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Marwa Koumaiha

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**Thèse de doctorat de l'Université Paris-Est  
Cotutelle avec l'Université Libanaise**

*Pour l'obtention du grade de*  
**DOCTEUR EN MATHÉMATIQUES**

École Doctorale Mathématiques et Sciences et Technologies de l'Informatique et de  
la Communication et École Doctorale des Sciences et de Technologie

Présentée par

**Marwa KOUMAIHA**

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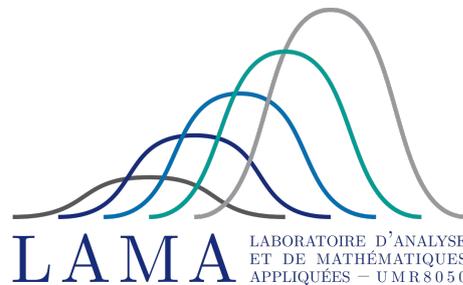
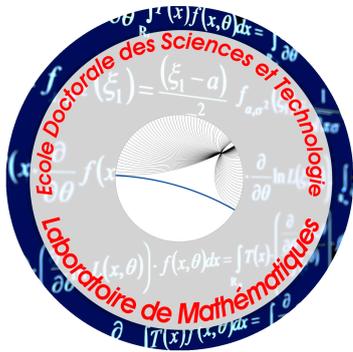
**Analyse numérique pour les équations de Hamilton-Jacobi  
sur réseaux et contrôlabilité / stabilité indirecte d'un système  
d'équations des ondes 1D**

---

Thèse dirigée par **Cyril IMBERT** et **Ali WEHBE**

Soutenue publiquement le 19 Juillet 2017 devant le Jury composé de :

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Thèse effectuée au sein du **Laboratoire d'Analyse et de Mathématiques Appliquées, UMR 8050** de l'Université Paris-Est et du **Laboratoire de Mathématiques de l'Ecole Doctorale des Sciences et de Technologie** de l'Université Libanaise.



**Université Libanaise**



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*Aux étoiles qui éclairent ma vie, mes parents, Mohamad et Mariam*

إلى أبي بطلي الأوحـد واستقامة ظهري؛  
إلى أمي التي أنارت بدعواتها قنديل دربي؛  
إليكما يا نبض قلبي..

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# Résumé

Cette thèse est composée de deux parties dans lesquelles nous étudions d'une part des estimations d'erreurs pour des schémas numériques associés à des équations de Hamilton-Jacobi du premier ordre. D'autre part, nous nous intéressons à l'étude de la stabilité et de la contrôlabilité exacte frontière indirecte des équations d'onde couplées.

Dans un premier temps, en utilisant la technique de Crandall-Lions, nous établissons une estimation d'erreur d'un schéma numérique aux différences finies pour des conditions de jonction dites à flux limité maximal, pour une équation de Hamilton-Jacobi du premier ordre. Ensuite, nous montrons que ce schéma numérique peut être généralisé à des conditions de jonction générales. Nous établissons alors la convergence de la solution discrétisée vers la solution de viscosité du problème continu. Enfin, nous proposons une nouvelle approche, à la Crandall-Lions, pour améliorer les estimations d'erreur déjà obtenues, pour une classe des Hamiltoniens bien choisis. Cette approche repose sur l'interprétation du type contrôle optimal de l'équation de Hamilton-Jacobi considérée.

Dans un second temps, nous étudions la stabilisation et la contrôlabilité exacte frontière indirecte d'un système mono-dimensionnel d'équations d'ondes couplées. D'abord, nous considérons le cas d'un couplage avec termes de vitesses, et par une méthode spectrale, nous montrons que le système est exactement contrôlable moyennant un seul contrôle à la frontière. Les résultats obtenus dépendent de la nature arithmétique du quotient des vitesses de propagation et de la nature algébrique du terme de couplage. Ensuite, nous considérons le cas d'un couplage d'ordre zéro et nous établissons un taux polynômial optimal de la décroissance de l'énergie. Enfin, nous montrons que le système est exactement contrôlable moyennant un seul contrôle à la frontière.

## Mots-clé:

Equations de Hamilton-Jacobi, estimation d'erreur, schéma numérique monotone, jonction, équations d'ondes couplées, contrôlabilité exacte, stabilité, approche spectrale.

**Numerical analysis for  
Hamilton-Jacobi equations on  
networks and indirect  
controllability/stability of a 1D  
system of wave equations**

# Abstract

The aim of this work is mainly to study on the one hand a numerical approximation of a first order Hamilton-Jacobi equation posed on a junction. On the other hand, we are concerned with the stability and the exact indirect boundary controllability of coupled wave equations in a one-dimensional setting.

Firstly, using the Crandall-Lions technique, we establish an error estimate of a finite difference scheme for maximal flux-limited junction conditions, associated to a first order Hamilton-Jacobi equation. We prove afterwards that the scheme can generally be extended to general junction conditions. We prove then the convergence of the numerical solution towards the viscosity solution of the continuous problem. We adopt afterwards a new approach, using the Crandall-Lions technique, in order to improve the error estimates for the finite difference scheme already introduced, for a class of well chosen Hamiltonians. This approach relies on the optimal control interpretation of the Hamilton-Jacobi equation under consideration.

Secondly, we study the stabilization and the indirect exact boundary controllability of a system of weakly coupled wave equations in a one-dimensional setting. First, we consider the case of coupling by terms of velocities, and by a spectral method, we show that the system is exactly controllable through one single boundary control. The results depend on the arithmetic property of the ratio of the propagating speeds and on the algebraic property of the coupling parameter. Furthermore, we consider the case of zero coupling parameter and we establish an optimal polynomial energy decay rate. Finally, we prove that the system is exactly controllable through one single boundary control.

**Keywords:**

Hamilton-Jacobi equations, error estimate, monotone numerical scheme, junction, coupled wave equations, exact controllability, stability, spectral approach.

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# Introduction

This thesis focuses on two main subjects. On the one hand, we are interested in numerical approximation of first order Hamilton-Jacobi equations posed on a domain containing one single singularity: a junction. The aim of our mathematical analysis of such equations is to show the convergence properties of the numerical solution to the exact solution of the problem, and to derive error estimates in the case of numerical schemes. On the other hand, we implement the semi-group theory in the spirit of spectral approach to study the controllability and stabilization of coupled wave equations. Roughly speaking, the concept of controllability is described as the ability to steer our evolution system, whether described in terms of partial or ordinary differential equations, from any initial state to any desired final state in a finite time interval by means of a suitable control (boundary control, internal control, etc). As for stabilization, it is defined as the ability to find an input control that requires the state response to approach zero as time tends to infinity.

## 0.1 Numerical analysis for Hamilton-Jacobi equations on networks

We begin by introducing the first part related to the study of a finite difference scheme associated to a first order Hamilton- Jacobi equation posed on a junction of roads.

### 0.1.1 Motivation

In the 1950s James Lighthill and Gerald Whitham, two experts in fluid dynamics, (and independently P. Richards) thought that the equations describing the flow of a fluid could also describe the flow of car traffic [78]. The basic idea is to look at large scales so to consider cars as small particles and their density as the main quantity to be considered. In any case, it is reasonable to assume the conservation of the number of cars, thus leading again to a conservation law, described by the equation:

$$\partial_t \rho + \partial_x f(\rho) = 0 \tag{0.1.1}$$

where  $\rho = \rho(t, x)$  is the density of cars,  $f(\rho)$  is the flux which can be written  $f(\rho) = \rho v$ , where  $v$  is the average velocity of the cars. The main assumption of

the LWR model is that the average velocity of  $v$  depends only on the density of the cars. A reasonable property of  $v$  is that  $v$  is a decreasing function of the density. The law giving the flux as a function of the density is called fundamental diagram, which is assumed to be a concave function. In [51], Garavello and Piccoli proved the existence of entropy admissible solutions of (0.1.1) posed on a network, while submitted to an initial condition. Using the wave front tracking method, it is proved in [52], by providing a formulation in terms of Riemann solvers, uniqueness result for  $N = 2$  where  $N$  denotes the number of the branches of the junction.

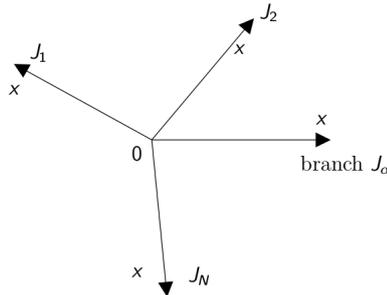


Figure 1: The junction

### 0.1.2 Physical interpretation of Hamilton-Jacobi equations

In this subsection, we present a simplified overview on Hamilton-Jacobi equations. Readers wishing to deepen some concepts mentioned in the following are referred to the book of Evans [47] on partial differential equations, to Bardi and Capuzzo-Dolcetta [19] and to Barles [24] presenting the notion of viscosity solutions of Hamilton-Jacobi equations. We recall that this notion of solutions was introduced in 1980's by Crandall and Lions [42] (see also Crandall et al. [43]).

Hamilton-Jacobi equations form a class of nonlinear partial differential equations. They can be written under the following form:

$$u_t + H(D_x u) = 0 \quad [0, +\infty) \times \mathbb{R}^n, \quad (0.1.2)$$

where  $u : [0, +\infty) \times \mathbb{R}^n \rightarrow \mathbb{R}$  is the unknown,  $D_x u = (u_{x_1}, \dots, u_{x_n}) \in \mathbb{R}^n$  its gradient and  $H : \mathbb{R}^n \rightarrow \mathbb{R}$  denotes the Hamiltonian.

Let us consider the following equation with its corresponding Dirichlet boundary conditions, in one dimension, on  $I = ]0, 1[$ :

$$\begin{cases} |u'(x)| = 1, & \forall x \in I, \\ u(0) = u(1) = 0. \end{cases} \quad (0.1.3)$$

If one looks for  $C^1$  solutions, we directly encounter many problems. We can see that if there exists a smooth solution  $u$ , there will exist some  $x_0 \in ]0, 1[$  such that  $u'(x_0) = 0$ . Hence, (0.1.3) is no longer satisfied in a neighborhood of  $x_0$ . The highly

non-linear character of the equation, precisely the absolute value on the derivative, prevents from using the theory of distributions. If we look for Lipschitz solutions on  $I$ , that is  $u \in W^{1,\infty}(I)$ , we can define solutions of (0.1.3) in the sense almost everywhere by Rademacher's Theorem. However, as shown in the picture below, all solutions verifying the boundary conditions with slopes varying between the two values  $+1$  and  $-1$  are solutions. We hence loose the uniqueness of the solution.

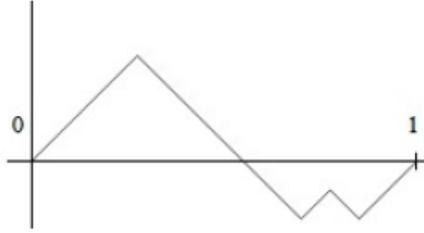


Figure 2: Example of Lipschitz solution

Using the *vanishing viscosity* method, that is, adding a term of the form  $-\epsilon u''(x)$  in the left hand side of (0.1.3), without changing the boundary conditions, we can obtain a unique smooth solution,  $u_\epsilon$  say, for each  $\epsilon > 0$ . A natural question is: does  $u_\epsilon$  converge as  $\epsilon \rightarrow 0$ ? If yes, in which sense, and is the possible limit a solution of (0.1.3) in a sense that could ensure uniqueness? A notion of weak solutions is then introduced by Crandall, Lions in 1980's [42]: the viscosity solution theory will allow us to pass to the limit in a precise sense, for a large class of equations. It will then provide a correct framework to obtain existence and uniqueness for solutions and define the derivatives for non-smooth solutions.

The relationship between Hamiltonian and Lagrangian formulations, which are two different views of the same problem is provided by the Legendre transformation.

**Definition 0.1.1** (Legendre-Fenchel Transform). If the Lagrangian  $L$  does not depend on the space variable  $x$ , and if  $q \mapsto L(q)$  is convex, and if the Lagrangian is coercive i.e.

$$\lim_{|q| \rightarrow +\infty} \frac{L(q)}{|q|} \rightarrow +\infty,$$

the Legendre-Fenchel transform of the Lagrangian  $L$  is given by

$$L^*(p) = \sup_{q \in \mathbb{R}^n} \{p \cdot q - L(q)\}, \quad \text{for } p \in \mathbb{R}^n,$$

where the dot denotes the scalar product in  $\mathbb{R}^n$ .

### 0.1.3 Viscosity solutions, optimal control and representation formula

It is possible to get representation formulas for solutions for Hamilton-Jacobi equations when the Hamiltonian of the equation can be written as the Hamiltonian of an optimal control problem. In this case where the Hamiltonian  $H(p)$  is convex in  $p$ , the *value function* of the control problem is a (the unique) solution of the Hamilton-Jacobi equation. We have already defined the Legendre-Fenchel transform for a continuous, convex, coercive function  $L$  by

$$L^*(p) = \sup_{q \in \mathbb{R}^n} \{p \cdot q - L(q)\}, \quad \text{for } p \in \mathbb{R}^n.$$

We then recover  $H^* = L$  and  $L^* = H$ .

We can then rewrite the Hamilton-Jacobi equation (0.1.2) as:

$$u_t + \sup_{b \in \mathbb{R}^n} \{-u_x \cdot b - L(b)\} = 0.$$

This is the version of the Bellman finite horizon control problem. The dynamic is given by the following ordinary differential equation

$$\frac{\partial X}{\partial s} = b(s),$$

and the instantaneous cost by the Lagrangian  $L$ . The solution of the following Cauchy problem

$$\begin{cases} u_t + H(D_x u) = 0, & \text{on } (0, +\infty) \times \mathbb{R}^n, \\ u(0, x) = u_0(x) & \text{on } \mathbb{R}^n, \end{cases}$$

is given by

$$u(t, x) = \inf_{X(\cdot)} \left[ \int_0^t L(\dot{X}(s)) ds + u_0(X(0)) \right],$$

where  $X(\cdot)$  denotes the trajectory such that

$$\left\{ X \in W^{1,1}([s, t], \mathbb{R}^2) : \begin{array}{l} X(\tau) \in J \quad \text{for all } \tau \in (s, t), \\ X(s) = y \quad \text{and } X(t) = x \end{array} \right\}.$$

It is possible to prove that (see [24]) the constant control  $v = \frac{x - X(0)}{t}$  is optimal. And hence, the optimal trajectory is the line segment which links the starting point  $(0, X(0))$  at the end point  $(t, x)$ . We then deduce the Lax-Oleinik formula

$$u(t, x) = \inf_{y \in \mathbb{R}^n} \left[ t H^* \left( \frac{x - y}{t} \right) + u_0(y) \right].$$

More generally, the Hamilton-Jacobi equation with a Hamiltonian depending on the spatial variable  $x$  but always satisfying the assumptions of continuity (that is precisely this assumption that is not satisfied in the case of a traffic junction), convexity and coercivity in  $p$  uniformly with respect to  $x$ , is always associated with a control problem where the dynamics are  $b$  and the running cost is  $L(x, b)$ .

### 0.1.4 Results

In this subsection, we present the problems that have been studied in this thesis, mainly in Chapters 1 and 2, and the results that were obtained.

The study of dynamical systems on networks (and their control) has attracted the attention of mathematicians especially in the last decade, linked to a variety of applications including road traffic, but also for the study of fluids on other types of networks such as gas networks, telecommunications networks, blood vessels or the networks in economy. The reader could refer to the book of Garavello and Piccoli [51], and to Bressan et al. in [30] for different applications for hyperbolic systems on networks.

**Literature.** There are many papers in the literature dealing with Hamilton Jacobi equations with discontinuous Hamiltonians in space posed on networks. Overall, the difficulty lies in the choice of a concept of viscosity solutions at the discontinuity point of the Hamiltonians. The problem at the node lies not only in the definition of test functions to the right and the left of the junction point, knowing the differential irregularity at this point, but in finding the right junction conditions. Apart from this, the authors use the classical notion of viscosity solutions à la Crandall, Evans and Lions. Different approaches to address discontinuity have been conducted so far. The proposed solution concepts are generally associated with a particular motivation, which make them, a priori, different from each to others. The first results for Hamilton-Jacobi equations on networks were obtained in [102] for eikonal equations. Some years later, the results were extended in [2, 67, 103]. Many new results were obtained since then, see for instance [63, 65] and references therein. Note for example that the article of Achdou, Camilli, Cutri and Tchou [2], is interested in a control problem of a network and in the Bellman equation that follows. In order to handle the discontinuity of the Hamiltonian, the authors use the modified distance that gives them an admissible test function. It has been shown by Camilli and Marchi [32] that this notion of viscosity solution is equivalent to the definition introduced by Imbert, Monneau and Zidani [67] when frameworks coincide. It is also possible to show that in the case of a network in one dimension space, the solutions constructed in [53] to these Hamilton-Jacobi equations with discontinuous Hamiltonians fall into the family of built solutions in the paper of Imbert and Monneau [65]. The first results of Achdou et al. [2] and Imbert et al. [67] were completed by a newer and more general work [3]. In [22, 23], the authors study regional control, i.e. control with dynamics and costs which are regular on either side of a hyperplane but with no compatibility or continuity assumption along the hyperplane. They identify the maximal and minimal Ishii solutions as value functions of two different optimal control problems. They also use the vanishing viscosity limit on a 1D example in order to prove that the two Ishii solutions can be different. The recent papers of Imbert and Monneau [63, 65] and Barles et al. [26] notably contain general results of uniqueness and comparison (relaxed assumptions on the Hamiltonians for example, quasi-convexity condition), with purely PDE proofs, in contrast to proofs of

optimal control type developed in [2, 3, 22, 23]. In [26], Barles et al. compared two different approaches for regional control problems. In fact, they showed how the results of the classical approach, using a standard notion of viscosity solutions, can be interpreted in terms of flux-limited solutions. In particular, they gave a simpler proof of the comparison principle, avoiding in particular the use of the vertex test function  $G$ . In his lectures at Collège de France [80], Lions also treats problems related to Hamilton- Jacobi equations with discontinuities. Very recently, Guerand in [58] on the one hand, and Lions and Souganidis in [81, 82] on the other hand studied well-posedness and stability properties for Hamilton-Jacobi equations posed on junctions with coercive Hamiltonians that are possibly not convex.

We are interested in the following Hamilton-Jacobi model proposed in [65], which is written as follows,

$$\begin{cases} u_t + H_\alpha(u_x) = 0 & \text{in } (0, T) \times J_\alpha \setminus \{0\}, \\ u_t + F_A\left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_N}\right) = 0 & \text{in } (0, T) \times \{0\} \end{cases} \quad (0.1.4)$$

subject to the initial condition

$$u(0, x) = u_0(x) \quad \text{for } x \in J, \quad (0.1.5)$$

where

$$F_A(p) = \max\left(A, \max_{\alpha=1, \dots, N} H_\alpha^-(p_\alpha)\right) \quad \text{for } p = (p_1, \dots, p_N), \quad (0.1.6)$$

is the  $A$ -flux limited junction function.  $H_\alpha$  is the Hamiltonian defined on the branch  $\alpha, \alpha = 1, \dots, N$ , and  $H_\alpha^-$  is the decreasing part of the Hamiltonian for some minimizing  $H_\alpha$  in  $\mathbb{R}$ . For  $\alpha = 1, \dots, N$ , each branch  $J_\alpha$  is assumed to be isometric to  $[0, +\infty)$  and we define the junction as

$$J = \bigcup_{\alpha=1, \dots, N} J_\alpha \quad \text{with} \quad J_\alpha \cap J_\beta = \{0\} \quad \text{for } \alpha \neq \beta.$$

We point out that all the junction functions  $F_A$  associated with  $A \in (-\infty; A_0]$  coincide if one chooses

$$A_0 = \max_{\alpha=1, \dots, N} \min_{\mathbb{R}} H_\alpha. \quad (0.1.7)$$

We consider the important case of quasi-convex Hamiltonians satisfying the following conditions:

$$\text{There exists } p_0^\alpha \in \mathbb{R} \text{ such that } \begin{cases} H_\alpha \in C^2(\mathbb{R}) \text{ and } H_\alpha''(p_0^\alpha) > 0 \\ \pm H_\alpha'(p) > 0 \text{ for } \pm(p - p_0^\alpha) > 0 \\ \lim_{|p| \rightarrow +\infty} H_\alpha(p) = +\infty. \end{cases} \quad (0.1.8)$$

In particular,  $H_\alpha$  is non-increasing in  $(-\infty, p_0^\alpha]$  and non-decreasing in  $[p_0^\alpha, +\infty)$ , and we set

$$H_\alpha^-(p) = \begin{cases} H_\alpha(p) & \text{for } p \leq p_0^\alpha, \\ H_\alpha(p_0^\alpha) & \text{for } p \geq p_0^\alpha, \end{cases} \quad H_\alpha^+(p) = \begin{cases} H_\alpha(p_0^\alpha) & \text{for } p \leq p_0^\alpha, \\ H_\alpha(p) & \text{for } p \geq p_0^\alpha. \end{cases}$$

In [65], it is proved that general junction conditions are reduced to special ones of optimal control type. In fact, when replacing  $F_A$  in (0.1.4) by a general junction function  $F$ , which satisfies the following conditions

$$\begin{cases} \text{(Continuity)} & F \in C(\mathbb{R}^n) \\ \text{(Monotonicity)} & \forall \alpha, p_\alpha \mapsto F(p_1, \dots, p_N) \text{ is non-increasing,} \end{cases} \quad (0.1.9)$$

Imbert and Monneau proved that every relaxed viscosity solution (see below for a definition) of the obtained problem

$$\begin{cases} u_t + H_\alpha(u_x) = 0 & \text{in } (0, T) \times J_\alpha \setminus \{0\}, \\ u_t + F\left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_N}\right) = 0 & \text{in } (0, T) \times \{0\}, \end{cases} \quad (0.1.10)$$

is a viscosity solution of (0.1.4) for some  $A \in \mathbb{R}$ . Moreover, it is now understood that under quasi-convexity condition on Hamiltonians, the solution of problem (0.1.4) is unique, and hence the solution of the  $F$  problem (0.1.10) is unique. Furthermore, well-posedness and stability properties for Hamilton-Jacobi equations with non convex coercive Hamiltonians have been studied in [58].

As explained in [65], it is difficult to construct viscosity solutions in the classical sense because of the junction condition. It is in fact possible in the case of the flux limited junction conditions  $F_A$ . For general junction conditions, the Perron process generates a viscosity solution in the following relaxed sense.

We introduce the *space of test functions*

$$C^1(J_T) = \{u \in C(J_T) : \forall \alpha = 1, \dots, N, u^\alpha \in C^1([0, T) \times J_\alpha)\},$$

where  $u^\alpha$  denotes the restriction of  $u$  to  $[0, T) \times J_\alpha$ .

In order to define classical viscosity solutions, we recall the definition of upper and lower semi-continuous envelopes  $u^*$  and  $u_*$  of a (locally bounded) function  $u$  defined on  $[0, T) \times J$ :

$$u^*(t, x) = \limsup_{(s, y) \rightarrow (t, x)} u(s, y) \quad u_*(t, x) = \liminf_{(s, y) \rightarrow (t, x)} u(s, y).$$

**Definition 0.1.2 (Relaxed viscosity solution).** We say that  $u$  is a *relaxed sub-solution* (resp. *relaxed super-solution*) of (0.1.10) in  $(0, T) \times J$  if for all test function  $\varphi \in C^1(J_T)$  such that

$$u^* \leq \varphi \quad (\text{resp. } u_* \geq \varphi) \quad \text{in a neighborhood of } (t_0, x_0) \in J_T$$

with equality at  $(t_0, x_0)$  for some  $t_0 > 0$ , we have

$$\varphi_t + H_\alpha(\varphi_x) \leq 0 \quad (\text{resp. } \geq 0) \quad \text{at } (t_0, x_0)$$

if  $x_0 \neq 0$  and,

$$\begin{cases} \text{either } \varphi_t + F(\varphi_x) \leq 0 \quad (\text{resp. } \geq 0) & \text{at } (t_0, x_0) \\ \text{or } \varphi_t + H_\alpha(\partial_\alpha \varphi) \leq 0 \quad (\text{resp. } \geq 0) & \text{at } (t_0, x_0) \quad \text{for some } \alpha. \end{cases}$$

We say that  $u$  is a *relaxed (viscosity) solution* of (0.1.10) if  $u$  is both a sub-solution and a super-solution of (0.1.10).

The Hamilton-Jacobi model on which we depended [65], does not benefit from the optimal control formulation of equations and does not display the Lax-Oleinik representation formulas, thus the solution can not be simply computed. And hence the numerical scheme does not depend on this formulation. There are many examples of numerical schemes for Hamilton-Jacobi equations with continuous Hamiltonians. For example,

- the semi-Lagrangian schemes (see [36, 38, 48, 50]) that use the outflow of optimal control type and the principle of dynamic programming of Bellman. The technique is a method of coupling integration for ODEs and an interpolation method. These schemes do not require the introduction of a sort of Courant-Friedrichs-Lewy condition (CFL).
- the finite difference method. These examples were primarily used by Crandall and Lions [42].

However, there are only few existing works dealing with numerical approximation for Hamilton-Jacobi equations on networks. In [41], Costeseque, Lebacque and Monneau introduced a finite difference scheme for (0.1.4), and proved the convergence for the discretized solution towards the continuous viscosity solution in the sense of [67]. We cite also the following works: a convergent semi-Lagrangian scheme is introduced in [33] for equations of eikonal type. In [57], an adapted Lax-Friedrichs scheme is used to solve a traffic model; it is worth mentioning that this discretization implies to pass from the scalar conservation law to the associated Hamilton-Jacobi equation at each time step.

We introduce discrete steps in time and space  $h = (\Delta t, \Delta x)$  who have to satisfy a stability condition, for explicit schemes. The discretized junction is hence denoted by  $\mathcal{G}_h$ . We consider  $(U_i^{\alpha,n})_{i,\alpha,n}$  the solution of the following numerical scheme which approximates the continuous solution of model (0.1.10) for discrete time steps  $n\Delta t$  and space steps  $i\Delta x$ . Hence, we generalize the finite difference scheme introduced in [41] for general junction conditions:

$$\begin{cases} \frac{U_i^{\alpha,n+1} - U_i^{\alpha,n}}{\Delta t} + \max\{H_\alpha^+(p_{i,-}^{\alpha,n}), H_\alpha^-(p_{i,+}^{\alpha,n})\} = 0, & i \geq 1, \quad \alpha = 1, \dots, N, \\ U_0^{\beta,n} := U_0^n, & i = 0, \quad \beta = 1, \dots, N, \\ \frac{U_0^{n+1} - U_0^n}{\Delta t} + F(p_{0,+}^{1,n}, \dots, p_{0,+}^{N,n}) = 0, \end{cases} \quad (0.1.11)$$

where  $p_{i,\pm}^{\alpha,n}$  are the discrete (space) gradients defined by

$$p_{i,+}^{\alpha,n} := \frac{U_{i+1}^{\alpha,n} - U_i^{\alpha,n}}{\Delta x}, \quad p_{i,-}^{\alpha,n} := \frac{U_i^{\alpha,n} - U_{i-1}^{\alpha,n}}{\Delta x}, \quad (0.1.12)$$

with the initial condition

$$U_i^{\alpha,0} = u_0(x_i^\alpha), \quad i \geq 0, \quad \alpha = 1, \dots, N. \quad (0.1.13)$$

**Contribution.** We prove the convergence of the numerical solution towards the relaxed viscosity solution of the general continuous model (0.1.10) as the mesh size

$h = (\Delta t, \Delta x)$  tends to zero. The following Courant-Friedrichs-Lewy (CFL) condition ensures that the explicit scheme is monotone,

$$\frac{\Delta x}{\Delta t} \geq \max \left\{ \max_{\substack{\alpha=1,\dots,N, \\ p_\alpha \leq p \leq \bar{p}_\alpha}} |H'_\alpha(p)|; \max_{p_\alpha^0 \leq p_\alpha \leq \bar{p}_\alpha} \left\{ (-\nabla \cdot F)(p_1, \dots, p_N) \right\} \right\} \quad (0.1.14)$$

for some  $p_\alpha, \bar{p}_\alpha, p_\alpha^0 \in \mathbb{R}$ , depending only on  $u_0, H$  and  $F$ . We assume additional conditions on  $F$  than the ones considered in [65]. In fact, for  $F : \mathbb{R}^n \rightarrow \mathbb{R}$  satisfying

$$\begin{cases} F \text{ is continuous and piecewise } C^1(\mathbb{R}^n), \\ \forall \alpha, \forall p = (p_1, \dots, p_N) \in \mathbb{R}^n, \frac{\partial F}{\partial p_\alpha}(p) < 0, \\ F(p_1, \dots, p_N) \rightarrow +\infty \text{ as } \min_{i \in \{1, \dots, N\}} p_i \rightarrow -\infty, \end{cases} \quad (0.1.15)$$

we prove the following convergence result.

**Theorem 0.1.3 (Convergence for general junction conditions).** *Let  $T > 0$  and  $u_0$  be Lipschitz continuous. There exist  $p_\alpha, \bar{p}_\alpha, p_\alpha^0 \in \mathbb{R}$ ,  $\alpha = 1, \dots, N$ , depending only on the initial data, the Hamiltonians and the junction function  $F$ , such that, if the mesh size  $h = (\Delta t, \Delta x)$  satisfies the CFL condition (0.1.14), then the numerical solution  $u^h$  defined by (0.1.11), (0.1.13) converges locally uniformly as  $h$  goes to zero to the unique weak (relaxed viscosity) solution  $u$  of (0.1.10), (0.1.5), on any compact set  $\mathcal{K} \subset [0, T) \times J$ , i.e.*

$$\limsup_{|h| \rightarrow 0} \sup_{(t,x) \in \mathcal{K} \cap \mathcal{G}_h} |u^h(t, x) - u(t, x)| = 0. \quad (0.1.16)$$

We hence need to prove discrete (time and gradient) estimates in order to ensure the monotonicity of the scheme and, in turn, its convergence.

For the flux limited junction conditions, as we mentioned above, Costeseque, Lebacque and Monneau considered the following scheme,

$$\begin{cases} \frac{U_i^{\alpha, n+1} - U_i^{\alpha, n}}{\Delta t} + \max\{H_\alpha^+(p_{i,-}^{\alpha, n}), H_\alpha^-(p_{i,+}^{\alpha, n})\} = 0, & i \geq 1, \quad \alpha = 1, \dots, N, \\ U_0^{\beta, n} := U_0^n, & i = 0, \quad \beta = 1, \dots, N, \\ \frac{U_0^{n+1} - U_0^n}{\Delta t} + F_A(p_{0,+}^{1,n}, \dots, p_{0,+}^{N,n}) = 0, \end{cases} \quad (0.1.17)$$

and proved that its discretized solution converges locally uniformly towards the viscosity solution of (0.1.4), under the following CFL condition

$$\frac{\Delta x}{\Delta t} \geq \max_{\substack{\alpha=1,\dots,N, \\ p_\alpha \leq p \leq \bar{p}_\alpha}} |H'_\alpha(p)|, \quad (0.1.18)$$

where  $p_\alpha, \bar{p}_\alpha \in \mathbb{R}$  depend only on the Hamiltonians and the initial condition.

We assume in addition that,

$$\text{For all } \alpha, \beta \in \{1, \dots, N\}, \quad \min H_\alpha = \min H_\beta. \quad (0.1.19)$$

Our second result, is thus an error estimate for the flux limited junction conditions if the minima of the Hamiltonians on different branches are equal. More precisely, we prove the following result.

**Theorem 0.1.4 (Error estimates for flux-limited junction conditions).** *Assume that the Hamiltonians  $H_\alpha$  satisfy (0.1.8),(0.1.19). Let  $u_0$  be Lipschitz continuous,  $u^h$  be the solution of the associated numerical scheme (0.1.17),(0.1.13) and  $u$  be the weak (viscosity) solution of (0.1.4)-(0.1.5) for some  $A = A_0 \in \mathbb{R}$ . If the CFL condition (0.1.18) is satisfied for  $\underline{p}_\alpha, \bar{p}_\alpha$  given in Theorem 0.1.3 then there exists  $C > 0$  (independent of  $h$ ) such that*

$$\sup_{[0,T) \times J \cap \mathcal{G}_h} |u^h(t, x) - u(t, x)| \leq C(\Delta x)^{1/2}. \quad (0.1.20)$$

In fact, it is explained in [44] that the proof of the comparison principle between sub- and super-solutions of the continuous Hamilton-Jacobi equation can be adapted in order to derive error estimates between the numerical solution associated with monotone (stable and consistent) schemes and the continuous solution. In the Euclidian case, the comparison principle is proved thanks to the technique of doubling variables; it relies on the classical penalization term  $\frac{(x-y)^2}{\epsilon}$ . Such a penalization procedure is known to fail in general if the equation is posed on a junction; it is explained in [65] that it has to be replaced with a *vertex test function*  $G$ .

In order to derive error estimates as in [42], it is important to study the regularity of the vertex test function. More precisely, we prove (Proposition 1.5.1) that it can be constructed in such a way that its gradient is locally Lipschitz continuous, at least if the minima of the Hamiltonians on different branches are equal. In fact, we see that the second order derivatives of the regularized vertex test function explode near the origin when the minima are not equal, that is why we restrict ourselves to the case where no regularization is needed.

**Comments.** Let us note that the derivative of a weak (viscosity) solution of a Hamilton-Jacobi equation posed on the real line is known to coincide with the entropy solution of the corresponding scalar conservation law. It is therefore reasonable to expect that the error between the viscosity solution of the Hamilton-Jacobi equation and its approximation is as good as the one obtained between the entropy solution of the scalar conservation law and its approximation. We would like then to compare our result to some existing results in the literature. In the scalar case, it is proved in [46] that the error between the solution of the regularized equation with a vanishing viscosity coefficient equal to  $\epsilon$  and the entropy solution of the conservation law (which is merely of bounded variation in space) is of order  $\epsilon^{\frac{1}{3}}$  (in  $L_t^\infty L_x^1$  norm). In [93], Ohlberger and Vovelle derive error estimates for finite volume schemes associated with such boundary value problems and prove that it is of order  $(\Delta x)^{\frac{1}{6}}$  (in  $L_{t,x}^1$  norm). In [103], assuming that the flux is bell-shaped, that is to say the opposite is quasi-convex, it is proved that the error between the finite volume

scheme and the entropy solution is of order  $(\Delta x)^{\frac{1}{3}}$  and that it can be improved to  $(\Delta x)^{\frac{1}{2}}$  under an additional condition on the traces of the BV entropy solution.

**Contribution.** In Chapter 2, we adapt a new approach in deriving error estimates for flux limited junction conditions for well chosen Hamiltonians (in fact a larger class of Hamiltonians), by replacing the vertex test function by the *reduced minimal action*  $\mathcal{D}$  following the Oleinik-Lax representation formula introduced in [67, 80]. We thus prove, in  $L_{loc}^\infty$ , that the error is of order  $(\Delta x)^{\frac{1}{2}}$  if the flux is strictly limited.

It is well known that the Legendre-Fenchel conjugate is crucial in establishing a link between the general problem (0.1.4) and a control problem. Through this link, we obtain the representation formula for the exact solution. Before treating the case where the Hamiltonians  $H_\alpha$  satisfy (0.1.8), we first consider the case of Hamiltonians satisfying the hypotheses of [67] i.e.,

$$\left\{ \begin{array}{l} \text{(Regularity)} \quad H_\alpha \text{ is continuous} \\ \text{(Coercivity)} \quad \lim_{|p| \rightarrow +\infty} H_\alpha(p) = +\infty \\ \text{(Convexity)} \quad H_\alpha \text{ is convex and is the Legendre Fenchel transform of } L_\alpha \\ \quad \text{where } L_\alpha \text{ is of class } C^2 \text{ and satisfies (B0).} \end{array} \right. \quad (0.1.21)$$

We recall that

$$H_\alpha(p) = L_\alpha^*(p) = \sup_{q \in \mathbb{R}} (pq - L_\alpha(q)). \quad (0.1.22)$$

We consider the following hypothesis for  $L_\alpha$ ,

(B0) There exists a constant  $\gamma > 0$  such that for all  $\alpha = 1, \dots, N$ , the  $C^2(\mathbb{R})$  functions  $L_\alpha$  satisfy  $L_\alpha'' \geq \gamma > 0$ .

**Theorem 0.1.5 (Error estimates for flux-limited junction conditions).** *Let  $u_0$  be Lipschitz continuous,  $u^h$  be the solution of the associated numerical scheme (0.1.17)-(0.1.13) and  $u$  be the viscosity solution of (0.1.4)-(0.1.5) for some  $A \in \mathbb{R}$ . If the CFL condition (0.1.18) is satisfied, then there exists  $C > 0$  (independent of  $h$ ) such that*

$$\sup_{[0,T) \times J \cap \mathcal{G}_h} |u^h(t, x) - u(t, x)| \leq \begin{cases} C(\Delta x)^{1/2} & \text{if } A > A_0, \\ C(\Delta x)^{2/5} & \text{if } A = A_0. \end{cases} \quad (0.1.23)$$

In order to derive error estimates as in [42], it is important to study the regularity of the vertex test function. More precisely, we prove (Proposition 2.3.12) that its gradient is locally Lipschitz continuous, at least if the flux is “strictly limited” and far away from a special curve. But we also see that the reduced minimal action is not of class  $C^1$  on this curve. However we can get “weaker” viscosity inequalities thanks to a result in [65] (see Proposition 2.2.3).

**Comments.** For optimal control problems, the numerical approximation of Hamilton Jacobi equations has already been studied using schemes based on the discrete dynamic programming principle. Essentially, these schemes are built by replacing the continuous optimal control problem by its discrete time version. We refer to Capuzzo Dolcetta [35], Capuzzo Dolcetta-Ishii [37] for the results concerning the convergence of  $u_h$  to  $u$  and the a priori estimates (of order  $\Delta x$ ), in the  $L^\infty$ , giving the order of convergence of the discrete-time approximation. We refer to Falcone [48] for the results related to the order of convergence of the fully discrete (i.e. in space and time) approximation and for the construction of the algorithm, we mention that under a semi-concavity assumption the rate of convergence is of order 1. We do not know hence if we should obtain an error estimate of order 1, due to technical difficulties implied by the discontinuity. In [49], Falcone and Ferretti showed that in one dimension, the first-order semi-Lagrangian scheme coincides with the integration of the Godunov scheme for the corresponding conservation laws. We cite also [50] and references therein for discrete time high order schemes for Hamilton-Jacobi Bellman equations.

## 0.2 Indirect controllability/stability of a 1D system of coupled wave equations

We introduce now the second part of this work, related to the study of controllability and stability of weakly coupled wave equations in the one dimensional setting.

### 0.2.1 Motivation

Control theory can be described as the process of influencing the behavior of a physical system to achieve a desired goal, primarily through the use of feedback which monitors the effect of a system and modifies its output. Its application ranges widely from earthquake engineering and seismology to fluid transfer, cooling water and noise reduction in cavities, vehicles, such as pipe systems, the regulation of biological systems like human cardiovascular system, the design of robotic systems, and laser control in quantum mechanical systems. Roughly speaking, the concept of controllability is described as the ability to steer our evolution system, whether described in terms of partial or ordinary differential equations, from any initial state to any desired final state in a finite time interval by means of a suitable control (boundary control, internal control, controls localized on open subsets of bounded sets, etc...).

Observability is a measure for how well internal states of a system can be inferred by knowledge of its external outputs. The duality between the controllability and observability of systems of partial differential equations in Banach spaces has been examined in many works such as those of Lions [79] where Hilbert uniqueness theorem HUM is explained, and the works of Russell [99, 100]. Various methods could

be used to prove observability inequalities such as Carleman estimates, micro-local analysis and the multiplier method. For more details on the treatment of observability problems and proving observability inequalities for linear systems, we refer the reader to [79], and [104].

As for stabilization, it is defined as the ability to find an input control that requires the state response to approach zero as time  $T \rightarrow \infty$ . Different types of stability also occur. We only encounter polynomial stability in our case.

In this thesis, we implement the semi-group theory in the spirit of spectral theory to study the exact controllability and stabilization of some evolution equations. We use detailed spectral analysis in order to establish indirect exact controllability and stabilization for coupled wave equations. In fact, we consider two types of coupling: a zero order coupling parameter, and a coupling by means of velocities. In chapter 3, we consider a one dimensional setting of coupled wave equations, coupled by means of velocities, with only one boundary control acted on one of the two equations. The second equation is hence controlled indirectly by means of coupling. We consider afterwards waves propagating with same and different speeds, for which we establish indirect exact controllability using the spectral approach, and hence Ingham type inequalities hold [71, 73]. Whereas, in chapter 4, we consider a zero order coupling parameter for a one dimensional setting of coupled wave equations propagating with equal speeds, subject to only one boundary control on one of the two equations. Using the Riesz basis approach, we establish an optimal decay rate of the energy of the associated problem. Afterwards, using the Ingham type inequalities, we derive observability spaces in order to conclude exact controllability of the system in consideration.

In order to introduce the main theme of our study, and the obtained results let us recall some of the fundamental definitions that are being used throughout this thesis.

**Definition 0.2.1.** Let  $X$  be a Banach space. A one parameter family  $(S(t))_{t \geq 0}$  of bounded linear operators defined from  $X$  into  $X$  is a strongly continuous semigroup of bounded linear operators on  $X$  if:

1.  $S(0) = I$ , ( $I$  identity operator on  $X$ ).
2.  $S(t + s) = S(t)S(s)$  for every  $t, s \geq 0$ .
3.  $S(t)x \rightarrow x$ , as  $t \rightarrow \infty$ ,  $\forall x \in X$ .

Such a semigroup is called a  $C_0$ -semigroup

**Definition 0.2.2.** The infinitesimal generator  $\mathcal{A}$  of the semigroup  $(S(t))_{t \geq 0}$  is defined by:

$$D(\mathcal{A}) = \{x \in X, \lim_{t \rightarrow \infty} \frac{S(t)x - x}{t} \text{ exists}\}.$$

and

$$\mathcal{A}x = \lim_{t \rightarrow \infty} \frac{S(t)x - x}{t}, x \in D(\mathcal{A}).$$

**Definition 0.2.3.** Let  $\mathcal{H}$  be a Hilbert space. An operator  $(\mathcal{A}, D(\mathcal{A}))$  on  $\mathcal{H}$  satisfying

$$\mathcal{R}(\mathcal{A}u, u) \leq 0, \quad \forall u \in D(\mathcal{A}),$$

is said to be a dissipative operator. A maximal dissipative operator  $(\mathcal{A}, D(\mathcal{A}))$  on  $\mathcal{H}$  is a dissipative operator for which  $\mathcal{R}(\lambda I - \mathcal{A}) = \mathcal{H}$ , for some  $\lambda > 0$ . A maximal dissipative operator is also called m-dissipative operator.

Generally speaking, the first step in dealing with the study of the stability of the solution is to rewrite our evolution system of partial differential equations as a Cauchy problem on some appropriate Hilbert space  $\mathcal{H}$  called the energy space

$$\partial_t U = \mathcal{A}U, \quad U(0) = U_0. \quad (0.2.1)$$

where  $\mathcal{A}$  is an unbounded operator on  $\mathcal{H}$ . Then we prove that  $\mathcal{A}$  is the infinitesimal generator of a  $C_0$ -semi-group of contractions  $(S(t))_{t \geq 0}$  on  $\mathcal{H}$  in order to deduce the existence of a solution in a certain Hilbert space. The solution is hence of the form  $U(t) = S(t)U_0$ . We mention here Lumer-Phillips theorem (see [91]) which is applied to justify the existence and uniqueness of solutions of some partial differential equations.

**Theorem 0.2.4** (Lumer-Phillips Theorem). *Let  $\mathcal{A}$  be a linear operator with dense domain  $D(\mathcal{A})$  in a Banach space  $X$ .*

1. *If  $\mathcal{A}$  is dissipative and there exists  $\lambda_0 > 0$  such that  $\mathcal{R}(\lambda_0 I - \mathcal{A}) = X$  then  $\mathcal{A}$  is the infinitesimal generator of a  $C_0$ -semi-group of contractions on  $X$ .*
2. *If  $\mathcal{A}$  is the infinitesimal generator of a  $C_0$ -semi-group of contractions on  $X$  then  $\mathcal{R}(\lambda I - \mathcal{A}) = X$  for all  $\lambda > 0$  and  $\mathcal{A}$  is dissipative.*

Consequently,  $\mathcal{A}$  is maximal dissipative on a Hilbert space  $\mathcal{H}$  if and only if it generates a  $C_0$ - semi-group of contractions on  $\mathcal{H}$  and thus the existence of the solution is justified by the following corollary which follows from Lumer-Phillips theorem.

**Corollary 0.2.5.** *Let  $\mathcal{H}$  be a Hilbert space and let  $\mathcal{A}$  be a linear operator defined from  $D(\mathcal{A}) \subset \mathcal{H}$  into  $\mathcal{H}$ . If  $\mathcal{A}$  is maximal dissipative then the initial value problem*

$$\frac{du}{dt}(t) = \mathcal{A}u(t), \quad t > 0, u(0) = u_0$$

*has a unique solution  $u \in C([0, +\infty), \mathcal{H})$ , for each initial datum  $u_0 \in \mathcal{H}$ . Moreover, if  $u_0 \in D(\mathcal{A})$ , then*

$$u \in C([0, +\infty), D(\mathcal{A})) \cap C^1([0, +\infty), \mathcal{H}).$$

We aim now to discuss the notions of stability of semi-groups.

Assume that  $\mathcal{A}$  is a generator of a strongly continuous semi-group of contractions on a Hilbert space  $\mathcal{H}$ . We say that the semi-group  $(S(t))_{t \geq 0}$  generated by  $\mathcal{A}$  is

- Strongly (asymptotically) stable if for all  $U_0 \in \mathcal{H}$

$$\|S(t)U_0\|_{\mathcal{H}} \rightarrow 0.$$

- Exponentially stable if there exist two positive constants  $C, \omega$  such that

$$\|S(t)U_0\|_{\mathcal{H}} \leq Ce^{-\omega t} \|U_0\|_{\mathcal{H}}, \quad \forall t \geq 0, \forall U_0 \in \mathcal{H}.$$

- Polynomially stable if there exist constants  $\alpha, \beta, C > 0$  such that

$$\|S(t)(d - \mathcal{A})^\alpha\| \leq Ct^{-\beta}, \quad t > 0,$$

for some  $d > 0$ .

## 0.2.2 Results

In this subsection, we present the problems that have been studied in this thesis, mainly in Chapters 3 and 4, and the results that were obtained.

**Literature.** The mechanism of indirect dynamic controls has been introduced by Russel [101]. Since then, many works have been established on the stabilization and the exact controllability of hyperbolic systems with feedback, or systems subject to internal or indirect boundary controls. Exact controllability and observability of coupled systems either for hyperbolic-hyperbolic type or hyperbolic-parabolic type had been earlier investigated by Lions [79]. These results assume that the coupling parameter is sufficiently small. They have been extended in [71] to the cases of arbitrary coupling parameters (assuming bounded coupling operators). For both references, the multiplier technique was the fundamental element for acquiring the desired estimates. Observability and exact controllability results have also been obtained by Alabau [7, 9], Liu and Rao in [86], Wehbe [105–107] and Komornik and Loretto in [71, 72]. We cite also [75] for Carleman estimates for exact boundary controllability for hyperbolic equations. One can also look at [8, 10, 87, 88, 97] for stabilization results.

We are interested in coupled wave equations, coupled through velocity terms, or through zero order terms, in the one dimensional setting.

### 0.2.2.1 Wave equations coupled through velocity terms

Firstly, in chapter 3 we are interested in the following system of wave equations coupled by velocities, in the one dimensional setting.

$$\begin{cases} u_{tt} - u_{xx} + by_t = 0 & \text{on } (0, 1) \times (0, T), \\ y_{tt} - ay_{xx} - bu_t = 0 & \text{on } (0, 1) \times (0, T), \\ u(1, t) = u(0, t) = y(0, t) = 0 & \text{for all } t \in (0, T), \\ y(1, t) = v(t) & \text{for all } t \in (0, T). \end{cases} \quad (0.2.2)$$

The system (0.2.2) is a classic model for the motions of two stacked elastic bodies. The equations are coupled by means of velocities. We denote by  $a > 0$  the ratio of the velocities of the equations and by  $b$  the coupling parameter. The control  $v \in L^2(0, T)$  is applied only at the right boundary of the second equation. The first equation is controlled indirectly by means of coupling of the equations.

We consider the indirect boundary exact controllability problem: For given  $T > 0$  and initial data  $(u_0, u_1, y_0, y_1)$  belonging to a suitable space, does there exist a suitable control  $v$  that brings back the solution to equilibrium at time  $T$ , such that the solution of (0.2.2) satisfies

$$u(x, T) = u_t(x, T) = y(x, T) = y_t(x, T) = 0 \quad \text{on} \quad (0, 1).$$

**Literature.** In [13], Ammar-khodja and Bader proved that the internal damping applied to only one of the equations never gives exponential stability if the wave speeds are different. If the wave speeds are the same, i.e. if  $a = 1$ , they present necessary and sufficient conditions for stability. In a generalized setting, Toufayli in [104] proved, using the spectral method of Benchimol [28], that the system is strongly stable for usual initial values. Afterwards, by [95] and [61], she established exponential stability of the problem. Furthermore, using the multiplier method, she established exact controllability for a boundary control acted only on one equation. In [68] and [69], wave-wave systems having the same principal part are coupled through velocity terms. Therefore, the coupling is not compact. Indirect observability and uniform stabilization results are established. Recently, Najdi and Wehbe in [92] considered system (0.2.2) with only one boundary damping and established exponential stability for  $b \neq k\pi, k \in \mathbb{Z}^*$ , when waves propagate with the same speed. For two waves with different speeds of propagation, polynomial stability of type  $\frac{1}{t}$  was proved only for  $a \in \mathbb{Q}$ .

**Contribution.** We prove exact controllability of system (0.2.2). Indeed, we establish observability inequalities using Ingham's theorem ([73]) while distinguishing the cases of different or equal speeds of propagation of the coupled waves. In order to do so, we consider respectively the associated homogeneous problem (for  $a \neq 1$ , and  $a = 1$ ), that is to say, the null boundary acted control system. Hence, using the Hilbert Uniqueness Method ([79]), we deduce the indirect exact controllability of the considered system.

Unlike the spectral method, the multiplier method used in [6, 9, 12, 104] requires some technical conditions on the coefficients of the system. Then, our aim is to establish observability inequalities using the Ingham's theorem while distinguishing

the following cases:

$$\text{(Case 1)} \quad a = 1 \text{ and } b \notin \pi\mathbb{Z},$$

$$\text{(Case 2)} \quad a = 1 \text{ and } b \in \pi\mathbb{Z},$$

$$\text{(Case 3)} \quad a \neq 1 \text{ such that } a \in \mathbb{Q},$$

$$\text{(Case 4)} \quad a \neq 1 \text{ such that } a \in \mathbb{R} \setminus \mathbb{Q}.$$

Let us denote by  $\lambda_{1,m}, \lambda_{2,n}$  the two branches of eigenvalues of the homogeneous systems corresponding respectively for the cases  $a \neq 1$  and  $a = 1$ , and by

$$\begin{cases} E_{1,n} = (\phi_{1,n}, \lambda_{1,n}\phi_{1,n}, \psi_{1,n}, \lambda_{1,n}\psi_{1,n}), \\ E_{2,n} = (\phi_{2,n}, \lambda_{2,n}\phi_{2,n}, \psi_{2,n}, \lambda_{2,n}\psi_{2,n}) \end{cases} \quad (0.2.3)$$

the corresponding eigenvectors. Our main results are the following.

**Case 1.** The eigenvalues  $\lambda_{1,m}, \lambda_{2,n}$  satisfy an uniform gap condition. Then using the usual Ingham's theorem, and if

$$b^2 \neq \frac{(k_1^2 - k_2^2)^2}{2\pi^2(k_1^2 + k_2^2)}, \quad \forall k_1, k_2 \in \mathbb{Z}^*, \quad (0.2.4)$$

we prove the following result.

**Theorem 0.2.6.** *Assume that  $a = 1$ , condition (0.2.4) holds and  $b \notin \pi\mathbb{Z}$ . Let  $0 < |\hat{b}| < \pi$  be the resulting quantity of  $b - \hat{k}\pi$ , where  $\hat{k} \in \mathbb{N}^*$ , such that  $\hat{k}\pi < |b| < (\hat{k}+1)\pi$  and let*

$$T > \frac{2\pi}{|\hat{b}|}. \quad (0.2.5)$$

*Then there exists a constant  $c_1 > 0$  such that the following direct inequality holds*

$$\int_0^T |\psi_x(1, t)|^2 dt \leq c_1 \|(\phi_0, \phi_1, \psi_0, \psi_1)\|_{\mathcal{H}}^2, \quad (0.2.6)$$

*for all solution  $\Phi = (\phi, \phi_t, \psi, \psi_t)$  solving the corresponding homogeneous Cauchy problem.*

*Moreover, there exists a constant  $0 < c_2 < c_1$  depending only on  $b$ , such that the following inverse observability inequality holds*

$$c_2 \|(\phi_0, \phi_1, \psi_0, \psi_1)\|_{\mathcal{H}}^2 \leq \int_0^T |\psi_x(1, t)|^2 dt. \quad (0.2.7)$$

**Case 2.** The eigenvalues  $\lambda_{1,m}, \lambda_{2,n}$  are asymptotically close. To be precise, following proposition 3.2.7 in Chapter 3, we prove that there exist infinitely many  $m \sim n + k_0$

$$\lambda_{1,m} - \lambda_{2,n} \sim \frac{O(1)}{m^2}, \quad \lambda_{1,m} - \lambda_{2,n} \sim \frac{O(1)}{n^2}. \quad (0.2.8)$$

Then, the usual Ingham's theorem which requires an uniform gap condition does not work in this case and consequently, the observability inequalities are not true in the energy space. In order to get the inverse observability inequality, we will use a general Ingham-type theorem based on the divided differences, which tolerates asymptotically and even multiple eigenvalues. On the other hand, the observation is on the second components of the eigenfunctions  $E_{1,n}, E_{2,n}$ . And, from (3.2.11)-(3.2.12) from Proposition 3.2.2, we see that

$$(\psi_{1,n})_x(1) = O(1), \quad (\psi_{2,n})_x(1) = O(1). \quad (0.2.9)$$

So, a natural space for the observability inequalities is the following weighted spectral space

$$D_2 = \{(\phi_0, \phi_1, \psi_0, \psi_1) = \sum_{n \neq 0} (\alpha_{1,n} E_{1,n} + \alpha_{2,n} E_{2,n}) n^2\},$$

where the factor  $n^2$  is due to the closeness of eigenvalues (0.2.8).

We prove afterwards the following result.

**Theorem 0.2.7.** *Assume that  $a = 1$ , condition (0.2.4) and  $b = k_0\pi$  for some  $k_0 \in \mathbb{Z}^*$ . Let*

$$T > 4.$$

*Then there exists a constant  $c_3 > 0$  such that the following direct inequality holds*

$$\int_0^T |\psi_x(1, t)|^2 dt \leq c_3 \|(\phi_0, \phi_1, \psi_0, \psi_1)\|_{\mathcal{H}}^2, \quad (0.2.10)$$

*for all solution  $\Phi = (\phi, \phi_t, \psi, \psi_t)$  solving the corresponding homogeneous Cauchy problem.*

*Moreover, there exists a constant  $0 < c_4 < c_3$ , such that the following inverse observability inequality holds*

$$c_4 \|(\phi_0, \phi_1, \psi_0, \psi_1)\|_{D_2}^2 \leq \int_0^T |\psi_x(1, t)|^2 dt. \quad (0.2.11)$$

**Cases 3 and 4.** We prove in Proposition 3.3.4 of Chapter 3, that there exist infinitely many  $\sqrt{am} \sim n$  such that the eigenvalues are asymptotically close

$$\lambda_{1,m} - \lambda_{2,n} \sim \frac{O(1)}{m}, \quad \lambda_{1,m} - \lambda_{2,n} \sim \frac{O(1)}{n}, \quad (0.2.12)$$

where we distinguish cases of the ratio of the wave speeds, as  $0 < a \neq 1$  is a rational number or an irrational number. Then, we will use a general Ingham-type

theorem. On the other hand, the observation is on the second components of the eigenfunctions  $E_{1,n}, E_{2,n}$ . From (3.3.12)-(3.3.13) from Proposition 3.3.2, we see that

$$(\psi_{1,n})_x(1) = O(1), \quad (\psi_{2,n})_x(1) = \frac{O(1)}{n}. \quad (0.2.13)$$

So, if we want to observe the first equation via the second one, we have to use a weaker norm so that the observation on the second component  $(\psi_{1,n})_x(1)$  does not disappear. It seems that the following weighted spectral space

$$D_1 = \{(\phi_0, \phi_1, \psi_0, \psi_1) = \sum_{n \neq 0} (\alpha_{1,n} E_{1,n} + \alpha_{2,n} n E_{2,n}) n\} \quad (0.2.14)$$

is the natural space for the observability. In (0.2.14), the factor  $n$  before the eigenvector  $E_{2,n}$  is due to the transmission of the modes between the two equations (0.2.13), and the factor  $n$  is due to the closeness of the eigenvalues (0.2.12). If

$$b^2 \neq \frac{(k_1^2 - ak_2^2)(ak_1^2 - k_2^2)}{\pi^2(a+1)(k_1^2 + k_2^2)}, \quad \forall k_1, k_2 \in \mathbb{Z}^*, \quad (0.2.15)$$

we have the following result.

