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Groupes quantiques : actions sur des modules hilbertiens et calculs différentiels

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THÈSE

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Spécialité : Mathématiques fondamentales

Par **Manon Thibault de Chanvalon**

GROUPES QUANTIQUES : ACTIONS SUR DES MODULES HILBERTIENS ET CALCULS DIFFÉRENTIELS

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Fini, c'est fini, ça va finir, ça va peut-être finir. Les grains s'ajoutent aux grains, un à un, et un jour, soudain, c'est un tas, un petit tas, l'impossible tas.

Samuel Beckett, *Fin de partie*.

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Groupes quantiques : actions sur des modules hilbertiens et calculs différentiels

Résumé La première partie de cette thèse consiste en la construction du groupe quantique de symétrie d'un module hilbertien équipé d'une filtration orthogonale. Elle est principalement motivée par la construction du groupe quantique d'isométries d'un triplet spectral admissible faite par Goswami, ainsi que par celle du groupe quantique de symétrie d'une C^* -algèbre équipée d'une filtration orthogonale faite par Banica et Skalski. Notre construction généralise et unifie les deux précédentes, et permet d'autre part une approche alternative au groupe quantique d'isométries de certains triplets spectraux, basée non plus sur le spectre du « laplacien non-commutatif » du triplet spectral, mais sur celui de l'opérateur de Dirac. Pour illustrer cette construction, on détermine le groupe quantique de symétrie de certains modules hilbertiens équipés d'une filtration orthogonale. On s'intéresse notamment à la variété $[0, 1] \times \{1, \dots, n\}$ (à laquelle on associe un module hilbertien muni d'une filtration orthogonale), et à certain quotients de cette variété.

La seconde partie de cette thèse consiste en la classification des calculs différentiels bicovariants sur les algèbres de Hopf libres orthogonales. On commence par montrer que si deux algèbres de Hopf sont monoïdalement équivalentes, alors il existe une équivalence entre leurs catégories de calculs différentiels bicovariants. Une algèbre de Hopf libre orthogonale étant monoïdalement équivalente à $\mathcal{O}_q(SL_2)$ pour $q \in \mathbb{C}^*$, on est alors amené à classifier les calculs différentiels bicovariants sur $\mathcal{O}_q(SL_2)$. Pour ce faire, on montre que si la catégorie des modules de Yetter-Drinfeld de dimension finie sur une algèbre de Hopf est semi-simple, alors les calculs différentiels bicovariants de dimension finie sur cette algèbre de Hopf sont tous intérieurs. La classification des modules de Yetter-Drinfeld de dimension finie sur $\mathcal{O}_q(SL_2)$ faite par Takeuchi nous permet de conclure que tout calcul différentiel bicovariant sur $\mathcal{O}_q(SL_2)$, et donc tout calcul différentiel bicovariant sur une algèbre de Hopf libre orthogonale, est intérieur.

Mots-clefs Groupes quantiques de symétrie – Triplets spectraux – Modules hilbertiens – Algèbres de Hopf – Calculs différentiels bicovariants.

Quantum groups: actions on Hilbert modules and differential calculi

Abstract The first part of this dissertation consists in the construction of the quantum symmetry group of a Hilbert module equipped with an orthogonal filtration. It is mainly motivated by Goswami’s construction of the quantum isometry group of an admissible spectral triple, and by Banica and Skalski’s quantum symmetry group of a C^* -algebra equipped with an orthogonal filtration. Our construction generalizes and unifies those two. It provides furthermore an alternative approach to the quantum isometry group of some spectral triples, which is no longer based on the spectrum of the “non-commutative Laplacian” of the spectral triple, but on the spectrum of its Dirac operator. As an illustration, we compute the quantum symmetry group of some Hilbert modules, with a particular focus on the manifold $[0, 1] \times \{1, \dots, n\}$ (to which is associated a Hilbert module endowed with an orthogonal filtration), and on some quotients of this manifold.

The second part of this dissertation consists in the classification of bicovariant differential calculi over free orthogonal Hopf algebras. We firstly show that if two Hopf algebras are monoidally equivalent, then there exists an equivalence between their categories of bicovariant differential calculi. Since a free orthogonal Hopf algebra is monoidally equivalent to $\mathcal{O}_q(SL_2)$ for a given $q \in \mathbb{C}^*$, this leads us to the question of classifying bicovariant differential calculi over $\mathcal{O}_q(SL_2)$. To this end, we show that if the category of finite dimensional Yetter-Drinfeld modules over a Hopf algebra is semisimple, then the bicovariant differential calculi over this Hopf algebra are inner. The classification of finite dimensional Yetter-Drinfeld over $\mathcal{O}_q(SL_2)$ made by Takeuchi allows to conclude that each bicovariant differential calculus over $\mathcal{O}_q(SL_2)$, hence over a free orthogonal Hopf algebra, is inner.

Keywords Quantum symmetry groups – Spectral triples – Hilbert modules – Hopf algebras – Bicovariant differential calculi.

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Introduction

Cette thèse s’articule en deux parties relativement distinctes, gravitant toutes deux autour du thème des groupes quantiques et leur lien à la géométrie différentielle.

On s’intéresse dans la première partie de cette thèse (Chapitre 2) à la notion de groupe quantique de symétrie de certains modules hilbertiens. Les groupes prenant leur essence dans le concept de *symétrie*, une démarche naturelle est d’étendre cette notion dans le cadre des groupes quantiques, et de chercher à mesurer les symétries quantiques de différents objets. Cette notion de *symétrie quantique* n’étant cependant pas aussi aisée à appréhender que celle de symétrie au sens classique, elle a mis quelque temps à apparaître : alors que les groupes quantiques ont vu le jour au milieu des années 80, apparaissant d’abord comme des déformations de groupes de Lie, ce n’est que plus d’une décennie plus tard, dans l’article [Wan98], que sont définies les premières symétries quantiques. L’idée de Wang pour définir l’analogie quantique du groupe de permutations S_n a été de remarquer que S_n (et plus généralement le groupe d’automorphismes $Aut(X)$ d’un ensemble donné X) peut être caractérisé par une propriété universelle. Si G est un groupe, et $\xi : X \times G \rightarrow X$ est une action de G sur X , alors il existe un unique morphisme $\mu : G \rightarrow Aut(X)$ qui entrelace les actions respectives de G et $Aut(X)$. Ce dernier est donné par $\mu(g) = x \mapsto \xi(x, g)$. Cette propriété exprime le fait que $Aut(X)$ est un objet final dans la catégorie des groupes de transformations de X , et caractérise $Aut(X)$ à isomorphisme près. Cette caractérisation s’adapte en fait dans un contexte plus général : mesurer les symétries d’un espace donné revient à déterminer l’objet final dans la catégorie des groupes de transformations de cet espace. Le groupe d’isométries d’un espace métrique est, par exemple, l’objet final dans la catégorie des groupes agissant isométriquement sur cet espace. Cette vision des groupes de symétrie a l’avantage de ne pas faire intervenir la notion de *point*, et d’être très naturellement adaptable dans un cadre quantique : le groupe quantique de symétrie d’un espace (éventuellement « quantique ») est, s’il existe, l’objet final dans la catégorie des groupes de transformations quantiques de cet espace.

La notion d’*action* d’un groupe quantique compact sur un espace quantique compact (une C^* -algèbre unifère) avait été définie par Podleś dans [Pod95].

Définition. Soit Q une C^* -algèbre de Woronowicz (c'est-à-dire une « algèbre de fonctions continues sur un groupe quantique compact »), et soit A une C^* -algèbre. Une *coaction de Q sur A* est un $*$ -morphisme $\alpha : A \rightarrow A \otimes Q$ vérifiant :

- $(\alpha \otimes id_Q) \circ \alpha = (id_A \otimes \Delta) \circ \alpha$,
- $\text{vect}\{\alpha(A).(1 \otimes Q)\}$ est dense dans $A \otimes Q$.

Wang a vérifié dans [Wan98] que la catégorie des groupes quantiques compacts agissant sur $\{1, \dots, n\}$ admettait bien un objet final, définissant ainsi le groupe quantique de permutations S_n^+ . Il a cependant montré dans ce même article que la catégorie des groupes quantiques compacts agissant sur la C^* -algèbre de matrices $\mathcal{M}_n(\mathbb{C})$ n'admettait pas d'objet final, et que l'on était donc amené rapidement, même en dimension finie, à considérer des catégories plus petites que celle de tous les groupes quantiques de transformations (notamment des catégories de groupes quantiques de transformations préservant une trace donnée).

Suite à cet article, les groupes quantiques de symétrie d'espaces munis de structures additionnelles ont été définies et étudiées. On peut noter en particulier l'article [Bic03a] où est construit le groupe quantique d'automorphismes de graphes finis, ou encore l'article [Ban05] où est défini le groupe quantique d'isométries d'espaces métriques finis.

Mais, bien qu'étant l'une des principales motivations dans le développement d'une théorie des *groupes quantiques de symétrie*, la question d'adapter cette construction aux objets de la géométrie non commutative (à la Connes [Con94]) est longtemps restée ouverte. Ce n'est que récemment, dans [Gos09], qu'est défini le groupe quantique d'isométries de certains triplets spectraux (incluant les triplets spectraux provenant de variétés riemanniennes compactes). Cet article ainsi que son successeur [BG09a] ont donné lieu à de nombreux autres articles, consacrés à la détermination des groupes quantiques de symétrie de divers variétés riemanniennes compactes et triplets spectraux, voir entre autres : [Bho09, BG09b, BG10, BS10, LDS12, BS12].

L'article [Gos09] a été le point de départ de cette thèse. Pour définir le groupe quantique d'isométries d'une variété riemannienne compacte M , Goswami utilise le fait qu'un C^∞ -difféomorphisme $f : M \rightarrow M$ est une isométrie si et seulement si l'application $f^* : C^\infty(M) \rightarrow C^\infty(M)$ commute avec le laplacien \mathcal{L} de M . Ceci lui permet de défini-

$$g \mapsto g \circ f$$

nir une action isométrique d'un groupe quantique compact sur M , comme une coaction $\alpha : C(M) \rightarrow C(M) \otimes Q$ d'une C^* -algèbre de Woronowicz Q , laissant stable les sous-espaces propres de \mathcal{L} . Plus généralement, il adapte cette construction aux triplets spectraux dits *admissibles* (possédant un « bon laplacien »).

La première de nos motivations dans notre construction était de donner une définition alternative au groupe quantique d'isométries d'un triplet spectral, invoquant non plus les sous-espaces propres du « laplacien non commutatif » mais directement ceux de l'opérateur de Dirac du triplet spectral. Une action isométrique $\xi : M \times G \rightarrow M$ d'un groupe compact G sur M peut en effet être également caractérisée par le fait que l'action induite sur le fibré des formes extérieures :

$$\begin{aligned} \Gamma(\Lambda^*M) &\rightarrow \Gamma(\Lambda^*M) \otimes C(G) \cong C(G, \Gamma(\Lambda^*M)) \\ \omega &\mapsto (g \mapsto \xi_g^*(\omega)) \end{aligned}$$

où ξ_g^* désigne le tiré en arrière de l'application $\xi_g = \xi(\cdot, g)$, laisse stable les sous-espaces propres de l'opérateur de de Rham $D = d + d^*$.

Un autre de nos objectifs était de donner une construction relativement générale, à l'instar de la construction de Banica et Skalski [BS13]. Cela nous a menés à la construction du *groupe quantique de symétrie d'un module hilbertien muni d'une filtration orthogonale*. Elle est une généralisation naturelle de [Gos09] et [BS13], et ouvre la voie vers d'autres exemples.

Définition. Soient A une C^* -algèbre, τ un état fidèle sur A et E un A -module hilbertien. Une *filtration orthogonale* $(\tau, (V_i)_{i \in \mathcal{I}}, J, W)$ de E consiste en :

- une famille $(V_i)_{i \in \mathcal{I}}$ de sous-espaces vectoriels de E tels que :
 1. pour tous $i, j \in \mathcal{I}$ distincts, $\forall \xi \in V_i$ et $\forall \eta \in V_j$, $\tau(\langle \xi | \eta \rangle_A) = 0$, où $\langle \cdot | \cdot \rangle_A$ désigne le A -produit scalaire sur E ,
 2. l'espace $\mathcal{E}_0 = \sum_{i \in \mathcal{I}} V_i$ est dense dans E muni de la A -norme,
- un opérateur antilinéaire injectif $J : \mathcal{E}_0 \rightarrow \mathcal{E}_0$,
- un sous-espace vectoriel de dimension finie W de E .

Le premier exemple de filtration orthogonale sur un module hilbertien provient de la géométrie riemannienne : si M est une variété riemannienne compacte, l'espace des sections continues du fibré des formes extérieures sur M , $\Gamma(\Lambda^*M)$, est un $C(M)$ -module hilbertien. Il est naturellement muni d'une filtration orthogonale, avec $\tau = \int \cdot d\text{vol}$ (où $d\text{vol}$ désigne la densité riemannienne sur M), W est l'espace de dimension 1 engendré par l'unité ($m \mapsto 1_{\Lambda_m^*M}$), $J : \Gamma(\Lambda^*M) \rightarrow \Gamma(\Lambda^*M)$ est la conjugaison complexe et $(V_i)_{i \in \mathbb{N}}$ est la famille des sous-espaces propres de l'opérateur de de Rham $D = d + d^*$.

D'autre part, si A est une C^* -algèbre munie d'une filtration orthogonale dans le sens de [BS13], elle est un A -module hilbertien muni d'une filtration orthogonale au sens que

l'on vient de donner. Dans ce cas, le groupe quantique de symétrie de ce A -module hilbertien coïncide avec le groupe quantique de symétrie construit dans [BS13] et notre construction généralise donc celle de [BS13].

Pour définir le groupe quantique de symétrie d'un module hilbertien E équipé d'une filtration orthogonale $(\tau, (V_i)_{i \in \mathcal{I}}, J, W)$, il faut avant tout décrire les groupes quantiques de transformations de E , c'est-à-dire les groupes quantiques agissant sur E tout en préservant sa structure. Les groupes quantiques de transformations d'un module hilbertien avaient déjà été définis précédemment par Baaj et Skandalis [BS89] :

Définition. Soient A une C^* -algèbre et E un A -module hilbertien. Une *coaction* d'une C^* -algèbre de Woronowicz Q sur E est la donnée :

- d'une coaction de Q sur A , $\alpha : A \rightarrow A \otimes Q$,
- d'une application linéaire $\beta : E \rightarrow E \otimes Q$ satisfaisant :
 1. $\text{vect}\{\beta(E).(1 \otimes Q)\}$ est dense dans $E \otimes Q$,
 2. $(\beta \otimes id_Q) \circ \beta = (id_E \otimes \Delta) \circ \beta$,
 3. $\forall \xi, \eta \in E, \langle \beta(\xi) | \beta(\eta) \rangle_{A \otimes Q} = \alpha(\langle \xi | \eta \rangle_A)$,
 4. $\forall \xi \in E, \forall a \in A, \beta(\xi.a) = \beta(\xi).\alpha(a)$.

Partant de là, la définition des groupes quantiques de transformations d'un module hilbertien muni d'une filtration orthogonale s'ensuit naturellement :

Définition. Soient A une C^* -algèbre et E un A -module hilbertien muni d'une filtration orthogonale $(\tau, (V_i)_{i \in \mathcal{I}}, J, W)$. Une *coaction préservant la filtration* d'une C^* -algèbre de Woronowicz Q sur E est une coaction (α, β) de Q sur E vérifiant :

- $(\tau \otimes id_Q) \circ \alpha = \tau(\cdot)1_Q$,
- $\forall i \in \mathcal{I}, \beta(V_i) \subset V_i \odot Q$,
- $(J \otimes *) \circ \beta = \beta \circ J$ sur \mathcal{E}_0 , où $*$ est l'involution de Q ,
- $\forall \xi \in W, \beta(\xi) = \xi \otimes 1_Q$.

Les C^* -algèbres de Woronowicz coagissant sur un module hilbertien donné E muni d'une filtration orthogonale forment une catégorie. Le résultat principal du chapitre 2 est le suivant :

Théorème (2.3.1). Soient A une C^* -algèbre et E un A -module hilbertien plein, muni d'une filtration orthogonale $(\tau, (V_i)_{i \in \mathcal{I}}, J, W)$. La catégorie des C^* -algèbres de Woronowicz coagissant sur E en préservant la filtration admet un objet initial (c'est-à-dire qu'il existe une C^* -algèbre de Woronowicz universelle qui coagit sur E en préservant la filtration). Le groupe quantique correspondant à cette C^* -algèbre de Woronowicz est appelé le **groupe quantique de symétrie** de $(E, \tau, (V_i)_{i \in \mathcal{I}}, J, W)$.

La preuve de ce résultat suit la même idée que celle de Goswami [Gos09]. On en esquisse ici les grandes lignes. La première étape de la preuve consiste à montrer que si (α, β) est une coaction d'une C^* -algèbre de Woronowicz Q sur E préservant la filtration, et si pour $i \in \mathcal{I}$, $(e_{ij})_{1 \leq j \leq d_i}$ est une base orthonormale de V_i , alors les matrices $v^{(i)} = (v_{kj}^{(i)})_{1 \leq k, j \leq d_i}$, caractérisées par $\beta(e_{ij}) = \sum_{k=1}^{d_i} e_{ik} \otimes v_{kj}^{(i)}$, sont unitaires et vérifient de plus

$$v^{(i)t} s^{(i)} \overline{v^{(i)}}(s^{(i)})^{-1} = s^{(i)} \overline{v^{(i)}}(s^{(i)})^{-1} v^{(i)t} = I_{d_i},$$

pour certaines matrices $s^{(i)} \in GL_{d_i}(\mathbb{C})$ ne dépendant que de la filtration orthogonale de E . On considère alors la C^* -algèbre de Woronowicz $\mathcal{U} = \bigstar_{i \in \mathcal{I}} \mathcal{A}_u(s^{(i)})$, où $\mathcal{A}_u(s^{(i)})$ est la C^* -algèbre de Woronowicz universelle engendrée par une matrice multiplicative et unitaire $u^{(i)} = (u_{kj}^{(i)})_{1 \leq k, j \leq d_i}$, satisfaisant :

$$u^{(i)t} s^{(i)} \overline{u^{(i)}}(s^{(i)})^{-1} = s^{(i)} \overline{u^{(i)}}(s^{(i)})^{-1} u^{(i)t} = I_{d_i}.$$

Cette C^* -algèbre \mathcal{U} admet une sorte de coaction partielle sur le sous-espace \mathcal{E}_0 de E définie par :

$$\begin{aligned} \beta_u : \mathcal{E}_0 &\rightarrow \mathcal{E}_0 \otimes \mathcal{U} \\ e_{ij} &\mapsto \sum_{k=1}^{d_i} e_{ik} \otimes u_{kj}^{(i)}. \end{aligned}$$

Cette « coaction » est universelle en ce sens que si (Q, α, β) est une coaction sur E préservant la filtration, alors il existe un unique morphisme de C^* -algèbres de Woronowicz $\mu : \mathcal{U} \rightarrow Q$ tel que $(id \otimes \mu) \circ \beta_u = \beta$. En d'autres termes, toute coaction fidèle sur E préservant la filtration est de la forme $(\mathcal{U}/I, \alpha_I, \beta_I)$, où $I \subset \mathcal{U}$ est un idéal de Woronowicz et $\beta_I : E \rightarrow E \otimes \mathcal{U}/I$ prolonge $(id \otimes \pi_I) \circ \beta_u$ (où $\pi_I : \mathcal{U} \rightarrow \mathcal{U}/I$ désigne la projection). Le problème de trouver un objet initial dans la catégorie des C^* -algèbres de Woronowicz coagissant sur E en préservant sa filtration est donc ramené à celui de trouver un idéal minimal parmi les idéaux I de \mathcal{U} satisfaisant la propriété précédente, c'est-à-dire tels qu'il existe un prolongement $\beta_I : E \rightarrow E \otimes \mathcal{U}/I$ de $(id \otimes \pi_I) \circ \beta_u$ et une application $\alpha_I : A \rightarrow A \otimes \mathcal{U}/I$ faisant de $(\mathcal{U}/I, \alpha_I, \beta_I)$ une coaction sur E préservant sa filtration. La fin de la preuve consiste donc à vérifier que

l'intersection de tous les idéaux satisfaisant cette propriété, satisfait lui aussi cette propriété.

Comme expliqué précédemment, ce résultat généralise la construction de [BS13]. Il généralise également la construction de [Gos09] en ce sens que si $(\mathcal{A}, \mathcal{H}, D)$ est un triplet spectral admissible au sens de Goswami, on peut lui associer de manière naturelle un module hilbertien muni d'une filtration orthogonale, dont le groupe quantique de symétries coïncide avec le groupe quantique d'isométries de $(\mathcal{A}, \mathcal{H}, D)$ tel que défini dans [Gos09]. D'autre part, sous certaines hypothèses (dont la plupart sont issues du théorème de reconstruction de Connes [Con13] : hypothèses de dimension, de finitude et d'absolue continuité), un module hilbertien muni d'une filtration orthogonale peut être associé de façon naturelle à un triplet spectral donné.

Dans la dernière section du chapitre 2, on calcule le groupe quantique de symétrie sur certains exemples concrets :

- Le groupe quantique de symétrie du \mathbb{C} -module hilbertien \mathbb{C}^n , muni de la filtration orthogonale $(id_{\mathbb{C}}, (\mathbb{C}), J, (0))$ où $J : \mathbb{C}^n \rightarrow \mathbb{C}^n$ est une application antilinéaire injective quelconque, est le groupe quantique libre orthogonal $\mathcal{A}_o(P)$, où P est la matrice de J dans la base canonique.
- Le groupe quantique de symétrie d'un certain module hilbertien muni d'une filtration orthogonale décrivant la variété $[0, 1] \times \{1, \dots, n\}$, est le groupe quantique hyperoctaédral $\mathcal{A}_h(n)$ (c'est-à-dire le produit libre en couronne du groupe $\mathbb{Z}/2\mathbb{Z}$ par le groupe quantique de permutations $\mathcal{A}_s(n)$).
- On s'intéresse ensuite à certains « quotients » du module hilbertien associé à $[0, 1] \times \{1, \dots, n\}$. On montre que le groupe quantique de symétrie du quotient correspondant à n segments $[0, 1]$ recollés ensemble au point 0 et au point 1, est le produit du groupe $\mathbb{Z}/2\mathbb{Z}$ avec le groupe quantique de permutations $\mathcal{A}_s(n)$.

Le seconde partie de cette thèse (Chapitre 3) porte sur la classification des calculs différentiels bicovariants sur les algèbres de Hopf libres orthogonales.

La notion de *calcul différentiel bicovariant* sur une algèbre de Hopf a été introduite par Woronowicz dans [Wor89], dans le but d'adapter au cadre quantique le calcul différentiel sur les groupes de Lie, motivé notamment par l'exemple de $SU_q(2)$ [Wor87b].

Définition. Soit H une algèbre de Hopf. Un *calcul différentiel bicovariant* (du premier ordre) (M, d) sur H est un bimodule de Hopf M , muni d'un morphisme de comodules (à droite et à gauche) $d : H \rightarrow M$ tel que $\forall x, y \in H, d(xy) = x.d(y) + d(x).y$ et tel que $M = \text{vect}\{x.d(y) ; x, y \in H\}$.

Dans cette définition, H joue donc le rôle de l'algèbre des fonctions lisses sur un groupe quantique, tandis que M joue le rôle de l'espace de ses 1-formes. Il n'y a pas en général de calcul différentiel bicovariant canonique sur une algèbre de Hopf donnée, et la question de la classification de ses calculs différentiels bicovariants se pose donc. Différents auteurs se sont penchés sur cette question : Heckenberger, Schmüdgen, Schüler entre autres (voir par exemple [SS95a, SS95b, BS98, HS98]).

Nous nous intéressons ici à la classification des calculs différentiels bicovariants (de dimension finie) sur les algèbres de Hopf libres orthogonales, c'est-à-dire les algèbres de Hopf sous-jacentes aux classiques groupes quantiques libres orthogonaux. Une propriété fondamentale de ces algèbres de Hopf, est que leurs catégories de comodules sont semblables à celle des comodules sur l'algèbre de Hopf $\mathcal{O}_q(SL_2)$, pour un certain $q \in \mathbb{C}^*$. Plus précisément : si H est une algèbre de Hopf libre orthogonale, il existe q dans \mathbb{C}^* tel que H et $\mathcal{O}_q(SL_2)$ soient monoïdalement équivalentes (c'est-à-dire qu'il existe une équivalence linéaire monoïdale entre leurs catégories de comodules à droites).

La première étape dans la classification des calculs différentiels bicovariants sur les algèbres de Hopf libres orthogonales a été de vérifier que les calculs différentiels bicovariants sont préservés par équivalence monoïdale :

Théorème (3.2.6). *Soient H et L deux algèbres de Hopf monoïdalement équivalentes. Il existe alors une équivalence entre leurs catégories de calculs différentiels bicovariants $\mathcal{DC}(H)$ et $\mathcal{DC}(L)$, induisant une équivalence entre leurs catégories de calculs différentiels bicovariants de dimension finie $\mathcal{DC}_f(H)$ et $\mathcal{DC}_f(L)$.*

La preuve de ce résultat repose sur le fait que, d'après Schauenburg [Sch96], se donner une équivalence monoïdale $\mathcal{F} : \mathcal{M}^H \rightarrow \mathcal{M}^L$ entre les catégories de comodules de deux algèbres de Hopf H et L , revient à se donner un objet H - L -bigaloisien, qui fournit une description « explicite » de l'équivalence \mathcal{F} . Cette explicitation nous permet de vérifier que tout calcul différentiel bicovariant sur H se transporte en un calcul différentiel bicovariant sur

L (Lemme 3.2.4) et le théorème 3.2.6 s'ensuit aisément. D'après ce théorème, la classification des calculs différentiels bicovariants sur les algèbres de Hopf libres orthogonales est donc ramenée à l'étude, relativement plus simple, des calculs différentiels bicovariants sur l'algèbre de Hopf $\mathcal{O}_q(SL_2)$.

La classification des calculs différentiels bicovariants sur $\mathcal{O}_q(SL_2)$ a été faite par Heckenberger et Schmüdgen dans [HS98], pour les valeurs transcendentes de q . Ceci ne correspond cependant pas à des valeurs de q où $\mathcal{O}_q(SL_2)$ est monoïdalement équivalente à une algèbre de Hopf libre orthogonale, et ne permet donc pas de conclure notre étude. Leur classification semble pouvoir s'adapter dans le cas où la condition sur q se réduit à ne pas être racine de l'unité, mais nous avons utilisé ici une approche différente pour classer les calculs différentiels bicovariants sur $\mathcal{O}_q(SL_2)$. Elle est basée sur l'article [Tak92], où sont classifiés les modules de Yetter-Drinfeld (de dimension finie) sur $\mathcal{O}_q(SL_2)$. Puisque la catégorie des modules de Yetter-Drinfeld sur une algèbre de Hopf H est monoïdalement équivalente à celle de ses bimodules de Hopf (d'après [Sch94]), la notion de calcul différentiel bicovariant se transporte via cette équivalence, et l'on obtient la notion de *calcul différentiel réduit* sur H .

Définition. Soit H une algèbre de Hopf. Un *calcul différentiel réduit* sur H est un couple (V, ω) , où V est un module de Yetter-Drinfeld sur H , et $\omega : H \rightarrow V$ est une application surjective vérifiant :

$$\begin{aligned} \forall x, y \in H, \quad \omega(xy) &= \omega(x).y + \varepsilon(x)\omega(y) \\ \text{et} \quad \sum \omega(x)_{(0)} \otimes \omega(x)_{(1)} &= \sum \omega(x_{(2)}) \otimes S(x_{(1)})x_{(3)}. \end{aligned}$$

Un tel calcul différentiel est dit *intérieur* s'il existe un élément coinvariant $\theta \in V$ tel que pour tout $x \in H$, $\omega(x) = \theta.x - \varepsilon(x)\theta$.

On montre ensuite que si la catégorie des modules de Yetter-Drinfeld (de dimension finie) sur une algèbre de Hopf est semi-simple, alors tous ses calculs différentiels réduits (de dimension finie) sont intérieurs. La preuve de ce résultat part du fait bien connu que, étant donnée une algèbre augmentée (H, ε) et V un H -module, on a une correspondance entre ε -dérivations de la forme $\omega : H \rightarrow V$, et suites exactes courtes de H -modules de la forme :

$$0 \longrightarrow V \xrightarrow{j} \tilde{V} \xrightarrow{p} \mathbb{C}_\varepsilon \longrightarrow 0$$

où \mathbb{C}_ε est le H -module de dimension 1 tel que : $\forall \lambda \in \mathbb{C}_\varepsilon, \forall x \in H, \lambda.x = \lambda\varepsilon(x)$.

Partant de cette idée, on associe à tout calcul différentiel réduit (V, ω) sur une algèbre

de Hopf H , une suite exacte courte de modules de Yetter-Drinfeld :

$$0 \longrightarrow V \xrightarrow{j} V_\omega \xrightarrow{p} \mathbb{C}_\varepsilon \longrightarrow 0$$

qui est scindée si et seulement si ω est intérieur.

Takeuchi a montré dans [Tak92] que la catégorie des modules de Yetter-Drinfeld de dimension finie sur $\mathcal{O}_q(SL_2)$ est semi-simple, et on obtient donc que tout calcul différentiel réduit de dimension finie sur $\mathcal{O}_q(SL_2)$ (pour q pas racine de l'unité) est intérieur. Il a également montré que, comme comodules, les modules de Yetter-Drinfeld simples sont de la forme $V_n \odot V_m$, où pour $k \in \mathbb{N}$, V_k désigne le $\mathcal{O}_q(SL_2)$ -comodule simple de dimension $k + 1$. Connaissant les règles de fusion des $\mathcal{O}_q(SL_2)$ -comodules simples, on peut déterminer les éléments coinvariants des $\mathcal{O}_q(SL_2)$ -modules de Yetter-Drinfeld simples, et en déduire la classification des $\mathcal{O}_q(SL_2)$ -calculs différentiels réduits simples : ils sont de la forme $(V_n^\varepsilon, \omega_n^\varepsilon)$ où $V_n^\varepsilon \cong V_n \odot V_n$ comme comodule, $\varepsilon \in \{-1, 1\}$ paramétrise la structure de module, et $\omega_n^\varepsilon : \mathcal{O}_q(SL_2) \rightarrow V_n^\varepsilon$ est le calcul différentiel réduit intérieur associé à un élément coinvariant de $V_n \odot V_n$. On obtient la classification suivante :

Théorème (3.3.7). *Tout calcul différentiel réduit simple de dimension finie sur $\mathcal{O}_q(SL_2)$ est de la forme $(V_n^\varepsilon, \omega_n^\varepsilon)$, où $n \in \mathbb{N}$, $\varepsilon \in \{-1, 1\}$ et $(n, \varepsilon) \neq (0, 1)$.*

De plus tout calcul différentiel réduit de dimension finie (V, ω) sur $\mathcal{O}_q(SL_2)$ se décompose en une somme directe :

$$(V, \omega) \cong \bigoplus_{i=1}^d (V_{n_i}^{\varepsilon_i}, \omega_{n_i}^{\varepsilon_i}),$$

où $(n_1, \dots, n_d) \in \mathbb{N}^d$, $(\varepsilon_1, \dots, \varepsilon_d) \in \{-1, 1\}^d$ vérifie $(n_i, \varepsilon_i) \neq (0, 1)$ pour tout $i \in \{1, \dots, d\}$ et $(n_i, \varepsilon_i) \neq (n_j, \varepsilon_j)$ pour tous $i \neq j$.

