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Olivier Arrigoni

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# THÈSE de DOCTORAT

présentée par

**Olivier ARRIGONI**

*Pour obtenir le grade de  
Docteur en Mathématiques et Applications  
de l'Université Bourgogne Franche-Comté*

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## Familles commutantes d'opérateurs sectoriels et d'opérateurs de Ritt.

Calcul fonctionnel joint  $H^\infty$ , dilatations et fonctions  
carré associés.

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Soutenue publiquement le 9 février 2021 à 14h, devant le jury composé de :

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# Remerciements

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# Préambule

Ce manuscrit de thèse est issue d'articles publiés ou en cours de publication dans des revues spécialisées en Analyse Fonctionnelle, dont les références sont les suivantes.

- O. Arrigoni and C. Le Merdy,  *$H^\infty$ -functional calculus for commuting families of Ritt operators and sectorial operators*, Oper. Matrices 13 (2019), no. 4, 1055–1090.
- O. Arrigoni and C. Le Merdy, *New properties of the multivariable  $H^\infty$  functional calculus of sectorial operators*, arXiv:2007.04580.
- O. Arrigoni, *Square functions for commuting families of Ritt operators*, arXiv:2009.02270.

L'ensemble est donc rédigé en anglais, dans un souci d'universalité, de sorte que tous puissent avoir accès aux résultats.

Il traite du calcul  $H^\infty$  de  $d$ -uplets commutants d'opérateurs sectoriels ou opérateurs de Ritt. Les outils qui interviennent sont essentiellement les moyennes probabilistes, la géométrie des espaces de Banach, les décompositions de fonctions holomorphes et les fonctions carré associées aux opérateurs considérés.



# Notations

$\mathbb{N} = \{0, 1, 2, \dots\}$ , the set of nonnegative integers.

$\mathbb{Z}$  : set of integers.

$\mathbb{R}$  : set of real numbers.

$\mathbb{R}_+ = [0, \infty)$ .

$\mathbb{C}$  : set of complex numbers.

$E^* = E \setminus \{0\}$ , for any set  $E$  of numbers.

$D(a, r)$  : the open disc centered at  $a \in \mathbb{C}$  with radius  $r > 0$ .

$\mathbb{D} = D(0, 1)$ , the open unit disc of  $\mathbb{C}$ .

$\mathbb{T} = \overline{\mathbb{D}} \setminus \mathbb{D}$ .

$d$  : an integer of  $\mathbb{N}^*$ .

$\mathcal{O}$  : an open subset of  $\mathbb{C}^d$ .

$\partial\mathcal{O}$  : the boundary of  $\mathcal{O}$  in  $\mathbb{C}^d$ .

$X$  : a Banach space.

$B(X)$  : the space of bounded operators acting on  $X$ .

$I_X$  : the identity operator of  $X$ .

$A$  : a (possibly unbounded) operator acting on  $X$ .

$\text{Ker}(A)$  : the kernel of  $A$ .

$\text{Ran}(A)$  : the range of  $A$ .

$\sigma(A)$  : the spectrum of  $A$ .

$R(\lambda, A) = (\lambda I_X - A)^{-1}$  : the resolvent operator of  $A$  ( $\lambda \in \mathbb{C} \setminus \sigma(A)$ ).



$H^\infty(\mathcal{O}; X)$  : the set of bounded holomorphic functions  $f : \mathcal{O} \rightarrow X$ .

$$H^\infty(\mathcal{O}) = H^\infty(\mathcal{O}; \mathbb{C}).$$

$$\|f\|_{\infty, \mathcal{O}} = \sup \{ \|f(z_1, \dots, z_d)\| : (z_1, \dots, z_d) \in \mathcal{O} \},$$

for any  $f$  in  $H^\infty(\mathcal{O}; X)$ .

$p$  : a number of  $[1, \infty)$  or  $(1, \infty)$  if stated.

$(\Omega, \mu)$  : a measure space.

$L_p(\Omega; X)$  or  $L^p(\Omega; X)$  : the Bochner space of measurable functions  $f : \Omega \rightarrow X$  such that  $\int_\Omega \|f(\omega)\|^p d\mu(\omega) < \infty$ .

$$\|f\|_p = \left( \int_\Omega \|f(\omega)\|^p d\mu(\omega) \right)^{\frac{1}{p}}, \text{ for } f \text{ in } L_p(\Omega; X).$$

$$L_p(\Omega) = L_p(\Omega; \mathbb{C}).$$

$S^p(H)$  : the Schatten classes of an Hilbert space  $H$ , may be denoted as  $S^p$  (see e.g [35, Appendix D]).

$l^p$  : the space of complex sequences  $(u_n)$  for which  $\sum_{n=0}^{+\infty} |u_n|^p < \infty$  with its corresponding norm.

$H^\infty(\mathcal{O})$  is a Banach algebra for the pointwise multiplication. We refer the reader e.g. to [34] for more details on Bochner spaces.

In certain proofs, we use the notation  $\lesssim$  to indicate an inequality valid up to a constant which does not depend on the particular elements to which it applies. We use as well notation  $A \simeq B$  to say that we have both  $A \lesssim B$  and  $B \lesssim A$ .

$\Lambda$  : a finite set.

$|\Lambda|$  : the number of elements of  $\Lambda$ .

$$(\Lambda')^c = \Lambda \setminus \Lambda', \text{ for } \Lambda' \subset \Lambda.$$

$\mathcal{I}$  : a countable non empty set.

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# Chapitre 1

## Introduction (français)

### 1.1 Introduction générale

Soit  $\mathcal{A}$  une algèbre de Banach unitaire. Soit  $d \geq 1$  un entier naturel.

Soit  $\mathcal{P}_d$  l'algèbre des fonctions polynomiales sur  $\mathbb{C}^d$ . Soit  $x_1, \dots, x_d$  des éléments commutants de  $\mathcal{A}$ . On peut définir un *calcul polynomial* sur  $\mathcal{P}_d$  en associant à toute fonction de  $\mathcal{P}_d$  écrite sous la forme  $f = \sum a_{k_1, \dots, k_d} z_1^{k_1} \cdots z_d^{k_d}$  (avec une somme finie) l'élément

$$\Phi(f) = f(x_1, \dots, x_d) = \sum a_{k_1, \dots, k_d} x_1^{k_1} \cdots x_d^{k_d}.$$

Soit maintenant  $\sigma(x_k) \subset \mathbb{C}$  le spectre de  $x_k$ ,  $k = 1, \dots, d$ . Soit  $\mathcal{O}_k$  un voisinage ouvert de  $\sigma(x_k)$  dans  $\mathbb{C}$  et  $\Gamma_k$  un contour de  $\sigma(x_k)$  dans  $\mathcal{O}_k$  orienté dans le sens trigonométrique,  $k = 1, \dots, d$ . Soit  $\mathcal{H}(\mathcal{O}_1 \times \cdots \times \mathcal{O}_d)$  l'algèbre des fonctions holomorphes sur  $\mathcal{O}_1 \times \cdots \times \mathcal{O}_d$ . On peut définir un *calcul fonctionnel holomorphe* sur  $\mathcal{H}(\mathcal{O}_1 \times \cdots \times \mathcal{O}_d)$  en posant pour toute fonction  $f$  de  $\mathcal{H}(\mathcal{O}_1 \times \cdots \times \mathcal{O}_d)$ ,

$$\Psi(f) = \frac{1}{(2i\pi)^d} \int_{\Gamma_1 \times \cdots \times \Gamma_d} f(z_1, \dots, z_d) \prod_{k=1}^d R(z_k, x_k) \prod_{k=1}^d dz_k \quad (1.1.1)$$

où  $R(z_k, x_k) = (z_k \mathbf{1}_{\mathcal{A}} - x_k)^{-1}$  est la résolvante de  $x_k$  en tout  $z_k$  de  $\mathbb{C} \setminus \sigma(x_k)$ ,  $k = 1, \dots, d$ . Cette définition ne dépend pas du choix des  $\Gamma_k$  selon la formule de Cauchy. Quand  $d = 1$ , cette définition coïncide avec le calcul de Dunford classique. De plus, pour toute fonction polynomiale  $f$ , on a  $\Psi(f) = \Phi(f)$ , ce qui signifie que  $\Psi$  étend  $\Phi$ . Ces deux éléments seront désormais notés communément  $f(x_1, \dots, x_d)$ .

On en vient à présent à  $\mathcal{A} = B(X)$ , l'algèbre de Banach des opérateurs bornés sur un espace de Banach  $X$ , munie de la norme classique d'opérateurs. Soit  $E_k$  un sous-ensemble de  $\mathbb{C}$  tel que  $\sigma(x_k) \subset E_k \subset \mathcal{O}_k$ ,  $k = 1, \dots, d$  (en prenant les notations ci-dessus). Si  $f$  est une fonction holomorphe sur  $\mathcal{O}_1 \times \cdots \times \mathcal{O}_d$  et bornée sur  $E = E_1 \times \cdots \times E_d$ , on pose

$$\|f\|_{\infty, E} = \sup \{|f(z)| : z \in E\}.$$

Une question classique du calcul fonctionnel est de savoir s'il existe une constante  $C > 0$  telle que

$$\|f(x_1, \dots, x_d)\| \leq C \|f\|_{\infty, E}, \quad (1.1.2)$$

pour  $f$  parcourant une certaine sous-algèbre de fonctions holomorphes et bornées sur  $\mathcal{O}_1 \times \cdots \times \mathcal{O}_d$ .

Des résultats classiques en théorie des opérateurs hilbertiens sont des réponses à ce problème générique. L'inégalité de von Neumann assure que pour toute contraction  $T$  sur un espace de Hilbert  $H$  et pour toute fonction polynomiale  $P$  à une variable on a

$$\|P(T)\| \leq \|P\|_{\infty, \mathbb{D}}.$$

Ensuite, l'inégalité de Ando, qui généralise celle de von Neumann, affirme que pour toute paire  $(T_1, T_2)$  de contractions qui commutent sur un même espace de Hilbert  $H$  et pour toute fonction polynomiale  $Q$  de deux variables on a

$$\|Q(T_1, T_2)\| \leq \|Q\|_{\infty, \mathbb{D}^2}. \quad (1.1.3)$$

Un contre-exemple de Crabb-Davie montre qu'une telle inégalité ne peut être satisfaite pour trois contractions qui commutent sur un espace de Hilbert. Aussi, en fixant  $T_1, \dots, T_d$  des contractions commutantes sur un même espace de Hilbert, la question de l'existence d'une constante  $C > 0$  telle que pour toute fonction polynomiale  $p$  de  $d$  variables on ait  $\|p(T_1, \dots, T_d)\| \leq C_d \|p\|_{\infty, \mathbb{D}^d}$  est toujours une question ouverte (voir [69] pour tous les résultats qui précèdent). Néanmoins, une telle inégalité se produit si  $T_1, \dots, T_d$  sont des opérateurs normaux qui commutent, selon le calcul fonctionnel sur les  $C^*$ -algèbres.

Il est aussi possible d'étudier des inégalités similaires sur les espaces non-hilbertiens, ce qui engendre d'autres difficultés. On peut également traiter le cas d'opérateurs non bornés.

Le fait que l'inégalité d'Ando ne s'étende pas à  $d \geq 3$  donne lieu au type de question suivant :

**(Q)** Si  $T_1, \dots, T_d$  sont des opérateurs commutants chacun vérifiant une inégalité du type  $\|f(T_k)\| \leq C_k \|f\|_{\infty, E_k}$ ,  $k = 1, \dots, d$ , sous quelle condition le  $d$ -uplet  $(T_1, \dots, T_d)$  satisfait une estimation du type (1.1.2), à savoir l'existence d'une constante  $C > 0$  telle que pour toute fonction adéquate  $f$  on ait  $\|f(T_1, \dots, T_d)\| \leq C \|f\|_{\infty, E}$  ?

Cette discussion illustre le fait que l'étude du calcul fonctionnel d'un  $d$ -uplet ne se restreint pas à l'étude du calcul fonctionnel individuel de chaque élément du  $d$ -uplet.

Notre travail concerne le calcul fonctionnel des opérateurs de Ritt et des opérateurs sectoriels. Chacun de ces opérateurs vérifie une condition sur sa résolvante qui permet de définir un calcul fonctionnel avec une formule comme en (1.1.1), en choisissant judicieusement les contours des spectres des opérateurs.

Les opérateurs sectoriels apparaissent dans nombre de problèmes de l'analyse fonctionnelle. Ils apparaissent comme les générateurs négatifs de semi-groupes analytiques bornés, qui interviennent dans la résolution d'équations elliptiques (voir [42], [10] et [47]), en analyse harmonique (voir [16], [33] et [73]) et dans les problèmes de régularité maximale (voir [3], [13], [21], [22], [38], [78] et [79]). Pour des informations générales sur les semi-groupes d'opérateurs, on renvoie à [64].

Les opérateurs de Ritt jouent un rôle dans la résolution de certaines équations linéaires par des méthodes d'itérations (voir [63]). Les problèmes de régularité maximale font également appel à ces opérateurs (voir [11]).

Les opérateurs de Ritt et opérateurs sectoriels ont fait l'objet d'études du point de vue de leur calcul fonctionnel dans maintes situations, comme on peut le voir dans [5], [14], [15], [39], [43], [40], [41], [46], [53], [60], [61], [75], [76] et [77]. Pour un large résumé sur le calcul fonctionnel des opérateurs sectoriels, on renvoie à [31]. De plus, la subordination de semi-groupes continus ou discrets met en jeu le calcul fonctionnel des opérateurs de Ritt ou des opérateurs sectoriels (voir [23], [28] et [29]).

Le point de départ des travaux mentionnés ci-dessus est le calcul fonctionnel  $H^\infty$  élaboré par Alan McIntosh (voir [60]). Dans cette dernière référence, Alan McIntosh a mis en relief le rôle des *fonctions carré*. Il a montré des caractérisations du calcul fonctionnel  $H^\infty$  des opérateurs sectoriels sur les espaces de Hilbert en terme de fonctions carré. Suivant ce chemin, Cowling, Doust, McIntosh et Yagi ont trouvé des fonctions carré pour caractériser le calcul fonctionnel de ces opérateurs sur les espaces  $L_p$  (voir [15]). Ceci admet une généralisation naturelle aux treillis de Banach. Un cadre plus général a été donné par Kalton et Weis dans [41], lorsqu'ils ont utilisé les espaces  $\gamma$  dans le but de définir des fonctions carré sur les espaces de Banach généraux. Le cas des espaces  $L_p$  non commutatifs est traité par Junge, Le Merdy et Xu dans [56]. Un résumé de ces méthodes est donné dans [52] et le cas des opérateurs de Ritt est traité dans [53].

Un autre sujet proche du calcul fonctionnel est celui des *dilatations*. Premièrement, l'inégalité de von Neumann repose entre autre sur le Théorème de dilatation de Nagy. Ce dernier résultat affirme que pour une contraction  $T$  sur un espace de Hilbert  $H$ , il existe un opérateur unitaire  $U$  sur un espace de Hilbert  $K$  contenant  $H$  tels que

$$T^n = P_H U^n|_H, \quad n \in \mathbb{N},$$

où  $P_H$  désigne la projection orthogonale de  $K$  sur  $H$ . Une des conséquences d'un tel résultat est que le générateur négatif d'un semi-groupe de contractions sur un Hilbert admet un calcul fonctionnel  $H^\infty$ .

Deuxièmement, l'inégalité de Matsaev, qui repose sur le Théorème de dilatation d'Akcoglu-Sucheston pour les contractions positives sur un espace  $L_p$  (voir [1]), entraîne que le générateur négatif d'un semi-groupe de contractions positives sur un espace  $L_p$  admet un calcul fonctionnel  $H^\infty$ .

On peut également trouver des résultats de dilatations sur les espaces  $L_p$  non commutatifs dans [56] qui fournissent des propriétés de calcul fonctionnel  $H^\infty$ .

La question réciproque, qui recherche des dilatations à partir d'un calcul fonctionnel  $H^\infty$ , admet également des réponses remarquables. En particulier, un tournant sur ce sujet est un article de Fröhlich et Weis, qui ont montré qu'un calcul fonctionnel  $H^\infty$  implique une dilatation d'opérateurs sectoriels en des isomorphismes (voir [27]). Cet article a mis en évidence les liens très étroits entre les résultats de dilatations et le calcul fonctionnel  $H^\infty$ . Plus récemment, Arhancet, Fackler et Le Merdy ont étudié dans [4] des caractérisations du calcul fonctionnel  $H^\infty$  pour les opérateurs de Ritt et pour les semi-groupes en termes de dilatation sur des espaces ayant des propriétés spéciales, comme la propriété UMD. En particulier, les auteurs ont obtenu l'équivalence entre un calcul

fonctionnel  $H^\infty$  d'opérateurs de Ritt et leur dilatation en contractions de Ritt sur un espace de Bochner.

Cette thèse a pour sujet l'étude du *calcul fonctionnel joint*  $H^\infty$  d'un  $d$ -uplet commutant d'opérateurs de Ritt ou d'opérateurs sectoriels. Dans un premier temps, en prenant  $(T_1, \dots, T_d)$  un  $d$ -uplet d'opérateurs commutants, on étudie la différence entre le calcul fonctionnel  $H^\infty$  de chaque élément du  $d$ -uplet et le calcul fonctionnel joint  $H^\infty$  du  $d$ -uplet. On exprimera ceci à l'aide d'une question du même style que **(Q)**. Ensuite, nous nous fixerons pour objectif d'étudier les relations entre le calcul fonctionnel et les notions discutées ci-dessus (fonctions carré, propriétés de dilatation) pour un  $d$ -uplet, dans le but de dépasser le cas d'un seul opérateur.

## 1.2 Contenu de la thèse

### Première partie

Le premier chapitre de cette partie donne les bases nécessaires sur les moyennes de Rademacher et les moyennes gaussiennes. Considérant des variables indépendantes de Rademacher  $r_i : \Omega \rightarrow \{-1, 1\}$  définies sur un même espace probabilisé  $\Omega$  et indexées par un ensemble dénombrable  $\mathcal{I}$ , on regarde

$$\sum_{i \in \mathcal{I}} r_i \otimes x_i,$$

où  $(x_i)_{i \in \mathcal{I}}$  est une famille finie d'un espace de Banach  $X$ .

De la même façon, si  $g_i : \Omega \rightarrow \mathbb{C}$ , sont des gaussiennes standards et indépendantes, on regardera

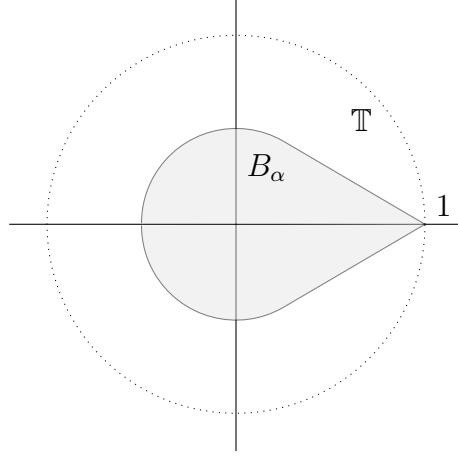
$$\sum_{i \in \mathcal{I}} g_i \otimes x_i.$$

En particulier, on définit les espaces  $\text{Rad}(X)$  et  $\text{Gauss}(X)$  sur tout espace de Banach  $X$ . Des notions sur les opérateurs basées sur ces moyennes, telles que celles d'opérateurs  $R$ -bornés ou  $\gamma$ -bornés, sont introduites. Enfin, on rappelle les propriétés fondamentales de géométrie des espaces de Banach que sont le type et le cotype.

Le deuxième chapitre présente les notions de géométrie des espaces de Banach qui interviennent dans des questions variées sur le calcul fonctionnel des opérateurs de Ritt et des opérateurs sectoriels. En particulier, on traitera les espaces  $K$ -convexes, les espaces UMD et les espaces bénéficiant de la propriété  $(\alpha)$  ou  $(\Delta)$ . On donne les liens de base entre ces propriétés et on présente des exemples simples d'espaces de Banach disposant de telle ou telle propriété.

On termine cette partie par les définitions élémentaires sur les opérateurs de Ritt et opérateurs sectoriels. On choisit également des définitions appropriées du calcul fonctionnel joint  $H^\infty$  de  $d$ -uplets de tels opérateurs. Le cas des  $d$ -uplets des opérateurs sectoriels avait été pour partie traité par Albrecht dans [2].

Dans la suite, on utilise des objets et notations introduits dans cette partie et on y réfère pour les détails et les définitions précises. Ceci inclut les *domains de Stolz*  $B_\alpha$ ,  $\alpha \in (0, \frac{\pi}{2})$ , dont voici une figure dans le plan complexe.



On considère également les secteurs  $\Sigma_\theta = \{z \in \mathbb{C}^* : |\text{Arg}(z)| < \theta\}$ ,  $\theta \in (0, \pi)$ .

