Contraction de cônes complexes multidimensionnels
Maxence Novel

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THÈSE DE DOCTORAT

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Contraction de cônes complexes multidimensionnels

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Après avis des rapporteurs : PHILIPPE THIEULLEN (Université Bordeaux 1)
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Contents

Chapter 1

Introduction

1.1 Introduction au sujet et résultats existants

Commençons par présenter quelques résultats connus sur les cônes réels et complexes, et sur les taux de croissance exponentielle associés à des cocycles.

1.1.1 Cônes et trou spectral : de Perron-Frobenius à Birkhoff

Le théorème de Perron-Frobenius

En 1907 et 1908, Perron et Frobenius ont prouvé le théorème suivant [Per07, Fro12] :

Théorème (Perron, Frobenius). *Soit $A \in M_N(\mathbb{R})$ à coefficients strictement positifs. Alors A admet un trou spectral : A a une valeur propre simple $\lambda > 0$, associée à un vecteur propre u à coordonnées strictement positives, telle que $\forall \mu \in \mathrm{Sp}_{\mathbb{C}}(A), |\mu| < \lambda$.*

Autrement dit, une telle matrice admet une valeur propre simple dominante. Ce théorème possède de nombreuses applications, dont la plus connue est sans doute l'existence d'une mesure invariante pour les chaînes de Markov irréductibles. Le point clé de la preuve est de considérer le cône $\mathbb{R}_+^N$ des points à coordonnées positives. Ce cône est envoyé par A dans lui-même, et même dans son intérieur. La technique standard consiste alors à considérer une section de ce cône et à étudier le maximum de la fonction de dilatation $x \mapsto \frac{\|Ax\|}{\|x\|}$ sur cette section.

Cependant, il semble naturel de vouloir étudier une notion de contraction du cône lui-même, plutôt que de se restreindre à une section.

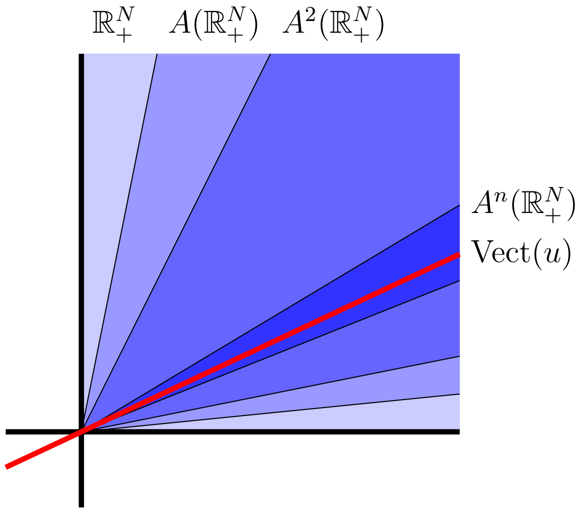

Figure 1.1: Contraction et convergence dans le cadre du théorème de Perron-Frobenius

À chaque itération de A, le cône $A^n(\mathbb{R}_+^N)$ se contracte un peu plus, et "converge" en un certain sens vers une droite, qui est la droite propre associée à la valeur propre dominante λ. Cette convergence ne peut pas être euclidienne : étant donné un point x, la suite $A^n(x)$ peut tout à fait diverger, pendant que sa direction se rapproche de la droite propre. C'est donc une convergence au sens projectif qui entre en jeu ici.

Les cônes de Birkhoff

Cette idée que le trou spectral provient fondamentalement d'une contraction "projective" d'un cône a été mise en évidence très concrètement par Birkhoff [Bir57] à la fin des années 1950. Il introduit une métrique projective de Hilbert sur des cônes et démontre un principe de contraction.

Définition. *Soit E un espace de Banach réel. On appelle cône de E tout sous-ensemble non vide de E, stable par multiplication par $\mathbb{R}^+$. On étudie généralement des cônes fermés convexes. Un cône fermé convexe est dit propre si $\mathcal{C} \cap -\mathcal{C} = \{0\}$.*

On dira qu'un cône fermé convexe $\mathcal{C}$ est borné s'il existe un hyperplan affine fermé H tel que H intersecte chaque demi-droite vectorielle de $\mathcal{C}$ exactement en un point et tel que $H \cap \mathcal{C}$ soit borné. En dimension finie, cette condition est équivalente à $\mathcal{C} \cap -\mathcal{C} = \{0\}$.

Étant donné un cône $\mathcal{C}$, on note $\mathcal{C}^* := \mathcal{C} \setminus \{0\}$ son cône épointé.

Définition. *Soit $\mathcal{C}$ un cône convexe fermé propre. Soit $x, y \in \mathcal{C}^*$. La droite affine projective passant par x et y intersecte alors $\mathcal{C} \cup -\mathcal{C}$ en un segment. On définit la distance projective $d_{\mathcal{C}}(x, y)$ entre x et y relative au cône $\mathcal{C}$ par le logarithme du birapport de x et y relatif à ce segment. Autrement dit, c'est la distance hyperbolique entre x et y dans la section du cône par la droite complexe passant par x et y (voir Figures 1.2 et 1.3).*

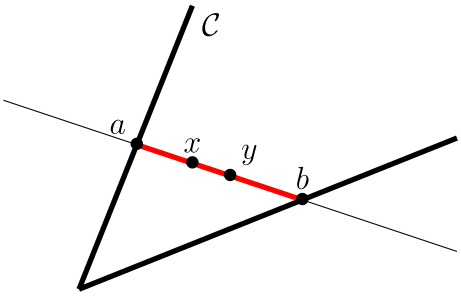

Figure 1.2: Dans cette configuration, $d_{\mathcal{C}}(x,y) = \log \dfrac{\|x-b\|.\|y-a\|}{\|x-a\|.\|y-b\|}$.

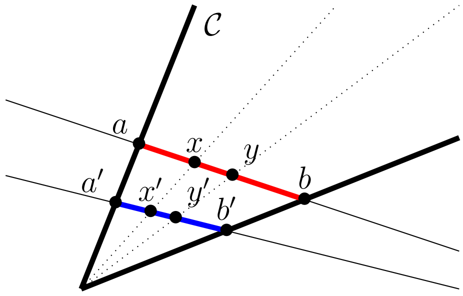

Figure 1.3: $d_{\mathcal{C}}(x,y) = d_{\mathcal{C}}(x',y')$ car le birapport est conservé par la projection relative à l'origine. La distance $d_{\mathcal{C}}$ est bien projective.

Cela définit une métrique sur les droites vectorielles du cône $\mathcal{C}$, et on a :

Théorème (Principe de contraction de Birkhoff)**.** *Soit $\mathcal{C}$ un cône convexe fermé propre et $T : E \to E$ une application linéaire continue qui envoie $\mathcal{C}^*$ dans lui-même. Si le diamètre Δ de $T(\mathcal{C}^*)$ est fini pour la distance $d_{\mathcal{C}}$, alors*

$$\forall x, y \in \mathcal{C}^*, d_{\mathcal{C}}(Tx, Ty) \leq \underbrace{\tanh\left(\frac{\Delta}{4}\right)}_{<1} d_{\mathcal{C}}(x,y).$$

De plus, pour tout $x \in \mathcal{C}^*$, la suite des $T^n(\mathrm{Vect}(x))$ converge géométriquement vers une unique droite $D \subset \mathcal{C}$ fixe par T.

Lorsque le cône est en plus borné et d'intérieur non vide, ce principe amène à montrer qu'un tel opérateur T admet un trou spectral. Dans le contexte du théorème de Perron-Frobenius, une matrice A à coefficients strictement positifs envoie le cône $\mathcal{C} = \mathbb{R}_+^N$ dans son intérieur, et cette condition suffit en dimension finie pour que le diamètre de $A(\mathcal{C})$ pour $d_{\mathcal{C}}$ soit fini. Le cône $\mathbb{R}_+^N$ étant borné et d'intérieur non vide, on obtient alors un trou spectral pour A : c'est le théorème de Perron-Frobenius.

Le cadre plus large de cette théorie permet de généraliser le théorème de Perron-Frobenius, en démontrant l'existence de trou spectral pour les opérateurs uniformément positifs. Appliqué à des opérateurs de transfert, cela fournit des propriétés ergodiques fortes pour les systèmes dynamiques correspondants, telles que la décroissance exponentielle de corrélation. En voici un exemple dont le lecteur pourra trouver les détails dans [Wal82, Bal99, Bal00] :

Proposition. *On considère le cercle $\mathbb{S}^1$ et $T : \mathbb{S}^1 \to \mathbb{S}^1$ une application C^2 uniformément expansive, i.e. telle qu'il existe $C > 0, \lambda > 1$ tels que $\forall n \in \mathbb{N}, |(T^n)'| \geq C\lambda^n$ (par exemple, T peut être une petite perturbation de l'application de doublement de l'angle). On considère l'opérateur de transfert $\mathcal{L} : C^1(\mathbb{S}^1) \to C^1(\mathbb{S}^1)$ défini par :*

$$\forall \phi \in C^1(\mathbb{S}^1), (\mathcal{L}\phi)(x) = \sum_{y \in T^{-1}(x)} \frac{\phi(y)}{|T'(y)|}.$$

Cet opérateur envoie le cône $\{\phi \geq 0\}$ dans lui-même. Il est même uniformément positif, et donc possède un trou spectral. Cela permet d'obtenir des constantes $K > 0, \eta < 1$ telles que:

$$\forall \phi, \psi \in C^1(\mathbb{S}^1), \left| \int_{\mathbb{S}^1} \phi(T^n(x))\psi(x)dx - \int_{\mathbb{S}^1} \phi(x)dx \int_{\mathbb{S}^1} \psi(x)dx \right| \leq K\eta^n \int_{\mathbb{S}^1} |\phi(x)|dx.$$

Ce type d'approche est particulièrement efficace pour étudier des petites perturbations d'un opérateur. Si l'opérateur contracte un cône de manière facilement descriptible, on peut en général montrer que des perturbations de cet opérateur contracte ce même cône et donc obtenir les résultats précédents pour ces opérateurs perturbés.

1.1.2 Le cas complexe : un cas... plus complexe

Définition et exemple de cône complexe

La question se pose alors de savoir si des constructions similaires sont possibles dans un espace de Banach E complexe. La théorie des perturbations requiert en général des estimations complexes uniformes (voir e.g. [Rue79, Rug02] ou encore [NN87, BRS05] dans le cadre de processus additifs de Markov), au lieu d'une simple condition de diamètre fini dans un cône. Ainsi, Rugh donne à la fin des années 2000 une définition de cône complexe et construit une jauge projective qui vérifie un principe de contraction et fournit un trou spectral [Rug10].

Pour définir naturellement les cônes complexes, on essaye d'adapter la définition du cas réel :

Cône réel		Cône complexe
$\mathcal{C} \neq \emptyset$	$\longleftrightarrow$	$\mathcal{C} \neq \emptyset$
$\mathcal{C}$ fermé	$\longleftrightarrow$	$\mathcal{C}$ fermé
$\mathbb{R}^+\mathcal{C} = \mathcal{C}$	$\longleftrightarrow$	$\mathbb{C}\mathcal{C} = \mathcal{C}$
$\mathcal{C}$ convexe et $\mathcal{C} \cap -\mathcal{C} = \{0\}$	$\longleftrightarrow$	$\mathcal{C}$ ne contient pas de $\mathbb{C}$-s.e.v. de dimension 2

La notion de "demi-droite complexe" n'ayant pas de sens, les cônes complexes sont en fait réunion de droites complexes, et non de demi-droites comme dans le cas réel. Toute hypothèse de convexité est alors compromise ; on remplace cette hypothèse et l'hypothèse de propreté $\mathcal{C} \cap -\mathcal{C} = \{0\}$ par une condition qui traduit la même idée : $\mathcal{C}$ est dit propre si $\mathcal{C}$ ne contient pas de s.e.v. de dimension 2. C'est ce fait qui nous permet dans le cas réel de bien définir $d_{\mathcal{C}}$: lorsqu'on réalise la section du cône par une droite affine D, cette section est un segment dans la droite projective $\widehat{D}$ correspondante. Le même type de phénomène se produit dans le cas complexe et permet de définir une jauge sur les cônes complexes.

Voici un exemple simple d'un tel cône. Soit D une droite complexe de E et H un hyperplan fermé, supplémentaire de D dans E. On note π la projection (continue) sur D parallèlement à H. L'ensemble $\mathcal{C}_{\pi,a} = \{x \in E \mid \|(\mathrm{Id} - \pi)(x)\| \leq a\|\pi(x)\|\}$ est alors un cône complexe propre et borné. C'est l'analogue complexe du cône standard réel en forme de sablier (voir Figure 1.8).

Nous présentons maintenant les jauges sur les cônes complexes.

La jauge de Rugh et le principe de contraction

Étant donné deux points x, y non nuls d'un cône complexe $\mathcal{C}$, nous allons nous inspirer de la métrique de Hilbert réelle pour construire $d_{\mathcal{C}}(x, y)$. Essentiellement, on veut définir $d_{\mathcal{C}}$ comme la distance hyperbolique (voir e.g. [CG93]) entre x et y dans la section du cône par la droite complexe passant par x et y.

Définition. *Soit $S(x, y)$ la section du cône par la droite affine complexe $\mathbb{C}_{x,y}$ passant par x et y.*

- *Si x et y sont colinéaires, cette section est la droite toute entière, et on pose alors $d_{\mathcal{C}}(x, y) = 0$.*
- *Sinon, l'hypothèse de fermeture et "$\mathcal{C}$ ne contient pas de s.e.v. de dimension 2" nous assure que $S(x, y)$, vu comme sous-ensemble de la sphère de Riemann $\widehat{\mathbb{C}_{x,y}}$, est hyperbolique, i.e. évite au moins trois points de $\widehat{\mathbb{C}_{x,y}}$. Son intérieur $S^o(x, y)$ est alors une surface de Riemann hyperbolique. Deux cas se présentent :*
  - *Si les points x et y sont dans la même composante connexe U de $S^o(x, y)$, on pose $d_{\mathcal{C}} = d_U(x, y)$, où d_U est la métrique hyperbolique sur U (voir Figure 1.4).*
  - *Sinon, on pose $d_{\mathcal{C}} = \infty$.*

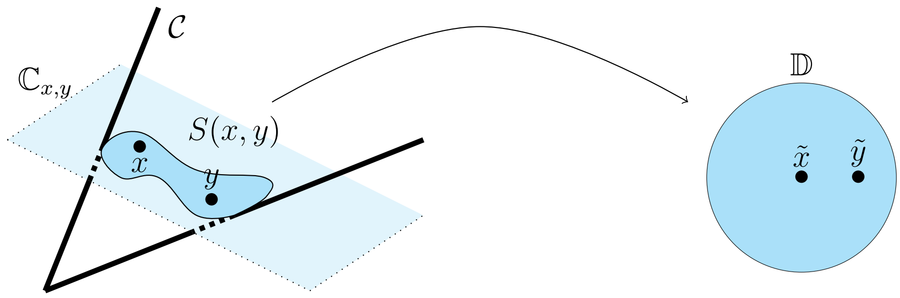

Figure 1.4: Quand la section $S(x, y)$ est simplement connexe, elle est biholomorphe au disque de Poincaré $\mathbb{D}$. Dans ce cas, $d_{\mathcal{C}}(x, y)$ est égale à la distance $d_{\mathbb{D}}(\tilde{x}, \tilde{y})$ entre les images de x et y pour la métrique de Poincaré $d_{\mathbb{D}}$.

Cette jauge est projective : si x' est colinéaire à x et y' est colinéaire à y, alors $S(x', y')$ est l'image de $S(x, y)$ par la projection $p : \widehat{\mathbb{C}_{x,y}} \to \widehat{\mathbb{C}_{x',y'}}$ relative à l'origine (voir Figure 1.5). Comme cette projection est une application conforme, la métrique hyperbolique est conservée et donc $d_{\mathcal{C}}(x', y') = d_{\mathcal{C}}(x, y)$.

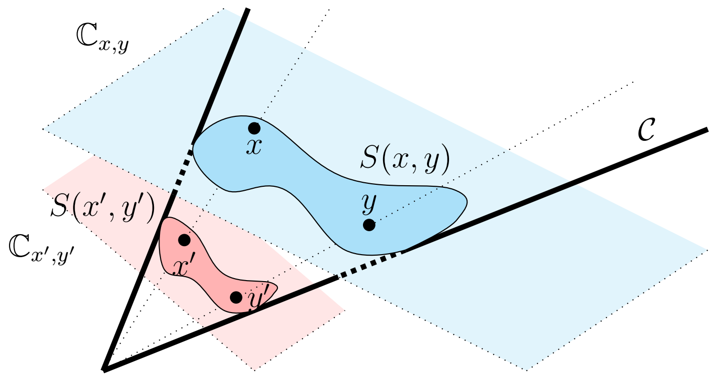

Figure 1.5: Le domaine rouge $S(x',y')$ est le projeté du domaine bleu $S(x,y)$ relativement à l'origine.

Cette jauge n'est pas une métrique projective, car elle ne vérifie pas l'inégalité triangulaire. Elle vérifie cependant la condition de séparation : si x et y ne sont pas colinéaires, $d_{\mathcal{C}}(x,y) > 0$. Mais elle reste tout à fait utilisable pour obtenir des résultats similaires au cas réel. En définissant une notion de cône borné analogue à celle des cônes réels, Rugh obtient le résultat suivant :

Théorème (Rugh)**.** *Soit $\mathcal{C}$ un cône complexe propre et $T : E \to E$ une application linéaire continue qui envoie $\mathcal{C}^*$ dans lui-même. Si le diamètre Δ de $T(\mathcal{C}^*)$ est fini pour la jauge $d_{\mathcal{C}}$, alors il existe $\eta = \eta(\Delta) < 1$ tel que*

$$\forall x, y \in \mathcal{C}^*, d_{\mathcal{C}}(Tx, Ty) \leq \eta . d_{\mathcal{C}}(x,y).$$

Si de plus le cône $\mathcal{C}$ est d'intérieur non vide et borné, alors T admet un trou spectral, de rapport η.

Une application simple de cette théorie est une généralisation du théorème de Perron-Frobenius :

Théorème (Rugh)**.** *Soit $A \in M_N(\mathbb{C})$. Supposons qu'il existe $c > 0$ telle que*

$$\forall i,j,k,l \in [\![1,N]\!], |\text{Im } A_{ij}\bar{A}_{kl}| < c \leq \text{Re } A_{ij}\bar{A}_{kl}.$$

Alors A a un trou spectral.

La jauge de Dubois

Bien que cette jauge $d_{\mathcal{C}}$ soit définie selon le même principe que la métrique de Hilbert et fournisse les résultats désirés dans les espaces de Banach complexes, elle a l'inconvénient d'être difficile à évaluer ou à estimer. En effet, le domaine $S(x,y)$ n'est pas toujours facilement descriptible et les calculs de distance hyperbolique peuvent rapidement devenir compliqués. De plus, comme déjà mentionné, ce n'est pas une métrique.

Dans cette optique, Loïc Dubois définit en 2009 [Dub09] une nouvelle jauge $\delta_{\mathcal{C}}$, elle aussi comparable dans sa définition avec la métrique de Hilbert.

Définition. *Soit $\mathcal{C}$ un cône complexe propre. Soit $x, y \in \mathcal{C}^*$. On considère les ensembles $E_{\mathcal{C}}(x,y) = \{z \in \mathbb{C} \mid zx - y \notin \mathcal{C}\}$ et $|E_{\mathcal{C}}(x,y)| = \{|z| \mid z \in E_{\mathcal{C}}(x,y)\}$. On définit alors*

- $\delta_{\mathcal{C}}(x,y) = 0$ *si x et y sont colinéaires ;*
- $\delta_{\mathcal{C}}(x,y) = \log\left(\dfrac{\sup|E_{\mathcal{C}}(x,y)|}{\inf|E_{\mathcal{C}}(x,y)|}\right)$ *sinon.*

Cette jauge vérifie les propriétés présentées pour $d_{\mathcal{C}}$. De plus, si le cône $\mathcal{C}$ est linéairement convexe - tout $x \notin \mathcal{C}$ est contenu dans un hyperplan fermé ne rencontrant pas $\mathcal{C}$ - alors cette jauge est une métrique projective. Si le cône est de plus borné, cette métrique est complète. Enfin, on a de nouveau le principe de contraction, avec constante de contraction $\eta = \tanh\left(\frac{\Delta}{4}\right)$, comme dans le cas réel.

Dubois introduit également une notion de cône dual. Pour un cône $\mathcal{C}$ linéairement convexe, son dual est

$$\mathcal{C}' := \{l \in E' \mid \forall x \in \mathcal{C}^*, l(x) \neq 0\}.$$

Dubois peut ainsi exprimer la jauge $\delta_{\mathcal{C}}$ à l'aide des éléments de $\mathcal{C}'$. Dans le cas réel, le cône dual est défini par $\mathcal{C}' := \{l \in E' \mid \forall x \in \mathcal{C}, l(x) \geq 0\}$ (voir e.g. [LN14]) et n'a pas d'équivalent complexe. La construction de Dubois correspond à une généralisation du cône dual réel ouvert $\{l \in E' \mid \forall x \in \mathcal{C}^*, l(x) > 0\}$: il faut alors vérifier si l'adhérence du cône dual complexe satisfait de bonnes propriétés.

1.1.3 Cocycles et taux de croissance exponentielle

Le théorème d'Oseledec et ses extensions

Le théorème d'Oseledec [Ose68], aussi appelé théorème multiplicatif ergodique, décrit le comportement asymptotique de cocycles agissant sur un fibré vectoriel vis-à-vis d'un système dynamique.

Définition. *Soit $(\Omega, \mathcal{F}, \mu, T)$ un système dynamique ergodique et E un espace de Banach. On appelle cocycle une application mesurable $M : (\omega, n) \in \Omega \times \mathbb{N} \mapsto M_\omega^n \in \mathcal{L}(E)$ vérifiant pour presque tout ω :*

$$M_\omega^0 = \mathrm{Id}_E \ ;$$
$$\forall m, n \geq 0, M_\omega^{m+n} = M_{T^n\omega}^m M_\omega^n.$$

Cela est équivalent à se donner une famille mesurable $(M_\omega)_{\omega\in\Omega}$ de $\mathcal{L}(E)$ et définir $M_\omega^n := M_{T^{n-1}\omega} \dots M_{T\omega} M_\omega$. On peut définir de manière semblable un cocycle dans le cadre d'un espace topologique Ω et d'une application continue T.

Une version de ce théorème est donnée ci-dessous :

Théorème (Oseledec)**.** *Soit $(\Omega, \mathcal{F}, \mu, T)$ un système dynamique ergodique inversible et E un espace vectoriel de dimension d. Soit M un cocycle vérifiant*

$$\forall(\omega, n) \in \Omega \times \mathbb{N}, M_\omega^n \in GL(E) \ ;$$
$$\textstyle\int_\Omega \log\|M_\omega^1\|\, d\mu(\omega) < \infty \text{ et } \int_\Omega \log\left\|(M_\omega^1)^{-1}\right\| d\mu(\omega) < \infty.$$

Alors il existe des réels $\lambda_1 > \cdots > \lambda_k$, et des applications mesurables $E_1, \ldots, E_k$ de Ω dans la grassmannienne $\mathcal{G}(E)$ (l'ensemble des sous-espaces de E) tels que pour μ-presque tout $\omega \in \Omega$:

$$E = E_1(\omega) \oplus \cdots \oplus E_k(\omega) \ ; \ M_\omega(E_i(\omega)) = E_i(T\omega) \ \forall i \in [\![1, k]\!] \ ;$$

$$\forall i \in [\![1, k]\!], \forall v \in E_i(\omega), \lim_{n \to +\infty} \frac{1}{n} \log \|M_\omega^n(v)\| = \lambda_i.$$

Pour presque tout $\omega \in \Omega$, l'espace E admet une décomposition en sous-espaces, préservée par l'application T, telle que le cocycle M_ω^n ait une croissance exponentielle sur chacun de ses sous-espaces, de l'ordre de $e^{\lambda_i n}$ sur le i-ième. Les réels $\lambda_1, \ldots, \lambda_k$ sont appelés exposants caractéristiques, ou exposants de Lyapunov.

Le cas le plus parlant d'utilisation est de considérer une variété riemannienne Ω et une application T de classe $C^{1+\varepsilon}$. Le cocycle $M_\omega^n = D(T^n)_\omega$ décrit alors l'action de T sur l'espace tangent. La décomposition obtenue est alors une décomposition de l'espace tangent en sous-espaces correspondant à différents taux de croissance exponentielle par la transformation T. La direction E_1 est la direction "dominante", c'est-à-dire celle avec le taux de croissance le plus élevé. Lorsque les angles entre les différents sous-espaces peuvent être contrôlés, on se retrouve dans une configuration similaire à celle des sections précédentes, où les points de l'espace tangent se rapprochent d'une certaine direction - qui cette fois-ci est mobile avec le temps.

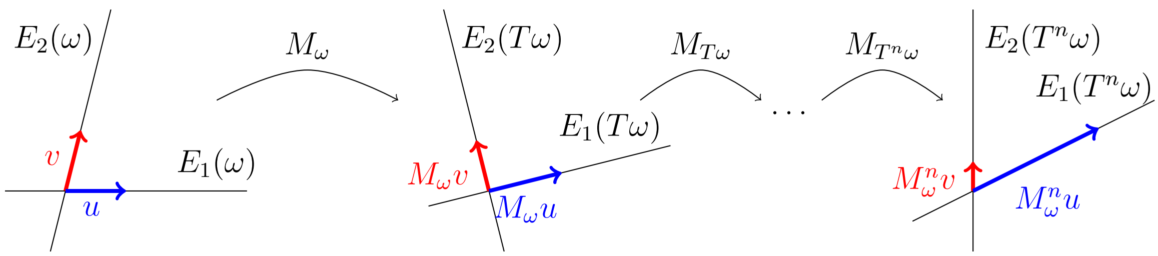

Figure 1.6: Le théorème d'Oseledec, avec $\lambda_1 > 0$ et $\lambda_2 < 0$.

Ce résultat clé pour les systèmes dynamiques a par la suite été profondément étudié et généralisé sur de nombreux points : retrait de l'hypothèse d'inversibilité de la dynamique et du cocycle, régularité en ω des familles $E_i(\omega)$, évolution des exposants pour des perturbations du cocycle, cas d'espaces de dimension infinie, etc.

En 1979, Ruelle démontre que lorsque le cocycle M_ω^n contracte une famille de cônes de l'espace reél E de dimension finie, l'espace E_1 est de dimension 1 et l'exposant de Lyapunov dominant λ_1 est analytique. Autrement dit, en considérant une application $t \mapsto (M_\omega^n(t))$ analytique et vérifiant les conditions précédentes, l'exposant $\lambda_1(t)$ correspondant au cocycle $(M_\omega^n(t))$ est analytique en t. Pour la régularité des exposants suivants, Ruelle applique ce résultat au produit extérieur $\bigwedge^p M_\omega$: si ces opérateurs contractent une famille de cônes réels dans l'espace $\bigwedge^p E$, alors la somme des p premiers exposants de Lyapunov est analytique. Cependant, cette dernière condition est en pratique difficile à vérifier.

Vient ensuite la conquête des espaces de dimension infinie. Ruelle étend en 1982 le théorème d'Oseledec à des opérateurs compacts dans les espaces de Hilbert [Rue82].

En ce qui concerne les espaces de Banach, c'est Mañe qui démontre un an plus tard le théorème pour des cocycles compacts et injectifs, vérifiant de plus une condition de continuité [Man83]. Dans ces cas-là, l'espace E se décompose en $E = E_1(\omega) \oplus \cdots \oplus E_k(\omega) \oplus F_{k+1}(\omega)$, où les espaces $E_i(\omega)$ sont des espaces rapides de dimension finie, à croissance exponentielle correspondant à l'exposant de Lyapunov λ_i, et $F_{k+1}(\omega)$ est un espace lent de codimension finie, dont la croissance est exponentiellement plus lente que celle des $E_i(\omega)$.

Une des principales difficultés à relever fut alors de se défaire le plus possible des hypothèses contraignantes sur le cocycle : compacité, injectivité, condition de continuité. Dans un article de 1987, Thieullen [Thi87] arrive à se libérer de la condition de continuité en ω et affaiblit l'hypothèse de compacité en considérant une condition de compacité asymptotique. Il étudie également le cas d'opérateurs non-injectifs : dans ce contexte, il démontre l'existence des espaces lents de codimension finie et mesurable en ω. De plus, il existe alors une suite (λ_i) d'exposants de Lyapunov qui sont les seuls taux de croissance exponentielle possibles. Ces dernières années, les travaux de Quas, Froyland, Lloyd et Gonzáles-Tokman fournissent l'existence d'une décomposition d'Oseledec pour des cocycles d'opérateurs quasi-compacts non inversibles, et appliquent ce résultat à l'étude d'opérateurs de transfert ([FLQ13, GTQ14]).

Ce théorème d'Oseledec et ses extensions continuent encore d'être étudiés. Le question de la régularité des exposants et de la décomposition en sous-espaces a été abordée par Rugh [Rug10] pour des familles d'opérateurs sur un espace de Banach complexe contractant des cônes - étendant ainsi le résultat de Ruelle dans [Rue79] pour des cônes réels en dimension finie. Sous certaines conditions d'uniformité dans la contraction des cônes, il démontre que le premier exposant de Lyapunov, ainsi que l'espace associé, sont analytiques.

Plus récemment, Froyland, Gonzalez-Tokman et Quas ont démontré un principe de stabilité des exposants de Lyapunov par ajout de bruit, ainsi de convergence en probabilité des espaces associés : d'abord en dimension finie [FGQ15], puis pour des opérateurs compacts sur un espace de Hilbert [FGQ17].

Les décompositions dominées

La décomposition précise en sous-espaces n'étant pas définie dans tous les cas, et les taux de croissance n'étant eux non plus pas toujours bien définis, une notion proche de la décomposition d'Oseledec a fait récemment l'objet d'études plus poussées : la notion de *décomposition dominée* (en anglais, dominated splitting). Plutôt que de trouver une décomposition très précise isolant chacun des taux de croissance possibles dans un sous-espace E_i, on cherche à mettre en évidence un "trou de croissance", similaire à un trou spectral, et une décomposition de l'espace qui y est adaptée : $E = V_\omega \oplus W_\omega$, où tous les vecteurs de V_ω ont une croissance par le cocycle exponentiellement plus grande que celle des vecteurs de W_ω.

Définition. *Soit $(\Omega, \mathcal{F}, \mu, T)$ un système dynamique ergodique inversible et E un espace de Banach. Soit (M^n_ω) un cocycle sur E. On note $\mathcal{G}(E)$ la grassmannienne de E, c'est-à-dire l'ensemble des sous-espaces vectoriels fermés de E. On dit que le cocycle admet une décomposition dominée de taux $\eta \in]0,1[$ s'il existe une constante $C > 0$ et une application mesurable $\omega \mapsto (V_\omega, W_\omega) \in \mathcal{G}(E) \times \mathcal{G}(E)$ telle que, pour presque tout $\omega \in \Omega$:*

$$E = V_\omega \oplus W_\omega \ ; \ M_\omega(V_\omega) = V_{\tau\omega} \ ; \ M_\omega(W_\omega) \subset W_{\tau\omega};$$

$$\forall n \in \mathbb{N}^*, \sup_{\substack{v \in V_\omega, w \in W_\omega \\ \|v\|=\|w\|=1}} \frac{\|M_\omega^n(w)\|}{\|M_\omega^n(v)\|} \leq C\eta^n.$$

Une définition semblable peut être donnée dans le cas où Ω est un espace topologique et T est une application continue. Cette notion a des applications profondes en théorie ergodique et en analyse matricielle ([BM15], [BV05], [GW06], [Mor10], [PS09]).

Dans le cas de la dimension finie, Bochi et Gourmelon [BG09] caractérisent l'existence d'une décomposition dominée, d'espace rapide de dimension k, par une décroissance exponentielle du rapport entre la k-ième et la $(k+1)$-ième valeur singulière du cocycle :

Théorème (Bochi, Gourmelon)**.** *Soit Ω un espace topologique compact et $T : \Omega \to \Omega$ un homéomorphisme. Pour une matrice $A \in M_d(\mathbb{R})$, on note $\sigma_1(A) \geq \cdots \geq \sigma_d(A)$ ses valeurs singulières, qui sont les racines carrées des valeurs propres de la matrice positive $A^\top A$.*

Soit $M : \Omega \times \mathbb{N} \to GL_d(\mathbb{R})$ un cocycle. Alors il y a équivalence entre :

- *Il existe $C > 0$, $\eta \in]0,1[$ tels que*

$$\forall \omega \in \Omega, n \geq 0, \sigma_{k+1}(M_\omega^n) \leq C\eta^n \sigma_k(M_\omega^n) \ ;$$

- *Le cocycle (M_ω^n) admet une décomposition dominée $E = V_\omega \oplus W_\omega$ telle que* $\dim V_\omega = k$ *pour tout $\omega \in \Omega$.*

Blumenthal et Morris généralisent ce théorème à des cocycles d'opérateurs injectifs sur des espaces de Banach dans [BlM15]. Dans le cas de la dimension finie, ils en obtiennent une version quantitative ne nécessitant pas la compacité de l'espace Ω.

Dans le cadre de notre approche avec des cônes, c'est la deuxième caractérisation des décompositions dominées donnée par Bochi et Gourmelon [BG09] qui se révèle la plus intéressante.

Théorème (Bochi, Gourmelon)**.** *Soit Ω un espace topologique compact, $T : \Omega \to \Omega$ un homéomorphisme et $M : \Omega \times \mathbb{N} \to GL_d(\mathbb{R})$ à valeurs dans un compact de $GL_d(\mathbb{R})$. On note $\mathcal{G}_k(\mathbb{R}^d)$ l'espace topologique des sous-espaces de dimension k de $\mathbb{R}^d$. Alors il y a équivalence entre :*

- *Le cocycle (M_ω^n) admet une décomposition dominée $E = V_\omega \oplus W_\omega$ telle que* $\dim V_\omega = k$ *pour tout $\omega \in \Omega$.*

- *Il existe un sous-ensemble fermé $C \subset \mathcal{G}_k(\mathbb{R}^d)$ envoyé dans son intérieur par tous les opérateurs du cocycle et un espace de dimension $d-k$ transverse à tous les éléments de C.*

- *Il existe un sous-ensemble fermé $C \subset P(\mathbb{R}^d)$ envoyé dans son intérieur par tous les opérateurs du cocycle, qui contient la projection PE d'un espace de dimension k, et qui n'intersecte pas la projection PF d'un espace de dimension $d-k$.*

L'existence d'une décomposition dominée se traduit par la contraction d'un ensemble C constitué d'espaces de dimension k, qui pourrait ainsi généraliser la contraction de cône formé par une réunion de droites. Ces objets, parfois appelés multicones, ont été étudiés dans le cas de $SL_2(\mathbb{R})$ dans [ABY10], avec une description précise pour des multicones associés à une paire de matrices. Plus récemment, ils ont été utilisés dans [BoM15] pour étudier la régularité du rayon spectral inférieur de sous-ensembles de $GL_d(\mathbb{R})$.

1.2 Plan et contributions de la thèse

Dans la plupart des travaux mentionnés précédemment, les résultats de trou spectral et de régularité sont obtenus pour isoler une valeur propre simple dominante ou le plus grand exposant de Lyapunov, associé à un espace de dimension 1. Les cônes utilisés sont des réunions de droites ou de demi-droites qui ne contiennent pas d'espace de dimension supérieure.

L'objet principal de cette thèse et de définir et d'utiliser des cônes complexes constitués d'espaces de dimension p dans un espace de Banach pour isoler p valeurs propres dominantes ou les p premiers exposants de Lyapunov (avec multiplicité). Le Chapitre 2 rappelle certains résultats connus sur les grassmanniennes des espaces de Banach et les complète pour préparer leur utilisation dans la suite. Nous définissons dans le chapitre 3 les cônes complexes p-dimensionnels ainsi que'une jauge sur ces cônes, et énonçons un principe de contraction. Cela nous amène à démontrer, pour un opérateur contractant un tel cône, l'existence d'un trou spectral séparant les p valeurs propres dominantes du reste du spectre. Cette théorie fournit également un théorème de régularité analytique pour les exposants de Lyapunov d'un produit aléatoire d'opérateurs contractant un même cône. Enfin, nous développons dans le Chapitre 4 une notion de cône dual pour ces cônes p-dimensionnels. Le théorème de régularité précédent est alors complété et nous donne l'existence et l'analyticité d'une décomposition de l'espace en "espace lent" et "espace rapide".

Le lecteur trouvera dans l'Appendice les démonstrations des résultats utilisés pour manipuler les grassmanniennes et l'agèbre extérieure. Nous avons aussi fait le choix de reporter dans cette appendice des énoncés plus techniques sur les cônes complexes p-dimensionnels mais néanmoins importants pour les théorèmes principaux. En particulier, nous y démontrons un résultat d'équivalence entre la distance de Hausdorff et la jauge définie sur les cônes.

1.2.1 Grassmanniennes

Le premier chapitre commence par rappeler des résultats standards (voir par exemple [Ber63, Kat95]) sur l'ensemble $\mathcal{G}(E)$ des sous-espaces fermés d'un espace de Banach E, appelé grassmannienne de E. La distance standard sur $\mathcal{G}(E)$ est la distance de Hausdorff d_H, définie par

$$\forall V, W \in \mathcal{G}(E), d_H(V,W) = \max\left(\sup_{x\in S_V} d(x, S_W), \sup_{y\in S_V} d(x, S_V)\right),$$

où S_V et S_W sont les sphères unités de V et W. Cette distance fait de $\mathcal{G}(E)$ un espace métrique complet. Cependant, pour les estimations faites par la suite, les calculs seront généralement faits dans l'algèbre extérieure. Étant donné un espace $V = \mathrm{Vect}(x_1, \dots, x_p)$ de dimension p, on peut le représenter dans l'algèbre extérieure $\bigwedge^p E$ par le vecteur $x_1\wedge\cdots\wedge x_p$. Une autre base $(x_1', \dots, x_p')$ donnera un vecteur $x_1'\wedge\cdots\wedge x_p'$ colinéaire à $x_1\wedge\cdots\wedge x_p$. On étudie donc tout d'abord deux normes sur $\bigwedge^p E$, définies par

Définition. *Si $l_1, \dots, l_p \in E'$ et $x_1, \dots, x_p \in E$, on définit*

$$\langle l_1 \wedge\cdots\wedge l_p, x_1\wedge\cdots\wedge x_p\rangle = \det((\langle l_i, x_j\rangle)_{i,j}).$$

Cette formule s'étend en une forme linéaire sur $\bigwedge^p E$*, encore notée* $l_1 \wedge \cdots \wedge l_p$*. On définit la norme* $\|.\|_{\wedge l,p}$ *sur* $\bigwedge^p E$ *par*

$$\|x\|_{\wedge l,p} = \sup_{\substack{l_1,\ldots,l_p \in E' \\ \|l_i\|=1}} |\langle l_1 \wedge \cdots \wedge l_p, x\rangle|$$

On définit la norme $\|.\|_{\wedge d,p}$ *sur* $\bigwedge^p E$ *par*

$$\|x\|_{\wedge d,p} = \inf \sum_i \|x_1^i\| \cdots \|x_p^i\|\,,$$

où l'infimum est pris sur toutes les décompositions $x = \sum_i x_1^i \wedge \cdots \wedge x_p^i$.

Les résultats principaux sur ces normes et leurs relations entre elles sont compilés et démontrés dans l'appendice. Le lecteur y trouvera également une comparaison avec les normes définis sur l'algèbre tensorielle $\bigotimes^p E$.

Ces deux normes ne sont pas équivalentes sur $\bigwedge^p E$, mais elles le sont si on se restreint à des éléments de $\bigwedge^p E$ qui s'expriment comme somme d'un nombre k fixé de tenseurs purs. On définit alors une distance extérieure sur l'ensemble $\mathcal{G}_p(E)$ des sous-espaces de dimension p de E :

Définition. *Pour* $V, W \in \mathcal{G}_p(E)$ *:*

$$d_{\wedge,p}(V,W) = \inf\{\|v - w\|_{\wedge l,p} \mid (v,w) \in \bigwedge^p V \times \bigwedge^p W \text{ with } \|v\|_{\wedge l,p} = \|w\|_{\wedge l,p} = 1\}$$

C'est la distance de Hausdorff entre $\bigwedge^p V$ *et* $\bigwedge^p W$ *dans la grassmannienne de dimension 1 de* $(\bigwedge^p E, \|.\|_{\wedge l,p})$*.*

Cette distance est équivalente à la distance analogue définie avec la norme $\|.\|_{\wedge d,p}$. On montre alors le résultat suivant :

Théorème A. *La distance* $d_{\wedge,p}$ *est équivalente à la distance de Hausdorff* d_H *sur* $\mathcal{G}_p(E)$*.*

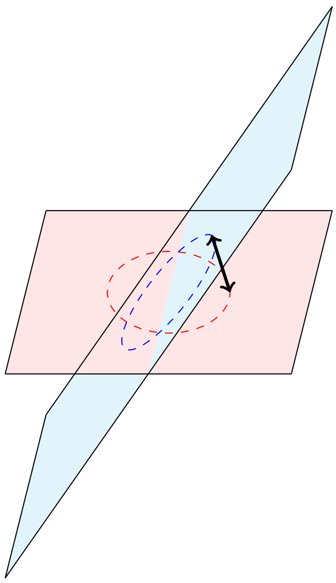

Figure 1.7: La distance de Hausdorff entre deux sous-espaces.

La preuve de ce théorème repose sur l'utilisation de "bonnes décompositions" qui permettent d'avoir des estimées sur $\|.\|_{\wedge l,p}$.

Définition. *Soit V un espace vectoriel de dimension p. On dit qu'une famille $(x_1, \dots, x_p)$ de vecteurs unitaires de V est une bonne décomposition de V s'il existe des formes linéaires $l_1, \dots, l_p$ de norme 1 sur V telles que*

$$\begin{gathered}\forall i \in [\![1,p]\!], l_i(x_i) = 1 \ ; \\ \forall i \in [\![1,p]\!], \forall j > i, x_j \in \ker l_i.\end{gathered}$$

Cette définition remplace la notion d'orthogonalité existant dans les espaces de Hilbert. Pour une telle famille $(x_1, \dots, x_p)$, la norme $\|x_1 \wedge \cdots \wedge x_p\|_{\wedge l,p}$ est minorable. On raisonne alors avec de telles bases pour obtenir nos résultats sur les espaces de dimension p.

1.2.2 Définition des cônes p-dimensionnels complexes et premières applications

Dans un espace de Banach complexe E, on définit une notion de cône p-dimensionnel, réunion d'espaces p-dimensionnels, par analogie avec les cônes complexes décrits précédemment qui sont réunions de droites complexes.

Cône complexe "classique"		Cône complexe p-dimensionnel, ou "p-cone"
$\mathcal{C} \neq \emptyset$, $\mathcal{C}$ fermé	$\longleftrightarrow$	$\mathcal{C} \neq \emptyset$, $\mathcal{C}$ fermé
$\mathbb{C}\mathcal{C} = \mathcal{C}$	$\longleftrightarrow$	$\mathcal{C}$ est réunion de s.e.v. de dim. p
$\mathcal{C}$ ne contient pas de s.e.v. de dim. 2	$\longleftrightarrow$	$\mathcal{C}$ ne contient pas de s.e.v. de dim. $p+1$

Un exemple de tel cône peut être obtenu à partir d'une projection sur un espace de dimension p, comme dans le cas des cônes classiques. Si l'espace E se décompose en $E = F \oplus G$ avec $\dim F = p$ et G fermé, on note π la projection sur F parallèlement à G et pour $a > 0$, l'ensemble

$$\mathcal{C}_{\pi,a} = \{x \in E \mid \|(\mathrm{Id} - \pi)(x)\| \leq a\|\pi(x)\|\}$$

est un p-cône de E. La figure suivante en donne une illustration dans le cadre réel :

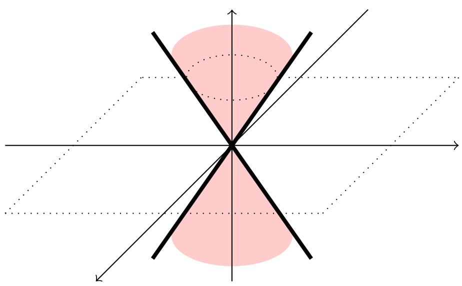

Figure 1.8: En rouge, le cône standard associé à la projection orthogonale selon (Oz). Son complémentaire est un 2-cône, associé à la projection orthogonale sur le plan (xOy).

Il s'agit ensuite d'obtenir une jauge relative au p-cône $\mathcal{C}$, qui s'applique aux sous-espaces vectoriels de $\mathcal{C}$ de dimension p. Si V et W sont deux sous-espaces p-dimensionnels de $\mathcal{C}$, on définit

$$d_{\mathcal{C}}(V,W) = \sup_{x\in V^*, y\in W^*} \delta_{\mathcal{C}}(x,y),$$

où $\delta_{\mathcal{C}}$ est définie (au choix) comme la jauge de Rugh ou la jauge de Dubois pour les cônes complexes classiques. Les conditions sur le cône $\mathcal{C}$ nous assurent que cette définition a bien un sens et que $d_{\mathcal{C}}(V,W) > 0$ si $V \neq W$.

On montre alors le principe de contraction suivant :

Théorème B. *Soient E_1, E_2 deux espaces de Banach complexes. Soit $T : E_1 \to E_2$ une application linéaire continue et $\mathcal{C}_1 \subset E_1$, $\mathcal{C}_2 \subset E_2$ deux p-cônes tels que $T(\mathcal{C}_1^*) \subset \mathcal{C}_2^*$. Alors pour tous espaces V, W de dimension p inclus dans $\mathcal{C}_1$, on a $d_{\mathcal{C}_2}(TV, TW) \leq d_{\mathcal{C}_1}(V,W)$.*

De plus, si $\Delta = \mathrm{diam}_{\mathcal{C}_2}(T\mathcal{C}_1) < \infty$, il existe $\eta < 1$, ne dépendant que de Δ, tel que pour tous V, W de dimension p inclus dans $\mathcal{C}_1$, on a

$$d_{\mathcal{C}_2}(TV,TW) \leq \eta . d_{\mathcal{C}_1}(V,W).$$

On définit alors une notion d'ouverture bornée similaire à celle des cônes classiques, et on obtient un théorème de trou spectral :

Théorème C. *Soit $\mathcal{C} \subset E$ un p-cône d'intérieur non vide dans $\mathcal{G}_p(E)$ et d'ouverture bornée. Soit $T \in \mathcal{L}(E)$ tel que $T(\mathcal{C}^*) \subset \mathcal{C}^*$ et $diam_{\mathcal{C}}(T\mathcal{C}) = \Delta < \infty$. On note $\eta < 1$ le coefficient de contraction donné par le théorème qui précède. Alors T a un trou spectral, i.e. il existe un sous-espace V de dimension p et un supplémentaire fermé W de V, stables par T, tels que*

$$\sup \left|Sp(T_{|W})\right| \leq \eta \inf \left|Sp(T_{|V})\right|,$$

où $\left|Sp(T_{|V})\right|$ (resp. W) désigne l'ensemble des modules des éléments du spectre de $T_{|V}$ (resp. W).

1.2.3 Analyticité de la somme des p premiers exposants de Lyapunov

Pour démontrer l'analyticité du taux de croissance correspondant aux p directions les plus rapides pour un cocycle, nous avons besoin d'une uniformité de la contraction de cône considérée. Étant donné un cône $\mathcal{C}$ d'ouverture bornée et d'intérieur non vide dans $\mathcal{G}_p(E)$, on introduit l'ensemble

$$\mathscr{M} := \{M \in \mathcal{L}(E) \mid M(\mathcal{C}^*) \subset \mathcal{C}^*, \mathrm{diam}_{\mathcal{C}}(M(\mathcal{C}^*)) \leq \Delta, M(\mathcal{C}) \subset \mathcal{C}[\rho]\},$$

où $\mathcal{C}[\rho] := \{x \in \mathcal{C} \mid B(x, \rho\|x\|) \subset \mathcal{C}\}$ est un sous-ensemble de $\mathcal{C}$, qui contient un sous-espace de dimension p pour ρ assez petit. Soit $(\Omega, \mathcal{F}, \mu, T)$ un système dynamique ergodique (pas nécessairement inversible) et notons $\mathscr{E}(\Omega, \mathscr{M})$ l'ensemble des applications Bochner-mesurables de Ω dans $\mathscr{M}$. Lorsque la dynamique n'est pas inversible, il est plus pratique d'utiliser des cocycles à gauche, définis par $M_\omega^{(n)} = M_\omega M_{\tau\omega} \dots M_{\tau^{n-1}\omega}$ pour une famille d'opérateurs $(M_\omega)_{\omega\in\Omega}$. On démontre le résultat de régularité suivant, qui généralise le théorème 10.2 de [Rug10] :

Théorème D. *Soit $t \in \mathbb{D} \mapsto M(t) \in \mathscr{E}(\Omega, \mathscr{M})$ une application telle que*

1. *$(t, \omega) \in \mathbb{D} \times \Omega \mapsto M_\omega(t) \in \mathcal{L}(E)$ est mesurable et pour tout $\omega \in \Omega$, l'application $t \mapsto M_\omega(t)$ est analytique.*
2. $\sup \left\{ \dfrac{\| \bigwedge^{p-1} M_\omega(t) \|}{\| \bigwedge^p M_\omega(t) \|} \| \frac{d}{dt} M_\omega(t) \| : \omega \in \Omega, t \in \mathbb{D} \right\} < \infty.$
3. $\int_\Omega |\log \|M_\omega(0)\|| < \infty.$

Alors pour tout $t \in \mathbb{D}$, la limite

$$\chi_p(t) = \lim_{n \to \infty} \frac{1}{n} \log \left\| \bigwedge^p \left(M_\omega^{(n)}(t) \right) \right\|$$

existe pour μ-a.e. $\omega \in \Omega$ and est (μ-a.e.) indépendante de ω. De plus, l'application $t \mapsto \chi_p(t)$ est analytique.

