Around the Zilber-Pink Conjecture for Shimura Varieties

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Résumé

Dans cette thèse, nous nous intéressons à l'étude de l'arithmétique et de la géometrie des variétés de Shimura. Cette thèse s'est essentiellement organisée autour de trois volets.

Dans la première partie, on étudie certaines applications de la théorie des modèles en théorie des nombres. En 2014, Pila et Tsimerman ont donné une preuve de la conjecture d'Ax-Schanuel pour la fonction j et, avec Mok, ont récemment annoncé une preuve de sa généralisation à toute variété de Shimura. Nous nous référons à cette généralisation comme à la conjecture d'Ax-Schanuel hyperbolique. Dans ce projet, nous cherchons à généraliser les idées de Habegger et Pila pour montrer que, sous un certain nombre d'hypothèses arithmétiques, la conjecture d'Ax-Schanuel hyperbolique implique, par une extension de la stratégie de Pila-Zannier, la conjecture de Zilber-Pink pour les variétés de Shimura. Nous concluons en vérifiant toutes ces hypothèses arithmétiques à l'exception d'une seule dans le cas d'un produit de courbes modulaires, en admettant la conjecture dite des grandes orbites de Galois. Il s'agit d'un travail en commun avec Christopher Daw.

La seconde partie est consacrée à un résultat cohomologique en direction de la conjecture de Zilber-Pink. Étant donné un groupe algébrique semi-simple sur un corps de nombres F contenu dans $\mathbb{R}$, nous démontrons que deux sous-groupes algébriques semi-simples définis sur F sont conjugués sur $\bar{F}$, si et seulement s'il le sont sur une extension réelle finie de F de degré majoré indépendamment des sous-groupes choisis. Il s'agit d'un travail en commun avec Mikhail Borovoi et Christopher Daw.

La troisième partie étudie la distribution des variétés de Shimura compactes. On rappelle qu'une variété de Shimura S de dimension 1 est toujours compacte sauf si S est une courbe modulaire. Nous généralisons cette observation en définissant une fonction de hauteur dans l'espace des variétés de Shimura associées à un groupe réductif réel donné. Dans le cas des groupes unitaires, on prouve que la densité des variétés de Shimura non-compactes est nulle.

Mots clés : variété de Shimura, O-minimalité, géométrie diophantienne, cohomologie galoisienne, groupe dual, application de Kottwitz.

Abstract

In this thesis, we study some arithmetic and geometric problems for Shimura varieties. This thesis consists of three parts.

In the first part, we study some applications of model theory to number theory. In 2014, Pila and Tsimerman gave a proof of the Ax-Schanuel conjecture for the j-function and, with Mok, have recently announced a proof of its generalization to any (pure) Shimura variety. We refer to this generalization as the hyperbolic Ax-Schanuel conjecture. In this article, we show that the hyperbolic Ax-Schanuel conjecture can be used to reduce the Zilber-Pink conjecture for Shimura varieties to a problem of point counting. We further show that this point counting problem can be tackled in a number of cases using the Pila-Wilkie counting theorem and several arithmetic conjectures. Our methods are inspired by previous applications of the Pila-Zannier method and, in particular, the recent proof by Habegger and Pila of the Zilber-Pink conjecture for curves in abelian varieties. This is joint work with Christopher Daw.

The second part is devoted to a Galois cohomological result towards the proof of the Zilber-Pink conjecture. Let G be a linear algebraic group over a field k of characteristic 0. We show that any two connected semisimple k-subgroups of G that are conjugate over an algebraic closure of k are actually conjugate over a finite field extension of k of degree bounded independently of the subgroups. Moreover, if k is a *real number field*, we show that any two connected semisimple k-subgroups of G that are conjugate over the field of real numbers $\mathbb{R}$ are actually conjugate over a finite *real* extension of k of degree bounded independently of the subgroups. This is joint work with Mikhail Borovoi and Christopher Daw.

Finally, in the third part, we consider the distribution of compact Shimura varieties. We recall that a Shimura variety S of dimension 1 is always compact unless S is a modular curve. We generalise this observation by defining a height function in the space of Shimura varieties attached to a fixed real reductive group. In the case of unitary groups, we prove that the density of non-compact Shimura varieties is zero.

Keywords: Shimura variety, O-minimality, Diophantine geometry, Galois cohomology, dual group, Kottwitz map.

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Organization of the thesis

As this thesis is composed of several articles, I decide to write this short paragraph to indicate origins of these chapters.

Chapter 2 is based on the following work:
C.Daw and J.Ren *Applications of the hyperbolic Ax-Schanuel conjecture*. arXiv:1703.08967, submitted.

Chapter 3 is based on the following work:
M. Borovoi, C. Daw and J.Ren *Conjugation of semisimple subgroups over real number fields of bounded degree*, arXiv:1802.05894, submitted.

Chapter 4 is based on the following work:
J. Ren *Counting compact Shimura varieties*, preprint.

Contents

Chapter 1

Introduction (version française)

1.1 Exposé général de la géométrie diophantienne des variétés de Shimura et des problèmes corrélatifs

1.1.1 Généralités sur les intersections exceptionnelles

Une intersection exceptionnelle se présente lorsque l'intersection de deux sous-variétés comporte une composante irréductible de dimension excédant ce que l'on pourrait attendre d'un simple comptage de dimension. Le premier problème non trivial concernant les intersections exceptionnelles est le théoréme suivant, répondant à une question de Lang.

Théorème 1.1.1 (Ihara-Serre-Tate). *Soit $f(X,Y) \in \mathbb{C}[X,Y]$ un polynôme irréductible possédant une infinité de racines (α, β) telles que α et β sont des racines de l'unité. Alors f est de l'une des formes suivantes :*

- $a(X^m - \eta Y^n)$,
- $a(X^m Y^n - \eta)$,

oÃź dans les deux cas, $a \in \mathbb{C}^\times$, $m,n \in \mathbb{N}$ et η est une racine de l'unité.

Dans un langage plus abstrait, le théorème 1.1.1 peut être reformulé comme suit :

Théorème 1.1.2 (Manin-Mumford pour $\mathbb{G}_m^2$). *Soit $V \subset \mathbb{C}^2$ une courbe algb́rique irréductible dans $\mathbb{G}_m^2$. Si V une infinité de points de torsion, alors V est le translaté d'un sous-groupe algébrique irréductible de $\mathbb{G}_m^2$ par un point de torsion.*

Ce théorème admet naturellement la généralisation suivante, démontrée par Mann :

Théorème 1.1.3 (Manin-Mumford pour $\mathbb{G}_m^n$, cf. [50]). *Soit $V \subset \mathbb{C}^n$ une sous-variété algébrique irréductible de $\mathbb{G}_m^n$. Si V contient un ensemble Zariski-dense de points de torsion, alors V est le translaté d'un sous-groupe algébrique irréductible de $\mathbb{G}_m^n$ par un point de torsion.*

On peut considérer le problème analogue pour les variétés abéliennes, i.e. la conjecture de Manin-Mumford.

Théorème 1.1.4 (Raynaud, etc.). *Soit A une variété abélienne sur un corps k de caractéristique zéro, et $V \subset A$ une sous-variété algébrique irréductible de A.*

Si V contient un ensemble Zariski-dense de points de torsion, alors V est le translaté d'une sous-variété abélienne de A par un point de torsion.

La première démonstration de cette conjecture est due à Raynaud [66], dans le cas de $k = \mathbb{C}$ et $\dim V = 1$. Par la suite, d'autres démonstrations en ont été données, notamment par Hrushovski, Ullmo, Zhang, ainsi que Pila et Zannier, cf. [39, 78, 90, 62].

Nous arrivons maintenant à la question des intersections exceptionnelles dans les variétés de Shimura. On note que si la variété algébrique ambiante est de dimension un, alors il n'y a pas d'intersections exceptionnelles non triviales. Considérons ainsi la plus simple des variétés de Shimura de dimension deux, $\mathbb{C}^2$, considérée comme le produit de deux courbes modulaires. L'analogue du théorème 1.1.3 dans cette situation est alors l'instance non triviale la plus simple de la conjecture d'André-Oort.

Théorème 1.1.5 (André). *Soit $V \subset \mathbb{C}^2$ une courbe algébrique irréductible contenant une infinité de points (x_1, x_2) tels que chaque x_i est le j-invariant d'une courbe elliptique CM. Alors V est de l'une des formes suivantes :*

- *$\{x_1\} \times \mathbb{C}$ où x_1 est le j-invariant d'une courbe elliptique CM,*
- *$\mathbb{C} \times \{x_2\}$ où x_2 est le j-invariant d'une courbe elliptique CM,*
- *une correspondance de Hecke, i.e. l'image de*

$$\mathbb{H} \to \mathbb{H} \times \mathbb{H} \to \mathbb{C} \times \mathbb{C}, \quad z \mapsto (z, nz) \mapsto (j(z), j(nz))$$

  pour un certain $n \in \mathbb{N}$.

On note que le théorème ci-dessus implique que si une courbe dans $\mathbb{C}^2$ contient une infinité de points CM, alors cette courbe est elle-même une courbe modulaire. Cette perspective nous permet de formuler une conjecture plus générale pour toutes les variétés de Shimura.

Conjecture 1.1.1 (André-Oort). *Soit $S \subset \mathrm{Sh}_K(G, \mathfrak{X})$ une composante connexe d'une variété de Shimura, et $V \subset S$ une sous-variété qui n'est pas une sous-variété de Shimura de S. Alors l'ensemble des points CM contenus dans V n'est pas Zariski-dense dans V.*

Remarque 1.1.2. Dans la littérature, les problèmes d'intersections exceptionnelles sont généralement formulés en utilisant la terminologie des "sous-variétés spéciales". Dans notre contexte, notre variété ambiante est une variété Shimura connexe, et les sous-variétés spéciales sont exactement les sous-variétés de Shimura. Il est à noter que les sous-variétés de Shimura de dimension zéro ne sont autres que les points spéciaux, i.e. les points CM.

De nombreux résultats partiels remarquables ont déjà été donnés. En supposant l'hypothèse de Riemann généralisée, il a été démontré que la conjecture d'André-Oort est vraie pour $\mathbb{C}^n$, cf. Edixhoven [26], et même que la conjecture d'André-Oort est entièrement valide, cf. Klingler, Ullmo and Yafaev [46, 85].

Ultérieurement, grâce à la "O-minimalité", une notion provenant de la théorie des modèles, et grâce à la stratégie de Pila-Zannier, il a été démontré que la conjecture d'André-Oort est vraie pour $\mathbb{C}^n$, cf. Pila [59], ainsi que pour les surfaces modulaires de Hilbert, cf. Daw et Yafaev [22], et pour les variétés modulaires de Siegel, cf. Tsimerman [77], Yuan-Zhang [89] et Andreatta-Goren-Howard-Madapusi Pera [1]. Cette méthode donne également une nouvelle démonstration de la conjecture d'André-Oort pour toutes les variétés de Shimura sous l'hypothèse de Riemann généralisée. Gao a généralisé les résultats ci-dessus à toutes les variétés de Shimura mixtes, cf. [30].

Puisque la conjecture d'André-Oort traite de l'intersection d'une sous-variété stricte d'une variété de Shimura avec une famille de sous-variétés de dimension zéro (les points CM), il s'agit d'une question typique concernant les intersections exceptionnelles. Si nous remplaçons à présent l'ensemble des points CM par une famille plus riche de sous-variétés de Shimura qui sont peu susceptibles d'intersecter une sous-variété donnée, nous obtenons la conjecture suivante de Zilber et Pink.

Conjecture 1.1.3 (cf. [64], Conjecture 1.3)**.** *Soit $S \subset \mathrm{Sh}_K(G,\mathfrak{X})$ une composant connexe d'une variété de Shimura et, pour tout entier d, on note $S^{[d]}$ la réunion des sous-variétés de Shimura de S ayant codimension au moins d. Soit V une sous-variÃl'tÃl'* **Hodge-générique** *de S, i.e. V n'est pas contenue dans une sous-variété de Shimura stricte de S. Alors*

$$V \cap S^{[1+\dim V]}$$

n'est pas Zariski-dense dan V.

On note également que dans l'esprit du travail de Bombieri-Masser-Zannier [7], la conjecture de Zilber-Pink a une formulation équivalente en termes de sous-variétés anormales spéciales, voir le chapitre 2, section 15 de cette thèse pour de plus amples détails.

En développant la stratégie de Pila-Zannier, cf. [21], nous sommes capables de ramener la conjecture 1.1.3 à plusieurs problèmes géométriques et/ou arithmétiques pour les variétés de Shimura, qui sont listés dans la section 2. L'étude de ces problèmes servira de moyen raisonnable de prouver la Conjecture de Zilber-Pink.

1.1.2 Variétés de Shimura

Puisque les variétés de Shimura sont l'objet d'étude principal de cette thèse, il n'est pas inutile de passer ici en revue leurs propriétés de base. Pour une introduction plus détaillée, on se rapportera aux notes de Milne [52] ou aux articles originaux de Deligne [23, 24].

Définition 1.1.4. On défini $\mathbb{R}$-tore de Deligne $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{m,\mathbb{C}}$. Une **donnée de Shimura** est un couple (G,X) où G est un groupe réductif sur $\mathbb{Q}$ et X est une $G(\mathbb{R})$-orbite dans $\mathrm{Hom}(\mathbb{S},G_{\mathbb{R}})$, tel qu'il existe $h \in X$ avec les propriétés suivantes.

1. L'algèbre de Lie $\mathrm{Lie}(G)$ est de type $\{(-1,1),(0,0),(1,-1)\}$, où la structure de Hodge de $\mathrm{Lie}(G)$ est donnée par la composition de h avec la représentation adjointe de G.
2. L'application $\mathrm{int}(h(i))$ est une involution de Cartan de $G^{\mathrm{ad}}_{\mathbb{R}}$, i.e. $\{g \in G^{\mathrm{ad}}(\mathbb{C}); h(i)\bar{g}h(i)^{-1} = g\}$ est compact.
3. Pour tous les facteur $\mathbb{Q}$-simple H de G^{ad}, la composition de h et de $G_{\mathbb{R}} \to G^{\mathrm{ad}}_{\mathbb{R}} \to H_{\mathbb{R}}$ n'est pas triviale.

Définition 1.1.5. Soit (G,X) une donnée de Shimura et K un sous-groupe compact ouvert de $G(\mathbb{A}_f)$. Alors la **variété de Shimura** $\mathrm{Sh}_K(G,X)$ associée à (G,X) et K est le quotient double

$$\mathrm{Sh}_K(G,X) := G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K,$$

où les actions de $G(\mathbb{Q})$ et K sont données par $q(x,a)k = (qx,qak)$ pour tous $q \in G(\mathbb{Q})$ et $k \in K$.

Exemple 1.1.6. Soit $G = \mathrm{GL}_{2,\mathbb{Q}}$, et considérons $h_0 : a+bi \mapsto \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. Alors (G,X) est une donnée de Shimura. Notons que X peut être identifié naturellement avec $\mathbb{H}^{\pm} = \mathbb{C}\backslash\mathbb{R}$ en associant gh_0g^{-1} à $g\sqrt{-1}$ pour tous $g \in G(\mathbb{R}) = \mathrm{GL}_2(\mathbb{R})$. Choisissons $K = \prod_p \mathrm{GL}(\mathbb{Z}_p)$ pour sous-groupe compact ouvert de $G(\mathbb{A}_f) = \mathrm{GL}(\mathbb{A}_f)$. On peut montrer que $\mathrm{Sh}_K(\mathrm{GL}_{2,\mathbb{Q}},\mathbb{H}^{\pm})$ est isomorphe au produit de deux copies de $Y(1) = \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$.

L'exemple ci-dessus implique que la courbe modulaire $Y(1)$ est une composante connexe d'une variété de Shimura, i.e. une variété de Shimura connexe.

Dans le cadre des conjectures d'André-Oort et de Zilber-Pink, les notions de sous-variét'es spéciales et faiblement spéciales jouent un rôle important.

Définition 1.1.7. Soit $\mathrm{Sh}_K(G,X)$ une variété de Shimura. On dit qu'une sous-variété Z est une **sous-variété spéciale** s'il existe un triple (H,Y,g) où $(H,Y) \subset (G,X)$ est une sous-donnée de Shimura et g est un élément de $G(\mathbb{A}_f)$, tel que Z est une composante irréductible de l'image de $Y \times \{g\}$ dans $\mathrm{Sh}_K(G,X)$.

Remarque 1.1.8. Les sous-variétés spéciales sont parfois aussi appelées **sous-variétés de type Hodge**.

Une variété de Shimura est aussi équipée d'une collection bien plus grande de sous-variétés faiblement spéciales. Pour simplifier, nous donnons seulement ici une définition des sous-variétés faiblement spéciales aux structures de niveau près. Pour une définition précise, voir le chapitre 2, section 2.

Définition 1.1.9. Soit S une composante connexe d'une variété de Shimura $\mathrm{Sh}_K(G,X)$. Une sous-variété Z de S est appelée **faiblement spéciale** si S a une sous-variété spéciale S' avec une decomposition en tant que produit de deux variétés de Shimura $S' = S_1 \times S_2$, telle que que $Z = S_1 \times \{z_2\}$ pour un certain $z_2 \in S_2$.

Pour appliquer la stratégie de Pila-Zannier, il faut aussi se donner un analogue de l'application exponentielle $\mathbb{C}^n \to (\mathbb{C}^{\times})^n$, une application d'uniformisation pour les variétés de Shimura.

Définition 1.1.10. Soit $\mathrm{Sh}_K(G,X)$ une variété de Shimura, et X^+ une composante connexe de X. On note

$$G(\mathbb{Q})_+ := \{g \in G(\mathbb{Q}); gX^+ = X^+\}, \quad \Gamma = G(\mathbb{Q})_+ \cap K.$$

Alors $S = \Gamma \backslash X^+$ est une composante connexe de $\mathrm{Sh}_K(G,X)$ et l'application d'uniformisation est définie par l'application quotient

$$\pi : X^+ \longrightarrow S.$$

Parfois, nous écrivons aussi l'application d'uniformisation comme $\pi : X \longrightarrow S$ par un léger abus de notation.

1.1.3 Transcendance fonctionnelle

Un ingrédient clé vers la démonstration d'une conjecture de type Zilber-Pink est l'analogue de certains problèmes classiques de la théorie des nombres transcendants.

Théorème 1.1.6 (Lindemann-Weierstraß)**.** *Soient $\alpha_1,...,\alpha_n$ des nombres alébriques qui sont linéairement indépendants sur $\mathbb{Q}$. Alors les nombres $e^{\alpha_1},...,e^{\alpha_n}$ sont algébriquement indépendants sur $\mathbb{Q}$.*

Conjecture 1.1.11 (Schanuel)**.** *Soient $\alpha_1,...,\alpha_n$ des nombres alébriques qui sont linéairement indépendants sur $\mathbb{Q}$, alors*

$$\mathrm{tr.deg}_{\mathbb{Q}}(\alpha_1,...,\alpha_n,e^{\alpha_1},...,e^{\alpha_n}) \geq n.$$

Les premiers analogues fonctionnels des deux assertions ci-dessus sont donnés par Ax [3].

Définition 1.1.12. Soit A un sous-ensemble analytique complexe de $\mathbb{C}^n$. On dit des fonctions $f_1,...,f_r : A \to \mathbb{C}$ sont linéairement indépendants sur $\mathbb{Q}$ modulo $\mathbb{C}$ s'il n'y a pas de relation non triviale $q_1 f_1 + ... + q_r f_r = c$ pour $q_i \in \mathbb{Q}$ et $c \in \mathbb{C}$.

Théorème 1.1.7 (Ax-Lindemann-Weierstraß pour les tores)**.** *Soit A un sous-ensemble analytique complexe de $\mathbb{C}^n$. On note α_i la i-ième fonction de coordonnées $A \to \mathbb{C}, (z_1,..,z_n) \mapsto z_i$. On suppose que les fonctions α_i sont linéairement indépendantes sur $\mathbb{Q}$ modulo $\mathbb{C}$. Alors $e^{\alpha_1},...,e^{\alpha_n}$ sont algébriquement indépendants sur $\mathbb{C}$.*

Théorème 1.1.8 (Ax-Schanuel pour tores)**.** *Sous les hypothèses ci-dessus, nous avons*

$$\mathrm{tr.deg}_{\mathbb{C}}(\alpha_1,...,\alpha_n,e^{\alpha_1},...,e^{\alpha_n}) \geq n + \dim A.$$

Dans le même esprit, des conjectures d'Ax-Lindemann-Weierstraß et d'Ax-Schanuel peuvent être formulées de manière similaire dans le contexte des variétés abéliennes et des variétés de Shimura. Ici, nous ne listons que le cas qui sera utilisé dans cette thèse, voir le chapitre 2, section 5 pour plus de détails.

Conjecture 1.1.13 (Ax-Schanuel pour les variétés de Shimura)**.** *Soit S une composante connexe d'une variété de Shimura $\mathrm{Sh}_K(G,X)$ et Z une sous-variété algébrique de $X \times S$. Notons D_S le graphe de l'application d'uniformisation $\pi : X \to S$, et U une composante analytique complexe de $Z \cap D_S$. Si $\pi_S(U)$ n'est pas contenue dans une sous-variété faiblement spéciale de S, alors on a*

$$\dim Z = \dim U + \dim S.$$

Au cours de la préparation de cette thèse, Mok, Pila et Tsimerman ont annoncé une démonstration de cette conjecture pour toutes les variétés de Shimura pures, cf [53]. Plus récemment, Bakker et Tsimerman ont généralisé cette assertion aux variations de structures de Hodge, voir [6]. Nous remarquons que Klingler travaille également dans cette direction, voir [45]. Il est également intéressant de noter que Chambert-Loir et Loeser ont construit un analogue p-adique du résultat de transcendance fonctionnel, voir [13].

1.1.4 Théorème de comptage de Pila-Wilkie

La stratégie de Pila-Zannier fait usage d'un outil puissant de géométrie diophantienne, un théorème de comptage nommé d'après Pila et Wilkie, pour attaquer différents types de problèmes concernant les intersections exceptionnelles.

Le théorème de comptage de Pila-Wilkie a été motivé par l'intuition simple et naturelle qu'une courbe $\mathcal{C} \subset \mathbb{R}^2$ contient peu de points rationnels, sauf si cette courbe est algébrique. Par exemple, la courbe transcendante $\mathcal{C}_1 = \{(x,y); y = e^x\}$ contient un unique point rationnel $(0,1)$. D'autre part, la parabole $\mathcal{C}_2 = \{(x,y); y = x^2\}$ contient de nombreux points rationnels, i.e. $(q,q^2) \in \mathcal{C}_2$ pour tout $q \in \mathbb{Q}$.

Plus généralement, soit $A \subset \mathbb{R}^n$ un sous-ensemble avec de bonnes propriétés, i.e. définissable dans une certaine structure O-minimale. Si A contient beaucoup de points rationnels, alors on peut s'attendre à ce que A contienne une courbe algébrique. Nous donnons une formulation précise du théorème de comptage comme suit.

Définition 1.1.14. La hauteur $H(q)$ d'un nombre rationnel q est égale à $\max\{|m|,|n|\}$, où m,n sont des entiers premiers entre eux tels que $q = \frac{m}{n}$. Pour un n-uplet $q = (q_1,\ldots,q_n) \in \mathbb{Q}^n$, nous définissons sa hauteur $H(q)$ comme $\max\{H(q_1),\ldots,H(q_n)\}$.

Nous pouvons compter les points rationnels dans tout sous-ensemble de $\mathbb{R}^n$ par rapport à cette fonction de hauteur.

Définition 1.1.15. Soit $Y \subset \mathbb{R}^n$, pour tout $T \geq 1$, nous désignons par

$$N(Y,T) := \{y \in Y; H(y) < T\}.$$

Définition 1.1.16. Un **ensemble semi-algébrique** dans $\mathbb{R}^n$ est ou bien un sous-ensemble défini par une conjonction finie d'équations polynomiales, i.e. $P(X_1,\ldots,X_n) = 0$ et d'inégalités polynomiales, i.e. $Q(X_1,\ldots,X_n) > 0$, où $P,Q \in \mathbb{R}[X_1,\ldots,X_n]$, ou bien une combinaison booléenne de tels ensembles. Un ensemble semi-algébrique est appelé **irréductible** si son adhérence de Zariski dans $\mathbb{R}^n$ est irréductible. La **dimension** d'un sous-ensemble semi-algébrique irréductible est la dimension réelle de sa clôture de Zariski.

En ce qui concerne la distribution des points rationnels dans un ensemble qui est définissable dans une structure O-minimale, par exemple, $\mathbb{R}_{\text{an}}$ ou $\mathbb{R}_{\text{an,exp}}$, nous avons le théorème de comptage suivant, cf. [58], qui joue un rôle crucial dans cette thèse. Pour une introduction détaillée à la O-minimalité, voir [86].

Theorem 1.1.17 (Pila-Wilkie)**.** *Soit $A \subset \mathbb{R}^n$ un sous-ensemble définissable dans une structure O-minimale. Alors pour tout $\varepsilon > 0$, il existe une constante $c = c(A, \varepsilon) > 0$ avec la propriété suivante : pour tout sous-ensemble $D \subset A$, s'il y a un $T > 1$ avec $\#N(D, T) \geq cT^{\varepsilon}$, alors A contient un sous-ensemble semi-algébrique de dimension strictement positive Z tel que $D \cap Z$ est non vide.*

Remarque 1.1.18. Le théorème de comptage ci-dessus est écrit sous une forme qui semble être d'usage commode. Dans l'article original de Pila-Wilkie, on en trouve une formulation légèrement différente, utilisant la notion de **partie transcendante** d'un ensemble définissable, i.e. le sous-ensemble d'un ensemble définissable obtenu en supprimant tous les sous-ensembles semi-algébrique connexes de dimension strictement positive. Leur théorème affirme que cette partie transcendante contient toujours peu de points rationnels.

Dans le contexte de la conjecture de Zilber-Pink, on a besoin d'une version en famille du théorème ci-dessus, et on devrait compter les nombres algébriques de degré borné plutôt que se restreindre aux nombres rationnels. Dans cette thèse, la version formulée par Habegger-Pila est suffisante, cf. Corollaire 7.2 de [35], voir aussi le chapitre 2, section 9 de cette thèse. Nous remarquons également que Cluckers, Comte et Loeser ont démontré une version p-adique du théorème de comptage de Pila-Wilkie, cf. [16].

1.1.5 Premier aperçu de la stratégie de Pila-Zannier : la conjecture de Lang

Afin d'illustrer comment le théorème de comptage de Pila-Wilkie et le théorème de transcendance fonctionnel nous permettent d'attaquer des problèmes d'intersections exceptionnelles, nous esquissons une démonstration de la conjecture de Lang, i.e. la conjecture de Manin-Mumford multiplicative pour $\mathbb{G}_m^2$, comme mentionné au début de l'introduction. Soit $V \subset \mathbb{G}_m^2$ une sous-variété algébrique irréductible qui contient une infinité de points de torsion.

Définition 1.1.19. Nous considérons l'application d'uniformisation

$$\pi : \mathbb{C}^2 \to (\mathbb{C}^{\times})^2, \quad (z_1, z_2) \mapsto (e^{2\pi i z_1}, e^{2\pi i z_2}),$$

et nous posons$\mathcal{F} := [0,1)^2$. Pour tout $x \in (\mathbb{C}^{\times})^2$, on note $\tilde{x}$ l'unique élément de $\pi^{-1}(x) \cap \mathcal{F}$.

Démonstration du Théorème 1.1.2. Supposons que V ne soit pas un le translaté d'un sous-groupe algébrique par un point de torsion; il suffit de montrer que V ne contient qu'un nombre fini de points de torsion. On écrit $\mathcal{V}$ l'intersection $\mathcal{F} \cap \pi^{-1}(V)$.

Puisque $\mathcal{F}$ est semi-algébrique, $\mathcal{V}$ est définissable dans $\mathbb{R}_{\text{an}}$, et nous considérons $c = c(\mathcal{V}, \frac{1}{2})$ la constante fournie par le théorème de comptage de Pila-Wilkie pour $\varepsilon = \frac{1}{2}$. Par interpolation de Lagrange, V est définie sur un corps de nombres K, et on note Γ_K son groupe de Galois absolu. Pour tout point de torsion $P \in V$, nous définissons

$$D_P = \{\widetilde{\sigma(P)}; \sigma \in \Gamma_K\} \subset \mathcal{F}.$$

Nous affirmons que $\#D_P = \#N(D_P, H(P)) \leq cH(P)^{\frac{1}{2}}$. En effet, le théorème de comptage impliquerait sinon que $\mathcal{F}$ contient une courbe semi-algébraique Z qui a une intersection non vide avec D_P. Par le théorème d'Ax-Lindemann-Weierstraß, $V = \overline{\pi(Z)}^{\text{Zar}}$ est un translaté d'un sous-groupe algébrique. Puisque V contient un point de torsion P, c'est en en fait le translaté d'un sous-groupe algébrique par un point de torsion, une contradiction. D'un autre côté, on a $\#D_P = \#\Gamma_K P \geq H(P)$. En combinant les deux assertions ci-dessus, nous avons $H(P) \leq cH(P)^{\frac{1}{2}}$, i.e., $H(P) \leq c^2$, ce qui donne la finitude de l'ensemble des points de torsions de V. $\square$

Cette preuve contient déjà l'idée essentielle de la stratégie de Pila-Zannier, c'est-à-dire la conjonction du théorème de transcendance algébrique, d'une minoration de la taille de certaines orbites galoisiennes, et du théorème de comptage. Une des tâches essentielles de cette thèse est de généraliser la démonstration ci-dessus à des variétés de Shimura arbitraires et à un contexte plus général d'intersections exceptionnelles.

1.2 Ingrédients arithmétiques

En vue d'appliquer la stratégie de Pila-Zannier à la conjecture de Zilber-Pink, de nombreux problèmes arithmétiques et géométriques apparaissent naturellement. Ces conjectures sont précisément les obstructions à une démonstration inconditionnelle de la conjecture de Zilber-Pink (voir la section 3.3).

Outre leurs applications potentielles à la conjecture de Zilber-Pink, ces problèmes sont par ailleurs intrinsèquements intéressants.

1.2.1 Une propriété de finitude pour la complexité

En utilisant la conjecture d'Ax-Schanuel (dont une preuve a été annoncée par Mok, Pila et Tsimerman), nous sommes en mesure de ramener la conjecture de Zilber-Pink à un problème de comptage de points, cf. Section 8 de [21].

Afin de résoudre ce problème de comptage, nous devons compter les sous-variétés spéciales d'une variété de Shimura de manière raisonnable. Cela nous amène à la notion de "complexité" pour les sous-variétés spéciales. Pour ce faire, rappelons d'abord la définition de la complexité dans le contexte de variétés abéliennes.

Définition 1.2.1 (cf. Définition 3.4 de [35])**.** Soit A une variété abélienne sur un corps de nombres K, munie d'un fibré en droites ample $\mathcal{L}$. Si Z est le translaté d'une sous-variété abélienne B de A par un point de torsion, nous définissons sa complexité arithmétique comme

$$\Delta_{\text{arith}}(Z) = \min\{\text{ordre de T};\ Z = T + B \text{ et } T \text{ est d'ordre fini}\},$$

et sa complexité

$$\Delta(A) = \max\{\deg_{\mathcal{L}} B, \Delta_{\text{arith}}(A)\},$$

où $\deg_{\mathcal{L}} B$ est le degré of B par rapport à $\mathcal{L}$.

De même, cf. [35], dans le cas des variétés de Shimura, on souhaite que notre "complexité" se décompose encore en une partie arithmétique et une partie géométrique. Afin d'éviter les détails techniques, nous nous contentons de définir la complexité des sous-variétés de Shimura de $\mathcal{A}_g$.

Définition 1.2.2. Pour un point CM $x \in \mathcal{A}_g$, soit A_x la variété abélienne principalement polarisée correspondante à x, $R_x := Z(\text{End}(A_x))$ le centre de l'anneau des endomorphismes de A_x, et $\Delta(x) := \text{Disc}(R_x)$ son discriminant. La complexité arithmétique d'une sous-variété spéciale Z de $\mathcal{A}_g$ est définie par

$$\Delta_{\text{arith}}(Z) = \min\{\Delta(x);\ x \in Z \text{ et } x \text{ est un point CM}\}.$$

Nous remarquons qu'une sous-variété spéciale d'une variétd́ee Shimura contient toujours des points spéciaux, donc cette définition a bien un sens. La "complexité arithmétique" peut être définie pour des variétés de Shimura arbitraires, cf. Section 10 de [21].

Pour la partie géométrique de la complexité, rappelons qu'une variété de Shimura a une compactification de Baily-Borel $\overline{S}$ par rapport à un fibré en droites ample $\mathcal{L}$. Pour une sous-variété W de S, nous notons $\deg_{\mathcal{L}} W$ le degré de la clôture de Zariski de W dans $\overline{S}$.

Définition 1.2.3. La complexité d'une sous-variété spéciale Z d'une variété de Shimura S est l'entier positif

$$\Delta(Z) := \max\{\deg_{\mathcal{L}} Z, \Delta_{\text{arith}}(Z)\}.$$

Il est naturel de conjecturer que la complexité est en effet une bonne fonction de comptage pour les sous-variétés spéciales:

Conjecture 1.2.4. *Pour tout $b > 1$, il n'y a qu'un nombre fini de sous-variétés spéciales Z d'une variété de Shimura S donnée telles que $\Delta(Z) < b$.*

Nous pensons qu'obtenir une bonne majoration pour le degré d'une sous-variété de Shimura est nécessaire pour cette conjecture.

1.2.2 Corps de définition pour sous-variétiés spéciales

Selon le théorème de réduction semi-stable pour les variétés abéliennes, nous avons le résultat classique suivant.

Proposition 1.2.5. *Soit A une variété abélienne sur un corps de nombres K. Alors il y a une extension finie L de K telle que toute sous-variété abélienne de A soit définies sur L.*

Cette proposition est une observation importante en vue de démontrer un résultat conditionnel sur la conjecture de Zilber-Pink pour les variétés abéliennes sur un corps de nombre, voir le théorème 9.8 de [35]. Mais l'analogue de la proposition 1.2.5 pour les variétés Shimura est fausse, puisque l'orbite galoisienne d'un point CM peut être grande. Cependant, si notre sous-variété spéciale n'a pas de "partie CM", on peut tout de même espérer que son corps de définition ne croisse pas trop vite. En fait, nous émettons la conjecture suivante.

Conjecture 1.2.6. *Soit $S \subset \mathrm{Sh}_K(G,\mathfrak{X})$ une composante connexe d'une variété de Shimura définie sur un corps de nombre k (une extension finie du corps reflex $E = E(G,\mathfrak{X})$). Alors pour tout $\varepsilon > 0$, il existe une constante uniforme $c = c(S,\varepsilon) > 0$ avec la propriété suivante.*

Pour toute sous-variété spéciale $Z \subset S$ qui n'est pas de la forme $S_1 \times \{s_2\} \subset S_1 \times S_2 \subset S$, où S_i sont deux variétés de Shimura de dimensions positives, il existe une extension finie L de k avec $[L:k] \leq c\Delta(Z)^{\varepsilon}$ tel que Z est définie sur L.

Cette conjecture est varie pour $S = \mathbb{C}^n$, voir la section 13 de [21]. Il ne devrait pas être trop difficile de la démontrer pour un produit arbitraire des courbes de Shimura (non nécessairement deux à deux isomorphes) et il est peut être possible de la démontrer pour toutes les variétés de Shimura.

L'estimation du nombre de classes de sous-groupes semi-simples et une étude détaillée de la cohomologie galoisienne peuvent jouer un rôle important dans la résolution de ce type de problèmes.

1.2.3 Une version quantitative de la propriété de Siegel en théorie de la réduction

La construction d'un domaine fondamental par rapport à l'action à gauche de $\mathrm{SL}_2(\mathbb{Z})$ sur le demi-plan de Poincaré (ou de façon équivalente, $\mathrm{SL}_2(\mathbb{R})$) est classique. Plus généralement, la théorie de la réduction des groupes arithmétiques donne :

Proposition 1.2.7. *(cf. Théorème 15.4 [9]) Soit G un groupe réductif sur $\mathbb{Q}$, et $\Gamma \subset G(\mathbb{Q})$ un sous-groupe arithmétique. Alors il existe un ensemble semi-algébrique $\mathfrak{S} \subset G(\mathbb{R})$ et un sous-ensemble fini C de $G(\mathbb{Q})$ tel que $\mathcal{F} := C \cdot \mathfrak{S}$ satisfait*

1. $\Gamma\mathcal{F} = G(\mathbb{R})$
2. ***Propriété de Siegel** : Pour tout $\alpha \in G(\mathbb{Q})$, l'ensemble $\{\gamma \in \Gamma; \gamma\mathcal{F} \cap \alpha\mathcal{F} \neq \emptyset\}$ est fini.*
3. *Il existe un sous-groupe compact maximal $K_\infty \subset G(\mathbb{R})$ tel que $\mathcal{F}K_\infty = \mathcal{F}$.*

L'ensemble semi-algébrique $\mathfrak{S}$ (resp. $\mathcal{F}$) dans la proposition 1.2.7 est appelé un **ensemble de Siegel** (resp. **ensemble fundamental**) de $G(\mathbb{R})$ pour l'action de Γ.

Il est intéressant de considérer une version "relative" du théorème ci-dessus. Pour des raisons techniques, nous travaillons maintenant au niveau des domaines symétriques hermitiens plutôt qu'au niveau des groupes eux-mêmes. La proposition suivante est une conséquence de la théorie de la réduction de Borel.

Proposition 1.2.8. *Soit (G,X) une donnée de Shimura, et $\Gamma \subset G$ un sous-groupe arithmétique. On se donne un ensemble fondamental $\mathcal{F} \subset X$ pour l'action de Γ. On prend une sous-donnée de Shimura $(H,X_H) \subset (G,X)$. Alors il existe un ensemble fondamental $\mathcal{F}_H \subset X_H$ pour l'action de $\Gamma_H := \Gamma \cap H(\mathbb{R})$ tel que l'ensemble*

$$\{\gamma \in \Gamma; \mathcal{F} \cap \gamma.\mathcal{F}_H \neq \emptyset\}$$

est fini.

On peut également formuler une version quantitative de la conjecture ci-dessus. Afin d'éviter les détails techniques, nous formulons seulement la conjecture dans le cas où H est semi-simple.

Conjecture 1.2.9. *Soit (G,X) une donnée de Shimura, et $\Gamma \subset G$ un sous-groupe arithmétique. On se donne un ensemble fondamental $\mathcal{F} \subset X$ pour l'action de Γ. Alors il existe des constantes $c = c(G,\Gamma) > 0$ et $\delta = \delta(G,\Gamma) > 0$ telles que la propriété suivante est vraie.*

Soit $(H,X_H) \subset (G,X)$ une sous-donnée de Shimura. Alors il existe un ensemble fondamental $\mathcal{F}_H \subset X_H$ pour l'action de $\Gamma_H := \Gamma \cap H(\mathbb{R})$ tel que pour tout $\gamma \in \Gamma$ avec $\mathcal{F} \cap \gamma.\mathcal{F}_H \neq \emptyset$, on a

$$H(\gamma) \leq c \cdot \mathrm{vol}(\mathcal{F}_H)^{\delta}.$$

On a ici noté $\mathrm{vol}(\mathcal{F}_H)$ le volume de $\mathcal{F}$ pour la mesure de Haar normalisé sur $H(\mathbb{R})$, comme dans [17], Section 4. La conjecture 1.2.9 est vraie pour $\mathbb{C}^n$, cf. Section 13 de [21]. L'analogue de la conjecture 1.2.9 est également vrai pour les variétés abéliennes, cf. [35], Section 3.

1.2.4 Les hauteurs des éléments de conjuguants

Nous donnons ici une version quantitative du théorème 1.3.6. Nous espérons en effet trouver des éléments petite hauteur, conjuguant un sous-groupe sur un autre. Rappelons d'abord la fonction de hauteur que nous utilisons.

Soit $d \geq 1$ un nombre entier. Pour un nombre réel y, on défini sa **d-hauteur** comme

$$\mathrm{H}_d(y) := \min\{\max\{|a_0|,...,|a_d|\} : a_i \in \mathbb{Z},\ \gcd\{a_0,...,a_d\} = 1,\ a_0y^d + ... + a_d = 0\},$$

où nous utilisons la convention que, si l'ensemble est vide i.e. si y n'est pas algébrique de degré au plus d, alors $\mathrm{H}_d(y)$ vaut $+\infty$. For $y = (y_1,...,y_m) \in \mathbb{R}^m$, on définit

$$\mathrm{H}_d(y) := \max\{\mathrm{H}_k(y_1),...,\mathrm{H}_d(y_m)\}.$$

En vue d'utiliser le théorème de comptage de Pila-Wilkie, nous espérons que nos éléments conjuguants dans le théorème 1.3.6 ont la propriété suivante :

Conjecture 1.2.10. *Soit G un groupe semi-simple connexe défini sur un corps de nombres $k \subset \mathbb{R}$. Alors il y a un ensemble fini $\Omega = \{H_1,\ldots,H_r\}$ de sous-groupes semi-simples sur k, un entier positif $d = d(G_{\bar{k}},\Omega)$, deux constantes strictement positives $c = c(G_{\bar{k}},\Omega)$*

et $\delta = \delta(G_{\bar{k}}, \Omega)$ avec la propriété suivante: si H est un sous-groupe semi-simple de G sur k, alors il est conjugué à un certain $H_i \in \Omega$ sur $\bar{k}$, et il existe une extension finie $K = K(H, H_i) \subset \mathbb{R}$ of k de degré $[K:k] \leq d$ et un élément $g \in G(K)$ tel que $H_K = gH_{i,K}g^{-1}$ avec

$$\mathrm{H}_d(g) \leq c \cdot \mathrm{vol}(\mathcal{F}_H)^{\delta}$$

Ici la hauteur $\mathrm{H}_d(g)$ signifie $\mathrm{H}_d(\rho(g))$ pour une représentation fidèle fixée $\rho : G \hookrightarrow \mathrm{GL}_N$, et H_d est la hauteur d'un nombre algébrique de degré au plus d. La fonction de volume est la même que celle de la conjecture 1.2.9.

La conjecture 1.2.10 est vraie pour $\mathbb{C}^n$, cf. Lemma 5.2 dans [34]. Il est peut être possible de prouver le même résultat pour un produit de courbes de Shimura en appliquant un résultat récent de Orr, cf. [57].

1.2.5 Minoration des orbites galoisiennes des points optimaux

La conjecture des orbites galoisiennes est un sujet plus difficile. Il s'agit d'une généralisation de la conjecture des orbites galoisiennes apparaissant dans le contexte de la conjecture d'André-Oort.

Conjecture 1.2.11 (cf. Conjecture 8.2 de [35])**.** *Soit V une sous-variété dúne variété de Shimura connexe S. Soit $L \subset \mathbb{C}$ un corps de type fini sur $\mathbb{Q}$ tel que S et V sont définies sur L. Alors il existe des constantes $c, \delta > 0$ telles que pour tout point optimal $P \in V$, on a*

$$\#\mathrm{Gal}(\mathbb{C}/L) \cdot P \geq c\Delta(\langle P \rangle)^{\delta}$$

où $\langle P \rangle$ est la plus petite sous-variété spéciale de S contenant P.

Habegger et Pila ont démontré la conjecture 1.2.11 pour des "courbes asymétriques" dans $\mathbb{C}^n$. Plus tard, généralisant cette idée, Orr a démontré un résultat partiel pour certaines courbes dans $\mathcal{A}_g \times \mathcal{A}_g$, cf. [56]. Une des raisons pour lesquelles nous croyons que la conjecture 1.2.11 est beaucoup plus difficile que sa contrepartie pour la conjecture d'André-Oort est que, dans ce cas, les points optimaux P n'ont guère à voir avec les points CM, et il n'y en a donc pas de description arithmétique; c'est pourquoi nous pensons que plus d'idées géométriques sont nécessaires pour démontrer cette conjecture.

1.3 Résultats principaux de cette thèse

Le travail de cette thèse est formé de cinq parties. Les trois premières parties sont un travail commun avec Christopher Daw, la quatrième partie est un travail en commun avec Mikhail Borovoi et Christopher Daw et la cinquième partie est due au présent auteur.

1.3.1 Un théorème de structure pour les sous-variétés anormales

Soit S une variété de Shimura connexe, et V une sous-variété algébrique irréductible de S. Nous définissons son **lieu faiblement spécial** $\mathrm{SpL}(V)$ comme la réunion de toutes les sous-variétés faiblement spéciales de dimension strictement positive de S contenu dans V. L'une des étapes importantes de la stratégie de Pila-Zannier en vue de la démonstration de la conjecture d'André-Oort consiste à étudier la structure de ce lieu faiblement spécial. En fait, nous avons la propriété suivante, cf. Théorème 4.1 de [80].

Théorème 1.3.1. *Soit V une sous-variété Hodge-générique de S tel que pour toute décomposition possible de S en un produit de variétés de Shimura de dimension positives $S = S_1 \times S_2$, V n'est pas de la forme $S_1 \times V'$ pour une sous-variété V' de S_2. Alors $\mathrm{SpL}(V)$ est Zariski fermée dan V et $\mathrm{SpL}(V) \neq V$.*

Ce théorème permet de réduire la conjecture d'André-Oort au problème de comptage des orbites de Galois de points spéciaux.

Cependant, dans le contexte de la conjecture de Zilber-Pink, considérer seulement les sous-variétés faiblement spéciales contenues dans V ne suffit pas. Nous avons également besoin d'étudier toutes les sous-variétés faiblement spéciales qui ont une grande intersection avec la sous-variété donnée. Cela nous amène à la définition des sous-variétés anormales.

La notion de sous-variété anormale est définie par Bombieri, Masser et Zannier dans [7] pour les sous-variétés de tores algébriques. Son analogue pour les variétés de Shimura mérite également d'être étudié.

Pour tout sous-ensemble $W \subset S$, nous désignons par $\langle W \rangle_{\mathrm{ws}}$ la plus petite sous-variété faiblement spéciale de S contenant W. Cela nous amène à la définition des sous-variétés anormales.

Définition 1.3.1. Soit V une sous-variété dúne variété de Shimura connexe S. Une sous-variété $W \subset V$ est appelée **anormale** si

$$\dim W > \max\{0, \dim\langle W \rangle_{\mathrm{ws}} + \dim V - \dim S\}.$$

En utilisant la "O-minimality" et la conjecture d'Ax-Schanuel (dont une démonstration a été annoncÃl'e par Mok, Pila et Tsimerman), nous avons démontré le théorème de structure suivant pour les sous-variétés anormales, qui est un analogue d'un des résultats principaux de [7]. Notre théorème est aussi une contrepartie d'un théorème de la section 6.1 de [35], qui concerne les sous-variétés anormales dans les variétés abéliennes.

Théorème 1.3.2 (Daw-Ren)**.** *Soit V une sous-variété dúne variété de Shimura connexe S.*

1. *Pour toute sous-variété spéciale S' avec une décomposition des variétés de Shimura $S' = S_1 \times S_2$, la réunion de toutes les sous-variétés de dimensions positives Y of V contenu dans certains $S_1 \times \{x_2\}$ avec $\dim Y > \dim V - \dim S_2$ est un fermé de Zariski dans V.*
2. *Il y a une collection finie Φ de sous-variétés spéciales munies de décompositions $S' = S_1 \times S_2$, avec la propriété suivante. Pour chaque sous-variété anormale maximale W de V, il existe $S' = S_1 \times S_2 \in \Phi$ tel que W est une composante de $V \cap (S_1 \times \{x_2\})$ pour un certain $x_2 \in S_2$.*
3. *La réunion V^{a} des sous-variétés anormales dans V est Zariski fermée.*

4. *Supposons que V est Hodge-générique dans S. Le complément $V^{\mathrm{oa}} := V \setminus V^{\mathrm{a}}$ est vide si et seulement si S s'écrit comme un produit $S_1 \times S_2$ de deux variétés de Shimura tel que*
$$\dim \overline{\phi(V)}^{\mathrm{Zar}} < \min\{\dim V, \dim S_2\},$$
où ϕ est la réstriction de la projection $S \to S_2$ à V.

On remarque que le dernier résultat de ce théorème de structure implique que V^{oa} est vide pour une sous-variété V en "position générale". Nous avons aussi la conséquence suivante.

Corollaire 1.3.2. *Soit V une sous-variété Hodge-génériqe de $\mathcal{A}_g$. Alors V^{oa} est toujours non-vide.*

1.3.2 Réduire la conjecture de Zilber-Pink à un problème de comptage de points

Nous considérons à présent une autre formulation de la conjecture de Zilber-Pink.

Définition 1.3.3. Soit V une sous-variété dúne variété de Shimura connexe S. Pour tout sous-ensemble $W \subset V$, on note $\langle W \rangle$ la plus petite sous-variété spéciale contenant W. Une sous-variété W est réputée être **optimale** dans V si pour toute sous-variété W' de V contenant strictement W, on a

$$\dim\langle W' \rangle - \dim W' > \dim\langle W \rangle - \dim W.$$

Une sous -variété optimale de dimension zéro est appelée un **point optimal**.