Pour  $\theta_1, \dots, \theta_d$  dans  $(0, \pi)$ , on considère alors les classes de fonctions holomorphes bornées de  $d$  variables ne dépendant que des  $(z_i)_{i \in \Lambda}$ ,  $\Lambda \subset \{1, \dots, d\}$ , vérifiant une inégalité du type

$$|f(z_1, \dots, z_d)| \lesssim \prod_{i \in \Lambda} \frac{|z_i|^{s_i}}{1 + |z_i|^{2s_i}}, \quad (z_i)_{i \in \Lambda} \in \prod_{i \in \Lambda} \Sigma_{\theta_i}$$

et on note  $H_0^\infty(\prod_{i \in \Lambda} \Sigma_{\theta_i})$  l'algèbre de fonctions correspondantes.

On définit ensuite  $f(A_1, \dots, A_d)$  pour toute  $f$  de  $H_0^\infty(\prod_{i \in \Lambda} \Sigma_{\theta_i})$  et  $(A_1, \dots, A_d)$  un  $d$ -uplet d'opérateurs sectoriels commutants par une formule du type (1.1.1), soit

$$f(A_1, \dots, A_d) = \left(\frac{1}{2\pi i}\right)^{|\Lambda|} \int_{\prod_{i \in \Lambda} \partial \Sigma_{\nu_i}} f(z_1, \dots, z_d) \prod_{i \in \Lambda} R(z_i, A_i) \prod_{i \in \Lambda} dz_i.$$

Ceci permet de définir un calcul fonctionnel sur la classe

$$H_{0,1}^\infty(\Sigma_{\theta_1} \times \dots \times \Sigma_{\theta_d}) = \bigoplus_{\Lambda \subset \{1, \dots, d\}} H_0^\infty\left(\prod_{i \in \Lambda} \Sigma_{\theta_i}\right),$$

de sorte que l'on considère par ce biais toutes les sous-familles de  $(A_1, \dots, A_d)$  et notamment les singletons  $A_1, \dots, A_d$ .

**Définition 1.2.1 (Calcul fonctionnel joint  $H^\infty$  des opérateurs sectoriels)** *On dit que  $(A_1, \dots, A_d)$  admet un calcul fonctionnel joint  $H^\infty(\Sigma_{\theta_1} \times \dots \times \Sigma_{\theta_d})$  si  $f \mapsto f(A_1, \dots, A_d)$  est borné, c'est-à-dire qu'il existe une constante  $K > 0$  telle que pour toute  $f$  dans  $H_{0,1}^\infty(\Sigma_{\theta_1} \times \dots \times \Sigma_{\theta_d})$ ,*

$$\|f(A_1, \dots, A_d)\| \leq K \|f\|_{\infty, \Sigma_{\theta_1} \times \dots \times \Sigma_{\theta_d}}.$$

On parle simplement de calcul fonction joint  $H^\infty$  si la définition précédente est vérifiée pour certains  $\theta_1, \dots, \theta_d$  de  $(0, \pi)$ .

Les définitions sont identiques pour les opérateurs de Ritt en remplaçant les  $\Sigma_\theta$  par les domaines de Stolz  $B_\alpha$ .



## Deuxième partie

La deuxième partie s'occupe de la décomposition de Franks et McIntosh des fonctions holomorphes bornées. Ces auteurs ont construit une telle décomposition dans [26] pour des fonctions d'une ou plusieurs variables, dans le but d'étudier des formes variées de calculs fonctionnels.

Cette partie consiste en trois étapes. Premièrement, on présente le travail issu de [26] en lien avec les secteurs  $\Sigma_\theta$ ,  $\theta \in (0, \pi)$ . Deuxièmement, on donne les résultats analogues pour les domaines de Stolz, élaborés en vue d'applications aux opérateurs de Ritt. Enfin, on donne une décomposition de l'unité dans les deux cas.

Pour le cas sectoriel, les estimations reposeront sur les deux théorèmes à venir. Le théorème suivant apparaît implicitement dans [26].

**Théorème 1.2.2** *Soit  $d \geq 1$  un entier, soit  $\nu_k$  dans  $(0, \pi)$  et  $\mu_k$  dans  $(0, \nu_k)$ ,  $k = 1, \dots, d$ . Il existe des suites  $(\Psi_{k,i_k})_{i_k \geq 1}$  et  $(\tilde{\Psi}_{k,i_k})_{i_k \geq 1}$  dans  $H_0^\infty(\Sigma_{\mu_k})$  vérifiant les propriétés suivantes.*

(1) *Pour tout réel  $p > 0$  et pour tout  $k = 1, \dots, d$ ,*

$$\sup \left\{ \sum_{i_k=1}^{\infty} |\Psi_{k,i_k}(\zeta_k)|^p : \zeta_k \in \Sigma_{\mu_k} \right\} < \infty \quad \text{et} \quad \sup \left\{ \sum_{i_k=1}^{\infty} |\tilde{\Psi}_{k,i_k}(\zeta_k)|^p : \zeta_k \in \Sigma_{\mu_k} \right\} < \infty.$$

(2) *Il existe une constante  $C > 0$  telle que pour tout  $h$  dans  $H^\infty(\Sigma_{\nu_1} \times \dots \times \Sigma_{\nu_d})$ , il existe une famille  $(a_{i_1, \dots, i_d})_{i_1, \dots, i_d \geq 1}$  de nombres complexes tels que*

$$|a_{i_1, \dots, i_d}| \leq C \|h\|_{\infty, \Sigma_{\nu_1} \times \dots \times \Sigma_{\nu_d}}, \quad (i_1, \dots, i_d) \in \mathbb{N}^{*d},$$

*et pour tout  $(\zeta_1, \dots, \zeta_d)$  dans  $\prod_{k=1}^d \Sigma_{\mu_k}$ ,*

$$h(\zeta_1, \dots, \zeta_d) = \sum_{i_1, \dots, i_d \geq 1} a_{i_1, \dots, i_d} \Psi_{1,i_1}(\zeta_1) \tilde{\Psi}_{1,i_1}(\zeta_1) \cdots \Psi_{d,i_d}(\zeta_d) \tilde{\Psi}_{d,i_d}(\zeta_d).$$

La proposition ci-dessous fournit une décomposition de l'unité sur les secteurs et provient de principes analogues. Les détails seront donnés au Chapitre 6.

**Proposition 1.2.3** *Soit  $\mu$  dans  $(0, \pi)$ . Il existe des suites  $(\Delta_i)_{i \geq 1}$ ,  $(\psi_i)_{i \geq 1}$  et  $(\tilde{\psi}_i)_{i \geq 1}$  de  $H_0^\infty(\Sigma_\mu)$  satisfaisant les propriétés suivantes.*

(1) *Il existe une constante  $C > 0$  telle que*

$$\forall z \in \Sigma_\mu, \quad \sum_{i=1}^{\infty} |\psi_i(z)| \leq C, \quad \sum_{i=1}^{\infty} |\tilde{\psi}_i(z)| \leq C.$$

(2) *Pour tout  $\nu \in (0, \mu)$ , il existe une constante  $K \geq 0$  telle que*

$$\forall i \geq 1, \quad \int_{\partial \Sigma_\nu} \frac{|\Delta_i(z)|}{|z|} |dz| \leq K.$$

(3) Il existe une constante  $C \geq 0$  telle que

$$\forall i \geq 1, \forall z \in \Sigma_\mu, \quad |\Delta_i(z)| \leq C$$

et

$$\forall z \in \Sigma_\mu, \quad 1 = \sum_{i=1}^{\infty} \Delta_i(z) \psi_i(z) \tilde{\psi}_i(z).$$

Dans l'optique d'obtenir des résultats sur le calcul fonctionnel joint  $H^\infty$  des opérateurs de Ritt, on apporte une preuve détaillée de la décomposition de Franks-McIntosh sur les domaines de Stolz. Le résultat ci-après est implicite dans [26, Section 4], pour autant aucune preuve n'a été écrite jusqu'ici. Celle que nous donnons dans notre travail est proche de celle pour les secteurs que l'on trouve dans [26, Section 3], et plus simple que celle ébauchée dans [26, Section 4] pour des domaines à plusieurs points de contact.

**Théorème 1.2.4** *Soit  $d \geq 1$  un entier, soit  $\beta_k$  dans  $(0, \frac{\pi}{2})$  et  $\alpha_k$  dans  $(0, \beta_k)$ ,  $k = 1, \dots, d$ . Il existe des suites  $(\Psi_{k,i_k})_{i_k \geq 1}$  et  $(\tilde{\Psi}_{k,i_k})_{i_k \geq 1}$  dans  $H_0^\infty(B_{\alpha_k})$  vérifiant les propriétés suivantes.*

(1) Pour tout réel  $p > 0$  et pour tout  $k = 1, \dots, d$ ,

$$\sup \left\{ \sum_{i_k=1}^{\infty} |\Psi_{k,i_k}(\zeta_k)|^p : \zeta_k \in B_{\alpha_k} \right\} < \infty \quad \text{et} \quad \sup \left\{ \sum_{i_k=1}^{\infty} |\tilde{\Psi}_{k,i_k}(\zeta_k)|^p : \zeta_k \in B_{\alpha_k} \right\} < \infty.$$

(2) Il existe une constante  $C > 0$  telle que pour tout  $h$  dans  $H^\infty(B_{\beta_1} \times \dots \times B_{\beta_d})$ , il existe une famille  $(a_{i_1, \dots, i_d})_{i_1, \dots, i_d \geq 1}$  de nombres complexes tels que

$$|a_{i_1, \dots, i_d}| \leq C \|h\|_{\infty, B_{\beta_1} \times \dots \times B_{\beta_d}}, \quad (i_1, \dots, i_d) \in \mathbb{N}^{*d},$$

et pour tout  $(\zeta_1, \dots, \zeta_d)$  dans  $\prod_{k=1}^d B_{\alpha_k}$ ,

$$h(\zeta_1, \dots, \zeta_d) = \sum_{i_1, \dots, i_d \geq 1} a_{i_1, \dots, i_d} \Psi_{1,i_1}(\zeta_1) \tilde{\Psi}_{1,i_1}(\zeta_1) \cdots \Psi_{d,i_d}(\zeta_d) \tilde{\Psi}_{d,i_d}(\zeta_d).$$

L'énoncé qui suit est un analogue de la Proposition 1.2.3.

**Théorème 1.2.5** *Soit  $\alpha$  dans  $(0, \frac{\pi}{2})$ . Il existe des suites  $(\phi_i)_{i \geq 1}$ ,  $(\varphi_i)_{i \geq 1}$ ,  $(\theta_i)_{i \geq 1}$  et  $(\psi_i)_{i \geq 1}$  de  $H_0^\infty(B_\alpha)$ , telles que*

(i) Pour tout  $i \geq 1$ , on a  $\phi_i = \theta_i \varphi_i \psi_i$ ;

(ii) Il existe une constante  $c > 0$  telle que pour tout  $z$  de  $B_\alpha$

$$\sum_{i=1}^{\infty} |\varphi_i(z)| \leq c, \quad \sum_{i=1}^{\infty} |\psi_i(z)| \leq c;$$

(iii) Pour tout  $\gamma$  de  $(0, \alpha)$ , il existe une constante  $e > 0$  telle que pour tout  $i \geq 1$

$$\int_{\partial B_\gamma} \frac{|\theta_i(z)|}{|1-z|} |dz| \leq e;$$

(iv) Pour tout  $z$  dans  $B_\alpha$ , la série  $\sum_{i \geq 1} \phi_i(z)$  converge absolument et il existe une constante  $c' > 0$  telle que

$$\sup \left\{ \sum_{i=1}^{\infty} |\phi_i(z)| : z \in B_\alpha \right\} \leq c'.$$

De plus, on a

$$\sum_{i=1}^{\infty} \phi_i(z) = 1, \quad z \in B_\alpha.$$

On notera que les théorèmes 1.2.2 et 1.2.4 sont vrais pour les fonctions holomorphes  $h$  à valeurs dans un espace de Banach  $Z$  quelconque.

### Troisième partie

La troisième partie traite trois questions indépendantes en lien avec le calcul fonctionnel joint  $H^\infty$  de  $d$ -uplets commutants d'opérateurs de Ritt ou d'opérateurs sectoriels.

Le premier chapitre de cette partie donne une réponse positive à la question de l'automaticité du calcul fonctionnel joint  $H^\infty$  sous l'hypothèse que chaque élément du  $d$ -uplet a un calcul fonctionnel  $H^\infty$ . Il s'agit d'une propriété clef pour de nombreux résultats de cette thèse. Elle consiste en le théorème suivant.

**Théorème 1.2.6** *Soit  $X$  un espace de Banach. Supposons que soit  $X$  est un treillis de Banach, soit que  $X$  ou  $X^*$  a la propriété  $(\alpha)$ . Soit  $d \geq 2$  un entier. Alors les deux propriétés suivantes sont vraies :*

**(P1)** *Soit  $(T_1, \dots, T_d)$  un  $d$ -uplet d'opérateurs de Ritt commutants sur  $X$  et supposons que pour certains  $0 < \gamma_1, \dots, \gamma_d < \frac{\pi}{2}$ ,  $T_k$  admet un calcul fonctionnel  $H^\infty(B_{\gamma_k})$  pour tout  $k = 1, \dots, d$ . Alors pour tout  $\gamma'_k \in (\gamma_k, \frac{\pi}{2})$ ,  $k = 1, \dots, d$ ,  $(T_1, \dots, T_d)$  admet un calcul fonctionnel joint  $H^\infty(B_{\gamma'_1} \times \dots \times B_{\gamma'_d})$ .*

**(P2)** *Soit  $(A_1, \dots, A_d)$  un  $d$ -uplet d'opérateurs sectoriels commutants sur  $X$  et supposons que pour certains  $0 < \theta_1, \dots, \theta_d < \pi$ ,  $A_k$  admet un calcul fonctionnel  $H^\infty(\Sigma_{\theta_k})$  pour tout  $k = 1, \dots, d$ . Alors pour tout  $\theta'_k \in (\theta_k, \pi)$ ,  $k = 1, \dots, d$ ,  $(A_1, \dots, A_d)$  admet un calcul fonctionnel joint  $H^\infty(\Sigma_{\theta'_1} \times \dots \times \Sigma_{\theta'_d})$ .*

La propriété **(P2)** pour  $d = 2$  est prouvée dans [48]. La preuve pour  $d \geq 3$  est une simple adaptation de l'argument élaboré dans ce dernier article. Dans le cas spécifique où  $X$  est un espace  $L_p$ ,  $p \in [1, \infty)$ , la propriété **(P2)** renvoie à [2]. La preuve de **(P1)** nécessitera la décomposition de Franks-McIntosh que l'on a décrit dans la Deuxième Partie.

Ensuite, le deuxième chapitre de cette partie considère les combinaisons convexes d'opérateurs de Ritt.

**Proposition 1.2.7** *Soit  $d \geq 2$  un entier. Soit  $X$  un espace de Banach. Soit  $T_1, \dots, T_d$  dans  $B(X)$  des opérateurs de Ritt commutants. Soit  $c_1, \dots, c_d$  des réels positifs tels que  $\sum_{k=1}^d c_k = 1$ . Alors  $\sum_{k=1}^d c_k T_k$  est un opérateur de Ritt.*

On obtient alors le calcul fonctionnel d'une telle combinaison, si l'espace de Banach sous-jacent a la propriété  $(\Delta)$ , une propriété plus faible que la propriété  $(\alpha)$ .

**Théorème 1.2.8** *Soit  $X$  un espace de Banach ayant la propriété  $(\Delta)$  et  $d \geq 2$  un entier. Soit  $T_1, \dots, T_d$  des opérateurs de Ritt commutants sur  $X$ . Soit  $c_1, \dots, c_d$  des réels positifs tel que  $\sum_{k=1}^d c_k = 1$ . Supposons que tous les  $T_k$  admettent un calcul fonctionnel  $H^\infty$ ,  $k = 1, \dots, d$ . Alors  $T = \sum_{k=1}^d c_k T_k$  admet un calcul fonctionnel  $H^\infty$ .*

Le dernier chapitre de cette partie donne un résultat de réduction d'angle du calcul fonctionnel  $H^\infty$ . Rappelons que si  $(A_1, \dots, A_d)$  est un  $d$ -uplet d'opérateurs sectoriels qui commutent sur un espace de Banach  $X$  ayant un calcul fonctionnel joint  $H^\infty(\Sigma_{\theta_1} \times \dots \times \Sigma_{\theta_d})$ , alors il admet aussi un calcul fonctionnel joint  $H^\infty(\Sigma_{\theta'_1} \times \dots \times \Sigma_{\theta'_d})$  pour tout  $\theta'_k \geq \theta_k$ ,  $k = 1, \dots, d$ . De même, si  $(T_1, \dots, T_d)$  est un  $d$ -uplet d'opérateurs de Ritt qui commutent sur un espace de Banach  $X$  ayant un calcul fonctionnel joint  $H^\infty(B_{\gamma_1} \times \dots \times B_{\gamma_d})$ , alors il admet aussi un calcul fonctionnel joint  $H^\infty(B_{\gamma'_1} \times \dots \times B_{\gamma'_d})$  pour tout  $\gamma'_k \geq \gamma_k$ ,  $k = 1, \dots, d$ . Le but est d'avoir une réciproque de tels résultats.

McIntosh a prouvé dans [60] que le calcul fonctionnel  $H^\infty(\Sigma_\theta)$  d'un opérateur sectoriel sur un espace de Hilbert ne dépend pas de  $\theta$ , tant que celui-ci est bien défini. Cependant, Kalton a montré que cette propriété n'est pas vraie sur des espaces de Banach généraux (voir [37]). Enfin, Kalton et Weis ont montré une extension du résultat de McIntosh dans les espaces de Banach généraux sous une condition de  $R$ -sectorialité de l'opérateur considéré. Le théorème ci-dessous est une généralisation au cas multivarié du Théorème de Kalton-Weis. La partie (2) est une adaptation au cas des opérateurs de Ritt et la partie (3) est une variante de cette dernière.

**Théorème 1.2.9** *Soit  $X$  un espace de Banach. Soit  $d \geq 2$  un entier. Alors les deux propriétés suivantes sont vraies :*

- (1) *Soit  $(A_1, \dots, A_d)$  un  $d$ -uplet d'opérateurs  $R$ -sectoriels commutants sur  $X$ , de  $R$ -types respectifs  $\omega_1, \dots, \omega_d$ . Soit  $\theta_k \in (\omega_k, \pi)$ , pour  $k = 1, \dots, d$ , et supposons que  $(A_1, \dots, A_d)$  admet un calcul fonctionnel joint  $H^\infty(\Sigma_{\theta_1} \times \dots \times \Sigma_{\theta_d})$ . Alors pour tout  $\theta'_k \in (\omega_k, \pi)$ ,  $k = 1, \dots, d$ , le  $d$ -uplet  $(A_1, \dots, A_d)$  admet un calcul fonctionnel joint  $H^\infty(\Sigma_{\theta'_1} \times \dots \times \Sigma_{\theta'_d})$ .*
- (2) *Soit  $(T_1, \dots, T_d)$  un  $d$ -uplet d'opérateurs  $R$ -Ritt commutants sur  $X$ , de  $R$ -types respectifs  $a_1, \dots, a_d$ . Soit  $\gamma_k \in (a_k, \frac{\pi}{2})$ , pour  $k = 1, \dots, d$ , et supposons que  $(T_1, \dots, T_d)$  admet un calcul fonctionnel joint  $H^\infty(B_{\gamma_1} \times \dots \times B_{\gamma_d})$ . Alors pour tout  $\gamma'_k \in (a_k, \frac{\pi}{2})$ ,  $k = 1, \dots, d$ , le  $d$ -uplet  $(T_1, \dots, T_d)$  admet un calcul fonctionnel joint  $H^\infty(\Sigma_{\gamma'_1} \times \dots \times \Sigma_{\gamma'_d})$ .*
- (3) *Soit  $(T_1, \dots, T_d)$  un  $d$ -uplets d'opérateurs commutants sur  $X$  tel que chaque  $T_k$  est un opérateur  $R$ -Ritt,  $k = 1, \dots, d$ . Supposons que  $(T_1, \dots, T_d)$  est polynomialement*

borné, c'est-à-dire qu'il existe une constante  $C \geq 1$  telle que pour toute fonction polynomiale  $h$  de  $d$  variables on ait

$$\|h(T_1, \dots, T_d)\| \leq C \sup \{|h(z_1, \dots, z_d)| : (z_1, \dots, z_d) \in \mathbb{T}^d\}. \quad (1.2.1)$$

Alors  $(T_1, \dots, T_d)$  admet un calcul fonctionnel joint  $H^\infty$ .

