which is not a leaf. In all cases, if u was not a leaf then \hat{p} or e may be extended to a simple path, connecting \hat{x} to a leaf because G is a cycle-free¹. This extended path is still denoted by \hat{p} and u the leaf which is its other extremity. This path cannot intersect p since G has no cycle. Thus the concatenation of $p \setminus \hat{e}$ and \hat{p} is a simple path joining a pair of leaves and which has strictly more vertices than p because \hat{p} has at least 3 vertices: a contradiction. We denote by A the leaves in \mathcal{C} and we remove \mathcal{C} and \hat{e} from G , thus obtaining a new graph \hat{G} . By construction, $\hat{G} : \hat{\mu} \rightarrow \hat{\nu}$ where $\hat{\mu}$ and $\hat{\nu}$ are obtained from μ, ν by removing the atoms in A and possibly adding an atom at \hat{x} . Moreover this graph still enjoys properties (i) and (iv). Consequently we have removed $\text{card } A + 1$ leaves (those in A , and x) while adding a possible one at \hat{x} , hence there are at most $L - \ell$ leaves where $\ell = \text{card } A \geq 1$, which implies by the induction hypothesis:

$$\begin{aligned} \text{card } V(G) &= \text{card } V(\hat{G}) + \ell + 1 \\ &\leq f(L - \ell) + \ell + 1 \\ &\leq 2(L - \ell - 1) + \ell + 1 \leq 2(L - 1), \end{aligned}$$

thus

$$f(L) \leq 2(L - 1)$$

for all $L \geq 2$. Consequently, the number of edges and vertices of the graphs G_n is uniformly bounded, and they are all contained in a common compact set, so that one may extract a subsequence making all vertices and edges converge, leading to a limit graph $G : \mu \rightarrow \nu$ (even though some edges may disappear in the limit). The continuity of the cost in the weights and vertices allows us to say that $\mathbf{E}^\alpha(G) = \min(\mathbf{DI}_\alpha)$.

Suppose G optimal. The three first paragraphs imply that G is supported in the convex envelope of the sources and targets, that G has no circuit and no (non-trivial) inner vertex of degree 2. Moreover if $\alpha > 1$, G has no cycle. \square

Remark 1.1.3. In the case $\alpha = 1$, it is possible to have optimizers with cycles: consider for example a square with two sources of mass $1/2$ at vertices of a diagonal, and two targets of mass $1/2$ at vertices of the other diagonal. An optimizer consists in 4 segments connecting each source to the two targets, with mass $1/4$. However, it is possible to remove the cycles with unchanged cost (destroying the symmetry).

Examples In the case when there is only one source s and two targets t_1, t_2 , an optimal graph is made of at most three edges. If the targets are equidistant from s with same mass $1/2$, then there are two possible topologies: Y-shaped or V-shaped. When t_1, t_2 are far enough from s , compared to their mutual distance $|t_1 - t_2|$, the Y-shaped graph is the cheapest for $\alpha < 1$, i.e. there is branching. The optimal angle can be explicitly computed in terms of α .

¹One iteratively add a new vertex to the path until there is no more, which happens because G is cycle-free. This last vertex is necessarily a source or target, i.e. a leaf.

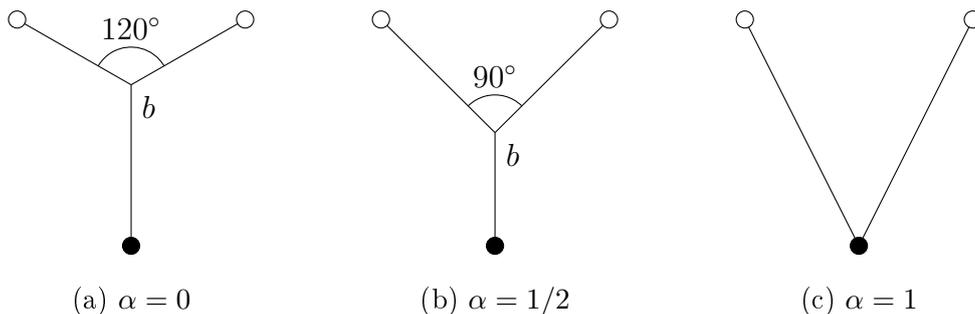


Figure 1.1: Optimal graphs with one source and two targets of equal mass.

Continuous extensions Several continuous extensions to the discrete model have been proposed, which are all equivalent or can be suitably modified to be so. Two dynamical models were proposed, one by Brasco and Santambrogio [BS11] based on a path functional formulation in some Wasserstein space and another one by Brasco, Buttazzo, Santambrogio [BBS11] based on a Benamou-Brenier formulation (see [BB00] for the seminal work on the dynamical formulation of optimal transport). However, we will concentrate on the two main static models: the first one, which dates back to 2003 and is the earliest continuous model of branched transport, has been introduced by Xia in [Xia03]. We will call it the *Eulerian model* as it is based on vector measures which can be viewed as a velocity field associated to moving particles. The other one is based on measures on paths: the traffic plan formulation introduced by Bernot, Caselles, Morel [BCM09], and the pattern formulation introduced by Maddalena, Morel, Solimini [MSM03], which are almost identical and which we will call *Lagrangian model*. We are going to give a description of the Eulerian and Lagrangian models, with an emphasis on the last one, as it allows us to define a “landscape function” which is now a central tool in branched transport.

1.2 The Lagrangian model

The first lagrangian model of branched transport was introduced by Morel, Maddalena, Solimini in [MSM03] using what they called irrigation patterns, which essentially consist of collections of curves χ_ω labeled by some index ω living in a probability space (Ω, \mathbb{P}) and which represent the trajectory of each particle. Later, Bernot, Caselles and Morel [BCM05] proposed a formulation based on measures on paths which they called traffic plans, thus getting rid of the parameter ω by considering only $\chi_{\#\mathbb{P}}$ instead of χ . However, many proofs still used such parameterizations χ which induced some complications (measurability issues, extensive use of Skorokhod’s Theorem for instance), which can actually be avoided. We remark that in [MS13] the authors go further in that direction when proving existence of minimizers. That is why in this section we define the Lagrangian model using measures on paths, which we call here *irrigation plans*, and give simple proofs of elementary results, namely existence of minimizers and basic structure properties of optimizers (single-path property, cycle-free property), without resorting to parameterizations, thus avoiding unnecessary technicalities and measurability issues. Also notice that, in this way, the Lagrangian branched transport

model fits the general framework of dynamical transport problems with measures on curves, as in [Bre11] for Incompressible Euler and in [BCS10; CJS08] for traffic congestion. This exposition and its proofs are the object of [Peg16, Sections 1–2], extended here to the non-compact case.

1.2.1 Notations and general framework

Let X be subset of \mathbb{R}^d . We denote by $\Gamma(X)$ the space of 1-Lipschitz curves in X parameterized on \mathbb{R}_+ , embedded with the topology of uniform convergence on compact sets. We write for short $\Gamma = \Gamma(\mathbb{R}^d)$, but we will often take X to be some compact subset K of \mathbb{R}^d . If X is closed (for example a compact set or \mathbb{R}^d) then $\Gamma(X)$ is a Polish space, which can be metrized by $d(\gamma_1, \gamma_2) = \sup_{n \in \mathbb{N}^*} \frac{1}{n} \min(1, \|\gamma_1 - \gamma_2\|_{L^\infty([0, n])})$. If $\phi : X \rightarrow \mathbb{R}$ and $\psi : X \rightarrow \mathbb{R}^d$ are Borel functions on X , $\gamma \in \Gamma(X)$, we set

$$\int_\gamma \phi(x) dx := \int_0^\infty \phi(\gamma(t)) |\dot{\gamma}(t)| dt \quad \text{and} \quad \int_\gamma \psi(x) \cdot dx := \int_0^\infty \psi(\gamma(t)) \cdot \dot{\gamma}(t) dt$$

provided these integrals exist.

Length and stopping time If $\gamma \in \Gamma(X)$, we define its *stopping time* and its *length* respectively by

$$T(\gamma) = \inf\{t \geq 0 : \gamma \text{ is constant on } [t, +\infty[),$$

$$L(\gamma) = \int_0^\infty |\dot{\gamma}(t)| dt,$$

which are valued in $[0, +\infty]$. Since curves are 1-Lipschitz, $L(\gamma) \leq T(\gamma)$. Moreover, one may prove that T and L are both lower semicontinuous functions and, as such, are Borel. We denote by $\Gamma^1(X)$ the set of curves of finite length.

Irrigation plans We call *irrigation plan on \mathbb{R}^d* any probability measure $\eta \in \text{Prob}(\Gamma(\mathbb{R}^d))$ satisfying

$$\mathbf{L}(\eta) := \int_{\Gamma(\mathbb{R}^d)} L(\gamma) d\eta(\gamma) < +\infty, \tag{1.2.1}$$

and denote by $\mathbf{IP}(\mathbb{R}^d)$ the set of irrigation plans on \mathbb{R}^d . Notice that any irrigation plan is concentrated on $\Gamma^1(\mathbb{R}^d)$. Since this set is not closed in $\text{Prob}(\Gamma(\mathbb{R}^d))$, we will introduce some closed or compact subsets later. If μ and ν are two probability measures on \mathbb{R}^d , one says that $\eta \in \mathbf{IP}(\mathbb{R}^d)$ irrigates ν from μ if one recovers the measures μ and ν by sending the mass of each curve respectively to its initial point and to its final point, which means that

$$(\pi_0)_\# \eta = \mu, \quad (\pi_\infty)_\# \eta = \nu,$$

where $\pi_0(\gamma) = \gamma(0)$, $\pi_\infty(\gamma) = \gamma(\infty) := \lim_{t \rightarrow +\infty} \gamma(t)$ and $f_\# \eta$ denotes the push-forward of η by f whenever f is a Borel map². We denote by $\mathbf{IP}(\mu, \nu)$ the set of irrigation plans irrigating ν from μ :

$$\mathbf{IP}(\mu, \nu) = \{\eta \in \mathbf{IP}(\mathbb{R}^d) : (\pi_0)_\# \eta = \mu, (\pi_\infty)_\# \eta = \nu\}.$$

²Notice that $\lim_{t \rightarrow \infty} \gamma(t)$ exists if $\gamma \in \Gamma^1(\mathbb{R}^d)$, and this is all we need since any irrigation plan is concentrated on $\Gamma^1(K)$.

Arc length parameterization We say that a curve $\gamma \in \Gamma$ is *parameterized by arc length* if it has unit speed until it stops, i.e. $|\dot{\gamma}(t)| = 1$ for a.e. $t \in [0, T(\gamma)[$. If an irrigation plan $\eta \in \mathbf{IP}(\mathbb{R}^d)$ is such that η -a.e. curve γ is parameterized by arc length, we say that η is itself *parameterized by arc length*. Set $\phi : \gamma \mapsto \tilde{\gamma}$ the map which associates to each curve $\gamma \in \Gamma(K)$ its arc-length parameterization³. One may define the arc length parameterization of η as $\tilde{\eta} := \phi_{\#}\eta$ and check that $(\pi_0)_{\#}\eta = (\pi_0)_{\#}\tilde{\eta}$ and $(\pi_{\infty})_{\#}\eta = (\pi_{\infty})_{\#}\tilde{\eta}$.

Multiplicity Given an irrigation plan $\eta \in \mathbf{IP}(\mathbb{R}^d)$, let us define the multiplicity $\theta_{\eta} : \mathbb{R}^d \rightarrow [0, \infty]$ as

$$\theta_{\eta}(x) = \eta(\gamma \in \Gamma(\mathbb{R}^d) : x \in \gamma).$$

It represents the amount of mass which flows by x through curves of η . We call *network* associated to η the set N_{η} of points with positive multiplicity, i.e. points that are really visited by η :

$$N_{\eta} := \{x \in \mathbb{R}^d : \theta_{\eta}(x) > 0\}.$$

Simplicity If $\gamma \in \Gamma$, we denote by

$$m(x, \gamma) = \#\{t \in [0, T(\gamma)] \cap \mathbb{R}_+ : \gamma(t) = x\} \in \mathbb{N} \cup \{\infty\}$$

the multiplicity of x on γ , that is the number of times the curve γ visits x . We call *simple points* of $\gamma \in \Gamma$ those which are visited only once, i.e. such that $m(x, \gamma) = 1$ and denote by S_{γ} the set of such points. We say that γ is *simple* if $\gamma \setminus S_{\gamma} = \emptyset$ and *essentially simple* if $\mathcal{H}^1(\gamma \setminus S_{\gamma}) = 0$. As usual we extend these definitions to irrigation plans, saying that η is *simple* (resp. *essentially simple*) if η -a.e. curve is simple (resp. essentially simple). Finally we set

$$m_{\eta}(x) := \int_{\Gamma} m(x, \gamma) d\eta(\gamma)$$

which represents the mean number of times curves visit x . Notice that

$$\theta_{\eta}(x) = \int_{\Gamma} \mathbf{1}_{x \in \gamma} d\eta(\gamma) \leq \int_{\Gamma} m(x, \gamma) d\eta(\gamma) = m_{\eta}(x)$$

so that $m_{\eta}(x)$ is in a way the “full” multiplicity at x .

1.2.2 The Lagrangian irrigation problem

For $\alpha \in [0, 1]$ we consider the *irrigation cost* $\mathbf{I}^{\alpha} : \mathbf{IP}(\mathbb{R}^d) \rightarrow [0, \infty]$ defined by

$$\mathbf{I}^{\alpha}(\eta) := \int_{\Gamma} \int_{\gamma} \theta_{\eta}(x)^{\alpha-1} dx d\eta(\gamma),$$

with the conventions $0^{\alpha-1} = \infty$ if $\alpha < 1$, $0^{\alpha-1} = 1$ otherwise, and $\infty \times 0 = 0$. If μ, ν are two probability measures on \mathbb{R}^d , the Lagrangian irrigation problems consists in minimizing the cost \mathbf{I}^{α} on the set of irrigation plans which send μ to ν , which reads

$$\min_{\eta \in \mathbf{IP}(\mu, \nu)} \int_{\Gamma} \int_{\gamma} \theta_{\eta}(x)^{\alpha-1} dx d\eta(\gamma). \quad (\text{LI}_{\alpha})$$

³One may check that it is Borel.

Notice that \mathbf{I}^α is invariant by arc length reparameterization, thus we will often assume without loss of generality that irrigation plans are parameterized by arc length.

First variation The cost \mathbf{I}^α may be written as

$$\mathbf{I}^\alpha(\eta) = \int_{\Gamma} Z_\eta^\alpha(\gamma) d\eta(\gamma) \quad \text{where} \quad Z_\eta^\alpha(\gamma) := \int_{\gamma} \theta_\eta(x)^{\alpha-1} dx$$

is the α -cost the curve $\gamma \in \Gamma$ w.r.t. η . This map Z_η is a first variation of \mathbf{I}^α in the following sense:

Proposition 1.2.1 (First variation inequality for \mathbf{I}_α). *If η is an irrigation plan with finite α -cost, then for all irrigation plan $\tilde{\eta}$ the following holds:*

$$\mathbf{I}_\alpha(\tilde{\eta}) \leq \mathbf{I}_\alpha(\eta) + \alpha \int Z_\eta(\gamma) d(\tilde{\eta} - \eta). \quad (1.2.2)$$

Notice that the integral $\int Z_\eta d(\tilde{\eta} - \eta)$ is well-defined since $\int Z_\eta d\eta < \infty$ and Z_η is nonnegative, though it may be infinite.

This result may be obtained by adapting [San07, Theorem 3.1].

1.2.3 Existence of minimizers

In this section we prove the lower semicontinuity and compactness results leading to the proof of existence of minimizers by the direct method in the calculus of variations [Giu03]. We recall here that, unless stated otherwise, continuity properties on Γ relate to the topology of uniform convergence on compact sets and on $\mathbf{IP}(\mathbb{R}^d)$ to the weak- \star topology in the duality with $\mathcal{C}_b(\Gamma(\mathbb{R}^d))$.

Closedness and compactness results For $M > 0$, we set

$$\Gamma_M(X) := \{\gamma \in \Gamma(X) : T(\gamma) \leq M\}.$$

Notice that if $K \subseteq \mathbb{R}^d$ is a compact set then $\Gamma_M(K)$ is a compact set for the uniform topology by Ascoli's Theorem. For $C > 0$, we define $\mathbf{IP}_C(X)$ as the set of irrigation plans η on X which satisfy:

$$\mathbf{T}(\eta) := \int_{\Gamma(X)} T(\gamma) d\eta(\gamma) \leq C,$$

and we set $\mathbf{IP}_C(\mu, \nu) := \mathbf{IP}_C(\mathbb{R}^d) \cap \mathbf{IP}(\mu, \nu)$. By lower semicontinuity of \mathbf{T} on $\text{Prob}(\Gamma(\mathbb{R}^d))$ and because $\mathbf{L} \leq \mathbf{T}$, it is easy to check that $\mathbf{IP}_C(\mathbb{R}^d)$ is a closed subset of $\text{Prob}(\Gamma(\mathbb{R}^d))$. Now let us investigate the closedness of $\mathbf{IP}_C(\mu, \nu)$, via the continuity of the maps $\pi_0 : \eta \mapsto (\pi_0)_\# \eta$ and $\pi_\infty : \eta \mapsto (\pi_\infty)_\# \eta$ on $\mathbf{IP}(\mathbb{R}^d)$.

Proposition 1.2.2. *The maps $\pi_0, \pi_\infty : \mathbf{IP}_C(\mathbb{R}^d) \rightarrow \text{Prob}(\mathbb{R}^d)$ are continuous⁴. In particular $\mathbf{IP}_C(\mu, \nu)$ is a closed subset of $\text{Prob}(\Gamma(\mathbb{R}^d))$.*

⁴Recall that we always endow spaces of measures with their weak- \star topology in the duality with \mathcal{C}_b .

Proof. Clearly the map $\pi_0 : \gamma \mapsto \gamma(0)$ is continuous on $\Gamma(\mathbb{R}^d)$, hence $\boldsymbol{\pi}_0$ is continuous on $\mathbf{IP}(\mathbb{R}^d)$. However $\pi_\infty : \gamma \mapsto \gamma(\infty)$ defined on Γ^1 is not necessarily continuous thus $\boldsymbol{\pi}_\infty$ needs not be continuous on $\mathbf{IP}(\mathbb{R}^d)$. Nevertheless, given any $\eta \in \mathbf{IP}_C(\mu, \nu)$, by Markov's inequality:

$$\eta(\gamma : T(\gamma) > M) \leq \frac{C}{M}, \quad (1.2.3)$$

which rewrites

$$\eta(\Gamma(\mathbb{R}^d) \setminus \Gamma_M(\mathbb{R}^d)) \leq \frac{C}{M}. \quad (1.2.4)$$

Since π_∞ is continuous on all the sets $\Gamma_M(\mathbb{R}^d)$ for $M > 0$, inequality (1.2.4) allows us to conclude. Indeed take $\eta_n \rightharpoonup \eta$ in $\mathbf{IP}_C(\mathbb{R}^d)$. Take $\epsilon > 0$ and M large enough so that $\frac{C}{M} \leq \epsilon$. For any $\phi \in \mathcal{C}_b(\mathbb{R}^d)$ one has

$$\int_{\Gamma} \phi(\gamma(\infty)) d\eta_n(\gamma) = \int_{\Gamma} \phi(\gamma(M)) d\eta_n(\gamma) + \int_{\Gamma \setminus \Gamma_M} (\phi(\gamma(\infty)) - \phi(\gamma(M))) d\eta_n(\gamma)$$

and

$$\int_{\Gamma} \phi(\gamma(\infty)) d\eta(\gamma) = \int_{\Gamma} \phi(\gamma(M)) d\eta(\gamma) + \int_{\Gamma \setminus \Gamma_M} (\phi(\gamma(\infty)) - \phi(\gamma(M))) d\eta(\gamma)$$

thus

$$\begin{aligned} \left| \int_{\Gamma} \phi(\gamma(\infty)) \eta_n(d\gamma) - \int_{\Gamma} \phi(\gamma(\infty)) d\eta(\gamma) \right| \\ \leq \left| \int_{\Gamma} \phi(\gamma(M)) \eta_n(d\gamma) - \int_{\Gamma} \phi(\gamma(M)) d\eta(\gamma) \right| + 4\epsilon \|\phi\|_{\infty}. \end{aligned}$$

We pass to the \limsup_n using the continuity of $\pi_M : \gamma \mapsto \gamma(M)$ on $\Gamma(\mathbb{R}^d)$, then take the limit as $\epsilon \rightarrow 0$ to get

$$\int_{\mathbb{R}^d} \phi(x) (\pi_\infty)_\# \eta_n(dx) \rightarrow \int_{\mathbb{R}^d} \phi(x) (\pi_\infty)_\# d\eta(x),$$

which means that $\boldsymbol{\pi}_\infty$ is continuous on $\mathbf{IP}_C(\mathbb{R}^d)$. Since $\mathbf{IP}_C(\mathbb{R}^d)$ is a closed subset of $\text{Prob}(\mathbb{R}^d)$, then so is $\mathbf{IP}_C(\mu, \nu)$. \square

Now we just have to investigate its tightness to show that it is compact, as done in the next lemma.

Lemma 1.2.3 (Compactness). *Given $\mu, \nu \in \text{Prob}(\mathbb{R}^d)$ and $C > 0$, the set $\mathbf{IP}_C(\mu, \nu)$ is a compact subset of $\text{Prob}(\Gamma(\mathbb{R}^d))$.*

Proof. Thus it remains to prove that it is tight. We know that there exists a compact set $X \subseteq \mathbb{R}^d$ such that

$$\mu(\mathbb{R}^d \setminus X) \leq \epsilon. \quad (1.2.5)$$

Notice that any curve $\gamma \in \Gamma(\mathbb{R}^d)$ which starts in X and such that $T(\gamma) \leq M$ stays inside the compact set $K = X_M := \{x : d(x, X) \leq M\}$ because it is 1-Lipschitz.

Consequently, using inequality (1.2.5) together with (1.2.4):

$$\begin{aligned} \eta(\Gamma(\mathbb{R}^d) \setminus \Gamma_M(K)) &\leq \eta(\{\gamma : \gamma(0) \notin X\}) + \eta(\{\gamma : T(\gamma) > M\}) \\ &= \mu(\mathbb{R}^d \setminus X) + \eta(\{\gamma : T(\gamma) > M\}) \\ &\leq \varepsilon + \frac{C}{M} \leq 2\varepsilon \end{aligned}$$

if $M \geq C/\varepsilon$. This holds for all $\eta \in \mathbf{IP}_C(\mu, \nu)$ hence $\mathbf{IP}_C(\mu, \nu)$ is a tight subset of $\text{Prob}(\mathbb{R}^d)$ since $\Gamma_M(K)$ is compact (it is actually compact for the uniform topology). \square

Continuity results When A is a closed subset of K , we set

$$[A] = \{\gamma \in \Gamma^1 : \gamma \cap A \neq \emptyset\}, \quad |A|_\eta = \eta([A]),$$

so that $\theta_\eta(x) = |\{x\}|_\eta =: |x|_\eta$. One may show that $[A]$ is Borel. Our first continuity result is that of $|\cdot|_\eta$ along decreasing sequences of closed sets.

Lemma 1.2.4. *If $(A_n)_{n \in \mathbb{N}}$ is a decreasing sequence of closed sets and $A = \bigcap_n^\downarrow A_n$ then*

$$|A|_\eta = \lim_{n \rightarrow \infty} |A_n|_\eta.$$

Proof. Let us prove that $[A] = \bigcap_n^\downarrow [A_n]$. The inclusion $[A] \subseteq \bigcap_n [A_n]$ is clear since $[A] \subseteq [A_n]$ for all n . Let us take $\gamma \in \bigcap_n [A_n]$. Since belonging to some $[B]$ only depends on the trajectory of γ , we may assume that it is parameterized by arc length. Because γ has finite length L , there is a sequence $(A_n)_n$ and a sequence $(t_n)_n \in [0, L]$ such that $\gamma(t_n) \in A_n$ for all n . One may extract a converging subsequence, still denoted $(t_n)_n$, such that $t_n \xrightarrow{n \rightarrow \infty} t \in [0, L]$. Since the (A_n) 's are decreasing closed sets, $\gamma(t)$ belongs to their intersection A , hence $\gamma \in [A]$. By the monotone convergence theorem

$$\lim_n |A_n|_\eta = \lim_n \eta([A_n]) = \eta([A]) = |A|_\eta. \quad \square$$

Proposition 1.2.5. *For $C > 0$, the map*

$$\theta : \begin{cases} \mathbb{R}^d \times \mathbf{IP}_C(\mathbb{R}^d) & \longrightarrow [0, 1] \\ (x, \eta) & \longmapsto \theta_\eta(x) \end{cases}$$

is upper semicontinuous.

Proof. Given $x_n \rightarrow x$ and $\eta_n \rightarrow \eta$, take $\varepsilon > 0$. If n is large enough, $x_n \in \bar{B}(x, \varepsilon)$, hence $\limsup_n |x_n|_{\eta_n} \leq \limsup_n |\bar{B}(x, \varepsilon)|_{\eta_n}$. Besides, using (1.2.3) one gets

$$\begin{aligned} |\bar{B}(x, \varepsilon)|_{\eta_n} &\leq \eta_n(\{T > M\}) + \eta_n(\{T \leq M\} \cap [\bar{B}(x, \varepsilon)]) \\ &\leq C/M + \eta_n(A) \end{aligned}$$

where we have set $A := \Gamma_M(\mathbb{R}^d) \cap [\bar{B}(x, \varepsilon)]$. It is easy to check A is closed since T is lower semicontinuous and the ball is closed. Hence passing to the lim sup in n yields

$$\limsup_n |\bar{B}(x, \varepsilon)|_{\eta_n} \leq C/M + \eta(A) \leq C/M + |\bar{B}(x, \varepsilon)|_\eta. \quad (1.2.6)$$

Taking $M \rightarrow \infty$, one gets

$$\limsup_n |x_n|_{\eta_n} \leq \limsup_n |\bar{B}(x, \epsilon)|_{\eta_n} \leq |\bar{B}(x, \epsilon)|_{\eta},$$

then we pass to the limit $\epsilon \rightarrow 0$ using Lemma 1.2.4:

$$\limsup_n |x_n|_{\eta_n} \leq \lim_{\epsilon \rightarrow 0} |\bar{B}(x, \epsilon)|_{\eta} = \theta_{\eta}(x). \quad \square$$

Recall that

$$Z_{\eta}^{\alpha}(\gamma) := \int_{\gamma} \theta_{\eta}(x)^{\alpha-1} dx$$

is the α -cost the curve $\gamma \in \Gamma$ w.r.t. η . Thus we set

$$Z^{\alpha} : \begin{cases} \Gamma(X) \times \mathbf{IP}(X) & \longrightarrow \mathbb{R} \\ (\gamma, \eta) & \longmapsto Z_{\eta}^{\alpha}(\gamma) \end{cases}.$$

Proposition 1.2.6. *For any $C > 0$, the function Z^{α} is lower semicontinuous on $\Gamma(\mathbb{R}^d) \times \mathbf{IP}_C(\mathbb{R}^d)$.*

Proof. The case $\alpha = 1$ is clear since $Z^1(\gamma, \eta) = L(\gamma)$ hence we assume $\alpha < 1$. We know that the map $f : (x, \eta) \mapsto \theta_{\eta}(x)^{\alpha-1}$ is lower semicontinuous on $\mathbb{R}^d \times \mathbf{IP}_C(\mathbb{R}^d)$ since $\theta : (x, \eta) \mapsto \theta_{\eta}(x)$ is upper semicontinuous by Proposition 1.2.5 and because $\alpha - 1 < 0$. Since T is lower semicontinuous, for $\epsilon > 0$ and n large enough we have $T(\gamma) \leq T(\gamma_n) + \epsilon$, which implies that

$$\int_0^{T(\gamma_n)} f(\gamma_n(t), \eta_n) |\dot{\gamma}_n(t)| dt \geq \int_0^{T(\gamma)-\epsilon} f(\gamma_n(t), \eta_n) |\dot{\gamma}_n(t)| dt. \quad (1.2.7)$$

Recall that $\mathbf{IP}_C(\mathbb{R}^d)$ is metrizable⁵ for the narrow convergence as a subset of $\text{Prob}(\Gamma(\mathbb{R}^d))$ where $\Gamma(\mathbb{R}^d)$ is a Polish space, hence $\mathbb{R}^d \times \mathbf{IP}_C(\mathbb{R}^d)$ is metrizable by some distance d . Suppose for a moment that f is Lipschitz continuous on $\mathbb{R}^d \times \mathbf{IP}_C(\mathbb{R}^d)$ equipped with this distance. Since γ_n converges uniformly to γ on $[0, T(\gamma) - \epsilon]$, the functions $g_n : t \mapsto f(\gamma_n(t), \eta_n)$ converge uniformly to $g : t \mapsto f(\gamma(t), \eta)$ on $[0, T(\gamma) - \epsilon]$. Now we have to take care of the $|\dot{\gamma}_n(t)|$ factor. Since the sequence $(\dot{\gamma}_n)_n$ is bounded in $L^{\infty}([0, T(\gamma) - \epsilon])$ one may extract a subsequence $(\dot{\gamma}_{n_k})_k$ such that $\dot{\gamma}_{n_k} \xrightarrow{L^{\infty}} \dot{\gamma}$ and $|\dot{\gamma}_{n_k}| \xrightarrow{L^{\infty}} u$. It is a classical result that $|\dot{\gamma}(t)| \leq u(t)$ almost everywhere on $[0, T(\gamma) - \epsilon]$. Denoting by $\langle \cdot, \cdot \rangle$ the duality bracket $L^1 - L^{\infty}$ on $[0, T(\gamma) - \epsilon]$, we have

$$\begin{aligned} \int_0^{T(\gamma)-\epsilon} f(\gamma_{n_k}(t), \eta_{n_k}) |\dot{\gamma}_{n_k}(t)| dt &= \langle g_{n_k}, |\dot{\gamma}_{n_k}| \rangle \\ &\xrightarrow{k \rightarrow \infty} \langle g, u \rangle \geq \int_0^{T(\gamma)-\epsilon} f(\gamma(t), \eta) |\dot{\gamma}(t)| dt \end{aligned}$$

since $g_{n_k} \rightarrow g$ uniformly hence strongly in L^1 . Prior to extracting the subsequence $(\dot{\gamma}_{n_k})_k$ we could have taken first a subsequence of $\dot{\gamma}_n$ such that the left hand-side converged to $\liminf_n \int_0^{T(\gamma)-\epsilon} f(\gamma_n(t), \eta_n) |\dot{\gamma}_n(t)| dt$. Thus we actually have

$$\liminf_n \int_0^{T(\gamma)-\epsilon} f(\gamma_n(t), \eta_n) |\dot{\gamma}_n(t)| dt \geq \int_0^{T(\gamma)-\epsilon} f(\gamma(t), \eta) |\dot{\gamma}(t)| dt.$$

⁵E.g. by the Lévy-Prokhorov metric.

Finally, we use this inequality together with (1.2.7) and pass to the limit $\epsilon \rightarrow 0$ thanks to the monotone convergence theorem, which yields

$$\liminf_n \int_0^{T(\gamma_n)} f(\gamma_n(t), \eta_n) |\dot{\gamma}_n(t)| dt \geq \int_0^{T(\gamma)} f(\gamma(t), \eta) |\dot{\gamma}(t)| dt. \quad (1.2.8)$$

In general f is not Lipschitz continuous but only lower semicontinuous. Nevertheless (1.2.8) still holds for our function f because it may be written as an increasing sequence of Lipschitz continuous functions $f_k \uparrow f$, hence writing the inequality with f_k and using the monotone convergence theorem as $k \rightarrow \infty$ proves that,

$$\liminf_n Z^\alpha(\gamma_n, \eta_n) \geq Z^\alpha(\gamma, \eta)$$

thus Z^α is lower semicontinuous on $\Gamma(\mathbb{R}^d) \times \mathbf{IP}_C(\mathbb{R}^d)$. \square

Recall that

$$\mathbf{I}^\alpha(\eta) = \int_\Gamma Z^\alpha(\gamma, \eta) d\eta(\gamma),$$

hence the lower semicontinuity of \mathbf{I}^α on $\mathbf{IP}_C(\mathbb{R}^d)$ will be obtained as a corollary to the following lemma.

Lemma 1.2.7. *Let X be a subset of $\text{Prob}(\Gamma(\mathbb{R}^d))$.*

(i) *If $f : \Gamma(\mathbb{R}^d) \times X \rightarrow \mathbb{R}$ is uniformly continuous and bounded, then the functional $F : \eta \mapsto \int_\Gamma f(\gamma, \eta) d\eta(\gamma)$ is continuous on X .*

(ii) *If $f : \Gamma(\mathbb{R}^d) \times X \rightarrow [0, +\infty]$ is lower semicontinuous, then $F : \eta \mapsto \int_\Gamma f(\gamma, \eta) d\eta(\gamma)$ is lower semicontinuous on X .*

Proof. Let us prove the first item. Take $\eta_n \rightarrow \eta$. We set $g_n(\gamma) = f(\gamma, \eta_n)$ and $g(\gamma) = f(\gamma, \eta)$. The fact that f is uniformly continuous on $\Gamma \times X$ implies that $g_n \rightarrow g$ strongly in $\mathcal{C}_b(\Gamma)$. Since $\eta_n \rightarrow \eta$, we have $\int_\Gamma f(\gamma, \eta_n) \eta_n(d\gamma) = \langle \eta_n, g_n \rangle \rightarrow \langle \eta, g \rangle = \int_\Gamma f(\gamma, \eta) \eta(d\gamma)$ where $\langle \cdot, \cdot \rangle$ denotes the $\mathcal{C}_b(\Gamma) - \text{Prob}(\Gamma)$ duality bracket.

The second item is a straightforward consequence of the fact that f can be written as the increasing limit of Lipschitz bounded functions f_k , and of the monotone convergence theorem. \square

Theorem 1.2.8. *For $\alpha \in [0, 1]$, the irrigation functional \mathbf{I}^α is lower semicontinuous on $\mathbf{IP}_C(\mathbb{R}^d)$.*

Proof. We just put together Proposition 1.2.6 which states that Z^α is lower semicontinuous on $\Gamma(\mathbb{R}^d) \times \mathbf{IP}_C(\mathbb{R}^d)$ and the previous lemma with $X = \mathbf{IP}_C(\mathbb{R}^d)$ and $f = Z^\alpha$, recalling that

$$\mathbf{I}^\alpha(\eta) = \int_\Gamma Z^\alpha(\gamma, \eta) d\eta(\gamma). \quad \square$$

The existence theorem We are now able to prove the existence theorem for the Lagrangian irrigation problem (\mathbf{LI}_α) .

Theorem 1.2.9. *If μ, ν are probability measures on \mathbb{R}^d , there exists a minimizer η for the problem*

$$\min_{\eta \in \mathbf{IP}(\mu, \nu)} \left\{ \mathbf{I}^\alpha(\eta) \doteq \int_{\Gamma} \int_{\gamma} \theta_{\eta}(x)^{\alpha-1} dx d\eta(\gamma) \right\}. \quad (\mathbf{LI}_\alpha)$$

Proof. We assume $\mathbf{I}^\alpha \not\equiv +\infty$, otherwise there is nothing to prove. Let us take a minimizing sequence η_n , which we may assume to be parameterized by arc length. In particular $\mathbf{I}^\alpha(\eta_n) \leq C$ for some $C > 0$. Consequently

$$\begin{aligned} \mathbf{T}(\eta_n) = \mathbf{L}(\eta_n) &\doteq \int_{\Gamma} \int_{\gamma} dx d\eta_n(\gamma) \\ &\leq \int_{\Gamma} \int_{\gamma} \theta_{\eta}(x)^{\alpha-1} dx d\eta_n(\gamma) \\ &\doteq \mathbf{I}^\alpha(\eta_n) \leq C, \end{aligned}$$

which implies that $\eta_n \in \mathbf{IP}_C(\mu, \nu)$. Thanks to Proposition 1.2.2, $\mathbf{IP}_C(\mu, \nu)$ is a compact (and metrizable) subset of $\text{Prob}(\Gamma(\mathbb{R}^d))$ hence we can extract a converging sequence $\eta_n \rightharpoonup \eta \in \mathbf{IP}_C(\mu, \nu)$ up to some renaming. By Theorem 1.2.8, \mathbf{I}^α is lower semicontinuous on $\mathbf{IP}_C(\mathbb{R}^d)$, thus

$$\mathbf{I}^\alpha(\eta) \leq \liminf_n \mathbf{I}^\alpha(\eta_n) = \inf (\mathbf{LI}_\alpha),$$

which shows that η is a minimizer for the problem (\mathbf{LI}_α) . \square

Notice that we do not claim that for given $\mu, \nu \in \text{Prob}(\mathbb{R}^d)$ and $\alpha \in [0, 1]$, the minimum is finite, i.e. that there is an irrigation plan in $\mathbf{IP}(\mu, \nu)$ with finite α -cost. Hence in the sequel we will say that $\eta \in \mathbf{IP}(\mu, \nu)$ is optimal for \mathbf{I}^α if it is a minimizer of (\mathbf{LI}_α) and if it has finite α -cost, in which case we write $\eta \in \mathbf{OIP}^\alpha(\mu, \nu)$.

Existence for the H -cost Let us just remark that all the computations and the proofs done from Section 1.2.2 hold if we replace the map $x \mapsto x^\alpha$ by $H : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ and $x^{\alpha-1}$ with $H(x)/x$, where H satisfies:

- (i) $H(0) = 0$,
- (ii) H is continuous and nondecreasing,
- (iii) $x \mapsto H(x)/x$ is nonincreasing,
- (iv) $\lim_{x \rightarrow 0} \frac{H(x)}{x} = +\infty$.

Notice that items (ii)-(iii) hold if H is concave and increasing, and that (iii) implies that H is subadditive.

1.2.4 Operations on irrigation plans

In this section we define restrictions, cuts and concatenations of irrigation plans. These standard operations were defined in [BCM09], although there was a minor glitch in the construction of a concatenation. In [Peg14], we defined it via approximation by finite graphs, while here we propose to prove a “gluing lemma” (analogous to the one in optimal transport, see [San15, Lemma 5.5]) by disintegration of measures. In this section X is a closed subset of \mathbb{R}^d .

Restriction. Given an irrigation plan $\eta \in \mathbf{IP}(X)$ and a Borel set $E \subseteq \Gamma(X)$, the restriction of η to E is merely the measure-theoretic restriction: for all Borel set $A \subseteq \Gamma(X)$,

$$\eta \llcorner E(A) = \eta(E \cap A).$$

Cut. If χ is a Borel map which associates to each curve $\gamma \in \Gamma(X)$ a probability measure $\chi_\gamma \in \text{Prob}(\mathbb{R}_+^2)$ concentrated on $H = \{(a, b) : a \leq b\}$, then we define the cut $\eta[\chi]$ of η with respect to χ by:

$$\langle \phi, \eta[\chi] \rangle = \int_{\Gamma(X)} \int_H \phi(\gamma|_{[a,b]}(\cdot - a)) \, d\chi_\gamma(a, b) \, d\eta(\gamma),$$

for all $\phi \in \mathcal{C}_0(\Gamma)$. It is a way to restrict each curve γ to some interval $[a, b]$, but each curve may be restricted in possibly infinitely many ways, under a law given by χ_γ . A possibility is for example to cut all curves γ between times $a(\gamma)$ and $b(\gamma)$ where $a, b : \Gamma(X) \rightarrow \mathbb{R}_+$ are Borel and $a \leq b$. The resulting irrigation plan is denoted by $\eta[a, b]$ and corresponds to $\eta[\chi]$ where $\chi_\gamma = \delta_{(a(\gamma), b(\gamma))}$. If τ is a Borel map which associates to each $\gamma \in \Gamma(X)$ a probability measure $\tau_\gamma \in \text{Prob}(\mathbb{R}_+)$, then the cuts denoted by $\eta[0, \tau]$ and $\eta[\tau, +]$ associated to $\chi_\gamma = \delta_0 \otimes \tau_\gamma$ and $\chi_\gamma = \tau_\gamma \otimes \delta_{T(\gamma)}$ respectively are called complementary cuts (respectively initial and final cut) associated to τ .

Gluing. We are given two irrigation plans η_1, η_2 such that $\eta_1^+ = \eta_2^-$ and we want to glue them together.

Lemma 1.2.10 (Gluing lemma). *There exists an irrigation plan η_{12} and a Borel map $\tau : \Gamma(X) \rightarrow \text{Prob}(\mathbb{R}_+)$, such that*

$$\eta_1 = \eta_{12}[0, \tau] \quad \text{and} \quad \eta_2 = \eta_{12}[\tau, +].$$

The plan η_{12} is called a concatenation of η_1 and η_2 and τ a recovery cutting time.

Proof. Let us disintegrate η_1 and η_2 with respect to π_∞ and π_0 respectively, recalling that $(\pi_\infty)_\# \eta_1 = (\pi_0)_\# \eta_2 =: \sigma$, so that:

$$d\eta_1(x, y) = d\eta_1^y(x) \, d\sigma(y) \quad \text{and} \quad d\eta_2(y, z) = d\eta_2^y(z) \, d\sigma(y),$$

where η_1^y is concentrated on curves which stop at y and η_2^y on curves that start at y . We set⁶ $\pi^y = \eta_1^y \otimes \eta_2^y$ and define for all pair (γ_1, γ_2) of curves which start and stop at

⁶Any coupling π in $\Pi(\eta_1^y, \eta_2^y)$ would do.

a common point:

$$\gamma_1 : \gamma_2(t) = \begin{cases} \gamma_1(t) & \text{if } t \leq T(\gamma_1), \\ \gamma_2(t - T(\gamma_1)) & \text{if } t > T(\gamma_1). \end{cases}$$

We define η_{12} in the following way:

$$\langle \eta_{12}, \phi \rangle = \int_{\mathbb{R}^d} \int_{\Gamma(X) \times \Gamma(X)} \phi(\gamma_1 : \gamma_2) d\pi^y(\gamma_1, \gamma_2) d\sigma(y),$$

for all $\phi \in \mathcal{C}_0(\Gamma(X))$. Now let us build τ . We start by defining a measure $\rho \in \mathcal{M}^+(\mathbb{R}_+ \times \Gamma(X))$:

$$\langle \rho, \psi \rangle = \int_{\mathbb{R}^d} \int_{\Gamma(X) \times \Gamma(X)} \psi(T(\gamma_1), \gamma_1 : \gamma_2) d\pi^y(\gamma_1, \gamma_2) d\sigma(y),$$

for all $\psi \in \mathcal{C}_0(\mathbb{R}_+ \times \Gamma(X))$. Then we disintegrate ρ with respect to the second projection π_2 , noticing that $(\pi_2)_\# \rho = \eta_{12}$, thus getting:

$$d\rho(t, \gamma) = d\tau_\gamma(t) d\eta_{12}(\gamma).$$

Now let us check that the desired properties hold. Take a test function $\phi \in \mathcal{C}_0(\Gamma(X))$:

$$\begin{aligned} \langle \eta_{12}[0, \tau], \phi \rangle &= \int_{\Gamma(X)} \int_0^\infty \phi(\gamma|_{[0,t]}) d\tau_\gamma(t) d\eta_{12}(\gamma) \\ &= \int_{\mathbb{R}_+ \times \Gamma(X)} \phi(\gamma|_{[0,t]}) d\rho(t, \gamma) \\ &= \int_{\mathbb{R}^d} \int_{\Gamma(X) \times \Gamma(X)} \phi((\gamma_1 : \gamma_2)|_{[0, T(\gamma_1)]}) d\pi^y(\gamma_1, \gamma_2) d\sigma(y) \\ &= \int_{\mathbb{R}^d} \int_{\Gamma(X) \times \Gamma(X)} \phi(\gamma_1) d\pi^y(\gamma_1, \gamma_2) d\sigma(y) \\ &= \int_{\mathbb{R}^d} \int_{\Gamma(X)} \phi(\gamma_1) d\eta_1^y(\gamma_1) d\sigma(y) = \int_{\Gamma(X)} \phi(\gamma) d\eta_1(\gamma), \end{aligned}$$

and $\eta_1 = \eta_{12}[0, \tau]$. A similar computation shows that $\eta_2 = \eta_{12}[\tau, +]$, which ends the proof. \square

Remark 1.2.11. The result generalizes easily with n irrigation plans η_1, \dots, η_n such that $\eta_i^+ = \eta_{i+1}^-$: one may find an irrigation plan η and recovery cutting times $\tau_i : \Gamma(X) \rightarrow \text{Prob}(\mathbb{R}_+)$ for $0 < i < n$ such that

$$\eta_i = \eta[\tau_{i-1}, \tau_i],$$

with $\tau_0 = \delta_0$ and $\tau_n = \delta_{T(\cdot)}$. We write $\eta \in \eta_1 : \eta_2 : \dots : \eta_n$ to say that η is such a gluing.

