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Some observer and feedback designs for finite dimensional nonlinear systems

Vincent Andrieu

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Vincent Andrieu. Some observer and feedback designs for finite dimensional nonlinear systems. Optimization and Control [math.OC]. Université Claude Bernard Lyon 1, 2018. tel-01711965

HAL Id: tel-01711965

<https://theses.hal.science/tel-01711965>

Submitted on 19 Feb 2018

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Habilitation à diriger la recherche

présentée à

L'Université Claude Bernard Lyon 1

par

Vincent ANDRIEU

Some observer and feedback designs for finite dimensional nonlinear systems

Soutenue le 16 janvier 2018 devant le Jury composé de

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Thanks

I would first like to thank Alessandro Astolfi, Gildas Besançon and Henk Nijmeijer, who were kind enough to agree to report on this manuscript.

I would also like to thank Hassan Hammouri for agreeing to be a member of my jury.

I want to thank Laurent Praly for agreeing to be a member of my jury of course, but also to have introduced me to research in control theory. I enjoyed working with Laurent as a supervisor but I enjoyed even better working with him as a colleague and friend.

I want to thank all my collaborators who often became friends: Bayu, Christophe, Madiha, Pascal, Sophie, Ulysse and I forget for sure some of them.

I don't forget the young guard who did me the honor to work with me on these subjects: Sofiane, Humberto, Chouaib, Swann, Pauline and Tu.

I want also to thank my colleagues from the LAGEP, from the funktional analysis group in Wuppertal and the MAC group in Toulouse.

Finally, I want to thank my wife Anne-Sophie who is my collaborator on my (by far) main result "Lucile, Loïc and Lise".

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A brief summary

To study physical and biological phenomena, the first step is to provide a mathematical model of this phenomenon in the form of an ordinary differential equation. From this model, the aim of a control scientist is to answer two fundamental questions :

- What is the actual state of the system ? Can we estimate it ?
- In which way do I have to act on some physical parameters of the system so that it achieves a given objective ?

The answer to these questions is expected to be in the form of an algorithm. This algorithm must provide a response with maximum accuracy keeping in mind modeling imperfections.

The central property which is hidden in this requirement is the Lyapunov stability property. This stability property can concern a set point (or equilibrium) or sets (as it is the case for instance when dealing with estimation problem). In this thesis I present some of the works I have done to analyze stability properties. I show also how they can be employed in the design of estimation and control algorithms.

In a first part of the thesis, I present analysis tools to characterize the stability property of invariant manifold and/or equilibria for uncontrolled dynamics. A very interesting analysis tool is the linearization approach and some homogeneous extensions of these approximations. I will present some of the works I have done in this topic.

In a second chapter, we consider the state estimation problem. Employing some of the analysis tools we have developed we derive some necessary conditions for the design of a particular estimation algorithm which is the asymptotic observer. From these necessary conditions, we give some sufficient conditions and consider several types of observer designs.

In a third chapter, we consider the feedback design for stabilization. Several control objectives are considered. In a first step, we consider the problem of designing a locally optimal stabilizing state feedback law. In a second step, we consider the problem of designing a stabilizing output feedback law forgetting the optimality requirement. Finally, we consider the case of the synthesis of synchronizing control laws.

In a conclusion, I present very briefly some other works on which I have been involved. I give also some perspectives of research.

Preamble

All along the manuscript, we shall consider a model that can be represented by an equation in the form :

$$\dot{x} = f(x, u) \quad , \quad y = h(x) \quad ,$$

where x is the state of the system which belongs to \mathbb{R}^n (sometimes x is decomposed in two parts x and z), y in \mathbb{R}^p is a vector which contains the available measurements and u in $\mathcal{U} \subset \mathbb{R}^q$ is a control input which characterizes how it is possible to modify the trajectories of the system. Also, to allow deeper analysis, we may consider systems in the form

$$\dot{x} = f(x) + g(x)u.$$

with g the controlled vector field or simply

$$\dot{x} = f(x) \quad ,$$

when the system is autonomous.

Note however, that some times, we may need to rewrite this system in a specific form (via change of coordinates or embedding for instance) in which the structure is more fruitful for the design. In that case, we will denote the system in the form

$$\dot{\xi} = \varphi(\xi, y, u) \quad , \quad y = \mathfrak{h}(\xi) \quad , \quad \xi \in \mathbb{R}^m \quad .$$

This emphasizes the fact that all properties which concern the vector field φ are coordinates dependent and strongly rely on the structure of φ and \mathfrak{h} .

I decided not to be exhaustive in the presentation of my work and to simplify some of the results which are given. This was done in order to ensure accessibility to the document. Thus, for example, this document does not deal with the results in which I participated in the field of infinite dimensional systems. Also, some results are not written with the maximum of generality. However, at the end of the document, I present very succinctly the other research topics on which I have been working on. Moreover, of course, references are given all along the manuscript in order to get access to the most general version of the results.

My publications quoted in the text appear with a "V" in the front and are reported at the end of the document in a separated reference list.

Chapter 1

Analysis of nonlinear systems based on homogeneous approximations

In a first step, we present some analysis tools that are instrumental to introduce some solutions to the state estimation and control design problems for nonlinear dynamical systems.

The main notion that will be studied all along this document is the notion of stability of trajectories or of invariant sets of dynamical systems. The most popular and efficient tool to address this notion is the use of Lyapunov functions. The use of Lyapunov functions in the study of stability properties has a long history. It can be traced back to Lyapunov himself who has introduced this concept in its dissertation in 1892 (see [79] for an english translation). The primary objective of a Lyapunov function is to analyze the behavior of trajectories of a dynamical systems and express how this behavior is preserved after perturbations. However, this tool is also very efficient to synthesize control algorithms as for instance stabilizing control laws, regulators, asymptotic observers (see for instance [48, 119, 60, 101]). In Chapters 2 and 3 estimation and stabilization problems are addressed most of the time employing these tools.

This is why the study of converse Lyapunov theorem have received a huge attention from the nonlinear control community. One of the first major contribution to the problem of existence of a Lyapunov function can be attributed to Massera [83]. This results have then been subsequently improved over the years (see [84, 70]) and we can quote Teel and Praly who established a theorem of existence of a Lyapunov function in a very general framework in [129]. However, despite the rise of a general theory to infer the existence of a Lyapunov function, its construction in practice appears to be a very difficult task.

On another hand, using a first order approximation to analyze the local stability of a nonlinear system is the most commonly used approach. Indeed, a first order analysis deals intrinsically with linear systems tools and it provides a simple way to construct local Lyapunov functions for a nonlinear system.

In this chapter, the *linearization approach* which is one particular case of *homogeneous approximation* is extended in three directions. The first extension is the case in which the stability studied concerns a simple invariant manifold and not an equilibrium point. This extension has been recently published in [V7] and [V8] and in this document we briefly rephrase these results. The two other extensions show that when dealing with equilibrium

points, global property may be characterized from homogeneous approximations. Indeed, the second extension is to show that if we consider first order approximations *along solutions* (and not only around the attractor) it is possible to construct global Lyapunov functions. This result appeared in [V5]. Finally, we introduce the notion of homogeneous in the bi-limit approximation in which not only local homogeneous approximations are considered but also some homogeneous approximations *at infinity*. This allows also to establish some global stability property. These results have been published in [V19].

1.1 Local transverse exponential stability of a manifold

1.1.1 Transverse local uniform exponential stability

Throughout this section, we consider an autonomous system in the following form

$$\dot{x} = f(x, z) , \quad \dot{z} = g(x, z) , \quad f(0, z) = 0 , \quad (1.1)$$

where x is in \mathbb{R}^{n_x} , z is in \mathbb{R}^{n_z} and the functions $f : \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \rightarrow \mathbb{R}^{n_x}$ and $g : \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \rightarrow \mathbb{R}^{n_z}$ are C^2 . We denote by $(X(x, z, t), Z(x, z, t))$ the (unique) solution which goes through (x, z) in $\mathbb{R}^{n_x} \times \mathbb{R}^{n_z}$ at time $t = 0$. It is assumed that these solutions are defined for all positive times, i.e. the system is *forward complete*.

For this system, the manifold $\mathcal{X} = \{(x, z), x = 0\}$ is an invariant manifold. The purpose of this section is to characterize exponential stability property of \mathcal{X} via a certain linearization. This will allow us to introduce (local) homogeneous Lyapunov function of order two (in x).

The local exponential stability of \mathcal{X} is called the *transverse local exponential stability* and is defined as follows.

Definition 1 (Transversal unif. loc. exp. stab. (TULES-NL)) *The system (1.1) is forward complete and there exist strictly positive real numbers r, k and λ such that we have, for all (x_0, z_0, t) in $\mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \times \mathbb{R}_{\geq 0}$ with $|x| \leq r$,*

$$|X(x_0, z_0, t)| \leq k|x_0| \exp(-\lambda t) . \quad (1.2)$$

In other words, the system (1.1) is said to be TULES-NL if the manifold $\mathcal{X} := \{(x, z) : x = 0\}$ is exponentially stable for the system (1.1), locally in x and uniformly in z .

1.1.2 Exponential stability of a linearized system

A linearized system "around" the invariant manifold can be considered. In this case, this one may be defined as :

$$\dot{\tilde{x}} = \frac{\partial f}{\partial x}(0, z)\tilde{x} , \quad \dot{z} = \tilde{g}(z) , \quad (1.3)$$

where $\tilde{g}(z) = g(0, z)$.

When considering the particular case in which there is no z dynamics and \mathcal{X} is simply the origin, it is well known that exponential stability of \mathcal{X} implies exponential stability of the origin of the first order approximation. It is possible to show that the same result holds in this more general context assuming some bounds on derivatives.

Proposition 1 ([V8]) *If Property TULES-NL holds and there exist positive real numbers ρ , μ and c such that, for all z in \mathbb{R}^{n_z} ,*

$$\left| \frac{\partial f}{\partial x}(0, z) \right| \leq \mu, \quad \left| \frac{\partial g}{\partial z}(0, z) \right| \leq \rho \quad (1.4)$$

and, for all (x, z) in $\mathbb{R}^{n_x} \times \mathbb{R}^{n_z}$ with $|x| \leq kr$,

$$\left| \frac{\partial^2 f}{\partial x \partial x}(x, z) \right| \leq c, \quad \left| \frac{\partial^2 f}{\partial z \partial x}(x, z) \right| \leq c, \quad \left| \frac{\partial g}{\partial x}(x, z) \right| \leq c, \quad (1.5)$$

then the system (1.3) is forward complete and there exist strictly positive real numbers \tilde{k} and $\tilde{\lambda}$ such that any solution $(\tilde{X}(\tilde{x}_0, z_0, t), Z(z_0, t))$ initiated from (\tilde{x}_0, z_0) at $t = 0$ of the transversally linear system (1.3) satisfies, for all (\tilde{x}_0, z_0, t) in $\mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \times \mathbb{R}_{\geq 0}$,

$$|\tilde{X}(\tilde{x}_0, z, t)| \leq \tilde{k} \exp(-\tilde{\lambda}t) |\tilde{x}_0|. \quad (1.6)$$

The proof of this proposition given in [V8] is based on the comparison between a given \tilde{x} -component of a solution $\tilde{X}(\tilde{x}_0, z_0, t)$ of (1.3) with pieces of x -component of solutions of (1.1) denoted $X(\tilde{x}_i, z_i, t - t_i)$ where \tilde{x}_i, z_i are sequences of points defined on $\tilde{X}(\tilde{x}_0, z_0, t)$. Thanks to the bounds (1.4) and (1.5), it is possible to show that \tilde{X} and X remain sufficiently closed so that \tilde{X} inherit the convergence property of the solution X .

In [V8], the exponential stability of the manifold $\tilde{\mathcal{X}} := \{(z, \tilde{x}) : \tilde{x} = 0\}$ of the linearized system transversal to \mathcal{X} in (1.3) is named property UES-TL.

1.1.3 Lyapunov matrix inequality

The \tilde{x} dynamics of the system (1.3) is a parametrized time varying linear system. Hence, the solutions $\tilde{X}(x, z, t)$, can be written as :

$$\tilde{X}(\tilde{x}, z, t) = \mathbb{M}(z, t) \tilde{x},$$

where \mathbb{M} is the transition matrix defined as a solution to the following $\mathbb{R}^{n_x \times n_x}$ dynamical system :

$$\dot{\mathbb{M}}(z, t) = \frac{\partial f}{\partial x}(0, \tilde{Z}(z, t)) \mathbb{M}(z, t), \quad \mathbb{M}(z, 0) = I.$$

An important point that has to be noticed is that due to equation (1.6), each element of the (matrix) time function $t \mapsto \mathbb{M}(z, t)$ is in $L^2([0, +\infty))$. Consequently, for all positive definite matrices Q in $\mathbb{R}^{n_x \times n_x}$, the matrix function

$$P(z) = \lim_{T \rightarrow +\infty} \int_0^T \mathbb{M}(z, s)^\top Q \mathbb{M}(z, s) ds \quad (1.7)$$

is well defined.

By computing the Lie derivative of the components of the matrix P given in (1.7) along \tilde{g} , it is possible to show that this one satisfies a particular partial differential equation which shows that this function may be used to construct a quadratic Lyapunov function in \tilde{x} of the linearized system.

Proposition 2 ([V8]) *Assume Property UES-TL holds, i.e. there exist \tilde{k} and $\tilde{\lambda}$ such that any solution $(\tilde{X}(\tilde{x}_0, z_0, t), Z(z_0, t))$ of the transversally linear system (1.3) satisfies, 1.6. Assume moreover, that there exists a positive real number μ such that*

$$\left| \frac{\partial f}{\partial x}(0, z) \right| \leq \mu \quad \forall z \in \mathbb{R}^{n_z}, \quad (1.8)$$

then for all positive definite matrix Q , the function $P : \mathbb{R}^{n_z} \rightarrow \mathbb{R}^{n_x \times n_x}$ defined in (1.7) is well defined and continuous and there exist strictly positive real numbers \underline{p} and \bar{p} such that P has a derivative $\mathfrak{d}_{\tilde{g}}P$ along \tilde{g} in the following sense

$$\mathfrak{d}_{\tilde{g}}P(\tilde{z}) := \lim_{h \rightarrow 0} \frac{P(\tilde{Z}(z, h)) - P(z)}{h}, \quad (1.9)$$

and we have, for all z in \mathbb{R}^{n_z} ,

$$\mathfrak{d}_{\tilde{g}}P(z) + P(z) \frac{\partial f}{\partial x}(0, z) + \frac{\partial f}{\partial x}(0, z)^\top P(z) \leq -Q, \quad (1.10)$$

$$\underline{p}I \leq P(z) \leq \bar{p}I. \quad (1.11)$$

When looking at the time derivative of the function $(\tilde{x}, z) \mapsto \tilde{x}^\top P(z) \tilde{x}$ along the solution of the system (1.3), it yields from (1.10) :

$$\overbrace{\tilde{x}^\top P(z) \tilde{x}}^{\cdot} = -\tilde{x}^\top Q \tilde{x}.$$

Hence, $(\tilde{x}, z) \mapsto \tilde{x}^\top P(z) \tilde{x}$ is a Lyapunov function associated to the \tilde{x} component of the linearized system (1.3).

The assumption (1.8) is used to show that P satisfies the left inequality in (1.11). Nevertheless this inequality holds without (1.8) provided the function $s \mapsto \left| \frac{\partial \tilde{X}}{\partial x}(0, z, s) \right|$ does not go too fast to zero.

1.1.4 Construction of a Lyapunov function

From the matrix function P obtained previously, it is possible to define a Lyapunov function which allows to characterize the property of local exponential stability of \mathcal{X} .

Proposition 3 ([V8]) *Assume there exists a matrix function P which satisfies (1.11) and for which the derivative in the sense of (1.9) exists and satisfies (1.10) and there exist positive real numbers η and c such that, for all (x, z) in $\mathbb{R}^{n_x} \times \mathbb{R}^{n_z}$ with $|x| \leq \eta$,*

$$\left| \frac{\partial P}{\partial z}(z) \right| \leq c, \quad (1.12)$$

$$\left| \frac{\partial^2 f}{\partial z \partial x}(x, z) \right| \leq c, \quad \left| \frac{\partial^2 f}{\partial z \partial x}(x, z) \right| \leq c, \quad \left| \frac{\partial g}{\partial x}(x, z) \right| \leq c, \quad (1.13)$$

then Property TULES-NL holds.

This is a direct consequence of the use of $V(x, z) = x^\top P(z)x$ as a Lyapunov function. The bounds (1.12) and (1.13) are used to show that, with equation (1.10), the time derivative of this Lyapunov function is negative in a (uniform) tubular neighborhood of the manifold $\mathcal{X} = \{(x, z), x = 0\}$.

1.1.5 Conclusion on transverse exponential stability

All this framework that, together with Bayu Jayawardhana and Laurent Praly, we have introduced in [V8] has been employed as a design tool in different research topics.

- It has been employed to construct a Lyapunov function which characterizes the property of exponential incremental stability in [V8].
- It can be used to show that a detectability property introduced by Laurent Praly and Ricardo Sanfelice in [114] is a necessary condition to the existence of an exponential full order observer. This is done in a slightly more general context in Chapter 2 employing some results of [V6] (see also [V8]).
- It has been employed by Wang, Ortega and coworkers in [130] as a design tool in parameter identifications.
- With L. Praly, B. Jayawardhana and S. Tarbouriech we have employed it in [V8, V9, V10] in the context of nonlinear synchronization. See also Chapter 3.

All results written so far concern local properties. The following two sections are concerned with global stability properties of an equilibrium point.

1.2 Global stability properties based on first order approximation

1.2.1 Local exponential stability and global attractivity

In this Section, we consider an autonomous system in the form

$$\dot{x} = f(x), \quad f(0) = 0, \quad (1.14)$$

with x in \mathbb{R}^n and we assume that it is forward complete and that the origin is locally exponentially stable. We assume also that the origin is *globally attractive* in the following sense.

Definition 2 (Local Exponential Stability and Global Attractivity (LES-GA))

The origin of system (1.14) is forward complete and

1. *there exist strictly positive real numbers r , k and λ such that we have, for all (x, t) in $\mathbb{R}^n \times \mathbb{R}_{\geq 0}$ with $|x| \leq r$,*

$$|X(x, t)| \leq k|x| \exp(-\lambda t). \quad (1.15)$$

2. *for all x in \mathbb{R}^n*

$$\lim_{t \rightarrow +\infty} |X(x, t)| = 0. \quad (1.16)$$

Note that global attractivity in combination with the local exponential stability of the origin implies that the system is globally and asymptotically stable. However, it is not globally exponentially stable in the usual sense (see [60, definition 4.5 p.150]). It may be not possible to find positive real numbers k and λ such that (1.15) holds for all x in \mathbb{R}^n (see for instance [95] for a counter example). Nevertheless the following characterization can be simply obtained.

Proposition 4 ([V5]) *The origin of (1.14) is locally exponentially stable and globally attractive if and only if there exist a positive real number λ and a continuous strictly increasing function $k : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that :*

$$|X(x, t)| \leq k(|x|) \exp(-\lambda t) |x| . \quad (1.17)$$

1.2.2 Global Lyapunov functions based on first order approximations

To oppose to the former section in which only local approximation of dynamics could be considered, in this section, since we are dealing with global properties, the linearized system along all solutions have to be considered. Assuming that f is C^1 everywhere, the linearized system along trajectories is defined as :

$$\dot{\tilde{x}} = \frac{\partial f}{\partial x}(x) \tilde{x} , \quad \dot{x} = f(x) , \quad (1.18)$$

with (x, \tilde{x}) in $\mathbb{R}^n \times \mathbb{R}^n$. This system is also called the *lifted* system in [35] or the *variational system* in [31].

Note that the \tilde{x} -components of this system may be rewritten as follows :

$$\underbrace{\dot{\tilde{x}} = \frac{\partial f}{\partial x}(0) \tilde{x}}_{\text{(LES)} \Rightarrow \text{goes exp. to zero}} + \underbrace{\left[\frac{\partial f}{\partial x}(x) - \frac{\partial f}{\partial x}(0) \right]}_{\text{(Glob. Attract.)} \Rightarrow \text{goes to zero}} \tilde{x} . \quad (1.19)$$

The following proposition shows that if the x components go exponentially to zero, then the \tilde{x} components do the same.

Proposition 5 ([V5]) *Let f be C^1 in \mathbb{R}^n and C^2 around the origin. Assume the origin of (1.14) is locally exponentially stable and globally attractive, then there exist a positive real number $\tilde{\lambda}$ and a strictly increasing function $\tilde{k} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that :*

$$|\tilde{X}(x, t)| \leq \tilde{k}(|x|) \exp(-\tilde{\lambda} t) |\tilde{x}| . \quad (1.20)$$

By linearity the \tilde{x} components of the linearized system (1.18) can be written :

$$\tilde{X}(\tilde{x}, x, t) = \mathbb{M}(x, t) \tilde{x} ,$$

where \mathbb{M} is the transition matrix. This transition matrix is defined as the solution of the following $\mathbb{R}^{n \times n}$ dynamical system :

$$\dot{\mathbb{M}}(x, t) = \frac{\partial f}{\partial x}(X(x, t)) \mathbb{M}(x, t) , \quad \mathbb{M}(x, 0) = I .$$

An important point that has to be noticed is that due to equation (1.20), each element of the (matrix) time function $t \mapsto \mathbb{M}(x, t)$ is in $L^2([0, +\infty))$. Consequently, for all positive definite matrix Q in $\mathbb{R}^{n \times n}$, the matrix function :

$$P(x) = \lim_{T \rightarrow +\infty} \int_0^T \mathbb{M}(x, s)^\top Q \mathbb{M}(x, s) ds , \quad (1.21)$$

is well defined. Moreover, it can be shown that the following proposition holds.

Proposition 6 ([V5]) Assume that there exist function (k, \tilde{k}) and positive real numbers $(\lambda, \tilde{\lambda})$ such that (1.17) and (1.20) are satisfied. Then, the matrix function $P : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$ defined in (1.21) is well defined, continuous, and there exist a non increasing function \underline{p} and a non decreasing function \bar{p} such that

$$0 < \underline{p}(|x|)I \leq P(x) \leq \bar{p}(|x|)I, \quad \forall x \in \mathbb{R}^n. \quad (1.22)$$

Moreover¹,

$$\underbrace{\mathfrak{d}_f P(x) + P(x) \frac{\partial f}{\partial x}(x) + \frac{\partial f}{\partial x}(x)^\top P(x)}_{=L_f P(x)} \leq -Q, \quad \forall x \in \mathbb{R}^n. \quad (1.23)$$

Finally, if the vector field f is C^3 then P is C^2 .

With the matrix function P defined for instance in (1.21) which Lie derivative satisfies inequality (1.23), it yields that along the solution of the linearized system (1.18) :

$$\overline{\tilde{x}^\top P(x) \tilde{x}} \leq -\tilde{x}^\top Q \tilde{x}.$$

In other words, the mapping $(\tilde{x}, x) \mapsto \tilde{x}^\top P(x) \tilde{x}$ is a global Lyapunov function for the \tilde{x} components of the linearized system (1.21).

However, $x \mapsto x^\top P(x)x$ is not a global Lyapunov function for $\dot{x} = f(x)$. Indeed, a simple computation gives :

$$\overline{x^\top P(x)x} \leq 2x^\top P(x) \left[f(x) - \frac{\partial f}{\partial x}(x)x \right] - x^\top Qx.$$

This is negative definite if $f(x) - \frac{\partial f}{\partial x}(x)x$ is small. However, there is no guarantee that this is the case away from the origin.

Nevertheless, it is still possible to construct a Lyapunov function for the system (1.14). Indeed, the matrix function P may be used to define a Riemannian metric on \mathbb{R}^n which may be used as a Lyapunov function. Precisely, if P is a C^2 function the values of which are symmetric matrices satisfying (1.22), the associated Riemannian length of any piecewise C^1 path $\gamma : [s_1, s_2] \rightarrow \mathbb{R}^n$ between two arbitrary points $x_1 = \gamma(s_1)$ and $x_2 = \gamma(s_2)$ in \mathbb{R}^n is defined as :

$$L(\gamma) \Big|_{s_1}^{s_2} = \int_{s_1}^{s_2} \sqrt{\frac{d\gamma}{ds}(\sigma)^\top P(\gamma(\sigma)) \frac{d\gamma}{ds}(\sigma)} d\sigma. \quad (1.24)$$

By minimizing along all such path we get the Riemannian distance $d_P(x_1, x_2)$ associated to the metric defined from P .

Then, thanks to the well established relation between (geodesically) monotone vector field (semi-group generator) (operator) and contracting (non-expansive) flow (semi-group) (see [75, 41, 20, 47] and many others), we know that if P is C^2 and the metric space is complete, this distance between any two solutions of (1.14) is exponentially decreasing to 0 as time goes on forward if (1.23) is satisfied with Q a positive definite symmetric matrix. For a proof, see for example [75, Theorem 1] or [47, Theorems 5.7 and 5.33] or [110, Lemma 3.3] (replacing $f(x)$ by $x + hf(x)$).

¹See the notation (1.9).

From this fact, a candidate Lyapunov function is the Riemannian distance to the origin. Hence we introduce the function $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$

$$V(x) = d_P(x, 0) . \quad (1.25)$$

In the following proposition it is shown that this function is indeed a good Lyapunov function candidate and moreover that it admits an upper Dini derivative along the solution of the system (1.14) which is negative definite.

Proposition 7 ([V5]) *Assume f is C^2 . Assume moreover that there exists a C^2 matrix function P such that equations (1.22) and (1.23) hold and that the function \underline{p} satisfies the following property*

$$\lim_{r \rightarrow +\infty} \underline{p}(r)r^2 = +\infty . \quad (1.26)$$

Then the function V defined in (1.25) is a Lyapunov function for the system (1.14). More precisely V admits an upper Dini derivative along the solutions of system (1.14) defined as

$$D_f^+ V(x) := \limsup_{h \searrow 0} \frac{V(X(x, h)) - V(x)}{h} ,$$

which satisfies

$$D_f^+ V(x) \leq -\frac{\mu_{\min}\{Q\}}{\bar{p}(|x|)} V(x) .$$

Hence the origin is locally exponentially stable and globally attractive (LES-GA).

The requirement (1.26) is essential to make sure that \mathbb{R}^n endowed with the Riemannian metric P is complete. It is also essential to make sure that the obtained Lyapunov function is proper. It imposes that the mapping \underline{p} doesn't vanish too quickly as $|x|$ goes to infinity. Going back to a definition of the mapping \underline{p} obtained in the proof of Proposition 6 in [V5], it yields that if the vector field f is globally Lipschitz then \underline{p} is a constant. In other words, in the globally Lipschitz context this assumption is trivially satisfied.

Another solution to make sure that this assumption is satisfied is to modify the function P to ensure that this one is lower bounded by a positive real number. Following this route, note that the trajectories of the system

$$\dot{x} = \frac{f(x)}{1 + \left| \frac{\partial f}{\partial x}(x) \right|^3} , \quad \dot{\tilde{x}} = \frac{\frac{\partial f}{\partial x}(x)}{1 + \left| \frac{\partial f}{\partial x}(x) \right|^3} \tilde{x}$$

are the same than the one of the lifted system (1.18) (this system is obtained after a time rescaling). Consequently, the origin is globally attractive. Moreover, it is not difficult to show that its origin is also locally exponentially stable. Finally, if f is C^4 then the vector field $x \mapsto \frac{f(x)}{1 + \left| \frac{\partial f}{\partial x}(x) \right|^3}$ is C^3 . Let $\tilde{\mathbb{M}}$ be the transition matrix defined as the solution of the following $\mathbb{R}^{n \times n}$ dynamical system :

$$\frac{d}{dt} \tilde{\mathbb{M}}(x, t) = \frac{\frac{\partial f}{\partial x}(X(x, t))}{1 + \left| \frac{\partial f}{\partial x}(X(x, t)) \right|^3} \tilde{\mathbb{M}}(x, t) , \quad \tilde{\mathbb{M}}(x, 0) = I .$$

Again, each element of the (matrix) time function $t \mapsto \tilde{\mathbb{M}}(x, t)$ is in $L^2([0, +\infty))$. Consequently, for all positive definite matrix Q in $\mathbb{R}^{n \times n}$, the matrix function :

$$\tilde{P}(x) = \lim_{T \rightarrow +\infty} \int_0^T \tilde{\mathbb{M}}(x, s)^\top Q \tilde{\mathbb{M}}(x, s) ds , \quad (1.27)$$

is well defined. With this mapping, the following property may be obtained.

