



HAL
open science

Controlled structures for partial differential equations

Marco Furlan

► **To cite this version:**

Marco Furlan. Controlled structures for partial differential equations. General Mathematics [math.GM]. Université Paris sciences et lettres, 2018. English. NNT : 2018PSLED008 . tel-01864398

HAL Id: tel-01864398

<https://theses.hal.science/tel-01864398v1>

Submitted on 29 Aug 2018

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

THÈSE DE DOCTORAT

de l'Université de recherche Paris Sciences et Lettres
PSL Research University

Préparée à l'Université Paris Dauphine

Controlled Structures for Partial Differential Equations

École doctorale n°543

ÉCOLE DOCTORALE DE DAUPHINE

Spécialité SCIENCES

Soutenue par **Marco FURLAN**
le 26 juin 2018

Dirigée par **Massimiliano GUBINELLI**

COMPOSITION DU JURY :

M. Massimiliano GUBINELLI
IAM Bonn - Université Paris-Dauphine
Directeur de thèse

M. Hendrik WEBER
University of Warwick
Rapporteur

M. Lorenzo ZAMBOTTI
LPSM - Sorbonne Université
Rapporteur

M. Ivan NOURDIN
Université du Luxembourg
Président du jury

M. Ismaël BAILLEUL
Université Rennes 1
Membre du jury

Mme Anne DE BOUARD
CMAP - École Polytechnique
Membre du jury

Mme Martina HOFMANOVA
Bielefeld University
Membre du jury

M. Cyril LABBÉ
Université Paris-Dauphine
Membre du jury

M. Jean-Christophe MOURRAT
ENS Paris
Membre du jury

Remerciements

First of all, I would like to thank my PhD supervisor, Massimiliano Gubinelli, for all the encouragement and illuminating suggestions he gave me during these four years, and for the many insightful discussions we had. Working together on different research subjects has been a fascinating and deeply inspiring experience. I am very thankful to Hendrik Weber and Lorenzo Zambotti for the interest they demonstrated in reviewing this thesis, and to Ismaël Bailleul, Anne de Bouard, Martina Hofmanová, Cyril Labbé, Jean-Christophe Mourrat and Ivan Nourdin for accepting to take part in the jury committee. I would like to thank in particular Jean-Christophe Mourrat for the fruitful and motivating collaboration started during my first year of PhD, that led to my first scientific result, and Ismaël Bailleul for the interesting discussions and the hospitality in Brest.

Je tiens à remercier aussi toutes les personnes qui m'ont soutenu directement ou indirectement pendant mon doctorat. Merci à tout·e·s les doctorant·e·s de Dauphine qui ont partagé une partie de leur parcours avec moi: Arnaud, Michaël, Qun, Laurent, William, Luca, Marco, Amine, Camille, Aude, Raphaël B., Raphaël D., Amal, Roméo, Maxime, Jean, Thibaut, Fang et tou·te·s les autres que je n'ai pas pu citer. Les sorties après le séminaire et les petites conversations au quotidien ont contribué à me rendre plus légère la tâche. Merci aussi aux ami·e·s parisien·ne·s, en particulier à Irène et Arnold qui ont accepté de corriger mes fautes d'orthographe dans l'introduction de cette thèse.

Merci surtout à Andreea, sans laquelle les derniers trois ans de ma vie auraient été très différents. Elle n'a jamais arrêté de croire en moi avec une certitude et une énergie incroyables. Merci encore.

Grazie agli amici che sono venuti a trovarmi a Parigi e a quelli che sono sempre pronti a organizzare un'uscita quando torno in Italia, grazie in particolare allo scaltro Stefano per le discussioni sulle crypto e la corsa, e a Codri che condivide a distanza le soddisfazioni e le frustrazioni di un PhD. Grazie ai fisici della Bicocca per le nostre reunion natalizie. Un'attività che mi ha permesso di distrarmi dal lavoro di dottorato è stata quella di partire per destinazioni improbabili. Ringrazio Luca e Luca per i viaggi degli ultimi anni, affrontati con lo spirito di chi prepara un grafo dell'itinerario prima di partire e smette di seguirlo alla prima tappa.

Infine, il ringraziamento più sentito va ai miei genitori che da sempre mi hanno motivato a seguire le mie aspirazioni, e mi hanno sostenuto incondizionatamente in tutte le mie scelte. Grazie di tutto.

Table of contents

Remerciements	3
Résumé	7
Introduction	20
Prerequisites	29
1 Besov spaces and paracontrolled calculus	31
1.1 Besov spaces	31
1.1.1 Characterization of Besov spaces via Littlewood-Paley theory	31
Littlewood-Paley decomposition	32
Distributions with compactly supported Fourier transform	34
Besov spaces: definition and first properties	36
1.1.2 Characterization of local Besov spaces via wavelets	39
1.1.2.1 Tightness and continuity criteria for random fields	49
1.1.3 Equivalence of norms	52
1.2 Paracontrolled calculus	54
1.2.1 Bony's paraproducts and parilinearization	54
1.2.2 Parabolic and time-weighted spaces	57
1.2.3 Schauder estimates	58
1.2.4 Time-smoothed paraproduct	60
1.2.5 Commutator estimates and parilinearization	60
2 Some stochastic calculus	63
2.1 White noise, Wiener chaos and Wick products	63
2.2 Estimation of finite chaos stochastic terms	65
2.2.1 Estimation of finite chaos diagrams	68
2.3 Introduction to Malliavin calculus	75
2.3.1 Basic definitions and notations	75
2.3.2 Partial chaos expansion	76
3 Brief introduction to FK percolation and the Ising model	81
3.1 The random cluster model	81
3.2 Relation with the 2-d Ising model	82
Results	85
4 Tightness of the 2-d Ising magnetization field at criticality	87
4.1 Absence of tightness in higher-order spaces	93
5 Quasi-linear paracontrolled PDEs	95

5.1	Introduction	95
5.2	Nonlinear paracontrolled calculus	96
5.2.1	Nonlinear paraproducts	96
5.2.2	Nonlinear commutator	99
5.2.3	Approximate paradifferential problem	102
5.3	Solution theory for quasi-linear equations	105
5.3.1	Paracontrolled structure	105
5.3.2	Local well-posedness	108
5.3.3	Renormalization	110
5.4	Nonlinear source terms	111
5.5	Full generality	113
6	Weak universality for a class of 3d stochastic reaction-diffusion models	117
6.1	Introduction	117
6.2	Analysis of the mesoscopic model	120
6.2.1	Paracontrolled structure	120
6.2.2	Identification of the limit	123
6.3	Convergence of the enhanced noise	126
6.3.1	An example of convergence	126
6.3.2	Main theorem and overview of the proof	128
6.3.3	Analysis of simple trees	131
	Time regularity of trees	132
6.3.4	Analysis of composite trees	133
6.3.4.1	Renormalisation of composite trees	134
6.3.4.2	Estimation of renormalised composite trees	140
6.4	Convergence of the remainder and a priori bounds	144
6.4.1	Boundedness of the remainder	144
6.4.2	A priori bounds on the solution	145
6.4.3	Convergence of the remainder	149
	Bibliography	151

Résumé

Le sujet principal de mon travail de thèse sont les équations aux dérivées partielles stochastiques (EDPS), en particulier les équations paraboliques dans 2 ou 3 dimensions avec termes de perturbation très irréguliers. La difficulté qui se présente dans l'étude de ce type d'équations est due au fait qu'on s'attend à ce que les solutions soient des distributions tempérées dans l'espace. En fait, on sait que même si des opérations linéaires comme la différentiation, la transformée de Fourier ou la multiplication par des fonctions suffisamment régulières sont bien posées sur l'espace des distributions de Schwartz, il n'y a pas de façon canonique de définir un produit de distributions, et donc les non-linéarités dans les EDPS traitées dans ce travail sont a-priori mal posées. Un sujet complémentaire de mon travail de thèse est l'étude de la tension du champ de magnétisation d'Ising en 2 dimensions, qui est fait en utilisant des techniques en commun avec le sujet principal.

L'analyse des EDPS dans cette thèse est basée sur l'ensemble de techniques développées par Gubinelli, Imkeller et Perkowski dans [GIP15], appelées *calcul paradifférentiel*, *calcul paracontrôlé* ou technique des *distributions contrôlées*. Leur travail peut être considéré comme une généralisation de l'idée de chemins rugueux contrôlés introduite par Gubinelli dans [Gub04], qui à son tour est une généralisation de la théorie des chemins rugueux de Lyons [Lyo98].

Dans la suite on donne un exemple d'application des techniques du calcul paradifférentiel introduites dans [GIP15] à une EDPS appelée Modèle d'Anderson Parabolique (MAP) :

$$\partial_t u - \Delta u = u \xi \quad u(t=0) = 0, \quad (1)$$

où ∂_t est la dérivée par rapport au temps et Δ le laplacien. Le bruit blanc ξ sur le tore 2-dimensionnel $\mathbb{T}^2 = (\mathbb{R}/2\pi\mathbb{Z})^2$ peut être considéré formellement comme le champ aléatoire gaussien centré qui a une covariance donnée par $\mathbb{E}[\xi(x)\xi(y)] = \delta(x-y)$. Les trajectoires de ξ sont presque sûrement des distributions tempérées dans l'espace $\mathcal{S}'(\mathbb{T}^2)$. Pour chaque trajectoire de ξ , soit ξ_ε une régularisation du bruit blanc par convolution avec une fonction lisse à support compact dans une boule de taille $\varepsilon > 0$ (de façon à ce que $\xi_\varepsilon \rightarrow \xi$ pour $\varepsilon \rightarrow 0$). Il n'est pas difficile d'obtenir des solutions u_ε pour chaque régularisation de l'équation (1) à l'échelle $\varepsilon > 0$ (par exemple en utilisant un théorème de point fixe), et si la famille $(u_\varepsilon)_{\varepsilon \in (0,1]}$ admet une limite u (dans un sens à préciser) on peut définir cette limite u comme la solution de (1).

Néanmoins, il n'est pas possible d'obtenir une solution "trajectoire par trajectoire" avec cette méthode, parce qu'on ne s'attend pas à ce qu'une solution u soit plus régulière que $\vartheta := \Delta^{-1}\xi$, et sa régularité n'est pas suffisante à rendre le produit $u\xi$ bien posé. On s'attend donc à ce que la suite $u_\varepsilon \xi_\varepsilon$ soit divergente.

Pour faire face à ce problème, les auteurs de [GIP15] utilisent d'abord des techniques d'analyse que l'on a rappelées dans le chapitre 1. Leur première observation est que le produit $u\xi$ peut se décomposer formellement en somme de paraproducts de Bony [Bon81] :

$$u\xi = u \prec \xi + u \succ \xi + u \circ \xi$$

où les paraproducts $u \prec \xi$ et $u \succ \xi$ sont bien posés pour chaque couple de distributions tempérées $u, \xi \in \mathcal{S}'$, et seulement le dernier terme (qu'on appelle produit résonant) est mal posé si la distribution u n'est pas assez régulière, ce qui est notre cas (une caractérisation exacte de la régularité de u en fonction de celle de ξ afin de définir $u \circ \xi$ est donnée dans le chapitre 1). La décomposition en paraproducts est basée sur une partition de l'unité de la transformée de Fourier de u et ξ : on peut en fait voir $u \prec \xi$ comme le produit de ξ avec les petites fréquences de u , ce qui donne une modulation d'amplitude de ξ avec un "filtrage passe-bas" de u . Le terme $u \circ \xi$ représente un phénomène de résonance dans lequel des fréquences comparables de u et ξ interagissent et causent une explosion. Le problème donc devient celui de contrôler la limite $u_\varepsilon \circ \xi_\varepsilon$ pour $\varepsilon \rightarrow 0$.

La deuxième observation de [GIP15], qui trouve son origine dans l'idée de chemin rugueux contrôlé et est appelée *Ansatz paracontrôlé*, est une conjecture sur la structure de la solution de (1). En particulier, on suppose que u ait la même régularité que $\vartheta = \Delta^{-1}\xi$ et donc qu'il doive pouvoir s'estimer par une modulation de ϑ avec une fonction plus régulière u' . On pourrait assumer donc $u = u' \prec \vartheta + u^\sharp$ avec u' et u^\sharp à déterminer en remplaçant u dans (1), et u^\sharp plus régulier que u , mais pour des raisons techniques on pose

$$u = u' \ll \vartheta + u^\sharp, \tag{2}$$

où le paraproduct modifié \ll coïncide avec \prec mais avec une régularisation dans la variable temporelle du premier terme u' , qui est nécessaire pour gérer la dérivée temporelle dans (1).

La troisième observation analytique apportée par les auteurs de [GIP15] est que grâce à l'Ansatz (2), le produit résonant $u_\varepsilon \circ \xi_\varepsilon$ peut se décomposer de façon à ce que tous les termes mal posés soient des produits d'intégrales du bruit blanc (on les appelle *enhanced noises* ou *bruits augmentés*), qui de conséquence ne contiennent pas des termes $u_\varepsilon, u'_\varepsilon, u^\sharp_\varepsilon$ à déterminer en résolvant une régularisation de (1). L'idée est donc de *definir* une limite par les termes de bruit augmenté (qui dépendent seulement de ξ_ε mais pas de la solution), et vu que tous les autres termes qu'on obtient sont bien définis (grâce aux estimations du chapitre 1) on peut de cette façon obtenir une limite $u_\varepsilon \rightarrow u$. Plus en détail, la décomposition (2) donne

$$u_\varepsilon \circ \xi_\varepsilon = (u'_\varepsilon \ll \vartheta_\varepsilon) \circ \xi_\varepsilon + u^\sharp_\varepsilon \circ \xi_\varepsilon$$

où on assume que u_ε^\sharp soit maintenant assez régulier dans la limite pour pouvoir borner $u_\varepsilon^\sharp \circ \xi_\varepsilon$ dans \mathcal{S}' uniformément sur $\varepsilon \in (0, 1]$ (parce que en remplaçant (2) dans (1) on peut obtenir que u_ε^\sharp résout une équation différentielle avec terme de droite plus régulier que celui de (1)). Grâce au fondamental *lemma de commutation* de [GIP15] on écrit

$$(u'_\varepsilon \ll \vartheta_\varepsilon) \circ \xi_\varepsilon = C(u'_\varepsilon, \vartheta_\varepsilon, \xi_\varepsilon) + u'_\varepsilon(\vartheta_\varepsilon \circ \xi_\varepsilon)$$

où le commutateur tri-linéaire $C(\cdot, \cdot, \cdot)$ a une extension bien définie sur \mathcal{S}' et le seul terme qui ne peut pas être contrôlé uniformément sur $\varepsilon \in (0, 1]$ avec des estimations d'analyse fonctionnelle est $\vartheta_\varepsilon \circ \xi_\varepsilon$. Si on postule que $\vartheta_\varepsilon \circ \xi_\varepsilon$ ait une limite bien définie dans \mathcal{S}' et assez régulière pour définir le produit $u'_\varepsilon(\vartheta_\varepsilon \circ \xi_\varepsilon)$ à la limite $\varepsilon \rightarrow 0$, on peut donner du sens au terme de droite de (1). On obtien donc, pour chaque $\varepsilon > 0$, le couple d'équations *paracontrôlées*

$$\begin{cases} u_\varepsilon = u'_\varepsilon \ll \vartheta + u_\varepsilon^\sharp \\ (\partial_t - \Delta)u_\varepsilon = u \prec \xi + u \succ \xi + C(u'_\varepsilon, \vartheta_\varepsilon, \xi_\varepsilon) + u'_\varepsilon(\vartheta_\varepsilon \circ \xi_\varepsilon) + u_\varepsilon^\sharp \circ \xi_\varepsilon \end{cases}$$

dont la famille de solutions $(u'_\varepsilon, u_\varepsilon^\sharp)$ a une limite $(u', u^\sharp) \in \mathcal{S}' \times \mathcal{S}'$ si on suppose que $\mathbb{X}_\varepsilon = (\xi_\varepsilon, \vartheta_\varepsilon \circ \xi_\varepsilon)$ converge vers une distribution tempérée \mathbb{X} qui soit assez régulière pour donner un sens aux équations ci-dessus.

Cela conclut la partie analytique de la méthode de distributions paracontrôlées de [GIP15]. Ce qui reste à montrer, avec des techniques de théorie des probabilités, est que le terme \mathbb{X}_ε , quand on lui applique une renormalisation, converge en probabilité vers une distribution tempérée qu'on appelle \mathbb{X} .

Pour résumer la situation : il n'existe pas d'espaces de Banach tels que la fonction de solution $\Phi: \xi_\varepsilon \rightarrow u_\varepsilon$ soit continue, à cause du fait que le produit de distributions n'est pas bien défini. On peut quand même séparer Φ dans 2 fonctions comme suit :

$$\xi_\varepsilon \xrightarrow{J} \mathbb{X}_\varepsilon \xrightarrow{\Psi} u_\varepsilon$$

où Ψ est la fonction de solution de l'équation paracontrôlée, qui associe à chaque bruit $(\xi_\varepsilon, \vartheta_\varepsilon \circ \xi_\varepsilon)$ la solution $(u'_\varepsilon, u_\varepsilon^\sharp)$, et qui est déterministe et continue. Malheureusement le terme $\mathbb{X}_\varepsilon = (\xi_\varepsilon, \vartheta_\varepsilon \circ \xi_\varepsilon)$ ne converge pas, mais on peut démontrer que sa renormalisation $\hat{\mathbb{X}}_\varepsilon = (\xi_\varepsilon, \vartheta_\varepsilon \circ \xi_\varepsilon - \mathbb{E}(\vartheta_\varepsilon \circ \xi_\varepsilon))$ (où on soustrait un terme divergent $\mathbb{E}(\vartheta_\varepsilon \circ \xi_\varepsilon)$) converge en probabilité vers une distribution \mathbb{X} . On a donc

$$\xi_\varepsilon \xrightarrow{\hat{J}} \hat{\mathbb{X}}_\varepsilon \xrightarrow{\Psi} \hat{u}_\varepsilon$$

En remplaçant $\hat{\mathbb{X}}_\varepsilon$ dans l'équations paracontrôlée on peut retrouver l'équation renormalisée suivante pour chaque $\varepsilon \in (0, 1]$:

$$\partial_t \hat{u}_\varepsilon - \Delta \hat{u}_\varepsilon = \hat{u}_\varepsilon \xi_\varepsilon - \hat{u}_\varepsilon \mathbb{E}(\vartheta_\varepsilon \circ \xi_\varepsilon).$$

Grâce à la continuité de Ψ et à la convergence en probabilité $\hat{\mathbb{X}}_\varepsilon \rightarrow \mathbb{X}$, les solutions \hat{u}_ε de l'équation ci-dessus convergent en probabilité vers une distribution u qui est définie comme la solution de (1). On remarque enfin que pour chaque $C \in \mathbb{R}$ la renormalisation $\hat{\mathbb{X}}_\varepsilon = (\xi_\varepsilon, \vartheta_\varepsilon \circ \xi_\varepsilon - \mathbb{E}(\vartheta_\varepsilon \circ \xi_\varepsilon) + C)$ a une limite convergente : on a en fait une famille infinie de solutions de (1) qui dépend d'un paramètre.

Cela conclut l'exemple d'application de la méthode paracontrôlée à (1). La fonction Ψ et le terme $\hat{\mathbb{X}}_\varepsilon$ dépendent évidemment de l'équation considérée, et on doit montrer la continuité de Ψ et la convergence de $\hat{\mathbb{X}}_\varepsilon$ pour chaque EDPS qu'on étudie.

La technique de distributions paracontrôlées de [GIP15] a été utilisée avec succès pour étudier des EDPS différentes, en commençant par le modèle d'Anderson parabolique généralisé, l'équation de Burgers rugueuse et d'autres équations différentielles dans [GIP15]. Le calcul paracontrôlé a été utilisé aussi pour étudier l'équation de Kardar-Parisi-Zhang (KPZ) [GP17], le modèle dynamique Φ_3^4 [CC13] et l'existence et unicité d'une solution pour tous les temps [MW17b], l'existence et unicité globale d'une solution de Φ_2^4 [MW17a] et le spectre de l'opérateur hamiltonien de Anderson en dimension 2 [AC15]. En utilisant des techniques liées au semi-groupe de la chaleur, un calcul paracontrôlé sur des variétés a été développé par Bailleul et Bernicot [BB15].

Dans le cadre du calcul paracontrôlé "du premier ordre" développé dans [GIP15], la décomposition paracontrôlée (par exemple (2) dans le cas de l'équation (1)) est typiquement limitée à un développement au premier ordre, c'est à dire que la solution est paracontrôlée par un seul terme stochastique. Cela limite le gain de régularité qu'on peut obtenir dans le terme de reste (u^\sharp dans (2)) par rapport à la solution, et par conséquent limite la singularité des équations qui peuvent être traitées avec cette méthode. Par exemple l'équation (1) en dimension 3, ou avec un bruit blanc espace-temps, ne peut pas être résolue avec les techniques de [GIP15]. Récemment, Bailleul et Bernicot [BB16] ont développé un calcul paracontrôlé d'ordre supérieur qui permet de traiter des équations moins régulières.

Cependant, outre ce progrès récent, la théorie plus générale pour les EDPS singulières a été développée par Hairer [Hai14a, Hai14b, FH14] sous le nom de théorie des structures de régularité. Les structures de régularité sont une généralisation de la théorie des chemins rugueux de Lyons, qui donnent des outils efficaces pour analyser les opérations non-linéaires qui agissent sur certains espaces de distributions, leur renormalisation par soustraction des singularités locales et leur utilisation pour résoudre des EDPS singulières. La théorie des structures de régularité a été appliquée avec succès à tous les modèles répertoriés ci-dessus [Hai14a, Hai13], à d'autres modèles comme le modèle de Sine-Gordon [HS16] (qui peut être néanmoins résolu avec des techniques de calcul paradifférentiel) et à l'étude de l'universalité faible de certains modèles [HQ15, HX16].

Partie I - Connaissances préalables

Dans la première partie de ma thèse, qui comprend les chapitres 1,2 et 3, j'introduis des concepts et des techniques qui seront largement utilisés dans le reste du travail.

Dans le chapitre 1 je présente la décomposition de Littlewood-Paley (L-P), qui permet de développer une distribution tempérée f en somme de *blocs* $\Delta_j f$ qui ont une transformée de Fourier à support compact de l'ordre de $\simeq 2^j$. En particulier, soit χ, ρ une partition de l'unité de \mathbb{R}^d formée par des fonctions lisses à support compact, avec χ à support dans une boule et ρ à support dans un anneau, tels que

$$1 = \chi(x) + \sum_{j \geq 0} \rho(2^{-j}x)$$

et les supports de $\rho(2^{-j}\cdot)$ et $\rho(2^{-k}\cdot)$ soient disjoints pour tous $|k-j| \geq 2$. On définit alors pour tous $j \geq 0$ le bloc de Littlewood-Paley j -ième d'une distribution tempérée f comme

$$\Delta_j f = \mathcal{F}^{-1}(\rho(2^{-j}\cdot)\mathcal{F}f)$$

et le bloc $\Delta_{-1}f$ comme

$$\Delta_{-1}f = \mathcal{F}^{-1}(\chi\mathcal{F}f)$$

où \mathcal{F} et \mathcal{F}^{-1} sont respectivement la transformée de Fourier et la transformée inverse. On remarque que \mathcal{F} et \mathcal{F}^{-1} sont bien définies sur les distributions tempérées $\mathcal{S}'(\mathbb{R}^d)$ et que les blocs sont des fonctions analytiques (en ayant une transformée de Fourier à support compact). On peut prouver facilement que l'équivalence suivante est vérifiée dans le sens des distributions :

$$f = \sum_{j=-1}^{+\infty} \Delta_j f.$$

J'utilise ensuite les blocs de Littlewood-Paley pour définir les espaces de Besov $B_{p,q}^\alpha$, $\forall \alpha \in \mathbb{R}$, $p, q \in [1, \infty]$. Ces espaces sont des généralisations des espaces de Sobolev, et une façon de les caractériser est de mesurer la vitesse de divergence à l'infini de la transformée de Fourier, à l'aide des blocs de Littlewood-Paley. On définit la norme

$$\|f\|_{B_{p,q}^\alpha} = \|(2^{j\alpha} \|\Delta_j f\|_{L^p})_{j \geq -1}\|_{\ell^q}$$

où pour une séquence $a_j: \mathbb{N} \cup \{-1\} \rightarrow \mathbb{R}$ on pose la norme

$$\|(a_j)_{j \geq -1}\|_{\ell^q}^q = \sum_{j \geq -1} a_j^q.$$

On définit l'espace de Besov $B_{p,q}^\alpha$ comme l'ensemble des distributions tempérées qui ont une norme $\|\cdot\|_{B_{p,q}^\alpha}$ finie. Dans le cas $p = q = \infty$ on pose $\mathcal{C}^\alpha := B_{\infty,\infty}^\alpha$ et la norme devient plus simplement

$$\|f\|_{\mathcal{C}^\alpha} = \sup_{j \geq -1} 2^{j\alpha} \sup_{x \in \mathbb{R}^d} |\Delta_j f(x)|,$$

ce qui met en évidence comme le paramètre $\alpha \in \mathbb{R}$ mesure la divergence à l'infini de la transformée de Fourier. Pour $\alpha \in \mathbb{R}^+ \setminus \mathbb{N}$ les espaces de Besov \mathcal{C}^α coïncident avec les espaces de Hölder, et on peut voir par des résultats bien connus (estimations de Bernstein) que le laplacien Δf d'une distribution $f \in \mathcal{C}^\alpha$ appartient à l'espace $\mathcal{C}^{\alpha-2}$. Cela donne une interprétation pour les espaces de Besov avec exposant α négatif.

Les résultats de la section 1.1.1 sur les espaces de Besov constituent la base pour l'analyse des EDPS avec la méthode des distributions paracontrôlées.

Dans la section 1.1.2 je donne une autre caractérisation des espaces de Besov de régularité α négative, en utilisant la norme suivante :

$$\|f\|_{\mathcal{C}^\alpha} := \sup_{\lambda \in (0,1]} \sup_{x \in \mathbb{R}^d} \sup_{\eta \in \mathcal{B}^{r_0}} \lambda^{-\alpha} \int_{\mathbb{R}^d} f \lambda^{-d} \eta\left(\frac{\cdot - x}{\lambda}\right)$$

où η est une fonction $C^r(\mathbb{R}^d)$ (avec $r = -\lfloor \alpha \rfloor$, $\alpha < 0$) à support compact sur une boule de rayon 1. Cette définition est similaire à celle utilisée par Hairer dans [Hai14a], avec la différence que dans ma thèse j'utilise la convention d'appeler $\bar{\mathcal{C}}^\alpha$ la fermeture de \mathcal{C}_c^∞ par rapport à la norme ci-dessus.

Je montre ensuite qu'on peut caractériser ce type d'espaces (et plus en général des espaces de Besov $\bar{\mathcal{B}}_{p,q}^\alpha$ pour $p, q \neq \infty$) en utilisant des ondelettes à support compact de Daubechies. Un des avantages de travailler avec des fonctions test à support compact, et donc avec une caractérisation locale des espaces de Besov, est la possibilité de définir des espaces de Besov locaux sur un domaine ouvert quelconque $U \subset \mathbb{R}^d$ pour des distributions qui peuvent avoir une régularité différente proche du bord de U . Pour faire ça je montre que pour caractériser un espace de Besov local il suffit d'utiliser seulement des ondelettes qui ne touchent pas le bord de U , en prenant des ondelettes centrées à une distance du bord plus grande que le diamètre de leur support. Une définition des espaces de Besov locaux de ce type est particulièrement utile quand on considère la convergence d'une famille de distributions dans un domaine $U' \subset U$ qui tend vers U , dans le cas où la famille ne converge pas dans un domaine $U \setminus U'$ asymptotiquement proche du bord de U .

À la fin de la section 1.1.2 je donne un critère de tension pour des familles de distributions aléatoires dans les espaces de Besov locaux présentés ici. Ce critère est l'instrument principal dans la preuve de la tension du champ de magnétisation d'Ising du chapitre 4. Je donne aussi une formulation du théorème de continuité de Kolmogorov adaptée à la

décomposition en ondelettes présentée dans cette section. Le matériel rédigé dans la section 1.1.2 est déjà apparu dans l'article [FM17], qui est un travail joint avec J-C.Mourrat.

Dans la section 1.1.3 je rappelle des résultats d'équivalence entre les différentes définitions d'espaces de Besov citées précédemment. En particulier, l'espace $\bar{B}_{p,q}^\alpha$ de la section 1.1.2 étant défini comme fermeture de fonctions C_c^∞ , il ne peut pas coïncider avec l'espace $B_{p,q}^\alpha$ de la section 1.1.1 que quand p et q sont tous les deux finis. Néanmoins, les normes de $\bar{B}_{p,q}^\alpha$ et $B_{p,q}^\alpha$ sont équivalentes sur des fonctions suffisamment lisses.

La section 1.2 recueille les outils analytiques principaux du calcul paracontrôlé. Une des notions plus importantes est celle de paraproduit de Bony [Bon81]. Pour chaque couple de distributions tempérées $u, v \in \mathcal{S}'$ on peut écrire leur décomposition de Littlewood-Paley comme

$$u = \sum_{j \geq -1} \Delta_j u \quad v = \sum_{j \geq -1} \Delta_j v .$$

On peut donc écrire *formellement*

$$uv = \sum_{j \geq -1} \sum_{k < j-1} \Delta_j u \Delta_k v + \sum_{k \geq -1} \sum_{j < k-1} \Delta_j u \Delta_k v + \sum_{|k-j| \leq 1} \Delta_j u \Delta_k v$$

mais $\Delta_j u \Delta_k v$ n'est plus un bloc de Littlewood-Paley d'une distribution : il n'est donc pas garanti qu'il converge. D'autre part, l'intuition est que grâce aux propriétés des supports de la partition de l'unité (χ, ρ) utilisée dans la définition de blocs de Littlewood-Paley, pour chaque $j \geq 1$ le terme $\sum_{k < j-1} \Delta_j u \Delta_k v$ est une approximation suffisamment précise d'un bloc de L-P $\Delta_j f$ (pour une certaine distribution f) afin que la série $\sum_{j \geq -1} \sum_{k < j-1} \Delta_j u \Delta_k v$ converge dans le sens des distributions. Par symétrie on obtient le même résultat pour le deuxième terme ci-dessus. Le paraproduit est donc

$$u \prec v := \sum_{j \geq -1} \sum_{k < j-1} \Delta_j u \Delta_k v$$

qui est bien posé pour chaque $u, v \in \mathcal{S}'$ et, sachant que le bloc j -ième localise la transformée de Fourier autour d'une valeur 2^j , on dit que $u \prec v$ est *paracontrôlé* par v dans le sens que à petite échelle la contribution de u dans $u \prec v$ est négligeable par rapport à celle de v .

Bony a aussi prouvé que si $u \in \mathcal{C}^\alpha$, $v \in \mathcal{C}^\beta$ avec $\alpha + \beta > 0$ (\mathcal{C}^α est l'espace de Besov défini dans la section 1.1.1) la série $\sum_{|k-j| \leq 1} \Delta_j u \Delta_k v$ converge vers une distribution $u \circ v$ qui appartient à $\mathcal{C}^{\alpha+\beta}$, appelée *produit résonant*. Ce résultat est fondamental dans le calcul paracontrôlé : le produit résonant, s'il est bien défini, est plus régulier du produit uv .

Dans le reste de la section 1.2 je présente les résultats qui sont utilisés dans la partie analytique de la méthode des distributions paracontrôlées.

Le chapitre 2 contient des techniques d'estimation des champs aléatoires qui sont nécessaires à l'étude de la convergence des termes de bruit augmenté (\hat{X}_ε dans l'exemple (1)) dans le cadre des distributions paracontrôlées. Un des concepts principaux utilisés à cette fin est la décomposition en chaos de Wiener.

Le bruit blanc sur \mathbb{R}^d défini sur un espace de probabilité $(\Omega, \mathcal{F}, \mathbb{P})$ est un processus aléatoire Gaussien centré $\xi = \{\xi(f) | f \in L^2(\mathbb{R}^d)\}$ indexé par les fonctions $f \in L^2(\mathbb{R}^d)$, ayant une covariance déterminée par

$$\mathbb{E}(\xi(f), \xi(g)) = \langle f, g \rangle_{L^2(\mathbb{R}^d)}.$$

La décomposition en chaos de Wiener est la suivante :

$$L^2(\Omega, \sigma(\xi), \mathbb{P}) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n$$

où $\sigma(\xi)$ est la tribu engendrée par le bruit blanc et \mathcal{H}_n est l'espace engendré par les variables aléatoires $\{H_n(\xi(f)) | f \in L^2(\mathbb{R}^d), \|f\|_{L^2(\mathbb{R}^d)} = 1\}$ avec H_n les polynômes de Hermite. L'intérêt de la décomposition en chaos de Wiener, au delà du fait qu'elle est orthogonale dans $L^2(\Omega)$, est que quand une variable aléatoire X a une décomposition en chaos de Wiener finie, on peut utiliser l'importante propriété d'*hypercontractivité* de Nelson :

$$\|X\|_{L^p(\Omega)} \lesssim_p \|X\|_{L^2(\Omega)}.$$

Cette propriété est fondamentale dans l'estimation des bruits augmentés (enhanced noises) qui apparaissent dans l'étude d'une EDPS à l'aide de la méthode des distributions paracontrôlées. La procédure expliquée dans la section 2.2 peut se résumer comme suit : soit par exemple X_ε un champ aléatoire stationnaire lisse sur \mathbb{T}^d qu'on veut borner uniformément (sur $\varepsilon \in (0, 1]$) dans un espace de Besov $\mathcal{C}^{\alpha-\delta}(\mathbb{T}^d)$ pour chaque $\delta > 0$ et $\alpha \in \mathbb{R}$ fixé. Le théorème de plongement d'espaces de Besov (qui est une généralisation du théorème de plongement de Sobolev) donne :

$$\|\cdot\|_{\mathcal{C}^{\alpha-d/p}}^p \lesssim \|\cdot\|_{B_{p,p}^\alpha}^p.$$

On peut donc borner X_ε comme suit :

$$\mathbb{E}(\|X_\varepsilon\|_{\mathcal{C}^{\alpha-d/p}}^p) \lesssim \mathbb{E}(\|X_\varepsilon\|_{B_{p,p}^\alpha}^p) \lesssim \sum_{j \geq -1} 2^{\alpha p j} \int_{\mathbb{T}^d} \mathbb{E}(|\Delta_j X_\varepsilon(x)|^p) dx.$$

Si la variable aléatoire $\Delta_j X_\varepsilon(x)$ a une décomposition en chaos de Wiener finie, grâce à la propriété d'hypercontractivité, il suffit donc d'estimer sa norme $L^2(\Omega)$. Cette norme, à cause du fait que la décomposition en chaos de Wiener est orthogonale pour le produit scalaire de $L^2(\Omega)$, est une somme finie de covariances de variables aléatoires qui

appartiennent à \mathcal{H}_n . Chaque covariance obtenue par cette procédure est sous la forme d'une convolution de fonctions singulières dont on connaît le degré d'homogénéité (on appelle cette convolution *diagramme* en analogie avec les diagrammes de Feynman). Les diagrammes peuvent être bornés avec des lemmes bien connus sur l'estimation des convolutions de fonctions singulières (qui sont rappelés dans la section 2.2.1).

La section 2.3 contient quelques résultats de calcul de Malliavin qui permettent d'estimer des champs aléatoires qui ont une décomposition en chaos de Wiener infinie. En particulier cette section contient une décomposition partielle en chaos de Wiener dans laquelle le reste est écrit sous forme d'intégrale itérée de Skorohod (qui est une généralisation de l'intégrale de Itô). Ce reste peut être estimé dans la norme $L^p(\Omega)$ en utilisant des résultats de calcul de Malliavin qui sont rappelés dans la section 2.3. Le produit d'intégrales de Skorohod possède une *formule du produit* (analogue à celle de l'intégrale de Itô et qu'on trouve par exemple dans [Üst14] et [Nua06]) qui est obtenue dans la section 2.3 en partant des résultats de [NN10].

Le chapitre 3 contient une introduction rapide à deux modèles probabilistes bien connus : la percolation FK et le modèle d'Ising en 2 dimensions. La mesure de Edwards et Sokal donne un couplage entre les mesures du modèle d'Ising et FK, ce qui permet d'examiner le système en utilisant à la fois des techniques issues de la théorie de la percolation ou de l'étude du modèle d'Ising. Le contenu de ce chapitre est tiré de la monographie [Gri09] et a la seule fonction de présenter le cadre dans lequel les résultats du chapitre 4 sont énoncés.

Partie II - Résultats

Chapitre 4

Le contenu de ce chapitre est pris de l'article [FM17] qui a été écrit en collaboration avec J-C. Mourrat. On y applique le critère de tension pour les champs aléatoires (qui a été développé dans la section 1.1.2) à l'étude du champ de magnétisation d'Ising à température critique. Soit $U \subset \mathbb{R}^2$ un ensemble ouvert, et pour $a > 0$ soit $U_a := U \cap (a\mathbb{Z}^2)$. On désigne comme $(\sigma_y)_{y \in U_a}$ les spins du modèle d'Ising à température critique avec (par exemple) spin + sur le bord de U_a , et on définit le champ de magnétisation

$$\Phi_a := a^{-\frac{1}{8}} \sum_{y \in U_a} \sigma_y \mathbb{1}_{S_a(y)}$$

où $S_a(y)$ est le carré centré dans y avec côtés de longueur a . Dans [CGN15] les auteurs ont montré que si $U = [0, 1]^2$, pour tous $\varepsilon > 0$ la famille $(\Phi_a)_{a \in (0, 1]}$ est tendue dans l'espace de Besov $B_{2,2}^{-1-\varepsilon}(U)$, et ils ont analysé aussi des domaines plus généraux. Une question ouverte dans leur travail était de déterminer précisément dans quel espace la famille $(\Phi_a)_{a \in (0, 1]}$

est tendue. On a répondu à cette question en montrant que pour chaque ouvert $U \subset \mathbb{R}^2$ et chaque $\varepsilon > 0$ le champ de magnétisation est tendu dans $B_{p,q}^{-1/8-\varepsilon,\text{loc}}(U) \forall p, q \in [1, \infty]$, et qu'il n'est pas tendu dans $B_{p,q}^{-1/8+\varepsilon,\text{loc}}(\mathbb{R}^2)$.

L'ingrédient principal de la preuve de la tension de $(\Phi_a)_{a \in (0,1]}$ est la borne supérieure sur les corrélations du modèle d'Ising bidimensionnel obtenue par Onsager [Ons44] dans sa solution exacte du modèle, qui permet d'obtenir, grâce à l'application de l'inégalité FKG, une estimation de la somme des corrélations à p points sur un sous-ensemble borné de \mathbb{Z}^2 de taille N . Cette quantité peut être utilisée directement pour déterminer l'espace de Besov $B_{p,q}^{-1/8-\varepsilon,\text{loc}}(U)$ dans lequel $(\Phi_a)_{a \in (0,1]}$ est tendue, à l'aide du critère de tension de la section 1.1.2.

Pour démontrer que $(\Phi_a)_{a \in (0,1]}$ n'est pas tendue dans $B_{p,q}^{-1/8+\varepsilon,\text{loc}}(\mathbb{R}^2)$ on utilise une borne inférieure sur les corrélations du modèle d'Ising, obtenue par Hongler, Duminil-Copin et Nolin dans [HDN11] comme corollaire de leurs estimations du type Russo-Seymour-Welsh sur la probabilité de percolation pour le modèle FK. Pour pouvoir montrer qu'aucun point limite de $(\Phi_a)_{a \in (0,1]}$ n'appartient à $B_{p,q}^{-1/8+\varepsilon,\text{loc}}(\mathbb{R}^2)$, il est nécessaire d'utiliser une caractérisation d'espaces de Besov basée sur des fonctions test positives.

Il a été démontré récemment qu'il existe une limite unique de la famille $(\Phi_a)_{a \in (0,1]}$ [CGN15, CHI15]. Grâce à notre résultat il est clair que cette limite est singulière (aussi dans des domaines compacts) par rapport au champ libre gaussien, parce que celui-là prend des valeurs dans $B_{p,q}^{-\varepsilon,\text{loc}}(\mathbb{R}^2)$ pour chaque $\varepsilon > 0, p, q \in [1, \infty]$.

Chapitre 5

Dans ce chapitre, qui contient les résultats obtenus dans [FG16] en collaboration avec M.Gubinelli, on développe un calcul paracontrôlé non-linéaire afin de montrer l'existence locale de solutions à certaines EDPS quasi-linéaires uniformément paraboliques.

On considère principalement les deux équations

$$\partial_t u(t, x) - a_1(u(t, x)) \Delta u(t, x) = \xi(x), \quad u(0, x) = u_0(x), \quad x \in \mathbb{T}^2, t \geq 0,$$

et la plus générale

$$\partial_t u(t, x) - a_1(u(t, x)) \Delta u(t, x) = a_2(u(t, x)) \xi(x), \quad u(0, x) = u_0(x), \quad x \in \mathbb{T}^2, t \geq 0$$

avec $a_1: \mathbb{R} \rightarrow [\lambda, 1]$, $a_2: \mathbb{R} \rightarrow [-L, L]$ ($\lambda, L > 0$) qui sont des coefficients C^3 uniformément bornés, et $\|a_i^{(k)}\|_{L^\infty} \leq 1$ for $k = 0, \dots, 3$. Pour les deux équations on suppose que le bruit aléatoire ξ prenne des valeurs dans l'espace de Besov $\mathcal{C}^{\alpha-2}(\mathbb{T}^2)$ avec $2/3 < \alpha < 1$: cela s'applique par exemple au bruit blanc en espace sur \mathbb{T}^2 .

Néanmoins, on montre dans la section 5.5 que le calcul paracontrôlé non-linéaire développé dans ce chapitre peut s'appliquer aussi à une classe d'équations de la forme

$$a_3(u(t, x))\partial_t u(t, x) - a_1(u(t, x))\Delta u(t, x) = \xi(a_2(u(t, x)), t, x), \quad x \in \mathbb{T}^2, t \geq 0,$$

où a_1, a_2, a_3 sont des coefficients non-dégénérés assez réguliers et $\xi(z, t, x)$ est un processus gaussien avec covariance

$$\mathbb{E}[\xi(z, t, x)\xi(z', t', x')] = F(z, z')Q(t - t', x - x'), \quad x, x' \in \mathbb{T}^2, t, t', z, z' \in \mathbb{R},$$

où F est une fonction lisse et Q une distribution de régularité parabolique $\rho > -4/3$. Cela inclut comme cas particulier le bruit cité précédemment, mais on pourrait considérer un bruit blanc en temps qui soit régulier dans sa variable spatiale, ou bien un bruit qui soit plus faiblement irrégulier dans l'espace et le temps.

Récemment Otto et Weber [OW16] et Bailleul, Debussche et Hofmanová [BDH16] ont étudié des EDPS quasi-linéaires dans le contexte des méthodes "trajectoire par trajectoire" et dans une gamme de régularités superposables à celles qu'on considère dans ce chapitre.

Dans [OW16] les auteurs ont montré que l'équation

$$\partial_t u(t, x) - a(u(t, x))\partial_x^2 u(t, x) = f(u(t, x))\xi(t, x), \quad t \geq 0$$

est localement bien posée, avec une variable spatiale périodique de dimension 1 et un bruit qui peut être blanc en espace mais coloré dans le temps, et fonctionne essentiellement comme une distribution de régularité parabolique $(-4/3, 1)$. Pour obtenir ce résultat ils introduisent une notion spécifique de fonction *modélée* et des estimations conséquentes, basée sur la théorie des chemins rugueux contrôlés.

Leur *Ansatz paramétrique* est la source principale d'inspiration pour notre travail [FG16]. D'ailleurs, cette observation fondamentale ne nécessite pas le développement d'une théorie alternative des EDPS singulières, qui est le sujet plus important dans leur article, mais peut être mise en place dans le cadre des distributions paracontrôlées. En effet, dans le chapitre 5 on montre que une extension relativement directe de la méthode paracontrôlée (appelée calcul paracontrôlé non-linéaire) est suffisante pour obtenir plutôt rapidement les résultats sur les équations quasi-linéaires qui sont contenus dans leur article.

Bailleul, Debussche et Hofmanová dans [BDH16] ont montré que l'équation du modèle d'Anderson parabolique généralisé

$$\partial_t u(t, x) - a(u(t, x))\Delta u(t, x) = g(u(t, x))\xi(x) \quad t \geq 0, x \in \mathbb{T}^2.$$

est bien posée. Les auteurs ont obtenu le même résultat qu'on présente dans la section 5.4 de cette thèse, sans utiliser le calcul paracontrôlé non-linéaire introduit dans le chapitre 5 mais seulement des techniques du calcul paracontrôlé standard et des transformations

astucieuses. Cette simplification a comme inconvénient une certaine perte de généralité de la méthode : par exemple une EDPS quasi-linéaire avec coefficients de diffusion $(a_{ij})_{i,j}$ à valeurs dans des matrices, c'est à dire une équation sous la forme

$$\partial_t u(t, x) - a_{ij}(u(t, x)) \frac{\partial^2}{\partial x_i \partial x_j} u(t, x) = g(u(t, x)) \xi, \quad t \geq 0, x \in \mathbb{T}^2,$$

ne peut pas être traitée par les techniques utilisées dans [BDH16] mais elle peut être traitée avec notre calcul paracontrôlé non-linéaire et avec les techniques de Otto et Weber.

Plus récemment, Hairer et Gerencsér ont développé dans [GH17a] une théorie des structures de régularité paramétriques, et ils ont montré la convergence des modèles aléatoires associés, pour résoudre une large gamme d'EDPS quasi-linéaires (y compris les équations avec bruit trop irrégulier pour pouvoir être analysées avec les méthodes de [FG16], [OW16] et [BDH16]). Leur théorie généralise considérablement les travaux précédents sur les EDPS quasi-linéaires.

Chapitre 6

Les résultats de ce chapitre ont été obtenus en collaboration avec M. Gubinelli et ils sont apparus en premier dans [FG18]. On y étudie la famille d'EDPS sur $[0, T] \times \mathbb{T}^3$ et indexée par $\varepsilon \in (0, 1]$ suivante :

$$\mathcal{L}u_\varepsilon(t, x) = -\varepsilon^{-\frac{3}{2}} F_\varepsilon(\varepsilon^{\frac{1}{2}} u_\varepsilon(t, x)) + \eta_\varepsilon(t, x) \quad (3)$$

où η_ε est un champ aléatoire gaussien sur $\mathbb{R} \times \mathbb{T}^3$ qui converge en loi vers le bruit blanc espace-temps pour $\varepsilon \rightarrow 0$, et $(F_\varepsilon)_\varepsilon \subseteq C^9(\mathbb{R})$ avec toutes ses dérivées ayant une croissance au maximum exponentielle. On se demande si les solutions (3) convergent en loi vers une limite $u_\varepsilon \rightarrow u$, et si cette limite peut s'identifier avec un objet universel.

Soit Y_ε la solution stationnaire de $\mathcal{L}Y_\varepsilon = -Y_\varepsilon + \eta_\varepsilon$. Avec des hypothèses de convergence des premiers 4 coefficients de la décomposition en chaos de $\varepsilon^{-3/2} F_\varepsilon(\varepsilon^{1/2} Y_\varepsilon)$ (et aussi des hypothèses sur la convergence de la condition initiale $u_{0,\varepsilon}$) on montre dans ce chapitre que les solutions de (3) convergent en loi vers une limite $u_\varepsilon \rightarrow u$ qui dépend seulement de 4 paramètres. Ce type de résultat s'appelle d'*universalité faible* : le mot *universalité* vient du fait que, même si les solutions de (3) dépendent de la fonction F_ε entière (c'est à dire par exemple qu'elles dépendent de toute la décomposition en chaos de F_ε), la limite $u_\varepsilon \rightarrow u$ dépend seulement d'un nombre fini de paramètres qui sont obtenus des premiers termes de la décomposition en chaos de F_ε . L'adjectif *faible* vient du fait que la non-linéarité doit être asymptotiquement petite par rapport au bruit η_ε (ou bien le bruit doit être asymptotiquement petit par rapport à la non-linéarité, mais ce cas est plus facile à traiter et moins intéressant pour ce modèle). Le même résultat de convergence a été obtenu en premier par Hairer et Xu dans [HX16], mais avec l'importante restriction de supposer que F_ε soit un polynôme $\forall \varepsilon \in (0, 1]$.

On peut identifier la limite pour $\varepsilon \rightarrow 0$ des solutions de (3) avec la limite des solutions de

$$\mathcal{L}u_\varepsilon = -\lambda_3 u_\varepsilon^3 - \lambda_2 u_\varepsilon^2 - c_\varepsilon u_\varepsilon - d_\varepsilon + \eta_\varepsilon \quad (4)$$

avec $\lambda_3, \lambda_2 \in \mathbb{R}$ et des constantes de renormalisation divergentes $c_\varepsilon, d_\varepsilon$. Cette équation s'appelle *modèle* Φ_3^4 . Dans 2 dimensions le modèle Φ_2^4 a été le sujet de différents travaux pendant plus de trente ans [JM85, AR91, DD03]. Dans le cas tridimensionnel Φ_3^4 , la convergence des solutions de (4) a été obtenue d'abord par Hairer [Hai14a, Hai15] et constitue une première application révolutionnaire de sa théorie des structure de régularité. Un résultat similaire a été obtenu après par Catellier et Chouk dans [CC13] en utilisant le calcul paracontrôlé développé par Gubinelli, Imkeller et Perkowski [GIP15]. Kupiainen dans [Kup14] a aussi résolu cette équation en utilisant des techniques de groupe de renormalisation.

Le premier résultat d'universalité faible pour une EDPS singulière a été donné par Hairer et Quastel [HQ15] dans le cadre de l'équation de Kardar-Parisi-Zhang (KPZ) avec une non-linéarité donnée par un polynôme pair. En utilisant la théorie développée dans ce dernier article, Hairer et Xu ont montré dans [HX16] l'universalité faible d'une famille d'équations de réaction-diffusion tridimensionnelles dans le cas d'un bruit gaussien et d'une non-linéarité polynomiale. L'universalité faible pour des équations de réaction-diffusion perturbées par un bruit non-gaussien a été traitée par Shen et Xu dans [SX16]. La pré-publication [OGK17] analyse une version hyperbolique de l'équation de quantification stochastique bidimensionnelle, en incluant un résultat d'universalité faible (pour un bruit asymptotiquement petit).

Récemment, Hairer et Xu [HX18] ont obtenu une généralisation du résultat précédent de Hairer et Quastel sur la convergence d'une certaine classe de modèles vers la solution de l'équation KPZ [HQ15], avec des hypothèses plus générales sur la non-linéarité et une méthode essentiellement différente de celle introduite d'abord dans [FG18]. Les techniques que je présente dans le chapitre 6 de ma thèse s'appliquent aussi bien aux modèles considérées dans [HX18], et vice-versa la technique de [HX18] pourrait en principe retrouver les mêmes résultats qui sont énoncés ici.

L'article [FG18], dont le chapitre 6 de ma thèse tire son origine, est le premier à considérer dans tous les détails un problème d'universalité faible en utilisant la technique des distributions paracontrôlées. Cela permet de montrer que les résultats analytiques peuvent s'obtenir facilement avec le calcul paracontrôlé standard qui a été présenté dans le chapitre 1 (en particulier dans [FG18] on se base sur la construction de Catellier et Chouk [CC13]), et que la seule difficulté est de prouver la convergence d'un certain nombre de champs aléatoires, ayant un développement en chaos infini, vers des objets limites universels. La nouveauté plus importante de notre article [FG18] est le fait d'utiliser le calcul de Malliavin [Nua06, NN10, NP12] pour analyser ces termes stochastiques sans devoir supposer que la non-linéarité soit polynomiale comme dans [HX16].

Introduction

The main subject of this work are Stochastic Partial Differential Equations (SPDEs), in particular 2 and 3-dimensional nonlinear parabolic equations with very singular stochastic forcing terms. The difficulty arising when studying the well-posedness of such equations is that their solutions are expected to be tempered distributions in the space variable. Indeed, it is well-known that although linear operations on Schwartz' distributions (such as differentiation, Fourier transform or multiplication with a smooth functions) are well defined, there is no canonical way of defining the product of two distributions, and therefore any nonlinearity in the equations we consider is ill-defined. Another subject of this work is the study of the tightness of the 2-dimensional Ising magnetization field, which is performed with techniques similar to those used for SPDEs.

The analysis of SPDEs in this work is based on the *paracontrolled distributions* framework (also called *paracontrolled calculus* or *paradifferential calculus*), introduced by Gubinelli, Imkeller and Perkowski in [GIP15]. Their work can be thought of as a generalization and improvement upon the idea of *controlled rough paths* introduced by Gubinelli in [Gub04], which builds on the theory of rough paths, introduced by Lyons in the seminal article [Lyo98].

We give an example of the main points of the paracontrolled distributions technique [GIP15] by considering the following equation, called continuous Parabolic Anderson Model (PAM):

$$\partial_t u - \Delta u = u \xi \quad u(t=0) = 0, \quad (5)$$

where ∂_t is the time derivative and Δ the Laplacian. ξ is the space *white noise* on the two dimensional torus $\mathbb{T}^2 = (\mathbb{R}/2\pi\mathbb{Z})^2$, which can be identified formally as the centered Gaussian random field with covariance $\mathbb{E}[\xi(x)\xi(y)] = \delta(x-y)$. The trajectories of ξ are almost surely tempered distributions in the space $\mathcal{S}'(\mathbb{T}^2)$. For every trajectory of ξ , let ξ_ε be a regularization of the white noise by convolution with a smooth function compactly supported at scale $\varepsilon > 0$ (such that $\xi_\varepsilon \rightarrow \xi$ as $\varepsilon \rightarrow 0$). It is not difficult to obtain solutions u_ε to a regularized version of (5) for every scale ε (for example by a Picard fixed-point argument), and if the family $(u_\varepsilon)_{\varepsilon \in (0,1]}$ has a limit in some sense, we define it to be the solution of (5).

Nevertheless, it is not possible to obtain a pathwise solution with this method, since a solution u of (5) is not expected to be more regular than $\vartheta = \Delta^{-1}\xi$, and this is not enough to make the product $u\xi$ well defined. Therefore, $u_\varepsilon \xi_\varepsilon$ is expected to have a divergent limit.

To overcome this difficulty, the authors of [GIP15] employ a set of functional analysis tools that we have recalled in Chapter 1. Their first important observation is that the ill-posed product $u\xi$ can be (formally) decomposed as a sum of Bony’s paraproducts [Bon81]:

$$u\xi = u \prec \xi + u \succ \xi + u \circ \xi$$

where the paraproducts $u \prec \xi$, $u \succ \xi$ can be defined for every couple of tempered distributions $u, \xi \in \mathcal{S}'$ and the last term (called *resonant product*) is ill-defined when the sum of the regularities of u and ξ is below a certain threshold, which is our case (for precise bounds on the regularities of u and ξ in a resonant paraproduct $u \circ \xi$ see Chapter 1). The paraproduct decomposition above is based on a partition of unity in Fourier space of u and ξ : we can indeed interpret $u \prec \xi$ as the product of ξ with a high-frequencies cutoff of u , which results in some sort of “amplitude modulation”. On the other hand, the term $u \circ \xi$ models a resonance phenomenon in which two similar frequencies interact catastrophically. The problem now becomes that of controlling the limit $u_\varepsilon \circ \xi_\varepsilon$ for $\varepsilon \rightarrow 0$.

The second crucial observation of [GIP15], that originates from the notion of controlled rough paths and is called accordingly *paracontrolled distribution* Ansatz, is a guess on the structure of the distribution u that could solve (5). In particular, we can postulate that u should “look like” the integral of ξ at small scales (high frequencies), i.e. it should be approximated by a modulation of $\vartheta = \Delta^{-1}\xi$ with a more regular function u' . The Ansatz would then be $u = u' \prec \vartheta + u^\sharp$ with u^\sharp, u' to be established by equation (5) and the remainder term u^\sharp more regular than u , but for technical reasons we set instead

$$u = u' \llcorner \vartheta + u^\sharp, \tag{6}$$

where the modified paraproduct \llcorner coincides with \prec apart from a time-smoothing of the first variable u' to cope with the time derivative in (5).

The third point put forward in [GIP15] is that, given the Ansatz (6), $u_\varepsilon \circ \xi_\varepsilon$ can be decomposed in such a way that the ill-posedness appears only in correspondence of products of explicit integrals of the white noise (called *enhanced noises*), which do not depend on the solution itself but just on ξ_ε . The idea is then to *define* a limit for the enhanced noise terms and since every other operation in (5) is well defined (thanks to the estimations in Chapter 1), it is possible to obtain a limit $u_\varepsilon \rightarrow u$ in this way. More precisely, the Ansatz (6) yields

$$u_\varepsilon \circ \xi_\varepsilon = (u'_\varepsilon \llcorner \vartheta_\varepsilon) \circ \xi_\varepsilon + u^\sharp_\varepsilon \circ \xi_\varepsilon$$

and we assume that u^\sharp_ε is regular enough in the limit $\varepsilon \rightarrow 0$ to obtain a well behaved limit of $u^\sharp_\varepsilon \circ \xi_\varepsilon$ in \mathcal{S}' (indeed, by substituting (6) into (5) it can be seen that u^\sharp_ε solves a differential equation with a more regular r.h.s, but we skip the technical details here). Thanks to the fundamental *commutator lemma* of [GIP15] we can write

$$(u'_\varepsilon \llcorner \vartheta_\varepsilon) \circ \xi_\varepsilon = C(u'_\varepsilon, \vartheta_\varepsilon, \xi_\varepsilon) + u'_\varepsilon(\vartheta_\varepsilon \circ \xi_\varepsilon)$$

where the trilinear commutator $C(\cdot, \cdot, \cdot)$ can be extended on \mathcal{S}' and the only term that can't be controlled analytically as $\varepsilon \rightarrow 0$ is $\vartheta_\varepsilon \circ \xi_\varepsilon$. By postulating that the term $\vartheta_\varepsilon \circ \xi_\varepsilon$ has a limit in \mathcal{S}' regular enough to make sense of the product $u'_\varepsilon(\vartheta_\varepsilon \circ \xi_\varepsilon)$ as $\varepsilon \rightarrow 0$, we can make sense of the r.h.s. of (5). Indeed, we have for every $\varepsilon > 0$ the couple of paracontrolled equations

$$\begin{cases} u_\varepsilon = u'_\varepsilon \llcorner \vartheta + u_\varepsilon^\sharp \\ (\partial_t - \Delta)u_\varepsilon = u \prec \xi + u \succ \xi + C(u'_\varepsilon, \vartheta_\varepsilon, \xi_\varepsilon) + u'_\varepsilon(\vartheta_\varepsilon \circ \xi_\varepsilon) + u_\varepsilon^\sharp \circ \xi_\varepsilon \end{cases}$$

which have a solution couple $(u'_\varepsilon, u_\varepsilon^\sharp)$. If we assume that $\mathbb{X}_\varepsilon = (\xi_\varepsilon, \vartheta_\varepsilon \circ \xi_\varepsilon)$ converges to a tempered distribution \mathbb{X} which is regular enough to make sense of the equations above, we have a limit $(u'_\varepsilon, u_\varepsilon^\sharp) \rightarrow (u', u^\sharp) \in \mathcal{S}' \times \mathcal{S}'$.

This concludes the analytic part of the theory. What it is left to do, in the probabilistic part of the theory, is to exploit stochastic cancellations on $\vartheta_\varepsilon \circ \xi_\varepsilon$ to show that (some renormalized version of) \mathbb{X}_ε converges in probability to a limit distribution \mathbb{X} .

We resume the situation as follows: there is no choice of Banach spaces that makes the solution map $\Phi: \xi_\varepsilon \mapsto u_\varepsilon$ continuous, because of the ill-posedness of the product of distributions. However, we can split Φ as follows:

$$\xi_\varepsilon \xrightarrow{J} \mathbb{X}_\varepsilon \xrightarrow{\Psi} u_\varepsilon$$

where Ψ is a the continuous “paracontrolled” solution map which associates \mathbb{X}_ε to the solutions $(u'_\varepsilon, u_\varepsilon^\sharp)$ of the system of equations above. Unfortunately \mathbb{X}_ε does not converge, but after defining the renormalized enhanced noise $\hat{\mathbb{X}}_\varepsilon = (\xi_\varepsilon, \vartheta_\varepsilon \circ \xi_\varepsilon - \mathbb{E}(\vartheta_\varepsilon \circ \xi_\varepsilon))$ (where $\mathbb{E}(\vartheta_\varepsilon \circ \xi_\varepsilon)$ diverges as $\varepsilon \rightarrow 0$) it is possible to show that $\hat{\mathbb{X}}_\varepsilon$ converges in probability to a distribution \mathbb{X} . We have then

$$\xi_\varepsilon \xrightarrow{\hat{J}} \hat{\mathbb{X}}_\varepsilon \xrightarrow{\Psi} \hat{u}_\varepsilon$$

By retracing the constant $\mathbb{E}(\vartheta_\varepsilon \circ \xi_\varepsilon)$ added in the paracontrolled equations we recover a renormalized version of (5) which is:

$$\partial_t \hat{u}_\varepsilon - \Delta \hat{u}_\varepsilon = \hat{u}_\varepsilon \xi_\varepsilon - \hat{u}_\varepsilon \mathbb{E}(\vartheta_\varepsilon \circ \xi_\varepsilon).$$

From the continuity of Ψ together with the convergence in probability of $\hat{\mathbb{X}}_\varepsilon$ we obtain that the regular solutions \hat{u}_ε of the equation above converge to a distribution u which is defined to be a solution of (5). Note that, since for every $C \in \mathbb{R}$ the renormalization $\hat{\mathbb{X}}_\varepsilon = (\xi_\varepsilon, \vartheta_\varepsilon \circ \xi_\varepsilon - \mathbb{E}(\vartheta_\varepsilon \circ \xi_\varepsilon) + C)$ has a convergent limit, there is actually a one-parameter family of solutions to (5). This concludes our example of application of the paracontrolled method to a sample equation. Both the function Ψ and the term $\hat{\mathbb{X}}_\varepsilon$ are specific to the

SPDE to be treated, and have to be shown respectively to be continuous and convergent in probability for every model being considered.

The paracontrolled distributions technique recalled above and first introduced in [GIP15] (see also the lecture notes [GP15]) has been used successfully to study a variety of SPDEs, starting from generalized PAM, rough Burgers' equation and other rough differential equations in [GIP15]. Paracontrolled calculus has been used also to study the KPZ equation [GP17], the dynamic Φ_3^4 model [CC13] and its global well-posedness in time [MW17b], the global well-posedness of Φ_2^4 model [MW17a], and the spectrum of the continuous Anderson Hamiltonian in $d = 2$ [AC15]. By using heat-semigroup techniques, paracontrolled calculus has been extended to the manifold context by Bailleul and Bernicot [BB15].

In the framework of the “first order” paracontrolled calculus developed in [GIP15], the paracontrolled Ansatz (for example (6) in our equation above) is typically limited to a first order expansion, i.e. the solution can only be para-controlled by a single stochastic term. This limits the amount of regularity that the remainder (u^\sharp in eq. (6)) can gain, and consequently puts a lower bound on the regularity of the random driving term that is allowed in the equation. For example eq. (5) in $d = 3$, or with a time-space white noise, is out of scope of the theory of [GIP15]. Recently, Bailleul and Bernicot [BB16] developed a higher order version of paracontrolled calculus that allows to treat more singular equations.

However, apart from this recent development, the most general theory for singular SPDEs has been developed by Hairer [Hai14a, Hai14b, FH14] under the name of *regularity structures* theory. Regularity structures are a vast generalisation of Lyons' rough paths which give effective tools to describe non-linear operations acting on certain spaces of distributions, their renormalization by subtraction of local singularities and their use to solve singular SPDEs. Regularity structures have been successfully applied to all the models mentioned so far [Hai14a, Hai13], to other models like the Sine-Gordon model [HS16] (which however can also be handled via paracontrolled techniques) and to study weak universality conjectures [HQ15, HX16].

Part I - Prerequisites

In the first part of this work, which spans chapters 1,2,3, we introduce some basic concepts and develop some techniques that are widely used in the rest of the work.

In Chapter 1 we begin with introducing the Littlewood-Paley (L-P) decomposition, that is a way of decomposing a tempered distribution f in *blocks* $\Delta_i f$ with compactly supported Fourier transform of magnitude $\simeq 2^i$. The L-P blocks are then used to define Besov spaces $B_{p,q}^\alpha \forall \alpha \in \mathbb{R}, p, q \in [1, \infty]$ and some basic properties are proved. The well-known results of Section 1.1.1 constitute the basis for our analysis of SPDEs via paracontrolled calculus.

A characterization of local Besov spaces on bounded domains $U \subset \mathbb{R}^d$, based on compactly supported wavelets, is given in Section 1.1.2. We discuss also the equivalence between the definition of Besov spaces given in this section and similar ones given in [Hai14a, HL16]. Finally we enounce a tightness criterion for families of random distributions that will be used to study the Ising magnetization field, and give a formulation of Kolmogorov's continuity theorem in this setting. An advantage of working with compactly supported wavelets is to be able to define local Besov spaces on U that are tolerant to bad behaviour close to the boundary. This is useful for example when considering families of distributions converging in a domain $U' \subset U$ that converges to U . Section 1.1.2 appeared as part of [FM17], a joint work with J-C.Mourrat.

In Section 1.1.3 we discuss the equivalence between the two definitions of Besov spaces presented before. Section 1.2 contains the main analytic tools of paracontrolled calculus.

Chapter 2 deals with the techniques needed to estimate random fields for showing the convergence of the enhanced noise (\mathbb{X}_ε in our example eq. (5)). One of the main tools for this task is the Wiener chaos decomposition, in which a square-integrable random variable which is measurable with respect to the white noise, has an orthogonal decomposition as a sum of random variables belonging to subspaces $\mathcal{H}_n \forall n \geq 0$. For random fields with a finite chaos decomposition, one uses Nelson's hypercontractivity property to estimate L^p norms by L^2 norms. The covariance of the random field obtained in this way takes the form of a convolution of singular functions with known degree of homogeneity (and we call it *diagram* in analogy with Feynman's diagrams). This well-known procedure is explained in Section 2.2.

In Section 2.3 we recall some results of Malliavin calculus that make it possible to estimate random fields with an infinite chaos decomposition.

Finally, Chapter 3 contains an introduction on two closely related models: FK percolation and the 2-d Ising model.

Part II - Results

Chapter 4

The content of this chapter first appeared in [FM17] as a joint work with J-C.Mourrat. We apply the tightness criterion for random fields developed in Section 1.1.2 to study the magnetization field of the two-dimensional Ising model at critical temperature. Let $U \subseteq \mathbb{R}^2$ be an open set, and for $a > 0$, let $U_a := U \cap (a\mathbb{Z}^2)$. Denote by $(\sigma_y)_{y \in U_a}$ the Ising spin system at the critical temperature, with, say, + boundary condition, and define the magnetization field

$$\Phi_a := a^{-\frac{1}{8}} \sum_{y \in U_a} \sigma_y \mathbb{1}_{S_a(y)}$$

where $S_a(y)$ is the square centered at y of side length a . In [CGN15], the authors showed that for $U = [0, 1]^2$ and every $\varepsilon > 0$, the family $(\Phi_a)_{a \in (0,1]}$ is tight in the Besov space $B_{2,2}^{-1-\varepsilon}(U)$ and proceeded to discuss similar results in more general domains. They asked in which precise function spaces the family $(\Phi_a)_{a \in (0,1]}$ is tight. We answer this question by showing that for every open set $U \subset \mathbb{R}^2 \forall \varepsilon > 0$ the magnetization field is tight in $B_{p,q}^{-1/8-\varepsilon, \text{loc}}(U) \forall p, q \in [1, \infty]$, and that it is not tight in $B_{p,q}^{-1/8+\varepsilon, \text{loc}}(\mathbb{R}^2)$.

It was shown recently that there exists a unique limit point to the family $(\Phi_a)_{a \in (0,1]}$, see [CGN15, CHI15]. Our result makes it clear that this limit is singular (even on compact subsets) with respect to the planar Gaussian free field, since the latter takes values in $B_{p,q}^{-\varepsilon, \text{loc}}(\mathbb{R}^2)$ for every $\varepsilon > 0$ and $p, q \in [1, \infty]$.

Chapter 5

This chapter draws from the paper [FG16] which is a joint work with M.Gubinelli. We develop a nonlinear paracontrolled calculus in order to show the local well-posedness of some quasi-linear uniformly parabolic SPDEs. We will consider mainly the two equations

$$\partial_t u(t, x) - a_1(u(t, x))\Delta u(t, x) = \xi(x), \quad u(0, x) = u_0(x), \quad x \in \mathbb{T}^2, t \geq 0,$$

and the slightly more general

$$\partial_t u(t, x) - a_1(u(t, x))\Delta u(t, x) = a_2(u(t, x))\xi(x), \quad u(0, x) = u_0(x), \quad x \in \mathbb{T}^2, t \geq 0$$

with $a_1: \mathbb{R} \rightarrow [\lambda, 1]$, $a_2: \mathbb{R} \rightarrow [-L, L]$ for $\lambda, L > 0$ uniformly bounded C^3 diffusion coefficients, and $\|a_i^{(k)}\|_{L^\infty} \leq 1$ for $k = 0, \dots, 3$. We assume that the random noise ξ takes values in the Besov space $\mathcal{C}^{\alpha-2}(\mathbb{T}^2)$ with $2/3 < \alpha < 1$: this would apply for example to the space white noise on \mathbb{T}^2 .

However, we show in Section 5.5 that the nonlinear paracontrolled calculus developed in this chapter allows to deal with a class of equations of the form

$$a_3(u(t, x))\partial_t u(t, x) - a_1(u(t, x))\Delta u(t, x) = \xi(a_2(u(t, x)), t, x), \quad x \in \mathbb{T}^2, t \geq 0,$$

where a_1, a_2, a_3 are sufficiently smooth non-degenerate coefficients and $\xi(z, t, x)$ is a Gaussian process with covariance

$$\mathbb{E}[\xi(z, t, x)\xi(z', t', x')] = F(z, z')Q(t - t', x - x'), \quad x, x' \in \mathbb{T}^2, t, t', z, z' \in \mathbb{R},$$

with F a smooth function and Q a distribution of parabolic regularity $\rho > -4/3$. This includes as a special case the space white noise discussed before, but we could consider a time white noise with a regular dependence on the space variable, or some noise which is mildly irregular in space and time.

Recently Otto and Weber [OW16] and Bailleul, Debussche and Hofmanová [BDH16] investigated quasilinear SPDEs in the context of pathwise methods and in a range of regularities compatible with the ones we consider in this chapter.

In [OW16] the authors obtained a local well-posedness result for equations of the form

$$\partial_t u(t, x) - a(u(t, x)) \partial_x^2 u(t, x) = f(u(t, x)) \xi(t, x), \quad t \geq 0$$

where the space variable belongs to a one dimensional periodic domain and the noise can be white in time but colored in space, essentially behaving like a distribution of parabolic regularity in $(-4/3, 1)$. In order to do that they introduce a specific notion of *modelled* function and related estimates. Their *parametric* controlled Ansatz is the main source of inspiration for the work [FG16]. However, this fundamental observation does not necessitate the development of an alternative theory for singular SPDEs, which is the main aim of their work, and can be employed in a paracontrolled distributions framework. Indeed, in Chapter 5 (as done in [FG16]) we show that a relatively straightforward extension of the paracontrolled approach is sufficient to retrieve quite directly the results on quasilinear equations contained in their paper.

Bailleul, Debussche and Hofmanová in [BDH16] obtained local well-posedness for the generalised parabolic Anderson model equation

$$\partial_t u(t, x) - a(u(t, x)) \Delta u(t, x) = g(u(t, x)) \xi(x) \quad t \geq 0, x \in \mathbb{T}^2.$$

The authors obtained the same result as the one presented in Section 5.4 of our work, without the machinery of nonlinear paraproducts introduced here, but using only the basic tools of paracontrolled analysis and some clever transformations. This simplicity comes with a loss of generality, for example a quasilinear SPDE with matrix-valued diffusion coefficients $(a_{ij})_{i,j}$, i.e. an equation of the form

$$\partial_t u(t, x) - a_{ij}(u(t, x)) \frac{\partial^2}{\partial x_i \partial x_j} u(t, x) = g(u(t, x)) \xi, \quad t \geq 0, x \in \mathbb{T}^2,$$

is out of reach of the techniques used in [BDH16], while can be treated with our techniques and by Otto and Weber's approach.

More recently, Hairer and Gerencsér in [GH17a] developed a theory of parameter-dependent regularity structures, and the corresponding convergence of random models, to solve a wide range of quasilinear SPDEs (including in particular those with noise regularity that is out of reach of the methods of [FG16], [OW16] and [BDH16]). Their theory generalizes greatly the previous work on quasilinear SPDEs.

Chapter 6

The content of this chapter first appeared in [FG18] (joint work with M.Gubinelli). We study the following family of SPDEs on $[0, T] \times \mathbb{T}^3$ indexed on $\varepsilon \in (0, 1]$

$$\mathcal{L}u_\varepsilon(t, x) = -\varepsilon^{-\frac{3}{2}} F_\varepsilon(\varepsilon^{\frac{1}{2}} u_\varepsilon(t, x)) + \eta_\varepsilon(t, x) \quad (7)$$

with η_ε a stationary centered Gaussian field on $\mathbb{R} \times \mathbb{T}^3$ converging in law to the time-space white noise for $\varepsilon \rightarrow 0$, and $(F_\varepsilon)_\varepsilon \subseteq C^9(\mathbb{R})$ with all derivatives having at most exponential growth.

Let Y_ε be the stationary solution to $\mathcal{L}Y_\varepsilon = -Y_\varepsilon + \eta_\varepsilon$. Under some assumptions on the convergence of the first 4 Wiener chaos decomposition coefficients of $\varepsilon^{-3/2} F_\varepsilon(\varepsilon^{1/2} Y_\varepsilon)$ (and the convergence of the initial conditions $u_{0,\varepsilon}$), we show that the solutions of (7) have a limit $u_\varepsilon \rightarrow u$ in law that depends just on 4 parameters. This kind of result is called *weak universality*: the universality comes from the fact that solutions of (7) for a finite ε depend on the whole function F_ε (for example from its complete chaos decomposition), but the limit $u_\varepsilon \rightarrow u$ depends only on a finite number of parameters that are obtained from the first chaos coefficients of F_ε . The adjective *weak* comes from the fact that we are restricted to nonlinearities that become asymptotically small with respect to the noise η_ε . This convergence result was first obtained by Hairer and Xu [HX16], but with the important restriction of assuming F_ε to be a polynomial $\forall \varepsilon > 0$.

The limit of the solutions u_ε of (7) coincides with the limit of the solution of

$$\mathcal{L}u_\varepsilon = -\lambda_3 u_\varepsilon^3 - \lambda_2 u_\varepsilon^2 - c_\varepsilon u_\varepsilon - d_\varepsilon + \eta_\varepsilon \quad (8)$$

for $\lambda_3, \lambda_2 \in \mathbb{R}$ and diverging renormalization constants $c_\varepsilon, d_\varepsilon$. This equation is called the Φ_3^4 model. In two dimensions, the Φ_2^4 model has been subject of various studies for more than thirty years [JM85, AR91, DD03]. For the three dimensional case, the convergence of the solutions to (8) is originally due to Hairer [Hai14a, Hai15] and constitute one of the first groundbreaking applications of his theory of regularity structures. A similar result was later obtained by Catellier and Chouk [CC13] using the paracontrolled approach of Gubinelli, Imkeller and Perkowski [GIP15]. Kupiainen [Kup14] described a third approach using renormalization group ideas.

The first result of weak universality for a singular SPDE has been given by Hairer and Quastel [HQ15] in the context the Kardar–Parisi–Zhang (KPZ) equation in the case of a polynomial non-linearity. Using the machinery developed there, Hairer and Xu [HX16] proved a weak universality result for three dimensional reaction–diffusion equations in the case of Gaussian noise and a polynomial non-linearity, within the context of regularity structures. Weak universality for reaction–diffusion equations driven by non Gaussian noise is analysed by Shen and Xu [SX16]. The recent preprint [OGK17] analyzes an hyperbolic version of the stochastic quantisation equation in two dimensions, including the associated universality in the small noise regime.

Recently, Hairer and Xu [HX18] generalized the result of Hairer et Quastel on the convergence of a certain class of models to the solution of the KPZ equation [HQ15]. They use more general hypotheses for the nonlinearity (in particular eliminating the need for it to be polynomial). Their approach differs substantially from ours, but could in principle retrieve the same results as [FG18]. Conversely, it is possible to show the convergence of the family of equations considered in [HX18] to the KPZ equation using the theory we first developed in [FG18].

The paper [FG18], from which this chapter is derived, is the first to consider in detail the weak universality problem with the technique of paracontrolled distributions, showing that on the analytic side the a priori estimates can be obtained via standard arguments (we rely in particular on the paracontrolled construction of [CC13]) and that the major difficulty is related to showing the convergence of a finite number of random fields with infinite chaos decomposition to universal limiting objects. The main novelty of our work is the use of Malliavin calculus [Nua06, NN10, NP12] to perform the analysis of these stochastic terms without requiring the non-linearity to be polynomial as in [HX16].

Prerequisites

Chapter 1

Besov spaces and paracontrolled calculus

1.1 Besov spaces

In this chapter we present two different (but equivalent) characterizations of nonhomogeneous Besov spaces $B_{p,q}^\alpha$ (with $p, q \in [1, \infty]$, $\alpha \in \mathbb{R}$) and recall some well-known properties of these spaces. A good book on Littlewood-Paley theory applied to Besov spaces is the one by Bahouri, Chemin and Danchin [BCD11]. The discussion on Besov spaces defined via compactly supported wavelets is taken from the article [FM17], which draws some results from [HL16], [Hai14a], [Mey92]. We will consider Besov spaces on either \mathbb{R}^d , an open bounded domain $U \subset \mathbb{R}^d$ or the d -dimensional torus $\mathbb{T}^d := (\mathbb{R}^d / 2\pi\mathbb{Z})^d$. All the results for Besov spaces on \mathbb{R}^d also hold *mutatis mutandis* for \mathbb{T}^d .

We begin by introducing some general notation. If $u = (u_n)_{n \in I}$ is a family of real numbers indexed by a countable set I , and $p \in [1, \infty]$, we write

$$\|u\|_{\ell^p} = \left(\sum_{n \in I} |u_n|^p \right)^{1/p},$$

with the usual interpretation as a supremum when $p = \infty$. We write $B(x, R)$ for the open Euclidean ball centred at x and of radius R . For every open set $U \subseteq \mathbb{R}^d$ and $n \in \mathbb{N} \cup \{\infty\}$, we write $C^n(U)$ to denote the set of n times continuously differentiable functions on U , and $C_c^n(U)$ the subset of $C^n(U)$ of functions with compact support. We simply write C^n and C_c^n for $C^n(\mathbb{R}^d)$ and $C_c^n(\mathbb{R}^d)$ respectively. For $f \in C^n$, we write

$$\|f\|_{C^n} := \sum_{|a| \leq n} \|\partial^a f\|_{L^\infty},$$

where the sum is over multi-indices $a \in \mathbb{N}^d$.

1.1.1 Characterization of Besov spaces via Littlewood-Paley theory

In this section we present a definition of Besov spaces based on Littlewood-Paley decomposition. Due to its localization in Fourier space, this characterization is best suited to introduce Bony's paraproducts (Section 1.2).

Let's start with showing that it is possible to construct a smooth partition of unity in \mathbb{R}^d made of functions with support in concentric annuli, such that in every point there are at most two overlapping functions. An annulus $\mathcal{A}(r, R)$ in \mathbb{R}^d is defined $\forall 0 < r < R$ as follows:

$$\mathcal{A}(r, R) := \{x \in \mathbb{R}^d \mid r < |x| < R\}.$$

Proposition 1.1. (dyadic partition of unity)

There exist radial functions $\chi, \rho \in C_c^\infty$ with values in $[0, 1]$ and constants $r, R, R' \in \mathbb{R}^+$

such that

$$\begin{aligned} \text{supp } \rho &\subset \mathcal{A}(r, R), \\ \text{supp } \chi &\subset B(0, R') \end{aligned}$$

and for every $x \in \mathbb{R}^d$:

$$1 = \chi(x) + \sum_{j \geq 0} \rho(2^{-j}x)$$

with a locally finite sum. Also, if $|j - j'| \geq 2$ then:

$$\text{supp } \rho(2^{-j} \cdot) \cap \text{supp } \rho(2^{-j'} \cdot) = \emptyset \quad (1.1)$$

and if $j \geq 1$:

$$\text{supp } \chi \cap \text{supp } \rho(2^{-j} \cdot) = \emptyset \quad (1.2)$$

In particular, we can take $r = 3/4$, $R = 8/3$, $R' = 4/3$.

Proof. The proof is well known, here we follow [BCD11]. Fix $\alpha \in (1, 4/3)$. Choose a radial function $\theta \in C_c^\infty(\mathbb{R}^d)$ such that $\text{supp } \theta \subseteq \mathcal{A}(3/4, 4/3)$ and $\theta|_{\mathcal{A}(\alpha^{-1}, 2\alpha)} = 1$ (such a function trivially exists). The property (1.1) is verified by the different scalings of θ , and defining the function $\Theta(x) := \sum_{j \in \mathbb{Z}} \theta(2^{-j}x)$ we see that this sum is locally finite on $\mathbb{R}^d - \{0\}$. We have also $\bigcup_{j \in \mathbb{Z}} 2^j \mathcal{A}(\alpha^{-1}, 2\alpha) = \mathbb{R}^d - \{0\}$, and then $\Theta(x) \neq 0$ for every $x \in \mathbb{R}^d - \{0\}$. Thus, by the continuity of Θ , we can affirm that $\Theta > 0$ on \mathbb{R}^d . We define for every $j \geq 0$:

$$\begin{aligned} \rho &:= \frac{\theta}{\Theta} \\ \chi &:= 1 - \sum_{j \geq 0} \rho(2^{-j} \cdot) \end{aligned}$$

It is now straightforward to show that ρ and χ have the desired properties. \square

Remark 1.2. It is worth noting that for all the results of Littlewood-Paley theory it is not important what the functions ρ and χ are, as long as they satisfy the conditions given in Proposition 1.1.

We adopt the following normalization for the Fourier transform $\mathcal{F}: L^1(\mathbb{R}^d) \rightarrow L^\infty(\mathbb{R}^d)$:

$$\begin{aligned} \hat{f}(\xi) &:= \mathcal{F}f(\xi) := \int_{\mathbb{R}^d} e^{-i\langle \xi, x \rangle} f(x) dx \\ \mathcal{F}^{-1}g(x) &:= (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i\langle \xi, x \rangle} g(\xi) d\xi \end{aligned}$$

and we extend it by duality to the space of tempered distributions $\mathcal{S}'(\mathbb{R}^d)$.