On en déduit un théorème semblable (Théorème 3.3.12) classifiant les calculs différentiels réduits de dimension finie sur les algèbres de Hopf libres orthogonales.

1

Préliminaires

Notations et conventions

- Dans tout ce manuscrit, on entendra par *algèbre*, une algèbre unifère sur le corps \mathbb{C} .
- Hormis dans le chapitre 3, le symbole \odot désignera le produit tensoriel algébrique d'espaces vectoriels, tandis que le symbole \otimes désignera le produit tensoriel d'applications, le produit tensoriel spatial de C^* -algèbres, ou encore le produit tensoriel extérieur de modules hilbertiens.
Dans le chapitre 3, seul le produit tensoriel algébrique intervenant, il sera noté \otimes .
- Pour X un espace compact, $C(X)$ désignera l'espace des fonctions continues sur X à valeurs dans \mathbb{C} .
- Pour \mathcal{H} un espace de Hilbert, $\mathcal{L}(\mathcal{H})$ désignera l'espace des applications linéaires continues de \mathcal{H} dans lui-même.
- Si A est une C^* -algèbre, $\mathcal{M}_n(A)$ désignera la C^* -algèbre des matrices carrées de taille n à coefficients dans A . Pour $a = (a_{ij})_{1 \leq i, j \leq n}$ dans $\mathcal{M}_n(A)$, on notera $a^t = (a_{ji})_{1 \leq i, j \leq n}$ sa transposée, $\bar{a} = (a_{ij}^*)_{1 \leq i, j \leq n}$ sa matrice conjuguée, et $a^* = (a_{ji}^*)_{1 \leq i, j \leq n}$ sa matrice adjointe.

- Pour deux entiers $i, j \in \mathbb{Z}$, $\delta_{i,j}$ désignera le symbole de Kronecker.

1.1 Géométrie non commutative

On rappelle dans cette partie les résultats fondateurs de la géométrie (ou topologie) non commutative que sont les théorèmes de Gelfand-Naimark et de Serre-Swan, ainsi que certaines définitions relatives aux triplets spectraux.

L'idée de la géométrie non commutative consiste à ne plus considérer un espace comme un ensemble de points, mais à considérer à la place son algèbre de fonctions (fonctions continues pour faire de la topologie, fonctions lisses pour faire de la géométrie différentielle etc.). On est alors amené à traduire dans ce contexte algébrique les notions géométriques usuelles, de manière à les adapter pour des algèbres non nécessairement commutatives. Le point de départ de cette géométrie non commutative est le théorème de Gelfand-Naimark.

Théorème 1.1.1 (de Gelfand-Naimark). *Soit A une C^* -algèbre commutative. Il existe un espace compact X tel que A soit isomorphe à la C^* -algèbre $C(X)$.*

Ce théorème permet en fait de vérifier que le foncteur, qui à un espace compact X associe sa C^* -algèbre de fonctions continues $C(X)$, est une anti-équivalence entre la catégorie des espaces compacts et celle des C^* -algèbres commutatives. Ce résultat justifie le fait de voir les C^* -algèbres (plus nécessairement commutatives) comme des algèbres de fonctions sur des espaces dits *quantiques* : on définit la catégorie des espaces quantiques compacts comme la catégorie opposée à celle des C^* -algèbres.

Dans le même esprit que le théorème précédent, le théorème de Serre-Swan permet de donner un équivalent non commutatif au concept de fibré vectoriel (localement trivial et de dimension finie) sur un espace compact.

Théorème 1.1.2 (de Serre-Swan). *Soit X un espace compact, la catégorie des fibrés vectoriels (localement triviaux) de dimension finie est équivalente à celle des $C(X)$ -modules projectifs de type fini.*

L'équivalence précédente est fournie par le foncteur qui à un fibré vectoriel E associe le $C(X)$ -module $\Gamma(E)$ des sections continues de E . Le théorème de Serre-Swan s'adapte également dans un contexte de modules hilbertiens : il est montré dans [Tak79] que la catégorie des modules hilbertiens sur des C^* -algèbres commutatives est équivalente à celle des fibrés hilbertiens (plus nécessairement localement triviaux) sur des espaces compacts. Ceci permet donc de voir les modules hilbertiens comme des sections de fibrés hilbertiens sur des espaces quantiques compacts.

Les variétés riemanniennes compactes étant des espaces compacts munis de structures additionnelles, une question fondamentale en géométrie non commutative est de trouver un pendant au théorème de Gelfand-Naimark, pour les variétés riemanniennes compactes.

La théorie des triplets spectraux a été développée par Connes (entre autres dans [Con94]), et généralise la notion de variété riemannienne compacte.

Définition 1.1.3. Soient \mathcal{A} un algèbre involutive, \mathcal{H} un espace de Hilbert tel qu'il existe une représentation fidèle $\rho : \mathcal{A} \rightarrow \mathcal{L}(\mathcal{H})$, et soit D un opérateur non borné autoadjoint sur \mathcal{H} . On dit que $(\mathcal{A}, \mathcal{H}, D)$ est un *triplet spectral* si les conditions suivantes sont vérifiées :

- D est à résolvante compacte.
- Pour tout $a \in \mathcal{A}$, le commutateur $[D, \rho(a)]$ de D avec $\rho(a)$ est un opérateur borné.

Dans la suite, on identifie les éléments a de \mathcal{A} avec leur image $\rho(a)$ dans $\mathcal{L}(\mathcal{H})$.

Exemple 1.1.4. Il existe plusieurs façons d'associer à une variété riemannienne compacte M un triplet spectral, l'une des plus classiques étant de lui associer le triplet spectral $(\mathcal{A}, \mathcal{H}, D)$ où \mathcal{A} est l'algèbre des fonctions lisses $C^\infty(M)$, \mathcal{H} est l'espace de Hilbert des formes extérieures de carré sommable $L^2(\Lambda^*M)$, et D est l'opérateur de de Rham $d + d^*$.

Une autre façon, lorsque la variété M est à spin, est de lui associer le triplet spectral $(\mathcal{A}, \mathcal{H}, D)$, où \mathcal{A} est toujours l'algèbre $C^\infty(M)$, \mathcal{H} est l'espace de Hilbert $L^2(S)$ des sections de carré sommable du fibré des spineurs S , et D est l'opérateur de Dirac associé à la connexion de Levi-Civita spinorielle.

Exemple 1.1.5 (Le tore non-commutatif). Soit θ un réel. L'algèbre du tore non-commutatif A_θ est la C^* -algèbre universelle engendrée par deux éléments unitaires U et V , soumis aux relations : $VU = e^{2i\pi\theta}UV$. Il existe sur A_θ un état fidèle et tracial τ , caractérisé par $\tau(U^m V^n) = \delta_{m,0} \delta_{n,0}$. On note \mathcal{H}_τ l'espace de Hilbert obtenu par la construction GNS associée à τ , et $\mathcal{H} = \mathcal{H}_\tau \oplus \mathcal{H}_\tau$. Puisque τ est fidèle, la représentation GNS $\rho : A_\theta \rightarrow \mathcal{H}_\tau$ est fidèle, et on obtient donc une représentation fidèle de A_θ sur \mathcal{H} . Le triplet spectral du tore non-commutatif est le triplet $(A_\theta, \mathcal{H}, D)$, avec

$$D = -i \begin{pmatrix} 0 & d_1 - id_2 \\ d_1 + id_2 & 0 \end{pmatrix}.$$

où $d_1 : \mathcal{H}_\tau \rightarrow \mathcal{H}_\tau$ est l'opérateur (non borné) fermé caractérisé par $d_1(U^n V^m) = 2i\pi n U^n V^m$ et $d_2 : \mathcal{H}_\tau \rightarrow \mathcal{H}_\tau$ est l'opérateur fermé caractérisé par $d_2(U^n V^m) = 2i\pi m U^n V^m$.

On rencontre dans la littérature de nombreuses conditions additionnelles sur les triplets spectraux (triplets spectraux réels, pairs, θ -sommables *etc.*). On rappelle ici celles que l'on sera amené à considérer dans le chapitre 2.

Définition 1.1.6. Un triplet spectral $(\mathcal{A}, \mathcal{H}, D)$ est dit :

- *réel* si il est muni d'un opérateur anti-unitaire $\mathcal{J} : \mathcal{H} \rightarrow \mathcal{H}$ tel que :
 - (1) $\mathcal{J}(\text{Dom}(D)) \subset \text{Dom}(D)$,
 - (2) $\mathcal{J}^2 = \varepsilon$ et $\mathcal{J}D = \varepsilon'D\mathcal{J}$, où $\varepsilon, \varepsilon' \in \{-1, 1\}$,
 - (3) $\forall a, b \in \mathcal{A}, [a, \mathcal{J}b^*\mathcal{J}^*] = 0$,
- *régulier* si pour tout $a \in \mathcal{A}$ et tout $n \in \mathbb{N}$, a et $[D, a]$ sont dans le domaine de l'opérateur non borné δ^n sur $\mathcal{L}(\mathcal{H})$, où $\delta = [|D|, \cdot]$,
- *finiment sommable* s'il existe $p \in \mathbb{N}$ tel que l'opérateur $|D|^{-p}$ admette une trace de Dixmier $Tr_\omega(|D|^{-p})$, qui soit non nulle.

Si $(\mathcal{A}, \mathcal{H}, D)$ est régulier, l'espace $\mathcal{H}^\infty = \bigcap_{k \in \mathbb{N}} \text{Dom}(D^k)$ est laissé stable par les éléments de \mathcal{A} , et \mathcal{H}^∞ est donc un \mathcal{A} -module à gauche. On dit qu'un triplet spectral régulier et finiment sommable $(\mathcal{A}, \mathcal{H}, D)$ satisfait la condition de *finitude et d'absolue continuité* si le \mathcal{A} -module \mathcal{H}^∞ est projectif de type fini, et s'il existe $q \in \mathcal{M}_n(\mathcal{A})$ tel que $q = q^2 = q^*$ et :

- (1) $\mathcal{H}^\infty \cong \mathcal{A}^n q$,
- (2) le \mathcal{A} -produit scalaire à gauche ${}_{\mathcal{A}}\langle \cdot | \cdot \rangle$ induit par l'isomorphisme précédent sur \mathcal{H}^∞ satisfait :

$$\forall \xi, \eta \in \mathcal{H}^\infty, \frac{Tr_\omega({}_{\mathcal{A}}\langle \xi | \eta \rangle |D|^{-p})}{Tr_\omega(|D|^{-p})} = (\eta | \xi)_{\mathcal{H}}.$$

Le théorème de reconstruction de Connes [Con13] fournit un équivalent au théorème de Gelfand-Naimark pour les variétés riemanniennes compactes orientées : si $(\mathcal{A}, \mathcal{H}, D)$ est un triplet spectral commutatif (c'est-à-dire dont l'algèbre \mathcal{A} est commutative) satisfaisant la condition de finitude et d'absolue continuité, et vérifiant quelques conditions supplémentaires, alors il existe une variété riemannienne compacte orientée M telle que l'algèbre \mathcal{A} soit isomorphe à $C^\infty(M)$. Dans une version légèrement modifiée, il donne également les conditions pour que $\mathcal{A} \cong C^\infty(M)$ où M est une variété riemannienne compacte à spin.

1.2 Groupes quantiques

On rappelle dans cette partie certaines définitions et certains résultats sur les groupes quantiques importants pour la suite. On s'intéressera d'une part aux groupes quantiques algébriques, correspondants aux algèbres de Hopf, et d'autre part aux groupes quantiques compacts, correspondants aux C^* -algèbres de Woronowicz.

1.2.1 Algèbres de Hopf

Définition 1.2.1. Une *cogèbre* est un triplet (H, Δ, ε) , où H est un espace vectoriel, $\Delta : H \rightarrow H \odot H$ et $\varepsilon : H \rightarrow \mathbb{C}$ sont des applications linéaires telles que les diagrammes suivants commutent :

Axiome de coassociativité :

$$\begin{array}{ccc} H & \xrightarrow{\Delta} & H \odot H \\ \Delta \downarrow & & \downarrow id_H \otimes \Delta \\ H \odot H & \xrightarrow{\Delta \otimes id_H} & H \odot H \odot H \end{array}$$

Axiome de counité :

$$\begin{array}{ccccc} & & H \odot H & & \\ & \xleftarrow{id_H \otimes \varepsilon} & & \xrightarrow{\varepsilon \otimes id_H} & \\ H & & & & H \\ & \searrow id_H & \uparrow \Delta & \swarrow id_H & \\ & & H & & \end{array}$$

L'application Δ est appelée la *comultiplication* de H , et ε sa *counité*.

Un *morphisme* entre deux cogèbres (H, Δ, ε) et $(H', \Delta', \varepsilon')$, est une application $\mu : H \rightarrow H'$ satisfaisant :

$$\Delta' \circ \mu = (\mu \otimes \mu) \circ \Delta \quad \text{et} \quad \varepsilon' \circ \mu = \varepsilon.$$

On utilisera souvent dans la suite les *notations de Sweedler* : Pour x dans une algèbre de Hopf H , on écrira $\Delta(x) = \sum x_{(1)} \otimes x_{(2)}$, de sorte que les axiomes de coassociativité et de counité se traduisent respectivement par les écritures :

$$\begin{aligned} \sum \Delta(x_{(1)}) \otimes x_{(2)} &= \sum x_{(1)} \otimes \Delta(x_{(2)}) = \sum x_{(1)} \otimes x_{(2)} \otimes x_{(3)} \\ \text{et} \quad \sum \varepsilon(x_{(1)})x_{(2)} &= \sum x_{(1)}\varepsilon(x_{(2)}) = x. \end{aligned}$$

Définition 1.2.2. Une *algèbre de Hopf* est un sextuplet $H = (H, m, u, \Delta, \varepsilon, S)$, où

- (H, m, u) est une algèbre (m désigne la multiplication de H , $u : \mathbb{C} \rightarrow H$ son unité),
- (H, Δ, ε) est une cogèbre,
- $\Delta : H \rightarrow H \odot H$ et $\varepsilon : H \rightarrow \mathbb{C}$ sont des morphismes d'algèbres,
- $S : H \rightarrow H$ est une application linéaire telle que les diagrammes suivants soient commutatifs :

$$\begin{array}{ccc} H & \xrightarrow{\varepsilon} & \mathbb{C} \xrightarrow{u} & H \\ \Delta \downarrow & & & \uparrow m \\ H \odot H & \xrightarrow{S \otimes id_H} & H \odot H \end{array} \quad \begin{array}{ccc} H & \xrightarrow{\varepsilon} & \mathbb{C} \xrightarrow{u} & H \\ \Delta \downarrow & & & \uparrow m \\ H \odot H & \xrightarrow{id_H \otimes S} & H \odot H \end{array}$$

C'est-à-dire, avec les notations de Sweedler : $\sum S(x_{(1)})x_{(2)} = \sum x_{(1)}S(x_{(2)}) = \varepsilon(x)1_H$ pour tout x dans H .

L'application S est appelée *l'antipode* de H . Elle est automatiquement un anti-morphisme d'algèbre, et satisfait $\sum S(x)_{(1)} \otimes S(x)_{(2)} = \Delta(S(x)) = \sum S(x_{(2)}) \otimes S(x_{(1)})$.

Un *morphisme* entre deux algèbres de Hopf $(H, m, u, \Delta, \varepsilon, S)$ et $(H', m', u', \Delta', \varepsilon', S')$, est une application $\mu : H \rightarrow H'$, qui est à la fois un morphisme d'algèbres et de cogèbres. Il satisfait alors automatiquement $\mu \circ S = S' \circ \mu$.

Exemple 1.2.3. Soit $E \in GL_n(\mathbb{C})$ une matrice inversible. L'algèbre universelle $\mathcal{B}(E)$ engendrée par des éléments $(a_{ij})_{1 \leq i, j \leq n}$ soumis aux relations :

$$E^{-1} a^t E a = I_n = a E^{-1} a^t E,$$

où $a = (a_{ij})_{1 \leq i, j \leq n}$, admet une structure d'algèbre de Hopf définie par :

$$\begin{aligned} \Delta(a_{ij}) &= \sum_{k=1}^n a_{ik} \otimes a_{kj}, \\ \varepsilon(a_{ij}) &= \delta_{ij}, \\ S(a_{ij}) &= (E^{-1} a^t E)_{ij}. \end{aligned}$$

Elle est appelée *l'algèbre de Hopf libre orthogonale associée à E* . Pour $q \in \mathbb{C}^*$, $\mathcal{B}(E_q)$ où

$$E_q = \begin{pmatrix} 0 & 1 \\ -q^{-1} & 0 \end{pmatrix},$$

est la classique q -déformation de l'algèbre de Hopf $\mathcal{O}(SL_2)$, elle est généralement notée $\mathcal{O}_q(SL_2)$.

En dualisant la notion de *module* sur une algèbre, on obtient la notion de *comodule* sur une algèbre de Hopf.

Définition 1.2.4. Soit $H = (H, m, u, \Delta, \varepsilon, S)$ une algèbre de Hopf. Un H -comodule à droite est un couple (V, ρ) où V est un espace vectoriel, et $\rho : V \rightarrow V \otimes H$ est une application linéaire telle que les diagrammes suivants commutent :

$$\begin{array}{ccc} V & \xrightarrow{\rho} & V \otimes H \\ \rho \downarrow & & \downarrow id_V \otimes \Delta \\ V \otimes H & \xrightarrow{\rho \otimes id_H} & V \otimes H \otimes H \end{array} \quad \begin{array}{ccc} V \otimes H & \xrightarrow{id_V \otimes \varepsilon} & V \\ \rho \uparrow & \nearrow id_V & \\ V & & \end{array}$$

De façon analogue, un H -comodule à gauche est un couple (V, λ) où V est un espace vectoriel, et $\lambda : V \rightarrow H \otimes V$ est une application linéaire telle que les diagrammes suivants

commutent :

$$\begin{array}{ccc}
 V & \xrightarrow{\lambda} & H \odot V \\
 \lambda \downarrow & & \downarrow \Delta \otimes id_V \\
 H \odot V & \xrightarrow{id_H \otimes \lambda} & H \odot H \odot V
 \end{array}
 \qquad
 \begin{array}{ccc}
 H \odot V & \xrightarrow{\varepsilon \otimes id_V} & V \\
 \lambda \uparrow & \nearrow id_V & \\
 V & &
 \end{array}$$

Sauf mention du contraire, on entendra par comodule, un comodule à droite.

Si H et L sont deux algèbres de Hopf, un H - L -cobimodule est un triplet (V, λ, ρ) , où (V, λ) est un H -comodule à gauche, (V, ρ) est un L -comodule à droite, et tel que les coactions à gauche et à droite soient compatibles, c'est-à-dire tel que le diagramme suivant commute :

$$\begin{array}{ccc}
 V & \xrightarrow{\lambda} & H \odot V \\
 \rho \downarrow & & \downarrow id_H \otimes \rho \\
 V \odot L & \xrightarrow{\lambda \otimes id_L} & H \odot V \odot L
 \end{array}$$

Un *morphisme* entre deux comodules sur H , $\rho : V \rightarrow V \odot H$ et $\gamma : W \rightarrow W \odot H$ est une application linéaire $\mu : V \rightarrow W$ vérifiant $(\mu \otimes id_H) \circ \rho = \gamma \circ \mu$.

La catégorie des H -comodules (à droite), notée \mathcal{M}^H , est une catégorie monoïdale. Le produit tensoriel entre deux comodules $\rho : V \rightarrow V \odot H$ et $\gamma : W \rightarrow W \odot H$ étant défini par :

$$V \odot W \xrightarrow{\rho \otimes \gamma} V \odot H \odot W \odot H \xrightarrow{id_V \otimes \tau_{H,W} \otimes id_H} V \odot W \odot H \odot H \xrightarrow{id_V \otimes id_W \otimes m} V \odot W \odot H,$$

où $\tau_{H,W} = H \odot W \rightarrow W \odot H$.

$$x \otimes v \mapsto v \otimes x$$

On dit que deux algèbres de Hopf H et L sont *monoïdalement équivalentes*, s'il existe une équivalence monoïdale (linéaire) entre leurs catégories de comodules \mathcal{M}^H et \mathcal{M}^L .

On utilisera également des notations de Sweedler pour les comodules. Si (V, ρ) est un H -comodule, on notera $\rho(v) = \sum v_{(0)} \otimes v_{(1)}$ pour v dans V , de sorte que les axiomes de comodules deviennent :

$$\sum \rho(v_{(0)}) \otimes v_{(1)} = \sum v_{(0)} \otimes \Delta(v_{(1)}) = \sum v_{(0)} \otimes v_{(1)} \otimes v_{(2)}$$

et $\sum v_{(0)} \varepsilon(v_{(1)}) = v$.

Soit (V, ρ) un comodule de dimension finie sur une algèbre de Hopf H , et soit (e_1, \dots, e_n) une base de V . Alors, d'après les axiomes de comodules, la matrice $(x_{ij})_{1 \leq i, j \leq n} \in \mathcal{M}_n(H)$

caractérisée par $\rho(e_i) = \sum_{j=1}^n e_j \otimes x_{ji}$, satisfait $\Delta(x_{ij}) = \sum_{k=1}^n x_{ik} \otimes x_{kj}$ et $\varepsilon(x_{ij}) = \delta_{ij}$. Une telle matrice est dite *multiplicative*.

On rappelle dans la suite de cette section la correspondance entre équivalences monoïdales d'algèbres de Hopf, et objets bigaloisiers [Sch96]. On rappelle également la définition des cogroupoïdes, qui permet de reformuler cette correspondance dans un contexte pratique et élégant.

Définition 1.2.5. Soient H une algèbre de Hopf, (V, ρ) un H -comodule à droite, et (W, λ) un H -comodule à gauche. Le *produit cotensoriel* de V et W , noté $V \square_H W$, est le noyau de l'application $\rho \otimes id_W - id_V \otimes \lambda : V \odot W \rightarrow V \odot H \odot W$.

Si de plus L est une autre algèbre de Hopf, et que (W, λ_W, ρ_W) est un H - L -cobimodule, alors $V \square_H W$ est un comodule à droite sur L (un sous-comodule de $V \odot W$, équipé de la coaction $id_V \otimes \rho_W : V \odot W \rightarrow V \odot W \odot L$).

Si W est un H - L -cobimodule, le produit cotensoriel fournit en fait un foncteur linéaire

$$\begin{aligned} \mathcal{F}_W : \mathcal{M}^H &\rightarrow \mathcal{M}^L \\ V &\mapsto V \square_H W. \end{aligned}$$

Sous certaines conditions sur W , ce foncteur est une équivalence monoïdale.

Définition 1.2.6. Soit H une algèbre de Hopf. Une *algèbre H -comodule* (à droite) est un H -comodule (A, ρ) où A est une algèbre telle que la coaction $\rho : A \rightarrow A \odot H$ soit un morphisme d'algèbres.

On définit de manière semblable la notion d'algèbre H -comodule à gauche.

Définition 1.2.7. Soit H une algèbre de Hopf. Un *objet H -galoisien à droite* est une algèbre H -comodule à droite (A, ρ) non nulle, telle que l'application linéaire :

$$A \odot A \xrightarrow{id_A \otimes \rho} A \odot A \odot H \xrightarrow{m \otimes id_H} A \odot H$$

soit un isomorphisme (où m désigne la multiplication sur A). De même, un *objet H -galoisien à gauche* est une algèbre H -comodule à gauche (A, λ) non nulle, telle que l'application :

$$A \odot A \xrightarrow{\lambda \otimes id_A} H \odot A \odot A \xrightarrow{id_H \otimes m} H \odot A$$

soit un isomorphisme.

Définition 1.2.8. Soient H et L deux algèbres de Hopf. Un objet H - L -bigaloisien est une algèbre A qui est à la fois un H - L -cobimodule, un objet H -galoisien à gauche et un objet L -galoisien à droite.

Si W est un objet H - L -bigaloisien, alors le foncteur

$$\begin{aligned} \mathcal{F}_W : \mathcal{M}^H &\rightarrow \mathcal{M}^L \\ V &\mapsto V \square_H W. \end{aligned}$$

est une équivalence monoïdale. Réciproquement, Schauenburg a montré dans [Sch96] que toute équivalence monoïdale entre algèbres de Hopf est de cette forme. Deux algèbres de Hopf H et L sont donc monoïdalement équivalentes si et seulement si il existe un objet H - L -bigaloisien.

On donne maintenant la définition des cogroupoïdes, introduits dans [Bic], et l'on reformule le théorème de Schauenburg dans ce contexte.

Définition 1.2.9. Une *cocatégorie* \mathcal{C} est la donnée :

- d'un ensemble d'objets $\text{ob}(\mathcal{C})$,
- pour tous $X, Y \in \text{ob}(\mathcal{C})$, d'une algèbre $\mathcal{C}(X, Y)$,
- pour tous $X, Y, Z \in \text{ob}(\mathcal{C})$, de morphismes d'algèbres $\Delta_{X,Y}^Z : \mathcal{C}(X, Y) \rightarrow \mathcal{C}(X, Z) \odot \mathcal{C}(Z, Y)$ et $\varepsilon_X : \mathcal{C}(X, X) \rightarrow \mathbb{C}$ tels que pour tous $X, Y, Z, T \in \text{ob}(\mathcal{C})$, les diagrammes suivants soient commutatifs :

$$\begin{array}{ccc} \mathcal{C}(X, Y) & \xrightarrow{\Delta_{X,Y}^Z} & \mathcal{C}(X, Z) \odot \mathcal{C}(Z, Y) \\ \Delta_{X,Y}^T \downarrow & & \downarrow id \otimes \Delta_{Z,Y}^T \\ \mathcal{C}(X, T) \odot \mathcal{C}(T, Y) & \xrightarrow{\Delta_{X,T}^Z \otimes id} & \mathcal{C}(X, Z) \odot \mathcal{C}(Z, T) \odot \mathcal{C}(T, Y) \end{array}$$

$$\begin{array}{ccc} \mathcal{C}(X, Y) & \xrightarrow{\Delta_{X,Y}^X} & \mathcal{C}(X, X) \odot \mathcal{C}(X, Y) \\ & \searrow & \downarrow \varepsilon_X \otimes id \\ & & \mathcal{C}(X, Y) \end{array} \quad \begin{array}{ccc} \mathcal{C}(X, Y) & \xrightarrow{\Delta_{X,Y}^Y} & \mathcal{C}(X, Y) \odot \mathcal{C}(Y, Y) \\ & \searrow & \downarrow id \otimes \varepsilon_Y \\ & & \mathcal{C}(X, Y) \end{array}$$

Une cocatégorie est dite *connexe* si pour tous $X, Y \in \text{ob}(\mathcal{C})$, l'algèbre $\mathcal{C}(X, Y)$ est non nulle.

Définition 1.2.10. Un *cogroupoïde* \mathcal{C} est une cocatégorie munie d'applications linéaires $S_{X,Y} : \mathcal{C}(X, Y) \rightarrow \mathcal{C}(Y, X)$ telles que pour tous $X, Y \in \text{ob}(\mathcal{C})$, les diagrammes suivants com-

mutent :

$$\begin{array}{ccc}
 \mathcal{C}(X, X) & \xrightarrow{\varepsilon_X} & \mathbb{C} \xrightarrow{u} \mathcal{C}(Y, X) \\
 \Delta_{X,X}^Y \downarrow & & \uparrow m \\
 \mathcal{C}(X, Y) \odot \mathcal{C}(Y, X) & \xrightarrow{S_{X,Y} \otimes id} & \mathcal{C}(Y, X) \odot \mathcal{C}(Y, X)
 \end{array}$$

$$\begin{array}{ccc}
 \mathcal{C}(X, X) & \xrightarrow{\varepsilon_X} & \mathbb{C} \xrightarrow{u} \mathcal{C}(X, Y) \\
 \Delta_{X,X}^Y \downarrow & & \uparrow m \\
 \mathcal{C}(X, Y) \odot \mathcal{C}(Y, X) & \xrightarrow{id \otimes S_{Y,X}} & \mathcal{C}(X, Y) \odot \mathcal{C}(X, Y)
 \end{array}$$

où m désigne la multiplication, et u l'unité.

Les axiomes définissant un cogroupoïde sont donc très similaires à ceux des algèbres de Hopf, et l'on peut également utiliser des notations de Sweedler dans ce contexte : pour $a^{X,Y} \in \mathcal{C}(X, Y)$, on notera $\Delta_{X,Y}^Z(a^{X,Y}) = \sum a_{(1)}^{X,Z} \otimes a_{(2)}^{Z,Y}$. Les axiomes de cogroupoïde deviennent :

$$\begin{aligned}
 \sum \Delta_{X,T}^Z(a_{(1)}^{X,T}) \otimes a_{(2)}^{T,Y} &= \sum a_{(1)}^{X,Z} \otimes \Delta_{Z,Y}^T(a_{(2)}^{Z,Y}) = \sum a_{(1)}^{X,Z} \otimes a_{(2)}^{Z,T} \otimes a_{(3)}^{T,Y}, \\
 \sum \varepsilon_X(a_{(1)}^{X,X}) a_{(2)}^{X,Y} &= \sum a_{(1)}^{X,Y} \varepsilon_Y(a_{(2)}^{Y,Y}) = a^{X,Y} \\
 \text{et } \sum S_{X,Y}(a_{(1)}^{X,Y}) a_{(2)}^{Y,X} &= \sum a_{(1)}^{X,Y} S_{Y,X}(a_{(2)}^{Y,X}) = \varepsilon_X(a^{X,X}) 1_{\mathcal{C}(X,Y)}
 \end{aligned}$$

Soient \mathcal{C} un cogroupoïde et $X, Y \in \text{ob}(\mathcal{C})$.

- L'algèbre $\mathcal{C}(X, X)$, équipée de $\Delta_{X,X}^X$, ε_X et $S_{X,X}$ est une algèbre de Hopf,
- $(\mathcal{C}(X, Y), \Delta_{X,Y}^X, \Delta_{X,Y}^Y)$ est un $\mathcal{C}(X, X)$ - $\mathcal{C}(Y, Y)$ -cobimodule,
- si de plus \mathcal{C} est connexe, alors $\mathcal{C}(X, Y)$ est un objet $\mathcal{C}(X, X)$ - $\mathcal{C}(Y, Y)$ -bigaloisien.

Le théorème de Schauenburg se traduit alors dans ce contexte par le théorème suivant.

Théorème 1.2.11 ([Bic, Théorème 2.10]). *Deux algèbres de Hopf H et L sont monoïdalement équivalentes si et seulement si il existe un cogroupoïde connexe \mathcal{C} , et deux objets X, Y de \mathcal{C} tels que $H \cong \mathcal{C}(X, X)$ et $L \cong \mathcal{C}(Y, Y)$.*

Dans ce cas, l'équivalence monoïdale est donnée par l'objet H - L -bigaloisien $\mathcal{C}(X, Y)$:

$$\begin{array}{ccc}
 \mathcal{F}_X^Y : \mathcal{M}^{\mathcal{C}(X,X)} \rightarrow \mathcal{M}^{\mathcal{C}(Y,Y)} & \text{de quasi-inverse} & \mathcal{F}_Y^X : \mathcal{M}^{\mathcal{C}(Y,Y)} \rightarrow \mathcal{M}^{\mathcal{C}(X,X)} \\
 V \mapsto V \square_{\mathcal{C}(X,X)} \mathcal{C}(X, Y), & & V \mapsto V \square_{\mathcal{C}(Y,Y)} \mathcal{C}(Y, X).
 \end{array}$$

Exemple 1.2.12. Le cogroupoïde bilinéaire \mathcal{B} est défini comme suit.