Lorsque la dynamique est inversible, ce théorème permet de montrer l'analycité de la somme des p premiers exposants de Lyapunov associés au cocycle (dans le sens habituel).

Exemple. Soit $(\xi_n)_{n \in \mathbb{Z}}$ des variables aléatoires i.i.d à valeurs dans $\mathbb{D}$. Pour $t \in \mathbb{D}$, soit

$$M_n(t) = \begin{pmatrix} 10 + t\xi_n & t + \xi_n & it \\ t + \xi_n & 6 & \xi_n \\ i\xi_n & 0 & 1 \end{pmatrix}$$

Alors les trois exposants caractéristiques (avec multiplicité) du produit aléatoire des $(M_n(t))$ sont analytiques.

1.2.4 Cônes duaux

Dans cette dernière partie, on définit une notion de cône dual d'un p-cône $\mathcal{C}$. Pour commencer, on introduit l'ensemble des supplémentaires topologiques de sous-espaces du cônes, $GC(\mathcal{C}) := \{V \in \mathcal{G}^p(E) \mid \forall W \in \mathcal{G}_p(E) \text{ t.q. } W \subset \mathcal{C}, V \oplus W = E\}$, appelé supplémentaire grassmannien du cône $\mathcal{C}$. Pour V sous-espace de E, on note $V^\perp \subset E'$ l'orthogonal de V, défini par $V^\perp = \{l \in E' \mid \forall x \in V, l(x) = 0\}$. Le cône dual de $\mathcal{C}$ est alors un sous-ensemble du dual E' :

$$\mathcal{C}' := \bigcup_{V \in GC(\mathcal{C})} V^\perp.$$

Il est constitué de sous-espaces de dimension p de E' : Si $V' = \mathrm{Vect}(l_1, \ldots, l_p)$ est dans $\mathcal{C}'$ et W est un sous-espace de dimension p de $\mathcal{C}$, alors les $(l_{i|W})$ forment une base du dual de W. Cette définition généralise celle de Dubois dans le cadre des cônes complexes classiques.

On démontre des correspondances entre les propriétés du cône $\mathcal{C}$ et de son dual $\mathcal{C}'$:

Proposition E. *Le p-cône $\mathcal{C}$ est d'intérieur non vide dans $\mathcal{G}_p(E)$ si et seulement si son cône dual $\mathcal{C}'$ est d'ouverture bornée.*

Le p-cône $\mathcal{C}$ est d'ouverture bornée si et seulement si son cône dual $\mathcal{C}'$ est d'intérieur non vide dans $\mathcal{G}_p(E')$.

Dans ces conditions, $\mathcal{C}'$ est presque un p-cône : il n'est pas fermé. En considérant sa fermeture, on obtient un ensemble qui est presque un cône. Il en a toutes les propriétés, sauf celle d'être réunion d'espaces de dimension p : il peut y avoir des éléments "isolés" qui ne font pas partie d'un p-espace inclus dans le cône. Cependant, dans le cas des cônes de projections, ces problèmes disparaissent :

Proposition F. *Soit $a > 0$, $E = F \oplus G$ une décomposition topologique de E avec $\dim F = p$, et $\pi : E \to F$ la projection sur F parallèlement à G. Alors*

$$\overline{(\mathcal{C}_{\pi,a})'} = \mathcal{C}_{\pi^\top,1/a},$$

i.e. le cône dual de $\mathcal{C}_{\pi,a}$ est le cône associé à la projection $\pi^\top$ dans le dual, avec coefficient $1/a$.

Dans ce cadre, on peut obtenir un principe de contraction et un théorème de trou spectral pour les opérateurs duaux. Mais plus important, l'utilisation des cônes duaux permet d'obtenir un théorème de décomposition dominée. En se plaçant dans le cadre d'opérateurs envoyant un cône $\mathcal{C}_{\pi,a}$ dans $\mathcal{C}_{\pi,b}$ avec $b < a$, on obtient :

Théorème G. *Soit $\mathscr{M} = \{M \in \mathcal{L}(E) \mid M(\mathcal{C}^*_{\pi,a}) \subset \mathcal{C}^*_{\pi,b}\}$. Soit $\in \mathbb{D} \mapsto M(t) \in \mathscr{E}(\Omega, \mathscr{M})$ une application vérifiant les hypothèses du théorème D.*

Alors il existe une application $(t, \omega) \in \mathbb{D} \times \Omega \mapsto (V_\omega(t), W_\omega(t)) \in \mathcal{G}_p(E) \times \mathcal{G}^p(E)$, mesurable en ω et analytique en t, telle que $E = V_\omega(t) \oplus W_\omega(t)$ soit une décomposition dominée de E de taux $\eta = \frac{a}{b}$ pour tout $t \in \mathbb{D}$.

Enfin, nous proposons des pistes pour généraliser ce théorème dans le cas de p-cônes d'ouverture bornée et d'intérieur non-vide.

1.2.5 Appendice

Pour rendre la lecture plus cohérente, les résultats et démonstrations concernant la topologie des grassmanniennes, les normes sur l'algèbre extérieure $\bigwedge^p E$ et leurs estimées sont réunis dans l'appendice.

Nous y comparons également la distance de Hausdorff d_H avec notre jauge $d_{\mathcal{C}}$ lorsque les espaces considérés sont "loins" du bord du cône :

Proposition H. *Soit $0 < a < b$. Alors d_H et $d_{\mathcal{C}_{\pi,b}}$ sont équivalentes sur le sous-cône $\mathcal{C}_{\pi,a}$. En particulier, si un cône $\mathcal{C}$ contient $\mathcal{C}_{\pi,b}$, alors $d_{\mathcal{C}}$ et d_H sont équivalentes sur $\mathcal{C}_{\pi,a}$.*

Pour mieux comprendre les p-cones $\mathcal{C}$, nous établissons des liens entre $\mathcal{C}[\rho]$, les cônes de projections $\mathcal{C}_{\pi,a}$, et la topologie de $\mathcal{C}$ dans la grassmannienne. En particulier, nous montrons qu'un cône est d'intérieur non vide dans $\mathcal{G}_p(E)$ si et seulement si il contient un cône de projection.

1.3 Pistes de recherche

1.3.1 Développements autour des cônes duaux

L'utilisation des cônes duaux se révèle pour l'instant assez difficile dans le cas de p-cônes quelconques. Plus particulièrement, le fait que le dual d'un p-cone ne soit pas nécessairement un p-cone est un obstacle majeur. Il serait donc intéressant de trouver des conditions sur le cône $\mathcal{C}$ pour que son dual $\mathcal{C}'$ soit lui aussi un p-cone.

Une autre approche possible serait de définir une notion plus large de cône multidimensionnel, qui engloberait les duaux de p-cones. Cette piste est peut-être la plus prometteuse : nous fournissons à la fin du Chapitre 4 deux propositions qui poussent à penser que les cônes duaux ont de bonnes propriétés pour adapter les preuves présentées ici, en particulier pour généraliser le Théorème G.

Dans [Dub09], Dubois relie la gauge $d_{\mathcal{C}}$ d'un cône complexe $\mathcal{C}$ de dimension 1 avec son dual $\mathcal{C}'$ via la formule suivante :

Proposition (Dubois). *Soit $\mathcal{C}$ un cône complexe. Supposons que tout $v \notin \mathcal{C}$ est contenu dans un hyperplan fermé H_v ne rencontrant pas $\mathcal{C}$. Notons $\delta_{\mathcal{C}}$ la jauge de Dubois et $\mathcal{C}'$ le cône dual de $\mathcal{C}$. Alors*

$$\forall x, y \in \mathcal{C}^*, \delta_{\mathcal{C}}(x, y) = \sup_{l,m \in \mathcal{C}'} \log \left| \frac{l(y)m(x)}{l(x)m(y)} \right|.$$

En particulier, sous ces hypothèses, cette proposition permet de montrer que $\delta_{\mathcal{C}}$ est une métrique. Une formule analogue est valable pour la métrique de Hilbert sur les cônes réels convexes fermés (voir par exemple [LN14]). Il est donc naturel de se demander si notre jauge $d_{\mathcal{C}}$ sur les p-cones peut s'exprimer à l'aide du cône dual.

Enfin, cette approche pourrait permettre de répondre à la question suivante : étant donné deux p-cônes $\mathcal{C}_1 \subset \mathcal{C}_2$ tels que $\mathrm{diam}_{\mathcal{C}_2}(\mathcal{C}_1) < \infty$, est-ce que $\mathrm{diam}_{\mathcal{C}_1'}(\mathcal{C}_2') < \infty$? Si la réponse est affirmative, cela pourra fournir un résultat de trou spectral pour les opérateurs duaux et être utilisé pour obtenir des décompositions dominées.

1.3.2 Applications à des opérateurs de transfert

Dans le cas d'opérateurs de transfert, les cônes standards permettent d'obtenir des informations sur la première valeur propre d'un opérateur. Cette démarche est intéressante notamment parce qu'elle s'applique aux petites perturbations d'opérateurs connus : nous avons cité précédemment les perturbations de l'opérateur de transfert $\mathcal{L}$ associé à l'application T de doublement de l'angle sur le cercle $\mathbb{S}^1$ (voir e.g. [Bal99]).

L'utilisation de p-cônes pourrait donner une description plus précise du spectre de tels opérateurs. Dans l'exemple précédent, une approche naturelle serait de considérer des cônes associés aux projections spectrales de l'opérateur de transfert $\mathcal{L}$. La difficulté est alors de considérer les bons espaces fonctionnels et d'obtenir des estimations raisonnables sur les perturbations de T qui répondent au critère de contraction des cônes.

1.3.3 Cônes de dimension infinie - variétés stables et instables

Les cônes considérés dans cette thèse sont des réunions d'espaces de dimension p ne contenant pas d'espace de dimension $p + 1$. Une telle condition n'est pas applicable si on cherche à définir des cônes qui soient réunions d'espaces de dimension infinie. On peut tout de même définir un cône associé à une projection. Si $E = F \oplus G$ est une somme topologique de sous-espaces et si π désigne la projection sur F parallèlement à G, l'ensemble

$$\mathcal{C}_{\pi,a} = \{x \in E \mid \|(\mathrm{Id} - \pi)(x)\| \leq a\|\pi(x)\|\}$$

est bien défini pour $a > 0$. Cependant, une contraction d'un tel cône ne s'accompagne pas forcément d'un trou spectral. Considérons par exemple l'opérateur de décalage

$$T : \begin{array}{ccc} l^2(\mathbb{N}) & \to & l^2(\mathbb{N}) \\ (a_n)_{n\in\mathbb{N}} & \mapsto & (0, a_0, a_1, a_2, \dots). \end{array}$$

En posant $F \coloneqq \{(a_n)_{n\in\mathbb{N}} \mid a_0 = 0\}$, $G \coloneqq \mathrm{Vect}((1, 0, 0, \cdots))$ et π la projection sur F parallèlement à G, l'opérateur T envoie $\mathcal{C}^*_{\pi,1}$ dans $\mathcal{C}^*_{\pi,1/2}$ mais n'a pas de trou spectral. Les conditions de contraction seront donc plus difficiles à exprimer et à manipuler pour de tels cônes de dimension infinie.

Un autre moyen de considérer ces cônes $\mathcal{C}_{\pi,a}$ est de regarder les sous-espaces qui sont obtenus comme des graphes d'applications linéaires de F dans G. Pour les cônes de dimension finie p, cela correspond exactement aux sous-espaces de $\mathcal{C}_{\pi,a}$ de dimension p. On pourrait alors utiliser des techniques de transformée de graphe pour étudier les cônes de dimension infinie ; cependant, les méthodes présentées dans cette thèse semblent difficiles à transposer dans ce cadre. En particulier, le recours de l'algèbre extérieure ne peut s'appliquer.

Enfin, il serait intéressant de voir si des conditions de contraction de cônes au sens de la jauge $d_{\mathcal{C}}$ peuvent suffire pour obtenir l'existence de variétés stables et instables. L'approche standard pour ce type de question (voir par exemple [KH95]) utilise des cônes mais repose sur des estimations lipschitziennes et sur des transformées de graphes, qui demandent des conditions assez fortes d'hyperbolicité sur les opérateurs. L'utilisation des notions introduites dans cette thèse pourrait permettre de réduire les hypothèses assurant l'existence de ces variétés.

Chapter 2

Grassmannians

In this chapter, we look at the Grassmannian of a Banach space E. The Grassmannian $\mathcal{G}(E)$ is the set of closed subspaces of E. In finite dimensions, the grassmanian is easy to study thanks to the compactness of the unit sphere. Equipped with the Hausdorff metric, it is a compact manifold; its connected components are exactly the sets $\mathcal{G}_p(E)$ of p-dimensional subspaces of E for $p \in \mathbb{N}$ [Bo07V].

One can also embed $\mathcal{G}_p(E)$ into $P(\bigwedge^p E)$, mapping a subspace $V = \mathrm{Span}(x_1, \dots, x_p)$ to the line generated by $x_1 \wedge \cdots \wedge x_p$. The coordinates in $\bigwedge^p E$ of $x_1 \wedge \cdots \wedge x_p$ are called the Plücker coordinates of V. More details on this can be found in [Har92].

In infinite dimensions, several complications arise. First, the unit sphere is no longer compact: the Hausdorff distance between two unit spheres S_V and S_W can be only a supremum and not a maximum. However, it still defines a distance on $\mathcal{G}(E)$ and makes $\mathcal{G}(E)$ a complete metric space. General results on this distance are available in [Ber63] and [Kat95].

Looking at the exterior algebra $\bigwedge^p E$, the difficulties pile up. As $\bigwedge^p E$ is no longer finite-dimensional, the choice of the norm on $\bigwedge^p E$ is important. In the case of Hilbert spaces, a canonical inner product can be defined on $\bigwedge^p E$; but in the Banach spaces case, there is no canonical norm. One can define several "natural" norms on this space: however, they are not equivalent. And even when they can be compared, the constants linking them explode when p goes to infinity.

In dynamical systems, both approaches have been used to obtain results on characteristic exponents and their associated spaces, and on dominated splittings. Raghunathan gave another proof of the original Oseledec theorem using the exterior algebra [Rag79]. In the Hilbert space setting, Ruelle also used tensors to prove the multiplicative ergodic theorem [Rue82]. These methods heavily rely on the use of singular values: in these cases, the product of the p largest singular values of a linear operator A can be more easily studied through the largest eigenvalue of $\bigwedge^p(A^*A)$. The associated eigenvector in $\bigwedge^p E$ then gives the corresponding subspace in $\mathcal{G}_p(E)$.

More recently, the preferred approach for Banach spaces uses the Grassmannian equipped with the Hausdorff metric and the gap between two spaces to realize the computations. See e.g. [Blu16] and [GTQ15] where Oseledec's theorem is proved via volume calculations on Banach spaces. In [GTQ14], González-Tokman and Quas use a notion of "nice basis", which has some properties in common with the "right decomposition" we introduce here.

This latter notion is more convenient for our proofs and computations.

The first section of this chapter goes over $\mathcal{G}(E)$ and the Hausdorff distance and recalls some of its properties. The second section introduces some natural norms on $\bigwedge^p E$ and defines the notion of *right decomposition* $(x_1, \ldots, x_p)$ for a p-dimensional space V. This allows us to obtain an element $x_1 \wedge \cdots \wedge x_p$ in $\bigwedge^p E$ representing V whose norm can be controlled. We then define a distance on $\mathcal{G}_p(E)$ using the norms on the exterior algebra, and show that this distance is equivalent to the Hausdorff distance on $\mathcal{G}_p(E)$.

Background material for Grassmannians and some related technical proofs have been relegated to the Appendix. In particular, we study precisely the norms on $\bigwedge^p E$: we show that they can be seen as the quotient of corresponding norms on the tensor algebra $\bigotimes^p E$, and we compare these different norms. We also prove that these norms are equivalent when looking at elements which are sums of a fixed finite number of tensors.

2.1 Hausdorff distance and gap for Grassmannians

In this chapter, $(E, \|.\|)$ is a normed vector space and E' its dual. For $x \in E$ and $A \subset E$, we denote $d(x, A)$ the distance from x to A, defined by $d(x, A) = \inf\{\|x - y\| \mid y \in A\}$. For any subspace V of E, we denote S_V the unit sphere of V. We first recall the notion of topological complement (see e.g. [Bre10], Section 2.4), which is used throughout the thesis.

Definition 2.1.1. *Let V be a closed subspace of E. We say that a subspace W of E is a topological complement of V if $E = V \oplus W$ and W is closed. This is equivalent to: $E = V \oplus W$ and the projection onto V parallel to W is a bounded linear map. In this case, we say that $E = V \oplus W$ is a topological decomposition of E.*

The usual reference for the geometry of Grassmannians is [Kat95]. We recall here some of the results. However, some statements in [Kat95] are not proved or are left to the reader. We therefore offer a proof for these for completeness in Section 5.1 of the Appendix.

First, let us define the Hausdorff distance d_H on $\mathcal{G}(E)$.

Definition 2.1.2. *The Grassmannian $\mathcal{G}(E)$ of E is the set of closed subspaces of E. For $p \in \mathbb{N}^*$, $\mathcal{G}_p(E)$ denotes the set of p-dimensional subpaces of E and $\mathcal{G}^p(E)$ denotes the set of closed p-codimensional subpaces of E. We define :*

$$\begin{array}{rccc} d_H: & \mathcal{G}(E) \times \mathcal{G}(E) & \longrightarrow & \mathbb{R}^+ \\ & (V, W) & \longmapsto & \max\left(\sup_{x \in S_V} d(x, S_W), \sup_{y \in S_W} d(x, S_V)\right) \end{array}$$

i.e. $d_H(V, W)$ is the Hausdorff distance between the unit spheres of V and W.

The application d_H defines a metric on $\mathcal{G}(E)$ and $(\mathcal{G}(E), d_H)$ is a complete metric space when E is a Banach space (see e.g. [Ber63]). However, the distance d_H can sometimes be troublesome to compute. In most cases, we actually compute the gap between two spaces V and W and use the following lemma to get $d_H(V, W)$.

Definition 2.1.3. *Let $V, W \in \mathcal{G}(E)$. The gap $\delta(V, W)$ between V and W is defined as*

$$\delta(V, W) = \sup_{x \in S_V} d(x, W).$$

Lemma 2.1.4 ([Kat95], Chapter IV, Section 2.1). *Let $x \in E$ with $\|x\| = 1$ and V, W be elements of $\mathcal{G}(E)$. Then $d(x, V) \leq d(x, S_V) \leq 2d(x, V)$ and*

$$\max(\delta(V, W), \delta(W, V)) \leq d_H(V, W) \leq 2\max(\delta(V, W), \delta(W, V)).$$

We now define the minimal angle between two subspaces $V, W \in \mathcal{G}(E)$.

Definition 2.1.5. *Let $V, W \in G(E)$. The minimal angle $\theta(V, W) \in [0, \frac{\pi}{2}]$ from V to W is defined by*

$$\sin\theta(V, W) = \inf\{\|v - w\| \mid v \in V, \|v\| = 1, w \in W\}.$$

Unlike the Hausdorff distance d_H which helps estimate how close two subspaces are, the minimal angle tells us how far they are to each other. This can also be seen with the following proposition.

Proposition 2.1.6 ([BlM15, Blu16]). *Let $V, W \in G(E)$ be topological complements and π be the projection onto V parallel to W. Then*

$$\sin\theta(V, W) = \frac{1}{\|\pi\|}$$

and $\sin\theta(W, V) \leq 2\sin\theta(V, W)$.

Finally, the Grassmannian $\mathcal{G}(E)$ has the structure of an analytic manifold. One can find details in [Bo07V].

Definition 2.1.7. *We define charts $(U_{F,G}, \phi_{F,G})$ on $\mathcal{G}(E)$ as follows. If $E = F \oplus G$ is a topological decomposition of E, let $U_{F,G}$ be the set of topological complements of G and*

$$\phi_{F,G} : \begin{array}{ccc} \mathcal{L}(F, G) & \to & U_{F,G} \\ f & \mapsto & \operatorname{graph} f. \end{array}$$

This forms an analytic atlas of $\mathcal{G}(E)$ compatible with the topology given by d_H, making $\mathcal{G}(E)$ an analytic Banach manifold.

2.2 Right decompositions and metrics on the Grassmannian

In this section, we first introduce two norms $\|.\|_{\wedge d,p}$ and $\|.\|_{\wedge l,p}$ on the exterior algebra $\bigwedge^p E$. We compare these norms and give ways to get upper bounds on them. To ease the reading, we just state these results; the norms on the exterior algebra, their relationship with norms on the tensor algebra, and the proof of these results are compiled in Section 5.2 of the Appendix. When referring to either of $\|.\|_{\wedge d,p}$ or $\|.\|_{\wedge l,p}$ indifferently, we use $\|.\|_{\wedge,p}$.

One obstacle to get lower bounds of the norm of $x_1 \wedge \cdots \wedge x_p$ is that the $\|x_i\|$ can be big while $\|x_1 \wedge \cdots \wedge x_p\|_{\wedge,p}$ is small. To prevent this from happening, we define a notion which resembles orthogonality: the right decomposition. Given a p-dimensional space V, we first take $x_1 \in V$ of norm 1 and l_1 a linear form on V of norm 1 such that $l_1(x_1) = 1$. Then we take x_2 of norm 1 in the kernel of l_1 and l_2 a linear form on V of norm 1 such that

$l_2(x_2) = 1$. Iterating this process, we take x_{i+1} of norm 1 in the kernel of $l_1, \dots, l_i$ and l_{i+1} a linear form on V of norm 1 such that $l_{i+1}(x_{i+1}) = 1$. Such a basis $(x_1, \dots, x_p)$ of V is called a right decomposition, and the linear forms l_i are used to get bounds on $\|x_1 \wedge \cdots \wedge x_p\|_{\wedge,p}$. We use this to prove that the norms $\|.\|_{\wedge d,p}$ and $\|.\|_{\wedge l,p}$ are equivalent when restricted to the subset of $\bigwedge^p E$ contaning the sums of a fixed finite number of simple tensors.

Finally, we define a new distance on $\mathcal{G}_p(E)$. Given two subspaces $V = \mathrm{Span}(x_1, \dots, x_p)$ and $W = \mathrm{Span}(y_1, \dots, y_p)$ in $\mathcal{G}_p(E)$, we look at the distance between $x_1 \wedge \cdots \wedge x_p$ and $y_1 \wedge \cdots \wedge y_p$, or more precisely at the distance between the lines of $\bigwedge^p E$ generated by these two elements. Using well-chosen right decompositions, we show that this new distance is equivalent to d_H on $\mathcal{G}_p(E)$.

2.2.1 Norms on the exterior algebra

For the definition and basic properties of the exterior algebra $\bigwedge^p E$, we refer to [Bo07A]. We define our norms on $\bigwedge^p E$. The norm $\|.\|_{\wedge d,p}$ looks at the different decompositions of an element x in simple tensors: the norm is then given by taking the infimum, over all decompositions of x, of the sum of the norms of these simple tensors. The norm $\|.\|_{\wedge l,p}$ is defined using the dual E': the norm of an element x is the supremum of the $|\langle l, x\rangle|$, where l is taken in a specific subset of $\bigwedge^p E'$.

Definition 2.2.1. *For $x \in \bigwedge^p E$, we define*

$$\|x\|_{\wedge d,p} = \inf \sum_i \|x_1^i\| \cdots \|x_p^i\|$$

where the infimum is taken over all (finite) decompositions $x = \sum_i x_1^i \wedge \cdots \wedge x_p^i$; and

$$\|x\|_{\wedge l,p} = \sup_{\substack{l_1,\dots,l_p \in E' \\ \|l_i\|=1}} \langle l_1 \wedge \cdots \wedge l_p, x\rangle.$$

We have the following properties for the norms $\|.\|_{\wedge d,p}$ and $\|.\|_{\wedge l,p}$. For the proofs of these results, see Section 5.2 of the Appendix.

Proposition 2.2.2. *On $\bigwedge^p E$, we have $\|.\|_{\wedge l,p} \leq (\sqrt{p})^p \|.\|_{\wedge d,p}$.*

Lemma 2.2.3. *Let $x \in E$ and $u \in \bigwedge^p E$. Then*

(i) $\|x \wedge u\|_{\wedge l,p+1} \leq (p+1)\|x\|.\|u\|_{\wedge l,p}$,

(ii) $\|x \wedge u\|_{\wedge d,p+1} \leq \|x\|.\|u\|_{\wedge d,p}$.

Lemma 2.2.4. *Let $x_1, \dots, x_p, y_1, \dots, y_p \in E$ such that $\|x_i\| = \|y_i\| = 1$ for all $i \in [\![1,p]\!]$. Then*

(i) $\|x_1 \wedge \cdots \wedge x_p - y_1 \wedge \cdots \wedge y_p\|_{\wedge l,p} \leq p.p!.\max(\|x_i - y_i\|)$,

(ii) $\|x_1 \wedge \cdots \wedge x_p - y_1 \wedge \cdots \wedge y_p\|_{\wedge d,p} \leq p.\max(\|x_i - y_i\|)$.

2.2.2 Right decompositions

We now introduce the notion of adapted linear form and right decomposition. The main interest of these notions is the following: if $(x_1, \dots, x_p)$ is a right decomposition, then the x_i are "far" from each other and create a volume, and therefore $\|x_1 \wedge \cdots \wedge x_p\|$ cannot be too small. Then in practice, for arbitrary $x'_1 \wedge \cdots \wedge x'_p$, we first find a right decomposition $(x_1, \dots, x_p)$ of the subspace $\mathrm{Span}(x'_1, \dots, x'_p)$, compute $\|x'_1 \wedge \cdots \wedge x'_p\|$ and then compare the element $x'_1 \wedge \cdots \wedge x'_p$ to $x_1 \wedge \cdots \wedge x_p$.

Definition 2.2.5. *Let E be a normed vector space and $x \in E$. We say that a linear form $l : E \to \mathbb{C}$ is adapted to x if $\|l\| = 1$ and $l(x) = \|x\|$, which is equivalent to $\|l\| = 1$ and $\forall y \in \ker l, \|x + y\| \geq \|x\|$. The hyperplane $\ker l$ is said to be adapted to x.*

Such a linear form always exists (as a consequence of the Hahn-Banach theorem).

Remark. A natural way to think about this is to see $\ker l$ as a set of vectors "orthogonal" to x. The difference with the euclidian case is that given such a z "orthogonal" to x, x is not necessarily "orthogonal" to z.

Definition 2.2.6. *Let E be a normed vector space and V be a p-dimensional subspace of E. We say that $(x_1, \dots, x_p)$ is a right decomposition of V if $\|x_i\| = 1$ for all i and there exists linear forms $l_1, \dots, l_p$ such that*

- *l_i is adapted to x_i for all i,*
- *$x_j \in \ker l_i$ for all $j > i$.*

We say that the l_i are linear forms adapted to the decomposition $(x_1, \dots, x_p)$.

Remark. This definition means that the vectors $x_2, \dots, x_p$ are "orthogonal" to x_1 ; the vectors $x_3, \dots, x_p$ are "orthogonal" to x_2 ; etc. This would be our equivalent of an orthogonal basis.

The next lemma show that having a right decomposition $(x_1, \dots, x_p)$ gives a lower bound on $\|x_1 \wedge \cdots \wedge x_p\|_{\wedge l,p}$.

Lemma 2.2.7. *If $(x_1, \dots, x_p)$ is a right decomposition of V and $(l_1, \dots, l_p)$ are adapted linear forms, then $(x_1, \dots, x_p)$ is a basis of V and the l_i are independent (and therefore are forming a basis of V'). Moreover, $\|x_1 \wedge \cdots \wedge x_p\|_{\wedge l,p} \geq 1$.*

Proof. Let $(x_1, \dots, x_p)$ be a right decomposition of V and $(l_1, \dots, l_p)$ adapted linear forms. Then

$$\begin{aligned}
\langle l_1 \wedge \cdots \wedge l_p, x_1 \wedge \cdots \wedge x_p \rangle &= \det(\langle l_i, x_j \rangle) \\
&= \begin{vmatrix} 1 & 0 & \cdots & 0 \\ & 1 & \ddots & \vdots \\ & (*) & \ddots & 0 \\ & & & 1 \end{vmatrix} \quad \text{as } \forall i, \langle l_i, x_i \rangle = 1 \text{ and } \forall j > i, x_j \in \ker l_i \\
&= 1
\end{aligned}$$

So the l_i are independent, the x_i are independent and $\|x_1 \wedge \cdots \wedge x_p\|_{\wedge l,p} \geq 1$. $\square$

Remark. If $(x_1, \dots, x_p)$ is a right decomposition of V in V, we can extend to E the adapted linear forms (using the Hahn-Banach theorem), and therefore $(x_1, \dots, x_p)$ is a right decomposition of V in E. Being a right decomposition only depends on the x_i.

We now present two ways to get a right decomposition. The first one is to get a right decomposition as the completion of the right decomposition of a smaller subspace. The second one is the Auerbach basis. Auerbach bases are useful to get estimates on coordinates of a point, given its norm. However, there is a drawback: if $V_1 \subset V_2$, we usually cannot complete an Auerbach basis of V_1 into an Auerbach basis of V_2. The broader notion of right decomposition is thus necessary to get the results we desire; the estimations we get on the coordinates are however not as powerful.

Proposition 2.2.8. (completion of a right decomposition). *Let E be a normed vector space and V be a p-dimensional subspace of E. Given $x_1, \dots, x_k \in V$ such that $(x_1, \dots, x_k)$ is a right decomposition of* $\mathrm{Span}(x_1, \dots, x_k)$*, there exists $x_{k+1}, \dots, x_p \in V$ such that $(x_1, \dots, x_p)$ is a right decomposition of V. In particular, V has a right decomposition.*

Proof. Let $k < p$ and let $l_1, \dots, l_k$ be linear forms on V adapted to the decomposition $(x_1, \dots, x_k)$. As $\dim(\bigcap_{i=1}^k \ker l_i) \geq p-k > 0$, we can take $x_{k+1} \in \bigcap_{i=1}^k \ker l_i$ with $\|x_{k+1}\| = 1$ and let l_{k+1} be a linear form adapted to x_{k+1}. Then $(x_1, \dots, x_{k+1})$ is a right decomposition of $\mathrm{Span}(x_1, \dots, x_{k+1})$. By iteration, we get: $(x_1, \dots, x_p)$ is a right decomposition of $\mathrm{Span}(x_1, \dots, x_p)$, and given Lemma 2.2.7 and $\dim(V) = p$, the p-tuple $(x_1, \dots, x_p)$ is a right decomposition of V. $\square$

Proposition 2.2.9 (Auerbach basis, see e.g. [Thi87])**.** *Given V a p-dimensional space, there exists a right decomposition $(x_1, \dots, x_p)$ of V with adapted linear forms $(l_1, \dots, l_p)$ such that $\|x_i\| = \|l_i\| = 1$ for all i, and $l_i(x_j) = \delta_{i,j}$ for all i, j (where $\delta_{i,j}$ is the Kronecker delta). Such a basis is called an Auerbach basis.*

Lemma 2.2.10. *Consider $(x_1, \dots, x_p)$ an Auerbach basis of a p-dimensional space V. If $x = \sum_{i=1}^{p} a_i x_i \in V$, then $|a_i| \leq \|x\|$ for all $i \in [\![1, p]\!]$.*

Proof.
$$\forall i \in [\![1, p]\!], |a_i| = |l_i(x)| \leq \|l_i\|.\|x\| = \|x\|. \qquad \square$$

Example. Consider $\mathbb{R}^2$ equipped with $\|.\|_\infty$ and denote (e_1, e_2) its canonical basis. Then (e_1, e_2) is an Auerbach basis. The vector $e_1 + \frac{1}{2}e_2$ (of norm 1) can not be completed into an Auerbach basis; it can, however, be completed into a right decomposition by $(e_1 + \frac{1}{2}e_2, e_2)$.

Proof. Let (e_1^*, e_2^*) denote the dual basis of (e_1, e_2). One can easily check that (e_1, e_2) is an Auerbach basis, with associated linear forms (e_1^*, e_2^*).

The vector $x = e_1 + \frac{1}{2}e_2$ is a unit vector. Consider l_1 a linear form of norm 1 such that $l_1(x) = 1$. As the only affine line tangent to the unit sphere of $\|.\|_\infty$ in x is $x + \mathrm{Span}(e_2)$, l_1 must have its kernel equal to $\mathrm{Span}(e_2)$ (see Figure 2.1). Then l_1 is collinear to e_2^*. By contradiction, assume x can be completed into an Auerbach basis (x, y). Then we have $y = \pm e_2$. As before, any linear form l_2 of norm 1 such that $l_2(y) = 1$ must have its kernel equal to $\mathrm{Span}(e_1)$. Then $x \notin \ker l_2$, which is impossible since (x, y) is an Auerbach basis. However, (x, e_2) is a right decomposition, as it does not require $x \in \ker l_2$. $\square$

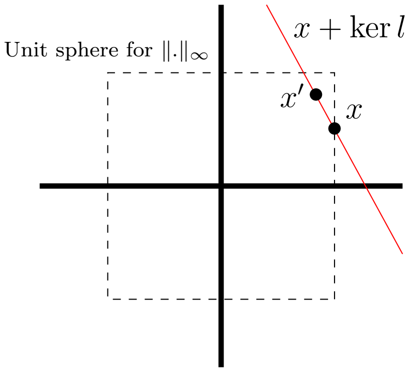

Figure 2.1: If $l(x) = 1$ and $x + \ker l$ is the red line, then $l(x') = 1$ and $\|x'\| < 1$, preventing l from having norm 1.

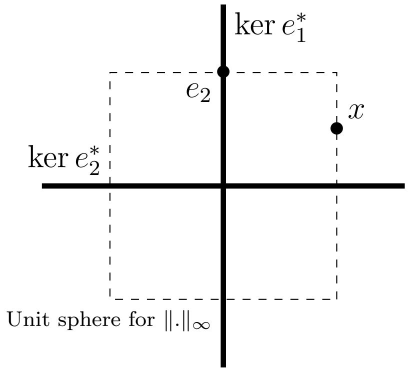

Figure 2.2: $(x, e2)$ is a right decomposition, but is not an Auerbach basis as $x \notin \ker e_2^*$.

Lemma 2.2.11. *Let $(x_1, \dots, x_p)$ be a right decomposition of V and $x = \sum_{i=1}^{p} a_i x_i \in V$. Then $|a_i| \leq 2^{i-1}\|x\|$ for all $i \in [\![1, p]\!]$.*

Proof. As $\langle l_i, x\rangle = \sum_{k=1}^{i-1} a_k \langle l_i, x_k\rangle + a_i$ and $\|l_i\| = 1$, we get $|a_i| \leq \|x\| + \sum_{k=1}^{i-1} |a_k|$ and therefore $|a_i| \leq 2^{i-1}\|x\|$ by induction. $\square$

Finally, we show that when we only look at sums of a fixed finite number of tensors, the norms $\|.\|_{\wedge l,p}$ and$\|.\|_{\wedge d,p}$ are equivalent.

Lemma 2.2.12. *Let $k \in \mathbb{N}$. The norms $\|.\|_{\wedge l,p}$ and $\|.\|_{\wedge d,p}$ are equivalent on the subset S_k of $\bigwedge^p E$ containing the sums of at most k simple tensors.*

Proof. From Proposition 2.2.2, on $\bigwedge^p E$, we have $\|.\|_{\wedge l,p} \leq (\sqrt{p})^p \|.\|_{\wedge d,p}$.

Let $x = \sum_{i=1}^{k} x_1^i \wedge \dots \wedge x_p^i \in S_k$ and denote $F = \operatorname{Span}(x_k^i)$. Then $\dim F \leq kp$. We consider $(e_1, \dots, e_d)$ a right decomposition of F and $(l_1, \dots, l_d)$ the corresponding linear forms ($d \leq kp$). In this basis, we can write $x = \sum_{i_1 < \dots < i_p} \lambda_I e_{i_1} \wedge \dots \wedge e_{i_p}$. Then

$$\|x\|_{\wedge l,p} \geq |\langle l_{i_1} \wedge \dots \wedge l_{i_p}, x\rangle| = |\lambda_I|$$

which gives

$$\|x\|_{\wedge d,p} \leq \sum_{i_1 < \dots < i_p} |\lambda_I| \|e_{i_1}\| \cdots \|e_{i_p}\| \leq \sum_{i_1 < \dots < i_p} \|x\|_{\wedge l,p} \leq \binom{kp}{p} \|x\|_{\wedge l,p}.$$

$\square$

2.2.3 Equivalence between the Hausdorff distance and the exterior algebra metric

Now we have all the tools to define a new metric on $\mathcal{G}_p(E)$ and compare it to the metric d_H. For that, see that if $(x_1, \dots, x_p)$ and $(y_1, \dots, y_p)$ are bases of the same p-dimensional space V of E, then $x_1 \wedge \cdots \wedge x_p$ and $y_1 \wedge \cdots \wedge y_p$ are collinear in $\bigwedge^p E$. We can thus identify the p-dimensional space V of E to the vector line $\mathrm{Span}(x_1 \wedge \cdots \wedge x_p)$ of $\bigwedge^p E$. We say that $x_1 \wedge \cdots \wedge x_p$ represents V, or is a representative of V.

The following definition, proposition and theorem are stated and proved for the norm $\|.\|_{\wedge,p} = \|.\|_{\wedge l,p}$. They remain true for $\|.\|_{\wedge,p} = \|.\|_{\wedge d,p}$, thanks to Lemma 2.2.12.

Definition 2.2.13. *For $V \in \mathcal{G}_p(E)$, $\bigwedge^p V$ is a 1-dimensional subspace of $\bigwedge^p E$. Hence we define for $V, W \in \mathcal{G}_p(E)$:*

$$d_{\wedge,p}(V,W) = \inf\{\|v - w\|_{\wedge,p} \mid (v,w) \in \bigwedge^p V \times \bigwedge^p W \text{ with } \|v\|_{\wedge,p} = \|w\|_{\wedge,p} = 1\}$$

This corresponds to the Hausdorff distance between the complex lines $\bigwedge^p V$ and $\bigwedge^p W$ in $(\bigwedge^p E, \|.\|_{\wedge,p})$.

Proposition 2.2.14. *If $x \in \bigwedge^p V, y \in \bigwedge^p W$ with $\|x\|_{\wedge,p} = \|y\|_{\wedge,p} = 1$, then*

$$d_{\wedge,p}(V,W) = \inf_{|\lambda|=1} \|x - \lambda y\|_{\wedge,p} = \min_{|\lambda|=1} \|x - \lambda y\|_{\wedge,p}.$$

Moreover, $d(x, \mathrm{Span}(y)) \leq d_{\wedge,p}(V,W) \leq 2\max(d(x,\mathrm{Span}(y)), d(y,\mathrm{Span}(x)))$.

Proof. The first part is a consequence of $\dim \bigwedge^p V = \dim \bigwedge^p V = 1$. Lemma 2.1.4 applied to $\bigwedge^p V$ and $\bigwedge^p W$ gives us the second part. □

Theorem 2.2.15. *For every $p \in \mathbb{N}^*$, there exists a constant $c_p > 0$ such that: for every normed vector space E, for all $V, W \in \mathcal{G}_p(E)$,*

$$\frac{1}{c_p} d_H(V,W) \leq d_{\wedge,p}(V,W) \leq c_p d_H(V,W).$$

In other words, the metrics d_H and $d_{\wedge,p}$ are equivalent on $\mathcal{G}_p(E)$.

Proof. Let $p \in \mathbb{N}^*$ and E be a normed vector space. Let V, W be p-dimensional spaces in E and set $d \coloneqq d_H(V,W)$.

As mentionned before, since Lemma 2.2.12 states that the norms $\|.\|_{\wedge l,p}$ and $\|.\|_{\wedge d,p}$ are equivalent when restricted to sum of at most two simple tensors, we only have to prove the theorem for $\|.\|_{\wedge d,p}$. We thus show the following inequalities:

$$\frac{1}{3(\sqrt{p})^p} d_H(V,W) \leq d_{\wedge l,p}(V,W) \leq (2p.p!) d_H(V,W).$$

Let us first prove the second inequality. Let $x_1 \wedge \cdots \wedge x_p$ be a right decomposition of V. For all i, there exists $y_i \in W$ such that $\|x_i - y_i\| \leq d$ and $\|y_i\| = 1$. Then, by Lemma 2.2.4, $\|x_1 \wedge \cdots \wedge x_p - y_1 \wedge \cdots \wedge y_p\| \leq d.p.p!$. As $x_1 \wedge \cdots \wedge x_p$ is a right decomposition of V, the inequality $\|x_1 \wedge \cdots \wedge x_p\| \geq 1$ holds by Lemma 2.2.7. Let

$$x \coloneqq \frac{x_1 \wedge \cdots \wedge x_p}{\|x_1 \wedge \cdots \wedge x_p\|} \in \bigwedge^p V \text{ and } y \coloneqq \frac{y_1 \wedge \cdots \wedge y_p}{\|x_1 \wedge \cdots \wedge x_p\|} \in \bigwedge^p W.$$

We have $\|x\| = 1$ and $\|x - y\| \leq d.p.p!$. Thus $d(x, \mathrm{Span}(y)) \leq d.p.p!$. By an argument of symmetry, Proposition 2.2.14 shows that $d_{\wedge,p}(V,W) \leq (2.p.p!).d = (2p.p!) d_H(V,W)$.

Now we prove the remaining inequality of our proposition. Let $x \in V$ with $\|x\| = 1$ such that $d(x, W) \geq d/3$. Let l be the linear form on $W + \mathrm{Span}(x)$ such that $l(x) = 1$ and $l_{|W} = 0$. We first estimate $\|l\|$. For $\lambda \in \mathbb{C}^*$ and $w \in W$, we have

$$\|\lambda x + w\| = |\lambda|.\|x + \frac{1}{\lambda}w\| \geq |\lambda| d(x, W) \geq \frac{|\lambda| d}{3} = \frac{d}{3}|l(\lambda x + w)|.$$

Therefore $\|l\| \leq 3/d$. Using the Hahn-Banach theorem, we can extend l to a linear form on E, still called l, with $\|l\| \leq 3/d$. Let $l_1 := l/\|l\|$. Using Proposition 2.2.8, let $(x, x_2, \ldots, x_n)$ be a right decomposition of V with adapted linear forms $(l_1, l_2, \ldots, l_p)$. Let $(y_1, \ldots, y_d)$ be a basis of W.

Then for all $\lambda \in \mathbb{C}$,

$$\begin{aligned}
\|x \wedge x_2 \wedge \cdots \wedge x_p - \lambda y_1 \wedge \cdots \wedge y_p\| &\geq |\langle l_1 \wedge \cdots \wedge l_p, x \wedge x_2 \wedge \cdots \wedge x_p - \lambda y_1 \wedge \cdots \wedge y_p\rangle| \\
&\geq |\langle l_1 \wedge \cdots \wedge l_p, x \wedge x_2 \wedge \cdots \wedge x_p\rangle| \\
&\geq \begin{vmatrix} \frac{1}{\|l\|} & 0 & \cdots & 0 \\ & 1 & \ddots & \vdots \\ & (*) & \ddots & 0 \\ & & & 1 \end{vmatrix} \\
&\geq \frac{1}{\|l\|}
\end{aligned}.$$

As $x, x_2, \ldots, x_p$ are unit vectors, $\|x \wedge x_2 \wedge \cdots \wedge x_p\| \leq (\sqrt{p})^p$ by Proposition 2.2.2 and therefore

$$d\left(\frac{x \wedge x_2 \wedge \cdots \wedge x_p}{\|x \wedge x_2 \wedge \cdots \wedge x_p\|}, \mathrm{Span}(y_1 \wedge \cdots \wedge y_p)\right) \geq \frac{1}{(\sqrt{p})^p \|l\|}.$$

By Proposition 2.2.14, we get

$$d_{\wedge,p}(V, W) \geq \frac{1}{(\sqrt{p})^p \|l\|} \geq \frac{d}{3(\sqrt{p})^p} = \frac{1}{3(\sqrt{p})^p} d_H(V, W).$$ □

Finally, the following propositions show the completeness of $\mathcal{G}_p(E)$ and of the set of simple tensors in $\bigwedge^p E$.

Proposition 2.2.16. *Let $(V_n)_{n \in \mathbb{N}} \in \mathcal{G}_p(E)^{\mathbb{N}}$ such that $V_n \to V \in \mathcal{G}(E)$. Then $V \in \mathcal{G}_p(E)$. In particular, $\mathcal{G}_p(E)$ is closed in $\mathcal{G}(E)$ and thus is a complete metric space.*

Proof. Let $x_1, \ldots, x_{p+1} \in S_V$. There exists sequences $(x_1^{(n)})_{n \geq 0}, \ldots, (x_{p+1}^{(n)})_{n \geq 0}$ satisfying

$$\forall i, \forall n \geq 0, x_i^{(n)} \in V_n$$

such that $x_i^{(n)} \underset{n \to \infty}{\longrightarrow} x_i$. Using Lemma 2.2.4, we get $x_1^{(n)} \wedge \cdots \wedge x_{p+1}^{(n)} \underset{n \to \infty}{\longrightarrow} x_1 \wedge \cdots \wedge x_{p+1}$. Since $\dim(V_n) = p$, we have $x_1^{(n)} \wedge \cdots \wedge x_{p+1}^{(n)} = 0$ for all n, so $x_1 \wedge \cdots \wedge x_{p+1} = 0$ and therefore $\dim(V) \leq p$.

We now prove that $\dim(V) \geq p$. There exists an integer N such that $d_H(V_N, V) \leq \frac{1}{4p.p!}$. Let $(y_1, \ldots, y_p)$ be a right decomposition of V_N. There exists $x_1, \ldots, x_p \in V$ such that $\|y_i - x_i\| \leq \frac{1}{2p.p!}$ for all i, and using Lemma 2.2.4, we get $\|x_1 \wedge \cdots \wedge x_p - y_1 \wedge \cdots \wedge y_p\|_{\wedge l, p} \leq \frac{1}{2}$. As $(y_1, \ldots, y_p)$ is a right decomposition, $\|y_1 \wedge \cdots \wedge y_p\|_{\wedge l, p} \geq 1$ and therefore we have $\|x_1 \wedge \cdots \wedge x_p\|_{\wedge l, p} \geq \frac{1}{2}$. In particular, $x_1 \wedge \cdots \wedge x_p \neq 0$, hence $\dim(V) \geq p$. □

Remark. A similar proof using the exterior product shows that if $V_n \to V$ and $\dim(V) = p$, then $\dim(V_n) = p$ for n large enough. Also, if $V_n \to V$ and $\dim(V_n) < \infty$ for all n, then $\dim(V) < \infty$. In particular, the set of all finite-dimensional subspaces of E is a closed subset of $\mathcal{G}(E)$ (and therefore is a complete metric space).

Proposition 2.2.17. *The subset $\{x_1 \wedge \cdots \wedge x_p \mid x_1, \ldots, x_p \in E\}$ of $\bigwedge^p E$ is a complete metric space for $\|.\|_{\wedge,p}$.*

Proof. Let $\alpha_n = x_1^n \wedge \cdots \wedge x_p^n$ be a Cauchy sequence for $\|.\|_{\wedge,p}$. If there is a subsequence α_{n_k} such that $\alpha_{n_k} \to 0$, then $\alpha_n \underset{n\to+\infty}{\longrightarrow} 0$. Otherwise, for n big enough we have $\|\alpha_n\|_{\wedge,p} > 1/c$ for some constant $c > 0$, hence $V_n \coloneqq \mathrm{Span}(x_1^n, \ldots, x_p^n)$ is a p-dimensional subspace of E. Since $d_{\wedge,p}$ and d_H are equivalent, (V_n) is a Cauchy sequence in the complete metric space $(\mathcal{G}_p(E), d_H)$ and therefore converges to some $V \in \mathcal{G}_p(E)$. Let $(x_1, \ldots, x_p)$ be a basis of V such that $\alpha = x_1 \wedge \cdots \wedge x_p$ satisfies $\|\alpha\|_{\wedge,p} = 1$. Proposition 2.2.14 and the fact that $d_{\wedge,p}$ and d_H are equivalent show that there exists a sequence $\lambda_n \in \mathbb{C}$ with $|\lambda_n| = 1$ such that

$$\frac{\lambda_n}{\|\alpha_n\|_{\wedge,p}}\alpha_n - \alpha \underset{n\to+\infty}{\longrightarrow} 0.$$

Let $\mu_n = \|\alpha_n\|_{\wedge,p}/\lambda_n$. Since (α_n) is a Cauchy sequence, it is bounded and we can assume $\|\alpha_n\|_{\wedge,p} \leq c$ for all n. Hence $1/c \leq |\mu_n| \leq c$ and therefore $(\alpha_n - \mu_n\alpha) \underset{n\to+\infty}{\longrightarrow} 0$. Then

$$|\mu_n - \mu_p| = \|\mu_n\alpha - \mu_p\alpha\|_{\wedge,p} \leq \|\mu_n\alpha - \alpha_n\|_{\wedge,p} + \|\alpha_n - \alpha_p\|_{\wedge,p} + \|\mu_p\alpha - \alpha_p\|_{\wedge,p}.$$

This shows that (μ_n) is a Cauchy sequence and therefore converges to some $\mu \in \mathbb{C}^*$. This gives us $\alpha_n \underset{n\to+\infty}{\longrightarrow} \mu\alpha$ and achieves the proof. $\square$

Chapter 3

p-dimensional cones and applications

3.1 Preliminaries

3.1.1 Introduction

The Perron-Frobenius Theorem [Per07, Fro12] asserts that a real square matrix with strictly positive entries has a spectral gap, i.e. it has a positive simple eigenvalue and the remaining part of the spectrum is strictly smaller in modulus.