**Theorem 0.2.8.** *Let  $0 < a \neq 1$ , and let  $b$  a real number satisfying (0.2.15). Assume that*

$$T > 2 \left( 1 + \frac{1}{\sqrt{a}} \right).$$

*Then there exists a constant  $c_1 > 0$  such that the following direct inequality holds*

$$\int_0^T |\psi_x(1, t)|^2 dt \leq c_1 \|(\phi_0, \phi_1, \psi_0, \psi_1)\|_{\mathcal{H}}^2, \quad (0.2.16)$$

*for all solution  $\Phi = (\phi, \phi_t, \psi, \psi_t)$  solving the corresponding homogeneous Cauchy problem.*

*Moreover, there exists a constant  $0 < c < c_1$  depending only on  $a$  and  $b$  such that the following inverse observability inequalities hold true:*

1. *If  $a$  is rational number, then*

$$c \|(\phi_0, \phi_1, \psi_0, \psi_1)\|_{D_1}^2 \leq \int_0^T |\psi_x(1, t)|^2 dt. \quad (0.2.17)$$

2. *For almost all irrational number  $a > 0$ , we have*

$$c \|(\phi_0, \phi_1, \psi_0, \psi_1)\|_{D_1}^2 \leq \int_0^T |\psi_x(1, t)|^2 dt. \quad (0.2.18)$$

It is well known that the observability of the corresponding homogeneous Cauchy problem leads to the exact controllability of the considered systems [79]. After characterizing the corresponding weighted spectral spaces, we deduce the following exact controllability results.

**Case 1.** We have the following exact controllability result.

**Theorem 0.2.9.** *Let  $a = 1$ , and suppose that condition (0.2.4) is satisfied. Assume that there exist no integers  $k \in \mathbb{Z}$ , such that  $b = k\pi$ . Let  $0 < |\hat{b}| < \pi$  the resulting quantity of  $b - \hat{k}\pi$ , where  $\hat{k} \in \mathbb{N}^*$ , such that  $|b| < (\hat{k} + 1)\pi$ . Assume that*

$$T > \frac{2\pi}{|\hat{b}|}.$$

Let

$$(u_0, u_1, y_0, y_1) \in (L^2(0, 1) \times H^{-1}(0, 1))^2,$$

then there exists a control function  $v \in L^2(0, T)$  such that the solution of the non homogeneous system (0.2.2) satisfies the null final conditions:

$$u(x, T) = u_t(x, T) = y(x, T) = y_t(x, T).$$

**Case 2.** We have the following exact controllability result.

**Theorem 0.2.10.** *Let  $a = 1$  and suppose that there exists some integer  $k_0 \in \mathbb{Z}^*$  such that  $b = k_0\pi$ . Assume that*

$$T > 4.$$

Let

$$(u_0, u_1, y_0, y_1) \in (H^2(0, 1) \times H_0^1(0, 1))^2,$$

then there exists  $v \in L^2(0, T)$  such that the solution of the non homogeneous system (0.2.2) satisfies the null final conditions

$$u(x, t) = u_t(x, t) = y(x, t) = y_t(x, t) = 0, \quad \forall t \geq T.$$

**Cases 3 and 4.** We have the following exact controllability result.

**Theorem 0.2.11.** *Let  $b \neq 0$  be a real number satisfying (0.2.15) and  $0 < a \neq 1$ . Assume that*

$$T > 2 \left( 1 + \frac{1}{\sqrt{a}} \right).$$

1. If  $a$  is a rational number, let

$$(u_0, u_1, y_0, y_1) \in (H^3 \times H^2 \times H^2 \times H^1)(0, 1).$$

2. For almost all irrational number  $a > 0$ , let

$$(u_0, u_1, y_0, y_1) \in (H^3 \times H^2 \times H^2 \times H^1)(0, 1).$$

Then there exists a control function  $v \in L^2(0, T)$  such that the solution of the non-homogeneous system (0.2.2) satisfies the null conditions:

$$u(x, T) = u_t(x, T) = y(x, T) = y_t(x, T) = 0.$$

**Comments.** In [89], Liu and Rao considered the one dimensional setting of two coupled wave equations with a coupling parameter of order zero, propagating with different speeds. Depending on the arithmetic property of  $a$ , they obtained different spaces of observability. In the multi-dimensional setting, Alabau on the other hand, proved in [9] with the multiplier method, that the observability holds with a stronger norm than the one obtained in our case as in Theorem 3.1.1, for waves propagating with equal speeds.

### 0.2.2.2 Wave equations coupled through zero order terms

In chapter 4, we are interested in the following system of coupled wave equations through zero order terms posed in the one dimensional setting

$$\begin{cases} u_{tt} - u_{xx} + \alpha y = 0 & \text{on } (0, 1) \times (0, T), \\ y_{tt} - y_{xx} + \alpha u = 0 & \text{on } (0, 1) \times (0, T), \\ u(1, t) = y(0, t) = y(1, t) = 0 & \text{for all } t \in (0, T), \\ u_x(1, t) + \gamma u_t(1, t) = 0 & \text{for all } t \in (0, T). \end{cases} \quad (0.2.19)$$

We denote by  $\alpha$  the coupling parameter, which is assumed to be a real number small enough, and  $\gamma$  a positive number. The damping  $u_t$  is only applied at the right boundary of the first equation. The second equation is indirectly damped through the coupling between the two equations.

We consider the indirect boundary exact controllability problem: For given  $T > 0$  and initial data  $(u_0, u_1, y_0, y_1)$  belonging to a suitable space, does there exist a suitable control  $v$  that brings back the solution to equilibrium at time  $T$ , such that the solution of (0.2.19) satisfies

$$u(x, T) = u_t(x, T) = y(x, T) = y_t(x, T) = 0 \quad \text{on } (0, 1).$$

**Literature.** The polynomial energy decay rate occurs in many control problems where the open-loop systems are strongly stable, but not exponentially stable (hybrid systems, partially or locally damped systems), see [74] and references therein. The majority of the works in establishing polynomial energy decay rate has been based on the spectral method, frequency domain method, time domain multiplier and weak observability methods. We quote [84, 85, 96, 97] for hybrid systems, [76, 77, 88] for wave equations with local internal or boundary damping, [5, 14, 16, 90] for second order systems with partial internal damping, [27, 87] for abstract systems and [101] for systems of coupled wave-heat equations. We also mention [8, 10] for coupled hyperbolic systems, and [11] for coupled wave type systems. For a general formulation of partially damped systems see [98] and references therein. In [8] (see also references therein), polynomial decay estimates in the case of indirect internal stabilization are given.

In [10], Alabau-Boussouira considered more general systems of coupled second order evolution equations (wave-wave, Kirchoff-Petrowsky, wave-Petrowsky). In the multi-dimensional case of the wave-wave coupling, the lack of uniform stability was proved

by the compact perturbation argument, and the polynomial energy decay rate  $\frac{1}{\sqrt{t}}$  was established by a general integral inequality, for a star shaped domain of  $\mathbb{R}^n$ . While when considering different speeds of propagation (particularly the ratio of speeds equal to  $k_0^2, k_0 \in \mathbb{Z}^*$ ) the same energy decay rate was established in a  $N$ -dimensional interval where  $N \leq 3$ . These results are very interesting but are not optimal.

Hereafter in [11], Alabau and Léautaud considered a coupled system of wave equations, with variable coefficients, with one boundary feedback, and proved a polynomial energy decay rate of order  $\frac{1}{\sqrt{t}}$  for initial data in  $D(\mathcal{A})$ . And thus, the aim of this chapter is to improve the energy decay rate and to establish an optimal polynomial decay of type  $\frac{1}{t}$  for initial data in  $D(\mathcal{A})$  by the Riesz basis approach.

**Contribution.** We prove by the spectral approach, that the solution of (0.2.19) decays with an optimal polynomial rate of order  $\frac{1}{t}$ . More precisely, we prove the following result.

**Theorem 0.2.12.** *Assume that  $\gamma \neq 1$ . For all initial data  $U_0 \in D(\mathcal{A})$ , there exists a constant  $c > 0$  independent of  $U_0$ , such that the energy of the problem (0.2.19) satisfies the following estimation:*

$$E(t) \leq \frac{c}{t} \|U_0\|_{D(\mathcal{A})}^2. \quad (0.2.20)$$

Moreover, the energy decay rate (0.2.20) is optimal.

In order to establish this result, using a result from [90], it is sufficient to prove that the real part, and the imaginary part of the corresponding eigenvalues are in fact bounded. Moreover, one has to prove that the corresponding eigenvectors form a Riesz basis of the energy space in consideration [90].

Hereafter, we study the exact controllability of the following system of weakly coupled wave equations with Neumann boundary conditions, propagating with equal speeds, described by

$$\begin{cases} u_{tt} - u_{xx} + \alpha y = 0 & \text{on } (0, 1) \times (0, T), \\ y_{tt} - y_{xx} + \alpha u = 0 & \text{on } (0, 1) \times (0, T), \\ u(0, t) = y(0, t) = y(1, t) = 0 & \text{for all } t \in (0, T), \\ u_x(1, t) = v(t) & \text{for all } t \in (0, T). \end{cases} \quad (0.2.21)$$

The control  $v$  is applied only at the right boundary of the first equation. The second equation is indirectly controlled by means of the coupling between the equations. We consider the indirect boundary exact controllability problem: For given  $T > 0$  and initial data  $(u_0, u_1, y_0, y_1)$  belonging to a suitable space, is it possible to find a suitable control  $v$  such that the solution of system (0.2.21)  $(u, u_t, y, y_t)$  is driven to zero at time  $T$ , i.e.,

$$u(x, T) = u_t(x, T) = y(x, T) = y_t(x, T) = 0 \quad \text{on } (0, 1).$$

**Literature.** Concerning coupled hyperbolic-hyperbolic systems, several results concerning both stabilization and observability via two control forces have been obtained. Complete and partial observability (respectively, controllability) results for coupled systems either of hyperbolic-hyperbolic type or of hyperbolic-parabolic type can be found in [79]. These results assume that the coupling parameter is sufficiently small. They have been extended in [71] to the cases of arbitrary coupling parameters (assuming bounded coupling operators). For both references, the multiplier method was the main ingredient for obtaining the desired estimates. Complete observability (respectively, controllability) results have also been obtained in [75] for systems of coupled second order hyperbolic equations containing first order terms in both the original and the coupled unknowns. These results are based on Carleman estimates. Stabilization and observability results for hyperbolic-hyperbolic systems via a single control force have been considered more recently. In [8] (see also references therein), polynomial decay estimates in the case of indirect internal stabilization case are given. These results have been extended to several (wave-wave coupling, Petrowsky-Petrowsky coupling) for the locally distributed indirect stabilization case in [29]. Moreover, Alabau in [6], proved that it is possible to reach any target state in  $(H_0^1 \times L^2)^2(\Omega)$ , for a sufficiently large time. These results have been generalized in [12] and [9], where she considered a multidimensional setting of two weakly coupled wave equations, and proved its exact controllability using a so called indirect observability method: that is to observe only one component of the unknown on the boundary and know whether this observation can give back the initial energy of components of the solution.

**Contribution.** We prove exact controllability for system (0.2.21). Indeed, we establish observability inequalities using Ingham’s theorem ([73]). In order to do so, we consider the associated homogeneous problem, that is to say, the null boundary acted control system. Hence using the Hilbert Uniqueness Method ([79]), we deduce the indirect exact controllability of the considered system.

Let us denote by  $\lambda_{1,m}, \lambda_{2,n}$  the two branches of eigenvalues of the corresponding homogeneous system associated to (0.2.21), and denote by

$$\begin{cases} E_{1,n} = (\phi_{1,n}, \lambda_{1,n}\phi_{1,n}, \psi_{1,n}, \lambda_{1,n}\psi_{1,n}), \\ E_{2,n} = (\phi_{2,n}, \lambda_{2,n}\phi_{2,n}, \psi_{2,n}, \lambda_{2,n}\psi_{2,n}) \end{cases} \quad (0.2.22)$$

the corresponding eigenvectors. We prove in Proposition 4.4.1 of Chapter 4, that there exist infinitely many  $m \sim n$  such that the eigenvalues satisfy a standard gap

$$\lambda_{1,m} - \lambda_{2,n} \sim O(1). \quad (0.2.23)$$

But the observation is on the first components of the corresponding eigenvectors  $E_{1,n}, E_{2,n}$ . Following Proposition 4.4.2 in Chapter 4, we have

$$\phi_{1,n}(1) = O\left(\frac{1}{n}\right), \quad \phi_{2,n}(1) = O\left(\frac{1}{n^2}\right). \quad (0.2.24)$$

Hence, the following weighted spectral space is the natural space for observability

$$\mathcal{D} = \{(\phi_0, \phi_1, \psi_0, \psi_1) = \sum_{n \neq 0} (\alpha_{1,n} n E_{1,n} + \alpha_{2,n} n^2 E_{2,n})\}. \quad (0.2.25)$$

In (0.2.25), the factors  $n$  and  $n^2$  are due to the transmission of the modes between the two equations.

We prove the following result.

**Theorem 0.2.13** (Observability inequalities for Neumann boundary control). *Let  $\alpha \neq 0$  be a real number small enough. Assume that*

$$T > 4.$$

*Then there exists a constant  $c_1 > 0$  such that the direct observability inequality holds*

$$\int_0^T |\phi(1, t)|^2 dt \leq c_1 \|(\phi_0, \phi_1, \psi_0, \psi_1)\|_{\mathcal{H}}^2, \quad (0.2.26)$$

*for all solution  $\Phi = (\phi, \phi_t, \psi, \psi_t)$  solving the corresponding homogeneous Cauchy problem.*

*Moreover, there exists a constant  $0 < c < c_1$ , such that the following inverse observability inequality holds*

$$c \|(\phi_0, \phi_1, \psi_0, \psi_1)\|_{\mathcal{D}}^2 \leq \int_0^T |\phi(1, t)|^2 dt. \quad (0.2.27)$$

It is well known that the observability of the corresponding homogeneous Cauchy problem leads to the exact controllability of the considered systems [79]. After characterizing the corresponding weighted spectral space, we deduce the following exact controllability result.

**Theorem 0.2.14.** *Let  $\alpha \neq 0$  be a real number small enough. Assume that*

$$T > 4.$$

*Let*

$$(u_0, u_1, y_0, y_1) \in (H^1 \times L^2 \times H^2 \times H^1)(0, 1),$$

*then there exists  $v \in L^2(0, T)$  such that the solution of the non homogeneous system (0.2.21) satisfies the null final conditions*

$$u(x, t) = u_t(x, t) = y(x, t) = y_t(x, t) = 0, \quad \forall t \geq T.$$

## 0.3 Organization of the thesis

This thesis is constituted of four chapters organized into two parts. Some chapters of this manuscript are from items being prepared, or submitted.

Chapters 1 and 2 of Part I correspond to the study of numerical approximations of Hamilton-Jacobi equations posed on a junction. The first chapter deals with finite difference schemes of a first order Hamilton-Jacobi equation posed on a junction, where we derive error estimates for flux limited junction conditions, and prove convergence under general junction conditions. We ameliorate in the second chapter, by using a new approach, the result of the error estimate obtained in the first chapter, for a larger class of Hamiltonians.

Chapters 3 and 4 of Part II are concerned with the study of indirect controllability and stability of coupled wave equations. The third chapter is dedicated to the proof of the exact controllability of a system of wave equations coupled through velocity terms, in the one dimensional setting, while propagating with equal or different speeds of propagation. While in the fourth chapter, we study stabilization and exact controllability of coupled wave equations coupled through zero order terms, in the one dimensional setting.



# Part I

## Numerical analysis for Hamilton-Jacobi equations on networks



# Chapter 1

## Error estimates for finite difference schemes associated with Hamilton-Jacobi equations on a junction

*Ce chapitre vise à étudier un schéma monotone de différences finies associé à une équation de Hamilton-Jacobi du premier ordre. Ce schéma est une extension du schéma déjà introduit par Costeseque, Lebacque et Monneau (2013) pour des conditions de jonctions générales. D'une part, nous démontrons la convergence de la solution numérique vers la solution de l'équation de Hamilton-Jacobi, quand la taille du maillage tend vers zéro, pour les conditions de jonction générales. D'autre part, nous dérivons des estimations d'erreurs d'ordre  $(\Delta x)^{\frac{1}{2}}$  dans  $L_{loc}^{\infty}$  pour des conditions de jonction du type contrôle optimal si les minimums des Hamiltoniens sont égaux.*

**Abstract.** This chapter is concerned with monotone finite difference schemes associated with first order Hamilton-Jacobi equations posed on a junction. They extend the schemes recently introduced by Costeseque, Lebacque and Monneau (2013) to general junction conditions. On one hand, we prove the convergence of the numerical solution towards the weak (viscosity) solution of the Hamilton-Jacobi equation as the mesh size tends to zero for general junction conditions. On the other hand, we derive error estimates of order  $(\Delta x)^{\frac{1}{2}}$  in  $L_{\text{loc}}^{\infty}$  for junction conditions of optimal-control type if the minima of the Hamiltonians are equal.

## 1.1 Introduction

This chapter is concerned with numerical approximation of first order Hamilton-Jacobi equations posed on a junction, that is to say a network made of one node and a finite number of edges.

The theory of weak (viscosity) solutions for such equations on such domains has reached maturity by now [2, 65, 67, 102, 103]. In particular, it is now understood that general junction conditions reduce to special ones of optimal-control type [64–66]. Roughly speaking, it is proved in [65] that imposing a junction condition ensuring the existence of a continuous viscosity solution and a comparison principle is equivalent to imposing a junction condition obtained by “limiting the flux” at the junction point. Very recently, Lions and Souganidis [81, 82] wrote a note about a new approach for Hamilton-Jacobi equations posed on junctions with coercive Hamiltonians that are possibly not convex.

For the “maximal” *flux-limited* junction conditions, Costeseque, Lebacque and Monneau [41] introduced a monotone numerical scheme and proved its convergence. Their scheme can be naturally extended to general junction conditions and our first contribution is to introduce such a generalization and to prove its convergence.

Our second and main result is an error estimate à la Crandall-Lions [42] in the case of flux-limited junction conditions. It is explained in [42] that the proof of the comparison principle between sub- and super-solutions of the continuous Hamilton-Jacobi equation can be adapted in order to derive error estimates between the numerical solution associated with monotone (stable and consistent) schemes and the continuous solution. In the Euclidean case, the comparison principle is proved thanks to the technique of doubling variables; it relies on the classical penalization term  $\varepsilon^{-1}|x - y|^2$ . Such a penalization procedure is known to fail in general if the equation is posed in a junction; it is explained in [65] that it has to be replaced with a *vertex test function*.

In order to derive error estimates as in [42], it is important to study the regularity of the vertex test function. More precisely, we prove (Proposition 1.5.1) that it can be constructed in such a way that its gradient is locally Lipschitz continuous, at least if the minima of the Hamiltonians on different branches are equal. In fact, we see that the second order derivatives of the regularized vertex test function explode

near the origin when the minima are not equal, that is why we restrict ourselves to the case where no regularization is needed.

### 1.1.1 Hamilton-Jacobi equations posed on junctions

A *junction* is a network made of one node and a finite number of infinite edges. It can be viewed as the set of  $N$  distinct copies ( $N \geq 1$ ) of the half-line which are glued at the origin. For  $\alpha = 1, \dots, N$ , each branch  $J_\alpha$  is assumed to be isometric to  $[0, +\infty)$  and

$$J = \bigcup_{\alpha=1, \dots, N} J_\alpha \quad \text{with} \quad J_\alpha \cap J_\beta = \{0\} \quad \text{for} \quad \alpha \neq \beta$$

where the origin 0 is called the *junction point*. For points  $x, y \in J$ ,  $d(x, y)$  denotes the geodesic distance on  $J$  defined as

$$d(x, y) = \begin{cases} |x - y| & \text{if } x, y \text{ belong to the same branch,} \\ |x| + |y| & \text{if } x, y \text{ belong to different branches.} \end{cases}$$

With such a notation in hand, we consider the following Hamilton-Jacobi equation posed on the junction  $J$ ,

$$\begin{cases} u_t + H_\alpha(u_x) = 0 & \text{in } (0, T) \times J_\alpha \setminus \{0\}, \\ u_t + F\left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_N}\right) = 0 & \text{in } (0, T) \times \{0\}, \end{cases} \quad (1.1.1)$$

submitted to the initial condition

$$u(0, x) = u_0(x), \quad \text{for } x \in J \quad (1.1.2)$$

where  $u_0$  is globally Lipschitz in  $J$ . The second equation in (1.1.1) is referred to as the *junction condition*. We consider the important case of Hamiltonians  $H_\alpha$  satisfying the following conditions:

$$\text{There exists } p_0^\alpha \in \mathbb{R} \text{ such that } \begin{cases} H_\alpha \in C^2(\mathbb{R}) \text{ and } H_\alpha''(p_0^\alpha) > 0 \\ \pm H_\alpha'(p) > 0 \text{ for } \pm(p - p_0^\alpha) > 0 \\ \lim_{|p| \rightarrow +\infty} H_\alpha(p) = +\infty. \end{cases} \quad (1.1.3)$$

In particular  $H_\alpha$  is non-increasing in  $(-\infty, p_0^\alpha]$  and non-decreasing in  $[p_0^\alpha, +\infty)$ , and we set

$$H_\alpha^-(p) = \begin{cases} H_\alpha(p) & \text{for } p \leq p_0^\alpha \\ H_\alpha(p_0^\alpha) & \text{for } p \geq p_0^\alpha \end{cases} \quad \text{and} \quad H_\alpha^+(p) = \begin{cases} H_\alpha(p_0^\alpha) & \text{for } p \leq p_0^\alpha \\ H_\alpha(p) & \text{for } p \geq p_0^\alpha \end{cases}$$

where  $H_\alpha^-$  is non-increasing and  $H_\alpha^+$  is non-decreasing.

We assume in addition that,

$$\text{For all } \alpha, \beta \in \{1, \dots, N\}, \quad \min H_\alpha = \min H_\beta. \quad (1.1.4)$$

We next introduce a one-parameter family of junction conditions: given a *flux limiter*  $A \in \mathbb{R} \cup \{-\infty\}$ , the  $A$ -limited flux junction function is defined for  $p = (p_1, \dots, p_N)$  as,

$$F_A(p) = \max \left( A, \max_{\alpha=1, \dots, N} H_\alpha^-(p_\alpha) \right) \quad (1.1.5)$$

for some given  $A \in \mathbb{R} \cup \{-\infty\}$  where  $H_\alpha^-$  is non-increasing part of  $H_\alpha$ .

We now consider the following important special case of (1.1.1),

$$\begin{cases} u_t + H_\alpha(u_x) = 0 & \text{in } (0, T) \times J_\alpha \setminus \{0\}, \\ u_t + F_A(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_N}) = 0 & \text{in } (0, T) \times \{0\}. \end{cases} \quad (1.1.6)$$

We point out that all the junction functions  $F_A$  associated with  $A \in [-\infty, A_0]$  coincide if one chooses

$$A_0 = \max_{\alpha=1, \dots, N} \min_{\mathbb{R}} H_\alpha. \quad (1.1.7)$$

As far as general junction conditions are concerned, we assume that the junction function  $F : \mathbb{R}^n \mapsto \mathbb{R}$  satisfies

$$\begin{cases} F \text{ is continuous and piecewise } C^1(\mathbb{R}^n), \\ \forall \alpha, \forall p = (p_1, \dots, p_N) \in \mathbb{R}^N, \frac{\partial F}{\partial p_\alpha}(p) < 0, \\ F(p_1, \dots, p_N) \rightarrow +\infty \text{ as } \min_{i \in \{1, \dots, N\}} p_i \rightarrow -\infty. \end{cases} \quad (1.1.8)$$

### 1.1.2 Presentation of the scheme

The domain  $(0, +\infty) \times J$  is discretized with respect to time and space. We choose a regular grid in order to simplify the presentation but it is clear that more general meshes could be used here. The space step is denoted by  $\Delta x$  and the time step by  $\Delta t$ . If  $h$  denotes  $(\Delta t, \Delta x)$ , the mesh (or grid)  $\mathcal{G}_h$  is chosen as

$$\mathcal{G}_h = \{n\Delta t : n \in \mathbb{N}\} \times J^{\Delta x}$$

where

$$J^{\Delta x} = \bigcup_{\alpha=1, \dots, N} J_\alpha^{\Delta x} \quad \text{with} \quad J_\alpha \supset J_\alpha^{\Delta x} \simeq \{i\Delta x : i \in \mathbb{N}\}.$$

It is convenient to write  $x_i^\alpha$  for  $i\Delta x \in J_\alpha$ .

A numerical approximation  $u^h$  of the solution  $u$  of the Hamilton-Jacobi equation is defined on  $\mathcal{G}_h$ ; the quantity  $u^h(n\Delta t, x_i^\alpha)$  is simply denoted by  $U_i^{\alpha, n}$ . We want it to be an approximation of  $u(n\Delta t, x_i^\alpha)$  for  $n \in \mathbb{N}$ ,  $i \in \mathbb{N}$ , where  $\alpha$  stands for the index of the branch.

We consider the following time-explicit scheme: for  $n \geq 0$ ,

$$\begin{cases} \frac{U_i^{\alpha, n+1} - U_i^{\alpha, n}}{\Delta t} + \max\{H_\alpha^+(p_{i,-}^{\alpha, n}), H_\alpha^-(p_{i,+}^{\alpha, n})\} = 0, & i \geq 1, \quad \alpha = 1, \dots, N \\ U_0^{\beta, n} := U_0^n, & i = 0, \quad \beta = 1, \dots, N \\ \frac{U_0^{n+1} - U_0^n}{\Delta t} + F(p_{0,+}^{1, n}, \dots, p_{0,+}^{N, n}) = 0, \end{cases} \quad (1.1.9)$$

where  $p_{i,\pm}^{\alpha,n}$  are the discrete (space) gradients defined by

$$p_{i,+}^{\alpha,n} := \frac{U_{i+1}^{\alpha,n} - U_i^{\alpha,n}}{\Delta x}, \quad p_{i,-}^{\alpha,n} := \frac{U_i^{\alpha,n} - U_{i-1}^{\alpha,n}}{\Delta x} \quad (1.1.10)$$

with the initial condition

$$U_i^{\alpha,0} = u_0(x_i^\alpha), \quad i \geq 0, \quad \alpha = 1, \dots, N. \quad (1.1.11)$$

The following Courant-Friedrichs-Lewy (CFL) condition ensures that the explicit scheme is monotone,

$$\frac{\Delta x}{\Delta t} \geq \max \left\{ \max_{\substack{i \geq 0, \\ \alpha=1, \dots, N, \\ 0 \leq n \leq n_T}} |H'_\alpha(p_{i,+}^{\alpha,n})|; \max_{0 \leq n \leq n_T} \left\{ (-\nabla \cdot F)(p_{0,+}^{1,n}, \dots, p_{0,+}^{N,n}) \right\} \right\} \quad (1.1.12)$$

where the integer  $n_T$  is the integer part of  $\frac{T}{\Delta t}$  for a given  $T > 0$ .

### 1.1.3 Main results

As previously noticed in [41] in the special case  $F = F_{A_0}$ , it is not clear that the time step  $\Delta t$  and space step  $\Delta x$  can be chosen in such a way that the CFL condition (1.1.12) holds true since the discrete gradients  $p_{i,+}^{\alpha,n}$  depend itself on  $\Delta t$  and  $\Delta x$  (through the numerical scheme). We thus impose a more stringent CFL condition,

$$\frac{\Delta x}{\Delta t} \geq \max \left\{ \max_{\substack{\alpha=1, \dots, N, \\ \underline{p}_\alpha \leq p \leq \bar{p}_\alpha}} |H'_\alpha(p)|; \max_{\underline{p}_\alpha \leq p_\alpha \leq \bar{p}_\alpha} \left\{ (-\nabla \cdot F)(p_1, \dots, p_N) \right\} \right\} \quad (1.1.13)$$

for some  $\underline{p}_\alpha, \bar{p}_\alpha, p_\alpha^0 \in \mathbb{R}$  to be fixed (only depending on  $u_0$ ,  $H$ , and  $F$ ). We can argue as in [41] and prove that  $\underline{p}_\alpha, \bar{p}_\alpha, p_\alpha^0 \in \mathbb{R}$  can be chosen in such a way that the CFL condition (1.1.13) implies (1.1.12) and, in turn, the scheme is monotone (Lemma 1.4.1 in Section 1.4). We will also see that it is stable (Lemma 1.4.5) and consistent (Lemma 1.4.6). It is thus known that it converges [21, 42]. Notice that taking  $F = F_A$ , gives the following CFL condition

$$\frac{\Delta x}{\Delta t} \geq \max_{\substack{\alpha=1, \dots, N, \\ \underline{p}_\alpha \leq p \leq \bar{p}_\alpha}} |H'_\alpha(p)|. \quad (1.1.14)$$

**Theorem 1.1.1 (Convergence for general junction conditions).** *Let  $T > 0$  and  $u_0$  be Lipschitz continuous. There exist  $\underline{p}_\alpha, \bar{p}_\alpha, p_\alpha^0 \in \mathbb{R}$ ,  $\alpha = 1, \dots, N$ , depending only on the initial data, the Hamiltonians and the junction function  $F$ , such that, if  $h = (\Delta t, \Delta x)$  satisfies the CFL condition (1.1.13), then the numerical solution  $u^h$  defined by (1.1.9), (1.1.11) converges locally uniformly as  $h$  goes to zero to the unique weak (relaxed viscosity) solution  $u$  of (1.1.1)-(1.1.2), on any compact set  $\mathcal{K} \subset [0, T) \times J$ , i.e.*

$$\limsup_{|h| \rightarrow 0} \sup_{(t,x) \in \mathcal{K} \cap \mathcal{G}_h} |u^h(t, x) - u(t, x)| = 0. \quad (1.1.15)$$

*Remark 1.1.2.* We know from [65] that the equation (1.1.1)-(1.1.2) have a unique relaxed viscosity solution in the sense of Definition 1.2.2. For particular junction conditions, i.e., replacing  $F$  by  $F_A$  the unique viscosity solution is defined in the sense of Definition 1.2.1. Notice that the scheme has a junction condition which is not relaxed. However the solution of the scheme converges to the unique relaxed solution of the associated Hamilton-Jacobi equation.

The main result of this chapter lies in getting error estimates in the case of flux-limited junction conditions, under the assumption (1.1.4).

**Theorem 1.1.3 (Error estimates for flux-limited junction conditions).** *Assume that the Hamiltonians  $H_\alpha$  satisfy (1.1.3)-(1.1.4). Let  $u_0$  be Lipschitz continuous,  $u^h$  be the solution of the associated numerical scheme (1.1.9)-(1.1.11) and  $u$  be the weak (viscosity) solution of (1.1.6)-(1.1.2) for some  $A = A_0 \in \mathbb{R}$ . If the CFL condition (1.1.14) is satisfied, then there exists  $C > 0$  (independent of  $h$ ) such that*

$$\sup_{[0,T) \times J \cap \mathcal{G}_h} |u^h(t, x) - u(t, x)| \leq C(\Delta x)^{1/2}. \quad (1.1.16)$$

### 1.1.4 Related results

**Numerical schemes for Hamilton-Jacobi equations on networks.** The discretization of weak (viscosity) solutions of Hamilton-Jacobi equations posed on networks has been studied in few papers only. Apart from [41] mentioned above, we are only aware of three other works. A convergent semi-Lagrangian scheme is introduced in [33] for equations of eikonal type. In [57], an adapted Lax-Friedrichs scheme is used to solve a traffic model; it is worth mentioning that this discretization implies to pass from the scalar conservation law to the associated Hamilton-Jacobi equation at each time step. In [59], Guerand and Koumaiha (see Chapter 2) improved the error estimate for a larger class of Hamiltonians. Their approach is slightly different from our approach, they use a function relative to the optimal control interpretation of the problem, in the penalization procedure.

**Link with monotone schemes for scalar conservation laws.** We first follow [41] by emphasizing that the convergence result, Theorem 1.1.1, implies the convergence of a monotone scheme for scalar conservation laws (in the sense of distributions). In fact, this scheme recovers the classical Godunov scheme [55].

In order to introduce the scheme, it is useful to introduce a notation for the numerical Hamiltonian  $\mathcal{H}_\alpha$ ,

$$\mathcal{H}_\alpha(p^+, p^-) = \max\{H_\alpha^-(p^+), H_\alpha^+(p^-)\}.$$

The discrete solution  $(V^n)$  of the scalar conservation law is defined as follows,

$$V_{i+\frac{1}{2}}^{\alpha,n} = \begin{cases} \frac{U_{i+1}^{\alpha,n} - U_i^{\alpha,n}}{\Delta x} & \text{if } i \geq 1 \\ \frac{U_1^{\alpha,n} - U_0^n}{\Delta x} & \text{if } i = 0. \end{cases}$$

In view of (1.1.9), it satisfies for all  $\alpha = 1, \dots, N$ ,

$$\begin{cases} \frac{V_{i+\frac{1}{2}}^{\alpha,n+1} - V_{i+\frac{1}{2}}^{\alpha,n}}{\Delta t} + (\Delta x)^{-1} \left( \mathcal{H}_\alpha(V_{i+\frac{3}{2}}^{\alpha,n}, V_{i+\frac{1}{2}}^{\alpha,n}) - \mathcal{H}_\alpha(V_{i+\frac{1}{2}}^{\alpha,n}, V_{i-\frac{1}{2}}^{\alpha,n}) \right) = 0, & i \geq 1, \\ \frac{V_{\frac{1}{2}}^{\alpha,n+1} - V_{\frac{1}{2}}^{\alpha,n}}{\Delta t} + (\Delta x)^{-1} \left( \mathcal{H}_\alpha(V_{\frac{3}{2}}^{\alpha,n}, V_{\frac{1}{2}}^{\alpha,n}) - F(V_{\frac{1}{2}}^{1,n}, \dots, V_{\frac{1}{2}}^{N,n}) \right) = 0 \end{cases}$$

submitted to the initial condition

$$V_{i+\frac{1}{2}}^{\alpha,0} = \frac{u_0(x_i^\alpha) - u_0(0)}{\Delta x}, \quad i \geq 0, \quad \alpha = 1, \dots, N.$$

It is worth mentioning that our scheme In view of Theorem 1.1.1, we thus can conclude that the discrete solution  $v^h$  constructed from  $(V^n)$  converges towards  $u_x$  in the sense of distributions, at least far from the junction point.

**Scalar conservation laws with Dirichlet boundary conditions and constrained fluxes.** We would like next to explain why our result can be seen as the Hamilton-Jacobi counterpart of the error estimates obtained by Ohlberger and Vovelle [93] for scalar conservation laws submitted to Dirichlet boundary conditions.

On the one hand, it is known since 1979 and Bardos, Le Roux and Nedelec [20] that Dirichlet boundary conditions imposed to scalar conservation laws should be understood in a generalized sense. This can be seen by studying the parabolic regularization of the problem. A boundary layer analysis can be performed for systems if the solution of the conservation law is smooth; see for instance [54, 60]. Depending on the fact that the boundary is characteristic or not, the error is  $\varepsilon^{\frac{1}{2}}$  or  $\varepsilon$ . In the scalar case, it is proved in [46] that the error between the solution of the regularized equation with a vanishing viscosity coefficient equal to  $\varepsilon$  and the entropy solution of the conservation law (which is merely of bounded variation in space) is of order  $\varepsilon^{1/3}$  (in  $L_t^\infty L_x^1$  norm). In [93], the authors derive error estimates for finite volume schemes associated with such boundary value problems and prove that it is of order  $(\Delta x)^{1/6}$  (in  $L_{t,x}^1$  norm). More recently, scalar conservation laws with flux constraints were studied [39, 40] and some finite volume schemes were built [17]. In [34], assuming that the flux is bell-shaped, that is to say the opposite is quasi-convex, it is proved that the error between the finite volume scheme and the entropy solution is of order  $(\Delta x)^{\frac{1}{3}}$  and that it can be improved to  $(\Delta x)^{\frac{1}{2}}$  under an additional condition on the traces of the BV entropy solution. It is not known if the estimates from [34] are optimal or not.

On the other hand, the derivative of a weak (viscosity) solution of a Hamilton-Jacobi equation posed on the real line is known to coincide with the entropy solution of the corresponding scalar conservation law. It is therefore reasonable to expect that the error between the viscosity solution of the Hamilton-Jacobi equation and its approximation is as good as the one obtained between the entropy solution of the scalar conservation law and its approximation.

Moreover, it is explained in [67] that the junction conditions of optimal-control type are related to the BLN condition mentioned above; such a correspondence is recalled

in Appendix 1.9. It is therefore interesting to get an error estimate of order  $(\Delta x)^{1/2}$  for the Hamilton-Jacobi problem.

### 1.1.5 Open problems

Let us first mention that it is not known if the error estimate between the (entropy) solution of the scalar conservation law with Dirichlet boundary condition and the solution of the parabolic approximation [46] or with the numerical scheme [93] is optimal or not. Here, we prove an optimal error estimate for  $A = A_0$ , for Hamiltonians satisfying (1.1.4). As the second order derivatives of the vertex test function  $G$  are not bounded near the diagonal for  $x = y = 0$ , we can not derive error estimates for  $A > A_0$ , or in the case where (1.1.4) is not satisfied.

Deriving error estimates for general junction conditions seems difficult to us. The main difficulty is the singular geometry of the domain. The vertex test function, used in deducing the error estimates with flux limited solutions, is designed to compare flux limited solutions. Consequently, when applying the reasoning of Section 1.6, the discrete viscosity inequality cannot be combined with the continuous one. We expect that a layer develops between the continuous solution and the discrete scheme at the junction point.

**Organization of the chapter.** The remaining of the chapter is organized as follows. In Section 1.2, we recall definitions and results from [65] about viscosity solutions for (1.1.1)-(1.1.2) and the so-called vertex test function. Section 1.3 is dedicated to the derivation of discrete gradient estimates for the numerical scheme. In Section 1.4, the convergence result, Theorem 1.1.1 is proved. In Section 1.5, it is proved that the vertex test function constructed in [65] can be chosen so that the gradient is locally Lipschitz continuous if  $A = A_0$  and if (1.1.4) holds true. The final section, Section 1.6, is dedicated to the proof of the error estimates.

## 1.2 Preliminaries

### 1.2.1 Viscosity solutions

We introduce the main definitions related to viscosity solutions for Hamilton-Jacobi equations that are used in the remainder. For a more general introduction to viscosity solutions, the reader is referred to Barles [24] and to Crandall, Ishii, Lions [45].

**Space of test functions.** For a real valued function  $u$  defined on  $J$ , we denote by  $u^\alpha$  the restriction of  $u$  to  $(0, T) \times J_\alpha$ .

Then we define the natural space of functions on the junction:

$$C^1(J_T) = \{u \in C(J_T) : \forall \alpha = 1, \dots, N, u^\alpha \in C^1((0, T) \times J_\alpha)\}.$$

**Viscosity solutions.** In order to define classical viscosity solutions, we recall the definition of upper and lower semi-continuous envelopes  $u^*$  and  $u_*$  of a (locally bounded) function  $u$  defined on  $[0, T) \times J$ :

$$u^*(t, x) = \limsup_{(s, y) \rightarrow (t, x)} u(s, y) \quad u_*(t, x) = \liminf_{(s, y) \rightarrow (t, x)} u(s, y).$$

**Definition 1.2.1 (Viscosity solution).** Assume that the Hamiltonians satisfy (1.1.3) and that  $F$  satisfies (1.1.8) and let  $u: (0, T) \times J \rightarrow \mathbb{R}$ .

(i) We say that  $u$  is a *sub-solution* (resp. *super-solution*) of (1.1.1) in  $(0, T) \times J$  if for all test function  $\varphi \in C^1(J_T)$  such that

$$u^* \leq \varphi \quad (\text{resp. } u_* \geq \varphi) \quad \text{in a neighborhood of } (t_0, x_0) \in J_T$$

with equality at  $(t_0, x_0)$  for some  $t_0 > 0$ , we have

$$\varphi_t + H(\varphi_x) \leq 0 \quad (\text{resp. } \geq 0) \quad \text{at } (t_0, x_0)$$

if  $x_0 \neq 0$ , else

$$\varphi_t + F\left(\frac{\partial \varphi}{\partial x_1}, \dots, \frac{\partial \varphi}{\partial x_N}\right) \leq 0 \quad (\text{resp. } \geq 0) \quad \text{at } (t_0, x_0) = (t_0, 0).$$

(ii) We say that  $u$  is a *sub-solution* (resp. *super-solution*) of (1.1.1)-(1.1.2) on  $[0, T) \times J$  if additionally

$$u^*(0, x) \leq u_0(x) \quad (\text{resp. } u_*(0, x) \geq u_0(x)) \quad \text{for all } x \in J.$$

(iii) We say that  $u$  is a (viscosity) solution of (1.1.1)-(1.1.2) if  $u$  is both a sub-solution and a super-solution of (1.1.1)-(1.1.2).

As explained in [65], it is difficult to construct viscosity solutions in the sense of Definition 1.2.1 because of the junction condition. It is possible in the case of the flux-limited junction conditions  $F_A$ . For general junction conditions, the Perron process generates a viscosity solution in the following relaxed sense [65].

**Definition 1.2.2 (Relaxed viscosity solution).** Assume that the Hamiltonians satisfy (1.1.3) and that  $F$  satisfies (1.1.8) and let  $u: (0, T) \times J \rightarrow \mathbb{R}$ .

( i ) We say that  $u$  is a *relaxed sub-solution* (resp. *relaxed super-solution*) of (1.1.1) in  $(0, T) \times J$  if for all test function  $\varphi \in C^1(J_T)$  such that

$$u^* \leq \varphi \quad (\text{resp. } u_* \geq \varphi) \quad \text{in a neighborhood of } (t_0, x_0) \in J_T$$

with equality at  $(t_0, x_0)$  for some  $t_0 > 0$ , we have

$$\varphi_t + H_\alpha(\varphi_x) \leq 0 \quad (\text{resp. } \geq 0) \quad \text{at } (t_0, x_0)$$

if  $x_0 \neq 0$ , else

$$\begin{cases} \text{either } \varphi_t + F(\frac{\partial \varphi}{\partial x_1}, \dots, \frac{\partial \varphi}{\partial x_N}) \leq 0 & (\text{resp. } \geq 0) & \text{at } (t_0, x_0) = (t_0, 0) \\ \text{or } \varphi_t + H_\alpha(\frac{\partial \varphi}{\partial x_\alpha}) \leq 0 & (\text{resp. } \geq 0) & \text{at } (t_0, x_0) = (t_0, 0) \quad \text{for some } \alpha. \end{cases}$$

( ii ) We say that  $u$  is a *relaxed (viscosity) solution* of (1.1.1) if  $u$  is both a sub-solution and a super-solution of (1.1.1).

Let us recall some theorems in [65].

**Theorem 1.2.3 (Comparison principle on a junction).** *Let  $A \in \mathbb{R} \cup \{-\infty\}$ . Assume that the Hamiltonians satisfy (1.1.3) and the initial datum  $u_0$  is uniformly continuous. Then for all sub-solution  $u$  and super-solution  $v$  of (1.1.6),(1.1.2) satisfying for some  $T > 0$  and  $C_T > 0$*

$$u(t, x) \leq C_T(1 + d(0, x)), \quad v(t, x) \geq -C_T(1 + d(0, x)), \quad \text{for all } (t, x) \in [0, T) \times J,$$

we have

$$u \leq v \quad \text{in } [0, T) \times J.$$

**Theorem 1.2.4 (General junction conditions reduce to flux-limited ones).** *Assume that the Hamiltonians satisfy (1.1.3) and that  $F$  satisfies (1.1.8). Then there exists  $A_F \in \mathbb{R}$  such that any relaxed viscosity (sub-/super-)solution of (1.1.1) is in fact a viscosity (sub-/super-)solution of (1.1.6) with  $A = A_F$ .*

**Theorem 1.2.5 (Existence and uniqueness on a junction).** *Assume that the Hamiltonians satisfy (1.1.3) and that  $F$  satisfies (1.1.8) and that the initial datum  $u_0$  is Lipschitz continuous. Then there exists a unique relaxed viscosity solution  $u$  of (1.1.1)-(1.1.2), such that*

$$|u(t, x) - u_0(x)| \leq Ct \quad \text{for all } (t, x) \in [0, T) \times J$$

for some constant  $C$  only depending on  $H$  and  $u_0$ . Moreover, it is Lipschitz continuous with respect to time and space, in particular,

$$\|\nabla u\|_\infty \leq C.$$

## 1.2.2 Vertex test function

In this subsection, we recall what a *vertex test function* is. It is introduced in [65] in order to prove a comparison principle for (1.1.1). This function  $G$  plays the role of  $|x - y|^2$  in the classical “doubling variables” method [45].

Let us just mention that on the one hand Guerand in [58] and Lions and Souganidis in [81, 82] on the other hand studied well-posedness and stability properties for Hamilton-Jacobi equations with non convex coercive Hamiltonians. Moreover, in [26], Barles et al. compared two different approaches for regional control problems. In fact, they showed how the results of the classical approach, using a standard notion of viscosity solutions, can be interpreted in terms of flux-limited solutions. In particular, they gave a simpler proof of the comparison principle, avoiding in particular the use of the vertex test function  $G$ . We wonder thus if we adapt their reasoning in the proof of the error estimate, whether we have to restrict ourselves in the case of Hamiltonians satisfying (1.1.4) for  $A = A_0$ , or not. And what error estimate could we obtain while considering non convex Hamiltonians as in [82].

We recall now the vertex test function  $G$  from [65].

**Theorem 1.2.6** (Vertex test function - the general case- [65]). *Let  $A \in \mathbb{R} \cup \{-\infty\}$  and  $\gamma > 0$ . Assume the Hamiltonians satisfy (1.1.3) and  $p_0^\alpha = 0$ , that is to say*

$$\min H_\alpha = H_\alpha(0). \quad (1.2.1)$$

*Then there exists a function  $G : J^2 \rightarrow \mathbb{R}$  enjoying the following properties.*

(i) (Regularity)

$$G \in C(J^2) \quad \text{and} \quad \begin{cases} G(x, \cdot) \in C^1(J) & \text{for all } x \in J, \\ G(\cdot, y) \in C^1(J) & \text{for all } y \in J. \end{cases}$$

(ii) (Bound from below)  $G \geq 0 = G(0, 0)$ .

(iii) (Compatibility condition on the diagonal) For all  $x \in J$

$$G(x, x) \leq \gamma. \quad (1.2.2)$$

(iv) (Compatibility condition on the gradients) For all  $(x, y) \in J^2$

$$\left\{ \begin{array}{l} H_\beta(-G_y(x, y)) - H_\alpha(G_x(x, y)) \leq \gamma \\ \hspace{15em} \text{if } x \in J_\alpha \setminus \{0\}, y \in J_\beta \setminus \{0\}, \\ H_\beta(-G_y(x, y)) - F_A\left(\frac{\partial G}{\partial x_1}(x, y), \dots, \frac{\partial G}{\partial x_N}(x, y)\right) \leq \gamma \\ \hspace{15em} \text{if } x = 0, y \in J_\beta \setminus \{0\}, \\ F_A\left(-\frac{\partial G}{\partial y_1}(x, y), \dots, -\frac{\partial G}{\partial y_N}(x, y)\right) - H_\alpha(G_x(x, y)) \leq \gamma \\ \hspace{15em} \text{if } x \in J_\alpha \setminus \{0\}, y = 0, \\ F_A\left(-\frac{\partial G}{\partial y_1}(x, y), \dots, -\frac{\partial G}{\partial y_N}(x, y)\right) - F_A\left(\frac{\partial G}{\partial x_1}(x, y), \dots, \frac{\partial G}{\partial x_N}(x, y)\right) \leq \gamma \\ \hspace{15em} \text{if } x = 0, y = 0. \end{array} \right. \quad (1.2.3)$$

(v) (Superlinearity) There exists  $g: [0, +\infty) \rightarrow \mathbb{R}$  non-decreasing and s.t. for  $(x, y) \in J^2$

$$g(d(x, y)) \leq G(x, y) \quad \text{and} \quad \lim_{a \rightarrow +\infty} \frac{g(a)}{a} = +\infty. \quad (1.2.4)$$

(vi) (Gradient bounds) For all  $K \geq 0$ , there exists  $C_K > 0$  such that for all  $(x, y) \in J^2$ ,

$$d(x, y) \leq K \quad \Rightarrow \quad |G_x(x, y)| + |G_y(x, y)| \leq C_K. \quad (1.2.5)$$

It is mentioned in [65] that the vertex test function  $G$  is obtained as a regularized version of  $A + G^0$  where  $G^0$  is defined, for  $\alpha, \beta = 1, \dots, N$ , by

$$G^0(x, y) := \sup_{(p, \lambda) \in \mathcal{G}(A)} (p_\alpha x - p_\beta y - \lambda) \quad \text{if} \quad (x, y) \in J_\alpha \times J_\beta \quad (1.2.6)$$

where  $\mathcal{G}(A)$  is referred to as the germ and is defined as follows

$$\mathcal{G}(A) = \begin{cases} \{(p, \lambda) \in \mathbb{R}^N \times \mathbb{R}, H_\alpha(p_\alpha) = F_A(p) = \lambda \quad \text{for} \quad \alpha = 1, \dots, N\} & \text{if} \quad N \geq 2, \\ \{(p_1, \lambda) \in \mathbb{R} \times \mathbb{R}, H_1(p_1) = \lambda \geq A\} & \text{if} \quad N = 1. \end{cases}$$

We recall that  $G^0$  is a  $C^1$  function except on the diagonal. Under assumptions (1.1.4) and (1.2.1),  $G^0 \in C^1(J^2)$ , and thus the vertex test function  $G$  is equal to  $A + G^0$ , for all  $(x, y) \in J_\alpha \times J_\beta$ . Hereafter,  $G^0$  satisfies the following properties.

**Theorem 1.2.7.** Assume the Hamiltonians satisfy (1.1.3), (1.1.4) and assume that  $p_0^\alpha = 0$ , that is to say,

$$\min H_\alpha = H_\alpha(0) = A_0, \quad \forall \alpha = 1, \dots, N. \quad (1.2.7)$$

Then the function  $G^0: J^2 \rightarrow \mathbb{R}$  enjoys the following properties.

(i) (Regularity)

$$G^0 \in C^1(J^2)$$

(ii) (Bound from below)  $G^0 \geq G^0(0, 0) = -A_0$ .

(iii) (Compatibility condition on the diagonal) For all  $x \in J$

$$G^0(x, x) = -A_0.$$

(iv) (Compatibility condition on the gradients) For all  $(x, y) \in J^2$

$$\left\{ \begin{array}{l} H_\beta(-G_y^0(x, y)) - H_\alpha(G_x^0(x, y)) \leq 0 \\ \hspace{15em} \text{if } x \in J_\alpha \setminus \{0\}, y \in J_\beta \setminus \{0\}, \\ H_\beta(-G_y^0(x, y)) - F_A(\frac{\partial G^0}{\partial x_1}(x, y), \dots, \frac{\partial G^0}{\partial x_N}(x, y)) \leq 0 \\ \hspace{15em} \text{if } x = 0, y \in J_\beta \setminus \{0\}, \\ F_{A_0}(-\frac{\partial G^0}{\partial y_1}(x, y), \dots, -\frac{\partial G^0}{\partial y_N}(x, y)) - H_\alpha(G_x^0(x, y)) \leq 0 \\ \hspace{15em} \text{if } x \in J_\alpha \setminus \{0\}, y = 0, \\ F_{A_0}(-\frac{\partial G^0}{\partial y_1}(x, y), \dots, -\frac{\partial G^0}{\partial y_N}(x, y)) - F_{A_0}(\frac{\partial G^0}{\partial x_1}(x, y), \dots, \frac{\partial G^0}{\partial x_N}(x, y)) \leq 0 \\ \hspace{15em} \text{if } x = 0, y = 0. \end{array} \right.$$

(v) (*Superlinearity*) There exists  $g^0: [0, +\infty) \rightarrow \mathbb{R}$  non-decreasing and s.t. for  $(x, y) \in J^2$

$$g^0(d(x, y)) \leq G^0(x, y) \quad \text{and} \quad \lim_{a \rightarrow +\infty} \frac{g^0(a)}{a} = +\infty.$$

(vi) (*Gradient bounds*) For all  $K \geq 0$ , there exists  $C_K > 0$  such that for all  $(x, y) \in J^2$ ,

$$d(x, y) \leq K \quad \Rightarrow \quad |G_x^0(x, y)| + |G_y^0(x, y)| \leq C_K.$$

*Remark 1.2.8.* We remark as in [65] that we can assume without loss of generality that the Hamiltonians satisfy the additional conditions (1.1.4) and (1.2.1). Indeed, if  $u$  solves (1.1.1) then

$$\tilde{u}(t, x) = u(t, x) - p_0^\alpha x \quad \text{for} \quad x \in J^\alpha$$

solves the same equation in which  $H_\alpha$  replaced with

$$\tilde{H}_\alpha(p) = H_\alpha(p_0^\alpha + p).$$

### 1.3 Gradient estimates for the scheme

This section is devoted to the proofs of the discrete (time and space) gradient estimates. These estimates ensure the monotonicity of the scheme and, in turn, its convergence.

**Theorem 1.3.1 (Discrete gradient estimates).** *If  $u^h = (U_i^{\alpha, n})$  is the numerical solution of (1.1.9)-(1.1.11) and if the CFL condition (1.1.13) is satisfied and if*

$$m^0 = \inf_{\substack{\beta=1, \dots, N, \\ i \in \mathbb{N}}} W_i^{\beta, 0} \tag{1.3.1}$$

*is finite, then the following two properties hold true for any  $n \geq 0$ .*

(i) (*Gradient estimate*) There exist  $\underline{p}_\alpha, \bar{p}_\alpha, \underline{p}_\alpha^0$  (only depending on  $H_\alpha, u_0$  and  $F$ ) such that

$$\begin{cases} \underline{p}_\alpha \leq p_{i,+}^{\alpha, n} \leq \bar{p}_\alpha & i \geq 1, \alpha = 1, \dots, N, \\ \underline{p}_\alpha^0 \leq p_{0,+}^{\alpha, n} \leq \bar{p}_\alpha & i = 0, \alpha = 1, \dots, N. \end{cases} \tag{1.3.2}$$

(ii) (*Time derivative estimate*) The discrete time derivative defined as

$$W_i^{\alpha, n} := \frac{U_i^{\alpha, n+1} - U_i^{\alpha, n}}{\Delta t}$$

satisfies

$$m^0 \leq m^n \leq m^{n+1} \leq M^{n+1} \leq M^n \leq M^0$$

where

$$m^n := \inf_{\alpha, i} W_i^{\alpha, n}, \quad M^n := \sup_{\alpha, i} W_i^{\alpha, n}.$$

In the proofs of discrete gradient estimates, as well as in the construction of the vertex test functions, “generalized” inverse functions of  $H_\alpha^\pm$  are needed; they are defined as follows:

$$\begin{cases} \pi_\alpha^+(a) := \sup\{p : H_\alpha^+(p) = \max(a, A_\alpha)\} \\ \pi_\alpha^-(a) := \inf\{p : H_\alpha^-(p) = \max(a, A_\alpha)\} \end{cases} \quad (1.3.3)$$

with the additional convention that  $(H_\alpha^\pm)^{-1}(+\infty) = \pm\infty$ , where

$$A_\alpha := \min_{\mathbb{R}} H_\alpha.$$

In order to define a “generalized” inverse function of  $F$ , we remark that (1.1.8) implies that

for all  $K \in \mathbb{R}$ , there exists  $\underline{\rho}(K) = (\rho_1(K), \dots, \rho_N(K)) \in \mathbb{R}^N$  such that

$$F(p_1, \dots, p_N) \leq K \Rightarrow p_\alpha \geq \underline{\rho}_\alpha(K).$$

Remark that the functions  $\underline{\rho}_\alpha$  can be chosen non-increasing.

*Remark 1.3.2.* The quantities  $\underline{p}_\alpha, \bar{p}_\alpha, \underline{p}_\alpha^0$  are defined as follows

$$\begin{cases} \underline{p}_\alpha = \begin{cases} \pi_\alpha^-(-m^0) & \text{if } -m_0 > A_\alpha \\ \pi_\alpha^-(-m^0 + 1) & \text{if } -m_0 = A_\alpha \end{cases} \\ \bar{p}_\alpha = \begin{cases} \pi_\alpha^+(-m^0) & \text{if } -m_0 > A_\alpha \\ \pi_\alpha^+(-m^0 + 1) & \text{if } -m_0 = A_\alpha \end{cases} \\ \underline{p}_\alpha^0 = \begin{cases} \underline{\rho}_\alpha(-m^0) & \text{if } \underline{\rho}_\alpha(-m^0) < \bar{p}_\alpha \\ \underline{\rho}_\alpha(-m^0 + 1) & \text{if } \underline{\rho}_\alpha(-m^0) = \bar{p}_\alpha \end{cases} \end{cases} \quad (1.3.4)$$

where  $m^0$  is defined in (1.3.1).

In order to establish Theorem 1.3.1, we first prove two auxiliary results. In order to state them, some notation should be introduced.