- $\text{ob}(\mathcal{B}) = \{E \in GL_n(\mathbb{C}) ; n \geq 1\}$,
- Pour $E, F \in \text{ob}(\mathcal{B})$, et $m, n \geq 1$ tels que $E \in GL_m(\mathbb{C})$ et $F \in GL_n(\mathbb{C})$, $\mathcal{B}(E, F)$ est l'algèbre universelle engendrée par des éléments $(a_{ij})_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}}$ soumis aux relations :

$$F^{-1}a^t E a = I_n \quad \text{et} \quad a F^{-1}a^t E = I_m,$$

où $a = (a_{ij})_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}}$.

- Pour $E, F, G \in \text{ob}(\mathcal{B})$, $\Delta_{E,F}^G : \mathcal{B}(E, F) \rightarrow \mathcal{B}(E, G) \otimes \mathcal{B}(G, F)$, $\varepsilon_E : \mathcal{B}(E, E) \rightarrow \mathbb{C}$ et $S_{E,F} : \mathcal{B}(E, F) \rightarrow \mathcal{B}(F, E)$ sont caractérisés par :

$$\begin{aligned} \Delta_{E,F}^G(a_{ij}) &= \sum_{k=1}^n a_{ik} \otimes a_{kj}, \text{ où } n \geq 1 \text{ est tel que } G \in GL_n(\mathbb{C}), \\ \varepsilon_E(a_{ij}) &= \delta_{ij}, \\ S_{E,F}(a_{ij}) &= (E^{-1}a^t F)_{ij}. \end{aligned}$$

Pour $E \in GL_n(\mathbb{C})$, $\mathcal{B}(E, E)$ coïncide avec l'algèbre de Hopf $\mathcal{B}(E)$ de l'exemple 1.2.3.

1.2.2 C^* -algèbres de Woronowicz

On rappelle ici les définitions de bases sur les C^* -algèbres de Woronowicz, c'est-à-dire les algèbres que l'on peut voir comme des « algèbres de fonctions continues sur des groupes quantiques compacts ». On rappelle également leur lien avec les algèbres de Hopf.

Définition 1.2.13. Une C^* -algèbre de Woronowicz est un couple (Q, Δ) , où Q est une C^* -algèbre et $\Delta : Q \rightarrow Q \otimes Q$ est un $*$ -morphisme, appelé le *coproduit* de Q , vérifiant :

- $(\Delta \otimes id_Q) \circ \Delta = (id_Q \otimes \Delta) \circ \Delta$,
- les espaces vectoriel $\text{vect}\{\Delta(Q).(Q \otimes 1_Q)\}$ et $\text{vect}\{\Delta(Q).(1_Q \otimes Q)\}$ sont tous deux denses dans $Q \otimes Q$.

Exemple 1.2.14. Soit $E \in GL_n(\mathbb{C})$. La C^* -algèbre universelle engendrée par une matrice unitaire $u = (u_{ij})_{1 \leq i, j \leq n}$, satisfaisant la relation $u = E \bar{u} E^{-1}$ admet une structure de C^* -algèbre de Woronowicz. Son coproduit est caractérisé par $\Delta(u_{ij}) = \sum_{k=1}^n u_{ik} \otimes u_{kj}$. Elle est notée $\mathcal{A}_o(E)$ et appelée la C^* -algèbre de Woronowicz du *groupe quantique libre orthogonal* associé à E .

Exemple 1.2.15. Soit $F \in GL_n(\mathbb{C})$. La C^* -algèbre universelle engendrée par une matrice unitaire $u = (u_{ij})_{1 \leq i, j \leq n}$, satisfaisant la relation $u^t F \bar{u} F^{-1} = F \bar{u} F^{-1} u^t = I_n$, admet une structure de C^* -algèbre de Woronowicz. Son coproduit est lui-aussi caractérisé par $\Delta(u_{ij}) = \sum_{k=1}^n u_{ik} \otimes u_{kj}$. Elle est notée $\mathcal{A}_u(F)$ et appelée la C^* -algèbre de Woronowicz universelle associée à F .

Définition 1.2.16. Soient (Q_0, Δ_0) et (Q_1, Δ_1) deux algèbres de Woronowicz. Un *morphisme d'algèbres de Woronowicz* de Q_0 vers Q_1 est un $*$ -morphisme :

$$\mu : Q_0 \rightarrow Q_1 \quad \text{vérifiant} \quad (\mu \otimes \mu) \circ \Delta_0 = \Delta_1 \circ \mu$$

On définit un *groupe quantique compact* comme un objet de la catégorie opposée à celle des C^* -algèbres de Woronowicz. Ce terme se justifie par le fait que le théorème de Gelfand-Naimark s'adapte pour les groupes compacts : la catégorie des C^* -algèbres de Woronowicz commutatives est anti-équivalente à celle des groupes compacts. L'équivalence est donné de la façon suivante : si G est un groupe compact, la C^* -algèbre de Woronowicz qui lui est associée est le couple $(C(G), \mu^*)$, où $\mu : G \rightarrow G \times G$ désigne la multiplication de G , et

$$\begin{aligned} \mu^* : C(G) &\rightarrow C(G \times G) \cong C(G) \otimes C(G) \\ f &\mapsto f \circ \mu \end{aligned}$$

est son application duale.

Il existe un lien fort entre certaines algèbres de Hopf, dites *compactes*, et les C^* -algèbres de Woronowicz.

Définition 1.2.17. Une algèbre de Hopf H est dite *compacte* s'il existe une structure d' $*$ -algèbre de Hopf sur H (c'est-à-dire une involution $*$ sur H telle que H soit une $*$ -algèbre et sa comultiplication soit un $*$ -morphisme) telle que pour toute matrice multiplicative $x = (x_{ij})_{1 \leq i, j \leq n} \in \mathcal{M}_n(H)$, il existe une matrice $F \in GL_n(\mathbb{C})$ telle que $F x F^{-1}$ soit unitaire.

Toute algèbre de Hopf compacte H admet une C^* -algèbre enveloppante $C^*(H)$, et sa comultiplication se prolonge en une comultiplication $\Delta : C^*(H) \rightarrow C^*(H) \otimes C^*(H)$, faisant de $(C^*(H), \Delta)$ une C^* -algèbre de Woronowicz.

Réciproquement, pour toute C^* -algèbre de Woronowicz (Q, Δ) il existe une unique $*$ -algèbre dense $\mathcal{Q} \subset Q$ satisfaisant $\Delta(\mathcal{Q}) \subset \mathcal{Q} \otimes \mathcal{Q}$, et admettant une structure d'algèbre de Hopf compacte, dont la comultiplication soit la restriction de Δ à \mathcal{Q} . Mais cette correspondance n'est pas bijective : une algèbre de Hopf compacte peut être complétée de plusieurs façons en une C^* -algèbre de Woronowicz.

Exemple 1.2.18. Soit $E \in GL_n(\mathbb{C})$. D'après [Bic03b, Proposition 6.1], l'algèbre de Hopf $\mathcal{B}(E)$ définie dans l'exemple 1.2.3 est compacte si et seulement si il existe $\lambda \in \mathbb{R}^*$, $\mu \in \mathbb{C}^*$ et une matrice $M \in GL_n(\mathbb{C})$ telle que

$$M^t E^* M = E, \quad \overline{M} M = \lambda I_n \quad \text{et} \quad \mu(M^t)^{-1} E \text{ soit une matrice positive.}$$

Dans ce cas, l'involution sur $\mathcal{B}(E)$ est caractérisée par $u_{ij}^* = (MuM^{-1})_{ij}$. En particulier, si E vérifie $E\overline{E} = \lambda I_n$ avec $\lambda \in \mathbb{C}^*$. On a alors $EE^*E^t = E(E\overline{E})^t = \lambda E$, et on en déduit que les conditions précédentes sont vérifiées, avec M proportionnelle à E^t . Il s'ensuit que $\mathcal{B}(E)$ admet une structure d'algèbre de Hopf compacte, avec $\bar{u} = E^t u (E^t)^{-1}$. On a alors

$$(E^t)^{-1} \bar{u} E^t = (E^t)^{-1} E^t u (E^t)^{-1} E^t = u$$

et $u^* u = E^{-1} u^t E u = I_n = u E^{-1} u^t E = u u^*$.

Ceci permet d'identifier l'algèbre de Hopf $\mathcal{B}(E)$ comme l'algèbre de Hopf compacte sous-jacente à la C^* -algèbre de Woronowicz $\mathcal{A}_o((E^t)^{-1})$.

Ce chapitre est constitué du texte de l'article [TdC14], publié dans le *Journal of Functional Analysis*.

2

Quantum symmetry groups of Hilbert modules equipped with orthogonal filtrations

We define and show the existence of the quantum symmetry group of a Hilbert module equipped with an orthogonal filtration. Our construction unifies the constructions of Banica-Skalski's quantum symmetry group of a C^* -algebra equipped with an orthogonal filtration and Goswami's quantum isometry group of an admissible spectral triple.

Introduction

The quantum isometry group of a noncommutative Riemannian compact manifold (an admissible spectral triple) was defined and constructed by Goswami in [Gos09]. His breakthrough construction, technically more involved than the previous approaches to quantum symmetry groups in the case of finite structures [Wan98, BB07], provides a very natural direct link between Connes' noncommutative geometry [Con94] and the theory of compact quantum groups introduced by Woronowicz in the eighties [Wor87a]. We refer the reader to the introduction and bibliography of [BS13] for an overview of the several developments since Goswami's paper.

Motivated by the work of Goswami, Banica and Skalski define and construct in [BS13] the quantum symmetry group of a C^* -algebra endowed with an orthogonal filtration. Their construction provides a general powerful tool to define and check the existence of quantum symmetry groups of various mathematical systems and unifies several known quantum symmetry groups constructions. The work of Banica and Skalski also has the merit to clearly exhibit some of the structures needed to enable one to prove the existence of a compact quantum symmetry group, see [BS13] for details. However, although Goswami's work was one of the inspirations for [BS13], it seems that Goswami's quantum isometry group in [Gos09] cannot, in general, be seen as a particular case of the quantum symmetry groups defined in [BS13] (because the subspace spanned by the eigenvalues of Goswami's Laplacian does not seem to form a subalgebra in general).

It is the purpose of the present paper to propose a construction that simultaneously generalizes the quantum symmetry groups of Goswami and of Banica-Skalski. We define and construct the quantum symmetry group of a Hilbert module endowed with an orthogonal filtration. The concept of Hilbert module endowed with an orthogonal filtration is inspired by Banica-Skalski's notion of C^* -algebra equipped with an orthogonal filtration, and is a natural generalization of it. Also, to an admissible spectral triple in the sense of [Gos09], one can associate an appropriate Hilbert module endowed with an orthogonal filtration, and our quantum symmetry group coincides with the quantum isometry group in [Gos09].

The concept also has the interest to provide an alternative approach to the quantum isometry group of a spectral triple. The main difference with the approach of [Gos09] is that, instead of extracting from the spectral triple an analogue of the Laplacian on functions (the so-called "noncommutative Laplacian") and making appropriate assumptions on its spectrum, we directly use the Dirac operator of the spectral triple, its spectrum and its natural domain. We then add assumptions to these data to get the desired orthogonally filtered Hilbert module. In the case of ordinary compact Riemannian manifolds, our exterior forms based quantum isometry group coincides with the one of Goswami in [Gos09].

The paper is organized as follows. In Section 2.1, we briefly recall some basic definitions about compact quantum groups. Then we introduce in Section 2.2 the concept of Hilbert module endowed with an orthogonal filtration, and define the category of "quantum transformation groups" for a Hilbert module equipped with an orthogonal filtration (our starting point being the notion of action of a compact quantum group on a Hilbert module given in [BS89]). Section 2.3 is devoted to the proof of the existence of a universal object in this category. In Section 2.4 we discuss some examples and compare our construction with the ones of Goswami and Banica-Skalski mentioned previously.

Notations and conventions

By *algebra* we will always mean *unital algebra*. So that algebra morphisms are assumed to preserve the units. The symbol \odot will denote the algebraic tensor product, while \otimes will denote tensor product of maps, spatial tensor product of C^* -algebras, or exterior tensor product of Hilbert modules.

2.1 Compact quantum groups

We recall here some basic definitions on compact quantum groups. See [Wor87a, Wor98, Wan95] and [Pod95] for more details.

Definition 2.1.1. A *Woronowicz C^* -algebra* is a couple (Q, Δ) , where Q is a C^* -algebra and $\Delta : Q \rightarrow Q \otimes Q$ is a $*$ -morphism such that:

- $(\Delta \otimes id_Q) \circ \Delta = (id_Q \otimes \Delta) \circ \Delta$,
- the spaces $\text{span}\{\Delta(Q).(Q \otimes 1_Q)\}$ and $\text{span}\{\Delta(Q).(1_Q \otimes Q)\}$ are both dense in $Q \otimes Q$.

Definition 2.1.2. Let (Q_0, Δ_0) and (Q_1, Δ_1) be Woronowicz C^* -algebras. A *morphism of Woronowicz C^* -algebras* from Q_0 to Q_1 is a $*$ -morphism:

$$\mu : Q_0 \rightarrow Q_1 \quad \text{such that} \quad (\mu \otimes \mu) \circ \Delta_0 = \Delta_1 \circ \mu.$$

The category of compact quantum groups is then defined to be the opposite category of the category of Woronowicz C^* -algebras.

Definitions 2.1.3. Let $Q = (Q, \Delta)$ be a Woronowicz C^* -algebra.

- A *Woronowicz C^* -ideal* of Q is a C^* -ideal I of Q such that $\Delta(I) \subset \text{Ker}(\pi \otimes \pi)$, where $\pi : Q \rightarrow Q/I$ is the canonical quotient map.
- A *Woronowicz C^* -subalgebra* of Q is a C^* -subalgebra Q' of Q such that $\Delta(Q') \subset Q' \otimes Q'$.

Definition 2.1.4. Let Q be a Woronowicz C^* -algebra. A matrix $(v_{ij})_{1 \leq i, j \leq n} \in \mathcal{M}_n(Q)$ is called *multiplicative* if we have $\Delta(v_{ij}) = \sum_{k=1}^n v_{ik} \otimes v_{kj}$ for all i, j .

The concept of an action of a quantum group on a C^* -algebra is formalized as follows.

Definition 2.1.5. Let Q be a Woronowicz C^* -algebra and let A be a C^* -algebra. A *coaction of Q on A* is a $*$ -morphism $\alpha : A \rightarrow A \otimes Q$ satisfying:

- $(\alpha \otimes id_Q) \circ \alpha = (id_A \otimes \Delta) \circ \alpha$,
- $\text{span}\{\alpha(A).(1 \otimes Q)\}$ is dense in $A \otimes Q$.

We say that a coaction α of Q on A is *faithful* if there exists no nontrivial Woronowicz C^* -subalgebra Q' of Q such that $\alpha(A) \subset A \otimes Q'$. Furthermore if τ is a continuous linear functional on A , we say that α preserves τ if $(\tau \otimes id_Q) \circ \alpha = \tau(\cdot)1_Q$.

2.2 Quantum groups actions on Hilbert modules

We recall now the definition of an action of a compact quantum group on a Hilbert module (see [Lan95] for background material on Hilbert modules). Then we introduce the notion of orthogonal filtration on a Hilbert module, give some natural examples of such objects, and define what we mean by *preserving the filtration* for an action of a compact quantum group on a Hilbert module endowed with an orthogonal filtration.

Definition 2.2.1. Let A be a C^* -algebra. A (right) pre-Hilbert A -module is a vector space E , equipped with a (right) A -module structure together with an A -valued inner product $\langle \cdot | \cdot \rangle_A$, that is to say:

- $\forall \xi, \eta, \zeta \in E, \forall a, b \in A, \langle \xi | \eta a + \zeta b \rangle_A = \langle \xi | \eta \rangle_A a + \langle \xi | \zeta \rangle_A b$,
- $\forall \xi, \eta \in E, \langle \xi | \eta \rangle_A^* = \langle \eta | \xi \rangle_A$,
- $\forall \xi \in E, \langle \xi | \xi \rangle_A \geq 0$ and if $\langle \xi | \xi \rangle_A = 0$ then $\xi = 0$.

We define a norm $\| \cdot \|_A$ on E by setting for $\xi \in E, \|\xi\|_A = \|\langle \xi | \xi \rangle_A\|_A^{\frac{1}{2}}$. If furthermore E is complete with respect to this norm, we say that E is a (right) Hilbert A -module.

We say that E is *full* if the space $\langle E | E \rangle_A = \text{span}\{\langle \xi | \eta \rangle_A ; \xi, \eta \in E\}$ is dense in A .

Left Hilbert A -modules are defined analogously, except that the A -valued inner product ${}_A \langle \cdot | \cdot \rangle$ has to be linear in the first variable and antilinear in the second one. In what follows we will mostly consider right Hilbert modules. Of course, the construction can be adapted for left Hilbert modules.

The notion of coaction on a Hilbert module is due to Baaj and Skandalis [BS89, Definition 2.2]. But working with Woronowicz C^* -algebras instead of Hopf C^* -algebras simplifies the original definition:

Definition 2.2.2. Let A be a C^* -algebra and let E be a Hilbert A -module. A coaction of a Woronowicz C^* -algebra Q on E consists of:

- a coaction $\alpha : A \rightarrow A \otimes Q$,
- a linear map $\beta : E \rightarrow E \otimes Q$ satisfying:
 1. $\text{span}\{\beta(E).(1 \otimes Q)\}$ is dense in $E \otimes Q$,
 2. $(\beta \otimes id_Q) \circ \beta = (id_E \otimes \Delta) \circ \beta$,
 3. $\forall \xi, \eta \in E, \langle \beta(\xi) | \beta(\eta) \rangle_{A \otimes Q} = \alpha(\langle \xi | \eta \rangle_A)$,
 4. $\forall \xi \in E, \forall a \in A, \beta(\xi.a) = \beta(\xi).\alpha(a)$.

We say that the coaction (α, β) of Q on E is faithful if there exists no nontrivial Woronowicz C^* -subalgebra Q' of Q such that $\beta(E) \subset E \otimes Q'$ (note that we do not require α to be faithful).

Remark 2.2.3. If (α, β) is a coaction of a Woronowicz C^* -algebra Q on a Hilbert A -module E , then $\beta : E \rightarrow E \otimes Q$ is necessarily continuous. Indeed:

$$\text{For all } \xi \in E, \|\beta(\xi)\|_{A \otimes Q}^2 = \|\langle \beta(\xi) | \beta(\xi) \rangle_{A \otimes Q}\| = \|\alpha(\langle \xi | \xi \rangle_A)\| \leq \|\langle \xi | \xi \rangle_A\| = \|\xi\|_A^2.$$

Also note that $\beta \otimes id_Q : E \otimes Q \rightarrow E \otimes Q \otimes Q$ extends continuously on $E \otimes Q$, so that the equality 2 of the previous definition makes sense. Indeed, we have for an element $x = \sum_i \xi_i \otimes a_i \in E \otimes Q$,

$$\begin{aligned} \|(\beta \otimes id_Q)(x)\|_{A \otimes Q \otimes Q}^2 &= \left\| \sum_{i,j} \langle \beta(\xi_i) \otimes a_i | \beta(\xi_j) \otimes a_j \rangle_{A \otimes Q \otimes Q} \right\| \\ &= \left\| \sum_{i,j} \langle \beta(\xi_i) | \beta(\xi_j) \rangle_{A \otimes Q} \otimes a_i^* a_j \right\| = \left\| \sum_{i,j} \alpha(\langle \xi_i | \xi_j \rangle_A) \otimes a_i^* a_j \right\| \\ &= \|(\alpha \otimes id_Q)(\langle x | x \rangle_{A \otimes Q})\| \leq \|\langle x | x \rangle_{A \otimes Q}\| = \|x\|_{A \otimes Q}^2. \end{aligned}$$

Definition 2.2.4. Let A be a C^* -algebra, let τ be a faithful state on A and let E be a Hilbert A -module. An *orthogonal filtration* $(\tau, (V_i)_{i \in \mathcal{I}}, J, W)$ of E consists of:

- a family $(V_i)_{i \in \mathcal{I}}$ of finite-dimensional subspaces of E such that:
 1. for all $i, j \in \mathcal{I}$ with $i \neq j$, $\forall \xi \in V_i$ and $\forall \eta \in V_j$, $\tau(\langle \xi | \eta \rangle_A) = 0$,
 2. the space $\mathcal{E}_0 = \sum_{i \in \mathcal{I}} V_i$ is dense in $(E, \|\cdot\|_A)$,
- a one-to-one antilinear operator $J : \mathcal{E}_0 \rightarrow \mathcal{E}_0$,
- a finite-dimensional subspace W of E .

Examples 2.2.5. (1) Let M be a compact Riemannian manifold. The space of continuous sections of the bundle of exterior forms on M , $\Gamma(\Lambda^* M)$, is a Hilbert $C(M)$ -module. We can equip it with an orthogonal filtration by taking $\tau = \int \cdot dvol$ (where $dvol$ denotes

the Riemannian density of M), $W = \mathbb{C} \cdot (m \mapsto 1_{\Lambda_m^* M})$, $J : \Gamma(\Lambda^* M) \rightarrow \Gamma(\Lambda^* M)$ the canonical involution and $(V_i)_{i \in \mathbb{N}}$ the family of eigenspaces of the de Rham operator $D = \overline{d} + d^*$.

- (2) We recall from [BS13] the definition of a C^* -algebra equipped with an orthogonal filtration:

Definition 2.2.6. Let A be a C^* -algebra, τ be a faithful state on A and $(V_i)_{i \in \mathcal{I}}$ be a family of finite-dimensional subspaces of A (with the index set \mathcal{I} containing a distinguished element 0). We say that $(\tau, (V_i)_{i \in \mathcal{I}})$ is an *orthogonal filtration* of A if:

- $V_0 = \mathbb{C} \cdot 1_A$,
- $\forall i, j \in \mathcal{I}$ such that $i \neq j$, $\forall a \in V_i$ and $\forall b \in V_j$, $\tau(a^* b) = 0$,
- the space $\mathcal{A}_0 = \sum_{i \in \mathcal{I}} V_i$ is a dense $*$ -subalgebra of A .

Setting $E = A$ (with its canonical Hilbert A -module structure), $W = \mathbb{C} \cdot 1_A$ and $J = a \mapsto a^*$, then $(\tau, (V_i)_{i \in \mathcal{I}}, J, W)$ is an orthogonal filtration of E .

- (3) Recall that a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is said to be *finitely summable* if there exists $p \in \mathbb{N}$ such that $|D|^{-p}$ is an infinitesimal operator of order 1. In that case, the smallest p satisfying the previous condition is called the *metric dimension* of $(\mathcal{A}, \mathcal{H}, D)$. (In other words, the metric dimension p is characterized by the fact that $|D|^{-p}$ admits a Dixmier trace, which is nonzero.) Let $(\mathcal{A}, \mathcal{H}, D)$ be an admissible spectral triple in the sense of [Gos09] (in particular, $(\mathcal{A}, \mathcal{H}, D)$ is finitely summable). We set:

(a) $E = A = \overline{\mathcal{A}}^{\mathcal{L}(\mathcal{H})}$,

(b) $\tau = \begin{cases} a \mapsto \frac{Tr_\omega(a|D|^{-p})}{Tr_\omega(|D|^{-p})} & \text{if } \mathcal{H} \text{ is infinite dimensional,} \\ \text{the usual trace} & \text{otherwise,} \end{cases}$

where Tr_ω denotes the Dixmier trace and p is the metric dimension of $(\mathcal{A}, \mathcal{H}, D)$,

(c) the $(V_i)_{i \in \mathbb{N}}$ are the eigenspaces of the ‘noncommutative Laplacian’,

(d) J is the involution of A and $W = \mathbb{C} \cdot 1_A$.

The couple $(\tau, (V_i)_{i \in \mathbb{N}})$ does not seem in general to be an orthogonal filtration of A in the sense of [BS13], since $\sum_{i \in \mathbb{N}} V_i$ does not appear to be a $*$ -subalgebra of A . However, $(\tau, (V_i)_{i \in \mathbb{N}}, J, W)$ is an orthogonal filtration of A , seen as a Hilbert A -module.

- (4) Let us recall some common conditions on spectral triples.

Definition 2.2.7. Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple.

- $(\mathcal{A}, \mathcal{H}, D)$ is said to be *regular* if for all $a \in \mathcal{A}$ and all $n \in \mathbb{N}$, a and $[D, a]$ are in the domain of the unbounded operator δ^n on $\mathcal{L}(\mathcal{H})$, where $\delta = [|D|, \cdot]$.
- We say that $(\mathcal{A}, \mathcal{H}, D)$ is *real* if it is equipped with an antiunitary operator $\mathcal{J} : \mathcal{H} \rightarrow \mathcal{H}$ such that:
 - (a) $\mathcal{J}(\text{Dom}(D)) \subset \text{Dom}(D)$,
 - (b) $\mathcal{J}^2 = \varepsilon$ and $\mathcal{J}D = \varepsilon' D \mathcal{J}$, where $\varepsilon, \varepsilon' \in \{-1, 1\}$,
 - (c) $\forall a, b \in \mathcal{A}, [a, \mathcal{J}b^* \mathcal{J}^*] = 0$.

- We say that $(\mathcal{A}, \mathcal{H}, D)$ *satisfies the finiteness and absolute continuity condition* if it is of finite metric dimension p , if the space $\mathcal{H}^\infty = \bigcap_{k \in \mathbb{N}} \text{Dom}(D^k)$ is a finitely generated projective left \mathcal{A} -module, and if there exists $q \in \mathcal{M}_n(\mathcal{A})$ with $q = q^2 = q^*$ such that:

- (a) $\mathcal{H}^\infty \cong \mathcal{A}^n q$,
- (b) the left \mathcal{A} -scalar product ${}_{\mathcal{A}}\langle \cdot | \cdot \rangle$ induced on \mathcal{H}^∞ by the previous isomorphism satisfies:

$$\frac{\text{Tr}_\omega({}_{\mathcal{A}}\langle \xi | \eta \rangle |D|^{-p})}{\text{Tr}_\omega(|D|^{-p})} = (\eta | \xi)_{\mathcal{H}}.$$

(Note that if $(\mathcal{A}, \mathcal{H}, D)$ is regular then \mathcal{H}^∞ is automatically a left \mathcal{A} -module.)

If $(\mathcal{A}, \mathcal{H}, D)$ satisfies the finiteness and absolute continuity condition it is natural to consider $A = \overline{\mathcal{A}}^{\mathcal{L}(\mathcal{H})}$ and the Hilbert A -module E obtained by completing \mathcal{H}^∞ (for the A -norm). The eigenspaces $(V_i)_{i \in \mathbb{N}}$ of D are two by two orthogonal in \mathcal{H} , thus $\forall \xi \in V_i, \forall \eta \in V_j$ such that $i \neq j$, we get $\tau({}_{\mathcal{A}}\langle \xi | \eta \rangle) = 0$ where $\tau = a \mapsto \frac{\text{Tr}_\omega(a|D|^{-p})}{\text{Tr}_\omega(|D|^{-p})}$. If τ is faithful and \mathcal{E}_0 is dense in E , then E can be equipped with an orthogonal filtration (with $J : \mathcal{E}_0 \rightarrow \mathcal{E}_0$ any one-to-one antilinear map and e.g. $W = (0)$).

If we assume furthermore that $(\mathcal{A}, \mathcal{H}, D)$ is real, then a natural choice is to set $J = \mathcal{J}|_{\mathcal{E}_0}$.

Notation. Let A be a C^* -algebra and let E be a Hilbert A -module endowed with an orthogonal filtration $(\tau, (V_i)_{i \in \mathbb{I}}, J, W)$. We define on E a scalar product by:

$$\forall \xi, \eta \in E, (\xi | \eta)_\tau = \tau(\langle \xi | \eta \rangle_A).$$

We denote by \mathcal{H} the completion of E with respect to this scalar product and by $\|\cdot\|_\tau$ the norm associated with it. Remark that the V_i 's are pairwise orthogonal in \mathcal{H} , and since

$\|\xi\|_\tau^2 = \tau(\langle \xi | \xi \rangle_A) \leq \|\langle \xi | \xi \rangle_A\| = \|\xi\|_A^2$ for all $\xi \in E$, we have a continuous injection $E \hookrightarrow \mathcal{H}$ with dense image.

We will define now the coactions that preserve the structure of a given Hilbert module equipped with an orthogonal filtration. This will allow us to describe the category of its “quantum transformation groups”.

Definition 2.2.8. Let A be a C^* -algebra and let E be a Hilbert A -module endowed with an orthogonal filtration $(\tau, (V_i)_{i \in \mathcal{I}}, J, W)$. A *filtration-preserving coaction* of a Woronowicz C^* -algebra Q on E is a coaction (α, β) of Q on E satisfying:

- $(\tau \otimes id_Q) \circ \alpha = \tau(\cdot)1_Q$,
- $\forall i \in \mathcal{I}, \beta(V_i) \subset V_i \odot Q$,
- $(J \otimes *) \circ \beta = \beta \circ J$ on \mathcal{E}_0 , where $*$ denotes the involution of Q ,
- $\forall \xi \in W, \beta(\xi) = \xi \otimes 1_Q$.

In that case, we will also say that Q coacts on E in a *filtration-preserving way*.

Remark 2.2.9. The compatibility condition between β and J allows to see that the left scalar product defined by ${}_\tau(\xi | \eta) = \tau(\langle J(\xi) | J(\eta) \rangle_A)$ for $\xi, \eta \in \mathcal{E}_0$ is compatible with the coaction, which will be fundamental in the construction of the next section. In contrast, the fact that β leaves W invariant is not necessary. We make this assumption because in some cases, it seems natural to ask that β leaves invariant some elements of E (e.g. in the case of Example 2.2.5.(1)) and in other cases, setting $W = (0)$ allows to drop this extra assumption.

Definition 2.2.10. Let A be a C^* -algebra and let E be a Hilbert A -module equipped with an orthogonal filtration $(\tau, (V_i)_{i \in \mathcal{I}}, J, W)$. We will denote by $\mathcal{C}(E, \tau, (V_i)_{i \in \mathcal{I}}, J, W)$ the category of Woronowicz C^* -algebras coacting on E in a filtration-preserving way. If (α_0, β_0) and (α_1, β_1) are filtration-preserving coactions of Woronowicz C^* -algebras Q_0 and Q_1 on E , then a morphism from Q_0 to Q_1 in that category is a morphism of Woronowicz C^* -algebras $\mu : Q_0 \rightarrow Q_1$ satisfying:

$$\alpha_1 = (id_A \otimes \mu) \circ \alpha_0 \quad \text{and} \quad \beta_1 = (id_E \otimes \mu) \circ \beta_0.$$

Remark 2.2.11. If E is full and $\mu : Q_0 \rightarrow Q_1$ is a morphism of Woronowicz C^* -algebras satisfying $\beta_1 = (id_E \otimes \mu) \circ \beta_0$, then μ automatically satisfies $\alpha_1 = (id_A \otimes \mu) \circ \alpha_0$.

Indeed, we have for all $\xi, \eta \in E$:

$$\alpha_1(\langle \xi | \eta \rangle_A) = \langle \beta_1(\xi) | \beta_1(\eta) \rangle_{A \otimes Q_1} = \langle (id_E \otimes \mu) \circ \beta_0(\xi) | (id_E \otimes \mu) \circ \beta_0(\eta) \rangle_{A \otimes Q_1}$$

$$= (id_A \otimes \mu)(\langle \beta_0(\xi) | \beta_0(\eta) \rangle_{A \otimes Q_0}) = (id_A \otimes \mu) \circ \alpha_0(\langle \xi | \eta \rangle_A).$$

And since E is full, we get $\alpha_1 = (id_A \otimes \mu) \circ \alpha_0$.