Conjecture 1.3.4. *[Zilber-Pink] Soit V une sous-variété dúne variété de Shimura connexe S. Alors V ne contient qu'un nombre fini de sous-variétés optimales.*

On peut montrer que la conjecture 1.3.4 implique la conjecture 1.1.3. De plus, comme les sous-variétés de spéciale maximales dans V sont optimales, on peut en déduire la conjecture 1.3.4 que V a seulement un nombre fini de sous-variétés spéciales maximales, ce qui n'est autre qu'une reformulation de la conjecture d'André-Oort.

Rappelons que l'estimation des tailles de certaines orbites galoisiennes joue un rôle important dans la stratégie de Pila-Zannier. Cependant, la minoration des orbites galoisiennes nous permet seulement de montrer que certains ensembles sont finis, et nous ne pouvons donc pas espérer démontrer la conjecture 1.3.4 en utilisant les orbites galoisiennes directement. Néanmoins, par un résultat de cette thèse, il suffit en fait de résoudre un problème de comptage de points.

Conjecture 1.3.5. *Soit V une sous-variété dúne variété de Shimura connexe S. Alors V ne contient qu'un nombre fini de points optimaux.*

Ceci est manifestement une conséquence de la conjecture 1.3.4, et ces deux conjectures sont en fait équivalentes :

Théorème 1.3.3 (Daw-Ren). *La conjecture 1.3.5 implique la conjecture 1.3.4.*

La preuve de ce théorème utilise la conjecture d'Ax-Schanuel et une étude détaillée des sous-variétés faiblement spéciales d'une variété de Shimura.

Ainsi, nous n'aurons besoin que de prouver Conjecture 1.3.5.

1.3.3 Résultats partiels vers le problème de comptage de points

Le travail de Habegger et Pila dans [34, 35] pour $\mathbb{C}^n$ peut être généralisé à une variété de Shimura quelconque en supposant certaines conjectures arithmétiques additionnelles. Dans le cas de $\mathbb{C}^n$, ces conjectures arithmétiques sont ou bien des évidences, ou bien des théorèmes, cf. [21]. Cependant, celles-ci deviennent bien moins évidentes pour des variétés de Shimura arbitraires. Ces conjectures sont formulées dans la section 2.

Notre premier résultat est une preuve conditionnelle de la conjecture de Zilber-Pink pour les courbes.

Théorème 1.3.4 (Daw-Ren). *Supposons que la conjecture de l'orbite galoisienne 1.2.11 soit valide, et supposons la validité de l'une des assertions suivantes :*

- *Les conjectures 1.2.4, 1.2.6 et 1.2.9 sont vraies.*
- *La conjecture 1.2.10 est vraie.*

Alors la conjecture 1.3.4 est vraie pour les courbes, i.e. si V est une courbe contenue dans S, alors l'ensemble des points optimaux dans V est fini.

Un autre résultat principal est une preuve conditionnelle de la conjecture de Zilber-Pink complète pour une sous-variété "générique" dans V.

Théorème 1.3.5 (Daw-Ren). *Soit V est une sous-variété Hodge-générique dans une variété de Shimura connexe S. Avec les mêmes suppositions que dans la conjecture 1.3.4, l'ensemble*

$$V^{\mathrm{oa}} \cap S^{[1+\dim V]}$$

est fini.

En particulier, sous ces hypothèses, si V est Hodge-générique dans S et si, pour toute décomposition possible de S en un produit de deux variétés de Shimura $S = S_1 \times S_2$, on a

$$\overline{\phi(V)}^{\mathrm{Zar}} = \min\{\dim V, \dim S_2\},$$

où ϕ est la réstriction de la projection $S \to S_2$ à V, i.e., V^{oa} n'est pas vide, cf. Théorème 1.3.2, alors

$$V \cap S^{1+\dim V}$$

n'est pas Zariski-dense dans V.

1.3.4 Conjugaison de sous-groupes semi-simples sur des corps de nombres réels de degré borné

Rappelons que dans la stratégie de Pila-Zannier pour la conjecture d'André-Oort, nous devons construire un ensemble définissable qui paramètre les sous-variétés faiblement spéciales d'une variété de Shimura donnée. La proposition suivante est nécessaire dans notre construction.

Théorème 1.3.6. *Soit G un groupe semi-simple sur $\mathbb{R}$. Alors il existe un ensemble fini $\Omega = \{H_1, ..., H_r\}$ de sous-groupes semi-simple tel que pour tout sous-groupe semi-simple réel H de G, il existe $g \in G(\mathbb{R})$ et un certain $H_i \in \Omega$ tel que $H = gH_ig^{-1}$.*

Dans le contexte de la conjecture de Zilber-Pink, cependant, la propriété de finitude ci-dessus n'est pas suffisante. Pour utiliser le théorème de comptage de Pila-Wilkie pour les points algébriques, il faut que l'élément conjuguant g soit défini sur une extension finie de $\mathbb{Q}$ de degré borné. Nous avons ainsi démontré le résultat suivant.

Proposition 1.3.6 (Borovoi-Daw-Ren)**.** *Soit G un groupe semi-simple connexe défini sur un corps de nombres $k \subset \mathbb{R}$. Alors il y a un ensemble fini $\Omega = \{H_1, \ldots, H_r\}$ de sous-groupes semi-simple sur k et un entier positif $d = d(G, \Omega)$ avec la propriété suivante : si H est un sous-groupe semi-simple connexe de G sur k, alors il est conjugé sur $\mathbb{R}$ à un certain $H_i \in \Omega$, et il existe une extension finie $K = K(H, H_i) \subset \mathbb{R}$ de k de degré $[K : k] \leq d$ telle que H et H_i sont conjugués sur K.*

La démonstration consiste en une étude détaillée de la cohomologie galoisienne et des suites spectrales.

On remarque que ce théorème est une étape intermédiaire dans l'utilisation du théorème de comptage de Pila-Wilkie puisqu'il faut aussi majorer la hauteur de l'élément conjuguant g.

1.3.5 Comptage de variétés de Shimura compactes

Rappelons que les seules variétés de Shimura non compactes de dimension un sont des courbes modulaires. Nous aimerions généraliser cette observation à toutes les variétés de Shimura, i.e. pour montrer qu'en un certain sens, la plupart des variétés de Shimura sont compactes.

Rappelons d'abord un critère de compacité d'une variété de Shimura, cf. Théorème 4.12, p. 210, [65].

Théorème 1.3.7. *Une variété de Shimura $\mathrm{Sh}_K(G, X)$ est compacte si et seulement si G^{ad} est $\mathbb{Q}$-anisotrope, i.e. $G^{\mathrm{ad}}(\mathbb{Q})$ ne contient aucun élément unipotent non trivial, ou, de manière équivalente, $G^{\mathrm{ad}}_{\mathbb{Q}}$ ne contient pas de tore déployé sur $\mathbb{Q}$ et de dimension strictement positive.*

Ceci nous amène naturellement à la question suivante : étant donné un groupe de Lie réel $G(\mathbb{R})$, combien de sous-groupes arithmétiques cocompact a-t-il? Plus précisément, si G est un groupe algébrique sur $\mathbb{Q}$, quel est la proportion d'éléments dans $H^1(\mathrm{Aut}(\mathbb{R}/\mathbb{Q}), \mathrm{Aut}(G))$ qui correspondent à des formes anisotropes de G?

Dans cette thèse, on étudie la proportion de formes intérieures anisotropes de G, lorsque G est le groupe général linéaire ou un groupe unitaire. De plus, nous permettons à G d'être défini sur un corps de nombres arbitraire plutôt que seulement sur $\mathbb{Q}$.

D'abord, on définit une fonction de hauteur naturelle $h : H^1(F, G^{\mathrm{ad}}) \longrightarrow \mathbb{R}_{\geq 0}$ tel que pour tout $b > 0$, il n'y a qu'un nombre fini de $\eta \in H^1(F, G^{\mathrm{ad}})$ avec $H(\eta) < b$. L'ingrédient clé dans la définition de cette fonction de hauteur est l'application de Kottwitz.

Définition 1.3.7. On note p_∞ l'application de projection naturelle

$$p_\infty : H^1(F, G) \to \bigoplus_{v|\infty} H^1(F_v, G),$$

et pour tout $\theta \in \bigoplus_{v|\infty} H^1(F_v, G)$, la fibre $p_\infty^{-1}(\theta)$ est notée $H^1(F, G)_\theta$.

Définition 1.3.8. Soit G un groupe réductif sur un corps de nombres F, et $h : H^1(F, G) \to \mathbb{R}_{\geq 0}$ une fonction. Pour tout $T > 1$, et tout $\theta \in \bigoplus_{v|\infty} H^1(F_v, G)$, on définit

$$N_F(h, G, T)_\theta := \{\eta \in H^1(F, G)_\theta; h(\eta) < T\},$$

et on note $N_F^c(h, G, T)_\theta$ (respectivement $N_F^{nc}(h, G, T)_\theta$) le sous-ensemble de $N_F(h, G, T)_\theta$ constitué des η tels que la forme intérieure correspondante G_ν est F-anisotrope (respectivement F-isotrope).

On remarque que les deux ensembles ci-dessus comptent des formes intérieure sur F et des formes anisotropes intérieure sur F de hauteurs bornées et un comportement fixe aux places infinis. Le résultat principal est le suivant.

Théorème 1.3.8. *Soit G un groupe général linéaire ou un groupe unitaire sur un corps de nombres F. Alors il existe une fonction de hauteur naturelle $h : H^1(F, G) \to \mathbb{R}_{\geq 0}$ telle que pour tout $\theta \in \bigoplus_{v|\infty} H^1(F_v, G)$, la densité*

$$\lim_{T\to\infty} \frac{\# N_F^{nc}(h, G, T)_\theta}{\# N_F(h, G, T)_\theta} = 0.$$

En d'autres termes, pour cette fonction de hauteur naturelle, presque toutes les formes de G sur F sont F-anisotropes.

Chapter 2

Introduction (English version)

2.1 Overview of Diophantine geometry of Shimura varieties and surrounding problems

2.1.1 Generalities about unlikely intersections

An unlikely intersection occurs when two subvarieties intersect in a component of larger dimension than would be expected from dimension counting. The first non-trivial problem in unlikely intersection is the following question asked by Lang.

Theorem 2.1.1 (Ihara-Serre-Tate). *Let $f(X,Y) \in \mathbb{C}[X,Y]$ be an irreducible polynomial with infinitely many roots (α, β) such that α and β are both roots of unity. Then f is one of the following:*

- $a(X^m - \eta Y^n)$
- $a(X^m Y^n - \eta)$

Where in both cases, $a \in \mathbb{C}^\times$, $m,n \in \mathbb{N}$ and η is a root of unity.

In a more abstract language, Theorem 2.1.1 can be translated into the following equivalent form.

Theorem 2.1.2 (Multiplicative-Manin-Mumford for $\mathbb{G}_m^2$). *Let $V \subset \mathbb{C}^2$ be an irreducible algebraic curve contained in $\mathbb{G}_m^2$. If V contains infinitely many torsion points, then V is a translation of an irreducible algebraic subgroup of $\mathbb{G}_m^2$ by a torsion point.*

The following generalization, proved by Mann, is natural.

Theorem 2.1.3 (Multiplicative-Manin-Mumford for $\mathbb{G}_m^n$, cf. [50]). *Let $V \subset \mathbb{C}^n$ be an irreducible algebraic variety contained in $\mathbb{G}_m^n$. If V contains a Zariski dense subset of torsion points, then V is a translation of an irreducible algebraic subgroup of $\mathbb{G}_m^n$ by a torsion point.*

One can consider a similar question for abelian varieties, i.e., the Manin-Mumford conjecture.

Theorem 2.1.4 (Raynaud, etc.). *Let A be an abelian variety over a field k of characteristic zero, and $V \subset A$ be an irreducible algebraic subvariety contained in A.*

If V contains a Zariski dense subset of torsion points, then V is a translation of an abelian subvariety by a torsion point.

The first proof of this conjecture is given by Raynaud [66] in the case of $k = \mathbb{C}$ and $\dim V = 1$. After that, people from different areas, including Hrushovski, Ullmo, Zhang, Pila and Zannier have given other proofs, cf. [39, 78, 90, 62].

Now we come to the situation of unlikely intersections for Shimura varieties. Notice that if the ambient algebraic variety has dimension one, then non-trivial unlikely intersections will not occur. So let us take the simplest Shimura variety of dimension two, $\mathbb{C}^2$, considered as the product of two modular curves, then the analogue of Theorem 2.1.3 becomes the simplest non-trivial case of the André-Oort conjecture.

Theorem 2.1.5 (André). *Let $V \subset \mathbb{C}^2$ be an irreducible algebraic curve containing infinitely many points (x_1, x_2) such that the x_i are both j-invariant of CM elliptic curves, then V is one of the following:*

- *$\{x_1\} \times \mathbb{C}$ with x_1 CM,*
- *$\mathbb{C} \times \{x_2\}$ with x_2 CM,*
- *Hecke correspondence, i.e. the image of*

$$\mathbb{H} \to \mathbb{H} \times \mathbb{H} \to \mathbb{C} \times \mathbb{C}, \quad z \mapsto (z, nz) \mapsto (j(z), j(nz))$$

  for some $n \in \mathbb{N}$.

Notice that the above theorem tells us that if a curve in $\mathbb{C}^2$ contains infinitely many CM points, then this curve is itself a modular curve. This perspective allows us to formulate a more general conjecture for all Shimura varieties.

Conjecture 2.1.6 (André-Oort). *Let $S \subset \mathrm{Sh}_K(G, \mathfrak{X})$ be a connected component of a Shimura variety, and let $V \subset S$ be a subvariety which is not a Shimura subvariety. Then the set of special points contained in V is not Zariski dense in V.*

Remark 2.1.7. In the literature, problems about unlikely intersections are usually formulated by using the terminology of "special subvarieties". In our context, our ambient variety is a connected Shimura variety, so special subvarieties are exactly Shimura subvarieties. It is also worth observing that zero-dimensional Shimura subvarieties are just the so-called special points or CM points.

A lot of beautiful partial results are given. By assuming the generalized Riemann hypothesis, it was proven that the André-Oort conjecture is true for $\mathbb{C}^n$, cf. Edixhoven [26], and the full André-Oort conjecture, cf. Klingler, Ullmo and Yafaev [46, 85].

Later, through 'O-minimality' in model theory and the Pila-Zannier strategy, it was shown that the André-Oort conjecture is true for $\mathbb{C}^n$, cf. Pila [59], and for Hilbert modular surfaces, cf. Daw and Yafaev [22] and Siegel modular varieties , cf. Tsimerman [77], Yuan-Zhang, cf. [89] and Andreatta-Goren-Howard-Madapusi Pera, cf [1]. This method also gives a new proof of the André-Oort conjecture for all Shimura varieties under the generalized Riemann hypothesis. Gao has generalised above results to all the mixed Shimura varieties, cf. [30].

Since the André-Oort conjecture considers the intersection of a proper subvariety of a Shimura variety with a family of zero dimensional subvarieties (i.e. CM points), this is a typical question in the frame of unlikely intersections. Now if we replace the set of CM points by a larger family of Shimura subvarieties that are unlikely to intersect a given subvariety, we obtain the following conjecture of Zilber and Pink.

Conjecture 2.1.8 (cf. [64], Conjecture 1.3)**.** *Let $S \subset \mathrm{Sh}_K(G,\mathfrak{X})$ be a connected component of a Shimura variety and, for any integer d, let $S^{[d]}$ denote the union of the Shimura subvarieties of S having codimension at least d. Let V be a* **Hodge generic** *subvariety of S, i.e. V is not contained in any proper Shimura subvariety of S. Then*

$$V \cap S^{[1+\dim V]}$$

is not Zariski dense in V.

We also notice that in the spirit of the work by Bombieri-Masser-Zannier, [7], the Zilber-Pink conjecture has an equivalent formulation in the terminology of Special anomalous subvarieties, see Chapter 2, Section 15 of this thesis for details.

By developing the Pila-Zannier strategy, cf. [21], we are able to reduce Conjecture 3.1.1 to several geometric and/or arithmetic problems in Shimura varieties, which are listed in Section 2. Studying these problems may serve as a reasonable way towards the proof of the Zilber-Pink conjecture.

2.1.2 Shimura varieties

Since Shimura varieties are the key object in this thesis, it is worthwhile to review their basic properties here. For a more detailed introduction, see Milne's lecture note [52] or Deligne's original papers [23, 24].

Definition 2.1.9. We denote by $\mathbb{S}$ the Delighe torus $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{m,\mathbb{C}}$. A **Shimura datum** is a pair (G,X) where G is a reductive algebraic group over $\mathbb{Q}$ and X is a $G(\mathbb{R})$-orbit of some element in $\mathrm{Hom}(\mathbb{S},G_{\mathbb{R}})$, such that there is some $h \in X$ with the following property.

1. The Lie algebra $\mathrm{Lie}(G)$ is of type $\{(-1,1),(0,0),(1,-1)\}$, here the Hodge structure of $\mathrm{Lie}(G)$ is given by the composition of h with the adjoint representation of G.
2. The map $\mathrm{int}(h(i))$ is a Cartan involution of $G^{\mathrm{ad}}_{\mathbb{R}}$, i.e., $\{g \in G^{\mathrm{ad}}(\mathbb{C}); h(i)\bar{g}h(i)^{-1} = g\}$ is compact.
3. For every $\mathbb{Q}$-simple factor H of G^{ad}, the composition of h and $G_{\mathbb{R}} \to G^{\mathrm{ad}}_{\mathbb{R}} \to H_{\mathbb{R}}$ is not trivial.

Definition 2.1.10. Let (G,X) be a Shimura datum and K be a compact open subgroup of $G(\mathbb{A}_f)$. Then the **Shimura variety** $\mathrm{Sh}_K(G,X)$ attached to (G,X) and K is the double quotient

$$\mathrm{Sh}_K(G,X) := G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K,$$

where the actions of $G(\mathbb{Q})$ and K are given by $q(x,a)k = (qx, qak)$ for all $q \in G(\mathbb{Q})$ and $k \in K$.

Example 2.1.11. Let $G = \mathrm{GL}_{2,\mathbb{Q}}$ and pick $h_0 : a+bi \mapsto \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. Then (G,X) is a Shimura datum. Notice that X can be naturally identified with $\mathbb{H}^{\pm} = \mathbb{C}\backslash\mathbb{R}$ by maping gh_0g^{-1} to $g\sqrt{-1}$ for all $g \in G(\mathbb{R}) = \mathrm{GL}_2(\mathbb{R})$. Now we take $K = \prod_p \mathrm{GL}(\mathbb{Z}_p)$ as a compact open subgroup of $G(\mathbb{A}_f) = \mathrm{GL}(\mathbb{A}_f)$. One can show that $\mathrm{Sh}_K(\mathrm{GL}_{2,\mathbb{Q}}, \mathbb{H}^{\pm})$ is isomorphic to two copies of $Y(1) = \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$.

The above example implies that the modular curve $Y(1)$ is a connected component of a Shimura variety, i.e., a connected Shimura variety.

In the context of the André-Oort and Zilber-Pink conjectures, the notions of special and weakly special subvarieties play an important role.

Definition 2.1.12. Let $\mathrm{Sh}_K(G,X)$ be a Shimura variety, we say a subvariety Z is a **special subvariety** if there is a triple (H,Y,g) where $(H,Y) \subset (G,X)$ is a sub-Shimura datum and $g \in G(\mathbb{A}_f)$, such that Z is an irreducible component of the image of $Y \times \{g\}$ in $\mathrm{Sh}_K(G,X)$.

Remark 2.1.13. Special subvarieties are sometimes also called **subvarieties of Hodge type**.

A Shimura variety is also equipped with a much larger collection of weakly special subvarieties. For simplicity, here, we only give a definition for weakly special subvarieties up to level structures. For a precise definition, see Chapter 2, Section 2.

Definition 2.1.14. Let S be a connected component of a Shimura variety $\mathrm{Sh}_K(G,X)$. A subvariety Z of S is called **weakly special** if S has a special subvariety S' with a splitting as a product of two Shimura varieties $S' = S_1 \times S_2$, such that $Z = S_1 \times \{z_2\}$ for some $z_2 \in S_2$.

In order to apply the Pila-Zannier strategy, one also needs to formulate an analogue of the exponential map $\mathbb{C}^n \to (\mathbb{C}^{\times})^n$, a uniformization map for Shimura varieties.

Definition 2.1.15. Let $\mathrm{Sh}_K(G,X)$ be a Shimura variety, and X^+ be a connected component of X and we denote

$$G(\mathbb{Q})_+ := \{g \in G(\mathbb{Q}); gX^+ = X^+\}, \quad \Gamma = G(\mathbb{Q})_+ \cap K.$$

Then $S = \Gamma\backslash X^+$ is a connected component of $\mathrm{Sh}_K(G,X)$ and the uniformization map is defined to be the quotient map

$$\pi : X^+ \longrightarrow S.$$

Sometimes, we also write the uniformization map as $\pi : X \longrightarrow S$ by a slight abuse of notation.

2.1.3 Functional Transcendence

A key ingredient towards the proof of a Zilber-Pink type conjecture is the analogue of some classical problems in transcendental number theory.

Theorem 2.1.16 (Lindemann-Weierstraß)**.** *Let $\alpha_1,...,\alpha_n$ be algebraic numbers that are $\mathbb{Q}$-linearly independent, then $e^{\alpha_1},...,e^{\alpha_n}$ are algebraically independent over $\mathbb{Q}$.*

Conjecture 2.1.17 (Schanuel)**.** *Let $\alpha_1,...,\alpha_n$ be complex numbers that are $\mathbb{Q}$-linearly independent, then*

$$\mathrm{tr.deg}_{\mathbb{Q}}(\alpha_1,...,\alpha_n,e^{\alpha_1},...,e^{\alpha_n}) \geq n.$$

The first functional analogues of the above two assertions are given by Ax [3].

Definition 2.1.18. Let A be a complex analytic subset $\mathbb{C}^n$, we say the functions $f_1,...,f_r : A \to \mathbb{C}$ are $\mathbb{Q}$-linearly independent modulo $\mathbb{C}$ if there are no non-trivial relations $q_1 f_1 + ... + q_r f_r = c$ for $q_i \in \mathbb{Q}$ and $c \in \mathbb{C}$.

Theorem 2.1.19 (Ax-Lindemann-Weierstraß for tori)**.** *Let A be a complex analytic subset of $\mathbb{C}^n$, we denote by α_i the i-th coordinate function $z_i : A \to \mathbb{C}, (z_1,..,z_n) \mapsto z_i$. Suppose these α_i are $\mathbb{Q}$-linearly independent modulo $\mathbb{C}$, then $e^{\alpha_1},...,e^{\alpha_n}$ are algebraically independent over $\mathbb{C}$.*

Theorem 2.1.20 (Ax-Schanuel for tori)**.** *Under the same hypothesis as above, we have*

$$\mathrm{tr.deg}_{\mathbb{C}}(\alpha_1,...,\alpha_n,e^{\alpha_1},...,e^{\alpha_n}) \geq n + \dim A.$$

Under the same spirit, the Ax-Lindemann-Weierstraß and Ax-Schanuel conjectures can be formulated similarly in the context of abelian varieties and Shimura varieties. Here we only list the case which will be used in the thesis, see Chapter 2, Section 5 for more details.

Conjecture 2.1.21 (Ax-Schanuel for Shimura varieties)**.** *Let S be a connected component of a Shimura variety $\mathrm{Sh}_K(G,X)$ and let Z be an algebraic subvariety of $X \times S$. We denote by D_S the graph of the uniformisation map $\pi : X \to S$. Let U be an irreducible analytic component of $V \cap D_S$ and suppose that $\pi_S(U)$ is not contained in any proper weakly special subvariety of S. Then we have*

$$\dim Z = \dim U + \dim S.$$

During the preparation of the thesis, Mok, Pila and Tsimerman have announced a proof of this conjecture for all the pure Shimura varieties, cf [53]. More recently, Bakker and Tsimerman have generalized this assertion to variations of Hodge structures, see [6]. We notice that Klingler has similar work in this direction as well, see [45]. It is also worth noting that Chambert-Loir and Loeser have proved a p-adic analogue of the functional transcendental result, see [13].

2.1.4 The Pila-Wilkie counting theorem

As a powerful tool in Diophantine geometry, the Pila-Zannier strategy uses a counting theorem, which is named after Pila and Wilkie, to attack different kinds of problems in unlikely intersections.

The Pila-Wilkie counting theorem was initiated by the natural and simple intuition that a curve $\mathcal{C} \subset \mathbb{R}^2$ contains little rational points unless this curve is algebraic. For example, the transcendental curve $\mathcal{C}_1 = \{(x,y); y = e^x\}$ contains only one rational point $(0,1)$. On the other hand, the parabola $\mathcal{C}_2 = \{(x,y); y = x^2\}$ contains many rational points, i.e., $(q,q^2) \in \mathcal{C}_2$ for all $q \in \mathbb{Q}$.

More generally, let $A \subset \mathbb{R}^n$ be a subset with some good property, i.e., definable in a certain O-minimal structure. If A contains many rational points, then one may expect that A contains an algebraic curve. Let us formulate the counting theorem in a precise way as following.

Definition 2.1.22. The height $H(q)$ of a rational number q is equal to $\max\{|m|,|n|\}$, where m,n are coprime integers such that $q = \frac{m}{n}$. For a tuple $q = (q_1,\ldots,q_n) \in \mathbb{Q}^n$, we define its height $H(q)$ to be $\max\{H(q_1),\ldots,H(q_n)\}$.

We can count rational points in any subset of $\mathbb{R}^n$ with respect to this height function.

Definition 2.1.23. Let $Y \subset \mathbb{R}^n$, for any $T \geq 1$, we denote by

$$N(Y,T) := \{y \in Y; H(y) < T\}.$$

Definition 2.1.24. A **semialgebraic set** in $\mathbb{R}^n$ is a subset defined by a finite sequence of polynomial equations, i.e., $P(X_1,\ldots,X_n) = 0$ and polynomial inequalities, i.e., $Q(X_1,\ldots,X_n) > 0$, where $P,Q \in \mathbb{R}[X_1,\ldots,X_n]$, or any boolean combination of such sets. A semialgebraic set is called **irreducible** if its Zariski closure in $\mathbb{R}^n$ is irreducible. The **dimension** of an irreducible semialgebraic subset is the real dimension of its Zariski closure.

Regarding the distribution of rational points in a set which is definable in some O-minimal structure, e.g., $\mathbb{R}_{\mathrm{an}}$ or $\mathbb{R}_{\mathrm{an,exp}}$, we have the following counting theorem, cf. [58], which plays a crucial role in this thesis. For a detailed introduction to O-minimality, see [86].

Theorem 2.1.25 (Pila-Wilkie)**.** *Let $A \subset \mathbb{R}^n$ be a subset which is definable in an O-minimal structure. Then for any $\varepsilon > 0$, there is a constant $c = c(A,\varepsilon) > 0$ with the following property: for any subset $D \subset A$, if there is a $T > 1$ with $\#N(D,T) \geq cT^{\varepsilon}$, then A contains a positive dimensional semialgebraic subset Z such that $D \cap Z$ is non-empty.*

Remark 2.1.26. The counting theorem above is written in a form which seems more convenient to use in some sense. In the Pila-Wilkie's original paper, they use a slightly different formulation by introducing the notion of transcendental part of a definable set, i.e., a subset of a definable set obtained by removing all the positive dimensional irreducible positive dimensional semialgebraic subsets. Their theorem asserts that this deprived set always contains little rational points.

In the context of the Zilber-Pink conjecture, one needs a family version of the above counting theorem, and one should count algebraic numbers with bounded degree instead of only rational numbers. In this thesis, the version formulated by Habegger-Pila is sufficient, cf. Corollary 7.2 of [35], see also Chapter 2, Section 9 of this thesis. We also remark that Cluckers, Comte and Loeser have proved a p-adic version of the Pila-Wilkie counting theorem, cf. [16].

2.1.5 First sight of the Pila-Zannier strategy: the Lang's conjecture

In order to illustrate how the Pila-Wilkie counting theorem and the functional transcendental theorem enable us to attack unlikely intersection problems, in this subsection we sketch a proof of the Lang's conjecture, i.e., the multiplicative Manin-Mumford conjecture for $\mathbb{G}_m^2$, as mentioned in the beginning of the introduction. Let $V \subset \mathbb{G}_m^2$ be an irreducible algebraic subvariety which contains infinitely many torsion points.

Definition 2.1.27. We write the uniformization map

$$\pi : \mathbb{C}^2 \to (\mathbb{C}^\times)^2, \quad (z_1, z_2) \mapsto (e^{2\pi i z_1}, e^{2\pi i z_2}),$$

and we denote $\mathcal{F} := [0,1)^2$. For any $x \in (\mathbb{C}^\times)^2$, we denote $\tilde{x}$ be the unique element in $\pi^{-1}(x) \cap \mathcal{F}$.

Proof of Theorem 2.1.3. Suppose V is not a torsion coset, it suffices to show that V contains only finitely many torsion points. We denote by $\mathcal{V}$ the intersection $\mathcal{F} \cap \pi^{-1}(V)$.

Since $\mathcal{F}$ is semialgebraic, $\mathcal{V}$ is definable in $\mathbb{R}_{\mathrm{an}}$, we let $c = c(\mathcal{V}, \frac{1}{2})$ be the constant in the Pila-Wilkie counting theorem, i.e., for $\varepsilon = \frac{1}{2}$. By the Lagrange interpolation, V is defined over a number field K, we denote by Γ_K its absolute Galois group. For any torsion point $P \in V$, we define

$$D_P = \{\widetilde{\sigma(P)}; \sigma \in \Gamma_K\} \subset \mathcal{V}.$$

We claim that $\#D_P = \#N(D_P, H(P)) \leq cH(P)^{\frac{1}{2}}$, otherwise, by the counting theorem, $\mathcal{F}$ contains a semialgebraic curve Z which has a non-empty intersection with D_P. By the Ax-Lindemann-Weierstraß theorem, one can show that $V = \overline{\pi(Z)}^{\mathrm{Zar}}$ is a coset. Since V contains a torsion point P, it is further a torsion coset, a contradiction. On the other hand, one has $\#D_P = \#\Gamma_K P \geq H(P)$. Combining the above two assertions, we have $H(P) \leq cH(P)^{\frac{1}{2}}$, i.e., $H(P) \leq c^2$, which yields the finiteness of P. □

This proof already contains the essential idea of the Pila-Zannier strategy, i.e., the algebraic transcendental theorem, the lower bound of the Galois orbit and the counting theorem. One of the major task of this thesis is to generalise the above proof to arbitrary Shimura varieties and to a more general context of unlikely intersections.

2.2 Arithmetic ingredients

In the process of applying the Pila-Zannier strategy to the Zilber-Pink conjecture, a lot of arithmetic and geometric problems appear naturally. These conjectures are precisely the obstructions to an unconditional proof of the Zilber-Pink conjecture. (see Section 3.3)

Besides their potential applications to the Zilber-Pink conjecture, these problems are also very interesting in themselves.

2.2.1 A finiteness property for complexity

By using the Ax-Schanuel conjecture (a proof of which has been announced by Mok, Pila and Tsimerman), we are able to reduce the Zilber-Pink conjecture to a point-counting problem, cf. Section 8 of [21].

In order to solve this counting problem, we need to find a way to count Shimura subvarieties of a given Shimura variety in a reasonable way. This leads us to the notion of "complexity" for Shimura subvarieties. To do this, let us first recall the definition of complexity in abelian varieties.

Definition 2.2.1 (cf. Definition 3.4 of [35])**.** Let A be an abelian variety over a number field K with an ample line bundle $\mathcal{L}$. If Z is a torsion coset of A which is the translate of an abelian subvariety B of A by a torsion point, we define its arithmetic complexity as

$$\Delta_{\text{arith}}(Z) = \min\{\text{order of T};\ Z = T + B \text{ and } T \text{ has finite order}\}$$

and its complexity as

$$\Delta(A) = \max\{\deg_{\mathcal{L}} B, \Delta_{\text{arith}}(A)\}$$

where $\deg_{\mathcal{L}} B$ is the degree of B with respect to $\mathcal{L}$.

Similarly, cf. [35], in the case of Shimura varieties, we still wish our "complexity" consists of an arithmetic part and a geometric part. In order to avoid technical details, we only give the definition of arithmetic complexities in $\mathcal{A}_g$.

Definition 2.2.2. For a CM point $x \in \mathcal{A}_g$, let A_x denote the corresponding g-dimensional principally polarized abelian variety, $R_x := Z(\text{End}(A_x))$ the centre of the endomorphism ring of A_x, and $\Delta(x) := \text{Disc}(R_x)$ its discriminant. The arithmetic complexity of a Shimura subvariety Z of $\mathcal{A}_g$ is defined by

$$\Delta_{\text{arith}}(Z) = \min\{\Delta(x);\ x \in Z \text{ and } x \text{ is a CM point}\}.$$

Notice that a Shimura subvariety of a Shimura variety always contains special points, so this definition makes sense. For simplicity, we just mention that "arithmetic complexity" can be defined for arbitrary Shimura varieties, cf. Section 10 of [21].

For the geometric part of complexity, recall that a Shimura variety has a Baily-Borel compactification $\overline{S}$ given by an ample line bundle $\mathcal{L}$. For a subvariety W of S, we denote by $\deg_{\mathcal{L}} W$ the degree of the Zariski closure of W in $\overline{S}$.

Definition 2.2.3. The complexity of a Shimura subvariety Z of a Shimura variety S is the positive integer

$$\Delta(Z) := \max\{\deg_{\mathcal{L}} Z, \Delta_{\mathrm{arith}}(Z)\}.$$

A natural thing we would like to show is that complexity is indeed a good counting function for Shimura subvarieties.

Conjecture 2.2.4. *For any $b > 1$, there are only finitely many Shimura subvarieties Z of a given Shimura variety S such that $\Delta(Z) < b$.*

We believe that obtaining an good upper bound for the degree of a Shimura subvariety is necessary for this conjecture.

2.2.2 Fields of definition of Shimura subvarieties

According to the semi-stableness of abelian varieties, we have the following classical result.

Proposition 2.2.5. *Let A be an abelian variety over a number field K. Then there is a finite extension L of K such that all the abelian subvarieties of A are defined over L.*

This proposition is an important observation in proving a conditional result on the Zilber-Pink conjecture for abelian varieties over number fields, see Theorem 9.8 of [35]. However, the analogue of Proposition 2.2.5 for Shimura varieties doesn't hold, since the Galois orbit of a CM point can be large. However, if our Shimura subvariety has no "CM part", one can still hope that the field of definition does not grow too fast. In fact, we make the following conjecture.

Conjecture 2.2.6. *Let $S \subset \mathrm{Sh}_K(G, \mathfrak{X})$ be a connected component of a Shimura variety defined over a number field k (a finite extension of the reflex field $E = E(G, \mathfrak{X})$). Then for any $\varepsilon > 0$, there is a uniform constant $c = c(S, \varepsilon) > 0$ with the following property.*

For any Shimura subvariety $Z \subset S$ which is not of the form $S_1 \times \{s_2\} \subset S_1 \times S_2 \subset S$, where S_i are both positive dimensional Shimura varieties, there is a finite extension L of k with $[L : k] \leq c\Delta(Z)^{\varepsilon}$ such that Z is defined over L.

This conjecture holds for $S = \mathbb{C}^n$, see Section 13 of [21]. It should not be too hard to prove it for an arbitrary product of Shimura curves (not necessarily pairwise isomorphic) and it may be possible to prove it for all Shimura varieties.

The estimation of the class number of semi-simple subgroups, and a detailed study of Galois cohomology may play an important role in solving these kind of problems.

2.2.3 A quantitative version of the Siegel property in reduction theory

The construction of a fundamental domain with respect to the left action of $\mathrm{SL}_2(\mathbb{Z})$ on the Poincaré upper half plane (or equivalently, $\mathrm{SL}_2(\mathbb{R})$) is classical. More generally, from the reduction theory of arithmetic groups we have:

Proposition 2.2.7. *(cf. Theorem 15.4 [9])Let G be a reductive group over $\mathbb{Q}$, and let $\Gamma \subset G(\mathbb{Q})$ be an arithmetic subgroup. Then there is a semi-algebraic subset $\mathfrak{S} \subset G(\mathbb{R})$ and a finite subset C of $G(\mathbb{Q})$ such that $\mathcal{F} := C \cdot \mathfrak{S}$ satisfies*

1. $\Gamma\mathcal{F} = G(\mathbb{R})$
2. ***Siegel Property**: For all $\alpha \in G(\mathbb{Q})$, the set $\{\gamma \in \Gamma; \gamma\mathcal{F} \cap \alpha\mathcal{F} \neq \emptyset\}$ is finite.*
3. *There exists a maximal compact subgroup $K_\infty \subset G(\mathbb{R})$ such that $\mathcal{F}K_\infty = \mathcal{F}$.*

The semi-algebraic set $\mathfrak{S}$ (resp. $\mathcal{F}$) in Proposition 2.2.7 is called a **Siegel set** (resp. **fundamental set**) of $G(\mathbb{R})$ with respect to the action of Γ.

It is interesting to consider a "relative" version of the above theorem. For technical reasons, instead of groups, now we work on the level of Hermitian symmetric domains. The following proposition is a consequence of Borel's reduction theory.

Proposition 2.2.8. *Let (G,X) be a Shimura datum, and let $\Gamma \subset G$ be an arithmetic subgroup. We fix a fundamental set $\mathcal{F} \subset X$ with respect to the action of Γ. Take a Shimura subdatum $(H,X_H) \subset (G,X)$ over $\mathbb{Q}$. Then there is a fundamental set $\mathcal{F}_H \subset X_H$ with respect to the action of $\Gamma_H := \Gamma \cap H(\mathbb{R})$ such that the set*

$$\{\gamma \in \Gamma; \mathcal{F} \cap \gamma.\mathcal{F}_H \neq \emptyset\}$$

is finite.

One can even formulate a quantitative version of the above conjecture. In order to avoid technical details, we only formulate the conjecture when H is semi-simple.

Conjecture 2.2.9. *Let (G,X) be a Shimura datum, and let $\Gamma \subset G$ be an arithmetic subgroup. We fix a fundamental set $\mathcal{F} \subset X$ with respect to the action of Γ. Then there exist constants $c = c(G,\Gamma) > 0$ and $\delta = \delta(G,\Gamma) > 0$ such that the following property holds.*

Let $(H,X_H) \subset (G,X)$ be a Shimura subdatum. Then there is a fundamental set Then there is a fundamental set $\mathcal{F}_H \subset X_H$ with respect to the action of $\Gamma_H := \Gamma \cap H(\mathbb{R})$ such that for any $\gamma \in \Gamma$ with $\mathcal{F} \cap \gamma.\mathcal{F}_H \neq \emptyset$, we have

$$H(\gamma) \leq c \cdot \mathrm{vol}(\mathcal{F}_H)^{\delta}.$$

Here we let $\mathrm{vol}(\mathcal{F}_H)$ denote the volume of $\mathcal{F}$ with respect to the Haar measure on $H(\mathbb{R})$ normalized as in [17], Section 4. Conjecture 2.2.9 holds for $\mathbb{C}^n$, cf. Section 13 of [21]. The analogue of Conjecture 2.2.9 is also true for abelian varieties, cf. [35], Section 3.

2.2.4 Heights of conjugation elements

I would like to give a quantitative version of Theorem 2.3.12, i.e. we hope to find a conjugation element of small height. Let us first recall the height function we are using.

Let $d \geq 1$ be an integer. For any real number y, we define its d**-height** as

$$\mathrm{H}_d(y) := \min\{\max\{|a_0|,...,|a_d|\} : a_i \in \mathbb{Z},\ \gcd\{a_0,...,a_d\} = 1,\ a_0 y^d + ... + a_d = 0\},$$

where we use the convention that, if the set is empty i.e., y is not algebraic of degree at most d, then $\mathrm{H}_d(y)$ is $+\infty$. For $y = (y_1,...,y_m) \in \mathbb{R}^m$, we set

$$\mathrm{H}_d(y) := \max\{\mathrm{H}_k(y_1),...,\mathrm{H}_d(y_m)\}.$$

In order to use the Pila-Wilkie counting theorem in a most natural way, we hope our conjugation elements in Theorem 2.3.12 have the following behaviour.

Conjecture 2.2.10. *Let G be a connected semisimple linear algebraic group defined over a number field $k \subset \mathbb{R}$, then there is a finite set $\Omega = \{H_1, \ldots, H_r\}$ of semi-simple subgroups over k, a natural number $d = d(G_{\bar{k}}, \Omega)$ and two positive constants $c = c(G_{\bar{k}}, \Omega)$ and $\delta = \delta(G_{\bar{k}}, \Omega)$ with the following property: let H be a connected semisimple k-subgroup of G, then it is conjugate over $\bar{k}$ to some $H_i \in \Omega$. Further, for this H_i, there exist a finite field extension $K = K(H, H_i) \subset \mathbb{R}$ of k of degree $[K : k] \leq d$ and an element $g \in G(K)$ such that $H_K = gH_{i,K}g^{-1}$ with*

$$\mathrm{H}_d(g) \leq c \cdot \mathrm{vol}(\mathcal{F}_H)^\delta$$

Here the height $\mathrm{H}_d(g)$ means $\mathrm{H}_d(\rho(g))$ for a fixed faithful representation $\rho : G \hookrightarrow \mathrm{GL}_N$, and H_d is the height of an algebraic number of degree at most d. The volume function is the same as the one in Conjecture 2.2.9.

Conjecture 2.2.10 holds for $\mathbb{C}^n$, cf. Lemma 5.2 in [34]. It may be possible to prove the same result for a product of Shimura curves by applying a recent result of Orr, cf. [57].

2.2.5 Lower bound for Galois orbits of optimal points

The Galois orbit conjecture a more challenging subject. This is a generalization of the Galois orbit conjecture involved in the André-Oort conjecture.

Conjecture 2.2.11 (cf. Conjecture 8.2 of [35])**.** *Let V be a subvariety of a connected Shimura variety S. Let $L \subset \mathbb{C}$ be a field which is finitely generated over $\mathbb{Q}$ such that S and V are both defined over L. Then there exist constants $c, \delta > 0$ such that for any optimal point $P \in V$, we have*

$$\#\mathrm{Gal}(\mathbb{C}/L) \cdot P \geq c\Delta(\langle P \rangle)^\delta$$

where $\langle P \rangle$ is the smallest Shimura subvariety of S containing P.

Habegger and Pila proved Conjecture 2.2.11 for an "asymmetric curve" in $\mathbb{C}^n$. Later, generalizing this idea, Orr proved a partial result for certain curves in $\mathcal{A}_g \times \mathcal{A}_g$, cf. [56]. One reason why we believe Conjecture 2.2.11 is much more difficult than the counterpart for the André-Oort conjecture is that, in this case, the optimal points P have nothing to do with CM points, so there is no number theoretical description of these points. That is also why we believe more geometry is needed in proving this conjecture.

2.3 Main results of the thesis

The work in this thesis consists of five parts. The first three parts are joint work with Christopher Daw, the forth part is joint work with Mikhail Borovoi and Christopher Daw and the fifth part is by the author alone.

2.3.1 A structure theorem for anomalous subvarieties

Let S be a connected Shimura variety, and V be an irreducible algebraic subvariety of S. We define its **weakly special locus** $\mathrm{SpL}(V)$ as the union of all the positive dimensional weakly special subvarieties of S contained in V. Then one of the important step in the Pila-Zannier strategy towards the proof of André-Oort conjecture is to study the structure of this weakly special locus. In fact, we have the following property, cf. Theorem 4.1 of [80].

Theorem 2.3.1. *Let V be a Hodge generic subvariety of S such that for any possible decomposition of S into a product of positive dimensional Shimura varieties $S = S_1 \times S_2$, V is not of the form $S_1 \times V'$ for a subvariety V' of S_2. Then* $\mathrm{SpL}(V)$ *is Zariski closed in V and* $\mathrm{SpL}(V) \neq V$.

This theorem enables us to reduce the André-Oort conjecture to the counting problem of Galois orbits of special points.

However, in the context of the Zilber-Pink conjecture, it is not enough to only consider all the weakly special subvarieties contained in V. We also need to study those weakly special subvarieties which have large intersection with the given subvariety. This leads us to the definition of anomalous subvarieties.

The notion of anomalous subvarieties is defined by Bombieri, Masser, and Zannier in [7] for subvarieties of algebraic tori. Its analogue in Shimura varieties is also worth studying.

For any subset $W \subset S$, we denote by $\langle W \rangle_{\mathrm{ws}}$ the smallest weakly special subvariety of S containing W. This leads us to the definition of an anomalous subvariety.

Definition 2.3.2. Let V be a subvariety of a connected Shimura variety S. A subvariety $W \subset V$ is called **anomalous** if

$$\dim W > \max\{0, \dim\langle W \rangle_{\mathrm{ws}} + \dim V - \dim S\}.$$

By using 'O-minimality' and the Ax-Schanuel conjecture (a proof of which has been announced by Mok, Pila, and Tsimerman), we proved the following structure theorem for anomalous subvarieties, which is an analogue of a main result in [7]. Our theorem is also a counterpart of a theorem in Section 6.1 of [35], which considers anomalous subvarieties in abelian varieties.

Theorem 2.3.3 (Daw-Ren)**.** *Let V be a subvariety of a connected Shimura variety S.*

1. *For any Shimura subvariety S' together with a splitting of Shimura varieties $S' = S_1 \times S_2$, the union of all positive dimensional subvarieties Y of V contained in some $S_1 \times \{x_2\}$ with $\dim Y > \dim V - \dim S_2$ is Zariski closed in V.*
2. *There is a finite collection Φ of Shimura subvarieties equipped with a splitting $S' = S_1 \times S_2$ with the following property. For every maximal anomalous subvariety W of V, there is an $S' = S_1 \times S_2 \in \Phi$ such that W is a component of $V \cap (S_1 \times \{x_2\})$ for some $x_2 \in S_2$.*
3. *The union V^{a} of the anomalous subvarieties in V is Zariski closed.*
4. *Suppose that V is Hodge generic in S. The complement $V^{\mathrm{oa}} := V \backslash V^{\mathrm{a}}$ if empty if and only if S can be written as a product $S_1 \times S_2$ of two Shimura varieties such that*

$$\dim \overline{\phi(V)}^{\mathrm{Zar}} < \min\{\dim V, \dim S_2\},$$

where ϕ is the restriction of the projection map $S \to S_2$ to V.

Notice that the last item of the structure theorem tells us that V^{oa} is empty for a subvariety V in "general position". We also have the following consequence.

Corollary 2.3.4. *Let V be a Hodge generic subvariety of $\mathcal{A}_g$, then V^{oa} is always non-empty.*

2.3.2 Reducing the Zilber-Pink conjecture to a point counting problem

Now we consider another formulation of the Zilber-Pink conjecture.

Definition 2.3.5. Let V be a subvariety of a connected Shimura variety S. For any subset $W \subset V$, we denote by $\langle W \rangle$ the smallest Shimura subvariety containing W. A subvariety W is said to be **optimal** in V if for any subvariety W' of V strictly containing W, we have

$$\dim\langle W' \rangle - \dim W' > \dim\langle W \rangle - \dim W.$$

A zero dimensional optimal subvariety is called an **optimal point**.

Conjecture 2.3.6. *[Zilber-Pink] Let V be a subvariety of a connected Shimura variety S. Then V contains only finitely many optimal subvarieties.*

It can be shown that Conjecture 2.3.6 implies Conjecture 3.1.1. Moreover, since maximal Shimura subvarieties in V are optimal, we can infer from Conjecture 2.3.6 that V has only finitely many maximal Shimura subvarieties, which is just an equivalent formulation of the André-Oort conjecture.

Recall that the estimation of Galois orbits of suitable objects plays an important role in the Pila-Zannier strategy. However, the lower bound of Galois orbits only enables us to show that some set is finite, so we cannot expect to prove Conjecture 2.3.6 using Galois orbits alone. Nonetheless, by a result in my thesis, it indeed suffices to solve a point counting problem.

Conjecture 2.3.7. *Let V be a subvariety of a connected Shimura variety S. Then V contains only finitely many optimal points.*

This is obviously a consequence of Conjecture 2.3.6, and in fact it is equivalent to Conjecture 2.3.6.

Theorem 2.3.8 (Daw-Ren)**.** *Conjecture 2.3.7 implies Conjecture 2.3.6.*

The proof of this theorem uses the Ax-Schanuel conjecture and a detailed study of the weakly special subvarieties of a Shimura variety.

Thus we shall only need to prove Conjecture 2.3.7.

2.3.3 Partial results toward the point counting problem

The work of Habegger and Pila in [34, 35] for $\mathbb{C}^n$ can be generalized for general Shimura varieties by assuming some extra arithmetic conjectures. These arithmetic conjectures are either evident or can be proved in [21] for $\mathbb{C}^n$. However, they are far from evident for arbitrary Shimura varieties. These conjectures have been formulated in section 2.

Our first result is a conditional proof of the Zilber-Pink conjecture for curves.

Theorem 2.3.9 (Daw-Ren)**.** *Assume that the Galois orbit Conjecture 2.2.11 holds and assume that either*

- *Conjectures 2.2.4, 2.2.6 and 2.2.9 hold or that*
- *Conjecture 2.2.10 holds.*

Then Conjecture 2.3.6 is true for curves, i.e., if V is a curve contained in S, then the set of optimal points in V is finite.

Another main result is a conditional proof of the full Zilber-Pink conjecture for a "generic" subvariety V.