### 1.3.1 Discrete time derivative estimates

In order to state the first one, Proposition 1.3.3 below, we introduce some notation. For  $\sigma \in \{+, -\}$ , we set

$$I_{i,\sigma}^{\alpha,n} := [\min(p_{i,\sigma}^{\alpha,n}, p_{i,\sigma}^{\alpha,n+1}), \max(p_{i,\sigma}^{\alpha,n}, p_{i,\sigma}^{\alpha,n+1})]$$

with  $p_{i,\sigma}^{\alpha,n}$  defined in (1.1.10) and

$$D_{i,+}^{\alpha,n} := \sup \left\{ \sup_{p_\alpha \in I_{i,+}^{\alpha,n}} |H'_\alpha(p_\alpha)|, \sup_{p_\alpha \in I_{0,+}^{\alpha,n}} \left\{ -(\nabla \cdot F)(p_1, \dots, p_N) \right\} \right\}. \quad (1.3.5)$$

The following proposition asserts that if the discrete space gradients enjoy suitable estimates, then the discrete time derivative is controlled.

**Proposition 1.3.3 (Discrete time derivative estimate).** *Let  $n \geq 0$  be fixed and  $\Delta x, \Delta t > 0$ . Let us consider  $(U_{i,\alpha}^{\alpha,n})_{\alpha,i}$  satisfying for some constant  $C^n > 0$ :*

$$|p_{i,+}^{\alpha,n}| \leq C^n \quad \text{for } i \geq 0, \alpha = 1, \dots, N.$$

We also consider  $(U_i^{\alpha,n+1})_{\alpha,i}$  and  $(U_i^{\alpha,n+2})_{\alpha,i}$  computed using the scheme (1.1.9). If

$$D_{i,+}^{\alpha,n} \leq \frac{\Delta x}{\Delta t} \quad \text{for } i \geq 0, \alpha = 1, \dots, N, \quad (1.3.6)$$

then

$$m^n \leq m^{n+1} \leq M^{n+1} \leq M^n.$$

**Proof.** For  $\sigma = +$  (resp.  $\sigma = -$ ),  $-\sigma$  denotes  $-$  (resp.  $+$ ). We introduce for  $n \geq 0$ ,  $\alpha \in \{1, \dots, N\}$ ,  $i \in \{1, \dots, N\}$ ,  $\sigma \in \{+, -\}$ ,

$$\begin{aligned} C_{i,\sigma}^{\alpha,n} &:= -\sigma \int_0^1 (H_\alpha^{-\sigma})' (p_{i,\sigma}^{\alpha,n+1} + \tau(p_{i,\sigma}^{\alpha,n} - p_{i,\sigma}^{\alpha,n+1})) d\tau \geq 0, \\ C_{0,+}^{\alpha,n} &:= -\int_0^1 \frac{\partial F}{\partial p_\alpha} \left( \{p_{0,+}^{\beta,n+1} + \tau(p_{0,+}^{\beta,n} - p_{0,+}^{\beta,n+1})\}_\beta \right) d\tau \geq 0. \end{aligned} \quad (1.3.7)$$

Notice that for  $i \geq 1$ ,  $C_{i,\sigma}^{\alpha,n}$  is defined as the integral of  $(H_\alpha^{-\sigma})'$  over a convex combination of  $p \in I_{i,\sigma}^{\alpha,n}$ . Similarly for  $C_{0,+}^{\alpha,n}$  which is defined as the integral of  $F'$  on a convex combination of  $p \in I_{0,+}^{\alpha,n}$ . Hence, in view of (1.3.6), we have for any  $n \geq 0$ ,  $\alpha = 1, \dots, N$  and for any  $\sigma \in \{+, -\}$  or for  $i = 0$  and  $\sigma = +$ , we can check that

$$\begin{cases} C_{i,\sigma}^{\alpha,n} \leq \frac{\Delta x}{\Delta t} & \text{if } i \geq 1, \sigma \in \{-, +\} \\ \sum_{\beta=1}^N C_{0,+}^{\beta,n} \leq \frac{\Delta x}{\Delta t}. \end{cases} \quad (1.3.8)$$

We can also underline that for any  $n \geq 0$ ,  $\alpha = 1, \dots, N$  and for any  $i \geq 1$ ,  $\sigma \in \{+, -\}$  or for  $i = 0$  and  $\sigma = +$ , we have the following relationship:

$$\frac{p_{i,\sigma}^{\alpha,n} - p_{i,\sigma}^{\alpha,n+1}}{\Delta t} = -\sigma \frac{W_{i+\sigma}^{\alpha,n} - W_i^{\alpha,n}}{\Delta x}. \quad (1.3.9)$$

Let  $n \geq 0$  be fixed and consider  $(U_i^{\alpha,n})_{\alpha,i}$  with  $\Delta x, \Delta t > 0$  given. We compute  $(U_i^{\alpha,n+1})_{\alpha,i}$  and  $(U_i^{\alpha,n+2})_{\alpha,i}$  using the scheme (1.1.9).

**Step 1:  $(m^n)_n$  is non-decreasing.** We want to show that  $W_i^{\alpha,n+1} \geq m^n$  for  $i \geq 0$  and  $\alpha = 1, \dots, N$ . Let  $i \geq 0$  be fixed and let us distinguish two cases.

**Case 1:**  $i \geq 1$ . Let a branch  $\alpha$  be fixed and let  $\sigma(i, \alpha, n+1) = \sigma \in \{+, -\}$  be such that

$$\max \left\{ H_\alpha^+(p_{i,-}^{\alpha,n+1}), H_\alpha^-(p_{i,+}^{\alpha,n+1}) \right\} = H_\alpha^{-\sigma}(p_{i,\sigma}^{\alpha,n+1}). \quad (1.3.10)$$

We have

$$\begin{aligned} \frac{W_i^{\alpha,n+1} - W_i^{\alpha,n}}{\Delta t} &= \frac{1}{\Delta t} \left( \max \left\{ H_\alpha^+(p_{i,-}^{\alpha,n}), H_\alpha^-(p_{i,+}^{\alpha,n}) \right\} - \max \left\{ H_\alpha^+(p_{i,-}^{\alpha,n+1}), H_\alpha^-(p_{i,+}^{\alpha,n+1}) \right\} \right) \\ &\geq \frac{1}{\Delta t} \left( H_\alpha^{-\sigma}(p_{i,\sigma}^{\alpha,n}) - H_\alpha^{-\sigma}(p_{i,\sigma}^{\alpha,n+1}) \right) \\ &= \int_0^1 (H_\alpha^{-\sigma})'(p_{i,\sigma}^{\alpha,n+1} + \tau(p_{i,\sigma}^{\alpha,n} - p_{i,\sigma}^{\alpha,n+1})) \left( \frac{p_{i,\sigma}^{\alpha,n} - p_{i,\sigma}^{\alpha,n+1}}{\Delta t} \right) d\tau \\ &= C_{i,\sigma}^{\alpha,n} \left( \frac{W_{i+\sigma}^{\alpha,n} - W_i^{\alpha,n}}{\Delta x} \right) \end{aligned}$$

where we used (1.3.7) and (1.3.9) in the last line. Using (1.3.8), we thus get

$$\begin{aligned} W_i^{\alpha,n+1} &\geq \left( 1 - C_{i,\sigma}^{\alpha,n} \frac{\Delta t}{\Delta x} \right) W_i^{\alpha,n} + C_{i,\sigma}^{\alpha,n} \frac{\Delta t}{\Delta x} W_{i+\sigma}^{\alpha,n} \\ &\geq m^n. \end{aligned}$$

**Case 2:**  $i = 0$ . We recall that in this case, we have  $U_0^{\beta,n} := U_0^n$  and  $W_0^{\beta,n} := W_0^n = \frac{U_0^{n+1} - U_0^n}{\Delta t}$  for any  $\beta = 1, \dots, N$ . We compute in this case:

$$\begin{aligned} \frac{W_0^{n+1} - W_0^n}{\Delta t} &= \frac{1}{\Delta t} \left( -F(\{p_{0,+}^{\alpha,n+1}\}_\alpha) + F(\{p_{0,+}^{\alpha,n}\}_\alpha) \right) \\ &= \frac{1}{\Delta t} \int_0^1 \sum_{\beta=1}^N p_\beta \frac{\partial F}{\partial p_\beta} (\{p_{0,+}^{\alpha,n+1} + \tau p_\alpha\}_\alpha) d\tau \quad \text{with } p = (\{p_{0,+}^{\alpha,n} - p_{0,+}^{\alpha,n+1}\}_\alpha) \\ &= - \int_0^1 \sum_{\beta=1}^N \frac{\partial F}{\partial p_\beta} (\{p_{0,+}^{\alpha,n+1} + \tau p_\alpha\}_\alpha) d\tau \left( \frac{W_1^{\beta,n} - W_0^n}{\Delta x} \right) \\ &= \sum_{\beta=1}^N C_{0,+}^{\beta,n} \left( \frac{W_1^{\beta,n} - W_0^n}{\Delta x} \right). \end{aligned}$$

Using (1.3.8), we argue like in Case 1 and get

$$W_0^{n+1} \geq m^n.$$

**Step 2:**  $(M^n)_n$  is non-increasing. We want to show that  $W_i^{\alpha,n+1} \leq M^n$  for  $i \geq 0$  and  $\alpha = 1, \dots, N$ . We argue as in Step 1 by distinguishing two cases.

**Case 1:**  $i \geq 1$ . We simply choose  $\sigma = \sigma(i, \alpha, n)$  (see (1.3.10)) and argue as in Step 1.

**Case 2:**  $i = 0$ . Using (1.3.6), we can argue exactly as in Step 1. The proof is now complete.  $\square$

### 1.3.2 Gradient estimates

The second result needed in the proof of Theorem 1.3.1 is the following one. It asserts that if the discrete time derivative is controlled from below, then a discrete gradient estimate holds true.

**Proposition 1.3.4 (Discrete gradient estimate).** *Let  $n \geq 0$  be fixed, consider that  $(U_i^{\alpha,n})_{\alpha,i}$  is given and compute  $(U_i^{\alpha,n+1})_{\alpha,i}$  using the scheme (1.1.9)-(1.1.10). If there exists a constant  $K \in \mathbb{R}$  such that for any  $i \geq 0$  and  $\alpha = 1, \dots, N$ ,*

$$K \leq W_i^{\alpha,n} := \frac{U_i^{\alpha,n+1} - U_i^{\alpha,n}}{\Delta t}$$

then

$$\begin{cases} \pi_\alpha^-(-K) \leq p_{i,+}^{\alpha,n} \leq \pi_\alpha^+(-K), & \alpha = 1, \dots, N, \quad i \geq 1, \\ \underline{\rho}_\alpha(-K) \leq p_{0,+}^{\alpha,n} \leq (H_\alpha^+)^{-1}(-K), & \alpha = 1, \dots, N \end{cases}$$

where  $p_{i,+}^{\alpha,n}$  is defined in (1.1.10) and  $\pi_\alpha^\pm$  and  $\underline{\rho}$  are the “generalized” inverse functions of  $H_\alpha$  and  $F$ , respectively.

**Proof.** Let  $n \geq 0$  be fixed and consider  $(U_i^{\alpha,n})_{\alpha,i}$  with  $\Delta x, \Delta t > 0$  given. We compute  $(U_i^{\alpha,n+1})_{\alpha,i}$  using the scheme (1.1.9). Let us consider any  $i \geq 0$  and  $\alpha = 1, \dots, N$ .

If  $i \geq 1$ , the result follows from

$$K \leq W_i^{\alpha,n} = - \max_{\sigma=+,-} H_\alpha^\sigma(p_{i,\sigma}^{\alpha,n}).$$

If  $i = 0$ , the result follows from

$$K \leq W_0^n = -F\left(\{p_{0,+}^{\alpha,n}\}_\alpha\right).$$

This achieves the proof of Proposition 1.3.4  $\square$

### 1.3.3 Proof of gradient estimates

**Proof of Theorem 1.3.1.** The idea of the proof is to introduce new Hamiltonians  $\tilde{H}_\alpha$  and a new junction function  $\tilde{F}$  for which it is easier to derive gradient estimates but whose corresponding numerical scheme in fact coincides with the original one.

**Step 1: Modification of the Hamiltonians and the junction function.** Let the new Hamiltonians  $\tilde{H}_\alpha$  for all  $\alpha = 1, \dots, N$  be defined as

$$\tilde{H}_\alpha(p) = \begin{cases} H_\alpha(\underline{p}_\alpha) - \frac{C_\alpha}{2}(p - \underline{p}_\alpha) & \text{if } p \leq \underline{p}_\alpha \\ H_\alpha(p) & \text{if } p \in [\underline{p}_\alpha, \bar{p}_\alpha] \\ H_\alpha(\bar{p}_\alpha) + \frac{C_\alpha}{2}(p - \bar{p}_\alpha) & \text{if } p \geq \bar{p}_\alpha \end{cases} \quad (1.3.11)$$

where  $\underline{p}_\alpha$  and  $\bar{p}_\alpha$  are defined in (1.3.4) respectively, and

$$C_\alpha = \sup_{p_\alpha \in [\underline{p}_\alpha, \bar{p}_\alpha]} |H'_\alpha(p_\alpha)|.$$

These new Hamiltonians are now globally Lipschitz continuous: their derivatives are bounded. More precisely, the  $\tilde{H}_\alpha$  satisfy (1.1.3) and

$$\tilde{H}_\alpha \equiv H_\alpha \text{ in } [\underline{p}_\alpha, \bar{p}_\alpha]$$

and

$$\forall p \in \mathbb{R}, \quad |\tilde{H}'_\alpha(p)| \leq \sup_{p_\alpha \in [\underline{p}_\alpha, \bar{p}_\alpha]} |H'_\alpha(p_\alpha)|. \quad (1.3.12)$$

Let the new  $\tilde{F}$  satisfy (1.1.8), be such that

$$\tilde{F} \equiv F \text{ in } Q_0 := \prod_{\alpha=1}^N [\underline{p}_\alpha^0, \bar{p}_\alpha]$$

and (See Appendix 1.8)

$$\forall p \in \mathbb{R}^N, \quad (-\nabla \cdot \tilde{F})(p) \leq \sup_{Q_0} (-\nabla \cdot F). \quad (1.3.13)$$

In the remainder of the proof, when notation contains a tilde, it is associated with the new Hamiltonians  $\tilde{H}_\alpha$  and the new non-linearity  $\tilde{F}$ . We then consider the new numerical scheme

$$\begin{cases} \frac{\tilde{U}_i^{\alpha,n+1} - \tilde{U}_i^{\alpha,n}}{\Delta t} + \max\{\tilde{H}_\alpha^+(\tilde{p}_{i,-}^{\alpha,n}), \tilde{H}_\alpha^-(\tilde{p}_{i,+}^{\alpha,n})\} = 0, & i \geq 1, \quad \alpha = 1, \dots, N \\ \tilde{U}_0^{\beta,n} := U_0^n, & i = 0, \quad \beta = 1, \dots, N \\ \frac{\tilde{U}_0^{n+1} - \tilde{U}_0^n}{\Delta t} + \tilde{F}(\tilde{p}_{0,+}^{1,n}, \tilde{p}_{0,+}^{2,n}, \dots, \tilde{p}_{0,+}^{N,n}) = 0 \end{cases}$$

with the same initial condition, namely,

$$\tilde{U}_i^{\alpha,0} = u_0^\alpha(i\Delta x), \quad i \geq 0, \quad \alpha = 1, \dots, N.$$

In view of (1.3.12) and (1.3.13), the CFL condition (1.1.13) gives that for any  $i \geq 0$ ,  $n \geq 0$ , and  $\alpha = 1, \dots, N$

$$\tilde{D}_{i,+}^{\alpha,n} \leq \sup \left\{ \sup_{\underline{p}_\alpha \leq p \leq \bar{p}_\alpha} |H'_\alpha(p)|; \sup_{\tilde{I}_{0,+}^{\alpha,n}} (-\nabla \cdot F) \right\} \leq \frac{\Delta x}{\Delta t} \quad (1.3.14)$$

where  $\tilde{D}_{i,+}^{\alpha,n}$  is given by (1.3.5) after replacing  $H_\alpha$  and  $F$  with  $\tilde{H}_\alpha$  and  $\tilde{F}$ .

**Step 2: First gradient bounds.** Let  $n \geq 0$  be fixed. If  $\tilde{m}^n$  and  $\tilde{M}^n$  are finite, we have

$$\tilde{m}^n \leq \tilde{W}_i^{\alpha,n} \quad \text{for any } i \geq 0, \quad \alpha = 1, \dots, N.$$

Proposition 1.3.4 implies that

$$\begin{cases} \tilde{\pi}_\alpha^-(-\tilde{m}^n) \leq \tilde{p}_{i,+}^{\alpha,n} \leq \tilde{\pi}_\alpha^+(-\tilde{m}^n), & i \geq 1, \quad \alpha = 1, \dots, N, \\ \tilde{\rho}_\alpha(-\tilde{m}^n) \leq \tilde{p}_{0,+}^{\alpha,n} \leq \tilde{\pi}_\alpha^+(-\tilde{m}^n), & i \geq 0, \quad \alpha = 1, \dots, N. \end{cases}$$

In particular, we get that

$$|\tilde{p}_{i,+}^{\alpha,n}| \leq C^n \quad \text{for } i \geq 0, \quad \alpha = 1, \dots, N$$

with

$$C^n = \max_\alpha \left( \max \left( |\tilde{\pi}_\alpha^-(-\tilde{m}^n)|, |\tilde{\pi}_\alpha^+(-\tilde{m}^n)|, |\tilde{\rho}_\alpha(-\tilde{m}^n)| \right) \right).$$

In view of (1.3.14), Proposition 1.3.3 implies that

$$\tilde{m}^n \leq \tilde{m}^{n+1} \leq \tilde{M}^{n+1} \leq \tilde{M}^n \quad \text{for any } n \geq 0. \quad (1.3.15)$$

In particular,  $\tilde{m}^{n+1}$  is also finite. Since  $\tilde{m}^0 = m^0$  and  $\tilde{M}^0 = M^0$  are finite, we conclude that  $\tilde{m}^n$  and  $\tilde{M}^n$  are finite for all  $n \geq 0$  and for all  $n \geq 0$ ,

$$m^0 \leq \tilde{m}^n \leq \tilde{M}^n \leq M^0. \quad (1.3.16)$$

**Step 3: Time derivative and gradient estimates.** Now we can repeat the same reasoning but applying Proposition 1.3.4 with  $K = m^0$  and get

$$\begin{cases} \underline{p}_\alpha \leq \tilde{p}_{i,+}^{\alpha,n} \leq \bar{p}_\alpha, & i \geq 1, \quad \alpha = 1, \dots, N, \\ \underline{p}_\alpha^0 \leq \tilde{p}_{0,+}^{\alpha,n} \leq \bar{p}_\alpha, & i \geq 0, \quad \alpha = 1, \dots, N. \end{cases} \quad (1.3.17)$$

This implies that  $\tilde{U}_i^{\alpha,n} = U_i^{\alpha,n}$  for all  $i \geq 0, n \geq 0, \alpha = 1, \dots, N$ . In view of (1.3.15), (1.3.16) and (1.3.17), the proof is now complete.  $\square$

## 1.4 Convergence for general junction conditions

This section is devoted to the convergence of the scheme defined by (1.1.9)-(1.1.10). In order to do so, we first make precise how to choose  $\bar{p}_\alpha, \underline{p}_\alpha$  and  $\underline{p}_\alpha^0$  in the CFL condition (1.1.13).

### 1.4.1 Monotonicity of the scheme

In order to prove the convergence of the numerical solution as the mesh size tends to zero, we need first to prove a monotonicity result. It is common to write the scheme defined by (1.1.9)-(1.1.10) under the compact form

$$u^h(t + \Delta t, x) = S^h[u^h(t)](x)$$

where the operator  $S^h$  is defined on the set of functions defined in  $J^h$ . The scheme is monotone if

$$u \leq v \quad \Rightarrow \quad S^h[u] \leq S^h[v].$$

In our cases, if  $t = n\Delta t$  and  $x = i\Delta x \in J^\alpha$  and  $U(t, x) = U_i^{\alpha, n}$  for  $x \in J^\alpha$ , then  $S^h[U]$  is defined as follows,

$$\begin{cases} U_i^{\alpha, n+1} = S_\alpha[U_{i-1}^{\alpha, n}, U_i^{\alpha, n}, U_{i+1}^{\alpha, n}] & \text{for } i \geq 1, \alpha = 1, \dots, N, \\ U_0^{n+1} = S_0[U_0^n, (U_1^{\beta, n})_{\beta=1, \dots, N}] \end{cases}$$

where

$$\begin{cases} S_\alpha[U_{i-1}^{\alpha, n}, U_i^{\alpha, n}, U_{i+1}^{\alpha, n}] := U_i^{\alpha, n} - \Delta t \max \left\{ H_\alpha^+ \left( \frac{U_i^{\alpha, n} - U_{i-1}^{\alpha, n}}{\Delta x} \right), H_\alpha^- \left( \frac{U_{i+1}^{\alpha, n} - U_i^{\alpha, n}}{\Delta x} \right) \right\}, \\ S_0[U_0^n, (U_1^{\beta, n})_{\beta=1, \dots, N}] := U_0^n - \Delta t F(p_{0,+}^{1, n}, \dots, p_{0,+}^{N, n}). \end{cases} \quad (1.4.1)$$

Checking the monotonicity of the scheme reduces to checking that  $S_\alpha$  and  $S_0$  are non-decreasing in all their variables.

**Lemma 1.4.1 (Monotonicity of the numerical scheme).** *Let  $(U^n) := (U_i^{\alpha, n})_{\alpha, i}$  the numerical solution of (1.1.9)-(1.1.11). Under the CFL condition (1.1.12) the scheme is monotone.*

**Proof.** We distinguish two cases.

**Case 1:**  $i \geq 1$ . It is straightforward to check that, for any  $\alpha = 1, \dots, N$ , the function  $S_\alpha$  is non-decreasing with respect to  $U_{i-1}^{\alpha, n}$  and  $U_{i+1}^{\alpha, n}$ . Moreover,

$$\frac{\partial S_\alpha}{\partial U_i^{\alpha, n}} = \begin{cases} 1 - \frac{\Delta t}{\Delta x} (H_\alpha^+)'(p_{i,-}^{\alpha, n}) & \text{if } \max\{H_\alpha^+(p_{i,-}^{\alpha, n}), H_\alpha^-(p_{i,+}^{\alpha, n})\} = H_\alpha^+(p_{i,-}^{\alpha, n}) \\ 1 + \frac{\Delta t}{\Delta x} (H_\alpha^-)'(p_{i,+}^{\alpha, n}) & \text{if } \max\{H_\alpha^+(p_{i,-}^{\alpha, n}), H_\alpha^-(p_{i,+}^{\alpha, n})\} = H_\alpha^-(p_{i,+}^{\alpha, n}) \end{cases}$$

which is non-negative if the CFL condition (1.1.12) is satisfied.

**Case 2:**  $i = 0$ . Similarly it is straightforward to check that  $S_0$  is non-decreasing with respect to  $U_1^{\beta, n}$  for  $\beta = 1, \dots, N$ . Moreover,

$$\frac{\partial S_0}{\partial U_0^n} = 1 + \frac{\Delta x}{\Delta t} \sum_{\beta=1}^N \frac{\partial F}{\partial p_\beta} \{(p_{0,+}^{\alpha, n})_{\alpha=1}^N\}$$

which is non-negative due to the CFL condition. The proof is now complete.  $\square$

A direct consequence of the previous lemma is the following elementary but useful discrete comparison principle.

**Lemma 1.4.2 (Discrete Comparison Principle).** *Let  $(U^n) := (U_i^{\alpha,n})_{\alpha,i}$  and  $(V^n) := (V_i^{\alpha,n})_{\alpha,i}$  be such that*

$$\forall n \geq 1, \quad U^{n+1} \leq S^h[U^n] \quad \text{and} \quad V^{n+1} \geq S^h[V^n].$$

*If the CFL condition (1.1.12) is satisfied and if  $U^0 \leq V^0$ , then  $U^n \leq V^n$  for all  $n \in \mathbb{N}$ .*

*Remark 1.4.3.* The discrete function  $(U^n)$  (resp.  $(V^n)$ ) can be seen as a *super-scheme* (resp. *sub-scheme*).

We finally recall how to derive discrete viscosity inequalities for monotone schemes.

**Lemma 1.4.4 (Discrete viscosity inequalities).** *Let  $u^\varepsilon$  be a solution of (1.1.9)-(1.1.11) with  $F = F_A$  defined in (1.1.5). If  $u^h - \varphi$  has a global maximum (resp. global minimum) on  $\mathcal{G}_h$  at  $(\bar{t} + \Delta t, \bar{x})$ , then*

$$\delta_t \varphi(\bar{t}, \bar{x}) + \mathcal{H}(\bar{x}, D_+ \varphi(\bar{t}, \bar{x}), D_- \varphi(\bar{t}, \bar{x})) \leq 0. \quad (\text{resp. } \geq 0)$$

where

$$\mathcal{H}(x, p_+, p_-) = \begin{cases} \max\{H_\alpha^+(p_-), H_\alpha^-(p_+)\} & \text{if } \bar{x} \neq 0 \\ \max\{A, \max_\alpha H_\alpha^-(p_\alpha^+)\} & \text{if } \bar{x} = 0 \end{cases}$$

and

$$\begin{aligned} D_+ \varphi(\bar{t}, \bar{x}) &= \begin{cases} \frac{1}{\Delta x} \{\varphi(\bar{t}, \bar{x} + \Delta x) - \varphi(\bar{t}, \bar{x})\} & \text{if } \bar{x} \neq 0 \\ \left( \frac{1}{\Delta x} \{\varphi^\alpha(\bar{t}, \Delta x) - \varphi^\alpha(\bar{t}, 0)\} \right)_\alpha & \text{if } \bar{x} = 0 \end{cases} \\ D_- \varphi(\bar{t}, \bar{x}) &= \frac{1}{\Delta x} \{\varphi(\bar{t}, \bar{x}) - \varphi(\bar{t}, \bar{x} - \Delta x)\} \\ \delta_t \varphi(\bar{t}, \bar{x}) &= \frac{1}{\Delta t} \{\varphi(\bar{t} + \Delta t, \bar{x}) - \varphi(\bar{t}, \bar{x})\}. \end{aligned}$$

## 1.4.2 Stability and Consistency of the scheme

We first derive a local  $L^\infty$  bound for the solution of the scheme.

**Lemma 1.4.5 (Stability of the numerical scheme).** *Assume that the CFL condition (1.1.13) is satisfied and let  $u^h$  be the solution of the numerical scheme (1.1.9)-(1.1.11). There exists a constant  $C_0 > 0$ , such that for all  $(t, x) \in \mathcal{G}_h$ ,*

$$|u^h(t, x) - u_0(x)| \leq C_0 t. \quad (1.4.2)$$

*In particular, the scheme is (locally) stable.*

**Proof.** If  $C_0$  large enough so that

$$\begin{cases} C_0 + \max\{H_\alpha^+(p_{i,-}^{\alpha,0}), H_\alpha^-(p_{i,+}^{\alpha,0})\} \geq 0, & i \geq 1, \quad \alpha = 1, \dots, N \\ C_0 + F(p_{0,+}^{1,0}, p_{0,+}^{2,0}, \dots, p_{0,+}^{N,0}) \geq 0, \end{cases}$$

and

$$\begin{cases} -C_0 + \max\{H_\alpha^+(p_{i,-}^{\alpha,0}), H_\alpha^-(p_{i,+}^{\alpha,0})\} \leq 0, & i \geq 1, \quad \alpha = 1, \dots, N \\ -C_0 + F(p_{0,+}^{1,0}, p_{0,+}^{2,0}, \dots, p_{0,+}^{N,0}) \leq 0, \end{cases}$$

then  $\bar{U}_i^{\alpha,n} = U_i^{\alpha,0} + C_0 n \Delta t$  is a super-scheme and  $\bar{U}_i^{\alpha,n} = U_i^{\alpha,0} - C_0 n \Delta t$  is a sub-scheme (see Remark 1.4.3). The discrete comparison principle, Proposition 1.4.2, then implies

$$|U_i^{\alpha,n} - U_i^{\alpha,0}| \leq C_0 n \Delta t$$

which is the desired inequality. This achieves the proof.  $\square$

Another condition to satisfy convergence of the numerical scheme (1.1.9) towards the continuous solution of (1.1.6) is the consistency of the scheme (which is obvious in our case). In the statement below, we use the short hand notation (1.5.11) introduced in section 1.5.

**Lemma 1.4.6 (Consistency of the numerical scheme).** *Under the assumptions on the Hamiltonians (1.1.3), the finite difference scheme is consistent with the continuous problem (1.1.6), that is to say for any smooth function  $\varphi(t, x)$ , we have*

$$\frac{S^h[\varphi](s, y) - \varphi(s, y)}{\Delta t} \rightarrow H_\alpha(\varphi_x(t, x)) \quad \text{as } \mathcal{G}_h \ni (s, y) \rightarrow (t, x)$$

if  $x \in J_\alpha \setminus \{0\}$ , and

$$\frac{S^h[\varphi](s, y) - \varphi(s, y)}{\Delta t} \rightarrow F\left(\frac{\partial \varphi}{\partial x_1}, \dots, \frac{\partial \varphi}{\partial x_N}(t, 0)\right) \quad \text{as } \mathcal{G}_h \ni (s, y) \rightarrow (t, 0).$$

### 1.4.3 Convergence of the numerical scheme

In this subsection, we present a sketch of the proof of Theorem 1.1.1.

*Sketch of the proof of Theorem 1.1.1.* Let  $T > 0$  and  $h := (\Delta t, \Delta x)$  satisfying the CFL condition (1.1.13). We recall that

$$u^h(0, x) = u(0, x) \quad \text{for } x \in \mathcal{G}_h.$$

We consider  $\bar{u}$  and  $\underline{u}$  respectively defined as

$$\bar{u}(t, y) = \limsup_{\substack{h \rightarrow 0 \\ \mathcal{G}_h \ni (t', y') \rightarrow (t, y)}} u^h(t', y'), \quad \underline{u}(t, y) = \liminf_{\substack{h \rightarrow 0 \\ \mathcal{G}_h \ni (t', y') \rightarrow (t, y)}} u^h(t', y').$$

By construction, we have  $\underline{u} \leq \bar{u}$ . Since the scheme is monotone (Lemma 1.4.1), stable (Lemma 1.4.5) and consistent (Lemma 1.4.6), we can follow [21, 41, 42] we can show that  $\underline{u}$  (resp.  $\bar{u}$ ) is a relaxed viscosity super-solution (resp. viscosity sub-solution) of equation (1.1.1)-(1.1.2). Using Theorem 1.2.4, we know that  $\underline{u}$  (resp.  $\bar{u}$ ) is a viscosity super-solution (resp. sub-solution) of (1.1.6)-(1.1.2). Moreover, (1.4.2) implies that

$$\bar{u}(0, x) \leq u_0(x) \leq \underline{u}(0, x).$$

The comparison principle (see Theorem 1.2.3) then implies that

$$\bar{u} \leq u \leq \underline{u}$$

which achieves the proof.  $\square$

## 1.5 $C^{1,1}$ estimates for the vertex test function

In this section, we study the Lipschitz regularity of the gradient of the vertex test function constructed in [65]. It turns out that its gradient is indeed Lipschitz if the flux limiter  $A$  is equal to  $A_0$ , the minimal flux limiter, and if (1.1.4) is satisfied. Such a technical result will be used when deriving error estimates.

**Proposition 1.5.1** ( *$C^{1,1}$  estimates for the vertex test function*). *Assume that the Hamiltonians satisfy (1.1.3), and  $\min H_\alpha = \min H_\beta, \forall \alpha, \beta = 1, \dots, N$ . The vertex test function  $G^0$  associated with the flux limiter  $A_0$  obtained from Theorem 1.2.7 can be chosen  $C^{1,1}(J_K^2)$  for any  $K > 0$  where  $J_K^2 = \{(x, y) \in J^2 : d(x, y) \leq K\}$ . Moreover, there exists  $C_K$  such that*

$$\|D^2 G^0\|_{L^\infty(J_K^2)} \leq C_K;$$

the constant  $C_K$  depends only on  $K$  and (1.1.3).

**Proof.** We first get the desired estimate in the smooth convex case and then derive it in the general case.

**Step 1: the smooth convex case.** We first assume that Hamiltonians satisfy

$$\begin{cases} H_\alpha \in C^2(\mathbb{R}) \\ \min H_\alpha = H_\alpha(0) = A_0 \quad \forall \alpha = 1, \dots, N \\ \min H_\alpha'' =: m_\alpha > 0, \end{cases} \quad (1.5.1)$$

where  $A_0$  is defined in (1.1.7).

We recall that the vertex test function  $G$  is a regularized version of  $A + G^0$  where  $G^0$  is defined in [65] by the following formula,

$$G^0(x, y) = \sup_{\lambda \geq A_0} \{\pi_\alpha^+(\lambda)x - \pi_\beta^-(\lambda)y - \lambda\}, \quad \text{for } (x, y) \in J_\alpha \times J_\beta. \quad (1.5.2)$$

But  $G^0 \in C^1(\{(x, y) \in J \times J, x \neq y\})$  and  $G^0 \in C^1(J^2)$  if and only if  $\pi_\alpha^+(A) = 0 = \pi_\alpha^-(A)$ , where  $\pi_\alpha^\pm$  are the generalized inverse functions defined in (1.3.4). Thus, under (1.5.1), there is no need to regularize  $G^0$  on the diagonal, and thus

$$G(x, y) = A + G^0(x, y) \quad \text{for } (x, y) \in J_\alpha \times J_\beta.$$

The supremum in (1.5.2) is reached for some  $\lambda \geq A_0$  which depends on  $x$  and  $y$ . In the region where  $\lambda = A_0$ , the function  $G^0$  is linear and there is nothing to prove. In  $\{\lambda > A_0\}$ , the function  $\lambda(x, y)$  is implicitly defined by the following equation

$$(\pi_\alpha^+(\lambda))'x - (\pi_\beta^-(\lambda))'y = 1 \quad (1.5.3)$$

and the gradient of  $G^0$  is given by

$$\partial_x G^0(x, y) = \pi_\alpha^+(\lambda), \quad \partial_y G^0(x, y) = -\pi_\beta^-(\lambda)$$

with  $\lambda = \lambda(x, y)$ . We thus can easily compute the second order derivatives of  $G^0$ ,

$$\begin{cases} \partial_x^2 G^0(x, y) = \frac{1}{\frac{x}{H'_\alpha} H''_\alpha + \frac{y}{-H'_\beta} H''_\beta \frac{(H'_\alpha)^2}{(H'_\beta)^2}} \\ \partial_y^2 G^0(x, y) = \frac{1}{\frac{x}{H'_\alpha} H''_\alpha \frac{(H'_\beta)^2}{(H'_\alpha)^2} + \frac{y}{-H'_\beta} H''_\beta} \\ \partial_{xy}^2 G^0(x, y) = \frac{1}{-x H'_\beta \frac{H''_\alpha}{(H'_\alpha)^2} + H'_\alpha \frac{H''_\beta}{(H'_\beta)^2} y} \end{cases} \quad (1.5.4)$$

where  $H''_\alpha$  and  $H''_\beta$  respectively denote  $H''_\alpha(\pi_\alpha^+(\lambda))$  and  $H''_\beta(\pi_\beta^-(\lambda))$ .

We recall that  $\min H_\alpha = \min H_\beta = A_0$ . Then

$$H'_\alpha(\pi_\alpha^+(A_0)) = 0 \quad \text{and} \quad H'_\beta(\pi_\beta^-(A_0)) = 0.$$

Using a second order Taylor expansion for  $H_\alpha$  and  $H_\beta$  respectively we prove that

$$\begin{aligned} H'_\beta(\pi_\beta^-(A_0 + \gamma)) &\sim \sqrt{2\gamma H''_\beta(0)}, \\ H'_\alpha(\pi_\alpha^+(A_0 + \gamma)) &\sim \sqrt{2\gamma H''_\alpha(0)}. \end{aligned} \quad (1.5.5)$$

Indeed, on the one hand we have

$$H'_\beta(\pi_\beta^-(A_0 + \gamma)) = H''_\beta(0)[\pi_\beta^-(A_0 + \gamma)] + o(\pi_\beta^-(A_0 + \gamma)). \quad (1.5.6)$$

On the other hand, we have

$$H_\beta(\pi_\beta^-(A_0 + \gamma)) = H_\beta(0) + \frac{H''_\beta(0)}{2}[\pi_\beta^-(A_0 + \gamma)]^2 + o(\pi_\beta^-(A_0 + \gamma))^2.$$

Using the fact that  $H_\beta(\pi_\beta^-(A_0 + \gamma)) = H_\beta^-(\pi_\beta^-(A_0 + \gamma)) = A_0 + \gamma$ , and that  $H_\beta(0) = A_0$ , one can deduce

$$[\pi_\beta^-(A_0 + \gamma)]^2 = \frac{2\gamma}{H''_\beta(0)} + o(\pi_\beta^-(A_0 + \gamma))^2, \quad (1.5.7)$$

which implies

$$\pi_\beta^-(A_0 + \gamma) \sim \sqrt{\frac{2\gamma}{H''_\beta(0)}}. \quad (1.5.8)$$

Substituting (1.5.8) in (1.5.6) leads to (1.5.5).

Moreover, we have

$$\frac{x}{H'_\alpha} H''_\alpha + \frac{y}{-H'_\beta} H''_\beta \frac{(H'_\alpha)^2}{(H'_\beta)^2} \geq \min \left( H''_\alpha, H''_\beta \frac{(H'_\alpha)^2}{(H'_\beta)^2} \right) \underbrace{\left( \frac{x}{H'_\alpha} - \frac{y}{H'_\beta} \right)}_{=1}. \quad (1.5.9)$$

using the fact that  $H''_\alpha$  are bounded from below,

$$|\partial_x^2 G^0| \leq \frac{1}{\min \left( H''_\alpha, H''_\beta \frac{(H'_\alpha)^2}{(H'_\beta)^2} \right)} = \mathcal{O}(1).$$

Similarly,

$$\frac{x}{H'_\alpha} H''_\alpha \frac{(H'_\beta)^2}{(H'_\alpha)^2} + \frac{y}{-H'_\beta} H''_\beta \geq \min \left( H''_\alpha \frac{(H'_\beta)^2}{(H'_\alpha)^2}, H''_\beta \right) \underbrace{\left( \frac{x}{H'_\alpha} - \frac{y}{H'_\beta} \right)}_{=1}$$

implies

$$|\partial_y^2 G^0| \leq \frac{1}{\min \left( H''_\alpha \frac{(H'_\beta)^2}{(H'_\alpha)^2}, H''_\beta \right)} = \mathcal{O}(1)$$

and

$$\begin{aligned} -x H'_\beta \frac{H''_\alpha}{(H'_\alpha)^2} + H'_\alpha \frac{H''_\beta}{(H'_\alpha)^2} y &= \frac{x}{H'_\alpha} \frac{-H'_\beta}{H'_\alpha} H''_\alpha + \frac{-y}{H'_\beta - H'_\alpha} \frac{H'_\alpha}{H'_\beta} H''_\beta \\ &\geq \min \left( \frac{-H'_\beta}{H'_\alpha} H''_\alpha, \frac{H'_\alpha}{-H'_\beta} H''_\beta \right) \underbrace{\left( \frac{x}{H'_\alpha} - \frac{y}{H'_\beta} \right)}_{=1} \end{aligned}$$

implies

$$|\partial_{xy}^2 G^0| \leq \frac{1}{\min \left( \frac{-H'_\beta}{H'_\alpha} H''_\alpha, \frac{H'_\alpha}{-H'_\beta} H''_\beta \right)} = \mathcal{O}(1).$$

**Step 2: the smooth case.** We now weaken (1.5.1) as

$$\begin{cases} H_\alpha \in C^2(\mathbb{R}), & H''_\alpha(0) := m_\alpha > 0 \\ \pm H'_\alpha(p) > 0 & \text{for } \pm p > 0 \\ H_\alpha(p) \rightarrow +\infty & \text{as } |p| \rightarrow +\infty. \end{cases} \quad (1.5.10)$$

In this case, it is explained in [65] that the smooth convex case can be used by considering  $\hat{H}_\alpha = \beta \circ H_\alpha$  for some  $C^2$  convex function  $\beta$  such that  $\beta(0) = 0$  and

$\beta' \geq \delta$  for some  $\delta > 0$ . Indeed,

$$\begin{aligned} (\hat{H}_\alpha)''(p) &= \beta''(\lambda)(H'_\alpha)^{\prime 2}(p) + \beta'(\lambda)H''_\alpha(p) \\ &\geq \frac{m_\alpha}{2}\beta'(\lambda) \\ &\geq m_\alpha \end{aligned}$$

with  $\lambda = H_\alpha(p)$  and  $\beta$  such that

$$\forall \lambda > H_\alpha(0), \quad \begin{cases} \frac{\beta''}{\beta'}(\lambda) \geq -\frac{H''_\alpha}{(H'_\alpha)^2} \circ \pi_\alpha^\pm(\lambda) + \frac{m_\alpha}{2(H'_\alpha)^2} \circ \pi_\alpha^\pm(\lambda), \\ \beta' \geq 2 > 0. \end{cases}$$

In this case, the vertex test function studied in Step 1 and associated with Hamiltonians  $\hat{H}_\alpha$  satisfies

$$\hat{H}(y, -G_y(x, y)) \leq \hat{H}(x, G_x(x, y))$$

which implies that, since  $0 \leq (\beta^{-1})' \leq 1/2$ ,

$$H(y, -G_y(x, y)) \leq \beta^{-1}(\beta H(x, G_x(x, y))) = H(x, G_x(x, y))$$

where the short hand notation  $H(x, p)$  is given by

$$H(x, p) = \begin{cases} H_\alpha(p) & \text{for } p = p_\alpha & \text{if } x \in J_\alpha^*, \\ F_{A_0}(p) & \text{for } p = (p_1, \dots, p_N) & \text{if } x = 0. \end{cases} \quad (1.5.11)$$

We proved in Step 1 that

$$\|D^2G^{A_0}\|_{L^\infty(K)} \leq C_K.$$

The proof is now complete. □

## 1.6 Error estimates

### 1.6.1 Proof of the error estimates

To prove Theorem 1.1.3, we will need the following result whose classical proof is given in Appendix for the reader's convenience.

**Lemma 1.6.1 (A priori control).** *Let  $T > 0$  and let  $u^h$  be a solution of the numerical scheme (1.1.9)-(1.1.11) and  $u$  a super-solution of (1.1.1)-(1.1.2) satisfying for some  $C_T > 0$ ,*

$$u(t, x) \geq -C_T(1 + d(0, x)) \quad \text{for } t \in (0, T).$$

*Then there exists a constant  $C = C(T) > 0$  such that for all  $(t, x) \in \mathcal{G}_h$ ,  $t \leq T$ , and  $(s, y) \in [0, T) \times J$ , we have*

$$u^h(t, x) \leq u(s, y) + C(1 + d(x, y)). \quad (1.6.1)$$

We now turn to the proof of the error estimates in the case of flux-limited junction conditions.

**Proof of Theorem 1.1.3.** Before deriving the error estimate, we remark as in [65] that we can assume without loss of generality that the Hamiltonians satisfy the additional condition (1.2.7). Indeed, if  $u$  solves (1.1.1) then

$$\tilde{u}(t, x) = u(t, x) - p_0^\alpha x \quad \text{for } x \in J^\alpha$$

solves the same equation in which  $H_\alpha$  replaced with

$$\tilde{H}_\alpha(p) = H_\alpha(p_0^\alpha + p).$$

We next remark that the solution  $\tilde{u}^h$  of the associated scheme satisfies

$$\tilde{u}^h(t, x) = u^h(t, x) - p_0^\alpha x \quad \text{for } (t, x) \in \mathcal{G}_h.$$

Hence, if

$$\sup_{[0, T) \times J \cap \mathcal{G}_h} |\tilde{u}^h - \tilde{u}| \leq C(\Delta x)^{1/2}$$

then the same estimate between  $u^h$  and  $u$  holds true. We thus assume from now on that (1.2.7) holds true.

In order to get (1.1.16), we only prove that

$$u^h(t, x) - u(t, x) \leq C_T(\Delta x)^{1/2} \quad \text{in } [0, T) \times J \cap \mathcal{G}_h$$

since the proof of the other inequality is very similar. We are going to prove that

$$u^h(t, x) - u(t, x) \leq \mathcal{O}\left(\frac{\Delta t}{\nu}\right) + \mathcal{O}\left(\frac{\Delta x}{\epsilon}\right) + \mathcal{O}(\epsilon) + \mathcal{O}(\nu) \quad (1.6.2)$$

which yields the desired inequality by minimizing the right hand side with respect to  $\epsilon$  and  $\nu$ . Let

$$M = \sup_{[0, T) \times J \cap \mathcal{G}_h} \{u^h(t, x) - u(t, x)\}.$$

The remaining of the proof proceeds in several steps.

**Step 1: Penalization procedure.** From Theorem 1.2.7, we recall the properties of the vertex test function  $G^0$ , mainly the fact that  $G^0(0, 0) = G^0(x, x) = -A$ .

Let  $G = G^0 + A$ , we have

$$G(x, x) = 0.$$

For  $\eta > 0$ ,  $\delta > 0$  let us define

$$M_{\epsilon, \delta} = \sup_{\substack{(t, x) \in \mathcal{G}_h, \\ (s, y) \in [0, T) \times J}} \left\{ u^h(t, x) - u(s, y) - \epsilon G\left(\frac{x}{\epsilon}, \frac{y}{\epsilon}\right) - \frac{(t-s)^2}{2\nu} - \frac{\delta}{2} d^2(y, 0) - \frac{\eta}{T-s} - \sigma s \right\} \quad (1.6.3)$$

where the vertex test function  $G^0$  is given by Theorem 1.2.7 associated with the flux limiter  $A = A_0$ . In this step, we assume that  $M_{\epsilon,\delta} > 0$ . Thanks to Lemma 1.6.1 and the superlinearity of  $G^0$  (see Theorem 1.2.7), we deduce that for  $(x, y)$  such that the quantity in the supremum is larger than  $\frac{M_{\epsilon,\delta}}{2}$ ,

$$0 < \frac{M_{\epsilon,\delta}}{2} \leq C(1 + d(x, y)) - \epsilon g^0\left(\frac{d(x, y)}{\epsilon}\right) - \frac{(t-s)^2}{2\nu} - \frac{\delta}{2}d^2(y, 0) - \frac{\eta}{T-s} - \sigma s$$

which implies in particular that the supremum is reached at some point  $(t, x, s, y)$  and

$$\delta d^2(y, 0) \leq 2 \sup_{d>0} (C(1+d) - \epsilon g^0(d/\epsilon)) \leq \tilde{C}$$

where  $\tilde{C}$  only depends on  $g^0$  (in particular, it does not depend on  $\epsilon$ ). This estimate together with the fact that  $-G_y(x/\epsilon, y/\epsilon) - \delta d(y, 0)$  lies in the viscosity subdifferential of  $u(t, \cdot)$  at  $x$  implies that there exists  $K > 0$  only depending on  $\|\nabla u\|_\infty$  (see Theorem 1.2.5) and  $g^0$  such that the point  $(t, x, s, y)$  realizing the maximum satisfies

$$d\left(\frac{x}{\epsilon}, \frac{y}{\epsilon}\right) + \left|G_x\left(\frac{x}{\epsilon}, \frac{y}{\epsilon}\right)\right| + \left|G_y\left(\frac{x}{\epsilon}, \frac{y}{\epsilon}\right)\right| \leq K. \quad (1.6.4)$$

We want to prove that for  $\sigma > \sigma^*$  (to be determined) that the supremum in (1.6.3) is attained for  $t = 0$  or  $s = 0$ , or that we have  $M_{\epsilon,\delta} \leq 0$ . We assume that  $t > 0$  and  $s > 0$  and we prove that  $\sigma \leq \sigma^*$ .

**Step 2: Viscosity inequalities.** Since  $t > 0$  and  $s > 0$ , we can use Lemma 1.4.4 and get the following viscosity inequalities.

If  $x \neq 0$ , then

$$\frac{t-s}{\nu} - \frac{\Delta t}{2\nu} + \max \left\{ H_\alpha^- \left( \frac{\epsilon}{\Delta x} \left\{ G\left(\frac{x+\Delta x}{\epsilon}, \frac{y}{\epsilon}\right) - G\left(\frac{x}{\epsilon}, \frac{y}{\epsilon}\right) \right\} \right), \right. \\ \left. H_\alpha^+ \left( \frac{\epsilon}{\Delta x} \left\{ G\left(\frac{x}{\epsilon}, \frac{y}{\epsilon}\right) - G\left(\frac{x-\Delta x}{\epsilon}, \frac{y}{\epsilon}\right) \right\} \right) \right\} \leq 0.$$

If  $x = 0$ , then

$$\frac{t-s}{\nu} - \frac{\Delta t}{2\nu} + \max \left\{ A, \max_\beta \left\{ H_\beta^- \left( \frac{\epsilon}{\Delta x} \left\{ G^\beta\left(\frac{\Delta x}{\epsilon}, \frac{y}{\epsilon}\right) - G^\beta\left(0, \frac{y}{\epsilon}\right) \right\} \right) \right\} \right\} \leq 0.$$

If  $y \neq 0$ , then

$$-\frac{\eta}{(T-s)^2} + \frac{t-s}{\nu} + H\left(-G_y\left(\frac{x}{\epsilon}, \frac{y}{\epsilon}\right) - \delta d(y, 0)\right) \geq \sigma.$$

If  $y = 0$ , then

$$-\frac{\eta}{(T-s)^2} + \frac{t-s}{\nu} + F_A\left(-G_y\left(\frac{x}{\epsilon}, 0\right)\right) \geq \sigma.$$

Thanks to the  $C^{1,1}$  regularity of the function  $G^0$ , see Proposition 1.5.1 and Estimate (1.6.4), we obtain,

$$\text{if } x \neq 0, \quad \frac{t-s}{\nu} - \frac{\Delta t}{2\nu} + H_\alpha \left( G_x \left( \frac{x}{\epsilon}, \frac{y}{\epsilon} \right) \right) + \mathcal{O} \left( \frac{\Delta x}{\epsilon} \right) \leq 0 \quad (1.6.5)$$

$$\text{if } x = 0, \quad \frac{t-s}{\nu} - \frac{\Delta t}{2\nu} + F_A \left( G_x \left( 0, \frac{y}{\epsilon} \right) \right) + \mathcal{O} \left( \frac{\Delta x}{\epsilon} \right) \leq 0 \quad (1.6.6)$$

$$\text{if } y \neq 0, \quad \frac{t-s}{\nu} + H_\beta \left( -G_y \left( \frac{x}{\epsilon}, \frac{y}{\epsilon} \right) \right) + \mathcal{O}(\sqrt{\delta}) \geq \sigma \quad (1.6.7)$$

$$\text{if } y = 0, \quad \frac{t-s}{\nu} + F_A \left( -G_y \left( \frac{x}{\epsilon}, 0 \right) \right) \geq \sigma. \quad (1.6.8)$$

Combining these viscosity inequalities, we get in all cases:

$$\sigma \leq \frac{\Delta t}{2\nu} + \mathcal{O} \left( \frac{\Delta x}{\epsilon} \right) + \mathcal{O}(\sqrt{\delta}) =: \sigma^*. \quad (1.6.9)$$

**Step 3: Estimate of the supremum.** We proved in the previous step that, if  $\sigma > \sigma^*$  with  $\sigma^*$  defined in (1.6.9), then either  $M_{\epsilon,\delta} \leq 0$  or  $M_{\epsilon,\delta}$  is reached either for  $t = 0$  or  $s = 0$ .

If  $t = 0$ , then

$$M_{\epsilon,\delta} \leq u_0(x) - u_0(y) + Cs - \frac{s^2}{2\nu}.$$

Using the fact that  $u_0$  is  $L_0$ -Lipschitz and  $d(x, y) = \mathcal{O}(\epsilon)$  (see (1.6.4)) one can deduce

$$\begin{aligned} M_{\epsilon,\delta} &\leq L_0 d(x, y) + \sup_{r>0} \left( Cr - \frac{r^2}{2\nu} \right) \\ &\leq \mathcal{O}(\epsilon) + \mathcal{O}(\nu). \end{aligned}$$

If  $s = 0$ , then we can argue similarly (by using (1.4.2)) and get

$$M_{\epsilon,\delta} \leq \mathcal{O}(\epsilon) + \mathcal{O}(\nu).$$

**Step 4: Conclusion.** We proved that for  $\sigma > \sigma^*$  with  $\sigma^*$  defined in (1.6.9) that  $M_{\epsilon,\delta} \leq \mathcal{O}(\epsilon) + \mathcal{O}(\nu)$ . This implies that for all  $(t, x) \in \mathcal{G}_h$ ,  $t \leq T/2$ , we have

$$u^h(t, x) - u(t, x) \leq \epsilon G \left( \frac{x}{\epsilon}, \frac{x}{\epsilon} \right) + \frac{\delta}{2} d^2(x, 0) + \frac{2\eta}{T} + \sigma t + \mathcal{O}(\epsilon) + \mathcal{O}(\nu).$$

Replacing  $\sigma$  by  $2\sigma^*$ , say, and recalling that  $G(x, x) = 0$  for all  $x \in J$ , we deduce that for  $(t, x) \in \mathcal{G}_h$  and  $t \leq T/2$  (after letting  $\delta \rightarrow 0$  and  $\eta \rightarrow 0$ ), we get (1.6.2). Using the CFL condition (1.1.13) and optimizing with respect to  $\epsilon$  and  $\nu$  yields the desired result.  $\square$

## 1.7 Proofs of some technical results

In order to prove Lemma 1.6.1, we need the following one.

**Lemma 1.7.1 (A priori control at the same time).** *Assume that  $u_0$  is Lipschitz continuous. Let  $T > 0$  and let  $u^h$  be a sub-solution of (1.1.9)-(1.1.11) and  $u$  be a super-solution of (1.1.1)-(1.1.2). Then there exists a constant  $C = C_T > 0$  such that for all  $t \in [0, T)$ ,  $x, y \in J$ , we have*

$$u^h(t, x) \leq u(t, y) + C_T(1 + d(x, y)). \quad (1.7.1)$$

We first derive Lemma 1.6.1 from Lemma 1.7.1.

**Proof of Lemma 1.6.1.** Let us fix some  $h$  and let us consider the sub-solution  $u^-$  of (1.1.9) and the super-solution  $u^+$  of (1.1.1) defined as :

$$u^+(t, x) = u_0(x) + C_0 t$$

$$u^-(n\Delta t, i\Delta x) = u_0(i\Delta x) - C_0 n\Delta t$$

where

$$C_0 = \max \left\{ |A|, \max_{\alpha=1, \dots, N} \max_{|p_\alpha| \leq L_0} |H_\alpha(p_\alpha)|; \max_{|p_\alpha| \leq L_0} F(p_1, \dots, p_N) \right\}$$

and  $L_0$  denotes the Lipschitz constant of  $u_0$ . We have for all  $(t, x) \in [0, T) \times J$ ,  $(s, y) \in \mathcal{G}_h$

$$u^-(t, x) - u^+(s, y) \leq 2C_0 T + L_0 d(x, y).$$

We first apply Lemma 1.7.1 to control  $u^h(t, x) - u^-(t, x)$  and then apply Lemma 1.6.1 to control  $u^+(s, y) - u(s, y)$ . Finally we get the control on  $u^h(t, x) - u(s, y)$ .  $\square$

We can now prove Lemma 1.7.1.

**Proof of Lemma 1.7.1.** We define  $\varphi$  in  $J^2$  as

$$\varphi(x, y) = \sqrt{1 + d^2(x, y)}.$$

Since,

$$d^2(x, y) = \begin{cases} (x - y)^2 & \text{if } (x, y) \in J_\alpha \times J_\alpha \\ (x + y)^2 & \text{if } (x, y) \in J_\alpha \times J_\beta \end{cases}$$

we see that  $d^2$  (and consequently  $\varphi$ ) is in  $C^{1,1}$  in  $J^2$ . Moreover  $\varphi$  satisfies

$$|\varphi_x(x, y)|, |\varphi_y(x, y)| \leq 1. \quad (1.7.2)$$

For constants  $C_1, C_2 > 0$  to be chosen let us consider

$$M = \sup_{t \in [0, T), x \in \mathcal{G}_h, y \in J} (u^h(t, x) - u(t, y) - C_2 t - C_1 \varphi(x, y)).$$

The result follows if we show that  $M$  is non-positive for  $C_1$  and  $C_2$  large enough. Assume by contradiction that  $M > 0$  for any  $C_1$  and  $C_2$ . Then for  $\eta, \delta > 0$  small enough, we have  $M_{\eta, \delta} \geq \frac{M}{2} > 0$  with

$$M_{\eta, \delta} = \sup_{t \in [0, T], x \in \mathcal{G}_h, y \in J} \left( u^h(t, x) - u(t, y) - C_2 t - C_1 \varphi(x, y) - \frac{\eta}{T-t} - \frac{\delta}{2} d^2(y, 0) \right) \quad (1.7.3)$$

Recalling that there exists  $C > 0$  such that

$$|u^h(t, x) - u_0(x)| \leq Ct \quad \text{and} \quad |u(t, y) - u_0(y)| \leq Ct$$

(see Theorem 1.2.5 and (1.4.2)) and using that  $u_0$  is Lipschitz continuous, we see that  $M_{\eta, \delta}$  is reached for  $C_1$  large enough (larger than the Lipschitz constant of  $u_0$ ) and

$$\delta d^2(y, 0) \leq C. \quad (1.7.4)$$

We introduce the short hand notation

$$H(x, p) = \begin{cases} H_\alpha(p) & \text{for } p = p_\alpha & \text{if } x \in J_\alpha^*, \\ F(p) & \text{for } p = (p_1, \dots, p_N) & \text{if } x = 0. \end{cases} \quad (1.7.5)$$

Then the classical time penalization (or doubling variable technique) implies the existence of  $a, b \in \mathbb{R}$  such that

$$\begin{aligned} a + H(x, C_1 \varphi_x(x, y)) &\leq C \Delta x \\ b + H(y, -C_1 \varphi_y(x, y) - \delta d(0, y)) &\geq 0 \end{aligned}$$

with

$$a - b = C_2 + \eta(T-t)^{-2} \geq C_2.$$

Subtracting these inequalities yields

$$C_2 \leq H(y, -C_1 \varphi_y(x, y) - \delta d(0, y)) - H(x, C_1 \varphi_x(x, y)) + S \Delta x$$

Using bounds (1.7.2) and (1.7.4) yields to a contradiction for  $C_2$  large enough.  $\square$

## 1.8 Construction of $\tilde{F}$

**Lemma 1.8.1.** *There exists  $\tilde{F}$ , such that*

1.  $\tilde{F}$  satisfies (1.1.8);
2.  $F = \tilde{F}$  in  $Q_0$ ;
3. For a.e.  $p \in \mathbb{R}^N$ ,  $(-\nabla \cdot \tilde{F})(p) \leq \sup_{Q_0} (-\nabla \cdot F)$ .