Remark 2.2.12. When $E = \Gamma(\Lambda^*M)$ is equipped with the orthogonal filtration $(\tau, (V_i)_{i \in \mathbb{N}}, J, W)$ described in Example 2.2.5.(1), the full subcategory of $\mathcal{C}(E, \tau, (V_i)_{i \in \mathbb{N}}, J, W)$ consisting of the commutative Woronowicz C^* -algebras coacting on $\Gamma(\Lambda^*M)$ in a filtration-preserving way is antiequivalent to the category of compact groups acting isometrically on M (see section 2.4.2 for more details). This explains our choice of seeing the opposite category of $\mathcal{C}(E, \tau, (V_i)_{i \in \mathbb{I}}, J, W)$ as the category of quantum transformation groups of E . Moreover since the isometry group of M is a universal object in the category of compact groups acting isometrically on M , we will define the quantum symmetry group of E as a universal object in $\mathcal{C}(E, \tau, (V_i)_{i \in \mathbb{I}}, J, W)$. Proving the existence of such a universal object is the aim of the next section.

Remark 2.2.13. We have not been able to adapt the algebraic viewpoint of [BS13] in our context. A reason for this is that it seems difficult to give an “algebraic version” of our category of transformation groups, without making the extra assumption that $\mathcal{E}_0 \cdot \langle \mathcal{E}_0, \mathcal{E}_0 \rangle_A \subset \mathcal{E}_0$ (in which case $\langle \mathcal{E}_0, \mathcal{E}_0 \rangle_A$ would be an algebra, and the coactions on E could restrict to “algebraic coactions” on the $\langle \mathcal{E}_0, \mathcal{E}_0 \rangle_A$ -module \mathcal{E}_0).

2.3 Construction of the quantum symmetry group of a Hilbert module equipped with an orthogonal filtration

The following theorem generalizes the results of Goswami [Gos09] and Banica-Skalski [BS13].

Theorem 2.3.1. *Let A be a C^* -algebra and let E be a full Hilbert A -module endowed with an orthogonal filtration $(\tau, (V_i)_{i \in \mathbb{I}}, J, W)$. The category $\mathcal{C}(E, \tau, (V_i)_{i \in \mathbb{I}}, J, W)$ admits an initial object, which means that there exists a universal Woronowicz C^* -algebra coacting on E in a filtration-preserving way. The quantum group corresponding to that universal object will be called the **quantum symmetry group** of $(E, \tau, (V_i)_{i \in \mathbb{I}}, J, W)$.*

Examples will be discussed in the next section. This section is devoted to the proof of Theorem 2.3.1. The proof mostly consists in carefully adapting Goswami’s arguments in [Gos09, Section 4]. In what follows E denotes a full Hilbert module over a given C^* -algebra A , equipped with an orthogonal filtration $(\tau, (V_i)_{i \in \mathbb{I}}, J, W)$.

Lemma 2.3.2. *Let (α, β) be a filtration-preserving coaction of a Woronowicz C^* -algebra Q on E . The Q -linear map $\bar{\beta} : \mathcal{E}_0 \otimes Q \rightarrow \mathcal{E}_0 \otimes Q$ given by $\bar{\beta}(\xi \otimes x) = \beta(\xi)(1 \otimes x)$ extends to a unitary of the Hilbert Q -module $\mathcal{H} \otimes Q$.*

PROOF. We have for $\xi, \eta \in \mathcal{E}_0$ and $x, y \in Q$:

$$\begin{aligned} \langle \bar{\beta}(\xi \otimes x) | \bar{\beta}(\eta \otimes y) \rangle_Q &= x^* \langle \beta(\xi) | \beta(\eta) \rangle_Q y \\ &= x^* (\tau \otimes id) (\langle \beta(\xi) | \beta(\eta) \rangle_{A \otimes Q}) y \quad (\text{by definition of } (\cdot | \cdot)_\tau = \tau(\langle \cdot | \cdot \rangle_A)) \\ &= x^* (\tau \otimes id) \circ \alpha (\langle \xi | \eta \rangle_A) y \\ &= x^* \tau (\langle \xi | \eta \rangle_A) y \quad (\text{since } (\tau \otimes id) \circ \alpha = \tau(\cdot) 1_Q) \\ &= (\xi | \eta)_\tau x^* y = \langle \xi \otimes x | \eta \otimes y \rangle_Q. \end{aligned}$$

In particular $\bar{\beta}$ is isometric and thus extends to a Q -linear isometric operator still denoted by $\bar{\beta} : \mathcal{H} \otimes Q \rightarrow \mathcal{H} \otimes Q$. To show that $\bar{\beta}$ is unitary, it is enough to check that $\bar{\beta}$ has dense image (see [Lan95, Theorem 3.5]). Since $\text{span}\{\beta(E).(1 \otimes Q)\}$ is dense in $E \otimes Q$ and \mathcal{E}_0 is dense in E , it follows that $\text{span}\{\beta(\mathcal{E}_0).(1 \otimes Q)\}$ is dense in $E \otimes Q$. Moreover the canonical injection $E \otimes Q \hookrightarrow \mathcal{H} \otimes Q$ has dense image, so that $\text{span}\{\beta(\mathcal{E}_0).(1 \otimes Q)\}$ is also dense in $\mathcal{H} \otimes Q$. \square

Notation. We define on \mathcal{E}_0 a left scalar product by:

$${}_\tau(\xi | \eta) = \tau(\langle J(\xi) | J(\eta) \rangle_A).$$

For each $i \in \mathcal{I}$ we set $d_i = \dim(V_i)$ and we fix:

- an orthonormal basis $(e_{ij})_{1 \leq j \leq d_i}$ of V_i for the right scalar product $(\cdot | \cdot)_\tau$,
- an orthonormal basis $(f_{ij})_{1 \leq j \leq d_i}$ of V_i for the left scalar product ${}_\tau(\cdot | \cdot)$.

We denote by $p^{(i)} \in GL_{d_i}(\mathbb{C})$ the change of basis matrix from (f_{ij}) to the basis (e_{ij}) of V_i and we set $s^{(i)} = p^{(i)t} \overline{p^{(i)}}$.

Lemma 2.3.3. *Let (α, β) be a filtration-preserving coaction of a Woronowicz C^* -algebra Q on E . For all $i \in \mathcal{I}$, we denote by $v^{(i)}$ the multiplicative matrix associated with the basis $(e_{ij})_{1 \leq j \leq d_i}$ of the Q -comodule V_i (in other words, $v^{(i)}$ is characterized by: $\forall j, \beta(e_{ij}) = \sum_{k=1}^{d_i} e_{ik} \otimes v_{kj}^{(i)}$).*

- For all $i \in \mathcal{I}$, the matrix $v^{(i)} = (v_{kj}^{(i)})_{1 \leq k, j \leq d_i}$ is unitary and

$$v^{(i)t} s^{(i)} \overline{v^{(i)}} (s^{(i)})^{-1} = s^{(i)} \overline{v^{(i)}} (s^{(i)})^{-1} v^{(i)t} = I_{d_i}$$

- The unital C^* -subalgebra Q' of Q generated by $\{v_{kj}^{(i)} ; i \in \mathcal{I}, j, k \in \{1, \dots, d_i\}\}$ is a Woronowicz C^* -subalgebra of Q satisfying $\alpha(A) \subset A \otimes Q'$ and $\beta(E) \subset E \otimes Q'$. Furthermore (α, β) is a faithful filtration-preserving coaction of Q' on E .

PROOF. First let us check that the $v^{(i)}$'s are unitary matrices.

Consider the unitary $\bar{\beta} : \mathcal{H} \otimes Q \rightarrow \mathcal{H} \otimes Q$ of the Hilbert Q -module $\mathcal{H} \otimes Q$ constructed in the previous lemma. For all i, j, m, n , we have

$$\langle e_{ij} \otimes 1 | \bar{\beta}(e_{mn} \otimes 1) \rangle_Q = \delta_{im} v_{jn}^{(i)} = \sum_{k=1}^{d_i} \langle e_{ik} \otimes v_{jk}^{(i)*} | e_{mn} \otimes 1 \rangle_Q.$$

Thus for all i, j , we have $\bar{\beta}^*(e_{ij} \otimes 1) = \sum_{k=1}^{d_i} e_{ik} \otimes v_{jk}^{(i)*}$. Then we get:

$$e_{ij} \otimes 1 = \bar{\beta} \circ \bar{\beta}^*(e_{ij} \otimes 1) = \sum_{k=1}^{d_i} \beta(e_{ik})(1 \otimes v_{jk}^{(i)*}) = \sum_{k,l=1}^{d_i} e_{il} \otimes v_{lk}^{(i)} v_{jk}^{(i)*},$$

which shows that for all $l, j \in \{1, \dots, d_i\}$, $\sum_{k=1}^{d_i} v_{lk}^{(i)} v_{jk}^{(i)*} = \delta_{lj}$, i.e. $v^{(i)} v^{(i)*} = I_{d_i}$.

Similarly (using $e_{ij} \otimes 1 = \bar{\beta}^* \circ \bar{\beta}(e_{ij} \otimes 1)$) we get $v^{(i)*} v^{(i)} = I_{d_i}$. Thus the matrices $v^{(i)}$ are unitary.

★ Let us show now that $v^{(i)t} s^{(i)} \overline{v^{(i)}} (s^{(i)})^{-1} = s^{(i)} \overline{v^{(i)}} (s^{(i)})^{-1} v^{(i)t} = I_{d_i}$.

For $i \in \mathcal{I}$, since $v^{(i)}$ is a multiplicative and unitary matrix in a Woronowicz C^* -algebra, the matrix $v^{(i)t}$ is invertible in $\mathcal{M}_{d_i}(Q)$ (cf. [MVD98]), so it is enough to prove that $v^{(i)t} s^{(i)} \overline{v^{(i)}} (s^{(i)})^{-1} = I_{d_i}$.

Using Sweedler's notations, we get for $\xi, \eta \in \mathcal{E}_0$:

$$\begin{aligned} {}_Q \langle \beta(\xi) | \beta(\eta) \rangle &= \sum \tau \left(\langle J(\xi_{(0)}) | J(\eta_{(0)}) \rangle_A \right) \xi_{(1)} \eta_{(1)}^* \\ &= \sum (\tau \otimes id) \left(\langle J(\xi_{(0)}) \otimes \xi_{(1)}^* | J(\eta_{(0)}) \otimes \eta_{(1)}^* \rangle_{A \otimes Q} \right) \\ &= (\tau \otimes id) \left(\langle (J \otimes *) \circ \beta(\xi) | (J \otimes *) \circ \beta(\eta) \rangle_{A \otimes Q} \right) \\ &= (\tau \otimes id) \left(\langle \beta \circ J(\xi) | \beta \circ J(\eta) \rangle_{A \otimes Q} \right) \\ &= (\tau \otimes id) \circ \alpha \left(\langle J(\xi) | J(\eta) \rangle_A \right) = \tau \left(\langle J(\xi) | J(\eta) \rangle_A \right) 1_Q = \tau(\xi | \eta) 1_Q. \end{aligned}$$

Moreover, since $\beta(V_i) \subset V_i \odot Q$, there exists $w^{(i)} \in \mathcal{M}_{d_i}(Q)$ such that

$$\forall j, \beta(f_{ij}) = \sum_{k=1}^{d_i} f_{ik} \otimes w_{kj}^{(i)}.$$

Then we get ${}_Q\langle\beta(f_{ij})|\beta(f_{ik})\rangle = \sum_{l,m=1}^{d_i} \tau(f_{il}|f_{im})w_{lj}^{(i)}w_{mk}^{(i)*} = \sum_{l=1}^{d_i} w_{lj}^{(i)}w_{lk}^{(i)*} = \tau(f_{ij}|f_{ik})1_Q = \delta_{jk}$.

This shows:

$$w^{(i)t}\overline{w^{(i)}} = I_{d_i}. \quad (2.1)$$

Furthermore we have for all j , $e_{ij} = \sum_{k=1}^{d_i} p_{kj}^{(i)}f_{ik}$, thus $w^{(i)} = p^{(i)}v^{(i)}(p^{(i)})^{-1}$. Then replacing $w^{(i)}$ in the equality (2.1) we get $(p^{(i)t})^{-1}v^{(i)t}p^{(i)t}\overline{p^{(i)}v^{(i)}(p^{(i)})^{-1}} = I_{d_i}$, which shows that $v^{(i)t}s^{(i)}\overline{v^{(i)}}(s^{(i)})^{-1} = I_{d_i}$, where $s^{(i)} = p^{(i)t}\overline{p^{(i)}}$.

★ It then follows easily that Q' is a Woronowicz C^* -subalgebra of Q satisfying $\alpha(A) \subset A \otimes Q'$ and $\beta(E) \subset E \otimes Q'$.

Indeed, since the $v^{(i)}$'s are multiplicative matrices, we have $\Delta(Q') \subset Q' \otimes Q'$ so that Q' is a Woronowicz C^* -subalgebra of Q . Moreover $\beta(\mathcal{E}_0) \subset \mathcal{E}_0 \otimes Q'$, thus $\beta(E) \subset E \otimes Q'$. Then for all $\xi, \eta \in \mathcal{E}_0$, $\alpha(\langle\xi|\eta\rangle_A) = \langle\beta(\xi)|\beta(\eta)\rangle_{A \otimes Q'} \subset \langle E \otimes Q' | E \otimes Q' \rangle_{A \otimes Q'} \subset A \otimes Q'$. This shows $\alpha(A) \subset A \otimes Q'$ since $\langle \mathcal{E}_0 | \mathcal{E}_0 \rangle_A$ is dense in A .

★ It remains to check that (α, β) is a faithful filtration-preserving coaction of Q' on E .

We only show that $\text{span}\{\alpha(A).(1 \otimes Q')\}$ and $\text{span}\{\beta(E).(1 \otimes Q')\}$ are respectively dense in $A \otimes Q'$ and $E \otimes Q'$ (the other conditions that (α, β) must satisfy to be a filtration-preserving coaction of Q' on E directly follow from the fact that it is a filtration-preserving coaction of Q on E).

We have for all i, j :

$$\begin{aligned} \sum_{k=1}^{d_i} \beta(e_{ik})(1 \otimes v_{jk}^{(i)*}) &= \sum_{k,l=1}^{d_i} (e_{il} \otimes v_{lk}^{(i)})(1 \otimes v_{jk}^{(i)*}) = \sum_{k,l=1}^{d_i} e_{il} \otimes (v_{lk}^{(i)}v_{jk}^{(i)*}) \\ &= \sum_{l=1}^{d_i} e_{il} \otimes \delta_{lj} \quad (\text{since } v^{(i)}v^{(i)*} = I_{d_i}) \\ &= e_{ij} \otimes 1. \end{aligned}$$

This implies that $\text{span}\{\beta(E).(1 \otimes Q')\}$ is dense in $E \otimes Q'$. Moreover, we have seen that for all $i \in \mathcal{I}$, $v^{(i)}$ is invertible in $\mathcal{M}_{d_i}(Q')$ with inverse $x^{(i)} = (s^{(i)})^{-1}v^{(i)t}s^{(i)}$.

Let $i, j \in \mathcal{I}$, $m \in \{1, \dots, d_i\}$ and $n \in \{1, \dots, d_j\}$:

$$\begin{aligned} \sum_{k=1}^{d_i} \sum_{l=1}^{d_j} \alpha(\langle e_{ik}|e_{jl}\rangle_A) \cdot (1 \otimes v_{nl}^{(j)*}x_{km}^{(i)}) &= \sum_{k=1}^{d_i} \sum_{l=1}^{d_j} \langle\beta(e_{ik})|\beta(e_{jl})\rangle_{A \otimes Q'} \cdot (1 \otimes v_{nl}^{(j)*}x_{km}^{(i)}) \\ &= \sum_{k=1}^{d_i} \langle\beta(e_{ik})| \underbrace{\sum_{l=1}^{d_j} \beta(e_{jl}).(1 \otimes v_{nl}^{(j)*})}_{e_{jn} \otimes 1} \rangle_{A \otimes Q'} \cdot (1 \otimes x_{km}^{(i)}) \end{aligned}$$

$$\begin{aligned}
 &= \sum_{k,l=1}^{d_i} \langle e_{il} \otimes v_{lk}^{(i)} | e_{jn} \otimes 1 \rangle_{A \otimes Q} \cdot (1 \otimes x_{km}^{(i)}) \\
 &= \sum_{k,l=1}^{d_i} \langle e_{il} | e_{jn} \rangle_A \otimes (v_{lk}^{(i)*} x_{km}^{(i)}) \\
 &= \sum_{l=1}^{d_i} \langle e_{il} | e_{jn} \rangle_A \otimes \delta_{lm} \quad (\text{since } \overline{v^{(i)}} x^{(i)} = I_{d_i}) \\
 &= \langle e_{im} | e_{jn} \rangle_A \otimes 1.
 \end{aligned}$$

Thus for all i, j, m, n , $\langle e_{im} | e_{jn} \rangle_A \otimes 1$ is in $\text{span}\{\alpha(A) \cdot (1 \otimes Q')\}$. By density of $\langle \mathcal{E}_0 | \mathcal{E}_0 \rangle_A$ in A , this shows that $\text{span}\{\alpha(A) \cdot (1 \otimes Q')\}$ is dense in $A \otimes Q'$. \square

Notation. For all $i \in \mathcal{I}$, we consider $\mathcal{A}_u(s^{(i)})$ the universal Woronowicz C^* -algebra of Van Daele and Wang (see [VDW96]) associated with $s^{(i)}$. That is, $\mathcal{A}_u(s^{(i)})$ is the universal Woronowicz C^* -algebra generated by a multiplicative and unitary matrix $u^{(i)} = (u_{kj}^{(i)})_{1 \leq k, j \leq d_i}$, satisfying the following relations:

$$u^{(i)t} s^{(i)} \overline{u^{(i)}} (s^{(i)})^{-1} = s^{(i)} \overline{u^{(i)}} (s^{(i)})^{-1} u^{(i)t} = I_{d_i}.$$

We set $\mathcal{U} = \bigstar_{i \in \mathcal{I}} \mathcal{A}_u(s^{(i)})$ and $\beta_u : \mathcal{E}_0 \rightarrow \mathcal{E}_0 \odot \mathcal{U}$ the linear map given by:

$$\beta_u(e_{ij}) = \sum_{k=1}^{d_i} e_{ik} \otimes u_{kj}^{(i)}.$$

See [Wan95] for the construction of free product of compact quantum groups.

In the following, if (Q, Δ) is a Woronowicz C^* -algebra and I is a Woronowicz C^* -ideal of Q , we will denote by $\pi_I : Q \rightarrow Q/I$ the canonical projection and by Δ_I the canonical coproduct of Q/I (i.e. $\Delta_I : Q/I \rightarrow Q/I \otimes Q/I$ is the unique $*$ -morphism satisfying $\Delta_I \circ \pi_I = (\pi_I \otimes \pi_I) \circ \Delta$).

Lemma 2.3.4. *Let (α, β) be a faithful filtration-preserving coaction of a Woronowicz C^* -algebra Q on E . There exists a Woronowicz C^* -ideal $I \subset \mathcal{U}$ and a faithful filtration-preserving coaction (α_I, β_I) of \mathcal{U}/I on E such that:*

- \mathcal{U}/I and Q are isomorphic in $\mathcal{C}(E, \tau, (V_i)_{i \in \mathcal{I}}, J, W)$,
- β_I extends $(id \otimes \pi_I) \circ \beta_u$.

PROOF. For all $i \in \mathcal{I}$, we denote by $v^{(i)}$ the multiplicative matrix associated with the basis $(e_{ij})_{1 \leq j \leq d_i}$ of the Q -comodule V_i . In virtue of Lemma 2.3.3, we know that $v^{(i)}$ is unitary

and satisfies $v^{(i)t} s^{(i)} \overline{v^{(i)}} (s^{(i)})^{-1} = s^{(i)} \overline{v^{(i)}} (s^{(i)})^{-1} v^{(i)t} = I_{d_i}$. So by universal property of \mathcal{U} there exists a morphism of Woronowicz C^* -algebras $\mu : \mathcal{U} \rightarrow Q$ such that for all i, p, q , $\mu(u_{pq}^{(i)}) = v_{pq}^{(i)}$. Then $\text{Im } \mu$ is a Woronowicz C^* -subalgebra of Q and for all i, j :

$$\beta(e_{ij}) = \sum_{k=1}^{d_i} e_{ik} \otimes v_{kj}^{(i)} = (id \otimes \mu) \circ \beta_u(e_{ij}). \quad (2.2)$$

Thus the inclusion $\beta(E) \subset E \otimes (\text{Im } \mu)$ holds, so that μ is necessarily onto (since the coaction is faithful). We set $I = \text{Ker } \mu$, we denote by $\widehat{\mu} : \mathcal{U}/I \rightarrow Q$ the isomorphism of Woronowicz C^* -algebras such that $\widehat{\mu} \circ \pi_I = \mu$ and we set $\alpha_I = (id \otimes \widehat{\mu}^{-1}) \circ \alpha$ and $\beta_I = (id \otimes \widehat{\mu}^{-1}) \circ \beta$. It is then easy to see that (α_I, β_I) is a filtration-preserving coaction, and that:

$$\widehat{\mu} : (\mathcal{U}/I, \Delta_I, \alpha_I, \beta_I) \rightarrow (Q, \Delta, \alpha, \beta) \text{ is an isomorphism.}$$

Thanks to (2.2), we see that $\beta_I = (id \otimes \widehat{\mu}^{-1}) \circ \beta$ extends $(id \otimes \pi_I) \circ \beta_u$. \square

Before proving Theorem 2.3.1 we need a last lemma.

Lemma 2.3.5. *Let A and B be C^* -algebras and let \mathcal{I} be a nonempty family of C^* -ideals of B . Set $I_0 = \bigcap_{I \in \mathcal{I}} I$, and for $I \in \mathcal{I}$, set $p_I : B/I_0 \rightarrow B/I$ the unique $*$ -morphism such that:*

$$\begin{array}{ccc} & B & \\ \pi_0 \swarrow & & \searrow \pi_I \\ B/I_0 & \xrightarrow{p_I} & B/I \end{array}$$

where π_0 and π_I denote the canonical projections. Then we have for all $x \in A \otimes (B/I_0)$,

$$\|x\| = \sup_{I \in \mathcal{I}} \|id \otimes p_I(x)\|.$$

PROOF. Let $\rho_A : A \hookrightarrow \mathcal{L}(H)$ be a faithful representation of A on a Hilbert space H , and for $I \in \mathcal{I}$, let $\rho_I : B/I \hookrightarrow \mathcal{L}(K_I)$ be a faithful representation of B/I on K_I . The ρ_I 's induce a faithful representation of B/I_0 on $K = \bigoplus_{I \in \mathcal{I}} K_I$:

$$\begin{aligned} \rho_0 : B/I_0 &\rightarrow \mathcal{L}(K) \\ x &\mapsto \bigoplus_{I \in \mathcal{I}} \rho_I \circ p_I(x), \end{aligned}$$

By definition of the spatial norms, we have for $x \in A \otimes B/I_0$, $\|x\| = \|\rho_A \otimes \rho_0(x)\|$, and for

$y \in A \otimes B/I$, $\|y\| = \|\rho_A \otimes \rho_I(y)\|$. Furthermore, we have canonical injections:

$$\bigoplus_{I \in \mathcal{I}} \mathcal{L}(H \otimes K_I) \hookrightarrow \mathcal{L}\left(\bigoplus_{I \in \mathcal{I}} H \otimes K_I\right) \cong \mathcal{L}(H \otimes K),$$

and since injective morphisms of C^* -algebras are isometric, we have for $x \in A \otimes B/I_0$:

$$\|\rho_A \otimes \rho_0(x)\| = \sup_{I \in \mathcal{I}} \|(\rho_A \otimes (\rho_I \circ p_I))(x)\|,$$

so that $\|x\| = \sup_{I \in \mathcal{I}} \|(id \otimes p_I)(x)\|$. □

We are now ready to prove Theorem 2.3.1.

PROOF OF THEOREM 2.3.1. We denote by \mathcal{I} the set of all C^* -ideals $I \subset \mathcal{U}$ such that:

$(id \otimes \pi_I) \circ \beta_u$ extends to a continuous linear map $\beta_I : E \rightarrow E \otimes \mathcal{U}/I$ such that there exists a $*$ -morphism $\alpha_I : A \rightarrow A \otimes \mathcal{U}/I$ preserving τ and satisfying:

- $\forall \xi, \eta \in E, \langle \beta_I(\xi) | \beta_I(\eta) \rangle_{A \otimes \mathcal{U}/I} = \alpha_I(\langle \xi | \eta \rangle_A)$,
- $\forall \xi \in E, \forall a \in A, \beta_I(\xi \cdot a) = \beta_I(\xi) \cdot \alpha_I(a)$,
- $(J \otimes *) \circ \beta_I = \beta_I \circ J$ on \mathcal{E}_0 ,
- $\forall \xi \in W, \beta_I(\xi) = \xi \otimes 1$.

The set \mathcal{I} is nonempty, since it contains the kernel of the counit $\varepsilon : \mathcal{U} \rightarrow \mathbb{C}$ (this can be directly checked, or seen by applying Lemma 2.3.4 to the trivial coaction $A \rightarrow A \otimes \mathbb{C}$, $E \rightarrow E \otimes \mathbb{C}$).

We denote by I_0 the intersection of all elements of \mathcal{I} , by $Q_0 = \mathcal{U}/I_0$ and by $\pi_0 : \mathcal{U} \rightarrow Q_0$ the canonical projection (as intersection of C^* -ideals, I_0 is a C^* -ideal, so π_0 is a $*$ -morphism). Let us show that $I_0 \in \mathcal{I}$.

★ First let us check that $(id \otimes \pi_0) \circ \beta_u$ extends to a continuous linear map $\beta_0 : E \rightarrow E \otimes Q_0$. Note that for $x \in E \otimes Q_0$, $\|x\|_{A \otimes Q_0} = \sup_{I \in \mathcal{I}} \|id_E \otimes p_I(x)\|_{A \otimes \mathcal{U}/I}$. Indeed:

$$\begin{aligned} \|x\|_{A \otimes Q_0}^2 &= \|\langle x | x \rangle_{A \otimes Q_0}\| = \sup_{I \in \mathcal{I}} \|(id_A \otimes p_I)(\langle x | x \rangle_{A \otimes Q_0})\| \quad (\text{by the previous lemma}) \\ &= \sup_{I \in \mathcal{I}} \|\langle id_E \otimes p_I(x) | id_E \otimes p_I(x) \rangle_{A \otimes \mathcal{U}/I}\| = \sup_{I \in \mathcal{I}} \|id_E \otimes p_I(x)\|_{A \otimes \mathcal{U}/I}^2. \end{aligned}$$

Furthermore we have for all $\xi \in \mathcal{E}_0$ and all $I \in \mathcal{I}$:

$$\|(id_E \otimes \pi_I) \circ \beta_u(\xi)\|_{A \otimes \mathcal{U}/I}^2 = \|\langle \beta_I(\xi) | \beta_I(\xi) \rangle_{A \otimes \mathcal{U}/I}\| = \|\alpha_I(\langle \xi | \xi \rangle_A)\| \leq \|\langle \xi | \xi \rangle_A\| = \|\xi\|_A^2$$

since α_I is a $*$ -morphism. Hence for all $\xi \in \mathcal{E}_0$:

$$\begin{aligned} \|(id_E \otimes \pi_0) \circ \beta_u(\xi)\|_{A \otimes Q_0} &= \sup_{I \in \mathcal{I}} \|(id_E \otimes (p_I \circ \pi_0)) \circ \beta_u(\xi)\|_{A \otimes \mathcal{U}/I} \\ &= \sup_{I \in \mathcal{I}} \|(id_E \otimes \pi_I) \circ \beta_u(\xi)\|_{A \otimes \mathcal{U}/I} \leq \|\xi\|_A, \end{aligned}$$

which shows that $(id \otimes \pi_0) \circ \beta_u$ extends to a continuous linear map $\beta_0 : E \rightarrow E \otimes Q_0$.

★ Next let us show that there exists a linear map $\alpha_0 : \langle \mathcal{E}_0 | \mathcal{E}_0 \rangle_A \rightarrow A \otimes Q_0$ such that

$$\forall \xi, \eta \in \mathcal{E}_0, \alpha_0(\langle \xi | \eta \rangle_A) = (id \otimes \pi_0) (\langle \beta_u(\xi) | \beta_u(\eta) \rangle_{A \otimes \mathcal{U}}).$$

Let ξ_1, \dots, ξ_n and η_1, \dots, η_n be elements of \mathcal{E}_0 such that $\sum_{i=1}^n \langle \xi_i | \eta_i \rangle_A = 0$.

Then for all $I \in \mathcal{I}$:

$$\begin{aligned} \sum_{i=1}^n (id \otimes \pi_I) (\langle \beta_u(\xi_i) | \beta_u(\eta_i) \rangle_{A \otimes \mathcal{U}}) &= \sum_{i=1}^n \langle \beta_I(\xi_i) | \beta_I(\eta_i) \rangle_{A \otimes \mathcal{U}/I} \\ &= \sum_{i=1}^n \alpha_I(\langle \xi_i | \eta_i \rangle_A) = \alpha_I(0) = 0. \end{aligned}$$

Thus we have:

$$\left\| \sum_{i=1}^n (id \otimes \pi_0) (\langle \beta_u(\xi_i) | \beta_u(\eta_i) \rangle_{A \otimes \mathcal{U}}) \right\| = \sup_{I \in \mathcal{I}} \left\| \sum_{i=1}^n (id \otimes \pi_I) (\langle \beta_u(\xi_i) | \beta_u(\eta_i) \rangle_{A \otimes \mathcal{U}}) \right\| = 0.$$

This shows that we can define a linear map $\alpha_0 : \langle \mathcal{E}_0 | \mathcal{E}_0 \rangle_A \rightarrow A \otimes Q_0$ by the formula

$$\alpha_0 \left(\sum_{i=1}^n \langle \xi_i | \eta_i \rangle_A \right) = \sum_{i=1}^n (id \otimes \pi_0) (\langle \beta_u(\xi_i) | \beta_u(\eta_i) \rangle_{A \otimes \mathcal{U}}).$$

★ Let us check that $\alpha_0 : \langle \mathcal{E}_0 | \mathcal{E}_0 \rangle_A \rightarrow A \otimes Q_0$ extends to a $*$ -morphism $\alpha_0 : A \rightarrow A \otimes Q_0$ preserving τ . We get for all $\xi, \eta \in \mathcal{E}_0$ and all I in \mathcal{I} :

$$\begin{aligned} (id \otimes p_I) \circ \alpha_0(\langle \xi | \eta \rangle_A) &= (id \otimes p_I) \circ (id \otimes \pi_0) (\langle \beta_u(\xi) | \beta_u(\eta) \rangle_{A \otimes \mathcal{U}}) \\ &= (id \otimes \pi_I) (\langle \beta_u(\xi) | \beta_u(\eta) \rangle_{A \otimes \mathcal{U}}) = \alpha_I(\langle \xi | \eta \rangle_A). \end{aligned}$$

Hence $(id \otimes p_I) \circ \alpha_0$ and α_I coincide on $\langle \mathcal{E}_0 | \mathcal{E}_0 \rangle_A$. Consequently we have for $x \in \langle \mathcal{E}_0 | \mathcal{E}_0 \rangle_A$:

$$\|\alpha_0(x)\| = \sup_{I \in \mathcal{I}} \|(id \otimes p_I) \circ \alpha_0(x)\| = \sup_{I \in \mathcal{I}} \|\alpha_I(x)\| \leq \|x\|.$$

Thus α_0 extends continuously to A . Moreover for all $a, b \in A$, we have

$$\begin{aligned} (id \otimes p_I)(\alpha_0(ab) - \alpha_0(a)\alpha_0(b)) &= \alpha_I(ab) - \alpha_I(a)\alpha_I(b) = 0 \\ \text{and } (id \otimes p_I)(\alpha_0(a^*) - \alpha_0(a)^*) &= \alpha_I(a^*) - \alpha_I(a)^* = 0. \end{aligned}$$

Hence $\|\alpha_0(ab) - \alpha_0(a)\alpha_0(b)\| = \sup_{I \in \mathcal{I}} \|(id \otimes p_I)(\alpha_0(ab) - \alpha_0(a)\alpha_0(b))\| = 0$.