Theorem 2.3.10 (Daw-Ren)**.** *Let V be a Hodge generic subvariety of a connected Shimura variety S. By assuming the same hypothesis as in Conjecture 3.14.1, the set*

$$V^{\mathrm{oa}} \cap S^{[1+\dim V]}$$

is finite.

In particular, under these hypotheses, if V is Hodge generic in S and, for any possible splitting of S into a product of two Shimura varieties $S = S_1 \times S_2$, we have

$$\overline{\phi(V)}^{\mathrm{Zar}} = \min\{\dim V, \dim S_2\},$$

where ϕ is the restriction of the projection map $S \to S_2$ to V, i.e. V^{oa} is non-empty, cf. Theorem 2.3.3, then

$$V \cap S^{1+\dim V}$$

is not Zariski dense in V.

2.3.4 Conjugation of semisimple subgroups over real number fields of bounded degree

Recall that in the Pila-Zannier strategy for the Zilber-Pink conjecture, we need to build a definable set which parameterizes weakly special subvarieties of a given Shimura variety. The following proposition is necessary in our construction.

Theorem 2.3.11. *Let G be a semi-simple algebraic group over $\mathbb{R}$. Then there exists a finite set $\Omega = \{H_1, ..., H_r\}$ of real semi-simple algebraic subgroups such that for any real semi-simple subgroup H of G, there is an element $g \in G(\mathbb{R})$ and some $H_i \in \Omega$ such that $H = gH_ig^{-1}$.*

In the context of the Zilber-Pink conjecture, however, the above finiteness property is not enough. In order to use the Pila-Wilkie counting theorem for algebraic points, we need that the conjugation element g is defined over a finite extension of $\mathbb{Q}$ of bounded degree. We thus proved the following result.

Proposition 2.3.12 (Borovoi-Daw-Ren)**.** *Let G be a connected semisimple linear algebraic group defined over a number field $k \subset \mathbb{R}$. Then there is a finite set $\Omega = \{H_1, \dots, H_r\}$ of semi-simple subgroups over k and a natural number $d = d(G, \Omega)$ with the following property: let H be a connected semisimple k-subgroup of G, then it is conjugate over $\mathbb{R}$ to some $H_i \in \Omega$, and there exists a finite field extension $K = K(H, H_i) \subset \mathbb{R}$ of k of degree $[K : k] \leq d$ such that H and H_i are conjugate over K.*

The proof requires a detailed study of Galois cohomology and spectral sequences.

Notice that this theorem is an intermediate step in using the Pila-Wilkie counting theorem since we also need to know the height of the element g.

2.3.5 Counting compact Shimura varieties

We recall that the only non-compact one-dimensional Shimura variety are modular curves. We would like to generalise this observation to all Shimura varieties, i.e. to show that "most" Shimura varieties are compact in a certain sense.

First let us recall a criterion for compactness of a Shimura variety, cf. Theorem 4.12, p. 210, [65].

Theorem 2.3.13. *A Shimura variety $\mathrm{Sh}_K(G, X)$ is compact if and only if G is $\mathbb{Q}$-anisotropic, i.e., $G(\mathbb{Q})$ contains no non-trivial unipotent elements, or, equivalently, $G_{\mathbb{Q}}$ contains no positive dimensional $\mathbb{Q}$-split torus.*

This leads us to the following natural question: given a real Lie group $G(\mathbb{R})$, how many cocompact arithmetic subgroups does it have? More precisely, let G be an algebraic group over $\mathbb{Q}$, what is the percentage of elements in $H^1(\mathrm{Aut}(\mathbb{R}/\mathbb{Q}), \mathrm{Aut}(G))$ that correspond to anisotropic forms of G?

In this thesis, one studies the percentage of anisotropic inner forms of G where G is the general linerar group or a unitary group. And further, we allow G to be over an arbitrary number field rather than only over $\mathbb{Q}$.

First, we define a natural height function $h : H^1(F, G^{\mathrm{ad}}) \longrightarrow \mathbb{R}_{\geq 0}$ such that for any $b > 0$, there are only finitely many $\eta \in H^1(F, G^{\mathrm{ad}})$ with $H(\eta) < b$. The key ingredient in defining this height function is the Kottwitz map.

Definition 2.3.14. We denote by p_∞ the natural projection map

$$p_\infty : H^1(F, G) \to \bigoplus_{v|\infty} H^1(F_v, G),$$

and for any $\theta \in \bigoplus_{v|\infty} H^1(F_v, G)$, we write the fibre $p_\infty^{-1}(\theta)$ by $H^1(F, G)_\theta$.

Definition 2.3.15. Let G be a reductive algebraic group over a number field F, and let $h : H^1(F, G) \to \mathbb{R}_{\geq 0}$ be a function. Then for any $T > 1$, and $\theta \in \bigoplus_{v|\infty} H^1(F_v, G)$, we write

$$N_F(h, G, T)_\theta := \{\eta \in H^1(F, G)_\theta; h(\eta) < T\},$$

and let $N_F^c(h, G, T)_\theta$ (respectively $N_F^{nc}(h, G, T)_\theta$) be the subset of $N_F(h, G, T)_\theta$ consisting those η such that the corresponding inner form G_v is F-anisotropic (respectively F-isotropic).

Notice that the above two sets count F-inner forms and F-inner anisotropic forms with a bounded height and a fixed behaviour at infinite places. The main result is the following.

Theorem 2.3.16. *Let G be a general linear group or a unitary group over a number field F, then there exists a natural height function $h : H^1(F, G) \to \mathbb{R}_{\geq 0}$ such that for any $\theta \in \bigoplus_{v|\infty} H^1(F_v, G)$, the density*

$$\lim_{T \to \infty} \frac{\# N_F^{nc}(h, G, T)_\theta}{\# N_F(h, G, T)_\theta} = 0.$$

In other words, with respect to this natural height function, almost all F-inner forms of G are F-anisotropic.

Chapter 3

Applications of the hyperbolic Ax-Schanuel conjecture

3.1 Introduction

The Ax-Schanuel theorem [3] is a result regarding the transcendence degrees of fields generated over the complex numbers by power series and their exponentials. Formulated geometrically for the uniformization maps of algebraic tori, it has inspired analogous statements for the uniformization maps of abelian varieties and Shimura varieties. The former, following from another theorem of Ax [4], has recently been used by Habegger and Pila in their proof of the Zilber-Pink conjecture for curves in abelian varieties [35].

Habegger and Pila also extended the Pila-Zannier strategy to the Zilber-Pink conjecture for products of modular curves. Their method relies on an Ax-Schanuel conjecture for the j-function and is conditional on their so-called large Galois orbits conjecture. The purpose of this paper is to show that the Pila-Zannier strategy can be extended to the Zilber-Pink conjecture for general Shimura varieties.

This conjecture can just as easily be stated in the generality of **mixed** Shimura varieties but, in this article, we will restrict our attention to **pure** Shimura varieties, though we have no explicit reason to believe that the methods presented here will not extend to the mixed setting. We begin by stating a conjecture of Pink. We note that, throughout this article, unless preceded by the word **Shimura**, varieties (and, indeed, subvarieties) will be assumed **geometrically irreducible**.

Conjecture 3.1.1 (cf. [64], Conjecture 1.3)**.** *Let* $\mathrm{Sh}_K(G,\mathfrak{X})$ *be a Shimura variety and, for any integer d, let* $\mathrm{Sh}_K(G,\mathfrak{X})^{[d]}$ *denote the union of the special subvarieties of* $\mathrm{Sh}_K(G,\mathfrak{X})$ *having codimension at least d. Let V be a* **Hodge generic** *subvariety of* $\mathrm{Sh}_K(G,\mathfrak{X})$. *Then*

$$V\cap \mathrm{Sh}_K(G,\mathfrak{X})^{[1+\dim V]}$$

is not Zariski dense in V.

The heuristics of this conjecture are as follows. For two subvarieties V and W of $\mathrm{Sh}_K(G,\mathfrak{X})$, such that the codimension of W is at least $1+\dim V$, we expect $V\cap W=\emptyset$.

Even if we fix V and take the union of $V \cap W$ for countably many W of codimension at least $1 + \dim V$, the resulting set should still be rather small in V unless, of course, V was not sufficiently generic in $\mathrm{Sh}_K(G, \mathfrak{X})$. Pink's conjecture turns this expectation into an explicit statement about the intersection of Hodge generic subvarieties with the special subvarieties of small dimension.

Conjecture 3.1.1 can also be formulated for algebraic tori, abelian varieties, or even semi-abelian varieties, though Conjecture 3.1.1 for mixed Shimura varieties implies all of these formulations (see [64]). When V is a curve, defined over a number field, and contained in an algebraic torus, we obtain a theorem of Maurin [51]. We also note that Capuano, Masser, Pila, and Zannier have recently applied the Pila-Zannier method in this setting [11]. When V is a curve, defined over a number field, and contained in an abelian variety, we obtain the recent theorem of Habegger and Pila [35], and it is the ideas presented there that form the basis for this article. Habegger and Pila had given some partial results when V is a curve, defined over a number field, and contained in the Shimura variety $\mathbb{C}^n$ [34], and Orr has recently generalized their results to a curve contained in $\mathcal{A}_g^2$ (see [56] for more details).

We should point out that Conjecture 3.1.1 implies the André-Oort conjecture.

Conjecture 3.1.2 (André-Oort)**.** *Let $\mathrm{Sh}_K(G, \mathfrak{X})$ be a Shimura variety and let V be a subvariety of $\mathrm{Sh}_K(G, \mathfrak{X})$ such that the special points of $\mathrm{Sh}_K(G, \mathfrak{X})$ in V are Zariski dense in V. Then V is a special subvariety of $\mathrm{Sh}_K(G, \mathfrak{X})$.*

To see this, we may assume that V is Hodge generic in $\mathrm{Sh}_K(G, \mathfrak{X})$. Then, since special points have codimension $\dim \mathrm{Sh}_K(G, \mathfrak{X})$, Conjecture 3.1.1 implies that, either $\dim V = \dim \mathrm{Sh}_K(G, \mathfrak{X})$, in which case V is a connected component of $\mathrm{Sh}_K(G, \mathfrak{X})$ and, in particular, a special subvariety of $\mathrm{Sh}_K(G, \mathfrak{X})$, or the set of special points of $\mathrm{Sh}_K(G, \mathfrak{X})$ in V are not Zariski dense in V.

In precisely the same fashion, the Zilber-Pink conjecture for abelian varieties implies the Manin-Mumford conjecture.

The André-Oort conjecture has a rich history of its own. Here, we simply recall that it was recently settled for $\mathcal{A}_g$ by Pila and Tsimerman [60, 77], thanks to recent progress on the Colmez conjecture due to Andreatta, Goren, Howard, Madapusi Pera [1] and Yuan and Zhang [89], and it is known to hold for all Shimura varieties under conjectural lower bounds for Galois orbits of special points due to the work of Orr, Klingler, Ulmo, Yafaev, and the first author [20, 44, 85]. Furthermore, Gao has generalized these proofs to all mixed Shimura varieties [30, 31].

In his work on Schanuel's conjecture, Zilber made his own conjecture on unlikely intersections [91], which was closely related to the independent work of Bombieri, Masser, and Zannier [7]. To describe Zilber's formulation, we require the following definition.

Definition 3.1.3. Let $\mathrm{Sh}_K(G, \mathfrak{X})$ be a Shimura variety and let V be a subvariety of $\mathrm{Sh}_K(G, \mathfrak{X})$. A subvariety W of V is called **atypical** with respect to V if there is a special subvariety T of $\mathrm{Sh}_K(G, \mathfrak{X})$ such that W is an irreducible component of $V \cap T$ and

$$\dim W > \dim V + \dim T - \dim \mathrm{Sh}_K(G, \mathfrak{X}).$$

We denote by $\mathrm{Atyp}(V)$ the union of the subvarieties of V that are atypical with respect to V.

Zilber's conjecture, formulated for Shimura varieties, is then as follows.

Conjecture 3.1.4 (cf. [35], Conjecture 2.2)**.** *Let* $\mathrm{Sh}_K(G,\mathfrak{X})$ *be a Shimura variety and let V be a subvariety of* $\mathrm{Sh}_K(G,\mathfrak{X})$*. Then* $\mathrm{Atyp}(V)$ *is equal to a finite union of atypical subvarieties of V.*

Since there are only countably many special subvarieties of $\mathrm{Sh}_K(G,\mathfrak{X})$, the conjecture is equivalent to the statement that V contains only finitely many subvarieties that are atypical with respect to V and maximal with respect to this property.

We will see that Conjecture 3.1.4 strengthens Conjecture 3.1.1 and, therefore, it is Conjecture 3.1.4 that we refer to as the **Zilber-Pink** conjecture. Habegger and Pila obtained a proof of the Zilber-Pink conjecture for products of modular curves assuming the weak complex Ax conjecture and the large Galois orbits conjecture. Subsequently, Pila and Tsimerman obtained the weak complex Ax conjecture as a corollary to their proof of the Ax-Schanuel conjecture for the j-function [61]. Habegger and Pila had previously verified the large Galois orbits conjecture for so-called asymmetric curves [34].

This article seeks to generalize the ideas of [35] to general Shimura varieties. Hence, we will have to make generalizations of the previously mentioned hypotheses. The foremost of which will be the statement from functional transcendence, namely, the hyperbolic Ax-Schanuel conjecture that generalizes the Ax-Schanuel conjecture for the j-function to general Shimura varieties. Our main result (Theorem 3.8.3) is that, under the hyperbolic Ax-Schanuel conjecture, the Zilber-Pink conjecture can be reduced to a problem of point counting. However, given that Mok, Pila, and Tsimerman have recently announced a proof of the hyperbolic Ax-Schanuel conjecture [53], this result is now very likely unconditional. Besides the hyperbolic Ax-Schanuel conjecture, our main input will be the theory of o-minimality and, in particular, the fact that the uniformization map of a Shimura variety is definable in $\mathbb{R}_{\mathrm{an,exp}}$ when it is restricted to an appropriate fundamental domain.

After establishing the main result, we attempt to tackle the point counting problem using the now famous Pila-Wilkie counting theorem. To do so, we formulate several arithmetic conjectures that are inspired by previous applications of the Pila-Zannier strategy. In this vein, our paper is very much in the spirit of [80], which, at the time, reduced the André-Oort conjecture to a point counting problem and then explained how various conjectural ingredients, namely, the hyperbolic Ax-Lindemann conjecture, lower bounds for Galois orbits of special points, upper bounds for the heights of pre-special points, and the definability of the uniformization map, could be combined to deliver a proof of the André-Oort conjecture.

Our arithmetic hypotheses are (1) lower bounds for Galois orbits of so-called optimal points (see Definition 3.3.2), which we also refer to as the large Galois orbits conjecture, and (2) upper bounds for the heights of pre-special subvarieties. Hypothesis (1) generalizes the (in some cases still conjectural) lower bounds for Galois orbits of special points (when such special points are also maximal special subvarieties), and also generalizes the large Galois orbits conjecture of Habegger and Pila. Hypothesis (2) generalizes the upper bounds for heights of pre-special points, which were proved by Orr and the first author [20]. However, we also show that it is possible to replace hypothesis (2) with two other arithmetic hypotheses, namely, (3) upper bounds for the degrees of fields associated with special subvarieties, and (4) upper bounds for the heights of lattice elements. Hypothesis (3) is a replacement for the fact that, for an abelian variety, its abelian subvarieties can be defined over a fixed finite extension of the base field. Hypothesis (4) is an analogue of a known result for abelian varieties. We verify hypotheses (2), (3), and (4) for a product of modular curves.

Conventions

- Throughout this paper, **definable** means definable in the o-minimal structure $\mathbb{R}_{\mathrm{an,exp}}$.
- Unless preceded by the word **Shimura**, varieties (and, indeed, subvarieties) will be assumed **geometrically irreducible**.
- By a subvariety, we will always mean a **closed** subvariety.

Index of notations We collect here the main symbols appearing in this article.

- $\langle W\rangle$ is the smallest special subvariety containing W.
- $\langle W\rangle_{\mathrm{ws}}$ is the smallest weakly special subvariety containing W.
- $\langle A\rangle_{\mathrm{Zar}}$ is the smallest algebraic subvariety containing A.
- $\langle A\rangle_{\mathrm{geo}}$ is the smallest totally geodesic subvariety containing A.
- $\delta(W) := \dim\langle W\rangle - \dim W$
- $\delta_{\mathrm{ws}}(W) := \dim\langle W\rangle_{\mathrm{ws}} - \dim W$
- $\delta_{\mathrm{Zar}}(A) := \dim\langle A\rangle_{\mathrm{Zar}} - \dim A$
- $\delta_{\mathrm{geo}}(A) := \dim\langle A\rangle_{\mathrm{geo}} - \dim A$
- $\mathrm{Opt}(V)$ is the set of subvarieties of V that are optimal in V.
- $\mathrm{Opt}_0(V)$ is the set of points of V that are optimal in V.
- G^{ad} is the adjoint group of G i.e. the quotient of G by its centre.
- G^{der} is derived group of G.
- G° is the (Zariski) connected component of G containing the identity.
- $G_H := HZ_G(H)^\circ$ whenever H is a subgroup of G.
- $G(\mathbb{R})^+$ is the (archimedean) connected component of $G(\mathbb{R})$ containing the identity.

3.2 Special and weakly special subvarieties

Let $(G,\mathfrak{X})$ be a Shimura datum and let K be a compact open subgroup of $G(\mathbb{A}_f)$, where $\mathbb{A}_f$ will henceforth denote the finite rational adèles. Let $\mathrm{Sh}_K(G,\mathfrak{X})$ denote the corresponding **Shimura variety**. By this, we mean the complex quasi-projective algebraic variety such that $\mathrm{Sh}_K(G,\mathfrak{X})(\mathbb{C})$ is equal to the image of

$$G(\mathbb{Q})\backslash[\mathfrak{X}\times(G(\mathbb{A}_f)/K)] \tag{3.2.1}$$

under the canonical embedding into complex projective space given by Baily and Borel [5]. We will identify (3.2.1) with $\mathrm{Sh}_K(G,\mathfrak{X})(\mathbb{C})$. We recall that, on $\mathfrak{X}\times(G(\mathbb{A}_f)/K)$, the action of $G(\mathbb{Q})$ is the diagonal one.

Let X be a connected component of $\mathfrak{X}$ and let $G(\mathbb{Q})_+$ be the subgroup of $G(\mathbb{Q})$ acting on it. For any $g \in G(\mathbb{A}_f)$, we obtain a **congruence** subgroup Γ_g of $G(\mathbb{Q})_+$ by intersecting it with gKg^{-1}. Furthermore, the locally symmetric variety $\Gamma_g \backslash X$ is contained in (3.2.1) via the map that sends the class of x to the class of (x,g). If we take the disjoint union of the $\Gamma_g \backslash X$ over a (finite) set of representatives for

$$G(\mathbb{Q})_+ \backslash G(\mathbb{A}_f)/K,$$

the corresponding inclusion map is a bijection.

Definition 3.2.1. For any compact open subgroup K' of $G(\mathbb{A}_f)$ contained in K, we obtain a finite morphism

$$\mathrm{Sh}_{K'}(G,\mathfrak{X}) \to \mathrm{Sh}_K(G,\mathfrak{X}),$$

given by the natural projection. Furthermore, for any $a \in G(\mathbb{A}_f)$, we obtain an isomorphism

$$\mathrm{Sh}_K(G,\mathfrak{X}) \to \mathrm{Sh}_{a^{-1}Ka}(G,\mathfrak{X})$$

sending the class of (x,g) to the class of (x,ga). We let $T_{K,a}$ denote the map on algebraic cycles of $\mathrm{Sh}_K(G,\mathfrak{X})$ given by the algebraic correspondence

$$\mathrm{Sh}_K(G,\mathfrak{X}) \leftarrow \mathrm{Sh}_{K\cap aKa^{-1}}(G,\mathfrak{X}) \to \mathrm{Sh}_{a^{-1}Ka\cap K}(G,\mathfrak{X}) \to \mathrm{Sh}_K(G,\mathfrak{X}),$$

where the outer arrows are the natural projections and the middle arrow is the isomorphism given by a. We refer to a map of this sort as a **Hecke correspondence**.

Definition 3.2.2. Let $(H,\mathfrak{X}_H)$ be a Shimura subdatum of $(G,\mathfrak{X})$ and let K_H denote a compact open subgroup of $H(\mathbb{A}_f)$ contained in K. The natural map

$$H(\mathbb{Q})\backslash[\mathfrak{X}_H \times (H(\mathbb{A}_f)/K_H)] \to G(\mathbb{Q})\backslash[\mathfrak{X} \times (G(\mathbb{A}_f)/K)]$$

yields a finite morphism of Shimura varieties

$$\mathrm{Sh}_{K_H}(H,\mathfrak{X}_H) \to \mathrm{Sh}_K(G,\mathfrak{X})$$

(see, for example, [63], Facts 2.6), and we refer to the image of any such morphism as a **Shimura subvariety** of $\mathrm{Sh}_K(G,\mathfrak{X})$.

For any Shimura subvariety Z of $\mathrm{Sh}_K(G,\mathfrak{X})$ and any $a \in G(\mathbb{A}_f)$, we refer to any irreducible component of $T_{K,a}(Z)$ as a **special subvariety** of $\mathrm{Sh}_K(G,\mathfrak{X})$.

Recall that, by definition, $\mathfrak{X}$ is a $G(\mathbb{R})$ conjugacy class of morphisms from $\mathbb{S}$ to $G_{\mathbb{R}}$ and the **Mumford-Tate group** $\mathrm{MT}(x)$ of $x \in \mathfrak{X}$ is defined as the smallest $\mathbb{Q}$-subgroup H of G such that x factors through $H_{\mathbb{R}}$. If we let $\mathfrak{X}_M$ denote the $M(\mathbb{R})$ conjugacy class of $x \in X$, where $M := \mathrm{MT}(x)$, then $(M,\mathfrak{X}_M)$ is a Shimura subdatum of $(G,\mathfrak{X})$. In particular, if we let X_M denote a connected component of $\mathfrak{X}_M$ contained in X, then the image of X_M

in $\Gamma_g\backslash X$, for any $g \in G(\mathbb{A}_f)$, is a special subvariety of $\mathrm{Sh}_K(G,\mathfrak{X})$, and it is easy to see that every special subvariety of $\mathrm{Sh}_K(G,\mathfrak{X})$ arises this way.

Of course, if $x \in X_M$, then X_M is equal to the $M(\mathbb{R})^+$ conjugacy class of x. Furthermore, the action of $M(\mathbb{R})$ on $\mathfrak{X}_M$ factors through $M^{\mathrm{ad}}(\mathbb{R})$ and the group M^{ad} is equal to the direct product of its $\mathbb{Q}$-simple factors. Therefore, we can write M^{ad} as a product

$$M^{\mathrm{ad}} = M_1 \times M_2$$

of two normal $\mathbb{Q}$-subgroups, either of which may (by choice or necessity) be trivial, and we thus obtain a corresponding splitting

$$X_M = X_1 \times X_2.$$

For any such splitting, and any $x_1 \in X_1$ or $x_2 \in X_2$, we refer to the image of $\{x_1\} \times X_2$ or $X_1 \times \{x_2\}$ in $\Gamma_g\backslash X$, for any $g \in G(\mathbb{A}_f)$, as a **weakly special subvariety** of $\mathrm{Sh}_K(G,\mathfrak{X})$. In particular, every special subvariety of $\mathrm{Sh}_K(G,\mathfrak{X})$ is a weakly special subvariety of $\mathrm{Sh}_K(G,\mathfrak{X})$. By [55], Section 4, the weakly special subvarieties of $\mathrm{Sh}_K(G,\mathfrak{X})$ are precisely those subvarieties of $\mathrm{Sh}_K(G,\mathfrak{X})$ that are totally geodesic in $\mathrm{Sh}_K(G,\mathfrak{X})$. Furthermore, a weakly special subvariety of $\mathrm{Sh}_K(G,\mathfrak{X})$ is a special subvariety of $\mathrm{Sh}_K(G,\mathfrak{X})$ if and only if it contains a special subvariety of dimension zero, henceforth known as a **special point**.

Remark 3.2.3. The following observations will facilitate various reductions.

- Let K' be a compact open subgroup of $G(\mathbb{A}_f)$ contained in K. By definition, a subvariety Z of $\mathrm{Sh}_K(G,\mathfrak{X})$ is a (weakly) special subvariety of $\mathrm{Sh}_K(G,\mathfrak{X})$ if and only if any irreducible component of the inverse image of Z in $\mathrm{Sh}_{K'}(G,\mathfrak{X})$ is a (weakly) special subvariety of $\mathrm{Sh}_{K'}(G,\mathfrak{X})$.
- For any $a \in G(\mathbb{A}_f)$, a subvariety Z of $\mathrm{Sh}_K(G,\mathfrak{X})$ is a (weakly) special subvariety of $\mathrm{Sh}_K(G,\mathfrak{X})$ if and only if any irreducible component of $T_{K,a}(Z)$ is a (weakly) special subvariety of $\mathrm{Sh}_{K'}(G,\mathfrak{X})$.
- If we let G^{ad} denote the adjoint group of G i.e. the quotient of G by its centre, we obtain another Shimura datum $(G^{\mathrm{ad}},\mathfrak{X}^{\mathrm{ad}})$, known as the **adjoint Shimura datum** associated with $(G,\mathfrak{X})$. For any compact open subgroup K^{ad} of $G^{\mathrm{ad}}(\mathbb{A}_f)$ containing the image of K, we obtain a finite morphism

  $$\mathrm{Sh}_K(G,\mathfrak{X}) \to \mathrm{Sh}_{K^{\mathrm{ad}}}(G^{\mathrm{ad}},\mathfrak{X}^{\mathrm{ad}}).$$

  As in [27], Proposition 2.2, a subvariety Z of $\mathrm{Sh}_{K^{\mathrm{ad}}}(G^{\mathrm{ad}},\mathfrak{X}^{\mathrm{ad}})$ is a special subvariety of $\mathrm{Sh}_{K^{\mathrm{ad}}}(G^{\mathrm{ad}},\mathfrak{X}^{\mathrm{ad}})$ if and only if any irreducible component of its inverse image in $\mathrm{Sh}_K(G,\mathfrak{X})$ is a special subvariety of $\mathrm{Sh}_K(G,\mathfrak{X})$.

By [63], Remark 4.9, for any subvariety W of $\mathrm{Sh}_K(G,\mathfrak{X})$, there exists a **smallest** weakly special subvariety $\langle W\rangle_{\mathrm{ws}}$ of $\mathrm{Sh}_K(G,\mathfrak{X})$ containing W and a **smallest** special subvariety $\langle W\rangle$ of $\mathrm{Sh}_K(G,\mathfrak{X})$ containing W. We note that here, and throughout, our notations and terminology regarding subvarieties often differ from those found in [35].

3.3 The Zilber-Pink conjecture

For the remainder of this article, we fix a Shimura datum $(G,\mathfrak{X})$ and we let X be a connected component of $\mathfrak{X}$. We fix a compact open subgroup K of $G(\mathbb{A}_f)$ and we let

$$\Gamma := G(\mathbb{Q})_+ \cap K,$$

where $G(\mathbb{Q})_+$ is the subgroup of $G(\mathbb{Q})$ acting on X. We denote by S the connected component $\Gamma\backslash X$ of $\mathrm{Sh}_K(G,\mathfrak{X})$.

As in [35], we will consider an equivalent formulation of Conjecture 3.1.4 using the language of optimal subvarieties.

Definition 3.3.1. Let W be a subvariety of S. We define the **defect** of W to be

$$\delta(W) := \dim\langle W\rangle - \dim W.$$

Definition 3.3.2. Let V be a subvariety of S and let W be a subvariety of V. Then W is called **optimal** in V if, for any subvariety Y of S,

$$W \subsetneq Y \subseteq V \implies \delta(Y) > \delta(W).$$

We denote by $\mathrm{Opt}(V)$ the set of all subvarieties of V that are optimal in V.

Remark 3.3.3. Let V be a subvariety of S. First note that $V \in \mathrm{Opt}(V)$. Secondly, if $W \in \mathrm{Opt}(V)$, then W is an irreducible component of

$$\langle W\rangle \cap V.$$

Conjecture 3.3.4 (cf. [35], Conjecture 2.6)**.** *Let V be a subvariety of S. Then $\mathrm{Opt}(V)$ is finite.*

Observe that a maximal special subvariety of V is an optimal subvariety of V. Therefore, Conjecture 3.3.4 immediately implies that V contains only finitely many maximal special subvarieties, which is another formulation of the André-Oort conjecture for V.

Lemma 3.3.5. *The Zilber-Pink conjecture (Conjecture 3.1.4) is equivalent to Conjecture 3.3.4.*

Proof. Consider the situation described in the statement of Conjecture 3.1.4. By Remark 3.2.3, we suffer no loss in generality if we assume that V is contained in S. Then the result follows from [35], Lemma 2.7. $\square$

Lemma 3.3.6. *The Zilber-Pink conjecture implies Conjecture 3.1.1.*

Proof. By Lemma 3.3.5, it suffices to show that Conjecture 3.3.4 implies Conjecture 3.1.1.

Consider the situation described in Conjecture 3.1.1. By Remark 3.2.3, we suffer no loss in generality if we assume that V is contained in S. Let P be a point belonging to

$$V \cap \mathrm{Sh}_K(G,\mathfrak{X})^{[1+\dim V]}.$$

Let W be a subvariety of V that is optimal in V and contains P such that

$$\delta(W) \leq \delta(P) = \dim\langle P\rangle.$$

Since P belongs to a special subvariety of codimension at least $\dim V + 1$ and V is Hodge generic in $\mathrm{Sh}_K(G,\mathfrak{X})$, we have

$$\dim\langle P\rangle \leq \dim S - \dim V - 1 = \dim\langle V\rangle - \dim V - 1 < \delta(V).$$

Therefore, $\delta(W) < \delta(V)$ and we conclude that W is not V. According to Conjecture 3.3.4, the union of the subvarieties belonging to $\mathrm{Opt}(V) \setminus V$ is not Zariski dense in V. $\square$

3.4 The defect condition

In this section, we prove Habegger and Pila's defect condition (Proposition 3.4.4) for Shimura varieties, and thus show that a subvariety that is optimal is weakly optimal.

Definition 3.4.1. Let W be a subvariety of S. We define the **weakly special defect** of W to be

$$\delta_{\mathrm{ws}}(W) := \dim\langle W\rangle_{\mathrm{ws}} - \dim W.$$

We note that, in [35], this notion was referred to as geodesic defect.

Definition 3.4.2. If V is a subvariety of S and W a subvariety of V, then W is called **weakly optimal** in V if, for any subvariety Y of S,

$$W \subsetneq Y \subseteq V \implies \delta_{\mathrm{ws}}(Y) > \delta_{\mathrm{ws}}(W).$$

Remark 3.4.3. Let V be a subvariety of S and W a subvariety of V. If W is weakly optimal in V, then W is an irreducible component of

$$\langle W\rangle_{\mathrm{ws}} \cap V.$$

Proposition 3.4.4 (cf. [35], Proposition 4.3)**.** *The following* **defect condition** *holds.*

Let $W \subseteq Y$ be two subvarieties of S. Then

$$\delta(Y) - \delta_{\mathrm{ws}}(Y) \leq \delta(W) - \delta_{\mathrm{ws}}(W).$$

Proof. We need to show that

$$\dim\langle Y\rangle - \dim\langle Y\rangle_{\mathrm{ws}} \leq \dim\langle W\rangle - \dim\langle W\rangle_{\mathrm{ws}}.$$

By Remark 3.2.3, we can and do assume that G is the generic Mumford-Tate group on X, that it is equal to G^{ad}, and that Y is Hodge generic in S. By definition, there exists a decomposition

$$G = G_1 \times G_2,$$

which induces a splitting

$$X = X_1 \times X_2,$$

such that $\langle Y \rangle_{\mathrm{ws}}$ is equal to the image of $X_1 \times \{x_2\}$ in S, for some $x_2 \in X_2$.

Let $\Gamma_1 := \mathrm{p}_1(\Gamma)$ and $\Gamma_2 := \mathrm{p}_2(\Gamma)$, where p_1 and p_2 are the projections from G to G_1 and G_2, respectively. Then $\Gamma' := \Gamma_1 \times \Gamma_2$ is a congruence subgroup of $G(\mathbb{Q})_+$ containing Γ as a finite index subgroup. Let $\phi : \Gamma\backslash X \to \Gamma'\backslash X$ denote the natural (finite) morphism. Then $\phi(W) \subseteq \phi(Y) \subseteq S' := \Gamma'\backslash X$, and we have

$$\begin{aligned}
\dim\langle Y \rangle &= \dim\langle \phi(Y) \rangle, \\
\dim\langle W \rangle &= \dim\langle \phi(W) \rangle, \\
\dim\langle Y \rangle_{\mathrm{ws}} &= \dim\langle \phi(Y) \rangle_{\mathrm{ws}}, \\
\dim\langle W \rangle_{\mathrm{ws}} &= \dim\langle \phi(W) \rangle_{\mathrm{ws}}.
\end{aligned}$$

Therefore, after replacing Y, W, and S by $\phi(Y)$, $\phi(W)$, and S', respectively, we may assume that Γ is of the form $\Gamma_1 \times \Gamma_2$, and $S = \Gamma_1\backslash X_1 \times \Gamma_2\backslash X_2 = S_1 \times S_2$.

Thus, $\langle Y \rangle_{\mathrm{ws}} = S_1 \times \{s_2\}$, where s_2 is the image of x_2 in S_2, $Y = Y_1 \times \{s_2\}$, where Y_1 is the projection of Y to S_1, and $W = W_1 \times \{s_2\}$, where W_1 is the projection of W to S_1. In particular, we can take

$$x := (x_1, x_2) \in X_1 \times X_2$$

such that $\langle W \rangle$ is equal to the image in S of the $M(\mathbb{R})^+$ conjugacy class X_M of x, where $M := \mathrm{MT}(x)$.

Again, there exists a decomposition

$$M^{\mathrm{ad}} = M_1 \times M_2,$$

which induces a splitting

$$X_M = X_{M_1} \times X_{M_2} \subseteq X = X_1 \times X_2$$

such that $\langle W \rangle_{\mathrm{ws}}$ is equal to the image in S of $X_{M_1} \times \{y_2\}$, for some $y_2 \in X_{M_2}$.

Since $\mathrm{MT}(x_2)$ is equal to G_2, it follows that M is a subgroup of $G_1 \times G_2$ that surjects on to the second factor. In particular,

$$X_M = M^{\mathrm{der}}(\mathbb{R})^+ x$$

surjects on to X_2. Therefore, let M_1' and M_2' be two normal semisimple subgroups of M^{der} corresponding to M_1 and M_2, respectively, so that

$$M^{\mathrm{der}}(\mathbb{R})^+ x = M_1'(\mathbb{R})^+ M_2'(\mathbb{R})^+ x.$$

Since W is contained in $S_1 \times \{s_2\}$, the projection of M_1' to G_2 must be trivial. Hence, $M_1'(\mathbb{R})^+x$ is contained in $X_1 \times \{x_2\}$ and we conclude that $M_2'(\mathbb{R})^+x$ surjects on to X_2. Since

$$M_2'(\mathbb{R})^+x = \{y_1\} \times X_{M_2},$$

for some $y_1 \in X_{M_1}$, we have

$$\dim\langle W\rangle - \dim\langle W\rangle_{\mathrm{ws}} = \dim X_{M_2} \geq \dim X_2 = \dim\langle Y\rangle - \dim\langle Y\rangle_{\mathrm{ws}},$$

as required. □

Corollary 3.4.5 (cf. [35], Proposition 4.5)**.** *Let V be a subvariety of S. A subvariety of V that is optimal in V is weakly optimal in V.*

3.5 The hyperbolic Ax-Schanuel conjecture

In this section, we formulate various conjectures about Shimura varieties that are analogous to the original Ax-Schanuel theorem from functional transcendence theory.

Theorem 3.5.1 (cf. [3], Theorem 1)**.** *Let $f_1,...,f_n \in \mathbb{C}[[t_1,...,t_m]]$ be power series that are $\mathbb{Q}$-linearly independent modulo $\mathbb{C}$. Then we have the following inequality*

$$\mathrm{tr.deg}_{\mathbb{C}}\mathbb{C}(f_1,...,f_n,e(f_1),...,e(f_n)) \geq n + \mathrm{rank}\left(\frac{\partial f_i}{\partial t_j}\right)_{\substack{i=1,...,n\\ j=1,...,m}}$$

where $e(f) = e^{2\pi i f} \in \mathbb{C}[[t_1,...,t_m]]$.

The following theorem is then an immediate corollary.

Theorem 3.5.2. *Let $f_1,...,f_n \in \mathbb{C}[[t_1,...,t_m]]$ as above. Then*

$$\mathrm{tr.deg}_{\mathbb{C}}\mathbb{C}(f_1,...,f_n) + \mathrm{tr.deg}_{\mathbb{C}}\mathbb{C}(e(f_1),...,e(f_n)) \geq n + \mathrm{rank}\left(\frac{\partial f_i}{\partial t_j}\right)_{\substack{i=1,...,n\\ j=1,...,m}}.$$

Let π denote the uniformization map

$$\mathbb{C}^n \to (\mathbb{C}^\times)^n : (x_1,...,x_n) \mapsto (e(x_1),...,e(x_n))$$

and let D_n denote its graph in $\mathbb{C}^n \times (\mathbb{C}^\times)^n$. We can rephrase Theorem 3.5.1 as follows.

Theorem 3.5.3 (cf. [76], Theorem 1.2)**.** *Let V be a subvariety of $\mathbb{C}^n \times (\mathbb{C}^\times)^n$ and let U be an irreducible analytic component of $V \cap D_n$. Assume that the projection of U to $(\mathbb{C}^\times)^n$ is not contained in a coset of a proper subtorus of $(\mathbb{C}^\times)^n$. Then*

$$\dim V \geq \dim U + n.$$

Similarly, we can rephrase Theorem 3.5.2 as follows.

Theorem 3.5.4. *Let W be a subvariety of $\mathbb{C}^n$ and V a subvariety of $(\mathbb{C}^\times)^n$. Let A be an irreducible analytic component of $W \cap \pi^{-1}(V)$. If A is not contained in $b+L$, for any proper $\mathbb{Q}$-linear subspace L of $\mathbb{C}^n$ and any $b \in \mathbb{C}^n$, then*

$$\dim V + \dim W \geq \dim A + n.$$

Recall that X is naturally endowed with the structure of a hermitian symmetric domain. In particular, it is a complex manifold. We define an (irreducible algebraic) **subvariety** of X as in Appendix B of [44]. In particular, we consider the Harish-Chandra realization of X, which is a bounded domain in $\mathbb{C}^N$, for some $N \in \mathbb{N}$, and we define an (irreducible algebraic) subvariety of X to be an irreducible analytic component of the intersection of X with an algebraic subvariety of $\mathbb{C}^N$. We define an (irreducible algebraic) subvariety of $X \times S$ to be an irreducible analytic component of the intersection of $X \times S$ with an algebraic subvariety of $\mathbb{C}^N \times S$. We note, however, that, by [44], Corollary B.2, the algebraic structure that we are putting on X and $X \times S$ does not depend on our particular choice of the Harish-Chandra realization of X; any realization of X would yield the same algebraic structures.

We are, therefore, able to formulate conjectures for Shimura varieties that are analogous to those above. Let π henceforth denote the uniformization map

$$X \to S$$

and let D_S denote the graph of π in $X \times S$. The following conjecture generalizes Conjecture 1.1 of [61].

Conjecture 3.5.5 (hyperbolic Ax-Schanuel)**.** *Let V be a subvariety of $X \times S$ and let U be an irreducible analytic component of $V \cap D_S$. Assume that the projection of U to S is not contained in a weakly special subvariety of $\mathrm{Sh}_K(G,\mathfrak{X})$ strictly contained in S. Then*

$$\dim V \geq \dim U + \dim S.$$

For $S = \mathbb{C}^n$, Conjecture 3.5.5 and its generalization involving derivatives were obtained in [61]. Mok, Pila, and Tsimerman have very recently announced a proof of Conjecture 3.5.5 in full [53].

For applications to the Zilber-Pink conjecture, only the following weaker version will be needed.

Conjecture 3.5.6 (cf. [35], Conjecture 5.10)**.** *Let W be a subvariety of X and let V be a subvariety of S. Let A be an irreducible analytic component of $W \cap \pi^{-1}(V)$ and assume that $\pi(A)$ is not contained in a weakly special subvariety of $\mathrm{Sh}_K(G,\mathfrak{X})$ strictly contained in S. Then*

$$\dim V + \dim W \geq \dim A + \dim S.$$

Proof that Conjecture 3.5.5 implies Conjecture 3.5.6. Consider the situation described in the statement of Conjecture 3.5.6. Then $Y := W \times V$ is an algebraic subvariety of $X \times S$ and

$$U := \{(a, \pi(a)); a \in A\}$$

is an irreducible analytic component of $Y \cap D_S$. Clearly, the projection of U to S is not contained in a weakly special subvariety of $\mathrm{Sh}_K(G,\mathfrak{X})$ strictly contained in S. Therefore, by Conjecture 3.5.5,

$$\dim Y \geq \dim U + \dim S$$

and the result follows since $\dim U = \dim A$ and $\dim Y = \dim W + \dim V$. $\square$

In our applications, we will use a reformulation of Conjecture 3.5.6. For this reformulation, we will need the following definitions.

Fix a subvariety V of S.

Definition 3.5.7. An **intersection component** of $\pi^{-1}(V)$ is an irreducible analytic component of the intersection of $\pi^{-1}(V)$ with a subvariety of X.

For any intersection component A of $\pi^{-1}(V)$, there exists a smallest subvariety of X containing A; we denote it $\langle A\rangle_{\mathrm{Zar}}$. It follows that A is an irreducible analytic component of

$$\langle A\rangle_{\mathrm{Zar}} \cap \pi^{-1}(V).$$

Definition 3.5.8. Let A be an intersection component of $\pi^{-1}(V)$. We define the **Zariski defect** of A to be

$$\delta_{\mathrm{Zar}}(A) := \dim\langle A\rangle_{\mathrm{Zar}} - \dim A.$$

Definition 3.5.9. We say that an intersection component A of $\pi^{-1}(V)$ is **Zariski optimal** in $\pi^{-1}(V)$ if, for any intersection component B of $\pi^{-1}(V)$,

$$A \subsetneq B \subseteq \pi^{-1}(V) \implies \delta_{\mathrm{Zar}}(B) > \delta_{\mathrm{Zar}}(A).$$

Definition 3.5.10. Let $x \in X$ and let X_M denote the $M(\mathbb{R})^+$ conjugacy class of x in X, where $M := \mathrm{MT}(x)$. Write M^{ad} as a product

$$M^{\mathrm{ad}} = M_1 \times M_2$$

of two normal $\mathbb{Q}$-subgroups, either of which may be trivial, thus inducing a splitting

$$X_M = X_1 \times X_2.$$

For any $x_1 \in X_1$ or $x_2 \in X_2$, we obtain a subvariety $\{x_1\} \times X_2$ or $X_1 \times \{x_2\}$ of X. We refer to any subvariety of X taking this form as a **pre-weakly special subvariety** of X. That is, a weakly special subvariety of $\mathrm{Sh}_K(G,\mathfrak{X})$ contained in S is, by definition, the image in S of a pre-weakly special subvariety of X.

Remark 3.5.11. Note that pre-weakly special subvarieties of X are indeed subvarieties of X (see [30], Lemma 6.2, for example). In particular, they are irreducible analytic subsets of X. As explained in [55], pre-weakly special subvarieties of X are totally geodesic subvarieties of X.

Definition 3.5.12. An intersection component A of $\pi^{-1}(V)$ is called **pre-weakly special** if $\langle A\rangle_{\mathrm{Zar}}$ is a pre-weakly special subvariety of X.

Conjecture 3.5.13 (weak hyperbolic Ax-Schanuel)**.** *Let A be an intersection component of $\pi^{-1}(V)$ that is Zariski optimal in $\pi^{-1}(V)$. Then A is pre-weakly special.*

Note that Conjecture 3.5.13 is a direct generalization of the hyperbolic Ax-Lindemann theorem.

Theorem 3.5.14 (hyperbolic Ax-Lindemann)**.** *The maximal subvarieties contained in $\pi^{-1}(V)$ are pre-weakly special.*

Proof that Conjecture 3.5.13 implies Theorem 3.5.14. The maximal subvarieties contained in $\pi^{-1}(V)$ are precisely the intersection components of $\pi^{-1}(V)$ that are Zariski optimal in $\pi^{-1}(V)$ and whose Zariski defect is zero. □

Although [35], Section 5.2 is dedicated to products of modular curves, the proof that Formulations A and B of *Weak Complex Ax* are equivalent is completely general and, when translated into our terminology, yields the following.

Lemma 3.5.15. *Conjecture 3.5.6 and Conjecture 3.5.13 are equivalent.*

We conclude this section with the following consequence of the weak hyperbolic Ax-Schanuel conjecture. Here and elsewhere, we will tacitly make use of the following remark.

Remark 3.5.16. Let W be a subvariety of S and let A denote an irreducible analytic component of $\pi^{-1}(W)$ in X. Then, since W is analytically irreducible, every irreducible analytic component of $\pi^{-1}(W)$ is equal to a Γ-translate of A (as mentioned in [82], Section 4, for example). In particular, $\pi(A)$ is equal to W.

Lemma 3.5.17. *Assume that the weak hyperbolic Ax-Schanuel conjecture is true for V. Let A be a Zariski optimal intersection component of $\pi^{-1}(V)$. Then $\pi(A)$ is a closed irreducible subvariety of V and, as such, is weakly optimal in V.*

Proof. Clearly, the Zariski closure $\overline{\pi(A)}$ of $\pi(A)$ is irreducible. Therefore, let W be a subvariety of V containing $\overline{\pi(A)}$ such that $\delta_{\mathrm{ws}}(W)\leq\delta_{\mathrm{ws}}(\overline{\pi(A)})$. We can and do assume that W is weakly optimal in V. Let B be an irreducible analytic component of $\pi^{-1}(W)$ containing A. We have

$$\begin{aligned}\delta_{\mathrm{Zar}}(B)=\dim\langle B\rangle_{\mathrm{Zar}}-\dim B&=\dim\langle B\rangle_{\mathrm{Zar}}-\dim W\\&\leq\dim\langle W\rangle_{\mathrm{ws}}-\dim W=\delta_{\mathrm{ws}}(W)\leq\delta_{\mathrm{ws}}(\overline{\pi(A)})\\&\leq\dim\langle A\rangle_{\mathrm{Zar}}-\dim A=\delta_{\mathrm{Zar}}(A),\end{aligned}$$

where we use the fact that, by the weak hyperbolic Ax-Schanuel conjecture, $\langle A\rangle_{\mathrm{Zar}}$ is pre-weakly special. Therefore, we conclude that $B=A$. Hence, $\pi(A)=\pi(B)=W$.

□

3.6 A finiteness result for weakly optimal subvarieties

In this section, we deduce from the weak hyperbolic Ax-Schanuel conjecture a finiteness statement for the weakly optimal subvarieties of a given subvariety V.

Definition 3.6.1. Let $x \in X$ and let X_M denote the $M(\mathbb{R})^+$ conjugacy class of x in X, where $M := \mathrm{MT}(x)$. Then X_M is a subvariety of X and we refer to any subvariety of X taking this form as a **pre-special subvariety** of X. In particular, a pre-special subvariety of X is a pre-weakly special subvariety of X. If X_M is a point, that is, if M is a torus, we refer to X_M as a **pre-special point** of X. A special subvariety of $\mathrm{Sh}_K(G, \mathfrak{X})$ contained in S is, by definition, the image in S of a pre-special subvariety of X.

Definition 3.6.2. Let $x \in X$ and let X_M denote the $M(\mathbb{R})^+$ conjugacy class of x in X, where $M := \mathrm{MT}(x)$. Decomposing M^{ad} as a product

$$M^{\mathrm{ad}} = M_1 \times M_2$$

of two normal $\mathbb{Q}$-subgroups, either of which may be trivial, induces a splitting

$$X_M = X_1 \times X_2.$$

For any such splitting, and any $x_1 \in X_1$ or $x_2 \in X_2$, we refer to the pre-weakly special subvariety $\{x_1\} \times X_2$ or $X_1 \times \{x_2\}$ as a **fiber of (the pre-special subvariety) X_M**. In particular, the points of X_M are all fibers of X_M, and so too is X_M itself.

The main result of this section is the following.

Proposition 3.6.3 (cf. [35], Proposition 6.6)**.** *Let V be a subvariety of S and assume that the weak hyperbolic Ax-Schanuel conjecture is true for V. Then there exists a finite set Σ of pre-special subvarieties of X such that the following holds.*

Let W be a subvariety of V that is weakly optimal in V. Then there exists $Y \in \Sigma$ such that $\langle W \rangle_{\mathrm{ws}}$ is equal to the image in S of a fiber of Y.

Note that similar theorems also hold for abelian varieties (see [35], Proposition 6.1 and [67], Proposition 3.2).

Now fix a subvariety V of S. Given an intersection component A of $\pi^{-1}(V)$, there is a smallest totally geodesic subvariety $\langle A \rangle_{\mathrm{geo}}$ of X that contains A. In particular, we may make the following definition.