Proposition 8 (Lower bounded P [V5]) *Assume that there exist function (k, \tilde{k}) and positive real numbers $(\lambda, \tilde{\lambda})$ such that (1.17) and (1.20) are satisfied. Then, the matrix function $P : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$ defined in (1.27) is well defined, continuous, and there exist a positive real number \underline{p} and a non decreasing function \bar{p} such that*

$$0 < \underline{p}I \leq \tilde{P}(x) \leq \bar{p}(|x|)I, \quad \forall x \in \mathbb{R}^n. \quad (1.28)$$

Moreover,

$$\mathfrak{d}_f \tilde{P}(x) + \tilde{P}(x) \frac{\partial f}{\partial x}(x) + \frac{\partial f}{\partial x}(x)^\top \tilde{P}(x) \leq -Q \left(1 + \left| \frac{\partial f}{\partial x}(x) \right|^3 \right), \quad \forall x \in \mathbb{R}^n. \quad (1.29)$$

Finally, if the vector field f is C^4 then P is C^2 .

Since the matrix function P is lower bounded, we can define a Lyapunov function following the proposition 7. Roughly speaking we have established the following Lyapunov inverse result : *Assuming some regularity on the system, if the origin is locally exponentially stable and globally attractive then there exists a strictly decreasing Lyapunov function given as a Riemannian distance to the origin.*

This property gives a new insight on the link between global asymptotic stability and incrementally stable systems as already studied in [93].

1.2.3 Conclusion on global stability property from linearization

In this section we have shown how Lyapunov functions can be obtained from first order approximation. Of course the local exponential stability property is essential. Note that in [37], is shown that up to a change of coordinates (which is not a diffeomorphism since it is not smooth at the origin) it is possible to transform any system which origin is asymptotically stable into a system which origin is exponentially stable. This implies that up to a change of variable, it is always possible to consider Lyapunov function coming from a Riemannian distance.

This facts motivates the study of first order analysis for nonlinear dynamics in order to construct global control algorithms.

All the results presented in this Section have been proved in [V5]. Note however that most of the result are not completely new. For instance using Riemannian metric as Lyapunov function to be used in control has been done (for instance) recently by F. Forni and R. Sepulchre in [30] (see also [31]). Moreover, it has also been used by R. Sanfelice and L. Praly in the context of observer design in [113] or [114]. Note also that with B. Jayawardhana and L. Praly, we have employed it in the context of synchronization in [V8] (see also Section 3.4).

1.3 Homogeneous in the bi-limit approximations

Another way to obtain global characterizations from homogeneous approximation is to consider homogeneous in the bi-limit approximations. To introduce the tools that can be considered to analyze stability properties of solutions of a nonlinear system, let us considered the particular case in which system (1.14) is

$$\dot{x}_1 = x_2 - x_1, \quad \dot{x}_2 = -x_1 - x_2 + x_2^q \quad (1.30)$$

For this system, the origin is an equilibrium point and to analyze its stability, the usual approach consists in considering the local homogeneous approximation around the origin. In this case and when $q > 1$ it is the first order approximation, i.e. the linear system

$$\dot{x}_1 = x_2 - x_1 \quad , \quad \dot{x}_2 = -x_2 - x_1 \quad .$$

The origin of this system is asymptotically stable, and we can conclude to the local and asymptotic stability of the origin of the system (1.30).

Note however that for large value of x_1 and x_2 the linear approximation is no longer valid. In fact, as we have shown in [V19] together with Laurant Praly and Alessandro Astolfi, it is possible to consider another homogeneous approximation which will characterize the behavior of the solutions for large values of the state. This new framework is named homogeneity in the bi-limit. The aim of this section is to present this tool.

1.3.1 Definition of homogeneity in the bi-limit

The use of homogeneous approximations has a long history in the study of stability of an equilibrium. It can be traced back to Lyapunov first order approximation theorem and has been pursued by many authors, for example Massera [84], Hahn [38], Hermes [44], Rosier [112]. Similarly this technique has been used to investigate the behavior of the solutions of dynamical systems at infinity, see for instance Lefschetz in [71, IX.5] and Orsi, Praly and Mareels in [91]. In this section, we recall the definitions of homogeneous approximation at the origin and at infinity and restate and/or complete some related results.

The local homogeneous approximation of a function or of a dynamical system can be defined as follows.

Definition 3 (Homogeneity in the 0-limit)

1. A function $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$ is said homogeneous in the 0-limit with associated triple $(r_0, \mathfrak{d}_0, \phi_0)$, where r_0 in $(\mathbb{R}_+ \setminus \{0\})^n$ is the weight, \mathfrak{d}_0 in \mathbb{R}_+ the degree and $\phi_0 : \mathbb{R}^n \rightarrow \mathbb{R}$ the approximating function, if ϕ is continuous, ϕ_0 is continuous and not identically zero and, for each compact set C in $\mathbb{R}^n \setminus \{0\}$ and each $\varepsilon > 0$, there exists λ_0 such that :

$$\max_{x \in C} \left| \frac{\phi(\lambda^{r_0,1} x_1, \dots, \lambda^{r_0,n} x_n)}{\lambda^{\mathfrak{d}_0}} - \phi_0(x) \right| \leq \varepsilon \quad , \quad \forall \lambda \in (0, \lambda_0] \quad .$$

2. A vector field $f = \sum_{i=1}^n f_i \frac{\partial}{\partial x_i}$ is said homogeneous in the 0-limit with associated triple $(r_0, \mathfrak{d}_0, f_0)$, where r_0 in $(\mathbb{R}_+ \setminus \{0\})^n$ is the weight, \mathfrak{d}_0 in \mathbb{R} is the degree and $f_0 = \sum_{i=1}^n f_{0,i} \frac{\partial}{\partial x_i}$ the approximating vector field, if, for each i in $\{1, \dots, n\}$, $\mathfrak{d}_0 + r_{0,i} \geq 0$ and the function f_i is homogeneous in the 0-limit with associated triple $(r_0, \mathfrak{d}_0 + r_{0,i}, f_{0,i})$.

This notion of local approximation of a function or of a vector field can be found in [44, 112, 12, 45].

Similarly, we can define the notion of homogeneity at infinity as follows.

Definition 4 (Homogeneity in the ∞ -limit)

1. A function $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$ is said homogeneous in the ∞ -limit with associated triple $(r_\infty, d_\infty, \phi_\infty)$, where r_∞ in $(\mathbb{R}_+ \setminus \{0\})^n$ is the weight, d_∞ in \mathbb{R}_+ the degree and $\phi_\infty : \mathbb{R}^n \rightarrow \mathbb{R}$ the approximating function, if ϕ is continuous, ϕ_∞ is continuous and not identically zero and, for each compact set C in $\mathbb{R}^n \setminus \{0\}$ and each $\varepsilon > 0$, there exists λ_∞ such that :

$$\max_{x \in C} \left| \frac{\phi(\lambda^{r_\infty, 1} x_1, \dots, \lambda^{r_\infty, n} x_n)}{\lambda^{d_\infty}} - \phi_\infty(x) \right| \leq \varepsilon, \quad \forall \lambda \geq \lambda_\infty.$$

2. A vector field $f = \sum_{i=1}^n f_i \frac{\partial}{\partial x_i}$ is said homogeneous in the ∞ -limit with associated triple $(r_\infty, \mathfrak{d}_\infty, f_\infty)$, where r_∞ in $(\mathbb{R}_+ \setminus \{0\})^n$ is the weight, \mathfrak{d}_∞ in \mathbb{R} the degree and $f_\infty = \sum_{i=1}^n f_{\infty, i} \frac{\partial}{\partial x_i}$ the approximating vector field, if, for each i in $\{1, \dots, n\}$, $\mathfrak{d}_\infty + r_{\infty, i} \geq 0$ and the function f_i is homogeneous in the ∞ -limit with associated triple $(r_\infty, \mathfrak{d}_\infty + r_{\infty, i}, f_{\infty, i})$.

Merging the last two definitions, we obtain homogeneity in the bi-limit.

Definition 5 (Homogeneity in the bi-limit) A function $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$ (or a vector field $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$) is said homogeneous in the bi-limit if it is homogeneous in the 0-limit and homogeneous in the ∞ -limit.

Example 1 When $0 < \mathfrak{q} < 2$ the vector field f given in (1.30) is homogeneous in the bi-limit. When $1 < \mathfrak{q} < 2$ the weights are $r_0 = (1, 1)$ and $r_\infty = (2 - \mathfrak{q}, 1)$, the degrees $\mathfrak{d}_0 = 0$ and $\mathfrak{d}_\infty = 2 - \mathfrak{q}$ and approximating vector fields :

$$f_0(x) = \begin{bmatrix} x_2 - x_1 \\ -x_1 - x_2 \end{bmatrix}, \quad f_\infty(x) = \begin{bmatrix} x_2 \\ x_2^\mathfrak{q} \end{bmatrix}. \quad (1.31)$$

Note that when $0 < \mathfrak{q} < 1$, homogeneous approximations are exchanged. In other words, $\mathfrak{d}_\infty = 0$ and $\mathfrak{d}_0 = 2 - \mathfrak{q}$ and approximating vector fields :

$$f_0(x) = \begin{bmatrix} x_2 \\ x_2^\mathfrak{q} \end{bmatrix}, \quad f_\infty(x) = \begin{bmatrix} x_2 - x_1 \\ -x_1 - x_2 \end{bmatrix}.$$

1.3.2 Stability and homogeneous approximation

A very basic property of asymptotic stability is its robustness. This fact was already known to Lyapunov who proposed his second method, (local) asymptotic stability of an equilibrium is established by looking at the first order approximation of the system. This is what has been done in the first section of this chapter when dealing with transverse properties. The case of local homogeneous approximations of higher degree has been investigated by Massera [84], Hermes [44] and Rosier [112].

Proposition 9 ([112]) Consider a homogeneous in the 0-limit vector field $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ with associated triple $(r_0, \mathfrak{d}_0, f_0)$. If the origin of the system :

$$\dot{x} = f_0(x)$$

is locally asymptotically stable then the origin of $\dot{x} = f(x)$ is locally asymptotically stable.

Consequently, a natural strategy to study local asymptotic stability of an equilibrium of a system is to focus on homogeneous approximation in the 0 limit. This has been employed in control design (see [44, 57, 24] for instance).

In the context of homogeneity in the ∞ -limit, we have introduced the following result.

Proposition 10 ([V19]) Consider a homogeneous in the ∞ -limit vector field $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ with associated triple $(r_\infty, \mathfrak{d}_\infty, f_\infty)$. If the origin of the system :

$$\dot{x} = f_\infty(x) ,$$

is globally asymptotically stable then there exists an invariant compact subset of \mathbb{R}^n , denoted \mathcal{C}_∞ , which is globally asymptotically stable for the system :

$$\dot{x} = f(x) .$$

The key step in the proof of Propositions 9 and 10 is the converse Lyapunov theorem given by Rosier in [112]. Indeed, Rosier has shown that given a homogeneous vector field such that the origin is asymptotically stable, then there exists a homogeneous Lyapunov function. In [V19], we have extended this result to the case of homogeneity in the bi-limit.

Theorem 1 (Homogeneous in the bi-limit Lyapunov functions [V19]) Consider a homogeneous in the bi-limit vector field $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$, with associated triples $(r_\infty, \mathfrak{d}_\infty, f_\infty)$ and $(r_0, \mathfrak{d}_0, f_0)$ such that the origins of the systems :

$$\dot{x} = f(x) \quad , \quad \dot{x} = f_\infty(x) \quad , \quad \dot{x} = f_0(x) \quad (1.32)$$

are globally asymptotically stable equilibria. Let d_{V_∞} and d_{V_0} be real numbers such that $d_{V_\infty} > \max_{1 \leq i \leq n} r_{\infty,i}$ and $d_{V_0} > \max_{1 \leq i \leq n} r_{0,i}$. Then there exists a C^1 , positive definite and proper function $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$ such that, for each i in $\{1, \dots, n\}$, the functions $x \mapsto \frac{\partial V}{\partial x_i}$ is homogeneous in the bi-limit with associated triples $(r_0, d_{V_0} - r_{0,i}, \frac{\partial V_0}{\partial x_i})$ and $(r_\infty, d_{V_\infty} - r_{\infty,i}, \frac{\partial V_\infty}{\partial x_i})$ and the function $x \mapsto \frac{\partial V}{\partial x}(x) f(x)$, $x \mapsto \frac{\partial V_0}{\partial x}(x) f_0(x)$ and $x \mapsto \frac{\partial V_\infty}{\partial x}(x) f_\infty(x)$ are negative definite.

A direct consequence of this result is an Input-to-State Stability (ISS) property with respect to disturbances (see [124]). To illustrate this property, consider the system with exogenous disturbance $\delta = (\delta_1, \dots, \delta_m)$ in \mathbb{R}^m :

$$\dot{x} = f(x, \delta) , \quad (1.33)$$

with $f : \mathbb{R}^n \times \mathbb{R}^m$ a continuous vector field homogeneous in the bi-limit with associated triples $(\mathfrak{d}_0, (r_0, \mathfrak{r}_0), f_0)$ and $(\mathfrak{d}_\infty, (r_\infty, \mathfrak{r}_\infty), f_\infty)$ where \mathfrak{r}_0 and \mathfrak{r}_∞ in $(\mathbb{R}_+ \setminus \{0\})^m$ are the weights associated to the disturbance δ .

Corollary 1 (ISS Property [V19]) If the origins of the systems :

$$\dot{x} = f(x, 0) \quad , \quad \dot{x} = f_0(x, 0) \quad , \quad \dot{x} = f_\infty(x, 0)$$

are globally asymptotically stable equilibria, then under the hypotheses of Theorem 1 the function V given by Theorem 1 satisfies for all $\delta = (\delta_1, \dots, \delta_m)$ in \mathbb{R}^m and x in \mathbb{R}^n :

$$(1.34) \quad \frac{\partial V}{\partial x}(x) f(x, \delta) \leq -c_V \mathfrak{H} \left(V(x)^{\frac{d_{V_0} + \mathfrak{d}_0}{d_{V_0}}}, V(x)^{\frac{d_{V_\infty} + \mathfrak{d}_\infty}{d_{V_\infty}}} \right) + c_\delta \sum_{j=1}^m \mathfrak{H} \left(|\delta_j|^{\frac{d_{V_0} + \mathfrak{d}_0}{\mathfrak{r}_{0,j}}}, |\delta_j|^{\frac{d_{V_\infty} + \mathfrak{d}_\infty}{\mathfrak{r}_{\infty,j}}} \right) ,$$

where c_V and c_δ are positive real numbers and the function $\mathfrak{H} : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ is defined as

$$\mathfrak{H}(a, b) = \frac{a}{1+a} [1 + b] . \quad (1.35)$$

In other words, system (1.33) with δ as input is ISS.

Finally, we have also the following small-gain result for homogeneous in the bi-limit vector fields.

Corollary 2 (Small-Gain [V19]) *Under the hypotheses of Corollary 1, there exists a real number $c_G > 0$ such that, for each class \mathcal{K} function γ_z and \mathcal{KL} function β_δ , there exists a class \mathcal{KL} function β_x such that, for each function $t \in [0, T) \mapsto (x(t), \delta(t), z(t))$, $T \leq +\infty$, with x C^1 and δ and z continuous, which satisfies, both (1.33) on $[0, T)$ and, for all $0 \leq s \leq t \leq T^2$,*

$$|z(t)| \leq \max \left\{ \beta_\delta(|z(s)|, t-s), \sup_{s \leq \kappa \leq t} \gamma_z(|x(\kappa)|) \right\} , \quad (1.36)$$

$$|\delta_i(t)| \leq \max \left\{ \beta_\delta(|z(s)|, t-s), c_G \sup_{s \leq \kappa \leq t} \left\{ \mathfrak{H}(|x(\kappa)|_{r_0}^{\mathfrak{r}_{0,i}}, |x(\kappa)|_{r_\infty}^{\mathfrak{r}_{\infty,i}}) \right\} \right\} , \quad (1.37)$$

we have

$$|x(t)| \leq \beta_x(|(x(s), z(s))|, t-s) \quad 0 \leq s \leq t \leq T . \quad (1.38)$$

An interesting case which can be dealt with by Corollary 2 is when the δ_i 's are outputs of auxiliary systems with state z_i in \mathbb{R}^{n_i} , i.e :

$$\delta_i(t) := \delta_i(z_i(t), x(t)) \quad , \quad \dot{z}_i = g_i(z_i, x) . \quad (1.39)$$

It can be checked that the bounds (1.37) and (1.36) are satisfied by all the solutions of (1.33) and (1.39) if there exist positive definite and radially unbounded functions $Z_i : \mathbb{R}^{n_i} \rightarrow \mathbb{R}_+$, class \mathcal{K} functions ω_1 , ω_2 and ω_3 , a positive real number ϵ in $(0, 1)$ such that for all x in \mathbb{R}^n , for all i in $\{1, \dots, m\}$ and z_i in \mathbb{R}^{n_i} , we have :

$$|\delta_i(z_i, x)| \leq \omega_1(|x|) + \omega_2(Z_i(z_i)) \quad , \quad \frac{\partial Z_i}{\partial z_i}(z_i) g_i(z_i, x) \leq -Z_i(z_i) + \omega_3(|x|) ,$$

$$\omega_1(|x|) + \omega_2([1 + \epsilon] \omega_3(|x|)) \leq c_G \mathfrak{H}(|x|_{r_0}^{\mathfrak{r}_{0,i}}, |x|_{r_\infty}^{\mathfrak{r}_{\infty,i}}) .$$

Another important result exploiting Theorem 1 deals with finite time convergence of solutions to the origin when this is a globally asymptotically stable equilibrium (see [16]). It is well known that when the origin of the homogeneous approximation in the 0-limit is globally asymptotically stable and with a strictly negative degree then solutions converge to the origin in finite time (see [17]). We extend this result by showing that if, furthermore the origin of the homogeneous approximation in the ∞ -limit is globally asymptotically stable with strictly positive degree then the convergence time doesn't depend on the initial condition. This is expressed by the following corollary.

Corollary 3 (Uniform and Finite Time Convergence [V19]) *Under the hypotheses of Theorem 1, if we have $\mathfrak{d}_\infty > 0 > \mathfrak{d}_0$, then all solutions of the system $\dot{x} = f(x)$ converge in finite time to the origin, uniformly in the initial condition.*

²The function \mathfrak{H} is defined in (1.35).

1.3.3 Conclusion on homogeneity in the bi-limit

This framework of homogeneity in the bi-limit has been used in different control problem. With Laurent Praly and Alessandro Astolfi, we have used it to design homogeneous in the bi-limit observer and output feedback in [V19, V22, V21, V20, V18, V17]. In this document we will use it for instance in the following contexts.

1. When dealing with high-gain observers with updated correction terms (see Section 2.2.3).
2. To design observers for systems with nonlocally Lipschitz nonlinearities with hölder restrictions (see Section 2.2.4).
3. To obtain finite time observers for general continuous triangular systems (see Section 2.2.5).
4. When dealing with output feedback designs (see Section 3.2.4).

Chapter 2

State estimation

In this chapter we consider the problem of state estimation of a nonlinear systems. In a first step, based on the tools we have introduced in the first Chapter we introduce some necessary conditions for the existence of an asymptotic observer. In a second step, we present some designs approaches allowing the construction of an asymptotic observer. In this regards, homogeneity in the bi-limit introduced in the former chapter is instrumental.

2.1 Necessary conditions for observers

2.1.1 Necessary conditions for an asymptotic observer

State observers have been largely studied and developed since they were first introduced together with the state-space representation in the 1960's (see [78, 53]). It is known that their possible design is related to some appropriate *observability* property of the considered representation. In particular for linear systems, the existence of an asymptotic observer is obtained if the system is *detectable* (see [132] for instance). Moreover, it is also well-known that for such systems, if the poles of the estimation error dynamics can be arbitrarily tuned, then the system is observable.

In this first section, the purpose is to investigate in which aspect this type of properties can be obtained for *nonlinear systems*. The main part of this work has been published in a conference paper [V6] in collaboration with Gildas Besançon and Ulysse Serres. I present some new developments which have not been published elsewhere.

From the early observability characterization of [43] for instance, sufficient conditions for possible observer constructions have indeed been more and more investigated for nonlinear systems, together with related actual designs (see e.g. [35, 14] and references therein). In this section, we are interested in *necessary* conditions of this type. More precisely, two cases are distinguished: the existence of an observer with an asymptotically decaying estimation error, corresponding to the usual notion of *asymptotic observer*, and the case of an observer with a convergence rate for the estimation error which can be tuned, and which has been called *tunable observers* in [14]. In each case, a special attention will be given to the stronger property of so-called *exponential observers*, for which the asymptotic decay of the estimation error is exponential. The existence of asymptotic observers will then be related to notions of detectability and observability defined in a quite natural way, while conditions for exponential observers will be given in terms of *infinitesimal* versions of such properties, following the terminology of [35] and employing the tools developed in Chapter 1.

Consider an autonomous nonlinear system given as

$$\dot{x} = f(x) , y = h(x) , \quad (2.1)$$

with $x \in \mathbb{R}^n$ being the state variable, $y \in \mathbb{R}^p$ being the measurement (also called the measured output), f being a smooth vector field on \mathbb{R}^n , and $h : \mathbb{R}^n \rightarrow \mathbb{R}^p$ being a smooth map.

Assume now that for system (2.1), an asymptotic state observer associated to a given open set \mathcal{A} of \mathbb{R}^n is available. Such an observer is a dynamical system driven by the output y and described by a smooth y -parametrized family of vector fields $\varphi(\cdot, y) : \mathbb{R}^m \rightarrow \mathbb{R}^m$, and a smooth mapping $\tau : \mathbb{R}^m \times \mathbb{R}^p \rightarrow \mathbb{R}^n$

$$\dot{\hat{\xi}} = \varphi(\hat{\xi}, y) , \hat{x} = \tau(\hat{\xi}, y) , \quad (2.2)$$

with $\hat{\xi} \in \mathbb{R}^m$ being the observer state.

The solution of the coupled dynamical systems (2.1)-(2.2) initiated from $(x, \hat{\xi})$ in $\mathcal{A} \times \mathbb{R}^m$ at $t = 0$, will be denoted by $(X(x, t), \hat{\Xi}(\hat{\xi}, x, t))$. Moreover, for any $(x, \hat{\xi})$, we introduce the notation $\hat{X}(x, \hat{\xi}, t) = \tau(\hat{\Xi}(\hat{\xi}, x, t), h(X(x, t)))$ which makes sense as long as the solutions are defined.

Finally, the two functions φ and τ are such that the output \hat{x} of this dynamical system asymptotically estimates the state of system. More precisely, an asymptotic observer is defined as follows¹.

Definition 6 (Asymptotic observer in \mathcal{A}) *The couple (φ, τ) defines an observer in the set \mathcal{A} if for any x in \mathcal{A} for which $\sigma_{\mathcal{A}}^+(x) = +\infty$, and for any $\hat{\xi}$ in \mathbb{R}^m , the solution of the coupled dynamical system (2.1)-(2.2) initiated from $(x, \hat{\xi})$ is defined on $[0, +\infty)$ and*

$$\lim_{t \rightarrow +\infty} |X(x, t) - \hat{X}(x, \hat{\xi}, t)| = 0 . \quad (2.3)$$

Then, based on this definition, we have the following necessary condition for the existence of an observer for the system:

Proposition 11 ([V6]) *If there exists an observer, then for any x_a and x_b in \mathcal{A} such that $\sigma_{\mathcal{A}}^+(x_a) = \sigma_{\mathcal{A}}^+(x_b) = +\infty$, and such that*

$$h(X(x_a, t)) = h(X(x_b, t)) , \forall t \geq 0 ,$$

one has

$$\lim_{t \rightarrow +\infty} |X(x_a, t) - X(x_b, t)| = 0 . \quad (2.4)$$

◇

This result is easily proved just rewriting the observer definition. When f and h are linear, the obtained property clearly reduces to the usual notion of *detectability* available for linear systems. In this sense, Proposition 11 generalizes detectability as a necessary condition for the existence of an observer.

¹Given a set \mathcal{A} , we denote $\sigma_{\mathcal{A}}^+(x)$ is the maximal time of existence of the solution initiated from x forward in time.

When \mathcal{A} is a forward invariant set for system (2.1), this also establishes that the set $\{(x_a, x_b) \in \mathcal{A}^2, x_a = x_b\}$ attracts all solutions of the implicit system defined on \mathcal{A}^2 as

$$\dot{x}_a = f(x_a) , \dot{x}_b = f(x_b) , h(x_a) = h(x_b) . \quad (2.5)$$

With Assumption 6, another property can be obtained on the observer when dealing with bounded trajectories:

Proposition 12 (Invariant & attractive zero error set [V6]) *Assume that an asymptotic observer as in definition 6 is given. Moreover, assume that there exists a compact forward invariant set with $\mathcal{C} = \mathcal{C}_x \times \mathcal{C}_\xi \subset \mathcal{A} \times \mathbb{R}^m$. In this case, there exists a closed forward invariant subset $\mathcal{C}_2 \subseteq \mathcal{C}_x$ and a closed set valued map, which maps $x \in \mathcal{C}_2 \mapsto \tau^*(x) \subset \mathcal{C}_\xi$ such that if we consider its graph:*

$$\mathcal{E} = \left\{ (x, \hat{\xi}) \in \mathcal{C}_2 \times \mathcal{C}_\xi : \hat{\xi} \in \tau^*(x) \right\}$$

then we have that:

1. for all $(x, \hat{\xi})$ in \mathcal{E}

$$\tau(\hat{\xi}, h(x)) = x ; \quad (2.6)$$

2. the set \mathcal{E} is forward invariant;

3. the set \mathcal{E} is attractive in \mathcal{C} . More precisely, for all $(x, \hat{\xi})$ in \mathcal{C} we have,

$$\lim_{t \rightarrow +\infty} \left| \left(X(x, t), \hat{\Xi}(x, \hat{\xi}, t) \right) \right|_{\mathcal{E}} = 0 .$$

where,

$$\left| (x, \hat{\xi}) \right|_{\mathcal{E}} = \min_{(x_0, \hat{\xi}_0) \in \mathcal{E}} |x - x_0, \hat{\xi} - \hat{\xi}_0| .$$

This proposition follows from Birkhoff's theorem (see e.g. [49, p. 517]). Note that in most of the approaches to design an observer available in the literature, the observer is designed from the mapping τ^* which is taken as a single valued function. For instance, in the case of the high gain observer (see [35], [60]), or Section 2.2

$$\tau^*(x) = (h(x), L_f h(x), \dots, L_f^{m-1} h(x)) , x \in \mathbb{R}^n ,$$

where m is a parameter to be designed and L_f denotes the Lie derivative along f . In the case of the nonlinear Luenberger observer (see [122, 58] [V13] and Section 2.3) this mapping is selected to satisfy

$$L_f \tau^*(x) = \Lambda \tau^*(x) + B(h(x)) , x \in \mathbb{R}^n ,$$

where Λ is a Hurwitz matrix and B is a function. This is also the case in the immersion and invariance principle of [11] or in the work of Besançon in [15]. Also, when considering observer designs based on some contraction property (see [7, 114]) then we may simply take $\tau^*(x) = \text{Id}$.