*Proof.* Let  $I_\alpha$  denote  $[\underline{p}_\alpha^0; \bar{p}_\alpha]$  so that  $Q_0 = \prod_\alpha I_\alpha$ . We first define  $\tilde{F}$  for  $p$  in the following set

$$D_\alpha = \{p \in \mathbb{R}^N : \exists \alpha \in \{1, \dots, N\} \text{ such that } p_\alpha \notin I_\alpha \text{ and } \forall \beta \neq \alpha, p_\beta \in I_\beta\}.$$

For  $p \in D_\alpha$ , we then define

$$\tilde{F}(p) = F(p_1, \dots, \mathcal{P}_\alpha(p_\alpha), \dots, p_N) - C_\alpha(p_\alpha - \mathcal{P}_\alpha(p_\alpha))$$

where

$$C_\alpha = \min_{p_\alpha \in I_\alpha} \left( -\frac{\partial F}{\partial p_\alpha}(p_1, \dots, \mathcal{P}_\alpha(p_\alpha), \dots, p_N) \right),$$

and

$$\mathcal{P}_\alpha(r) = \begin{cases} \underline{p}_\alpha^0 & \text{if } r < \underline{p}_\alpha^0, \\ r & \text{if } r \in I_\alpha, \\ \bar{p}_\alpha & \text{if } r > \bar{p}_\alpha. \end{cases}$$

Remark that in view of the assumptions made on  $F$ , we have  $C_\alpha^r > 0$  which will ensure that (1.1.8) holds true.

For  $p \notin \cup_{\beta=1}^N D_\beta$ , let  $\bar{p}_\alpha$  denote  $p_\alpha - \mathcal{P}_\alpha(p_\alpha)$  and  $\bar{p} = (\bar{p}_1, \dots, \bar{p}_N)$ . We next define

$$\lambda_\alpha = \frac{|\bar{p}_\alpha|}{|\bar{p}_1| + \dots + |\bar{p}_N|}.$$

We first remark that  $\lambda_\alpha = 0$  if  $p_\alpha \in I_\alpha$ . We next remark that for all  $\alpha$ , there exist  $P_\alpha \in D_\alpha$  such that

$$p = \sum_{\alpha=1}^N \lambda_\alpha P_\alpha.$$

Moreover,  $P_\alpha$  is unique if  $\lambda_\alpha \neq 0$ . We thus define

$$\tilde{F}(p) = \sum_{\alpha=1}^N \lambda_\alpha \tilde{F}(P_\alpha).$$

It is now easy to check that (1.1.8) and Item 3 are satisfied. This ends the proof of the Lemma.  $\square$

## 1.9 Relation between the junction and BLN conditions

Consider the following scalar conservation law posed on  $(0, +\infty)$ ,

$$\begin{cases} \partial_t v + \partial_x(H(v)) = 0, & t > 0, x > 0, \\ v(t, 0) = v_b(t), & t > 0, \\ v(0, x) = v_0(x), & x > 0. \end{cases}$$

The usual BLN condition asserts that the trace  $v_\tau$  of the entropy solution at  $x = 0$  (if it exists) of the previous scalar conservation law should satisfy

$$\forall \kappa \in [\min(v_b, v_\tau), \max(v_b, v_\tau)], \quad \text{sgn}(v_\tau - v_b)(H(v_\tau) - H(\kappa)) \leq 0.$$

If  $H$  is quasi-convex, this reduces to

$$H(v_\tau) = \max(H^-(v_\tau), H^+(v_b)).$$

This corresponds to a flux limiter  $A = H^+(v_b)$ .



## Chapter 2

# New approach to error estimates for finite difference schemes associated with Hamilton-Jacobi equations on a junction

*Ce chapitre vise à étudier un schéma numérique montone aux différences finies associé à des équations de Hamilton-Jacobi posées sur une jonction. En utilisant une nouvelle approche, nous améliorons les résultats obtenus dans le Chapitre 1, et nous prouvons des estimations d'erreurs d'ordre  $(\Delta x)^{\frac{1}{2}}$  dans  $L_{loc}^{\infty}$  pour des conditions de jonction du type contrôle optimal.*

**Abstract.** In this chapter, we derive error estimates for monotone (time explicit) finite difference schemes associated with first order Hamilton-Jacobi equations posed on a junction. Using a new approach, we improve the results obtained in chapter 1, and we prove, for a larger class of Hamiltonians, that error estimates are of order  $(\Delta x)^{\frac{1}{2}}$  in  $L_{loc}^{\infty}$  for junction conditions of optimal-control type at least if the flux is "strictly limited".

## 2.1 Introduction

In this chapter we are interested in numerical approximation of first order Hamilton Jacobi equations posed on a one dimensional domain containing one single singularity. Such a domain is referred to as a junction: a network made of a node and a finite number of infinite edges. The theory of viscosity solutions for such equations on such domains has reached maturity by now [2, 65, 67, 102, 103]. In particular, it is now understood that general junction conditions reduce to special ones of optimal-control type [64–66]. For the maximal flux-limited junction conditions, Costeseque, Lebacque and Monneau [41] introduced a monotone numerical scheme and proved its convergence. It is explained in [42] that the proof of the comparison principle between sub- and super-solutions of the continuous Hamilton-Jacobi equation can be adapted in order to derive error estimates between the numerical solution associated with monotone (stable and consistent) schemes and the continuous solution. Hereafter in Chapter 1, we derived an error estimate à la Crandall-Lions [42] of order  $(\Delta x)^{1/2}$  in  $L_{loc}^{\infty}$ , if the minima of the Hamiltonians are equal, using a so called vertex test function introduced in [65], because the penalization procedure lying on the classical penalization term  $\varepsilon^{-1}|x - y|^2$  is known to fail at a junction.

Our main result is to introduce a new approach in deriving error estimates à la Crandall-Lions for flux-limited junction conditions, by replacing the vertex test function by the *reduced minimal action*  $\mathcal{D}$  following the *Oleinik-Lax representation formula* introduced in [67]. Thus we improve the results obtained in Chapter 1 to  $(\Delta x)^{\frac{1}{2}}$  in  $L_{loc}^{\infty}$ , for a larger class of Hamiltonians. In order to derive error estimates as in [42], it is important to study the regularity of the test function. More precisely, we prove (Proposition 2.3.12) that its gradient is locally Lipschitz continuous, at least if the flux is "strictly limited" and far away from a special curve. But we also see that the reduced minimal action is not of class  $C^1$  on this curve. However we can get "weaker" viscosity inequalities thanks to a result in [65] (see Proposition 2.2.3). Such a regularity result is of independent interest.

### 2.1.1 Setting of the problem

**The junction.** It can be viewed as the set of  $N$  distinct copies ( $N \geq 1$ ) of the half-line which are glued at the origin. For  $\alpha = 1, \dots, N$ , each branch  $J_{\alpha}$  is assumed

to be isometric to  $[0, +\infty)$  and

$$J = \bigcup_{\alpha=1, \dots, N} J_\alpha \quad \text{with} \quad J_\alpha \cap J_\beta = \{0\} \quad \text{for} \quad \alpha \neq \beta$$

where the origin 0 is called the *junction point*. For points  $x, y \in J$ ,  $d(x, y)$  denotes the geodesic distance on  $J$  defined as

$$d(x, y) = \begin{cases} |x - y| & \text{if } x, y \text{ belong to the same branch,} \\ |x| + |y| & \text{if } x, y \text{ belong to different branches.} \end{cases}$$

For a real-valued function  $u$  defined on  $J$ ,  $\partial_\alpha u(x)$  denotes the (spatial) derivative of  $u$  at  $x \in J_\alpha \setminus \{0\}$  and the gradient of  $u$  is defined as follows,

$$u_x(x) := \begin{cases} \partial_\alpha u(x) & \text{if } x \in J_\alpha^*, \\ (\partial_1 u(0), \dots, \partial_N u(0)) & \text{if } x = 0. \end{cases}$$

**HJ equation on a junction.** We consider the following Hamilton-Jacobi equation posed on the junction  $J$ ,

$$\begin{cases} u_t + H_\alpha(u_x) = 0 & \text{in } (0, T) \times J_\alpha \setminus \{0\}, \\ u_t + F_A\left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}\right) = 0 & \text{in } (0, T) \times \{0\}, \end{cases} \quad (2.1.1)$$

submitted to the initial condition

$$u(0, x) = u_0(x), \quad \text{for } x \in J \quad (2.1.2)$$

where  $u_0$  is Lipschitz continuous in  $J$ .

We consider Hamiltonians  $H_\alpha$  satisfying the following conditions

$$\text{There exists } p_0^\alpha \in \mathbb{R} \text{ such that } \begin{cases} H_\alpha \in C^2(\mathbb{R}) \text{ and } H_\alpha''(p_0^\alpha) > 0 \\ \pm H_\alpha'(p) > 0 \text{ for } \pm(p - p_0^\alpha) > 0 \\ \lim_{|p| \rightarrow +\infty} H_\alpha(p) = +\infty. \end{cases} \quad (2.1.3)$$

In particular  $H_\alpha$  is non-increasing in  $(-\infty, p_0^\alpha]$  and non-decreasing in  $[p_0^\alpha, +\infty)$ , and we set

$$H_\alpha^-(p) = \begin{cases} H_\alpha(p) & \text{for } p \leq p_0^\alpha \\ H_\alpha(p_0^\alpha) & \text{for } p \geq p_0^\alpha \end{cases} \quad \text{and} \quad H_\alpha^+(p) = \begin{cases} H_\alpha(p_0^\alpha) & \text{for } p \leq p_0^\alpha, \\ H_\alpha(p) & \text{for } p \geq p_0^\alpha \end{cases}$$

where  $H_\alpha^-$  is non-increasing and  $H_\alpha^+$  is non-decreasing.

The second equation in (2.1.1) is referred to as the *junction condition*, where we introduce a one-parameter family of junction conditions. Given a *flux limiter*  $A \in \mathbb{R} \cup \{-\infty\}$ , the  $A$ -limited flux junction function is defined for  $p = (p_1, \dots, p_N)$  as,

$$F_A(p) = \max \left( A, \max_{\alpha=1, \dots, N} H_\alpha^-(p_\alpha) \right) \quad (2.1.4)$$

for some given  $A \in \mathbb{R} \cup \{-\infty\}$  where  $H_\alpha^-$  is the non-increasing part of  $H_\alpha$ .

We point out that all the junction functions  $F_A$  associated with  $A \in [-\infty, A_0]$  coincide if one chooses

$$A_0 = \max_{\alpha=1, \dots, N} \min_{\mathbb{R}} H_\alpha. \quad (2.1.5)$$

Without loss of generality (see [65, Lemma 3.1]), we consider in this chapter that  $p_0^\alpha = 0$  for  $\alpha = 1, \dots, N$ , i.e

$$\min H_\alpha = H_\alpha(0). \quad (2.1.6)$$

Indeed,  $u$  solves (2.1.1) if and only if  $\tilde{u}(t, x) := u(t, x) - p_0^\alpha x$  for  $x \in J_\alpha$  solves the same equation in which  $H_\alpha$  is replaced by  $\tilde{H}_\alpha(p) = H_\alpha(p + p_0^\alpha)$ . We have the same result for  $u^h$  the solution of the scheme (2.1.12).

**The optimal control framework.** It is well known that the Legendre-Fenchel conjugate is crucial in establishing a link between the general Cauchy problem (2.1.1)-(2.1.2) and a control problem [83]. Through this link, we obtain the representation formula for the exact solution. Before treating the case where the Hamiltonians  $H_\alpha$  satisfy (2.1.3), we first consider the case of Hamiltonians satisfying the hypotheses of [67] i.e.,

$$\left\{ \begin{array}{l} \text{(Regularity)} \quad H_\alpha \text{ is of class } C^2 \\ \text{(Coercivity)} \quad \lim_{|p| \rightarrow +\infty} H_\alpha(p) = +\infty \\ \text{(Convexity)} \quad H_\alpha \text{ is convex and is the Legendre Fenchel transform of } L_\alpha \\ \quad \text{where } L_\alpha \text{ is of class } C^2 \text{ and satisfies (B0).} \end{array} \right. \quad (2.1.7)$$

We recall that

$$H_\alpha(p) = L_\alpha^*(p) = \sup_{q \in \mathbb{R}} (pq - L_\alpha(q)). \quad (2.1.8)$$

We consider the following hypothesis for  $L_\alpha$ ,

$$\text{(B0)} \quad \text{There exists a constant } \gamma > 0 \text{ such that for all } \alpha = 1, \dots, N, \text{ the } C^2(\mathbb{R}) \text{ functions } L_\alpha \text{ satisfy } L_\alpha'' \geq \gamma > 0.$$

An optimal control interpretation of the Hamilton-Jacobi equation (2.1.1) is given in [18, 25, 80, 83]. We define the set of admissible controls at a point  $x \in J$  by

$$\mathcal{U}(x) = \begin{cases} \mathbb{R}e_{\alpha_0} & \text{if } x \in J_{\alpha_0}^*, \\ \cup_{\alpha=1, \dots, N} \mathbb{R}^+ e_\alpha & \text{if } x = 0. \end{cases}$$

For  $(s, y), (t, x) \in [0, T] \times J$  with  $s \leq t$ , we define the set of admissible trajectories from  $(s, y)$  to  $(t, x)$  by

$$\mathcal{A}(s, y; t, x) = \left\{ X \in W^{1,1}([s, t], \mathbb{R}^2) : \left. \begin{array}{ll} X(\tau) \in J & \text{for all } \tau \in (s, t) \\ \dot{X}(\tau) \in \mathcal{U}(X(\tau)) & \text{for a.e } \tau \in (s, t) \\ X(s) = y & \text{and } X(t) = x \end{array} \right\}. \quad (2.1.9)$$

For  $P = pe_i \in \mathcal{U}(x)$  with  $p \in \mathbb{R}$ , we define the Lagrangian on the junction

$$L(x, p) = \begin{cases} L_\alpha(p) & \text{if } x \in J_\alpha^*, \\ L_A(p) & \text{if } x = 0, \end{cases} \quad (2.1.10)$$

with

$$L_A(p) = \min \left( -A, \min_{\alpha=1, \dots, N} L_\alpha(p) \right).$$

The Hopf-Lax representation formula of the solution of (2.1.1)-(2.1.2) is given in [4, 67] by

$$u_{oc}(t, x) = \inf_{y \in J} \{u_0(y) + \mathcal{D}(0, y; t, x)\} \quad (2.1.11)$$

with

$$\mathcal{D}(0, y; t, x) = \inf_{X \in \mathcal{A}(0, y; t, x)} \left\{ \int_0^t L(X(\tau), \dot{X}(\tau)) d\tau \right\}.$$

## 2.1.2 Presentation of the scheme

The domain  $(0, +\infty) \times J$  is discretized with respect to time and space. We choose a regular grid in order to simplify the presentation. The space step is denoted by  $\Delta x$  and the time step by  $\Delta t$ . If  $h$  denotes  $(\Delta t, \Delta x)$ , the mesh (or grid)  $\mathcal{G}_h$  is chosen as

$$\mathcal{G}_h = \{n\Delta t : n \in \mathbb{N}\} \times J^{\Delta x}$$

where

$$J^{\Delta x} = \bigcup_{\alpha=1, \dots, N} J_\alpha^{\Delta x} \quad \text{with} \quad J_\alpha \supset J_\alpha^{\Delta x} \simeq \{i\Delta x : i \in \mathbb{N}\}.$$

It is convenient to write  $x_i^\alpha$  for  $i\Delta x \in J_\alpha$ .

A numerical approximation  $u^h$  of the solution  $u$  of the Hamilton-Jacobi equation is defined in  $\mathcal{G}_h$ ; the quantity  $u^h(n\Delta t, x_i^\alpha)$  is simply denoted by  $U_i^{\alpha, n}$ . We want it to be an approximation of  $u(n\Delta t, x_i^\alpha)$  for  $n \in \mathbb{N}$ ,  $i \in \mathbb{N}$ , where  $\alpha$  stands for the index of the branch.

We consider the following time-explicit scheme, introduced in [41], for  $n \geq 0$ ,

$$\begin{cases} \frac{U_i^{\alpha, n+1} - U_i^{\alpha, n}}{\Delta t} + \max\{H_\alpha^+(p_{i,-}^{\alpha, n}), H_\alpha^-(p_{i,+}^{\alpha, n})\} = 0, & i \geq 1, \quad \alpha = 1, \dots, N \\ U_0^{\beta, n} := U_0^n, & i = 0, \quad \beta = 1, \dots, N \\ \frac{U_0^{n+1} - U_0^n}{\Delta t} + F_A(p_{0,+}^{1, n}, \dots, p_{0,+}^{N, n}) = 0, \end{cases} \quad (2.1.12)$$

where  $p_{i,\pm}^{\alpha, n}$  are the discrete (space) gradients defined by

$$p_{i,+}^{\alpha, n} := \frac{U_{i+1}^{\alpha, n} - U_i^{\alpha, n}}{\Delta x}, \quad p_{i,-}^{\alpha, n} := \frac{U_i^{\alpha, n} - U_{i-1}^{\alpha, n}}{\Delta x} \quad (2.1.13)$$

with the initial condition

$$U_i^{\alpha,0} = u_0(x_i^\alpha), \quad i \geq 0, \quad \alpha = 1, \dots, N. \quad (2.1.14)$$

The following Courant-Friedrichs-Lewy (CFL) condition ensures that the explicit scheme is monotone,

$$\frac{\Delta x}{\Delta t} \geq \max_{\substack{\alpha=1,\dots,N, \\ i \geq 0, 0 \leq n \leq n_T}} |H'_\alpha(p_{i,+}^{\alpha,n})| \quad (2.1.15)$$

where the integer  $n_T$  is assumed to be defined as  $n_T = \lfloor \frac{T}{\Delta t} \rfloor$  for a given  $T > 0$ .

### 2.1.3 Main result

The main result of this chapter lies in getting error estimates in the case of flux-limited junction conditions.

**Theorem 2.1.1 (Error estimates for flux-limited junction conditions).** *Let  $u_0$  be Lipschitz continuous,  $u^h$  be the solution of the associated numerical scheme (2.1.12)-(2.1.14) and  $u$  be the viscosity solution of (2.1.1)-(2.1.2) for some  $A \in \mathbb{R}$ . If the CFL condition (2.1.15) is satisfied, then there exists  $C > 0$  (independent of  $h$ ) such that*

$$\sup_{[0,T) \times J \cap \mathcal{G}_h} |u^h(t,x) - u(t,x)| \leq \begin{cases} C(\Delta x)^{1/2} & \text{if } A > A_0, \\ C(\Delta x)^{2/5} & \text{if } A = A_0. \end{cases} \quad (2.1.16)$$

### 2.1.4 Comments

**Numerical schemes for Hamilton-Jacobi equations.** Up to our knowledge, there are only few papers dealing with numerical schemes for HJ equations on junctions or networks. Apart from [41], we mention [33], where a convergent semi-Lagrangian scheme is introduced for equations of eikonal type. In [57], an adapted Lax-Friedrichs scheme is used to solve a traffic model.

For optimal control problems, the numerical approximation of (HJ) has already been studied using schemes based on the discrete dynamic programming principle. Essentially, these schemes are built by replacing the continuous optimal control problem by its discrete time version. We refer to Capuzzo Dolcetta [35], Capuzzo Dolcetta-Ishii [37] for the results concerning the convergence of  $u_h$  to  $u$  and the a priori estimates (of order  $\Delta x$ ), in the  $L^\infty$ , giving the order of convergence of the discrete-time approximation. We refer to Falcone [48] for the results related to the order of convergence of the fully discrete (i.e. in space and time) approximation and for the construction of the algorithm, we mention that under a semiconcavity assumption the rate of convergence is of order 1. We cite also [50] and references therein for discrete time high order schemes for Hamilton Jacobi Bellman equations.

**Link with monotone schemes for scalar conservation laws.** Following [41], one can deduce that the convergence result of the finite difference scheme (2.1.12), implies the convergence of a monotone scheme for scalar conservation laws (in the sense of distributions).

On one hand, the derivative of a viscosity solution of a Hamilton-Jacobi equation posed on the real line is known to coincide with the entropy solution of the corresponding scalar conservation law. It is therefore reasonable to expect that the error between the viscosity solution of the Hamilton-Jacobi equation and its approximation is as good as the one obtained between the entropy solution of the scalar conservation law and its approximation. It is known since [20] that Dirichlet boundary conditions imposed to scalar conservation laws should be understood in a generalized sense. This can be seen by studying the parabolic regularization of the problem. A boundary layer analysis can be performed for systems if the solution of the conservation law is smooth. Depending on the fact that the boundary is characteristic or not, the error is  $\epsilon^{\frac{1}{2}}$  or  $\epsilon$ .

**The contribution of the chapter.** We improve for quasi-convex Hamiltonians, the error estimates obtained in Chapter 1 and we prove, if the flux is well chosen, that the error is of order  $(\Delta x)^{\frac{1}{2}}$  in  $L_{loc}^{\infty}$ . Our approach is slightly different from their approach, we use the function  $\mathcal{D}_0$  relative to the optimal control interpretation of the problem [83], in the penalization procedure. We emphasize that the key point that allowed us to improve the error estimate is based on a work of Imbert and Monneau ([65]), where comparison principle on networks, and particularly on a junction, is proved via piecewise  $C^1$  test functions.

**Organization of the chapter.** The remaining of the chapter is organized as follows. In Section 2.2, the definition of viscosity solutions is made precise. In Section 2.3, the important properties of optimal trajectories are given. More precisely, we study the reduced minimal action for a "strictly" limited flux and prove that the gradient is locally Lipschitz continuous (at least if the flux is strictly limited). We prove also the compatibility condition between Hamiltonians, a crucial step in order to derive error estimates. Section 2.4 is devoted to the proof of the main result of the chapter, the error estimates.

## 2.2 Preliminaries

### 2.2.1 Viscosity solutions

We give first a definition of viscosity solutions for (2.1.1). For a more general introduction to viscosity solutions, the reader could refer to Barles [25] and to Crandall, Ishii, Lions [45]. The reader can also refer to [67] for application to the modeling of traffic flows.

**Space of test functions.** For a smooth real valued function  $u$  defined on  $J$ , we denote by  $u^\alpha$  the restriction of  $u$  to  $(0, T) \times J_\alpha$ . We define  $J_T = (0, T) \times J$  for  $T > 0$ .

Then we define the natural space of functions on the junction,

$$C^1(J_T) = \{u \in C(J_T) : \forall \alpha = 1, \dots, N, u^\alpha \in C^1((0, T) \times J_\alpha)\}.$$

**Viscosity solutions.** In order to define classical viscosity solutions, we recall the definition of upper and lower semi-continuous envelopes  $u^*$  and  $u_*$  of a (locally bounded) function  $u$  defined on  $[0, T) \times J$  :

$$u^*(t, x) = \limsup_{(s, y) \rightarrow (t, x)} u(s, y) \quad u_*(t, x) = \liminf_{(s, y) \rightarrow (t, x)} u(s, y).$$

It is convenient to introduce the following shorthand notation

$$H(x, p) = \begin{cases} H_\alpha(p) & \text{for } p = p_\alpha \in \mathbb{R} & \text{if } x \in J_\alpha^*, \\ F_A(p) & \text{for } p = (p_1, \dots, p_N) \in \mathbb{R}^N & \text{if } x = 0. \end{cases} \quad (2.2.1)$$

**Definition 2.2.1 (Viscosity solution).** Assume that the Hamiltonians satisfy (2.1.3), and let  $u : [0, T) \times J \rightarrow \mathbb{R}$ .

( i ) We say that  $u$  is a *sub-solution* (resp. *super-solution*) of (2.1.1) in  $J_T$ , if for all  $(t_0, x_0) \in J_T$  and for all test function  $\varphi \in C^1(J_T)$  such that

$$u^* \leq \varphi \quad (\text{resp. } u_* \geq \varphi) \quad \text{in a neighborhood of } (t_0, x_0) \in J_T$$

with equality at  $(t_0, x_0)$ , we have

$$\varphi_t + H(x, \varphi_x) \leq 0 \quad (\text{resp. } \geq 0) \quad \text{at } (t_0, x_0).$$

( ii ) We say that  $u$  is a *sub-solution* (resp. *super-solution*) of (2.1.1)-(2.1.2) on  $[0, T) \times J$  if additionally

$$u^*(0, x) \leq u_0(x) \quad (\text{resp. } u_*(0, x) \geq u_0(x)) \quad \text{for all } x \in J.$$

( iii ) We say that  $u$  is a (viscosity) solution of (2.1.1)-(2.1.2) if  $u$  is both a sub-solution and a super-solution (2.1.1)-(2.1.2).

We recall the following result extracted from [67].

**Theorem 2.2.2** (Existence and uniqueness [67]). *Assume that the Hamiltonians satisfy (B0) and that the initial datum  $u_0$  is Lipschitz continuous and let  $T > 0$ . Then there exists a unique viscosity solution  $u$  of (2.1.1)-(2.1.2) on  $J_T$  in the sense of Definition 2.2.1, satisfying for some constant  $C_T > 0$  only depending on  $H$ ,  $u_0$  and  $T$ ,*

$$|u(t, x) - u_0(x)| \leq C_T t \quad \text{for all } (t, x) \in J_T.$$

*Moreover, the function  $u$  is Lipschitz continuous with respect to  $(t, x)$  on  $J_T$ . In particular, we have*

$$\|\nabla u\|_\infty \leq C_T.$$

Under the general assumptions on the Hamiltonians (2.1.3), the uniqueness of a solution  $u$  of (2.1.1)-(2.1.2) is proved in [65] using a special function denoted by the vertex test function. In fact, Imbert and Monneau proved that general junction conditions can be reduced to flux limited ones, for some  $A \in \mathbb{R}$ . Now, if we replace the condition (2.1.3) by a stronger assumption (2.1.7), defined in Subsection 2.1.1, the uniqueness of the solution is proved otherwise in [67] by combining a super-optimality principle for super-solutions and a direct comparison principle for sub-solutions (the proof relies on an optimal control interpretation).

The following proposition is a main tool in the proof of error estimates. Indeed, we use a test function which is not  $C^1$  with respect to the gradient variable at one point and this proposition allows us to get a “weak viscosity inequality”. We don’t give the proof since it is the same as the proof of [65, Proposition 2.16].

**Proposition 2.2.3** (Non  $C^1$  test function at one point [65]). *Assume that  $H$  satisfies (2.1.3) and let  $u$  be a solution of*

$$u_t + H_\alpha(u_x) = 0 \quad \text{in } (0, T) \times J_\alpha \setminus \{0\}.$$

For all  $x_0 \in J_\alpha \setminus \{0\}$  and all test function  $\varphi \in C^1((0, T) \times J_\alpha \setminus \{0, x_0\})$

$$u^* \leq \varphi \quad (\text{resp. } u_* \geq \varphi) \quad \text{in a neighborhood of } (t_0, x_0) \in (0, T) \times J_\alpha \setminus \{0\}$$

with equality at  $(t_0, x_0)$ , we have

$$\varphi_t(t_0, x_0) + \max \{H_\alpha^+(\varphi_x(t_0, x_0^-), H_\alpha^-(\varphi_x(t_0, x_0^+))\} \leq 0 \quad (\text{resp. } \geq 0).$$

## 2.2.2 Convergence result

Under the CFL condition (2.1.15), the convergence result of the numerical scheme (2.1.12) as the mesh size tends to zero, was established in [41]. It is thus known that the scheme converges if it is stable consistent and monotone [21, 42].

We recall now how to derive discrete viscosity inequalities for monotone schemes.

**Lemma 2.2.4 (Discrete viscosity inequalities).** *Let  $u^h$  be a solution of (2.1.12)-(2.1.14) with  $F_A$  defined in (2.1.4). If  $u^h - \varphi$  has a global maximum (resp. global minimum) on  $\mathcal{G}_h$  at  $(\bar{t} + \Delta t, \bar{x})$ , then*

$$\delta_t \varphi(\bar{t}, \bar{x}) + \mathcal{H}(\bar{x}, D_+ \varphi(\bar{t}, \bar{x}), D_- \varphi(\bar{t}, \bar{x})) \leq 0. \quad (\text{resp. } \geq 0)$$

where

$$\mathcal{H}(x, p_+, p_-) = \begin{cases} \max\{H_\alpha^+(p_-), H_\alpha^-(p_+)\} & \text{if } \bar{x} \neq 0 \\ \max\{A, \max_\alpha H_\alpha^-(p_+)\} & \text{if } \bar{x} = 0 \end{cases}$$

and

$$\begin{aligned} D_+ \varphi(\bar{t}, \bar{x}) &= \begin{cases} \frac{1}{\Delta x} \{\varphi(\bar{t}, \bar{x} + \Delta x) - \varphi(\bar{t}, \bar{x})\} & \text{if } \bar{x} \neq 0, \\ \left(\frac{1}{\Delta x} \{\varphi^\alpha(\bar{t}, \Delta x) - \varphi^\alpha(\bar{t}, 0)\}\right)_\alpha & \text{if } \bar{x} = 0, \end{cases} \\ D_- \varphi(\bar{t}, \bar{x}) &= \frac{1}{\Delta x} \{\varphi(\bar{t}, \bar{x}) - \varphi(\bar{t}, \bar{x} - \Delta x)\}, \\ \delta_t \varphi(\bar{t}, \bar{x}) &= \frac{1}{\Delta t} \{\varphi(\bar{t} + \Delta t, \bar{x}) - \varphi(\bar{t}, \bar{x})\}. \end{aligned}$$

**Inverse functions of Hamiltonians.** In the proof of the error estimate, “generalized” inverse functions of  $H_\alpha^\pm$  are needed. They are defined as follow

$$\begin{cases} \pi_\alpha^+(a) := \sup\{p : H_\alpha^+(p) = \max(a, A_\alpha)\} \\ \pi_\alpha^-(a) := \inf\{p : H_\alpha^-(p) = \max(a, A_\alpha)\} \end{cases} \quad (2.2.2)$$

with the additional convention that  $(H_\alpha^\pm)^{-1}(+\infty) = \pm\infty$ , where

$$A_\alpha := \min_{\mathbb{R}} H_\alpha.$$

## 2.3 Study of the reduced minimal action

In this section, we consider that the Hamiltonians  $H_\alpha$  satisfy (2.1.7). We study the reduced minimal action  $\mathcal{D}_0$  which replace the classical term  $\frac{(x-y)^2}{2\epsilon}$  in the doubling variable method. This function allows us to prove that the error estimate is of order  $(\Delta x)^{\frac{1}{2}}$ .

### 2.3.1 Reduction of the study

We start this section by the following remark, the analysis can be reduced to the case  $(s, t) = (0, 1)$ . Precisely, using the fact that the Hamiltonian does not depend on time and is homogeneous with respect to the state, the reader can check that a change of variables in time yields the following Lemma.

**Lemma 2.3.1.** *For all  $y, x \in J$  and  $s < t$ , we have*

$$\mathcal{D}(s, y; t, x) = (t - s)\mathcal{D}\left(0, \frac{y}{t - s}; 1, \frac{x}{t - s}\right).$$

where

$$\mathcal{D}(s, y; t, x) = \inf_{X \in \mathcal{A}(s, y; t, x)} \left\{ \int_s^t L(X(\tau), \dot{X}(\tau)) d\tau \right\}.$$

This is the reason why we consider the reduced minimal action  $\mathcal{D}_0 : J^2 \rightarrow \mathbb{R}$  defined by

$$\mathcal{D}_0(y, x) = \mathcal{D}(0, y; 1, x). \quad (2.3.1)$$

We also need the following lower bound on  $\mathcal{D}$ .

**Lemma 2.3.2.** *Assume (B0). Then*

$$\mathcal{D}(s, y; t, x) \geq \frac{\gamma}{2(t - s)} d^2(x, y) - A(t - s)$$

where  $\gamma$  is defined in (B0).

Moreover,

$$\mathcal{D}(s, x; t, x) \leq L_A(0)(t - s).$$

*Proof.* We only prove the first inequality since the other inequality is elementary. As  $L'_\alpha(0) = 0$ , and  $L_\alpha(0) \geq L_A(0) = -A$ , we have

$$L_\alpha(p) \geq \frac{\gamma}{2}p^2 + L'_\alpha(0)p + L_\alpha(0) \geq \frac{\gamma}{2}p^2 - A.$$

Thus, we can write for  $X(\cdot) \in \mathcal{A}(s, y; t, x)$ ,

$$\int_s^t L(X(\tau), \dot{X}(\tau))d\tau \geq -A(t-s) + \frac{\gamma}{2} \int_s^t (\dot{X}(\tau))^2 d\tau.$$

Then Jensen's inequality allows us to conclude.  $\square$

### 2.3.2 Piecewise linear trajectories

We are going to see that the infimum defining the minimal action can be computed among piecewise linear trajectories. In order to state a precise statement, we first introduce that optimal curves are of two types depending on the position of  $y$  and  $x$  on the same branch or not: if they are, then the trajectories are of two types: either they reach the junction point, or they stay in a branch and are straight lines. For  $y \in J_\beta$ ,  $x \in J_\alpha$  with  $\beta \neq \alpha$ , the trajectories can spend some time at the junction point.

**Lemma 2.3.3.** *The infimum defining the reduced minimal action  $\mathcal{D}_0$  can be computed among piecewise linear trajectories; more precisely for all  $y, x \in J$ ,*

$$\mathcal{D}_0(y, x) = \begin{cases} \mathcal{D}_{junction}(y, x) & \text{if } \alpha \neq \beta, \\ \min(L_\alpha(x-y), \mathcal{D}_{junction}(y, x)) & \text{if } \alpha = \beta, \end{cases} \quad (2.3.2)$$

where for  $x \in J_\alpha, y \in J_\beta$

$$\mathcal{D}_{junction}(y, x) = \inf_{0 \leq t_1 \leq t_2 \leq 1} \left\{ t_1 L_\beta \left( \frac{-y}{t_1} \right) + (t_2 - t_1) L_A(0) + (1 - t_2) L_\alpha \left( \frac{x}{1 - t_2} \right) \right\}. \quad (2.3.3)$$

*Proof.* We write  $\mathcal{D}_0 = \inf_{X \in \mathcal{A}_0(y, x)} \Lambda(X)$ , where  $\Lambda(X) = \int_0^1 L(X(\tau), \dot{X}(\tau))d\tau$ . In order to prove the lemma, it is enough to consider a curve  $X \in \mathcal{A}(0, y; 1, x)$  and prove that

$$\Lambda(X) \geq \min(L_\alpha(x-y), \mathcal{D}_{junction}(y, x)).$$

For  $\alpha \neq \beta$ , the trajectories can spend some time at the junction point, hence we can write

$$\begin{aligned} \mathcal{D}_0(y, x) &= \inf_{\substack{X(0)=y \\ X(1)=x}} \left\{ \int_0^{t_1} L_\beta(\dot{X}(\tau))d\tau + \int_{t_1}^{t_2} L(X(\tau), \dot{X}(\tau))d\tau + \int_{t_2}^1 L_\alpha(\dot{X}(\tau))d\tau \right\} \\ &\geq \inf_{0 \leq t_1 \leq t_2 \leq 1} \left\{ \inf_{\substack{X(0)=y \\ X(t_1)=x}} \int_0^{t_1} L_\beta(\dot{X}(\tau))d\tau + \inf_{\substack{X(t_1)=0 \\ X(t_2)=0}} \int_{t_1}^{t_2} L(X(\tau), \dot{X}(\tau))d\tau \right. \\ &\quad \left. + \inf_{\substack{X(t_2)=0 \\ X(1)=x}} \int_{t_2}^1 L_\alpha(\dot{X}(\tau))d\tau \right\} \end{aligned}$$

then using that  $L \geq L_A$  for the second term and Jensen's inequality for all terms, we conclude that

$$\mathcal{D}_0(y, x) \geq \mathcal{D}_{junction}(y, x).$$

Now for  $\alpha = \beta$ , we can deduce from the preceding that

$$\mathcal{D}_0(y, x) \geq \min \left( \mathcal{D}_{junction}(y, x), \inf_{\substack{X(0)=y \\ X(1)=x}} \int_0^1 L_\alpha(\dot{X}(\tau)) d\tau \right).$$

Then, by Jensen's inequality once again, we can deduce (2.3.2). This ends the proof.  $\square$

In view of (2.3.2), we see that the study of  $\mathcal{D}_0$  can now be reduced to the study of  $\mathcal{D}_{junction}$ .

### 2.3.3 Study of $\mathcal{D}_{junction}$

We introduce a simpler notation of  $\mathcal{D}_{junction}$  defined in (2.3.3),

$$\mathcal{D}_{junction}(y, x) = \inf_{0 \leq t_1 \leq t_2 \leq 1} G(t_1, t_2, y, x), \quad (2.3.4)$$

where

$$G(t_1, t_2, y, x) = t_1 L_\beta \left( \frac{-y}{t_1} \right) + (t_2 - t_1) L_A(0) + (1 - t_2) L_\alpha \left( \frac{x}{1 - t_2} \right).$$

As in [67], for  $(y, x) \in J_\beta^* \times J_\alpha^*$  the function  $(t_1, t_2) \rightarrow G(t_1, t_2, y, x)$  is strictly convex on  $(0, 1) \times (0, 1)$ . Indeed, for  $t_1, t_2 \in (0, 1)$ , we compute

$$D^2 G(t_1, t_2, y, x) = \frac{L''_\beta \left( \frac{-y}{t_1} \right)}{t_1} V_y^T V_y + \frac{L''_\alpha \left( \frac{x}{1-t_2} \right)}{1-t_2} V_x^T V_x \geq 0,$$

where  $V_y = \left( \frac{-y}{t_1}, 0, 1, 0 \right)$  and  $V_x = \left( 0, \frac{x}{1-t_2}, 0, 1 \right)$  and in particular, we have

$$\frac{\partial^2}{\partial t_1^2} G(t_1, t_2, y, x) = \frac{y^2}{t_1^3} L''_\beta \left( \frac{-y}{t_1} \right) > 0,$$

and

$$\frac{\partial^2}{\partial t_2^2} G(t_1, t_2, y, x) = \frac{x^2}{(1-t_2)^3} L''_\alpha \left( \frac{x}{1-t_2} \right) > 0.$$

So we deduce that for  $(y, x) \in J_\beta^* \times J_\alpha^*$ , if the function  $(t_1, t_2) \rightarrow G(t_1, t_2, y, x)$  admits a critical point, then it reaches its infimum at this point, else it reaches its infimum at the boundary.

**Lemma 2.3.4.** *Let  $(y, x) \in J$ , and  $D_{junction}(y, x)$  as in (2.3.4). We have the following equivalences for the infimum,*

$$\begin{cases} x = 0 \Leftrightarrow t_2 = 1, \\ y = 0 \Leftrightarrow t_1 = 0. \end{cases}$$

*Proof.* It is a direct consequence of the expression (2.3.3).  $\square$

**Definition 2.3.5** (Numbers  $\xi_l^+, \xi_l^-$ ). We define  $\xi_l^-, \xi_l^+$  thanks to the following function (for  $l \in \{1, \dots, N\}$ )

$$K_l(x) = L_l(x) - xL'_l(x) - L_A(0). \quad (2.3.5)$$

We define  $(K_l^-)^{-1}$  (resp.  $(K_l^+)^{-1}$ ) as the inverse of the function  $K_l$  restricted to  $(-\infty, 0]$  (resp.  $[0, +\infty)$ ), in fact one can write

$$K'_l(x) = -xL''_l(x) < 0 \text{ on } (0, +\infty) \quad (\text{resp. } > 0 \text{ on } (-\infty, 0)).$$

More precisely, we define  $\xi_l^\pm = (K_l^\pm)^{-1}(0)$ .

**Lemma 2.3.6 (Explicit expression of  $\mathcal{D}_{junction}(y, x)$ ).** *There exists a unique function  $\tau : J \times J \rightarrow (0, 1)$  of class  $\mathcal{C}^1$  such that for  $(y, x) \in J_\beta \times J_\alpha$ , we have*

$$\mathcal{D}_{junction}(y, x) = \begin{cases} \tau(y, x)L_\beta\left(\frac{-y}{\tau(y, x)}\right) + (1 - \tau(y, x))L_\alpha\left(\frac{x}{1 - \tau(y, x)}\right) & \text{if } (y, x) \in (J_\beta^* \times J_\alpha^*) \setminus \Delta_{\beta\alpha}, \\ -yL'_\beta(\xi_\beta^-) + xL'_\alpha(\xi_\alpha^+) + L_A(0) & \text{if } (y, x) \in \Delta_{\beta\alpha}, \\ L_\alpha(x) & \text{if } y = 0 \text{ and } x > \xi_\alpha^+, \\ L_\beta(y) & \text{if } x = 0 \text{ and } y > -\xi_\beta^-, \end{cases} \quad (2.3.6)$$

where

$$\Delta_{\beta\alpha} = \left\{ (y, x) \in J_\beta \times J_\alpha, \quad \frac{x}{\xi_\alpha^+} - \frac{y}{\xi_\beta^-} \leq 1 \right\}.$$

We have a different expression of  $\mathcal{D}_{junction}$  on each subset of the previous Lemma (see Figure 2.1).

*Proof.* Writing the optimal conditions of  $G$  associated with the infimum in (2.3.4), we have

$$\begin{cases} \frac{y}{t_1}L'_\beta\left(\frac{-y}{t_1}\right) - L_A(0) + L_\beta\left(\frac{-y}{t_1}\right) = 0, \\ -\frac{x}{1-t_2}L'_\alpha\left(\frac{x}{1-t_2}\right) - L_A(0) + L_\alpha\left(\frac{x}{1-t_2}\right) = 0, \end{cases} \quad (2.3.7)$$

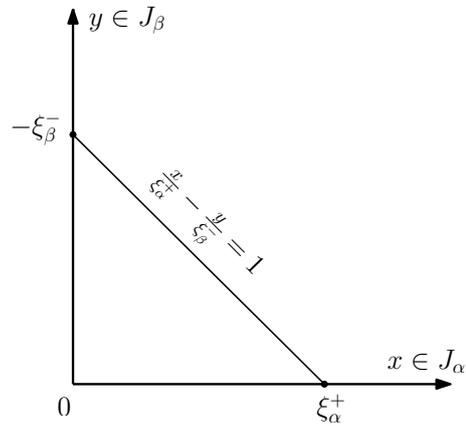


Figure 2.1: Illustration of the several subsets for  $\mathcal{D}_{junction}$  for  $\alpha \neq \beta$ .

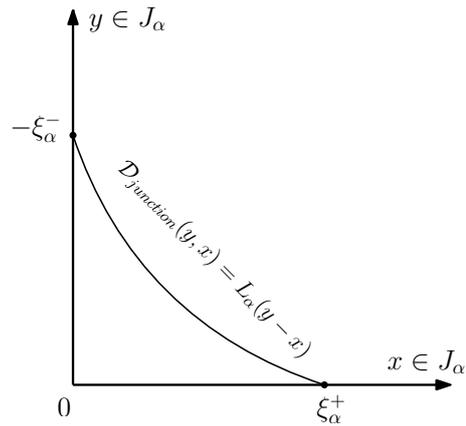


Figure 2.2: Illustration of the several subsets for  $\mathcal{D}_0$  for  $\alpha = \beta$ .

where  $t_1$  and  $t_2$  are the quantities realizing the minimum. Hence from (2.3.7), we deduce

$$K_\beta\left(-\frac{y}{t_1}\right) = 0 = K_\alpha\left(\frac{x}{1-t_2}\right).$$

But  $K_\beta$  is a bijection on  $(-\infty, 0)$ , and so is  $K_\alpha$  on  $(0, +\infty)$ . Therefore, setting  $(K_\beta^-)^{-1}(0) := \xi_\beta^-$ , and  $(K_\alpha^+)^{-1}(0) := \xi_\alpha^+$ , we deduce for  $(y, x) \in \Delta_{\beta\alpha} \setminus \{xy = 0\}$ ,

$$\begin{aligned} \mathcal{D}_{junction}(y, x) &= \frac{-y}{\xi_\beta^-} L_\beta(\xi_\beta^-) + \frac{x}{\xi_\alpha^+} L_\alpha(\xi_\alpha^+) + \left(1 - \frac{x}{\xi_\alpha^+} + \frac{y}{\xi_\beta^-}\right) L_A(0) \\ &= -yL'_\beta(\xi_\beta^-) + xL'_\alpha(\xi_\alpha^+) + L_A(0). \end{aligned}$$

Now, for  $x = 0$  and  $y < -\xi_\beta^-$ , using the first condition of (2.3.7), we deduce that

$$\mathcal{D}_{junction}(y, 0) = -yL'_\beta(\xi_\beta^-) + L_A(0).$$

For  $x = 0$  and  $y \geq -\xi_\beta^-$ , we deduce from Lemma 2.3.4, that  $t_2 = 1$ . Using the first optimal condition in (2.3.7), we have  $K_\beta(\frac{-y}{t_1}) = 0$  so  $t_1 = \frac{-y}{\xi_\beta^-} \geq 1$ . We deduce that the optimal condition must be satisfied at the boundary of the set  $\{0 \leq t_1 \leq 1\}$ . Here using (2.3.3), we have  $t_1 = 1$ , so

$$\mathcal{D}_{junction}(y, 0) = L_\beta(-y).$$

Similarly, for  $y = 0$  and  $x < \xi_\alpha^+$ ,

$$\mathcal{D}_{junction}(y, x) = xL'_\alpha(\xi_\alpha^+) + L_A(0).$$

For  $y = 0$  and  $x \geq \xi_\alpha^+$ , we deduce that

$$\mathcal{D}_{junction}(0, x) = L_\alpha(x).$$

In all other cases, that is to say for  $(y, x) \in (J_\beta^* \times J_\alpha^*) \setminus \Delta_{\beta\alpha}$ , the infimum of  $G$  is attained at the boundary of  $\{0 \leq t_1 \leq t_2 \leq 1\}$ , here for some  $t_1 = t_2 = \tau \in (0, 1)$ . Hence we have

$$\mathcal{D}_{junction}(y, x) = \inf_{0 < \tau < 1} \left\{ \tau L_\beta\left(\frac{-y}{\tau}\right) + (1 - \tau) L_\alpha\left(\frac{x}{1 - \tau}\right) \right\}$$

Once again, writing the optimal conditions for  $G(\tau, \tau, y, x)$ , we deduce that

$$K_\beta\left(\frac{-y}{\tau}\right) = K_\alpha\left(\frac{x}{1 - \tau}\right). \quad (2.3.8)$$

We define

$$\tilde{G}(\tau, y, x) = K_\beta\left(\frac{-y}{\tau}\right) - K_\alpha\left(\frac{x}{1 - \tau}\right).$$

Deriving

$$\frac{\partial \tilde{G}}{\partial \tau} = K'_\beta\left(\frac{-y}{\tau}\right) \frac{y}{\tau^2} - K'_\alpha\left(\frac{x}{1 - \tau}\right) \frac{x}{(1 - \tau)^2} > 0 \quad \text{for } (y, x) \in (J_\beta^* \times J_\alpha^*) \setminus \Delta_{\beta\alpha},$$

by implicit function theorem, we deduce that there exists a unique  $\tilde{\tau} \in C^1(0, 1)$  satisfying  $\tilde{G}(\tilde{\tau}, y, x) = 0$ . The proof is thus complete.  $\square$

**Lemma 2.3.7 (Continuity of  $\mathcal{D}_{\text{junction}}$ ).** *The function  $\mathcal{D}_{\text{junction}}$  is continuous in  $J^2$ .*

*Proof.* From (2.3.6), we already know that  $\mathcal{D}_{\text{junction}} \in C((J_\beta^* \times J_\alpha^*) \setminus \Delta_{\beta\alpha}) \cup C(\Delta_{\beta\alpha} \cup \{x = 0\} \cup \{y = 0\})$ . Therefore in order to prove that  $\mathcal{D}_{\text{junction}} \in C(J_\beta \times J_\alpha)$ , it is sufficient to prove that for any given sequence  $(y^k, x^k) \in (J_\beta^* \times J_\alpha^*) \setminus \Delta_{\beta\alpha}$  such that  $(y^k, x^k) \rightarrow (y, x)$ , where  $(y, x) \in \bar{\Delta} := \{\frac{x}{\xi_\alpha^+} - \frac{y}{\xi_\beta^-} = 1\} \cup \{x \geq \xi_\alpha^+\} \cup \{y \geq -\xi_\beta^-\}$ , we have

$$\mathcal{D}_{\text{junction}}(y^k, x^k) \rightarrow \mathcal{D}_{\text{junction}}(y, x).$$

Since the sequence  $\{\tau(y^k, x^k)\}$  is bounded, we can deduce that there exists a subsequence such that  $\tau(y^k, x^k) \rightarrow \tau^0$ . We distinguish the following cases.

**Case 1:**  $\tau^0 \in (0, 1)$ . By continuity of  $K_l$ , we have

$$K_\alpha \left( \frac{x}{1 - \tau^0} \right) = K_\beta \left( \frac{-y}{\tau^0} \right). \quad (2.3.9)$$

If  $x = 0$ , we have as  $K_\alpha(0) > 0$  and  $(K_\beta^-)^{-1}$  is increasing

$$\frac{y}{\tau^0} = -(K_\beta^-)^{-1}(K_\alpha)(0) < -(K_\beta^-)^{-1}(0) = -\xi_\beta^-,$$

hence deduce that  $(y, 0) \notin \bar{\Delta}$ , so this case is not possible.

Similarly, if  $y = 0$ , we have

$$\frac{x}{1 - \tau^0} = (K_\alpha^+)^{-1}(K_\beta)(0) < (K_\alpha^+)^{-1}(0) = \xi_\alpha^+,$$

hence deduce that  $(0, x) \notin \bar{\Delta}$ , so this case is not possible.

Now if  $(y, x) \in (J_\beta^* \times J_\alpha^*) \cap \bar{\Delta}$ , then  $\frac{x}{\xi_\alpha^+} - \frac{y}{\xi_\beta^-} = 1$  and passing to the limit, we have (2.3.9). We know that  $K_\alpha(\xi_\alpha^+) = K_\beta(\xi_\beta^-) = 0$ , so if we set  $\bar{\tau} = -\frac{y}{\xi_\beta^-} = 1 - \frac{x}{\xi_\alpha^+}$  so  $1 - \bar{\tau} = \frac{x}{\xi_\alpha^+}$ , we have

$$K_\beta \left( \frac{-y}{\bar{\tau}} \right) = 0 = K_\alpha \left( \frac{x}{1 - \bar{\tau}} \right).$$

By uniqueness of  $\tau$  satisfying (2.3.8), we deduce that  $\tau^0 = \bar{\tau}$ . So we have

$$\mathcal{D}_{\text{junction}}(y^k, x^k) \rightarrow -yL'_\beta(\xi_\beta^-) + xL'_\alpha(\xi_\alpha^+) + L_A(0) = \mathcal{D}_{\text{junction}}(y, x).$$

**Case 2:**  $\tau^0 = 0$ . In this case, using Lemma 2.3.4,  $y^k \rightarrow y = 0$ , so  $x \geq \xi_\alpha^+$  and with (2.3.8) we deduce that

$$\frac{-y^k}{\tau(y^k, x^k)} = (K_\beta^-)^{-1} \left( K_\alpha \left( \frac{x^k}{1 - \tau(y^k, x^k)} \right) \right) \rightarrow (K_\beta^-)^{-1} (K_\alpha(x)). \quad (2.3.10)$$

Therefore  $\mathcal{D}_{\text{junction}}(y^k, x^k) \rightarrow L_\alpha(x) = \mathcal{D}_{\text{junction}}(0, x)$ .

**Case 3:**  $\tau^0 = 1$ . In this case,  $x^k \rightarrow x = 0$ . Arguing as in Case 2, we deduce that  $y \geq \xi_\beta^-$ , and

$$\frac{x^k}{1 - \tau(y^k, x^k)} = (K_\alpha^+)^{-1} \left( K_\beta \left( \frac{-y^k}{\tau(y^k, x^k)} \right) \right) \rightarrow (K_\alpha^+)^{-1} (K_\beta(-y)). \quad (2.3.11)$$

Therefore,  $\mathcal{D}_{\text{junction}}(y^k, x^k) \rightarrow L_\beta(-y) = \mathcal{D}_{\text{junction}}(y, x)$ .

The proof is thus complete.  $\square$

**Lemma 2.3.8.** *The function  $\mathcal{D}_{\text{junction}}$  is  $C^1$  in  $J^2$  and for  $(y, x) \in J_\beta \times J_\alpha$ , we have*

$$\partial_x \mathcal{D}_{\text{junction}}(y, x) = \begin{cases} L'_\alpha\left(\frac{x}{1-\tau}\right) & \text{if } (y, x) \in (J_\beta^* \times J_\alpha^*) \setminus \Delta_{\beta\alpha}, \\ L'_\alpha(\xi_\alpha^+) & \text{if } (y, x) \in \Delta_{\beta\alpha}, \\ L'_\alpha(x) & \text{if } y = 0 \text{ and } x > \xi_\alpha^+, \\ L'_\alpha \circ (K_\alpha^+)^{-1} \circ K_\beta(-y) & \text{if } x = 0 \text{ and } y > -\xi_\beta^-, \end{cases} \quad (2.3.12)$$

and

$$\partial_y \mathcal{D}_{\text{junction}}(y, x) = \begin{cases} -L'_\beta\left(\frac{-y}{\tau}\right) & \text{if } (y, x) \in (J_\beta^* \times J_\alpha^*) \setminus \Delta_{\beta\alpha}, \\ -L'_\beta(\xi_\beta^-) & \text{if } (y, x) \in \Delta_{\beta\alpha}, \\ -L'_\beta \circ (K_\beta^-)^{-1} \circ K_\alpha(x) & \text{if } y = 0 \text{ and } x > \xi_\alpha^+, \\ -L'_\beta(-y) & \text{if } x = 0 \text{ and } y > -\xi_\beta^-. \end{cases} \quad (2.3.13)$$

*Proof.* We compute the partial derivatives in domains where the function is naturally of class  $C^1$  using that the function  $\tau$  is continuously differentiable in  $(0, 1)^2$  and using (2.3.9). We prove the continuity of the partial derivatives using the same proof as Lemma 2.3.7.  $\square$

### 2.3.4 Compatibility condition

In this subsection, we prove a compatibility result, which will be used in deriving error estimates.

Let us introduce the following shorthand notation

$$H(x, p) = \begin{cases} H_\alpha(p) & \text{if } x \in J_\alpha^* \\ F_A(p) & \text{if } x = 0. \end{cases}$$

*Remark 2.3.9.* In  $J_\alpha \times J_\alpha$ , we give a description of  $\{\mathcal{D}_{\text{junction}}(y, x) = L_\alpha(y-x)\} \cap \Delta_{\beta\alpha}$  using [67], see Figure 2.2. We have

$$\begin{cases} \mathcal{D}_{\text{junction}}(0, \xi_\alpha^+) = \xi_\alpha^+ L'_\alpha(\xi_\alpha^+) + L_A(0) = L_\alpha(\xi_\alpha^+) = L_\alpha(0, \xi_\alpha^+), \\ \mathcal{D}_{\text{junction}}(-\xi_\beta^-, 0) = \xi_\beta^- L'_\beta(\xi_\beta^-) + L_A(0) = L_\beta(\xi_\beta^-) = L_\beta(-\xi_\beta^-, 0). \end{cases}$$

This means that the functions  $\mathcal{D}_{\text{junction}}$  and  $(y, x) \mapsto L_\alpha(x-y)$  coincide at the same points  $X_\alpha = (0, \xi_\alpha^+)$  and  $Y_\alpha = (-\xi_\alpha^-, 0)$ . Therefore we have

$$L_\alpha(x-y) < \mathcal{D}_{\text{junction}}(y, x) \quad \text{on the open line segment } ]X_\alpha, Y_\alpha[$$

because  $\mathcal{D}_{junction}$  is linear and  $L_\alpha$  is strictly convex as a function of  $y - x$ .