In a seminal paper [Bir57], Birkhoff generalized this theory: he showed a contraction principle for projective cones $\mathcal{C}$ equipped with the Hilbert metric. If a linear map T satisfies $T(\mathcal{C}^*) \subset \mathcal{C}^*$, then it is a contraction for this metric. Moreover, if the diameter of $T(\mathcal{C}^*)$ is finite for this metric, then T is a strict contraction.

These notions for real cones were extended to complex cones, as introduced by Rugh in [Rug10]. The gauge used is no longer necessarily a metric, and relies on the hyperbolic metric on sections of the cones. Using this theory, he obtained a contraction principle for complex cones, a spectral gap theorem, and regularity results for the first characteristic exponent of random products of operators preserving a complex cone. This was further studied by Dubois [Dub09], with a new gauge which is easier to use and which is a metric for some specific complex cones.

The aformentionned works only concern the leading eigenvalue or characteristic exponent. In [Rue79], Ruelle remarks that such analytic properties can be obtained for the p first characteristic exponents for some families of operators, by applying the result he had for the first exponent to the p-exterior product of operators $T \wedge \cdots \wedge T$. However, cone contraction can be difficult to check for p-exterior products, especially as there is no canonical norm on the exterior product $\bigwedge^p E$ of a Banach space E. More recently, Blumenthal, Quas and Gonzalez-Tokman have used several tools on Grassmannian spaces to get more concise and intuitive proofs for the Oseledec theorem [Blu16, GTQ15]. They also obtained, with Morris and Froyland, results on splittings and stability of characteristic exponents [BlM15, FGQ15]. This approach avoids the use of exterior algebra, making it more geometric and natural. However, it does not give rise to analytic properties for the exponents of the splitting. In this chapter, we address this issue by using similar tools and also the exterior algebra to get our results.

Specifically, we use complex multidimensional cones. Real multidimensional cones were notably used to characterize dominated splitting for cocycles on finite-dimensional vector bundles by Bochi and Gourmelon in [BG09]. In [ABY10], the case of $SL_2(\mathbb{R})$ is studied in detail and a complete description of multidimensional cones associated with a pair of matrices of $SL_2(\mathbb{R})$ is obtained. More recently they have been used in [BoM15] to study the regularity of the lower spectral radius of subsets of $GL_d(\mathbb{R})$. Our approach is more general as we study multidimensional cones in Banach complex spaces and operators which are not necessarily invertible.

In Section 3.1.2, we recall some known results on one-dimensional complex cones from [Rug10] and [Dub09]. We introduce the notion of complex cone, and a projective gauge $\delta_{\mathcal{C}}$, giving rise to a contraction principle. Under some regularity assumptions on the complex cone, such a contraction has a spectral gap. In Section 3.2, we define the notion of p-dimensional cone and construct a cone gauge $d_{\mathcal{C}}(V,W)$ between p-dimensional subspaces V and W contained in the cone. We then deduce from the contraction principle on one-dimensional complex cones a contraction principle for p-dimensional cones. In Section 3.3, we generalize the regularity notions from complex cones to p-dimensional cones to obtain a spectral gap theorem for p-cone contractions. Finally, in Section 3.4, we prove a theorem on the analyticity of characteristic exponents of random products of cone contractions. We illustrate this result on the following example: if (ξ_n) is a sequence of independent and identically distributed random variables with values in the Poincaré disk $\mathbb{D}$, then for $t \in \mathbb{D}$ the characteristic exponents of the product of

$$M_n(t) = \begin{pmatrix} 10 + t\xi_n & t + \xi_n & it \\ t + \xi_n & 6 & \xi_n \\ i\xi_n & 0 & 1 \end{pmatrix}$$

is analytic in t.

3.1.2 Definitions and results for classic complex cones

In this section, we recall some definitions and results from [Rug10] and [Dub09], which serve as a foundation for our work. In the sequel, E denotes a complex Banach space. For a subset $X \subset E$, we write $X^* = X \setminus \{0\}$.

Definition 3.1.1. *We say that a subset $\mathcal{C} \subset E$ is a closed complex cone if it is closed in E, $\mathbb{C}$-invariant and $\mathcal{C} \neq \{0\}$. Such a cone is said to be proper if it contains no complex subpaces of dimension 2. We refer to proper closed complex cones as $\mathbb{C}$-cones.*

Definition 3.1.2. *Given a $\mathbb{C}$-cone $\mathcal{C}$, we define the gauge of Rugh $\delta_{\mathcal{C}} : \mathcal{C}^* \times \mathcal{C}^* \to [0, \infty]$ between two points $x, y \in \mathcal{C}^*$ as follows:*

- *If x and y are co-linear we set $\delta_{\mathcal{C}}(x, y) = 0$,*
- *If x and y are linearly independent, we define $S(x, y) := \mathcal{C} \cap (x + \mathbb{C}(y - x))$ as the section of $\mathcal{C}$ by the affine complex plane passing through x and y, and:*
  - *If x and y are in the same connected component U of the interior $S^o(x, y)$ of this section, then we set $\delta_{\mathcal{C}}(x, y) = d_U(x, y)$ where d_U is the hyperbolic metric on U;*

– *otherwise, we set* $\delta_{\mathcal{C}}(x,y)=\infty$.

This gauge is symmetric and projective, i.e. $\delta_{\mathcal{C}}(y,x)=\delta_{\mathcal{C}}(x,y)=\delta_{\mathcal{C}}(\lambda x,y)=\delta_{\mathcal{C}}(x,\lambda y)$.

For information on the hyperbolic metric on Riemann surfaces, we refer to [CG93].

Definition 3.1.3. *Given a* $\mathbb{C}$*-cone* $\mathcal{C}$ *and* $x,y\in\mathcal{C}^*$, *we consider the set*

$$E_{\mathcal{C}}(x,y)=\{z\in\mathbb{C}\mid zx-y\notin\mathcal{C}\}$$

and denote $|E_{\mathcal{C}}(x,y)|=\{|z|\mid z\in E_{\mathcal{C}}(x,y)\}$. *The gauge of Dubois* $\delta_{\mathcal{C}}:\mathcal{C}^*\times\mathcal{C}^*\to[0,\infty]$ *between* x *and* y *is defined as follows:*

- *If* x *and* y *are co-linear we set* $\delta_{\mathcal{C}}(x,y)=0$
- *If* x *and* y *are linearly independent, we set* $\delta_{\mathcal{C}}(x,y)=\log(b/a)$, *where* $b=\sup|E_{\mathcal{C}}(x,y)|$ *and* $a=\inf|E_{\mathcal{C}}(x,y)|$.

This gauge is symmetric and projective.

The following results are valid for each of these gauges.

Proposition 3.1.4 (Rugh, Dubois)**.** *Let* E_1,E_2 *be complex Banach spaces. Let* $T\colon E_1\to E_2$ *be a continuous linear map and* $\mathcal{C}_1\subset E_1$, $\mathcal{C}_2\subset E_2$ *two* $\mathbb{C}$*-cones such that* $T(\mathcal{C}_1^*)\subset\mathcal{C}_2^*$. *Then for all* $x,y\in\mathcal{C}_1^*$, *we have* $\delta_{\mathcal{C}_2}(Tx,Ty)\le\delta_{\mathcal{C}_1}(x,y)$.

Moreover, if $\Delta=\mathrm{diam}_{\mathcal{C}_2}(T\mathcal{C}_1^*)<\infty$, *there exists* $\eta<1$ *depending only on* Δ *such that for all* $x,y\in\mathcal{C}_1^*$ *we have*

$$\delta_{\mathcal{C}_2}(Tx,Ty)\le\eta.\delta_{\mathcal{C}_1}(x,y)\,.$$

We introduce the notion of bounded aperture and reproducing cone and recall a theorem from [Rug10].

Definition 3.1.5. *For* $K>0$, *we say that a* $\mathbb{C}$*-cone* $\mathcal{C}$ *is of* K*-bounded aperture if there exists a linear form* $m:E\to\mathbb{C}$ *such that* $\forall x\in\mathcal{C},\|m\|\|x\|\le K|m(x)|$.

Moreover, we say that the cone is of K*-bounded sectional aperture if for all* $x,y\in E$, *the cone* $\mathcal{C}(x,y):=\mathcal{C}\cap\mathrm{Span}(x,y)$ *is of* K*-bounded aperture, i.e. there exists a linear form* $m_{x,y}:\mathrm{Span}(x,y)\to\mathbb{C}$ *such that*

$$\forall u\in\mathcal{C}(x,y),\|m_{x,y}\|\|u\|\le K|m_{x,y}(u)|.$$

Definition 3.1.6. *We say that a* $\mathbb{C}$*-cone* $\mathcal{C}$ *is reproducing if there exists a constant* $A>0$ *and* $k\in\mathbb{N}^*$ *such that for all* $x\in E$, *there exists* $x_1,\dots,x_k\in\mathcal{C}$ *satisfying* $x=x_1+\cdots+x_k$ *and* $\|x_1\|+\cdots+\|x_k\|\le A\|x\|$.

Theorem 3.1.7 (Rugh)**.** *Let* $T\in\mathcal{L}(E)$ *and let* $\mathcal{C}$ *be a reproducing* $\mathbb{C}$*-cone of bounded sectional aperture. Suppose that* T *is a strict cone contraction, i.e.* $T(\mathcal{C}^*)\subset\mathcal{C}^*$ *and* $\mathrm{diam}_{\mathcal{C}}(T\mathcal{C}^*)<\infty$ *(as in Proposition 3.1.4). Then* T *has a spectral gap.*

3.2 Multidimensional cones and contraction principle

3.2.1 Definition of p-cones

In [Rug10], Rugh defines a notion of proper complex cone which can be seen as an union of $\mathbb{C}$-lines which does not contain any 2-dimensional subspace. We extend this definition to introduce higher dimensional cones. Then we use the gauge/metric defined in [Rug10] and [Dub09] to define a "distance" on our cones.

In the remainder of this chapter, p denotes a positive integer.

Definition 3.2.1. *We say that a subset $\mathcal{C} \subset E$ is a closed p-dimensional cone if $\mathcal{C}$ is a union of p-dimensional subspaces of E, is closed, and $\mathcal{C} \neq \emptyset$.*

We say that a closed p-dimensional cone $\mathcal{C}$ is proper if it does not contains any $(p+1)$-dimensional subspace of E. We refer to proper closed p dimensional cones as p-cones.

Such a cone $\mathcal{C}$ can thus also been seen as a collection of p-dimensional subspaces, instead of a subset of E. We denote

$$\mathcal{G}_p(\mathcal{C}) = \{V \in \mathcal{G}_p(E) \mid V \subset \mathcal{C}\}$$

the subset of the Grassmannian of E of p-dimensional spaces contained in $\mathcal{C}$. In the sequel, we sometimes abuse the notation and use $\mathcal{C}$ to denote $\mathcal{G}_p(\mathcal{C})$.

Remark. A p-cone is always $\mathbb{C}$-invariant. For $p = 1$ we get the usual definition of proper closed complex cone.

Remark. This definition is more general than the one for multicones in [BG09], where a transverse subspace of codimension p is required.

Example. Let F be a p-dimensional subspace of E, and G a closed complement of F in E. Let $\pi : E \to F$ be the projection on F parallel to G. Then for all $a > 0$, the set

$$\mathcal{C}_{\pi,a} = \{x \in E \mid \|(\mathrm{Id} - \pi)(x)\| \leq a\|\pi(x)\|\}$$

is a p-cone of E. Such a cone is called a projection cone.

We now define a "distance" on the p-dimensional spaces of a p-cone, using the gauge defined on 1-dimensional complex cones in [Rug10] or in [Dub09]. If $\mathscr{C}$ is a classic complex cone, we denote by $\delta_{\mathscr{C}}$ either of these 1-dimensional gauges on $\mathscr{C}$. We use quotation marks on the word "distance" as this object may not satisfy the triangle inequality.

Definition 3.2.2. *Let $\mathcal{C} \subset E$ be a p-cone and $x, y \in \mathcal{C}^*$. We define the 1-dimensional "distance" $d_{1,\mathcal{C}}(x, y)$ by :*

- $d_{1,\mathcal{C}}(x, y) = 0$ *if* $\mathrm{Span}(x, y) \subset \mathcal{C}$,
- $d_{1,\mathcal{C}}(x, y) = \delta_{\mathcal{C} \cap \mathrm{Span}(x,y)}(x, y)$ *otherwise (in this case, $\mathcal{C} \cap \mathrm{Span}(x, y)$ is a proper complex cone).*

Let $\mathcal{C}$ be a p-cone and $V, W \subset \mathcal{C}$ be p-dimensional subspaces of E. We define the "cone distance" between V and W as :

$$d_{\mathcal{C}}(V, W) = \sup_{(x,y) \in (V^*, W^*)} d_{1,\mathcal{C}}(x, y).$$

If $\mathcal{C}_1$ is a p-cone contained in $\mathcal{C}$, the diameter of $\mathcal{C}_1$ in $\mathcal{C}$ is

$$\operatorname{diam}_{\mathcal{C}}(\mathcal{C}_1) = \sup\{d_{\mathcal{C}}(V,W) \mid V,W \text{ } p\text{-dimensional subspaces contained in } \mathcal{C}_1\}.$$

Remark. If $\operatorname{Span}(x,y) \nsubseteq \mathcal{C}$, then $d_{1,\mathcal{C}}(x,y) > 0$.

Let $V \neq W$ be p-dimensional spaces contained in a p-cone $\mathcal{C}$. The cone $\mathcal{C}$ is proper so we have $V + W \nsubseteq \mathcal{C}$. Hence there exist $x \in V$, $y \in W$ such that $\operatorname{Span}(x,y) \nsubseteq \mathcal{C}$. This implies that $d_{\mathcal{C}}(V,W) > 0$.

Example. Let $0 < a < b$. Defining our "distance" with respect to the gauge for [Dub09], we have $\mathcal{C}_{\pi,a} \subset \mathcal{C}_{\pi,b}$ and $\operatorname{diam}_{\mathcal{C}_{\pi,b}}(\mathcal{C}_{\pi,a}) \leq 2\log\left(\frac{b+a}{b-a}\right)$.

Proof. Let $x,y \in \mathcal{C}_{\pi,a}$ with decompositions $x = x_F + x_G$ and $y = y_F + y_G$ with respect to $E = F \oplus G$. To estimate $\delta_{\mathcal{C}_{\pi,b}}(x,y)$ we must find an upper and lower bound for the set $E_{\mathcal{C}_{\pi,b}}(x,y) = \{z \in \mathbb{C} \mid zx - y \notin \mathcal{C}_{\pi,b}\}$. The gauge is projective, so we can assume that $\|x_F\| = \|y_F\| = 1$, and therefore we have $\|x_G\|, \|y_G\| \leq a$.

Let $z \in \mathbb{C}$ with $|z| \geq \frac{b+a}{b-a}$. Then

$$\|z.x_G - y_G\| \leq |z|a + a \leq b(|z|-1) \leq b\|z.x_F - y_F\|,$$

so $zx - y \in \mathcal{C}_{\pi,b}$. Similarly, if $|z| \leq \frac{b-a}{b+a}$, then

$$\|z.x_G - y_G\| \leq |z|a + a \leq b(1-|z|) \leq b\|z.x_F - y_F\|,$$

so $zx - y \in \mathcal{C}_{\pi,b}$. This gives $\inf|E_{\mathcal{C}_{\pi,b}}(x,y)| \geq \frac{b-a}{b+a}$ and $\sup|E_{\mathcal{C}_{\pi,b}}(x,y)| \leq \frac{b+a}{b-a}$, hence

$$\delta_{\mathcal{C}_{\pi,b}}(x,y) \leq \log\left((\frac{b+a}{b-a})^2\right) = 2\log\left(\frac{b+a}{b-a}\right).$$

□

3.2.2 Contraction principle for p-cones

We can now formulate the following contraction principle:

Theorem 3.2.3. *Let E_1, E_2 be complex Banach spaces. Let $T : E_1 \to E_2$ a continuous linear map and $\mathcal{C}_1 \subset E_1$, $\mathcal{C}_2 \subset E_2$ two p-cones such that $T(\mathcal{C}_1^*) \subset \mathcal{C}_2^*$. Then for every p-dimensional spaces V,W contained in $\mathcal{C}_1$, we have $d_{\mathcal{C}_2}(TV,TW) \leq d_{\mathcal{C}_1}(V,W)$.*

Moreover, if $\Delta = \operatorname{diam}_{\mathcal{C}_2}(T\mathcal{C}_1) < \infty$, there exists $\eta < 1$ depending only on Δ such that for all p-dimensional spaces V,W contained in $\mathcal{C}_1$, we have

$$d_{\mathcal{C}_2}(TV,TW) \leq \eta . d_{\mathcal{C}_1}(V,W).$$

Proof. The proof relies on the result known for the 1-dimensional cones. Recall that in the definition of our "distance", the only points (x,y) contributing to the supremum are the ones for which $\operatorname{Span}(x,y) \nsubseteq \mathcal{C}$. For these points, $\operatorname{Span}(x,y) \cap \mathcal{C}$ is a proper closed 1-dimensional cone, allowing us to use the results of [Rug10] and [Dub09]. When we have $\operatorname{Span}(x,y) \nsubseteq \mathcal{C}$, we use the notation $\mathcal{C}(x,y) := \operatorname{Span}(x,y) \cap \mathcal{C}$.

Let V,W be p-dimensional subspaces of E_1 contained in $\mathcal{C}_1$. Let $x \in V^*$, $y \in W^*$. If $\operatorname{Span}(x,y) \subset \mathcal{C}_1$, then $\operatorname{Span}(Tx,Ty) = T\operatorname{Span}(x,y) \subset T\mathcal{C}_1 \subset \mathcal{C}_2$. As $T(\mathcal{C}_1^*) \subset \mathcal{C}_2^*$, we have

$(TV)^* = T(V^*)$, $(TW)^* = T(W^*)$, and TV and TW are p-dimensional subspaces of E_2. The definition of $d_{\mathcal{C}_2}$ gives us:

$$\begin{aligned} d_{\mathcal{C}_2}(TV,TW) &= \sup\{d_{1,\mathcal{C}_2}(x,y) \mid (x,y) \in (TV^*,TW^*)\} \\ &= \sup\{d_{1,\mathcal{C}_2}(Tx,Ty) \mid (x,y) \in (V^*,W^*)\} \\ &= \sup\{d_{1,\mathcal{C}_2}(Tx,Ty) \mid (x,y) \in (V^*,W^*), \mathrm{Span}(Tx,Ty) \nsubseteq \mathcal{C}_2\} \\ &= \sup\{\delta_{\mathcal{C}_2(Tx,Ty)}(Tx,Ty) \mid (x,y) \in (V^*,W^*), \mathrm{Span}(Tx,Ty) \nsubseteq \mathcal{C}_2\}. \end{aligned}$$

For all $(x,y) \in (V^*,W^*)$ such that $\mathrm{Span}(Tx,Ty) \nsubseteq \mathcal{C}_2$, the map $T_{|\,\mathrm{Span}(x,y)} : \mathrm{Span}(x,y) \to \mathrm{Span}(Tx,Ty)$ satisfies $T(\mathcal{C}_1(x,y)^*) \subset \mathcal{C}_2(Tx,Ty)^*$. Using Proposition 3.1.4, we get

$$\delta_{\mathcal{C}_2(Tx,Ty)}(Tx,Ty) \leq \delta_{\mathcal{C}_1(x,y)}(x,y).$$

Hence

$$\begin{aligned} d_{\mathcal{C}_2}(TV,TW) &\leq \sup\{\delta_{\mathcal{C}_1(x,y)}(x,y) \mid (x,y) \in (V^*,W^*), \mathrm{Span}(Tx,Ty) \nsubseteq \mathcal{C}_2\} \\ &\leq \sup\{\delta_{\mathcal{C}_1(x,y)}(x,y) \mid (x,y) \in (V^*,W^*), \mathrm{Span}(x,y) \nsubseteq \mathcal{C}_1\} \\ &\leq \sup\{d_{1,\mathcal{C}_1}(x,y) \mid (x,y) \in (V^*,W^*), \mathrm{Span}(x,y) \nsubseteq \mathcal{C}_1\} \\ &\leq \sup\{d_{1,\mathcal{C}_1}(x,y) \mid (x,y) \in (V^*,W^*)\} \\ &\leq d_{\mathcal{C}_1}(V,W). \end{aligned}$$

For the second part of the statement, we show that $\mathrm{diam}_{\mathcal{C}_2(Tx,Ty)}(T(\mathcal{C}_1(x,y))) \leq \Delta$, where the diameter is taken with respect to the gauge $\delta_{\mathcal{C}_2(Tx,Ty)}$ on classic complex cones. Let $u,v \in \mathcal{C}_1(x,y)$. If u and v are collinear, $\delta_{\mathcal{C}_2(Tx,Ty)}(Tu,Tv) = 0$. Otherwise, we have $\mathrm{Span}(u,v) = \mathrm{Span}(x,y)$, hence $\delta_{\mathcal{C}_2(Tx,Ty)}(Tu,Tv) = \delta_{\mathcal{C}_2(Tu,Tv)}(Tu,Tv) \leq d_{1,\mathcal{C}_2}(Tu,Tv)$. As $Tu,Tv \in \mathcal{C}_1$, there exists p-dimensional subspaces Z_u and Z_v such that $u \in Z_u \subset \mathcal{C}_1$ (and similarly for v). Then $TZ_u \subset \mathcal{C}_2$ (and similarly for v) and we have

$$\delta_{\mathcal{C}_2(Tx,Ty)}(Tu,Tv) \leq d_{\mathcal{C}_2}(TZ_u,TZ_v) \leq \Delta.$$

Using Proposition 3.1.4, we get a constant $\eta < 1$ such that: for all $(x,y) \in (V^*,W^*)$ such that $\mathrm{Span}(Tx,Ty) \nsubseteq \mathcal{C}_2$, $\delta_{\mathcal{C}_2(Tx,Ty)}(Tx,Ty) \leq \eta.\delta_{\mathcal{C}_1(x,y)}(x,y)$. The same reasoning as before then provides the desired result.

□

Remark. If E is finite-dimensional, the finite-diameter condition is satisfied when the inclusion $T(\mathcal{C}_1^*) \subset \mathrm{int}\ \mathcal{C}_2^*$ holds. In this specific case, the arguments employed can be compared to the ones used in the proof of Theorem B in [BG09].

Example. For $0 < a < b$, recall $\mathcal{C}_{\pi,a} \subset \mathcal{C}_{\pi,b}$. if $T \in \mathcal{L}(E)$ satisfies $T(\mathcal{C}_{\pi,b}^*) \subset \mathcal{C}_{\pi,a}^*$, then T satisfies the assumptions of Theorem 3.2.3 and we have (using the Dubois gauge from [Dub09]): For all p-dimensional spaces V,W contained in $\mathcal{C}_{\pi,b}$,

$$d_{\mathcal{C}_{\pi,b}}(TV,TW) \leq \frac{a}{b}.d_{\mathcal{C}_{\pi,b}}(V,W),$$

i.e. T is a contraction on the set of p-dimensional spaces of $\mathcal{C}_{\pi,b}$ with respect to the "distance" $d_{\mathcal{C}_{\pi,b}}$.

3.3 Regularity of p-cones and spectral gap theorem

We first define a notion of bounded aperture for p-cones, then use it to compare norms with our cone "distance". We finally use these comparisons to get a spectral gap result. In this section, we denote by $\|.\|_{\wedge,p}$ any of the previous norms on $\bigwedge^p E$ defined in Chapter 2: as we never make computations on more than 2 simple tensors, Lemma 2.2.12 ensures that these norms are equivalent. We may also use the notation $\|.\|$ for $\|.\|_{\wedge,p}$. We denote by $|.|$ the standard hermitian norm on $\mathbb{C}^p$.

3.3.1 Aperture of a p-cone

Definition 3.3.1. *We say that a p-cone $\mathcal{C} \subset E$ is of aperture bounded by $K > 0$ if there exists a continuous linear map $m : E \to \mathbb{C}^p$ such that $\forall x \in \mathcal{C}, \|m\|.\|x\| \leq K|m(x)|$.*
We say that a p-cone $\mathcal{C} \subset E$ is of sectional aperture bounded by $K > 0$ if for any two p-dimensional spaces V and W contained in $\mathcal{C}$, there exists a linear map $m_{V,W}$: $\mathrm{Span}(V,W) \to \mathbb{C}^p$ such that $\forall x \in \mathcal{C} \cap \mathrm{Span}(V,W), \|m_{V,W}\|.\|x\| \leq K|m_{V,W}(x)|$.

Lemma 3.3.2. *Let $\mathcal{C} \subset E$ be a p-cone of sectional aperture bounded by $K > 0$. Let V, W be p-dimensional spaces contained in $\mathcal{C}$ and $m = m_{V,W}$ a linear map satisfying the inequality from Definition 3.3.1. Then for all $x \in V^*$, there exists $y \in W^*$ such that*

$$\left\| \frac{x}{|m(x)|} - \frac{y}{|m(y)|} \right\| \leq \frac{K}{\|m\|} d_{\mathcal{C}}(V,W).$$

Moreover,

$$\|x - y\| \leq \frac{K}{\|m\|} |m(x)| d_{\mathcal{C}}(V,W).$$

Proof. We normalize m so that $\|m\| = K$. Then for all $x \in V^*$, $|m(x)| \geq \|x\| > 0$, so $m_{|V} : V \to \mathbb{C}^p$ is injective, and therefore bijective. Similarly, $m_{|W} : W \to \mathbb{C}^p$ is bijective. Let $x \in V^*$. Then there exists $y \in W^*$ such that $m(y) = m(x) \neq 0$. In particular $m(x)$ and $m(y)$ are collinear. This allows us to use the computations from [Rug10] and [Dub09]. Let $x' = \frac{x}{|m(x)|}$ and $y' = \frac{y}{|m(y)|}$.

Case 1: with the gauge from [Rug10]
Let $u_\lambda = (1+\lambda)x' + (1-\lambda)y'$ pour $\lambda \in \mathbb{C}$. Then $|m(u_\lambda)| = |(1+\lambda)\frac{m(x)}{|m(x)|} + (1-\lambda)\frac{m(x)}{|m(x)|}| = 2$. When $u_\lambda \in \mathcal{C}$, we get (using Definition 3.3.1):

$$|\lambda|\|x' - y'\| \leq \|u_\lambda\| + (\|x'\| + \|y'\|) \leq 2 + 1 + 1 \leq 4.$$

We mimic the proof of Lemma 3.4 in [Rug10] to get our result: let $R := \frac{4}{\|x'-y'\|}$. If $u_\lambda \in \mathcal{C}$, then $|\lambda| \leq R$ and therefore $S(x', y') \subset B(0, R)$ (see Definition 3.1.2). As a bigger set implies a smaller hyperbolic distance, we get:

$$d_{\mathcal{C}}(x,y) = d_{S^o(x',y')}(-1,1) \geq d_{B(0,R)}(-1,1) = d_{\mathbb{D}}(-\frac{1}{R}, \frac{1}{R}) = 2\log\frac{1+\frac{1}{R}}{1-\frac{1}{R}}.$$

This shows $\frac{\|x'-y'\|}{4} = \frac{1}{R} \leq \tanh(\frac{d_{\mathcal{C}}(x,y)}{4}) \leq \frac{d_{\mathcal{C}}(x,y)}{4} \leq \frac{d_{\mathcal{C}}(V,W)}{4}$, which gives the desired results as $m(x) = m(y)$.

Case 2: with the gauge from [Dub09]
Here we mimic the proof and computations in [Dub09]. If x and y are collinear, then $\|x' - y'\| = 0$ and the proof is over. Otherwise, let $E_C(x', y') = \{z \in \mathbb{C} \mid zx' - y' \notin \mathcal{C}\}$. As $|m(x' - y')| = 0 < \|x' - y'\|$, we get $x' - y' \notin \mathcal{C}$ and therefore $1 \in E_C(x', y')$. Let $0 < a < \inf |E_C(x', y')|$ and $b > \sup |E_C(x', y')|$. Clearly, we have $a < 1 < b$.

$$\begin{aligned}
\|x' - y'\| &\leq \frac{b-1}{b-a}\|ax' - y'\| + \frac{1-a}{b-a}\|bx' - y'\| \\
&\leq \frac{b-1}{b-a}|m(ax' - y')| + \frac{1-a}{b-a}|m(bx' - y')| \\
&\leq 2\frac{(b-1)(1-a)}{b-a} \\
&\leq 2\frac{\sqrt{b} - \sqrt{a}}{\sqrt{b} + \sqrt{a}} \\
&= 2\tanh\left(\frac{\log(b/a)}{4}\right) \leq \frac{1}{2}\log(b/a).
\end{aligned}$$

Hence $\|x' - y'\| \leq \frac{1}{2} d_{\mathcal{C}}(x, y) \leq d_{\mathcal{C}}(V, W)$. □

We now extend this result to representatives $x_1 \wedge \cdots \wedge x_p$ of p-dimensional spaces.

Definition 3.3.3. *If V is a complex Banach space and $m : V \to \mathbb{C}^p$ is a continuous linear map, we consider the map $\hat{m} : \bigwedge^p V \to \mathbb{C}$ obtained from the alternating p-linear map $(x_1, \ldots, x_p) \mapsto \det_{\mathbb{C}^p}(m(x_1), \ldots, m(x_p))$.*

The following lemma gives bounds on the norm $\|\hat{m}\|$.

Lemma 3.3.4. *Let $\mathcal{C}$ be p-cone of E of aperture bounded by $K > 0$ with $m : E \to \mathbb{C}^p$ a continuous linear map such that $\forall x \in \mathcal{C}, \|m\|.\|x\| \leq K|m(x)|$. Then*

- *For the norm $\|.\|_{\wedge d,p}$, we have $\|\hat{m}\| \leq \|m\|^p$ and*

$$\forall\, \mathrm{Span}(x_1, \ldots, x_p) \subset \mathcal{C}, \|x_1 \wedge \cdots \wedge x_p\|_{\wedge d,p} \leq \frac{K^p}{\|m\|^p}|\hat{m}(x_1 \wedge \cdots \wedge x_p)|.$$

- *For the norm $\|.\|_{\wedge l,p}$, we have*

$$\begin{gathered}
\forall (x_1, \ldots, x_p) \in E^p, |\hat{m}(x_1 \wedge \cdots \wedge x_p)| \leq \|m\|^p \|x_1 \wedge \cdots \wedge x_p\|_{\wedge l,p} \;; \\
\forall\, \mathrm{Span}(x_1, \ldots, x_p) \subset \mathcal{C}, \|x_1 \wedge \cdots \wedge x_p\|_{\wedge l,p} \leq (\sqrt{p})^p \frac{K^p}{\|m\|^p}|\hat{m}(x_1 \wedge \cdots \wedge x_p)|.
\end{gathered}$$

This lemma is proved in Section 5.3 of the Appendix.

Lemma 3.3.5. *Let $\mathcal{C} \subset E$ be a p-cone of sectional aperture bounded by $K > 0$. Let V, W be p-dimensional spaces contained in $\mathcal{C}$ and $m = m_{V,W}$ a linear map satisfying the inequality from Definition 3.3.1. If $(x_1, \ldots, x_p)$ is a basis of V and $(y_1, \ldots, y_p)$ is a basis of W, then*

$$\left\| \frac{x_1 \wedge \ldots \wedge x_p}{\hat{m}(x_1 \wedge \ldots \wedge x_p)} - \frac{y_1 \wedge \ldots \wedge y_p}{\hat{m}(y_1 \wedge \ldots \wedge y_p)} \right\| \leq a_p \frac{K^p}{\|m\|^p} d_{\mathcal{C}}(V, W)$$

where $a_p > 0$ is a constant depending only on p.

Proof. By Lemma 2.2.12, it suffices to show this for one of the two norms $\|.\|_{\wedge l,p}$ and $\|.\|_{\wedge d,p}$. Here, we use the norm $\|.\|_{\wedge d,p}$.

Given two bases $(x_1, \dots, x_p)$ and $(x'_1, \dots, x'_p)$ of V, $x_1 \wedge \dots \wedge x_p$ and $x'_1 \wedge \dots \wedge x'_p$ are collinear (as it is for two bases of W). Hence we only have to show the result for one particular basis of V and one particular basis of W.

Let $(e_1, \dots, e_p)$ be the canonical basis of $\mathbb{C}^p$. The map $m_{|V} : V \to \mathbb{C}^p$ is bijective, as proved for Lemma 3.3.2. Then there exists a basis $(x_1, \dots, x_p)$ of V such that $(m(x_1), \dots, m(x_p)) = (e_1, \dots, e_p)$. Similarly to the proof of Lemma 3.3.2, we choose $(y_1, \dots, y_p)$ in W such that $m(x_i) = m(y_i) = e_i$ for $1 \leq i \leq p$. In particular $(y_1, \dots, y_p)$ is a basis of W. Note that $|m(x_i)| = |m(y_i)| = 1$ for all i, and

$$\hat{m}(x_1 \wedge \cdots \wedge x_p) = \hat{m}(y_1 \wedge \cdots \wedge y_p) = 1.$$

Using Lemma 3.3.2, we get: for all i, $\|x_i - y_i\| \leq \dfrac{K}{\|m\|} d_{\mathcal{C}}(V, W)$. Using the latter inequality, the fact that $\|x_i\| \leq \frac{K}{\|m\|}$ and Lemma 3.3.4, we get:

$$\begin{aligned}
&\|x_1 \wedge \dots \wedge x_p - y_1 \wedge \dots \wedge y_p\| \\
&\quad \leq \|(x_1 - y_1) \wedge x_2 \wedge \cdots \wedge x_p\| + \|y_1 \wedge (x_2 \wedge \cdots \wedge x_p - y_2 \wedge \cdots \wedge y_p)\| \\
&\quad \leq \|(x_1 - y_1)\| \|x_2 \wedge \cdots \wedge x_p\| + \|y_1\| \|x_2 \wedge \cdots \wedge x_p - y_2 \wedge \cdots \wedge y_p\| \\
&\quad \leq \frac{K}{\|m\|} d_{\mathcal{C}}(V, W) \frac{K^{p-1}}{\|m\|^{p-1}} + \frac{K}{\|m\|} \|x_2 \wedge \cdots \wedge x_p - y_2 \wedge \cdots \wedge y_p\| \\
&\quad \leq \frac{K^p}{\|m\|^p} d_{\mathcal{C}}(V, W) + \frac{K}{\|m\|} \|x_2 \wedge \cdots \wedge x_p - y_2 \wedge \cdots \wedge y_p\|
\end{aligned}$$

By induction on p, we get

$$\|x_1 \wedge \dots \wedge x_p - y_1 \wedge \dots \wedge y_p\| \leq p \frac{K^p}{\|m\|^p} d_{\mathcal{C}}(V, W). \qquad \square$$

Example. Let F be a p-dimensional subspace of E, and G a closed complement of F in E. Let $\pi : E \to F$ be the projection on F parallel to G and $a > 0$. Then the cone $\mathcal{C}_{\pi,a}$ is of bounded aperture. Indeed, let $\phi : F \to \mathbb{C}^p$ be an isomorphism and define the continuous linear map $m : E \to \mathbb{C}^p$ as $m = \phi \circ \pi$. For $x = x_F + x_G \in \mathcal{C}_{\pi,a}$, $|m(x)| = |\phi(x_F)| \geq \|\phi^{-1}\|^{-1} \|x_F\| \geq \frac{\|\phi^{-1}\|^{-1}}{1+a} \|x\|$.

Remark. In the previous example, the isomorphism ϕ can be chosen such that $\|\phi\|$ and $\|\phi^{-1}\|$ only depends on the dimension p.

3.3.2 Spectral gap theorem

We now use the estimates given by the previous lemmas to prove the following proposition:

Proposition 3.3.6. *Let $\mathcal{C} \subset E$ be a p-cone of sectional aperture bounded by $K > 0$. Let $T \in \mathcal{L}(E)$ such that $T(\mathcal{C}^*) \subset \mathcal{C}^*$ and $diam_{\mathcal{C}}(T(\mathcal{C}^*)) = \Delta < \infty$. Let $\eta < 1$ be the contraction coefficient given by Theorem 3.2.3. Denote $\hat{T}$ the operator of $\mathcal{L}(\bigwedge^p E)$ induced by T. Then:*

(i) *There exists a unique p-dimensional subspace V contained in $\mathcal{C}$ fixed by T. Given a representative $h \in \bigwedge^p E$ of V with norm 1, h is an eigenvector for $\hat{T}$ for some eigenvalue $\lambda \in \mathbb{C}^*$: $\hat{T}h = \lambda h$.*

(ii) *There exist a map $c : \{x_1 \wedge \cdots \wedge x_p \mid \mathrm{Span}(x_1, \ldots, x_p) \subset \mathcal{C}\} \to \mathbb{C}$ and constants $R, C < \infty$ such that for all $(x_1, \ldots, x_p)$ with $\mathrm{Span}(x_1, \ldots, x_p) \subset \mathcal{C}$ and all $n \in \mathbb{N}^*$,*

$$\|\lambda^{-n}\hat{T}^n(x_1 \wedge \cdots \wedge x_p) - c(x_1 \wedge \cdots \wedge x_p).h\| \leq C\eta^n \|x_1 \wedge \cdots \wedge x_p\|,$$

$$|c(x_1 \wedge \cdots \wedge x_p)| \leq R\|x_1 \wedge \cdots \wedge x_p\|.$$

Remark. Given a p-dimensional subspace W contained in $\mathcal{C}$, the conditions $TW \subset W$ and $TW = W$ are equivalent since $T(\mathcal{C}^*) \subset \mathcal{C}^*$.

Proof. In this proof, we use the norm $\|.\|_{\wedge,p} = \|.\|_{\wedge l,p}$.

(i). Let $V_0 = \mathrm{Span}(x_1^0, \ldots, x_p^0)$ be a p-dimensional subspace contained in $\mathcal{C}$ and let $\hat{x}_0 = x_1^0 \wedge \cdots \wedge x_p^0 / \|x_1^0 \wedge \cdots \wedge x_p^0\|$. Let $V_n = T^n V_0$. We construct recursively a sequence $(\hat{x}_n)$ of elements of $\bigwedge^p E$. Given $\hat{x}_n$, Definition 3.3.1 gives us a functional m_n of norm K adapted with V_n and V_{n+1}. Let $\lambda_n = \dfrac{\widehat{m_n}(\hat{T}\hat{x}_n)}{\widehat{m_n}(\hat{x}_n)}$. By Lemma 3.3.4, we have $\lambda_n \in]0, \|\hat{T}\|(\sqrt{p}K)^p]$. We then define:

$$\hat{x}_{n+1} = \frac{\lambda_n^{-1}\hat{T}\hat{x}_n}{\|\lambda_n^{-1}\hat{T}\hat{x}_n\|} \in \mathrm{Span}(\hat{T}^{n+1}\hat{x}_0).$$

It is a representative of V_{n+1} in $\bigwedge^p E$. By Lemma 3.3.5 and the contraction principle in Theorem 3.2.3, we have:

$$\left\| \frac{\hat{x}_n}{\widehat{m_n}(\hat{x}_n)} - \frac{\hat{T}\hat{x}_n}{\widehat{m_n}(\hat{T}\hat{x}_n)} \right\| \leq a_p d_{\mathcal{C}}(V_n, V_{n+1}) \leq a_p \Delta \eta^{n-1}$$

As $|\widehat{m_n}(\hat{x}_n)| \leq K^p$ by Lemma 3.3.4, we get

$$\|\hat{x}_n - \lambda_n^{-1}\hat{T}\hat{x}_n\| \leq a_p \Delta K^p \eta^{n-1}.$$

Combined with the fact that $\|\hat{x}_n\| = 1$, we get $\|\hat{x}_n - \hat{x}_{n+1}\| \leq 2a_p\Delta K^p \eta^{n-1}$. Thus $(\hat{x}_n)$ is a Cauchy sequence and so is (V_n). By Proposition 2.2.17, there exists $V \in \mathcal{G}_p(E)$ and $h \in \bigwedge^p E$ representing V such that $\hat{x}_n \underset{n\to\infty}{\to} h$ and $V_n \underset{n\to\infty}{\to} V$. As $\mathcal{C}$ is closed, V is contained in $\mathcal{C}$.

We now show that V is a fixed space. We have:

$$|\lambda_n - \lambda_{n+1}| \leq \|(\hat{T} - \lambda_n)\hat{x}_n\| + \|(\lambda_{n+1} - \hat{T})\hat{x}_{n+1}\| + \|(\hat{T} - \lambda_n)(\hat{x}_{n+1} - \hat{x}_n)\|.$$

Given our inequalities and the fact that $|\lambda_n| \leq \|\hat{T}\|(\sqrt{p}K)^p$, we get

$$\|(\hat{T} - \lambda_n)\hat{x}_n\| = |\lambda_n|.\|\hat{x}_n - \lambda_n^{-1}\hat{T}\hat{x}_n\| \leq \|\hat{T}\|(\sqrt{p}K)^p a_p \Delta K^p \eta^{n-1} \ ;$$
$$\|(\lambda_{n+1} - \hat{T})\hat{x}_{n+1}\| = |\lambda_{n+1}|.\|\hat{x}_{n+1} - \lambda_{n+1}^{-1}\hat{T}\hat{x}_{n+1}\| \leq \|\hat{T}\|(\sqrt{p}K)^p a_p \Delta K^p \eta^n \ ;$$
$$\|(\hat{T} - \lambda_n)(\hat{x}_{n+1} - \hat{x}_n)\| \leq \|\hat{T} - \lambda_n\|.\|\hat{x}_{n+1} - \hat{x}_n\| \leq 2(1 + (\sqrt{p}K)^p)\|\hat{T}\| a_p \Delta K^p \eta^{n-1}.$$

Summing these three inequalities shows that

$$|\lambda_n - \lambda_{n+1}| \leq a_p((\sqrt{p}K)^p + \eta(\sqrt{p}K)^p + (2 + 2(\sqrt{p}K)^p))\Delta\|\hat{T}\|K^p\eta^{n-1}$$

so that $\lambda_n \underset{n\to\infty}{\to} \lambda \in \mathbb{C}$. As $\|\hat{x}_n - \lambda_n^{-1}\hat{T}\hat{x}_n\| \underset{n\to\infty}{\to} 0$, we get $\|\lambda h - \hat{T}h\| = 0$, hence $\hat{T}h = \lambda h$. As $T\mathcal{C}^* \subset \mathcal{C}^*$, this proves $\lambda \neq 0$ and V is fixed by T.

The space V is unique: if W satisfies $TW = W$, then $d_{\mathcal{C}}(V, W) = d_{\mathcal{C}}(TV, TW) \leq \eta d_{\mathcal{C}}(V, W)$, hence $d_{\mathcal{C}}(V, W) = 0$ and $W = V$.

(ii). We now prove the second part of our proposition. We follow the method and computations from [Rug10]. Let $\hat{y}_n = \hat{T}^n\hat{x}_0$. For this computation, we choose a functional $m_n : E \to \mathbb{C}^p$ of norm K adapted with V and V_n and set $c_n = \dfrac{\widehat{m_n}(\lambda^{-n}\hat{y}_n)}{\widehat{m_n}(h)}$. In particular, $|c_n| \leq (\sqrt{p})^pK^p\|\lambda^{-n}\hat{y}_n\|$ by Lemma 3.3.4. Using Lemma 3.3.5, we get:

$$\left\|\frac{\hat{y}_n}{\widehat{m_n}(\hat{y}_n)} - \frac{h}{\widehat{m_n}(h)}\right\| \leq a_p d_{\mathcal{C}}(V_n, V) \leq a_p\Delta\eta^{n-1}.$$

Multiplying by $|\widehat{m_n}(\lambda^{-n}\hat{y}_n)|$, we obtain

$$\left\|\lambda^{-n}\hat{y}_n - c_nh\right\| \leq |\widehat{m_n}(\lambda^{-n}\hat{y}_n)|a_p\Delta\eta^{n-1} \leq a_pK^p\|\lambda^{-n}\hat{y}_n\|\Delta\eta^{n-1} \qquad (*)$$

which then gives

$$\begin{aligned}
\|\lambda^{-n-1}\hat{y}_{n+1} - \lambda^{-n}\hat{y}_n\| &= \left\|\lambda^{-1}\lambda^{-n}\hat{y}_{n+1} - \lambda^{-1}\hat{T}(c_nh) - \lambda^{-n}\hat{y}_n + c_nh\right\| \\
&= \left\|\lambda^{-1}\hat{T}(\lambda^{-n}\hat{y}_n - c_nh) - (\lambda^{-n}\hat{y}_n - c_nh)\right\| \\
&\leq (1 + \lambda^{-1}\|\hat{T}\|)a_pK^p\|\lambda^{-n}\hat{y}_n\|\Delta\eta^{n-1}.
\end{aligned}$$

This shows $\|\lambda^{-n-1}\hat{y}_{n+1}\| \leq \left(1 + (1 + \lambda^{-1}\|\hat{T}\|)a_pK^p\Delta\eta^{n-1}\right)\|\lambda^{-n}\hat{y}_n\|$. Iterating this process, we get for $n \geq 1$:

$$\begin{aligned}
\|\lambda^{-n}\hat{y}_n\| &\leq \prod_{k=0}^{n-2}\left(1 + (1 + \lambda^{-1}\|\hat{T}\|)a_pK^p\Delta\eta^k\right)\|\lambda^{-1}\hat{y}_1\| \\
&\leq \prod_{k=0}^{\infty}\left(1 + (1 + \lambda^{-1}\|\hat{T}\|)a_pK^p\Delta\eta^k\right)\|\lambda^{-1}\hat{y}_1\| \\
&\leq \underbrace{\exp\left((1 + \lambda^{-1}\|\hat{T}\|)\frac{a_pK^p\Delta}{1-\eta}\right)\|\lambda^{-1}\hat{T}\|}_{=:R}\|\hat{x}_0\|.
\end{aligned}$$

We now show that the sequence (c_n) converges. We have, using $(*)$ for the second inequality:

$$\begin{aligned}
|c_{n+1} - c_n| &= \|(c_{n+1} - c_n)h\| \\
&= \|c_{n+1}h - \lambda^{-n-1}\hat{y}_{n+1} - c_nh + \lambda^{-1}\hat{T}(\lambda^{-n}\hat{y}_n)\| \\
&\leq \|c_{n+1}h - \lambda^{-n-1}\hat{y}_{n+1}\| + \|\lambda^{-1}\hat{T}(\lambda^{-n}\hat{y}_n - c_nh)\| \\
&\leq a_pK^p\|\lambda^{-n-1}\hat{y}_{n+1}\|\Delta\eta^n + \|\lambda^{-1}\hat{T}\|a_pK^p\|\lambda^{-n}\hat{y}_n\|\Delta\eta^{n-1} \\
&\leq (\eta + \|\lambda^{-1}\hat{T}\|)a_pK^p\Delta\eta^{n-1}R\|\hat{x}_0\|.
\end{aligned}$$

The sequence (c_n) is therefore a Cauchy sequence and converges to $c^* \in \mathbb{C}$. Summing the previous inequality from n to infinity, one gets

$$|c_n - c^*| \leq \frac{\eta + \|\lambda^{-1}\hat{T}\|}{1-\eta} a_p K^p \Delta \eta^{n-1} R \|\hat{x}_0\|.$$

Using $(*)$ again, this shows

$$\left\|\lambda^{-n}\hat{y}_n - c^* h\right\| \leq \underbrace{\frac{1 + \|\lambda^{-1}\hat{T}\|}{1-\eta} a_p K^p \Delta R}_{=:C\eta} \eta^{n-1}\|\hat{x}_0\| \leq C\eta^n \|\hat{x}_0\|.$$

Thus, the limit c^* depends only on $\hat{x}_0$ and not on the choice of the $\widehat{m_n}$: we can note this limit $c(\hat{x}_0)$. Moreover, $|c(\hat{x}_0)| = \|c(\hat{x}_0)h\| = \lim_{n\to\infty} \|\lambda^{-n}\hat{y}_n\| \leq R\|\hat{x}_0\|$.