Remark 1.2.12. From two irrigation plans $\eta \in \mathbf{IP}(\mu, \nu)$ and $\eta' \in \mathbf{IP}(\nu, \xi)$ one may build an irrigation plan $\eta'' \in \mathbf{IP}(\mu, \xi)$ with cost

$$\mathbf{I}^\alpha(\eta'') \leq \mathbf{I}^\alpha(\eta) + \mathbf{I}^\alpha(\eta').$$

Indeed, just take η'' to be a gluing of η and η' , with τ a recovery cutting time. Then one has:

$$\begin{aligned} \mathbf{I}^\alpha(\eta'') &= \int_{\Gamma(X)} \int_{\gamma} \theta_{\eta''}(x)^{\alpha-1} dx d\eta''(\gamma) \\ &= \int_{\Gamma(X)} \int_{\mathbb{R}^+} \left(\int_{\gamma|_{[0,b]}} \theta_{\eta''}(x)^{\alpha-1} dx + \int_{\gamma|_{[b,\infty]}} \theta_{\eta''}(x)^{\alpha-1} dx \right) d\tau_\gamma(b) d\eta''(\gamma) \\ &= \int_{\Gamma(X)} \int_{\mathbb{R}^+} \int_{\gamma|_{[0,b]}} \theta_{\eta''}(x)^{\alpha-1} dx d\tau_\gamma(b) d\eta''(\gamma) \\ &\quad + \int_{\Gamma(X)} \int_{\mathbb{R}^+} \int_{\gamma|_{[b,\infty]}} \theta_{\eta''}(x)^{\alpha-1} dx d\tau_\gamma(b) d\eta''(\gamma). \end{aligned}$$

Notice that $\theta_{\eta''}(x) \geq \theta_\eta(x)$ and $\theta_{\eta''}(x) \geq \theta_{\eta'}(x)$, hence

$$\begin{aligned} \int_{\Gamma(X)} \int_{\mathbb{R}^+} \int_{\gamma|_{[0,b]}} \theta_{\eta''}(x)^{\alpha-1} dx d\tau_\gamma(b) d\eta''(\gamma) &\leq \int_{\Gamma(X)} \int_{\mathbb{R}^+} Z_\eta(\gamma|_{[0,b]}) d\tau_\gamma(b) d\eta''(\gamma) \\ &= \int Z_\eta(\gamma) d\eta(\gamma) = \mathbf{I}^\alpha(\eta), \end{aligned}$$

and the same goes for the other term, leading to

$$\mathbf{I}^\alpha(\eta'') \leq \mathbf{I}^\alpha(\eta) + \mathbf{I}^\alpha(\eta').$$

Moreover it is clear that $\eta'' \in \mathbf{IP}(\mu, \xi)$.

1.2.5 Irrigability and irrigation distance

We have already noticed that a pair of probability measures cannot necessarily be connected with finite α -cost. However, according to the previous subsection, the relation “ μ and ν can be connected with finite α -cost” is transitive, thus every pair of measures can be connected with finite α -cost if and only if every measure μ be connected with finite α -cost to δ_0 , in which case we say that μ is α -irrigable. The next result from [Xia03] provides a sufficient condition on α and d for irrigability of compactly supported measures.

Proposition 1.2.13 (Irrigability). *If $\alpha > 1 - \frac{1}{d}$ then for every pair of compactly supported measures $(\mu, \nu) \in \text{Prob}(\mathbb{R}^d)$, there is an irrigation plan $\eta \in \mathbf{IP}(\mu, \nu)$ of finite α -cost, i.e. such that $\mathbf{I}^\alpha(\eta) < \infty$.*

Remark 1.2.14. One can actually show that the uniform measure on a unit cube can be irrigated from a unit Dirac mass if and only if $\alpha > 1 - \frac{1}{d}$, hence this condition is necessary for an arbitrary pair (μ, ν) to be irrigable with finite α -cost.

For $\mu, \nu \in \text{Prob}(\mathbb{R}^d)$, we set $d_\alpha(\mu, \nu)$ to be the optimal α -cost connecting the two, namely:

$$d_\alpha(\mu, \nu) = \min \{ \mathbf{I}^\alpha(\eta) : \eta \in \mathbf{IP}(\mu, \nu) \} \in \mathbb{R} \cup \{+\infty\}.$$

The lower semicontinuity of \mathbf{I}^α and the continuity of π_0, π_∞ on $\mathbf{IP}_C(\mathbb{R}^d)$ readily imply that d_α is lower semicontinuous. From Remark 1.2.12 we get the triangle inequality, thus d_α is a distance on compactly supported measures, and we may state the following (proven in [Xia03]).

Proposition 1.2.15. *Fix a compact subset $X \subseteq \mathbb{R}^d$ and $\alpha > 1 - \frac{1}{d}$. Then $d_\alpha(\mu, \nu) < \infty$ for every pair of measures $\mu, \nu \in \text{Prob}(X)$ and d_α is a distance on $\text{Prob}(X)$ which metrizes the weak- \star convergence of measures in the duality with $\mathcal{C}(X)$.*

Actually, sharp inequalities comparing d_α and Wasserstein distances W_p have been established in [BBS11] and [MS07]:

$$cW_{1/\alpha}(\mu, \nu) \leq d_\alpha(\mu, \nu) \leq W_1(\mu, \nu)^\beta,$$

where $\beta = 1 + \alpha d - d$.

1.2.6 Basic structure properties of optimizers

From now on, we assume that we are in the branched case $\alpha < 1$. We show that optimizers of the Lagrangian irrigation problem (LI_α) enjoy some tree-like properties: the single-path property and the cycle-free property. The first one is actually a first step to prove the second, which is a stronger property. In order to do so, let us anticipate a bit of content from Chapter 2, that is:

- (i) any $\eta \in \mathbf{OIP}^\alpha(\mu, \nu)$ is *simple*, meaning that it is concentrated on curves γ which are injective on $[0, T(\gamma)]$,
- (ii) any $\eta \in \mathbf{OIP}^\alpha(\mu, \nu)$ is *rectifiable*, i.e. there is a 1-rectifiable set R such that η -a.e. curve γ stays essentially in R : $\mathcal{H}^1(\gamma \setminus R) = 0$,
- (iii) the following energy formula (see Theorem 2.1.9) is true for all simple and rectifiable irrigation plans (hence for optimal ones):

$$\mathbf{I}^\alpha(\eta) = \int_{\mathbb{R}^d} \theta_\eta(x)^\alpha \, d\mathcal{H}^1(x), \tag{EF}$$

- (iv) if $\eta \in \mathbf{OIP}^\alpha(\mu, \nu)$, and $\eta' \in \mathbf{IP}(\mu, \nu)$ is a candidate irrigation plan which is rectifiable, then

$$\mathbf{I}^\alpha(\eta) \leq \int_{\mathbb{R}^d} \theta_{\eta'}(x)^\alpha \, d\mathcal{H}^1(x) \leq \mathbf{I}^\alpha(\eta'),$$

even if η' is not simple and last inequality is strict, thanks to the simple replacement lemma (Lemma 2.1.12).

Let us start with a few notations. Given n points $(x_1, \dots, x_n) \in \mathbb{R}^d$, we denote by $\Gamma(x_1, \dots, x_n)$ the set of curves $\gamma \in \Gamma(\mathbb{R}^d)$ which visit points (x_1, \dots, x_n) in that order, i.e. there are times $t_1 \leq \dots \leq t_n$ such that $\gamma(t_i) = x_i$ for all i . Also, we denote by $\Gamma[x_1, \dots, x_n]$ the set of curves which visit (x_1, \dots, x_n) no matter the order of visit.

Proposition 1.2.16 (Single path property). *Assume that $\eta \in \mathbf{OIP}^\alpha(\mu, \nu)$ with $\alpha < 1$. For all pair of points x, y such that $\eta(\Gamma(x, y)) > 0$, there is a curve parameterized by arc length on a segment $[0, \ell]$ and denoted by $\gamma_\eta(x, y)$ such that η -a.e. curve $\gamma \in \Gamma(x, y)$ coincide in its trajectory with $\gamma_\eta(x, y)$.*

Remark 1.2.17. Let us clarify a bit. We say that γ_1 and γ_2 coincide in their trajectory if they can be reparameterized in $\tilde{\gamma}_1, \tilde{\gamma}_2$ so that $\tilde{\gamma}_1 = \tilde{\gamma}_2$, or equivalently if this equality holds with $\tilde{\gamma}_1, \tilde{\gamma}_2$ being the arc-length parameterization of γ_1, γ_2 .

Proof. Let A and B be two subsets of $\Gamma(x, y)$ with positive mass $\eta(A), \eta(B) > 0$. For $\varepsilon > 0$ small, we define another irrigation plan η_ε obtained from η by replacing a proportion ε of curves of B by curves where the path between x and y has been replaced by the paths of those in A . We shall be more precise. We set

$$\eta_B = \frac{\eta \llcorner B}{\eta(B)} \quad \text{and} \quad \eta_A = \frac{\eta \llcorner A}{\eta(A)},$$

and $t_x(\gamma), t_y(\gamma)$ to be the unique times⁷ for which $\gamma(t_x(\gamma)) = x, \gamma(t_y(\gamma)) = y$. Following Remark 1.2.11 we define a gluing $\tilde{\eta}_A \in \eta_B[0, t_x] : \eta_A[t_x, t_y] : \eta_B[t_y, +]$, which is possible after checking the compatibility of the consecutive marginals. It is clear that the recovery cutting times are t_x, t_y . Then we set

$$\eta_\varepsilon = \eta - \varepsilon \eta_B + \varepsilon \tilde{\eta}_A.$$

Obviously $\eta_\varepsilon \in \mathbf{IP}(\mu, \nu)$ because by construction $(\pi_0)_\# \tilde{\eta}_A = (\pi_0)_\# \eta_B$ and $(\pi_\infty)_\# \tilde{\eta}_A = (\pi_\infty)_\# \eta_B$. Given $u \in \mathbb{R}^d$, notice that

$$\begin{aligned} \theta_{\tilde{\eta}_A}(u) &= \int_{\Gamma(\mathbb{R}^d)} (\mathbf{1}_{u \in \gamma([0, t_x])} + \mathbf{1}_{u \in \gamma([t_x, t_y])} + \mathbf{1}_{u \in \gamma([t_y, T])}) d\tilde{\eta}_A(\gamma) \\ &= \theta_{\eta_B}(\Gamma(u, x, y)) + \theta_{\eta_A}(\Gamma(x, u, y)) + \theta_{\eta_B}(\Gamma(x, y, u)), \end{aligned}$$

so that the multiplicity $\theta_{\eta_\varepsilon}(u)$ can be expressed as

$$\theta_{\eta_\varepsilon}(u) = \theta_\eta(u) - \varepsilon \theta_\eta^B(u) + \varepsilon \theta_\eta^A(u)$$

where

$$\theta_\eta^A(u) = \frac{\eta(A \cap \Gamma(x, u, y))}{\eta(A)} \quad \text{and} \quad \frac{\eta(B \cap \Gamma(x, u, y))}{\eta(B)}.$$

We know that η_ε is rectifiable, hence by optimality of η (recall (iv) at the beginning of this section), one has

$$\int_{\mathbb{R}^d} \theta_{\eta_\varepsilon}(u)^\alpha d\mathcal{H}^1(u) \geq \mathbf{I}^\alpha(\eta),$$

or equivalently

$$\int_{\mathbb{R}^d} (\theta_\eta(u) + \varepsilon (\theta_\eta^A(u) - \theta_\eta^B(u)))^\alpha d\mathcal{H}^1(u) \geq \int_{\mathbb{R}^d} \theta_\eta(u)^\alpha d\mathcal{H}^1(u).$$

By interchanging the roles of A and B , one gets the same inequality but for ε of arbitrary sign. Notice that the map $\varepsilon \mapsto (\theta_\eta(u) + \varepsilon \rho)^\alpha$ is strictly concave if $\rho \neq 0$ hence the map

$$\varepsilon \mapsto \int_{\mathbb{R}^d} (\theta_\eta(u) + \varepsilon (\theta_\eta^A(u) - \theta_\eta^B(u)))^\alpha d\mathcal{H}^1(u)$$

is strictly concave if $\theta_\eta^A(u) - \theta_\eta^B(u)$ is not identically 0. Since it is minimal at $\varepsilon = 0$, it cannot be strictly concave, therefore

$$\theta_\eta^A(u) = \theta_\eta^B(u)$$

⁷It is well defined for simple curves, which is true η -a.e.

for \mathcal{H}^1 -a.e. $u \in \mathbb{R}^d$. For any nonnegative function f defined on \mathbb{R}^d , one has

$$\int_{\mathbb{R}^d} f(u) \theta_\eta^A(u) \, d\mathcal{H}^1(u) = \int_{\mathbb{R}^d} f(u) \theta_\eta^B(u) \, d\mathcal{H}^1(u). \quad (1.2.9)$$

By Fubini's Theorem, we can give another expression for these:

$$\begin{aligned} \int_{\mathbb{R}^d} f(u) \theta_\eta^A(u) \, d\mathcal{H}^1(u) &= \frac{1}{\eta(A)} \int_{\mathbb{R}^d} \int_A \mathbf{1}_{\gamma \in \Gamma(x,u,y)} f(u) \, d\eta(\gamma) \, d\mathcal{H}^1(u) \\ &= \frac{1}{\eta(A)} \int_A \int_\gamma f(u) \mathbf{1}_{\gamma \in \Gamma(x,u,y)} \, du \, d\eta(\gamma) \\ &= \int_A \int_{\gamma[x,y]} f(u) \, du \, d\eta(\gamma), \end{aligned}$$

where we have set $\gamma[x, y] = \gamma|_{[t_x(\gamma), t_y(\gamma)]}$, thus (1.2.9) rewrites:

$$\int_A \int_{\gamma[x,y]} f(u) \, du \, d\eta(\gamma) = \int_B \int_{\gamma[x,y]} f(u) \, du \, d\eta(\gamma). \quad (1.2.10)$$

Now we consider a countable basis of open sets $(O_n)_{n \in \mathbb{N}}$ in \mathbb{R}^d and we take $f = \mathbf{1}_{O_n}$. For all $n \in \mathbb{N}$ and all $A, B \in \Gamma(x, y)$, one has:

$$\int_A \mathcal{H}^1(\gamma[x, y] \cap O_n) \, d\eta(\gamma) = \int_B \mathcal{H}^1(\gamma[x, y] \cap O_n). \quad (1.2.11)$$

As a consequence, the mean value of $\mathcal{H}^1(\gamma[x, y] \cap O_n)$ on all subsets of $\Gamma(x, y)$ is constant, which implies that outside an η -negligible set E , for all curve $\gamma \in \Gamma(x, y) \setminus E$, the quantity $\mathcal{H}^1(\gamma[x, y] \cap O_n)$ does not depend on γ . This implies that for all curves in $\gamma \in \Gamma(x, y) \setminus E$, the trajectory of $\gamma[x, y]$ is the same. By contradiction, if there were two curves $\gamma, \tilde{\gamma} \in \Gamma(x, y) \setminus E$ such that the trajectories of $\gamma[x, y], \tilde{\gamma}[x, y]$ were different, we could find an open set O_n which is disjoint from $\gamma[x, y]$ and which intersects $\tilde{\gamma}[x, y]$, thus $\mathcal{H}^1(\gamma[x, y] \cap O_n) = 0 \neq \mathcal{H}^1(\tilde{\gamma}[x, y] \cap O_n)$: a contradiction. As a consequence all curves $\gamma \in \Gamma(x, y) \setminus E$ coincide in their trajectory between x and y . We denote this curve, parameterized by arc length on an interval $[0, \ell]$, by $\gamma_\eta(x, y)$. \square

We are now able to prove the stronger *cycle-free property*, which is a Lagrangian counterpart to the cycle-free property in the discrete case ((iv) of Theorem 1.1.2).

Theorem 1.2.18 (Cycle-free property). *If $\eta \in \mathbf{OIP}^\alpha(\mu, \nu)$ with $\alpha < 1$ then there is no loop sequence $x_1, x_2, \dots, x_n, x_{n+1} = x_1$ such that $\eta(\Gamma[x_i, x_{i+1}]) > 0$ for $1 \leq i \leq n$.*

Remark 1.2.19. Recall that $\Gamma[x, y]$ is the set of curves which visit x and y , no matter in which order. Thus $\eta(\Gamma[x_i, x_{i+1}]) > 0$ means that either $\eta(\Gamma(x_i, x_{i+1})) > 0$ or $\eta(\Gamma(x_{i+1}, x_i)) > 0$.

Proof. The idea is similar to the proof of the previous proposition, thus we are going to give fewer details. For all i , we set $s_i = 1$, $A_i = \Gamma(x_i, x_{i+1})$ if $\eta(\Gamma(x_i, x_{i+1})) > 0$, and $s_i = -1$, $A_i = \Gamma(x_{i+1}, x_i)$ otherwise, in which case $\eta(\Gamma(x_{i+1}, x_i)) > 0$. We know by the single-path property that all curves in A_i follow a common trajectory γ_i between x_i and x_{i+1} (from x_i to x_{i+1} if $s_i = 1$, from x_{i+1} to x_i if $s_i = -1$). We are going to remove

a small mass $\varepsilon > 0$ from the γ_i 's such that $s_i > 0$ and add ε to the other ones. Here again, the construction is based on the gluing lemma. We set $\eta_{A_i} = \frac{\eta \llcorner A_i}{\eta(A_i)}$. We consider a maximal (i.e. longest) sequence of points $\sigma = x_k, x_{k+1}, \dots, x_l$ such that $s_i = 1$ for all $i = k, \dots, (l-1)$, and we take $\eta_\sigma = \eta_{A_{k-1}}[0, t_{x_k}] : \delta_{\gamma_\sigma} : \eta_{A_l}[t_{x_l}, +]$ (or just $\eta_\sigma = \delta_{\gamma_\sigma}$ if σ is the whole sequence), where $\gamma_\sigma = \gamma_k : \dots : \gamma_l$ is the concatenation of the consecutive curves associated to σ . We denote by U the set of all such maximal sequences σ . Now for each consecutive triple $\tau = x_{k-1}, x_k, x_{k+1}$ such that $s_{k-1} = s_k = -1$ we take $\eta_\tau \in \eta_{A_{k-1}}[0, t_{x_k}] : \eta_{A_k}[t_{x_k}, +]$: these triples form a set V . We define the competitor

$$\eta_\varepsilon = \eta - \varepsilon \sum_{s_i = -1} \eta_{A_i} + \varepsilon \sum_{\tau \in V} \eta_\tau + \varepsilon \sum_{\sigma \in U} \eta_\sigma.$$

The first sum corresponds to the mass we remove, the second is a reconnection term which we must add due to this removal, while the third sum corresponds to the mass that we want to add. We leave it to the reader to check that $\eta_\varepsilon \in \mathbf{IP}(\mu, \nu)$ and that

$$\theta_{\eta_\varepsilon}(u) = \theta_\eta(u) + \varepsilon \Delta\theta(u),$$

where $\Delta\theta(u) = s_i \theta(u)$ if $u \in \gamma_i$. By optimality of η , we have

$$\int_{\mathbb{R}^d} \theta_\eta(u)^\alpha \, d\mathcal{H}^1(u) \leq \int_{\mathbb{R}^d} (\theta_\eta(u) + \varepsilon \Delta\theta(u))^\alpha \, d\mathcal{H}^1(u),$$

which holds for $\varepsilon \geq 0$, but also for $\varepsilon \leq 0$, had we reversed the order of the loop sequence. Here again we are faced with a concave function of ε which is minimal at the interior point 0. This can only happen if $\theta(u) = 0$ for \mathcal{H}^1 -a.e. $u \in \bigcup_i \gamma_i$, which is a contradiction since on each γ_i , $\theta_\eta(u) \geq \eta(A_i) > 0$. \square

1.3 The Eulerian model

We keep assuming till the end of this chapter, that $\alpha \in [0, 1]$.

Irrigation flows We call *irrigation flow* on \mathbb{R}^d any vector measure $v \in \mathcal{M}^d(\mathbb{R}^d)$ such that $\nabla \cdot v \in \mathcal{M}^1(\mathbb{R}^d)$, where $\nabla \cdot v$ is the divergence of v in the sense of distribution. We denote by $\mathbf{IF}(\mathbb{R}^d)$ the set of irrigation flows, and we equip it with the \mathcal{M}_{div} topology, that is $v_n \xrightarrow{\mathcal{M}_{\text{div}}} v$ if v_n and $\nabla \cdot v_n$ converge to v and $\nabla \cdot v$ weakly as measures, in the duality with $\mathcal{C}_0(\mathbb{R}^d, \mathbb{R}^d)$ and $\mathcal{C}_0(\mathbb{R}^d, \mathbb{R})$ respectively.

Remark 1.3.1. These objects have several names. They are called 1-dimensional normal currents in the terminology of Geometric Measure Theory, and are called traffic paths by Xia in [Xia03].

If $E \subseteq \mathbb{R}^d$ is an \mathcal{H}^1 -measurable set, $\tau : E \rightarrow \mathbb{S}^{d-1}$ is \mathcal{H}^1 -measurable and $\theta : E \rightarrow \mathbb{R}_+$ is \mathcal{H}^1 -integrable, we define the vector measure $\llbracket E, \tau, \theta \rrbracket \in \mathcal{M}^d(K)$ by

$$\langle \llbracket E, \tau, \theta \rrbracket, \psi \rangle = \int_E \theta(x) \psi(x) \cdot \tau(x) \, d\mathcal{H}^1(x),$$

for all $\psi \in \mathcal{C}_0(\mathbb{R}^d, \mathbb{R}^d)$. In other terms $\llbracket E, \tau, \theta \rrbracket \doteq \theta \tau \mathcal{H}^1 \llcorner E$.

Rectifiable irrigation flow Recall that if E is a 1-rectifiable set, at \mathcal{H}^1 -a.e $x \in E$ there is an *approximate tangent line* (see [Mat95, Chapter 17]) denoted by $\text{Tan}(x, E)$. An irrigation flow of the form $v = \llbracket E, \tau, \theta \rrbracket$ where E is 1-rectifiable and $\tau(x) \in \text{Tan}(x, E)$ for \mathcal{H}^1 -a.e. $x \in E$ is termed *rectifiable*⁸.

From discrete to continuous Consider an irrigation graph $G \in \mathbf{IG}(\mathbb{R}^d)$. One can define the vector measure v_G by

$$v_G := \sum_{e \in E(G)} \llbracket e, \hat{e}, \theta_e \rrbracket.$$

One can check that $\nabla \cdot v_G = \sum_{e \in E(G)} \theta_e (\delta_{e^-} - \delta_{e^+}) \in \mathcal{M}^1(\mathbb{R}^d)$ and that v_G is a rectifiable irrigation flow on \mathbb{R}^d . Also, both the cost \mathbf{E}^α and the constraint $G \in \mathbf{IG}(\mu, \nu)$ can be expressed solely in terms of v_G . Indeed Kirchhoff's law can actually be expressed in terms of the divergence $\nabla \cdot v_G$:

Proposition 1.3.2. *If G is a graph, $G \in \mathbf{IG}(\mu, \nu)$ if and only if $\nabla \cdot v_G = \mu - \nu$.*

Consequently, we say that v sends μ to ν if $\nabla \cdot v = \mu - \nu$ and denote by $\mathbf{IF}(\mu, \nu)$ the set of such irrigation flows.

Now on the generalization of Gilbert Energy, if we identify v_G with its \mathcal{H}^1 -density, one has

$$\mathbf{E}^\alpha(G) \doteq \sum_{e \in E(G)} \theta_e^\alpha |e| = \int_{\mathbb{R}^d} |v_G(x)|^\alpha \, d\mathcal{H}^1(x).$$

This leads to defining the following cost on $\mathbf{IF}(\mathbb{R}^d)$:

$$\mathbf{M}^\alpha(v) = \begin{cases} \int |v(x)|^\alpha \, d\mathcal{H}^1(x) & \text{if } v \text{ is rectifiable,} \\ +\infty & \text{otherwise,} \end{cases}$$

which is called the α -mass of v .

On the definition and semicontinuity of the α -mass. Several definitions of α -mass have been given depending on the setting (currents or vector measures, euclidean or metric setting). In [DH03] it was defined in a euclidean setting for integer rectifiable currents as the integral of the multiplicity to the α (actually they consider more general costs than $x \mapsto x^\alpha$ but concave integrands H , and define a so-called H -mass). Lower semicontinuity is proven in this space of integer rectifiable currents for a α -flat norm \mathbf{F}_α . A similar definition is used in a metric setting in [Ste10]. Though lower semicontinuity is not explicitly proven, the same definition for real rectifiable currents (in a Euclidean setting) is used by Stepanov and Paolini in [PS06] and extended to flat chains. In the works [Xia03] and [Xia04], two different definitions of \mathbf{M}^α are used, both by relaxation: either in the space \mathcal{M}_{div} of vector measures v whose divergence are measures (for the weak convergence of v and $\nabla \cdot v$), or in the space of flat chains (for the usual flat

⁸A rectifiable irrigation flow is, in the language of currents, a rectifiable normal 1-current on \mathbb{R}^d .

norm). Things are clarified in [Col+17] where it is proven that the lower semicontinuous envelope of the α -mass defined on real-valued polyhedral chains is precisely

$$\mathbf{M}^\alpha(v) = \begin{cases} \int |v(x)|^\alpha \mathcal{H}^1(dx) & \text{if } v \text{ is rectifiable,} \\ +\infty & \text{otherwise,} \end{cases}$$

on the space of real flat chains. Since this space is larger than $\mathbf{IF}(\mathbb{R}^d)$, and the topology weaker than the weak convergence in \mathcal{M}_{div} , all these definitions coincide and in particular \mathbf{M}^α is lower semicontinuous on $\mathbf{IF}(\mathbb{R}^d)$.

Eulerian irrigation problem We are now able to formulate an Eulerian irrigation problem in a continuous setting. Given two probability measures $\mu, \nu \in \text{Prob}(\mathbb{R}^d)$, we want to find an irrigation flow v sending μ to ν which has minimal α -mass. This reads

$$\min \{ \mathbf{M}^\alpha(v) : v \in \mathbf{IF}(\mu, \nu) \}. \quad (\text{EI}_\alpha)$$

Xia proved the following existence theorem in [Xia03].

Theorem 1.3.3 (Existence of minimizers). *For any compactly supported probability measures $\mu, \nu \in \text{Prob}(\mathbb{R}^d)$, there is a minimizer v to the Eulerian irrigation problem (EI_α) . Moreover if $1 - \frac{1}{d} < \alpha < 1$ the minimum is always finite.*

As for lagrangian irrigation problem, we call optimal irrigation flow for \mathbf{M}^α with prescribed source and target μ, ν a minimizer of (EI_α) which has a finite cost, and we denote by $\mathbf{OIF}^\alpha(\mu, \nu)$ the set of such optimal irrigation flows.

1.4 The single source case

In this section, we place ourselves in the case of a single source by assuming that the initial measure is a unit Dirac: $\mu = \delta_0$, and we assume $\alpha < 1$. This is the most studied case in branched transport, either from the point of view of Maddalena, Morel, and Solimini's patterns defined in [MSM03], studied also by Devilanova, Solimini in [DS07a; DS07b], or in the traffic plan framework (the closest to our irrigation plans framework) by Bernot, Caselles, Morel [BCM05; BCM08; BCM09]. In this case, Santambrogio was able to define a *landscape function* in [San07], a terminology borrowed from geophysics ([RR01]), which can be thought of as a "supergradient" of the cost \mathbf{I}^α and has become an important tool to study optimal networks or variational problems related to branched transport, as done in Chapter 3.

1.4.1 Tree property

Optimal networks in the single source case have a tree structure stemming from a root located at 0, as stated in the next proposition.

Proposition 1.4.1 (Tree property). *Let $\eta \in \mathbf{IP}(\delta_0, \nu)$ be an optimal irrigation plan with finite α -cost. For all $x \in N_\eta$, η -a.e. curve γ passing through x coincides in its trajectory with a common curve $\gamma_\eta(x) : [0, L(\gamma_\eta(x))] \rightarrow \mathbb{R}^d$ which goes from 0 to x .*

Proof. We simply use Proposition 1.2.16, noticing that $\theta_\eta(x) > 0$ if and only if $\eta(\Gamma(0, x)) > 0$ since η -a.e. curve starts at 0, and defining $\gamma_\eta(x) = \gamma_\eta(0, x)$. \square

For $\gamma \in \Gamma(\mathbb{R}^d)$ and $t \geq 0$, we define the joint multiplicity of γ at t w.r.t. the irrigation plan η by

$$\theta_\eta(\gamma[0, t]) = \eta(\tilde{\gamma} \in \Gamma(\mathbb{R}^d) : \tilde{\gamma}|_{[0, t]} = \gamma|_{[0, t]}).$$

The previous proposition yields the following immediate corollary.

Corollary 1.4.2. *Let η as in the previous proposition and parameterized by arc length. If $x \in N_\eta$ and $\gamma = \gamma_\eta(x)$ then:*

$$\forall t \leq L(\gamma), \quad \theta_\eta(\gamma(t)) = \theta_\eta(\gamma[0, t]).$$

1.4.2 The landscape function

The landscape function associated to an optimal irrigation plan was defined by Santambrogio in [San07] with some inspiration from optimal channel networks in geophysics (see [RR01]). The terminology and the results of this section are extracted from [San07] and given without proofs, for which we refer to the original paper, or to Chapter 4 which generalizes these results to the multiple sources case.

To define the landscape function associated to an optimal network η , we need to define a large class of curves which follow the network in a sense.

Definition 1.4.3 (η -good curves). We say that a curve γ is η -good if

$$\int_0^{T(\gamma)} \theta_\eta(\gamma[0, t])^{\alpha-1} dt < \infty$$

which is equivalent to the two assertions:

- (i) $Z_\eta(\gamma) < \infty$,
- (ii) $\theta_\eta(\gamma(t)) = \theta_\eta(\gamma[0, t])$ for all $t < T(\gamma)$.

Proposition 1.4.4. *If η is an optimal irrigation plan, the following assertions hold:*

- (i) η -good curves γ are injective⁹ on $[0, T(\gamma)[$,
- (ii) they are parameterized by arc length if η is itself parameterized by arc length,
- (iii) η is concentrated on the set of η -good curves.

One may view the set of η -good curves as the right generalization of support for irrigation plans. The following proposition is key to defining the landscape function.

Theorem 1.4.5 (Well-definedness property). *If η is optimal and γ_1, γ_2 are two η -good curves with the same endpoint x , then $Z_\eta(\gamma_1) = Z_\eta(\gamma_2)$.*

⁹Actually, it is injective on the whole $[0, T(\gamma)]$ but this is a consequence of Theorem 1.4.5.

Definition 1.4.6 (Landscape function). If η is optimal, the previous proposition allows us to define the landscape function z_η by:

$$z_\eta(x) = \begin{cases} Z_\eta(\gamma) & \text{if } \gamma \text{ is an } \eta\text{-good curve such that } \gamma(\infty) = x, \\ +\infty & \text{otherwise.} \end{cases}$$

The following first variation formula is fundamental. A slightly weaker version was established in [San07], the proof of this one is given in Chapter 3, Proposition 3.1.5.

Proposition 1.4.7 (First variation). *Suppose that η is an optimal irrigation plan between δ_0 and ν , with landscape function z_η . The following holds:*

$$d_\alpha(\delta_0, \tilde{\nu}) \leq d_\alpha(\delta_0, \nu) + \alpha \int_{\mathbb{R}^d} z_\eta d(\tilde{\nu} - \nu),$$

for all $\tilde{\nu} \in \text{Prob}(\mathbb{R}^d)$.

Therefore, z_η may be thought as a “supergradient” of $\tilde{\nu} \mapsto d_\alpha(\delta_0, \tilde{\nu})$. Let us state the other main properties of landscape functions.

Proposition 1.4.8 (Lower semicontinuity). *The landscape function associated to an optimal irrigation plan is lower semicontinuous.*

Proposition 1.4.9 (Steepest descent). *The network N_η follows the direction of steepest descent of z_η , in the sense that for all $x_0 \in N_\eta$:*

$$\forall x \in N_\eta, \quad z_\eta(x) \geq z_\eta(x_0) - \theta_\eta(x_0)^{\alpha-1} |x - x_0| + o(|x - x_0|).$$

With some extra hypotheses on ν , one may get more regularity of z_η .

Proposition 1.4.10 (Hölder continuity). *We assume that $\alpha > 1 - \frac{1}{d}$. Suppose that ν is of the form:*

$$\nu = f \mathcal{L} \llcorner E \quad \text{with} \quad f \geq c > 0 \text{ on } E,$$

and E is regular in the sense that

$$\forall x \in E, \forall r \leq \text{diam}(E), \quad \Theta_E(x, r) := \frac{|B_r(x) \cap E|}{|B_r(x)|} \geq c',$$

for some $c' > 0$. Then the landscape function z_η associated to any optimal irrigation plan $\eta \in \mathbf{OIP}^\alpha(\delta_0, \nu)$ is β -Hölder continuous on E where

$$\beta = d \left(\alpha - \left(1 - \frac{1}{d} \right) \right) = 1 + d\alpha - d.$$

Equivalence of the models

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The goal of this chapter is to give a complete and rigorous proof of the equivalence between the Lagrangian and Eulerian irrigation problems (\mathbf{LI}_α) and (\mathbf{EI}_α) . This is a fact which has been known for some time and was stated in the book [BCM09]. The first connection between the two models was perhaps being made in [PS06], where the Lagrangian model is set through what they call “transports” which correspond to our irrigation plans with a preliminary quotient with respect to reparameterization, and where the Eulerian model is cast in the framework of flat chains. An ad hoc Smirnov decomposition theorem was proved for flat chains (Smirnov originally proved it for normal currents in [Smi93]), as a way to build a transport from a flat chain, while a flat chain is clearly induced by a transport. Although it is not explicitly stated, the way to pass from Eulerian to Lagrangian (our Section 2.3) can be extracted from their work. For the converse way, one needs to show some simplicity of optimizers (done by some recursive loop-removal in [BCM09]), and use an energy formula which expresses the Lagrangian cost as a generalized Gilbert energy. This is done in [BCM09] via Fubini-Tonelli’s theorem also the σ -finiteness is not explicitly stated. To prove it, one should prove first the rectifiability of optimal irrigation plans, as done in Bernot’s PhD thesis ([Ber05]), which we prove differently in Section 2.1.1. The same fact was noticed in the recent paper [BW15] where they prove the equivalence between the Lagrangian (patterns) and Eulerian (mass flux) models with more general costs, replacing the α -cost and α -mass with H -costs and H -masses for a large class of functions H .

In Section 2.4 the equivalence between the two models is established, using the energy formula and a Smirnov decomposition (see [Smi93]) for vector measures obtained by Santambrogio in [San14] via a Dacorogna-Moser approach, as stated in Theorem 2.4.1. In short, both Eulerian and Lagrangian irrigation problems they have same minimal value, and one can build minimizers of one problem from minimizers of the other. In this chapter we assume $\alpha < 1$, and that we are on a compact set $K \subseteq \mathbb{R}^d$ for simplicity.

2.1 The energy formula

The aim of this section is to establish an energy formula stated in [BCM09, Section 4.3], expressing the Lagrangian irrigation cost as follows:

$$\int_{\Gamma} \int_{\gamma} \theta_{\eta}(x)^{\alpha-1} dx d\eta(\gamma) = \int_K \theta_{\eta}(x)^{\alpha} d\mathcal{H}^1(x), \quad (\text{EF})$$

provided η satisfies some hypotheses (namely essential simplicity and rectifiability). The term on the right-hand side is the so-called *Gilbert Energy* denoted by $\mathbf{E}^{\alpha}(\eta)$. The proof relies solely on the correct use of Fubini-Tonelli’s theorem, which requires σ -finiteness of measures. The next subsection is therefore devoted to the rectifiability of irrigation plans.

2.1.1 Rectifiable irrigation plans

Intensity and flow We define the *intensity* $i_\eta \in \mathcal{M}^+(K)$ and *flow* $v_\eta \in \mathcal{M}^d(K)$ of an irrigation plan $\eta \in \mathbf{IP}(\mu, \nu)$ by the action on continuous maps:

$$\begin{aligned}\langle i_\eta, \phi \rangle &= \int_\Gamma \int_\gamma \phi(x) \, dx \, d\eta(\gamma), \\ \langle v_\eta, \psi \rangle &= \int_\Gamma \int_\gamma \psi(x) \cdot dx \, d\eta(\gamma),\end{aligned}$$

for all $\phi \in \mathcal{C}(K)$, $\psi \in \mathcal{C}(K)^d$. The quantity $di_\eta(x)$ represents the total circulation at x and $dv_\eta(x)$ the total flow at x .

Concentration and rectifiability Let A be a Borel subset of K . By definition of i_η , the following assertions are equivalent:

- (i) i_η is concentrated on A , i.e. $i_\eta(A^c) = 0$,
- (ii) for η -a.e. $\gamma \in \Gamma$, $\gamma \subseteq A$ up to an \mathcal{H}^1 -null set, i.e. $\mathcal{H}^1(\gamma \setminus A) = 0$.

In that case we say (with a slight abuse) that η is *concentrated on* A . An irrigation plan $\eta \in \mathbf{IP}(K)$ is termed *σ -finite* if it is concentrated on a σ -finite set w.r.t. \mathcal{H}^1 and *rectifiable* if it is concentrated on a 1-rectifiable set¹.

The intensity i_η has a simple expression when η is σ -finite, as shown below.

Proposition 2.1.1. *If η is an irrigation plan concentrated on a σ -finite set A , then $i_\eta = m_\eta \mathcal{H}^1 \llcorner A$.*

Proof. For any Borel set B , one has

$$\begin{aligned}i_\eta(B) &= i_\eta(A \cap B) = \int_\Gamma \int_\gamma \mathbf{1}_{A \cap B} \, dx \, d\eta(\gamma) \\ &= \int_\Gamma \int_B m(x, \gamma) \, d\mathcal{H}^1 \llcorner A(x) \, d\eta(\gamma) \\ &= \int_B \int_\Gamma m(x, \gamma) \, d\eta(\gamma) \, d\mathcal{H}^1 \llcorner A(x) \\ &= \int_B m_\eta(x) \, d\mathcal{H}^1 \llcorner A(x).\end{aligned}$$

The equality on the second line follows from the coarea formula, the next from Fubini-Tonelli's theorem which holds because measures are σ -finite, and the last one from the definition of m_η . \square

Now we would like to prove that irrigation plans η are concentrated on their network $N_\eta \doteq \{x \in K : \theta_\eta(x) > 0\}$ and that the latter is a rectifiable set. The first assertion is true provided η has finite α -cost, as we shall see later, and the second item holds in all generality, as we shall prove know. Our proof follows an idea of [BCM09, Theorem 4.10],

¹A 1-rectifiable set is the union of an \mathcal{H}^1 -null set with a countable union of Lipschitz curves.

though we cannot use the energy formula yet (as its proof requires said rectifiability or σ -finiteness) and take care of the fact that the rectifiability criterion of Lemma 2.1.3 (as well as Lemma 2.1.4) applies to sets of finite \mathcal{H}^1 measure and not of σ -finite \mathcal{H}^1 measure. In [BW15, Lemma 3.18], another proof is provided adapting a proof of Bernot ([Ber05, Lemma 4.6.4] or [BCM09, Lemma 6.3]). With our approach, we will need a few notions and lemmas from geometric measure theory.

If A is a subset of \mathbb{R}^d we define the upper and lower 1-density of A at x as

$$\bar{\Theta}(x, A) = \limsup_{r \downarrow 0} \frac{\mathcal{H}^1(A \cap B(x, r))}{2r}, \quad \underline{\Theta}(x, A) = \liminf_{r \downarrow 0} \frac{\mathcal{H}^1(A \cap B(x, r))}{2r}.$$

When these quantities are equal, we call their common value $\Theta(x, A)$ the 1-density of A at x .

The first lemma we will need is proved in [Mat95, Chapter 8].

Lemma 2.1.2. *If $B \subseteq \mathbb{R}^d$,*

$$\mathcal{H}^1(B) = \sup\{\mathcal{H}^1(K) : K \subseteq B \text{ compact such that } \mathcal{H}^1(K) < \infty\}.$$

The second is due to Besicovitch and may be obtained as a particular case of [Mat95, Theorem 17.6].

Lemma 2.1.3. *Let E be an \mathcal{H}^1 -measurable set such that $\mathcal{H}^1(E) < \infty$. If its 1-density $\Theta(x, E)$ exists and is equal to 1 for \mathcal{H}^1 -a.e. x in E then it is 1-rectifiable.*

Finally, we will need the following result which is included in [Mat95, Theorem 6.2].

Lemma 2.1.4. *If E is a set such that $\mathcal{H}^1(E) < \infty$, then the upper 1-density $\bar{\Theta}(x, E)$ is less than 1 for \mathcal{H}^1 -a.e. x in E .*

Proposition 2.1.5 (Rectifiability of the network). *If $\eta \in \mathbf{IP}(K)$ is an irrigation plan, its network N_η is 1-rectifiable.*

Proof. First of all, since the network does not change under normalization, we may assume that η is parameterized by arc length. We have $N_\eta = \bigcup_{n>0} D^n$ where

$$D^n := \left\{ x : \theta_\eta(x) > \frac{1}{n} \right\}.$$

Let us show that that $\mathcal{H}^1(D^n) < \infty$. By contradiction assume that for some n , $\mathcal{H}^1(D^n) = \infty$, hence thanks to Lemma 2.1.2 one can find for $M > 0$ as large as we want a compact subset $K' \subseteq D^n$ such that $M \leq \mathcal{H}^1(K') < \infty$. Since $\mathcal{H}^1(K') < \infty$ one can use Fubini-Tonelli's theorem to get

$$\begin{aligned} \mathbf{L}(\eta) &\doteq \int_{\Gamma} L(\gamma) \, d\eta(\gamma) \geq \int_{\Gamma} \mathcal{H}^1(\gamma \cap K') \, d\eta(\gamma) = \int_{\Gamma} \int_{K'} \mathbf{1}_{x \in \gamma} \, d\mathcal{H}^1(x) \, d\eta(\gamma) \\ &= \int_{K'} \int_{\Gamma} \mathbf{1}_{x \in \gamma} \, d\eta(\gamma) \, d\mathcal{H}^1(x) \\ &= \int_{K'} \theta_\eta(x) \, d\mathcal{H}^1(x) > \frac{M}{n}. \end{aligned}$$

The inequality $\mathbf{L}(\eta) > \frac{M}{n}$ must be true for all $M > 0$, i.e. $\mathbf{L}(\eta) = \infty$, which contradicts the definition of an irrigation plan, hence $\mathcal{H}^1(D^n) < \infty$.