The function  $(y, x) \mapsto L_\alpha(x - y) - \mathcal{D}_{junction}(y, x)$  being convex because  $\mathcal{D}_{junction}(y, x)$  is linear, we can consider the convex set

$$K^\alpha = \{(y, x) \in J_\alpha \times J_\alpha, \quad L_\alpha(x - y) \leq \mathcal{D}_{junction}(y, x)\}.$$

Then the set

$$\Gamma^\alpha = \{(y, x) \in \Delta_{\alpha\alpha}, \quad \mathcal{D}_{junction}(y, x) = L_\alpha(x - y)\}$$

is contained in the boundary of the convex set  $K^\alpha$ . More precisely, we have

$$\Gamma^\alpha = ((\partial K^\alpha) \cap \Delta_{\alpha\alpha}) \subset J_\alpha \times J_\alpha$$

which shows that  $\Gamma^\alpha$  is a curve which contains the points  $X_\alpha$  and  $Y_\alpha$ .

**Theorem 2.3.10.** *Assume the Hamiltonians are convex, with Legendre Fenchel transform satisfying (B0). Then for all  $(x, y) \in J \times J \setminus \bigcup_{\alpha \in \{1, \dots, N\}} \Gamma^\alpha$ , (i.e., everywhere except on the curves where  $\mathcal{D}_0$  is not  $C^1$ ), we have*

$$H(y, -\partial_y \mathcal{D}_{junction}) = H(x, \partial_x \mathcal{D}_{junction}).$$

*Proof of Theorem 2.3.10.* First, notice that in the interior of  $K^\alpha$  (i.e., in the regions where  $\mathcal{D}_0(y, x) = L_\alpha(x - y)$ ), we have the result as

$$H(y, -\partial_y \mathcal{D}_0(y, x)) = H_\alpha(L'_\alpha(x - y)) = H(x, \partial_x \mathcal{D}_0(y, x)).$$

Now we prove the result in the regions where  $\mathcal{D}_0 = \mathcal{D}_{junction}$  defined in the expressions of  $\partial_x \mathcal{D}_{junction}$  and  $\partial_y \mathcal{D}_{junction}$  in (2.3.12)-(2.3.13). Let us first point out that we have the following assertion

$$H_\alpha(p) + L_\alpha(q) = pq \Leftrightarrow q \in \partial H_\alpha(p), \tag{2.3.14}$$

where  $\partial H_\alpha(p)$  is the convex subdifferential of  $H_\alpha(p)$ .

We distinguish several cases.

**Case 1**  $(y, x) \in (J_\beta^* \times J_\alpha^*) \setminus \Delta_{\beta\alpha}$ . From (2.3.14), on the one hand, and from (2.3.13) we have

$$H_\beta \left( L'_\beta \left( \frac{-y}{\tau} \right) \right) = \frac{-y}{\tau} L'_\beta \left( \frac{-y}{\tau} \right) - L_\beta \left( \frac{-y}{\tau} \right).$$

From (2.3.5), we have then  $H_\beta \left( L'_\beta \left( \frac{-y}{\tau} \right) \right) = -K_\beta \left( \frac{-y}{\tau} \right) - L_A(0)$ .

On the other hand, and from (2.3.12)

$$H_\alpha \left( L'_\alpha \left( \frac{x}{1-\tau} \right) \right) = \frac{x}{1-\tau} L'_\alpha \left( \frac{x}{1-\tau} \right) - L_\alpha \left( \frac{x}{1-\tau} \right),$$

similarly, from (2.3.5), we deduce that  $H_\alpha \left( L'_\alpha \left( \frac{x}{1-\tau} \right) \right) = -K_\alpha \left( \frac{x}{1-\tau} \right) - L_A(0)$ . Hence, from (2.3.8), the compatibility condition.

**Case 2**  $(y, x) \in (J_\beta^* \times J_\alpha^*) \cap \Delta_{\beta\alpha}$ . We argue as in Case 1, one can deduce that

$$\begin{aligned} H_\beta(L'_\beta(\xi_\beta^-)) &= -K_\beta(\xi_\beta^-) - L_A(0) = A \\ H_\alpha(L'_\alpha(\xi_\alpha^+)) &= -K_\alpha(\xi_\alpha^+) - L_A(0) = A. \end{aligned}$$

From the definition of  $\xi_\alpha^+$  and  $\xi_\beta^-$ , one can deduce the compatibility condition.

*Remark 2.3.11.* We deduce that the functions  $\pi_\alpha^+, \pi_\beta^-$  defined in [65] satisfy

$$\pi_\alpha^+(A) = L'_\alpha(\xi_\alpha^+) \quad \text{and} \quad \pi_\beta^-(A) = L'_\beta(\xi_\beta^-).$$

**Case 3**  $y = 0$  and  $x > \xi_\alpha^+$ . Let us check the following equality

$$\max \left( A, \max_{\beta=1, \dots, N} H_\beta^- \left( L'_\beta \left( (K_\beta^-)^{-1} \circ K_\alpha(x) \right) \right) \right) = H_\alpha(L'_\alpha(x)).$$

On the one hand, from the definition of  $K_\beta^-$ , we deduce that

$$H_\beta^- \left( L'_\beta \left( (K_\beta^-)^{-1} \circ K_\alpha(x) \right) \right) = H_\beta \left( L'_\beta \left( (K_\beta^-)^{-1} \circ K_\alpha(x) \right) \right),$$

and arguing as previously, we deduce that

$$H_\beta \left( L'_\beta \left( (K_\beta^-)^{-1} \circ K_\alpha(x) \right) \right) = -K_\beta \left( (K_\beta^-)^{-1} \circ K_\alpha(x) \right) - L_A(0) = -K_\alpha(x) - L_A(0).$$

On the other hand from (2.3.14), we have  $H_\alpha(L'_\alpha(x)) = -K_\alpha(x) - L_A(0)$ .

And for  $x > \xi_\alpha^+$ , we have  $H_\alpha(L'_\alpha(x)) > H_\alpha(L'_\alpha(\xi_\alpha^+)) = H_\alpha(\pi_\alpha^+(A)) = A$ . So one can deduce the compatibility condition.

**Case 4**  $x = 0$  and  $y > -\xi_\beta^-$ . Let us check the following equality

$$\max \left( A, \max_{\substack{\alpha=1, \dots, N \\ \alpha \neq \beta}} H_\alpha^- \left( L'_\alpha \left( (K_\alpha^+)^{-1} \circ K_\beta(-y) \right) \right) \right) = H_\beta(L'_\beta(-y)).$$

Similarly, as in the previous case, one can deduce that

$$\max_{\substack{\alpha=1, \dots, N \\ \alpha \neq \beta}} H_\alpha^- \left( L'_\alpha \left( (K_\alpha^+)^{-1} \circ K_\beta(-y) \right) \right) = A_0 \leq A.$$

And for  $y > \xi_\beta^-$ , we have  $H_\beta^-(L'_\beta(-y)) > H_\beta^-(\pi_\beta^-(A)) = A$ .

**Case 5**  $y = 0$  and  $0 < x \leq \xi_\alpha^+$ . Let us check the following equality

$$\max(A, \max_{\beta=1, \dots, N} H_\beta^-(L'_\beta(\xi_\beta^-))) = H_\alpha(L'_\alpha(\xi_\alpha^+)).$$

On the one hand, from (2.3.14)  $H_\alpha(L'_\alpha(\xi_\alpha^+)) = -K_\alpha(\xi_\alpha^+) - L_A(0) = -L_A(0) = A$ .

On the other hand,

$$\max_{\beta=1, \dots, N} H_\beta^-(L'_\beta(\xi_\beta^-)) = \max_{\beta=1, \dots, N} H_\beta^-(\pi_\beta^-(A)) = A.$$

**Case 6**  $x = 0$  and  $0 < y \leq -\xi_\beta^-$ . Let us check the following equality

$$\max(A, \max_{\alpha=1, \dots, N} H_\alpha^-(L'_\alpha(\xi_\alpha^+)) = H_\beta(L'_\beta(\xi_\beta^-)).$$

Similarly, as in Case 5, one can deduce the compatibility condition.

**Case 7**  $x = 0$  and  $y = 0$ . Let us check the following equality

$$\max(A, \max_{\beta=1, \dots, N} H_\beta^-(L'_\beta(\xi_\beta^-)) = \max(A, \max_{\alpha=1, \dots, N} H_\alpha^-(L'_\alpha(\xi_\alpha^+)).$$

In fact, it follows directly from Case 5 and Case 6.

The proof is thus complete. □

### 2.3.5 $C^{1,1}$ estimates for the reduced minimal action

In this section, we study the Lipschitz regularity of the gradient of the reduced minimal action  $\mathcal{D}_0$ . It turns out that its gradient is indeed Lipschitz if the flux limiter  $A$  is not equal to  $A_0$ , the minimal flux limiter. Such a technical result will be used when deriving error estimates. It is also of independent interest.

**Proposition 2.3.12** ( $C^{1,1}$  estimates for the reduced minimal action). *Let  $\rho > 0$  and assume that the Hamiltonians satisfy (2.1.7) and (2.1.6). The function  $\mathcal{D}_0$  associated with the flux limiter  $A_0 + \rho$  can be chosen  $C^{1,1}(J_K^2)$  for any  $K > 0$  where  $J_K^2 = \{(x, y) \in J^2 : d(0, x) \leq K \text{ and } d(0, y) \leq K\}$ . Moreover, there exists  $C_K$  and  $C'_K$  such that*

$$\|\partial_{xx} \mathcal{D}_{junction}\|_{L^\infty(J_K^2)} \leq \frac{C_K}{\min(1, \rho)}; \quad (2.3.15)$$

and

$$\|H'_\alpha(\partial_x \mathcal{D}_{junction}) \partial_{xx} \mathcal{D}_{junction}\|_{L^\infty(J_K^2)} \leq \frac{C'_K}{\min(1, \sqrt{\rho})}. \quad (2.3.16)$$

The constants  $C_K$  and  $C'_K$  depend only on  $K$  and (2.1.7).

Moreover, in the case where for all  $\alpha \in \{1, \dots, N\}$ ,  $\min H_\alpha = A_0$ , we have

$$\|\partial_{xx} \mathcal{D}_{junction}\|_{L^\infty(J_K^2)} \leq C_K. \quad (2.3.17)$$

**Proof.** In the following  $A$  denotes  $A_0 + \rho$ . Using (2.3.12), we see that  $\partial_{xx}\mathcal{D}_{junction} = 0$  on  $\Delta_{\beta\alpha}$  for all  $(\beta, \alpha) \in \{1, \dots, N\}^2$  and  $\partial_{xx}\mathcal{D}_{junction}(y, x) = L''_{\alpha}(x)$  on  $\{0\} \times \{x \in J_{\alpha} \mid x > \xi_{\alpha}^+\}$ . So it is sufficient to prove (2.3.15) and (2.3.16) on  $T := J_{\beta}^* \times J_{\alpha}^* \setminus \Delta_{\beta\alpha}$  for all  $(\beta, \alpha) \in \{1, \dots, N\}^2$ . By (2.3.12), we deduce that on  $T$ ,

$$\partial_{xx}\mathcal{D}_{junction}(y, x) = \left( \frac{1}{1 - \tau(y, x)} + \frac{x}{(1 - \tau(y, x))^2} \frac{\partial\tau}{\partial x}(y, x) \right) L''_{\alpha} \left( \frac{x}{1 - \tau(y, x)} \right).$$

Let us compute also  $\frac{\partial\tau}{\partial x}$  using (2.3.8),

$$\frac{\partial\tau}{\partial x}(y, x) = \frac{\frac{1}{1 - \tau(y, x)} K'_{\alpha} \left( \frac{x}{1 - \tau(y, x)} \right)}{\frac{y}{\tau(y, x)^2} K'_{\beta} \left( \frac{-y}{\tau(y, x)} \right) - \frac{x}{(1 - \tau(y, x))^2} K'_{\alpha} \left( \frac{x}{1 - \tau(y, x)} \right)}.$$

And as  $K'_{\beta} \left( \frac{-y}{\tau} \right) = \frac{y}{\tau} L''_{\beta} \left( \frac{-y}{\tau} \right) \geq 0$  and  $K'_{\alpha} \left( \frac{x}{1 - \tau} \right) = \frac{-x}{1 - \tau} L''_{\alpha} \left( \frac{x}{1 - \tau} \right) \leq 0$  we deduce that

$$\frac{\partial\tau}{\partial x}(y, x) = \frac{\frac{-x}{(1 - \tau(y, x))^2} L''_{\alpha} \left( \frac{x}{1 - \tau(y, x)} \right)}{\frac{y^2}{\tau(y, x)^3} L''_{\beta} \left( \frac{-y}{\tau(y, x)} \right) + \frac{x^2}{(1 - \tau(y, x))^3} L''_{\alpha} \left( \frac{x}{1 - \tau(y, x)} \right)}. \quad (2.3.18)$$

So we have on  $T$

$$\partial_{xx}\mathcal{D}_{junction}(y, x) = \frac{\frac{y^2}{(1 - \tau(y, x))\tau(y, x)^3} L''_{\alpha} \left( \frac{x}{1 - \tau(y, x)} \right) L''_{\beta} \left( \frac{-y}{\tau(y, x)} \right)}{\frac{y^2}{\tau(y, x)^3} L''_{\beta} \left( \frac{-y}{\tau(y, x)} \right) + \frac{x^2}{(1 - \tau(y, x))^3} L''_{\alpha} \left( \frac{x}{1 - \tau(y, x)} \right)} \geq 0. \quad (2.3.19)$$

As the denominator is a sum of two positive functions,  $\partial_{xx}\mathcal{D}_{junction}$  from above by the same numerator over only one term of the denominator. We deduce in these two cases that,

$$\partial_{xx}\mathcal{D}_{junction}(y, x) \leq \begin{cases} 2L''_{\alpha} \left( \frac{x}{1 - \tau(y, x)} \right) & \text{if } \tau(y, x) \leq \frac{1}{2} \\ \frac{8y^2}{\left( \frac{x}{1 - \tau(y, x)} \right)^2} L''_{\beta} \left( \frac{-y}{\tau(y, x)} \right) & \text{if } \tau(y, x) \geq \frac{1}{2}. \end{cases} \quad (2.3.20)$$

Moreover, we have on  $T$ ,

$$H'_{\alpha}(\partial_x \mathcal{D}_{junction}(y, x)) = H'_{\alpha} \left( L'_{\alpha} \left( \frac{x}{1 - \tau(y, x)} \right) \right) = \frac{x}{1 - \tau(y, x)},$$

and

$$\frac{x}{1 - \tau(y, x)} \partial_{xx}\mathcal{D}_{junction}(y, x) \leq \begin{cases} 4x^2 L''_{\alpha} \left( \frac{x}{1 - \tau(y, x)} \right) & \text{if } \tau(y, x) \leq \frac{1}{2} \\ \frac{8y^2}{\frac{x}{1 - \tau(y, x)}} L''_{\beta} \left( \frac{-y}{\tau(y, x)} \right) & \text{if } \tau(y, x) \geq \frac{1}{2}, \end{cases}$$

In the case  $\tau(y, x) \leq \frac{1}{2}$ , as  $0 \leq \frac{x}{1 - \tau(y, x)} \leq 2x$ , we get the inequality (2.3.15) and (2.3.16). Let us prove the following lower bound for  $(y, x) \in T$ ,

$$\frac{x}{1 - \tau(y, x)} \geq \xi_{\alpha}^+, \quad (2.3.21)$$

which helps us for the second case. For  $y \in J_\beta$ , we see that  $x \rightarrow \frac{x}{1-\tau(y,x)}$  has a non-negative derivative using (2.3.18), so it is a non-decreasing function. Therefore to prove (2.3.21), it is sufficient to show it on  $\partial T$ . Let  $(y, x)$  be in  $\partial T$ . We distinguish three cases.

In the case where  $y = 0$ , necessarily  $x \geq \xi_\alpha^+$  and as  $\tau(y, x) \in [0, 1]$ , we deduce (2.3.21).

In the case where  $y \in ]0, -\xi_\beta^-[$ , we have  $(y, x) \in \left\{ (y, x) \in J_\beta \times J_\alpha, \quad \frac{x}{\xi_\alpha^+} - \frac{y}{\xi_\beta^-} = 1 \right\}$ .

So by (2.3.9) we deduce that  $\frac{x}{1-\tau(y,x)} = \xi_\alpha^+$ .

In the case where  $y \geq -\xi_\beta^-$ , we have  $x = 0$ . It is enough to prove that

$$\liminf_{x' \rightarrow 0} \frac{x'}{1 - \tau(y, x')} \geq \xi_\alpha^+. \quad (2.3.22)$$

We have for  $(y, x') \in T$ ,

$$K_\alpha \left( \frac{x'}{1 - \tau(y, x')} \right) = K_\beta \left( \frac{-y}{\tau(y, x')} \right) \leq K_\beta \left( \frac{-\xi_\beta^-}{\tau(y, x')} \right),$$

as  $K_\beta$  is non-decreasing on  $] -\infty, 0]$ . We deduce that

$$\frac{x'}{1 - \tau(y, x')} \geq (K_\alpha^+)^{-1} \circ K_\beta \left( \frac{-\xi_\beta^-}{\tau(y, x')} \right),$$

as  $(K_\alpha^+)^{-1}$  is non-increasing. As  $\lim_{x' \rightarrow 0} \tau(y, x') = 1$ , taking the limit inferior in the preceding inequality gives (2.3.22). So we deduce (2.3.21) and

$$\partial_{xx} \mathcal{D}_{junction}(y, x) \leq \frac{8y^2}{(\xi_\alpha^+)^2} L_\beta'' \left( \frac{-y}{\tau(y, x)} \right) \quad \text{if } \tau(y, x) \geq \frac{1}{2},$$

$$\frac{x}{1 - \tau(y, x)} \partial_{xx} \mathcal{D}_{junction}(y, x) \leq \frac{8y^2}{\xi_\alpha^+} L_\beta'' \left( \frac{-y}{\tau(y, x)} \right) \quad \text{if } \tau(y, x) \geq \frac{1}{2}.$$

If  $\xi_\alpha^+ > 1$ , we deduce (2.3.15). If  $\xi_\alpha^+ \leq 1$ , let us prove that it exists a constant  $C > 0$  only depending on (2.1.7) such that

$$(\xi_\alpha^+)^2 \geq C\rho. \quad (2.3.23)$$

As  $A = A_0 + \rho$  we have

$$K_\alpha(\xi) = L_\alpha(\xi) - \xi L_\alpha'(\xi) + A_0 + \rho,$$

and

$$K_\alpha'(\xi) = -\xi L_\alpha''(\xi).$$

The function  $L''_\alpha$  is bounded on  $[0, 1]$ , it exists  $M > 0$  such that

$$\gamma \leq L''_\alpha \leq M.$$

So we have  $K'_\alpha(\xi) \geq -M\xi$ . We integrate from 0 to  $\xi \geq 0$  and get

$$K_\alpha(\xi) - K_\alpha(0) \geq -M\frac{\xi^2}{2}. \quad (2.3.24)$$

Taking  $\xi = \xi_\alpha^+$ , as  $K_\alpha(\xi_\alpha^+) = 0$  and as  $L_\alpha(0) + A_0 \geq 0$ , we deduce that

$$(\xi_\alpha^+)^2 \geq \frac{2}{M}(L_\alpha(0) + A_0 + \rho) \geq \frac{2}{M}\rho.$$

So we get (2.3.23) and we deduce (2.3.15) and (2.3.16).

In the case where for all  $\alpha \in \{1, \dots, N\}$ ,  $\min H_\alpha = A_0$ , we only have to consider the case  $\tau(y, x) \geq \frac{1}{2}$  in (2.3.20) since the case  $\tau(y, x) \leq \frac{1}{2}$  gives already the bound (2.3.20). In order to get a bound for the term  $\frac{8y^2}{(1-\tau(y,x))^2} = \frac{8y^2}{((K_\alpha^+)^{-1} \circ K_\beta(-\frac{y}{\tau(y,x)}))^2}$ , let us prove that for all  $\xi \in [-2K, 2K]$ , we have

$$\frac{\xi^2}{((K_\alpha^+)^{-1} \circ K_\beta(-\xi))^2} \leq C_{2K}, \quad (2.3.25)$$

where  $C_{2K} > 0$  is a constant which depends on  $K$ . Let  $M_{2K}$  be such that on  $[-2K, 2K]$  we have for all  $\alpha \in \{1, \dots, N\}$ ,

$$\gamma \leq L''_\alpha \leq M_{2K}.$$

Replacing  $\xi$  by  $(K_\alpha^+)^{-1}(\xi)$  in (2.3.24), we deduce that

$$M_{2K} \frac{((K_\alpha^+)^{-1}(\xi))^2}{2} \geq -\xi + K_\alpha(0).$$

So we have

$$M_{2K} \frac{((K_\alpha^+)^{-1} \circ K_\beta(-\xi))^2}{2} \geq -K_\beta(-\xi) + K_\alpha(0).$$

As for (2.3.24), we have the following inequality

$$K_\beta(0) - K_\beta(-\xi) \geq \gamma \frac{\xi^2}{2}.$$

So as  $K_\alpha(0) = K_\beta(0) = \rho$  we deduce that

$$M_{2K} \frac{((K_\alpha^+)^{-1} \circ K_\beta(-\xi))^2}{2} \geq \gamma \frac{\xi^2}{2} + K_\alpha(0) - K_\beta(0) \geq \gamma \frac{\xi^2}{2}.$$

That gives (2.3.25) and we deduce (2.3.17).  $\square$

## 2.4 Error estimates

To prove Theorem 2.1.1, we will need the following result whose proof is given in the Appendix of Chapter 1 for the reader's convenience.

**Lemma 2.4.1 (A priori control).** *Let  $T > 0$  and let  $u^h$  be a solution of the numerical scheme (2.1.12), (2.1.14) and  $u$  a super-solution of (2.1.1)-(2.1.2) satisfying for some  $C_T > 0$ ,*

$$u(t, x) \geq -C_T(1 + d(0, x)) \quad \text{for } t \in (0, T).$$

*Then there exists a constant  $C = C(T) > 0$  such that for all  $(t, x) \in \mathcal{G}_h$ ,  $t \leq T$ , and  $(s, y) \in [0, T) \times J$ , we have*

$$u^h(t, x) \leq u(s, y) + C(1 + d(x, y)). \quad (2.4.1)$$

We also need the following result [65, Lemma 4.4] where the proof is given in [65].

**Lemma 2.4.2 (From non-convex to convex Hamiltonians).** *Let  $K \in (0, +\infty)$ . Given Hamiltonians  $H_\alpha : [-K, K] \rightarrow \mathbb{R}$  satisfying (2.1.3), there exists a function  $\beta : \mathbb{R} \rightarrow \mathbb{R}$  such that the functions  $\beta \circ H_\alpha$  satisfy (2.1.7) for  $\alpha = 1, \dots, N$ . Moreover, we can choose  $\beta$  such that  $\beta \in C^2(\mathbb{R})$  and  $\beta' > 1$ .*

*Remark 2.4.3.* In [65, Lemma 4.4], the functions  $\beta \circ H_\alpha$  satisfy in fact the following assumptions

$$\begin{cases} H_\alpha \in C^2(\mathbb{R}) \text{ with } H_\alpha'' > 0 \text{ on } \mathbb{R}, \\ H_\alpha' < 0 \text{ on } (-\infty, 0) \text{ and } H_\alpha' > 0 \text{ on } (0, +\infty), \\ \lim_{|p| \rightarrow +\infty} \frac{H_\alpha(p)}{|p|} = +\infty. \end{cases} \quad (2.4.2)$$

which implies (2.1.7). Indeed, in the next proof on error estimates, we only need to consider Hamiltonians on a compact set which only depends on  $u_0$  and the Hamiltonians  $H_\alpha$ , thanks to the fact that the solution is Lipschitz continuous, see Theorem 2.2.2 and the fact that the discrete gradients are bounded (see Chapter 1, Proposition 1.3.1). So on  $[-K, K]$ , the functions  $(\beta \circ H_\alpha)''$  are bounded by some constant  $C > 0$ . We deduce that the functions  $L_\alpha$  are of class  $C^2$  and satisfy  $L_\alpha'' \geq \gamma = \frac{1}{C}$ . Indeed, from the relation  $H_\alpha(p) + L_\alpha(q) = pq$  with  $q = H_\alpha'(p)$ , one can deduce that  $L_\alpha'(H_\alpha'(q)) = q$ , so

$$L_\alpha''(q) = \frac{1}{H_\alpha'' \circ (H_\alpha')^{-1}(q)} \geq \gamma.$$

We now turn to the proof of the error estimates.

**Proof of Theorem 2.1.1.** We assume that the Hamiltonians  $H_\alpha$  satisfy (2.1.3) and  $A > A_0$ . Let  $u$  be the solution of (2.1.1) and  $u^h$  the solution of the corresponding scheme (2.1.12). In order to get (2.1.16), we only prove that

$$u^h(t, x) - u(t, x) \leq \begin{cases} C_T(\Delta x)^{1/2} & \text{if } A > A_0, \\ C_T(\Delta x)^{2/5} & \text{if } A = A_0 \end{cases} \quad \text{in } [0, T) \times J \cap \mathcal{G}_h$$

since the proof of the other inequality is very similar. We are going to prove that

$$u^h(t, x) - u(t, x) \leq \begin{cases} \mathcal{O}\left(\frac{\Delta t}{\nu}\right) + \mathcal{O}\left(\frac{\Delta x}{\varepsilon}\right) + \mathcal{O}(\varepsilon) + \mathcal{O}(\nu) & \text{if } A > A_0, \\ \mathcal{O}\left(\frac{\Delta t}{\nu}\right) + \mathcal{O}\left(\frac{\Delta x}{\varepsilon\sqrt{\rho}}\right) + \mathcal{O}\left(\frac{(\Delta x)^2}{(\varepsilon\rho)^2}\right) + \mathcal{O}(\rho) + \mathcal{O}(\varepsilon) + \mathcal{O}(\nu) & \text{if } A = A_0, \end{cases} \quad (2.4.3)$$

which yields the desired inequality by minimizing the right hand side with respect to  $\varepsilon$  and  $\nu$  in the case  $A > A_0$  and with respect to  $\rho, \varepsilon$  and  $\nu$  in the case  $A = A_0$ . Let  $\beta$  be the function defined in Lemma 2.4.2 such that the functions  $\beta \circ H_\alpha$  satisfy (2.1.7). In the following, we consider that the function  $\mathcal{D}_0$  is associated to the Hamiltonians  $\beta \circ H_\alpha$  and to the flux limiter  $\beta(A)$  which satisfies  $\beta(A) > \beta(A_0)$  in the case  $A > A_0$ . The remaining of the proof proceeds in several steps.

**Step 1: Penalization procedure.** Using the expression of  $\mathcal{D}_0$  in (2.3.2) and  $\mathcal{D}_{junction}$  in (2.3.6), we deduce that it exists  $C > 0$ , such that  $\forall x \in J$

$$\mathcal{D}_0(0, 0) = L_A(0) = -A \leq \mathcal{D}_0(x, x) \leq C.$$

Let  $\tilde{\mathcal{D}}_0 = \mathcal{D}_0 + A$ , we have that

$$0 \leq \tilde{\mathcal{D}}_0(x, x) \leq C + A.$$

For  $\eta, \delta, \varepsilon, \nu$  positive constants, let us define

$$M_{\varepsilon, \delta} = \sup_{\substack{(t, x) \in \mathcal{G}_h, \\ (s, y) \in [0, T] \times J}} \left\{ u^h(t, x) - u(s, y) - \varepsilon \tilde{\mathcal{D}}_0\left(\frac{y}{\varepsilon}, \frac{x}{\varepsilon}\right) - \frac{(t-s)^2}{2\nu} - \frac{\delta}{2} d^2(y, 0) - \frac{\eta}{T-s} \right\} \quad (2.4.4)$$

where the test function  $\mathcal{D}_0$  is given in (2.3.2). In this step, we assume that  $M_{\varepsilon, \delta} > 0$ . Thanks to Lemma 2.4.1 and the superlinearity of  $\mathcal{D}_0$  (see Lemma 2.3.2), we deduce that for  $(x, y)$  such that the quantity in the supremum is larger than  $\frac{M_{\varepsilon, \delta}}{2}$ ,

$$0 < \frac{M_{\varepsilon, \delta}}{2} \leq C(1 + d(y, x)) - \varepsilon \frac{\gamma}{2} d^2\left(\frac{y}{\varepsilon}, \frac{x}{\varepsilon}\right) - \frac{(t-s)^2}{2\nu} - \frac{\delta}{2} d^2(y, 0) - \frac{\eta}{T-s}$$

which implies in particular

$$\frac{\gamma}{2\varepsilon} d^2(y, x) \leq C(1 + d(y, x)),$$

and

$$\frac{\delta}{2} d^2(y, 0) \leq C(1 + d(y, x)).$$

Notice that in the following, we use the notation  $\mathcal{D}_0$  instead of  $\tilde{\mathcal{D}}_0$ . Indeed we deal only with partial derivatives of  $\mathcal{D}_0$  which are equal to partial derivatives of  $\tilde{\mathcal{D}}_0$  and differences between two values of  $\mathcal{D}_0$  at two points which are equal to differences between two values of  $\tilde{\mathcal{D}}_0$  at these two points.

We deduce from the two last inequalities that  $d(y, x)$  is bounded and  $d(y, 0)$  is bounded, so the supremum is reached at some point  $(t, x, s, y)$  where  $y \in J_\beta$  and  $x \in J_\alpha$ . This estimate together with the fact that  $-\partial_y \mathcal{D}_0(\frac{y}{\varepsilon}, \frac{x}{\varepsilon}) - \delta d(y, 0)$  lies in the viscosity subdifferential of  $u(t, \cdot)$  at  $x$  and the fact that  $\delta d(y, 0)$  is bounded, implies that there exists  $K > 0$  only depending on  $\|\nabla u\|_\infty$  (see Theorem 2.2.2) such that the point  $(t, x, s, y)$  realizing the maximum satisfies

$$\left| \partial_y \mathcal{D}_0 \left( \frac{y}{\varepsilon}, \frac{x}{\varepsilon} \right) \right| \leq K. \quad (2.4.5)$$

If  $\alpha = \beta$ , for  $\frac{y}{\varepsilon}$  or  $\frac{x}{\varepsilon}$  large, then (2.4.5) implies

$$\left| L'_\alpha \left( \frac{y}{\varepsilon} - \frac{x}{\varepsilon} \right) \right| \leq K.$$

As  $L_\alpha$  is superlinear, it implies that  $d(\frac{y}{\varepsilon}, \frac{x}{\varepsilon}) \leq C$ , for  $C > 0$  which is sufficient for the use in step 2 of the  $C^{1,1}$  estimates as  $\mathcal{D}_0$  only depends on  $d(\frac{y}{\varepsilon}, \frac{x}{\varepsilon})$  for  $\frac{y}{\varepsilon}$  or  $\frac{x}{\varepsilon}$  large. If  $\alpha \neq \beta$ , assume by contradiction that  $\frac{y}{\varepsilon}$  or  $\frac{x}{\varepsilon}$  are not bounded when  $\varepsilon \rightarrow 0$ . Then using (2.3.13) and (2.3.8) we get a contradiction with (2.4.5). So  $\frac{y}{\varepsilon}$  and  $\frac{x}{\varepsilon}$  are bounded by a constant which only depends on  $\|\nabla u\|_\infty$  and on the Hamiltonians  $H_\alpha$ .

We want to prove that for  $\eta > \eta^*$  (to be determined) that the supremum in (2.4.4) is attained for  $t = 0$  or  $s = 0$ . We assume that  $t > 0$  and  $s > 0$  and we prove that  $\eta \leq \eta^*$ .

**Step 2: Viscosity inequalities.** Since  $t > 0$  and  $s > 0$ , we can use Lemma 2.2.4 and get the following viscosity inequalities.

If  $x \neq 0$ , then

$$\frac{t-s}{\nu} - \frac{\Delta t}{2\nu} + \max \left\{ H_\alpha^- \left( \frac{\varepsilon}{\Delta x} \left\{ \mathcal{D}_0 \left( \frac{y}{\varepsilon}, \frac{x+\Delta x}{\varepsilon} \right) - \mathcal{D}_0 \left( \frac{y}{\varepsilon}, \frac{x}{\varepsilon} \right) \right\} \right), \right. \\ \left. H_\alpha^+ \left( \frac{\varepsilon}{\Delta x} \left\{ \mathcal{D}_0 \left( \frac{y}{\varepsilon}, \frac{x}{\varepsilon} \right) - \mathcal{D}_0 \left( \frac{y}{\varepsilon}, \frac{x-\Delta x}{\varepsilon} \right) \right\} \right) \right\} \leq 0.$$

If  $x = 0$ , then

$$\frac{t-s}{\nu} - \frac{\Delta t}{2\nu} + \max \left( A, \max_\beta \left\{ H_\beta^- \left( \frac{\varepsilon}{\Delta x} \left\{ \mathcal{D}_0 \left( \frac{y}{\varepsilon}, \frac{\Delta x}{\varepsilon} \right) - \mathcal{D}_0 \left( \frac{y}{\varepsilon}, 0 \right) \right\} \right) \right\} \right) \leq 0.$$

If  $y \neq 0$ , then

$$-\frac{\eta}{(T-s)^2} + \frac{t-s}{\nu} + H_\alpha \left( -\partial_y \mathcal{D}_0 \left( \frac{y}{\varepsilon}, \frac{x}{\varepsilon} \right) - \delta d(y, 0) \right) \geq 0.$$

If  $y = 0$ , then

$$-\frac{\eta}{(T-s)^2} + \frac{t-s}{\nu} + F_A \left( -\partial_y \mathcal{D}_0 \left( 0, \frac{x}{\varepsilon} \right) \right) \geq 0.$$

We now distinguish the case  $A > A_0$  and  $A = A_0$ .

**Case  $A > A_0$ .** Thanks to the  $C^{1,1}$  regularity of the function  $\mathcal{D}_0$ , see Proposition 2.3.12, and the fact that the functions  $H_\alpha^\pm, H_\alpha$  are locally Lipschitz we obtain, for  $x \in J_\alpha$  and  $y \in J_\beta$  with  $\alpha \neq \beta$ ,

$$\text{if } x \neq 0, \quad \frac{t-s}{\nu} - \frac{\Delta t}{2\nu} + H_\alpha \left( \partial_x \mathcal{D}_0 \left( \frac{y}{\varepsilon}, \frac{x}{\varepsilon} \right) \right) + O \left( \frac{\Delta x}{\varepsilon} \right) \leq 0 \quad (2.4.6)$$

$$\text{if } x = 0, \quad \frac{t-s}{\nu} - \frac{\Delta t}{2\nu} + F_A \left( \partial_x \mathcal{D}_0 \left( \frac{y}{\varepsilon}, 0 \right) \right) + O \left( \frac{\Delta x}{\varepsilon} \right) \leq 0 \quad (2.4.7)$$

$$\text{if } y \neq 0, \quad \frac{t-s}{\nu} + H_\beta \left( -\partial_y \mathcal{D}_0 \left( \frac{y}{\varepsilon}, \frac{x}{\varepsilon} \right) \right) + O(\sqrt{\delta}) \geq \frac{\eta}{2T^2} \quad (2.4.8)$$

$$\text{if } y = 0, \quad \frac{t-s}{\nu} + F_A \left( -\partial_y \mathcal{D}_0 \left( 0, \frac{x}{\varepsilon} \right) \right) \geq \frac{\eta}{2T^2}. \quad (2.4.9)$$

Now for  $(y, x) \in J_\alpha \times J_\alpha$ , from (2.3.2) and (2.3.6), one can deduce that  $\mathcal{D}_0$  is in fact  $C^2$  far away from the curve  $\Gamma^\alpha$  defined in Remark 2.3.9, hence the viscosity inequalities (2.4.6)-(2.4.9) remain true.

Now we treat the case where  $(\frac{y}{\varepsilon}, \frac{x}{\varepsilon})$  is near the curve  $\Gamma^\alpha$ , but not on it.

First if  $(\frac{y}{\varepsilon}, \frac{x}{\varepsilon})$  is such that  $(\frac{y}{\varepsilon}, \frac{x}{\varepsilon}) \in K^\alpha \setminus \Gamma^\alpha$  and  $(\frac{y}{\varepsilon}, \frac{x-\Delta x}{\varepsilon}) \notin K^\alpha$ , we have

$$\mathcal{D}_0 \left( \frac{y}{\varepsilon}, \frac{x-\Delta x}{\varepsilon} \right) \leq L_\alpha \left( \frac{x-\Delta x-y}{\varepsilon} \right).$$

So as  $H_\alpha^+$  is non-decreasing, we deduce that

$$\begin{aligned} & H_\alpha^+ \left( \frac{\varepsilon}{\Delta x} \left\{ L_\alpha \left( \frac{x-y}{\varepsilon} \right) - L_\alpha \left( \frac{x-\Delta-yx}{\varepsilon} \right) \right\} \right) \\ & \leq H_\alpha^+ \left( \frac{\varepsilon}{\Delta x} \left\{ \mathcal{D}_0 \left( \frac{y}{\varepsilon}, \frac{x}{\varepsilon} \right) - \mathcal{D}_0 \left( \frac{y}{\varepsilon}, \frac{x-\Delta x}{\varepsilon} \right) \right\} \right). \end{aligned}$$

Hence the viscosity inequalities (2.4.6)-(2.4.9) remain true. If  $(\frac{y}{\varepsilon}, \frac{x}{\varepsilon})$  is such that  $(\frac{y}{\varepsilon}, \frac{x}{\varepsilon}) \notin K^\alpha$  and  $(\frac{y}{\varepsilon}, \frac{x+\Delta x}{\varepsilon}) \in K^\alpha \setminus \Gamma^\alpha$ , we have

$$\mathcal{D}_0 \left( \frac{y}{\varepsilon}, \frac{x-\Delta x}{\varepsilon} \right) \leq \mathcal{D}_{\text{junction}} \left( \frac{y}{\varepsilon}, \frac{x-\Delta x}{\varepsilon} \right).$$

So as  $H_\alpha^-$  is non-increasing, we deduce that

$$\begin{aligned} & H_\alpha^- \left( \frac{\varepsilon}{\Delta x} \left\{ \mathcal{D}_{\text{junction}} \left( \frac{y}{\varepsilon}, \frac{x}{\varepsilon} \right) - \mathcal{D}_{\text{junction}} \left( \frac{y}{\varepsilon}, \frac{x+\Delta x}{\varepsilon} \right) \right\} \right) \\ & \leq H_\alpha^- \left( \frac{\varepsilon}{\Delta x} \left\{ \mathcal{D}_0 \left( \frac{y}{\varepsilon}, \frac{x}{\varepsilon} \right) - \mathcal{D}_0 \left( \frac{y}{\varepsilon}, \frac{x+\Delta x}{\varepsilon} \right) \right\} \right). \end{aligned}$$

Hence the viscosity inequalities (2.4.6)-(2.4.9) remain true.

Now for  $(\frac{y}{\varepsilon}, \frac{x}{\varepsilon})$  on the curve  $\Gamma^\alpha$ , we get the following viscosity inequalities, using Proposition 2.2.3.

If  $x \neq 0$ , then

$$\frac{t-s}{\nu} - \frac{\Delta t}{2\nu} + \max \left\{ H_{\alpha}^{-} \left( \frac{\varepsilon}{\Delta x} \left\{ L_{\alpha} \left( \frac{x + \Delta x - y}{\varepsilon} \right) - L_{\alpha} \left( \frac{x - y}{\varepsilon} \right) \right\} \right), \right. \\ \left. H_{\alpha}^{+} \left( \frac{\varepsilon}{\Delta x} \left\{ L'_{\alpha}(\xi_{\alpha}^{+}) \frac{x}{\varepsilon} - L'_{\alpha}(\xi_{\alpha}^{+}) \left( \frac{x - \Delta x}{\varepsilon} \right) \right\} \right) \right\} \leq 0.$$

If  $x = 0$ , then

$$\frac{t-s}{\nu} - \frac{\Delta t}{2\nu} + \max \left( A, \max_{\alpha} \left\{ H_{\alpha}^{-} \left( \frac{\varepsilon}{\Delta x} \left\{ L_{\alpha} \left( \frac{\Delta x - y}{\varepsilon} \right) - L_{\alpha} \left( -\frac{y}{\varepsilon} \right) \right\} \right) \right\} \right) \leq 0.$$

If  $y \neq 0$ , then

$$-\frac{\eta}{(T-s)^2} + \frac{t-s}{\nu} + \max \left\{ H_{\alpha}^{-} \left( L'_{\alpha} \left( \frac{x-y}{\varepsilon} \right) - \delta d(y, 0) \right), \right. \\ \left. H_{\alpha}^{+} \left( L'_{\alpha}(\xi_{\alpha}^{-}) - \delta d(y, 0) \right) \right\} \geq 0.$$

If  $y = 0$ , then

$$-\frac{\eta}{(T-s)^2} + \frac{t-s}{\nu} + \max \left( A, \max_{\alpha} H_{\alpha}^{-} \left( L'_{\alpha} \left( \frac{x}{\varepsilon} \right) \right) \right) \geq 0.$$

We now simplify the above inequalities,

$$\text{if } x \neq 0, \quad \frac{t-s}{\nu} - \frac{\Delta t}{2\nu} + \max \left\{ H_{\alpha}^{-} \left( L'_{\alpha} \left( \frac{x-y}{\varepsilon} \right) \right), H_{\alpha}^{+}(L'_{\alpha}(\xi_{\alpha}^{+})) \right\} \\ + O\left(\frac{\Delta x}{\varepsilon}\right) \leq 0 \quad (2.4.10)$$

$$\text{if } x = 0, \quad \frac{t-s}{\nu} - \frac{\Delta t}{2\nu} + \max \left( A, \max_{\alpha} H_{\alpha}^{-} \left( L'_{\alpha} \left( -\frac{y}{\varepsilon} \right) \right) \right) + O\left(\frac{\Delta x}{\varepsilon}\right) \leq 0 \quad (2.4.11)$$

$$\text{if } y \neq 0, \quad \frac{t-s}{\nu} + \max \left\{ H_{\alpha}^{-} \left( L'_{\alpha} \left( \frac{x-y}{\varepsilon} \right) \right), H_{\alpha}^{+}(L'_{\alpha}(\xi_{\alpha}^{-})) \right\} + O(\sqrt{\delta}) \geq \frac{\eta}{2T^2} \quad (2.4.12)$$

$$\text{if } y = 0, \quad \frac{t-s}{\nu} + \max \left( A, \max_{\alpha} H_{\alpha}^{-} \left( L'_{\alpha} \left( \frac{x}{\varepsilon} \right) \right) \right) \geq \frac{\eta}{2T^2}. \quad (2.4.13)$$

Combining these viscosity inequalities and using Theorem 2.3.10 with the Hamiltonians  $\beta \circ H_{\alpha}$ , we deduce the same equalities for the Hamiltonians  $H_{\alpha}$  as  $\beta$  is a bijection. We use also the fact that  $H_{\alpha}^{+}(L'_{\alpha}(\xi_{\alpha}^{+})) = A$  and  $H_{\alpha}^{+}(L'_{\alpha}(\xi_{\alpha}^{-})) = A_0$ , we get in all cases

$$\eta \leq \mathcal{O}\left(\frac{\Delta t}{\nu}\right) + \mathcal{O}\left(\frac{\Delta x}{\varepsilon}\right) + \mathcal{O}(\sqrt{\delta}) =: \eta^*.$$

**Case  $A = A_0$ .** In this case the function  $\mathcal{D}_{junction}$  is not of class  $C^{1,1}$ , see Proposition 2.3.12. So we consider the function  $\mathcal{D}_0$  associated with  $A = A_0 + \rho$  where  $\rho$  is a small parameter. The only difference with the case  $A > A_0$  is in the case  $x \in J_\alpha$  and  $y \in J_\beta$  with  $\alpha \neq \beta$ . We only treat the case  $x \in J_\alpha \setminus \{0\}$  and  $y \in J_\beta$  with  $\alpha \neq \beta$  since in the other cases the arguments are the same as the proof of the case  $A > A_0$ . Since  $\mathcal{D}_0(\frac{y}{\varepsilon}, \cdot)$  is non-decreasing and  $H_\alpha^-(p) = A_0$  for  $p \geq 0$ , and  $H_\alpha^+(p) = H_\alpha(p)$  for  $p \geq 0$ , we have

$$\frac{t-s}{\nu} - \frac{\Delta t}{2\nu} + H_\alpha \left( \frac{\varepsilon}{\Delta x} \left\{ \mathcal{D}_0 \left( \frac{y}{\varepsilon}, \frac{x}{\varepsilon} \right) - \mathcal{D}_0 \left( \frac{y}{\varepsilon}, \frac{x - \Delta x}{\varepsilon} \right) \right\} \right) \leq 0. \quad (2.4.14)$$

By using the Taylor expansion of the function  $\mathcal{D}_0(\frac{y}{\varepsilon}, \cdot)$  of class  $C^1$ , there exists  $\theta_1 \in [0, 1]$  such that

$$\begin{aligned} H_\alpha \left( \frac{\varepsilon}{\Delta x} \left\{ \mathcal{D}_0 \left( \frac{y}{\varepsilon}, \frac{x}{\varepsilon} \right) - \mathcal{D}_0 \left( \frac{y}{\varepsilon}, \frac{x - \Delta x}{\varepsilon} \right) \right\} \right) \\ = H_\alpha \left( \partial_x \mathcal{D}_0 \left( \frac{y}{\varepsilon}, \frac{x}{\varepsilon} \right) - \frac{\Delta x}{2\varepsilon} \partial_{xx} \mathcal{D}_0 \left( \frac{y}{\varepsilon}, \frac{x - \theta_1 \Delta x}{\varepsilon} \right) \right) \end{aligned}$$

Using now a Taylor expansion of the function  $H_\alpha$  of class  $C^2$ , there exists  $\theta_2 \in [0, 1]$  such that

$$\begin{aligned} H_\alpha \left( \frac{\varepsilon}{\Delta x} \left\{ \mathcal{D}_0 \left( \frac{y}{\varepsilon}, \frac{x}{\varepsilon} \right) - \mathcal{D}_0 \left( \frac{y}{\varepsilon}, \frac{x - \Delta x}{\varepsilon} \right) \right\} \right) = \\ H_\alpha \left( \partial_x \mathcal{D}_0 \left( \frac{y}{\varepsilon}, \frac{x}{\varepsilon} \right) \right) - \frac{\Delta x}{2\varepsilon} \partial_{xx} \mathcal{D}_0 \left( \frac{y}{\varepsilon}, \frac{x - \theta_1 \Delta x}{\varepsilon} \right) H'_\alpha \left( \partial_x \mathcal{D}_0 \left( \frac{y}{\varepsilon}, \frac{x}{\varepsilon} \right) \right) \\ + \frac{1}{8} \left( \frac{\Delta x}{\varepsilon} \right)^2 \partial_{xx} \mathcal{D}_0 \left( \frac{y}{\varepsilon}, \frac{x - \theta_1 \Delta x}{\varepsilon} \right)^2 H''_\alpha \left( \partial_x \mathcal{D}_0 \left( \frac{y}{\varepsilon}, \frac{x}{\varepsilon} \right) - \frac{\theta_2 \Delta x}{2\varepsilon} \partial_{xx} \mathcal{D}_0 \left( \frac{y}{\varepsilon}, \frac{x - \theta_1 \Delta x}{\varepsilon} \right) \right). \end{aligned} \quad (2.4.15)$$

Using Taylor expansion for  $\partial_x \mathcal{D}_0(\cdot, \frac{y}{\varepsilon})$  and  $H'_\alpha$  of class  $C^1$  there exists  $\theta_3, \theta_4 \in [0, 1]$  such that

$$\begin{aligned} H'_\alpha \left( \partial_x \mathcal{D}_0 \left( \frac{y}{\varepsilon}, \frac{x}{\varepsilon} \right) \right) = H'_\alpha \left( \partial_x \mathcal{D}_0 \left( \frac{y}{\varepsilon}, \frac{x - \theta_1 \Delta x}{\varepsilon} \right) + \theta_1 \frac{\Delta x}{\varepsilon} \partial_{xx} \mathcal{D}_0 \left( \frac{y}{\varepsilon}, \frac{x - \theta_3 \Delta x}{\varepsilon} \right) \right) \\ = H'_\alpha \left( \partial_x \mathcal{D}_0 \left( \frac{y}{\varepsilon}, \frac{x - \theta_1 \Delta x}{\varepsilon} \right) \right) \\ + \theta_1 \frac{\Delta x}{\varepsilon} \partial_{xx} \mathcal{D}_0 \left( \frac{y}{\varepsilon}, \frac{x - \theta_3 \Delta x}{\varepsilon} \right) H''_\alpha \left( \partial_x \mathcal{D}_0 \left( \frac{y}{\varepsilon}, \frac{x - \theta_1 \Delta x}{\varepsilon} \right) + \theta_4 \frac{\Delta x}{\varepsilon} \partial_{xx} \mathcal{D}_0 \left( \frac{y}{\varepsilon}, \frac{x - \theta_3 \Delta x}{\varepsilon} \right) \right). \end{aligned} \quad (2.4.16)$$

Notice that the terms in  $H''_\alpha$  are bounded since  $\frac{x}{\varepsilon}, \frac{y}{\varepsilon}$  and  $\frac{\Delta x}{\varepsilon\rho}$  are bounded independently of  $\Delta x \leq 1$  as we take  $\varepsilon = \rho = (\Delta x)^{\frac{2}{5}}$ .

So combining (2.4.15) and (2.4.16), thanks to the  $C^{1,1}$  regularity of the function  $\mathcal{D}_0$ , see Proposition 2.3.12, we deduce that

$$\begin{aligned} & H_\alpha \left( \frac{\varepsilon}{\Delta x} \left\{ \mathcal{D}_0 \left( \frac{y}{\varepsilon}, \frac{x}{\varepsilon} \right) - \mathcal{D}_0 \left( \frac{y}{\varepsilon}, \frac{x - \Delta x}{\varepsilon} \right) \right\} \right) \\ &= H_\alpha \left( \partial_x \mathcal{D}_0 \left( \frac{y}{\varepsilon}, \frac{x}{\varepsilon} \right) \right) + \mathcal{O} \left( \frac{\Delta x}{\varepsilon \sqrt{\rho}} \right) + \mathcal{O} \left( \left( \frac{\Delta x}{\varepsilon \rho} \right)^2 \right). \end{aligned}$$

So combining the viscosity inequality and using the fact that  $|F_A - F_{A_0}| \leq \rho$  we have

$$\eta \leq \mathcal{O} \left( \frac{\Delta t}{\nu} \right) + \mathcal{O} \left( \frac{\Delta x}{\varepsilon \sqrt{\rho}} \right) + \mathcal{O} \left( \left( \frac{\Delta x}{\varepsilon \rho} \right)^2 \right) + \mathcal{O}(\sqrt{\delta}) + \rho =: \eta^*. \quad (2.4.17)$$

**Step 3: Estimate of the supremum.** We proved in the previous step that, if  $\eta > \eta^*$ , then either  $M_{\varepsilon, \delta} \leq 0$  or  $M_{\varepsilon, \delta}$  is reached either for  $t = 0$  or  $s = 0$ .

If  $t = 0$ , then using Theorem 2.2.2, we have

$$M_{\varepsilon, \delta} \leq u_0(x) - u_0(y) - \frac{\gamma}{2\varepsilon} d^2(y, x) + C_T s - \frac{s^2}{2\nu}.$$

Using the fact that  $u_0$  is  $L_0$ -Lipschitz, one can deduce

$$\begin{aligned} M_{\varepsilon, \delta} &\leq \sup_{r \geq 0} \left( L_0 r - \frac{\gamma}{2\varepsilon} r^2 \right) + \sup_{r > 0} \left( C r - \frac{r^2}{2\nu} \right) \\ &\leq O(\varepsilon) + O(\nu). \end{aligned}$$

If  $s = 0$ , then we can argue similarly (by using the stability of the numerical scheme) and get

$$M_{\varepsilon, \delta} \leq O(\varepsilon) + O(\nu).$$

**Step 4: Conclusion.** We proved that for  $\eta > \eta^*$ ,  $M_{\varepsilon, \delta} \leq O(\varepsilon) + O(\nu)$ . This implies that for all  $(t, x) \in \mathcal{G}_h$ ,  $t \leq T/2$ , we have

$$u^h(t, x) - u(t, x) \leq \varepsilon \tilde{\mathcal{D}}_0 \left( \frac{x}{\varepsilon}, \frac{x}{\varepsilon} \right) + \frac{\delta}{2} d^2(x, 0) + \frac{2\eta}{T} + O(\varepsilon) + O(\nu).$$

Replacing  $\eta$  by  $2\eta^*$  and recalling that  $\tilde{\mathcal{D}}_0(x, x) \leq C + A$  for all  $x \in J$ , we deduce that for  $(t, x) \in \mathcal{G}_h$  and  $t \leq T/2$  (after letting  $\delta \rightarrow 0$ ),

$$u^h(t, x) - u(t, x) \leq \mathcal{O} \left( \frac{\Delta t}{\nu} \right) + \mathcal{O} \left( \frac{\Delta x}{\varepsilon} \right) + O(\varepsilon) + O(\nu).$$

Using the CFL condition (2.1.15) and optimizing with respect to  $\varepsilon$  and  $\nu$  yields the desired result.  $\square$

*Remark 2.4.4.* If for all  $\alpha \in \{1, \dots, N\}$ ,  $\min H_\alpha = A_0$ , then in the case where  $A = A_0$ , thanks to the  $C^{1,1}$  regularity of the function  $D_0$ , see Proposition 2.3.12, we can conclude as the case  $A > A_0$  that the error estimate is of order  $\Delta x^{\frac{1}{2}}$ .

$\Delta x$	$\ u(T, \cdot) - u^h(T, \cdot)\ _\infty$
0.00250	$1,192 \times 10^{-4}$
0.00100	$0,753 \times 10^{-4}$
0.00075	$0,644 \times 10^{-4}$
0.00050	$0,503 \times 10^{-4}$
0.00025	$0,329 \times 10^{-4}$

Figure 2.3: Error estimates for  $A = A_0 = 0$ 

$\Delta x$	$\ u(T, \cdot) - u^h(T, \cdot)\ _\infty$
0.00250	$1,266 \times 10^{-4}$
0.00100	$0,719 \times 10^{-4}$
0.00075	$0,616 \times 10^{-4}$
0.00050	$0,511 \times 10^{-4}$
0.00025	$0,350 \times 10^{-4}$

Figure 2.4: Error estimates for  $A = 0.1 > A_0$ 

### 2.4.1 Numerical simulations

In this subsection, we give a numerical example which illustrates the convergence rate we obtained in the previous subsection. In the case  $A > A_0$ , we get an optimal error estimate of order  $\Delta x^{\frac{1}{2}}$ . But in the case  $A = A_0$  we only have examples with an error estimate of order  $\Delta x^{\frac{1}{2}}$  when in the proof we have  $\Delta x^{\frac{2}{5}}$ . So we wonder if the error estimate obtained in the proof is optimal for the case  $A = A_0$ .

Here we consider a junction with two branches  $J_1 = J_2 = [0, X]$ . We have the two following Hamiltonians,

$$\begin{aligned} H_1(p) &= p^2, \\ H_2(p) &= p^2 - 1, \end{aligned}$$

and the initial data

$$u_0(x) = \begin{cases} \sin(0.2x) & \text{if } x \in J_1, \\ \sin(x) & \text{if } x \in J_2. \end{cases}$$

In the simulation we take  $X = 0.1$  and we give the error  $\|u(T, \cdot) - u^h(T, \cdot)\|_\infty$  at time  $T = 0.01$ . Here we have  $A_0 = 0$ .

For  $A = 0$ ,  $A = 0.1 > A_0$  and  $\Delta t = \frac{\Delta x}{10}$  we get the following result, see Figure 2.5 and 2.6 plotted in logarithmic scale and the error values in Table 2.3 and 2.4.