□

Remark. We present here another method to show that V is a fixed space in the proof of (i). Let $x \in V$. There exist $x_n \in V_n$ such that $x_n \underset{n\to\infty}{\to} x$, and this sequence is bounded by some $M > 0$. Then $Tx_n \underset{n\to\infty}{\to} Tx$ and therefore

$$\begin{aligned} d(Tx, V) &\leq d(Tx, Tx_n) + d(Tx_n, V) \\ &\leq d(Tx, Tx_n) + \|Tx_n\| d_H(V_{n+1}, V) \\ &\leq \underbrace{d(Tx, Tx_n)}_{\underset{n\to\infty}{\to} 0} + \|T\| M \underbrace{d_H(V_{n+1}, V)}_{\underset{n\to\infty}{\to} 0}. \end{aligned}$$

Thus $d(Tx, V) = 0$ and $Tx \in V$, proving that V is fixed by T.

The spectral gap theorem requires another regularity assumption on our cone $\mathcal{C}$. For example, for $\mathcal{C}$ reduced to a p-dimensional space V, the identity map satisfies the hypotheses of Proposition 3.3.6 but does not have a spectral gap. We therefore introduce the notion of p-reproducing cone, which means the cone must give us information about all p-dimensional directions.

Definition 3.3.7. *We say that a p-cone $\mathcal{C} \subset E$ is p-reproducing if there exists $k \in \mathbb{N}^*$ and $A > 0$ such that for all $(x_1, \ldots, x_p) \in E^p$, there exists $((x_1^i, \ldots, x_p^i))_{1\leq i\leq k}$ such that*

$$\begin{gathered} \forall i \in [\![1, k]\!],\ \ \mathrm{Span}(x_1^i, \ldots, x_p^i) \subset \mathcal{C}\ ; \\ x_1 \wedge \cdots \wedge x_p = \sum_{i=1}^{k} x_1^i \wedge \cdots \wedge x_p^i\ ; \\ \sum_{i=1}^{k} \|x_1^i \wedge \cdots \wedge x_p^i\| \leq A \|x_1 \wedge \cdots \wedge x_p\|. \end{gathered}$$

Proposition 3.3.8. *Let $a > 0$ and F, G be topological complements: $E = F \oplus G$ and the projection π onto F along G is continuous. Then the cone $\mathcal{C}_{\pi,a}$ is reproducing.*

The proof of this proposition is technical and can be found in Section 5.4.1 of the Appendix. We also prove in Section 5.4.3 the following criterion implying reproductibility (see Proposition 5.4.11).

Proposition 3.3.9. *Let $\mathcal{C}$ be a p-cone such that $\mathcal{G}_p(\mathcal{C}) = \{W \in \mathcal{G}_p(E) \mid W \subset \mathcal{C}\}$ has non-empty interior in $\mathcal{G}_p(E)$. Then $\mathcal{C}$ is reproducing.*

Remark. We often abuse the notation and write "$\mathcal{C}$ has non-empty interior in $\mathcal{G}_p(E)$" for "$\mathcal{G}_p(\mathcal{C})$ has non-empty interior in $\mathcal{G}_p(E)$".

We now state our spectral gap theorem:

Theorem 3.3.10. *Let $\mathcal{C} \subset E$ be a reproducing p-cone of sectional aperture bounded by $K > 0$. Let $T \in \mathcal{L}(E)$ such that $T(\mathcal{C}^*) \subset \mathcal{C}^*$ and $diam_{\mathcal{C}}(T(\mathcal{C}^*)) = \Delta < \infty$. Let $\eta < 1$ be the contraction coefficient given by Theorem 3.2.3. Then T has a spectral gap, i.e. there exists a p-dimensional subspace V and a closed complement W of V, both invariant under T, such that*

$$\sup \left|Sp(T_{|W})\right| \leq \eta. \inf \left|Sp(T_{|V})\right|,$$

where $\left|Sp(T_{|V})\right|$ (resp. W) denotes the set of moduli of elements in the spectrum of $T_{|V}$ (resp. W).

Proof. By Proposition 3.3.6, there exists a unique p-dimensional subspace V contained in $\mathcal{C}$ fixed by T. We chose a representative $h \in \bigwedge^p E$ of V: h is an eigenvector for $\hat{T}$ for some eigenvalue $\lambda \in \mathbb{C}^*$, i.e. $\hat{T}h = \lambda h$.

Let $(x_1, \dots, x_p) \in E^p$ and $\hat{x} = x_1 \wedge \cdots \wedge x_p$. Then Definition 3.3.7 gives us a decomposition $\hat{x} = \sum_{i=1}^{k} \hat{x^i}$ where each $\hat{x^i} = x_1^i \wedge \cdots \wedge x_p^i$ represents a subspace of $\mathcal{C}$. By Proposition 3.3.6, we get a constant $C > 0$ and a map c such that

$$\|\lambda^{-n}\hat{T}(\hat{x^i}) - c(\hat{x^i}).h\| \leq C\eta^n \|\hat{x^i}\|,$$

hence

$$\|\lambda^{-n}\hat{T}(\hat{x}) - (c(\hat{x^1}) + \cdots + c(\hat{x^k})).h\| \leq C\eta^n \sum_{i=1}^{k} \|\hat{x^i}\| \leq C.A\eta^n \|\hat{x}\|$$

where A is the constant given by Definition 3.3.7. We define $c(\hat{x}) \coloneqq c(\hat{x^1}) + \cdots + c(\hat{x^k})$. Then $|c(\hat{x})| \leq R.A.\|\hat{x}\|$, and letting $n \to \infty$ in the previous inequality shows that the value $c(\hat{x})$ only depends on $\hat{x}$ and not on the decomposition. Given the linearity of $\hat{T}$, we can extend c to a *linear* map $c : \bigwedge^p E \to \mathbb{C}$ such that for all $\hat{x} = x_1 \wedge \cdots \wedge x_p$,

$$\|\lambda^{-n}\hat{T}(\hat{x}) - c(\hat{x}).h\| \leq CA\eta^n \|\hat{x}\|. \tag{$*$}$$

Let $V \subset \mathcal{C}$ be the p-dimensional space fixed by T given by Proposition 3.3.6. Let $(h_1, \dots, h_p)$ be a basis of V in which $T_{|V}$ is triangular, and $(\lambda_1, \dots, \lambda_p)$ the diagonal coefficients (which are the eigenvalues of T with multiplicity), ordered by decreasing modulus: $|\lambda_1| \geq \cdots \geq |\lambda_p|$. Since h is a representative of V, it is a complex multiple of $h_1 \wedge \cdots \wedge h_p$. Up to dividing the map c in $(*)$ by this complex value, we can assume $h = h_1 \wedge \cdots \wedge h_p$. As $\hat{T}h = Th_1 \wedge \cdots \wedge Th_p = \lambda_1 \dots \lambda_p.h_1 \wedge \cdots \wedge h_p = \lambda_1 \dots \lambda_p.h$ and $\hat{T}h = \lambda h$ by Proposition 3.3.6, we get $\lambda = \lambda_1 \dots \lambda_p$. Taking $\hat{x} = h$ and letting n go to infinity in $(*)$ shows that $c(h) = 1$.

Consider the linear maps $\pi_i : E \mapsto V$ defined by

$$\pi_i(x) = c(h_1 \wedge \cdots h_{i-1} \wedge x \wedge h_{i+1} \wedge h_p)\, h_i \;.$$

The map π_i is a projection onto $\mathrm{Span}(h_i)$, and furthermore $\pi_i \circ \pi_j = 0$ if $i \neq j$. Thus $\pi = \pi_1 + \cdots + \pi_p$ is a projection onto V parallel to

$$\ker(\pi) = \{x \in E \mid \forall i, \pi_i(x) = 0\} = \{x \in E \mid \forall i, c(h_1 \wedge \cdots h_{i-1} \wedge x \wedge h_{i+1} \wedge h_p) = 0\}.$$

From $(*)$, we get

$$\forall (x_1, \ldots, x_p) \in E^p, c(Tx_1 \wedge \cdots \wedge Tx_p) = \lambda_1 \ldots \lambda_p c(x_1 \wedge \cdots \wedge x_p).$$

This gives for all $x \in E$:

$$\begin{aligned} & c(h_1 \wedge \cdots h_{i-1} \wedge Tx \wedge h_{i+1} \wedge h_p) \\ = \ & \lambda_1^{-1} \ldots \lambda_{i-1}^{-1} \lambda_{i+1}^{-1} \ldots \lambda_p^{-1} c(Th_1 \wedge \cdots Th_{i-1} \wedge Tx \wedge Th_{i+1} \wedge Th_p) \\ = \ & \lambda_i c(h_1 \wedge \cdots h_{i-1} \wedge x \wedge h_{i+1} \wedge h_p). \end{aligned}$$

In particular, this shows that $\ker(\pi)$ is stable by T. Let $x \in E$. We have:

$$\begin{aligned} & \|h_1 \wedge \cdots \wedge h_{p-1} \wedge (T^n(\mathrm{Id} - \pi)x)\| \\ & = \|h_1 \wedge \cdots \wedge h_{p-1} \wedge T^n x - c(h_1 \wedge \cdots \wedge h_{p-1} \wedge x)\lambda_p^n h_1 \wedge \cdots \wedge h_p\| \\ & = \|\lambda_1^{-n} \ldots \lambda_{p-1}^{-n} T^n h_1 \wedge \cdots \wedge T^n h_{p-1} \wedge T^n x - \lambda_p^n c(h_1 \wedge \cdots \wedge h_{p-1} \wedge x)h\| \\ & = |\lambda_p|^n . \|\lambda^{-n} \widehat{T}^n (h_1 \wedge \cdots \wedge h_{p-1} \wedge x) - c(h_1 \wedge \cdots \wedge h_{p-1} \wedge x)h\| \\ & \leq CA|\lambda_p|^n \eta^n \|h_1 \wedge \cdots \wedge h_{p-1} \wedge x\|, \end{aligned} \qquad (**)$$

where the last inequality comes from $(*)$. To conclude our estimation we need the following lemma, proved in Section 5.3 of the Appendix:

Lemma 3.3.11. *Let V be a p-dimensional subset of E and W a closed complement of E. Let $h_1, \ldots, h_{p-1}$ be independent vectors of V. Then there exists $B > 0$ such that:*

$$\forall y \in W, \|h_1 \wedge \cdots \wedge h_{p-1} \wedge y\| \geq B\|y\|.$$

We apply this lemma to $(**)$ with $W = \ker(\pi)$ and $y = (T^n(\mathrm{Id} - \pi)x)$:

$$\|T^n(\mathrm{Id} - \pi)x\| \leq \frac{1}{B}\|h_1 \wedge \cdots \wedge h_{p-1} \wedge (T^n(\mathrm{Id} - \pi)x)\| \leq \frac{CA}{B}(\eta|\lambda_p|)^n \|h_1 \wedge \cdots \wedge h_{p-1} \wedge x\|.$$

This proves that $\sup \left|Sp(T_{|W})\right| \leq \eta|\lambda_p| = \eta. \inf \left|Sp(T_{|V})\right|$. $\square$

Proposition 3.3.12. *The application $c : \bigwedge^p E \to \mathbb{C}$ which satisfies*

$$\forall \hat{x} = x_1 \wedge \cdots \wedge x_p \in \bigwedge^p E, \forall n \geq 0, \|\lambda^{-n} \hat{T}(\hat{x}) - c(\hat{x}).h\| \leq CA\eta^n \|\hat{x}\|$$

can be written as a simple tensor $l_1 \wedge \cdots \wedge l_p$, where $l_i : x \mapsto c(h_1 \wedge \cdots \wedge h_{i-1} \wedge x \wedge h_{i+1} \wedge \cdots \wedge h_p)$.

Proof. As seen in the previous proof, we have for all $x \in E$, $l_i(Tx) = \lambda_i l_i(x)$. Hence for all $\hat{x} \in \bigwedge^p E$, $l_1 \wedge \cdots \wedge l_p(\hat{T}\hat{x}) = \lambda_1 \ldots \lambda_p . l_1 \wedge \cdots \wedge l_p(\hat{x}) = \lambda . l_1 \wedge \cdots \wedge l_p(\hat{x})$, and therefore $l_1 \wedge \cdots \wedge l_p(\lambda^{-n}\hat{T}^n\hat{x}) = l_1 \wedge \cdots \wedge l_p(\hat{x})$. As $\|\lambda^{-n}\hat{T}(\hat{x}) - c(\hat{x}).h\| \underset{n \to +\infty}{\longrightarrow} 0$, we get

$$l_1 \wedge \cdots \wedge l_p(\hat{x}) = l_1 \wedge \cdots \wedge l_p(\lambda^{-n}\hat{T}^n\hat{x}) \underset{n \to +\infty}{\longrightarrow} l_1 \wedge \cdots \wedge l_p(c(\hat{x}).h) = c(\hat{x}). \qquad \square$$

3.4 Cone contraction for random products of operators

In this section, we prove the analyticity of the sum of the p first characteristic exponents of a random products of operators contracting a cone. First we show that the Grassmannian of p-dimensional subspaces of a p-cone and the corresponding set in the exterior algebra share the same analytic structure. We then state our theorem and introduce some tools to begin our proof. We provide bounds for the norms of our operators given by their value on elements of the cone. Finally, we use these bounds to show the analyticity of the subspaces corresponding to the characteristic exponents, and achieve the proof.

In this section, $\|.\|_{\wedge,p}$ denotes the norm $\|.\|_{\wedge d,p}$ on $\bigwedge^p E$. We consider a p-cone $\mathcal{C}$ of aperture bounded by $K>0$ and with non-empty interior in $\mathcal{G}_p(E)$. We set a linear map $m : E \to \mathbb{C}^p$ such that $\|m\| = 1$ and for all $x \in \mathcal{C}$, $\|x\| \leq K\|m(x)\|$. Given a Banach space F, $x \in F$ and $r \geq 0$, we denote by $B(x,r)$ the ball of center x and radius r relative to the norm of F.

Definition 3.4.1. *Given a p-cone $\mathcal{C}$, recall*

$$\mathcal{G}_p(\mathcal{C}) := \{W \subset \mathcal{C} \mid W \text{ is a subspace of dimension } p\}.$$

We define the subset $\widehat{\mathcal{C}}_{m=1}$ of $\bigwedge^p E$ by

$$\widehat{\mathcal{C}}_{m=1} := \{x_1 \wedge \cdots \wedge x_p \mid \widehat{m}(x_1 \wedge \cdots \wedge x_p) = 1 \text{ and } \operatorname{Span}(x_1, \cdots, x_p) \subset \mathcal{C}\},$$

where $\widehat{m}(x_1 \wedge \cdots \wedge x_p) = \det(m(x_1), \ldots, m(x_p))$.

3.4.1 Identifying $\mathcal{G}_p(\mathcal{C})$ and $\widehat{\mathcal{C}}_{m=1}$

In the next propositions, we prove that we can identify $\mathcal{G}_p(\mathcal{C})$ and $\widehat{\mathcal{C}}_{m=1}$. More precisely, $\mathcal{G}_p(\mathcal{C})$ considered as a submanifold of $\mathcal{G}_p(E)$ and $\widehat{\mathcal{C}}_{m=1}$ considered as a submanifold of $\bigwedge^p E$ share the same analytic structure. Then the p-dimensional subspaces in $\mathcal{C}$ can be considered as elements of $\widehat{\mathcal{C}}_{m=1}$ for analytic purposes, and as elements of $\mathcal{G}_p(\mathcal{C})$ for geometric purposes. Another way to consider the cone would be to see it as a subset of the projective space $\mathbb{P}(E)$, as in [BG09]. However, the set $\widehat{\mathcal{C}}_{m=1}$ allows us to do explicit and direct estimates which would be difficult to do without the norm on $\bigwedge^p E$.

Proposition 3.4.2. *The map*

$$G : \begin{cases} \widehat{\mathcal{C}}_{m=1} & \longrightarrow & \mathcal{G}_p(\mathcal{C}) \\ x_1 \wedge \cdots \wedge x_p & \mapsto & \operatorname{Span}(x_1, \ldots, x_p) \end{cases}$$

is a homeomorphism.

Proof. If $\operatorname{Span}(x_1, \ldots, x_p) = \operatorname{Span}(y_1, \ldots, y_p)$, then $x_1 \wedge \cdots \wedge x_p = \lambda y_1 \wedge \cdots \wedge y_p$ for some $\lambda \in \mathbb{C}$. Then, as $\widehat{m}(x_1 \wedge \cdots \wedge x_p) = \widehat{m}(y_1 \wedge \cdots \wedge y_p) = 1$, we get $\lambda = 1$ and $x_1 \wedge \cdots \wedge x_p = y_1 \wedge \cdots \wedge y_p$. Hence G is injective.

Let $W \in \mathcal{G}_p(\mathcal{C})$ and $x_1, \ldots, x_p$ such that $W = \operatorname{Span}(x_1, \ldots, x_p)$. Then $x_1 \wedge \cdots \wedge x_p \neq 0$, and, since $\operatorname{Span}(x_1, \ldots, x_p) \subset \mathcal{C}$, we get $\widehat{m}(x_1 \wedge \cdots \wedge x_p) \neq 0$ (as in Lemma 3.3.5). Hence $W = G(\frac{x_1}{\widehat{m}(x_1 \wedge x_2 \wedge \cdots \wedge x_p)} \wedge \cdots \wedge x_p)$ and G is surjective.

Let $x_1 \wedge \cdots \wedge x_p$ and $y_1 \wedge \cdots \wedge y_p$ be elements of $\widehat{\mathcal{C}}_{m=1}$. Then $\frac{1}{\|\widehat{m}\|} \leq \|x_1 \wedge \cdots \wedge x_p\|_\wedge$. Denoting by C the constant given by Theorem 2.2.15 and using Proposition 2.2.14, we get:

$$\begin{aligned}
d_H(G(x_1 \wedge \cdots \wedge x_p), G(y_1 \wedge \cdots \wedge y_p)) &\leq C d_{\wedge,p}(G(x_1 \wedge \cdots \wedge x_p), G(y_1 \wedge \cdots \wedge y_p)) \\
&\leq 2C. \inf_{\lambda \in \mathbb{C}} \left\| \frac{x_1 \wedge \cdots \wedge x_p}{\|x_1 \wedge \cdots \wedge x_p\|} - \lambda \frac{y_1 \wedge \cdots \wedge y_p}{\|y_1 \wedge \cdots \wedge y_p\|} \right\| \\
&\leq 2C\|\widehat{m}\|. \inf_{\lambda \in \mathbb{C}} \|x_1 \wedge \cdots \wedge x_p - \lambda y_1 \wedge \cdots \wedge y_p\| \\
&\leq 2C\|\widehat{m}\|. \|x_1 \wedge \cdots \wedge x_p - y_1 \wedge \cdots \wedge y_p\|,
\end{aligned}$$

hence G is Lipschitz.

We denote $x = x_1 \wedge \cdots \wedge x_p$, $y = y_1 \wedge \cdots \wedge y_p$, $x' = \frac{x_1 \wedge \cdots \wedge x_p}{\|x_1 \wedge \cdots \wedge x_p\|}$ and $y' = \frac{y_1 \wedge \cdots \wedge y_p}{\|y_1 \wedge \cdots \wedge y_p\|}$. Note that $x = \frac{x'}{\widehat{m}(x')}$ and $y = \frac{y'}{\widehat{m}(y')}$. By Theorem 2.2.15 and Proposition 2.2.14, there exist a constant $C > 0$ and $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ such that $\|x' - \lambda y'\| \leq C d_H(G(x), G(y))$. Hence, using Lemma 3.3.4,

$$\begin{aligned}
\|x - y\| &\leq \left\| \frac{x' - \lambda y'}{\widehat{m}(x')} \right\| + \left\| \frac{\lambda y'}{\widehat{m}(x')} - \frac{y'}{\widehat{m}(y')} \right\| \\
&\leq \frac{K^p}{\|m\|^p} \|x' - \lambda y'\| + \left| \frac{\lambda \widehat{m}(y') - \widehat{m}(x')}{\widehat{m}(x')\widehat{m}(y')} \right| . \|y'\| \\
&\leq \frac{K^p}{\|m\|^p} \|x' - \lambda y'\| + \left| \frac{\widehat{m}(\lambda y' - x')}{\widehat{m}(x')\widehat{m}(y')} \right| \\
&\leq \frac{K^p}{\|m\|^p} \|x' - \lambda y'\| + \frac{K^{2p}}{\|m\|^p} \|x' - \lambda y'\| \\
&\leq \frac{K^p(K^p + 1)}{\|m\|^p} C.d_H(G(x), G(y))
\end{aligned}$$

and G^{-1} is Lipschitz. Therefore G is a homeomorphism. □

We now show that these two sets also share the same analytic structure.

Proposition 3.4.3. *The map $G^{-1} : \mathcal{G}_p(\mathcal{C}) \to \bigwedge^p E$ is an embedding, where $\mathcal{G}_p(\mathcal{C})$ is given its analytic structure from $\mathcal{G}_p(E)$.*

Proof. Let $W_0 \in \mathcal{G}_p(\mathcal{C})$, and $(x_1, \cdots, x_p)$ a right decomposition of W_0. Given F_0 a closed complement of W_0 in E, the map

$$L \in \mathcal{L}(W_0, F_0) \mapsto \operatorname{graph} L$$

is a local chart around W_0. In particular, $W_0 = \operatorname{graph}\ L_0$ where $L_0 \equiv 0$. Let $H \in \mathcal{L}(W_0, F_0)$ and $W = \operatorname{graph}(L_0 + H)$. Using the notations $x = x_1 \wedge \cdots \wedge x_p$ and $h_i = x_1 \wedge \cdots \wedge Hx_i \wedge \cdots \wedge x_p$, we get:

$$\begin{aligned}
G^{-1}(W)-G^{-1}(W_0) &= \frac{(x_1+Hx_1)\wedge\cdots\wedge(x_p+Hx_p)}{\widehat{m}((x_1+Hx_1)\wedge\cdots\wedge(x_p+Hx_p))}-\frac{x}{\widehat{m}(x)}\\
&= \frac{x+\sum_i h_i+o(H)}{\widehat{m}(x)+\sum_i \widehat{m}(h_i)+o(H)}-\frac{x}{\widehat{m}(x)}\\
&= \left(1-\sum_i \frac{\widehat{m}(h_i)}{\widehat{m}(x)}+o(H)\right)\frac{x}{\widehat{m}(x)}+\sum_i \frac{h_i}{\widehat{m}(x)}+o(H)-\frac{x}{\widehat{m}(x)}\\
&= -\left(\sum_i \frac{\widehat{m}(h_i)}{\widehat{m}(x)^2}\right)x+\sum_i \frac{1}{\widehat{m}(x)}h_i+o(H)
\end{aligned}$$

Hence G^{-1} is differentiable with analytic derivative $H \mapsto -\left(\sum_i \frac{\widehat{m}(h_i)}{\widehat{m}(x)^2}\right)x+\sum_i \frac{1}{\widehat{m}(x)}h_i$.

We now show that G^{-1} is an immersion. Let $H \in \mathcal{L}(W_0, F_0)$ in the kernel of the derivative, i.e. satisfying $-\left(\sum_i \frac{\widehat{m}(h_i)}{\widehat{m}(x)^2}\right)x+\sum_i \frac{1}{\widehat{m}(x)}h_i = 0$. Denoting I the set of indices i such that $h_i \neq 0$, we verify easily that $\{h_i\}_{i\in I} \cup \{x\}$ is a linearly independent set of vectors of $\bigwedge^p E$, thus $I = \emptyset$. Therefore for all i, $Hx_i \in W_0$ and $Hx_i \in F_0$, hence $Hx_i = 0$ and $H = 0$. The derivative of G^{-1} is injective and G^{-1} is an homeomorphism onto its image by Proposition 3.4.2, hence G^{-1} is an embedding.

□

3.4.2 Statement of the theorem

Recall that the p-cone $\mathcal{C}$ has aperture bounded by $K > 0$ and non-empty interior in $\mathcal{G}_p(E)$. We see later that such a cone has a transverse subspace of codimension p, in a similar fashion to the cones in $\mathbb{R}^d$ in [BG09]. For $\rho > 0$ small enough, the set

$$\mathcal{C}[\rho] := \{x \in \mathcal{C} \mid B(x, \rho\|x\|) \subset \mathcal{C}\}$$

contains a p-dimensional subspace. We fix such a value of $\rho > 0$. This notion is studied in more detail in Section 5.4 of the Appendix. We also fix $\Delta < +\infty$ and write η for the contraction constant given by Theorem 3.2.3.

Definition 3.4.4. *We define the set*

$$\mathscr{M} := \{M \in \mathcal{L}(E) \mid M(\mathcal{C}^*) \subset \mathcal{C}^*, \mathrm{diam}_{\mathcal{C}}(M(\mathcal{C}^*)) \leq \Delta, M(\mathcal{C}) \subset \mathcal{C}[\rho]\}.$$

We consider (Ω, μ) a probability space and $\tau : \Omega \to \Omega$ a measurable μ-ergodic application. Given a Banach space F, we denote by $\mathscr{B}(\Omega, F)$ the Banach space of (μ-essentially) bounded mesurable maps equipped with the (μ-essentially) uniform norm on F. If A is a subset of F, we denote by $\mathscr{E}(\Omega, A)$ the set of Bochner-measurable maps from Ω into A.

For $(M_\omega)_{\omega\in\Omega} \in \mathscr{E}(\Omega, \mathscr{M})$, we denote by $M_\omega^{(n)} := M_\omega \cdots M_{\tau^{n-1}\omega}$ the product of operators along the orbit of $\omega \in \Omega$. The goal of this section is to prove the following theorem:

Theorem 3.4.5. *Let $t \in \mathbb{D} \mapsto M(t) \in \mathscr{E}(\Omega, \mathscr{M})$ be a map on the Poincaré disk such that*

1. *$(t,\omega) \in \mathbb{D} \times \Omega \mapsto M_\omega(t) \in \mathcal{L}(E)$ is measurable and the map $t \mapsto M_\omega(t)$ is analytic for all $\omega \in \Omega$;*
2. $\sup \left\{ \dfrac{\| \bigwedge^{p-1} M_\omega(t)\|}{\| \bigwedge^{p} M_\omega(t)\|} \| \frac{d}{dt} M_\omega(t)\| : \omega \in \Omega, t \in \mathbb{D} \right\} < \infty$ *;*
3. $\int_\Omega \left| \log \|M_\omega(0)\| \right| < \infty$.

Then for all $t \in \mathbb{D}$, the limit

$$\chi_p(t) = \lim_{n\to\infty} \frac{1}{n} \log \left\| \bigwedge^p M_\omega^{(n)} \right\|$$

exists for μ-a.e. $\omega \in \Omega$ and is (μ-a.e.) independent of ω. Moreover the map $t \mapsto \chi_p(t)$ is real-analytic and harmonic.

Remark. We can compare this theorem to Theorem B from [BG09]. In [BG09], a contraction of a multicone gives rise to a dominated splitting for the cocycle; in our case, a p-cone contraction gives rise to analyticity for the growth rate of dimension p. Moreover, in Theorem 3.4.5, we do not require E to be finite-dimensional nor the operators to be invertible, and the compacity of the base space Ω and the invertibility of the dynamic are not needed.

Remark. This theorem applies in particular if we consider π a continuous projection onto a p-dimensional space and operators mapping $\mathcal{C}_{\pi,a}$ into $\mathcal{C}_{\pi,b}$ with $0 < b < a$. See Section 5.4 of the Appendix for more details.

We now begin the proof of Theorem 3.4.5. In the remainder of this chapter, for any linear operator M on E, $\widehat{M}$ denotes the induced operator on $\bigwedge^p E$.

First, we describe how the set $\mathscr{M}$ acts on the p-cone $\mathcal{C}$.

Definition 3.4.6. *For $M \in \mathcal{L}$ such that $M(\mathcal{C}^*) \subset \mathcal{C}^*$, we define*

$$\pi_M : \left\{ \begin{array}{ccc} \widehat{\mathcal{C}}_{m=1} & \longrightarrow & \widehat{\mathcal{C}}_{m=1} \\ x & \mapsto & \frac{\widehat{M}x}{\widehat{m}(\widehat{M}x)} \end{array} \right.$$

which is analytic (as a restriction of an analytic map). By the identification $\widehat{\mathcal{C}}_{m=1} \simeq \mathcal{G}_p(\mathcal{C})$, π_M can be seen as

$$\pi_M : \left\{ \begin{array}{ccc} \mathcal{G}_p(\mathcal{C}) & \longrightarrow & \mathcal{G}_p(\mathcal{C}) \\ W & \mapsto & M(W) \end{array} \right.$$

Lemma 3.4.7. *Let $(M_n)_{n\in\mathbb{N}} \in \mathscr{M}^{\mathbb{N}}$. Let $M^{(n)} = M_n \circ \cdots \circ M_1$ and $\pi^{(n)} := \pi_{M^{(n)}} = \pi_{M_n} \circ \cdots \circ \pi_{M_1}$. Let $v, w \in \widehat{\mathcal{C}}_{m=1}$. Then*

$$\|\pi^{(n)}(v) - \pi^{(n)}(w)\|_{\wedge,p} \leq C\eta^n$$

where $C > 0$ only depends on p, K and on the definition of $\mathscr{M}$.

Proof. Let $V, W \in \mathcal{G}_p(\mathcal{C})$ be the p-dimensional subspaces represented by v and w. Using Lemma 3.3.5 and $\|m\| = 1$, we get a constant $a_p > 0$ such that

$$\|\pi^{(n)}(v) - \pi^{(n)}(w)\|_{\wedge,p} \leq a_p K^p d_{\mathcal{C}}(\pi^{(n)}(V), \pi^{(n)}(W)) \leq a_p K^p \eta^n \Delta. \qquad \square$$

3.4.3 Upper and lower bounds for the elements of the cone

Our next step is to understand and control $\|\widehat{M}\|_{\wedge,p}$ for $M \in \mathscr{M}$. To do that, we use estimates on $\widehat{m}(\widehat{M}(v))$, where v is representing a subspace $V \subset \mathcal{C}[\rho]$. We first state two lemmas which give ways to estimate $\|\widehat{M}\|_{\wedge,p}$ with simple tensors and $\|M\|$. Their proofs can be found in Section 5.3 of the Appendix.

Lemma 3.4.8. *Let $M \in \mathcal{L}(E)$ and consider the operator norm of $\widehat{M}$ on $\bigwedge^p E$ given by $\|\widehat{M}\|_{\wedge,p} := \sup\limits_{\|x\|_{\wedge,p}=1} \|\widehat{M}(x)\|_{\wedge,p}$. Then this supremum is given by its value on the set of simple tensors, i.e.*

$$\|\widehat{M}\|_{\wedge,p} = \sup_{\|x_1\wedge\cdots\wedge x_p\|_{\wedge,p}=1} \|\widehat{M}(x_1 \wedge \cdots \wedge x_p)\|_{\wedge,p}.$$

Lemma 3.4.9. *Let $M \in \mathcal{L}(E)$. We have*

$$\|\widehat{M}\|_{\wedge,p} \leq \left\|\bigwedge^{p-1} M\right\|_{\wedge,p-1} \|M\| \leq \|M\|^p.$$

The next two lemmas allow us to use the fact that some space W is in $\mathcal{C}[\rho]$ to get estimates on what happens when we move around it. This is then used to prove the estimates on $\|\widehat{M}\|_{\wedge,p}$ in Lemma 3.4.12.

Lemma 3.4.10 asserts that under some conditions, if we can move around a p-space $W \in \mathcal{C}$ following directions $y_1, \dots, y_p$ for some time $r > 0$ and stay in $\mathcal{C}$, then $\|y_1 \wedge \cdots \wedge y_p\|$ cannot be too large and is controlled by r.

Lemma 3.4.10. *Let $r > 0$, $x_1, \dots, x_p, y_1, \dots, y_p \in E$ such that*

- *for all $i \in [\![1,p]\!]$, $m(y_i) \in \mathrm{Span}(m(x_1), \dots, m(x_i))$;*
- *for all $(t_1, \dots, t_p) \in \mathbb{C}^p$ with $|t_i| \leq r$, $\mathrm{Span}(x_1 + t_1 y_1, \dots, x_p + t_p y_p) \subset \mathcal{C}$.*

Then

$$\|y_1 \wedge \cdots \wedge y_p\| \leq \frac{K^p}{r^p} \|x_1 \wedge \cdots \wedge x_p\|.$$

Proof. We first observe that given $a_1, \dots, a_p, b_1, \dots, b_p \in E$, we have

$$\sum_{\varepsilon_1,\dots,\varepsilon_p \in \{-1,1\}} (a_1 + \varepsilon_1 b_1) \wedge \cdots \wedge (a_p + \varepsilon_p b_p) = 2^p a_1 \wedge \cdots \wedge a_p.$$

Then

$$\begin{aligned}
2^p r^p \|y_1 \wedge \cdots \wedge y_p\| &= 2^p \|r y_1 \wedge \cdots \wedge r y_p\| \\
&\leq \sum_{\varepsilon_1,\dots,\varepsilon_p \in \{-1,1\}} \|(r y_1 + \varepsilon_1 x_1) \wedge \cdots \wedge (r y_p + \varepsilon_p x_p)\| \\
&\leq \sum_{\varepsilon_1,\dots,\varepsilon_p \in \{-1,1\}} \|(x_1 + \varepsilon_1 r y_1) \wedge \cdots \wedge (x_p + \varepsilon_p r y_p)\| \\
&\leq K^p \sum_{\varepsilon_1,\dots,\varepsilon_p \in \{-1,1\}} |\widehat{m}((x_1 + \varepsilon_1 r y_1) \wedge \cdots \wedge (x_p + \varepsilon_p r y_p))|
\end{aligned}$$

where the last inequality comes from Lemma 3.3.4 (recall that $\|m\| = 1$).

For $i \in [\![1,p]\!]$, set $f_i = m(x_i) \in \mathbb{C}^p$. Possibly after multiplicating the x_i by complex phases, we may assume that $\widehat{m}(x_1 \wedge \cdots \wedge x_p) = \det(f_1, \ldots, f_p) > 0$. Possibly after multiplicating the y_i by complex phases, we may further assume that the matrix of $(m(y_1), \ldots, m(y_n))$ in this basis $(f_1, \ldots, f_n)$ is of the form

$$\begin{pmatrix} a_{11} & * & * \\ 0 & \ddots & * \\ 0 & 0 & a_{pp} \end{pmatrix}$$

with $a_{ii} > 0$. Then for $(t_1, \ldots, t_p) \in [-r, r]^p$, the complex value

$$\widehat{m}((x_1 + t_1 y_1) \wedge \cdots \wedge (x_p + t_p y_p)) = \widehat{m}(x_1 \wedge \cdots \wedge x_p) \prod_{1 \le i \le p} (1 + t_i a_{ii})$$

is in fact a real number. As this quantity is never equal to 0, it is even in $\mathbb{R}_+$. We finally get

$$\begin{aligned} 2^p r^p \|y_1 \wedge \cdots \wedge y_p\| &\le K^p \sum_{\varepsilon_1, \cdots, \varepsilon_p \in \{-1,1\}} \left| \widehat{m}((x_1 + \varepsilon_1 r y_1) \wedge \cdots \wedge (x_p + \varepsilon_p r y_p)) \right| \\ &= K^p \sum_{\varepsilon_1, \cdots, \varepsilon_p \in \{-1,1\}} \widehat{m}((x_1 + \varepsilon_1 r y_1) \wedge \cdots \wedge (x_p + \varepsilon_p r y_p)) \\ &= K^p \widehat{m} \left(\sum_{\varepsilon_1, \cdots, \varepsilon_p \in \{-1,1\}} (x_1 + \varepsilon_1 r y_1) \wedge \cdots \wedge (x_p + \varepsilon_p r y_p) \right) \\ &= K^p \widehat{m} \left(2^p x_1 \wedge \cdots \wedge x_p \right) \\ &\le 2^p K^p \|x_1 \wedge \cdots \wedge x_p\|. \end{aligned}$$

where the last inequality is given by Lemma 3.3.4. We thus have

$$\|y_1 \wedge \cdots \wedge y_p\| \le \frac{K^p}{r^p} \|x_1 \wedge \cdots \wedge x_p\|. \qquad \square$$

In the next lemma, given a p-space $W \in \mathcal{C}[\rho]$, we find $x = x_1 \wedge \cdots \wedge x_p$ representing W and $y = y_1 \wedge \cdots \wedge y_p$ verifying properties similar to the ones from the previous lemma.

Lemma 3.4.11. *Let $W \in \mathcal{C}[\rho]$ and $M \in \mathscr{M}$. Let $x \in \bigwedge^p E$ representing W and $y \in \bigwedge^p E$ be a simple tensor. Then there exists decompositions $x = x_1 \wedge \cdots \wedge x_p$ and $y = y_1 \wedge \cdots \wedge y_p$ verifying the following properties (compare with Lemma 3.4.10) with $r := \dfrac{\rho}{p\sqrt{p}} \left(\dfrac{\|x\|}{\|y\|} \right)^{\frac{1}{p}}$:*

- *$m(My_i) \in \mathrm{Span}(m(Mx_1), \cdots, m(Mx_i))$ for all $i \in [\![1,p]\!]$;*
- *for all $(t_1, \cdots, t_p) \in \mathbb{C}^p$ with $|t_i| \le r$, $\mathrm{Span}(x_1 + t_1 y_1, \cdots, x_p + t_p y_p) \subset \mathcal{C}$.*

Proof. Write $x = x_1 \wedge \cdots \wedge x_p$, where $\|x_1\| = \cdots = \|x_p\|$ and $(\frac{x_1}{\|x_1\|}, \ldots, \frac{x_p}{\|x_p\|})$ is an Auerbach basis of W. We denote $(f_1, \cdots, f_p) := (m(Mx_1), \cdots, m(Mx_p))$; it is a basis of $\mathbb{C}^p$ as $W \subset \mathcal{C}$. If $y = 0$ then the decomposition $y = 0 \wedge \cdots \wedge 0$ works. Otherwise, consider V the p-dimensional space represented by y, and let us build recursively a p-tuple $(y_1, \cdots, y_p)$ such that

- $y = y_1 \wedge \cdots \wedge y_p$;
- $(y_1/\|y_1\|, \cdots, y_p/\|y_p\|)$ is a right decomposition of V ;
- $m(My_i) \in \mathrm{Span}(f_1, \cdots, f_i)$ for all $i \in [\![1, p]\!]$.

If $y_1, \cdots, y_i$ are built with linear forms $(l_1, \cdots, l_i)$ on V, we consider the restriction m_i of $m \circ M$ to $V \cap \ker l_1 \cap \cdots \cap \ker l_i$:

$$m_i : V \cap \ker l_1 \cap \cdots \cap \ker l_i \to \mathrm{Span}(f_1, \cdots, f_p).$$

If m_i is not injective, then we take $y_{i+1} \in \ker m_i \setminus \{0\}$ and l_{i+1} a linear form associated with y_{i+1}. If m_i is injective, then $\dim m_i(V \cap \ker l_1 \cap \cdots \cap \ker l_i) \geq p - i$, therefore we have $m_i(V \cap \ker l_1 \cap \cdots \cap \ker l_i) \cap \mathrm{Span}(f_1, \cdots, f_{i+1}) \neq \{0\}$. Thus we can take $y_{i+1} \neq 0$ in $V \cap \ker l_1 \cap \cdots \cap \ker l_i$ such that $m \circ M(y_{i+1}) \in \mathrm{Span}(f_1, \cdots, f_{i+1})$, and l_{i+1} a linear form associated with y_{i+1}.

We can now write $y = y_1 \wedge \cdots \wedge y_p$ with $(y_1/\|y_1\|, \cdots, y_p/\|y_p\|)$ a right decomposition of V such that $m(My_i) \in \mathrm{Span}(f_1, \cdots, f_i)$ for all $i \in [\![1, p]\!]$. Without loss of generality, we may assume $\|y_1\| = \cdots = \|y_p\|$. Let $(t_1, \cdots, t_p) \in \mathbb{C}^p$ with $|t_i| \leq r$ for all i, and $u = \sum_i \lambda_i(x_i + t_i y_i)$ an element of $\mathrm{Span}(x_1 + t_1 y_1, \cdots, x_p + t_p y_p)$. We want to show that $u \in \mathcal{C}$, by using the fact that $\mathrm{Span}(x_1, \cdots, x_p) \subset \mathcal{C}[\rho]$. Since $(\frac{x_1}{\|x_1\|}, \dots, \frac{x_p}{\|x_p\|})$ is an Auerbach basis, by Lemma 2.2.10 we have $|\lambda_i|.\|x_i\| \leq \|\sum_j \lambda_j x_j\|$, and then

$$\left\|\sum_i \lambda_i t_i y_i\right\| \leq \sum_i |\lambda_i|.|t_i|.\|y_i\| \leq \sum_i \left(|t_i|.\frac{\|y_i\|}{\|x_i\|}.\left\|\sum_j \lambda_j x_j\right\|\right). \qquad (*)$$

We have $\|y_1\| = \cdots = \|y_p\|$ and $(y_1/\|y_1\|, \cdots, y_p/\|y_p\|)$ is a right decomposition of V. By Proposition 2.2.7 and Proposition 2.2.2, we get

$$\|y_i\|^p = \|y_1\| \cdots \|y_p\| \leq (\sqrt{p})^p \|y\|.$$

Since $r = \dfrac{\rho}{p\sqrt{p}}\left(\dfrac{\|x\|}{\|y\|}\right)^{\frac{1}{p}}$ (by definition) and $\|x_1\| = \cdots = \|x_p\|$, this becomes

$$\|y_i\|^p \leq \left(\frac{\rho}{rp}\right)^p \|x\| \leq \left(\frac{\rho}{rp}\right)^p \|x_1\| \cdots \|x_p\| \leq \left(\frac{\rho}{rp}\|x_i\|\right)^p.$$

Thus we have $\|y_i\| \leq \frac{\rho}{rp}\|x_i\|$. Combining the latter inequality with $(*)$ gives:

$$\|\sum_i \lambda_i t_i y_i\| \;\; \leq \sum_i \frac{|t_i|}{r}\frac{\rho}{p}\|\sum_j \lambda_j x_j\| \leq \rho\|\sum_j \lambda_j x_j\|.$$

Using $\sum_j \lambda_j x_j \in \mathcal{C}[\rho]$, we therefore get $u = \sum_i \lambda_i(x_i + t_i y_i) \in \mathcal{C}$. This proves $\mathrm{Span}(x_1 + t_1 y_1, \cdots, x_p + t_p y_p) \subset \mathcal{C}$, which achieves the proof. □

We can now use the two previous lemmas to get bounds on $\|\widehat{M}\|$ given by elements of $\mathcal{C}[\rho]$.

Lemma 3.4.12. *Let $M \in \mathscr{M}$, $W \subset \mathcal{C}[\rho]$ a p-dimensional space and $x \in \bigwedge^p E$ representing W. Then*

$$\frac{1}{K^p}\left|\frac{\widehat{m}(\widehat{M}x)}{\widehat{m}(x)}\right| \leq \|\widehat{M}\| \leq \left(\frac{p\sqrt{p}K^2}{\rho}\right)^p \left|\frac{\widehat{m}(\widehat{M}x)}{\widehat{m}(x)}\right|$$

Proof. Let $y \in \bigwedge^p E$ be a simple tensor. By Lemma 3.4.11, we get decompositions $x = x_1 \wedge \cdots \wedge x_p$ and $y = y_1 \wedge \cdots \wedge y_p$ such that the decompositions $\widehat{M}x = Mx_1 \wedge \cdots \wedge Mx_p$ and $\widehat{M}y = My_1 \wedge \cdots \wedge My_p$ satisfy the hypothesis of Lemma 3.4.10 for $r = \frac{\rho}{p\sqrt{p}}\left(\frac{\|x\|}{\|y\|}\right)^{\frac{1}{p}}$. This gives

$$\|\widehat{M}y\| \leq \left(\frac{p\sqrt{p}K}{\rho}\right)^p \frac{\|y\|}{\|x\|}.\|\widehat{M}x\|\,.$$

The properties of $\widehat{m}$ given in Lemma 3.3.4 then show:

$$\|\widehat{M}y\| \leq \left(\frac{p\sqrt{p}K}{\rho}\right)^p K^p \left|\frac{\widehat{m}(\widehat{M}x)}{\widehat{m}(x)}\right|.\|y\|\,.$$

By Lemma 3.4.8, $\|\widehat{M}\|$ is given by its supremum over the simple tensors, so we have

$$\|\widehat{M}\| \leq \left(\frac{p\sqrt{p}K^2}{\rho}\right)^p \left|\frac{\widehat{m}(\widehat{M}x)}{\widehat{m}(x)}\right|.$$

Lemma 3.3.4 also gives

$$|\widehat{m}(\widehat{M}x)| \leq \|\widehat{M}\|.\|\widehat{m}\|.\|x\| \leq \|\widehat{M}\|.K^p|\widehat{m}(x)|.$$ □

3.4.4 Fixed subspace, analyticity and end of the proof

Before continuing the proof of Theorem 3.4.5, we need to define $\mathscr{E}(\Omega, \mathcal{G}_p(\mathcal{C}))$ (the set of mesurable maps from Ω to $\mathcal{G}_p(\mathcal{C})$) and what it means to be analytic on this space. Since neither $\mathcal{G}_p(E)$ or $\bigwedge^p E$ are Banach spaces, the analyticity has to come from the manifold structure, i.e. from the charts on $\mathcal{G}_p(E)$. Fortunately in our case, when studying $\mathcal{G}_p(\mathcal{C})$, we can restrict the space we look at to a single chart.

We have something even better: there exists a chart of $\mathcal{G}_p(E)$ which covers $\mathcal{G}_p(\mathcal{C})$ and in which the image of $\mathcal{G}_p(\mathcal{C})$ is bounded. Indeed, consider $V_0 = \ker m$ and $W_0 \in \mathcal{G}_p(\mathcal{C})$. Then $W_0 \cap V_0 = \{0\}$ and given the dimension and codimension of these objects, we have $E = W_0 \oplus V_0$ (and they are topological complements since they are closed). For $W \in \mathcal{G}_p(\mathcal{C})$, W is a complement of V_0, so the chart $A \in \mathcal{L}(W_0, V_0) \mapsto \operatorname{graph} A \in \mathcal{G}_p(E)$ covers $\mathcal{G}_p(\mathcal{C})$. Moreover, if $\operatorname{graph} A \in \mathcal{G}_p(\mathcal{C})$, then for $x \in W_0$ we have $x + Ax \in \mathcal{C}$ and

$$\|Ax\| \leq \|x + Ax\| + \|x\| \leq K|m(x + Ax)| + \|x\| = K|m(x)| + \|x\| \leq (K+1)\|x\|.$$

Therefore the image of $\mathcal{G}_p(\mathcal{C})$ in the chart $\mathcal{L}(W_0, V_0)$ is bounded by $(K+1)$. Now let $\mathscr{E}(\Omega, \mathcal{G}_p(\mathcal{C}))$ be the set of measurable maps from Ω to $\mathcal{G}_p(\mathcal{C})$. Then by identifying $\mathcal{G}_p(\mathcal{C})$ and $\mathcal{L}(W_0, V_0)$, we can consider $\mathscr{E}(\Omega, \mathcal{G}_p(\mathcal{C}))$ as a subset of $\mathscr{B}(\Omega, \mathcal{L}(W_0, V_0))$, thus giving it an analytic structure. We define the analytic structure of $\mathscr{E}(\Omega, \widehat{\mathcal{C}}_{m=1})$ by the identification $\widehat{\mathcal{C}}_{m=1} \simeq \mathcal{G}_p(\mathcal{C})$.

We can now define our objects properly and formulate a fixed point theorem.

Definition 3.4.13. *For $t \in \mathbb{D}$, we define:*

$$\pi_t : \begin{cases} \mathscr{E}(\Omega, \mathcal{G}_p(\mathcal{C})) & \to & \mathscr{E}(\Omega, \mathcal{G}_p(\mathcal{C})) \\ (V_\omega)_{\omega\in\Omega} & \mapsto & \pi_t(V) : \omega \mapsto \pi_{M_\omega(t)}(V_{\tau\omega}) = M_\omega(t)(V_{\tau\omega}) \end{cases}$$

Or equivalently,

$$\pi_t : \begin{cases} \mathscr{E}(\Omega, \widehat{\mathcal{C}}_{m=1}) & \to & \mathscr{E}(\Omega, \widehat{\mathcal{C}}_{m=1}) \\ (v_\omega)_{\omega\in\Omega} & \mapsto & \pi_t(v) : \omega \mapsto \pi_{M_\omega(t)}(v_{\tau\omega}) = \dfrac{\widehat{M_\omega}(t)(v_{\tau\omega})}{\widehat{m}(\widehat{M_\omega}(t)(v_{\tau\omega}))} \end{cases}$$

Lemma 3.4.14. *For $t \in \mathbb{D}$, π_t has a unique fixed point $V^*(t)$ in $\mathscr{E}(\Omega, \mathcal{G}_p(\mathcal{C}))$ - or equivalently, $v^*(t)$ in $\mathscr{E}(\Omega, \widehat{\mathcal{C}}_{m=1})$.*

Proof. Let $t \in \mathbb{D}$. There exists $C > 0$ such that $\operatorname{diam}(\pi_t^n(\mathscr{E}(\Omega, \mathcal{G}_p(\mathcal{C}))) \leq C\eta^n$ by Lemma 3.4.7. Pick $V_0 \in \mathcal{G}_p(\mathcal{C})$ and define $V_\omega^0(t) = V_0$ for all $\omega \in \Omega$. Then the sequence defined by $V^{n+1}(t) = \pi_t(V^n(t))$ is a Cauchy sequence. As $\mathcal{C}$ is closed, $\mathcal{G}_p(\mathcal{C})$ is a closed subset of $\mathcal{G}_p(E)$, hence a complete metric space. Therefore $\mathscr{E}(\Omega, \mathcal{G}_p(\mathcal{C}))$ is complete and $V^n(t)$ converges to some $V^*(t) \in \mathscr{E}(\Omega, \mathcal{G}_p(\mathcal{C}))$. As π_t is continuous, $V^*(t)$ is a fixed point and the uniqueness follows from Lemma 3.4.7. $\square$

We now use this fixed point and the previous lemmas to compute $\chi_p(t)$ as defined in Theorem 3.4.5. Then we show that $t \mapsto v^*(t)$ is analytic and conclude with the real-harmonicity of $t \mapsto \chi_p(t)$.