## Quatrième partie

La quatrième partie propose des caractérisations du calcul fonctionnel joint  $H^\infty$  sur les espaces UMD ayant la propriété  $(\alpha)$ . Ce travail utilise celui initié dans [4].

Le premier chapitre traite des espaces UMD et  $L_p$ . Dans la mesure où l'on utilisera des dilatations de chaque opérateurs d'un  $d$ -uplet commutant, il apparaît nécessaire d'envisager la combinaison de celles-ci.

On établit le lemme suivant, qui permet une telle combinaison de dilatations.

**Lemme 1.2.10** *Soit  $d \geq 2$  un entier, soit  $T_1, \dots, T_d$  des opérateurs commutants sur une espace de Banach  $X$  et soit  $p \in [1, \infty)$ . Soit  $1 \leq m \leq d$ . Supposons que :*

- (1) *Pour tout  $k = 1, \dots, m$ , il existe un opérateur positif  $V_k$  sur un espace  $L^p(\Omega)$  et deux opérateurs bornés  $J_k: X \rightarrow L^p(\Omega; X)$  et  $Q_k: L^p(\Omega; X) \rightarrow X$  tels que*

$$T_k^{n_k} = Q_k(V_k \overline{\otimes} I_X)^{n_k} J_k, \quad n_k \in \mathbb{N}.$$

- (2) *Si  $m < d$ , il existe un espace de Banach  $Y$ , deux opérateurs bornés  $J_{m+1}: X \rightarrow Y$  et  $Q_{m+1}: Y \rightarrow X$  ainsi que des opérateurs commutants  $V_{m+1}, \dots, V_d$  sur  $Y$  tels que*

$$T_{m+1}^{n_{m+1}} \cdots T_d^{n_d} = Q_{m+1} V_{m+1}^{n_{m+1}} \cdots V_d^{n_d} J_{m+1}, \quad (n_{m+1}, \dots, n_d) \in \mathbb{N}^{d-m}.$$

- (3) *Pour tout  $i = 1, \dots, m$  et  $j = 1, \dots, d$ , on a*

$$J_i T_j = (I_{L^p(\Omega)} \overline{\otimes} T_j) J_i.$$

Alors il existe deux opérateurs bornés  $J: X \rightarrow L^p(\Omega^m; Y)$  et  $Q: L^p(\Omega^m; Y) \rightarrow X$  tels que

$$T_1^{n_1} \cdots T_d^{n_d} = Q U_1^{n_1} \cdots U_d^{n_d} J, \quad (n_1, \dots, n_d) \in \mathbb{N}^d,$$

où les opérateurs  $U_1, \dots, U_d: L^p(\Omega^m; Y) \rightarrow L^p(\Omega^m; Y)$  sont donnés par

$$U_k = I^{\otimes k-1} \overline{\otimes} V_k \overline{\otimes} I^{\otimes m-k} \overline{\otimes} I_Y, \quad k = 1, \dots, m;$$

$$U_k = I^{\otimes m} \overline{\otimes} V_k, \quad k = m+1, \dots, d.$$

Ici,  $I = I_{L^p(\Omega)}$  et  $I^{\otimes l} = \underbrace{I \overline{\otimes} \cdots \overline{\otimes} I}_{l \text{ facteurs}}$  pour tout entier  $l \geq 1$ .

La première étape vers la caractérisation sur les espaces UMD se trouve dans le théorème suivant.

**Théorème 1.2.11** *Soit  $p \in (1, \infty)$ . Soit  $X$  un espace de Banach réflexif tel que  $X$  et  $X^*$  ont chacun un cotype fini. Soit  $T_1, \dots, T_d$  des opérateurs de Ritt commutants sur  $X$  tels que tout  $T_k$  admet un calcul fonctionnel  $H^\infty(B_{\gamma_k})$  pour certains  $\gamma_k \in (0, \frac{\pi}{2}), k = 1, \dots, d$ . Alors il existe un espace mesuré  $\Omega$ , des isomorphismes isométriques commutants  $U_1, \dots, U_d$  sur  $L^p(\Omega^d; X)$ , et deux opérateurs bornés  $J: X \rightarrow L^p(\Omega^d; X)$  et  $Q: L^p(\Omega^d; X) \rightarrow X$  tels que*

$$T_1^{n_1} \cdots T_d^{n_d} = QU_1^{n_1} \cdots U_d^{n_d} J, \quad (n_1, \dots, n_d) \in \mathbb{N}^d.$$

L'application du théorème précédent et du théorème 1.2.6 sur les espaces  $(\alpha)$  donne alors la caractérisation suivante du calcul fonctionnel joint  $H^\infty$  en termes de dilatation.

**Théorème 1.2.12** *Soit  $X$  un espace de Banach ayant les propriétés UMD et  $(\alpha)$  et soit  $d \geq 1$  un entier. Soit  $T_1, \dots, T_d$  des opérateurs de Ritt commutants sur  $X$  et soit  $p \in (1, \infty)$ . Les deux assertions suivantes sont équivalentes.*

- (1)  $(T_1, \dots, T_d)$  admet un calcul fonctionnel joint  $H^\infty(B_{\gamma_1} \times \cdots \times B_{\gamma_d})$  pour certains  $\gamma_k \in (0, \frac{\pi}{2}), k = 1, \dots, d$ .
- (2) Il existe un espace mesuré  $\Omega$ , des opérateurs de Ritt commutants et contractants  $R_1, \dots, R_d$  sur  $L^p(\Omega; X)$  tels que tout  $R_k$  admet un calcul fonctionnel  $H^\infty(B_{\gamma'_k})$  pour certains  $\gamma'_k \in (0, \frac{\pi}{2}), k = 1, \dots, d$ , ainsi que deux opérateurs bornés  $J: X \rightarrow L^p(\Omega; X)$  et  $Q: L^p(\Omega; X) \rightarrow X$  tels que

$$T_1^{n_1} \cdots T_d^{n_d} = QR_1^{n_1} \cdots R_d^{n_d} J, \quad (n_1, \dots, n_d) \in \mathbb{N}^d.$$

L'implication principale est "(1)  $\Rightarrow$  (2)". Le fait remarquable dans la propriété de dilatation est que les opérateurs  $R_1, \dots, R_d$  sont contractants.

Le théorème prend une forme spéciale lorsque  $X$  est un espace  $L_p$ , donnant lieu à une caractérisation en terme de contractions positives sur  $L_p$ .

**Theorem 1.2.13** *Soit  $\Sigma$  un espace mesuré et soit  $p \in (1, \infty)$ . Soit  $T_1, \dots, T_d$  des opérateurs de Ritt commutants sur  $L^p(\Sigma)$ . Les deux assertions suivantes sont équivalentes.*

- (1)  $(T_1, \dots, T_d)$  admet un calcul fonctionnel joint  $H^\infty(B_{\gamma_1} \times \cdots \times B_{\gamma_d})$  pour certains  $\gamma_k \in (0, \frac{\pi}{2}), k = 1, \dots, d$ .
- (2) Il existe un espace mesuré  $\Omega$ , des opérateurs de Ritt commutants, contractants et positifs  $R_1, \dots, R_d$  sur  $L^p(\Omega)$ , et deux opérateurs bornés  $J: L^p(\Sigma) \rightarrow L^p(\Omega)$  et  $Q: L^p(\Omega) \rightarrow L^p(\Sigma)$  tels que

$$T_1^{n_1} \cdots T_d^{n_d} = QR_1^{n_1} \cdots R_d^{n_d} J, \quad (n_1, \dots, n_d) \in \mathbb{N}^d.$$

Ensuite, ces résultats permettent de trouver une généralisation du théorème de dilatation d'Akcoglu-Sucheston, affirmant qu'une contraction positive sur un espace  $L_p$  se dilate en un isomorphisme isométrique. On obtient une telle dilatation pour un  $d$ -uplet de contractions positives commutantes pourvu que  $d - 1$  parmi celles-ci soient des opérateurs de Ritt.

**Théorème 1.2.14** *Soit  $\Sigma$  un espace mesuré et soit  $p \in (1, \infty)$ . Soit  $T_1, \dots, T_d$  des contractions positives commutantes sur  $L^p(\Sigma)$ . Supposons de plus que  $T_1, \dots, T_{d-1}$  sont des opérateurs de Ritt.*

*Alors il existe un espace mesuré  $\Omega$ , deux opérateurs bornés  $J: L^p(\Sigma) \rightarrow L^p(\Omega)$  et  $Q: L^p(\Omega) \rightarrow L^p(\Sigma)$ , ainsi que des isomorphismes isométriques  $U_1, \dots, U_d: L^p(\Omega) \rightarrow L^p(\Omega)$  tels que*

$$T_1^{n_1} \cdots T_d^{n_d} = QU_1^{n_1} \cdots U_d^{n_d}J, \quad (n_1, \dots, n_d) \in \mathbb{N}^d.$$

L'autre section du chapitre consacrée aux espaces UMD donne les analogues en termes de semi-groupes et de leur générateurs infinitésimaux, supposés être des opérateurs sectoriels. Rappelons à ce sujet qu'un opérateur  $A$  sur  $X$  est sectoriel de type  $\omega < \frac{\pi}{2}$  si et seulement si  $-A$  engendre un semi-groupe analytique borné  $(e^{-tA})_{t \geq 0}$  sur  $X$ .

Comme pour les combinaisons de dilatations pour les opérateurs de Ritt, on effectuera des combinaisons de dilatations de semi-groupes. Ainsi, on fera appel au lemme suivant.

**Lemme 1.2.15** *Soit  $d \geq 2$  un entier, soit  $(T_{1,t})_{t \geq 0}, \dots, (T_{d,t})_{t \geq 0}$  des  $C_0$ -semi-groupes commutants sur un espace de Banach  $X$  et soit  $p \in [1, \infty)$ . Soit  $1 \leq m \leq d$ . Supposons que :*

- (1) *Pour tout  $k = 1, \dots, m$ , il existe un  $C_0$ -semi-groupe  $(V_{k,t})_{t \geq 0}$  d'opérateurs positifs sur un certain espace  $L^p(\Omega)$  et deux opérateurs bornés  $J_k: X \rightarrow L^p(\Omega; X)$  et  $Q_k: L^p(\Omega; X) \rightarrow X$  tels que*

$$T_{k,t} = Q_k(V_{k,t} \overline{\otimes} I_X)J_k, \quad t \geq 0.$$

- (2) *Si  $m < d$ , il existe un espace de Banach  $Y$ , deux opérateurs bornés  $J_{m+1}: X \rightarrow Y$  et  $Q_{m+1}: Y \rightarrow X$  ainsi que des  $C_0$ -semi-groupes commutants  $(V_{m+1,t})_{t \geq 0}, \dots, (V_{d,t})_{t \geq 0}$  sur  $Y$  tels que*

$$T_{m+1,t_{m+1}} \cdots T_{d,t_d} = Q_{m+1}V_{m+1,t_{m+1}} \cdots V_{d,t_d}J_{m+1}, \quad t_{m+1} \geq 0, \dots, t_d \geq 0.$$

- (3) *Pour tout  $i = 1, \dots, m$  et  $j = 1, \dots, d$ , et pour tout  $t \geq 0$ , on a*

$$J_i T_{j,t} = (I_{L^p(\Omega)} \overline{\otimes} T_{j,t})J_i.$$

*Alors il existe deux opérateurs bornés  $J: X \rightarrow L^p(\Omega^m; Y)$  et  $Q: L^p(\Omega^m; Y) \rightarrow X$  tels que*

$$T_{1,t_1} \cdots T_{d,t_d} = QU_{1,t_1} \cdots U_{d,t_d}J, \quad t_1 \geq 0, \dots, t_d \geq 0,$$

où  $(U_{1,t})_{t \geq 0}, \dots, (U_{d,t})_{t \geq 0}$  sont des  $C_0$ -semi-groupes sur  $L^p(\Omega^m; Y)$  donnés par

$$U_{k,t} = I^{\otimes k-1} \overline{\otimes} V_{k,t} \overline{\otimes} I^{\otimes m-k} \overline{\otimes} I_Y, \quad k = 1, \dots, m;$$

$$U_{k,t} = I^{\otimes m} \overline{\otimes} V_{k,t}, \quad k = m+1, \dots, d.$$

Ici,  $I = I_{L^p(\Omega)}$  et  $I^{\otimes l} = \underbrace{I \overline{\otimes} \dots \overline{\otimes} I}_{l \text{ facteurs}}$  pour tout entier  $l \geq 1$ .

Tenant compte des résultats obtenus sur les dilatations d'opérateurs de Ritt (et donc sur les semi-groupes discrets  $(T^n)_{n \in \mathbb{N}^*}$ ), on obtient les résultats analogues sur les semi-groupes continus du type  $(e^{-tA})_{t \geq 0}$ .

**Théorème 1.2.16** *Soit  $p \in (1, \infty)$ . Soit  $X$  un espace de Banach réflexif tel que  $X$  et  $X^*$  ont chacun un cotype fini. Soit  $A_1, \dots, A_d$  des opérateurs sectoriels commutants sur  $X$  tels que chaque  $A_k$  admet un calcul fonctionnel  $H^\infty(\Sigma_{\theta_k})$  pour certains  $\theta_k$  dans  $(0, \frac{\pi}{2})$ ,  $k = 1, \dots, d$ .*

*Alors il existe un espace mesuré  $\Omega$ , des  $C_0$ -groupes commutants d'isométries  $(U_{1,t})_{t \in \mathbb{R}}, \dots, (U_{d,t})_{t \in \mathbb{R}}$  sur  $L^p(\Omega; X)$ , et deux opérateurs bornés  $J: X \rightarrow L^p(\Omega; X)$  et  $Q: L^p(\Omega; X) \rightarrow X$  tels que*

$$e^{-t_1 A_1} \dots e^{-t_d A_d} = Q U_{1,t_1} \dots U_{d,t_d} J, \quad t_1 \geq 0, \dots, t_d \geq 0.$$

Ce théorème ainsi que le théorème 1.2.6 permettent d'obtenir la caractérisation suivante du calcul fonctionnel joint  $H^\infty$  en terme de dilatation de semi-groupes.

**Théorème 1.2.17** *Soit  $X$  un espace de Banach ayant les propriétés UMD et  $(\alpha)$  et soit  $d \geq 1$  un entier. Soit  $A_1, \dots, A_d$  des opérateurs sectoriels commutants et soit  $p \in (1, \infty)$ . Les deux assertions suivantes sont équivalentes.*

- (1)  $(A_1, \dots, A_d)$  admet un calcul fonctionnel joint  $H^\infty(\Sigma_{\theta_1} \times \dots \times \Sigma_{\theta_d})$  pour certains  $\theta_k \in (0, \frac{\pi}{2})$ ,  $k = 1, \dots, d$ .
- (2) Il existe un espace mesuré  $\Omega$ , des opérateurs sectoriels commutants  $B_1, \dots, B_d$  sur  $L^p(\Omega; X)$  tels que tout  $B_k$  admet un calcul fonctionnel  $H^\infty(\Sigma_{\theta'_k})$  pour certains  $\theta'_k \in (0, \frac{\pi}{2})$ ,  $k = 1, \dots, d$ , ainsi que deux opérateurs bornés  $J: X \rightarrow L^p(\Omega; X)$  et  $Q: L^p(\Omega; X) \rightarrow X$  tels que

$$e^{-t_1 A_1} \dots e^{-t_d A_d} = Q e^{-t_1 B_1} \dots e^{-t_d B_d} J, \quad t_1 \geq 0, \dots, t_d \geq 0,$$

et tous les  $(e^{-t B_k})_{t \geq 0}$  sont des semi-groupes de contractions.

On retrouve dans le théorème précédent le fait remarquable d'avoir des opérateurs contractants (ici, des semi-groupes de contractions).

On a alors la caractérisation suivante sur les espaces  $L_p$ .

**Théorème 1.2.18** *Soit  $\Sigma$  un espace mesuré et soit  $p \in (1, \infty)$ . Soit  $A_1, \dots, A_d$  des opérateurs sectoriels sur  $L^p(\Sigma)$ . Les deux assertions suivantes sont équivalentes.*



- (1)  $(A_1, \dots, A_d)$  admet un calcul fonctionnel joint  $H^\infty(\Sigma_{\theta_1} \times \dots \times \Sigma_{\theta_d})$  pour certains  $\theta_k \in (0, \frac{\pi}{2})$ ,  $k = 1, \dots, d$ .
- (2) Il existe un espace mesuré  $\Omega$ , des opérateurs sectoriels commutants  $B_1, \dots, B_d$  sur  $L^p(\Omega)$  de type  $< \frac{\pi}{2}$ , et deux opérateurs bornés  $J: L^p(\Sigma) \rightarrow L^p(\Omega)$  et  $Q: L^p(\Omega) \rightarrow L^p(\Sigma)$  tels que

$$e^{-t_1 A_1} \dots e^{-t_d A_d} = Q e^{-t_1 B_1} \dots e^{-t_d B_d} J, \quad t_1 \geq 0, \dots, t_d \geq 0,$$

et tous les  $(e^{-t B_k})_{t \geq 0}$  sont des semi-groupes de contractions positives sur  $L^p(\Omega)$ .

**Théorème 1.2.19** Soit  $\Sigma$  un espace mesuré et soit  $p \in (1, \infty)$ . Soit  $(T_{1,t})_{t \geq 0}, \dots, (T_{d,t})_{t \geq 0}$  des  $C_0$ -semi-groupes de contractions positives sur  $L^p(\Sigma)$ . Supposons de plus que  $(T_{1,t})_{t \geq 0}, \dots, (T_{d-1,t})_{t \geq 0}$  sont des semi-groupes analytiques bornés.

Alors il existe un espace mesuré  $\Omega$ , deux opérateurs bornés  $J: L^p(\Sigma) \rightarrow L^p(\Omega)$  et  $Q: L^p(\Omega) \rightarrow L^p(\Sigma)$ , ainsi que des  $C_0$ -groupes commutants  $(U_{1,t})_{t \geq 0}, \dots, (U_{d,t})_{t \geq 0}$  d'isomorphismes isométriques sur  $L^p(\Omega)$  tels que

$$T_{1,t_1} \dots T_{d,t_d} = Q U_{1,t_1} \dots U_{d,t_d} J, \quad t_1 \geq 0, \dots, t_d \geq 0.$$

Le deuxième chapitre de cette partie traite du cas des espaces de Hilbert. Rappelons qu'un opérateur de Ritt sur un espace de Hilbert admet un calcul fonctionnel  $H^\infty$  si et seulement s'il est semblable à une contraction (voir [53, Théorème 8.1]). On étendra cette caractérisation aux  $d$ -uplets.

La question de la similarité jointe de plusieurs opérateurs commutants n'est pas simple. En effet, Pisier a montré dans [68] l'existence d'une paire  $(T_1, T_2)$  d'opérateurs commutants sur un Hilbert  $H$  tels que  $T_1$  et  $T_2$  sont tous deux semblables à des contractions (ce qui signifie qu'il existe des opérateurs inversibles  $S_1, S_2: H \rightarrow H$  tels que  $S_1^{-1} T_1 S_1$  et  $S_2^{-1} T_2 S_2$  sont des contractions sur  $H$ ) mais il n'existe pas d'opérateur inversible  $S: H \rightarrow H$  tel que  $S^{-1} T_1 S$  et  $S^{-1} T_2 S$  soient des contractions sur  $H$  (*similarité jointe*).

On obtient des caractérisations du calcul fonctionnel  $H^\infty$  joint de  $d$ -uplets d'opérateurs de Ritt commutants en terme de similarité jointe à des contractions commutantes.

**Théorème 1.2.20** Soit  $d \geq 3$  un entier et soit  $H$  un espace de Hilbert. Soit  $T_1, \dots, T_d$  des opérateurs commutants sur  $H$  tels que :

- (i) Pour tout  $j$  dans  $\{1, \dots, d-2\}$ ,  $T_j$  est un opérateur de Ritt semblable à une contraction.
- (ii) Il existe un opérateur borné et inversible  $S: H \rightarrow H$  tel que  $S^{-1} T_{d-1} S$  et  $S^{-1} T_d S$  sont tous deux des contractions.

Alors on a les propriétés suivantes :

- (1) Il existe un espace de Hilbert  $K$ , deux opérateurs bornés  $J: H \rightarrow K$  et  $Q: K \rightarrow H$  et des opérateurs unitaires et commutants  $U_1, \dots, U_d$  sur  $K$  tels que

$$T_1^{n_1} \dots T_d^{n_d} = Q U_1^{n_1} \dots U_d^{n_d} J, \quad (n_1, \dots, n_d) \in \mathbb{N}^d.$$

(2) Il existe  $C \geq 1$  tel que pour toute fonction polynomiale de  $d$  variables  $\phi$ ,

$$\|\phi(T_1, \dots, T_d)\| \leq C \|\phi\|_{\infty, \mathbb{D}^d}.$$

(3) Il existe un opérateur borné et inversible  $S: H \rightarrow H$  tels que pour tout  $j = 1, \dots, d$ ,  $S^{-1}T_jS$  est une contraction.