2.1.2 Necessary conditions for an exponentially stable observer

As seen previously, if we have an invariant compact set and an asymptotic observer, then we get a specific structure on the observer. To be more precise on the property of the system assuming the existence of an observer, we assume the following²:

- The mapping τ^* is a single valued function defined for all x in \mathcal{A} . More precisely, the set \mathcal{E} defined previously satisfies

$$\mathcal{E} = \{(x, \hat{\xi}) \in \mathcal{A} \times \mathbb{R}^m, \hat{\xi} = \tau^*(x)\} .$$

Note that this implies that for any x in \mathcal{A}

$$\tau(\tau^*(x), h(x)) = x . \quad (2.7)$$

In the following, we assume that the set \mathcal{E} is *exponentially* stable, and we will then show that the system satisfies an *infinitesimal detectability* property. Let us thus consider the following assumption:

Assumption 1 (observer with exponentially stable zero error set):

Let \mathcal{A} be a forward invariant, open and relatively compact³ subset of \mathbb{R}^n . Assume that the functions φ and τ of (2.2) can be selected such that for all x in \mathcal{A} the solution to the coupled dynamical systems (2.1)-(2.2) is defined for all $t \geq 0$. Moreover, assume that there exists a C^2 mapping $\tau^*: \mathcal{A} \rightarrow \mathbb{R}^m$ such that :

1. for all x in \mathcal{A} equality (2.7) is satisfied;
2. there exist two positive real numbers k and λ such that for all $(\hat{\xi}, x)$ in $\mathbb{R}^m \times \mathcal{A}$, we have

$$\left| \hat{\Xi}(\hat{\xi}, x, t) - \tau^*(X(x, t)) \right| \leq k \exp(-\lambda t) \left| \hat{\xi} - \tau^*(x) \right| . \quad (2.8)$$

This assumption implies that the observer has an exponential state reconstruction rate. Indeed, we have the following proposition.

Proposition 13 ([V6]) *If Assumption 1 holds, there exists a positive real number c such that for all x in \mathcal{A} , we get*

$$|\hat{X}(x, 0, t) - X(x, t)| \leq c \exp(-\lambda t) , \forall t \geq 0 . \quad (2.9)$$

Moreover, employing some Lyapunov arguments, it may be possible to show that this observer has some robustness property.

With Assumption 1 we get a tighter observability property on system (2.1). To introduce this one, we need to consider the lift of system (2.1) as in (1.18) for which we add also a linearized output map. The lift of system (2.1) is then given as the system

$$\dot{x} = f(x) , \quad \dot{\tilde{x}} = \frac{\partial f}{\partial x}(x)\tilde{x} , \quad \tilde{y} = \frac{\partial h}{\partial x}(x)\tilde{x} , \quad (2.10)$$

Given (x, \tilde{x}) , we denote $(X(x, t), \tilde{X}(x, \tilde{x}, t))$ the solution to system (2.10) which is defined in $[0, \sigma_{\mathbb{R}^n}^+(x))$.

We can now define some new notions of detectability.

²This assumption is a restriction. Indeed, if we consider the simple system $\dot{x} = -x, y = h(x) = 0$, then $\hat{\xi} = 0, \hat{x} = \tau(\hat{\xi}) = 0$, is a state observer. However, for $x \neq 0$ there does not exist $\tau^*(x)$ such that $\tau(\tau^*(x), h(x)) = x$.

³Recall that a subset in a topological space is relatively compact if its closure is compact

Definition 7 (Infinitesimal detectability) We say that the system (2.1) is infinitesimally detectable if every solution of (2.10) such that $\frac{\partial h}{\partial x}(x)\tilde{x} = 0$ defined on $[0, +\infty)$ satisfies $\lim_{t \rightarrow +\infty} |\tilde{X}(\tilde{x}, x, t)| = 0$.

Finally, we define a Lyapunov characterization of this property.

Definition 8 (R-Detectability) We say that the system (2.1) is R-detectable if there exist a continuous function $P : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$ and positive real numbers $0 < \underline{p} \leq \bar{p}$ and $0 < \underline{q}$ such that P has a derivative $\mathfrak{d}_f P$ along f in the sense of (1.36) and we have

$$\underline{p}I \leq P(x) \leq \bar{p}I \quad \forall x \in \mathbb{R}^n \quad (2.11)$$

and

$$v' \mathfrak{d}_f P(\tilde{x})v + 2v' P(\tilde{x}) \frac{\partial f}{\partial x}(x)v \leq -\underline{q} v' P(x)v \quad (2.12)$$

holds for all (x, v) in $\mathbb{R}^n \times \mathbb{R}^n$ satisfying $\frac{\partial h}{\partial x}(x)v = 0$.

Based on those definitions, and inspired by [114], we can show the following property on the system.

Proposition 14 Assume that Assumption 1 is satisfied. Assume moreover that the functions φ , τ^* , and f have bounded first and second derivative. Assume moreover that τ is C^1 and has bounded first derivative. Then system (2.1) is infinitesimally detectable and also R-detectable.

This result is an improvement of the proposition obtained with Gildas Besançon and Ulysse Serres in [V6] in which it was assumed the existence of a quadratic Lyapunov function. This improvement is made possible thanks to the analysis tools we have presented in Chapter 1 and more precisely Propositions 1 and 2. A similar result has also been obtained in [V8] when considering observer of dimension n which states is directle \hat{x} . Since this proof can't be found elsewhere, we give it in the appendix (see Chapter 4).

2.1.3 Necessary conditions for tunable observers

Another property of interest when dealing with observers is the fact that their convergence rate can be tuned. This corresponds to what has been called in [14] a *tunable* observer. More precisely, for \mathcal{A} a relatively compact open subset of \mathbb{R}^n , let us here consider the following:

Assumption 2 (Tunable asymptotic observer in \mathcal{A}) For any $\epsilon > 0$, and for any time t_e in \mathbb{R}_+ , there exist a locally Lipschitz vector field $\varphi : \mathbb{R}^m \times \mathbb{R}^p \rightarrow \mathbb{R}^m$ and a continuous mapping $\tau : \mathbb{R}^m \times \mathbb{R}^p \rightarrow \mathcal{M}$ such that the dynamical system (2.2) satisfies the following two properties:

1. For any x in \mathcal{A} , the function $t \mapsto (X(x, t), \hat{\Xi}(x, \hat{\xi}, t))$, solution of system (2.1)-(2.2) is well defined for all t in $] \sigma_{\mathcal{A}}^-(x), \sigma_{\mathcal{A}}^+(x) [$.
2. For any x in \mathcal{A} such that $\sigma_{\mathcal{A}}^+(x) \geq t_e$, we have,

$$\left| X(x, t) - \hat{X}(x, \hat{\xi}, t) \right| \leq \epsilon, \quad \forall t \in [t_e, \sigma_{\mathcal{A}}^+(x)).$$

It is well-known that this property is obtained with the celebrated *high-gain observer* assuming differential observability (see for instance [35, 14, 60]). This kind of property is typically the one needed when dealing with output feedback design based on some *separation principle* paradigm. This is for instance used in [127] (see also [V14]). In [V3, V4], it was also shown that this tunable aspect is obtained for the nonlinear Luenberger observer assuming differential observability.

Based on this assumption, we would like to emphasize the following property on the system.

Proposition 15 ([V6]) *If Assumption 2 is satisfied, for any x_a and x_b in \mathcal{A}^2 such that there exists t_d in $[0, \min \{\sigma_{\mathcal{A}}^+(x_a), \sigma_{\mathcal{A}}^+(x_b)\})$ with*

$$h(X(x_a, t)) = h(X(x_b, t)) , \forall t \in [0, t_d) ,$$

we have $x_a = x_b$. ◇

The property which is now obtained corresponds to the basic notion of *observability* for nonlinear systems [43], here satisfied over any time interval for which the solutions exist, in a similar way as it is assumed for the classical high-gain observer design [34].

A last assumption we can make on the observer is the one combining tunable and exponential convergence.

More precisely, let us finally consider the following:

Assumption 3 (Tunable exponential observer with stable invariant manifold): *Given \mathcal{A} a forward invariant, open and relatively compact subset of \mathbb{R}^n , for any $\lambda > 0$ there exist a C^1 vector field $\varphi : \mathbb{R}^m \times \mathbb{R}^p \rightarrow \mathbb{R}^m$ and a C^2 function $\tau : \mathbb{R}^m \times \mathbb{R}^p \rightarrow \mathbb{R}^n$ such that the following properties hold:*

1. *there exists a C^2 function $\tau^* : \mathcal{A} \rightarrow \mathbb{R}^m$ such that for any x in \mathcal{A} such that (2.7) holds.*
2. *φ, τ^* is an exponential observer with exponential rate λ . More precisely there exists k such that inequality (2.8) is satisfied for the given λ .*

In [35], it is shown that this property for the high-gain observer is obtained assuming infinitesimal differential observability. Moreover, it is shown in [V4] that the same property holds for the nonlinear Luenberger observer, under the same assumption.

We have the following converse result:

Proposition 16 ([V6]) *If Assumption 3 is satisfied, then for any x in \mathcal{A} , the time-varying linear system (2.10) is instantaneously observable. More precisely, for any \tilde{x}_a and \tilde{x}_b in \mathbb{R}^n such that*

$$\frac{\partial h}{\partial x}((X(x, t)))\tilde{X}(\tilde{x}_a, x, t) = \frac{\partial h}{\partial x}(X(x, t))\tilde{X}(\tilde{x}_b, x, t) ,$$

on an interval, we have $\tilde{x}_a = \tilde{x}_b$.

The obtained property now corresponds to a notion of *infinitesimal observability* [35].

2.1.4 Conclusion on the necessary conditions

In this section, necessary conditions for asymptotic and exponential (resp. tunable) observers have been inspected, and as a summary, the following four properties have been established:

Observer	\Rightarrow	Detectability.
Exp. observer	\Rightarrow	Infinitesimal detectability.
Tunable observer	\Rightarrow	Observability.
Exp. tunable observer	\Rightarrow	Infinitesimal observability.

At this stage, the study has been limited to autonomous systems, but the extension of such results to more general classes of systems will be part of future works.

2.1.5 From necessary conditions to sufficient conditions

It is interesting to remark that we can go from necessary conditions to sufficient conditions. Indeed, consider now a controlled nonlinear system given by

$$\dot{x} = f(x, u) , \quad y = h(x) , \quad (2.13)$$

In the remaining part of this section, we assume that there exists an uniformly injective immersion $\tau^* : x \mapsto \xi \in \mathbb{R}^m$ which sends the given nonlinear systems (2.13) into a system living in \mathbb{R}^m and which is written as

$$\dot{\xi} = \varphi(\xi, y, u) , \quad y = \mathfrak{h}(\xi) . \quad (2.14)$$

This new set of coordinates has to be selected such that the following theorem applies.

Theorem 2 *If there exist two positive definite matrices P and Q in $\mathbb{R}^{m \times m}$ and a smooth function K taking value in \mathbb{R}^m such that*

$$K(\mathfrak{h}(\xi), \xi) = 0 , \quad \forall \xi \in \mathbb{R}^m , \quad (2.15)$$

and,

$$P \left[\frac{\partial \varphi}{\partial \xi}(\xi, y, u) + \frac{\partial K}{\partial \xi}(y, \xi) \right] + \left[\frac{\partial \varphi}{\partial \xi}(\xi, y, u) + \frac{\partial K}{\partial \xi}(y, \xi) \right]^\top P < -Q , \quad \forall (\xi, y, u) , \quad (2.16)$$

then with the mapping τ defines as a uniformly continuous left inverse of the mapping τ^* the system

$$\dot{\hat{\xi}} = \varphi(\hat{\xi}, y, u) + K(y, \hat{\xi}) , \quad \hat{x} = \tau(\hat{\xi}) , \quad (2.17)$$

defines a (global) observer for the system (2.13) in the sens of definition 6 with $\mathcal{A} = \mathbb{R}^n$.

The proof of this result can be easily deduced from the fact that due to (2.16) the system (2.17) defines a (uniform in y and u) contraction and that due to (2.15), the system solution is also a solution to the observer solution.

It can be noticed that along the solution of the system and the observer dynamics the zero error set

$$\mathcal{E} = \{(x, \hat{\xi}), \tau^*(x) = \hat{\xi}\}$$

is exponentially stable. In other words, the observer obtained satisfies Assumption 1 we have introduced before.

Asking for contraction may be too much. As shown in [114], what is really needed is the asymptotic stability of the state trajectory in the ξ coordinates along the coupled observer and system solutions. Note however that obtaining checkable sufficient conditions to get this property is not an easy task. Moreover, most of the commonly employed approaches follow this strategy. For instance,

- Picking first

$$\tau^*(x) = (h(x), L_f h(x), \dots, L_f^{m-1} h(x)) , \quad x \in \mathbb{R}^n ,$$

and assuming some observability property, we can follow the high-gain observer strategy to design in a second step the mapping K such that (2.15) and (2.16) are satisfied. This is the subject of the following section.

- Another approach is to restrict ourselves to the case in which φ and \mathfrak{h} take the form

$$\varphi(\xi, y, u) = A\xi + B(y, u) , \quad y = \mathfrak{h}(\xi) = C\xi ,$$

with (A, C) is observable. In that case, k will be a simple linear correction terms but an important effort has to be made on τ^* to construct this one. This is the approach followed for instance in [65] and also in [111]. It can also be extended to the case in which A depends on known time function (y or u) (see [14] or [15]).

- Picking φ as (linear) incrementally stable mapping in a first step (with $k = 0$) and looking for τ^* in a second step is the nonlinear Luenberger observer strategy (see [122, 58] [V13] and Section 2.3).
- A last approach is to consider τ^* directly given and to work directly in the ξ coordinates assuming some monotonicity on the nonlinearities (see [5, 67]).

In the following, we present some variations on the high-gain observer strategy and also the nonlinear Luenberger approach.

Moreover, in the following chapter when dealing with output feedback design, this condition is also considered. We see that in order to be able to design an output feedback, depending on the properties of the state feedback some variations on (2.16) have to be required.

2.2 High-gain observers for upper triangular systems

2.2.1 Triangular canonical form

Our starting point of this analysis is that we assume that a mapping $\tau^* : x \in \mathbb{R}^n \mapsto \xi \in \mathbb{R}^m$ is given which sends the given controlled nonlinear system into a system in the form

$$\begin{cases} \dot{\xi}_1 &= \xi_2 + \Phi_1(\xi_1, u) \\ \dot{\xi}_2 &= \xi_3 + \Phi_2(\xi_1, \xi_2, u) \\ &\vdots \\ \dot{\xi}_m &= \Phi_m(\xi_1, \dots, \xi_m, u) \end{cases}, y = \xi_1, \quad (2.18)$$

where ξ is the state in \mathbb{R}^m , u is a known input in \mathbb{R}^q , y is a measured output in \mathbb{R} (i.e. $p = 1$), and Φ is a continuous function which may not be locally Lipschitz. Solutions to this triangular system are denoted $\Xi(\xi, t; u)$ and the purpose of this section is to design an observer for this system.

This kind of triangular form appears when we consider systems which are uniformly observable and differentially observable⁴. The typical algorithm which is employed in order to construct an observer for upper triangular systems is a high-gain observer approach. To explain this approach, let us first rewrite system (2.18) as follow.

$$\dot{\xi} = A\xi + \Phi(\xi, u), \quad y = C\xi,$$

where $A \in \mathbb{R}^{m \times m}$ is the upper shift matrix, i.e. $A\xi = (\xi_2, \dots, \xi_m, 0)^T$ and $C = (1, \dots, 0) \in \mathbb{R}^m$.

If one wants to follow a high-gain methodology, we have to consider a two steps design.

1. In a first step, we consider only the linear part and we design an observer for the system

$$\dot{\xi} = A\xi, \quad y = C\xi$$

In other words, we design an output dependant vector field $e_1 \mapsto K(e_1)$ in \mathbb{R}^m such that the origin of the error system

$$\dot{e} = Ae + K(e_1),$$

is globally and asymptotically stable.

2. In a second step we increase the robustness of the observer by modifying the corection term via a high-gain parameter. This modification has to be made accordingly to the properties on the nonlinearities Φ .

Note that in this two steps design, there is flexibility in the way K is computed. Indeed, If K is a linear corection term, then it will give robustness with respect to terms which are linear in the error. This explains the usual global Lipschitz assumption. However, if we consider correction terms which include high-order terms, we may get a robustness with respect to rational power of the error.

⁴This is for instance shown in [34] in the case in which τ^* is a diffeomorphism obtained from the differential observability property of order n . Actually, with Pauline Bernard and Laurent Praly, we have shown recently that in the case in which the differential observability rank is larger than the given system dimension a triangular form is possible imposing some topological assumptions. Note however that we may loose the local Lipschitz properties of the nonlinearities. See [V35] for more details.

In this Section, we investigate which properties on the functions Φ have to be imposed to be able to construct a high-gain observer. To get a better grip on our approaches we first consider an illustrative second order nonlinear example which can be rewritten in some specific coordinates :

$$\dot{\xi}_1 = \xi_2 + u , \quad \dot{\xi}_2 = \Phi_2(\xi_1, \xi_2, u) , \quad y = \xi_1 . \quad (2.19)$$

with

$$\Phi_2(\xi_1, \xi_2, u) = g(\xi_1)\xi_2 + \xi_2^q + u .$$

This system is in observability canonical form and consequently it may be suitable for the design of a nominal high-gain observer provided the nonlinearities involved in the dynamics satisfy a global Lipschitz property. So following the high-gain methodology we need to upper bound the function

$$(\xi_1, \xi_2, e_1, e_2, u) \mapsto |\Phi_2(\xi_1 + e_1, \xi_2 + e_2, u) - \Phi_2(\xi_1, \xi_2, u)| .$$

However, note that this term comes from the fact that there is a mismatch between the model of the observer and the system. Nevertheless it has to be noticed that ξ_1 is known. Consequently, instead of upper-bounding the previous term, we could upper bound the function

$$(y, \xi_2, e_2, u) \mapsto |\Phi_2(y, \xi_2 + e_2, u) - \Phi_2(y, \xi_2, u)| .$$

In this case, we will have to modify the dynamic of the observer adequately.

Moreover, when we compute this upperbound several cases may be distinguished depending on the value of the function g and the parameter q

1. When $q = 1$ and the function g is bounded, i.e there exists $M > g(y)$. In this case, we get that an upper bound on the local incremental rate of the nonlinearity can be computed as :

$$|\Phi_2(y, \xi_2 + e_2, u) - \Phi_2(y, \xi_2, u)| \leq |g(y) + 1||e_2| \leq c_L|e_2| .$$

with $c_L = M + 1$. In this case, we are back to the global Lipschitz framework. Hence, we know from [35] that an observer can be computed as,

$$\dot{\hat{\xi}}_1 = \hat{\xi}_2 + Lk_1(\hat{\xi}_1 - y) , \quad \dot{\hat{\xi}}_2 = (g(y) + 1)\hat{\xi}_2 + u + L^2k_2(\hat{\xi}_2 - y), \quad (2.20)$$

where $L > L_{\min}$ and k_1, k_2 are real number selected in such a way that the matrix

$$\begin{bmatrix} k_1 & 1 \\ k_2 & 0 \end{bmatrix}$$

is stable. Here, the use of the high-gain parameter L sufficiently "high" allows to increase the robustness of the observer in order to converge despite the mismatch between the nonlinearities.

2. In a second case, we still consider the case in which the parameter $q = 1$. However we relax the assumption on the bound of the function g . In this case we can no longer employ the high gain observer as developed previously. Indeed, the nonlinearity is no longer globally Lipschitz. However, we have

$$|\Phi_2(y, \xi_2 + e_2, u) - \Phi_2(y, \xi_2, u)| \leq |g(y) + 1||e_2| . \quad (2.21)$$

The Lipschitz constant depends on the output only. The idea in this case will be to update the high-gain parameter online with respect to the output. So the obtained observer will be one similar to (2.20) with a high-gain parameter L obtained by integrating an ordinary differential equation depending on the output as

$$\dot{L} = L [a_1(a_2 - L) + a_3 \Gamma(y)] , \quad \Gamma(y) = |g(y) + 1| . \quad (2.22)$$

The form of this dynamics will be discussed in the following section. Note that similar to a Riccati equation it is composed of a quadratic stabilizing terms and that the output dependent local incremental rate (i.e. $\Gamma(y)$) appears in its definition. Note also that since the observer is composed of extra dynamics the analysis of the stability of the algorithm is more involved than the usual high-gain approach.

3. When $q > 1$, the previous inequality becomes :

$$|\Phi_2(y, \xi_2 + e_2, u) - \Phi_2(y, \xi_2, u)| \leq \left(|g(y)| + q|\hat{\xi}_2|^{q-1} \right) |e_2| + |e_2|^q . \quad (2.23)$$

The first term in the right hand side exhibits the bound $\left(|g(y)| + q|\hat{\xi}_2|^{q-1} \right)$ on the local incremental rate. It can be handled in a very similar way to the one in (2.21) and (2.22), although it depends on $\hat{\xi}_2$ (i.e. $\Gamma(\hat{\xi}, y) = |g(y)| + q|\hat{\xi}_2|^{q-1}$).

The term, $|e_2|^q$ is a rational power of the norm of the error $|e_2|$ with a degree strictly larger than one. To deal with this term with Laurent Praly and Alessandro Astolfi, we have used the homogeneous in the bi-limit framework introduced in [V19] and that is presented in Section 1.3. In this case, instead of using linear correction terms, we employ homogeneous in the bi-limit function of the output.

4. A last framework of interest is the one in which g is again bounded and moreover, $q < 1$. In that case the nonlinearity is not locally Lipschitz and we may wonder if we are still able to design an observer. This has been the study of recent works in collaboration with Pauline Bernard and Laurent Praly. Assuming a known bound on the state trajectory and employing homogeneous (and not locally Lipschitz) correction terms, it is possible to design an observer based on cascade of finite time observers.

The last two contexts shows that in order to go beyond the usual global Lipschitz properties one needs to consider nonlinear correction terms. This aspect is the topic of the following three subsections in which we show that it is possible to construct homogeneous in the bi-limit correction terms suitable for the design of observers. The last subsection concerns the design of an event-triggered observer with linear correction terms but with a continuous discrete dynamics scaling.

2.2.2 Homogeneous in the bi-limit observer for a chain of integrator

It is at the beginning of the century that researchers started to consider homogeneous observers with various motivations: exact differentiators ([72, 73, 74]), domination as a tool for designing stabilizing output feedback ([133], [106], [108] and references therein... The advantage of this type of observers is their ability to face non globally Lipschitz perturbation (it can for instance deal with Hölder perturbations).

Throughout this subsection we consider the first step of the high-gain methodology summarized in the previous section. Hence a system given as a measured chain of integrators, with state $\xi = (\xi_1, \dots, \xi_m)$ in \mathbb{R}^m and output y in \mathbb{R} , namely :

$$\dot{\xi} = A\xi, y = C\xi \quad (2.24)$$

where A is the shift matrix of order m , and $C = (1, 0, \dots, 0)$. By selecting arbitrary vector field degrees \mathfrak{d}_0 and \mathfrak{d}_∞ in $\left[-1, \frac{1}{m-1}\right)$, we see that, to possibly obtain homogeneity in the bi-limit of the vector field associated with (2.24) we must choose the weights $r_0 = (r_{0,1}, \dots, r_{0,m})$ and $r_\infty = (r_{\infty,1}, \dots, r_{\infty,m})$ as :

$$r_{b,m} = 1, \quad r_{b,i} = r_{b,i+1} - \mathfrak{d}_b = 1 - \mathfrak{d}_b(m-i), \quad (2.25)$$

where the letter "b" stands for "0" or " ∞ ".

The goal of this section is to introduce a global homogeneous in the bi-limit observer for the system (2.24). The design presented in [V19] follows a recursive method and concerns the case in which $-1 < \mathfrak{d}_0 \leq \mathfrak{d}_\infty < \frac{1}{m-1}$. This recursive design has been extended in [V34] to the case in which $\mathfrak{d}_0 = \mathfrak{d}_\infty = -1$. The observer is given by the system :

$$\dot{\hat{\xi}} = A\hat{\xi} + K(\hat{\xi}_1 - y), \quad (2.26)$$

with state $\hat{\xi} = (\hat{\xi}_1, \dots, \hat{\xi}_m)$, and where $K : \mathbb{R} \rightarrow \mathbb{R}^m$ is a homogeneous in the bi-limit vector field with weights r_0 and r_∞ , and degrees \mathfrak{d}_0 and \mathfrak{d}_∞ . If we introduce the estimation error vector e in \mathbb{R}^m defined as $e = \hat{\xi} - \xi$, the output injection vector field K has to be selected such that the origin is a globally asymptotically stable equilibrium for the system :

$$\dot{e} = Ae + K(e_1), \quad e = (e_1, \dots, e_m)^T, \quad (2.27)$$

and also for its homogeneous approximations.

Theorem 3 (Homogeneous in the bi-limit observer [V19]) *With $\mathfrak{d}_0, \mathfrak{d}_\infty$ such that $-1 < \mathfrak{d}_0 \leq \mathfrak{d}_\infty < \frac{1}{m-1}$ and r_0 and r_∞ given in (2.25), there exists a homogeneous in the bi-limit vector field $K : \mathbb{R} \rightarrow \mathbb{R}^n$, with associated triples $(r_0, \mathfrak{d}_0, K_0)$ and $(r_\infty, \mathfrak{d}_\infty, K_\infty)$, such that, the origin is a globally asymptotically stable equilibrium for the system (2.27) and its homogeneous approximation :*

$$\dot{e} = Ae + K_0(e_1), \quad \dot{e} = Ae + K_\infty(e_1).$$

In the case in which $\mathfrak{d}_0 = \mathfrak{d}_\infty = -1$, this result has been complemented in [V34] as the following.

Theorem 4 (Non locally Lipschitz observer [V34]) *With \mathfrak{d} in $\left[-1, \frac{1}{m-1}\right)$ and r given in (2.25), there exists (k_1, \dots, k_m) such that the homogeneous (set-valued) vector field*

$$K(e_1) = \begin{bmatrix} k_1 \text{sign}(e_1) |e_1|^{\frac{r_2}{r_1}} \\ \vdots \\ k_m \text{sign}(e_1) |e_1|^{\frac{r_{m+1}}{r_1}} \end{bmatrix}, \quad (2.28)$$

is such that the origin is a globally asymptotically stable equilibrium for the system (2.27).

Note that when $\mathfrak{d} = -1$ the former vector field is a set valued vector field which has been introduced by A. Levant in [72]. Note however that our constructive procedure allows the construction of a homogeneous Lyapunov function from which we get robustness properties. These properties are instrumental when designing high-gain observers since we need to analyze the robustness with respect to the nonlinearities.

2.2.3 Homogeneous in the bi-limit observer for system (2.18)

When dealing with locally Lipschitz nonlinearities the following result has been obtained in [V22] in collaboration with Alessandro Astolfi and Laurent Praly.