Now we shall prove that $\Theta(x, D^n) = 1$ a.e. on D^n . Since D^n has finite \mathcal{H}^1 -measure, we already know $\bar{\Theta}(x, D^n) \leq 1$ for \mathcal{H}^1 -a.e. $x \in D^n$ by Lemma 2.1.4, thus it remains only to prove $\underline{\Theta}(x, D^n) \geq 1$. If A is a Borel subset of \mathbb{R} we denote $\text{Leb}(A)$ the set of Lebesgue points of A , which are points t such that

$$\lim_{r \downarrow 0} \frac{|A \cap [t-r, t+r]|}{2r} = 1,$$

where $|X|$ denotes the Lebesgue measure of $X \subseteq \mathbb{R}$. Recall that by Lebesgue's theorem we have $|A \setminus \text{Leb}(A)| = 0$. For any $\gamma \in \Gamma^1$, we set

$$\begin{aligned} A_\gamma &= \left\{ t : t \in \text{Leb} \left(s : |\dot{\gamma}(s)|_\eta > \frac{1}{n} \right) \text{ for all } n \text{ s.t. } |\dot{\gamma}(t)|_\eta > \frac{1}{n} \right\}, \\ B_\gamma &= \{ t \in]0, T(\gamma)[: \dot{\gamma}(t) \text{ exists} \}, \\ D_\gamma &= \gamma(A_\gamma \cap B_\gamma). \end{aligned}$$

Notice that $|[0, T(\gamma)] \setminus (A_\gamma \cup B_\gamma)| = 0$ hence $\mathcal{H}^1(\gamma \setminus D_\gamma) = 0$ since γ is Lipschitz. Finally we set

$$D' = \bigcup_{\gamma \in \Gamma^1} D_\gamma.$$

Let us check that $\mathcal{H}^1(N_\eta \setminus D') = 0$. We have

$$\begin{aligned} \int_{N_\eta \setminus D'} \theta_\eta(x) \, d\mathcal{H}^1(x) &= \int_{N_\eta \setminus D'} \int_\Gamma \mathbf{1}_{x \in \gamma} \, d\eta(\gamma) \, d\mathcal{H}^1(x) \\ &= \int_\Gamma \int_{N_\eta \setminus D'} \mathbf{1}_{x \in \gamma} \, d\mathcal{H}^1(x) \, d\eta(\gamma) \\ &= \int_{\Gamma^1} \mathcal{H}^1(N_\eta \cap \gamma \setminus D') \, d\eta(\gamma) \\ &= 0. \end{aligned}$$

The use of Fubini-Tonelli's theorem is justified since $N_\eta = \bigcup_n D^n$ is σ -finite and the last equality follows from $\mathcal{H}^1(N_\eta \cap \gamma \setminus D') \leq \mathcal{H}^1(\gamma \setminus D_\gamma) = 0$. This implies $\mathcal{H}^1(N_\eta \setminus D') = 0$ since $\theta_\eta > 0$ on N_η . Now take any $x \in D^n \cap D'$. By construction of D' there is a curve $\gamma \in \Gamma^1$ and a $t \in A_\gamma \cap B_\gamma$ such that $x = \gamma(t)$, which implies that

$$\frac{|s \in [t-r, t+r] : \gamma(s) \in D^n|}{2r} \xrightarrow{r \downarrow 0} 1 \quad (2.1.1)$$

and

$$\frac{\mathcal{H}^1(\gamma([t-r, t+r]) \setminus D^n)}{2r} \xrightarrow{r \downarrow 0} 0. \quad (2.1.2)$$

It follows from (2.1.2) and the fact that $\gamma([t-r, t+r]) \subseteq \bar{B}(x, r)$ because γ is 1-Lipschitz that

$$\underline{\Theta}(x, D^n) \doteq \liminf_{r \downarrow 0} \frac{\mathcal{H}^1(B(x, r) \cap D^n)}{2r} \geq \liminf_{r \downarrow 0} \frac{\mathcal{H}^1(\gamma([t-r, t+r]))}{2r}.$$

But γ has a derivative e at t which has unit norm. Moreover the \mathcal{H}^1 -measure of $\gamma([t-r, t+r])$, which is a compact connected set, is larger than the distance between $\gamma(t-r)$ and $\gamma(t+r)$, and since $\gamma(t \pm r) = x \pm re + o(r)$ one has

$$\mathcal{H}^1(\gamma([t-r, t+r])) \geq |\gamma(t+r) - \gamma(t-r)| = 2r + o(r),$$

which yields $\Theta(x, D^n) \geq 1$. This proves that $\Theta(x, D^n)$ exists and is equal to 1 for \mathcal{H}^1 -a.e. $x \in D^n$ hence D^n is 1-rectifiable by Lemma 2.1.3 and $N_\eta = \bigcup_n D^n$ as well. \square

At this stage we have shown that the network of any irrigation plan is rectifiable, yet this does not mean that any irrigation plan is rectifiable (this is obviously not the case) since η needs not be concentrated on N_η . However, it is essentially the only candidate rectifiable set (or even candidate σ -finite set) on which η could be concentrated, as stated below.

Corollary 2.1.6. *Given $\eta \in \mathbf{IP}(K)$ an irrigation plan, the following assertions are equivalent:*

- (i) η is concentrated on N_η ,
- (ii) η is rectifiable,
- (iii) η is σ -finite.

Proof. It is enough to prove (iii) \Rightarrow (i) by the previous proposition. If η is concentrated on a σ -finite set A , we know by Proposition 2.1.1 that $i_\eta = m_\eta \mathcal{H}^1 \llcorner A$. Therefore i_η is also concentrated on $\{x : m_\eta(x) > 0\} = N_\eta$. \square

Remark 2.1.7. From this and Proposition 2.1.1 we get that $i_\eta = m_\eta \mathcal{H}^1$ if η is rectifiable.

The most important consequence of Proposition 2.1.5 is the following rectifiability result.

Theorem 2.1.8 (Rectifiability). *If η has finite α -cost with $\alpha \in [0, 1[$, it is rectifiable.*

Proof. Because of the previous statement, we need only show that η is concentrated on N_η . We have $\mathbf{I}^\alpha(\eta) \doteq \int_\Gamma \int_\gamma \theta_\eta(x)^{\alpha-1} dx d\eta(\gamma) < \infty$ hence for η -almost every curve γ , for \mathcal{H}^1 -almost every x in γ , $\theta_\eta(x)^{\alpha-1} < \infty$, which implies that $\theta_\eta(x) > 0$ i.e. $x \in N_\eta$. By definition, it means that η is concentrated on N_η . \square

2.1.2 Proof of the energy formula

We define Gilbert energy $\mathbf{E}^\alpha : \mathbf{IP}(K) \rightarrow [0, \infty]$ as

$$\mathbf{E}^\alpha(\eta) = \begin{cases} \int_K \theta_\eta(x)^\alpha d\mathcal{H}^1(x) & \text{if } \eta \text{ is rectifiable,} \\ +\infty & \text{otherwise.} \end{cases}$$

and a variant $\bar{\mathbf{E}}^\alpha : \mathbf{IP}(K) \rightarrow [0, \infty]$, which is kind of a “full” Gilbert energy:

$$\bar{\mathbf{E}}^\alpha(\eta) = \begin{cases} \int_K \theta_\eta^{\alpha-1}(x) m_\eta(x) \, d\mathcal{H}^1(x) & \text{if } \eta \text{ is rectifiable,} \\ +\infty & \text{otherwise.} \end{cases}$$

Assuming $\alpha \in [0, 1[$, we would like to establish the energy formula

$$\mathbf{I}^\alpha(\eta) = \mathbf{E}^\alpha(\eta). \quad (\text{EF})$$

This does not hold in general. Actually we are going to show that $\mathbf{I}^\alpha(\eta) = \bar{\mathbf{E}}^\alpha(\eta)$ for all irrigation plan $\eta \in \mathbf{IP}(K)$ and that $\bar{\mathbf{E}}^\alpha(\eta) = \mathbf{E}^\alpha(\eta)$ provided η is essentially simple.

Theorem 2.1.9 (Energy formula). *Assuming $\alpha \in [0, 1[$, the following formula holds:*

$$\mathbf{I}^\alpha(\eta) = \bar{\mathbf{E}}^\alpha(\eta). \quad (\text{EF}')$$

Moreover, if η is essentially simple this rewrites

$$\mathbf{I}^\alpha(\eta) = \mathbf{E}^\alpha(\eta). \quad (\text{EF})$$

Proof. By Theorem 2.1.8, if η is not rectifiable then $\mathbf{I}^\alpha(\eta) = \mathbf{E}^\alpha(\eta) = \bar{\mathbf{E}}^\alpha(\eta) = \infty$ and the result is clear. Now we assume that η is rectifiable, which means that it is concentrated on the rectifiable set N_η , according to Theorem 2.1.8 and Corollary 2.1.6. Notice that by the coarea formula we have

$$\int_\Gamma \int_\gamma \theta_\eta(x)^{\alpha-1} \, dx \, d\eta(\gamma) = \int_\Gamma \int_K \theta_\eta(x)^{\alpha-1} m(x, \gamma) \, d\mathcal{H}^1(x) \, d\eta(\gamma),$$

thus the goal is to reverse the order of integration. Here Fubini-Tonelli’s theorem applies because η is concentrated on its network, which is rectifiable, which yields

$$\begin{aligned} \mathbf{I}^\alpha(\eta) &= \int_\Gamma \int_{N_\eta} \theta_\eta(x)^{\alpha-1} m(x, \gamma) \, d\mathcal{H}^1(x) \, d\eta(\gamma) \\ &= \int_{N_\eta} \theta_\eta(x)^{\alpha-1} m_\eta(x) \, d\mathcal{H}^1(x) \\ &= \int_K \theta_\eta(x)^{\alpha-1} m_\eta(x) \, d\mathcal{H}^1(x) = \bar{\mathbf{E}}^\alpha(\eta). \end{aligned}$$

and (EF’) holds. Now if η is essentially simple then in all the previous calculations $m_\eta(x) = \theta_\eta(x)$ so that

$$\begin{aligned} \mathbf{I}^\alpha(\eta) &= \int_{N_\eta} \int_\Gamma \theta_\eta(x)^{\alpha-1} \theta_\eta(x) \, d\eta(\gamma) \, d\mathcal{H}^1(x) \\ &= \int_K \theta_\eta(x)^\alpha \, d\mathcal{H}^1(x) = \mathbf{E}^\alpha(\eta), \end{aligned}$$

thus getting (EF). □

Remark 2.1.10. Actually, the proof shows that the equality $\mathbf{I}^\alpha(\eta) = \bar{\mathbf{E}}^\alpha(\eta)$ (and $\mathbf{I}^\alpha(\eta) = \mathbf{E}^\alpha(\eta)$ if η is essentially simple) holds also for $\alpha = 1$ provided η is rectifiable. However, one may find η non-rectifiable such that $\mathbf{I}_1(\eta) \in]0, \infty[$ while $\mathcal{H}^1(N_\eta) = 0$. In that case one has $0 = \int_K m_\eta(x) \, d\mathcal{H}^1(x) < \mathbf{I}_1(\eta) < \bar{\mathbf{E}}_1(\eta) = \infty$. Notice also that for such η one has $0 = \int_K \theta_\eta(x)^{\alpha-1} m_\eta(x) \, d\mathcal{H}^1(x) < \mathbf{I}^\alpha(\eta) = \infty$ for $\alpha \in [0, 1[$, which explains why we imposed $\bar{\mathbf{E}}^\alpha(\eta) = \bar{\mathbf{E}}^\alpha(\eta) = \infty$ if η is not rectifiable.

2.1.3 Simplicity of optimal irrigation plans

In this section we shall prove that optimal irrigation plans are necessary simple using the energy formula.

“Reduced” intensity We associate to any irrigation plan $\eta \in \mathbf{IP}(K)$ a “reduced” intensity j_η by

$$\langle j_\eta, \phi \rangle = \int_{\Gamma} \int_{\gamma} \phi(x) \, d\mathcal{H}^1(x) \, d\eta(\gamma),$$

for all $\phi \in \mathcal{C}(K)$. It is a positive finite measure, since the total mass is

$$\|j_\eta\| = \int_{\Gamma} \mathcal{H}^1(\gamma) \, d\eta(\gamma) \leq \int_{\Gamma} L(\gamma) \, d\eta(\gamma) = \mathbf{L}(\eta) < \infty.$$

Remark 2.1.11. Notice that if A is a Borel set, $j_\eta(A) = 0 \Leftrightarrow i_\eta(A) = 0$ hence by definition η is rectifiable if and only if j_η is concentrated on a rectifiable set, in which case it is concentrated on the rectifiable network N_η and one has $j_\eta = \theta_\eta \mathcal{H}^1$ using Fubini-Tonelli’s theorem.

Lemma 2.1.12 (Simple replacement). *Let $\eta \in \mathbf{IP}(\mu, \nu)$ be an irrigation plan. Consider the minimization problem*

$$\min \{ \mathbf{L}(\zeta) : j_\zeta \leq j_\eta \text{ and } \zeta \in \mathbf{IP}(\mu, \nu) \}. \quad (\text{LEN}_\eta)$$

Then

- (i) this problem admits minimizers which are all simple,
- (ii) if η is rectifiable, all minimizers ζ are also rectifiable and $j_\zeta \leq j_\eta$ rewrites

$$\theta_\zeta \leq \theta_\eta \quad \mathcal{H}^1 \text{-almost everywhere.} \quad (2.1.3)$$

Any minimizer of (LEN_η) is called a simple replacement of η .

Proof. Let us call m the infimum of (LEN_η) and show that it admits a minimizer. Take a minimizing sequence $(\zeta_n)_n$ such that every ζ_n is normalized, in particular $\zeta_n \in \mathbf{IP}_C(K)$ for some $C > 0$. Up to extraction we have convergence $\zeta_n \rightarrow \zeta$, and since $\mathbf{IP}_C(\mu, \nu)$ is closed by Proposition 1.2.2, $\zeta \in \mathbf{IP}_C(\mu, \nu)$. Moreover $\mathbf{L}(\zeta) = m$ by lower semicontinuity of $\mathbf{L} \doteq \mathbf{I}_1$ on $\mathbf{IP}_C(K)$, which we proved in Corollary 1.2.8. Now in order to show that ζ is a solution of (LEN_η) we only have to check the last constraint $j_\zeta \leq j_\eta$. Take any open set O . One has

$$j_{\zeta_n}(O) = \int_{\Gamma} \mathcal{H}^1(\gamma \cap O) \zeta_n(d\gamma).$$

By a generalization of Golab’s Theorem (see [BB05]), the following holds

$$\mathcal{H}^1(\gamma \cap O) \leq \liminf_n \mathcal{H}^1(\gamma_n \cap O)$$

if $\gamma_n \rightarrow \gamma$ uniformly on compact sets, which means that $\gamma \mapsto \mathcal{H}^1(\gamma \cap O)$ is lower semicontinuous on Γ . Consequently $\zeta \mapsto \int_{\Gamma} \mathcal{H}^1(\gamma \cap O) \zeta(d\gamma)$ is lower semicontinuous and one gets

$$j_{\zeta}(O) \doteq \int_{\Gamma} \mathcal{H}^1(\gamma \cap O) \zeta(d\gamma) \leq \liminf_n \int_{\Gamma} \mathcal{H}^1(\gamma \cap O) \zeta_n(d\gamma) \leq j_{\eta}(O)$$

for all open set O . This implies that $j_{\zeta} \leq j_{\eta}$ by regularity of finite measures hence ζ is a minimizer of [\(LEN\) \$_{\eta}\$](#) .

Let us check that any minimizer ζ is simple. By contradiction, if it was not simple there would be a set $\Gamma' \subseteq \Gamma$ such that $\zeta(\Gamma') > 0$ and every $\gamma \in \Gamma'$ has a loop. One may define a Borel map $r : \gamma \mapsto r(\gamma)$ which removes from $\gamma \in \Gamma'$ the loop with maximal length (the first one in case there are several), and is identical on $\Gamma \setminus \Gamma'$. Then set $\bar{\zeta} := r(\zeta)$. Obviously one has $\mathbf{L}(\bar{\zeta}) < \mathbf{L}(\zeta)$, $\bar{\zeta} \in \mathbf{IP}(\mu, \nu)$ and $j_{\bar{\zeta}} \leq j_{\zeta}$, which contradicts the optimality of ζ in [\(LEN\) \$_{\eta}\$](#) .

Finally, suppose η is rectifiable and take ζ a minimizer of our problem. According to Remark [2.1.11](#), the inequality $j_{\zeta} \leq j_{\eta}$ implies that ζ is rectifiable and $j_{\eta} = \theta_{\eta} \mathcal{H}^1$, $j_{\zeta} = \theta_{\zeta} \mathcal{H}^1$, which yields [\(ii\)](#). \square

Proposition 2.1.13. *Given $\alpha \in [0, 1]$, if $\eta \in \mathbf{IP}(\mu, \nu)$ is optimal with finite α -cost, then it is simple.*

Proof. The case $\alpha = 1$ is straightforward from Lemma [2.1.12](#) since $\mathbf{L} = \mathbf{I}_1$. Now we assume that $\alpha < 1$ and take η optimal, in which case the finiteness of the α -cost implies the rectifiability of η by Theorem [2.1.8](#). We need only show that η is a minimizer of [\(LEN\) \$_{\eta}\$](#) . Take $\tilde{\eta}$ a simple replacement of η . Then since $\eta, \tilde{\eta}$ are rectifiable and $\theta_{\tilde{\eta}} \leq \theta_{\eta}$ \mathcal{H}^1 -a.e., one has

$$\mathbf{I}^{\alpha}(\eta) = \int_K \theta_{\eta}^{\alpha-1} m_{\eta} d\mathcal{H}^1 \geq \int_K \theta_{\eta}^{\alpha} d\mathcal{H}^1 \geq \int_K \theta_{\tilde{\eta}}^{\alpha} d\mathcal{H}^1 = \mathbf{I}^{\alpha}(\tilde{\eta})$$

Since η is optimal we have equality everywhere, which means that $m_{\eta} = \theta_{\eta} = \theta_{\tilde{\eta}} = m_{\tilde{\eta}}$ \mathcal{H}^1 -a.e. Consequently

$$\mathbf{L}(\eta) = \int_K m_{\eta}(x) d\mathcal{H}^1(x) = \int_K m_{\tilde{\eta}}(x) d\mathcal{H}^1(x) = \mathbf{L}(\tilde{\eta})$$

hence η minimizes [\(LEN\) \$_{\eta}\$](#) and is as such simple by Lemma [2.1.12](#). \square

2.2 From Lagrangian to Eulerian

Recall that we have associated to any irrigation plan $\eta \in \mathbf{IP}(K)$ an intensity $i_{\eta} \in \mathcal{M}^+(K)$ and a flow $v_{\eta} \in \mathcal{M}^d(K)$. We will show that v_{η} is an irrigation flow sending μ to ν and satisfying $\mathbf{M}^{\alpha}(v_{\eta}) \leq \mathbf{I}^{\alpha}(\eta)$ under some hypotheses.

Proposition 2.2.1. *If $\eta \in \mathbf{IP}(\mu, \nu)$ then $v_{\eta} \in \mathbf{IF}(\mu, \nu)$.*

Proof. Let us calculate the distributional divergence of v_η . For $\phi \in \mathcal{C}(K)$, we have

$$\begin{aligned} \langle \nabla \cdot v_\eta, \phi \rangle &= -\langle v_\eta, \nabla \phi \rangle = -\int_\Gamma \int_0^\infty \nabla \phi(\gamma(t)) \cdot \dot{\gamma}(t) \, dt \, d\eta(\gamma) \\ &= \int_{\Gamma^1} (\phi(\gamma(0)) - \phi(\gamma(\infty))) \, d\eta(\gamma) \\ &= \int_K \phi(x) \mu(dx) - \int_K \phi(x) \nu(dx), \end{aligned}$$

thus $\nabla \cdot v_\eta = \mu - \nu \in \mathcal{M}^1(K)$, which implies that $v_\eta \in \mathbf{IF}(\mu, \nu)$. \square

Proposition 2.2.2. *If η is rectifiable, then v_η is a rectifiable irrigation flow.*

Proof. Without loss of generality we may assume that η is parameterized by arc length. We know that η is concentrated on its network N_η , which is rectifiable. Given $\psi \in \mathcal{C}(K)^d$, by the area formula one has

$$\begin{aligned} \int_\Gamma \int_0^\infty \psi(\gamma(t)) \cdot \dot{\gamma}(t) \, dt \, d\eta(\gamma) &= \int_\Gamma \int_\gamma \psi(x) \cdot \sum_{\gamma(t)=x} \dot{\gamma}(t) \, dx \, d\eta(\gamma) \\ &= \int_K \psi(x) \cdot \int_{\Gamma(x)} \sum_{\gamma(t)=x} \dot{\gamma}(t) \, d\eta(\gamma) \, d\mathcal{H}^1(x) \\ &= \int_K \psi(x) \cdot v(x) \, d\mathcal{H}^1(x), \end{aligned}$$

where we have set for \mathcal{H}^1 -almost every $x \in N_\eta$:

$$v(x) = \int_{\Gamma(x)} \sum_{\gamma(t)=x} \dot{\gamma}(t) \, d\eta(\gamma).$$

Let us denote by $L(x)$ the approximate tangent line at \mathcal{H}^1 -almost every x . Since a.e. γ stays in N_η up to a \mathcal{H}^1 -null set, we know that for η -almost every γ , for \mathcal{H}^1 -almost every $x \in \gamma$, if $x = \gamma(t)$ then $\dot{\gamma}(t)$ (is well-defined and) belongs to $L(x)$. By Fubini's theorem, one can reverse the order of "almost everywhere"'s and state that for \mathcal{H}^1 -a.e. $x \in N_\eta$, for η -a.e. $\gamma \in \Gamma(x)$, $\sum_{\gamma(t)=x} \dot{\gamma}(t) \in L(x)$ hence $v(x) \in L(x)$. Consequently v_η is the rectifiable irrigation flow $v_\eta = \llbracket N_\eta, \tau, \theta \rrbracket$ where $\tau = v(x)/|v(x)|$ (whatever value in \mathbb{S}^{d-1} if $v(x) = 0$) and $\theta(x) = |v(x)|$. \square

Proposition 2.2.3. *If η is an essentially simple and rectifiable irrigation plan, in particular if η is optimal, we have*

$$\mathbf{M}^\alpha(v_\eta) \leq \mathbf{I}^\alpha(\eta).$$

Proof. By Remark 2.1.7 we know that $i_\eta = m_\eta \mathcal{H}^1 = \theta_\eta \mathcal{H}^1$. Since $|v_\eta| \leq i_\eta = \theta_\eta \mathcal{H}^1$, v_η has an \mathcal{H}^1 -density which is less than θ_η . Using the energy formula, we have

$$\mathbf{I}^\alpha(\eta) = \mathbf{E}^\alpha(\eta) = \int_K \theta_\eta(x)^\alpha \mathcal{H}^1(dx) \geq \int_K |v_\eta(x)|^\alpha \, d\mathcal{H}^1(x) = \mathbf{M}^\alpha(v_\eta),$$

the last equality because v_η rectifiable as state in the previous proposition. \square

We have therefore proven that we have

$$\inf_{\mathbf{IF}(\mu, \nu)} \mathbf{M}^\alpha \leq \inf_{\mathbf{IP}(\mu, \nu)} \mathbf{I}^\alpha,$$

and that if η is optimal, v_η is a good optimal candidate for the Eulerian problem (\mathbf{EI}_α).

2.3 From Eulerian to Lagrangian

Given an irrigation flow $v \in \mathbf{IF}(\mu, \nu)$ of finite cost \mathbf{M}^α , we would like to build an irrigation plan $\eta \in \mathbf{IP}(\mu, \nu)$ such that $v = v_\eta$ (and whose cost is less than v). This is not true in general (because of the presence of cycles) but a Smirnov decomposition gives the result if v is optimal for (\mathbf{EI}_α) .

Cycle If $v \in \mathbf{IF}(K)$, we say that $w \in \mathbf{IF}(K)$ is a *cycle* of v if $|v| = |w| + |v - w|$ and $\nabla \cdot w = 0$. It is easy to check that if v is rectifiable then w and $v - w$ are also rectifiable. The following Smirnov decomposition is proved by Santambrogio via a Dacorogna-Moser approach in [San14].

Theorem 2.3.1 (Irrigation flow decomposition). *Given an irrigation flow $v \in \mathbf{IF}(\mu, \nu)$, there is an irrigation plan $\eta \in \mathbf{IP}(\mu, \nu)$ and a cycle $w \in \mathbf{IF}(K)$ satisfying*

$$(i) \quad v = v_\eta + w,$$

$$(ii) \quad i_\eta \leq |v|.$$

From this we deduce:

Corollary 2.3.2. *If v is an optimal irrigation flow in $\mathbf{IF}(\mu, \nu)$, there is an irrigation plan $\eta \in \mathbf{IP}(\mu, \nu)$ such that*

$$(i) \quad v = v_\eta,$$

$$(ii) \quad |v_\eta| = i_\eta.$$

Proof. Let us take v_η, w as in the previous theorem. Since $\mathbf{M}^\alpha(v) < \infty$, v and v_η are rectifiable, and by optimality of v one has

$$\int_K |v_\eta(x)|^\alpha \, d\mathcal{H}^1(x) = M_\alpha(v_\eta) \geq M_\alpha(v) = \int_K (|v_\eta(x)| + |w(x)|)^\alpha \, d\mathcal{H}^1(x),$$

thus we must have $|w(x)| = 0$ \mathcal{H}^1 -a.e., which means $w = 0$, thus $v = v_\eta$ and (i) holds. This implies $|v_\eta| \leq i_\eta \leq |v| = |v_\eta|$ and thus we have the equality $|v_\eta| = i_\eta$ claimed in (ii). \square

Proposition 2.3.3. *If v is an optimal irrigation flow in $\mathbf{IF}(\mu, \nu)$, one can find an irrigation plan $\eta \in \mathbf{IP}(\mu, \nu)$ such that*

$$\mathbf{I}^\alpha(\eta) \leq \mathbf{M}^\alpha(v).$$

Proof. Take η as in the previous corollary. Since $\mathbf{M}^\alpha(v) < \infty$, v is rectifiable and $i_\eta = |v|$ is concentrated on a rectifiable set, which means by definition that η is rectifiable. As a consequence $|v| = i_\eta = m_\eta \mathcal{H}^1$ and we have

$$\mathbf{M}^\alpha(v) = \int_K |v(x)|^\alpha \, d\mathcal{H}^1(x) = \int_K m_\eta(x)^\alpha \, d\mathcal{H}^1(x),$$

while

$$\mathbf{I}^\alpha(\eta) = \bar{\mathbf{E}}^\alpha(\eta) = \int_K \theta_\eta(x)^{\alpha-1} m_\eta(x) \, d\mathcal{H}^1(x).$$

We would like $\mathbf{I}^\alpha(\eta) \leq \mathbf{M}^\alpha(v)$, which is a priori not necessarily the case for the η we have constructed, since it is not necessarily essentially simple. Instead, take a simple replacement $\tilde{\eta} \in \mathbf{IP}(\mu, \nu)$ satisfying $m_{\tilde{\eta}} = \theta_{\tilde{\eta}} \leq \theta_\eta \leq m_\eta$. Then we get

$$\mathbf{I}^\alpha(\tilde{\eta}) = \int_K \theta_{\tilde{\eta}}(x)^\alpha \, d\mathcal{H}^1(x) \leq \int_K m_\eta(x)^\alpha \, d\mathcal{H}^1(x) = \mathbf{M}^\alpha(v)$$

which yields the result. \square

Remark 2.3.4. Since the minima in the Eulerian and Lagrangian problems are actually the same as we shall see in Theorem 2.4.1, the previous inequality is an equality, which implies that $\theta_\eta = \theta_{\tilde{\eta}} = m_\eta$ \mathcal{H}^1 -a.e. thus η was actually optimal hence simple.

2.4 The equivalence theorem

We are now able to formulate the equivalence between the Lagrangian and Eulerian models.

Theorem 2.4.1 (Equivalence theorem). *If $\alpha < 1$ and $\mu, \nu \in \text{Prob}(K)$, the Eulerian problem (\mathbf{EI}_α) and the Lagrangian problem (\mathbf{LI}_α) are equivalent in the following sense:*

(i) *the minima are the same*

$$\min_{\eta \in \mathbf{IP}(\mu, \nu)} \mathbf{I}^\alpha(\eta) = \min_{v \in \mathbf{IF}(\mu, \nu)} \mathbf{M}^\alpha(v),$$

(ii) *if v is optimal in $\mathbf{IF}(\mu, \nu)$, it can be represented by an optimal irrigation plan, i.e. $v = v_\eta$ for some optimal $\eta \in \mathbf{IP}(\mu, \nu)$,*

(iii) *if η is optimal in $\mathbf{IP}(\mu, \nu)$, then v_η is optimal in $\mathbf{IF}(\mu, \nu)$ and $i_\eta = |v_\eta|$.*

Proof. It all follows from Proposition 2.2.3 and Proposition 2.3.3. The equality $i_\eta = |v_\eta|$ comes from

$$\mathbf{I}^\alpha(\eta) = \int_K i_\eta^\alpha(x) \, d\mathcal{H}^1(x) \geq \int_K |v_\eta(x)|^\alpha \, d\mathcal{H}^1(x) = \mathbf{M}^\alpha(v_\eta),$$

since we have equality everywhere by optimality of v_η and η . \square

Remark 2.4.2. Notice in particular that the equality $i_\eta = |v_\eta|$ implies that curves of η have the same tangent vectors when they coincide. To be more precise, one may use the proof of Proposition 2.2.2 to state that there is an \mathcal{H}^1 -a.e. defined function $\tau : N_\eta \mapsto \mathbb{S}^{d-1}$ such that for η -a.e. $\gamma \in \Gamma$, for \mathcal{H}^1 -a.e. $x \in \gamma$, $\dot{\gamma}(t) = |\dot{\gamma}(t)|\tau(x)$ whenever $\gamma(t) = x$.

Extension to the H -mass We assume that H satisfies:

- (i) $H(0) = 0$,
- (ii) H is lower semicontinuous and nondecreasing,
- (iii) H is subadditive,
- (iv) $\lim_{x \rightarrow 0} \frac{H(x)}{x} = +\infty$.

These hypotheses are weaker than those needed to prove lower semicontinuity of the Lagrangian H -cost, as remarked in the last paragraph of Section 1.2.3. Under these hypotheses, it was proven in [Col+17] that the H -mass is lower semicontinuous and it is not difficult to show existence for the Eulerian problem with such H -mass. Without any changes other than replacing x^α and $x^{\alpha-1}$ with $H(x)$ and $H(x)/x$ respectively, one may check that the equivalence between the Lagrangian and Eulerian irrigation problems hold. As a byproduct, we obtain existence of minimizers for the Lagrangian problem (built from a minimizer of the Eulerian problem), although we could not prove it directly since the lower semicontinuity of the H -cost with these hypotheses on H is not guaranteed, as noticed in [BW15]. In this paper, they prove in particular existence and equivalence between the Eulerian and Lagrangian problems in a more general case, dropping the last condition $\lim_{x \rightarrow 0} \frac{H(x)}{x} = +\infty$. In that case optimizers are not necessarily rectifiable and the Eulerian cost can be expressed as

$$\mathbf{M}^H(v) = \int_{\mathbb{R}^d} H(|\bar{v}(x)|) \, d\mathcal{H}^1(x) + H'(0)\|v_s\|,$$

where $v = \bar{v} \mathcal{H}^1 + v_s$ and v_s is the singular part w.r.t \mathcal{H}^1 .

A fractal optimal shape

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Looking at the optimal branching structures computed numerically in [OS11] (in some non-atomic cases), or at natural drainage networks and their irrigations basins, one is tempted to describe them as fractal (see [RR01]). Actually, even though the underlying network has infinitely many branching points, it is still a 1-rectifiable set, hence it is not clear in what sense fractality appears. Fractality is a notion which usually relates either to self-similarity properties of non-smooth objects, or to non-integer dimension of sets. A first rigorous result which would fall in the first category is proven by Brancolini and Solimini in [BS14]: for sufficiently diffuse measures (for example the Lebesgue measure restricted to a Lipschitz open set), the number of branches of length $\sim \varepsilon$ stemming from a branch of length l is of order l/ε . This may read as a self-similarity property since in a way the total length is preserved when looking at subbranches at all scales.

The present chapter leans towards the other notion of fractality, that is towards “fractal” dimension. Some sets in branched transport have already been proposed as candidates to exhibit non-integer dimension, for instance the boundary of adjacent irrigation basins (an open conjecture by J.-M. Morel). Here we are interested in another candidate which is related to branched transport: the boundary of what we call unit balls for branched transport. With the results of the present chapter, we can only prove an upper bound on the dimension, which is non-integer, and conjecture that this upper bound is actually sharp.

The chapter is divided into five parts. In a preliminary section we briefly recall the Lagrangian framework of branched transport set in Chapter 1, and we formulate our question as a shape optimization problem involving the irrigation distance. Section 3.2 is devoted to the proof of existence of minimizers and to elementary properties of minimizers. In Section 3.3 we prove the β -Hölder regularity of the landscape function, which appears in the description of optimizers, and use it to derive an upper bound on the Minkowski dimension of the boundary of the optimizers in Section 3.4. The final section is an attempt at computing optimizers numerically by adapting the Modica-Mortola approach introduced by [OS11], where we provide some computer visualizations.

3.1 Preliminaries

Let us make a very quick recall of the Lagrangian framework we have introduced in Chapter 1 and its main features.

3.1.1 The irrigation problem

We denote by $\Gamma(\mathbb{R}^d)$ (or Γ for short) the set of 1-Lipschitz curves in \mathbb{R}^d parameterized on $[0, \infty]$, endowed with the topology of uniform convergence on compact sets.

Irrigation plans We call *irrigation plan* any probability measure $\eta \in \text{Prob}(\Gamma)$ satisfying the following finite-length condition

$$\mathbf{L}(\eta) := \int_{\Gamma} L(\gamma) \, d\eta(\gamma) < +\infty, \quad (3.1.1)$$

where $L(\gamma) = \int_0^\infty |\dot{\gamma}(t)| dt$. Notice that any irrigation plan is concentrated on $\Gamma^1(\mathbb{R}^d) := \{\gamma : L(\gamma) < \infty\}$. We denote by $\mathbf{IP}(\mathbb{R}^d)$ the set of all irrigation plans $\eta \in \text{Prob}(\Gamma)$. If μ and ν are two probability measures on \mathbb{R}^d , one says that $\eta \in \mathbf{IP}(\mathbb{R}^d)$ irrigates ν from μ if one recovers the measures μ and ν by sending the mass of each curve respectively to its initial point and to its final point, which means that

$$(\pi_0)_\# \eta = \mu \text{ and } (\pi_\infty)_\# \eta = \nu,$$

where $\pi_0(\gamma) = \gamma(0)$, $\pi_\infty(\gamma) = \gamma(\infty) := \lim_{t \rightarrow +\infty} \gamma(t)$ and $f_\# \eta$ denotes the push-forward of η by f whenever f is a Borel map¹. We denote by $\mathbf{IP}(\mu, \nu)$ the set of irrigation plans irrigating ν from μ :

$$\mathbf{IP}(\mu, \nu) = \{\eta \in \mathbf{IP}(\mathbb{R}^d) : (\pi_0)_\# \eta = \mu, (\pi_\infty)_\# \eta = \nu\}.$$

If η is a given irrigation plan, we define the multiplicity at x , that is the total mass passing by x , as

$$\theta_\eta(x) = \eta(\{\gamma \in \Gamma : x \in \gamma\}),$$

where $x \in \gamma$ means that x belongs to the image of the curve γ . Finally, for any nonnegative function f , we denote by $\int_\gamma f(x) dx$ the line integral of f along $\gamma \in \Gamma$:

$$\int_\gamma f(x) dx := \int_0^{+\infty} f(\gamma(t)) |\dot{\gamma}(t)| dt.$$

Irrigation costs For $\alpha \in [0, 1]$ we consider the *irrigation cost* $\mathbf{I}_\alpha : \mathbf{IP}(\mathbb{R}^d) \rightarrow [0, \infty]$ defined by

$$\mathbf{I}_\alpha(\eta) := \int_\Gamma \int_\gamma \theta_\eta(x)^{\alpha-1} dx d\eta(\gamma),$$

with the conventions $0^{\alpha-1} = \infty$ if $\alpha < 1$, $0^{\alpha-1} = 1$ otherwise, and $\infty \times 0 = 0$. If μ, ν are two probability measures on \mathbb{R}^d , the irrigation (or branched transport) problem consists in minimizing the cost \mathbf{I}_α on the set of irrigation plans which send μ to ν , which reads

$$\min_{\eta \in \mathbf{IP}(\mu, \nu)} \int_\Gamma \int_\gamma \theta_\eta(x)^{\alpha-1} dx d\eta(\gamma). \quad (\text{LI}_\alpha)$$

We set $Z_\eta(\gamma) = \int_\gamma \theta_\eta(x)^{\alpha-1} dx$ so that the cost may expressed as

$$\mathbf{I}_\alpha(\eta) = \int_\Gamma Z_\eta(\gamma) d\eta(\gamma).$$

Proposition 3.1.1 (First variation inequality for \mathbf{I}_α). *If η is an irrigation plan with $\mathbf{I}_\alpha(\eta)$ finite, then for all irrigation plan $\tilde{\eta}$ the following holds:*

$$\mathbf{I}_\alpha(\tilde{\eta}) \leq \mathbf{I}_\alpha(\eta) + \alpha \int Z_\eta(\gamma) d(\tilde{\eta} - \eta). \quad (3.1.2)$$

Theorem 3.1.2 (Existence of minimizers.). *For any pair of probability measures $\mu, \nu \in \text{Prob}(\mathbb{R}^d)$ with compact support, the problem (LI_α) admits a minimizer.*

Theorem 3.1.3 (Irrigability). *If $1 - \frac{1}{d} < \alpha < 1$, for any $\mu, \nu \in \text{Prob}(\mathbb{R}^d)$ with compact support there exists some $\eta \in \mathbf{IP}(\mu, \nu)$ such that $\mathbf{I}_\alpha(\eta)$ is finite.*

From now on we assume that $\alpha \in]1 - \frac{1}{d}, 1[$.

¹Notice that $\lim_{t \rightarrow \infty} \gamma(t)$ exists if $\gamma \in \Gamma^1(K)$, and this is all we need since any irrigation plan is concentrated on $\Gamma^1(K)$.

Irrigation distance Let us set

$$d_\alpha(\mu, \nu) = \min\{\mathbf{I}_\alpha(\eta) : \eta \in \mathbf{IP}(\mu, \nu)\}$$

for any pair μ, ν of probability measures on \mathbb{R}^d . For any compact $K \subseteq \mathbb{R}^d$, it induces a distance on $\text{Prob}(K)$ which metrizes the weak- \star convergence of measures in the duality with $\mathcal{C}(K)$. On non-compact subsets of \mathbb{R}^d , the distance d_α is lower semicontinuous w.r.t. the weak- \star convergence of measures in the duality with bounded and continuous functions (narrow convergence)².

The following result comes from [BCM09; Xia03].

Proposition 3.1.4 (Scaling law). *For any compactly supported measures μ, ν with equal mass, there is an upper bound on the irrigation distance depending on the mass and the diameter. We set $\mu' = \mu - \mu \wedge \nu, \nu' = \nu - \mu \wedge \nu$ the disjoint parts of the measures and $m = |\mu'| = |\nu'|$ their common mass. Then:*

$$d_\alpha(\mu, \nu) \leq Cm^\alpha \text{diam}(\text{supp } \mu' \cup \text{supp } \nu').$$

Landscape function Given an optimal irrigation plan $\eta \in \mathbf{IP}(\delta_0, \nu)$, we say that a curve γ is η -good if

$$\int_0^{T(\gamma)} \theta_\eta(\gamma[0, t])^{\alpha-1} dt < \infty$$

which is equivalent to the two assertions:

- (i) $Z_\eta(\gamma) < \infty$,
- (ii) $\theta_\eta(\gamma(t)) = \theta_\eta(\gamma[0, t])$ for all $t < T(\gamma)$.

One may prove by optimality that η is concentrated on the set of η -good curves. Moreover it is proven in [San07] that for all η -good curves γ , the quantity $Z_\eta(\gamma)$ depends only on the final point $\gamma(\infty)$ of the curve, thus we may define the landscape function z_η as follows:

$$z_\eta(x) = \begin{cases} Z_\eta(\gamma) & \text{if } \gamma \text{ is an } \eta\text{-good curve s.t. } x = \gamma(\infty), \\ +\infty & \text{otherwise.} \end{cases}$$

Notice that for an optimal η the cost may be expressed in terms of z_η :

$$\mathbf{I}_\alpha(\eta) = \int_\Gamma Z_\eta(\gamma) d\eta(\gamma) = \int_{\mathbb{R}^d} z_\eta(x) d\nu(x).$$

Finally, one may show that z_η is lower semicontinuous and that the inequality $z_\eta(x) \geq |x|$ holds.

²Proving this is just an adaptation of the proof on compact sets. If μ is fixed (for example) and $\nu_n \rightarrow \nu$ with $\eta_n \in \mathbf{IP}(\mu, \nu_n)$ optimal and parameterized by arc length, assuming that the cost is bounded, the irrigation plans η_n are tight and one may extract a subsequence converging to some η which irrigates ν and whose cost is less than $\liminf d_\alpha(\mu, \nu_n)$ by lower semicontinuity of \mathbf{I}_α .

3.1.2 The shape optimization problem

We ask ourselves the following question: what is the set of unit volume which is closest to the origin in the sense of irrigation? To give this a precise meaning, we embed everything in the space of probability measures; hence we want to minimize the d_α distance between the unit Dirac mass at $0 \in \mathbb{R}^d$ and sets E of unit volume, seen as the uniform measure on E . This problem reads

$$\min \{d_\alpha(\delta_0, \mathbf{1}_E \mathcal{L}) : |E| = 1\}, \quad (\text{S}_\alpha)$$

where \mathcal{L} denotes the Lebesgue measure on \mathbb{R}^d . We relax this problem by minimizing on a larger set, which is the set of probability measures with Lebesgue density bounded by 1, thus getting:

$$\min \{\mathbf{X}_\alpha(\nu) : \nu \leq 1, \nu \in \text{Prob}(\mathbb{R}^d)\}, \quad (\text{R}_\alpha)$$

where $\mathbf{X}_\alpha(\nu) = d_\alpha(\delta_0, \nu)$.

In the following, we will sometimes encounter positive measures which do not have unit mass, thus we extend the functional by setting $\mathbf{X}_\alpha(\nu) := d_\alpha(|\nu|\delta_0, \nu)$ for any finite measure ν .

A key tool in the analysis of this problem lies in the following lemma.

Proposition 3.1.5 (First variation inequality for \mathbf{X}_α). *Suppose that $\nu \in \text{Prob}(\mathbb{R}^d)$ with $\mathbf{X}_\alpha(\nu) < \infty$. Suppose also that η is an optimal irrigation plan between $|\nu|\delta_0$ and ν , with landscape function z_η . The following holds:*

$$\mathbf{X}_\alpha(\tilde{\nu}) \leq \mathbf{X}_\alpha(\nu) + \alpha \int z_\eta d(\tilde{\nu} - \nu)$$

for any $\tilde{\nu} \in \text{Prob}(\mathbb{R}^d)$.

Notice also that the integral $\int z_\eta d(\tilde{\nu} - \nu)$ is well-defined since $\int z_\eta d\nu = \mathbf{I}_\alpha(\eta) = \mathbf{X}_\alpha(\nu) < \infty$ and z_η is non-negative, though it may be infinite.

Proof. If $\int z_\eta d\tilde{\nu} = \infty$ then there is nothing to prove. Otherwise for ν -a.e. x , $z_\eta(x)$ is finite hence there are η -good curves reaching x and one can find a measurable³ map $g : \mathbb{R}^d \rightarrow \Gamma$ which associates with every x an η -good curve reaching x . Let us build an irrigation plan $\tilde{\eta} \in \mathbf{IP}(|\tilde{\nu}|\delta_0, \tilde{\nu})$ which is concentrated on η -good curves, by setting $\tilde{\eta} = g\#\nu$, so that

$$\int_\Gamma Z_\eta d\tilde{\eta} = \int_\Gamma z_\eta(\gamma(\infty)) d\tilde{\eta}(\gamma) = \int_{\mathbb{R}^d} z_\eta(x) d\tilde{\nu}.$$

Then, by the first variation inequality for \mathbf{I}_α , we get:

$$\mathbf{X}_\alpha(\tilde{\nu}) \doteq d_\alpha(|\tilde{\nu}|\delta_0, \tilde{\nu}) \leq \mathbf{I}_\alpha(\tilde{\eta}) \leq \mathbf{I}_\alpha(\eta) + \alpha \int_\Gamma Z_\eta d(\tilde{\eta} - \eta) = \mathbf{X}_\alpha(\nu) + \alpha \int_{\mathbb{R}^d} z_\eta d(\tilde{\nu} - \nu). \quad \square$$

³One can characterize η -good curves as those γ such that $\tilde{Z}_\eta(\gamma) < \infty$ where $\tilde{Z}_\eta(\gamma) := \int_0^\infty |\dot{\gamma}|_{t,\eta} dt$ is a slight variation of Z_η defined in [Sam07] which is also lower semicontinuous. Hence the multifunction associating to every x the set of η -good curves reaching x can be written as $\bigcup_{\ell \in \mathbb{Q}} \{\gamma \in \Gamma : \tilde{Z}_\eta(\gamma) \leq \ell, \gamma(\infty) = x\}$, i.e. as a countable union of multifunctions with closed graph. This means that this multifunction is measurable and admits a measurable selection (see e.g. [CV77]).

3.2 Existence and first properties

We will often denote by $C = C(\alpha, d)$ or $c = c(d)$ different positive constants which depend only on α, d or d respectively.