On note que la propriété (ii) est vérifiée si  $T_{d-1}$  et  $T_d$  sont des contractions. En particulier, la propriété (2) ci-dessus fournit une généralisation de l'inégalité de Ando et répond dans le cas d'opérateurs de Ritt au problème ouvert de l'existence d'une constante  $C > 0$  telle que  $\|\phi(T_1, \dots, T_d)\| \leq C \|\phi\|_{\infty, \mathbb{D}^d}$  pour toute fonction polynomiale  $\phi$ . De plus, toujours dans le cas d'opérateurs de Ritt, on obtient une similarité jointe à des contractions sous condition de similarité individuelle de chaque opérateurs à des contractions. On l'exprime avec le corollaire suivant.

**Corollaire 1.2.21** *Soit  $d \geq 2$  un entier et soit  $(T_1, \dots, T_d)$  des opérateurs de Ritt commutants sur un espace de Hilbert  $H$ . Les deux assertions suivantes sont équivalentes.*

(1)  $(T_1, \dots, T_d)$  admet un calcul fonctionnel joint  $H^\infty(B_{\gamma_1} \times \dots \times B_{\gamma_d})$  pour certains  $\gamma_k \in (0, \frac{\pi}{2})$ ,  $k = 1, \dots, d$ .

(2) Il existe un opérateur borné inversible  $S: H \rightarrow H$  tels que pour tout  $k = 1, \dots, d$ ,  $S^{-1}T_kS$  est une contraction.

On donne aussi des résultats analogues pour les semi-groupes d'opérateurs sur un Hilbert. Ils sont dans la continuité du travail initié dans [49].

## Cinquième partie

On note que des résultats variés sur les dilatations des opérateurs de Ritt ou sectoriels reposent sur des *fonctions carré* sur un espace de Banach. On ambitionne d'étudier les propriétés de dilatations pour les  $d$ -uplets d'opérateurs sur un espace de Banach ne vérifiant pas les hypothèses du Théorème 1.2.6. Pour ce faire, on introduit un outil incontournable : les fonctions carré relatives au  $d$ -uplets d'opérateurs de Ritt ou sectoriels. Ces définitions généralisent celles connues pour un seul opérateur.

Pour les opérateurs de Ritt, la fonction carré associée à un  $d$ -uplets  $T = (T_1, \dots, T_d)$  d'opérateurs commutants et à un  $d$ -uplet  $\alpha = (\alpha_1, \dots, \alpha_d)$  de réels strictement positifs s'exprime par

$$\|x\|_{T, \alpha} = \left\| \sum_{k_1, \dots, k_d \geq 1} \prod_{i=1}^d k_i^{\alpha_i - \frac{1}{2}} r_{k_1, \dots, k_d} \otimes \left( \prod_{i=1}^d T_i^{k_i} (I_X - T_i)^{\alpha_i} x \right) \right\|_{\text{Rad}((\mathbb{N}^*)^d; X)},$$

pour tout  $x$  de  $X$ . On s'intéresse alors aux  $d$ -uplets  $(\alpha_1, \dots, \alpha_d)$  de telle sorte que pour tout  $x$  de l'espace  $X$  on ait

$$\|x\|_{T, \alpha} \lesssim \|x\|.$$

Les fonctions carré correspondantes pour les  $d$ -uplets d'opérateurs sectoriels  $(A_1, \dots, A_d)$  se définissent par des fonctions  $F$  de  $H_0^\infty(\Sigma_{\theta_1} \times \dots \times \Sigma_{\theta_d})$  et non avec des  $d$ -uplets  $(\alpha_1, \dots, \alpha_d)$ . Elles sont plus abstraites et n'admettent pas d'expression explicite contrairement à celles relatives aux opérateurs de Ritt. On s'intéresse également aux fonctions  $F$  de telle sorte que pour tout  $x$  de l'espace  $X$  on ait

$$\|x\|_{A,F} \lesssim \|x\|.$$

Dans un premier temps, on étudie les liens possibles entre le calcul fonctionnel joint  $H^\infty$  et ces fonctions carré.

On montre d'abord que le calcul fonctionnel joint  $H^\infty$  entraîne des estimations carré, pourvu que l'espace sous-jacent soit de cotype fini. Ce résultat fait appel pour les opérateurs sectoriels comme pour les opérateurs de Ritt à la notion de calcul quadratique  $H^\infty$ .

**Définition 1.2.22** *Soit  $X$  un espace de Banach. Soit  $T = (T_1, \dots, T_d)$  un  $d$ -uplet d'opérateurs de Ritt commutants sur  $X$  tels que  $T_k$  est de type  $a_k$  pour  $k = 1, \dots, d$ . Soit  $\gamma_k \in (a_k, \frac{\pi}{2})$  pour  $k = 1, \dots, d$ . On dit que  $T$  admet un calcul fonctionnel quadratique  $H_{0,1}^\infty(B_{\gamma_1} \times \dots \times B_{\gamma_d})$  s'il existe une constante  $C > 0$  telle que pour toute famille finie  $(\varphi_i)_{i \in \mathcal{I}}$  dans  $H_0^\infty(B_{\gamma_1} \times \dots \times B_{\gamma_d})$  et  $x$  dans  $X$ ,*

$$\left\| \sum_{i \in \mathcal{I}} r_i \otimes \varphi_i(T_1, \dots, T_d)(x) \right\|_{\text{Rad}(\mathcal{I}, X)} \leq C \|x\| \left\| \left( \sum_{i \in \mathcal{I}} |\varphi_i|^2 \right)^{\frac{1}{2}} \right\|_{\infty, B_{\gamma_1} \times \dots \times B_{\gamma_d}},$$

avec  $(r_i)_{i \in \mathcal{I}}$  une famille de variables de Rademacher indépendantes.

**Proposition 1.2.23** *Soit  $X$  un espace de Banach de cotype fini. Soit  $T = (T_1, \dots, T_d)$  un  $d$ -uplet d'opérateurs de Ritt commutants sur  $X$ . Supposons que  $T$  admet un calcul fonctionnel joint  $H^\infty(B_{b_1} \times \dots \times B_{b_d})$  pour certains  $b_1, \dots, b_d$  dans  $(0, \frac{\pi}{2})$ .*

*Alors pour tous  $\gamma_1, \dots, \gamma_d$  tels que  $\frac{\pi}{2} > \gamma_k > b_k$  for  $k = 1, \dots, d$ ,  $T$  admet un calcul fonctionnel quadratique  $H^\infty(B_{\gamma_1} \times \dots \times B_{\gamma_d})$ .*

**Proposition 1.2.24** *Soit  $X$  un espace de Banach ne contenant pas  $c_0$ . Soit  $T = (T_1, \dots, T_d)$  un  $d$ -uplet d'opérateurs de Ritt commutants sur  $X$ . Supposons que  $T$  admet un calcul fonctionnel quadratique  $H^\infty(B_{\gamma_1} \times \dots \times B_{\gamma_d})$  pour certains  $\gamma_1, \dots, \gamma_d$  dans  $(0, \frac{\pi}{2})$ . Soit  $\alpha = (\alpha_1, \dots, \alpha_d)$  dans  $(\mathbb{R}_+^*)^d$ . Alors  $T$  satisfait une estimation carré, c'est-à-dire qu'il existe une constante  $K > 0$  telle que*

$$\|x\|_{T,\alpha} \leq K \|x\|.$$

On combine les deux propositions précédentes pour obtenir le théorème suivant.

**Théorème 1.2.25** *Soit  $X$  un espace de Banach de cotype fini. Supposons que  $T = (T_1, \dots, T_d)$  est un  $d$ -uplet d'opérateurs de Ritt commutants sur  $X$  qui admet un calcul fonctionnel joint  $H^\infty(B_{\gamma_1} \times \dots \times B_{\gamma_d})$  pour certains  $\gamma_1, \dots, \gamma_d$  dans  $(0, \frac{\pi}{2})$ . Soit  $\alpha = (\alpha_1, \dots, \alpha_d)$  dans  $(\mathbb{R}_+^*)^d$ . Alors  $T$  vérifie une estimation carré, c'est-à-dire qu'il existe une constante  $K > 0$  telle que pour tout  $x$  dans  $X$  on a*

$$\|x\|_{T,\alpha} \leq K \|x\|.$$

Les résultats analogues pour les sectoriels sont alors les suivants.

**Proposition 1.2.26** *Soit  $X$  un espace de Banach de cotype fini et soit  $(A_1, \dots, A_d)$  un  $d$ -uplet d'opérateurs sectoriels commutants admettant un calcul fonctionnel joint  $H^\infty(\Sigma_{\theta_1} \times \dots \times \Sigma_{\theta_d})$ . Alors pour tout  $\nu_k \in (\theta_k, \pi)$ ,  $k = 1, \dots, d$ , il existe une constante  $K \geq 0$  telle que*

$$\left\| \sum_{j=1}^n r_j \otimes F_j(A_1, \dots, A_d)x \right\|_{\text{Rad}(X)} \leq K \|x\| \left\| \left( \sum_{j=1}^n |F_j|^2 \right)^{\frac{1}{2}} \right\|_{\infty, \Sigma_{\nu_1} \times \dots \times \Sigma_{\nu_d}},$$

pour tout  $n \geq 1$ , pour toutes  $F_1, \dots, F_n$  dans  $H_0^\infty(\Sigma_{\nu_1} \times \dots \times \Sigma_{\nu_d})$  et pour tout  $x \in X$ .

**Théorème 1.2.27** *Soit  $X$  un espace de Banach de cotype fini. Soit  $A = (A_1, \dots, A_d)$  un  $d$ -uplet d'opérateurs sectoriels commutants sur  $X$ , admettant un calcul fonctionnel joint  $H^\infty(\Sigma_{\theta_1} \times \dots \times \Sigma_{\theta_d})$ . Alors pour toute fonction  $F \in H_{0,1}^\infty(\Sigma_{\nu_1} \times \dots \times \Sigma_{\nu_d})$ , avec  $\nu_k \in (\theta_k, \pi)$ ,  $(A_1, \dots, A_d)$  admet une estimation carré relative à  $F$ , c'est-à-dire qu'il existe une constante  $K > 0$  telle que pour tout  $x$  de  $X$  on a*

$$\|x\|_{A,F} \leq K \|x\|.$$

En référence aux résultats connus dans le cas d'un seul opérateur, on étudie une réciproques des théorèmes précédents dans le cas où l'on considère des opérateurs  $R$ -Ritt ou  $R$ -sectoriels. Si  $(T_1, \dots, T_d)$  est un  $d$ -uplet d'opérateurs de Ritt,  $(\alpha_1, \dots, \alpha_d) \in (0, \infty)^d$  et  $\Lambda$  est une partie de  $\{1, \dots, d\}$ , il est nécessaire de considérer  $\|x\|_{T, \alpha_\Lambda}$ , la fonction carré relative à la sous famille  $(T_k)_{k \in \Lambda}$  et  $(\alpha_k)_{k \in \Lambda}$  ( $x$  dans  $X$ ).

**Théorème 1.2.28** *Soit  $X$  un espace de Banach réflexif tel que  $X$  et  $X^*$  ont chacun un cotype fini. Soit  $T = (T_1, \dots, T_d)$  des opérateurs de Ritt commutants sur  $X$  tels que tous les  $T_k$  sont  $R$ -Ritt de  $R$ -type  $\delta_k \in (0, \frac{\pi}{2})$  pour  $k = 1, \dots, d$ . Soit  $\alpha = (\alpha_1, \dots, \alpha_d)$  et  $\beta = (\beta_1, \dots, \beta_d)$  des  $d$ -uplets de  $(\mathbb{R}_+^*)^d$ . Supposons qu'il existe une constante  $C > 0$  telle que pour tout  $\Lambda$ , partie de  $\{1, \dots, d\}$ , il existe  $\alpha_\Lambda = (\alpha_k)_{k \in \Lambda}$  et  $\beta_\Lambda = (\beta_k)_{k \in \Lambda}$  dans  $(\mathbb{R}_+^*)^\Lambda$  tels que*

$$\|x\|_{T, \alpha_\Lambda} \leq C \|x\|, \quad x \in X$$

et

$$\|y\|_{T, \beta_\Lambda} \leq C \|y\|, \quad y \in X^*.$$

Alors  $(T_1, \dots, T_d)$  admet un calcul fonctionnel joint  $H^\infty(B_{\gamma_1} \times \dots \times B_{\gamma_d})$  pour tout  $\gamma_k \in (\delta_k, \frac{\pi}{2})$ .

L'hypothèse de réflexivité assure l'équivalence des fonctions carré sur  $X$ , ce qui autorise à se ramener aux  $d$ -uplets  $\alpha = \beta = (1, \dots, 1)$ . Sans cette hypothèse, le théorème persiste en fixant lesdits  $d$ -uplets à  $(1, \dots, 1)$ .

On utilise alors pleinement ce théorème pour obtenir une caractérisation du calcul fonctionnel joint  $H^\infty$  sur les espaces ayant la propriété  $(\Delta)$ . Cette classe contient plus d'espaces que ceux ayant la propriété  $(\alpha)$ , notamment les espaces  $L_p$  non commutatifs. En particulier, on obtient une caractérisation du calcul fonctionnel joint  $H^\infty$  sur des espaces pour lesquels l'automatisme d'un tel calcul n'est pas nécessairement assurée (Théorème 1.2.6).

**Corollaire 1.2.29** *Soit  $X$  un espace de Banach ayant la propriété  $(\Delta)$ . Soit  $T_1, \dots, T_d$  des opérateurs de Ritt commutants sur  $X$ . Les deux assertions suivantes sont équivalentes.*

- i)  $(T_1, \dots, T_d)$  admet un calcul fonctionnel joint  $H^\infty(B_{\gamma_1} \times \dots \times B_{\gamma_d})$  pour certains  $\gamma_k \in (0, \frac{\pi}{2})$ ,  $k = 1, \dots, d$ .*
- ii) Chaque  $T_k$  est  $R$ -Ritt et il existe une constante  $C > 0$  telle que pour tout  $\Lambda$  partie de  $\{1, \dots, d\}$ , il existe  $\alpha_\Lambda$  et  $\beta_\Lambda$  dans  $(\mathbb{R}_+^*)^\Lambda$  tels que*

$$\|x\|_{T, \alpha_\Lambda} \leq C \|x\|, \quad x \in X,$$

$$\|y\|_{T^*, \beta_\Lambda} \leq C \|y\|, \quad y \in X^*.$$

Les résultats analogues pour les opérateurs sectoriels s'expriment comme suit.

**Théorème 1.2.30** *Supposons que  $X$  est réflexif de cotype fini et que pour tout  $k = 1, \dots, d$ ,  $A_k$  est  $R$ -sectoriel de  $R$ -type  $\omega_k \in (0, \pi)$  d'image dense dans  $X$ . Supposons de plus qu'il existe des fonctions non nulles  $F_1, F_2 \in H_0^\infty(\Sigma_{\nu_1} \times \dots \times \Sigma_{\nu_d})$ , pour certains  $\nu_k \in (\omega_k, \pi)$ , telles que  $(A_1, \dots, A_d)$  admet une estimation carré relative à  $F_1$ , soit*

$$\|x\|_{A, F_1} \leq K \|x\|, \quad x \in X, \quad K > 0$$

*et que  $A^* = (A_1^*, \dots, A_d^*)$  admet une estimation carré relative à  $F_2$ , soit*

$$\|y\|_{A^*, F_2} \leq K \|y\|, \quad y \in X^*, \quad K > 0.$$

*Alors pour tout  $\theta_k \in (\omega_k, \pi)$ ,  $k = 1, \dots, d$ ,  $(A_1, \dots, A_d)$  admet un calcul fonctionnel joint  $H^\infty(\Sigma_{\theta_1} \times \dots \times \Sigma_{\theta_d})$ .*

**Corollaire 1.2.31** *Soit  $X$  un espace de Banach réflexif avec la propriété  $(\Delta)$  et tel que  $X^*$  est de cotype fini. Les deux assertions suivantes sont équivalentes.*

- (i) Pour tout  $\theta_k \in (\omega_k, \pi)$ ,  $k = 1, \dots, d$ ,  $(A_1, \dots, A_d)$  admet un calcul fonctionnel joint  $H^\infty(\Sigma_{\theta_1} \times \dots \times \Sigma_{\theta_d})$ .*
- (ii) Pour tout  $\theta_k \in (\omega_k, \pi)$ ,  $k = 1, \dots, d$ , et pour toute  $F \in H_{0,1}^\infty(\Sigma_{\theta_1} \times \dots \times \Sigma_{\theta_d})$ ,  $(A_1, \dots, A_d)$  et  $A^* = (A_1^*, \dots, A_d^*)$  admettent tous deux une estimation carré relative à  $F$ , c'est-à-dire qu'il existe une constante  $K > 0$  telle que*

$$\|x\|_{A, F} \leq K \|x\|, \quad x \in X$$

*et*

$$\|y\|_{A^*, F} \leq K \|y\|, \quad y \in X^*.$$

## Sixième partie

Le sujet de cette partie est de trouver des connexions entre le calcul fonctionnel joint  $H^\infty$  et les propriétés de dilatations. Premièrement, on exploite les fonctions carré relatives aux  $d$ -uplets pour montrer que le calcul fonctionnel joint  $H^\infty$  implique des résultats de dilatations. Le résultat ci-dessous utilise les fonctions carré à travers une hypothèse de calcul fonctionnel joint  $H^\infty$  et non un calcul fonctionnel  $H^\infty$  de chaque opérateur.

**Théorème 1.2.32** *Soit  $X$  un espace de Banach réflexif et  $K$ -convexe et  $p$  dans  $(1, \infty)$ . Soit  $T = (T_1, \dots, T_d)$  un  $d$ -uplet d'opérateurs de Ritt commutants sur  $X$ . Supposons de plus que  $T$  admet un calcul fonctionnel joint  $H^\infty(B_{\gamma_1} \times \dots \times B_{\gamma_d})$  pour certains  $\gamma_1, \dots, \gamma_d$  dans  $(0, \frac{\pi}{2})$ .*

*Alors il existe un espace mesuré  $\Sigma$ , un  $d$ -uplet d'isomorphismes commutants  $(U_1, \dots, U_d)$  sur  $L_p(\Sigma; X)$  et deux opérateurs bornés  $J : X \rightarrow L_p(\Sigma; X)$  et  $Q : L_p(\Sigma; X) \rightarrow X$  tels que*

a)  $(U_1, \dots, U_d)$  admet un  $C(\mathbb{T}^d)$  calcul borné.

b) Pour tous entiers  $n_1, \dots, n_d \geq 0$ , on a

$$T_1^{n_1} \dots T_d^{n_d} = Q U_1^{n_1} \dots U_d^{n_d} J, \quad (n_1, \dots, n_d) \in \mathbb{N}^d. \quad (1.2.2)$$

On montre aussi dans une certaine mesure que la propriété de dilatation énoncée précédemment implique un calcul fonctionnel joint  $H^\infty$ . La réciproque que l'on obtient repose sur le Théorème 1.2.9.

**Théorème 1.2.33** *Soit  $X$  un espace de Banach et  $p \in (1, \infty)$ . Soit  $(T_1, \dots, T_d)$  un  $d$ -uplet d'opérateurs commutants sur  $X$  tel que tout  $T_k$  est un opérateur  $R$ -Ritt,  $k = 1, \dots, d$ . Supposons qu'il existe un espace mesuré  $\Sigma$ , un  $d$ -uplet d'isomorphismes  $(U_1, \dots, U_d)$  sur  $L_p(\Sigma; X)$  admettant un  $C(\mathbb{T}^d)$  calcul borné et deux opérateurs bornés  $J : X \rightarrow L_p(\Sigma; X)$ ,  $Q : L_p(\Sigma; X) \rightarrow X$  tels que (1.2.2) est vérifié.*

*Alors il existe  $\gamma_1, \dots, \gamma_d$  dans  $(0, \frac{\pi}{2})$  tel que  $(T_1, \dots, T_d)$  admet un calcul fonctionnel joint  $H^\infty(B_{\gamma_1} \times \dots \times B_{\gamma_d})$ .*

Les résultats analogues pour les opérateurs sectoriels s'expriment comme suit. On remarquera quelques différences dans l'expression des résultats, qui s'explique par une approche différente du travail par deux articles séparés, l'un concernant exclusivement les opérateurs de Ritt et l'autre les opérateurs sectoriels.