Theorem 5 (High-gain observer for non globally Lipschitz Systems [V22]) *Suppose there exist a real number \mathfrak{d}_∞ in $[0, \frac{1}{m-1})$, a positive real number c_∞ , a continuous function Ω and real numbers v_j in $[0, \frac{1}{j-1})$, for $j = 2, \dots, m$, such that, for all i in $\{2, \dots, m\}$ and all $(\hat{\xi}, \xi, y, u)$ in $\mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^d$, we have:*

$$\begin{aligned} & |\Phi_i(y, \hat{\xi}_2, \dots, \hat{\xi}_1, u) - \Phi_i(y, \xi_2, \dots, \xi_i, u)| \\ & \leq \Omega(u, y) \left(1 + \sum_{j=2}^m |\hat{\xi}_j|^{v_j} \right) \sum_{j=2}^i |\hat{\xi}_j - \xi_j| + c_\infty \sum_{j=2}^i |\hat{\xi}_j - \xi_j|^{\frac{1-d_\infty(m-i-1)}{1-d_\infty(m-j)}}. \end{aligned} \quad (2.29)$$

Then, for all sufficiently small strictly positive real numbers b , there exists a function K such that, for all sufficiently small strictly positive real number a_1 and sufficiently large real numbers a_2 and a_3 , we can find functions β_W and β_L of class \mathcal{KL} and functions γ_W and γ_L of class \mathcal{K} such that the observer

$$\dot{\hat{\xi}} = A\hat{\xi} + \Phi(y, \hat{\xi}_2, \dots, \hat{\xi}_m, u) + L^{b-1} \mathfrak{L}(L) K \left(\frac{\hat{\xi}_1 - y}{L^b} \right), \quad (2.30)$$

$$\dot{L} = L \left[a_1(a_2 - L) + a_3 \Gamma(u, y, \hat{\xi}) \right], \quad (2.31)$$

where :

$$\Gamma(u, y, \hat{\xi}) = \Omega(u, y) \left(1 + \sum_{j=2}^m |\hat{\xi}_j|^{v_j} \right), \quad (2.32)$$

$$\mathfrak{L}(L) = \text{diag}(L, \dots, L^m), \quad (2.33)$$

initialized with $L(0) \geq a_2$, has the following property :

For each solution $t \mapsto \Xi(\xi, t, u)$ of (2.18) right maximally defined on $[0, \sigma_{\mathbb{R}^m, u}^+(\xi))$, the observer solution is defined on the same interval and the error estimate $e = \hat{\xi} - \xi$ satisfies :

$$|L(t)^{b-1} \mathfrak{L}(L(t))^{-1} e(t)| \leq \beta_W \left(L(0)^{b-1} \mathfrak{L}(L(0))^{-1} e(0), t \right), \quad \forall t \in [0, \sigma_{\mathbb{R}^m, u}^+(\xi)) \quad (2.34)$$

where L satisfies :

$$L(t) \leq 4a_2 + \beta_L \left(\begin{pmatrix} e(0) \\ L(0) \end{pmatrix}, t \right) + \sup_{s \in [0, t]} \gamma_L \left(\left| \begin{pmatrix} \Gamma(u(s), y(s)) \\ \Xi(\xi, s, u) \end{pmatrix} \right| \right). \quad (2.35)$$

With the form (2.29), the main assumption of Theorem 5 is the inequality (2.29) which extends usual global Lipschitz property. To understand the meaning of (2.29), we observe that, for any C^1 function f , there exist always two functions \mathfrak{U} and $\mathfrak{\Delta}$ such that we have :

$$|f(a, b+c) - f(a, b)| \leq \mathfrak{U}(a, b) |c| + \mathfrak{\Delta}(c).$$

with

$$\begin{aligned} \mathfrak{U}(a, b) &= \sup_{|s| \leq |(a, b)|} |\partial_b f(a, b, s)|, \\ \mathfrak{\Delta}(c) &= |c| \sup_{|(s, r)| \leq |c|} |\partial_b f(s, r, c)| \end{aligned}$$

where $\partial_b f$, characterizing the local incremental rate of f with respect to b , is defined as :

$$\partial_b f(a, b, c) = \sup_{s \in [0,1]} \left| \frac{\partial f}{\partial b}(a, b + sc) \frac{c}{|c|} \right| .$$

Hence, in essence, (2.29) imposes two restrictions :

1. the function Γ , defined in (2.32), is a bound on the local incremental rate \mathfrak{U} .
2. It is also setting a fractional power limitation, $\frac{1-d_\infty(m-i-1)}{1-d_\infty(m-j)}$ with d_∞ in $[0, \frac{1}{m-1})$, on the growth of Δ which bounds function increments for large argument increments. A motivation for this rational power is that, by following arguments similar to those used in [86], it can be proved that, for $d_\infty > \frac{1}{m-1}$, there is no continuous function K such that the origin of the following system is globally asymptotically stable. Concerning the restriction $v_j < \frac{1}{j-1}$, at this time we have no “natural” justification. It is a technical assumption we need to get (2.35).

For the illustrative system (see equation (2.19)), the incremental bound on the nonlinearity (i.e. inequality (2.23)) is in the form (2.29) with $d_\infty = \mathfrak{q} - 1$, $\Gamma(u, y) = (|g(y)| + q)$ and $v_2 = \mathfrak{q} - 1$. Hence, Theorem 5 applies when q is in the interval $[1, 2)$. Actually, when $\mathfrak{q} > 2$ and $u = 0$, there does not exist any observer guaranteeing convergence of the estimation error within the domain of existence of the solutions (see [10, Proposition 1]).

With (2.34) and (2.35) but with the presence of $\sup_s |\Xi(\xi, s, u)|$, Theorem 5 says that the observer (2.30),(2.31) gives, at least for bounded solutions, an estimation error converging to the origin asymptotically.

Actually, it can be shown with the same observer, convergence still holds for state solutions which escape to infinity but not too fast. However we have to be careful with convergence for unbounded solutions since it is a coordinate dependent notion which extends to other coordinates in general only through uniformly continuous diffeomorphisms.

Although we restrict our attention to bounded solutions, the reader may think that we are back to the global Lipschitz case. This is not completely true since the “Lipschitz constant” is solution dependent and therefore unavailable for observer design. It has to be learned on line and this is what L is doing in (2.31). The update law for L is very similar to the one introduced in [100]. The difference is in the fact that (2.31) depends also on $\hat{\xi}$ and u and not only on y and we need the restrictions on v_j to deal with this dependence on $\hat{\xi}$.

If Γ were differentiable along the solutions, the update law (2.31) would give :

$$\overbrace{L - \left(a_2 + \frac{a_3}{a_1} \Gamma \right)}^{\dot{}} = \frac{a_3}{a_1} \dot{\Gamma} - a_1 L \left[L - \left(a_2 + \frac{a_3}{a_1} \Gamma \right) \right] .$$

This says that L would track $a_2 + \frac{a_3}{a_1} \Gamma$ up to an error proportional to the magnitude of $\dot{\Gamma}$. We expect improved performance from this tracking property.

In the case of the second order system, Theorem 5 gives the following observer :

$$\begin{cases} \dot{\hat{\xi}}_1 &= \hat{\xi}_2 - L^{1+b} k_1 \left(\ell_1 \frac{[\hat{\xi}_1 - y]}{L^b} \right) , \\ \dot{\hat{\xi}}_2 &= \Phi_2(y, \hat{\xi}_2, u) - L^{2+b} k_2 \left(\ell_2 k_1 \left(\ell_1 \frac{[\hat{\xi}_1 - y]}{L^b} \right) \right) , \\ \dot{L} &= L [a_1 (a_2 - L) + a_3 \Gamma_2(u, y, \hat{\xi}_2)] , \end{cases}$$

where $k_1(s) = s + s^{\frac{1}{2-\mathfrak{q}}}$, $k_2(s) = s + s^\mathfrak{q}$ and b, a_i and ℓ_i are parameters to be chosen.

2.2.4 High-gain observers for non locally Lipschitz triangular systems

Motivated by the fact that in general, for differentially observable nonlinear and uniform observable systems, the obtained triangular form is not locally Lipschitz (see [V35]), with Pauline Bernard and Laurent Praly, we have derived the following high-gain observer.

Proposition 17 (Observer for non locally Lipschitz triangular systems, [V11])

Assume that there exist \mathfrak{d}_0 in $[-1, 0]$ and a positive real number Γ such that

$$|\Phi_i(y, \hat{\xi}_2, \dots, \hat{\xi}_1, u) - \Phi_i(y, \xi_2, \dots, \xi_i, u)| \leq \Gamma \sum_{j=2}^i |\hat{\xi}_j - \xi_j|^{\alpha_{ij}}. \quad (2.36)$$

with α verifying

$$\alpha_{ij} = \frac{1 - \mathfrak{d}_0(m - i - 1)}{1 - \mathfrak{d}_0(m - j)} = \frac{r_{i+1}}{r_j}, \quad 1 \leq j \leq i \leq m. \quad (2.37)$$

There exist (k_1, \dots, k_m) and a class \mathcal{KL} function β such that for all locally bounded time function u and all $(\xi, \hat{\xi})$ in $\mathbb{R}^m \times \mathbb{R}^m$ the system

$$\dot{\hat{\xi}} = A\hat{\xi} + \Phi(y, \hat{\xi}_2, \dots, \hat{\xi}_m, u) + \mathfrak{L}(L)K(\hat{\xi}_1 - y), \quad (2.38)$$

where K is the vector field defined in (2.28) admits absolutely continuous solutions $\hat{\Xi}(\xi, \hat{\xi}, t)$ defined on \mathbb{R}_+ and for any such solution

$$|\hat{\Xi}_i(\hat{\xi}, \xi, t) - \Xi_i(\xi, t)| \leq \beta(|\xi - \hat{\xi}|, t). \quad (2.39)$$

Moreover, when $\mathfrak{d}_0 < 0$ and for $j = 1, \dots, m$, there exists T such that $\hat{\Xi}(\hat{\xi}, \xi, t) = \Xi(\xi, t)$ for all $t \geq T$.

The proof of Propositions 17 for the case $\mathfrak{d}_0 \in]-1, 0]$ and without disturbances is given for example in [V19], via a Lyapunov design. In the limit case (i.e. $\mathfrak{d}_0 = -1$) the observer (2.38) is a differential inclusion corresponding to the exact differentiator studied in [72], where convergence is established in the particular case in which $\Phi_i = 0$ for $j = 1, \dots, m-1$ and Φ_m is bounded. With Pauline Bernard and Laurent Praly in [V34] we have constructed a Lyapunov function which enables to extend this result and show that the observer (2.38) still converges with the Φ_i 's.

The less restrictive conditions one may ask for are obtained for $\mathfrak{d}_0 = -1$ and are summed up in Table 2.1. On top of that, finite time estimation may be obtained.

2.2.5 Cascade of homogeneous observer for continuous triangular structure

In the case where the nonlinearities do not satisfy (17), we could lose the convergence of observer. But, we can still take advantage of the fact that when $\mathfrak{d}_0 = -1$, there are no restriction besides boundedness on the last functions Φ_m (see Table 2.1).

From the remark that observer (2.38)

i	j	1	2	...	$m-2$	$m-1$	m
1		$\frac{m-1}{m}$					
2		$\frac{m-2}{m}$	$\frac{m-2}{m-1}$				
\vdots	$\alpha_{ij} \geq$	\vdots	\vdots	\ddots			
$m-2$		$\frac{2}{m}$	$\frac{2}{m-1}$...	$\frac{2}{3}$		
$m-1$		$\frac{1}{m}$	$\frac{1}{m-1}$	$\frac{1}{2}$	
m		0	0	0

Table 1 : Hölder restrictions on Φ for a homogeneous observer with $\mathfrak{d}_0 = -1$

1. can be used for the system

$$\begin{aligned}
\dot{\xi}_1 &= \xi_2 + \psi_1(t) \\
&\vdots \\
\dot{\xi}_{k-1} &= \xi_k + \psi_{k-1}(t) \\
\dot{\xi}_k &= \varphi_k(t)
\end{aligned}$$

provided the functions ψ_i are known and the function φ_k is unknown but bounded, with known bound.

2. gives estimates of the ξ_i 's in finite time,

we see that it can be used as a preliminary step to deal with the system

$$\begin{aligned}
\dot{\xi}_1 &= \xi_2 + \psi_1(t) \\
&\vdots \\
\dot{\xi}_{k-1} &= \xi_k + \psi_{k-1}(t) \\
\dot{\xi}_k &= \xi_{k+1} + \Phi_k(u, \xi_1, \dots, \xi_k) \\
\dot{\xi}_{k+1} &= \varphi_{k+1}(u, \xi_1, \dots, \xi_{k+1})
\end{aligned}$$

Indeed, thanks to the above observer we know in finite time the values of ξ_1, \dots, ξ_k , so that the function $\Phi_k(u, \xi_1, \dots, \xi_k)$ becomes a known signal $\psi_k(t)$.

From this, we can propose the following observer made of a cascade of homogeneous

discretization was studied in [6] (see also [97]). In [87], a Newton observer is provided which estimates the state at time t_k from N consecutive measurements of outputs and inputs; in [18], the authors show how this method can be implemented in the case where the sampled system is not known analytically. In [55] observers were designed from an output predictor (see also related works in [2]). Some other approaches based on time delayed techniques have also been considered in [109]. Recently, in collaboration with Dinh Thach, Madiha Nadri and Ulysse Serres, we have introduced a new continuous-discrete observer design methodology for Lipschitz nonlinear systems based on reachability analysis in [V36] (see also our improvement of this approach with Frederic Mazenc and Mickael Malisoff in [V40]).

In this section, we consider the result we have obtained with Jean-Claude Vivalda, Madiha Nadri and Ulysse Serres in [V11] where we have also considered the design of a continuous discrete time observer. However, in opposition to former results, we consider the case in which the sampling time is variable and used as a tuning parameter. More precisely, we consider that the quantity $t_{k+1} - t_k$ is a part of the design of the continuous discrete observer. Hence, in the proposed algorithm, the measurement time is computed online. In fact, the use of sensors follows an event based on an extended observer state component. This may be related to the event-triggered control methodology (see for instance [120, 125]).

As we have seen in the high-gain design methodology, the asymptotic convergence of the estimate to the state is obtained by dominating the Lipschitz nonlinearities with high-gain techniques. However, there is a trade-off between the high-gain parameter and the measurement step size. This can lead to restrictive design conditions on the sampling measurement time (see also [88]). Inspired by [V22], the extra observer state component estimates the local Lipschitz constant in order to maximize the measurement sampling interval.

We consider the case in which the nonlinearities $\Phi : \mathbb{R}^m \times \mathbb{R}^q \rightarrow \mathbb{R}^m$ satisfies the following assumption.

Assumption 4 (Nonlinear bound) *The function $\Phi = (\Phi_1, \dots, \Phi_m)'$ is such that the following incremental bound is satisfied for all $(\xi, e, u) \in \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^q$,*

$$|\Phi_j(\xi + e, u) - \Phi_j(\xi, u)| \leq \Gamma(\xi, u) \sum_{i=1}^j |e_i|, \quad (2.41)$$

where $\Gamma : \mathbb{R}^m \times \mathbb{R}^q \rightarrow \mathbb{R}_+$ is a continuous function which satisfies the following bound

$$\Gamma(\xi, u) \leq c(u), \quad \forall (\xi, u) \in \mathbb{R}^\xi \times \mathbb{R}^p, \quad (2.42)$$

where $c : \mathbb{R}^p \rightarrow \mathbb{R}_+$.

Note that in the case in which we know a bound on the input u , we come back to the globally Lipschitz context. However, even in this case, we believe that employing a tighter bound in term of a state-dependent function Γ implies that the sensors are less used than they would be if we were considering directly the Lipschitz bound.

The continuous-discrete time observer with updated sampling period is given by⁵

$$\begin{cases} \dot{\hat{\xi}} = A\hat{\xi} + \Phi(\hat{\xi}, u), & t \in [t_k, t_{k+1}) \\ \hat{\xi}(t_{k+1}) = \hat{\xi}(t_{k+1}^-) + \delta_k \mathcal{L}(t_{k+1}^-) K (C\hat{\xi}(t_{k+1}^-) - y_{k+1}) \end{cases}, \quad (2.43)$$

⁵The solution $\hat{\xi}$ is a right-continuous function. Given a right-continuous function $\phi : \mathbb{R} \rightarrow \mathbb{R}^m$, the notation $\phi(t^-)$ stands for $\phi(t^-) = \lim_{h \rightarrow 0, h < 0} \phi(t + h)$.

where K in \mathbb{R}^m is a gain matrix. The matrix function $\mathfrak{L} : \mathbb{R}_+ \rightarrow \mathbb{R}^{m \times m}$ is defined as $\mathfrak{L} = \text{Diag}(L, \dots, L^m)$ where $L : \mathbb{R}_+ \rightarrow \mathbb{R}$ is given as a solution to the following system of continuous discrete differential equations

$$\dot{L} = a_2 L M \Gamma(\hat{\xi}, u), \quad t \in [t_k, t_{k+1}) \quad (2.44a)$$

$$\dot{M} = a_3 M \Gamma(\hat{\xi}(t), u(t)), \quad t \in [t_k, t_{k+1}) \quad (2.44b)$$

$$L(t_{k+1}) = L(t_{k+1}^-)(1 - a_1 \alpha) + a_1 \alpha \quad (2.44c)$$

$$M(t_{k+1}) = 1, \quad (2.44d)$$

initiated from $L(0) \geq 1$ and with $a_1 \alpha < 1$. We have for all k ,

$$y_k = C \xi(t_k),$$

where the t_k 's, k in \mathbb{N} are given by the following relations,

$$\begin{aligned} t_0 &= 0, \quad t_{k+1} = t_k + \delta_k, \\ \delta_k &= \min\{s \in \mathbb{R}_+ \mid sL((t_k + s)^-) = \alpha\}, \end{aligned} \quad (2.45)$$

where α , a_1 , a_2 and a_3 are positive real numbers to be chosen.

To understand the motivation of this update law note that a first order approximation gives

$$L(t_{k+1}^-) = L(t_k) + a_2 L(t_k) \Gamma(\hat{x}(t_k), u(t_k)) \delta_k + o(\delta_k).$$

Hence, taking into account that $\alpha = \delta_k L(t_{k+1}^-)$, it yields,

$$\frac{L(t_{k+1}) - L(t_k)}{\delta_k} = L(t_k) \left[a_1 (1 - L(t_k)) + a_2 \Gamma(\hat{\xi}(t_k), u(t_k)) \right] + o(1).$$

We recognize here the same update law structure than the one introduced in equation (2.31) which was motivated by a Riccati equation and comes from [100].

The sampling time interval which depends on L is well defined as this is shown in the following proposition.

Proposition 19 (Sequence $(\delta_k)_{k \in \mathbb{N}}$ well defined [V11]) *If u is in $\mathcal{L}^\infty(\mathbb{R}_+, \mathbb{R}^p)$ then there exists a positive real number δ_{\min} depending on the initial condition $L(0)$ such that for all k in \mathbb{N} there exists δ_k such that $\delta_{\min} \leq \delta_k \leq \alpha$.*

Note that if we know a bound on u , the function $\Gamma(\hat{\xi}, u)$ in (2.44) could simply be replaced by a constant depending on the function c . Note however that in this case, \dot{L} becomes larger which reduces the duration of each sampling period $(\delta_k)_{k \in \mathbb{N}}$. Consequently, the sensors are more frequently employed which is something we would like to avoid.

With the property given above in hand, we are now able to state the main result of [V11].

Theorem 6 (Self-triggered continuous-discrete time observer [V11]) *There exist a gain matrix K and $\alpha_m > 0$ such that for all α in $(0, \alpha_m]$, there exist positive numbers a_1, a_2 and a_3 such that for every essentially bounded input functions the estimation error obtained using the observer (2.43)-(2.44) converges asymptotically toward zero. More precisely, for every initial condition $(\xi, \hat{\xi})$ in $\mathbb{R}^m \times \mathbb{R}^m$ and $L(0) \geq 1$, for every input function u in $\mathcal{L}^\infty(\mathbb{R}_+, \mathbb{R}^p)$ the associated solution to system (2.18), (2.43)-(2.44) satisfies*

$$\lim_{t \rightarrow +\infty} \left| \Xi(\xi, t; u) - \hat{\Xi}(\xi, \hat{\xi}, t; u) \right| = 0.$$

2.2.7 Conclusion on high-gain observers

These extensions to the classical high-gain observer presented in this section have two features :

1. The high-gain parameter is obtained from a dynamical system;
2. The correction terms may be homogeneous in the bi-limit.

In fact, these two aspects of our observers have been employed in some other works. For instance, employing homogeneous in the bi-limit corection terms has been done in [V23] in order to obtain an observer for global Lipschitz system which convergence is obtained in finite time uniformly with respect to initial conditions. Moreover, when considering homogeneous observer only (and not homogeneous in the bi-limit observer), there have been some results related to output feedback designs (see [107] and coworkers). In Section 3.2 we employ a homogeneous in the bi-limit high-gain observer to obtain a general result in stabilization via output feedback designs.

This high-gain event-triggered observer approached has been extended to consider the case of event-triggered output feedback design. This result is presented in Section 3.3.

2.3 Nonlinear Luenberger observer

Since 1964 and the seminal work of Luenberger in [78], designing an observer for observable linear systems is now well understood. The approach of Luenberger can be decomposed into two steps. In the first one, a linear dynamic extension which defines a contraction, uniform in the measured output of the system, is introduced. In the second step, based on some observability properties, a linear map is obtained such that, when applied to the state of the dynamic extension, a state observer is obtained. With this approach, the estimate converges asymptotically toward the state of the system.

For nonlinear models, the problem is much more complicated and many different routes have been followed in order to extend this strategy. For instance we can refer to the popular linearization up to output injection (see for instance [64, 65, 111]). Few years back, Shoshitaishvili in [122] and more recently Kazantzis and Kravaris in [58] (see also [66]) have introduced a nonlinear local extension of the linear Luenberger observer. With their approach, it was shown that the existence of an observer around an equilibrium was obtained assuming local observability.

The purpose of this Section is to introduce the non local version of this approach as introduced in [V13] and [V4] (see also [63]).

2.3.1 Structure of a Nonlinear Luenberger observer

Consider a nonlinear system described by the following equation⁶:

$$\dot{x} = f(x) \quad , \quad y = h(x) \quad , \quad (2.46)$$

where $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $h : \mathbb{R}^n \rightarrow \mathbb{R}$ are two sufficiently smooth functions. For all x in \mathbb{R}^n , the solution of System (2.46) initiated from x at time 0 is denoted $X(x, t)$.

For all x in a given open set \mathcal{A} in \mathbb{R}^n , the maximal time interval of definition in \mathcal{A} is denoted $(\sigma_{\mathcal{A}}^-(x), \sigma_{\mathcal{A}}^+(x))$. More precisely, for all x in \mathcal{A} , $X(x, t)$ is in \mathcal{A} for all t in $(\sigma_{\mathcal{A}}^-(x), \sigma_{\mathcal{A}}^+(x))$. And if $X(x, \sigma_{\mathcal{A}}^-(x))$ (respectively $X(x, \sigma_{\mathcal{A}}^+(x))$) exists, then $X(x, \sigma_{\mathcal{A}}^-(x)) \notin \mathcal{A}$ (resp. $X(x, \sigma_{\mathcal{A}}^+(x)) \notin \mathcal{A}$).

Following [122, 58, 63] [V13] and [V4] a nonlinear Luenberger observer is a dynamical system of the form:

$$\dot{\hat{\xi}} = \varphi(\hat{\xi}, y) = \Lambda \hat{\xi} + B_{1m} y \quad , \quad \hat{x} = \tau(z) \quad , \quad (2.47)$$

with state z (a complex vector) in \mathbb{C}^m , Λ is a diagonal Hurwitz matrix in $\mathbb{C}^{m \times m}$, B_{1m} in \mathbb{R}^m is defined as

$$B_{1m} = (1, \dots, 1)' \quad , \quad (2.48)$$

and $\tau : \mathbb{C}^m \rightarrow \mathbb{R}^n$ is a continuous functions. The motivation for this structure is to design the mapping τ as the left inverse of a C^1 mapping⁷ $\tau^* : \mathcal{A} \rightarrow \mathbb{R}^m$ satisfying

$$\frac{\partial \tau^*}{\partial x}(x) f(x) = \Lambda \tau^*(x) + B_{1m} h(x) \quad . \quad (2.49)$$

⁶In this section, for the sake of clarity only mono output time invariant systems are considered. However, following [99] it is possible to extend some of these results to time varying systems provided all Assumptions imposed are uniform in the time. The extension to the multi-output case can also be performed following [V13] in which the state $\hat{\xi}$ of the observer is now seen as a matrix. Recently Pauline Bernard has developed extension to the context with controlled input (see [13]).

⁷As shown in [V13], we don't need τ^* to be C^1 as long as the Lie derivative of τ^* along f exists.

Indeed, it is easy to show that since Λ is Hurwitz, then there exists $P > 0$ such that

$$P \left[\frac{\partial \varphi}{\partial \xi}(\xi, y) \right] + \left[\frac{\partial \varphi}{\partial \xi}(\xi, y) \right]^\top P < -Q, \quad \forall (\xi, y).$$

Hence, equation (2.16) is trivially satisfied since the observer dynamics define a contraction uniform in the output. Moreover, if τ^* is a uniformly injective mapping solution to (2.49) then there exists \mathfrak{h} such that system (2.46) is sent in the form (2.14). Hence, as a direct application of Theorem 2, it yields⁸ for all x such that $\sigma_{\mathcal{A}}^+(x) = +\infty$:

$$\lim_{t \rightarrow +\infty} \left| \hat{\Xi}(\hat{\xi}, x, t) - \tau^*(X(x, t)) \right| = 0. \quad (2.50)$$

Designing the map τ as a uniformly continuous left inverse of the mapping τ^* gives the asymptotic estimation of the solution with our observer.

Note that in [V13], the nonlinear Luenberger observer considered is slightly more general since the function $y \mapsto B_{1m}y$ is a nonlinear function of the output. However, since in this section we consider bounded sets, we can restrict ourselves to this specific case.

2.3.2 Existence and construction of a nonlinear Luenberger observer

As shown in [V13], one of the the main interests of this approach is that its existence is guaranteed under some weak observability assumption. Indeed, assume that the past output path $t \mapsto h(X(x, t))$ restricted to the time in which the trajectory remains in a certain set is injective in x . Then, it is sufficient to choose $m = n + 1$ generic complex eigenvalues for Λ to get the existence of a function τ making System (2.47) an observer. The specific observability condition made is :

Assumption 5 ((\mathcal{O}, δ_d)-Backward distinguishability Property) *There exists a bounded open set \mathcal{O} of \mathbb{R}^n and a strictly positive real number δ_d such that, for each pair of distinct points x_1 and x_2 in \mathcal{O} , there exists a negative time t in⁹ $\left(\max \left\{ \sigma_{\mathcal{O} + \delta_d}^-(x_1), \sigma_{\mathcal{O} + \delta_d}^-(x_2) \right\}, 0 \right]$ such that :*

$$h(X(x_1, t)) \neq h(X(x_2, t)).$$

This distinguishability assumption says that the present state x can be distinguished from other states in an open set containing \mathcal{O} by looking at the past output path restricted to the time in which the solution remains in $\mathcal{O} + \delta_d$.

One of the results obtained in [V13] can be reformulated as follows.

Theorem 7 ([V13] Generic existence of Luenberger observer) *Assume System (2.46) satisfies Assumption 5. Then, for all bounded open set \mathcal{A} such that $\mathcal{c}\mathcal{L}(\mathcal{A}) \subset \mathcal{O}$, there exist a negative real number ρ and zero Lebesgue measure subset \mathcal{I}_d of¹⁰ $(\mathbb{C}_\rho)^{n+1}$ such that for*

⁸Actually, we don't need τ^* to be injective to get the convergence property (2.50).

⁹ Given a subset $S \subseteq \mathbb{R}^n$ and a strictly positive real number δ , $\mathbb{S} + \delta$ is the open set defined as ,

$$\mathbb{S} + \delta = \{x \in \mathbb{R}^n, \exists x_S \in \mathbb{S}, |x - x_S| < \delta\}. \quad (2.51)$$

¹⁰ \mathbb{C}_ρ is the open subset of \mathbb{C} defined as

$$\mathbb{C}_\rho = \{\lambda \in \mathbb{C} : \Re(\lambda) < \rho\}, \quad (2.52)$$

where \Re is the real part.

each $(\lambda_1, \dots, \lambda_{n+1})$ in $(\mathbb{C}_\rho)^{n+1} \setminus \mathcal{I}_d$ the following holds. There exists a continuous function τ and a continuous function β which is decreasing in its second component such that for all x in \mathcal{A} and all $\hat{\xi}$ in \mathbb{C}^{n+1}

$$\lim_{t \rightarrow +\infty} \beta(\hat{\xi}, t) = 0 ,$$

and,

$$|\hat{X}(x, \hat{\xi}, t) - X(x, t)| \leq \beta(\hat{\xi}, t) , \quad \forall t \in [0, \sigma_{\mathcal{A}}^+(x)) , \quad (2.53)$$

where,

$$\hat{X}(x, z, t) = \tau(\hat{\Xi}(x, \hat{\xi}, t)) ,$$

and where $(\hat{\Xi}(x, \hat{\xi}, t), X(x, t))$ is the solution of System (2.46) and (2.47) with

$$\Lambda = \text{Diag}\{\lambda_1, \dots, \lambda_{n+1}\} .$$

In [V13], this result was not stated in this way. However, it is a direct consequence of the fact that we restrict our analysis to a bounded set \mathcal{A} .

The β function in the previous equation allows us to assess the quality of the estimate on the time of existence of the solution of the model in the set \mathcal{A} . As long as the solution remains in the set \mathcal{A} , the quality of the estimate increases due to the fact that this function is strictly decreasing with time.

Moreover, if for a given initial condition x in \mathcal{A} , the corresponding state trajectory $X(x, t)$ remains in \mathcal{A} in forward time (i.e. $\sigma_{\mathcal{A}}^+(x) = +\infty$), the estimation asymptotically converges to the state.

Note moreover that as shown in [V13], the bound exhibits the distance between z and $\tau^*(x)$. Hence, if $\hat{\xi}$ was initiated at $\tau^*(x)$ then the estimation $\hat{x}(t)$ would follow the true state of the system.