Similarly, we get $\|\alpha_0(a^*) - \alpha_0(a)^*\| = 0$. So α_0 is indeed a $*$ -morphism. Moreover for all $a \in A$ and all $I \in \mathcal{I}$, we have

$$p_I \circ (\tau \otimes id)(\alpha_0(a)) = (\tau \otimes p_I)(\alpha_0(a)) = (\tau \otimes id) \circ \alpha_I(a) = \tau(a)1_{\mathcal{U}/I} = p_I(\tau(a)1_{Q_0}).$$

Thus α_0 preserves τ .

\star We are now ready to check that $I_0 \in \mathcal{I}$. For all $\xi, \eta \in \mathcal{E}_0$ we have (by construction of α_0) that $\alpha_0(\langle \xi | \eta \rangle_A) = \langle \beta_0(\xi) | \beta_0(\eta) \rangle_{A \otimes Q_0}$, and this equality extends by continuity for $\xi, \eta \in E$. Since $(id_A \otimes p_I) \circ \alpha_0 = \alpha_I$ and $(id_E \otimes p_I) \circ \beta_0 = \beta_I$ for all $I \in \mathcal{I}$, we get for all $\xi \in E$ and $a \in A$:

$$\begin{aligned} &\|\beta_0(\xi.a) - \beta_0(\xi).\alpha_0(a)\| \\ &= \sup_{I \in \mathcal{I}} \|(id_E \otimes p_I) \circ \beta_0(\xi.a) - (id_E \otimes p_I)(\beta_0(\xi).\alpha_0(a))\| \\ &= \sup_{I \in \mathcal{I}} \|(id_E \otimes p_I) \circ \beta_0(\xi.a) - [(id_E \otimes p_I) \circ \beta_0(\xi)].[(id_A \otimes p_I) \circ \alpha_0(a)]\| \\ &= \sup_{I \in \mathcal{I}} \|\beta_I(\xi.a) - \beta_I(\xi).\alpha_I(a)\| = 0. \end{aligned}$$

Thus $\beta_0(\xi.a) = \beta_0(\xi).\alpha_0(a)$.

Similarly for $\xi \in \mathcal{E}_0$, $\|(J \otimes *) \circ \beta_0(\xi) - \beta_0 \circ J(\xi)\| = \sup_{I \in \mathcal{I}} \|(J \otimes *) \circ \beta_I(\xi) - \beta_I \circ J(\xi)\| = 0$ and for $\xi \in W$, $\|\beta_0(\xi) - \xi \otimes 1_{Q_0}\| = \sup_{I \in \mathcal{I}} \|\beta_I(\xi) - \xi \otimes 1_{\mathcal{U}/I}\| = 0$. Thus we have $(J \otimes *) \circ \beta_0 = \beta_0 \circ J$ on \mathcal{E}_0 and $\beta_0(\xi) = \xi \otimes 1$ for all $\xi \in W$. We conclude that $I_0 \in \mathcal{I}$.

We set $K = \text{Ker}((\pi_0 \otimes \pi_0) \circ \Delta_{\mathcal{U}})$. In order to show that I_0 is a Woronowicz C^* -ideal we have to check that $I_0 \subset K$, and by definition of I_0 it is enough to show that $K \in \mathcal{I}$.

Denote by $\mu : \mathcal{U}/K \rightarrow \text{Im}((\pi_0 \otimes \pi_0) \circ \Delta_{\mathcal{U}})$ the C^* -isomorphism that satisfies $(\pi_0 \otimes \pi_0) \circ \Delta_{\mathcal{U}} = \mu \circ \pi_K$.

Then for all i, j :

$$\begin{aligned} (id \otimes \mu) \circ (id \otimes \pi_K) \circ \beta_u(e_{ij}) &= (id \otimes \pi_0 \otimes \pi_0) \circ (id \otimes \Delta_{\mathcal{U}}) \circ \beta_u(e_{ij}) \\ &= \sum_{k,l=1}^{d_i} e_{il} \otimes \pi_0(u_{lk}^{(i)}) \otimes \pi_0(u_{kj}^{(i)}) = \sum_{k=1}^{d_i} (\beta_0 \otimes id)(e_{ik} \otimes \pi_0(u_{kj}^{(i)})) \end{aligned}$$

$$= (\beta_0 \otimes id) \circ \beta_0(e_{ij}). \quad (2.3)$$

Thus we have $(\beta_0 \otimes id) \circ \beta_0(E) \subset E \otimes \text{Im } \mu$, and we set $\beta_K = (id \otimes \mu^{-1}) \circ (\beta_0 \otimes id) \circ \beta_0 : E \rightarrow E \otimes \mathcal{U}/K$. We get for all i, j, m, n :

$$\begin{aligned} (\alpha_0 \otimes id) \circ \alpha_0(\langle e_{ij} | e_{mn} \rangle_A) &= \alpha_0 \otimes id \left(\langle \sum_{k=1}^{d_i} e_{ik} \otimes \pi_0(u_{kj}^{(i)}) | \sum_{l=1}^{d_m} e_{ml} \otimes \pi_0(u_{ln}^{(m)}) \rangle_{A \otimes Q_0} \right) \\ &= \alpha_0 \otimes id \left(\sum_{k,l} \langle e_{ik} | e_{ml} \rangle_A \otimes \pi_0(u_{kj}^{(i)*} u_{ln}^{(m)}) \right) \\ &= \sum_{k,l} \langle \beta_0(e_{ik}) | \beta_0(e_{ml}) \rangle_{A \otimes Q_0} \otimes \pi_0(u_{kj}^{(i)*} u_{ln}^{(m)}) \\ &= \sum_{k,l} \langle \sum_{p=1}^{d_i} e_{ip} \otimes \pi_0(u_{pk}^{(i)}) | \sum_{q=1}^{d_m} e_{mq} \otimes \pi_0(u_{ql}^{(m)}) \rangle_{A \otimes Q_0} \otimes \pi_0(u_{kj}^{(i)*} u_{ln}^{(m)}) \\ &= \sum_{k,l,p,q} \langle e_{ip} | e_{mq} \rangle_A \otimes \pi_0(u_{pk}^{(i)*} u_{ql}^{(m)}) \otimes \pi_0(u_{kj}^{(i)*} u_{ln}^{(m)}) \\ &= \langle (\beta_0 \otimes id) \circ \beta_0(e_{ij}) | (\beta_0 \otimes id) \circ \beta_0(e_{ml}) \rangle_{A \otimes Q_0 \otimes Q_0} \\ &= (id \otimes \mu)(\langle \beta_K(e_{ij}) | \beta_K(e_{ml}) \rangle). \end{aligned} \quad (2.4)$$

Hence for $\xi, \eta \in E$, we have

$$(\alpha_0 \otimes id) \circ \alpha_0(\langle \xi | \eta \rangle_A) = \langle (\beta_0 \otimes id) \circ \beta_0(\xi) | (\beta_0 \otimes id) \circ \beta_0(\eta) \rangle_{A \otimes Q_0 \otimes Q_0} \in A \otimes \text{Im } \mu.$$

Thus we also have $(\alpha_0 \otimes id) \circ \alpha_0(A) \subset A \otimes \text{Im } \mu$, and we define:

$$\alpha_K = (id \otimes \mu^{-1}) \circ (\alpha_0 \otimes id) \circ \alpha_0 : A \rightarrow A \otimes \mathcal{U}/K.$$

We know from (2.3) that β_K extends $(id \otimes \pi_K) \circ \beta_u$ and from (2.4) that for all $\xi, \eta \in E$, $\alpha_K(\langle \xi | \eta \rangle_A) = \langle \beta_K(\xi) | \beta_K(\eta) \rangle_{A \otimes \mathcal{U}/K}$.

★ Let us check that α_K preserves τ . We have for all $a \in A$:

$$\begin{aligned} (\tau \otimes id) \circ \alpha_K(x) &= (\tau \otimes id_{\mathcal{U}/K}) \circ (id_A \otimes \mu^{-1}) \circ (\alpha_0 \otimes id_{Q_0}) \circ \alpha_0(x) \\ &= \mu^{-1} \circ (\tau \otimes id_{Q_0} \otimes id_{Q_0}) \circ (\alpha_0 \otimes id_{Q_0}) \circ \alpha_0(x) \\ &= \mu^{-1} \circ (\tau(\cdot) 1_{Q_0} \otimes id_{Q_0}) \circ \alpha_0(x) = \mu^{-1} (1_{Q_0} \otimes (\tau \otimes id_{Q_0}) \circ \alpha_0(x)) \\ &= \mu^{-1} (1_{Q_0} \otimes \tau(x) 1_{Q_0}) = \tau(x) 1_{\mathcal{U}/K}. \end{aligned}$$

★ We have for $\xi \in E$ and $a \in A$:

$$\begin{aligned}\beta_K(\xi.a) &= (id \otimes \mu^{-1}) \circ (\beta_0 \otimes id)(\beta_0(\xi.a)) = (id \otimes \mu^{-1}) \circ (\beta_0 \otimes id)(\beta_0(\xi).\alpha_0(a)) \\ &= (id \otimes \mu^{-1})((\beta_0 \otimes id) \circ \beta_0(\xi).(\alpha_0 \otimes id) \circ \alpha_0(a)) = \beta_K(\xi).\alpha_K(a).\end{aligned}$$

★ Moreover, we have on \mathcal{E}_0 :

$$\begin{aligned}(J \otimes *) \circ \beta_K &= (J \otimes *) \circ (id_E \otimes \mu^{-1}) \circ (\beta_0 \otimes id) \circ \beta_0 \\ &= (id_E \otimes \mu^{-1}) \circ (J \otimes * \otimes *) \circ (\beta_0 \otimes id) \circ \beta_0 \\ &= (id_E \otimes \mu^{-1}) \circ (\beta_0 \otimes id) \circ (J \otimes *) \circ \beta_0 \\ &= (id_E \otimes \mu^{-1}) \circ (\beta_0 \otimes id) \circ \beta_0 \circ J = \beta_K \circ J.\end{aligned}$$

$$\begin{aligned}\text{and for } \xi \in W, \beta_K(\xi) &= (id_E \otimes \mu^{-1}) \circ (\beta_0 \otimes id) \circ \beta_0(\xi) = (id_E \otimes \mu^{-1}) \circ (\beta_0 \otimes id)(\xi \otimes 1) \\ &= (id_E \otimes \mu^{-1})(\xi \otimes 1 \otimes 1) = \xi \otimes 1.\end{aligned}$$

So $K \in \mathcal{I}$ and I_0 is indeed a Woronowicz C^* -ideal. We denote by Δ_0 the coproduct on Q_0 . In order to show that $Q_0 \in \mathcal{C}(E, \tau, (V_i)_{i \in \mathcal{I}}, J, W)$ it only remains to check that α_0 and β_0 are coassociative and that $\text{span}\{\alpha_0(A).(1 \otimes Q_0)\}$ and $\text{span}\{\beta_0(E).(1 \otimes Q_0)\}$ are respectively dense in $A \otimes Q_0$ and $E \otimes Q_0$.

★ We have seen (cf. (2.3)) that for all i, j , $(\beta_0 \otimes id) \circ \beta_0(e_{ij}) = (id \otimes \pi_0 \otimes \pi_0) \circ (id \otimes \Delta_U) \circ \beta_u(e_{ij})$. But $(\pi_0 \otimes \pi_0) \circ \Delta_U = \Delta_0 \circ \pi_0$. Thus:

$$(\beta_0 \otimes id) \circ \beta_0(e_{ij}) = (id \otimes \Delta_0) \circ (id \otimes \pi_0) \circ \beta_u(e_{ij}) = (id \otimes \Delta_0) \circ \beta_0(e_{ij}).$$

We deduce that $(\beta_0 \otimes id) \circ \beta_0 = (id \otimes \Delta_0) \circ \beta_0$ on E . Hence for $\xi, \eta \in E$:

$$\begin{aligned}(\alpha_0 \otimes id_{Q_0}) \circ \alpha_0(\langle \xi | \eta \rangle_A) &= \langle (\beta_0 \otimes id_{Q_0}) \circ \beta_0(\xi) | (\beta_0 \otimes id_{Q_0}) \circ \beta_0(\eta) \rangle_{A \otimes Q_0 \otimes Q_0} \\ &= \langle (id_E \otimes \Delta_0) \circ \beta_0(\xi) | (id_E \otimes \Delta_0) \circ \beta_0(\eta) \rangle_{A \otimes Q_0 \otimes Q_0} \\ &= (id_A \otimes \Delta_0) \left(\langle \beta_0(\xi) | \beta_0(\eta) \rangle_{A \otimes Q_0} \right) \\ &= (id_A \otimes \Delta_0) \circ \alpha_0(\langle \xi | \eta \rangle_A),\end{aligned}$$

which shows (by density of $\langle E | E \rangle_A$ in A) that α_0 is coassociative as well.

★ Finally, to show that $\text{span}\{\alpha_0(A).(1 \otimes Q_0)\}$ and $\text{span}\{\beta_0(E).(1 \otimes Q_0)\}$ are respectively dense in $A \otimes Q_0$ and $E \otimes Q_0$, we can proceed in the same way as in the proof of Lemma 2.3.3, by checking that for all i, j :

$$\sum_{k=1}^{d_i} \beta_0(e_{ik})(1 \otimes \pi_0(u_{jk}^{(i)*})) = e_{ij} \otimes 1$$

and for all i, j, m, n :

$$\sum_{k=1}^{d_i} \sum_{l=1}^{d_j} \alpha_0(\langle e_{ik} | e_{jl} \rangle_A) \cdot \left(1 \otimes \pi_0 \left(u_{nl}^{(j)*} x_{mk}^{(i)} \right) \right) = \langle e_{im} | e_{jn} \rangle_A \otimes 1,$$

where $x^{(i)} = (s^{(i)})^{-1} u^{(i)t} s^{(i)}$ is the inverse of $\overline{u^{(i)}}$. Thus (α_0, β_0) is a filtration-preserving coaction of Q_0 on E .

It remains to see that Q_0 is in fact an initial object in the category $\mathcal{C}(E, \tau, (V_i)_{i \in \mathcal{I}}, J, W)$. Let $I \subset \mathcal{U}$ be a Woronowicz C^* -ideal such that there exists a filtration-preserving coaction (α_I, β_I) of \mathcal{U}/I on E such that β_I extends $(id \otimes \pi_I) \circ \beta_u$. We get in particular $I \in \mathcal{S}$, thus $I_0 \subset I$ and $p_I : Q_0 \rightarrow \mathcal{U}/I$ is then a morphism in $\mathcal{C}(E, \tau, (V_i)_{i \in \mathcal{I}}, J, W)$.

Such a morphism is unique. Indeed, if η is a morphism from Q_0 to \mathcal{U}/I then $(id \otimes \eta) \circ \beta_0 = \beta_I$, so for all i, j, k , $\eta \circ \pi_0(u_{ij}^{(k)}) = \pi_I(u_{ij}^{(k)})$. Hence $\eta \circ \pi_0 = \pi_I$, and $\eta = p_I$ follows from uniqueness in the factorization theorem. Finally, according to Lemmas 2.3.3 and 2.3.4, we conclude that $(Q_0, \Delta_0, \alpha_0, \beta_0)$ is an initial object in $\mathcal{C}(E, \tau, (V_i)_{i \in \mathcal{I}}, J, W)$. \square

Remarks 2.3.6. As in [BS13], we can make the following remarks:

- If $Q \in \mathcal{C}(E, \tau, (V_i)_{i \in \mathcal{I}}, J, W)$ coacts faithfully on E , then the morphism $\mu : Q_0 \rightarrow Q$ is onto. So that the quantum group associated with Q is a quantum subgroup of the one associated with Q_0 .
- If $(W_j)_{j \in \mathcal{J}}$ is a subfiltration of $(V_i)_{i \in \mathcal{I}}$ (that is $(W_j)_{j \in \mathcal{J}}$ is an orthogonal filtration of E , such that $\forall j \in \mathcal{J}$, there exists $i \in \mathcal{I}$ such that $W_j \subset V_i$) then the quantum symmetry group of $(E, \tau, (W_j)_{j \in \mathcal{J}}, J, W)$ is a quantum subgroup of the quantum symmetry group of $(E, \tau, (V_i)_{i \in \mathcal{I}}, J, W)$.

2.4 Examples

2.4.1 Example of a C^* -algebra equipped with an orthogonal filtration

We recall from [BS13] the construction of the quantum symmetry group of a C^* -algebra equipped with an orthogonal filtration.

Definition 2.4.1. Let $(A, \tau, (V_i)_{i \in \mathcal{I}})$ be a C^* -algebra equipped with an orthogonal filtration (see Example 2.2.5.(2) for the definition). We say that a Woronowicz C^* -algebra Q coacting on A coacts in a filtration-preserving way, if the coaction $\alpha : A \rightarrow A \otimes Q$ of Q on A satisfies for all $i \in \mathcal{I}$, $\alpha(V_i) \subset V_i \odot Q$.

Theorem 2.4.2 ([BS13]). *Let $(A, \tau, (V_i)_{i \in \mathcal{I}})$ be a C^* -algebra equipped with an orthogonal filtration. The category of Woronowicz C^* -algebras coacting on A in a filtration-preserving way admits an initial object. The quantum group corresponding to that universal object is called the **quantum symmetry group** of $(A, \tau, (V_i)_{i \in \mathcal{I}})$.*

Setting $E = A$, $W = \mathbb{C} \cdot 1_A$ and $J = a \mapsto a^*$, it is easy to see that the quantum symmetry group of $(E, \tau, (V_i)_{i \in \mathcal{I}}, J, W)$ coincides with the one constructed in the previous theorem (if (Q, Δ, α) coacts on $(A, \tau, (V_i)_{i \in \mathcal{I}})$ in a filtration-preserving way then $(\tau \otimes id) \circ \alpha = \tau(\cdot)1_A$ is automatic since $V_0 = \mathbb{C} \cdot 1_A$).

In fact our construction allows to see that the category of Woronowicz C^* -algebras coacting on $(A, \tau, (V_i)_{i \in \mathcal{I}})$ in a filtration-preserving way admits an initial object, even when the assumption “ \mathcal{A}_0 is a $*$ -subalgebra of A ” is dropped.

In particular, we see that our construction generalizes the one of [Gos09] in the sense that if $(\mathcal{A}, \mathcal{H}, D)$ is an admissible spectral triple and if we set:

- $E = A = \overline{\mathcal{A}}^{\mathcal{L}(\mathcal{H})}$,
- $\tau = \begin{cases} a \mapsto \frac{Tr_\omega(a|D|^{-p})}{Tr_\omega(|D|^{-p})} & \text{if } \mathcal{H} \text{ is infinite dimensional,} \\ \text{the usual trace} & \text{otherwise,} \end{cases}$
where Tr_ω denotes the Dixmier trace and p is the metric dimension of $(\mathcal{A}, \mathcal{H}, D)$,
- the $(V_i)_{i \in \mathbb{N}}$ are the eigenspaces of the ‘noncommutative Laplacian’,
- J is the involution of A and $W = \mathbb{C} \cdot 1_A$,

then we recover the quantum isometry group of $(\mathcal{A}, \mathcal{H}, D)$ in the sense of [Gos09].

Given a spectral triple, we have seen in Example 2.2.5.(4) another way to attach a Hilbert module equipped with an orthogonal filtration to it (induced by D instead of the Laplacian). For an admissible spectral triple $(\mathcal{A}, \mathcal{H}, D)$ satisfying conditions of Example 2.2.5.(4), the quantum symmetry group of \mathcal{H}^∞ and the quantum isometry group of $(\mathcal{A}, \mathcal{H}, D)$ in the sense of Goswami both exist. We do not know if they coincide in that situation. But in the case of the spectral triple of a Riemannian compact manifold the question is solved in the next paragraph.

2.4.2 Example of the bundle of exterior forms on a Riemannian manifold

Let M be a compact Riemannian manifold. Set $A = C(M)$, $\tau = \int \cdot dvol$ where $dvol$ denotes the Riemannian density of M and set $E = \Gamma(\Lambda^*M)$ equipped with its canonical

Hilbert $C(M)$ -module structure. We denote by $D = \overline{d + d^*} : L^2(\Lambda^*M) \rightarrow L^2(\Lambda^*M)$ the de Rham operator. D is self-adjoint and has compact resolvent. So that $\text{Sp}(D)$ can be written as: $\text{Sp}(D) = \{\lambda_i ; i \in \mathbb{N}\}$, with $\lim_{i \rightarrow +\infty} |\lambda_i| = +\infty$ and where each λ_i is a real eigenvalue of D with finite multiplicity. For $i \in \mathbb{N}$ we denote by V_i the subspace associated with λ_i and by d_i the dimension of V_i . Note that $V_i \subset \Gamma^\infty(\Lambda^*M)$, so $V_i \subset E$.

Clearly, the family $(V_i)_{i \in \mathbb{N}}$ is an orthogonal filtration of E , E is full and $\mathcal{H} = L^2(\Lambda^*M)$.

We denote by $W = \mathbb{C}.\xi_0$, where $\xi_0 = m \mapsto 1_{\Lambda_m^*M} \in E$ and by $J : E \rightarrow E$ the canonical involution.

Comparison with the quantum isometry group of M as defined in [Gos09]

Let (α, β) be a filtration-preserving coaction of a Woronowicz C^* -algebra Q on E . For ϕ a state on Q , we set $\beta_\phi = (id \otimes \phi) \circ \beta : L^2(\Lambda^*M) \rightarrow L^2(\Lambda^*M)$. Since β preserves the filtration, β_ϕ commutes with D on \mathcal{E}_0 . This implies that $\forall k \in \mathbb{N}$, $\beta_\phi(\text{Dom}(D^k)) \subset \text{Dom}(D^k)$ and $\beta_\phi \circ D^k = D^k \circ \beta_\phi$ on $\text{Dom}(D^k)$. Thus $\beta_\phi(\Gamma^\infty(\Lambda^*M)) \subset \Gamma^\infty(\Lambda^*M)$ and β_ϕ commutes with D^2 on $\Gamma^\infty(\Lambda^*M)$. Now for $f \in C^\infty(M)$, we have $\beta_\phi(f) = \beta_\phi(f.\xi_0) = \alpha_\phi(f).\xi_0 = \alpha_\phi(f)$, where $\alpha_\phi = (id \otimes \phi) \circ \alpha$. Thus $\alpha_\phi(C^\infty(M)) \subset C^\infty(M)$ and α_ϕ commutes with \mathcal{L} (the Laplacian on functions) on $C^\infty(M)$. This shows that α is an isometric coaction of Q on $C(M)$ in the sense of [Gos09].

Thus we have a forgetful functor \mathcal{F} from $\mathcal{C}(\Gamma(\Lambda^*M), \tau, (V_i)_{i \in \mathbb{N}}, J, W)$ to the category of Woronowicz C^* -algebras coacting isometrically on M in the sense of [Gos09] defined by $\mathcal{F}(Q, \alpha, \beta) = (Q, \alpha)$. This functor is in fact an equivalence of categories. To see this, we show that it is fully faithful and essentially surjective.

Let α be an isometric coaction of a Woronowicz C^* -algebra Q on M , in the sense of Goswami. It can be seen, along the lines of [DGJ], that there is a well defined map $\bar{\alpha} : \Gamma(\Lambda^*M) \rightarrow \Gamma(\Lambda^*M) \otimes Q$ satisfying

$$\bar{\alpha}(f_0 df_1 \wedge \dots \wedge df_k) = \alpha(f_0)(d \otimes id) \circ \alpha(f_1) \dots (d \otimes id) \circ \alpha(f_k)$$

for all $f_0 \in C(M)$, $f_1, \dots, f_k \in C^\infty(M)$, and that $(\alpha, \bar{\alpha})$ is a coaction of Q on $\Gamma(\Lambda^*M)$. In order to show that for all $k \in \mathbb{Z}$, $\bar{\alpha}(V_k) \subset V_k \otimes Q$, we check that $(D \otimes id) \circ \bar{\alpha} = \bar{\alpha} \circ D$ on $\Gamma^\infty(\Lambda^*M)$. The equality $(d \otimes id) \circ \bar{\alpha} = \bar{\alpha} \circ d$ holds by definition of $\bar{\alpha}$. Now for all $\omega, \eta \in \Gamma^\infty(\Lambda^*M)$, all $x \in Q$ we have

$$\begin{aligned} \langle (d^* \otimes id) \circ \bar{\alpha}(\omega) | \bar{\alpha}(\eta).(1 \otimes x) \rangle_Q &= \langle \bar{\alpha}(\omega) | (d \otimes id) \circ \bar{\alpha}(\eta) \rangle_Q . x = \langle \bar{\alpha}(\omega) | \bar{\alpha}(d\eta) \rangle_Q . x \\ &= (\tau \otimes id)(\langle \bar{\alpha}(\omega) | \bar{\alpha}(d\eta) \rangle_{A \otimes Q}) . x = (\omega | d\eta)_{\mathcal{H}} . x \\ &= (d^* \omega | \eta)_{\mathcal{H}} . x = \langle \bar{\alpha}(d^* \omega) | \bar{\alpha}(\eta) \rangle_Q . x \\ &= \langle \bar{\alpha}(d^* \omega) | \bar{\alpha}(\eta).(1 \otimes x) \rangle_Q, \end{aligned}$$

and since $\text{span}\{\bar{\alpha}(\Gamma^\infty(\Lambda^*M)).(1 \otimes Q)\}$ is dense in $\Gamma(\Lambda^*M) \otimes Q$, we have $(d^* \otimes id) \circ \bar{\alpha} = \bar{\alpha} \circ d^*$ on $\Gamma^\infty(\Lambda^*M)$. This allows to see that $(\alpha, \bar{\alpha})$ is a filtration-preserving coaction of Q on $\Gamma(\Lambda^*M)$ and $\mathcal{F}(Q, \alpha, \bar{\alpha}) = (Q, \alpha)$, so \mathcal{F} is essentially surjective.

Let (α, β) be a filtration-preserving coaction of a Woronowicz C^* -algebra Q on E . Let us show by induction on $k \in \mathbb{N}$ that $\beta = \bar{\alpha}$ on $\Gamma(\Lambda^k M)$. The fact that $\beta = \bar{\alpha} = \alpha$ on $C(M)$ is clear. Let $k \in \mathbb{N}$ such that for all $l \leq k$, $\beta = \bar{\alpha}$ on $\Gamma(\Lambda^l M)$. Let f_0 be in $C(M)$ and f_1, \dots, f_{k+1} be in $C^\infty(M)$. Then:

$$\begin{aligned} \beta(df_1 \wedge \dots \wedge df_{k+1}) &= \beta(D(f_1 df_2 \wedge \dots \wedge df_{k+1})) - \beta(d^*(f_1 df_2 \wedge \dots \wedge df_{k+1})) \\ &= (D \otimes id) \circ \beta(f_1 df_2 \wedge \dots \wedge df_{k+1}) - \bar{\alpha}(d^*(f_1 df_2 \wedge \dots \wedge df_{k+1})) \\ &= (D \otimes id) \circ \bar{\alpha}(f_1 df_2 \wedge \dots \wedge df_{k+1}) - (d^* \otimes id) \circ \bar{\alpha}(f_1 df_2 \wedge \dots \wedge df_{k+1}) \\ &= (d \otimes id) \circ \bar{\alpha}(f_1 df_2 \wedge \dots \wedge df_{k+1}) = \bar{\alpha}(d(f_1 df_2 \wedge \dots \wedge df_{k+1})) \\ &= \bar{\alpha}(df_1 \wedge \dots \wedge df_{k+1}). \end{aligned}$$

Thus we have $\beta(f_0 df_1 \wedge \dots \wedge df_{k+1}) = \alpha(f_0) \bar{\alpha}(df_1 \wedge \dots \wedge df_{k+1}) = \bar{\alpha}(f_0 df_1 \wedge \dots \wedge df_{k+1})$, which ends the induction. Now the fact that β necessarily coincides with $\bar{\alpha}$ allows to see that a morphism of Woronowicz C^* -algebras $\mu : (Q_0, \alpha_0, \beta_0) \rightarrow (Q_1, \alpha_1, \beta_1)$, between Woronowicz C^* -algebras coacting in a filtration-preserving way on $\Gamma(\Lambda^*M)$, that satisfies $(id \otimes \mu) \circ \alpha_0 = \alpha_1 \circ \mu$, automatically satisfies $(id \otimes \mu) \circ \beta_0 = \beta_1 \circ \mu$. This shows that \mathcal{F} is fully faithful, so \mathcal{F} is indeed an equivalence of categories. In particular, \mathcal{F} preserves the initial object, so our quantum isometry group of M coincides with the one of Goswami.

Note that in case (α, β) is a filtration-preserving coaction of $C(G)$ on $\Gamma(\Lambda^*M)$, for G a given compact group, then there exists an isometric action $\gamma : M \times G \rightarrow M$ of G on M such that

$$\begin{aligned} \alpha &= C(M) \rightarrow C(M \times G) \cong C(M) \otimes C(G) \\ f &\mapsto f \circ \gamma \end{aligned}$$

and

$$\begin{aligned} \beta &= \Gamma(\Lambda^*M) \rightarrow C(G, \Gamma(\Lambda^*M)) \cong \Gamma(\Lambda^*M) \otimes C(G) \\ \omega &\mapsto (g \mapsto \gamma_g^*(\omega)), \end{aligned}$$

where $\gamma_g = m \mapsto \gamma(m, g)$ and $\gamma_g^* : \Gamma(\Lambda^*M) \rightarrow \Gamma(\Lambda^*M)$ denotes the pullback by γ_g .