Definition 3.4.15. *We define the map*

$$p : \begin{cases} \mathbb{D} & \to & \mathscr{E}(\Omega, \mathbb{C}) \\ t & \mapsto & \left(\dfrac{\widehat{m}(\widehat{M_\omega}(t)v^*_{\tau\omega}(t))}{\widehat{m}(v^*_\omega(t))} \right)_{\omega\in\Omega} = \left(\widehat{m}(\widehat{M_\omega}(t)v^*_{\tau\omega}(t)) \right)_{\omega\in\Omega} =: (p_\omega(t))_{\omega\in\Omega} \end{cases}$$

Lemma 3.4.16. *For $t \in \mathbb{D}$, $\chi_p(t) = \int \log|p_\omega(t)| d\mu(\omega)$.*

Proof. As $v^*(t)$ is a fixed point of π_t, we have $V^*_\omega(t) \in \mathscr{M}(\mathcal{C}) \subset \mathcal{C}[\rho]$ for all $\omega \in \Omega$. By Lemma 3.4.12 and since $\hat{m}(v^*_{\tau\omega}(t)) = 1$ by Definition 3.4.13, there exist $C_1, C_2 > 0$ such that for all $\omega \in \Omega$ and $t \in \mathbb{D}$,

$$C_1 \left| \widehat{m}\left(\widehat{M_\omega(t)} v^*_{\tau\omega}(t) \right) \right| \leq \|\widehat{M_\omega}(t)\| \leq C_2 \left| \widehat{m}\left(\widehat{M_\omega(t)} v^*_{\tau\omega}(t) \right) \right|.$$

This means that $|p_\omega(t)|$ is equivalent to $\|\widehat{M_\omega}(t)\| \leq \|M_\omega(t)\|^p$. By Assumption 2 of Theorem 3.4.5, this is dominated by $\|M_\omega(0)\|^p$, which is log-integrable by Assumption 3 of the same theorem. This proves $(\omega \mapsto \log|p_\omega(t)|) \in L^1(\Omega, \mu)$. Applying Lemma 3.4.12 to $M_\omega^{(n)}(t)$ and $v^*_{\tau^n\omega}(t)$, we get

$$C_1 \left| \widehat{m}\left(\widehat{M_\omega^{(n)}(t)} v^*_{\tau^n\omega}(t) \right) \right| \leq \left\| \widehat{M_\omega^{(n)}(t)} \right\| \leq C_2 \left| \widehat{m}\left(\widehat{M_\omega^{(n)}(t)} v^*_{\tau^n\omega}(t) \right) \right|.$$

Then

$$\begin{aligned}
\frac{1}{n}\log\left\|\widehat{M_\omega^{(n)}(t)}\right\| &= \frac{1}{n}\log\left|\widehat{m}\left(\widehat{M_\omega^{(n)}(t)}v^*_{\tau^n\omega}(t)\right)\right| + O\left(\frac{1}{n}\right)\\
&= \frac{1}{n}\log\left|\prod_{k=0}^{n-1}\widehat{m}\left(\widehat{M_\omega^{(k+1)}}(t)v^*_{\tau^{k+1}\omega}(t)\right)\right| + O\left(\frac{1}{n}\right)\\
&= \frac{1}{n}\sum_{k=0}^{n-1}\log|p_{\tau^k\omega}(t)| + O\left(\frac{1}{n}\right).
\end{aligned}$$

As $(\omega \mapsto \log|p_\omega(t)|) \in L^1(\Omega,\mu)$ and τ is ergodic, Birkhoff's ergodic theorem gives $\frac{1}{n}\log\left\|\widehat{M_\omega^{(n)}(t)}\right\| \underset{n\to+\infty}{\longrightarrow} \int \log|p_\omega(t)|d\mu(\omega)$, which is the desired result. □

Lemma 3.4.17. *Let $W \in \mathcal{C}[\rho]$ and $w = G^{-1}(W) \in \bigwedge^p E$ representing W (recall G is the correspondance map defined in Proposition 3.4.2). Consider $h \in \bigwedge^p E$ such that $w+h$ is in $G^{-1}(\mathcal{G}_p(\mathcal{C}))$. Then*

$$\|\pi^{(n)}(w+h) - \pi^{(n)}(w)\| \le C\eta^n\|h\| + o(\|h\|)$$

as $h \to 0$, where C is a constant depending only on p, K, and ρ.

Proof. Let W and W_h be the vector spaces represented by w and $w+h$. As $W \subset \mathcal{C}[\rho]$, Proposition 5.4.12 of the Appendix gives us constants $r, C_1 > 0$ (depending on ρ) such that $d_{\mathcal{C}}(W,W_h) \le C_1 d_H(W,W_h)$ when $d_H(W,W_h) \le r$. By Proposition 3.4.2, the condition $d_H(W,W_h) \le r$ is satisfied when h is small enough. Then, recalling that $\|m\| = 1$,

$$\begin{aligned}
\|\pi^{(n)}(w+h) - \pi^{(n)}(w)\| &= \left\|\frac{\widehat{M}^{(n)}(w+h)}{\widehat{m}(\widehat{M}^{(n)}(w+h))} - \frac{\widehat{M}^{(n)}w}{\widehat{m}(\widehat{M}^{(n)}w)}\right\| \\
&\le a_p K^p d_{\mathcal{C}}(M^{(n)}W, M^{(n)}W_h) && \text{by Lemma 3.3.5}\\
&\le a_p K^p \eta^n d_{\mathcal{C}}(W,W_h) && \text{by Theorem 3.2.3}\\
&\le a_p K^p C_1 \eta^n d_H(W,W_h)\\
&\le a_p K^p C_1 C_2 \eta^n d_\wedge(W,W_h) && (C_2 \text{ from Theorem 2.2.15})\\
&\le a_p K^p C_1 C_2 \eta^n \left\|\tfrac{w+h}{\|w+h\|} - \tfrac{w}{\|w\|}\right\|\\
&\le C\eta^n(\|h\| + o(\|h\|)).
\end{aligned}$$

As $w \in \widehat{\mathcal{C}}_{m=1}$, Lemma 3.3.4 shows that $1 \le \|w\| \le K^p$. Therefore the constant C only depends on p, K, and ρ. □

Lemma 3.4.18. *The map $t \in \mathbb{D} \mapsto v^*(t) \in \mathscr{E}(\Omega, \mathcal{G}_p(\mathcal{C}))$ is analytic.*

Proof. If $0 < \rho' < \rho$, the set $\mathcal{C}[\rho']$ is a neighborhood of $\mathcal{C}[\rho]$ in $\mathcal{G}_p(E)$. Indeed, let $W \subset \mathcal{C}[\rho]$, set $d := \frac{\rho-\rho'}{2(1+\rho')}$ and consider $W' \in \mathcal{G}_p(E)$ such that $d_H(W,W') < d$. Let $x' \in W' \setminus \{0\}$ and $y' \in E$ such that $\|y'\| \le \rho'\|x'\|$. As $d_H(W,W') < d$, there exists $x \in W$ such that $\|x - x'\| < d\|x\|$. Then

$$\|(x'+y')-x\| \le \|x'-x\|+\|y'\| < d\|x\|+\rho'\|x'\| \le d\|x\|+\rho'(d+1)\|x\| \le (d+\rho'(d+1))\,\|x\|.$$

Since $d = \frac{\rho-\rho'}{2(1+\rho')}$, we get $d + \rho'(d+1) = \frac{\rho+\rho'}{2} < \rho$ which gives $\|(x'+y') - x\| < \rho\|x\|$ and therefore $x' + y' \in \mathcal{C}$. This shows that $x' \in \mathcal{C}[\rho']$ hence $W' \subset \mathcal{C}[\rho']$. We proved $B_{d_H}(W,d) \subset \mathcal{C}[\rho']$ for all $W \subset \mathcal{C}[\rho]$, therefore $\mathcal{C}[\rho']$ is a neighborhood of $\mathcal{C}[\rho]$ in $\mathcal{G}_p(E)$.

Consider the map

$$\Phi : \begin{array}{ccc} \mathbb{D} \times \mathscr{E}(\Omega, \mathcal{G}_p(\mathcal{C}[\rho/2])) & \longrightarrow & \mathscr{E}(\Omega, \mathcal{G}_p(\mathcal{C})) \\ (t, v : \omega \mapsto v_\omega) & \mapsto & \pi_t(v) \end{array}$$

To see if this map is analytic, we remember that there is a chart of $\mathcal{G}_p(E)$ which covers $\mathcal{G}_p(\mathcal{C})$ and in which the image of $\mathcal{G}_p(\mathcal{C})$ is bounded: the image of $\mathcal{G}_p(\mathcal{C})$ in some chart $\mathcal{L}(W_0, V_0)$ is bounded by $(K+1)$. In this chart, we can write Φ as

$$\tilde{\Phi} : \begin{array}{ccc} \mathbb{D} \times B\left(\mathscr{B}(\Omega, \mathcal{L}(W_0, V_0)), K+1\right) & \longrightarrow & \mathscr{B}(\Omega, \mathcal{L}(W_0, V_0)) \\ (t, A : \omega \mapsto A_\omega) & \mapsto & \tilde{\pi}_t(A) \end{array}$$

where $\tilde{\pi}_t$ is the map π_t seen in this chart.

We denote $A^*(t)$ the element of $B\left(\mathscr{B}(\Omega, \mathcal{L}(W_0, V_0)), K+1\right)$ corresponding to the fixed point $v^*(t) \in \mathscr{E}(\Omega, \mathcal{G}_p(\mathcal{C}))$. Let $t_0 \in \mathbb{D}$ and denote by $T_0 = D\tilde{\pi_{t_0}}(A^*(t_0))$ the derivative of $\tilde{\pi_{t_0}}$ at the fixed point $A^*(t_0)$. By Lemma 3.4.17 and Proposition 3.4.3, $T_0^n = D\widetilde{\pi_{t_0}^{(n)}}(A^*(t_0))$ verifies $\|T_0^n\| \le C\eta^n$ for some constant C. Then the spectral radius of T_0 is less that η and therefore the derivative $\mathrm{Id} - T_0$ of $v \mapsto A - \tilde{\pi}_t(A)$ is invertible at the fixed point $A^*(t_0)$.

We can now apply the implicit function theorem and conclude there is an analytic function $t \mapsto A^*(t)$ defined on a neighborhood of t_0 for which $A^*(t) - \tilde{\pi}_t(A^*(t)) = 0$. This proves the analyticity of $t \mapsto A^*(t)$ and therefore the analyticity of $t \mapsto v^*(t)$. $\square$

The following lemma gives an upper-bound for the derivative of $\widehat{M_\omega}(t)$ and is proved in Section 5.3 of the Appendix.

Lemma 3.4.19. *For $t \in \mathbb{D}$,*

$$\left\| \frac{d}{dt} \widehat{M_\omega}(t) \right\| \le p \left\| \bigwedge^{p-1} M_\omega(t) \right\| \left\| \frac{d}{dt} M_\omega(t) \right\| .$$

Under the assumptions of Theorem 3.4.5, this gives

$$\left\| \frac{d}{dt} \widehat{M_\omega}(t) \right\| \le pC \left\| \widehat{M_\omega}(t) \right\| ,$$

where $C = \sup \left\{ \dfrac{\| \bigwedge^{p-1} M_\omega(t)\|}{\| \bigwedge^{p} M_\omega(t)\|} \| \tfrac{d}{dt} M_\omega(t)\| : \omega \in \Omega, t \in \mathbb{D} \right\}$.

We now have everything in place to achieve the proof of our main theorem.

End of the proof of Theorem 3.4.5. Let $\omega \in \Omega$. The map $t \mapsto p_\omega(t)$ is analytic by Definition 3.4.15 and Lemma 3.4.18. In order to prove the analyticity of $t \mapsto \log p_\omega(t)$, we need to define a complex logarithm in a consistent way. Let $\varepsilon > 0$ and $t_0 \in \mathbb{D}$. As $t \mapsto v^*(t)$ is analytic, there exist $\delta > 0$ such that $\|v^*_{\tau\omega}(t) - v^*_{\tau\omega}(t_0)\| \le K^p \varepsilon$ for $|t - t_0| < \delta$. By Lemma 3.4.19, we can choose δ small enough such that $\|\widehat{M_\omega}(t) - \widehat{M_\omega}(t_0)\| \le \varepsilon \|\widehat{M_\omega}(t_0)\|$. Then for $|t - t_0| < \delta$, we get

$$\begin{array}{rcl} |p_\omega(t) - p_\omega(t_0)| & = & \left| \widehat{m} \left(\widehat{M_\omega}(t) v^*_{\tau\omega}(t) - \widehat{M_\omega}(t_0) v^*_{\tau\omega}(t_0) \right) \right| \\ & \le & \|\widehat{M_\omega}(t) v^*_{\tau\omega}(t) - \widehat{M_\omega}(t_0) v^*_{\tau\omega}(t_0)\| \\ & \le & \|\widehat{M_\omega}(t) - \widehat{M_\omega}(t_0)\| \|v^*_{\tau\omega}(t)\| + \|\widehat{M_\omega}(t_0)\| \|v^*_{\tau\omega}(t) - v^*_{\tau\omega}(t_0)\| \\ & \le & \varepsilon \|\widehat{M_\omega}(t_0)\| K^p + \|\widehat{M_\omega}(t_0)\| \|v^*_{\tau\omega}(t) - v^*_{\tau\omega}(t_0)\| \\ & \le & 2K^p \|\widehat{M_\omega}(t_0)\| \varepsilon. \end{array}$$

By Lemma 3.4.12, $|p_\omega(t_0)| \geq C_1 \|\widehat{M}_\omega(t_0)\|$ for some constant $C_1 > 0$, hence

$$\left|\frac{p_\omega(t)}{p_\omega(t_0)} - 1\right| \leq 2K^p\varepsilon/C_1.$$

Fixing ε small enough such that $2K^p\varepsilon/C_1 < 1/2$, we have $\left|\frac{p_\omega(t)}{p_\omega(t_0)} - 1\right| \leq 1/2$. Then, for $|t-t_0| \leq \delta$,

$$\chi_p(t) - \chi_p(t_0) = \int \log\left|\frac{p_\omega(t)}{p_\omega(t_0)}\right| d\mu(\omega) = \operatorname{Re}\int \log\frac{p_\omega(t)}{p_\omega(t_0)} d\mu(\omega)$$

where log is the usual logarithm on $\mathbb{C} \setminus \mathbb{R}_-$. Thus the map $t \mapsto \chi_p(t)$ is real-analytic. **This concludes the proof of Theorem 3.4.5.** □

Remark. When the dynamic is invertible, one can see that we obtain the same result for $M_\omega^n := M_{\tau^{n-1}\omega} \cdots M_\omega$. The theorem then shows the analyticity of the sum of the p largest Lyapunov exponents (with multiplicity). These Lyapunov exponents exist by standard results as we have exhibited a vector bundle $V_\omega^*(t)$ of finite dimension p invariant by $M_\omega(t)$.

Example. Let $(\xi_n)_{n\geq 0}$ be a sequence of independent and identically distributed random variables with values in $\mathbb{D}$. Consider for $t \in \mathbb{D}$

$$M_n(t) = \begin{pmatrix} 10 + t\xi_n & t+\xi_n & it \\ t+\xi_n & 6 & \xi_n \\ i\xi_n & 0 & 1 \end{pmatrix}$$

One can check that $M_n(t)$ maps $\mathcal{C}_{\pi,1}$ into $\mathcal{C}_{\pi,0.85}$, where π is the orthogonal projection on the first two coordinates. Conditions 1 and 3 of Theorem 3.4.5 are clearly satisfied, and condition 2 is also satisfied as $\det M_n(t) \geq 43$ and $\|M_n(t)\|$ and $\|\frac{d}{dt}M_n(t)\|$ are bounded. By Theorem 3.4.5,

$$\chi_2(t) = \lim \frac{1}{n} \log \|(M_1(t) \wedge M_1(t)) \cdots (M_n(t) \wedge M_n(t))\|$$

exists almost surely and defines a harmonic function of $t \in \mathbb{D}$. We can do a similar reasoning to show that $\chi_1(t) = \lim \frac{1}{n} \log \|M_1(t) \cdots M_n(t)\|$ exists and defines a harmonic function of $t \in \mathbb{D}$. Thus the first two characteristic exponents of the random product of $t \mapsto (M_n(t))$ are harmonic. Moreover, by Birkhoff's theorem, we get

$$\chi_3(t) = \lim \frac{1}{n} \log\det\left(M_1(t) \cdots M_n(t)\right) = \mathbb{E}\left[\log\left|\begin{matrix} 10 + t\xi_0 & t+\xi_0 & it \\ t+\xi_0 & 6 & \xi_0 \\ i\xi_0 & 0 & 1 \end{matrix}\right|\right],$$

thus all three characteristic exponents of the random product of $t \mapsto (M_n(t))$ are harmonic.

Chapter 4

Dual cones

In this chapter, we introduce dual cones for p-cones in a complex Banach space. Some notions of dual cones have been introduced for real and complex cones of dimension 1.

If $\mathcal{C}$ is a proper closed convex cone on a real Banach space, its dual cone $\mathcal{C}'$ is defined as (see e.g. [LN14])

$$\mathcal{C}' \coloneqq \{l \in E' \mid \forall x \in \mathcal{C}, l(x) \geq 0\}.$$

If $\mathcal{C}$ is a proper complex cone on a real Banach space, Dubois introduces a notion of convexity in [Dub09]: a complex cone $\mathcal{C}$ is said to be linearly convex if for all $x \notin \mathcal{C}$, there exists a closed hyperplane H_x such that $H_x \cap \mathcal{C} = \{0\}$. For such a cone, he defines its dual cone as

$$\mathcal{C}' \coloneqq \{l \in E' \mid \forall x \in \mathcal{C}^*, l(x) \neq 0\}.$$

Here we are interested in making transposed operators act on dual cones to obtain dominated splittings.

In Section 4.1, we recall some results on orthogonal spaces in the dual space E' and define for p-cones in E a notion of dual cone as an union of p-dimensional spaces in E'. Then we study in Section 4.2 some properties of the dual cones. We show that it has almost all the features of a p-cone. Moreover, we prove that a p-cone $\mathcal{C}$ has bounded aperture (resp. has non-empty interior in $\mathcal{G}_p(E)$) if and only if its dual $\mathcal{C}'$ has non-empty interior in $\mathcal{G}_p(E')$ (resp. has bounded aperture). We also study in detail the case of projection cones $\mathcal{C}_{\pi,a}$. In Section 4.3, we show that an inclusion of cones in E implies the inverse inclusion for the dual cones, and use this to apply the results from Chapter 3 to transposed operators. Finally, this notion of dual cones allows us to prove a dominated splitting result in Section 4.4, extending Theorem 3.4.5: a cocycle contracting reasonable p-cones admits a dominated splitting with fast space of dimension p, and this splitting varies analytically with the cocycle.

4.1 Orthogonals in E' and dual cones

We now introduce a notion of dual cone for p-cones. If $\mathcal{C}$ is a p-cone of E, there are two ways to consider its dual cone: as a union of subspaces of codimension p in E, or as a

p-cone in the dual space E' of E. Recall that $\mathcal{G}_p(E)$ (resp. $\mathcal{G}_p(E')$) denotes the set of p-dimensional spaces of E (resp. E'), and $\mathcal{G}^p(E)$ (resp. $\mathcal{G}^p(E')$) denotes the set of closed p-codimensional spaces of E (resp. E').

Definition 4.1.1. *Given $V \subset E$, we define $V^\perp \subset E'$ by $V^\perp = \{l \in E' \mid \forall x \in V, l(x) = 0\}$. Given $V' \subset E'$, we define $V'^\perp \subset E$ by $V'^\perp = \{x \in E \mid \forall l \in V', l(x) = 0\}$.*

Definition 4.1.2. *Let $\mathcal{C}$ be a p-cone. We define:*

- $GC(\mathcal{C}) := \{V \in \mathcal{G}^p(E) \mid \forall W \in \mathcal{G}_p(E) \text{ s.t. } W \subset \mathcal{C}, V \oplus W = E\}$ *the grassmannian complement of $\mathcal{C}$;*
- $\tilde{\mathcal{C}} := \bigcup_{V \in GC(\mathcal{C})} V$ *the co-cone of $\mathcal{C}$ in E;*
- $\mathcal{C}' := \bigcup_{V \in GC(\mathcal{C})} V^\perp$ *the dual cone of $\mathcal{C}$ in E'.*

Remark. Note that the sets $GC(\mathcal{C})$, $\tilde{\mathcal{C}}$ and $\mathcal{C}'$ may be empty. However, this is never the case when $\mathcal{C}$ has bounded aperture.

We first identify the closed p-codimensional suspaces of $\tilde{\mathcal{C}}$.

Lemma 4.1.3. *Let $V \in \mathcal{G}^p(E)$. If $V \subset \tilde{\mathcal{C}}$, then $V \in GC(\mathcal{C})$; i.e. every closed subspace V of codimension p in $\tilde{\mathcal{C}}$ is a subspace satisfying the condition*

$$\forall W \in \mathcal{G}_p(E) \text{ s.t. } W \subset \mathcal{C}, V \oplus W = E.$$

Proof. Let $V \in \mathcal{G}^p(E)$ such that $V \subset \tilde{\mathcal{C}}$ and let $W \in \mathcal{G}_p(E)$ such that $W \subset \mathcal{C}$. Let $x \in V \cap W$. Then $x \in \tilde{\mathcal{C}} \cap W$. By definition of $\tilde{\mathcal{C}}$, there exists $V_1 \in GC(\mathcal{C})$ such that $x \in V_1$. As $V_1 \cap W = \{0\}$ (by definition of $GC(\mathcal{C})$), we get $x = 0$ and thus $V \cap W = \{0\}$. As $V \in \mathcal{G}^p(E)$ and $W \in \mathcal{G}_p(E)$, the subspaces V and W are topological complements. This proves $V \in GC(\mathcal{C})$. $\square$

We now show some standard results on orthogonals for the Grassmannians of E and E'.

Lemma 4.1.4. *We have the following implications:*

(i) If $V' \in \mathcal{G}_p(E')$, then $V'^\perp \in \mathcal{G}^p(E)$.

(ii) If $V \in \mathcal{G}^p(E)$, then $V^\perp \in \mathcal{G}_p(E')$.

Proof. (i). Let $V' = \mathrm{Span}(l_1, \ldots, l_p)$ be a p-dimensional space of E'. Consider the linear map

$$L : \begin{array}{ccc} E & \to & \mathbb{C}^p \\ x & \mapsto & (l_1(x), \ldots, l_p(x)). \end{array}$$

Assume that L is not surjective. Then its image is contained in a subspace H of $\mathbb{C}^p$ of dimension $p-1$, and therefore there exists $(a_1, \ldots, a_p) \in \mathbb{C}^p \setminus \{0\}$ such that

$$\forall x \in E, \sum_{i=1}^{p} a_i l_i(x) = 0.$$

This contradicts the fact that $l_1, \dots, l_p$ are independent. So the map L is surjective, hence there exist vectors $e_1, \dots, e_p$ of E such that $l_i(e_j) = \delta_{i,j}$ for all $i, j \in [\![1, p]\!]$.

We now show that $E = \ker L \oplus \mathrm{Span}(e_1, \dots, e_p)$. By definition of the e_i, we have $\ker L \cap \mathrm{Span}(e_1, \dots, e_p) = \{0\}$. Moreover, for every $x \in E$, we have

$$x = \underbrace{\left(x - \sum_{i=1}^{p} l_i(x) e_i\right)}_{\in \ker L} + \underbrace{\sum_{i=1}^{p} l_i(x) e_i}_{\in \mathrm{Span}(e_1,\dots,e_p)} ,$$

which proves that $E = \ker L + \mathrm{Span}(e_1, \dots, e_p)$ and therefore $E = \ker L \oplus \mathrm{Span}(e_1, \dots, e_p)$. As $V'^{\perp} = \ker L$, the subspace $V'^{\perp}$ is closed and has codimension p.

(ii). Let $V \in \mathcal{G}^p(E)$: there exists $W \in \mathcal{G}_p(E)$ such that V and W are topological complements. Let $W' := \mathcal{L}(W, \mathbb{C})$ be the dual space of W. Then the map

$$\mathrm{Restr}_W : \begin{array}{ccc} V^{\perp} & \to & W' \\ l & \mapsto & l_W \end{array}$$

is an isomorphism. Indeed, if $l \in V^{\perp}$ satisfies $l_{|W} = 0$, then $l_{|V} = 0$ and $l_{|W} = 0$, hence $l = 0$ and the map Restr_W is injective. Given $l_0 \in W'$, we define $l \in E'$ by $l_{|V} = 0$ and $l_{|W} = l_0$. This linear form is continuous (therefore in E') as V and W are topological complements. This proves the surjectivity of the map Restr_W, which gives $\dim V^{\perp} = \dim W' = p$. □

Lemma 4.1.5. *We have the following implications:*

(i) If $V \in \mathcal{G}^p(E)$, then $(V^{\perp})^{\perp} = V$.

(ii) If $V' \in \mathcal{G}_p(E')$, then $(V'^{\perp})^{\perp} = V'$.

Proof. Given $V \in \mathcal{G}^p(E)$, we immediately have $V \subset (V^{\perp})^{\perp}$. As $\mathrm{codim}(V^{\perp})^{\perp} = p = \mathrm{codim}\, V$ by the previous lemmas, we get $(V^{\perp})^{\perp} = V$.

Similarly for $V' \in \mathcal{G}_p(E')$, we have $V' \subset (V'^{\perp})^{\perp}$ and the dimensions are equal, therefore $(V'^{\perp})^{\perp} = V'$. □

The following propositions estimate the gap and the Hausdorff distance between orthogonals.

Proposition 4.1.6. *Let $V, W \in \mathcal{G}^p(E)$, and $V', W' \in \mathcal{G}_p(E')$. Then*

$$\delta(V, W) = \delta(V^{\perp}, W^{\perp}),$$

$$\delta(V', W') = \delta(V'^{\perp}, W'^{\perp}).$$

Proof. The proof of the first inequality can be found in Chapter IV, section 2.3 of [Kat95]. The second one is just this first one applied to $V = V'^{\perp}$ and $W = W'^{\perp}$, using Lemma 4.1.5. □

Corollary 4.1.7. *Let $V, W \in \mathcal{G}^p(E)$, and $V', W' \in \mathcal{G}_p(E')$. Then*

$$d_H(V, W) \leq 2 d_H(V^{\perp}, W^{\perp}),$$

$$d_H(V', W') \leq 2 d_H(V'^{\perp}, W'^{\perp}).$$

Proof. This is a consequence of Proposition 4.1.6 and Lemma 2.1.4. □

We now prove that the only p-dimensional subspaces of a dual cone $\mathcal{C}'$ are the ones which define it, i.e. orthogonals of elements of $GC(\mathcal{C})$.

Proposition 4.1.8. *Let $\mathcal{C}$ be a p-cone and $\mathcal{C}'$ be its dual cone. Let $V' \in \mathcal{G}_p(E')$ such that $V' \subset \mathcal{C}'$. Then there exists (a unique) $V \in GC(\mathcal{C})$ such that $V' = V^\perp$.*

Proof. Let $V' = \mathrm{Span}(l_1, \ldots, l_p)$ be a p-dimensional space of E' such that $V' \subset \mathcal{C}'$, and set $V \coloneqq (V')^\perp$ (which immediately gives $V' = V^\perp$). Let us show that $V \in GC(\mathcal{C})$. Let $W \in \mathcal{G}_p(E)$ such that $W \subset \mathcal{C}$, and $W' \coloneqq \mathcal{L}(W, \mathbb{C})$ be the dual of W. Consider the map

$$\mathrm{Restr}_W : \begin{array}{ccc} V' & \to & W' \\ l & \mapsto & l_{|W}. \end{array}$$

Let $l \in \ker \mathrm{Restr}_W$. We have $l \in \mathcal{C}'$, so there exists $V_l \in GC(\mathcal{C})$ such that $l \in V_l^\perp$. Then $E = V_l \oplus W$, $l_{|V_l} = 0$ and $l_{|W} = 0$, hence $l = 0$. This shows that the map Restr_W is injective, therefore surjective given the dimensions.

Let $x \in V \cap W$. Then for all $l \in V'$, $l(x) = 0$. As $x \in W$ and Restr_W is surjective, this shows that $l(x) = 0$ for all $l \in W'$, hence $x = 0$. We thus have $V \cap W = \{0\}$, $V \in \mathcal{G}^p(E)$ and $W \in \mathcal{G}_p(E)$, therefore $V \oplus W = E$. As this holds for all $W \in \mathcal{G}_p(E)$ such that $W \subset \mathcal{C}$, this proves that $V \in GC(\mathcal{C})$. □

4.2 Properties of dual cones

In this section, we look at some properties of the co-cone and the dual cone, which makes the latter "almost a p-cone" of E'. Properties of bounded aperture for $\mathcal{C}$ or openness of $\mathcal{C}$ in $\mathcal{G}_p(E)$ translates into properties for $\mathcal{C}'$. Finally, we look at the particular example of p-cones $\mathcal{C}_{\pi,a}$ defined with respect to a projection π.

Proposition 4.2.1. *Let $\mathcal{C}$ be a p-cone. The co-cone $\tilde{\mathcal{C}}$ does not contain any subspace of codimension $p-1$.*

Proof. If there exists $V \subset \tilde{\mathcal{C}}$ with $\operatorname{codim} V = p-1$, then for any $W \subset \mathcal{C}$ with $\dim W = p$, we have $V \cap W \neq 0$. Therefore there exists $x \in V \cap W \setminus \{0\}$. As $x \in \tilde{\mathcal{C}}$, there exists $V_x \in \mathcal{G}^p(E)$ such that $x \in V_x$ and $V_x \oplus W = E$. This contradicts $x \in W \setminus \{0\}$. □

Proposition 4.2.2. *Let $\mathcal{C}$ be a p-cone. The dual cone $\mathcal{C}'$ does not contain any subspace of dimension $p+1$.*

Proof. Assume there exists $V' \subset \tilde{\mathcal{C}}$ with $\dim V' = p+1$. Let $(l_1, \ldots, l_{p+1})$ be a basis of V', and let

$$V_0 = \bigcup_{(\lambda,\mu) \in \mathbb{C}^2 \setminus \{(0,0)\}} \left(\mathrm{Span}(\lambda l_1 + \mu l_2, l_3, \ldots, l_p, l_{p+1})\right)^\perp.$$

Let us show that V_0 is a subspace of E, of codimension $\leq p-1$. The set V_0 contains 0 and is clearly stable by scalar multiplication. Let $x, y \in V_0$. By definition of V_0, there exists

$(\lambda, \mu) \in \mathbb{C}^2 \setminus \{(0,0)\}$, $(\lambda', \mu') \in \mathbb{C}^2 \setminus \{(0,0)\}$ such that $\lambda l_1(x) + \mu l_2(x) = \lambda' l_1(y) + \mu' l_2(y) = 0$. We denote $l = \lambda l_1 + \mu l_2$ and $m = \lambda' l_1 + \mu' l_2$.

If l and m are collinear, say for example $m = \alpha l$, then $x, y \in \operatorname{Span}(l, l_3, \ldots, l_p, l_{p+1})^{\perp}$, and therefore we get $x + y \in \operatorname{Span}(l, l_3, \ldots, l_p, l_{p+1})^{\perp} \subset V_0$.

Otherwise, we have $(m(x)l - l(y)m)(x + y) = 0$ and $m(x)l - l(y)m \in \operatorname{Span}(l_1, l_2) \setminus \{0\}$, which leads to $x+y \in \operatorname{Span}(m(x)l - l(y)m, l_3, \ldots, l_p, l_{p+1})^{\perp} \subset V_0$. Therefore V_0 is a subspace of E.

Since $(l_1, \ldots, l_{p+1})$ are independent, the subspace $\operatorname{Span}(l_1, l_3, \ldots, l_p, l_{p+1})^{\perp}$ is strictly contained in V_0, hence $\operatorname{codim} V_0 < p$. The space V_0 is an union of subspaces of the co-cone $\tilde{\mathcal{C}}$, so we have $V_0 \subset \tilde{\mathcal{C}}$. This contradicts Proposition 4.2.1 which states that $\tilde{\mathcal{C}}$ does not contain any subspace of codimension $p - 1$. □

Remark. However, this proposition does not ensure that $\mathcal{C}'$ is a p-cone of E'. Indeed, $\mathcal{C}'$ is not closed. Moreover, its closure is not necessarily a p-cone of E'. For example, if $\mathcal{C} = W$ is a single p-dimensional subspace of E, then $\mathcal{C}' \setminus \{0\} = \{l \in E' \mid l_{|W} \neq 0\}$, giving $\overline{\mathcal{C}'} = E'$.

Proof. Let $l \in E'$ such that $l_{|W} \neq 0$. Then there exists $(l_2, \ldots, l_p)$ linear forms on W such that $(l_{|W}, l_2, \ldots, l_p)$ is a basis of $\mathcal{L}(W, \mathbb{C})$. Using the Hahn-Banach theorem, we extend these linear forms into forms of E'. The p-tuple $(l, l_2, \ldots, l_p)$ is independent in E' as the restrictions on W are independent. Therefore $V := \ker l \cap \bigcap_{i=2}^{p} \ker l_i$ is a closed subspace of E of codimension p, with $V \cap W = \{0\}$. This shows $V \oplus W = E$, hence $V \in GC(\mathcal{C})$ and $l \in V^{\perp} \subset \mathcal{C}'$.

Conversely, if $l \in \mathcal{C}'$, there exists $V \in \mathcal{G}^p(E)$ such that $V \oplus W = E$ and $l_{|V} = 0$. In this case, if $l_{|W} = 0$, then $l = 0$. This achieves the proof of $\mathcal{C}' \setminus \{0\} = \{l \in E' \mid l_{|W} \neq 0\}$. □

So it is possible for the closure of the dual of a p-cone to contain a $(p + 1)$-dimensional subspace. Looking at the example above, it seems that this phenomenon appears when the p-cone $\mathcal{C}$ of E is not "wide enough": this results in the dual cone being "too wide". We remedy this problem with the next propositions.

Definition 4.2.3. *Let $\mathcal{C}$ be a p-cone. We say that $\mathcal{C}$ is linearly convex if for all $x \notin \mathcal{C}$, there exists $V_x \in \mathcal{G}^p(E)$ such that $V_x \in GC(\mathcal{C})$.*

Remark. For $p = 1$, this gives the definition of linear convexity for classic complex cones introduced by Dubois in [Dub09].

Remark. When $\mathcal{C}$ is linearly convex, the co-cone $\tilde{\mathcal{C}} \subset E$ is equal to $E \setminus \mathcal{C}$.

Proposition 4.2.4. *Let $\mathcal{C}$ be a p-cone. The interior of $\{W \in \mathcal{G}_p(E) \mid W \subset \mathcal{C}\}$ in $\mathcal{G}_p(E)$ is non-empty if and only if the dual cone $\mathcal{C}'$ has bounded aperture (and so does its closure).*

Proof. Assume that the interior of $\{W \in \mathcal{G}_p(E) \mid W \subset \mathcal{C}\}$ in $\mathcal{G}_p(E)$ is non-empty, i.e. there exists $W_0 \in \mathcal{G}_p(E)$ and $r > 0$ such that $W \subset \mathcal{C}$ for all $W \in B_{d_H}(W_0, r)$. Consider the map

$$\operatorname{Restr}_{W_0} : \begin{array}{ccc} E' & \to & W_0' := \mathcal{L}(W_0, \mathbb{C}) \\ l & \mapsto & l_{|W_0}. \end{array}$$

Let $l \in \mathcal{C}'$. There exists $V \in GC(\mathcal{C})$ such that $l \in V^{\perp}$. We want to bound $\|l\|$ using $\|l_{|W_0}\|$. Let $x_V \in V$. As V is a complement to all p-dimensional subspaces of $\mathcal{C}$, the vector

x_V is not in $\mathcal{C}$. If $d(x_V, W_0) < r/8$, then by Proposition 5.4.7 of the Appendix there exists a p-dimensional subspace W_1 such that $x_V \in W_1$ and $d(W_1, W_0) < r$. We deduce that $W_1 \subset \mathcal{C}$, which contradicts $x_V \notin \mathcal{C}$. Hence $d(x_V, W_0) \geq r/8$. By Lemma 2.1.6, this shows that $\sin\theta(V, W_0) \geq r/8$, hence $\|\pi_{V//W_0}\| \leq 8/r$ (where $\pi_{V//W_0}$ is the projection on V parallel to W_0).

Let $x \in E$ with $\|x\| = 1$ and with decomposition $x = x_V + x_W$ with respect to $E = V \oplus W_0$. Then

$$\|l(x)\| = \|l(x_W)\| \leq \|l_{|W_0}\|.\|x_W\| \leq \|l_{|W_0}\|.\|x - x_V\| \leq \|l_{|W_0}\|.(1 + 8/r)\|x\|,$$

giving $\|l\| \leq \|l_{|W_0}\|.(1 + 8/r)$. We have

$$\forall l \in \mathcal{C}', \|l\| \leq (1 + 8/r)\| \operatorname{Restr}_{W_0}(l)\|.$$

Since W_0' is p-dimensional and the linear map $\operatorname{Restr}_{W_0} : E' \to W_0'$ is continuous, this proves that the dual cone $\mathcal{C}'$ has bounded aperture.

Conversely, assume the dual cone $\mathcal{C}'$ has bounded aperture and let $m : E' \to \mathbb{C}^p$ be an operator of norm 1 such that $\forall l \in \mathcal{C}', \|l\| \leq K|m(l)|$. Let $W_0' := \ker m \in \mathcal{G}(E')$ and let $W_0 := W_0'^{\perp} \in \mathcal{G}_p(E)$. Let $W \in \mathcal{G}_p(E)$ with $d_H(W, W_0) < \frac{1}{4K}$. We want to prove that $W \subset \mathcal{C}$.

First, we show that for all $V \in GC(\mathcal{C})$, $W^{\perp} \cap V^{\perp} = \{0\}$. Let $l \in W^{\perp} \cap V^{\perp}$. As $d_H(W, W_0) < \frac{1}{4K}$, we have $d_H(W^{\perp}, W_0^{\perp}) < \frac{1}{2K}$ by Corollary 4.1.7. Thus there exists a linear form $l_0 \in W_0^{\perp} = \ker m$ such that $\|l - l_0\| \leq \frac{1}{2K}\|l\|$. We can now estimate $|m(l)|$:

$$\frac{1}{K}\|l\| \leq |m(l)| = |m(l - l_0)| \leq \|l - l_0\| \leq \frac{1}{2K}\|l\|,$$

where the first inequality stems from the bounded aperture of $\mathcal{C}'$, as $l \in V^{\perp} \subset \mathcal{C}'$. Thus $l = 0$ and $W^{\perp} \cap V^{\perp} = \{0\}$.

By contradiction, assume $W \not\subset \mathcal{C}$: there exists $x \in W$ with $x \notin \mathcal{C}$. As $\mathcal{C}$ is linearly convex, there exists $V \in GC(\mathcal{C})$ such that $x \in V$. Then we can decompose W into the sum $W = W_1 \oplus (W \cap V)$, with $\dim W_1 < p$. The sum $W + V$ is a closed subspace of E, and $W + V = W_1 \oplus V \subsetneq E$, as $\dim W_1 < p$ and $\operatorname{codim} V = p$. We can therefore find a linear form $l \neq 0$ such that $l_{|V+W} = 0$: this contradicts $W^{\perp} \cap V^{\perp} = \{0\}$.

We have shown that if $W \in \mathcal{G}_p(E)$ satisfies $d_H(W, W_0) < \frac{1}{4K}$, then $W \subset \mathcal{C}$. This means $B(W_0, \frac{1}{4K}) \subset \{W \in \mathcal{G}_p(E) \mid W \subset \mathcal{C}\}$. □

Proposition 4.2.5. *Let $\mathcal{C}$ be a p-cone. Then $\mathcal{C}$ has bounded aperture if and only if its dual cone $\mathcal{C}'$ has non-empty interior in $\mathcal{G}_p(E')$ (and so does its closure).*

Proof. Assume $\mathcal{C}$ has aperture bounded by $K > 0$ and let $m : E \to \mathbb{C}^p$ such that $\|m\| = 1$ and $\|x\| \leq K|m(x)|$ for all $x \in \mathcal{C}$. We set $V_0 := \ker m$ and $V_0' := V_0^{\perp}$. Let $r > 0$ and $V' \in \mathcal{G}_p(E')$ such that $d_H(V_0', V') \leq r$. We set $V := V'^{\perp}$; by Lemma 4.1.5, $V^{\perp} = V'$. By Lemmas 4.1.6 and 2.1.4, we have $\delta(V, V_0) = \delta(V', V_0') \leq 2r$.

Let $x \in V$ and $x_0 \in V_0$ such that $\|x - x_0\| \leq 3r\|x\|$. Then

$$|m(x)| = |m(x - x_0)| \leq \|m\|\|x - x_0\| \leq 3r\|x\|.$$

Setting $r = \frac{1}{4K}$, this shows that $x \notin \mathcal{C}$ (if $x \neq 0$). This proves that for all $W \in \mathcal{G}_p(E)$ such that $W \subset \mathcal{C}$, we have $V \cap W = \{0\}$. As $V \in \mathcal{G}^p(E)$, we get $V \in GC(\mathcal{C})$. Thus $V' = V^\perp \subset \mathcal{C}'$, and therefore the interior of $\mathcal{C}'$ in $\mathcal{G}_p(E')$ is non-empty since it contains $B_{d_H}(V_0, \frac{1}{4K})$.

Conversely, assume the dual cone $\mathcal{C}'$ has non-empty interior in $\mathcal{G}_p(E')$. Let $V_0' \in \mathcal{G}_p(E')$ such that: for all $V' \in B_{d_H}(V_0', r)$, $V' \subset \mathcal{C}'$. By Proposition 4.1.8, we have: $V'^\perp \in GC(\mathcal{C})$ for all $V' \in B_{d_H}(V_0', r)$. Using Corollary 4.1.7, we have $B_{d_H}(V_0, r/2) \subset GC(\mathcal{C})$, where $V_0 = V_0'^\perp$. Let $W \subset \mathcal{G}_p(E)$ with $W \subset \mathcal{C}$ and let $m : E \to W$ be the (continuous) projection onto W (of dimension p) parallel to V_0.

Let $x \in \mathcal{C}^*$, with decomposition $x = x_{V_0} + x_W$ with respect to $E = V_0 \oplus W$. By Lemma 5.4.7, there exists a subspace $V_x \in \mathcal{G}^p(E)$ such that $d_H(V_x, V_0) \leq 8 d_H(\frac{x}{\|x\|}, V_0)$. Then

$$\|m(x)\| = \|x_W\| \geq d(x, V_0) \geq \frac{1}{8}\|x\| d_H(V_x, V_0) \geq \frac{r}{16}\|x\|,$$

as V_x cannot be in $B(V_0, r/2)$ because $x \in V_x \cap \mathcal{C}$. This shows that $\mathcal{C}$ is of bounded aperture. □

Proposition 4.2.6. *Let $\mathcal{C}$ be a p-cone. If the interior of $\{W \in \mathcal{G}_p(E) \mid W \subset \mathcal{C}\}$ in $\mathcal{G}_p(E)$ is non-empty, then the closure of the dual cone $\mathcal{C}'$ does not contain any $(p+1)$-dimensional space.*

Proof. The closure $\overline{\mathcal{C}'}$ of the dual cone $\mathcal{C}'$ is obviously closed. By Proposition 4.2.4, there exists $m : E' \to \mathbb{C}^p$ and $K > 0$ such that $\forall l \in \mathcal{C}', \|l\| \leq K|m(l)|$. This inequality extends to $\overline{\mathcal{C}'}$: since m takes its values in a p-dimensional space and is non-vanishing on $\overline{\mathcal{C}'} \setminus \{0\}$, the set $\overline{\mathcal{C}'}$ can not contain any $(p+1)$-dimensional space. □

In the latter case, one can wonder if the closure $\overline{\mathcal{C}'}$ of the dual cone $\mathcal{C}'$ is a p-cone of E'. It is closed and does not contain any any $(p+1)$-dimensional space. However, $\overline{\mathcal{C}'}$ may not be an union of p-dimensional spaces anymore.

There is a particular case in which this holds true and that we study in more detail in the following sections: the case of cones $\mathcal{C}_{\pi,a}$ associated with a projection π. Here, the dual cone is much more explicit.

Definition 4.2.7. *Let E_1, E_2 be Banach spaces. For $T : E_1 \to E_2$ a continuous linear map, we denote by $T^\top$ the operator*

$$T^\top : \begin{array}{ccc} E_2' & \to & E_1' \\ l & \mapsto & l \circ T. \end{array}$$

Lemma 4.2.8. *Let $E = F \oplus G$ be a topological decomposition of E and and $\pi : E \to F$ be the projection onto F parallel to G. Then $E' = G^\perp \oplus F^\perp$ is a topological decomposition of E', and the projection onto $G^\perp$ parallel to $F^\perp$ is $\pi^\top$.*

Proof. Let $l \in E'$. Since π is continuous and $\pi^\top \circ \pi^\top = (\pi \circ \pi)^\top = \pi^\top$, the map $\pi^\top$ is a continuous projection. Then $l = l \circ \pi + (l - l \circ \pi)$ and we have:

$$l \in \ker \pi^\top \Leftrightarrow \forall x \in E, l(\pi(x)) = 0 \Leftrightarrow \forall x \in F, l(x) = 0 \Leftrightarrow l \in F^\perp;$$

$$l \in \ker(\mathrm{Id} - \pi^\top) \Leftrightarrow \forall x \in E, l(x - \pi(x)) = 0 \Leftrightarrow \forall x \in G, l(x) = 0 \Leftrightarrow l \in G^\perp.$$

Therefore $\pi^\top$ is a projection onto $G^\perp$ parallel to $F^\perp$ and $E' = G^\perp \oplus F^\perp$. □

Proposition 4.2.9. *Let $a > 0$, $E = F \oplus G$ be a topological decomposition of E with $\dim F = p$, and $\pi : E \to F$ be the (continuous) projection onto F parallel to G. Then*

$$\underbrace{\{l \in E' \mid \|(\mathrm{Id} - \pi^\top)(l)\| < \frac{1}{a}\|\pi^\top(l)\|\}}_{\text{essentially } \mathcal{C}_{\pi^\top,1/a} \text{ with a strict inequality}} \subset (\mathcal{C}_{\pi,a})' \subset \mathcal{C}_{\pi^\top,1/a}.$$

In particular, $\overline{(\mathcal{C}_{\pi,a})'} = \mathcal{C}_{\pi^\top,1/a}$ is a p-cone of E'.

Proof. First, we show that $\{l \in E' \mid \|(\mathrm{Id} - \pi^\top)(l)\| < \frac{1}{a}\|\pi^\top(l)\|\} \subset (\mathcal{C}_{\pi,a})'$. Let V' be a p-dimensional subspace of $\{l \in E' \mid \|(\mathrm{Id} - \pi^\top)(l)\| < \frac{1}{a}\|\pi^\top(l)\|\}$ and $F' \coloneqq \mathcal{L}(F, \mathbb{C})$ be the dual space of F. The map

$$\mathrm{Restr}_F : \begin{array}{ccc} V' & \to & F' \\ l & \mapsto & l_{|F} \end{array}$$

is an isomorphism. Indeed, if $l \in V'$ verifies $l_{|F} = 0$, then $\|l \circ \pi\| = 0$, and as $l \in \mathcal{C}_{\pi^\top,1/a}$, $l = 0$. The considered map Restr_F is injective, and bijective as V' and F' have same dimension p.

Let $W \in \mathcal{G}_p(E)$ such that $W \subset \mathcal{C}_{\pi,a}$. By contradiction, let us assume there exists $x \in V'^\perp \cap W \setminus \{0\}$ and denote (x_F, x_G) its components with respect to $E = F \oplus G$. As $x \in \mathcal{C}_{\pi,a}$, we have $x_F \neq 0$, and we may assume for simplicity that $\|x_F\| = 1$. Then there exists $l_0 \in F'$ such that $\|l_0\| = 1$ and $l_0(x_F) = 1$. The map Restr_F is an isomorphism, so there exists $l \in V'$ such that $l_{|F} = l_0$; in particular, $l(x_F) = 1$ and $\|l \circ \pi\| = 1$.