Definition 3.6.4. Let A be an intersection component of $\pi^{-1}(V)$. We define the **geodesic defect** of A to be

$$\delta_{\mathrm{geo}}(A) := \dim \langle A \rangle_{\mathrm{geo}} - \dim A.$$

We note that, in [35], this notion was referred to as the Möbius defect of A.

Definition 3.6.5. We say that an intersection component A of $\pi^{-1}(V)$ is **geodesically optimal** in $\pi^{-1}(V)$ if, for any intersection component B of $\pi^{-1}(V)$,

$$A \subsetneq B \subseteq \pi^{-1}(V) \implies \delta_{\text{geo}}(B) > \delta_{\text{geo}}(A).$$

We note that the terminology geodesically optimal has a different meaning in [35].

Remark 3.6.6. Let A be an intersection component of $\pi^{-1}(V)$. If A is geodesically optimal in $\pi^{-1}(V)$, then A is an irreducible analytic component of

$$\langle A\rangle_{\text{geo}} \cap \pi^{-1}(V).$$

Lemma 3.6.7. *Assume that the weak hyperbolic Ax-Schanuel conjecture is true for V and let A be an intersection component of $\pi^{-1}(V)$. If A is geodesically optimal in $\pi^{-1}(V)$, then A is Zariski optimal in $\pi^{-1}(V)$.*

Proof. Suppose that B is an intersection component of $\pi^{-1}(V)$ containing A such that

$$\delta_{\text{Zar}}(B) \leq \delta_{\text{Zar}}(A).$$

We can and do assume that B is Zariski optimal and so, by the weak hyperbolic Ax-Schanuel conjecture, it is pre-weakly special. In particular, $\langle B\rangle_{\text{Zar}}$ is a pre-weakly special subvariety of X and, therefore, equal to $\langle B\rangle_{\text{geo}}$. Then

$$\delta_{\text{geo}}(B) = \delta_{\text{Zar}}(B) \leq \delta_{\text{Zar}}(A) \leq \delta_{\text{geo}}(A),$$

and, since A is geodesically optimal in $\pi^{-1}(V)$, we conclude that $B = A$. □

Lemma 3.6.8. *Assume that the weak hyperbolic Ax-Schanuel conjecture is true for V and let W be a subvariety of V that is weakly optimal in V. Let A be an irreducible analytic component of $\pi^{-1}(W)$. Then A is an intersection component of $\pi^{-1}(V)$ and is geodesically optimal in $\pi^{-1}(V)$.*

Proof. Clearly, A is an intersection component of $\pi^{-1}(V)$ since W is an irreducible component of $\langle W\rangle_{\text{ws}} \cap V$ and $\pi^{-1}\langle W\rangle_{\text{ws}}$ is equal to the Γ-orbit of a pre-weakly special subvariety of X.

Therefore, let B be an intersection component of $\pi^{-1}(V)$ containing A such that

$$\delta_{\text{geo}}(B) \leq \delta_{\text{geo}}(A).$$

We can and do assume that B is geodesically optimal in $\pi^{-1}(V)$ and so, by Lemma 3.6.7, B is Zariski optimal in $\pi^{-1}(V)$. Therefore, by the weak hyperbolic Ax-Schanuel conjecture, B is pre-weakly special i.e. $\langle B\rangle_{\text{Zar}}$ is a pre-weakly special subvariety of X.

Let $Z := \pi(B)$ (which is a closed irreducible subvariety of V by Lemma 3.5.17). We claim that $\langle Z\rangle_{\text{ws}} = \pi(\langle B\rangle_{\text{Zar}})$. To see this, note that Z is contained in $\pi(\langle B\rangle_{\text{Zar}})$ and so

$\langle Z\rangle_{\mathrm{ws}}$ is contained in $\pi(\langle B\rangle_{\mathrm{Zar}})$. On the other hand, $\langle B\rangle_{\mathrm{Zar}}$ is contained in $\pi^{-1}(\langle Z\rangle_{\mathrm{ws}})$ and so $\pi(\langle B\rangle_{\mathrm{Zar}})$ is contained in $\langle Z\rangle_{\mathrm{ws}}$, which proves the claim. Therefore,

$$\begin{aligned}\delta_{\mathrm{ws}}(Z) &= \dim \pi(\langle B\rangle_{\mathrm{Zar}}) - \dim Z \\ &= \dim\langle B\rangle_{\mathrm{Zar}} - \dim Z \\ &= \dim\langle B\rangle_{\mathrm{Zar}} - \dim B = \delta_{\mathrm{geo}}(B) \leq \delta_{\mathrm{geo}}(A) \\ &\leq \dim\langle W\rangle_{\mathrm{ws}} - \dim W = \delta_{\mathrm{ws}}(W).\end{aligned}$$

Since W is weakly optimal in V and contained in Z, we conclude that $Z = W$. In particular, B is contained in $\pi^{-1}(W)$ and, therefore, $B = A$. □

Let us briefly summarize the relationship between Zariski optimal and weakly optimal.

Proposition 3.6.9. *Assume that the weak hyperbolic Ax-Schanuel conjecture is true for V.*

If A is a Zariski optimal intersection component of $\pi^{-1}(V)$, then $\pi(A)$ is a closed irreducible subvariety of V that is weakly optimal in V.

On the other hand, if W is a subvariety of V that is weakly optimal in V, and A is an irreducible analytic component of $\pi^{-1}(W)$, then A is a Zariski optimal intersection component of $\pi^{-1}(V)$.

Proof. The first claim is Lemma 3.5.17, whereas the second claim is Lemma 3.6.8 and Lemma 3.6.7. □

As explained in [44], there exists an open semialgebraic fundamental set $\mathcal{F}$ in X for the action of Γ such that the set $\mathcal{V} := \pi^{-1}(V) \cap \mathcal{F}$ is definable.

Definition 3.6.10. Once and for all, let $\mathcal{F}$ denote an open semialgebraic fundamental set $\mathcal{F}$ in X for the action of Γ, as above, and let $\mathcal{V}$ denote the definable set $\pi^{-1}(V) \cap \mathcal{F}$.

Recall from [86], 1.17 that the **local dimension** $\dim_x A$ of a definable set A at a point $x \in A$ is definable. By [35], Lemma 6.2, if A is also a (complex) analytic set, then this dimension is exactly twice the local analytic dimension at x. Furthermore, if A is analytically irreducible, then its local dimension at the points of A is constant. For the remainder of this section, dimensions will be taken in the sense of definable sets. The key step in the proof of Proposition 3.6.3 is the following.

Proposition 3.6.11. *Assume that the weak hyperbolic Ax-Schanuel theorem is true for V. There exists a finite set Σ of pre-special subvarieties of X such that the following holds.*

Let A be an intersection component of $\pi^{-1}(V)$ that is pre-weakly special such that, for some $x \in \langle A\rangle_{\mathrm{Zar}} \cap \mathcal{V}$,

$$\dim A = \dim_x(\langle A\rangle_{\mathrm{Zar}} \cap \mathcal{V}).$$

Then there exists $Y \in \Sigma$ such that $\langle A\rangle_{\mathrm{Zar}}$ is equal to a fiber of Y.

In order to prove Proposition 3.6.11, we require some further preparations.

Definition 3.6.12. We say that a real semisimple algebraic group F is **without compact factors** if it is equal to an almost direct product of almost simple subgroups whose underlying real Lie groups are not compact. We allow the product to be trivial i.e. we consider the trivial group as a real semisimple algebraic group without compact factors.

Lemma 3.6.13. *A subvariety of X that is totally geodesic in X is of the form*

$$F(\mathbb{R})^+x,$$

where F is a semisimple algebraic subgroup of $G_{\mathbb{R}}$ without compact factors and $x \in X$ factors through

$$G_F := FZ_{G_{\mathbb{R}}}(F)^{\circ}.$$

Conversely, if F is a semisimple algebraic subgroup of $G_{\mathbb{R}}$ without compact factors and $x \in X$ factors through G_F, then $F(\mathbb{R})^+x$ is a subvariety of X that is totally geodesic in X.

Proof. See [81], Proposition 2.3. □

We let Ω denote a set of representatives for the $G(\mathbb{R})$-conjugacy classes of semisimple algebraic subgroups of $G_{\mathbb{R}}$ that are without compact factors. Note that Ω is a finite set (see [10], Corollary 0.2, for example), and it is clear that the set

$$\Pi_0 := \{(x,g,F) \in \mathcal{V} \times G(\mathbb{R}) \times \Omega : x(\mathbb{S}) \subseteq gG_Fg^{-1}\},$$

parametrising (albeit in a many-to-one fashion) the totally geodesic subvarieties of X passing through $\mathcal{V}$, is definable. Consider the two functions

$$\begin{aligned} d(x,g,F) &:= \dim_x(gF(\mathbb{R})^+g^{-1}x) = \dim(gF(\mathbb{R})^+g^{-1}x) \\ d_{\mathcal{V}}(x,g,F) &:= \dim_x(\mathcal{V} \cap gF(\mathbb{R})^+g^{-1}x), \end{aligned}$$

and let Π_1 denote the definable set

$$\begin{aligned} \{(x,g,F) \in \Pi_0 : (x,g_1,F_1) \in \Pi_0,\ & gF(\mathbb{R})^+g^{-1}x \subsetneq g_1F_1(\mathbb{R})^+g_1^{-1}x \\ & \implies d(x,g,F) - d_{\mathcal{V}}(x,g,F) < d(x,g_1,F_1) - d_{\mathcal{V}}(x,g_1,F_1)\}. \end{aligned}$$

Finally, let Π_2 denote the definable set

$$\begin{aligned} \{(x,g,F) \in \Pi_1 : (x,g_1,F_1) \in \Pi_0,\ & g_1F_1(\mathbb{R})^+g_1^{-1}x \subsetneq gF(\mathbb{R})^+g^{-1}x \\ & \implies d_{\mathcal{V}}(x,g_1,F_1) < d_{\mathcal{V}}(x,g,F)\}. \end{aligned}$$

The proof of Proposition 3.6.11 will require the following three lemmas.

Lemma 3.6.14. *Let A be an intersection component of $\pi^{-1}(V)$ that is pre-weakly special such that, for some $x \in \langle A\rangle_{\mathrm{Zar}} \cap \mathcal{V}$,*

$$\dim A = \dim_x(\langle A\rangle_{\mathrm{Zar}} \cap \mathcal{V}).$$

Then we can write

$$\langle A\rangle_{\mathrm{Zar}} = gF(\mathbb{R})^+g^{-1}x,$$

where $(x,g,F) \in \Pi_2$.

Proof. By Lemma 3.6.13, we can write

$$\langle A\rangle_{\text{Zar}} = gF(\mathbb{R})^+g^{-1}x$$

for some $F\in\Omega$ and some $x\in\mathcal{V}$ that factors through gG_Fg^{-1}. In particular, $(x,g,F)\in\Pi_0$. By assumption, we can and do choose $x\in\langle A\rangle_{\text{Zar}}\cap\mathcal{V}$ such that

$$\dim A = \dim_x(\langle A\rangle_{\text{Zar}}\cap\mathcal{V}) = d_{\mathcal{V}}(x,g,F).$$

Suppose that (x,g,F) does not belong to Π_1 i.e. that there exists $(x,g_1,F_1)\in\Pi_0$ such that

$$gF(\mathbb{R})^+g^{-1}x \subsetneq g_1F_1(\mathbb{R})^+g_1^{-1}x,$$

and

$$d(x,g,F)-d_{\mathcal{V}}(x,g,F) \geq d(x,g_1,F_1)-d_{\mathcal{V}}(x,g_1,F_1). \tag{3.6.1}$$

Let B be an irreducible analytic component of

$$g_1F_1(\mathbb{R})^+g_1^{-1}x\cap\pi^{-1}(V)$$

passing through x such that

$$\dim B = d_{\mathcal{V}}(x,g_1,F_1).$$

From (3.6.1), we obtain $\delta_{\text{Zar}}(B)\leq\delta_{\text{Zar}}(A)$.

On the other hand, the Intersection Inequality (see [32], Chapter 5, §3) yields

$$\dim_x(B\cap\langle A\rangle_{\text{Zar}}) \geq \dim B+\dim\langle A\rangle_{\text{Zar}} - d(x,g_1,F_1)$$

and, from (3.6.1), we obtain

$$\dim_x(B\cap\langle A\rangle_{\text{Zar}}) \geq \dim A.$$

It follows that $B\cap\langle A\rangle_{\text{Zar}}$, and hence B itself, contains a complex neighbourhood of x in A, which implies that A is contained in B.

Since A is Zariski optimal, we conclude that $A=B$. However, this implies that

$$d(x,g_1,F_1)-d_{\mathcal{V}}(x,g_1,F_1) > 2\delta_{\text{Zar}}(B) = 2\delta_{\text{Zar}}(A) = d(x,g,F)-d_{\mathcal{V}}(x,g,F),$$

which contradicts (3.6.1). Therefore, $(x,g,F)\in\Pi_1$.

Now suppose that (x,g,F) does not belong to Π_2 i.e. that there exists $(x,g_1,F_1)\in\Pi_0$ such that

$$g_1F_1(\mathbb{R})^+g_1^{-1}x \subsetneq gF(\mathbb{R})^+g^{-1}x,$$

and

$$d_{\mathcal{V}}(x,g_1,F_1) = d_{\mathcal{V}}(x,g,F) = \dim A.$$

But then A is contained in

$$g_1F_1(\mathbb{R})^+g_1^{-1}x \subsetneq \langle A\rangle_{\text{Zar}},$$

which contradicts the definition of $\langle A\rangle_{\text{Zar}}$. $\square$

Lemma 3.6.15. *Assume that the weak hyperbolic Ax-Schanuel conjecture is true for V. Then, if $(x,g,F) \in \Pi_2$, there exists a semisimple subgroup F' of G defined over $\mathbb{Q}$ such that gFg^{-1} is equal to the almost direct product of the almost simple factors of $F'_{\mathbb{R}}$ whose underlying real Lie groups are non-compact.*

Proof. By [80], Proposition 3.1, it suffices to show that $gF(\mathbb{R})^+g^{-1}x$ is a pre-weakly special subvariety of X. Therefore, let A be an irreducible analytic component of

$$gF(\mathbb{R})^+g^{-1}x \cap \pi^{-1}(V)$$

passing through x such that

$$\dim A = d_{\mathcal{V}}(x,g,F).$$

Let B be an intersection component of $\pi^{-1}(V)$ containing A such that $\delta_{\mathrm{Zar}}(B) \leq \delta_{\mathrm{Zar}}(A)$. We can and do assume that B is Zariski optimal and, therefore, by the weak hyperbolic Ax-Schanuel conjecture, pre-weakly special i.e. B is an irreducible component of

$$\langle B\rangle_{\mathrm{Zar}} \cap \pi^{-1}(V)$$

and $\langle B\rangle_{\mathrm{Zar}}$ is a pre-weakly special subvariety of X.

Therefore, A is contained in

$$gF(\mathbb{R})^+g^{-1}x \cap \langle B\rangle_{\mathrm{Zar}}$$

and we let Y be an irreducible analytic component of this intersection containing A. Then Y is a subvariety of X that is totally geodesic is X and, hence, equal to $g_1F_1(\mathbb{R})^+g_1^{-1}x$ for some $(x,g_1,F_1) \in \Pi_0$. Furthermore,

$$d_{\mathcal{V}}(x,g_1,F_1) = \dim A = d_{\mathcal{V}}(x,g,F)$$

and, since $(x,g,F) \in \Pi_2$, we conclude that

$$g_1F_1(\mathbb{R})^+g_1^{-1}x = gF(\mathbb{R})^+g^{-1}x \subseteq \langle B\rangle_{\mathrm{Zar}}.$$

We also have

$$\begin{aligned}\dim\langle B\rangle_{\mathrm{Zar}} - \dim_x(\langle B\rangle_{\mathrm{Zar}} \cap \mathcal{V}) &\leq \delta_{\mathrm{Zar}}(B) \leq \delta_{\mathrm{Zar}}(A)\\ &\leq d(x,g,F) - \dim A\\ &= d(x,g,F) - d_{\mathcal{V}}(x,g,F),\end{aligned}$$

and so, since $(x,g,F) \in \Pi_1$, we conclude that

$$gF(\mathbb{R})^+g^{-1}x = \langle B\rangle_{\mathrm{Zar}}.$$

□

Lemma 3.6.16. *Assume that the weak hyperbolic Ax-Schanuel conjecture is true for V. Then, the set*

$$\{gFg^{-1} : (x,g,F) \in \Pi_2\}$$

is finite.

Proof. Decompose Π_2 as the finite union of the Π_F, varying over the members F of Ω, where Π_F denotes the set of tuples $(x,g,F) \in \Pi_2$. For each $F \in \Omega$, consider the map

$$\Pi_F \to G(\mathbb{R})/N_{G(\mathbb{R})}(F),$$

defined by

$$(x,g,F) \mapsto gN_{G(\mathbb{R})}(F),$$

whose image, therefore, is in bijection with $\{gFg^{-1} : (x,g,F) \in \Pi_2\}$. It is also definable and, by Lemma 3.6.15, it is countable. Hence, it is finite. $\square$

Proof of Proposition 3.6.11. Let A be an intersection component of $\pi^{-1}(V)$ that is pre-weakly special such that, for some $x \in \langle A\rangle_{\mathrm{Zar}} \cap \mathcal{V}$,

$$\dim A = \dim_x(\langle A\rangle_{\mathrm{Zar}} \cap \mathcal{V}).$$

Then, by Lemma 3.6.14, we can write

$$\langle A\rangle_{\mathrm{Zar}} = gF(\mathbb{R})^+g^{-1}x$$

where $(x,g,F) \in \Pi_2$. By Lemma 3.6.15, there exists a semisimple subgroup F' of G defined over $\mathbb{Q}$ such that gFg^{-1} is equal to the almost direct product of the almost simple factors of $F'_{\mathbb{R}}$ whose underlying real Lie groups are non-compact. In fact, by [80], Proposition 3.1, F' is the smallest subgroup of G defined over $\mathbb{Q}$ containing gFg^{-1}. Since, by Lemma 3.6.16, gFg^{-1} comes from a finite set, so too does F'. Therefore, the reductive algebraic group

$$M := F'Z_G(F')^\circ$$

is defined over $\mathbb{Q}$ and belongs to a finite set.

If we write M^{nc} for the almost direct product of the almost $\mathbb{Q}$-simple factors of M whose underlying real Lie groups are not compact, then x factors through $M'_{\mathbb{R}} := Z(M)^\circ_{\mathbb{R}}M^{\mathrm{nc}}_{\mathbb{R}}$ and, if we write $\mathfrak{X}_M$ for the $M'(\mathbb{R})$ conjugacy class of x in $\mathfrak{X}$, then, by [79], Lemme 3.3, $(M',\mathfrak{X}_M)$ is a Shimura subdatum of $(G,\mathfrak{X})$. Furthermore, by [83], Lemma 3.7, the number of Shimura subdatum $(M',\mathfrak{Y})$ is finite. Therefore, since the $M'(\mathbb{R})^+$ conjugacy class X_M of x in X is a pre-special subvariety of X and $\langle A\rangle_{\mathrm{Zar}}$ is a fiber of X_M, the proof is complete. $\square$

Proof of Proposition 3.6.3. Let A be an irreducible analytic component of $\pi^{-1}(W)$. By Proposition 3.6.9, A is an intersection component of $\pi^{-1}(V)$ and is Zariski optimal in $\pi^{-1}(V)$. Therefore, by the weak hyperbolic Ax-Schanuel conjecture, A is pre-weakly special. It follows that the image of $\langle A\rangle_{\mathrm{Zar}}$ in S is equal to $\langle W\rangle_{\mathrm{ws}}$.

After possibly replacing A by a γA, for some $\gamma \in \Gamma$, we can and do assume that there exists $x \in \langle A\rangle_{\mathrm{Zar}} \cap \mathcal{V}$ such that

$$\dim(A) = \dim_x(\langle A\rangle_{\mathrm{Zar}} \cap \mathcal{V}).$$

By Proposition 3.6.11, $\langle A\rangle_{\mathrm{Zar}}$ is a fiber of $Y \in \Sigma$, where Σ is a finite set of pre-special subvarieties of X depending only on V. $\square$

3.7 Anomalous subvarieties

In this section, we recall the notion of an anomalous subvariety, which is defined by Bombieri, Masser, and Zannier in [7] for subvarieties of algebraic tori. In fact, we give the more general notion of an r-anomalous subvariety, as introduced by Rémond [67].

Let V be a subvariety of S. We will use Proposition 3.6.3 to show that, under the weak hyperbolic Ax-Schanuel conjecture, the union of the subvarieties of V that are r-anomalous in V constitutes a Zariski closed subset of V. We will then give a criterion for when it is a proper subset.

Definition 3.7.1. Let $r \in \mathbb{Z}$. A subvariety W of V is called ***r*-anomalous** in V if

$$\dim W \geq \max\{1, r + \dim\langle W\rangle_{\mathrm{ws}} - \dim S\}.$$

A subvariety of V is **maximal *r*-anomalous** in V if it is r-anomalous in V and not strictly contained in another subvariety of V that is also r-anomalous in V.

We denote by $\mathrm{an}(V,r)$ the set of subvarieties of V that are maximal r-anomalous in V and by $V^{\mathrm{an},r}$ the union of the elements of $\mathrm{an}(V,r)$, which is then the union of all the subvarieties of V that are r-anomalous in V.

We say that a subvariety of V is **anomalous** if it is $(1+\dim V)$-anomalous. We write $\mathrm{an}(V)$ for $\mathrm{an}(V, 1+\dim V)$ and V^{an} for $V^{\mathrm{an},1+\dim V}$.

Theorem 3.7.2. *Assume that the weak hyperbolic Ax-Schanuel conjecture is true for V and let $r \in \mathbb{Z}$. Then $V^{\mathrm{an},r}$ is a Zariski closed subset of V.*

We refer the reader to [7], [67], and [35] for similar results on algebraic tori and abelian varieties. We will require the following facts.

Proposition 3.7.3 (cf. [37], Chapter 2, Exercise 3.22 (d))**.** *Let $f : W \to Y$ be a dominant morphism between two integral schemes of finite type over a field and let*

$$e := \dim W - \dim Y$$

denote the **relative dimension**. *For $h \in \mathbb{Z}$, let E_h denote the set of points $x \in W$ such that the fibre $f^{-1}(f(x))$ possesses an irreducible component of dimension at least h that contains x. Then*

(1) E_h is a Zariski closed subset of W,

(2) $E_e = W$, and

(3) if $h > e$, E_h is not Zariski dense in W.

Lemma 3.7.4. *Let $W \in \mathrm{an}(V,r)$. Then W is weakly optimal in V.*

Proof. Let Y be a subvariety of V containing W such that $\delta_{\mathrm{ws}}(Y) \leq \delta_{\mathrm{ws}}(W)$. We can and do assume that Y is weakly optimal. Then

$$\begin{aligned}\dim Y &= \dim\langle Y\rangle_{\mathrm{ws}} - \delta_{\mathrm{ws}}(Y)\\ &\geq \dim\langle Y\rangle_{\mathrm{ws}} - \delta_{\mathrm{ws}}(W)\\ &= \dim\langle Y\rangle_{\mathrm{ws}} - (\dim\langle W\rangle_{\mathrm{ws}} - \dim W)\\ &\geq \dim\langle Y\rangle_{\mathrm{ws}} + r - \dim S.\end{aligned}$$

Since Y contains W, we know that $\dim Y \geq 1$, and so Y is r-anomalous in V. Since W is maximal r-anomalous in V, we conclude that Y must be equal to W. Therefore, W is weakly optimal. □

Proof of Theorem 3.7.2. Let Σ be a finite set of pre-special subvarieties of X (whose existence is ensured by Proposition 3.6.3) such that, if W is a subvariety of V that is weakly optimal in V, then there exists $x \in X$ such that, if $M := \mathrm{MT}(x)$, the $M(\mathbb{R})^+$ conjugacy class X_M of x in X belongs to Σ and $\langle W\rangle_{\mathrm{ws}}$ is equal to the image in S of a fiber of X_M. That is, we may write M^{ad} as a product $M_1 \times M_2$ of two normal $\mathbb{Q}$-subgroups, which induces a splitting $X = X_1 \times X_2$, such that $\langle W\rangle_{\mathrm{ws}}$ is equal to the image in S of $\{x_1\} \times X_2$, for some $x_1 \in X_1$.

Let $W \in \mathrm{an}(V,r)$. By Lemma 3.7.4, there exists $X_M \in \Sigma$ such that $\langle W\rangle_{\mathrm{ws}}$ is equal to the image in S of $\{x_1\} \times X_2$, for some $x_1 \in X_1$, where $X_M = X_1 \times X_2$, as above.

Let Γ_M be a congruence subgroup of $M(\mathbb{Q})_+$ contained in Γ, where $M(\mathbb{Q})_+$ denotes the subgroup of $M(\mathbb{Q})$ acting on X_M, and let Γ_1 denote the image of Γ under the natural maps

$$M(\mathbb{Q}) \to M^{\mathrm{ad}}(\mathbb{Q}) \to M_1(\mathbb{Q}).$$

We denote by f the restriction of

$$\Gamma_M\backslash X_M \to \Gamma_1\backslash X_1$$

to an irreducible component $\widetilde{V}$ of $\phi^{-1}(V)$, such that $\dim\widetilde{V} = \dim V$, where ϕ denotes the natural map

$$\Gamma_M\backslash X_M \to \Gamma\backslash X = S.$$

In particular, $\phi(\widetilde{V}) = V$. Therefore, by Proposition 3.7.3 (1), the set E_h of points z in $\widetilde{V}$ such that the fibre $f^{-1}(f(z))$ possesses an irreducible component of dimension at least $h \in \mathbb{Z}$ that contains z is a Zariski closed subset of $\widetilde{V}$. Since ϕ is a closed morphism, $\phi(E_h)$ is Zariski closed in V.

We claim that W is contained in $\phi(E_h)$, where

$$h := \max\{1, r + \dim X_2 - \dim S\}.$$

To see this, fix an irreducible component $\widetilde{W}$ of $\phi^{-1}(W)$ contained in $\widetilde{V}$ such that $\dim \widetilde{W} = \dim W$. Then $\langle \widetilde{W} \rangle_{\mathrm{ws}}$ is equal to the image of $\{x_1\} \times X_2$ in $\Gamma_M \backslash X_M$ and so $\widetilde{W}$ lies in a fiber of f. Since

$$\dim \widetilde{W} = \dim W \geq \max\{1, r + \dim \langle W \rangle_{\mathrm{ws}} - \dim S\} = \max\{1, r + \dim X_2 - \dim S\},$$

$\widetilde{W}$ is contained in E_h, which implies that W is contained in $\phi(E_h)$.

On the other hand, we claim that $\phi(E_h)$ is contained in $V^{\mathrm{an},r}$. To see this, let $z \in E_h$ and let Y be an irreducible component of the fibre $f^{-1}(f(z))$ of dimension at least h containing z. Then Y is contained in the image of $\{x_1\} \times X_2$ in $\Gamma_M \backslash X_M$, where $x_1 \in X_1$ lies above $f(z) \in \Gamma_1 \backslash X_1$, and so

$$\dim \langle Y \rangle_{\mathrm{ws}} \leq \dim X_2.$$

Therefore,

$$\dim \phi(Y) = \dim Y \geq h = \max\{1, r + \dim X_2 - \dim S\} \geq \max\{1, r + \dim \langle \phi(Y) \rangle_{\mathrm{ws}} - \dim S\}$$

and so $\phi(Y)$ is r-anomalous in V.

Hence, if we let E denote the union of the $\phi(E_h)$ as we vary over the finitely many maps f obtained from the $X_M \in \Sigma$ and their possible splittings, we conclude that $E = V^{\mathrm{an},r}$, which finishes the proof. □

We denote by V^{oa} the complement in V of V^{an}. By Theorem 3.7.2, this is a (possibly empty) open subset of V. In the literature, it is sometimes referred to as the open-anomalous locus, hence the subscript. We conclude this section by showing that, when V is sufficiently generic, V^{oa} is not empty.

Proposition 3.7.5. *Suppose that V is Hodge generic in S. Then $V^{\mathrm{an}} = V$ if and only if we can write $G^{\mathrm{ad}} = G_1 \times G_2$, and thus $X = X_1 \times X_2$, such that*

$$\dim f(V) < \min\{\dim V, \dim X_1\},$$

where f denotes the projection map

$$\Gamma \backslash X \to \Gamma_1 \backslash X_1,$$

and Γ_1 denotes the image of Γ under the natural maps

$$G(\mathbb{Q}) \to G^{\mathrm{ad}}(\mathbb{Q}) \to G_1(\mathbb{Q}).$$

Proof. First suppose that $V^{\mathrm{an}} = V$. Then, for any set Σ as in the proof of Theorem 3.7.2, V is contained in the (finite) union of the images in S of the $X_M \in \Sigma$. Therefore, since V is assumed to be Hodge generic in S, it must be that $X \in \Sigma$ and, furthermore, that there exists $W \in \mathrm{an}(V)$ such that $G^{\mathrm{ad}} = G_1 \times G_2$, and thus $X = X_1 \times X_2$, such that $\langle W \rangle_{\mathrm{ws}}$ is equal to the image in S of $\{x_1\} \times X_2$, for some $x_1 \in X_1$.

Let f denote the projection map

$$\Gamma \backslash X \to \Gamma_1 \backslash X_1$$

and consider its restriction

$$V \to \overline{f(V)},$$

where $\overline{f(V)}$ denotes the Zariski closure of $f(V)$ in $\Gamma_1 \backslash X_1$. Since $V^{\mathrm{an}} = V$, it follows from Proposition 3.7.3 (3), that

$$h := \max\{1, 1 + \dim V + \dim X_2 - \dim X\} \leq \dim V - \dim f(V).$$

Hence,

$$\dim f(V) < \dim X - \dim X_2 = \dim X_1$$

and

$$\dim f(V) \leq \dim V - h \leq \dim V - 1 < \dim V.$$

Conversely, suppose that $G^{\mathrm{ad}} = G_1 \times G_2$, and thus $X = X_1 \times X_2$, such that

$$\dim f(V) < \min\{\dim V, \dim X_1\},$$

where f again denotes the projection map

$$\Gamma \backslash X \to \Gamma_1 \backslash X_1.$$

Restricting f to

$$V \to \overline{f(V)},$$

as before, we see from Proposition 3.7.3 (2) that the set E_h of points z in V such that the fibre $f^{-1}(f(z))$ possesses an irreducible component of dimension at least

$$h := \max\{1, 1 + \dim V - \dim X_1\} \leq \dim V - \dim f(V) = \dim V - \dim \overline{f(V)}$$

that contains z is equal to V. However, from the proof of Theorem 3.7.2, we have seen that E_h is contained in V^{an}, so the claim follows. $\square$

Corollary 3.7.6. *If G^{ad} is $\mathbb{Q}$-simple and V is a Hodge generic subvariety in S, then V^{an} is strictly contained in V. In particular, V^{an} is strictly contained in V whenever V is a Hodge generic subvariety of $\mathcal{A}_g$.*

3.8 Main results (part 1): Reductions to point counting

In this section, we prove our main theorem: under the weak hyperbolic Ax-Schanuel conjecture, the Zilber-Pink conjecture can be reduced to a problem of point counting. We also give a reduction of Pink's conjecture in the case when the open-anomalous locus is non-empty.

Definition 3.8.1. Let V be a subvariety of S. We denote by $\mathrm{Opt}_0(V)$ the set of all points in V that are optimal in V.

Consider the following corollary of the Zilber-Pink conjecture.

Conjecture 3.8.2. *Let V be a subvariety of S. Then $\mathrm{Opt}_0(V)$ is finite.*

We will later show that, under certain arithmetic hypotheses, one can prove Conjecture 3.8.2 when V is a curve. Our main result in this section is that (under the weak hyperbolic Ax-Schanuel conjecture), Conjecture 3.8.2 implies the Zilber-Pink conjecture.

Theorem 3.8.3. *Assume that the weak hyperbolic Ax-Schanuel conjecture is true and assume that Conjecture 3.8.2 holds.*

Let V be a subvariety of S. Then $\mathrm{Opt}(V)$ is finite.

Proof. We prove Theorem 3.8.3 by induction on $\dim V$. Of course, Theorem 3.8.3 is trivial when $\dim V = 0$ or $\dim V = 1$. Therefore, we assume that $\dim V \geq 2$ and that Theorem 3.8.3 holds whenever the subvariety in question is of lower dimension.

We need to show that the induction hypothesis implies that there are only finitely many subvarieties of positive dimension belonging to $\mathrm{Opt}(V)$.

Let Σ be a finite set of pre-special subvarieties of X, as in the proof of Theorem 3.7.2, and let $W \in \mathrm{Opt}(V)$ be of positive dimension.

By Corollary 3.4.5, W is weakly optimal and, therefore, there exists $x \in X$ such that, if $M := \mathrm{MT}(x)$, the $M(\mathbb{R})^+$ conjugacy class X_M of x in X belongs to Σ and $\langle W \rangle_{\mathrm{ws}}$ is equal to the image in S of a fiber of X_M. That is, we may write M^{ad} as a product

$$M^{\mathrm{ad}} = M_1 \times M_2$$

of two normal $\mathbb{Q}$-subgroups, thus inducing a splitting

$$X_M = X_1 \times X_2,$$

such that $\langle W \rangle_{\mathrm{ws}}$ is equal to the image in S of $\{x_1\} \times X_2$, for some $x_1 \in X_1$.

Let Γ_M be a congruence subgroup of $M(\mathbb{Q})_+$ contained in Γ, where $M(\mathbb{Q})_+$ denotes the subgroup of $M(\mathbb{Q})$ acting on X_M, such that the image of Γ_M under the natural map

$$M(\mathbb{Q}) \to M^{\mathrm{ad}}(\mathbb{Q}) = M_1(\mathbb{Q}) \times M_2(\mathbb{Q})$$

is equal to a product $\Gamma_1 \times \Gamma_2$. We denote by f the natural morphism

$$\Gamma_M \backslash X_M \to \Gamma_1 \backslash X_1,$$

and by ϕ the finite morphism

$$\Gamma_M \backslash X_M \to \Gamma \backslash X = S.$$

Let $\widetilde{V}$ be an irreducible component of $\phi^{-1}(V)$ such that $\dim \widetilde{V} = \dim V$, and let $\widetilde{W}$ denote an irreducible component of $\phi^{-1}(W)$ contained in $\widetilde{V}$ such that $\dim \widetilde{W} = \dim W$. Then $\widetilde{W}$ is optimal in $\widetilde{V}$. On the other hand, by the generic smoothness property, there exists a dense open subset V_0 of $\widetilde{V}$ such that the restriction f_0 of f to V_0 is a smooth morphism of relative dimension ν. We denote by V_1 the Zariski closure of $f(V_0)$ in $\Gamma_1 \backslash X_1$.

Now suppose that

$$\widetilde{W} \cap V_0 = \emptyset. \tag{3.8.1}$$

Then $\widetilde{W}$ is a subvariety of some irreducible component V^0 of $\widetilde{V} \setminus V_0$. Furthermore, $\widetilde{W}$ is optimal in V^0. However, since $\dim V^0$ is strictly less than $\dim V$, our induction hypothesis implies that $\mathrm{Opt}(V^0)$ is finite.

Therefore, we assume that (3.8.1) does not hold. As an irreducible component of the fibre $f_0^{-1}(z)$, where z denotes the image of x_1 in V_1, its dimension is equal to ν. In particular,

$$\dim \widetilde{W} = \nu.$$

We claim that z is optimal in V_1. To see this, note that $f(\langle \widetilde{W} \rangle)$ contains z and is a special subvariety of dimension

$$\dim \langle \widetilde{W} \rangle - \dim X_2 = \dim \widetilde{W} + \delta(\widetilde{W}) - \dim X_2 = \nu + \delta(\widetilde{W}) - \dim X_2.$$

Therefore, let A be a subvariety of V_1 containing z such that

$$\delta(A) \leq \delta(z) \leq \nu + \delta(\widetilde{W}) - \dim X_2,$$

and let B be an irreducible component of $f^{-1}(A)$ containing $\widetilde{W}$. Since V_0 is open in $\widetilde{V}$ and A is contained in V_1,

$$\dim B = \dim A + \nu.$$

Therefore,

$$\delta(B) \leq \dim \langle A \rangle + \dim X_2 - \dim B = \delta(A) + \dim A + \dim X_2 - \dim B \leq \delta(\widetilde{W})$$

and, since $\widetilde{W}$ is optimal in $\widetilde{V}$, we conclude that B is equal to $\widetilde{W}$. In particular, $\widetilde{W}$ is an irreducible component of $f^{-1}(A)$ but, since it is also contained in $f^{-1}(z)$, it must be that A is equal to z, proving the claim.

Since W was assumed to be of positive dimension, so too must be X_2. It follows that $\dim V_1$ is strictly less than $\dim V$ and so, by the induction hypothesis, $\mathrm{Opt}(V_1)$ is finite. Since $z \in \mathrm{Opt}(V_1)$ and since Σ and the number of splittings are finite, we are done.

$\square$

We will later prove that the following conjecture is a consequence of the weak hyperbolic Ax-Schanuel conjecture and our arithmetic conjectures. It is inspired by the cited theorem of Habegger and Pila.

Conjecture 3.8.4 (cf. [35], Theorem 9.15 (iii))**.** *Let V be a subvariety of S. Then the set $V^{\mathrm{oa}} \cap S^{[1+\dim V]}$ is finite.*

The importance of Conjecture 3.8.4 for us is that, when V is suitably generic, Conjecture 3.8.4 implies Pink's conjecture (assuming the weak hyperbolic Ax-Schanuel conjecture).

Theorem 3.8.5. *Assume that the weak hyperbolic Ax-Schanuel conjecture is true and that Conjecture 3.8.4 holds.*

Let V be a Hodge generic subvariety of S such that (even after replacing Γ) S cannot be decomposed as a product $S_1 \times S_2$ such that V is contained in $V' \times S_2$, where V' is a proper subvariety of S_1 of dimension strictly less than the dimension of V. Then

$$V \cap S^{[1+\dim V]}$$

is not Zariski dense in V.

Proof. We claim that the assumptions guarantee that V^{an} is strictly contained in V. Otherwise, by Proposition 3.7.5, we can write $G^{\mathrm{ad}} = G_1 \times G_2$, and thus $X = X_1 \times X_2$, such that

$$\dim f(V) < \min\{\dim V, \dim X_1\},$$

where f denotes the projection map

$$\Gamma \backslash X \to \Gamma_1 \backslash X_1,$$

and Γ_1 denotes the image of Γ under the natural maps

$$G(\mathbb{Q}) \to G^{\mathrm{ad}}(\mathbb{Q}) \to G_1(\mathbb{Q}).$$

Therefore, after replacing Γ, we can write S as a product $S_1 \times S_2$ so that f is simply the projection on to the first factor and V is contained in $V' \times S_2$, where V' is Zariski closure of $f(V)$ in S_1. However, since

$$\dim V' = \dim f(V),$$

this is a contradiction.

Therefore, by Theorem 3.7.2, V^{an} is a proper Zariski closed subset of V. On the other hand, $V \cap S^{[1+\dim V]}$ is contained in

$$V^{\mathrm{an}} \cup \left[V^{\mathrm{oa}} \cap S^{[1+\dim V]} \right]$$

and so the theorem follows from Conjecture 3.8.4. □

3.9 The counting theorem

Henceforth, we turn our attention to the counting problems themselves. We will approach these problems using a theorem of Pila and Wilkie concerned with counting points in definable sets. We first recall the notations.

Let $k \geq 1$ be an integer. For any real number y, we define its **k-height** as

$$\mathrm{H}_k(y) := \min\{\max\{|a_0|, \dots, |a_k|\} : a_i \in \mathbb{Z},\ \gcd\{a_0, \dots, a_k\} = 1,\ a_0 y^k + \dots + a_k = 0\},$$

where we use the convention that, if the set is empty i.e. y is not algebraic of degree at most k, then $\mathrm{H}_k(y)$ is $+\infty$. For $y = (y_1, \dots, y_m) \in \mathbb{R}^m$, we set

$$\mathrm{H}_k(y) := \max\{\mathrm{H}_k(y_1), \dots, \mathrm{H}_k(y_m)\}.$$

For any set $A \subseteq \mathbb{R}^m \times \mathbb{R}^n$, and for any real number $T \geq 1$, we define

$$A(k,T) := \{(y,z) \in Z : \mathrm{H}_k(y) \leq T\}.$$

The counting theorem of Pila and Wilkie is stated as follows.

Theorem 3.9.1 (cf. the proof of [35], Corollary 7.2)**.** *Let $D \subseteq \mathbb{R}^l \times \mathbb{R}^m \times \mathbb{R}^n$ be a definable family parametrised by $\mathbb{R}^l$, let k be a positive integer, and let $\varepsilon > 0$. There exists a constant $c := c(D,k,\varepsilon) > 0$ with the following properties.*

Let $x \in \mathbb{R}^l$ and let

$$D_x := \{(y,z) \in \mathbb{R}^m \times \mathbb{R}^n : (x,y,z) \in D\}.$$

Let π_1 and π_2 denote the projections $\mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$ and $\mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$, respectively. If $T \geq 1$ and $\Sigma \subseteq D_x(k,T)$ satisfies

$$\#\pi_2(\Sigma) > cT^{\varepsilon},$$

there exists a continuous and definable function $\beta : [0,1] \to D_x$ such that the following properties hold.

1. *The composition $\pi_1 \circ \beta : [0,1] \to \mathbb{R}^m$ is semi-algebraic.*
2. *The composition $\pi_2 \circ \beta : [0,1] \to \mathbb{R}^n$ is non-constant.*
3. *We have $\beta(0) \in \Sigma$.*
4. *The restriction $\beta_{|(0,1)}$ is real analytic.*

Note that, although the conclusion $\beta(0) \in \Sigma$ does not appear in the statement of [35], Corollary 7.2, it is, indeed, established in its proof. The final property holds because $\mathbb{R}_{\mathrm{an,exp}}$ admits analytic cell decomposition (see [87]).

3.10 Complexity

In order to apply the counting theorem, we will need a way of counting special points and, more generally, special subvarieties. Recall that S is a connected component of the Shimura variety $\mathrm{Sh}_K(G,\mathfrak{X})$ defined by the Shimura datum (G,X) and the compact open subgroup K of $G(\mathbb{A}_f)$.

Let P be a special point in S and let $x \in X$ be a pre-special point lying above P. In particular, $T := \mathrm{MT}(x)$ is a torus and we denote by D_T the absolute value of the discriminant of its splitting field. We let K_T^m denote the maximal compact open subgroup of $T(\mathbb{A}_f)$ and we let $K_T \subseteq K_T^m$ denote $K \cap T(\mathbb{A}_f)$.

Definition 3.10.1. The **complexity** of P is the natural number

$$\Delta(P) := \max\{D_T, [K_T^m : K_T]\}.$$

Note that this does not depend on the choice of x.

Now let Z be a special subvariety of S. There exists a Shimura subdatum $(H,\mathfrak{X}_H)$ of $(G,\mathfrak{X})$, such that H is the generic Mumford-Tate group on $\mathfrak{X}_H$, and a connected component X_H of $\mathfrak{X}_H$ contained in X such that Z is the image of X_H in $\Gamma\backslash X$. In fact, these choices are well-defined up to conjugation by Γ.

By the **degree** $\deg(Z)$ of Z, we refer to the degree (in the sense of [**?**], Section 5.1) of the Zariski closure of Z in the Baily-Borel compactification of S with respect to the line bundle L_K defined in [**?**], Proposition 5.3.2 (1).

Definition 3.10.2. The **complexity** of Z is the natural number

$$\Delta(Z) := \max\{\deg(Z), \min\{\Delta(P) : P \in Z \text{ is a special point}\}\}.$$

Note that when Z is a special point, this complexity coincides with the former.

This is a natural generalization of the complexities given in [35], Definition 3.4 and Definition 3.8. In order to count special subvarieties, however, it is crucial that the complexity of Z satisfies the following property.

Conjecture 3.10.3. *For any $b \geq 1$, we have*

$$\#\{Z \subseteq S : Z \text{ is special and } \Delta(Z) \leq b\} < \infty.$$

The obstruction to proving that this property holds for a general Shimura variety can be expressed as follows.

Conjecture 3.10.4. *For any $b \geq 1$, there exists a finite set Ω of semisimple subgroups of G defined over $\mathbb{Q}$ such that, if Z is a special subvariety of S, and $\deg(Z) \leq b$, then*

$$H^{\mathrm{der}} = \gamma F \gamma^{-1},$$

for some $\gamma \in \Gamma$ and some $F \in \Omega$.

We will later verify Conjecture 3.10.4 for a product of modular curves.

Proof that Conjecture 3.10.4 implies Conjecture 3.10.3. Let Z be a special subvariety of S such that $\Delta(Z) \leq b$. By Conjecture 3.10.4, there exists a finite set Ω of semisimple subgroups of G defined over $\mathbb{Q}$, independent of Z, such that

$$H^{\mathrm{der}} = \gamma F \gamma^{-1},$$

for some $\gamma \in \Gamma$ and some $F \in \Omega$.

Let $P \in Z$ be a special point such that $\Delta(P)$ is minimal among all special points in Z and let $x \in X$ be a point lying above P such that $\mathrm{MT}(x)$ is contained in H. Therefore, Z is equal to the image of $F(\mathbb{R})^+\gamma^{-1}x$ in $\Gamma\backslash X$. Furthermore, $\mathrm{MT}(\gamma^{-1}x)$ is contained in

$$G_F := FZ_G(F)^\circ$$

and, by [79], Lemme 3.3, if we denote by $\mathfrak{X}'$ the $G_F(\mathbb{R})$ conjugacy class of $\gamma^{-1}x$, we obtain a Shimura subdatum $(G_F, \mathfrak{X}')$ of $(G, \mathfrak{X})$.

Therefore, let X' denote the connected component $G_F(\mathbb{R})^+\gamma^{-1}x$ of $\mathfrak{X}'$ and let Γ' denote $\Gamma \cap G_F(\mathbb{Q})_+$, where $G_F(\mathbb{Q})_+$ denotes the subgroup of $G_F(\mathbb{Q})$ acting on X'. By [83], Proposition 3.21 and its proof, there exist only finitely many Γ' orbits of pre-special points in X' whose image in $\Gamma'\backslash X'$ has complexity at most b. Therefore, there exists $\lambda \in \Gamma'$ such that $\gamma^{-1}x = \lambda y$, where $y \in X'$ belongs to a finite set. We conclude that Z is equal to the image of

$$\Gamma F(\mathbb{R})^+\gamma^{-1}x = \Gamma F(\mathbb{R})^+\lambda y = \Gamma\lambda F(\mathbb{R})^+ y = \Gamma F(\mathbb{R})^+ y$$

in $\Gamma\backslash X$, which concludes the proof. $\square$

3.11 Galois orbits

In [35], Habegger and Pila formulated a conjecture about Galois orbits of optimal points in $\mathbb{C}^n$ that in [34] they had been able to prove for so-called asymmetric curves. In [56], Orr generalized the result to asymmetric curves in $\mathcal{A}_g^2$.

Recall that $\mathrm{Sh}_K(G, \mathfrak{X})$ possesses a canonical model, defined over a number field E, which depends only on $(G, \mathfrak{X})$. Furthermore, S is defined over a finite abelian extension F of E. In particular, for any extension L of F contained in $\mathbb{C}$, it makes sense to say that a subvariety V of S is defined over L. Moreover, if V is such a subvariety, then $\mathrm{Aut}(\mathbb{C}/L)$ acts on the points of V.

If Z is a special subvariety of S and $\sigma \in \mathrm{Aut}(\mathbb{C}/F)$, then $\sigma(Z)$ is also a special subvariety of S and its complexity is also $\Delta(Z)$. In particular, if V is a subvariety of S, as above, then $\mathrm{Aut}(\mathbb{C}/L)$ acts on $\mathrm{Opt}(V)$ and its orbits are finite.

Conjecture 3.11.1 (large Galois orbits)**.** *Let V be a subvariety of S, defined over a finitely generated extension L of F contained in $\mathbb{C}$. There exist positive constants c_G and δ_G such that the following holds.*

If $P \in \mathrm{Opt}_0(V)$, *then*

$$\#\mathrm{Aut}(\mathbb{C}/L) \cdot P \geq c_G \Delta(\langle P \rangle)^{\delta_G}.$$

Remark 3.11.2. In the context of the André-Oort conjecture, there is the pioneering hypothesis that Galois orbits of special points should be large. See [28], Problem 14 for the formulation for special points in $\mathcal{A}_g$ and see [88], Theorem 2.1 for special points in a general Shimura variety. This hypothesis, which was verified by Tsimerman for special points of $\mathcal{A}_g$ [77] via progress on the Colmez conjecture due to Andreatta, Goren, Howard, Madapusi Pera [1] and Yuan and Zhang [89], is now the only obstacle in an otherwise unconditional proof of the André-Oort conjecture. The conjecture is that there exist positive constants c and δ such that, for any special point $P \in S$,

$$\#\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \cdot P \geq c\Delta(P)^{\delta}.$$

Of course, this conjecture does not follow from Conjecture 3.11.1 because special points lying in V need not be optimal in V. However, the proof of the André-Oort conjecture only requires the bound for special points that are not contained in the positive dimensional special subvarieties contained in V i.e. special points contained in $\mathrm{Opt}_0(V)$ (see [19] for more details). Furthermore, since special points are defined over number fields, we may also assume in that case that V is defined over a finite extension of F. It follows that Conjecture 3.11.1 is sufficient to prove the André-Oort conjecture.