Littlewood-Paley decomposition

Definition 1.3. (Littlewood-Paley blocks)

Let $u \in \mathcal{S}'(\mathbb{R}^d)$, for every $j \in \mathbb{N}$ the Littlewood-Paley blocks are defined as follows:

$$\begin{aligned} \Delta_{-1}u &:= \mathcal{F}^{-1}(\chi \hat{u}) \\ \Delta_j u &:= \mathcal{F}^{-1}(\rho(2^{-j} \cdot) \hat{u}) \end{aligned}$$

We also define the high-frequency cut-off operator $S_j: \mathcal{S}'(\mathbb{R}^d) \rightarrow \mathcal{S}'(\mathbb{R}^d)$ for $j \in \mathbb{N}$ as

$$S_j u := \sum_{k < j-1} \Delta_k u \quad (1.3)$$

Remark 1.4. Note that $\forall j \in \mathbb{N}$:

$$\Delta_j f(x) = \int_{\mathbb{R}^d} K_{j,x}(y) f(y) dy, \quad S_j f(x) = \int_{\mathbb{R}^d} P_{j,x}(y) f(y) dy \quad (1.4)$$

with

$$K_{j,x}(y) := 2^{jd} K(2^j(x-y)), \quad P_{j,x}(y) := \sum_{j'=-1}^{j-2} K_{j',x}(y)$$

where $K = \mathcal{F}^{-1}\rho$ and $K_{-1,x}(y) = \mathcal{F}^{-1}\chi(x-y)$. Here K and $\mathcal{F}^{-1}\chi$ are radially symmetric and belong to the Schwartz space $\mathcal{S}(\mathbb{R})$ of smooth rapidly decreasing functions (since it can be seen easily that the Fourier transform is a continuous endomorphism on $\mathcal{S}(\mathbb{R}^d) \subset C_c^\infty(\mathbb{R}^d)$).

From Plancherel's theorem one obtains then $\forall x \in \mathbb{R}^d, j \in \mathbb{N}$:

$$\begin{aligned} \int_{\mathbb{R}^d} K_{j,x}(y) dy &= \int_{\mathbb{R}^d} 2^{jd} K(2^j y) dy \\ &= (2\pi)^d \langle \delta, \rho(2^{-j}\cdot) \rangle \\ &= 0 \end{aligned}$$

since ρ is supported away from zero (and $\delta \in \mathcal{S}'(\mathbb{R}^d)$ is the Dirac's delta), and

$$\begin{aligned} \int_{\mathbb{R}^d} P_{j,x}(y) dy &= \int_{\mathbb{R}^d} \mathcal{F}^{-1}\chi(y) dy \\ &= 1 \end{aligned}$$

since $\chi(0) = 1$ by construction.

Remark 1.5. From Young's inequality for convolutions it is easy to see that there exists a positive constant $M \in \mathbb{R}$ such that $\forall j \geq -1, \forall p \in [1, \infty]$

$$\begin{aligned} \|\Delta_j f\|_{L^p} &\leq M \|f\|_{L^p} \\ \|S_j f\|_{L^p} &\leq M \|f\|_{L^p} \end{aligned}$$

since $\|K_{j,x}\|_{L^1} = \|K\|_{L^1}$ and $\|P_{j,x}\|_{L^1} = \|P\|_{L^1} \forall j \geq 0$.

Lemma 1.6. (\mathcal{S}' convergence of L-P decomposition)

Let $u \in \mathcal{S}'(\mathbb{R}^d)$, then $\lim_{n \rightarrow +\infty} \sum_{j \geq -1}^n \Delta_j u = u$.

Proof.

The operator Δ_j is self-adjoint since the Fourier transform is self-adjoint. Therefore we obtain

$$|\langle S_n u - u, \varphi \rangle| = |\langle u, S_n \varphi - \varphi \rangle| \lesssim \|S_n \varphi - \varphi\|_{\mathcal{S},k}$$

$\forall k \in \mathbb{N}$, where we introduced the seminorm

$$\|\varphi\|_{\mathcal{S},k} := \sup_{x \in \mathbb{R}^d} \sup_{|a| \leq k} (1+|x|)^k |\partial^a \varphi(x)|.$$

with the derivative on a multi index $a = (a_1, \dots, a_d)$ as usual: $\partial^a = \partial^{a_1} \dots \partial^{a_d}$. Since the Fourier transform is a continuous operator over \mathcal{S} it is enough to show that $\mathcal{F}(S_n \varphi)$ converges to $\mathcal{F}(\varphi)$ in \mathcal{S} . From the definition of the L-P blocks Δ_j we have

$$\mathcal{F}(S_n \varphi) = \chi \varphi + \sum_{j=0}^n \rho(2^{-j}\cdot) \hat{\varphi}.$$

From the properties of the dyadic partition of unity given in Proposition 1.1 we see that there exists $R > 0$ such that

$$\begin{aligned} \mathcal{F}(S_n \varphi)(x) &= 0 & \forall x \in B^c(0, 2^n R), \\ \mathcal{F}(S_n \varphi)(x) &= 1 & \forall x \in B(0, 2^{n-1} R), \\ \chi(x) + \sum_{j=0}^n \rho(2^{-j} x) &= \rho(2^{-n} x) & \forall x \in B(0, 2^n R) \setminus B(0, 2^{n-1} R). \end{aligned}$$

Within the proof we note $B_{2^n} := B(0, 2^n R)$. We proceed to estimate $\mathcal{F}(S_n \varphi) - \mathcal{F}(\varphi)$ on the three separate regions $B_{2^{n-1}}$, $B_{2^n} \setminus B_{2^{n-1}}$ and $(B_{2^n})^c$. On $B_{2^{n-1}}$ this term is zero, on $(B_{2^n})^c$ it is equal to $\hat{\varphi} \in \mathcal{S}(\mathbb{R}^d)$ and thus decreases more rapidly than any rational function at infinity, and on $B_{2^n} \setminus B_{2^{n-1}}$ its derivatives are bounded by a 2^{-n} coefficient (because the annuli $2^j \mathcal{A}$ become wider as well as larger). We obtain then for every $k \in \mathbb{N}$:

$$\begin{aligned} & \|\mathcal{F}(S_n \varphi - \varphi)\|_{\mathcal{S}, k} = \\ &= \sup_{x \in (B_{2^{n-1}})^c} \sup_{|a| \leq k} (1 + |x|)^k |\partial^a [\rho(2^{-n} x) \hat{\varphi}(x) + \rho(2^{-n} x) \hat{\varphi}(x) - \hat{\varphi}(x)]| \\ &\leq \sup_{x \in (B_{2^n})^c} \sup_{|a| \leq k} (1 + |x|)^k |\partial^a \hat{\varphi}(x)| + \\ &\quad + \sup_{x \in B_{2^n} \setminus B_{2^{n-1}}} \sup_{|a| \leq k} (1 + |x|)^k |\partial^a [\rho(2^{-n} x) \hat{\varphi}(x) - \hat{\varphi}(x)]| \\ &\leq \sup_{x \in (B_{2^n})^c} \sup_{|a| \leq k} (1 + |x|)^k |\partial^a \hat{\varphi}(x)| + \sup_{x \in B_{2^n} \setminus B_{2^{n-1}}} \sup_{|a| \leq k} (1 + |x|)^k |\partial^a \hat{\varphi}(x)| \\ &\quad + \sup_{x \in B_{2^n} \setminus B_{2^{n-1}}} \sup_{|a| \leq k} (1 + |x|)^k \sum_{|b| < |a|} C_b \partial^{a-b} \rho(2^{-n} x) \partial^b \hat{\varphi}(x) \\ &\lesssim \sup_{x \in (B_{2^{n-1}})^c} (1 + |x|)^{-1} \|\hat{\varphi}\|_{\mathcal{S}, k+1} + 2^{-(n-1)} \|\hat{\varphi}\|_{\mathcal{S}, k} \end{aligned}$$

which converges to zero for $n \rightarrow +\infty$. \square

Remark 1.7. The Littlewood-Paley blocks $\Delta_j u$ are smooth functions for every $u \in \mathcal{S}'(\mathbb{R}^d)$, $j \geq -1$. This follows from the Paley-Wiener-Schwartz theorem (see [Hör03, Th.7.1.14 and 7.3.1]) since the blocks $\Delta_j u$ have compactly supported Fourier transform. Another way to see that $\Delta_j u$ is smooth is by noting that for every $u \in \mathcal{S}'(\mathbb{R}^d)$ there exists a continuous function f and $a \in \mathbb{N}^d$ such that $\forall \varphi \in \mathcal{S}(\mathbb{R}^d)$:

$$\langle u, \varphi \rangle = \int (-1)^{|a|} f(x) (\partial^a \varphi)(x) dx$$

(see [RS80, Th.V.10] for the proof). Then since $\Delta_j u = \eta_j * u$ with $\eta_j \in C^\infty(\mathbb{R}^d)$ it is easy to see that $\forall b \in \mathbb{N}^d$

$$\partial^b \Delta_j u(y) = \partial^b \eta_j * u(y) = \int (-1)^{|a|} f(x) \partial^{a+b} \eta_j(y-x).$$

Distributions with compactly supported Fourier transform

We recall here some properties of functions with compactly supported Fourier transform, which will be useful in the following.

Lemma 1.8. (Bernstein inequalities)

Let \mathcal{A} be an annulus and $B(0, R)$ a ball in \mathbb{R}^d (for some $R > 0$). Then there exists $C > 0$ such that $\forall k \in \mathbb{N}$, $\forall \lambda > 0$ and for every $1 \leq p \leq q \leq +\infty$:

1. if $f \in L^p$ and $\text{supp } \hat{f} \subset \lambda B(0, R)$ then:

$$\sup_{|a| \leq k} \|\partial^a f\|_{L^q} \leq C^{k+1} \lambda^{k+d\left(\frac{1}{p}-\frac{1}{q}\right)} \|f\|_{L^p}$$

2. if $f \in L^p$ and $\text{supp } \hat{f} \subset \lambda \mathcal{A}$ then:

$$\lambda^k \|f\|_{L^p} \leq C^{k+1} \sup_{|a| \leq k} \|\partial^a f\|_{L^p}$$

Proof. The idea of the proof is that, when deriving k times the function f , one obtains a term of order x^k in its Fourier transform, which has to be roughly the same size of the support of \hat{f} (which is contained in either $\lambda B(0, R)$ or $\lambda \mathcal{A}$) and this explains the factor λ^k . The factor $\lambda^{d/p-d/q}$ is simply due to the different scaling behaviour of the L^p and L^q norms. To prove point 1, let $\lambda = 1$. Let $\psi \in C_c^\infty$ supported on a ball of \mathbb{R}^d such that $\hat{f} = \hat{f}\psi$, and thus $f = g * f$ with $g = \mathcal{F}^{-1}(\psi)$ as well as $\partial^a f = \partial^a g * f$ for every multi-index $a \in \mathbb{N}^d$. From Young's inequality $\|\partial^a f\|_{L^q} \leq \|\partial^a g\|_{L^r} \|f\|_{L^p}$ with $1/q + 1 = 1/p + 1/r$. From Hölder's inequality $\|\partial^a g\|_{L^r} \leq \|(1 + |\cdot|^2)^{-d}\|_{L^r} \|(1 + |\cdot|^2)^d \partial^a g\|_{L^\infty}$. By interpolation and Young's inequality for products $\exists C > 0$ such that:

$$\begin{aligned} \|\partial^a g\|_{L^r} &\leq \|\partial^a g\|_{L^1} + \|\partial^a g\|_{L^\infty} \\ &\leq C \|(1 + |\cdot|^2)^d \partial^a g\|_{L^\infty} \\ &\leq C \|(\text{id} - \Delta)^d (\cdot)^a \psi\|_{L^1} \\ &\leq C^{k+1} \end{aligned}$$

since x^a can be estimated in L^∞ on the support of ψ . Let now $\text{supp } \hat{f} \subset \lambda B(0, R)$ with $\lambda \neq 1$. Defining $h(x) = f(\lambda^{-1}x)$ we obtain $\mathcal{F}(h) = \lambda^d(\hat{f})(\lambda \cdot)$ and then by applying on h the inequality above we obtain the first claim.

To show point 2, let as before $\lambda = 1$. We have for every $x \in \mathbb{R}^d$:

$$\begin{aligned} \hat{f}(x) &= |x|^{-2k} \hat{f}(x) [(ix_1)(-ix_1) + \dots + (ix_d)(-ix_d)]^k \\ &= |x|^{-2k} \hat{f}(x) \sum_{|a|=k} A_a (ix)^a (-ix)^a \end{aligned} \quad (1.5)$$

with $A_a = \binom{k}{a_1, \dots, a_d}$. If $\psi \in C_c^\infty$ with support contained in an annulus is chosen such that $\hat{f} = \hat{f}\psi$, then:

$$\hat{f}(x) = \sum_{|a|=k} [(ix)^a \hat{f}(x)] [A_a \frac{(-ix)^a}{|x|^{2k}} \psi(x)] \quad (1.6)$$

and applying the inverse Fourier transform we obtain

$$f(x) = \sum_{|a|=k} \mathcal{F}^{-1}[(ix)^a \hat{f}] * \mathcal{F}[A_a \frac{(-ix)^a}{|x|^{2k}} \psi(x)] = \sum_{|a|=k} \partial^a f * g_a$$

where $g_a = \mathcal{F}^{-1}(A_a \frac{(-ix)^a}{|x|^{2k}} \psi(x))$. We can estimate as before (since $\sum_{|a|=k} A_a = d^k$)

$$\begin{aligned} \sum_{|a|=k} \|g_a\|_{L^1} &\leq C \|(1 + |\cdot|^2)^d g_a\|_{L^\infty} \\ &\leq C \sum_{|a|=k} A_a \|(\text{id} - \Delta)^d \frac{(-ix)^a}{|x|^{2k}} \psi(x)\|_{L^\infty} \\ &\leq C^{k+1} \end{aligned}$$

to obtain

$$\|f\|_{L^p} \leq C^{k+1} \sup_{|a|=k} \|\partial^a f\|_{L^p}$$

and the second claim follows from the same scaling argument as above. \square

One can ask when a collection of smooth functions with compactly supported Fourier transforms converges in \mathcal{S}' . The following lemma answers this question.

Lemma 1.9.

Let $(u_j)_{j \in \mathbb{N}}$ be a collection of C^∞ functions with Fourier transform supported on rescaled annuli $2^j \mathcal{A}$. If there exists $n \in \mathbb{N}$ such that:

$$\sup_{j \in \mathbb{N}} (2^{-jn} \|u_j\|_{L^\infty}) < +\infty$$

then there exists $u \in \mathcal{S}'$ such that

$$\sum_{j=0}^n u_j \xrightarrow{\mathcal{S}'} u \quad \text{as } n \rightarrow +\infty.$$

Proof. As in the proof of Lemma 1.8 we can write $\forall k \in \mathbb{N}$:

$$\begin{aligned} u_j(x) &= \sum_{|a|=k} \mathcal{F}^{-1}[(ix)^a \hat{u}_j] * 2^{j(d-k)} g_a(2^j x) \\ &= \sum_{|a|=k} 2^{j(d-k)} g_a(2^j x) * \partial^a u_j \end{aligned}$$

We have for every $\varphi \in \mathcal{S}(\mathbb{R}^d)$:

$$\langle u_j, \varphi \rangle = 2^{-jk} \sum_{|a|=k} \langle u_j, 2^{jd} g_a(-2^j \cdot) * (-\partial)^a \varphi \rangle$$

and then:

$$\begin{aligned} |\langle u_j, \varphi \rangle| &\leq 2^{-jk} \sum_{|a|=k} \|u_j\|_{L^\infty} \|g_a\|_{L^1} \|\partial^a \varphi\|_{L^1} \\ &\lesssim 2^{-jk} 2^{jn} \sum_{|a|=k} \|\partial^a \varphi\|_{L^1}. \end{aligned}$$

Choosing $k > n$ the sequence converges. \square

Besov spaces: definition and first properties

The reason we introduced the Littlewood-Paley decomposition is to be able to define Besov spaces as follows:

Definition 1.10. (Besov spaces via L-P decomposition)

Let $\alpha \in \mathbb{R}$, $1 \leq p, q \leq \infty$. A tempered distribution $u \in \mathcal{S}'(\mathbb{R}^d)$ belongs to the Besov space $B_{p,q}^\alpha(\mathbb{R}^d)$ if the norm

$$\|u\|_{B_{p,q}^\alpha} = \|(2^{j\alpha} \|\Delta_j u\|_{L^p})_{j \geq -1}\|_{\ell^q} \quad (1.7)$$

is finite. We note $\mathcal{C}^\alpha := B_{\infty,\infty}^\alpha$ with the following norm:

$$\|u\|_{\mathcal{C}^\alpha} = \sup_{j \geq -1} 2^{j\alpha} \|\Delta_j u\|_{L^\infty}. \quad (1.8)$$

The local Besov space $B_{p,q}^{\alpha,\text{loc}}(U)$ is the completion of $C^\infty(U)$ with respect to the family of semi-norms

$$f \mapsto \|\chi f\|_{B_{p,q}^\alpha}$$

indexed by $\chi \in C_c^\infty(U)$.

Remark 1.11. Besov spaces are Banach spaces [BCD11, Th.2.72] and for $p, q \in [1, +\infty)$ $C_c^\infty(\mathbb{R}^d)$ is dense in $B_{p,q}^\alpha$ for every $\alpha \in \mathbb{R}$ [BCD11, Prop.2.74]. In particular $B_{p,q}^\alpha(\mathbb{R}^d)$ is separable for p, q both finite. It is easy to see that if $p = \infty$ or $q = \infty$ the space $B_{p,q}^\alpha(\mathbb{R}^d)$ is not separable (since neither L^∞ nor ℓ^∞ are separable).

Remark 1.12. and it is easy to show that they are independent from the choice of a dyadic partition of unity. Indeed, let (χ', ρ') a partition of unity that satisfies (1.1) and (1.2) and Δ'_j its corresponding Littlewood-Paley blocks for $j \geq -1$. Then there exists $n_0 \in \mathbb{N}$ such that for every $|j - j'| > n_0$:

$$\begin{aligned} \text{supp } \rho(2^{-j} \cdot) \cap \text{supp } \rho'(2^{-j} \cdot) &= \emptyset, \\ \text{supp } \chi \cap \text{supp } \rho'(2^{-j} \cdot) &= \emptyset, \\ \text{supp } \rho(2^{-j} \cdot) \cap \text{supp } \chi' &= \emptyset. \end{aligned}$$

Then $\forall u \in \mathcal{S}'$

$$\begin{aligned} 2^{j\alpha} \|\Delta'_j u\|_{L^p} &\lesssim 2^{j\alpha} \sum_{|k-j| \leq n_0} \|\Delta_k u\|_{L^p} \\ &\lesssim \sum_{k \geq -1} \mathbb{1}_{[-n_0, n_0]}(j-k) \|\Delta_k u\|_{L^p} \end{aligned}$$

and therefore by Young's inequality

$$\|(2^{j\alpha} \|\Delta'_j u\|_{L^p})_{j \geq -1}\|_{\ell^q} \lesssim \|(2^{k\alpha} \|\Delta_k u\|_{L^p})_{k \geq -1}\|_{\ell^q}.$$

We remark that $B_{\infty, \infty}^\alpha$ coincides with the usual Hölder space when α is positive and not integer. Spaces $B_{\infty, \infty}^\alpha$ for $\alpha > 0$ are called Zygmund spaces [Tri92].

Proposition 1.13. *Let $\alpha \in (0, 1)$, $n \in \mathbb{N}$. Then*

$$B_{\infty, \infty}^{n+\alpha}(\mathbb{R}^d) = C^{n, \alpha}(\mathbb{R}^d)$$

where $C^{n, \alpha}(\mathbb{R}^d)$ is the Hölder space of $C^n(\mathbb{R}^d)$ functions such that

$$\|f\|_{C^{n, \alpha}} = \sup_{|a| \leq n} (\|\partial^a f\|_{L^\infty} + \sup_{x \neq y} \frac{|\partial^a f(x) - \partial^a f(y)|}{|x - y|^\alpha}) < +\infty$$

If $\alpha \in \mathbb{N}$ we have $C^\alpha(\mathbb{R}^d) \subset B_{\infty, \infty}^\alpha(\mathbb{R}^d)$ and the inclusion is strict.

Proof. We start proving the second claim. Let $f \in C^n(\mathbb{R}^d)$. By Bernstein inequalities:

$$\|f\|_{B_{\infty, \infty}^n} = \sup_{j \geq -1} 2^{jn} \|\Delta_j f\|_{L^\infty} \lesssim \sup_{|a| \leq n} \|\partial^a f\|_{L^\infty} = \|f\|_{C^n}$$

To show that $C^n \subsetneq B_{\infty, \infty}^n$, let $\forall x \in \mathbb{R}^d$ and

$$u_j^n(x) = \frac{1}{i^n 2^{nj}} [e^{i2^j x_1} + (-1)^n e^{-i2^j x_1}].$$

Then $\sup_{j \geq -1} \|u_j^n\|_{L^\infty} \leq 2^{-nj+1}$ and $\forall j \geq 0$ u_j^n has Fourier transform supported in a rescaled annulus $2^j \mathcal{A}$. Then by Lemma 1.9 there exists $u \in \mathcal{S}'(\mathbb{R}^d)$ such that $u = \sum_j u_j^n$, and $u \in B_{\infty, \infty}^n(\mathbb{R}^d)$, but

$$\left\| \frac{d}{dx_1} u_j^n \right\|_{L^\infty} \geq \left| \sum_j u_j^n(0) \right| = +\infty.$$

Let now $f \in B_{\infty, \infty}^{n+\alpha}$ with $\alpha \in (0, 1)$. Then for every multi-index b such that $|b| \leq n$ we have, again by Bernstein inequalities:

$$\|\partial^b \Delta_j f\|_{L^\infty} \lesssim 2^{-\alpha j} \|f\|_{B_{\infty, \infty}^{n+\alpha}} \quad (1.9)$$

and then $\sum_j \partial^b \Delta_j f$ converges in L^∞ , and as the derivative is a continuous operator on \mathcal{S}' we obtain that $f \in C^n$. We obtain:

$$\begin{aligned} \frac{|\partial^b f(y+h) - \partial^b f(y)|}{|h|^\alpha} &\lesssim \sum_{j=-1}^k \frac{|\partial^b \Delta_j f(y+h) - \partial^b \Delta_j f(y)|}{|h|^\alpha} \\ &\quad + \sum_{j \geq k+1} \frac{\|\partial^b \Delta_j f\|_{L^\infty}}{|h|^\alpha} \end{aligned}$$

with $k \geq -1$ such that $2^{-(k+1)} \leq |h| \leq 2^{-k}$ (if $|h| \geq 2$ just choose $j = -2$). Then using the bound (1.9) for the second term, and

$$|\partial^b \Delta_j f(y+h) - \partial^b \Delta_j f(y)| \lesssim 2^{j(n+1)} \|\Delta_j f\|_{L^\infty} |h|$$

on the first term we obtain:

$$\begin{aligned} \frac{|\partial^b f(y+h) - \partial^b f(y)|}{|h|^\alpha} &\lesssim \|f\|_{B_{\infty, \infty}^{n+\alpha}} \left(\sum_{j=-1}^k 2^{(1-\alpha)j} |h|^{1-\alpha} + \sum_{j \geq k+1} 2^{-j(n+\alpha)} |h|^{-\alpha} \right) \\ &\lesssim \|f\|_{B_{\infty, \infty}^{n+\alpha}} \end{aligned}$$

Conversely, let $f \in C^{n, \alpha}$. We have $\|\Delta_{-1} f\|_{L^\infty} \leq \|f\|_{L^\infty}$ and for every $j \geq 0$ again by Bernstein inequalities one obtains $\|\Delta_j f\|_{L^\infty} \lesssim 2^{-jn} \sup_{|b| \leq n} \|\partial^b \Delta_j f\|_{L^\infty}$. As noted in Remark 1.4 the kernel K_j associated to Δ_j has zero mass, and then:

$$\begin{aligned} \partial^b \Delta_j f(x) &= \int K_{j,x}(y) [\partial^b f(y) - \partial^b f(x)] dy \\ &\lesssim \|f\|_{C^{n, \alpha}} 2^{jd} \int K(2^j(x-y)) |x-y|^\alpha dy \\ &\lesssim 2^{-\alpha j} \|f\|_{C^{n, \alpha}}. \end{aligned}$$

This yields $\|f\|_{B_{\infty, \infty}^{n+\alpha}} \lesssim \|f\|_{C^{n, \alpha}}$. □

Another well-known characterization of Besov spaces worth noting is their relation with Sobolev spaces. We define the Sobolev space $H^\alpha(\mathbb{R}^d)$ as the tempered distributions $u \in \mathcal{S}'(\mathbb{R}^d)$ such that $\hat{u} \in L_{\text{loc}}^2(\mathbb{R}^d)$ and

$$\|u\|_{H^\alpha}^2 := \int_{\mathbb{R}^d} (1+|x|^2)^\alpha |\hat{u}(x)|^2 dx < +\infty.$$

Proposition 1.14. $H^\alpha(\mathbb{R}^d) = B_{2,2}^\alpha(\mathbb{R}^d)$.

Proof. Let $u \in H^\alpha(\mathbb{R}^d)$. Then by Plancherel's theorem

$$\begin{aligned} \|\Delta_j u\|_{L^2} &= \left[\int_{\mathbb{R}^d} (1+|x|^2)^{-\alpha} (1+|x|^2)^\alpha |\rho(2^{-j}x) \hat{u}(x)|^2 dx \right]^{\frac{1}{2}} \\ &\lesssim 2^{-\alpha j} \|u\|_{H^\alpha}. \end{aligned}$$

Conversely, let $u \in B_{2,2}^\alpha(\mathbb{R}^d)$. Then $\hat{u} \in L_{\text{loc}}^2(\mathbb{R}^d)$ and denoting as $\rho_j = \rho(2^{-j}\cdot)$ for $j \geq 0$ and $\rho_{-1} = \chi$ the dyadic partition of unity we have:

$$\begin{aligned} \|u\|_{H^\alpha}^2 &= \int_{\mathbb{R}^d} \left| \sum_{j \geq -1} \rho_j(x) \hat{u}(x) \right|^2 (1 + |x|^2)^\alpha dx \\ &\lesssim \sum_{j \geq -1} 2^{j\alpha} \|\Delta_j u\|_{L^2}^2. \end{aligned}$$

□

Remark 1.15. One can see immediately that $\forall 1 \leq p, q \leq +\infty$ and $\alpha < \beta$

$$\|\cdot\|_{B_{p,q}^\alpha} \lesssim \|\cdot\|_{B_{p,q}^\beta}$$

and for $q < r$ from the continuous embedding $\ell^q \hookrightarrow \ell^r$ it follows

$$\|\cdot\|_{B_{p,r}^\alpha} \lesssim \|\cdot\|_{B_{p,q}^\alpha}.$$

The following result generalizes Sobolev embeddings to the Besov setting:

Proposition 1.16. (Besov embedding)

Let $1 \leq p_1 \leq p_2 \leq +\infty$, $1 \leq q_1 \leq q_2 \leq +\infty$ and $\alpha \in \mathbb{R}$. Then the space $B_{p_1, q_1}^\alpha(\mathbb{R}^d)$ is continuously embedded in the space $B_{p_2, q_2}^{\alpha - d(\frac{1}{p_1} - \frac{1}{p_2})}(\mathbb{R}^d)$.

Proof. The proof follows from Bernstein inequalities (Lemma 1.8) and the fact that ℓ^{q_1} is continuously embedded in ℓ^{q_2} . □

1.1.2 Characterization of local Besov spaces via wavelets

In this section we give an alternative definition of Besov spaces, based on a decomposition on compactly supported wavelets (Daubechies' wavelets). Due to its localization, this characterization is best suited to define local Besov spaces on a bounded domain $U \subset \mathbb{R}^d$ and prove a tightness criterion (Theorem 1.46) in this framework. We also give a continuity criterion (analogous to Kolmogorov's continuity theorem, Proposition 1.49).

We define the Hölder space of exponent $\alpha < 0$ very similarly to [Hai14a, Definition 3.7].

Definition 1.17. (Besov-Hölder spaces of negative regularity)

Let $\alpha < 0$, $r_0 := -\lfloor \alpha \rfloor$, and

$$\mathcal{B}^{r_0} := \left\{ \eta \in C^{r_0} : \|\eta\|_{C^{r_0}} \leq 1 \text{ and } \text{supp } \eta \subseteq B(0, 1) \right\}.$$

For every $f \in C_c^\infty$, denote

$$\|f\|_{C^\alpha} := \sup_{\lambda \in (0,1]} \sup_{x \in \mathbb{R}^d} \sup_{\eta \in \mathcal{B}^{r_0}} \lambda^{-\alpha} \int_{\mathbb{R}^d} f \lambda^{-d} \eta\left(\frac{\cdot - x}{\lambda}\right). \quad (1.10)$$

The Hölder space $\bar{C}^\alpha(\mathbb{R}^d)$ is the completion of $C_c^\infty(\mathbb{R}^d)$ with respect to the norm $\|\cdot\|_{C^\alpha}$. For every open set $U \subseteq \mathbb{R}^d$, the local Hölder space $\bar{C}_{\text{loc}}^\alpha(U)$ is the completion of $C^\infty(U)$ with respect to the family of seminorms

$$f \mapsto \|\chi f\|_{C^\alpha},$$

where χ ranges in $C_c^\infty(U)$.

Remark 1.18. By definition, an element of \bar{C}^α defines a continuous mapping on

$$\{\eta(\cdot - x) \in C^{r_0}: x \in \mathbb{R}^d, \|\eta\|_{C^{r_0}} \leq 1 \text{ and } \text{supp } \eta \subseteq B(0, 1)\}$$

and taking values in \mathbb{R} . It is straightforward to extend this mapping to a linear form on $C_c^{r_0}$. In particular, we may and will think of \bar{C}^α as a subset of the dual of C_c^∞ . Similarly, the space $\bar{C}_{\text{loc}}^\alpha(U)$ can be seen as a subset of the dual of $C_c^\infty(U)$.

Remark 1.19. As will be seen shortly, the topology of $\bar{C}_{\text{loc}}^\alpha(U)$ is metrisable.

Definition 1.20. A multiresolution analysis of $L^2(\mathbb{R}^d)$ is an increasing sequence $(V_n)_{n \in \mathbb{Z}}$ of subspaces of $L^2(\mathbb{R}^d)$, together with a scaling function $\phi \in L^2(\mathbb{R}^d)$, such that

- $\bigcup_{n \in \mathbb{Z}} V_n$ is dense in $L^2(\mathbb{R}^d)$, $\bigcap_{n \in \mathbb{Z}} V_n = \{0\}$;
- $f \in V_n$ if and only if $f(2^{-n} \cdot) \in V_0$;
- $(\phi(\cdot - k))_{k \in \mathbb{Z}^d}$ is an orthonormal basis of V_0 .

Definition 1.21. A multiresolution analysis is called r -regular ($r \in \mathbb{N}$) if its scaling function ϕ can be chosen in such a way that

$$|\partial^k \phi(x)| \leq C_m (1 + |x|)^{-m}$$

for every integer m and for every multi-index $k \in \mathbb{N}^d$ with $|k| \leq r$.

While a given sequence (V_n) can be associated with several different scaling functions to form a multiresolution analysis, a multiresolution analysis is entirely determined by the knowledge of its scaling function. We denote by W_n the orthogonal complement of V_n in V_{n+1} .

Theorem 1.22. (compactly supported wavelets) For every positive integer r , there exist $\phi, (\psi^{(i)})_{1 \leq i < 2^d}$ such that

- $\phi, (\psi^{(i)})_{i < 2^d}$ all belong to C_c^r ;
- ϕ is the scaling function of a multiresolution analysis (V_n) ;
- $(\psi^{(i)}(\cdot - k))_{i < 2^d, k \in \mathbb{Z}^d}$ is an orthonormal basis of W_0 .

This result is due to [Dau88] (see also e.g. [Pin01, Chapter 6]). We recall that a wavelet basis on \mathbb{R}^d can be constructed from one on \mathbb{R} by taking products of wavelet functions for each coordinate. We also recall from [Mey92, Theorem 2.6.4] that for every multi-index $\beta \in \mathbb{N}^d$ such that $|\beta| < r$ and every $i < 2^d$, we have

$$\int x^\beta \psi^{(i)}(x) dx = 0. \quad (1.11)$$

Except for Theorem 1.22, (1.11) and Proposition 1.35, this section is self-contained. From now on, we fix both $r \in \mathbb{N}$ and a wavelet basis $\phi, (\psi^{(i)})_{i < 2^d} \in C_c^r$, as obtained with Theorem 1.22. Let R be such that

$$\text{supp } \phi \subseteq B(0, R), \quad \text{supp } \psi^{(i)} \subseteq B(0, R) \quad (i < 2^d). \quad (1.12)$$

For any $n \in \mathbb{Z}$ and $x \in \mathbb{R}^d$, if we define

$$\phi_{n,x}(y) := 2^{dn/2} \phi(2^n(y - x)) \quad (1.13)$$

and $\Lambda_n = \mathbb{Z}^d / 2^n$, then $(\phi_{n,x})_{x \in \Lambda_n}$ is an orthonormal basis of V_n . Similarly, we define

$$\psi_{n,x}^{(i)}(y) := 2^{dn/2} \psi^{(i)}(2^n(y-x)),$$

so that $(\psi_{n,x}^{(i)})_{i < 2^d, x \in \Lambda_n, n \in \mathbb{Z}}$ is an orthonormal basis of $L^2(\mathbb{R}^d)$. For $f \in L^2(\mathbb{R}^d)$, we set

$$v_{n,x} f := \langle f, \phi_{n,x} \rangle, \quad w_{n,x}^{(i)} f := \langle f, \psi_{n,x}^{(i)} \rangle, \quad (1.14)$$

where $\langle \cdot, \cdot \rangle$ is the scalar product on $L^2(\mathbb{R}^d)$. Denoting by \mathcal{V}_n and \mathcal{W}_n the orthogonal projections on V_n, W_n respectively, we have

$$\mathcal{V}_n f = \sum_{x \in \Lambda_n} v_{n,x}(f) \phi_{n,x}, \quad \mathcal{W}_n f = \sum_{i < 2^d, x \in \Lambda_n} w_{n,x}^{(i)}(f) \psi_{n,x}^{(i)}, \quad (1.15)$$

and for every $k \in \mathbb{Z}$,

$$f = \mathcal{V}_k f + \sum_{n=k}^{+\infty} \mathcal{W}_n f \quad (1.16)$$

in $L^2(\mathbb{R}^d)$.

Definition 1.23. (Besov spaces via wavelet decomposition)

Let $\alpha \in \mathbb{R}$, $|\alpha| < r$ and $p, q \in [1, \infty]$. The Besov space $\bar{B}_{p,q}^\alpha$ is the completion of C_c^∞ with respect to the norm

$$\|f\|_{\bar{B}_{p,q}^\alpha} := \|\mathcal{V}_0 f\|_{L^p} + \|(2^{\alpha n} \|\mathcal{W}_n f\|_{L^p})_{n \in \mathbb{N}}\|_{\ell^q}. \quad (1.17)$$

The local Besov space $\bar{B}_{p,q}^{\alpha, \text{loc}}(U)$ is the completion of $C^\infty(U)$ with respect to the family of semi-norms

$$f \mapsto \|\chi f\|_{\bar{B}_{p,q}^\alpha}$$

indexed by $\chi \in C_c^\infty(U)$.

Remark 1.24. By taking the closure of a family of smooth compactly supported functions, the present definition has the advantage of making the space separable for every p, q . However, in Section 1.1.3 we outline a proof of the equivalence between the norms of Definition 1.17 and Definition 1.10. Thus, assuming that $\exists C > 0$ such that

$$C^{-1} \|f\|_{B_{p,q}^\alpha} \leq \|f\|_{\bar{B}_{p,q}^\alpha} \leq C \|f\|_{B_{p,q}^\alpha}$$

for every $f \in C_c^\infty$, one can see that $B_{p,q}^\alpha$ coincides with $\bar{B}_{p,q}^\alpha$ if both p and q are finite, because C_c^∞ is dense in $B_{p,q}^\alpha$ as discussed in Remark 1.11.

Remark 1.25. A different notion of Hölder space on a bounded domain $U \subset \mathbb{R}^d$, encoding more precise weighted information on the size of the distribution as one gets closer and closer to the boundary of the domain, has been introduced in the very recent work [GH17b].

Remark 1.26. The space $\bar{B}_{p,q}^\alpha$ of Definition 1.23 does not depend on the choice of the multiresolution analysis, in the sense that for any $r > |\alpha|$, any different r -regular multiresolution analysis yields an equivalent norm. This will be made explicit in Section 1.1.3 by comparing $\bar{B}_{p,q}^\alpha$ with the space $B_{p,q}^\alpha$ given in Definition 1.10. Recall that throughout this section we fix $r \in \mathbb{N}$, and consider Besov spaces $\bar{B}_{p,q}^\alpha$ with $\alpha \in \mathbb{R}$, $|\alpha| < r$.

Remark 1.27. We can obtain inequalities between norms as done in the previous section. It is clear from Definition 1.23 that if $\alpha_1 \leq \alpha_2 \in \mathbb{R}$ and $q_1 \geq q_2 \in [1, \infty]$, then

$$\|f\|_{\bar{B}_{p,q_1}^{\alpha_1}} \leq C \|f\|_{\bar{B}_{p,q_2}^{\alpha_2}},$$

where C is independent of $f \in C_c^\infty$. In particular, the space $\bar{\mathcal{B}}_{p,q_2}^{\alpha_2}$ is continuously embedded in $\bar{\mathcal{B}}_{p,q_1}^{\alpha_1}$. Similarly, for $p_1 \leq p_2$, $q \in [1, \infty]$ and for a given $\chi \in C_c^\infty$, there exists a constant $C < \infty$ such that for every $f \in C_c^\infty$,

$$\|\chi f\|_{\mathcal{B}_{p_1,q}^\alpha} \leq C \|\chi f\|_{\mathcal{B}_{p_2,q}^\alpha}.$$

Indeed, this is a consequence of Jensen's inequality and the fact that for each $n \in \mathbb{N}$, the support of $\mathscr{W}_n(\chi f)$ is contained in the bounded set $2R + \text{supp}\chi$. Hence, the space $\bar{\mathcal{B}}_{p_2,q}^{\alpha,\text{loc}}(U)$ is continuously embedded in $\bar{\mathcal{B}}_{p_1,q}^{\alpha,\text{loc}}(U)$.

The finiteness of $\|f\|_{\mathcal{B}_{p,q}^\alpha}$ can be expressed in terms of the magnitude of the coefficients $v_{n,x}(f)$ and $w_{n,x}^{(i)}(f)$.

Proposition 1.28. (Besov spaces via wavelet coefficients) *For every $p \in [1, \infty]$, there exists $C \in (0, \infty)$ such that for every $f \in C_c^\infty$ and every $n \in \mathbb{Z}$,*

$$C^{-1} \|\mathscr{V}_n f\|_{L^p} \leq 2^{dn\left(\frac{1}{2}-\frac{1}{p}\right)} \|(v_{n,x} f)_{x \in \Lambda_n}\|_{\ell^p} \leq C \|\mathscr{V}_n f\|_{L^p}, \quad (1.18)$$

$$C^{-1} \|\mathscr{W}_n f\|_{L^p} \leq 2^{dn\left(\frac{1}{2}-\frac{1}{p}\right)} \|(w_{n,x}^{(i)} f)_{i < 2^d, x \in \Lambda_n}\|_{\ell^p} \leq C \|\mathscr{W}_n f\|_{L^p}. \quad (1.19)$$

Proof. We will prove just (1.18) in detail, since (1.19) follows in the same way. (See also [Mey92, Proposition 6.10].) Recalling the definition of the constant $R > 0$ in (1.12), we have $\text{supp}\phi_{n,x} \subseteq B(x, 2^{-n}R)$ and thus, for every $y \in \mathbb{R}$,

$$\mathscr{V}_n f(y) = \sum_{x \in \Lambda_n, x \in B(y, 2^{-n}R)} v_{n,x}(f) \phi_{n,x}(y). \quad (1.20)$$

Let $p < +\infty$. Since the sum $\sum_{x \in \Lambda_n, x \in B(y, 2^{-n}R)}$ is finite uniformly over n , we can use Jensen's inequality to obtain:

$$\begin{aligned} \|\mathscr{V}_n f\|_{L^p}^p &= \int \left| \sum_{x \in \Lambda_n, x \in B(y, 2^{-n}R)} v_{n,x}(f) \phi_{n,x}(y) \right|^p dy \\ &\lesssim \int \sum_{x \in \Lambda_n, x \in B(y, 2^{-n}R)} |v_{n,x}(f) \phi_{n,x}(y)|^p dy \\ &\lesssim \sum_{x \in \Lambda_n} |v_{n,x}(f)|^p \int_{B(x, 2^{-n}R)} |\phi_{n,x}(y)|^p dy \\ &\lesssim \|(v_{n,x} f)_{x \in \Lambda_n}\|_{\ell^p}^p \|\phi_{n,0}\|_{L^p}^p. \end{aligned}$$

The leftmost inequality of (1.18) follows from the scaling properties of $\phi_{n,0}$, namely:

$$\|\phi_{n,x}\|_{L^p} = 2^{dn\left(\frac{1}{2}-\frac{1}{p}\right)} \|\phi_{0,x}\|_{L^p}. \quad (1.21)$$

For $p = +\infty$ we estimate $\|\mathscr{V}_n f\|_{L^\infty}$ using

$$|\mathscr{V}_n f(y)| \lesssim R^d \sup_{x \in \Lambda_n} |v_{n,x} f| |\phi_{n,x}(y)| \lesssim \|\phi_{n,0}(y)\|_{L^\infty} \sup_{x \in \Lambda_n} |v_{n,x} f|.$$

This yields the upper bound for $\|\mathscr{V}_n f\|_{L^p}$.

As for the rightmost inequality, notice that $v_{n,x}(\mathscr{V}_n f) = v_{n,x} f$, that is,

$$v_{n,x} f = \int \phi_{n,x}(y) \mathscr{V}_n f(y) dy.$$

Let $p < +\infty$ and p' be its conjugate exponent. By Hölder's inequality

$$|v_{n,x} f| \leq \|\phi_{n,x}\|_{L^{p'}} \|\mathscr{V}_n f \mathbb{1}_{B(x, 2^{-n}R)}\|_{L^p},$$

and moreover,

$$\sum_{x \in \Lambda_n} \int |\mathcal{V}_n f(y)|^p \mathbb{1}_{B(x, 2^{-n}R)}(y) dy = \int |\mathcal{V}_n f(y)|^p \sum_{x \in \Lambda_n} \mathbb{1}_{B(x, 2^{-n}R)}(y) dy \lesssim \|\mathcal{V}_n f\|_{L^p}^p.$$

By (1.21), we have $\|\phi_{n,x}\|_{L^{p'}} \lesssim 2^{dn(\frac{1}{2}-\frac{1}{p'})} = 2^{-dn(\frac{1}{2}-\frac{1}{p})}$, and this concludes the proof for the case $p < +\infty$. For $p = +\infty$, we just notice that $|v_{n,x} f| \leq \|\mathcal{V}_n f\|_{L^\infty} \|\phi_{n,x}\|_{L^1}$. \square

Remark 1.29. For each $k \in \mathbb{Z}$, the norm

$$\|f\|_{\mathcal{B}_{p,q}^{\alpha,k}} = |(v_{k,x} f)_{x \in \Lambda_k}|_{\ell^p} + \left| \left(2^{\alpha n} 2^{dn(\frac{1}{2}-\frac{1}{p})} |(w_{n,x}^{(i)} f)_{i < 2^d, x \in \Lambda_n}|_{\ell^p} \right)_{n \geq k} \right|_{\ell^q}$$

is equivalent to that in (1.17). This is easy to show using Proposition 1.28 and the definition of multiresolution analysis.

As we now show, for $\alpha < 0$, the Besov space $\bar{\mathcal{B}}_{\infty,\infty}^\alpha$ of Definition 1.23 coincides with the Besov-Hölder space $\bar{\mathcal{C}}^\alpha$ given by Definition 1.17.

Proposition 1.30. *Let $\alpha < 0$. There exist $C_1, C_2 \in (0, \infty)$ such that for every $f \in C_c^\infty$, we have*

$$C_1 \|f\|_{\mathcal{C}^\alpha} \leq \|f\|_{\bar{\mathcal{B}}_{\infty,\infty}^\alpha} \leq C_2 \|f\|_{\mathcal{C}^\alpha}. \quad (1.22)$$

Proof. The result is classical and proved e.g. in [Hai14a, Proposition 3.20]. We recall the proof for the reader's convenience. One can check that there exists $C \in \mathbb{R}$ such that for every $f \in C_c^\infty$, $n \in \mathbb{Z}$ and $x \in \mathbb{R}^d$,

$$2^{\alpha n} 2^{\frac{dn}{2}} |w_{n,x}^{(i)} f| \leq C \|f\|_{\mathcal{C}^\alpha}, \quad (1.23)$$

and this yields the second inequality in (1.22).

Conversely, we let $f \in C^\infty$ satisfy $\|f\|_{\bar{\mathcal{B}}_{\infty,\infty}^\alpha} \leq 1$. We aim to show that there exists a constant $C \in \mathbb{R}$ (independent of f) such that for every $y \in \mathbb{R}^d$, $\eta \in \mathcal{B}^{r_0}$ (with $r_0 = -\lfloor \alpha \rfloor$) and $\lambda \in (0, 1]$, we have

$$\lambda^{-\alpha-d} \int_{\mathbb{R}^d} f \eta \left(\frac{\cdot - y}{\lambda} \right) \leq C.$$

We write $\eta_{\lambda,y} := \lambda^{-d} \eta((\cdot - y)/\lambda)$, and observe that

$$\int f \eta_{\lambda,y} = \sum_{x \in \Lambda_0} (v_{0,x} f) (v_{0,x} \eta_{\lambda,y}) + \sum_{i < 2^d} \sum_{n \geq 0} \sum_{x \in \Lambda_n} (w_{n,x}^{(i)} f) (w_{n,x}^{(i)} \eta_{\lambda,y}).$$

We consider only the second term of the sum above, as the first one can be obtained with the same technique. By the definition of $\|f\|_{\bar{\mathcal{B}}_{\infty,\infty}^\alpha}$, for every $n \geq 0$, we have

$$2^{\frac{dn}{2}} |w_{n,x}^{(i)} f| \leq C 2^{-\alpha n}. \quad (1.24)$$

By a Taylor expansion of η of order $r_0 \leq r$ around $x - y$, and recalling (1.11), we obtain

$$2^{-n} \leq \lambda \implies 2^{\frac{dn}{2}} |w_{n,x}^{(i)} \eta_{\lambda,y}| \leq C 2^{-r_0 n} \lambda^{-d-r_0}, \quad (1.25)$$

and by the boundedness of the L^1 norm of $\eta_{\lambda,y}$ we have

$$2^{-n} \geq \lambda \implies 2^{\frac{dn}{2}} |w_{n,x}^{(i)} \eta_{\lambda,y}| \leq C 2^{dn}. \quad (1.26)$$

The same bounds hold for $2^{\frac{dn}{2}} |v_{0,x} \eta_{\lambda,y}|$. In order for $w_{n,x}^{(i)} \eta_{\lambda,y}$ to be non-zero, we must have $|x - y| \leq C (\lambda \vee 2^{-n})$. By splitting the sum over $n \geq 0$ one can observe that if $2^{-n} \leq \lambda$ the sum $\sum_{x \in \Lambda_n}$ has less than $C \lambda^d 2^{dn}$ terms, and if $2^{-n} \geq \lambda$ it has less than C terms, and therefore the results follows from the estimations (1.25) and (1.26). \square

Remark 1.31. Notice that we can replace $r_0 = -\lfloor \alpha \rfloor$ by a generic integer $r > |\alpha|$ in Definition 1.17, obtaining an equivalent norm. Indeed, Proposition 1.30 shows that it suffices to control the behavior of f against shifted and rescaled versions of the wavelet functions ϕ and $\psi^{(i)}$.

We now present an equivalent norm $\|\cdot\|_{\mathcal{E}_p^\alpha}$ for Besov spaces, which reduces to Definition 1.17 in the case $p = \infty$.

Definition 1.32. ([HL16, Definition 2.5]) *Let $f \in C_c^\infty$. For every $\alpha < 0$ and $p \in [1, \infty]$ we introduce the norm*

$$\|f\|_{\mathcal{E}_p^\alpha} := \sup_{\lambda \in (0,1]} \lambda^{-\alpha} \left\| \sup_{\eta \in \mathcal{B}_{r_0}} |\langle f, \eta_{\lambda,x} \rangle| \right\|_{L^p(dx)}$$

with $\eta_{\lambda,x} := \lambda^{-d} \eta(\lambda^{-1}(\cdot - x))$ and \mathcal{B}_{r_0} as in Definition 1.17.

The following is a straightforward generalization of Proposition 1.30.

Proposition 1.33. ([HL16, Proposition 2.6]) *Let $\alpha < 0$. There exist $C_1, C_2 \in (0, \infty)$ such that for every $f \in C_c^\infty$, we have*

$$C_1 \|f\|_{\mathcal{E}_p^\alpha} \leq \|f\|_{\mathcal{B}_{p,\infty}^\alpha} \leq C_2 \|f\|_{\mathcal{E}_p^\alpha}. \quad (1.27)$$

Remark 1.34. In view of propositions 1.30 and 1.33, when $\alpha < 0$, we write

$$\begin{aligned} \bar{\mathcal{C}}^\alpha &= \bar{\mathcal{B}}_{\infty,\infty}^\alpha, \\ \bar{\mathcal{C}}_{\text{loc}}^\alpha(U) &= \bar{\mathcal{B}}_{\infty,\infty}^{\alpha,\text{loc}}(U), \end{aligned}$$

and we set $\forall \alpha \in \mathbb{R}$

$$\begin{aligned} \bar{\mathcal{C}}^\alpha &:= \bar{\mathcal{B}}_{\infty,\infty}^\alpha \\ \bar{\mathcal{C}}_{\text{loc}}^\alpha(U) &:= \bar{\mathcal{B}}_{\infty,\infty}^{\alpha,\text{loc}}(U) \\ \bar{\mathcal{E}}_p^\alpha &:= \bar{\mathcal{B}}_{p,\infty}^\alpha, \\ \bar{\mathcal{E}}_p^{\alpha,\text{loc}}(U) &:= \bar{\mathcal{B}}_{p,\infty}^{\alpha,\text{loc}}(U). \end{aligned}$$

The following proposition is a weak manifestation of the multiplicative structure of Besov spaces, which is exposed in more details in Section 1.2.1.

Proposition 1.35. (multiplication by a smooth function)

Let $r > |\alpha|$ and $p, q \in [1, \infty]$. For every $\chi \in C_c^r$, the mapping $f \mapsto \chi f$ extends to a continuous functional from $\bar{\mathcal{B}}_{p,q}^\alpha$ to itself.

Proof. We give a complete proof just for $\alpha < 0$ and $p = q = \infty$. The general case is a consequence of the paraproduct estimates of Proposition 1.56 and the equivalence between wavelet-Besov norms $\|\cdot\|_{\mathcal{B}_{p,q}^\alpha}$ and Littlewood-Paley-Besov norms $\|\cdot\|_{\mathcal{B}_{p,q}^\alpha}$, discussed in Section 1.1.3.

Let then $f \in C_c^\infty$ and consider the integral

$$\lambda^{-d} \int f(y) \chi(y) \eta\left(\frac{y-x}{\lambda}\right) dy.$$

For every $\lambda > 0$ and $x \in \mathbb{R}^d$, define $\tilde{\eta}$ as: $\tilde{\eta}_{\lambda,x}\left(\frac{y-x}{\lambda}\right) = \chi(y)\eta\left(\frac{y-x}{\lambda}\right)$. Then $\tilde{\eta}_{\lambda,x}(z) = \chi(z\lambda+x)\eta(z)$ for $z \in \mathbb{R}^d$. One can notice that $\tilde{\eta}_{\lambda,x} \in C_c^{r_0}$ and $\text{supp } \tilde{\eta}_{\lambda,x} \subseteq \text{supp } \eta$. Hence, by Proposition 1.30, there exists $C > 0$ (possibly different in every line) such that:

$$\begin{aligned} \lambda^{-d} \int f(y) \chi(y) \eta\left(\frac{y-x}{\lambda}\right) dy &\leq C \lambda^\alpha \|f\|_{C^\alpha} \|\tilde{\eta}_{\lambda,x}\|_{C_c^{r_0}} \\ &\leq C \lambda^\alpha \|f\|_{C^\alpha} \|\chi(\lambda \cdot)\|_{C_c^{r_0}} \\ &\leq C \lambda^\alpha \|f\|_{C^\alpha} \|\chi\|_{C_c^{r_0}}, \end{aligned}$$

uniformly over $f \in C_c^\infty$, $\lambda \in (0, 1]$, $\eta \in \mathcal{B}^{r_0}$ and $x \in \mathbb{R}^d$. The result follows by the fact that C_c^∞ is dense in \bar{C}^α . \square

Remark 1.36. The notion of a complete space makes sense for arbitrary topological vector spaces, since a description of neighbourhoods of the origin is sufficient for defining what a Cauchy sequence is. Yet, in our present setting, the topology of $\bar{\mathcal{B}}_{p,q}^{\alpha,\text{loc}}(U)$ is in fact metrisable. To see this, note that there is no loss of generality in restricting the range of χ indexing the semi-norms to a countable subset of $C_c^\infty(U)$, e.g. $\{\chi_n, n \in \mathbb{N}\}$ such that for every compact $K \subseteq U$, there exists n such that $\chi_n = 1$ on K . Indeed, it is then immediate from Proposition 1.35 that if χ has support in K , then $\|\chi f\|_{\mathcal{B}_{p,q}^\alpha} \leq C \|\chi_n f\|_{\mathcal{B}_{p,q}^\alpha}$ for some C not depending on f . Hence, we can view $\bar{\mathcal{B}}_{p,q}^{\alpha,\text{loc}}(U)$ as a complete (Fréchet) space equipped with the metric

$$d_{\bar{\mathcal{B}}_{p,q}^{\alpha,\text{loc}}(U)}(f, g) = \sum_{n=0}^{+\infty} 2^{-n} \|\chi_n(f - g)\|_{\mathcal{B}_{p,q}^\alpha} \wedge 1. \quad (1.28)$$

We now give an alternative family of semi-norms, based on wavelet coefficients, that is equivalent to the family given in Definition 1.23 or Remark 1.36.

Definition 1.37. (spanning sequence) Let $R > 0$ such that (1.12) holds. Let $K \subseteq U$ be compact and $k \in \mathbb{N}$. We say that the pair (K, k) is adapted if

$$2^{-k} R < \text{dist}(K, U^c). \quad (1.29)$$

We say that the set \mathcal{K} is a spanning sequence if it can be written as

$$\mathcal{K} = \{(K_n, k_n), n \in \mathbb{N}\},$$

where $(K_n)_{n \in \mathbb{N}}$ is an increasing sequence of compact subsets of U such that $\bigcup_n K_n = U$ and for every n , the pair (K_n, k_n) is adapted.

For every adapted pair (K, k) , $f \in C_c^\infty(U)$ and $n \geq k$, we let

$$v_{n,K,p} f = 2^{dn\left(\frac{1}{2}-\frac{1}{p}\right)} \|(v_{n,x} f)_{x \in \Lambda_n \cap K}\|_{\ell^p}, \quad (1.30)$$

$$w_{n,K,p} f = 2^{dn\left(\frac{1}{2}-\frac{1}{p}\right)} \|(w_{n,x}^{(i)} f)_{i < 2^d, x \in \Lambda_n \cap K}\|_{\ell^p}, \quad (1.31)$$

and we define the semi-norm

$$\|f\|_{\mathcal{B}_{p,q}^{\alpha,K,k}} = v_{k,K,p} f + \|(2^{\alpha n} w_{n,K,p} f)_{n \geq k}\|_{\ell^q}. \quad (1.32)$$

Proposition 1.38. (Local Besov spaces via wavelet coefficients) Let $p, q \in [1, \infty]$.

1. For every adapted pair (K, k) , the mapping $f \mapsto \|f\|_{\mathcal{B}_{p,q}^{\alpha,K,k}}$ extends to a continuous semi-norm on $\bar{\mathcal{B}}_{p,q}^{\alpha,\text{loc}}(U)$.

2. The topology induced by the family of semi-norms $\|\cdot\|_{\mathcal{B}_{p,q}^{\alpha,K,k}}$, indexed by adapted pairs (K, k) , is that of $\bar{\mathcal{B}}_{p,q}^{\alpha,\text{loc}}(U)$.
3. Let \mathcal{K} be a spanning sequence. Part (2) above remains true when considering only the seminorms indexed by pairs in \mathcal{K} .

Remark 1.39. Another metric that is compatible with the topology on $\bar{\mathcal{B}}_{p,q}^{\alpha,\text{loc}}(U)$ is thus given by

$$d'_{\mathcal{B}_{p,q}^{\alpha,\text{loc}}(U)}(f, g) = \sum_{n=0}^{+\infty} 2^{-n} \|f - g\|_{\mathcal{B}_{p,q}^{\alpha,K_n,k_n}} \wedge 1,$$

where $\mathcal{K} = \{(K_n, k_n), n \in \mathbb{N}\}$ is any given spanning sequence.

Proof. (of Proposition 1.38) In order to prove parts (1-2) of the proposition, it suffices to show the following two statements.

$$\begin{aligned} \text{For every adapted pair } (K, k), \text{ there exists } \chi \in C_c^\infty(U) \text{ and } C < \infty \text{ s.t.} \\ \forall f \in C^\infty(U), \|f\|_{\mathcal{B}_{p,q}^{\alpha,K,k}} \leq C \|\chi f\|_{\mathcal{B}_{p,q}^\alpha}; \end{aligned} \quad (1.33)$$

$$\begin{aligned} \text{For every } \chi \in C_c^\infty(U), \text{ there exists } (K, k) \text{ adapted pair and } C < \infty \text{ s.t.} \\ \forall f \in C^\infty(U), \|\chi f\|_{\mathcal{B}_{p,q}^\alpha} \leq C \|f\|_{\mathcal{B}_{p,q}^{\alpha,K,k}}. \end{aligned} \quad (1.34)$$

We begin with (1.33). Let (K, k) be an adapted pair, and let $\chi \in C_c^\infty(U)$ be such that $\chi = 1$ on $K + \bar{B}(2^{-k}R)$. For every $n \geq k$ and $x \in \Lambda_n \cap K$,

$$v_{n,x} f = v_{n,x}(\chi f), \quad w_{n,x}^{(i)} f = w_{n,x}^{(i)}(\chi f) \quad (i < 2^d),$$

and as a consequence,

$$v_{n,K,p}(f) \leq 2^{dn\left(\frac{1}{2}-\frac{1}{p}\right)} \|(|v_{n,x}(\chi f)|)_{x \in \Lambda_n}\|_{\ell^p} \leq C \|\mathcal{V}_n(\chi f)\|_{L^p}$$

(where we used (1.18) in the last step), and similarly with $v_{n,K,p}$, $v_{n,x}$ and \mathcal{V}_n replaced by $w_{n,K,p}$, $w_{n,x}^{(i)}$ and \mathcal{W}_n respectively. We thus get that

$$\begin{aligned} \|f\|_{\mathcal{B}_{p,q}^{\alpha,K,k}} &= v_{k,K,p} f + \|(2^{\alpha n} w_{n,K,p} f)_{n \geq k}\|_{\ell^q} \\ &\leq C [\|\mathcal{V}_k(\chi f)\|_{L^p} + \|(2^{\alpha n} \|\mathcal{W}_n(\chi f)\|_{L^p})_{n \geq n_0}\|_{\ell^q}] \leq C \|\chi f\|_{\mathcal{B}_{p,q}^\alpha}. \end{aligned}$$

We now turn to (1.34). In order to also justify part (3), we will show that we can in fact pick the adapted pair in $\mathcal{K} = \{(K_n, k_n), n \in \mathbb{N}\}$.

Let (K, k) be an adapted pair. For every $f \in C^\infty(U)$, we define

$$f_K = \sum_{x \in \Lambda_k \cap K} v_{k,x}(f) \phi_{k,x} + \sum_{\substack{n \geq k, i < 2^d \\ x \in \Lambda_n \cap K}} w_{n,x}^{(i)}(f) \psi_{n,x}^{(i)}. \quad (1.35)$$

The functions f and f_K coincide on

$$K' := \{x \in \mathbb{R}^d: \text{dist}(x, K^c) \geq 2^{-k}R\}. \quad (1.36)$$

(Although the notation is not explicit in this respect, we warn the reader that f_K and K' are defined in terms of the pair (K, k) rather than in terms of K only.) Let $\chi \in C_c^\infty(U)$ with compact support $L \subseteq U$. Assuming that

$$\text{there exists } n \in \mathbb{N} \text{ such that } L \subseteq K'_n, \quad (1.37)$$

we see that for such an n ,

$$\|\chi f\|_{\mathcal{B}_{p,q}^\alpha} = \|\chi f_{K_n}\|_{\mathcal{B}_{p,q}^\alpha} \leq C \|f_{K_n}\|_{\mathcal{B}_{p,q}^\alpha} \leq C \|f\|_{\mathcal{B}_{p,q}^{\alpha,K_n,k_n}}$$

by Proposition 1.35 and (1.32). Hence, it suffices to justify (1.37). Let $d = \text{dist}(L, U^c)$. Since $x \mapsto \text{dist}(x, U^c)$ is positive and continuous on L , we obtain $d > 0$. If U is bounded, then there exists $n \in \mathbb{N}$ such that K_n contains the compact set $\{x: \text{dist}(x, U^c) \geq d/2\}$. We must then have $2^{-k_n} R < d/2$, so that

$$\begin{aligned} x \in L &\Rightarrow \text{dist}(x, K_n^c) \geq \text{dist}(x, U^c) - \frac{d}{2} \geq \frac{d}{2} > 2^{-k_n} R \\ &\Rightarrow x \in K_n'. \end{aligned}$$

If U is unbounded, we can do the same reasoning with U replaced by

$$U \cap (L + B(0, R)),$$

so the proof is complete. \square

Remark 1.40. For any adapted pair (K, k) , the quantity $\|f\|_{\mathcal{B}_{p,q}^{\alpha,K,k}}$ is well defined as an element of $[0, +\infty]$ as soon as f is a linear form on $C_c^r(U)$, through the interpretation of $v_{k,x} f$ and $w_{n,x}^{(i)} f$ in (1.14) as a duality pairing.

The characterization of Proposition 1.38 yields another straightforward proof of the Besov embedding (Proposition 1.16).

Proposition 1.41. (Local Besov embedding) *Let $1 \leq p_2 \leq p_1 \leq +\infty$, $1 \leq q_2 \leq q_1 \leq +\infty$, $\alpha \in \mathbb{R}$ and*

$$\beta = \alpha + d \left(\frac{1}{p_2} - \frac{1}{p_1} \right).$$

If $|\alpha|, |\beta| < r$ and (K, k) is an adapted pair, then there exists $C < \infty$ such that for every linear form f on $C_c^r(U)$,

$$\|f\|_{\mathcal{B}_{p_1,q_1}^{\alpha,K,k}} \leq C \|f\|_{\mathcal{B}_{p_2,q_2}^{\beta,K,k}}.$$

In particular, we have $\bar{\mathcal{B}}_{p_2,q_2}^{\beta,\text{loc}}(U) \subseteq \bar{\mathcal{B}}_{p_1,q_1}^{\alpha,\text{loc}}(U)$.

Proof. We write the norm (1.32), recall (1.30) and (1.31), and use the fact that $\|\cdot\|_{\ell^{p_1}} \leq \|\cdot\|_{\ell^{p_2}}$ if $p_1 \geq p_2$. \square

Due to our definition of the space $\bar{\mathcal{B}}_{p,q}^{\alpha,\text{loc}}(U)$ as a completion of $C^\infty(U)$, the fact that $\|f\|_{\mathcal{B}_{p,q}^{\alpha,K,k}}$ is finite for every adapted pair (K, k) does not necessarily imply that $f \in \bar{\mathcal{B}}_{p,q}^{\alpha,\text{loc}}(U)$. We have nonetheless the following result.

Proposition 1.42. *Let $|\alpha'| < r$ and let $p, q \in [1, \infty]$. Let f be a linear form on $C_c^r(U)$, and let \mathcal{K} be a spanning sequence. If for every $(K, k) \in \mathcal{K}$,*

$$\|f\|_{\mathcal{B}_{p,q}^{\alpha',K,k}} < \infty,$$

then for every $\alpha < \alpha'$, the form f belongs to $\bar{\mathcal{B}}_{p,1}^{\alpha,\text{loc}}(U)$.

Proof. We first check that for every $(K, k) \in \mathcal{K}$, there exists a sequence $(f_{N,k})_{N \in \mathbb{N}}$ in $C_c^r(U)$ such that $\|f - f_{N,k}\|_{\mathcal{B}_{p,1}^{\alpha,K,k}}$ tends to 0 as N tends to infinity. The functions

$$f_{N,k} := \sum_{x \in \Lambda_k \cap K} v_{k,x}(f) \phi_{k,x} + \sum_{\substack{k \leq n \leq N, i < 2^d \\ x \in \Lambda_n \cap K}} w_{n,x}^{(i)}(f) \psi_{n,x}^{(i)}$$

satisfy this property. Now notice that for $(\tilde{k}, \tilde{K}) \in \mathcal{K}$ such that $\tilde{K} \supset K$, the function $f_{N, \tilde{k}}$ coincides with $f_{N, k}$ on the set K' of (1.36). Then defining $f_N = f_{N, N}$, we obtain that for every $\chi \in C_c^\infty(U)$, there exists $n_0, N_0(n_0)$ such that for every $n \geq n_0$ and $N \geq N_0$,

$$\|(f_N - f) \chi\|_{\mathcal{B}_{p,1}^\alpha} = \|(f_{N, k_n} - f) \chi\|_{\mathcal{B}_{p,1}^\alpha},$$

where we have indexed the spanning sequence as $\mathcal{K} = (k_n, K_n)_{n \in \mathbb{N}}$. By (1.34), there exist $(k_m, K_m) \in \mathcal{K}$, $C > 0$ with m large enough, such that:

$$\|(f_{N, k_n} - f) \chi\|_{\mathcal{B}_{p,1}^\alpha} \leq C \|f_{N, k_n} - f\|_{\mathcal{B}_{p,1}^{\alpha, K_m, k_m}}$$

We can eventually choose $m = n$ to obtain $\|(f_N - f) \chi\|_{\mathcal{B}_{p,1}^\alpha} \rightarrow 0$ for every $\chi \in C_c^\infty(U)$, which by Proposition 1.38 is the needed result. \square

Naturally, tightness criteria rely on the identification of compact subsets of the space of interest.

Proposition 1.43. (Compact embedding) *Let U be an open subset of \mathbb{R}^d . For every $\alpha < \alpha'$ and $p, q, s \in [1, +\infty]$, the embedding $\bar{\mathcal{B}}_{p,q}^{\alpha', \text{loc}}(U) \subseteq \bar{\mathcal{B}}_{p,s}^{\alpha, \text{loc}}(U)$ is compact.*

Proof. By Proposition 1.38 and the definition of boundedness in Fréchet spaces, a sequence $(f_m)_{m \in \mathbb{N}}$ of elements of $\bar{\mathcal{B}}_{p,q}^{\alpha', \text{loc}}(U)$ is bounded in $\bar{\mathcal{B}}_{p,q}^{\alpha', \text{loc}}(U)$ if and only if for every adapted pair (K, k) , we have

$$\sup_{m \in \mathbb{N}} \|f_m\|_{\mathcal{B}_{p,q}^{\alpha', K, k}} < \infty.$$

We show that for every adapted pair (K, k) , there exists a subsequence $(m_{n_k})_{n_k \in \mathbb{N}}$ and $f^{(K)}$ in $\bar{\mathcal{B}}_{p,s}^{\alpha, \text{loc}}(U)$ such that $\|f_{m_{n_k}} - f^{(K)}\|_{\mathcal{B}_{p,s}^{\alpha', K, k}}$ converges to 0 as n goes to infinity. The assumption that $\sup_m \|f_m\|_{\mathcal{B}_{p,q}^{\alpha', K, k}} < \infty$ can be rewritten as

$$\begin{aligned} & \|(v_{k,x} f_m)_{x \in \Lambda_k \cap K}\|_{\ell^p} + \\ & + \left\| \left(2^{n\alpha' + nd\left(\frac{1}{2} - \frac{1}{p}\right)} \|(w_{n,x}^{(i)} f_m)_{i < 2^d, x \in \Lambda_n \cap K}\|_{\ell^p} \right)_{n \geq k} \right\|_{\ell^q} \leq C \end{aligned}$$

uniformly over $m \in \mathbb{N}$. By a diagonal extraction argument, there exist a subsequence, which we still denote (f_m) for convenience, and numbers $\tilde{v}_{k,x}, \tilde{w}_{n,x}^{(i)}$ such that

$$\begin{aligned} & \|(v_{k,x} f_m - \tilde{v}_{k,x})_{x \in \Lambda_k \cap K}\|_{\ell^p} + \\ & + \left\| \left(2^{n\alpha + nd\left(\frac{1}{2} - \frac{1}{p}\right)} \|(w_{n,x}^{(i)} f_m - \tilde{w}_{n,x}^{(i)})_{i < 2^d, x \in \Lambda_n \cap K}\|_{\ell^p} \right)_{n \geq k} \right\|_{\ell^s} \xrightarrow{m \rightarrow \infty} 0 \end{aligned}$$

Defining

$$f^{(K)} = \sum_{x \in \Lambda_k \cap K} \tilde{v}_{k,x} \phi_{k,x} + \sum_{\substack{n \geq k, i < 2^d \\ x \in \Lambda_n \cap K}} \tilde{w}_{n,x}^{(i)} \psi_{n,x}^{(i)},$$

we have $f^{(K)} \in \bar{\mathcal{B}}_{p,s}^{\alpha, \text{loc}}(U)$ and $\|f_m - f^{(K)}\|_{\mathcal{B}_{p,s}^{\alpha, K, k}} \rightarrow 0$ as m goes to infinity. The subsequence (f_m) is Cauchy in $\bar{\mathcal{B}}_{p,s}^{\alpha, \text{loc}}(U)$. Indeed, for every $(K, k) \in \mathcal{K}$, there exists $n_0 = n_0(K)$ such that for every $n, m \geq n_0$,

$$\|f_n - f_m\|_{\mathcal{B}_{p,s}^{\alpha, K, k}} \leq \|f_n - f^{(K)}\|_{\mathcal{B}_{p,s}^{\alpha, K, k}} + \|f^{(K)} - f_m\|_{\mathcal{B}_{p,s}^{\alpha, K, k}} < \varepsilon.$$

This completes the proof. \square

Remark 1.44. Proposition 1.43 would not be true if $\bar{\mathcal{B}}_{p,q}^{\alpha',\text{loc}}(U)$ and $\bar{\mathcal{B}}_{p,s}^{\alpha,\text{loc}}(U)$ were replaced by their global counterparts, respectively $\bar{\mathcal{B}}_{p,q}^{\alpha'}(\mathbb{R}^d)$ and $\bar{\mathcal{B}}_{p,s}^{\alpha}(\mathbb{R}^d)$. Indeed, one can take for example a non-zero function $f \in C_c^\infty$ and consider the sequence

$$[f(\cdot - np)]_{n \geq 1} \quad \text{with } p \in \mathbb{R}^d \setminus \{0\}.$$

This sequence is bounded in every global Besov space $\bar{\mathcal{B}}_{p,q}^{\alpha}(\mathbb{R}^d)$, but has no convergent subsequence in any of these spaces.

An immediate consequence of Propositions 1.42 and 1.43 is the following:

Corollary 1.45. *Let $|\alpha'| < r$, $p, q \in [1, \infty]$, let \mathcal{K} be a spanning sequence, and for every $(K, k) \in \mathcal{K}$, let $M_K \in [0, \infty)$. For every $\alpha < \alpha'$, $s \in [1, \infty]$, the set*

$$\left\{ f \text{ linear form on } C_c^r(U) \text{ such that } \forall (K, k) \in \mathcal{K}, \|f\|_{\bar{\mathcal{B}}_{p,q}^{\alpha',K,k}} \leq M_K \right\} \quad (1.38)$$

is compact in $\bar{\mathcal{B}}_{p,s}^{\alpha,\text{loc}}(U)$.

1.1.2.1 Tightness and continuity criteria for random fields

Theorem 1.46. (Tightness criterion)

Recall that $\phi, (\psi^{(i)})_{1 \leq i < 2^d}$ are in C_c^r and such that (1.12) holds, and fix $p \in [1, \infty)$, $q \in [1, \infty]$ and $\alpha, \beta \in \mathbb{R}$ satisfying $|\alpha|, |\beta| < r$, $\alpha < \beta$. Let $(f_m)_{m \in \mathbb{N}}$ be a family of random linear forms on $C_c^r(U)$, and let \mathcal{K} be a spanning sequence (see Definition 1.37). Assume that for every $(K, k) \in \mathcal{K}$, there exists $C = C(K, k) < \infty$ such that for every $m \in \mathbb{N}$,

$$\sup_{x \in \Lambda_k \cap K} \mathbb{E}[|\langle f_m, \phi(2^k(\cdot - x)) \rangle|^p]^{1/p} \leq C, \quad (1.39)$$

and

$$\sup_{x \in \Lambda_n \cap K} 2^{dn} \mathbb{E}[|\langle f_m, \psi^{(i)}(2^n(\cdot - x)) \rangle|^p]^{1/p} \leq C 2^{-n\beta}, \quad (i < 2^d, n \geq k). \quad (1.40)$$

Then the family (f_m) is tight in $\bar{\mathcal{B}}_{p,q}^{\alpha,\text{loc}}(U)$. If moreover $\alpha < \beta - d/p$, then the family is also tight in $\bar{\mathcal{C}}_{\text{loc}}^{\alpha}(U)$.

Remark 1.47. Note that the assumption in Theorem 1.46 simplifies when the field under consideration is stationary, since the suprema in (1.39) and (1.40) can be removed. Although we are primarily motivated by applications of this result for negative exponents of regularity, the statements we prove are insensitive to the sign of this exponent. Naturally, such tightness statements can then be lifted to statements of convergence in $\bar{\mathcal{B}}_{p,q}^{\alpha,\text{loc}}(U)$ provided that one verifies that the sequence (f_m) has a unique possible limit point (and the latter can be accomplished by checking that for each test function $\chi \in C_c^\infty(U)$ the random variable $\langle f_m, \chi \rangle$ converges in law as m tends to infinity).

Proof. (Th. 1.46) By (1.13) and (1.14), we have for every $(K, k) \in \mathcal{K}$, uniformly over m that

$$\begin{aligned} \sup_{x \in \Lambda_k \cap K} \mathbb{E}[|v_{k,x} f_m|^p] &\lesssim 1, \\ \sup_{x \in \Lambda_n \cap K} 2^{\frac{dnp}{2}} \mathbb{E}[|w_{n,x}^{(i)} f_m|^p] &\lesssim 2^{-np\beta} \quad (i < 2^d, n \geq k). \end{aligned}$$

Recalling the definition of $v_{k,K,p}$ and $w_{n,K,p}$ in (1.30) and (1.31) respectively, we have

$$|v_{k,K,p} f_m|^p \lesssim \sum_{x \in \Lambda_k \cap K} |v_{k,x} f_m|^p,$$

so that

$$\mathbb{E}[|v_{k,K,p} f_m|^p] \lesssim 1.$$

Similarly,

$$|w_{n,K,p} f_m|^p \lesssim 2^{dn(\frac{p}{2}-1)} \sum_{i < 2^d, x \in \Lambda_k \cap K} |w_{n,x}^{(i)} f_m|^p,$$

so that

$$\mathbb{E}[|w_{n,K,p} f_m|^p] \lesssim 2^{-np\beta}.$$

It follows from these two observations and from (1.32) that

$$\sup_{m \in \mathbb{N}} \mathbb{E} \left[\|f_m\|_{\mathcal{B}_{p,\infty}^{\beta,K,k}}^p \right] < \infty. \quad (1.41)$$

By Chebyshev's inequality, for any given $\varepsilon > 0$, there exist (M_K) such that if we set

$$\mathcal{E} := \left\{ f \text{ linear form on } C_c^r(U) \text{ such that } \forall (K, k) \in \mathcal{K}, \|f\|_{\mathcal{B}_{p,\infty}^{\beta,K,k}} \leq M_K \right\},$$

then for every m ,

$$\mathbb{P}[f_m \in \mathcal{E}] \geq 1 - \varepsilon.$$

By Corollary 1.45, this implies the tightness result in $\bar{\mathcal{B}}_{p,q}^{\alpha,\text{loc}}(U)$. For the second statement, we note that (1.41) and Proposition 1.41 imply that

$$\sup_{m \in \mathbb{N}} \mathbb{E} \left[\|f_m\|_{\mathcal{B}_{\infty,\infty}^{\beta-d/p,K,k}}^p \right] < \infty.$$

The conclusion then follows in the same way. \square

Remark 1.48. We can also infer from the proof that for each $\chi \in C_c^\infty(U)$, there exists a constant \tilde{C}_χ such that under the assumption of Theorem 1.46, we have

$$\sup_{m \in \mathbb{N}} \mathbb{E} \left[\|\chi f_m\|_{\mathcal{B}_{p,\infty}^\beta}^p \right] < \tilde{C}_\chi C,$$

as well as

$$\sup_{m \in \mathbb{N}} \mathbb{E} [\|\chi f_m\|_{C^{\beta-d/p}}^p] < \tilde{C}_\chi C.$$

We conclude this section by proving a statement analogous to Kolmogorov's continuity theorem.

Proposition 1.49. (Continuity criterion) *Let $(f(\eta), \eta \in C_c^r(U))$ be a family of random variables such that, for every $\eta, \eta' \in C_c^r(U)$ and every $\mu \in \mathbb{R}$, there exists a measurable set $A = A(\mu, \eta, \eta')$ with $\mathbb{P}(A) = 1$ such that*

$$f(\mu\eta + \eta')(\omega) = \mu f(\eta)(\omega) + f(\eta')(\omega) \quad \forall \omega \in A. \quad (1.42)$$

Assume also the following weak continuity property: for each compact $K' \subseteq U$ and each sequence $\eta_n, \eta \in C_c^r(U)$ with $\text{supp } \eta_n \subseteq K'$, we have

$$\eta_n \xrightarrow[\text{in } C_c^{r-1}]{n \rightarrow \infty} \eta \implies f(\eta_n) \xrightarrow[\text{prob.}]{n \rightarrow \infty} f(\eta).$$

Let $p \in [1, \infty)$, $q \in [1, \infty]$, and let $\alpha, \beta \in \mathbb{R}$ be such that $|\alpha|, |\beta| < r$ and $\alpha < \beta$. Let \mathcal{K} be a spanning sequence, and assume finally that, for every $(K, k) \in \mathcal{K}$, there exists $C > 0$ such that for every $n \geq k$,

$$\sup_{x \in \Lambda_k \cap K} \mathbb{E}[|f(\phi(2^k(\cdot - x)))|^p]^{1/p} \leq C$$

and

$$\sup_{x \in \Lambda_n \cap K} 2^{dn} \mathbb{E}[|f(\psi(2^n(\cdot-x)))|^p]^{1/p} \leq C 2^{-n\beta}.$$

Then there exists a random distribution \tilde{f} taking values in $\bar{\mathcal{B}}_{p,q}^{\alpha,\text{loc}}(U)$ such that for every $\eta \in C_c^r(U)$,

$$(\tilde{f}, \eta) = f(\eta) \quad \text{a.s.} \quad (1.43)$$

Moreover, if $\alpha < \beta - \frac{d}{p}$, then \tilde{f} takes values in $\bar{\mathcal{C}}_{\text{loc}}^\alpha(U)$ with probability one.

Proof. For every $(K, k) \in \mathcal{K}$ and $N \in \mathbb{N}$, we define

$$\tilde{f}_{N,k} := \sum_{x \in \Lambda_k \cap K} v_{k,x}(f) \phi_{k,x} + \sum_{\substack{k \leq n \leq N, i < 2^d \\ x \in \Lambda_n \cap K}} w_{n,x}^{(i)}(f) \psi_{n,x}^{(i)},$$

where we set

$$v_{k,x}(f) := f(\phi_{k,x}) \quad \text{and} \quad w_{n,x}^{(i)}(f) = f(\psi_{n,x}^{(i)}).$$

Clearly, $\tilde{f}_{N,k}$ is almost surely in C_c^r . Following the proof of Theorem 1.46, we get:

$$\mathbb{E} \left[2^{dn(p/2-1)} \sum_{x \in \Lambda_n \cap K, i < 2^d} |w_{n,x}^{(i)}(f)|^p \right] \lesssim 2^{-np\beta},$$

where the implicit constant does not depend on n . Hence, for each $\beta' < \beta$ and each fixed integer k , we deduce by Chebyshev's inequality and Borel-Cantelli's lemma that $(\tilde{f}_{N,k})_N$ is a Cauchy sequence in $\bar{\mathcal{B}}_{p,\infty}^{\beta'}$ with probability one. We denote the limit by \tilde{f}_k . It is clear that \tilde{f}_k converges to some element \tilde{f} of $\bar{\mathcal{B}}_{p,\infty}^{\beta',\text{loc}}(U)$ as k tends to infinity, since for each $\chi \in C_c^r$ with compact support in U , the sequence $\chi \tilde{f}_k$ is eventually constant as k tends to infinity. By Proposition 1.41, if $\alpha < \beta - \frac{d}{p}$, then $\tilde{f} \in \bar{\mathcal{C}}_{\text{loc}}^\alpha(U)$ with probability one. There remains to check that for every $\eta \in C_c^r(U)$, the identity (1.43) holds. By the orthogonality properties of $\phi_{k,x}$, $\psi_{n,x}^{(i)}$ and the fact that η has compact support in U , we have, for k sufficiently large,

$$\eta = \sum_{x \in \Lambda_k \cap K} \langle \phi_{k,x}, \eta \rangle \phi_{k,x} + \lim_{N \rightarrow +\infty} \sum_{\substack{k \leq n \leq N, i < 2^d \\ x \in \Lambda_n \cap K}} \langle \psi_{n,x}^{(i)}, \eta \rangle \psi_{n,x}^{(i)},$$

where we recall that $\langle \cdot, \cdot \rangle$ denotes the scalar product of $L^2(\mathbb{R}^d)$. We fix such k sufficiently large, and denote

$$\eta_N := \sum_{x \in \Lambda_k \cap K} \langle \phi_{k,x}, \eta \rangle \phi_{k,x} + \sum_{\substack{k \leq n \leq N, i < 2^d \\ x \in \Lambda_n \cap K}} \langle \psi_{n,x}^{(i)}, \eta \rangle \psi_{n,x}^{(i)}.$$

By a Taylor expansion of η and (1.11), one can check that there exists $C(d, \eta) < \infty$ such that

$$2^{\frac{dn}{2}} |\langle \psi_{n,x}^{(i)}, \eta \rangle| \leq C 2^{-rn}.$$

From this, together with the expressions for η_N and η above, we obtain that $\exists C(d, \eta) < \infty$ such that for any multi-index $\alpha \leq |r|$

$$\|\partial^\alpha \eta - \partial^\alpha \eta_N\|_{L^\infty} < C \sum_{n > N} 2^{-rn} 2^{|\alpha|n}$$

and thus

$$\eta_N \xrightarrow[\text{in } C_c^{r-1}]{} \eta.$$

Therefore by the weak continuity assumption, we deduce that

$$f(\eta_N) \xrightarrow[\text{prob.}]{N \rightarrow \infty} f(\eta).$$

In order to conclude, there remains to verify that

$$\langle \tilde{f}, \eta_N \rangle = f(\eta_N) \quad \text{almost surely}$$

This follows from the assumption (1.42). \square

1.1.3 Equivalence of norms

In this section we discuss the equivalence between the characterization of Besov spaces via Littlewood-Paley decomposition of Section 1.1.1 and the characterization via compactly supported wavelets of Section 1.1.2.