Given that $x \in V^\perp$, we have $0 = l(x) = 1 + l(x_G)$, i.e. $l(x_G) = -1$ (and $x_G \neq 0$). It follows that

$$1 = |l(x_G)| \leq \|(\mathrm{Id} - \pi^\top)(l)\|.\|x_G\| < \frac{1}{a}\|\pi^\top(l)\|.\|x_G\| = \frac{1}{a}\|x_G\|.$$

This shows that $\|x_G\| > a = a\|x_F\|$, which is absurd since $x \in W \subset \mathcal{C}_{\pi,a}$. Therefore, we get that $V'^\perp \cap W = \{0\}$, and , as it holds for all $W \subset \mathcal{C}_{\pi,a}$, this proves that $V'^\perp \in GC(\mathcal{C}_{\pi,a})$. Hence $V' = (V'^\perp)^\perp$ is a subspace of $(\mathcal{C}_{\pi,a})'$, which achieves the proof of the first inclusion of our proposition.

Now we show the second inclusion of our proposition. Let $l \in (\mathcal{C}_{\pi,a})'$. By definition, there exists $V \in GC(\mathcal{C}_{\pi,a})$ such that $l \in V^\perp$. Let $x_G \in G$. As $V \oplus F = E$, there exists $x_V \in V$, $x_F \in F$ such that $x_G = x_V + x_F$. Then $x_G - x_F \in V$, and since $V \cap \mathcal{C}_{\pi,a} = \{0\}$, we get $a\|x_F\| < \|x_G\|$.

Moreover, $x_G - x_F \in V$ implies $l(x_G - x_F) = 0$, hence:

$$|l(x_G)| = |l(x_F)| \leq \|l \circ \pi\|.\|x_F\| \leq \frac{1}{a}\|l \circ \pi\|.\|x_G\|$$

This shows that $\|(\mathrm{Id} - \pi^\top)(l)\| \leq \frac{1}{a}\|\pi^\top(l)\|$, which means that $l \in \mathcal{C}_{\pi^\top,1/a}$.

□

Lemma 4.2.8 also allows us to show analyticity for the applications mapping a space to its orthogonal:

Proposition 4.2.10. *The maps $\perp : \mathcal{G}^p(E) \to \mathcal{G}_p(E')$ and $\perp : \mathcal{G}_p(E') \to \mathcal{G}^p(E)$ are reciprocal analytic maps.*

Proof. The fact that they are reciprocal maps is simply a reformulation of Lemma 4.1.5. One can also see that Corollary 4.1.7 implies that these are homeomorphisms.

Let $F \in \mathcal{G}^p(E)$ and G be a topological complement of F. Let $f \in \mathcal{L}(F,G)$ and denote $V := \operatorname{graph} f \in \mathcal{G}^p(E)$. Let denote π_F (resp. π_G) the projection onto F parallel to G (resp. onto G parallel to F). Let also $l \in E'$. With respect to the decomposition $E' = F^\perp \oplus G^\perp$, we have $l = l_{F^\perp} + l_{G^\perp} = l \circ \pi_G + l \circ \pi_F$. Then

$$\begin{aligned}
l \in V^\perp &\iff l \circ \pi_F + l \circ \pi_G \in V^\perp \\
&\iff \forall x \in F, l \circ \pi_F(x + f(x)) + l \circ \pi_G(x + f(x)) = 0 \\
&\iff \forall x \in F, l \circ \pi_F(x) + l \circ \pi_G(f(x)) = 0 \\
&\iff \forall x \in E, l \circ \pi_F(x) + l \circ \pi_G(f(\pi_F(x))) = 0 \\
&\iff \forall x \in E, l \circ \pi_F(x) = -(f \circ \pi_F)^\top (l \circ \pi_G)(x) \\
&\iff \forall x \in E, l_{G^\perp}(x) = -(f \circ \pi_F)^\top (l_{F^\perp})(x).
\end{aligned}$$

This proves $V^\perp = \operatorname{graph}(-(f \circ \pi_F)^\top : F^\perp \to G^\perp)$. The map

$$f \in \mathcal{L}(F,G) \mapsto (f \circ \pi_F)^\top \in \mathcal{L}(F^\perp, G^\perp)$$

is analytic, therefore the map $V \in \mathcal{G}^p(E) \mapsto V^\perp \in \mathcal{G}_p(E')$ is also analytic.

Let $F' \in \mathcal{G}_p(E')$ and G' be a topological complement of F' in E'. Setting $F = F'^\perp$ and $G = G'^\perp$, we get $F' = F^\perp$ and $G' = G^\perp$ and have a setting similar to the previous one. Let $g \in \mathcal{L}(F', G')$ and $V' = \operatorname{graph} g \in \mathcal{G}_p(E')$.

Let $(e_1, \dots, e_p)$ be a basis of G and consider the dual applications $e_i^* \in E'$ associated with the decomposition $E = \operatorname{Span}(e_1, \dots, e_p) \oplus F$. Note that $(e_1^*, \dots, e_p^*)$ is a basis of $F^\perp$. Let $x \in E$ and $x = x_G + x_F$ its decomposition with respect to $E = F \oplus G$. Then

$$\begin{aligned}
x \in V'^\perp &\Leftrightarrow \forall l \in V', l(x) = 0 \\
&\Leftrightarrow \forall l \in F^\perp, l(x) + g(l)(x) = 0 \\
&\Leftrightarrow \forall l \in F^\perp, l(x_G) + g(l)(x_F) = 0 \\
&\Leftrightarrow \forall i \in [\![1, p]\!], e_i^*(x_G) + g(e_i^*)(x_F) = 0 \\
&\Leftrightarrow x_G = -\sum_{i=1}^{p} g(e_i^*)(x_F) e_i.
\end{aligned}$$

Denote by $A(g)$ the linear map in $\mathcal{L}(F,G)$ defined by $A(g)(x_F) = -\sum_{i=1}^{p} -g(e_i^*)(x_F) e_i$. We get $V'^\perp = \operatorname{graph}(A(g))$. The map $g \in \mathcal{L}(F', G') \mapsto A(g) \in \mathcal{L}(F,G)$ is analytic, therefore the map $V' \in \mathcal{G}_p(E') \to V'^\perp \in \mathcal{G}^p(E)$ is analytic.

□

4.3 Applications of previous results to dual cones

In this section, we look at operators acting on cones and shows how it translates into transposed operators acting on the dual cones. We then apply results from Chapter 3 to transposed operators.

Lemma 4.3.1. *Let $\mathcal{C}_1$, $\mathcal{C}_2$ be p-cones. If $\mathcal{C}_1 \subset \mathcal{C}_2$, then $GC(\mathcal{C}_2) \subset GC(\mathcal{C}_1)$, $\tilde{\mathcal{C}}_2 \subset \tilde{\mathcal{C}}_1$, and $\mathcal{C}_2' \subset \mathcal{C}_1'$.*

Proof. It is a direct consequence of the definitions. □

Proposition 4.3.2. *Let $T : E \to E$ be a continuous linear map. Then*

(i) for all $V \in \mathcal{G}(E), (T^\top)^{-1}(V^\perp) = (T(V))^\perp$,

(ii) for all $V' \in \mathcal{G}(E'), T^{-1}(V'^\perp) = \left(T^\top(V')\right)^\perp$,

(iii) for all $V' \in \mathcal{G}(E'), (T^\top(V'))^\perp = T^{-1}(V'^\perp)$.

Proof. Let V be a closed subspace of E. Then

$$\begin{aligned} l \in (T^\top)^{-1}(V^\perp) &\Leftrightarrow l \circ T \in V^\perp \\ &\Leftrightarrow \forall x \in V, l(T(x)) = 0 \\ &\Leftrightarrow \forall y \in T(V), l(y) = 0 \\ &\Leftrightarrow l \in (T(V))^\perp \end{aligned}$$

which proves (i).

Let $x \in E$. Then

$$\begin{aligned} x \in T^{-1}(V'^\perp) &\Leftrightarrow T(x) \in V'^\perp \\ &\Leftrightarrow \forall l \in V', l(T(x)) = 0 \\ &\Leftrightarrow \forall l \in T^\top(V'), l(x) = 0 \\ &\Leftrightarrow x \in \left(T^\top(V')\right)^\perp. \end{aligned}$$

which ends the proof of (ii). Moreover, we have

$$\begin{aligned} x \in (T^\top(V'))^\perp &\Leftrightarrow \forall l \in T^\top(V'), l(x) = 0 \\ &\Leftrightarrow \forall l \in V', l(T(x)) = 0 \\ &\Leftrightarrow T(x) \in V'^\perp \\ &\Leftrightarrow x \in T^{-1}(V'^\perp), \end{aligned}$$

and this achieves the proof of (iii). □

Proposition 4.3.3. *Let $\mathcal{C}$ be a p-cone and $T : E \to E$ be a continuous linear map such that $\mathcal{C} \cap \ker T = \{0\}$. Then $(T(\mathcal{C}))' \subset (T^\top)^{-1}(\mathcal{C}')$. If T is surjective, then $(T(\mathcal{C}))' = (T^\top)^{-1}(\mathcal{C}')$.*

Proof. Consider $V' \in \mathcal{G}_p(E')$. Then

$$\begin{aligned} V' \subset (T\mathcal{C})' &\Leftrightarrow V'^\perp \in GC(T\mathcal{C}) && \text{by Proposition 4.1.8} \\ &\Leftrightarrow \forall W \in \mathcal{G}_p(\mathcal{C}), V'^\perp \oplus TW = E \\ &\Rightarrow \forall W \in \mathcal{G}_p(\mathcal{C}), T^{-1}(V'^\perp) \oplus W = E && \text{(this implication is proved below)} \\ &\Leftrightarrow \forall W \in \mathcal{G}_p(\mathcal{C}), (T^\top(V'))^\perp \oplus W = E && \text{by } (ii) \text{ of Proposition 4.3.2} \\ &\Leftrightarrow (T^\top(V'))^\perp \in GC(\mathcal{C}) \\ &\Leftrightarrow T^\top(V') \subset \mathcal{C}' && \text{by Proposition 4.1.8} \\ &\Leftrightarrow V' \subset (T^\top)^{-1}(\mathcal{C}'), \end{aligned}$$

which shows the desired results.

We now prove the implication noted earlier: assume that for all p-dimensional subspaces $W \subset \mathcal{C}$, we have $V'^{\perp} \oplus TW = E$. Let $W \subset \mathcal{C}$ be a p-dimensional subspace. The subspace $T^{-1}(V'^{\perp})$ is closed as T is continuous and W is closed as it is finite-dimensional. Let $x \in T^{-1}(V'^{\perp}) \cap W$. Then $T(x) \in V'^{\perp} \cap TW = \{0\}$, hence $x \in \ker T$. As $\mathcal{C} \cap \ker T = \{0\}$ and $x \in W \subset \mathcal{C}$, we get $x = 0$, i.e. $T^{-1}(V'^{\perp}) \cap W = \{0\}$.

Let $x \in E$. Then $T(x)$ can be decomposed in $T(x) = v + T(w)$ with $v \in V'^{\perp}$ and $w \in W$. Then $T(x-w) = v \in V'^{\perp}$, i.e. $x - w \in T^{-1}(V'^{\perp})$, hence $x \in T^{-1}(V'^{\perp}) + W$. This proves that $E = T^{-1}(V'^{\perp}) \oplus W$ is a topological decomposition. In particular, $T^{-1}(V'^{\perp}) = (T^{\top}(V'))^{\perp}$ is a subspace of codimension p, so it belongs to $GC(\mathcal{C})$.

If T is surjective, we now show the converse. Assume that for all p-dimensional subspaces $W \subset \mathcal{C}$, $T^{-1}(V'^{\perp}) \oplus W = E$. Let $W \subset \mathcal{C}$ be a p-dimensional subspace. Let $x \in V'^{\perp} \cap TW$. There exists $w \in W$ such that $x = T(w)$. As $x \in V'^{\perp}$, we get $w \in T^{-1}(V'^{\perp}) \cap W = \{0\}$, hence $x = 0$ and $V'^{\perp} \cap TW = \{0\}$.

Let $x \in E$. As T is surjective, there exists $y \in E$ such that $x = T(y)$. Let $y = z + w$ be a decomposition with respect to $T^{-1}(V'^{\perp}) \oplus W = E$; then $x = T(z) + T(w) \in V'^{\perp} + TW$. Hence $E = V'^{\perp} \oplus TW$, and it is a topological decomposition as both sets are closed. □

This allows us to obtain the following proposition:

Proposition 4.3.4. *Let $\mathcal{C}_1, \mathcal{C}_2$ be p-cones of E and $T : E \to E$ be a continuous linear map such that $T(\mathcal{C}_1^*) \subset \mathcal{C}_2^*$. Then $T^{\top}(\mathcal{C}_2'^*) \subset \mathcal{C}_1'^*$.*

Proof. By Lemma 4.3.1, we have $C_2' \subset (T\mathcal{C}_1)'$. By Proposition 4.3.3, we get the inclusion $(T\mathcal{C}_1)' \subset (T^{\top})^{-1}(\mathcal{C}_1')$, therefore $C_2' \subset (T^{\top})^{-1}(\mathcal{C}_1')$ and $T^{\top}(\mathcal{C}_2') \subset \mathcal{C}_1'$.

We now show that $\ker T^{\top} \cap \mathcal{C}_2' = \{0\}$. By contradiction, assume that there exists $l \in \mathcal{C}_2'$ such that $T^{\top}(l) = 0$ and $l \neq 0$. Let $W \in \mathcal{G}_p(E)$ be a subspace of $\mathcal{C}_1$. As $T(\mathcal{C}_1^*) \subset \mathcal{C}_2^*$, TW is a p-dimensional subspace of $\mathcal{C}_2$. As $l \circ T = 0$, we have $l_{|TW} = \{0\}$. Given that $l \in \mathcal{C}_2'$, there exists $V \in GC(\mathcal{C}_2)$ such that $l \in V^{\perp}$; we can write $V = \ker l \cap \bigcap_{i=2}^{p} \ker l_i$ where $(l, l_2, \ldots, l_p)$ are independent linear forms. Then

$$TW \cap V = TW \cap \ker l \cap \bigcap_{i=2}^{p} \ker l_i = \underbrace{TW}_{\dim = p} \cap \underbrace{\bigcap_{i=2}^{p} \ker l_i}_{\text{codim}=p-1},$$

hence $TW \cap V \neq \{0\}$, with is impossible as TW is a p-dimensional subspace of $\mathcal{C}_2$ and $V \in GC(\mathcal{C}_2)$. Therefore $\ker T^{\top} \cap \mathcal{C}_2' = \{0\}$ and $T^{\top}(\mathcal{C}_2'^*) \subset \mathcal{C}_1'^*$. □

This last proposition prompts us to extend our contraction principle stated in Theorem 3.2.3 to dual cones. Unfortunately, we run into two difficulties : the dual cones (or more importantly their closure) may not be p-cones, and the hypothesis of finite diameter for the cones may not translate into finite diameter for the dual cones. The former obstacle is avoided by only considering p-dimensional spaces contained in the dual cone, but the latter has yet to be solved.

Proposition 4.3.5. *Let E_1, E_2 be complex Banach spaces. Let $T : E_1 \to E_2$ a continuous linear map and $\mathcal{C}_1 \subset E_1$, $\mathcal{C}_2 \subset E_2$ two p-cones such that $T(\mathcal{C}_1^*) \subset \mathcal{C}_2^*$. Assume that $\mathcal{C}_1$ and*

$\mathcal{C}_2$ have non-empty interior in $\mathcal{G}_p(E_1)$ and $\mathcal{G}_p(E_2)$. Then we can define cone gauges on the closure of dual cones $\overline{\mathcal{C}_1'}$ and $\overline{\mathcal{C}_2'}$, and for every p-dimensional spaces V', W' contained in $\overline{\mathcal{C}_2'}$, we have $d_{\overline{\mathcal{C}_1'}}(T^\top V', T^\top W') \leq d_{\overline{\mathcal{C}_2'}}(V', W')$.

Proof. By Proposition 4.2.6, we know that $\overline{\mathcal{C}_1'}$ and $\overline{\mathcal{C}_2'}$ are closed and do not contain any $(p+1)$-dimensional spaces. This is enough to define the "distances" $d_{\overline{\mathcal{C}_1'}}$ and $d_{\overline{\mathcal{C}_2'}}$ as in Definition 3.2.2. Denote by δ the gauge of Rugh or the gauge of Dubois for proper complex cones (see Section 3.1.2).

Let $V', W' \subset \overline{\mathcal{C}_1'}$ be p-dimensional subspaces of E'. We define $d_{\overline{\mathcal{C}_1'}}$ as :

$$d_{\overline{\mathcal{C}_1'}}(V', W') = \sup_{(l,m)\in(V'^*, W'^*)} d_{1,\overline{\mathcal{C}_1'}}(l, m),$$

where $d_{1,\overline{\mathcal{C}_1'}}(l,m) = 0$ if $\mathrm{Span}(l,m) \subset \overline{\mathcal{C}_1'}$, and else $d_{1,\overline{\mathcal{C}_1'}}(l,m) = \delta_{\overline{\mathcal{C}_1'}\cap \mathrm{Span}(l,m)}(l,m)$. This last one is well-defined since $\overline{\mathcal{C}_1'} \cap \mathrm{Span}(l,m)$ is a proper complex cone (of dimension 1): it is non-empty closed, does not contain any 2-dimensional space and is stable by scalar multiplication (this last property holds for $\overline{\mathcal{C}_1'}$ since it is satisfied by $\mathcal{C}_1'$).

For the same reason as in Definition 3.2.2, if $V' \neq W'$, then $d_{\overline{\mathcal{C}_1'}}(V', W') > 0$. By Proposition 4.3.4, we have $T^\top(\mathcal{C}_2'^*) \subset \mathcal{C}_1'^*$. We can now follow the exact same proof of Theorem 3.2.3, applied to $\overline{\mathcal{C}_2'}$, $\overline{\mathcal{C}_1'}$ and operator $T^\top$, to get $d_{\overline{\mathcal{C}_1'}}(T^\top V', T^\top W') \leq d_{\overline{\mathcal{C}_2'}}(V', W')$. □

In the present setting, a strict contraction principle would require the implication: if $\mathrm{diam}_{\mathcal{C}_2}(T\mathcal{C}_1) < \infty$, then $\mathrm{diam}_{\overline{\mathcal{C}_1'}}(T^\top \overline{\mathcal{C}_2'}) < \infty$. This remains an open problem. Of course, in the case of cones associated with projections, we can get those results:

Proposition 4.3.6. *Let $E = F \oplus G$ be a topological decomposition with $\dim F = p$ and let π be the projection onto F parallel to G. Let $0 < a < b$. Let $T : E \to E$ a continuous linear map such that $T(\mathcal{C}_{\pi,b}^*) \subset \mathcal{C}_{\pi,a}^*$. Then for all p-dimensional spaces V, W contained in $\mathcal{C}_{\pi,b}$, we have*

$$d_{\mathcal{C}_{\pi,b}}(TV, TW) \leq \frac{a}{b}.d_{\mathcal{C}_{\pi,b}}(V, W),$$

and for all p-dimensional spaces V', W' contained in $\mathcal{C}_{\pi^\top,1/a} \subset E'$, we have

$$d_{\mathcal{C}_{\pi,1/a}}(T^\top V', T^\top W') \leq \frac{a}{b}.d_{\mathcal{C}_{\pi,1/a}}(V', W').$$

Proof. The first inequality is the one from the example following Theorem 3.2.3. Using Propositions 4.3.4 and 4.2.9, we get that $T^\top(\mathcal{C}_{\pi^\top,1/a}^*) \subset \mathcal{C}_{\pi^\top,1/b}^*$, and applying the same result to E', $T^\top$ and $\pi^\top$, we get for all p-dimensional spaces V', W' contained in $\mathcal{C}_{\pi^\top,1/a} \subset E'$:

$$d_{\mathcal{C}_{\pi,1/a}}(T^\top V', T^\top W') \leq \frac{1/b}{1/a}.d_{\mathcal{C}_{\pi,1/a}}(V', W') = \frac{a}{b}.d_{\mathcal{C}_{\pi,1/a}}(V', W'). \qquad \square$$

This reasoning can also apply to get a spectral gap theorem for $T^\top$:

Theorem 4.3.7. *Under the hypotheses of Proposition 4.3.6, $T^\top$ has a spectral gap, i.e. there exists a p-dimensional subspace V' of E' and a closed complement W' of V', both stable by $T^\top$, such that $\sup |Sp(T^\top_{|W'})| \leq \frac{a}{b} \inf |Sp(T^\top_{|V'})|$.*

Proof. We apply Theorem 3.3.10 to E' and $T^\top$, as $T^\top(\mathcal{C}_{\pi^\top,1/a}^*) \subset \mathcal{C}_{\pi^\top,1/b}^*$ and $\mathcal{C}_{\pi^\top,1/a}$ has bounded aperture and is reproducing. □

4.4 Dominated splitting for random products of operators contracting a cone

Even if the dual cones gives us results on the operator $T^\top$ on E', this is not where their true usefulness resides. The main point of using dual cones is to obtain a dominated splitting for T or for a random product of operators, i.e. to decompose E into a "fast space" (the one corresponding to the p largest Lyapunov exponents) and a "slow space" with an estimate of the speed ratio on these two subspaces.

Moreover, under assumptions similar to the ones in Theorem 3.4.5, we obtain analytic regularity for this dominated splitting. As we do not know whether finite diameter for p-cones implies finite-diameter for the associated dual cones and whether the dual cones are p-cones, we have to consider cones associated with projections (for which we know this fact) to state the following theorem. However, since these two problems are the only issue, we do the proof of Theorem 4.4.3 with the most general tools possible. We offer at the end of this chapter potential ways to get around these obstacles.

Definition 4.4.1. *Let $\pi \in \mathcal{L}(E)$ be a continuous projection of E onto a p-dimensional space, and $0 < a < b$. We define $\mathscr{M} := \{M \in \mathcal{L}(E) \mid M(\mathcal{C}^*_{\pi,b}) \subset \mathcal{C}^*_{\pi,a}\}$.*

We denote by Δ the diameter of the cone $\mathcal{C}_{\pi,a}$ in the cone $\mathcal{C}_{\pi,b}$, by η the contraction constant associated with Δ, by $m : E \to \mathbb{C}^p$ a linear map of norm 1 and $K > 0$ the constant making $\mathcal{C}_{\pi,b}$ a cone with aperture bounded by K. In order to keep notations similar to Definition 3.4.4, we set $\rho > 0$ such that $\mathcal{C}_{\pi,a} \subset \mathcal{C}_{\pi,b}[\rho]$. In particular, we have $M(\mathcal{C}_{\pi,b}) \subset \mathcal{C}_{\pi,b}[\rho]$ for all $M \in \mathscr{M}$.

We recall the notations we used to state Theorem 3.4.5. We consider here (Ω, μ) a probability space and $\tau : \Omega \to \Omega$ an *invertible* measurable μ-ergodic application. Given a Banach space F, we denote by $\mathscr{B}(\Omega, F)$ the Banach space of (μ-essentially) bounded mesurable maps equipped with the (μ-essentially) uniform norm on F. If A is a subset of F, we denote by $\mathscr{E}(\Omega, A)$ the set of Bochner-measurable maps from Ω into A. For $(M_\omega)_{\omega\in\Omega} \in \mathscr{E}(\Omega, \mathscr{M})$, we denote by $M^n_\omega := M_{\tau^{n-1}\omega} \cdots M_\omega$ the product of operators along the orbit of $\omega \in \Omega$. Note the difference of order when comparing to Theorem 3.4.5, and that this theorem and its proof still applies as τ in invertible. We specify how in the proof of Theorem 4.4.3.

Definition 4.4.2. *Let $(M_\omega)_{\omega\in\Omega} \in \mathscr{E}(\Omega, \mathscr{M})$. We say that a Bochner-measurable map $\omega \mapsto (V_\omega, W_\omega) \in \mathcal{G}_p(E) \times \mathcal{G}^p(E)$ is a dominated splitting of ratio $\alpha \in]0,1[$ if there exists $C > 0$ such that:*

$$\forall \omega \in \Omega,\ E = V_\omega \oplus W_\omega\ ,\ M_\omega(V_\omega) = V_{\tau\omega}\ ,\ M_\omega(W_\omega) \subset W_{\tau\omega}\ ;$$

$$\forall \omega \in \Omega, \forall n \in \mathbb{N}^*, \sup_{\substack{v\in V_\omega, w\in W_\omega \\ \|v\|=\|w\|=1}} \frac{\|M^n_\omega(w)\|}{\|M^n_\omega(v)\|} \leq C\alpha^n.$$

Theorem 4.4.3. *Let $t \in \mathbb{D} \mapsto M(t) \in \mathscr{E}(\Omega, \mathscr{M})$ be a map such that*

1. *$(t, \omega) \in \mathbb{D} \times \Omega \mapsto M_\omega(t) \in \mathcal{L}(E)$ is measurable and the map $t \mapsto M_\omega(t)$ is analytic for all $\omega \in \Omega$.*

2. $\sup\left\{\dfrac{\|\bigwedge^{p-1} M_\omega(t)\|}{\|\bigwedge^{p} M_\omega(t)\|}\|\frac{d}{dt}M_\omega(t)\| : \omega\in\Omega, t\in\mathbb{D}\right\}<\infty.$

3. $\int_\Omega |\log\|M_\omega(0)\|| < \infty$.

Then there exists a map $(t,\omega)\in\mathbb{D}\times\Omega\mapsto(V_\omega(t),W_\omega(t))\in\mathcal{G}_p(E)\times\mathcal{G}^p(E)$, *measurable in* ω *and analytic in* t, *such that* $\omega\mapsto(V_\omega(t),W_\omega(t))$ *is a dominated splitting of* E *with ratio* $\eta=\frac{a}{b}$ *for all* $t\in\mathbb{D}$.

Proof. First, we get our "fast space" from the fixed p-dimensional space we have built in the proof of Theorem 3.4.5. We have considered the map

$$\pi_t : \left\{\begin{array}{ccc} \mathscr{E}(\Omega,\mathcal{G}_p(\mathcal{C})) & \to & \mathscr{E}(\Omega,\mathcal{G}_p(\mathcal{C})) \\ (V_\omega)_{\omega\in\Omega} & \mapsto & \pi_t(V) : \omega\mapsto \pi_{M_\omega(t)}(V_{\tau\omega}) = M_\omega(t)(V_{\tau\omega}) \end{array}\right.$$

and have shown it has a fixed point. As we reversed the order of the product in the definition of M^n used in Theorem 3.4.5, the π_t we consider here is defined by $\pi_t((V_\omega)_{\omega\in\Omega}) : \omega\mapsto M_{\tau^{-1}\omega}(t)(V_{\tau^{-1}\omega})$. Like in Lemma 3.4.14, for $t\in\mathbb{D}$, π_t has a unique fixed point $V(t)$ in $\mathscr{E}(\Omega,\mathcal{G}_p(\mathcal{C}_{\pi,b}))$, meaning that:

$$\forall\omega\in\Omega, M_\omega(t)(V_\omega(t)) = V_{\tau\omega}(t).$$

Like in Lemma 3.4.18, the map $t\mapsto V(t)$ is analytic. We show later that $V(t)$ is the fast space of our dominated splitting.

We now aim to construct a subspace $W(t)$ wich acts as the slow space for our dominated splitting. To achieve this, we find a fast space in E' for the transposed operators and the dual cone, and take its orthogonal to get the slow space in E.

Let $\mathscr{M}^\top := \{M^\top \mid M\in\mathscr{M}\}$. Then we have $\mathscr{M}^\top\subset\{A\in\mathcal{L}(E')\mid A(\mathcal{C}^*_{\pi,1/a})\subset\mathcal{C}^*_{\pi,1/b}\}$. The set $\mathscr{M}^\top$ is a subset of $\mathcal{L}(E')$ satisfying Definition 3.4.4 (for cones in E'). Thus we apply the results obtained in the proof of Theorem 3.4.5 to the map $t\mapsto M(t)^\top$, which first requires verifying the assumptions of this theorem:

1. As $M\in\mathcal{L}(E)\mapsto M^\top\in\mathcal{L}(E')$ is analytic and measurable, Assumption 1 in Theorem 4.4.3 shows that $(t,\omega)\in\mathbb{D}\times\Omega\mapsto M_\omega(t)^\top\in\mathcal{L}(E)$ is measurable and the map $t\mapsto M_\omega(t)^\top$ is analytic for all $\omega\in\Omega$.

2. For every Banach space F, $M\in\mathcal{L}(F)$ and $k\in\mathbb{N}^*$, we have $\bigwedge^k(M^\top) = \left(\bigwedge^k M\right)^\top$ and $\|M^\top\| = \|M\|$. Since the transposition commutes with $\frac{d}{dt}$, Assumption 1 in Theorem 4.4.3 shows that $\sup\left\{\dfrac{\|\bigwedge^{p-1} M_\omega(t)^\top\|}{\|\bigwedge^{p} M_\omega(t)^\top\|}\|\frac{d}{dt}M_\omega(t)^\top\| : \omega\in\Omega, t\in\mathbb{D}\right\}<\infty.$

3. As $\|M_\omega^\top\| = \|M_\omega\|$ for all $\omega\in\Omega$, we have $\int_\Omega \left|\log\|M_\omega(0)^\top\|\right| < \infty$.

All the conditions of Theorem 3.4.5 are satisfied for the cocycle of transposed operators. As before, there exists an analytic map $W'(t)$ in $\mathscr{E}(\Omega,\mathcal{G}_p(\mathcal{C}_{\pi^\top,1/a}))\subset\mathscr{E}(\Omega,E')$ such that:

$$\forall\omega\in\Omega, M_\omega(t)^\top(W'_\omega(t)) = W'_{\tau\omega}(t).$$

Let $W_\omega(t) := W'_{\tau\omega}(t)^\perp$. As the map $W' \in \mathcal{G}_p(E') \mapsto W'^\perp \in \mathcal{G}^p(E)$ is analytic by Proposition 4.2.10, the map $(t,\omega) \mapsto W_\omega(t)$ is Bochner-measurable, and analytic in t. The following computations give the inclusion needed with respect to $(M_\omega(t))$. Let $\omega \in \Omega$, $t \in \mathbb{D}$. Using Lemma 4.1.5, we have

$$\begin{array}{rcll} W'_{\tau\omega}(t) & = & M_\omega(t)^\top(W'_\omega(t)) & \\ W_{\tau\omega}(t)^\perp & = & M_\omega(t)^\top(W_\omega(t)^\perp) & \\ W_{\tau\omega}(t) & = & \left(M_\omega(t)^\top(W_\omega(t)^\perp)\right)^\perp & \\ W_{\tau\omega}(t) & = & M_\omega(t)^{-1}\left((W_\omega(t)^\perp)^\perp\right) & \text{By Lemma 4.3.2} \\ W_{\tau\omega}(t) & = & M_\omega(t)^{-1}(W_\omega(t)), & \end{array}$$

hence $M_\omega(t)(W_{\tau\omega}(t)) \subset W_\omega(t)$. Finally, since $W'_\omega(t) \subset C_{\pi^\top,1/a} \subset (C_{\pi,b})'$, Lemma 4.1.8 shows that $W_\omega(t) \in GC(\mathcal{C}_{\pi,b})$. As $V_\omega(t) \subset \mathcal{C}_{\pi,b}$, we have $E = V_\omega(t) \oplus W_\omega(t)$ for all $(t,\omega) \in \mathbb{D} \times \Omega$.

To show that this decomposition is a dominated splitting, it remains to estimate the speed ratio between the two subspaces. Since the following reasoning does not depend on the parameter t, we omit it for the rest of the proof. In order to lighten the notations, we also set $\mathcal{C} := \mathcal{C}_{\pi,b}$ (and therefore $\mathcal{C}[\rho] = \mathcal{C}_{\pi,b}[\rho]$).

Let $n \in \mathbb{N}^*$ and $\omega \in \Omega$. Let $v \in V_\omega$ and $w \in W_\omega$ with $\|v\| = \|w\| = 1$. Let $x = v + \frac{\rho^2}{2}w$. Denoting P the projection onto V_ω parallel to W_ω, we have $x \in \mathcal{C}_{P,\frac{\rho^2}{2}}$. As $V_\omega \subset \mathcal{C}[\rho]$, we get from Lemma 5.4.9 of the Appendix the inclusion $\mathcal{C}_{P,\frac{\rho^2}{2}} \subset \mathcal{C}$. Let $V^x \subset \mathcal{C}_{P,\frac{\rho^2}{2}} \subset \mathcal{C}$ be a p-dimensional subspace containing x. By our contraction principle from Theorem 3.2.3,

$$d_\mathcal{C}(M^n_\omega V^x, M^n_\omega V_\omega) \leq \eta^n d_\mathcal{C}(V^x, V_\omega) \leq \Delta\eta^n$$

Using an approach similar to Proposition 3.3.6, let $x_1 \wedge \cdots \wedge x_p$ be a representative of $M^n_\omega V^x$ of norm 1, and $y_1 \wedge \cdots \wedge y_p$ be a representative of $M^n_\omega V_\omega$ of norm 1. By Lemma 3.3.5,

$$\left\| \frac{x_1 \wedge \ldots \wedge x_p}{\hat{m}(x_1 \wedge \ldots \wedge x_p)} - \frac{y_1 \wedge \ldots \wedge y_p}{\hat{m}(y_1 \wedge \ldots \wedge y_p)} \right\| \leq a_p K^p d_\mathcal{C}(M^n_\omega V^x, M^n_\omega V_\omega) \leq a_p K^p \Delta\eta^n.$$

Then, as $\|\hat{m}\| = 1$ and $\|x_1 \wedge \cdots \wedge x_p\| = 1$,

$$\left\| x_1 \wedge \ldots \wedge x_p - \frac{\hat{m}(x_1 \wedge \ldots \wedge x_p) y_1 \wedge \ldots \wedge y_p}{\hat{m}(y_1 \wedge \ldots \wedge y_p)} \right\| \leq a_p K^p \Delta\eta^n,$$

therefore $d(x_1 \wedge \ldots \wedge x_p, \mathrm{Span}(y_1 \wedge \ldots \wedge y_p)) \leq a_p K^p \Delta\eta^n$. By Proposition 2.2.14 and an argument of symmetry, we have $d_\wedge(M^n_\omega V^x, M^n_\omega V_\omega) \leq 2a_p K^p \Delta\eta^n$. Using Theorem 2.2.15, we get some constant $C > 0$, depending only on p, K and Δ, such that

$$d_H(M^n_\omega V^x, V_{\tau^n\omega}) \leq C\eta^n.$$

This shows that $d\left(\frac{M^n_\omega(x)}{\|M^n_\omega(x)\|}, V_{\tau^n\omega}\right) \leq C\eta^n$. As $x = v + \frac{\rho^2}{2}w \in V_\omega \oplus W_\omega$, we have $M^n_\omega(x) = M^n_\omega(v) + \frac{\rho^2}{2}M^n_\omega(w) \in V_{\tau^n\omega} \oplus W_{\tau^n\omega}$. Hence

$$d\left(\frac{\rho^2 M^n_\omega(w)}{2\|M^n_\omega(x)\|}, V_{\tau^n\omega}\right) \leq C\eta^n,$$

i.e.

$$d\left(\underbrace{\frac{M_\omega^n(w)}{\|M_\omega^n(w)\|}}_{\in W_{\tau^n\omega}}, V_{\tau^n\omega}\right) \le C\frac{2\|M_\omega^n(x)\|}{\rho^2\|M_\omega^n(w)\|}\eta^n$$

Since $V_{\tau^n\omega} \subset \mathcal{C}[\rho]$ and $W_{\tau^n\omega} \subset GC(\mathcal{C})$, we have $\sin\theta(V_{\tau^n\omega}, W_{\tau^n\omega}) \ge \rho$ by Lemma 5.4.9 of the Appendix. We deduce that $\sin\theta(W_{\tau^n\omega}, V_{\tau^n\omega}) \ge \rho/2$ by Lemma 2.1.6, hence

$$\frac{\rho}{2} \le C\frac{2\|M_\omega^n(x)\|}{\rho^2\|M_\omega^n(w)\|}\eta^n$$

Since $M_\omega^n V^x$ is contained in $\mathcal{C}[\rho]$ and $W_{\tau^n\omega} \subset GC(\mathcal{C})$, we get $\sin\theta(M_\omega^n V^x, W_{\tau^n\omega}) \ge \rho$ by Lemma 5.4.9 of the Appendix. Thus we have $d(M_\omega^n(x), W_{\tau^n\omega}) \ge \rho\|M_\omega^n(x)\|$. This gives $d(M_\omega^n(v), W_{\tau^n\omega}) \ge \rho\|M_\omega^n(x)\|$, showing that $\|M_\omega^n(v)\| \ge \rho\|M_\omega^n(x)\|$. Finally, we obtain the following inequality:

$$\frac{\rho}{2} \le C\frac{2\|M_\omega^n(v)\|}{\rho^3\|M_\omega^n(w)\|}\eta^n$$

i.e.

$$\frac{\|M_\omega^n(w)\|}{\|M_\omega^n(v)\|} \le \frac{\rho^5}{4}C\eta^n.$$

Thus the decomposition $E = V_\omega \oplus W_\omega$ is a dominated splitting, with speed ratio η. $\square$

4.4.1 Discussion overcoming obstacles to generalization

The key point of the proof of Theorem 4.4.3 is to apply the results from Theorem 3.4.5 to the transposed operators acting on dual cones. But in the general case, the dual cones may not be p-cones. Moreover, finiteness of the diameter $d_{\mathcal{C}}(M(\mathcal{C}))$ does not ensure finiteness of the diameter for the corresponding dual cones. To generalize Theorem 4.4.3 to sets $\mathscr{M}$ defined as in Definition 3.4.4, we would also need to prove that the inclusion $M(\mathcal{C}) \subset \mathcal{C}[\rho]$ gives rise to a similar inclusion for the dual cones.

The next propositions partially address the latter two obstacles.

Proposition 4.4.4. *Let $\mathcal{C}$ be a p-cone and $0 < \rho < 1$ such that $\mathcal{C}[\rho]$ contains a p-dimensional space. We define the dual $\mathcal{C}[\rho]'$ of $\mathcal{C}[\rho]$ as in Definition 4.1.2, and by extension of our previous notations, we write:*

$$(\mathcal{C}[\rho'])\,[\rho/8] \coloneqq \{l \in E' \mid B(l, \frac{\rho}{8}\|l\|) \subset \mathcal{C}[\rho]'\}.$$

Then $\mathcal{C}' \subset (\mathcal{C}[\rho]')\,[\rho/8]$.

Proof. Let $V' \subset \mathcal{C}'$ be a p-dimensional subspace of $\mathcal{C}'$. Consider $V'' \in \mathcal{G}_p(E')$ such that the gap between V' and V'' satisfies $\delta(V', V'') < \rho/2$. Then, by Corollary 4.1.7, we get $\delta(V'^{\perp}, V''^{\perp}) < \rho/2$.

Let $W \subset \mathcal{C}[\rho]$ be a p-dimensional space. By contradiction, assume that there exists $x \in V''^{\perp} \cap W$ with $\|x\| = 1$. Then $d(x, V'^{\perp}) < \rho/2$, so there exists $v \in V'^{\perp}$ such that

$\|x - v\| < \rho/2$. So $v \in B(x, \rho)$, and as $x \in W \subset \mathcal{C}[\rho]$, we have $v \in \mathcal{C}$. This is impossible by Definition 4.1.2, as $V' \subset \mathcal{C}'$.

Therefore $V''^{\perp} \cap W = \{0\}$ and $E = V''^{\perp} \oplus W$ is a topological decomposition. As this holds for all $W \subset \mathcal{C}[\rho]$, we get $V'' \in \mathcal{C}[\rho]'$.

Now, let $l \in \mathcal{C}'$ and $m \in E'$ such that $\|l - m\| < (\rho/8)\|l\|$. By definition of the dual cone, there exists a p-dimensional space $V' \subset \mathcal{C}'$ containing l. By Proposition 5.4.7 of the Appendix, there exists a p-dimensional subspace V'' of E' containing m such that $\delta(V', V'') < \rho/2$. Then by the previous argument we have $V'' \in \mathcal{C}[\rho]'$, hence $m \in \mathcal{C}[\rho]'$. This shows that $B(l, \frac{\rho}{8}\|l\|) \subset \mathcal{C}[\rho]'$, i.e. $l \in (\mathcal{C}[\rho]')\,[\rho/8]$. □

Proposition 4.4.5. *Let $\mathcal{C}$ be a closed subset of E, stable by scalar multiplication, and not containing any $p+1$-dimensional subspace. Let $0 < \rho < 1$. Using the gauge of Dubois (see Definition 3.1.3), we define a gauge $d_{\mathcal{C}}$ on the set $\mathcal{G}_p(\mathcal{C})$ of p-dimensional subspaces of $\mathcal{C}$ as in Definition 3.2.2. Then*

$$\operatorname{diam}_{\mathcal{C}}(\mathcal{C}[\rho]) \leq 2\log(1/\rho),$$

where $\operatorname{diam}_{\mathcal{C}}(\mathcal{C}[\rho]) = \sup\{d_{\mathcal{C}}(V, W) \mid V, W \in \mathcal{G}_p(\mathcal{C}[\rho])\}$. *In particular, this holds for the closure of dual cones of p-cones with non-empty interior in $\mathcal{G}_p(E)$.*

Proof. Let $V, W \in \mathcal{G}_p(\mathcal{C})$ and consider x, y non-zero elements of $\mathcal{C}[\rho]$. The gauge of Dubois is projective, so we can assume $\|x\| = \|y\| = 1$. Let $z \in \mathbb{C}$ with $|z| < \rho$. Then $zx - y \in B(y, \rho)$ hence $zx - y \in \mathcal{C}$. Similarly, if $|z| > 1/\rho$, $zx - y \in \mathcal{C}$. Hence $\inf\{|z| \mid zx - y \notin \mathcal{C}\} \geq \rho$ and $\sup\{|z| \mid zx - y \notin \mathcal{C}\} \leq 1/\rho$, and $d_{1,\mathcal{C}}(x, y) \leq \log \frac{1/\rho}{\rho} = 2\log(1/\rho)$.

This holds for all $x, y \in \mathcal{C}[\rho]^*$, so we get $d_{\mathcal{C}}(V, W) \leq 2\log(1/\rho)$, which achieves the proof. □

As the sets $\mathcal{C}[\rho]'$ and $\mathcal{C}'$ are not p-cones, this makes a potential proof of Theorem 4.4.3 in the general case very tedious. We believe this obstacle may be overcome by studying in detail the sets of p-dimensional subspaces contained in $\mathcal{C}[\rho]'$ and $\mathcal{C}'$.

Chapter 5

Appendix

5.1 Topology of the Grassmannian

In this section, we prove the results stated in Section 2.1 of Chapter 2.

Proposition 5.1.1. *The application d_H defines a metric on $\mathcal{G}(E)$.*

Proof. The symmetry and positivity of d_H is immediate. Let $V, W \in \mathcal{G}(E)$ such that $d_H(V,W) = 0$. Let $x \in S_V$. Then $d(x, S_W) = 0$, and as W is a closed subspace of E, we have $x \in S_W$. Thus $S_V \subset S_W$ and similarly $S_W \subset S_V$, hence $S_W = S_V$ and $V = W$.

Let $U, V, W \in \mathcal{G}(E)$. Let $\varepsilon > 0$. Let $x \in S_U$. There exists a vector $y \in S_V$ such that $\|x - y\| \leq d_H(U,V) + \varepsilon$. Then there exists $z \in S_W$ such that $\|y - z\| \leq d_H(V,W) + \varepsilon$. Thus $\|x - z\| \leq d_H(U,V) + d_H(V,W) + 2\varepsilon$ and we get $d(x, S_W) \leq d_H(U,V) + d_H(V,W) + 2\varepsilon$. Similarly, for all $z \in S_W$, $d(z, S_U) \leq d_H(U,V) + d_H(V,W) + 2\varepsilon$. This proves

$$d_H(U,W) \leq d_H(U,V) + d_H(V,W) + 2\varepsilon,$$

and the triangle inequality follows as $\varepsilon \to 0$.

$\square$

Lemma 2.1.4 ([Kat95], Chapter IV, Section 2.1)**.** *Let $x \in E$ with $\|x\| = 1$ and V, W be elements of $\mathcal{G}(E)$. Then $d(x,V) \leq d(x, S_V) \leq 2d(x,V)$ and*

$$\max(\delta(V,W), \delta(W,V)) \leq d_H(V,W) \leq 2\max(\delta(V,W), \delta(W,V)).$$

Proof. The inequality $d(x,V) \leq d(x,S_V)$ is immediate. Let $\varepsilon > 0$ and set $d := d(x,V)$ and $\varepsilon > 0$. There exists $y \in V$ such that $\|x - y\| \leq d + \varepsilon$. By slightly changing y if we must, we can assume $y \neq 0$. Let $y_0 = \frac{y}{\|y\|}$. If $\|y\| \leq 1$, we have:

$$\|y_0 - y\| = \|y_0\| - \|y\| = 1 - \|x - (x - y)\| \leq 1 - (1 - (d + \varepsilon)) = d + \varepsilon$$

If $\|y\| \geq 1$, we have:

$$\|y_0 - y\| = \|y\| - \|y_0\| = \|x - (x - y)\| - 1 \leq 1 + d + \varepsilon - 1 = d + \varepsilon$$

Thus, in all cases, $\|x - y_0\| \leq \|x - y\| + \|y - y_0\| \leq 2d + 2\varepsilon$, hence $d(x, S_V) \leq 2d + 2\varepsilon$. Letting $\varepsilon \to 0$, we get $d(x, S_V) \leq 2d$.

Applied to all $x \in S_W$, we get

$$\delta(W,V) \leq d_H(V,W) \leq 2\delta(W,V).$$

By symmetry, we obtain the desired inequalities. $\square$

Proposition 5.1.2. *Let E be a Banach Space. Then $(\mathcal{G}(E), d_H)$ is a complete metric space.*

Proof. Let $(V_n)_{n\in\mathbb{N}} \in (\mathcal{G}(E))^{\mathbb{N}}$ be a Cauchy sequence for d_H. Let

$$V := \left\{ x \in E \mid \text{There exists } (x_n)_{n\in\mathbb{N}} \text{ with } \forall n \in \mathbb{N}, x_n \in V_n \text{ and } x_n \underset{n\to\infty}{\longrightarrow} x \right\}$$

It is clear that V is a subspace of E. Let us show that $d_H(V_n, V) \to 0$ (here we extend the definition of d_H to all subspaces). Let $\varepsilon > 0$. There exists an increasing sequence of indices $(n_i)_{i\geq 0}$ such that for all $i, m, n \in \mathbb{N}$, we have $d_H(V_m, V_n) \leq \frac{\min(\varepsilon,1)}{2^{i+1}}$ when $m, n \geq n_i$. Let $N \geq n_0$ and $y \in S_{V_N}$. Then there exists $x_{n_0} \in S_{V_{n_0}}$ such that $\|y - x_{n_0}\| \leq \varepsilon$. Given the definition of the n_i, there exists $x_{n_0+1}, \dots, x_{n_1} \in S_{V_{n_0+1}}, \dots, S_{V_{n_1}}$ such that $\forall k \in [\![n_0+1, n_1]\!], \|x_k - x_{n_0}\| \leq \varepsilon$. Iterating this process, we build a sequence (x_k) such that for all $i \geq 0$ and for all $k \in [\![n_i + 1, n_{i+1}]\!], x_k \in V_k$ and $\|x_k - x_{n_i}\| \leq \frac{\varepsilon}{2^i}$.

We now show that it is a Cauchy sequence in E. For all $i \geq 0$, for all $m, n \geq n_i$, defining k_1, k_2 by $n \in [\![n_{k_1} + 1, n_{k_1+1}]\!]$ and $m \in [\![n_{k_2} + 1, n_{k_2+1}]\!]$, we have :

$$\begin{aligned}
\|x_m - x_n\| &\leq \|x_n - x_{n_{k_1}}\| + \|x_m - x_{n_{k_2}}\| + \sum_{k=k_1}^{k_2} \|x_{n_k} - x_{n_{k+1}}\| \\
&\leq \tfrac{\varepsilon}{2^i} + \tfrac{\varepsilon}{2^i} + \sum_{k=i}^{\infty} \tfrac{\varepsilon}{2^k} && \text{as } k_1, k_2 \geq i \\
&\leq \tfrac{4\varepsilon}{2^i} && (*)
\end{aligned}$$

Hence (x_k) is a Cauchy sequence in E and therefore there exists $x \in E$ such that $x_k \to x$. Since $\|x_k\| = 1$ for all k, we get $\|x\| = 1$, hence $x \in S_V$. Letting $m = n_0$ and $n \to \infty$ in the previous inequality, we get $\|x_{n_0} - x\| \leq 4\varepsilon$, so $\|y - x\| \leq 5\varepsilon$ (where $y \in S_{V_{n_0}}$ is the element from which we built our sequence). Hence

$$\sup_{y \in S_{V_N}} d(y, S_V) \leq 5\varepsilon \tag{$**$}$$

and this holds for all $N \geq n_0$.