**Théorème 1.2.34** *Soit  $X$  un espace de Banach réflexif et  $K$ -convexe. Soit  $p$  dans  $(1, \infty)$ . Soit  $((T_t^1)_{t \geq 0}, \dots, (T_t^d)_{t \geq 0})$  un  $d$ -uplet de semi-groupes analytiques bornés qui commutent et soit  $(A_1, \dots, A_d)$  leur générateurs négatifs respectifs. Supposons que  $(A_1, \dots, A_d)$  admet un calcul fonctionnel joint  $H^\infty(\Sigma_{\theta_1} \times \dots \times \Sigma_{\theta_d})$ , pour certains  $\theta_k$  de  $(0, \frac{\pi}{2})$ ,  $k = 1, \dots, d$ . Alors il existe un espace mesuré  $(\Sigma, dm)$ , deux opérateurs bornés  $J : X \rightarrow L_p(\Sigma; X)$  et  $Q : L_p(\Sigma; X) \rightarrow X$ , ainsi qu'un  $d$ -uplet  $((U_t^1)_{t \geq 0}, \dots, (U_t^d)_{t \geq 0})$  de  $C_0$ -semi-groupes sur  $L_p(\Sigma; X)$  tel que*

- a) Le  $d$ -uplet  $(B_1, \dots, B_d)$ , où  $B_k$  est le générateur négatif de  $(U_t^k)_{t \geq 0}$  pour tout  $k = 1, \dots, d$ , admet un calcul fonctionnel joint  $H^\infty(\Sigma_{\frac{\pi}{2}} \times \dots \times \Sigma_{\frac{\pi}{2}})$ .
- b) Pour tout  $t_1, \dots, t_d$  de  $\mathbb{R}_+$ , on a

$$T_{t_1}^1 \dots T_{t_d}^d = QU_{t_1}^1 \dots U_{t_d}^d J. \quad (1.2.3)$$

**Theorem 1.2.35** Soit  $p$  dans  $(1, \infty)$ . Soit  $((T_t^1)_{t \geq 0}, \dots, (T_t^d)_{t \geq 0})$  un  $d$ -uplet de semi-groupes analytiques bornés qui commutent sur un espace de Banach  $X$ , et soit  $(A_1, \dots, A_d)$  leurs générateurs négatifs respectifs. Supposons que chaque  $A_k$  soit  $R$ -sectoriel de  $R$ -type  $\omega_k < \frac{\pi}{2}$ ,  $k = 1, \dots, d$ . Supposons également qu'il existe un espace mesuré  $(\Sigma, dm)$ , deux opérateurs bornés  $J : X \rightarrow L_p(\Sigma; X)$  et  $Q : L_p(\Sigma; X) \rightarrow X$ , ainsi qu'un  $d$ -uplet  $((U_t^1)_{t \geq 0}, \dots, (U_t^d)_{t \geq 0})$  de  $C_0$ -groupes bornés qui commutent tels que (1.2.3) est vérifié. Supposons de plus que le  $d$ -uplet  $(B_1, \dots, B_d)$ , où  $B_k$  est le générateur négatif de  $(U_t^k)_{t \geq 0}$  pour  $k = 1, \dots, d$ , admet un calcul fonctionnel joint  $H^\infty(\Sigma_{\frac{\pi}{2}} \times \dots \times \Sigma_{\frac{\pi}{2}})$ .

Alors  $(A_1, \dots, A_d)$  admet un calcul fonctionnel joint  $H^\infty(\Sigma_{\theta'_1} \times \dots \times \Sigma_{\theta'_d})$ , pour tout  $\theta'_k$  de  $(\omega, \frac{\pi}{2})$ ,  $k = 1, \dots, d$ .

# Chapter 2

## Introduction

### 2.1 General introduction

Let  $\mathcal{A}$  be a unital Banach algebra. Let  $d \geq 1$  be an integer.

Let  $\mathcal{P}_d$  be the algebra of polynomial functions on  $\mathbb{C}^d$ . Let  $x_1, \dots, x_d$  be commuting elements of  $\mathcal{A}$ . One can define a *polynomial calculus* on  $\mathcal{P}_d$ , associating to each function of  $\mathcal{P}_d$  the form  $f = \sum a_{k_1, \dots, k_d} z_1^{k_1} \cdots z_d^{k_d}$  (with a finite sum) the element

$$\Phi(f) = f(x_1, \dots, x_d) = \sum a_{k_1, \dots, k_d} x_1^{k_1} \cdots x_d^{k_d}.$$

Let now  $\sigma(x_k) \subset \mathbb{C}$  be the spectrum of  $x_k$ ,  $k = 1, \dots, d$ . Let  $\mathcal{O}_k$  be an open neighbourhood of  $\sigma(x_k)$  and  $\Gamma_k$  a contour of  $\sigma(x_k)$  in  $\mathcal{O}_k$ ,  $k = 1, \dots, d$ . Let  $\mathcal{H}(\mathcal{O}_1 \times \cdots \times \mathcal{O}_d)$  be the algebra of holomorphic function on  $\mathcal{O}_1 \times \cdots \times \mathcal{O}_d$ . One can define a *holomorphic calculus* on  $\mathcal{H}(\mathcal{O}_1 \times \cdots \times \mathcal{O}_d)$ , setting for any function  $f$  of  $\mathcal{H}(\mathcal{O}_1 \times \cdots \times \mathcal{O}_d)$ ,

$$\Psi(f) = \frac{1}{(2i\pi)^d} \int_{\Gamma_1 \times \cdots \times \Gamma_d} f(z_1, \dots, z_d) \prod_{k=1}^d R(z_k, x_k) \prod_{k=1}^d dz_k, \quad (2.1.1)$$

where  $R(z_k, x_k) = (z_k \mathbf{1}_{\mathcal{A}} - x_k)^{-1}$  denotes the resolvent of  $x_k$ ,  $k = 1, \dots, d$ . The definition does not depend on the choice of the  $\Gamma_k$ 's. When  $d = 1$ , this definition coincides with the Dunford calculus. Moreover, for any polynomial function  $f$ , we have  $\Phi(f) = \Psi(f)$ , meaning that  $\Psi$  extends the homomorphism  $\Phi$ . Both of these latter elements will be denoted by  $f(x_1, \dots, x_d)$ .

We turn now to the Banach algebra  $\mathcal{A} = B(X)$ , the algebra of all bounded operators on a Banach space  $X$  equipped with the operator norm. Let  $E_k$  be a subset of  $\mathbb{C}$  such that  $\sigma(x_k) \subset E_k \subset \mathcal{O}_k$ ,  $k = 1, \dots, d$  (taking notations above). If  $f$  is an analytic function on  $\mathcal{O}_1 \times \cdots \times \mathcal{O}_d$  which is bounded on  $E = E_1 \times \cdots \times E_d$ , we let

$$\|f\|_{\infty, E} = \sup \{|f(z)| : z \in E\}.$$

A classical question in functional calculus is to know whether there exists a constant  $C > 0$  such that

$$\|f(x_1, \dots, x_d)\| \leq C \|f\|_{\infty, E}, \quad (2.1.2)$$



where  $f$  runs over a certain subalgebra of bounded holomorphic functions on  $\mathcal{O}_1 \times \cdots \times \mathcal{O}_d$ .

Some classical results on Hilbertian operators theory are answers to this generic question. First, von Neumann's inequality says that for any contraction  $T$  on an Hilbert space, for any  $P$  a polynomial function of one variable we have

$$\|P(T)\| \leq \|P\|_{\infty, \mathbb{D}}.$$

Next, Ando's inequality, which generalises the von Neumann's inequality, claims that for any two commuting contractions  $T_1, T_2$  on an Hilbert space  $H$  and any polynomial function  $Q$  of two variables we have

$$\|Q(T_1, T_2)\| \leq \|Q\|_{\infty, \mathbb{D}^2}. \quad (2.1.3)$$

A counterexample of Crabb-Davie shows that this inequality fails for three commuting contractions on an Hilbert space. However, fixing commuting contractions  $T_1, \dots, T_d$  on an Hilbert space  $H$ , the existence of a constant  $C > 0$  such that for any polynomial function  $p$  of  $d$  variables  $\|p(T_1, \dots, T_d)\| \leq C \|p\|_{\infty, \mathbb{D}^d}$  is still an open question (see e.g [69] for all the facts above). Nevertheless, such an inequality happens if  $T_1, \dots, T_d$  are commuting normal operators, according to functional calculus on  $C^*$ -algebras.

It is also possible to study similar inequalities on non Hilbertian spaces, which generates other difficulties. One can treat the case of unbounded operators as well.

The fact that Ando's inequality does not extend to  $d \geq 3$  leads to the following type of question :

**(Q)** If  $T_1, \dots, T_d$  are commuting operators each verifying an inequality of type  $\|f(T_k)\| \leq C_k \|f\|_{\infty, E_k}$ , when does the  $d$ -tuple  $(T_1, \dots, T_d)$  satisfy an estimate (2.1.2) for  $E = E_1 \times \cdots \times E_d$ , that is there exists a constant  $C > 0$  such that for any suitable  $f$  we have  $\|f(T_1, \dots, T_d)\| \leq C \|f\|_{\infty, E_1 \times \cdots \times E_d}$  ?

In a nutshell, we claim that the study of the functional calculus of a  $d$ -tuple does not reduce to the study of the individual functional calculus of each of the element of the  $d$ -tuple.

Our work is devoted to the functional calculus of Ritt operators and sectorial operators. Both of these operators verify a resolvent condition which allows to define a rather general functional calculus with formula as (2.1.1), taking suitable contours of their spectra.

The sectorial operators appear in many areas of functional analysis. They appear as negative generator of analytic bounded semigroups, which intervene in resolution of elliptic equations (see [42], [10] and [47]), harmonic analysis (see [16], [33] and [73]), maximal regularity (see [3], [13], [21], [22], [38], [78] and [79]). For general informations about semigroups of operators, see [64].

The Ritt operators play a role in the resolution of certain linear equations using iterations (see [63]). Problems of maximal regularity use these operators as well as sectorial operators (see [11]).

Ritt and sectorial operators have been studied from the standpoint of functional calculus in many situations, as one can see in [5], [14], [15], [39], [43], [40], [41], [46],

[53], [60], [61], [75], [76] and [77]. For a large survey on functional calculus of sectorial operators, see [31]. Furthermore, subordination of discrete or continuous semigroups appeal to functional calculus of Ritt or sectorial operators (see [23], [28], [29]).

The starting point of the works mentioned above is the  $H^\infty$  functional calculus devised by Alan McIntosh (see [60]). In the latter reference, Alan McIntosh emphasized the role of *square functions*. He showed characterisations of the  $H^\infty$  functional calculus of sectorial operators acting on Hilbert spaces in terms of square function estimates. Later on, Cowling, Doust, McIntosh and Yagi found square functions to characterise the functional calculus of sectorial operators (see [15]) on (commutative)  $L_p$ -spaces. Their results admit natural generalisations to Banach lattices. A much more general context was given by Kalton and Weis in [41], when they used  $\gamma$ -spaces to define square functions in general Banach spaces. The case of non commutative  $L_p$ -spaces was treated by Junge, Le Merdy and Xu in [56]. A survey of all these methods is given in [52] and the case of square functions related to Ritt operators is treated in [53].

Another topic closely related to  $H^\infty$  functional calculus is *dilations*. First, von Neumann's inequality relies on Nagy's dilation Theorem. The latter result says that for a contraction  $T$  on an Hilbert space  $H$ , there exists a unitary element  $U$  of  $B(K)$  where  $K \supset H$  an Hilbert space, such that

$$T^n = P_H U^n|_H, \quad n \in \mathbb{N},$$

where  $P_H$  denotes the orthogonal projection from  $K$  onto  $H$ . One of the consequences of von Neumann's inequality is that the generator of a semigroup of contractions on an Hilbert space admits an  $H^\infty$  functional calculus.

Second, Matsaev's inequality, which relies on the Akcoglu-Sucheston dilation Theorem for positive contractions on  $L_p$ -space (see [1]), implies that the negative generator of a semigroup of positive contractions on an  $L_p$ -space admits an  $H^\infty$  functional calculus.

One can find examples of dilations on noncommutative  $L_p$ -spaces in [56, 3.2] which also yield  $H^\infty$  functional calculus properties.

The converse question, which searches dilation results from an  $H^\infty$  functional calculus, admits some remarkable answers. A turning point on this subject is the work of Fröhlich and Weis, who showed that  $H^\infty$  functional calculus implies dilation of sectorial operators into isomorphisms (see [27]). This paper brought to light the close relationship between dilation results and  $H^\infty$  functional calculus. More recently, Arhancet, Fackler and Le Merdy studied in [4] characterisations of  $H^\infty$  functional calculus for Ritt operators and semigroups in terms of dilations on space with special geometric properties (as UMD property). In particular, the authors obtain equivalence between  $H^\infty$  functional calculus of Ritt operators and dilation into Ritt contractions on a Bochner space.

This thesis deals about  $H^\infty$  *joint functional calculus* of a  $d$ -tuple of either commuting Ritt operators or sectorial operators. First, taking  $(T_1, \dots, T_d)$  a  $d$ -tuple of commuting operators, we will study the differences between  $H^\infty$  functional calculus of each of the  $T_k$ 's and the  $H^\infty$  *joint* functional calculus of the  $d$ -tuple. This will be expressed by a question in the style of **(Q)**. Next we aim to study the relationship between functional calculus and the notions we discussed above (square functions, dilation properties) for a  $d$ -tuple of commuting operators, in order to override the case of a single operator.

## 2.2 Contents of the thesis

### First part

The first chapter is devoted to the averages upon Rademacher variables and Gaussian variables. In particular, we will define the spaces  $\text{Rad}(X)$  and  $\text{Gauss}(X)$  on some Banach space  $X$ . Notions on operators based on Rademacher and Gaussian averages, such as  $R$ -boundedness and  $\gamma$ -boundedness, are introduced. Finally, we recall the fundamental notions of type and cotype of a Banach space.

The second chapter presents notions of Banach space geometry which play a role in various questions concerning functional calculus of Ritt operators and sectorial operators. In particular we discuss  $K$ -convex spaces, UMD-spaces and spaces having property  $(\alpha)$  or  $(\Delta)$ . We give the basic comparisons between these notions and we present large classes of Banach spaces which satisfy or do not satisfy these properties.

We end this part giving basic definitions of Ritt operators and sectorial operators and we give definitions of  $H^\infty$  joint functional calculus of a commuting  $d$ -tuple of these operators. The case of a  $d$ -tuple of sectorial operators was addressed by Albrecht in [2]. We naturally generalise it to  $d$ -tuples of Ritt operators, considering a suitable class of bounded analytic functions.

In the sequel, we use objects and notations introduced in this part, we refer to it for details and precise definitions. This includes the *Stolz domains*  $B_\alpha$ ,  $\alpha \in (0, \frac{\pi}{2})$  of  $\mathbb{C}$ , sectors  $\Sigma_\theta = \{z \in \mathbb{C}^* : |\text{Arg}(z)| < \theta\}$ ,  $\theta \in (0, \pi)$  as well as  $H_0^\infty(\Sigma_{\theta_1} \times \cdots \times \Sigma_{\theta_d})$  (resp.  $H_0^\infty(B_{\alpha_1} \times \cdots \times B_{\alpha_d})$ ) and  $H_{0,1}^\infty(\Sigma_{\theta_1} \times \cdots \times \Sigma_{\theta_d})$  (resp.  $H_{0,1}^\infty(B_{\alpha_1} \times \cdots \times B_{\alpha_d})$ ), which are subalgebras of bounded holomorphic functions on the corresponding domains.

### Second part

The second part deals with Franks-McIntosh decompositions of holomorphic functions. These authors constructed in [26] such decompositions for holomorphic functions of either one or several variables, in order to study various forms of functional calculi.

This part is divided into three steps. Firstly, we present the work from [26] related to sectors. Secondly, we give analogous results for Stolz domains, devised for applications to Ritt operators. Finally, we give decomposition of the unit in both cases.

For the sectorial case, all the useful estimations rely on the two theorem below. The following theorem implicitly appears in [26].

**Theorem 2.2.1** *Let  $d \geq 1$  be an integer, let  $\nu_k$  in  $(0, \pi)$  and  $\mu_k$  in  $(0, \nu_k)$ ,  $k = 1, \dots, d$ . There exist sequences  $(\Psi_{k,i_k})_{i_k \geq 1}$  and  $(\tilde{\Psi}_{k,i_k})_{i_k \geq 1}$  in  $H_0^\infty(\Sigma_{\mu_k})$  verifying the following properties.*

(1) *For every real number  $p > 0$  and for any  $k = 1, \dots, d$ ,*

$$\sup \left\{ \sum_{i_k=1}^{\infty} |\Psi_{k,i_k}(\zeta_k)|^p : \zeta_k \in \Sigma_{\mu_k} \right\} < \infty \quad \text{and} \quad \sup \left\{ \sum_{i_k=1}^{\infty} |\tilde{\Psi}_{k,i_k}(\zeta_k)|^p : \zeta_k \in \Sigma_{\mu_k} \right\} < \infty.$$

(2) *There exists a constant  $C > 0$  such that for every  $h$  in  $H^\infty(\Sigma_{\nu_1} \times \cdots \times \Sigma_{\nu_d})$ , there exists a family  $(a_{i_1, \dots, i_d})_{i_1, \dots, i_d \geq 1}$  of complex numbers such that*

$$|a_{i_1, \dots, i_d}| \leq C \|h\|_{\infty, \Sigma_{\nu_1} \times \cdots \times \Sigma_{\nu_d}}, \quad (i_1, \dots, i_d) \in \mathbb{N}^{*d},$$

*and for every  $(\zeta_1, \dots, \zeta_d)$  in  $\prod_{k=1}^d \Sigma_{\mu_k}$ ,*

$$h(\zeta_1, \dots, \zeta_d) = \sum_{i_1, \dots, i_d \geq 1} a_{i_1, \dots, i_d} \Psi_{1, i_1}(\zeta_1) \tilde{\Psi}_{1, i_1}(\zeta_1) \cdots \Psi_{d, i_d}(\zeta_d) \tilde{\Psi}_{d, i_d}(\zeta_d).$$

The proposition below yields a decomposition of the unit on sectors and follows from analogous principles. Details will be supplied in Chapter 6.

**Proposition 2.2.2** *Let  $\mu \in (0, \pi)$ . There exist three sequences  $(\Delta_i)_{i \geq 1}$ ,  $(\psi_i)_{i \geq 1}$  and  $(\tilde{\psi}_i)_{i \geq 1}$  in  $H_0^\infty(\Sigma_\mu)$  satisfying the following properties.*

(1) *There exists a constant  $C \geq 0$  such that*

$$\forall z \in \Sigma_\mu, \quad \sum_{i=1}^{\infty} |\psi_i(z)| \leq C \quad \text{and} \quad \sum_{i=1}^{\infty} |\tilde{\psi}_i(z)| \leq C.$$

(2) *For any  $\nu \in (0, \mu)$ , there exists a constant  $K \geq 0$  such that*

$$\forall i \geq 1, \quad \int_{\partial \Sigma_\nu} |\Delta_i(z)| \left| \frac{dz}{z} \right| \leq K.$$

(3) *There exists a constant  $C \geq 0$  such that*

$$\forall i \geq 1, \quad \forall z \in \Sigma_\mu, \quad |\Delta_i(z)| \leq C,$$

*and*

$$\forall z \in \Sigma_\mu, \quad 1 = \sum_{i=1}^{\infty} \Delta_i(z) \psi_i(z) \tilde{\psi}_i(z).$$

For the purpose of having results for  $H^\infty$  joint functional calculus of Ritt operators, we provide a detailed proof of the Franks-McIntosh decomposition on Stolz domains. The result below is implicit in [26, Section 4], however no proof has been written yet. The one we provide in our work is close to the one for sectors given in [26, Section 3], and simpler than the one which is sketched in [26, Section 4] for domains having several points of contact.

**Theorem 2.2.3** *Let  $d \geq 1$  be an integer, let  $\beta_k$  in  $(0, \frac{\pi}{2})$  and  $\alpha_k$  in  $(0, \beta_k)$ ,  $k = 1, \dots, d$ . There exist sequences  $(\Psi_{k, i_k})_{i_k \geq 1}$  and  $(\tilde{\Psi}_{k, i_k})_{i_k \geq 1}$  in  $H_0^\infty(B_{\alpha_k})$  verifying the following properties.*

(1) For every real number  $p > 0$  and for any  $k = 1, \dots, d$ ,

$$\sup \left\{ \sum_{i_k=1}^{\infty} |\Psi_{k,i_k}(\zeta_k)|^p : \zeta_k \in B_{\alpha_k} \right\} < \infty \quad \text{and} \quad \sup \left\{ \sum_{i_k=1}^{\infty} |\tilde{\Psi}_{k,i_k}(\zeta_k)|^p : \zeta_k \in B_{\alpha_k} \right\} < \infty.$$

(2) There exists a constant  $C > 0$  such that for every  $h$  in  $H^\infty(B_{\beta_1} \times \dots \times B_{\beta_d})$ , there exists a family  $(a_{i_1, \dots, i_d})_{i_1, \dots, i_d \geq 1}$  of complex numbers such that

$$|a_{i_1, \dots, i_d}| \leq C \|h\|_{\infty, B_{\beta_1} \times \dots \times B_{\beta_d}}, \quad (i_1, \dots, i_d) \in \mathbb{N}^{*d},$$

and for every  $(\zeta_1, \dots, \zeta_d)$  in  $\prod_{k=1}^d B_{\alpha_k}$ ,

$$h(\zeta_1, \dots, \zeta_d) = \sum_{i_1, \dots, i_d \geq 1} a_{i_1, \dots, i_d} \Psi_{1,i_1}(\zeta_1) \tilde{\Psi}_{1,i_1}(\zeta_1) \cdots \Psi_{d,i_d}(\zeta_d) \tilde{\Psi}_{d,i_d}(\zeta_d).$$

The following statement is an analogue of Proposition 2.2.2.