Proposition 1.50. ([Mey92, Proposition 2.9.4]) *Let $\alpha > 0$, $p, q \in [1, \infty]$ and $f \in L^p(\mathbb{R}^d)$. The following two properties are equivalent.*

1. *Let $r > \alpha$ be an integer and $\phi, (V_n)_{n \in \mathbb{Z}}$ be a r -regular multiresolution analysis as of Definition 1.20. Then the sequence $(2^{n\alpha} \|\mathscr{W}_n f\|_{L^p})_{n \in \mathbb{N}}$ belongs to ℓ^q and $\mathscr{V}_0 f$ belongs to $L^p(\mathbb{R}^d)$.*
2. *There exists a sequence of positive numbers $(\varepsilon_n)_{n \in \mathbb{N}} \in \ell^q$ and a sequence of functions $f_0, g_0, g_1, \dots \in L^p(\mathbb{R}^d)$ such that $f = f_0 + \sum_{n \geq 0} g_n$, $\|g_n\|_{L^p} \leq \varepsilon_n 2^{-n\alpha}$ for $n \geq 0$ and $\|\partial^k g_n\|_{L^p} \leq \varepsilon_n 2^{(m-\alpha)n}$ for some integer $m > \alpha$ and every multi-index $k \in \mathbb{N}^d$ such that $|k| = m$.*

In particular, the functions $f_0 = \mathscr{V}_0 f$, $g_n = \mathscr{W}_n f$ verify (2). Moreover, the norms $\|f_0\|_{L^p} + \|2^{n\alpha} \|g_n\|_{L^p}\|_{\ell^q}$ and $\|\mathscr{V}_0 f\|_{L^p} + \|2^{n\alpha} \|\mathscr{W}_n f\|_{L^p}\|_{\ell^q}$ are equivalent.

A consequence of this result is the fact that the Besov spaces defined in Section 1.1.2 are independent from the choice of a particular wavelet basis or multiresolution analysis.

Lemma 1.51. (Equivalence of multiresolution analyses) *For any $\alpha \in \mathbb{R} \setminus \{0\}$ and any positive integer r such that $r > |\alpha|$, the norm $\|\cdot\|_{\mathcal{B}_{p,q}^\alpha}$ of Definition 1.23 does not depend on the given r -regular multiresolution analysis, i.e. every r -regular multiresolution analysis yields an equivalent norm.*

Proof. Proposition 1.50 gives the equivalence of norms for $\alpha > 0$.

For $\alpha < 0$, $p, q \in [1, \infty]$, define $\alpha' = -\alpha$, $1/p + 1/p' = 1$ and $1/q + 1/q' = 1$. We introduce the following norm which is clearly independent from the choice of multiresolution analysis:

$$\|f\|_{\tilde{\mathcal{B}}_{p,q}^\alpha} = \sup_{\substack{g \in L^{p'} \\ \|g\|_{\mathcal{B}_{p',q'}^{\alpha'}} \leq 1}} \langle f, g \rangle$$

(notice that this norm is slightly different from the norm of the dual of $\mathcal{B}_{p',q'}^{\alpha'}$, because we chose $\mathcal{B}_{p',q'}^{\alpha'}$ to be the completion of C_c^∞ with respect to the norm $\|\cdot\|_{\mathcal{B}_{p',q'}^{\alpha'}}$).

We want to show that $\|\cdot\|_{\tilde{\mathcal{B}}_{p,q}^\alpha}$ and $\|\cdot\|_{\mathcal{B}_{p,q}^\alpha}$ are equivalent. Let $f \in C_c^\infty$. The bound $\|f\|_{\tilde{\mathcal{B}}_{p,q}^\alpha} \lesssim \|f\|_{\mathcal{B}_{p,q}^\alpha}$ is straightforward: by Proposition 1.50 we can write $g = \mathscr{V}_0 g + \sum_n \mathscr{W}_n g$ and obtain

$$\langle f, g \rangle = \langle \mathscr{V}_0 f, \mathscr{V}_0 g \rangle + \sum_{n \geq 0} \langle \mathscr{W}_n f, \mathscr{W}_n g \rangle \leq \|f\|_{\mathcal{B}_{p,q}^\alpha} \|g\|_{\mathcal{B}_{p',q'}^{\alpha'}}$$

thanks to the orthogonality in L^2 between spaces W_n and Hölder's inequality.

To show that $\|f\|_{\mathcal{B}_{p,q}^\alpha} \lesssim \|f\|_{\tilde{\mathcal{B}}_{p,q}^\alpha}$, recall that if $f \in L^p(\mu)$ then

$$\|f\|_{L^p(\mu)} = \sup_{g \in L^{p'}(\mu), \|g\|_{L^{p'}} \leq 1} \int f(x) g(x) \mu(dx)$$

(see e.g. Lemma 1.2 of [BCD11]). Then for every $\delta > 0$ there exists $h_0 \in L^{p'}$ such that $\|h_0\|_{L^{p'}} \leq 1$ and $\|\mathcal{V}_0 f\|_{L^p} \leq \int \mathcal{V}_0 f(x) h_0(x) dx + \delta$. Let

$$Q_N^{q'} = \{(a_n)_{n \geq 0} \in \ell^{q'} \mid \|a_n\|_{\ell^{q'}} \leq 1, a_n = 0 \text{ for } n > N\}.$$

We have

$$\|f\|_{\mathcal{B}_{p,q}^\alpha} = \|\mathcal{V}_0 f\|_{L^p} + \sup_{N \in \mathbb{N}} \sup_{(a_n) \in Q_N^{q'}} \sum_{n=0}^N a_n 2^{n\alpha} \|\mathcal{W}_n f\|_{L^p}.$$

As above, for every $n \geq 0$ there exist $g_n \in L^{p'}$ such that $\|g_n\|_{L^{p'}} \leq 1$ and $\|\mathcal{W}_n f\|_{L^p} \leq \int \mathcal{W}_n f(x) g_n(x) dx + \varepsilon_n$. Now we can estimate the norm

$$\begin{aligned} \|f\|_{\mathcal{B}_{p,q}^\alpha} &\leq \langle \mathcal{V}_0 f, \mathcal{V}_0 h_0 \rangle + \sup_{N \in \mathbb{N}} \sup_{(a_n) \in Q_N^{q'}} \sum_{n=0}^N \langle \mathcal{W}_n f, 2^{n\alpha} a_n \mathcal{W}_n g_n \rangle + \varepsilon \\ \varepsilon &= \delta + \sup_{N \in \mathbb{N}} \sup_{(a_n) \in Q_N^{q'}} \sum_{n=0}^N 2^{n\alpha} a_n \varepsilon_n \end{aligned}$$

where we used the fact that the spaces W_n are orthogonal in L^2 . The remainder ε can be made arbitrarily small: indeed $\sum_{n=0}^N 2^{n\alpha} a_n \varepsilon_n \leq \|2^{n\alpha}\|_{\ell^q} \sup_{n \geq 0} \varepsilon_n$ (recall that $\alpha < 0$). Define

$$g_N = \mathcal{V}_0 h_0 + \sum_{n=0}^N 2^{n\alpha} a_n \mathcal{W}_n g_n.$$

The operators $\mathcal{V}_n: L^p \rightarrow L^p$ and $\mathcal{W}_n: L^p \rightarrow L^p$ are uniformly bounded: we can estimate the norm of g_N as

$$\|g_N\|_{\mathcal{B}_{p',q'}^{\alpha'}} \leq \|h_0\|_{L^{p'}} + \|2^{n\alpha'} 2^{n\alpha} a_n\| \|g_n\|_{L^{p'}} \|_{\ell^{q'}} \leq C$$

and then

$$\|f\|_{\mathcal{B}_{p,q}^\alpha} \leq \sup_{N \in \mathbb{N}} \sup_{(a_n) \in Q_N^{q'}} \langle f, g_N \rangle + \varepsilon = \sup_{\substack{g_N \in L^{p'} \\ \|g_N\|_{\mathcal{B}_{p',q'}^{\alpha'}} \leq C}} \langle f, g_N \rangle + \varepsilon \lesssim \|f\|_{\tilde{\mathcal{B}}_{p,q}^\alpha} + \varepsilon.$$

This completes the proof of the result for $\alpha \neq 0$. The case $\alpha = 0$ can then be recovered by interpolation. \square

Proposition 1.52. (Equivalence of LP-wavelet Besov spaces) *For any $\alpha \in \mathbb{R} \setminus \{0\}$ there exist $C_1, C_2 \in (0, \infty)$ such that for every $f \in C_c^r$, we have*

$$C_1 \|f\|_{\mathcal{B}_{p,q}^\alpha} \leq \|f\|_{B_{p,q}^\alpha} \leq C_2 \|f\|_{\mathcal{B}_{p,q}^\alpha}$$

with $\|\cdot\|_{\mathcal{B}_{p,q}^\alpha}$ given by Definition 1.23 and $\|\cdot\|_{B_{p,q}^\alpha}$ given by Definition 1.10

Proof. For $\alpha > 0$ we follow the proof of Proposition 1.50 given in [Mey92] to show that the following two properties are equivalent:

1. The sequence $(2^{j\alpha} \|\Delta_j f\|_{L^p})_{j \in \mathbb{N}}$ belongs to ℓ^q and $\Delta_{-1} f$ belongs to $L^p(\mathbb{R}^d)$.

2. $\exists(\varepsilon_k)_{k \in \mathbb{N}} \in \ell^q$, $\varepsilon_k \geq 0$ and $\exists f_0, g_0, g_1, \dots \in L^p(\mathbb{R}^d)$ such that $f = f_0 + \sum_{k \geq 0} g_k$, $\|g_k\|_{L^p} \leq \varepsilon_k 2^{-k\alpha}$ for $k \geq 0$ and $\|\partial^r g_k\|_{L^p} \leq \varepsilon_k 2^{(m-\alpha)k}$ for some integer $m > \alpha$ and every multi-index $r \in \mathbb{N}^d$ such that $|r| = m$.

The fact that in point 2 one can take $g_k = \Delta_k f$ is clear from Bernstein inequalities (Lemma 1.8). To show (2) \Rightarrow (1) we write

$$\Delta_j f = \Delta_j f_0 + \sum_{k=0}^{\infty} \Delta_j g_k.$$

Then since $\|\Delta_j g_k\|_{L^p} \leq C \|g_k\|_{L^p}$ uniformly on $j \in \mathbb{N}$ (see Remark 1.5) we estimate

$$\left\| \sum_{k > j} \Delta_j g_k \right\|_{L^p} \leq C \sum_{k > j} \varepsilon_k 2^{-\alpha k}$$

and for $k \leq j$ we have, again by Bernstein inequalities $\|\Delta_j g_k\|_{L^p} \leq C 2^{-mj} 2^{(m-\alpha)k} \varepsilon_k$ and then

$$\|\Delta_j f\|_{L^p} \leq C 2^{-mj} \sum_{k \leq j} \varepsilon_k 2^{(m-\alpha)k} + C \sum_{k > j} \varepsilon_k 2^{-\alpha k} \leq C' \varepsilon'_j 2^{-\alpha j}$$

with $\varepsilon'_j = \sum_k \varepsilon_k 2^{-(m-\alpha)|j-k|}$ and then $(\varepsilon'_j)_{j \in \mathbb{N}} \in \ell^q$. For $\alpha < 0$ we proceed as in the proof of Lemma 1.51. \square

From Proposition 1.52 follows immediately that $\bar{\mathcal{B}}_{p,q}^\alpha \subset B_{p,q}^\alpha$ and $B_{p,q}^{\alpha,\text{loc}}(U) = \bar{\mathcal{B}}_{p,q}^{\alpha,\text{loc}}(U)$.

Remark 1.53. The equivalence between Littlewood-Paley-Besov norms and wavelets-Besov norms holds for $\alpha = 0$ as well, as can be shown by interpolation techniques (for a discussion on interpolation of Besov space we refer to [BCD11]).

1.2 Paracontrolled calculus

In this section we first introduce Bony's paraproducts [Bon81], and then recall the some basic results of paracontrolled calculus introduced by Gubinelli, Imkeller and Perkowski in [GIP15] to deal with parabolic SPDEs. We will focus on distributions on the d -dimensional torus $\mathbb{T}^d = (\mathbb{R}^d / 2\pi\mathbb{Z})^d$, denoted as $\mathcal{D}'(\mathbb{T}^d) = \mathcal{S}'(\mathbb{T}^d)$, and write $\mathcal{C}^\alpha := \mathcal{C}^\alpha(\mathbb{T}^d)$, $L^p := L^p(\mathbb{T}^d)$, $C^n = C^n(\mathbb{T}^d)$. However, the results presented here can be easily carried over to the whole d -dimensional space \mathbb{R}^d .

1.2.1 Bony's paraproducts and parilinearization

The theory of paraproducts, developed by J. M. Bony in 1981 [Bon81], yields a way to define the product of two tempered distributions under appropriate assumptions. Let $u \in \mathcal{C}^\alpha$ and $v \in \mathcal{C}^\beta$ as in Definition 1.10 for $\alpha, \beta \in \mathbb{R}$. We can decompose $u, v \in \mathcal{S}'$ as

$$u = \sum_{j \geq -1} \Delta_j u, \quad v = \sum_{j \geq -1} \Delta_j v$$

thanks to Lemma 1.6. For every $j, k \geq -1$, the product $\Delta_j u \Delta_k v$ is well defined as a C^∞ function: showing the convergence of $\sum_{j,k} \Delta_j u \Delta_k v$ in \mathcal{S}' would allow to *define* the product uv as this limit. We start splitting this sum as:

$$\sum_{j,k \geq -1} \Delta_j u \Delta_k v = \sum_{k \geq 1} S_k u \Delta_k v + \sum_{k \geq 1} \Delta_k u S_k v + \sum_{j \geq -1} \sum_{k: |k-j| \leq 1} \Delta_j u \Delta_k v$$

with $S_k u = \sum_{j < k-1} \Delta_j u$ as in (1.3). Note that $\forall k \geq 1, \forall j \geq -1$ the products

$$S_k u \Delta_k v, \quad \Delta_k u S_k v, \quad \sum_{k: |k-j| \leq 1} \Delta_j u \Delta_k v \quad (1.44)$$

are well defined as finite sums of smooth functions. As a consequence of the disjointness properties (1.1),(1.2) of the dyadic partition of unity of Proposition 1.1, we obtain that there exists an annulus $\mathcal{A} \subset \mathbb{R}^d$ and a ball $B(0, R) \subset \mathbb{R}^d$ such that $\forall k \geq 1, \forall j \geq -1$:

$$\begin{aligned} \text{supp } \mathcal{F}(S_k u \Delta_k v) &\subset 2^k \mathcal{A}, \\ \text{supp } \mathcal{F}(\Delta_k u S_k v) &\subset 2^k \mathcal{A}, \\ \text{supp } \mathcal{F}\left(\sum_{k: |j-k| \leq 1} \Delta_j u \Delta_k v\right) &\subset 2^k B(0, R). \end{aligned}$$

Therefore, by Lemma 1.9 it is immediate to see that $\sum_{k \geq 1} S_k u \Delta_k v$ and $\sum_{k \geq 1} \Delta_k u S_k v$ are well defined elements of \mathcal{S}' for every $u \in \mathcal{C}^\alpha, v \in \mathcal{C}^\beta, \alpha, \beta \in \mathbb{R}$. On the other hand we cannot apply Lemma 1.9 to the term $\sum_{j \geq -1} \sum_{k: |k-j| \leq 1} \Delta_j u \Delta_k v$, and this gives some restrictions on α, β as shown in the following proposition.

Proposition 1.54. (paraproduct estimates)

1. Let $u \in \mathcal{C}^\alpha, v \in \mathcal{C}^\beta, \alpha, \beta \in \mathbb{R}$. Then the sum

$$u \prec v := \sum_{k \geq 1} S_k u \Delta_k v \quad (1.45)$$

belongs to \mathcal{S}' .

2. Let $u \in L^\infty$ and $v \in \mathcal{C}^\beta$ with $\beta \in \mathbb{R}$. Then $u \prec v \in \mathcal{C}^\beta$ and

$$\|u \prec v\|_{\mathcal{C}^\beta} \lesssim \|u\|_{L^\infty} \|v\|_{\mathcal{C}^\beta} \quad (1.46)$$

3. Let $u \in \mathcal{C}^\alpha$ and $v \in \mathcal{C}^\beta$ with $\alpha < 0, \beta \in \mathbb{R}$. Then $u \prec v \in \mathcal{C}^{\alpha+\beta}$ and

$$\|u \prec v\|_{\mathcal{C}^{\alpha+\beta}} \lesssim \|u\|_{\mathcal{C}^\alpha} \|v\|_{\mathcal{C}^\beta} \quad (1.47)$$

4. Let $u \in \mathcal{C}^\alpha$ and $v \in \mathcal{C}^\beta$ with $\alpha + \beta > 0$. Then the sum

$$u \circ v := \sum_{j \geq -1} \sum_{k: |j-k| \leq 1} \Delta_j u \Delta_k v \quad (1.48)$$

belongs to L^∞ . Moreover, $u \circ v \in \mathcal{C}^{\alpha+\beta}$ and

$$\|u \circ v\|_{\mathcal{C}^{\alpha+\beta}} \lesssim \|u\|_{\mathcal{C}^\alpha} \|v\|_{\mathcal{C}^\beta} \quad (1.49)$$

Proof.

1. Since $S_k u \Delta_k v$ has Fourier transform supported in an annulus $2^k \mathcal{A}$, we can apply Lemma 1.9 with $n = \lceil -\alpha - \beta \rceil \vee 0$ to obtain $u \prec v \in \mathcal{S}'$.
2. Since $\text{supp } \mathcal{F}(S_k u \Delta_k v) \subset 2^k \mathcal{A}$ there exists $N_0 \in \mathbb{N}$ such that

$$\begin{aligned} \rho(2^{N_0-k} \cdot) \mathcal{F}(S_k u \Delta_k v) &= 0, \\ \rho(2^{-N_0-k} \cdot) \mathcal{F}(S_k u \Delta_k v) &= 0, \end{aligned}$$

and then for every $\ell \in \mathbb{N}$ large enough we have

$$\begin{aligned} \|\Delta_\ell \sum_{k \geq 1} S_k u \Delta_k v\|_{L^\infty} &\lesssim \|S_\ell u \Delta_\ell v\|_{L^\infty} \\ &\lesssim \|u\|_{L^\infty} 2^{-\beta \ell} \|v\|_\beta \end{aligned}$$

where we used the fact that the operators $S_\ell: L^\infty \rightarrow L^\infty$ $\Delta_\ell: L^\infty \rightarrow L^\infty$ with norms independent of ℓ , as shown in Remark 1.5.

3. We have for every $k \geq 1$:

$$\begin{aligned} \left\| \sum_{j < k-1} \Delta_j u \Delta_k v \right\|_{L^\infty} &\lesssim 2^{-k\beta} \|u\|_{\mathcal{C}^\alpha} \|v\|_{\mathcal{C}^\beta} \sum_{j < k-1} 2^{-j\alpha} \\ &\lesssim 2^{-k(\alpha+\beta)} \|u\|_{\mathcal{C}^\alpha} \|v\|_{\mathcal{C}^\beta} \end{aligned}$$

and bounding as before $\|\Delta_\ell \sum_{k \geq 1} S_k u \Delta_k v\|_{L^\infty}$ we obtain the result.

4. We have $\forall j \geq -1$

$$\left\| \sum_{k: |j-k| \leq 1} \Delta_j u \Delta_k v \right\|_{L^\infty} \lesssim 2^{-(\alpha+\beta)j} \|u\|_{\mathcal{C}^\alpha} \|v\|_{\mathcal{C}^\beta}$$

and since $\alpha + \beta > 0$ the sum converges in L^∞ . Since there exists $R > 0$ such that $\text{supp } \mathcal{F}(\sum_{k: |k-j| \leq 1} \Delta_j u \Delta_k v) \subset 2^k B(0, R)$, then $\exists N_0 \in \mathbb{N}$ such that

$$\Delta_\ell \sum_{j \geq -1} \sum_{k: |j-k| \leq 1} \Delta_j u \Delta_k v = \Delta_\ell \sum_{j > \ell + N_0} \sum_{k: |j-k| \leq 1} \Delta_j u \Delta_k v$$

and estimating

$$\left\| \sum_{k: |j-k| \leq 1} \Delta_j u \Delta_k v \right\|_{L^\infty} \lesssim 2^{-j(\alpha+\beta)} \|u\|_{\mathcal{C}^\alpha} \|v\|_{\mathcal{C}^\beta}$$

we obtain

$$\|\Delta_\ell(u \circ v)\|_{L^\infty} \lesssim 2^{-\ell(\alpha+\beta)} \|u\|_{\mathcal{C}^\alpha} \|v\|_{\mathcal{C}^\beta}. \quad \square$$

Remark 1.55. Proposition 1.54 allows to define the product between $u \in \mathcal{C}^\alpha$ and $v \in \mathcal{C}^\beta$ when $\alpha + \beta > 0$ as

$$uv := u \prec v + v \prec u + u \circ v.$$

In order for this to be a good definition, the paraproduct should verify the Leibniz' property for the derivative of a product. But this is obviously true, knowing that the partial sums (1.44) converge in \mathcal{S}' and that the derivative is a continuous operator $\mathcal{S}' \rightarrow \mathcal{S}'$.

Bony's paraproducts estimates hold for general Besov spaces $B_{p,q}^\alpha$ as shown in Theorem 2.82 and 2.85 of [BCD11]. We recall here the result:

Proposition 1.56. (Multiplicative inequalities) *Let $p, p_1, p_2, q, q_1, q_2 \in [1, \infty]$ be such that*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \quad \text{and} \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}.$$

1. *If $\alpha > 0$, then the mapping*

$$(f, g) \mapsto f \prec g + g \prec f + f \circ g \tag{1.50}$$

defined on $f, g \in C_c^\infty$ extends to a bilinear continuous functional from $B_{p_1, q_1}^\alpha \times B_{p_2, q_2}^\alpha$ to $B_{p, q}^\alpha$.

2. *If $\alpha < 0 < \beta$ with $\alpha + \beta > 0$, then the mapping (1.50) extends to a bilinear continuous functional from $B_{p_1, q_1}^\alpha \times B_{p_2, q_2}^\beta$ to $B_{p, q}^\alpha$.*

1.2.2 Parabolic and time-weighted spaces

When dealing with paraproducts in the context of parabolic equations (as will be the case in Chapters 5 and 6) it is possible to define parabolic Besov spaces on $\mathbb{R}^+ \times \mathbb{T}^d$ and related paraproducts. Anyway, in order to keep the setting simpler, we choose to introduce distribution-valued functions of time as follows. Given a Banach space X with norm $\|\cdot\|_X$ and $T > 0$, we note

$$C_T X := C([0, T], X)$$

for the space of continuous maps from $[0, T]$ to X , equipped with the supremum norm $\|\cdot\|_{C_T X}$, and we set $CX = C(\mathbb{R}_+, X)$. For $\alpha \in (0, 1)$ we also define $C_T^\alpha X$ as the space of α -Hölder continuous functions from $[0, T]$ to X , endowed with the norm

$$\|f\|_{C_T^\alpha X} = \sup_{0 \leq t \leq T} \|f(t)\|_X + \sup_{0 \leq s < t \leq T} \frac{\|f(t) - f(s)\|_X}{|t - s|^\alpha}$$

and we write $C_{\text{loc}}^\alpha X$ for the space of locally α -Hölder continuous functions from \mathbb{R}_+ to X . Moreover for convenience we denote

$$\mathcal{C}_T^\alpha := C_T \mathcal{C}^\alpha(\mathbb{T}^d).$$

We will avoid to write explicitly the time span T whenever this does not cause ambiguities.

We introduce the following time-weighted spaces, also called *explosive spaces* for the fact that they allow the X norm to diverge as $t \rightarrow 0$ at a speed prescribed by the parameter γ .

Definition 1.57. (time-weighted spaces) *Let X be a Banach space and $C_T X$ as above. For $\gamma > 0$, $p \in [1, \infty)$, we define*

$$\begin{aligned} \mathcal{M}_T^{\gamma, p} X &= \{v: L^p((0, T], X) : \|v\|_{\mathcal{M}_T^{\gamma, p} X} = \|t \mapsto t^\gamma v(t)\|_{L^p((0, T], X)} < \infty\}, \\ \mathcal{M}_T^\gamma X &= \{v: C((0, T], X) : \|v\|_{\mathcal{M}_T^\gamma X} = \|t \mapsto t^\gamma v(t)\|_{C_T X} < \infty\}. \end{aligned} \quad (1.51)$$

The following spaces are the best suited in the framework of parabolic PDEs, and will be useful in the next sections.

Definition 1.58. (parabolic spaces) *Let $\alpha \in (0, 2)$ and \mathcal{C}^α as in Definition 1.10. Define the space*

$$\mathcal{L}_T^\alpha = C_T^{\alpha/2} L^\infty \cap C_T \mathcal{C}^\alpha \quad (1.52)$$

equipped with the norm

$$\|f\|_{\mathcal{L}_T^\alpha} = \max \left\{ \|f\|_{C_T^{\alpha/2} L^\infty}, \|f\|_{C_T \mathcal{C}^\alpha} \right\}.$$

Define also the space

$$\dot{\mathcal{L}}_T^\alpha = C_T^{\alpha/2} \mathcal{C}^0 \cap C_T \mathcal{C}^\alpha \quad (1.53)$$

equipped with the norm

$$\|f\|_{\dot{\mathcal{L}}_T^\alpha} = \max \left\{ \|f\|_{C_T^{\alpha/2} \mathcal{C}^0}, \|f\|_{C_T \mathcal{C}^\alpha} \right\}.$$

Obviously, $\|\cdot\|_{\dot{\mathcal{L}}_T^\alpha} \lesssim \|\cdot\|_{\mathcal{L}_T^\alpha}$ since $\|\cdot\|_{\mathcal{C}^0} \lesssim \|\cdot\|_{L^\infty}$ (as discussed in Section 1.1.1). On the other hand, we have the following interpolation result.

Lemma 1.59. *Let $\alpha \in (0, 2)$, $0 < \varepsilon < \alpha$ and $u \in \dot{\mathcal{L}}_T^\alpha$. Then*

$$\|u\|_{C_T^{\alpha/2 - \varepsilon/2} L^\infty} \lesssim \|u\|_{\dot{\mathcal{L}}_T^\alpha}$$

and

$$\|u\|_{\mathcal{L}_T^{\alpha-\varepsilon}} \lesssim \|u\|_{\dot{\mathcal{L}}_T^\alpha}$$

Proof. We start showing the first inequality:

$$\sup_{s \neq t} \frac{\|u_t - u_s\|_{L^\infty}}{|t-s|^{\alpha/2-\varepsilon/2}} \leq \sup_{s \neq t} \left[\sum_{i \leq n} \frac{\|\Delta_i u_t - \Delta_i u_s\|_{L^\infty}}{|t-s|^{\alpha/2-\varepsilon/2}} + \sum_{i > n} \frac{\|\Delta_i u_t - \Delta_i u_s\|_{L^\infty}}{|t-s|^{\alpha/2-\varepsilon/2}} \right]$$

and choosing $2^{-n-1} \leq |t-s|^{1/2} \leq 2^{-n}$ we obtain

$$\begin{aligned} \sum_{i < n} \frac{\|\Delta_i u_t - \Delta_i u_s\|_{L^\infty}}{|t-s|^{\alpha/2-\varepsilon/2}} &\lesssim \|u\|_{C_T^{\alpha/2} \mathcal{L}^0} \sum_{i \leq n} |t-s|^{\varepsilon/2} \\ \sum_{i \geq n} \frac{\|\Delta_i u_t - \Delta_i u_s\|_{L^\infty}}{|t-s|^{\alpha/2-\varepsilon/2}} &\lesssim \|u\|_{\mathcal{L}_T^\alpha} \sum_{i > n} 2^{-\alpha i} 2^{(\alpha-\varepsilon)n} \end{aligned}$$

and this yields the result. The second inequality is a direct consequence of the first one together with the Besov norm inequality $\|\cdot\|_{C_T \mathcal{L}^{\alpha-\varepsilon}} \lesssim \|\cdot\|_{C_T \mathcal{L}^\alpha}$ seen in Section 1.1.1. \square

When working with irregular initial conditions, we need to consider explosive spaces of parabolic type. For $\gamma \geq 0$, $\alpha \in (0, 2)$, and $T > 0$ we define the norm

$$\|f\|_{\mathcal{L}_T^{\gamma,\alpha}} = \max \left\{ \|t \mapsto t^\gamma f(t)\|_{C_T^{\alpha/2} L^\infty}, \|f\|_{\mathcal{M}_T^{\gamma,\alpha}} \right\}, \quad (1.54)$$

and the space $\mathcal{L}_T^{\gamma,\alpha} = \{f: [0, T] \rightarrow \mathbb{R}: \|f\|_{\mathcal{L}_T^{\gamma,\alpha}} < \infty\}$. In particular $\mathcal{L}_T^{0,\alpha} = \mathcal{L}_T^\alpha$. The following lemma is a direct consequence of the definition (1.54).

Lemma 1.60. *For all $\alpha \in (0, 2)$, $\gamma \in [0, 1)$, $\varepsilon \in [0, \alpha \wedge 2\gamma)$, $T > 0$ and $f \in \mathcal{L}_T^{\gamma,\alpha}$ with $f(0) = 0$ we have*

$$\|f\|_{\mathcal{L}_T^{\gamma-\varepsilon/2, \alpha-\varepsilon}} \lesssim \|f\|_{\mathcal{L}_T^{\gamma,\alpha}}. \quad (1.55)$$

Let $\alpha \in (0, 2)$, $\gamma \in (0, 1)$, $T > 0$, and let $f \in \mathcal{L}_T^\alpha$. Then for all $\delta \in (0, \alpha]$ we have

$$\begin{aligned} \|f\|_{\mathcal{L}_T^\delta} &\lesssim \|f(0)\|_{\mathcal{L}^\delta} + T^{(\alpha-\delta)/2} \|f\|_{\mathcal{L}_T^\alpha}, \\ \|f\|_{\mathcal{L}_T^{\gamma,\delta}} &\lesssim T^{(\alpha-\delta)/2} \|f\|_{\mathcal{L}_T^{\gamma,\alpha}}. \end{aligned} \quad (1.56)$$

1.2.3 Schauder estimates

Let

$$\mathcal{L} := \partial_t - \Delta \quad (1.57)$$

denote the heat operator with periodic boundary conditions on \mathbb{T}^d . We introduce the linear operator $I: C(\mathbb{R}_+, \mathcal{D}'(\mathbb{T}^d)) \rightarrow C(\mathbb{R}_+, \mathcal{D}'(\mathbb{T}^d))$ given by

$$If(t) := \int_0^t P_{t-s} f(s) ds,$$

where $(P_t)_{t \geq 0}$ is the heat semigroup with kernel $P_t(x) := \frac{1}{(4\pi t)^{3/2}} e^{-\frac{|x|^2}{4t}} \mathbb{1}_{t \geq 0}$. The integral above has of course to be interpreted as testing $f \in \mathcal{D}'(\mathbb{T}^d)$ against the kernel $P_t(\cdot)$. We have then the following well-known Schauder estimates (see Chapter 2 of [BCD11] and [GIP15, GP15]):

Lemma 1.61. *Let $\alpha \in (0, 2)$ and $\gamma \in [0, 1)$. Then*

$$\|If\|_{\mathcal{L}_T^{\gamma, \alpha}} \lesssim \|f\|_{\mathcal{M}_T^{\gamma} \mathcal{G}^{\alpha-2}}, \quad (1.58)$$

for all $T > 0$. If further $\beta \geq -\alpha$, then $\forall s \in [0, T]$:

$$\|s \mapsto P_s u_0\|_{\mathcal{L}_T^{(\beta+\alpha)/2, \alpha}} \lesssim \|u_0\|_{\mathcal{G}^{-\beta}}. \quad (1.59)$$

For all $\alpha \in \mathbb{R}$, $\gamma \in [0, 1)$, and $T > 0$ we have

$$\|If\|_{\mathcal{M}_T^{\gamma} \mathcal{G}^{\alpha}} \lesssim \|f\|_{\mathcal{M}_T^{\gamma} \mathcal{G}^{\alpha-2}}. \quad (1.60)$$

We give also some bounds for the solutions of the heat equation with sources in space–time Lebesgue spaces.

Lemma 1.62. *Let $\beta \in \mathbb{R}$ and $f \in L_T^p B_{p, \infty}^{\beta}$, then for every $\kappa \in [0, 1]$ we have $If \in C_T^{\kappa/q} \mathcal{G}^{\beta+2(1-\kappa)-(2-2\kappa+d)/p}$ with*

$$\|If\|_{C_T^{\kappa/q} \mathcal{G}^{\beta+2(1-\kappa)-(2-2\kappa+d)/p}} \lesssim T \|f\|_{L_T^p B_{p, \infty}^{\beta}},$$

with $\frac{1}{q} + \frac{1}{p} = 1$. Moreover, for every $\gamma < \gamma' < 1 - 1/p$ and every $0 < \alpha < (2 - 5/p + \beta) \wedge 2$ we have

$$\|If\|_{\mathcal{L}_T^{\gamma', \alpha}} \lesssim T \|f\|_{\mathcal{M}_T^{\gamma, p} B_{p, \infty}^{\beta}}.$$

Proof. We only show the second inequality as the first one is easier and obtained with similar techniques. Let $u = If$, we have

$$\begin{aligned} t^{\gamma} \|\Delta_i u(t)\|_{L^{\infty}} &\leq t^{1/q} 2^{di/p} \left[\int_0^1 s^{-\gamma q} e^{-cq2^{2i}t(1-s)} ds \right]^{1/q} \left[\int_0^t s^{\gamma p} \|\Delta_i f(s)\|_{L^p}^p ds \right]^{1/p} \\ &\lesssim_{\gamma, q} 2^{id/p} 2^{-2i/q} \left[\int_0^t s^{\gamma p} \|\Delta_i f(s)\|_{L^p}^p ds \right]^{1/p} \end{aligned}$$

which allows us to bound $\|If\|_{\mathcal{M}_T^{\gamma} \mathcal{G}^{\alpha}}$. In order to estimate $\|t \mapsto t^{\gamma'} If\|_{C_T^{\alpha/2} L^{\infty}}$ we write

$$\begin{aligned} \|t^{\gamma'} \Delta_i u(t) - s^{\gamma'} \Delta_i u(s)\|_{L^{\infty}} &\lesssim \int_s^t v^{\gamma'-1} \|\Delta_i u(v)\|_{L^{\infty}} dv \\ &\quad + |t-s| 2^{i(d+2)/p} \|\Delta_i f\|_{\mathcal{M}_T^{\gamma, p} L^p(\mathbb{T}^3)} \\ &\quad + \left\| \int_s^t v^{\gamma'} \Delta_i f(v) dv \right\|_{L^{\infty}}. \end{aligned}$$

We can estimate the first term as

$$\int_s^t v^{\gamma'-1} \|\Delta_i u(v)\|_{L^{\infty}} dv \lesssim 2^{i(d+2)/p} \|\Delta_i f\|_{\mathcal{M}_T^{\gamma, p} L^p(\mathbb{T}^3)} \int_s^t v^{\gamma'-\gamma-1} dv.$$

For the third term we have

$$\begin{aligned} \left\| \int_s^t v^{\gamma'} \Delta_i f(v) dv \right\|_{L^{\infty}} &\lesssim \left[\int_s^t dv \right]^{1/q} \left[\int_s^t v^{\gamma p} \|\Delta_i f(s)\|_{L^{\infty}}^p dv \right]^{1/p} \\ &\lesssim 2^{id/p} |t-s|^{1/q} \|\Delta_i f\|_{\mathcal{M}_T^{\gamma, p} L^p(\mathbb{T}^3)} \end{aligned}$$

We obtain then if $2^{2i}|t-s| \leq 1$

$$\|t^{\gamma'} \Delta_i u(t) - s^{\gamma'} \Delta_i u(s)\|_{L^{\infty}} \lesssim 2^{id/p} |t-s|^{1/q} \|\Delta_i f\|_{\mathcal{M}_T^{\gamma, p} L^p(\mathbb{T}^3)}$$

and if $2^{2i}|t-s| > 1$ we just use the trivial estimate

$$\begin{aligned} \|t^{\gamma'} \Delta_i u(t) - s^{\gamma'} \Delta_i u(s)\|_{L^\infty} &\lesssim 2^{id/p} 2^{-2i/q} \|\Delta_i f\|_{\mathcal{M}_T^{\gamma,p} L^p(\mathbb{T}^3)} \\ &\lesssim 2^{id/p} |t-s|^{1/q} \|\Delta_i f\|_{\mathcal{M}_T^{\gamma,p} L^p(\mathbb{T}^3)}. \end{aligned}$$

Therefore, for every $\kappa \in [0, 1]$:

$$\|t^{\gamma'} \Delta_i u(t) - s^{\gamma'} \Delta_i u(s)\|_{L^\infty} \lesssim 2^{\binom{d+2}{p}-2i} 2^{2\kappa i/q} |t-s|^{\kappa/q} \|\Delta_i f\|_{\mathcal{M}_T^{\gamma,p} L^p(\mathbb{T}^3)}.$$

Choosing $\kappa/q = \alpha/2$ we obtain the desired estimate. \square

1.2.4 Time-smoothed paraproduct

In order to deal with parabolic equations involving the heat operator \mathcal{L} of (1.57) we need to introduce a modified paraproduct with a smoothing in the time variable that is tuned to the parabolic scaling. Let $\varphi \in C^\infty(\mathbb{R}, \mathbb{R}_+)$ be a non-negative function with $\text{supp } \varphi \subset \mathbb{R}_+$ and such that $\int \varphi(x) dx = 1$, and define for all $k \geq -1$ the operator

$$Q_k: C\mathcal{E}^\alpha \rightarrow C\mathcal{E}^\alpha, \quad Q_k f(t) = \int_0^\infty 2^{-2k} \varphi(2^{2k}(t-s)) f(s) ds.$$

We can extend functions $f \in C_T \mathcal{E}^\alpha$ to functions $g \in f \in C\mathcal{E}^\alpha$ by taking $g(t) := f(t \wedge T)$ $\forall t \geq 0$ and thus apply the operator Q_k defined on $C\mathcal{E}^\alpha$. We define then the paraproduct

$$f \llcorner g := \sum_{k \geq 1} (Q_k S_k f) \Delta_k g \quad (1.61)$$

for $f, g \in C(\mathbb{R}_+, \mathcal{D}'(\mathbb{T}^d))$. We collect in the following lemma some standard estimates for the modified paraproduct $f \llcorner g$, whose proof can be found in [GP17].

Lemma 1.63.

a) For every $\beta \in \mathbb{R}$ and $\gamma \in [0, 1)$ we have

$$t^\gamma \|f \llcorner g(t)\|_{\mathcal{E}^\beta} \lesssim \|f\|_{\mathcal{M}_t^\gamma L^\infty} \|g(t)\|_{\mathcal{E}^\beta}, \quad (1.62)$$

for every $t > 0$, and for $\alpha < 0$ furthermore

$$t^\gamma \|f \llcorner g(t)\|_{\mathcal{E}^{\alpha+\beta}} \lesssim \|f\|_{\mathcal{M}_t^\gamma \mathcal{E}^\alpha} \|g(t)\|_{\mathcal{E}^\beta}. \quad (1.63)$$

b) Let $\alpha, \delta \in (0, 2)$, $\gamma \in [0, 1)$, $T > 0$, and let $f \in \mathcal{L}_T^{\gamma, \delta}$, $g \in C_T \mathcal{E}^\alpha$, and $\mathcal{L}g \in C_T \mathcal{E}^{\alpha-2}$. Then

$$\|f \llcorner g\|_{\mathcal{L}_T^{\gamma, \alpha}} \lesssim \|f\|_{\mathcal{L}_T^{\gamma, \delta}} (\|g\|_{C_T \mathcal{E}^\alpha} + \|\mathcal{L}g\|_{C_T \mathcal{E}^{\alpha-2}}). \quad (1.64)$$

1.2.5 Commutator estimates and parilinearization

The key ingredient at the core of paracontrolled calculus is the following commutation result, first proved in [GIP15]. See also [GP15, Lemma 14] for the slightly different statement recalled here.

Lemma 1.64. (commutator) Assume that $\alpha, \beta, \gamma \in \mathbb{R}$ are such that $\alpha + \beta + \gamma > 0$ and $\beta + \gamma \neq 0$. Then for $f, g, h \in C^\infty$ the trilinear operator

$$\text{com}_1(f, g, h) := ((f \llcorner g) \circ h) - f(g \circ h) \quad (1.65)$$

allows for the bound

$$\|\text{com}_1(f, g, h)\|_{\mathcal{C}^{\beta+\gamma}} \lesssim \|f\|_{\mathcal{C}^\alpha} \|g\|_{\mathcal{C}^\beta} \|h\|_{\mathcal{C}^\gamma}, \quad (1.66)$$

and can thus be uniquely extended to a bounded trilinear operator

$$\text{com}_1: \mathcal{C}^\alpha \times \mathcal{C}^\beta \times \mathcal{C}^\alpha \rightarrow \mathcal{C}^{\beta+\gamma}.$$

If $\alpha + \beta + \gamma > 0$ with $\alpha \in (0, 1)$, then we have the bound

$$\|\text{com}_1(f, g, h)\|_{\mathcal{C}^{\alpha+\beta+\gamma}} \lesssim \|f\|_{\mathcal{C}^\alpha} \|g\|_{\mathcal{C}^\beta} \|h\|_{\mathcal{C}^\gamma}, \quad (1.67)$$

and com_1 can be extended to a bounded trilinear operator

$$\text{com}_1: \mathcal{C}^\alpha \times \mathcal{C}^\beta \times \mathcal{C}^\alpha \rightarrow \mathcal{C}^{\alpha+\beta+\gamma}.$$

The following commutator is due to Bony [Bon81]:

Lemma 1.65. *Let $\alpha \in (0, 1)$, $\beta \in \mathbb{R}$, and let $g \in \mathcal{C}^\alpha$, $f \in L^\infty$, and $h \in \mathcal{C}^\beta$. Then*

$$\|f \prec (g \prec h) - (fg) \prec h\|_{\mathcal{C}^{\alpha+\beta}} \lesssim \|f\|_{L^\infty} \|g\|_{\mathcal{C}^\alpha} \|h\|_{\mathcal{C}^\beta},$$

if $f \in \mathcal{C}^\alpha$, $g \in L^\infty$, $h \in \mathcal{C}^\beta$ we obtain the estimation

$$\|f \prec (g \prec h) - (fg) \prec h\|_{\mathcal{C}^{\alpha+\beta}} \lesssim \|f\|_{\mathcal{C}^\alpha} \|g\|_{L^\infty} \|h\|_{\mathcal{C}^\beta}.$$

Proof. From the Fourier support properties of the L-P blocks we obtain easily

$$\begin{aligned} \|\Delta_j[f \prec (g \prec h) - (fg) \prec h]\|_{L^\infty} &\lesssim \|S_j f S_j g - S_j(fg)\|_{L^\infty} \|\Delta_j h\|_{L^\infty} \\ &\lesssim 2^{-\beta j} \|h\|_{\mathcal{C}^\beta} \|S_j f S_j g - S_j(fg)\|_{L^\infty} \end{aligned}$$

And since $\forall x \in \mathbb{T}^d$

$$\begin{aligned} |[S_j f S_j g - S_j(fg)](x)| &= \left| \int P_{j,x}(y) P_{j,x}(z) f(y) [g(z) - g(y)] dy dz \right| \\ &\lesssim \|g\|_{\mathcal{C}^\alpha} \|f\|_{L^\infty} \int |P_{j,x}(y) P_{j,x}(z)| |y - z| dy dz \\ &\lesssim 2^{-j\alpha} \|g\|_{\mathcal{C}^\alpha} \|f\|_{L^\infty} \end{aligned}$$

we obtain the first estimation. The second one follows in the same way. \square

Bony proved also a basic *paralinearisation* result, soon after improved by Meyer. We give here a particular version suited to our purposes.

Proposition 1.66. (paralinearization)

Let $\alpha \in (0, 1)$, $f \in (\mathcal{C}^\alpha)^d$ and $F \in C^3(\mathbb{R}^d; \mathbb{R})$ then

$$R_F(f) := F(f) - F'(f) \prec f \in \mathcal{C}^{2\alpha}$$

with

$$\|R_F(f)\|_{\mathcal{C}^{2\alpha}} \lesssim \|F\|_{C^2} (1 + \|f\|_{\mathcal{C}^\alpha})^2.$$

Moreover the map $f \mapsto R_F(f)$ is locally Lipschitz and

$$\|R_F(f) - R_F(\tilde{f})\|_{\mathcal{C}^{2\alpha}} \lesssim \|F\|_{C^3} (1 + \|f\|_{\mathcal{C}^\alpha} + \|\tilde{f}\|_{\mathcal{C}^\alpha})^2 \|\tilde{f} - f\|_{\mathcal{C}^\alpha}.$$

We introduce some commutators involving the time-smoothed paraproduct, which allow to control its interaction with the heat operator (1.57). The proof can be found in [GP17].

Lemma 1.67.

- a) For $\alpha, \beta, \gamma \in \mathbb{R}$ such that $\alpha + \beta + \gamma > 0$ and $\alpha \in (0, 1)$ there exists a bounded trilinear map

$$\overline{\text{com}}_1: \mathcal{C}^\alpha \times \mathcal{C}^\beta \times \mathcal{C}^\gamma \rightarrow \mathcal{C}^{\alpha+\beta+\gamma},$$

such that for smooth f, g, h :

$$\overline{\text{com}}_1(f, g, h) = (f \llcorner g) \circ h - f(g \circ h). \quad (1.68)$$

- b) Let $\alpha \in (0, 2)$, $\beta \in \mathbb{R}$, and $\gamma \in [0, 1)$. Then the bilinear maps

$$\text{com}_2(f, g) := f \prec g - f \llcorner g. \quad (1.69)$$

$$\text{com}_3(f, g) := [\mathcal{L}, f \llcorner]g := \mathcal{L}(f \llcorner g) - f \llcorner \mathcal{L}g. \quad (1.70)$$

have the bounds

$$t^\gamma \|\text{com}_2(f, g)(t)\|_{\alpha+\beta} \lesssim \|f\|_{\mathcal{L}_t^{\gamma, \alpha}} \|g(t)\|_{\mathcal{C}^\beta}, \quad t > 0. \quad (1.71)$$

as well as

$$t^\gamma \|\text{com}_3(f, g)(t)\|_{\alpha+\beta-2} \lesssim \|f\|_{\mathcal{L}_t^{\gamma, \alpha}} \|g(t)\|_{\mathcal{C}^\beta}, \quad t > 0. \quad (1.72)$$

Chapter 2

Some stochastic calculus

2.1 White noise, Wiener chaos and Wick products

In this section we recall some basic stochastic calculus results, following closely [Jan97] and [Nua06].

Definition 2.1. *Let H be a real separable Hilbert space. We say that a stochastic process $W = \{W(h), h \in H\}$ defined in a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is an isonormal Gaussian process indexed by H if W is a family of centered Gaussian random variables such that $\mathbb{E}(W(g)W(h)) = \langle g, h \rangle_H$. We call white noise on $\mathbb{R} \times \mathbb{T}^d$ the isonormal Gaussian process ξ indexed by $L^2(\mathbb{R} \times \mathbb{T}^d)$.*

By Kolmogorov's extension theorem, for every H there exists a probability space and a Gaussian process W defined on it which verifies the properties above. For the existence of an $\mathcal{S}'(\mathbb{R} \times \mathbb{T}^d)$ valued measure ξ with the properties above, see also [Jan97]. We define Hermite polynomials with leading coefficient = 1, i.e.

$$\begin{aligned} H_n(x) &:= (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2}}, \\ H_n(x, \sigma^2) &:= \sigma^n H_n\left(\frac{x}{\sigma}\right). \end{aligned} \tag{2.1}$$

Observing that $e^{tx - t^2/2} = \sum_{n \geq 0} \frac{t^n}{n!} H_n(x)$ we obtain easily the relations

$$\begin{aligned} H'_n(x, \sigma^2) &= n H_{n-1}(x, \sigma^2), \\ H_{n+1}(x, \sigma^2) &= x H_n(x, \sigma^2) - n \sigma^2 H_{n-1}(x, \sigma^2) \end{aligned}$$

Lemma 2.2. ([Nua06, Lemma 1.1.1]) *Let X, Y two random variables with joint centered Gaussian distribution with $\mathbb{E}(X^2) = \mathbb{E}(Y^2) = \sigma^2$. Then $\forall n, m \geq 0$*

$$\mathbb{E}(H_n(X, \sigma^2) H_m(Y, \sigma^2)) = \mathbb{1}_{m=n} n! [\mathbb{E}(XY)]^n.$$

Let \mathcal{H}_0 be the set of constants and \mathcal{H}_n for every $n \geq 1$ the closure in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ of the vector space generated by the random variables $\{H_n(W(h)) | h \in H, \|h\|_H = 1\}$.

Proposition 2.3. ([Nua06, Th.1.1.1]) *The space $L^2(\Omega, \sigma(W), \mathbb{P})$ has the orthogonal decomposition*

$$L^2(\Omega, \sigma(W), \mathbb{P}) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n.$$

The decomposition above is called *Wiener chaos* decomposition. For random variables belonging to a finite chaos decomposition we have the following *hypercontractivity* property:

Proposition 2.4. *Let $p > 1$, $X \in \bigoplus_{n=0}^N \mathcal{H}_n$ for some $N \in \mathbb{N}$ and $X \in L^p(\Omega)$. Then $\forall q > p$ there exists a constant $C = C(p, q, N)$ such that*

$$\|X\|_{L^q(\Omega)} \leq C \|X\|_{L^p(\Omega)}.$$

This result is due to E.Nelson. For a proof which uses Itô's lemma see [Nua06, Th.1.4.1].

Let $F: \mathbb{R} \rightarrow \mathbb{R}$ be a measurable function such that $F(W(h)) \in L^2(\Omega, \sigma(W), \mathbb{P})$. Choosing an orthogonal basis of H that contains h , we can write the chaos decomposition of $F(W(h))$ as

$$F(W(h)) = \sum_{n \geq 0} f_n H_n(W(h), \|h\|_H^2) \quad (2.2)$$

and by Lemma 2.2 $f_n = \frac{1}{n! \|h\|_H^{2n}} \mathbb{E}[F(\|h\|_H G) H_n(G)]$ with $G \sim N(0, 1)$. We note J_n the projection on \mathcal{H}_n , and this yields $J_0(X) = \mathbb{E}(X)$.

Definition 2.5. *Let $W(h_1) \cdots W(h_n) \in \{W(h), h \in H\}$. The Wick product is defined as*

$$\llbracket W(h_1) \cdots W(h_n) \rrbracket := J_n(W(h_1) \cdots W(h_n))$$

Following [Jan97] we call *Feynman diagram* a graph with n vertices and $r \leq n/2$ edges without common endpoints. The graph $\gamma = (V(\gamma), E(\gamma))$ has then $n - 2r$ unconnected vertices, which we call $V_0(\gamma)$, and $2r$ vertices that are connected with exactly one other vertex, called $V_c(\gamma)$. We call *contractions* the edges of γ , and note $n(\gamma)$ and $r(\gamma)$ respectively the *order* and *rank* of γ . There are $\frac{n!}{2^r r! (n-2r)!}$ diagrams of order n and rank r . Each vertex is labelled with a random variable $W(h_i)$, $1 \leq i \leq n$ (and the labels are kept distinguished even when two random variables are equal). The value $v(\gamma)$ of a Feynman diagram γ labelled by $(W(h_1), \dots, W(h_n))$ is defined as

$$v(\gamma) = \prod_{\langle i, j \rangle \in E(\gamma)} \mathbb{E}[W(h_i)W(h_j)] \prod_{k \in V_0(\gamma)} W(h_k).$$

We list some results of [Jan97, Ch. III] in the following.

Lemma 2.6. *Let $W(h_1), \dots, W(h_n) \in \{W(h), h \in H\}$. Then*

$$\begin{aligned} \llbracket W(h_1) \cdots W(h_n) \rrbracket &= \sum_{\gamma} (-1)^{r(\gamma)} v(\gamma), \\ W(h_1) \cdots W(h_n) &= \sum_{\gamma} \llbracket v(\gamma) \rrbracket. \end{aligned}$$

where the sum is over all Feynman diagrams labelled by $(W(h_1), \dots, W(h_n))$. We used above the notation

$$\llbracket v(\gamma) \rrbracket := \prod_{\langle i, j \rangle \in E(\gamma)} \mathbb{E}[W(h_i)W(h_j)] \llbracket \prod_{k \in V_0(\gamma)} W(h_k) \rrbracket.$$

Remark 2.7. From (2.2) and the definition of Wick product, it follows that

$$\llbracket W(h)^n \rrbracket = H_n(W(h), \|h\|_H^2),$$

indeed in this case

$$f_n = \frac{1}{n!} \mathbb{E}[G^n H_n(G)] = \frac{(-1)^n}{n! \sqrt{2\pi}} \int_{\mathbb{R}} x^n \frac{d^n}{dx^n} e^{-\frac{x^2}{2}} dx = 1,$$

since integrating by parts it is immediate to verify $\mathbb{E}[G^n H_n(G)] = n\mathbb{E}[G^{n-1} H_{n-1}(G)]$ and $\mathbb{E}(G H_1(G)) = \mathbb{E}(G^2) = 1$.

Proposition 2.8. ([Jan97, Th.3.9])

Let $W(h_1), \dots, W(h_n), W(g_1), \dots, W(g_m) \in \{W(h), h \in H\}$ which is the isonormal Gaussian process of Definition 2.1. Then

$$\begin{aligned} \mathbb{E}(\llbracket W(h_1) \cdots W(h_n) \rrbracket \llbracket W(g_1) \cdots W(g_m) \rrbracket) &= \mathbb{1}_{m=n} \sum_{\sigma \in S_n} \prod_{i=1}^n \mathbb{E}[W(h_i) W(g_{\sigma(i)})] \\ &= \mathbb{1}_{m=n} n! \langle h_1 \odot \cdots \odot h_n, g_{\sigma(1)} \odot \cdots \odot g_{\sigma(n)} \rangle_{H^{\otimes n}} \end{aligned}$$

where S_n is the symmetric group of the permutations of $\{1, \dots, n\}$. We defined the symmetrization $h_1 \odot \cdots \odot h_n = \frac{1}{n!} \sum_{\sigma \in S_n} h_1 \otimes \cdots \otimes h_n$ and used the notation:

$$\langle h_1 \otimes \cdots \otimes h_n, g_1 \otimes \cdots \otimes g_n \rangle_{H^{\otimes n}} := \prod_{i=1}^n \langle h_i, g_i \rangle_H.$$

With the results given above, it is not difficult to obtain a product formula for Wick products (Proposition 2.9 below), i.e. a formula that lets us write products of Wick products as a linear combination of Wick products. This can be seen as a finite-chaos equivalent of Lemma 2.28. Note also that Proposition 2.9 can be written in the form of stochastic integrals [Nua06, Proposition 1.1.3].

Proposition 2.9. (product formula, [Jan97, Th.3.15])

Let $(W(h_{i,j}))_{1 \leq i \leq k, 1 \leq j \leq \ell}$ belong to the isonormal Gaussian process of Definition 2.1. Define $Y_i := \llbracket W(h_{i,1}) \cdots W(h_{i,\ell_i}) \rrbracket$. Then

$$Y_1 \cdots Y_k = \sum_{\gamma} \llbracket v(\gamma) \rrbracket$$

where the sum is over all Feynman diagrams γ labelled by $(W(h_{i,j}))_{i,j}$ such that no edge joins two variables $W(h_{i_1,j_1})$ and $W(h_{i_2,j_2})$ when $i_1 = i_2$.

Heuristically, noting that for the Feynman diagram γ with no edges $\llbracket v(\gamma) \rrbracket$ is just the “full” Wick product $\llbracket \prod_{i,j} W(h_{i,j}) \rrbracket$ where all contractions are considered, we interpret Proposition 2.9 as the operation of splitting the family of random variables $(W(h_{i,j}))_{i,j}$ in blocks $(Y_i)_{1 \leq i \leq k}$ and considering separately contractions inside blocks and contractions between blocks. This same reasoning motivates Lemma 2.28 (see Remark 2.31).

2.2 Estimation of finite chaos stochastic terms

In this section we present some techniques for estimating the stochastic noise terms (also called *enhanced noises*) appearing in both paracontrolled distributions theory [GIP15] and regularity structures theory [Hai14a]. Due to our treatment of a class of models rescaling to Φ_3^4 in Chapter 6, we will focus here on the Φ_3^4 noise terms, although many of the results presented below apply as well to other finite chaos terms with similar structure. For a complete treatment of Φ_3^4 noise terms using a Fourier transform approach, we refer to [MWX16]. We will work instead in real space variables because this technique is better adapted to the calculations of Chapter 6.

We start taking η_ε to be a space-time mollification of the Gaussian white noise on $\mathbb{R} \times \mathbb{T}^3$ of Definition 2.1. Its stationary covariance is

$$\mathfrak{C}_\varepsilon(t-s, x-y) := \mathbb{E}[\eta_\varepsilon(t, x) \eta_\varepsilon(s, y)]$$

such that $\forall \varepsilon \in (0, 1]$

$$\mathfrak{C}_\varepsilon(t, x) = \varepsilon^{-5} \Sigma(\varepsilon^{-2}t, \varepsilon^{-1}x) \quad (2.3)$$

where Σ is a smooth even function compactly supported in $[-1, 1] \times B_{\mathbb{R}^3}(0, 1)$. Indeed, let $\psi \in C_c^\infty(\mathbb{R} \times \mathbb{T}^d)$ compactly supported on $[-1/2, 1/2] \times B_{\mathbb{R}^3}(0, 1/2)$ and let $\psi_\varepsilon(t, x) = \varepsilon^{-5/2} \psi(\varepsilon^{-2}t, \varepsilon^{-1}x)$. If we define $\eta_\varepsilon = \psi_\varepsilon * \xi$ (where ξ is the white noise on $\mathbb{R} \times \mathbb{T}^d$) we obtain $\Sigma = \psi * \psi$. Let P be the heat kernel on $\mathbb{R} \times \mathbb{R}^3$, i.e.

$$P(t, x) = \frac{1}{(4\pi t)^{3/2}} e^{-\frac{|x|^2}{4t}} \mathbb{1}_{t \geq 0} \quad (2.4)$$

and call $P(t, x)$ also its periodized version on $\mathbb{R} \times \mathbb{T}^3$ (with an abuse of notation) which for $(t, x) \in \mathbb{R} \times \mathbb{T}^3$ can be obtained as $\sum_{j \in \mathbb{Z}^d} P(t, x + 2\pi j)$. Let X_ε be the stationary Gaussian field which solves

$$\mathcal{L} X_\varepsilon = -X_\varepsilon + \eta_\varepsilon. \quad (2.5)$$

In order for X_ε to satisfy eq. (2.5) we introduce the following modified heat kernel for $(t, x) \in \mathbb{R} \times \mathbb{T}^d$

$$\check{P}(t, x) = \sum_{k \in \mathbb{Z}^d} \frac{1}{(4\pi t)^{d/2}} e^{-\frac{|x+2\pi k|^2}{4t}} e^{-t} \mathbb{1}_{t \geq 0} \quad (2.6)$$

and take

$$X_\varepsilon(t, x) = \int_{-\infty}^t \int_{\mathbb{T}^d} \check{P}(t-s, x-y) \eta_\varepsilon(s, y) ds dy. \quad (2.7)$$

From Definition 2.1 we see that $X_\varepsilon(t, x)$ is a centered Gaussian random variable and that there exists $h_{\varepsilon, (t, x)} \in H$ with $H := L^2(\mathbb{R} \times \mathbb{T}^3)$, such that $X_\varepsilon(t, x) = W(h_{\varepsilon, (t, x)})$. The exact form of $h_{\varepsilon, (t, x)}$ can be read in (2.7), but we are not going to use it here. We call C_ε the covariance of X_ε , and then:

$$C_\varepsilon(t-s, x-y) := \mathbb{E}[X_\varepsilon(t, x) X_\varepsilon(s, y)] = \langle h_{\varepsilon, (t, x)}, h_{\varepsilon, (s, y)} \rangle_H.$$

The Φ_3^4 noise terms are the limits for $\varepsilon \rightarrow 0$ of:

$$\begin{aligned} X_\varepsilon^\vee &:= \llbracket X_\varepsilon^2 \rrbracket \\ X_\varepsilon^\Psi(t, x) &:= \int_0^t \int_{\mathbb{T}^3} P(t-s, x-y) \llbracket X_\varepsilon^3(s, y) \rrbracket ds dy \\ X_\varepsilon^{\Psi\vee} &:= X_\varepsilon^\Psi \circ X_\varepsilon \\ X_\varepsilon^{\Psi\Psi} &:= X_\varepsilon^\Psi \circ X_\varepsilon^\vee - \mathbb{E}(X_\varepsilon^\Psi \circ X_\varepsilon^\vee) \\ X_\varepsilon^{\Psi\Psi\vee} &:= X_\varepsilon^\Psi \circ X_\varepsilon^\vee - \mathbb{E}(X_\varepsilon^\Psi \circ X_\varepsilon^\vee) X_\varepsilon \end{aligned} \quad (2.8)$$

with the resonant paraproduct \circ given in Section 1.2.1. The convergence in probability of

$$\begin{aligned} \mathbb{X}_\varepsilon &:= \left(X_\varepsilon, X_\varepsilon^\vee, X_\varepsilon^\Psi, X_\varepsilon^{\Psi\vee}, X_\varepsilon^{\Psi\Psi}, X_\varepsilon^{\Psi\Psi\vee} \right) \\ &\in C_T \mathcal{L}^{-1/2-\kappa} \times C_T \mathcal{L}^{-1-\kappa} \times \mathcal{L}_T^{1/2-\kappa} \times C_T \mathcal{L}^{-\kappa} \times C_T \mathcal{L}^{-1/2-\kappa} \end{aligned}$$

$\forall \kappa \in (0, 1)$ to a distribution $\mathbb{X} = (X, X^\vee, X^\Psi, X^{\Psi\vee}, X^{\Psi\Psi}, X^{\Psi\Psi\vee})$ was first established in [CC13] and is thoroughly explained in [MWX16]. In this section we just give a quick overview on uniform (in ε) bounds for the quantity \mathbb{X}_ε in $L^p(\Omega)$, in order to show how the diagrams $I_\alpha, \bar{I}_\beta, I_{k, \ell, m, n}$ of Lemmas 2.20 and 2.21 arise in the estimation of \mathbb{X}_ε .

It is not difficult to see (for example using time-space Littlewood-Paley blocks on $\mathbb{R} \times \mathbb{T}^3$) that for every $N \leq 4$ the term $\llbracket X_\varepsilon^N \rrbracket$ converges in probability in $\mathcal{S}'(\mathbb{R} \times \mathbb{T}^3)$ to a well-defined tempered distribution that we call $\llbracket X^N \rrbracket$. From the results presented in this section the uniform bound on $\llbracket X_\varepsilon^N \rrbracket$ is apparent (and so is the reason why $N \leq 4$).

Recall from Section 1.1.1 that, for every $q \in \mathbb{N}$, $K_{q,x}(y) = 2^{3q}K(2^q(x-y))$ is the kernel associated to the q -th Littlewood-Paley block Δ_q on \mathbb{R}^3 , i.e.

$$\Delta_q f(x) = \int_{\mathbb{R}^d} K_{q,x}(y) f(y) dy$$

with $K \in \mathcal{S}(\mathbb{R}^3)$ $f: \mathbb{R}^3 \rightarrow \mathbb{R}$. For a function f defined on the torus \mathbb{T}^3 we still write $\Delta_q f(x) = \int_{\mathbb{T}^3} K_{q,x}(y) f(y) dy$ where with an abuse of notation $K_{q,x}$ stands for the kernel on \mathbb{T}^3 , which is $\sum_{j \in \mathbb{Z}^3} K_{q,x}(y + 2\pi j)$. For $X_\varepsilon^\tau \in \mathbb{X}_\varepsilon$, $|\tau| \in \mathbb{R}$ and fixed $t > 0$, we first use Besov embedding (Lemma 1.16) to bound

$$\mathbb{E}(\|X_\varepsilon^\tau(t, \cdot)\|_{\mathcal{C}^{|\tau|-3/p}}^p) \lesssim \mathbb{E}(\|X_\varepsilon^\tau(t, \cdot)\|_{B_{p,p}^{|\tau|}}^p) \lesssim \sum_q 2^{|\tau|pq} \int_{\mathbb{T}^3} \mathbb{E}(|\Delta_q X_\varepsilon^\tau(t, x)|^p) dx.$$

Due to the stationarity in space of X_ε^τ , it suffices to bound $\mathbb{E}(|\Delta_q X_\varepsilon^\tau(t, \bar{x})|^p)$ for some fixed $\bar{x} \in \mathbb{T}^3$. From Proposition 2.9 it is clear that every $X_\varepsilon^\tau \in \mathbb{X}_\varepsilon$ belongs to the finite chaos decomposition $\bigoplus_{n=0}^5 \mathcal{H}_n$. In this case one can use hypercontractivity (Proposition 2.4) to estimate the p -th moment of $\Delta_q X_\varepsilon^\tau(t, \bar{x})$ with its second moment:

$$\|\Delta_q X_\varepsilon^\tau(t, \bar{x})\|_{L^p(\Omega)} \lesssim \|\Delta_q X_\varepsilon^\tau(t, \bar{x})\|_{L^2(\Omega)}.$$

Let us proceed therefore with the estimation of the second moment of $\Delta_q X_\varepsilon^\tau(t, \bar{x})$. We have

$$\begin{aligned} \Delta_q X_\varepsilon^\vee(t, \bar{x}) &= \int K_{q,\bar{x}}(y) [X_\varepsilon^2(t, y)] dy, \\ \Delta_q X_\varepsilon^\forall(t, \bar{x}) &= \int K_{q,\bar{x}}(x) P(t-s, x-y) [X_\varepsilon^3(s, y)] ds dx dy. \end{aligned}$$

The resonant products $X_\varepsilon^\forall \circ X_\varepsilon$, $X_\varepsilon^\forall \circ X_\varepsilon^\vee$, $X_\varepsilon^\forall \circ X_\varepsilon^\forall$ can be written as:

$$\Delta_q(X_\varepsilon^\tau \circ X_\varepsilon^{\tau'}) = \int [X_\varepsilon^n(\zeta_1)] [X_\varepsilon^m(\zeta_2)] \mu_{q,\zeta_1,\zeta_2}$$

with $\zeta_1 = (s_1, x_1)$, $\zeta_2 = (s_2, x_2)$ and the measure μ_{q,ζ_1,ζ_2} on $(\mathbb{R} \times \mathbb{T}^3)^2$ given by

$$\mu_{q,\zeta_1,\zeta_2} := \left[\int_{x,y} K_{q,\bar{x}}(x) \sum_{i \sim j} K_{i,x}(y) K_{j,x}(x_2) P_{t-s_1}(y-x_1) \mathbb{1}_{[0,+\infty)}(s_1) \delta(t-s_2) d\zeta_1 d\zeta_2 \right]$$

Proposition 2.8 allows to bound the second moment of a Wick product of random variables with products of their covariances. In order to be able to apply this result to terms of the type $[X_\varepsilon^n(\zeta_1)] [X_\varepsilon^m(\zeta_2)]$, we develop them as linear combinations of Wick products using Proposition 2.9. We obtain respectively for $X_\varepsilon^\forall \circ X_\varepsilon$, $X_\varepsilon^\forall \circ X_\varepsilon^\vee$, $X_\varepsilon^\forall \circ X_\varepsilon^\forall$:

$$\begin{aligned} [X_{\varepsilon,\zeta_1}^3] [X_{\varepsilon,\zeta_2}] &= [X_{\varepsilon,\zeta_1}^3 X_{\varepsilon,\zeta_2}] + 3C_\varepsilon(\zeta_1, \zeta_2) [X_{\varepsilon,\zeta_1}^2 X_{\varepsilon,\zeta_2}], \\ [X_{\varepsilon,\zeta_1}^2] [X_{\varepsilon,\zeta_2}^2] &= [X_{\varepsilon,\zeta_1}^2 X_{\varepsilon,\zeta_2}^2] + 4C_\varepsilon(\zeta_1, \zeta_2) [X_{\varepsilon,\zeta_1} X_{\varepsilon,\zeta_2}] + 2C_\varepsilon^2(\zeta_1, \zeta_2), \\ [X_{\varepsilon,\zeta_1}^3] [X_{\varepsilon,\zeta_2}^2] &= [X_{\varepsilon,\zeta_1}^3 X_{\varepsilon,\zeta_2}^2] + 6C_\varepsilon(\zeta_1, \zeta_2) [X_{\varepsilon,\zeta_1}^2 X_{\varepsilon,\zeta_2}] + 6C_\varepsilon^2(\zeta_1, \zeta_2) [X_{\varepsilon,\zeta_1}], \end{aligned}$$

with $C_\varepsilon(\zeta_1, \zeta_2) = \langle h_{\varepsilon,\zeta_1}, h_{\varepsilon,\zeta_2} \rangle$ being the covariance of X_ε as before. Graphically, we represent this decomposition as

$$\begin{aligned} X_\varepsilon^\forall &= [X_\varepsilon^\forall] + 3X_\varepsilon^\forall \\ X_\varepsilon^\forall &= [X_\varepsilon^\forall] + 4X_\varepsilon^\forall + [2X_\varepsilon^\forall - \mathbb{E}(X_\varepsilon^\forall \circ X_\varepsilon^\forall)] = [X_\varepsilon^\forall] + 4X_\varepsilon^\forall \\ X_\varepsilon^\forall &= [X_\varepsilon^\forall] + 6X_\varepsilon^\forall + [6X_\varepsilon^\forall - \mathbb{E}(X_\varepsilon^\forall \circ X_\varepsilon^\forall)] X_\varepsilon \end{aligned} \tag{2.9}$$

Having obtained a Wiener chaos decomposition of $\Delta_q X_\varepsilon^T(t, \bar{x})$, we proceed to bound separately the L^2 norm of each term using Proposition 2.8. The terms $\llbracket X_\varepsilon^{\vee_Y} \rrbracket$, $\llbracket X_\varepsilon^{\vee_Y} \rrbracket$, $\llbracket X_\varepsilon^{\vee_Y} \rrbracket$ can be estimated as:

$$\begin{aligned} \|J_{m+n}[\Delta_q X_\varepsilon^T(t, \bar{x})]\|_{L^2(\Omega)}^2 &= \left\langle \int h_{\varepsilon, \zeta_1}^{\odot m} \odot h_{\varepsilon, \zeta_2}^{\odot n} \mu_{q, \zeta_1, \zeta_2}, \int h_{\varepsilon, \zeta_1}^{\odot m} \odot h_{\varepsilon, \zeta_2}^{\odot n} \mu_{q, \zeta_1, \zeta_2} \right\rangle_{H^{\otimes m+n}} \\ &\leq (m+n)! \left\| \int h_{\varepsilon, \zeta_1}^{\otimes m} \otimes h_{\varepsilon, \zeta_2}^{\otimes n} \mu_{q, \zeta_1, \zeta_2} \right\|_{H^{\otimes m+n}}^2 \\ &\lesssim \int |C_\varepsilon(\zeta_1, \zeta_1)|^m |C_\varepsilon(\zeta_2, \zeta_2)|^n |\mu_{q, \zeta_1, \zeta_2}| |\mu_{q, \zeta_1, \zeta_2}| \end{aligned} \quad (2.10)$$

by Jensen's inequality. In next section we present some tools to bound terms like (2.10), which we call Φ_3^4 diagrams. All the relevant quantities for the estimation of \mathbb{X}_ε can be bounded by Φ_3^4 diagrams as (2.10): we give the details (for a more general case) in Section 6.3.

2.2.1 Estimation of finite chaos diagrams

In this section we recall some results from [Hai14a, Chap.10] that can be used to estimate finite chaos diagrams like the one in (2.10).

First of all, we need to estimate the covariance C_ε of X_ε . One can see easily that

$$C_\varepsilon = \check{P}(-\cdot) * \check{P} * \mathfrak{C}_\varepsilon \quad (2.11)$$

with time-space convolutions in $\mathbb{R} \times \mathbb{T}^d$ and the kernel $\check{P}(-\cdot)$ that has reversed time. We are going to estimate the convolution above using some results from [Hai14a].

In the following we use the notations

$$\begin{aligned} |k|_{\mathfrak{s}} &= 2k_1 + \sum_{j=2}^{d+1} k_j \quad \text{for } k \in \mathbb{N}^{d+1} \\ \|\zeta\|_{\mathfrak{s}} &= |t|^{1/2} + \sum_{j=1}^d |x_j| \quad \text{for } \zeta = (t, x) \in \mathbb{R} \times \mathbb{T}^d \end{aligned}$$

where the index \mathfrak{s} stands for the fact that we are using a non-euclidean scaling (in this case a parabolic one).

Lemma 2.10. (estimation of heat kernel) *Let $P(t, x)$ be defined in (2.4). Then $\forall (t, x) \in \mathbb{R} \times \mathbb{R}^d$*

$$|P(t, x)| \lesssim (|t|^{1/2} + |x|)^{-d}.$$

Moreover, for every multi-index $k \in \mathbb{N}^{d+1}$ with $|k|_{\mathfrak{s}} = 2k_1 + k_2 + \dots + k_{d+1}$ we have:

$$|\partial^k P(t, x)| \lesssim (|t|^{1/2} + |x|)^{-d - |k|_{\mathfrak{s}}}$$

Proof.

$$|P(t, x)| (|t|^{1/2} + |x|)^d \lesssim \left[1 + \left(\frac{|x|}{|t|^{1/2}} \right)^d \right] e^{-\frac{|x|^2}{4|t|}}$$

and $\forall d \in \mathbb{N} \exists C > 0$ such that $(1 + |\alpha|^d) e^{-\frac{|\alpha|}{4}} \leq C$ for every $\alpha \in \mathbb{R}$. Calling $k_t = k_1$ and $k_x = (k_2, \dots, k_{d+1})$ one can see directly by taking derivatives of $P(t, x)$ that

$$|\partial^{k_t} \partial^{k_x} P(t, x)| \lesssim \sum_{j=0}^{|k_t|} \frac{|x|^{2j}}{|t|^j} |t|^{-\frac{d + |k_x| + 2|k_t|}{2}} e^{-\frac{|x|^2}{4t}}$$

and then $|\partial^{k_t} \partial^{k_x} P(t, x)| (|t|^{1/2} + |x|)^{d+|k_x|+2|k_t|} \leq C$. \square

Remark 2.11. It is immediate to note that the estimation of Lemma 2.10 holds as well for the kernel $P(t, x)$ on the torus $\mathbb{R} \times \mathbb{T}^3$ and for the stationary kernel $\check{P}(t, x)$.

We recall [Hai14a, Lemma 10.14] in a restricted formulation that is enough for our purposes.

Lemma 2.12. (estimation of convolutions) *Let $K_1, K_2: \mathbb{R}^{d+1} \setminus \{0\} \rightarrow \mathbb{R}$ smooth and such that $\exists \alpha, \beta \in (-d-2, 0)$, $\exists m \in \mathbb{N}$ such that $\forall |k|_s \leq m$, $\forall \zeta \in \mathbb{R}^{d+1} \setminus \{0\}$*

$$|\partial^k K_1(\zeta)| \lesssim \|\zeta\|_s^{\alpha-|k|_s} \quad \text{and} \quad |\partial^k K_2(\zeta)| \lesssim \|\zeta\|_s^{\beta-|k|_s}. \quad (2.12)$$

Let $\gamma = \alpha + \beta + d + 2$. If $\gamma < 0$ then

$$|\partial^k (K_1 * K_2)(\zeta)| \lesssim \|\zeta\|_s^{\gamma-|k|_s} \quad \forall |k|_s \leq m, \zeta \in \mathbb{R}^{d+1} \setminus \{0\}.$$

Proof. (sketch) The idea behind the proof is easy and gives a good intuition of the problem: we recall it briefly.

Let $B_s(\zeta, r) = \{\zeta' \in \mathbb{R}^{d+1} \mid \|\zeta - \zeta'\|_s < r\}$ a parabolic ball. For each $\zeta = (t, x)$ the domain of integration of the convolution

$$K_1 * K_2(\zeta) = \int_{\mathbb{R}^{d+1}} K_1(\zeta - \zeta') K_2(\zeta') d\zeta' \quad (2.13)$$

can be split as

$$\mathbb{R}^{d+1} = B_s(0, r) \cup B_s(\zeta, r) \cup [\mathbb{R}^{d+1} \setminus (B_s(0, r) \cup B_s(\zeta, r))]$$

with $r < \|\zeta\|_s / 4$ such that $\|\zeta - \zeta'\|_s$ and $\|\zeta'\|_s$ are small respectively for $\zeta' \in B_s(\zeta, r)$ and $\zeta' \in B_s(0, r)$. Then $\int_{B_s(0, r)} K_1(\zeta - \zeta') K_2(\zeta') d\zeta'$ can be bounded by the L^∞ norm of K_1 (which is of order $\|\zeta\|_s^\alpha$) times the L^1 norm of K_2 (which is of order $\|\zeta\|_s^{\beta+d+2}$). By symmetry the term $\int_{B_s(\zeta, r)} K_1(\zeta - \zeta') K_2(\zeta') d\zeta'$ yields the same estimation.

Consider now the case $\zeta' \in \mathbb{R}^{d+1} \setminus (B_s(0, r) \cup B_s(\zeta, r))$. First we note that if both $\|\zeta'\|_s \geq \|\zeta\|_s / 4$ and $\|\zeta - \zeta'\|_s \geq \|\zeta\|_s / 4$ we must have $\|\zeta'\|_s \gtrsim \|\zeta - \zeta'\|_s$, indeed $\|\zeta - \zeta'\|_s \leq \|\zeta\|_s + \|\zeta'\|_s \leq 5\|\zeta'\|_s$. Then $|K_2(\zeta')| \gtrsim \|\zeta - \zeta'\|_s^\beta$ and

$$\int_{\mathbb{R}^{d+1} \setminus (B_s(0, r) \cup B_s(\zeta, r))} K_1(\zeta - \zeta') K_2(\zeta') d\zeta' \lesssim \int_{\|\zeta'\|_s \geq \|\zeta\|_s / 4} \|\zeta'\|_s^{\alpha+\beta} d\zeta' \lesssim \|\zeta\|_s^{\alpha+\beta+d+2}$$

In order to consider derivatives $\partial^k (K_1 * K_2)$ it suffices to take a smooth partition of unity to split the domains of integration as above. \square

Remark 2.13. It is clear that the argument of Lemma 2.12 works as well for space-periodic kernels $K_1, K_2: \mathbb{R} \times \mathbb{T}^d \setminus \{0\} \rightarrow \mathbb{R}$.

We recall now [Hai14a, Lemma 10.17].