In order to give an explicit realization of the observer, we need first to find a solution to the partial differential equation (2.49). This equation is the corner stone of the nonlinear Luenberger observer methodology. Despite the fact that its existence is ensured with some weak assumptions, obtaining an explicit solution may be a hard task. Note however that on some examples this partial differential equation may be solved as [V2], [99, 102]. Note moreover, that it is possible to perform a numerical approximation of this mapping. Indeed, it has been shown in [V13] that an approximation of the mapping τ^* could be used provided the dynamics of the observer is modified. Following this remark, a numerical scheme has been introduced in [81] to construct a suitable approximation of the mapping τ^* given a generic nonlinear model.

As an illustrative example of the nonlinear Luenberger observer and following [102], consider the model of a harmonic oscillator with unknown angular velocity (see [22] for recent results on the same topic with some other approaches). This one is given as the following nonlinear model with state $x := (x_1, x_2, x_3)$ satisfying

$$\dot{x}_1 = -x_2 , \quad \dot{x}_2 = x_3 x_1 , \quad \dot{x}_3 = 0 , \quad y = x_1 . \quad (2.54)$$

Given an initial condition $x_a := (x_{a1}, x_{a2}, x_{a3})$ with $x_{a3} > 0$, the measured output is the time function defined for all positive time¹¹ as $y(t) := h(X(x_a, t)) = x_{a1} \cos(\sqrt{x_{a3}}t) - \frac{x_{a2}}{\sqrt{x_{a3}}} \sin(\sqrt{x_{a3}}t)$. It can be easily checked that the set $\mathcal{O} = \{x_1^2 + x_2^2 > 0, x_3 > 0\}$ is backward invariant. Moreover, consider two initial conditions denoted (x_a, x_b) in the open

¹¹This time function can be written in the form $y(t) = \mathbb{A} \sin(\omega t + \Phi)$ with \mathbb{A} the amplitude of the signal and w and Φ respectively frequency and phase.

set \mathcal{O} . If $h(X(x_a, t)) = h(X(x_b, t))$ on a time interval then it yields that $x_a = x_b$. Thus Assumption 5 is satisfied for all positive real number δ_d . Hence we know with Theorem 7 that there exists a nonlinear Luenberger observer for the harmonic oscillator provided the initial condition is in a given bounded open subset of \mathcal{O} .

For instance, following [102] (see also [V2] and [V1]) the set \mathcal{A} is selected as $\mathcal{A} := \{x_3 x_1^2 + x_2^2 > \epsilon_1, \epsilon_2 > x_3 > \epsilon_3\}$. This selection guarantees that the set \mathcal{A} is forward invariant. An (asymptotic) observer for this system is given as,

$$\hat{x} = \tau(z) , \hat{\xi} = [\hat{\xi}_1, \hat{\xi}_2, \hat{\xi}_3, \hat{\xi}_4]^T , \dot{\hat{\xi}}_i = \lambda_i \hat{\xi}_i + y , i = 1, \dots, 4 , \quad (2.55)$$

where λ_i are four distinct negative real numbers and $\tau : \mathbb{R}^4 \rightarrow \mathcal{A}$ is a left inverse of the mapping $\tau^* : \mathcal{A} \rightarrow \mathbb{R}^4$ defined as

$$\tau^*(x) = [\tau^*_1(x), \tau^*_2(x), \tau^*_3(x), \tau^*_4(x)]^T , \tau^*_i(x) = \frac{-\lambda_i x_1 + x_2}{\lambda_i^2 + x_3} . \quad (2.56)$$

In [102] a continuous left inverse of this mapping is explicitly given.

Despite the fact that a nonlinear Luenberger observer may ensure asymptotic estimation of the state of the system, nothing is said concerning its convergence speed. In the next section, a sufficient condition is given under which exponential convergence of the estimation error towards the origin is obtained. In other words, the function β given in Theorem 7 is given as

$$\beta(\hat{\xi}, t) = M(\hat{\xi}) \exp(-ct) ,$$

where c is a positive real number and the function M is a continuous given later on.

2.3.3 Exponential convergence

In this section, a sufficient condition guaranteeing exponential convergence of the observer (2.47) is given. More precisely, in this section, the following two observability assumptions are imposed on the system (2.46).

Assumption 6 (\mathcal{O} -Backward distinguishability Property) *There exists an open set \mathcal{O} of \mathbb{R}^n such that, for each pair of distinct points x_1 and x_2 in \mathcal{O} , there exists a negative time t in $(\max \{\sigma_{\mathbb{R}^n}^-(x_1), \sigma_{\mathbb{R}^n}^-(x_2)\}, 0]$ such that :*

$$h(X(x_1, t)) \neq h(X(x_2, t)) .$$

Note that this Assumption is weaker than Assumption 5 previously defined. Indeed, now, nothing is said on how the output distinguishes two given initial conditions. Moreover, the set \mathcal{O} may not be bounded. Note however that by taking \mathcal{O} bounded and $\delta_d = +\infty$, we recover Assumption 5.

The second sufficient condition is an observability assumption which characterizes how a small change of the state modifies the backward output path on the time of existence of the solutions.

This assumption is related to the backward distinguishability of the following time varying linear system with output defined for all x in \mathbb{R}^n and t in $(\sigma_{\mathbb{R}^n}^-(x), \sigma_{\mathbb{R}^n}^+(x))$ as

$$\dot{\zeta} = \frac{\partial f}{\partial x}(X(x, t))\zeta , y_z = \frac{\partial h}{\partial x}(X(x, t))\zeta , \quad (2.57)$$

whose solutions initiated from ζ at $t = 0$ for a given x is written $\mathcal{M}(x, \zeta, t)$.

Assumption 7 (\mathcal{O} -Backward Infinitesimal distinguishability property) *Given an open set \mathcal{O} of \mathbb{R}^n , for all x in \mathcal{O} , for all ζ in $\mathbb{R}^n \setminus \{0\}$, there exists a negative time t in $(\sigma_{\mathbb{R}^n}^-(x), 0]$ such that*

$$y_z(t) := \frac{\partial h}{\partial x}(X(x, t))\mathcal{M}(\zeta, x, t) \neq 0. \quad (2.58)$$

The following result has been obtained in [V4]

Theorem 8 (Exponential Luenberger observers [V4]) *Let \mathcal{O} be an open set of \mathbb{R}^n . Assume System (2.46) satisfies Assumptions 6 and 7 with the same observability set \mathcal{O} . Then for all bounded open set \mathcal{A} of \mathbb{R}^n such that $\text{cl}(\mathcal{A}) \subset \mathcal{O}$, there exist a negative real number ρ , a zero Lebesgue measure subset \mathcal{I}_e of $(\mathbb{C}_\rho)^{n+1}$ such that for each $(\lambda_1, \dots, \lambda_{n+1})$ in $(\mathbb{C}_\rho)^{n+1} \setminus \mathcal{I}_e$ the following holds. There exists a continuous function $\tau : \mathbb{C}^{n+1} \rightarrow \mathbb{R}^n$ and a continuous function $M : \mathbb{C}^{n+1} \rightarrow \mathbb{R}_+$ such that for all $(x, \hat{\xi})$ in $\mathcal{A} \times \mathbb{C}^{n+1}$*

$$|\tau(\hat{\Xi}(x, \hat{\xi}, t) - X(x, t))| \leq M(\hat{\xi}) \exp\left(\max_i \{\text{Re}(\lambda_i)\} t\right), \quad \forall t \in [0, \sigma_{\mathcal{A}}^+(x)), \quad (2.59)$$

and where $(X(x, t), \hat{\Xi}(x, \hat{\xi}, t))$ is the solution of System (2.46) and (2.47) initiated from $(x, \hat{\xi})$ at $t = 0$ with $\Lambda = \text{Diag}\{\lambda_1, \dots, \lambda_{n+1}\}$.

Note that if we consider the model of the harmonic oscillator given in (2.54), the associated linear time varying system defined in (2.57) becomes simply,

$$\dot{\zeta} = \begin{bmatrix} 0 & -1 & 0 \\ X_3(x, t) & 0 & X_1(x, t) \\ 0 & 0 & 0 \end{bmatrix} \zeta, \quad y_z = \zeta_1,$$

where X_1 and X_3 are respectively the first and the third component of the state trajectory $X(x, t)$. Consider x_a in the open set \mathcal{A} , and assume there exists ζ in \mathbb{R}^3 such that $y_z(t) := \mathcal{M}_1(\zeta, x, t) = 0$ for all negative time. It yields, $\zeta_1 = 0$ and $\mathcal{M}_1(\zeta, x, t) = -\mathcal{M}_2(\zeta, x, t) = 0 \quad \forall t \leq 0$. Hence, it yields $\zeta_2 = 0$ and $X_1(x, t)\zeta_3 = 0 \quad \forall t \leq 0$. Note that x being in \mathcal{A} , we know that there exists $t < 0$ such that $X_1(x, t) \neq 0$. This implies that $\zeta_3 = 0$. Consequently, Assumption 7 is satisfied.

With Theorem 8, it yields the existence of an exponential Luenberger observer for the harmonic oscillator. Actually, on this particular example it can be shown that provided $\lambda_1, \lambda_2, \lambda_3$ and λ_4 are 4 different negative real numbers, the observer given in (2.55) has an exponential convergence rate.

2.3.4 Nonlinear Luenberger observer for parameterized linear systems

It is interesting to remark that example (2.54) which has been considered in the previous section can be rewritten as :

$$\dot{\chi} = F(\theta)\chi, \quad \dot{\theta} = 0, \quad y = H\chi, \quad (2.60)$$

where $\chi = (x_1, x_2)$ is in \mathbb{R}^2 and $F(\theta) = \begin{bmatrix} 0 & -1 \\ \theta & 0 \end{bmatrix}$ where $\theta = x_3$. This is the structure that is encountered when dealing with adaptive observer. Inspired by the former result, with Chouaib Afri, Laurent Bako and Pascal Dufour, we have considered in [V1] the state estimation for parameterized linear system described by the following equations :

$$\dot{\chi} = F(\theta)\chi + G(\theta)u, \quad y = H(\theta)\chi, \quad (2.61)$$

where θ in $\Theta \subset \mathbb{R}^q$ is a vector of unknown constant parameters and Θ is a known set, u in \mathbb{R} is a control input. The state vector χ is in \mathbb{R}^n and y is the measured output in \mathbb{R} . Mappings $F : \Theta \rightarrow \mathbb{R}^{n \times n}$, $G : \Theta \rightarrow \mathbb{R}^{n \times 1}$ and $H : \Theta \rightarrow \mathbb{R}^{1 \times n}$ are known C^1 matrix valued functions.

Following the approach of nonlinear Luenberger observer introduced previously, the first step is to design a C^1 function $(\chi, \theta, w) \mapsto \tau^*(\chi, \theta, w)$ such that the following equation is satisfied:

$$\frac{\partial \tau^*}{\partial \chi}(\chi, \theta, w)[F(\theta)\chi + G(\theta)u] + \frac{\partial \tau^*}{\partial w}(\chi, \theta, w)g(w, u) = \Lambda \tau^*(\chi, \theta, w) + LH(\theta)\chi \quad (2.62)$$

where Λ is a Hurwitz squared matrix, L a column vector and g is a controlled vector field which is a degree of freedom added to take into account the control input. The dimensions of the matrices and of the vector field g must be chosen consistently. This will be precisely defined in the sequel.

Let m be a positive integer. For all m -uplet of negative real numbers $(\lambda_1, \dots, \lambda_m)$ such that, for all θ in Θ we have

$$\lambda_i \notin \left(\bigcup_{\theta \in \Theta} \sigma\{F(\theta)\} \right), \quad i = 1, \dots, m, \quad (2.63)$$

we can introduce the matrix $M_i(\theta)$ in $\mathbb{R}^{1 \times n}$ defined by

$$M_i(\theta) = H(\theta)(F(\theta) - \lambda_i I_n)^{-1}$$

for all i in $\{1, \dots, m\}$. Let $\tau^*_i : \mathbb{R}^n \times \Theta \times \mathbb{R} \rightarrow \mathbb{R}$ be defined as:

$$\tau^*_i(\chi, \theta, w_i) = M_i(\theta)[\chi - G(\theta)w_i]. \quad (2.64)$$

Let also the vector field $g_i : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be defined as

$$g_i(w_i, u) = \lambda_i w_i + u. \quad (2.65)$$

It can be noticed that τ^*_i is solution to the PDE

$$\begin{aligned} \frac{\partial \tau^*_i}{\partial x}(\chi, \theta, w_i) [F(\theta)\chi + G(\theta)u] + \frac{\partial \tau^*_i}{\partial w_i}(\chi, \theta, w_i)g_i(w_i, u) \\ = \lambda_i \tau^*_i(\chi, \theta, w_i) + H(\theta)\chi. \end{aligned}$$

Hence, the solution of the PDE (2.62) is simply taken as

$$\tau^*(\chi, \theta, w) = [\tau^*_1(\chi, \theta, w_1) \quad \dots \quad \tau^*_m(\chi, \theta, w_m)]^\top. \quad (2.66)$$

The second step of the design is to construct the mapping τ , left inverse of the mapping τ^* which existence is obtained via differential observability assumptions. Note however that this observer relies on an explicit construction of a mapping τ which requires a nonlinear (and probably non convex) optimization.

In [V1], a particular canonical structure for system (2.61) is considered. This allows to give an explicit construction of a mapping τ left inverse of τ^* . Moreover, it allows to give a complete characterization of the dimension of the observer and the class of inputs which guarantee that the differential observability property holds.

The considered particular canonical structure for the matrix-valued functions A, B, C is given as follows.

$$F(\theta) = \begin{bmatrix} \theta_a & I_{n-1} \\ \vdots & 0 \end{bmatrix}, G(\theta) = \theta_b, H = [1 \quad 0 \quad \dots] \quad (2.67)$$

where

$$\theta = [\theta_a^\top \quad \theta_b^\top]^\top \in \mathbb{R}^{2n \times 1}$$

Note that assuming the structures (2.67) for F, G, H is without loss of generality: any input-output behavior of a linear SISO system can be described with a model of this structure (maybe after a linear change of coordinates). Such a realization is observable for any vector θ .

The interest of this structure is twofold:

1. it is possible to select m and to characterize the class of input such that a differential observability property is satisfied;
2. it is possible to give explicitly a candidate for the mapping τ which allows us to implement the algorithm on practical examples.

Hence, together with Chouaib Afri, Laurent Bako and Pascal Dufour, we have given a new identification algorithm for linear systems.

2.4 Conclusion on the observers

In this section we have presented some necessary conditions for the design of asymptotic observers. From these necessary conditions, we have introduced sufficient conditions for the existence of observers. These sufficient conditions have been employed to introduce different type of designs. We have presented some improvements to the high-gain observer and the nonlinear Luenberger observers.

All these designs rely on the possibility to send the dynamical nonlinear system in some new coordinates via an injective mapping τ^* . Of course, a crucial step to implement this algorithm is to compute τ the left inverse of the mapping τ^* . And there is no general way of designing such mapping (even so in the former section when dealing with identification algorithms it was possible to construct such mapping explicitly).

Note that a possible expression can be given as the solution to a global optimization procedure :

$$\hat{x} = \tau(z) := \mathbf{Argmin}_{x \in \text{cl}(\mathcal{A})} |z - \tau^*(x)|^2 .$$

In [81], some numerical constructions of this left inverse have been given. Recently in [V33] we have given a new approach to address this problem which is based on the writing of the observer directly in the x coordinates.

A very important topic which is far from being understood is the robust design of these algorithms. For instance, in the case of the nonlinear Luenberger observer and knowing some model of uncertainties, what would be an optimal selection of the matrix Λ ? Is there an interest in selecting more eigenvalues ? These questions should be addressed in the future.

Chapter 3

Stabilization via feedback

In this chapter we discuss the problem of designing a feedback ensuring some stability properties. We will discuss different control objectives and different control constraints. In a first part, we consider the problem of designing a stabilizing state feedback law which ensures the global asymptotic stabilization of an equilibrium point. We consider in this context the problem of optimality. In a second part of this chapter, forgetting this optimality requirement, we consider the same stabilization objective with the constraint that only a part of the system state can be used to design a stabilizing control law. In a third part, we consider the case in which the measurement and the control are discrete in time. Finally, we consider the problem of stabilization of a set in the context of feedback synchronization.

3.1 Global stabilization with local optimal behavior

The synthesis of a stabilizing control law for systems described by nonlinear differential equations has been the subject of great interest by the nonlinear control community during the last three decades. Depending on the structure of the model, some techniques are now available to synthesize control laws ensuring global and asymptotic stabilization of the equilibrium point.

For instance, we can refer to the popular backstepping approach (see [69], [V15] and references therein), or the forwarding approach (see [85, 50, 103]) and some others based on energy considerations or dissipativity properties (see [62] for a survey of the available approaches).

Although the global asymptotic stability of the steady state can be achieved in some specific cases, it remains difficult to address in the same control objective performances issues of a nonlinear system in a closed loop. However, when the first order approximation of the non-linear model is considered, some performances aspects can be addressed by using linear optimal control techniques (using LQ controller for instance).

Hence, it is interesting to raise the question of synthesizing a nonlinear control law which guarantees the global asymptotic stability of the origin while ensuring a prescribed local linear behavior. For instance, this problem has been addressed in [29] where local optimal control laws are designed for systems which admit the existence of a backstepping.

In a first part of this section we consider a strategy based on the design of a uniting control Lyapunov function. We show that this is related to an equivalent problem which is the design of a control Lyapunov function with a specific property on the quadratic

approximation around the origin. In a second part, we consider the case in which the prescribed local behavior is an optimal LQ controller. In this framework, we investigate what type of performances is achieved by the control solution to the stabilization with prescribed local behavior. In a third part we consider two specific classes of systems and show how the control with prescribed local behavior can be solved.

3.1.1 Stabilization with prescribed local behavior

In this section, we restrict our attention to the particular case in which the system is input affine. More precisely we consider systems in the form

$$\dot{x} = f(x) + g(x)u, \quad (3.1)$$

with the state x in \mathbb{R}^n , the two C^1 functions $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $g : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times q}$. For this system, we can introduce the two matrices A in $\mathbb{R}^{n \times n}$ and B in $\mathbb{R}^{n \times q}$ describing its first order approximation : $A = \frac{\partial f}{\partial x}(0)$ and $B = g(0)$.

For system (3.1), the problem we intend to solve can be described as follows:

Global asymptotic stabilization with prescribed local behavior: *Let a linear state feedback law $u = K_o x$ with K_o in $\mathbb{R}^{q \times n}$ which stabilizes the first order approximation of system (3.1) (i.e. $A + BK_o$ is Hurwitz) be given. We are looking for a stabilizing control law $u = \alpha_o(x)$, with $\alpha_o : \mathbb{R}^n \rightarrow \mathbb{R}^q$, a locally Lipschitz map differentiable at 0 such that:*

1. *The origin of the closed-loop system $\dot{x} = f(x) + g(x)\alpha_o(x)$ is globally and asymptotically stable ;*
2. *The first order approximation of the control law α_o satisfies the following equality.*

$$\frac{\partial \alpha_o}{\partial x}(0) = K_o. \quad (3.2)$$

In this section, we present some results we have obtained on this topic in [V24, V26, V25, V45, V30, V31] This problem had already been addressed in the literature. For instance, it is the topic of the papers [29]. Note moreover that this subject can be related to the problem of uniting a local and a global control laws as introduced in [128] (see also [104] and some of my contributions on this topic [V24, V26, V25, V27]).

Employing the tools that, together with Christophe Prieur, we have developed in [V25], it is possible to show that merging control Lyapunov functions may solve the problem of stabilization with prescribed local behavior. With Sofiane Benachour and Humberto Stein Shiromoto, we have shown in [V32] that working with the control Lyapunov function is indeed equivalent to address this problem.

Theorem 9 ([V32]) *Given a linear state feedback law $u = K_o x$ with K_o in $\mathbb{R}^{p \times n}$ which stabilizes the first order approximation of system (3.1). The following two statements are equivalent.*

1. *There exists a locally Lipschitz function $\alpha_o : \mathbb{R}^n \rightarrow \mathbb{R}^p$ solution to the global asymptotic stabilization with prescribed local behavior problem.*
2. *There exists a C^2 proper, positive definite function $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$ such that the following two properties are satisfied.*

- If we denote¹ $P := \frac{1}{2}H(V)(0)$, then P is a positive definite matrix. Moreover this inequality holds.

$$(A + BK_o)^\top P + P(A + BK_o) < 0 ; \quad (3.3)$$

- Artstein condition is satisfied. More precisely, this implication holds for all x in $\mathbb{R}^n \setminus \{0\}$,

$$L_f V(x) = 0 \Rightarrow L_g V(x) < 0. \quad (3.4)$$

This proof is based on the uniting of control Lyapunov functions as developed in [V25]. It can be found in [V29].

Theorem 9 establishes that looking for a global control Lyapunov function locally assigned by the prescribed local behavior and looking for the controller itself are equivalent problems.

3.1.2 Locally optimal and globally inverse optimal control laws

If one wants to guarantee a specific behavior on the closed loop system, one might want to find a control law which minimizes a specific cost function. More precisely, we may look for a stabilizing control law which minimizes the criterium

$$J(x; u) = \int_0^{+\infty} \mathbf{q}(X(x, t; u)) + u(t)^\top \mathbf{r}(X(x, t; u))u(t) dt , \quad (3.5)$$

where $X(x, t; u)$ is the solution of the system (3.1) initiated from $x_0 = x$ at $t = 0$ and employing the control $u : \mathbb{R}_+ \rightarrow \mathbb{R}^q$, $\mathbf{q} : \mathbb{R}^n \rightarrow \mathbb{R}_+$ is a continuous function and r is a continuous function which values $\mathbf{r}(x)$ are symmetric positive definite matrices.

The control law which solves this minimization problem (see [119]) is given as a state feedback

$$u = -\frac{1}{2}\mathbf{r}(x)^{-1}L_b V(x)^\top , \quad (3.6)$$

where $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$ is the solution with $V(0) = 0$ to the following Hamilton-Jacobi-Bellman equation for all x in \mathbb{R}^n

$$\mathbf{q}(x) + L_f V(x) - \frac{1}{4}L_g V(x)\mathbf{r}(x)^{-1}L_g V(x)^\top = 0 . \quad (3.7)$$

Given a function \mathbf{q} and a function \mathbf{r} , it is in general difficult or impossible to solve the so called HJB equation. However, for linear system, this might be solved easily. If we consider the first order approximation of the system (3.1), and given a positive definite matrix R and a positive semi definite matrix Q we can introduce the quadratic cost:

$$J(x; u) = \int_0^{+\infty} \left[X(x, t; u)^\top Q X(x, t; u) + u(t)^\top R u(t) \right] dt , \quad (3.8)$$

In this context, solving the HJB equation can be rephrased in solving the algebraic Riccati equation given as :

$$PA + A^\top P - PBR^{-1}B^\top P + Q = 0 . \quad (3.9)$$

¹In the following, given a C^2 function $V : \mathbb{R}^n \rightarrow \mathbb{R}$, the notation $H(V)(x)$ is the Hessian matrix in $\mathbb{R}^{n \times n}$ evaluated at x of the function V . More precisely, it is the matrix $(H(V))_{i,j}(x) = \frac{\partial^2 V}{\partial x_i \partial x_j}(x)$.

It is well known that provided, the couple (A, B) is controllable, it is possible to find a solution to this equation. Hence, for the first order approximation, it is possible to solve the optimal control problem when considering a cost in the form of (3.8).

From this discussion, we see that an interesting control strategy is to solve the stabilization with prescribed local behavior with the local behavior obtained solving LQ control strategy. Note however that once we have solved this problem, one may wonder what type of performances has been achieved by this new control law. The following Theorem that we have established with Sofiane Benachour and Humberto Stein Shiromoto addresses this point and is inspired from [119] (see also [101]). Following Theorem 9, this one is given in terms of control Lyapunov functions.

Theorem 10 (Local optimality and global inverse optimality [V32]) *Given two positive definite matrices R and Q . Assume there exists a C^2 proper positive definite function $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$ such that the following two properties hold.*

- *The matrix $P := H(V)(0)$ is positive definite matrix and satisfies the following equality.*

$$PA + A^\top P - PBR^{-1}B^\top P + Q = 0 ; \quad (3.10)$$

- *Equation (3.4) is satisfied.*

Then there exist $\mathfrak{q} : \mathbb{R}^n \rightarrow \mathbb{R}_+$ a continuous function, C^2 at zero and \mathfrak{r} a continuous function whose values $\mathfrak{r}(x)$ are symmetric positive definite matrices such that the following properties are satisfied.

- *The function \mathfrak{q} and \mathfrak{r} satisfy*

$$H(\mathfrak{q})(0) = 2Q , \quad \mathfrak{r}(0) = R ; \quad (3.11)$$

- *The function V is a value function associated to the cost (3.5). More precisely, V satisfies the HJB equation (3.7).*

This proof is inspired from some of the results of [101] and can be found in [V29].

This Theorem establishes that if we solve the stabilization with a prescribed local behavior, we may design a control law $u = \alpha_o(x)$ such that this one is solution to an optimal control problem and such that the local approximation of the associated cost is exactly the one of the local system. This framework has already been studied in the literature in [29] when dealing with design of a backstepping with a prescribed local optimal control law. In our context we get a Lyapunov sufficient condition to design a globally and asymptotically stabilizing optimal control law with prescribed local cost function.

3.1.3 Some sufficient conditions

In this section we give some sufficient conditions allowing us to solve the stabilization with prescribed local behavior problem. The first result is obtained from the tools developed in [V25]. It assumes the existence of a global control Lyapunov function and a sufficient condition is given in terms of a matrix inequality. In the second and third results we give some structural conditions on the vector field to avoid a matrix inequality that we have published in [V31] and [V32].

Based on matrix inequalities

The first solution to solve the stabilization with prescribed local behavior is to follow the result of [V25] and to assume that there exists a global control Lyapunov function which can be modified locally in order to fit in the context of Theorem 9.

Assumption 8 *There exists a positive definite and C^2 function $V_\infty : \mathbb{R}^n \rightarrow \mathbb{R}_+$ such that the following holds.*

1. *The implication (3.4) is satisfied.*
2. *The function V_∞ is locally quadratic. i.e. $P_\infty = H(V)(0)$ is a positive definite matrix.*

In this context the result obtained from [V25] may be formalized as follows.

Theorem 11 ([V25],[V45], [V44]) *Let Assumption 8 be satisfied. Let K_o in $\mathbb{R}^{p \times n}$ be a matrix such that $A + BK_o$ is Hurwitz with A and B defined in (3.14). If there exists K_u in $\mathbb{R}^{p \times n}$ and a positive definite matrix P in $\mathbb{R}^{n \times n}$ such that these matrix inequalities are satisfied*

$$\begin{aligned} (A + BK_o)^\top P + P(A + BK_o) &< 0, \\ (A + BK_u)^\top P + P(A + BK_u) &< 0, \\ (A + BK_u)^\top P_\infty + P_\infty(A + BK_u) &< 0, \end{aligned} \quad (3.12)$$

then there exists a smooth function $\alpha_o : \mathbb{R}^n \rightarrow \mathbb{R}^p$ which solves the global asymptotic stabilization with prescribed local behavior.

The proof of this result is a direct consequence of the tools related to the uniting of control lyapunov function developed in collaboration with Christophe Prieur in [V25].

In inequalities (3.12), P and K_u are the unknown. This implies that this inequality is not linear. However by introducing some new variables, it is possible to give a (conservative) linear relaxation which allows the use of the tools devoted to solve linear matrix inequalities (see our work with Christophe Prieur, Sophie Tarbouriech and Denis Arzelier in [V27] for instance).

Strict feedback form

Following the work of [29], consider the case in which system (3.1) may be written in some coordinates (ξ_1, ξ_2) such that the dynamics take the following structure :

$$\dot{\xi}_1 = h_1(\xi_1) + h_2(\xi_1)\xi_2, \quad \dot{\xi}_2 = f(\xi_1, \xi_2) + g(\xi_1, \xi_2)u. \quad (3.13)$$

with ξ_1 in \mathbb{R}^{n_1} , ξ_2 in \mathbb{R} and $g(\xi_1, \xi_2) \neq 0$ for all (ξ_1, ξ_2) .

In this case, the first order approximation of the system is

$$A = \begin{bmatrix} H_1 & H_2 \\ F_1 & F_2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ G \end{bmatrix}, \quad (3.14)$$

with $H_1 = \frac{\partial h_1}{\partial x_1}(0)$, $H_2 = h_2(0)$, $F_1 = \frac{\partial f}{\partial x_1}(0, 0)$, $F_2 = \frac{\partial f}{\partial x_2}(0, 0)$, $G = g(0, 0)$.