**Theorem 2.2.4** *Let  $\alpha$  in  $(0, \frac{\pi}{2})$ . There exist sequences  $(\phi_i)_{i \geq 1}$ ,  $(\varphi_i)_{i \geq 1}$ ,  $(\theta_i)_{i \geq 1}$  and  $(\psi_i)_{i \geq 1}$  of  $H_0^\infty(B_\alpha)$  such that*

(i) *For every  $i \geq 1$ , we have  $\phi_i = \theta_i \varphi_i \psi_i$ ;*

(ii) *There exists a constant  $c > 0$  such that for every  $z$  in  $B_\alpha$*

$$\sum_{i=1}^{\infty} |\varphi_i(z)| \leq c, \quad \sum_{i=1}^{\infty} |\psi_i(z)| \leq c;$$

(iii) *For every  $\gamma \in (0, \alpha)$ , there exists a constant  $e > 0$  such that for every  $i \geq 1$*

$$\int_{\partial B_\gamma} \frac{|\theta_i(z)|}{|1-z|} |dz| \leq e;$$

(iv) *For every  $z$  in  $B_\alpha$ , the series  $\sum_{i \geq 1} \phi_i(z)$  absolutely converges and there exists a constant  $c'$  such that*

$$\sup \left\{ \sum_{i=1}^{\infty} |\phi_i(z)| : z \in B_\alpha \right\} \leq c'.$$

Moreover, we have

$$\sum_{i=1}^{\infty} \phi_i(z) = 1, \quad z \in B_\alpha.$$

Note that Theorem 2.2.1 and 2.2.3 hold true for holomorphic functions  $h$  taking values in some Banach space  $Z$ .

### Third part

The third part treats three independent questions related to the  $H^\infty$  joint functional calculus of a commuting  $d$ -tuple of Ritt operators or sectorial operators.

The first chapter gives an answer to the question of the automaticity of  $H^\infty$  joint functional calculus of a  $d$ -tuple under the assumption that each element of the  $d$ -tuple has an  $H^\infty$  functional calculus. This is a key property for lots of the result of this thesis. It consists in proving the following theorem.

**Theorem 2.2.5** *Let  $X$  be a Banach space. Assume that either  $X$  is a Banach lattice, or  $X$  or  $X^*$  has property  $(\alpha)$ . Let  $d \geq 2$  be an integer. Then the following two properties hold :*

- (P1) *Let  $(T_1, \dots, T_d)$  be a commuting  $d$ -tuple of Ritt operators on  $X$  and assume that for some  $0 < \gamma_1, \dots, \gamma_d < \frac{\pi}{2}$ ,  $T_k$  has an  $H^\infty(B_{\gamma_k})$  functional calculus for any  $k = 1, \dots, d$ . Then for any  $\gamma'_k \in (\gamma_k, \frac{\pi}{2})$ ,  $k = 1, \dots, d$ ,  $(T_1, \dots, T_d)$  admits an  $H^\infty(B_{\gamma'_1} \times \dots \times B_{\gamma'_d})$  joint functional calculus.*
- (P2) *Let  $(A_1, \dots, A_d)$  be a commuting  $d$ -tuple of sectorial operators on  $X$  and assume that for some  $0 < \theta_1, \dots, \theta_d < \pi$ ,  $A_k$  has an  $H^\infty(\Sigma_{\theta_k})$  functional calculus for any  $k = 1, \dots, d$ . Then for any  $\theta'_k \in (\theta_k, \pi)$ ,  $k = 1, \dots, d$ ,  $(A_1, \dots, A_d)$  admits an  $H^\infty(\Sigma_{\theta'_1} \times \dots \times \Sigma_{\theta'_d})$  joint functional calculus.*

Property (P2) for  $d = 2$  was proved in [48]. The proof for  $d \geq 3$  is a simple adaptation of the argument devised in the latter paper. In the special case when  $X$  is an  $L^p$ -space for  $p \in [1, \infty)$ , property (P2) goes back to [2]. Proving property (P1) will require the Franks-McIntosh decomposition we described in the Second Part.

Next, the second chapter of this part considers the convex combinations of Ritt operators. We prove that a convex combination of Ritt operators is still a Ritt operator. We obtain a functional calculus property of such a convex combination, if the underlying Banach space has property  $(\Delta)$ , a weaker property than property  $(\alpha)$ .

**Theorem 2.2.6** *Let  $X$  be a Banach space with property  $(\Delta)$  and  $d \geq 2$  be an integer. Let  $T_1, \dots, T_d$  be commuting Ritt operators acting on  $X$ . Let  $c_1, \dots, c_d$  be positive real numbers such that  $\sum_{k=1}^d c_k = 1$ . Suppose that any  $T_k$  admits an  $H^\infty$  functional calculus,  $k = 1, \dots, d$ . Then  $T = \sum_{k=1}^d c_k T_k$  admits an  $H^\infty$  functional calculus.*

The last chapter provides a result of angle reduction of  $H^\infty$  functional calculus. Recall that if  $(A_1, \dots, A_d)$  is a  $d$ -tuple of commuting sectorial operators on some Banach space  $X$  having an  $H^\infty(\Sigma_{\theta_1} \times \dots \times \Sigma_{\theta_d})$  joint functional calculus, then it also admits an  $H^\infty(\Sigma_{\theta'_1} \times \dots \times \Sigma_{\theta'_d})$  joint functional calculus whenever  $\theta'_k \geq \theta_k$ ,  $k = 1, \dots, d$ . Likewise, if  $(T_1, \dots, T_d)$  is a  $d$ -tuple of Ritt operators having an  $H^\infty(B_{\gamma_1} \times \dots \times B_{\gamma_d})$  joint functional calculus, then it also admits an  $H^\infty(B_{\gamma'_1} \times \dots \times B_{\gamma'_d})$  for any  $\gamma'_k \geq \gamma_k$ ,  $k = 1, \dots, d$ . The aim is to have a converse property.

McIntosh proved in [60] that the  $H^\infty(\Sigma_\theta)$  functional calculus of a sectorial operator acting on Hilbert does not depend on  $\theta$ , as long as it is well defined. However, Kalton showed that this property does not hold true on general Banach spaces (see

[37]). Finally, Kalton and Weis showed an extension of McIntosh's result in general Banach spaces provided they consider an  $R$ -sectorial operator. The theorem below is a generalisation to the multivariate case of the Kalton-Weis theorem. Part (2) is the adaptation to the Ritt case and part (3) is a variant of the latter.

**Theorem 2.2.7** *Let  $X$  be a Banach space. Let  $d \geq 2$  be an integer. Then the following three properties hold :*

- (1) *Let  $(A_1, \dots, A_d)$  be a  $d$ -tuple of commuting  $R$ -sectorial operators on  $X$ , of respective  $R$ -types  $\omega_1, \dots, \omega_d$ . Let  $\theta_k \in (\omega_k, \pi)$ , for  $k = 1, \dots, d$ , and assume that  $(A_1, \dots, A_d)$  admits an  $H^\infty(\Sigma_{\theta_1} \times \dots \times \Sigma_{\theta_d})$  joint functional calculus. Then for any  $\theta'_k \in (\omega_k, \pi)$ ,  $k = 1, \dots, d$ , the family  $(A_1, \dots, A_d)$  admits an  $H^\infty(\Sigma_{\theta'_1} \times \dots \times \Sigma_{\theta'_d})$  joint functional calculus.*
- (2) *Let  $(T_1, \dots, T_d)$  be a  $d$ -tuple of commuting  $R$ -Ritt operators on  $X$ , of respective  $R$ -types  $a_1, \dots, a_d$ . Let  $\gamma_k \in (a_k, \frac{\pi}{2})$ , for  $k = 1, \dots, d$ , and assume that  $(T_1, \dots, T_d)$  admits an  $H^\infty(B_{\gamma_1} \times \dots \times B_{\gamma_d})$  joint functional calculus. Then for any  $\gamma'_k \in (a_k, \frac{\pi}{2})$ ,  $k = 1, \dots, d$ , the family  $(T_1, \dots, T_d)$  admits an  $H^\infty(B_{\gamma'_1} \times \dots \times B_{\gamma'_d})$  joint functional calculus.*
- (3) *Let  $(T_1, \dots, T_d)$  be a  $d$ -tuple of commuting operators acting on  $X$  such that every  $T_k$  is a  $R$ -Ritt operator,  $k = 1, \dots, d$ . Suppose that  $(T_1, \dots, T_d)$  is polynomially bounded, that is there exists a constant  $C \geq 1$  such that for any polynomial function  $h$  of  $d$  variables we have*

$$\|h(T_1, \dots, T_d)\| \leq C \sup \{ |h(z_1, \dots, z_d)| : (z_1, \dots, z_d) \in \mathbb{T}^d \}. \quad (2.2.1)$$

*Then  $(T_1, \dots, T_d)$  admits a  $H^\infty$  joint functional calculus.*

## Fourth part

The fourth part explores characterisations of  $H^\infty$  joint functional calculus on UMD-spaces having property  $(\alpha)$  in terms of dilations. This uses the work initiated in [4] and generalises it. Theorem 2.2.5 will be useful here.

**Theorem 2.2.8** *Let  $X$  be a UMD Banach space with property  $(\alpha)$  and let  $d \geq 1$  be an integer. Let  $T_1, \dots, T_d$  be commuting Ritt operators on  $X$  and let  $p \in (1, \infty)$ . The following two conditions are equivalent.*

- (1)  *$(T_1, \dots, T_d)$  admits an  $H^\infty(B_{\gamma_1} \times \dots \times B_{\gamma_d})$  joint functional calculus for some  $\gamma_k \in (0, \frac{\pi}{2})$ ,  $k = 1, \dots, d$ .*
- (2) *There exist a measure space  $\Omega$ , commuting contractive Ritt operators  $R_1, \dots, R_d$  on  $L^p(\Omega; X)$  such that every  $R_k$  admits an  $H^\infty(B_{\gamma'_k})$  functional calculus for some  $\gamma'_k \in (0, \frac{\pi}{2})$ ,  $k = 1, \dots, d$ , as well as two bounded operators  $J: X \rightarrow L^p(\Omega; X)$  and  $Q: L^p(\Omega; X) \rightarrow X$  such that*

$$T_1^{n_1} \dots T_d^{n_d} = QR_1^{n_1} \dots R_d^{n_d} J, \quad (n_1, \dots, n_d) \in \mathbb{N}^d.$$

The main implication is "(1)  $\Rightarrow$  (2)". The remarkable fact in the dilation property (2) is that  $R_1, \dots, R_d$  are contractive.

This theorem takes a special form if  $X$  is an  $L_p$ -space, leading to a characterisation in terms of positive contractions on  $L_p$ .

**Theorem 2.2.9** *Let  $\Sigma$  be a measure space and let  $p \in (1, \infty)$ . Let  $T_1, \dots, T_d$  be commuting Ritt operators on  $L^p(\Sigma)$ . The following two conditions are equivalent.*

- (1)  $(T_1, \dots, T_d)$  admits an  $H^\infty(B_{\gamma_1} \times \dots \times B_{\gamma_d})$  joint functional calculus for some  $\gamma_k \in (0, \frac{\pi}{2})$ ,  $k = 1, \dots, d$ .
- (2) There exist a measure space  $\Omega$ , commuting positive contractive Ritt operators  $R_1, \dots, R_d$  on  $L^p(\Omega)$ , and two bounded operators  $J: L^p(\Sigma) \rightarrow L^p(\Omega)$  and  $Q: L^p(\Omega) \rightarrow L^p(\Sigma)$  such that

$$T_1^{n_1} \dots T_d^{n_d} = QR_1^{n_1} \dots R_d^{n_d} J, \quad (n_1, \dots, n_d) \in \mathbb{N}^d.$$

Next, we show that the Akcoglu-Sucheston dilation of one positive contraction on a  $L_p$ -space admits a generalisation to  $d$ -tuples of commuting positive contractions provided  $d - 1$  of them are Ritt operators.

**Theorem 2.2.10** *Let  $\Sigma$  be a measure space and let  $p \in (1, \infty)$ . Let  $T_1, \dots, T_d$  be commuting positive contractions on  $L^p(\Sigma)$ . Assume further that  $T_1, \dots, T_{d-1}$  are Ritt operators.*

*Then there exist a measure space  $\Omega$ , two bounded operators  $J: L^p(\Sigma) \rightarrow L^p(\Omega)$  and  $Q: L^p(\Omega) \rightarrow L^p(\Sigma)$ , as well as commuting isometric isomorphisms  $U_1, \dots, U_d: L^p(\Omega) \rightarrow L^p(\Omega)$  such that*

$$T_1^{n_1} \dots T_d^{n_d} = QU_1^{n_1} \dots U_d^{n_d} J, \quad (n_1, \dots, n_d) \in \mathbb{N}^d.$$

There are adaptations of these results to the semigroups. Recall that an operator  $A$  on  $X$  is sectorial of type  $\omega < \frac{\pi}{2}$  if and only if  $-A$  generates a bounded analytic semigroup  $(e^{-tA})_{t \geq 0}$  on  $X$ . We cite the main results.

**Theorem 2.2.11** *Let  $X$  be a UMD Banach space with property  $(\alpha)$  and let  $d \geq 1$  be an integer. Let  $A_1, \dots, A_d$  be commuting sectorial operators and let  $p \in (1, \infty)$ . The following two conditions are equivalent.*

- (1)  $(A_1, \dots, A_d)$  admits an  $H^\infty(\Sigma_{\theta_1} \times \dots \times \Sigma_{\theta_d})$  joint functional calculus for some  $\theta_k \in (0, \frac{\pi}{2})$ ,  $k = 1, \dots, d$ .
- (2) There exist a measure space  $\Omega$ , commuting sectorial operators  $B_1, \dots, B_d$  on  $L^p(\Omega; X)$  such that every  $B_k$  admits an  $H^\infty(\Sigma_{\theta'_k})$  functional calculus for some  $\theta'_k \in (0, \frac{\pi}{2})$ ,  $k = 1, \dots, d$ , as well as two bounded operators  $J: X \rightarrow L^p(\Omega; X)$  and  $Q: L^p(\Omega; X) \rightarrow X$  such that

$$e^{-t_1 A_1} \dots e^{-t_d A_d} = Qe^{-t_1 B_1} \dots e^{-t_d B_d} J, \quad t_1 \geq 0, \dots, t_d \geq 0,$$

and all the  $(e^{-tB_k})_{t \geq 0}$  are semigroups of contractions.



Consequences of the theorem above are similar to the ones for Ritt operators, as follows.

**Theorem 2.2.12** *Let  $\Sigma$  be a measure space and let  $p \in (1, \infty)$ . Let  $A_1, \dots, A_d$  be commuting sectorial operators on  $L^p(\Sigma)$ . The following conditions are equivalent.*

- (1)  $(A_1, \dots, A_d)$  admits an  $H^\infty(\Sigma_{\theta_1} \times \dots \times \Sigma_{\theta_d})$  joint functional calculus for some  $\theta_k \in (0, \frac{\pi}{2})$ ,  $k = 1, \dots, d$ .
- (2) There exist a measure space  $\Omega$ , commuting sectorial operators  $B_1, \dots, B_d$  on  $L^p(\Omega)$  of type  $< \frac{\pi}{2}$ , and two bounded operators  $J: L^p(\Sigma) \rightarrow L^p(\Omega)$  and  $Q: L^p(\Omega) \rightarrow L^p(\Sigma)$  such that

$$e^{-t_1 A_1} \dots e^{-t_d A_d} = Q e^{-t_1 B_1} \dots e^{-t_d B_d} J, \quad t_1 \geq 0, \dots, t_d \geq 0,$$

and all the  $(e^{-t B_k})_{t \geq 0}$  are semigroups of positive contractions.

**Theorem 2.2.13** *Let  $\Sigma$  be a measure space and let  $p \in (1, \infty)$ . Let  $(T_{1,t})_{t \geq 0}, \dots, (T_{d,t})_{t \geq 0}$  be  $C_0$ -semigroups of positive contractions on  $L^p(\Sigma)$ . Assume further that  $(T_{1,t})_{t \geq 0}, \dots, (T_{d-1,t})_{t \geq 0}$  are bounded analytic semigroups.*

*Then there exist a measure space  $\Omega$ , two bounded operators  $J: L^p(\Sigma) \rightarrow L^p(\Omega)$  and  $Q: L^p(\Omega) \rightarrow L^p(\Sigma)$ , as well as commuting  $C_0$ -groups  $(U_{1,t})_{t \geq 0}, \dots, (U_{d,t})_{t \geq 0}$  of isometric isomorphisms on  $L^p(\Omega)$  such that*

$$T_{1,t_1} \dots T_{d,t_d} = Q U_{1,t_1} \dots U_{d,t_d} J, \quad t_1 \geq 0, \dots, t_d \geq 0.$$

The second chapter of this part is devoted to the Hilbert space case. Recall that a Ritt operator acting on a Hilbert space admits an  $H^\infty$  functional calculus if and only if it is similar to a contraction (see [53, Theorem 8.1]). We will extend this characterisation to  $d$ -tuples of Ritt operators.

The question of the joint similarity of several commuting operators is not straightforward. Indeed, Pisier showed in [68] the existence of a pair  $(T_1, T_2)$  of commuting operators on Hilbert space  $H$  such that  $T_1$  and  $T_2$  are both similar to contractions (that is, there exist bounded invertible operators  $S_1, S_2: H \rightarrow H$  such that  $S_1^{-1} T_1 S_1$  and  $S_2^{-1} T_2 S_2$  are contractions) but there is no common bounded invertible  $S: H \rightarrow H$  such that  $S^{-1} T_1 S$  and  $S^{-1} T_2 S$  are contractions.

We obtain the following properties of  $d$ -tuples of Ritt operators having an  $H^\infty$  joint functional calculus.

**Theorem 2.2.14** *Let  $d \geq 3$  be an integer and let  $H$  be a Hilbert space. Let  $T_1, \dots, T_d$  be commuting operators on  $H$  such that:*

- (i) *For every  $j$  in  $\{1, \dots, d-2\}$ ,  $T_j$  is a Ritt operator which is similar to a contraction.*
- (ii) *There exists a bounded invertible operator  $S: H \rightarrow H$  such that  $S^{-1} T_{d-1} S$  and  $S^{-1} T_d S$  are both contractions.*

Then we have the following three properties:

- (1) There exist a Hilbert space  $K$ , two bounded operators  $J: H \rightarrow K$  and  $Q: K \rightarrow H$  and commuting unitary operators  $U_1, \dots, U_d$  on  $K$  such that

$$T_1^{n_1} \cdots T_d^{n_d} = QU_1^{n_1} \cdots U_d^{n_d} J, \quad (n_1, \dots, n_d) \in \mathbb{N}^d.$$

- (2) There exists  $C \geq 1$  such that for any polynomial  $\phi$  in  $\mathcal{P}_d$ ,

$$\|\phi(T_1, \dots, T_d)\| \leq C \|\phi\|_{\infty, \mathbb{D}^d}.$$

- (3) There exists a bounded invertible operator  $S: H \rightarrow H$  such that for any  $j = 1, \dots, d$ ,  $S^{-1}T_jS$  is a contraction.

Note that property (ii) is satisfied if  $T_{d-1}, T_d$  are contractions. Thus, this theorem yields a generalisation of Ando's inequality for a  $d$ -tuple of commuting contractions provided that  $d - 2$  of them are Ritt operators. Further, we obtain that if  $T_1, \dots, T_d$  are commuting Ritt operators on an Hilbert space each similar to a contraction, then they are simultaneously similar to a contraction.

We also obtain analogous results for semigroups on Hilbert space. This is a continuation of the work initiated in [49].

## Fifth part

We note that various results of dilations of a Ritt operator or a sectorial operator rely on square function related to this operator. We aim to study dilation properties for  $d$ -tuples of operators acting on Banach spaces which do not verify the assumption of Theorem 2.2.5. In order to do that, we introduce a tool which is inescapable : the square functions related to a  $d$ -tuple of either commuting Ritt operators or sectorial operators.