2.4.3 Basic example: free orthogonal quantum groups

Let n be in \mathbb{N} . We set $A = \mathbb{C}$, $E = \mathbb{C}^n$ equipped with its canonical Hilbert space structure, $W = (0)$ and $V_0 = \mathbb{C}^n$. Let $J : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be any invertible antilinear map. We denote by P the matrix of J in the canonical basis and by $\mathcal{A}_0(P)$ the universal Woronowicz C^* -

algebra generated by a multiplicative and unitary matrix $u = (u_{ij})_{1 \leq i, j \leq n}$, satisfying the relation $u = P\bar{u}P^{-1}$ (the quantum group associated with $\mathcal{A}_o(P)$ is a so-called *free orthogonal quantum group*, see [Ban96]). We denote by $\alpha_p : \mathbb{C} \rightarrow \mathbb{C} \otimes \mathcal{A}_o(P)$ the trivial coaction and by $\beta_p : \mathbb{C}^n \rightarrow \mathbb{C}^n \otimes \mathcal{A}_o(P)$ the linear map given by $\beta(e_i) = \sum_{k=1}^n e_k \otimes u_{ki}$ where $(e_k)_{1 \leq k \leq n}$ is the canonical basis of \mathbb{C}^n . We can easily check that $(id_{\mathbb{C}}, (V_0), J, W)$ is an orthogonal filtration of E and that (α_p, β_p) is a coaction of $\mathcal{A}_o(P)$ on E . To see that (α_p, β_p) is a filtration-preserving coaction, the only nontrivial point is to check that $(J \otimes *) \circ \beta_p = \beta_p \circ J$. We have for all i in $\{1, \dots, n\}$:

$$\begin{aligned} (J \otimes *) \circ \beta_p(e_i) &= \sum_{k=1}^n J(e_k) \otimes u_{ki}^* = \sum_{k,l=1}^n P_{lk} e_l \otimes u_{ki}^* = \sum_{l=1}^n e_l \otimes \left(\sum_{k=1}^n P_{lk} u_{ki}^* \right) \\ &= \sum_{l=1}^n e_l \otimes \left(\sum_{k=1}^n u_{lk} P_{ki} \right) \quad (\text{since } P\bar{u} = uP) \\ &= \sum_{k=1}^n P_{ki} \beta_p(e_k) = \beta_p \circ J(e_i). \end{aligned}$$

So (α_p, β_p) is a filtration-preserving coaction of $\mathcal{A}_o(P)$ on \mathbb{C}^n . Now we show that it is a universal object in the category $\mathcal{C}(E, id_{\mathbb{C}}, (V_0), J, W)$. Let (α, β) be a filtration-preserving coaction of a Woronowicz C^* -algebra Q on E and let $v = (v_{ij})_{1 \leq i, j \leq n} \in \mathcal{M}_n(Q)$ be characterized by $\beta(e_i) = \sum_{k=1}^n e_k \otimes v_{ki}$. By Lemma 2.3.3 we already know that v is unitary. Furthermore, by a similar computation to the previous one, we see that $(J \otimes *) \circ \beta = \beta \circ J$ leads to the equality $P\bar{v} = vP$. Thus by universal property of $\mathcal{A}_o(P)$ we get the existence of a morphism $\mu : \mathcal{A}_o(P) \rightarrow Q$ such that for all $i, j \in \{1, \dots, n\}$, $\mu(u_{ij}) = v_{ij}$, which is clearly a morphism in the category $\mathcal{C}(E, id_{\mathbb{C}}, (V_0), J, W)$. Consequently the quantum symmetry group of E is the free orthogonal quantum group associated with P .

2.4.4 Example built on segments

Huang [Hua13] has constructed examples of faithful actions of non-classical quantum groups on connected metric spaces. We now examine some of his examples and put them into our framework.

We first describe in this paragraph the Hilbert module endowed with an orthogonal filtration we associate to d disjoint copies of $[0, 1]$. Then we compute its quantum symmetry group, which appears to be the hyperoctahedral quantum group. Finally, we show the existence of a universal object in a certain subcategory of the one of Woronowicz C^* -algebras coacting on this Hilbert module in a filtration-preserving way. The quantum group asso-

ciated with that universal object might be seen as a quantum isometry group of a certain quotient of $[0, 1] \times \{1, \dots, d\}$.

Hilbert module associated with $[0, 1] \times \{1, \dots, d\}$

We set $I = [0, 1]$ and we denote by $\delta_+ : L^2(I) \rightarrow L^2(I)$ the operator $\frac{d}{dx}$ with domain:

$$\text{Dom}(\delta_+) = \{f \in H^1(I) ; f(0) = f(1) = 0\}.$$

Its adjoint operator is $\delta_- = -\frac{d}{dx}$ with domain $H^1(I)$.

We define $D_0 : L^2(\Lambda^*(I)) \rightarrow L^2(\Lambda^*(I)) \cong L^2(I) \oplus L^2(I)$ by:

$$D_0 = \begin{pmatrix} 0 & \delta_- \\ \delta_+ & 0 \end{pmatrix}.$$

D_0 is a self-adjoint operator with compact resolvent. It can be checked that the eigenvectors of D_0 are the $(\sin(\pi k \cdot), \cos(\pi k \cdot))$ with $k \in \mathbb{Z}$. We set $A = C(I)^d \cong C(I \times \{1, \dots, d\})$, $E = (C_0(I) \oplus C(I))^d$. For $i \in \{1, \dots, d\}$ and $n \in \mathbb{Z}$, we denote by $e_{ni} \in E$ the vector whose components are zero, except the i -th one whose value is $(\sin(\pi n \cdot), \cos(\pi n \cdot))$. We set $V_n = \text{span}\{e_{ni} ; i \in \{1, \dots, d\}\}$, $\tau = \sum_{i=1}^d \int \cdot dx_i$, $W = \mathbb{C} \cdot \xi_0$ where $\xi_0 = \underbrace{((0, 1), \dots, (0, 1))}_{d \text{ times}}$ and we denote by $J : E \rightarrow E$ the complex conjugation operator. Then $(E, \tau, (V_n)_{n \in \mathbb{Z}}, J, W)$ is a Hilbert A -module equipped with an orthogonal filtration. To see that this Hilbert module is a good description of $[0, 1] \times \{1, \dots, d\}$, just remark that it is obtained from the spectral triple (A, H, D) , where $H = L^2(\Lambda^*(I))^d$ and $D = \text{diag}(\underbrace{D_0, \dots, D_0}_{d \text{ times}})$ (cf. Example 2.2.5.(4) to see how we associate a Hilbert module endowed with an orthogonal filtration to a suitable spectral triple).

Computation of its quantum symmetry group

Let (α, β) be a filtration-preserving coaction of a Woronowicz C^* -algebra Q on E . For $n \in \mathbb{Z}$, we denote by $v^{(n)} \in \mathcal{M}_d(Q)$ the unitary matrix characterized by:

$$\forall i \in \{1, \dots, d\}, \beta(e_{ni}) = \sum_{j=1}^d e_{nj} \otimes v_{ji}^{(n)}.$$

For $i \in \{1, \dots, d\}$, let $e_i \in A$ denote the vector whose components are zero except the i -th one which equals 1, and let v and w respectively denote $v^{(0)}$ and $v^{(1)}$.

We have $\beta(e_{ni}) = \beta(\overline{e_{ni}}) = (J \otimes *) \circ \beta(e_{ni}) = \sum_{k=1}^d e_{nk} \otimes v_{ki}^{(n)*}$, thus $v_{ki}^{(n)} = v_{ki}^{(n)*}$. And

$$\begin{aligned} \alpha(e_i) &= \alpha(\langle e_{0i} | \xi_0 \rangle_A) = \langle \beta(e_{0i}) | \xi_0 \otimes 1 \rangle_{A \otimes Q} = \sum_{k=1}^d \langle e_{0k} | \xi_0 \rangle_A \otimes v_{ki}^* = \sum_{k=1}^d e_k \otimes v_{ki} \\ &= \alpha(e_i^2) = \alpha(e_i)^2 = \sum_{k=1}^d e_k \otimes v_{ki}^2. \end{aligned}$$

Thus $v_{ki}^2 = v_{ki}$ for all i, k . We get $\sum_{k=1}^d v_{ik} = \sum_{k=1}^d v_{ik} v_{ik}^* = 1$ and $\sum_{k=1}^d v_{ki} = \sum_{k=1}^d v_{ki}^* v_{ki} = 1$ since v is unitary. We have furthermore

$$\langle e_{ni} | e_{mj} \rangle_A = \delta_{ij} (\sin(\pi n \cdot) \sin(\pi m \cdot) + \cos(\pi n \cdot) \cos(\pi m \cdot)) e_i = \delta_{ij} \cos(\pi(n - m) \cdot) e_i.$$

Thus we get

$$\begin{aligned} \alpha(\langle e_{ni} | e_{mj} \rangle_A) &= \delta_{ij} \alpha(\cos(\pi(n - m) \cdot) e_i) = \langle \beta(e_{ni}) | \beta(e_{mj}) \rangle_{A \otimes Q} = \sum_{k,l=1}^d \langle e_{nk} | e_{ml} \rangle_A \otimes v_{ki}^{(n)} v_{lj}^{(m)} \\ &= \sum_{k=1}^d \cos(\pi(n - m) \cdot) e_k \otimes v_{ki}^{(n)} v_{kj}^{(m)} \end{aligned}$$

We deduce that:

- for all $i \neq j$, $\alpha(\langle e_{ni} | e_{mj} \rangle_A) = 0 = \sum_{k=1}^d \cos(\pi(n - m) \cdot) e_k \otimes v_{ki}^{(n)} v_{kj}^{(m)}$, so that for all n, m, k , $v_{ki}^{(n)} v_{kj}^{(m)} = 0$,
- $\alpha(\langle e_{ni} | e_{mi} \rangle_A) = \alpha(\cos(\pi(n - m) \cdot) e_i) = \sum_{k=1}^d \cos(\pi(n - m) \cdot) e_k \otimes v_{ki}^{(n)} v_{ki}^{(m)}$.

Thus for all i, k , and all n, m, n', m' such that $|n - m| = |n' - m'|$, $v_{ki}^{(n)} v_{ki}^{(m)} = v_{ki}^{(n')} v_{ki}^{(m')}$.

Therefore we have for all i, j, n , $v_{ij}^{(n)} = v_{ij}^{(n)} \left(\sum_{k=1}^d v_{ik} \right) = v_{ij}^{(n)} v_{ij}$ and similarly $v_{ij}^{(n)} = v_{ij} v_{ij}^{(n)}$.

Consequently, $v_{ij}^{(n)} = v_{ij}^{(n)} v_{ij} = v_{ij} v_{ij}^{(-n)} = v_{ij}^{(-n)}$ (since $|n - 0| = |0 - n|$), thus $v_{ij}^{(n+1)} = v_{ij}^{(n)} v_{ij}^{(-1)} = v_{ij}^{(n)} w_{ij}$. Then the fact that $w_{ij}^2 = v_{ij}^{(1)} v_{ij}^{(1)} = v_{ij}$ allows to see by immediate induction that $v_{ij}^{(n)} = w_{ij}^{r(n)}$ for all i, j, n , where $r(n) = 2$ if n is even and $r(n) = 1$ otherwise.

We obtain finally:

- for $j \neq k$, $w_{ij} w_{ik} = 0$ and $w_{ji} w_{ki} = S(w_{ij}) S(w_{ik}) = S(w_{ik} w_{ij}) = 0$ (here S denotes the antipode of Q).

- $\sum_{l=1}^n w_{il}^2 = \sum_{l=1}^n v_{il} = 1$ and similarly $\sum_{l=1}^n w_{li}^2 = 1$.

This leads to the existence of a unique morphism of Woronowicz C^* -algebras $\mu : \mathcal{A}_h(d) \rightarrow Q$ such that for all i, j , $\mu(u_{ij}) = w_{ij}$ (where $\mathcal{A}_h(d)$ is the hyperoctahedral quantum group and the u_{ij} 's are the canonical generators of $\mathcal{A}_h(d)$ - see [BBC07, Bic04]).

In order to conclude that $\mathcal{A}_h(d)$ is a universal object in the category of Woronowicz C^* -algebras coacting in a filtration-preserving way on E , it only remains to check that there exists a filtration-preserving coaction (α, β) of $\mathcal{A}_h(d)$ on E such that for all $n \in \mathbb{Z}$ and $i \in \{1, \dots, d\}$, $\beta(e_{ni}) = \sum_{k=1}^d e_{nk} \otimes u_{ki}^{r(n)}$ (so that the morphism $\mu : \mathcal{A}_h(d) \rightarrow Q$ constructed previously automatically intertwines the coactions).

For $f \in C([0, 1])$, we set

$$p(f) = x \mapsto \frac{1}{2}(f(x) + f(1-x)) \quad \text{and} \quad q(f) = x \mapsto \frac{1}{2}(f(x) - f(1-x)).$$

We define $\alpha : C([0, 1])^d \rightarrow C([0, 1])^d \otimes \mathcal{A}_h(d)$ by:

$$\alpha \left(\sum_{i=1}^d f_i \cdot e_i \right) = \sum_{i,k=1}^d p(f_i) \cdot e_k \otimes u_{ki}^2 + q(f_i) \cdot e_k \otimes u_{ki}$$

for all $f_1, \dots, f_d \in C([0, 1])$. Let us check that α is a $*$ -morphism. The fact that α preserves the involution is straightforward, and we have for all $f_1, \dots, f_d, g_1, \dots, g_d \in C([0, 1])$

$$\begin{aligned} & \alpha \left(\sum_{i=1}^d f_i \cdot e_i \right) \alpha \left(\sum_{j=1}^d g_j \cdot e_j \right) \\ &= \left(\sum_{i,k=1}^d p(f_i) \cdot e_k \otimes u_{ki}^2 + q(f_i) \cdot e_k \otimes u_{ki} \right) \left(\sum_{j,l=1}^d p(g_j) \cdot e_l \otimes u_{lj}^2 + q(g_j) \cdot e_l \otimes u_{lj} \right) \\ &= \sum_{i,j,k=1}^d (p(f_i) \cdot e_k \otimes u_{ki}^2 + q(f_i) \cdot e_k \otimes u_{ki}) (p(g_j) \cdot e_k \otimes u_{kj}^2 + q(g_j) \cdot e_k \otimes u_{kj}) \\ &= \sum_{i,k=1}^d (p(f_i)p(g_i) + q(f_i)q(g_i)) \cdot e_k \otimes u_{ki}^2 + (p(f_i)q(g_i) + q(f_i)p(g_j)) \cdot e_k \otimes u_{ki}^2 \end{aligned}$$

(the last equality holds since $u_{ki}u_{kj} = 0$ for $i \neq j$ and $u_{ij}^3 = u_{ij}$).

But we have $p(f)p(g) + q(f)q(g) = p(fg)$ and $p(f)q(g) + q(f)p(g) = q(fg)$ for all

$f, g \in C([0, 1])$. We thus get

$$\alpha \left(\sum_{i=1}^d f_i \cdot e_i \right) \alpha \left(\sum_{j=1}^d g_j \cdot e_j \right) = \alpha \left(\sum_{i=1}^d f_i g_i \cdot e_i \right) = \alpha \left(\left(\sum_{i=1}^d f_i \cdot e_i \right) \left(\sum_{j=1}^d g_j \cdot e_j \right) \right).$$

Now we have for $f \in C([0, 1])$ and $i \in \{1, \dots, d\}$

$$\begin{aligned} (id \otimes \Delta) \circ \alpha(f \cdot e_i) &= \sum_{k=1}^d p(f) \cdot e_k \otimes \Delta(u_{ki})^2 + q(f) \cdot e_k \otimes \Delta(u_{ki}) \\ &= \sum_{k,l=1}^d p(f) \cdot e_k \otimes u_{kl}^2 \otimes u_{li}^2 + q(f) \cdot e_k \otimes u_{kl} \otimes u_{li}, \end{aligned}$$

and since $q \circ p = p \circ q = 0$, $p^2 = p$ and $q^2 = q$, we also have

$$\begin{aligned} (\alpha \otimes id) \circ \alpha(f \cdot e_i) &= \sum_{l=1}^d \alpha(p(f) \cdot e_l) \otimes u_{li}^2 + \alpha(q(f) \cdot e_l) \otimes u_{li} \\ &= \sum_{k,l=1}^d p(f) \cdot e_k \otimes u_{kl}^2 \otimes u_{li}^2 + q(f) \cdot e_k \otimes u_{kl} \otimes u_{li}. \end{aligned}$$

Thus α is coassociative.

We have $p(\cos(\pi n \cdot)) = \frac{1 + (-1)^n}{2} \cos(\pi n \cdot)$ and $q(\cos(\pi n \cdot)) = \frac{1 + (-1)^{n+1}}{2} \cos(\pi n \cdot)$, so that $\alpha(\cos(\pi n \cdot) e_i) = \sum_{k=1}^d \cos(\pi n \cdot) \cdot e_k \otimes u_{ki}^{r(n)}$ (where $r(n)$ is still equal to 2 when n is even and to 1 otherwise). Therefore, we have for all $n \in \mathbb{Z}$ and all $i \in \{1, \dots, d\}$,

$$\begin{aligned} \sum_{k=1}^d \alpha(\cos(\pi n \cdot) \cdot e_k) (1 \otimes u_{ik}^{r(n)}) &= \sum_{k,l=1}^d \cos(\pi n \cdot) \cdot e_l \otimes u_{lk}^{r(n)} u_{ik}^{r(n)} = \sum_{k=1}^d \cos(\pi n \cdot) \cdot e_i \otimes u_{ik}^{2r(n)} \\ &= \cos(\pi n \cdot) \cdot e_i \otimes \left(\sum_{k=1}^d u_{ik}^2 \right) = \cos(\pi n \cdot) \cdot e_i \otimes 1, \end{aligned}$$

which allows us to see that $\text{span}\{\alpha(C(I)^d) \cdot (1 \otimes \mathcal{A}_h(d))\}$ is dense in $C(I)^d \otimes \mathcal{A}_h(d)$, so that α is a coaction of $\mathcal{A}_h(d)$ on $C(I)^d$. Then we define $\beta : E \rightarrow E \otimes \mathcal{A}_h(d)$ by:

$$\beta((f, g) \cdot e_i) = \sum_{k=1}^d (q(f), p(g)) \cdot e_k \otimes u_{ki}^2 + (p(f), q(g)) \cdot e_k \otimes u_{ki}$$

for all $f \in C_0([0, 1])$, $g \in C([0, 1])$ and $i \in \{1, \dots, d\}$.

We have for $f, f' \in C_0([0, 1])$, $g, g' \in C([0, 1])$ and $i, j \in \{1, \dots, d\}$

$$\begin{aligned}
 & \langle \beta((f, g).e_i | \beta((f', g').e_j)) \rangle_{A \otimes \mathcal{A}_h(d)} \\
 &= \delta_{ij} \sum_{k=1}^d (q(\bar{f})q(f') + p(\bar{g})p(g') + p(\bar{f})p(f') + q(\bar{g})q(g')).e_k \otimes u_{ki}^2 \\
 & \quad + (q(\bar{f})p(f') + p(\bar{g})q(g') + p(\bar{f})q(f') + q(\bar{g})p(g')).e_k \otimes u_{ki} \\
 &= \delta_{ij} \sum_{k=1}^d (p(\bar{f}f') + p(\bar{g}g')).e_k \otimes u_{ki}^2 + (q(\bar{f}f') + q(\bar{g}g')).e_k \otimes u_{ki} \\
 &= \alpha(\langle (f, g).e_i | (f', g').e_j \rangle_A).
 \end{aligned}$$

Then by similar calculations as the ones done on α , we obtain that (α, β) is a coaction of $\mathcal{A}_h(d)$ on E . Furthermore we have for all $n \in \mathbb{Z}$ and all $i \in \{1, \dots, d\}$,

$$\beta(e_{ni}) = \sum_{k=1}^d e_{nk} \otimes u_{ki}^{r(k)}$$

since $p(\sin(\pi n \cdot)) = \frac{1 + (-1)^{n+1}}{2} \sin(\pi n \cdot)$ and $q(\sin(\pi n \cdot)) = \frac{1 + (-1)^n}{2} \sin(\pi n \cdot)$. Consequently, we have $\beta(V_n) \subset V_n \odot \mathcal{A}_h(d)$.

We have moreover

$$\beta(\xi_0) = \sum_{i=1}^d \beta((0, 1).e_i) = \sum_{i,k=1}^d (0, 1).e_k \otimes u_{ki}^2 + (0, 0).e_k \otimes u_{ki} = \sum_{k=1}^d (0, 1).e_k \otimes 1 = \xi_0 \otimes 1,$$

and it is clear that $(J \otimes *) \circ \beta = \beta \circ J$. Lastly, we have $\tau(p(f).e_i) = \tau(f.e_i) = \int_0^1 f$ and

$\tau(q(f).e_i) = 0$ since $\int_0^1 f(1-x)dx = \int_0^1 f(x)dx$ for all $f \in C([0, 1])$. Thus

$$\begin{aligned}
 (\tau \otimes id) \circ \alpha(f.e_i) &= \sum_{k=1}^d \tau(p(f).e_k)u_{ki}^2 + \tau(q(f).e_k)u_{ki} = \left(\int_0^1 f \right) \cdot \left(\sum_{k=1}^d u_{ki}^2 \right) \\
 &= \int_0^1 f = \tau(f.e_i),
 \end{aligned}$$

which lets us conclude that (α, β) is a filtration-preserving coaction of $\mathcal{A}_h(d)$ on E , so that the quantum symmetry group of E is the hyperoctahedral quantum group.

“Quantum isometry groups” of quotients of $[0, 1] \times \{1, \dots, d\}$

Now we look at what might be the quantum isometry group of d segments, all joined together in the point 0 and in the point 1. The idea is to check the existence of a universal object in the full subcategory of $\mathcal{C}(E, \tau, (V_i)_{i \in \mathbb{Z}}, J, W)$, consisting of the Woronowicz C^* -algebras Q whose coaction (α, β) on E satisfies $\alpha(B) \subset B \otimes Q$, where $B \subset A = C(I)^d$ is the C^* -algebra of continuous functions on the quotient of $[0, 1] \times \{1, \dots, d\}$, where for all $i, j \in \{1, \dots, d\}$, the points $(0, i)$ and $(0, j)$ are identified, and the points $(1, i)$ and $(1, j)$ are identified. That is to say:

$$B = \{(f_1, \dots, f_d) \in C(I)^d ; \forall i, j \in \{1, \dots, d\}, f_i(0) = f_j(0) \text{ and } f_i(1) = f_j(1)\}.$$

First notice that there is a filtration-preserving coaction (α_0, β_0) of $\mathcal{A}_s(d) \otimes C(\mathbb{Z}_2)$ on E (where $\mathcal{A}_s(d)$ is the quantum permutation group on d points [Wan98]) characterized for $f \in C_0(I)$, $g \in C(I)$ and $i \in \{1, \dots, d\}$ by:

$$\begin{aligned} \alpha_0(g.e_i) &= \sum_{k=1}^d p(g).e_k \otimes v_{ki} \otimes 1 + q(g).e_k \otimes v_{ki} \otimes z \\ \text{and } \beta_0((f, g).e_i) &= \sum_{k=1}^d (q(f), p(g)).e_k \otimes v_{ki} \otimes 1 + (p(f), q(g)).e_k \otimes v_{ki} \otimes z, \end{aligned}$$

where the v_{ij} 's are the canonical generators of $\mathcal{A}_s(d)$ and $z \in C(\mathbb{Z}_2)$ is the function such that $z(0) = 1$ and $z(1) = -1$.

Now let (α, β) be a filtration-preserving coaction of a Woronowicz C^* -algebra Q on E , and satisfying $\alpha(B) \subset B \otimes Q$. Recall that there is a multiplicative and unitary matrix $w \in \mathcal{M}_d(Q)$ such that for all i, n , $\beta(e_{ni}) = \sum_{k=1}^d e_{nk} \otimes w_{ki}^{r(n)}$, $\alpha(\cos(\pi n \cdot).e_i) = \sum_{k=1}^d \cos(\pi n \cdot).e_k \otimes w_{ki}^{r(n)}$, and satisfying:

- $w_{ij}^* = w_{ij}$,
- for $j \neq k$, $w_{ij}w_{ik} = w_{ji}w_{ki} = 0$,
- $\sum_{l=1}^n w_{il}^2 = \sum_{l=1}^n w_{li}^2 = 1$.

This leads to the existence of a unique morphism of Woronowicz C^* -algebras $\mu : \mathcal{A}_s(d) \rightarrow Q$ such that $\mu(v_{ij}) = w_{ij}^2$. Moreover, the condition $\alpha(B) \subset B \otimes Q$ means that

$$\forall a \in B, \forall \varepsilon \in \{0, 1\}, \forall i, j \in \{1, \dots, d\}, (ev_{(\varepsilon, i)} \otimes id) \circ \alpha(a) = (ev_{(\varepsilon, j)} \otimes id) \circ \alpha(a).$$

In particular, $(ev_{(0, i)} \otimes id) \circ \alpha(\cos(\pi \cdot).1_A) = (ev_{(0, j)} \otimes id) \circ \alpha(\cos(\pi \cdot).1_A)$ for all $i, j \in \{1, \dots, d\}$.

But we have

$$\alpha(\cos(\pi \cdot).1_A) = \alpha \left(\cos(\pi \cdot). \left(\sum_{k=1}^d e_k \right) \right) = \sum_{k,l=1}^d \cos(\pi \cdot) e_l \otimes w_{lk} = \sum_{l=1}^d \cos(\pi \cdot) e_l \otimes \left(\sum_{k=1}^d w_{lk} \right),$$

thus $(e_{V_{(0,i)}} \otimes id) \circ \alpha(\cos(\pi \cdot).1_A) = \sum_{k=1}^d w_{ki}$. Therefore we must have $\sum_{k=1}^d w_{ki} = \sum_{k=1}^d w_{kj}$ for

all $i, j \in \{1, \dots, d\}$. A simple calculation shows that $\omega = \sum_{k=1}^d w_{ki}$ satisfies $\omega^2 = 1$, $\omega^* = \omega$ and $\Delta(\omega) = \omega \otimes \omega$, so that there exists a unique morphism of Woronowicz C^* -algebras $\nu : C(\mathbb{Z}_2) \rightarrow Q$ sending z to ω (recall that $C(\mathbb{Z}_2)$ is isomorphic to the universal Woronowicz C^* -algebra generated by a unitary self-adjoint element). Since ω clearly commutes with the w_{ij}^2 's, the morphism $\mu \otimes \nu : \mathcal{A}_s(d) \otimes C(\mathbb{Z}_2) \rightarrow Q$ is well defined, and it is easy to see that it intertwines the coactions. Thus $\mathcal{A}_s(d) \otimes C(\mathbb{Z}_2)$ is a universal object in the subcategory of $\mathcal{C}(E, \tau, (V_i)_{i \in \mathbb{Z}}, J, W)$ we considered.

Note that the quantum isometry groups of other quotients of $[0, 1] \times \{1, \dots, d\}$ can be computed in a similar fashion. For example, the universal object in the category of Woronowicz C^* -algebras coacting on E in a filtration-preserving way that additionally preserve the C^* -algebra $C = \{(f_1, \dots, f_d) ; \forall i, j \in \{1, \dots, d\}, f_i(0) = f_j(0) = f_i(1) = f_j(1)\}$ (respectively the C^* -algebra $D = \{(f_1, \dots, f_d) ; \forall i, j \in \{1, \dots, d\}, f_i(0) = f_j(0)\}$) is $\mathcal{A}_h(d)$ (respectively $\mathcal{A}_s(d)$).

More generally, one may wonder whether given a Hilbert A -module equipped with an orthogonal filtration $(E, \tau, (V_i)_{i \in \mathcal{I}}, J, W)$, and given a C^* -subalgebra $B \subset A$, the full subcategory of $\mathcal{C}(E, \tau, (V_i)_{i \in \mathcal{I}}, J, W)$ consisting of the Woronowicz C^* -algebras coacting on E in a filtration-preserving way that additionally preserve B admits a universal object. Unfortunately, we have not been able to show this in the general case.

3

Classification of bicovariant differential calculi over free orthogonal Hopf algebras

We show that if two Hopf algebras are monoidally equivalent, then their categories of bicovariant differential calculi are equivalent. We then classify, for $q \in \mathbb{C}^*$ not a root of unity, the finite dimensional bicovariant differential calculi over the Hopf algebra $\mathcal{O}_q(SL_2)$. Using a monoidal equivalence between free orthogonal Hopf algebras and $\mathcal{O}_q(SL_2)$ for a given q , this leads us to the classification of finite dimensional bicovariant differential calculi over free orthogonal Hopf algebras.

Introduction

The notion of *differential calculus* over a Hopf algebra has been introduced by Woronowicz in [Wor89], with the purpose of giving a natural adaptation of differential geometry over groups, in the context of quantum groups. An important question in this topic, is the classification of bicovariant differential calculi over a given Hopf algebra, see for example [BS98], [Maj98] or [HS98].

The aim of the present paper is to classify the finite dimensional (first order) bicovariant differential calculi over an important class of Hopf algebras, namely the *free orthogonal Hopf algebras*, also called *Hopf algebras associated to non-degenerate bilinear forms* [DVL90].

Given an invertible matrix $E \in GL_n(\mathbb{C})$ with $n \geq 2$, the free orthogonal Hopf algebra $\mathcal{B}(E)$ associated with E is the universal Hopf algebra generated by a family of elements $(a_{ij})_{1 \leq i, j \leq n}$ submitted to the relations:

$$E^{-1}a^t E a = I_n = a E^{-1}a^t E,$$

where a is the matrix $(a_{ij})_{1 \leq i, j \leq n}$. Its coproduct, counit and antipode are defined by:

$$\Delta(a_{ij}) = \sum_{k=1}^n a_{ik} \otimes a_{kj}, \quad \varepsilon(a) = I_n, \quad S(a) = E^{-1}a^t E.$$

The Hopf algebra $\mathcal{B}(E)$ can also be obtained as an appropriate quotient of the FRT bialgebra associated to Yang-Baxter operators constructed by Gurevich [Gur91].

If $E\bar{E} = \lambda I_n$, with $\lambda \in \mathbb{R}^*$, there exists an involution $*$ on $\mathcal{B}(E)$ defined by $a_{ij}^* = (E^{-1}a^t E)_{ji}$, endowing $\mathcal{B}(E)$ with a Hopf $*$ -algebra structure. This Hopf $*$ -algebra corresponds to a free orthogonal compact quantum group as defined in [VDW96] or [Ban96], and is generally denoted by $\mathcal{A}_o((E^t)^{-1})$. This justifies the term “free orthogonal Hopf algebra” for $\mathcal{B}(E)$.

The starting point of our classification is a result of [Bic03b], which states that if $q \in \mathbb{C}^*$ satisfies $q^2 + \text{tr}(E^{-1}E^t)q + 1 = 0$, then the Hopf algebras $\mathcal{B}(E)$ and $\mathcal{O}_q(SL_2)$ are monoidally equivalent, *i.e.* their categories of comodules are monoidally equivalent. The proof of [Bic03b], is based on a deep result of Schauenburg [Sch96], and gives an explicit description of the correspondence between $\mathcal{B}(E)$ -comodules and $\mathcal{O}_q(SL_2)$ -comodules. We use here similar arguments to show that if two Hopf algebras are monoidally equivalent, then their categories of bicovariant differential calculi are equivalent (Theorem 3.2.6). This theorem generalizes a result of [MO99], where the two monoidally equivalent Hopf algebras are assumed to be related by a cocycle twist. Applying Theorem 3.2.6 to the Hopf algebras $\mathcal{B}(E)$ and $\mathcal{O}_q(SL_2)$, the study of bicovariant differential calculi over the Hopf algebra $\mathcal{B}(E)$ is simplified, and therefore reduces to the study of bicovariant differential calculi over $\mathcal{O}_q(SL_2)$.

This classification has been made over $\mathcal{O}_q(SL_2)$ in [HS98] for transcendental values of q (which is not the case here since q has to satisfy $q^2 + \text{tr}(E^{-1}E^t)q + 1 = 0$). Our classification uses a different approach than in [HS98], and is based on the classification of the finite dimensional $\mathcal{O}_q(SL_2)$ -Yetter-Drinfeld modules made in [Tak92].

The paper is organized as follows. We gather in Section 3.1 some known results about bicovariant differential calculi over Hopf algebras, and their formulation in terms of Yetter-Drinfeld modules. Furthermore, we show that if the category of Yetter-Drinfeld modules over a Hopf algebra H is semisimple, then the bicovariant differential calculi over H are inner. In Section 3.2, using the language of cogroupoids [Bic], we prove that two monoidally

equivalent Hopf algebras have equivalent categories of bicovariant differential calculi. We finally classify in Section 3.3 the finite dimensional bicovariant differential calculi over the Hopf algebra $\mathcal{O}_q(SL_2)$ for $q \in \mathbb{C}^*$ not a root of unity, using the fact that by [Tak92] the category of finite dimensional Yetter-Drinfeld modules over $\mathcal{O}_q(SL_2)$ is semisimple. This allows to classify the finite dimensional bicovariant differential calculi over $\mathcal{B}(E)$, provided that the solutions of the equation $q^2 + \text{tr}(E^{-1}E^t)q + 1 = 0$ are not roots of unity.