Let $x \in S_V$. By definition of V, there exists a sequence $(x_k)_{k\geq 0}$ with $x_k \in V_k$ for all $k \geq 0$ such that $x_k \to x$. By normalizing the x_k, we can assume $\|x_k\| = 1$. As $x_k \to x$, there exists a $k_0 \geq n_0$ such that $\|x - x_{k_0}\| \leq \varepsilon$. Let $N \geq n_0$. As $d_H(V_N, V_{k_0}) \leq \varepsilon$ (by definition of n_0), there exists $y \in S_{V_N}$ such that $\|x_{k_0} - y\| \leq 2\varepsilon$. Then $\|x - y\| \leq \varepsilon + 2\varepsilon \leq 3\varepsilon$, and therefore $d(x, S_{V_N}) \leq 3\varepsilon$.

This gives us : $\forall N \geq n_0, \sup\limits_{x\in S_V} d(x, S_{V_N}) \leq 3\varepsilon$. Combining this with $(**)$ we get

$$\forall N \geq n_0, d_H(V_N, V) \leq 5\varepsilon.$$

The last point left to show is that V is closed. Let $(y_k)_{k\in\mathbb{N}} \in V^{\mathbb{N}}$ such that $y_k \to y \in E$. If $y = 0$, $y \in V$. Otherwise, we can normalize y and the y_k and suppose $\|y\| = \|y_k\| = 1$.

We must build a sequence (x_k) with $x_k \in V_k$ and $x_k \to y$. Let (n_i) be, as before, a sequence of indices such that $\forall m, n \geq n_i, d_H(W_n, W_m) \leq \frac{1}{2^i}$. We construct the sequence (x_k) step by step, for $k \in [\![n_i, n_{i+1} - 1]\!]$. For $i \in \mathbb{N}$, there exists $p \in \mathbb{N}$ such that $\|y_p - y\| \leq \frac{1}{2^i}$. Using the reasoning we did to get $(*)$, there exists $x_{n_i} \in S_{W_{n_i}}, \ldots, x_{n_{i+1}-1} \in S_{W_{n_{i+1}-1}}$ such that $\forall k \in [\![n_i, n_{i+1} - 1]\!], \|y_p - x_k\| \leq \frac{4}{2^i}$, and then we get:

$$\forall k \in [\![n_i, n_{i+1} - 1]\!], \|y - x_k\| \leq \frac{5}{2^i}.$$

The sequence $(x_k)_{k\geq 0}$ then satisfies $\forall k \geq n_i, \|y - x_k\| \leq \frac{5}{2^i}$ so $x_k \to y$. As $x_k \in V_k$ and $x_k \to y$, we get $y \in V$ and so V is closed. $\square$

Proposition 2.1.6 ([BlM15, Blu16]). *Let $V, W \in G(E)$ be topological complements and π be the projection onto V parallel to W. Then*

$$\sin\theta(V, W) = \frac{1}{\|\pi\|}$$

and $\sin\theta(W, V) \leq 2 \sin\theta(V, W)$.

Proof. We have

$$\begin{aligned}
\|\pi\| &= \sup_{v\in V, w\in W, v+w\neq 0} \frac{\|v\|}{\|v + w\|} \\
&= \sup_{v\in V, \|v\|=1, w\in W} \frac{1}{\|v + w\|} \\
&= \frac{1}{\inf\{\|v - w\| \mid v \in V, \|v\| = 1, w \in W\}} \\
&= \frac{1}{\sin\theta(V, W)}.
\end{aligned}$$

The second equality follows since

$$\sin\theta(W, V) = \frac{1}{\|\operatorname{Id} - \pi\|} \geq \frac{1}{1 + \|\pi\|} \geq \frac{1}{2\|\pi\|} = \frac{1}{2}\sin\theta(V, W).$$ $\square$

5.2 Norms on the exterior algebra

In this section, we consider $(E, \|.\|)$ a complex Banach space and we study norms on the tensor algebra $\bigotimes^p E$ and the exterior algebra $\bigwedge^p E$. We show that the norms defined on the exterior algebra $\bigwedge^p E$ can be obtained as quotient norms of corresponding norms on $\bigotimes^p E$. We show that in non-finite dimension, these norms are not equivalent. However, they are equivalent if we only look at elements which can be decomposed into a sum of k tensors, where k is a fixed integer.

The norms on $\bigotimes^p E$ have been thoroughly studied in [Gro52]. In particular, $\bigotimes^p E$ is never a complete space when E is of non-finite dimension. We invite the interested reader to read [Gro52] for more details and for information on the completions of $\bigotimes^p E$.

Here we look at three natural norms. The first one looks at the different decomposition of an element x in simple tensors: the norm is then given by taking the infimum, over all decompositions of x, of the sum of the norms of these simple tensors. The second one is defined thanks to the dual E': the norm of an element x is the supremum of the $|\langle l, x\rangle|$, where l is taken in a specific subset of $\bigotimes^p E'$ or $\bigwedge^p E'$. The last norm is used in the case of Hilbert spaces: it rises from the canonical inner product defined on $\bigotimes^p E'$ or $\bigwedge^p E'$.

Definition 5.2.1. *For $x \in \bigotimes^p E$, we define*

$$\|x\|_{\otimes,dec} = \inf \sum_i \|x_1^i\| \cdots \|x_p^i\|$$

where the infimum is taken over all (finite) decompositions $x = \sum\limits_i x_1^i \otimes \cdots \otimes x_p^i$; and

$$\|x\|_{\otimes,l} = \sup_{\substack{l_1,\dots,l_p \in E' \\ \|l_i\|=1}} \langle l_1 \otimes \cdots \otimes l_p, x\rangle.$$

We denote by $\|.\|_{\otimes,pr}$ the associated norm.

Definition 5.2.2. *For $x \in \bigwedge^p E$, we define*

$$\|x\|_{\wedge d,p} = \|x\|_{\wedge,dec} = \inf \sum_i \|x_1^i\| \cdots \|x_p^i\|$$

where the infimum is taken over all (finite) decompositions $x = \sum\limits_i x_1^i \wedge \cdots \wedge x_p^i$; and

$$\|x\|_{\wedge l,p} = \|x\|_{\wedge,l} = \sup_{\substack{l_1,\dots,l_p \in E' \\ \|l_i\|=1}} \langle l_1 \wedge \cdots \wedge l_p, x\rangle.$$

If E has an inner product $\langle .|.\rangle$, then we define an inner product on $\bigwedge^p E$ by extending the formula

$$\langle x_1 \wedge \cdots \wedge x_p \mid y_1 \wedge \cdots \wedge y_p\rangle_\wedge = \det\left((\langle x_i \mid y_j\rangle)_{1\le i,j\le p}\right)$$

We denote by $\|.\|_{\wedge,pr}$ the associated norm.

We also define on $\bigwedge^p E$ the norms $\|.\|_{\otimes q,dec}$, $\|.\|_{\otimes q,l}$ and $\|.\|_{\otimes q,pr}$ obtained as the quotient norms of $\|.\|_{\otimes,dec}$, $\|.\|_{\otimes,l}$ and $\|.\|_{\otimes,pr}$.

Proof. First, we prove that $\|.\|_{\wedge,dec}$ is a norm. Let $x, y \in \bigwedge^p E$ and $\lambda \in \mathbb{C}$. If $x = \sum\limits_i x_1^i \wedge \cdots \wedge x_p^i$ is a decomposition of x, then $\lambda x = \sum\limits_i \lambda x_1^i \wedge \cdots \wedge x_p^i$ is a decomposition of λx. Taking the infimum over all decompositions of x, we get $\|\lambda x\|_{\wedge,dec} \le |\lambda|\|x\|_{\wedge,dec}$. If $\lambda = 0$, then $\|0.x\| = 0 = 0.\|x\|_{\wedge,dec}$. If $\lambda \neq 0$, then our reasoning gives $\|x\|_{\wedge,dec} \le \frac{1}{|\lambda|}\|\lambda x\|_{\wedge,dec}$, hence $\|\lambda x\|_{\wedge,dec} = |\lambda|\|x\|_{\wedge,dec}$.
If $x = \sum\limits_i x_1^i \wedge \cdots \wedge x_p^i$ and $y = \sum\limits_j y_1^j \wedge \cdots \wedge y_p^j$, then $x + y = \sum\limits_i x_1^i \wedge \cdots \wedge x_p^i + \sum\limits_j y_1^j \wedge \cdots \wedge y_p^j$ is a decomposition of $x + y$, thus $\sum\limits_i \|x_1^i\| \cdots \|x_p^i\| + \sum\limits_j \|y_1^j\| \cdots \|y_p^j\| \ge \|x + y\|_{\wedge,dec}$. Taking the infimum over all decompositions of x and y, we get $\|x\|_{\wedge,dec} + \|y\|_{\wedge,dec} \ge \|x + y\|_{\wedge,dec}$. The separation comes from the Proposition 2.2.2 shown later in this section.

We now prove that $\|.\|_{\wedge,l}$ is a norm. The homogeneity and the triangle inequality are immediate. Let $x \in \bigwedge^p E$ such that $\|x\|_{\wedge,l} = 0$. Let $x = \sum_i x_1^i \wedge \cdots \wedge x_p^i$ be a decomposition of x. If we take $(f_1, \ldots, f_k)$ a basis of $\mathrm{Span}((x_j^i)_{i,j})$, then we can write

$$x = \sum_{1 \leq i_1 < \cdots < i_p \leq k} a_{i_1,\ldots,i_p} f_{i_1} \wedge \cdots \wedge f_{i_p}.$$

Taking for $(l_1, \ldots, l_k)$ the dual basis of $(f_1, \ldots, f_k)$ and extending these linear forms to E with the Hahn-Banach theorem, we get $0 = \langle l_{i_1} \wedge \cdots \wedge l_{i_p}, x\rangle = a_{i_1,\ldots,i_p}$, hence $x = 0$. This proves the separation of the norm $\|.\|_{\wedge,l}$.

Finally, one can easily check that the formula above defines an inner product on $\bigwedge^p E$. $\square$

We now prove that our norms on $\bigwedge^p E$ are, up to a constant, the quotient norms obtained from the norms on $\bigotimes^p E$.

Proposition 5.2.3. *On $\bigwedge^p E$, we have $\|.\|_{\otimes q,dec} = \|.\|_{\wedge,dec}$.*

Proof. Let $x \in \bigwedge^p E$ and $x = \sum_i x_1^i \wedge \cdots \wedge x_p^i$ a decomposition of x. Then $u = \sum_i x_1^i \otimes \cdots \otimes x_p^i$ represents x in $\bigotimes^p E$. Hence

$$\|x\|_{\otimes q,dec} \leq \|u\|_{\otimes,dec} \leq \sum_i \|x_1^i\| \cdots \|x_p^i\|.$$

Taking the infimum over all decompositions for x, we have $\|x\|_{\otimes q,dec} \leq \|x\|_{\wedge,dec}$.

Let $x \in \bigwedge^p E$ and let u represent x in $\bigotimes^p E$. Let $u = \sum_i x_1^i \otimes \cdots \otimes x_p^i$ be a decomposition of u. Then $x = \sum_i x_1^i \wedge \cdots \wedge x_p^i$, so $\|x\|_{\wedge,dec} \leq \sum_i \|x_1^i\| \cdots \|x_p^i\|$. Taking the infimum over all decompositions for u, we have $\|x\|_{\wedge,dec} \leq \|u\|_{\otimes,dec}$. As it holds for every u representing x, we have $\|x\|_{\wedge,dec} \leq \|x\|_{\otimes q,dec}$. $\square$

Proposition 5.2.4. *Assume E is an Hilbert space. Let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal family of E. Then $\{e_{i_1} \otimes \cdots \otimes e_{i_p} \mid i_1, \ldots, i_p \in \mathbb{N}\}$ is an orthonormal family of $\bigotimes^p E$ and $\{e_{i_1} \wedge \cdots \wedge e_{i_p} \mid i_1 < \cdots < i_p\}$ is an orthonormal family of $\bigwedge^p E$.*

Proof. It is an immediate verification. $\square$

Proposition 5.2.5. *Assume E is an Hilbert space. On $\bigwedge^p E$, we have $\|.\|_{\otimes q,pr} = \frac{1}{\sqrt{p!}}\|.\|_{\wedge,pr}$.*

Proof. We denote $\mathfrak{S}_p$ the symmetric group of degree p and $\varepsilon : \mathfrak{S}_p \to \{-1, 1\}$ the sign application. Let $x \in \bigwedge^p E$ and $u = \sum_i u_1^i \otimes \cdots \otimes u_p^i$ representing x in $\bigotimes^p E$. Then $x = \sum_i u_1^i \wedge \cdots \wedge u_p^i$. We denote $x^i := u_1^i \wedge \cdots \wedge u_p^i$, $u^i := u_1^i \otimes \cdots \otimes u_p^i$ and $u_\sigma^i := u_{\sigma(1)}^i \otimes \cdots \otimes u_{\sigma(p)}^i$ for $\sigma \in \mathfrak{S}_p$. These notations are made such that $x = \sum_i x^i$ and $u = \sum_i u^i$. We set $u_\sigma := \sum_i u_\sigma^i$.

First we show that $\|u_\sigma\|_{\otimes,pr} = \|u\|_{\otimes,pr}$:

$$\begin{aligned}
\langle u_\sigma \mid u_\sigma\rangle_\otimes &= \sum_{i,j}\langle u^i_\sigma \mid u^j_\sigma\rangle_\otimes \\
&= \sum_{i,j}\langle u^i_{\sigma(1)} \mid u^j_{\sigma(1)}\rangle \cdots \langle u^i_{\sigma(p)} \mid u^j_{\sigma(p)}\rangle \\
&= \sum_{i,j}\langle u^i_1 \mid u^j_1\rangle \cdots \langle u^i_p \mid u^j_p\rangle \\
&= \langle u \mid u\rangle_\otimes \,.
\end{aligned}$$

Now we link $\|x\|_{\wedge,pr}$ to $(u_\sigma)_{\sigma\in\mathfrak{S}_p}$:

$$\begin{aligned}
\langle x \mid x\rangle_\wedge &= \sum_{i,j}\langle x^i \mid x^j\rangle_\wedge \\
&= \sum_{i,j}\det((\langle u^i_k \mid u^j_l\rangle)_{1\le k,l\le p}) \\
&= \sum_{i,j}\sum_{\sigma\in\mathfrak{S}_p}\varepsilon(\sigma)\langle u^i_1 \mid u^j_{\sigma(1)}\rangle \cdots \langle u^i_p \mid u^j_{\sigma(p)}\rangle \\
&= \sum_{i,j}\sum_{\sigma\in\mathfrak{S}_p}\varepsilon(\sigma)\langle u^i \mid u^j_\sigma\rangle_\otimes \\
&= \sum_{\sigma\in\mathfrak{S}_p}\varepsilon(\sigma)\sum_{i,j}\langle u^i \mid u^j_\sigma\rangle_\otimes \\
&= \sum_{\sigma\in\mathfrak{S}_p}\varepsilon(\sigma)\langle u \mid u_\sigma\rangle_\otimes \,.
\end{aligned}$$

So $\|x\|^2_{\wedge,pr} \le \sum_{\sigma\in\mathfrak{S}_p}|\langle u \mid u_\sigma\rangle_\otimes| \le \sum_{\sigma\in\mathfrak{S}_p}\|u\|_{\otimes,pr}.\|u_\sigma\|_{\otimes,pr} \le \sum_{\sigma\in\mathfrak{S}_p}\|u\|^2_{\otimes,pr} = p!\|u\|_{\otimes,pr}$. Taking the infimum over all u representing x, we get $\|x\|_{\wedge,pr} \le \sqrt{p!}\|x\|_{\otimes q,pr}$.

Let $x = \sum_i x^i_1 \wedge \cdots \wedge x^i_p \in \bigwedge^p E$. Then $u = \frac{1}{p!}\sum_i\sum_{\sigma\in\mathfrak{S}_p}\varepsilon(\sigma)x^i_{\sigma(1)} \otimes \cdots \otimes x^i_{\sigma(p)} \in \bigotimes^p E$ represents x. We note $u = \sum_{i,\sigma\in\mathfrak{S}_p} u^{i,\sigma}$ this decomposition.

Using the same notations as before for this particular decomposition, we now show that for all $\sigma' \in \mathfrak{S}_p$, $u_{\sigma'} = \varepsilon(\sigma')u$. For $\sigma, \sigma' \in \mathfrak{S}_p$ and index i, we have:

$$\begin{aligned}
u^{i,\sigma}_{\sigma'} &= \tfrac{1}{p!}\varepsilon(\sigma)x^i_{\sigma'\circ\sigma(1)} \otimes \cdots \otimes x^i_{\sigma'\circ\sigma(p)} \\
&= \tfrac{1}{p!}\varepsilon(\sigma')\varepsilon(\sigma'\circ\sigma)x^i_{\sigma'\circ\sigma(1)} \otimes \cdots \otimes x^i_{\sigma'\circ\sigma(p)} \\
&= \varepsilon(\sigma')u^{i,\sigma'\circ\sigma}.
\end{aligned}$$

Hence $u_{\sigma'} = \sum_{i,\sigma\in\mathfrak{S}_p} u^{i,\sigma}_{\sigma'} = \sum_{i,\sigma\in\mathfrak{S}_p}\varepsilon(\sigma')u^{i,\sigma'\circ\sigma} = \varepsilon(\sigma')u$.

Then $\langle x \mid x\rangle_\wedge = \sum_{\sigma\in\mathfrak{S}_p}\varepsilon(\sigma)\langle u \mid u_\sigma\rangle_\otimes = \sum_{\sigma\in\mathfrak{S}_p}\langle u \mid u\rangle_\otimes = p!\langle u \mid u\rangle_\otimes$, hence the inequality $\|x\|_{\wedge,pr} \ge \sqrt{p!}\|x\|_{\otimes q,pr}$. □

Proposition 5.2.6. *On $\bigwedge^p E$, we have $\|.\|_{\otimes q,l} = (1/p!)\|.\|_{\wedge,l}$.*

Proof. Let $x \in \bigwedge^p E$ and $u = \sum_i u^i_1 \otimes \cdots \otimes u^i_p$ representing x in $\bigotimes^p E$. Then we have $x = \sum_i u^i_1 \wedge \cdots \wedge u^i_p$. Let $l_1, \cdots, l_p \in E'$ with $\|l_1\| = \cdots = \|l_p\| = 1$. We get (using the definition of the determinant):

$$\begin{aligned}
\langle l_1 \wedge \cdots \wedge l_p, x\rangle &= \sum_i \sum_{\sigma\in\mathfrak{S}_p} \varepsilon(\sigma)\langle l_{\sigma(1)}, x_1^i\rangle \cdots \langle l_{\sigma(p)}, x_p^i\rangle \\
&= \sum_i \sum_{\sigma\in\mathfrak{S}_p} \varepsilon(\sigma)\langle l_{\sigma(1)} \otimes \cdots \otimes l_{\sigma(p)}, x_1^i \otimes \cdots \otimes x_p^i\rangle \\
&= \sum_{\sigma\in\mathfrak{S}_p} \varepsilon(\sigma)\langle l_{\sigma(1)} \otimes \cdots \otimes l_{\sigma(p)}, \sum_i x_1^i \otimes \cdots \otimes x_p^i\rangle \\
&= \sum_{\sigma\in\mathfrak{S}_p} \varepsilon(\sigma)\langle l_{\sigma(1)} \otimes \cdots \otimes l_{\sigma(p)}, u\rangle\,.
\end{aligned}$$

Therefore we have $\|x\|_{\wedge,l} \leq p!\|u\|_{\otimes,l}$, and taking the infimum over all u representing x we get $\|x\|_{\wedge,l} \leq p!\|x\|_{\otimes q,l}$.

Let $x = \sum_i x_1^i \wedge \cdots \wedge x_p^i \in \bigwedge^p E$. Then $u := \frac{1}{p!}\sum_i \sum_{\sigma\in\mathfrak{S}_p} \varepsilon(\sigma) x_{\sigma(1)}^i \otimes \cdots \otimes x_{\sigma(p)}^i \in \bigotimes^p E$ represents x. Let $l_1, \cdots, l_p \in E'$ with $\|l_1\| = \cdots = \|l_p\| = 1$. We have:

$$\begin{aligned}
\langle l_1 \otimes \cdots \otimes l_p, u\rangle &= \frac{1}{p!}\sum_i \sum_{\sigma\in\mathfrak{S}_p} \varepsilon(\sigma)\langle l_1, x_{\sigma(1)}^i\rangle \cdots \langle l_p, x_{\sigma(p)}^i\rangle \\
&= \frac{1}{p!}\sum_i \langle l_1 \wedge \cdots \wedge l_p, x_1^i \wedge \cdots \wedge x_p^i\rangle \\
&= \frac{1}{p!}\langle l_1 \wedge \cdots \wedge l_p, x\rangle\,.
\end{aligned}$$

Hence $\|u\|_{\otimes,l} = \frac{1}{p!}\|x\|_{\wedge,l}$ and therefore $\|x\|_{\otimes q,l} \leq \frac{1}{p!}\|x\|_{\wedge,l}$. □

We now compare these norms. First, we prove a lemma which is useful when comparing the inner product norm to the other ones. It is a simple extension of the Cauchy-Schwarz inequality.

Lemma 5.2.7. *Let* $(a_{k,l})_{\substack{0\leq k\leq n-1\\ 1\leq l\leq p}} \in \mathbb{C}^{np}$, *with* $p \geq 2$. *Then*

$$\left|\sum_{k=0}^{n-1} a_{k,1}\cdots a_{k,p}\right| \leq \sqrt{\sum_{k=0}^{n-1}|a_{k,1}|^2}\cdots\sqrt{\sum_{k=0}^{n-1}|a_{k,p}|^2}\,.$$

Proof. For $p = 2$, it's the Cauchy-Schwarz inequality. Assume the lemma is true for $p \geq 2$. Let $(a_{k,l})_{\substack{0\leq k\leq n-1\\ 1\leq l\leq p+1}} \in \mathbb{C}^{n(p+1)}$. Applying the result for p, we get

$$\left|\sum_{k=0}^{n-1} a_{k,1}\cdots a_{k,p}.a_{k,p+1}\right| \leq \sqrt{\sum_{k=0}^{n-1}|a_{k,1}|^2}\cdots\sqrt{\sum_{k=0}^{n-1}|a_{k,p-1}|^2}.\sqrt{\sum_{k=0}^{n-1}|a_{k,p}|^2|a_{k,p+1}|^2}\,.$$

Since $\sum_{k=0}^{n-1}|a_{k,p}|^2.\sum_{k=0}^{n-1}|a_{k,p+1}|^2 \geq \sum_{k=0}^{n-1}|a_{k,p}|^2|a_{k,p+1}|^2$, we get the final result for $p+1$. □

Proposition 5.2.8. *Assume E is an Hilbert space. On $\bigotimes^p E$, we have $\|.\|_{\otimes,pr} \leq \|.\|_{\otimes,dec}$. If E is not finite dimensional, these two norms are not equivalent.*

Proof. Let $x = \sum\limits_i x_1^i \otimes \cdots \otimes x_p^i = \sum\limits_i x^i \in \bigotimes^p E$. Then

$$\begin{aligned}
|\langle x \mid x \rangle_\otimes| &= \left| \sum_{i,j} \langle x^i \mid x^j \rangle_\otimes \right| \\
&\leq \sum_{i,j} |\langle x_1^i \mid x_1^j \rangle| \cdots |\langle x_p^i \mid x_p^j \rangle| \\
&\leq \sum_{i,j} \|x_1^i\| \|x_1^j\| \cdots \|x_p^i\| \|x_p^j\| \\
&= \left(\sum_i \|x_1^i\| \cdots \|x_p^i\| \right) \left(\sum_j \|x_1^j\| \cdots \|x_p^j\| \right).
\end{aligned}$$

Hence $\|x\|_{\otimes,pr} \leq \sum\limits_i \|x_1^i\| \cdots \|x_p^i\|$. Taking the infimum over all decompositions of x, we get $\|x\|_{\otimes,pr} \leq \|x\|_{\otimes,dec}$.

We suppose that E is not finite-dimensional. Let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal family of E and let $x_n = \sum\limits_{k=0}^{n-1} e_{kp+1} \otimes e_{kp+2} \otimes \cdots \otimes e_{kp+p}$. We have $\|x_n\|_{\otimes,pr} = \sqrt{n}$ and $\|x_n\|_{\otimes,dec} \leq n$. Let $x_n = \sum\limits_i x_1^i \otimes \cdots \otimes x_p^i$ be a decomposition of x_n. Let $F = \mathrm{Span}((x_k^i)) \subset E$ and G be the orthogonal complement of $\mathrm{Span}((e_n)_{n\in\mathbb{N}}) \cap F$ in F. Let $(g_1, \cdots, g_d)$ be an orthonormal basis of G. We write, for $l \in [\![1, p]\!]$, $x_l^i = \sum\limits_n a_n^{i,l} e_n + \sum\limits_j b_j^{i,l} g_j$. Then $\|x_l^i\|^2 \geq \sum\limits_n |a_n^{i,l}|^2$ and

$$x_n = \sum_i x_1^i \otimes \cdots \otimes x_p^i = \sum_{0 \leq k_1, \cdots, k_p} \left(\sum_i a_{k_1}^{i,1} \cdots a_{k_p}^{i,p} \right) e_{k_1} \otimes \cdots \otimes e_{k_p} + \text{tensors involving some } g_j.$$

As $x_n = \sum\limits_{k=0}^{n-1} e_{kp+1} \otimes e_{kp+2} \otimes \cdots \otimes e_{kp+p}$, we can identify the two formulas. The terms involving some g_j are thus equal to zero and

$$x_n = \sum_{k=0}^{n-1} \underbrace{\left(\sum_i a_{kp+1}^{i,1} \cdots a_{kp+p}^{i,p}\right)}_{=1} e_{kp+1} \otimes \cdots \otimes e_{kp+p}\,.$$

Hence

$$\begin{aligned}
n &= \sum_{k=0}^{n-1} \sum_i a_{kp+1}^{i,1} \cdots a_{kp+p}^{i,p} \\
&= \sum_i \sum_{k=0}^{n-1} a_{kp+1}^{i,1} \cdots a_{kp+p}^{i,p} \\
&\leq \sum_i \sqrt{\sum_{k=0}^{n-1} |a_{kp+1}^{i,1}|^2} \cdots \sqrt{\sum_{k=0}^{n-1} |a_{kp+p}^{i,p}|^2} \qquad \text{by Lemma 5.2.7} \\
&\leq \sum_i \|x_1^i\| \cdots \|x_p^i\|\,.
\end{aligned}$$

Therefore $\|x_n\|_{\otimes,dec} \geq n$. As we proved $\|x_n\|_{\otimes,dec} \leq n$, we get $\|x_n\|_{\otimes,dec} = n$ and $\|x_n\|_{\otimes,dec} = \sqrt{n} \|x_n\|_{\otimes,pr}$. $\square$

Proposition 5.2.9. *Assume E is an Hilbert space. On $\bigwedge^p E$, we have $\|.\|_{\wedge,pr} \leq \sqrt{p!}\|.\|_{\wedge,dec}$. If E is not finite dimensional, these two norms are not equivalent.*

Proof. This is a consequence of Proposition 5.2.8, Proposition 5.2.5 and Proposition 5.2.3. □

Proposition 5.2.10. *Assume E is an Hilbert space. On $\bigotimes^p E$, we have $\|.\|_{\otimes,l} \leq \|.\|_{\otimes,pr}$. If E is not finite dimensional, these two norms are not equivalent.*

Proof. Assume that the space E is a Hilbert space. Let $x \in \bigotimes^p E$ and $l_1, \dots, l_p \in E'$ with $\|l_1\| = \cdots = \|l_p\| = 1$. By Riesz's representation theorem, there exists $y_1, \dots, y_n \in E$ with $\|y_1\| = \cdots = \|y_p\| = 1$ such that $l_i = \langle y_i \mid . \rangle$ for all $i \in [\![1, p]\!]$. Then for all $x_1, \dots, x_p \in E$,

$$\begin{aligned} \langle l_1 \otimes \cdots \otimes l_p, x_1 \otimes \cdots \otimes x_p \rangle &= \langle l_1, x_1 \rangle \cdots \langle l_p, x_p \rangle \\ &= \langle y_1 \mid x_1 \rangle \cdots \langle y_p \mid x_p \rangle \\ &= \langle y_1 \otimes \cdots \otimes y_p \mid x_1 \otimes \cdots \otimes x_p \rangle_\otimes . \end{aligned}$$

Hence $|\langle l_1 \otimes \cdots \otimes l_p, x \rangle| = |\langle y_1 \otimes \cdots \otimes y_p \mid x \rangle_\otimes| \leq \|y_1 \otimes \cdots \otimes y_p\|_{\otimes,pr} \|x\|_{\otimes,pr} \leq \|x\|_{\otimes,pr}$. Taking the supremum over all $l_1, \dots, l_p$, we get $\|x\|_{\otimes,l} \leq \|x\|_{\otimes,pr}$.

We suppose that E is not finite-dimensional. Let us show that the two norms are not equivalent. Let $(e_n)_{n \in \mathbb{N}}$ be an orthonormal family of E and consider

$$x_n = \sum_{k=0}^{n-1} e_{kp+1} \otimes e_{kp+2} \otimes \cdots \otimes e_{kp+p}.$$

We have $\|x_n\|_{\otimes,pr} = \sqrt{n}$ and, taking $l_i = \langle e_i \mid . \rangle$, we get $\|x_n\|_{\otimes,l} \geq 1$.

Let $l_1, \dots, l_p \in E'$ with $\|l_1\| = \cdots = \|l_p\| = 1$. By Riesz's representation theorem, there exists $y_1, \dots, y_n \in E$ with $\|y_1\| = \cdots = \|y_p\| = 1$ such that $l_i = \langle y_i \mid . \rangle$ for all $i \in [\![1, p]\!]$. Let $F = \mathrm{Span}((y_k)_{1 \leq k \leq p})$ and G be the orthogonal complement of $\mathrm{Span}((e_n)_{n \in \mathbb{N}}) \cap F$ in F. Let $(g_1, \dots, g_d)$ be an orthonormal basis of G. We write, for $l \in [\![1, p]\!]$, $y_l = \sum\limits_n y_{l,n} e_n + \sum\limits_j z_{l,j} g_j$. We have

$$\begin{aligned} |\langle l_1 \otimes \cdots \otimes l_p, x \rangle| &= |\langle y_1 \otimes \cdots \otimes y_p \mid x \rangle_\otimes| \\ &= |\sum_{k=0}^{n-1} \langle y_1 \otimes \cdots \otimes y_p \mid e_{kp+1} \otimes e_{kp+2} \otimes \cdots \otimes e_{kp+p} \rangle_\otimes| \\ &= |\sum_{k=0}^{n-1} y_{1,kp+1} \cdots y_{p,kp+p}| \\ &\leq \sqrt{\sum_{k=0}^{n-1} |y_{1,kp+1}|^2} \cdots \sqrt{\sum_{k=0}^{n-1} |y_{p,kp+p}|^2} \qquad \text{by Lemma 5.2.7} \\ &\leq \|y_1\| \cdots \|y_p\| \\ &= 1, \end{aligned}$$

hence $\|x_n\|_{\otimes,l} \leq 1$. Since we have also shown $\|x_n\|_{\otimes,l} \geq 1$, we get $\|x_n\|_{\otimes,l} = 1$ and $\|x_n\|_{\otimes,pr} = \sqrt{n}\|x_n\|_{\otimes,l}$. □

Proposition 5.2.11. *Assume E is an Hilbert space. On $\bigwedge^p E$, we have $\|.\|_{\wedge,l} \leq \sqrt{p!}\|.\|_{\wedge,pr}$. If E is not finite dimensional, these two norms are not equivalent.*

Proof. This is a consequence of Proposition 5.2.10, Proposition 5.2.5 and Proposition 5.2.6. ☐

Proposition 5.2.12. *On $\bigotimes^p E$, we have $\|.\|_{\otimes,l} \leq \|.\|_{\otimes,dec}$.*

Proof. Let $x = \sum\limits_i x_1^i \otimes \cdots \otimes x_p^i$ and let $l_1, \cdots, l_p \in E'$ with $\|l_1\| = \cdots = \|l_p\| = 1$. Then $|\langle l_1 \otimes \cdots \otimes l_p, x\rangle| = |\sum\limits_i \langle l_1, x_1^i\rangle \cdots \langle l_p, x_p^i\rangle| \leq \sum\limits_i |\langle l_1, x_1^i\rangle| \cdots |\langle l_p, x_p^i\rangle| \leq \sum\limits_i \|x_1^i\| \cdots \|x_p^i\|$. Taking the supremum over all l_i and the infimum over all decompositions of x, we get $\|x\|_{\otimes,l} \leq \|x\|_{\otimes,dec}$. ☐

Proposition 5.2.13. *On $\bigwedge^p E$, we have $\|.\|_{\wedge,l} \leq p!\|.\|_{\wedge,dec}$.*

Proof. This is a consequence of Proposition 5.2.12, Proposition 5.2.3 and Proposition 5.2.6. ☐

A refinement of this estimate is given by the following proposition, stated earlier in Chapter 2:

Proposition 2.2.2. *On $\bigwedge^p E$, we have $\|.\|_{\wedge l,p} \leq (\sqrt{p})^p \|.\|_{\wedge d,p}$.*

Proof. Let $x = \sum\limits_i x_1^i \wedge \cdots \wedge x_p^i$ and let $l_1, \ldots, l_p \in E'$ with $\|l_1\| = \cdots = \|l_p\| = 1$. Then

$$\begin{aligned}
|\langle l_1 \wedge \cdots \wedge l_p, x\rangle| &= |\sum_i \langle l_1 \wedge \cdots \wedge l_p, x_1^i \wedge \cdots \wedge x_p^i\rangle| \\
&\leq \sum_i |\langle l_1 \wedge \cdots \wedge l_p, x_1^i \wedge \cdots \wedge x_p^i\rangle| \\
&\leq \sum_i \left| \det \begin{pmatrix} \langle l_1, x_1^i\rangle & \cdots & \langle l_p, x_1^i\rangle \\ \vdots & & \vdots \\ \langle l_1, x_p^i\rangle & \cdots & \langle l_p, x_p^i\rangle \end{pmatrix} \right| \\
&\leq \sum_i (\sqrt{p})^p \|x_1^i\| \cdots \|x_p^i\|,
\end{aligned}$$

where the last line is given by Hadamard's inequality. Taking the supremum over all l_i and the infimum over all decompositions of x, we get $\|x\|_{\wedge,l} \leq (\sqrt{p})^p \|x\|_{\wedge,dec}$. ☐

Remark. The previous method to prove the non-equivalence of the norms does not work without the inner product setting:
Let us consider $l^1(\mathbb{N})$, $(e_n)_{n\in\mathbb{N}}$ the canonical family of $l^1(\mathbb{N})$ and $l : (u_n)_{n\in\mathbb{N}} \in l^1(\mathbb{N}) \mapsto \sum\limits_{n\geq 0} u_n$.
As before, we note $x_n = \sum\limits_{k=0}^{n-1} e_{kp+1} \otimes e_{kp+2} \otimes \cdots \otimes e_{kp+p}$. Then we have

$$\langle l \otimes \cdots \otimes l, x_n\rangle = \sum_{k=0}^{n-1} 1 \times \cdots \times 1 = n,$$

hence $n \leq \|x_n\|_{\otimes,l} \leq \|x_n\|_{\otimes,dec} \leq n$, giving $\|x_n\|_{\otimes,l} = \|x_n\|_{\otimes,dec} = n$.

We finally state two lemmas which gives bounds on $\|.\|_{\wedge l,p}$ with lower-dimensional norms.

Lemma 2.2.3. *Let $x \in E$ and $u \in \bigwedge^p E$. Then* *(i)* $\|x \wedge u\|_{\wedge l,p+1} \leq (p+1)\|x\|.\|u\|_{\wedge l,p}$,

(ii) $\|x \wedge u\|_{\wedge d,p+1} \leq \|x\|.\|u\|_{\wedge d,p}$.

Proof. (i). Let $x \in E$ and let $l_1, l_2, \cdots, l_{p+1}$ be independent linear forms of norm 1 on E. Let $\hat{l_i} = l_1 \wedge \cdots l_{i-1} \wedge l_{i+1} \wedge \cdots \wedge l_{p+1}$ and $f : u \in \bigwedge^p E \mapsto \sum_{i=1}^{p+1}(-1)^{i+1} l_i(x)\hat{l_i}(u)$. For all $(x_1, \cdots, x_p) \in E^p$ we have $l_1 \wedge \cdots \wedge l_{p+1}(x \wedge x_1 \wedge \cdots \wedge x_p) = f(x_1 \wedge \cdots \wedge x_p)$. Then for all $u \in \bigwedge^p E$:

$$l_1 \wedge \cdots \wedge l_{p+1}(x \wedge u) = f(u) = \sum_{i=1}^{p+1}(-1)^{i+1} l_i(x)\hat{l_i}(u).$$

Hence $|l_1 \wedge \cdots \wedge l_{p+1}(x \wedge u)| \leq \sum_{i=1}^{p+1} |l_i(x)|.|\hat{l_i}(u)| \leq (p+1)\|x\|.\|u\|_{\wedge l,p}$ and therefore

$$\|x \wedge u\|_{\wedge l,p+1} \leq (p+1)\|x\|.\|u\|_{\wedge l,p}.$$

(ii). Let $u = \sum_i u_1^i \wedge \cdots \wedge u_p^i$. Then $\sum_i x \wedge u_1^i \wedge \cdots \wedge u_p^i$ is a decomposition of $x \wedge u$, hence $\|x \wedge u\|_{\wedge d,p+1} \leq \sum_i \|x\|.\|u_1^i\| \cdots \|u_p^i\|$. Taking the infimum over all decompositions of u, we get $\|x \wedge u\|_{\wedge d,p+1} \leq \|x\|.\|u\|_{\wedge d,p}$.

□

Lemma 2.2.4. *Let $x_1, \ldots, x_p, y_1, \ldots, y_p \in E$ such that $\|x_i\| = \|y_i\| = 1$ for all $i \in [\![1, p]\!]$. Then*

(i) $\|x_1 \wedge \cdots \wedge x_p - y_1 \wedge \cdots \wedge y_p\|_{\wedge l,p} \leq p.p!. \max(\|x_i - y_i\|)$,

(ii) $\|x_1 \wedge \cdots \wedge x_p - y_1 \wedge \cdots \wedge y_p\|_{\wedge d,p} \leq p. \max(\|x_i - y_i\|)$.

Proof. For $k \in [\![1, p]\!]$, we set $\alpha_k := x_1 \wedge \cdots \wedge x_k$ and $\beta_k := y_1 \wedge \cdots \wedge y_k$. Note that $\alpha_k = \alpha_{k-1} \wedge x_k$ and $\beta_k = \beta_{k-1} \wedge y_k$.

Let $d := \max(\|x_i - y_i\|)$. Using Lemma 2.2.3, we get

$$\begin{aligned} \|\alpha_p - \beta_p\|_{\wedge l,p} &\leq \|\alpha_{p-1} \wedge (x_p - y_p)\|_{\wedge l,p} + \|(\alpha_{p-1} - \beta_{p-1}) \wedge y_p\|_{\wedge l,p} \\ &\leq p.d.\|\alpha_{p-1}\|_{\wedge l,p-1} + p.\|\alpha_{p-1} - \beta_{p-1}\|_{\wedge l,p-1} \\ &\leq d.p! + p.\|\alpha_{p-1} - \beta_{p-1}\|_{\wedge l,p-1}\ , \end{aligned}$$

and the desired result follows by recursion. This reasoning is easily adjusted to get the result for $\|.\|_{\wedge d,p}$. □

5.3 Estimates on the norms of induced operators on $\bigwedge^p E$

In this section, we provide the proofs for the bounds on norms of operators on $\bigwedge^p E$ used in Chapter 3.

Lemma 3.3.4. *Let $\mathcal{C}$ be p-cone of E of aperture bounded by $K > 0$ with $m : E \to \mathbb{C}^p$ a continuous linear map such that $\forall x \in \mathcal{C}, \|m\|.\|x\| \leq K|m(x)|$. Then*

- *For the norm $\|.\|_{\wedge d,p}$, we have $\|\hat{m}\| \leq \|m\|^p$ and*

$$\forall \operatorname{Span}(x_1, \dots, x_p) \subset \mathcal{C}, \|x_1 \wedge \dots \wedge x_p\|_{\wedge d,p} \leq \frac{K^p}{\|m\|^p}|\hat{m}(x_1 \wedge \dots \wedge x_p)|.$$

- *For the norm $\|.\|_{\wedge l,p}$, we have*

$$\forall (x_1, \dots, x_p) \in E^p, |\hat{m}(x_1 \wedge \dots \wedge x_p)| \leq \|m\|^p \|x_1 \wedge \dots \wedge x_p\|_{\wedge l,p} \;;$$
$$\forall \operatorname{Span}(x_1, \dots, x_p) \subset \mathcal{C}, \|x_1 \wedge \dots \wedge x_p\|_{\wedge l,p} \leq (\sqrt{p})^p \frac{K^p}{\|m\|^p}|\hat{m}(x_1 \wedge \dots \wedge x_p)|.$$

Proof. We first prove these inequalities for the norm $\|.\|_{\wedge d,p}$. Let $u \in \bigwedge^p E$ and consider $u = \sum x_1^i \wedge \dots \wedge x_p^i$ a decomposition of u. For $m : E \to \mathbb{C}^p$, we get:

$$\begin{array}{rcl} |\widehat{m}(u)| & \leq & \sum |\hat{m}(x_1^i \wedge \dots \wedge x_p^i)| \\ & \leq & \sum |\det(m(x_1^i), \dots, m(x_p^i))| \\ & \leq & \sum \|m(x_1^i)\| \cdots \|m(x_p^i)\| \\ & \leq & \|m\|^p \sum \|x_1^i\| \cdots \|x_p^i\| \,. \end{array}$$

Taking the infimum over all decompositions of u, we get $|\widehat{m}(u)| \leq \|m\|^p \|u\|_{\wedge d,p}$ hence $\|\hat{m}\| \leq \|m\|^p$.

Now, let $\mathcal{C}$ be p-cone of E of bounded aperture with $m : E \to \mathbb{C}^p$ such that for all $u \in \mathcal{C}$, $\|m\|.\|u\| \leq K|m(u)|$. Let $\operatorname{Span}(x_1, \dots, x_p) \subset \mathcal{C}$. As in the proof of Lemma 3.3.5, we can find $x_1', \dots, x_p' \in \mathcal{C}$ such that $x_1 \wedge \dots \wedge x_p = x_1' \wedge \dots \wedge x_p'$ and $(m(x_1), \dots, m(x_p))$ is an orthogonal basis of $\mathbb{C}^p$. Then

$$\|x_1 \wedge \dots \wedge x_p\| \leq \|x_1'\| \dots \|x_p'\| \leq \left(\frac{K}{\|m\|}\right)^p |m(x_1')| \dots |m(x_p')| = \frac{K^p}{\|m\|^p}|\hat{m}(x_1 \wedge \dots \wedge x_p)|.$$

We now prove the inequalities for the norm $\|.\|_{\wedge l,p}$. Let $x_1, \dots, x_p \in E$. By linearity of m and homoegeneity of the norm, it suffices to prove the inequality for any non-zero vector of $\bigwedge^p E$ collinear to $x_1 \wedge \dots \wedge x_p$. Let $(e_1, \dots, e_p)$ be the canonical basis of $\mathbb{C}^p$. As in the proof of Lemma 3.3.2, $m_{|V} : V \to \mathbb{C}^p$ is bijective. Then there exists a basis $(x_1', \dots, x_p')$ of $\operatorname{Span}(x_1, \dots, x_p)$ such that $(m(x_1'), \dots, m(x_p')) = (e_1, \dots, e_p)$. Note that $(x_1', \dots, x_p')$ is an Auerbach basis of $\operatorname{Span}(x_1, \dots, x_p)$, where the associated linear forms are the coordinate functionals of $m : E \to \mathbb{C}^p$. Then $\hat{m}(x_1' \wedge \dots \wedge x_p') = 1$ and $(x_1'/\|x_1'\|, \dots, x_p'/\|x_p'\|)$ is a right decomposition. Thus

$$\begin{array}{rcl} |\hat{m}(x_1' \wedge \dots \wedge x_p')| & \leq & \|m(x_1')\| \dots \|m(x_p')\| \\ & \leq & \|m\|^p \|x_1'\| \cdots \|x_p'\| \\ & \leq & \|m\|^p \|x_1'\| \cdots \|x_p'\| \left\| \frac{x_1'}{\|x_1'\|} \wedge \dots \wedge \frac{x_p'}{\|x_p'\|} \right\|_{\wedge l,p} \\ & \leq & \|m\|^p \|x_1' \wedge \dots \wedge x_p'\|_{\wedge l,p} \,. \end{array}$$

The second equality is a consequence of the corresponding inequality for $\|.\|_{\wedge d,p}$ and Proposition 2.2.2.

□

Lemma 5.3.1. *Let V be a p-dimensional subset of E and W a closed complement of E. Let $h_1,\dots,h_{p-1}$ be independent vectors of V. Then there exists $B>0$ such that:*

$$\forall y \in W, \|h_1 \wedge \dots \wedge h_{p-1} \wedge y\| \geq B\|y\|.$$

Proof. We prove this result for the norm $\|.\|_{\wedge l,p}$, which is enough by Lemma 2.2.12.

Let $l_1,\dots,l_{p-1}$ be linear forms such that $l_i(h_i)=1$, $l_i(h_j)=0$ if $i\neq j$, and $l_{i|W}=0$. Let $y\in W$ and let l be a linear form on $V+\mathrm{Span}(y)$ such that $l_{|V}=0$ and $l(y)=\|y\|$.

Then

$$\begin{aligned}
|h_1 \wedge \dots \wedge h_{p-1} \wedge y\| &\geq \langle \tfrac{l_1}{\|l_1\|} \wedge \tfrac{l_{p-1}}{\|l_{p-1}\|} \wedge \tfrac{l}{\|l\|}, h_1 \wedge \dots \wedge h_{p-1} \wedge y\rangle \\
&\geq \frac{1}{\|l_1\|\dots\|l_{p-1}\|\|l\|}\left|\det\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & 1 & 0 \\ 0 & \cdots & 0 & \|y\| \end{pmatrix}\right| \\
&\geq \frac{1}{\|l_1\|\dots\|l_{p-1}\|\|\pi_{W/\!/V}\|}\|y\|,
\end{aligned}$$

where $\pi_{W/\!/V}$ is the projection onto W parallel to V. We get the desired result by setting $B=\dfrac{1}{\|l_1\|\dots\|l_{p-1}\|\|\pi_{W/\!/V}\|}$ (independent of $y\in W$).

□

The remainder of the lemmas of this section use the norm $\|.\|_{\wedge d,p}$, as in Section 3.4 of Chapter 3.

Lemma 3.4.8. *Let $M\in\mathcal{L}(E)$ and consider the operator norm of $\widehat{M}$ on $\bigwedge^p E$ given by $\|\widehat{M}\|_{\wedge,p} := \sup_{\|x\|_{\wedge,p}=1}\|\widehat{M}(x)\|_{\wedge,p}$. Then this supremum is given by its value on the set of simple tensors, i.e.*

$$\|\widehat{M}\|_{\wedge,p} = \sup_{\|x_1\wedge\dots\wedge x_p\|_{\wedge,p}=1}\|\widehat{M}(x_1\wedge\dots\wedge x_p)\|_{\wedge,p}.$$

Proof. Let $\|\widehat{M}\|_0 := \sup_{\|x_1\wedge\dots\wedge x_p\|_{\wedge,p}=1}\|\widehat{M}(x_1\wedge\dots\wedge x_p)\|_{\wedge,p}$. By definition of $\|\widehat{M}\|$ we have $\|\widehat{M}\|_{\wedge,p}\geq\|\widehat{M}\|_0$. Let $u\in\bigwedge^p E$ and write $u=\sum x_1^i\wedge\dots\wedge x_p^i$ a decomposition of u. Then

$$\|\widehat{M}u\| \leq \sum\|\widehat{M}(x_1^i\wedge\dots\wedge x_p^i)\| \leq \|\widehat{M}\|_0\sum\|x_1^i\wedge\dots\wedge x_p^i\| \leq \|\widehat{M}\|_0\sum\|x_1^i\|\dots\|x_p^i\|.$$

Taking the infimum over all decompositions of u, we get $\|\widehat{M}u\|\leq\|\widehat{M}\|_0.\|u\|$, which shows that $\|\widehat{M}\|\leq\|\widehat{M}\|_0$.