Given  $T = (T_1, \dots, T_d)$  a  $d$ -tuple of commuting Ritt operators, and  $\alpha = (\alpha_1, \dots, \alpha_d)$  in  $(\mathbb{R}_+^*)^d$ , we give a suitable definition of square functions  $\|x\|_{T, \alpha}$  for  $x$  in the underlying Banach space  $X$ . This generalises the case of a single operator given in [6].

Likewise, we generalise the square function  $\|x\|_{A, F}$  for a sectorial operator  $A$  given in [40] to a  $d$ -tuple of commuting sectorial operators and a multivariate function  $F$ .

Then we study the links between  $H^\infty$  joint functional calculus and these square functions. We show that  $H^\infty$  functional calculus implies square function estimates, only supposing the finite cotype of the underlying Banach space.

**Theorem 2.2.15** *Let  $X$  be a Banach space with a finite cotype. Suppose that  $T = (T_1, \dots, T_d)$  is a  $d$ -tuple of commuting Ritt operators on  $X$  which has an  $H^\infty(B_{\gamma_1} \times \cdots \times B_{\gamma_d})$  joint functional calculus for some  $\gamma_1, \dots, \gamma_d$  in  $(0, \frac{\pi}{2})$ . Let  $\alpha = (\alpha_1, \dots, \alpha_d)$  in  $(\mathbb{R}_+^*)^d$ . Then  $T$  verifies a square function estimate, that is there exists a constant  $K > 0$  such that for any  $x$  in  $X$  we have*

$$\|x\|_{T, \alpha} \leq K \|x\|.$$

The analogous result for sectorial operators is the following.

**Theorem 2.2.16** *Assume that  $X$  has finite cotype and that  $(A_1, \dots, A_d)$  admits an  $H^\infty(\Sigma_{\theta_1} \times \dots \times \Sigma_{\theta_d})$  joint functional calculus. Then for any  $F \in H_{0,1}^\infty(\Sigma_{\nu_1} \times \dots \times \Sigma_{\nu_d})$ , with  $\nu_k \in (\theta_k, \pi)$ ,  $(A_1, \dots, A_d)$  admits a square function estimate with respect to  $F$ .*

In analogy with the case of a single operator, we study a converse result for either  $R$ -Ritt operators or  $R$ -sectorial operators. If  $(T_1, \dots, T_d)$  is a  $d$ -tuple of commuting Ritt operators,  $(\alpha_1, \dots, \alpha_d) \in (0, \infty)$  and  $\Lambda$  is a subset of  $\{1, \dots, d\}$ , it is necessary to consider  $\|x\|_{T, \alpha_\Lambda}$ , the square function related to the subfamily  $(T_k)_{k \in \Lambda}$  and  $(\alpha_k)_{k \in \Lambda}$  ( $x \in X$ ).

**Theorem 2.2.17** *Let  $X$  be a reflexive Banach space such that  $X$  and  $X^*$  have finite cotype. Let  $T = (T_1, \dots, T_d)$  be commuting Ritt operators on a reflexive Banach space  $X$  such that every  $T_k$  is  $R$ -Ritt of  $R$ -type  $\delta_k \in (0, \frac{\pi}{2})$  for  $k = 1, \dots, d$ . Suppose that there exists a constant  $C > 0$  such that for any  $\Lambda$ , subset of  $\{1, \dots, d\}$ , there exist  $\alpha_\Lambda = (\alpha_k)_{k \in \Lambda}$  and  $\beta_\Lambda = (\beta_k)_{k \in \Lambda}$  in  $(\mathbb{R}_+^*)^\Lambda$  such that*

$$\|x\|_{T, \alpha_\Lambda} \leq C \|x\|, \quad x \in X$$

and

$$\|y\|_{T, \beta_\Lambda} \leq C \|y\|, \quad y \in X^*.$$

Then  $(T_1, \dots, T_d)$  admits a  $H^\infty(B_{\gamma_1} \times \dots \times B_{\gamma_d})$  joint functional calculus for every  $\gamma_k \in (\delta_k, \frac{\pi}{2})$ .

We use of the preceding theorem to obtain a characterisation of  $H^\infty$  joint functional calculus in spaces having property  $(\Delta)$ .

**Corollary 2.2.18** *Let  $X$  be a Banach space with property  $(\Delta)$ . Let  $T_1, \dots, T_d$  be commuting Ritt operators on  $X$ . The two assertions are equivalent :*

- i)  $(T_1, \dots, T_d)$  admits a  $H^\infty(B_{\gamma_1} \times \dots \times B_{\gamma_d})$  joint functional calculus for some  $\gamma_k \in (0, \frac{\pi}{2})$ ,  $k = 1, \dots, d$ .
- ii) Every  $T_k$  is  $R$ -Ritt and there exists a constant  $C > 0$  such that for every  $\Lambda$  subset of  $\{1, \dots, d\}$ , there exist  $\alpha_\Lambda$  and  $\beta_\Lambda$  in  $(\mathbb{R}_+^*)^\Lambda$  such that

$$\|x\|_{T, \alpha_\Lambda} \leq C \|x\|, \quad x \in X,$$

$$\|y\|_{T^*, \beta_\Lambda} \leq C \|y\|, \quad y \in X^*.$$

The analogous results for sectorial operators consist in the following two results.

**Theorem 2.2.19** *Assume that  $X$  is reflexive with finite cotype, and that for any  $k = 1, \dots, d$ ,  $A_k$  is  $R$ -sectorial of  $R$ -type  $\omega_k \in (0, \pi)$  and has dense range. Assume further that there exist non zero functions  $F_1, F_2 \in H_0^\infty(\Sigma_{\nu_1} \times \dots \times \Sigma_{\nu_d})$ , for some  $\nu_k \in (\omega_k, \pi)$ , such that  $(A_1, \dots, A_d)$  admits a square function estimate with respect to  $F_1$  and  $(A_1^*, \dots, A_d^*)$  admits a square function estimate with respect to  $F_2$ . Then for any  $\theta_k \in (\omega_k, \pi)$ ,  $k = 1, \dots, d$ ,  $(A_1, \dots, A_d)$  admits an  $H^\infty(\Sigma_{\theta_1} \times \dots \times \Sigma_{\theta_d})$  joint functional calculus.*

**Corollary 2.2.20** *Assume that  $X$  is a reflexive Banach space with property  $(\Delta)$  and that  $X^*$  has finite cotype. The following are equivalent.*

- (i) *For any  $\theta_k \in (\omega_k, \pi)$ ,  $k = 1, \dots, d$ ,  $(A_1, \dots, A_d)$  admits an  $H^\infty(\Sigma_{\theta_1} \times \dots \times \Sigma_{\theta_d})$  joint functional calculus.*
- (ii)  *$A_k$  is  $R$ -sectorial of  $R$ -type  $\omega_k$  and for any  $\theta_k \in (\omega_k, \pi)$ ,  $k = 1, \dots, d$ , and for any  $F \in H_{0,1}^\infty(\Sigma_{\theta_1} \times \dots \times \Sigma_{\theta_d})$ ,  $(A_1, \dots, A_d)$  and  $(A_1^*, \dots, A_d^*)$  both admit a square function estimate with respect to  $F$ .*

## Sixth part

The purpose of this part is to find connections between  $H^\infty$  joint functional calculus and dilation properties. Firstly, we exploit square functions for  $d$ -tuples to show that  $H^\infty$  joint functional calculus implies dilation results.

**Theorem 2.2.21** *Let  $X$  be a reflexive  $K$ -convex Banach space and  $p$  in  $(1, \infty)$ . Let  $T = (T_1, \dots, T_d)$  be a  $d$ -tuple of commuting Ritt operators on  $X$ . Suppose moreover that  $T$  admits an  $H^\infty(B_{\gamma_1} \times \dots \times B_{\gamma_d})$  joint functional calculus for some  $\gamma_1, \dots, \gamma_d$  in  $(0, \frac{\pi}{2})$ .*

*Then there exists a measure space  $\Sigma$ , a  $d$ -tuple of commuting isomorphisms  $(U_1, \dots, U_d)$  on  $L_p(\Sigma; X)$  and two bounded operators  $J : X \rightarrow L_p(\Sigma; X)$  and  $Q : L_p(\Sigma; X) \rightarrow X$  such that*

- a)  *$(U_1, \dots, U_d)$  admits a  $C(\mathbb{T}^d)$  bounded calculus.*
- b) *For any integers  $n_1, \dots, n_d \geq 0$ , we have*

$$T_1^{n_1} \dots T_d^{n_d} = QU_1^{n_1} \dots U_d^{n_d}J. \quad (2.2.2)$$

We also show that to some extent the dilation property stated in the previous theorem implies  $H^\infty$  joint functional calculus. The following converse result relies on Theorem 2.2.7.

**Theorem 2.2.22** *Let  $X$  be a Banach space and  $p \in (1, \infty)$ . Let  $(T_1, \dots, T_d)$  be a  $d$ -tuple of commuting operators acting on  $X$  such that every  $T_k$  is a  $R$ -Ritt operator,  $k = 1, \dots, d$ . Suppose that there exist a measure space  $\Sigma$ , a  $d$ -tuple of isomorphisms  $(U_1, \dots, U_d)$  acting on  $L_p(\Sigma; X)$  having a  $C(\mathbb{T}^d)$  bounded calculus and two bounded operator  $J : X \rightarrow L_p(\Sigma; X)$ ,  $Q : L_p(\Sigma; X) \rightarrow X$  such that (2.2.2) is verified.*

*Then there exist  $\gamma_1, \dots, \gamma_d$  in  $(0, \frac{\pi}{2})$  such that  $(T_1, \dots, T_d)$  admits an  $H^\infty(B_{\gamma_1} \times \dots \times B_{\gamma_d})$  joint functional calculus.*

Again, we obtain analogous results for semigroups.

**Theorem 2.2.23** *Assume that  $X$  is reflexive and  $K$ -convex. Let  $p$  in  $(1, \infty)$ . Let  $((T_t^1)_{t \geq 0}, \dots, (T_t^d)_{t \geq 0})$  be a  $d$ -tuple of commuting bounded analytic semigroups, and let  $A_1, \dots, A_d$  denote their negative generators. Assume that  $(A_1, \dots, A_d)$  admits an  $H^\infty(\Sigma_{\theta_1} \times \dots \times \Sigma_{\theta_d})$  joint functional calculus, for some  $\theta_k \in (0, \frac{\pi}{2})$ ,  $k = 1, \dots, d$ . Then there exist a measure space  $(\Sigma, dm)$ , two bounded operators  $J : X \rightarrow L_p(\Sigma; X)$  and  $Q : L_p(\Sigma; X) \rightarrow X$ , as well as a  $d$ -tuple  $((U_t^1)_{t \in \mathbb{R}}, \dots, (U_t^d)_{t \in \mathbb{R}})$  of commuting bounded  $C_0$ -groups on  $L_p(\Sigma; X)$  such that:*

- (a) The  $d$ -tuple  $(B_1, \dots, B_d)$ , where  $B_k$  is the negative generator of  $(U_t^k)_t$  for all  $k = 1, \dots, d$ , admits an  $H^\infty(\Sigma_{\frac{\pi}{2}} \times \dots \times \Sigma_{\frac{\pi}{2}})$  joint functional calculus;
- (b) For any  $t_1, \dots, t_d$  in  $\mathbb{R}_+$ , we have

$$T_{t_1}^1 \dots T_{t_d}^d = QU_{t_1}^1 \dots U_{t_d}^d J. \quad (2.2.3)$$

**Theorem 2.2.24** *Let  $p$  in  $(1, \infty)$ . Let  $((T_t^1)_{t \geq 0}, \dots, (T_t^d)_{t \geq 0})$  be a  $d$ -tuple of commuting bounded analytic semigroups on some Banach space  $X$ , and let  $A_1, \dots, A_d$  denote their negative generators. Assume that each  $A_k$  is  $R$ -sectorial of  $R$ -type  $\omega_k < \frac{\pi}{2}$ . Assume further that there exists a measure space  $(\Sigma, dm)$ , two bounded operators  $J: X \rightarrow L_p(\Sigma; X)$  and  $Q: L_p(\Sigma; X) \rightarrow X$ , as well as a  $d$ -tuple  $((U_t^1)_{t \in \mathbb{R}}, \dots, (U_t^d)_{t \in \mathbb{R}})$  of commuting bounded  $C_0$ -groups on  $L_p(\Sigma; X)$  such that (2.2.3) is verified. Suppose moreover that the  $d$ -tuple  $(B_1, \dots, B_d)$ , where  $B_k$  is the negative generator of  $(U_t^k)_t$  for all  $k = 1, \dots, d$ , admits an  $H^\infty(\Sigma_{\frac{\pi}{2}} \times \dots \times \Sigma_{\frac{\pi}{2}})$  joint functional calculus.*

*Then  $(A_1, \dots, A_d)$  admits an  $H^\infty(\Sigma_{\theta'_1} \times \dots \times \Sigma_{\theta'_d})$  joint functional calculus, for all  $\theta'_k \in (\omega_k, \frac{\pi}{2})$ ,  $k = 1, \dots, d$ .*

# Part I

## Tools



# Chapter 3

## Rademacher averages and Gaussian averages

### 3.1 Definitions and basic properties

In this subpart, we recall elementary background on Rademacher, Gaussian variables and their averages.

**Proposition 3.1.1** *Let  $\mathcal{I}$  be a nonempty set. Let  $\Omega = \{-1, 1\}^{\mathcal{I}}$  equipped with its normalised Haar measure  $\mu$ . Let  $j$  be an element of  $\mathcal{I}$ . Let  $r_j$  be the fonction from  $\Omega$  into  $\{-1, 1\}$  defined by*

$$r_j(\omega) = \omega_j, \quad \omega = (\omega_i)_{i \in \mathcal{I}} \in \Omega, \quad \omega_i = \pm 1.$$

*Then  $(r_j)_{j \in \mathcal{I}}$  is a sequence of independent random variables.*

The system  $(r_j)_{j \in \mathcal{I}}$  is an orthonormal system of  $L_2(\Omega)$ .

**Definition 3.1.2** *The family of functions  $(r_j)_{j \in \mathcal{I}}$  is called the **Rademacher functions** for  $\Omega$ .*

Frequently, we will use  $\mathcal{I} = \mathbb{N}^*$  in order to consider a sequence of Rademacher functions or even a countable set  $\mathcal{I}$  (as  $\mathcal{I} = \mathbb{Z}$ ).

Let now  $X$  be a Banach space,  $(r_j)_{j \in \mathcal{I}}$  a family of Rademacher averages and  $(x_j)_{j \in \mathcal{I}}$  a family of  $X$ . We will identify the element  $r_j \otimes x_j$  with the function from  $\Omega$  into  $X$  defined as

$$(r_j \otimes x_j)(\omega) = r_j(\omega)x_j, \quad \omega = (\omega_i)_{i \in \mathcal{I}}, \quad \omega_i = \pm 1.$$

It is clear that every function  $r_j \otimes x_j$  is a strongly measurable function from  $\Omega$  into  $X$  and belongs to every Bochner space  $L_p(\Omega; X)$ ,  $p \in [1, \infty]$ . We take the following definition for the linear span of these functions.

**Definition 3.1.3** *Let  $p \in [1, \infty)$ . The closed linear span of the  $r_j \otimes x_j$ 's,  $j \in \mathcal{I}$ , considered as a subspace of  $L_p(\Omega; X)$  with the corresponding norm, will be denoted by  $\text{Rad}_p(\mathcal{I}; X)$ , that is*

$$\text{Rad}_p(\mathcal{I}; X) = \overline{\left\{ \sum_{j \in J} r_j \otimes x_j : \emptyset \subsetneq J \subset \mathcal{I}, J \text{ finite}, (x_j)_{j \in \mathcal{I}} \subset X \right\}}^{\|\cdot\|_p}.$$



In general, we will call a *Rademacher average* of a family  $(x_j)_{j \in J}$  with  $J$  a non empty finite subset of  $\mathcal{I}$  the quantity

$$\left\| \sum_{j \in J} r_j \otimes x_j \right\|_{\text{Rad}_p(\mathcal{I}; X)} = \left( \int_{\Omega} \left\| \sum_{j \in J} r_j(\omega) \otimes x_j \right\|_X^p d\mu(\omega) \right)^{\frac{1}{p}}$$

for some  $p$  in  $[1, \infty)$ . We note that if  $X = H$  is an Hilbert space and  $p = 2$ , we have

$$\left\| \sum_{j \in J} r_j \otimes x_j \right\|_{\text{Rad}_2(\mathcal{I}; X)} = \left( \sum_{j \in J} \|x_j\|_H^2 \right)^{\frac{1}{2}}$$

as the  $r_j$ 's are an orthonormal system of  $L_2(\{-1, 1\}^{\mathcal{I}})$ .

If  $\mathcal{I} = \mathbb{N}^*$ , we will denote  $\text{Rad}_p(X) = \text{Rad}_p(\mathbb{N}^*; X)$ .

A very important fact is the Khintchine-Kahane's inequality (see [57, Theorem 1.e.13]), which says that the Rademacher averages are equivalent with constant which does not depend on the Banach space  $X$ .

**Theorem 3.1.4** *Let  $p, q$  in  $[1, \infty)$ . There exists a constant  $K_{p,q} > 0$  such that for every Banach space  $X$  and for any finite family  $(x_j)_{j \in J}$  we have*

$$(K_{p,q})^{-1} \left\| \sum_{j \in J} r_j \otimes x_j \right\|_{\text{Rad}_q(\mathcal{I}; X)} \leq \left\| \sum_{j \in J} r_j \otimes x_j \right\|_{\text{Rad}_p(\mathcal{I}; X)} \leq K_{p,q} \left\| \sum_{j \in J} r_j \otimes x_j \right\|_{\text{Rad}_q(\mathcal{I}; X)}. \quad (3.1.1)$$

A consequence of the Khintchine-Kahane's inequalities is that the Banach spaces  $\text{Rad}_p(\mathcal{I}; X)$  are canonically isomorphic for  $p \in [1, \infty)$ . We take the following notation.

**Definition 3.1.5** *We will denote by  $\text{Rad}(\mathcal{I}; X)$  the space  $\text{Rad}_2(\mathcal{I}; X)$  with its underlying norm  $\|\cdot\|_{\text{Rad}_2(\mathcal{I}; X)}$ . In particular, we set  $\text{Rad}(X) = \text{Rad}(\mathbb{N}^*; X)$ .*

We turn now to the case of Banach lattices. Let  $x_1, \dots, x_n$  be elements of a Banach lattice  $X$ . We refer the reader to [57][1.d] for the definition of square function  $\left\| \left( \sum_{k=1}^n |x_k|^2 \right)^{\frac{1}{2}} \right\|$ . It turns out that we have for any family  $x_1, \dots, x_n$  of such a Banach lattice the inequality

$$\left\| \left( \sum_{k=1}^n |x_k|^2 \right)^{\frac{1}{2}} \right\| \lesssim \left\| \sum_{k=1}^n r_k \otimes x_k \right\|_{\text{Rad}(X)}. \quad (3.1.2)$$

We now pass to the Gaussian averages. Let  $\mathcal{I}$  be a nonempty set. We consider a family  $(g_j)_{j \in \mathcal{I}}$  of independant standard normal complex-valued Gaussian variables on some probability space  $(\Sigma, \lambda)$ . Every function  $g_j$  belongs to  $L_p(\Sigma)$  for every  $p \in [1, \infty)$ , but not for  $p = \infty$ .

If  $X$  is a Banach space and  $(x_j)_{j \in \mathcal{I}}$  is a family of  $X$ , we also identify the element  $g_j \otimes x_j$  with the function  $\sigma \mapsto g_j(\sigma)x_j$  from  $\Sigma$  into  $X$ .

Then we take the following definition, similar to the one for the spaces  $\text{Rad}(\mathcal{I}; X)$ .

**Definition 3.1.6** Let  $p$  in  $[1, \infty)$ . The closed linear span of the  $g_j \otimes x_j$ 's,  $j \in \mathcal{I}$ , considered as a subspace of  $L_p(\Sigma; X)$  with the corresponding norm, will be denoted by  $\text{Gauss}_p(\mathcal{I}; X)$ , that is

$$\text{Gauss}_p(\mathcal{I}; X) = \overline{\left\{ \sum_{j \in J} g_j \otimes x_j : \emptyset \subsetneq J \subset \mathcal{I} \text{ finite}, (x_j) \subset X \right\}}^{\|\cdot\|_p}.$$

Then we call *gaussian average* the quantity

$$\left\| \sum_{j \in J} g_j \otimes x_j \right\|_{\text{Gauss}_p(\mathcal{I}; X)} = \left( \int_{\Sigma} \left\| \sum_{j \in J} g_j(\sigma) \otimes x_j \right\|_X^p d\lambda(\sigma) \right)^{\frac{1}{p}}$$

for some  $p$  in  $[1, \infty)$ . If  $X = H$  is an Hilbert space, we have

$$\left\| \sum_{j \in J} g_j \otimes x_j \right\|_2 = \left( \sum_{j \in J} \|x_j\|_H^2 \right)^{\frac{1}{2}}$$

as  $(g_j)$  is an orthonormal system of  $L_2(\Sigma)$ .