3.2.1 Existence of minimizers

Theorem 3.2.1. *The relaxed shape optimization problem (\mathbf{R}_α) admits at least a minimizer.*

Proof. The existence of a minimizer follows from the lower semicontinuity and tightness. Indeed, any minimizing sequence ν_n must have bounded first moment since

$$\int |x| d\nu(x) \leq \int z_\eta(x) d\nu(x) = d_\alpha(\delta_0, \nu).$$

A bound on the first moment implies tightness of the sequence and, up to extracting a subsequence, one has $\nu_n \rightharpoonup \nu$. The condition $\nu_n \leq 1$ implies $\nu \leq 1$ and the lower semicontinuity of d_α provides the optimality of ν . \square

For $1 > \alpha > 1 - \frac{1}{d}$, we will denote the optimal value for the relaxed shape optimization problem (\mathbf{R}_α) by:

$$e_\alpha := \min\{d_\alpha(\delta_0, \nu) : \nu \leq 1 \text{ and } \nu \in \text{Prob}(\mathbb{R}^d)\}.$$

Lemma 3.2.2 (Scaling lemma). *For any finite measure ν we have*

$$\mathbf{X}_\alpha(\nu) \geq e_\alpha |\nu|^{\alpha + \frac{1}{d}}.$$

Proof. For $\lambda = |\nu|^{-1/d}$, let $\tilde{\nu} = \lambda^d \varphi_\#(\nu)$ be a scaling of ν under the map $\varphi(x) = \lambda x$ in \mathbb{R}^d . Then, $\int_{\mathbb{R}^d} d\tilde{\nu} = \lambda^d \int_{\mathbb{R}^d} d\nu = \lambda^d |\nu| = 1$ and $\tilde{\nu} \leq 1$. Thus,

$$e_\alpha \leq d_\alpha(\tilde{\nu}, \delta_0) = \lambda^{\alpha d + 1} d_\alpha(\nu, |\nu| \delta_0) = |\nu|^{-(\alpha + \frac{1}{d})} \mathbf{X}_\alpha(\nu). \quad \square$$

For any ν , we say that z is a landscape function of ν if it is the landscape function z_η associated with some optimal irrigation plan $\eta \in \mathbf{IP}(\delta_0, \nu)$.

Theorem 3.2.3. *Let ν be a minimizer of (\mathbf{R}_α) and z a landscape function of ν . Then ν is the indicator of a set A which is a sublevel set of z :*

$$A = \{x : z(x) \leq z^*\}, \text{ with } z^* = \frac{e_\alpha}{\alpha} \left(\alpha + \frac{1}{d}\right). \quad (3.2.1)$$

In particular, A is a solution to problem (\mathbf{S}_α) and it is a compact and path-connected set.

Proof. We show that ν also minimizes the first variation of \mathbf{X}_α , that is $\mu \mapsto \int z d\mu$. Take $\tilde{\nu}$ a competitor for (\mathbf{R}_α) . By Proposition 3.1.5, one has:

$$\mathbf{X}_\alpha(\tilde{\nu}) \leq \mathbf{X}_\alpha(\nu) + \alpha \int z d(\tilde{\nu} - \nu),$$

but $\mathbf{X}_\alpha(\nu) \leq \mathbf{X}_\alpha(\tilde{\nu})$, thus

$$\int z \, d\nu \leq \int z \, d\tilde{\nu}$$

for any $\tilde{\nu}$. So as to minimize this quantity, ν must concentrate its mass on the points where z takes its lowest values. More precisely, there is a value $z^* \in [0, \infty]$ such that

$$\nu(x) \begin{cases} = 1 & \text{if } z(x) < z^*, \\ \in [0, 1] & \text{if } z(x) = z^*, \\ = 0 & \text{if } z(x) > z^*. \end{cases}$$

Indeed, we just take $z^* = \sup\{t \in \mathbb{R} : |\{z(x) \leq t\}| < 1\}$. Since $\int z \, d\nu = e_\alpha > 0$, necessarily $z^* > 0$. This kind of arguments is typical in optimization problems under an upper density constraint, as it was for instance done for crowd motion applications in [MRS14].

Step 1: $z^* \leq \frac{e_\alpha}{\alpha} \left(\alpha + \frac{1}{d}\right)$

For $0 \leq k < z^*$, we consider the competitor $\tilde{\nu} = \mathbf{1}_{\{z \leq k\}}$ and set $|\tilde{\nu}| = 1 - m$, noting that $m > 0$ by definition of z^* . Using Lemma 3.2.2 and Proposition 3.1.5, one gets

$$e_\alpha(1 - m)^{\alpha + \frac{1}{d}} \leq \mathbf{X}_\alpha(\tilde{\nu}) \leq \mathbf{X}_\alpha(\nu) + \alpha \int z \, d(\tilde{\nu} - \nu) = e_\alpha - \alpha \int_{\{z > k\}} z \, d\nu.$$

Since $\nu(\{z > k\}) = 1 - |\{z \leq k\}| = m$, it follows that

$$e_\alpha(1 - m)^{\alpha + \frac{1}{d}} \leq e_\alpha - \alpha k m. \quad (3.2.2)$$

As $\alpha + \frac{1}{d} > 1$, the map $t \mapsto t^{\alpha + \frac{1}{d}}$ is (strictly) convex, thus

$$e_\alpha \left(1 - \left(\alpha + \frac{1}{d}\right) m\right) \leq e_\alpha(1 - m)^{\alpha + \frac{1}{d}} \leq e_\alpha - \alpha k m,$$

hence forgetting the middle term, subtracting e_α and dividing by m :

$$\alpha k \leq e_\alpha \left(\alpha + \frac{1}{d}\right).$$

Taking the limit $k \rightarrow z^*$ yields:

$$z^* \leq \frac{e_\alpha}{\alpha} \left(\alpha + \frac{1}{d}\right). \quad (3.2.3)$$

Step 2: $\nu = \mathbf{1}_A$ where $A = \{z \leq z^*\}$

Take the competitor $\tilde{\nu} = \mathbf{1}_{\{z \leq z^*\}}$ and set $|\tilde{\nu}| = 1 + m$, $m \geq 0$. Using again the scaling lemma and the first variation of \mathbf{X}_α one gets:

$$e_\alpha(1 + m)^{\alpha + \frac{1}{d}} \leq e_\alpha + \alpha \int_{z=z^*} z \, d(\tilde{\nu} - \nu) = e_\alpha + \alpha z^* m.$$

Now by strict convexity of $t \mapsto t^{\alpha + \frac{1}{d}}$, if $m > 0$ then one has $e_\alpha(1 + m)^{\alpha + \frac{1}{d}} > e_\alpha \left(1 + \left(\alpha + \frac{1}{d}\right) m\right)$, thus

$$e_\alpha \left(\alpha + \frac{1}{d}\right) m < \alpha z^* m,$$

which contradicts (3.2.3). Consequently $m = 0$, hence $\nu = \tilde{\nu} = \mathbf{1}_{\{z \leq z^*\}}$.

Step 3: Compactness and connectedness

A is closed since z is lower semicontinuous and bounded since $z(x) \geq |x|$ for all $x \in \mathbb{R}^d$. It is path-connected since any point x with $z(x) \leq z^*$ is the endpoint of an η -good curve γ starting from 0 and $\gamma \subseteq A$ because z is increasing along this curve.

Step 4: $z^* \geq \frac{e_\alpha}{\alpha} \left(\alpha + \frac{1}{d} \right)$

Take $x_0 \in A$ with maximal Euclidean norm. Then the half ball $H_r(x_0) := B_r(x_0) \cap \{x : \langle x - x_0, x_0 \rangle > 0\}$ is included in A^c . We consider the competitor $\tilde{\nu} = \mathbf{1}_{A \sqcup H_r(x_0)}$, with mass $|\tilde{\nu}| = 1 + m$, where $m = |H_r(x_0)| = cr^d$ for some constant $c = c(d)$. To irrigate $\tilde{\nu}$, we pay at most the cost of irrigation of ν , plus the price for moving an extra mass m from 0 to x_0 along the irrigation plan, plus the cost for moving this mass to $B_r(x_0) \setminus A$, which we can bound by $Cm^\alpha r$, as follows:

$$\begin{aligned} \mathbf{X}_\alpha(\tilde{\nu}) &= d_\alpha((1+m)\delta_0, \tilde{\nu}) \leq d_\alpha((1+m)\delta_0, \nu + m\delta_{x_0}) + d_\alpha(\nu + m\delta_{x_0}, \nu + \mathbf{1}_{H_r(x_0)}) \\ &= \mathbf{X}_\alpha(\nu + m\delta_{x_0}) + d_\alpha(m\delta_{x_0}, \mathbf{1}_{H_r(x_0)}) \\ &\leq e_\alpha + \alpha m z(x_0) + Cr^{1+d\alpha}, \end{aligned}$$

where $C = C(\alpha, d)$ is some positive constant. Since x_0 is not a Lebesgue point of A and $x_0 \in A$, by Lemma 3.2.4 below, it follows that $z(x_0) = z^*$. Putting this in the previous inequality, combining it with the convexity inequality

$$\mathbf{X}_\alpha(\tilde{\nu}) \geq e_\alpha(1+m)^{\alpha+\frac{1}{d}} \geq e_\alpha \left(1 + \left(\alpha + \frac{1}{d} \right) m \right),$$

and dividing by $m > 0$, one gets:

$$e_\alpha \left(\alpha + \frac{1}{d} \right) \leq \alpha z^* + Cr^{1+d\alpha-d}.$$

Passing to the limit $r \rightarrow 0$, we obtain

$$z^* \geq \frac{e_\alpha}{\alpha} \left(\alpha + \frac{1}{d} \right). \quad \square$$

3.2.2 Lebesgue points

For any set K , let us set $\Theta_K(x, r) := \frac{|K \cap B(x, r)|}{|B_r(x)|}$ the fraction of mass of $B_r(x)$ lying in K . We also set

$$\beta = d \left(\alpha - \left(1 - \frac{1}{d} \right) \right) = 1 + d\alpha - d,$$

a number which is strictly between 0 and 1 as $1 > \alpha > 1 - \frac{1}{d}$, which will appear as in the first chapter as a Hölder exponent of the landscape function.

Lemma 3.2.4. *If $z(x) < z^*$, then x is a Lebesgue point of A .*

Proof. Consider the competitor $\tilde{\nu} = \mathbf{1}_{A \cup B_r(x)}$ with mass $|\tilde{\nu}| = 1 + m$ where $m = |B_r(x) \setminus A|$. To irrigate $\tilde{\nu}$ from 0 one may irrigate $\nu + m\delta_x$ from 0, then $\mathbf{1}_{B_r(x) \setminus A}$ from x . The first cost may be estimated by the first variation of \mathbf{X}_α , and the second one

may be bounded knowing that irrigating a mass m at distance r costs less than $Cm^\alpha r$ for some constant $C = C(\alpha, d)$. This writes rigorously as:

$$\begin{aligned} \mathbf{X}_\alpha(\tilde{\nu}) &= d_\alpha((1+m)\delta_0, \tilde{\nu}) \leq d_\alpha((1+m)\delta_0, \nu + m\delta_x) + d_\alpha(\nu + m\delta_x, \nu + \mathbf{1}_{B_r(x)\setminus A}) \\ &= \mathbf{X}_\alpha(\nu + m\delta_x) + d_\alpha(m\delta_x, \mathbf{1}_{B_r(x)\setminus A}) \\ &\leq e_\alpha + \alpha m z(x) + Cm^\alpha r, \end{aligned}$$

On the other hand, by the scaling lemma and by convexity, one has $\mathbf{X}_\alpha(\tilde{\nu}) \geq e_\alpha (1 + (\alpha + \frac{1}{d})m) \geq e_\alpha + \alpha m z^*$, the last inequality resulting from Theorem 3.2.3. Combining this with the previous series of inequalities yields:

$$m^{1-\alpha}(z^* - z(x)) \leq Cr,$$

which rewrites

$$\Theta_{A^c}^{1-\alpha}(x, r)(z^* - z(x)) \leq Cr^\beta, \quad (3.2.4)$$

where we recall $\beta = 1 + d\alpha - d$ is a number strictly between 0 and 1. As a consequence, if $z(x) < z^*$ then $\Theta_{A^c}(x, r) \xrightarrow{r \rightarrow 0} 0$ and x is a Lebesgue point of A . \square

3.3 Hölder continuity of the landscape function

We set $A_r(x) := A \cap B_r(x)$, $\bar{z}_r(x)$ the central median of z on the set $A_r(x)$ and $z_r(x)$ its mean. We are going to show that z is β -Hölder continuous using Campanato estimates (Campanato spaces were introduced in [Cam63], see [Giu03, Section 2.3] for a modern exposition), as it is done in [San07]. More precisely, we are going to prove the following inequality, for arbitrary $r > 0$:

$$\int_{A_r(x)} |z - z_r(x)| \leq Cr^\beta, \quad (3.3.1)$$

$$|z_r(x) - z_{r/2}(x)| \leq Cr^\beta, \quad (3.3.2)$$

$$z_r(x) - z(x) \leq Cr^\beta, \quad (3.3.3)$$

$$|z(x) - z_r(x)| \leq Cr^\beta, \quad (3.3.4)$$

$$|z_{|y-x|}(x) - z_{|y-x|}(y)| \leq C|y - x|^\beta. \quad (3.3.5)$$

Notice that the two last inequalities imply that z is indeed β -Hölder continuous:

$$\begin{aligned} |z(y) - z(x)| &\leq |z(y) - z_{|y-x|}(y)| + |z_{|y-x|}(x) - z_{|y-x|}(y)| + |z(x) - z_{|y-x|}(x)| \\ &\leq C|y - x|^\beta. \end{aligned}$$

The main difficulty we will encounter is that we will quite easily obtain estimates of the form

$$\dots \leq C \frac{r^\beta}{\Theta_A(x, r)^{1-\alpha}},$$

and will need to get rid of the term $\Theta_A(x, r)^{1-\alpha}$, i.e. treat the case when it becomes small.

3.3.1 Main lemmas

We will make use of the following lemmas.

Lemma 3.3.1 (Maximum deviation). *There is a constant $C = C(d, \alpha) > 0$ such that the following holds:*

$$\forall y \in A_r(x), \quad z^* - z(y) \leq C \frac{r^\beta}{\Theta_{A^c}(x, r)^{1-\alpha}}. \quad (3.3.6)$$

Remark 3.3.2. One can see that if $\Theta_A(x, r)$ becomes small, then $\Theta_{A^c}(x, r)$ is large (close to 1), and actually all values of z in $A_r(x)$ become close to the same value z^* up to Cr^β .

Proof. We consider the competitor $\tilde{\nu} = \mathbf{1}_{A \cup B_r(x)}$, with mass $|\tilde{\nu}| = 1 + m$ where $m = |B_r(x) \setminus A|$. For any $y \in A_r(x)$, let us irrigate $\tilde{\nu}$ from 0 by irrigating ν from 0, moving an extra mass m from 0 to y along the irrigation plan, then irrigating $\mathbf{1}_{B_r(x) \setminus A}$ from this mass at y . Using Lemma 3.2.2, we have

$$e_\alpha(1+m)^{\alpha+\frac{1}{d}} \leq \mathbf{X}_\alpha(\tilde{\nu}) \leq \mathbf{X}_\alpha(\nu) + \alpha \int z \, d(m\delta_y) + Crm^\alpha.$$

By convexity,

$$e_\alpha \left(1 + \left(\alpha + \frac{1}{d} \right) m \right) \leq (1+m)^{\alpha+\frac{1}{d}} e_\alpha \leq e_\alpha + \alpha m z(y) + Crm^\alpha,$$

thus, knowing that $e_\alpha \left(\alpha + \frac{1}{d} \right) = \alpha z^*$ by (3.2.1):

$$\alpha m z^* \leq \alpha m z(y) + Crm^\alpha.$$

By definition, $m = \omega_d r^d \Theta_{A^c}(x, r)$ where ω_d is the volume on the unit d -dimensional ball, hence

$$z^* - z(y) \leq Cr (r^d \Theta_{A^c}(x, r))^{\alpha-1} = C \frac{r^\beta}{\Theta_{A^c}(x, r)^{1-\alpha}}. \quad \square$$

Lemma 3.3.3 (Mean deviation). *There is some constant $C = C(d, \alpha) > 0$ such that*

$$\int_{A_r(x)} |z(y) - z_r(x)| \, dy \leq Cr^\beta$$

for all $r > 0$ and all $x \in A$.

Proof. We will first show that

$$\int_{A_r(x)} |z(y) - z_r(x)| \, dy \leq C \frac{r^\beta}{\Theta_A(x, r)^{1-\alpha}}.$$

There is a disjoint union $A_r(x) = A^- \sqcup A^+$ such that $|A^-| = |A^+| = \frac{|A_r(x)|}{2}$ and $z \leq \bar{z}_r(x)$ on A^- , $z \geq \bar{z}_r(x)$ on A^+ . Let us consider the competitor $\tilde{\nu} = \mathbf{1}_A - \mathbf{1}_{A^+} + \mathbf{1}_{A^-}$. By the first variation lemma:

$$\mathbf{X}_\alpha(\tilde{\nu}) \leq \mathbf{X}_\alpha(\nu) + \alpha \int z \, d(\tilde{\nu} - \nu).$$

Recall that $\mathbf{X}_\alpha(\rho) = d_\alpha(\delta_0, \rho)$ when ρ is a probability measure, which is the case for ν and $\tilde{\nu}$, and that d_α is a distance. Thus by the triangle inequality:

$$\alpha \int z \, d(\nu - \tilde{\nu}) \leq d_\alpha(\delta_0, \nu) - d_\alpha(\delta_0, \tilde{\nu}) \leq d_\alpha(\nu, \tilde{\nu}).$$

We know that $d_\alpha(\nu, \tilde{\nu}) \leq C|\tilde{\nu} - \nu|^\alpha \text{diam}(\text{supp}(\tilde{\nu} - \nu)) \leq C|A_r(x)|^\alpha r$ for some $C = C(\alpha, d) > 0$. Moreover notice that

$$\begin{aligned} \int z \, d(\nu - \tilde{\nu}) &= \int_{A^+} z(y) \, dy - \int_{A^-} z(y) \, dy \\ &= \int_{A^+} (z(y) - \bar{z}_r(x)) \, dy + \int_{A^-} (\bar{z}_r(x) - z(y)) \, dy \\ &= \int_{A_r(x)} |z(y) - \bar{z}_r(x)| \, dy. \end{aligned}$$

Consequently:

$$\int_{A_r(x)} |z(y) - \bar{z}_r(x)| \, dy \leq C|A_r(x)|^{\alpha-1} r \leq C \frac{|A_r(x)|^{\alpha-1}}{|B_r(x)|^{\alpha-1}} r^{1+d(\alpha-1)} = C \frac{r^\beta}{\Theta_A(x, r)^{1-\alpha}}.$$

Moreover, one has

$$|z_r(x) - \bar{z}_r(x)| = \left| \int_{A_r(x)} z(y) - \bar{z}_r(x) \, dy \right| \leq \int_{A_r(x)} |z(y) - \bar{z}_r(x)| \, dy \leq C \frac{r^\beta}{\Theta_A(x, r)^{1-\alpha}}$$

which leads to

$$\int_{A_r(x)} |z(y) - z_r(x)| \, dy = \int_{A_r(x)} |z(y) - \bar{z}_r(x)| \, dy + |z_r(x) - \bar{z}_r(x)| \leq C \frac{r^\beta}{\Theta_A(x, r)^{1-\alpha}}.$$

Now we get rid of $\Theta_A(x, r)^{1-\alpha}$. If $\Theta_A(x, r) \geq 1/2$, we get the desired inequality. On the other hand, if $\Theta_{A^c}(x, r) \geq 1/2$, by Lemma 3.3.1, we have

$$0 \leq z^* - z(y) \leq Cr^\beta, \quad \forall y \in A_r(x),$$

which also implies that

$$0 \leq z^* - z_r(x) \leq Cr^\beta.$$

By these two inequalities, we have

$$|z(y) - z_r(x)| \leq Cr^\beta, \quad \forall y \in A_r(x).$$

Now, taking the mean over $A_r(x) \ni y$ leads to the wanted inequality as well:

$$\int_{A_r(x)} |z(y) - z_r(x)| \, dy \leq Cr^\beta. \quad \square$$

Remark 3.3.4. Notice that the estimate

$$\int_{A_r(x)} |z(y) - z_r(x)| \, dy \leq C \frac{r^\beta}{\Theta_A(x, r)^{1-\alpha}}$$

is valid in general: we only use the fact that ν is an indicator function (a density bounded from below would suffice). The optimality of ν comes into play to get rid of $\Theta_A(x, r)$.

3.3.2 Hölder regularity

Proposition 3.3.5 (Small-scale difference). *For all $x \in A$ and all $r > 0$ one has*

$$|z_r(x) - z_{r/2}(x)| \leq Cr^\beta.$$

Proof. First we show that

$$|z_r(x) - z_{r/2}(x)| \leq C \frac{r^\beta}{\Theta_A(x, r)^{1-\alpha}}.$$

Indeed, by Lemma 3.3.3,

$$\begin{aligned} |z_r(x) - z_{r/2}(x)| &\leq \frac{\int_{A_{r/2}(x)} |z(y) - z_r(x)| \, dy}{|A_{r/2}(x)|} \leq \frac{|A_r(x)|}{|A_{r/2}(x)|} \int_{A_r(x)} |z(y) - z_r(x)| \, dy \\ &\leq 2^d \frac{\Theta_A(x, r)}{\Theta_A(x, r/2)} Cr^\beta \leq C \frac{r^\beta}{\Theta_A(x, r/2)}. \end{aligned}$$

As before, if $\Theta_A(x, r/2) \geq 1/2$ we get the desired estimate. Otherwise, we have $\Theta_{A^c}(x, r/2) \geq 1/2$ and $\Theta_{A^c}(x, r) \geq 2^{-d}\Theta_{A^c}(x, r/2) \geq 2^{-1-d}$. Now, by Lemma 3.3.1,

$$0 \leq z^* - z(y) \leq C \frac{r^\beta}{\Theta_{A^c}(x, r)} \leq Cr^\beta, \quad \forall y \in A_r(x).$$

Consequently $0 \leq z^* - z_{r/2}(x) \leq Cr^\beta$ and $0 \leq z^* - z_r(x) \leq Cr^\beta$ which implies that

$$|z_r(x) - z_{r/2}(x)| \leq Cr^\beta. \quad \square$$

Lemma 3.3.6 (Lower deviation to the mean). *There is a constant $C = C(d, \alpha) > 0$ such that for all $x \in A$ and all $r > 0$ one has:*

$$\forall y \in A_r(x), \quad z_r(x) - z(y) \leq Cr^\beta. \quad (3.3.7)$$

Proof. First we show that

$$z_r(x) - z(y) \leq C \frac{r^\beta}{\Theta_A(x, r)^{1-\alpha}}.$$

Remove the mass $m = |A_r(x)|$ going to $A_r(x)$ from the irrigation plan, make it travel along the plan to any fixed $y \in A_r(x)$ and then send it to $A_r(x)$: this should cost more. This implies

$$\alpha m z(y) - \alpha \int_{A_r(x)} z + Cm^\alpha r \geq 0,$$

which may be rewritten as

$$z_r(x) - z(y) \leq Cm^{\alpha-1} r \leq C \frac{r^\beta}{\Theta_A(x, r)^{1-\alpha}}.$$

Now if $\Theta_A(x, r) \geq 1/2$ one gets the desired result. Otherwise $\Theta_{A^c}(x, r) \geq 1/2$ and Lemma 3.3.1 yields:

$$0 \leq z^* - z(y) \leq Cr^\beta, \quad \forall y \in A_r(x).$$

Thus $0 \leq z^* - z_r(x) \leq Cr^\beta$ and for any fixed $y \in A_r(x)$,

$$|z_r(x) - z(y)| \leq |z_r(x) - z^*| + |z^* - z(y)| \leq Cr^\beta,$$

from which we also get the wanted inequality. \square

Lemma 3.3.7 (Deviation to the mean). *For all $x \in A$ and all $r > 0$, one has*

$$|z(x) - z_r(x)| \leq Cr^\beta.$$

Proof. By Proposition 3.3.5, one has

$$|z(x) - z_r(x)| \leq |z(x) - z_{r/2}(x)| + |z_{r/2}(x) - z_r(x)| \leq |z(x) - z_{r/2}(x)| + Cr^\beta,$$

which means by setting $f(r) = |z(x) - z_r(x)|$ for $r > 0$ that:

$$f(r) \leq f(r/2) + Cr^\beta.$$

Consequently for all $k \in \mathbb{N}$

$$f(r) \leq f(r \cdot 2^{-(k+1)}) + Cr^\beta \sum_{i=0}^k 2^{-i\beta}$$

thus

$$f(r) \leq \limsup_{\varepsilon \rightarrow 0} f(\varepsilon) + Cr^\beta \sum_{i=0}^{\infty} 2^{-i\beta} \leq \limsup_{\varepsilon \rightarrow 0} f(\varepsilon) + Cr^\beta.$$

Now let us prove that $f(\varepsilon) \rightarrow 0$ when $\varepsilon \rightarrow 0$, i.e. $z_\varepsilon(x) \xrightarrow{\varepsilon \rightarrow 0} z(x)$. We already know that z is lower semi-continuous hence $z(x) \leq \liminf_{\varepsilon \rightarrow 0} z_\varepsilon(x)$. Moreover using (3.3.7), we have

$$\limsup_{\varepsilon \rightarrow 0} z_\varepsilon(x) \leq \limsup_{\varepsilon \rightarrow 0} (z(x) + C\varepsilon^\beta) = z(x),$$

which implies that $z_\varepsilon(x) \rightarrow z(x)$ when $\varepsilon \rightarrow 0$. Therefore the inequality $f(r) \leq Cr^\beta$ holds, that is to say:

$$|z(x) - z_r(x)| \leq Cr^\beta. \quad \square$$

Lemma 3.3.8 (Large scale difference). *For any $x, y \in A$, one has:*

$$|z_{|y-x|}(x) - z_{|y-x|}(y)| \leq C|y-x|^\beta.$$

Proof. Set $r = |y-x|$, and $\Delta_r = B_r(x) \cap B_r(y)$. Notice that, Δ_r being a fixed fraction of $B_r(x)$ (independent of r), $|\Delta_r| = c|B_r|$ for some $c = c(d) \in]0, 1[$.

If both $\Theta_{A^c}(x, r) \geq \frac{c}{2}$ and $\Theta_{A^c}(y, r) \geq \frac{c}{2}$, then by Lemma 3.3.1 one has:

$$0 \leq z^* - z_r(x) \leq C \frac{r^\beta}{\Theta_{A^c}(x, r)^{1-\alpha}} \leq Cr^\beta, \text{ and } 0 \leq z^* - z_r(y) \leq C \frac{r^\beta}{\Theta_{A^c}(y, r)^{1-\alpha}} \leq Cr^\beta,$$

which implies the desired inequality

$$|z_r(x) - z_r(y)| \leq Cr^\beta. \quad (3.3.8)$$

3.3. HÖLDER CONTINUITY OF THE LANDSCAPE FUNCTION

On the other hand, if either $\Theta_{A^c}(x, r)$ or $\Theta_{A^c}(y, r)$ is less than $c/2$, say $\Theta_{A^c}(x, r) \leq \frac{c}{2}$, we claim the desired inequality (3.3.8) still holds. Indeed, for all $u \in A_r(x) \cap A_r(y)$ one has

$$|z_r(x) - z_r(y)| \leq |z_r(x) - z(u)| + |z_r(y) - z(u)|$$

thus integrating over $A_r(x) \cap A_r(y)$ in u one gets:

$$\begin{aligned} |z_r(x) - z_r(y)| &\leq \frac{1}{|A_r(x) \cap A_r(y)|} \left[\int_{A_r(x)} |z(u) - z_r(x)| \, du + \int_{|A_r(y)|} |z(u) - z_r(y)| \, du \right] \\ &\leq Cr^\beta \frac{|A_r(x)| + |A_r(y)|}{|A_r(x) \cap A_r(y)|}, \end{aligned}$$

the last inequality resulting from Lemma 3.3.3. Note that

$$|A_r(x) \cap A_r(y)| = |\Delta_r \cap A| \geq |\Delta_r| - |B_r(x) \setminus A| = c|B_r(x)| - |B_r(x) \setminus A|$$

which implies that

$$\frac{|A_r(x)| + |A_r(y)|}{|A_r(x) \cap A_r(y)|} \leq \frac{2|B_r(x)|}{c|B_r(x)| - |B_r(x) \setminus A|} = \frac{2}{c - \Theta_{A^c}(x, r)} \leq \frac{4}{c}.$$

Thus, in this case, we still have

$$|z_r(x) - z_r(y)| \leq Cr^\beta \frac{|A_r(x)| + |A_r(y)|}{|A_r(x) \cap A_r(y)|} \leq Cr^\beta. \quad \square$$

Theorem 3.3.9 (Hölder continuity). *The function z is β -Hölder continuous on A . More precisely:*

$$\forall x, y \in A, \quad |z(y) - z(x)| \leq C|y - x|^\beta,$$

for some constant $C = C(\alpha, d)$.

Proof. By Lemma 3.3.7 and Lemma 3.3.8,

$$|z(y) - z(x)| \leq |z(y) - z_{|y-x|}(y)| + |z_{|y-x|}(y) - z_{|y-x|}(x)| + |z(y) - z_{|y-x|}(y)| \leq 3C|y - x|^\beta. \quad \square$$

Let us state a consequence of this result which is a refinement of the previous proposition on Lebesgue points. Now we may quantify the minimal size of a ball one can put inside A around x in terms of $z^* - z(x)$.

Proposition 3.3.10 (Interior points). *For some constant $C = C(\alpha, d)$ the following holds:*

$$\forall x \in A, \quad B_{r(x)}(x) \subseteq A, \quad (3.3.9)$$

where $r(x) = C(z^* - z(x))^{1/\beta} \geq 0$. In particular

$$\{x \in A : z(x) < z^*\} \subseteq \overset{\circ}{A} \text{ and } \partial A \subseteq \{x \in A : z(x) = z^*\}.$$

Proof. It suffices to prove (3.3.9) for $x_0 \in A$ satisfying $z(x_0) < z^*$. Consider a point $x \in A^c$. Take a point $y \in A$ which is closest to x : it is possible since A is compact. By construction y is not a Lebesgue point of A , thus by Lemma 3.2.4, $z(y) = z^*$. By the Hölder continuity of z stated in Theorem 3.3.9,

$$z^* - z(x_0) = |z(y) - z(x_0)| \leq C|y - x_0|^\beta \leq C|x - x_0|^\beta,$$

where the last inequality follows from the fact that $|y - x_0| \leq |y - x| + |x - x_0| \leq 2|x - x_0|$ because y minimizes the distance from x . Hence, for all $x \in A^c$, $|x - x_0| \geq C(z^* - z(x_0))^{1/\beta} = r(x_0)$, which implies the desired result. \square

3.4 On the dimension of the boundary

We are interested in the dimension of the boundary ∂A , our guess being that it should be non-integer, and lie between $d-1$ and d . Here we look at the Minkowski dimension (also called box-counting dimension). Given a set X , we denote by $N_\varepsilon(X)$ the maximum amount of disjoint balls of radius ε centered at points of X .

Definition 3.4.1 (Minkowski dimension). We define the upper Minkowski dimension of X by

$$\overline{\dim}_M(X) = \limsup_{\varepsilon \rightarrow 0} \frac{\log(N_\varepsilon(X))}{-\log \varepsilon},$$

and the lower Minkowski dimension by

$$\underline{\dim}_M(X) = \liminf_{\varepsilon \rightarrow 0} \frac{\log(N_\varepsilon(X))}{-\log \varepsilon}.$$

When these coincide we just call it the Minkowski dimension and denote it by $\dim_M(X)$.

We shall get an upper bound on the upper Minkowski dimension. We say that X is of dimension smaller than δ if $\overline{\dim}_M X \leq \delta$.

Lemma 3.4.2. *There is a constant $C = C(\alpha, d)$ such that for all $k \leq z^*$,*

$$|\{x \in A : k < z(x) \leq z^*\}| \leq C(z^* - k).$$

Proof. Consider the competitor $\tilde{\nu} = \mathbf{1}_{\{z \leq k\}}$ with total mass $|\tilde{\nu}| = 1 - m$, where $m = |\{x \in A : k < z(x) \leq z^*\}|$. As in (3.2.2), one has

$$e_\alpha(1 - m)^{\alpha+1/d} \leq e_\alpha - \alpha km$$

hence knowing that $\alpha z^* = (\alpha + 1/d)e_\alpha$ and developing the term on the left-hand side at order 2, we obtain:

$$-\alpha m z^* + \frac{e_\alpha}{2} \left(\alpha + \frac{1}{d}\right) \left(\alpha + \frac{1}{d} - 1\right) m^2 \leq -\alpha km$$

Thus

$$m \leq C(z^* - k)$$

with $1/C = e_\alpha(\alpha + 1/d)(\alpha + 1/d - 1)/(2\alpha)$. \square

Theorem 3.4.3. *The set ∂A is of dimension less than $d - \beta$.*

Proof. For $\varepsilon > 0$ fixed, take disjoint balls $(B_i)_{i \in I}$ of radius ε , where $N := |I| = N_\varepsilon(\partial A)$. We set $B_i^+ = B_i \setminus A$, $B_i^- = B_i \cap A$. We split the set of balls into two parts: those which have a larger intersection with A rather than A^c , and vice-versa. Namely, we set

$$\begin{aligned} I^+ &= \{i \in I : |B_i^+| \geq |B_i|/2\}, & N^+ &= |I^+|, \\ I^- &= \{i \in I : |B_i^-| \geq |B_i|/2\}, & N^- &= |I^-|, \end{aligned}$$

so that $I = I^+ \cup I^-$ and $N \leq N^+ + N^-$. We are going to bound N^+ and N^- by some power of ε .

Step 1: Bound on N^-

Since z is β -Hölder continuous on A , one has for each $B_i = B_\varepsilon(x_i)$:

$$\forall x \in B_i \cap A, \quad |z(x) - z^*| < C\varepsilon^\beta,$$

since the center x_i lies in $\partial A \subseteq \{z = z^*\}$ according to Proposition 3.3.10. Consequently

$$(\partial A)_\varepsilon \cap A \subseteq \{z^* - C\varepsilon^\beta < z \leq z^*\},$$

thus because of Lemma 3.4.2:

$$|(\partial A)_\varepsilon \cap A| \leq |\{z^* - C\varepsilon^\beta < z \leq z^*\}| \leq C\varepsilon^\beta.$$

Using the previous inequality and the fact that $|B_i^-| \geq |B_i|/2 \geq C\varepsilon^d$ for $i \in I^-$, one has:

$$CN^{-\varepsilon^d} \leq \sum_{i \in I^-} |B_i^-| \leq |(\partial A)_\varepsilon \cap A| \leq C\varepsilon^\beta,$$

which implies:

$$N^- \leq C\varepsilon^{-(d-\beta)}. \quad (3.4.1)$$

Step 2: Bound on N^+

We consider the competitor $\tilde{\nu} = \mathbf{1}_{\tilde{A}}$ where $\tilde{A} = A \cup \bigcup_{i \in I^+} B_i^+$. It has a mass $|\tilde{\nu}| = 1 + m$ where $m = \sum_{i \in I^+} |B_i^+|$. To irrigate $\tilde{\nu}$, we send an extra mass $|B_i^+|$ to each center x_i along the irrigation plan, which costs $\alpha|B_i^+|z^*$, then we send this mass towards B_i^+ , which costs at most $C|B_i^+|^\alpha\varepsilon$. But one should get a cost no less than $e_\alpha(1+m)^{\alpha+1/d}$ by the scaling lemma. Moreover, with a development of order 2 one has:

$$\begin{aligned} (1+m)^{\alpha+1/d} &\geq 1 + (\alpha+1/d)m + 1/2 \cdot (\alpha+1/d)(\alpha+1/d-1)(1+m)^{\alpha+1/d-2}m^2 \\ &\geq 1 + (\alpha+1/d)m + Cm^2 \end{aligned}$$

because for ε small, $1+m$ is less than 2 for example. Consequently one may say:

$$e_\alpha(1 + (\alpha+1/d)m + Cm^2) \leq e_\alpha + \alpha mz^* + \sum_{i \in I^+} C\varepsilon|B_i^+|^\alpha.$$

Recall that $\alpha z^* = e_\alpha(\alpha+1/d)$, thus after simplifying one gets for some $C > 0$:

$$m^2 \leq C \sum_{i \in I^+} |B_i^+|^\alpha \varepsilon \leq CN^+ \varepsilon^{1+ad}. \quad (3.4.2)$$

Notice that for $i \in I^+$, $|B_i^+| \geq |B_i|/2 \geq C\varepsilon^d$, so that

$$m = \sum_{i \in I^+} |B_i^+| \geq CN^+ \varepsilon^d.$$

Injecting this into (3.4.2), one gets:

$$(N^+ \varepsilon^d)^2 \leq CN^+ \varepsilon^{1+ad},$$

thus

$$N^+ \leq C\varepsilon^{1+\alpha d-2d} = C\varepsilon^{-(d-\beta)}. \quad (3.4.3)$$

Putting (3.4.1) and (3.4.3) together yields:

$$N_\varepsilon(\partial A) = N \leq N^+ + N^- \leq \frac{C}{\varepsilon^{d-\beta}},$$

and

$$\overline{\dim}_M(\partial A) = \limsup_{\varepsilon \rightarrow 0} \frac{\log(N_\varepsilon(\partial A))}{-\log(\varepsilon)} \leq d - \beta,$$

which means that ∂A is of dimension smaller than $d - \beta$. \square

This result pushes us to propose the following conjecture:

Conjecture 3.4.4. *The boundary ∂A is of dimension $d - \beta$, in the sense that:*

$$\dim_H(\partial A) = \dim_M(\partial A) = d - \beta.$$

Proving this requires to establish the inequality $\dim_H(\partial A) \geq d - \beta$, for which we do not have a working strategy yet.

3.5 Numerical simulations

Our goal now is to compute solutions to our shape optimization problem numerically. To perform numerical simulations, we use the Eulerian framework of branched transport, first defined by Xia in [Xia03]. This framework is based on vector measures with a measure divergence, i.e. measures $v \in \mathcal{M}^d(\mathbb{R}^d)$ such that $\nabla \cdot v \in \mathcal{M}(\mathbb{R}^d)$, the set of such measures being denoted by $\mathcal{M}_{div}(\mathbb{R}^d)$. The cost is the so-called α -mass:

$$M_\alpha(v) = \begin{cases} \int \left| \frac{dv}{d\mathcal{H}^1}(x) \right|^\alpha d\mathcal{H}^1(x) & \text{if } v \text{ is 1-rectifiable,} \\ +\infty & \text{otherwise.} \end{cases}$$

An elliptic approximation of this functional was introduced by Oudet and Santambrogio in [OS11], in the spirit of Modica and Mortola [MM77]. The approximate functional is defined for $\varepsilon > 0$ by:

$$M_\varepsilon^\alpha(v) = \varepsilon^{-\sigma_1} \int |v(x)|^\sigma dx + \varepsilon^{\sigma_2} \int \frac{|v(x)|^2}{2} dx$$

for suitably chosen $\sigma, \sigma_1, \sigma_2$. It is proven in [OS11] that $\mathbf{M}_\varepsilon^\alpha$ Γ -converges to M^α as ε goes to 0, for a suitable topology on $\mathcal{M}_{div}(\mathbb{R}^d)$. Moreover, the Γ -convergence result also holds imposing an equality constraint on the divergence $\nabla \cdot v = f_\varepsilon$, for a suitable sequence $f_\varepsilon \rightharpoonup f$, as proven in [Mon17]. The results of [OS11] are proven in dimension $d = 2$, but in [Mon15] there is a proof of how to extend to higher dimension, in the case $\alpha > 1 - 1/d$ (in dimension $d = 2$ there is also a version of the Γ -convergence result for $\alpha \leq 1/2$). Also note that, recently, other phase-field approximations for

branched transport or other network problems have been studied, see for instance [BOO16; FCM16; BLS15].

Here we adapt the approach of [OS11] to our shape optimization problem by adding this time an inequality constraint on the divergence.

Recall that the Lagrangian and Eulerian frameworks are equivalent [Peg16], so that the irrigation distance may be computed in the following way:

$$d_\alpha(\mu, \nu) = \inf_v \{M^\alpha(v) \quad : \quad \nabla \cdot v = \mu - \nu\}.$$

Consequently the shape optimization problem (\mathbf{R}_α) rewrites, in relaxed form, as:

$$\min_v \{M^\alpha(v) \quad : \quad \mu - 1 \leq \nabla \cdot v \leq \mu\} \quad \text{where } \mu = \delta_0. \quad (\text{ES})$$

Setting $a = \mu - 1, b = \mu$ and some mollified versions $a_\varepsilon = \mu_\varepsilon - 1, b_\varepsilon = \mu_\varepsilon$, for example a convolution of μ with the standard mollifier of suitable size r_ε (e.g. $\varepsilon^{\sigma_2} r_\varepsilon^{-d} = o(1)$ as in [Mon17]), we define the following approximate problem, for $\varepsilon > 0$:

$$\min_v \{M_\varepsilon^\alpha(v) \quad : \quad a_\varepsilon \leq \nabla \cdot v \leq b_\varepsilon\}. \quad (\text{AS})$$

Let us remark that the above-mentioned Γ -convergence results do not allow us to say that this problem approximates (ES), as the inequality constraint on the divergence is not directly in these works. We leave this question for further investigation, as our aim is for now to make a first attempt to compute numerically an optimal shape for the original problem (\mathbf{S}_α) .

3.5.1 Optimization methods

We tackle problem (AS) by descent methods. Two difficulties arise: first of all, the functional M_ε^α is not convex hence there is no guarantee that the methods converge, and if they do, they may converge to a local minimizer which is not necessarily a global minimizer ; secondly, this is a constrained problem, hence we will need to compute projections or resort to proximal methods to handle the constraint. The simplest approach is to use a first-order method, for instance to perform a projected gradient descent on the functional M_ε^α for ε fixed (but small):

The projected gradient method.

$$\begin{cases} v_0 \in C \\ v_{n+1} = p_C(v_n - \tau_n \nabla M_\varepsilon^\alpha(v_n)), \end{cases}$$

where

$$C = \{v : a_\varepsilon \leq \nabla \cdot v \leq b_\varepsilon\}$$

is the convex set of admissible vector fields for (AS).

Computing the projection p_C is not an easy task, even more so as we want fast computations since this projection should be done at each step of the algorithm. Actually, this projection step will be quite costly (at least in our approach), hence we need to pass to a higher order method to get to an approximate minimizer in a reasonable number of iterations.

Recall that the projected gradient method is a particular case of the proximal gradient method, which we describe briefly. Consider a problem of the form

$$\min_v f(v) + g(v)$$

where f is smooth and g “proximable”, in the sense that one may easily compute its proximal operator

$$\text{prox}_g^\tau(v) = \operatorname{argmin}_{v'} g(v') + \frac{1}{2\tau}|v' - v|^2.$$

The proximal gradient method consists in doing at each step an explicit descent for f and an implicit descent for g :

The proximal gradient method.

$$\left| \begin{array}{l} v_0 \text{ given} \\ v_{n+1} = \text{prox}_g^{\tau_n}(v_n - \tau_n \nabla f(v_n)). \end{array} \right.$$

The projected gradient method corresponds to the case

$$g(v) = \begin{cases} 0 & \text{if } v \in C, \\ +\infty & \text{otherwise.} \end{cases}$$

If there was no function g , we recover the classical gradient descent method. Notice that there is an implicit choice in this method, since we compute gradients which depend on the scalar product. There is no reason that the canonical scalar product is well adapted to the function we want to minimize. Following the work of Lee, Sun and Saunders [LSS14] on Newton-type proximal methods, one may “twist” the scalar product, leading to the more general method:

A “twisted” proximal gradient method.

$$\left| \begin{array}{l} v_0 \text{ given} \\ v_{n+1} = \text{prox}_g^{\tau_n, H_n}(v_n - \tau_n \nabla_{H_n} f(v_n)), \end{array} \right. \quad (3.5.1)$$

where $\nabla_H f(x)$ is the gradient of f with respect to the scalar product $\langle x, y \rangle_H = \langle Hx, y \rangle$ for H an invertible self-adjoint operator, and

$$\text{prox}_g^{\tau, H}(v) = \operatorname{argmin}_{v'} g(v') + \frac{1}{2\tau} \|v' - v\|_H^2.$$

The best quadratic model of f around a point x_0 is

$$Q_{f, x_0}(x) = f(x_0) + \langle \nabla_H f(x_0), x \rangle_H + 1/2 \langle x, x \rangle_H,$$

with $H = H_f(x_0)$ being the Hessian of f at x , thus it is natural to consider (3.5.1) with $H_n = H_f(x_n)$. Notice indeed that if g is zero, the proximal operator is the identity and that $\nabla_H f(v) = H^{-1} \nabla f$, so that one recovers Newton’s method:

$$v_{n+1} = v_n - \tau_n H_n^{-1} \nabla f(v_n),$$

which is known to converge quadratically for smooth enough f . This is why this method is called *proximal Newton method*. However, for large-scale problems, computing and storing the Hessian is very costly, thus an alternative is to set H_n to be an approximation of the Hessian of f at v_n , thus leading to *proximal quasi-Newton methods*. These methods were introduced in [LSS14], which we refer to for further detail and theoretical results of convergence.