To prove Conjecture 3.8.4, however, one only requires the following hypothesis.

Conjecture 3.11.3. *Let V be a subvariety of S, defined over a finitely generated extension L of F contained in $\mathbb{C}$. There exist positive constants c_G and δ_G such that the following holds.*

If $P \in V^{\mathrm{oa}} \cap S^{[1+\dim V]}$, *then*

$$\#\mathrm{Aut}(\mathbb{C}/L) \cdot P \geq c_G \Delta(\langle P \rangle)^{\delta_G}.$$

Remark 3.11.4. Note that, if $P \in V^{\mathrm{oa}} \cap S^{[1+\dim V]}$, then $P \in \mathrm{Opt}_0(V)$. To see this, let W be a subvariety of V containing P such that $\delta(W) \leq \delta(P)$ i.e.

$$\dim\langle W \rangle - \dim W \leq \dim\langle P \rangle \leq \dim S - 1 - \dim V.$$

In fact, we can and do assume that W is optimal. We have

$$\dim W \geq 1 + \dim V + \dim\langle W \rangle - \dim S \geq 1 + \dim V + \dim\langle W \rangle_{\mathrm{ws}} - \dim S,$$

and so $\dim W = 0$, as $P \notin V^{\mathrm{an}}$, which implies that $W = P$, proving the claim. Therefore, Conjecture 3.11.3 follows from Conjecture 3.11.1, but the former may turn out to be more tractable. It is worth recalling that, when S is an abelian variety and V is a subvariety defined over $\bar{\mathbb{Q}}$, Habegger [33] famously showed that the Néron-Tate height is bounded on $\bar{\mathbb{Q}}$-points of $V^{\mathrm{oa}} \cap S^{[\dim V]}$.

3.12 Further arithmetic hypotheses

The principal obstruction to applying the Pila-Wilkie counting theorems to our point counting problems (except for the availability of lower bounds for Galois orbits) is the ability to parametrize pre-special subvarieties of S using points of bounded height in a definable set.

Definition 3.12.1. We say that a semisimple algebraic group defined over $\mathbb{Q}$ is of non-compact type if its almost-simple factors all have the property that their underlying real Lie group is not compact.

Let Ω be a set of representatives for the semisimple subgroups of G defined over $\mathbb{Q}$ of non-compact type modulo the equivalence relation

$$H_2 \sim H_1 \iff H_{2,\mathbb{R}} = gH_{1,\mathbb{R}}g^{-1} \text{ for some } g \in G(\mathbb{R}).$$

Then Ω is a finite set (see [10], Corollary 0.2, for example). Add the trivial group to Ω. Recall that we realise X as a bounded symmetric domain in $\mathbb{C}^N$ for some $N \in \mathbb{N}$, which we identify with $\mathbb{R}^{2N}$. We fix an embedding of G into GL_n such that Γ is contained in $\mathrm{GL}_n(\mathbb{Z})$. We consider $\mathrm{GL}_n(\mathbb{R})$ as a subset of $\mathbb{R}^{n^2}$ in the natural way. Recall the definition of $\mathcal{F}$ (Definition 3.6.10).

Conjecture 3.12.2 (cf. [35], Proposition 6.7)**.** *There exist positive constants d, $c_{\mathcal{F}}$, and $\delta_{\mathcal{F}}$ such that, if $z \in \mathcal{F}$, then the smallest pre-special subvariety of X containing z can be written $gF(\mathbb{R})^+g^{-1}x$, where $F \in \Omega$, and $g \in G(\mathbb{R})$ and $x \in X$ satisfy*

$$\mathrm{H}_d(g,x) \leq c_{\mathcal{F}}\Delta(\langle\pi(z)\rangle)^{\delta_{\mathcal{F}}}.$$

This is seemingly a natural generalization of the following theorem due to Orr and the first author on the heights of pre-special points, which plays a crucial role in the proof of the André-Oort conjecture.

Theorem 3.12.3 (cf. [20], Theorem 1.4)**.** *There exist positive constants d, $c_{\mathcal{F}}$ and $\delta_{\mathcal{F}}$ such that, if $z \in \mathcal{F}$ is a pre-special, then*

$$\mathrm{H}_d(z) \leq c_{\mathcal{F}}\Delta(\pi(z))^{\delta_{\mathcal{F}}}.$$

We remark that the problem of finding d as in Conjecture 3.12.2 poses no obstacle in itself. Indeed a proof of the following theorem will appear in a forthcoming article of Borovoi and the authors.

Theorem 3.12.4 (cf. [10], Corollary 0.7)**.** *There exists a positive constant d such that, for any two semisimple subgroups H_1 and H_2 of G defined over $\mathbb{Q}$ that are conjugate by an element of $G(\mathbb{R})$, there exists a number field K contained in $\mathbb{R}$ of degree at most d, and an element $g \in G(K)$, such that*

$$H_{2,K} = gH_{1,K}g^{-1}.$$

A nice feature of Conjecture 3.12.2 is that it implies Conjecture 3.10.3 that there are only finitely many special subvarieties of bounded complexity.

Lemma 3.12.5. *Conjecture 3.12.2 implies Conjecture 3.10.3.*

Proof. Let Z be a special subvariety of S such that $\Delta(Z) \leq b$ and let $P \in Z$ be such that $\langle P \rangle = Z$. Let $z \in \mathcal{F}$ be such that $\pi(z) = P$ and let X_H be the smallest pre-special subvariety of X containing z. Then $\pi(X_H) = Z$ and, by Conjecture 3.12.2, $X_H = gFg^{-1}x$, where $F \in \Omega$, and $g \in G(\mathbb{R})$ and $x \in X$ satisfy

$$\mathrm{H}_d(g,x) \leq c_{\mathcal{F}} \Delta(Z)^{\delta_{\mathcal{F}}} \leq c_{\mathcal{F}} b^{\delta_{\mathcal{F}}}.$$

The claim follows, therefore, from the fact that there are only finitely many algebraic numbers of bounded degree and height i.e. Northcott's property.

□

Another, albeit longer, approach to our point counting problems can be given by replacing Conjecture 3.12.2 with two related conjectures, although we will have to additionally assume Conjecture 3.10.3 in this case. We will also rely on the fact that Theorem 3.9.1 is uniform in families. The advantage is that the following two conjectures are seemingly more accessible.

Conjecture 3.12.6. *For any $\kappa > 0$, there exists a positive constant c_κ such that, if Z is a special subvariety of S, then there exists a semisimple subgroup H of G defined over $\mathbb{Q}$ of non-compact type, and an extension L of F satisfying*

$$[L : F] \leq c_\kappa \Delta(Z)^\kappa,$$

such that, for any $\sigma \in \mathrm{Aut}(\mathbb{C}/L)$,

$$\sigma(Z) = \pi(H(\mathbb{R})^+ x_\sigma),$$

where $H(\mathbb{R})^+ x_\sigma$ is a pre-special subvariety of X intersecting $\mathcal{F}$.

Recall that, for an abelian variety A, defined over a field K, every abelian subvariety of A can be defined over a fixed, finite extension of K. The analogue of Conjecture 3.12.6 is, therefore, trivial. In a Shimura variety, one hopes that the degrees of fields of definition of strongly special subvarieties grow as in Conjecture 3.12.6. If this were true, Conjecture 3.12.6 for strongly special subvarieties would follow easily.

Our final conjecture is also inspired by the abelian setting.

Conjecture 3.12.7 (cf. [35], Lemma 3.2)**.** *There exist positive constants c_Γ and δ_Γ such that, if X_H is a pre-special subvariety of X intersecting $\mathcal{F}$ and $z \in \mathcal{F}$ belongs to ΓX_H, then $z \in \gamma X_H$, where $\gamma \in \Gamma$ satisfies*

$$\mathrm{H}_1(\gamma) \leq c_\Gamma \deg(\pi(X_H))^{\delta_\Gamma}.$$

Conjecture 3.12.7 has the following useful consequence.

Lemma 3.12.8. *Assume that Conjecture 3.12.7 holds.*

There exist positive constants d, c_{H}, and δ_{H} such that, if $H(\mathbb{R})^{+}x$ is a pre-special subvariety of X intersecting $\mathcal{F}$, then

$$H(\mathbb{R})^{+}x = H(\mathbb{R})^{+}y$$

where $\mathrm{H}_d(y) \leq c_{\mathrm{H}}\Delta(\pi(H(\mathbb{R})^{+}x))^{\delta_{\mathrm{H}}}$.

Proof. Let d, $c_{\mathcal{F}}$, and $\delta_{\mathcal{F}}$ be the positive constants afforded to us by Theorem 3.12.3, and let c_{Γ} and δ_{Γ} be the positive constants afforded to us by Conjecture 3.12.7.

Let $x' \in \Gamma H(\mathbb{R})^{+}x \cap \mathcal{F}$ denote a pre-special point such that $\pi(x')$ is of minimal complexity among the special points of $\pi(H(\mathbb{R})^{+}x)$. By Theorem 3.12.3, we have

$$\mathrm{H}_d(x') \leq c_{\mathcal{F}}\Delta(\pi(H(\mathbb{R})^{+}x))^{\delta_{\mathcal{F}}}.$$

On the other hand, by Conjecture 3.12.7, $x' \in \gamma H(\mathbb{R})^{+}x$, where

$$\mathrm{H}_1(\gamma) \leq c_{\Gamma}\Delta(\pi(H(\mathbb{R})^{+}x))^{\delta_{\Gamma}}.$$

It follows easily from the properties of heights that there exist positive constants c and δ depending only on the fixed data such that

$$\mathrm{H}_d(\gamma^{-1}x') \leq c\mathrm{H}_1(\gamma)^{\delta}\mathrm{H}_d(x')^{\delta}.$$

Therefore, the previous remarks show that

$$y := \gamma^{-1}x' \in H(\mathbb{R})^{+}x$$

satisfies the requirements of the lemma. □

We will now verify the arithmetic conjectures stated above in an arbitrary product of modular curves.

3.13 Products of modular curves

Our definition of a Shimura variety allows for the possibility that S might be a product of modular curves. In that case $G = \mathrm{GL}_2^n$, where n is the number of modular curves, and $\mathfrak{X}$ is the $G(\mathbb{R})$ conjugacy class of the morphism $\mathbb{S} \to G_{\mathbb{R}}$ given by

$$a+ib \mapsto \left(\begin{pmatrix} a & b \\ -b & a \end{pmatrix}, \ldots, \begin{pmatrix} a & b \\ -b & a \end{pmatrix}\right).$$

We let X denote the $G(\mathbb{R})^{+}$ conjugacy class of this morphism, which one identifies with the n-th cartesian power $\mathbb{H}^n$ of the upper half-plane $\mathbb{H}$.

For our purposes, we can and do suppose that Γ is equal to $\mathrm{SL}_2(\mathbb{Z})^n$ and we let $\mathcal{F}$ denote a fundamental set in X for the action of Γ, equal to the n-th cartesian power of a fundamental set $\mathcal{F}_{\mathbb{H}}$ in $\mathbb{H}$ for the action of $\mathrm{SL}_2(\mathbb{Z})$. Note that, as explained in [57], Section 1.3, we can and do choose $\mathcal{F}_{\mathbb{H}}$ in the image of a Siegel set. Via the j-function applied to each factor of $\mathbb{H}^n$, the quotient $\Gamma\backslash X$ is isomorphic to the algebraic variety $\mathbb{C}^n$. Special subvarieties have the following well-documented description.

Proposition 3.13.1 (cf. [26], Proposition 2.1)**.** *Let $I = \{1,\dots,n\}$. A subvariety Z of $\mathbb{C}^n$ is a special subvariety if and only if there exists a partition $\Omega = (I_1,\dots,I_t)$ of I, with $|I_i| = n_i$, such that Z is equal to the product of subvarieties Z_i of $\mathbb{C}^{n_i}$, where, either*

- *I_i is a one element set and Z_i is a special point, or*
- *Z_i is the image of $\mathbb{H}$ in $\mathbb{C}^{n_i}$ under the map sending $\tau \in \mathbb{H}$ to the image of $(g_j\tau)_{j\in I_i}$ in $\mathbb{C}^{n_i}$ for elements $g_j \in \mathrm{GL}_2(\mathbb{Q})^+$.*

First note that Conjecture 3.12.2 for $\mathbb{C}^n$ follows from Proposition 6.7 of [35]. Hence, we will now verify Conjecture 3.12.6 and Conjecture 3.12.7 in that setting.

Proof of Conjecture 3.12.6 for $\mathbb{C}^n$. Let Z be a special subvariety of $\mathbb{C}^n$, equal to a product of special subvarieties Z_i of $\mathbb{C}^{n_i}$, as above. Without loss of generality, we may assume that the product contains only one factor and, by Theorem 3.12.3, we may assume that it is not a special point. Therefore, Z is equal to the image of $\mathbb{H}$ in $\mathbb{C}^n$ under the map sending $\tau \in \mathbb{H}$ to the image of $(g_j\tau)_{j=1}^n$ in $\mathbb{C}^n$ for elements $g_j \in \mathrm{GL}_2(\mathbb{Q})^+$.

In other words, we have a morphism of Shimura data from $(\mathrm{GL}_2,\mathbb{H}^\pm)$ to $(G,\mathfrak{X})$, where $\mathbb{H}^\pm$ is the union of the upper and lower half-planes (or, rather, the conjugacy class we associate with it, as above), induced by the morphism

$$\mathrm{GL}_2 \to \mathrm{GL}_2^n : g \mapsto (g_j g g_j^{-1})_{j=1}^n,$$

such that Z is equal to the image of $\mathbb{H}\times\{1\}$ under the corresponding morphism

$$\mathrm{Sh}_K(\mathrm{GL}_2,\mathbb{H}^\pm) \to \mathrm{Sh}_{\mathrm{GL}_2(\hat{\mathbb{Z}})^n}(G,\mathfrak{X}), \tag{3.13.1}$$

where K is the product of the groups

$$K_p := g_1\mathrm{GL}_2(\mathbb{Z}_p)g_1^{-1}\cap\cdots\cap g_n\mathrm{GL}_2(\mathbb{Z}_p)g_n^{-1}$$

over all primes p.

Since (3.13.1) is defined over $E(\mathrm{GL}_2,\mathbb{H}^\pm) = \mathbb{Q}$, it suffices to bound the size of

$$\pi_0(\mathrm{Sh}_K(\mathrm{GL}_2,\mathbb{H}^\pm)),$$

which, by [52], Theorem 5.17, is in bijection with

$$\mathbb{Q}_{>0}\backslash\mathbb{A}_f^\times/\nu(K),$$

where ν is the determinant map on GL_2. However, since $\mathbb{A}_f^\times$ is equal to the direct product $\mathbb{Q}_{>0}\hat{\mathbb{Z}}^\times$, it suffices to bound the size of $\hat{\mathbb{Z}}^\times/\nu(K)$.

To that end, let Σ denote the (finite) set of primes p such that $g_j \notin \mathrm{GL}_2(\mathbb{Z}_p)$, for some $j \in \{1,\dots,n\}$. In particular,

$$[\hat{\mathbb{Z}}^\times : \nu(K)] = \prod_{p\in\Sigma} [\mathbb{Z}_p^\times : \nu(K_p)]$$

and, since K_p contains the elements $\mathrm{diag}(a,a)$, where $a \in \mathbb{Z}_p^\times$,

$$[\mathbb{Z}_p^\times : \nu(K_p)] \leq [\mathbb{Z}_p^\times : \mathbb{Z}_p^{\times 2}] \leq 4,$$

where $\mathbb{Z}_p^{\times 2}$ denotes the squares in $\mathbb{Z}_p^\times$.

On the other hand, by [18],

$$\deg(Z) \geq \prod_{p\in\Sigma} p,$$

and the conjecture follows easily from the following classical fact regarding primorials.

Lemma 3.13.2. *Let $n \in \mathbb{N}$. The product of the first n prime numbers is equal to*

$$e^{(1+o(1))n\log n}.$$

$\square$

Proof of Conjecture 3.12.7 for $\mathbb{C}^n$. Let X_H be a product of spaces $X_i \subseteq \mathbb{H}^{n_i}$ each equal to either a pre-special point or to the image of $\mathbb{H}$ given by the map sending τ to $(g_j\tau)_{j=1}^{n_i}$ for elements $g_j \in \mathrm{GL}_2(\mathbb{Q})^+$. Without loss of generality, we may assume that the product contains only one factor. If X_H is a pre-special point contained in $\mathcal{F}$, then the claim follows from the fact that

$$\{\gamma \in \Gamma : \gamma\mathcal{F} \cap \mathcal{F} \neq \emptyset\}$$

is finite. Therefore, assume that X is equal to the image of $\mathbb{H}$ in $\mathbb{H}^n$ given by the map sending τ to $(g_j\tau)_{j=1}^n$ for elements $g_j \in \mathrm{GL}_2(\mathbb{Q})^+$. We can and do assume that g_1 is equal to the identity element and that all of the g_j have coprime integer entries.

As in the statement of Conjecture 3.12.7, we assume that X intersects $\mathcal{F}$, and we let $x \in \mathcal{F} \cap X$. Therefore,

$$x = (g_j\tau_x)_{j=1}^n,$$

where $\tau_x \in \mathcal{F}_{\mathbb{H}}$. By [57], Theorem 1.2 (cf. [34], Lemma 5.2), $\mathrm{H}_1(g_j) \leq c_1 \det(g_j)^2$, for all $j \in \{1,\dots,n\}$, where c_1 is a positive constant not depending on Z. In particular,

$$\mathrm{H}_1(g_j^{-1}) \leq \det(g_j)\mathrm{H}_1(g_j) \leq c_1 \det(g_j)^3,$$

for each $j \in \{1,\dots,n\}$.

Now let $z := (z_j)_{j=1}^n \in \mathcal{F}$ be a point belonging to ΓX. For each $j \in \{1,\dots,n\}$,

$$z_j = \gamma_j g_j g \tau_x,$$

for some $g \in \mathrm{GL}_2(\mathbb{R})^+$ and some $\gamma_j \in \mathrm{SL}_2(\mathbb{Z})$. Therefore, let

$$\Lambda := \bigcap_{j=1}^{n} g_j^{-1}\mathrm{SL}_2(\mathbb{Z})g_j$$

and let $\mathcal{C}$ denote a set of representatives in $\mathrm{SL}_2(\mathbb{Z})$ for $\Lambda\backslash\mathrm{SL}_2(\mathbb{Z})$. Note that, if we define

$$m_j := \det(g_j),$$

then, for any multiple N of m_j, the principal congruence subgroup $\Gamma(N)$ is contained in $g_j^{-1}\mathrm{SL}_2(\mathbb{Z})g_j$. In particular, if we define N to be the lowest common multiple of the m_j, then $\Gamma(N)$ is contained in Λ. It follows that any subset of $\mathrm{SL}_2(\mathbb{Z})$ mapping bijectively to $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ contains a set $\mathcal{C}$, as above. Via the procedure outlined in [25], Exercise 1.2.2, it is straightforward to verify that we can (and do) choose $\mathcal{C}$ such that, for any $c \in \mathcal{C}$,

$$H(c) \leq 7N^5$$

(though we certainly do not claim that this is the best possible bound; any polynomial bound would suffice for our purposes).

The union

$$\bigcup_{c\in\mathcal{C}} c\mathcal{F}_{\mathbb{H}}$$

constitutes a fundamental set in $\mathbb{H}$ for the action of Λ. Hence, there exists $c \in \mathcal{C}$ and $\lambda_g \in \Lambda$ such that

$$c^{-1}\lambda_g g\tau_x \in \mathcal{F}_{\mathbb{H}}.$$

Furthermore, for each $j \in \{1,\ldots,n\}$, we can write $\lambda_g = g_j^{-1}\lambda_j g_j$, for some $\lambda_j \in \mathrm{SL}_2(\mathbb{Z})$ and, hence,

$$z_j = \gamma_j g_j g\tau_x = \gamma_j g_j \lambda_g^{-1} c \cdot c^{-1}\lambda_g g\tau_x = \gamma_j\lambda_j^{-1} g_j c \cdot c^{-1}\lambda_g g\tau_x. \tag{3.13.2}$$

Therefore, by [57], Theorem 1.2, we have $\mathrm{H}_1(\gamma_j\lambda_j^{-1}g_jc) \leq c_1 m_j^2$, for all $j \in \{1,\ldots,n\}$.

We write

$$\mathrm{H}_1(\gamma_j\lambda_j^{-1}) = \mathrm{H}_1(\gamma_j\lambda_j^{-1}g_jc\cdot c^{-1}g_j^{-1}) \leq c_2\mathrm{H}_1(\gamma_j\lambda_j^{-1}g_jc)^{\delta}\mathrm{H}_1(c^{-1})^{\delta}\mathrm{H}_1(g_j^{-1})^{\delta},$$

where c_2 and δ are positive constants not depending on Z, and we obtain

$$\mathrm{H}_1(\gamma_j\lambda_j^{-1}) \leq c_3 N^{10\delta},$$

for each $j \in \{1,\ldots,n\}$, where c_3 is a positive constant not depending on Z.

Conversely, by [18], §2,

$$\deg(Z) \geq [\mathrm{SL}_2(\mathbb{Z}) : \Lambda]$$

and, by writing the g_j in Smith normal form i.e.

$$g_j = \gamma_{j,1} \begin{pmatrix} 1 & 0 \\ 0 & m_j \end{pmatrix} \gamma_{j,2},$$

where $\gamma_{j,1}, \gamma_{j,2} \in \mathrm{SL}_2(\mathbb{Z})$, we conclude that

$$\mathrm{SL}_2(\mathbb{Z}) \cap g_j^{-1} \mathrm{SL}_2(\mathbb{Z}) g_j = \gamma_{j,2}^{-1} \Gamma_0(m_j) \gamma_{j,2},$$

whose index in $\mathrm{SL}_2(\mathbb{Z})$ is the same as $\Gamma_0(m_j)$, which is $m_j \varphi(m_j)$. It follows that

$$[\mathrm{SL}_2(\mathbb{Z}) : \Lambda] \geq N,$$

and so

$$\mathrm{H}_1(\gamma_j \lambda_j^{-1}) \leq c_3 \deg(Z)^{10\delta}.$$

Therefore, we let $\gamma := (\gamma_j \lambda_j^{-1})_{j=1}^n \in \Gamma$. By (3.13.2), $z \in \gamma X$, and the result follows.

□

Finally, we will verify Conjecture 3.10.4 in this case.

Proof of Conjecture 3.10.4 for $\mathbb{C}^n$. Let Z be a special subvariety of $\mathbb{C}^n$. Then Z is a product of special subvarieties. Since there are only finitely many partitions of $\{1, \ldots, n\}$, we may assume that the product contains only one factor. If Z is a special point, H^{der} is trivial. Therefore, we assume that Z is equal to the image of $\mathbb{H}$ in $\mathbb{C}^n$ under the map sending $\tau \in \mathbb{H}$ to the image of $(g_j \cdot \tau)_{j=1}^n$ in $\mathbb{C}^n$ for elements $g_j \in \mathrm{GL}_2(\mathbb{Q})^+$. Then H^{der} is the image of SL_2 under the morphism

$$\mathrm{SL}_2 \to \mathrm{GL}_2^n : g \mapsto (g_j g g_j^{-1})_{j=1}^n.$$

We see from the calculations in the previous proof that the bound $\deg(Z) \leq b$ implies that the g_j come from the union of finitely many double cosets $\Gamma g \Gamma$ for $g \in \mathrm{GL}_2(\mathbb{Q})^+$. Since each such double coset is equal to a finite union of single cosets Γh for $h \in \mathrm{GL}_2(\mathbb{Q})^+$, the result follows.

□

3.14 Main results (part 2): Conditional solutions to the counting problems

We conclude by demonstrating how our arithmetic conjectures might be used to resolve the counting problems stated in Section 3.8. In our applications of the counting theorem, we will need the following.

Lemma 3.14.1. *Let $\beta : [0,1] \to G(\mathbb{R}) \times X$ be semi-algebraic. Then $\Im(\beta)$ is contained in a complex algebraic subset B of $G(\mathbb{C}) \times \mathbb{C}^N$ of dimension at most* 1.

Proof. Let Y denote the real Zariski closure of $\mathfrak{I}(\beta)$ in $G(\mathbb{R})\times\mathbb{R}^{2N}$. In particular, $\dim Y \leq 1$. Without loss of generality, we can and do assume that Y is irreducible. If Y is a point then there is nothing to prove. Therefore, we can and do assume that Y is an irreducible real algebraic curve. In particular, the complexification $Y_{\mathbb{C}}$ of Y in $G(\mathbb{C})\times\mathbb{C}^{2N}$ is an irreducible complex algebraic curve.

Let $g_1,\ldots,g_{n^2},x_1,y_1,\ldots,x_N,y_N$ denote the real coordinate functions on $G(\mathbb{R})\times X$ and let $z_j = x_j + iy_j$ denote the coordinate functions on $\mathbb{C}^N = \mathbb{R}^{2N}$. If all of the coordinates functions on $\mathbb{R}^{2N}$ are constant on Y, the result is obvious. Therefore, without loss of generality, we can and do assume that x_1 is not constant on Y.

We claim that each of the coordinate functions $x_2,y_2,\ldots,x_N,y_N$ on $\mathbb{C}^{2N}$ is algebraic over the field $\mathbb{C}(z_1)$, considered as a field of functions on $Y_{\mathbb{C}}$. To see this, note that z_1 is non-constant on $Y_{\mathbb{C}}$, and so $\mathbb{C}(z_1)$ has transcendence degree at least 1. On the other hand, $\mathbb{C}(z_1)$ is contained in $\mathbb{C}(x_1,y_1)$, which is algebraic over $\mathbb{C}(x_1)$.

In particular, each of the functions $x_2+iy_2,\ldots,x_N+iy_N$ is algebraic over the field $\mathbb{C}(z_1)$. It follows that, for each $j\geq 2$, there exists a polynomial $f_j(z_1,z_j)\in\mathbb{C}[z_1,z_j]$, non-trivial in z_j, such that $f_j(z_1,z_j)=0$ on Y. Similarly, for each $k=1,\ldots,n^2$, there exists a polynomial $f_j(z_1,g_k)\in\mathbb{C}[z_1,g_k]$, non-trivial in g_k, such that $f_k(z_1,g_k)=0$ on Y. In particular, Y is contained in the vanishing locus of the f_j and the f_k, which define a complex algebraic curve in $G(\mathbb{C})\times\mathbb{C}^N$. $\square$

We denote by $X^\vee$ the compact dual of X, which is a complex algebraic variety on which $G(\mathbb{C})$ acts via an algebraic morphism

$$G(\mathbb{C})\times X^\vee \to X^\vee.$$

Furthermore, X naturally embeds into $X^\vee$ and the embedding factors through an embedding of $\mathbb{C}^N$ i.e. the Harish-Chandra realization, into $X^\vee$. We could have defined subvarieties of X using $X^\vee$ in the place of $\mathbb{C}^N$ but, as mentioned previously, the two notions coincide. If we have a decomposition $G^{\mathrm{ad}} = G_1\times G_2$, and thus $X = X_1\times X_2$, we have a natural decomposition

$$X^\vee = X_1^\vee \times X_2^\vee.$$

Furthermore, if $(H,\mathfrak{X}_H)$ denotes a Shimura subdatum of $(G,\mathfrak{X})$ and X_H is a connected component of $\mathfrak{X}_H$ contained in X, then $X_H^\vee$ is naturally contained in $X^\vee$. We refer the reader to [82], Section 3 for more details.

Theorem 3.14.2. *Assume that Conjecture 3.11.1 holds and assume that either*

- *Conjecture 3.12.2 holds or*
- *Conjectures 3.10.3, 3.12.6, and 3.12.7 hold.*

Then Conjecture 3.8.2 is true for curves i.e. if V is a curve contained in S, then the set $\mathrm{Opt}_0(V)$ *is finite.*

Proof. We will assume that Conjecture 3.12.2 holds. The proof in the case that Conjectures 3.10.3, 3.12.6, and 3.12.7 hold is very similar, hence we omit it. To elucidate the use of Conjectures 3.10.3, 3.12.6, and 3.12.7 we will use them in the proof of Theorem

3.14.3, at the expense of making the proof longer. We suffer no loss of generality if we assume, as we will, that V is Hodge generic.

Let Ω denote a finite set of semisimple subgroups of G defined over $\mathbb{Q}$ as in Section 3.12 and let d, $c_{\mathcal{F}}$, and $\delta_{\mathcal{F}}$ be the constants afforded to us by Conjecture 3.12.2. Let L be a finitely generated extension of F contained in $\mathbb{C}$ over which V is defined and let c_G and δ_G be the constants afforded to us by Conjecture 3.11.1. Let $\kappa := 2\delta_G/3\delta_{\mathcal{F}}$.

We claim that there exists a positive constant c such that, for any $P \in \mathrm{Opt}_0(V)$, we have

$$\#\mathrm{Aut}(\mathbb{C}/L)\cdot P \leq c c_{\mathcal{F}}^{\kappa}\Delta(\langle P\rangle)^{\kappa\delta_{\mathcal{F}}}.$$

This would be sufficient to prove Theorem 3.14.2 since then, by Conjecture 3.11.1, we obtain

$$c_G\Delta(\langle P\rangle)^{\delta_G} \leq \#\mathrm{Aut}(\mathbb{C}/L)\cdot P \leq c c_{\mathcal{F}}^{\kappa}\Delta(\langle P\rangle)^{\kappa\delta_{\mathcal{F}}}$$

and, rearranging this expression, we obtain

$$\Delta(\langle P\rangle) \leq (c c_{\mathcal{F}}^{\kappa} c_G^{-1})^{3/\delta_G},$$

which is a bound independent of P. We remind the reader that P is one of only finitely many irreducible components of $\langle P\rangle \cap V$. Hence, Theorem 3.14.2 would follow from Lemma 3.12.5 and, therefore, it remains only to prove the claim.

To that end, for each $\sigma \in \mathrm{Gal}(\mathbb{C}/L)$, let $z_\sigma \in \mathcal{V}$ be a point in $\pi^{-1}(\sigma(P))$. Therefore, by Conjecture 3.12.2, the smallest pre-special subvariety of X containing z_σ can be written $g_\sigma F_\sigma(\mathbb{R})^+ g_\sigma^{-1} x_\sigma$, where $F_\sigma \in \Omega$, and $g_\sigma \in G(\mathbb{R})$ and $x_\sigma \in X$ satisfy

$$\mathrm{H}_d(g_\sigma, x_\sigma) \leq c_{\mathcal{F}}\Delta(\langle\sigma(P)\rangle)^{\delta_{\mathcal{F}}} = c_{\mathcal{F}}\Delta(\langle P\rangle)^{\delta_{\mathcal{F}}}.$$

Without loss of generality, we can and do assume that $F := F_\sigma$ is fixed. Therefore, for each $\sigma \in \mathrm{Gal}(\mathbb{C}/L)$, the tuple $(g_\sigma, x_\sigma, z_\sigma)$ belongs to the definable set D of tuples

$$(g,x,z) \in G(\mathbb{R}) \times X \times X \subseteq \mathbb{R}^{n^2+2N} \times \mathbb{R}^{2N},$$

such that $z \in \mathcal{V} \cap gF(\mathbb{R})^+ g^{-1} x$ and $x(\mathbb{S}) \subseteq gG_F g^{-1}$. We consider D as a family over a point in an omitted parameter space and choose for c the constant $c(D,d,\kappa)$ afforded to us by Theorem 3.9.1 applied to D. Since Ω is finite, we can and do assume that c does not depend on F. We let Σ denote the union over $\mathrm{Aut}(\mathbb{C}/L)$ of the tuples $(g_\sigma, x_\sigma, z_\sigma) \in D$. In particular, Σ is contained in the subset

$$D(d, c_{\mathcal{F}}\Delta(\langle P\rangle)^{\delta_{\mathcal{F}}}).$$

Let π_1 and π_2 be the projection maps from $\mathbb{R}^{n^2+2N} \times \mathbb{R}^{2N}$ to $\mathbb{R}^{n^2+2N}$ and $\mathbb{R}^{2N}$, respectively, and suppose, for the sake of obtaining a contradiction, that

$$\#\mathrm{Aut}(\mathbb{C}/L)\cdot P = \#\pi_2(\Sigma) > c c_{\mathcal{F}}^{\kappa}\Delta(\langle P\rangle)^{\kappa\delta_{\mathcal{F}}}.$$

Then, by Theorem 3.9.1, there exists a continuous definable function

$$\beta : [0,1] \to D,$$

such that $\beta_1 := \pi_1 \circ \beta$ is semi-algebraic, $\beta_2 := \pi_2 \circ \beta$ is non-constant, $\beta(0) \in \Sigma$, and $\beta_{|(0,1)}$ is real analytic. Let $z_0 := \beta_2(0)$ and let $P_0 := \pi(z_0)$. To obtain a contradiction, we will closely imitate arguments found in [56].

It follows from the Global Decomposition Theorem (see [32], p172) that there exists $0 < t \leq 1$ such that $\beta_2([0,t))$ intersects only finitely many of the irreducible analytic components of $\pi^{-1}(V)$. In fact, since $\beta_{2|(0,t)}$ is real analytic, $\beta_2((0,t))$ must be wholly contained in one such component V_1. Since V_1 is closed, we conclude from the fact that β is continuous that V_1 contains $\beta_2([0,t])$.

By [81], Theorem 1.3 (the inverse Ax-Lindemann conjecture), $\langle V_1 \rangle_{\mathrm{Zar}}$ is pre-weakly special and so, since V is Hodge generic in S, we can decompose $G^{\mathrm{ad}} = G_1 \times G_2$, and thus $X = X_1 \times X_2$, so that

$$\langle V_1 \rangle_{\mathrm{Zar}} = X_1 \times \{x_2\},$$

where $x_2 \in X_2$ is Hodge generic. By abuse of notation, we denote by π_2 both the projection from G to G_2 and from $X^\vee$ to $X_2^\vee$.

Note that, for any $(g,x) \in \mathfrak{I}(\beta_1)$, we have $(g^{-1}x)(\mathbb{S}) \subseteq G_{F,\mathbb{R}}$. If we write G'_F for the largest normal subgroup of G_F of non-compact type, then the properties of Shimura data imply that $g^{-1}x$ factors through $G'_{F,\mathbb{R}}$ and, if we write $\mathfrak{X}'$ for the $G'_F(\mathbb{R})$ conjugacy class of $g^{-1}x$ in $\mathfrak{X}$, then, by [79], Lemme 3.3, $(G'_F, \mathfrak{X}')$ is a Shimura subdatum of $(G, \mathfrak{X})$. Furthermore, by [83], Lemma 3.7, the number of Shimura subdata $(G'_F, \mathfrak{Y})$ of $(G, \mathfrak{X})$ is finite and, by [52], Corollary 5.3, the number of connected components Y of $\mathfrak{Y}$ is also finite. It follows that, after possibly replacing t, we can and do assume that $g^{-1}x$ belongs to one such component Y, which we write as $Y_1 \times Y_2$, such that $F(\mathbb{R})^+$ acts transitively on Y_1. In particular,

$$\dim Y_1 = \dim \langle P_0 \rangle.$$

We let p_2 denote the projection from $Y^\vee$ to $Y_2^\vee$.

Let B denote the complex algebraic subset of $G(\mathbb{C}) \times X^\vee$ of dimension at most 1 containing $\mathfrak{I}(\beta_1)$ afforded to us by Lemma 3.14.1. For any $(g,x) \in B$, we have $g^{-1}x \in Y^\vee$.

Let $\overline{V}_1$ denote the Zariski closure of V_1 in $X^\vee$ and consider the complex algebraic set

$$W_B := \{(g,x,y) \in B \times Y^\vee : p_2(y) = p_2(g^{-1}x),\ gy \in \overline{V}_1\}.$$

Let V_B denote the Zariski closure in $X^\vee$ of the set

$$\{gy : (g,x,y) \in W_B\}.$$

Since the latter is the image of W_B under an algebraic morphism, we have $\dim V_B \leq \dim W_B$.

Since V_1 is an irreducible complex analytic curve having uncountable intersection with V_B, it follows that V_1 is contained in V_B. Therefore, $\langle V_1\rangle_{\mathrm{Zar}}$ is contained in V_B also, and so

$$\dim X_1 = \dim\langle V_1\rangle_{\mathrm{Zar}} \leq \dim V_B \leq \dim W_B. \tag{3.14.1}$$

Now, for each $(g,x) \in B$, consider the fibre $W_{(g,x)}$ of W_B over (g,x) i.e. the set

$$\{y \in Y^\vee : p_2(y) = p_2(g^{-1}x),\ \pi_2(y) = \pi_2(g)^{-1}x_2\}.$$

Since $P_0 \in V$, it follows that $\pi_2(F) = G_2$ and so, for any $y \in Y_2^\vee$, the natural projection

$$Y_1^\vee \times \{y\} \to X_2^\vee$$

is an equivariant morphism of $F(\mathbb{C})$-homogeneous spaces. In particular, its fibres are equidimensional of dimension

$$\dim Y_1^\vee - \dim X_2^\vee = \dim Y_1 - \dim X_2.$$

Since $W_{(g,x)}$ is contained in such a fibre, we have

$$\dim W_{(g,x)} \leq \dim Y_1 - \dim X_2 \leq \dim X - 2 - \dim X_2 = \dim X_1 - 2,$$

where we use the fact that $P_0 \in \mathrm{Opt}_0(V)$, hence,

$$\dim Y_1 = \delta(P_0) \leq \delta(V) - 1 = \dim X - 2.$$

Since this holds for all $(g,x) \in B$ and $\dim B \leq 1$, we conclude that

$$\dim W_B \leq \dim X_1 - 1,$$

which contradicts (3.14.1). □

Of course, Theorem 3.14.2 is not really satisfactory in the sense that it only deals with curves. One would hope that, for V of arbitrary dimension, a path such as β would yield, via the weak hyperbolic Ax-Schanuel conjecture, a positive dimensional subvariety of V, containing a conjugate of P, having defect at most $\delta(P)$, thus contradicting the optimality of P. However, the authors haven't been able to carry out this procedure. Instead, the very same idea appears to work when one attempts to contradict the membership of a point in the open-anomalous locus. The difference is that we are only required to bound the weakly special defect, as opposed to the defect itself.

Theorem 3.14.3. *Assume that Conjecture 3.11.3 holds and assume that the weak hyperbolic Ax-Schanuel conjecture is true. Assume also that, either*

- *Conjecture 3.12.2 holds, or*
- *Conjectures 3.10.3, 3.12.6, and 3.12.7 hold.*

Then, Conjecture 3.8.4 is true i.e. if V is a subvariety of S, then the set

$$V^{\mathrm{oa}} \cap S^{[1+\dim V]}$$

is finite.

Proof. We will assume that Conjectures 3.10.3, 3.12.6, and 3.12.7 hold. The proof in the case that Conjecture 3.12.2 holds is very similar, hence we omit it. We used Conjecture 3.12.2 in the proof of Theorem 3.14.2.

Let Ω denote a finite set of semisimple subgroups of G defined over $\mathbb{Q}$ as in Section 3.12. Let c_Γ and δ_Γ be the constants afforded to us by Conjecture 3.12.7, let d, c_{H}, and δ_{H} be the constants afforded to us by Lemma 3.12.8, and let

$$c := \max\{c_{\mathrm{H}}, c_\Gamma\} \text{ and } \delta := \max\{\delta_{\mathrm{H}}, \delta_\Gamma\}.$$

Let L' be a finitely generated extension of F contained in $\mathbb{C}$ over which V is defined and let c_G and δ_G be the constants afforded to us by Conjecture 3.11.1. Let $\kappa := \delta_G/3$, and let c_κ be the constant afforded to us by Conjecture 3.12.6. Let

$$P \in V^{\mathrm{oa}} \cap S^{[1+\dim V]}$$

and let L and H be, respectively, the finite field extension of F and the semisimple subgroup of G defined over $\mathbb{Q}$ of non-compact type afforded to us by Conjecture 3.12.6 applied to $\langle P \rangle$. Replacing L by its compositum with L', we have

$$[L : L'] \leq c_\kappa \Delta(\langle P \rangle)^\kappa.$$

We claim that there exists a positive constant c_3, independent of P, such that

$$\#\mathrm{Aut}(\mathbb{C}/L) \cdot P \leq c_3 c^{\frac{\kappa}{\delta}} \Delta(\langle P \rangle)^\kappa.$$

This would be sufficient to prove Theorem 3.14.3 since then, by Conjecture 3.11.3, we obtain

$$\frac{c_G}{c_\kappa} \Delta(\langle P \rangle)^{2\kappa} \leq \frac{1}{[L : L']} \#\mathrm{Aut}(\mathbb{C}/L') \cdot P = \#\mathrm{Aut}(\mathbb{C}/L) \cdot P \leq c_3 c^{\frac{\kappa}{\delta}} \Delta(\langle P \rangle)^\kappa$$

and, rearranging this expression, we obtain

$$\Delta(\langle P \rangle) \leq (c_3 c^{\frac{\kappa}{\delta}} c_\kappa c_G^{-1})^{\frac{1}{\kappa}},$$

which is a bound independent of P. We remind the reader that, as explained in Remark 3.11.4, $P \in \mathrm{Opt}_0(V)$ and, therefore, P is one of only finitely many irreducible components of $\langle P \rangle \cap V$. Hence, Theorem 3.14.3 would follow from Conjecture 3.10.3 and it remains only, therefore, to prove the claim.

By Conjecture 3.12.6, for each $\sigma \in \mathrm{Aut}(\mathbb{C}/L)$,

$$\sigma(\langle P \rangle) = \pi(H(\mathbb{R})^+ x_\sigma),$$

where $H(\mathbb{R})^+ x_\sigma$ is a pre-special subvariety of X intersecting $\mathcal{F}$. By Lemma 3.12.8, we can and do assume that

$$\mathrm{H}_d(x_\sigma) \leq c_{\mathrm{H}} \Delta(\sigma(\langle P \rangle))^{\delta_{\mathrm{H}}} = c_{\mathrm{H}} \Delta(\langle P \rangle)^{\delta_{\mathrm{H}}}.$$

We let $z_\sigma \in \mathcal{V}$ be a point in $\pi^{-1}(\sigma(P))$, so that

$$z_\sigma \in \Gamma H(\mathbb{R})^+ x_\sigma$$

and so, by Conjecture 3.12.7, there exists $\gamma_\sigma \in \Gamma$ satisfying

$$\mathrm{H}_1(\gamma_\sigma) \leq c_\Gamma \Delta(\langle P \rangle)^{\delta_\Gamma}$$

such that $z_\sigma \in \gamma_\sigma H(\mathbb{R})^+ x_\sigma$.

By definition, there exists $F \in \Omega$ and $g \in G(\mathbb{R})$ such that $H_\mathbb{R}$ is equal to $gF_\mathbb{R}g^{-1}$. In particular, for each $\sigma \in \mathrm{Aut}(\mathbb{C}/L)$, the tuple $(g,(\gamma_\sigma,x_\sigma),z_\sigma)$ belongs to the definable family D of tuples

$$(g,(\gamma,x),z) \in G(\mathbb{R}) \times [G(\mathbb{R}) \times X] \times X \subseteq \mathbb{R}^{n^2} \times \mathbb{R}^{n^2+2N} \times \mathbb{R}^{2N},$$

parametrised by $G(\mathbb{R})$, such that

$$z \in \mathcal{V} \cap \gamma g F(\mathbb{R})^+ g^{-1} x, \text{ and } x(\mathbb{S}) \subseteq g G_F g^{-1}.$$

We choose, then, for c_3 the constant $c(D,d,\kappa/\delta)$ afforded to us by Theorem 3.9.1 applied to D. Since, Ω is finite, we can and do assume that c_3 does not depend on F. We let Σ denote the union over $\mathrm{Aut}(\mathbb{C}/L)$ of the tuples $((\gamma_\sigma,x_\sigma),z_\sigma) \in D_g$ (to use the notation of Section 3.9). In particular, Σ is contained in the subset

$$D_g(d, c\Delta(Z)^\delta).$$

Let π_1 and π_2 be the projection maps from $\mathbb{R}^{n^2+2N} \times \mathbb{R}^{2N}$ to $\mathbb{R}^{n^2+2N}$ and $\mathbb{R}^{2N}$, respectively, and suppose, for the sake of obtaining a contradiction, that

$$\#\mathrm{Aut}(\mathbb{C}/L) \cdot P = \#\pi_2(\Sigma) > c_3 c^{\frac{\kappa}{\delta}} \Delta(Z)^\kappa.$$

Then, by Theorem 3.9.1, there exists a continuous definable function

$$\beta : [0,1] \to D_g,$$

such that $\beta_1 := \pi_1 \circ \beta$ is semi-algebraic, $\beta_2 := \pi_2 \circ \beta$ is non-constant, $\beta(0) \in \Sigma$, and $\beta_{|(0,1)}$ is real analytic. Let $z_0 := \beta_2(0)$ and $(\gamma_0, x_0) := \beta_1(0)$. Denote by P_0 the point $\pi(z_0)$ and denote by X_0 the pre-special subvariety $\gamma_0 H(\mathbb{R})^+ x_0$.

We claim that there exists a positive dimensional intersection component A of $\pi^{-1}(V)$ containing z_0. To see this, let W denote the union of the totally geodesic subvarieties $\gamma H(\mathbb{R})^+ x$ of X, where (γ,x) varies over $\Im(\beta_1)$, and let $\overline{W}$ denote the Zariski closure of W in $X^\vee$. The irreducible analytic components of $\overline{W} \cap \pi^{-1}(V)$ are, by definition, intersection components of $\pi^{-1}(V)$. It follows from the Global Decomposition Theorem (see [32], p172) that there exists $0 < t \leq 1$ such that $\beta_2([0,t))$ intersects only finitely many of said components. In fact, since $\beta_{2|(0,t)}$ is real analytic, $\beta_2((0,t))$ must be wholly contained in one such component A. Since A is closed, we conclude from the fact that β is continuous that A contains $\beta_2([0,t])$, which proves the claim.

Let B denote a Zariski optimal intersection component of $\pi^{-1}(V)$ containing A such that

$$\delta_{\text{Zar}}(B) \leq \delta_{\text{Zar}}(A),$$

and let Z denote the Zariski closure of $\pi(B)$ in S. By the weak hyperbolic Ax-Schanuel conjecture, $\langle B\rangle_{\text{Zar}}$ is pre-weakly special and, as in the proof of Lemma 3.6.8,

$$\langle Z\rangle_{\text{ws}} = \pi(\langle B\rangle_{\text{Zar}}).$$

Therefore, we have $\dim Z \geq 1$ and, also,

$$\dim Z \geq \dim B \geq \dim\langle B\rangle_{\text{Zar}} - \delta_{\text{Zar}}(A) \geq \dim\langle Z\rangle_{\text{ws}} - (\dim \overline{W} - 1),$$

where we use the fact that $\delta_{\text{Zar}}(A)$ is at most $\dim \overline{W} - 1$. We claim that $\dim \overline{W} - 1 \leq \dim X_0$, which would conclude the proof as

$$\dim X_0 \leq \dim S - \dim V - 1$$

and this would imply that $Z \in \text{an}(V)$, which is not allowed as $P_0 \in Z$.

Therefore, it remains to prove the claim. However, this is easy to prove working with complex duals and using the methods explained in the proof of Theorem 3.14.2.

□

3.15 A brief note on special anomalous subvarieties

In their paper [7], Bombieri, Masser, and Zannier also defined what they referred to as a torsion anomalous subvariety. We will make the analogous definition in the context of Shimura varieties. Let V be a subvariety of S.

Definition 3.15.1. A subvariety W of V is called **special anomalous** in V if

$$\dim W \geq \max\{1, 1 + \dim V + \dim\langle W\rangle - \dim S\}.$$

A subvariety of V is **maximal special anomalous** in V if it is special anomalous in V and not strictly contained in another subvariety of V that is also special anomalous in V.

The similarity with the definition of an atypical subvariety is clear. Indeed, it is immediate that a positive dimensional subvariety of V that is atypical with respect to V is special anomalous in V. However, since a subvariety W of V that is special anomalous in V is not necessarily an irreducible component of $V \cap \langle W\rangle$, it is not necessarily the case that W is atypical with respect to V. Nonetheless, it follows that the properties of being maximal special anomalous and atypical are equivalent for positive dimensional subvarieties of V. In particular, Conjecture 3.1.4 implies the following anologue of the Torsion Openness conjecture of Bombieri, Masser, and Zannier.

Conjecture 3.15.2. *There are only finitely many subvarieties of V that are maximal special anomalous in V.*

Of course, one would more naturally translate the Torsion Openness conjecture as follows.

Conjecture 3.15.3. *The complement V^{sa} in V of the subvarieties of V that are special anomalous in V is open in V.*

However, these two formulations are equivalent. Indeed, the statement that Conjecture 3.15.2 implies Conjecture 3.15.3 is obvious. On the other hand, suppose that Conjecture 3.15.3 were true. Then the union of all subvarieties of V that are positive dimensional and atypical with respect to V would be closed in V. In particular, we could write it as a finite union of subvarieties of V. However, since there are only countably many subvarieties of V that are atypical with respect to V, it follows that each member of the aforementioned union would be atypical with respect to V.