For this class of system we make the following assumption.

Assumption 9 For all couples (K_1, P_1) with K_1 in \mathbb{R}^{n_1} and P_1 a positive definite matrix in $\mathbb{R}^{n_1 \times n_1}$ such that $P_1(H_1 + H_2 K_1) + (H_1 + H_2 K_1)^\top P_1 < 0$, there exists a smooth function $V_1 : \mathbb{R}^{n_1} \rightarrow \mathbb{R}_+$ such that $H(V_1)(0) = 2P_1$ and such that for all $\xi_1 \neq 0$

$$L_{h_2} V_1(\xi_1) = 0 \Rightarrow L_{h_1} V_1(\xi_1) < 0 . \quad (3.15)$$

With Theorem 9, this assumption establishes that the stabilization with prescribed local behavior is satisfied for the ξ_1 subsystem seeing ξ_2 as the control input.

For this class of system, we have obtained in [V32] the following theorem which can already be found in [29] when restricted to locally optimal controllers.

Theorem 12 (Backstepping Case [V32], [29]) Let Assumption 9 be satisfied. Let K_o in \mathbb{R}^n be a matrix such that $A + BK_o$ is Hurwitz with A and B defined in (3.14). Then there exists a smooth function $\alpha_o : \mathbb{R}^n \rightarrow \mathbb{R}$ which solves the global asymptotic stabilization with prescribed local behavior.

Note that with Theorem 10, this theorem establishes that given Q , a positive definite matrix in $\mathbb{R}^{n_1 \times n_1}$, and R , a positive real number, then there exist \mathbf{q} , \mathbf{r} and α_o which is solution to an optimal control problem with cost $J(\xi, u)$ defined in (3.5), with \mathbf{q} and \mathbf{r} which satisfy (3.11). In other words we can design a globally and asymptotically stabilizing optimal control law with prescribed local cost function as already seen in [29].

Feedforward form

Following the results we have obtained in [V31], consider the case in which the system with state $\xi = (\xi_1, \xi_2)$ can be written in the form

$$\dot{\xi}_1 = h(\xi_2) , \quad \dot{\xi}_2 = f(\xi_2) + g(\xi_2)u , \quad (3.16)$$

with ξ_1 in \mathbb{R} , ξ_2 in \mathbb{R}^{n_2} . Note that to oppose to what has been done in the previous subsection, now the state component ξ_1 is a scalar and ξ_2 is a vector. Note moreover that the functions h , f and g do not depend of ξ_1 . This restriction on h has been partially removed in [V31].

The first order approximation of the system is denoted by

$$A = \begin{bmatrix} 0 & H \\ 0 & F \end{bmatrix} , \quad B = \begin{bmatrix} 0 \\ G \end{bmatrix} , \quad (3.17)$$

with $H = \frac{\partial h}{\partial x}(0)$, $F = \frac{\partial f}{\partial x}(0)$, $G = g(0)$.

For this class of system we make the following assumption.

Assumption 10 For all couples (K_2, P_2) with K_2 in $\mathbb{R}^{p \times n_2}$ and P_2 a positive definite matrix in $\mathbb{R}^{n_2 \times n_2}$ such that $P_2(F + GK_2) + (F + GK_2)^\top P_2 < 0$, there exists a smooth function $V_2 : \mathbb{R}^{n_2} \rightarrow \mathbb{R}_+$ such that $H(V_2)(0) = 2P_2$ and such that for all $\xi_2 \neq 0$

$$L_g V_2(\xi_2) = 0 \Rightarrow L_f V_2(\xi_2) < 0 . \quad (3.18)$$

This assumption establishes that the stabilization with prescribed local behavior is satisfied for the ξ_2 subsystem. With this Assumption we have the following theorem whose proof can be found in [V31] (in a more general form).

Theorem 13 (Forwarding Case, [V31]) *Let Assumption 10 be satisfied. Let K_o in \mathbb{R}^n be a vector such that the matrix $A+BK_o$ is Hurwitz with A and B defined in (3.17). Then there exists a smooth function $\alpha_o : \mathbb{R}^n \rightarrow \mathbb{R}$ which solves the global asymptotic stabilization with prescribed local behavior.*

Similarly to the backstepping case this theorem with Theorem 10 establish that given Q , a positive definite matrix in $\mathbb{R}^{n \times n}$, and R , a positive real number, there exists \mathfrak{q} , \mathfrak{r} and α_o which is solution to an optimal control problem with cost $J(\xi; u)$ defined in (3.5), with \mathfrak{q} and \mathfrak{r} which satisfy (3.11). Consequently, similarly to the backstepping case, we can design a globally and asymptotically stabilizing optimal control law with prescribed local cost function.

3.1.4 Conclusion on stabilization with prescribed local behaviors

In this Section we have considered the stabilization with prescribed local behavior. We have shown that for the two classical nonlinear designs of control laws (backstepping and forwarding) it is possible to select properly parameters in order to obtain local optimality. In some sens, it shows that these nonlinear designs are not restrictive.

We have employed this framework to the case of the orbital transfer which stabilization algorithm was already developped in [59]. In [V32], we have introduced a class of costs that could be locally optimized for this system.

It would be interesting to see if the same type of properties hold when dealing with infinite dimensional system for which backstepping techniques apply.

What would be interesting to look for would be a counter example. More precisely, an open question related to this topic could be the following. *Would it be possible to find a locally stabilizing behavior which can't be reproduced by a stabilizable nonlinear dynamical system ?*

Employing some of the results we have found in [V25], we know that if such counter example exists it has to be a system in dimension larger then 2.

3.2 Output feedback stabilization

3.2.1 Introduction

Problem statement.

We now relax the qualitative behavior of the closed loop system and we consider again the stabilization problem of the origin. Note however that we consider the case in which only an output measurement is allowed in the control loop. Indeed, we now consider the problem of stabilization via output feedback designs. More precisely, we are interested in studying the solutions which have been proposed to the following stabilization problem.

Stabilization by dynamic output feedback : *Given two continuous functions $f : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$ and $h : \mathbb{R}^n \rightarrow \mathbb{R}$, find an integer m and continuous functions $\nu : \mathbb{R}^m \times \mathbb{R} \rightarrow \mathbb{R}^m$ and $\varpi : \mathbb{R}^m \times \mathbb{R} \rightarrow \mathbb{R}$ such that the origin is a globally asymptotically stable equilibrium of the system :*

$$\begin{aligned} \dot{x} &= f(x, u), \quad y = h(x) & x \in \mathbb{R}^n, \quad y \in \mathbb{R}, \\ \dot{w} &= \nu(w, y), \quad u = \varpi(w, y) & w \in \mathbb{R}^m, \quad u \in \mathbb{R}, \end{aligned} \tag{3.19}$$

where x is the state of a dynamical system to be controlled, y is a measured output, u is the control and w is the state of a controller to be designed.

We restrict our attention here to the global case for two reasons :

1. We want the domain of attraction to be a *given* open set which, in the coordinates of (3.19), is the whole space,
2. For the sake of possibly achieving better performance, we address the non linear terms as they are, forbidding for instance the possibility of dominating them by functions with linear growth as typically done in the design of high gain output feedback addressing the semi-global case.

But the global case with non linear dynamics is difficult. It is known (see [86]) that stabilizability and observability are not sufficient for the existence of a global solution, as opposed, for instance, to the semi-global case [127] or the local case [23].

To overcome these difficulties many different routes have been investigated by different authors and schools. Getting a complete view of all the literature is very difficult, because of its variety and its dispersion. In this section we propose a framework for studying output feedback designs, in a unified way. This is a summary of the survey we have published in [V16].

We rely on the distinction of two classes of designs :

1. the direct approach, also called control error model analysis, in which the attention is focused on directly estimating a stabilizer,
2. the indirect approach, also called dynamic error model analysis, in which the stabilization task is fulfilled for an estimated model of the system and not directly for the system itself.

Such a classification and the terminology we are using are not new. They are borrowed from the literature on adaptive linear control (see [46]) and have been used in the non-linear context in [96].

Our contribution in [V16] is to provide new ways of proving and viewing existing results. Also, it is certainly not a compilation of the existing literature. Note that on the way, we give some results that we have obtained in [V15] and [V19] to solve this problem for some class of systems.

An illuminating detour.

To motivate our forthcoming classification of output feedback designs, we consider a general interconnected system²

$$\dot{\eta}_s = f_s(\eta_s, \eta_e) \quad , \quad \dot{\eta}_e = f_e(\eta_s, \eta_e) \quad (3.20)$$

with f_s and f_e two continuous functions. As we shall see, writing the closed loop system (3.19) as system (3.20) leads to distinct interpretations depending on which part of the state (x, w) is named η_s or η_e .

Assume the origin is a globally asymptotically stable equilibrium for system (3.20). Then there exists a C^∞ positive definite and radially unbounded function V whose derivative along the solutions of the system is negative definite. It follows that $\eta_s \mapsto \underset{\eta_e}{\text{Argmin}} V(\eta_s, \eta_e)$ is a set valued map with non-empty values. We have (see [105];[92, Section III]) :

Lemma 1 ([V16]) *If there exists a selection $\eta_s \mapsto \psi(\eta_s) \in \underset{\eta_e}{\text{Argmin}} V(\eta_s, \eta_e)$ which is locally Hölder³ of order strictly larger than $\frac{1}{2}$, then the following holds :*

1. $U(\eta_s) = V(\eta_s, \psi(\eta_s))$ is a C^1 control Lyapunov function (CLF) for the system :

$$\dot{\eta}_s = f_s(\eta_s, u)$$

whose derivative is made negative definite by the feedback⁴ $u = \psi(\eta_s)$. Precisely,

$$\eta_s \mapsto \mathcal{U}(\eta_s) = \frac{\partial U}{\partial \eta_s}(\eta_s) f_s(\eta_s, \psi(\eta_s)) \quad (3.21)$$

is a negative definite function.

2. There exists a continuous function H satisfying :

$$V(\eta_s, \eta_e) = U(\eta_s) + (\eta_e - \psi(\eta_s))^T H(\eta_s, \eta_e) (\eta_e - \psi(\eta_s)) . \quad (3.22)$$

Hence with an extra condition – Hölder selection – global asymptotic stability of the origin of system (3.23) gives rise to the decomposition (3.22) which exhibits :

1. a CLF for the η_s sub-system associated to the stabilizing state feedback ψ ;
2. a quadratic term in $\eta_e - \psi(\eta_s)$ that, in the present context, it is tempting to interpret as an estimation error, with η_e playing the role of an estimation of the stabilizer $\psi(\eta_s)$.

²Index “s” is to be thought as “stabilize” and index “e” as “estimating”.

³A function f is said Hölder of order α if there exists a real number k such that we have $|f(x_1) - f(x_2)| \leq k|x_1 - x_2|^\alpha$, for all (x_1, x_2) .

⁴In this case, we say that the feedback ψ is associated to the CLF U .

We have also the following decomposition for the time derivative of V along (3.20) :

$$\dot{V}(\eta_s, \eta_e) = \mathcal{U}(\eta_s) + (\eta_e - \psi(\eta_s))^T [\mathfrak{A}(\eta_s, \eta_e)\dot{\eta}_e + \mathfrak{B}(\eta_s, \eta_e)] , \quad (3.23)$$

with the function \mathcal{U} defined in (3.21) and some functions \mathfrak{A} and \mathfrak{B} . Since \dot{V} is negative definite, $\dot{\eta}_e$ must be such that the positive part of :

$$(\eta_e - \psi(\eta_s))^T [\mathfrak{A}(\eta_s, \eta_e)\dot{\eta}_e + \mathfrak{B}(\eta_s, \eta_e)]$$

is canceled or dominated by the negative definite function $\mathcal{U}(\eta_s)$.

The decomposition (3.22) is the basis of the classification we propose for output feedback designs. Specifically,

1. when the role of η_s is played by the system state x and the one of η_e by the controller state w , i.e. :

$$\eta_s = x \quad , \quad \eta_e = w \quad ,$$

then we have what we call a direct design, or a control error model analysis.

2. Instead, when :

$$\eta_s = w \quad , \quad \eta_e = x$$

or⁵

$$\eta_s = (w, y) \quad , \quad \eta_e = x \pmod{y = h(x)} \quad ,$$

then we have what we call an indirect design, or a dynamic error model analysis.

In each of these two classes, variations are possible depending on how much the stability margin (for instance quantified by \mathcal{U}) is used in designing $\dot{\eta}_e$, as discussed about (3.23) above.

System in normal form.

To illustrate our presentation we will consider systems in a controlled triangular form. In other word, we assume that there exists a state transformation (diffeomorphism, injective immersion...) such that the system can be written

$$\begin{aligned} \dot{\xi}_z &= \Phi_z(\xi_z, \xi_1) , \\ \dot{\xi}_1 &= \xi_2 + \Phi_1(\xi_z, \xi_1) \quad , \quad y = \xi_1 \quad , \\ &\vdots \\ \dot{\xi}_{n-1} &= \xi_n + \Phi_{n-1}(\xi_z, \xi_1, \dots, \xi_{n-1}) \quad , \\ \dot{\xi}_n &= u + \Phi_n(\xi_z, \xi_1, \dots, \xi_n) \quad , \end{aligned} \quad (3.24)$$

or, in compact form, as :

$$\dot{\xi} = \varphi(\xi, y, u) \quad , \quad y = \xi_1 \quad (3.25)$$

A complete coordinate-free characterization of system which can be rewritten via diffeomorphism in this form is given in [21, Corollary 5.7]. This is one of the most general (nominal) structure for which we know how to design a globally asymptotically stabilizing output feedback and whose study has been initiated by [54] and [82] and further developed for instance in [V19], [33, 67, 56, 107] (see also the references therein).

⁵ $x \pmod{y = h(x)}$ means that η_e is made of the components of x that are not directly given by the knowledge of $y = h(x)$. This notion makes full sense when $h(x)$ can be used as a coordinate, i.e. when the function h is injective with $|\frac{\partial h}{\partial x}|$ never zero.

3.2.2 Direct design = control error model analysis.

The design is approached by viewing the closed loop system (3.19) as system (3.20) with the following identification :

$$\eta_s = x \quad , \quad \eta_e = w \quad .$$

Lemma 1 says (ignoring the requirement of a Hölder selection!) that, if the stabilization problem is solved, then there exist a function ψ and a Lyapunov function V such that we have :

$$V(x, w) = U(x) + (w - \psi(x))^T H(x, w) (w - \psi(x)) \quad (3.26)$$

$$\dot{V}(x, w) = \mathcal{U}(x) + (w - \psi(x))^T [\mathfrak{A}(x, w)\dot{w} + \mathfrak{B}(x, w)] < 0 \quad \forall (x, w) \neq 0 \quad (3.27)$$

where,

$$\mathcal{U}(x) = \frac{\partial U}{\partial x}(x) f(x, \varpi(\psi(x), h(x))) < 0 \quad \forall x \neq 0. \quad (3.28)$$

These three relations can be interpreted as follows.

1. As mentioned above, we can view w as an estimator of $\psi(x)$, whose meaning is clarified below.
2. (3.28) says that, for the system :

$$\dot{x} = f(x, u) \quad , \quad (3.29)$$

we have a CLF U to which is associated the state feedback :

$$u = \phi(x) = \varpi(\psi(x), h(x)) \quad .$$

Embedded here is a control reparameterization :

$$u = \varpi(v, y) \quad ,$$

with v the new control. This operation allows us to go from the estimated ψ to the state feedback ϕ . For instance, in the case where the function ψ is the identity map, w should be an estimation of the state x itself.

3. Finally, (3.27) says that \dot{w} must be designed to get \dot{V} negative definite. This can be done by using or not the already negative term \mathcal{U} , i.e. by exploiting or not the stability margin of the state feedback.

We call this approach direct design since w is “directly” estimating $\psi(x)$, the reparameterized state feedback, which is the only information we need for the stabilization of (3.29). But w is only an estimation, hence, when implementing the control as (see (3.19)) :

$$u = \varpi(w, y) = \varpi(w, h(x)) \quad ,$$

we are introducing the control error $e = \psi(x) - w$ as a disturbance. This explains why we call also this method control error model analysis.

According to this direct approach, an output feedback design consists of the following steps :

step 1 : Design a stabilizing state feedback $\phi(x)$ for system (3.29),

step 2 : Do a control reparameterization of this state feedback as :

$$\phi(x) = \varpi(\psi(x), h(x)) ,$$

step 3 : Design an observer, i.e. \dot{w} , for the reparameterized control law ψ which also guarantees the negativeness of \dot{V} in (3.27).

We can re-interpret along these lines what is proposed for instance in [8, 94] and in our paper [V15].

If one wants to introduce a general approach which follow this strategy the idea is to exploit the possibility that we can find a state feedback making the system input-to-state stable (ISS) or integral input-to-state stable (iISS) with respect to an input disturbance (see [V19] [32]). This is formalized in the following statement.

Proposition 20 (ISS or iISS domination,[V16]) *The output feedback stabilization problem is solved if the integer m and the continuous functions $\nu : \mathbb{R}^m \times \mathbb{R} \rightarrow \mathbb{R}^q$ and $\varpi : \mathbb{R}^m \times \mathbb{R} \rightarrow \mathbb{R}$ are such that the following holds :*

1. *There exist a control reparameterization $u = \varpi(v, y)$ and a corresponding state feedback ψ making the system :*

$$\dot{x} = f(x, \varpi(\psi(x) + e, h(x)))$$

(γ)-iISS (respectively ISS) with e as input i.e. there exist a C^1 , positive definite and radially unbounded function U and a continuous function γ , zero at zero, satisfying :

$$\dot{U}(x) \leq \mathcal{U}(x) + \gamma(|e|) \quad \forall (x, e) ,$$

with \mathcal{U} negative definite (respectively, and radially unbounded);

2. *The state w of :*

$$\dot{w} = \nu(w, y)$$

is an estimate of $\psi(x)$ such that $\gamma(|w - \psi(x)|)$ is integrable (respectively, bounded and converges to 0) along any solution of the closed loop system.

A straightforward application of this design via ISS domination yields the following result that we have established in [V15] for systems in the normal form (3.24).

Proposition 21 ([V15]) *If :*

1. *the sub-system $\dot{\xi}_z = \Phi_z(\xi_z, \xi_1)$ is linear in y_1 and feedback linearizable;*
2. *there exist a continuous function $y \mapsto K(y)$ and a positive definite symmetric matrix P satisfying, for all (ξ, y, u)*

$$P \left[\frac{\partial \varphi}{\partial \xi}(\xi, y, u) + K(y)C \right] + \left[\frac{\partial \varphi}{\partial \xi}(\xi, y, u) + K(y)C \right]^\top P < 0 ; \quad (3.30)$$

then we can solve the output feedback stabilization problem for system (3.24).

The first condition guarantees the existence of a continuous function ϕ such that the system (see (3.25)) :

$$\dot{\xi} = \varphi(\xi, \xi_1, \phi(\xi + e, \xi_1))$$

is ISS with e as input. This has been established in [32]. The second condition is a slightly weaker version of the sufficient condition that we have introduced in (2.16). It implies that, by selecting :

$$\dot{\hat{\xi}} = \varphi(\hat{\xi}, y, u) + K(y)(C\hat{\xi} - y) ,$$

we get that $e = \xi - \hat{\xi}$ is bounded and converges to 0 along any (bounded) solution⁶. Hence a direct design via ISS domination (Proposition 20) can be done with the control reparameterization :

$$u = \varpi(v, y) = \phi(v, y)$$

and $\psi(\xi, y_1) = \xi$.

The first condition in Proposition 21 is not satisfactory. It remains an open problem to know if it could be replaced by the more “natural” one :

There exists a sufficiently many times differentiable function ϕ_z such that the system :

$$\dot{\xi}_z = \Phi_z(\xi_z, \phi_z(\xi_z + d))$$

is ISS, with d as input.

There are many cases where the stability margin is unknown, though it exists. This is typically the case when we have only a weak CLF. To proceed in such cases a possibility is to apply in a straightforward manner what we learned in Lemma 1, i.e. to go with a Lyapunov design for \dot{w} . This approach which does not rely on a robustness analysis has been followed for mechanical system and it has been extended in [105]. In collaboration with Helene Piet Lahanier from ONERA (a french aerospace company) we have employed this approach to develop an output feedback law solving an exoatmospheric interception problem (see [V12]).

3.2.3 Indirect design = Dynamic error model analysis.

To introduce in a simple way the indirect design based on dynamic error model analysis, we assume that the output can be taken as one coordinate. This means that x can be decomposed as $x = (\chi, y)$ and the dynamic is (see (3.25) for an illustration) :

$$\dot{\chi} = A(\chi, y, u) \quad , \quad \dot{y} = C(\chi, y, u) . \quad (3.31)$$

The design is approached by viewing the system (3.19) as system (3.20) with the following identification :

$$\eta_s = (w, y) \quad , \quad \eta_e = \chi \quad (= x \text{ (mod } y = h(x))) .$$

Lemma 1 says (ignoring again the extra condition) that, if the stabilization problem is solved then there exists a Lyapunov function V admitting the decomposition :

$$V((w, y), \chi) = U(w, y) + (\chi - \psi(w, y))^T H(\chi, (w, y)) (\chi - \psi(w, y)) .$$

Considering $\chi - \psi(w, y)$ as an estimation error leads us to interpret the equations :

⁶The stronger version (2.16) implies convergence of the estimation error also for unbounded solution since we ask for a uniform decrease of the estimation error Lyapunov function via the Q matrix.

$$\dot{w} = \nu(w, y) , \quad \hat{\chi} = \psi(w, y) , \quad (3.32)$$

as those of an observer of the unmeasured state components χ .

Lemma 1 says also that U is a CLF for the $\eta_s = (w, y)$ sub-system when $\chi = \psi(w, y)$. This sub-system is :

$$\dot{w} = \nu(w, y) , \quad \dot{y} = C(\chi, y, \varpi(w, y)) , \quad (3.33)$$

with therefore χ as control and $\chi = \psi(w, y)$ as a stabilizing state feedback.

To get a better grasp on the above two comments, it is informative to rewrite (3.32) and (3.33) as :

$$\begin{aligned} \dot{w} &= \nu(w, y) , \\ \dot{y} &= C(\hat{\chi}, y, \varpi(w, y)) + d_y(\chi, w, y) , \\ \hat{\chi} &= \psi(w, y) . \end{aligned} \quad (3.34)$$

This shows a system with $\hat{\chi}$ as both input and output, and disturbed by :

$$d_y(\chi, w, y) = C(\chi, y, \varpi(w, y)) - C(\hat{\chi}, y, \varpi(w, y)) .$$

The presence of d_y explains why we call (3.33) a model with an error in its dynamic or shortly dynamic error model. As U is a CLF for the undisturbed part of (3.34), with the input $\hat{\chi} = \psi(w, y)$ the associated stabilizing state feedback, one task in designing the controller functions ν , ϖ and indirectly ψ , is to achieve stabilization in spite of the presence of d_y . This has motivated many specific contributions on state feedbacks providing larger stability margin. See [V15], [32][54, 68, 52, 76] for instance.

On the other hand the disturbance d_y is necessary for guaranteeing the convergence of the output $\hat{\chi}$ of (3.34) towards χ which is needed to transfer the stabilization property obtained for (w, y) to χ . It is because stabilization for χ is obtained in this indirect way that we call this design indirect.

Furthermore, we remark that, if (3.32) is indeed an observer of χ , then the set $\{(\chi, y, w) : \chi = \psi(w, y)\}$ should be invariant for the coupled system (3.31),(3.33). In other words, we should have the identity :

$$A(\psi(w, y), y, \varpi(w, y)) = \frac{\partial \psi}{\partial w}(w, y) \nu(w, y) + \frac{\partial \psi}{\partial y}(w, y) C(\psi(w, y), y, \varpi(w, y)) .$$

But then this implies that we have (differentiate on both side the equation $\hat{\chi} = \psi(w, y)$) :

$$\begin{aligned} \dot{y} &= C(\hat{\chi}, y, \varpi(w, y)) + d_y(\chi, w, y) , \\ \dot{\hat{\chi}} &= \underbrace{A(\hat{\chi}, y, \varpi(w, y))}_{\text{Copy of the system}} + \underbrace{\frac{\partial \psi}{\partial y}(w, y) d_y(\chi, w, y)}_{\text{Dynamic error}} . \end{aligned} \quad (3.35)$$

From all these arguments, we conclude that an output feedback design according to this indirect approach consists in the following two steps :

step 1 : Design an observer for the state unmeasured part χ , i.e. a correction term (d_y, ψ) in (3.35), with the objective that any good property (e.g. convergence) obtained for $\hat{\chi}$ is transferred to χ ,

step 2 : Design a control law ϖ ensuring good properties for $\hat{\chi}$ in spite of the presence of the correction term.

Most of the publications on global stabilization by output feedback can be re-interpreted along these lines. In particular this is the case of our result together with Laurent Praly in [V15].

As opposed to the case of the direct approach, in the indirect approach, the observer is designed first and then the state feedback is designed for this observer.

We can formalize the following statement.

Proposition 22 (ISS or iISS domination, [V16]) *The output feedback stabilization problem is solved if we can find three functions k_l , k_r and ϖ such that :*

1. *the system :*

$$\dot{\hat{x}} = f(\hat{x}, \varpi(\hat{x}, y)) + k_l(\hat{x}, y) d$$

is (γ) iISS (resp. ISS) with d as input;

2. *along the solutions of :*

$$\begin{aligned} \dot{x} &= f(x, \varpi(\hat{x}, y)) , \\ \dot{\hat{x}} &= f(\hat{x}, \varpi(\hat{x}, y)) + k_l(\hat{x}, y) k_r(\hat{x}, y) , \end{aligned}$$

$\gamma(|k_r(\hat{x}, y)|)$ is integrable (resp. bounded) and $\hat{x} - x$ converges to 0.

In the context of this proposition, we have $q = n$ and :

$$\nu(w, y) = f(w, \varpi(w, y)) + k_l(w, y) k_r(w, y) .$$

For example, a straightforward application of this design via iISS domination yields the following result that we have established in [V15] for systems in the normal form (3.24).

Proposition 23 ([V15]) *If :*

1. *there exist a continuous function $y \mapsto K(y)$ and a positive definite symmetric matrix P satisfying, for all (χ, y) (see notation in (3.25)),*

$$P \left[\frac{\partial \varphi}{\partial \chi}(\chi, y, u) + K(y)C \right] + \left[\frac{\partial \varphi}{\partial \chi}(\chi, y, u) + K(y)C \right]^\top P < -C^\top C ; \quad (3.36)$$

2. *there exists a sufficiently many times differentiable function ϕ_z such that the system :*

$$\dot{\xi}_z = \Phi_z(\xi_z, \phi_z(\xi_z)) + K_z(y) d$$

is (γ) iISS, with d as input and $\gamma(s) = ks^2$, and where K_z is the ξ_z -component of K above,

then we can solve the output feedback stabilization problem for system (3.24).

The first condition guarantees that :

$$d = C(\hat{\xi} - \xi),$$

is square integrable along any solution of the system

$$\begin{aligned} \dot{\xi} &= \varphi(\xi, \xi_1, u) , \\ \dot{\hat{\xi}} &= \varphi(\hat{\xi}, \xi_1, u) + K(\xi_1)[\hat{\xi}_1 - \xi_1] \end{aligned}$$

and for any input $t \mapsto u(t)$. The second condition guarantees the existence of a continuous function ϕ such that the system :

$$\dot{\xi} = \varphi(\xi, \xi_1, \phi(\xi)) + K(\xi_1)d ,$$

is (γ) iISS, with $\gamma(s) = ks^2$, and with d as input.

Proposition 23 follows the route of domination expressed as a property of (γ) iISS, with $\gamma(s) = ks^2$.

3.2.4 Design via a dominant model.

Up to now, both for the direct and indirect case, we have mentioned designs following a domination approach where we exploit the negativeness of \mathcal{U} , obtained for a CLF U for the nominal system :

$$\dot{x} = f(x, u) .$$

or the one given in other structured coordinates

$$\dot{\xi} = \varphi(\xi, \xi_1, u)$$

We can push this strategy further by working only with a “dominant” approximation of this nominal system. In the linear case, the archetype of such an approach says that, by designing a high gain linear output feedback for the chain of integrators :

$$\dot{\xi}_1 = \xi_2 , \dots , \dot{\xi}_{n-1} = \xi_n , \dot{\xi}_n = u , \quad y = \xi_1 , \quad (3.37)$$

and by adjusting the gain, we can solve the stabilization problem by output feedback for any minimum phase linear system with relative degree n (see [61] for instance).