A very popular choice for H_n is given by the L-BFGS method (see [LN89]), which is a quasi-Newton method building in some sense the “best” approximation of the Hessian at v_n using only the information of the points v_k and the gradients $\nabla f(v_k)$ for a fixed number of previous steps $k = n, n-1, \dots, n-L+1$. The interest is that no matrix is stored, and there is a very efficient way to compute the matrix-vector product $H_n^{-1} \cdot v$ using simple algebra. Therefore, we decided to implement a proximal L-BFGS method, which in our case reads:

Projected L-BFGS method.

$$\begin{cases} v_0 \text{ given} \\ v_{n+1} = p_C^{\tilde{H}_n}(v_n - \tau_n \tilde{H}_n^{-1} \nabla f(v_n)), \end{cases} \quad (3.5.2)$$

where \tilde{H}_n is the approximate Hessian computed with the L-BFGS method with L steps and $p_C^{\tilde{H}_n}$ is the projection on C with respect to the norm $\|\cdot\|_{\tilde{H}_n}$.

The algorithm to compute the matrix-vector product $\tilde{H}_n^{-1} \cdot x$ is given in Section 3.5.3.

3.5.2 Computing the projection

The difficulty lies in the computation of the projection, that is on the proximal operator. A box constraint on the variable is very easy to deal with, but here we are faced with box constraints on $\nabla \cdot v$, that is on a linear operator applied to v . Moreover, we want to compute a projection with respect to a twister scalar product $\langle \cdot, \cdot \rangle_H$, which adds some extra difficulty. For simplicity of notations, we rename $a_\varepsilon, b_\varepsilon$ as a, b . By definition, finding the projection $p_C^H(v_0)$ of v_0 amounts to solving the optimization problem:

$$\min \left\{ \frac{\|v - v_0\|_H^2}{2} : a \leq \nabla \cdot v \leq b, v // \partial\Omega \right\}. \quad (\text{P})$$

Note that, when one considers the divergence operator as an operator acting on vector fields defined on the whole \mathbb{R}^d (extended to 0 outside Ω), the Neumann boundary condition above exactly corresponds to the fact that the divergence has no mass on $\partial\Omega$, which can be considered as included in the inequality constraints.

As a convex optimization, such a problem admits a dual problem, which we are going to use. We set

$$\psi(w) = \begin{cases} 0 & \text{if } a \leq w \leq b, \\ +\infty & \text{if not,} \end{cases}.$$

whose Legendre transform is

$$g(u) = \psi^*(u) = \int bu_+ - \int au_-,$$

so that $\psi = \psi^{**} = g^*$. Let us derive formally the dual problem by an inf – sup exchange:

$$\begin{aligned}
 \inf_{v // \partial \Omega} \left\{ \frac{\|v - v_0\|_H^2}{2} : a \leq \nabla \cdot v \leq b \right\} &= \inf_v \frac{1}{2} \|v - v_0\|_H^2 + \psi(\nabla \cdot v) \\
 &= \inf_v \frac{1}{2} \|v - v_0\|_H^2 + \sup_u -\langle \nabla u, v \rangle - g(u) \\
 &= \inf_v \sup_u \frac{1}{2} \|v - v_0\|_H^2 - \langle \nabla_H u, v \rangle_H - g(u) \\
 &\geq \sup_u -g(u) + \inf_v \frac{1}{2} \|v - v_0\|_H^2 - \langle \nabla_H u, v \rangle_H \\
 &= -\inf_u g(u) + \sup_v \langle \nabla_H u, v \rangle_H - \frac{1}{2} \|v - v_0\|_H^2 \\
 &= -\inf_u g(u) + \frac{\|\nabla_H u\|_H^2}{2} + \langle \nabla_H u, v_0 \rangle_H \\
 &= -\inf_u g(u) + \frac{1}{2} \int H^{-1} \nabla u \cdot \nabla u - \int u(\nabla \cdot v_0).
 \end{aligned}$$

Hence the dual problem reads:

$$\min_u \underbrace{\frac{1}{2} \int H^{-1} \nabla u \cdot \nabla u - \int u(\nabla \cdot v_0)}_{f(u)} + \underbrace{\int b u_+ - \int a u_-}_{g(u)}. \quad (\text{D})$$

The inf – sup exchange can be justified with equality via Fenchel's duality [Bre11, Chapter 1] in a well-chosen Banach space. Hence there is no duality gap:

$$\min(\text{P}) + \min(\text{D}) = 0.$$

As a consequence solving the dual problem provides a solution to the primal one. Indeed if u is optimal for (D) then $v = v_0 + \nabla_H u$ is optimal for (P). Now let us justify why it was interesting to pass by the resolution of a dual problem. Such a problem is of the form

$$\min_u f(u) + g(u), \quad (3.5.3)$$

where f is smooth, with gradient $\nabla f(u) = -\nabla \cdot (H^{-1} \nabla u) - \nabla \cdot v_0$, and g is proximal:

$$\text{prox}_g^\tau(u)(x) = \begin{cases} u(x) - \tau a & \text{if } u(x) < \tau a, \\ 0 & \text{if } \tau a \leq u(x) \leq \tau b, \\ u(x) - \tau b & \text{if } u(x) > \tau b. \end{cases}$$

We know how to compute the proximal operator and the gradient of f , since L-BFGS provides a simple method to compute the product $H^{-1}x$. Problems of the form (3.5.3) with f smooth (and computable gradient) and g proximal can be tackled with first-order methods such as the proximal gradient method described in the previous section (also called ISTA) or a fast proximal gradient method called FISTA, introduced in

[BT09]. We opted for the latter, which is a slight modification of the proximal gradient method using an intermediary point:

$$\begin{cases} u_0 \in H^1(\mathbb{R}^d), \\ \tilde{u}_n = u_n + \lambda_n(u_n - u_{n-1}), \\ u_{n+1} = \text{prox}_g^\tau(\tilde{u}_n - \tau \nabla f(\tilde{u}_n)), \end{cases} \quad (\text{FISTA})$$

where λ_n is given by some recursive formula (we refer to [BT09] for the details). It enjoys a theoretical and practical rate of convergence which is higher than ISTA and which is that of the classical gradient method:

$$f(u_n) - f_{opt} \leq \frac{2L_f |u_0 - u_{opt}|^2}{(n+1)^2}.$$

3.5.3 Algorithms and numerical experiments

Following the work of [OS11], we discretize our problem on a staggered grid : we divide the cube $Q = [-1, 1]^2$ into M^2 subcubes of side $2/M$, the functions U are defined at the center of the small cubes, while the x component V^x of a vector fields V is defined on the vertical edges of the grid and the y component V^y on the horizontal edges of the grid. This is quite convenient to compute the discrete divergence of a vector field and the discrete gradient of a function.

- Unknowns: $(V_{i,j}^x)_{\substack{1 \leq i \leq M \\ 1 \leq j \leq M+1}}$, $(V_{i,j}^y)_{\substack{1 \leq i \leq M+1 \\ 1 \leq j \leq M}}$, with

$$V_{1,j}^x = V_{M+1,j}^x = V_{i,1}^y = V_{i,M+1}^y = 0,$$

which means that V is parallel to the boundary.

- Objective function:

$$F(V) = \varepsilon^{-\sigma_1} h^2 \sum_{i,j} N(\hat{V}_{i,j})^\sigma + \varepsilon^{\sigma_2} h^2 / 2 \left(\sum_{i,j} |\nabla_{i,j} V^x|^2 + \sum_{i,j} |\nabla_{i,j} V^y|^2 \right).$$

There are several definitions to give to make sense of F . First of all N is a smooth approximation of the norm, of the form

$$N(x) = (|x|^2 + \varepsilon_s^2)^{1/2} \quad \text{for } \varepsilon_s \text{ small.}$$

The discrete vector field $\hat{V}_{i,j} = (\hat{V}_{i,j}^x, \hat{V}_{i,j}^y)$ is an interpolation of (V^x, V^y) defined at the centers of the cubes:

$$\hat{V}_{i,j}^x = \frac{V_{i,j}^x + V_{i+1,j}^x}{2}, \quad \hat{V}_{i,j}^y = \frac{V_{i,j}^y + V_{i,j+1}^y}{2}, \quad 1 \leq i, j \leq M.$$

Finally the discrete gradient is defined as usual by

$$\begin{aligned} \nabla_{i,j} V^x &= ((V_{i,j+1}^x - V_{i,j}^x)/h, (V_{i+1,j}^x - V_{i,j}^x)/h), \quad 1 \leq i \leq M-1, 1 \leq j \leq M, \\ \nabla_{i,j} V^y &= ((V_{i,j+1}^y - V_{i,j}^y)/h, (V_{i+1,j}^y - V_{i,j}^y)/h), \quad 1 \leq j \leq M-1, 1 \leq i \leq M. \end{aligned}$$

We may now give the main algorithm and its sub-methods.

Algorithm 1 Proximal L-BFGS for F

Data: tolerance tol , initial vector field V_0 , step τ_0 , source δ
 $V \leftarrow V_0, U \leftarrow U_0$
compute $error$
while $error > tol$ **do**
 $\tau \leftarrow \tau_0$
 repeat
 $G \leftarrow \text{MULTIPLYBFGS}(\nabla F(V))$
 $V, U \leftarrow \text{PROJECT}(V - \tau G, U, \delta, \tau)$
 $\tau \leftarrow \tau/2$
 until $F(V)$ has decreased
 update L-BFGS data
 compute $error$
end while

The update step for L-BFGS data consists in storing in Y, Z, r the points and gradients of the L previous steps, so that at step n :

$$Y_{L-k} = \nabla F(V_{n-k}) - \nabla F(V_{n-k-1}), \quad Z_{L-k} = V_{n-k} - V_{n-k-1}$$

for all $k = 0, \dots, L-1$, and $r_k = 1/(Y_k \cdot Z_k)$ for all $k = 0, \dots, L-1$. Notice here that we do a simple backtracking line search by reducing the step size τ until the energy has decreased, for example until it has sufficiently decreased and satisfies the Armijo rule. Also, notice that the potential U computed at step n is used at the next step as initial data ; this trick extensively speeds up the computation of the projection. Finally, we took as error measurement some relative difference between two consecutive steps.

Now, as stated in Section 3.5.2, the projection on C with respect to $\|\cdot\|_H$ is computed via the FISTA method, as follows:

Algorithm 2 Project V_0 on C with respect to $\|\cdot\|_H$

Data: tolerance tol_p , step τ_p
function $\text{PROJECT}(V_0, U_0, \delta, \tau)$
 $D_0 \leftarrow \nabla \cdot V_0$
 $U \leftarrow U_0$
 while $error > tol_p$ **do**
 $t_p \leftarrow t; t \leftarrow (1 + \sqrt{1 + 4t_p^2})/2; s \leftarrow (t_p - 1)/t$
 $G \leftarrow \text{MULTIPLYBFGS}(\nabla U)$
 $U_i \leftarrow U + s(U - U_{old})$
 $U_{old} \leftarrow U$
 $U \leftarrow \text{PROX}(U_i - \tau_p(\nabla \cdot G - D_0), \delta, \tau)$
 compute $error$
 end while
 $V \leftarrow V_0 + \text{MULTIPLYBFGS}(\nabla U)$
 return V, U
end function

The PROX function is just the proximal operator associated with the discrete counterpart of $g : u \mapsto \int bu_+ - \int au_-$ where $a = \delta - 1, b = \delta$. Thus $P = \text{PROX}(U, \delta, \tau)$ is defined by:

$$P_i = \begin{cases} U_i - \tau(\delta_i - 1) & \text{if } U_i < \tau(\delta_i - 1), \\ 0 & \text{if } \tau(\delta_i - 1) \leq U_i \leq \tau\delta_i, \\ U_i - \tau\delta_i & \text{if } U_i > \tau\delta_i. \end{cases}$$

For the sake of completeness, we give a simple method to compute the L-BFGS multiplication $H^{-1}X$ (see [LN89; Noc80] for details).

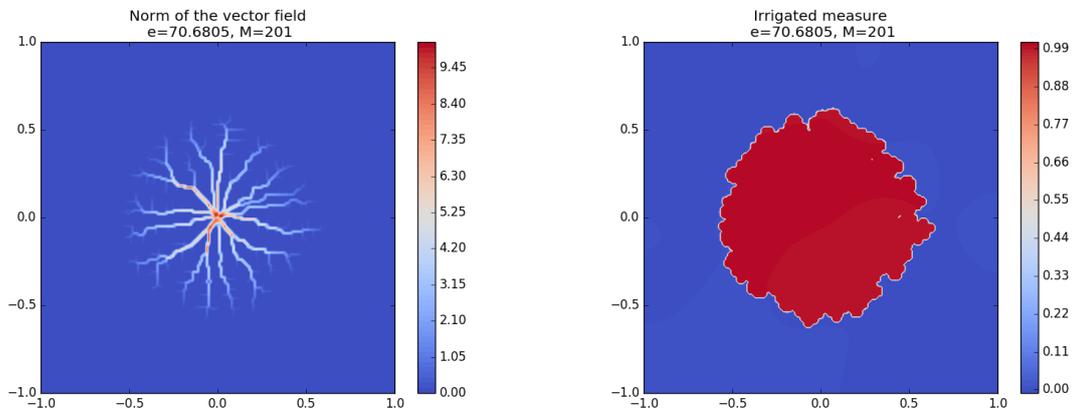
Algorithm 3 L-BFGS multiplication $H^{-1}X$

```

function MULTIPLYBFGS( $X$ )
   $G \leftarrow X$ 
  for  $i = L, \dots, 1$  do
     $s_i \leftarrow r_i Z_i \cdot G$ 
     $G \leftarrow G - s_i Y_i$ 
  end for
   $G \leftarrow (Z_L \cdot Y) / (Y_L \cdot Y_L) G$ 
  for  $i = 1, \dots, L$  do
     $t \leftarrow r_i Y_i \cdot G$ 
     $G \leftarrow G + (s_i - t) Z_i$ 
  end for
  return  $G$ 
end function

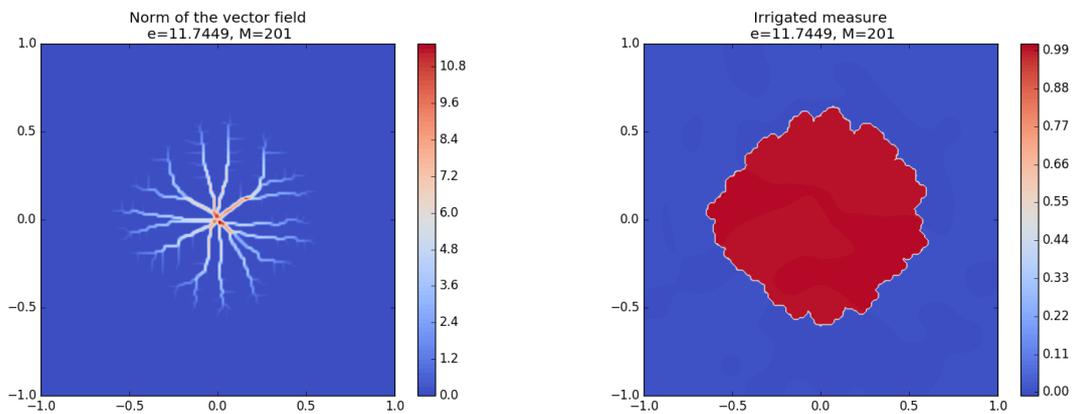
```

We present some numerical results obtained with $\varepsilon_s = 10^{-4}$, on a $M \times M$ grid with $M = 201$ and $\varepsilon = 3h$ where $h = 2/M$, the code being written in Julia. We have started with random initial values for V and a smooth approximation δ of the Dirac δ_0 . After some days of computation on a standard laptop, one gets the following shapes and underlying networks.



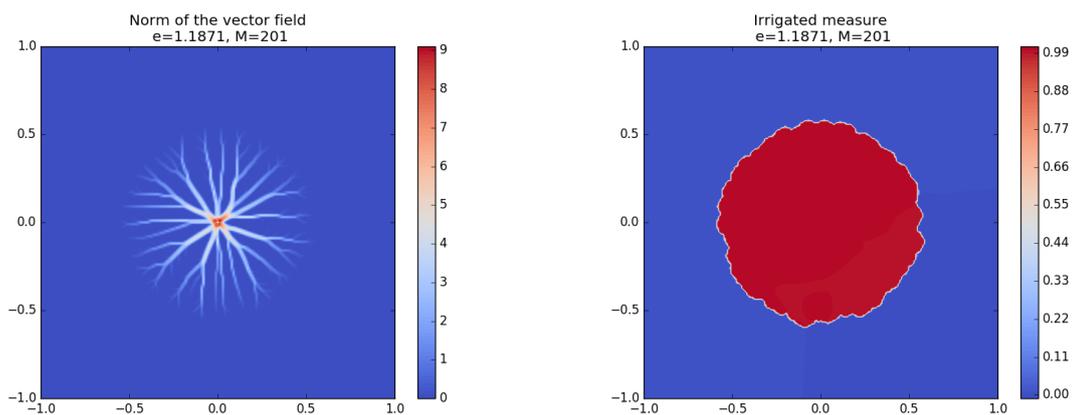
(a) Norm of the vector field, $\alpha = 0.55$

(b) Irrigated measure, $\alpha = 0.55$



(c) Norm of the vector field, $\alpha = 0.65$

(d) Irrigated measure, $\alpha = 0.65$



(e) Norm of the vector field, $\alpha = 0.85$

(f) Irrigated measure, $\alpha = 0.85$

Figure 3.1: Algorithm output for different α 's after ~ 15000 – 25000 iterations.

The landscape function with multiple sources

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The landscape function is now a central tool in branched transport. It was first defined by Santambrogio in [San07] as a generalization of the landscape function which was very well known by geophysicists (see [RR01]) though it was considered on a discretized grid. It represents the elevation of the landscape on which water flows to irrigate water basins such as lakes. The definition was given in the single-source case, and regularity results of Hölder type were established first in [San07], then by Brancolini and Solimini in [BS11] under more general hypotheses and with a different approach. An attempt at defining the landscape function for multiple sources was made by Xia in [Xia14] in the discrete case. The present chapter aims at defining the landscape function in the multiple source case but with non atomic measures, under an hypothesis on the source and target measures μ, ν , namely that they have disjoint support:

$$\text{supp}(\mu) \cap \text{supp}(\nu) = \emptyset.$$

From this definition we are able to extend many nice properties already known in the single-source case: the landscape function is the first variation of the function \mathbf{X}^α , and it exhibits Hölder regularity under some regularity hypotheses on the measures. A direct use of the landscape function can be made to obtain the uniform boundedness of the length of the fibers. Other applications already known in the single source case hopefully also hold in the multiple source case, but this is left for further investigation: the regularity of the fibers (locally finite curvature) proved in [MS10], and the fractal branching behaviour established in [BS14].

4.1 Framework definition

We make slight modifications to the standard framework described in the first chapter, in order to simplify notations and proofs. The main difference will be that curves are no longer defined on \mathbb{R}^+ but on \mathbb{R} instead.

The set of curves We set $\Gamma = \text{Lip}_1(\mathbb{R}, \mathbb{R}^d)$, and define for all $\gamma \in \Gamma$:

$$\begin{aligned} T^-(\gamma) &= \sup\{t : \gamma \text{ is constant on }]-\infty, t[\}, \\ T^+(\gamma) &= \inf\{t : \gamma \text{ is constant on }]t, +\infty[\}. \end{aligned}$$

For a curve $\gamma \in \Gamma$, we write $\gamma : [T^-, T^+] \rightarrow \mathbb{R}^d$ to mean that $T^-(\gamma) = T^-$ and $T^+(\gamma) = T^+$, and we denote by $\gamma^- = \gamma(T^-(\gamma))$, $\gamma^+ = \gamma(T^+(\gamma))$ the starting point and stopping point of γ . We will write $x \xrightarrow{\gamma} y$ if γ is a curve starting at x and stopping at y , and $x \xrightarrow{\gamma} y$ if either $x \xrightarrow{\gamma} y$ or $x \xleftarrow{\gamma} y$.

Restrictions and inclusions If γ is simple and $x \in \gamma$, we denote by t_x the unique $t \in [T^-(\gamma), T^+(\gamma)]$ such that $\gamma(t) = x$. For $x, y \in \gamma$, $\gamma[x, y]$ denotes the usual restriction¹ $\gamma|_{[t_x, t_y]}$ if $t_x \leq t_y$ or $\gamma|_{[t_y, t_x]}$ if $t_y \leq t_x$, and $\gamma[x, +]$, $\gamma[x, -]$ denote respectively $\gamma|_{[x, \gamma^+]}$ and $\gamma|_{[x, \gamma^-]}$. We also define the obvious variants $\gamma]a, b[$, $\gamma[a, b[$ and so on excluding some

¹For us, if I is an interval $\gamma|_I$ is a map defined only on I , but it induces a unique Lipschitz extension to the whole \mathbb{R} which is constant before and after I and we will make the slight abuse of not distinguishing the two.

extremities. We say that γ_1 is included in γ_2 and we write $\gamma_1 \subseteq \gamma_2$ if the trajectory of γ_1 is part of that of γ_2 .

Paths A path p between x and y is a sequence of curves γ_i of the form

$$p : x = x_1 \xrightarrow{\gamma_1} x_2 \xrightarrow{\gamma_2} \dots x_{n-1} \xrightarrow{\gamma_{n-1}} x_n = y,$$

and an oriented path p from x to y is a sequence of curves γ_i of the form

$$p : x = x_1 \xrightarrow{\gamma_1} x_2 \xrightarrow{\gamma_2} \dots x_{n-1} \xrightarrow{\gamma_{n-1}} x_n = y.$$

For such an oriented path p we will denote by $\gamma_p : [0, T^+] \rightarrow \mathbb{R}^d$ the curve, parameterized by arc length, obtained by concatenating the γ_i 's, so that $x \xrightarrow{\gamma_p} y$.

Curve multiplicity Let η be an irrigation plan. For $\gamma \in \Gamma$ we set

$$\theta_\eta(\gamma) = \eta(\tilde{\gamma} : \gamma \subseteq \tilde{\gamma}),$$

and we say that γ is a η -branch if $\theta_\eta(\gamma) > 0$. An η -branch path, oriented or not, is a path made of η -branches. When η is fixed and there is no confusion, we will omit the prefix η - and talk about branches and branch paths.

Cycle-free property We recall that an irrigation plan η is said cycle-free if there is no loop sequence $x_1, x_2, \dots, x_n, x_{n+1} = x_1$ such that $\eta(\Gamma[x_i, x_{i+1}]) > 0$ for $1 \leq i \leq n$, where $\Gamma[x, y]$ is the set of curves which visit x and y , no matter the order. In particular if $\eta(\Gamma[x, y]) > 0$ then there is a unique² curve $\gamma_\eta(x, y)$ joining x and y (either from x to y or the contrary), such that η -almost every curve $\gamma \in \Gamma[x, y]$ follows $\gamma_\eta(x, y)$ in its trajectory.

4.1.1 Good curves and paths

We need a notion of curve which is more general than branches: we would like to consider curves that do not stay entirely in the network but can go to its “boundary”, but which nevertheless follow the irrigation plan. This is the role played by what we call η -good curves.

Definition 4.1.1 (η -good curve). We say that $\gamma : [T^-, T^+] \rightarrow \mathbb{R}^d$ is an η -good curve stemming from $\sigma \in \gamma$ if

$$\int_\gamma \theta_\eta(\gamma[\sigma, x])^{\alpha-1} dx < \infty.$$

Moreover, if γ starts at the stem σ (resp. if it stops at the stem), we say that γ is right-sided (resp. left-sided). We say that it is one-sided if it is left- or right-sided.

Remark 4.1.2. In the single-source case $\mu = \delta_s$, η -good curves stemming from s correspond to the definition of η -good curves given in Definition 1.4.3. Moreover it is possible to show that if γ stems from another point, then it is the restriction of an η -good curves stemming from s .

²Parameterized by arc length on an interval $[0, \ell]$.

The definition of η -good curves induces a notion of η -good path.

Definition 4.1.3 (η -good path). We say that a path p , oriented or not, is η -good if it is made of one-sided η -good curves.

Now We would like to prove that optimal irrigation plans η with finite cost are concentrated on the set of η -good curves: in this way this set could be thought as some kind of support for η . For now, we are able to prove it in the case of optimal irrigation plans which have a finite root system (defined in Subsection 4.1.2), though it should be true in all generality. Let us establish first a general lemma which does not require this extra hypothesis.

Lemma 4.1.4. *Let η be a simple irrigation plan which has finite \mathbf{I}^α cost and which satisfies the cycle-free property. Fix $x_0 \in N_\eta$. Then for η -almost every curve $\gamma \in \Gamma(x_0)$,*

$$\int_\gamma \theta_\eta(\gamma[x_0, x])^{\alpha-1} dx < \infty.$$

Proof. Let us denote by $\Gamma(x_0, x)$ the set of curves γ such that $x_0, x \in \gamma$ and

$$\theta_{x_0}(x) = \eta(\Gamma(x_0, x)).$$

Recall that η satisfies the cycle-free property. Consequently, for all x such that $\theta_{x_0}(x) > 0$, almost all curves γ in $\Gamma(x_0, x)$ follow a common trajectory $\gamma_{x_0, x}$, i.e. $\gamma[x_0, x] = \gamma_{x_0, x}$. This implies that for almost every curve $\gamma \in \Gamma(x_0, x)$:

$$\theta_{x_0}(x) = \theta_\eta(\gamma[x_0, x]).$$

Taking the power $\alpha - 1$ and integrating of $\Gamma(x_0, x)$ yields:

$$\theta_{x_0}(x)^\alpha = \int_{\Gamma(x_0, x)} \theta_\eta(\gamma[x_0, x])^{\alpha-1} d\eta(\gamma).$$

This equality also holds trivially if $\theta_{x_0}(x) = 0$. Then integrating in x :

$$\int_{\mathbb{R}^d} \int_{\Gamma(x_0, x)} \theta_\eta(\gamma[x_0, x])^{\alpha-1} d\eta(\gamma) d\mathcal{H}^1(x) = \int_{\mathbb{R}^d} \theta_{x_0}(x)^\alpha d\mathcal{H}^1(x) \leq \mathbf{I}^\alpha(\eta) < \infty.$$

Thus by Fubini's theorem one gets

$$\begin{aligned} \infty &> \int_{\Gamma(x_0)} \int_{\mathbb{R}^d} \mathbf{1}_{x \in \gamma} \theta_\eta(\gamma[x_0, x])^{\alpha-1} d\mathcal{H}^1(x) d\eta(\gamma) \\ &= \int_{\Gamma(x_0)} \int_\gamma \theta_\eta(\gamma[x_0, x])^{\alpha-1} dx d\eta(\gamma). \end{aligned}$$

Consequently for η -almost every curve γ in $\Gamma(x_0)$,

$$\int_\gamma \theta_\eta(\gamma[x_0, x])^{\alpha-1} dx < \infty,$$

which is what we wanted to prove. \square

Remark 4.1.5. This shows in particular that if $\theta_\eta(x_0) > 0$, then either x_0 is an atom of both μ and ν , or one can find a branch containing x_0 . Indeed, since $\Gamma(x_0)$ has positive measure, either almost all curves are constant equal to x_0 and x_0 is an atom of μ and ν , or there is a positive quantity of non-trivial curves in $\Gamma(x_0)$. Then for sure there is a non-trivial curve $\gamma \in \Gamma(x_0)$ such that

$$\int_\gamma \theta_\eta(\gamma[x_0, x])^{\alpha-1} dx < \infty,$$

and $\gamma[x_0, x]$ for some x in the interior of the curve is a branch containing x_0 .

We end this subsection by giving a definition of *irrigation basins* using η -good curves.

Definition 4.1.6 (Irrigation basins). If $\eta \in \mathbf{IP}(\mu, \nu)$ and $x \in N_\eta$ we define the irrigation basins of x as:

$$\text{Bas}_\eta^+(x) := \{y : \text{there is an } \eta\text{-good curve stemming from } x \text{ s.t. } \gamma^+ = y,$$

$$\text{Bas}_\eta^-(x) := \{y : \text{there is an } \eta\text{-good curve stemming from } x \text{ s.t. } \gamma^- = y,$$

and

$$\text{Bas}_\eta(x) := \text{Bas}_\eta^+(x) \cup \text{Bas}_\eta^-(x).$$

4.1.2 Root systems

The role of the origin in the single-source case is replaced by the notion of root system.

Definition 4.1.7 (Root system). We call root system for η any set of points $R \subseteq N_\eta$ such that η -almost every curve γ passes through some point $\rho \in R$. The elements $\rho \in R$ are called roots.

Remark 4.1.8. In the single-source case $\mu = \delta_s$, there is obviously a root system made of a single point: $R = \{s\}$.

The following lemma states that for disjointly supported source and target measures, optimal irrigation plans have a finite root system. It follows from the fact that the irrigation network has a finite graph structure away from the support of the measures, which is proven in [Xia04] where it is termed ‘‘interior regularity’’.

Lemma 4.1.9 (Finite root system). *If $\eta \in \mathbf{IP}(\mu, \nu)$ is an optimal irrigation plan with finite \mathbf{I}^α cost, it admits a finite root system $R \subseteq N_\eta$. Moreover one can assume that all points $\rho \in R$ are at positive distance from the supports of μ and ν .*

Proof. We cut the curves of the irrigation plan η between their first exit time from $(\text{supp } \mu)_\varepsilon$ and their first entry time into $(\text{supp } \nu)_\varepsilon$ for ε small, getting a irrigation plan $\tilde{\eta} \in \mathbf{IP}(\tilde{\mu}, \tilde{\nu})$. Since by optimality η is known to have a finite graph structure far from the supports of the measures (by [BCM09, Theorem 4.7]), the marginal measures $\tilde{\mu}$ and $\tilde{\nu}$ are atomic. We take R to be the set of atoms of $\tilde{\mu}$ for example. \square

Now we are able to prove that optimal irrigation plans are concentrated on the set of good curves, provided that they have a finite root system.

Proposition 4.1.10. *Suppose η is a simple irrigation plan with finite \mathbf{I}^α cost which satisfies the cycle-free property and admits a finite root system R . Then η is concentrated on the set of η -good curves passing through points of R .*

Proof. By Lemma 4.1.4, if $\rho \in R$ is fixed, outside an η -negligible set E_ρ , all curves $\gamma \in \Gamma(\rho)$ satisfy

$$\int_\gamma \theta_\eta(\gamma[\rho, x])^{\alpha-1} dx < \infty.$$

Since almost all curves pass through an element $\rho \in R$, η is concentrated on $\bigcup_{\rho \in R} (\Gamma(\rho) \setminus E_\rho)$ which proves the result. \square

However, considering η -good curves reaching a root will be too restrictive to give a definition of a landscape function which satisfies the desired properties (for example semicontinuity). This justifies the following definition.

Definition 4.1.11 (Root network). We denote by \bar{R} the set of points x which belong to a simple branch path connecting two roots, and call it the root network.

Remark 4.1.12. If η has the cycle-free property and R is finite, then \bar{R} is a compact set since it is made of a finite union of compact curves.

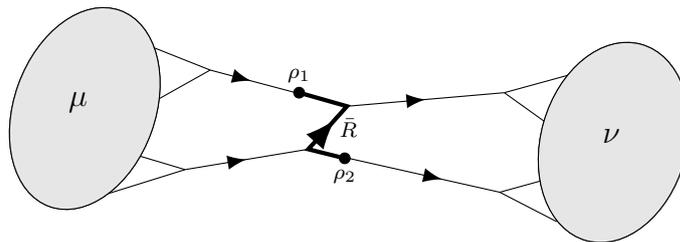


Figure 4.1: A root network induced by a system of two roots.

If γ is a curve intersecting \bar{R} , we denote by $\rho^-(\gamma), \rho^+(\gamma)$ the first and last points of γ belonging to \bar{R} . Next lemma will relate η -good curves stemming from arbitrary points $\sigma \in \mathbb{R}^d$ to η -good curves stemming from points ρ of the root network \bar{R} .

Lemma 4.1.13. *Let η be simple irrigation plan with finite \mathbf{I}^α cost which is cycle-free and has a finite root system R . If γ is a right-sided η -good curve stemming from $\sigma \in \mathbb{R}^d$ and reaching x , then we are in one of the three situations:*

- (i) γ intersects \bar{R} : γ can be cut in two, a branch $\gamma[\sigma, \rho_+(\gamma)]$ and a right-sided η -good curve $\gamma[\rho^+(\gamma), x]$,
- (ii) γ is after \bar{R} : $\gamma \cap \bar{R} = \emptyset$ and there is a right-sided η -good curve $\bar{\gamma}$ starting at $\rho = \rho^+(\bar{\gamma})$ and stopping at x such that $\gamma = \bar{\gamma}[\sigma, x]$,
- (iii) γ is before \bar{R} : $\gamma \cap \bar{R} = \emptyset$ and there is a branch $\bar{\gamma}$ starting at σ and stopping at $\rho = \rho^-(\bar{\gamma})$ such that $\gamma = \bar{\gamma}[\sigma, x]$.

The analogous statement for left-sided η -good curves holds as well.

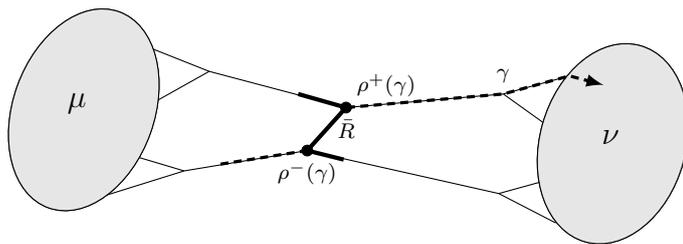


Figure 4.2: A right-sided η -good curve entering and leaving \bar{R} .

Proof. If γ intersects \bar{R} and $\rho^+(\gamma) \neq x$, then we are clearly in situation (i). Now we assume that that γ does not intersect \bar{R} except possibly at x . We proceed in two steps.

Step 1. Let ξ be a branch such that $\xi \not\subseteq \bar{R}$ and consider the curves containing ξ . We know almost all of them pass through a root. We want to show that almost all of them may only pass through a root after following ξ , or almost all of them may only pass through a root before following ξ . Suppose on the contrary that there are $\rho_1, \rho_2 \in R$ and positive amounts of curves A_1, A_2 such that all curves in A_1 pass through ρ_1 then ξ along a common trajectory γ_1 , and all curves in A_2 pass through ξ then ρ_2 along a trajectory γ_2 . Since η has no circuit, the branch path

$$\rho_1 \xrightarrow{\gamma_1[\rho_1, \xi^-]} \xi^- \xrightarrow{\xi} \xi^+ \xrightarrow{\gamma_2[\xi^+, \rho_2]} \rho_2$$

is a simple branch path joining $\rho_1 \neq \rho_2 \in R$ hence by definition $\xi \subseteq \bar{R}$ which is a contradiction. This proves what we claimed.

Step 2. Consider $y \in \gamma[\sigma, x[$: it is clear that $\gamma[\sigma, y]$ is a branch which does not intersect \bar{R} , hence by the first step, we know that either almost all curves containing this branch reach a root before, or they all reach a root after it. By taking a sequence $y_n \in \gamma[\sigma, x[$ such that $y_n \rightarrow x$, one can state that there are two possibilities.

The first possibility is that except for a negligible set U , all curves $\tilde{\gamma}$ satisfy the following: if $\tilde{\gamma}$ contains a piece $\gamma[\sigma, y]$ for some $y \in \gamma[\sigma, x[$ and a root $\rho \in R$, then ρ is reached before γ , following a unique trajectory γ_ρ from σ to ρ . We set $\bar{\rho}$ the first point where these trajectories part when we follow the curves backwards starting at σ , and set γ' the common trajectory from $\bar{\rho}$ to σ . We set $\bar{\gamma} = \gamma' \cup \gamma$ the concatenation of γ' and γ . By definition all the curves following $\gamma[\sigma, y]$ contain $\bar{\gamma}[\bar{\rho}, y]$, hence $\bar{\gamma}$ is a right-sided η -good curve stemming from $\bar{\rho}$. Let us show that $\bar{\rho} \in \bar{R}$. If $\bar{\rho} \in R$, we are done. Otherwise there are two trajectories $\gamma_\rho, \gamma_{\rho'}$ which parted at $\bar{\rho}$. Since they could not meet again by the cycle-free property, $\rho \xrightarrow{\gamma_\rho[\rho, \bar{\rho}]} \bar{\rho} \xrightarrow{\gamma_{\rho'}[\bar{\rho}, \rho']} \rho'$ is a simple branch path, and by construction $\bar{\rho} \in \bar{R}$. Then we replace $\bar{\gamma}$ with its restriction $\bar{\gamma}[\rho^+(\bar{\gamma}), x]$ and we are in situation (ii).

The second possibility is that except for a negligible set U , all curves $\tilde{\gamma}$ satisfy the following: if $\tilde{\gamma}$ contains a piece $\gamma[\sigma, y]$ for some $y \in \gamma[\sigma, x[$ and a root $\rho \in R$, then ρ is reached after $\gamma[\sigma, y]$. Since the amount of roots is finite and the cycle-free property holds, there is a point $y \in \gamma[\sigma, x[$ close enough to x such that for all the trajectories between y and ρ of the curves containing $\gamma[\sigma, y]$ cannot part from γ . Hence almost all

curves containing $\gamma[\sigma, y]$ contain γ and γ is actually a branch. If $\gamma^+ \in \bar{R}$ then we are in situation (i) (with $\gamma[\rho^+(\gamma), x]$ being trivial), while if $\gamma^+ \notin \bar{R}$, we are in situation (iii).

The case of γ left-sided is obtained in a similar manner. \square

Remark 4.1.14. Notice that this shows that any η -good curve (not necessarily one-sided) can be decomposed into a path made of curves which are restrictions of η -good curves stemming from points of \bar{R} .

Next lemma will be essential, in that if a good curve stems from a point $\rho \in \bar{R}$, there is a point after which (and a point before which) the multiplicity and the “joint multiplicity” coincide.

Lemma 4.1.15 (Good stem). *Let η be a simple irrigation plan satisfying the cycle-free property, and R be a finite root system for η . If γ is an η -good curve stemming from a point of the root network, then*

$$\forall y \in \gamma[\rho^+(\gamma), +[, \quad \theta_\eta(y) = \theta_\eta(\gamma[\rho^+(\gamma), y]),$$

and

$$\forall y \in \gamma]-, \rho^-(\gamma)[, \quad \theta_\eta(y) = \theta_\eta(\gamma[y, \rho^-(\gamma)]).$$

Proof. Take such a curve γ . The curve $\gamma[\rho^+(\gamma), +]$ is an η -good curve stemming from $\rho^+(\gamma) \in \bar{R}$ which is right-sided. Take $y \in \gamma[\rho^+(\gamma), +[$. For sure $\theta_\eta(y) \geq \theta_\eta(\gamma[\rho^+(\gamma), y]) > 0$. We look at the curves passing through y . We know by hypothesis that almost all curves pass through a root, and the amount of roots is finite. Take A a positive amount of curves passing through y and by a root $\rho \in R$, then all curves in A follow a common trajectory $\gamma' = \gamma_{y,\rho}$, and $\rho^+(\gamma) \xrightarrow{\gamma[\rho^+(\gamma), y]} y \xrightarrow{\gamma_{y,\rho}} \rho$ is a branch path joining $\rho^+(\gamma)$ and ρ . If one trajectory is included into the other, since $\rho \notin \gamma[\rho^+(\gamma), y[$ it must be $\gamma[\rho^+(\gamma), y] \subseteq \gamma_{y,\rho}$. By the cycle-free property, these trajectories cannot part and meet again. Thus if we follow the curves $\gamma[\rho^+(\gamma), y]$ and γ' from the common point y , if those trajectories part at some point y' , we know that $\rho^+(\gamma) \xrightarrow{\gamma[\rho^+(\gamma), y']} y' \xrightarrow{\gamma'[y', \rho]} \rho$ would be a simple branch path joining $\rho^+(\gamma)$ and ρ , thus they would belong to the root network, which is a contradiction to the definition of $\rho^+(\gamma)$. By way of consequence, almost every curve passing through y contains $\gamma[\rho^+(\gamma), y]$ and

$$\theta_\eta(y) = \theta_\eta(\gamma[\rho^+(\gamma), y]).$$

The same reasoning shows that

$$\forall y \in \gamma]-, \rho^-(\gamma)[, \quad \theta_\eta(y) = \theta_\eta(\gamma[y, \rho^-(\gamma)]). \quad \square$$

4.2 Definition of the landscape function

We assume that μ and ν have compact and disjoint supports:

$$\text{supp } \mu \cap \text{supp } \nu = \emptyset.$$

By Lemma 4.1.9, the situation is the following: each point $\rho \in \bar{R}$ will give rise to a cost function Z_ρ defined on curves with extremity ρ as the integral³ along the curve of $\theta_\eta^{\alpha-1}$. The goal will be to prove that with a suitable choice of an additive constant to each Z_ρ , which amounts to assigning a reference value at ρ , these maps take the same value when needed: namely when a point is reached by different curves, possibly joining different roots. This will allow us to define a global landscape function $z : \mathbb{R}^d \rightarrow \mathbb{R}$.

Recall that $Z_\eta(\gamma) = \int_\gamma \theta_\eta(x)^{\alpha-1} dx$. We define a signed variant when a reference point σ is fixed.

Definition 4.2.1. Let $\sigma \in \mathbb{R}^d$ be fixed. For all simple curve γ such that $\sigma = \gamma^\pm$ we define

$$Z_\sigma(\gamma) = \begin{cases} + \int_\gamma \theta_\eta(x)^{\alpha-1} dx & \text{if } \sigma = \gamma^-, \\ - \int_\gamma \theta_\eta(x)^{\alpha-1} dx & \text{if } \sigma = \gamma^+. \end{cases}$$

4.2.1 Balance property

The following proposition is the multiple source counterpart to the well-definedness property established in Proposition 1.4.5, and will be key to defining the landscape function.

Proposition 4.2.2 (Balance property). *We assume that η is an optimal irrigation plan with finite \mathbf{I}^α cost. Consider a good loop, that is a loop made of one-sided⁴ η -good curves:*

$$p_1 \xrightarrow{\gamma_1} p_2 \xrightarrow{\gamma_2} \dots \xrightarrow{\gamma_n} p_{n+1} = p_1$$

where $p_i \xrightarrow{\gamma_i} p_{i+1}$ is either $p_i \xrightarrow{\gamma_i} p_{i+1}$ or $p_i \xleftarrow{\gamma_i} p_{i+1}$. Then one has

$$\sum_{p_i \xrightarrow{\gamma_i} p_{i+1}} Z_\eta(\gamma_i) = \sum_{p_i \xleftarrow{\gamma_i} p_{i+1}} Z_\eta(\gamma_i),$$

or equivalently

$$\sum_i Z_{p_i}(\gamma_i) = 0.$$

³Actually, up to a sign depending on the orientation of the curve, just as in the definition of the integral on the real line $\int_a^b f = \pm \int_{[a,b]} f$ depending on whether $a \leq b$ or $b \leq a$.

⁴Since two-sided good curves can be cut in two one-sided good curves, it does not matter.

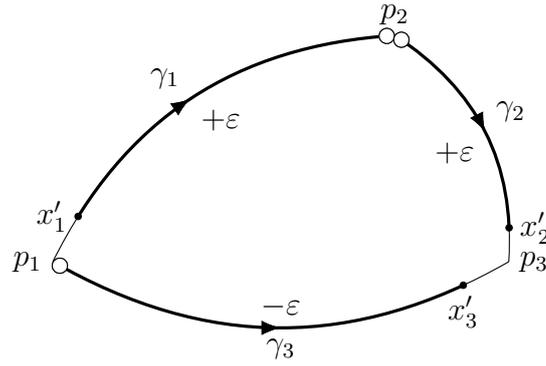


Figure 4.3: A good loop made of three η -good curves. Their stems are signaled by a large white circle.

Proof. We denote by σ_i the stem of the i -th η -good curve, and x_i its other extremity. Choose an orientation for the loop and a small $\varepsilon > 0$. We want to add a small mass ε to all the curves which are oriented accordingly, and remove ε from all the other ones. We cannot do it directly to the whole curves because they are not necessarily branches: there may be no mass left at the point x_i . Therefore for all i , we consider a small $\varepsilon_i \geq 0$ and we cut the curve γ_i at a time $t_i = T^\mp(\gamma_i) \pm \varepsilon_i$ before the extremity x_i , setting $x'_i = \gamma_i(t_i)$. We are going to remove and add the mass to the restrictions $\gamma'_i = \gamma_i[\sigma_i, x'_i]$ of the original curves γ_i . For example, say the chosen orientation is such that the loop should be read left to right. Then one adds ε to the curves γ'_i such that $p_i \xrightarrow{\gamma_i} p_{i+1}$ and one removes ε to the other curves γ'_i such that $p_i \xleftarrow{\gamma_i} p_{i+1}$. Of course, ε must be small enough so as to be able to remove ε to those curves, but we should choose ε and ε_i even more carefully. We do the following:

- if γ_i is a branch we do not restrict the curve, i.e. we set $\varepsilon_i = 0$,
- if γ_i is not a branch, since $\theta_\eta(\gamma_i[\sigma_i, x'_i])$ continuously decreases to 0 as $\varepsilon_i \rightarrow 0$, we may assume that $\theta_\eta(\gamma_i[\sigma_i, x'_i]) = \varepsilon$, with ε smaller than all the masses $\theta_\eta(\gamma_j) > 0$ which correspond to the branches γ_j 's.