Of course, if one could prove Conjecture 3.15.2, one would reduce the Zilber-Pink conjecture to the following analogue of the Torsion Finiteness conjecture of Bombieri, Masser, and Zannier.

Conjecture 3.15.4. *There are only finitely many points in V that are maximal atypical with respect to V.*

In this article, we have concerned ourselves with optimal subvarieties. Now, it is straightforward to verify that a subvariety W of V that is optimal in V is atypical with respect to V. However, it is not necessarily the case that W is maximal atypical. On the other hand, a subvariety of V that is maximal atypical with respect to V is optimal. In particular, the points in V that are maximal atypical with respect to V constitute a (possibly proper) subset of $\mathrm{Opt}_0(V)$.

Again, it would be more natural to translate the Torsion Finiteness conjecture as follows.

Conjecture 3.15.5. *For any integer d, let $S^{[d]}$ denote the union of the special subvarieties contained in S having codimension at least d. Then*

$$V^{\mathrm{sa}} \cap S^{[1+\dim V]}$$

is finite. Equivalently, there are only finitely many points $P \in V^{\mathrm{sa}}$ such that

$$\dim\langle P\rangle \leq \dim S - \dim V - 1.$$

However, the two formulations are also equivalent. Indeed, Conjecture 3.15.5 implies Conjecture 3.15.4 because a point $P \in V$ that is maximal atypical with respect to V is contained in V^{sa} and

$$\dim\langle P\rangle \leq \dim S - \dim V - 1.$$

On the other hand, suppose that Conjecture 3.15.4 were true and consider a point $P \in V^{\mathrm{sa}}$ such that

$$\dim\langle P\rangle \leq \dim S - \dim V - 1.$$

Then P is a component of $\langle P\rangle \cap V$. Otherwise, such a component W containing P would be special anomalous in V, which would contradict the fact that $P \in V^{\mathrm{sa}}$. Therefore, P is atypical with respect to V and, in fact, maximal atypical with respect to V.

In their article [7], Bombieri, Masser, and Zannier showed that, in fact, the Torsion Openness conjecture implies the Torsion Finiteness conjecture. We imitate their argument to show the following.

Proposition 3.15.6. *Let $Y(1)$ denote the modular curve associated with $\mathrm{SL}_2(\mathbb{Z})$. If Conjecture 3.15.3 is true for $S \times Y(1)$, then Conjecture 3.15.5 is true for S.*

Proof. We denote by U the complement in V^{sa} of $V^{\mathrm{sa}} \cap S^{[\dim V+1]}$. Regarding $V \times Y(1)$ as a subvariety of the Shimura variety $S \times Y(1)$, we claim that

$$(V \times Y(1))^{\mathrm{sa}} = U \times Y(1). \tag{3.15.1}$$

To see this, first let $\hat{Y}$ be a special anomalous subvariety of $V \times Y(1)$. Then

$$\begin{aligned}
\dim \hat{Y} &\geq 1 + \dim\langle\hat{Y}\rangle + \dim(V \times Y(1)) - \dim(S \times Y(1)) \\
&= 1 + \dim\langle\hat{Y}\rangle + (1 + \dim V) - (1 + \dim S) \\
&= 1 + \dim\langle\hat{Y}\rangle + \dim V - \dim S.
\end{aligned}$$

We denote by π_S the projection $\pi_S : S \times Y(1) \to S$. Then the Zariski closure Y of $\pi_S(\hat{Y})$ lies in the special subvariety $\pi_S(\langle\hat{Y}\rangle)$ of S. If $\dim Y = \dim \hat{Y}$, then

$$\begin{aligned}
\dim Y &= \dim \hat{Y} \\
&\geq 1 + \dim\langle\hat{Y}\rangle + \dim V - \dim S \\
&\geq 1 + \dim \pi_S(\langle\hat{Y}\rangle) + \dim V - \dim S \\
&\geq 1 + \dim\langle Y\rangle + \dim V - \dim S
\end{aligned}$$

and Y is special anomalous in V. If $1 \leq \dim Y < \dim \hat{Y}$, then $\dim Y = \dim \hat{Y} - 1$ and $\hat{Y} = Y \times Y(1)$, which implies $\langle\hat{Y}\rangle = \langle Y\rangle \times Y(1)$. Therefore,

$$\begin{aligned}
\dim Y &= \dim \hat{Y} - 1 \\
&\geq 1 + \dim\langle\hat{Y}\rangle + \dim V - \dim S - 1 \\
&= 1 + \dim\langle Y\rangle + 1 + \dim V - \dim S - 1 \\
&= 1 + \dim\langle Y\rangle + \dim V - \dim S
\end{aligned}$$

and Y is special anomalous in V. In both cases, $\hat{Y}$ is contained in

$$Y \times Y(1) \subseteq (V \backslash V^{\mathrm{sa}}) \times Y(1) \subseteq (V \backslash U) \times Y(1) = (V \times Y(1)) \backslash (U \times Y(1)).$$

Finally, if $\dim Y = 0$, then $\hat{Y} = \{P\} \times Y(1)$ for some $P \in S$. Since $\hat{Y}$ is special anomalous, we have

$$\begin{aligned}
1 &= \dim \hat{Y} \\
&\geq 1 + \dim\langle \hat{Y} \rangle + (1 + \dim V) - (1 + \dim S) \\
&= 1 + \dim\langle P \rangle + 1 + \dim V - \dim S
\end{aligned}$$

and $P \in S^{[1+\dim V]}$. Therefore

$$\begin{aligned}
\hat{Y} &\subseteq V \backslash (V - S^{[1+\dim V]}) \times Y(1) \\
&\subseteq V \backslash (V^{\mathrm{sa}} - S^{[1+\dim V]}) \times Y(1) \\
&= (V \backslash U) \times Y(1) \\
&= (V \times Y(1)) \backslash (U \times Y(1)).
\end{aligned}$$

We conclude that $(V \times Y(1))^{\mathrm{sa}} \subseteq (U \times Y(1))$.

On the other hand, for any $P \in V \backslash U = V \backslash (V^{\mathrm{sa}} - S^{[1+\dim V]})$, we have either $P \notin V^{\mathrm{sa}}$ or $P \in V^{\mathrm{sa}} \cap S^{[1+\dim V]}$.

If $P \notin V^{\mathrm{sa}}$, then P is contained in a special anomalous Y of V. Therefore,

$$\begin{aligned}
\dim(Y \times Y(1)) &= 1 + \dim Y \\
&\geq 1 + 1 + \dim\langle Y \rangle + \dim V - \dim S \\
&= 1 + \dim\langle Y \times Y(1) \rangle + \dim(V \times Y(1)) - \dim(S \times Y(1))
\end{aligned}$$

and $Y \times Y(1) \subset (V \times Y(1)) \backslash (U \times Y(1))$ is special anomalous.

If $P \in V^{\mathrm{sa}} \cap S^{[1+\dim V]}$, we let $\hat{Y} = \{P\} \times Y(1)$. Then

$$\begin{aligned}
\dim \hat{Y} &= 1 \\
&\geq 1 + 1 + \dim\langle P \rangle + \dim V - \dim S \\
&= 1 + \dim\langle \hat{Y} \rangle + \dim(V \times Y(1)) - \dim(S \times Y(1))
\end{aligned}$$

and $\hat{Y} \subseteq (V \times Y(1)) \backslash (U \times Y(1))$ is special anomalous. We conclude that $(U \times Y(1)) \subseteq (V \times Y(1))^{\mathrm{sa}}$, which proves (3.15.1).

Therefore, if Conjecture 3.15.3 is true for $S \times Y(1)$, we conclude that $U \times Y(1)$ is open in $V \times Y(1)$. Therefore, U is open in V. On the other hand, V^{sa} is also open in V, whereas $V^{\mathrm{sa}} \cap S^{[1+\dim V]}$ is at most countable since there are only countably many special subvarieties and each point in $V^{\mathrm{sa}} \cap S^{[1+\dim V]}$ is an irreducible component of $V \cap Z$ for some special subvariety Z of codimension at most $1 + \dim V$. It follows that $V^{\mathrm{sa}} \cap S^{[1+\dim V]}$ is finite. $\square$

Chapter 4

Conjugation of semisimple subgroups over real number fields of bounded degree

4.1 Introduction

In this article, by a semisimple algebraic group, we always mean a *connected* semisimple algebraic group.

Let k be a field and let G be a linear algebraic group over k. Let H and H' be two algebraic k-subgroups of G. Let K/k be a field extension. We say that H and H' are *conjugate over* K if there exists an element $g \in G(K)$ such that

$$H'_K = g \cdot H_K \cdot g^{-1}.$$

We start with the following finiteness results for the set of conjugacy classes of semisimple subgroups:

Proposition 4.1.1. *Let G be a linear algebraic group over an algebraically closed field k of characteristic 0. Then the set $C(G)$ of $G(k)$-conjugacy classes of semisimple k-subgroups of G is finite.*

This proposition is essentially due to Richardson.

Corollary 4.1.2. *Let k be either the field of real numbers $\mathbb{R}$ or a p-adic field (that is, a finite extension of the field of p-adic numbers $\mathbb{Q}_p$). Let G be a linear algebraic group over k. Then the set $C(G)$ of $G(k)$-conjugacy classes of semisimple k-subgroups of G is finite.*

Proposition 4.1.1 and Corollary 4.1.2 will be proved in Section 4.2.

Our main results are Theorems 4.1.3 and 4.1.7 below.

Theorem 4.1.3. *Let k be a field of characteristic 0 and let $\bar{k}$ be a fixed algebraic closure of k. Let G be a linear algebraic group over k. There exists a natural number d depending only on $G_{\bar{k}}$ with the following property: if H and H' are two semisimple k-subgroups of G that are conjugate over $\bar{k}$, then they are conjugate over a finite extension K/k of degree $[K:k] \leq d$.*

In the rest of this chapter, for an algebraic group G over a field k, we denote by $H^i(k,G)$ the i-th cohomology set $H^i(\mathrm{Gal}(\bar{k}/k), G(\bar{k})), i = 0,1,2$. Let us also introduce a notion in Galois cohomology.

Definition 4.1.4. Let G be an algebraic group over a field k, and $\eta \in H^i(k,G)$. We say that η is killed by or vanished over a field extension k' of k if the image of η in $H^i(k',G)$ under the restriction map

$$\mathrm{Res}_k^{k'} : H^i(k,G) \to H^i(k',G)$$

is the distinguished element.

In Section 4.3, we shall deduce Theorem 4.1.3 from Proposition 4.1.1 and the following theorem:

Theorem 4.1.5. *Let k be a field of characteristic 0 and let $\bar{k}$ be a fixed algebraic closure of k. Let N be a linear algebraic group over k. Then there exists a natural number $d = d(N_{\bar{k}})$ such that any cohomology class $\xi \in H^1(k,N)$ can be killed by a finite field extension of degree at most d (that is, there exists a field extension $K = K(N,\xi)$ of k of degree $[K:k] \leq d$ such that the image of ξ in $H^1(K,N)$ is 1).*

We shall prove Theorem 4.1.3 in Section 4.3 by applying Theorem 4.1.5 to the normalizer $N = \mathcal{N}_G(H)$ of H in G. For a proof of Theorem 4.1.5, see Section 4.4.

Next we state another main result.

Definition 4.1.6. A **real number field** is a pair (k,ι), where k is a number field and $\iota : k \hookrightarrow \mathbb{R}$ is an embedding. Given two real number fields $(k_i,\iota_i), i = 1,2$, we say that (k_2,ι_2) is a **real number field extension** of (k_1,ι_1) if there is an injective field homomorphism $f : k_1 \hookrightarrow k_2$ such that $\iota_2 \circ f = \iota_1$.

In this chapter, for simplicity, we omit the symbol ι in the above definition if there is no risk of confusion.

Theorem 4.1.7. *Let G be a linear algebraic group over a* real number field *$k \subset \mathbb{R}$. There exists a natural number $d_{\mathbb{R}}$ depending only on $G_{\mathbb{C}}$ with the following property: if H and H' are two semisimple k-subgroups of G that are conjugate over $\mathbb{R}$, then they are conjugate over a finite* real extension *K/k, $k \subset K \subset \mathbb{R}$, of degree $[K:k] \leq d_{\mathbb{R}}$.*

In Section 4.3, we shall deduce Theorem 4.1.7 from Corollary 4.1.2 and the following theorem:

Theorem 4.1.8. *Let $k \subset \mathbb{R}$ be a real number field. Let N be a linear algebraic group over k. Then there exists a natural number $d_{\mathbb{R}} = d_{\mathbb{R}}(N_{\mathbb{C}})$ such that any cohomology class $\xi \in H^1(k,N)$ vanishing over $\mathbb{R}$ can be killed by a finite real extension of degree at most $d_{\mathbb{R}}$.*

We shall prove Theorem 4.1.8 in Section 4.7.

Now let k be a real number field and let G be a linear algebraic group over k. Consider the set $C(G_{\mathbb{R}})$ of $G(\mathbb{R})$-conjugacy classes of semisimple $\mathbb{R}$-subgroups of G; by Corollary 4.1.2, this set is finite. Let $C_0 \subset C(G_{\mathbb{R}})$ denote the set of those $c \in C(G_{\mathbb{R}})$ for which there exists a semisimple subgroup in c *defined over k*. For each $c \in C_0$, let us choose such a semisimple k-subgroup $H_c \subset G$ in c. We obtain a finite set of subgroups $\Omega = \{H_c \mid c \in C_0\}$ with the following property: any semisimple k-subgroup of G is conjugate over $\mathbb{R}$ to some $H_c \in \Omega$. The next corollary follows immediately from Theorem 4.1.7.

Corollary 4.1.9. *Let k be a real number field and let G be a linear algebraic group over k. Let $\Omega = \{H_c \mid c \in C_0\}$ be a finite set of semisimple k-subgroups of G as above. Then there exists a natural number $d = d(G_{\mathbb{C}})$ such that any semisimple k-subgroup $H \subset G$ is conjugate to some H_c ($c \in C_0$) over a finite real extension K/k, $k \subset K \subset \mathbb{R}$, of degree $[K : k] \leq d$.*

This paper was motivated by earlier work of Daw and Ren on the Zilber-Pink conjecture for Shimura varieties. The relevance of Corollary 4.1.9 is explained in Section 12 of their article [21].

4.2 Conjugation over an algebraically closed field and over a local field

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Proof of Proposition 4.1.1. Write $\mathfrak{g} = \mathrm{Lie}(G)$; the group $G(k)$ acts on $\mathfrak{g}$ by the adjoint representation. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Lie subalgebra; $\mathfrak{h}$ acts on itself, on $\mathfrak{g}$, and on the quotient space $\mathfrak{g}/\mathfrak{h}$, so we may consider the cohomology space $H^1(\mathfrak{h}, \mathfrak{g}/\mathfrak{h})$. By Richardson [68], Corollary (b) in the introduction, there exist only finitely many $G(k)$-conjugacy classes of Lie subalgebras $\mathfrak{h} \subset \mathfrak{g}$ such that $H^1(\mathfrak{h}, \mathfrak{g}/\mathfrak{h}) = 0$.

If $\mathfrak{h}$ is semisimple, then for any $\mathfrak{h}$-module M, we have $H^1(\mathfrak{h}, M) = 0$ (see Chevalley and Eilenberg [14], Theorem 25.1). In particular, we have $H^1(\mathfrak{h}, \mathfrak{g}/\mathfrak{h}) = 0$. It follows that there exist only finitely many $G(k)$-conjugacy classes of *semisimple* Lie subalgebras of $\mathfrak{g}$.

Now let H_1 and H_2 be two semisimple subgroups of G. Let $\mathfrak{h}_1$ and $\mathfrak{h}_2$ denote their respective Lie algebras. Assume that $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are conjugate under $G(k)$, that is, there exists $g \in G(k)$ such that $\mathrm{Ad}(g)(\mathfrak{h}_1) = \mathfrak{h}_2$). Since $\mathrm{char}(k) = 0$, a connected algebraic subgroup $H \subset G$ is uniquely determined by its Lie algebra $\mathrm{Lie}(H) \subset \mathfrak{g}$ (see Humphreys [40], Section 13.1). It follows that $gH_1g^{-1} = H_2$. We see that if $\mathrm{Lie}(H_1)$ and $\mathrm{Lie}(H_2)$ are conjugate, then H_1 and H_2 are conjugate. We conclude that there are only finitely many conjugacy classes of semisimple subgroups of G, which proves the proposition. $\square$

Let k be a field of characteristic 0 and let $\bar{k}$ be a fixed algebraic closure of k. Let G be a linear algebraic group over k. Let C be a conjugacy class of connected $\bar{k}$-subgroups $\overline{H}'$ of $G_{\bar{k}}$ that contains a subgroup $H_{\bar{k}}$ defined over k. Then C is the set of $\bar{k}$-points of a k-variety V on which G acts transitively by

$$g * \overline{H}' = g \cdot \overline{H}' \cdot g^{-1}.$$

The stabilizer of the k-point H of this variety is $N = \mathcal{N}_G(H)$. Therefore, we may identify V with G/N. We identify the set of k-subgroups $H' \subset G$ that are conjugate to H over k with $(G/N)(k)$, and we identify the set of $G(k)$-conjugacy classes of such k-subgroups with the set of orbits of $G(k)$ in $(G/N)(k)$.

Proof of Corollary 4.1.2. Let k be either the field of real numbers $\mathbb{R}$ or a p-adic field. By Serre [72], III.4.4, Theorem 5, the set of orbits of $G(k)$ in $(G/N)(k)$ is finite. Therefore, the set of $G(k)$-conjugacy classes of connected k-subgroups in a $G(\bar{k})$-conjugacy class is finite.

Let C_k denote the set of $G(k)$-conjugacy classes of connected algebraic k-subgroups and let $C_{\bar{k}}$ denote the corresponding set for $\bar{k}$. Then we have a canonical map

$$C_k \to C_{\bar{k}}.$$

As explained above, since k is either the field of real numbers $\mathbb{R}$ or a p-adic field, all the fibers of this map are finite. By Proposition 4.1.1, the set of $G(\bar{k})$ conjugacy classes of *semisimple* $\bar{k}$-subgroups is finite, and the corollary follows. $\square$

4.3 Reductions

We show that, in order to prove Theorem 4.1.3, it suffices to prove Theorem 4.1.5. Indeed, by Proposition 4.1.1, the set C of $G(\bar{k})$-conjugacy classes of semisimple subgroups of $G_{\bar{k}}$ is finite. Therefore, it suffices to show that, if $H, H' \subset G$ are two k-subgroups *in a given $G(\bar{k})$-conjugacy class $c \in C$*, then they are conjugate over a finite extension K of k of degree at most d_c, where d_c depends only on $G_{\bar{k}}$ and c.

Set

$$Y = \{g \in G(\bar{k}) \mid gHg^{-1} = H'\}$$

and write $N = \mathcal{N}_G(H)$. Then Y is a variety defined over k on which the k-group N acts on the right by

$$g \mapsto gn, \qquad (g \in G(\bar{k}),\ n \in N(\bar{k})).$$

This action is transitive (over $\bar{k}$). Hence, Y is a principal homogeneous space (torsor) of N, and so we obtain a cohomology class $\xi = \xi(H, H') \in H^1(k, N)$. The two subgroups H and H' are conjugate over a field extension K of k if and only if Y has a K-point, hence, if and only if the extension K/k kills ξ. Since, by Theorem 4.1.5, the class ξ can be killed by a finite extension K/k of degree at most $d(N_{\bar{k}})$, the subgroups H and H' are conjugate over this field K. Clearly, $N_{\bar{k}}$ depends (up to isomorphism) only on $G_{\bar{k}}$ and c.

Further, we show that, in order to prove Theorem 4.1.7, it suffices to prove Theorem 4.1.8. Indeed, by Corollary 4.1.2, the set $C_{\mathbb{R}}$ of $G(\mathbb{R})$-conjugacy classes of semisimple subgroups of $G_{\mathbb{R}}$ is finite. Therefore, it suffices to show that, if $H, H' \subset G$ are two semisimple k-subgroups of G *in a given* $G(\mathbb{R})$*-conjugacy class* $c \in C_{\mathbb{R}}$, then they are conjugate over a finite real extension K of k of degree at most $d_{\mathbb{R},c}$, where $d_{\mathbb{R},c}$ depends only on $G_{\mathbb{C}}$ and c.

As above, starting from two semisimple k-subgroups $H, H' \subset G$, which are conjugate over $\mathbb{R}$, we obtain a cohomology class

$$\xi \in \ker[H^1(k,N) \to H^1(\mathbb{R},N)],$$

where $N = \mathcal{N}_G(H)$. By Theorem 4.1.8, the class ξ can be killed by a finite real extension K/k of degree at most $d_{\mathbb{R}}(N_{\mathbb{C}})$. Then the subgroups H and H' are conjugate over this field K. Clearly, $N_{\mathbb{C}}$ depends (up to isomorphism) only on $G_{\mathbb{C}}$ and the conjugacy class of H over $\mathbb{C}$.

4.4 Killing a first cohomology class over an arbitrary field of characteristic 0

In this section, we prove Theorem 4.1.5, which we state here in a more precise form.

Theorem 4.4.1. *Let $\bar{k}$ be an algebraically closed field of characteristic 0. Let $\overline{G}$ be a linear algebraic group over $\bar{k}$. Then there exists a natural number $d = d(\overline{G})$ with the following property:*

$(*)$ *For any subfield $k \subset \bar{k}$ such that $\bar{k}$ is an algebraic closure of k, for any k-form G of $\overline{G}$, and for any cohomology class $\xi \in H^1(k,G)$, the class ξ can be killed by a finite field extension K/k of degree at most d.*

First, we prove Theorem 4.4.1 for finite groups.

Lemma 4.4.2. *Let $\overline{G}$ be a finite $\bar{k}$-group. Then $\overline{G}$ satisfies $(*)$ with $d = |\overline{G}|$.*

Proof. Let G be a k-form of $\overline{G}$. A cohomology class $\xi \in H^1(k,G)$ defines a torsor $\mathcal{T} = \mathcal{T}_\xi$ of G. By definition, we have $|\mathcal{T}(\bar{k})| = |\overline{G}|$. For any point $t \in \mathcal{T}(\bar{k})$, the $\mathrm{Gal}(\bar{k}/k)$-orbit of t has cardinality at most $|\overline{G}|$, hence, t is defined over a finite extension of k in $\bar{k}$ of degree at most $|G|$. Since this extension kills ξ, the proof is complete. □

In what follows, we shall need two known results.

Proposition 4.4.3 (Sansuc [70], Lemme 1.13)**.** *Let G be a connected linear algebraic group over a perfect field k and let U be its unipotent radical. Then the canonical map $H^1(k,G) \to H^1(k,G/U)$ is bijective.*

Proposition 4.4.4 (Minkowski, cf. Friedland [29])**.** *For any natural number r, there is a constant $\beta = \beta(r) > 0$ such that every finite subgroup of $\mathrm{GL}_r(\mathbb{Z})$ has cardinality at most β.*

Now we deal with the case when $\overline{G}$ is connected.

Proposition 4.4.5. *Let $\overline{G}$ be a connected $\bar{k}$-group of rank r (that is, the dimension of a maximal torus of $\overline{G}$ is r). Then $\overline{G}$ satisfies $(*)$ with $d = \beta(r)^2$.*

Proof. (due to Vladimir Chernousov, private communication). By Proposition 4.4.3, we may and shall assume that $\overline{G}$ is reductive. Let G be a k-form of $\overline{G}$ and let $\xi \in H^1(k,G)$ be a cohomology class. Let $T \subset G$ be a maximal torus defined over k and let $\mathsf{X}(T) = \mathrm{Hom}(T_{\bar{k}}, \mathbb{G}_{m,\bar{k}})$ denote the character group of T (as an abelian group it is isomorphic to $\mathbb{Z}^r$). The k-torus T defines a homomorphism

$$\mathrm{Gal}(\bar{k}/k) \to \mathrm{Aut}\,\mathsf{X}(T) \simeq \mathrm{GL}_r(\mathbb{Z})$$

with finite image. By Proposition 4.4.4, this image is of cardinality at most $\beta(r)$. It follows that T and G split over some finite extension K/k of degree at most $\beta(r)$.

Set $\xi_K = \mathrm{Res}^K_k(\xi) \in H^1(K,G) = H^1(K,G_K)$. By Steinberg's theorem ([73], Theorem 11.1), the cohomology class ξ_K of the split reductive K-group comes from some maximal torus (that is, there exists a maximal torus $T' \subset G_K$ defined over K and a cohomology class $\xi' \in H^1(K,T')$ whose image in $H^1(K,G_K)$ is ξ_K). Arguing as above, we conclude that T' splits over a finite extension L/K of degree at most $\beta(r)$. Since T'_L is a split torus, we have $H^1(L,T') = 1$ by Hilbert's Theorem 90 (see Serre [71], X.1, Proposition 2). Hence, $\mathrm{Res}^L_K(\xi') = 1$, and so $\mathrm{Res}^L_K(\xi_K) = 1$. Thus, $\mathrm{Res}^L_k(\xi) = 1$, and $[L:k] \le \beta(r)^2$. ∎

4.4.6. Proof of Theorem 4.4.1. *We argue by dévissage. Let G be a k-form of $\overline{G}$. Write $G' = G^0$ (that is, the identity component of G) and $G'' = G/G^0$. We have a short exact sequence*

$$1 \to G' \to G \to G'' \to 1, \tag{4.4.1}$$

which induces a cohomological exact sequence

$$H^1(k,G') \to H^1(k,G) \to H^1(k,G'').$$

We have $G'_{\bar{k}} = \overline{G}^0$ and $G''_{\bar{k}} = \overline{G}/\overline{G}^0$. Let $\xi \in H^1(k,G)$ and let ξ'' denote the image of ξ in $H^1(k,G'')$. By Lemma 4.4.2 applied to the finite $\bar{k}$-group $\overline{G}/\overline{G}^0$, the class ξ'' can be killed by a finite extension K/k of degree at most $|\overline{G}/\overline{G}^0|$. We consider the restriction $\mathrm{Res}^K_k(\xi) \in H^1(K,G)$. We have the image of $\mathrm{Res}^K_k(\xi)$ in $H^1(K,G'')$ is $\mathrm{Res}^K_k(\xi'') = 1$. From the cohomological exact sequence

$$H^1(K,G') \to H^1(K,G) \to H^1(K,G'')$$

induced by the short exact sequence (4.4.1), we see that $\mathrm{Res}^K_k(\xi)$ comes from some $\xi'_K \in H^1(K,G')$. By Proposition 4.4.5, there exists a finite extension L/K of degree at most $\beta(r)^2$ killing ξ'_K, where r is the rank of $\overline{G}^0$ and $\beta(r)$ is the constant afforded to us by Proposition 4.4.4. Therefore, the extension L/K kills $\mathrm{Res}^K_k(\xi)$. We conclude that the extension L/k of degree at most $|\overline{G}/\overline{G}^0| \cdot \beta(r)^2$ kills ξ. This completes the proofs of Theorems 4.4.1 and 4.1.3. ∎

4.5 Killing a first cohomology class over a real number field

Our aim is to prove Theorem 4.1.8, which we state here in a more precise form.

Theorem 4.5.1. *Let $G_{\mathbb{C}}$ be a linear algebraic group over the field of complex numbers $\mathbb{C}$. Then there exists a natural number $d_{\mathbb{R}} = d_{\mathbb{R}}(G_{\mathbb{C}})$ with the following property:*

$(*_{\mathbb{R}})$ *For any real number field $k \subset \mathbb{R}$, any k-form G of $G_{\mathbb{C}}$, and any cohomology class $\xi \in H^1(k,G)$ vanishing over $\mathbb{R}$, the class ξ can be killed by a finite real extension K/k of degree at most $d_{\mathbb{R}}$.*

First, we introduce some notation:

We denote by $\overline{\mathbb{Q}}$ the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$.

Let k be a number field. We denote by $\mathcal{V}(k)$ the set of all places of k, by $\mathcal{V}_\infty(k)$ the set of all infinite places of k, by $\mathcal{V}_{\mathbb{R}}(k)$ the set of all real places of k, and by $\mathcal{V}_f(k)$ the set of all finite places of k. For $v \in \mathcal{V}(k)$, we denote by k_v the completion of k at v.

We shall need the following result:

Proposition 4.5.2 (Weak approximation theorem)**.** *Let $S \subset \mathcal{V}(k)$ be a finite subset and, for any $v \in S$, let $U_v \subset k_v$ be a nonempty open subset. Then there exists $\alpha \in k$ such that $\mathrm{loc}_v(\alpha) \subset U_v$, for all $v \in S$, where $\mathrm{loc}_v \colon k \hookrightarrow k_v$ is the canonical embedding.*

Proof. See Cassels [12, Section 15, Theorem] for strong approximation, which implies weak approximation. □

We shall prove Theorem 4.5.1 in Section 4.7 by dévissage. For this, we shall need the following results:

Lemma 4.5.3. *Let $G_{\mathbb{C}}$ be a finite algebraic group over $\mathbb{C}$. Then $G_{\mathbb{C}}$ satisfies $(*_{\mathbb{R}})$ with $d_{\mathbb{R}} = |G_{\mathbb{C}}|$.*

Proof. Let $k \subset \mathbb{R}$ be a real number field, let G be a k-form of $G_{\mathbb{C}}$, and let $\xi \in H^1(k,G)$ be a cohomology class vanishing over $\mathbb{R}$. Then ξ defines a k-torsor $\mathcal{T} = \mathcal{T}_\xi$ of G such that $\mathcal{T}(\mathbb{R})$ is nonempty. Let $t \in \mathcal{T}(\mathbb{R})$. Since $\mathcal{T}(\mathbb{C})$ is finite, t is defined over $\overline{\mathbb{Q}}$. Let k_t denote the fixed field of the stabilizer of t in $\mathrm{Gal}(\overline{\mathbb{Q}}/k)$. Then $k_t \subset \mathbb{R}$ and

$$[k_t : k] = |\mathrm{Gal}(\overline{\mathbb{Q}}/k) \cdot t| \le |\mathcal{T}(\mathbb{C})| = |G_{\mathbb{C}}|.$$

Since t is defined over k_t, the class ξ is killed by k_t. □

Proposition 4.5.4. *Let $G_{\mathbb{C}}$ be an algebraic torus over $\mathbb{C}$ of rank r and let $\beta = \beta(r)$ be the constant afforded to us by Proposition 4.4.4. Then $G_{\mathbb{C}}$ satisfies $(*_{\mathbb{R}})$ with $d_{\mathbb{R}} = 2^r\beta$.*

Proof. Let $k \subset \mathbb{R}$ be a real number field and let G be a k-form of $G_{\mathbb{C}}$. Let L be a subfield of $\mathbb{C}$ containing k over which the torus G splits. Arguing as in the proof of Proposition 4.4.5, we can construct L such that that $[L:k] \leq \beta$. By Hilbert's Theorem 90 we have $H^1(L,G) = 1$, hence, the extension L/k kills all cohomology classes $\xi \in H^1(k,G)$. In the case when $L \subset \mathbb{R}$, the proposition is proved. Therefore, we assume that $L \not\subset \mathbb{R}$, and we set $K = L \cap \mathbb{R}$. Then K/k is a real extension of degree at most $\beta/2 < \beta$. The field L is an imaginary quadratic extension of K, that is, $L = K(\sqrt{-\alpha})$, where $\alpha \in K \subset \mathbb{R}$ is positive. Since the K-torus G_K splits over the quadratic extension L/K, by Platonov and Rapinchuk [65], §2.2, p. 76, we have

$$G_K \simeq \mathbb{G}_{m,K}^l \times \mathrm{Res}_{L/K}(\mathbb{G}_m)^m \times \mathrm{Res}_{L/K}^{(1)}(\mathbb{G}_m)^n,$$

for some natural numbers l, m, and n such that $l + 2m + n = r$. However, by Hilbert's Theorem 90, $H^1(K,\mathbb{G}_m) = 1$, and, by the Faddeev-Shapiro lemma (see Serre [72], I.2.5, Proposition 10),

$$H^1(K,\mathrm{Res}_{L/K}(\mathbb{G}_m)) = 1$$

as well. It follows that

$$H^1(K,G) = H^1(K,G')^n,$$

where $G' = \mathrm{Res}_{L/K}^{(1)}(\mathbb{G}_m)$. Since $n \leq r$, the proposition will follow if we can show that any nonzero cohomology class $\xi \in H^1(K,G')$ that vanishes over $\mathbb{R}$ can be killed by a real quadratic extension of K.

As a variety over K, the one-dimensional torus G' is defined by the equation

$$x^2 + \alpha y^2 = 1.$$

By Platonov and Rapinchuk [65], Lemma 2.5, p. 73,

$$H^1(K,G') = H^1(L/K,G') = K^\times / N_{L/K}(L^\times),$$

where $N_{L/K}$ denotes the norm map from L to K. If $\xi \in H^1(K,G')$ is the class of a cocycle associated with $c \in K^\times$, then ξ is the class of the G'-torsor $\mathcal{T}_c$ given by the equation

$$w^2 + \alpha z^2 = c.$$

Since ξ vanishes over $\mathbb{R}$, the torsor $\mathcal{T}_c$ has an $\mathbb{R}$-point (w_0,z_0), say. Then $c = w_0^2 + \alpha z_0^2 > 0$. We write $\mathbb{R}_{>0}$ for the set of positive real numbers. We take $K_1 = K(\sqrt{c})$, where $\sqrt{c} \in \mathbb{R}_{>0}$. Then K_1 is a real extension of K of degree $[K_1 : K] \leq 2$. Clearly, $\mathcal{T}_c$ has a K_1-point, namely, $(\sqrt{c},0)$. This completes the proof of the proposition. $\square$

Now let us recall the Hasse principle for simply connected algebraic groups, cf. Platonov and Rapinchuk [65], p 286.

Theorem 4.5.5. *Let k be a number field and we denote by V_∞^k (respectively V_f^k) the set of archimedean (respectively non-archimedean) places of K. Thus for any simply connected algebraic group G over k, we have*

1. $H^1(k,G)=1$ for $v \in V_f^k$;

2. The map $\theta : H^1(k,G) \longrightarrow \prod_{v\in V_\infty^k} H^1(k_v,G)$ is bijective.

Proposition 4.5.6. *Let $G_\mathbb{C}$ be a simply connected semisimple group over $\mathbb{C}$. Then $G_\mathbb{C}$ satisfies $(*_\mathbb{R})$ with $d_\mathbb{R}=2$.*

Proof. Let $k \subset \mathbb{R}$ be a real number field, let G be a k-form of $G_\mathbb{C}$, and let $\xi \in H^1(k,G)$ be a cohomology class vanishing over $\mathbb{R}$. Let $\sigma_0 \in V_\mathbb{R}(k)$ denote the inclusion embedding $k \hookrightarrow \mathbb{R}$. If k has only one real embedding σ_0, then, by Theorem 4.5.5, $\xi = 0$, and there is nothing to prove. Therefore, we assume that there exists $\sigma_1 \in \mathcal{V}_\mathbb{R}(k)$ such that $\sigma_1 \neq \sigma_0$. By Proposition 4.5.2 (weak approximation), there exists an element $\alpha \in k$ such that $\sigma_0(\alpha) > 0$ and such that $\sigma(\alpha) < 0$, for all $\sigma \in V_\mathbb{R}^k \setminus \{\sigma_0\}$. Since $\sigma_1(\alpha) < 0$, we see that the polynomial $X^2-\alpha$ is irreducible over k. We set $K = k(\sqrt{\alpha})$, where $\sqrt{\alpha} \in \mathbb{R}_{>0}$. Then K is a real quadratic extension of k with precisely two real embeddings (namely, the two extensions of σ_0).

We obtain a commutative diagram:

$$\begin{tikzcd}
H^1(k,G) \arrow[r, "\mathrm{Res}_k^K"] \arrow[d, "\mathrm{loc}_\infty^k"'] & H^1(K,G) \arrow[d, "\mathrm{loc}_\infty^K"] \\
\prod_{v\in\mathcal{V}_\infty(k)} H^1(k_v,G) \arrow[r] & \prod_{w\in\mathcal{V}_\infty(K)} H^1(K_w,G).
\end{tikzcd}$$

By Theorem 4.5.5, the vertical arrows in the diagram are bijective. The bottom arrow takes $(\xi_v) \in \prod_{v\in\mathcal{V}_\infty(k)} H^1(k_v,G)$ to $(\xi_w) \in \prod_{w\in\mathcal{V}_\infty(K)} H^1(K_w,G)$, where for a place w of K extending a place v of k, the cohomology class ξ_w is the restriction of ξ_v from k_v to K_w.

Now let $(\xi_v) \in \prod_{v\in\mathcal{V}_\infty(k)} H^1(k_v,G)$ be the image of $\xi \in H^1(k,G)$ under loc_∞^k and let $(\xi_w) \in \prod_{w\in\mathcal{V}_\infty(K)} H^1(K_w,G)$ be the image of (ξ_v) as above. Let $v_0 \in \mathcal{V}_\infty(k)$ correspond to σ_0. By assumption, $\xi_{v_0}=1$, and so, if $w_0, w_0' \in \mathcal{V}_\infty(K)$ are the two extensions of v_0 to K, then ξ_{w_0} and $\xi_{w_0'}$ are both equal to 1. All of the other $w \in \mathcal{V}_\infty(K)$ are complex, hence, $H^1(K_w,G)=\{1\}$ and $\xi_w = 1$. Therefore, $(\xi_w)=1$ and, from the injectivity of loc_∞^K, we conclude that $\mathrm{Res}_k^K \xi = 1$, as required. □

4.6 Killing a second cohomology class of a finite abelian group

Another key ingredient in the dévissage will be the following proposition.

Proposition 4.6.1. *Let $A_\mathbb{C}$ be a finite abelian algebraic group over $\mathbb{C}$. Then, for any real number field $k \subset \mathbb{R}$, any k-form A of $A_\mathbb{C}$, and any cohomology class $\eta \in H^2(k,A)$ vanishing over $\mathbb{R}$, the class η can be killed by a finite real extension K/k of degree at most*

$$e_\mathbb{R}(A_\mathbb{C}) = |A_\mathbb{C}| \cdot |A_\mathbb{C}|!\,.$$

We need a lemma.

Lemma 4.6.2 (cf. Sansuc [70], Lemma 1.6)**.** *Let k denote a real number field. If G is a finite abelian k-group, then the localization map*

$$H^1(k,G) \to H^1(\mathbb{R},G)$$

is surjective.

4.6.3. *Let $k \subset \mathbb{C}$ be a number field and let A be a k-form of $A_{\mathbb{C}}$. Since $A_{\mathbb{C}}$ is finite, $A(\mathbb{C}) = A(\overline{\mathbb{Q}})$. We write $|A|$ for $|A(\mathbb{C})|$ and we write*

$$A' = \mathsf{X}^*(A) := \mathrm{Hom}_{\overline{\mathbb{Q}}}(A, \mathbb{G}_m).$$

Then $\mathrm{Gal}(\overline{\mathbb{Q}}/k)$ naturally acts on A'. In this article, we say that A is split *if $\mathrm{Gal}(\overline{\mathbb{Q}}/k)$ acts on A' trivially. For example, μ_n is a split group over $\mathbb{Q}$. We say that A* splits *over an extension L/k contained in $\overline{\mathbb{Q}}$, or that L splits A, if $\mathrm{Gal}(\overline{\mathbb{Q}}/L)$ acts trivially on A'.*

Let Γ be a group. By a finite Γ-module *we mean a finite abelian group B with an action of Γ. We say that a Γ-module B is* simple *if it has no proper Γ-submodules.*

Let A be a finite abelian algebraic group over a field $k \subset \overline{\mathbb{Q}}$. In this article, we say that A is a simple abelian k-group *if A has no proper k-subgroups. Clearly, a finite abelian k-group is simple if and only if A' is a simple $\mathrm{Gal}(\overline{\mathbb{Q}}/k)$-module.*

Lemma 4.6.4. *Let $A_{\mathbb{C}}$ be a finite abelian algebraic group over $\mathbb{C}$. Then, for any real number field $k \subset \mathbb{R}$ and any k-form A over k, there exists a* real *extension K/k of degree at most $|A|!$ and an extension L/K of degree at most 2 such that L splits A.*

Proof. Let $\mathrm{Aut}(A')$ denote the group of automorphisms of $A'(\mathbb{C}) = A'(\bar{\mathbb{Q}})$ as an abelian group and let $\mathrm{Sym}(A')$ denote the group of permutations of the finite set A'. Then we have a homomorphism

$$\rho_{A'} \colon \mathrm{Gal}(\overline{\mathbb{Q}}/k) \to \mathrm{Aut}(A') \subset \mathrm{Sym}(A').$$

Let $L \subset \overline{\mathbb{Q}}$ denote the field corresponding to $\ker(\rho_{A'})$. Then L splits A and

$$[L:k] = |\mathrm{im}(\rho_{A'})| \leq |\mathrm{Aut}(A')| \leq |\mathrm{Sym}(A')| = |A'|! = |A|!.$$

We set $K = L \cap \mathbb{R}$. Then K is a real number field of degree $[K:k] \leq |A|!$. Furthermore, $[L:K] \leq 2$ since L is a Galois extension of k, which completes the proof. □

Now Proposition 4.6.1 follows from the next proposition.

Proposition 4.6.5. *Let $A_{\mathbb{C}}$ be a finite abelian group over $\mathbb{C}$. Then, for any real number field $k \subset \mathbb{R}$, any k-form A of $A_{\mathbb{C}}$ such that A splits over a finite extension L/k of degree ≤ 2, and for any cohomology class $\eta \in H^2(k,A)$ vanishing over $\mathbb{R}$, the class η can be killed by a finite real extension K/k of degree at most $|A|$.*

4.6.6. Reduction of Proposition 4.6.5 to the case when A is simple. *Let $k \subset \mathbb{R}$ be a real number field, let A be a k-form of $A_{\mathbb{C}}$ splitting over an extension L/k of degree $[L:k] \le 2$, and let $\eta \in H^2(k,A)$ be a cohomology class vanishing over $\mathbb{R}$. Since A splits over L, the Galois group* $\mathrm{Gal}(\overline{\mathbb{Q}}/L)$ *acts on A' trivially. Hence,* $\mathrm{Gal}(\overline{\mathbb{Q}}/k)$ *acts on A' via* $\Gamma := \mathrm{Gal}(L/k)$. *We prove by dévissage that η can be killed by a real extension of degree at most $|A|$, assuming that this is true in the case when A is a* simple abelian finite k-group.

We argue by induction on $|A|$. Let $A'_2 \subset A'$ be a minimal *nonzero Γ-submodule. Then A'_2 is a simple Γ-module. Set*

$$A_1 = \{a \in A \mid \chi(a) = 1 \;\forall \chi \in A'_2\}.$$

Then A_1 is a k-subgroup of A. Set $A_2 = A/A_1$. Then $A'_2 = \mathsf{X}^*(A_2)$. Since A'_2 is a simple *Γ-module, the finite abelian k-group A_2 is a* simple *abelian k-group. We have a short exact sequence of finite k-groups*

$$0 \to A_1 \to A \to A_2 \to 0. \tag{4.6.1}$$

Let $\eta \in H^2(k,A)$ be a cohomology class such that $\mathrm{loc}_{\mathbb{R}}(\eta) = 0$. Let η_2 denote the image of η in $H^2(k,A_2)$. Then $\mathrm{loc}_{\mathbb{R}}(\eta_2) = 0$. Since A_2 is a simple abelian finite k-group, there exists (by assumption) a real extension k'/k of degree at most $|A_2|$ that kills η_2. From the short exact sequence (4.6.1), we obtain a commutative diagram with exact rows:

$$\begin{tikzcd}
H^1(k,A_2) \arrow[r, "\Delta"] \arrow[d, "\mathrm{Res}_k^{k'}"] & H^2(k,A_1) \arrow[r, "\varphi"] \arrow[d, "\mathrm{Res}_k^{k'}"] & H^2(k,A) \arrow[r, "\psi"] \arrow[d, "\mathrm{Res}_k^{k'}"] & H^2(k,A_2) \arrow[d, "\mathrm{Res}_k^{k'}"] \\
H^1(k',A_2) \arrow[r, "\Delta"] \arrow[d, "\mathrm{loc}_{\mathbb{R}}"] & H^2(k',A_1) \arrow[r, "\varphi"] \arrow[d, "\mathrm{loc}_{\mathbb{R}}"] & H^2(k',A) \arrow[r, "\psi"] \arrow[d, "\mathrm{loc}_{\mathbb{R}}"] & H^2(k',A_2) \arrow[d, "\mathrm{loc}_{\mathbb{R}}"] \\
H^1(\mathbb{R},A_2) \arrow[r, "\Delta_{\mathbb{R}}"] & H^2(\mathbb{R},A_1) \arrow[r, "\varphi_{\mathbb{R}}"] & H^2(\mathbb{R},A) \arrow[r, "\psi_{\mathbb{R}}"] & H^2(k,A_2).
\end{tikzcd}$$

By construction,

$$\psi(\mathrm{Res}_k^{k'}(\eta)) = \mathrm{Res}_k^{k'}(\eta_2) = 0.$$

It follows that $\mathrm{Res}_k^{k'}(\eta) = \varphi(\eta_1)$ *for some $\eta_1 \in H^2(k',A_1)$. Set* $\eta_{1,\mathbb{R}} = \mathrm{loc}_{\mathbb{R}}(\eta_1)$. *Since* $\mathrm{loc}_{\mathbb{R}}(\eta) = 0$, *we know that $\varphi_{\mathbb{R}}(\eta_{1,\mathbb{R}}) = 0$. It follows that $\eta_{1,\mathbb{R}} = \Delta_{\mathbb{R}}(\xi_{2,\mathbb{R}})$, for some $\xi_{2,\mathbb{R}} \in H^1(\mathbb{R},A_2)$. Since A_2 is finite and abelian, by Lemma 4.6.2, there exists $\xi_2 \in H^1(k',A_2)$ such that $\xi_{2,\mathbb{R}} = \mathrm{loc}_{\mathbb{R}}(\xi_2)$. We set $\eta'_1 = \eta_1 - \Delta(\xi_2)$. Then* $\mathrm{Res}_k^{k'}(\eta) = \varphi(\eta'_1)$ *and* $\mathrm{loc}_{\mathbb{R}}(\eta'_1) = 0$. *Moreover, A_1 splits over the extension Lk'/k' of degree at most 2. Since $|A_1| < |A|$, the induction hypothesis implies that there exists a real extension k''/k' of degree at most $|A_1|$ that kills η'_1. Then the extension k''/k' also kills* $\mathrm{Res}_k^{k'}(\eta)$ *and the extension k''/k kills η. We have*

$$[k'':k] = [k'':k'][k':k] \le |A_1| \cdot |A_2| = |A|,$$

which proves Proposition 4.6.5 for A (assuming that it holds for simple abelian finite k-groups*).* □

Now we must prove Proposition 4.6.5 for any simple abelian k-group A. We need the following elementary lemma:

Lemma 4.6.7. *Let Γ be a group of order at most 2 and let B be a finite simple Γ-module. Then $|B|$ is a prime number.*

Proof. The case $|\Gamma| = 1$ being trivial, we assume that $\Gamma = \{1, \tau\}$ is a group of order 2. By the structure theorem for finite abelian groups, B is the direct product of its p-power torsion subgroups (Sylow p-subgroups), all of which are Γ-submodules of B. Since B is simple, it is equal to such a subgroup, that is, B is a p-group for some prime p. Since the kernel of multiplication by p is a Γ-submodule of B, it must be all of B. Therefore, $B = (\mathbb{Z}/p\mathbb{Z})^r$ for some natural number r.

The action of Γ on B is given by an irreducible representation

$$\rho : \Gamma \longrightarrow \mathrm{GL}_r(\mathbb{F}_p).$$

For $v \in (\mathbb{Z}/p\mathbb{Z})^r$, $v \neq 0$, let W be the (nonzero) $\mathbb{F}_p$-subspace of B spanned by v and $\rho(\tau)v$. Then W is Γ-invariant and, since ρ is irreducible, we conclude that $W = B$ and, hence, $r = \dim_{\mathbb{F}_p} W \leq 2$.

Now suppose that $r = 2$. Then v and $\rho(\tau)v$ are linearly independent over $\mathbb{F}_p$ and $v + \rho(\tau)v$ is nonzero. It follows that its $\mathbb{F}_p$-span is a one-dimensional Γ-invariant $\mathbb{F}_p$-subspace of W, which contradicts the assumption that ρ is irreducible. Therefore, $r = 1$, which concludes the proof. $\square$

The reduction in Subsection 4.6.6, together with Lemma 4.6.7, show that Propositions 4.6.5, hence Proposition 4.6.1 follow from the next lemma.

Lemma 4.6.8. *Let A be a finite abelian algebraic group* of prime order p *over a real number field $k \subset \mathbb{R}$ and let $\eta \in H^2(k, A)$ be a cohomology class. In the case $p = 2$ assume that η vanishes over $\mathbb{R}$. In the case when p is odd assume that there exists an extension L/k of degree at most 2 that splits A. Then there exists a* real *extension k'/k of degree at most $|A|$ that kills η.*

4.6.9. *In order to prove Lemma 4.6.8, we need some results from local and global class field theory.*

Let $A = \mu_n$ and let $\eta \in H^2(k, A) = H^2(k, \mu_n)$. The Kummer exact sequence

$$1 \to \mu_n \to \mathbb{G}_{m,k} \xrightarrow{n} \mathbb{G}_{m,k} \to 1$$

induces the exact sequence in Galois cohomology

$$0 = H^1(k, \mathbb{G}_{m,k}) \to H^2(k, \mu_n) \to \mathrm{Br}(k) \xrightarrow{n} \mathrm{Br}(k),$$

which shows that there is a canonical isomorphism

$$H^2(k, \mu_n) = \mathrm{Br}(k)_n := \ker[\mathrm{Br}(k) \xrightarrow{n} \mathrm{Br}(k)].$$

Thus, we may assume that $\eta \in \mathrm{Br}(k)_n$. *We would like to construct an extension* K/k *of degree* n *that kills* η.