Lemma 2.14. *Let $K: \mathbb{R}^{d+1} \setminus \{0\} \rightarrow \mathbb{R}$ smooth and such that $\exists m \in \mathbb{N}$, $\exists \alpha \in (-d-2, 0)$ such that*

$$|\partial^k K(\zeta)| \lesssim \|\zeta\|_s^{\alpha-|k|_s} \quad \forall |k|_s \leq m, \zeta \in \mathbb{R}^{d+1} \setminus \{0\}.$$

Let $\psi \in C_c^\infty(\mathbb{R}^{d+1})$ with unit mass. Let $\psi_\varepsilon(t, x) = \varepsilon^{-d-2} \psi(\varepsilon^{-2}t, \varepsilon^{-1}x)$ and define $K_\varepsilon = K * \psi_\varepsilon$. Then

$$|\partial^k K_\varepsilon(\zeta)| \lesssim (\|\zeta\|_s + \varepsilon)^{\alpha-|k|_s} \quad \forall |k|_s \leq m, \zeta \in \mathbb{R}^{d+1}.$$

Proof. Let w.l.o.g. ψ_ε be supported on $B_s(0, \varepsilon) = \{\zeta \in \mathbb{R} \times \mathbb{R}^d \mid \|\zeta\|_s < \varepsilon\}$. For $\|\zeta\|_s > 2\varepsilon$ we bound $\partial^k K_\varepsilon$ as

$$\begin{aligned} \left| \int \partial^k K(\zeta - \zeta') \psi_\varepsilon(\zeta') d\zeta' \right| &\leq \sup_{\|\zeta'\| < \varepsilon} |\partial^k K(\zeta - \zeta')| \int |\psi_\varepsilon(\zeta')| d\zeta' \\ &\lesssim \|\zeta\|_s^{\alpha - |k|_s} \lesssim \varepsilon^{\alpha - |k|_s} \end{aligned}$$

since for $\|\zeta'\| < \varepsilon$ we have $\|\zeta - \zeta'\| \geq \|\zeta\| - \varepsilon \geq \|\zeta\|/2$. For $\|\zeta\|_s \leq 2\varepsilon$ we bound

$$\begin{aligned} \left| \int K(\zeta - \zeta') \partial^k \psi_\varepsilon(\zeta') d\zeta' \right| &\lesssim \varepsilon^{-d-2-|k|_s} \int_{\|\zeta - \zeta'\| \leq 3\varepsilon} |K(\zeta - \zeta')| d\zeta' \\ &\lesssim \varepsilon^{\alpha - |k|_s} \lesssim \|\zeta\|_s^{\alpha - |k|_s}. \end{aligned}$$

□

Lemma 2.15. (covariance estimation) *The covariance C_ε on $\mathbb{R} \times \mathbb{T}^3$ has the bound, for every multi-index $k \in \mathbb{N}^4$:*

$$|\partial^k C_\varepsilon(t, x)| \lesssim (|t|^{1/2} + |x| + \varepsilon)^{-1 - |k|_s} \quad \forall (t, x) \in \mathbb{R} \times \mathbb{T}^3, \varepsilon \in (0, 1].$$

Proof. Note that from Lemma 2.10 it follows immediately that \check{P} satisfies the assumptions of Lemma 2.12. We obtain then the estimation

$$|\partial^k [\check{P}(-) * \check{P}](\zeta)| \lesssim \|\zeta\|^{-1 - |k|_s}$$

for every multi-indexed $k \in \mathbb{N}^4$, and from Lemma 2.14 we obtain the result. □

Lemma 2.16. *We have for every $\sigma \in [0, 1]$*

$$\sup_{x \in \mathbb{T}^3} |C_\varepsilon(t, x) - C_\varepsilon(0, x)| \lesssim \varepsilon^{-1 - 2\sigma} |t|^\sigma$$

Proof. Since for every $\varepsilon \in (0, 1]$ C_ε is smooth, the result is immediately obtained by Taylor expansion and interpolation from the bound of Lemma 2.15. □

Lemma 2.17. *For every $t > 0$ and every $\varepsilon > 0$ we have*

$$A_{2, \varepsilon} := \int_{\mathbb{R} \times \mathbb{T}^3} P_s(x) [C_\varepsilon(s, x)]^2 \mathbb{1}_{[0, t]}(s) dx ds \lesssim |\log \varepsilon|,$$

and for every $n \geq 3$

$$A_n := \varepsilon^{n-2} \int_{\mathbb{R} \times \mathbb{T}^3} P_s(x) |C_\varepsilon(s, x)|^n \mathbb{1}_{[0, t]}(s) dx ds \lesssim 1.$$

Proof. From the estimations of Lemma 2.10 and Lemma 2.15 we have

$$\begin{aligned} A_{2, \varepsilon} &\lesssim \int_{\mathbb{R} \times \mathbb{T}^3} \frac{1}{(|s|^{1/2} + |x|)^3} \frac{1}{(|s|^{1/2} + |x| + \varepsilon)^2} \mathbb{1}_{[0, t]}(s) dx ds \\ &\lesssim \int_{\mathbb{R} \times (\varepsilon^{-1} \mathbb{T})^3} \frac{1}{(|s|^{1/2} + |x|)^3} \frac{1}{(|s|^{1/2} + |x| + 1)^2} \mathbb{1}_{[0, \varepsilon^{-2} t]}(s) dx ds \\ &\lesssim \int_{\mathbb{R}^4} \frac{1}{(|s|^{1/2} + |x|)^3} \frac{1}{(|s|^{1/2} + |x| + 1)^2} \mathbb{1}_{[0, \varepsilon^{-2} t]}(s) \mathbb{1}_{B(0, \varepsilon^{-1})}(x) dx ds \\ &\lesssim |\log(\varepsilon)|. \end{aligned}$$

In the same way for $n \geq 3$

$$\begin{aligned} A_n &\lesssim \varepsilon^{n-2} \int_{\mathbb{R} \times \mathbb{T}^3} \frac{1}{(|s|^{1/2} + |x|)^3} \frac{1}{(|s|^{1/2} + |x| + \varepsilon)^n} \mathbb{1}_{[0,t]}(s) \, dx ds \\ &\lesssim \int_{\mathbb{R}^4} \frac{1}{(|s|^{1/2} + |x|)^3} \frac{1}{(|s|^{1/2} + |x| + 1)^n} \, dx ds \\ &\lesssim 1. \end{aligned}$$

□

Lemma 2.18. *Let K_i be the kernel associated to the L-P block $\Delta_i \forall i \in \mathbb{N}$. We have for every $\alpha \in (0, 3)$*

$$\int \frac{|K_{i,x}(y)|}{(|y| + |t|^{1/2})^\alpha} \, dy \lesssim (|x| + |t|^{1/2} + 2^{-i})^{-\alpha}, \quad (2.14)$$

$$\int \frac{|K_{i,x}(y)|}{|y|^\alpha} \, dy \lesssim (|x| + 2^{-i})^{-\alpha}. \quad (2.15)$$

with the integral over \mathbb{T}^3 or \mathbb{R}^3 . Moreover for $P: \mathbb{R}^4 \rightarrow \mathbb{R}$ as in (2.4) we have $\forall \delta \in [0, 1]$

$$\left| \int_{\mathbb{R}^3} K_{i,x}(y) P(t, y) \, dy \right| \lesssim \frac{2^{-\delta i}}{(|x| + |t|^{1/2} + 2^{-i})^{3+\delta}}. \quad (2.16)$$

Proof. We start with inequality (2.14), which can be obtained in a similar way as Lemma 2.14. When $|t|^{1/2} \geq 2^{-i} \vee |x|$ we have

$$\int_{\mathbb{T}^3} \frac{|K_{i,x}(y)|}{(|y| + |t|^{1/2})^\alpha} \, dy \lesssim \frac{1}{t^{\alpha/2}} \int_{\mathbb{T}^3} |K_{i,x}(y)| \, dy \lesssim \frac{1}{t^{\alpha/2}} \lesssim (|x| + |t|^{1/2} + 2^{-i})^{-\alpha}.$$

When $2^{-i} \geq |t|^{1/2} \vee |x|$ we estimate for $\alpha \in (0, 3)$

$$\begin{aligned} \int_{\mathbb{T}^3} \frac{|K_i(x-y)|}{(|y| + |t|^{1/2})^\alpha} \, dy &\lesssim 2^{\alpha i} \int_{\mathbb{T}^3} \frac{|K(y)|}{|2^i x - y|^\alpha} \, dy \\ &\lesssim 2^{\alpha i} \sup_{z \in (2^i \mathbb{T})^3} \int_{(2^i \mathbb{T})^3} \frac{|K(y)|}{|z - y|^\alpha} \, dy \\ &\lesssim 2^{\alpha i} \lesssim (|x| + |t|^{1/2} + 2^{-i})^{-\alpha}. \end{aligned}$$

Finally, when $|x| \geq 2^{-i} \vee |t|^{1/2}$ we split the domains $|x| \geq 2^{-i+1}|y|$ or $|x| < 2^{-i+1}|y|$. In the first region $|x - 2^{-i}y| \gtrsim |x|$ so

$$\int_{\mathbb{T}^3} \frac{|K_i(x-y)|}{(|y| + |t|^{1/2})^\alpha} \, dy \lesssim \int_{(2^i \mathbb{T})^3} \frac{|K(y)|}{|x - 2^{-i}y|^\alpha} \, dy \lesssim |x|^{-\alpha} \lesssim (|x| + |t|^{1/2} + 2^{-i})^{-\alpha},$$

while in the second region $|y| \geq 2^i|x|/2$, then $|K(y)| \leq |K(y)|^{1/2} f(2^i|x|/2)$ where f is another rapidly decreasing function which can be defined on the torus as $f(\cdot) = \sum_{j \in \mathbb{Z}^3} f(\cdot + 2\pi j)$ by an abuse of notation. Then for $\alpha \in (0, 3)$

$$\begin{aligned} \int_{\mathbb{T}^3} \frac{|K_i(x-y)|}{(|y| + |t|^{1/2})^\alpha} \, dy &\lesssim f(2^i|x|/2) \int_{(2^i \mathbb{T})^3} \frac{|K(y)|^{1/2}}{|2^{-i}y|^\alpha} \, dy \\ &\lesssim 2^{\alpha i} f(2^i|x|/2) \\ &\lesssim |x|^{-\alpha} \lesssim (|x| + |t|^{1/2} + 2^{-i})^{-\alpha}, \end{aligned}$$

concluding the argument. Taking the integral over \mathbb{R}^3 in (2.14) does not change the estimations, and the second inequality (2.15) is obtained in the same way.

Let us show (2.16). Note that since $\forall i \geq 0, \forall x \in \mathbb{R}^3 \int K_{i,x}(y)dy = 0$ (obvious from its Fourier transform) we have

$$\begin{aligned} I &= \int_{\mathbb{R}^3} K_{i,x}(y)P(t,y)dy = \int_{\mathbb{R}^3} K_{i,x}(y)[P(t,y) - P(t,x)]dy \\ &= \int_0^1 d\tau \int_{\mathbb{R}^3} K_{i,x}(y)[P'(t,x+\tau(y-x))(y-x)]dy \\ |I| &\lesssim \int_0^1 d\tau \int_{\mathbb{R}^3} |(y-x)K_i(x-y)||P'(t,x+\tau(y-x))|dy \\ &\lesssim 2^{-i} \int_0^1 d\tau \int_{\mathbb{R}^3} |yK(y)||P'(t,x+\tau 2^{-i}y)|dy \\ &\lesssim 2^{-i} \int_0^1 d\tau \int_{\mathbb{R}^3} |yK(y)| \frac{1}{(|t|^{1/2} + |x + \tau 2^{-i}y|)^4} dy \end{aligned}$$

where P' denotes the derivative of P with respect to the space variable and can be estimated with Lemma 2.10. As before we can bound $(|t|^{1/2} + |x + \tau 2^{-i}y|)^{-4}$ by considering three separate regions: when $|t|^{1/2} \geq 2^{-i} \vee |x|$ we have

$$|I| \lesssim 2^{-i} |t|^{-2} \leq 2^{-i\delta} |t|^{-\frac{3-\delta}{2}} \lesssim 2^{-i\delta} (|x| + |t|^{1/2} + 2^{-i})^{-3-\delta}.$$

When $2^{-i} \geq |t|^{1/2} \vee |x|$ we estimate simply

$$|I| \lesssim \int_{\mathbb{R}^3} |K_i(x-y)||P(t,y)|dy \lesssim 2^{3i} \lesssim 2^{-i\delta} (|x| + |t|^{1/2} + 2^{-i})^{-3-\delta}$$

since $P(2^{-2i}t, 2^i y) = 2^{-3i}P(t, y)$.

When $|x| \geq 2^{-i} \vee |t|^{1/2}$ we have instead that either $|x| \geq 2\tau 2^{-i}|y|$ or $|x| < 2\tau 2^{-i}|y|$. In the first region $|x + \tau 2^{-i}y| \gtrsim |x|$ so

$$\begin{aligned} |I| &\lesssim 2^{-i} \int_0^1 d\tau \int_{\mathbb{R}^3} |yK(y)| \frac{1}{|x|^4} dy \\ &\lesssim 2^{-i} |x|^{-4} \lesssim 2^{-i\delta} (|x| + |t|^{1/2} + 2^{-i})^{-3-\delta}. \end{aligned}$$

In the region $|y| > 2^i|x|/(2\tau)$ we have $|yK(y)| \lesssim (2^i|x|/(2\tau))K(2^i|x|/(2\tau))$ and then the integral can be estimated as follows:

$$\begin{aligned} |I| &\lesssim 2^{-i} \int_0^1 d\tau \int_{\mathbb{R}^3} |yK(y)| \frac{1}{|x + \tau 2^{-i}y|^4} dy \\ &\lesssim \frac{2^{-i}}{|x|^4} \int_0^1 d\tau (2^i|x|/(2\tau))K(2^i|x|/(2\tau)) \int_{\mathbb{R}^3} (1 + \frac{\tau 2^{-i}}{|x|}|y|)^{-4} dy \\ &\lesssim \frac{2^{-i}}{|x|^4} \int_0^1 d\tau (2^i|x|/(2\tau))^4 K(2^i|x|/(2\tau)) \int_{\mathbb{R}^3} \frac{1}{(1+|y|)^4} dy \\ &\lesssim 2^{-i} |x|^{-4} \lesssim 2^{-i\delta} (|x| + |t|^{1/2} + 2^{-i})^{-3-\delta}. \end{aligned}$$

This concludes the proof for (2.16). \square

Lemma 2.19. *We have for every $\alpha \in (0, 3)$*

$$\sum_{i \sim j} \left| \int K_{i,x}(y)P(t,y)dy \right| \int \frac{|K_{j,x}(y)|}{(|y| + |t|^{1/2})^\alpha} dy \lesssim \frac{1}{(|x| + |t|^{1/2})^{3+\alpha}},$$

with the notation $\sum_{i \sim j} = \sum_i \sum_{j=i-1}^{i+1}$.

Proof. From (2.14) and (2.16) we deduce that

$$\sum_{i \sim j} \left| \int K_{i,x}(y) P(t, y) dy \right| \int \frac{|K_{j,x}(y)|}{(|y| + |t|^{1/2})^\alpha} dy \lesssim \sum_i \frac{2^{-i}}{(|x| + |t|^{1/2} + 2^{-i})^{4+\alpha}}.$$

Bounding the sum over i with an integral, we conclude that

$$\begin{aligned} \int_0^1 \frac{d\lambda}{\lambda} \frac{\lambda}{(|x| + |t|^{1/2} + \lambda)^{4+\alpha}} &= \frac{1}{(|x| + |t|^{1/2})^{3+\alpha}} \int_0^{1/(|x| + |t|^{1/2})} \frac{d\lambda}{(1 + \lambda)^{4+\alpha}} \\ &\lesssim \frac{1}{(|x| + |t|^{1/2})^{3+\alpha}}. \end{aligned}$$

□

Lemma 2.20. (estimation of simple diagrams)

For a fixed $\bar{\zeta} = (t, \bar{x}) \in \mathbb{R} \times \mathbb{T}^3$ and $\forall q \in \mathbb{Z}, q \geq -1$ define the measures

$$\begin{aligned} \mu_{q,\zeta} &:= K_{q,\bar{x}}(y) \delta(t-s) d\zeta, & \text{with } \zeta = (s, y), \\ \tilde{\mu}_{q,\zeta} &:= \left[\int_{\mathbb{R}^3} K_{q,\bar{x}}(x) P(t-s, x-y) dx \right] \mathbb{1}_{[0,+\infty)}(s) d\zeta, & \text{with } \zeta = (s, y). \end{aligned}$$

Let $C_\varepsilon(\zeta)$ for $\zeta \in \mathbb{R} \times \mathbb{R}^3$ be the (\mathbb{T}^3 -periodic) covariance of Y_ε . Then for every $\alpha < 3$

$$I_\alpha(t, \bar{x}) := \int_{(\mathbb{R} \times \mathbb{T}^3)^2} |C_\varepsilon(\zeta - \zeta')|^\alpha |\mu_{q,\zeta}| |\mu_{q,\zeta'}| \lesssim 2^{\alpha q}.$$

For every $\beta \in (3, 5) \setminus \{4\}$

$$\bar{I}_\beta(t, \bar{x}) := \int_{(\mathbb{R} \times \mathbb{T}^3)^2} |C_\varepsilon(\zeta - \zeta')|^\beta |\tilde{\mu}_{q,\zeta}| |\tilde{\mu}_{q,\zeta'}| \lesssim 2^{(\beta-4)q}$$

Proof. We have

$$I_\alpha(t, \bar{x}) = \int |K_{q,\bar{x}}(y) K_{q,\bar{x}}(y')| |y - y'|^{-\alpha} dy dy'$$

by Lemma 2.15. By (2.15) we have

$$\int |K_{q,\bar{x}}(y)| |y - y'|^{-\alpha} dy \lesssim (|\bar{x} - y'| + 2^{-q})^{-\alpha} \lesssim |\bar{x} - y'|^{-\alpha}$$

and using again (2.15) we obtain the result. For the second estimation we obtain from (2.16) that

$$|\tilde{\mu}_{q,\zeta}| \lesssim 2^{-q} (|t-s|^{1/2} + |\bar{x} - y| + 2^{-q})^4 ds dy,$$

and from Lemma 2.10

$$|\tilde{\mu}_{q,\zeta'}| \lesssim \int_{\mathbb{R}^3} |K_{q,\bar{x}}(x')| (|t-s'|^{1/2} + |\bar{x} - y'|)^3 dx' ds' dy'.$$

By the estimation on convolutions of Lemma 2.12 we obtain then

$$\begin{aligned} \bar{I}_\beta(t, \bar{x}) &\lesssim 2^{-q} \int |K_{q,\bar{x}}(x')| |\bar{x} - x'|^{3-\beta} dx' \\ &\lesssim 2^{(\beta-4)q} \end{aligned}$$

by (2.15), and this concludes the proof. □

Lemma 2.21. (estimation of composite diagrams)

For a fixed $\bar{\zeta} = (t, \bar{x}) \in \mathbb{R} \times \mathbb{T}^3$ and $\forall q \in \mathbb{Z}, q \geq -1$ define the measure

$$\mu_{q, \zeta_1, \zeta_2} := \left[\int_{\mathbb{R}^6} K_{q, \bar{x}}(x) \sum_{i \sim j} K_{i, x}(y) K_{j, x}(x_2) P(t - s_1, y - x_1) dx dy \right] \mathbb{1}_{[0, +\infty)}(s_1) \delta(t - s_2) d\zeta_1 d\zeta_2.$$

with the notation $\zeta_i = (s_i, x_i) \in \mathbb{R} \times \mathbb{T}^d$ for $i = 1, 2$. For $k, \ell \in [0, 2)$ and $m \in (0, 5)$, $n \in (0, 3)$ define

$$I_{k, \ell, m, n} := \int_{(\mathbb{R} \times \mathbb{T}^d)^2} |C_\varepsilon(\zeta_1 - \zeta_2)|^k |C_\varepsilon(\zeta'_1 - \zeta'_2)|^\ell |C_\varepsilon(\zeta_1 - \zeta'_1)|^m |C_\varepsilon(\zeta_2 - \zeta'_2)|^n |\mu_{q, \zeta_1, \zeta_2}| |\mu_{q, \zeta'_1, \zeta'_2}| \quad (2.17)$$

If $k + m - 1 \in (0, 5)$ or $\ell + m - 1 \in (0, 5)$, $k + \ell + m - 4 \in (-2, 5)$ we have the bound

$$I_{k, \ell, m, n} \lesssim 2^{(k + \ell + m + n - 4)q}.$$

Proof. Note from the Fourier support properties of the L-P decomposition (Prop.1.1) that there exists $N \in \mathbb{N}$ such that if $i < q - N$ we have

$$\int_{\mathbb{R}^6} K_{q, \bar{x}}(x) \sum_{j=i-1}^{i+1} K_{i, x}(y) K_{j, x}(x_2) P(t - s_1, y - x_1) dx dy = 0$$

and then the sum in $\mu_{q, \zeta_1, \zeta_2}$ becomes $\sum_{i \sim j \gtrsim q}$ (a notation to indicate that we sum over $i \gtrsim q - N$ and $i - 1 \leq j \leq i + 1$). Moreover, there exists $N \in \mathbb{N}$ (which we can take w.l.o.g. to be the same as before) such that if $i > q + N$ we have

$$\int_{\mathbb{R}^6} K_{q, \bar{x}}(x) \sum_{j > i+1, j < i-1} K_{i, x}(y) K_{j, x}(x_2) P(t - s_1, y - x_1) dx dy = 0.$$

Therefore, since $\sum_j K_{j, x}(x_2) = \delta(x - x_2)$ (as is readily seen again in Fourier space) we can write

$$\begin{aligned} \mu_{q, \zeta_1, \zeta_2} &= \left[\int_{\mathbb{R}^6} K_{q, \bar{x}}(x) \sum_{i \sim j \sim q} K_{i, x}(y) K_{j, x}(x_2) P(t - s_1, y - x_1) dx dy \right] \mathbb{1}_{[0, +\infty)}(s_1) \delta(t - s_2) d\zeta_1 d\zeta_2 \\ &+ \left[\int_{\mathbb{R}^6} K_{q, \bar{x}}(x) \sum_{i \gtrsim q} K_{i, x}(y) P(t - s_1, y - x_1) \delta(x - x_2) dx dy \right] \mathbb{1}_{[0, +\infty)}(s_1) \delta(t - s_2) d\zeta_1 d\zeta_2 \\ &= \bar{\mu}_{q, \zeta_1, \zeta_2} + \tilde{\mu}_{q, \zeta_1, \zeta_2}. \end{aligned}$$

By Cauchy-Schwarz inequality it suffices then to bound the terms $\bar{I}_{k, \ell, m, n}$ and $\tilde{I}_{k, \ell, m, n}$ where $\mu_{q, \zeta_1, \zeta_2}$ in (2.17) is replaced respectively by $\bar{\mu}_{q, \zeta_1, \zeta_2}$ and $\tilde{\mu}_{q, \zeta_1, \zeta_2}$.

The first term can be estimated by repeated change of variables, using the fact that $P(2^{-2i}s, 2^{-i}y) = 2^{-3i}P(s, y)$ and $|C_\varepsilon(2^{-2i}s, 2^{-i}y)| \lesssim 2^i(|s|^{1/2} + |y|)^{-1}$. Note that although after rescaling $s \in [0, 2^{2i}t]$, to integrate at infinity one can just bound P in $L^\infty(\mathbb{R}^3 \setminus B(0, 1))$ and obtain $P(s, y) \lesssim |s|^{-3/2}$. The space integrals are bounded thanks to the rapid decreasing of the kernel K .

To bound the term $\tilde{\mu}_{q, \zeta_1, \zeta_2}$ note that from (2.16)

$$\left| \int \sum_{i \gtrsim q} K_{i, x}(y) P(t - s_1, y - x_1) dy \right| \lesssim \sum_{i \gtrsim q} 2^{-\delta i} (|t - s_1|^{1/2} + |x - x_1| + 2^{-i})^{-3 - \delta}$$

and we can choose $\delta \in [0, 1]$ such that the hypotheses of Lemma 2.12 are satisfied. A repeated application of its convolution estimations yields the result. \square

2.3 Introduction to Malliavin calculus

We recall here some tools of Malliavin calculus that can be used to estimate stochastic terms that do not belong to any finite Wiener chaos decomposition. A more detailed introduction to Malliavin calculus, and the proofs of some results of this section, can be found in [Nua06, NP12, Shi04]. Lemma 2.28 was inspired by the calculations of [NN10].

2.3.1 Basic definitions and notations

Let $\{W(h)\}_{h \in H}$ be an isonormal Gaussian process indexed by a real separable Hilbert space H . Let $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space with \mathcal{F} generated by the isonormal Gaussian process W . We note $L^2(\Omega, \mathcal{F}, \mathbb{P}) = \bigoplus_{n \geq 0} \mathcal{H}_n$ the Wiener chaos decomposition of $L^2(\Omega)$ and write $J_n \Psi$ the projection on \mathcal{H}_n of the random variable $\Psi \in L^2(\Omega)$.

Let V be a real separable Hilbert space. For smooth functions $F \in L^p(\Omega; V)$ of the form

$$F = \sum_{j=1}^m F_j v_j \quad \text{with } v_j \in V, F_j = f(W(h_1), \dots, W(h_n)),$$

$f \in C^\infty(\mathbb{R}^n)$ with polynomial growth of all the derivatives, we define the Malliavin derivative

$$DF = \sum_{j=1}^m \sum_{i=1}^n \partial_i f(W(h_1), \dots, W(h_n)) h_i \otimes v_j.$$

For every $k \in \mathbb{N}$ the k -th derivative $D^k F$ can be defined consequently. For $p \geq 1$ we will write $\mathbb{D}^{k,p}(V) \subset L^p(\Omega; V)$ for the closure of smooth random variables with respect to the norm

$$\|\Psi\|_{\mathbb{D}^{k,p}(V)} = [\mathbb{E}(\|\Psi\|_V^p) + \sum_{j=1}^k \mathbb{E}(\|D^j \Psi\|_{H^{\otimes j} \otimes V}^p)]^{1/p}$$

with the notation $\mathbb{D}^{k,p} := \mathbb{D}^{k,p}(\mathbb{R})$. The Malliavin derivative can be extended to an operator

$$D^k: \mathbb{D}^{k,p}(V) \subset L^p(\Omega; V) \rightarrow L^p(\Omega; H^{\otimes k} \otimes V).$$

For $p = 2$ we denote as δ^k the unbounded operator on $L^2(\Omega; H^{\otimes k} \otimes V)$ with values on $L^2(\Omega; V)$ defined as the adjoint of D^k : i.e. $\delta(u)$ is the element of $L^2(\Omega; V)$ such that

$$\mathbb{E}(\langle F, \delta^k(u) \rangle_V) = \mathbb{E}(\langle D^k F, u \rangle_{H^{\otimes k} \otimes V}) \quad \forall F \in \mathbb{D}^{k,2}(V)$$

and the domain of δ^k consists of all $u \in L^2(\Omega; H^{\otimes k} \otimes V)$ such that the expression above makes sense. δ is called *divergence operator* or *Skorohod integral*. Let $\{P_t\}_{t \in \mathbb{R}^+}$ the Ornstein-Uhlenbeck semigroup defined as

$$P_t \Psi = \sum_{n=0}^{\infty} e^{-nt} J_n \Psi$$

$\forall \Psi \in L^2(\Omega)$ and $L: L^2(\Omega) \rightarrow L^2(\Omega)$ defined as

$$L \Psi = \sum_{n=0}^{\infty} -n J_n \Psi$$

be its infinitesimal generator, i.e. $e^{tL} = P_t$ [Nua06, Prop.1.4.2]. Following [Shi04] we introduce the Green operator

$$G_j = (j - L)^{-1}$$

with the notation

$$G_{[j]}^{[m]} := \prod_{k=j}^m G_k \quad \text{for } 1 \leq j \leq m \quad (2.18)$$

so that $G_{[j]}^{[j]} = G_j$. To avoid confusion, it is worth stressing that $G_{[j]}^{[m]}$ is *not* the m -th power of the operator G_j but just a shortcut for $\prod_{k=j}^m G_k$.

2.3.2 Partial chaos expansion

Let $\Psi \in L^2(\Omega)$ which has the Wiener chaos decomposition $\Psi = \sum_{n \geq 0} J_n \Psi$. Then by Proposition 1.2.2 of [Nua06]:

$$DJ_n \Psi = J_{n-1} D\Psi,$$

and knowing that $LJ_n \Psi = -nJ_n \Psi$ we obtain the commutation property

$$D(j-L)^{-\alpha} \Psi = D \sum_{n \geq 0} \frac{1}{(j+n)^\alpha} J_n \Psi = \sum_{n \geq 1} \frac{1}{(j+n)^\alpha} J_{n-1} D\Psi = (j+1-L)^{-\alpha} D\Psi \quad (2.19)$$

for every $\alpha > 0$, $j > 0$. The above formula holds also for $j = 0$ if $\mathbb{E}(\Psi) = 0$.

The results we have recalled so far enable us to write a partial chaos expansion to the n th-order for a random variable in $\mathbb{D}^{n,2}$:

Lemma 2.22. *Let $\Psi \in \mathbb{D}^{n,2}$ and $G_{[j]}^{[m]}$ as in (2.18). Then for every $n \in \mathbb{N} \setminus \{0\}$ $G_{[1]}^{[n]} D^n \Psi \in \text{Dom } \delta^n$, $J_0 D^k \Psi \in \text{Dom } \delta^k \forall 0 \leq k < n$ and*

$$\delta^n G_{[1]}^{[n]} D^n \Psi = (\text{id} - J_0 - \dots - J_{n-1}) \Psi = \Psi - \sum_{k=0}^{n-1} \frac{1}{k!} \delta^k J_0 D^k \Psi. \quad (2.20)$$

Proof. We have for any $\Psi \in L^2(\Omega)$, since $L = -\delta D$ ([Nua06], Proposition 1.4.3):

$$\Psi - \mathbb{E}(\Psi) = LL^{-1}(\Psi - J_0 \Psi) = -\delta D L^{-1}(\Psi - J_0 \Psi) = \delta (1 - L)^{-1} D\Psi$$

where we used (2.19), and the fact that $(1 - L)^{-1} D\Psi \in \text{Dom } \delta$ is obvious by construction. This yields the first order expansion

$$\Psi = \mathbb{E}(\Psi) + \delta (1 - L)^{-1} D\Psi.$$

Iterating this formula up to order n we obtain (2.20). It is clear that $J_0 D^k \Psi \in \text{Dom } \delta^k$ since $J_0 D^k \Psi$ is constant with values in $H^{\otimes k}$. The second equality comes from the fact that $\delta^k J_0 D^k \Psi \in \mathcal{H}_k \forall k \in \mathbb{N}$, indeed $\forall \Psi' \in \mathbb{D}^{k,2}$:

$$\mathbb{E}(\delta^k (J_0 D^k \Psi) J_k \Psi') = \langle J_0 D^k \Psi, J_0 D^k \Psi' \rangle_{L^2(\Omega; H^{\otimes k})} = \mathbb{E}(\delta^k (J_0 D^k \Psi) \Psi') \quad \square$$

In order to obtain L^p estimations of the remainder term $\delta^n G_{[1]}^{[n]} D^n \Psi$ generated by expansion (2.20), one can use the following lemmas:

Lemma 2.23. ([Nua06], Prop. 1.5.7) *Let V be a real separable Hilbert space. For every $p > 1$ and every $q \in \mathbb{N}$, $k \geq q$ and every $\Psi \in \mathbb{D}^{k,p}(H^q \otimes V)$ we have*

$$\|\delta^q(\Psi)\|_{\mathbb{D}^{k-q,p}(V)} \lesssim_{k,p} \|\Psi\|_{\mathbb{D}^{k,p}(H^q \otimes V)}$$

Remark 2.24. Let V be a real separable Hilbert space. For every $v \in V$ and every $\Psi \in \mathbb{D}^{q,2}(H^{\otimes q})$ with $q \in \mathbb{N}$ we have $\Psi \otimes v \in \text{Dom } \delta^q$ and

$$\delta^q(\Psi)v = \delta^q(\Psi \otimes v).$$

Indeed, notice that for every smooth $\Psi' \in \mathbb{D}^{q,2}(V)$ and every smooth $\Psi \in \mathbb{D}^{q,2}(H^{\otimes q})$ we have

$$\mathbb{E}(\langle \delta^q(\Psi \otimes v), \Psi' \rangle_V) = \mathbb{E}(\langle \Psi \otimes v, D^q \Psi' \rangle_{H^{\otimes q} \otimes V}) = \mathbb{E}(\langle \delta^q(\Psi)v, \Psi' \rangle_V).$$

Now since $D^q(\Psi \otimes v) = D^q \Psi \otimes v$ and $\Psi \in \mathbb{D}^{q,2}(H^{\otimes q})$, we have $\Psi \otimes v \in \mathbb{D}^{q,2}(H^{\otimes q} \otimes V)$. Lemma 2.23 yields the bound $\|\delta^q(\Psi \otimes v)\|_{L^2(V)} \lesssim \|\Psi \otimes v\|_{\mathbb{D}^{q,2}(H^{\otimes q} \otimes V)}$ which allows to pass to the limit for Ψ' and $\delta^q(\Psi \otimes v)$ in $L^2(V)$.

Lemma 2.25. ([Shi04], Prop. 4.3) *For every $j > 0$ the operator $(j - L)^{-1/2}$ is bounded in L^p for every $1 \leq p < \infty$.*

Lemma 2.26. *Let $j \in \mathbb{N} \setminus \{0\}$ and V a real separable Hilbert space. There exists a finite constant c_p such that for every $\Psi \in L^p(\Omega, V)$:*

$$\|D(j - L)^{-1/2} \Psi\|_{L^p(\Omega, H \otimes V)} \leq c_p \|\Psi\|_{L^p(\Omega, V)}$$

(where the operator $D(j - L)^{-1/2}$ is defined on every Ψ which is polynomial in $W(h_1), \dots, W(h_n)$ and can be extended by density on L^p).

Proof. First notice that we can suppose w.l.o.g. $\mathbb{E}(\Psi) = 0$ thanks to (2.19). Therefore we can write $D(j - L)^{-1/2}$ as

$$D(j - L)^{-1/2} = D(-C)^{-1}(-C)(j - L)^{-1/2}$$

with $C = -\sqrt{-L}$. We decompose the second part as

$$-C(j - L)^{-1/2} \Psi = \sum_{n=1}^{\infty} \left(\frac{n}{j+n} \right)^{1/2} J_n \Psi = T_\phi \Psi$$

with $T_\phi \Psi := \sum_{n=0}^{\infty} \phi(n) J_n \Psi$. We apply Theorem 1.4.2 of [Nua06] to show that T_ϕ is bounded in L^p , indeed $\phi(n) = h(1/n)$ and $h(x) = (jx + 1)^{-1/2}$ which is analytic in a neighbourhood of 0. Finally, we can apply Proposition 1.5.2 of [Nua06] to show that DC^{-1} is bounded in L^p , thus concluding the proof. \square

The two lemmas above give the following immediate corollary:

Corollary 2.27. *For every $1 \leq m \leq n$ the operator $G_{[m]}^{[n]} := \prod_{j=m}^n (j - L)^{-1}$ is bounded in L^p for every $1 \leq p < \infty$.*

Moreover, Let $j \in \mathbb{N} \setminus \{0\}$ and V a real separable Hilbert space. Then for every $\Psi \in L^p(\Omega, V)$ we have:

$$\|D(j - L)^{-1} \Psi\|_{L^p(\Omega, H \otimes V)} \lesssim \|\Psi\|_{L^p(\Omega, V)}.$$

Moreover, for every $0 \leq k \leq 2m$, $i \geq 0$ we have

$$\|D^k G_{[i+1]}^{[i+m]} \Psi\|_{L^p(\Omega, H^{\otimes k} \otimes V)} \lesssim \|\Psi\|_{L^p(\Omega, V)}.$$

The next lemma allows to write products of decompositions of the type (2.20) as sums of iterated Skorohod integrals. From now on we will note $\langle \cdot, \cdot \rangle_{H^{\otimes r}}$ or $\langle \cdot, \cdot \rangle_r$ the r -th contraction, which to avoid inconsistency has to be taken between symmetric tensors. We also note $h_{v_1, \dots, v_n}^{\odot n} := h_{v_1} \odot \dots \odot h_{v_n}$ for $h_{v_1}, \dots, h_{v_n} \in H$.

Lemma 2.28. (product formula) *Let $u = f(W(h_u))h_u^{\otimes m}$ and $v = F h_{v_1, \dots, v_n}^{\otimes n}$ with $f \in C^{m+n}(\mathbb{R})$ and $F \in \mathbb{D}^{m+n, 2}$. Then*

$$\delta^m(u)\delta^n(v) = \sum_{(q,r,i) \in I_{m,n}} C_{m,n,q,r,i} \delta^{m+n-q-r} [f^{(r-i)}(W(h_u)) \langle h_u^{\otimes m-i}, D^{q-i}F \rangle_{q-i} \langle h_u^{\otimes r}, h_{v_1, \dots, v_n}^{\otimes n} \rangle_r] \quad (2.21)$$

with

$$C_{m,n,q,r,i} := \binom{m}{q} \binom{n}{r} \binom{q}{i} \binom{r}{i} i!, \\ I_{m,n} := \{(q,r,i) \in \mathbb{N}^3 : 0 \leq q \leq m, 0 \leq r \leq n, 0 \leq i \leq q \wedge r\}.$$

A trivial change of variables gives also

$$\delta^m(u)\delta^n(v) = \sum_{I'_{m,n}} C_{m,n,q+i,r+i,i} \delta^{m+n-q-r-2i} [f^{(r)}(W(h_u)) \langle h_u^{\otimes m-i}, D^q F \rangle_q \langle h_u^{\otimes r+i}, h_{v_1, \dots, v_n}^{\otimes n} \rangle_{r+i}]$$

with $I'_{m,n} := \{(i,q,r) \in \mathbb{N}^3 : 0 \leq i \leq m \wedge n, 0 \leq q \leq m-i, 0 \leq r \leq n-i\}$.

Remark 2.29. In the special case $v = g(W(h_v))h_v^{\otimes n}$ eq. (2.21) takes the form

$$\delta^m(u)\delta^n(v) = \sum_{(q,r,i) \in I_{m,n}} C_{m,n,q,r,i} \delta^{m+n-q-r} (\langle D^{r-i}u, D^{q-i}v \rangle_{H^{\otimes q+r-i}}) \quad (2.22)$$

which is just a generalization to Skorohod integrals of the multiplication formula for multiple Wiener integrals ([Shi80], [Üst14]). We can write the above formula more explicitly as

$$\delta^m(u)\delta^n(v) = \sum_{I_{m,n}} C_{m,n,q,r,i} \delta^{m+n-q-r} [f^{(r-i)}(W(h_u)) g^{(q-i)}(W(h_v)) h_u^{\otimes m-q} \otimes h_v^{\otimes n-r}] \langle h_u, h_v \rangle^{q+r-i}.$$

Remark 2.30. Note that one can assume w.l.o.g. the argument of $\delta^{m+n-q-r}$ in (2.21) to be symmetric, and this would allow to iterate Lemma 2.28.

Remark 2.31. We can give the following intuition for the second formula in Lemma 2.28. The random variables $\delta^m(u)$ and $\delta^n(v)$ have an infinite chaos decomposition, and following the tree notation of (2.8) (see also [GIP15], [Hai14a]) they could be represented as trees with at least m (respectively n) leaves, which need to be contracted with each other as in (2.9) when decomposing the product $\delta^m(u)\delta^n(v)$. One can either contract the existing leaves or perform the Malliavin derivative that makes a new leaf appear in the notation.

It is therefore clear that the index i in the second equation of Lemma 2.28 denotes contractions between the already existing leaves of the trees u, v . The indexes r and q count new leaves in each vertex that are created by the Malliavin derivatives, which are then contracted with other leaves from the other tree. There are then $m+n-r-q-2i$ overall unmatched leaves which are arguments to the iterated Skorohod integral.

The more intuitive interpretation of the second equation in Lemma 2.28 is the reason why we gave two distinct expression for the same quantity. Nevertheless, the formula (2.21) is more practical in the calculations and is more widely used throughout this work.

Proof. (Lemma 2.28) Using Cauchy-Schwarz inequality and Lemma 2.23 we can show that $\langle D^r \delta^n(v), \delta^j(u) \rangle_{H^{\otimes r}} \in L^2(\Omega, H^{\otimes m-j-r})$ for every $0 \leq r+j \leq m$. Then we apply Lemma 2.32 to get:

$$\delta^m(u)\delta^n(v) = \sum_{r=0}^n \binom{n}{r} \delta^{n-r} (\langle D^r \delta^m(u), v \rangle_{H^{\otimes r}}).$$

Using the commutation formula (2.23) we rewrite the r.h.s. as

$$\delta^m(u)\delta^n(v) = \sum_{r=0}^n \binom{n}{r} \sum_{i=0}^{r \wedge m} \binom{r}{i} \binom{m}{i} i! \delta^{n-r}(\langle \delta^{m-i}(D^{r-i}u), v \rangle_{H^{\otimes r}}).$$

We obtain

$$\langle \delta^{m-i}(D^{r-i}u), v \rangle_{H^{\otimes r}} = \delta^{m-i}(f^{(r-i)}(W(h_u)) h_u^{\otimes m-i}) F \langle h_u^{\otimes r}, h_{v_1, \dots, v_n}^{\odot n} \rangle_{H^{\otimes r}}$$

and using again Lemma 2.32 we obtain

$$\begin{aligned} & \langle \delta^{m-i}(D^{r-i}u), v \rangle_{H^{\otimes r}} = \\ & \sum_{\ell=0}^{m-i} \binom{m-i}{\ell} \delta^{m-i-\ell}(f^{(r-i)}(W(h_u)) \langle h_u^{\otimes m-i}, D^\ell F \rangle_\ell \langle h_u^{\otimes r}, h_{v_1} \odot \dots \odot h_{v_n} \rangle_r) \end{aligned}$$

where we used $\delta^k(\Psi)h^{\otimes n-r} = \delta^k(\Psi \otimes h^{\otimes n-r})$ for $\Psi \in \text{Dom } \delta^k$, as seen in Remark 2.24. Substituting this expression into $\delta^m(u)\delta^n(v)$ we get

$$\begin{aligned} & \delta^m(u)\delta^n(v) = \\ & \sum_{J_{m,n}} A_{m,n,r,i,\ell} \delta^{m+n-r-i-\ell} [f^{(r-i)}(W(h_u)) \langle h_u^{\otimes m-i}, D^\ell F \rangle_\ell \langle h_u^{\otimes r}, h_{v_1} \odot \dots \odot h_{v_n} \rangle_r] \end{aligned}$$

where we set

$$\begin{aligned} A_{m,n,r,i,\ell} & := \binom{m}{i} \binom{n}{r} \binom{m-i}{\ell} \binom{r}{i} i!, \\ J_{m,n} & := \{(r, i, \ell) \in \mathbb{N}^3 : 0 \leq r \leq n, 0 \leq i \leq r \wedge n, 0 \leq \ell \leq m-i\}. \end{aligned}$$

In order to complete the proof we just have to perform some basic changes of indexes. Taking $q = \ell + i$ and noting that $\binom{m}{i} \binom{m-i}{q-i} = \binom{q}{i} \binom{m}{q}$ we have

$$A_{m,n,r,i,\ell} = \binom{m}{q} \binom{q}{i} \binom{n}{r} \binom{r}{i} i!$$

and this yields (2.21). Finally, we perform the change of variables $q - i \rightarrow q$, $r - i \rightarrow r$ to get the second formula. \square

We list below the tools we used to prove Lemma 2.28.

Lemma 2.32. ([NN10], Lemma 2.1) *Let $q \in \mathbb{N} \setminus \{0\}$, $\Psi \in \mathbb{D}^{q,2}$, $u \in \text{Dom } \delta^q$ and symmetric. Assume also that $\forall 0 \leq r + j \leq q$*

$$\langle D^r \Psi, \delta^j(u) \rangle_{H^{\otimes r}} \in L^2(\Omega, H^{\otimes q-r-j}).$$

Then $\forall 0 \leq r < q$ $\langle D^r \Psi, u \rangle_r \in \text{Dom } \delta^{q-r}$ and

$$\Psi \delta^q(u) = \sum_{r=0}^q \binom{q}{r} \delta^{q-r}(\langle D^r \Psi, u \rangle_{H^{\otimes r}}).$$

Remark 2.33. Note that $\delta^n(h^{\otimes n}) = \llbracket W^n(h) \rrbracket$ with the Wick product $\llbracket \cdot \rrbracket$ defined in Section 2.1. Indeed $\forall \Psi \in \mathbb{D}^{1,2}$ we know that $\mathbb{E}[\delta(h^{\otimes n})\Psi] = \mathbb{E}[W(h) h^{\otimes n-1} \Psi]$ using the facts that $\delta(h) = W(h)$ and $\delta(h^{\otimes n}) = \delta(h)h^{\otimes n-1}$, therefore $\delta^n(h^{\otimes n}) = \delta^{n-1}(W(h)h^{\otimes n-1})$. From Lemma 2.32 we have, since $DW(h) = h$:

$$\delta^{n-1}(W(h)h^{\otimes n-1}) = \delta^{n-1}(h^{\otimes n-1})W(h) - (n-1)\langle h, h \rangle \delta^{n-2}(h^{\otimes n-2})$$

which gives by induction $\delta^n(h^{\otimes n}) = \llbracket W^n(h) \rrbracket$.

Lemma 2.34. *Let $j, k \in \mathbb{N}$, $u \in \mathbb{D}^{j+k,2}(H^{\otimes j})$ symmetric and such that all its derivatives are symmetric. We have*

$$D^k \delta^j(u) = \sum_{i=0}^{k \wedge j} \binom{k}{i} \binom{j}{i} i! \delta^{j-i}(D^{k-i} u) \quad (2.23)$$

Proof. If $j = 0$, $k = 1$ or $k = 0$, $j = 1$ eq. (2.23) is trivial. Let $j = k = 1$ and $u \in \mathbb{D}^{2,2}(H) \subset \mathbb{D}^{1,2}(H)$. We can apply Proposition 1.3.2 of [Nua06] to obtain $\langle D\delta(u), h \rangle = \langle u, h \rangle + \delta(\langle Du, h \rangle) \forall h \in H$. Since by hypothesis Du is symmetric we have $\delta(\langle Du, h \rangle) = \langle \delta Du, h \rangle$, and then $D\delta(u) = u + \delta Du$. The proof by induction is easy noticing that $D\delta^j = \delta^j D + j\delta^{j-1}$. \square

Chapter 3

Brief introduction to FK percolation and the Ising model

3.1 The random cluster model

The random cluster model, or FK percolation model, was first introduced by Fortuin and Kasteleyn in [FK72]. We refer to comprehensive book on the subject [Gri09], and introduce here just some notations that will be used in Chapter 4.

Let $\Lambda \subset \mathbb{Z}^d$ bounded, $\Omega = \{0, 1\}^{\mathbb{E}^d}$ with \mathbb{E}^d the set of edges of the graph \mathbb{Z}^d , and \mathcal{F} be the σ -algebra generated by cylinder sets. For $\omega \in \Omega$, let ω_e be the component of ω at $e \in \mathbb{E}^d$. Let

$$E_\Lambda = \{e = \langle x, y \rangle \in \mathbb{E}^d \mid x \in \Lambda, y \in \Lambda\} \quad (3.1)$$

the set of edges with both endpoints in Λ . For $\xi \in \Omega$, define the following finite subset of Ω :

$$\Omega_\Lambda^\xi = \{\omega \in \Omega \mid \omega_e = \xi_e \quad \forall e \in \mathbb{E}^d \setminus E_\Lambda\}.$$

Definition 3.1. Let $p \in [0, 1]$, $q \in (0, \infty)$. The FK probability measure on (Ω, \mathcal{F}) with boundary condition ξ is

$$\phi_{\Lambda, p, q}^\xi(\omega) = \begin{cases} \frac{1}{Z_{\xi, \Lambda}} [\prod_{e \in E_\Lambda} p^{\omega_e} (1-p)^{1-\omega_e}] q^{k(\omega)} & \text{if } \omega \in \Omega_\Lambda^\xi \\ 0 & \text{otherwise} \end{cases} \quad (3.2)$$

with $Z_{\xi, \Lambda}(p, q) = \sum_{\omega \in \Omega_\Lambda^\xi} [\prod_{e \in E_\Lambda} p^{\omega_e} (1-p)^{1-\omega_e}] q^{k(\omega)}$ and $k(\omega)$ the number of connected components of the graph $(\mathbb{Z}^d, \eta(\omega))$, with $\eta(\omega) = \{e \in \mathbb{E}^d \mid \omega_e = 1\}$.

We will call the edge e *open* if $\omega_e = 1$, and *closed* otherwise. We call *open clusters* the connected components of $(\mathbb{Z}^d, \eta(\omega))$, and write $x \leftrightarrow y$ if x, y are in the same open cluster, $x \nleftrightarrow y$ otherwise. An *open path* is a (possibly infinite) sequence (e_i) of edges belonging to $\eta(\omega)$. The boundary condition is *free* if $\xi_e = 0 \quad \forall e \in \mathbb{E}^d$ and *wired* if $\xi_e = 1 \quad \forall e \in \mathbb{E}^d$. One can see from Definition 3.1 that for $q = 1$ the model corresponds to the standard Bernoulli bond percolation, with edges being independently open or closed at random.

Remark 3.2. For both free and wired boundary conditions, if the domain Λ is the union of two subsets Λ_1 and Λ_2 such that $E_{\Lambda_1} \cap E_{\Lambda_2} = \emptyset$, then the configurations on Λ_1 and Λ_2 are independent. Indeed, calling $\overline{k(\omega, \mathbb{E}^d \setminus E_\Lambda)}$ the number of open clusters of ω that do not intersect $\mathbb{E}^d \setminus E_\Lambda$, we have $\overline{k(\omega, \mathbb{E}^d \setminus E_\Lambda)} = \overline{k(\omega, \mathbb{E}^d \setminus E_{\Lambda_1})} + \overline{k(\omega, \mathbb{E}^d \setminus E_{\Lambda_2})}$.

Although in general the states on two different edges are not independent, the model exhibits a “domain Markov” [HDN11] or “nesting” [Gri09] property. Let \mathcal{F}_Λ (respectively \mathcal{F}_{Λ^c}) be the σ -algebra generated by the states of edges in E_Λ (respectively in $\mathbb{E}^d \setminus E_\Lambda$). We have the following result.

Lemma 3.3. ([Gri09, Lemma 4.13]) *Let $p \in [0, 1]$, $q \in (0, \infty)$, and let Λ, Δ be finite subsets of \mathbb{Z}^d with $\Lambda \subseteq \Delta$. For every $\xi \in \Omega$, every event $A \in \mathcal{F}_\Lambda$ and every $\omega \in \Omega_\Delta^\xi$:*

$$\phi_{\Delta,p,q}^\xi(A | \mathcal{F}_\Lambda)(\omega) = \phi_{\Lambda,p,q}^\omega(A). \quad (3.3)$$

The set $\Omega = \{0, 1\}^{\mathbb{E}^d}$ has a partial ordering given by $\omega \leq \omega'$ if $\forall e \in \mathbb{E}^d \omega_e \leq \omega'_e$. A function $X: \Omega \rightarrow \mathbb{R}$ is called *increasing* if $\omega \leq \omega' \Rightarrow X(\omega) \leq X(\omega')$. Likewise, an event $A \in \mathcal{F}$ is called increasing if the random variable $\mathbb{1}_A$ is increasing.

Definition 3.4. (FKG inequality) *A probability measure μ on Ω is said to be positively associated, or to verify the FKG inequality, if for every increasing events A, B we have*

$$\mu(A \cap B) \geq \mu(A) \mu(B)$$

The probability measure $\phi_{\Lambda,p,q}^\xi$ of Definition 3.1 is positively associated [Gri09, Lemma 4.14]. As a direct consequence we have the following monotonicity properties.

Lemma 3.5. *Let $p \in [0, 1]$, $q \geq 1$ and $\Lambda \subseteq \Delta \subseteq \mathbb{Z}^d$ finite sets. Then:*

- *For every $\eta \leq \xi \in \Omega$ and for every increasing event $A \in \mathcal{F}_\Lambda$:*

$$\phi_{\Lambda,p,q}^\eta(A) \leq \phi_{\Lambda,p,q}^\xi(A).$$

- *For every increasing event $A \in \mathcal{F}_\Lambda$:*

$$\phi_{\Delta,p,q}^1(A) \leq \phi_{\Lambda,p,q}^1(A).$$

For $p \in [0, 1]$, $q \geq 1$, the random cluster measure $\phi_{\Lambda,p,q}^\xi$ for both free and wired boundary conditions admits a thermodynamic limit as $\Lambda \rightarrow \mathbb{Z}^d$ [Gri09, Theorems 4.17 and 4.19], which we call $\phi_{p,q}^\xi$. For every boundary condition ξ such that $\phi_{\Lambda,p,q}^\xi$ admits a limit and every increasing event A , we have:

$$\phi_{p,q}^0(A) \leq \phi_{p,q}^\xi(A) \leq \phi_{p,q}^1(A).$$

3.2 Relation with the 2-d Ising model

We introduce now the Ising-Potts model on a finite set $\Lambda \subseteq \mathbb{Z}^d$. Take a configuration space $\Sigma_\Lambda^0 = \{-1, 1\}^\Lambda$ with \mathcal{F} the σ -algebra generated by cylinder events. For $\sigma \in \Sigma_\Lambda^0$ and $x \in \Lambda$ we call σ_x the function that associates σ with its value in x , and $\sigma_e = \sigma_x \sigma_y$ with $e = \langle x, y \rangle \in E_\Lambda$. The Ising probability measure on $(\Sigma_\Lambda^0, \mathcal{F})$ with *free* boundary condition is defined $\forall \sigma \in \Sigma_\Lambda^0$ by

$$\pi_\Lambda^0(\sigma) = \frac{1}{Z_I^0} e^{-\beta H(\sigma)}, \quad H^0(\sigma) = - \sum_{e \in E_\Lambda^0} \mathbb{1}_{\sigma_e=1} \quad (3.4)$$

with $E_\Lambda^0 = \{e = \langle x, y \rangle \in \mathbb{E}^d \mid x \in \Lambda, y \in \Lambda\}$, $\beta > 0$, $\sigma_e = \sigma_x \sigma_y$ and $Z_I^0(\beta) = \sum_{\sigma \in \Sigma_\Lambda^0} e^{-\beta H^0(\sigma)}$.

Similarly, let $\Sigma_\Lambda^1 = \{\sigma \in \{-1, 1\}^{\Lambda \cup \partial\Lambda} \mid \sigma_x = 1 \forall x \in \partial\Lambda\}$. The Ising probability measure with *+* boundary condition on Λ is defined $\forall \sigma \in \Sigma_\Lambda^1$ by

$$\pi_\Lambda^1(\sigma) = \frac{1}{Z_I^1} e^{-\beta H^1(\sigma)} \mathbb{1}_{\Sigma_\Lambda^1}(\sigma), \quad H^1(\sigma) = - \sum_{e \in E_\Lambda^1} \mathbb{1}_{\sigma_e=1} \quad (3.5)$$

with $E_\Lambda^1 = \{e = \langle x, y \rangle \in \mathbb{E}^d \mid x \in \Lambda \vee y \in \Lambda\}$ and $Z_I^1(\beta) = \sum_{\sigma \in \Sigma_\Lambda^1} e^{-\beta H^1(\sigma)}$. From now on, random variables σ_x for $x \in \mathbb{Z}^d$ are (unsurprisingly) called *spins*.

Remark 3.6. Traditionally, the Hamiltonian of the Ising model is written as

$$H'(\sigma) = - \sum_{x \sim y} \sigma_x \sigma_y$$

with x, y nearest neighbours. Defining $\lambda_{\beta'}(\sigma) \propto e^{-\beta' H'(\sigma)}$ for the usual Ising measure, we recover it as $\lambda_{\beta/2} = \pi_\beta$.

The Edwards-Sokal coupling on Λ with boundary condition $\xi \in \{0, 1\}$ consists of defining a probability measure on $\Sigma_\Lambda^\xi \times \Omega$ as

$$\mu_\Lambda^\xi(\sigma, \omega) = \frac{1}{Z_{\text{es}}^\xi} \prod_{e \in E_\Lambda^\xi} [(1-p) \mathbb{1}_{\omega_e=0} + p \mathbb{1}_{\omega_e=1} \mathbb{1}_{\sigma_e=1}] \mathbb{1}_{\Omega_\Lambda^\xi}(\omega) \quad (3.6)$$

with Z_{es}^ξ such that $\sum_{(\sigma, \omega) \in \Sigma_\Lambda^\xi \times \Omega} \mu_\Lambda^\xi(\sigma, \omega) = 1$. From now on we fix

$$e^{-\beta} = 1 - p \quad \text{and} \quad q = 2. \quad (3.7)$$

It is easy to obtain the following lemma (see [Gri09]).

Lemma 3.7. *Let $p \in [0, 1]$, $e^{-\beta} = (1 - p)$, $q = 2$ and $\xi \in \{0, 1\}$. Let μ_Λ^ξ be defined as in (3.6). Then:*

- *The marginal of μ_Λ^ξ on Σ_Λ^ξ is π_Λ^ξ .*
- *The marginal of μ_Λ^ξ on Ω is $\phi_{\Lambda, p, 2}^\xi$.*

In order to characterize the regularity of the Ising magnetization field on an unbounded domain $U \subseteq \mathbb{R}^2$, in Chapter 4 we will use the well-known FK-Ising coupling for infinite volume measures.

Theorem 3.8. ([Gri09, Theorem 4.91]) *Let $p \in [0, 1]$, $q = 2$, $e^{-\beta} = (1 - p)$.*

- *Let ω be sampled from $\Omega = \{0, 1\}^{\mathbb{E}^2}$ with law $\phi_{p, q}^1$. Conditionally on ω , each vertex is assigned a random spin $\sigma_x \in \{-1, +1\}$ such that:*
 1. *$\sigma_x = 1$ if $x \leftrightarrow \infty$ (i.e. if x belongs to an infinite open cluster)*
 2. *σ_x takes values in $\{-1, 1\}$ with probability $\frac{1}{2}$ if $x \nleftrightarrow \infty$*
 3. *$\sigma_x = \sigma_y$ if $x \leftrightarrow y$*
 4. *spins in different open clusters are independent.*

Then the configuration $\sigma = \{\sigma_x\}_{x \in \mathbb{Z}^d}$ is distributed according to the weak limit π^1 of Ising measures with + boundary condition.

- *Let σ be sampled from $\Sigma = \{-1, +1\}^{\mathbb{Z}^d}$ with the Ising limit law π^1 . Conditionally on σ , each edge is assigned a random state $\omega_e \in \{0, 1\}$ such that:*
 1. *the states of different edges are independent*
 2. *$\omega_e = 0$ if $\sigma_x \neq \sigma_y$*
 3. *if $\sigma_x = \sigma_y$, then $\omega_e = 1$ with probability p and 0 otherwise.*

Then the edge configuration $\omega = \{\omega_e\}_{e \in \mathbb{E}^2}$ has law $\phi_{p, q}^1$.

A similar argument is valid for $\phi_{p, q}^0$ and the infinite-volume Ising measure π^0 , with the difference that no fixed value is assigned to σ_x in the case $x \leftrightarrow \infty$.

Results

Chapter 4

Tightness of the 2-d Ising magnetization field at criticality

In this chapter we study the magnetization field of the two-dimensional Ising model introduced in Chapter 3, at critical (inverse) temperature $\beta_c = \ln(1 + \sqrt{2})$ and associated FK percolation parameter $p_c = 1 - e^{-\beta_c}$. The problem of finding the exact regularity for the magnetization field was first posed in [CGN15]. Let $U \subseteq \mathbb{R}^2$ be an open set, and for $a > 0$, let $U_a := U \cap (a\mathbb{Z}^2)$. Denote by $(\sigma_y)_{y \in U_a}$ the Ising spin system at critical temperature, with ξ boundary condition (either + or free), and define the magnetization field

$$\hat{\Phi}_a := a^{\frac{15}{8}} \sum_{y \in U_a} \sigma_y \delta_y, \quad (4.1)$$

where δ_y is the Dirac mass at y . It is easy to see that Dirac masses do not belong to $B_{2,2}^{-1}(U)$, and thus prevent the family $(\hat{\Phi}_a)_{a \in (0,1]}$ from being tight in this space. However, we can define the piecewise constant random field

$$\Phi_a := a^{-\frac{1}{8}} \sum_{y \in U_a} \sigma_y \mathbb{1}_{S_a(y)}, \quad (4.2)$$

(as done in [CGN15]) where $S_a(y)$ is the square centered at y of side length a . We note that the set of limit points of $(\hat{\Phi}_a)_{a \in (0,1]}$ and of $(\Phi_a)_{a \in (0,1]}$ coincide. Indeed, one can check using Definition 1.17 that the difference $\Phi_a - \hat{\Phi}_a$ converges to zero almost surely in, say, $C_{\text{loc}}^\alpha(U)$, for every $\alpha < -3$.

In [CGN15], the authors showed that for $U = [0, 1]^2$ and every $\varepsilon > 0$, the family $(\Phi_a)_{a \in (0,1]}$ is tight in $B_{2,2}^{-1-\varepsilon}(U)$, and proceeded to discuss similar results in more general domains. (In Proposition 1.14 we showed the equivalence between Sobolev spaces $H^{-1-\varepsilon}$ used in [CGN15] and Besov spaces $B_{2,2}^{-1-\varepsilon}$). They asked in which precise function spaces the family $(\Phi_a)_{a \in (0,1]}$ is tight.

Using the Onsager correlation bounds (Lemma 4.4) and the tightness criterion of Theorem 1.46 (together with the equivalence of norms of Proposition 1.52), we prove the following result.

Theorem 4.1. *Fix an open set $U \subseteq \mathbb{R}^2$. For every $\varepsilon > 0$ and $p, q \in [1, \infty]$, the family of Ising magnetization fields $(\Phi_a)_{a \in (0,1]}$ on U is tight in $B_{p,q}^{-\frac{1}{8}-\varepsilon, \text{loc}}(U)$.*

We also prove, in Section 4.1 that the previous result is essentially sharp, when $U = \mathbb{R}^2$:

Theorem 4.2. *Let $\varepsilon > 0$ and $p, q \in [1, \infty]$. If Φ is a limit point of the family of Ising magnetization fields $(\Phi_a)_{a \in (0,1]}$ on \mathbb{R}^2 , then $\Phi \notin B_{p,q}^{-\frac{1}{8}+\varepsilon, \text{loc}}(\mathbb{R}^2)$ with positive probability. In particular, the family $(\Phi_a)_{a \in (0,1]}$ is not tight in $B_{p,q}^{-\frac{1}{8}+\varepsilon, \text{loc}}(\mathbb{R}^2)$.*

It was shown recently that there exists a unique limit point to the family $(\Phi_a)_{a \in (0,1]}$, see [CGN15, CHI15]. Theorem 4.2 gives a characterization of the regularity of this limit.

In order to prove Theorem 4.1, we choose a spanning sequence \mathcal{K} of U (as specified in Definition 1.37) and bound (1.39), (1.40) for the family of random linear forms $(\Phi_a)_{a \in (0,1]}$. For $p \in [2, \infty)$ the terms (1.39), (1.40) become:

$$a^{-\frac{1}{8}} \sup_{x \in \Lambda_k \cap K} \left[\sum_{y_1 \dots y_p \in U_a} \mathbb{E}_{U_a}^\xi(\sigma_{y_1} \dots \sigma_{y_p}) \prod_{j=1}^p \int_{S_a(y_j)} \varphi(2^k(z-x)) dz \right]^{\frac{1}{p}}, \quad (4.3)$$

$$a^{-\frac{1}{8}} 2^{2n} \sup_{x \in \Lambda_n \cap K} \left[\sum_{y_1 \dots y_p \in U_a} \mathbb{E}_{U_a}^\xi(\sigma_{y_1} \dots \sigma_{y_p}) \prod_{j=1}^p \int_{S_a(y_j)} \psi^{(i)}(2^n(z-x)) dz \right]^{\frac{1}{p}}, \quad (4.4)$$

with $(K, k) \in \mathcal{K}$. Here $\mathbb{E}_{U_a}^\xi(\sigma_{y_1} \dots \sigma_{y_p})$ is the expectation with respect to the Ising-Potts measure $\pi_{U_a}^\xi$ at critical temperature with either free or + boundary condition (see (3.4) and (3.5)).

In the following discussion we will exploit the Ising-FK relation discussed in Section 3.2 and introduce some lemmas which are useful to prove Theorem 4.1.

Let $\Lambda \subseteq \mathbb{Z}^2$ be a finite set. Define $A_{y_1 \dots y_n}^1 \subseteq \{0, 1\}^{E_\Lambda}$ the event that each open cluster of the FK model on Λ contains an even number of the points y_1, \dots, y_n , or is connected to the boundary $\partial\Lambda$. Define also $A_{y_1 \dots y_n}^{1, \infty} \subseteq \{0, 1\}^{E^2}$ the event that each open cluster of the FK model on \mathbb{Z}^2 contains an even number of the points y_1, \dots, y_n , or is infinite. Finally, let $A_{y_1 \dots y_n}^0$ be the event that each open cluster contains an even number of the points y_1, \dots, y_n . It is easy to notice that all these events are increasing, i.e. they are preserved when switching any ω_e from 0 to 1.

Lemma 4.3. *Let ϕ be the FK probability measure with $p \in [0, 1]$ and $q = 2$, and take $e^{-\beta} = (1-p)$. Then for any $n \geq 1$:*

1. $\mathbb{E}_\Lambda^+(\sigma_{y_1} \dots \sigma_{y_n}) = \phi_\Lambda^1(A_{y_1 \dots y_n}^1)$.
2. $\mathbb{E}_{\mathbb{Z}^2}^+(\sigma_{y_1} \dots \sigma_{y_n}) = \phi_{\mathbb{Z}^2}^1(A_{y_1 \dots y_n}^{1, \infty})$.
3. $\mathbb{E}_\Lambda^{\text{free}}(\sigma_{y_1} \dots \sigma_{y_n}) = \phi_\Lambda(A_{y_1 \dots y_n}^0)$.
4. $\mathbb{E}_{\mathbb{Z}^2}^{\text{free}}(\sigma_{y_1} \dots \sigma_{y_n}) = \phi_{\mathbb{Z}^2}^0(A_{y_1 \dots y_n}^0)$.

Proof. We only prove the first point in this lemma, as the other equalities can be obtained with the same arguments, using Theorem 3.8. Let $f: \sigma \mapsto \sigma_{y_1} \dots \sigma_{y_n}$, from Lemma 3.7 and (3.6) we can write

$$\begin{aligned} \mathbb{E}_\Lambda^+[f] &= \sum_{\sigma \in \Sigma_\Lambda^1} f(\sigma) \sum_{\omega \in \Omega} \mu_\Lambda^1(\sigma, \omega) \\ &= \frac{1}{Z_{\text{es}}^1} \sum_{\sigma \in \Sigma_\Lambda^1} f(\sigma) \sum_{\omega \in \Omega_\Lambda^1} \prod_{e \in E_\Lambda} [(1-p) \mathbb{1}_{\omega_e=0} + p \mathbb{1}_{\omega_e=1} \mathbb{1}_{\sigma_e=1}] \\ &= \frac{1}{Z_{\text{es}}^1} \sum_{\omega \in \Omega_\Lambda^1} (1-p)^{|E_\Lambda \setminus \eta_\Lambda(\omega)|} p^{|\eta_\Lambda(\omega)|} \sum_{\sigma \in \Sigma_\Lambda^1} f(\sigma) \prod_{e \in \eta_\Lambda(\omega)} \mathbb{1}_{\sigma_e=1} \end{aligned}$$

with $\eta_\Lambda(\omega) = \{e \in E_\Lambda \mid \omega_e = 1\}$.

Now take $\omega \in \Omega_\Lambda^1$ such that one or more of its clusters contain an odd number of points in $y_1 \dots y_n$. The sum $\sum_{\sigma \in \Sigma_\Lambda^1} f(\sigma) \prod_{e \in \eta_\Lambda(\omega)} \mathbb{1}_{\sigma_e=1}$ is zero (indeed, each odd cluster takes the values $+1$ and -1 and all terms cancel out). Conversely, if $\omega \in A_{y_1 \dots y_n}^1$, the product $\sigma_{y_1} \dots \sigma_{y_{2k}}$ in the same cluster is equal to 1. We can write then:

$$\begin{aligned} \mathbb{E}_\Lambda^+[f] &= \frac{1}{Z_{\text{es}}^1} \sum_{\omega \in \Omega_\Lambda^1} \mathbb{1}_{A_{y_1 \dots y_n}^1}(\omega) (1-p)^{|E_\Lambda \setminus \eta_\Lambda(\omega)|} p^{|\eta_\Lambda(\omega)|} \sum_{\sigma \in \Sigma_\Lambda^1} \prod_{e \in \eta_\Lambda(\omega)} \mathbb{1}_{\sigma_e=1} \\ &= \frac{1}{Z_{\text{es}}^+} \sum_{\omega \in \Omega_\Lambda^1} \mathbb{1}_{A_{y_1 \dots y_n}^1}(\omega) (1-p)^{|E_\Lambda \setminus \eta_\Lambda(\omega)|} p^{|\eta_\Lambda(\omega)|} 2^{\overline{k(\omega, \mathbb{E}^2 \setminus E_\Lambda)}} \end{aligned}$$

Here $\overline{k(\omega, \mathbb{E}^2 \setminus E_\Lambda)}$ is the number of connected clusters of ω that do not intersect $\mathbb{E}^2 \setminus E_\Lambda$.

The following equivalence between partition functions yields the result:

$$\begin{aligned} Z_{\text{es}}^1 &= \sum_{\omega \in \Omega_\Lambda^1} (1-p)^{|E_\Lambda \setminus \eta_\Lambda(\omega)|} p^{|\eta_\Lambda(\omega)|} \sum_{\sigma \in \Sigma_\Lambda^1} \prod_{e \in \eta_\Lambda(\omega)} \mathbb{1}_{\sigma_e=1} \\ &= \frac{1}{2} \sum_{\omega \in \Omega_\Lambda^1} (1-p)^{|E_\Lambda \setminus \eta_\Lambda(\omega)|} p^{|\eta_\Lambda(\omega)|} 2^{k(\omega, E_\Lambda)} = \frac{1}{2} Z_{\text{FK}}^{1, \Lambda}(p, 2). \end{aligned}$$

□

We are going to need a well-known inequality for the 2-d Ising model of Onsager [Ons44], formulated using connection probabilities for the FK model. See also [HDN11, Lemma 5.4].

Lemma 4.4. *Let $m \in \mathbb{N}$ and $B_m = [-m, m]^2 \cap \mathbb{Z}^2$. At critical temperature $p_c = 1 - e^{-\beta_c}$, there exists $C > 0$ such that:*

$$\phi_{B_m, p_c, q=2}^1(0 \leftrightarrow \partial B_m) \leq C m^{-\frac{1}{8}}.$$

The following proposition is known (see [CGN15, Prop. 3.9] for a sketch of the proof), but we give here a different (and complete) proof which employs the *pin and sum argument with hairy cycles* of A. Abdesselam [Abd16].

Proposition 4.5. *Let $p \in \mathbb{N}$. There exists $C > 0$ such that, for every $N \in \mathbb{N}$:*

$$\sum_{y_1, \dots, y_p \in U_N} \mathbb{E}_{U_N(\mathbb{Z}^2)}^\xi(\sigma_{y_1} \dots \sigma_{y_p}) \leq C (N+1)^{\frac{15}{8}p} \quad (4.5)$$

with $U_N = [0, N]^2 \cap \mathbb{Z}^2$ and $\mathbb{E}_{U_N(\mathbb{Z}^2)}^\xi$ being the expectation on either U_N or \mathbb{Z}^2 at critical temperature β_c .

Proof. The events $A_{y_1 \dots y_p}$ are increasing, and we have $A_{y_1 \dots y_p}^0 \subseteq A_{y_1 \dots y_p}^1$ when the events are on the same domain (finite or infinite). From the coupling of Lemma 4.3, and using the monotonicity properties of Lemma 3.5 it is easy to obtain $\mathbb{E}_{U_N(\mathbb{Z}^2)}^\xi \leq \mathbb{E}_{U_N}^+$. We are then left to show the inequality for this term.

We start by showing that

$$\sum_{\substack{y_1, \dots, y_p \in U_N \\ y_i \neq y_j \forall i \neq j}} \mathbb{E}_{U_N}^+(\sigma_{y_1} \dots \sigma_{y_p}) \leq C N^{\frac{15}{8}p}. \quad (4.6)$$

The event $A_{y_1 \dots y_p}^1$ of Lemma 4.3 implies that every point in $\{y_1, \dots, y_p\}$ is connected by an open path to another point in $\{y_1, \dots, y_p\}$ or to the boundary ∂U_N , which from now on is identified with the point y_0 . For every $1 \leq i \leq p$, call $\ell_i = \min_{j \geq 0, j \neq i} d(y_i, y_j)$ where $d(y_i, y_j)$ is the \mathbb{Z}^2 distance between y_i and y_j , and define $B_i = y_i + [-\ell_i/4, \ell_i/4]^2 \cap \mathbb{Z}^2$, $F = \bigcup_{i=1}^p B_i$. Notice that the graph $F \subseteq \mathbb{Z}^2$ has p disjoint components.

From Lemma 3.3, Remark 3.2 and since $\phi_{U_N}^+(A) = \sum_{\omega} \phi_{U_N}^+(A | \mathcal{F}_F)(\omega) \phi_{U_N}^+(\omega)$, we obtain

$$\mathbb{E}_{U_N}^+(\sigma_{y_1} \cdots \sigma_{y_p}) \leq \phi_{U_N}^+ \left(\bigcap_{i=1}^p \{y_i \leftrightarrow \partial B_i\} \right) \leq \prod_{i=1}^p \phi_{B_i}^+(y_i \leftrightarrow \partial B_i),$$

where we used the monotonicity property of Lemma 3.5 in the second inequality. Lemma 4.4 yields:

$$\begin{aligned} \sum_{\substack{y_1, \dots, y_p \in U_N \\ y_i \neq y_j \forall i \neq j}} \mathbb{E}_{U_N}^+(\sigma_{y_1} \cdots \sigma_{y_p}) &\lesssim \sum_{\substack{y_1, \dots, y_p \in U_N \\ y_i \neq y_j \forall i \neq j}} \prod_{i=1}^p \left[\min_{j \geq 0, j \neq i} d(y_i, y_j) \right]^{-\frac{1}{8}} \\ &\lesssim \sum_{\substack{y_1, \dots, y_p \in U_N \\ y_i \neq y_j \forall i \neq j}} \prod_{i=1}^p \sum_{\substack{j=0 \\ j \neq i}}^p d(y_i, y_j)^{-\frac{1}{8}} \\ &\lesssim \sum_{\substack{j_1 \dots j_p = 0 \\ j_i \neq i}}^p \sum_{\substack{y_1, \dots, y_p \in U_N \\ y_i \neq y_j \forall i \neq j}} d(y_1, y_{j_1})^{-\frac{1}{8}} \cdots d(y_p, y_{j_p})^{-\frac{1}{8}} \end{aligned}$$

It is easy to see that for $i \in \{1 \dots p\}$, $j \in \{0 \dots p\}$

$$\sum_{\substack{y_i \in U_N \\ y_i \neq y_j}} d(y_i, y_j)^{-\frac{1}{8}} \lesssim N^{\frac{15}{8}}, \quad (4.7)$$

there are indeed $\sim k$ points at distance k from y_j .

To estimate the term

$$\sum_{\substack{y_1, \dots, y_p \in U_N \\ y_i \neq y_j \forall i \neq j}} d(y_1, y_{j_1})^{-\frac{1}{8}} \cdots d(y_p, y_{j_p})^{-\frac{1}{8}} \quad (0 \leq j_i \leq p, j_i \neq i) \quad (4.8)$$

we need to find the right order in which to compute the sums \sum_{y_i} . We associate then (4.8) to a graph with $p+1$ vertices labelled $\{0, 1, \dots, p\}$ and p directed edges, such that $d(y_i, y_{j_i})$ corresponds an edge going from i to j_i .

Notice that every vertex in $\{1, \dots, p\}$ has exactly one edge going to a vertex in $\{0, 1, \dots, p\}$ and the vertex 0 (which is a notation for the boundary ∂U_N) has no outgoing edges. Therefore, following the directed edges starting from any vertex in $\{1, \dots, p\}$ one either ends up at the vertex 0, or enters a cycle (because every vertex except 0 has an outgoing edge). This cycle cannot be escaped, again because vertices in $\{1, \dots, p\}$ have only one outgoing edge (indeed, to every y_i there is only one y_{j_i} associated to it).

This said, we can conclude that our graph has one or more connected components, each of which can be of two distinct types:

- a tree with root in the vertex 0
- a cycle, possibly with branches attached to it (i.e. each point of the cycle can be the root of a tree).

We can then proceed to estimate every sum in (4.8) in the order given by the oriented graph, starting from the leaves. This is just a repeated application of (4.7), until we reach the root (0) or a circle. Hence every connected component with root in 0 and k edges gives a term of order $N^{\frac{15}{8}k}$. For example we can estimate the following term as follows (starting from the leaves y_1 and y_3):

$$\begin{aligned} & \sum_{\substack{y_1, y_2, y_3 \in U_N \\ y_i \neq y_j \forall i \neq j}} d(y_1, y_2)^{-\frac{1}{8}} d(y_2, y_0)^{-\frac{1}{8}} d(y_3, y_2)^{-\frac{1}{8}} \\ & \leq \sum_{y_2 \in U_N} d(y_2, y_0)^{-\frac{1}{8}} \sum_{\substack{y_1 \in U_N \\ y_1 \neq y_2}} d(y_1, y_2)^{-\frac{1}{8}} \sum_{\substack{y_3 \in U_N \\ y_3 \neq y_2}} d(y_2, y_3)^{-\frac{1}{8}} \\ & \lesssim N^{\frac{45}{8}} \end{aligned}$$

Summing on circles does not pose any additional problem: indeed one can just choose a point within the circle (call it \hat{y}_2) and sum keeping fixed both the ‘‘inbound’’ point \hat{y}_1 and the ‘‘outbound’’ point \hat{y}_3 :

$$\begin{aligned} \sum_{\substack{\hat{y}_2 \in U_N \\ \hat{y}_2 \neq \hat{y}_1, \hat{y}_2 \neq \hat{y}_3}} d(\hat{y}_1, \hat{y}_2)^{-\frac{1}{8}} d(\hat{y}_2, \hat{y}_3)^{-\frac{1}{8}} & \leq \sum_{\substack{\hat{y}_2 \in U_N \\ \hat{y}_2 \neq \hat{y}_1}} \frac{d(\hat{x}_1, \hat{x}_2)^{-\frac{1}{4}}}{2} + \sum_{\substack{\hat{y}_2 \in U_N \\ \hat{y}_2 \neq \hat{y}_3}} \frac{d(\hat{y}_2, \hat{y}_3)^{-\frac{1}{4}}}{2} \\ & \lesssim N^{2-\frac{1}{4}} \end{aligned}$$

where we used Young inequality. Then (for a circle with k edges) the sum over the remaining vertices $\hat{y}_3 \dots \hat{y}_k$ gives an estimation of order $N^{\frac{15}{8}(k-2)}$. This proves (4.6).

Now consider the general case in which two or more points coincide. At the price of a factor $p!$ we can reorder the points, and take the last $p-k$ points to be all different from each other (with $2 \leq k \leq p$). Conversely, $\{y_1, \dots, y_k\}$ can be partitioned in m subsets such that all the points in the same subset are equal: we call k_i the number of points in the i -th subset with $k = k_1 + \dots + k_m$, and therefore $m \leq k/2$. We want to show that:

$$\sum_{\substack{\bar{y}_1, \dots, \bar{y}_m \in U_N, \\ \bar{y}_i \neq y_j, i \leq m, j \in [k+1, p]}} \sum_{\substack{y_{k+1}, \dots, y_p \in U_N \\ y_i \neq y_j}} \mathbb{E}_{U_N}^+ (\sigma_{\bar{y}_1}^{k_1} \dots \sigma_{\bar{y}_m}^{k_m} \sigma_{y_{k+1}} \dots \sigma_{y_p}) \leq C N^{\frac{15}{8}p}.$$

As before we define $\ell_i = \min_{j \geq 0, j \neq i} d(y_i, y_j)$ for every $k+1 \leq i \leq p$ and $B_i = y_i + [-\ell_i/4, \ell_i/4]^2 \cap \mathbb{Z}^2$. Notice that the event $A_{y_1 \dots y_p}^1$ implies that every y_i with $i \geq k+1$ is connected by an open path to the boundary of B_i . Then using the results already obtained:

$$\begin{aligned} & \sum_{\substack{\bar{y}_1, \dots, \bar{y}_m \in U_N \\ \bar{y}_i \neq y_j, i \leq m, k+1 \leq j \leq p}} \sum_{\substack{y_{k+1}, \dots, y_p \in U_N \\ y_i \neq y_j}} \mathbb{E}_{U_N}^+ (\sigma_{\bar{y}_1}^{k_1} \dots \sigma_{\bar{y}_m}^{k_m} \sigma_{y_{k+1}} \dots \sigma_{y_p}) \\ & \lesssim N^{2m} \phi_{U_N}^+ \left(\bigcap_{i=k+1}^p \{y_i \leftrightarrow \partial B_i\} \right) \\ & \lesssim N^{2m} N^{\frac{15}{8}(p-k)} \leq N^{\frac{15}{8}p}. \end{aligned}$$

□

We are now ready to prove Theorem 4.1.

Proof. (Theorem 4.1)

By Theorem 1.46, the result is proved as soon as we can bound (4.3) and (4.4) for any even $p \geq 2$. If the domain U is bounded, we choose $\mathcal{K} = (K_n, n)_{n \in \mathbb{N}}$ as its spanning sequence, with:

$$K_n = \{x \in \mathbb{R}^2 \mid \text{dist}(x, U^c) \geq (2 + \delta) R 2^{-n}\} \quad (4.9)$$

for $\delta > 0$ and R such that (1.12) holds. If U is unbounded, it suffices to take

$$\hat{K}_n = K_n \cap \bar{B}(0, n):$$

in both cases we have a valid spanning sequence according to Definition 1.37.

We first consider (4.4). From the support properties of $\psi^{(i)}(2^n(\cdot - x))$ given in (1.12) we can restrict the sum over y_j to the set

$$\Omega_{n,x} = \{y \in U_a \mid \text{d}(y, x) < 2^{-n} R + a/\sqrt{2}\}.$$

Now we bound (4.4) separately for small and large values of n . If $2^n \geq R a^{-1}$ we have

$$\begin{aligned} & \sum_{y_1 \dots y_p \in U_a} \mathbb{E}_{U_a}^\xi(\sigma_{y_1} \dots \sigma_{y_p}) \prod_{j=1}^p \int_{S_a(y_j)} \psi^{(i)}(2^n(z-x)) \, dz \\ & \leq \sum_{y_1 \dots y_p \in \Omega_{n,x}} \prod_{j=1}^p \int_{S_a(y_j)} |\psi^{(i)}(2^n(z-x))| \, dz \\ & \leq \sum_{y_1 \dots y_p \in \Omega_{n,x}} 2^{-2pn} \lesssim 2^{-2pn}. \end{aligned}$$

This gives the estimation

$$a^{-\frac{1}{8}} 2^{2n} \sup_{x \in \Lambda_n \cap K} \left[\sum_{y_1 \dots y_p \in U_a} \mathbb{E}_{U_a}^\xi(\sigma_{y_1} \dots \sigma_{y_p}) \prod_{j=1}^p \int_{S_a(y_j)} \psi^{(i)}(2^n(z-x)) \, dz \right]^{\frac{1}{p}} \lesssim 2^{\frac{1}{8}n}.$$

Conversely, if $2^n < R a^{-1}$ we first notice that

$$\Omega_{n,x} \subseteq \tilde{U}_{a,x} = [x - 2R2^{-n}, x + 2R2^{-n}]^2 \cap a\mathbb{Z}^2$$

and then using Lemma 3.5:

$$\begin{aligned} \sum_{y_1 \dots y_p \in \tilde{U}_{a,x}} \mathbb{E}_{U_a}^\xi(\sigma_{y_1} \dots \sigma_{y_p}) & \lesssim \sum_{y_1 \dots y_p \in \tilde{U}_{a,x}} \mathbb{E}_{U_a}^+(\sigma_{y_1} \dots \sigma_{y_p}) \\ & \lesssim \sum_{y_1 \dots y_p \in \tilde{U}_{a,x}} \mathbb{E}_{\tilde{U}_{a,x}}^+(\sigma_{y_1} \dots \sigma_{y_p}) \\ & \lesssim \sum_{y_1 \dots y_p \in [-N, N]^2 \cap \mathbb{Z}^2} \mathbb{E}_{[-N, N]^2 \cap \mathbb{Z}^2}^+(\sigma_{y_1} \dots \sigma_{y_p}) \end{aligned}$$

with $N = \lfloor \frac{2R2^{-n}}{a} \rfloor$. By Proposition 4.5, we finally obtain

$$\sum_{y_1 \dots y_p \in \tilde{U}_{a,x}} \mathbb{E}_{U_a}^\xi(\sigma_{y_1} \dots \sigma_{y_p}) \lesssim a^{-\frac{15}{8}p} 2^{-\frac{15}{8}pn}$$

uniformly over x . As a result, (4.4) can be bound from above by $C 2^{\frac{1}{8}n}$ for some $C > 0$. Using the same techniques it is easy to obtain the following bound for (4.3):

$$a^{-\frac{1}{8}} \sup_{x \in \Lambda_k \cap K} \left[\sum_{y_1 \dots y_p \in U_a} \mathbb{E}_{U_a}^\xi(\sigma_{y_1} \dots \sigma_{y_p}) \prod_{j=1}^p \int_{S_a(y_j)} \varphi(2^k(z-x)) \, dz \right]^{\frac{1}{p}} \lesssim 1.$$

Therefore, by the tightness criterion of Theorem 1.46 we have shown that the family $(\Phi_a)_{a \in (0,1]}$ is tight in $\bar{B}_{p,q}^{-\frac{1}{8}-\varepsilon, \text{loc}}(U)$ for $p \geq 2$ and even, and by the equivalence of norms of Proposition 1.52 it is tight in $B_{p,q}^{-\frac{1}{8}-\varepsilon, \text{loc}}(U)$. The embedding described in Remark 1.27 yields the result for all $p \in [1, \infty]$. \square

4.1 Absence of tightness in higher-order spaces

In this section, we prove Theorem 4.2. The proof is based on the following lemma, which is a consequence of the RSW-type bounds for the FK model obtained in [HDN11].

Lemma 4.6. ([HDN11, Proposition 27]) *There exists $c > 0$ such that for any $y_1, y_2 \in \mathbb{Z}^2$ with $d(y_1, y_2) > 0$:*

$$\mathbb{E}_{\mathbb{Z}^2}^{\xi}(\sigma_{y_1} \sigma_{y_2}) \geq c d(y_1, y_2)^{-\frac{1}{4}}$$

for any boundary condition ξ .

In order to show the absence of tightness we only need the following partial converse to Proposition 4.5 for two-points correlations.

Lemma 4.7. *There exists $c > 0$ such that, for every $N \in \mathbb{N}$:*

$$\sum_{y_1, y_2 \in U_N} \mathbb{E}_{\mathbb{Z}^2}^{\xi}(\sigma_{y_1} \sigma_{y_2}) \geq c(N+1)^{\frac{15}{4}}$$

with $U_N = [0, N]^2 \cap \mathbb{Z}^2$ and $\mathbb{E}_{\mathbb{Z}^2}^{\xi}$ being the expectation on \mathbb{Z}^2 with arbitrary boundary conditions.