Again, as it has been done in the high-gain observer context, it is possible to extend this result to nonlinear dynamics based on homogeneous in the bi-limit tools that we have introduced in Chapter 1.

The chain above giving the dominant part of the model, the actual system can take the form :

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 + \delta_1 , \\ &\vdots \\ \dot{\xi}_{n-1} &= \xi_n + \delta_{n-1} , \\ \dot{\xi}_n &= u + \delta_n , \end{aligned} \quad , \quad y = \xi_1 ,$$

where the perturbations δ_i are handled via robustness. Typically, they are considered as outputs of ISS systems with the ξ_i as inputs. A standard way to formalize this is to assume the following inequalities hold :

$$|\delta_i| \leq \sqrt{V_i} + \mathbf{b}_i(\xi_1, \dots, \xi_n)$$

where, along each solution of the system, V_i satisfies the differential inequality :

$$\dot{V}_i \leq - (V_i - \mathbf{b}_i(\xi_1, \dots, \xi_n)^2) .$$

With an indirect approach, employing a homogeneous in the bi-limit state feedback in combination with the homogeneous in the bi-limit observer (introduced in Section 2.2.2)

we considered the cases in [V19]

$$\mathbf{b}_i(\xi_1, \dots, \xi_n) = \mu \sum_{j=1}^i |\xi_j|^{\frac{1-d_0(n-i-1)}{1-d_0(n-j)}} + |\xi_j|^{\frac{1-d_\infty(n-i-1)}{1-d_\infty(n-j)}}$$

or

$$\mathbf{b}_i(\xi_1, \dots, \xi_n) = \mu \sum_{j=i+2}^n |\xi_j|^{\frac{1-d_0(n-i-1)}{1-d_0(n-j)}} + |\xi_j|^{\frac{1-d_\infty(n-i-1)}{1-d_\infty(n-j)}}$$

with $-1 < d_0 \leq d_\infty < \frac{1}{n-1}$ and μ a positive real number.

3.2.5 Conclusion on output feedback designs

The literature on output feedback is so rich that there is a need to clarify and compare the various contributions. To address this point, we have proposed in [V16] a framework for studying, in a unified way, the proposed globally stabilizing output feedback designs. The core is a classification in direct versus indirect approach where *direct* means that the design is done to directly address the stabilization problem whereas *indirect* says that this problem is solved only because some kind of observer converges. By far, the indirect approach is the most frequently exploited in the theoretical contributions. Instead the direct approach is likely to be the most frequently used by control designers. In [V16] we have also seen that, within the same class – direct or indirect – a wide variety of designs is possible depending on how much the stability margin of a state feedback or the convergence margin of an observer is exploited. In particular a full exploitation of these margins allows to develop further the approach to make it applicable, not to the given system, but only to a “dominant” approximation of it.

Although a lot of effort has been devoted to this output feedback stabilization problem, there are still many open problems. For instance, the need of observers is apparent, to reconstruct only a function of the state (a reparameterized version of the state feedback) or the state itself. To answer this need new observers going significantly beyond the linear paradigm have to be considered. However we are still limited with some necessary conditions introduced for instance in [86].

Also we have motivated the restriction of our attention here to the global asymptotic stabilization case in particular for forcing the designer to address the non linear terms as they are. It turns out that this specific point is hardly addressed by the available designs and certainly not by those going with a dominant approximation as those mentioned right above. This leads to question their interest for practice. In these regards direct designs not relying on any stability margin are very appealing since they exploit more the peculiarity of the system. Unfortunately they have received very little attention up to now and strongly rely on very particular structure.

3.3 Event-triggered high gain feedback

3.3.1 Introduction and problem statement

The implementation of a control law on a process requires the use of an appropriate sampling scheme. In this regards, periodic control (with a constant sampling period) is the usual approach that is followed for practical implementation on digital platforms. However, the recent growth of shared networked control systems for which communication and energy resources are often limited goes with an increasing interest in aperiodic control design. This can be observed in the comprehensive overview on event-triggered and self-triggered control presented in [42]. Event-triggered control strategies introduce a triggering condition assuming a continuous monitoring of the plant (that requires a dedicated hardware).

Most of the existing results on event-triggered control for nonlinear systems are based on the existence of a continuous time feedback control law ensuring an input-to-state stability (ISS) property with respect to measurement errors (see [125, 4, 1, 98] and also [121]). However, as it is mentioned in the previous section, this requirement is difficult to address in general. In this ISS framework, an emulation approach is followed: the knowledge of an existing robust feedback law in continuous time is assumed, and some triggering conditions are proposed to preserve stability under sampling. Another proposed approach consists in the redesign of a continuous time stabilizing control. For instance, the authors in [80] adapted the original *universal formula* introduced by Sontag for nonlinear control affine systems.

Although aperiodic control literature has demonstrated an interesting potential, important fields still need to be further investigated to allow a wider practical deployment. In particular, literature on event-triggered output feedback control for nonlinear systems is scarce ([134], [77], [126]) whereas, in many control applications, the full state information is not available for measurement.

As we have already seen, the use of high-gain approach via a dominant model is very efficient to address the output feedback stabilization problem in the continuous time case (see Section 3.2.4). It has the advantage to allow uncertainties in the model and to remain simple. In the context of observer design, we have already followed this approach in Section 2.2.6 to design a continuous discrete time observer, revisiting high-gain techniques in order to give an adaptive sampling stepsize.

In this section we present the result we have obtained following this route in the context of the output feedback designs as it has been published in [V42] (see also the state-feedback design in [V43]) Consequently, again, we consider systems with an upper triangular structure (as in (2.18))

$$\begin{cases} \dot{\xi}_1 &= \xi_2 + \Phi_1(\xi_1) \\ \dot{\xi}_2 &= \xi_3 + \Phi_2(\xi_1, \xi_2) \\ &\vdots \\ \dot{\xi}_m &= u(t) + \Phi_m(\xi_1, \dots, \xi_m) \end{cases}, y = \xi_1, \quad (3.38)$$

where the state $\xi \in \mathbb{R}^m$, the control signal $u \in \mathbb{L}^\infty(\mathbb{R}_+, \mathbb{R})$ and the output $y \in \mathbb{R}$. The dynamics of this system can be rewritten as

$$\dot{\xi} = A\xi + Bu + \Phi(\xi), \quad (3.39)$$

where $A \in \mathbb{R}^{n \times n}$ is the upper shift matrix, $B = (0, \dots, 0, 1) \in \mathbb{R}^n$ and $\Phi = (\Phi_1, \dots, \Phi_m)$ is a vector field on \mathbb{R}^n .

Note that to oppose to the system we have considered in the continuous time output feedback (see system 3.24) we assume that there is no inverse dynamics. I believe that this strong assumption could be removed. Note however that in that case computation might be really messy.

Notation. In this section to simplify the presentation, we introduce the following notations: $\xi_k = \xi(t_k)$ and $\xi_k^- = \xi(t_k^-)$, where $\xi(t^-) = \lim_{\substack{\tau \rightarrow t \\ \tau < t}} \xi(\tau)$.

Since we follow a high-gain approach (with linear correction terms and state-feedback), we have to restrict ourselves to nonlinearities Φ_i verifying some kind of linear growth conditions. More precisely, we consider the following assumption.

Assumption 11 (Nonlinear bound) *There exist positive real numbers c_0, c_1, q such that for all $\xi \in \mathbb{R}^m$, we have*

$$|\Phi_j(\xi(t))| \leq \Gamma(\xi_1) (|\xi_1| + |\xi_2| + \dots + |\xi_j|), \quad (3.40)$$

where c is function defined by

$$\Gamma(\xi_1) = c_0 + c_1 |\xi_1|^q. \quad (3.41)$$

Notice that Assumption 11 is more general than the incremental property employed in the observer design context in Assumption 4 since the function Γ is not upper bounded but depends on x_1 . This bound could be also related to [100] in which continuous output feedback laws were designed. Note however that in these works no polynomial bounds were imposed on the function Γ .

In the sequel, we restrict ourselves to a sample-and-hold implementation, i.e. the input is assumed to be constant between any two execution times. The control input u is defined through a sequence $(t_k, u_k)_{k \in \mathbb{N}}$ in $\mathbb{R}_+ \times \mathbb{R}$ in the following way

$$u(t) = u_k, \quad \forall t \in [t_k, t_{k+1}), \quad k \in \mathbb{N}. \quad (3.42)$$

It can be noticed that for u to be well defined for all positive time, we need that $\lim_{k \rightarrow +\infty} t_k = +\infty$.

Our control objective is to design the sequence $(t_k, u_k)_{k \in \mathbb{N}}$ such that the origin of the obtained closed loop system is asymptotically stable. This sequence depends only on the system output. In the same spirit as for the sample-and-hold control, only the sequence of output values

$$y_k = C\xi(t_k), \quad C = [1 \quad 0 \quad \dots \quad 0], \quad (3.43)$$

will enter the output feedback.

In addition to a feedback controller that computes the control input, event-triggered control systems need a *triggering mechanism* that determines when a new measurement occurs and when the control input has to be updated again. This rule is said to be *static* if it only involves the current state of the system, and *dynamic* if it uses an additional internal dynamic variable.

3.3.2 Preliminary result: linear case

As we have already seen, in high-gain design, the idea is to consider the nonlinear terms (the Φ_i 's) as disturbances. A first step consists in synthesizing a robust control for the

linear part of the system, neglecting the effects of the nonlinearities. Then, convergence and robustness are amplified through a high-gain parameter to deal with the nonlinearities.

Therefore, let us first focus on a general linear dynamical system

$$\dot{\xi} = A\xi + Bu, \quad (3.44)$$

where the state ξ evolves in \mathbb{R}^m and the control u is in \mathbb{R} . The matrix A is in $\mathbb{R}^{m \times m}$ and the matrix B is in \mathbb{R}^m . The measured output is given as a sequence of values $y_k = y(t_k) \in \mathbb{R}$ as in (3.43) where $(t_k)_{k \in \mathbb{N}}$ is a sequence of times to be selected.

In this preliminary case, we review a well known result concerning periodic sampling approaches. Indeed, an emulation approach is used for the stabilization of the linear part: a feedback law is designed in continuous time and a triggering condition is chosen to preserve stability under sampling.

It is well known that if there exists a continuous time dynamical output feedback control law that asymptotically stabilizes the system, then there exists a positive inter-execution time $\delta = t_{k+1} - t_k$ such that the sampled control law renders the system asymptotically stable. This result is rephrased in Lemma 2 below.

Lemma 2 *Assume that K_c and K_o are such that $A + BK_c$ and $A + K_oC$ are Hurwitz. Then there exists a positive real number δ^* such that for all δ in $[0, \delta^*)$ and any sequence $(t_k, u_k)_{k \in \mathbb{N}}$ defined as*

$$t_0 = 0, \quad t_{k+1} = t_k + \delta, \quad u_k = K_c \hat{x}_k, \quad \forall k \in \mathbb{N}, \quad (3.45)$$

where \hat{x}_0 is in \mathbb{R}^n and for k in \mathbb{N}^*

$$\dot{\hat{\xi}} = A\hat{\xi} + Bu_k, \quad \forall t \in [t_k, t_{k+1}), \quad (3.46)$$

$$\hat{\xi}_k = \hat{\xi}_k^- + \delta K_o (C\hat{\xi}_k^- - y_k), \quad (3.47)$$

$(\xi, \hat{\xi}) = 0$ is a globally and asymptotically stable (GAS) equilibrium for the dynamical system defined by (3.42)–(3.47).

This result which is based on robustness is valid for general matrices A , B and C . We want to point out that the proof of Lemma 2 is based on the fact that if $A + BK_c$ and $A + K_oC$ are Hurwitz, the origin of the discrete time linear system defined for all k in \mathbb{N} as

$$\begin{bmatrix} \hat{\xi}_{k+1} \\ e_{k+1} \end{bmatrix} = \begin{bmatrix} F_c(\delta) & \delta K_o C \exp(A\delta) \\ 0 & F_o(\delta) \end{bmatrix} \begin{bmatrix} \hat{\xi}_k \\ e_k \end{bmatrix} \quad (3.48)$$

where $e = \hat{\xi} - \xi$ is the estimation error, and

$$F_c(\delta) = \exp(A\delta) + \int_0^\delta \exp(A(\delta - s))BK_c ds \quad (3.49)$$

$$F_o(\delta) = (I + \delta K_o C) \exp(A\delta) \quad (3.50)$$

is asymptotically stable for δ sufficiently small. However, when we consider the particular case in which (A, B, C) are as in (3.44)–(3.43) (i.e. a chain of integrators), it is shown in the following theorem that the inter-execution time can be selected arbitrarily large as long as the control is modified.

Theorem 14 (Chain of integrators [V42]) *Suppose A , B and C have the structure stated in (3.44)-(3.43). Let K_c and K_o , be such that $A + BK_c$ and $A + K_oC$ are Hurwitz. Then there exists a positive real number α^* such that for all $\alpha \in [0, \alpha^*)$, all $\delta > 0$ and any sequence $(t_k, u_k)_{k \in \mathbb{N}}$ defined as*

$$t_0 = 0, \quad t_{k+1} = t_k + \delta, \quad u_k = K_c L^{n+1} \mathcal{L}^{-1} \hat{\xi}_k, \quad (3.51)$$

where $L = \frac{\alpha}{\delta}$, $\hat{\xi}_0 \in \mathbb{R}^n$ and

$$\dot{\hat{\xi}} = A\hat{\xi} + Bu_k, \quad \forall t \in [t_k, t_{k+1}), \quad (3.52)$$

$$\hat{\xi}_k = \hat{\xi}_k^- + \delta \mathcal{L} K_o (C \hat{\xi}_k^- - y_k), \quad k \in \mathbb{N}, \quad (3.53)$$

with \mathcal{L} defined as before as $\mathcal{L} = \text{Diag}(L, \dots, L^m)$, $(\xi, \hat{\xi}) = 0$ is a GAS equilibrium for the dynamical system defined by (3.42), (3.44), (3.51), (3.52) and (3.53).

Remark 1 *Note that the difference between equation (3.47) and equation (3.53) is the \mathcal{L} factor that appears in the latter.*

Remark 2 *Note that in the particular case of the chain of integrators the sampling period δ can be selected arbitrarily large. To obtain this result the two gains K_c and K_o have to be modified as seen in equations (3.51) and (3.53). In the following section, when dealing with nonlinear systems, the matrix \mathcal{L} is modified via a dynamical event-triggered mechanism. This aspect implies that compared to usual emulation approaches, a co-design approach has to be performed in order to obtain asymptotic stabilization.*

To understand the proof of this result which is given in [V42] let us consider the change of coordinates:

$$\hat{Z} = \mathcal{L}^{-1} \hat{\xi}, \quad E = \mathcal{L}^{-1} e. \quad (3.54)$$

it yields that in the new coordinates the closed-loop dynamics satisfies for all k in \mathbb{N} :

$$\hat{Z}_{k+1} = F_c(\alpha) \hat{Z}_k + \alpha K_o C \exp(A\alpha) E_k.$$

and

$$E_{k+1} = F_o(\alpha) E_k.$$

In other words, this is the same discrete dynamic as the one given in (3.48). Consequently, from Lemma 2, there exists a positive real number α^* such that $(\hat{Z}, E) = 0$ (and thus $(\xi, \hat{\xi}) = 0$) is a GAS equilibrium for the system provided $L\delta$ is in $[0, \alpha^*)$.

3.3.3 The nonlinear case

We now consider the full nonlinear system (3.39) together with the growth condition on the nonlinearities in Assumption 11. Following what we have already done in the context of high-gain observer we define the high-gain parameter to be the L coordinate of the solution of the following continuous discrete dynamical system:

$$\dot{L} = a_2 L M \Gamma(\xi_1), \quad \forall t \in [t_k, t_k + \delta_k), k \in \mathbb{N} \quad (3.55)$$

$$\dot{M} = a_3 M \Gamma(\xi_1), \quad \forall t \in [t_k, t_k + \delta_k), k \in \mathbb{N} \quad (3.56)$$

$$L_k = L_k^- (1 - a_1 \alpha) + a_1 \alpha, \quad \forall k \geq 1 \quad (3.57)$$

$$L_0 \geq 1 \quad (3.58)$$

$$M_k = 1 \quad \forall k \in \mathbb{N} \quad (3.59)$$

where a_1, a_2, a_3, α are positive real numbers to be chosen and with the event triggering mechanism formally defined by

$$t_0 = 0, \quad t_{k+1} = t_k + \delta_k, \quad (3.60)$$

$$\delta_k = \min \{s \in \mathbb{R}_+ \mid sL((t_k + s)^-) = \alpha\}, \quad \forall k \in \mathbb{N}. \quad (3.61)$$

In [V42], we have shown that the above triggering mechanism is well-posed, i.e. for all k , the set $\{s \in \mathbb{R}_+ \mid sL((t_k + s)^-) = \alpha\}$ defined in (3.61) is non empty and admits a positive minimum. Finally, the control sequence is defined as

$$u_k = K_c L_k^{n+1} \mathcal{L}_k^{-1} \hat{\xi}_k, \quad k \in \mathbb{N}, \quad (3.62)$$

where $\hat{x}_0 \in \mathbb{R}^n$ and

$$\dot{\hat{\xi}} = A\hat{\xi} + Bu_k, \quad \forall t \in [t_k, t_{k+1}), k \in \mathbb{N} \quad (3.63)$$

$$\hat{\xi}_k = \hat{\xi}_k^- + \delta_{k-1}(\mathcal{L}_k^-)K_o(C\hat{\xi}_k^- - y_k). \quad (3.64)$$

with $\mathcal{L}_k^- = \text{Diag}(L_k^-, \dots, (L_k^-)^n)$. We are now ready to state our output feedback result which has been established in [V42].

Theorem 15 (Stabilization via event-triggered output feedback control, [V42])

Let Assumption 11 hold. Then there exist positive real numbers a_1, a_2, a_3, α^ and two gain matrices K_c, K_o such that, for all α in $[0, \alpha^*]$, there exists a positive real number ℓ_{\max} such that the set*

$$\{\xi = 0, \hat{\xi} = 0, L \leq \ell_{\max}\} \subset \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R},$$

is GAS along the solution of system (3.39) with the event-triggered feedback (3.55)-(3.64). More precisely, there exists a class \mathcal{KL} function β such that the solution initiated from $(\xi, \hat{\xi}, L(0))$ with $L(0) \geq 1$ is defined for all $t \geq 0$ and satisfies

$$|\Xi(t)| + |\hat{\Xi}(t)| + |\tilde{L}(t)| \leq \beta(|\xi| + |\hat{\xi}| + |\tilde{L}(0)|, t), \quad (3.65)$$

where $\tilde{L}(t) = \max\{L(t) - \ell_{\max}\}$. Moreover, there exists a positive real number δ_{\min} such that $\delta_k > \delta_{\min}$ for all k and so ensures the existence of a minimal inter-execution time.

Even so the result seems quite similar to the one which has been obtained in the context of observer design, the technicalities employed to show that this result have some important differences. This is due to the fact that to oppose to the observer design case, since the L -dynamics depend on Γ which is not a priori bounded, a small gain approach needs to be carried out. Since at the same time the dynamics are continuous discrete, computations become quite lengthy. Everything can be found in [V42].

3.3.4 Conclusion on high-gain event-triggered feedbacks

In this Section, we have presented a new event-triggered output feedback for a class of nonlinear systems. As we have already done for high-gain observer designs, the triggered mechanism depends on an additional dynamic. This additional dynamic depends on the output measurement which makes the analysis involved. In collaboration with Johan Peralez, Madiha Nadri and Ulysse Serres, we have also published some related results in the context of state-feedback in [V43].

3.4 Synchronization problems

Controlled synchronization, as a coordinated control problem of a group of autonomous systems, has been regarded as one of important group behaviors. It has found its relevance in many engineering applications, such as, the distributed control of (mobile) robotic systems, the control and reconfiguration of devices in the context of internet-of-things, and the synchronization of autonomous vehicles (see, for example, [90]).

For linear systems, the solvability of this problem and, as well as, the design of controller, have been thoroughly studied in literature. To name a few, we refer to the classical work on the nonlinear Goodwin oscillators [36], to the synchronization of linear systems in [131, 118] and to the recent works in nonlinear systems [116, 27, 19, 117]. For linear systems, the solvability of synchronization problem reduces to the solvability of stabilization of individual systems by either an output or state feedback. It has recently been established in [131] that for linear systems, the solvability of the output synchronization problem is equivalent to the existence of an internal model, which is a well-known concept in the output regulation theory.

The generalization to nonlinear systems has appeared recently in the literature (see, for example, [116, 27, 26, 25, 117, 40]). In these works, based on the concept of passivity theory (or, the weakened notions of co-coercive systems), some sufficient conditions are proposed that solve the synchronization problem. For such a class of systems, the synchronizer is constructed based on the relative output/state measurement, as in the linear systems case. In [39], small-gain theorem is used to construct a synchronization control law for L_2 -stable systems. If we restrict ourselves to the class of incremental ISS, as discussed in [3], the synchronizer can again be based on the relative output/state measurement.

In general, with the lack of characterization of controlled synchronization for general nonlinear systems, it is difficult to conclude on the generality of the synchronizer as proposed in the aforementioned works. Our work on transverse exponential stability presented in Chapter 1 is instrumental to address this type of problem. In this section, we present some of the result we have obtained in [V8, V9] and [V10]. In these works, We have established some necessary and sufficient conditions for the solvability of a (locally) exponential synchronization. We show that a necessary condition for achieving synchronization is the existence of a symmetric covariant tensor field of order two whose Lie derivative has to satisfy a Control Matrix Function (CMF) inequality, which is similar to the Control Lyapunov Function.

3.4.1 Problem definition

In this Section, we consider the problem of synchronizing N identical nonlinear systems with $N \geq 2$. For every $i = 1, \dots, N$, the i -th system Σ_i is described by

$$\dot{x}_i = f(x_i) + g(x_i)u_i, \quad i = 1, \dots, N \quad (3.66)$$

where $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^q$ and the functions f and g are assumed to be C^2 . In this setting, all systems has the same drift vector field f and the same control vector field $g : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times q}$, but not the same controls in \mathbb{R}^q . For simplicity of notation, we denote the complete state variables by $x = [x_1^\top, \dots, x_N^\top]^\top$ in \mathbb{R}^{Nn} .

The synchronization manifold \mathcal{D} , where the state variables of different systems agree with each other, is defined by

$$\mathcal{D} = \{(x_1, \dots, x_N) \in \mathbb{R}^{Nn} \mid x_1 = x_2 = \dots = x_N\}.$$

For every x in \mathbb{R}^{Nn} , we denote the Euclidean distance to the set \mathcal{D} by $|x|_{\mathcal{D}}$.

The communication graph \mathcal{G} , which is used for synchronizing the state through distributed control u_i , $i = 1, \dots, N$, is assumed to be an undirected graph and is defined by $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where \mathcal{V} is the set of N nodes (where the i -th node is associated to the system Σ_i) and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is a set of M edges that define the pairs of communicating systems. Moreover we assume that the graph \mathcal{G} is connected.

Let us, for every edge k in \mathcal{G} connecting node i to node j , label one end (e.g., the node i) by a positive sign and the other end (e.g., the node j) by a negative sign. The incidence matrix D that corresponds to \mathcal{G} is an $N \times M$ matrix such that

$$d_{i,k} = \begin{cases} +1 & \text{if node } i \text{ is the positive end of edge } k \\ -1 & \text{if node } i \text{ is the negative end of edge } k \\ 0 & \text{otherwise} \end{cases}$$

Using D , the Laplacian matrix L can be given by $L = DD^{\top}$ whose kernel, by the connectedness of \mathcal{G} , is spanned by $\mathbf{1}_N$.

Using the description of the interconnected systems via \mathcal{G} , the state synchronization control problem is defined as follows.

Definition 9 *The control laws $u_i = \phi_i(x)$, $i = 1 \dots, N$ solve the local uniform exponential synchronization problem of (3.66) if the following conditions hold:*

1. *For all non-communicating pair (i, j) (i.e., $(i, j) \notin \mathcal{E}$),*

$$\frac{\partial \phi_i}{\partial x_j}(x) = \frac{\partial \phi_j}{\partial x_i}(x) = 0, \quad \forall x \in \mathbb{R}^{Nn};$$

2. *For all $x \in \mathcal{D}$, $\phi(x) = 0$ (i.e., ϕ is zero on \mathcal{D}); and*
3. *The manifold \mathcal{D} of the closed-loop system*

$$\dot{x}_i = f(x_i) + g(x_i)\phi_i(x), \quad i = 1, \dots, N \quad (3.67)$$

is uniformly exponentially stable, i.e., there exist positive constants r , k and $\lambda > 0$ such that for all x in \mathbb{R}^{Nn} satisfying $|x|_{\mathcal{D}} < r$,

$$|X(x, t)|_{\mathcal{D}} \leq k \exp(-\lambda t) |x|_{\mathcal{D}}, \quad (3.68)$$

where $X(x, t)$ denotes the solution initiated from x , holds for all t in the time domain of existence of solution.

When $r = \infty$, it is called the global uniform exponential synchronization problem. \triangle

In this definition, the condition 1) implies that the solution u_i is a distributed control law that requires only a local state measurement from its neighbors in the graph \mathcal{G} .

An important feature of our study is that we focus on exponential stabilization of the synchronizing manifold. This allows us to rely on the study introduced in Section 1.1 in which an infinitesimal characterization of exponential stability of a transverse manifold is given. As it will be shown in the following section this allows us to formalize some necessary and sufficient conditions in terms of matrix functions ensuring the existence of a synchronizing control law.

3.4.2 Necessary conditions

In Section 2.1.2, we have shown that necessary conditions for the existence of an exponential observer are infinitesimal detectability properties and its Lyapunov characterization respectively given in definitions 7 and 8. Similarly, in this section we show that infinitesimal characterization of stabilizability are necessary conditions for the existence of a synchronizing control law.

Definition 10 (Infinitesimal stabilizability (IS)) *The couple (f, g) is said infinitesimally stabilizable (IS) if the n -dimensional manifold $\{\tilde{z} = 0\}$ of the transversally linear system*

$$\dot{\tilde{z}} = \frac{\partial f}{\partial \tilde{x}}(\tilde{x})\tilde{z} + g(\tilde{x})\tilde{u} \quad (3.69a)$$

$$\dot{\tilde{x}} = f(\tilde{x}) \quad (3.69b)$$

with \tilde{z} in \mathbb{R}^n and \tilde{x} in \mathbb{R}^n is stabilizable by a state feedback that is linear in \tilde{z} (i.e., $\tilde{u} = h(\tilde{x})\tilde{z}$ for some function $h : \mathbb{R}^n \rightarrow \mathbb{R}^{q \times n}$).

Definition 11 (Control Matrix Function (CMF)) *For all positive definite matrix $Q \in \mathbb{R}^{n \times n}$, there exist a continuous function $P : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$ which values are symmetric positive definite matrices and strictly positive real numbers \underline{p} and \bar{p} such that*

$$\underline{p}\text{Id}_n \leq P(\tilde{x}) \leq \bar{p}\text{Id}_n \quad (3.70)$$

holds for all $\tilde{x} \in \mathbb{R}^n$, and the inequality⁷

$$v^\top L_f P(\tilde{x})v \leq -v^\top Qv \quad (3.73)$$

holds for all (v, \tilde{x}) in $\mathbb{R}^n \times \mathbb{R}^n$ satisfying $v^\top P(\tilde{x})g(\tilde{x}) = 0$.

An important feature of properties IS and CMF comes from the fact that they are properties of each individual agent, independent of the network topology. The first one is a local stabilizability property. The second one establishes that there exists a symmetric covariant tensor field of order two denoted by P whose Lie derivative satisfies a certain inequality in some specific directions. This type of condition can be related to the notion of control Lyapunov function which is a characterization of stabilizability as studied by Artstein in [9] or Sontag in [123]. This property can be regarded as an Artstein like condition.