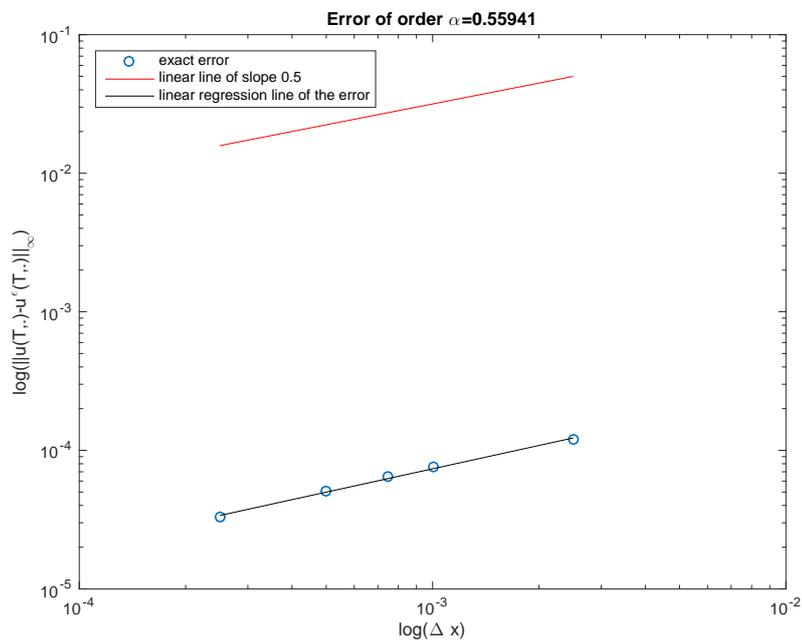


Figure 2.5: Error estimates for  $A = A_0 = 0$

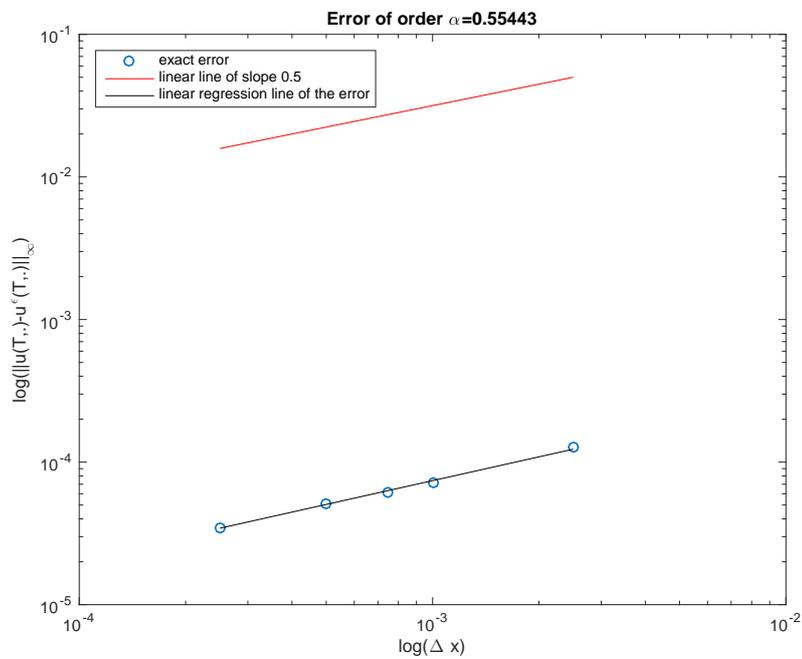


Figure 2.6: Error estimates for  $A = 0.1 > A_0$

## Part II

Indirect controllability/stability of  
a 1D system of wave equations



## Chapter 3

# Influence of the coefficients of coupled wave equations on their indirect exact boundary controllability

*Le taux de décroissance de l'énergie d'un système d'équations d'ondes couplées dépend du type de couplage, de la nature algébrique du paramètre du couplage et de la propriété arithmétique du rapport des vitesses de propagation des ondes (voir [68], [13], [104], [92]). Dans ce chapitre, nous étudions la contrôlabilité exacte indirecte d'un système d'équations d'ondes couplées par des termes de vitesse, dans le cas mono-dimensionnel. Nous considérons les cas où les ondes se propagent avec des vitesses égales ou différentes. Tout d'abord, en utilisant l'analyse non harmonique, nous établissons les inégalités d'observabilité, qui sont influencées par la nature du paramètre du couplage et par la propriété arithmétique du rapport des vitesses de propagation des ondes. Ensuite, en utilisant la méthode d'unicité de Hilbert, nous démontrons que le système est exactement contrôlable, et que le temps de contrôle peut être petit.*

**Abstract.** The energy decay rate of a system of coupled wave equations depends on the type of damping, the type of coupling, the algebraic nature of the coupling parameter and the arithmetic property of the ratio of the wave propagation speeds (see [68], [13], [104], [92]). In this chapter, we study the indirect boundary exact controllability of a system of wave equations coupled through velocity terms in one dimensional space. We consider the cases where waves propagate with equal or different speeds. First, using the non harmonic analysis, we establish the weak observability inequalities, which are greatly influenced by the nature of the coupling parameter and are sensitive to the arithmetic property of the ratio of the wave propagation speeds. Next, using the HUM method, we prove that the system is exactly controllable, and that the control time can be small.

### 3.1 Introduction

In [92], Najdi and Wehbe studied the indirect boundary stabilization of a system of two wave equations coupled through velocity terms. The system is described by:

$$\begin{cases} u_{tt} - u_{xx} + by_t = 0 & \text{on } (0, 1) \times (0, +\infty), \\ y_{tt} - ay_{xx} - bu_t = 0 & \text{on } (0, 1) \times (0, +\infty), \\ y_x(0, t) - y_t(0, t) = 0 & \text{on } (0, +\infty), \\ u(1, t) = y(1, t) = u(0, t) = 0 & \text{on } (0, +\infty), \end{cases} \quad (3.1.1)$$

where  $a > 0$  and  $b \in \mathbb{R}^*$  are constants. First, they proved that system (3.1.1) is strongly stable if and only if the coupling parameter  $b$  is outside a well determined discrete set  $S_s$  of exceptional values. Consequently, the strong stability does not hold in general. Next, for  $b \notin S_s$ , they showed that the energy decay rate of system (3.1.1) is greatly influenced by the nature of the coupling parameter  $b$  (an additional condition on  $b$ ) and by the arithmetic property of the ratio of the wave propagation speeds  $a$ . Indeed, in the case of  $a = 1$  when the waves propagate at the same speed and if there exist no  $k \in \mathbb{Z}$  such that  $b = k\pi$ , they established exponential stability of system (3.1.1). Otherwise, they proved the lack of exponential stability of the system and established a polynomial energy decay rate also depending on the nature of  $b$  and on the arithmetic property of  $a$ . Roughly speaking, if  $a = 1$  and  $b$  is of the form  $k\pi$  for  $k \in \mathbb{Z}$ , an optimal energy decay rate of type  $\frac{1}{t}$  is established, and if  $a \neq 1$  and  $\sqrt{a} \in \mathbb{Q}$ , they obtained the same polynomial energy decay rate. In this chapter, we are interested in the influence of the nature of the coefficients  $a$  and  $b$  on the indirect boundary exact controllability of a one dimensional setting of wave equations coupled through velocity terms. The system is described by:

$$\begin{cases} u_{tt} - u_{xx} + by_t = 0 & \text{on } (0, 1) \times (0, T), \\ y_{tt} - ay_{xx} - bu_t = 0 & \text{on } (0, 1) \times (0, T), \\ y(1, t) = v(t) & \text{for all } t \in (0, T), \\ u(1, t) = u(0, t) = y(0, t) = 0 & \text{for all } t \in (0, T), \end{cases} \quad (3.1.2)$$

where  $a > 0$  is the ratio of the speeds of the two equations,  $b \in \mathbb{R}^*$  is the coupling parameter, and  $v \in L^2(0, T)$  is the control. The system (3.1.2) is a classic

model for the motions of two stacked elastic bodies. The control  $v$  is applied only at the right boundary of the second equation. The first equation is indirectly controlled by means of the coupling between the equations. We consider the following indirect boundary exact controllability problem: For given  $T > 0$  and initial data  $(u_0, u_1, y_0, y_1)$  belonging to a suitable space, is it possible to find a suitable control  $v$  so that the solution of system (3.1.2)  $(u, u_t, y, y_t)$  is driven to zero in time  $T$ ? *i.e.*

$$u(x, T) = u_t(x, T) = y(x, T) = y_t(x, T) = 0 \quad \text{on} \quad (0, 1).$$

In [104], Toufayli studied the exact controllability of system (3.1.2) in a star-shaped domain  $\Omega$  in  $\mathbb{R}^N$ . Using a multiplier method, she proved that by observing only one component of the associated homogeneous system, one can get back a full energy of both components in the case where  $a = 1$  and  $b$  small enough. Consequently, using Hilbert Uniqueness Method, she established an indirect exact controllability result. It seems that the conditions  $a = 1$  and  $b$  small enough are technical for the multiplier method. The natural question is then whether or not the exact controllability still holds in the natural physical cases  $a = 1$  and  $b$  large or  $a \neq 1$ . The aim of this chapter is to give a complete answer to this interesting question. For this goal, we will use a spectral approach to investigate how the interaction between the modes of the first equation and the modes of the second equation is influenced by the algebraic nature of the coupling parameter  $b$  and how it is sensitive to the arithmetic property of the ratio of the wave propagation speeds  $a$ . We use it also to get the suitable controllability spaces (see Theorems 3.1.3, 3.1.1 and 3.1.2).

**Main results.** Unlike the spectral method, the multiplier method used in [6, 9, 12, 104] requires some technical conditions on the coefficients of the system. Then, our aim is to establish observability inequalities using the Ingham's theorem while distinguishing the following cases:

$$\text{(Case 1)} \quad a = 1 \text{ and } b \notin \pi\mathbb{Z},$$

$$\text{(Case 2)} \quad a = 1 \text{ and } b \in \pi\mathbb{Z},$$

$$\text{(Case 3)} \quad a \neq 1 \text{ such that } a \in \mathbb{Q},$$

$$\text{(Case 4)} \quad a \neq 1 \text{ such that } a \in \mathbb{R} \setminus \mathbb{Q}.$$

In order to do so, we consider respectively the associated homogeneous problem (for  $a \neq 1$  and  $a = 1$ ), that is to say, the null boundary acted control system. Hence, using the Hilbert Uniqueness Method (see [79]), we deduce the indirect exact controllability of the considered system.

Let us denote by  $\lambda_{1,m}, \lambda_{2,n}$  the two branches of eigenvalues of the homogeneous systems (3.2.1) and (3.3.1) respectively, and by

$$\begin{cases} E_{1,n} = (\phi_{1,n}, \lambda_{1,n}\phi_{1,n}, \psi_{1,n}, \lambda_{1,n}\psi_{1,n}), \\ E_{2,n} = (\phi_{2,n}, \lambda_{2,n}\phi_{2,n}, \psi_{2,n}, \lambda_{2,n}\psi_{2,n}) \end{cases} \quad (3.1.3)$$

the corresponding eigenvectors. Our main results are the following.

**Case 1.** The eigenvalues  $\lambda_{1,m}, \lambda_{2,n}$  satisfy an uniform gap condition. Then using the usual Ingham's theorem, we prove the following result.

**Theorem 3.1.1.** *Assume that  $a = 1$ , condition (3.2.8) holds and  $b \notin \pi\mathbb{Z}$ . Let  $0 < |\hat{b}| < \pi$  be the resulting quantity of  $b - \hat{k}\pi$ , where  $\hat{k} \in \mathbb{N}^*$ , such that  $\hat{k}\pi < |b| < (\hat{k}+1)\pi$  and let*

$$T > \frac{2\pi}{|\hat{b}|}. \quad (3.1.4)$$

Then there exists a constant  $c_1 > 0$  such that the following direct inequality holds

$$\int_0^T |\psi_x(1, t)|^2 dt \leq c_1 \|(\phi_0, \phi_1, \psi_0, \psi_1)\|_{\mathcal{H}}^2, \quad (3.1.5)$$

for all solution  $\Phi = (\phi, \phi_t, \psi, \psi_t)$  solving the corresponding homogeneous Cauchy problem.

Moreover, there exists a constant  $0 < c_2 < c_1$  depending only on  $b$ , such that the following inverse observability inequality holds

$$c_2 \|(\phi_0, \phi_1, \psi_0, \psi_1)\|_{\mathcal{H}}^2 \leq \int_0^T |\psi_x(1, t)|^2 dt. \quad (3.1.6)$$

**Case 2.** The eigenvalues  $\lambda_{1,m}, \lambda_{2,n}$  are asymptotically close. To be precise, following proposition 3.2.7, we prove that there exist infinitely many  $m \sim n + k_0$

$$\lambda_{1,m} - \lambda_{2,n} \sim \frac{O(1)}{m^2}, \quad \lambda_{1,m} - \lambda_{2,n} \sim \frac{O(1)}{n^2}.$$

Then, the usual Ingham's theorem which requires an uniform gap condition does not work in this case and consequently, the observability inequalities are not true in the energy space. In order to get the inverse observability inequality, we will use a general Ingham-type theorem based on the divided differences, which tolerates asymptotically and even multiple eigenvalues. On the other hand, the observation is on the second components of the eigenfunctions  $E_{1,n}, E_{2,n}$ . From (3.2.11)-(3.2.12), we see that

$$(\psi_{1,n})_x(1) = O(1), \quad (\psi_{2,n})_x(1) = O(1). \quad (3.1.7)$$

So, a natural space for the observability inequalities is the following weighted spectral space

$$D_2 = \{(\phi_0, \phi_1, \psi_0, \psi_1) = \sum_{n \neq 0} (\alpha_{1,n} E_{1,n} + \alpha_{2,n} E_{2,n}) n^2\}.$$

We prove afterwards the following result.

**Theorem 3.1.2.** *Assume that  $a = 1$ , condition (3.2.8) and  $b = k_0\pi$  for some  $k_0 \in \mathbb{Z}^*$ . Let*

$$T > 4.$$

Then there exists a constant  $c_3 > 0$  such that the following direct inequality holds

$$\int_0^T |\psi_x(1, t)|^2 dt \leq c_3 \|(\phi_0, \phi_1, \psi_0, \psi_1)\|_{\mathcal{H}}^2, \quad (3.1.8)$$

for all solution  $\Phi = (\phi, \phi_t, \psi, \psi_t)$  solving the corresponding homogeneous Cauchy problem.

Moreover, there exists a constant  $0 < c_4 < c_3$ , such that the following inverse observability inequality holds

$$c_4 \|(\phi_0, \phi_1, \psi_0, \psi_1)\|_{D_2}^2 \leq \int_0^T |\psi_x(1, t)|^2 dt. \quad (3.1.9)$$

**Cases 3 and 4.** We prove in Proposition 3.3.4, that there exist infinitely many  $\sqrt{am} \sim n$  such that the eigenvalues are asymptotically close

$$\lambda_{1,m} - \lambda_{2,n} \sim \frac{O(1)}{m}, \quad \lambda_{1,m} - \lambda_{2,n} \sim \frac{O(1)}{n}, \quad (3.1.10)$$

where we distinguish cases of the ratio of the wave speeds, as  $0 < a \neq 1$  is a rational number or an irrational number. Then, we will use a general Ingham-type theorem. On the other hand, the observation is on the second components of the eigenfunctions  $E_{1,n}, E_{2,n}$ . From (3.3.12)-(3.3.13), we see that

$$(\psi_{1,n})_x(1) = O(1), \quad (\psi_{2,n})_x(1) = \frac{O(1)}{n}. \quad (3.1.11)$$

So, if we want to observe the first equation via the second one, we have to use a weaker norm so that the observation on the second component  $(\psi_{1,n})_x(1)$  does not disappear. It seems that the following weighted spectral space

$$D_1 = \{(\phi_0, \phi_1, \psi_0, \psi_1) = \sum_{n \neq 0} (\alpha_{1,n} E_{1,n} + \alpha_{2,n} n E_{2,n}) n\} \quad (3.1.12)$$

is the natural space for the observability. In (3.1.12), the factor  $n$  before the eigenvector  $E_{2,n}$  is due to the transmission of the modes between the two equations (3.1.11), and the factor  $n$  is due to the closeness of the eigenvalues (3.1.10). We then prove the following result.

**Theorem 3.1.3.** *Let  $0 < a \neq 1$ , and let  $b$  a real number satisfying (3.3.9). Assume that*

$$T > 2 \left( 1 + \frac{1}{\sqrt{a}} \right).$$

*Then there exists a constant  $c_1 > 0$  such that the following direct inequality holds*

$$\int_0^T |\psi_x(1, t)|^2 dt \leq c_1 \|(\phi_0, \phi_1, \psi_0, \psi_1)\|_{\mathcal{H}}^2, \quad (3.1.13)$$

for all solution  $\Phi = (\phi, \phi_t, \psi, \psi_t)$  solving the corresponding homogeneous Cauchy problem.

Moreover, there exists a constant  $0 < c < c_1$  depending only on  $a$  and  $b$  such that the following inverse observability inequalities hold true:

1. If  $a$  is rational number, then

$$c\|(\phi_0, \phi_1, \psi_0, \psi_1)\|_{D_1}^2 \leq \int_0^T |\psi_x(1, t)|^2 dt. \quad (3.1.14)$$

2. For almost all irrational number  $a > 0$ , we have

$$c\|(\phi_0, \phi_1, \psi_0, \psi_1)\|_{D_1}^2 \leq \int_0^T |\psi_x(1, t)|^2 dt. \quad (3.1.15)$$

**Brief review of the literature.** Observability and exact controllability have been studied in an extensive number of publications. In [79], Lions studied the complete and partial observability and controllability of coupled systems of either hyperbolic-hyperbolic type or hyperbolic-parabolic type. These results assume that the coupling parameter is sufficiently small. In [71], Komornik and Lorette extended the study to the case of arbitrary coupling parameters. The null controllability of the reaction diffusion system has been studied in [15], by deriving an observability estimate for the linearized problem. In [108], Zhang and Zuazua obtained the exact controllability for one-dimensional system of coupled heat-wave equations by proving the observability estimate with a new type of Ingham inequality. In [9], (see also [6] and [12]) Alabau studied the boundary exact controllability of an abstract system of two second order evolution equations coupled through the zero order terms wherein only one of the equations is controlled. She proved that, by observing only one component of the associated homogeneous system, one can get back a full weakened energy of both components under a compatibility condition linking the operators of each equation and for small coupling parameter. Consequently, using Hilbert Uniqueness Method, she established an indirect exact controllability result for small coupling parameter. Liu and Rao in [89], studied the exact controllability of a system of one-dimensional wave equations coupled through the zero order terms with boundary control acted on only one equation. Using the non harmonic analysis, they established the weak observability inequalities, which depend on the ratio of the wave propagation speeds.

**Organization of the chapter.** The remaining of the chapter is organized as follows. Section 3.2 is dedicated to the proof of exact controllability while waves propagate with equal speeds. Depending on the algebraic property of the coupling parameter  $b$ , we deduce the corresponding observability spaces. In Section 3.3, we consider the case where the waves propagate with different speeds. Using the divided difference technique, the suitable weighted observability spaces are deduced.

## 3.2 Observability and exact controllability under equal speeds waves propagation condition. The case $a = 1$

In this section, we study exact controllability of a system of coupled wave equations propagating with equal speeds, which corresponds to the case  $a = 1$ .

### 3.2.1 Observability and exact controllability in spectral spaces

The aim of this subsection is to establish some observability inequalities by spectral approach. We consider the following associated homogeneous system

$$\begin{cases} \phi_{tt} - \phi_{xx} + b\psi_t = 0 & \text{on } (0, 1) \times (0, T), \\ \psi_{tt} - \psi_{xx} - b\phi_t = 0 & \text{on } (0, 1) \times (0, T), \\ \phi(0, t) = \phi(1, t) = 0 & \text{for all } t \in (0, T), \\ \psi(0, t) = \psi(1, t) = 0 & \text{for all } t \in (0, T), \end{cases} \quad (3.2.1)$$

where  $b \neq 0$ , denoted as the coupling parameter, is a real number.

Let us recall the energy space  $\mathcal{H}$  defined in (3.3.2) endowed with the inner product

$$(\Phi, \Psi)_{\mathcal{H}} = \int \left( \phi_x \bar{\phi}_x + \theta \bar{\theta} + \psi_x \bar{\psi}_x + \eta \bar{\eta} \right) dx \quad \forall \Phi = (\phi, \theta, \psi, \eta), \Psi = (\tilde{\phi}, \tilde{\theta}, \tilde{\psi}, \tilde{\eta}) \in \mathcal{H}. \quad (3.2.2)$$

Now we define a linear unbounded operator  $\mathcal{A} : D(\mathcal{A}) \rightarrow \mathcal{H}$  by

$$D(\mathcal{A}) = (H^2 \cap H_0^1)(0, 1) \times H_0^1(0, 1) \times (H^2 \cap H_0^1)(0, 1) \times H_0^1(0, 1),$$

and

$$\mathcal{A}(\phi, \tilde{\phi}, \psi, \tilde{\psi}) = (\tilde{\phi}, \phi_{xx} - b\tilde{\psi}, \tilde{\psi}, \psi_{xx} + b\tilde{\phi}).$$

Then setting  $\Phi = (\phi, \phi_t, \psi, \psi_t)$  a regular solution of (3.2.1), we rewrite it into an evolution equation

$$\begin{cases} \Phi_t & = \mathcal{A}\Phi(t), \\ \Phi(0) & = \Phi_0 \in \mathcal{H}. \end{cases} \quad (3.2.3)$$

Since  $\mathcal{A}$  is a skew adjoint operator with a compact resolvent, then, by a corollary of the Lumer Philips's theorem [94],  $\mathcal{A}$  is the infinitesimal generator of a  $C_0$  semi-group of contractions  $e^{t\mathcal{A}}$  on  $\mathcal{H}$ .

Let us consider the corresponding eigenvalue problem

$$\begin{cases} \lambda^2 \phi - \phi_{xx} + b\lambda \psi = 0, \\ \lambda^2 \psi - \psi_{xx} - b\lambda \phi = 0, \\ \phi(0) = \phi(1) = 0, \\ \psi(0) = \psi(1) = 0. \end{cases} \quad (3.2.4)$$

For some constants  $C, D$  let

$$\phi(x) = C \sin(n\pi x), \quad \psi(x) = D \sin(n\pi x) \quad (3.2.5)$$

be a solution of (3.2.4). It follows that

$$\begin{cases} (\lambda^2 + (n\pi)^2)C + b\lambda D = 0, \\ (\lambda^2 + (n\pi)^2)D - b\lambda C = 0, \end{cases}$$

which has a non-trivial solution if and only if

$$\lambda^4 + \lambda^2(2(n\pi)^2 + b^2) + (n\pi)^4 = 0. \quad (3.2.6)$$

*Remark 3.2.1* (Condition on the coupling parameter). Denoting  $\lambda = i\mu$  in system (3.2.4), the previous system has a non trivial solution if and only if  $\sinh(r_1) = 0$  and/or  $\sinh(r_2) = 0$  where  $r_1$  and  $r_2$  are defined as:

$$r_1 = \sqrt{\frac{-2\mu^2 - \mu\sqrt{4ab^2}}{2a}}, \quad r_2 = \sqrt{\frac{-2\mu^2 + \mu\sqrt{4ab^2}}{2a}}. \quad (3.2.7)$$

Taking

$$b^2 \neq \frac{(k_1^2 - k_2^2)^2}{2\pi^2(k_1^2 + k_2^2)}, \quad \forall k_1, k_2 \in \mathbb{Z}^*. \quad (3.2.8)$$

one can write the expressions of  $\phi$  and  $\psi$  defined in (3.2.5).

**Proposition 3.2.2.** *Assume that  $a = 1$  and condition (3.2.8) holds. Then, the following asymptotic expansions hold*

$$\lambda_{1,n} = in\pi + i\frac{b}{2} + i\frac{b^2}{8n\pi} + \frac{O(b^4)}{n^3}, \quad (3.2.9)$$

$$\lambda_{2,n} = in\pi - i\frac{b}{2} + i\frac{b^2}{8n\pi} + \frac{O(b^4)}{n^3}, \quad (3.2.10)$$

with the corresponding eigenfunctions

$$\phi_{1,n}(x) = \frac{\sin(n\pi x)}{n\pi}, \quad \psi_{1,n}(x) = \frac{-i \sin(n\pi x)}{n\pi}, \quad (3.2.11)$$

$$\phi_{2,n}(x) = \frac{-i \sin(n\pi x)}{n\pi}, \quad \psi_{2,n}(x) = \frac{\sin(n\pi x)}{n\pi}. \quad (3.2.12)$$

*Proof.* First solving the equation (3.2.6), we get

$$\lambda_n^2 = \frac{-2(n\pi)^2 - b^2 \pm \sqrt{b^4 + 4b^2(n\pi)^2}}{2}. \quad (3.2.13)$$

Using asymptotic expansion in (3.2.13), we get

$$\lambda_{1,n}^2 = -(n\pi)^2 - n\pi b - \frac{b^2}{2} - \frac{b^3}{8n\pi} + \frac{O(b^5)}{n^3}, \quad (3.2.14)$$

$$\lambda_{2,n}^2 = -(n\pi)^2 + n\pi b - \frac{b^2}{2} + \frac{b^3}{8n\pi} + \frac{O(b^5)}{n^3}. \quad (3.2.15)$$

Once again using asymptotic expansion in (3.2.14)-(3.2.15), we get (3.2.9)-(3.2.10). Next for  $\lambda = \lambda_{1,n}$ , setting

$$C_{1,n} = \frac{1}{n\pi}, \quad D_{1,n} = -\frac{C_{1,n}(\lambda_{1,n}^2 + (n\pi)^2)}{b\lambda_{1,n}} = \left( \frac{-i}{n\pi} + \frac{O(b^2)}{n^3} \right),$$

in (3.2.5), we get the corresponding eigenfunctions (3.2.11).

Similarly for  $\lambda = \lambda_{2,n}$ , setting

$$D_{2,n} = \frac{1}{n\pi}, \quad C_{2,n} = \frac{D_{2,n}(\lambda_{2,n}^2 + (n\pi)^2)}{b\lambda_{2,n}} = \left( \frac{-i}{n\pi} + \frac{O(b^2)}{n^3} \right)$$

in (3.2.5), we get the corresponding eigenfunctions (3.2.12). The proof is thus complete.  $\square$

*Remark 3.2.3.* It is easy to check that all the roots  $\lambda_{1,n}, \lambda_{2,n}$  of equation (3.2.6) are simple and different from zero. Then we set the eigenfunctions of the operator  $\mathcal{A}$  as

$$\begin{cases} E_{1,n} = (\phi_{1,n}, \lambda_{1,n}\phi_{1,n}, \psi_{1,n}, \lambda_{1,n}\psi_{1,n}), \\ E_{2,n} = (\phi_{2,n}, \lambda_{2,n}\phi_{2,n}, \psi_{2,n}, \lambda_{2,n}\psi_{2,n}). \end{cases} \quad (3.2.16)$$

Using the asymptotic expansions (3.2.9)-(3.2.10) and (3.2.11)-(3.2.12), we can easily prove that  $E_{1,n}, E_{2,n}$ ,  $n \in \mathbb{Z}^*$ , form a Riesz basis in the energy space  $\mathcal{H}$ . We distinguish different types of observability inequalities, depending on the algebraic property of the coupling parameter  $b$ . In fact, as we will see in Proposition 3.2.4, if there exist no integers  $k \in \mathbb{Z}^*$  such that  $b = k\pi$ , the observability holds in the energy space  $\mathcal{H}$ . In fact, the eigenvalues of the same branch satisfy a uniform gap condition, and also do the eigenvalues of different branches. So, we will apply the usual Ingham's theorem (see [73], Theorem 4.3), in order to get observability inequalities. Nevertheless, if there exists some integer  $k_0 \in \mathbb{Z}^*$  such that  $b = k_0\pi$ , the eigenvalues of the same branch satisfy a uniform gap condition, while on different branches they can be asymptotically close at a rate of order  $\frac{1}{n^2}$  (see Proposition 3.2.7). So the usual Ingham's theorem used in the case  $b \neq k\pi$  is no longer valid. In order to get the observability inequalities, a general Ingham's theorem based on divided differences will be used, which tolerates asymptotically close eigenvalues (see Theorem 9.4 in [73])

According to the asymptotic behavior (3.2.9)-(3.2.10), we distinguish two cases.

### 3.2.1.1 The first case: $a = 1$ and $b \notin \pi\mathbb{Z}$

In this part, we assume that there exists no  $k \in \mathbb{Z}^*$  such that  $b = k\pi$ . Then, we have the following result

**Proposition 3.2.4.** *Assume that  $a = 1$ , condition (3.2.8) holds and  $b \notin \pi\mathbb{Z}$ . Then there exists a constant  $\gamma > 0$  depending only on  $b$  such that the two branches of eigenvalues of  $\mathcal{A}$  satisfy a uniform gap condition*

$$\gamma := \inf_{m,n} |\lambda_{1,m} - \lambda_{2,n}| > 0. \quad (3.2.17)$$

Moreover, the eigenvalues of the same branch satisfy an uniform gap condition.

*Proof.* Using the asymptotic expansions (3.2.9)-(3.2.10), we have

$$|\lambda_{1,m} - \lambda_{2,n}| = |\lambda_{2,m} - \lambda_{1,n}| = |\pi(m - n) + b| + \frac{O(b)}{|m|} + \frac{O(b)}{|n|}.$$

Since there exists no integer  $k \in \mathbb{Z}$  such that  $b = k\pi$ , then there exists  $\hat{b}$  such that  $0 < |\hat{b}| < \pi$  in a way that  $|\pi(m - n) + b| \geq |\hat{b}|$ . Hence, for  $m, n \in \mathbb{Z}^*$  one can deduce that there exists  $\gamma = \gamma(\hat{b}) > 0$  such that

$$\inf_{m \neq n} |\lambda_{1,m} - \lambda_{2,n}| > \gamma.$$

Similarly, using (3.2.9)-(3.2.10), we have

$$|\lambda_{j,m} - \lambda_{j,n}| = |\pi(m - n)| + \frac{O(b)}{|m|} + \frac{O(b)}{|n|}, \quad \text{for } j = 1, 2. \quad (3.2.18)$$

It follows that

$$\inf_{m \neq n} |\lambda_{j,m} - \lambda_{j,n}| \geq \pi.$$

Hence, one can deduce the uniform gap for eigenvalues laying on the same branch. The proof is thus complete.  $\square$

The observation is on the third components of the eigenfunctions  $E_{1,n}, E_{2,n}$ . From (3.2.11)-(3.2.12), we see that

$$(\psi_{1,n})_x(1) \sim O(1), \quad (\psi_{2,n})_x(1) \sim O(1). \quad (3.2.19)$$

Because of the uniform gap condition between the eigenvalues, the usual space of the observability is the energy space  $\mathcal{H}$ . Therefore, we will use the general Ingham's theorem which requires an uniform gap condition (see [94], Theorem 4.3).

We can now prove our first main result.

**Proof of Theorem 3.1.1.** For all  $n \in \mathbb{Z}^*$ , all eigenvalues are different from zero and are all algebraically simple. Given any initial data  $(\phi_0, \phi_1, \psi_0, \psi_1) \in \mathcal{H}$ , using the Riesz basis property one can write

$$(\phi_0, \phi_1, \psi_0, \psi_1) = \sum_{n \neq 0} (\alpha_{1,n} E_{1,n} + \alpha_{2,n} E_{2,n}).$$

Moreover, the solution of (3.2.3) can be written a

$$(\phi(x, t), \phi_t(x, t), \psi(x, t), \psi_t(x, t)) = \sum_{n \neq 0} (\alpha_{1,n} E_{1,n} e^{\lambda_{1,n} t} + \alpha_{2,n} E_{2,n} e^{\lambda_{2,n} t}).$$

It follows that

$$\psi_x(1, t) = \sum_{n \neq 0} (\alpha_{1,n} (\psi_{1,n})_x(1) e^{\lambda_{1,n} t} + \alpha_{2,n} (\psi_{2,n})_x(1) e^{\lambda_{2,n} t}). \quad (3.2.20)$$

But, back to (3.2.19), we can rewrite (3.2.20) as

$$\psi_x(1, t) \sim \sum_{n \neq 0} (\alpha_{1,n} e^{\lambda_{1,n} t} + \alpha_{2,n} e^{\lambda_{2,n} t}).$$

Following a generalization of Ingham's theorem (see [73], Theorem 9.1), the sequence  $(e^{in\pi t})_{n \neq 0}$  forms a Riesz basis in  $L^2(0, T)$  provided that

$$T > 2\pi/\gamma$$

where  $\gamma$  is the uniform gap between the eigenvalues. It follows that

$$\int_0^T |\psi_x(1, t)|^2 dt \sim \sum_{n \neq 0} (|\alpha_{1,n}|^2 + |\alpha_{2,n}|^2).$$

This yield inequalities (3.1.5) and (3.1.6). The proof is now complete.  $\square$

Now, let  $(\phi, \phi_t, \psi, \psi_t)$  be a solution of the homogeneous problem (3.2.1) and let  $v(t) \in L^2(0, T)$ . Then, multiplying the first and the second equation of (3.1.2) by  $\phi$  and  $\psi$  respectively, and integrating by parts, we obtain formally

$$\left\{ \begin{array}{l} \int_0^1 u_t(T)\phi(T)dx + \int_0^1 y_t(T)\psi(T)dx - \int_0^1 u(T)\phi_t(T)dx \\ - \int_0^1 y(T)\psi_t(T)dx - \int_0^1 bu(T)\psi(T)dx + \int_0^1 by(T)\phi(T)dx \\ = \int_0^1 u_t(0)\phi(0)dx + \int_0^1 y_t(0)\psi(0)dx - \int_0^1 \phi_t(0)u(0)dx \\ - \int_0^1 \psi_t(0)y(0)dx - \int_0^1 bu(0)\psi(0)dx + \int_0^1 by(0)\phi(0)dx \\ - \int_0^T \psi_x(1, t)v(t)dt. \end{array} \right. \quad (3.2.21)$$

We define the linear form  $L$  by

$$L(\Phi_0) = \langle (u_1, -u_0, y_1, -y_0), \Phi_0 \rangle_{\mathcal{H}' \times \mathcal{H}} - \int_0^T \psi_x(1, t)v(t)dt. \quad (3.2.22)$$

Then, we obtain a weak formulation of system (3.1.2)

$$\langle (u_t(T, x), -u(T, x), y_t(T, x), -y(T, x)), \Phi(T) \rangle_{\mathcal{H}' \times \mathcal{H}} = L(\Phi_0), \quad \forall \Phi_0 \in \mathcal{H}. \quad (3.2.23)$$

**Theorem 3.2.5.** *Assume that  $a = 1$ , condition (3.2.8) holds and  $b \notin \pi\mathbb{Z}$ . Let  $0 < |\hat{b}| < \pi$  be the resulting quantity of  $b - \hat{k}\pi$ , where  $\hat{k} \in \mathbb{N}^*$ , such that  $\hat{k}\pi < |b| < (\hat{k}+1)\pi$  and let*

$$T > \frac{2\pi}{|\hat{b}|}.$$

Then, for all initial data  $(u_0, u_1, y_0, y_1) \in (L^2(0, 1) \times H^{-1}(0, 1))^2$  and all  $v \in L^2(0, T)$ , system (3.1.2) admits a unique weak solution

$$(u, u_t, y, y_t) \in C^0([0, T]; (L^2(0, 1) \times H^{-1}(0, 1))^2)$$

in the sense that the variational equation (3.2.23) is satisfied for all  $\Phi_0 \in \mathcal{H}$  on the interval  $[0, T]$ . Moreover, the linear mapping

$$(U_0, v) \longrightarrow U$$

is continuous from  $(L^2(0, 1) \times H^{-1}(0, 1))^2 \times L^2(0, T)$  into  $(L^2(0, 1) \times H^{-1}(0, 1))^2$  with the corresponding strong topology.

*Proof.* Using the direct inequality (3.1.5), we deduce that

$$\|L\|_{\mathcal{L}(\mathcal{H}, \mathbb{R})} \leq \|v\|_{L^2(0, T)} + \|U_0\|_{\mathcal{H}'} . \quad (3.2.24)$$

By virtue of Riesz-Fréchet's representation theorem, for each  $0 \leq t \leq T$ , there exists a unique element  $\mathcal{Z}(x, t) \in \mathcal{H}'$  such that

$$L(\Phi_0) = \langle \mathcal{Z}, \Phi_0 \rangle_{\mathcal{H}' \times \mathcal{H}}, \quad \forall \Phi_0 \in \mathcal{H}. \quad (3.2.25)$$

Then, setting  $\mathcal{Z}(x, t) = e^{-tA}U(x, t)$  in (3.2.25) we get (3.2.23). Moreover, we have

$$\|U(t)\|_{\mathcal{H}'} \leq c \left( \|v\|_{L^2(0, T)} + \|U_0\|_{\mathcal{H}'} \right), \quad \forall t \in [0, T]. \quad (3.2.26)$$

This implies the continuity of the linear mapping. The proof is thus complete.  $\square$

It is well known that the observability of system (3.2.1) implies the exact controllability of system (3.1.2) (see [70, 79]). We can now state the following exact controllability result.

**Theorem 3.2.6.** *Assume that  $a = 1$ , condition (3.2.8) holds and there exist no integers  $k \in \mathbb{Z}$ , such that  $b = k\pi$ . Let  $0 < |\hat{b}| < \pi$  the resulting quantity of  $b - \hat{k}\pi$ , where  $\hat{k} \in \mathbb{N}^*$ , such that  $\hat{k}\pi < |b| < (\hat{k} + 1)\pi$  and let*

$$T > \frac{2\pi}{|\hat{b}|}.$$

*Then system (3.1.2) is exactly controllable at the moment  $T$ . More precisely, for any initial data  $(u_0, u_1, y_0, y_1) \in (L^2(0, 1) \times H^{-1}(0, 1))^2$ , there exists a control function  $v \in L^2(0, T)$  such that the solution  $(u, u_t, y, y_t) \in (L^2(0, 1) \times H^{-1}(0, 1))^2$ , of the controlled system (3.1.2) satisfies the null final conditions*

$$u(x, T) = u_t(x, T) = y(x, T) = y_t(x, T) = 0.$$

*Proof.* Let  $\Phi_0 = (\phi_0, \phi_1, \psi_0, \psi_1) \in D(\mathcal{A})$  and  $\Phi = (\phi, \phi_t, \psi, \psi_t) \in D(\mathcal{A})$  be the corresponding solution of problem (3.2.1). Thanks to the observability inequality (3.1.6), the semi-norm

$$\|\Phi_0\|_{\mathcal{F}} = \left( \int_0^T |\psi_x(1, t)|^2 dt \right)^{1/2}, \quad (3.2.27)$$

is a norm on  $D(\mathcal{A})$ . We denote by  $\mathcal{F}$  the completion of  $D(\mathcal{A})$  with respect to this norm, thus we obtain an Hilbert space. Thanks to the direct and inverse observability inequalities, we have the following continuous and dense imbeddings:

$$(H_0^1(0, 1) \times L^2(0, 1))^2 \subset \mathcal{F} \subset (L^2(0, 1) \times H^{-1}(0, 1))^2.$$

Consequently, by duality, we have the following continuous embedding:

$$(H_0^1(0, 1) \times L^2(0, 1))^2 \subset \mathcal{F}' \subset (L^2(0, 1) \times H^{-1}(0, 1))^2.$$

Now, by choosing the control  $v(t) = \psi_x(1, t)$ , we solve the backward problem

$$\begin{cases} \chi_{tt} - \chi_{xx} + \alpha\zeta = 0 & \text{on } (0, 1) \times (0, T), \\ \zeta_{tt} - \zeta_{xx} + \alpha\chi = 0 & \text{on } (0, 1) \times (0, T), \\ \chi(0, t) = \zeta(0, t) = \zeta(1, t) = 0 & \text{for all } t \in (0, T), \\ \zeta(1, t) = -\psi_x(1, t) & \text{for all } t \in (0, T), \\ \chi(x, T) = \chi_t(x, T) = \zeta(x, T) = \zeta_t(x, T) = 0 & \text{on } (0, 1). \end{cases} \quad (3.2.28)$$

Following Theorem 3.2.5, the backward problem (3.2.28) admits a unique weak solution  $(\chi, \chi_t, \zeta, \zeta_t) \in C^0([0, T]; (L^2(0, 1) \times H^{-1}(0, 1))^2)$ . Next, we define the operator

$$\Lambda : (H_0^1(0, 1) \times L^2(0, 1))^2 \rightarrow (L^2(0, 1) \times H^{-1}(0, 1))^2$$

$$\Lambda(\phi_0, \phi_1, \psi_0, \psi_1) = (\chi_t(0), -\chi(0), \zeta_t(0), -\zeta(0)).$$

From (3.2.23) and (3.2.28), it follows that

$$\langle \Lambda\Phi_0, \tilde{\Phi}_0 \rangle = \int_0^T \psi_x(1, t) \tilde{\psi}_x(1, t) dt \quad \forall \Phi_0, \tilde{\Phi}_0 \in \mathcal{H}. \quad (3.2.29)$$

Therefore, we have

$$|\langle \Lambda\Phi_0, \tilde{\Phi}_0 \rangle| \leq \|\Phi_0\|_{\mathcal{F}} \|\tilde{\Phi}_0\|_{\mathcal{F}} \quad \forall \Phi_0, \tilde{\Phi}_0 \in \mathcal{H}. \quad (3.2.30)$$

Hence, since  $\mathcal{H}$  is dense in  $\mathcal{F}$  by definition of  $\mathcal{F}$ , the linear map  $\Lambda\Phi_0$  can be extended in a unique way to a continuous map on  $\mathcal{F}$  and consequently  $\Lambda\Phi_0 \in \mathcal{F}'$ . Moreover, using (3.2.30) we deduce that the linear map  $\Lambda$  that maps  $\Phi_0 \in \mathcal{H}$  to  $\Lambda\Phi_0 \in \mathcal{F}'$  is continuous when  $\mathcal{H}$  is equipped with the norm  $\|\cdot\|_{\mathcal{F}}$ . Hence, since  $\mathcal{H}$  is dense in  $\mathcal{F}$ , the linear map  $\Lambda$  can be extended in a unique way to a continuous linear map, still denoted by  $\Lambda$ , from  $\mathcal{F}$  to  $\mathcal{F}'$ . In addition, we have

$$\langle \Lambda\Phi_0, \tilde{\Phi}_0 \rangle_{\mathcal{F}', \mathcal{F}} = (\Phi_0, \tilde{\Phi}_0)_{\mathcal{F}} \quad \forall \Phi_0, \tilde{\Phi}_0 \in \mathcal{H} \quad (3.2.31)$$

where  $(\cdot, \cdot)_{\mathcal{F}}$  denotes the scalar product associated with the norm  $\|\cdot\|_{\mathcal{F}}$ . The continuity of  $\Lambda$  follows from the well-posedness of the problem (3.1.2) and (3.2.28). The coercivity of  $\Lambda$  comes from the inverse observability inequality. Thanks to the Lax-Milgram theorem,  $\Lambda$  is an isomorphism from  $\mathcal{F}$  onto  $\mathcal{F}'$ . Let  $U_0 = (u_0, u_1, y_0, y_1) \in (H_0^1(\Omega) \times L^2(\Omega))^2 \subset \mathcal{F}'$ , and define  $(\chi(0), \chi_t(0), \zeta(0), \zeta_t(0)) = U_0$ . Then equation

$$\Lambda(\phi_0, \phi_1, \psi_0, \psi_1) := (\chi_t(0), -\chi(0), \zeta_t(0), -\zeta(0))$$

has a unique solution  $(\phi_0, \phi_1, \psi_0, \psi_1) \in \mathcal{F}$ . But, according to the uniqueness of the solution of the problem (3.1.2), we have

$$u(x, T) = u_t(x, T) = y(x, T) = y_t(x, T) = 0.$$

The proof is thus complete. □

### 3.2.1.2 The second case: $a = 1$ and $b \in \pi\mathbb{Z}^*$

It is natural to ask, what happens if condition  $b \notin \pi\mathbb{Z}$  does not hold *i.e.* if there exists some integer  $k_0 \in \mathbb{Z}^*$  such that  $b = k_0\pi$ . Indeed, from the asymptotic expansions (3.2.9)-(3.2.10), the eigenvalues on the same branch satisfy an uniform gap condition, while on different branches they can be asymptotically close. So, the usual Ingham's theorem does not work in this situation. We will prove the following result.

**Proposition 3.2.7.** *Assume that  $a = 1$ , condition (3.2.8) holds and there exists some integer  $k_0 \in \mathbb{Z}^*$  such that  $b = k_0\pi$ . Then, there exist constants  $c_2 > c_1 > 0$  depending only on  $b$  such that for all  $|m|, |n| \geq N$  we have*

$$|\lambda_{1,m} - \lambda_{2,n}| \geq \frac{c_1}{m^2} \quad \text{and} \quad |\lambda_{1,m} - \lambda_{2,n}| \geq \frac{c_1}{n^2}, \quad (3.2.32)$$

and there exist infinitely many integers  $m, n$  such that

$$|\lambda_{1,m} - \lambda_{2,n}| \leq \frac{c_2}{m^2} \quad \text{and} \quad |\lambda_{1,m} - \lambda_{2,n}| \leq \frac{c_2}{n^2}. \quad (3.2.33)$$

Moreover, there exists a constant  $\gamma > 0$  depending only on  $b$  such that the eigenvalues of the same branch satisfy an uniform gap condition

$$|\lambda_{j,m} - \lambda_{l,n}| \leq 2\gamma \implies j \neq l. \quad (3.2.34)$$

*Proof.* Using the asymptotic expansions (3.2.9)-(3.2.10), and for  $b = k_0\pi$ , we have

$$|\lambda_{j,m} - \lambda_{l,n}| = \left| (m-n)\pi + k_0\pi + \frac{b^2}{8m\pi} - \frac{b^2}{8n\pi} + \frac{O(b)}{m^3} + \frac{O(b)}{n^3} \right|. \quad (3.2.35)$$

We distinguish cases:

**Case 1**  $m = n$ . We have

$$|\lambda_{j,m} - \lambda_{l,m}| = \pi|k_0| + \frac{O(b)}{|m|^3} \geq \frac{\pi|k_0|}{m^2}. \quad (3.2.36)$$

**Case 2**  $m - n + k_0 = 0$ . We have,

$$|\lambda_{j,m} - \lambda_{l,n}| = \frac{k_0\pi}{8} \left| \frac{1}{m} - \frac{1}{n} \right| + \frac{O(b)}{|n|^3} + \frac{O(b)}{|m|^3} = \frac{k_0\pi}{8} \left| \frac{1}{n - k_0} - \frac{1}{n} \right| + \frac{O(b)}{|n|^3} \geq \frac{k_0^2\pi}{8n^2}.$$

Let us now consider the leading term in the previous inequality, for all  $m \in \mathbb{Z}^*$  such that  $|m| \geq 1$ , and if  $|m| > |n|$  we have  $\frac{1}{|mn|} \leq \frac{1}{n^2}$ .

Conversely, if  $|m| < |n|$ , we have  $\frac{1}{|mn|} \leq \frac{1}{m^2}$ . Hence one can deduce (3.2.33).

The proof is thus complete.  $\square$

**Proposition 3.2.8.** *Assume that  $a = 1$ , condition (3.2.8) holds and there exists some integer  $k_0 \in \mathbb{Z}^*$  such that  $b = k_0\pi$ . We rearrange the two branches of eigenvalues into one sequence  $(\lambda_n)_{n \neq 0}$  such that  $(Im\lambda_n)_{n \neq 0}$  is strictly increasing. Assume that*

$$0 < Im\lambda_{n+1} - Im\lambda_n \leq \gamma, \quad (3.2.37)$$

then we have

$$Im\lambda_n - Im\lambda_{n-1} > \gamma, \quad Im\lambda_{n+2} - Im\lambda_{n+1} > \gamma. \quad (3.2.38)$$

We say that  $Im\lambda_n, Im\lambda_{n+1}$  is a chain of close exponents relative to  $\gamma$  of length 2.

*Proof.* The conditions (3.2.37) and (3.2.17) imply that  $\lambda_n, \lambda_{n+1}$  belong to different branches of eigenvalues. If  $\lambda_{n-1}, \lambda_n$  belong to the same branch of eigenvalues, then (3.2.17) implies that

$$Im\lambda_n - Im\lambda_{n-1} > 2\gamma.$$

In the opposite case,  $\lambda_{n-1}, \lambda_{n+1}$  must belong to the same branch of eigenvalues. Once again (3.2.17) implies that

$$Im\lambda_{n+1} - Im\lambda_{n-1} > 2\gamma. \quad (3.2.39)$$

Then, from (3.2.37) and (3.2.39), we get

$$Im\lambda_n - Im\lambda_{n-1} = (Im\lambda_{n+1} - Im\lambda_{n-1}) - (Im\lambda_{n+1} - Im\lambda_n) > 2\gamma - \gamma = \gamma.$$

In a similar way, one can show that

$$Im\lambda_{n+2} - Im\lambda_{n+1} > \gamma.$$

The proof is complete.  $\square$

Due to the fact that the eigenvalues can be asymptotically close, the inverse observability inequality is not true in the energy space  $\mathcal{H}$ . That is why we define the following weighted spectral space

$$D_2 = \left\{ (\phi_0, \phi_1, \psi_0, \psi_1) = \sum_{n \neq 0} (\alpha_{1,n} E_{1,n} + \alpha_{2,n} E_{2,n}) n^2 \right\}.$$

Since the set  $\{E_{1,n}, E_{2,n}\}$  is a Riesz basis in the energy space  $\mathcal{H}$ , the space  $D_2$  is obviously a Hilbert space equipped with the norm

$$\sum_{n \neq 0} (|\alpha_{1,n}|^2 + |\alpha_{2,n}|^2).$$

We can therefore proceed to the proof of our second main result.

**Proof of Theorem 3.1.2.** The proof is divided into two steps.

**Step 1: Inverse observability inequality.** From (3.2.6), one can deduce that all eigenvalues are different from zero and are all algebraically simple. Using the Riesz basis property, given any initial data  $(\phi_0, \phi_1, \psi_0, \psi_1)$ , we have

$$(\phi_0, \phi_1, \psi_0, \psi_1) = \sum_{n \neq 0} (\alpha_{1,n} E_{1,n} + \alpha_{2,n} E_{2,n}) n^2.$$

Using again the Riesz property the solution of (3.2.3) can be written as

$$(\phi(x, t), \phi_t(x, t), \psi(x, t), \psi_t(x, t)) = \sum_{n \neq 0} (\alpha_{1,n} E_{1,n} e^{\lambda_{1,n} t} + \alpha_{2,n} E_{2,n} e^{\lambda_{2,n} t}) n^2.$$

It follows that

$$\psi_x(1, t) = \sum_{n \neq 0} (\alpha_{1,n} (\psi_{1,n})_x(1) e^{\lambda_{1,n} t} + \alpha_{2,n} (\psi_{2,n})_x(1) e^{\lambda_{2,n} t}) n^2. \quad (3.2.40)$$

Now we rearrange the two branches of eigenvalues  $(\lambda_{1,n})_{n \neq 0}, (\lambda_{2,n})_{n \neq 0}$  into one sequence  $(\lambda_n)_{n \neq 0}$  such that the sequence  $(Im \lambda_n)_{n \neq 0}$  is strictly increasing. Following Proposition 3.2.8, all chain  $Im \lambda_n, Im \lambda_{n+1}$  of close exponents relative to  $\gamma$  is of length 2. Let  $A$  denote the set of integers  $n \in \mathbb{Z}^*$  such that the condition (3.2.37) holds true and let

$$B = \mathbb{Z}^* \setminus \{n, n+1 : n \in A\}.$$

We denote by  $a_n$  the coefficient before  $e^{\lambda_{1,n} t}$  or  $e^{\lambda_{2,n} t}$  in (3.2.40). We can rewrite it into

$$\begin{aligned} \psi_x(1, t) &= \sum_{n \in B} a_n e^{\lambda_n t} + \sum_{n \in A} a_n e^{\lambda_n t} + a_{n+1} e^{\lambda_{n+1} t} \\ &= \sum_{n \in B} a_n e^{\lambda_n t} + \sum_{n \in A} ((a_n + a_{n+1}) e^{\lambda_n t} + (\lambda_{n+1} - \lambda_n) a_{n+1} e_{n+1}(t)), \end{aligned}$$

where  $e_{n+1}(t)$  denotes the divided difference of the chain of close exponents  $\lambda_n, \lambda_{n+1}$  relative to  $\gamma$

$$e_{n+1}(t) = \frac{e^{\lambda_{n+1} t} - e^{\lambda_n t}}{\lambda_{n+1} - \lambda_n}.$$

Following Theorem 9.4 in [73], the sequence

$$(e^{\lambda_n t})_{n \in B}, (e^{\lambda_n t}, e_{n+1}(t))_{n \in A}$$

forms a Riesz sequence in  $L^2(0, T)$  provided that  $T > 2\pi D^+$ , where  $D^+$  is the upper density of the sequence  $(\lambda_n)_{n \in \mathbb{Z}^*}$ , defined as

$$D^+(r) = \lim_{n \rightarrow \infty} \frac{n^+(r)}{r},$$

where  $n^+(r)$  denotes the largest number of terms of the sequence  $(\lambda_n)_{n \geq 0}$  contained in an interval of length  $r$ . To be more precise,

$$D^+ = \frac{2}{\pi}.$$

It follows that

$$\int_0^T |\psi_x(1, t)|^2 dt \sim \sum_{n \in B} |a_n|^2 + \sum_{n \in A} (|a_n + a_{n+1}|^2 + |\lambda_{n+1} - \lambda_n|^2 |a_{n+1}|^2). \quad (3.2.41)$$

The right hand side of the assertion (3.2.32) of Proposition 3.2.7 implies that

$$|a_n + a_{n+1}|^2 + |\lambda_{n+1} - \lambda_n|^2 |a_{n+1}|^2 \geq c \left( \frac{|a_n|^2}{|n|^4} + \frac{|a_{n+1}|^2}{|n+1|^4} \right) \quad (3.2.42)$$

Inserting (3.2.42) into (3.2.41) and returning to the previous notations, we get

$$\begin{aligned} \int_0^T |\psi_x(1, t)|^2 dt &\geq c \sum_{n \in B} \frac{|a_n|^2}{|n|^4} + \sum_{n \in A} \left( \frac{|a_n|^2}{|n|^4} + \frac{|a_{n+1}|^2}{|n+1|^4} \right) \geq c \sum_{n \in \mathbb{Z}^*} \frac{|a_n|^2}{|n|^4} \\ &= c \sum_{n \neq 0} (|\alpha_{1,n}|^2 + |\alpha_{2,n}|^2). \end{aligned} \quad (3.2.43)$$

This leads to the desired inverse observability inequality.

**Step 2: Direct inequality.** We consider the eigenvectors defined in (3.2.16), hence we have

$$(\phi_0, \phi_1, \psi_0, \psi_1) = \sum_{n \neq 0} (\alpha_{1,n} E_{1,n} + \alpha_{2,n} E_{2,n}).$$

Thus

$$(\phi(x, t), \phi_t(x, t), \psi(x, t), \psi_t(x, t)) = \sum_{n \neq 0} (\alpha_{1,n} E_{1,n} e^{\lambda_{1,n} t} + \alpha_{2,n} E_{2,n} e^{\lambda_{2,n} t}),$$

and consequently,

$$\psi_x(1, t) = \sum_{n \neq 0} (\alpha_{1,n} (\psi_{1,n})_x(1) e^{\lambda_{1,n} t} + \alpha_{2,n} (\psi_{2,n})_x(1) e^{\lambda_{2,n} t}).$$

By (3.2.19), we recall that

$$(\psi_{1,n})_x(1) \sim O(1), \quad (\psi_{2,n})_x(1) \sim O(1).$$

Consequently,

$$\begin{aligned}\psi_x(1, t) &= \sum_{n \neq 0} (\tilde{\alpha}_{1,n} e^{\lambda_{1,n} t} + \tilde{\alpha}_{2,n} e^{\lambda_{2,n} t}) \\ &= \sum_{n \neq 0} (\tilde{\alpha}_{1,n} + \tilde{\alpha}_{2,n}) e^{\lambda_{1,n} t} + \tilde{\alpha}_{2,n} (e^{\lambda_{2,n} t} - e^{\lambda_{1,n} t}) \\ &= \sum_{n \neq 0} (\tilde{\alpha}_{1,n} + \tilde{\alpha}_{2,n}) e^{\lambda_{1,n} t} + \tilde{\alpha}_{2,n} e_n(t) (\lambda_{2,n} - \lambda_{1,n})\end{aligned}$$

with

$$e_n(t) = \frac{e^{\lambda_{2,n} t} - e^{\lambda_{1,n} t}}{\lambda_{2,n} - \lambda_{1,n}}.$$

Since the sequence  $e^{\lambda_{1,n} t}$ ,  $e_n(t)$  forms a Riesz basis in  $L^2(0, T)$  provided that  $T > 4$ , we deduce the following direct observability inequality

$$\begin{aligned}\int_0^T |\psi_x(1, t)|^2 dt &\sim \sum_{n \neq 0} (\tilde{\alpha}_{1,n}^2 + \tilde{\alpha}_{2,n}^2) + \frac{\tilde{\alpha}_{2,n}^2}{n^4} \\ &\leq \tilde{c} \sum_{n \neq 0} (\tilde{\alpha}_{1,n}^2 + \tilde{\alpha}_{2,n}^2) \\ &\leq c_1 \sum_{n \neq 0} (\alpha_{1,n}^2 + \alpha_{2,n}^2)\end{aligned}$$

This gives (3.1.13). The proof is thus complete.  $\square$

*Remark 3.2.9.* From inequality (3.2.33), there exist an infinity of  $(\phi_0, \phi_1, \psi_0, \psi_1) \in D_2$  such that the corresponding inverse inequality of (3.1.9) holds.

Similarly as in the first case, using the HUM method (see [70, 79]), we have the following exact controllability result.

**Theorem 3.2.10.** *Assume that  $a = 1$ , condition (3.2.8) holds and there exists some integer  $k_0 \in \mathbb{Z}$ , such that  $b = k_0 \pi$ . Let  $T > 4$ , then system (3.1.2) is exactly controllable at the moment  $T$ . More precisely, for any initial data  $(u_0, u_1, y_0, y_1) \in D'_2$ , there exists a control function  $v \in L^2(0, T)$  such that the solution  $(u, u_t, y, y_t) \in D'_2$ , of the controlled system (3.1.2) satisfies the null final conditions*

$$u(x, T) = u_t(x, T) = y(x, T) = y_t(x, T) = 0.$$

It is natural to think about the characterization of the spectral space  $D_2$ . For this aim, we have the following new subsection.