Proof. The result is immediate since there are $(N+1)^4$ terms in the sum, each being larger than $c(N+1)^{-\frac{1}{4}}$ for some fixed constant $c > 0$. \square

Note that in the proof of Theorem 4.2 we make use of Proposition 1.33, which allows us to easily obtain lower bounds on the Besov norm of some distribution by testing against a non-negative function.

Proof. (Proof of Theorem 4.2) We decompose the proof into three steps.

1. In this first step, we recall that for a non-negative random variable X , we have

$$\mathbb{P}\left[X > \frac{\mathbb{E}[X]}{2}\right] \geq \frac{(\mathbb{E}[X])^2}{4\mathbb{E}[X^2]}. \quad (4.10)$$

Indeed, this follows from

$$\begin{aligned} \mathbb{E}[X] &= \mathbb{E}[X \mathbb{1}_{X \leq \mathbb{E}[X]/2}] + \mathbb{E}[X \mathbb{1}_{X > \mathbb{E}[X]/2}] \\ &\leq \mathbb{E}[X]/2 + \mathbb{E}[X^2]^{\frac{1}{2}} \mathbb{P}[X > \mathbb{E}[X]/2]^{\frac{1}{2}}, \end{aligned}$$

by the Cauchy-Schwarz inequality.

2. Let η be a smooth non-negative function supported on the ball $B(0, 1)$ and such that $\eta \equiv 1$ on $B(0, 1/2)$. We set $\eta_{\lambda, x} := \lambda^{-2} \eta(\lambda^{-1}(\cdot - x))$ and

$$X_{a, \lambda} := \int_{B(0,1)} |\langle \Phi_a, \eta_{\lambda, x} \rangle| dx.$$

In this step, we show that there exists a constant $c > 0$ such that $\forall a < \lambda \in (0, 1]$,

$$\mathbb{P}\left[X_{a,\lambda} \geq c \lambda^{-\frac{1}{8}}\right] \geq c. \quad (4.11)$$

As in the proof of Theorem 4.1, we can use Proposition 4.5 to show that there exists a constant $C < \infty$ such that for every $p \in \{2, 4\}$, $a < \lambda \in (0, 1]$ and $x \in \mathbb{R}^2$,

$$\mathbb{E}[(\langle \Phi_a, \eta_{\lambda,x} \rangle)^p] \leq C \lambda^{-\frac{p}{8}}. \quad (4.12)$$

By a similar reasoning, we obtain from Lemma 4.7 that there exists a constant $c > 0$ such that for every $a < \lambda \in (0, 1]$ and $x \in \mathbb{R}^2$,

$$\mathbb{E}[(\langle \Phi_a, \eta_{\lambda,x} \rangle)^2] \geq c \lambda^{-\frac{1}{4}}. \quad (4.13)$$

Combining (4.10), (4.12) with $p = 4$ and (4.13), we deduce that $\forall a < \lambda \in (0, 1]$ and $x \in \mathbb{R}^2$,

$$\mathbb{P}\left[|\langle \Phi_a, \eta_{\lambda,x} \rangle| \geq \frac{\sqrt{c}}{2} \lambda^{-\frac{1}{8}}\right] \geq \frac{c}{C}.$$

In particular, after reducing the constant $c > 0$ as necessary, we obtain that for every $a < \lambda \in (0, 1]$ and $x \in \mathbb{R}^2$,

$$\mathbb{E}[|\langle \Phi_a, \eta_{\lambda,x} \rangle|] \geq c \lambda^{-\frac{1}{8}}, \quad (4.14)$$

and thus that

$$\mathbb{E}[X_{a,\lambda}] \geq c \lambda^{-\frac{1}{8}}.$$

Using (4.12) with $p = 2$ and Jensen's inequality, we also have, for every $a < \lambda \in (0, 1]$,

$$\mathbb{E}[X_{a,\lambda}^2] \leq C \lambda^{-\frac{1}{4}}.$$

We therefore obtain (4.11) by another application of (4.10).

3. Let $\alpha > -\frac{1}{8}$, and let $\bar{\Phi}$ be a possible limit point of the family $(\Phi_a)_{a \in (0,1]}$. Passing to the limit along a subsequence in (4.11), we get that for every $\lambda \in (0, 1]$,

$$\mathbb{P}\left[\int_{B(0,1)} |\langle \bar{\Phi}, \eta_{\lambda,x} \rangle| dx \geq c \lambda^{-\frac{1}{8}}\right] \geq c. \quad (4.15)$$

By the embedding of Besov spaces discussed in Remark 1.27 and Proposition 1.52, in order to prove Theorem 4.2 it suffices to show that $\bar{\Phi} \notin \bar{\mathcal{B}}_{1,\infty}^{\alpha,\text{loc}}(\mathbb{R}^2)$ with positive probability.

Let then χ be a non-negative smooth function of compact support such that $\chi \equiv 1$ on $B(0, 2)$. By Proposition 1.33, there exists a constant $c' > 0$ such that for every $\lambda \in (0, 1]$,

$$\|\chi \bar{\Phi}\|_{B_{1,\infty}^\alpha} \geq c' \lambda^{-\alpha} \int_{\mathbb{R}^2} |\langle \chi \bar{\Phi}, \eta_{\lambda,x} \rangle| dx \geq c' \lambda^{-\alpha} \int_{B(0,1)} |\langle \bar{\Phi}, \eta_{\lambda,x} \rangle| dx.$$

Combining this with (4.15) yields

$$\mathbb{P}\left[\|\chi \bar{\Phi}\|_{B_{1,\infty}^\alpha} \geq c c' \lambda^{-\alpha - \frac{1}{8}}\right] \geq c.$$

Since $\alpha > -\frac{1}{8}$, letting λ tend to 0 gives

$$\mathbb{P}[\|\chi \bar{\Phi}\|_{B_{1,\infty}^\alpha} = +\infty] \geq c > 0,$$

which completes the proof. \square

Chapter 5

Quasi-linear paracontrolled PDEs

5.1 Introduction

In this chapter we study a class of general quasilinear equations of which one of the simplest examples is the following parabolic SPDE:

$$\partial_t u(t, x) - a(u(t, x))\Delta u(t, x) = \xi(x), \quad u(0, x) = u_0(x), \quad x \in \mathbb{T}^2, t \geq 0, \quad (5.1)$$

with $a: \mathbb{R} \rightarrow [\lambda, 1]$ for $\lambda > 0$ a uniformly bounded C^3 diffusion coefficient, and $\|a^{(k)}\|_{L^\infty} \leq 1$ for $k=0, \dots, 3$. We assume that $\xi \in \mathcal{C}^{\alpha-2}(\mathbb{T}^2)$ with $2/3 < \alpha < 1$ where $\mathcal{C}^\alpha(\mathbb{T}^2)$ is the Besov space introduced in Chapter 1 (this would apply for example to the space white noise on \mathbb{T}^2). In this case, we expect (by analogy with the Schauder estimates of Section 1.2.3) that $u(t, \cdot) \in \mathcal{C}^\alpha(\mathbb{T}^2)$. Therefore, as one can see from the paraproduct theory introduced in Section 1.2.1, the product $a(u(t, \cdot))\Delta u(t, \cdot)$ cannot be well posed whenever $2\alpha - 2 < 0$. Equation (5.1) is a quasilinear generalisation of the two-dimensional periodic parabolic Anderson model (PAM).

The techniques developed in this section allow to deal with a class of equations of the form

$$\partial_t u(t, x) - a_1(u(t, x))\Delta u(t, x) = a_2(u(t, x))\xi(x) \quad t \geq 0, x \in \mathbb{T}^2, \quad (5.2)$$

and the more general equations

$$a_3(u(t, x))\partial_t u(t, x) - a_1(u(t, x))\Delta u(t, x) = \xi(a_2(u(t, x)), t, x), \quad x \in \mathbb{T}^2, t \geq 0, \quad (5.3)$$

where a_1, a_2, a_3 are sufficiently smooth non-degenerate coefficients, $a_1, a_3: \mathbb{R} \rightarrow [\lambda, 1]$, $a_2: \mathbb{R} \rightarrow [-L, L]$ $\lambda \in (0, 1), L > 0$ and $\xi(z, t, x)$ is a Gaussian process with covariance

$$\mathbb{E}[\xi(z, t, x)\xi(z', t', x')] = F(z, z')Q(t - t', x - x'), \quad x, x' \in \mathbb{T}^2, t, t', z, z' \in \mathbb{R},$$

with F a smooth function and Q a distribution of parabolic regularity $\rho > -4/3$. This includes as a special case the space white noise discussed before, but we could consider a time white noise with a regular dependence on the space variable, or some noise which is mildly irregular in space and time. Moreover, with our technique we are able to treat equations with matrix-valued coefficients like

$$\partial_t u(t, x) - a_{ij}(u(t, x))\frac{\partial^2}{\partial x_i \partial x_j} u(t, x) = g(u(t, x))\xi, \quad t \geq 0, x \in \mathbb{T}^2, \quad (5.4)$$

provided the template problem associated to it (see (5.5) and Remark 5.17) is uniformly parabolic.

For the sake of clarity and simplicity we will discuss mainly the basic example (5.1) since this contains already most of the technical difficulties. The study of this equation will be carried on in full detail in Section 5.3, employing the theory developed in Section 5.2. We will be able to treat equation (5.2) in Section 5.4 and equation (5.3) in Section 5.5 with slight modifications of the results of Section 5.2.

Our approach can be described as follows. For an equation of the form (5.2) we consider at first a *parametric* “template” problem, with *constant* coefficients η_1, η_2 replacing $a_1(u), a_2(u)$:

$$\partial_t \vartheta(\eta, t, x) - \eta_1 \Delta \vartheta(\eta, t, x) = \eta_2 \xi(t, x) \quad (5.5)$$

with $\eta = (\eta_1, \eta_2) \in \mathbb{R}^2$, $\eta_1 \in [\lambda, 1]$, $\eta_2 \in [-L, L]$ for $\lambda \in (0, 1)$, $L > 0$. We introduce a *nonlinear* paraproduct Π_{\prec} and formulate the Ansatz:

$$u = \Pi_{\prec}(a(u), \vartheta) + u^\sharp,$$

where it will be shown that u^\sharp defines a more regular remainder term which solves a standard PDE. With this decomposition the equation can be treated along the lines of standard paracontrolled calculus presented in Section 1.2, and all the arguments introduced there can be extended in a straightforward manner to the quasilinear setting.

A notation shortcut widely used in this chapter is to write $\int_{x,y}$ for integrals on \mathbb{T}^2 or \mathbb{R} with respect to the measures dx and dy without specifying the integration bounds, whenever this does not create ambiguity. We will write as usual $\sum_{i \sim j} = \sum_{i \geq -1} \sum_{j=i-1}^{i+1}$ and $\sum_{i \gtrsim j} = \sum_{i > j+1}$ when summing over Littlewood-Paley blocks. We will also use the notation $\delta f_{sy}^{tx} = f(t, x) - f(s, y)$ and $\delta_\tau f_{sy}^{tx} = f(s, y) + \tau(f(t, x) - f(s, y))$ for $\tau \in [0, 1]$.

5.2 Nonlinear paracontrolled calculus

In this section we present the main tools of nonlinear paradifferential calculus, which is an extension of linear paracontrolled calculus presented in Section 1.2.

5.2.1 Nonlinear paraproducts

Let $g^{[i]}: [0, T] \times \mathbb{T}^2 \rightarrow \mathbb{R}$ for $i \in \{1, \dots, n\}$ and $h: \mathbb{R}^n \times [0, T] \times \mathbb{T}^2 \rightarrow \mathbb{R}$ be smooth functions. We write $g = (g^{[1]}, \dots, g^{[n]})$ and decompose $h(g(\cdot), \cdot)$ via *nonlinear paraproducts* as follows. Recall from Remark 1.4 that K_q is the kernel associated to the Littlewood-Paley block Δ_q and P_j is the kernel associated with the operator $\sum_{q < j-1} \Delta_q$. Define

$$\Pi_{\prec}(g, h)(t, x) := \sum_{q \geq 1} \int_{y,z} P_{q,x}(y) K_{q,x}(z) h(g(t, y), t, z) \quad (5.6)$$

$$\Pi_{\circ}(g, h)(t, x) := \sum_{k \geq -1} \sum_{|k-q| \leq 1} \int_{y,z} K_{k,x}(y) K_{q,x}(z) h(g(t, y), t, z) \quad (5.7)$$

$$\Pi_{\succ}(g, h)(t, x) := \sum_{k \geq 1} \int_{y,z} K_{k,x}(z) P_{k,x}(y) h(g(t, z), t, y). \quad (5.8)$$

Let V be a Banach space of functions on $[0, T] \times \mathbb{T}^2$ (typically $V = \mathcal{C}_T^\alpha$ or $V = \dot{\mathcal{L}}_T^\alpha$ as defined in Section 1.2.2). We fix a compact set $K \subset \mathbb{R}^n$ and introduce the notation

$$\|F\|_{C_\eta^k V} = \sup_{\eta \in K} \sup_{|a| \leq k} \|\partial_\eta^a F(\eta, \cdot)\|_V, \quad (5.9)$$

for functions $F: (\eta, t, x) \rightarrow \mathbb{R}$ depending on $(t, x) \in \mathbb{R} \times \mathbb{T}^2$ and on an additional parameter $\eta = (\eta_1, \dots, \eta_n)$. We note $C_\eta^k V$ the Banach space with norm (5.9). For $g = (g^{[1]}, \dots, g^{[n]})$ we use the notation $\|g\|_V := \sum_j \|g^{[j]}\|_V$.

Definitions (5.6), (5.7), (5.8) yield a map

$$(g, h) \mapsto \Pi_\diamond(g, h) := \Pi_\prec(g, h) + \Pi_\circ(g, h) + \Pi_\succ(g, h) = h(g(\cdot), \cdot) \quad (5.10)$$

that can be uniquely extended to

$$\Pi_\diamond: \mathcal{C}_T^\rho \times C_\eta^2 \mathcal{C}_T^\gamma \rightarrow \mathcal{C}_T^{\gamma \wedge \rho} \quad \rho \in (0, 1), \gamma \in \mathbb{R}, \rho + \gamma > 0$$

thanks to the following bounds.

Lemma 5.1. (Nonlinear paraproduct estimates)

Let $g, g_1, g_2: [0, T] \times \mathbb{T}^2 \rightarrow K \subset \mathbb{R}^n$, $g, g_1, g_2 \in \mathcal{C}_T^\rho$ for some $\rho \in (0, 1)$, and $h \in C_\eta^2 \mathcal{C}_T^\gamma$ for any $\gamma \in \mathbb{R}$. Then

$$\|\Pi_\prec(g, h)\|_{\mathcal{C}_T^\gamma} \lesssim \|h\|_{C_\eta \mathcal{C}_T^\gamma}, \quad \|\Pi_\succ(g, h)\|_{\mathcal{C}_T^{\rho \wedge (\rho + \gamma)}} \lesssim \|g\|_{\mathcal{C}_T^\rho} \|h\|_{C_\eta^1 \mathcal{C}_T^\gamma},$$

and

$$\begin{aligned} \|\Pi_\prec(g_1, h) - \Pi_\prec(g_2, h)\|_{\mathcal{C}_T^\gamma} &\lesssim \|g_1 - g_2\|_{C_T L^\infty} \|h\|_{C_\eta^1 \mathcal{C}_T^\gamma}, \\ \|\Pi_\succ(g_1, h) - \Pi_\succ(g_2, h)\|_{\mathcal{C}_T^{\rho \wedge (\rho + \gamma)}} &\lesssim \|g_1 - g_2\|_{C_T L^\infty} (\|g_1\|_{\mathcal{C}_T^\rho} + \|g_2\|_{\mathcal{C}_T^\rho}) \|h\|_{C_\eta^2 \mathcal{C}_T^\gamma} \\ &\quad + \|g_1 - g_2\|_{\mathcal{C}_T^\rho} \|h\|_{C_\eta^1 \mathcal{C}_T^\gamma}. \end{aligned}$$

Moreover if $\rho + \gamma > 0$ we have also

$$\begin{aligned} \|\Pi_\circ(g, h)\|_{\mathcal{C}_T^{\rho + \gamma}} &\lesssim \|h\|_{C_\eta^1 \mathcal{C}_T^\gamma} \|g\|_{\mathcal{C}_T^\rho}, \\ \|\Pi_\circ(g_1, h) - \Pi_\circ(g_2, h)\|_{\mathcal{C}_T^{\rho + \gamma}} &\lesssim \|g_1 - g_2\|_{C_T L^\infty} (\|g_1\|_{\mathcal{C}_T^\rho} + \|g_2\|_{\mathcal{C}_T^\rho}) \|h\|_{C_\eta^2 \mathcal{C}_T^\gamma} \\ &\quad + \|g_1 - g_2\|_{\mathcal{C}_T^\rho} \|h\|_{C_\eta^1 \mathcal{C}_T^\gamma}. \end{aligned}$$

In particular if $\rho + \gamma > 0$ the composition $\Pi_\diamond(g, h) = h(g(\cdot), \cdot)$ is linear in h and locally Lipschitz in g :

$$\begin{aligned} \|\Pi_\diamond(g, h)\|_{\mathcal{C}_T^\gamma} &\lesssim \|h\|_{C_\eta^1 \mathcal{C}_T^\gamma} \|g\|_{\mathcal{C}_T^\rho}, \\ \|\Pi_\diamond(g_1, h) - \Pi_\diamond(g_2, h)\|_{\mathcal{C}_T^\gamma} &\lesssim \|g_1 - g_2\|_{\mathcal{C}_T^\rho} (1 + \|g_1\|_{\mathcal{C}_T^\rho} + \|g_2\|_{\mathcal{C}_T^\rho}) \|h\|_{C_\eta^2 \mathcal{C}_T^\gamma}. \end{aligned}$$

Remark 5.2. The inequalities above are a direct generalization of Bony's paraproduct estimations of Proposition 1.54.

Proof. (Lemma 5.1) We consider $g, g_1, g_2: [0, T] \times \mathbb{T}^2 \rightarrow K \subset \mathbb{R}$, as the extension to vector-valued g, g_1, g_2 is straightforward. Due to the support properties of the Fourier transforms of the kernels K_q and P_q , it is easy to see that

$$\int_{y,z} P_{q,x}(y) K_{q,x}(z) h(g(t, y), t, z)$$

has Fourier transform compactly supported in an annulus $2^q \mathcal{A}$, and the same holds for

$$\int_{y,z} K_{k,x}(z) P_{k,x}(y) h(g(t, z), t, y),$$

while the resonant term

$$\sum_{q=k-1}^{k+1} \int_{y,z} K_{k,x}(y)K_{q,x}(z)h(g(t,y),t,z)$$

has Fourier transform supported in a ball of radius 2^q . This allows us to estimate Besov norms of paraproducts via L^∞ norms of Littlewood-Paley blocks in the usual way (as done in Section 1.2). Using the fact that

$$\begin{aligned} \|\Delta_k h(g(t,y),t,\cdot)\|_{L^\infty} &\lesssim 2^{-\gamma k} \|h\|_{C_\eta^\gamma \mathcal{L}_T^\gamma}, \\ \|\Delta_k h(g(t,y),t,\cdot) - \Delta_k h(g(t,y'),t,\cdot)\|_{L^\infty} &\lesssim 2^{-\gamma k} \|h\|_{C_\eta^1 \mathcal{L}_T^\gamma} \|g\|_{\mathcal{L}_T^\rho} |y - y'|^\rho, \end{aligned}$$

and

$$\|\Delta_k h(g_1(t,y),t,\cdot) - \Delta_k h(g_2(t,y),t,\cdot)\|_{L^\infty} \lesssim 2^{-\gamma k} \|h\|_{C_\eta^1 \mathcal{L}_T^\gamma} \|g_1 - g_2\|_{C_T L^\infty}.$$

we obtain the bounds on $\Pi_{<}(g,h)$, $\Pi_{>}(g,h)$, $\Pi_{\circ}(g,h)$ and $\Pi_{<}(g_1,h) - \Pi_{<}(g_2,h)$.

We proceed to estimate the term $\Pi_{>}(g_1,h) - \Pi_{>}(g_2,h)$. We will use the following notation for brevity:

$$\delta g_{2z}^{1y} := g_1(t,y) - g_2(t,z) \quad \text{and} \quad \delta_\tau g_{2z}^{1y} := g_2(t,z) + \tau(g_1(t,y) - g_2(t,z)).$$

Then

$$\begin{aligned} & \left| \int_{y,z} K_{k,x}(z)P_{k,x}(y)[h(g_1(t,z),t,y) - h(g_2(t,z),t,y)] \right| \\ &= \left| \int_{y,z,\tau \in [0,1]} K_{k,x}(z)P_{k,x}(y)[\partial_\eta h(\delta_\tau g_{2z}^{1z},t,y)\delta g_{2z}^{1z} - \partial_\eta h(\delta_\tau g_{2x}^{1x},t,y)\delta g_{2x}^{1x}] \right| \\ &\lesssim \left| \int_{y,z,t \in [0,1], \sigma \in [0,1]} K_{k,x}(z)P_{k,x}(y)\partial_\eta^2 h(\delta_\sigma(\delta_\tau g_2^1)^z,t,y)(\delta g_{1x}^{1z} - \delta g_{2x}^{2z})\delta g_{2z}^{1z} \right| \\ &\quad + \left| \int_{y,z,\tau \in [0,1]} K_{k,x}(z)P_{k,x}(y)\partial_\eta h(\delta_\tau g_{2x}^{1x},t,y)(\delta g_{2z}^{1z} - \delta g_{2x}^{1x}) \right| \\ &\lesssim \|g_1 - g_2\|_{C_T L^\infty} (\|g_1\|_{C_T \mathcal{L}^\rho} + \|g_2\|_{C_T \mathcal{L}^\rho}) \|h\|_{C_\eta^2 C_T \mathcal{L}^\gamma} 2^{-\rho k} \sum_{q < k-1} 2^{-\gamma q} \\ &\quad + \|g_1 - g_2\|_{C_T \mathcal{L}^\rho} \|h\|_{C_\eta^1 C_T \mathcal{L}^\gamma} 2^{-\rho k} \sum_{q < k-1} 2^{-\gamma q}. \end{aligned}$$

With the same reasoning we can bound the norm of $\Pi_{\circ}(g_1,h) - \Pi_{\circ}(g_2,h)$. \square

We introduce a time-smoothed version of $\Pi_{<}$, in the exact same way as done with linear paraproducts in Section 1.2.4. Let

$$\Pi_{\leftarrow}(g,h)(t,x) := \sum_{i \geq 1} \int_{y,s} Q_{i,t}(s)P_{i,x}(y)(\Delta_i h(g(s,y),t,\cdot))(x), \quad (5.11)$$

with $Q \in C_c^\infty(\mathbb{R})$ with total mass 1, and $Q_{i,t}(s) := 2^{2i}Q(2^i(t-s))$. In (5.11) we use the convention that a continuous function $t \mapsto g(t)$ on \mathbb{R}_+ is extended to \mathbb{R} by defining $g(t) = g(0)$ for $t \leq 0$. This preserves the Hölder norms of index in $[0,1]$. The modified nonlinear paraproduct enjoys similar bounds to the regular one.

Lemma 5.3. *Let $g, g_1, g_2: [0, T] \times \mathbb{T}^2 \rightarrow K \subset \mathbb{R}^n$, $g, g_1, g_2 \in C_T L^\infty$ and $h \in C_\eta^1 \mathcal{L}_T^\gamma$ for $\gamma \in (0, 2)$. Then*

$$\|\Pi_{\leftarrow}(g,h)\|_{\mathcal{L}_T^\gamma} \lesssim \|h\|_{C_\eta^\gamma \mathcal{L}_T^\gamma} \quad \text{and} \quad \|\Pi_{\leftarrow}(g,h)\|_{\mathcal{L}_T^\gamma} \lesssim \|h\|_{C_\eta^\gamma \mathcal{L}_T^\gamma}.$$

Moreover, $\Pi_{\leftarrow}(g, h)$ is linear in h and:

$$\begin{aligned}\|\Pi_{\leftarrow}(g_1, h) - \Pi_{\leftarrow}(g_2, h)\|_{\mathcal{E}_T^\gamma} &\lesssim \|g_1 - g_2\|_{C_T L^\infty} \|h\|_{C_\eta^1 \mathcal{E}_T^\gamma}, \\ \|\Pi_{\leftarrow}(g_1, h) - \Pi_{\leftarrow}(g_2, h)\|_{\dot{\mathcal{Z}}_T^\gamma} &\lesssim \|g_1 - g_2\|_{C_T L^\infty} \|h\|_{C_\eta^1 \dot{\mathcal{Z}}_T^\gamma}.\end{aligned}$$

Proof. In order to bound $\|\Pi_{\leftarrow}(g, h)\|_{\dot{\mathcal{Z}}_T^\gamma}$, we note that the norm $\|\Pi_{\leftarrow}(g, h)\|_{C_T \mathcal{E}^\gamma}$ can be treated in the same way as in Lemma 5.1, and $\|\Pi_{\leftarrow}(g, h)\|_{C_T^{\gamma/2} \mathcal{E}^0}$ can be estimated as follows:

$$\begin{aligned}&\|\Delta_j \Pi_{\leftarrow}(g, h)(t_1) - \Delta_j \Pi_{\leftarrow}(g, h)(t_2)\|_{L^\infty} \\ &\lesssim \sup_x \left| \int_z K_{j,x}(z) \sum_{i \sim j} \int_{y,s} Q_{i,t_1}(s) P_{i,z}(y) [\Delta_i h(g(s, y), t_1, z) - \Delta_i h(g(s, y), t_2, z)] \right| \\ &\quad + \sup_x \left| \int_z K_{j,x}(z) \sum_{i \sim j} \int_{y,s} [Q_{i,t_1}(s) - Q_{i,t_2}(s)] P_{i,z}(y) \Delta_i h(g(s, y), t_2, z) \right| \\ &\lesssim \|h(\cdot, t_1, \cdot) - h(\cdot, t_2, \cdot)\|_{C_\eta \mathcal{E}^0} + |t_1 - t_2|^{\gamma/2} \|h\|_{C_\eta \mathcal{E}_T^\gamma}.\end{aligned}$$

The second inequality can be obtained easily with the same techniques used so far. \square

Remark 5.4. Using the Fourier support properties of the kernel $P_{q,x}(\cdot)$ it is easy to see that $\forall x \in \mathbb{T}^2, \forall q \geq 0: \int_y P_{q,x}(y) = \int_y K_{-1,x}(y) = 1$ and $\int_y K_{q,x}(y) = 0$. Then for a constant function $g(t, x) = \bar{g}$ one can write

$$\Pi_{\diamond}(\bar{g}, h) = \Pi_{\leftarrow}(\bar{g}, h) + \sum_{q \leq 0} \Delta_q h(\bar{g}, \cdot)$$

and using the fact that the kernel Q in the definition of Π_{\leftarrow} has mass 1, we have

$$\Pi_{\leftarrow}(\bar{g}, h) = \Pi_{\leftarrow}(\bar{g}, h).$$

5.2.2 Nonlinear commutator

The next technical ingredient is a commutator lemma between the non-linear paraproduct of (5.11) and the standard resonant product. Since it will be needed below to analyse a term of the form $\Pi_{\leftarrow}(g, h) \circ \Delta \Pi_{\leftarrow}(g, h)$, we will specialise our discussion to this specific structure. Notice that in the following the various space-time operators act pointwise in the parameter η , in the sense that, for example:

$$(h \circ \Delta h)(\eta, t, x) = (h(\eta, t, \cdot) \circ \Delta h(\eta, t, \cdot))(x).$$

Lemma 5.5. Define $\Lambda: C^\infty([0, T], \mathbb{T}^2) \times C_\eta^2 C^\infty([0, T], \mathbb{T}^2) \rightarrow C^\infty([0, T], \mathbb{T}^2)$ by

$$\Lambda(g, h) := [\Pi_{\leftarrow}(g, h) \circ \Delta \Pi_{\leftarrow}(g, h)] - \Pi_{\diamond}(g, h \circ \Delta h).$$

Then for all $\rho \in (0, 1), \gamma < 1, \varepsilon > 0$ such that $2\gamma - 2 + \rho - \varepsilon > 0$ and $g, g_1, g_2: [0, T] \times \mathbb{T}^2 \rightarrow K \subset \mathbb{R}^n$, we have

$$\begin{aligned}\|\Lambda(g, h)\|_{\mathcal{E}_T^{2\gamma-2+\rho-\varepsilon}} &\lesssim (1 + \|g\|_{\dot{\mathcal{L}}_T^\rho}) \|h\|_{C_\eta^1 \mathcal{E}_T^\gamma}^2, \\ \|\Lambda(g_1, h) - \Lambda(g_2, h)\|_{\mathcal{E}_T^{2\gamma-2+\rho-\varepsilon}} &\lesssim \|g_1 - g_2\|_{C_T L^\infty} (\|g_1\|_{\dot{\mathcal{L}}_T^\rho} + \|g_2\|_{\dot{\mathcal{L}}_T^\rho}) \|h\|_{C_\eta^2 \mathcal{E}_T^\gamma} \\ &\quad + \|g_1 - g_2\|_{\dot{\mathcal{L}}_T^\rho} \|h\|_{C_\eta^1 \mathcal{E}_T^\gamma}^2.\end{aligned}$$

As a consequence Λ can be uniquely extended to a locally Lipschitz function

$$\Lambda: \dot{\mathcal{L}}_T^\rho \times C_\eta^2 \mathcal{E}_T^\gamma \rightarrow \mathcal{E}_T^{2\gamma-2+\rho-\varepsilon}.$$

Proof. As for the proof of Lemma 5.3, we consider scalar-valued functions g, g_1, g_2 not to burden the notation. We can approximate $\Pi_{\leftarrow}(g, h) \circ \Delta \Pi_{\leftarrow}(g, h)$ with its value for a fixed $g = g(t, z)$, to obtain

$$\begin{aligned} & \Delta_q \Lambda(g, h)(t, x) = \\ &= \int_z K_{q,x}(z) (\Pi_{\leftarrow}(g(t, z), h) \circ \Delta \Pi_{\leftarrow}(g(t, z), h))(t, z) \\ & \quad - \int_z K_{q,x}(z) \Pi_{\diamond}(g, h \circ \Delta h)(t, z) \\ & \quad + \int_z K_{q,x}(z) (\Pi_{\leftarrow}(g, h) \circ \Delta (\Pi_{\leftarrow}(g, h) - \Pi_{\leftarrow}(g(t, z), h)))(t, z) \end{aligned} \quad (5.12)$$

$$+ \int_z K_{q,x}(z) ((\Pi_{\leftarrow}(g, h) - \Pi_{\leftarrow}(g(t, z), h)) \circ \Delta \Pi_{\leftarrow}(g(t, z), h))(t, z). \quad (5.13)$$

We start considering the first two terms in the expression above. Note that

$$\Delta_j \Delta \Pi_{\leftarrow}(g(t, z), h)(t, z) = \Delta_j \Pi_{\leftarrow}(g(t, z), \Delta h)(t, z)$$

and that by Remark 5.4 $\forall i \geq 2$

$$\Delta_i \Pi_{\leftarrow}(g(t, z), h)(t, z) = \Delta_i h(g(t, z), t, \cdot)(z).$$

This yields

$$\begin{aligned} & (\Pi_{\leftarrow}(g(t, z), h) \circ \Delta \Pi_{\leftarrow}(g(t, z), h))(t, z) - \Pi_{\diamond}(g, h \circ \Delta h)(t, z) \\ &= \sum_{i \sim j} \Delta_i \Pi_{\leftarrow}(g(t, z), h)(t, z) \Delta_j \Pi_{\leftarrow}(g(t, z), \Delta h)(t, z) \\ & \quad - \sum_{i \sim j} \Delta_i h(g(t, z), t, \cdot)(z) \Delta_j \Delta h(g(t, z), t, \cdot)(z) \\ &= \sum_{q=1}^2 \Delta_1 \Delta_q h(g(t, z), t, \cdot)(z) \sum_{\ell=1}^2 \Delta_1 \Delta_\ell \Delta h(g(t, z), t, \cdot)(z) \end{aligned} \quad (5.14)$$

$$- \sum_{i \sim j} \mathbb{1}_{i \leq 1} \mathbb{1}_{j \leq 1} \Delta_i h(g(t, z), t, \cdot)(z) \Delta_j \Delta h(g(t, z), t, \cdot)(z). \quad (5.15)$$

Note that fixing the value of the function g in (5.14) makes it localized in Fourier space: i.e. $\exists n_0 \in \mathbb{N}$ such that $\forall k \geq n_0$:

$$\int_z K_{k,x}(z) \sum_{q=1}^2 \Delta_1 \Delta_q h(g(t, x), t, \cdot)(z) \sum_{\ell=1}^2 \Delta_1 \Delta_\ell \Delta h(g(t, x), t, \cdot)(z) = 0$$

and this last term can be easily bound in L^∞ by $\|h\|_{C_n \mathcal{C}_T^\gamma}^2$. Thus, we add and subtract to (5.14) the terms

$$\begin{aligned} & \sum_{q=1}^2 \Delta_1 \Delta_q h(g(t, x), t, \cdot)(z) \sum_{\ell=1}^2 \Delta_1 \Delta_\ell \Delta h(g(t, x), t, \cdot)(z), \\ & \sum_{q=1}^2 \Delta_1 \Delta_q h(g(t, z), t, \cdot)(z) \sum_{\ell=1}^2 \Delta_1 \Delta_\ell \Delta h(g(t, x), t, \cdot)(z) \end{aligned}$$

and we are left estimating

$$\int_z K_{k,x}(z) \sum_{q=1}^2 \Delta_1 \Delta_q h(g(t, z), t, \cdot)(z) \sum_{\ell=1}^2 \Delta_1 [\Delta_\ell \Delta h(g(t, z), t, \cdot) - \Delta_\ell \Delta h(g(t, x), t, \cdot)](z)$$

and

$$\int_z K_{k,x}(z) \sum_{q=1}^2 \Delta_1 [\Delta_q h(g(t,z), t, \cdot) - \Delta_q h(g(t,x), t, \cdot)](z) \sum_{\ell=1}^2 \Delta_1 \Delta_\ell \Delta h(g(t,x), t, \cdot)(z)$$

As already noted in the proof of Lemma 5.1 we have

$$\begin{aligned} \|\Delta_q h(g(t,z), t, \cdot) - \Delta_q h(g(t,x), t, \cdot)\|_{L^\infty} &\lesssim 2^{-\gamma q} \|h\|_{C_\eta^1 \mathcal{E}_T^\gamma} \|g\|_{\mathcal{E}_T^\rho} |z-x|^\rho, \\ \|\Delta_\ell \Delta h(g(t,z), t, \cdot) - \Delta_\ell \Delta h(g(t,x), t, \cdot)\|_{L^\infty} &\lesssim 2^{(2-\gamma)\ell} \|h\|_{C_\eta^1 \mathcal{E}_T^\gamma} \|g\|_{\mathcal{E}_T^\rho} |z-x|^\rho \end{aligned}$$

and this, together with the estimation

$$\int_z |K_{k,x}(z)| |z-x|^\rho \lesssim 2^{-k\rho}$$

allows to bound the terms above in L^∞ with $2^{-k\rho} \|g\|_{\mathcal{E}_T^\rho} \|h\|_{C_\eta^1 \mathcal{E}_T^\gamma}^2$. Summing up, we have seen that (5.14) can be bound in \mathcal{E}_T^ρ by $(1 + \|g\|_{\mathcal{E}_T^\rho}) \|h\|_{C_\eta^1 \mathcal{E}_T^\gamma}^2$, and this gives a bound on $\mathcal{E}_T^{2\gamma-2+\rho-\varepsilon}$ for $\gamma < 1$, $\varepsilon > 0$. The exact same reasoning can be applied to (5.15) to obtain the same estimation.

Consider now (5.12) and (5.13). We obtain

$$\begin{aligned} &\int_z K_{q,x}(z) [(\Pi_{\leftarrow}(g, h) - \Pi_{\leftarrow}(g(t,z), h)) \circ \Delta \Pi_{\leftarrow}(g(t,z), h)](t, z) \\ &= \int_z K_{q,x}(z) \sum_{i \sim j \geq q} (\Delta_i \Pi_{\leftarrow}(g, h)(t, z) - \Delta_i \Pi_{\leftarrow}(g(t,z), h)(t, z)) \Delta_j \Delta \Pi_{\leftarrow}(g(t,z), h)(t, z). \end{aligned}$$

Using Lemma 5.3 we have

$$|\Delta \Delta_j \Pi_{\leftarrow}(g(t,z), h)(t, z)| \lesssim 2^{(2-\gamma)j} \|h\|_{C_\eta \mathcal{E}_T^\gamma}.$$

Lemma 5.6 gives

$$|\Delta_i (\Pi_{\leftarrow}(g, h) - \Pi_{\leftarrow}(g(t,z), h))(t, z)| \lesssim 2^{-(\gamma+\rho-\varepsilon)i} \|g\|_{\mathcal{E}_T^\rho} \|h\|_{C_\eta^1 \mathcal{E}_T^\gamma},$$

and thus (5.13) is bounded by $2^{-(2\gamma+\rho-2-\varepsilon)q} \|g\|_{\mathcal{E}_T^\rho} \|h\|_{\mathcal{E}_T^\gamma}^2$. We can easily bound (5.12) in the same way, and this proves the first inequality.

The second result of this lemma can be obtained following the same reasoning as above, noting that $\forall k, q \geq -1$:

$$\begin{aligned} &\int_z K_{k,x}(z) [\Delta_q h(g_1(t,z), t, \cdot)(z) - \Delta_q h(g_1(t,x), t, \cdot)(z) + \\ &\quad + \Delta_q h(g_2(t,x), t, \cdot)(z) - \Delta_q h(g_2(t,z), t, \cdot)(z)] \\ &\lesssim 2^{-\rho k} (\|g_1 - g_2\|_{\mathcal{E}_T^\rho} \|h\|_{C_\eta^1 \mathcal{E}_T^\gamma} + \|g_1 - g_2\|_{C_T L^\infty} (\|g_1\|_{\mathcal{E}_T^\rho} + \|g_2\|_{\mathcal{E}_T^\rho})) \|h\|_{C_\eta^2 \mathcal{E}_T^\gamma} \end{aligned}$$

and using the estimations of Lemma 5.6.

The extension of Λ to $\mathcal{E}_T^\rho \times C_\eta^2 \mathcal{E}_T^\gamma$ is standard (see e.g. the proof of the commutator lemma [GIP15, Lemma 2.4]). \square

Lemma 5.6. *Let us introduce the shortcut notation*

$$\wp_i(g, h)(t, z) := \Delta_i \Pi_{\leftarrow}(g, h)(t, z) - \Delta_i \Pi_{\leftarrow}(g(t,z), h)(t, z)$$

Then, with the same assumptions of Lemma 5.5, we have

$$|\wp_i(g, h)(t, z)| \lesssim 2^{(\varepsilon-\rho-\gamma)i} \|g\|_{\mathcal{E}_T^\rho} \|h\|_{C_\eta^1 \mathcal{E}_T^\gamma}$$

and

$$\lesssim 2^{(\varepsilon-\rho-\gamma)i} \left[\|g_1 - g_2\|_{\dot{\mathcal{L}}_T^\rho} \|h\|_{C_\eta^1 \mathcal{E}_T^\gamma} + \|g_1 - g_2\|_{C_T L^\infty} (\|g_1\|_{\dot{\mathcal{L}}_T^\rho} + \|g_2\|_{\dot{\mathcal{L}}_T^\rho}) \|h\|_{C_\eta^2 \mathcal{E}_T^\gamma} \right].$$

Proof.

$$\begin{aligned} & [\Delta_i \Pi_{\prec}(g, h) - \Delta_i \Pi_{\prec}(g(t, z), h)](t, z) \\ &= \sum_{k \sim i} \int_{\substack{x, y \\ s, \tau}} K_{i, z}(x) Q_{k, t}(s) P_{k-1, x}(y) \partial_\eta \Delta_k h(\delta_\tau g_{tz}^{sy}, t, x) (\delta g_{ty}^{sy} + \delta g_{tz}^{ty}) \\ &\lesssim \sum_{k \sim i} \int_{\substack{x, y \\ s, \tau}} |K_{i, z}(x) Q_{k, t}(s) P_{k-1, x}(y)| \|\partial_\eta \Delta_k h\|_{C_T L^\infty} |t - s|^{(\rho-\varepsilon)/2} \|g\|_{C_T^{\rho/2-\varepsilon/2} L^\infty} \\ &\quad + \sum_{k \sim i} \int_{\substack{x, y \\ s, \tau}} |K_{i, z}(x) Q_{k, t}(s) P_{k-1, x}(y)| \|\partial_\eta \Delta_k h\|_{C_T L^\infty} |y - z|^\rho \|g\|_{\mathcal{E}_T^\rho} \\ &\lesssim 2^{-(\rho-\varepsilon)i} 2^{-\gamma i} \|\partial_\eta h\|_{\mathcal{E}_T^\gamma} \left(\|g\|_{C_T^\rho \mathcal{E}^0} + \|g\|_{\mathcal{E}_T^\rho} \right) \end{aligned}$$

where we used the notation $\delta g_{tz}^{sy} = g(s, y) - g(t, z)$, $\delta_\tau g_{tz}^{sy} = g(t, z) + \tau(g(s, y) - g(t, z))$ and Lemma 1.59. This proves the first bound.

The second inequality can be obtained in the same way with the techniques already used here and in Lemma 5.1. \square

5.2.3 Approximate paradifferential problem

In this section we present the last ingredient of our theory, i.e. the construction of an approximate solution to the equation

$$(\partial_t - g \prec \Delta)u = f, \quad u(0, \cdot) = 0, \quad (5.16)$$

with data $f \in \mathcal{E}^{\gamma-2}$ and $g: [0, T] \times \mathbb{T}^2 \rightarrow [\lambda, 1]$ $\lambda \in (0, 1)$, $g \in \dot{\mathcal{L}}_T^\rho$, for some fixed $\rho, \gamma \in (0, 1)$.

We first introduce the operator \mathcal{L} acting on functions of (η, t, x) with $\eta = (\eta_1, \dots, \eta_m)$, $\eta_1 \in [\lambda, 1]$ as

$$(\mathcal{L}U)(\eta, t, x) := \partial_t U(\eta, t, x) - \eta_1 \Delta U(\eta, t, x). \quad (5.17)$$

We will also use the notation $\mathcal{L}_{\eta_1} := \partial_t - \eta_1 \Delta$.

Observe that if u does not depend on η we can define

$$\Pi_{\prec}(g, \mathcal{L})u := \Pi_{\prec}(g, \mathcal{L}u) \quad (5.18)$$

and from definition (5.6) with $h = \mathcal{L}u$ we obtain $\Pi_{\prec}(g, \mathcal{L})u = \partial_t u - g \prec \Delta u$ that is the operator appearing in (5.16).

We can describe the commutation between the differential operator \mathcal{L} and the paraproduct $\Pi_{\prec}(g, \cdot)$ via the following estimate:

Lemma 5.7. *Let $\rho \in (0, 1)$, $\gamma \in \mathbb{R}$. Let $U \in C_\eta^2 \mathcal{E}_T^\gamma$ and $g: [0, T] \times \mathbb{T}^2 \rightarrow [\lambda, 1]$, $g \in \dot{\mathcal{L}}_T^\rho$. Define*

$$\Psi(g, U) := R_1 + R_2$$

with R_1 and R_2 as in (5.21), (5.22). Then for every $\varepsilon > 0$

$$\|\Psi(g, U)\|_{\mathcal{E}_T^{\rho+\gamma-2-\varepsilon}} \lesssim (1 + \|g\|_{C_T L^\infty}) \|g\|_{\dot{\mathcal{L}}_T^\rho} \|U\|_{C_\eta^1 \mathcal{E}_T^\gamma}. \quad (5.19)$$

Moreover, $\Psi(g, U)$ is linear in U and

$$\|\Psi(g_1, U) - \Psi(g_2, U)\|_{\mathcal{E}_T^{\rho+\gamma-2-\varepsilon}} \lesssim \|g_1 - g_2\|_{\dot{\mathcal{L}}_T^\rho} (1 + \|g_1\|_{\dot{\mathcal{L}}_T^\rho} + \|g_2\|_{\dot{\mathcal{L}}_T^\rho}) \|U\|_{C_\eta^2 \mathcal{E}_T^\gamma}.$$

In particular, we have

$$\Psi(g, U) = \Pi_{\leftarrow}(g, \mathcal{L}U) - \Pi_{\leftarrow}(g, \mathcal{L})\Pi_{\leftarrow}(g, U) \in C_T \mathcal{L}^{\rho+\gamma-2-\varepsilon} \quad (5.20)$$

whenever this expression makes sense.

Proof. We start considering $g \in C^\infty([0, T], \mathbb{T}^2)$ and $U \in C_\eta^2 C^\infty([0, T], \mathbb{T}^2)$, and prove (5.20) in this setting. Note that $\Pi_{\leftarrow}(g(t, y), \mathcal{L}_{g(t, y)}U) = \mathcal{L}U(g(t, y))$. As a consequence, we can estimate

$$\begin{aligned} & \Pi_{\leftarrow}(g, \mathcal{L}U)(t, x) - \Pi_{\leftarrow}(g, \mathcal{L}\Pi_{\leftarrow}(g, U))(t, x) \\ &= \Pi_{\leftarrow}(g, \mathcal{L}U)(t, x) - \sum_k \int_y P_{k, x}(y) (\mathcal{L}_{g(t, y)} \Delta_k \Pi_{\leftarrow}(g, U))(t, x) \\ &= \Pi_{\leftarrow}(g, \mathcal{L}U)(t, x) - \sum_k \int_y P_{k, x}(y) (\partial_t \Delta_k \Pi_{\leftarrow}(g, U))(t, x) \\ & \quad + \sum_k \int_y P_{k, x}(y) g(t, y) (\Delta \Delta_k \Pi_{\leftarrow}(g, U))(t, x) \\ &= \Pi_{\leftarrow}(g, \mathcal{L}U - \partial_t U)(t, x) \\ & \quad + \sum_k \int_y P_{k, x}(y) g(t, y) (\Delta_k \Pi_{\leftarrow}(g, \Delta U))(t, x) \\ & \quad + \sum_k \int_y P_{k, x}(y) g(t, y) (\Delta_k [\Delta, \Pi_{\leftarrow}(g, \cdot)]U)(t, x) \\ & \quad - \sum_k \int_y P_{k, x}(y) (\Delta_k [\partial_t, \Pi_{\leftarrow}(g, \cdot)]U)(t, x) \end{aligned}$$

with the commutators

$$\begin{aligned} [\Delta, \Pi_{\leftarrow}(g, \cdot)]U &:= \Delta \Pi_{\leftarrow}(g, U) - \Pi_{\leftarrow}(g, \Delta U), \\ [\partial_t, \Pi_{\leftarrow}(g, \cdot)]U &:= \partial_t \Delta_k \Pi_{\leftarrow}(g, U) - \Delta_k \Pi_{\leftarrow}(g, \partial_t U). \end{aligned}$$

We have

$$\Pi_{\leftarrow}(g, \mathcal{L}U - \partial_t U)(t, x) + \sum_k \int_y P_{k, x}(y) g(t, y) (\Delta_k \Pi_{\leftarrow}(g, \Delta U))(t, x) = R_1(t, x)$$

with the definition

$$R_1(t, x) :=$$

$$\sum_{k, i} \int_{\substack{y', z \\ y, z}} P_{k, x}(y) K_{k, x}(z) P_{i, z}(y') Q_{i, t}(s) [g(t, y) - g(s, y')] \Delta \Delta_i U(g(s, y'), t, z) \quad (5.21)$$

and

$$\sum_k \int_y P_{k, x}(y) [g(t, y) (\Delta_k [\Delta, \Pi_{\leftarrow}(g, \cdot)]U)(t, x) - (\Delta_k [\partial_t, \Pi_{\leftarrow}(g, \cdot)]U)(t, x)] = R_2(t, x)$$

with the definition

$$\begin{aligned} R_2(t, x) &:= \sum_{k, i} \int_{y, y', s} P_{k, x}(y) K_{k, x}(z) Q_{i, t}(s) g(t, y) \Delta P_{i, z}(y') \Delta_i U(g(s, y'), t, z) \\ & \quad + 2 \sum_{k, i} \int_{y, y', s} P_{k, x}(y) K_{k, x}(z) Q_{i, t}(s) g(t, y) \nabla P_{i, z}(y') \nabla \Delta_i U(g(s, y'), t, z) \\ & \quad - \sum_{k, i} \int_{y, y', s} P_{k, x}(y) K_{k, x}(z) \partial_t Q_{i, t}(s) P_{i, z}(y') \Delta_i U(g(s, y'), t, z). \end{aligned} \quad (5.22)$$

Indeed:

$$\begin{aligned}
([\partial_t, \Pi_{\leftarrow}(g, \cdot)]U)(t, x) &= \sum_i \int_{y,s} (\partial_t Q_{i,t})(s) P_{i,x}(y) (\Delta_i U(g(s, y), t, x)), \\
([\Delta, \Pi_{\leftarrow}(g, \cdot)]U)(t, x) &= \sum_i \int_{y,s} Q_{i,t}(s) \Delta P_{i,x}(y) (\Delta_i U(g(s, y), t, x)) \\
&\quad + 2 \sum_i \int_{y,s} Q_{i,t}(s) \nabla P_{i,x}(y) (\nabla \Delta_i U(g(s, y), t, x)).
\end{aligned} \tag{5.23}$$

This shows that (5.20) holds for smooth functions.

With the techniques used in Lemma 5.6 we can estimate

$$|\Delta_q R_1(t, x)| \lesssim \sum_{k \sim q} (2^{-(\rho-\varepsilon)k} \|g\|_{C_T^{\rho/2} \mathcal{C}_0} + 2^{-\rho k} \|g\|_{\mathcal{C}_T^\rho}) 2^{(2-\gamma)k} \|U\|_{C_\eta \mathcal{C}_T^\gamma}.$$

By the spectral support properties of the commutators we have that

$$\|[\Delta, \Pi_{\leftarrow}(g, \cdot)]U\|_{\mathcal{C}_T^{\gamma+\rho-2}} \lesssim \|g\|_{\mathcal{C}_T^\rho} \|U\|_{C_\eta^1 \mathcal{C}_T^\gamma},$$

and

$$\|\Delta_q [\partial_t, \Pi_{\leftarrow}(g, \cdot)]U\|_{C_T L^\infty} \lesssim (2^{(2+\varepsilon-\rho-\gamma)q} \|g\|_{C_T^{\rho/2} \mathcal{C}_0} + 2^{(2-\rho-\gamma)q} \|g\|_{\mathcal{C}_T^\rho}) \|U\|_{C_\eta^1 \mathcal{C}_T^\gamma}.$$

This yields

$$\|R_2\|_{\mathcal{C}_T^{\gamma+\rho-2-\varepsilon}} \lesssim (1 + \|g\|_{C_T L^\infty}) \|g\|_{\mathcal{C}_T^\rho} \|U\|_{C_\eta^1 \mathcal{C}_T^\gamma}.$$

We have so far proved (5.19) and then (5.20) follows by continuity. The local Lipschitz dependence on g can be obtained via similar computations. \square

Remark 5.8. If f does not depend on η we can take $\eta = \eta_1 \in [\lambda, 1]$ and consider the parametric problem

$$(\partial_t - \eta_1 \Delta) U_f(\eta_1, t) = f, \quad U_f(\eta_1, 0) = 0, \tag{5.24}$$

which is solved by

$$U_f(\eta_1, t) = \int_0^t e^{\eta_1 \Delta(t-s)} f ds.$$

Remark that

$$\begin{aligned}
\partial_{\eta_1} U_f(\eta_1, t) &= \int_0^t e^{\eta_1 \Delta(t-s)} (t-s) \Delta f ds \quad \text{and} \\
\partial_{\eta_1}^2 U_f(\eta_1, t) &= \int_0^t e^{\eta_1 \Delta(t-s)} (t-s)^2 \Delta^2 f ds.
\end{aligned}$$

We have, thanks to the well-known Schauder estimates of Lemma 1.61 (since $\eta_1 \geq \lambda$):

$$\|U_f\|_{C_{\eta_1}^2 \dot{\mathcal{C}}_T^\gamma} := \sup_{k=0,1,2} \sup_{\eta_1 \in [\lambda, 1]} \|\partial_{\eta_1}^k U_f(\eta_1)\|_{\dot{\mathcal{C}}_T^\gamma} \lesssim \|f\|_{\mathcal{C}_T^{\gamma-2}} \tag{5.25}$$

We define then

$$u(t, x) := \Pi_{\leftarrow}(g, U_f)(t, x) \tag{5.26}$$

and observe that $u(t, x)$ is an approximate solution of equation (5.16), indeed

$$\begin{aligned}
(\partial_t - g \prec \Delta)u &= \Pi_{\leftarrow}(g, \mathcal{L} \Pi_{\leftarrow}(g, U_f)) = \Pi_{\leftarrow}(g, \mathcal{L} U_f) - \Psi(g, U_f) \\
&= f - \Psi(g, U_f)
\end{aligned}$$

and the estimation in Lemma 5.7 together with the bound (5.25) yield immediately the following inequality:

$$\|\Psi(g, U_f)\|_{\mathcal{C}_T^{\rho+\gamma-2-\varepsilon}} \lesssim \|g\|_{\dot{\mathcal{L}}_T^\rho} (1 + \|g\|_{C_T L^\infty}) \|f\|_{\mathcal{C}_T^{\gamma-2}}. \quad (5.27)$$

5.3 Solution theory for quasi-linear equations

Let us state one simple result on equation (5.1) that can be obtained via the theory developed in the previous section:

Theorem 5.9. *Fix $2/3 < \alpha < 1$. Let $\xi \in \mathcal{C}^{\alpha-2}(\mathbb{T}^2)$ be a space white noise with zero average on the torus, $u_0 \in \mathcal{C}^\alpha$ an initial condition and $a: \mathbb{R} \rightarrow [\lambda, 1]$ for some $\lambda > 0$, $a \in C^3(\mathbb{R})$ and $\|a^{(k)}\|_{L^\infty} \leq 1 \ \forall k = 0, \dots, 3$. Let $(\xi_\varepsilon, u_{0,\varepsilon})_{\varepsilon>0}$ be a family of smooth approximations of ξ, u_0 obtained by convolution with a rescaled smoothing kernel and u_ε the classical solution to the Cauchy problem*

$$\partial_t u_\varepsilon - a(u_\varepsilon) \Delta u_\varepsilon = \xi_\varepsilon + \sigma_\varepsilon \frac{a'(u_\varepsilon)}{a(u_\varepsilon)^2}, \quad u(0) = u_{0,\varepsilon}. \quad (5.28)$$

Then we can choose a sequence of constants $(\sigma_\varepsilon)_{\varepsilon>0}$ and a random time $T > 0$ in such a way that the family of r.v. $(u_\varepsilon)_{\varepsilon>0} \subseteq \dot{\mathcal{L}}_T^\alpha(\mathbb{T}^2)$ converges almost surely as $\varepsilon \rightarrow 0$ to a random element $u \in \dot{\mathcal{L}}_T^\alpha(\mathbb{T}^2)$, where $\dot{\mathcal{L}}_T^\alpha$ is the parabolic space $C([0, T], \mathcal{C}^\alpha(\mathbb{T}^2)) \cap C^{\alpha/2}([0, T], \mathcal{C}^0(\mathbb{T}^2))$.

This element can be characterised as the solution to a paracontrolled singular SPDE (see Theorem 5.11 for more details).

In order to devise a suitable formulation of eq. (5.1) and obtain a theory with $u \in \mathcal{C}^\alpha$ we start decomposing the non-linear diffusion term in the l.h.s. with the help of Bony's paraproducts and write

$$\partial_t u - a(u) \prec \Delta u = \xi + \Phi(u) \quad (5.29)$$

with

$$\Phi(u) := a(u) \circ \Delta u + a(u) \succ \Delta u \quad (5.30)$$

where \prec, \succ are standard paraproducts and \circ denotes the resonant product (see Section 1.2 for the definitions). By Proposition 1.54 the l.h.s. of (5.29) is always well defined, irrespectively of the regularity of the function u , and the problem becomes that of controlling the resonant product $a(u) \circ \Delta u$ appearing in the r.h.s. . The two key points of the analysis put forward below is that this term can be expected to be of regularity $2\alpha - 2 > \alpha - 2$, better than the leading term $\xi \in \mathcal{C}^{\alpha-2}$, and that the differential operator of (5.29) can be approximately inverted using Lemma 5.7.

5.3.1 Paracontrolled structure

In order to give a meaning to the PDE in (5.29) with initial condition $u_0 \in \mathcal{C}^\alpha$, our initial goal will be to get informations on solutions $\theta = \theta(g)$ of the equation

$$\partial_t \theta - g \prec \Delta \theta = \xi,$$

for a fixed $g \in \mathcal{C}_T^\alpha$, $2/3 < \alpha < 1$, $g \in [\lambda, 1]$. Using the results of Section 5.2.3, we consider to this effect the parametric problem

$$(\partial_t - \eta\Delta)\vartheta(\eta, t) = \xi,$$

for $\eta \in [\lambda, 1]$. We will consider the stationary solution of this problem which has the form

$$\vartheta(\eta, x) = \int_0^\infty e^{\eta\Delta s} \xi \, ds = (-\eta\Delta)^{-1} \xi \quad (5.31)$$

and in order for (5.31) to be well defined we impose that the noise ξ has zero mean on \mathbb{T}^2 (this is a simplifying assumption which can be easily removed, e.g. at the price of adding a linear term to the equation). We can control (5.31) by bounding its Littlewood-Paley blocks with a Bernstein lemma for distributions with compactly supported Fourier transform (Lemma 1.8) to obtain:

$$\|\vartheta\|_{C_\eta^2 \dot{\mathcal{C}}_T^\alpha} = \|\vartheta\|_{C_\eta^2 \mathcal{C}_T^\alpha} \lesssim \|\xi\|_{\mathcal{C}^{\alpha-2}}. \quad (5.32)$$

We define now for every $t \in [0, T]$

$$\theta(t, x) := \Pi_{\leftarrow}(a(u), \vartheta).$$

Thanks to paraproducts estimates of Lemma 5.3 we have the bound

$$\|\theta\|_{\dot{\mathcal{C}}_T^\alpha} \lesssim \|\vartheta\|_{C_\eta^2 \dot{\mathcal{C}}_T^\alpha} \lesssim \|\xi\|_{\mathcal{C}^{\alpha-2}}.$$

We observe that this definition together with Lemma 5.7 gives

$$\partial_t \theta - a(u) \prec \Delta \theta = \xi - \Psi(a(u), \vartheta)$$

with $\|\Psi(a(u), \vartheta)\|_{\mathcal{C}_T^{2\alpha-2-\varepsilon}} \lesssim \|a(u)\|_{\dot{\mathcal{C}}_T^\alpha}^2 \|\xi\|_{\mathcal{C}_T^{\alpha-2}}$. We expect then $\Psi(a(u), \vartheta)$ to be bounded in $\mathcal{C}_T^{2\alpha-2-\varepsilon}$ for any $\varepsilon > 0$. At this point let us introduce the Ansatz

$$\begin{aligned} u &= \theta + u^\sharp \\ &= \Pi_{\leftarrow}(a(u), \vartheta) + u^\sharp. \end{aligned} \quad (5.33)$$

Remark 5.10. Notice that we are not making any assumption on the existence of such u , which is the subject of Section 5.3.2. Our aim here is to find the equation that a couple $(u, u^\sharp) \in \mathcal{C}_T^\alpha \times \mathcal{C}_T^{2\alpha}$ verifying (5.33) must solve, in order for u to solve (5.29).

Observe that

$$\begin{aligned} \partial_t u - a(u) \prec \Delta u &= (\partial_t - a(u) \prec \Delta)\theta + (\partial_t - a(u) \prec \Delta)u^\sharp \\ &= \xi + (\partial_t - a(u) \prec \Delta)u^\sharp - \Psi(a(u), \vartheta). \end{aligned}$$

It follows that u^\sharp must solve

$$\begin{cases} (\partial_t - a(u) \prec \Delta)u^\sharp = \Phi(u) + \Psi(a(u), \vartheta) \\ u^\sharp(t=0) = u_0^\sharp := u_0 - \Pi_{\leftarrow}(a(u_0), \vartheta)(t=0) \in \mathcal{C}^\alpha \end{cases} \quad (5.34)$$

with $\Phi(u) = a(u) \circ \Delta u + a(u) \succ \Delta u$, and if we can make sense of the resonant term $a(u) \circ \Delta u$, it is reasonable to expect $u^\sharp(t, \cdot) \in \mathcal{C}^{2\alpha} \forall t \in (0, T]$. Indeed, take $U^\sharp := U_Q$ to be the solution of

$$\mathcal{L}U^\sharp(\eta) := (\partial_t - \eta\Delta)U^\sharp(\eta) = Q, \quad U^\sharp(\eta, t=0) = 0 \quad (5.35)$$

for some $Q = Q(u^\sharp)$ to be determined and $\eta \in [\lambda, 1]$. Using again Lemma 5.7 as shown in Remark 5.8 we have

$$(\partial_t - a(u) \prec \Delta) \Pi_{\prec}(a(u), U^\sharp) = Q(u^\sharp) - \Psi(a(u), U^\sharp).$$

For $\eta \in [\lambda, 1]$ we define $\mathcal{P}_t u_0^\sharp(\eta) := e^{\eta \Delta t} u_0^\sharp$ so that $\mathcal{L}(\mathcal{P}_t u_0^\sharp) = 0$, with \mathcal{L} as in (5.17).

We set

$$u^\sharp := \Pi_{\prec}(a(u), U^\sharp) + \Pi_{\prec}(a(u), \mathcal{P} u_0^\sharp). \quad (5.36)$$

Taking

$$Q(u^\sharp) := \Phi(u) + \Psi(a(u), \vartheta) + \Psi(a(u), U^\sharp) + \Psi(a(u), \mathcal{P} u_0^\sharp),$$

we obtain that U^\sharp solves equation (5.35) if and only if u^\sharp solves equation (5.34). As we will see, $Q(u^\sharp)(t)$ belongs to $\mathcal{C}^{2\alpha-2} \forall t \in (0, T]$ but not uniformly as $t \rightarrow 0$. However it belongs to $\mathcal{C}^{\alpha-2}$ uniformly as $t \rightarrow 0$.

It remains to control the resonant term $a(u) \circ \Delta u$ appearing in $\Phi(u)$. We have

$$a(u) \circ \Delta u = a(u) \circ \Delta \theta + a(u) \circ \Delta u^\sharp.$$

By parilinearization (see Theorem 1.66) $a(u) = a'(u) \prec u + R_a(u)$ with

$$\|R_a(u)\|_{\mathcal{C}^{2\alpha}} \lesssim 1 + \|u\|_{\mathcal{C}^{\frac{\alpha}{T}}},$$

and then

$$a(u) \circ \Delta \theta = (a'(u) \prec u) \circ \Delta \theta + R_a(u) \circ \Delta \theta.$$

We can write

$$a(u) \circ \Delta \theta = a'(u)(u \circ \Delta \theta) + C(a'(u), u, \Delta \theta) + R_a(u) \circ \Delta \theta$$

where $C(a'(u), u, \Delta \theta) := ((a'(u) \prec u) \circ \Delta \theta) - a'(u)(u \circ \Delta \theta)$ is the commutator of Lemma 1.64. In order to estimate $C(a'(u), u, \Delta \theta)$ we can bound $a'(u)$, recalling that $\alpha \in (0, 1)$, as

$$\|a'(u)\|_{\mathcal{C}^{\frac{\alpha}{T}}} \lesssim \|a''\|_{L^\infty} \|u\|_{\mathcal{C}^{\frac{\alpha}{T}}},$$

Ansatz (5.33) gives

$$a(u) \circ \Delta \theta = a'(u)(\theta \circ \Delta \theta) + a'(u)(u^\sharp \circ \Delta \theta) + C(a'(u), u, \Delta \theta) + R_a(u) \circ \Delta \theta.$$

Summarizing, we have:

$$\begin{aligned} \Phi(u) &= a'(u)(\theta \circ \Delta \theta) + a(u) \succ \Delta u + a'(u)(u^\sharp \circ \Delta \theta) \\ &\quad + C(a'(u), u, \Delta \theta) + R_a(u) \circ \Delta \theta + a(u) \circ \Delta u^\sharp \end{aligned}$$

Thanks to the nonlinear commutator (Lemma 5.5), we can decompose the resonant term $\theta \circ \Delta \theta$ to obtain

$$\begin{aligned} \Phi(u) &= a(u) \succ \Delta u + a'(u)(u^\sharp \circ \Delta \theta) + C(a'(u), u, \Delta \theta) + R_a(u) \circ \Delta \theta \\ &\quad + a'(u) \Lambda(a(u), \vartheta) + a'(u) \Pi_{\diamond}(a(u), \Theta_2) + a(u) \circ \Delta u^\sharp \end{aligned}$$

and $\Lambda(a(u), \vartheta) \in C_T L^\infty$ if $u \in \dot{\mathcal{L}}_T^\alpha$. Here we defined

$$\Theta_2(\eta, x) := (\vartheta \circ \Delta \vartheta)(\eta, x) = \sum_{i \sim j} \Delta_i \vartheta(\eta, \cdot)(x) \Delta_j [\Delta \vartheta(\eta, \cdot)](x) \quad (5.37)$$

Finally, recalling the decomposition of u^\sharp in two terms (5.36) we obtain

$$\Phi(u) = a'(u) \Pi_{\diamond}(a(u), \Theta_2) + \Phi_1(u) + \Phi_2(u)$$

where

$$\begin{aligned}\Phi_1(u) &:= a(u) \succ \Delta u + C(a'(u), u, \Delta\theta) + R_a(u) \circ \Delta\theta + a'(u)\Lambda(a(u), \vartheta) \\ &\quad + a'(u)(\Pi_{\leftarrow}(a(u), U^\sharp) \circ \Delta\theta) + a(u) \circ \Delta\Pi_{\leftarrow}(a(u), U^\sharp), \\ \Phi_2(u) &:= a'(u)(\Pi_{\leftarrow}(a(u), \mathcal{P}u_0^\sharp) \circ \Delta\theta) + a(u) \circ \Delta\Pi_{\leftarrow}(a(u), \mathcal{P}u_0^\sharp).\end{aligned}$$

Thanks to Lemma 5.1 the terms $a'(u)\Pi_{\diamond}(a(u), \Theta_2)$ and $\Phi_1(u)$ can be estimated in $\mathcal{C}_T^{2\alpha-2}$, provided $\Theta_2 \in C_\eta^2 \mathcal{C}_T^{2\alpha-2}$ (see Section 5.3.3). On the other hand the term $\Phi_2(u)(t)$ can be estimated in $\mathcal{C}^{2\alpha-2}$ only for strictly positive times $t > 0$ due to the lack of regularity of the initial condition u_0^\sharp which a priori lives only in \mathcal{C}^α .

Note moreover that the specific form of Φ allows to deduce that if we replace Θ_2 by $\tilde{\Theta}_2 = \Theta_2 - H$ with $H \in C_\eta^2 \mathcal{C}_T^{2\alpha-2}$ then this is equivalent to consider an equation for u of the form

$$\partial_t u(t, x) - a(u(t, x))\Delta u(t, x) = \xi(x) - a'(u(t, x))H(a(u(t, x)), t, x).$$

Let us resume this long discussion in the following theorem:

Theorem 5.11. (paracontrolled equation) *Assume that $\xi \in \mathcal{C}^0$, $u_0 \in \mathcal{C}^2$, $H \in C_\eta^2 \mathcal{C}_T^0$. $u \in C_T^1 \mathcal{C}^2$ is the classical solution to the equation*

$$\partial_t u(t, x) - a(u(t, x))\Delta u(t, x) = \xi(x) - a'(u(t, x))H(a(u(t, x)), t, x), \quad u(0) = u_0, \quad (5.38)$$

up to time $T > 0$ if

$$u = \Pi_{\leftarrow}(a(u), \vartheta + U^\sharp + \mathcal{P}u_0^\sharp),$$

where ϑ is the solution to equation (5.31) and U^\sharp is the solution to the PDE

$$(\partial_t - \eta\Delta)U^\sharp(\eta) = F(u, U^\sharp, u_0^\sharp) \quad U^\sharp(\eta, 0) = 0 \quad \eta \in [\lambda, 1] \quad (5.39)$$

with

$$\begin{aligned}F(u, U^\sharp, u_0) &= a'(u)\Pi_{\diamond}(a(u), \Theta_2) + \Phi_1(u) + \Phi_2(u) + \Psi(a(u), \vartheta) \\ &\quad + \Psi(a(u), U^\sharp) + \Psi(a(u), \mathcal{P}u_0^\sharp)\end{aligned}$$

and $\Theta_2 = \vartheta \circ \Delta\vartheta - H$.

Definition 5.12. (enhanced noises) *For any $\alpha \in \mathbb{R}$ we define the enhanced noise $\mathcal{X}^\alpha \subseteq C_\eta^2 \mathcal{C}^\alpha \times C_\eta^2 \mathcal{C}^{2\alpha-2}$ as the closure of the image of the map*

$$(\rho, H) \in C_\eta^2 \mathcal{C}^2 \times C_\eta^2 \mathcal{C}^0 \mapsto J(\rho, H) = (\rho, \rho \circ \Delta\rho - H) \in C_\eta^2 \mathcal{C}^2 \times C_\eta^2 \mathcal{C}^0$$

(in the topology of $C_\eta^2 \mathcal{C}^\alpha \times C_\eta^2 \mathcal{C}^{2\alpha-2}$).

5.3.2 Local well-posedness

The main result of this section is the local well-posedness for equations (5.33) and (5.39) when $(\vartheta, \Theta_2) \in \mathcal{X}^\alpha$ and $u_0 \in \mathcal{C}^\alpha$ for $2/3 < \alpha < 1$. This yields a unique solution to (5.38), thanks to Theorem 5.11.

Theorem 5.13. *Let $\alpha \in (2/3, 1)$. Then for any $(\vartheta, \Theta_2) \in \mathcal{X}^\alpha$ and $u_0 \in \mathcal{C}^\alpha$ there exists a time $T > 0$ depending only on $\|(\vartheta, \Theta_2)\|_{\mathcal{X}^\alpha}$ and $\|u_0\|_\alpha$ up to which the system of equations (5.33) and (5.39) has a unique solution $(u, U^\sharp) \in \dot{\mathcal{Z}}_T^\alpha \times C_\eta^2 \dot{\mathcal{Z}}_T^{2\delta}$ for all $\delta < \alpha$ such that $2\delta + \alpha > 2$. For any fixed $\tau > 0$ there exist a ball $B_\tau \subseteq \mathcal{C}^\alpha \times \mathcal{X}^\alpha$ such that the solution map*

$$\Sigma_\tau: (u_0, \vartheta, \Theta_2) \in B_\tau \mapsto (u, U^\sharp) \in \dot{\mathcal{Z}}_\tau^\alpha \times C_\eta^2 \dot{\mathcal{Z}}_\tau^{2\delta}$$

is well defined and Lipschitz continuous in the data.

Remark 5.14. The proof is based on a Picard fixed point argument. In order to have a contraction map on a small time interval $[0, T]$, we carry on our analysis of U^\sharp in the space $C_\eta^2 \dot{\mathcal{L}}_T^{2\delta} \supset C_\eta^2 \dot{\mathcal{L}}_T^{2\alpha}$ and make use of the estimates on time-weighted spaces of Lemma 1.61 to obtain a factor T^ε for some $\varepsilon > 0$ small enough.

Proof. (Th. 5.13) Let $\mathcal{G}_T = \dot{\mathcal{L}}_T^\alpha \times C_\eta^2 \dot{\mathcal{L}}_T^{2\delta}$. We introduce the map

$$\Gamma: (u, U^\sharp) \in \mathcal{G}_T \mapsto (\Gamma_u(u, U^\sharp), \Gamma_{U^\sharp}(u, U^\sharp)) \in \mathcal{G}_T$$

by

$$\Gamma_u(u, U^\sharp) := \Pi_{\leftarrow}(a(u), \vartheta) + \Pi_{\leftarrow}(a(u), \Gamma_{U^\sharp}(u, U^\sharp)) + \Pi_{\leftarrow}(a(u), \mathcal{P}u_0^\sharp)$$

and

$$(\partial_t - \eta\Delta)\Gamma_{U^\sharp}(u, U^\sharp)(\eta) = F(u, U^\sharp, u_0^\sharp), \quad \Gamma_{U^\sharp}(u, U^\sharp)(\eta)(0) = 0, \quad \eta \in [\lambda, 1],$$

We will establish that this map is a contraction in the space \mathcal{G}_T .

First, we have to show that there exists a ball $B \subset \mathcal{G}_T$ such that $\Gamma(B) \subseteq B$. We have the bound $\|\mathcal{P}u_0^\sharp\|_{C_\eta^1 \dot{\mathcal{L}}_T^\alpha} \lesssim \|u_0^\sharp\|_{\mathcal{E}_T^\alpha}$. It is easy to obtain, using the estimates of Section 5.2.1 and Lemma 1.61:

$$\begin{aligned} & \left\| \int_0^T e^{-\eta\Delta(t-s)} [\Phi_1(u) + \Psi(a(u), \vartheta) + \Psi(a(u), U^\sharp) + \Psi(a(u), \mathcal{P}u_0^\sharp)]_s ds \right\|_{C_\eta^2 \dot{\mathcal{L}}_T^{2\delta}} \\ & \lesssim T^\kappa (1 + \|u\|_{\dot{\mathcal{L}}_T^\alpha})^4 (1 + \|\xi\|_{\mathcal{E}^{\alpha-2}})^2 \|u_0^\sharp\|_{\mathcal{E}^\alpha} \left(1 + \|U^\sharp\|_{C_\eta^2 \dot{\mathcal{L}}_T^{2\delta}} \right) \end{aligned}$$

for some $\kappa > 0$.

By the assumption that $(\vartheta, \Theta_2) \in \mathcal{X}^\alpha$ we deduce that there exists $M > 0$ such that $\|\Theta_2\|_{C_\eta^2 \mathcal{E}_T^{2\alpha-2}} \leq M$. We have

$$\left\| \int_0^T e^{-\eta\Delta(t-s)} [a'(u)\Pi_\diamond(a(u), \Theta_2)]_s ds \right\|_{C_\eta^2 \dot{\mathcal{L}}_T^{2\delta}} \lesssim T^{\alpha-\delta} (1 + \|u\|_{\mathcal{E}_T^\alpha})^2 \|\Theta_2\|_{C_\eta^2 \mathcal{E}_T^{2\alpha-2}}.$$

To bound the term $\Phi_2(u)$ we observe that $\|\mathcal{P}t u_0^\sharp\|_{C_\eta^2 \mathcal{E}^{2\alpha}} \lesssim t^{-\frac{\alpha}{2}} \|u_0^\sharp\|_{\mathcal{E}^\alpha}$ thanks to Lemma 1.61. This gives

$$\left\| \int_0^T e^{-\eta\Delta(t-s)} \Phi_2(u)_s ds \right\|_{C_\eta^2 \dot{\mathcal{L}}_T^{2\delta}} \lesssim T^{\alpha-\delta} (1 + \|u\|_{\mathcal{E}_T^\alpha}) (1 + \|\xi\|_{\mathcal{E}^{\alpha-2}}) \|u_0^\sharp\|_{\mathcal{E}^\alpha}$$

and then $\Gamma_{U^\sharp}(u, U^\sharp)$ is bounded in $C_\eta^2 \dot{\mathcal{L}}_T^{2\delta}$ for T small enough. We have also

$$\begin{aligned} \|\Gamma_u(u, U^\sharp)\|_{\dot{\mathcal{L}}_T^\alpha} & \lesssim \|\xi\|_{\mathcal{E}^{\alpha-2}} + \|u_0^\sharp\|_{\mathcal{E}^\alpha} + \|\Gamma_{U^\sharp}(u, U^\sharp)\|_{C_\eta \dot{\mathcal{L}}_T^\alpha} \\ & \lesssim \|\xi\|_{\mathcal{E}^{\alpha-2}} + \|u_0^\sharp\|_{\mathcal{E}^\alpha} + T^{\frac{2\delta-\alpha}{2}} \|\Gamma_{U^\sharp}(u, U^\sharp)\|_{C_\eta^2 \dot{\mathcal{L}}_T^{2\delta}} \end{aligned}$$

and these bounds show that $\Gamma(B) \subseteq B$. The contractivity of $\Gamma_{U^\sharp}(u, U^\sharp)$ can be obtained in the same way. Now consider $\Gamma_u(u, U^\sharp)$: we have

$$\begin{aligned} & \|\Pi_{\leftarrow}(a(u_1), U_1^\sharp) - \Pi_{\leftarrow}(a(u_2), U_2^\sharp)\|_{\dot{\mathcal{L}}_T^\alpha} \\ & \lesssim T^{\frac{2\delta-\alpha}{2}} \left(\|U_1^\sharp - U_2^\sharp\|_{C_\eta \dot{\mathcal{L}}_T^{2\delta}} + \|u_1 - u_2\|_{C_T L^\infty} \|U_2^\sharp\|_{C_\eta^1 \dot{\mathcal{L}}_T^{2\delta}} \right) \end{aligned}$$

while for the other terms in $\Gamma_u(u_1, U_1^\sharp) - \Gamma_u(u_2, U_2^\sharp)$ we remark that

$$\sup_{s \in [0, t]} \|u_{1,s} - u_0 - u_{2,s} + u_0\|_{L^\infty} \lesssim t^{\varepsilon/2} \|u_1 - u_2\|_{C_{[0, t]}^{\varepsilon/2} L^\infty}.$$

Then $\forall 0 < \varepsilon < \alpha$, using Lemma 5.3 and Lemma 1.59:

$$\begin{aligned} \|\Pi_{\leftarrow}(a(u_1), \vartheta) - \Pi_{\leftarrow}(a(u_2), \vartheta)\|_{\dot{\mathcal{L}}_T^\alpha} &\lesssim \|a(u_1) - a(u_2)\|_{C_T L^\infty} \|\vartheta\|_{C_\eta^1 \dot{\mathcal{L}}_T^\alpha} \\ &\lesssim \|u_1 - u_2\|_{C_T L^\infty} \|\xi\|_{\mathcal{C}^{\alpha-2}} \\ &\lesssim T^{\varepsilon/2} \|u_1 - u_2\|_{C_T^{\varepsilon/2} L^\infty} \|\xi\|_{\mathcal{C}^{\alpha-2}} \\ &\lesssim T^{\varepsilon/2} \|u_1 - u_2\|_{\dot{\mathcal{L}}_T^\alpha} \|\xi\|_{\mathcal{C}^{\alpha-2}}. \end{aligned}$$

With the same reasoning we estimate

$$\begin{aligned} &\|\Pi_{\leftarrow}(a(u_1), \mathcal{P}u_0^\sharp) - \Pi_{\leftarrow}(a(u_2), \mathcal{P}u_0^\sharp)\|_{\dot{\mathcal{L}}_T^\alpha} \\ &\lesssim T^{\varepsilon/2} \|u_1 - u_2\|_{C_T^{\varepsilon/2} L^\infty} \|\mathcal{P}u_0^\sharp\|_{C_\eta^1 \dot{\mathcal{L}}_T^\alpha} \\ &\lesssim T^{\varepsilon/2} \|u_1 - u_2\|_{\dot{\mathcal{L}}_T^\alpha} \|u_0^\sharp\|_{\mathcal{C}_T^\alpha} \end{aligned}$$

and then Γ is a contraction for small times.

The uniqueness of the solution $(u, U^\sharp) \in \dot{\mathcal{L}}_T^\alpha \times C_\eta^2 \dot{\mathcal{L}}_T^{2\delta}$ and the Lipschitz continuity of the localized solution map Σ_τ can be proved along the same lines via standard arguments. \square

5.3.3 Renormalization

At this point we want to construct an enhanced noise $\Xi \in \mathcal{X}^\alpha$ associated to the white noise ξ . Already in the standard setting of the generalised PAM model with constant diffusion matrix, the construction of the enhancement requires a renormalization since the resonant product $\vartheta \circ \Delta \vartheta$ is not well defined.

Let $\psi \in \mathcal{S}(\mathbb{T}^2)$ be a cutoff function and let $\psi_\varepsilon(x) = \varepsilon^{-2} \psi(x/\varepsilon)$. Then define a regularised noise by $\xi_\varepsilon = \psi_\varepsilon * \xi$ and let $\vartheta_\varepsilon = (-\eta \Delta)^{-1} \xi_\varepsilon$. Notice that

$$\begin{aligned} H_\varepsilon(\eta) &:= \mathbb{E}[\vartheta_\varepsilon(\eta, x) \circ \Delta \vartheta_\varepsilon(\eta, x)] = \mathbb{E}[\vartheta_\varepsilon(\eta, x) \Delta \vartheta_\varepsilon(\eta, x)] \\ &= - \sum_{k \in \mathbb{Z}^2 \setminus \{0\}} \frac{\hat{\psi}_\varepsilon(k)^2}{\eta^2 |k|^2} = - \frac{\sigma_\varepsilon}{\eta^2} \end{aligned}$$

where

$$\sigma_\varepsilon := \sum_{k \in \mathbb{Z}^2 \setminus \{0\}} \frac{\hat{\psi}_\varepsilon(k)^2}{|k|^2} \simeq |\log \varepsilon|$$

as $\varepsilon \rightarrow 0$. Subtracting the diverging quantity H_ε to $\vartheta_\varepsilon \circ \Delta \vartheta_\varepsilon$ and then taking the limit as $\varepsilon \rightarrow 0$ delivers a finite result.

Theorem 5.15. *Take $\alpha < 1$ and let $\Xi_\varepsilon = (\xi_\varepsilon, \Xi_{2,\varepsilon}) := (\xi_\varepsilon, \vartheta_\varepsilon \circ \Delta \vartheta_\varepsilon - H_\varepsilon)$. Then the family $(\Xi_\varepsilon)_\varepsilon \subseteq \mathcal{X}^\alpha$ converges a.s. and in L^p to a random element $\Xi = (\xi, \Xi_2) \in \mathcal{X}^\alpha$.*

Proof. The proof is a mild modification of the proof for PAM found in [GIP15]. In order to establish the required $C_\eta^2 \mathcal{C}_T^{2\alpha-2}$ regularity for Ξ_2 we follow the computations for the case where the diffusion coefficient is constant. We only have to discuss the additional regularity in the parameter η . In order to do so observe that

$$\Xi_{2,\varepsilon}(\eta) = \sum_{i \sim j} [\Delta_i \vartheta_\varepsilon(\eta) \Delta_j \Delta \vartheta_\varepsilon(\eta)]$$

where $[\cdot]$ denotes the Wick product with respect to the Gaussian structure of ξ . Then we have

$$\partial_\eta \Xi_{2,\varepsilon}(\eta) = \sum_{i \sim j} [\Delta_i \partial_\eta \vartheta_\varepsilon(\eta) \Delta_j \Delta \vartheta_\varepsilon(\eta)] + \sum_{i \sim j} [\Delta_i \vartheta_\varepsilon(\eta) \Delta_j \Delta \partial_\eta \vartheta_\varepsilon(\eta)],$$

and

$$\begin{aligned} \partial_\eta^2 \Xi_{2,\varepsilon}(\eta) &= \sum_{i \sim j} \llbracket \Delta_i \partial_\eta^2 \vartheta_\varepsilon(\eta) \Delta_j \Delta \vartheta_\varepsilon(\eta) \rrbracket + \sum_{i \sim j} \llbracket \Delta_i \vartheta_\varepsilon(\eta) \Delta_j \Delta \partial_\eta^2 \vartheta_\varepsilon(\eta) \rrbracket \\ &\quad + \sum_{i \sim j} 2 \llbracket \Delta_i \partial_\eta \vartheta_\varepsilon(\eta) \Delta_j \Delta \partial_\eta \vartheta_\varepsilon(\eta) \rrbracket. \end{aligned}$$

Now the computations relative to the regularities of these additional stochastic objects are equivalent to those for the term $\Xi_{2,\varepsilon}$ where one or two instances of $\vartheta_\varepsilon(\eta)$ are replaced by Gaussian fields of similar regularities of the form $\partial_\eta \vartheta_\varepsilon(\eta)$ and $\partial_\eta^2 \vartheta_\varepsilon(\eta)$. A direct inspection of the proof contained in [GIP15] allows us to deduce that we have almost sure $\mathcal{C}^{2\alpha-2}$ regularity for these terms and also for random fields $\partial_\eta^n \Xi_{2,\varepsilon}$ for any finite n . This allows also to deduce that the random field is almost surely smooth in the parameter η . Similar computations allow to prove continuity in ε for $\varepsilon > 0$. The rest of the proof is standard. \square

In conclusion we see that in order to be able to use this convergence result we need to modify our approximate PDE and consider instead

$$\partial_t u_\varepsilon - a(u_\varepsilon) \Delta u_\varepsilon = \xi_\varepsilon - a'(u_\varepsilon) H_\varepsilon(a(u_\varepsilon))$$

which gives the renormalised equation (5.28).