To preserve the source and target measures μ, ν , we need to reconnect the point x_i to x'_i for all i . There are several possible cases depending if x_i is also a stem of the adjacent curve or not, and depending on their orientations. But in any case, this reconnection costs less than

$$C \sum_i \varepsilon^\alpha |x_i - x'_i|.$$

For i such that $p_i \xrightarrow{\gamma_i} p_{i+1}$, in which case we add ε , one should pay a cost no larger than

$$\alpha \varepsilon \int_{\gamma'_i} \theta_\eta(x)^{\alpha-1} dx = \alpha \varepsilon Z_\eta(\gamma'_i),$$

and for i such that $p_i \xleftarrow{\gamma_i} p_{i+1}$, we gain at least

$$-\alpha \varepsilon \int_{\gamma'_i} \theta_\eta(x)^{\alpha-1} dx = \alpha \varepsilon Z_\eta(\gamma'_i).$$

Since the total cost increment should be nonnegative by optimality of η , one gets

$$\alpha\varepsilon \sum_{p_i \xrightarrow{\gamma_i} p_{i+1}} Z_\eta(\gamma'_i) - \alpha\varepsilon \sum_{p_i \xleftarrow{\gamma_i} p_{i+1}} Z_\eta(\gamma'_i) + C\varepsilon^\alpha \sum_i |x_i - x'_i| \geq 0,$$

thus

$$\sum_{p_i \xleftarrow{\gamma_i} p_{i+1}} Z_\eta(\gamma'_i) \leq \sum_{p_i \xrightarrow{\gamma_i} p_{i+1}} Z_\eta(\gamma'_i) + C\varepsilon^{\alpha-1} \sum_i |x_i - x'_i|. \quad (4.2.1)$$

It is clear that for all i , $Z_\eta(\gamma'_i) \rightarrow Z_\eta(\gamma_i)$ as $\varepsilon \rightarrow 0$. Moreover since $x \mapsto \theta_\eta(\gamma_i[\sigma_i, x])$ decreases when x tends x_i along the curve and $\theta_\eta(\gamma_i[\sigma_i, x'_i]) = \varepsilon$, one has

$$\varepsilon^{\alpha-1} |x_i - x'_i| \leq \int_{\gamma_i[x'_i, x_i]} \theta_\eta(\gamma_i[\sigma_i, x])^{\alpha-1} dx,$$

which tends to 0 when $x'_i \rightarrow x_i$ because γ_i is η -good. Passing to the limit $\varepsilon \rightarrow 0$ in (4.2.1) yields:

$$\sum_{p_i \xleftarrow{\gamma_i} p_{i+1}} Z_\eta(\gamma_i) \leq \sum_{p_i \xrightarrow{\gamma_i} p_{i+1}} Z_\eta(\gamma_i).$$

Since the converse inequality holds, had we chosen the converse orientation for the loop, we obtain

$$\sum_{p_i \xrightarrow{\gamma_i} p_{i+1}} Z_\eta(\gamma_i) = \sum_{p_i \xleftarrow{\gamma_i} p_{i+1}} Z_\eta(\gamma_i).$$

Since $Z_{p_i}(\gamma_i)$ is $+Z_\eta(\gamma_i)$ in case $p_i \xrightarrow{\gamma_i} p_{i+1}$ and $-Z_\eta(\gamma_i)$ in case $p_i \xleftarrow{\gamma_i} p_{i+1}$, we get the equivalent equation

$$\sum_i Z_{p_i}(\gamma_i) = 0. \quad \square$$

Notice that if we have a good loop

$$\rho_1 \xrightarrow{\gamma_1} x_1 \xrightarrow{\gamma'_1} \rho_2 \xrightarrow{\gamma_2} x_2 \xrightarrow{\gamma'_2} \rho_3 - \dots - \rho_n \xrightarrow{\gamma_n} x_n \xrightarrow{\gamma'_n} \rho_{n+1} = \rho_1$$

where $\rho_1, \dots, \rho_n \in R$, one has by the balance property:

$$\sum_{j=1}^n (Z_{\rho_j}(\gamma_j) - Z_{\rho_{j+1}}(\gamma'_j)) = 0. \quad (4.2.2)$$

Notice that in this expression, each function Z_ρ appears an even number of times, half the time with a $+$ sign, and half the time with a $-$ sign, hence the same balance equation holds if one replaces Z_ρ by $\bar{Z}_\rho = c_\rho + Z_\rho$ for any choice of values $c_\rho \in \mathbb{R}$. Our goal now is to choose these constants c_ρ suitably so as to guarantee that *all terms* in the sum of (4.2.2) are 0:

$$\bar{Z}_\rho(\gamma) - \bar{Z}_\rho(\gamma') = 0,$$

for all good path $\rho \xrightarrow{\gamma} x \xrightarrow{\gamma'} \rho'$ where $\rho, \rho' \in R$ and γ, γ' are one-sided η -good curves.

4.2.2 Construction of the landscape function

For any pair of roots ρ, ρ' , we write $\rho \frown \rho'$ if $\rho \xrightarrow{\gamma} x \xrightarrow{\gamma'} \rho'$ holds for some one-sided η -good curves γ, γ' , and call it a link⁵. We say that ρ, ρ' are connected if there is a chain

$$\rho = \rho_1 \frown \rho_2 \frown \dots \frown \rho_n = \rho'.$$

The connectedness relation is an equivalence relation on R defining equivalence classes which we call connected components. The following lemma will provide us with a good enumeration of all possible links between roots.

Lemma 4.2.3 (Links enumeration). *For any connected component R' of R there is an enumeration of all links between elements of R' :*

$$\rho_1 \frown \rho'_1, \rho_2 \frown \rho'_2, \dots, \rho_n \frown \rho'_n,$$

such that

$$\forall j \leq n, \quad \rho_j \in \bigcup_{i < j} \{\rho_i, \rho'_i\}.$$

Remark 4.2.4. Obviously since all links $\rho \frown \rho$ are in this enumeration, all the roots $\rho \in R'$ appear in it.

Proof. We simply proceed by induction. We start by choosing a link $\rho_1 \frown \rho'_1$ arbitrarily, then having chosen the first $k - 1$ links, we choose the next one $\rho_k \frown \rho'_k$, distinct from the previous ones, such that ρ_k or ρ'_k is an extremity of one the previous links. We do this until there is none left. By construction the first item is satisfied. Moreover, all links have been enumerated: if there was another one left, since R' is a connected component, there would also be one which was not selected but which would share an extremity with one of the previous extremities. This contradicts the fact that we had completed the selection process. \square

Next proposition will define the numbers c_ρ for $\rho \in R'$ using this enumeration, while Proposition 4.2.6 will extend this choice to all $(c_\rho)_{\rho \in \bar{R}}$ in such a way that $\bar{Z}_\rho = c_\rho + Z_\rho$ satisfies the desired property: for x fixed, the quantity $\bar{Z}_\rho(\gamma)$ does not depend on the choice of ρ nor the curve γ with extremity ρ but only on the other extremity x of γ .

Proposition 4.2.5. *Let R' be a connected component of R , $t \in \mathbb{R}$. We denote by*

$$\rho_1 \frown \rho'_1, \rho_2 \frown \rho'_2, \dots, \rho_l \frown \rho'_l$$

the links enumeration of R' given by Lemma 4.2.3. For all i , we choose one-sided η -good curves ξ_i, ξ'_i such that $\rho_i \xrightarrow{\xi_i} x_i \xrightarrow{\xi'_i} \rho'_i$. Then there is a unique choice of values $(c_\rho)_{\rho \in R'}$ such that

$$(i) \quad c_{\rho_1} = t,$$

$$(ii) \quad Z_{\rho_i}(\xi_i) + c_{\rho_i} = Z_{\rho'_i}(\xi'_i) + c_{\rho'_i} \text{ for all } 1 \leq i \leq l.$$

⁵Of course $\rho \frown \rho$ is a link.

Proof. We set $c_{\rho_1} = t$ and proceed by induction. We consider the first link $\rho_1 \xrightarrow{\xi_1} x_1 \xrightarrow{\xi'_1} \rho'_1$. We want

$$c_{\rho_1} + Z_{\rho_1}(\xi_1) = c_{\rho'_1} + Z_{\rho'_1}(\xi'_1) \iff c_{\rho'_1} = c_{\rho_1} + (Z_{\rho_1}(\xi_1) - Z_{\rho'_1}(\xi'_1)).$$

If $\rho'_1 = \rho_1$, then $\rho_1 \xrightarrow{\xi_1} x_1 \xrightarrow{\xi'_1} \rho'_1$ is a good loop and by Corollary 4.2.2, relation (4.2.2) holds for our loop:

$$Z_{\rho_1}(\xi_1) - Z_{\rho'_1}(\xi'_1) = 0.$$

Hence (ii) holds for the first link, as $c_{\rho_1} = c_{\rho'_1}$. If $\rho'_1 \neq \rho_1$, then one may choose $c_{\rho'_1}$ so as to enforce it:

$$c_{\rho'_1} := c_{\rho_1} + (Z_{\rho_1}(\xi_1) - Z_{\rho'_1}(\xi'_1)).$$

And we go on by induction: let us assume that (ii) holds for $i \leq k-1 < l$. We pass on to the next: $\rho_k \xrightarrow{\xi_k} x_k \xrightarrow{\xi'_k} \rho'_k$. By hypothesis, either ρ_k or ρ'_k belongs to the roots already covered, say ρ_k . If $\rho'_k \notin \bigcup_{i < k} \{\rho_i, \rho'_i\}$ then one may enforce the desired property by setting:

$$c_{\rho'_k} := c_{\rho_k} + (Z_{\rho_k}(\xi_k) - Z_{\rho'_k}(\xi'_k)).$$

Now if both $\rho_k, \rho'_k \in \bigcup_{i < k} \{\rho_i, \rho'_i\}$, by hypothesis on the links enumeration, there is a chain made of the previous links with extremities ρ_k, ρ'_k :

$$\rho_k = \tilde{\rho}_1 \xrightarrow{\tilde{\xi}_1} \tilde{x}_1 \xrightarrow{\tilde{\xi}'_1} \tilde{\rho}'_1 - \dots - \tilde{\rho}_l \xrightarrow{\tilde{\xi}_l} \tilde{x}_l \xrightarrow{\tilde{\xi}'_l} \tilde{\rho}'_l = \rho'_k,$$

where each link $\tilde{\rho}_j \xrightarrow{\tilde{\xi}_j} \tilde{x}_j \xrightarrow{\tilde{\xi}'_j} \tilde{\rho}'_j$ is some link of the enumeration $\rho_i \xrightarrow{\xi_i} x_i \xrightarrow{\xi'_i} \rho'_i$ for $i < k$.

We complete the chain with $\rho'_k \xrightarrow{\xi'_k} x_k \xrightarrow{\xi_k} \rho_k$ to obtain a loop:

$$\rho_k = \tilde{\rho}_1 \xrightarrow{\tilde{\xi}_1} \tilde{x}_1 \xrightarrow{\tilde{\xi}'_1} \tilde{\rho}'_1 - \dots - \tilde{\rho}_l \xrightarrow{\tilde{\xi}_l} \tilde{x}_l \xrightarrow{\tilde{\xi}'_l} \tilde{\rho}'_l = \rho'_k \xrightarrow{\xi'_k} x_k \xrightarrow{\xi_k} \rho_k$$

and we apply the balance relation (4.2.2):

$$\begin{aligned} 0 &= \sum_{j \leq l} (Z_{\tilde{\rho}_j}(\tilde{\xi}_j) - Z_{\tilde{\rho}'_j}(\tilde{\xi}'_j)) + Z_{\rho'_k}(\xi'_k) - Z_{\rho_k}(\xi_k), \\ &= \sum_{j \leq l} \left((c_{\tilde{\rho}_j} + Z_{\tilde{\rho}_j}(\tilde{\xi}_j)) - (c_{\tilde{\rho}'_j} + Z_{\tilde{\rho}'_j}(\tilde{\xi}'_j)) \right) \\ &\quad + \left((c_{\rho'_k} + Z_{\rho'_k}(\xi'_k)) - (c_{\rho_k} + Z_{\rho_k}(\xi_k)) \right), \end{aligned}$$

Now by the induction hypothesis, since in the big sum we have differences of cost functions $(Z_{\rho} + c_{\rho}) - (Z_{\rho'} + c_{\rho'})$ associated to previous links $\rho_i \xrightarrow{\xi_i} x_i \xrightarrow{\xi'_i} \rho'_i$ for $i < k$, all these differences are 0. Consequently:

$$0 = (c_{\rho'_k} + Z_{\rho'_k}(\xi'_k)) - (c_{\rho_k} + Z_{\rho_k}(\xi_k)),$$

and (ii) holds for $i \leq k$. We conclude that we can choose the c_{ρ} 's inductively so that it holds for all $i \leq l$, and this choice is unique by construction. \square

Proposition 4.2.6. *There exists a choice of real numbers $(c_\rho)_{\rho \in \bar{R}}$ such that for all link $\rho \xrightarrow{\gamma} x \xrightarrow{\gamma'} \rho'$ with $\rho, \rho' \in \bar{R}$, one has*

$$\bar{Z}_\rho(\gamma) = \bar{Z}_{\rho'}(\gamma'),$$

where $\bar{Z}_\rho = Z_\rho + c_\rho$.

Remark 4.2.7. This choice is unique up to choosing a value c_{ρ_k} for a selection $(\rho_k)_k$ of representatives of the connected components of \bar{R} .

Proof. Consider a point $\bar{\rho} \in \bar{R}$. There is a branch path joining $\bar{\rho}$ to some root $\rho \in R$:

$$\rho = x_1 \xrightarrow{\gamma_1} x_2 \xrightarrow{\gamma_2} \dots x_{n+1} = \bar{\rho},$$

where we recall by definition that the γ_i are branches. We would like to set

$$c_{\bar{\rho}} = c_\rho + \sum_{i=1}^n Z_{x_i}(\gamma_i),$$

but first we need to check that this is independent of the chosen branch path. Take another one

$$\rho = x'_1 \xrightarrow{\gamma'_1} x'_2 \xrightarrow{\gamma'_2} \dots x'_{n'+1} = \bar{\rho},$$

which leads to a loop

$$\rho = x_1 \xrightarrow{\gamma_1} x_2 \xrightarrow{\gamma_2} \dots x_{n+1} = \bar{\rho} = x'_{n'+1} \xrightarrow{\gamma'_n} \dots x'_2 \xrightarrow{\gamma'_1} x'_1 = \rho.$$

By Corollary 4.2.2 one has

$$\sum_{i=1}^n Z_{x_i}(\gamma_i) = \sum_{i=1}^{n'} Z_{x'_i}(\gamma'_i),$$

which is what we wanted. Now let us show that with this choice of $(c_\rho)_{\rho \in \bar{R}}$, the desired equality holds. Take $\bar{\rho}, \bar{\rho}' \in \bar{R}$ and two one-sided η -good curves γ, γ' with extremities $\bar{\rho}, x$ and $\bar{\rho}', x$ respectively. We know that there are roots $\rho, \rho' \in R$ and branch paths such that:

$$\rho = x_1 \xrightarrow{\gamma_1} x_2 \xrightarrow{\gamma_2} \dots x_{n+1} = \bar{\rho} \quad \text{and} \quad \rho' = x'_1 \xrightarrow{\gamma'_1} x'_2 \xrightarrow{\gamma'_2} \dots x'_{n'+1} = \bar{\rho}'.$$

Thus, the roots ρ, ρ' are in a common connected component R' , and one may find a chain

$$\rho = \tilde{\rho}_1 \frown \tilde{\rho}_2 \frown \dots \frown \tilde{\rho}_{\tilde{n}} = \rho',$$

where for all $i \leq \tilde{n}$, $\tilde{\rho}_i \frown \tilde{\rho}_{i+1}$ is some link $\rho_{j_i} \xrightarrow{\xi_{j_i}} x_j \xrightarrow{\xi'_{j_i}} \rho'_{j_i}$ of the enumeration given by Proposition 4.2.5. Here again, we build a loop by concatenating the curves γ, γ' and three last chains together, which we write in short:

$$\rho' \xrightarrow{\gamma'_i} \bar{\rho}' \xrightarrow{\gamma'} x \xrightarrow{\gamma} \bar{\rho} \xrightarrow{\gamma_i} \rho \xrightarrow{\xi_{j_i} \xi'_{j_i}} \rho'.$$

By the balance property, one has:

$$\sum_i Z_{x'_i}(\gamma'_i) + Z_{\bar{\rho}'}(\gamma') - Z_{\bar{\rho}}(\gamma) + \sum_i Z_{x_{i+1}}(\gamma_i) + \sum_i (Z_{\rho_{j_i}}(\xi_{j_i}) - Z_{\rho'_{j_i}}(\xi'_{j_i})) = 0$$

which rewrites

$$c_{\bar{\rho}'} - c_{\rho'} + Z_{\bar{\rho}'}(\gamma') - Z_{\bar{\rho}}(\gamma) - (c_{\bar{\rho}} - c_{\rho}) + \sum_i (Z_{\rho_{j_i}}(\xi_{j_i}) - Z_{\rho'_{j_i}}(\xi'_{j_i})) = 0$$

or equivalently

$$\bar{Z}_{\bar{\rho}'}(\gamma') - \bar{Z}_{\bar{\rho}}(\gamma) = c_{\rho'} - c_{\rho} - \sum_i (Z_{\rho_{j_i}}(\xi_{j_i}) - Z_{\rho'_{j_i}}(\xi'_{j_i})). \quad (4.2.3)$$

Now notice that in the sum $Z_{\rho_{j_i}}$ appears the same amount of time with a + sign than with a - sign, except for the extreme terms $\bar{\rho}_{j_i} = \rho$ and $\bar{\rho}_{j_i} = \rho'$: Z_{ρ} appears once more with a + sign, and $Z_{\rho'}$ once more with a -. Hence recalling that $\bar{Z}_{\bar{\rho}} := c_{\bar{\rho}} + Z_{\bar{\rho}}$, (4.2.3) rewrites

$$\bar{Z}_{\bar{\rho}'}(\gamma') - \bar{Z}_{\bar{\rho}}(\gamma) = - \sum_i (\bar{Z}_{\rho_{j_i}}(\xi_{j_i}) - \bar{Z}_{\rho'_{j_i}}(\xi'_{j_i})),$$

but we know by Proposition 4.2.5 that $\bar{Z}_{\rho_{j_i}}(\xi_{j_i}) = \bar{Z}_{\rho'_{j_i}}(\xi'_{j_i})$ for all i , hence:

$$\bar{Z}_{\bar{\rho}}(\gamma) = \bar{Z}_{\bar{\rho}'}(\gamma'). \quad \square$$

Definition 4.2.8 (Landscape function). The previous proposition allows us to define for all optimal irrigation plan η with finite cost a landscape function $z : \text{Bas}_{\eta}(\bar{R}) \rightarrow \mathbb{R}$ by:

$$z(x) = \bar{Z}_{\rho}(\gamma)$$

if γ is a one-sided η -good curve joining $\rho \in \bar{R}$ and x .

4.3 Basic properties

In this section we prove the basic properties of the landscape function which were already known in the single-source case: semicontinuity, expression of the optimal cost and of the variation of d_{α} in terms of z .

4.3.1 Fundamental property

Beforehand, we establish what we call the fundamental property of the landscape function: it is in a sense a primitive of $\theta_{\eta}^{\alpha-1}$ along η .

Proposition 4.3.1 (Fundamental property). *If γ is an η -good curve then*

$$Z_{\eta}(\gamma) \doteq \int_{\gamma} \theta_{\eta}(x)^{\alpha-1} dx = z(\gamma^+) - z(\gamma^-).$$

Proof. The result holds by definition for one-sided η -good curves joining a point $\rho \in \bar{R}$ to any other point x . By subtraction it is clear that it therefore holds for any restriction of such curves. Noticing a telescopic sum, this shows that it holds also for curves which can be decomposed as a concatenation of such restrictions, thus by Lemma 4.1.13, it holds for any one-sided η -good curve, no matter the stem. Since a two-sided η -good curve is the concatenation of two one-sided η -good curves, we get the result for any η -good curve. \square

As a corollary, we get the same property for η -good paths. Recall that an oriented path p induces a curve γ_p obtained by concatenating the curves of p .

Corollary 4.3.2. *If p is an oriented η -good path, then*

$$Z_\eta(\gamma_p) \doteq \int_{\gamma_p} \theta_\eta(x)^{\alpha-1} dx = z(\gamma_p^+) - z(\gamma_p^-).$$

4.3.2 Semicontinuity

We define

$$\text{Bas}_\eta^+(\bar{R}) = \bigcup_{\rho \in \bar{R}} \text{Bas}_\eta^+(\rho) \quad \text{and} \quad \text{Bas}_\eta^-(\bar{R}) = \bigcup_{\rho \in \bar{R}} \text{Bas}_\eta^-(\rho).$$

We are going to prove that the landscape function z defined in the previous section is Lipschitz continuous on the root network \bar{R} , lower semicontinuous on the positive basin $\text{Bas}_\eta^+(\bar{R})$ and upper semicontinuous in the negative basin $\text{Bas}_\eta^-(\bar{R})$.

Lemma 4.3.3. *If η is an irrigation plan with finite \mathbf{I}^α cost, the map $(t, \gamma) \mapsto \theta_\eta(\gamma|_{[0,t]})$ is upper semicontinuous on $\mathbb{R}^+ \times \Gamma$.*

Proof. Take $t_n \rightarrow t$ and $\gamma_n \rightarrow \gamma$. Fix $\varepsilon > 0$. Since θ_η stays unchanged after reparameterization of η , we assume that for almost all curves γ , $T^-(\gamma) = 0$ and γ is parameterized by arc-length. We know that for n large enough $t_n \geq t - \varepsilon$, so that

$$\limsup_n \theta_\eta(\tilde{\gamma} : \gamma_n|_{[0,t_n]} \subseteq \tilde{\gamma}) \leq \limsup_n \eta(\tilde{\gamma} : \gamma_n|_{[0,t-\varepsilon]} \subseteq \tilde{\gamma}).$$

We shall prove that

$$\limsup_n \eta(\tilde{\gamma} : \gamma_n|_{[0,t-\varepsilon]} \subseteq \tilde{\gamma}) \leq \eta(\tilde{\gamma} : \gamma|_{[0,t-\varepsilon]} \subseteq \tilde{\gamma}).$$

Let us set

$$F_k = \{\tilde{\gamma} : \exists t_0, \forall s \in [0, t - \varepsilon], |\tilde{\gamma}(t_0 + s) - \gamma(s)| \leq 1/k\}.$$

It is a decreasing sequence of subsets of Γ . Let $\tilde{\gamma} \in \Gamma^1$ belonging to $\bigcap_k^\downarrow F_k$ and being parameterized by arc-length, with length T . There is a sequence $t_k \in [0, T]$ such that

$$\forall s \in [0, t - \varepsilon], \quad |\tilde{\gamma}(t_k + s) - \gamma(s)| \leq 1/k.$$

Up to subsequence $t_k \rightarrow \bar{t} \in [0, T]$, so that

$$\forall s \in [0, t - \varepsilon], \quad \tilde{\gamma}(\bar{t} + s) = \gamma(s),$$

and $\gamma|_{[0,t-\varepsilon]} \subseteq \tilde{\gamma}$, which means that

$$\bigcap_k^\downarrow F_k = F := \{\tilde{\gamma} : \gamma|_{[0,t-\varepsilon]} \subseteq \tilde{\gamma}\}. \quad (4.3.1)$$

Since $\gamma_n \rightarrow \gamma$, we know that $\gamma_n \in F_k$ for n large enough. Consequently

$$\limsup_n \eta(\tilde{\gamma} : \gamma_n|_{[0,t-\varepsilon]} \subseteq \tilde{\gamma}) \leq \eta(F_k).$$

Using the monotone convergence theorem and (4.3.1),

$$\limsup_n \eta(\tilde{\gamma} : \gamma_n|_{[0,t-\varepsilon]} \subseteq \tilde{\gamma}) \leq \lim_{k \rightarrow \infty} \eta(F_k) = \eta(F) = \eta(\tilde{\gamma} : \gamma|_{[0,t-\varepsilon]} \subseteq \tilde{\gamma}),$$

thus

$$\limsup_n \eta(\tilde{\gamma} : \gamma_n|_{[0,t_n]} \subseteq \tilde{\gamma}) \leq \eta(\tilde{\gamma} : \gamma|_{[0,t-\varepsilon]} \subseteq \tilde{\gamma}).$$

Passing to the limit $\varepsilon \rightarrow 0$ we get by the monotone convergence theorem :

$$\limsup_n \eta(\tilde{\gamma} : \gamma_n|_{[0,t_n]} \subseteq \tilde{\gamma}) \leq \eta(\tilde{\gamma} : \gamma|_{[0,t]} \subseteq \tilde{\gamma})$$

or equivalently

$$\limsup_n \theta_\eta(\gamma_n|_{[0,t_n]}) \leq \theta_\eta(\gamma|_{[0,t]}),$$

which is what we wanted to prove. \square

Lemma 4.3.4. *The map $\gamma \mapsto \int_0^{T^+(\gamma)} \theta_\eta(\gamma|_{[0,t]})^{\alpha-1} dt$ defined on curves $\Gamma \ni \gamma : [0, T^+(\gamma)] \rightarrow \mathbb{R}^d$ is lower semicontinuous.*

Proof. Set

$$f(\gamma) := \int_0^{T^+(\gamma)} \theta_\eta(\gamma|_{[0,t]})^{\alpha-1} dt.$$

The map $(t, \gamma) \mapsto \theta_\eta(\gamma|_{[0,t]})^{\alpha-1}$ being lower semicontinuous, as well as $\gamma \mapsto T^+(\gamma)$, this readily implies that f is lower semicontinuous. \square

Before proving the semicontinuity of the landscape function, we first need a continuity result on the root network.

Lemma 4.3.5. *The landscape function z is Lipschitz continuous on the root network \bar{R} .*

Proof. We know that \bar{R} is made of a finite union \mathcal{B} of branches (those belonging to a simple path joining two roots). We set $M > 0$ the minimal multiplicity of those branches. Take two points $x, y \in \bar{R}$. We know that there is a path

$$x = p_1 \xrightarrow{\gamma_1} p_2 \xrightarrow{\gamma_2} \dots p_{n+1} = y$$

where γ_i is a restriction of a branch belonging to \mathcal{B} for all i . We remove a mass M to all curves such that $p_i \xrightarrow{\gamma_i} p_{i+1}$, which is possible because all branch have a multiplicity larger than M , and add a mass M to those for which $p_i \not\xrightarrow{\gamma_i} p_{i+1}$, and sends a mass M

for x to y along a straight line, with cost no larger than $CM^\alpha|y - x|$. Since the cost increment must be nonnegative, one has

$$-M \sum_i Z_{p_i}(\gamma_i) + CM^\alpha|y - x| \geq 0.$$

Recalling that $z(y) - z(x) = \sum_i Z_{p_i}(\gamma_i)$, we obtain:

$$z(y) - z(x) \leq CM^{\alpha-1}|y - x|.$$

Interchanging the roles of x and y yields

$$|z(y) - z(x)| \leq CM^{\alpha-1}|y - x|,$$

and z is Lipschitz continuous on \bar{R} . □

Proposition 4.3.6 (Semicontinuity). *The landscape function is lower semicontinuous on $\text{Bas}_\eta^+(\bar{R})$ and upper semicontinuous on $\text{Bas}_\eta^-(\bar{R})$.*

Remark 4.3.7. If μ and ν have disjoint compact supports, one may define an extension of z in a neighbourhood of $\text{supp } \mu$ and $\text{supp } \nu$ which is upper semicontinuous on the neighborhood of $\text{supp } \mu$ and lower semicontinuous on the neighborhood of $\text{supp } \nu$.

Proof. We assume that η is parameterized by arc length. Take a sequence $x_n \rightarrow x \in \text{Bas}_\eta^+(\bar{R})$. We want to prove that

$$z(x) \leq \liminf_n z(x_n).$$

We assume that $\liminf_n z(x_n) < \infty$, otherwise there is nothing to prove. Also, up to subsequence, we may assume that $z(x_n) \rightarrow \ell = \liminf_n z(x_n)$ and $z(x_n) \leq M < \infty$. We know that for all n , x_n is reached by some right-sided η -good curve γ_n stemming from a point $\rho_n \in \bar{R}$. By Lemma 4.1.15 and by restricting the curve we may assume that $\rho_n = \rho^+(\gamma_n)$. Since \bar{R} is compact, up to subsequence we may assume that $\rho_n \rightarrow \rho \in \bar{R}$. Moreover up to reparameterization we may assume that $T^-(\gamma_n) = 0$ and that γ_n is parameterized by arc length. Consequently by the fundamental property of Proposition 4.3.1, one has

$$z(x_n) - z(\rho_n) = Z_\eta(\gamma_n).$$

Since z is continuous on \bar{R} and Z_η is lower semicontinuous one has

$$Z_\eta(\gamma) \leq \liminf_n Z_\eta(\gamma_n) = \liminf_n z(x_n) - z(\rho).$$

It is clear that $\gamma^- = \rho, \gamma^+ = x$. Now we want to check that γ is an η -good curve. Since $\theta_\eta(y) = \theta_\eta(\gamma_n[\rho_n, y])$ for $y \in \gamma_n[\rho_n, x]$ by Lemma 4.1.15, one has

$$Z_\eta(\gamma_n) = \int_{\gamma_n} \theta_\eta(y)^{\alpha-1} dy = \int_0^{T^+(\gamma_n)} \theta_\eta(\gamma_n|_{[0,t]})^{\alpha-1} dt.$$

As $z(\rho_n) \rightarrow z(\rho)$ and $z(x_n) \leq M$ we have $Z_\eta(\gamma_n) \leq C$ for some constant $C > 0$. By the previous proposition, the right-hand side is lower semicontinuous with respect to γ_n thus

$$\int_0^{T^+(\gamma)} \theta_\eta(\gamma|_{[0,t]})^{\alpha-1} dt \leq \liminf_n Z_\eta(\gamma_n) \leq C.$$

Since η is concentrated on simple curves, γ is injective on $[0, T^+(\gamma)[$, and if it is not injective on the whole $[0, T^+(\gamma)]$, we restrict γ to the first time it reaches γ^+ , obtaining a curve γ' such that

$$\int_{\gamma'} \theta_\eta(\gamma'[\rho, y])^{\alpha-1} dy \leq \int_0^{T^+(\gamma)} \theta_\eta(\gamma|_{[0,t]})^{\alpha-1} dt \leq C < \infty$$

and γ' is η -good. Therefore by Proposition 4.3.1 we obtain:

$$z(x) - z(\rho) = Z_\eta(\gamma) \leq \liminf_n z(x_n) - z(\rho)$$

and z is lower semicontinuous on $\text{Bas}_\eta^+(\bar{R})$. We can do the same reasoning to get the upper semicontinuity of z on $\text{Bas}_\eta^-(\bar{R})$, noticing that if γ is a left-sided η -good curve reaching x from a point ρ one has a minus sign:

$$z(x) - z(\rho) = -Z_\eta(\gamma). \quad \square$$

4.3.3 Formula for the optimal cost

Proposition 4.3.8. *If $\eta \in \mathbf{IP}(\mu, \nu)$ is an optimal irrigation plan with finite \mathbf{I}^α cost and z is an associated landscape function, then $z \in L^1(\mu) \cap L^1(\nu)$ and*

$$\mathbf{I}^\alpha(\eta) = \int_{\mathbb{R}^d} z d(\nu - \mu).$$

Proof. We know by Proposition 4.1.10 that η -almost every curve γ is an η -good curve, hence by the fundamental property stated in Proposition 4.3.1, one has:

$$Z_\eta(\gamma) = z(\gamma^+) - z(\gamma^-).$$

Moreover for all $\rho \in R$:

$$\begin{aligned} \int_{\Gamma(\rho)} (z(\gamma^+) - z(\rho)) d\eta(\gamma) &= \int_{\Gamma(\rho)} Z_\eta(\gamma[\rho, +]) d\eta(\gamma) \\ &\leq \int_{\Gamma(\rho)} Z_\eta(\gamma) d\eta(\gamma) \\ &= \mathbf{I}^\alpha(\eta) < \infty. \end{aligned}$$

Since R is a finite root system, this implies that $z \circ \pi_\infty$ is η -summable thus $z \in L^1(\nu)$. The same reasoning gives that $z \in L^1(\mu)$. Finally, again by Proposition 4.1.10, we have:

$$\begin{aligned} \mathbf{I}^\alpha(\eta) &= \int_\Gamma Z_\eta(\gamma) d\eta(\gamma) = \int_\Gamma (z(\gamma^+) - z(\gamma^-)) d\eta(\gamma) \\ &= \int_{\mathbb{R}^d} z d\nu - \int_{\mathbb{R}^d} z d\mu. \end{aligned} \quad \square$$

4.3.4 First variation formula

In the single-source case, z is the first variation of the functional $\mathbf{X}^\alpha = d_\alpha(\delta_0, \cdot)$. In the multiple source case, we want to show that it is in a way the first variation of d_α , when we do perturbations of μ or ν . However, we must restrict these perturbations to happen only in a given basin $\text{Bas}_\eta(\sigma)$. Indeed, to estimate the cost increment, we need to perturb the underlying irrigation plan $\eta \in \mathbf{OIP}^\alpha(\mu, \nu)$, but in the single-source case all curves passed by a same point $s = 0$, hence to perturb ν one could redirect the curves going to a set A to another set B using the common junction at s , without having to modify μ in consequence. In the multiple source case, this fails in general: changing the measure ν may force us to change the measure μ as well. If we impose that the changes are made inside a basin $\text{Bas}_\eta(\sigma)$, then we can use the common junction σ to do whatever we want with μ and ν independently.

Theorem 4.3.9 (First variation formula). *Let $\eta \in \mathbf{IP}(\mu, \nu)$ be an optimal irrigation plan with finite \mathbf{I}^α cost and z an associated landscape function. If $\mu', \nu' \in \mathcal{M}_+^1(\mathbb{R}^d)$ are such that $\mu - \mu'$ and $\nu - \nu'$ are concentrated in an irrigation basin $\text{Bas}_\eta(\sigma)$ then:*

$$d_\alpha(\mu', \nu') \leq d_\alpha(\mu, \nu) + \alpha \int_{\mathbb{R}^d} z d((\nu' - \nu)) - (\mu' - \mu),$$

provided $z \in L^1(\nu') \cap L^1(\mu')$.

Proof. We are going to use the first variation formula of \mathbf{I}^α stated in 1.2.1:

$$\mathbf{I}^\alpha(\tilde{\eta}) \leq \mathbf{I}^\alpha(\eta) + \alpha \int_{\Gamma} Z_\eta d(\tilde{\eta} - \eta). \quad (4.3.2)$$

Since $\mu - \mu', \nu - \nu'$ are concentrated in a common irrigation basin $\text{Bas}_\eta(\sigma)$, $|\mu - \mu'|$ -almost every points x and $|\nu - \nu'|$ -almost every points y are reached by right-sided η -good curve γ_x, γ_y respectively, which stem from σ . We can modify the irrigation plan η by removing, adding and redirecting mass which flows through curves visiting σ , using gluings as in the proofs of Proposition 1.2.16 and Theorem 1.2.18, to obtain an irrigation plan $\eta' \in \mathbf{IP}(\mu', \nu')$ which is concentrated on η -good paths.

As a consequence, by the fundamental property, one has $Z_\eta(\gamma) = z(\gamma^+) - z(\gamma^-)$ for η' -almost every γ . Since $z \in L^1(\mu') \cap L^1(\nu')$, it yields:

$$\int_{\Gamma} Z_\eta(\gamma) d\eta'(\gamma) = \int_{\Gamma} (z(\gamma^+) - z(\gamma^-)) d\eta'(\gamma) = \int_{\mathbb{R}^d} z d\nu' - \int_{\mathbb{R}^d} z d\mu'.$$

Finally, plugging this in (4.3.2), knowing that $\mathbf{I}^\alpha(\eta) = d_\alpha(\mu, \nu)$ and that $d_\alpha(\mu', \nu') \leq \mathbf{I}^\alpha(\eta')$, one gets

$$\begin{aligned} d_\alpha(\mu', \nu') &\leq \mathbf{I}^\alpha(\eta') \leq \mathbf{I}^\alpha(\eta) + \alpha \int_{\Gamma} Z_\eta d(\eta' - \eta) \\ &= d_\alpha(\mu, \nu) + \alpha \left(\int_{\mathbb{R}^d} z d\nu' - \int_{\mathbb{R}^d} z d\mu' \right) \\ &\quad - \alpha \left(\int_{\mathbb{R}^d} z d\nu - \int_{\mathbb{R}^d} z d\mu \right). \quad \square \end{aligned}$$

This theorem will be the essential tool in proving the Hölder regularity of the landscape function with some extra hypotheses, although other ingredients will be needed. Indeed, as we will do general perturbations which do not occur in a single basin, we will need to perform small transfers between basins with controlled cost.

4.4 Hölder regularity

In this section, we want to prove the regularity of the landscape function we have defined under some extra hypotheses on the measures μ and ν . Our assumption will be that μ and ν are probability measures of the form:

$$\mu = f \mathcal{L}|_U \quad \text{and} \quad \nu = g \mathcal{L}|_V,$$

where f and g are functions bounded from below by a constant $c_0 > 0$, and U, V are open connected sets which are Ahlfors regular in the sense that for some $r_0 > 0$ and all $r \leq r_0$:

$$\forall x \in U, \Theta_U(x, r) \geq c_0, \quad \text{and} \quad \forall y \in V, \Theta_V(y, r) \geq c_0.$$

Moreover, for this whole section we assume that $\eta \in \mathbf{OIP}^\alpha(\mu, \nu)$ is an optimal irrigation plan with $\alpha \in]1 - 1/d, 1[$ which has a finite root system R . We denote by z a landscape function as constructed in Section 4.2.

Before going further, we will need some new definitions. Recall that the root network \bar{R} is the union of all simple branch paths joining pairs of roots $\rho, \rho' \in R$. For each connected component P of \bar{R} , we pick once and for all a root $\rho_P \in P \cap R$ and we denote by \mathcal{P} the finite collection of these connected components. Each connected component $P \in \mathcal{P}$ is associated with a basin $\text{Bas}(P)$ and its signed counterparts $\text{Bas}(P)^\pm$ as follows:

$$\text{Bas}^+(P) = \bigcup_{\rho \in P} \text{Bas}_\eta^+(\rho), \quad \text{Bas}^-(P) = \bigcup_{\rho \in P} \text{Bas}_\eta^-(\rho), \quad \text{and} \quad \text{Bas}(P) = \bigcup_{\rho \in P} \text{Bas}_\eta(\rho).$$

These basins, which we call *rooted basins*, provide a decomposition of the measures μ, ν :

- (i) $\text{Bas}_\eta^+(\bar{R}) = \bigcup_{P \in \mathcal{P}} \text{Bas}^+(P)$ and $\text{Bas}_\eta^-(\bar{R}) = \bigcup_{P \in \mathcal{P}} \text{Bas}^-(P)$,
- (ii) for all $P \in \mathcal{P}$, $\mu(\text{Bas}^-(P)) > 0$ and $\nu(\text{Bas}^+(P)) > 0$,
- (iii) μ and ν are respectively concentrated on $\text{Bas}_\eta^-(\bar{R})$ and $\text{Bas}_\eta^+(\bar{R})$.

4.4.1 Transfer between basins

Definition 4.4.1. Let A, B be two Borel sets of positive measure $|A|, |B| > 0$. We call x a transfer point from A to B if for some constants $c, R > 0$ and all $r \leq R$:

$$\Theta_A(x, r) \geq c, \quad \text{and} \quad x \in \bar{B}.$$

4.4. HÖLDER REGULARITY

Lemma 4.4.2. *If $|U \setminus (A \cup B)| = 0$ with $|A|, |B| > 0$ and $|A \cap B| = 0$, then there is a point x such that*

$$\Theta_A(x, r) \geq c \quad \text{and} \quad \Theta_B(x, r) \geq c,$$

for $r \leq R$ and some constants $c = c_{A,B} > 0, R = R_{A,B} > 0$. In particular x is a transfer point from A to B and from B to A .

Proof. Consider two points x, y which are Lebesgue points of A and B respectively, and a small radius r_0 such that $\Theta_A^\square(x, r) := \frac{|Q_r(x) \cap A|}{(2r)^d} > 1/2$ and $\Theta_B^\square(x, r) := \frac{|Q_r(x) \cap B|}{(2r)^d} > 1/2$ for $r = r_0$, where $Q_r(x)$ is the cube centered at x with radius r . Since U is open and connected, there is a curve $\gamma : [0, 1] \rightarrow U$ which joins x to y . If the radius r_0 is small enough, the cubes $Q_{r_0}(\gamma(t))$ stay inside U as well, and the map

$$f(t) = \Theta_A^\square(\gamma(t), r_0)$$

is continuous on $[0, 1]$, with $f(0) > 1/2$ and $f(1) < 1/2$. Consequently there is a time t_0 and $x_0 = \gamma(t_0)$ such that

$$\Theta_A^\square(x_0, r_0) = \Theta_B^\square(x_0, r_0) = 1/2.$$

Now we take $r = r_0/2$ and we want to find a cube $Q_r(x) \subseteq Q_{r_0}(x_0)$ such that the same equality holds: $\Theta_A^\square(x, r) = \Theta_B^\square(x, r) = 1/2$. Cut the cube $Q_{r_0}(x_0)$ into 2^d subcubes Q^i of radius r and centers x^i . If the equality holds for one x^i we set $x = x^i$. Otherwise, one could find two cubes Q^i, Q^j with $i \neq j$ such that $\Theta_A^\square(x^i, r) > 1/2$ and $\Theta_A^\square(x^j, r) < 1/2$. Now we can reproduce the starting argument to find an x lying in the segment $[x^i, x^j]$ such that $\Theta_A^\square(x, r) = 1/2$. We set $x_1 = x, r_1 = r$, and we keep going to produce a sequence of cubes $Q^n = Q_{r_n}(x_n)$ such that each is included in the previous one, with $r_n = r_0 2^{-n}$ and such that

$$\Theta_A^\square(x_n, r_n) = \Theta_B^\square(x_n, r_n) = 1/2.$$

The sequence x_n is Cauchy and converges to a point x which is the intersection of all these cubes. We claim that there is a constant $c > 0$ such that $\Theta_A(x, r) \geq c$ and $\Theta_B(x, r) \geq c$ for all $r \leq R$ (recall that this density is computed with respect to balls and not cubes). Take n such that $r_{n+1} < r/2\sqrt{d} \leq r_n$. One knows that $x \in Q^{n+1}$ hence $B_r(x) \supseteq Q^{n+1}$, thus:

$$\Theta_A(x, r) \geq \frac{|A \cap Q^{n+1}|}{|B_r(x)|} = 1/2 \frac{|Q^{n+1}|}{|B_r(x)|} \geq 1/2 \frac{2^d}{\omega_d} \left(\frac{r_n/2}{r} \right)^d \geq c > 0.$$

The same holds for B and we are done. \square

We set \mathcal{X}^- (resp. \mathcal{X}^+) to be the collection of sets A made of a union of negative basins (resp. positive basins) associated to components $P \in \mathcal{P}$, that is sets of the form $A = \bigcup_{P \in \mathcal{P}'} \text{Bas}^-(P)$ (resp. $A = \bigcup_{P \in \mathcal{P}'} \text{Bas}^+(P)$) where $\mathcal{P}' \subseteq \mathcal{P}$. In the rest of this section, all results will be stated and proved for negative basins but their positive counterparts hold as well ; hence for this section we write $\mathcal{X} = \mathcal{X}^-$, $X = \text{Bas}_\eta^-(\bar{R})$ and $X_P = \text{Bas}^-(P)$ for $P \in \mathcal{P}$.

Definition 4.4.3 (Basin variation of μ). For each $P \in \mathcal{P}$, suppose that we have two nonnegative measures χ_P, ξ_P which are concentrated on X_P and which represent a mass that we want to remove from μ and add to μ respectively. Assume that $\sum_P \chi_P \leq \mu$ and set $\mu' = \mu + \sum_P (\xi_P - \chi_P)$ which is a nonnegative measure. Then the triple $(\mu', (\chi_P)_P, (\xi_P)_P)$ is called a basin variation of μ .