### 3.2.2 Observability and exact controllability in usual spaces.

#### The case $a = 1$ and $b \in \pi \mathbb{Z}^*$

The weighted spectral space  $D_2$  is defined by means of the eigenvectors  $(E_{1,n})_{n \neq 0}$  and  $(E_{2,n})_{n \neq 0}$  with weights. So, the four exponents  $(\phi_0, \phi_1, \psi_0, \psi_1)$  are a priori involved

together. In order to get the observability or exact controllability in usual energy spaces, we have to separate the components  $(\phi_0, \phi_1, \psi_0, \psi_1)$ . To do so, we will use the theorem below whose proof is established in [89].

**Theorem 3.2.11.** *Let  $(x_n)_{n \neq 0}$  and  $(y_n)_{n \neq 0}$  be Riesz basis of Hilbert spaces  $X$  and  $Y$  respectively, and  $(f_n)_{n \neq 0}$  and  $(g_n)_{n \neq 0}$  be Bessel sequences of  $X$  and  $Y$  with suitably small bounds respectively. Define*

$$D = \left\{ (x, y) = \alpha_n(x_n, g_n) + \beta_n(f_n, y_n) : \sum_{n \neq 0} (|\alpha_n|^2 + |\beta_n|^2) < \infty \right\}.$$

Then we have  $D = X \times Y$ .

Now, using the asymptotic expansions (3.2.11)-(3.2.12), we have

$$E_{1,n} = (x_n, g_n), \quad E_{2,n} = (f_n, y_n),$$

with

$$\begin{cases} x_n = \left( \frac{\sin(n\pi x)}{n\pi}, i \sin(n\pi x) \right), \\ g_n = \left( \frac{-i \sin(n\pi x)}{n\pi}, \sin(n\pi x) \right), \\ y_n = \left( \frac{-i \sin(n\pi x)}{n\pi}, \sin(n\pi x) \right), \\ f_n = \left( \frac{\sin(n\pi x)}{n\pi}, i \sin(n\pi x) \right). \end{cases} \quad (3.2.44)$$

Next, for any  $s \geq 0$ , we define the space

$$X_s = \left\{ (\hat{\phi}, \hat{\psi}) = \sum_{n \neq 0} \beta_n n^s y_n \right\}, \quad \|(\hat{\phi}, \hat{\psi})\|_{X_s}^2 = \sum_{n \neq 0} |\beta_n|^2. \quad (3.2.45)$$

Following Theorem 3.2.11, we can state the following result.

**Corollary 3.2.12.** *Let  $a = 1$ , and suppose that there exists some integer  $k_0 \in \mathbb{Z}^*$  such that  $b = k_0\pi$ . Then we have the following identification*

$$D_2 = X_2 \times X_2. \quad (3.2.46)$$

Furthermore, for any  $s \geq 0$ , we define

$$V_s = \left\{ f = \sum_{n > 0} \alpha_n \frac{\sin(n\pi x)}{n^s} \right\}, \quad \|f\|_{V_s}^2 = \sum_{n > 0} |\alpha_n|^2. \quad (3.2.47)$$

Thus with the pivot space  $L^2(0, 1)$ , we have

$$X_2 = V'_1 \times V'_2.$$

It follows then that,

$$D_2 = V'_1 \times V'_2 \times V'_1 \times V'_2.$$

Consequently, we have the following observability results.

**Theorem 3.2.13.** *Assume that  $a = 1$ , condition (3.2.8) holds and there exists some  $k_0 \in \mathbb{Z}^*$  such that  $b = k_0\pi$ . Let  $T > 4$ , then there exists a constant  $c_3 > 0$  such that the following direct inequality holds*

$$\int_0^T |\psi_x(1, t)|^2 dt \leq c_3 \|(\phi_0, \phi_1, \psi_0, \psi_1)\|_{\mathcal{H}}^2, \quad (3.2.48)$$

for all solution  $\Phi = (\phi, \phi_t, \psi, \psi_t)$  solving the corresponding homogeneous Cauchy problem.

Moreover, there exists a constant  $0 < c_4 < c_3$ , such that the following inverse observability inequality holds

$$c_4 \|(\phi_0, \phi_1, \psi_0, \psi_1)\|_{V'_1 \times V'_2 \times V'_1 \times V'_2}^2 \leq \int_0^T |\psi_x(1, t)|^2 dt. \quad (3.2.49)$$

We can deduce that the observability space is  $(H^{-1}(0, 1) \times H^{-2}(0, 1))^2$ . Finally, using the HUM method (see [70, 79] and the case 1), we have the following controllability result.

**Theorem 3.2.14.** *Assume that  $a = 1$ , condition (3.2.8) holds and there exists some integer  $k_0 \in \mathbb{Z}$ , such that  $b = k_0\pi$ . Let  $T > 4$ , then system (3.1.2) is exactly controllable at the moment  $T$ . More precisely, for any initial data  $(u_0, u_1, y_0, y_1) \in V_2 \times V_1 \times V_2 \times V_1$ , there exists a control function  $v \in L^2(0, T)$  such that the solution  $(u, u_t, y, y_t) \in V_2 \times V_1 \times V_2 \times V_1$ , of the controlled system (3.1.2) satisfies the null final conditions*

$$u(x, T) = u_t(x, T) = y(x, T) = y_t(x, T) = 0.$$

*Remark 3.2.15.* Unlike the first case, in the second case (where  $a = 1$  and  $b \in \pi\mathbb{Z}^*$ ), the two branches of eigenvalues are close in the order  $\frac{1}{n^2}$ . Because of the closeness of the eigenvalues, the observability space loses two derivatives and passes from the space of type

$$(H_0^1(0, 1) \times L^2(0, 1))^2$$

to the space of type

$$(H^{-1}(0, 1) \times H^{-2}(0, 1))^2.$$

Consequently, using the HUM method, the space of controlled initial data passes from the space of type

$$(L^2(0, 1) \times H^{-1}(0, 1))^2$$

to the space of type

$$(H^2(0, 1) \times H^1(0, 1))^2$$

with suitable boundary conditions.

### 3.3 Exact controllability under different speeds waves propagation condition. The case $a \neq 1$

In this section, the waves are considered to propagate with different speeds. This can be established by taking  $a \neq 1$  in the second equation of the main system (3.1.2). We will establish the exact controllability of the system in consideration.

#### 3.3.1 Observability inequalities and exact controllability in spectral spaces

The aim of this subsection is to establish some observability inequalities by spectral approach. We consider the following associated homogeneous system

$$\begin{cases} \phi_{tt} - \phi_{xx} + b\psi_t = 0 & \text{on } (0, 1) \times (0, T), \\ \psi_{tt} - a\psi_{xx} - b\phi_t = 0 & \text{on } (0, 1) \times (0, T), \\ \phi(0, t) = \phi(1, t) = 0 & \text{for } t \in (0, T), \\ \psi(0, t) = \psi(1, t) = 0 & \text{for } t \in (0, T). \end{cases} \quad (3.3.1)$$

First, we define the energy space  $\mathcal{H}$  by

$$\mathcal{H} = (H_0^1(0, 1) \times L^2(0, 1))^2, \quad (3.3.2)$$

endowed for all  $\Phi = (\phi, \theta, \psi, \eta), \Psi = (\tilde{\phi}, \tilde{\theta}, \tilde{\psi}, \tilde{\eta}) \in \mathcal{H}$ , with the inner product

$$(\Phi, \Psi)_{\mathcal{H}} = \int \left( \phi_x \tilde{\phi}_x + \theta \tilde{\theta} + a\psi_x \tilde{\psi}_x + \eta \tilde{\eta} \right) dx. \quad (3.3.3)$$

Next, we define the linear unbounded operator  $\mathcal{A} : D(\mathcal{A}) \rightarrow \mathcal{H}$  by

$$\begin{aligned} D(\mathcal{A}) &= ((H^2(0, 1) \cap H_0^1(0, 1)) \times H_0^1(0, 1))^2, \\ \mathcal{A}(\phi, \tilde{\phi}, \psi, \tilde{\psi}) &= (\tilde{\phi}, \phi_{xx} - b\tilde{\psi}, \tilde{\psi}, a\psi_{xx} + b\tilde{\phi}). \end{aligned}$$

Then setting  $\Phi = (\phi, \phi_t, \psi, \psi_t)$  a regular solution of (3.3.1), we rewrite it into an evolution equation

$$\begin{cases} \Phi_t &= \mathcal{A}\Phi(t), \\ \Phi(0) &= \Phi_0 \in \mathcal{H}. \end{cases} \quad (3.3.4)$$

It is easy to see that  $\mathcal{A}$  is a maximal dissipative operator, then it generates a  $C_0$  semi-group of contractions  $e^{t\mathcal{A}}$  on the Hilbert space  $\mathcal{H}$ .

Now let us consider the following corresponding eigenvalue problem

$$\begin{cases} \lambda^2 \phi - \phi_{xx} + b\lambda \psi = 0, \\ \lambda^2 \psi - a\psi_{xx} - b\lambda \phi = 0, \\ \phi(0) = \phi(1) = 0, \\ \psi(0) = \psi(1) = 0. \end{cases} \quad (3.3.5)$$

For some constants  $C, D$  let

$$\phi(x) = C \sin(n\pi x), \quad \psi(x) = D \sin(n\pi x), \quad (3.3.6)$$

be a solution of (3.3.5). We have thus

$$\begin{cases} (\lambda^2 + (n\pi)^2)C + b\lambda D = 0, \\ (\lambda^2 + a(n\pi)^2)D - b\lambda C = 0, \end{cases}$$

which has a non trivial solution if and only if

$$\lambda^4 + \lambda^2[(a+1)(n\pi)^2 + b^2] + a(n\pi)^4 = 0. \quad (3.3.7)$$

*Remark 3.3.1.* [Condition on the coupling parameter] Denoting  $\lambda = i\mu$  in system (3.3.5), the previous system has a non trivial solution if and only if  $\sinh(r_1) = 0$  and/or  $\sinh(r_2) = 0$  where  $r_1$  and  $r_2$  are defined as:

$$r_1 = \sqrt{\frac{-\mu^2(a+1) - \mu\sqrt{\mu^2(a-1)^2 + 4ab^2}}{2a}}, \quad r_2 = \sqrt{\frac{-\mu^2(a+1) + \mu\sqrt{\mu^2(a-1)^2 + 4ab^2}}{2a}}. \quad (3.3.8)$$

Taking

$$b^2 \neq \frac{(k_1^2 - ak_2^2)(ak_1^2 - k_2^2)}{\pi^2(a+1)(k_1^2 + k_2^2)}, \quad \forall k_1, k_2 \in \mathbb{Z}^*, \quad (3.3.9)$$

one can write the expressions of  $\phi$  and  $\psi$  defined in (3.3.6).

**Proposition 3.3.2.** *Assume that  $0 < a \neq 1$  and condition (3.3.9) is satisfied. Then the following asymptotic expansions hold*

$$\lambda_{1,n} = i\sqrt{a}n\pi + \frac{i\sqrt{a}b^2}{2(a-1)(n\pi)} - \frac{i\sqrt{a}b^4(a+3)}{8(n\pi)^3(a-1)^3} + \frac{O(a,b)}{n^5}, \quad (3.3.10)$$

$$\lambda_{2,n} = in\pi - \frac{ib^2}{2(a-1)n\pi} + \frac{ib^4(3a+1)}{8(n\pi)^3(a-1)^3} + \frac{O(a,b)}{n^5}, \quad (3.3.11)$$

with the corresponding eigenfunctions

$$\phi_{1,n}(x) = \frac{\sin(n\pi x)}{(n\pi)^2}, \quad \psi_{1,n}(x) = \frac{i(1-a)\sin(n\pi x)}{b\sqrt{a}n\pi}, \quad (3.3.12)$$

$$\phi_{2,n}(x) = \frac{i(1-a)\sin(n\pi x)}{b}, \quad \psi_{2,n}(x) = \frac{\sin(n\pi x)}{(n\pi)^2}. \quad (3.3.13)$$

*Proof.* First, solving equation (3.3.7), we get

$$\lambda_n^2 = \frac{-(a+1)(n\pi)^2 - b^2 \pm \sqrt{(a-1)^2(n\pi)^4 + 2(a+1)b^2(n\pi)^2 + b^4}}{2}. \quad (3.3.14)$$

Using the asymptotic expansion in (3.3.14) we get

$$\lambda_{1,n}^2 = \frac{-ab^2}{a-1} - a(n\pi)^2 + \frac{ab^4}{(a-1)^3(n\pi)^2} + \frac{O(a,b)}{n^4}, \quad (3.3.15)$$

$$\lambda_{2,n}^2 = \frac{b^2}{a-1} - (n\pi)^2 - \frac{ab^4}{(a-1)^3(n\pi)^2} + \frac{O(a,b)}{n^4}. \quad (3.3.16)$$

One again, using asymptotic expansion in (3.3.15)-(3.3.16) we get (3.3.10)-(3.3.11).

Next for  $\lambda = \lambda_{1,n}$ , setting

$$C_{1,n} = \frac{1}{(n\pi)^2}, \quad D_{1,n} = \frac{in\pi(1-a)}{b\sqrt{a}}C_{1,n} = \frac{i(1-a)}{b\sqrt{a}} \frac{1}{n\pi}$$

in (3.3.6), we get the corresponding eigenfunctions (3.3.12).

Similarly, for  $\lambda = \lambda_{2,n}$ , setting

$$D_{2,n} = \frac{1}{(n\pi)^2}, \quad C_{2,n} = \frac{in\pi(1-a)}{b}D_{2,n} = \frac{i(1-a)}{b} \frac{1}{n\pi}$$

in (3.3.6), we get the corresponding eigenfunctions (3.3.13). The proof is thus complete.  $\square$

*Remark 3.3.3.* It is easy to check that all the roots  $\lambda_{1,n}, \lambda_{2,n}$  of equation (3.3.7) are simple and different from zero. Then we set the eigenfunctions of the operator  $\mathcal{A}$  as

$$\begin{cases} E_{1,n} = (\phi_{1,n}, \lambda_{1,n}\phi_{1,n}, \psi_{1,n}, \lambda_{1,n}\psi_{1,n}), \\ E_{2,n} = (\phi_{2,n}, \lambda_{2,n}\phi_{2,n}, \psi_{2,n}, \lambda_{2,n}\psi_{2,n}). \end{cases} \quad (3.3.17)$$

Using the asymptotic expansions (3.3.10)-(3.3.11) and (3.3.12)-(3.3.13), we can easily prove that  $E_{1,n}, E_{2,n}, n \in \mathbb{Z}^*$ , form a Riesz basis in the energy space  $\mathcal{H}$ . But the observability is not true in this space because of the closeness of the eigenvalues. In fact, the eigenvalues of the same branch satisfy an uniform gap condition, but the eigenvalues of different branches can be asymptotically close at a rate which depends on the algebraic properties of the coupling parameter  $b$  and on the arithmetic property of the ratio of the speeds of propagation  $a$ . We will thus proceed as we did in the case the case  $a = 1$  and  $b \in \pi\mathbb{Z}^*$ .

**Proposition 3.3.4.** *Assume that  $0 < a \neq 1$  and condition (3.3.9) is satisfied. Then there exists a constant  $\gamma > 0$  depending only on  $a$  and  $b$  such that*

$$|\lambda_{j,m} - \lambda_{l,n}| \leq 2\gamma \implies j \neq l. \quad (3.3.18)$$

Moreover, there exist constants  $c' > c > 0$  depending only on  $a$  and  $b$  such that

1. *If  $a$  is a rational number different from  $p^2/q^2$  for all integers  $p, q$ , then for all  $|m|, |n| \geq N$ , for  $N$  large enough, we have*

$$|\lambda_{1,m} - \lambda_{2,n}| \geq \frac{c}{|m|} \quad \text{and} \quad |\lambda_{1,m} - \lambda_{2,n}| \geq \frac{c}{|n|}, \quad (3.3.19)$$

and there exist infinitely many integers  $m, n$  such that

$$|\lambda_{1,m} - \lambda_{2,n}| \leq \frac{c'}{|m|} \quad \text{and} \quad |\lambda_{1,m} - \lambda_{2,n}| \leq \frac{c'}{|n|}. \quad (3.3.20)$$

2. If  $a = p_0^2/q_0^2 \neq 1$  for some integers  $p_0, q_0$ , then for all  $|n|, |m| \geq N$ , for  $N$  large enough, we have

$$|\lambda_{1,m} - \lambda_{2,n}| \geq \frac{c}{|m|} \quad \text{and} \quad |\lambda_{1,m} - \lambda_{2,n}| \geq \frac{c}{|n|}, \quad (3.3.21)$$

and there exist infinitely many integers  $m, n$  such that

$$|\lambda_{1,m} - \lambda_{2,n}| \leq \frac{c'}{|m|} \quad \text{and} \quad |\lambda_{1,m} - \lambda_{2,n}| \leq \frac{c'}{|n|}. \quad (3.3.22)$$

3. For almost all positive irrational number  $a > 0$  and all  $|n|, |m| \geq N$ , for  $N$  large enough, we have

$$|\lambda_{1,m} - \lambda_{2,n}| \geq \frac{c}{|m|} \quad \text{and} \quad |\lambda_{1,m} - \lambda_{2,n}| \geq \frac{c}{|n|}, \quad (3.3.23)$$

and there exist infinitely many integers  $m, n$  such that

$$|\lambda_{1,m} - \lambda_{2,n}| \leq \frac{c'}{|m|} \quad \text{and} \quad |\lambda_{1,m} - \lambda_{2,n}| \leq \frac{c'}{|n|}. \quad (3.3.24)$$

*Proof.* The assertion (3.3.18) follows directly from the asymptotic expansions (3.3.10)-(3.3.11) and the fact that all the eigenvalues are geometrically simple.

Now using the asymptotic expansions (3.3.10)-(3.3.11), we have

$$\left| \frac{\lambda_{1,m} - \lambda_{2,n}}{m} \right| = \pi \left| \sqrt{a} - \frac{n}{m} \right| + \frac{O(a, b)}{m^2} + \frac{O(a, b)}{|mn|}. \quad (3.3.25)$$

If  $|\sqrt{a} - \frac{n}{m}| \geq \frac{1}{2}\sqrt{a}$ , then the estimates (3.3.19), (3.3.21) and (3.3.23) are trivial.

If  $|\sqrt{a} - \frac{n}{m}| \leq \frac{1}{2}\sqrt{a}$ , then  $m \sim n$  and (3.3.25) becomes

$$\left| \frac{\lambda_{1,m} - \lambda_{2,n}}{m} \right| = \pi \left| \sqrt{a} - \frac{n}{m} \right| + \frac{O(a, b)}{m^2}. \quad (3.3.26)$$

It is then sufficient to consider the leading term in (3.3.26).

**Case (1).** Let  $a = p_0/q_0$  be a reduced rational number. Then  $\sqrt{a}$  is a root of the integer polynomial  $q_0x^2 - p_0$  of second degree. Since  $a \neq p^2/q^2$  for all integers  $p, q$ , then the integer polynomial  $q_0x^2 - p_0$  is irreducible. This means that  $\sqrt{a}$  is a

quadratic algebraic number. Thanks to the Liouville's theorem on the approximation of algebraic numbers (see [31], Theorem 1.2), there exists a constant  $c_0 > 0$  depending only on  $a$  such that for all  $|n|, |m| \geq N$  we have

$$\left| \sqrt{a} - \frac{n}{m} \right| \geq \frac{c_0}{m^2}.$$

On the other hand, since  $\sqrt{a}$  is an irrational number, using the Dirichlet's classic theorem on number theory (see [31], Theorem 1.1), there exist infinitely many integers  $m, n$  such that

$$\left| \sqrt{a} - \frac{n}{m} \right| \leq \frac{1}{m^2}.$$

This gives the estimates (3.3.19)-(3.3.20).

**Case (2).** We return to (3.3.25), we get

$$|\lambda_{1,m} - \lambda_{2,n}| = \pi |\sqrt{am} - n| + \frac{O(a, b)}{|m|} + \frac{O(a, b)}{|n|}.$$

Since  $\sqrt{a} \in \mathbb{Q}$ , it is sufficient to consider the case  $\sqrt{am} = n$ . Using the fact that  $m \sim n$ , we deduce that

$$|\lambda_{1,m} - \lambda_{2,n}| \geq \frac{c}{|m|}.$$

On the other hand, by taking  $m = q_0 k$  and  $n = p_0 k$ ,  $k \in \mathbb{Z}^*$ , and using the asymptotic expansions (3.3.10)-(3.3.11), we easily get that

$$|\lambda_{1,q_0 k} - \lambda_{2,p_0 k}| \leq \frac{c'}{|m|}.$$

This gives the estimates (3.3.21)-(3.3.22).

**Case (3).** Firstly, following Khintchine's Theorem on Diophantine approximation (see [31], Theorem 1.10), there exist only finitely many integers  $m, n$  such that

$$\left| \sqrt{a} - \frac{n}{m} \right| \leq \frac{1}{m^2 (\ln|m|)^2}.$$

It follows thus from (3.3.26), that there exists a constant  $C > 0$  and  $N \in \mathbb{N}$ , large enough, such that, for all  $|m|, |n| \geq N$ , we have

$$\left| \frac{\lambda_{1,m} - \lambda_{2,n}}{m} \right| \geq \frac{c}{m^2}.$$

This gives the estimate (3.3.23). Secondly, following Hurwitz's Theorem (see [62]) for any irrational real number  $\sqrt{a}$ , there exist infinitely many integers  $m, n > 0$  such that

$$\left| \sqrt{a} - \frac{n}{m} \right| \leq \frac{1}{\sqrt{5}m^2}.$$

This gives the estimate (3.3.24). The proof is thus complete. □

**Proposition 3.3.5.** *Assume that  $0 < a \neq 1$  and condition (3.3.9) holds. We rearrange the two branches of eigenvalues into one sequence  $(\lambda_n)_{n \neq 0}$  such that  $(\text{Im}\lambda_n)_{n \neq 0}$  is strictly increasing. Assume that*

$$0 < \text{Im}\lambda_{n+1} - \text{Im}\lambda_n \leq \gamma, \quad (3.3.27)$$

then we have

$$\text{Im}\lambda_n - \text{Im}\lambda_{n-1} > \gamma, \quad \text{Im}\lambda_{n+2} - \text{Im}\lambda_{n+1} > \gamma. \quad (3.3.28)$$

We say that  $\text{Im}\lambda_n, \text{Im}\lambda_{n+1}$  is a chain of close exponents relative to  $\gamma$  of length 2.

By (3.3.12)-(3.3.13) we see that

$$(\psi_{1,n})_x(1) \sim O(1), \quad (\psi_{2,n})_x(1) \sim O\left(\frac{1}{n}\right). \quad (3.3.29)$$

Due to the fact that the eigenvalues can be asymptotically close, the inverse observability inequalities are not true in the energy space  $\mathcal{H}$ . That is why we define the following weighted spectral space

$$D_1 = \left\{ (\phi_0, \phi_1, \psi_0, \psi_1) = \sum_{n \neq 0} (\beta_{1,n} E_{1,n} + \beta_{2,n} n E_{2,n}) n \right\}. \quad (3.3.30)$$

Since the set  $\{E_{1,n}, E_{2,n}\}$  is a Riesz basis in the energy space  $\mathcal{H}$ , the space  $D_1$  is obviously a Hilbert space equipped with the norm

$$\sum_{n \neq 0} (|\beta_{1,n}|^2 + |\beta_{2,n}|^2).$$

We are now ready to prove our observability inequalities result.

**Proof of Theorem 3.1.3.** The proof is divided into two steps.

**Step 1: Inverse observability inequality.** From (3.3.7), we can deduce that all eigenvalues are different from zero and are all algebraically simple. Given any initial data such as

$$(\phi_0, \phi_1, \psi_0, \psi_1) = \sum_{n \neq 0} (\beta_{1,n} E_{1,n} + \beta_{2,n} n E_{2,n}) n \in D_1,$$

for the cases (1) and (2) respectively. Using the Riesz property the solution of (3.3.4) can be written as

$$(\phi(x, t), \phi_t(x, t), \psi(x, t), \psi_t(x, t)) = \sum_{n \neq 0} (\beta_{1,n} E_{1,n} e^{\lambda_{1,n} t} + \beta_{2,n} n E_{2,n} e^{\lambda_{2,n} t}) n.$$

It follows that

$$\psi_x(1, t) = \sum_{n \neq 0} (\beta_{1,n} (\psi_{1,n})_x(1) e^{\lambda_{1,n} t} + \beta_{2,n} n (\psi_{2,n})_x(1) e^{\lambda_{2,n} t}) n. \quad (3.3.31)$$

Now we rearrange the two branches of eigenvalues  $(\lambda_{1,n})_{n \neq 0}, (\lambda_{2,n})_{n \neq 0}$  into one sequence  $(\lambda_n)_{n \neq 0}$  such that the sequence  $(Im\lambda_n)_{n \neq 0}$  is strictly increasing. Following Proposition 3.3.5, all chain  $Im\lambda_n, Im\lambda_{n+1}$  of close exponents relative to  $\gamma$  is of length 2. Then, let  $A$  denotes the set of integers  $n \in \mathbb{Z}^*$  such that the condition (3.3.27) holds true and let

$$B = \mathbb{Z}^* \setminus \{n, n + 1 : n \in A\}.$$

We denote by  $a_n$  the coefficient before  $e^{\lambda_{1,n}t}$  or  $e^{\lambda_{2,n}t}$  in (3.3.31). We can rewrite it into

$$\begin{aligned} \psi_x(1, t) &= \sum_{n \in B} a_n e^{\lambda_n t} + \sum_{n \in A} a_n e^{\lambda_n t} + a_{n+1} e^{\lambda_{n+1} t} \\ &= \sum_{n \in B} a_n e^{\lambda_n t} + \sum_{n \in A} ((a_n + a_{n+1}) e^{\lambda_n t} + (\lambda_{n+1} - \lambda_n) a_{n+1} e_{n+1}(t)), \end{aligned}$$

where  $e_{n+1}(t)$  denotes the divided difference of the chain of exponents  $\lambda_n, \lambda_{n+1}$  relative to  $\gamma$

$$e_{n+1}(t) = \frac{e^{\lambda_{n+1}t} - e^{\lambda_n t}}{\lambda_{n+1} - \lambda_n}.$$

Following Theorem 9.4 in [73], the sequence

$$(e^{\lambda_n t})_{n \in B}, (e^{\lambda_n t}, e_{n+1}(t))_{n \in A}$$

forms a Riesz sequence in  $L^2(0, T)$  provided that  $T > 2\pi D^+$ , where  $D^+$  is the upper density of the sequence  $(\lambda_n)_{n \in \mathbb{Z}^*}$ , defined as

$$D^+(r) = \lim_{n \rightarrow \infty} \frac{n^+(r)}{r},$$

where  $n^+(r)$  denotes the largest number of terms of the sequence  $(\lambda_n)_{n \geq 0}$  contained in an interval of length  $r$ . To be more precise,

$$D^+ = \frac{1}{\pi} \left( 1 + \frac{1}{\sqrt{a}} \right).$$

Thus, it follows that

$$\int_0^T |\psi_x(1, t)|^2 dt \sim \sum_{n \in B} |a_n|^2 + \sum_{n \in A} (|a_n + a_{n+1}|^2 + |\lambda_{n+1} - \lambda_n|^2 |a_{n+1}|^2). \quad (3.3.32)$$

The assertions (3.3.19), (3.3.21) and (3.3.23) of Proposition 3.3.4 imply that

$$|a_n + a_{n+1}|^2 + |\lambda_{n+1} - \lambda_n|^2 |a_{n+1}|^2 \geq c \left( \frac{|a_n|^2}{|n|^2} + \frac{|a_{n+1}|^2}{|n+1|^2} \right). \quad (3.3.33)$$

Inserting (3.3.33) into (3.3.32) and returning to the previous notations, we get

$$\int_0^T |\psi_x(1, t)|^2 dt \geq c \sum_{n \in B} \frac{|a_n|^2}{|n|^2} + \sum_{n \in A} \left( \frac{|a_n|^2}{|n|^2} + \frac{|a_{n+1}|^2}{|n+1|^2} \right) \geq c \sum_{n \in \mathbb{Z}^*} \frac{|a_n|^2}{|n|^2}$$

$$= c \sum_{n \neq 0} (|\beta_{1,n}(\psi_{1,n})_x(1)|^2 + |\beta_{2,n}(\psi_{2,n})_x(1)|^2). \quad (3.3.34)$$

Then inserting (3.3.29) into (3.3.34), we get

$$\int_0^T |\psi_x(1, t)|^2 dt \geq c \sum_{n \neq 0} (|\beta_{1,n}|^2 + |\beta_{2,n}|^2), \quad (3.3.35)$$

which yields the inequalities (3.1.14)-(3.1.15).

**Step 2: Direct observability inequality.** We consider the eigenvectors defined in (3.3.17), hence we have

$$(\phi_0, \phi_1, \psi_0, \psi_1) = \sum_{n \neq 0} (\alpha_{1,n} E_{1,n} + \alpha_{2,n} E_{2,n}).$$

Thus

$$(\phi(x, t), \phi_t(x, t), \psi(x, t), \psi_t(x, t)) = \sum_{n \neq 0} (\alpha_{1,n} E_{1,n} e^{\lambda_{1,n} t} + \alpha_{2,n} E_{2,n} e^{\lambda_{2,n} t}),$$

and consequently,

$$\psi_x(1, t) = \sum_{n \neq 0} (\alpha_{1,n} (\psi_{1,n})_x(1) e^{\lambda_{1,n} t} + \alpha_{2,n} (\psi_{2,n})_x(1) e^{\lambda_{2,n} t}).$$

By (3.3.29), we recall that

$$(\psi_{1,n})_x(1) = O(1), \quad (\psi_{2,n})_x(1) = O\left(\frac{1}{n}\right).$$

Consequently,

$$\begin{aligned} \psi_x(1, t) &= \sum_{n \neq 0} (\tilde{\alpha}_{1,n} e^{\lambda_{1,n} t} + \tilde{\alpha}_{2,n} e^{\lambda_{2,n} t}) \\ &= \sum_{n \neq 0} (\tilde{\alpha}_{1,n} + \tilde{\alpha}_{2,n}) e^{\lambda_{1,n} t} + \tilde{\alpha}_{2,n} (e^{\lambda_{2,n} t} - e^{\lambda_{1,n} t}) \\ &= \sum_{n \neq 0} (\tilde{\alpha}_{1,n} + \tilde{\alpha}_{2,n}) e^{\lambda_{1,n} t} + \tilde{\alpha}_{2,n} e_n(t) (\lambda_{2,n} - \lambda_{1,n}) \end{aligned}$$

with

$$e_n(t) = \frac{e^{\lambda_{2,n} t} - e^{\lambda_{1,n} t}}{\lambda_{2,n} - \lambda_{1,n}}.$$

Since the sequence  $e^{\lambda_{1,n}t}$ ,  $e_n(t)$  forms a Riesz basis in  $L^2(0, T)$  provided that  $T > 2\left(1 + \frac{1}{\sqrt{a}}\right)$ , we deduce the following direct observability inequality

$$\begin{aligned} \int_0^T |\psi_x(1, t)|^2 dt &\sim \sum_{n \neq 0} (\tilde{\alpha}_{1,n}^2 + \tilde{\alpha}_{2,n}^2) + \frac{\tilde{\alpha}_{2,n}^2}{n^2} \\ &\leq \tilde{c} \sum_{n \neq 0} (\tilde{\alpha}_{1,n}^2 + \tilde{\alpha}_{2,n}^2) \\ &\leq c_1 \sum_{n \neq 0} (\alpha_{1,n}^2 + \alpha_{2,n}^2). \end{aligned}$$

This gives (3.1.13). The proof is thus complete. □

*Remark 3.3.6.* From inequalities (3.3.20), (3.3.22) and (3.3.24), there exist an infinity of  $(\phi_0, \phi_1, \psi_0, \psi_1) \in D_1$  such that the corresponding inverse inequalities of (3.1.14)-(3.1.15) hold.

Similarly as in the first case ( $a = 1$  and  $b \notin \pi\mathbb{Z}$ ), using the HUM method (see [70, 79]), we have the following exact controllability result.

**Theorem 3.3.7.** *Assume that  $0 < a \neq 1$  and condition (3.3.9) holds. Let  $T > 2\left(1 + \frac{1}{\sqrt{a}}\right)$ , then, if  $a$  or  $\sqrt{a}$  is rational given number and for almost irrational given number  $a$ , system (3.1.2) is exactly controllable at the moment  $T$ . More precisely, for any initial data  $(u_0, u_1, y_0, y_1) \in D'_1$ , there exists a control function  $v \in L^2(0, T)$  such that the solution  $(u, u_t, y, y_t) \in D'_1$ , of the controlled system (3.1.2) satisfies the null final conditions*

$$u(x, T) = u_t(x, T) = y(x, T) = y_t(x, T) = 0.$$

It is natural to think about the characterization of the spectral space  $D_1$ . For this aim, we have the following new subsection.

### 3.3.2 Observability and exact controllability in usual spaces. The case $a \neq 1$

The weighted spectral space  $D_1$  is defined by means of the eigenvectors  $(E_{1,n})_{n \neq 0}$  and  $(E_{2,n})_{n \neq 0}$  with weights. So, the four exponents  $(\phi_0, \phi_1, \psi_0, \psi_1)$  are a priori involved together. In order to get the observability or exact controllability in usual energy spaces, we have to separate the components  $(\phi_0, \phi_1, \psi_0, \psi_1)$ .

Now, using the asymptotic expansions (3.3.10)-(3.3.11) in (3.3.17), we can write

$$E_{1,n} = (x_n, g_n), \quad E_{2,n} = (f_n, y_n),$$

with

$$\begin{cases} x_n = \left( \frac{\sin(n\pi x)}{(n\pi)^2}, i\sqrt{a}\frac{\sin(n\pi x)}{(n\pi)} \right), \\ g_n = \left( \frac{i(1-a)\sin(n\pi x)}{b\sqrt{a}n\pi}, \frac{(a-1)}{b}\sin(n\pi x) \right), \\ y_n = \left( \frac{\sin(n\pi x)}{(n\pi)^2}, i\frac{\sin(n\pi x)}{n\pi} \right), \\ f_n = \left( \frac{i(1-a)\sin(n\pi x)}{bn\pi}, \frac{(a-1)}{b}\sin(n\pi x) \right). \end{cases} \quad (3.3.36)$$

Next, for any  $s \geq 0$ , we define the space

$$X_s = \left\{ (\hat{\phi}, \hat{\psi}) = \sum_{n \neq 0} \beta_n n^{s+1} y_n \right\}, \quad \|(\hat{\phi}, \hat{\psi})\|_{X_s}^2 = \sum_{n \neq 0} |\beta_n|^2. \quad (3.3.37)$$

Following Theorem 3.2.11, we can state the following result.

**Corollary 3.3.8.** *Assume that  $0 < a \neq 1$  and condition (3.3.9) holds. Then we have the following identification*

$$D_1 = X_2 \times X_1. \quad (3.3.38)$$

*Proof.* From (3.3.36) and (3.3.37), we see that  $(nx_n)_{n \neq 0}$  and  $(n^2 y_n)_{n \neq 0}$  are Riesz basis in  $X_2$  and  $X_1$  respectively. Moreover,  $(ng_n)_{n \neq 0}$  and  $(n^2 f_n)_{n \neq 0}$  are Bessel sequences in  $X_1$  and  $X_2$  respectively. Then (3.3.38) follows directly from Theorem 3.2.11.  $\square$

Furthermore, for any  $s \geq 0$ , we define the following space

$$V_s = \left\{ f = \sum_{n > 0} \alpha_n \frac{\sin(n\pi x)}{n^{s+1}} \right\}, \quad \|f\|_{V_s}^2 = \sum_{n > 0} |\alpha_n|^2. \quad (3.3.39)$$

With the pivot space  $L^2(0, 1)$ , we have

$$X_1 = V'_0 \times V'_1, \quad X_2 = V'_1 \times V'_2. \quad (3.3.40)$$

It follows from (3.3.38) that

$$D_1 = V'_1 \times V'_2 \times V'_0 \times V'_1. \quad (3.3.41)$$

We can now characterize the space of observability. We state thus the following result.

**Theorem 3.3.9.** *Assume that  $0 < a \neq 1$  and condition (3.3.9) holds. Let  $T > 2\left(1 + \frac{1}{\sqrt{a}}\right)$ , then there exists a constant  $c_1 > 0$  depending on  $a$  and  $b$  such that the following direct inequality holds*

$$\int_0^T |\psi_x(1, t)|^2 dt \leq c_1 \|(\phi_0, \phi_1, \psi_0, \psi_1)\|_{\mathcal{H}}^2,$$

for all solution  $\Phi = (\phi, \phi_t, \psi, \psi_t)$  solving the corresponding homogeneous Cauchy problem. Moreover, if  $a$  or  $\sqrt{a}$  is a given rational number and for almost irrational given number  $a$ , there exists a constant  $0 < c < c_1$  depending only on  $a$  and  $b$  such that the following observability inequality holds

$$c\|(\phi_0, \phi_1, \psi_0, \psi_1)\|_{V_1' \times V_2' \times V_0' \times V_1'}^2 \leq \int_0^T |\psi_x(1, t)|^2 dt.$$

*Proof.* The proof of the inverse observability inequality is a direct consequence of Theorem 3.1.3 and the identification (3.3.41). The proof of the direct observability inequality is a direct consequence of the divided difference technique.  $\square$

We deduce that the observability space is  $H^{-2}(0, 1) \times H^{-3}(0, 1) \times H^{-1}(0, 1) \times H^{-2}(0, 1)$ . Finally, using HUM method, we have the following controllability result

**Theorem 3.3.10.** *Assume that  $0 < a \neq 1$  and condition (3.3.9) holds. Let  $T > 2\left(1 + \frac{1}{\sqrt{a}}\right)$ , then, if  $a$  or  $\sqrt{a}$  is a given rational number and for almost irrational given number  $a$ , system (3.1.2) is exactly controllable at the moment  $T$ . More precisely, for any initial data  $(u_0, u_1, y_0, y_1) \in V_2 \times V_1 \times V_1 \times V_0$ , there exists a control function  $v \in L^2(0, T)$  such that the solution  $(u, u_t, y, y_t) \in V_2 \times V_1 \times V_1 \times V_0$ , of the controlled system (3.1.2) satisfies the null final conditions*

$$u(x, T) = u_t(x, T) = y(x, T) = y_t(x, T) = 0.$$

*Remark 3.3.11.* For the cases (1)-(2), the control space is of type

$$H^3(0, 1) \times H^2(0, 1) \times H^2(0, 1) \times H_0^1(0, 1),$$

with suitable boundary conditions.



## Chapter 4

# A spectral approach to the polynomial stability and to the indirect boundary control of weakly coupled wave equations

*Le taux de décroissance de l'énergie d'un système d'équations d'ondes couplées dépend du type de couplage, de la nature algébrique du paramètre du couplage et de la propriété arithmétique du rapport des vitesses de propagation des ondes (voir [85], [88], [87], [11]). Dans ce chapitre, nous sommes intéressés par l'étude d'un système mono-dimensionnel d'équations d'ondes couplées par des termes de couplage d'ordre zéro. D'abord, en utilisant l'approche spectrale, nous montrons que l'énergie décroît polynomialement avec un taux de décroissance optimal d'ordre  $\frac{1}{t}$ . Ensuite, nous étudions la contrôlabilité exacte indirecte et nous établissons les inégalités d'observabilité. Finalement, par la méthode d'unicité de Hilbert, nous démontrons que le système est exactement contrôlable.*

**Abstract.** The energy decay rate of a system of coupled wave equations depends on the type of the coupling, the algebraic nature of the coupling parameter and the arithmetic property of the ratio of the wave propagation speeds (see [85], [88], [87], [11]). In this chapter, we are interested in the study of a one dimensional setting of a system of wave equations coupled via zero order terms. Firstly, we prove optimal polynomial energy decay rate of order  $\frac{1}{t}$ , by using a spectral approach. Secondly, we study the indirect boundary exact controllability: using the non harmonic analysis, we establish the weak observability inequalities. Next, using the HUM method, we prove that the system is exactly controllable.

## 4.1 Introduction

In a recent paper of Alabau-Boussaouira [10], general systems of coupled second order evolution equations have been studied. In particular, she proved the lack of uniform stability, by a compact perturbation argument, of a system of wave equations which are weakly coupled and partially damped. The system is described by:

$$\begin{cases} u_{tt} - a\Delta u + \alpha y = 0 & \text{on } \Omega, \\ y_{tt} - \Delta y + \alpha u = 0 & \text{on } \Omega, \\ u = 0 & \text{on } \Gamma_0, \\ a\partial_\nu u + \gamma u + u_t = 0 & \text{on } \Gamma_1, \\ y = 0 & \text{on } \Gamma, \end{cases}$$

where  $\Omega \subset \mathbb{R}^N$  is a bounded domain with smooth boundary  $\Gamma$  of class  $C^2$  such that  $\Gamma = \Gamma_0 \cup \Gamma_1$  and  $\Gamma_0 \cap \Gamma_1 = \emptyset$ . Moreover, Alabau established the polynomial energy decay rate  $\frac{1}{\sqrt{t}}$  in the case where  $a = 1$  and  $\Omega$  is a star shaped domain of  $\mathbb{R}^N$ , or in the case where  $a = \frac{1}{k^2}$ , with  $k \in \mathbb{Z}^*$  and  $\Omega$  is a cubic domain of  $\mathbb{R}^3$ . Next, in [88] Liu and Rao established, by the frequency domain approach, polynomial decay rate of order  $\frac{\ln t}{t}$  for smooth initial data, while waves propagate with equal speeds. Moreover, while waves propagate with different speeds, i.e. the case  $a \neq 1$ , they proved that the energy decays at a rate which depends on the arithmetic property of the ratio of the wave speeds  $a$ . Later, in [11], Alabau and Léautaud considered a coupled system of wave equations, with variable coefficients, with one boundary feedback, and proved a polynomial energy decay rate of type  $\frac{1}{\sqrt{t}}$  for smooth initial data.

In this chapter, we firstly improve the energy decay rate in the one dimensional setting and we establish optimal polynomial decay of type  $\frac{1}{t}$  for smooth initial data, by the Riesz basis approach. More precisely, we study the stabilization of the following system of partially damped coupled wave equations propagating with equal speeds, described by

$$\begin{cases} u_{tt} - u_{xx} + \alpha y = 0 & \text{on } (0, 1) \times (0, T), \\ y_{tt} - y_{xx} + \alpha u = 0 & \text{on } (0, 1) \times (0, T), \\ u(1, t) = y(0, t) = y(1, t) = 0 & \text{for } t \in (0, T), \\ u_x(1, t) + \gamma u_t(1, t) = 0 & \text{for } t \in (0, T). \end{cases} \quad (4.1.1)$$

We denote by  $\alpha$  the coupling parameter, which is assumed to be real and small enough. We assume that  $\gamma > 0$ . The damping  $u_t$  is only applied at the right boundary of the first equation. The second equation is indirectly damped through the coupling between the two equations. We assume that the initial positions of the waves are given by

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x); \quad y(x, 0) = y_0(x), \quad y_t(x, 0) = y_1(x). \quad (4.1.2)$$

The energy of system (4.1.1) is given by

$$E(t) = \frac{1}{2} \int_0^1 (|u_t|^2 + |u_x|^2 + |y_t|^2 + |y_x|^2 + 2\alpha y u) dx.$$

A direct computation gives

$$\frac{d}{dt} E(t) = -\gamma |u_t(1, t)|^2.$$

Thus the system (4.1.1) is dissipative in the sense that its energy  $E(t)$  is a nonincreasing function with respect to the time  $t$ .

Secondly, we study the exact controllability of the following system of weakly coupled wave equations with Neumann boundary conditions, propagating with equal speeds, described by

$$\begin{cases} u_{tt} - u_{xx} + \alpha y = 0 & \text{on } (0, 1) \times (0, T), \\ y_{tt} - y_{xx} + \alpha u = 0 & \text{on } (0, 1) \times (0, T), \\ u(0, t) = y(0, t) = y(1, t) = 0 & \text{for } t \in (0, T), \\ u_x(1, t) = v(t) & \text{for } t \in (0, T). \end{cases} \quad (4.1.3)$$

The control  $v$  is applied only at the right boundary of the first equation. The second equation is indirectly controlled by means of the coupling between the equations. We consider the indirect boundary exact controllability problem: For given  $T > 0$  and initial data  $(u_0, u_1, y_0, y_1)$  belonging to a suitable space, is it possible to find a suitable control  $v$  such that the solution of system (4.1.3)  $(u, u_t, y, y_t)$  is driven to zero at time  $T$ , i.e.,

$$u(x, T) = u_t(x, T) = y(x, T) = y_t(x, T) = 0 \quad \text{on } (0, 1).$$

**Main results.** On the one hand, we prove that the energy of (4.1.1) decays at an optimal polynomial rate of type  $\frac{1}{t}$ , using the Riesz basis approach. In fact, the crucial part of the proof is to prove that the generalized eigenvectors of the associated operator, form a Riesz basis of the energy space, while using a new form of Bari's theorem (see [1]).

On the other hand, we prove exact controllability results for system 4.1.3. We use thus the Hilbert Uniqueness Method introduced by Lions [79]. We afterwards establish inverse and direct observability inequalities using Ingham's theorem [73].

In order to do so, we consider the associated homogeneous problem, that is to say, the null boundary acted control system.

Let us denote by  $\lambda_{1,m}, \lambda_{2,n}$  the two branches of eigenvalues for the corresponding homogeneous system (4.1.3), and denote by

$$\begin{cases} E_{1,n} = (\phi_{1,n}, \lambda_{1,n}\phi_{1,n}, \psi_{1,n}, \lambda_{1,n}\psi_{1,n}), \\ E_{2,n} = (\phi_{2,n}, \lambda_{2,n}\phi_{2,n}, \psi_{2,n}, \lambda_{2,n}\psi_{2,n}) \end{cases} \quad (4.1.4)$$

the corresponding eigenvectors. In Proposition 4.4.1, we prove that there exist infinitely many  $m \sim n$  such that the eigenvalues satisfy a standard gap

$$\lambda_{1,m} - \lambda_{2,n} \sim O(1). \quad (4.1.5)$$

But the observation is on the first components of the corresponding eigenvectors  $E_{1,n}, E_{2,n}$ . Following Proposition 4.4.2, we have

$$\phi_{1,n}(1) = O\left(\frac{1}{n}\right), \quad \phi_{2,n}(1) = O\left(\frac{1}{n^2}\right). \quad (4.1.6)$$

Hence, the following weighted spectral space is the natural space for observability

$$\mathcal{D} = \{(\phi_0, \phi_1, \psi_0, \psi_1) = \sum_{n \neq 0} (\alpha_{1,n} n E_{1,n} + \alpha_{2,n} n^2 E_{2,n})\}. \quad (4.1.7)$$

In (4.1.7), the factors  $n$  and  $n^2$  are due to the transmission of the modes between the two equations.

We prove the following results.

**Theorem 4.1.1.** *Assume that  $\gamma \neq 1$ . For all initial data  $U_0 \in D(\mathcal{A})$ , there exists a constant  $c > 0$  independent of  $U_0$ , such that the energy of the corresponding Cauchy problem associated to (4.1.1) satisfies the following estimation*

$$E(t) \leq \frac{c}{t} \|U_0\|_{D(\mathcal{A})}^2. \quad (4.1.8)$$

Moreover, the energy decay rate (4.1.8) is optimal.

**Theorem 4.1.2** (Observability inequalities for Neumann boundary control). *Let  $\alpha \neq 0$  be a real number small enough. Assume that*

$$T > 4.$$

*Then there exists a constant  $c_1 > 0$  such that the direct observability inequality holds*

$$\int_0^T |\phi(1, t)|^2 dt \leq c_1 \|(\phi_0, \phi_1, \psi_0, \psi_1)\|_{\mathcal{D}}^2, \quad (4.1.9)$$

*for all solution  $\Phi = (\phi, \phi_t, \psi, \psi_t)$  solving the corresponding homogeneous Cauchy problem.*

*Moreover, there exists a constant  $0 < c < c_1$ , such that the following inverse observability inequality holds*

$$c \|(\phi_0, \phi_1, \psi_0, \psi_1)\|_{\mathcal{D}}^2 \leq \int_0^T |\phi(1, t)|^2 dt. \quad (4.1.10)$$

**Brief review of the literature.** The polynomial energy decay rate occurs in many control problems where the open-loop systems are strongly stable, but not exponentially stable (hybrid systems, partially or locally damped systems), see [74] and references therein. The majority of the works in establishing polynomial energy decay rate has been based on the spectral method, frequency domain method, time domain multiplier and weak observability methods. We quote [84, 85, 96, 97] for hybrid systems, [76, 77, 88] for wave equations with local internal or boundary damping, [5, 14, 16, 90] for second order systems with partial internal damping, [27, 87] for abstract systems and [101] for systems of coupled wave-heat equations. We also mention [8, 10] for coupled hyperbolic systems, and [11] for coupled wave type systems. For a general formulation of partially damped systems see [98] and references therein. In [8] (see also references therein), polynomial decay estimates in the case of indirect internal stabilization are given. These results have been extended to several (wave-wave coupling, Petrowsky-Petrowsky coupling) in [29].

Complete and partial observability (respectively controllability) results for coupled systems either of hyperbolic-hyperbolic type or of hyperbolic-parabolic type can be found in [79], see also [75]. These results assume that the coupling parameter is sufficiently small. They have been extended in [71] to the cases of arbitrary coupling parameters (assuming bounded coupling operators). For both references, the multiplier method was the main ingredient for obtaining the desired estimates. Moreover, stabilization and observability results for hyperbolic-hyperbolic systems via a single control force were obtained in [68, 69], where wave-wave systems having the same principal part are coupled through velocity terms.

**Organization of the chapter.** The remaining of the chapter is organized as follows. In Section 4.2, we establish well-posedness of problem (4.1.1). Section 4.3 is devoted to the proof of the optimal decay rate of the energy of the system of coupled wave equations. The exact controllability for Neumann boundary control is proved in Section 4.4 by establishing the corresponding inverse and direct observability inequalities.

## 4.2 Well posedness and strong stability

In this section we study existence, uniqueness and strong stability of the solution of system (4.1.1). We start our study by formulating the problem as an abstract Cauchy problem in an appropriate Hilbert space. First, we introduce the following space

$$V = \{u \in H_0^1(0, 1); \quad u(0) = 0\}, \quad (4.2.1)$$

and the energy space as

$$\mathcal{H} = V \times L^2(0, 1) \times H_0^1(0, 1) \times L^2(0, 1), \quad (4.2.2)$$

endowed with the inner product

$$(U, \tilde{U})_{\mathcal{H}} = \int_0^1 \left( u_x \overline{\tilde{u}_x} + v \overline{\tilde{v}} + y_x \overline{\tilde{y}_x} + z \overline{\tilde{z}} + \alpha u \overline{\tilde{y}} + \alpha y \overline{\tilde{u}} \right) dx, \quad (4.2.3)$$

$$\forall U = (u, v, y, z), \tilde{U} = (\tilde{u}, \tilde{v}, \tilde{y}, \tilde{z}) \in \mathcal{H}.$$

It is easy to check that the inner product (4.2.3) is equivalent to the usual inner product in  $\mathcal{H}$  for small  $\alpha$ .

Now, we define a linear unbounded operator  $\mathcal{A} : D(\mathcal{A}) \mapsto \mathcal{H}$  by

$$D(\mathcal{A}) = \left\{ U = (u, v, y, z) \in \mathcal{H} \mid \begin{array}{l} u, y \in H^2(0, 1), v \in V, z \in H_0^1(0, 1) \\ u_x(1) = -\gamma v(1) \end{array} \right\} \quad (4.2.4)$$

and

$$\mathcal{A}U = (v, u_{xx} - \alpha y, z, y_{xx} - \alpha u). \quad (4.2.5)$$

Then setting  $U = (u, u_t, y, y_t)$  a regular solution of (4.1.1), we rewrite it into an evolution equation

$$\begin{cases} U_t &= \mathcal{A}U(t), \\ U(0) &= U_0, \end{cases} \quad (4.2.6)$$

where

$$U_0(x) = (u_0(x), u_1(x), y_0(x), y_1(x)).$$

**Proposition 4.2.1.** *The operator  $\mathcal{A}$  is maximal dissipative in the energy space  $\mathcal{H}$ , therefore it generates a  $C_0$ -semigroup  $e^{t\mathcal{A}}$  of contractions on  $\mathcal{H}$ . Moreover,  $\mathcal{A}$  is a densely defined operator with a compact resolvent in  $\mathcal{H}$ .*

*Proof.* Let  $U = (u, v, y, z) \in D(\mathcal{A})$ , using the definitions (4.2.3)- (4.2.4), it is easy to see that

$$\operatorname{Re}(\mathcal{A}U, U)_{\mathcal{H}} = -\gamma|v^2(1)| \leq 0, \quad (4.2.7)$$

which implies that  $\mathcal{A}$  is a dissipative operator in the energy space  $\mathcal{H}$ .

Now let  $F = (f, g, \xi, \theta) \in \mathcal{H}$ . We look for an element  $U = (u, v, y, z) \in D(\mathcal{A})$  such that  $(I - \mathcal{A})U = F$ . Equivalently, we have the following system

$$v = u - f, \quad (4.2.8)$$

$$z = y - \xi, \quad (4.2.9)$$

$$u - u_{xx} + \alpha y = g + f, \quad (4.2.10)$$

$$y - y_{xx} + \alpha u = \theta + \xi, \quad (4.2.11)$$

and

$$u(0) = y(0) = y(1) = 0, \quad u_x(1) = -\gamma(u - f)(1). \quad (4.2.12)$$

Using Lax-Milgram theorem, we deduce that (4.2.10)-(4.2.12) admits a unique solution  $(u, y) \in V \times H_0^1(0, 1)$ . Therefore, using (4.2.8)-(4.2.9) and the classical elliptic theory, we conclude that  $(I - \mathcal{A})U = F$  admits a unique solution  $U = (u, v, y, z) \in D(\mathcal{A})$ . Thus, by the resolvent identity, we have  $R(\lambda I - \mathcal{A}) = \mathcal{H}$ . Then,

the operator  $\mathcal{A}$  is a maximal dissipative operator (see [94], Theorem 1.4). Finally the Sobolev embedding theorem asserts that  $(\mathcal{A} - I)^{-1}$  is compact, thus  $\mathcal{A}$  is a densely defined operator in  $\mathcal{H}$  with a compact resolvent.  $\square$

Using the Lumer-Phillips theorem (see [94], Theorem 1.4.3),  $\mathcal{A}$  is the infinitesimal generator of a  $C_0$ -semi-group of contractions on  $\mathcal{H}$ .

Therefore, we have the following result concerning existence and uniqueness of the solution of the Cauchy problem (4.2.6).

**Theorem 4.2.2** (Existence and uniqueness). *For any initial data  $U_0 \in \mathcal{H}$ , the problem (4.2.6) has a unique weak solution*

$$U(t) = e^{t\mathcal{A}}U_0 \in C_0([0, \infty[, \mathcal{H}).$$

Moreover, if  $U_0 \in D(\mathcal{A})$ , then the problem (4.2.6) has a strong solution

$$U(t) = e^{t\mathcal{A}}U_0 \in C^1([0, \infty[, \mathcal{H}) \cap C^0([0, \infty[, D(\mathcal{A})).$$

In order to study the stability of problem (4.1.1), we have to study the asymptotic behavior of its solution.

### 4.2.1 Spectral analysis of the operator $\mathcal{A}$

Let  $\lambda$  be an eigenvalue of  $\mathcal{A}$  with its associated eigenvector  $U = (u, \lambda u, y, \lambda y)$ .

**Lemma 4.2.3.** *Let  $\alpha \neq 0$  be a real number small enough, and let*

$$f(\lambda) = \alpha[2\gamma \sinh(r_1) \sinh(r_3) + \frac{r_1}{\lambda} \sinh(r_3) \cosh(r_1) + \frac{r_3}{\lambda} \sinh(r_1) \cosh(r_3)], \quad (4.2.13)$$

where

$$r_1 = \sqrt{\lambda^2 + \alpha}, \quad r_3 = \sqrt{\lambda^2 - \alpha}. \quad (4.2.14)$$

Thus,  $\lambda$  is a root of  $f$ . Moreover, the expressions of  $u$  and  $y$  are given by

$$u(x) = C(\sinh(r_3) \sinh(r_1 x) + \sinh(r_1) \sinh(r_3 x)), \quad (4.2.15)$$

$$y(x) = C(\sinh(r_3) \sinh(r_1 x) - \sinh(r_1) \sinh(r_3 x)) \quad (4.2.16)$$

where  $C \in \mathbb{C}$ .