Our well-posedness results for the paracontrolled formulation of this equation together with the convergence result of Theorem 5.15 allow to deduce that $u_\varepsilon \rightarrow u$ in \mathcal{C}_T^δ for any $2/3 < \delta < \alpha < 1$ and that the limiting process u satisfies a modified version of eq. (5.1), namely

$$\partial_t u - a(u) \diamond \Delta u = \xi, \quad u(0) = u_0,$$

where $a(u) \diamond \Delta u$ denotes a *renormalized* diffusion term given by

$$a(u) \diamond \Delta u := a(u) \prec \Delta u + a'(u) \Pi_\diamond(a(u), \Xi_2) + \Phi_1(u) + \Phi_2(u). \quad (5.40)$$

5.4 Nonlinear source terms

In this section we want to solve equation (5.2):

$$\partial_t u - a_1(u) \Delta u = a_2(u) \xi$$

where $a_1: \mathbb{R} \rightarrow [\lambda, 1]$, $\lambda \in (0, 1)$ is a non-linear diffusion coefficient as before and $a_2: \mathbb{R} \rightarrow [-L, L]$, $L > 0$ is another bounded function with sufficiently many bounded derivatives. We rewrite this equation as

$$\Pi_\prec(a(u), \mathcal{L})u = a_2(u) \prec \xi + a_1(u) \circ \Delta u + a_2(u) \circ \xi + a_1(u) \succ \Delta u + a_2(u) \succ \xi$$

where now $a(u) = (a_1(u), a_2(u))$ is a vector valued non-linearity. Since we don't need u to depend on any parameter $\eta = (\eta_1, \eta_2)$, we have defined \mathcal{L} as

$$\mathcal{L}(\eta) := \partial_t - \eta_1 \Delta$$

and used the identity $\Pi_\prec(a(u), \mathcal{L})u = (\partial_t - a_1(u) \prec \Delta)u$, similarly to what done in (5.18).

As before we make the Ansatz

$$u = \Pi_\prec(a(u), \vartheta) + u^\sharp$$

where now $\vartheta(\eta)$ for $\eta = (\eta_1, \eta_2) \in [\lambda, 1] \times [-L, L]$ solves

$$\mathcal{L}(\eta)\vartheta(\eta) = (\partial_t - \eta_1\Delta)\vartheta(\eta) = \eta_2 \xi,$$

The bounded domain $[\lambda, 1] \times [-L, L]$ for η is important to be able to have uniform estimates and re-use the estimates proved in the past sections for the simple situation of $\eta_2 = 1$. The solution of this equation is

$$\vartheta(\eta, \cdot) = \eta_2 \int_0^\infty e^{\eta_1 \Delta s} \xi ds = -\frac{\eta_2}{\eta_1} \Delta^{-1} \xi.$$

Observe that

$$\Pi_{\prec}(a(u), \mathcal{L})u = \Pi_{\prec}(a(u), \mathcal{L})\Pi_{\ll}(a(u), \vartheta) + \Pi_{\prec}(a(u), \mathcal{L})u^\sharp$$

and recall that by Lemma 5.7

$$\Pi_{\prec}(a(u), \mathcal{L})\Pi_{\ll}(a(u), \vartheta) = \Pi_{\ll}(a(u), \mathcal{L}\vartheta) + \Psi(a(u), \vartheta).$$

Now

$$(\mathcal{L}\vartheta)(\eta) = (\partial_t - \eta_1\Delta)\vartheta(\eta, t, x) = \Xi(\eta), \quad \eta = (\eta_1, \eta_2) \in [\lambda, 1] \times [-L, L]$$

with the definition $\Xi(\eta)(t, x) := \eta_2 \xi(x)$ and then

$$\Pi_{\ll}(a(u), \mathcal{L}\vartheta) = \Pi_{\ll}(a(u), \Xi) = a_2(u) \ll \xi.$$

In conclusion

$$\Pi_{\prec}(a(u), \mathcal{L})\Pi_{\ll}(a(u), \vartheta) = a_2(u) \ll \xi + \Psi(a(u), \vartheta)$$

and the equation for u^\sharp reads

$$\begin{aligned} \Pi_{\prec}(a(u), \mathcal{L}u^\sharp) &= a_1(u) \circ \Delta u + a_2(u) \circ \xi + [a_2(u) \prec \xi - a_2(u) \ll \xi] \\ &\quad + a_1(u) \succ \Delta u + a_2(u) \succ \xi - \Psi(a(u), \vartheta) \end{aligned}$$

where now all the terms on the r.h.s. can be considered remainder terms. Let us just remark that the commutation term $a_2(u) \prec \xi - a_2(u) \ll \xi$ can be handled easily via Lemma 1.67. Of course, the first two terms in the equation above require to be treated as resonant terms. Note that, modulo terms of order $\mathcal{E}_T^{3\alpha-2}$ (or $\mathcal{M}_T^{\alpha/2} \mathcal{E}^{2\alpha-2}$ as defined in Lemma 1.61) the terms $a_1(u) \circ \Delta u + a_2(u) \circ \xi$ are equivalent to

$$a'_1(u)\Pi_{\diamond}(a(u), \vartheta \circ \Delta\vartheta) + a'_2(u)(\Pi_{\ll}(a(u), \vartheta) \circ \xi)$$

and that by Lemma 5.5 we have that

$$(\Pi_{\ll}(a(u), \vartheta) \circ \xi) = \Pi_{\diamond}(a(u), \vartheta \circ \xi) + \mathcal{E}_T^{3\alpha-2}$$

so the resonant terms are comparable to the sum of the two terms

$$a'_1(u)\Pi_{\diamond}(a(u), \vartheta \circ \Delta\vartheta) + a'_2(u)\Pi_{\diamond}(a(u), \vartheta \circ \xi)$$

which require renormalization of the form

$$\frac{a'_1(u)a_2(u)^2}{a_1(u)^2}\sigma_\varepsilon - \frac{a'_2(u)a_2(u)}{a_1(u)}\sigma_\varepsilon \quad (5.41)$$

and the convergence follows with the same arguments of Section 5.3.3.

Remark 5.16. The structure of the second renormalisation term, which is due to the r.h.s. in the equation, is the same of that found by Bailleul, Debussche and Hofmanová in [BDH16] with different methods.

Remark 5.17. Our approach works in the same way for equation (5.4), namely

$$\partial_t u(t, x) - a_{ij}(u(t, x)) \partial_{ij}^2 u(t, x) = g(u(t, x)) \xi$$

with $a: \mathbb{R} \rightarrow M_2(\mathbb{R})$ such that $\sum_{i,j} a(u)_{ij} x_i x_j \geq C|x|^2 \forall x \in \mathbb{R}^2$ for $C > 0$ and $\partial_{ij}^2 := \frac{\partial^2}{\partial x_i \partial x_j}$.

To see that, let $a(u) := (a_{ij}(u), g(u)) \in \mathbb{R}^5$ and $\eta = (\eta_{i,j}, \eta_g) \in \mathbb{R}^5$. Let

$$\begin{aligned} \mathcal{L}(\eta) &:= \partial_t - \eta_{ij} \partial_{ij}^2, \\ \Xi(\eta) &:= \eta_g \xi, \end{aligned}$$

with the uniform ellipticity condition $\sum_{i,j} \eta_{ij} x_i x_j \geq C|x|^2 \forall x \in \mathbb{R}^2$. It is easy to see that Lemma 5.7 and Lemma 5.5 hold within this setting, just considering nonlinear paraproducts for functions depending on 5 parameters. We have then:

$$u = \Pi_{\leftarrow}(a(u), \vartheta + U^\# + \mathcal{P}u_0^\#)$$

with $\vartheta(\eta)$ stationary solution of $\mathcal{L} \vartheta(\eta) = \Xi(\eta)$, $\mathcal{P}_t u_0^\# := e^{\eta_{ij} \partial_{ij}^2 t} u_0^\#$ and $U^\#(\eta)$ which solves

$$\mathcal{L} U^\#(\eta) = \Pi_{\diamond}((a(u), a'(u)), \Theta_1) + \Pi_{\diamond}((a(u), a'(u)), \Theta_2) + Q(u, U^\#)$$

with $Q(u, U^\#) \in \mathcal{C}^{2\alpha-2-\varepsilon}$, $\Theta_1 \in C_\eta^2 C_{\eta'}^2 \mathcal{C}^{2\alpha-2} = \vartheta(\eta) \circ \eta'_{ij} \partial_{ij}^2 \vartheta(\eta)$, $\Theta_2(\eta, \eta') = \vartheta(\eta) \circ \eta'_g \xi$ and $U^\#(t=0) = 0$. Note that we can write ϑ as

$$\vartheta(\eta) = \eta_g \int_0^\infty e^{t \eta_{ij} \partial_{ij}^2} \xi dt \quad \hat{\vartheta}(k) = \eta_g \frac{\hat{\xi}(k)}{\eta_{ij} k_i k_j}, \quad k \in \mathbb{Z}^2 \setminus \{0\}.$$

From the uniform ellipticity condition we have $\|\vartheta\|_{C_\eta^k \mathcal{C}^\alpha} \lesssim \|\xi\|_{\mathcal{C}^\alpha}$, and Schauder estimates analogous to those of Lemma 1.61 hold as well.

Now consider the renormalization. We have

$$\begin{aligned} H_1^\varepsilon(\eta, \eta') &:= \mathbb{E}(\Theta_1(\eta, \eta')) = -\eta_g^2 \sum_{k \in \mathbb{Z}^2 \setminus \{0\}} \hat{\psi}_\varepsilon(k)^2 \frac{\sum_{i,j} \eta'_{ij} k_i k_j}{(\sum_{i,j} \eta_{ij} k_i k_j)^2}, \\ H_2^\varepsilon(\eta, \eta') &:= \mathbb{E}(\Theta_2(\eta, \eta')) = \eta_g \eta'_g \sum_{k \in \mathbb{Z}^2 \setminus \{0\}} \frac{\hat{\psi}_\varepsilon(k)^2}{\sum_{i,j} \eta_{ij} k_i k_j}. \end{aligned}$$

We note that the convergence of $\Theta_1^\varepsilon - H_1^\varepsilon$, $\Theta_2^\varepsilon - H_2^\varepsilon$ in $C_{(\eta, \eta')}^k \mathcal{C}^{2\alpha-2}(\mathbb{T}^2)$ can be obtained with the same techniques as in [GIP15, Section 5.2], although we do not discuss it here.

5.5 Full generality

Within the framework of the present work we are actually able to treat equations of the form (5.3), which if a_3 takes values in $[\lambda, 1]$ for some $\lambda > 0$ is just:

$$\partial_t u(t, x) - a_1(u(t, x)) \Delta u(t, x) = \xi(a_2(u(t, x)), x) \quad (5.42)$$

where $\xi(\eta_2, x)$ is a Gaussian process with covariance

$$\mathbb{E}[\xi(\eta_2, x) \xi(\tilde{\eta}_2, \tilde{x})] = F(\eta_2, \tilde{\eta}_2) \delta(x - \tilde{x})$$

where F is a smooth covariance function. Let as before $2/3 < \alpha < 1$. In this case we can take as a parametric equation

$$\mathcal{L}(\eta) \vartheta := \partial_t \vartheta(\eta, t, x) - \eta_1 \Delta \vartheta(\eta, t, x) = \xi(\eta_2, x)$$

whose solution ϑ is a Gaussian process, smooth with respect to the variable $\eta = (\eta_1, \eta_2)$ which we assume taking value in a compact subset of \mathbb{R}^2 for which $\eta_1 \geq \lambda > 0$ with fixed λ . Letting $a(u) = (a_1(u), a_2(u))$ we can rewrite the l.h.s. of eq. (5.42) in the form

$$\partial_t u - a_1(u)\Delta u = \Pi_{\diamond}(a(u), \mathcal{L}u)$$

and the r.h.s. as

$$\xi(a_2(u(t, x)), x) = \Pi_{\diamond}(a(u), \Xi)$$

where $\Xi(\eta, x) = \xi(\eta_2, x)$. Now perform the paraproduct decomposition to get

$$\begin{aligned} \Pi_{\prec}(a(u), \mathcal{L}u) - \Pi_{\prec}(a(u), \Xi) &= \Pi_{\circ}(a(u), \Xi) + \Pi_{\circ}(a(u), \mathfrak{D}u) \\ &\quad + \Pi_{\succ}(a(u), \Xi) + \Pi_{\succ}(a(u), \mathfrak{D}u). \end{aligned}$$

We have introduced here the parametric differential operator

$$\mathfrak{D}(\eta) := \eta_1 \Delta$$

for $\eta = (\eta_1, \eta_2)$. Let $\mathcal{P}_t(\eta) := e^{t\eta_1 \Delta}$ as before, and set the paracontrolled Ansatz in the usual form

$$u = \Pi_{\prec}(a(u), \vartheta + U^{\sharp} + \mathcal{P}u_0^{\sharp}).$$

Using that

$$\begin{aligned} \Pi_{\prec}(a(u), \mathcal{L}\Pi_{\prec}(a(u), \vartheta + U^{\sharp} + \mathcal{P}u_0^{\sharp})) &= \Pi_{\prec}(a(u), \mathcal{L}(\vartheta + U^{\sharp} + \mathcal{P}u_0^{\sharp})) \\ &\quad + \Psi(a(u), \vartheta + U^{\sharp} + \mathcal{P}u_0^{\sharp}) \end{aligned}$$

and observing that we can take $\mathcal{L}\vartheta = \Xi$ and that $\mathcal{L}\mathcal{P}u_0^{\sharp} = 0$ we get

$$\Pi_{\prec}(a(u), \mathcal{L}U^{\sharp}) = F(u, U^{\sharp})$$

where

$$\begin{aligned} F(u, U^{\sharp}) &= \Pi_{\circ}(a(u), \Xi) + \Pi_{\circ}(a(u), \mathfrak{D}u) + \Pi_{\succ}(a(u), \Xi) + \Pi_{\succ}(a(u), \mathfrak{D}u) \\ &\quad + [\Pi_{\prec}(a(u), \Xi) - \Pi_{\prec}(a(u), \Xi)] - \Psi(a(u), \vartheta + U^{\sharp} + \mathcal{P}u_0^{\sharp}) \end{aligned}$$

which is solved by U^{\sharp} satisfying

$$\mathcal{L}U^{\sharp} = F(u, U^{\sharp}).$$

Indeed $\Pi_{\prec}(a(u), F(u, U^{\sharp})) = F(u, U^{\sharp})$, since $F(u, U^{\sharp})$ does not depend on the additional parameter. The term $\Pi_{\prec}(a(u), \Xi) - \Pi_{\prec}(a(u), \Xi)$, which does not appear in the simpler case, can be treated with the following lemma:

Lemma 5.18. *Let $g \in \dot{\mathcal{L}}_T^{\rho}$, $h \in C_{\eta}^1 \mathcal{E}_T^{\gamma}$ with $\rho \in (0, 1)$, $\gamma \in \mathbb{R}$. We have, $\forall \varepsilon > 0$*

$$\|\Pi_{\prec}(g, h) - \Pi_{\prec}(g, h)\|_{\mathcal{E}_T^{\rho+\gamma-\varepsilon}} \lesssim \|g\|_{\dot{\mathcal{L}}_T^{\rho}} \|h\|_{C_{\eta}^1 \mathcal{E}_T^{\gamma}}.$$

The proof consists in a straightforward adaptation to the nonlinear case of inequality (1.69).

It remains now to discuss the handling of the resonant products under the paracontrolled assumption, namely $\Pi_{\circ}(a(u), \Xi)$ and $\Pi_{\circ}(a(u), \mathfrak{D}u)$. Next lemma is a parilinearization result adapted to our non-linear context.

Lemma 5.19. *Assume that $u \in \mathcal{E}_T^{\rho}$ and $Z \in C_{\eta}^2 \mathcal{E}_T^{\gamma}$ then if $\gamma + 2\rho > 0$ we have*

$$C(u, Z) := \Pi_{\circ}(a(u), Z) - u \circ \Pi_{\prec}((a(u), a'(u)), \mathcal{D}Z) \in \mathcal{E}_T^{\gamma+2\rho}$$

where $\mathcal{D}Z((\eta, \eta'), t, x) := \sum_i \eta_i' \partial_{\eta_i} Z(\eta, t, x)$.

Proof.

$$\begin{aligned}
\Pi_o(a(u), Z)(t, x) &= \sum_{i \sim j} \int_{y, z} K_{i, x}(y) K_{j, x}(z) Z(a(u(t, y)), t, z) \\
&= \sum_{i \sim j, k} \int_{\substack{y, z' \\ z, z''}} K_{i, x}(y) K_{j, x}(z) P_{k, z}(z'') K_{k, z}(z') Z(a(u(t, y)), t, z') \\
&= \sum_{i \sim j, k} \int_{\substack{y, z' \\ z, z''}} K_{i, x}(y) K_{j, x}(z) P_{k, z}(z'') K_{k, z}(z') \times \\
&\quad \times [Z(a(u(t, y)), t, z') - Z(a(u(t, z'')), t, z')] \\
&= \sum_{i \sim j, k} \int_{\substack{y, z' \\ z, z''}} K_{i, x}(y) K_{j, x}(z) P_{k, z}(z'') K_{k, z}(z') \times \\
&\quad \times \left[\sum_{\ell} a'_{\ell}(u(t, z'')) \delta u_{t z''}^{t y} \partial_{a_{\ell}} Z(a(u(t, z'')), t, z') \right] \\
&\quad + \sum_{\substack{i \sim j \\ k \sim j}} \int_{\substack{y, z' \\ z, z''}} K_{i, x}(y) K_{j, x}(z) P_{k, z}(z'') K_{k, z}(z') \times \\
&\quad \times O((\delta u_{t z''}^{t y})^2) \partial_{\eta}^2 Z(a(u(t, y)), t, z')
\end{aligned}$$

and observe that the first term is equal to $u \circ \Pi_{\leftarrow}((a(u), a'(u)), \mathcal{D}Z)$ while the second term can be easily estimated in $\mathcal{E}_T^{\gamma+2\rho}$. \square

Using this result and Lemma 5.18 we can expand in the same way as done for Lemma 5.5 to obtain:

$$\begin{aligned}
\Pi_o(a(u), \Xi) &= u \circ \Pi_{\leftarrow}((a(u), a'(u)), \mathcal{D}\Xi) + \mathcal{E}_T^{3\alpha-2} \\
&= \Pi_{\leftarrow}((a(u), \vartheta) \circ \Pi_{\leftarrow}((a(u), a'(u)), \mathcal{D}\Xi) + \mathcal{E}_T^{3\alpha-2} \\
&= \Pi_{\diamond}((a(u), a'(u)), \vartheta \circ \mathcal{D}\Xi) + \mathcal{E}_T^{3\alpha-2}.
\end{aligned}$$

Similarly, noting that

$$\begin{aligned}
\Pi_{\leftarrow}((a(u), a'(u)), (\mathcal{D}\mathcal{D})u) &= \Pi_{\leftarrow}((a(u), a'(u)), (\mathcal{D}\mathcal{D})\Pi_{\leftarrow}(a(u), \vartheta)) + \mathcal{E}_T^{3\alpha-2} \\
&= \Pi_{\leftarrow}((a(u), a'(u)), (\mathcal{D}\mathcal{D})\vartheta) + \mathcal{E}_T^{3\alpha-2}
\end{aligned}$$

where $(\mathcal{D}\mathcal{D})(\eta, \eta') = \eta'_1 \Delta$, we have by a straightforward generalization of Lemma 5.5:

$$\begin{aligned}
\Pi_o(a(u), \mathcal{L}u) &= u \circ \Pi_{\leftarrow}((a(u), a'(u)), (\mathcal{D}\mathcal{D})u) + \mathcal{E}_T^{3\alpha-2} \\
&= \Pi_{\leftarrow}((a(u), \vartheta) \circ \Pi_{\leftarrow}((a(u), a'(u)), (\mathcal{D}\mathcal{D})\vartheta) + \mathcal{E}_T^{3\alpha-2} \\
&= \Pi_{\diamond}((a(u), a'(u)), \vartheta \circ (\mathcal{D}\mathcal{D})\vartheta) + \mathcal{E}_T^{3\alpha-2}.
\end{aligned}$$

Finally the equation for U^{\sharp} reads

$$\mathcal{L}U^{\sharp} = \Pi_{\diamond}((a(u), a'(u)), \vartheta \circ \mathcal{D}\Xi + \vartheta \circ (\mathcal{D}\mathcal{D})\vartheta) + \mathcal{E}_T^{3\alpha-2}.$$

This can be solved essentially as we did in the simpler context. We see that the general enhancement has the form

$$(\xi, \vartheta \circ \mathcal{D}\Xi + \vartheta \circ (\mathcal{D}\mathcal{D})\vartheta)$$

which of course will require renormalization like we did before. In particular

$$\begin{aligned}
(\vartheta \circ \mathcal{D}\Xi + \vartheta \circ (\mathcal{D}\mathcal{D})\vartheta)(\eta, \eta') &= \vartheta(\eta) \circ \eta'_2 \partial_{\eta_2} \xi(\eta_2, \cdot) + \vartheta(\eta) \circ \eta'_1 \Delta \vartheta(\eta) \\
&= -\frac{\eta'_2}{\eta_1} (\Delta^{-1} \xi(\eta_2, \cdot)) \circ \partial_{\eta_2} \xi(\eta_2, \cdot) \\
&\quad + \frac{\eta'_1}{\eta_1} (\Delta^{-1} \xi(\eta_2, \cdot)) \circ \xi(\eta_2, \cdot)
\end{aligned}$$

where we used that $\eta_1 \Delta \vartheta(\eta) = -\xi(\eta_2, \cdot)$. Now observe that

$$\mathbb{E}[(\Delta^{-1} \xi_\varepsilon(\eta_2, \cdot)) \circ \xi_\varepsilon(\eta_2, \cdot)] = -F(\eta_2, \eta_2) \sigma_\varepsilon$$

and that

$$\mathbb{E}[(\Delta^{-1} \xi_\varepsilon(\eta_2, \cdot)) \circ \partial_{\eta_2} \xi_\varepsilon(\eta_2, \cdot)] = -(\partial_1 F)(\eta_2, \eta_2) \sigma_\varepsilon$$

with $\partial_1 F$ denoting the derivative with respect to the first entry.

In the end the renormalised enhanced noise is obtained as the limit in \mathcal{X}^α of $(\xi_\varepsilon, \Xi_{2,\varepsilon})$ where

$$\Xi_{2,\varepsilon}(\eta, \eta') = -\frac{\eta'_2}{\eta_1} (\Delta^{-1} \xi_\varepsilon(\eta_2, \cdot)) \circ \partial_{\eta_2} \xi_\varepsilon(\eta_2, \cdot) + \frac{\eta'_1}{\eta_1^2} (\Delta^{-1} \xi_\varepsilon(\eta_2, \cdot)) \circ \xi_\varepsilon(\eta_2, \cdot) - H_\varepsilon(\eta, \eta')$$

with

$$H_\varepsilon(\eta, \eta') = \frac{\eta'_2}{\eta_1} (\partial_1 F)(\eta_2, \eta_2) \sigma_\varepsilon - \frac{\eta'_1}{\eta_1^2} F(\eta_2, \eta_2) \sigma_\varepsilon.$$

We do not discuss the convergence of the noise $(\xi_\varepsilon, \Xi_{2,\varepsilon})$ here, as the purpose of this section is just to show that a paracontrolled structure can be obtained for quasi-linear equations of the type (5.42) with the theory developed in Section 5.2. We just remark that if we take $F(\eta_2, \tilde{\eta}_2) = \eta_2 \tilde{\eta}_2$ we obtain again the situation treated in Section 5.4. Indeed in this case

$$\Pi_\diamond((a(u), a'(u)), H_\varepsilon) = \frac{a'_2(u) a_2(u)}{a_1(u)} \sigma_\varepsilon - \frac{a'_1(u) a_2(u)^2}{a_1(u)^2} \sigma_\varepsilon.$$

which coincides with (5.41).

Remark 5.20. Consider the more general equation (5.3), where the noise depends explicitly on time, e.g. with a covariance

$$\mathbb{E}[\xi(\eta, t, x) \xi(\eta', t', x')] = F(\eta, \eta') Q(t - t', x - x')$$

with F a smooth function and Q a distribution of parabolic regularity $\rho > -4/3$. Note that as before the coefficient $a_3(u) \in [\lambda, 1]$ in front of the time derivative can be eliminated trivially by dividing.

In order to handle the time dependence of the noise, the framework of Section 5.2 will still apply, provided we consider space–time paraproducts instead of paraproducts which act only on the space variable. However, the constraint of regularity $\rho > -4/3$ does allow to treat a noise which is white in time and smooth in space, but not a space–time white noise. It is well known that the first order paracontrolled approach that we presented in Section 1.2 does not allow to treat this kind of irregular signals in full generality.

Chapter 6

Weak universality for a class of 3d stochastic reaction-diffusion models

6.1 Introduction

Consider a family of stochastic reaction–diffusion equations in a *weakly nonlinear regime*:

$$\mathcal{L}u(t, x) = -\varepsilon^\alpha F_\varepsilon(u(t, x)) + \eta(t, x), \quad (t, x) \in [0, T/\varepsilon^2] \times (\mathbb{T}/\varepsilon)^3 \quad (6.1)$$

with $\varepsilon \in (0, 1]$, $T > 0$, initial condition $\bar{u}_{0,\varepsilon}: (\mathbb{T}/\varepsilon)^3 \rightarrow \mathbb{R}$, $F_\varepsilon \in C^9(\mathbb{R})$ with exponential growth at infinity, $\alpha > 0$ and $\mathcal{L} := (\partial_t - \Delta)$ the heat flow operator and $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$. Here η denotes a family of centered Gaussian noises on $[0, T/\varepsilon^2] \times (\mathbb{T}/\varepsilon)^3$ with stationary covariance

$$\mathbb{E}(\eta(t, x)\eta(s, y)) = \tilde{C}^\varepsilon(t - s, x, y)$$

such that $\tilde{C}^\varepsilon(t - s, x, y) = \Sigma(t - s, x - y)$ if $\text{dist}(x, y) \leq 1$ and 0 otherwise where $\Sigma: \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{R}^+$ is a smooth function compactly supported in $[-1, 1] \times B_{\mathbb{R}^3}(0, 1)$. We assume also that there exists a compactly supported function ψ such that $\psi * \psi = \Sigma$ (this is true e.g. when η is obtained by space-time convolution of the white noise with ψ).

We look for a large scale description of the solution to eq. (6.1) and we introduce the “mesoscopic” scale variable $u_\varepsilon(t, x) = \varepsilon^{-\beta}u(t/\varepsilon^2, x/\varepsilon)$ where $\beta > 0$. Substituting u_ε into (6.1) we get

$$\mathcal{L}u_\varepsilon(t, x) = -\varepsilon^{\alpha-2-\beta}F_\varepsilon(\varepsilon^\beta u_\varepsilon(t, x)) + \varepsilon^{-2-\beta}\eta\left(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}\right). \quad (6.2)$$

In order for the term $\varepsilon^{-2-\beta}\eta(t/\varepsilon^2, x/\varepsilon)$ to converge to a non-trivial random limit we need that $\beta = 1/2$. Indeed the Gaussian field $\eta_\varepsilon(t, x) := \varepsilon^{-5/2}\eta(t/\varepsilon^2, x/\varepsilon)$ has covariance $\varepsilon^{-5}\tilde{C}^\varepsilon(t/\varepsilon^2, x/\varepsilon)$ and converges in distribution to the space-time white noise on $\mathbb{R} \times \mathbb{T}^3$. For large values of α the non-linearity will be negligible with respect to the additive noise term. Heuristically, we can attempt an expansion of the reaction term around the stationary solution Y_ε to the linear equation

$$\mathcal{L}Y_\varepsilon = -Y_\varepsilon + \eta_\varepsilon, \quad (6.3)$$

i.e. $Y_\varepsilon(t, x) = \int_{-\infty}^t \check{P}(t - s, x - y)\eta_\varepsilon(s, y)dsdy$ with $\check{P}(t, x) = \frac{1}{(4\pi t)^{3/2}}e^{-\frac{|x|^2}{4t}}e^{-t}\mathbb{1}_{t \geq 0}$.

Let us denote with C_ε the covariance of Y_ε . We approximate the reaction term as

$$\varepsilon^{\alpha-5/2}F_\varepsilon(\varepsilon^{1/2}u_\varepsilon(t, x)) \simeq \varepsilon^{\alpha-5/2}F_\varepsilon(\varepsilon^{1/2}Y_\varepsilon(t, x)).$$

The Gaussian r.v. $\varepsilon^{1/2}Y_\varepsilon(t, x)$ has variance $\sigma_\varepsilon^2 = \varepsilon\mathbb{E}[(Y_\varepsilon(t, x))^2] = \varepsilon\mathbb{E}[(Y_\varepsilon(0, 0))^2] = \varepsilon C_\varepsilon(0, 0)$ independent of (t, x) . Although σ_ε^2 depends on ε , it can be bounded from above and below by two positive constants uniformly on $\varepsilon \in (0, 1]$. We can expand the random variable $F_\varepsilon(\varepsilon^{1/2}Y_\varepsilon(t, x))$ according to the chaos decomposition relative to $\varepsilon^{1/2}Y_\varepsilon(t, x)$ (as done in (2.2)) and obtain

$$F_\varepsilon(\varepsilon^{1/2}Y_\varepsilon(t, x)) = \sum_{n \geq 0} f_{n,\varepsilon} H_n(\varepsilon^{1/2}Y_\varepsilon(t, x), \sigma_\varepsilon^2), \quad (6.4)$$

where $H_n(x, \sigma_\varepsilon^2)$ are standard Hermite polynomials with variance σ_ε^2 and highest-order term normalized to 1. Note also that the coefficients $(f_{n,\varepsilon})_{n \geq 0}$ do not depend on (t, x) by stationarity of the law of $\varepsilon^{1/2} Y_\varepsilon(t, x)$ since they are given by the formula

$$f_{n,\varepsilon} = \frac{1}{n! \sigma_\varepsilon^{2n}} \mathbb{E}[F_\varepsilon(\varepsilon^{1/2} Y_\varepsilon(t, x)) H_n(\varepsilon^{1/2} Y_\varepsilon(t, x), \sigma_\varepsilon^2)] = \frac{1}{n! \sigma_\varepsilon^n} \mathbb{E}[F_\varepsilon(\sigma_\varepsilon G) H_n(G)]$$

where G is a standard Gaussian variable of unit variance.

Let X be the stationary solution to the equation

$$\mathcal{L}X = -X + \xi,$$

with ξ the space-time white noise on $\mathbb{R} \times \mathbb{T}^3$ and denote by $\llbracket X^N \rrbracket$ the generalized random fields given by the N -th Wick power of X , which are well defined as random elements of $\mathcal{S}'(\mathbb{R} \times \mathbb{T}^3)$ as long as $N \leq 4$. The Gaussian analysis which we set up in this chapter shows in particular that if $\varepsilon^{(n-N)/2} f_{n,\varepsilon} \rightarrow g_n$ as $\varepsilon \rightarrow 0$ for every $0 \leq n \leq N$, $N \leq 4$, and $(F_\varepsilon)_\varepsilon \subseteq C^{N+1}(\mathbb{R})$ with exponential growth, then the family of random fields

$$\mathbb{F}_\varepsilon^N: (t, x) \mapsto \varepsilon^{-N/2} F_\varepsilon(\varepsilon^{1/2} Y_\varepsilon(t, x)), \quad (t, x) \in \mathbb{R} \times \mathbb{T}^3,$$

converges in law in $\mathcal{S}'(\mathbb{R} \times \mathbb{T}^3)$ as $\varepsilon \rightarrow 0$ to $\sum_{n=0}^N g_n \llbracket X^n \rrbracket$.

Consider the smallest n such that $f_{n,\varepsilon}$ converges to a finite limit as $\varepsilon \rightarrow 0$. Since $H_n(\varepsilon^{1/2} Y_\varepsilon, \sigma_\varepsilon^2) = \varepsilon^{n/2} \llbracket Y_\varepsilon^n \rrbracket$, the n -th term in the expansion (6.4) is $f_{n,\varepsilon} \varepsilon^{\alpha+(n-5)/2} \llbracket Y_\varepsilon^n \rrbracket$. Therefore, the equation yields a non-trivial limit only if $\alpha = (5-n)/2$. We are interested mainly in the case $n=3 \Rightarrow \alpha=1$ and $n=1 \Rightarrow \alpha=2$. The case $\alpha=2$ gives rise to a Gaussian limit and its analysis is not very difficult. In the following we will concentrate on the analysis of the case $\alpha=1$ where the limiting behaviour of the model is the most interesting and given by the Φ_3^4 family of singular SPDEs. In this case we obtain the family of models

$$\mathcal{L}u_\varepsilon(t, x) = -\varepsilon^{-\frac{3}{2}} F_\varepsilon(\varepsilon^{\frac{1}{2}} u_\varepsilon(t, x)) + \eta_\varepsilon(t, x) \quad (6.5)$$

with initial condition $u_{0,\varepsilon}(\cdot) := \varepsilon^{-\frac{1}{2}} \bar{u}_{0,\varepsilon}(\varepsilon^{-1} \cdot)$ where $\bar{u}_{0,\varepsilon}$ is the initial condition of the microscopic model (6.1).

In order to state our main result, Theorem 6.3 below, let us introduce some notations and specify our assumptions. Let \tilde{F}_ε be the centering (up to the third Wiener chaos relative to $\varepsilon^{1/2} Y_\varepsilon(t, x)$) of the function F_ε , i.e.

$$\tilde{F}_\varepsilon(x) := F_\varepsilon(x) - f_{0,\varepsilon} - f_{1,\varepsilon} x - f_{2,\varepsilon} H_2(x, \sigma_\varepsilon^2) = \sum_{n \geq 3} f_{n,\varepsilon} H_n(x, \sigma_\varepsilon^2). \quad (6.6)$$

The decomposition of \tilde{F}_ε is obviously the same as in (6.4) except for the fact that we have discarded the orders 0,1,2. Let $\tilde{F}_\varepsilon^{(m)}$ be the m -th derivative of the function \tilde{F}_ε for $0 \leq m \leq 9$ and define the following ε -dependent constants:

$$\begin{aligned} d_\varepsilon^{\mathbb{Y}} &:= \frac{\varepsilon^{-2}}{9} \int_{s,x} P_s(x) \mathbb{E}[\tilde{F}_\varepsilon^{(1)}(\varepsilon^{1/2} Y_\varepsilon(s, x)) \tilde{F}_\varepsilon^{(1)}(\varepsilon^{1/2} Y_\varepsilon(0, 0))], \\ \bar{d}_\varepsilon^{\mathbb{Y}} &:= 2 \varepsilon^{-1/2} f_{3,\varepsilon} f_{2,\varepsilon} \int_{s,x} P_s(x) (C_\varepsilon(s, x))^2, \\ d_\varepsilon^{\mathbb{Y}} &:= \frac{\varepsilon^{-2}}{6} \int_{s,x} P_s(x) \mathbb{E}[\tilde{F}_\varepsilon^{(0)}(\varepsilon^{1/2} Y_\varepsilon(s, x)) \tilde{F}_\varepsilon^{(2)}(\varepsilon^{1/2} Y_\varepsilon(0, 0))], \\ d_\varepsilon^{\mathbb{Y}} &:= \frac{\varepsilon^{-5/2}}{3} \int_{s,x} P_s(x) \mathbb{E}[\tilde{F}_\varepsilon^{(0)}(\varepsilon^{1/2} Y_\varepsilon(s, x)) \tilde{F}_\varepsilon^{(1)}(\varepsilon^{1/2} Y_\varepsilon(0, 0))], \end{aligned} \quad (6.7)$$

where $P_s(x)$ is the heat kernel and $\int_{s,x}$ denotes integration on $\mathbb{R}^+ \times \mathbb{T}^3$.

Assumption 6.1. *All along this chapter we enforce the following assumptions:*

- a) $(u_{0,\varepsilon})_\varepsilon$ converges in law to a limit u_0 in $\mathcal{C}^{-1/2-\kappa}$ and is independent of η ;
- b) $(\bar{u}_{0,\varepsilon})_\varepsilon$ is uniformly bounded in L^∞ in probability, i.e. $\exists C > 0$ such that $\forall \varepsilon \in (0, 1]$ $\|\bar{u}_{0,\varepsilon}\|_{L^\infty((\mathbb{T}/\varepsilon)^3)} \leq C$;
- c) $(F_\varepsilon)_\varepsilon \subseteq C^9(\mathbb{R})$ and there exists constants $c, C > 0$ such that

$$\sup_{\varepsilon, x} \sum_{k=0}^9 |\partial_x^k F_\varepsilon(x)| \leq C e^{c|x|}, \quad (6.8)$$

- d) the family of vectors $\lambda_\varepsilon = (\lambda_{0,\varepsilon}, \lambda_{1,\varepsilon}, \lambda_{2,\varepsilon}, \lambda_{3,\varepsilon}) \in \mathbb{R}^4$ given by

$$\begin{aligned} \lambda_{3,\varepsilon} &:= f_{3,\varepsilon} & \lambda_{1,\varepsilon} &:= \varepsilon^{-1} f_{1,\varepsilon} - 9d_\varepsilon^{\mathbb{Y}} - 6d_\varepsilon^{\mathbb{Y}} \\ \lambda_{2,\varepsilon} &:= \varepsilon^{-1/2} f_{2,\varepsilon} & \lambda_{0,\varepsilon} &:= \varepsilon^{-3/2} f_{0,\varepsilon} - \varepsilon^{-1/2} f_{2,\varepsilon} d_\varepsilon^{\mathbb{Y}} - 3d_\varepsilon^{\mathbb{Y}} - 3\bar{d}_\varepsilon^{\mathbb{Y}} \end{aligned} \quad (6.9)$$

has a finite limit $\lambda = (\lambda_0, \lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^4$ as $\varepsilon \rightarrow 0$.

Remark 6.2. Note that under Assumption 6.1 we do not allow initial data $(u_{0,\varepsilon})_\varepsilon$ to be mollifications at size ε of a distribution $u_0 \in \mathcal{C}^{-1/2-\kappa}$. Indeed, that would verify 6.1.a) but not 6.1.b), as it is clear from the norm of $\mathcal{C}^{-1/2-\kappa}$ given in Definition 1.17. The reason we need the rescaling $\bar{u}_{0,\varepsilon} = \varepsilon^{1/2} u_{0,\varepsilon}(\varepsilon \cdot)$ to be bounded in probability in L^∞ is to be able to perform the estimations of Section 6.4.1 on the remainder R_ε (which accounts for the exponential divergence of F_ε). Simply assuming 6.1.a) would result on a non-integrable singularity in $t=0$, as noted in [HQ15] and [HX16].

Note also that our Assumption 6.1 does not allow to restart the solution in time, but since we are aiming for a local-in-time convergence of the solutions u_ε , this is not a big problem. It is possible to show that 6.1.b) follows from more specific assumptions on F_ε (for example by assuming $F_\varepsilon: \mathbb{R} \rightarrow \mathbb{R}$ coercive and using a maximum principle to obtain 6.1.b)), but since the main point of this chapter is the estimation of infinite-chaos stochastic terms (Section 6.3) we don't develop this idea further.

Theorem 6.3. (Convergence of the solution) *Under Assumption 6.1 the family of random fields $(u_\varepsilon)_\varepsilon$ given by the solution to eq. (6.5) converges in law and locally in time to a limiting random field $u(\lambda)$ in the space $C_T \mathcal{C}^{-\alpha}(\mathbb{T}^3)$ for every $1/2 < \alpha < 2/3$. The law of $u(\lambda)$ depends only on the value of λ and neither on the other details of the nonlinearity nor on the covariance of the noise term. We call this limit the dynamic Φ_3^4 model with parameter vector $\lambda \in \mathbb{R}^4$.*

Here $C_T \mathcal{C}^{-\alpha}(\mathbb{T}^3)$ denotes the space of continuous functions from $[0, T]$ to the Besov space $\mathcal{C}^{-\alpha}(\mathbb{T}^3) = B_{\infty,\infty}^{-\alpha}(\mathbb{T}^3)$ (see Chapter 1 for a brief introduction of Besov spaces and paraproducts). Theorem 6.3 is actually just a corollary of the more precise result Theorem 6.7, in which we identify the paracontrolled equation satisfied by the limiting random field $u(\lambda)$.

Remark 6.4. We are interested only in local-in-time convergence of u_ε , as a way to show the potential of our method for controlling stochastic terms with infinite chaos decomposition (developed in Section 6.3). Nevertheless, we expect it to be possible to obtain global-in-time convergence of the solution with more stringent assumptions on F_ε , although we do not treat this problem here.

Remark 6.5. As a special case we can take

$$F_\varepsilon(x) = \lambda_3 H_3(x, \sigma_\varepsilon^2) + \varepsilon^{1/2} \lambda_2 H_2(x, \sigma_\varepsilon^2) + \varepsilon(\lambda_1 + \gamma_{1,\varepsilon}) H_1(x, \sigma_\varepsilon^2) + \varepsilon^{3/2}(\lambda_0 + \gamma_{0,\varepsilon})$$

so that

$$f_{3,\varepsilon} = \lambda_3, \quad \varepsilon^{-1/2} f_{2,\varepsilon} = \lambda_2, \quad \varepsilon^{-1} f_{1,\varepsilon} = \lambda_1 + \gamma_{1,\varepsilon}, \quad \varepsilon^{-3/2} f_{0,\varepsilon} = \lambda_0 + \gamma_{0,\varepsilon},$$

and

$$d_\varepsilon^{\Psi} = (\lambda_3)^2 L_\varepsilon, \quad \bar{d}_\varepsilon^{\Psi} = \lambda_3 \lambda_2 L_\varepsilon, \quad d_\varepsilon^{\Psi} = d_\varepsilon^{\Psi} = 0,$$

where $L_\varepsilon := 2 \int_{s,x} P_s(x) (C_{Y,\varepsilon}(s,x))^2$. Choosing

$$\gamma_{1,\varepsilon} := 9 d_\varepsilon^{\Psi} = 9(\lambda_3)^2 L_\varepsilon, \quad \gamma_{0,\varepsilon} := 3 \bar{d}_\varepsilon^{\Psi} = 3 \lambda_3 \lambda_2 L_\varepsilon,$$

we obtain $\lambda_\varepsilon \rightarrow (\lambda_0, \lambda_1, \lambda_2, \lambda_3)$. This shows that all the possible limits $\lambda \in \mathbb{R}^4$ are attainable. In this case (6.5) takes the form

$$\mathcal{L}u_\varepsilon = -\lambda_3 u_\varepsilon^3 - \lambda_2 u_\varepsilon^2 - [\lambda_1 - 3\lambda_3 \varepsilon^{-1} \sigma_\varepsilon^2 + 9(\lambda_3)^2 L_\varepsilon] u_\varepsilon - \lambda_0 + \lambda_2 \sigma_\varepsilon^2 - 3\lambda_3 \lambda_2 L_\varepsilon + \eta_\varepsilon. \quad (6.10)$$

The name *dynamic Φ_3^4 equation* (or *stochastic quantisation equation*) derives from the fact that the simplest class of models which approximate the limiting random field $u(\lambda)$ is precisely obtained by choosing a cubic polynomial like in (6.10) as non-linear term (which is the gradient of a fourth order polynomial playing the role of local potential).

Weak universality is the observation that the same limiting object describes the large scale behaviour of solutions of more general equations, in particular that of the many parameters present in a general model, only a finite number of their combinations survive in the limit to describe the limiting object. The adjective “weak” is related to the fact that, in order to control the large scale limit, the non-linearity has to be very small in the microscopic scale. This sets up a perturbative regime which is well suited to the analysis via regularity structures or paracontrolled distributions.

6.2 Analysis of the mesoscopic model

The goal of this section is to obtain a paracontrolled structure for equation (6.5) analogous to that introduced by R.Catellier and K.Chouk [CC13] for the cubic polynomial case (the Φ_3^4 model), and use it to set up the limiting procedure. Convergence of the stochastic terms and some a priori estimates will be the subject of the following sections. Definitions and a reminder of the basic results of paradifferential calculus needed here can be found in Section 1.2.

6.2.1 Paracontrolled structure

Let us start our analysis by centering the reaction term $F_\varepsilon(\varepsilon^{1/2} u_\varepsilon)$ in (6.5) using decomposition (6.6) to obtain:

$$\begin{aligned} \mathcal{L}u_\varepsilon = & -\varepsilon^{-\frac{3}{2}} \tilde{F}_\varepsilon(\varepsilon^{\frac{1}{2}} u_\varepsilon(t,x)) + \eta_\varepsilon \\ & -\varepsilon^{-3/2} f_{0,\varepsilon} - \varepsilon^{-1} f_{1,\varepsilon} u_\varepsilon - \varepsilon^{-3/2} f_{2,\varepsilon} H_2\left(\varepsilon^{\frac{1}{2}} u_\varepsilon, \sigma_\varepsilon^2\right). \end{aligned}$$

We write $u_\varepsilon = Y_\varepsilon + v_\varepsilon$ with Y_ε as in (6.3), and perform a Taylor expansion of $\tilde{F}_\varepsilon(\varepsilon^{1/2}Y_\varepsilon + \varepsilon^{1/2}v_\varepsilon)$ around $\varepsilon^{1/2}Y_\varepsilon$ up to the third order to get

$$\begin{aligned} \mathcal{L}u_\varepsilon = & \eta_\varepsilon - \varepsilon^{-\frac{3}{2}}\tilde{F}_\varepsilon(\varepsilon^{\frac{1}{2}}Y_\varepsilon) - \varepsilon^{-1}\tilde{F}_\varepsilon^{(1)}(\varepsilon^{\frac{1}{2}}Y_\varepsilon)v_\varepsilon - \frac{1}{2}\varepsilon^{-\frac{1}{2}}\tilde{F}_\varepsilon^{(2)}(\varepsilon^{\frac{1}{2}}Y_\varepsilon)v_\varepsilon^2 - \frac{1}{6}\tilde{F}_\varepsilon^{(3)}(\varepsilon^{\frac{1}{2}}Y_\varepsilon)v_\varepsilon^3 \\ & - \varepsilon^{-3/2}f_{0,\varepsilon} - \varepsilon^{-1}f_{1,\varepsilon}(Y_\varepsilon + v_\varepsilon) - \varepsilon^{-1/2}f_{2,\varepsilon}(\llbracket Y_\varepsilon^2 \rrbracket + 2v_\varepsilon Y_\varepsilon + v_\varepsilon^2) - R_\varepsilon(v_\varepsilon). \end{aligned} \quad (6.11)$$

where $R_\varepsilon(v_\varepsilon)$ is the remainder of the Taylor series and we use the fact that $H_2(\varepsilon^{1/2}Y_\varepsilon, \sigma_\varepsilon^2) = \varepsilon\llbracket Y_\varepsilon^2 \rrbracket$. Notice that we stopped the Taylor expansion at the first term for which ε does not appear anymore with a negative exponent (that is $\tilde{F}_\varepsilon^{(3)}(\varepsilon^{\frac{1}{2}}Y_\varepsilon)$). One can then expect the remainder $R_\varepsilon(v_\varepsilon)$ to converge to zero in some sense. On the other hand, all the other terms except $\tilde{F}_\varepsilon^{(3)}(\varepsilon^{\frac{1}{2}}Y_\varepsilon)$ and $R_\varepsilon(v_\varepsilon)$ appear to diverge in the limit $\varepsilon \rightarrow 0$, but in analogy with well-known renormalization methods for random fields, we try to find a combination of them that can be made to converge in some function space. Define the following random fields:

$$\begin{aligned} \mathcal{L}Y_\varepsilon &:= -Y_\varepsilon + \eta_\varepsilon & \mathcal{L}\bar{Y}_\varepsilon^{\vee} &:= \bar{Y}_\varepsilon^{\vee}, \\ \bar{Y}_\varepsilon^{\vee} &:= \varepsilon^{-1/2}f_{2,\varepsilon}\llbracket Y_\varepsilon^2 \rrbracket & \mathcal{L}Y_\varepsilon^{\vee} &:= Y_\varepsilon^{\vee}, \\ Y_\varepsilon^{\vee} &:= \varepsilon^{-\frac{3}{2}}\tilde{F}_\varepsilon(\varepsilon^{\frac{1}{2}}Y_\varepsilon) & \mathcal{L}Y_\varepsilon^{\vee} &:= Y_\varepsilon^{\vee} \\ Y_\varepsilon^{\vee} &:= \frac{1}{3}\varepsilon^{-1}\tilde{F}_\varepsilon^{(1)}(\varepsilon^{\frac{1}{2}}Y_\varepsilon) & \mathcal{L}Y_\varepsilon^{\vee} &:= Y_\varepsilon^{\vee} \\ Y_\varepsilon^{\vee} &:= \frac{1}{6}\varepsilon^{-\frac{1}{2}}\tilde{F}_\varepsilon^{(2)}(\varepsilon^{\frac{1}{2}}Y_\varepsilon) & Y_\varepsilon^{\emptyset} &:= \frac{1}{6}\tilde{F}_\varepsilon^{(3)}(\varepsilon^{\frac{1}{2}}Y_\varepsilon) \\ \bar{Y}_\varepsilon^{\vee\vee} &:= \bar{Y}_\varepsilon^{\vee} \circ Y_\varepsilon^{\vee} - \bar{d}_\varepsilon^{\vee\vee} & Y_\varepsilon^{\vee\vee} &:= Y_\varepsilon^{\vee} \circ Y_\varepsilon^{\vee} - d_\varepsilon^{\vee\vee}, \\ Y_\varepsilon^{\vee\vee} &:= Y_\varepsilon^{\vee} \circ Y_\varepsilon^{\vee} - d_\varepsilon^{\vee\vee}, & Y_\varepsilon^{\vee\vee\vee} &:= Y_\varepsilon^{\vee} \circ Y_\varepsilon^{\vee} - d_\varepsilon^{\vee\vee\vee}Y_\varepsilon - d_\varepsilon^{\vee\vee\vee}, \end{aligned} \quad (6.12)$$

with Y_ε stationary solution, while $Y_\varepsilon, Y_\varepsilon^{\vee}, Y_\varepsilon^{\vee\vee}, \bar{Y}_\varepsilon^{\vee}$ have 0 initial condition in $t = 0$. The last four trees $\bar{Y}_\varepsilon^{\vee\vee}, Y_\varepsilon^{\vee\vee}, Y_\varepsilon^{\vee\vee\vee}, \bar{Y}_\varepsilon^{\vee\vee\vee}$ are obtained from the others via the resonant Bony's paraproduct \circ defined in Section 1.2, and $\bar{d}_\varepsilon^{\vee\vee}, d_\varepsilon^{\vee\vee}, d_\varepsilon^{\vee\vee\vee}, d_\varepsilon^{\vee\vee\vee\vee}, d_\varepsilon^{\vee\vee\vee\vee}$ are just ε -dependent constants whose exact value will matter only in Section 6.3. Indeed, in the scope of this section we only need the following relation to be verified:

$$d_\varepsilon^{\vee\vee\vee\vee} = 2d_\varepsilon^{\vee\vee\vee} + 3d_\varepsilon^{\vee\vee}. \quad (6.13)$$

The notation $\bar{Y}_\varepsilon^{\vee}$ denotes that this tree has finite chaos expansion and can be treated with the well-known techniques of [CC13] or [MWX16] (we put a bar on $\bar{Y}_\varepsilon^{\vee}$ just because it is the only tree obtained from $\bar{Y}_\varepsilon^{\vee}$).

With the definitions (6.12), equation (6.11) takes the form

$$\begin{aligned} \mathcal{L}v_\varepsilon = & Y_\varepsilon - \bar{Y}_\varepsilon^{\vee} - Y_\varepsilon^{\vee} - 3Y_\varepsilon^{\vee\vee}v_\varepsilon - 3Y_\varepsilon^{\vee\vee\vee}v_\varepsilon^2 - Y_\varepsilon^{\emptyset}v_\varepsilon^3 \\ & - \varepsilon^{-3/2}f_{0,\varepsilon} - \varepsilon^{-1}f_{1,\varepsilon}(Y_\varepsilon + v_\varepsilon) - \varepsilon^{-1/2}f_{2,\varepsilon}(2Y_\varepsilon v_\varepsilon + v_\varepsilon^2) - R_\varepsilon(v_\varepsilon). \end{aligned} \quad (6.14)$$

At this point it is worth noting that the trivial case $\tilde{F}_\varepsilon(x) = H_3(x, \sigma_\varepsilon^2)$ yields $Y_\varepsilon^{\vee} = \llbracket Y_\varepsilon^3 \rrbracket$, $Y_\varepsilon^{\vee\vee} = \llbracket Y_\varepsilon^2 \rrbracket$, $Y_\varepsilon^{\vee\vee\vee} = Y_\varepsilon$, $Y_\varepsilon^{\emptyset} = 1$. By comparing these random fields to the ones defined in [CC13] we can *guess* that $Y_\varepsilon^{\vee}, Y_\varepsilon^{\vee\vee}, Y_\varepsilon^{\vee\vee\vee}, Y_\varepsilon^{\emptyset}$ can be controlled respectively in $\mathcal{C}^{1/2-\kappa}$, $\mathcal{C}^{-1-\kappa}$, $\mathcal{C}^{-1/2-\kappa}$, $\mathcal{C}^{-\kappa}$ $\forall \kappa > 0$ for any F_ε satisfying Assumption 6.1, and carry on the paracontrolled analysis of (6.14) as if it were the case. Clearly, the paracontrolled structure is robust and does not depend on how the terms $Y_\varepsilon^{\vee}, Y_\varepsilon^{\vee\vee}, Y_\varepsilon^{\vee\vee\vee}, Y_\varepsilon^{\emptyset}$ are defined as long as they have the desired regularity.

From these observations, we do not expect to be able to control the products $Y_\varepsilon^\vee v_\varepsilon$, $Y_\varepsilon' v_\varepsilon^2$ and $Y_\varepsilon v_\varepsilon$ in eq. (6.14) uniformly in $\varepsilon > 0$. In order to proceed with the analysis we make the Ansatz:

$$\begin{aligned} u_\varepsilon &= Y_\varepsilon + v_\varepsilon, \\ v_\varepsilon &= -Y_\varepsilon^\Psi - \bar{Y}_\varepsilon^\Psi - 3v_\varepsilon \llcorner Y_\varepsilon^\Psi + v_\varepsilon^\natural \end{aligned} \quad (6.15)$$

and proceed to decompose the ill-defined products using the paracontrolled introduced in Section 1.2. We start by writing $v_\varepsilon Y_\varepsilon^\vee = v_\varepsilon \prec Y_\varepsilon^\vee + v_\varepsilon \succ Y_\varepsilon^\vee + v_\varepsilon \circ Y_\varepsilon^\vee$. The resonant term, together with Ansatz (6.15), yields:

$$\begin{aligned} v_\varepsilon \circ Y_\varepsilon^\vee &= -Y_\varepsilon^\Psi \circ Y_\varepsilon^\vee - \bar{Y}_\varepsilon^\Psi \circ Y_\varepsilon^\vee - 3v_\varepsilon (Y_\varepsilon^\Psi \circ Y_\varepsilon^\vee) \\ &\quad - 3\overline{\text{com}}_1(v_\varepsilon, Y_\varepsilon^\Psi, Y_\varepsilon^\vee) + v_\varepsilon^\natural \circ Y_\varepsilon^\vee, \end{aligned}$$

with the definition and bounds of $\overline{\text{com}}_1(\cdot, \cdot, \cdot)$ given in Lemma 1.67. Then we define

$$\begin{aligned} Y_\varepsilon^\vee \hat{\diamond} v_\varepsilon &:= v_\varepsilon Y_\varepsilon^\vee - v_\varepsilon \prec Y_\varepsilon^\vee + (3v_\varepsilon d_\varepsilon^{\Psi\Psi} + d_\varepsilon^{\Psi\Psi} Y_\varepsilon + d_\varepsilon^{\Psi\Psi} + \bar{d}_\varepsilon^{\Psi\Psi}) \\ &= v_\varepsilon \succ Y_\varepsilon^\vee - \bar{Y}_\varepsilon^{\Psi\Psi} - Y_\varepsilon^{\Psi\Psi} - 3v_\varepsilon Y_\varepsilon^{\Psi\Psi} + v_\varepsilon^\natural \circ Y_\varepsilon^\vee - 3\overline{\text{com}}_1(v_\varepsilon, Y_\varepsilon^\Psi, Y_\varepsilon^\vee). \end{aligned}$$

Moreover we have for $v_\varepsilon Y_\varepsilon$:

$$v_\varepsilon Y_\varepsilon = \varphi_\varepsilon Y_\varepsilon - Y_\varepsilon^\Psi \prec Y_\varepsilon - Y_\varepsilon^\Psi \succ Y_\varepsilon - Y_\varepsilon^\Psi \circ Y_\varepsilon,$$

where we introduced the shorthand $\varphi_\varepsilon := v_\varepsilon + Y_\varepsilon^\Psi$. So we let

$$v_\varepsilon \diamond Y_\varepsilon := v_\varepsilon Y_\varepsilon + d_\varepsilon^{\Psi\Psi} = \varphi_\varepsilon Y_\varepsilon - Y_\varepsilon^\Psi \prec Y_\varepsilon - Y_\varepsilon^\Psi \succ Y_\varepsilon - Y_\varepsilon^{\Psi\Psi},$$

Finally to analyse the product $Y_\varepsilon' v_\varepsilon^2$ we write

$$Y_\varepsilon' v_\varepsilon^2 = Y_\varepsilon' (Y_\varepsilon^\Psi)^2 - 2Y_\varepsilon' Y_\varepsilon^\Psi \varphi_\varepsilon + Y_\varepsilon' \varphi_\varepsilon^2,$$

and consider the products involving only Y^τ factors: first

$$Y_\varepsilon' Y_\varepsilon^\Psi = Y_\varepsilon' \succ Y_\varepsilon^\Psi + Y_\varepsilon' \prec Y_\varepsilon^\Psi + Y_\varepsilon^{\Psi\Psi} + d_\varepsilon^{\Psi\Psi} =: Y_\varepsilon' \diamond Y_\varepsilon^\Psi + d_\varepsilon^{\Psi\Psi},$$

and then we define the term $Y_\varepsilon' \diamond (Y_\varepsilon^\Psi)^2$ as follows:

$$\begin{aligned} Y_\varepsilon' \diamond (Y_\varepsilon^\Psi)^2 &:= Y_\varepsilon' (Y_\varepsilon^\Psi)^2 - 2d_\varepsilon^{\Psi\Psi} Y_\varepsilon^\Psi \\ &= Y_\varepsilon' \prec (Y_\varepsilon^\Psi)^2 + Y_\varepsilon' \succ (Y_\varepsilon^\Psi)^2 + Y_\varepsilon' \circ (Y_\varepsilon^\Psi \circ Y_\varepsilon^\Psi) \\ &\quad + 2\text{com}_1(Y_\varepsilon^\Psi, Y_\varepsilon^\Psi, Y_\varepsilon') + 2Y_\varepsilon^{\Psi\Psi} Y_\varepsilon^{\Psi\Psi}, \end{aligned}$$

so that

$$Y_\varepsilon' \diamond v_\varepsilon^2 := Y_\varepsilon' v_\varepsilon^2 + 2d_\varepsilon^{\Psi\Psi} v_\varepsilon = Y_\varepsilon' \diamond (Y_\varepsilon^\Psi)^2 - 2(Y_\varepsilon' \diamond Y_\varepsilon^\Psi) \varphi_\varepsilon + Y_\varepsilon' \varphi_\varepsilon^2.$$

We note also that

$$\mathcal{L}v_\varepsilon = -\mathcal{L}Y_\varepsilon^\Psi - \mathcal{L}\bar{Y}_\varepsilon^\Psi + \mathcal{L}v_\varepsilon^\natural - 3v_\varepsilon \prec \mathcal{L}Y_\varepsilon^\Psi - 3\text{com}_3(v_\varepsilon, Y_\varepsilon^\Psi) - 3\text{com}_2(v_\varepsilon, Y_\varepsilon^\vee),$$

with $\text{com}_2(\cdot, \cdot)$ and $\text{com}_3(\cdot, \cdot)$ specified in Lemma 1.67. Substituting these renormalized products into (6.14) and recalling the definition (6.9) for $\lambda_\varepsilon = (\lambda_{0,\varepsilon}, \lambda_{1,\varepsilon}, \lambda_{2,\varepsilon}, \lambda_{3,\varepsilon})$, we obtain the following equation for v_ε^\natural :

$$\begin{aligned} \mathcal{L}v_\varepsilon^\natural &= 3\text{com}_3(v_\varepsilon, Y_\varepsilon^\Psi) + 3\text{com}_2(v_\varepsilon, Y_\varepsilon^\vee) \\ &\quad - Y_\varepsilon^\emptyset v_\varepsilon^3 - 3Y_\varepsilon' \diamond v_\varepsilon^2 - 3Y_\varepsilon^\vee \diamond v_\varepsilon \\ &\quad + Y_\varepsilon - \lambda_{2,\varepsilon} (2v_\varepsilon \diamond Y_\varepsilon + v_\varepsilon^2) \\ &\quad - \lambda_{1,\varepsilon} (Y_\varepsilon + v_\varepsilon) + [9d_\varepsilon^{\Psi\Psi} + 6d_\varepsilon^{\Psi\Psi} - 3d_\varepsilon^{\Psi\Psi}] v_\varepsilon - \lambda_{0,\varepsilon} - R_\varepsilon(v_\varepsilon), \end{aligned}$$

where we can use the constraint (6.13) to remove the term proportional to v_ε . Summarizing, we obtain the following equation, together with Ansatz (6.15):

$$\begin{cases} v_\varepsilon = -Y_\varepsilon^\Upsilon - \bar{Y}_\varepsilon^\Upsilon - 3v_\varepsilon \llcorner Y_\varepsilon^\Upsilon + v_\natural \\ \mathcal{L}v_\varepsilon^\natural = U(\lambda_\varepsilon, \mathbb{Y}_\varepsilon; v_\varepsilon, v_\varepsilon^\natural) - R_\varepsilon(v_\varepsilon) \end{cases} \quad (6.16)$$

with initial condition $v_{\varepsilon,0} = u_{0,\varepsilon} - Y_\varepsilon(0)$ and U given by

$$\begin{aligned} U(\lambda_\varepsilon, \mathbb{Y}_\varepsilon; v_\varepsilon, v_\varepsilon^\natural) := & 3 \text{com}_3(v_\varepsilon, Y_\varepsilon^\Upsilon) + 3 \text{com}_2(v_\varepsilon, Y_\varepsilon^\vee) - Y_\varepsilon^\emptyset v_\varepsilon^3 \\ & - 3Y_\varepsilon^\natural \diamond v_\varepsilon^2 - 3Y_\varepsilon^\vee \diamond v_\varepsilon + Y_\varepsilon - \lambda_{2,\varepsilon}(2v_\varepsilon \diamond Y_\varepsilon + v_\varepsilon^2) \\ & - \lambda_{1,\varepsilon}(Y_\varepsilon + v_\varepsilon) - \lambda_{0,\varepsilon} - R_\varepsilon(v_\varepsilon). \end{aligned} \quad (6.17)$$

The enhanced noise vector \mathbb{Y}_ε is defined by

$$\begin{aligned} \mathbb{Y}_\varepsilon := & (Y_\varepsilon^\emptyset, Y_\varepsilon^\natural, Y_\varepsilon^\vee, \bar{Y}_\varepsilon^\vee, Y_\varepsilon^\Upsilon, Y_\varepsilon^\vee, Y_\varepsilon^{\vee\vee}, Y_\varepsilon^{\vee\vee}, Y_\varepsilon^{\vee\vee}) \\ \cap \\ \mathcal{X}_T := & C_T \mathcal{C}^{-\kappa} \times C_T \mathcal{C}^{-\frac{1}{2}-\kappa} \times (C_T \mathcal{C}^{-1-\kappa})^2 \times \mathcal{L}_T^{1/2-\kappa} \times (C_T \mathcal{C}^{-\kappa})^3 \times C_T \mathcal{C}^{-\frac{1}{2}-\kappa} \end{aligned} \quad (6.18)$$

for every $\kappa > 0$, $T > 0$. We use the notation $\|\mathbb{Y}_\varepsilon\|_{\mathcal{X}_T} = \sum_\tau \|\mathbb{Y}_\varepsilon^\tau\|_{\mathcal{X}^\tau}$ for the associated norm where Y_ε^τ is a generic tree in \mathbb{Y}_ε . The homogeneities $|\tau| \in \mathbb{R}$ are given by

Y_ε^τ	=	Y_ε^\emptyset	Y_ε^\natural	Y_ε^\vee	\bar{Y}_ε^\vee	Y_ε^Υ	$Y_\varepsilon^{\vee\vee}$	$Y_\varepsilon^{\vee\vee}$	$Y_\varepsilon^{\vee\vee}$
$ \tau $	=	0	-1/2	-1	-1	1/2	0	0	-1/2

Note that for every $\varepsilon > 0$ eq. (6.16) is equivalent to eq. (6.5).

Remark 6.6. The paracontrolled structure we developed in this section is the same as in the work of Catellier and Chouk [CC13], plus an additive source term (which is $R_\varepsilon(v_\varepsilon)$ in equation (6.16)). Therefore, there exists $T = T(\|\mathbb{Y}_\varepsilon\|_{\mathcal{X}_T}, \|u_{\varepsilon,0}\|_{\mathcal{C}^{-1/2-\kappa}}, |\lambda_\varepsilon|)$ such that we can define for $\alpha \in (1/2, 2/3)$, $p \in [4, \infty)$, $\gamma > \frac{1}{4} + \frac{3}{2}\kappa$ a solution map

$$\begin{aligned} \Gamma: \mathcal{C}^{-1/2-\kappa} \times \mathcal{X}^\tau \times \mathbb{R}^4 \times \mathcal{M}_T^{\gamma,p} L^p(\mathbb{T}^3) & \rightarrow C_T \mathcal{C}^{-\alpha}(\mathbb{T}^3) \\ (u_{\varepsilon,0}, \mathbb{Y}_\varepsilon, \lambda_\varepsilon, \mathcal{R}) & \mapsto u_\varepsilon \end{aligned}$$

so that $u_\varepsilon = \Gamma(u_{\varepsilon,0}, \mathbb{Y}_\varepsilon, \lambda_\varepsilon, \mathcal{R})$ with $u_\varepsilon = Y_\varepsilon + v_\varepsilon$ and v_ε that solves (6.16) with the remainder $R_\varepsilon(v_\varepsilon)$ replaced by \mathcal{R} . The space $\mathcal{M}_T^{\gamma,p} L^p(\mathbb{T}^3)$ is specified in Section 1.2.2. Indeed, we can use Lemma 1.60 and Lemma 1.62 to control $I\mathcal{R}$ as

$$\|I\mathcal{R}\|_{\mathcal{C}_T^{1/2, 1+2\kappa}} \lesssim T^\delta \|\mathcal{R}\|_{\mathcal{M}_T^{\gamma,p} L^p}$$

for $\delta > 0$ small enough, and thanks to this bound it is easy to see that the the fixed point procedure of Section 3 of [CC13] still holds with a *fixed* additive source term \mathcal{R} . In the same way, the continuity of the solution map Γ follows easily as in Theorem 1.2 of [CC13].

6.2.2 Identification of the limit

In order to identify interesting limits for equation (6.5), we introduce for every parameter $\lambda = (\lambda_0, \lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^4$ the enhanced noise $\mathbb{Y}(\lambda)$ which is constructed from universal noises X^τ as

$$\begin{aligned} \mathbb{Y}(\lambda) & \\ := & (Y^\emptyset(\lambda), Y^\natural(\lambda), Y^\vee(\lambda), \bar{Y}^\vee(\lambda), Y^\Upsilon(\lambda), Y^{\vee\vee}(\lambda), Y^{\vee\vee}(\lambda), Y^{\vee\vee}(\lambda), Y^{\vee\vee}(\lambda)) \\ := & (\lambda_3, \lambda_3 X, \lambda_3 X^\vee, \lambda_2 X^\vee, \lambda_3 X^\Upsilon, (\lambda_3)^2 X^{\vee\vee}, (\lambda_3)^2 X^{\vee\vee}, \lambda_3 \lambda_2 X^{\vee\vee}, (\lambda_3)^2 X^{\vee\vee}) \end{aligned} \quad (6.19)$$

where X is the stationary solution to the linear equation $\mathcal{L}X = -X + \xi$ and ξ is the time-space white noise on $\mathbb{R} \times \mathbb{T}^3$. We will sometimes use the shorter notation $\mathbb{Y}(\lambda) = (Y^\tau(\lambda))_\tau$ for (6.19).

We define the universal fields X^τ through their Littlewood-Paley decomposition $\forall(t, \bar{x}) \in \mathbb{R}^+ \times \mathbb{T}^3$ as:

$$\begin{aligned}
X^\vee &:= \llbracket X^3 \rrbracket, & \mathcal{L}X^\vee = X^\vee & \text{ with } X^\vee(t=0) = 0, \\
X^\nabla &:= \llbracket X^2 \rrbracket, \\
\Delta_q X^{\vee\vee}(t, \bar{x}) &:= \Delta_q(1 - J_0)(X^\vee \circ X^\nabla)(t, \bar{x}) = \int_{\zeta_1, \zeta_2} (1 - J_0)(\llbracket X^2(\zeta_1) \rrbracket \llbracket X^2(\zeta_2) \rrbracket) \mu_{q, \zeta_1, \zeta_2}, \\
\Delta_q X^{\vee\nabla}(t, \bar{x}) &:= \Delta_q(X^\vee \circ X)(t, \bar{x}) = \int_{\zeta_1, \zeta_2} \llbracket X^3(\zeta_1) \rrbracket X(\zeta_2) \mu_{q, \zeta_1, \zeta_2}, \\
\Delta_q X^{\vee\nabla\nabla}(t, \bar{x}) &:= \int_{\zeta_1, \zeta_2} (1 - J_1)(\llbracket X^3(\zeta_1) \rrbracket \llbracket X^2(\zeta_2) \rrbracket) \mu_{q, \zeta_1, \zeta_2} \\
&\quad + 6 \int_{s, x} [\Delta_q X(t+s, \bar{x}-x) - \Delta_q X(t, \bar{x})] P_s(x) [C(s, x)]^2,
\end{aligned} \tag{6.20}$$

where as before $\llbracket \cdot \rrbracket$ stands for the Wick product, $\zeta_i = (x_i, s_i) \in \mathbb{R} \times \mathbb{T}^3$, $C(\cdot, \cdot)$ is the covariance of X and $\mu_{q, \zeta_1, \zeta_2}$ is the measure

$$\mu_{q, \zeta_1, \zeta_2} := \mathbb{1}_{[0, +\infty)}(s_1) \delta(t - s_2) d\zeta_1 d\zeta_2 \int_{x, y} K_{q, \bar{x}}(x) \sum_{i \sim j} K_{i, x}(y) K_{j, x}(x_2) P_{t-s_1}(y - x_1)$$

with the usual heat kernel $P_t(x) = \frac{1}{(4\pi t)^{3/2}} e^{-\frac{|x|^2}{4t}} \mathbb{1}_{t \geq 0}$. We commit an abuse of notation by writing $X(\zeta)$ since X is actually a distribution in space: the integrals in (6.20) should obviously be intended as functionals.

Standard computations (see e.g. [CC13] or [MWX16]) show that, $\forall \lambda \in \mathbb{R}^4$ and for any $T > 0$, $0 < \kappa < \kappa'$

$$\mathbb{Y}(\lambda) \in C_T^\kappa \mathcal{C}^{-\frac{1}{2}-2\kappa'} \times (C_T^\kappa \mathcal{C}^{-1-2\kappa'})^2 \times C_T^\kappa \mathcal{C}^{\frac{1}{2}-2\kappa'} \times (C_T^\kappa \mathcal{C}^{0-2\kappa'})^2 \times C_T^\kappa \mathcal{C}^{\frac{1}{2}-2\kappa'},$$

almost surely.

Using the paracontrolled structure we developed in Section 6.2.1 we can identify the limiting solution $u(\lambda)$ introduced in Theorem 6.3.

Theorem 6.7. *The family of random fields u_ε given by the solutions of eq. (6.5) converges in law and locally in time to a limiting random field $u(\lambda)$ in the space $C_T \mathcal{C}^{-\alpha}(\mathbb{T}^3)$ for every $1/2 < \alpha < 2/3$. The limiting random field $u(\lambda)$ solves the paracontrolled equation*

$$\left\{ \begin{array}{l} u(\lambda) = X + v(\lambda) \\ v(\lambda) = -\lambda_3 X^\vee - \lambda_2 X^\nabla - 3\lambda_3 v(\lambda) \prec X^\vee + v^\natural(\lambda) \\ \mathcal{L}v^\natural(\lambda) = U(\lambda, \mathbb{Y}(\lambda); v(\lambda), v^\natural(\lambda)) \\ v^\natural(\lambda)(t=0) = v_0 + \lambda_3 X^\vee(t=0) + \lambda_2 X^\nabla(t=0) + 3\lambda_3 v_{\varepsilon, 0} \prec X^\vee(t=0) \end{array} \right. \tag{6.21}$$

with U defined in (6.17) and $v_0 = u_0 - X(t=0)$.

Proof. Fix $T > 0$. Let $u_\varepsilon = Y_\varepsilon + v_\varepsilon$ be the solution of eq. (6.5) for fixed $\varepsilon > 0$, which is seen to be unique in the (ε -dependent) time interval $[0, T_\varepsilon]$ by a usual fixed-point argument on the original equation (without resorting to the paracontrolled decomposition). Let $u_\varepsilon = \Gamma(u_{\varepsilon,0}, \mathbb{Y}_\varepsilon, \lambda_\varepsilon, R_\varepsilon(v_\varepsilon))$ on $[0, T_\varepsilon]$ with Γ defined as in Remark 6.6 and $R_\varepsilon(v_\varepsilon)$ seen as an exogenous source term. We know from the a priori estimations of Section 6.4.2 that there exists a time $T_\star = T_\star(\|\mathbb{Y}_\varepsilon\|_{\mathcal{X}_T}, \|u_{\varepsilon,0}\|_{\mathcal{C}^{-1/2-\kappa}}, |\lambda_\varepsilon|)$ and a family of events $(\mathcal{E}_\varepsilon)_{\varepsilon > 0}$ such that $\mathbb{P}(\mathcal{E}_\varepsilon) \rightarrow 1$ for $\varepsilon \rightarrow 0$ and we can control $\|v_\varepsilon\|_{\mathcal{M}_{T_\star}^{1/4+3\kappa/2}L^\infty}$. Thus, we can control the L^∞ norm of $v_\varepsilon(t)$ in $[T_\varepsilon/2, T_\varepsilon]$ and extend the solution v_ε on $[0, T_\star]$ for every ε . Denote by $u_\varepsilon^\star = \Gamma_\star(u_{\varepsilon,0}, \mathbb{Y}_\varepsilon, \lambda_\varepsilon, R_\varepsilon(v_\varepsilon))$ the process u_ε stopped at time T_\star and Γ_\star the corresponding stopped solution map.

Note that $u(\lambda)$ solves the same equation as u_ε^\star with \mathbb{Y}_ε replaced by $\mathbb{Y}(\lambda)$, $u_{\varepsilon,0}$ replaced by u_0 , λ_ε replaced by λ and $R_\varepsilon(v_\varepsilon) = 0$. So $u(\lambda) = u^\star = \Gamma_\star(u_0, \mathbb{Y}(\lambda), \lambda, 0)$ up to time T_\star . Let us introduce the random field $\bar{u}_\varepsilon^\star = \Gamma_\star(u_{\varepsilon,0}, \mathbb{Y}_\varepsilon, \lambda_\varepsilon, 0)$ with $\bar{v}_\varepsilon^\star = \bar{u}_\varepsilon^\star - Y_\varepsilon$ that solves the paracontrolled equation (6.16) but with remainder $R_\varepsilon(v_\varepsilon) = 0$.

Consider the n -tuple of random variables $(u_{\varepsilon,0}, \mathbb{Y}_\varepsilon, u_\varepsilon^\star, \bar{u}_\varepsilon^\star)$ and let μ_ε be its law on $\mathcal{Z} = \mathcal{C}^{-\alpha} \times \mathcal{X}_T \times (C_T \mathcal{C}^{-\alpha})^2$ conditionally on \mathcal{E}_ε . Observe that Γ_\star is continuous as discussed in Remark 6.6, and this gives that $\forall \delta > 0$, $\mu_\varepsilon(\|u_\varepsilon^\star - \bar{u}_\varepsilon^\star\|_{C_T \mathcal{C}^{-\alpha}} > \delta) \rightarrow 0$ as $\varepsilon \rightarrow 0$. Indeed $R_\varepsilon(v_\varepsilon) \rightarrow 0$ in probability in the space $\mathcal{M}_{T_\star}^{\gamma;p}L^p(\mathbb{T}^3)$ by Lemma 6.19. This shows that μ_ε concentrates on $\mathcal{C}^{-\alpha} \times \mathcal{X}_T \times \{(z, z) \in (C_T \mathcal{C}^{-\alpha})^2\}$. Let μ any accumulation point of $(\mu_\varepsilon)_\varepsilon$. Then $\mu(\mathcal{C}^{-\alpha} \times \mathcal{X}_T \times \{(z, z) \in (C_T \mathcal{C}^{-\alpha})^2\}) = 1$. The a priori estimations of Section 6.4.2 yield the tightness of μ_ε and from the concentration of μ on the diagonal we know that there exists a subsequence such that for any test function φ ,

$$\begin{aligned} \int_{\mathcal{Z}} \varphi(x, y, z, t) d\mu_\varepsilon(x, y, z, t) &\rightarrow \int_{\mathcal{Z}} \varphi(x, y, z, t) d\mu(x, y, z, t) \\ &= \int_{\mathcal{Z}} \varphi(x, y, t, t) d\mu(x, y, z, t). \end{aligned} \quad (6.22)$$

Moreover, still along subsequences we have that for any bounded continuous function φ

$$\begin{aligned} \mathbb{E}(\varphi(u_{\varepsilon,0}, \mathbb{Y}_\varepsilon, \bar{u}_\varepsilon^\star)) &= \mathbb{E}(\varphi(u_{\varepsilon,0}, \mathbb{Y}_\varepsilon, \Gamma_\star(u_{\varepsilon,0}, \mathbb{Y}_\varepsilon, \lambda_\varepsilon, 0))) \\ &\rightarrow \mathbb{E}(\varphi(u_0, \mathbb{Y}(\lambda), \Gamma_\star(u_0, \mathbb{Y}(\lambda), \lambda, 0))) \end{aligned}$$

since by Theorem 6.8 the vector \mathbb{Y}_ε converges in law to $\mathbb{Y}(\lambda)$, $u_{\varepsilon,0}$ to u_0 , and Γ_\star is a continuous function as discussed in Remark 6.6. This shows that

$$\int_{\mathcal{Z}} \varphi(x, y, t, t) d\mu_\varepsilon(x, y, z, t) \rightarrow \int_{\mathcal{Z}} \varphi(x, y, \Gamma_\star(x, y, \lambda, 0), \Gamma_\star(x, y, \lambda, 0)) d\mu(x, y, z, t) \quad (6.23)$$

and then by comparing (6.22) and (6.23) we can conclude that there exists a subsequence such that

$$\int_{\mathcal{Z}} \varphi(x, y, z, t) d\mu_\varepsilon(x, y, z, t) \rightarrow \int_{\mathcal{Z}} \varphi(x, y, \Gamma_\star(x, y, \lambda, 0), \Gamma_\star(x, y, \lambda, 0)) d\mu(x, y, z, t).$$

We can identify the limit distribution μ by noting that since $\mathbb{P}(\mathcal{E}_\varepsilon) \rightarrow 1$ we have

$$\mathbb{E}[\psi(u_{\varepsilon,0}, \mathbb{Y}_\varepsilon) | \mathcal{E}_\varepsilon] = \frac{\mathbb{E}[\psi(u_{\varepsilon,0}, \mathbb{Y}_\varepsilon) \mathbb{I}_{\mathcal{E}_\varepsilon}]}{\mathbb{P}(\mathcal{E}_\varepsilon)} \rightarrow \mathbb{E}[\psi(u_0, \mathbb{Y}(\lambda))]$$

for any test function ψ . So the first two marginals of μ have the law of $(u_0, \mathbb{Y}(\lambda))$ and they are independent since $(u_{\varepsilon,0}, \mathbb{Y}_\varepsilon)$ are independent for any ε . Calling ν the law of $(u_0, \mathbb{Y}(\lambda))$ we have that

$$\int_{\mathcal{Z}} \varphi(x, y, z, t) d\mu(x, y, z, t) = \int_{\mathcal{C}^{-\alpha} \times \mathcal{X}_T} \varphi(x, y, \Gamma_\star(x, y, \lambda, 0), \Gamma_\star(x, y, \lambda, 0)) d\nu(x, y)$$

which implies that μ is unique and that the whole family $(\mu_\varepsilon)_\varepsilon$ converges to μ . We can conclude that $u_\varepsilon^* \rightarrow u^*$ in law with $u^* = u(\lambda)$ up to the time $T_*(\|\mathbb{Y}(\lambda)\|_{\mathcal{X}_T}, \|u_0\|_{\mathcal{C}^{-1/2-\kappa}}, |\lambda|)$ since the function T^* is lower semicontinuous (as obtained from the a priori estimates). \square

6.3 Convergence of the enhanced noise

This is the most important section of this chapter, in which we present a new method to estimate certain random fields that do not have a finite chaos decomposition, and we apply it to the treatment of the random fields \mathbb{Y}_ε of (6.12).