Definition 4.4.4 (Transfer relation). We define a binary relation on \mathcal{X} as follows: $A \rightsquigarrow B$ if there exists constants $K = K_{A,B} > 0, \varepsilon = \varepsilon_{A,B} > 0$ such that for all $m \leq \varepsilon$ one can find a basin variation $(\mu', (\chi_P), (\xi_P))$ of μ such that:

- (i) $\sum_P \chi_P(X_P) + \xi_P(X_P) \leq Km$,
- (ii) for all P such that $X_P \not\subseteq A \cup B$, $\chi_P(X_P) = \xi_P(X_P)$,
- (iii) $\sum_{P: X_P \subseteq A} (\xi_P - \chi_P)(X_P) = -m$ and $\sum_{P: X_P \subseteq B} (\xi_P - \chi_P)(X_P) = +m$,
- (iv) $|\int z d(\mu' - \mu)| \leq cm^{1+\beta/d}$,
- (v) $d_\alpha(\mu, \mu') \leq cm^{1+\beta/d}$.

In that case we say that the basin variation $(\mu', (\chi_P), (\xi_P))$ transfers a mass m from A to B with controlled cost.

Lemma 4.4.5. *Let x be a transfer point from A to B . The following holds:*

- there exists $R = R_{A,B} > 0$ and $C = C_{A,B} > 0$ such that for all $r \leq R$,

$$\int_{B_r(x) \cap A} |z - z(x)| dx \leq Cr^\beta,$$

- if $A, B \in \mathcal{X}$ then $A \rightsquigarrow B$.

Remark 4.4.6. The basin variation $(\mu', (\chi_P), (\xi_P))$ satisfying the requirements for $A \rightsquigarrow B$ as in Definition 4.4.4 that we get from the proof is the following: $\chi_P = c\mathbf{1}_{X_P \cap B_r(x)}$ for all $P \in \mathcal{P}$ with $r^d \approx m$, $\xi_{P_0} = m\delta_x$, and $\xi_P = 0$ if $P \neq P_0$; P_0 being such that x belongs to $X_{P_0} \subseteq B$. Moreover $R_{A,B}^d \approx \varepsilon_{A,B}$.

Proof. We set $z_{r,A}(x) = \int_{B_r(x) \cap A} z(y) dy$ and $m_r = |B_r(x) \cap A|$. The proof strongly relies on computations made in [San07] to prove the Hölder continuity of the landscape function in the single-source case, and on calculations which are quite classical when working with Campanato spaces (see [Giu03, Section 2.3]). We divide the proof into two steps.

Step 1: $z(x) = z_A(x) := \lim_{r \rightarrow 0} z_{r,A}(x)$. Recalling that $\mu \geq c_0\mathbf{1}_U$, we remove $c_0\mathbf{1}_{B_r(x) \cap A}$ from μ and we send a mass c_0m_r to x , then we spread it again to $B_r(x) \cap A$. To be more precise, we set $\tilde{\mu} = \mu - c_0\mathbf{1}_{B_r(x) \cap A} + c_0m_r\delta_x$. Since $\text{diam}(\text{supp}(\mu - \tilde{\mu})) \leq r$ and $\|\mu - \tilde{\mu}\| \leq 2c_0m_r$, we know that

$$|d_\alpha(\tilde{\mu}, \nu) - d_\alpha(\mu, \nu)| \leq Crm_r^\alpha, \quad (4.4.1)$$

while on the other hand by the first variation formula of Theorem 4.3.9 one has:

$$\begin{aligned} d_\alpha(\tilde{\mu}, \nu) &\leq d_\alpha(\mu, \nu) + \alpha \int_{\mathbb{R}^d} -z \, d(\tilde{\mu} - \mu) \\ &= d_\alpha(\mu, \nu) - \alpha c_0 m_r z(x) + \alpha c_0 \int_{B_r(x) \cap A} z(y) \, dy. \end{aligned} \quad (4.4.2)$$

Putting (4.4.1) and (4.4.2) together yields:

$$\alpha c_0 m_r z(x) - \alpha c_0 \int_{B_r(x) \cap A} z(y) \, dy \leq C r m_r^\alpha$$

or equivalently

$$z(x) - \int_{B_r(x) \cap A} z(y) \, dy \leq C r m_r^{\alpha-1}. \quad (4.4.3)$$

But

$$C r m_r^{\alpha-1} = C \frac{r^\beta}{\Theta_A(x, r)^{1-\alpha}} \leq C r^\beta, \quad (4.4.4)$$

the last inequality holding for $r \leq R_{A,B}$ (with C depending on $c_{A,B}$) because x is a transfer point from A to B . By (4.4.3) and (4.4.4), one has

$$z(x) \leq z_{r,A}(x) + C r^\beta,$$

then passing to the limit $r \rightarrow 0$:

$$z(x) \leq \liminf_{r \rightarrow 0} z_{r,A}(x).$$

By upper semicontinuity of z on $\text{Bas}_\eta^-(\bar{R})$ one has the opposite inequality:

$$\limsup_{r \rightarrow 0} z_{r,A}(x) \leq z(x),$$

hence

$$z(x) = z_A(x) \doteq \lim_{r \rightarrow 0} z_{r,A}(x). \quad (4.4.5)$$

Step 2: $\int_{B_r(x) \cap A} |z - z(x)| \leq C r^\beta$. We cut $B_r(x) \cap A$ into two parts of equal measure:

$$B_r(x) \cap A = A_1 \sqcup A_2 \quad \text{where} \quad |A_1| = |A_2| = m_r/2.$$

Removing $c_0 \mathbf{1}_{A_1}$ and adding $c_0 \mathbf{1}_{A_2}$ to μ , one gets by the same kind of estimates we used in the first step:

$$C r m_r^\alpha + c_0 \alpha \int_{A_1} z - c_0 \alpha \int_{A_2} z \geq 0. \quad (4.4.6)$$

We denote by $\tilde{z}_{r,A}(x)$ the median of z on $B_r(x) \cap A$. Taking A_1, A_2 of equal measure such that $z \leq \tilde{z}_{r,A}(x)$ on A_1 and $z \geq \tilde{z}_{r,A}(x)$ on A_2 , we obtain

$$\int_{A_2} z - \int_{A_1} z = \int_{A_2} (z - \tilde{z}_{r,A}(x)) - \int_{A_1} (z - \tilde{z}_{r,A}(x)) = \int_{B_r(x) \cap A} |z - \tilde{z}_{r,A}(x)|,$$

and putting this into (4.4.6) implies:

$$\int_{B_r(x) \cap A} |z - \tilde{z}_{r,A}(x)| \leq Crm_r^{\alpha-1} \leq Cr^\beta,$$

where we have used (4.4.4) in the last inequality. It is clear that up to changing again C , the median may be replaced by the mean:

$$\int_{B_r(x) \cap A} |z - z_{r,A}(x)| \leq Cr^\beta. \quad (4.4.7)$$

The rest is just standard Campanato estimates, which we do here for the sake of completeness. Let us compare the quantities $z_{r,A}(x)$ and $z_{r/2,A}(x)$:

$$\begin{aligned} |z_{r/2,A}(x) - z_{r,A}(x)| &= \left| \int_{B_{r/2}(x) \cap A} (z - z_{r,A}(x)) \right| \\ &\leq \int_{B_{r/2}(x) \cap A} |z - z_{r,A}(x)| \\ &\leq \frac{|B_r(x) \cap A|}{|B_{r/2}(x) \cap A|} \int_{B_r(x) \cap A} |z - z_{r,A}(x)| \\ &\leq C \int_{B_r(x) \cap A} |z - z_{r,A}| \\ &\leq Cr^\beta, \end{aligned}$$

the last inequality following from (4.4.7), and the last but one from $|B_{r/2}(x) \cap A| \geq cr^d$ because x is a transfer point from A and B . Then one has

$$\begin{aligned} |z(x) - z_{r,A}(x)| &\leq |z(x) - z_{r/2,A}(x)| + |z_{r/2,A}(x) - z_{r,A}(x)| \\ &\leq |z(x) - z_{r/2,A}(x)| + Cr^\beta, \end{aligned}$$

which means by setting $f(r) = |z(x) - z_{r,A}(x)|$ for $r \in]0, R]$ that:

$$f(r) \leq f(r/2) + Cr^\beta.$$

Consequently for all $k \in \mathbb{N}$,

$$f(r) \leq f(r2^{-(k+1)}) + Cr^\beta \sum_{i=0}^k 2^{-i\beta}$$

thus by taking the limit $k \rightarrow \infty$:

$$f(r) \leq \limsup_{\varepsilon \rightarrow 0} f(\varepsilon) + Cr^\beta \sum_{i=0}^{\infty} 2^{-i\beta} \leq \limsup_{\varepsilon \rightarrow 0} f(\varepsilon) + Cr^\beta.$$

Now we know by (4.4.5) that $f(\varepsilon) \rightarrow 0$ when $\varepsilon \rightarrow 0$, which implies that $f(r) \leq Cr^\beta$, that is to say:

$$|z(x) - z_{r,A}(x)| \leq Cr^\beta.$$

By way of consequence

$$\int_{B_r(x) \cap A} |z - z(x)| \leq \int_{B_r(x) \cap A} |z - z_{r,A}(x)| + |z(x) - z_{r,A}(x)| \leq Cr^\beta.$$

Now set $\varepsilon_{A,B} = c_0^2 R_{A,B}^d$ and take $m \leq \varepsilon_{A,B}$. We know that $c_0 r^d \leq m_r \leq \omega_d r^d$ for all $r \leq R \leq r_0$. Set r such that $c_0^2 r^d = m$, so that $r \leq R_{A,B}$, and set

$$t_r = \frac{m}{c_0 m_r} = \frac{c_0 r^d}{m_r} \leq 1.$$

Choose P_0 such that $x \in X_{P_0} \subseteq A$. Consider a family of disjoint sets $\tilde{X}_P \subseteq X_P$ such that $\bigcup_P \tilde{X}_P = \bigcup_P X_P$. We define $\chi_P = t_r c_0 \mathbf{1}_{\tilde{X}_P \cap B_r(x) \cap A}$ for all $P \in \mathcal{P}$, $\xi_{P_0} = m \delta_x$ and $\xi_P = 0$ if $P \neq P_0$, and finally $\mu' = \mu + \sum_P \xi_P - \chi_P = \mu + m \delta_x - t_r c_0 \mathbf{1}_{A \cap B_r(x)}$. Since $t_r \leq 1$ and $t_r c_0 |A \cap B_r(x)| = t_r c_0 m_r = m$, μ' is a probability measure, and one may check that $(\mu', (\chi_P), (\xi_P))$ is a basin variation satisfying (i), (ii) and (iii) of Definition 4.4.4. Let us check (iv) and (v):

$$\left| \int z d(\mu - \mu') \right| = m \left| z(x) - \int_{A \cap B_r(x)} z \right| \leq C m r^\beta \leq C m^{1+\beta/d},$$

the last inequality coming from the fact that $m = c_0^2 r^d$. Moreover since $\|\mu - \mu'\| \leq 2m$ and $\text{diam supp}|\mu - \mu'| \leq 2r$, one has

$$d_\alpha(\mu, \mu') \leq C r m^\alpha \leq C m^{1+\beta/d}$$

because $\alpha + 1/d = 1 + \beta/d$. As a consequence $A \rightsquigarrow B$. □

Proposition 4.4.7. *The binary relation \rightsquigarrow on \mathcal{X} satisfies the following properties:*

(i) transitivity.

$$(A \rightsquigarrow B \text{ and } B \rightsquigarrow C) \implies A \rightsquigarrow C,$$

(ii) subset.

$$(A \subseteq A', B \subseteq B' \text{ and } A \rightsquigarrow B) \implies A' \rightsquigarrow B',$$

(iii) grouping.

$$\begin{aligned} (A_1 \rightsquigarrow A_2 \rightsquigarrow A_1 \text{ and } A_1 \cup A_2 \rightsquigarrow B) &\implies A_1 \rightsquigarrow B \text{ and } A_2 \rightsquigarrow B, \\ (A \rightsquigarrow B_1 \cup B_2 \text{ and } B_1 \rightsquigarrow B_2 \rightsquigarrow B_1) &\implies A \rightsquigarrow B_1 \text{ and } A \rightsquigarrow B_2, \end{aligned}$$

(iv) connection.

$$A \neq X \implies \exists P \in \mathcal{P}, (X_P \not\subseteq A \text{ and } A \rightsquigarrow X_P).$$

Proof. We prove the three items successively.

Proof of (i) We treat for example the first implication, the second one being handled symmetrically. Set $\varepsilon_{A,C} = 1/2 \cdot \min(\varepsilon_{A,B}, \varepsilon_{B,C})$ and take $m \leq \varepsilon_{A,C}$. We know there exists basin variations $(\mu^1, (\chi_P^1), (\xi_P^1))$ and $(\mu^2, (\chi_P^2), (\xi_P^2))$ transferring a mass $2m$ from A to B and from B to C respectively, with controlled cost. Let us build a new one as follows:

$$\chi_P = \frac{1}{2}(\chi_P^1 + \chi_P^2), \quad \xi_P = \frac{1}{2}(\xi_P^1 + \xi_P^2),$$

and

$$\mu' = \mu + \sum_P (\xi_P - \chi_P) = \frac{1}{2}(\mu^1 + \mu^2).$$

By construction $\sum_P \chi_P \leq \mu$ and μ' is a nonnegative measure, so that $(\mu', (\chi_P), (\xi_P))$ is a basin variation of μ , and it is easy to check that it transfers a mass m from A to C . Obviously, for all P one has $\sum_P \chi_P(X_P) + \xi_P(X_P) \leq (K_{A,B} + K_{B,C})m$. Moreover one can control the cost variation:

$$\begin{aligned} \left| \int z d(\mu' - \mu) \right| &\leq \frac{1}{2} \left| \int z d(\mu^1 - \mu) \right| + \frac{1}{2} \left| \int z d(\mu^2 - \mu) \right| \\ &\leq 2^{-1}(K_{A,B} + K_{B,C})(2m)^{1+\beta/d} \\ &\leq 2^{1/d+\alpha-1}(K_{A,B} + K_{B,C})m^{1+\beta/d}. \end{aligned}$$

Moreover by subadditivity of the d_α distance

$$\begin{aligned} d_\alpha(\mu, \mu') &= d_\alpha(\mu/2 + \mu/2, \mu^1/2 + \mu^2/2) \leq d_\alpha(\mu/2, \mu^1/2) + d_\alpha(\mu/2, \mu^2/2) \\ &\leq 2^{-\alpha}(K_{A,B} + K_{B,C})(2m)^{1+\beta/d} \\ &= 2^{1/d}(K_{A,B} + K_{B,C})m^{1+\beta/d} \end{aligned}$$

Thus setting $K_{A,C} = 2^{1/d}(K_{A,B} + K_{B,C})$ all the required properties to get $A \rightsquigarrow C$ are satisfied.

Proof of (iii) Consider a loop $A_1 \rightsquigarrow A_2 \rightsquigarrow A_1$ and B such that $A := A_1 \cup A_2 \rightsquigarrow B$. Let us establish for example $A_1 \rightsquigarrow B$. We consider a basin variation $(\hat{\mu}, (\hat{\chi}_P), (\hat{\xi}_P))$ of μ which transfers a mass m from A to B with controlled cost. Set $m_i = \hat{\xi}(A_i) - \hat{\chi}_P(A_i) \in \mathbb{R}$ for $i = 1, 2$. We take a basin variation $(\bar{\mu}, (\bar{\chi}_P), (\bar{\xi}_P))$ which sends a mass m_2 from A_2 to A_1 if $m_2 > 0$, or a mass $-m_2$ from A_1 to A_2 if $m_2 < 0$. This is possible if $|m_2| \leq \min(\varepsilon_{A_1, A_2}, \varepsilon_{A_2, A_1})$, but

$$|m_2| \leq \sum_{P: X_P \subseteq A_2} (\hat{\xi}_P(X_P) + \hat{\chi}_P(X_P)) \leq K_{A,B}m,$$

hence this condition is satisfied if

$$m \leq \frac{\min(\varepsilon_{A_1, A_2}, \varepsilon_{A_2, A_1})}{K_{A,B}}. \quad (4.4.8)$$

Define for all P

$$\chi_P = \frac{1}{2}(\bar{\chi}_P + \hat{\chi}_P), \quad \xi_P = \frac{1}{2}(\bar{\xi}_P + \hat{\xi}_P)$$

and

$$\mu' = \mu + \sum_P (\xi_P - \chi_P) = \frac{1}{2}(\bar{\mu} + \hat{\mu}).$$

It is easy to see that $(\mu', (\chi_P), (\xi_P))$ is a basin transfer of μ which transfers a mass $m/2$ from A_1 to B . Moreover one has

$$\sum_{P \in \mathcal{P}} \xi_P(X_P) + \chi_P(X_P) \leq \tilde{K}m,$$

as well as

$$\left| \int z d(\mu' - \mu) \right| \leq \tilde{K}m^{1+\beta/d}$$

and

$$d_\alpha(\mu, \mu') \leq \tilde{K}m^{1+\beta/d},$$

where $\tilde{K} := (K_{A,B} \max(K_{A_1, A_2}, c_{A_2, A_1}) + K_{A,B})/2$. Thus $(\mu', (\chi_P), (\xi_P))$ is a mass transfer of mass $m/2$ (and not m) from A_1 to B with controlled cost for constants

$$\varepsilon_{A_1, B} = \frac{\min(\varepsilon_{A_1, A_2}, \varepsilon_{A_2, A_1})}{2K_{A,B}} \quad \text{and} \quad K_{A_1, B} = \tilde{K}2^{1+\beta/d}.$$

We have proved that $A_1 \rightsquigarrow B$, and the same goes for A_2 . The analogous statement for $A \rightsquigarrow B_1 \cup B_2$ holds as well.

Proof of (iv) Take a set $A \in \mathcal{X}$ such that $A \neq X$. If there is a P such that $X_P \not\subseteq A$ and $|X_P \cap A| > 0$ then clearly $A \rightsquigarrow X_P$. Otherwise set $B = \bigcup_{P: X_P \not\subseteq A} X_P$. We know that $|A|, |B| > 0$, $|A \cap B| = 0$ and $|U \setminus (A \cup B)| = 0$, hence by Lemma 4.4.2 there is a frontier point x from A to B . Since B is closed, x actually belongs to B , and there is a basin X_P contained in B such that $x \in X_P$. By definition x is a frontier point from A to X_P and by Lemma 4.4.5 one has $A \rightsquigarrow X_P$. \square

Remark 4.4.8. Notice that using (i) successively, if one has a loop

$$A_1 \rightsquigarrow A_2 \rightsquigarrow \dots \rightsquigarrow A_n = A_1$$

then $A_i \rightsquigarrow A_j \rightsquigarrow A_i$ for all i, j , and using (iii) successively, it generalizes to arbitrary loops:

$$\begin{aligned} \left(A_1 \rightsquigarrow A_2 \rightsquigarrow \dots \rightsquigarrow A_n = A_1 \text{ and } \bigcup_i A_i \rightsquigarrow B \right) &\implies \forall i, A_i \rightsquigarrow B, \\ \left(A \rightsquigarrow \bigcup_i B_i \text{ and } B_1 \rightsquigarrow B_2 \rightsquigarrow \dots \rightsquigarrow B_n = B_1 \right) &\implies \forall i, A \rightsquigarrow B_i. \end{aligned}$$

Proposition 4.4.9. *The binary relation \rightsquigarrow is total in the sense that for all $A, B \in \mathcal{X}$, $A \rightsquigarrow B$.*

Proof. Let us define some terminology which will render the proof very easy. We say that two elements $A, B \in \mathcal{X}$ are equivalent if $A \rightsquigarrow B$ and $B \rightsquigarrow A$. We call blocks all elements $A \in \mathcal{X}$ which are unions of equivalent basins, that is sets of the form $A = \bigcup_{P \subseteq \mathcal{P}'} X_P$ where $\mathcal{P}' \subseteq \mathcal{P}$ and all the $(X_P)_{P \in \mathcal{P}'}$ are pairwise equivalent. We will consider block chains

$$\mathcal{C} = A_1 \rightsquigarrow A_2 \rightsquigarrow \dots A_{n-1} \rightsquigarrow A_n$$

which are simple in the sense that for $i \neq j$, A_i and A_j have no basin in common. The length $\ell(\mathcal{C})$ of \mathcal{C} is n and its size $s(\mathcal{C})$ is the total number of basins appearing in the A_j 's: $s(\mathcal{C}) = \text{card}\{P : \exists i, X_P \subseteq A_i\}$. We consider the simple chains of maximal size (it makes sense because there are only finitely many), and among them those with minimal length. Pick such a chain \mathcal{C} . If $\text{card}\{P : X_P \subseteq A_n\} < \text{card } \mathcal{P}$ then there exists a $P \in \mathcal{P}$ such that $A_n \rightarrow X_P$ and $X_P \not\subseteq A_n$ by (iv) of Proposition 4.4.7. Now since \mathcal{C} has maximal size, X_P must belong to a previous basin: $X_P \subseteq A_i$ for $i < n$. By (ii) one has $A_n \rightsquigarrow A_i$, thus the blocks A_i, A_{i+1}, \dots, A_n are equivalent. Consequently all the basins X_P belonging to the A_j 's for $i \leq j \leq n$ are equivalent, using (iii) successively as in Remark 4.4.8. Hence $\tilde{A}_i := \bigcup_{j \geq i} A_j$ is itself a block and one has a block chain

$$\tilde{\mathcal{C}} = A_1 \rightsquigarrow \dots A_{i-1} \rightsquigarrow \tilde{A}_i,$$

which is a simple block chain with $s(\tilde{\mathcal{C}}) = s(\mathcal{C})$ and $\ell(\tilde{\mathcal{C}}) < \ell(\mathcal{C})$: a contradiction. Consequently one had $\text{card}\{P : X_P \subseteq A_n\} = \text{card } \mathcal{P}$ and the chain was made of a single block $A_1 = A_n$ made of all basins. Therefore all basins are equivalent, and all elements $A \in \mathcal{X}$ as well by (ii). This is what we wanted. \square

We set $\bar{K} = \max_{A, B \in \mathcal{X}} K_{A, B}$ and $\bar{\varepsilon} = \max_{A, B \in \mathcal{X}} \varepsilon_{A, B}$.

Proposition 4.4.10 (Basin transfers). *There exists constants $K, \varepsilon > 0$ such that for any family of real numbers $(m_P)_{P \in \mathcal{P}}$ satisfying $\sum_{P \in \mathcal{P}} m_P = 0$ and $m := \sum_P |m_P| \leq \varepsilon$, one may find a basin variation $(\mu', (\chi_P), (\xi_P))$ of μ such that*

- (i) $\sum_P (\xi_P - \chi_P)(X_P) = m_P$,
- (ii) $|\int z d(\mu' - \mu)| \leq Km^{1+\beta/d}$,
- (iii) $d_\alpha(\mu, \mu') \leq Km^{1+\beta/d}$.

Proof. Assume that the m_P 's are not all 0, otherwise there is nothing to do. Take $P_1 \in \mathcal{P}$ such that $|m_{P_1}|$ is minimal. If $m_{P_1} < 0$ choose $Q_1 \in \mathcal{P}$ such that $m_{Q_1} > 0$; necessarily $|m_{P_1}| \leq m_{Q_1}$ and we know that there is a basin variation $V_1 = (\mu^1, (\chi_P^1), (\xi_P^1))$ of μ sending a mass $|m_{P_1}|$ from X_{P_1} to X_{Q_1} and with controlled cost, provided that $|m_{P_1}| \leq \bar{\varepsilon}$. If $m_{P_1} > 0$ we choose Q_1 such that $m_{Q_1} < 0$ and a basin variation which sends a mass $|m_{P_1}|$ from X_{Q_1} to X_{P_1} . Then we set for all $P \in \mathcal{P}$:

$$m_{1, P} = m_P - (\xi_P^1(X_P) - \chi_P^1(X_P)).$$

We know that $m_1 := \sum_P |m_{1, P}| \leq m + \bar{K}m_{P_1} \leq (1+c)m$ and $m_{1, P_1} = 0$. We consider the family of real numbers $(m_{1, P})_{P \in \mathcal{P}}$. We continue by induction, building sequences $P_i, Q_i, V_i, m_{i, P}$ until all the $(m_{i, P})_P$ are 0, and set I the number of steps. We obtain a sequence satisfying for all $i \leq I$:

- (i) for all P , $m_{i,P} = m_{i-1,P} - (\xi_P^i(X_P) - \chi_P^i(X_P))$,
- (ii) $m_i := \sum_P |m_{i,P}| \leq (1 + \bar{K})m_{i-1}$,
- (iii) $m_{i,P_j} = 0$ for all $j \leq i$.

Notice that this is justified provided $|m_i|$ is always less than $\bar{\varepsilon}$, which is guaranteed if $m(1 + \bar{K})^{1+\#\mathcal{P}} \leq \bar{\varepsilon}$. Then we define a basin variation $V = (\mu', (\chi_P), (\xi_P))$ by

$$\chi_P = 1/I \cdot \sum_i \chi_P^i, \quad \xi_P = 1/I \cdot \sum_i \xi_P^i$$

and

$$\mu' = \mu + \sum_P (\xi_P - \chi_P).$$

One has for all $P \in \mathcal{P}$:

$$0 = m_{I,P} = m_{I-1,P} - (\xi_P^I(X_P) - \chi_P^I(X_P)) = \dots = m_P - \sum_{i \leq I} (\xi_P^i(X_P) - \chi_P^i(X_P))$$

hence

$$(\xi_P - \chi_P)(X_P) = m_P.$$

Moreover one has

$$\left| \int z d(\mu' - \mu) \right| \leq \bar{K}(1 + \bar{K})^{I(1+\beta/d)} m^{1+\beta/d}$$

and

$$d_\alpha(\mu, \mu') \leq \bar{K}(1 + \bar{K})^{I(1+\beta/d)} m^{1+\beta/d}.$$

Setting $K = \bar{K}(1 + \bar{K})^{\#\mathcal{P}(1+\beta/d)}$ and $\varepsilon = \bar{\varepsilon}/(1 + \bar{K})^{\#\mathcal{P}}$, one gets the desired result. \square

4.4.2 A general first variation inequality

Proposition 4.4.11. *Let $\nu' \in \text{Prob}(\mathbb{R}^d)$ be a probability measure such that $|\nu' - \nu|$ is concentrated on $\text{Bas}_\eta^+(\bar{R})$ and has mass $m \leq \varepsilon$. Then there is a probability measure μ' and an irrigation plan $\eta' \in \mathbf{IP}(\mu', \nu')$ such that*

- (i) η' is concentrated on η -good paths,
- (ii) $\left| \int z d(\mu' - \mu) \right| \leq Cm^{1+\beta/d}$,
- (iii) $d_\alpha(\mu, \mu') \leq Cm^{1+\beta/d}$.

Proof. Notice that one may write:

$$\nu' = \nu + \bar{\xi} - \bar{\chi}$$

where $\bar{\xi}, \bar{\chi}$ are nonnegative measures concentrated on $\text{Bas}_\eta^+(\bar{R})$ which are mutually singular and have same mass, with $\bar{\chi} \leq \nu$. Setting for all $P \in \mathcal{P}$:

$$m_P = \bar{\xi}(\text{Bas}^+(P)) - \bar{\chi}(\text{Bas}^+(P)),$$

we know that $\sum_P m_P = 0$. By the previous proposition, one can find a basin variation $(\mu', (\xi_P), (\chi_P))$ such that

- (i) $\sum_P (\xi_P - \chi_P)(X_P) = m_P,$
- (ii) $|\int z d(\mu' - \mu)| \leq Km^{1+\beta/d},$
- (iii) $d_\alpha(\mu, \mu') \leq Km^{1+\beta/d}.$

It is possible to connect μ' and ν' using η -good paths, since for all P , $\xi_P - \chi_P$ and $(\bar{\xi} - \bar{\chi}) \llcorner \text{Bas}^+(P)$ are concentrated on a common basin $\text{Bas}_\eta(\rho_P)$ and have equal (signed) mass. Hence there exists an irrigation plan $\eta' \in \mathbf{IP}(\mu', \nu')$ obtained by adding, removing, and reconnecting η -good curves passing by a same connected component P , so that η' is concentrated on η -good paths. \square

From this result, we are able to prove a first variation formula for general variations of ν , with μ fixed, at the cost of an extra error term, which is of lower order in our case $\alpha \geq 1 - 1/d$.

Theorem 4.4.12. *Let $\eta \in \mathbf{IP}(\mu, \nu)$ be an optimal irrigation plan with finite \mathbf{I}^α cost and z a landscape function. If $\nu' \in \text{Prob}(\mathbb{R}^d)$ is such that $|\nu' - \nu|$ is concentrated on $\text{Bas}_\eta^+(\bar{R})$ and has mass m then the following holds:*

$$d_\alpha(\mu, \nu') \leq d_\alpha(\mu, \nu) + \alpha \int z d(\nu' - \nu) + Cm^{1+\beta/d},$$

for some constant C independent from ν' and m .

Remark 4.4.13. Notice that since z is bounded from below on $\text{Bas}_\eta+(\bar{R})$ and that $z \in L^1(\nu)$, the integral $\int z d(\nu' - \nu)$ is well-defined and valued in $\mathbb{R} \cup \{+\infty\}$.

Proof. If $\int z d(\nu' - \nu) = +\infty$ there is nothing to prove. Otherwise $z \in L^1(\nu')$ and we take μ' and $\eta' \in \mathbf{IP}(\mu', \nu')$ given by the previous proposition. One has

$$d_\alpha(\mu', \nu') \leq \mathbf{I}^\alpha(\eta') \leq \mathbf{I}^\alpha(\eta) + \alpha \int_\Gamma Z_\eta d(\eta' - \eta).$$

Now since $z \in L^1(\mu') \cap L^1(\nu')$ and that $Z_\eta(\gamma) = z(\gamma^+) - z(\gamma^-)$ for η' -almost every γ , one has

$$d_\alpha(\mu', \nu') \leq \mathbf{I}^\alpha(\eta) + \alpha \int z d(\nu' - \nu) - \alpha \int z d(\mu' - \mu),$$

hence using the triangle inequality and $\mathbf{I}^\alpha(\eta) = d_\alpha(\mu, \nu)$:

$$d_\alpha(\mu, \nu') \leq d_\alpha(\mu, \nu) + \alpha \int z d(\nu' - \nu) + d_\alpha(\mu, \mu') - \alpha \int z d(\mu' - \mu).$$

By the previous proposition, the two last terms are bounded by $Cm^{1+\beta/d}$, which implies that

$$d_\alpha(\mu, \nu') \leq d_\alpha(\mu, \nu) + \alpha \int z d(\nu' - \nu) + Cm^{1+\beta/d}. \quad \square$$

4.4.3 The regularity theorem

Theorem 4.4.14. *There exists a constant $C > 0$ such that for any $r \leq r_0$, $x \in \text{Bas}_\eta^-(\bar{R})$ and $y \in \text{Bas}_\eta^+(\bar{R})$, one has:*

$$\int_{U \cap B_r(x)} |z - z_r(x)| \leq Cr^\beta \quad \text{and} \quad \int_{V \cap B_r(y)} |z - z_r(y)| \leq Cr^\beta,$$

hence z is β -Hölder continuous on $\text{Bas}_\eta^-(\bar{R})$ and $\text{Bas}_\eta^+(\bar{R})$.

Proof. Consider a ball $B = B_r(x) \cap \text{Bas}_\eta^+(\bar{R})$ centered at some $x \in \text{Bas}_\eta^+(\bar{R})$. We partition it in two parts of equal measure:

$$B = B_1 \sqcup B_2 \quad \text{with} \quad |B_1| = |B_2| = |B|/2 =: m/2,$$

and we set $\nu' = \nu - c\mathbf{1}_{B_1} + c\mathbf{1}_{B_2} \in \text{Prob}(\mathbb{R}^d)$. Then we apply Theorem 4.4.12:

$$\begin{aligned} d_\alpha(\mu, \nu') &\leq d_\alpha(\mu, \nu) + \alpha \int_{\text{Bas}^-(\bar{R})} z \, d(\nu' - \nu) + Cm^{1+\beta/d}, \\ &= d_\alpha(\mu, \nu) + \alpha c \left(\int_{B_1} z - \int_{B_2} z \right) + Cm^{1+\beta/d}, \end{aligned} \tag{4.4.9}$$

but by the triangle inequality, one has

$$|d_\alpha(\mu, \nu) - d_\alpha(\mu, \nu')| \leq d_\alpha(\nu, \nu') \leq Cm^\alpha r, \tag{4.4.10}$$

the last inequality being true because $\|\nu - \nu'\| = m$ and $\text{diam}(\text{supp}|\nu' - \nu|) \leq 2r$. Combining (4.4.9) and (4.4.10), one gets

$$\int_{B_2} z - \int_{B_1} z \leq Cm^\alpha r + Cm^{1+\beta/d}.$$

Taking B_1, B_2 such that $z \geq \bar{z}_r(x)$ on B_2 and $z \leq \bar{z}_r(x)$ where $\bar{z}_r(x)$ is the median of z on $B_r(x) \cap \text{Bas}^+(\bar{R})$, and dividing by m we obtain

$$\int_{B_r(x) \cap \text{Bas}^+(\bar{R})} |z - \bar{z}_r(x)| \leq Cm^{\alpha-1}r + Cm^{\beta/d}.$$

Now since $m \approx r^d$, one has $m^{\alpha-1}r \approx r^\beta \approx r^\beta$, and replacing the median by a mean (which is easy to do), we finally get

$$\int_{B_r(x) \cap V} |z - z_r(x)| \leq Cr^\beta.$$

The same reasoning gives the corresponding inequality on $\text{Bas}_\eta^-(\bar{R})$. By the equivalence between Campanato and Hölder spaces, as U and V are Ahlfors regular, this means that z is β -Hölder continuous on U and V . \square

4.4.4 A straightforward consequence

Theorem 4.4.15. *Let μ, ν be probability measures of the form*

$$\mu = f \mathcal{L}|_U \quad \text{and} \quad \nu = g \mathcal{L}|_V,$$

where U, V are bounded, open and connected sets such that $\bar{U} \cap \bar{V} = \emptyset$ and which are Ahlfors regular, and $f, g \geq c_0 > 0$. If η is an optimal irrigation plan for the α -cost, where $\alpha > 1 - 1/d$, then η has equibounded fiber lengths, in the sense that:

$$\eta - \operatorname{ess\,sup}_{\gamma \in \Gamma} \mathbf{L}(\gamma) < \infty.$$

Perspectives

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5.1 A shape evolution problem

We want here to define a dynamical counterpart to the shape optimization problem studied in Chapter 3, so as to obtain a shape evolving continuously towards the optimal shape. We try to give a sketch of what one would like to accomplish regarding this problem. The calculations will be quite formal and most of the claims will not be proven but are left open for now, as their study is currently ongoing. We introduce the shape evolution problem via an implicit scheme of what should be its time discretization, in the spirit of gradient flows discretization (see [San15, Chapter 8] or [San14b; San17] for an overview of gradient flows, [AGS08] for the whole theory), or De Giorgi's minimizing movements [De 93; Amb95]. Actually, it may be seen as a geometric minimizing movement (see for example [Bra14, Chapters 6,8] or [ATW93] on this topic).

Given $\lambda > 0$, we define the functional $\mathcal{F} : \mathcal{M}^+(\mathbb{R}^d) \rightarrow \mathbb{R} \cup \{+\infty\}$ by

$$\mathcal{F}_{\alpha,\lambda}(\mu) := \mathbf{X}^\alpha(\mu) - \lambda\|\mu\|,$$

and identifying Lebesgue measurable sets E with $\mathbf{1}_E \mathcal{L}$, we also set

$$\mathcal{F}_{\alpha,\lambda}(E) := \mathbf{X}^\alpha(\mathbf{1}_E \mathcal{L}) - \lambda|E|.$$

The term $-\lambda|E|$ is a volume term which we take for simplicity as replacement for the constraint $|E| = 1$ which was imposed in Chapter 3.

Let $T > 0$ be a fixed duration and $\tau > 0$ be a small time step. Given a set E_0 , we look for a sequence of sets $(E_k^\tau)_k$ defined by

$$E_0^\tau = E_0, \tag{5.1.1}$$

and

$$E_{k+1}^\tau \in \operatorname{argmin}_E \mathcal{F}(E) + \int_{E \Delta E_k} \frac{d(x, \partial E_k)}{\tau} dx, \tag{5.1.2}$$

for $k \leq \lceil T/\tau \rceil$. The integral term behaves like squared distance (divided by τ), as it is more or less the integral of a distance on a small slice ; moreover if one considers a translation of a cube by a distance h this integral term indeed gives h^2 . Thus it is quite similar to minimizing movements, and very close to the variational approach for the motion by mean curvature (see [ATW93]). We would like to do the following:

- (i) (Well-definedness) Prove that one can define a sequence $(E_k^\tau)_k$ by (5.1.1)-(5.1.2), i.e. there exists minimizers at each step.
- (ii) (Continuous motion) Define a suitable set-valued map $\bar{E}^\tau(t)$ for $t \in [0, T]$ such that $\bar{E}(k\tau) = E_k^\tau$, for example $\bar{E}^\tau(t) = E_{\lceil t/\tau \rceil}^\tau$ and prove that it converges

$$\bar{E}^\tau \xrightarrow{\tau \rightarrow 0} \bar{E}$$

in some sense to some \bar{E} .

- (iii) (Evolution equation) Prove that the motion of \bar{E} is described by the evolution equation:

$$v_t(x) = \lambda - \alpha z_t(x) \quad \text{for } x \in \partial \bar{E}(t), \tag{5.1.3}$$

where $v_t(x)$ is the normal speed of $\bar{E}(t)$ at x and z_t is a landscape function associated to $E(t)$, which represents the marginal cost in irrigation. Recall that z is the first variation \mathbf{X}^α , so that the term on the right is actually the opposite of $\delta\mathcal{F}_{\alpha,\lambda}$, the first variation of $\mathcal{F}_{\alpha,\lambda}$. Of course, all these notions and equation (5.1.3) should be given a precise meaning first!

Before entering the study, let us rewrite (5.1.2). Notice that

$$\begin{aligned} \int_{E\Delta E_k} d(x, \partial E_k) dx &= \int_{E\setminus E_k} d(x, E_k) dx + \int_{E_k\setminus E} d(x, E_k^c) dx \\ &= \int d_s(x, E_k)(\mathbf{1}_{E\setminus E_k} - \mathbf{1}_{E_k\setminus E}) dx \\ &= \int_E d_s(x, E_k) - \int_{E_k} d_s(x, E_k) dx, \end{aligned}$$

where $d_s(x, E)$ is the signed distance to E . Since the second term is a constant, (5.1.2) is equivalent to

$$E_{k+1}^\tau \in \operatorname{argmin}_E \mathcal{F}_{\alpha,\lambda}(E) + \int_E \frac{d_s(x, E_k)}{\tau} dx. \quad (5.1.4)$$

Now let us try to discuss the issues in dealing with each step and propose sketches for some proofs.

Well-definedness. We are going to take minimizers of (5.1.4) in the larger class of positive measures ν such that $\nu \leq 1$. It is clear that $\nu \mapsto \mathcal{F}_{\alpha,\lambda}(\nu) + \int \frac{d_s(x, E_k^\tau)}{\tau} d\nu$ is lower semicontinuous since \mathbf{X}^α is lower semicontinuous and the remaining part

$$-\lambda|\nu| + \int \frac{d_s(x, E_k^\tau)}{\tau} d\nu(x) = \int \phi d\nu, \quad \text{where } \phi(x) = \frac{d_s(x, E_k^\tau)}{\tau} - \lambda,$$

is continuous because $\phi \in \mathcal{C}_b(\mathbb{R}^d)$. Assuming that E_k^τ compact, one may prove sufficient properties (lower bounded, tightness of minimizing sequences...) to guarantee that there is a minimizer

$$\nu_{k+1}^\tau \in \operatorname{argmin}_{\nu \leq 1} \mathcal{F}_{\alpha,\lambda}(\nu) + \int \frac{d_s(x, E_k^\tau)}{\tau} d\nu(x) =: \mathcal{G}(\nu).$$

Now let us prove that ν_{k+1}^τ is indeed the indicator of a set. Take a landscape function z_k^τ associated to $\nu_k^\tau = \mathbf{1}_{E_k^\tau}$. We know by Proposition 3.1.5 for any competitor $\nu \in \mathcal{M}^+(\mathbb{R}^d)$,

$$\mathbf{X}^\alpha(\nu) \leq \mathbf{X}^\alpha(\nu_{k+1}^\tau) + \alpha \int z_k^\tau d(\nu - \nu_{k+1}^\tau), \quad (5.1.5)$$

thus by adding $-\int \lambda d\nu + \int \frac{d_s(x, E_k^\tau)}{\tau} d\nu$:

$$\mathcal{G}(\nu) \leq \mathcal{G}(\nu_{k+1}^\tau) + \int_{\mathbb{R}^d} \left(\alpha z_k^\tau(x) + \frac{d_s(x, E_k^\tau)}{\tau} - \lambda \right) d(\nu - \nu_{k+1}^\tau). \quad (5.1.6)$$

By optimality of ν_{k+1}^τ , the integral on the right must be nonnegative for all competitor ν , which means that

$$\int_{\mathbb{R}^d} (\phi(x) - \lambda) d\nu_{k+1}^\tau(x) = \min_{\nu \leq 1} \int_{\mathbb{R}^d} (\phi(x) - \lambda) d\nu(x). \quad (5.1.7)$$

where we have set $\phi(x) = \alpha z_k^\tau(x) + \frac{d_s(x, E_k^\tau)}{\tau}$. Consequently, ν_{k+1}^τ should be equal to 1 on $\{\phi < \lambda\}$ and to 0 on $\{\phi > \lambda\}$. But using the strict concavity of the map $x \mapsto x^\alpha$ it is possible to show that inequality (5.1.5), hence inequality (5.1.6) as well, is actually strict if $\nu \neq \nu_{k+1}^\tau$. As a consequence (5.1.7) admits a unique minimizer, and $|\{\phi = \lambda\}| = 0$, which implies that

$$\nu_{k+1}^\tau = \mathbf{1}_{\phi \leq \lambda} \mathcal{L}.$$

Therefore the set $E_{k+1}^\tau := \{\alpha z_k^\tau(x) + \frac{d_s(x, E_k^\tau)}{\tau} \leq \lambda\}$ is a solution of (5.1.4), or equivalently of (5.1.2). Notice ϕ being lower semicontinuous and coercive, E_{k+1}^τ is a compact set (recall that we have assumed E_k^τ to be compact). We have proven if E_0 is compact, we may define a sequence $(E_k^\tau)_k$ as wanted, and it is made of compact sets.

Continuous motion. Now we would like to pass to the limit $\tau \rightarrow 0$ in the sequence $(E_k^\tau)_k$, but also in the sequence $(z_k^\tau)_k$, as this function will play a role in describing the motion, to obtain a continuum of sets $\bar{E}(t)$ and functions $\bar{z}(t, \cdot)$ for $t \in [0, T]$. To do that, we should place ourselves in some well-suited functional spaces where we can prove enough compactness to pass to the limit (at least up to subsequence).

Let us investigate the sequence $(E_k^\tau)_k$ first. We set \mathcal{K} the set of compact subsets of \mathbb{R}^d and define for all $A, B \in \mathcal{K}$:

$$\mathcal{D}(A, B) = \int_{A \Delta B} d(x, \partial A) dx = \int_{\mathbb{R}^d} d_s(x, A)(\mathbf{1}_B - \mathbf{1}_A) dx.$$

By optimality of E_{k+1}^τ , one has

$$\mathcal{F}_{\alpha, \lambda}(E_{k+1}^\tau) + \frac{\mathcal{D}(E_{k+1}^\tau, E_k^\tau)}{\tau} \leq \mathcal{F}_{\alpha, \lambda}(E_k^\tau)$$

or equivalently

$$\mathcal{D}(E_{k+1}^\tau, E_k^\tau) \leq \tau(\mathcal{F}_{\alpha, \lambda}(E_k^\tau) - \mathcal{F}_{\alpha, \lambda}(E_{k+1}^\tau)). \quad (5.1.8)$$

Thus by summation over k , one gets

$$\sum_{k=0}^{\lceil T/\tau \rceil - 1} \tau \frac{\mathcal{D}(E_{k+1}^\tau, E_k^\tau)}{\tau^2} \leq \mathcal{F}_{\alpha, \lambda}(E_0) - \mathcal{F}_{\alpha, \lambda}(E_{\lceil T/\tau \rceil}^\tau) \leq C, \quad (5.1.9)$$

for some constant $C > 0$. It seems that the quantity $\mathcal{D}(A, B)$ behaves like a squared distance between A and B (as said before, considering a translation of a cube by a distance h gives h^2), thus (5.1.9) should be some kind of H^1 bound and give some compactness.