*Proof.* Let us consider the corresponding eigenvalue problem

$$\lambda^2 u - u_{xx} + \alpha y = 0, \quad (4.2.17)$$

$$\lambda^2 y - y_{xx} + \alpha u = 0, \quad (4.2.18)$$

with the following boundary conditions

$$u(0) = y(0) = y(1) = u_x(1) + \gamma \lambda u(1) = 0. \quad (4.2.19)$$

Equivalently, we have

$$u_{xxxx} - 2\lambda^2 u_{xx} + (\lambda^4 - \alpha^2)u = 0, \quad (4.2.20)$$

and

$$u(0) = u_x(1) + \gamma\lambda u(1) = u_{xx}(0) = u_{xx}(1) - \lambda^2 u(1) = 0. \quad (4.2.21)$$

Hence, (4.2.20) has a non trivial solution satisfying the boundary conditions (4.2.21) if and only if  $\lambda \neq \pm i\sqrt{\alpha}$ .

Thus, the general solution of (4.2.20) is given by

$$u(x) = \sum_{i=1}^4 C_i e^{r_i x} \quad (4.2.22)$$

where  $C_i \in \mathbb{C}$ ,  $r_1, r_3$  are given by (4.2.14),  $r_2 = -r_1$ , and  $r_4 = -r_3$ .

From the boundary conditions at  $x = 0$  of (4.2.21) we can deduce that the solution of (4.2.20) can be written as

$$u(x) = B_1 \sinh(r_1 x) + B_2 \sinh(r_3 x), \quad (4.2.23)$$

where  $B_1, B_2 \in \mathbb{C}$ . In addition, from the boundary conditions at  $x = 1$  of (4.2.21), we get

$$Mb = 0,$$

where

$$M = \begin{pmatrix} \alpha \sinh(r_1) & -\alpha \sinh(r_3) \\ r_1 \cosh(r_1) + \gamma\lambda \sinh(r_1) & r_3 \cosh(r_3) + \gamma\lambda \sinh(r_3) \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}.$$

The determinant of  $M$  is given by

$$\det(M) = \alpha[2\gamma\lambda \sinh(r_1) \sinh(r_3) + r_1 \sinh(r_3) \cosh(r_1) + r_3 \sinh(r_1) \cosh(r_3)].$$

Hence a non trivial solution  $u$  exists if and only if  $\det(M) = 0$ , i.e, if and only if  $\lambda$  is a root of the function  $f$  defined in (4.2.13). Now, if  $f(\lambda) = 0$ , setting

$$B_1 = C \sinh(r_3) \quad \text{and} \quad B_2 = C \sinh(r_1)$$

in (4.2.23), we get (4.2.15). From (4.2.17) and (4.2.15), we get (4.2.16).

The proof is thus complete.  $\square$

*Remark 4.2.4.* Let  $\lambda$  be an eigenvalue of  $\mathcal{A}$  and  $U = (u, v, y, z)$  its normalized eigenvector. It is easy to see that  $Re(\lambda)$  is bounded. In fact, multiplying (4.2.17) (respectively (4.2.18)) by  $u$  (respectively by  $y$ ) and integrating by parts, one can deduce

$$1 = \|U\|_{\mathcal{H}}^2 = -\gamma Re(\lambda) |u(1)|^2.$$

In fact, using the trace theorem, and reasoning by contradiction we can see that  $Re(\lambda)$  is bounded.

In the sequel, since  $\mathcal{A}$  is dissipative, we study the asymptotic behavior of the large eigenvalues  $\lambda$  of  $\mathcal{A}$ .

**Proposition 4.2.5** (Spectrum of  $\mathcal{A}$ ). *Assume that  $\gamma \neq 1$ . There exists  $N \in \mathbb{N}^*$  sufficiently large enough such that the spectrum  $\sigma(\mathcal{A})$  of  $\mathcal{A}$  is given by*

$$\sigma(\mathcal{A}) = \sigma_0 \cup \sigma_1,$$

where

$$\sigma_0 = \{\hat{\lambda}_j\}_{j \in J} \cup \{\hat{\mu}_j\}_{j \in J}, \quad \sigma_1 = \{\lambda_k\}_{k \in \mathbb{Z}^*, |k| \geq N} \cup \{\mu_k\}_{k \in \mathbb{Z}^*, |k| \geq N}, \quad \sigma_0 \cap \sigma_1 = \emptyset,$$

where  $J$  is a finite set. The large eigenvalues of  $\mathcal{A}$  satisfy respectively the following asymptotic behaviors:

$$\mu_k = ik\pi + b + \frac{O(\alpha^2)}{k^2}, \quad (4.2.24)$$

where

$$b = \begin{cases} \frac{1}{2} \ln\left(\frac{\gamma-1}{\gamma+1}\right) & \text{if } \gamma > 1, \\ \frac{i\pi}{2} + \frac{1}{2} \ln\left(\frac{1-\gamma}{1+\gamma}\right) & \text{if } \gamma < 1, \end{cases} \quad (4.2.25)$$

and

$$\lambda_k = ik\pi - \frac{\gamma\alpha^2}{4(k\pi)^2} + \frac{O(\alpha^2)}{k^3}. \quad (4.2.26)$$

*Proof.* The proof is divided into several steps.

**Step 1.** Using (4.2.14), we get

$$\frac{r_1}{\lambda} = 1 + \frac{\alpha}{2\lambda^2} + \frac{O(\alpha^2)}{\lambda^4} \quad \text{and} \quad \frac{r_3}{\lambda} = 1 - \frac{\alpha}{2\lambda^2} + \frac{O(\alpha^2)}{\lambda^4}. \quad (4.2.27)$$

Inserting (4.2.27) in (4.2.13) and using the fact that real part of  $\lambda$  is bounded (see Remark 4.2.4), we get

$$f(\lambda) = (\cosh(r_1 + r_3) - \cosh(r_1 - r_3))\gamma + \sinh(r_1 + r_3) - \frac{\alpha \sinh(r_1 - r_3)}{2\lambda^2} + \frac{O(\alpha^2)}{\lambda^4}. \quad (4.2.28)$$

From (4.2.14), we get

$$r_1 + r_3 = 2\lambda - \frac{\alpha^2}{4\lambda^3} + \frac{O(\alpha^2)}{\lambda^4} \quad \text{and} \quad r_1 - r_3 = \frac{\alpha}{\lambda} + \frac{O(\alpha^2)}{\lambda^4}. \quad (4.2.29)$$

From (4.2.29), we get

$$\begin{cases} \sinh(r_1 + r_3) = \sinh(2\lambda) - \frac{\alpha^2 \cosh(2\lambda)}{4\lambda^3} + \frac{O(\alpha^2)}{\lambda^4}, \\ \cosh(r_1 + r_3) = \cosh(2\lambda) - \frac{\alpha^2 \sinh(2\lambda)}{4\lambda^3} + \frac{O(\alpha^2)}{\lambda^4} \end{cases} \quad (4.2.30)$$

and

$$\sinh(r_1 - r_3) = \frac{\alpha}{\lambda} + \frac{\alpha^3}{6\lambda^3} + \frac{O(\alpha^2)}{\lambda^4}, \quad \cosh(r_1 - r_3) = 1 + \frac{\alpha^2}{2\lambda^2} + \frac{O(\alpha^2)}{\lambda^4}. \quad (4.2.31)$$

Inserting (4.2.30) and (4.2.31) in (4.2.28), we get

$$f(\lambda) = f_0(\lambda) + \frac{f_1(\lambda)}{\lambda^2} + \frac{f_2(\lambda)}{\lambda^3} + \frac{O(\alpha^2)}{\lambda^4}, \quad (4.2.32)$$

where

$$f_0(\lambda) = 2 \sinh(\lambda) (\gamma \sinh(\lambda) + \cosh(\lambda)), \quad f_1(\lambda) = -\frac{\alpha^2 \gamma}{2}, \quad (4.2.33)$$

and

$$f_2(\lambda) = -\frac{\alpha^2 (\gamma \sinh(2\lambda) + \cosh(2\lambda) + 2)}{4}. \quad (4.2.34)$$

**Step 2.** We look for the roots of  $f_0(\lambda)$ . From (4.2.33), we deduce that the roots of  $f_0$  are given by

$$\mu_k^0 = ik\pi + b \quad \text{and} \quad \lambda_k^0 = ik\pi.$$

Next, since  $Re(\lambda)$  is bounded (see Remark 4.2.4) and thanks to Rouché's theorem, there exists  $N \in \mathbb{N}^*$  large enough, such that  $\forall |k| \geq N$ , the large roots of  $f$  (denoted by  $\mu_k$  and  $\lambda_k$  respectively) are simple and close to those of  $f_0$ , i.e.

$$\lambda_k = \lambda_k^0 + o(1), \quad |k| \rightarrow \infty, \quad \mu_k = \mu_k^0 + o(1), \quad |k| \rightarrow \infty.$$

Equivalently, we can write

$$\lambda_k = \lambda_k^0 + \zeta_k, \quad \lim_{|k| \rightarrow \infty} \zeta_k = 0, \quad (4.2.35)$$

and

$$\mu_k = \mu_k^0 + \epsilon_k, \quad \lim_{|k| \rightarrow \infty} \epsilon_k = 0. \quad (4.2.36)$$

**Step 3.** We seek to determine  $\zeta_k$  and  $\epsilon_k$ . First, substituting (4.2.33)-(4.2.34) into (4.2.32) for  $\lambda = \lambda_k$ , we obtain

$$f(\lambda_k) = 2 \sinh(\zeta_k) (\gamma \sinh(\zeta_k) + \cosh(\zeta_k)) + \frac{\alpha^2 \gamma}{2k^2} + \frac{O(\alpha^2)}{k^3} = 0. \quad (4.2.37)$$

On the other hand, since  $\lim_{|k| \rightarrow \infty} \zeta_k = 0$ , we have the following asymptotic expansion

$$\sinh(\zeta_k) = \zeta_k + O(\zeta_k^3) \quad \text{and} \quad \cosh(\zeta_k) = 1 + O(\zeta_k^2). \quad (4.2.38)$$

Using (4.2.38) in (4.2.37), and after several computations, we get

$$\zeta_k = -\frac{\gamma \alpha^2}{4(k\pi)^2} + \frac{O(\alpha^2)}{k^3}. \quad (4.2.39)$$

Inserting (4.2.39) in (4.2.35), we directly get (4.2.26). Similarly, substituting (4.2.33)-(4.2.34) in (4.2.32) for  $\lambda = \mu_k$ , we can show that

$$\epsilon_k = \frac{O(\alpha^2)}{k^2}. \quad (4.2.40)$$

Finally, inserting (4.2.40) in (4.2.36), we directly get (4.2.24).

The proof is thus complete.  $\square$

Let  $(E_{1,k})_{k \geq N}$ , and  $(E_{2,k})_{k \geq N}$  be the set of eigenvectors of  $\mathcal{A}$  associated to  $\lambda_k$  and  $\mu_k$  respectively, with

$$\begin{cases} E_{1,k} = (u_{1,k}, \lambda_k u_{1,k}, y_{1,k}, \lambda_k y_{1,k}), \\ E_{2,k} = (u_{2,k}, \mu_k u_{2,k}, y_{2,k}, \mu_k y_{2,k}). \end{cases} \quad (4.2.41)$$

We have the following result.

**Proposition 4.2.6.** *The eigenfunctions of the eigenvalue problem given by (4.2.20) have the following asymptotic expansions*

$$\begin{aligned} u_{1,k}(x) &= \frac{\sinh[(ik\pi + b)x]}{ik\pi + b} + \frac{O(\alpha)}{k^3}, \\ y_{1,k}(x) &= \frac{\alpha (\sinh[(ik\pi + b)x] \cosh(b) - x \cosh[(ik\pi + b)x] \sinh(b))}{2k^2\pi^2 \sinh(b)} + \frac{O(\alpha)}{k^3} \end{aligned} \quad (4.2.42)$$

and

$$u_{2,k}(x) = \frac{\alpha (\gamma \sin(k\pi x) + ix \cos(k\pi x))}{2k^2\pi^2} + \frac{O(\alpha^2)}{k^3}, \quad y_{2,k}(x) = \frac{\sin(k\pi x)}{k\pi} + \frac{O(\alpha^2)}{k^3}. \quad (4.2.43)$$

*Proof.* First, we determine the corresponding eigenfunctions of  $\mu_k$ . Let

$$C = C_{1,k} = \frac{1}{2(-1)^k \sinh(b) (ik\pi + b)}.$$

Inserting (4.2.24) in (4.2.14), we get

$$r_{1,k} = ik\pi + b + \frac{\alpha}{2ik\pi} + \frac{O(\alpha^2)}{k^2} \quad \text{and} \quad r_{3,k} = ik\pi + b - \frac{\alpha}{2ik\pi} + \frac{O(\alpha^2)}{k^2}. \quad (4.2.44)$$

Then it follows that

$$\begin{cases} \sinh(r_{1,k}x) = \sinh[(ik\pi + b)x] + \frac{\alpha x \cosh[(ik\pi + b)x]}{2i\pi k} + \frac{O(\alpha^2)}{k^2}, \\ \sinh(r_{3,k}x) = \sinh[(ik\pi + b)x] - \frac{\alpha x \cosh[(ik\pi + b)x]}{2i\pi k} + \frac{O(\alpha^2)}{k^2}, \end{cases} \quad (4.2.45)$$

and

$$\begin{cases} \sinh(r_{1,k}) = (-1)^k \left( \sinh(b) + \frac{\alpha \cosh(b)}{2ik\pi} \right) + \frac{O(\alpha)}{k^2}, \\ \sinh(r_{3,k}) = (-1)^k \left( \sinh(b) - \frac{\alpha \cosh(b)}{2ik\pi} \right) + \frac{O(\alpha)}{k^2}. \end{cases} \quad (4.2.46)$$

Inserting (4.2.45)-(4.2.46) into (4.2.15) and (4.2.16), we get (4.2.42).

Next, we look to determine the eigenfunctions of  $\lambda_k$ . Let

$$C = C_{2,k} = -\frac{1}{i\alpha(-1)^k}.$$

Inserting (4.2.26) in (4.2.14), we get

$$r_{1,k} = ik\pi + \frac{\alpha}{2ik\pi} - \frac{\alpha^2\gamma}{4k^2\pi^2} + \frac{O(\alpha)}{k^3}, \quad r_{3,k} = ik\pi - \frac{\alpha}{2ik\pi} - \frac{\alpha^2\gamma}{4k^2\pi^2} + \frac{O(\alpha)}{k^3}. \quad (4.2.47)$$

Then it follows that

$$\begin{cases} \sinh(r_{1,k}) = (-1)^k \left( \frac{\alpha}{2ik\pi} - \frac{\alpha^2\gamma}{4k^2\pi^2} \right) + \frac{O(\alpha)}{k^3}, \\ \sinh(r_{3,k}) = (-1)^k \left( -\frac{\alpha}{2ik\pi} - \frac{\alpha^2\gamma}{4k^2\pi^2} \right) + \frac{O(\alpha)}{k^3}, \end{cases} \quad (4.2.48)$$

and

$$\begin{cases} \sinh(r_{1,k}x) = i \sin(k\pi x) + \frac{\alpha x \cos(k\pi x)}{2ik\pi} - \frac{\alpha^2 x (ix \sin(k\pi x) + 2\gamma \cos(k\pi x))}{8\pi^2 k^2} + \frac{O(\alpha)}{k^3}, \\ \sinh(r_{3,k}x) = i \sin(k\pi x) - \frac{\alpha x \cos(k\pi x)}{2ik\pi} - \frac{\alpha^2 x (ix \sin(k\pi x) - 2\gamma \cos(k\pi x))}{8\pi^2 k^2} + \frac{O(\alpha)}{k^3}. \end{cases} \quad (4.2.49)$$

Finally, inserting (4.2.48)-(4.2.49) into (4.2.15) and (4.2.16), we get (4.2.43).

The proof is thus complete.  $\square$

*Remark 4.2.7.* [Non uniform exponential stability] The Cauchy problem (4.2.6) is not uniformly stable in the energy space  $\mathcal{H}$ , due to the fact that the large eigenvalues in (4.2.26) approach the imaginary axis.

### 4.3 Polynomial stability with optimal decay rate

Since the energy of system (4.2.6) has no uniform decay rate, we will look for polynomial decay rate for smooth initial data. Our main result is the optimal-type decay

estimate, stated in Theorem 4.1.1. The proof of this Theorem uses a spectral analysis approach, namely we show that the set of generalized eigenvectors of  $\mathcal{A}$  forms a Riesz basis of  $\mathcal{H}$ .

Let  $E_{1,k}$  and  $E_{2,k}$  be the eigenvectors associated with the eigenvalue  $\lambda_k$  and  $\mu_k$  respectively, defined in (4.2.41). Then using the asymptotic expansions (4.2.42)-(4.2.43), we can write

$$E_{1,k} = \begin{pmatrix} u_{1,k} \\ \lambda_k u_{1,k} \\ y_{1,k} \\ \lambda_k y_{1,k} \end{pmatrix} = \begin{pmatrix} \frac{\sinh[(ik\pi + b)x]}{ik\pi + b} \\ \sinh[(ik\pi + b)x] \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} O\left(\frac{1}{k^2}\right) \\ O\left(\frac{1}{k}\right) \\ O\left(\frac{1}{k^2}\right) \\ O\left(\frac{1}{k}\right) \end{pmatrix} \quad (4.3.1)$$

and

$$E_{2,k} = \begin{pmatrix} u_{2,k} \\ \mu_k u_{2,k} \\ y_{2,k} \\ \mu_k y_{2,k} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{\sin(k\pi x)}{ik\pi} \\ \sin(k\pi x) \end{pmatrix} + \begin{pmatrix} O\left(\frac{1}{k^2}\right) \\ O\left(\frac{1}{k}\right) \\ O\left(\frac{1}{k^2}\right) \\ O\left(\frac{1}{k}\right) \end{pmatrix} \quad (4.3.2)$$

Now, denoting by  $E_{1,k}^0, E_{2,k}^0$  the leading terms in the expansions (4.3.1)-(4.3.2), we have

$$\sum_{|k| \geq N} \|E_{1,k} - E_{1,k}^0\|_{\mathcal{H}}^2 + \|E_{2,k} - E_{2,k}^0\|_{\mathcal{H}}^2 < \infty \quad (4.3.3)$$

for some  $N$  large enough. We will prove in the following that the system of eigenvectors  $E_{1,k}, E_{2,k}$  forms a Riesz basis in the energy space  $\mathcal{H}$ . It is then sufficient to prove that  $E_{1,k}^0, E_{2,k}^0$  form a Riesz basis in  $\mathcal{H}$  (see Proposition 4.3.2 below).

In fact, this can be justified using the following theorem which is a new form of Bari's theorem (see Theorem 1.2.8 in [1]).

**Theorem 4.3.1.** *Let  $\mathcal{A}$  be a densely defined operator in a Hilbert space  $\mathcal{H}$  with a compact resolvent. Let  $\{\varphi_n\}_{n=1}^{\infty}$  be a Riesz basis of  $\mathcal{H}$ . If there are an integer  $N \geq 0$*

and a sequence of generalized eigenvectors  $\{\psi_n\}_{n=N+1}^\infty$  of  $\mathcal{A}$  such that

$$\sum_{n=N+1}^{\infty} \|\varphi_n - \psi_n\|^2 < \infty,$$

then the set of generalized eigenvectors of  $\mathcal{A}$ ,  $\{\psi_n\}_{n=1}^\infty$ , forms a Riesz basis of  $\mathcal{H}$ .

We can now state the following result.

**Proposition 4.3.2.** *The set of vectors  $E_{1,k}^0, E_{2,k}^0$  forms a Riesz basis of  $\mathcal{H}$ .*

In order to prove Proposition 4.3.2, we introduce the following auxiliary operator  $\mathcal{A}_0$  in  $\mathcal{H}$ , defined by

$$\mathcal{A}_0 U = (v, u_{xx}, z, y_{xx}), \forall U = (u, v, y, z) \in D(\mathcal{A}_0) = D(\mathcal{A}). \quad (4.3.4)$$

Let  $\mathcal{H}_j, j = 1, 2$ , be the subspaces of  $\mathcal{H}$  defined by

$$\begin{aligned} \mathcal{H}_1 &= \{F \in \mathcal{H} \mid F = (u, v, 0, 0)\}, \\ \mathcal{H}_2 &= \{G \in \mathcal{H} \mid G = (0, 0, y, z)\}. \end{aligned}$$

We state the following crucial result, whose proof is postponed.

**Lemma 4.3.3.** 1.  $\sigma(\mathcal{A}_1) = \{\widetilde{\mu}_k = ik\pi + b, k \in \mathbb{Z}\}$ , (where  $b$  is given in (4.2.25)) is the set of eigenvalues of  $\mathcal{A}_1 = \mathcal{A}_0 \mid \mathcal{H}_1$ , they are simple and the corresponding eigenvectors  $\{\widetilde{\phi}_k\}_{k \in \mathbb{Z}^*}$  are given by

$$\widetilde{\phi}_k = (\widetilde{\mu}_k^{-1} \sinh(\widetilde{\mu}_k x), \sinh(\widetilde{\mu}_k x), 0, 0), \quad \forall k \in \mathbb{Z},$$

and form an Riesz basis of  $\mathcal{H}_1$ .

2.  $\sigma(\mathcal{A}_2) = \{\widetilde{\lambda}_k = ik\pi\}_{k \in \mathbb{Z}^*}$  is the set of eigenvalues of  $\mathcal{A}_2 = \mathcal{A}_0 \mid \mathcal{H}_2$  they are simple and the corresponding eigenvectors  $\{\widetilde{\psi}_k\}_{k \in \mathbb{Z}^*}$  are given by

$$\widetilde{\psi}_k = (0, 0, \widetilde{\lambda}_k^{-1} \sinh(\widetilde{\lambda}_k x), \sinh(\widetilde{\lambda}_k x)), \quad \forall k \in \mathbb{Z}^*,$$

and form an orthogonal basis of  $\mathcal{H}_2$ .

We are now ready to prove Proposition 4.3.2.

*Proof of Proposition 4.3.2.* We remark that  $E_{1,k}^0 = \widetilde{\phi}_k$  and  $E_{2,k}^0 = \frac{1}{i} \widetilde{\psi}_k$ . Thus, from Lemma 4.3.3, a direct consequence of the direct decomposition  $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$  leads to the completion of the proof.  $\square$

**Theorem 4.3.4.** *The set of generalized eigenvectors associated to  $\sigma(\mathcal{A})$  forms a Riesz basis of  $\mathcal{H}$ .*

*Proof.* From (4.3.1)-(4.3.2), we have

$$\|E_{1,k} - E_{1,k}^0\|_{\mathcal{H}} = O\left(\frac{1}{k}\right) \quad \text{and} \quad \|E_{2,k} - E_{2,k}^0\|_{\mathcal{H}} = O\left(\frac{1}{k}\right).$$

We conclude the desired aim by Theorem 4.3.1. □

We are now ready to prove the optimal decay rate of the energy of system (4.1.1).

**Proof of Theorem 4.1.1.** First, using (4.2.26) we have  $Re(\lambda_k) \sim \frac{-(\gamma\alpha^2)}{4\pi^2k^2}$  and  $|Im\lambda_k| \sim k$ . Next, from Proposition 4.3.4, we know that the set of the generalized eigenvectors associated to  $\sigma(\mathcal{A})$  forms a Riesz basis in  $\mathcal{H}$ . Then by applying Theorem 2.1 in [90], we deduce the optimal polynomial energy decay rate (4.1.8) for smooth initial data. □

Now, in order to prove Lemma 4.3.3, we recall the following Bari's criterion (see Theorem 2.1 of Chapter 6 in [56].)

**Theorem 4.3.5.**  *$\{\phi_n\}$  is a Riesz basis of a Hilbert space  $\mathcal{H}$  if and only if  $\{\phi_n\}$  is complete in  $\mathcal{H}$ , and there corresponds to it a complete biorthogonal sequence  $\{\psi_n\}$  such that for any  $f \in \mathcal{H}$ , one has*

$$\sum_n |\langle \phi_n, f \rangle|^2 < \infty, \quad \sum_n |\langle \psi_n, f \rangle|^2 < \infty. \tag{4.3.5}$$

We are now ready to prove Lemma 4.3.3.

*Proof of Lemma 4.3.3.* In order to prove Point (1), we proceed by steps.

**Step 1.** We begin first by proving that  $\{\widetilde{\phi}_k\}_k$  is complete in  $\mathcal{H}_1$ . It suffices to show that any orthogonal element of  $\mathcal{H}_1$  to all the  $\widetilde{\phi}_k$  is zero. Hence, let  $(f, g)^\top$  be such that  $\langle (f, g)^\top, \widetilde{\phi}_k \rangle_{\mathcal{H}} = 0$  for all  $k \in \mathbb{Z}$ . Then we get

$$\begin{aligned} 0 &= 2 \int_0^1 (\overline{f_x} \cosh(\widetilde{\mu}_k x) + \overline{g} \sinh(\widetilde{\mu}_k x)) dx \\ &= \int_0^1 ((\overline{f_x + g}) e^{bx} e^{ik\pi x} + (\overline{f_x - g}) e^{-bx} e^{-ik\pi x}) dx, \quad \forall k \in \mathbb{Z}. \end{aligned} \tag{4.3.6}$$

In particular, for  $k = 0$ , we have

$$\int_0^1 ((\overline{f_x + g}) e^{bx} + (\overline{f_x - g}) e^{-bx}) dx = 0. \tag{4.3.7}$$

Moreover, for  $k < 0$ , we write  $k = -k'$  with  $k' \in \mathbb{N}^*$  to obtain

$$\int_0^1 ((\overline{f_x + g}) e^{bx} e^{-ik'\pi x} + (\overline{f_x - g}) e^{-bx} e^{ik'\pi x}) dx = 0, \quad \forall k' \in \mathbb{N}^*. \tag{4.3.8}$$

Adding (4.3.6) for  $k = k' > 0$  with (4.3.8) yields

$$\int_0^1 \overline{h(x)} \left( \frac{e^{ik\pi x} + e^{-ik\pi x}}{2} \right) dx = 0, \quad \forall k \in \mathbb{N}^*,$$

where

$$h(x) = (f_x + g)(x)e^{\overline{bx}} + (f_x - g)(x)e^{-\overline{bx}}.$$

Since  $\{\cos(k\pi x)\}_{k \in \mathbb{N}}$  is a basis in  $L^2(0, 1)$ , we get

$$h = 0. \tag{4.3.9}$$

Subtracting (4.3.6) from (4.3.8), we get

$$\int_0^1 \overline{K(x)} \left( \frac{e^{ik\pi x} - e^{-ik\pi x}}{2} \right) dx = 0, \quad \forall k \in \mathbb{N}^*,$$

where

$$K(x) = (f_x + g)(x)e^{\overline{bx}} - (f_x - g)(x)e^{-\overline{bx}}.$$

Since  $\{\sin(k\pi x)\}_{k \in \mathbb{N}^*}$  is a basis in  $L^2(0, 1)$ , we get

$$K = 0. \tag{4.3.10}$$

(4.3.9) and (4.3.10) imply that  $f_x = g = 0$  and so  $f = 0$  since  $f(0) = 0$ .

**Step 2.** We search for a sequence  $\{\psi_k\}_{k \in \mathbb{Z}}$  biorthogonal to  $\{\widetilde{\phi}_k\}_{k \in \mathbb{Z}}$ . Here we choose

$$\{\psi_k\}_{k \in \mathbb{Z}} = \left\{ \left( \frac{1}{\widetilde{\mu}_k} \sinh(\overline{\widetilde{\mu}_k \cdot}), -\sinh(\overline{\widetilde{\mu}_k \cdot}) \right) \right\},$$

where  $\overline{\widetilde{\mu}_k}$  is the conjugate of  $\widetilde{\mu}_k$ . The same arguments as before show that this set is complete. Indeed, for  $k \in \mathbb{Z}$ ,  $\psi_k$  is an eigenvector of the adjoint of  $\mathcal{A}_1$ .

**Step 3.** The set  $\{\psi_k\}_{k \in \mathbb{Z}}$  is biorthogonal to  $\{\widetilde{\phi}_k\}_{k \in \mathbb{Z}}$ .

Indeed, by definition we have

$$\begin{aligned} \langle \widetilde{\phi}_k, \psi_l \rangle_{\mathcal{H}_1} &= \int_0^1 (\cosh(\widetilde{\mu}_k x) \cosh(\widetilde{\mu}_l x) - \sinh(\widetilde{\mu}_k x) \sinh(\widetilde{\mu}_l x)) dx \\ &= \int_0^1 \cosh((\widetilde{\mu}_k - \widetilde{\mu}_l)x) dx \\ &= \int_0^1 \cos((k - l)\pi x) dx = \delta_{kl}. \end{aligned}$$

**Step 4.** Finally, in order to apply Bari's theorem, it remains to prove (4.3.5). Let  $(f, g)^\top \in \mathcal{H}_1$  and consider the following sum

$$\sum_{k \in \mathbb{Z}} \left| \langle (f, g)^\top, \tilde{\phi}_k \rangle_{\mathcal{H}_1} \right|^2 \lesssim \sum_{k \in \mathbb{Z}} \left| (f_x, \cosh(\tilde{\mu}_k \cdot))_{(0,1)} \right|^2 + \sum_{k \in \mathbb{Z}} \left| (g, \sinh(\tilde{\mu}_k \cdot))_{(0,1)} \right|^2. \quad (4.3.11)$$

By the definition of  $\tilde{\mu}_k$  given in Point (1) of Lemma 4.3.3, and by Parseval's identity,

$$\begin{aligned} \sum_{k \in \mathbb{Z}} \left| (f_x, \cosh(\tilde{\mu}_k \cdot))_{(0,1)} \right|^2 &\lesssim \sum_{k \in \mathbb{Z}} \left| (f_x e^{\tilde{\mu}_k \cdot}, e^{ik\pi \cdot})_{(0,1)} \right|^2 + \sum_{k \in \mathbb{Z}} \left| (f_x e^{-\tilde{\mu}_k \cdot}, e^{-ik\pi \cdot})_{(0,1)} \right|^2 \\ &\leq \left( \|f_x e^{\tilde{\mu}_k \cdot}\|_{(0,1)}^2 + \|f_x e^{-\tilde{\mu}_k \cdot}\|_{(0,1)}^2 \right) \\ &\lesssim \|f_x\|_{(0,1)}^2, \end{aligned} \quad (4.3.12)$$

and

$$\begin{aligned} \sum_{k \in \mathbb{Z}} \left| (g, \sinh(\tilde{\mu}_k \cdot))_{(0,1)} \right|^2 &\lesssim \sum_{k \in \mathbb{Z}} \left| (g e^{\tilde{\mu}_k \cdot}, e^{ik\pi \cdot})_{(0,1)} \right|^2 + \sum_{k \in \mathbb{Z}} \left| (g e^{-\tilde{\mu}_k \cdot}, e^{-ik\pi \cdot})_{(0,1)} \right|^2 \\ &\leq \left( \|g e^{\tilde{\mu}_k \cdot}\|_{(0,1)}^2 + \|g e^{-\tilde{\mu}_k \cdot}\|_{(0,1)}^2 \right) \\ &\lesssim \|g\|_{(0,1)}^2. \end{aligned} \quad (4.3.13)$$

(4.3.12) and (4.3.13) imply that the right hand side of (4.3.11) is finite. Similarly, we prove that

$$\sum_{k \in \mathbb{Z}} \left| \langle (f, g)^\top, \psi_k \rangle_{\mathcal{H}_1} \right|^2 < \infty.$$

In conclusion, by Theorem 4.3.5, the family  $\{\tilde{\phi}_k\}_{k \in \mathbb{Z}}$  forms a Riesz basis of  $\mathcal{H}_1$ .

Moreover, to prove Point (2),  $\{\tilde{\psi}_k\}_{k \in \mathbb{Z}^*}$  is an orthogonal basis of  $\mathcal{H}_2$ , since  $\mathcal{A}_2$  is a skew-adjoint operator. The proof is thus complete.  $\square$

## 4.4 Exact controllability with Neumann boundary control

In this section, we study the exact controllability of the following system

$$\begin{cases} u_{tt} - u_{xx} + \alpha y = 0 & \text{on } (0, 1) \times (0, T), \\ y_{tt} - y_{xx} + \alpha u = 0 & \text{on } (0, 1) \times (0, T), \\ u(0, t) = y(0, t) = y(1, t) = 0 & \text{for } t \in (0, T), \\ u_x(1, t) = v(t) & \text{for } t \in (0, T). \end{cases} \quad (4.4.1)$$

We will use the spectral approach to investigate how the modes of the second equation are influenced by the modes of the first equation. We denote by  $\alpha$  the coupling parameter, and  $v$  the control acted only on the right boundary of the first equation. The second equation is partially controlled via the coupling of the two waves.

### 4.4.1 Observability and exact controllability in spectral spaces

The aim of this subsection is to establish inverse and direct observability inequalities by the spectral approach. We consider the following associated homogeneous system

$$\begin{cases} \phi_{tt} - \phi_{xx} + \alpha\psi = 0 & \text{on } (0, 1) \times (0, T), \\ \psi_{tt} - \psi_{xx} + \alpha\phi = 0 & \text{on } (0, 1) \times (0, T), \\ \phi(0) = \phi_x(1) = 0 & \text{for } t \in (0, T), \\ \psi(0) = \psi(1) = 0 & \text{for } t \in (0, T). \end{cases} \quad (4.4.2)$$

Let us recall the energy space  $\mathcal{H}$  defined in (4.2.2) endowed for all  $U = (u, v, y, z), \tilde{U} = (\tilde{u}, \tilde{v}, \tilde{y}, \tilde{z}) \in \mathcal{H}$  with the inner product (4.2.3). Now we define a linear unbounded operator  $\mathcal{A}_0 : D(\mathcal{A}_0) \mapsto \mathcal{H}$  by

$$D(\mathcal{A}_0) = \left\{ \Phi = (\phi, \tilde{\phi}, \psi, \tilde{\psi}) \in \mathcal{H} \mid \phi, \psi \in H^2(0, 1), \tilde{\phi} \in V, \tilde{\psi} \in H_0^1(0, 1), \phi_x(1) = 0 \right\} \quad (4.4.3)$$

and

$$\mathcal{A}_0\Phi = (\tilde{\phi}, \phi_{xx} - \alpha\psi, \tilde{\psi}, \psi_{xx} - \alpha\phi). \quad (4.4.4)$$

Then setting  $\Phi = (\phi, \phi_t, \psi, \psi_t)$  a regular solution of (4.4.2), we rewrite it into an evolution equation

$$\begin{cases} \Phi_t(t) = \mathcal{A}_0\Phi(t), \\ \Phi(0) = \Phi_0 \in \mathcal{H}. \end{cases} \quad (4.4.5)$$

Since  $\mathcal{A}_0$  is a skew-adjoint operator with a compact resolvent, then, by a corollary of the Lumer Philips's Theorem [94],  $\mathcal{A}_0$  is the infinitesimal generator of a  $C_0$  semi-group of contractions  $e^{t\mathcal{A}_0}$  on  $\mathcal{H}$ .

Now, let us consider the corresponding eigenvalue problem, where  $\tilde{\lambda}$  denotes its associated eigenvalue

$$\begin{cases} \tilde{\lambda}^2\phi - \phi_{xx} + \alpha\psi = 0, \\ \tilde{\lambda}^2\psi - \psi_{xx} + \alpha\phi = 0, \\ \phi(0) = \phi_x(1) = 0, \\ \psi(0) = \psi(1) = 0. \end{cases} \quad (4.4.6)$$

Taking  $\gamma = 0$  in the spectral analysis of the non-conservative operator  $\mathcal{A}$ , (see Section 4.2.1) we can deduce the spectral analysis of the conservative operator  $\mathcal{A}_0$ . From Proposition 4.2.5 and Proposition 4.2.6, we have the following Propositions.

**Proposition 4.4.1** (Spectrum of  $\mathcal{A}_0$ ). *There exists  $k_0 \in \mathbb{N}^*$  sufficiently large enough such that the spectrum  $\sigma(\mathcal{A}_0)$  of  $\mathcal{A}_0$  is given by*

$$\sigma(\mathcal{A}_0) = \tilde{\sigma}_0 \cup \tilde{\sigma}_1,$$

where

$$\tilde{\sigma}_0 = \{\tilde{\kappa}_{1,j}\}_{j \in \tilde{\mathcal{J}}} \cup \{\tilde{\kappa}_{2,j}\}_{j \in \tilde{\mathcal{J}}}, \quad \tilde{\sigma}_1 = \{\tilde{\lambda}_{1,k}\}_{k \in \mathbb{Z}^*, |k| \geq k_0} \cup \{\tilde{\lambda}_{2,k}\}_{k \in \mathbb{Z}^*, |k| \geq k_0}, \quad \tilde{\sigma}_0 \cap \tilde{\sigma}_1 = \emptyset,$$

where  $\tilde{J}$  is a finite set. Moreover,  $\tilde{\lambda}_{j,k}$ , for  $j = 1, 2$  satisfy the following asymptotic behaviors:

$$\tilde{\lambda}_{1,k} = ik\pi + i\frac{\pi}{2} + \frac{O(\alpha)}{k^2} \quad (4.4.7)$$

and

$$\tilde{\lambda}_{2,k} = ik\pi + \frac{O(\alpha)}{k^3}. \quad (4.4.8)$$

*Proof.* Following the proof of Proposition 4.2.5, and taking  $\gamma = 0$  in (4.2.37), as well as in the expressions (4.2.33)-(4.2.34), we have the following asymptotic expansion

$$\tilde{f}(\tilde{\lambda}) = \tilde{f}_0(\tilde{\lambda}) + \frac{\tilde{f}_2(\tilde{\lambda})}{\tilde{\lambda}^3} + \frac{O(\alpha^2)}{\tilde{\lambda}^4}, \quad (4.4.9)$$

where

$$\tilde{f}_0(\tilde{\lambda}) = 2 \sinh \tilde{\lambda} \cosh \tilde{\lambda}, \quad \tilde{f}_2(\tilde{\lambda}) = -\frac{\alpha^2(\cosh(2\tilde{\lambda}) + 2)}{4}. \quad (4.4.10)$$

Next, we seek to determine the roots of  $\tilde{f}(\tilde{\lambda})$ . It is easy to check that the roots of  $\tilde{f}_0$  are simple and are given by

$$\tilde{\alpha}_{1,k} = ik\pi + i\frac{\pi}{2}, \quad \tilde{\alpha}_{2,k} = ik\pi. \quad (4.4.11)$$

Similarly, thanks to Rouché's theorem, there exists  $k_0 \in \mathbb{N}^*$  large enough, such that  $\forall |k| \geq k_0$ , the large eigenvalues of  $\mathcal{A}_0$  (denoted by  $\tilde{\lambda}_{1,k}, \tilde{\lambda}_{2,k}$ ) are simple and close respectively to  $\tilde{\alpha}_{1,k}, \tilde{\alpha}_{2,k}$  i.e.

$$\tilde{\lambda}_{1,k} = \tilde{\alpha}_{1,k} + o(1), \quad \tilde{\lambda}_{2,k} = \tilde{\alpha}_{2,k} + o(1), \quad \text{as } |k| \rightarrow \infty.$$

Equivalently, we can write

$$\tilde{\lambda}_{1,k} = \tilde{\alpha}_{1,k} + \tilde{\zeta}_{1,k}, \quad \tilde{\lambda}_{2,k} = \tilde{\alpha}_{2,k} + \tilde{\zeta}_{2,k}, \quad \lim_{|k| \rightarrow \infty} \tilde{\zeta}_{i,k} = 0, \quad i = 1, 2. \quad (4.4.12)$$

Following the proof of Proposition 4.2.5, and after several computations, we obtain

$$\tilde{\zeta}_{1,k} = \frac{O(\alpha)}{k^2}, \quad \tilde{\zeta}_{2,k} = \frac{O(\alpha)}{k^3}. \quad (4.4.13)$$

Finally, substituting (4.4.11) and (4.4.13) in (4.4.12), we directly get (4.4.7)-(4.4.8).

The proof is thus complete.  $\square$

**Proposition 4.4.2.** *The eigenfunctions of the eigenvalue problem given by (4.4.6) have the following asymptotic expansions*

$$\phi_{1,k}(x) = \frac{\sin\left(\frac{(2k+1)\pi x}{2}\right)}{k\pi} + \frac{O(\alpha^2)}{k^2}, \quad \psi_{1,k}(x) = -\frac{\alpha x \cos\left(\frac{(2k+1)\pi x}{2}\right)}{2k^2\pi^2} + \frac{O(\alpha^2)}{k^3} \quad (4.4.14)$$

and

$$\phi_{2,k}(x) = \frac{i\alpha x \cos(k\pi x)}{2k^2\pi^2} + \frac{O(\alpha^2)}{k^3}, \quad \psi_{2,k}(x) = \frac{\sin(k\pi x)}{ik\pi} + \frac{O(\alpha^2)}{k^3}. \quad (4.4.15)$$

*Proof.* First, we determine the corresponding eigenfunctions of  $\tilde{\lambda}_{1,k}$ . Let

$$C = C_{1,k} = -\frac{1}{2(-1)^k k\pi}.$$

Inserting (4.4.7) in (4.2.14), we get

$$r_{1,k} = ik\pi + i\frac{\pi}{2} - i\frac{\alpha}{2k} + \frac{O(\alpha^2)}{k^2}, \quad r_{3,k} = ik\pi + i\frac{\pi}{2} + i\frac{\alpha}{2k} + \frac{O(\alpha^2)}{k^2}. \quad (4.4.16)$$

Then it follows that

$$\begin{cases} \sinh(r_{1,k}x) = i \sin\left(\frac{(2k+1)x}{2}\right) - \frac{i\alpha x \cos\left(\frac{(2k+1)x}{2}\right)}{2k\pi} + \frac{O(\alpha^2)}{k^2}, \\ \sinh(r_{3,k}x) = i \sin\left(\frac{(2k+1)x}{2}\right) + \frac{i\alpha x \cos\left(\frac{(2k+1)x}{2}\right)}{2k\pi} + \frac{O(\alpha^2)}{k^2} \end{cases} \quad (4.4.17)$$

and

$$\left\{ \sinh(r_{1,k}) = \sinh(r_{3,k}) = i(-1)^k \left(1 - \frac{\alpha^2 x^2}{8k^2 \pi^2} + \frac{O(\alpha^2)}{k^4}\right) \right\}. \quad (4.4.18)$$

Inserting (4.4.17)-(4.4.18) into (4.2.15) and (4.2.16), we get (4.4.14).

Next, we look to determine the eigenfunctions of  $\tilde{\lambda}_{2,k}$ . Let

$$C = C_{2,k} = -\frac{1}{i\alpha(-1)^k}.$$

Inserting (4.4.8) in (4.2.14), we get

$$r_{1,k} = ik\pi + \frac{\alpha}{2ik\pi} + \frac{O(\alpha^2)}{k^3}, \quad r_{3,k} = ik\pi - \frac{\alpha}{2ik\pi} + \frac{O(\alpha^2)}{k^3}. \quad (4.4.19)$$

Then it follows that

$$\begin{cases} \sinh(r_{1,k}) = (-1)^k \frac{\alpha}{2ik\pi} + \frac{O(\alpha)}{k^3}, \\ \sinh(r_{3,k}) = -(-1)^k \frac{\alpha}{2ik\pi} + \frac{O(\alpha)}{k^3}, \end{cases} \quad (4.4.20)$$

and

$$\begin{cases} \sinh(r_{1,k}x) = i \sin(k\pi x) + \frac{\alpha x \cos(k\pi x)}{2ik\pi} - \frac{i\alpha^2 x^2 \sin(k\pi x)}{8k^2 \pi^2} + \frac{O(\alpha)}{k^3}, \\ \sinh(r_{3,k}x) = i \sin(k\pi x) - \frac{\alpha x \cos(k\pi x)}{2ik\pi} - \frac{i\alpha^2 x^2 \sin(k\pi x)}{8k^2 \pi^2} + \frac{O(\alpha)}{k^3}. \end{cases} \quad (4.4.21)$$

Finally, inserting (4.4.20)-(4.4.21) into (4.2.15) and (4.2.16), we get (4.4.15).

The proof is thus complete.  $\square$

*Remark 4.4.3.* If  $\lambda^4 \neq \alpha^2$ , it is easy to check that all the roots of (4.2.20) are simple and different from zero. Then we set the eigenfunctions of the conservative operator  $\mathcal{A}_0$  as

$$\begin{cases} \tilde{E}_{1,k} = (\phi_{1,k}, \tilde{\lambda}_{1,k}\phi_{1,k}, \psi_{1,k}, \tilde{\lambda}_{1,k}\psi_{1,k}), \\ \tilde{E}_{2,k} = (\phi_{2,k}, \tilde{\lambda}_{2,k}\phi_{2,k}, \psi_{2,k}, \tilde{\lambda}_{2,k}\psi_{2,k}). \end{cases} \quad (4.4.22)$$

Moreover, using the asymptotic expansions (4.4.7)-(4.4.8), and (4.4.14)-(4.4.15), and since  $\mathcal{A}_0$  is a skew-adjoint operator we can prove that  $\tilde{E}_{1,k}, \tilde{E}_{2,k}, k \in \mathbb{Z}^*$  constitute a Riesz basis in the energy space  $\mathcal{H}$ .

**Proposition 4.4.4.** *Let  $\alpha \neq 0$  be a real number small enough. Then there exists a constant  $\tilde{\gamma} > 0$  depending only on  $\alpha$  such that the two branches of eigenvalues of  $\mathcal{A}_0$  satisfy a uniform gap condition*

$$\tilde{\gamma} := \inf_{m,n} |\lambda_{1,m} - \lambda_{2,n}| > 0. \quad (4.4.23)$$

Moreover, the eigenvalues of the same branch satisfy a uniform gap condition.

*Proof.* Using the asymptotic expansions (4.4.7)-(4.4.8), we have

$$|\tilde{\lambda}_{1,m} - \tilde{\lambda}_{2,n}| = \left| \pi(m-n) + \frac{\pi}{2} + \frac{O(\alpha^2)}{m^2} + \frac{O(\alpha^2)}{n^3} \right|.$$

We distinguish cases:

**Case (1)**  $m = n$ . We have

$$|\tilde{\lambda}_{1,m} - \tilde{\lambda}_{2,n}| = \frac{\pi}{2} + \frac{O(\alpha^2)}{m^2},$$

we easily obtain the uniform gap condition (4.4.23).

**Case (2)**  $m \neq n$ . We have

$$|\tilde{\lambda}_{1,m} - \tilde{\lambda}_{2,n}| = \left| \pi(m-n) + \frac{\pi}{2} + \frac{O(\alpha^3)}{m^2} + \frac{O(\alpha^2)}{n^2} \right|. \quad (4.4.24)$$

Now let us consider the leading term in the previous identity. For all  $m, n \in \mathbb{Z}^*$ , we have  $\pi|m-n| + \frac{\pi}{2} \geq \frac{\pi}{2}$ . Similarly, using (4.4.7)-(4.4.8)

$$|\tilde{\lambda}_{j,m} - \tilde{\lambda}_{j,n}| = |\pi(m-n)| + \frac{O(|\alpha|)}{m^2} + \frac{O(|\alpha|)}{n^3}, \quad \text{for } j = 1, 2. \quad (4.4.25)$$

It follows that

$$\inf_{m \neq n} |\lambda_{j,m} - \lambda_{j,n}| \geq \pi.$$

Hence, one can deduce the uniform gap for eigenvalues laying on the same branch. The proof is thus complete.  $\square$

The observation is on the first components of the eigenfunctions  $\tilde{E}_{1,k}, \tilde{E}_{2,k}$  defined in (4.4.22). From (4.4.14)-(4.4.15), we see that

$$\phi_{1,k}(1) = O\left(\frac{1}{k}\right), \quad \phi_{2,k}(1) = O\left(\frac{1}{k^2}\right). \quad (4.4.26)$$

Due to the transmission between the nodes of the two equations, the inverse observability inequalities are not true in the energy space  $\mathcal{H}$ . That is why, we define the following weighted spectral space

$$\mathcal{D} = \left\{ (\phi_0, \phi_1, \psi_0, \psi_1) = \sum_{k \neq 0} (\alpha_{1,k} k \tilde{E}_{1,k} + \alpha_{2,k} k^2 \tilde{E}_{2,k}) \right\}.$$

Since system  $\tilde{E}_{1,k}, \tilde{E}_{2,k}$  is a Riesz basis in the energy space  $\mathcal{H}$ , the space  $\mathcal{D}$  is obviously a Hilbert space equipped with the norm

$$\sum_{k \neq 0} (|\alpha_{1,k}|^2 + |\alpha_{2,k}|^2).$$

We are now ready to prove our second main result.

**Proof of Theorem 4.1.2.** Assume first that  $\lambda^4 \neq \alpha^2$ . In this case, all eigenvalues are different from zero and are all algebraically simple. Given any initial data such as

$$(\phi_0, \phi_1, \psi_0, \psi_1) = \sum_{k \neq 0} (\alpha_{1,k} k \tilde{E}_{1,k} + \alpha_{2,k} k^2 \tilde{E}_{2,k}) \in \mathcal{D},$$

and using the Riesz property the solution of (4.4.5) can be written as

$$(\phi(x, t), \phi_t(x, t), \psi(x, t), \psi_t(x, t)) = \sum_{k \neq 0} (\alpha_{1,k} k \tilde{E}_{1,k} e^{\tilde{\lambda}_{1,k} t} + \alpha_{2,k} k^2 \tilde{E}_{2,k} e^{\tilde{\lambda}_{2,k} t}).$$

It follows that

$$\phi(1, t) = \sum_{n \neq 0} (\alpha_{1,k} k \phi_{1,k}(1) e^{\tilde{\lambda}_{1,k} t} + \alpha_{2,k} k^2 \phi_{2,k}(1) e^{\tilde{\lambda}_{2,k} t}). \quad (4.4.27)$$

We recall that the two branches of eigenvalues  $(\tilde{\lambda}_{1,k})_{k \neq 0}, (\tilde{\lambda}_{2,k})_{k \neq 0}$  satisfy an uniform gap condition. Back to (4.4.26), we can rewrite (4.4.27) as

$$\phi(1, t) \sim \sum_{k \neq 0} (\alpha_{1,k} e^{\tilde{\lambda}_{1,k} t} + \alpha_{2,k} e^{\tilde{\lambda}_{2,k} t}).$$

Following a generalization of Ingham's theorem (see [73], Theorem 9.1), the sequence  $(e^{ik\pi t})_{n \neq 0}$  forms a Riesz basis in  $L^2(0, T)$  provided that

$$T > 2\pi/\tilde{\gamma}$$

where  $\tilde{\gamma}$  is the uniform gap between the eigenvalues (4.4.23). It follows that

$$\int_0^T |\phi(1, t)|^2 dt \sim \sum_{k \neq 0} (|\alpha_{1,k}|^2 + |\alpha_{2,k}|^2).$$

This yield inequalities (4.1.9) and (4.1.10). The proof is now complete.  $\square$

Using the Hilbert Uniqueness Method ([79]), we have the following exact controllability result.

**Theorem 4.4.5.** *Let  $\alpha \neq 0$  be a real number small enough. Assume that  $T > 4$ , then system (4.4.1) is exactly controllable. More precisely, for any initial data  $(u_0, u_1, y_0, y_1) \in \mathcal{D}'$ , there exists a control function  $v \in L^2(0, T)$  such that the solution  $(u, u_t, y, y_t) \in \mathcal{D}'$ , and the controlled system (4.4.1) satisfies the null final conditions*

$$u(x, T) = u_t(x, T) = y(x, T) = y_t(x, T) = 0.$$

It is natural to think about the characterization of the spectral space  $\mathcal{D}$ . For this aim, we have this new subsection.

#### 4.4.2 Observability and exact controllability in usual spaces

The weighted spectral space  $\mathcal{D}$  is defined by means of the eigenfunctions  $(\tilde{E}_{1,k})_{k \neq 0}$  and  $(\tilde{E}_{2,k})_{k \neq 0}$  with weights. So, the four exponents  $(\phi_0, \phi_1, \psi_0, \psi_1)$  are a priori involved together. In order to get the observability or exact controllability in usual energy spaces, we have to separate the components  $(\phi_0, \phi_1, \psi_0, \psi_1)$ . In order to do so, we will use the Theorem below whose proof is established in [89].

**Theorem 4.4.6.** *Let  $(x_k)_{k \neq 0}$  and  $(y_k)_{k \neq 0}$  be Riesz basis of Hilbert spaces  $X$  and  $Y$  respectively, and  $(f_k)_{k \neq 0}$  and  $(g_k)_{k \neq 0}$  be Bessel sequences of  $X$  and  $Y$  with suitably small bounds respectively. Define*

$$D = \left\{ (x, y) = \alpha_k(x_k, g_k) + \beta_k(f_k, y_k) : \sum_{k \neq 0} (|\alpha_k|^2 + |\beta_k|^2) < \infty \right\}.$$

Then we have  $D = X \times Y$ .

Now using the asymptotic expansions (4.4.7)-(4.4.8) in (4.4.22), one can write

$$\tilde{E}_{1,k} = (x_k, g_k), \quad \tilde{E}_{2,k} = (f_k, y_k)$$

with

$$x_k = \left( \frac{\sin\left(\frac{\pi(2k+1)x}{2}\right)}{k\pi}, i \sin\left(\frac{\pi(2k+1)x}{2}\right) \right), \quad (4.4.28)$$

$$g_k = \left( -\frac{\alpha x \cos\left(\frac{\pi(2k+1)x}{2}\right)}{2k^2\pi^2}, -i\frac{\alpha x \cos\left(\frac{\pi(2k+1)x}{2}\right)}{2k\pi} \right), \quad (4.4.29)$$

$$f_k = \left( \frac{i\alpha x \cos(k\pi x)}{2k^2\pi^2}, \frac{\alpha x \cos(k\pi x)}{2k\pi} \right), \quad (4.4.30)$$

and

$$y_k = \left( \frac{\sin(k\pi x)}{ik\pi}, \sin(k\pi x) \right). \quad (4.4.31)$$

Next for any  $s \geq 0$ , we define the space

$$X_s = \left\{ (\hat{\phi}, \hat{\psi}) = \sum_{k \neq 0} \alpha_k k^s x_k \right\}, \quad \|(\hat{\phi}, \hat{\psi})\|_{X_s}^2 = \sum_{k \neq 0} |\alpha_k|^2. \quad (4.4.32)$$

Following Theorem 4.4.6, we can state the following result.

**Corollary 4.4.7.** *Let  $\alpha \neq 0$  be a real number small enough. We have the following identification*

$$\mathcal{D} = X_1 \times X_2. \quad (4.4.33)$$

*Proof.* From (4.4.28) and (4.4.31), we see that  $(kx_k)_{k \neq 0}$  and  $(k^2y_k)_{k \neq 0}$  are Riesz basis in  $X_1$  and  $X_2$  respectively. Moreover  $(kf_k)_{k \neq 0}$  and  $(k^2g_k)_{k \neq 0}$  are Bessel sequences in  $X_1$  and  $X_2$  respectively. Then (4.4.32) follows directly from Theorem 4.4.6.  $\square$

Furthermore for any  $s \geq 0$  we define the spaces

$$V_s = \left\{ f = \sum_{k > 0} \alpha_k \frac{\sin(k\pi x)}{k^s} \right\}, \quad \|f\|_{V_s}^2 = \sum_{n > 0} |\alpha_k|^2.$$

With the pivot space  $L^2(0, 1)$ , we have

$$X_s = V'_{s-1} \times V'_s. \quad (4.4.34)$$

It follows from (4.4.33)-(4.4.34) that

$$\mathcal{D} = V'_0 \times V'_1 \times V'_1 \times V'_2. \quad (4.4.35)$$

We can now characterize the space of observability. We state thus the following result.

**Theorem 4.4.8.** *Let  $\alpha$  be a real number small enough. Assume that*

$$T > 4.$$

*Then there exists a constant  $c_1 > 0$  such that the direct observability inequality holds*

$$\int_0^T |\phi(1, t)|^2 dt \leq c_1 \|(\phi_0, \phi_1, \psi_0, \psi_1)\|_{\mathcal{H}}^2,$$

for all solution  $\Phi = (\phi, \phi_t, \psi, \psi_t)$  solving the corresponding homogeneous Cauchy problem. Moreover, there exists a constant  $0 < c < c_1$  depending only on  $\alpha$  such that the following observability inequality holds true

$$c\|(\phi_0, \phi_1, \psi_0, \psi_1)\|_{V'_0 \times V'_1 \times V'_1 \times V'_2}^2 \leq \int_0^T |\phi(1, t)|^2 dt.$$

*Proof.* The proof of the inverse observability inequality is a direct consequence of Theorem 4.1.2 and the identification (4.4.35). The proof of the direct observability inequality is a direct consequence of the divided difference technique.  $\square$

We deduce that the observability space is

$$\mathcal{D} = L^2(0, 1) \times H^{-1}(0, 1) \times H^{-1}(0, 1) \times H^{-2}(0, 1).$$

Finally, using HUM method, we have the following controllability result.

**Theorem 4.4.9.** *Let  $\alpha \neq 0$  be a real number small enough. Assume that  $T > 4$ , then system (4.4.1) is exactly controllable. More precisely, for any initial data  $(u_0, u_1, y_0, y_1) \in V_1 \times V_0 \times V_2 \times V_1$ , there exists a control function  $v \in L^2(0, T)$  such that the solution  $(u, u_t, y, y_t) \in V_1 \times V_0 \times V_2 \times V_1$  and the controlled system (4.4.1) satisfies the null final conditions*

$$u(x, T) = u_t(x, T) = y(x, T) = y_t(x, T) = 0.$$

*Remark 4.4.10.* The control space is of type

$$H^1(0, 1) \times L^2(0, 1) \times H^2(0, 1) \times H^1(0, 1).$$



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