Notations and Conventions

Let H be a Hopf algebra. Its comultiplication, antipode and counit will respectively be denoted by Δ , S and ε . A coaction of a left (respectively right) H -comodule will generally be denoted by λ (respectively ρ).

We will use Sweedler's notations: $\Delta(x) = \sum x_{(1)} \otimes x_{(2)}$ for $x \in H$, and $\rho(v) = \sum v_{(0)} \otimes v_{(1)}$ for v in a right comodule V .

3.1 Bicovariant differential calculi

We start this section by recalling the definition of a bicovariant differential calculus, and of the equivalent notion, expressed in terms of Yetter-Drinfeld modules (called *reduced differential calculus* in this paper). We then prove some basic lemmas which will be useful in the sequel. The main result of this section states that if the category of (finite dimensional) Yetter-Drinfeld modules over a Hopf algebra H is semisimple, then the (finite dimensional) bicovariant differential calculi over H are inner.

We refer to [KS97] for background material on Hopf algebras and comodules.

Definition 3.1.1. Let H be a Hopf algebra. A Hopf bimodule M over H is an H -bimodule together with a left comodule structure $\lambda : M \rightarrow H \otimes M$ and a right comodule structure $\rho : M \rightarrow M \otimes H$ such that:

- $\forall x, y \in H, \forall v \in M, \lambda(x.v.y) = \Delta(x).\lambda(v).\Delta(y),$
- $\forall x, y \in H, \forall v \in M, \rho(x.v.y) = \Delta(x).\rho(v).\Delta(y),$
- $(id_H \otimes \rho) \circ \lambda = (\lambda \otimes id_H) \circ \rho.$

The category of Hopf bimodules over H , whose morphisms are the maps which are right and left linear and colinear over H , is denoted by ${}^H_H\mathcal{M}_H^H$.

Definition 3.1.2. Let H be a Hopf algebra. A (right) *Yetter-Drinfeld module* over H is a right H -module and a right H -comodule V such that:

$$\forall x \in H, \forall v \in V, \quad \sum (v \cdot x)_{(0)} \otimes (v \cdot x)_{(1)} = \sum v_{(0)} \cdot x_{(2)} \otimes S(x_{(1)})v_{(1)}x_{(3)}.$$

The category of Yetter-Drinfeld modules over H , whose morphisms are the maps which are both linear and colinear over H , is denoted by $\mathcal{YD}(H)$. The category of finite dimensional Yetter-Drinfeld modules over H is denoted by $\mathcal{YD}_f(H)$.

Example 3.1.3. Let H be a Hopf algebra. We denote by \mathbb{C}_ε the Yetter-Drinfeld module whose base-space is \mathbb{C} , with right coaction $\lambda \mapsto \lambda \otimes 1$ and right module structure defined by $\lambda \triangleleft x = \lambda \varepsilon(x)$ for $\lambda \in \mathbb{C}_\varepsilon$ and $x \in H$.

We recall from [Sch94] the correspondence between Yetter-Drinfeld modules and Hopf bimodules.

Theorem 3.1.4 ([Sch94, Theorem 5.7]). *Let H be a Hopf algebra. The categories ${}^H_H\mathcal{M}_H^H$ and $\mathcal{YD}(H)$ are equivalent.*

We describe for convenience the equivalence of categories involved in the previous theorem.

Let M be a Hopf bimodule over H , with right coaction ρ and left coaction λ . The space ${}^{inv}M = \{v \in M ; \lambda(v) = 1 \otimes v\}$ of left-coinvariant elements of M has a Yetter-Drinfeld module structure defined as follows. We have $\rho({}^{inv}M) \subset {}^{inv}M \otimes H$, and the right coaction of ${}^{inv}M$ is just the restriction of ρ to ${}^{inv}M$. The right module structure is defined by $w \triangleleft x = \sum S(x_{(1)}) \cdot w \cdot x_{(2)}$.

Conversely, given a Yetter-Drinfeld module V , then the space $H \otimes V$ can be equipped with a Hopf bimodule structure, with left and right actions given by:

$$x \cdot (y \otimes v) \cdot z = \sum x y z_{(1)} \otimes v \triangleleft z_{(2)},$$

and the right (ρ) and left (λ) coactions given by:

$$\begin{aligned} \rho(x \otimes v) &= \sum x_{(1)} \otimes v_{(0)} \otimes x_{(2)} v_{(1)}, \\ \lambda(x \otimes v) &= \sum x_{(1)} \otimes x_{(2)} \otimes v. \end{aligned}$$

We then have for $M \in {}^H_H\mathcal{M}_H^H$, $M \cong H \otimes {}^{inv}M$ and for $V \in \mathcal{YD}(H)$, $V \cong {}^{inv}(H \otimes V)$. The equivalence of categories between ${}^H_H\mathcal{M}_H^H$ and $\mathcal{YD}(H)$ is then:

$$\begin{array}{ccc} \mathcal{F} : {}^H_H\mathcal{M}_H^H & \rightarrow & \mathcal{YD}(H) & \text{with quasi-inverse } \mathcal{G} : \mathcal{YD}(H) & \rightarrow & {}^H_H\mathcal{M}_H^H \\ & & M & \mapsto & {}^{inv}M, & & V & \mapsto & H \otimes V. \end{array}$$

A morphism $f : M \rightarrow N$ in ${}^H_H\mathcal{M}_H^H$ automatically satisfies $f({}^{inv}M) \subset {}^{inv}N$, and $\mathcal{F}(f) : {}^{inv}M \rightarrow {}^{inv}N$ is just the restriction of f . Conversely, if $f : V \rightarrow W$ is a morphism of Yetter-Drinfeld modules, then $\mathcal{G}(f) = id_H \otimes f$.

Definition 3.1.5. Let H be a Hopf algebra. A (first order) *bicovariant differential calculus* (M, d) over H is a Hopf bimodule M together with a left and right comodule morphism $d : H \rightarrow M$ such that $\forall x, y \in H, d(xy) = x.d(y) + d(x).y$ and such that $M = \text{span}\{x.d(y) ; x, y \in H\}$.

A bicovariant differential calculus (M, d) is said *inner* if there exists a bi-coinvariant element $\theta \in M$ (i.e. satisfying $\rho(\theta) = \theta \otimes 1$ and $\lambda(\theta) = 1 \otimes \theta$) such that $\forall x \in H, d(x) = \theta.x - x.\theta$.

The *dimension* of a bicovariant differential calculus (M, d) is the dimension of the vector space ${}^{inv}M$.

A *morphism of bicovariant differential calculi* $f : (M, d_M) \rightarrow (N, d_N)$ is a morphism of Hopf bimodules such that $f \circ d_M = d_N$.

We denote by $\mathcal{DC}(H)$ the category of bicovariant differential calculi over H .

Bicovariant differential calculi were introduced by Woronowicz in [Wor89]. An overview is given in [KS97, Part IV]. The notion of bicovariant differential calculus has the following interpretation in terms of Yetter-Drinfeld modules.

Definition 3.1.6. Let H be a Hopf algebra. A *reduced differential calculus* over H is a Yetter-Drinfeld module V together with a surjective map $\omega : H \rightarrow V$ satisfying:

$$\begin{aligned} \forall x, y \in H, \omega(xy) &= \omega(x).y + \varepsilon(x)\omega(y) \\ \text{and } \sum \omega(x)_{(0)} \otimes \omega(x)_{(1)} &= \sum \omega(x_{(2)}) \otimes S(x_{(1)})x_{(3)}. \end{aligned}$$

A *morphism of reduced differential calculi* $f : (V, \omega_V) \rightarrow (W, \omega_W)$ is a morphism of Yetter-Drinfeld modules such that $f \circ \omega_V = \omega_W$.

We denote by $\mathcal{RDC}(H)$ the category of reduced differential calculi over H .

Lemma 3.1.7. *The equivalence of categories of Theorem 3.1.4 induces an equivalence between the categories $\mathcal{DC}(H)$ and $\mathcal{RDC}(H)$:*

$$\begin{aligned} \mathcal{F} : \mathcal{DC}(H) &\rightarrow \mathcal{RDC}(H) & \text{with quasi-inverse } \mathcal{G} : \mathcal{RDC}(H) &\rightarrow \mathcal{DC}(H) \\ (M, d) &\mapsto ({}^{inv}M, \omega_d) & (V, \omega) &\mapsto (H \otimes V, d_\omega) \end{aligned}$$

where for $x \in H, \omega_d(x) = \sum S(x_{(1)})d(x_{(2)})$ and $d_\omega(x) = \sum x_{(1)} \otimes \omega(x_{(2)})$.

PROOF. The one-to-one correspondence between bicovariant differential calculi and reduced differential calculi is described in [KS97, Section 14]. We may now focus on the functoriality of this correspondence.

If $f : (M, d_M) \rightarrow (N, d_N)$ is a morphism of bicovariant differential calculi, then the restriction of f , $\mathcal{F}(f) : {}^{inv}M \rightarrow {}^{inv}N$ satisfies for all $x \in H$,

$$\begin{aligned} \mathcal{F}(f) \circ \omega_{d_M}(x) &= \sum f(S(x_{(1)})d_M(x_{(2)})) = \sum S(x_{(1)})f(d_M(x_{(2)})) \\ &= \sum S(x_{(1)})d_N(x_{(2)}) = \omega_{d_N}(x). \end{aligned}$$

Hence $\mathcal{F}(f)$ is a morphism of reduced differential calculi.

Conversely, if $f : (V, \omega_V) \rightarrow (W, \omega_W)$ is a morphism of reduced differential calculi, then

$$(id_H \otimes f) \circ d_{\omega_V}(x) = \sum x_{(1)} \otimes f(\omega_V(x_{(2)})) = \sum x_{(1)} \otimes \omega_W(x_{(2)}) = d_{\omega_W}(x).$$

Thus $\mathcal{G}(f) = id_H \otimes f$ is a morphism of bicovariant differential calculi.

Since \mathcal{F} and \mathcal{G} are quasi-inverse to each other between the categories $\mathcal{YD}(H)$ and ${}^H_H\mathcal{M}_H^H$, it only remains to check that the natural transformation providing the equivalence $\mathcal{F} \circ \mathcal{G} \cong id$ (respectively $\mathcal{G} \circ \mathcal{F} \cong id$) consists of morphisms of reduced (respectively bicovariant) differential calculi. Let (V, ω) be a reduced differential calculus over H . The isomorphism of Yetter-Drinfeld modules

$$\begin{aligned} \theta : \mathcal{F} \circ \mathcal{G}(V) = {}^{inv}(H \otimes V) &\rightarrow V \\ x \otimes v &\mapsto \varepsilon(x)v \end{aligned}$$

satisfies for $x \in H$,

$$\begin{aligned} \theta \circ \omega_{d_\omega}(x) &= \theta \left(\sum S(x_{(1)})d_\omega(x_{(2)}) \right) = \theta \left(\sum S(x_{(1)})x_{(2)} \otimes \omega(x_{(3)}) \right) \\ &= \theta(1 \otimes \omega(x)) = \omega(x). \end{aligned}$$

Thus θ is an isomorphism of reduced differential calculi.

Conversely, let (M, d) be a bicovariant differential calculus over H . The isomorphism of Hopf bimodules

$$\begin{aligned} \gamma : \mathcal{G} \circ \mathcal{F}(M) = H \otimes {}^{inv}M &\rightarrow M \\ x \otimes v &\mapsto x.v \end{aligned}$$

satisfies for $x \in H$,

$$\gamma \circ d_{\omega_d}(x) = \gamma \left(\sum x_{(1)} \otimes \omega_d(x_{(2)}) \right) = \gamma \left(\sum x_{(1)} \otimes S(x_{(2)})d(x_{(3)}) \right)$$

$$= \sum \varepsilon(x_{(1)})d(x_{(2)}) = d(x).$$

Hence γ is a morphism of bicovariant differential calculi, which ends the proof. \square

Remark 3.1.8. Let V be a Yetter-Drinfeld module, and let $\omega : H \rightarrow V$ be a map satisfying all the axioms of a reduced differential calculus, except the surjectivity condition. Then $\text{Im}(\omega)$ is a Yetter-Drinfeld submodule of V . Indeed, we have $\omega(x).y = \omega(xy) - \varepsilon(x)\omega(y) = \omega(xy - \varepsilon(x)y) \in \text{Im}(\omega)$ for all $x, y \in H$, thus $\text{Im}(\omega)$ is a submodule of V , and $\sum \omega(x)_{(0)} \otimes \omega(x)_{(1)} = \sum \omega(x_{(2)}) \otimes S(x_{(1)})x_{(3)} \in \text{Im}(\omega) \otimes H$, thus $\text{Im}(\omega)$ is a subcomodule of V .

Definition 3.1.9. A reduced differential calculus (V, ω) is said *inner* if there exists a coinvariant element $\theta \in V$ (i.e. satisfying $\rho(\theta) = \theta \otimes 1$) such that $\forall x \in H, \omega(x) = \theta.x - \varepsilon(x)\theta$.

A reduced differential calculus $\omega : H \rightarrow V$ is said *simple* if V is a simple Yetter-Drinfeld module. That is to say, if there is no non-trivial subspace $W \subset V$, which is both a submodule and a subcomodule of V .

Let $(V, \omega), (W_1, \omega_1), (W_2, \omega_2)$ be reduced differential calculi. We say that (V, ω) is the direct sum of (W_1, ω_1) and (W_2, ω_2) and we write $(V, \omega) = (W_1, \omega_1) \oplus (W_2, \omega_2)$, if $V = W_1 \oplus W_2$ and if for all $x \in H, \omega(x) = (\omega_1(x), \omega_2(x))$.

Note that the direct sum of reduced differential calculi is not always well defined. The problem is that if (V, ω_V) and (W, ω_W) are reduced differential calculi over a Hopf algebra H , then the map

$$\begin{aligned} \omega : H &\rightarrow V \oplus W \\ x &\mapsto (\omega_V(x), \omega_W(x)) \end{aligned}$$

can fail to be surjective. We give in the next lemma a necessary and a sufficient condition for the existence of the direct sum of simple reduced differential calculi.

Lemma 3.1.10. *Let $(V_1, \omega_1), \dots, (V_n, \omega_n)$ be simple reduced differential calculi over a Hopf algebra H . We set*

$$\begin{aligned} \omega : H &\rightarrow V = \bigoplus_{i=1}^n V_i \\ x &\mapsto (w_1(x), \dots, w_n(x)) \end{aligned} .$$

If the V_i 's are two-by-two non isomorphic as Yetter-Drinfeld modules, then (V, ω) is a reduced differential calculus.

Conversely, if (V, ω) is a reduced differential calculus, then the reduced differential calculi (V_i, ω_i) are two-by-two non-isomorphic.

PROOF. The map ω clearly satisfies all the axioms of a reduced differential calculus, except the surjectivity condition. In order to prove the lemma, we thus have to examine

under which conditions ω is onto. Assume that the V_i 's are two-by-two non-isomorphic (as Yetter-Drinfeld modules). According to Remark 3.1.8, the image of ω is a Yetter-Drinfeld submodule of V . There is therefore a subset $\mathcal{I} \subset \{1, \dots, n\}$ such that there exists an isomorphism of Yetter-Drinfeld modules $f : \text{Im}(\omega) \rightarrow \bigoplus_{i \in \mathcal{I}} V_i$. For $k \in \{1, \dots, n\}$, we denote by $\pi_k : V \rightarrow V_k$ the canonical projection. The map $\pi_k \circ \omega = \omega_k$ is onto, thus the restriction of π_k to $\text{Im}(\omega)$ is also onto. This means that π_k induces a non-zero morphism of Yetter-Drinfeld modules $\bigoplus_{i \in \mathcal{I}} V_i \rightarrow V_k$, hence an isomorphism of Yetter-Drinfeld modules $V_l \cong V_k$, with $l \in \mathcal{I}$. Since by hypothesis the V_i 's are two-by-two non-isomorphic, we have $k = l$, hence $k \in \mathcal{I}$. Thus $\mathcal{I} = \{1, \dots, n\}$, $\text{Im}(\omega) = V$, and we conclude that $\omega : H \rightarrow V$ is a reduced differential calculus.

Assume now that there is an isomorphism $f : (V_j, \omega_j) \rightarrow (V_i, \omega_i)$ with $i \neq j$. We denote by $\eta : H \rightarrow V_i \oplus V_j$ the map defined by the composition

$$H \xrightarrow{\omega} V \xrightarrow{\pi_i \oplus \pi_j} V_i \oplus V_j \xrightarrow{id \oplus f} V_i \oplus V_i.$$

The map η is clearly not surjective, since $\eta(x) = (\omega_i(x), f \circ \omega_j(x)) = (\omega_i(x), \omega_i(x))$. This implies that ω is not surjective, since $\pi_i \oplus \pi_j$ and $id \oplus f$ are both surjective. \square

Lemma 3.1.11. *Let V be a simple Yetter-Drinfeld module over a Hopf algebra H , admitting a non-zero right-coinvariant element $\theta \in V$. If V is not isomorphic to the Yetter-Drinfeld module \mathbb{C}_ε (of Example 3.1.3), then the map*

$$\begin{aligned} \omega_\theta : H &\rightarrow V \\ x &\mapsto \theta.x - \varepsilon(x)\theta \end{aligned}$$

defines a reduced differential calculus over H .

PROOF. We have for $x \in H$,

$$\begin{aligned} \omega_\theta(x).y + \varepsilon(x)\omega_\theta(y) &= (\theta.x - \varepsilon(x)\theta).y + \varepsilon(x)(\theta.y - \varepsilon(y)\theta) \\ &= (\theta.x).y - \varepsilon(x)\varepsilon(y)\theta = \omega_\theta(xy) \end{aligned}$$

and

$$\begin{aligned} \rho \circ \omega_\theta(x) &= \rho(\theta.x) - \varepsilon(x)\theta \otimes 1 \\ &= \sum \theta.x_{(2)} \otimes S(x_{(1)}) \cdot 1.x_{(3)} - \varepsilon(x)\theta \otimes 1 \\ &= \sum \theta.x_{(2)} \otimes S(x_{(1)})x_{(3)} - \sum \varepsilon(x_{(2)})\theta \otimes S(x_{(1)})x_{(3)} \\ &= \sum \omega_\theta(x_{(2)}) \otimes S(x_{(1)})x_{(3)}. \end{aligned}$$

By Remark 3.1.8, the image of ω_θ is thus a Yetter-Drinfeld submodule of V . Since V is simple, the image of ω_θ is either V , in which case ω_θ is indeed a reduced differential calculus, or $\text{Im}(\omega_\theta) = (0)$. In that case, since θ is coinvariant and $\theta.x = \varepsilon(x)\theta$ for all $x \in H$, the map $\mu : \mathbb{C}_\varepsilon \rightarrow V$ given by $\mu(\lambda) = \lambda\theta$ is a non-zero morphism between simple Yetter-Drinfeld modules, hence an isomorphism. \square

The end of this section is devoted to the proof of the following lemma.

Lemma 3.1.12. *Let H be a Hopf algebra such that the category $\mathcal{YD}_f(H)$ is semisimple (i.e. each finite dimensional Yetter-Drinfeld module over H can be decomposed into a direct sum of simple Yetter-Drinfeld modules). Then each finite dimensional reduced differential calculus over H is inner.*

Definition 3.1.13. Let (V, ω) be a reduced differential calculus. We denote by V_ω the Yetter-Drinfeld module over H defined as follows. As a right comodule, $V_\omega = V \oplus \mathbb{C}$ (where the H -comodule structure on \mathbb{C} is the canonical one: $\lambda \rightarrow \lambda \otimes 1$). Its right module structure is defined for $v \in V$, $\lambda \in \mathbb{C}$ and $x \in H$ by: $(v, \lambda).x = (v.x + \lambda\omega(x), \lambda\varepsilon(x))$. Let us check that this formula defines an H -module structure on $V \oplus \mathbb{C}$. We have

$$\begin{aligned} ((v, \lambda).x).y &= (v.x + \lambda\omega(x), \lambda\varepsilon(x)).y = ((v.x + \lambda\omega(x)).y + \lambda\varepsilon(x)\omega(y), \lambda\varepsilon(x)\varepsilon(y)) \\ &= (v.(xy) + \lambda\omega(xy), \lambda\varepsilon(xy)) = (v, \lambda).(xy), \end{aligned}$$

and the other axioms of a right module are clearly satisfied. Before checking that the Yetter-Drinfeld condition is satisfied on V_ω , let us note that, denoting by $j : V \rightarrow V \oplus \mathbb{C}$ the canonical injection, and by $p : V \oplus \mathbb{C} \rightarrow \mathbb{C}_\varepsilon$ the canonical projection, then clearly j and p are both module and comodule maps, and the short sequence:

$$0 \longrightarrow V \xrightarrow{j} V_\omega \xrightarrow{p} \mathbb{C}_\varepsilon \longrightarrow 0$$

is exact. Since V is a Yetter-Drinfeld module and $j : V \rightarrow V_\omega$ is a module and comodule morphism, the Yetter-Drinfeld condition:

$$\forall x \in H, \rho(w.x) = \sum w_{(0)}.x_{(2)} \otimes S(x_{(1)})w_{(1)}x_{(3)}$$

is automatically satisfied for $w \in j(V)$. Hence it only remains to check that the Yetter-Drinfeld condition is also satisfied on \mathbb{C} , that is, that for all x in H , $\rho((0, 1).x) = \sum (0, 1).x_{(2)} \otimes S(x_{(1)})x_{(3)}$. We have for $x \in H$

$$\sum (0, 1).x_{(2)} \otimes S(x_{(1)})x_{(3)} = \sum (\omega(x_{(2)}), \varepsilon(x_{(2)})) \otimes S(x_{(1)})x_{(3)}$$

$$\begin{aligned}
 &= (j \otimes id) \left(\sum \omega(x_{(2)}) \otimes S(x_{(1)})x_{(3)} \right) + (0, 1) \otimes \varepsilon(x) \\
 &= (j \otimes id) \circ \rho(\omega(x)) + (0, 1) \otimes \varepsilon(x) \\
 &= \rho(j(\omega(x))) + \rho(0, \varepsilon(x)) = \rho(\omega(x), \varepsilon(x)) = \rho((0, 1).x),
 \end{aligned}$$

hence V_ω is indeed a Yetter-Drinfeld module, and

$$0 \longrightarrow V \xrightarrow{j} V_\omega \xrightarrow{p} \mathbb{C}_\varepsilon \longrightarrow 0$$

is a short exact sequence of Yetter-Drinfeld modules.

Lemma 3.1.14. *A reduced differential calculus (V, ω) is inner if and only if the short exact sequence of Yetter-Drinfeld modules*

$$0 \longrightarrow V \xrightarrow{j} V_\omega \xrightarrow{p} \mathbb{C}_\varepsilon \longrightarrow 0$$

splits.

PROOF. Assume first that (V, ω) is inner. Let $\theta \in V$ be a right-coinvariant element such that $\omega = x \mapsto \theta.x - \varepsilon(x)\theta$. We set

$$\begin{aligned}
 r : \quad V_\omega &\rightarrow V \\
 (v, \lambda) &\mapsto v + \lambda\theta.
 \end{aligned}$$

It is a comodule morphism since for $v \in V, \lambda \in \mathbb{C}$,

$$\begin{aligned}
 (r \otimes id) \circ \rho(v, \lambda) &= (r \otimes id) \circ \rho \circ j(v) + (r \otimes id)((0, \lambda) \otimes 1) \\
 &= ((r \circ j) \otimes id) \circ \rho(v) + \lambda\theta \otimes 1 = \rho(v) + \lambda\rho(\theta) = \rho \circ r(v, \lambda).
 \end{aligned}$$

And we have for $v \in V, \lambda \in \mathbb{C}$ and $x \in H$,

$$\begin{aligned}
 r((v, \lambda).x) &= r(v.x + \lambda\omega(x), \lambda\varepsilon(x)) = v.x + \lambda\theta.x - \lambda\varepsilon(x)\theta + \lambda\varepsilon(x)\theta \\
 &= (v + \lambda\theta).x = r(v, \lambda).x.
 \end{aligned}$$

Hence r is a Yetter-Drinfeld module morphism satisfying $r \circ j = id_V$, so that the above sequence splits.

Assume conversely that the short exact sequence of Yetter-Drinfeld modules associated to (V, ω) splits:

$$0 \longrightarrow V \xrightarrow{j} V_\omega = V \oplus \mathbb{C} \xrightarrow{p} \mathbb{C}_\varepsilon \longrightarrow 0.$$

We set $\theta = r(0, 1)$. Then $\rho(\theta) = \rho \circ r(0, 1) = (r \otimes id) \circ \rho(0, 1) = \theta \otimes 1$ and for $x \in H$,

$$\begin{aligned} \theta.x - \varepsilon(x)\theta &= r((0, 1).x) - r(0, \varepsilon(x)) = r((\omega(x), \varepsilon(x)) - (0, \varepsilon(x))) \\ &= r(j(\omega(x))) = \omega(x). \end{aligned}$$

Hence the result. □

Lemma 3.1.12 follows immediately.

3.2 Monoidal equivalence

We show in this section that if two Hopf algebras are monoidally equivalent, then their categories of bicovariant differential calculi are also equivalent. In order to describe the equivalence between the categories $\mathcal{DC}(H)$ and $\mathcal{DC}(L)$, when H and L are monoidally equivalent Hopf algebras, we will need some definitions and results about cogroupoids, which we recall here. We refer to [Bic] for a survey on the subject.

Definition 3.2.1. A cocategory \mathcal{C} consists of:

- a set of objects $\text{ob}(\mathcal{C})$,
- for all $X, Y \in \text{ob}(\mathcal{C})$, an algebra $\mathcal{C}(X, Y)$,
- for all $X, Y, Z \in \text{ob}(\mathcal{C})$, algebra morphisms $\Delta_{X,Y}^Z : \mathcal{C}(X, Y) \rightarrow \mathcal{C}(X, Z) \otimes \mathcal{C}(Z, Y)$ and $\varepsilon_X : \mathcal{C}(X, X) \rightarrow \mathbb{C}$ such that for all $X, Y, Z, T \in \text{ob}(\mathcal{C})$, the following diagrams commute:

$$\begin{array}{ccc} \mathcal{C}(X, Y) & \xrightarrow{\Delta_{X,Y}^Z} & \mathcal{C}(X, Z) \otimes \mathcal{C}(Z, Y) \\ \Delta_{X,Y}^T \downarrow & & \downarrow id \otimes \Delta_{Z,Y}^T \\ \mathcal{C}(X, T) \otimes \mathcal{C}(T, Y) & \xrightarrow{\Delta_{X,T}^Z \otimes id} & \mathcal{C}(X, Z) \otimes \mathcal{C}(Z, T) \otimes \mathcal{C}(T, Y) \end{array}$$

$$\begin{array}{ccc} \mathcal{C}(X, Y) & \xrightarrow{\Delta_{X,Y}^X} & \mathcal{C}(X, X) \otimes \mathcal{C}(X, Y) \\ \parallel & & \downarrow \varepsilon_X \otimes id \\ & & \mathcal{C}(X, Y) \end{array} \qquad \begin{array}{ccc} \mathcal{C}(X, Y) & \xrightarrow{\Delta_{X,Y}^Y} & \mathcal{C}(X, Y) \otimes \mathcal{C}(Y, Y) \\ \parallel & & \downarrow id \otimes \varepsilon_Y \\ & & \mathcal{C}(X, Y) \end{array}$$

A cocategory is said to be *connected* if for all $X, Y \in \text{ob}(\mathcal{C})$, $\mathcal{C}(X, Y)$ is a non-zero algebra.

Definition 3.2.2. A *cogroupoid* \mathcal{C} is a cocategory equipped with linear maps $S_{X,Y} : \mathcal{C}(X, Y) \rightarrow \mathcal{C}(Y, X)$ such that for all $X, Y \in \text{ob}(\mathcal{C})$, the following diagrams commute:

$$\begin{array}{ccccc} \mathcal{C}(X, X) & \xrightarrow{\varepsilon_X} & \mathbb{C} & \xrightarrow{u} & \mathcal{C}(Y, X) \\ \Delta_{X,X}^Y \downarrow & & & & m \uparrow \\ \mathcal{C}(X, Y) \otimes \mathcal{C}(Y, X) & \xrightarrow{S_{X,Y} \otimes \text{id}} & \mathcal{C}(Y, X) \otimes \mathcal{C}(Y, X) & & \end{array}$$

$$\begin{array}{ccccc} \mathcal{C}(X, X) & \xrightarrow{\varepsilon_X} & \mathbb{C} & \xrightarrow{u} & \mathcal{C}(X, Y) \\ \Delta_{X,X}^Y \downarrow & & & & m \uparrow \\ \mathcal{C}(X, Y) \otimes \mathcal{C}(Y, X) & \xrightarrow{\text{id} \otimes S_{Y,X}} & \mathcal{C}(X, Y) \otimes \mathcal{C}(X, Y) & & \end{array}$$

where m denotes the multiplication and u the unit.

We will use Sweedler notations for cogroupoids:

$$\text{for } a^{X,Y} \in \mathcal{C}(X, Y), \Delta_{X,Y}^Z(a^{X,Y}) = \sum a_{(1)}^{X,Z} \otimes a_{(2)}^{Z,Y}.$$

Theorem 3.2.3 ([Bic, Proposition 1.16 and Theorem 6.1]). *Let H and L be two Hopf algebras such that there exists a linear monoidal equivalence between their categories of right comodules \mathcal{M}^H and \mathcal{M}^L . Then there exists a linear monoidal equivalence between $\mathcal{YD}(H)$ and $\mathcal{YD}(L)$, inducing an equivalence between the categories of finite dimensional Yetter-Drinfeld modules $\mathcal{YD}_f(H)$ and $\mathcal{YD}_f(L)$.*

Let us recall the construction of this equivalence. As a consequence of [Sch96], restated in the context of cogroupoids, the existence of a linear monoidal equivalence between the categories \mathcal{M}^H and \mathcal{M}^L is equivalent to the existence of a connected cogroupoid \mathcal{C} and two objects $X, Y \in \text{ob}(\mathcal{C})$ such that $H \cong \mathcal{C}(X, X)$ and $L \cong \mathcal{C}(Y, Y)$ (see [Bic, Theorem 2.10]). Then the equivalence between the categories $\mathcal{YD}(H)$ and $\mathcal{YD}(L)$ is given by the functor:

$$\begin{aligned} \mathcal{F}_X^Y : \mathcal{YD}(\mathcal{C}(X, X)) &\rightarrow \mathcal{YD}(\mathcal{C}(Y, Y)) \\ V &\mapsto V \square_{\mathcal{C}(X, X)} \mathcal{C}(X, Y), \end{aligned}$$

where $V \square_{\mathcal{C}(X, X)} \mathcal{C}(X, Y)$ is the space:

$$\left\{ \sum_i v_i \otimes a_i^{X,Y} \in V \otimes \mathcal{C}(X, Y) ; \sum v_{i(0)} \otimes v_{i(1)}^{X,X} \otimes a_i^{X,Y} = \sum v_i \otimes a_{i(1)}^{X,X} \otimes a_{i(2)}^{X,Y} \right\}.$$

The right $L \cong \mathcal{C}(Y, Y)$ -module structure of $V \square_{\mathcal{C}(X, X)} \mathcal{C}(X, Y)$ is given by:

$$\left(\sum_i v_i \otimes a_i^{X, Y} \right) \triangleleft b^{Y, Y} = \sum_i v_i \cdot b_{(2)}^{X, X} \otimes S_{Y, X}(b_{(1)}^{Y, X}) a_i b_{(3)}^{X, Y}$$

and its right comodule structure is given by the map $id_V \otimes \Delta_{X, Y}^Y$. The quasi-inverse of \mathcal{F}_X^Y is the functor \mathcal{F}_Y^X . By [Bic, Proposition 1.16], the functor \mathcal{F}_X^Y induces an equivalence between the categories of finite dimensional Yetter-Drinfeld modules $\mathcal{YD}_f(H)$ and $\mathcal{YD}_f(L)$.