Let us investigate the sequence $(z_k^\tau)_k$. The situation is perhaps clearer here: the regularity one should expect is Hölder-continuity of exponent $\beta = 1 + d\alpha - d$. Indeed landscape functions associated to absolutely continuous measures μ are β -Hölder continuous as proven in [San07], with some extra regularity assumptions on μ . Moreover in Chapter 3 we prove the Hölder regularity (with same exponent) of the landscape function associated to optimal shapes. Thus what we are expecting here is that $z_k^\tau \in \mathcal{C}^{0, \beta}(E_k^\tau)$, or perhaps $z_k^\tau \in \mathcal{C}^{0, \beta}(X_k^\tau)$ for a large enough subset $X_k^\tau \subseteq E_k^\tau$. What

we would like is an upper bound on the Hölder constant to get compactness by Ascoli's theorem. But this will not be enough for the z_k^τ to converge, up to subsequence, to a curve $z(t) \in \mathcal{C}^{0,\beta}(E(t))$ for $t \in [0, T]$, as we need some regularity in t as well, that is we need a control of $|z_{k+1}^\tau - z_k^\tau|$ of some sort. This is something which may not be very easy for the following reason. We know that z_k^τ is a landscape function for ν_k^τ , that is a first variation of \mathbf{X}^α at ν_k^τ , and usually the regularity in time of the first variation is given by the fact that ν_{k+1} is close to ν_k , and that the first variation is unique and is somewhat regular as a function of ν (often it has an explicit in terms of ν). Here, we do not have uniqueness of the first variation in general, hence the one chosen at step $k+1$ may be very different from the one chosen at the previous step k . However since \mathbf{X}^α exhibits some kind of concave behaviour, we can hope that it is "often" unique (a concave function on \mathbb{R}^d being differential everywhere but in small set) and continuous. If uniqueness fails anyway in our case, a careful choice should be made to guarantee some continuity of the landscape functions between two consecutive time steps.

Evolution equation. As $E_k^\tau = \{x : \alpha z_k(x) + d_s(x, E_{k-1})/\tau \leq \lambda\}$, it is reasonable to claim that $\partial E_k^\tau \subseteq \{x : \alpha z_k(x) + d_s(x, E_{k-1})/\tau = \lambda\}$ as in the static shape optimization problem. Take a point $x \in \partial E_k^\tau$, then one has:

$$\frac{d_s(x, E_{k-1}^\tau)}{\tau} = \lambda - \alpha z_k(x).$$

The quotient $d_s(x, E_{k-1}^\tau)/\tau$ can be interpreted as the speed of the boundary at point x , thus the limit equation should be:

$$v_t(x) = \lambda - \alpha z_t(x) \text{ for } x \in \partial E(t), \quad (5.1.10)$$

where $v_t(x)$ is the normal speed of $\partial E(t)$ at x . But what sense should we give to (5.1.10)? This does not seem evident, as the sets $E(t)$ have no reason to be regular (even BV , since we suspect they may have fractal boundary), and as the regularity in time that one should be expect is unclear. A possible weak formulation of equation (5.1.10) could be the following: a curve of compact sets $E(t)$ moves with normal speed v if for all $t > 0$ and all $x \in \mathbb{R}^d \setminus E(t)$, the following holds:

$$\frac{d}{dt}d(x, E(t)) = v(\pi_{E(t)}(x)). \quad (5.1.11)$$

Difficulties arise because $\pi_{E(t)}$ may not be well-defined, even if x is taken close to the boundary $\partial E(t)$, and the strategy to prove this from the discrete scheme has yet to be elaborated. However let us give a reasoning which goes in that direction for a particular geometric situation. Take a point $x \in \mathbb{R}^d \setminus (E_k^\tau \cup E_{k+1}^\tau)$, denote by $x_k = \pi_{E_k^\tau}(x)$, $x_{k+1} = \pi_{E_{k+1}^\tau}(x)$ (assuming they are well defined). Suppose that set has "increased", in the sense that the segment $[x, x_{k+1}]$ does not intersect E_k^τ and that $[x, x_k]$ intersects E_{k+1}^τ (see Figure 5.1), and denote \tilde{x}_{k+1} the intersection point closest to x .

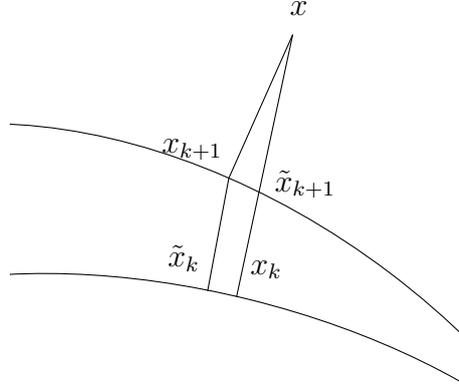


Figure 5.1: Projection on E_k^τ and E_{k+1}^τ .

On the one hand, one has

$$d(x, E_k^\tau) = |x - x_k| = |x - \tilde{x}_{k+1}| + |\tilde{x}_{k+1} - x_k| \geq d(x, E_{k+1}^\tau) + |\tilde{x}_{k+1} - x_k|,$$

but $|\tilde{x}_{k+1} - x_k| = d(\tilde{x}_{k+1}, E_k^\tau) = d_s(\tilde{x}_{k+1}, E_k^\tau)$ and $\alpha z_{k+1}(\tilde{x}_{k+1}) + d_s(\tilde{x}_{k+1}, E_k^\tau)/\tau = \lambda$ because $\tilde{x}_{k+1} \in \partial E_{k+1}^\tau$, which leads to

$$d(x, E_k^\tau) \geq d(x, E_{k+1}^\tau) + \tau(\lambda - \alpha z_{k+1}(\tilde{x}_{k+1})),$$

which rewrites

$$-\frac{d(x, E_{k+1}^\tau) - d(x, E_k^\tau)}{\tau} \geq \lambda - \alpha z_{k+1}(\tilde{x}_{k+1}). \quad (5.1.12)$$

On the other hand, setting $\tilde{x}_k = \pi_{E_k^\tau}(x_{k+1})$, one has

$$\begin{aligned} d(x, E_k^\tau) &\leq |x - \tilde{x}_k| \leq |x - x_{k+1}| + |x_{k+1} - \tilde{x}_k| \\ &= d(x, E_{k+1}^\tau) + d(x_{k+1}, E_k^\tau), \end{aligned}$$

but $d(x_{k+1}, E_k^\tau) = d_s(x_{k+1}, E_k^\tau)$ and $\alpha z_{k+1}(x_{k+1}) + d_s(x_{k+1}, E_k^\tau)/\tau = \lambda$ because $x_{k+1} \in \partial E_{k+1}^\tau$, which implies that

$$-\frac{d(x, E_{k+1}^\tau) - d(x, E_k^\tau)}{\tau} \leq \lambda - \alpha z_{k+1}(x_{k+1}). \quad (5.1.13)$$

When passing to the limit $\tau \rightarrow 0$, equations (5.1.12)-(5.1.13) may lead to the considered relation (5.1.11).

5.2 Towards proving fractality of optimal shapes

In this section we try to provide some ideas and formal computations in the direction of proving that the optimal shapes of Chapter 3 have a fractal boundary of non-integer

dimension $d - \beta$ where $\beta = 1 + d\alpha - d$. Again, we stress that the computations or proofs given here are quite sketchy and sometimes not proved in full detail as this is still ongoing work.

Let $\nu = \mathbf{1}_A$ be an optimizer of the shape optimization problem (\mathbf{R}_α) and $\eta \in \mathbf{OIP}^\alpha(\delta_0, \nu)$. We set $Z = Z_\eta$ and $z = z_\eta$ the landscape function associated to η . Then for any η -good curve γ such that $Z(\gamma) \geq t$ we define

$$\pi_{z \geq t}(\gamma) = \gamma_{z \geq t} := \gamma|_{[z^{-1}(t), +\infty[},$$

and we consider the tail of η at level t , defined by

$$\eta_t = (\pi_{z \geq t})_\#(\eta \llcorner \{Z > t\}).$$

Let us compute the initial and final measures $\mu_t := (\pi_0)_\# \eta_t$ and $\nu_t = (\pi_\infty)_\# \eta_t$:

$$\begin{aligned} \langle \mu_t, \phi \rangle &= \int_{Z > t} \gamma_{z \geq t}(0) \, d\eta(\gamma) = \int_{Z > t} \gamma(z^{-1}(t)) \, d\eta(\gamma). \\ \langle \nu_t, \phi \rangle &= \int_{Z > t} \phi(\gamma_{z \geq t}(\infty)) \, d\eta(\gamma) = \int_{Z > t} \phi(\gamma(\infty)) \, d\eta(\gamma) = \int_{z > t} \phi(x) \, d\nu(x). \end{aligned}$$

Notice that μ_t is atomic because if $Z(\gamma) > t$ then $\theta(\gamma(z^{-1}(t))) > 0$ hence there can only be countably many such $\gamma(z^{-1}(t))$, since they lie on different branches. Moreover the total mass of ν_t is $\|\nu_t\| = \nu(x : z(x) > t)$, but in our shape optimization problem $\nu = \mathbf{1}_{z \leq z^*}$, hence

$$\|\nu_t\| = \text{Vol}(x : z^* \geq z(x) > t).$$

We know by Lemma 3.4.2 that we have the upper bound:

$$\text{Vol}(x : z(x) > t) \leq C(z^* - t). \quad (5.2.1)$$

A conjecture on a lower bound of the volume In order to prove the lower bound on the Hausdorff dimension of the boundary ∂E that we seek, that is

$$\dim_H(\partial E) \geq d - \beta,$$

which is stated in Conjecture 3.4.4, a key ingredient would be the reverse inequality of (5.2.1), that is a lower bound on the volume of $\{z^* \geq z > t\}$ as conjectured below.

Conjecture 5.2.1. *There exists some $c > 0$ such that*

$$\text{Vol}(x : z^* \geq z(x) > t) \geq c(z^* - t).$$

We do not have a strategy to prove this (if true), but let us just relate this volume $|x : z^* \geq z(x) > t|$ to other expressions or quantities which might be used to estimate it from below. Let us compute the cost of η_t , observing that if $\infty > z(x) > t$ then

$\theta_{\eta_t}(x) = \theta(x)$. One has

$$\begin{aligned}
 \mathbf{I}^\alpha(\eta_t) &= \int_{Z>t} \int_\gamma \mathbf{1}_{z(x)>t} \theta_{\eta_t}(x)^{\alpha-1} d\mathcal{H}^1(x) d\eta(\gamma) \\
 &= \int_{Z>t} \int_\gamma \mathbf{1}_{z(x)>t} \theta(x)^{\alpha-1} d\mathcal{H}^1(x) d\eta(\gamma) \\
 &= \int_{Z>t} (z(\gamma(\infty)) - t) d\eta(\gamma) \\
 &= \int_{z>t} (z(x) - t) dx.
 \end{aligned}$$

This is a first simple expression for the cost of the tail of η cut for $z > t$. We may get another expression using Fubini-Tonelli's theorem:

$$\begin{aligned}
 \mathbf{I}^\alpha(\eta_t) &= \int_{Z>t} \int_\gamma \mathbf{1}_{z(x)>t} \theta_{\eta_t}(x)^{\alpha-1} d\mathcal{H}^1(x) d\eta(\gamma) \\
 &= \int_{z(x)>t} \theta_{\eta_t}(x)^\alpha d\mathcal{H}^1(x) \\
 &= \int_{z(x)>t} \theta_\eta(x)^\alpha d\mathcal{H}^1(x).
 \end{aligned}$$

We may relate the tail energy $\mathbf{I}^\alpha(\eta_t)$ to the volume $\text{Vol}(t < z \leq z^*)$.

Proposition 5.2.2. *The function $t \mapsto \mathbf{I}^\alpha(\eta_t)$ is Lipschitz and*

$$\frac{d^+}{dt} \mathbf{I}^\alpha(\eta_t) = -\text{Vol}(t < z \leq z^*) \quad \text{and} \quad \frac{d^-}{dt} \mathbf{I}^\alpha(\eta_t) = -\text{Vol}(t \leq z \leq z^*).$$

Proof. Take for example $\varepsilon > 0$ small. We have

$$\begin{aligned}
 \mathbf{I}^\alpha(\eta_{t+\varepsilon}) - \mathbf{I}^\alpha(\eta_t) &= \int_{z>t+\varepsilon} (z(x) - (t + \varepsilon)) dx - \int_{z>t} (z(x) - t) dx \\
 &= -\varepsilon \text{Vol}(t < z \leq z^*) \\
 &\quad + \left(\int_{z>t+\varepsilon} (z(x) - (t + \varepsilon)) dx - \int_{z>t} (z(x) - (t + \varepsilon)) dx \right) \\
 &= -\varepsilon \text{Vol}(t < z \leq z^*) - \varepsilon \int_{t < z \leq t+\varepsilon} \frac{(z(x) - (t + \varepsilon))}{\varepsilon} dx.
 \end{aligned}$$

But

$$\left| \int_{t < z \leq t+\varepsilon} \frac{(z(x) - (t + \varepsilon))}{\varepsilon} dx \right| \leq \text{Vol}(t < z \leq t + \varepsilon) \xrightarrow{\varepsilon \rightarrow 0} 0,$$

which shows that $t \mapsto \mathbf{I}^\alpha(\eta_t)$ is Lipschitz and implies that

$$\frac{d^+}{dt} \mathbf{I}^\alpha(\eta_t) = \lim_{\varepsilon \rightarrow 0^+} \frac{\mathbf{I}^\alpha(\eta_{t+\varepsilon}) - \mathbf{I}^\alpha(\eta_t)}{\varepsilon} = -\text{Vol}(t < z \leq z^*).$$

A similar computation with $\varepsilon < 0$ shows that

$$\frac{d^-}{dt} \mathbf{I}^\alpha(\eta_t) = -\text{Vol}(t \leq z \leq z^*). \quad \square$$

Now we may reformulate the desired lower bound on the volume as follows.

Proposition 5.2.3. *The following assertions are equivalent:*

- (i) $\exists c > 0$: $\text{Vol}(z > t) \geq c(z^* - t)$,
- (ii) $\exists c > 0$: $\mathbf{I}^\alpha(\eta_t) \doteq \int_{z>t} (z - t) \geq c(z^* - t)^2$,
- (iii) $\exists c > 0$: $\int_{z>t} (z^* - z) \geq c(z^* - t)^2$.

Proof. Let us prove (i) \Leftrightarrow (iii). We know by the first variation formula for \mathbf{X}^α and by the scaling Lemma that

$$e_\alpha - e_\alpha(\alpha + 1/d)m + e_\alpha \frac{\alpha(\alpha + 1/d)(\alpha + 1/d - 1)}{2} m^2 \leq e_\alpha - \alpha \int_{z>t} z(x) dx,$$

where $m = \text{Vol}(z > t)$. This rewrites, recalling that $e_\alpha(\alpha + 1/d) = \alpha z^*$:

$$cm^2 \leq \int_{z>t} (z^* - z(x)) dx,$$

where $c = c(\alpha, d)$ is some constant. This inequality shows that (i) \Rightarrow (iii). On the other hand one has

$$\int_{z>t} (z^* - z(x)) dx \leq \int_{z>t} (z^* - t) dx \leq (z^* - t)m,$$

which shows that (iii) \Rightarrow (i).

Similarly

$$\int_{z>t} (z(x) - t) dx \leq \int_{z>t} (z^* - t) dx \leq (z^* - t)m,$$

which shows that (ii) \Rightarrow (i). It remains to prove (i) \Rightarrow (ii). But we know that

$$-\frac{d}{dt} \mathbf{I}^\alpha(\eta_t) = \text{Vol}(t < z \leq z^*) \geq c(z^* - t),$$

which yields:

$$\mathbf{I}^\alpha(\eta_t) \geq c(z^* - t)^2. \quad \square$$

Lower dimension estimate from the lower estimate on the volume We want to show a way to prove $\dim_H\{z = z^*\} \geq d - \beta$ from Conjecture 5.2.1, that is from $\text{Vol}(z > t) \geq c(z^* - t)$, which we assume to be true in this paragraph. Notice that this would still not prove Conjecture 3.4.4 since we only know that $\partial A \subseteq \{z = z^*\}$.

The strategy is the following. The measure μ_t is an atomic measure, which we renormalize as an atomic probability measure $\sigma_t = \mu_t / \|\mu_t\|$. We write it as:

$$\sigma_t = \sum_{i \in \mathbb{N}} a_t^i \delta_{x_i^t}.$$

As $t \rightarrow z^*$ it converges weakly (up to subsequence) to a probability measure σ . We are going to prove that σ has a.e. a local dimension which is bounded from below by

$d - \beta$. Of course, this measure is concentrated on the set $\{z = z^*\}$ thus getting a lower bound on the dimension of this set by a mass distribution principle ([Fal97]).

Let us start with some notations. We write $a \approx b$ if there are constants c, C depending only on α, d such that $ca \leq b \leq Ca$. For all x we denote by $\text{Bas}(x)$ the basin of x , by $\delta(x)$ its diameter and we set

$$z^*(x) = \sup\{z(y) : x \leq y\}, \quad \rho(x) = \sup\{|y - x| : x \leq y\},$$

where by $x \leq y$ we mean that $y \in \text{Bas}(x)$. Finally we set

$$\ell(x, y) = \sup\{L(\gamma) : \gamma \text{ } \eta\text{-good from } x \text{ to } y\},$$

and

$$\ell(x) = \sup\{L(\gamma) : \gamma \text{ } \eta\text{-good starting at } x\}.$$

Lemma 5.2.4. *The following equivalences hold for all $x \in E$:*

$$|\text{Bas } x| \approx \ell(x)^d, \tag{5.2.2}$$

$$\rho(x) \approx \ell(x), \tag{5.2.3}$$

$$\delta(x) \approx \ell(x), \tag{5.2.4}$$

$$z^*(x) - z(x) \approx \ell(x)^\beta. \tag{5.2.5}$$

Moreover if $y \geq x$ is a point such that either $z(y) = z^*(x)$ or $\ell(x, y) = \ell(x)$ then

$$|y - x| \approx \ell(x). \tag{5.2.6}$$

Proof. Take a point $y \geq x$. Let us bound $\int_x^y \theta^{\alpha-1}$ from above and below, using the fact that z is Hölder-continuous and that $\theta(x) = |\text{Bas}(x)|$:

$$\begin{aligned} C\delta(x)^{d(\alpha-1)}\ell(x, y) &\leq |\text{Bas}(x)|^{\alpha-1}\ell(x, y) = \theta(x)^{\alpha-1}\ell(x, y) \\ &\leq \int_x^y \theta^{\alpha-1} \\ &= z(y) - z(x) \leq C|y - x|^\beta. \end{aligned}$$

Now take a point $y \geq x$ which has maximal z , that is $z(y) = z^*(x)$. One may write

$$z^*(x) - z(x) \leq C|y - x|^\beta \leq C\rho(x)^\beta \leq C\ell(x)^\beta.$$

Take a point y' which maximizes $\ell(x, y')$, that is $\ell(x, y') = \ell(x)$. One may write

$$C\ell(x)^\beta = C\ell(x)^{d(\alpha-1)}\ell(x) \leq C\delta(x)^{d(\alpha-1)}\ell(x, y') \leq z^*(x) - z(x).$$

Consequently one has approximate inequalities in the previous chain of inequalities:

$$\begin{aligned} \ell(x) &\approx \delta(x), \\ \ell(x)^\beta &\approx z^*(x) - z(x), \\ \rho(x) &\approx \delta(x), \\ |\text{Bas}(x)| &\approx \delta(x)^d. \end{aligned}$$

Moreover, for the chosen y and y' , one has moreover:

$$|y - x| \approx |y' - x| \approx \ell(x). \quad \square$$

Now notice that for all i , $\mu_t(x_i) = \text{Vol}(\text{Bas}(x_i)) \approx \delta(x_i)^d$, therefore:

$$\sigma_t(x_i) \approx \frac{\delta(x_i)^d}{\|\mu_t\|} \lesssim \frac{\delta(x_i)^d}{z^*(x_i) - t},$$

the last inequality following from the fact that

$$\|\mu_t\| = \text{Vol}(z > t) \geq c(z^* - t) \geq c(z^*(x_i) - t).$$

Since $\delta(x_i)^\beta \approx z^*(x_i) - z(x_i) = z^*(x_i) - t$, thus

$$\frac{\sigma_t(x_i)}{\delta(x_i)^{d-\beta}} \lesssim \frac{\delta(x_i)^\beta}{z^*(x_i) - t} \leq C.$$

We are not going to look at σ_t directly but at $\tilde{\sigma}_t$ defined by

$$\tilde{\sigma}_t = \sum_i a_i \frac{\mathcal{L}|_{\text{Bas}(x_i)}}{|\text{Bas}(x_i)|}.$$

The calculation we did before gives:

$$\tilde{\sigma}_t(\text{Bas}(x_i)) \leq C \text{diam}(\text{Bas}(x_i))^{d-\beta}.$$

When $t \rightarrow z^*$, the measures $\tilde{\sigma}_t$ converge (up to subsequence) to a measure $\tilde{\sigma}$. Notice that for all ϕ which is continuous:

$$\left| \int \phi d\tilde{\sigma}_t - \int \phi d\sigma_t \right| \leq \sum_i a_i^t \int_{\text{Bas}(x_i^t)} |\phi(x) - \phi(x_i^t)| dx \xrightarrow{t \rightarrow z^*} 0$$

thus σ_t and $\tilde{\sigma}_t$ have the same limit (up to subsequence).

Lemma 5.2.5. *The measure σ is concentrated on $\{x : z(x) = z^*\}$.*

Proof. Take $x \notin L$, that is to say $z(x) < z^*$. Consider $z^* > t > z(x) + \varepsilon$ with ε small. Then for any $y \in B_{C\varepsilon^{1/\beta}}(x)$, one has $z(y) < t$ by Hölder continuity of z and thus y does not belong to any basin of points x_t which satisfy $z(x_t) = t$. Consequently

$$\tilde{\sigma}_t(B_{C\varepsilon^{1/\beta}}(x)) = 0,$$

which is true for all t large enough, hence taking the limit $t \rightarrow z^*$ yields:

$$\tilde{\sigma}(B_{C\varepsilon^{1/\beta}}(x)) = 0.$$

Consequently $\text{supp } \sigma \subseteq \{x : z(x) = z^*\}$. □

Theorem 5.2.6. *The measure σ has lower local dimension at least $d - \beta$, in the sense that for σ -a.e. x , one has:*

$$\underline{\dim}_{\text{loc}} \sigma(x) := \liminf_{\varepsilon \rightarrow 0} \frac{\log \sigma(B_\varepsilon(x))}{\log \varepsilon} \geq d - \beta.$$

Lemma 5.2.7. *There exists some ε_0 small such that for all $\varepsilon < \varepsilon_0$ and all $x \in \{x : z(x) = z^*\}$, the set $B_\varepsilon(x)$ only intersects a number of basins $\text{Bas}(y)$ from points y such that $z(y) = z^* - c\varepsilon^{1/\beta}$ which is bounded from above by a constant $N = N(\alpha, d)$.*

Proof. It comes from the fact that all these basins are disjoint and occupy a volume $\approx \varepsilon^d$, and they must all lie in some ball $CB_\varepsilon(x)$, consequently their number is bounded by such a constant. \square

Corollary 5.2.8. *The level set of the landscape function $L := \{x : z(x) = z^*\}$ is of dimension at least $d - \beta$, in the sense that:*

$$\dim_H \{x : z(x) = z^*\} \geq d - \beta.$$

Proof of Theorem 5.2.6. Fix $\varepsilon > 0$ and x such that $z(x) = z^*$. We consider the ball $B_\varepsilon(x)$ and we fix a level $t < z^*$. For all $y \in B_\varepsilon(x) \cap L$, there is an ancestor y_t , that is $y_t \leq y$, such that $z(y_t) = t$. We know that along a curve reaching $x \in L$ we have $|y - x|^\beta \approx z^* - z(y)$ thus we may choose a t which is sufficiently close to z^* , namely

$$z^* - t \approx \varepsilon^\beta$$

to ensure that $y_t \in B_\varepsilon(x)$, independently of y . Let us call Y_t the set of such y_t 's. Thanks to the previous lemma, this set has a cardinal bounded from above by a constant $N = N(\alpha, d)$. Thus we write

$$Y_t = (y_t^i)_{i \leq N_t},$$

with $N_t \leq N$. Since

$$B_\varepsilon(x) \subseteq \bigcup_{i \leq N_t} \text{Bas}(y_t^i),$$

we may deduce that

$$\tilde{\sigma}_t(B_\varepsilon(x)) \leq \sum_{i \leq N_t} \tilde{\sigma}_t(\text{Bas}(y_t^i)) \leq C \sum_{i \leq N_t} \text{diam}(\text{Bas}(y_t^i))^{d-\beta}.$$

One has that $\text{diam}(\text{Bas}(y_t^i)) \approx (z^* - z(y_t^i))^{1/\beta} \approx \varepsilon$, which leads to

$$\tilde{\sigma}_t(B_\varepsilon(x)) \leq CN_t \varepsilon^{d-\beta} \leq C\varepsilon^{d-\beta}.$$

Since $B_\varepsilon(x)$ is open, passing to the limit as $t \rightarrow \infty$ yields:

$$\tilde{\sigma}(B_\varepsilon(x)) \leq \liminf_{t \rightarrow \infty} \tilde{\sigma}_t(B_\varepsilon(x)) \leq C\varepsilon^{d-\beta}.$$

Passing to the log and sending ε to 0 yields:

$$\underline{\dim}_{\text{loc}} \tilde{\sigma}(x) = \liminf_{\varepsilon \rightarrow 0} \frac{\log \tilde{\sigma}(B_\varepsilon(x))}{\log \varepsilon} \geq \liminf_{\varepsilon \rightarrow 0} \frac{\log(C\varepsilon^{d-\beta})}{\log \varepsilon} = d - \beta. \quad \square$$

5.3 Further perspectives

The aim of this final section is to formulate some remaining open questions, among many others of course, related to the work presented before and to propose some research perspectives in a few words.

The landscape function in the general case In Chapter 4 we have given a definition of landscape function when the optimal irrigation plan has a finite root system, and we have shown semicontinuity on the positive and negative basins, as well as Hölder regularity results with extra hypotheses on the source and target measures μ, ν . In general, there is only a countable (instead of finite) root system, hence there is still much work to be done to handle the general case. Many reasonings in Chapter 4 have been made by finite induction using the fact that there was a finite root system, thus some reasonings might be adapted to the general case by countable induction, perhaps to give a proper definition of the landscape function, although some more involved results like Hölder regularity or even semicontinuity should require more ideas (if even true).

A different approach to define a landscape function may come from the following remark. Notice that the map $H(x) = x \mapsto x^\alpha/\alpha$ is continuous and concave on \mathbb{R}_+ , hence it may be expressed as a Legendre transform:

$$H(x) = \inf_{w \geq 0} wx - H_*(w) \quad \text{where} \quad H_*(w) = \frac{w^p}{p}$$

where $p < 0$ is the conjugate exponent of α satisfying $1/\alpha + 1/p = 1$. The optimal irrigation problem may be rephrased formally as follows:

$$\begin{aligned} & \inf_{\substack{\eta \in \mathbf{IP}(\mu, \nu) \\ \eta \text{ rectifiable}}} \int_K H(\theta_\eta(x)) \, d\mathcal{H}^1(x) \\ &= \inf_{\substack{\eta \in \mathbf{IP}(\mu, \nu) \\ \eta \text{ rectifiable}}} \inf_{w > 0} \int_K \theta_\eta(x) w(x) \, d\mathcal{H}^1(x) - \int_K H_*(w(x)) \, d\mathcal{H}^1(x) \\ &= \inf_{w > 0} \int_K -H_*(w(x)) \, d\mathcal{H}^1(x) + \inf_{\substack{\eta \in \mathbf{IP}(\mu, \nu) \\ \eta \text{ rectifiable}}} \int_K \theta_\eta(x) w(x) \, d\mathcal{H}^1(x). \end{aligned}$$

We have merely proceeded by an inf-inf exchange, and we stress that the infimum for w is among any nonnegative Borel maps. Now notice that the last term rewrites by Fubini-Tonelli's theorem:

$$\inf_{\substack{\eta \in \mathbf{IP}(\mu, \nu) \\ \eta \text{ rectifiable}}} \int_K \theta_\eta(x) w(x) \, d\mathcal{H}^1(x) = \inf_{\substack{\eta \in \mathbf{IP}(\mu, \nu) \\ \eta \text{ rectifiable}}} \int_\Gamma L_w(\gamma) \, d\eta(\gamma),$$

where $L_w(\gamma) = \int_\gamma w(x) \, dx$ is the weighted length of γ w.r.t. w . Set a pseudo-distance $d_w(x, y) = \inf\{L_w(\gamma) : x \xrightarrow{\gamma} y\}$ and notice that if we want the righthand side to be (almost) minimal then the curves γ should be (almost) d_w -geodesics, hence

$$\inf_{\substack{\eta \in \mathbf{IP}(\mu, \nu) \\ \eta \text{ rectifiable}}} \int_\Gamma L_w(\gamma) \, d\eta(\gamma) = \inf_{\pi \in \Pi(\mu, \nu)} \int_{K \times K} d_w(x, y) \, d\pi(x, y),$$

having dropped the rectifiability requirement, which can be justified. Finally, the

optimal irrigation problems rewrites

$$\inf_{\substack{\eta \in \mathbf{IP}(\mu, \nu) \\ \eta \text{ rectifiable}}} \int_K H(\theta_\eta(x)) \, d\mathcal{H}^1(x) \quad (5.3.1)$$

$$= \inf_{w > 0} \int_K -H_\star(w(x)) \, d\mathcal{H}^1(x) + \underbrace{\inf_{\pi \in \Pi(\mu, \nu)} \int_{K \times K} d_w(x, y) \, d\pi(x, y)}_{\text{MK}_w(\mu, \nu)}. \quad (5.3.2)$$

The optimal irrigation problem can be seen as a Monge-Kantorovich problem with an unknown transport plan $\pi \in \Pi(\mu, \nu)$ but the cost is also part of the unknown: it is a distance d_w where w is not fixed but is penalized by $\int_K -H_\star(w(x)) \, d\mathcal{H}^1(x)$. This term forces w to be $+\infty$ everywhere but on a 1-dimensional set because $-H_\star(w) = -w^p/p$ with $p < 0$. This formulation might be worth looking at for itself, even more so as one can apply Monge-Kantorovich duality to the term $\text{MK}_w(\mu, \nu)$:

$$\inf_{\pi \in \Pi(\mu, \nu)} \int_{K \times K} d_w \, d\pi = \sup_{\phi \oplus \psi \leq d_w} \int \phi \, d\nu + \int \psi \, d\mu. \quad (5.3.3)$$

If d_w was the euclidean distance d , which is the Monge case, then the supremum could be simplified as

$$\sup_{\phi \oplus \psi \leq d_w} \int \phi \, d\mu + \int \psi \, d\nu = \sup_{\phi \in \text{Lip}_{1,d}} \int \phi \, d(\nu - \mu),$$

and one is tempted to write the same for d_w , leading to a primal-dual problem:

$$\inf_{\pi \in \Pi(\mu, \nu)} \int_{K \times K} d_w \, d\pi = \sup_{\phi \in \text{Lip}_{1,d_w}} \int \phi \, d(\nu - \mu), \quad (5.3.4)$$

but the meaning of this last problem is not so clear because d_w is not a classical distance as it takes $+\infty$ values and the choice of the functional space for ϕ is not so clear either. A possibility would be to look for solutions

$$\phi \in L^1(\mu) \cap L^1(\nu) \cap \text{Lip}_{1,d_w}(\mathbb{R}^d) \quad (5.3.5)$$

where

$$\text{Lip}_{1,d} = \{\phi \in \mathcal{B}(\mathbb{R}^d, \mathbb{R}) : |\phi(y) - \phi(x)| \leq d_w(x, y) \text{ for all } x, y\} \quad (5.3.6)$$

This setting gives a (proposed) a rigorous meaning to (5.3.4). But then what could we say about Kantorovich potentials, that is optimal ϕ 's? It turns out they could play the role of the landscape function. Indeed, in the single source case $\mu = \delta_0$, the primal problem is trivial since there is only one transport plan, and the primal-dual problem (5.3.4) may be written as

$$\int_K d_w(0, y) \, d\nu(y) = \sup_{\phi \leq d_w(0, \cdot)} \int_K \phi(y) \, d\nu(y),$$

and $d_w(0, \cdot)$ is a solution. Now if $\eta \in \mathbf{OIP}^\alpha(\delta_0, \nu)$, then $w(x) = \theta_\eta(x)^{\alpha-1}$ is a solution to (5.3.2) and one expects that η is concentrated on d_w -geodesics so that $d_w(0, x) = z_\eta(x)$ for ν -a.e. x .

The idea is the following: given $\eta \in \mathbf{OIP}^\alpha(\mu, \nu)$, set $w_\eta = \theta_\eta^{\alpha-1}$, consider the problem

$$\sup_{\phi \in \text{Lip}_{1, d_w}} \int \phi \, d(\nu - \mu), \quad (5.3.7)$$

for example in the setting of (5.3.5), and define landscape functions as solutions of this problem. The difficulty is then to show that there are solutions ! In classical optimal transport theory, the dual problem

$$\sup_{\psi \oplus \phi \leq c} \int \phi \, d\nu + \int \psi \, d\mu$$

has solutions $\phi \in L^1(\nu), \psi \in L^1(\mu)$ when c is lower semicontinuous, bounded from below and **real-valued**, which is not the case here with $c = d_w$. Notice that if ϕ was a solution to (5.3.7) and if we had duality in the sense of (5.3.4), then one should have

$$\int \phi \, d(\nu - \mu) = \int_{K \times K} d_w \, d\pi_\eta = \int_\Gamma L_w(\gamma) \, d\eta(\gamma)$$

where $\pi_\eta \in \Pi(\mu, \nu)$ is the transport plan associated to η . Since $|\phi(y) - \phi(x)| \leq d_w(x, y)$ and since integrals with respect to μ, ν, π_η may be rewritten as integrals with respect to η , the previous equality should imply that $L_w(\gamma) = d_w(\gamma(0), \gamma(\infty)) = \phi(\gamma(\infty)) - \phi(\gamma(0))$ for η -a.e. γ . Notice that $L_w(\gamma) = Z_\eta(\gamma)$ hence Z_η depends only on the endpoints of the curves via $Z_\eta(\gamma) = \phi(\gamma(\infty)) - \phi(\gamma(0))$. In particular the first variation formula for \mathbf{I}^α (Proposition 1.2.1) should yield a first variation formula for d_α , as done in the proof of Proposition 3.1.5 with $\mu = \delta_0$ fixed, of the form

$$d_\alpha(\mu', \nu') \leq d_\alpha(\mu, \nu) + \alpha \int \phi \, d(\nu' - \nu) - \alpha \int \phi \, d(\mu' - \mu),$$

for all μ', ν' . Notice however that such a general inequality for arbitrary μ', ν' may not be true if η is not single rooted, as noticed in Theorem 4.3.9. With such a definition of landscape function one could hope to recover the fact that it is a kind of (super) first variation for d_α , which is the main property that is asked to tackle variational problems involving this distance. Moreover, this is also the only property that is used on the landscape function to prove Hölder continuity. Finally, notice that in the single source case, problem (5.3.2) rewrites as

$$\min_{w > 0} \frac{1}{-p} \int_K w(x)^p \, d\mathcal{H}^1(x) + \int_K z_w(x) \, d\nu(x),$$

where $z_w = d_w(0, \cdot)$. Since $z_w(x)$ is obtained by integrating w along a geodesic γ from 0 to x , w can be thought as some kind a derivative z'_w of z_w along the network, and if one could rephrase the problem in terms of $z = z_w$, it would read as

$$\min_z \frac{1}{-p} \int_K |z'(x)|^p \, d\mathcal{H}^1(x) + \int_K z(x) \, d\nu(x),$$

so that z would satisfy in some vague sense $\Delta_p z = \nu$ but with negative p . This does not seem completely unreasonable since a connection between the branched transport problem and p -harmonic maps has been made in the discrete case by Xia in [Xia14].

On the frontier between irrigation basins We place ourselves in the single source case. Take $\eta \in \mathbf{OIP}^\alpha(\mu, \nu)$ and recall the definition of basin from a point $x \in N_\eta$:

$$\text{Bas}_\eta(x) = \{y : \text{there is an } \eta\text{-good curve } \gamma \text{ s.t. } x \in \gamma \text{ and } y = \gamma(\infty)\}.$$

One can show that it is a closed set. Consider two basins B, B' from two points x, x' such that $x' \notin B$ and $x \notin B'$: what can we say on $B \cap B'$ when the intersection is not empty? A conjecture by J.-M. Morel is that it should be fractal, perhaps in the sense that $\dim_H(B \cap B') > d - 1$, a very tempting conjecture being $\dim_H(B \cap B') = d - \beta$. This question was one of the initial questions posed at the beginning of this thesis, although the apparently simpler question regarding the fractality of the optimal shapes considered in Chapter 3 could not be given a satisfying answer yet, hence much work and new ideas are needed in order to answer this question. What can only be proven for now is the fact that $B \cap B'$ is Lebesgue-negligible.

Numerical refinements for the shape optimization problem Our numerical approach described in Chapter 3 to compute a candidate solution to our shape optimization problem does not allow us to refine the discretization grid and do the computations in a reasonable time. We would like to have a more precise picture of what an optimizer should look like, and to be able to compute a box dimension of the boundary so as to support Conjecture 3.4.4 stating that $\dim_H(\partial E) = d - \beta$. In order to do that, one should use more sophisticated tools to refine the grid, perhaps by multi grid methods (to refine the grid near the boundary) or moving grid techniques. This is in project with E. Oudet. Also, one should at some point prove rigorously the Γ -convergence of our approximate elliptic functional towards the branched transport functional with inequality constraint on the divergence.

Variational approximation of the H -mass We would like to consider branched transport problems with more general costs than $\int \theta_\eta(x)^\alpha \, d\mathcal{H}^1(x)$, namely to replace the map $x \mapsto x^\alpha$ with a function $H : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ leading to a so-called H -mass:

$$\mathbf{M}^H(v) = \begin{cases} \int_{\mathbb{R}^d} H(|v(x)|) \, d\mathcal{H}^1(x) & \text{if } v \text{ is rectifiable,} \\ +\infty & \text{otherwise.} \end{cases} \quad (5.3.8)$$

It seems the idea of considering such costs comes from the notion of flat chains with coefficients in normed groups, as considered by White in [Whi99], the H -mass being introduced for the first time, as far as we know, in [DH03]. The standard hypotheses for H are the following:

- (i) $H(0) = 0$,
- (ii) H is lower semicontinuous and nondecreasing,
- (iii) H is subadditive,
- (iv) $\lim_{x \rightarrow 0} \frac{H(x)}{x} = +\infty$.

More general, the last hypothesis can be dropped, in which case one should take another definition of H -mass by lower semicontinuous envelope:

$$\mathbf{M}^H(T) = \inf_{P_n \xrightarrow{\mathbf{F}} T} \liminf_{n \rightarrow \infty} \mathbf{M}^H(P_n),$$

for flat chains T , that is the completion of polyhedral chains with respect to the flat norm \mathbf{F} and \mathbf{M}^H is defined by expression 5.3.8 on polyhedral chains. The validity of (5.3.8) when (iv) is satisfied, as well as a general representation formula when it does not hold, was established recently in [Col+17]. A definition of H -cost in a Lagrangian setting was given by [BW15], together with the proof on general properties in this case. We have already remarked in Chapter 2 that the equivalence with the Lagrangian and Eulerian models holds in the case of the H -mass with almost no changes, with extra hypotheses on H (those also assumed [BW15]). Notice that removing hypothesis (iv) does not guarantee that v is rectifiable but it can have an absolutely continuous part. It covers in particular the case of the urban planning studied in [But+09; BW15] where $H(m) = \min(am, 1 + bm)$ with $0 < b < a$.

What we would like to do now is perform numerical computations for such problems. Can we adapt the variation model a la Modica-Mortola to the case of the H -mass? In dimension 2, given H , we would like to find a function $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that \mathbf{M}_ε^h Γ -converges to \mathbf{M}^H where

$$\mathbf{M}_\varepsilon^h(v) = \varepsilon^{-1} \int_{\mathbb{R}^2} h(\varepsilon|v(x)|) dx + \varepsilon^3 \int_{\mathbb{R}^2} |\nabla v(x)|^2 dx.$$

We expect that a rectifiable current v of mass m concentrated on a segment of length ℓ should be approximated by a diffuse vector field v_ε concentrated on a tube around this segment, oriented by the segment and invariant along it. The flux on a perpendicular section should be equal to m and v_ε should optimize \mathbf{M}_ε^h with cost $\ell H(m)$, hence it should be given by a radial profile $u : \mathbb{R} \rightarrow \mathbb{R}$ which optimizes:

$$\min \left\{ \int h(|u(x)|) dx + \int u'(x)^2 dx : \int u = H(m) \right\}.$$

Questions arise: can we invert this formula, i.e. find an h such that it holds for all m, H being given? Is there a formula or an expression in terms of H ? Can we prove Γ -convergence?

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Titre : Transport branché et structures fractales

Mots clefs : transport branché, transport optimal, calcul des variations, fractales, théorie géométrique de la mesure

Résumé : Cette thèse est consacrée à l'étude du transport branché, de problèmes variationnels qui y sont liés et de structures fractales qui peuvent y apparaître. Le problème du transport branché consiste à connecter deux mesures de même masse par le biais d'un réseau en minimisant un certain coût, qui sera pour notre étude proportionnel à mL^α afin de déplacer une masse m sur une distance L . Plusieurs modèles continus ont été proposés pour formuler le problème, et on s'intéresse plus particulièrement aux deux grands types de modèles statiques : le modèle Lagrangien et le modèle Eulérien, avec une emphase sur le premier. Après avoir posé proprement les bases de ces modèles, on établit rigoureusement leur équivalence en utilisant une décomposition de Smirnov des mesures vectorielles à divergence mesure. On s'intéresse par la suite à un problème d'optimisation de forme lié au transport branché qui consiste à déterminer les ensembles de volume 1 les plus proches de l'origine au sens du transport branché. On démontre l'existence d'une solution, décrite comme un ensemble de sous-niveau de la fonction paysage, désormais standard en transport branché. La régularité Hölder de la fonction paysage, obtenue ici sans hypothèse de régularité a priori sur la solution considérée, permet d'obtenir une borne supérieure sur la dimension de Minkowski de son bord, qui est non-entière et dont on conjecture qu'elle en est la dimension exacte. Des simulations numériques, basées sur une approximation variationnelle à la Modica-Mortola de la fonctionnelle du transport branché, ont été effectuées dans le but d'étayer cette conjecture. Une dernière partie de la thèse se concentre sur la fonction paysage, essentielle à l'étude de problèmes variationnels faisant intervenir le transport branché en ce sens qu'elle apparaît comme une variation première du coût d'irrigation. Le but est d'étendre sa définition et ses propriétés fondamentales au cas d'une source étendue, ce à quoi l'on parvient dans le cas d'un réseau possédant un système fini de racines, par exemple pour des mesures à supports disjoints. On donne une définition satisfaisante de la fonction paysage dans ce cas, qui vérifie en particulier la propriété de variation première et on démontre sa régularité Hölder sous des hypothèses raisonnables sur les mesures à connecter.

Title : Branched transport and fractal structures

Keys words : branched transport, optimal transport, calcul of variations, fractals, geometric measure theory

Abstract : This thesis is devoted to the study of branched transport, related variational problems and fractal structures that are likely to arise. The branched transport problem consists in connecting two measures of same mass through a network minimizing a certain cost, which in our study will be proportional to mL^α in order to move a mass m over a distance L . Several continuous models have been proposed to formulate this problem, and we focus on the two main static models : the Lagrangian and the Eulerian ones, with an emphasis on the first one. After setting properly the bases for these models, we establish rigorously their equivalence using a Smirnov decomposition of vector measures whose divergence is a measure. Secondly, we study a shape optimization problem related to branched transport which consists in finding the sets of unit volume which are closest to the origin in the sense of branched transport. We prove existence of a solution, described as a sublevel set of the landscape function, now standard in branched transport. The Hölder regularity of the landscape function, obtained here without a priori hypotheses on the considered solution, allows us to obtain an upper bound on the Minkowski dimension of its boundary, which is non-integer and which we conjecture to be its exact dimension. Numerical simulations, based on a variational approximation a la Modica-Mortola of the branched transport functional, have been made to support this conjecture. The last part of the thesis focuses on the landscape function, which is essential to the study of variational problems involving branched transport as it appears as a first variation of the irrigation cost. The goal is to extend its definition and fundamental properties to the case of an extended source, which we achieve in the case of networks with finite root systems, for instance if the measures have disjoint supports. We give a satisfying definition of the landscape function in that case, which satisfies the first variation property and we prove its Hölder regularity under reasonable assumptions on the measures we want to connect.