Let $v \in \mathcal{V}(k)$. *We have a canonical localization map*

$$\mathrm{loc}_v\colon \mathrm{Br}(k) \to \mathrm{Br}(k_v)$$

and, hence, a map

$$\mathrm{loc}\colon \mathrm{Br}(k) \to \prod_v \mathrm{Br}(k_v).$$

By the Brauer-Hasse-Noether theorem (see Harari [36, Theorem 14.11]), the latter map is injective and there exists a finite set $S = S(\eta)$ *such that* $\mathrm{loc}_v(\eta) = 0$ *if* $v \notin S$. *We let* $\eta_v \in \mathrm{Br}(k_v)_n$ *denote* $\mathrm{loc}_v(\eta)$.

Now let v *be a nonarchimedean place of* k. *From local class field theory, we know that there is a canonical isomorphism*

$$\mathrm{inv}_v\colon \mathrm{Br}(k_v) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}$$

(see Harari [36, Theorem 8.9]). If K_w/k_v *is a finite extension, we obtain a commutative diagram:*

$$\begin{tikzcd} \mathrm{Br}(k_v) \arrow[r, "\mathrm{inv}_v"] \arrow[d, "\mathrm{Res}^{K_w}_{k_v}"'] & \mathbb{Q}/\mathbb{Z} \arrow[d, "{[K_w:k_v]}"] \\ \mathrm{Br}(K_w) \arrow[r, "\mathrm{inv}_w"'] & \mathbb{Q}/\mathbb{Z}, \end{tikzcd} \tag{4.6.2}$$

where the right vertical arrow is multiplication by the degree $[K_w : k_v]$ *(again, see Harari [36, Theorem 8.9]). In particular, if* $n \mid [L_w : k_v]$, *then* $\mathrm{Res}^{L_w}_{k_v}(\eta_v) = 0$.

We need the following well-known result.

Lemma 4.6.10. *Let* k *be a number field and let* $n > 1$ *be a natural number. Let* $\eta \in H^2(k, \mu_n) = \mathrm{Br}(k)_n$, *and let* $S \subset \mathcal{V}(k)$ *be a finite subset such that* $\mathrm{loc}_v(\eta) = 0$ *for all* $v \notin S$. *Write* $S_f = S \cap \mathcal{V}_f(k)$ *and* $S_{\mathbb{R}} = S \cap \mathcal{V}_{\mathbb{R}}(k)$. *Let* K/k *be an extension of degree* n *such that:*
(i) for every $v \in S_f$ *and every place* w *of* K *over* v, *we have* $[K_w : k_v] = n$, *and*
(ii) for every $v \in S_{\mathbb{R}}$ *and every place* w *of* K *over* v, *we have* $K_w \simeq \mathbb{C}$.
Then the extension K/k *kills* η, *that is,* $\mathrm{Res}^K_k(\eta) = 0$.

Proof. We have a commutative diagram:

$$\begin{tikzcd} \mathrm{Br}(k)_n \arrow[r, "\mathrm{Res}^K_k"] \arrow[d, "\mathrm{loc}"'] & \mathrm{Br}(K)_n \arrow[d, "\mathrm{loc}"] \\ \prod_{v\in\mathcal{V}(k)} \mathrm{Br}(k_v)_n \arrow[r] & \prod_{w\in\mathcal{V}(K)} \mathrm{Br}(K_w)_n. \end{tikzcd}$$

Let $S_K \subset \mathcal{V}(K)$ denote the set of places of K over S. It follows from hypothesis (i) and the commutative diagram (4.6.2) that, for any $v \in S_f$ and for any w over v, the restriction map

$$\mathrm{Br}(k_v)_n \to \mathrm{Br}(K_w)_n \tag{4.6.3}$$

is the zero map. On the other hand, it follows from hypothesis (ii) that for any $v \in S_{\mathbb{R}}$ and for any w over v, the restriction map (4.6.3) is the zero map, because $\mathrm{Br}(K_w) = \mathrm{Br}(\mathbb{C}) = 0$. We see that the map

$$\textstyle\prod_{v\in S} \mathrm{Br}(k_v)_n \longrightarrow \prod_{w\in S_K} \mathrm{Br}(K_w)_n$$

is the zero map. It follows that the image of η in $\prod_{w\in\mathcal{V}(K)} \mathrm{Br}(K_w)_n$ is zero. Since the vertical arrows in the commutative diagram are injective, we conclude that $\mathrm{Res}_k^K(\eta) = 0$. □

Lemma 4.6.11. *Let k be a number field, let L/k be a finite extension, and let p be a prime number such that $[L:k] < p$. Then, for any nonarchimedean place v of k, there exists a nonempty open subset $U_v = U_v(k_v, p) \subset k_v$ such that, for each $\alpha \in U_v$, the polynomial $X^p - \alpha$ is irreducible over L_w, for each place w of L over v (in particular, it is irreducible over k_v).*

Proof. We divide the proof into two steps:

Step (1): Suppose that $L = k$. Let v be a nonarchimedean place of k. Since p is prime, by Karpilovsky [43, Theorem 1.6], the polynomial $X^p - \alpha$ is irreducible over k_v if and only if $\alpha \notin (k_v^\times)^p$ (that is, α is not a p-th power in k_v). The open subgroup $(k_v^\times)^p \subset k_v^\times$ is closed (because any open subgroup in a topological group is closed). We set $U_v = k_v^\times \smallsetminus (k_v^\times)^p$ (which is then an open subset of k_v). It remains to show that U_v is not empty. We have a *surjective* homomorphism

$$v\colon k_v^\times \to \mathbb{Z}$$

such that the norm of an element $x \in k_v^\times$ is $q^{-v(x)}$, where q is the cardinality of the residue field of k_v (see Serre [71], I, §1, and II, §1). Let $\pi \subset k_v^\times$ be an element such that $v(\pi) = 1$. Then it is clear that $\pi \notin (k_v^\times)^p$ (that is, $\pi \in U_v$), hence, the open subset $U_v \subset k_v$ is not empty, which finishes the proof in this case.

Step (2): Suppose now that $[L:k] < p$. Let v be a nonarchimedean place of k and let U_v be a nonempty open subset of k_v such that, for each $\alpha \in U_v$, the polynomial $X^n - \alpha$ is irreducible over k_v (such a set is afforded to us by Step (1)). We show that, for every $\alpha \in U_v$ and every place w of L over v, the polynomial $X^p - \alpha$ is irreducible over L_w. Indeed, suppose for the sake of contradiction that $X^p - \alpha$ is reducible over L_w. Since p is prime, by [43, Theorem 1.6], there exists $\beta \in L_w$ such that $\beta^p = \alpha$. Since $X^p - \alpha$ is irreducible over k_v, we conclude that $k_v(\beta)/k_v$ is a field extension of degree p contained in L_w. Then we obtain that

$$p = [k_v(\beta):k_v] \leq [L_w:k_v] \leq [L:k] < p,$$

which is clearly a contradiction. Therefore, we see that the polynomial $X^p - \alpha$ is irreducible over L_w, which concludes the proof. □

4.6.12. Proof of Lemma 4.6.8. *Let A be as in the lemma. Consider the localization map*

$$H^2(k,A) \to \prod_{v\in\mathcal{V}(k)} H^2(k_v,A).$$

There exists a finite set $S \subset \mathcal{V}(k)$ such that $\mathrm{loc}_v(\eta) = 0$ for $v \notin S$ (see Serre [72, II.6.1]). We set $S_{\mathbb{R}} = S \cap \mathcal{V}_{\mathbb{R}}(k)$ and $S_f = S \cap \mathcal{V}_f(k)$. We may and shall assume that the set S_f is not empty and we let $v_1 \in S_f$. We separate our arguments into three cases.

Case (1): *Suppose that p=2, in which case $A = \mu_2$. Let $\sigma_0 \colon k \hookrightarrow \mathbb{R}$ denote the inclusion embedding. For any real place $v \in \mathcal{V}_{\mathbb{R}}(k)$, let $\sigma_v \colon k \hookrightarrow \mathbb{R}$ denote the corresponding real embedding. Set*

$$\Sigma = \{\sigma_v \mid v \in S_{\mathbb{R}}\}.$$

Then, by hypothesis, $\sigma_0 \notin \Sigma$. By Proposition 4.5.2 (weak approximation), there exists $\alpha \in k$ such that $\sigma_0(\alpha) > 0$, $\sigma(\alpha) < 0$ for all $\sigma \in \Sigma$, and $\mathrm{loc}_v(\alpha) \in U_v(k_v, 2)$ for all $v \in S_f$, where the subsets $U_v(k_v, 2) \subset k_v$ are afforded to us by Lemma 4.6.11. Since the polynomial $X^2 - \alpha$ is irreducible over k_{v_1}, it is irreducible over k. Set $K = k(\sqrt{\alpha})$, where $\sqrt{\alpha} \in \mathbb{R}_{>0}$. Then K is a real quadratic extension of k that, by Lemma 4.6.10, kills η.

Case (2): *Suppose that p is odd and that A is split, that is, $A \simeq \mu_p$. Then $H^2(k_v, A) = 0$ for all $v \in \mathcal{V}_{\mathbb{R}}(k)$, hence, we may assume that $S = S_f$. By Proposition 4.5.2 (weak approximation), there exists $\alpha \in k$ such that $\mathrm{loc}_v(\alpha) \in U_v(k_v, p)$, for all $v \in S_f = S$, where the subsets $U_v(k_v, p) \subset k_v$ are those afforded to us by Lemma 4.6.11. Since the polynomial $X^p - \alpha$ is irreducible over k_{v_1}, it is irreducible over k. Set $K = k(\sqrt[p]{\alpha})$, where $\sqrt[p]{\alpha} \in \mathbb{R}$. Then K is real extension of degree p of k that, by Lemma 4.6.10, kills η.*

Case (3): *Suppose that p is odd and that A is not split, that is, $[L : k] = 2$ and $\Gamma = \mathrm{Gal}(L/k)$ acts nontrivially on $A' = \mathbb{Z}/p\mathbb{Z}$. We write $\Gamma = \{1, \tau\}$, where $\tau^2 = 1$. Then τ acts on on $A' = \mathbb{Z}/p\mathbb{Z}$ as multiplication by -1, because this is the only nontrivial involutive automorphism of $\mathbb{Z}/p\mathbb{Z}$. We have $H^2(k_v, A) = 0$, for every $v \in \mathcal{V}_{\mathbb{R}}(k)$, because the abelian group $H^2(k_v, A)$ is annihilated by multiplication by the odd number $p = |A|$ and it is annihilated by multiplication by $2 = |\mathrm{Gal}(\mathbb{C}/\mathbb{R})|$ (see Atiyah and Wall [2], Corollary 1 of Proposition 8). In other words, we may again assume that $S = S_f$. By weak approximation, there exists $\alpha \in k$ such that $\mathrm{loc}_v(\alpha) \in U_v(k_v, p)$ for all $v \in S_f = S$, where the subsets $U_v(k_v, p) \subset k_v$ are those afforded to us by Lemma 4.6.11. Since the polynomial $X^p - \alpha$ is irreducible over k_{v_1}, it is irreducible over k. Set $K = k(\sqrt[p]{\alpha})$, where $\sqrt[p]{\alpha} \in \mathbb{R}$. Then K is a real extension of k of degree $[K : k] = p$. Let w_1 be a place of L over v_1. Since the polynomial $X^p - \alpha$ is irreducible over L_{w_1}, it is irreducible over L. Consider $F = L(\sqrt[p]{\alpha})$. We have $[F : L] = p$, $[F : k] = 2p$, and $[F : K] = 2$. We obtain the following commutative diagram:*

$$\begin{tikzcd}
H^1(k,A) \arrow[r, "\mathrm{Res}^K_k"] \arrow[d, "\mathrm{Res}^L_k"'] & H^1(K,A) \arrow[d, "\mathrm{Res}^F_K"] \\
H^1(L,A) \arrow[r, "\mathrm{Res}^F_L"'] & H^1(F,A).
\end{tikzcd}$$

Let S_L denote the set of all places w of L extending those places $v \in S$ of k. The k-group A splits over L, that is, $A_L \simeq \mu_{p,L}$. Consider $\eta_L = \mathrm{Res}^L_k(\eta) \in H^2(L, A) = \mathrm{Br}(L)_p$. Then $\mathrm{loc}_w(\eta_L) = 0$ for all $w \notin S_L$. On the other hand, by construction, the polynomial $X^p - \alpha$ is irreducible over L_w for all $w \in S_L$. It follows that the field $F_u := L_w(\sqrt[p]{\alpha})$ is of degree p over L_w for all $w \in S_L$. By Lemma 4.6.10 the extension F/L kills η_L. This means that the quadratic extension F/K kills $\eta_K = \mathrm{Res}^K_k(\eta) \in H^2(K, A)$.

Now we consider the Hochschild-Serre spectral sequence (see Harari [36], Theorem 1.44, p. 32)

$$H^r(\Gamma,(H^s(F,A)) \Rightarrow H^{r+s}(K,A),$$

where $\Gamma = \mathrm{Gal}(F/K) = \mathrm{Gal}(L/k)$. *From the spectral sequence, we obtain an exact sequence of terms in low degrees*

$$H^2(\Gamma,A(F)) \to \ker\left[H^2(K,A) \to (H^2(F,A))^\Gamma\right] \to H^1(\Gamma,H^1(F,A))$$

(see Sansuc [70], proof of Lemma 6.3). Since $|A| = p$, *both* $H^2(\Gamma,A(F))$ *and* $H^1(\Gamma,H^1(F,A))$ *are annihilated by multiplication by p. Since* Γ *is of order 2, they are also annihilated by multiplication by 2 (see Atiyah and Wall [2], Corollary 1 of Proposition 8). Therefore,* $H^2(\Gamma,A(F)) = 0$ *and* $H^1(\Gamma,H^1(F,A)) = 0$. *From the exact sequence, we conclude that* $\ker[H^2(K,A) \to (H^2(F,A))^\Gamma] = 0$. *Since* $\eta_K \in \ker[H^2(K,A) \to (H^2(F,A))^\Gamma]$, *we conclude that* $\eta_K = 0$. *That is,* η *is killed by the real extension* K/k *of degree* $p = |A|$. *This completes the proofs of Lemma 4.6.8 and Proposition 4.6.1.* □

4.7 Dévissage

In this section, we carry out the dévissage and prove Theorem 4.5.1. Before doing so, we shall need three further results.

Lemma 4.7.1. *Let k denote a real number field and let G be a finite k-algebraic group. Then any* $\mathbb{R}$*-point of G is defined over a real number field whose degree over k is at most* $|G_{\mathbb{C}}|$.

Proof. Let $t \in G(\mathbb{R})$. Then $t \in G(\overline{\mathbb{Q}})$ and is defined over the fixed field k_t of its stabilizer in $\mathrm{Gal}(\overline{\mathbb{Q}}/k)$. It follows that $[k_t : k] \le |G_{\mathbb{C}}|$. □

The following well-known result is sometimes referred to as *real approximation*.

Lemma 4.7.2 (cf. Sansuc [70], Corollaire 3.5(iii))**.** *Let k denote a real number field. If G is a connected linear algebraic k-group, then* $G(k)$ *is dense in* $G(\mathbb{R})$.

We shall need some basic properties of nonabelian Galois cohomology. Let A and B be two k-groups such that A is normal in B. Set $C = B/A$. We obtain a cohomological exact sequence

$$H^1(k,A) \longrightarrow H^1(k,B) \longrightarrow H^1(k,C)$$

and we have a right action $*$ of $C(k)$ on $H^1(k,A)$ (see Serre [72], I, 5.5).

Lemma 4.7.3 (cf. Serre [72], I.5.5, Proposition 39 (ii))**.** *Two classes* $\xi_1, \xi_2 \in H^1(k,A)$ *have the same image in* $H^1(k,B)$ *if any only if they are in the same* $C(k)$*-orbit.*

If A is contained in the center of B, we have an action, also denoted $*$, of $H^1(k,A)$ on the set $H^1(k,B)$ (see Serre [72], I.5.7).

Lemma 4.7.4 (cf. Serre [72], Proposition 42 (ii), I.5.7)**.** *If $A \subset Z(B)$, two classes $\xi_1, \xi_2 \in H^1(k,B)$ have the same image in $H^1(k,C)$ if any only if they are in the same $H^1(k,A)$-orbit.*

Now we perform the dévissage. First, we reduce Theorem 4.5.1 to the case when G is connected.

Lemma 4.7.5. *Assume that Theorem 4.5.1 is true for connected groups and let $G_{\mathbb{C}}$ be a linear algebraic group over $\mathbb{C}$ (not necessarily connected). Then $G_{\mathbb{C}}$ satisfies $(*_{\mathbb{R}})$ with*

$$d_{\mathbb{R}} = d_{\mathbb{R}}(G^0_{\mathbb{C}}) \cdot |G_{\mathbb{C}}/G^0_{\mathbb{C}}|^2.$$

Proof. Let $k \subset \mathbb{R}$ be a real number field, let G be a k-form of $G_{\mathbb{C}}$, and let $\xi \in H^1(k,G)$ be a cohomology class vanishing over $\mathbb{R}$. From the short exact sequence

$$1 \longrightarrow G^0 \longrightarrow G \longrightarrow G/G^0 \longrightarrow 1,$$

we obtain a commutative diagram with exact rows:

$$\begin{tikzcd}
(G/G^0)(k) \arrow[r, "\delta"] \arrow[d, "\mathrm{loc}_{\mathbb{R}}"'] & H^1(k,G^0) \arrow[r, "\varphi"] \arrow[d, "\mathrm{loc}_{\mathbb{R}}"] & H^1(k,G) \arrow[r, "\psi"] \arrow[d, "\mathrm{loc}_{\mathbb{R}}"] & H^1(k,G/G^0) \arrow[d, "\mathrm{loc}_{\mathbb{R}}"] \\
(G/G^0)(\mathbb{R}) \arrow[r, "\delta_{\mathbb{R}}"] & H^1(\mathbb{R},G^0) \arrow[r, "\varphi_{\mathbb{R}}"] & H^1(\mathbb{R},G) \arrow[r, "\psi_{\mathbb{R}}"] & H^1(\mathbb{R},G/G^0).
\end{tikzcd}$$

By Proposition 4.5.3, after possibly replacing k by a real extension of degree at most $|G_{\mathbb{C}}/G^0_{\mathbb{C}}|$, we may and shall assume that $\xi = \varphi(\xi_0)$, for some $\xi_0 \in H^1(k,G^0)$. By assumption, the class $(\xi_0)_{\mathbb{R}} := \mathrm{loc}_{\mathbb{R}}(\xi_0) \in H^1(\mathbb{R},G^0)$ maps to $1 \in H^1(\mathbb{R},G)$. It follows that $(\xi_0)_{\mathbb{R}} = \delta_{\mathbb{R}}(g)$, for some $g \in (G/G^0)(\mathbb{R})$. By Lemma 4.7.1, after possibly replacing k by a real extension of degree at most $|G_{\mathbb{C}}/G^0_{\mathbb{C}}|$, we may and shall assume that $g \in (G/G^0)(k)$. By Lemma 4.7.3, the class $\xi_0' := \xi_0 * g^{-1} \in H^1(k,G^0)$ maps to ξ under φ and vanishes over $\mathbb{R}$. By assumption, ξ_0' can be killed by a real extension K/k of degree at most $d_{\mathbb{R}}(G^0_{\mathbb{C}})$. Clearly, K also kills ξ, which completes the proof. □

Next we reduce Theorem 4.5.1 to the case when G is reductive. We write $R_u(G)$ for the unipotent radical of G and we write $G^{\mathrm{red}} = G/R_u(G)$, which is a connected reductive k-group.

Lemma 4.7.6. *Assume that Theorem 4.5.1 is true for reductive groups and let $G_{\mathbb{C}}$ be a connected linear algebraic group over $\mathbb{C}$ (not necessarily reductive). Then $G_{\mathbb{C}}$ satisfies $(*_{\mathbb{R}})$ with*

$$d_{\mathbb{R}}(G_{\mathbb{C}}) = d_{\mathbb{R}}(G^{\mathrm{red}}_{\mathbb{C}}).$$

Proof. The lemma follows immediately from Proposition 4.4.3. □

Next we reduce Theorem 4.5.1 to the case when G is semisimple.

Lemma 4.7.7. *Assume that Theorem 4.5.1 is true for semisimple groups and let $G_{\mathbb{C}}$ be a reductive group over $\mathbb{C}$. Write $G^{\mathrm{ss}}_{\mathbb{C}} = [G_{\mathbb{C}}, G_{\mathbb{C}}]$ for the commutator subgroup of $G_{\mathbb{C}}$ (which is a semisimple $\mathbb{C}$-group). Write $G^{\mathrm{tor}}_{\mathbb{C}} = G_{\mathbb{C}}/G^{\mathrm{ss}}_{C}$ (which is a $\mathbb{C}$-torus). Then $G_{\mathbb{C}}$ satisfies $(*_{\mathbb{R}})$ with*

$$d_{\mathbb{R}} = d_{\mathbb{R}}(G^{\mathrm{ss}}_{\mathbb{C}}) \cdot d_{\mathbb{R}}(G^{\mathrm{tor}}_{\mathbb{C}}).$$

Proof. Let $k \subset \mathbb{R}$ be a real number field, let G be a k-form of $G_{\mathbb{C}}$, and let $\xi \in H^1(k,G)$ be a cohomology class vanishing over $\mathbb{R}$. Write $G^{\mathrm{ss}} = [G,G]$ for the commutator subgroup of G (which is a semisimple k-group and a k-form of $G^{\mathrm{ss}}_{\mathbb{C}}$). Write $G^{\mathrm{tor}} = G/G^{\mathrm{ss}}$ (which is a k-torus and a k-form of $G^{\mathrm{tor}}_{\mathbb{C}}$). From the short exact sequence

$$1 \to G^{\mathrm{ss}} \to G \xrightarrow{\pi} G^{\mathrm{tor}} \to 1,$$

we obtain a commutative diagram with exact rows:

$$\begin{tikzcd}
G(k) \arrow[r, "\pi"] \arrow[d, "\mathrm{loc}_{\mathbb{R}}"] & G^{\mathrm{tor}}(k) \arrow[r, "\delta"] \arrow[d, "\mathrm{loc}_{\mathbb{R}}"] & H^1(k,G^{\mathrm{ss}}) \arrow[r, "\varphi"] \arrow[d, "\mathrm{loc}_{\mathbb{R}}"] & H^1(k,G) \arrow[r, "\psi"] \arrow[d, "\mathrm{loc}_{\mathbb{R}}"] & H^1(k,G^{\mathrm{tor}}) \arrow[d, "\mathrm{loc}_{\mathbb{R}}"] \\
G(\mathbb{R}) \arrow[r, "\pi"] & G^{\mathrm{tor}}(\mathbb{R}) \arrow[r, "\delta_{\mathbb{R}}"] & H^1(\mathbb{R},G^{\mathrm{ss}}) \arrow[r, "\varphi_{\mathbb{R}}"] & H^1(\mathbb{R},G) \arrow[r, "\psi_{\mathbb{R}}"] & H^1(\mathbb{R},G^{\mathrm{tor}}).
\end{tikzcd}$$

Let $\xi \in H^1(k,G)$ be a cohomology class such that $\mathrm{loc}_{\mathbb{R}}(\xi) = 1$. Consider $\psi(\xi) \in H^1(k,G^{\mathrm{tor}})$. By Proposition 4.5.4, after possibly replacing k by a real extension of degree at most $d_{\mathbb{R}}(G^{\mathrm{tor}}_{\mathbb{C}})$, we may and shall assume that $\psi(\xi) = 1$. Thus, $\xi = \varphi(\xi^{\mathrm{ss}})$, for some cohomology class $\xi^{\mathrm{ss}} \in H^1(k,G^{\mathrm{ss}})$. If we set $\xi^{\mathrm{ss}}_{\mathbb{R}} = \mathrm{loc}_{\mathbb{R}}(\xi^{\mathrm{ss}})$, then $\phi_{\mathbb{R}}(\xi^{\mathrm{ss}}_{\mathbb{R}}) = 1$ and, hence, $\xi^{\mathrm{ss}}_{\mathbb{R}} = \delta_{\mathbb{R}}(t_{\mathbb{R}})$ for some $t_{\mathbb{R}} \in G^{\mathrm{tor}}(\mathbb{R})$.

Since the homomorphism $\pi\colon G \to G^{\mathrm{tor}}$ is surjective, the implicit function theorem implies that the subgroup $\pi(G(\mathbb{R})) \subset G^{\mathrm{tor}}(\mathbb{R})$ is open. Taking into account Lemma 4.7.2 (real approximation), we see that $G^{\mathrm{tor}}(k) \cdot \pi(G(\mathbb{R})) = G^{\mathrm{tor}}(\mathbb{R})$. Choose $t \in G^{\mathrm{tor}}(k)$ such that $t_{\mathbb{R}} \in t \cdot \pi(G(\mathbb{R}))$. Then $\delta_{\mathbb{R}}(t) = \delta_{\mathbb{R}}(t_{\mathbb{R}})$. By Lemma 4.7.4, the class $\xi^{\mathrm{ss}}_1 := t^{-1} * \xi^{\mathrm{ss}} \in H^1(k,G^{\mathrm{ss}})$ maps to ξ under φ and vanishes over $\mathbb{R}$. Since G^{ss} is semisimple, by assumption, ξ^{ss}_1 can be killed by a real extension K/k of degree at most $d_{\mathbb{R}}(G^{\mathrm{ss}}_{\mathbb{C}})$. Then K also kills ξ, which completes the proof. $\square$

Finally, we treat the case of a semisimple group.

Lemma 4.7.8. *Let $G_{\mathbb{C}}$ be a semisimple $\mathbb{C}$-group. Consider the (simply connected) universal covering $\widetilde{G}_{\mathbb{C}}$ of $G_{\mathbb{C}}$. Let $A_{\mathbb{C}}$ be the (central) kernel of the surjective homomorphism $\widetilde{G}_{\mathbb{C}} \to G_{\mathbb{C}}$ (which is a finite abelian $\mathbb{C}$-group). Then $G_{\mathbb{C}}$ satisfies $(*_{\mathbb{R}})$ with $d_{\mathbb{R}} = 2e_{\mathbb{R}}(A_{\mathbb{C}})$.*

Proof. Let $k \subset \mathbb{R}$ be a real number field, let G be a k-form of $G_{\mathbb{C}}$, and let $\xi \in H^1(k,G)$ be a cohomology class vanishing over $\mathbb{R}$. Consider the (simply connected) universal covering $\widetilde{G}$ of G. Let A be the (central) kernel of the surjective homomorphism $\widetilde{G} \to G$ (which is a finite abelian k-group and a k-form of $A_{\mathbb{C}}$). We have a commutative diagram with exact rows:

$$\begin{tikzcd}
H^1(k,A) \arrow[r, "\varphi"] \arrow[d, "\mathrm{loc}_{\mathbb{R}}"'] & H^1(k,\widetilde{G}) \arrow[r, "\psi"] \arrow[d, "\mathrm{loc}_{\mathbb{R}}"] & H^1(k,G) \arrow[r, "\Delta"] \arrow[d, "\mathrm{loc}_{\mathbb{R}}"] & H^2(k,A) \arrow[d, "\mathrm{loc}_{\mathbb{R}}"] \\
H^1(\mathbb{R},A) \arrow[r, "\varphi_{\mathbb{R}}"] & H^1(\mathbb{R},\widetilde{G}) \arrow[r, "\psi_{\mathbb{R}}"] & H^1(\mathbb{R},G) \arrow[r, "\Delta_{\mathbb{R}}"] & H^2(\mathbb{R},A).
\end{tikzcd}$$

By Proposition 4.6.1, after possibly replacing k with a finite real extension of degree at most $e_{\mathbb{R}}(A_{\mathbb{C}})$, we may and shall assume that $\Delta(\xi) \in H^2(k,A)$ is trivial. Therefore, $\xi = \psi(\tilde{\xi})$, for some $\tilde{\xi} \in H^1(k,\widetilde{G})$. By assumption, $(\tilde{\xi})_{\mathbb{R}} := \mathrm{loc}_{\mathbb{R}}(\tilde{\xi}) \in H^1(\mathbb{R},\widetilde{G})$ maps to $1 \in H^1(\mathbb{R},G)$. It follows that $(\tilde{\xi})_{\mathbb{R}} = \varphi_{\mathbb{R}}(a_{\mathbb{R}})$, for some $a_{\mathbb{R}} \in H^1(\mathbb{R},A)$. By Lemma 4.6.2, there exists $a \in H^1(k,A)$ such that $\mathrm{loc}_{\mathbb{R}}(a) = a_{\mathbb{R}}$. By Lemma 4.7.3, the element $\xi' = a^{-1} * \tilde{\xi}$ in $H^1(k,\widetilde{G})$ maps to ξ under ψ and vanishes over $\mathbb{R}$. Now, Proposition 4.5.6 implies that ξ' can be killed by a finite real extension K/k of degree at most 2. Then K also kills ξ, which completes the proof. □

Proof of Theorem 4.5.1. The theorem follows from Lemmas 4.7.5, 4.7.6, 4.7.7, and 4.7.8. □

Chapter 5

Counting compact Shimura varieties

5.1 Introduction

This paper starts with the classification of 1-dimensional Shimura varieties. Recall that a 1-dimensional connected Shimura variety is a quotient $S = \Gamma\backslash\mathcal{H}_1$, where $\mathcal{H}_1 := \{z \in \mathbb{C}; \mathrm{Im}(z) > 0\}$ is the Poincaré upper half plane, and $\Gamma \subset \mathrm{GL}_2(\mathbb{R})$ is a certain discrete subgroup. This Shimura variety is non-compact if and only if Γ is commensurable with $\mathrm{GL}_2(\mathbb{Z})$, i.e., S is a modular curve, cf. [84]. In other words, up to level structures, the modular curve $Y(1)$ is the only non-compact 1-dimensional connected Shimura variety.

One may consider similar questions for general Shimura varieties. Recall that in Deligne's language, a Shimura variety is a double quotient

$$\mathrm{Sh}_K(G,X) := G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K,$$

where G is a reductive group over $\mathbb{Q}$, X is conjugacy class of a morphism $h : \mathbb{S} \to G_{\mathbb{R}}$ and K is a compact open subgroup of $G(\mathbb{A}_f)$.

A Shimura variety as defined above is compact if and only if G^{ad} is $\mathbb{Q}$-anisotropic, i.e., G^{ad} does not contain any positive dimensional $\mathbb{Q}$-split torus, cf. Theorem 4.12, p. 210, [65]. So, the previous example just says $(\mathrm{PGL}_2, \mathbb{H}^{\pm})$ is the only adjoint Shimura datum which corresponds to a non-compact one dimensional Shimura variety. Thus counting compact Shimura varieties with a fixed real group amounts to describing all the anisotropic forms of a fixed reductive $\mathbb{Q}$-group with a prescribed behaviour at infinity. For simplicity, this paper only considers inner-forms of algebraic groups, which are determined by the first Galois cohomology of their adjoint groups. In the sequel the reductive algebraic groups can be over an arbitrary number field F, as at the level of group theory, the restriction to $\mathbb{Q}$ is not necessary. Also, it is not necessary to limit our attention to those groups G which correspond to Shimura varieties, instead, we may consider arbitrary reductive groups over number fields.

Definition 5.1.1. We denote by p_∞ the natural map

$$p_\infty : H^1(F, G^{\mathrm{ad}}) \to \bigoplus_{v|\infty} H^1(F_v, G^{\mathrm{ad}}),$$

and for any $\theta \in \bigoplus_{v|\infty} H^1(F_v, G^{\mathrm{ad}})$, we denote the fibre $p_\infty^{-1}(\theta)$ by $H^1(F, G^{\mathrm{ad}})_\theta$.

Definition 5.1.2. Let G be a reductive algebraic group over a number field F, and let $h : H^1(F, G^{\mathrm{ad}}) \to \mathbb{R}_{\geq 0}$ be a function. Then for any $T > 1$, and let $\theta \in \bigoplus_{v|\infty} H^1(F_v, G^{\mathrm{ad}})$, we define

$$N_F(h, G, T)_\theta := \left\{ \eta \in H^1(F, G^{\mathrm{ad}})_\theta ; h(\eta) < T \right\},$$

and $N_F^c(h, G, T)_\theta$ (respectively $N_F^{nc}(h, G, T)_\theta$) be the subset of $N_F(h, G, T)_\theta$ consisting those η such that for the corresponding inner form G_η, the adjoint group G_η^{ad} is F-anisotropic (respectively F-isotropic).

A crucial ingredient in our counting theorem is the height function on the Galois cohomology set.

Definition 5.1.3. A function $h : H^1(F, G^{\mathrm{ad}}) \to \mathbb{R}_{\geq 0}$ is called a **height function** if there exist

- a finite subset $M \subset \mathbb{R}_{\geq 0}$ containing 0;
- for each finite place v, an injective function $h_v : H^1(F_v, G^{\mathrm{ad}}) \to M$ which maps the distinguished element of $H^1(F_v, G^{\mathrm{ad}})$ to zero

such that for each $\eta \in H^1(F, G^{\mathrm{ad}})$, we have

$$h(\eta) = \sum_{v<\infty} h_v(\mathrm{loc}_v \eta) \log |\kappa_v|$$

where $\mathrm{loc}_v : H^1(F, G^{\mathrm{ad}}) \to H^1(F_v, G^{\mathrm{ad}})$ is the localisation map and κ_v is the residue field at v.

It is worthwhile to note the following easy but important property of the height function.

Proposition 5.1.4. *Let $h : H^1(F, G^{\mathrm{ad}}) \to \mathbb{R}_{\geq 0}$ be a height function, then for any $b > 0$, there are only finitely many elements $\eta \in H^1(F, G^{\mathrm{ad}})$ with $h(\eta) < b$.*

The proof will be given in the next section.

Remark 5.1.5. The above definition of height function is motivated by a general principle of decomposing a height function into local pieces, cf. Section B.8 of [38].

Conjecture 5.1.6. *Let G be a reductive algebraic group over a number field F, then there is a suitable height function $h : H^1(F, G^{\mathrm{ad}}) \to \mathbb{R}_{\geq 0}$ as in Definition 5.1.3, such that for all $\theta \in \bigoplus_{v|\infty} H^1(F_v, G^{\mathrm{ad}})$, we have*

$$\lim_{T\to\infty} \frac{\# N_F^{nc}(h, G, T)_\theta}{\# N_F(h, G, T)_\theta} = 0.$$

The main content of this paper is to prove this conjecture for $G = \mathrm{GL}_{n,F}$ or a unitary group over F.

Remark 5.1.7. In Conjecture 5.1.6, the reason why we fix this θ is that we want to count anisotropic forms G' of a group by fixing its group of $\mathbb{R}$-points.

Remark 5.1.8. Recall that for a Unitary Shimura variety $\mathrm{Sh}_K(G,X)$, the group of $\mathbb{R}$-points $G(\mathbb{R})$ is a product $G(\mathbb{R}) := \prod_\tau U(p,q)$, and X is consequently a product

$$\prod_\tau U(p,q)/(U(p)\times U(q)),$$

the above assertion for Unitary groups indicates that for almost all arithmetic lattices Γ, the Shimura variety $\Gamma\backslash X$ is compact.

Remark 5.1.9. One can consider a variation of the main theorem as follows: can we count Shimura varieties attached to a given Hermitian symmetric domain? The difference of this question and the one by fixing a real reductive group comes from the observation that many different real groups may give a same Hermitian symmetric space. For example, every quaternion algebra B over a totally real number field F such that $B\otimes_v\mathbb{R} = M_2(\mathbb{R})$ for exactly one prime v corresponds to the upper and lower Poincaré half plane $\mathbb{H}^{\pm}$, but the groups are different.

Despite the differences between these two questions, we remark that Conjecture 5.1.6 implies that for any given Hermitian symmetric space X, "most" Shimura varieties that are quotients of X are compact. Take Shimura curves in the previous paragraph for example, they are compact if $F\neq\mathbb{Q}$. In general, take G be the adjoint algebraic group over $\mathbb{Q}$ such that $G(\mathbb{R})$ is the isometry group of X. Then for any adjoint algebraic group H over $\mathbb{Q}$ such that X is isomorphic to $K_H\backslash H(\mathbb{R})$ for some maximal compact subgroup K_H of $H(\mathbb{R})$, we have

$$H_\mathbb{R}\simeq G_\mathbb{R}\times K'_H$$

for some compact Lie group K'_H. One notices that H is $\mathbb{Q}$-anisotropic if $K'_H\neq 1$. Thus the "density" of non-compact Shimura varieties associated to a given Hermitian symmetric space X is at most the "density" of the ones associated to a given adjoint algebraic group.

Remark 5.1.10. It is worth observing that although most Shimura varieties are compact, there are still many important Shimura varieties that are non-compact, for example, modular curves, Hilbert modular surfaces and Siegel modular varieties.

5.2 Preliminaries in Galois cohomology

In this section we gather some results of Galois cohomology which will be used later. One can consult [72] or Chapter 6 of [65] for a fuller account of this rich theory.

The central objects in this paper are forms of algebraic groups.

Definition 5.2.1. Let G be an algebraic group over a field k of characteristic 0, an F-group H is said to be an k-**form** of G if there is a $\bar{k}$-isomorphism $f: G\to H$.

We use Galois cohomology in this paper since they classify forms of algebraic groups.

Proposition 5.2.2. *Let G be an algebraic group over a field k of characteristic* 0*, then there is a one-to-one correspondence between the set of F-forms of G and $H^1(k,\mathrm{Aut}_{\bar{k}}(G))$.*

Next we discuss inner forms. Notice that the subgroup $\mathrm{Int}_{\bar{k}}(G)$ consisting of the inner automorphisms of G can be identified with the adjoint group G^{ad}. We have the split exact sequence of groups

$$1 \to G^{\mathrm{ad}}_{\bar{k}} \to \mathrm{Aut}_{\bar{k}}(G) \to \mathrm{Sym}(R) \to 1,$$

where $\mathrm{Sym}(R)$ denotes the group of symmetries of the Dynkin diagrams of the root system R of G. We have thus a corresponding exact sequence for cohomologies

$$H^1(k, G^{\mathrm{ad}}_{\bar{k}}) \to H^1(k, \mathrm{Aut}_{\bar{k}}(G)) \to H^1(k, \mathrm{Sym}(R)).$$

Definition 5.2.3. A k-form H of G is called an **inner-form** if the corresponding cohomology class η_H of H in $H^1(k, \mathrm{Aut}_{\bar{k}}(G))$ comes from an element in $H^1(k, G^{\mathrm{ad}}_{\bar{k}})$.

Thus in order to study the density of inner forms of an algebraic group G over a number field F, it suffices to construct a reasonable counting function on $H^1(F, G^{\mathrm{ad}}_{\bar{F}})$. A natural idea is that one first define the counting function "locally", as explained in the introduction. In order to do this, we need the notion of the Kottwitz map, as we will explain in the sequel.

Let k be a field, we denote by $\mathrm{Br}(k)$ the second cohomology set $H^2(k, \mathbb{G}_m)$. Recall that in the classical theory of central simple algebras, there is an exact sequence for Brauer groups, i.e., the theorem of Albert-Brauer-Hasse-Noether.

$$0 \longrightarrow \mathrm{Br}(F) \longrightarrow \bigoplus_{v \le \infty} \mathrm{Br}(F_v) \to \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

where F is a number field, the surjective map above is the sum (called the trace map) of the local invariant maps $\mathrm{inv}_v : \mathrm{Br}(F_v) \longrightarrow \mathbb{Q}/\mathbb{Z}$. Similarly, one can study reductive groups by looking at their local behaviors. This observation can be realized by constructing the Kottwitz map, [47] and [48], which we explain now.

Let k be a local or global field of characteristic zero, and G be a connected reductive group over k. We let $\hat{G} = {}^L G^\circ$ be the identity component of its Langlands dual group, cf. [8].

We denote

$$A(G) = A(G/k) = \pi_0(Z(\hat{G}))^D$$

where $Z(\cdot)$ is the center, $\pi_0(\cdot)$ is the set of connected components and $(\cdot)^D$ is the pontryagin dual. If k is a global field, we further denote

$$A_v(G) = A(G/k_v)$$

for any prime v of k.

For a global field k, Kottwitz has constructed a map $\alpha_v : H^1(F_v, G) \longrightarrow A_v(G)$ for all $v \le \infty$.

Proposition 5.2.4 (Kottwitz)**.** *The map α_v is bijective if v is finite.*

For each v, we denote by $\beta_v : H^1(k_v, G) \longrightarrow A(G)$ the composition of the Kottwitz map with the natural map $A_v(G) \longrightarrow A(G)$. By taking the sum of these β_v (which is always a finite sum), we further obtain a map $\beta : \bigoplus_{v \leq \infty} H^1(k_v, G) \longrightarrow A(G)$, which satisfies the following property.

Proposition 5.2.5. *We have the following exact sequence*

$$H^1(k,G) \longrightarrow \bigoplus_{v\leq\infty} H^1(k_v,G) \longrightarrow A(G).$$

When G is the adjoint group of GL_n or a unitary group, the Kottwitz map can be computed explicitly. This serves as an important step in defining the height function.

Finally, let us give a proof of Proposition 5.1.4. We need two facts.

Theorem 5.2.6. *[Hasse principle for adjoint groups] Let G be an adjoint group over a number field F, i.e., the center of G is trivial. Then the map*

$$H^1(F,G) \to \bigoplus_v H^1(F_v,G)$$

is injective.

Proof. See Theorem 6.22, Chapter 6, p336 of [65]. □

Lemma 5.2.7. *Let G be a reductive group over a number field F. Then the projection map*

$$p_f : \bigoplus_v H^1(F_v,G) \to \bigoplus_{v<\infty} H^1(F_v,G)$$

has the following property: there is a positive integer Δ depending only on G and F, such that for each $\eta_f \in \bigoplus_{v<\infty} H^1(F_v,G)$, the fiber $p_f^{-1}(\eta_f)$ has cardinality Δ.

In fact, $\Delta = \prod_{v|\infty} \#H^1(F_v,G)$, which is independent of η_f.

Proof of Proposition 5.1.4. Let M and h_v as guaranteed by the definition of height function. Then the condition $h(\eta) < b$ for each $b > 0$ yields finitely many $(\mathrm{loc}_v\eta)_{v<\infty} \in \bigoplus_{v<\infty} H^1(F_v,G)$, thus finitely many elements in $\bigoplus_v H^1(F_v,G)$ by Lemma 5.2. And therefore, gives finitely many $\eta \in H^1(F,G)$ by Theorem 5.2.6.

□

5.3 Bounds for Dedekind zeta functions

Similar to the case of Riemann zeta function, we are able to give some upper bound for all Dedekind zeta functions. In this section, let us fix a number field K of degree d

and discriminant D. Recall that the **Dedekind zeta function** of K is defined as a Dirichlet series

$$\zeta_K(s) := \sum_{\mathfrak{a}} \frac{1}{N(\mathfrak{a})^s}$$

where $\mathfrak{a}$ runs over non-zero integral ideals of $\mathcal{O}_K$. The Dedekind zeta function has an analytic continuation to a meromorphic functio on $\mathbb{C}$, with the only simple pole at $s = 1$. The region $0 < \mathrm{Re}(s) < 1$ is called the **critical strip**. The zeros of the Dedekind zeta function outside (resp. inside) the critical strip are called **trivial zeros** (resp. **non-trivial zeros**). In the following, for a complex number s, we denote by σ and τ its real part and imaginary part respectively.

Theorem 5.3.1. *There are positive constants c, d such that for $|\tau| \geq d$ and $\sigma \geq 1 - c/\log|\tau|$, we have*

1. $\zeta_K'(s)/\zeta_K(s) \ll \log|\tau|$,
2. $1/\zeta_K(s) \ll \log|\tau|$,
3. $|\log\zeta_K(s)| \leq \log\log|\tau| + O(1)$.

One notices that when $K = \mathbb{Q}$, this result is classical, cf. Chapter II, Theorem 3.22 of [74]. Here we use similar methods in proving our Theorem 5.3.1. First, we need a Hadamard product formula for Dedekind zeta functions.

Theorem 5.3.2. *For suitable constants a, b depending only on K, we have*

$$\zeta_K(s) = \frac{e^{a+bs}}{(s-1)\Gamma(\frac{s}{2}+1)\Gamma(\frac{s}{2})^{d-1}\Gamma(\frac{s+1}{2})^{r_2}} \prod_{\rho}(1-\frac{s}{\rho})e^{s/\rho}. \tag{5.3.1}$$

where r_1 is the number of real embeddings of K, r_2 is the number of conjugate pairs of complex embeddings of K, and the product is over non-trivial zeros of $\zeta_K(s)$.

Proof. Let us consider the precise statement about the analytic continuation of $\zeta_K(s)$. Let

$$\Lambda_K(s) = |D|^{\frac{s}{2}}(2\sqrt{\pi})^{-r_2}\Gamma\left(\frac{s}{2}\right)^d \Gamma\left(\frac{s+1}{2}\right)^{r_2}.$$

Then $\Lambda_K(s)$ admits an analytic continuation to $\mathbb{C}$ as a meromorphic functions with only two simple poles at $s = 0$ and at $s = 1$, such that $\Lambda_K(s) = \Lambda_K(1-s)$, cf. p10 of [49]. Thus

$$\xi_K(s) := s(1-s)\Lambda_K(s)$$

is a holomorphic function on $\mathbb{C}$ such that the zeros of which are exactly the non-trivial zeros of $\zeta_K(s)$. One can further show that this function has order one. Thus by Hadamard's factorisation theorem, cf. Section 8.24 of [75], there exist constants a', b' such that

$$\xi_K(s) = e^{a'+b's}\prod_{\rho}(1-\frac{s}{\rho})e^{s/\rho}.$$

By substituting the equation defining $\xi_K(s)$ into the above formula, we obtain (5.3.1). $\square$

We also need the notion of generalized von Mangoldt function.

Definition 5.3.3. For any ideal $\mathfrak{a}$ of $\mathcal{O}_K$, let

$$\Lambda_K(\mathfrak{a}) = \begin{cases} \log N\mathfrak{p}, & \text{if } \mathfrak{a} = \mathfrak{p}^k \text{ for some } k \geq 1; \\ 0, & \text{otherwise.} \end{cases}$$

These are coefficients in the Dirichlet series

$$-\frac{\zeta_K'(s)}{\zeta_K(s)} = \sum_{\mathfrak{a}} \frac{\Lambda_K(\mathfrak{a})}{N(\mathfrak{p})^s}.$$

Next, let us review a result about zero-free region of Dedekind zeta functions.

Theorem 5.3.4. *There exists an absolute constant $c = c(K) \in]0, \frac{1}{2}[$ such that $\zeta_K(s)$ has no zero in the region*

$$\sigma \geq 1 - \frac{c}{\log(|\tau|+2)}.$$

Proof. By Theorem 5.33 of [41], there exists an absolute constant $c' > 0$ such that $\zeta_K(s)$ has no zero in the region

$$\sigma \geq 1 - \frac{c'}{d^2 \log D(|\tau|+3)^d}$$

except possibly a simple real zero $\beta < 1$. Now it suffices to take c to be $\min\{\frac{1}{2}, \frac{1-\beta}{2}\log 2, \frac{c'\log 2}{d^2 \log D3^d}\}$ or $\min\{\frac{1}{2}, \frac{c'\log 2}{d^2\log D3^d}\}$, depending on whether or not the exceptional real zero β exists. □

Let us then recall a special case of the Borel-Carathéodory theorem in complex analysis.