□

Lemma 3.4.9. *Let $M\in\mathcal{L}(E)$. We have*

$$\|\widehat{M}\|_{\wedge,p} \leq \left\|\bigwedge^{p-1} M\right\|_{\wedge,p-1}\|M\| \leq \|M\|^p.$$

Proof. Let $u \in \bigwedge^p E$ and write $u = \sum x_1^i \wedge \cdots \wedge x_p^i$ a decomposition of u. Then $\widehat{M}u = \sum Mx_1^i \wedge \cdots \wedge Mx_p^i$ is a decomposition of Mu and

$$\|\widehat{M}u\|_{\wedge,p} \leq \sum \|Mx_1^i\| \left\|\bigwedge^{p-1} M(x_2^i \wedge \cdots \wedge x_p^i)\right\| \leq \sum \|M\| \left\|\bigwedge^{p-1} M\right\| \|x_1^i\| \ldots \|x_p^i\|.$$

Taking the infimum over all decompositions of u, we get $\|\widehat{M}u\|_{\wedge,p} \leq \|M\|.\|\bigwedge^{p-1} M\|.\|u\|_{\wedge,p}$, hence $\|\widehat{M}\|_{\wedge,p} \leq \|M\|.\|\bigwedge^{p-1} M\|$. An immediate induction gives the second inequality. □

Lemma 3.4.19. *For $t \in \mathbb{D}$,*

$$\left\|\frac{d}{dt}\widehat{M}_\omega(t)\right\| \leq p \left\|\bigwedge^{p-1} M_\omega(t)\right\| \left\|\frac{d}{dt}M_\omega(t)\right\| .$$

Under the assumptions of Theorem 3.4.5, this gives

$$\left\|\frac{d}{dt}\widehat{M}_\omega(t)\right\| \leq pC \left\|\widehat{M}_\omega(t)\right\| ,$$

where $C = \sup\left\{\dfrac{\|\bigwedge^{p-1} M_\omega(t)\|}{\|\bigwedge^{p} M_\omega(t)\|}\|\tfrac{d}{dt}M_\omega(t)\| : \omega \in \Omega, t \in \mathbb{D}\right\}$.

Proof. Let $x_1, \ldots, x_p \in E$.

$$\begin{aligned} \frac{d}{dt}\widehat{M}_\omega(t)x_1 \wedge \cdots \wedge x_p &= \frac{d}{dt}\left(M_\omega(t)x_1 \wedge \cdots \wedge M_\omega(t)x_p\right) \\ &= \sum_{k=1}^{p} M_\omega(t)x_1 \wedge \cdots \wedge \left(\frac{d}{dt}M_\omega(t)x_k\right) \wedge \cdots \wedge M_\omega(t)x_p\,, \end{aligned}$$

which gives

$$\left\|\frac{d}{dt}\widehat{M}_\omega(t)x_1 \wedge \cdots \wedge x_p\right\| \leq p \left\|\bigwedge^{p-1} M_\omega(t)\right\| . \left\|\frac{d}{dt}M_\omega(t)\right\| .\|x_1\| \ldots \|x_p\|.$$

Let $u \in \bigwedge^p E$ and write $u = \sum x_1^i \wedge \cdots \wedge x_p^i$ a decomposition of u. Then

$$\left\|\frac{d}{dt}\widehat{M}_\omega(t)u\right\| \leq p \left\|\bigwedge^{p-1} M_\omega(t)\right\| . \left\|\frac{d}{dt}M_\omega(t)\right\| . \sum \|x_1^i\| \ldots \|x_p^i\|.$$

Taking the infimum over all decompositions of u, we get

$$\left\|\frac{d}{dt}\widehat{M}_\omega(t)u\right\| \leq p \left\|\bigwedge^{p-1} M_\omega(t)\right\| . \left\|\frac{d}{dt}M_\omega(t)\right\| .\|u\|,$$

hence $\left\|\dfrac{d}{dt}\widehat{M}_\omega(t)\right\| \leq p \,\|\bigwedge^{p-1} M_\omega(t)\| \left\|\dfrac{d}{dt}M_\omega(t)\right\|$. □

5.4 Links between $\mathcal{C}$, $\mathcal{C}[\rho]$, $\mathcal{C}_{\pi,a}$ and the Hausdorff topology on $\mathcal{G}_p(E)$

In this section, we study the relation between $\mathcal{C}[\rho]$ and interior in $\mathcal{G}_p(E)$, between $\mathcal{C}[\rho]$ and projection cones, and between the Hausdorff distance d_H and the gauge $d_{\mathcal{C}}$. Those results give us a better intuition of how $\mathcal{C}[\rho]$, the projection cones and $d_{\mathcal{C}}$ behave concretely.

First, we give the proof of Proposition 3.3.8: a projection cone $\mathcal{C}_{\pi,a}$ is reproducing. Then, we connect the projection cone $\mathcal{C}_{\pi,a}$ to estimates on the Hausdorff distance d_H. We prove that if a cone $\mathcal{C}$ contains a projection cone $\mathcal{C}_{\pi,a}$, then the gauge $d_{\mathcal{C}}$ and the distance d_H are equivalent on the subcone $\mathcal{C}_{\pi,a'}$ for all $a' < a$.

Finally, we establish the equivalence between $\mathcal{C}$ being of non-empty interior in $\mathcal{G}_p(E)$, $\mathcal{C}$ containing a projection cone, and $\mathcal{C}[\rho]$ containing a p-dimensional subspace. This leads to a simple criterion for a cone $\mathcal{C}$ to be reproducing: satisfying the equivalent conditions from the previous sentence. We conclude by showing that if V is a p-dimensional subspace contained in $\mathcal{C}[\rho]$, then $d_{\mathcal{C}}$ and d_H are equivalent on a neighborhood of V.

5.4.1 Projection cones are reproducing

Here we prove Proposition 3.3.8 from Chapter 3:

Proposition 3.3.8. *Let $a > 0$ and F, G be topological complements: $E = F \oplus G$ and the projection π onto F along G is continuous. Then the cone $\mathcal{C}_{\pi,a}$ is reproducing.*

Proof. We use the norm $\|.\|_{\wedge d,p}$ for this proof. Let $(x_1, \dots, x_p) \in E^p$ be a right decomposition and let $x_k = x_k^F + x_k^G$ be the decomposition of x_k in $E = F \oplus G$. Then

$$x_1 \wedge \cdots \wedge x_p = \sum_{(S_1,\dots,S_p)\in\{F,G\}^p} x_1^{S_1} \wedge \cdots \wedge x_p^{S_p}.$$

We estimate the terms $x_1^{S_1} \wedge \cdots \wedge x_p^{S_p}$. Up to a change of sign, such a term can be written as $x_{i_1}^F \wedge \cdots \wedge x_{i_k}^F \wedge x_{j_1}^G \wedge \cdots \wedge x_{j_{p-k}}^G$. Let $k \in [\![0, p]\!]$ and let us prove by recursion the following:

For all $n \in [\![0, p-k]\!]$, the tensor $x_{i_1}^F \wedge \cdots \wedge x_{i_k}^F \wedge x_{j_1}^G \wedge \cdots \wedge x_{j_n}^G$ can be decomposed into a sum of 2^n simple tensors $\alpha_1, \dots, \alpha_{2^n}$ such that the subspaces represented by the α_i are in $\mathcal{C}_{\pi,\frac{a}{8^{p-n}}}$ and $\|\alpha_i\| \leq K_n \|x_{i_1}\| \dots \|x_{i_k}\| \|x_{j_1}\| \dots \|x_{j_n}\|$ for some constant K_n.

For clarity, we denote by $\mathcal{C}_n$ the cone $\mathcal{C}_{\pi,\frac{a}{8^{p-n}}}$. The result is clearly true for $n = 0$. Assume the result is true for $n \in [\![0, p-k-1]\!]$. By our recursion hypothesis, there exists tensors α_i representing subspaces in $\mathcal{C}_n$ such that $x_{i_1}^F \wedge \cdots \wedge x_{i_k}^F \wedge x_{j_1}^G \wedge \cdots \wedge x_{j_{n+1}}^G = \sum_{i=0}^{2^n} \alpha_i \wedge x_{j_{n+1}}^G$. We now show how to decompose $\alpha_i \wedge x_{j_{n+1}}^G$ into 2 simple tensors. Let $W_i \subset \mathcal{C}_n$ be the $(k+n)$-dimensional space represented by α_i. There exists a p-dimensional space $W \subset \mathcal{C}_n$ such that $W_i \subset W$. Let $y \in W$ with $\|y\| = 1$ such that $\sin\theta(W_i, \mathrm{Span}(y)) \geq 1/2$. Then

$$\alpha_i \wedge x_{j_{n+1}}^G = \alpha_i \wedge (x_{j_{n+1}}^G - \frac{8^{p-n}}{a} \|x_{j_{n+1}}^G\| y) + \alpha_i \wedge \frac{8^{p-n}}{a} \|x_{j_{n+1}}^G\| y.$$

The $(k+n+1)$-dimensional subspace represented by the second term is contained in $W \subset \mathcal{C}_n \subset \mathcal{C}_{n+1}$ and we have:

$$\begin{array}{rcl}\left\|\alpha_i \wedge \frac{8^{p-n}}{a}\|x_{j_{n+1}}\|y\right\| & \leq & \|\alpha_i\|.\frac{8^{p-n}}{a}\|x^G_{j_{n+1}}\| \\ & \leq & K_n\|x_{i_1}\|\dots\|x_{i_k}\|\|x_{j_1}\|\dots\|x_{j_n}\|.\frac{8^{p-n}}{a}\|x^G_{j_{n+1}}\| \\ & \leq & \frac{8^{p-n}}{a}K_n\|\operatorname{Id}-\pi\|\|x_{i_1}\|\dots\|x_{i_k}\|\|x_{j_1}\|\dots\|x_{j_n}\|\|x_{j_{n+1}}\|.\end{array}$$

We now prove that the first term $\alpha_i \wedge (x^G_{j_{n+1}} - \frac{8^{p-n}}{a}\|x^G_{j_{n+1}}\|y)$ represents a subspace contained in $\mathcal{C}_{n+1}$. First, let $u = u_i + \lambda(x^G_{j_{n+1}} - \frac{8^{p-n}}{a}\|x^G_{j_{n+1}}\|y)$ be a vector in this subspace, with $u_i \in W_i$ and $\lambda \in \mathbb{C}$. Since $W_i \subset \mathcal{C}_n$, we have $u_i = u_i^F + u_i^G$ with $\frac{a}{8^{p-n}}\|u_i^F\| \geq \|u_i^G\|$. Then

$$u = u_i + \lambda(x^G_{j_{n+1}} - \frac{8^{p-n}}{a}\|x^G_{j_{n+1}}\|y) = (u_i^F - \frac{8^{p-n}}{a}\lambda\|x^G_{j_{n+1}}\|y) + (u_i^G + \lambda x^G_{j_{n+1}}).$$

We estimate the norms of each of the latter terms. As $\sin\theta(W_i, \operatorname{Span}(y)) \geq 1/2$,

$$\left\|u_i^F - \frac{8^{p-n}}{a}\lambda\|x^G_{j_{n+1}}\|y\right\| \geq \frac{\|u_i^F\|}{4} \text{ and } \left\|u_i^F - \frac{8^{p-n}}{a}\lambda\|x^G_{j_{n+1}}\|y\right\| \geq \frac{8^{p-n}}{4a}|\lambda|\|x^G_{j_{n+1}}\|,$$

therefore

$$\left\|u_i^F - \frac{8^{p-n}}{a}\lambda\|x^G_{j_{n+1}}\|y\right\| \geq \frac{1}{8}\|u_i^F\| + \frac{1}{8}\frac{8^{p-n}}{a}|\lambda|\|x^G_{j_{n+1}}\|.$$

On the other hand, we have

$$\begin{array}{rcl}\left\|u_i^G + \lambda x^G_{j_{n+1}}\right\| & \leq & \left\|u_i^G\right\| + |\lambda|\|x^G_{j_{n+1}}\| \\ & \leq & \frac{8^{p-n}}{a}\|u_i^F\| + |\lambda|\|x^G_{j_{n+1}}\| \\ & \leq & \frac{8^{p-n-1}}{a}\left\|u_i^F - \frac{8^{p-n}}{a}\lambda\|x^G_{j_{n+1}}\|y\right\|.\end{array}$$

That shows that $u \in \mathcal{C}_{n+1}$ and thus $\alpha_i \wedge (x^G_{j_{n+1}} - \frac{8^{p-n}}{a}\|x^G_{j_{n+1}}\|y)$ represents a subspace contained in $\mathcal{C}_{n+1}$. A computation similar to the first case show that

$$\left\|\alpha_i \wedge (x^G_{j_{n+1}} - \frac{8^{p-n}}{a}\|x^G_{j_{n+1}}\|y)\right\| \leq \left(1 + \frac{8^{p-n}}{a}\right) K_n\|\operatorname{Id}-\pi\|\|x_{i_1}\|\dots\|x_{i_k}\|\|x_{j_1}\|\dots\|x_{j_n}\|\|x_{j_{n+1}}\|.$$

We have shown that $\alpha_i \wedge x^G_{j_{n+1}}$ can be decomposed in a sum of 2 simple tensors representing subspaces of $\mathcal{C}_{n+1}$. Moreover, we have proved that the norm of these tensors is bounded by

$$K_{n+1}\|x_{i_1}\|\dots\|x_{i_k}\|\|x_{j_1}\|\dots\|x_{j_n}\|\|x_{j_{n+1}}\|,$$

where $K_{n+1} = \left(1 + \frac{8^{p-n}}{a}\right) K_n\|\operatorname{Id}-\pi\|$. Since $x^F_{i_1} \wedge \dots \wedge x^F_{i_k} \wedge x^G_{j_1} \wedge \dots \wedge x^G_{j_{n+1}} = \sum_{i=0}^{2^n} \alpha_i \wedge x^G_{j_{n+1}}$, this achieves the recursion.

To conclude, the result of the recursion for $n = p - k$ shows that we can decompose

$$x^F_{i_1} \wedge \dots \wedge x^F_{i_k} \wedge x^G_{j_1} \wedge \dots \wedge x^G_{j_{p-k}} = \sum_{i=0}^{2^{p-k}} \alpha_i$$

where the α_i represent subspaces of $\mathcal{C}_{p-k} \subset \mathcal{C}_{\pi,a}$, and $\|\alpha_i\| \leq C\|x_1\| \dots \|x_p\|$ for some constant C (depending on p, π and a). Since we have shown

$$x_1 \wedge \cdots \wedge x_p = \sum_{(S_1,\dots,S_p)\in\{F,G\}^p} x_1^{S_1} \wedge \cdots \wedge x_p^{S_p},$$

$x_1 \wedge \cdots \wedge x_p$ can be decomposed into a giant sum $\sum_{i\in I}\alpha_i$ (with $|I| \leq 4^p$) where α_i represent subspaces of $\mathcal{C}_{\pi,a}$. As we can choose $(x_1/\|x_1\|, \dots, x_p/\|x_1\|)$ to be a right decomposition without changing $x_1 \wedge \cdots \wedge x_p$, we get by Lemma 2.2.7:

$$\sum_{i\in I}\|\alpha_i\| \leq |I|C\|x_1\| \dots \|x_p\| \leq 4^p C\|x_1 \wedge \cdots \wedge x_p\|_{\wedge l,p} \leq 4^p C b_p\|x_1 \wedge \cdots \wedge x_p\|,$$

where b_p is a constant given by Lemma 2.2.12. This proves that the cone $\mathcal{C}_{\pi,a}$ is reproducing. □

5.4.2 Comparing the Hausdorff metric and the gauge $d_{\mathcal{C}}$

In this section, we consider $E = F \oplus G$ a topological decomposition with $\dim F = p$, and π the continuous projection on F parallel to G. We first state a very simple lemma to estimate norms in projection cones. In the remainder of this chapter, we define our cone gauges with the gauge of Dubois [Dub09] (see Definitions 3.1.3 and 3.2.2).

Lemma 5.4.1. *Let $a > 0$. For every $x \in \mathcal{C}_{\pi,a}$, the inequalities $\|\pi(x)\| \geq \frac{1}{1+a}\|x\|$ and $\|(\mathrm{Id} - \pi)(x)\| \leq a\|\pi\|\|x\|$ hold.*

Proof.

$$\|x\| \leq \|\pi(x)\| + \|(\mathrm{Id} - \pi)(x)\| \leq \|\pi(x)\| + a\|\pi(x)\| = (1+a)\|\pi(x)\| \ ;$$

$$\|(\mathrm{Id} - \pi)(x)\| \leq a\|\pi(x)\| \leq a\|\pi\|\|x\|.$$ □

In the next lemma, we consider a subspace V and prove that V and F being close is equivalent to V being in a projection cone centered on F.

Lemma 5.4.2. *Let $0 < d < \|\mathrm{Id} - \pi\|^{-1}$, $a > 0$, and consider V a p-dimensional subspace of E. We have:*

- *If $d_H(V,F) \leq d$, then $V \subset \mathcal{C}_{\pi,\gamma}$, where $\gamma = \frac{\|\mathrm{Id} - \pi\|.d}{1-\|\mathrm{Id} - \pi\|.d}$;*
- *If $V \subset \mathcal{C}_{\pi,a}$, then $d_H(V,F) \leq 2a\|\pi\|$.*

Proof. Assume that $d_H(V,F) \leq d$. Let $x \in V$ with $\|x\| = 1$ and consider the decomposition $x = x_F + x_G$ given by π. Then for all $y_F \in F$, $x_G = (\mathrm{Id} - \pi)(x - y_F)$: this shows that $\|x_G\| \leq \|\mathrm{Id} - \pi\|d$. Then we have $\|x_F\| \geq 1 - \|\mathrm{Id} - \pi\|.d \geq \frac{1-\|\mathrm{Id} - \pi\|.d}{\|\mathrm{Id} - \pi\|.d}\|x_G\|$, and therefore $x \in \mathcal{C}_{\pi,\gamma}$ for $\gamma = \frac{\|\mathrm{Id} - \pi\|.d}{1-\|\mathrm{Id} - \pi\|.d}$. This proves $V \subset \mathcal{C}_{\pi,\gamma}$.

Assume that $V \in \mathcal{C}_{\pi,a}$. Let $x \in V$ with $\|x\| = 1$ and consider the decomposition $x = x_F + x_G$ given by π. Then $d(x, x_F) = \|x_G\| \leq a\|\pi\|$ hence $d(x, S_F) \leq 2a\|\pi\|$.

We claim that $\pi_{|V} : V \to F$ is an isomorphism. Indeed, it is injective as $V \subset \mathcal{C}_{\pi,a}$, and as $\dim V = \dim F$, it is bijective. Let $y \in F$ and $x \in V$ such that $\pi(x) = y$ (it exists as $\pi_{|V}$ is surjective). Then

$$d(y, V) \leq \|y - x\| = \|(\mathrm{Id} - \pi)(x)\| \leq a\|\pi\|$$

by Lemma 5.4.1. Thus $d(y, S_V) \leq 2a\|\pi\|$ and we finally get $d_H(V, F) \leq 2a\|\pi\|$. □

Given a projection cone $\mathcal{C}_{\pi,a}$, we now get an upper-bound for the cone gauge $d_{\mathcal{C}_{\pi,a}}$ given by the Hausdorff distance to the center-space F.

Lemma 5.4.3. *Assume* $\dim F = p < \infty$. *Let* $0 < a' < a$. *Then for every* p*-dimensional subspace* V *contained in* $\mathcal{C}_{\pi,a'}$, *we have*

$$d_{\mathcal{C}_{\pi,a}}(V, F) \leq \log\left(\frac{a + a'}{a - a'}\right) \frac{(1 + a')\|\mathrm{Id} - \pi\|}{a'} d_H(V, F),$$

where $d_{\mathcal{C}_{\pi,a}}$ *is defined with the gauge of Dubois* $\delta_{\mathcal{C}_{\pi,a}}$.

Proof. Let V be a p-dimensional subspace contained in $\mathcal{C}_{\pi,a'}$. Let $x \in V$ and $y \in F$ with $\|x\| = \|y\| = 1$. We write $x = x_F + x_G$ and $y = y_F + 0$ for the decompositions of x and y with respect to $E = F \oplus G$.

Let $z \in \mathbb{C}$. If $|z| \leq \frac{a\|y_F\|}{a\|x_F\| + \|x_G\|}$, then simple manipulations yield:

$$\begin{aligned} a|z|.\|x_F\| + |z|.\|x_G\| &\leq a\|y_F\| \\ |z|.\|x_G\| &\leq a\|y_F\| - a|z|.\|x_F\| \\ |z|.\|x_G\| &\leq a\|y_F - zx_F\|, \end{aligned}$$

so $zx - y \in \mathcal{C}_{\pi,a}$. Similarly, if $|z| \geq \frac{a\|y_F\|}{a\|x_F\| - \|x_G\|}$ (which is defined and positive as $x \in \mathcal{C}_{\pi,a'}$), then $zx - y \in \mathcal{C}_{\pi,a}$. We then have, using the gauge $\delta_{\mathcal{C}_{\pi,a}}$ of Dubois (see Definitions 3.1.3 and 3.2.2):

$$\delta_{\mathcal{C}_{\pi,a}}(x, y) \leq \log\left(\frac{\frac{a\|y_F\|}{a\|x_F\| - \|x_G\|}}{\frac{a\|y_F\|}{a\|x_{\|} + \|x_G\|}}\right) = \log\left(\frac{a\|x_F\| + \|x_G\|}{a\|x_F\| - \|x_G\|}\right) = \log\left(\frac{1 + \frac{\|x_G\|}{a\|x_F\|}}{1 - \frac{\|x_G\|}{a\|x_F\|}}\right).$$

As $\frac{\|x_G\|}{a\|x_F\|} \leq \frac{a'}{a}$ and $x \mapsto \log(\frac{1+x}{1-x})$ is convex on $[0, \frac{a'}{a}]$, we get

$$\delta_{\mathcal{C}_{\pi,a}}(x, y) \leq \log\left(\frac{a + a'}{a - a'}\right) \frac{a}{a'} \frac{\|x_G\|}{a\|x_F\|} \leq \log\left(\frac{a + a'}{a - a'}\right) \frac{\|x_G\|}{a'\|x_F\|}.$$

We now obtain an upper-bound on $\|x_G\|$ using the Hausdorff distance between the subspaces V and F:

$$\begin{aligned} d_H(V, F) &\geq d(x, F) && \text{by Lemma 2.1.4} \\ &\geq d(x_G, F) \\ &\geq \|x_G\| d(\tfrac{x_G}{\|x_G\|}, F) \\ &\geq \|x_G\| \sin\theta(G, F) && \text{(Definition 2.1.5)} \\ &\geq \frac{\|x_G\|}{\|\mathrm{Id} - \pi\|} && \text{by Lemma 2.1.6.} \end{aligned}$$

This proves that $\|x_G\| \leq \|\operatorname{Id} - \pi\| d_H(V, F)$. By Lemma 5.4.1, we have $\|x_F\| \geq 1/(1+a')$. Finally, we get

$$\delta_{\mathcal{C}_{\pi,a}}(x, y) \leq \log\left(\frac{a+a'}{a-a'}\right) \frac{(1+a')\|\operatorname{Id} - \pi\|}{a'} d_H(V, F).$$

Since the gauge of Dubois is projective, taking the supremum over all $(x, y) \in S_V \times S_W$ ends the proof of the lemma. □

Remark. For $a' = a/2$, this gives: for every p-dimensional subspace V contained in $\mathcal{C}_{\pi,a'}$,

$$d_{\mathcal{C}_{\pi,a}}(V, F) \leq \log(3) \frac{(2+a)\|\operatorname{Id} - \pi\|}{a} d_H(V, F)\,.$$

The next lemma states that, given positive real a and a' with $a' < a$, a small enough cone centered in $V \in \mathcal{C}_{\pi,a'}$ is a subset of $\mathcal{C}_{\pi,a}$.

Lemma 5.4.4. *Let $0 < a' < a$, $V \subset \mathcal{C}_{\pi,a'}$ of dimension p and π_V the projection on V parallel to G. Then*

$$\mathcal{C}_{\pi_V,\delta} \subset \mathcal{C}_{\pi,a}\ , \text{ where } \delta = \frac{a-a'}{1+a'}.$$

Proof. Let $x \in \mathcal{C}_{\pi_V, \frac{a-a'}{1+a'}}$ and $x = x_V + x_G$ its decomposition with respect to π_V. Consider the decomposition $x_V = x_F + x'_G$ associated with $E = F \oplus G$. Then $x = x_F + (x'_G + x_G)$, and, using the estimates from Lemma 5.4.1,

$$\|x'_G + x_G\| \leq \|x'_G\| + \|x_G\| \leq a'\|x_F\| + \frac{a-a'}{1+a'}\|x_V\| \leq a'\|x_F\| + (a-a')\|x_F\| \leq a\|x_F\|.$$

This proves that $x \in \mathcal{C}_{\pi,a}$, hence $\mathcal{C}_{\pi_V,\delta} \subset \mathcal{C}_{\pi,a}$. □

We now have all the tools to prove the equivalence between the cone gauge and the Hausdorff distance on specific subsets of our cone.

Proposition 5.4.5. *If $0 < a' < a$, then d_H and $d_{\mathcal{C}_{\pi,a}}$ are equivalent on $\mathcal{C}_{\pi,a'}$.*

Proof. Let $0 < a' < a$. Let V, W be subspaces of dimension p contained in $\mathcal{C}_{\pi,a'}$. As $\mathcal{C}_{\pi,a'}$ has aperture bounded by some $K > 0$, let $m : E \to \mathbb{C}^p$ be a linear map of norm 1 satisfying Definition 3.3.1. Let $\hat{x} = x_1 \wedge \cdots \wedge x_p$ be a representative of V and $\hat{y} = y_1 \wedge \cdots \wedge y_p$ be a representative of W, with $\|\hat{x}\| = \|\hat{y}\| = 1$. Then, by Lemma 3.3.5, there exists a constant $a_p > 0$ such that

$$\left\| \frac{\hat{x}}{\hat{m}(\hat{x})} - \frac{\hat{y}}{\hat{m}(\hat{y})} \right\| \leq a_p K^p d_{\mathcal{C}_{\pi,a}}(V, W).$$

Using Lemma 3.3.4, we get

$$\left\| \hat{x} - \frac{\hat{m}(\hat{x})}{\hat{m}(\hat{y})}\hat{y} \right\| \leq a_p K^p \|\hat{m}(\hat{x})\| d_{\mathcal{C}_{\pi,a}}(V, W) \leq a_p K^p d_{\mathcal{C}_{\pi,a}}(V, W),$$

showing that $d(\hat{x}, \operatorname{Span}(\hat{y})) \leq a_p K^p d_{\mathcal{C}_{\pi,a}}(V, W)$. By an argument of symmetry, Proposition 2.2.14 shows that $d_\wedge(V, W) \leq 2a_p K^p d_{\mathcal{C}_{\pi,a}}(V, W)$. Using Theorem 2.2.15, there exists a constant C such that $d_H(V, W) \leq C d_{\mathcal{C}_{\pi,a}}(V, W)$.

Now, we prove the converse inequality. Let π_V be the projection on V parallel to G. We want to estimate $\|\pi_V\|$. Consider $x \in E$, $x = x_V + x_G$ its decomposition with respect to $E = V \oplus G$ and $x_V = x_F + x'_G$ the decomposition of x_V with respect to $E = F \oplus G$. As $V \in \mathcal{C}_{\pi,a'}$, we have

$$\|x_V\| \leq (1+a')\|x_F\| \leq (1+a')\|\pi\|\|x\|.$$

Hence $\|\pi_V\| \leq (1+a')\|\pi\|$ and therefore

$$\|\operatorname{Id} - \pi_V\| \leq 1 + (1+a')\|\pi\|. \tag{$*$}$$

Set $\delta := \frac{a-a'}{1+a'}$ and $D > 0$ such that $\frac{(1+(1+a')\|\pi\|)D}{1-(1+(1+a')\|\pi\|)D} \leq \frac{\delta}{2}$.

- If $d_H(V,W) \leq D$, then using Lemma 5.4.2 and Inequality $(*)$ we get $W \subset \mathcal{C}_{\pi_V,\frac{\delta}{2}}$. By Lemma 5.4.4, we have $\mathcal{C}_{\pi_V,\delta} \subset \mathcal{C}_{\pi,a}$, hence $d_{\mathcal{C}_{\pi,a}}(V,W) \leq d_{\mathcal{C}_{\pi_V,\delta}}(V,W)$. Then, Lemma 5.4.3 shows that

$$d_{\mathcal{C}_{\pi,a}}(V,W) \leq \log(3)\frac{(2+\delta)\|\operatorname{Id} - \pi_V\|}{\delta} d_H(V,W),$$

  and the inequality $(*)$ gives

$$d_{\mathcal{C}_{\pi,a}}(V,W) \leq \log(3)\frac{(2+\delta)(1+(1+a')\|\pi\|)}{\delta} d_H(V,W).$$

- If $d_H(V,W) \geq D$, then let $\Delta \leq 2\log\left(\frac{a+a'}{a-a'}\right)$ be the diameter of $\mathcal{C}_{\pi,a'}$ for $d_{\mathcal{C}_{\pi,a}}$. Then

$$d_{\mathcal{C}_{\pi,a}}(V,W) \leq \frac{\Delta}{D}.D \leq \frac{\Delta}{D} d_H(V,W).$$

Thus we have

$$d_{\mathcal{C}_{\pi,a}}(V,W) \leq \min\left(\log(3)\frac{(2+\delta)(1+(1+a')\|\pi\|)}{\delta}, \frac{\Delta}{D}\right) d_H(V,W)$$

and d_H and $d_{\mathcal{C}_{\pi,a}}$ are equivalent on $\mathcal{C}_{\pi,a'}$. □

Corollary 5.4.6. *Let $\mathcal{C}$ be a p-cone of bounded aperture, containing a projection cone $\mathcal{C}_{\pi,a}$. For $0 < a' < a$, d_H and $d_{\mathcal{C}}$ are equivalent on $\mathcal{C}_{\pi,a'}$.*

Proof. The exact same argument as before shows that there exists a constant C such that $d_H(V,W) \leq C d_{\mathcal{C}}(V,W)$. Moreover, as $\mathcal{C}_{\pi,a} \subset \mathcal{C}$, we get $d_{\mathcal{C}} \leq d_{\mathcal{C}_{\pi,a}}$ and the previous proposition gives us the desired result. □

5.4.3 The sets $\mathcal{C}[\rho]$, $\mathcal{C}_{\pi,a}$, and the interior of $\mathcal{C}$ in $\mathcal{G}_p(E)$

We now study the set $\mathcal{C}[\rho]$ and its relationship with the interior of $\mathcal{C}$ in $\mathcal{G}_p(E)$. First, we prove a lemma stating that if x is close to a subspace $V \in \mathcal{G}_p(E)$, then there exists a subspace $V_x \in \mathcal{G}_p(E)$ containing x which is also close to V.

Proposition 5.4.7. *Let $r > 0$. Let $V \in \mathcal{G}_p(E)$ and $x \in E$ with $\|x\| = 1$ such that $d(x, V) \leq r$. Then there exists $V_x \in \mathcal{G}_p(E)$ containing x such that*

$$\delta(V_x, V) \leq 4r \text{ and } \delta(V, V_x) \leq 4r.$$

This also gives $d_H(V_x, V) \leq 8r$.

Proof. If $x \in V$, then $V_x = V$ immediately gives the desired result. When $r > \frac{1}{4}$, since the gap δ and the distance d_H are bounded respectively by 1 and 2, any subspace V_x containing x works.

Otherwise, let $u_1 \in S_V$ be a closest element to x in V (it exists as V is finite-dimensional). By Lemma 2.1.4, we have $\|x - u_1\| \leq 2r$. Using Proposition 2.2.8, we complete u_1 into a right decomposition $(u_1, u_2, \ldots, u_p)$ of V with adapted linear forms $(l_1, \ldots, l_p)$, and denote $V_2 = \text{Span}(u_2, \ldots, u_p)$. We set $V_x = \text{Span}(x, u_2, \ldots, u_p)$. It is a p-dimensional space as $x \notin V$. Now, we prove that this space V_x satisfies the stated inequalities.

Let $y = ax + \sum_{i=2}^{p} a_i u_i \in V_x$ with $\|y\| = 1$. First, see that

$$|l_1(x)| = |1 + l_1(x - u_1)| \geq 1 - \|x - u_1\| \geq 1 - 2r.$$

Then

$$|a| \leq \frac{|l_1(ax)|}{1 - 2r} = \frac{|l_1(y)|}{1 - 2r} \leq 2,$$

since $\|y\| = 1$, $\|l_1\| = 1$ and $r \leq 1/4$. We can now compute:

$$d(y, V) = \inf_{v \in V} \|y - v\| \leq \|a_1 x - a_1 u_1\| \leq |a_1| \|x - u_1\| \leq 4r.$$

This shows that $\delta(V_x, V) \leq 4r$. Now, let $y = a_1 u_1 + \sum_{i=2}^{p} a_i u_i \in V$ with $\|y\| = 1$. By Lemma 2.2.11, we have $|a_1| \leq 1$. Then we have

$$d(y, V_x) = \inf_{v \in V_x} \|y - v\| \leq \|a_1 u_1 - a_1 x\| \leq |a_1| \|x - u_1\| \leq 4r,$$

proving that $\delta(V_x, V) \leq 4r$. The inequality $d_H(V_x, V) \leq 8r$ follows from Lemma 2.1.4. □

Lemma 5.4.8. *Let $\mathcal{C}$ be a p-cone. Then $\mathcal{C}$ has non-empty interior in $\mathcal{G}_p(E)$ if and only if there exists $\rho > 0$ such that $\mathcal{C}[\rho] = \{x \in E \mid B(x, \rho\|x\|) \subset \mathcal{C}\}$ contains a p-dimensional subspace.*

Proof. Assume that $\mathcal{C}$ has non-empty interior in $\mathcal{G}_p(E)$, i.e. there exists a p-dimensional space $W_0 \subset \mathcal{C}$ and $r > 0$ such that $\forall W \in B_{d_H}(W_0, r), W \subset \mathcal{C}$. Let $x \in W_0$ with $\|x\| = 1$ and $y \in B(x, r/8)$. Then by Proposition 5.4.7, there exists $W \in \mathcal{G}_p(E)$ such that $y \in W$ and $d_H(W, W_0) < r$. Then $W \subset \mathcal{C}$ and $y \in \mathcal{C}$, which shows that $B(x, r/8) \subset \mathcal{C}$ and therefore $W_0 \subset \mathcal{C}[\rho]$.

Conversely, assume that there exists $\rho > 0$ such that $\mathcal{C}[\rho] = \{x \in E \mid B(x, \rho\|x\|) \subset \mathcal{C}\}$ contains a p-dimensional subspace W_0. Let $W \in B_{d_H}(W_0, \rho)$. For all $x \in W$ with $\|x\| = 1$, there exists $x_0 \in W_0$ with $\|x_0\| = 1$ such that $d(x, x_0) < \rho$. Then $x \in B(x_0, \rho) \subset \mathcal{C}$, hence $W \subset \mathcal{C}$ and $\mathcal{C}$ has non-empty interior in $\mathcal{G}_p(E)$.

□

The next lemma achieves to connect $\mathcal{C}[\rho]$ and the projection cones. These results are summed up in the subsequent proposition.

Lemma 5.4.9. *Let $\mathcal{C}$ be a p-cone and $\rho > 0$ such that $\mathcal{C}[\rho]$ contain a p-dimensional space. Let $V \in \mathcal{G}_p(E)$ such that $V \subset \mathcal{C}[\rho]$ and W a p-codimensional subspace in $GC(\mathcal{C})$ (see Definition 4.1.2). Let P be the projection onto V parallel to W. Then $\sin\theta(V,W) \geq \rho$. Moreover, the cone $\mathcal{C}_{P,\frac{\rho^2}{4}}$ associated with P satisfies: $\mathcal{C}_{P,\frac{\rho^2}{4}} \subset \mathcal{C}$.*

Proof. If $W \in GC(\mathcal{C})$, then $W \cap \mathcal{C} = \{0\}$. Since $V \subset \mathcal{C}[\rho]$, we get $W \cap B(x,\rho) = \emptyset$ for all $x \in V$ with $\|x\| = 1$. This proves $\sin\theta(V,W) = \inf\{\|x-y\| \mid x \in V, \|x\| = 1, y \in W\} \geq \rho$. By Lemma 2.1.6, we also have $\|P\| \leq \frac{1}{\rho}$.

Let $V_1 \in \mathcal{C}_{P,\frac{\rho^2}{4}}$. Then Lemma 5.4.2 shows that $d_H(V_1,V) \leq \frac{\rho^2}{2}\|P\| \leq \frac{\rho}{2}$. By the proof of Lemma 5.4.8, as $V \subset \mathcal{C}[\rho]$, we get $V_1 \subset \mathcal{C}$. This proves the inclusion $\mathcal{C}_{P,\frac{\rho^2}{4}} \subset \mathcal{C}$. □

Proposition 5.4.10. *Let $\mathcal{C}$ be a p-cone. The following assertions are equivalent:*

(i) $\mathcal{C}$ has non-empty interior in $\mathcal{G}_p(E)$;

(ii) There exists $\rho > 0$ such that $\mathcal{C}[\rho]$ contains a p-dimensional subspace;

(iii) there exists a continuous projection π onto a p-dimensional subspace and $a > 0$ such that $\mathcal{C}$ contains $\mathcal{C}_{\pi,a}$.

In particular, every projection cone satisfies these conditions.

Proof. The equivalence between (i) and (ii) is exactly Lemma 5.4.8. The implication $(ii) \Rightarrow (iii)$ is given by Lemma 5.4.9. Finally, assume there exists a continuous projection π onto a p-dimensional subspace F and $a > 0$ such that $\mathcal{C}$ contains $\mathcal{C}_{\pi,a}$. Let $d > 0$ such that $\frac{\|\operatorname{Id}-\pi\|.d}{1-\|\operatorname{Id}-\pi\|.d} < a$. Then Lemma 5.4.2 shows that for all $V \in B_{d_H}(F,d)$, $V \subset \mathcal{C}_{\pi,a} \subset \mathcal{C}$, hence $\mathcal{C}$ has non-empty interior in $\mathcal{G}_p(E)$. □

We now obtain a criteria to show that a cone is reproducing (see Definition 3.3.7).

Proposition 5.4.11. *Let $\mathcal{C}$ be a p-cone satisfying any of the assertions given in Proposition 5.4.10. Then $\mathcal{C}$ is reproducing.*

Proof. This is a direct consequence of Proposition 5.4.10 and the fact that every projection cone is reproducing (Proposition 3.3.8). □

We conclude this section by establishing the result used in the proof of Lemma 3.4.17 in Chapter 3.

Proposition 5.4.12. *Let $\mathcal{C}$ be a p-cone of bounded aperture and $\rho > 0$ such that $\mathcal{C}[\rho]$ contains a p-dimensional space V_0. Then there exists a constant $r > 0$, depending on ρ and p, such that the cone gauge $d_{\mathcal{C}}$ and the Hausdorff distance d_H are equivalent on the neighborhood $B_{d_H}(V_0,r)$ of V_0.*

Proof. Consider W a p-codimensional subspace in $GC(\mathcal{C})$ and denote by P the projection onto V_0 parallel to W. By Lemma 5.4.2, there exists a constant $r > 0$ such that: for every p-dimensional subspace V with $d_H(V_0, V) \leq r$, we have $V \subset \mathcal{C}_{P,\frac{\rho^2}{8}}$. Moreover, Lemma 5.4.9 states that $\mathcal{C}_{P,\frac{\rho^2}{4}} \subset \mathcal{C}$. Then Corollary 5.4.6 shows that $d_{\mathcal{C}}$ and d_H are equivalent on $\mathcal{C}_{P,\frac{\rho^2}{8}}$, therefore on $B_{d_H}(V_0, r)$.

One can check that the constant r given by Lemma 5.4.2 and the constants of equivalence from Corollary 5.4.6 only depends on ρ, p and $\|P\|$. Since $\|P\| \leq \frac{1}{\rho}$ by Lemma 5.4.9, they only depend on ρ and p. $\square$

Bibliography

[ABY10] A. Avila, J. Bochi, J.–C. Yoccoz, *Uniformly hyperbolic finite-valued SL(2, R)-cocycles*, Comment. Math. Helv. 85 , no. 4, pp. 813–884 (2010).

[Bal99] V. Baladi, *Decay of correlations*, 1999 AMS Summer Institute on Smooth ergodic theory and applications, Seattle, Proc. Symposia in Pure Math. Vol. 69, AMS, pp. 297-325 (2001)

[Bal00] V. Baladi, *Positive Transfer Operators and Decay of Correlations*, Book Advanced Series in Nonlinear Dynamics, Vol 16, World Scientific, Singapore (2000).

[Ber63] E. Berkson, *Some metrics on the subspaces of a Banach space*, Pacific J. Math 13, pp. 7–22 (1963).

[Bir57] G. Birkhoff, *Extensions of Jentzsch's theorem*, Trans. Amer. Math. Soc. 85, pp. 219-227 (1957).

[BG09] J. Bochi, N. Gourmelon, *Some characterizations of domination*, Mathematische Zeitschrift 263, no. 1, pp. 221-231 (2009).

[BM15] J. Bochi, I.D. Gourmelon, *Continuity properties of the lower spectral radius*, Proc. London Math. Soc. 110, pp. 477–509 (2015).

[Blu16] A. Blumenthal, *A volume-based approach to the multiplicative ergodic theorem on Banach spaces*, Discrete Contin. Dyn. Syst. A 36 (5), pp. 2377-2403 (2016).

[BlM15] A. Blumenthal, I. Morris, *Characterization of dominated splittings for operator cocycles acting on Banach spaces*, submitted.

[BoM15] J. Bochi, I. Morris, *Continuity properties of the lower spectral radius*, Proc. Lond. Math. Soc. (3), 110, no. 2, pp. 477–509 (2015).

[Bo07A] N. Bourbaki, *Éléments de mathématique. Algèbre. Chapitre 1 à 3*, Springer (2007).

[Bo07V] N. Bourbaki, *Éléments de mathématique. Variétés différentielles et analytiques*, Springer (2007).

[Bre10] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer Science and Business Media (2010).

[BRS05] J.D. Biggins, A. Rahimzadeh Sani, *Convergence results on multitype, multivariate branching random walks*, Adv. in Appl. Probab. 37, no. 3, pp. 681–705 (2005).

[BV05] J. Bochi, M. Viana, *The Lyapunov exponents of generic volume-preserving and symplectic maps*, Ann. of Math. (2), 161, pp. 1423–1485 (205).

[CG93] L. Carleson, T.W. Gamelin, *Complex Dynamics, Universitext: Tracts in Math.*, Springer-Verlag, New York (1993).

[Dub09] L. Dubois, *Projective metrics and contraction principles for complex cones*, J. Lond. Math. Soc. 79, pp. 719–737 (2009).

[FGQ15] F. Froyland, C. Gonzalez-Tokman, A. Quas, *Stochastic stability of Lyapunov exponents and Oseledets splittings for semi-invertible matrix cocycles*, Comm. Pure Appl. Math. 68, pp. 2052-2081 (2015).

[FGQ17] F. Froyland, C. Gonzalez-Tokman, A. Quas, *Hilbert Space Lyapunov Exponent stability*, Trans. Amer. Math. Soc. (to appear).

[FLQ13] G. Froyland, S. Lloyd, A. Quas, *A semi-invertible oseledets theorem with applications to transfer operator cocycles*, Discrete Contin. Dyn. Syst. 33 (2013).

[Fro12] G. Frobenius, *Über Matrizen aus nicht negativen Elementen*, S.-B. Preuss. Akad. Wiss., pp. 456-477 (1912).

[Gro52] A. Grothendieck, *Résumé des résultats essentiels dans la théorie des produits tensoriels topologiques et des espaces nucléaires*, Annales de l'institut Fourier, tome 4, pp. 73-112 (1952).

[GTQ14] C. Gonzalez-Tokman, A. Quas, *A semi-invertible operator oseledets theorem*, Ergodic Theory Dynam. Systems, Vol. 34 (04), pp. 1230–1272, (2014).

[GTQ15] C. Gonzalez-Tokman, A. Quas, *A concise proof of the Multiplicative Ergodic Theorem on Banach Spaces*, J. Mod. Dyn. 9, pp. 237-255 (2015).

[GW06] S. Gan, L. Wen, *Nonsingular star flows satisfy Axiom A and the no-cycle condition*, Invent. Math., 164, pp. 279–315 (2006).

[Har92] J. Harris, *Algebraic Geometry: A First Course*, Springer Science and Business Media (1992).

[Kat95] T. Kato, *Perturbation Theory for Linear Operators*, Springer (1995).

[KH95] A. Katok, B. Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems*, Cambridge University Press (1995).

[LN14] B. Lemmens, R. Nussbaum, *Birkhoff's version of Hilbert's metric and its applications in analysis*, Handbook of Hilbert Geometry, IRMA Lectures in Mathematics and Theoretical Physics No. 22, Chapter 10 (2014).

[Man83] R. Mañé, *Lyapounov exponents and stable manifolds for compact transformations*, Geometric dynamics, Springer, pp. 522-577 (1983).

[Mor10] I.D. Morris, *A rapidly-converging lower bound for the joint spectral radius via multiplicative ergodic theory*, Adv. Math., 225, pp. 3425–3445 (2010).

[NN87] P. Ney, E. Numellin, *Markov additive processes*, I and II, Ann. Probab. 15, pp. 561-592 (1987).

[Ose68] V.I. Oseledec, *A multiplicative ergodic theorem. Characteristic Ljapunov, exponents of dynamical systems*, Trudy Moskov. Mat. Obsc. 19, pp. 179–210 (1968).

[Per07] O. Perron, *Zur Theorie der Matrices*, Math. Ann. 64, pp. 248-263 (1907).

[PS09] E.R. Pujals, M. Sambarino, *On the dynamics of dominated splitting*, Ann. of Math. (2), 169, pp. 675–739 (2009).

[Rag79] M. Raghunathan, *A proof of Oseledec's multiplicative ergodic theorem*, Israel J. Math. 32(4), pp. 356–362 (1979).

[Rue79] D. Ruelle, *Analycity properties of the characteristic exponents of random matrix products*, Adv. Math. 32, pp. 68-80 (1979).

[Rue82] D. Ruelle, *Characteristic Exponents and Invariant Manifolds in Hilbert Space*, Ann. of Math. (2) , Vol. 115, No. 2, pp. 243-290 (1982).

[Rug02] H.H. Rugh, *Coupled maps and analytic function spaces.*, Ann. Sci. École Norm. Sup. 35, pp. 489–535 (2002).

[Rug10] H.H. Rugh, *Cones and gauges in complex spaces: Spectral gaps and complex Perron-Frobenius theory*, Ann. of Math. 171, pp. 1707–1752 (2010).

[Thi87] P. Thieullen, *Fibrés dynamiques asymptotiquement compacts exposants de lyapounov. Entropie. Dimension.*, Ann. Inst. H. Poincaré Anal. Non Linéaire, volume 4, pp. 49–97, Gauthier-Villars, 1987.

[Wal82] P. Walters, *An Introduction to Ergodic Theory*, Springer-Verlag, New York (1982).

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Titre : Contraction de cônes complexes multidimensionnels

Mots clés : contraction, cône, trou spectral, exposants de Lyapunov, grassmannienne, décomposition dominée.

Résumé : L'objet de cette thèse est l'introduction, l'étude et l'utilisation des cônes complexes multidimensionnels. Dans un premier temps, nous étudions la grassmannienne des espaces de Banach. Nous définissons une notion de bonne décomposition pour les espaces de dimension p et nous démontrons l'équivalence entre la distance de Hausdorff sur la grassmannienne et la distance fournie par une norme sur l'algèbre extérieure.
Dans un deuxième temps, nous définissons les cônes complexes p-dimensionnels ainsi qu'une jauge sur les sous-espaces de dimension p de ces cônes. Nous montrons alors un principe de contraction pour cette jauge. Cela nous permet de prouver, pour un opérateur contractant un tel cône, l'existence d'un trou spectral séparant les p valeurs propres dominantes du reste du spectre. Nous utilisons cette théorie pour démontrer un théorème de régularité analytique pour les exposants de Lyapunov d'un produit aléatoire d'opérateurs contractant un même cône. Nous donnons également une comparaison entre la distance de Hausdorff entre espaces vectoriels et notre jauge.
Enfin, nous introduisons une notion de cône dual pour les cônes p-dimensionnels. Dans ce cadre, nous prouvons que les propriétés topologiques d'un cône se traduisent en propriétés topologiques sur son dual, et réciproquement. Nous complétons le théorème de régularité précédent en démontrant l'existence et la régularité d'une décomposition de l'espace en "espace lent" et "espace rapide".

Title : Multidimensional complex cone contraction

Keywords : contraction, cone, spectral gap, Lyapunov exponents, grassmannian, dominated splitting.

Abstract : The object of this thesis is the introduction, the study and the applications of multidimensional complex cones. First, we study the grassmannian of Banach space. We define a notion of right decomposition for p-dimensional spaces and we prove the equivalence between the Hausdorff distance on the grassmannian and the distance given by a norm on the exterior algebra.
Then, we define p-dimensional complex cones and a gauge on the subspaces of dimension p of these cones. We show a contraction principle for this gauge. This allows us to prove, for an operator contracting such a cone, the existence of a spectral gap which isolates the p leading eigenvalues from the rest of the spectrum. We use this theory to prove a theorem of analytic regularity for Lyapunov exponents of a random product of operators contracting a cone. We also give a comparison between the Hausdorff distance for vector spaces and our gauge.
Finally, we introduce a notion of dual cone for p-dimensional cones. In this setting, we prove that the topological properties of a cone translate into topological properties for its dual and conversely. We complete the previous regularity theorem by proving the existence and the regularity of a dominated splitting of the space into a "fast space" and a "slow space".

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