6.3.1 An example of convergence

We choose to give first a complete example (the convergence of the tree Y_ε^\vee to $Y^\vee(\lambda)$) in order to put in evidence the main idea in the proof of Theorem 6.8. Recall its definition (6.12):

$$Y_\varepsilon^\vee = \frac{\varepsilon^{-1}}{3} \tilde{F}_\varepsilon^{(1)}(\varepsilon^{\frac{1}{2}} Y_\varepsilon),$$

with $\tilde{F}_\varepsilon^{(1)}$ being the first derivative of the centered function \tilde{F}_ε defined in (6.6). Since $\frac{d}{dx} H_n(x, \sigma_\varepsilon^2) = n H_{n-1}(x, \sigma_\varepsilon^2)$ the Wiener chaos decomposition of Y_ε^\vee reads:

$$\begin{aligned} \frac{\varepsilon^{-1}}{3} \tilde{F}_\varepsilon^{(1)}(\varepsilon^{\frac{1}{2}} Y_\varepsilon) &= \frac{\varepsilon^{-1}}{3} \sum_{n \geq 3} n f_{n,\varepsilon} H_{n-1}(\varepsilon^{1/2} Y_\varepsilon, \sigma_\varepsilon^2) \\ &= \varepsilon^{-1} f_{3,\varepsilon} H_2(\varepsilon^{1/2} Y_\varepsilon, \sigma_\varepsilon^2) + \frac{\varepsilon^{-1}}{3} \sum_{n \geq 4} n f_{n,\varepsilon} H_{n-1}(\varepsilon^{1/2} Y_\varepsilon, \sigma_\varepsilon^2) \\ &= f_{3,\varepsilon} \llbracket Y_\varepsilon^2 \rrbracket + \frac{\varepsilon^{-1}}{3} \sum_{n \geq 4} n f_{n,\varepsilon} H_{n-1}(\varepsilon^{1/2} Y_\varepsilon, \sigma_\varepsilon^2), \end{aligned} \quad (6.24)$$

where $\llbracket \cdot \rrbracket$ is the Wiener product and we used the fact that $\varepsilon^{-\frac{n}{2}} H_n(\varepsilon^{1/2} Y_\varepsilon, \sigma_\varepsilon^2) = \llbracket Y_\varepsilon^n \rrbracket$. Now one can use hypercontractivity (as done in [HQ15],[HX16]) to control the L^p norm of each chaos order by its L^2 norms. However this strategy does not give useful bounds for the infinite series in the second term of (6.24). Instead, we just observe that

$$\sum_{n \geq 4} n f_{n,\varepsilon} H_{n-1}(\varepsilon^{1/2} Y_\varepsilon, \sigma_\varepsilon^2) = (\text{id} - J_0 - \dots - J_2) \tilde{F}_\varepsilon^{(1)}(\varepsilon^{\frac{1}{2}} Y_\varepsilon),$$

where J_i is the projection of on the i -th chaos, and look for a different way to write this remainder. One of the main insights of this chapter's work is that we can write it as:

$$(\text{id} - J_0 - \dots - J_2) \tilde{F}_\varepsilon^{(1)}(\varepsilon^{\frac{1}{2}} Y_\varepsilon) = \delta^3 G_{[1]}^{[3]} D^3 \tilde{F}_\varepsilon^{(1)}(\varepsilon^{\frac{1}{2}} Y_\varepsilon)$$

where D , δ are the Malliavin derivative and divergence operators, and $G_{[1]}^{[3]} = (1 - L)^{-1} (2 - L)^{-1} (3 - L)^{-1}$ with L the Ornstein-Uhlenbeck operator. This is proven in Lemma 2.22.

To compute the Malliavin derivative of $\tilde{F}_\varepsilon^{(1)}(\varepsilon^{\frac{1}{2}} Y_\varepsilon)$ we observe that for every $\varepsilon > 0$ $(t, x) \in \mathbb{R} \times \mathbb{T}^3$ there exists $h_{(t,x)} \in L^2(\mathbb{R} \times \mathbb{T}^3)$ such that the Gaussian random variable $Y_{\varepsilon,(t,x)} := Y_\varepsilon(t, x)$ can be written as

$$Y_{\varepsilon,(t,x)} \stackrel{\text{law}}{=} \langle \xi, h_{(t,x)} \rangle. \quad (6.25)$$

Here ξ is the Gaussian white noise on $\mathbb{R} \times \mathbb{T}^3$, which can be seen as a Gaussian Hilbert space $\langle \xi, h \rangle_{h \in H} = \{W(h)\}_{h \in H}$ indexed by the Hilbert space $H := L^2(\mathbb{R} \times \mathbb{T}^3)$. This is the framework in which we apply the Malliavin calculus results of Section 2.3. Notice that by construction

$$\langle h_{(t,x)}, h_{(t',x')} \rangle = C_\varepsilon(t-t', x-x') := \mathbb{E}[Y_{\varepsilon,(t,x)} Y_{\varepsilon,(t',x')}]. \quad (6.26)$$

The function $h_{(t,x)}$ can actually be written as the space-time convolution

$$h_{(t,x)} = \check{P} * \psi_\varepsilon(t, x),$$

with $\check{P}(t, x) = \frac{1}{(4\pi t)^{3/2}} e^{-\frac{|x|^2}{4t}} e^{-t} \mathbb{1}_{t \geq 0}$ and ψ_ε s.t. $\eta_\varepsilon = \psi_\varepsilon * \xi$, $\psi_\varepsilon(t, x) = \varepsilon^{-5/2} \psi(\varepsilon^{-2}t, \varepsilon^{-1}x)$.

We omit the dependence on ε of $h_{(t,x)}$ not to burden the notation.

Going back to the calculations, we obtain from (6.25) that

$$D\tilde{F}_\varepsilon^{(1)}(\varepsilon^{\frac{1}{2}} Y_\varepsilon) = \varepsilon^{\frac{1}{2}} \tilde{F}_\varepsilon^{(2)}(\varepsilon^{\frac{1}{2}} Y_\varepsilon) h.$$

Then (noting that $\tilde{F}_\varepsilon^{(4)} = F_\varepsilon^{(4)}$):

$$\begin{aligned} Y_\varepsilon^\vee &= f_{3,\varepsilon}[\![Y_\varepsilon^2]\!] + \frac{\varepsilon^{1/2}}{3} \delta^3 G_{[1]}^{[3]} F_\varepsilon^{(4)}(\varepsilon^{\frac{1}{2}} Y_\varepsilon) h^{\otimes 3} \\ &= f_{3,\varepsilon}[\![Y_\varepsilon^2]\!] + \hat{Y}_\varepsilon^\vee, \end{aligned} \quad (6.27)$$

It can be easily seen from (6.25) that Y_ε has the same law of a time-space mollification of X by convolution (with X defined in Section 6.2.2). Then the convergence in law of $f_{3,\varepsilon}[\![Y_\varepsilon^2]\!]$ to $Y^\vee(\lambda)$ can be easily established by standard techniques (see [CC13] or [MWX16]). We are only left to show that \hat{Y}_ε^\vee in (6.27) converges to zero in $C_T \mathcal{C}^{-1-\kappa}$.

It is well known (see Section 6.3.2 for details) that in order to control the norm of $\hat{Y}_\varepsilon^\vee(t, \cdot)$ for $t \in (0, T]$ in the Besov space $\mathcal{C}^{-\alpha-\kappa} \forall \kappa > 0$ and in probability, it is enough to have suitable estimates for

$$\sup_{x \in \mathbb{T}^3} \|\Delta_q \hat{Y}_\varepsilon^\vee(t, \bar{x})\|_{L^p(\Omega)} = \|\Delta_q \hat{Y}_\varepsilon^\vee(t, \bar{x})\|_{L^p(\Omega)},$$

for any $\bar{x} \in \mathbb{T}^3$ since \hat{Y}_ε^\vee is stationary in space. We then proceed to compute:

$$\|\Delta_q \hat{Y}_\varepsilon^\vee(t, \bar{x})\|_{L^p(\Omega)} = \frac{\varepsilon^{1/2}}{3} \left\| \delta^3 G_{[1]}^{[3]} \int K_{q,\bar{x}}(x) F_\varepsilon^{(4)}(\varepsilon^{\frac{1}{2}} Y_{\varepsilon,(t,x)}) h_{(t,x)}^{\otimes 3} dx \right\|_{L^p(\Omega)}$$

where $K_{q,\bar{x}}(x)$ is the kernel associated to the Littlewood-Paley block Δ_q . Observe that $\|F_\varepsilon^{(4)}(\varepsilon^{1/2} Y_{\varepsilon,(t,x)})\|_{L^p}^p = \int_{\mathbb{R}} |F_\varepsilon^{(4)}(z)|^p \gamma(dz)$ where $\gamma(dz)$ is the density of a centered Gaussian with variance σ_ε^2 . This norm is then finite by the bound (6.8) of Assumption 6.1.

Another fundamental idea of this work is that we can “estimate out” the bounded term $\|F_\varepsilon^{(4)}(\varepsilon^{1/2} Y_{\varepsilon,(t,x)})\|_{L^p}$, which has an infinite chaos decomposition, and obtain a standard Φ_3^4 diagram that can be treated with well understood techniques. Using Lemma 2.23 and Corollary 2.27, one has

$$\begin{aligned} & \left\| \delta^3 G_{[1]}^{[3]} \int K_{q,\bar{x}}(x) F_\varepsilon^{(4)}(\varepsilon^{\frac{1}{2}} Y_{\varepsilon,(t,x)}) h_{(t,x)}^{\otimes 3} dx \right\|_{L^p(\Omega)} \\ & \lesssim \sum_{k=0}^3 \left\| D^k G_{[1]}^{[3]} \int K_{q,\bar{x}}(x) F_\varepsilon^{(4)}(\varepsilon^{\frac{1}{2}} Y_{\varepsilon,(t,x)}) h_{(t,x)}^{\otimes 3} dx \right\|_{L^p(\Omega, H^{\otimes 3+k})} \\ & \lesssim \left\| \int K_{q,\bar{x}}(x) F_\varepsilon^{(4)}(\varepsilon^{\frac{1}{2}} Y_{\varepsilon,(t,x)}) h_{(t,x)}^{\otimes 3} dx \right\|_{L^p(\Omega, H^{\otimes 3})}. \end{aligned}$$

Then note that we can decompose the norm $\|u\|_{L^p(\Omega, H^{\otimes 3})} = \| \|u\|_{H^{\otimes 3}}^2 \|u\|_{L^{p/2}}^{1/2}$ for $u \in L^p(\Omega, H^{\otimes 3})$ and since the norm of the Hilbert space $H^{\otimes 3}$ is given by the scalar product $\|h^{\otimes 3}\|_{H^{\otimes 3}}^2 = \langle h, h \rangle_H^3$ we obtain

$$\begin{aligned} \|\Delta_q \hat{Y}_\varepsilon^\vee(t, \bar{x})\|_{L^p} &\lesssim \varepsilon^{1/2} \left\| \int K_{q, \bar{x}}(x) F_\varepsilon^{(4)}(\varepsilon^{1/2} Y_{\varepsilon, (t, x)}) h_{(t, x)}^{\otimes 3} dx \right\|_{H^{\otimes 3}}^2 \Big\|_{L^{p/2}}^{1/2} \\ &\lesssim \left[\varepsilon \int |K_{q, \bar{x}}(x) K_{q, \bar{x}}(x')| \left\| F_\varepsilon^{(4)}(\varepsilon^{1/2} Y_{\varepsilon, (t, x)}) F_\varepsilon^{(4)}(\varepsilon^{1/2} Y_{\varepsilon, (t, x')}) \right\|_{L^{p/2}} |\langle h_{(t, x)}, h_{(t, x')} \rangle|^3 dx dx' \right]^{1/2}. \end{aligned}$$

The norm containing $\tilde{F}_\varepsilon^{(4)}$ can then be easily estimated using Hölder's inequality as

$$\left\| F_\varepsilon^{(4)}(\varepsilon^{1/2} Y_{\varepsilon, (t, x)}) F_\varepsilon^{(4)}(\varepsilon^{1/2} Y_{\varepsilon, (t, x')}) \right\|_{L^{p/2}} \leq \left\| F_\varepsilon^{(4)}(\varepsilon^{1/2} Y_{\varepsilon, (t, x)}) \right\|_{L^p} \left\| F_\varepsilon^{(4)}(\varepsilon^{1/2} Y_{\varepsilon, (t, x')}) \right\|_{L^p} \lesssim 1.$$

This yields

$$\|\Delta_q \hat{Y}_\varepsilon^\vee(t, \bar{x})\|_{L^p(\Omega)} \lesssim \left[\varepsilon \int |K_{q, \bar{x}}(x) K_{q, \bar{x}}(x')| |\langle h_{(t, x)}, h_{(t, x')} \rangle|^3 dx dx' \right]^{1/2}$$

which is a standard Φ_3^4 diagram that can be analysed with the techniques of [Hai14a] (recalled in Section 2.2.1). We just remark that $\langle h_{(t, x)}, h_{(t, x')} \rangle = C_\varepsilon(0, x - x')$ and from the bound $\varepsilon |C_\varepsilon(t, x)| \lesssim 1$ of Lemma 2.15 we obtain $\forall \delta \in (0, 1)$:

$$\begin{aligned} \|\Delta_q \hat{Y}_\varepsilon^\vee(t, \bar{x})\|_{L^p(\Omega)} &\lesssim \varepsilon^{\frac{\delta}{2}} \left[\int |K_{q, \bar{x}}(x) K_{q, \bar{x}}(x')| |C_\varepsilon(0, x - x')|^{2+\delta} dx dx' \right]^{1/2} \\ &\lesssim \varepsilon^{\frac{\delta}{2}} 2^{(1+\delta/2)q} \end{aligned}$$

and then $\hat{Y}_\varepsilon^\vee(t, \cdot)$ converges to zero in probability in the space $\mathcal{C}^{-1-\delta/2} \forall \delta \in (0, 1)$, as $\varepsilon \rightarrow 0$. The time regularity of \hat{Y}_ε^\vee needed to obtain the convergence in $C_T \mathcal{C}^{-1-\delta/2}$ does not need new ideas, and it is done in Section 6.3.3.

The method shown in this section is valid verbatim for the trees $Y_\varepsilon^\emptyset, Y_\varepsilon^!, Y_\varepsilon^\Psi$, while for the composite trees in (6.12) (namely $Y_\varepsilon^{\Psi\Psi}, Y_\varepsilon^{\Psi\Psi}, \bar{Y}_\varepsilon^{\Psi\Psi}, Y_\varepsilon^{\Psi\Psi}$) that are obtained via paraproducts of simple trees, one has to be able to write the remainder $\hat{Y}_\varepsilon^{\tau}$ as an iterated Skorohod integral $\delta^n(\dots)$ in order to exploit the boundedness of this operator. Moreover, these trees require a second renormalization (on top of the Wick ordering) which is not easy to control for infinite chaos decompositions. We deal with both these difficulties employing the product formula (2.22) of Section 2.3, which allows to write products of iterated Skorohod integrals as combinations of iterated Skorohod integrals. The details and calculations for composite trees can be found in Section 6.3.4.

6.3.2 Main theorem and overview of the proof

Theorem 6.8. *Under Assumption 6.1 there exists $C > 0$ such that for any $p \in [2, \infty)$ we have $\|\mathbb{Y}_\varepsilon\|_{\mathcal{X}_T} < C$ in $L^p(\Omega)$. Moreover, $\mathbb{Y}_\varepsilon \rightarrow \mathbb{Y}(\lambda) \in \mathcal{X}_T$ and $Y_\varepsilon \rightarrow X \in C_T \mathcal{C}^{-1/2-\kappa}$ in law.*

The rest of Section 6.3 is dedicated to the proof of Theorem 6.8.

From the definition $\mathbb{Y}(\lambda) = (Y^\tau(\lambda))_\tau$ of (6.19) it is clear that we need to prove that $Y_\varepsilon^\tau \rightarrow Y^\tau(\lambda)$ for every tree τ . Note that we can write each tree $Y^\tau(\lambda)$ as $Y^\tau(\lambda) = f_\tau(\lambda) K^\tau(X)$ for a measurable function K^τ of the Gaussian process $X \in C_T \mathcal{C}^{-1/2-\kappa}$ defined (6.20), and a suitable deterministic function $f_\tau(\lambda)$ of λ . For example, we can write $Y^\vee(\lambda) = \lambda_3 \llbracket X^2 \rrbracket$ with $K^\vee(X) = X^\vee = \llbracket X^2 \rrbracket$ and $f_\vee(\lambda) = \lambda_3$.

We will show (eqs. (6.36) and (6.44)) that every random field Y_ε^τ defined in (6.12) can be decomposed with the same functions $f_\tau(\cdot)$ and K^τ as

$$Y_\varepsilon^\tau = f_\tau(\lambda_\varepsilon)K^\tau(Y_\varepsilon) + Y_\varepsilon^{\hat{\tau}} \quad (6.28)$$

where $Y_\varepsilon^{\hat{\tau}}$ are suitable remainder terms. For all $p \geq 2$ it is well-known (see [CC13],[Hai14a]) that the term $f_\tau(\lambda_\varepsilon)K^\tau(Y_\varepsilon)$ is uniformly bounded in $L^p(\Omega; \mathcal{X}^\tau)$ (with \mathcal{X}^τ given by (6.18)). Thus, we will prove that $Y_\varepsilon^{\hat{\tau}}$ converges to zero in $L^p(\Omega; \mathcal{X}^\tau)$. This can be done by showing that, by Besov embedding, for $p \in [2, \infty)$ and $\forall \alpha < |\tau|$ we have

$$\mathbb{E}(\|Y_\varepsilon^{\hat{\tau}}(t)\|_{\mathcal{C}^{\alpha-3/p}}^p) \lesssim \mathbb{E}\left(\|Y_\varepsilon^{\hat{\tau}}(t)\|_{B_{p,p}^\alpha}^p\right) \leq \sum_q 2^{\alpha pq} \int_{\mathbb{T}^3} \|\Delta_q Y_\varepsilon^{\hat{\tau}}(t, x)\|_{L^p(\Omega)}^p dx \rightarrow 0 \quad (6.29)$$

thanks to the stationarity of the process $Y(t, x)$. In order to prove the bound (6.29) it suffices to show that $\forall t \in [0, T]$

$$\sum_q 2^{\alpha pq} \sup_x \|\Delta_q Y_\varepsilon^{\hat{\tau}}(t, x)\|_{L^p(\Omega)}^p \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0, \quad (6.30)$$

which is one of the key estimation of this chapter and will be performed in Sections 6.3.3 and 6.3.4.2.

In order to obtain uniform convergence for $t \in [0, T]$ it suffices to show that $\forall \sigma \in [0, 1/2]$, $q \geq -1$:

$$\sup_x \|\Delta_q Y_\varepsilon^{\hat{\tau}}(t, x) - \Delta_q Y_\varepsilon^{\hat{\tau}}(s, x)\|_{L^p(\Omega)}^p \leq C_\varepsilon |t - s|^{\sigma p} 2^{-(\alpha - 2\sigma)pq} \quad \text{with } C_\varepsilon \rightarrow 0. \quad (6.31)$$

Indeed, by the Garsia-Rodemich-Rumsey inequality we obtain for $\delta > 0$ small enough and p large enough

$$\begin{aligned} & \sup_\varepsilon \mathbb{E}\left(\|Y_\varepsilon^{\hat{\tau}}\|_{C_T^{\sigma-2/p} B_{p,p}^{\alpha-2\sigma-\delta}}^p\right) \\ & \leq T^2 \sum_q 2^{(\alpha-2\sigma-\delta)pq} \sup_{s < t \in [0, T]} \sup_x \frac{\|\Delta_q Y_\varepsilon^{\hat{\tau}}(t, x) - \Delta_q Y_\varepsilon^{\hat{\tau}}(s, x)\|_{L^p(\Omega)}^p}{|t - s|^{\sigma p}} \\ & \leq C_\varepsilon T^2 \sum_q 2^{-\delta pq} \end{aligned}$$

which by the standard Besov embedding (Proposition 1.16) yields an estimation on $\mathbb{E}\left(\|Y_\varepsilon^{\hat{\tau}}\|_{C_T^{\sigma-\kappa/2} \mathcal{C}^{\alpha-2\sigma-\kappa}}\right)$ for $\kappa > 0$ small enough. This gives us the necessary tightness to claim that \mathbb{Y}_ε has weak limits along subsequences.

The only thing left after proving (6.28), (6.30) and (6.31) is that for each τ we have $K^\tau(Y_\varepsilon) \rightarrow K^\tau(X)$ in law. However this is clear and already well-known, since by hypothesis we can introduce a space-time convolution regularisation of X (let's call it X_ε) which has the same law of Y_ε for any $\varepsilon > 0$. This yields immediately the convergence $Y_\varepsilon \rightarrow X$ in law. At this point an approximation argument gives that $K^\tau(Y_\varepsilon)$ has the same law of $K^\tau(X_\varepsilon)$. Transposing the regularisation to the kernels of the chaos expansion we can write $K^\tau(X_\varepsilon) = K_\varepsilon^\tau(X)$ and now it is easy to check that $K_\varepsilon^\tau(X) \rightarrow K^\tau(X)$ in probability (as done systematically in [CC13],[MWX16]). We can then conclude that $K^\tau(Y_\varepsilon) \rightarrow K^\tau(X)$ and therefore $Y_\varepsilon^\tau \rightarrow Y^\tau(\lambda)$ in law for every τ , since from Assumption 6.1 we have immediately $f_\tau(\lambda_\varepsilon) \rightarrow f_\tau(\lambda)$.

Let us give some more details on how to prove the decomposition (6.28) and the bounds (6.30) and (6.31). As seen in Section 6.3.1 we have $Y_{\varepsilon, \zeta} = \langle \xi, h_\zeta \rangle$ in law for $\zeta = (t, x) \in \mathbb{R} \times \mathbb{T}^3$ and this gives

$$D^n \tilde{F}_\varepsilon^{(m)}(\varepsilon^{1/2} Y_{\varepsilon, \zeta}) = \tilde{F}_\varepsilon^{(m+n)}(\varepsilon^{1/2} Y_{\varepsilon, \zeta}) h_\zeta^{\otimes n}.$$

We define for $m \in \mathbb{N}$, $\zeta \in \mathbb{R} \times \mathbb{T}^3$:

$$\Phi_\zeta^{[m]} := \varepsilon^{\frac{m-3}{2}} \tilde{F}_\varepsilon^{(m)}(\varepsilon^{1/2} Y_{\varepsilon, \zeta}) \quad (6.32)$$

Note that the term $\Phi_\zeta^{[m]}$ above is *not* the m -th derivative of some function Φ (we use the square parenthesis notation to emphasize this fact). It easy to see from (6.32) that $D^k \Phi_\zeta^{[m]} = \Phi_\zeta^{[m+k]} h_\zeta^{\otimes k}$. Therefore, the partial chaos expansion (2.20) takes a more explicit form when applied to $\Phi_\zeta^{[m]}$:

$$\begin{aligned} \Phi_\zeta^{[m]} &= \sum_{k=0}^{n-1} \frac{\mathbb{E}(\Phi_\zeta^{[m+k]})}{k!} \llbracket Y_{\varepsilon, \zeta}^k \rrbracket + \delta^n (G_{[1]}^{[n]} \Phi_\zeta^{[m+n]} h_\zeta^{\otimes n}) \\ &= \sum_{k=0}^{n-1} \varepsilon^{(m+k-3)/2} \frac{(m+k)!}{k!} \tilde{f}_{m+k, \varepsilon} \llbracket Y_{\varepsilon, \zeta}^k \rrbracket + \delta^n (G_{[1]}^{[n]} \Phi_\zeta^{[m+n]} h_\zeta^{\otimes n}) \end{aligned} \quad (6.33)$$

with $G_{[1]}^{[n]}$ defined in (2.18). Here we used the fact that $\delta^n(h_\zeta^{\otimes n}) = \llbracket Y_{\varepsilon, \zeta}^n \rrbracket$ (see Remark 2.33) and that by the definition of $\Phi_\zeta^{[m]}$ we obtain $\forall \zeta \in \mathbb{R} \times \mathbb{T}^3$:

$$\mathbb{E}(\Phi_\zeta^{[m+k]}) = \varepsilon^{(m+k-3)/2} (m+k)! \tilde{f}_{m+k, \varepsilon}$$

with $\tilde{f}_{n, \varepsilon}$ the coefficients in the decomposition $\tilde{F}_\varepsilon(\varepsilon^{1/2} Y_\varepsilon) := \sum_{n \geq 0} \tilde{f}_{n, \varepsilon} H_n(\varepsilon^{1/2} Y_\varepsilon, \sigma_\varepsilon^2)$. Choosing $n = 4 - m$ in eq. (6.33) we obtain

$$\Phi_\zeta^{[m]} = \frac{3!}{(3-m)!} f_{3, \varepsilon} \llbracket Y_{\varepsilon, \zeta}^{3-m} \rrbracket + \hat{\Phi}_\zeta^{[m]} \quad (6.34)$$

and a remainder with an infinite chaos decomposition strictly greater than $3 - m$:

$$\hat{\Phi}_\zeta^{[m]} = \delta^{4-m} (G_{[1]}^{[4-m]} \Phi_\zeta^{[4]} h_\zeta^{\otimes 4-m}). \quad (6.35)$$

This is a key step in the proof of Theorem 6.8. Indeed, it suffices to substitute (6.34) into definition (6.12) to identify the remainder \hat{Y}_ε^τ in decomposition (6.28) that has to converge to zero, and see that it always contains the term $\hat{\Phi}_\zeta^{[m]}$. Moreover, the structure (6.35) of $\hat{\Phi}_\zeta^{[m]}$ makes it possible to bound its L^p norm and obtain (6.30), (6.31) in the same way as done in Section 6.3.1 for Y_ε^\vee . We will consider separately *simple trees* (namely $Y_\varepsilon^\emptyset, Y_\varepsilon^!, Y_\varepsilon^\vee, Y_\varepsilon^{\vee\vee}, Y_\varepsilon^{\vee\vee\vee}$) which are linear functions of $\Phi_\zeta^{[m]}$ in Section 6.3.3, and *composite trees* (namely $Y_\varepsilon^{\vee\vee\vee}, Y_\varepsilon^{\vee\vee\vee\vee}, Y_\varepsilon^{\vee\vee\vee\vee\vee}, Y_\varepsilon^{\vee\vee\vee\vee\vee\vee}$) which are quadratic in simple trees and need to be further renormalized in order to converge to some limit as $\varepsilon \rightarrow 0$. We will show the decomposition (6.28) for composite trees in Section 6.3.4.1 and the bounds (6.30), (6.31) in Section 6.3.4.2.

Remark 6.9. We can easily estimate terms of the form $\varepsilon^{-(m-3)/2} \Phi_\zeta^{[m]}$ for $3 \leq m \leq 9$ and every $p \in [2, \infty)$. We have (as already observed for $F_\varepsilon^{(1)}$):

$$\left\| \varepsilon^{-\frac{m-3}{2}} \Phi_\zeta^{[m]} \right\|_{L^p}^p = \left\| F_\varepsilon^{(m)}(\varepsilon^{1/2} Y_{\varepsilon, \zeta}) \right\|_{L^p}^p = \int_{\mathbb{R}} |F_\varepsilon^{(m)}(x)|^p \gamma(dx)$$

where $\gamma(dx)$ is the density of a centered Gaussian with variance σ_ε^2 . The integral is finite by Assumption 6.1: in particular we only need to assume that the first m derivatives of F_ε have exponential growth (actually, it is easy to see that one can require even weaker growth conditions).

6.3.3 Analysis of simple trees

First of all note that the term \bar{Y}_ε^\vee has no remainder, and then it can be shown to converge in law to $\lambda^{(2)}Y^\vee$ by usual techniques (see [CC13]). In this section we show the convergence of the trees $Y_\varepsilon^\emptyset, Y_\varepsilon^!, Y_\varepsilon^\vee, Y_\varepsilon^\forall$. We obtain easily from (6.34):

$$\begin{aligned}\Delta_q Y_\varepsilon^\tau(t, x) &:= \frac{(3-m)!}{3!} \int_\zeta \Phi_\zeta^{[m]} \mu_{q,\zeta} \\ &= f_{3,\varepsilon} \int_\zeta \llbracket Y_{\varepsilon,\zeta}^{(3-m)} \rrbracket \mu_\zeta + \frac{(3-m)!}{3!} \int_\zeta \hat{\Phi}_\zeta^{[m]} \mu_{q,\zeta} \\ &= f_\tau(\lambda_\varepsilon) \Delta_q K^\tau(Y_\varepsilon)(t, x) + \Delta_q \hat{Y}_\varepsilon^\tau(t, x),\end{aligned}\tag{6.36}$$

with $\zeta = (s, y)$ and either

$$\begin{aligned}\mu_{q,\zeta} &= \delta(t-s) K_{q,x}(y) ds dy, & \text{for } \Delta_q Y_\varepsilon^\emptyset, \Delta_q Y_\varepsilon^!, \Delta_q Y_\varepsilon^\vee, \\ \mu_{q,\zeta} &= \mathbb{1}_{[0,+\infty)}(s) ds dy \int K_{q,x}(z) P_{t-s}(z-y) dz, & \text{for } \Delta_q Y_\varepsilon^\forall,\end{aligned}\tag{6.37}$$

where $K_{q,x}(y)$ is the kernel associated to the Littlewood-Paley block Δ_q and $P_t(x)$ is the heat kernel.

As said before, $f_{3,\varepsilon} \int_\zeta \llbracket Y_{\varepsilon,\zeta}^{(3-m)} \rrbracket \mu_{q,\zeta}$ converges in law in L^p for every $2 \leq p < +\infty$ to $\lambda_3 \int_\zeta \llbracket Y_\zeta^{(3-m)} \rrbracket \mu_{q,\zeta}$ since $f_{3,\varepsilon} \rightarrow \lambda_3$ by Assumption 6.1. We can bound the remainder term $\int_\zeta \hat{\Phi}_\zeta^{[m]} \mu_{q,\zeta}$ in $L^p(\Omega)$ using Lemma 2.23 and the definition of the norm $\|\cdot\|_{\mathbb{D}^{4-m,p}(H^{\otimes 4-m})}$ to obtain:

$$\begin{aligned}\left\| \int_\zeta \hat{\Phi}_\zeta^{[m]} \mu_{q,\zeta} \right\|_{L^p(\Omega)} &= \left\| \delta^{4-m} G_{[1]}^{[4-m]} \int_\zeta \Phi_\zeta^{[4]} h_\zeta^{\otimes 4-m} \mu_{q,\zeta} \right\|_{L^p(\Omega)} \\ &\lesssim \left\| G_{[1]}^{[4-m]} \int_\zeta \Phi_\zeta^{[4]} h_\zeta^{\otimes 4-m} \mu_{q,\zeta} \right\|_{\mathbb{D}^{4-m,p}(H^{\otimes 4-m})} \\ &\lesssim \sum_{k=0}^{4-m} \left\| D^k G_{[1]}^{[4-m]} \int_\zeta \Phi_\zeta^{[4]} h_\zeta^{\otimes 4-m} \mu_{q,\zeta} \right\|_{L^p(\Omega, H^{\otimes 4-m+k})}.\end{aligned}$$

From Corollary 2.27 we know that $(j-L)^{-1}$ and $D(j-L)^{-1}$ are bounded in L^p for every $p \in [2, \infty)$ and every $j \geq 1$. Applying repeatedly these estimations we obtain:

$$\left\| D^k G_{[1]}^{[4-m]} \int_\zeta \Phi_\zeta^{[4]} h_\zeta^{\otimes 4-m} \mu_{q,\zeta} \right\|_{L^p(\Omega, H^{\otimes 4-m+k})} \lesssim \left\| \int_\zeta \Phi_\zeta^{[4]} h_\zeta^{\otimes 4-m} \mu_{q,\zeta} \right\|_{L^p(\Omega, H^{\otimes 4-m})}.$$

Now we can proceed to implement the idea we already described in Section 6.3.1, i.e. estimating out the term $\left\| \varepsilon^{-\frac{1}{2}} \Phi_\zeta^{[4]} \right\|_{L^p(\Omega)}$ (which is bounded by Remark 6.9 but with infinite chaos decomposition) and considering the finite-chaos term that is left. We do this by decomposing the $L^p(\Omega, H^{\otimes 4-m})$ norm as norms on $H^{\otimes 4-m}$ and $L^{p/2}(\Omega)$ as follows:

$$\begin{aligned}\left\| \int_\zeta \Phi_\zeta^{[4]} h_\zeta^{\otimes 4-m} \mu_{q,\zeta} \right\|_{L^p(\Omega, H^{\otimes 4-m})} &\lesssim \left\| \left\| \int_\zeta \Phi_\zeta^{[4]} h_\zeta^{\otimes 4-m} \mu_{q,\zeta} \right\|_{H^{\otimes 4-m}} \right\|_{L^{p/2}(\Omega)}^{1/2} \\ &\lesssim \left\| \int_\zeta \Phi_\zeta^{[4]} \Phi_{\zeta'}^{[4]} \langle h_\zeta^{\otimes 4-m}, h_{\zeta'}^{\otimes 4-m} \rangle_{H^{\otimes 4-m}} \mu_{q,\zeta} \mu_{q,\zeta'} \right\|_{L^{p/2}(\Omega)}^{1/2} \\ &\lesssim \left[\int_{\zeta, \zeta'} \left\| \Phi_\zeta^{[4]} \Phi_{\zeta'}^{[4]} \right\|_{L^{p/2}(\Omega)} |\langle h_\zeta, h_{\zeta'} \rangle|^{4-m} |\mu_{q,\zeta} \mu_{q,\zeta'}| \right]^{1/2}.\end{aligned}$$

Finally, putting the estimations together and using Hölder's inequality, together with the bound $\varepsilon |\langle h_\zeta, h_{\zeta'} \rangle| = \varepsilon |C_\varepsilon(\zeta - \zeta')| \lesssim 1$ of Lemma 2.15, we obtain for every $\delta \in (0, 1]$:

$$\begin{aligned} \left\| \int_\zeta \hat{\Phi}_\zeta^{[m]} \mu_{q,\zeta} \right\|_{L^p(\Omega)} &\lesssim \left[\varepsilon \int_{\zeta, \zeta'} \left\| \varepsilon^{-\frac{1}{2}} \Phi_\zeta^{[4]} \right\|_{L^p(\Omega)} \left\| \varepsilon^{-\frac{1}{2}} \Phi_{\zeta'}^{[4]} \right\|_{L^p(\Omega)} |\langle h_\zeta, h_{\zeta'} \rangle|^{4-m} |\mu_{q,\zeta} \mu_{q,\zeta'}| \right]^{\frac{1}{2}} \\ &\lesssim \left[\varepsilon^\delta \int_{\zeta, \zeta'} \left\| \varepsilon^{-\frac{1}{2}} \Phi_\zeta^{[4]} \right\|_{L^p(\Omega)} \left\| \varepsilon^{-\frac{1}{2}} \Phi_{\zeta'}^{[4]} \right\|_{L^p(\Omega)} |\langle h_\zeta, h_{\zeta'} \rangle|^{3-m+\delta} |\mu_{q,\zeta} \mu_{q,\zeta'}| \right]^{\frac{1}{2}} \end{aligned}$$

Now using Remark 6.9 (note that to bound $\varepsilon^{-\frac{1}{2}} \Phi^{[4]}$ we only need to control the first 4 derivatives of F_ε) and the fact that $\langle h_\zeta, h_{\zeta'} \rangle_H = C_\varepsilon(\zeta - \zeta')$ we obtain as a final estimation

$$\left\| \int_\zeta \hat{\Phi}_\zeta^{(m)} \mu_{q,\zeta} \right\|_{L^p(\Omega)} \lesssim \varepsilon^{\frac{\delta}{2}} \left[\int |C_\varepsilon(\zeta - \zeta')|^{3-m+\delta} |\mu_{q,\zeta} \mu_{q,\zeta'}| \right]^{1/2}. \quad (6.38)$$

From the definition (6.37) of the measure $\mu_{q,\zeta}$, the l.h.s of (6.38) can be estimated in a standard way using Lemma 2.20 to obtain for every $x \in \mathbb{T}^3$, $q > 0$:

$$\begin{aligned} \|\Delta_q \hat{Y}_\varepsilon^\Psi(t, x)\|_{L^p(\Omega)} &\lesssim \varepsilon^{\frac{\delta}{2}} 2^{-\frac{1-\delta}{2}q} & \|\Delta_q \hat{Y}_\varepsilon^\dagger(t, x)\|_{L^p(\Omega)} &\lesssim \varepsilon^{\frac{\delta}{2}} 2^{\frac{1+\delta}{2}q}, \\ \|\Delta_q \hat{Y}_\varepsilon^\vee(t, x)\|_{L^p(\Omega)} &\lesssim \varepsilon^{\frac{\delta}{2}} 2^{\frac{2+\delta}{2}q} & \|\Delta_q \hat{Y}_\varepsilon^\emptyset(t, x)\|_{L^p(\Omega)} &\lesssim \varepsilon^{\frac{\delta}{2}} 2^{\frac{\delta}{2}q}. \end{aligned}$$

Time regularity of trees

We want to show (6.31). In order to do that, we compute in the same way as before:

$$\begin{aligned} \left\| \int_\zeta (\hat{\Phi}_{t,x}^{[m]} - \hat{\Phi}_{s,x}^{[m]}) \mu_{q,\zeta} \right\|_{L^p(\Omega)} &\lesssim \left\| \delta^{4-m} \int_\zeta G_{[1]}^{[4-m]} (\Phi_{t,x}^{[4]} h_{t,x}^{\otimes 4-m} - \Phi_{s,x}^{[4]} h_{s,x}^{\otimes 4-m}) \mu_{q,\zeta} \right\|_{L^p(\Omega)} \\ &\lesssim \left\| \int_\zeta (\Phi_{t,x}^{[4]} - \Phi_{s,x}^{[4]}) h_{s,x}^{\otimes 4-m} \mu_{q,\zeta} \right\|_{H^{\otimes 4-m}}^{2} \left\| \int_\zeta \Phi_{s,x}^{[4]} (h_{t,x}^{\otimes 4-m} - h_{s,x}^{\otimes 4-m}) \mu_{q,\zeta} \right\|_{H^{\otimes 4-m}}^{2} \right\|_{L^{p/2}(\Omega)}^{1/2}. \end{aligned}$$

We focus on the first term above to obtain that it is bounded by

$$\begin{aligned} &\left\| \int_\zeta (\Phi_{t,x}^{[4]} - \Phi_{s,x}^{[4]}) (\Phi_{t,x'}^{[4]} - \Phi_{s,x'}^{[4]}) \langle h_{s,x}^{\otimes 4-m}, h_{s,x'}^{\otimes 4-m} \rangle_{H^{\otimes 4-m}} \mu_{q,\zeta} \mu_{q,\zeta'} \right\|_{L^{p/2}(\Omega)}^{1/2} \\ &\lesssim \left[\int_{\zeta, \zeta'} \left\| (\Phi_{t,x}^{[4]} - \Phi_{s,x}^{[4]}) (\Phi_{t,x'}^{[4]} - \Phi_{s,x'}^{[4]}) \right\|_{L^{p/2}(\Omega)} |\langle h_{s,x}, h_{s,x'} \rangle|^{4-m} |\mu_{q,\zeta} \mu_{q,\zeta'}| \right]^{1/2} \\ &\lesssim \left[\varepsilon \int_{\zeta, \zeta'} \left\| \varepsilon^{-1} (\Phi_{t,x}^{[4]} - \Phi_{s,x}^{[4]}) (\Phi_{t,x'}^{[4]} - \Phi_{s,x'}^{[4]}) \right\|_{L^{p/2}(\Omega)} |\langle h_{s,x}, h_{s,x'} \rangle|^{4-m} |\mu_{q,\zeta} \mu_{q,\zeta'}| \right]^{\frac{1}{2}} \\ &\lesssim \left[\varepsilon^\delta \int_{\zeta, \zeta'} \left\| \varepsilon^{-1} (\Phi_{t,x}^{[4]} - \Phi_{s,x}^{[4]}) (\Phi_{t,x'}^{[4]} - \Phi_{s,x'}^{[4]}) \right\|_{L^{p/2}(\Omega)} |\langle h_{s,x}, h_{s,x'} \rangle|^{3-m+\delta} |\mu_{q,\zeta} \mu_{q,\zeta'}| \right]^{\frac{1}{2}}. \end{aligned}$$

Now note that

$$\begin{aligned} \varepsilon^{-\frac{1}{2}} (\Phi_{t,x}^{[4]} - \Phi_{s,x}^{[4]}) &= F^{(4)}(\varepsilon^{\frac{1}{2}} Y_\varepsilon(t, x)) - F^{(4)}(\varepsilon^{\frac{1}{2}} Y_\varepsilon(s, x)) \\ &= \varepsilon^{\frac{1}{2}} \int_0^1 F^{(5)}[\varepsilon^{\frac{1}{2}} Y_\varepsilon(s, x) + \tau \varepsilon^{\frac{1}{2}} (Y_\varepsilon(t, x) - Y_\varepsilon(s, x))] (Y_\varepsilon(t, x) - Y_\varepsilon(s, x)), \end{aligned}$$

and we can estimate $\left\| \varepsilon^{-\frac{1}{2}}(\Phi_{t,x}^{[4]} - \Phi_{s,x}^{[4]}) \right\|_{L^p(\Omega)}$ by hypercontractivity and using Lemma 2.16 as

$$\begin{aligned} &\lesssim_p \varepsilon^{1/2} \left\| \int_0^1 F^{(5)}[\varepsilon^{\frac{1}{2}} Y_\varepsilon(s, x) + \tau \varepsilon^{\frac{1}{2}}(Y_\varepsilon(t, x) - Y_\varepsilon(s, x))] \right\|_{L^{2p}(\Omega)} \|Y_\varepsilon(t, x) - Y_\varepsilon(s, x)\|_{L^2(\Omega)} \\ &\lesssim \varepsilon^{1/2} [C_\varepsilon(0, 0) - C_\varepsilon(t - s, 0)]^{1/2} \lesssim \varepsilon^{-2\sigma} |t - s|^\sigma \end{aligned}$$

for any $\sigma \in [0, 1/2]$. The other term can be estimated more easily by

$$\begin{aligned} &\left[\varepsilon^\delta \int_{\zeta, \zeta'} |\langle h_{s,x}, h_{s,x'} \rangle|^{2-m+\delta} |\langle h_{t,x} - h_{s,x}, h_{t,x'} - h_{s,x'} \rangle| |\mu_{q,\zeta} \mu_{q,\zeta'}| \right]^{\frac{1}{2}} \\ &\lesssim \varepsilon^{-2\kappa} |t - s|^\sigma \left[\varepsilon^\delta \int_{\zeta, \zeta'} |\langle h_{s,x}, h_{s,x'} \rangle|^{3-m+\delta+2\sigma} |\mu_{q,\zeta} \mu_{q,\zeta'}| \right]^{\frac{1}{2}}, \end{aligned}$$

and finally obtain

$$\left\| \int_\zeta (\hat{\Phi}_{t,x}^{[m]} - \hat{\Phi}_{s,x}^{[m]}) \mu_{q,\zeta} \right\|_{L^p(\Omega)} \lesssim \varepsilon^{\delta/2-2\kappa} |t - s|^\sigma \left[\int_{\zeta, \zeta'} |\langle h_{s,x}, h_{s,x'} \rangle|^{3-m+\delta+2\sigma} |\mu_{q,\zeta} \mu_{q,\zeta'}| \right]^{\frac{1}{2}}.$$

Which yields estimation (6.31) by applying Lemma 2.20 as before. This concludes the treatment of simple trees. Notice that in this section we only needed $F_\varepsilon \in C^5(\mathbb{R})$ with the first 5 derivatives having exponential growth: indeed we need to take 4 derivatives to bound $\varepsilon^{1/2} \Phi_\zeta^{[4]}$ as of Remark 6.9, plus one more derivative for the time regularity of $\varepsilon^{1/2} \Phi_\zeta^{[4]}$.

6.3.4 Analysis of composite trees

In this section we show the decomposition (6.28) and the bound (6.30) for the trees $Y_\varepsilon^{\forall}, Y_\varepsilon^{\forall}, Y_\varepsilon^{\forall}, Y_\varepsilon^{\forall}$. The time regularity (6.31) of Y_ε^τ can be obtained with the same technique as in the previous section, assuming that we can control one more derivative of F_ε than what is needed to prove the boundedness of Y_ε^τ (thus we will need $F_\varepsilon \in C^9(\mathbb{R})$ with exponential growth, as discussed in Remark 6.14). Looking at the definitions in (6.12) it is clear that we can write the q -th Littlewood-Paley blocks of $Y_\varepsilon^{\forall}, Y_\varepsilon^{\forall}, Y_\varepsilon^{\forall}$ and $Y_\varepsilon^{\forall} \forall \varepsilon > 0$ as:

$$\begin{aligned} \Delta_q Y_\varepsilon^{\forall}(\bar{\zeta}) &= \frac{1}{6} \int_{\zeta_1, \zeta_2} \Phi_{\zeta_1}^{[0]} \Phi_{\zeta_2}^{[2]} \mu_{q, \zeta_1, \zeta_2} - d_\varepsilon^{\forall} \Delta_q(1)(\bar{\zeta}), \\ \Delta_q Y_\varepsilon^{\forall}(\bar{\zeta}) &= \frac{1}{9} \int_{\zeta_1, \zeta_2} \Phi_{\zeta_1}^{[1]} \Phi_{\zeta_2}^{[1]} \mu_{q, \zeta_1, \zeta_2} - d_\varepsilon^{\forall} \Delta_q(1)(\bar{\zeta}), \\ \Delta_q Y_\varepsilon^{\forall}(\bar{\zeta}) &= \frac{1}{3} \int_{\zeta_1, \zeta_2} \bar{\Phi}_{\zeta_1}^{[1]} \Phi_{\zeta_2}^{[1]} \mu_{q, \zeta_1, \zeta_2} - \bar{d}_\varepsilon^{\forall} \Delta_q(1)(\bar{\zeta}), \\ \Delta_q Y_\varepsilon^{\forall}(\bar{\zeta}) &= \frac{1}{3} \int_{\zeta_1, \zeta_2} \Phi_{\zeta_1}^{[0]} \Phi_{\zeta_2}^{[1]} \mu_{q, \zeta_1, \zeta_2} - d_\varepsilon^{\forall} \Delta_q Y_\varepsilon(\bar{\zeta}) - d_\varepsilon^{\forall} \Delta_q(1)(\bar{\zeta}), \end{aligned} \tag{6.39}$$

for any time-space point $\bar{\zeta} = (t, \bar{x})$ that we keep fixed throughout this section. In order to keep the notation shorter we defined

$$\bar{\Phi}_{\zeta_1}^{[1]} := \varepsilon^{-1/2} f_{2,\varepsilon} \llbracket Y_\varepsilon^2(\zeta_1) \rrbracket,$$

which can be thought of as a finite-chaos equivalent of $\Phi_\varepsilon^{[1]}$ (modulo a constant $f_{2,\varepsilon}/f_{3,\varepsilon}$) in the same way as Y_ε^{\forall} is a finite-chaos equivalent of Y_ε^{\forall} . The measure $\mu_{q, \zeta_1, \zeta_2}$ on $(\mathbb{R} \times \mathbb{T}^3)^2$ is given by

$$\mu_{q, \zeta_1, \zeta_2} := \left[\int_{x, y} K_{q, \bar{x}}(x) \sum_{i \sim j} K_{i, x}(y) K_{j, x}(x_2) P_{t-s_1}(y-x_1) \right] \mathbb{1}_{[0, +\infty)}(s_1) \delta(t-s_2) d\zeta_1 d\zeta_2,$$

with $\zeta_i := (s_i, x_i)$ for $i = 1, 2$, K being the kernel associated to the Littlewood-Paley decomposition and P being the heat kernel. The first step for decomposing (6.39) is to expand them using the partial chaos expansion (2.20) to obtain

$$\begin{aligned}\Phi_{\zeta_1}^{[0]}\Phi_{\zeta_2}^{[2]} &= \mathbb{E}[\Phi_{\zeta_1}^{[0]}\Phi_{\zeta_2}^{[2]}] + \delta G_1 D(\Phi_{\zeta_1}^{[0]}\Phi_{\zeta_2}^{[2]}), \\ \Phi_{\zeta_1}^{[1]}\Phi_{\zeta_2}^{[1]} &= \mathbb{E}[\Phi_{\zeta_1}^{[1]}\Phi_{\zeta_2}^{[1]}] + \delta G_1 D(\Phi_{\zeta_1}^{[1]}\Phi_{\zeta_2}^{[1]}), \\ \Phi_{\zeta_1}^{[0]}\Phi_{\zeta_2}^{[1]} &= \mathbb{E}[\Phi_{\zeta_1}^{[0]}\Phi_{\zeta_2}^{[1]}] + \delta[J_0 D(\Phi_{\zeta_1}^{[0]}\Phi_{\zeta_2}^{[1]})] + \delta^2 G_{[1]}^{[2]} D^2(\Phi_{\zeta_1}^{[0]}\Phi_{\zeta_2}^{[1]}) \\ &= \mathbb{E}[\Phi_{\zeta_1}^{[0]}\Phi_{\zeta_2}^{[1]}] + Y_\varepsilon(\zeta_1)\mathbb{E}[\Phi_{\zeta_1}^{[1]}\Phi_{\zeta_2}^{[1]}] + Y_\varepsilon(\zeta_2)\mathbb{E}[\Phi_{\zeta_1}^{[0]}\Phi_{\zeta_2}^{[2]}] \\ &\quad + \delta^2 G_{[1]}^{[2]} D^2(\Phi_{\zeta_1}^{[0]}\Phi_{\zeta_2}^{[1]}).\end{aligned}\tag{6.40}$$

Like the trees appearing in the Φ_3^4 model, we expect composite trees to require a further renormalisation, on top of the Wick ordering. We developed (6.40) to the smallest order that allows us to see the effect of renormalization.

6.3.4.1 Renormalisation of composite trees

In this section we show how to renormalize (6.39) by estimating terms of the type $\mathbb{E}[\Phi_{\zeta_1}^{[m]}\Phi_{\zeta_2}^{[n]}]$ in expansion (6.40). This poses an additional difficulty, as in principle we would need to compute an infinite number of contractions between $\Phi_{\zeta_1}^{[m]}$ and $\Phi_{\zeta_2}^{[n]}$. However, we can again decompose $\Phi^{[m]}$ as in (6.34), and then the product formula (2.22) ensures that we only need to control a finite number of contractions. This is another important step in the proof and will be carried out in Lemma 6.12. First we need some preparatory results:

Lemma 6.10. *We have*

$$\begin{aligned}\text{and} \quad \int_{\zeta_1, \zeta_2} Y_\varepsilon(\zeta_1)\mathbb{E}[\Phi_{\zeta_1}^{[1]}\Phi_{\zeta_2}^{[1]}]\mu_{q, \zeta_1, \zeta_2} &= \int_{s, x} \Delta_q Y_\varepsilon(s, \bar{x} - x) G(t - s, x), \\ \int_{\zeta_1, \zeta_2} Y_\varepsilon(\zeta_2)\mathbb{E}[\Phi_{\zeta_1}^{[0]}\Phi_{\zeta_2}^{[2]}]\mu_{q, \zeta_1, \zeta_2} &= \int_x \Delta_q Y_\varepsilon(t, \bar{x} - x) H(t, x),\end{aligned}$$

where we introduced the kernels:

$$\begin{aligned}G(t - s, x) &:= \int_{x'_1, x'_2} \sum_{i \sim j} K_{i, x}(x'_1) K_{j, x}(x'_2) P_{t-s}(x'_1) \mathbb{E}[\Phi_0^{[1]}\Phi_{(t-s, x_2)}^{[1]}], \\ H(t, x) &:= \int_{s, x_1, x'_1} \sum_{i \sim j} K_{i, x}(x'_1) K_{j, x}(0) P_{t-s}(x'_1 - x_1) \mathbb{E}[\Phi_0^{[0]}\Phi_{(t-s, -x_1)}^{[2]}].\end{aligned}$$

Remark 6.11. Some caveat on the notation: although we use the same letter for the kernel $G(\cdot, \cdot)$ and the Green operator $G_{[m]}^{[n]}$, those two are not related in any possible way. It is always clear which one the notation refers to.

Proof. We have

$$\begin{aligned}&\int_{\zeta_1, \zeta_2} Y_\varepsilon(\zeta_1)\mathbb{E}[\Phi_{\zeta_1}^{[1]}\Phi_{\zeta_2}^{[1]}]\mu_{q, \zeta_1, \zeta_2} \\ &= \int_{s_1, x_1, x_2, x, x'_1} K_{q, \bar{x}}(x) \sum_{i \sim j} K_{i, x}(x'_1) K_{j, x}(x_2) P_{t-s_1}(x'_1 - x_1) Y_\varepsilon(s_1, x_1) \mathbb{E}[\Phi_0^{[1]}\Phi_{(t-s_1, x_2-x_1)}^{[1]}\end{aligned}$$

and by change of variables, exploiting the translation invariance of the problem we obtain:

$$= \int_{s_1, x_1, x} K_{q, \bar{x}}(x + x_1) Y(s_1, x_1) \int_{x'_1, x_2} \sum_{i \sim j} K_{i, x}(x'_1) K_{j, x}(x_2) P_{t-s_1}(x'_1) \mathbb{E}[\Phi_0^{[1]}\Phi_{(t-s_1, x_2)}^{[1]}].$$

Using the definition of K_q we have

$$= \int_{s_1, x} \Delta_q Y_\varepsilon(s_1, \bar{x} - x) \int_{x'_1, x_2} \sum_{i \sim j} K_{i,x}(x'_1) K_{j,x}(x_2) P_{t-s_1}(x'_1) \mathbb{E}[\Phi_0^{[1]} \Phi_{(t-s_1, x_2)}^{[1]}].$$

Finally we can write

$$\int_{\zeta_1, \zeta_2} Y_\varepsilon(\zeta_1) \mathbb{E}[\Phi_{\zeta_1}^{[1]} \Phi_{\zeta_2}^{[1]}] \mu_{q, \zeta_1, \zeta_2} = \int_{s_1, x} \Delta_q Y_\varepsilon(s_1, \bar{x} - x) G(t - s_1, x).$$

Similar computations holds for the other term, indeed

$$\begin{aligned} & \int_{\zeta_1, \zeta_2} Y_\varepsilon(\zeta_2) \mathbb{E}[\Phi_{\zeta_1}^{[0]} \Phi_{\zeta_2}^{[2]}] \mu_{q, \zeta_1, \zeta_2} \\ &= \int_{s_1, x_1, x_2, x, x'_1} K_{q, \bar{x}}(x) \sum_{i \sim j} K_{i,x}(x'_1) K_{j,x}(x_2) P_{t-s_1}(x'_1 - x_1) Y_\varepsilon(t, x_2) \mathbb{E}[\Phi_0^{[0]} \Phi_{(t-s_1, x_2-x_1)}^{[2]}] \\ &= \int_{x_2} K_{q, \bar{x}}(x + x_2) Y_\varepsilon(t, x_2) \int_{s_1, x_1, x, x'_1} \sum_{i \sim j} K_{i,x}(x'_1) K_{j,x}(0) P_{t-s_1}(x'_1 - x_1) \mathbb{E}[\Phi_0^{[0]} \Phi_{(t-s_1, -x_1)}^{[2]}] \\ &= \int_x \Delta_q Y_\varepsilon(t, \bar{x} - x) \int_{s_1, x_1, x'_1} \sum_{i \sim j} K_{i,x}(x'_1) K_{j,x}(0) P_{t-s_1}(x'_1 - x_1) \mathbb{E}[\Phi_0^{[0]} \Phi_{(t-s_1, -x_1)}^{[2]}] \\ &= \int_x \Delta_q Y_\varepsilon(t, \bar{x} - x) H(t, x) \end{aligned}$$

□

Substituting the lemma above and (6.40) in the expressions (6.39), we can write them as:

$$\begin{aligned} \Delta_q Y_\varepsilon^{\Psi}(\bar{\zeta}) &= \frac{1}{9} \int_{\zeta_1, \zeta_2} \delta G_1 D(\Phi_{\zeta_1}^{[1]} \Phi_{\zeta_2}^{[1]}) \mu_{q, \zeta_1, \zeta_2} + \Delta_q(1)(\bar{\zeta}) \left[\frac{1}{9} \int_{s, x} G(t-s, x) - d_\varepsilon^{\Psi} \right] \\ \Delta_q \bar{Y}_\varepsilon^{\Psi}(\bar{\zeta}) &= \frac{1}{3} \int_{\zeta_1, \zeta_2} \delta G_1 D(\bar{\Phi}_{\zeta_1}^{[1]} \bar{\Phi}_{\zeta_2}^{[1]}) \mu_{q, \zeta_1, \zeta_2} + \Delta_q(1)(\bar{\zeta}) \left[\frac{1}{3} \int_{s, x} \bar{G}(t-s, x) - \bar{d}_\varepsilon^{\Psi} \right] \\ \Delta_q Y_\varepsilon^{\Psi}(\bar{\zeta}) &= \frac{1}{6} \int_{\zeta_1, \zeta_2} \delta G_1 D(\Phi_{\zeta_1}^{[0]} \Phi_{\zeta_2}^{[2]}) \mu_{q, \zeta_1, \zeta_2} + \Delta_q(1)(\bar{\zeta}) \left[\frac{1}{6} \int_x H(t, x) - d_\varepsilon^{\Psi} \right] \\ \Delta_q Y_\varepsilon^{\Psi}(\bar{\zeta}) &= \frac{1}{3} \int_{\zeta_1, \zeta_2} \delta^2 G_{[1]}^{[2]} D^2(\Phi_{\zeta_1}^{[0]} \Phi_{\zeta_2}^{[1]}) \mu_{q, \zeta_1, \zeta_2} \\ &\quad + \Delta_q(1)(\bar{\zeta}) \left[\frac{1}{3} \int_{\zeta_1, \zeta_2} \mathbb{E}[\Phi_{\zeta_1}^{[0]} \Phi_{\zeta_2}^{[1]}] \mu_{q, \zeta_1, \zeta_2} - d_\varepsilon^{\Psi} \right] \\ &\quad + \Delta_q Y_\varepsilon(\bar{\zeta}) \left[\frac{1}{3} \int_{s, x} G(t-s, x) + \frac{1}{3} \int_x H(t, x) - d_\varepsilon^{\Psi} \right] \\ &\quad + \frac{1}{3} \Delta_q R_\varepsilon^{\Psi}(\bar{\zeta}) + \frac{1}{3} \Delta_q R_\varepsilon^{\Psi}(\bar{\zeta}) \end{aligned}$$

with the additional definitions

$$\begin{aligned} \bar{G}(t-s, x) &:= \int_{x_1, x'_1} \sum_{i \sim j} K_{i,x}(x'_1) K_{j,x}(0) P_{t-s_1}(x'_1 - x_1) \mathbb{E}[\bar{\Phi}_0^{[1]} \Phi_{(t-s, -x_1)}^{[1]}], \\ \Delta_q R_\varepsilon^{\Psi}(\bar{\zeta}) &:= \int_{s, x} [\Delta_q Y_\varepsilon(s, \bar{x} - x) - \Delta_q Y_\varepsilon(t, \bar{x})] G(t-s, x), \\ \Delta_q R_\varepsilon^{\Psi}(\bar{\zeta}) &:= \int_x [\Delta_q Y_\varepsilon(t, \bar{x} - x) - \Delta_q Y_\varepsilon(t, \bar{x})] H(t, x). \end{aligned}$$

Now we can characterise the local behaviour of $\mathbb{E}[\Phi_{\zeta_1}^{[m]}\Phi_{\zeta_2}^{[n]}]$ appearing in the integrals above. Decomposing separately $\Phi_{\zeta_1}^{[m]}$ and $\Phi_{\zeta_2}^{[n]}$ as in (6.34) we obtain:

$$\begin{aligned} \mathbb{E}[\Phi_{\zeta_1}^{[m]}\Phi_{\zeta_2}^{[n]}] &= \frac{3!^2}{(3-m)!(3-n)!} (f_{3,\varepsilon})^2 \mathbb{E}[\llbracket Y_{\varepsilon,\zeta_1}^{3-m} \rrbracket \llbracket Y_{\varepsilon,\zeta_2}^{3-n} \rrbracket] + \frac{3!}{(3-m)!} f_{3,\varepsilon} \mathbb{E}[\llbracket Y_{\varepsilon,\zeta_1}^{3-m} \rrbracket \hat{\Phi}_{\zeta_2}^{[n]}] \\ &\quad + \frac{3!}{(3-n)!} f_{3,\varepsilon} \mathbb{E}[\llbracket Y_{\varepsilon,\zeta_2}^{3-n} \rrbracket \hat{\Phi}_{\zeta_1}^{[m]}] + \mathbb{E}[\hat{\Phi}_{\zeta_1}^{[m]}\hat{\Phi}_{\zeta_2}^{[n]}], \end{aligned}$$

where $\mathbb{E}[\llbracket Y_{\varepsilon,\zeta_1}^{3-m} \rrbracket \llbracket Y_{\varepsilon,\zeta_2}^{3-n} \rrbracket] = (3-m)!\delta(3-m, 3-n)C_\varepsilon(\zeta_1 - \zeta_2)^{3-n}$ and to bound all other terms we introduce the following result.

Lemma 6.12. *Under Assumption 6.1 (in particular if $F \in C^8(\mathbb{R})$ with exponentially growing derivatives) we have, for every $0 \leq m, n \leq 3$ and $m \leq n$:*

$$|\mathbb{E}[\hat{\Phi}_{\zeta_1}^{[m]}\hat{\Phi}_{\zeta_2}^{[n]}]| \lesssim \sum_{i=0}^{4-n} \varepsilon^{1+\frac{n-m}{2}+i} |\langle h_{\zeta_1}, h_{\zeta_2} \rangle|^{4-m+i} \lesssim \varepsilon^\delta |\langle h_{\zeta_1}, h_{\zeta_2} \rangle|^{3-\frac{m+n}{2}+\delta}, \quad \forall \delta \in [0, 1].$$

Moreover for every $0 \leq m, n \leq 3$,

$$\begin{aligned} |\mathbb{E}[\llbracket Y_{\varepsilon,\zeta_1}^m \rrbracket \hat{\Phi}_{\zeta_2}^{[n]}]| &\lesssim \varepsilon^{\frac{m+n-3}{2}} |\langle h_{\zeta_1}, h_{\zeta_2} \rangle|^m \quad \text{if } m \geq 4-n, \\ \mathbb{E}[\llbracket Y_{\varepsilon,\zeta_1}^m \rrbracket \hat{\Phi}_{\zeta_2}^{[n]}] &= 0 \quad \text{if } m < 4-n. \end{aligned}$$

Proof. Using formula (2.21) we decompose

$$\begin{aligned} \mathbb{E}[\hat{\Phi}_{\zeta_1}^{[m]}\hat{\Phi}_{\zeta_2}^{[n]}] &= \mathbb{E}[\delta^{4-m}(G_{[1]}^{[4-m]}\Phi_{\zeta_1}^{[4]}h_{\zeta_1}^{\otimes 4-m})\delta^{4-n}(G_{[1]}^{[4-n]}\Phi_{\zeta_2}^{[4]}h_{\zeta_2}^{\otimes 4-n})] \\ &= \sum_{i=0}^{4-n} \binom{4-m}{i} \binom{4-n}{i} i! \mathbb{E}(G_{[5-n-i]}^{[8-m-n-i]}\Phi_{\zeta_1}^{[8-n-i]}G_{[5-m-i]}^{[8-m-n-i]}\Phi_{\zeta_2}^{[8-m-i]}) \langle h_{\zeta_1}, h_{\zeta_2} \rangle^{8-m-n-i}. \end{aligned}$$

We can bound the term

$$\begin{aligned} &\varepsilon^{\frac{m+n}{2}+i-5} \mathbb{E}(G_{[5-n-i]}^{[8-m-n-i]}\Phi_{\zeta_1}^{[8-n-i]}G_{[5-m-i]}^{[8-m-n-i]}\Phi_{\zeta_2}^{[8-m-i]}) \\ &\lesssim \left\| \varepsilon^{\frac{n+i-5}{2}} \Phi_{\zeta_1}^{[8-n-i]} \right\|_{L^2} \left\| \varepsilon^{\frac{m+i-5}{2}} \Phi_{\zeta_2}^{[8-m-i]} \right\|_{L^2} \end{aligned}$$

knowing $8-n-i \vee 8-m-i \leq 8$ derivatives of F_ε (see Remark 6.9) and using the bound on the covariance $\varepsilon|C_\varepsilon(\zeta_1 - \zeta_2)| \lesssim 1$ of Lemma 2.15 with $|\langle h_{\zeta_1}, h_{\zeta_2} \rangle| = |C_\varepsilon(\zeta_1 - \zeta_2)|$ we have:

$$\begin{aligned} |\mathbb{E}[\hat{\Phi}_{\zeta_1}^{[m]}\hat{\Phi}_{\zeta_2}^{[n]}]| &\lesssim \sum_{i=0}^{4-n} \varepsilon^{1+\frac{n-m}{2}+i} |\langle h_{\zeta_1}, h_{\zeta_2} \rangle|^{4-m+i} \\ &\lesssim \varepsilon^\delta |\langle h_{\zeta_1}, h_{\zeta_2} \rangle|^{3-\frac{m+n}{2}+\delta}. \end{aligned}$$

For the second bound we recall that $\llbracket Y_{\varepsilon,\zeta_1}^m \rrbracket = \delta^m(h_{\zeta_1}^{\otimes m})$ by construction (Remark 2.33) and compute:

$$\begin{aligned} \mathbb{E}[\llbracket Y_{\varepsilon,\zeta_1}^m \rrbracket \hat{\Phi}_{\zeta_2}^{[n]}] &= \mathbb{E}[\delta^m(h_{\zeta_1}^{\otimes m})\delta^{4-n}(G_{[1]}^{[4-n]}\Phi_{\zeta_2}^{[4]}h_{\zeta_2}^{\otimes 4-n})] \\ &= \sum_{i=0}^{m \wedge 4-n} \binom{m}{i} \binom{4-n}{i} i! \mathbb{E}(\langle D^{4-n-i}(h_{\zeta_1}^{\otimes m}), G_{[m+1-i]}^{[m+4-n-i]}\Phi_{\zeta_2}^{[4+m-i]}h_{\zeta_2}^{\otimes m+4-n-i} \rangle_{H^{\otimes m+4-n-i}}). \end{aligned}$$

Since $Dh_{\zeta_1}^{\otimes m} = 0$ we obtain $\mathbb{E}[\llbracket Y_{\varepsilon, \zeta_1}^m \rrbracket \hat{\Phi}_{\zeta_2}^{[n]}] = 0$ if $m < 4 - n$ and

$$|\mathbb{E}[\llbracket Y_{\varepsilon, \zeta_1}^m \rrbracket \hat{\Phi}_{\zeta_2}^{[n]}]| \lesssim \varepsilon^{\frac{m+n-3}{2}} \mathbb{E}[\varepsilon^{-\frac{3-m-n}{2}} G_{[m+n-3]}^{[m]} \Phi_{\zeta_2}^{[m+n]}] |\langle h_{\zeta_1}, h_{\zeta_2} \rangle|^m$$

if $m \geq 4 - n$, with

$$\mathbb{E}[\varepsilon^{-\frac{3-m-n}{2}} G_{[m+n-3]}^{[m]} \Phi_{\zeta_2}^{[m+n]}] \lesssim 1. \quad \square$$

Using Lemma 6.12 we obtain

$$\begin{aligned} \mathbb{E}[\Phi_{\zeta_1}^{[1]} \Phi_{\zeta_2}^{[1]}] &= 9 \mathbb{E}[(f_{3,\varepsilon} \llbracket Y_{\varepsilon, \zeta_1}^2 \rrbracket + \hat{\Phi}_{\varepsilon, \zeta_1}^{[1]})(f_{3,\varepsilon} \llbracket Y_{\varepsilon, \zeta_2}^2 \rrbracket + \hat{\Phi}_{\zeta_2}^{[1]})] \\ &= 18 (f_{3,\varepsilon})^2 [C_\varepsilon(\zeta_1 - \zeta_2)]^2 + \mathbb{E}[\hat{\Phi}_{\zeta_1}^{[1]} \hat{\Phi}_{\zeta_2}^{[1]}] \end{aligned}$$

and thus

$$\begin{aligned} G(t-s, x) &= 18 (f_{3,\varepsilon})^2 \int_{x'_1, x_2} \sum_{i \sim j} K_{i,x}(x'_1) K_{j,x}(x_2) P_{t-s}(x'_1) [C_\varepsilon(\zeta_1 - \zeta_2)]^2 \\ &\quad + \hat{G}(t-s, x) \end{aligned}$$

with the remainder term defined by

$$\hat{G}(t-s, x) := \int_{x'_1, x_2} \sum_{i \sim j} K_{i,x}(x'_1) K_{j,x}(x_2) P_{t-s}(x'_1) \mathbb{E}[\hat{\Phi}_0^{[1]} \hat{\Phi}_{(t-s, x_2)}^{[1]}].$$

We have the estimation

$$|\mathbb{E}[\hat{\Phi}_{\zeta_1}^{[1]} \hat{\Phi}_{\zeta_2}^{[1]}]| \lesssim \varepsilon^\delta C_\varepsilon(\zeta_1 - \zeta_2)^{2+\delta}. \quad (6.41)$$

Similarly

$$\begin{aligned} \mathbb{E}[\bar{\Phi}_{\zeta_1}^{[1]} \Phi_{\zeta_2}^{[1]}] &= 3 \varepsilon^{-1/2} f_{2,\varepsilon} \mathbb{E}[\llbracket Y_{\varepsilon, \zeta_1}^2 \rrbracket (f_{3,\varepsilon} \llbracket Y_{\varepsilon, \zeta_2}^2 \rrbracket + \hat{\Phi}_{\zeta_2}^{[1]})] \\ &= 6 \varepsilon^{-1/2} f_{2,\varepsilon} f_{3,\varepsilon} [C_\varepsilon(\zeta_1 - \zeta_2)]^2, \end{aligned}$$

and

$$\begin{aligned} |\mathbb{E}[\Phi_{\zeta_1}^{[0]} \Phi_{\zeta_2}^{[2]}]| &= |\mathbb{E}[(f_{3,\varepsilon} \llbracket Y_{\varepsilon, \zeta_1}^3 \rrbracket + \hat{\Phi}_{\zeta_1}^{[0]})(6 f_{3,\varepsilon} Y_{\varepsilon, \zeta_2} + \hat{\Phi}_{\zeta_2}^{[2]})]| \\ &\lesssim |f_{3,\varepsilon}| |\mathbb{E}[\llbracket Y_{\varepsilon, \zeta_1}^3 \rrbracket \hat{\Phi}_{\zeta_2}^{[2]}]| + |\mathbb{E}[\hat{\Phi}_{\zeta_1}^{[0]} \hat{\Phi}_{\zeta_2}^{[2]}]| \\ &\lesssim \varepsilon^\delta (|f_{3,\varepsilon}| + 1) C_\varepsilon(\zeta_1 - \zeta_2)^{2+\delta}, \end{aligned} \quad (6.42)$$

and

$$\begin{aligned} |\mathbb{E}[\Phi_{\zeta_1}^{[0]} \Phi_{\zeta_2}^{[1]}]| &= |\mathbb{E}[\Phi^{[0]}(3 f_{3,\varepsilon} \llbracket Y_{\varepsilon, \zeta_2}^2 \rrbracket + \hat{\Phi}^{[1]})]| \\ &\lesssim |f_{3,\varepsilon}| |\mathbb{E}[\llbracket Y_{\varepsilon, \zeta_1}^3 \rrbracket \hat{\Phi}_{\zeta_2}^{[1]}]| + \mathbb{E}[\hat{\Phi}_{\zeta_1}^{[0]} \hat{\Phi}_{\zeta_2}^{[1]}] \\ &\lesssim \varepsilon^{1/2} (|f_{3,\varepsilon}| + 1) C_\varepsilon(\zeta_1 - \zeta_2)^3. \end{aligned} \quad (6.43)$$

We have by Lemma 2.19 that for all $\delta \in (0, 1)$ $|\hat{G}(t-s, x)| \lesssim \varepsilon^\delta (|t-s|^{1/2} + |x|)^{-5-\delta}$. Using estimate (2.16) together with Lemma 2.12, we have that $\forall \delta \in (0, 1), \delta' \in (0, \delta)$ $|H(t, x)| \lesssim \varepsilon^{\delta'} |x|^{-\delta}$. Furthermore, we have

$$\begin{aligned} \frac{1}{3} \Delta_q R_\varepsilon^{\check{Y}} &= 6 (f_{3,\varepsilon})^2 \int_{s,x} [\Delta_q Y_\varepsilon(t+s, \bar{x} - x) - \Delta_q Y_\varepsilon(t, \bar{x})] P_s(x) [C_\varepsilon(s, x)]^2 \\ &\quad + \frac{1}{3} \Delta_q \hat{R}_\varepsilon^{\check{Y}} \end{aligned}$$

with the remainder term $\Delta_q \hat{R}_\varepsilon^{\check{Y}}$ given by

$$\Delta_q \hat{R}_\varepsilon^{\check{Y}} = \int_{s,x} [\Delta_q Y_\varepsilon(t, \bar{x} - x) - \Delta_q Y_\varepsilon(t, \bar{x})] \hat{G}(t-s, x).$$

The term

$$6(f_{3,\varepsilon})^2 \int_{s,x} [\Delta_q Y_\varepsilon(t+s, \bar{x}-x) - \Delta_q Y_\varepsilon(t, \bar{x})] P_s(x) [C_\varepsilon(s, x)]^2$$

can be shown to converge in law to

$$6(\lambda_3)^2 \int_{s,x} [\Delta_q Y(t+s, \bar{x}-x) - \Delta_q Y(t, \bar{x})] P_s(x) \mathbb{E}(Y(0,0)Y(s, x))^2$$

in $C_T^\kappa \mathcal{C}^{-1/2-2\kappa}$ with the standard techniques used in the analysis of the Φ_3^4 model. On the other hand, for all $\delta > 0$ sufficiently small we have the bounds

$$\begin{aligned} \left\| \Delta_q R_\varepsilon^{\Psi} \right\|_{L^\infty} + \left\| \Delta_q \hat{R}_\varepsilon^{\Psi} \right\|_{L^\infty} &\leq \varepsilon^\delta \|Y_\varepsilon\|_{C_T^\kappa \mathcal{C}^{-1/2-2\kappa}} 2^{q(1/2+2\kappa+2\delta)} \int_{s,x} (|x| + |t-s|^{1/2})^{\delta-5} \\ &\lesssim \varepsilon^\delta \|Y_\varepsilon\|_{C_T^\kappa \mathcal{C}^{-1/2-2\kappa}} 2^{q(1/2+2\kappa+2\delta)}, \end{aligned}$$

which shows that these remainders go to zero in $\mathcal{C}^{-1/2-2\kappa}$ as $\varepsilon \rightarrow 0$, since $\|Y_\varepsilon\|_{C_T^\kappa \mathcal{C}^{-1/2-2\kappa}}$ is bounded in $L^p(\Omega)$. Moreover, it is easy to see that

$$\left\| \Delta_q R_\varepsilon^{\Psi} - \Delta_q \hat{R}_\varepsilon^{\Psi} \right\|_{L^p(\Omega)} \sim O_{L^\infty}(2^{q(1/2+2\kappa+2\delta)}).$$

Note that

$$\begin{aligned} \int_{s,x} G(t-s, x) &= \int_{s,x} P_s(x) \mathbb{E}[\Phi_0^{[1]} \Phi_{(s,x)}^{[1]}] \\ &= 18(f_{3,\varepsilon})^2 \int_{s,x} P_s(x) [C_\varepsilon(s, x)]^2 + \int_{s,x} P_s(x) \mathbb{E}[\hat{\Phi}_0^{[1]} \hat{\Phi}_{(s,x)}^{[1]}], \\ \int_{s,x} H(t, x) &= \int_{s,x} P_s(x) \mathbb{E}[\Phi_0^{[0]} \Phi_{(s,x)}^{[2]}] = \int_{s,x} P_s(x) \mathbb{E}[\Phi_0^{[0]} \hat{\Phi}_{(s,x)}^{[2]}]. \end{aligned}$$

Here we used the fact that

$$\int_x \sum_{i \sim j} K_{i,x}(x'_1) K_{j,x}(0) = \int_x \sum_{i,j} K_{i,x}(x'_1) K_{j,x}(0) = \delta(x'_1),$$

since $\int_x K_{i,x}(x'_1) K_{j,x}(0) = 0$, where $|i-j| > 1$. This is readily seen in Fourier space taking into account the support properties of the Littlewood-Paley blocks. Now,

$$\int_{s,x} P_s(x) \mathbb{E}[\hat{\Phi}_0^{[1]} \hat{\Phi}_{(s,x)}^{[1]}], \quad \int_{s,x} P_s(x) \mathbb{E}[\Phi_0^{[0]} \hat{\Phi}_{(s,x)}^{[2]}],$$

converge to finite constants due to the bounds (6.41) and (6.42) and by Lemma 2.17 $\int_{s,x} P_s(x) C_\varepsilon(s, x)^2 \lesssim |\log \varepsilon|$.

Finally, from (6.43) we have

$$\int_{\zeta_1, \zeta_2} \mathbb{E}[\Phi_{\zeta_1}^{[0]} \Phi_{\zeta_2}^{[1]}] \mu_{q, \zeta_1, \zeta_2} = \int_{s,x} P_s(x) \mathbb{E}[\Phi_0^{[0]} \Phi_{(s,x)}^{[1]}] = O(\varepsilon^{-1/2}).$$

Indeed Lemma 2.17 again yields $\varepsilon \int_{s,x} P_s(x) C_\varepsilon(s, x)^3 \lesssim 1$.

Thus $\int_{\zeta_1, \zeta_2} \mathbb{E}[\Phi_{\zeta_1}^{[0]} \Phi_{\zeta_2}^{[1]}] \mu_{q, \zeta_1, \zeta_2}$ gives a diverging constant which depends on all the $(f_{n,\varepsilon})_n$. Making the choice to define the renormalisation constants d^r as in eq. (6.7) we cancel exactly these contributions which are either $(F_\varepsilon)_\varepsilon$ dependent and/or diverging. In particular we verify that we can satisfy the constraint (6.13).

Finally, noting that $\delta G_1 D = (1 - J_0)$ and $\delta^2 G_{[1]}^{[2]} D^2 = (1 - J_0 - J_1)$ (as seen in (2.20)), we can write the trees of (6.39) as

$$\begin{aligned}
\Delta_q Y_\varepsilon^{\Psi}(\bar{\zeta}) &= (f_{3,\varepsilon})^2 \int_{\zeta_1, \zeta_2} (1 - J_0)(\llbracket Y_{\varepsilon, \zeta_1}^2 \rrbracket \llbracket Y_{\varepsilon, \zeta_2}^2 \rrbracket) \mu_{q, \zeta_1, \zeta_2} \\
&\quad + \frac{f_{3,\varepsilon}}{3} \int_{\zeta_1, \zeta_2} \delta G_1 D (\hat{\Phi}_{\zeta_1}^{[1]} \llbracket Y_{\varepsilon, \zeta_2}^2 \rrbracket + \llbracket Y_{\varepsilon, \zeta_1}^2 \rrbracket \hat{\Phi}_{\zeta_2}^{[1]}) \mu_{q, \zeta_1, \zeta_2} \\
&\quad + \frac{1}{9} \int_{\zeta_1, \zeta_2} \delta G_1 D (\hat{\Phi}_{\zeta_1}^{[1]} \hat{\Phi}_{\zeta_2}^{[1]}) \mu_{q, \zeta_1, \zeta_2}, \\
\Delta_q \bar{Y}_\varepsilon^{\Psi}(\bar{\zeta}) &= \varepsilon^{-\frac{1}{2}} f_{2,\varepsilon} f_{3,\varepsilon} \int_{\zeta_1, \zeta_2} (1 - J_0)(\llbracket Y_{\varepsilon, \zeta_1}^2 \rrbracket \llbracket Y_{\varepsilon, \zeta_2}^2 \rrbracket) \mu_{q, \zeta_1, \zeta_2} \\
&\quad + \frac{1}{3} \int_{\zeta_1, \zeta_2} \delta G_1 D (\bar{\Phi}_{\zeta_1}^{[1]} \hat{\Phi}_{\zeta_2}^{[1]}) \mu_{q, \zeta_1, \zeta_2}, \\
\Delta_q Y_\varepsilon^{\Psi}(\bar{\zeta}) &= (f_{3,\varepsilon})^2 \int_{\zeta_1, \zeta_2} \llbracket Y_{\varepsilon, \zeta_1}^3 \rrbracket Y_{\zeta_2} \mu_{q, \zeta_1, \zeta_2} \\
&\quad + \frac{f_{3,\varepsilon}}{6} \int_{\zeta_1, \zeta_2} \delta G_1 D (6 \hat{\Phi}_{\zeta_1}^{[0]} Y_{\varepsilon, \zeta_2} + \llbracket Y_{\varepsilon, \zeta_1}^3 \rrbracket \hat{\Phi}_{\zeta_2}^{[2]}) \mu_{q, \zeta_1, \zeta_2} \\
&\quad + \frac{1}{6} \int_{\zeta_1, \zeta_2} \delta G_1 D (\hat{\Phi}_{\zeta_1}^{[0]} \hat{\Phi}_{\zeta_2}^{[2]}) \mu_{q, \zeta_1, \zeta_2}, \\
\Delta_q Y_\varepsilon^{\Psi}(\bar{\zeta}) &= (f_{3,\varepsilon})^2 \int_{\zeta_1, \zeta_2} (1 - J_1)(\llbracket Y_{\varepsilon, \zeta_1}^3 \rrbracket \llbracket Y_{\varepsilon, \zeta_2}^2 \rrbracket) \mu_{q, \zeta_1, \zeta_2} + \frac{1}{3} \Delta_q R_\varepsilon^{\Psi}(\bar{\zeta}) \\
&\quad + 6(f_{3,\varepsilon})^2 \int_{s,x} [\Delta_q Y_\varepsilon(t+s, \bar{x}-x) - \Delta_q Y_\varepsilon(t, \bar{x})] P_s(x) [C_\varepsilon(s, x)]^2 \\
&\quad + \frac{1}{3} \int_{\zeta_1, \zeta_2} \delta^2 G_{[1]}^{[2]} D^2 (3 \hat{\Phi}_{\zeta_1}^{[0]} \llbracket Y_{\varepsilon, \zeta_2}^2 \rrbracket + \llbracket Y_{\varepsilon, \zeta_1}^3 \rrbracket \hat{\Phi}_{\zeta_2}^{[1]}) \mu_{q, \zeta_1, \zeta_2} \\
&\quad + \frac{1}{3} \Delta_q \hat{R}_\varepsilon^{\Psi} + \frac{1}{3} \int_{\zeta_1, \zeta_2} \delta^2 G_{[1]}^{[2]} D^2 (\hat{\Phi}_{\zeta_1}^{[0]} \hat{\Phi}_{\zeta_2}^{[1]}) \mu_{q, \zeta_1, \zeta_2}.
\end{aligned} \tag{6.44}$$

Let us summarize our results so far. We have shown that

$$\Delta_q \hat{R}_\varepsilon^{\Psi}(\bar{\zeta}) + \Delta_q R_\varepsilon^{\Psi}(\bar{\zeta}) \sim O_{L^\infty}(\varepsilon^\delta 2^{q(1/2+2\kappa+2\delta)})$$

in $L^p(\Omega)$ and then these terms converge to 0 in the right topology as $\varepsilon \rightarrow 0$. As already mentioned, the convergence in law of

$$\begin{aligned}
(f_{3,\varepsilon})^2 \int_{\zeta_1, \zeta_2} (1 - J_0)(\llbracket Y_{\varepsilon, \zeta_1}^2 \rrbracket \llbracket Y_{\varepsilon, \zeta_2}^2 \rrbracket) \mu_{q, \zeta_1, \zeta_2} &\rightarrow (\lambda_3)^2 \Delta_q Y^{\Psi}(\bar{\zeta}) \\
\varepsilon^{-\frac{1}{2}} f_{2,\varepsilon} f_{3,\varepsilon} \int_{\zeta_1, \zeta_2} (1 - J_0)(\llbracket Y_{\varepsilon, \zeta_1}^2 \rrbracket \llbracket Y_{\varepsilon, \zeta_2}^2 \rrbracket) \mu_{q, \zeta_1, \zeta_2} &\rightarrow \lambda_3 \lambda_2 \Delta_q Y^{\Psi}(\bar{\zeta}) \\
(f_{3,\varepsilon})^2 \int_{\zeta_1, \zeta_2} \llbracket Y_{\varepsilon, \zeta_1}^3 \rrbracket Y_{\zeta_2} \mu_{q, \zeta_1, \zeta_2} &\rightarrow (\lambda_3)^2 \Delta_q Y^{\Psi}(\bar{\zeta})
\end{aligned}$$

$$\begin{aligned}
(f_{3,\varepsilon})^2 \int_{\zeta_1, \zeta_2} (1 - J_1)(\llbracket Y_{\varepsilon, \zeta_1}^3 \rrbracket \llbracket Y_{\varepsilon, \zeta_2}^2 \rrbracket) \mu_{q, \zeta_1, \zeta_2} \\
+ 6(f_{3,\varepsilon})^2 \int_{s,x} [\Delta_q Y_\varepsilon(t+s, \bar{x}-x) - \Delta_q Y_\varepsilon(t, \bar{x})] P_s(x) [C_\varepsilon(s, x)]^2 &\rightarrow (\lambda_3)^2 \Delta_q Y^{\Psi}(\bar{\zeta})
\end{aligned}$$

is easy to establish with standard techniques (as done in [CC13],[MWX16]) assuming the convergence of λ_ε as in (6.9) to λ . Then, comparing (6.44) with the canonical trees in (6.20) we can identify the remainder terms $\Delta_q Y_\varepsilon^{\hat{\tau}}$ that still need to be bounded, that are precisely those in which $\hat{\Phi}_\zeta^{[n]}$ appears. Estimating these terms is the content of next section.