Theorem 5.3.5. *Let f be a holomorphic function in $|s| \leq R$ with $f(0) = 0$. Suppose $\max_{|s|=R} \mathrm{Re} f(s) \leq A$, then*

$$|f(s)| \leq \frac{2AR}{R-|s|} \qquad |s| < R.$$

Proof of Theorem 5.3.1. We may assume $\tau > 0$. Taking the logarithmic derivative of the product formula (5.3.1), we have

$$-\frac{\zeta_K'(s)}{\zeta_K(s)} = -b + \frac{1}{s-1} + \frac{\Gamma'(\frac{s}{2}+1)}{2\Gamma(\frac{s}{2}+1)} + \frac{(d-1)\Gamma'(\frac{s}{2})}{2\Gamma(\frac{s}{2})} + \frac{r_2\Gamma'(\frac{s+1}{2})}{2\Gamma(\frac{s+1}{2})} - \sum_{\rho}(\frac{1}{\rho} + \frac{1}{s-\rho}). \tag{5.3.2}$$

By Theorem 5.3.4, there is a constant $c \in]0, \frac{1}{16}[$ such that each non-trivial zero $\rho = \beta + i\gamma$ of $\zeta_K(s)$ satisfies

$$\beta < 1 - \frac{8c}{\log(|\gamma|+2)}.$$

We claim that

$$\min_{\rho} \mathrm{Re}\{\frac{1}{\rho} + \frac{1}{s-\rho}\} \geq 0 \quad \text{for } \tau \geq 4, \sigma \geq 1 - \frac{4c}{\log\tau}. \tag{5.3.3}$$

In fact, if $|s-\rho| > \frac{1}{2}|\rho|$, the quantity inside the minimum function on the left hand side can be written as

$$\frac{\beta}{|\rho|^2} + \frac{\sigma-\beta}{|s-\rho|^2} = \frac{(\frac{2|s-\rho|}{|\rho|})^2\beta/4+\sigma-\beta}{|s-\rho^2|} \geq \frac{\sigma-3/4}{|s-\rho|^2} > 0.$$

If $|s-\rho| \leq \frac{1}{2}|\rho|$, we have

$$|\mathrm{Im}(s-\rho)| = |\tau-\gamma| \leq \frac{1}{2}|\rho| \leq \frac{1}{2}(|\beta|+|\gamma|) \leq \frac{1}{2}(|\gamma|+1)$$

therefore $|\gamma| \leq 2\tau+2$, so

$$\beta < 1 - \frac{8c}{\log(2\tau+4)} \leq 1 - \frac{4c}{\log\tau} \leq \sigma,$$

which yields (5.3.3).

Substituting in (5.3.2), we have

$$-\mathrm{Re}\frac{\zeta_K'(s)}{\zeta_K(s)} \leq M\log\tau \tag{5.3.4}$$

for some constant M.

We are now ready to establish (1) of the theorem. We may assume τ is sufficiently large. Let $s=\sigma+i\tau$ be a fixed complex number. Write

$$\eta := \frac{c}{\log\tau}, \quad s_0 := 1+\eta+i\tau.$$

Then for each w with $|w| \leq 4\eta$, the point $s_0+w := \sigma'+i\tau'$ satisfies . Thus

$$-\mathrm{Re}\frac{\zeta_K'(s_0+w)}{\zeta_K(s_0+w)} \leq 2M\log\tau.$$

Define the function

$$f(w) := \frac{\zeta_K'(s)}{\zeta_K(s)} - \frac{\zeta_K'(s_0+w)}{\zeta_K(s_0+w)}.$$

Then f satisfies the conditions in Theorem 5.3.5 for $A := 2M\log\tau + |\frac{\zeta_K'(s_0)}{\zeta_K(s_0)}|$ in $|w| \leq 4\eta$. Since $|s-s_0| \leq 2\eta$, Theorem implies

$$\left|\frac{\zeta_K'(s)}{\zeta_K(s)}\right| \leq 4M\log\tau + 3\left|\frac{\zeta_K'(s_0)}{\zeta_K(s_0)}\right|.$$

We observe that

$$\left|\frac{\zeta_K'(s_0)}{\zeta_K(s_0)}\right| = \sum_{\mathfrak{a}} \frac{\Lambda_K(\mathfrak{a})}{N(\mathfrak{a})^{1+\eta}} = \frac{1}{\eta} + O(1) \ll \log\tau$$

thus the inequality (1) holds.

The third inequality is a direct consequence of (2). Now we prove (2). Notice that

$$\log\frac{\zeta_K(s)}{\zeta_K(s_0)} = \int_{s_0}^{s} \frac{\zeta_K'(w)}{\zeta_K(w)}\mathrm{d}w \ll |s-s_0|\log\tau \ll 1.$$

The required inequality follows by observing that

$$|\log\zeta_K(s_0)| \ll \log\zeta_K(1+\eta) \leq \log(1/\eta) + O(1) \leq \log\log\tau + O(1).$$

□

5.4 The generalized Selberg-Delange method

In this section, we give a generalization of the Selberg-Delange method, which enables us to estimate the summatory functions of certain arithmetic functions that are sufficiently "near" certain powers of Dedekind zeta functions. For a detailed survey of this method for the Riemann zeta function, see Chapter II. 5 of [74] or Section 7.4 of [54].

Definition 5.4.1. Let $z \in \mathbb{C}, c_0 > 0, 0 < \delta \leq 1, M > 0$, and let K be a number field. We say that a Dirichlet series $F(s)$ has the property $\mathcal{P}(K; z; c_0, \delta, M)$ if the Dirichlet series

$$G(s,z) := F(s)\zeta(s)_K^{-z}$$

can be continued holomorphically as a function for $\sigma \geq 1 - c_0/(1+\log^+|\tau|)$, and, in this domain, satisfies the bound

$$G(s,z) \leq M(1+|\tau|)^{1-\delta}.$$

Theorem 5.4.2. *Let $F(s) := \sum_{n\geq 1} \frac{a_n}{n^s}$ be a Dirichlet series of type $\mathcal{P}(K; z; c_0, \delta, M)$ such that $a_n \geq 0$ for all n. Then we have*

$$\sum_{n\leq x} a_n \sim c_K^z \frac{G(1,z)}{\Gamma(z)} x(\log x)^{z-1}, \quad x \longrightarrow +\infty,$$

where $c_K = \mathrm{Res}_{s=1}\zeta_K$.

The proof is just a combination of Theorem 5.3.1 and the results of Theorem 5.4.2 for $K = \mathbb{Q}$. One may also write down the error terms explicitly.

5.5 Asymptotic density of k-free ideals

We give an asymptotic formula for ideals of a ring of integers of some number field with small power of prime factors, generalising the classical counting theorem for square-free numbers.

Let F be a fixed number field, $\mathcal{O}_F$ be its ring of integers. Then any integral ideal $\mathfrak{a} \neq 0$ admits a unique factorization into prime ideals

$$\mathfrak{a} = \prod_{\mathfrak{p}} \mathfrak{p}^{\alpha_{\mathfrak{p}}},$$

where $\mathfrak{p}$ runs over maximal ideals of $\mathcal{O}_F$, $\alpha_{\mathfrak{p}}$ are non-negative integers, and $\alpha_{\mathfrak{p}} = 0$ for almost all $\mathfrak{p}$. We also denote $\alpha_{\mathfrak{p}} = \mathrm{ord}_{\mathfrak{p}}\mathfrak{a}$.

Definition 5.5.1. Let $k > 1$ be an integer. An integral ideal $\mathfrak{a}$ of $\mathcal{O}_F$ is called k-free if $\mathrm{ord}_{\mathfrak{p}}\mathfrak{a} < k$ for all maximal ideals $\mathfrak{p}$. For any $x > 0$, we denote by $Q_k(x)$ the set of k-free ideals $\mathfrak{a}$ of $\mathcal{O}_F$ such that the norm $N(\mathfrak{a}) < x$.

Theorem 5.5.2. *We have the following asymptotic formula*

$$\#Q_k(x) = \frac{c_F}{\zeta_F(k)}x + o(x), \quad x \longrightarrow +\infty,$$

where $\zeta_F(t)$ is the Dedekind zeta function for F, and $c_F = \mathrm{Res}_{s=1}\zeta_F$.

In order to prove this, we introduce the generalised Möbius function.

Definition 5.5.3. Let F be a number field. The Möbius function for F is defined by the following relation.

$$\mu_F(\mathfrak{a}) = \begin{cases} 1, & \text{if } \mathfrak{a} = \mathcal{O}_F; \\ (-1)^r, & \text{if } \mathfrak{a} \text{ is a product of } r \text{ distinct prime ideals;} \\ 0, & \text{if } \mathrm{ord}_{\mathfrak{p}}\mathfrak{a} > 1 \text{ for some prime ideal } \mathfrak{p}. \end{cases}$$

Similarly to the classical case, this Möbius function has the following properties.

Lemma 5.5.4. *Let $\mathfrak{a}$ be an integral ideal of $\mathcal{O}_F$, then*

$$\sum_{\mathfrak{b}|\mathfrak{a}} \mu_F(\mathfrak{b}) = \begin{cases} 1, & \text{if } \mathfrak{a} = \mathcal{O}_F; \\ 0, & \text{otherwise.} \end{cases}$$

Corollary 5.5.5. *Let $\mathfrak{a}$ be an integral ideal of $\mathcal{O}_F$, and k be a positive integer, then*

$$\sum_{\mathfrak{b}^k|\mathfrak{a}} \mu_F(\mathfrak{b}) = \begin{cases} 1, & \text{if } \mathfrak{a} \text{ is } k\text{-free;} \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 5.5.6. *Let $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, then*

$$\sum_{\mathfrak{a}} \frac{\mu_F(\mathfrak{a})}{N(\mathfrak{a})^s} = \frac{1}{\zeta_F(s)} = \prod_{\mathfrak{p}} (1 - N(\mathfrak{p})^{-s}).$$

The proofs of these two lemmas above are similar to the classical cases.

Let us also recall a useful corollary of the Wiener-Ikehara Tauberian theorem, cf. Corollary 8.8 of [54].

Proposition 5.5.7. *We write $\zeta_F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$, then we may write*

$$\sum_{n<x} a_n = c_F x + \varepsilon_F(x)$$

such that $\lim_{x\to+\infty} \varepsilon(x)/x = 0$, here $c_F = \mathrm{Res}_{s=1}\zeta_F$.

Proof of Theorem 5.5.2. We have

$$
\begin{aligned}
\#Q_k(x) &= \sum_{\substack{\mathfrak{a}\subset\mathcal{O}_F\\ N(\mathfrak{a})<x}} \sum_{\substack{\mathfrak{b}\subset\mathcal{O}_F\\ \mathfrak{b}^k|\mathfrak{a}}} \mu_F(\mathfrak{b})\\
&= \sum_{\substack{\mathfrak{b}\subset\mathcal{O}_F\\ N(\mathfrak{b})<x^{\frac{1}{k}}}} \mu_F(\mathfrak{b}) \sum_{\substack{\mathfrak{c}\subset\mathcal{O}_F\\ N(\mathfrak{c})<\frac{x}{N(\mathfrak{b})^k}}} 1\\
&= \sum_{\substack{\mathfrak{b}\subset\mathcal{O}_F\\ N(\mathfrak{b})<x^{\frac{1}{k}}}} \mu_F(\mathfrak{b}) \sum_{i<\frac{x}{N(\mathfrak{b})^k}} a_i\\
&= \sum_{\substack{\mathfrak{b}\subset\mathcal{O}_F\\ N(\mathfrak{b})<x^{\frac{1}{k}}}} \mu_F(\mathfrak{b}) \left(c_F\frac{x}{N(\mathfrak{b})^k}+\varepsilon\left(\frac{x}{N(\mathfrak{b})^k}\right)\right)\\
&= \sum_{\mathfrak{b}}\mu_F(\mathfrak{b})c_F\frac{x}{N(\mathfrak{b})^k} - \sum_{\substack{\mathfrak{b}\subset\mathcal{O}_F\\ N(\mathfrak{b})\geq x^{\frac{1}{k}}}} \mu_F(\mathfrak{b})c_F\frac{x}{N(\mathfrak{b})^k} + \sum_{\substack{\mathfrak{b}\subset\mathcal{O}_F\\ N(\mathfrak{b})<x^{\frac{1}{k}}}} \mu_F(\mathfrak{b})\varepsilon\left(\frac{x}{N(\mathfrak{b})^k}\right)\\
&= \frac{c_F}{\zeta_F(k)}x + xO\left(\frac{1}{(x^{\frac{1}{k}})^{k-1}}\right) + \sum_{\substack{\mathfrak{b}\subset\mathcal{O}_F\\ N(\mathfrak{b})<x^{\frac{1}{k}}}} \mu_F(\mathfrak{b})\varepsilon\left(\frac{x}{N(\mathfrak{b})^k}\right)\\
&= \frac{c_F}{\zeta_F(k)}x + o(x) + \sum_{\substack{\mathfrak{b}\subset\mathcal{O}_F\\ N(\mathfrak{b})<x^{\frac{1}{k}}}} \mu_F(\mathfrak{b})\varepsilon\left(\frac{x}{N(\mathfrak{b})^k}\right), \quad x\longrightarrow+\infty.
\end{aligned}
$$

Further, we notice that the function $\varepsilon_1(y) := |\varepsilon(y)/y|$ in bounded in $[1,+\infty)$, say, with M be an upper bound. Then we have $\left|a_n\frac{1}{x}\varepsilon(\frac{x}{n^k})\right| < M\frac{a_n}{n_k}$. Since $\lim_{x\to+\infty} a_n\frac{1}{x}\varepsilon(\frac{x}{n^k}) = 0$ for all n, by Lebesgue's Dominated Convergence Theorem,

$$
\left|\frac{1}{x}\sum_{\mathfrak{b};N(\mathfrak{b})<x^{\frac{1}{k}}} \mu_F(\mathfrak{b})\varepsilon(\frac{x}{N(\mathfrak{b})^k})\right| \leq \sum_{n<x^{\frac{1}{k}}} \frac{a_n}{x}\varepsilon\left(\frac{x}{n^k}\right) = o(1), \quad x\to+\infty
$$

Therefore

$$
\#Q_k(x) = \frac{c_F}{\zeta_F(k)}x + o(x), \quad x\to+\infty.
$$

□

In our paper, we need a following variation of Theorem 5.5.2.

Definition 5.5.8. Let $\mathfrak{a}$ be an integral ideal of O_F with a prime ideal decomposition $\mathfrak{a} = \prod \mathfrak{p}^{\mathrm{ord}_\mathfrak{p}}$, then we write $\Omega(\mathfrak{a}) = \sum_\mathfrak{p} \mathrm{ord}_\mathfrak{p}$.

Theorem 5.5.9. *Let m,ℓ be two non-negative integers such that $0\leq\ell\leq m-1$, we denote by $Q_k(m,\ell,x)$ the subset of $Q_k(x)$ consisting of ideals $\mathfrak{a}$ with the condition $\Omega(\mathfrak{a}) \equiv$*

$\ell(\bmod m)$. *We have the following asymptotic formula*

$$Q_k(m,\ell,x) = \frac{c_F}{m\zeta_F(k)}x + o(x), \quad x \longrightarrow +\infty,$$

where $\zeta_F(t)$ *is the Dedekind zeta function for* F, *and* $c_F = \mathrm{Res}_{s=1}\zeta_F$.

The proof is a standard application of the Selberg-Delange method. Let us first introduce a notation.

Definition 5.5.10. Let F be a number field, m, ℓ be two non-negative integers such that $0 \le \ell \le m-1$. For each integral ideal $\mathfrak{a}$ of $\mathcal{O}_F$, let

$$\mathbb{1}_{m,\ell}(\mathfrak{a}) = \frac{1}{m}\sum_{j=0}^{m-1} \xi_m^{\Omega(\mathfrak{a})-\ell},$$

where $\xi_m = e^{\frac{2\pi\sqrt{-1}}{m}}$. Thus $\mathbb{1}_{m,\ell}(\mathfrak{a})$ if equal to 1 if $\Omega(\mathfrak{a}) \equiv \ell(\bmod m)$, and equals to 0 otherwise.

Proof of Theorem 5.5.9. For each $0 \le j \le m-1$, we define the Dirichlet series

$$F_j(s) = \sum_{\mathfrak{a}\ k \text{ free}} \frac{\xi_m^{j\Omega(\mathfrak{a})}}{N(\mathfrak{a})^s}.$$

Then

$$F_j(s) = \prod_{\mathfrak{p}}\left(1 + \frac{\xi_m^j}{N(\mathfrak{p})^s} + \dots + \frac{\xi_m^{j(k-1)}}{N(\mathfrak{p})^{s(k-1)}}\right) = \prod_{\mathfrak{p}}(1-N(\mathfrak{p})^{-s})^{-\xi_m^j}H_{\mathfrak{p},j}(s) = \zeta_F(s)^{\xi_m^j}H_j(s).$$

If one writes $F_j(s) = \sum_n \frac{a_{n,j}}{n^s}$, and $\widetilde{H}_j(s) = F_j(s)\zeta(s)^{\xi^j}$ then by Theorem 5.4.2,

$$\sum_{\mathfrak{a}\ k\text{-free},\ N(\mathfrak{a})<x} \xi_m^{j\Omega(\mathfrak{a})} = \sum_{n<x} a_{n,j} \sim \frac{\widetilde{H}_j(1)}{\Gamma(\xi^j)}x(\log x)^{\xi^j-1}$$

which is asymptotically equal to $\frac{c_F}{\zeta_F(k)}x$ if $j=0$, and equal to $o(x)$ otherwise. Thus

$$\begin{aligned}
\#Q_k(m,\ell,x) &= \sum_{\substack{\mathfrak{a}\ k\text{-free} \\ N(\mathfrak{a})<x}} \mathbb{1}_{m,\ell}(\mathfrak{a}) \\
&= \sum_{\substack{\mathfrak{a}\ k\text{-free} \\ N(\mathfrak{a})<x}} \frac{1}{m}\sum_{j=0}^{m-1}\xi_m^{\Omega(\mathfrak{a})-\ell} \\
&= \frac{1}{m}\sum_{j=0}^{m-1}\xi_m^{-j\ell}\sum_{\substack{\mathfrak{a}\ k\text{-free} \\ N(\mathfrak{a})<x}}\xi_m^{j\Omega(\mathfrak{a})} \\
&\sim \frac{c_F}{m\zeta_F(k)}x.
\end{aligned}$$

□

5.6 Distribution of k-free ideals with split prime factors

When dealing with unitary groups, we need a following variant of the density theorem above. Let F be a totally real number field, and E be a CM extension of F.

Definition 5.6.1. Let $\mathfrak{a}$ be an integral ideal of O_F, we say that $\mathfrak{a}$ is **split** (respectively **non-split**) in E if all the prime ideals dividing $\mathfrak{a}$ are split (respectively inert or ramified) in E. Thus any integral ideal $\mathfrak{a}$ of O_F has a unique decomposition of two ideals $\mathfrak{a} = \mathfrak{a}_{\mathrm{sp}}\mathfrak{a}_{\mathrm{ns}}$ where $\mathfrak{a}_{\mathrm{sp}}$ is split and $\mathfrak{a}_{\mathrm{ns}}$ is non-split.

Definition 5.6.2. Let $k > 1, m, \ell$ be three integers such that $0 \leq \ell \leq m-1$, for any $x > 0$, we denote by $Q_k^{\mathrm{sp}}(E/F, m, \ell, x)$ (respectively $Q_k^{\mathrm{ns}}(E/F, m, \ell, x)$) the set of k-free split (respectively non-split) ideals $\mathfrak{a}$ of $\mathcal{O}_F$ with the condition $N(\mathfrak{a}) < x$ and $\Omega(\mathfrak{a}) \equiv \ell (\mathrm{mod}\, m)$, such that all the non-zero prime ideals dividing $\mathfrak{a}$ are split (respectively non-split) in E.

Theorem 5.6.3. *There are constants $c, c' > 0$, depending only on E, F, k, m, ℓ, such that we have the following asymptotic formulae*

$$\#Q_k^{\mathrm{sp}}(E/F, m, \ell, x) \sim c\frac{x}{\sqrt{\log x}}, \quad x \longrightarrow +\infty$$

$$\#Q_k^{\mathrm{ns}}(E/F, m, \ell, x) \sim c'\frac{x}{\sqrt{\log x}}, \quad x \longrightarrow +\infty.$$

Again, the proof is a standard application of the Selberg-Delange method. First, let us introduce a generalisation of the Legendre symbol.

Definition 5.6.4. Let E/F be a quadratic extension of number fields, let

$$\chi_{E/F} : \{\text{integral ideals of } \mathcal{O}_F\} \to \mathbb{C}^\times$$

be the ray class character such that for any maximal ideal $\mathfrak{p}$ of $\mathcal{O}_F$,

$$\chi_{E/F}(\mathfrak{p}) = \begin{cases} 1, & \text{if } \mathfrak{p} \text{ is split in } E; \\ -1, & \text{if } \mathfrak{p} \text{ is inert in } E; \\ 0, & \text{if } \mathfrak{p} \text{ is ramified in } E. \end{cases}$$

Proof. We only prove the first equation, the second one is similar. For $0 \leq j \leq m-1$, we define the Dirichlet series

$$F_j^{\mathrm{sp}}(s) = \sum_{\mathfrak{a}\ k \text{ free and split}} \frac{\xi_m^{j\Omega(\mathfrak{a})}}{N(\mathfrak{a})^s}.$$

Recall that we have

$$\zeta_E(s) = \zeta_F(s)L(s, \chi_{E/F}) = \prod_{\mathfrak{p} \text{ split}} \left(1 - \frac{1}{N(\mathfrak{p})^s}\right)^{-2} \prod_{\mathfrak{p} \text{ inert}} \left(1 - \frac{1}{N(\mathfrak{p})^{2s}}\right)^{-1} \prod_{\mathfrak{p} \text{ ramified}} \left(1 - \frac{1}{N(\mathfrak{p})^s}\right)^{-1}$$

Then

$$
\begin{aligned}
F_j^{\mathrm{sp}}(s) &= \prod_{\mathfrak{p}\text{ split}} \left(1+\frac{\xi_m^j}{N(\mathfrak{p})^s}+\dots+\frac{\xi_m^{j(k-1)}}{N(\mathfrak{p})^{s(k-1)}}\right) \\
&= \prod_{\mathfrak{p}\text{ split}} \left(1-\frac{\xi_m^j}{N(\mathfrak{p})}\right) H_{\mathfrak{p},j}(s) \\
&= \zeta_E(s)^{\frac{\xi_m^j}{2}} \widetilde{H}_{\mathfrak{p},j}(s)
\end{aligned}
$$

If one writes $F_j^{\mathrm{sp}}(s) = \sum_n \frac{a_{n,j}^{\mathrm{sp}}}{n^s}$, then by Theorem 5.4.2,

$$
\sum_{\substack{\mathfrak{a}\ k\text{-free split} \\ N(\mathfrak{a})<x}} \xi_m^{j\Omega(\mathfrak{a})} = \sum_{n<x} a_{n,j}^{\mathrm{sp}} \sim \frac{\widetilde{H}_j(1)}{\Gamma(\xi^j)} x(\log x)^{\frac{\xi^j}{2}-1}
$$

which is asymptotically equal to $c\frac{x}{\sqrt{\log x}}$ for some constant c if $j=0$, and equal to $o(\frac{x}{\sqrt{\log x}})$ otherwise. Therefore

$$
\begin{aligned}
\#Q_k^{\mathrm{sp}}(E/F,m,\ell,x) &= \sum_{\substack{\mathfrak{a}\ k\text{-free split} \\ N(\mathfrak{a})<x}} \mathbb{1}_{m,\ell}(\mathfrak{a}) \\
&= \sum_{\substack{\mathfrak{a}\ k\text{-free split} \\ N(\mathfrak{a})<x}} \frac{1}{m}\sum_{j=0}^{m-1} \xi_m^{\Omega(\mathfrak{a})-\ell} \\
&= \frac{1}{m}\sum_{j=0}^{m-1} \xi_m^{-j\ell} \sum_{\substack{\mathfrak{a}\ k\text{-free split} \\ N(\mathfrak{a})<x}} \xi_m^{j\Omega(\mathfrak{a})} \\
&\sim \frac{c}{m}\frac{x}{\sqrt{\log x}} x.
\end{aligned}
$$

□

5.7 Inner forms of General linear groups

In this section we fix a number field F and prove Conjecture 5.1.6 for the group GL_n over F. We know that for k a local field or global field, the k-inner forms of GL_n are of the shape $G = \mathrm{Res}_{D/k}\mathrm{GL}_m$, where D is a division algebra over k satisfying $m\sqrt{\dim_k D} = n$. Observe that the adjoint group $G^{\mathrm{ad}} = (\mathrm{Res}_{D/F}\mathrm{GL}_m)^{\mathrm{ad}}$ is F-anisotropic if and only if $m=1$. In fact, if $m=1$, then G^{ad} has no positive dimensional split torus since F is the center of D; if $m>1$, then

$$
T = \{\mathrm{diag}(\lambda_1,\dots,\lambda_m);\lambda_i\in D^\times\}/\{\mathrm{diag}(\lambda,\dots,\lambda);\lambda\in D^\times\}
$$

is a positive dimensional split torus inside G^{ad}.

Now we define the height function $h: H^1(F, \mathrm{PGL}_n) \to \mathbb{R}_{\geq 0}$. For any $\eta \in H^1(F, \mathrm{PGL}_n)$ and $v < \infty$, we denote by $\mathrm{loc}_v \eta$ its image in $H^1(F_v, G)$ and $\mathrm{GL}_m(D_v)$ be the corresponding inner form, where D_v is a division algebra. We define

$$h_v : H^1(F_v, G) \to \{0, 1, ..., n-1\} \subset \mathbb{R}_{\geq 0}$$

by letting $h_v(\mathrm{loc}_v \eta)$ be the unique integer in $\{0, 1, ..., n-1\}$ such that $\frac{1}{n} h_v(\mathrm{loc}_v \eta) + \mathbb{Z} \in \frac{1}{n}\mathbb{Z}/\mathbb{Z}$ is equal to the invariant of D_v. Then if η corresponds to an F-isotropic inner form, there is a divisor $r > 1$ of n such that $r \mid h_v(\mathrm{loc}_v \eta)$ for all v.

Definition 5.7.1. We define the function $h: H^1(F, \mathrm{PGL}_n) \to \mathbb{R}_{\geq 0}$ by

$$h(\eta) = \sum_{v < \infty} h_v(\mathrm{loc}_v \eta) \log |\kappa_v|$$

where κ_v is the residue field.

Then this h is a height function in the sense of Definition 5.1.4.

Proof of Conjecture 5.1.6 for GL_n. Take the function h as above and fix $\theta \in \bigoplus_{v|\infty} H^1(F_v, \mathrm{PGL}_n)$. Let $\mathrm{inv}_\infty \theta = 0$ or $\frac{n}{2}$ (only happens if n is even) n times the sum of the invariants of θ for all $v \mid \infty$. We denote by A as the set

$$A = \{(a_v) \in \bigoplus_{v<\infty} \{0, 1, ..., n-1\}; \sum_v a_v \equiv \mathrm{inv}_\infty \theta (\mathrm{mod}\ n)\}.$$

We have

$$\begin{aligned}
N_F(h, \mathrm{PGL}_n, T)_\theta &= \{\eta \in A; \sum_{v<\infty} h_v(\mathrm{loc}_v \eta) \log |\kappa_v| < T\} \\
&= \{\eta \in A; \sum_{v<\infty} h_v(\mathrm{loc}_v \eta) \log N(\mathfrak{p}_v) < T\} \\
&= \{\eta \in A; \prod_{v<\infty} N(\mathfrak{p}_v)^{h_v(\mathrm{loc}_v \eta)} < e^T\} \\
&= \{\mathfrak{a}\ n\text{-free}, \Omega(\mathfrak{a}) \equiv \mathrm{inv}_\infty \theta (\mathrm{mod}\ n) \text{ and } N(\mathfrak{a}) < e^T\} \\
&= Q_n(n, \mathrm{inv}_\infty \theta, e^T)
\end{aligned}$$

and

$$\begin{aligned}
N_F^{nc}(h, \mathrm{PGL}_n, T)_\theta &\subset \bigcup_{r>1, r|n} \{\eta \in A; \sum_{v<\infty} h_v(\mathrm{loc}_v \eta) \log |\kappa_v| < T \text{ and } r \mid h_v(\mathrm{loc}_v \eta), \forall v\} \\
&= \bigcup_{r>1, r|n} \{\eta \in A; \sum_{v<\infty} h_v(\mathrm{loc}_v \eta) \log N(\mathfrak{p}_v) < T \text{ and } r \mid h_v(\mathrm{loc}_v \eta), \forall v\} \\
&= \bigcup_{r>1, r|n} \{\eta \in A; \prod_{v<\infty} N(\mathfrak{p}_v)^{h_v(\mathrm{loc}_v \eta)} < e^T \text{ and } r \mid h_v(\mathrm{loc}_v \eta), \forall v\} \\
&\subset \bigcup_{r>1, r|n} \{\mathfrak{a}\ n\text{-free}, N(\mathfrak{a}) < e^T \text{ and } r \mid \mathrm{ord}_{\mathfrak{p}_v} \mathfrak{a} \text{ for all } v\} \\
&= \bigcup_{r>1, r|n} \{\mathfrak{a} = \mathfrak{b}^r; \mathfrak{b} \in Q_{\frac{n}{r}}(e^{\frac{T}{r}})\}
\end{aligned}$$

Therefore

$$\begin{aligned}
\lim_{T\to\infty} \frac{\#N_F^{nc}(h,\mathrm{PGL}_n,T)_\theta}{\#N_F(h,\mathrm{PGL}_n,T)_\theta} &\le \sum_{r>1,r|n} \lim_{T\to\infty} \frac{\#Q_{\frac{n}{r}}(e^{\frac{T}{r}})}{\#Q_n(n,\mathrm{inv}_\infty\theta,e^T)} \\
&\le \sum_{r>1,r|n} n \lim_{T\to\infty} \frac{\frac{c_F}{\zeta(\frac{n}{r})}e^{\frac{T}{r}}+o(e^{\frac{T}{r}})}{\frac{c_F}{\zeta(n)}e^T+o(e^T)} \\
&= 0
\end{aligned}$$

as desired. □

5.8 Inner forms of Unitary groups

Now we prove Conjecture 5.1.6 for Unitary groups. We first fix some notations for Unitary groups, cf. §1.9 of [69]. Let F be a totally real number field and E is a totally imaginary quadratic (i.e., CM) extension of F. Let ι be the only non-trivial element of $\mathrm{Gal}(E/F)$, we also denote $\bar{x}$ for $\iota(x)$. Let $\Phi \in M_n(E)$ be a Hermitian matrix, i.e., ${}^t\bar{\Phi} = \Phi$, then we have an associated unitary group $U_{E/F}(n,\Phi)$ over F such that

$$U_{E/F}(n,\Phi)(A) := \{g \in \mathrm{GL}_n(E\otimes_F A); t\bar{g}\Phi g = \Phi\}$$

for any F-algebra A. In particular, if we take $\Phi = \Phi_n := \theta((-1)^{i-1}\delta_{i,n+1-j})$, where $\theta = 1$ if n is odd, and an arbitrary non-zero element in E of trace zero if n is even, and $\delta_{a,b}$ is the Kronecker symbol, we get the Unitary group $U(n) := U_{E/F}(n,\Phi)$. Notice that the choice of θ has no influence in defining our group. In this section, we denote by $G\,U(n)_{\mathrm{ad}}$.

Definition 5.8.1 (cf. Rogawski [69]). A **unitary group** over F is an F-inner form of $U(n)$.

We have the following more explicit description of Unitary groups.

Proposition 5.8.2. *Let G be a unitary group which is an F-inner form of $U(n)$, then there is a central simple algebra D over E together with an involution α of second kind, such that*

$$G(A) = \{g \in (D\otimes A)^\times; \alpha(g)g = 1\}$$

for any F-algebra A. In particular, G^{ad} is F-anisotropic if D is a division algebra.

In this case, the Kottwitz map can be calculated explicitly.

Proposition 5.8.3 (cf. Chapter 0 of [42] or Section 2 of [15]). *The Kottwitz map*

$$\alpha_v : H^1(F_v,U(n)_{\mathrm{ad}}) \longrightarrow A_v(U(n)_{\mathrm{ad}})$$

can be interpreted as follows

- *If v is finite and split in E, $A_v(U(n)_{\mathrm{ad}}) = A_v(\mathrm{PGL}_n) = \mathbb{Z}/n\mathbb{Z}$.*

- *If v is finite and non-split in E, then $A_v(U(n)_{\mathrm{ad}})$ is equal to $\mathbb{Z}/2\mathbb{Z}$ when n is even (in this case the only non quasi-split unitary group over F_v corresponds to $1+2\mathbb{Z}$), and 0 when n is odd.*
- *If v is real and split in E, then $A_v(U(n)_{\mathrm{ad}})$ is equal to $\mathbb{Z}/2\mathbb{Z}$ when n is even (in this case the group $\mathrm{GL}_{\frac{n}{2}}(\mathbb{H})$ corresponds to $1+2\mathbb{Z}$, where $\mathbb{H}$ is the usual quaternion algebra over $\mathbb{R}$), and 0 when n is odd.*
- *If v is real and not split in E, then $A_v(U(n)_{\mathrm{ad}})$ is equal to $\mathbb{Z}/2\mathbb{Z}$ when n is even (in this case the real unitary group $U(p,q)$ corresponds to $\frac{p-q}{2}+2\mathbb{Z}$), and 0 when n is odd.*

Again, for any $\eta \in H^1(F,U(n)_{\mathrm{ad}})$ and $v \leq \infty$, we write $\mathrm{loc}_v(\eta)$ as its image in $H^1(F_v,U(n)_{\mathrm{ad}})$ under the localization map. We further define

$$h_v : H^1(F_v,U(n)_{\mathrm{ad}}) \to \{0,1,...,n-1\}$$

by letting $h_v(\mathrm{loc}_v(\eta))$ be the unique integer in $\{0,1,...,n-1\}$ such that
- $h_v(\mathrm{loc}_v\eta)n\mathbb{Z} = \alpha_v(\mathrm{loc}_v\eta)$, if n is odd or n is even and $v < \infty$ is non-split
- $\frac{2}{n}h_v(\mathrm{loc}_v\eta)+2\mathbb{Z} = \alpha_v(\mathrm{loc}_v\eta)$ if n is even and v is infinite or not split

where α_v is the Kottwitz map.

The following proposition, which is a consequence of Proposition 5.2.5, gives a criterion of when a family of local inner forms come from a global inner form, which is also called the **trace condition**.

Proposition 5.8.4 (trace condition)**.** *Any local datum $(\eta_v)_v \in \bigoplus_{v\leq\infty} H^1(F_v,U(n)_{\mathrm{ad}})$ comes from a global unitary group if n is odd, and if n is even, a sufficient and necessary additional condition is that*

$$\sum_{v\leq\infty} h_v(\mathrm{loc}_v\eta) = 0 \text{ in } \mathbb{Z}/2\mathbb{Z}.$$

Definition 5.8.5. We define the function $h : H^1(F,U(n)_{\mathrm{ad}}) \to \mathbb{R}_{\geq 0}$ by

$$h(\eta) = \sum_{v<\infty} h_v(\mathrm{loc}_v\eta)\log|\kappa_v|$$

where κ_v is the residue field.

This h indeed gives a height function in the sense of definition 5.1.4.

Remark 5.8.6. In the definition of height h, for n even and v not split, we take $\frac{n}{2}$ times the local invariant in $\mathbb{Z}/2\mathbb{Z}$ instead of the local invariant itself. The reason is that we want non-split primes have "similar" amount of contributions to the height compare to split primes.

Now we are ready to show that "most" unitary groups are anisotropic.

Proof of Conjecture 5.1.6 for $U(n)$. Take the function h as above, and fix $\theta \in \bigoplus_{v|\infty} H^1(F_v,G)$. Since the trace condition depends on whether n is even or odd, we will prove the theorem in these two cases.

Case 1: n is odd. Then $A(G)=0$, the trace condition is empty, i.e., every almost everywhere trivial family of local data comes from a global inner form. We denote by B as the set

$$B = \bigoplus_{v<\infty, v \text{ splits}} \{0,1,...,n-1\}.$$

Thus

$$\begin{aligned}
N_F(h,G,T)_\theta &= \{\eta \in B; \sum_{v<\infty, v \text{ splits}} h_v(\mathrm{loc}_v\eta)\log|\kappa_v| < T\} \\
&= \{\eta \in B; \sum_{v<\infty, v \text{ splits}} h_v(\mathrm{loc}_v\eta)\log N(\mathfrak{p}_v) < T\} \\
&= \{\eta \in B; \prod_{v<\infty, v \text{ splits}} N(\mathfrak{p}_v)^{h_v(\mathrm{loc}_v\eta)} < e^T\} \\
&= \{\mathfrak{a} \text{ ideal of } O_F; \mathfrak{a} \text{ is } n\text{-free, split and } N(\mathfrak{a}) < e^T\} \\
&= Q_n^{\mathrm{sp}}(E/F, e^T)
\end{aligned}$$

and

$$\begin{aligned}
N_F^{nc}(h,G,T)_\theta &\subset \bigcup_{r>1, r|n} \{\eta \in B; \sum_{v<\infty, v \text{ splits}} h_v(\mathrm{loc}_v\eta)\log|\kappa_v| < T \text{ and } r \mid h_v(\mathrm{loc}_v\eta), \forall v\} \\
&= \bigcup_{r>1, r|n} \{\eta \in B; \sum_{v<\infty, v \text{ splits}} h_v(\mathrm{loc}_v\eta)\log N(\mathfrak{p}_v) < T \text{ and } r \mid h_v(\mathrm{loc}_v\eta), \forall v\} \\
&= \bigcup_{r>1, r|n} \{\eta \in B; \prod_{v<\infty, v \text{ splits}} N(\mathfrak{p}_v)^{h_v(\mathrm{loc}_v\eta)} < e^T \text{ and } r \mid h_v(\mathrm{loc}_v\eta), \forall v\} \\
&= \bigcup_{r>1, r|n} \{\mathfrak{a} \text{ ideal of } O_F; \mathfrak{a} \text{ is } n\text{-free, split and } N(\mathfrak{a}) < e^T \text{ and } r \mid \mathrm{ord}_{\mathfrak{p}_v}\mathfrak{a}, \forall v\} \\
&= \bigcup_{r>1, r|n} \{\mathfrak{a} = \mathfrak{b}^r \text{ be an ideal of } O_F; \mathfrak{b} \text{ is } \frac{n}{r}\text{-free, split and } N(\mathfrak{a}) < e^{\frac{n}{r}}\} \\
&= \bigcup_{r>1, r|n} \{\mathfrak{a} = \mathfrak{b}^r \text{ be an ideal of } O_F; \mathfrak{b} \in Q_{\frac{n}{r}}^{\mathrm{sp}}(E/F, e^{\frac{n}{r}})\}
\end{aligned}$$

We have

$$\begin{aligned}
\lim_{T\to\infty} \frac{\#N_F^{nc}(h,G,T)_\theta}{\#N_F(h,G,T)_\theta} &\le \lim_{T\to\infty} \sum_{r>1, r|n} \frac{\#Q_{\frac{n}{r}}^{\mathrm{sp}}(E/F, e^{\frac{T}{r}})}{\#Q_n^{\mathrm{sp}}(E/F, e^T)} \\
&= \lim_{T\to\infty} \sum_{r>1, r|n} \frac{c\frac{e^{\frac{T}{r}}}{\sqrt{\log e^{\frac{T}{r}}}} + o(\frac{e^{\frac{T}{r}}}{\sqrt{\log e^{\frac{T}{r}}}})}{\tilde{c}\frac{e^T}{\sqrt{\log e^T}} + o(\frac{e^T}{\sqrt{\log e^T}})} \\
&= 0
\end{aligned}$$

as desired.

Case 2: n is even. Then $A(G)=\mathbb{Z}/2\mathbb{Z}$, thus a family of local data comes from a global inner form if and only if their sum in $\mathbb{Z}/2\mathbb{Z}$ is zero. We denote by $B = B_\theta$ as the set

$$\left\{(a_v)_v \in \bigoplus_{v<\infty, v \text{ splits}} \{0,1,...,n-1\} \bigcup \bigoplus_{v<\infty, v \text{ non-split}} \{0, \frac{n}{2}\}; \sum_{v \text{ split}} a_v + \sum_{v \text{ non-split}} \frac{2}{n} a_v = \mathrm{inv}_\infty \theta \text{ in } \mathbb{Z}/2\mathbb{Z}\right\},$$

where $\mathrm{inv}_\infty \theta \in \mathbb{Z}/2\mathbb{Z}$ is the sum of the invariants of θ for all $v \mid \infty$ and the equation

$$\sum_{v \text{ split}} a_v + \sum_{v \text{ non-split}} \frac{2}{n} a_v = \mathrm{inv}_\infty \theta \text{ in } \mathbb{Z}/2\mathbb{Z}$$

is given by the trace condition. Thus

$$\begin{aligned}
N_F(h,G,T)_\theta &= \{\eta \in B; \sum_{v<\infty} h_v(\mathrm{loc}_v \eta) \log|\kappa_v| < T\} \\
&= \{\eta \in B; \sum_{v<\infty} h_v(\mathrm{loc}_v \eta) \log N(\mathfrak{p}_v) < T\} \\
&= \{\eta \in B; \prod_{v<\infty} N(\mathfrak{p}_v)^{h_v(\mathrm{loc}_v \eta)} < e^T\} \\
&= \{\mathfrak{a} \ n\text{-free}; N(\mathfrak{a}) < e^T, \frac{n}{2} \mid \mathrm{ord}_{\mathfrak{p}_v} \mathfrak{a} \text{ for } v \text{ non-split and trace condition holds}\} \\
&\supset \{\mathfrak{a} \ n\text{-free, split}; N(\mathfrak{a}) < e^T \text{ and trace condition holds}\}
\end{aligned}$$

and we have

$$\begin{aligned}
N_F^{nc}(h,G,T)_\theta &\subset \bigcup_{r>1, r|n} \{\eta \in B; \sum_{v<\infty} h_v(\mathrm{loc}_v \eta) \log|\kappa_v| < T \text{ and } r \mid h_v(\mathrm{loc}_v \eta), \forall v \text{ splits}\} \\
&= \bigcup_{r>1, r|n} \{\eta \in B; \sum_{v<\infty} h_v(\mathrm{loc}_v \eta) \log N(\mathfrak{p}_v) < T \text{ and } r \mid h_v(\mathrm{loc}_v \eta), \forall v \text{ splits}\} \\
&= \bigcup_{r>1, r|n} \{\eta \in B; \prod_{v<\infty} N(\mathfrak{p}_v)^{h_v(\mathrm{loc}_v \eta)} < e^T \text{ and } r \mid h_v(\mathrm{loc}_v \eta), \forall v \text{ splits}\} \\
&= \bigcup_{r>1, r|n} \left\{\mathfrak{a} \text{ ideal of } O_F \,\middle|\, \begin{array}{c} \mathfrak{a} \text{ is } n\text{-free}, N(\mathfrak{a}) < e^T \text{ and } r \mid \mathrm{ord}_{\mathfrak{p}_v} \mathfrak{a}, \forall v \text{ splits} \\ \text{and } \frac{n}{2} \mid \mathrm{ord}_{\mathfrak{p}_v} \mathfrak{a}, \forall v \text{ non-split} \end{array}\right\} \\
&\subset \bigcup_{r>1, r|n} \left\{\mathfrak{a} \,\middle|\, \begin{array}{c} \mathfrak{a} \text{ is } n\text{-free split}, N(\mathfrak{a}) < e^T \\ \text{and } r \mid \mathrm{ord}_{\mathfrak{p}_v} \mathfrak{a}, \forall v \end{array}\right\} \cdot \left\{\mathfrak{a} \,\middle|\, \begin{array}{c} \mathfrak{a} \text{ is } n\text{-free non-split}, \\ N(\mathfrak{a}) < e^T \text{ and } \frac{n}{2} \mid \mathrm{ord}_{\mathfrak{p}_v} \mathfrak{a}, \forall v \end{array}\right\} \\
&= \bigcup_{r>1, r|n} \left\{\mathfrak{a} = \mathfrak{b}^r \,\middle|\, \mathfrak{b} \text{ is } \frac{n}{r}\text{-free split}, N(\mathfrak{b}) < e^{\frac{T}{r}} \right\} \cdot \left\{\mathfrak{a} = \mathfrak{b}^{\frac{n}{2}} \,\middle|\, \begin{array}{c} \mathfrak{b} \text{ is squarefree} \\ \text{non-split}, N(\mathfrak{b}) < e^{\frac{2T}{n}} \end{array}\right\} \\
&= \bigcup_{r>1, r|n} \left\{\mathfrak{a} = \mathfrak{b}^r \,\middle|\, \mathfrak{b} \in Q^{\mathrm{sp}}_{\frac{n}{r}}(E/F, e^{\frac{n}{r}}) \right\} \cdot \left\{\mathfrak{a} = \mathfrak{b}^{\frac{n}{2}} \,\middle|\, \mathfrak{b} \in Q^{\mathrm{ns}}_2(E/F, e^{\frac{2T}{n}}) \right\}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\lim_{T\to\infty}\frac{\#N_F^{nc}(h,G,T)_\theta}{\#N_F(h,G,T)_\theta} &\le \lim_{T\to\infty}\sum_{r>1,r|n}\frac{\#Q^{\mathrm{sp}}_{\frac{n}{r}}(E/F,e^{\frac{T}{r}})\#Q^{\mathrm{ns}}_2(E/F,e^{\frac{2T}{n}})}{\frac{1}{2}\#Q^{\mathrm{sp}}_n(E/F,e^T)}\\
&= \lim_{T\to\infty}\sum_{r>1,r|n}\frac{c\frac{e^{\frac{T}{r}}}{\sqrt{\log e^{\frac{T}{r}}}}\cdot\frac{e^{\frac{2T}{n}}}{\sqrt{\log e^{\frac{2T}{n}}}}}{\tilde{c}\frac{e^T}{\sqrt{\log e^T}}+o(\frac{e^T}{\sqrt{\log e^T}})}\\
&= 0.
\end{aligned}$$

$\square$

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Titre : Autour de la conjecture de Zilber-Pink pour les Variétés de Shimura.

Mots-clefs : variété de Shimura, O-minimalité, géométrie diophantienne, cohomologie galoisienne, groupe dual, application de Kottwitz.

Résumé : Dans cette thèse, nous nous intéressons à l'étude de l'arithmétique et de la géometrie des variétés de Shimura. Cette thèse s'est essentiellement organisée autour de trois volets.

Dans la première partie, on étudie certaines applications de la théorie des modèles en théorie des nombres. En 2014, Pila et Tsimerman ont donné une preuve de la conjecture d'Ax-Schanuel pour la fonction j et, avec Mok, ont récemment annoncé une preuve de sa généralisation à toute variété de Shimura. Nous nous référons à cette généralisation comme à la conjecture d'Ax-Schanuel hyperbolique. Dans ce projet, nous cherchons à généraliser les idées de Habegger et Pila pour montrer que, sous un certain nombre d'hypothèses arithmétiques, la conjecture d'Ax-Schanuel hyperbolique implique, par une extension de la stratégie de Pila-Zannier, la conjecture de Zilber-Pink pour les variétés de Shimura. Nous concluons en vérifiant toutes ces hypothèses arithmétiques à l'exception d'une seule dans le cas d'un produit de courbes modulaires, en admettant la conjecture dite des grandes orbites de Galois. Il s'agit d'un travail en commun avec Christopher Daw.

La seconde partie est consacrée à un résultat cohomologique en direction de la conjecture de Zilber-Pink. Étant donné un groupe algébrique semi-simple sur un corps de nombres F contenu dans $\mathbb{R}$, nous démontrons que deux sous-groupes algébriques semi-simples définis sur F sont conjugués sur $\bar{F}$, si et seulement s'il le sont sur une extension réelle finie de F de degré majoré indépendamment des sous-groupes choisis. Il s'agit d'un travail en commun avec Mikhail Borovoi et Christopher Daw.

La troisième partie étudie la distribution des variétés de Shimura compactes. On rappelle qu'une variété de Shimura S de dimension 1 est toujours compacte sauf si S est une courbe modulaire. Nous généralisons cette observation en définissant une fonction de hauteur dans l'espace des variétés de Shimura associées à un groupe réductif réel donné. Dans le cas des groupes unitaires, on prouve que la densité des variétés de Shimura non-compactes est nulle.

Title: Around the Zilber-Pink conjecture for Shimura varieties.

Keywords: Shimura variety, O-minimality, Diophantine geometry, Galois cohomology, dual group, Kottwitz map.

Abstract: In this thesis, we study some arithmetic and geometric problems for Shimura varieties. This thesis consists of three parts.

In the first part, we study some applications of model theory to number theory. In 2014, Pila and Tsimerman gave a proof of the Ax-Schanuel conjecture for the j-function and, with Mok, have recently announced a proof of its generalization to any (pure) Shimura variety. We refer to this generalization as the hyperbolic Ax-Schanuel conjecture. In this article, we show that the hyperbolic Ax-Schanuel conjecture can be used to reduce the Zilber-Pink conjecture for Shimura varieties to a problem of point counting. We further show that this point counting problem can be tackled in a number of cases using the Pila-Wilkie counting theorem and several arithmetic conjectures. Our methods are inspired by previous applications of the Pila-Zannier method and, in particular, the recent proof by Habegger and Pila of the Zilber-Pink conjecture for curves in abelian varieties. This is joint work with Christopher Daw.

The second part is devoted to a Galois cohomological result towards the proof of the Zilber-Pink conjecture. Let G be a linear algebraic group over a field k of characteristic 0. We show that any two connected semisimple k-subgroups of G that are conjugate over an algebraic closure of k are actually conjugate over a finite field extension of k of degree bounded independently of the subgroups. Moreover, if k is a *real number field*, we show that any two connected semisimple k-subgroups of G that are conjugate over the field of real numbers $\mathbb{R}$ are actually conjugate over a finite *real* extension of k of degree bounded independently of the subgroups. This is joint work with Mikhail Borovoi and Christopher Daw.

Finally, in the third part, we consider the distribution of compact Shimura varieties. We recall that a Shimura variety S of dimension 1 is always compact unless S is a modular curve. We generalise this observation by defining a height function in the space of Shimura varieties attached to a fixed real reductive group. In the case of unitary groups, we prove that the density of non-compact Shimura varieties is zero.

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