In [V10], we show that properties IS and CMF are necessary conditions if one considers a network of agents with a communication graph \mathcal{G} . Hence, as this is already the case for linear system, we recover the paradigm, which establishes that a necessary condition for synchronization is a stabilizability property for each individual agent.

⁷We remind the reader that given a vector field f on \mathbb{R}^n and a covariant two tensor $P : \mathbb{R}^n \rightarrow \mathbb{R}^{m \times m}$, P is said to have a derivative along f denoted $\mathfrak{d}_f P$ if the following limit exists

$$\mathfrak{d}_f P(\tilde{x}) = \lim_{h \rightarrow 0} \frac{P(\tilde{X}(\tilde{x}, h)) - P(\tilde{x})}{h}, \quad (3.71)$$

where $\tilde{X}(\tilde{x}, \cdot)$ is the flow of the vector field f with an initial state \tilde{x} in \mathbb{R}^n . In that case and, when $m = n$ and f is C^1 $L_f P$ is the Lie derivative of the tensor along f which is defined as

$$L_f P(\tilde{x}) = \mathfrak{d}_f P(\tilde{x}) + P(x) \frac{\partial f}{\partial \tilde{x}}(\tilde{x}) + \frac{\partial f}{\partial \tilde{x}}(\tilde{x})^\top P(\tilde{x}). \quad (3.72)$$

Theorem 16 (Necessary condition ([V10])) Consider the interconnected systems in (3.66) with the communication graph \mathcal{G} and assume that there exists a control law $u = \phi(x)$ where $\phi(x) = [\phi_1^\top(x) \ \dots \ \phi_N^\top(x)]^\top$ in \mathbb{R}^{Np} that solves the local uniform exponential synchronization of (3.66). Assume that g is bounded, f , g and the ϕ_i 's have bounded first and second derivatives and the closed-loop system is complete. Then properties IS and CMF hold.

3.4.3 Sufficient conditions for local exponential synchronization

The interest of the Property CMF is to use the symmetric covariant tensor P in the design of a local synchronizing control law. Indeed, we get the following sufficient condition for the solvability of (local) uniform exponential synchronization problem. The first assumption is that, up to a scaling factor, the control vector field g is a gradient field with P as a Riemannian metric (see also [30] for similar integrability assumption). The second one is related to the CMF property.

Theorem 17 (Local sufficient condition [V10]) Assume that g is bounded and that f and g have bounded first and second derivatives. Assume that there exists a C^2 function $P : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$ which values are symmetric positive definite matrices and with a bounded derivative that satisfies the following two conditions.

1. There exist a C^2 function $U : \mathbb{R}^n \rightarrow \mathbb{R}$ which has bounded first and second derivatives, and a C^1 function $\alpha : \mathbb{R}^n \rightarrow \mathbb{R}^q$ which has bounded first and second derivatives such that

$$\frac{\partial U}{\partial \tilde{x}}(\tilde{x})^\top = P(\tilde{x})g(\tilde{x})\alpha(\tilde{x}), \quad (3.74)$$

holds for all \tilde{x} in \mathbb{R}^n ; and

2. There exist a symmetric positive definite matrix Q and positive constants \underline{p} , \bar{p} and $\rho > 0$ such that (3.70) holds and

$$L_f P(\tilde{x}) - \rho \frac{\partial U}{\partial \tilde{x}}(\tilde{x})^\top \frac{\partial U}{\partial \tilde{x}}(\tilde{x}) \leq -Q, \quad (3.75)$$

hold for all \tilde{x} in \mathbb{R}^n .

Then, given a connected graph \mathcal{G} with associated Laplacian matrix $L = (L_{ij})$, there exists a constant $\underline{\ell}$ such that the control law $u = \phi(x)$ with $\phi = [\phi_1^\top \ \dots \ \phi_N^\top]^\top$ given by

$$\phi_i(x) = -\ell \alpha(x_i) \sum_{j=1}^N L_{ij} U(x_j) \quad (3.76)$$

with $\ell \geq \underline{\ell}$ solves the local uniform exponential synchronization of (3.66).

As in the context of the observer design given in [114], a global result can be obtained by imposing a further constraint on P . Specifically, the notion we need to introduce is the following.

Definition 12 (Totally Geodesically Set) Given a C^2 function P defined on \mathbb{R}^n the values of which are symmetric positive definite matrices, a C^1 function $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}_+$ and a real number $\bar{\varphi}$, the (level) set $S = \{x \in \mathbb{R}^n, \varphi(x) = \bar{\varphi}\}$ is said to be totally geodesic with respect to P if, for any (x, v) in $S \times \mathbb{R}^n$ such that

$$\frac{\partial \varphi}{\partial x}(x)v = 0, v'P(x)v = 1,$$

any geodesic γ , i.e. a solution of

$$\frac{d}{ds} \left(\frac{d\gamma^*}{ds}(s)' P(\gamma^*(s)) \right) = \frac{1}{2} \frac{\partial}{\partial x} \left(\frac{d\gamma^*}{ds}(s)' P(x) \frac{d\gamma^*}{ds}(s) \right) \Big|_{x=\gamma^*(s)}, \quad (3.77)$$

with $\gamma(0) = x$ and $\frac{d\gamma}{ds}(0) = v$ satisfies $\frac{\partial \varphi}{\partial x}(\gamma(s)) \frac{d\gamma}{ds}(s) = 0$ for all s .

For the case of two agents only, we have the following the result which establishes that global synchronization may be achieved in the case in which the communication graph is trivial (There is only two agents which can communicate with each other).

Theorem 18 (Global sufficient condition for $m = 2$ [V8]) *Assume*

1. there exist a C^3 function $U : \mathbb{R}^n \rightarrow \mathbb{R}$ which has bounded first and second derivatives, and a C^1 function $\alpha : \mathbb{R}^n \rightarrow \mathbb{R}^q$ such that, for all x in \mathbb{R}^n , (3.74) is satisfied;
2. there exist a positive real number λ , a C^3 function $P : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$ and positive real numbers \underline{p} and \bar{p} , such that inequalities (3.70) hold and we have, for all (x, v) in $\mathbb{R}^n \times \mathbb{R}^n$ such that $\frac{\partial U}{\partial x}(x)' v = 0$

$$v^\top L_f P(x) v \leq -\lambda v^\top P(x) v, \quad (3.78)$$

3. For all \bar{U} in \mathbb{R} , the set $S = \{x \in \mathbb{R}^n, U(x) = \bar{U}\}$ is totally geodesic with respect to P . Then there exists a function $\ell : \mathbb{R}^{2n} \rightarrow \mathbb{R}_+$, invariant by permutation such that, with the controls given by

$$\phi_i(x) = \ell(x) \alpha(x_i) (U(x_j) - U(x_i)),$$

with $(i, j) \in \{(1, 2), (2, 1)\}$ the following holds and for all x in \mathbb{R}^{2n} ,

$$|X(x, t)|_{\mathcal{D}} \leq k|x|_{\mathcal{D}} \exp\left(-\frac{\lambda}{2}t\right), \quad (3.79)$$

where t is any positive real number in the time domain of definition of the closed loop solution.

The proof of this result is based on the study of the evolution with time of minimal geodesics. It borrows some ideas of [114] which was devoted to observer designs. However, different from [114], we have here a global convergence result. This follows from the fact that in the high gain parameter ℓ , the norm of the full state space can be used (and not only the norm of the estimate as in the observer case).

Note that nothing is said about the domain of existence of the solution.

It is still an open question to know if global synchronization may be achieved in the general nonlinear context with more than two agents.. However in the particular case in which the matrix $P(z)$ and the vector field g are constant, then global synchronization may be achieved as this is shown in the following theorem.

Theorem 19 (Global sufficient condition for larger dimension [V10]) *Assume that $g(z) = G$ and there exists a symmetric positive definite matrix P in $\mathbb{R}^{n \times n}$, a symmetric positive definite matrix Q and $\rho > 0$ such that*

$$P \frac{\partial f}{\partial \tilde{x}}(\tilde{x}) + \frac{\partial f}{\partial \tilde{x}}(\tilde{x})^\top P - \rho P G G^\top P \leq -Q. \quad (3.80)$$

Assume moreover that the graph is connected with Laplacian matrix L . Then there exist constants $\underline{\ell}$ and positive real numbers c_1, \dots, c_N such that the control laws $u = \phi(x)$ with $\phi = [\phi_1^\top \ \dots \ \phi_N^\top]^\top$ given by

$$\phi_i(x) = -\ell c_i \sum_{j=1}^N L_{ij} G^\top P x_j \quad (3.81)$$

with $\ell \geq \underline{\ell}$, solves the global uniform exponential synchronization of (3.66).

3.4.4 Construction of an admissible tensor via backstepping

As proposed in Theorem 17, a distributed synchronizing control law can be designed using a symmetric covariant tensor field of order 2, which satisfies (3.73). Given a general nonlinear system, the construction of such a matrix function P may be a hard task. In [115], a construction of the function P for observer based on the integration of a Riccati equation is introduced. Similar approach could be used in our synchronization problem. Note however that in our context an integrability condition (i.e. equation (3.74)) has to be satisfied by the function P . This constraint may be difficult to address when considering a Riccati equation approach.

In the following we present a constructive design of such a matrix P that resembles the backstepping method. This approach can be related to [136, 135] in which a metric is also constructed iteratively. We note that one of the difficulty we have here is that we need to propagate the integrability property given in equation (3.74).

For outlining the backstepping steps for designing P , we consider the case in which the vector fields (f, g) can be decomposed as follows

$$f(z) = \begin{bmatrix} f_a(z_a) + g_a(z_a)z_b \\ f_b(z_a, z_b) \end{bmatrix},$$

and,

$$g(z) = \begin{bmatrix} 0 \\ g_b(z) \end{bmatrix}, \quad 0 < \underline{g}_b \leq g_b(z) \leq \bar{g}_b$$

with $z = [z_a^\top \ z_b]^\top$, z_a in \mathbb{R}^{n_a} and z_b in \mathbb{R} . In other words,

$$\dot{z}_a = f_a(z_a) + g_a(z_a)z_b, \quad \dot{z}_b = f_b(z) + g_b(z)u. \quad (3.82)$$

Let \mathcal{C}_a be a compact subset of \mathbb{R}^{n_a} . As in the standard backstepping approach, we make the following assumptions on the z_a -subsystem where z_b is treated as a control input to this subsystem.

Assumption 12 (z_a -Synchronizability) Assume that there exists a C^∞ function $P_a : \mathbb{R}^{n_a} \rightarrow \mathbb{R}^{n_a \times n_a}$ that satisfies the following conditions.

1. There exist a C^∞ function $U_a : \mathbb{R}^{n_a} \rightarrow \mathbb{R}$ and a C^∞ function $\alpha_a : \mathbb{R}^{n_a} \rightarrow \mathbb{R}$ such that

$$\frac{\partial U_a}{\partial z_a}(z_a)^\top = \alpha_a(z_a) P_a(z_a) g_a(z_a) \quad (3.83)$$

holds for all z_a in \mathcal{C}_a ;

2. There exist a symmetric positive definite matrix Q_a and positive constants $\underline{p}_a, \bar{p}_a$ and $\rho_a > 0$ such that

$$\underline{p}_a \text{Id}_{n_a} \leq P_a(z_a) \leq \bar{p}_a \text{Id}_{n_a}, \quad \forall z_a \in \mathbb{R}^{n_a}, \quad (3.84)$$

holds and

$$L_{f_a} P_a(z_a) - \rho_a \frac{\partial U_a}{\partial z_a}(z_a)^\top \frac{\partial U_a}{\partial z_a}(z_a) \leq -Q_a, \quad (3.85)$$

holds for all z_a in \mathcal{C}_a .

As a comparison to the standard backstepping method for stabilizing nonlinear systems in the strict-feedback form, the z_a -synchronizability conditions above are akin to the stabilizability condition of the upper subsystem via a control Lyapunov function. However, for the synchronizer design as in the present context, we need an additional assumption to allow the recursive backstepping computation of the tensor P . Roughly speaking, we need the existence of a mapping q_a such that the metric P_a becomes invariant along the vector field $\frac{g_a}{q_a}$. In other words, $\frac{g_a}{q_a}$ is a Killing vector field.

Assumption 13 *There exists a non-vanishing smooth function $q_a : \mathbb{R}^{n_a} \rightarrow \mathbb{R}$ such that the metric obtained from P_a on \mathcal{C}_a is invariant along $\frac{g_a(z_a)}{q_a(z_a)}$. In other words, for all z_a in \mathcal{C}_a*

$$L_{\frac{g_a(z_a)}{q_a(z_a)}} P_a(z_a) = 0. \quad (3.86)$$

Similar assumption can be found in [31] in the characterization of differential passivity.

Based on the Assumptions 12 and 13, we have the following theorem on the backstepping method for constructing a symmetric covariant tensor field P_b of the complete system (3.82).

Theorem 20 *Assume that the z_a -subsystem satisfies Assumption 12 and Assumption 13 in the compact set \mathcal{C}_a with a $n_a \times n_a$ symmetric covariant tensor field P_a of order two and a non-vanishing smooth mapping $q_a : \mathbb{R}^{n_a} \rightarrow \mathbb{R}$. Then for all positive real number M_b , the system (3.82) with the state variables $z = (z_a, z_b) \in \mathbb{R}^{n_a+1}$ satisfies the Assumption 12 in the compact set $\mathcal{C}_a \times [-M_b, M_b] \subset \mathbb{R}^{n_a+1}$ with the symmetric covariant tensor field P_b be given by*

$$P_b(z) = \begin{bmatrix} P_a(z_a) + S_a(z) S_a(z)^\top & S_a(z) q_a(z_a) \\ S_a(z)^\top q_a(z_a) & q_a(z_a)^2 \end{bmatrix}$$

where

$$S_a(z) = \frac{\partial q_a}{\partial z_a}(z_a)^\top z_b + \eta \alpha_a(z_a) P_a(z_a) g_a(z_a)$$

and η is a positive real number. Moreover, there exists a non-vanishing mapping $q_b : \mathbb{R}^{n_a+1} \rightarrow \mathbb{R}$ such that P_b is invariant along $\frac{g}{q_b}$. In other words, Assumptions 12 and 13 hold for the complete system (3.82).

3.4.5 Conclusion on synchronization

These results related to the synchronization problem have been published in [V8] and [V9, V10]. Note however that some questions remain to be addressed.

- Assuming that all agents communicate with each other,

- Assuming that the couple (f, g) satisfy properties (IS) and (CMF) with a matrix function P which is not constant,
- Assuming that there exists U which satisfies the integrability condition,
- Assuming that the level set of U are totally geodesic with respect to the metric obtained from P ,

is it possible to design a synchronizing control law for $m \geq 3$ agents? This open question seems to be very difficult.

Recently, with Sophie Tarbouriech we have started to work on constructive procedure to design a (constant) matrix P such that the system satisfies the CMF properties. In the spirit of the work of Arcaj and coauthors (see [7, 5]), it is possible to give some sufficient conditions allowing the design of Synchronizing control law for nonlinear systems composed of monotonic or Lipschitz nonlinearities.

Chapter 4

Conclusion and Perspectives

In this section, I list some works I have done that I haven't presented in this thesis.

- **Rewriting an observer in the original coordinates.** I worked on this topic with Pauline Bernard and Laurent Praly. The problem is to find some means to rewrite a given observer directly in the x coordinates. The problem may be summarized as follows. Consider a system and an observer

$$\dot{x} = f(x, u) , y = h(x) , \dot{\hat{\xi}} = \varphi(\hat{\xi}, y, u) , \tau^*(\hat{x}) = \hat{\xi} .$$

Note that in the former observer the estimated value denoted \hat{x} is defined implicitly. The problem is to find two functions k and g such that the system

$$\dot{\hat{x}} = f(x, u) + k(y, u, \hat{x}, w) , \dot{w} = g(y, u, \hat{x}, w) ,$$

defines also an observer.

The practical interest we have in this new observer is that a part of the observer state is directly the data that needs to be estimated. Hence, we don't need to compute the mapping τ which is most of the time implicitly given. We have given in [V33] a set of sufficient conditions allowing to solve this problem.

- **Stabilization employing hybrid state feedback.** The use of hybrid control laws have received a very important interest by the control community in the last decades. I have also worked on this topic. For instance, with Humberto Stein Shiromoto, we have worked on some variations on the backstepping control law approaches in [V46] (see also [V47]). With Swan Marx and Christophe Prieur, we have also worked on some separation principles to design a stabilizing output feedback from a hybrid stabilizing control law. With Sophie Tarbouriech in [V28], we have introduced some sufficient conditions allowing the design of a stabilizing hybrid output feedback law for a class of bilinear systems.
- **Stabilization of some infinite dimensional systems with control constraints.** In the last three years, I have started to work on control problem for infinite dimension systems. The first one concerns the stabilization of linear and nonlinear PDEs with saturated (or cone bounded) nonlinearities. With Swan Marx and Christophe Prieur, we have given some sufficient conditions to obtain stabilization of the equilibrium for abstract Cauchy problems in [V38]. We have also considered the nonlinear Korteweg de Vries equation in [V39].

- **Regulation of some hyperbolic PDE systems employing Lyapunov techniques.** With Tu Ngoc Trinh and Cheng-Zhong Xu we have considered the problem of output regulation for nonlinear hyperbolic PDEs coming from conservation laws. We have been able to design PI controller based on Lyapunov functions for a general class of nonlinear hyperbolic PDEs. This has been published in [V48] and in [V49].
- With Bayu Jayawardhana we have worked on the **dissipativity analysis of some hysteresis phenomena** based on the Duhem model (see [V37] and [V41]).

Several aspects of this work deserves to be further investigated. In this chapter, I list some subjects I will consider in the following years.

- **Analysis of nonlinear systems**

- **Global transverse exponential stability.** As we have seen in the first chapter of this thesis, it is possible to characterize the local exponential stability of a specific manifold by introducing some specific tensor. Moreover, it is possible to characterize the global exponential stability of an equilibrium from a first order analysis. A natural question is to know if it is possible to characterize the global transverse exponential stability property from a first order analysis. Actually, together with Laurent Praly and Bayu Jayawardhana we are currently working on this topic. It turns out that it is also possible.
- **Transverse homogeneous approximation.** Another topic of interest would be to consider homogeneous transverse approximation. For instance, a typical case of study would be the following nonlinear dynamics.

$$\dot{z} = f(z, x) , \dot{x} = g(z, x).$$

The question I intend to study is the case in which the z dynamics is homogeneous in the bi-limit in z uniformly in x . Can we give some sufficient condition on the transverse stability of the manifold $z = 0$ based on these (transverse) homogeneous approximations ?

- **Observer design for nonlinear systems**

- **Robustness and observer designs.** As we have seen in the second Chapter of this thesis, there exists now a large variety of techniques to design an asymptotic observer. Note however that the qualitative aspect of the algorithm is difficult to address. Would there be a way to select the parameters of the observer in order to achieve some prescribed performances ?
- **Embedding and incremental approach.** As we have seen, a necessary condition to design an observer is the R -Detectability property. The R -Detectability property is not a sufficient condition to be able to construct a nonlocal state observer. As shown in [114], if the metric which shows up in the R -Detectability property is constant (the matrix function P is the same for all x) and if the output function is linear, it turns out that this condition becomes sufficient. A design approach could be to construct an isometric embedding which sends the system into a metric space which satisfies the former properties. This approach is very appealing and we have started to discuss on these aspects with Laurent Praly.

- Finding means to design observers for PDEs is also a very interesting topic. For instance, the use of the Luenberger approach is very interesting in the context of PDEs. With Christophe Prieur and Swann Marx, we have started to discuss on this topic.

- **Feedback designs**

- **Global synchronization for more than 2 agents.** As we have mentioned, finding general sufficient conditions to design a global synchronizing control law is still an open problem when there are more than two agents (without assuming a constant matrix P).
- **Constructive method for synchronization based on LMI's.** In the context of observer designs, there have been many publications considering monotonic nonlinearities and in which some linear matrix sufficient conditions were given. Due to the fact that synchronization problem follows the same type of routes (see for instance [89]), it might be very interesting to consider constructive method to obtain synchronization for nonlinear systems subject to monotonic nonlinearities. We have started to work on this topic with Sophie Tarbouriech.
- **Output regulation for some hyperbolic nonlinear systems.** Following the work we have done together with Tu Ngoc Trinh, we are considering the case of a general boundary conditions and the cancelation of more general disturbances. One very promising approach is the use of forwarding tools to construct Lyapunov functions for some hyperbolic PDEs to achieve regulation.

Appendices

Proof of Proposition 14

Proposition 14 Assume that Assumption 1 is satisfied. Assume moreover that the functions φ , τ^* , and f have bounded first and second derivative. Assume moreover that τ is C^1 and has bounded first derivative. Then system (2.1) is infinitesimally detectable and also R-detectable.

Proof : Let $z = \hat{\xi} - \tau^*(x)$. Note that it satisfies along solutions of the system (2.1) coupled with the observer :

$$\dot{z} = \varphi(z + \tau^*(x), h(x)) - \frac{\partial \tau^*}{\partial x}(x) f(x).$$

Moreover from equation (2.8), it yields that the zero error set $\{(x, z), z = 0\}$ is (at least on a small amount of time) invariant. Hence, for all x in \mathcal{A} , we have

$$L_f \tau^*(x) = \frac{\partial \tau^*}{\partial x}(x) f(x) = \varphi(\tau^*(x), h(x)). \quad (1)$$

Also, equation (2.8) and Proposition 1 imply that the set $(0, x)$ is (locally) exponentially stable along the solution of the system

$$\dot{\tilde{z}} = \frac{\partial \varphi}{\partial \tilde{z}}(\tau^*(x), h(x)) \tilde{z}, \quad \dot{x} = f(x),$$

uniformly in x . With Proposition 2, it implies that there exists a function $P : \mathbb{R}^n \rightarrow \mathbb{R}^{m \times m}$ such that

$$\mathfrak{d}_f P(x) + \frac{\partial \varphi}{\partial \tilde{z}}(\tau^*(x), h(x))^\top P(x) + P(x) \frac{\partial \varphi}{\partial \tilde{z}}(\tau^*(x), h(x)) \leq -\underline{q} P(x). \quad (2)$$

Let P_0 be defined as

$$P_0(x) = \frac{\partial \tau^*}{\partial x}(x)^\top P(x) \frac{\partial \tau^*}{\partial x}(x) + \rho \frac{\partial h}{\partial x}(x)' \frac{\partial \tau}{\partial y}(\tau^*(x), h(x))' \frac{\partial \tau}{\partial y}(\tau^*(x), h(x)) \frac{\partial h}{\partial x}(x).$$

where ρ is a positive real number to be tuned sufficiently small such that (2.11) holds. Indeed, note that the upper bound is trivially obtained from the bound on the derivative of h and τ and the upper bound on P . To establish the left hand side, note that by differentiating equation (2.7), it yields

$$\frac{\partial \tau}{\partial \hat{\xi}}(\tau^*(x), h(x)) \frac{\partial \tau^*}{\partial x}(x) + \frac{\partial \tau}{\partial y}(\tau^*(x), h(x)) \frac{\partial h}{\partial x} = \text{Id}.$$

Consequently

$$P_0(x) = \frac{\partial \tau^*}{\partial x}(x)^\top P(x) \frac{\partial \tau^*}{\partial x}(x) + \rho (\text{Id} - R(x))^\top (\text{Id} - R(x)),$$

with

$$R(x) = \frac{\partial \tau}{\partial \hat{\xi}}(\tau^*(x), h(x)) \frac{\partial \tau^*}{\partial x}(x).$$

Note that we have for all x

$$(\text{Id} - R(x))^\top (\text{Id} - R(x)) = \frac{1}{2} \text{Id} + \frac{1}{2} (\text{Id} - 2R(x))^\top (\text{Id} - 2R(x)) - R(x)^\top R(x).$$

Hence, we get,

$$\begin{aligned} P_0(x) &= \frac{\rho}{2} \text{Id} + \frac{\partial \tau^*}{\partial x}(x)^\top \left[P(x) - \rho \frac{\partial \tau}{\partial \hat{\xi}}(\tau^*(x), h(x))^\top \frac{\partial \tau}{\partial \hat{\xi}}(\tau^*(x), h(x)) \right] \frac{\partial \tau^*}{\partial x}(x) \\ &\quad + \rho \frac{1}{2} (\text{Id} - 2R(x))^\top (\text{Id} - 2R(x)). \end{aligned}$$

Note that by picking ρ sufficiently small en employing the lower bound on P , it yields that

$$P_0(x) \geq \frac{\rho}{2} \text{Id}.$$

Consequently (2.11) holds. Note that for all \tilde{x} in \mathbb{R}^n and x in \mathcal{A} , we have

$$\begin{aligned} \tilde{x}^\top L_f P_0(x) \tilde{x} &= \tilde{x}^\top \frac{\partial \tau^*}{\partial x}(x)^\top \mathfrak{d}_f P(x) \frac{\partial \tau^*}{\partial x}(x) \tilde{x} \\ &\quad + 2\tilde{x}^\top \frac{\partial \tau^*}{\partial x}(x)^\top P(x) \mathfrak{d}_f \frac{\partial \tau^*}{\partial x}(x) \tilde{x} + 2\tilde{x}^\top P_0(x) \frac{\partial f}{\partial x}(x) \tilde{x} \\ &\quad + \rho \tilde{x}^\top L_f \left\{ \frac{\partial h}{\partial x}(x)^\top \frac{\partial \tau}{\partial y}(\tau^*(x), h(x))^\top \frac{\partial \tau}{\partial y}(\tau^*(x), h(x)) \frac{\partial h}{\partial x}(x) \right\} \tilde{x}. \quad (3) \end{aligned}$$

Note that for all \tilde{x} such that $\frac{\partial h}{\partial x}(x) \tilde{x} = 0$, we have

$$\tilde{x}^\top L_f \left\{ \frac{\partial h}{\partial x}(x)^\top \frac{\partial \tau}{\partial y}(\tau^*(x), h(x))^\top \frac{\partial \tau}{\partial y}(\tau^*(x), h(x)) \frac{\partial h}{\partial x}(x) \right\} \tilde{x} = 0.$$

and,

$$\tilde{x}^\top P_0(x) = \tilde{x}^\top \frac{\partial \tau^*}{\partial x}(x)^\top P(x) \frac{\partial \tau^*}{\partial x}(x).$$

This gives with (2)

$$\begin{aligned} \tilde{x}^\top L_f P_0(x) \tilde{x} &\leq \\ &= \tilde{x}^\top \frac{\partial \tau^*}{\partial x}(x)^\top \left\{ \frac{\partial \varphi}{\partial \tilde{z}}(\tau^*(x), h(x))^\top P(x) + P(x) \frac{\partial \varphi}{\partial \tilde{z}}(\tau^*(x), h(x)) + \underline{q} P(x) \right\} \frac{\partial \tau^*}{\partial x}(x) \tilde{x} \\ &\quad + 2\tilde{x}^\top \frac{\partial \tau^*}{\partial x}(x)^\top P(x) \mathfrak{d}_f \frac{\partial \tau^*}{\partial x}(x) \tilde{x} + 2\tilde{x}^\top \frac{\partial \tau^*}{\partial x}(x)^\top P(x) \frac{\partial \tau^*}{\partial x}(x) \frac{\partial f}{\partial x}(x) \tilde{x}. \quad (4) \end{aligned}$$

Also, with (1), it yields for all \tilde{x} such that $\frac{\partial h}{\partial x}(x) \tilde{x} = 0$,

$$\frac{\partial \varphi}{\partial \tilde{z}}(\tau^*(x), h(x)) \frac{\partial \tau^*}{\partial x}(x) \tilde{x} = \frac{\partial L_f \tau^*}{\partial x}(x) \tilde{x},$$

On the other hand, we have for all x

$$\frac{\partial L_f \tau^*}{\partial x}(x) = \mathfrak{d}_f \frac{\partial \tau^*}{\partial x}(x) + \frac{\partial \tau^*}{\partial x}(x) \frac{\partial f}{\partial x}(x).$$

Consequently for all \tilde{x} such that $\frac{\partial h}{\partial x}(x) \tilde{x} = 0$, we have

$$\tilde{x}' L_f P_0(x) \tilde{x} \leq -\underline{q} \tilde{x}' \frac{\partial \tau^*}{\partial x}(x)' P(x) \frac{\partial \tau^*}{\partial x}(x) \tilde{x}$$

It implies that (2.12) holds. \square

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