6.3.4.2 Estimation of renormalised composite trees

In this section we prove the bound (6.30) for composite trees. The difficulty we encounter here is that the remainder $Y_\varepsilon^{\hat{\Psi}}$ cannot be written as an iterated Skorohod integral as in Section 6.3.1, but instead as a product of iterated Skorohod integrals. We will then use the product formula (2.22) to write the remainder in the desired form. We can write (6.44) in a much shorter way as:

$$\begin{aligned}\Delta_q Y_\varepsilon^{\hat{\Psi}}(\bar{\zeta}) &= \frac{1}{9} \delta G_1 \int_{\zeta_1, \zeta_2} D(\Phi_{\zeta_1}^{[1]} \Phi_{\zeta_2}^{[1]}) \mu_{q, \zeta_1, \zeta_2}, \\ \Delta_q \bar{Y}_\varepsilon^{\hat{\Psi}}(\bar{\zeta}) &= \frac{1}{3} \delta G_1 \int_{\zeta_1, \zeta_2} D(\bar{\Phi}_{\zeta_1}^{[1]} \Phi_{\zeta_2}^{[1]}) \mu_{q, \zeta_1, \zeta_2}, \\ \Delta_q Y_\varepsilon^{\Psi}(\bar{\zeta}) &= \frac{1}{6} \delta G_1 \int_{\zeta_1, \zeta_2} D(\Phi_{\zeta_1}^{[0]} \Phi_{\zeta_2}^{[2]}) \mu_{q, \zeta_1, \zeta_2}, \\ \Delta_q Y_\varepsilon^{\Psi}(\bar{\zeta}) &= \frac{1}{3} \delta^2 G_{[1]}^{[2]} \int_{\zeta_1, \zeta_2} D^2(\Phi_{\zeta_1}^{[0]} \Phi_{\zeta_2}^{[1]}) \mu_{q, \zeta_1, \zeta_2} + \frac{1}{3} \Delta_q R_\varepsilon^{\Psi}(\bar{\zeta}) + \frac{1}{3} \Delta_q \hat{R}_\varepsilon^{\Psi} \\ &\quad + 6(f_{3, \varepsilon})^2 \int_{s, x} [\Delta_q Y_\varepsilon(t+s, \bar{x}-x) - \Delta_q Y_\varepsilon(t, \bar{x})] P_s(x) [C_\varepsilon(s, x)]^2,\end{aligned}$$

just substituting again $\delta G_1 D = (1 - J_0)$ and $\delta^2 G_{[1]}^{[2]} D^2 = (1 - J_0 - J_1)$. In order to treat all trees at the same time, we can write the first terms in the r.h.s. above (modulo a constant that we discard) as:

$$\delta^r G_{[1]}^{[r]} \int_{\zeta_1, \zeta_2} D^r(\Phi_{\zeta_1}^{[i]} \Phi_{\zeta_2}^{[j]}) \mu_{q, \zeta_1, \zeta_2} \quad \text{with } r=1, i+j=2 \text{ or } r=2, i+j=1.$$

First notice that by Lemma 2.23 we have

$$\left\| \delta^r G_{[1]}^{[r]} \int_{\zeta_1, \zeta_2} D^r(\Phi_{\zeta_1}^{[i]} \Phi_{\zeta_2}^{[j]}) \mu_{q, \zeta_1, \zeta_2} \right\|_{L^p(\Omega)} \lesssim \sum_{\theta=0}^r \left\| D^\theta G_{[1]}^{[r]} \int_{\zeta_1, \zeta_2} D^r(\Phi_{\zeta_1}^{[i]} \Phi_{\zeta_2}^{[j]}) \mu_{q, \zeta_1, \zeta_2} \right\|_{L^p(H^{\otimes r+\theta})}$$

and from the boundedness of the operator $D^\theta G_{[1]}^{[r]}$ given by Corollary 2.27 we obtain:

$$\left\| \delta^r G_{[1]}^{[r]} \int_{\zeta_1, \zeta_2} D^r(\Phi_{\zeta_1}^{[i]} \Phi_{\zeta_2}^{[j]}) \mu_{q, \zeta_1, \zeta_2} \right\|_{L^p(\Omega)} \lesssim \left\| \int_{\zeta_1, \zeta_2} D^r(\Phi_{\zeta_1}^{[i]} \Phi_{\zeta_2}^{[j]}) \mu_{q, \zeta_1, \zeta_2} \right\|_{L^p(\Omega, H^{\otimes r})}.$$

Computing the r -th derivative of the integrand we obtain

$$\begin{aligned}& \Phi_{\zeta_1}^{[4-m]} \Phi_{\zeta_2}^{[4-n]} h_{\zeta_1}^{\otimes k} \otimes h_{\zeta_2}^{\otimes \ell} \\ = & \left[\frac{3! f_{3, \varepsilon}}{(3-m)!} \llbracket Y_{\zeta_1}^{m-1} \rrbracket + \hat{\Phi}_{\zeta_1}^{[4-m]} \right] \left[\frac{3! f_{3, \varepsilon}}{(3-n)!} \llbracket Y_{\zeta_2}^{n-1} \rrbracket + \hat{\Phi}_{\zeta_2}^{[4-n]} \right] h_{\zeta_1}^{\otimes k} \otimes h_{\zeta_2}^{\otimes \ell}\end{aligned} \quad (6.45)$$

for $m+n=5$ and $0 \leq k+\ell \leq 2$. The constraints on m, n, k, ℓ are related to the number of branches in the graphical notation of the trees: each tree has $m+k-1$ leaves with height 2 and $n+\ell-1$ leaves with height 1, as follows

$$\begin{aligned}Y_\varepsilon^{\hat{\Psi}} &\leftrightarrow m+k=3, \quad n+\ell=3 \\ \bar{Y}_\varepsilon^{\hat{\Psi}} &\leftrightarrow m=3, k=0, n=2, \ell=1 \\ Y_\varepsilon^{\Psi} &\leftrightarrow m+k=4, \quad n+\ell=2 \\ Y_\varepsilon^{\Psi} &\leftrightarrow m+k=4, \quad n+\ell=3.\end{aligned} \quad (6.46)$$

In (6.45), the terms which do not contain $\hat{\Phi}_\zeta^{[n]}$ will generate finite contributions in the limit, as seen in Section 6.3.4.1 by writing the decomposition (6.44). We just consider the terms proportional to $\hat{\Phi}_{\zeta_1}^{[4-m]}\hat{\Phi}_{\zeta_2}^{[4-n]}$, because all the other similar terms featuring at least one remainder $\hat{\Phi}_\zeta^{[m]}$ can be estimated with exactly the same technique, and are easily shown to be vanishing in the appropriate topology.

We can use one of the key observations of this chapter, the product formula (2.22), to rewrite products of Skorohod integrals in the form $\delta^m(u)\delta^n(v)$ as a sum of iterated Skorohod integrals $\delta^\ell(w)$, which are bounded in L^p by Lemma 2.23. We obtain

$$\begin{aligned}\hat{\Phi}_{\zeta_1}^{[4-m]}\hat{\Phi}_{\zeta_2}^{[4-n]} &= \delta^m(G_{[1]}^{[m]}\Phi_{\zeta_1}^{[4]}h_{\zeta_1}^{\otimes m})\delta^n(G_{[1]}^{[n]}\Phi_{\zeta_2}^{[4]}h_{\zeta_2}^{\otimes n}) \\ &= \sum_{(q,r,i) \in I} C_{q,r,i} \delta^{m+n-q-r} (\langle D^{r-i}G_{[1]}^{[m]}\Phi_{\zeta_1}^{[4]}h_{\zeta_1}^{\otimes m}, D^{q-i}G_{[1]}^{[n]}\Phi_{\zeta_2}^{[4]}h_{\zeta_2}^{\otimes n} \rangle_{H^{\otimes q+r-i}}) \\ &= \sum_I C_{q,r,i} \varepsilon^{\frac{2+r+q-2i}{2}} \delta^{m+n-q-r} (\langle \Theta_{[1+r-i]}^{[m+r-i]}(\zeta_1)h_{\zeta_1}^{\otimes m+r-i}, \Theta_{[1+q-i]}^{[n+q-i]}(\zeta_2)h_{\zeta_2}^{\otimes n+q-i} \rangle_{H^{\otimes q+r-i}})\end{aligned}$$

with $I = \{(q, r, i) \in \mathbb{N}^3: 0 \leq q \leq m, 0 \leq r \leq n, 0 \leq i \leq q \wedge r\}$ and the notation shortcut:

$$\Theta_{[i]}^{[j]}(\zeta) := \varepsilon^{-\frac{i}{2}} G_{[i]}^{[j]} \Phi_\zeta^{[3+i]}.$$

By Remark 2.24, for every $n, m \geq 1$ and $\Psi \in \text{Dom } \delta^n$ we can write $\delta^n(\Psi)h^{\otimes m} = \delta^n(\Psi \otimes h^{\otimes m})$, and therefore

$$\begin{aligned}& \int \hat{\Phi}_{\zeta_1}^{[4-m]}\hat{\Phi}_{\zeta_2}^{[4-n]}h_{\zeta_1}^{\otimes k} \otimes h_{\zeta_2}^{\otimes \ell} \mu_{q, \zeta_1, \zeta_2} \\ &= \sum_I C_{q,r,i} \varepsilon^{\frac{2+q+r-2i}{2}} \delta^{m+n-q-r} \\ & \int \Theta_{[1+r-i]}^{[m+r-i]}(\zeta_1)\Theta_{[1+q-i]}^{[n+q-i]}(\zeta_2) h_{\zeta_1}^{\otimes m-q} \otimes h_{\zeta_2}^{\otimes n-r} \otimes h_{\zeta_1}^{\otimes k} \otimes h_{\zeta_2}^{\otimes \ell} |\langle h_{\zeta_1}, h_{\zeta_2} \rangle|^{q+r-i} \mu_{q, \zeta_1, \zeta_2}\end{aligned}$$

With the term to estimate in this form, we can proceed as in Section 6.3.1 to estimate separately the terms $\Theta_{[i]}^{[j]}(\zeta) = \varepsilon^{-\frac{i}{2}} G_{[i]}^{[j]} \Phi_\zeta^{[3+i]}$ in $L^p(\Omega)$, which are bounded as discussed in Remark 6.9.

Lemma 6.13. *Under Assumption 6.1 (in particular if $F_\varepsilon \in C^8(\mathbb{R})$ and the first 8 derivatives have exponential growth) the norm $\|\cdot\|_{L^p(H^{\otimes k+\ell})}^2$ of the term*

$$\delta^{m+n-q-r} \int \Theta_{[1+r-i]}^{[m+r-i]}(\zeta_1)\Theta_{[1+q-i]}^{[n+q-i]}(\zeta_2) h_{\zeta_1}^{\otimes m-q} \otimes h_{\zeta_2}^{\otimes n-r} \otimes h_{\zeta_1}^{\otimes k} \otimes h_{\zeta_2}^{\otimes \ell} |\langle h_{\zeta_1}, h_{\zeta_2} \rangle|^{q+r-i} \mu_{q, \zeta_1, \zeta_2}^2$$

can be bounded as

$$\lesssim \int |\langle h_{\zeta_1}, h_{\zeta_1'} \rangle|^{m+k-q} |\langle h_{\zeta_2}, h_{\zeta_2'} \rangle|^{n+\ell-r} |\langle h_{\zeta_1}, h_{\zeta_2} \rangle|^{q+r-i} |\langle h_{\zeta_1'}, h_{\zeta_2'} \rangle|^{q+r-i} \mu_{q, \zeta_1, \zeta_2} \|\mu_{q, \zeta_1', \zeta_2'}\|.$$

Proof. Thanks to Lemma 2.23 the integral can be estimated with

$$\begin{aligned}& \sum_{j=0, h \leq j}^{m+n-q-r} \left\| \int D^h \Theta_{[1+r-i]}^{[m+r-i]}(\zeta_1) D^{j-h} \Theta_{[1+q-i]}^{[n+q-i]}(\zeta_2) \right. \\ & \quad \left. h_{\zeta_1}^{\otimes m-q} \otimes h_{\zeta_2}^{\otimes n-r} \otimes h_{\zeta_1}^{\otimes k} \otimes h_{\zeta_2}^{\otimes \ell} |\langle h_{\zeta_1}, h_{\zeta_2} \rangle|^{q+r-i} \mu_{q, \zeta_1, \zeta_2} \right\|_{L^p(V)}^2,\end{aligned}$$

with $V = H^{\otimes m+k+n+\ell-q-r+j}$. We have that $\|\cdot\|_{L^p(H^{\otimes k+\ell})}^2 = \|\|\cdot\|_{H^{\otimes k+\ell}}^2\|_{L^{p/2}}$ and therefore we can bound each term in the sum above as

$$\begin{aligned} &\lesssim \left(\int \|\langle D^h \Theta_{[1+r-i]}^{[m+r-i]}(\zeta_1) D^{j-h} \Theta_{[1+q-i]}^{[n+q-i]}(\zeta_2), D^h \Theta_{[1+r-i]}^{[m+r-i]}(\zeta'_1) D^{j-h} \Theta_{[1+q-i]}^{[n+q-i]}(\zeta'_2) \rangle_{H^{\otimes j}}\|_{L^{p/2}} \right. \\ &\quad \left. |\langle h_{\zeta_1}, h_{\zeta'_1} \rangle|^{m+k-q} |\langle h_{\zeta_2}, h_{\zeta'_2} \rangle|^{n+\ell-r} |\langle h_{\zeta_1}, h_{\zeta_2} \rangle|^{q+r-i} |\langle h_{\zeta'_1}, h_{\zeta'_2} \rangle|^{q+r-i} |\mu_{q,\zeta_1,\zeta_2}| |\mu_{q,\zeta'_1,\zeta'_2}| \right)^{1/2} \end{aligned}$$

Using Hölder's inequality we get the estimate

$$\begin{aligned} &\|\langle D^h \Theta_{[1+r-i]}^{[m+r-i]}(\zeta_1) D^{j-h} \Theta_{[1+q-i]}^{[n+q-i]}(\zeta_2), D^h \Theta_{[1+r-i]}^{[m+r-i]}(\zeta'_1) D^{j-h} \Theta_{[1+q-i]}^{[n+q-i]}(\zeta'_2) \rangle_{H^{\otimes j}}\|_{L^{p/2}} \\ &\lesssim \|\langle D^h \Theta_{[1+r-i]}^{[m+r-i]}(\zeta_1), D^h \Theta_{[1+r-i]}^{[m+r-i]}(\zeta'_1) \rangle_{H^{\otimes h}}\|_{L^p} \times \\ &\|\langle D^{j-h} \Theta_{[1+q-i]}^{[n+q-i]}(\zeta_2), D^{j-h} \Theta_{[1+q-i]}^{[n+q-i]}(\zeta'_2) \rangle_{H^{\otimes j-h}}\|_{L^p} \end{aligned}$$

Now to bound

$$\|\langle D^h \Theta_{[1+a]}^{[m+a]}(\zeta), D^h \Theta_{[1+a]}^{[m+a]}(\zeta') \rangle_{H^{\otimes h}}\|_{L^p}$$

(with $h \leq j \leq m+n-q-r$ and $a = r-i$) we can use the boundedness of the operator $D^h G_{[1+a]}^{[m+a]}$ for $h \leq 2m$ given by Corollary 2.27. Consider the two regions $h \leq 2m$ and $h > 2m$. In the first region we just use Corollary 2.27 to obtain:

$$\begin{aligned} &\|\langle D^h \Theta_{[1+a]}^{[m+a]}(\zeta), D^h \Theta_{[1+a]}^{[m+a]}(\zeta') \rangle_{H^{\otimes h}}\|_{L^p} \\ &\lesssim \left\| D^h G_{[1+a]}^{[m+a]} \varepsilon^{-\frac{1+a}{2}} \Phi_{\zeta_1}^{[4+a]} \right\|_{L^{4p}(H^{\otimes h})}^2 \left\| D^h G_{[1+a]}^{[m+a]} \varepsilon^{-\frac{1+a}{2}} \Phi_{\zeta'_1}^{[4+a]} \right\|_{L^{4p}(H^{\otimes h})} \\ &\lesssim \left\| \varepsilon^{-\frac{1+a}{2}} \Phi_{\zeta_1}^{[4+a]} \right\|_{L^{4p}}^2 \left\| \varepsilon^{-\frac{1+a}{2}} \Phi_{\zeta'_1}^{[4+a]} \right\|_{L^{4p}}^2. \end{aligned}$$

If $h > 2m$ we first use the bound $D^{2m} G_{[1+a]}^{[m+a]}$ and then take the remaining $h-2m$ derivatives on $\varepsilon^{-\frac{1+a}{2}} \Phi_{\zeta}^{[4+a]}$:

$$\begin{aligned} \left\| D^h G_{[1+a]}^{[m+a]} \varepsilon^{-\frac{1+a}{2}} \Phi_{\zeta_1}^{[4+a]} \right\|_{L^{4p}(H^{\otimes h})}^2 &\lesssim \left\| D^{h-2m} \varepsilon^{-\frac{1+a}{2}} \Phi_{\zeta_1}^{[4+a]} \right\|_{L^{4p}(H^{\otimes h-2m})}^2 \\ &\lesssim \left\| \varepsilon^{-\frac{1+a+h-2m}{2}} \Phi_{\zeta_1}^{[4+a+h-2m]} \right\|_{L^{4p}(H^{\otimes h-2m})}^2 \end{aligned}$$

From Remark 6.9 we see that this last term is bounded by a constant if $F \in C^{4+a+h-2m}(\mathbb{R})$ with the first $4+a+h-2m$ derivatives having an exponential growth, with $a = r-i$ and $h \leq m+n-q-r$.

Applying the same reasoning to $\|\langle D^{j-h} \Theta_{[1+q-i]}^{[n+q-i]}(\zeta_2), D^{j-h} \Theta_{[1+q-i]}^{[n+q-i]}(\zeta'_2) \rangle_{H^{\otimes j-h}}\|_{L^p}$ we conclude that we need to control $4+n \vee 4+m$ derivatives of F_ε in order to perform the estimates of this Lemma. From the constraints (6.46) we see that $4+n \vee 4+m \leq 8$. \square

From Lemma 6.13 we obtain $\forall \delta \in [0, 1/2)$: $\varepsilon^{\frac{2+q+r-2i}{2}}$

$$\begin{aligned} &\left\| \varepsilon^{\frac{2+q+r-2i}{2}} \delta^{m+n-q-r} \left[\int \Theta_{[1+r-i]}^{[m+r-i]}(\zeta_1) \Theta_{[1+q-i]}^{[n+q-i]}(\zeta_2) \right. \right. \\ &\quad \left. \left. h_{\zeta_1}^{\otimes m-q} \otimes h_{\zeta_2}^{\otimes n-r} \otimes h_{\zeta_1}^{\otimes k} \otimes h_{\zeta_2}^{\otimes \ell} |\langle h_{\zeta_1}, h_{\zeta_2} \rangle|^{q+r-i} \mu_{q,\zeta_1,\zeta_2} \right] \right\|_{L^p(H^{\otimes k+\ell})} \\ &\lesssim \varepsilon^\delta (\varepsilon^{2+q+r-2i-\delta} \int |\langle h_{\zeta_1}, h_{\zeta'_1} \rangle|^{m+k-q} |\langle h_{\zeta_2}, h_{\zeta'_2} \rangle|^{n+\ell-r} \\ &\quad |\langle h_{\zeta_1}, h_{\zeta_2} \rangle|^{q+r-i} |\langle h_{\zeta'_1}, h_{\zeta'_2} \rangle|^{q+r-i} |\mu_{q,\zeta_1,\zeta_2}| |\mu_{q,\zeta'_1,\zeta'_2}|)^{\frac{1}{2}} \\ &:= \varepsilon^{\frac{\delta}{2}} (\mathcal{J})^{\frac{1}{2}}. \end{aligned}$$

Our aim now is to estimate the quantity \mathfrak{I} . The idea is to use the bound $\varepsilon|\langle h_\zeta, h_{\zeta'} \rangle| = \varepsilon C_\varepsilon(\zeta - \zeta') \lesssim 1$ of Lemma 2.15 to cancel strategically some of the covariances $|\langle h_\zeta, h_{\zeta'} \rangle|$. We will consider three regions:

If $q + r \leq 2$ we use the bounds

$$\varepsilon^{q+r-2i} |\langle h_{\zeta_1}, h_{\zeta_2} \rangle|^{q+r-i} |\langle h_{\zeta'_1}, h_{\zeta'_2} \rangle|^{q+r-i} \lesssim \varepsilon^2 |\langle h_{\zeta_1}, h_{\zeta_2} \rangle|^q |\langle h_{\zeta'_1}, h_{\zeta'_2} \rangle|^r$$

and then (we suppose $r < 2$)

$$\varepsilon^{2-r-\delta} |\langle h_{\zeta_2}, h_{\zeta'_2} \rangle|^{n+\ell-r} \lesssim |\langle h_{\zeta_2}, h_{\zeta'_2} \rangle|^{n+\ell-2+\delta}$$

to obtain

$$\begin{aligned} \mathfrak{I} &\lesssim \varepsilon^{r-\delta} \int |\langle h_{\zeta_1}, h_{\zeta'_1} \rangle|^{m+k-q} |\langle h_{\zeta_2}, h_{\zeta'_2} \rangle|^{n+\ell-2} |\langle h_{\zeta_1}, h_{\zeta_2} \rangle|^q |\langle h_{\zeta'_1}, h_{\zeta'_2} \rangle|^r |\mu_{q, \zeta_1, \zeta_2}| |\mu_{q, \zeta'_1, \zeta'_2}| \\ &\lesssim \int |\langle h_{\zeta_1}, h_{\zeta'_1} \rangle|^{m+k-q} |\langle h_{\zeta_2}, h_{\zeta'_2} \rangle|^{n+\ell-2+\delta} |\langle h_{\zeta_1}, h_{\zeta_2} \rangle|^q |\mu_{q, \zeta_1, \zeta_2}| |\mu_{q, \zeta'_1, \zeta'_2}|. \end{aligned} \quad (6.47)$$

(If vice-versa $q < 2$ it suffices to put δ on the term $|\langle h_{\zeta_1}, h_{\zeta_2} \rangle|^{q+\delta}$.) Notice that in this case $m + k - q > 0$.

In the case $q + r = 3$ if $m + k - q \geq 2$ we estimate like before to obtain

$$\begin{aligned} \mathfrak{I} &\lesssim \varepsilon^{2-\delta} \int |\langle h_{\zeta_1}, h_{\zeta'_1} \rangle|^{m+k-q} |\langle h_{\zeta_2}, h_{\zeta'_2} \rangle|^{n+\ell-r} |\langle h_{\zeta_1}, h_{\zeta_2} \rangle|^{\frac{q+r}{2}} |\langle h_{\zeta'_1}, h_{\zeta'_2} \rangle|^{\frac{q+r}{2}} |\mu_{q, \zeta_1, \zeta_2}| |\mu_{q, \zeta'_1, \zeta'_2}| \\ &\lesssim \int |\langle h_{\zeta_1}, h_{\zeta'_1} \rangle|^{m+k-q} |\langle h_{\zeta_2}, h_{\zeta'_2} \rangle|^{n+\ell-r} |\langle h_{\zeta_1}, h_{\zeta_2} \rangle|^{1+\delta} |\mu_{q, \zeta_1, \zeta_2}| |\mu_{q, \zeta'_1, \zeta'_2}|. \end{aligned} \quad (6.48)$$

Note that $m + k - q + \delta - 1 > 0$ and $m + k - q + 2\delta - 3 > -1$ here. If $m + k - q = 1$ we bound

$$\mathfrak{I} \lesssim \int |\langle h_{\zeta_1}, h_{\zeta'_1} \rangle|^{m+k-q} |\langle h_{\zeta_2}, h_{\zeta'_2} \rangle|^{n+\ell-r-2} |\langle h_{\zeta_1}, h_{\zeta_2} \rangle|^{\frac{3+\delta}{2}} |\langle h_{\zeta'_1}, h_{\zeta'_2} \rangle|^{\frac{3+\delta}{2}} |\mu_{q, \zeta_1, \zeta_2}| |\mu_{q, \zeta'_1, \zeta'_2}| \quad (6.49)$$

and note that $m + k - q - 1/2 + \delta/2 > 0$, $m + k - q - 1 + \delta > 0$, $n + \ell - r - 2 \geq 0$. Finally if $m + k - q = 0$ we can only have $m + k = 3$, $q = 3$, $r = 0$, $i = 0$ and thus

$$\begin{aligned} \mathfrak{I} &\lesssim \varepsilon^{3-2\delta} \int |\langle h_{\zeta_2}, h_{\zeta'_2} \rangle|^{n+\ell} |\langle h_{\zeta_1}, h_{\zeta_2} \rangle|^{2-\delta} |\langle h_{\zeta'_1}, h_{\zeta'_2} \rangle|^{2-\delta} |\mu_{q, \zeta_1, \zeta_2}| |\mu_{q, \zeta'_1, \zeta'_2}| \\ &\lesssim \int |\langle h_{\zeta_2}, h_{\zeta'_2} \rangle|^{n+\ell+m+k-6} |\langle h_{\zeta_1}, h_{\zeta_2} \rangle|^{2-\delta} |\langle h_{\zeta'_1}, h_{\zeta'_2} \rangle|^{2-\delta} |\mu_{q, \zeta_1, \zeta_2}| |\mu_{q, \zeta'_1, \zeta'_2}| \end{aligned} \quad (6.50)$$

If $q + r \geq 4$ we bound first

$$\varepsilon^{2q+2r-2i+\delta-4} |\langle h_{\zeta_1}, h_{\zeta_2} \rangle|^{q+r-i} |\langle h_{\zeta'_1}, h_{\zeta'_2} \rangle|^{q+r-i} \lesssim |\langle h_{\zeta_1}, h_{\zeta_2} \rangle|^{2-\frac{\delta}{2}} |\langle h_{\zeta'_1}, h_{\zeta'_2} \rangle|^{2-\frac{\delta}{2}}$$

(note that $2q + 2r - 2i + \delta - 4 \geq \delta$) to obtain:

$$\begin{aligned} &\mathfrak{I} \lesssim \\ &\varepsilon^{6-q-r-\delta} \int |\langle h_{\zeta_1}, h_{\zeta'_1} \rangle|^{m+k-q} |\langle h_{\zeta_2}, h_{\zeta'_2} \rangle|^{n+\ell-r} |\langle h_{\zeta_1}, h_{\zeta_2} \rangle|^{2-\frac{\delta}{2}} |\langle h_{\zeta'_1}, h_{\zeta'_2} \rangle|^{2-\frac{\delta}{2}} |\mu_{q, \zeta_1, \zeta_2}| |\mu_{q, \zeta'_1, \zeta'_2}| \end{aligned}$$

Now in the cases $m + k = 3$, $n + \ell = 3$ and $m + k = 4$, $n + \ell = 2$ we can just write $\varepsilon^{6-q-r-\delta} = \varepsilon^{m+k-q} \varepsilon^{6-m-k-r-\delta}$ and cancel the corresponding number of covariances to obtain

$$\mathfrak{I} \lesssim \int |\langle h_{\zeta_2}, h_{\zeta'_2} \rangle|^\delta |\langle h_{\zeta_1}, h_{\zeta_2} \rangle|^{2-\frac{\delta}{2}} |\langle h_{\zeta'_1}, h_{\zeta'_2} \rangle|^{2-\frac{\delta}{2}} |\mu_{q, \zeta_1, \zeta_2}| |\mu_{q, \zeta'_1, \zeta'_2}| \quad (6.51)$$

while for the case $m+k=4, n+l=3$ we have either $l \geq 1$ or $k \geq 1$ and therefore with one of the following bounds

$$\begin{aligned} \varepsilon^{m+k-1-q} \varepsilon^{n+l-r-\delta} |\langle h_{\zeta_1}, h_{\zeta'_1} \rangle|^{m+k-q} |\langle h_{\zeta_2}, h_{\zeta'_2} \rangle|^{n+l-r} &\lesssim |\langle h_{\zeta_1}, h_{\zeta'_1} \rangle| |\langle h_{\zeta_2}, h_{\zeta'_2} \rangle|^\delta \\ \varepsilon^{m+k-q} \varepsilon^{n+l-1-r-\delta} |\langle h_{\zeta_1}, h_{\zeta'_1} \rangle|^{m+k-q} |\langle h_{\zeta_2}, h_{\zeta'_2} \rangle|^{n+l-r} &\lesssim |\langle h_{\zeta_2}, h_{\zeta'_2} \rangle|^{1+\delta} \end{aligned}$$

we obtain the estimates

$$\mathfrak{J} \lesssim \int |\langle h_{\zeta_2}, h_{\zeta'_2} \rangle|^{1+\delta} |\langle h_{\zeta_1}, h_{\zeta_2} \rangle|^{2-\frac{\delta}{2}} |\langle h_{\zeta'_1}, h_{\zeta'_2} \rangle|^{2-\frac{\delta}{2}} |\mu_{q, \zeta_1, \zeta_2}| |\mu_{q, \zeta'_1, \zeta'_2}| \quad (6.52)$$

$$\mathfrak{J} \lesssim \int |\langle h_{\zeta_1}, h_{\zeta'_1} \rangle| |\langle h_{\zeta_2}, h_{\zeta'_2} \rangle|^\delta |\langle h_{\zeta_1}, h_{\zeta_2} \rangle|^{2-\frac{\delta}{2}} |\langle h_{\zeta'_1}, h_{\zeta'_2} \rangle|^{2-\frac{\delta}{2}} |\mu_{q, \zeta_1, \zeta_2}| |\mu_{q, \zeta'_1, \zeta'_2}|. \quad (6.53)$$

We can use directly Lemma 2.21 to obtain a final estimate of (6.47), (6.48), (6.49), (6.52). For (6.50), (6.51) and (6.53) notice that the integral over ζ_1, ζ'_1 is finite and thus the whole quantity is proportional to $|\langle h_{\zeta_2}, h_{\zeta'_2} \rangle|^n$. Globally, we have

$$\mathfrak{J} \lesssim 2^{(m+k+n+l-6)q}$$

as needed to prove (6.30).

Remark 6.14. Finally, by controlling one more derivative of F_ε as done in Section 6.3.3, we can show (6.31) for $Y^\tau = Y_\varepsilon^{\nabla_Y}, Y_\varepsilon^{\nabla_{Y^*}}, Y_\varepsilon^{\nabla_{Y^*}}, Y_\varepsilon^{\nabla_{Y^*}}$, thus proving that $\hat{Y}^\tau \rightarrow 0$ in $C_T^{\kappa/2} \mathcal{C}^{\alpha-\kappa}$ in probability $\forall \alpha < |\tau|$. From the proof of Lemma 6.13 together with this observation, we conclude that we need to control the derivatives of F_ε up to order 9 to be able to show the convergence for composite trees.

6.4 Convergence of the remainder and a priori bounds

In this section we prove the convergence of the remainder (Lemma 6.19), as well as some technical results on the norm of the solution, needed in the proof of Theorem 6.7. In order to prove Lemma 6.19 we need first to prove Lemma 6.15, Lemma 6.16 and Lemma 6.17 in this order.

6.4.1 Boundedness of the remainder

We show that the remainder $R_\varepsilon(v_\varepsilon)$ that appears in equation (6.16) can be controlled by a stochastic term $M_{\varepsilon, \delta}$ that converges to zero in probability, times a function of the solution v_ε . Let.

$$M_{\varepsilon, \delta}(Y_\varepsilon, u_{0, \varepsilon}) := \varepsilon^{\delta/2} \|e^{c\varepsilon^{1/2}|Y_\varepsilon| + c\varepsilon^{1/2}|P_\cdot(u_{0, \varepsilon} - Y_\varepsilon(0))|}\|_{L^p[0, T]L^p(\mathbb{T}^3)}. \quad (6.54)$$

for $p \in [1, \infty)$, $\delta \in [0, 1]$ and define

$$v_\varepsilon^\square := v_\varepsilon - v_\varepsilon^\sharp \quad \text{with} \quad v_\varepsilon^\sharp: t \mapsto P_t(u_{0, \varepsilon} - Y_\varepsilon(0)). \quad (6.55)$$

Lemma 6.15. (Boundedness of remainder) *For every $\gamma \in (0, 1)$, $\delta \in [0, 1]$ we have*

$$\|R_\varepsilon(v_\varepsilon, v_\varepsilon^\flat, v_\varepsilon^\sharp)(t, x)\|_{\mathcal{M}_T^{\gamma, p}L^p(\mathbb{T}^3)} \lesssim M_\varepsilon(Y_\varepsilon, u_{0, \varepsilon}) \|v_\varepsilon\|_{\mathcal{M}_T^{\gamma/(3+\delta)}L^\infty(\mathbb{T}^3)}^{3+\delta} e^{c\varepsilon^{1/2}\|v_\varepsilon^\square\|_{C_T L^\infty}}$$

with M_ε as in (6.54), v_ε^\square as in (6.55) and $\mathcal{M}_T^{\gamma/(3+\delta), p}L^p(\mathbb{T}^3)$, $\mathcal{M}_T^{\gamma/(3+\delta)}L^\infty(\mathbb{T}^3)$ defined in (1.51).

Proof. We can write the remainder in two ways:

$$\begin{aligned} R_\varepsilon(v_\varepsilon) &= v_\varepsilon^3 \int_0^1 [F_\varepsilon^{(3)}(\varepsilon^{\frac{1}{2}} Y_\varepsilon + \tau \varepsilon^{\frac{1}{2}} v_\varepsilon) - F_\varepsilon^{(3)}(\varepsilon^{\frac{1}{2}} Y_\varepsilon)] \frac{(1-\tau)^2}{2!} d\tau \\ &= \varepsilon^{\frac{1}{2}} v_\varepsilon^4 \int_0^1 F_\varepsilon^{(4)}(\varepsilon^{\frac{1}{2}} Y_\varepsilon + \tau \varepsilon^{\frac{1}{2}} v_\varepsilon) \frac{(1-\tau)^3}{3!} d\tau. \end{aligned}$$

From assumption (6.8) on F_ε we obtain by interpolation of these two expressions, $\forall \delta \in [0, 1]$, $\forall t \geq 0, x \in \mathbb{T}^3$,

$$|R_\varepsilon(v_\varepsilon)(t, x)| \lesssim \varepsilon^{\delta/2} |v_\varepsilon(t, x)|^{3+\delta} e^{c\varepsilon^{\frac{1}{2}} |Y_\varepsilon(t, x)| + c\varepsilon^{\frac{1}{2}} |v_\varepsilon^\sharp(t, x)| + c|\varepsilon^{\frac{1}{2}} v_\varepsilon^\square(t, x)|},$$

and we estimate, $\forall \gamma \in [0, 1]$,

$$\begin{aligned} &\|t \mapsto t^\gamma R_\varepsilon(v_\varepsilon)(t, x)\|_{L^p((0, T], L^p(\mathbb{T}^3))} \\ &\lesssim \|t^{\frac{\gamma}{3+\delta}} v_\varepsilon(t)\|_{C_T L^\infty}^{3+\delta} e^{c\varepsilon^{1/2} \|v_\varepsilon^\square\|_{C_T L^\infty}} \left\| \varepsilon^{\frac{\delta}{2}} e^{c\varepsilon^{\frac{1}{2}} |Y_\varepsilon(t, x)| + c\varepsilon^{\frac{1}{2}} |v_\varepsilon^\sharp(t, x)|} \right\|_{L^p[0, T] L^p(\mathbb{T}^3)}. \end{aligned} \quad \square$$

We can also verify that $M_{\varepsilon, \delta} \rightarrow 0$ in probability for every $\delta > 0$:

Lemma 6.16. (Convergence of the stochastic term) *Under Assumption 6.1 the random variable $M_{\varepsilon, \delta}(Y_\varepsilon, u_{0, \varepsilon})$ defined in (6.54) converges to zero in probability $\forall \delta \in (0, 1]$.*

Proof. We can use Young's inequality to estimate $M_{\varepsilon, \delta}(Y_\varepsilon, u_{0, \varepsilon})$ for some $c' > 0$ as

$$\begin{aligned} M_{\varepsilon, \delta}(Y_\varepsilon, u_{0, \varepsilon}) &\lesssim \varepsilon^{\delta/2} \|e^{c'\varepsilon^{1/2} |Y_\varepsilon|}\|_{L^p[0, T] L^p(\mathbb{T}^3)} + \varepsilon^{\delta/2} \|e^{c'\varepsilon^{1/2} |P \cdot Y_\varepsilon(0)|}\|_{L^p[0, T] L^p(\mathbb{T}^3)} \\ &\quad + \varepsilon^{\delta/2} T^{1/p} e^{c'} \|\varepsilon^{1/2} u_{0, \varepsilon}\|_{L^\infty}. \end{aligned}$$

Under Assumptions 6.1 the term $\|\varepsilon^{1/2} u_{0, \varepsilon}\|_{L^\infty(\mathbb{T}^3)}$ is uniformly bounded, so the third term above converges to zero in probability. Note that $\varepsilon^{1/2} Y_\varepsilon(t, x)$ and $P_t \varepsilon^{1/2} Y_\varepsilon(t = 0)$ are centered Gaussian random variables, and then both $\mathbb{E} \|e^{c'\varepsilon^{1/2} |Y_\varepsilon|}\|_{L^p[0, T] L^p(\mathbb{T}^3)}^p$ and $\mathbb{E} \|e^{c'\varepsilon^{1/2} |P \cdot Y_\varepsilon(0)|}\|_{L^p[0, T] L^p(\mathbb{T}^3)}^p$ are uniformly bounded in $\varepsilon > 0$ for every $p \in [1, \infty)$. This yields the convergence in probability of $M_{\varepsilon, \delta}(Y_\varepsilon, u_{0, \varepsilon})$. \square

In order to show that $\|R_\varepsilon(v_\varepsilon)\|_{\mathcal{M}_T^{\gamma', p} L^p} \rightarrow 0$ in probability for $\gamma' > \frac{1}{4} + \frac{3}{2}\kappa$ as needed in the proof of Theorem 6.7, we still need to control the norms $\|v_\varepsilon\|_{\mathcal{M}_T^{\frac{3+\delta}{3+\delta} L^\infty}}$ and $\|v_\varepsilon^\square\|_{C_T L^\infty}$ that appear in Lemma 6.15. This is done in next section.

6.4.2 A priori bounds on the solution

Lemma 6.17. (A priori bound on the solution)

Fix $T > 0$. There exists $\kappa > 0$, $T_\star = T_\star(\|\mathbb{Y}_\varepsilon\|_{\mathcal{X}_T}, \|u_{\varepsilon, 0}\|_{\mathcal{G}^{-1/2-\kappa}}, |\lambda_\varepsilon|) \in (0, T]$ a lower semicontinuous function depending only on $(\|\mathbb{Y}_\varepsilon\|_{\mathcal{X}_T}, \|u_{\varepsilon, 0}\|_{\mathcal{G}^{-1/2-\kappa}}, |\lambda_\varepsilon|)$ and a collection of events $(\mathcal{E}_\varepsilon)_{\varepsilon > 0}$ such that

$$\mathbb{P}(\mathcal{E}_\varepsilon) \rightarrow 1 \quad \text{as } \varepsilon \rightarrow 0$$

and conditionally on \mathcal{E}_ε there exists a universal constant $C > 0$ such that:

$$\|v_\varepsilon\|_{C_{T_\star} \mathcal{G}^{-\frac{1}{2}-\kappa}} + \|v_\varepsilon\|_{\mathcal{M}_{T_\star}^{\frac{1}{4} + \frac{3\kappa}{2}} L^\infty} \leq C(1 + |\lambda_\varepsilon|) (1 + \|\mathbb{Y}_\varepsilon\|_{\mathcal{X}_T})^3 (1 + \|u_{\varepsilon, 0}\|_{\mathcal{G}^{-1/2-\kappa}})^3$$

for any v_ε that solves equation (6.16). Moreover, still conditionally on \mathcal{E}_ε we have

$$\|v_\varepsilon^\square\|_{C_{T_\star}L^\infty} \leq C(1 + |\lambda_\varepsilon|) (1 + \|\mathbb{Y}_\varepsilon\|_{\mathcal{X}_T})^3 (1 + \|u_{\varepsilon,0}\|_{\mathcal{C}^{-1/2-\kappa}})^3$$

with v_ε^\square as in (6.55).

Proof. We know from Lemma 6.18 that the bounds on $\|v_\varepsilon\|_{C_{T_\star}\mathcal{C}^{-\frac{1}{2}-\kappa}} + \|v_\varepsilon\|_{\mathcal{M}_{T_\star}^{\frac{1}{2}+\frac{3\kappa}{2}}L^\infty}$ and $\|v_\varepsilon^\square\|_{C_{T_\star}L^\infty}$ hold whenever $M_{\varepsilon,\delta} \leq T_\star^{\kappa/2}$. The event $\mathcal{E}_\varepsilon = \{M_{\varepsilon,\delta} \leq T_\star^{\kappa/2}\}$ has $\mathbb{P}(\mathcal{E}_\varepsilon) \rightarrow 1$ by Lemma 6.16 and this proves the result. \square

The only thing left to prove is Lemma 6.18, which just a standard application of some well-known bounds on paraproducts, that are introduced in Sections 1.2.3, and 1.2.1.

First observe that for $\varepsilon > 0$ a pair $(v_\varepsilon, v_\varepsilon^\sharp)$ solves the paracontrolled equation (6.16) if and only if $v_\varepsilon^\sharp = v_\varepsilon^b + v_\varepsilon^\#$ and $(v_\varepsilon, v_\varepsilon^b)$ solves:

$$\begin{cases} v_\varepsilon &= -Y_\varepsilon^\Psi - \bar{Y}_\varepsilon^\Psi - 3v_\varepsilon \llcorner Y_\varepsilon^\Psi + v_\varepsilon^b + v_\varepsilon^\# \\ \mathcal{L}v_\varepsilon^b &= U(\lambda_\varepsilon, \mathbb{Y}_\varepsilon; v_\varepsilon, v_\varepsilon^b + v_\varepsilon^\#) - R_\varepsilon(v_\varepsilon) \\ v_\varepsilon^b(0) &= Y_\varepsilon^\Psi(0) + \bar{Y}_\varepsilon^\Psi(0) + 3v_{\varepsilon,0} \prec Y_\varepsilon^\Psi(0) \end{cases} \quad (6.56)$$

Here U is the same as in (6.16). The initial condition of (6.56) is given by $v_{\varepsilon,0} := u_{0,\varepsilon} - Y_\varepsilon(0)$. The a priori bounds of Lemma 6.18 come from being able to find closed estimates for (6.56).

Let us specify now all the notations we are going to use in the rest of this section. We consider the spaces

$$\mathcal{V}_T^b := \mathcal{L}_T^{2\kappa} \cap \mathcal{L}_T^{1/4, 1/2+2\kappa} \cap \mathcal{L}_T^{1/2, 1+2\kappa}, \quad \mathcal{V}_T := \mathcal{L}_T^{1/2, 1/2-\kappa} \cap \mathcal{L}_T^{1/4+3\kappa/2, 2\kappa},$$

with the corresponding norms

$$\|v_\varepsilon^b\|_{\mathcal{V}_T^b} := \|v_\varepsilon^b\|_{\mathcal{L}_T^{2\kappa}} + \|v_\varepsilon^b\|_{\mathcal{L}_T^{1/4, 1/2+2\kappa}} + \|v_\varepsilon^b\|_{\mathcal{L}_T^{1/2, 1+2\kappa}}, \quad (6.57)$$

$$\|v_\varepsilon\|_{\mathcal{V}_T} := \|v_\varepsilon\|_{\mathcal{L}_T^{1/2, 1/2-\kappa}} + \|v_\varepsilon\|_{\mathcal{L}_T^{1/4+3\kappa/2, 2\kappa}}. \quad (6.58)$$

We refer to Section 1.2.2 for the definition of the parabolic spaces $\mathcal{L}_T^{\gamma, \alpha}$. We let

$$\begin{aligned} v_\varepsilon^\square &:= v_\varepsilon - v^\sharp &= -Y_\varepsilon^\Psi - \bar{Y}_\varepsilon^\Psi - 3(v_\varepsilon^\square + v_\varepsilon^\sharp) \llcorner Y_\varepsilon^\Psi + v_\varepsilon^b, \\ v_\varepsilon^\sharp &:= v_\varepsilon^\square + Y_\varepsilon^\Psi &= -\bar{Y}_\varepsilon^\Psi - 3(v_\varepsilon^\square + v_\varepsilon^\sharp) \llcorner Y_\varepsilon^\Psi + v_\varepsilon^b, \end{aligned}$$

and $v_\varepsilon^\square(t=0) = v_\varepsilon^\sharp(t=0) = 0$. We define also the norm

$$\|v_\varepsilon^\square\|_{\mathcal{V}_T^\square} := \|v_\varepsilon^\square\|_{\mathcal{L}_T^{2\kappa}} + \|v_\varepsilon^\square\|_{\mathcal{M}_T^{1/4}\mathcal{C}^{1/2+2\kappa}},$$

with $\mathcal{M}_T^{1/4}\mathcal{C}^{1/2+2\kappa}$ given in Section 1.2.2. In order not to get lost in these definitions the reader can keep in mind the following:

- v_ε is the solution without the linear term;
- v_ε^\sharp is the contribution of the initial condition, which give origin to some explosive norm (near the initial time);
- v_ε^b is the regular part of the solution;
- $v_\varepsilon^\square, v_\varepsilon^\sharp$ enter in the estimation of the remainder, they are just convenient shortcuts for certain contributions appearing in v_ε .

Lemma 6.18. *There exists $T_\star = T_\star(\|\mathbb{Y}_\varepsilon\|_{\mathcal{X}_T}, \|u_{\varepsilon,0}\|_{\mathcal{G}^{-1/2-\kappa}}, |\lambda_\varepsilon|) \in (0, T]$ a lower semicontinuous function depending only on $\|\mathbb{Y}_\varepsilon\|_{\mathcal{X}_T}$, $\|u_{\varepsilon,0}\|_{\mathcal{G}^{-1/2-\kappa}}$ and $|\lambda_\varepsilon|$, a constant $M_{\varepsilon,\delta} = M_{\varepsilon,\delta}(Y_\varepsilon, u_{0,\varepsilon}) > 0$ defined by (6.54), and a universal constant $C > 0$ such that, whenever $M_{\varepsilon,\delta} \leq T_\star^{\kappa/2}$ we have*

$$\begin{aligned} \|v_\varepsilon^b\|_{\mathcal{V}_{T_\star}^b} &\leq C(1 + |\lambda_\varepsilon|) (1 + \|\mathbb{Y}_\varepsilon\|_{\mathcal{X}_T})^3 (1 + \|u_{\varepsilon,0}\|_{\mathcal{G}^{-1/2-\kappa}})^3, \\ \|v_\varepsilon\|_{\mathcal{V}_{T_\star}} &\leq C(\|\mathbb{Y}_\varepsilon\|_{\mathcal{X}_T} + \|u_{\varepsilon,0}\|_{\mathcal{G}^{-1/2-\kappa}} + \|v_\varepsilon^b\|_{\mathcal{V}_{T_\star}^b}). \end{aligned}$$

Proof. Using the well-known Schauder estimates of Lemma 1.61 (and the fact that $\|f\|_{\mathcal{L}_T^{\kappa,\alpha}} \lesssim T^\kappa \|f\|_{\mathcal{L}_T^\alpha}$) we obtain for $\kappa, \theta > 0$ small enough

$$\begin{aligned} \|If\|_{\mathcal{L}_T^{-\theta+2\kappa,2\kappa}} + \|If\|_{\mathcal{L}_T^{1/4-\theta+2\kappa,1/2+2\kappa}} + \|If\|_{\mathcal{L}_T^{1/2-\theta+2\kappa,1+2\kappa}} \\ \lesssim T^{\frac{\kappa}{2}} (\|f\|_{\mathcal{M}_T^{1-\theta}\mathcal{G}^{-\kappa}} + \|f\|_{\mathcal{M}^{1/2+2\kappa}\mathcal{G}^{-1/2-2\kappa}}). \end{aligned} \quad (6.59)$$

We choose $\theta > 2\kappa$ small enough so that

$$\mathcal{L}_T^{-\theta+3\kappa/2,2\kappa} \cap \mathcal{L}_T^{1/4-\theta+3\kappa/2,1/2+2\kappa} \cap \mathcal{L}_T^{1/2-\theta+3\kappa/2,1+2\kappa} \subseteq \mathcal{V}_T^b.$$

Now

$$\begin{aligned} \|v_\varepsilon^\square\|_{\mathcal{V}_T^\square} &\lesssim \|Y_\varepsilon^\Psi + \bar{Y}_\varepsilon^{\Psi} \|_{\mathcal{V}_T^\square} + \|v_\varepsilon^\square\|_{C_T L^\infty} (\|Y_\varepsilon^\Psi\|_{C_T \mathcal{G}^{1-\kappa}} + \|Y_\varepsilon^\Psi\|_{C_T \mathcal{G}^{-1-\kappa}}) \\ &\quad + \left(\|v_\varepsilon^\sharp\|_{C_T \mathcal{G}^{-1/2-\kappa}} + \|v_\varepsilon^\sharp\|_{\mathcal{M}_T^{1/4}\mathcal{G}^{-\kappa}} \right) (\|Y_\varepsilon^\Psi\|_{C_T \mathcal{G}^{1-\kappa}} + \|Y_\varepsilon^\Psi\|_{C_T \mathcal{G}^{-1-\kappa}}) \\ &\quad + \|v_\varepsilon^b\|_{\mathcal{V}_T^\square} \\ &\lesssim \|\mathbb{Y}_\varepsilon\|_{\mathcal{X}_T} + T^\kappa \|v_\varepsilon^\square\|_{\mathcal{V}_T^\square} + \|v_{\varepsilon,0}\|_{\mathcal{G}^{-1/2-\kappa}} + \|v_\varepsilon^b\|_{\mathcal{V}_T^b} \end{aligned}$$

where we used that $v^\square(0) = 0$ and as a consequence that

$$\|v_\varepsilon^\square\|_{C_T L^\infty} \leq T^\kappa \|v_\varepsilon^\square\|_{C_T^\kappa L^\infty} \leq T^\kappa \|v_\varepsilon^\square\|_{\mathcal{V}_T^\square}$$

to gain a small power of T . So provided T is small enough (depending only on \mathbb{Y}_ε) this yields the following a priori estimation on v_ε^\square :

$$\|v_\varepsilon^\square\|_{C_T L^\infty} \lesssim \|v_\varepsilon^\square\|_{\mathcal{V}_T^\square} \lesssim \|\mathbb{Y}_\varepsilon\|_{\mathcal{X}_T} + \|v_{\varepsilon,0}\|_{\mathcal{G}^{-1/2-\kappa}} + \|v_\varepsilon^b\|_{\mathcal{V}_T^b}.$$

Therefore we have an estimation on v_ε :

$$\begin{aligned} \|v_\varepsilon\|_{\mathcal{V}_T} &\leq \|v_\varepsilon^\sharp\|_{\mathcal{V}_T} + \|v_\varepsilon^\square\|_{\mathcal{V}_T} \lesssim \|v_{\varepsilon,0}\|_{\mathcal{G}^{-1/2-\kappa}} + \|v_\varepsilon^\square\|_{\mathcal{V}_T^\square} \\ &\lesssim \|\mathbb{Y}_\varepsilon\|_{\mathcal{X}_T} + \|v_{\varepsilon,0}\|_{\mathcal{G}^{-1/2-\kappa}} + \|v_\varepsilon^b\|_{\mathcal{V}_T^b}. \end{aligned}$$

In order to estimate terms in $U(\lambda_\varepsilon, \mathbb{Y}_\varepsilon; v_\varepsilon, v_\varepsilon^b + v_\varepsilon^\sharp)$ we decompose the renormalised products as

$$\begin{aligned} Y_\varepsilon^\Psi \diamond v_\varepsilon &= v_\varepsilon \succ Y_\varepsilon^\Psi - \bar{Y}_\varepsilon^{\Psi} - Y_\varepsilon^{\Psi} - 3v_\varepsilon Y_\varepsilon^{\Psi} + v_\varepsilon^b \circ Y_\varepsilon^\Psi \\ &\quad + v_\varepsilon^\sharp \circ Y_\varepsilon^\Psi - 3\overline{\text{com}}_1(v_\varepsilon, Y_\varepsilon^\Psi, Y_\varepsilon^\Psi) \\ v_\varepsilon \diamond Y_\varepsilon &= -\bar{Y}_\varepsilon^{\Psi} Y_\varepsilon - 3(v_\varepsilon \ll Y_\varepsilon^\Psi) Y_\varepsilon + Y_\varepsilon \prec (v_\varepsilon^b + v_\varepsilon^\sharp) + Y_\varepsilon \succ (v_\varepsilon^b + v_\varepsilon^\sharp) \\ &\quad - Y_\varepsilon^\Psi \prec Y_\varepsilon - Y_\varepsilon^\Psi \succ Y_\varepsilon - Y_\varepsilon^{\Psi} \\ Y_\varepsilon' \diamond v_\varepsilon^2 &= Y_\varepsilon' \diamond (Y_\varepsilon^\Psi)^2 + 2(Y_\varepsilon' \diamond Y_\varepsilon^\Psi)(\bar{Y}_\varepsilon^{\Psi} + 3v_\varepsilon \ll Y_\varepsilon^\Psi) \\ &\quad - 2(Y_\varepsilon' \diamond Y_\varepsilon^\Psi) \prec (v_\varepsilon^b + v_\varepsilon^\sharp) + 2(Y_\varepsilon' \diamond Y_\varepsilon^\Psi) \succ (v_\varepsilon^b + v_\varepsilon^\sharp) \\ &\quad + Y_\varepsilon' \prec (v_\varepsilon^\square + v_\varepsilon^\sharp)^2 + Y_\varepsilon' \succ (v_\varepsilon^\square + v_\varepsilon^\sharp)^2. \end{aligned}$$

We decompose $U(\lambda_\varepsilon, \mathbb{Y}_\varepsilon; v_\varepsilon, v_\varepsilon^b + v_\varepsilon^\sharp)$ as

$$\begin{aligned}
U(\lambda_\varepsilon, \mathbb{Y}_\varepsilon; v_\varepsilon, v_\varepsilon^b + v_\varepsilon^\sharp) &= Q_{-1/2}(\lambda_\varepsilon, \mathbb{Y}_\varepsilon, v_{0,\varepsilon}, v_\varepsilon, v_\varepsilon^b) + Q_0(\lambda_\varepsilon, \mathbb{Y}_\varepsilon, v_{0,\varepsilon}, v_\varepsilon, v_\varepsilon^b) + Q_{\lambda_\varepsilon, \mathbb{Y}_\varepsilon} \\
Q_{-1/2} &:= -3[v_\varepsilon \succ Y_\varepsilon^\vee - 3\overline{\text{com}}_1(v_\varepsilon, Y_\varepsilon^\vee, Y_\varepsilon^\vee) + Y_\varepsilon' \succ (v_\varepsilon^\square + v_\varepsilon^\sharp)^2] \\
&\quad -6[(Y_\varepsilon' \diamond Y_\varepsilon^\vee)(3v_\varepsilon \ll Y_\varepsilon^\vee) + (Y_\varepsilon' \diamond Y_\varepsilon^\vee) \succ (v_\varepsilon^b + v_\varepsilon^\sharp)] \\
&\quad +2\lambda_{2,\varepsilon}(3(v_\varepsilon \ll Y_\varepsilon^\vee)Y_\varepsilon - Y_\varepsilon \succ (v_\varepsilon^b + v_\varepsilon^\sharp)) \\
&\quad +3\text{com}_3(v_\varepsilon, Y_\varepsilon^\vee) + 3\text{com}_2(v_\varepsilon, Y_\varepsilon^\vee) \\
Q_0 &:= 9v_\varepsilon Y_\varepsilon^{\vee\vee} - 3v_\varepsilon^b \circ Y_\varepsilon^\vee - 3v_\varepsilon^\sharp \circ Y_\varepsilon^\vee + 6(Y_\varepsilon' \diamond Y_\varepsilon^\vee) \preceq (v_\varepsilon^b + v_\varepsilon^\sharp) \\
&\quad -3Y_\varepsilon' \preceq (v_\varepsilon^\square + v_\varepsilon^\sharp)^2 - Y_\varepsilon^\emptyset v_\varepsilon^3 - \lambda_{2,\varepsilon}[v_\varepsilon^2 + 2Y_\varepsilon \preceq (v_\varepsilon^b + v_\varepsilon^\sharp)] \\
Q_{\lambda_\varepsilon, \mathbb{Y}_\varepsilon} &:= (1 - \lambda_{1,\varepsilon})Y_\varepsilon - \lambda_{0,\varepsilon} + 3[Y_\varepsilon^{\vee\vee} + Y_\varepsilon^{\vee\vee} - Y_\varepsilon' \diamond (Y_\varepsilon^\vee)^2 - 2(Y_\varepsilon' \diamond Y_\varepsilon^\vee)Y_\varepsilon^{\vee\vee}] \\
&\quad +2\lambda_{2,\varepsilon}(Y_\varepsilon^{\vee\vee}Y_\varepsilon + Y_\varepsilon^{\vee\vee} \prec Y_\varepsilon + Y_\varepsilon^{\vee\vee} \succ Y_\varepsilon + Y_\varepsilon^{\vee\vee}).
\end{aligned}$$

Here $Q_{\lambda_\varepsilon, \mathbb{Y}_\varepsilon}$ does not depend from the solution but only on $\lambda_\varepsilon, \mathbb{Y}_\varepsilon$ (as the notation suggests) and we have grouped the other terms which we expect to have regularity $\mathcal{C}^{-1/2-2\kappa}$ in $Q_{-1/2}$, (and the same for Q_0 and regularity \mathcal{C}^{-k}). With the same technique we used above for v_ε^\square , we obtain the following estimate on v_ε^\square

$$\|v_\varepsilon^\square\|_{\mathcal{L}_T^{1/2+3\kappa/2, 1/2+2\kappa}} + \|v_\varepsilon^\square\|_{\mathcal{L}_T^{1/4+\kappa, \kappa}} \lesssim \|\mathbb{Y}_\varepsilon\|_{\mathcal{X}_T} + \|v_{\varepsilon,0}\|_{\mathcal{C}^{-1/2-\kappa}} + \|v_\varepsilon^b\|_{\mathcal{V}_T^b}$$

and this yields

$$\|(v_\varepsilon^\square)^2\|_{\mathcal{L}_T^{3/4+5\kappa/2, 1/2+2\kappa}} + \|(v_\varepsilon^\square)^2\|_{\mathcal{L}_T^{1/2+2\kappa, \kappa}} \lesssim (\|\mathbb{Y}_\varepsilon\|_{\mathcal{X}_T} + \|v_{\varepsilon,0}\|_{\mathcal{C}^{-1/2-\kappa}} + \|v_\varepsilon^b\|_{\mathcal{V}_T^b})^2.$$

Then we are ready to bound $Q_{-1/2}, Q_0, Q_{\lambda_\varepsilon, \mathbb{Y}_\varepsilon}$ using the standard paraproducts estimations recalled in Section 1.2.1:

$$\begin{aligned}
&\|Q_{-1/2}\|_{\mathcal{M}_T^{1/2+2\kappa, \mathcal{C}^{-1/2-2\kappa}}} + \|Q_0\|_{\mathcal{M}_T^{1-\theta, \mathcal{C}^{-\kappa}}} \\
&\lesssim (1 + |\lambda_\varepsilon|)(1 + \|\mathbb{Y}_\varepsilon\|_{\mathcal{X}_T})^3 (1 + \|v_{\varepsilon,0}\|_{\mathcal{C}^{-1/2-\kappa}} + \|v_\varepsilon^b\|_{\mathcal{V}_T^b})^3 \\
&\|Q_{\lambda_\varepsilon, \mathbb{Y}_\varepsilon}\|_{C_T \mathcal{C}^{-1/2-\kappa}} \lesssim (1 + |\lambda_\varepsilon|) (1 + \|\mathbb{Y}_\varepsilon\|_{\mathcal{X}_T})^3.
\end{aligned}$$

In order to conclude the estimation of $\|v_\varepsilon^b\|_{\mathcal{V}_T^b}$ we have to control $\|IR_\varepsilon(v_\varepsilon)\|_{\mathcal{V}_T^b}$. This is achieved easily by the using the results of Section 6.4.1. Thanks to Lemma 6.15 $\forall \delta \in (0, 1)$, $\forall \theta > 0$ such that $\frac{1-\theta}{3+\delta} > \frac{1}{4} + \frac{3\kappa}{2}$ (note that it is possible to choose $\theta > 2\kappa$ that satisfies this property as long as k and δ are small enough) we have:

$$\|R_\varepsilon(v_\varepsilon)\|_{\mathcal{M}_T^{1-\theta, p} L^p(\mathbb{T}^3)} \lesssim M_{\varepsilon, \delta}(Y_\varepsilon, u_{0,\varepsilon}) \|v_\varepsilon\|_{\mathcal{V}_T^b}^{3+\delta} e^{c\varepsilon^{1/2}\|v_\varepsilon^\square\|_{\mathcal{V}_T^\square}}.$$

By Lemma 1.62 together with (1.55) we obtain then

$$\|IR_\varepsilon(v_\varepsilon)\|_{\mathcal{V}_T^b} \lesssim M_{\varepsilon, \delta}(Y_\varepsilon, u_{0,\varepsilon}) \|v_\varepsilon\|_{\mathcal{V}_T^b}^{3+\delta} e^{c\varepsilon^{1/2}\|v_\varepsilon^\square\|_{\mathcal{V}_T^\square}}.$$

Using that

$$\|P.v_\varepsilon^b(0)\|_{\mathcal{V}_T^b} \lesssim \|v_\varepsilon^b(0)\|_{C_T \mathcal{C}^{1/2-2\kappa}} \lesssim (1 + \|v_{\varepsilon,0}\|_{\mathcal{C}^{-1/2-\kappa}}) \|\mathbb{Y}_\varepsilon\|_{\mathcal{X}_T}$$

we obtain that $\exists C' > 0$ such that

$$\begin{aligned} \|v_{\varepsilon_n}^b\|_{\mathcal{V}_T^b} &\leq C'(1 + |\lambda_{\varepsilon_n}|)(1 + \|\mathbb{Y}_{\varepsilon_n}\|_{\mathcal{X}_T})^3(1 + \|v_{\varepsilon,0}\|_{\mathcal{G}^{-1/2-\kappa}})^3 \\ &\quad + C'T^{\kappa/2}(1 + |\lambda_{\varepsilon}|)(1 + \|\mathbb{Y}_{\varepsilon}\|_{\mathcal{X}_T})^3\|v_{\varepsilon}^b\|_{\mathcal{V}_T^b}^3 \\ &\quad + C'M_{\varepsilon,\delta}(Y_{\varepsilon}, u_{0,\varepsilon})e^{c\varepsilon^{1/2}(\|\mathbb{Y}_{\varepsilon}\|_{\mathcal{X}_T} + \|v_{\varepsilon,0}\|_{\mathcal{G}^{-1/2-\kappa}})}e^{c\varepsilon^{1/2}\|v_{\varepsilon}^b\|_{\mathcal{V}_T^b}}\|v_{\varepsilon}^b\|_{\mathcal{V}_T^b}^{3+\delta} \\ &\leq D + CM_{\varepsilon,\delta}(Y_{\varepsilon}, u_{0,\varepsilon})e^{c\varepsilon^{1/2}\|v_{\varepsilon}^b\|_{\mathcal{V}_T^b}} + CT^{\kappa/2}\|v_{\varepsilon}^b\|_{\mathcal{V}_T^b}^3 \\ &\quad + CM_{\varepsilon,\delta}(Y_{\varepsilon}, u_{0,\varepsilon})e^{c\varepsilon^{1/2}\|v_{\varepsilon}^b\|_{\mathcal{V}_T^b}}\|v_{\varepsilon}^b\|_{\mathcal{V}_T^b}^{3+\delta} \end{aligned}$$

with

$$\begin{aligned} C &:= C'(1 + |\lambda_{\varepsilon}|)(1 + \|\mathbb{Y}_{\varepsilon}\|_{\mathcal{X}_T})^3 \\ &\quad + C'e^{c\varepsilon^{1/2}(\|\mathbb{Y}_{\varepsilon}\|_{\mathcal{X}_T} + \|v_{\varepsilon,0}\|_{\mathcal{G}^{-1/2-\kappa}})}(1 + (\|\mathbb{Y}_{\varepsilon}\|_{\mathcal{X}_T} + \|v_{\varepsilon,0}\|_{\mathcal{G}^{-1/2-\kappa}})^{3+\delta}), \\ D &:= C'(1 + |\lambda_{\varepsilon_n}|)(1 + \|\mathbb{Y}_{\varepsilon_n}\|_{\mathcal{X}_T})^3(1 + \|v_{\varepsilon,0}\|_{\mathcal{G}^{-1/2-\kappa}})^3. \end{aligned}$$

Let $T_{\star} \in (0, T]$ such that:

$$CT_{\star}^{\kappa/2}[(5D)^2 + e^{c\varepsilon^{1/2}(5D)}(5D)^{2+\delta}] \leq \frac{1}{2}, \quad \text{and } CT_{\star}^{\kappa/2}e^{c\varepsilon^{1/2}(5D)} \leq D.$$

Assume that $M_{\varepsilon,\delta} \leq T_{\star}^{\kappa/2}$. Define a closed interval

$$[0, S] = \{t \in [0, T_{\star}]: \|v_{\varepsilon_n}^b\|_{\mathcal{V}_t^b} \leq 4D\} \subseteq [0, T_{\star}]$$

This interval is well defined and non-empty since $t \mapsto \|v_{\varepsilon_n}^b\|_{\mathcal{V}_t^b}$ is continuous and nondecreasing and $\|v_{\varepsilon_n}^b\|_{\mathcal{V}_0^b} \leq 4D$. Let us assume that $S < T_{\star}$, then we can take $\epsilon > 0$ small enough such that $S + \epsilon < T_{\star}$ and by continuity $\|v_{\varepsilon}^b\|_{\mathcal{V}_{S+\epsilon}^b} \leq 5D$, then

$$\begin{aligned} \|v_{\varepsilon_n}^b\|_{\mathcal{V}_{S+\epsilon}^b} &\leq D + CM_{\varepsilon,\delta}(Y_{\varepsilon}, u_{0,\varepsilon})e^{c\varepsilon^{1/2}\|v_{\varepsilon}^b\|_{\mathcal{V}_{S+\epsilon}^b}} + C(S + \epsilon)^{\kappa/2}\|v_{\varepsilon}^b\|_{\mathcal{V}_{S+\epsilon}^b}^3 \\ &\quad + CM_{\varepsilon,\delta}(Y_{\varepsilon}, u_{0,\varepsilon})e^{c\varepsilon^{1/2}\|v_{\varepsilon}^b\|_{\mathcal{V}_{S+\epsilon}^b}}\|v_{\varepsilon}^b\|_{\mathcal{V}_{S+\epsilon}^b}^{3+\delta} \\ &\leq D + CM_{\varepsilon,\delta}(Y_{\varepsilon}, u_{0,\varepsilon})e^{c\varepsilon^{1/2}(5D)} + CT_{\star}^{\kappa/2}(5D)^2\|v_{\varepsilon}^b\|_{\mathcal{V}_{S+\epsilon}^b} \\ &\quad + CT_{\star}^{\kappa/2}e^{c\varepsilon^{1/2}(5D)}(5D)^{2+\delta}\|v_{\varepsilon}^b\|_{\mathcal{V}_{S+\epsilon}^b} \\ &\leq 2D + \frac{1}{2}\|v_{\varepsilon}^b\|_{\mathcal{V}_{S+\epsilon}^b} \end{aligned}$$

which gives $\|v_{\varepsilon_n}^b\|_{\mathcal{V}_{S+\epsilon}^b} \leq 4D$. This implies $S = T_{\star}$ (by contradiction). From the construction of T_{\star} it is easy to see that $T_{\star}(\|\mathbb{Y}_{\varepsilon}\|_{\mathcal{X}_T}, \|u_{\varepsilon,0}\|_{\mathcal{G}^{-1/2-\kappa}}, |\lambda_{\varepsilon}|)$ is lower semicontinuous. \square

6.4.3 Convergence of the remainder

It suffices to put together the results obtained in Sections 6.4.1 and 6.4.2 to obtain the convergence of $R_{\varepsilon}(v_{\varepsilon})$:

Lemma 6.19. *The remainder $R_{\varepsilon}(v_{\varepsilon})$ that appears in equation (6.16) converges in probability to 0 as $\varepsilon \rightarrow 0$ in the space $\mathcal{M}_{T_{\star}}^{\gamma,p}L^p(\mathbb{T}^3)$.*

Proof. From the estimation on $R_{\varepsilon}(v_{\varepsilon})$ of Lemma 6.15, together with the fact that $M_{\varepsilon,\delta} \rightarrow 0$ in probability (Lemma 6.16) and the bounds on $\|v_{\varepsilon}\|_{\mathcal{M}_{T_{\star}}^{\frac{1}{2}+\frac{3\kappa}{2}}L^{\infty}}$ and $\|v_{\varepsilon}^{\square}\|_{C_T L^{\infty}}$ of Lemma 6.17 we see immediately that $\|R_{\varepsilon}(v_{\varepsilon})\|_{\mathcal{M}_{T_{\star}}^{\gamma,p}L^p} \rightarrow 0$ in probability. \square

Bibliography

- [**Abd16**] Abdelmalek Abdesselam. A second-quantized Kolmogorov-Chentsov Theorem. *ArXiv:1604.05259 [hep-th]*, Apr 2016.
- [**AC15**] Romain Allez and Khalil Chouk. The continuous Anderson hamiltonian in dimension two. *ArXiv:1511.02718 [math]*, Nov 2015.
- [**AR91**] Sergio Albeverio and Michael Röckner. Stochastic differential equations in infinite dimensions: solutions via Dirichlet forms. *Probability Theory and Related Fields*, 89(3):347–386, Sep 1991.
- [**BB15**] Ismaël Bailleul and Frédéric Bernicot. Heat semigroup and singular PDEs. *ArXiv:1501.06822 [math]*, Jan 2015.
- [**BB16**] Ismaël Bailleul and Frédéric Bernicot. Higher order paracontrolled calculus. *ArXiv:1609.06966 [math]*, Sep 2016.
- [**BCD11**] Hajer Bahouri, Jean-Yves Chemin, and Raphaël Danchin. *Fourier Analysis and Nonlinear Partial Differential Equations*. Springer Science Business Media, Jan 2011.
- [**BDH16**] Ismaël Bailleul, Arnaud Debussche, and Martina Hofmanová. Quasilinear generalized parabolic Anderson model equation. *ArXiv:1610.06726 [math]*, Oct 2016.
- [**Bon81**] Jean-Michel Bony. Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires. *Annales Scientifiques de l'École Normale Supérieure. Quatrième Série*, 14(2):209–246, 1981.
- [**CC13**] Rémi Catellier and Khalil Chouk. Paracontrolled distributions and the 3-dimensional stochastic quantization equation. *ArXiv:1310.6869 [math-ph]*, Oct 2013.
- [**CGN15**] Federico Camia, Christophe Garban, and Charles M. Newman. Planar Ising magnetization field I. uniqueness of the critical scaling limit. *The Annals of Probability*, 43(2):528–571, Mar 2015. Zbl: 06395208.
- [**CHI15**] Dmitry Chelkak, Clément Hongler, and Konstantin Izuyurov. Conformal invariance of spin correlations in the planar Ising model. *Annals of Mathematics*, 181(3):1087–1138, 2015.
- [**Dau88**] Ingrid Daubechies. Orthonormal bases of compactly supported wavelets. *Communications on Pure and Applied Mathematics*, 41(7):909–996, Oct 1988.
- [**DD03**] Giuseppe Da Prato and Arnaud Debussche. Strong solutions to the stochastic quantization equations. *The Annals of Probability*, 31(4):1900–1916, Oct 2003. Zbl: 02077580.
- [**FG16**] Marco Furlan and Massimiliano Gubinelli. Paracontrolled quasilinear SPDEs. *ArXiv:1610.07886 [math]*, Oct 2016. To appear in *Ann. Probab.*
- [**FG18**] M. Furlan and M. Gubinelli. Weak universality for a class of 3d stochastic reaction–diffusion models. *Probability Theory and Related Fields*, pages 1–66, May 2018.
- [**FH14**] Peter K. Friz and Martin Hairer. *A Course on Rough Paths: With an Introduction to Regularity Structures*. Springer, Aug 2014.
- [**FK72**] Cees M. Fortuin and Piet W. Kasteleyn. On the random-cluster model: I. introduction and relation to other models. *Physica*, 57(4):536–564, Feb 1972.
- [**FM17**] Marco Furlan and Jean-Christophe Mourrat. A tightness criterion for random fields, with application to the Ising model. *Electronic Journal of Probability*, 22:29, 2017.
- [**GH17a**] Máté Gerencsér and Martin Hairer. A solution theory for quasilinear singular SPDEs. *ArXiv:1712.01881 [math]*, Dec 2017.
- [**GH17b**] Máté Gerencsér and Martin Hairer. Singular SPDEs in domains with boundaries. *ArXiv:1702.06522 [math]*, Feb 2017.
- [**GIP15**] Massimiliano Gubinelli, Peter Imkeller, and Nicolas Perkowski. Paracontrolled distributions and singular PDEs. *Forum of Mathematics, Pi*, 3, Aug 2015.
- [**GP15**] Massimiliano Gubinelli and Nicolas Perkowski. Lectures on singular stochastic PDEs. *ArXiv:1502.00157 [math]*, Jan 2015.
- [**GP17**] Massimiliano Gubinelli and Nicolas Perkowski. KPZ reloaded. *Commun. Math. Phys.*, 349(1):165–269, Jan 2017.
- [**Gri09**] Geoffrey R. Grimmett. *The Random-Cluster Model*. Grundlehren der mathematischen Wissenschaften 333. Springer, 1st ed. 2006. Corr. 2nd printing edition, 2009.
- [**Gub04**] Massimiliano Gubinelli. Controlling rough paths. *Journal of Functional Analysis*, 216(1):86–140, Nov 2004.

- [Hai13] Martin Hairer. Solving the KPZ equation. *Annals of Mathematics*, 178(2):559–664, 2013.
- [Hai14a] Martin Hairer. A theory of regularity structures. *Inventiones mathematicae*, 198(2):269–504, 2014.
- [Hai14b] Martin Hairer. Singular stochastic PDEs. *ArXiv:1403.6353 [math]*, Mar 2014.
- [Hai15] Martin Hairer. Regularity structures and the dynamical Φ_3^4 model. *ArXiv:1508.05261 [math-ph]*, Aug 2015.
- [HDN11] Clément Hongler, Hugo Duminil-Copin, and Pierre Nolin. Connection probabilities and RSW-type bounds for the two-dimensional FK Ising model. *Communications on Pure and Applied Mathematics*, 64(9):1165–1198, 2011.
- [HL16] Martin Hairer and Cyril Labbé. Multiplicative stochastic heat equations on the whole space. *Journal of the European Mathematical Society*, Jun 2016.
- [Hör03] Lars Hörmander. *The Analysis of Linear Partial Differential Operators I: Distribution Theory and Fourier Analysis*. Classics in Mathematics. Springer-Verlag, 2 edition, 2003.
- [HQ15] Martin Hairer and Jeremy Quastel. A class of growth models rescaling to KPZ. *ArXiv:1512.07845 [math-ph]*, Dec 2015.
- [HS16] Martin Hairer and Hao Shen. The dynamical Sine-Gordon model. *Communications in Mathematical Physics*, 341(3):933–989, Feb 2016.
- [HX16] Martin Hairer and Weijun Xu. Large scale behaviour of 3d phase coexistence models. *ArXiv:1601.05138 [math-ph]*, Jan 2016.
- [HX18] Martin Hairer and Weijun Xu. Large-scale limit of interface fluctuation models. *ArXiv:1802.08192 [math-ph]*, Feb 2018.
- [Jan97] Svante Janson. *Gaussian Hilbert Spaces*. Cambridge University Press, Jun 1997.
- [JM85] Giovanni Jona-Lasinio and Pronob Mitter. On the stochastic quantization of field theory. *Communications in Mathematical Physics*, 101(3):409–436, 1985. Zbl: 0588.60054.
- [Kup14] Antti Kupiainen. Renormalization group and stochastic PDEs. *Annales Henri Poincaré*, pages 1–39, 2014.
- [Lyo98] Terry J. Lyons. Differential equations driven by rough signals. *Revista Matemática Iberoamericana*, 14(2):215–310, Aug 1998.
- [Mey92] Yves Meyer. *Wavelets and Operators*. Cambridge University Press, 1992.
- [MW17a] Jean-Christophe Mourrat and Hendrik Weber. Global well-posedness of the dynamic Φ^4 model in the plane. *The Annals of Probability*, 45(4):2398–2476, Jul 2017. Zbl: 06786085.
- [MW17b] Jean-Christophe Mourrat and Hendrik Weber. The dynamic Φ_3^4 model comes down from infinity. *Communications in Mathematical Physics*, 356(3):673–753, Dec 2017.
- [MWX16] Jean-Christophe Mourrat, Hendrik Weber, and Weijun Xu. Construction of Φ_3^4 diagrams for pedestrians. *ArXiv:1610.08897 [math-ph]*, Oct 2016.
- [NN10] Ivan Nourdin and David Nualart. Central limit theorems for multiple Skorokhod integrals. *Journal of Theoretical Probability*, 23(1):39–64, Mar 2010.
- [NP12] Ivan Nourdin and Giovanni Peccati. *Normal Approximations with Malliavin Calculus: From Stein’s Method to Universality*. Cambridge University Press, May 2012.
- [Nua06] David Nualart. *The Malliavin calculus and related topics*. Probability and its applications. Springer, 2nd ed edition, 2006.
- [OGK17] Tadahiro Oh, Massimiliano Gubinelli, and Herbert Koch. Renormalization of the two-dimensional stochastic nonlinear wave equations. *Transactions of the American Mathematical Society*, Oct 2017.
- [Ons44] Lars Onsager. Crystal statistics. I. a two-dimensional model with an order-disorder transition. *Physical Review*, 65(3–4):117–149, Feb 1944.
- [OW16] Felix Otto and Hendrik Weber. Quasilinear SPDEs via rough paths. *ArXiv:1605.09744 [math]*, May 2016.
- [Pin01] Mark A. Pinsky. *Introduction to Fourier Analysis and Wavelets*. Brooks/Cole, 2001.
- [RS80] Michael Reed and Barry Simon. *Functional Analysis*. Academic Press, Jan 1980.
- [Shi80] Ichiro Shigekawa. Derivatives of Wiener functionals and absolute continuity of induced measures. *Journal of Mathematics of Kyoto University*, 20(2):263–289, 1980. Zbl: 0476.28008.
- [Shi04] Ichiro Shigekawa. *Stochastic Analysis*. American Mathematical Society, May 2004.
- [SX16] Hao Shen and Weijun Xu. Weak universality of dynamical Φ_3^4 : non-Gaussian noise. *ArXiv:1601.05724 [math-ph]*, Jan 2016.
- [Tri92] Hans Triebel. *Theory of Function Spaces II*. Monographs in Mathematics. Birkhäuser Basel, 1992.
- [Üst14] Ali Süleyman Üstünel. A sophisticated proof of the multiplication formula for multiple Wiener integrals. *ArXiv:1411.4877 [math]*, Nov 2014.

Résumé

Le sujet principal de cette thèse sont les équations aux dérivées partielles stochastiques (EDPS) présentant des solutions distributionnelles. L'étude de ces équations se déroule dans le cadre du calcul paracontrôlé, qui comprend des techniques issues de l'analyse fonctionnelle et de la théorie des probabilités.

On développe un calcul paracontrôlé non-linéaire pour obtenir un résultat d'existence locale pour des EDPS quasi-linéaires. En utilisant le calcul paracontrôlé et des techniques de calcul de Malliavin pour borner des intégrales stochastiques généralisées, on étudie l'universalité faible d'une classe d'équations non-linéaires de réaction-diffusion qui convergent vers l'équation Φ_3^4 de quantification stochastique.

Une autre partie de cette thèse est dédiée à l'étude de la tension du champ de magnétisation d'Ising bidimensionnel à température critique, pour lequel on obtient une caractérisation précise dans des espaces de Besov locaux.

Mots Clés

Equations aux Dérivées Partielles, EDP Stochastiques, Paraproduct, Espace de Besov, Espace de Besov Local, Distributions Paracontrôlées, Calcul de Malliavin, Modèle d'Ising, EDP Quasi-linéaires, Universalité Faible, Equation de Quantisation Stochastique

Abstract

The main aim of this thesis is to investigate the well-posedness of distribution-valued stochastic partial differential equations (SPDEs) in the framework of paracontrolled distributions, which employs both functional analytic and probabilistic methods.

We develop a non-linear paracontrolled calculus to obtain a local well-posedness result for quasi-linear SPDEs. Using the paracontrolled approach and some techniques from Malliavin calculus to control generalized stochastic integrals, we are able to study the weak universality of a class of nonlinear reaction-diffusion models rescaling to the Φ_3^4 stochastic quantization equation.

Another subject of this thesis is the study of the tightness of the 2-dimensional Ising magnetization field at critical temperature, for which we obtain an essentially sharp characterization in local Besov spaces.

Keywords

Stochastic Partial Differential Equations, SPDEs, Paraproduct, Besov Space, Tightness Criterion, Local Besov Space, Paracontrolled Distributions, Malliavin Calculus, Ising Model, Quasi-linear PDEs, Weak Universality, Stochastic Quantization Equation