Lemma 3.2.4. *Let \mathcal{C} be a cogroupoid and let X, Y be in $\text{ob}(\mathcal{C})$ such that $\mathcal{C}(Y, X) \neq (0)$. Let $\omega : \mathcal{C}(X, X) \rightarrow V$ be a reduced differential calculus over $\mathcal{C}(X, X)$. The map*

$$\begin{aligned} \bar{\omega} : \mathcal{C}(Y, Y) &\rightarrow V \square_{\mathcal{C}(X, X)} \mathcal{C}(X, Y) \\ a^{Y, Y} &\mapsto \sum \omega(a_{(2)}^{X, X}) \otimes S_{Y, X}(a_{(1)}^{Y, X}) a_{(3)}^{X, Y} \end{aligned}$$

is a reduced differential calculus over $\mathcal{C}(Y, Y)$.

PROOF. We already know, by the previous theorem, that $V \square_{\mathcal{C}(X, X)} \mathcal{C}(X, Y)$ is a Yetter-Drinfeld module over $\mathcal{C}(Y, Y)$. We firstly have to check that the map $\bar{\omega}$ is well defined, which is to say, we have to check that

$$\sum \omega(a_{(2)}^{X, X})_{(0)} \otimes \omega(a_{(2)}^{X, X})_{(1)} \otimes S_{Y, X}(a_{(1)}^{Y, X}) a_{(3)}^{X, Y} = \sum \omega(a_{(2)}^{X, X}) \otimes \Delta_{X, Y}^X \left(S_{Y, X}(a_{(1)}^{Y, X}) a_{(3)}^{X, Y} \right).$$

On the one hand, we have:

$$\sum \omega(a_{(2)}^{X, X})_{(0)} \otimes \omega(a_{(2)}^{X, X})_{(1)} \otimes S_{Y, X}(a_{(1)}^{Y, X}) a_{(3)}^{X, Y} = \sum \omega(a_{(3)}^{X, X}) \otimes S_{X, X}(a_{(2)}^{X, X}) a_{(4)}^{X, X} \otimes S_{Y, X}(a_{(1)}^{Y, X}) a_{(5)}^{X, Y}.$$

And on the other hand,

$$\begin{aligned} \Delta_{X, Y}^X \left(S_{Y, X}(a^{Y, X}) b^{X, Y} \right) &= \Delta_{X, Y}^X(S_{Y, X}(a^{Y, X})) \Delta_{X, Y}^X(b^{X, Y}) \\ &= \sum \left(S_{X, X}(a_{(2)}^{X, X}) \otimes S_{Y, X}(a_{(1)}^{Y, X}) \right) \left(b_{(1)}^{X, X} \otimes b_{(2)}^{X, Y} \right) \\ &= \sum S_{X, X}(a_{(2)}^{X, X}) b_{(1)}^{X, X} \otimes S_{Y, X}(a_{(1)}^{Y, X}) b_{(2)}^{X, Y} \end{aligned}$$

so that

$$\sum \omega(a_{(2)}^{X, X}) \otimes \Delta_{X, Y}^X \left(S_{Y, X}(a_{(1)}^{Y, X}) a_{(3)}^{X, Y} \right) = \sum \omega(a_{(3)}^{X, X}) \otimes S_{X, X}(a_{(2)}^{X, X}) a_{(4)}^{X, X} \otimes S_{Y, X}(a_{(1)}^{Y, X}) a_{(5)}^{X, Y}$$

which shows that $\bar{\omega} : \mathcal{C}(Y, Y) \rightarrow V \square_{\mathcal{C}(X, X)} \mathcal{C}(X, Y)$ is well defined.

We have

$$\begin{aligned}\bar{\omega}(a^{Y,Y}) \triangleleft b^{Y,Y} &= \sum \left(\omega(a_{(2)}^{X,X}) \otimes S_{Y,X}(a_{(1)}^{Y,X}) a_{(3)}^{X,Y} \right) \triangleleft b^{Y,Y} \\ &= \sum \omega(a_{(2)}^{X,X}) \cdot b_{(2)}^{X,X} \otimes S_{Y,X}(b_{(1)}^{Y,X}) S_{Y,X}(a_{(1)}^{Y,X}) a_{(3)}^{X,Y} b_{(3)}^{X,Y}.\end{aligned}$$

Consequently, we have

$$\begin{aligned}\bar{\omega}(a^{Y,Y} b^{Y,Y}) &= \sum \omega(a_{(2)}^{X,X} b_{(2)}^{X,X}) \otimes S_{Y,X}(a_{(1)}^{Y,X} b_{(1)}^{Y,X}) a_{(3)}^{X,Y} b_{(3)}^{X,Y} \\ &= \sum \omega(a_{(2)}^{X,X}) \cdot b_{(2)}^{X,X} \otimes S_{Y,X}(a_{(1)}^{Y,X} b_{(1)}^{Y,X}) a_{(3)}^{X,Y} b_{(3)}^{X,Y} \\ &\quad + \sum \varepsilon_X(a_{(2)}^{X,X}) \omega(b_{(2)}^{X,X}) \otimes S_{Y,X}(a_{(1)}^{Y,X} b_{(1)}^{Y,X}) a_{(3)}^{X,Y} b_{(3)}^{X,Y} \\ &= \bar{\omega}(a^{Y,Y}) \triangleleft b^{Y,Y} + \sum \omega(b_{(2)}^{X,X}) \otimes S_{Y,X}(b_{(1)}^{Y,X}) S_{Y,X}(a_{(1)}^{Y,X}) a_{(2)}^{X,Y} b_{(3)}^{X,Y} \\ &= \bar{\omega}(a^{Y,Y}) \triangleleft b^{Y,Y} + \varepsilon_Y(a^{Y,Y}) \bar{\omega}(b^{Y,Y}).\end{aligned}$$

Denoting $\rho = id_V \otimes \Delta_{X,Y}^Y$ the $\mathcal{C}(Y, Y)$ -comodule structure of $V \square_{\mathcal{C}(X,X)} \mathcal{C}(X, Y)$, we have for all $a^{Y,Y} \in \mathcal{C}(Y, Y)$,

$$\begin{aligned}\rho \circ \bar{\omega}(a^{Y,Y}) &= \sum \omega(a_{(2)}^{X,X}) \otimes \Delta_{X,Y}^Y \left(S_{Y,X}(a_{(1)}^{Y,X}) a_{(3)}^{X,Y} \right) \\ &= \sum \omega(a_{(3)}^{X,X}) \otimes S_{Y,X}(a_{(2)}^{Y,X}) a_{(4)}^{X,Y} \otimes S_{Y,Y}(a_{(1)}^{Y,Y}) a_{(5)}^{Y,Y} \\ &= \sum \bar{\omega}(a_{(2)}^{Y,Y}) \otimes S_{Y,Y}(a_{(1)}^{Y,Y}) a_{(3)}^{Y,Y}.\end{aligned}$$

Now, in order to prove the lemma, it only remains to check that $\bar{\omega}$ is onto. Let $\sum_i v_i \otimes a_i^{X,Y}$ be in $V \square_{\mathcal{C}(X,X)} \mathcal{C}(X, Y)$ and let $\varphi : \mathcal{C}(Y, X) \rightarrow \mathbb{C}$ be a linear map satisfying $\varphi(1) = 1$. We have

$$\sum_i v_{i(0)} \otimes v_{i(1)}^{X,X} \otimes a_i^{X,Y} = \sum_i v_i \otimes a_{i(1)}^{X,X} \otimes a_{i(2)}^{X,Y}$$

since $\sum_i v_i \otimes a_i^{X,Y}$ is in $V \square_{\mathcal{C}(X,X)} \mathcal{C}(X, Y)$. Applying $id_V \otimes \Delta_{X,X}^Y \otimes id_{\mathcal{C}(X,Y)}$ on both sides, we find

$$\sum_i v_{i(0)} \otimes v_{i(1)}^{X,Y} \otimes v_{i(2)}^{Y,X} \otimes a_i^{X,Y} = \sum_i v_i \otimes a_{i(1)}^{X,Y} \otimes a_{i(2)}^{Y,X} \otimes a_{i(3)}^{X,Y}.$$

This shows that

$$\begin{aligned}\sum_i \varphi \left(v_{i(2)}^{Y,X} S_{X,Y}(a_{i(3)}^{X,Y}) \right) v_{i(0)} \otimes v_{i(1)}^{X,Y} &= \sum_i \varphi \left(a_{i(2)}^{Y,X} S_{X,Y}(a_{i(3)}^{X,Y}) \right) v_i \otimes a_{i(1)}^{X,Y} \\ &= \sum_i \varepsilon_Y(a_{i(2)}^{Y,Y}) v_i \otimes a_{i(1)}^{X,Y} = \sum_i v_i \otimes a_i^{X,Y}.\end{aligned}$$

Since $\omega : \mathcal{C}(X, X) \rightarrow V$ is onto, there exists $b_i^{X,X} \in \mathcal{C}(X, X)$ such that $\omega(b_i^{X,X}) = v_i$.

We have then

$$\sum v_{i(0)} \otimes v_{i(1)}^{X,X} = \sum \omega(b_i^{X,X})_{(0)} \otimes \omega(b_i^{X,X})_{(1)} = \sum \omega(b_{i(2)}^{X,X}) \otimes S_{X,X}(b_{i(1)}^{X,X})b_{i(3)}^{X,X},$$

so that

$$\sum v_{i(0)} \otimes v_{i(1)}^{X,Y} \otimes v_{i(2)}^{Y,X} = \sum \omega(b_{i(3)}^{X,X}) \otimes S_{Y,X}(b_{i(2)}^{Y,X})b_{i(4)}^{X,Y} \otimes S_{X,Y}(b_{i(1)}^{X,Y})b_{i(5)}^{Y,X}.$$

We have therefore

$$\begin{aligned} \sum_i v_i \otimes a_i^{X,Y} &= \sum_i \varphi \left(v_{i(2)}^{Y,X} S_{X,Y}(a_i^{X,Y}) \right) v_{i(0)} \otimes v_{i(1)}^{X,Y} \\ &= \sum_i \varphi \left(S_{X,Y}(b_{i(1)}^{X,Y}) b_{i(5)}^{Y,X} S_{X,Y}(a_i^{X,Y}) \right) \omega(b_{i(3)}^{X,X}) \otimes S_{Y,X}(b_{i(2)}^{Y,X}) b_{i(4)}^{X,Y} \\ &= \sum_i \varphi \left(S_{X,Y}(b_{i(1)}^{X,Y}) b_{i(3)}^{Y,X} S_{X,Y}(a_i^{X,Y}) \right) \overline{\omega}(b_{i(2)}^{Y,Y}) \end{aligned}$$

which allows to conclude that $\overline{\omega}$ is onto. \square

Remark 3.2.5. If $\omega : \mathcal{C}(X, X) \rightarrow V$ is an inner reduced differential calculus, let $\theta \in V$ be a right-coinvariant element such that $\forall a^{X,X} \in \mathcal{C}(X, X)$, $\omega(a^{X,X}) = \theta \cdot a^{X,X} - \varepsilon_X(a^{X,X})\theta$. We then have

$$\begin{aligned} \forall a^{Y,Y} \in \mathcal{C}(Y, Y), \overline{\omega}(a^{Y,Y}) &= \sum \omega(a_{(2)}^{X,X}) \otimes S_{Y,X}(a_{(1)}^{Y,X})a_{(3)}^{X,Y} \\ &= \sum (\theta \cdot a_{(2)}^{X,X} - \varepsilon_X(a_{(2)}^{X,X})\theta) \otimes S_{Y,X}(a_{(1)}^{Y,X})a_{(3)}^{X,Y} \\ &= \sum \theta \cdot a_{(2)}^{X,X} \otimes S_{Y,X}(a_{(1)}^{Y,X})a_{(3)}^{X,Y} - \theta \otimes \varepsilon_Y(a^{Y,Y}) \\ &= (\theta \otimes 1) \triangleleft a^{Y,Y} - \varepsilon_Y(a^{Y,Y})(\theta \otimes 1). \end{aligned}$$

Consequently, $\overline{\omega}$ is an inner reduced differential calculus, whose corresponding right-coinvariant element is $\theta \otimes 1$.

Combining the previous lemma with Theorem 3.2.3, we obtain the main result of this section. It generalizes a result of [MO99], where the two monoidally equivalent Hopf algebras are assumed to be related by a cocycle twist.

Theorem 3.2.6. *Let H and L be two Hopf algebras such that there exists a linear monoidal equivalence between their categories of right comodules \mathcal{M}^H and \mathcal{M}^L . Then there exists an equivalence between the categories:*

- of bicovariant differential calculi $DC(H)$ and $DC(L)$,

- of finite dimensional bicovariant differential calculi $\mathcal{DC}_f(H)$ and $\mathcal{DC}_f(L)$.

PROOF. Let \mathcal{C} be a connected cogroupoid such that there exist $X, Y \in \text{ob}(\mathcal{C})$ satisfying $\mathcal{C}(X, X) \cong H$ and $\mathcal{C}(Y, Y) \cong L$. We consider the functor induced by Theorem 3.2.3 and the previous lemma:

$$\begin{aligned} \mathcal{F}_X^Y : \mathcal{RDC}(H) &\rightarrow \mathcal{RDC}(L) \\ (V, \omega) &\mapsto (V \square_H \mathcal{C}(X, Y), \bar{\omega}) \end{aligned}$$

which sends a morphism $f : V \rightarrow W$ in $\mathcal{RDC}(H)$, to $\mathcal{F}_X^Y(f) = f \otimes id : V \square_H \mathcal{C}(X, Y) \rightarrow W \square_H \mathcal{C}(X, Y)$. It is known to be a morphism of Yetter-Drinfeld modules, and one easily checks that it is a morphism of reduced differential calculi.

Since \mathcal{F}_X^Y is an equivalence between the categories of Yetter-Drinfeld modules over H and L , with quasi-inverse \mathcal{F}_Y^X , we only have to check that the natural transformation providing the equivalence $\mathcal{F}_X^Y \circ \mathcal{F}_Y^X \cong id$ consists of morphisms of reduced differential calculi. In other words, we have to check that, for all $(V, \omega) \in \mathcal{RDC}(H)$, the morphism of Yetter-Drinfeld modules:

$$\begin{aligned} \theta_V : V &\rightarrow (V \square_H \mathcal{C}(X, Y)) \square_L \mathcal{C}(Y, X) \\ v &\mapsto \sum v_{(0)} \otimes v_{(1)}^{X,Y} \otimes v_{(2)}^{Y,X} \end{aligned}$$

is a morphism of reduced differential calculi. We have for $a^{X,X} \in H \cong \mathcal{C}(X, X)$,

$$\begin{aligned} \theta_V \circ \omega(a^{X,X}) &= \sum (id \otimes \Delta_{X,X}^Y) \left(\omega(a_{(2)}^{X,X}) \otimes S_{X,X}(a_{(1)}^{X,X}) a_{(3)}^{X,X} \right) \\ &= \sum \omega(a_{(3)}^{X,X}) \otimes S_{Y,X}(a_{(2)}^{Y,X}) a_{(4)}^{X,Y} \otimes S_{X,Y}(a_{(1)}^{X,Y}) a_{(5)}^{Y,X} \\ &= \sum \bar{\omega}(a_{(2)}^{Y,Y}) \otimes S_{X,Y}(a_{(1)}^{X,Y}) a_{(3)}^{Y,X} = \bar{\bar{\omega}}(a^{X,X}). \end{aligned}$$

Thus θ_V is a morphism of reduced differential calculi, and \mathcal{F}_X^Y is an equivalence of categories. Gathering this with Lemma 3.1.7, we obtain an equivalence $\mathcal{DC}(H) \cong \mathcal{RDC}(H) \cong \mathcal{RDC}(L) \cong \mathcal{DC}(L)$, inducing an equivalence $\mathcal{DC}_f(H) \cong \mathcal{RDC}_f(H) \cong \mathcal{RDC}_f(L) \cong \mathcal{DC}_f(L)$. \square

3.3 Classification of bicovariant differential calculi over free orthogonal Hopf algebras

In this section, we gather the results of the previous sections in order to classify the finite dimensional reduced differential calculi over the free orthogonal Hopf algebras. To this end, we start by classifying the finite dimensional reduced differential calculi over the Hopf algebra $\mathcal{O}_q(SL_2)$, when $q \in \mathbb{C}^*$ is not a root of unity. This classification is based on the

classification of finite dimensional $\mathcal{O}_q(SL_2)$ -Yetter-Drinfeld modules made in [Tak92], and Lemma 3.1.12.

Definition 3.3.1. Let $q \in \mathbb{C}^*$ be not a root of unity. $\mathcal{O}_q(SL_2)$ is the Hopf algebra generated by four elements a, b, c, d subject to the relations:

$$\begin{cases} ba = qab, ca = qac, db = qbd, dc = qcd, bc = cb, \\ ad - q^{-1}bc = da - qbc = 1. \end{cases}$$

Its comultiplication, counit and antipode are defined by:

$$\begin{aligned} \Delta(a) &= a \otimes a + b \otimes c, & \Delta(b) &= a \otimes b + b \otimes d, \\ \Delta(c) &= c \otimes a + d \otimes c, & \Delta(d) &= c \otimes b + d \otimes d, \\ S(a) &= d, & S(b) &= -qb, & \varepsilon(a) &= \varepsilon(d) = 1, \\ S(c) &= -q^{-1}c, & S(d) &= a, & \varepsilon(b) &= \varepsilon(c) = 0. \end{aligned}$$

Definition 3.3.2. Let n be in \mathbb{N} . We denote by V_n the simple right $\mathcal{O}_q(SL_2)$ -comodule with basis $(v_i^{(n)})_{0 \leq i \leq n}$, and coaction ρ_n defined by:

$$\rho_n(v_i^{(n)}) = \sum_{k=0}^n v_k^{(n)} \otimes \left(\sum_{\substack{r+s=k \\ 0 \leq r \leq i \\ 0 \leq s \leq n-i}} \binom{i}{r}_{q^2} \binom{n-i}{s}_{q^2} q^{(i-r)s} a^r b^s c^{i-r} d^{n-i-s} \right)$$

where $\binom{n}{k}_{q^2}$ denotes the q^2 -binomial coefficient. That is to say:

$$\binom{n}{k}_{q^2} = q^{k(n-k)} \frac{[n]_{q^2}!}{[n-k]_{q^2}! [k]_{q^2}!} \quad \text{with } [k]_q = \frac{q^k - q^{-k}}{q - q^{-1}} \quad \text{and } [k]_q! = [1]_q \cdot [2]_q \cdots [k]_q.$$

Definition 3.3.3. Let n, m be in \mathbb{N} and let $\epsilon \in \{-1, 1\}$. We denote by $V_{n,m}^\epsilon$ the $\mathcal{O}_q(SL_2)$ -Yetter-Drinfeld module $V_n \otimes V_m$ equipped with its canonical right coaction, and with right module structure defined by:

$$\begin{aligned} (v_i^{(n)} \otimes v_j^{(m)}) \cdot a &= \epsilon q^{\frac{m-n}{2} + i - j} v_i^{(n)} \otimes v_j^{(m)}, \\ (v_i^{(n)} \otimes v_j^{(m)}) \cdot b &= -\epsilon q^{-\frac{n+m}{2} + i + j + 1} (1 - q^{-2}) [j]_q v_i^{(n)} \otimes v_{j-1}^{(m)}, \\ (v_i^{(n)} \otimes v_j^{(m)}) \cdot c &= \epsilon q^{\frac{m+n}{2} - i - j} (1 - q^{-2}) [n - i]_q v_{i+1}^{(n)} \otimes v_j^{(m)}, \\ (v_i^{(n)} \otimes v_j^{(m)}) \cdot d &= \epsilon q^{\frac{n-m}{2} + j - i} (v_i^{(n)} \otimes v_j^{(m)}) - q(1 - q^{-2})^2 [j]_q [n - i]_q v_{i+1}^{(n)} \otimes v_{j-1}^{(m)}. \end{aligned}$$

$V_{n,n}^\epsilon$ will also be denoted by V_n^ϵ .

Remark 3.3.4. By [Tak92], every simple finite dimensional $\mathcal{O}_q(SL_2)$ -Yetter-Drinfeld module is of the form $V_{n,m}^\epsilon$, and each finite dimensional $\mathcal{O}_q(SL_2)$ -Yetter-Drinfeld module can be decomposed into a direct sum of simple Yetter-Drinfeld modules. To see that our description of $V_{n,m}^\epsilon$ coincides with the one given in [Tak92, (6.4)], just consider the basis $(v_{i,j})_{\substack{0 \leq i \leq n \\ 0 \leq j \leq m}}$ given by

$$v_{i,j} = \frac{1}{[n-i]_q![m-j]_q!} v_{n-i}^{(n)} \otimes v_j^{(m)}.$$

One can check that the $v_{i,j}$'s satisfy [Tak92, (6.4)] and that the map

$$v_{i,j} \mapsto \frac{1}{[n-i]_q![m-j]_q!} v_{n-i}^{(n)} \otimes v_j^{(m)}$$

is an isomorphism of Yetter-Drinfeld modules.

Remark 3.3.5. Let n, m be in \mathbb{N} and ϵ be in $\{-1, 1\}$. The Clebsch-Gordan formula for the decomposition of $V_n \otimes V_m$ into simple comodules ensures that the space of right-coinvariant elements of $V_{n,m}^\epsilon$ is one-dimensional if $n = m$, and zero-dimensional otherwise. Hence if $n \neq m$, there is no inner reduced differential calculus of the form $\omega : \mathcal{O}_q(SL_2) \rightarrow V_{n,m}^\epsilon$, and there is at most one (up to isomorphism) inner reduced differential calculus of the form $\omega : \mathcal{O}_q(SL_2) \rightarrow V_n^\epsilon$. If $(n, \epsilon) \neq (0, 1)$, then V_n^ϵ is not isomorphic to the Yetter-Drinfeld module \mathbb{C}_ϵ , and by Lemma 3.1.11, there indeed exists such an inner reduced differential calculus, which we denote by $\omega_n^\epsilon : \mathcal{O}_q(SL_2) \rightarrow V_n^\epsilon$.

As a direct consequence of Lemma 3.1.12, and the fact that by [Tak92], the category $\mathcal{YD}_f(\mathcal{O}_q(SL_2))$ is semisimple, we have the following result.

Proposition 3.3.6. *Each finite dimensional bicovariant differential calculus over $\mathcal{O}_q(SL_2)$ is inner.*

This allows to deduce the classification of finite dimensional reduced differential calculi over $\mathcal{O}_q(SL_2)$.

Theorem 3.3.7. *Every simple finite dimensional reduced differential calculus over $\mathcal{O}_q(SL_2)$ is of the form $(V_n^\epsilon, \omega_n^\epsilon)$, with $n \in \mathbb{N}$, $\epsilon \in \{-1, 1\}$ and $(n, \epsilon) \neq (0, 1)$.*

Furthermore, each finite dimensional reduced differential calculus (V, ω) over $\mathcal{O}_q(SL_2)$ can be decomposed into a direct sum:

$$(V, \omega) \cong \bigoplus_{i=1}^d (V_{n_i}^{\epsilon_i}, \omega_{n_i}^{\epsilon_i}),$$

where $(n_1, \dots, n_d) \in \mathbb{N}^d$, $(\epsilon_1, \dots, \epsilon_d) \in \{-1, 1\}^d$ satisfies $(n_i, \epsilon_i) \neq (0, 1)$ for all i in $\{1, \dots, d\}$ and $(n_i, \epsilon_i) \neq (n_j, \epsilon_j)$ for all $i \neq j$.

PROOF. Since each finite dimensional reduced differential calculus over $\mathcal{O}_q(SL_2)$ is inner, and each simple finite dimensional Yetter-Drinfeld module over $\mathcal{O}_q(SL_2)$ is of the form $V_{n,m}^\epsilon$, we conclude by Remark 3.3.5 that the simple finite dimensional reduced differential calculi over $\mathcal{O}_q(SL_2)$ are the $(V_n^\epsilon, \omega_n^\epsilon)$ with $(n, \epsilon) \neq (0, 1)$. Now if (V, ω) is a finite dimensional reduced differential calculus over $\mathcal{O}_q(SL_2)$, by [Tak92], we have an isomorphism of Yetter-Drinfeld modules $V \cong \bigoplus_{i=1}^d V_i$ where each V_i is a simple Yetter-Drinfeld module. One then easily checks that for $i \in \{1, \dots, d\}$, $\omega_i = \pi_i \circ \omega : \mathcal{O}_q(SL_2) \rightarrow V_i$ (where $\pi_i : V \rightarrow V_i$ is the canonical projection) is a reduced differential calculus. We thus have $(V_i, \omega_i) \cong (V_{n_i}^{\epsilon_i}, \omega_{n_i}^{\epsilon_i})$ for some $(n_i, \epsilon_i) \neq (0, 1)$. Then $(V, \omega) \cong \bigoplus_{i=1}^d (V_{n_i}^{\epsilon_i}, \omega_{n_i}^{\epsilon_i})$, and by Lemma 3.1.10, we have $(n_i, \epsilon_i) \neq (n_j, \epsilon_j)$ when $i \neq j$. \square

In order to give the classification of finite dimensional reduced differential calculi over free orthogonal Hopf algebras, we need the definition of the bilinear cogroupoid \mathcal{B} . It will provide an explicit description of the equivalence between the categories of reduced differential calculi over a free orthogonal Hopf algebra $\mathcal{B}(E)$ and $\mathcal{O}_q(SL_2)$, for a well chosen q .

Definition 3.3.8. The bilinear cogroupoid \mathcal{B} is defined as follows:

- $\text{ob}(\mathcal{B}) = \{E \in GL_n(\mathbb{C}) ; n \geq 1\}$,
- For $E, F \in \text{ob}(\mathcal{B})$, and $m, n \geq 1$ such that $E \in GL_m(\mathbb{C})$ and $F \in GL_n(\mathbb{C})$, $\mathcal{B}(E, F)$ is the universal algebra generated by elements $(a_{ij})_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}}$ submitted to the relations:

$$F^{-1} a^t E a = I_n \quad \text{and} \quad a F^{-1} a^t E = I_m,$$

where $a = (a_{ij})_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}}$.

- For $E, F, G \in \text{ob}(\mathcal{B})$, $\Delta_{E,F}^G : \mathcal{B}(E, F) \rightarrow \mathcal{B}(E, G) \otimes \mathcal{B}(G, F)$, $\varepsilon_E : \mathcal{B}(E, E) \rightarrow \mathbb{C}$ and $S_{E,F} : \mathcal{B}(E, F) \rightarrow \mathcal{B}(F, E)$ are characterized by:

$$\begin{aligned} \Delta_{E,F}^G(a_{ij}) &= \sum_{k=1}^n a_{ik} \otimes a_{kj}, \text{ where } n \geq 1 \text{ is such that } G \in GL_n(\mathbb{C}), \\ \varepsilon_E(a_{ij}) &= \delta_{ij}, \\ S_{E,F}(a_{ij}) &= (E^{-1} a^t F)_{ij}. \end{aligned}$$

For $E \in GL_n(\mathbb{C})$, $\mathcal{B}(E, E)$ is a Hopf algebra, which will also be denoted by $\mathcal{B}(E)$, and called the free orthogonal Hopf algebra associated with E .

Remark 3.3.9. One easily checks that $\mathcal{O}_q(SL_2) = \mathcal{B}(E_q)$, where

$$E_q = \begin{pmatrix} 0 & 1 \\ -q^{-1} & 0 \end{pmatrix}.$$

By [Bic, Corollary 3.5], for $\lambda \in \mathbb{C}$, the subcogroupoid \mathcal{B}^λ of \mathcal{B} defined by

$$\mathcal{B}^\lambda = \{E \in GL_m(\mathbb{C}) ; n \geq 2, \text{tr}(E^{-1}E^t) = \lambda\}$$

is connected (here “tr” denotes the usual trace).

In the following, $E \in GL_m(\mathbb{C})$ with $m \geq 2$, denotes a matrix such that any solution of the equation $q^2 + \text{tr}(E^{-1}E^t)q + 1 = 0$ is not a root of unity.

If q is a solution of this equation, we have $\text{tr}(E_q^{-1}E_q^t) = -q - q^{-1} = \text{tr}(E^{-1}E^t)$, thus E and E_q are in the connected cogroupoid \mathcal{B}^λ , where $\lambda = -q - q^{-1}$. The Hopf algebras $\mathcal{B}(E_q) = \mathcal{O}_q(SL_2)$ and $\mathcal{B}(E)$ are thus monoidally equivalent, and by Theorem 3.2.6, we have an equivalence between the categories of reduced differential calculi $\mathcal{RDC}(\mathcal{O}_q(SL_2))$ and $\mathcal{RDC}(\mathcal{B}(E))$ given by:

$$\begin{aligned} \mathcal{F}_{E_q}^E : \mathcal{RDC}(\mathcal{O}_q(SL_2)) &\rightarrow \mathcal{RDC}(\mathcal{B}(E)) \\ (V, \omega) &\mapsto (V \square_{\mathcal{O}_q(SL_2)} \mathcal{B}(E_q, E), \bar{\omega}). \end{aligned}$$

Definition 3.3.10. For n in \mathbb{N} and $\epsilon \in \{-1, 1\}$ such that $(n, \epsilon) \neq (0, 1)$, we denote by W_n^ϵ the $\mathcal{B}(E)$ -Yetter-Drinfeld module $V_n^\epsilon \square_{\mathcal{O}_q(SL_2)} \mathcal{B}(E_q, E)$. We fix a non-zero right-coinvariant element $\theta_n \in V_n \otimes V_n$ and we denote by $\eta_n^\epsilon : \mathcal{B}(E) \rightarrow W_n^\epsilon$ the inner reduced differential calculus defined by $\eta_n^\epsilon(x) = (\theta_n \otimes 1) \triangleleft x - \varepsilon(x)(\theta_n \otimes 1)$.

By Remark 3.2.5, $\mathcal{F}_{E_q}^E(V_n^\epsilon, \omega_n^\epsilon)$ is isomorphic to $(W_n^\epsilon, \eta_n^\epsilon)$ for all $n \in \mathbb{N}$ and all $\epsilon \in \{-1, 1\}$ such that $(n, \epsilon) \neq (0, 1)$. According to Theorems 3.2.6 and 3.3.7, we obtain the following classification of finite dimensional reduced differential calculi over $\mathcal{B}(E)$.

Proposition 3.3.11. *Each finite dimensional bicovariant differential calculus over $\mathcal{B}(E)$ is inner.*

Theorem 3.3.12. *Every simple finite dimensional reduced differential calculus over $\mathcal{B}(E)$ is of the form $(W_n^\epsilon, \eta_n^\epsilon)$, with $n \in \mathbb{N}$, $\epsilon \in \{-1, 1\}$ and $(n, \epsilon) \neq (0, 1)$.*

Furthermore, each finite dimensional reduced differential calculus (W, η) over $\mathcal{B}(E)$ can be decomposed into a direct sum:

$$(W, \eta) \cong \bigoplus_{i=1}^d (W_{n_i}^{\epsilon_i}, \eta_{n_i}^{\epsilon_i}),$$

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where $(n_1, \dots, n_d) \in \mathbb{N}^d$, $(\epsilon_1, \dots, \epsilon_d) \in \{-1, 1\}^d$ satisfies $(n_i, \epsilon_i) \neq (0, 1)$ for all i in $\{1, \dots, d\}$ and $(n_i, \epsilon_i) \neq (n_j, \epsilon_j)$ for all $i \neq j$.

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