In the case where  $\mathcal{I} = \mathbb{N}^*$ , we will set  $\text{Gauss}_p(\mathcal{I}; X) = \text{Gauss}_p(X)$ .

There exists a similar result to the Khintchine Kahane inequalities stated as follows.

**Theorem 3.1.7** Let  $p, q$  in  $[1, \infty)$ . There exists a constant  $K_{p,q} > 0$  such that for every Banach space  $X$  and for any finite family  $(x_j)_{j \in J}$  we have

$$(K_{p,q})^{-1} \left\| \sum_{j \in J} g_j \otimes x_j \right\|_{\text{Gauss}_q(\mathcal{I}; X)} \leq \left\| \sum_{j \in J} g_j \otimes x_j \right\|_{\text{Gauss}_p(\mathcal{I}; X)} \leq K_{p,q} \left\| \sum_{j \in J} g_j \otimes x_j \right\|_{\text{Gauss}_q(\mathcal{I}; X)}. \quad (3.1.3)$$

Thus, all the spaces  $\text{Gauss}_p(\mathcal{I}; X)$  are also canonically isomorphic for  $p \in [1, \infty)$ . We will denote  $\text{Gauss}(\mathcal{I}; X) = \text{Gauss}_2(\mathcal{I}; X) \subset L_2(\Sigma; X)$  and  $\text{Gauss}(X) = \text{Gauss}_2(\mathbb{N}^*; X)$ .

A very important property which benefits the  $g_j$ 's is the following.

**Proposition 3.1.8** Let  $(g_j)_{j \geq 1}$  be a sequence of independant standard normal complex valued Gaussian variables,  $M = [m_{i,j}]_{1 \leq i, j \leq n}$  be a matrix, with complex coefficients,  $n \in \mathbb{N}^*$ . Then for any  $x_1, \dots, x_n$  in  $X$

$$\left\| \sum_{i,j=1}^n m_{i,j} g_j \otimes x_i \right\|_{\text{Gauss}(\mathcal{I}; X)} \leq \|M\| \left\| \sum_{j=1}^n g_j \otimes x_j \right\|_{\text{Gauss}(\mathcal{I}; X)},$$

where  $\|M\|$  is the norm operator of  $M$ , considered as an element of  $B(l_n^2)$ .

To conclude this section, we give a comparison between the Rademacher averages and Gaussian averages. Indeed, there exists a constant  $C_p > 0$  (which does not depend on  $X$ ) such that for any finite subset  $J$  of  $\mathcal{I}$  and  $(x_j)_{j \in J}$  a family of  $X$ , we have

$$\left\| \sum_{j \in J} r_j \otimes x_j \right\|_{\text{Rad}_p(\mathcal{I}; X)} \leq C \left\| \sum_{j \in J} g_j \otimes x_j \right\|_{\text{Gauss}_p(\mathcal{I}; X)} \quad (3.1.4)$$

(see [35, Corollary 6.1.17]).

## 3.2 Type and cotype

Most of the results we present here may be found in [35, Chapter 7] or [19, Chapter 11].

Unless stated otherwise,  $(r_k)_{k \geq 1}$  will always denote a sequence of Rademacher independent variables defined on some probability space  $(\Omega, \mathbb{P})$ .

**Definition 3.2.1** *Let  $X$  be a Banach space, let  $p$  in  $[1, 2]$  and  $q$  in  $[2, \infty]$ .*

- (1) *The space  $X$  is said to have type  $p$  if there exists a constant  $c > 0$  such that for any finite family  $(x_k)_{1 \leq k \leq n}$ ,  $n \geq 1$ , we have*

$$\left\| \sum_{k=1}^n r_k \otimes x_k \right\|_{\text{Rad}(X)} \leq c \left( \sum_{k=1}^n \|x_k\|^p \right)^{\frac{1}{p}}.$$

- (2) *The space  $X$  is said to have cotype  $q$  if there exists a constant  $c > 0$  such that for any finite family  $(x_k)_{1 \leq k \leq n}$ ,  $n \geq 1$ , we have*

$$\left( \sum_{k=1}^n \|x_k\|^q \right)^{\frac{1}{q}} \leq c \left\| \sum_{k=1}^n r_k \otimes x_k \right\|_{\text{Rad}(X)}.$$

It is clear that type  $p$  and cotype  $q$  properties pass to closed subspaces.

By the Kahane-Khintchine inequalities, the exponents  $p$  and  $q$  in (1) and (2) (with the exception of  $q = \infty$ ) could be replaced by any exponent  $r \in [1, \infty)$ , which may change constant  $c > 0$ .

Taking  $X = \mathbb{C}$ , one can see that inequalities in (1) and (2) cannot be satisfied for  $p > 2$  and  $q < 2$ . Indeed, suppose that inequality in (1) is satisfied for some  $p > 2$ . Then Kahane-Khintchine inequalities implies that there exists a constant  $K > 0$  such that for any complex numbers  $(a_k)_{1 \leq k \leq n}$ ,  $n \geq 1$  we have

$$\left( \sum_{k=1}^n |a_k|^2 \right)^{\frac{1}{2}} \leq K \left( \sum_{k=1}^n |a_k|^p \right)^{\frac{1}{p}}.$$

If we take now  $a_k = 1$  for any  $k = 1, \dots, n$ , we obtain  $n^{\frac{1}{2}} \leq K n^{\frac{1}{p}}$ , which is impossible when  $n$  tends to  $\infty$ . It is the same for inequality (2).

It is easy to prove that any Banach space has type 1 and cotype  $\infty$ . We will talk about *non trivial type  $p$*  if (1) is satisfied for some  $p \in (1, 2]$  and *finite cotype  $q$*  if (2) is satisfied for some  $q \in [2, \infty)$ .

If we fix a family  $x_1, \dots, x_n$ ,  $n \geq 1$  in  $X$ , the function

$$t \mapsto \left( \sum_{k=1}^n \|x_k\|^t \right)^{\frac{1}{t}}$$

decreases on  $[1, \infty)$ . Thus, if  $X$  has non trivial type  $p \in (1, 2]$ , then it has type  $p'$  for any  $p' \in (1, p]$ . Likewise, if  $X$  has finite cotype  $q$ , then it has cotype  $q'$  for any  $q' \in [q, \infty)$ .

We give some examples of spaces with non-trivial type, finite cotype or not. We refer the reader to [35, Prop 7.1.4, Cor 7.1.5, 7.6.6 and 7.1.10] for the details.

**Examples 3.2.2** (1) *Any Hilbert space has type and cotype 2.*

(2) *If  $\Sigma$  is a measure space,  $r \in [1, \infty)$  and  $X$  has type  $p \in [1, 2]$  and cotype  $q \in [2, \infty]$ , the Bochner space  $L_r(\Sigma; X)$  has type  $\min(r, p)$  and cotype  $\max(r, q)$ .*

(3) *In particular,  $L_r(\Sigma) = L_r(\Sigma; \mathbb{C})$  has type  $\min(r, 2)$  and cotype  $\max(r, 2)$ , for any  $r$  in  $[1, \infty)$ .*

(4) *The spaces  $c_0$  and  $l^\infty$  do not have non-trivial type or finite cotype.*

Important results about Rademacher and Gaussian averages related to finite cotype are the following (see [35, 7.2.d]). We recall (3.1.4).

**Theorem 3.2.3** *Let  $X$  be a Banach space. The following two assertions are equivalent.*

(1)  *$X$  has finite cotype.*

(2) *The spaces  $\text{Rad}(X)$  and  $\text{Gauss}(X)$  are isomorphic. More precisely, there exists a constant  $C > 0$  such that for any finite family  $(x_k)_{1 \leq k \leq n}$  of  $X$  we have*

$$\left\| \sum_{k=1}^n g_k \otimes x_k \right\|_{\text{Gauss}(X)} \leq C \left\| \sum_{k=1}^n r_k \otimes x_k \right\|_{\text{Rad}(X)} .$$

A very useful result of Kwapien on convergence of series in  $\text{Rad}(X)$  apply to  $X$  with finite cotype.

**Theorem 3.2.4 (Kwapien, 1975)** *Let  $X$  be a Banach space which does not contain  $c_0$ . Let  $(r_k)_{k \geq 1}$  be a sequence of independent Rademacher functions and  $(x_k)_{k \geq 1}$  a sequence of  $X$ . The following two assertions are equivalent.*

(1) *The series  $\sum_{k \geq 1} r_k \otimes x_k$  converges in  $\text{Rad}(X)$ .*

(2) *The partial sums  $(\sum_{k=1}^n r_k \otimes x_k)_{n \geq 1}$  are uniformly bounded in  $\text{Rad}(X)$ .*

We recall results making links between type and cotype. According to [35, Prop 7.1.13], we have the duality result below.

**Proposition 3.2.5** *If  $X$  has non trivial type  $p \in (1, 2]$ , then  $X^*$  has finite cotype  $q$  with  $\frac{1}{p} + \frac{1}{q} = 1$ .*

A more difficult result of König and Tzafriri concerns a space itself (see [35, Theorem 7.1.14]).

**Theorem 3.2.6 (König-Tzafriri)** *If  $X$  has non-trivial type, then it has finite cotype.*

The converses of these two results above are wrong. Indeed,  $l^1$  has cotype 2 but neither its dual  $l^\infty$  nor its predual  $c_0$  has non-trivial type and  $l^1$  has non finite cotype.

Finally, we give a special result for Banach lattices with finite cotype  $X$ . Let  $x_1, \dots, x_n$  be a family of  $X$  ( $n \in \mathbb{N}^*$ ). We recall (3.1.2). If  $X$  has finite cotype, we have a converse inequality, that is,

$$\left\| \sum_{k=1}^n r_k \otimes x_k \right\|_{\text{Rad}(X)} \lesssim \left\| \left( \sum_{k=1}^n |x_k|^2 \right)^{\frac{1}{2}} \right\|_X.$$

In other words, we have for Banach lattice of finite cotype an equivalence

$$\left\| \sum_{k=1}^n r_k \otimes x_k \right\|_{\text{Rad}(X)} \simeq \left\| \left( \sum_{k=1}^n |x_k|^2 \right)^{\frac{1}{2}} \right\|_X. \quad (3.2.1)$$

As Rademacher and Gaussian averages are equivalent on spaces with finite cotype, these inequalities hold as well for Gaussian averages in the finite cotype case.

### 3.3 $R$ -boundedness and $\gamma$ -boundedness

In this section, we fix  $X$  a Banach space and  $E$  a subset of  $B(X)$ . In addition we take  $\mathcal{I}$  a countable non empty set,  $(r_j)_{j \in \mathcal{I}}$  and  $(g_j)_{j \in \mathcal{I}}$  some families of independent Rademacher variables and Gaussian variables respectively, being defined on some probability space  $(\Omega, \mathbb{P})$  for one or the other of the families.

**Definition 3.3.1 ( $R$ -boundedness,  $\gamma$ -boundedness)**

- (1) *We say that the subset  $E$  of  $B(X)$  is  $R$ -bounded if there exists a constant  $C > 0$  such that for any finite family  $(T_j)_{j \in J}$  of  $E$  and any finite family  $(x_j)_{j \in J}$  of  $X$  ( $J \neq \emptyset$  finite) we have*

$$\left\| \sum_{j \in J} r_j \otimes T_j(x_j) \right\|_{\text{Rad}(\mathcal{I}; X)} \leq C \left\| \sum_{j \in J} r_j \otimes x_j \right\|_{\text{Rad}(\mathcal{I}; X)}. \quad (3.3.1)$$

- (2) *We say that the subset  $E$  of  $B(X)$  is  $\gamma$ -bounded if there exists a constant  $C > 0$  such that for any finite family  $(T_j)_{j \in J}$  of  $E$  and any finite family  $(x_j)_{j \in J}$  of  $X$  ( $J \neq \emptyset$  finite) we have*

$$\left\| \sum_{j \in J} g_j \otimes T_j(x_j) \right\|_{\text{Gauss}(\mathcal{I}; X)} \leq C \left\| \sum_{j \in J} g_j \otimes x_j \right\|_{\text{Gauss}(\mathcal{I}; X)}. \quad (3.3.2)$$

*In each of these cases, we let  $\mathcal{R}(E)$  (respectively  $\gamma(E)$ ) be the infimum of all the  $C$  verifying (3.3.1) (resp. (3.3.2)).*

Every  $R$ -bounded (resp.  $\gamma$ -bounded) subset of  $B(X)$  is a bounded subset of  $B(X)$  (for the operator norm), taking a subfamily with only one operator in (3.3.1) (resp. (3.3.2)) and we have  $\|T\| \leq \mathcal{R}(E)$  (resp.  $\|T\| \leq \gamma(E)$ ) for any  $T$  in  $E$ . The converse is true for  $X = H$  an Hilbert space. Indeed, using that the  $r_j$ 's are orthonormal in  $L_2(\Omega; X)$  we obtain

$$\left\| \sum_{j \in J} r_j \otimes T_j(x_j) \right\|_{\text{Rad}(\mathcal{T}; H)}^2 = \sum_{j \in J} \|T_j(x_j)\|_H^2 \leq K^2 \sum_{j \in J} \|x_j\|_H^2 = K^2 \left\| \sum_{j \in J} r_j \otimes x_j \right\|_{\text{Rad}(\mathcal{T}; X)}^2$$

if we supposed that  $\|T\| \leq K$  for any  $T$  in  $E$ .

One can prove that a Banach space  $X$  such that any bounded subset of  $B(X)$  is a  $R$ -bounded subset of  $B(X)$  is isomorphic to an Hilbert space  $H$  (because it has type and cotype 2, see Kwapien's characterisation e.g [35, Theorem 7.3.1]).

A simple useful fact on product of  $R$ -bounded and  $\gamma$ -bounded family is the following technical lemma (see e.g [35, Proposition 8.1.19]).

**Lemma 3.3.2** *Let  $X$  be a Banach space and  $E_1, \dots, E_d$  be subset of  $B(X)$  such that every  $E_k$  is  $R$ -bounded (resp.  $\gamma$ -bounded),  $k = 1, \dots, d$ . Let*

$$E_1 \cdots E_d = \{S_1 \cdots S_d : S_k \in E_k, k = 1, \dots, d\}. \quad (3.3.3)$$

*Then  $E_1 \cdots E_d$  is  $R$ -bounded with  $\mathcal{R}(E_1 \cdots E_d) \leq \mathcal{R}(E_1) \cdots \mathcal{R}(E_d)$  (resp.  $\gamma(E_1 \cdots E_d) \leq \gamma(E_1) \cdots \gamma(E_d)$ ).*

We will also use the following result, which is taken from [35, 8.5.2].

**Lemma 3.3.3** *Let  $X$  be a Banach space and  $(\Sigma, \mu)$  a measure space. Let  $E$  be a  $R$ -bounded (resp.  $\gamma$ -bounded) subset of  $B(X)$  and  $K > 0$ . Let*

$$E_K = \left\{ \int_{\Sigma} h(\omega) F(\omega) d\mu(\omega) / F : \Sigma \rightarrow E, h \in L^1(\Sigma; \mu), \int_{\Sigma} |h(\omega)| d\mu(\omega) \leq K \right\}.$$

*Then  $E_K$  is  $R$ -bounded (resp.  $\gamma$ -bounded) with  $\mathcal{R}(E_K) \leq 2K\mathcal{R}(E)$  (resp.  $\gamma(E_K) \leq K\gamma(E)$ ).*

We recall that any  $\gamma$ -bounded family of  $B(X)$  is  $R$ -bounded by (3.1.4). On spaces having finite cotype, every  $R$ -bounded family of  $B(X)$  is  $\gamma$ -bounded. Thus, the two notions are equivalent on spaces  $X$  having finite cotype.

## 3.4 The $\gamma$ -spaces

We recall in this section the necessary background on  $\gamma$ -spaces in order to define square functions related to sectorial operators in many cases. For details and complements on this subject, see [35, Chapter 9].

We let  $H$  be an Hilbert space and  $X$  be a Banach space. We denote by  $\mathcal{F}(H; X)$  the space of all finite rank operators from  $H$  into  $X$ . An element of  $\mathcal{F}(H; X)$  can be written as

$$T = \sum_{k=1}^n h_k^* \otimes x_k$$

with  $h_1^*, \dots, h_n^*$  in  $H^*$  and  $x_1, \dots, x_n$ ,  $n \in \mathbb{N}$ . More precisely, the notation  $\sum_{k=1}^n h_k^* \otimes x_k$  names the operator

$$T : h \mapsto \sum_{k=1}^n \langle h, h_k^* \rangle x_k.$$

This expression allows us to identify  $\mathcal{F}(H; X)$  to  $H^* \otimes X$ .

We may suppose that  $h_1^*, \dots, h_n^*$  is an orthonormal system of  $H^*$  (in the sense of the natural structure of Hilbert space of  $H^*$ ) and we do as well for the sequel.

Let now  $(g_n)_{n \geq 1}$  be a sequence of independent complex-valued standard normal variables on some probability space  $S$ . Let  $p$  in  $[1, \infty)$ . For any operator  $T = \sum_{k=1}^n h_k^* \otimes x_k$  (with  $h_1^*, \dots, h_n^*$  is an orthonormal system), we let

$$\left\| \sum_{k=1}^n h_k^* \otimes x_k \right\|_{\gamma_\infty^p(H; X)} = \left\| \sum_{k=1}^n g_k \otimes x_k \right\|_{\text{Gauss}_p(X)} \quad (3.4.1)$$

which defines a norm on  $\mathcal{F}(H; X)$ . Note that the right hand side of the formula (3.4.1) does not depend of the choice of the expression  $\sum_{k=1}^n h_k^* \otimes x_k$  chosen for the operator  $T$ , as long as  $h_1^*, \dots, h_n^*$  is an orthonormal system. This is a consequence of Proposition 3.1.8.

Now we consider a larger class of operators form  $H$  into  $X$  containing  $\mathcal{F}(H; X)$  equipped with its new norm.

**Definition 3.4.1** *Let  $p$  in  $[1, \infty)$ . We let  $\gamma_\infty^p(H; X)$  be the Banach space of all the bounded operators  $T : H \rightarrow X$  such that*

$$\sup \left\| \sum_{k=1}^n g_k \otimes T(h_k) \right\|_{L_p(S; X)} < \infty,$$

where the supremum is taken over all finite orthonormal systems  $\{h_1, \dots, h_n\}$  of  $H$ ,  $n \in \mathbb{N}^*$ , equipped with the norm

$$\|T\|_{\gamma_\infty^p(H; X)} = \sup \left\| \sum_{k=1}^n g_k \otimes T(h_k) \right\|_{\text{Gauss}_p(X)}. \quad (3.4.2)$$

We denote by  $\gamma_\infty(H; X) = \gamma_\infty^2(H; X)$  the corresponding space for  $p = 2$ .

By the Khintchine-Kahane inaequalities (Theorem 3.1.7), all the norms  $\|\cdot\|_{\gamma_\infty^p(H; X)}$  are equivalent for  $p$  in  $[1, \infty)$  and then all the spaces  $\gamma_\infty^p(H; X)$  are naturally isomorphic.

It turns out that  $\mathcal{F}(H; X)$  is included in  $\gamma_\infty^p(H; X)$  for any  $p$  in  $[1, \infty)$  and the norms defined by (3.4.1) and (3.4.2) coincide on the space  $\mathcal{F}(H; X)$ .

**Definition 3.4.2** Let  $p$  in  $[1, \infty)$ . We let  $\gamma^p(H; X)$  be the closure of  $\mathcal{F}(H; X)$  in the  $\gamma_\infty^p(H; X)$  for the norm  $\|\cdot\|_{\gamma_\infty^p(H; X)}$ .

We note by [35, Proposition 9.1.6] that  $\gamma^p(H; X)$  is included in  $K(H; X)$ , the space of compact operators from  $H$  into  $X$ .

An important fact related to the geometry of Banach spaces is the following (see [35, Theorem 9.1.20]).

**Theorem 3.4.3** Let  $X$  be a Banach space which does not contain a closed subspace isomorphic to  $c_0$ . Then  $\gamma^p(H; X) = \gamma_\infty^p(H; X)$  for any  $p$  in  $[1, \infty)$ . In particular, this applies to Banach spaces with finite cotype.

We now give a fundamental tensor extension property, which is a direct consequence of Proposition 3.1.8.

**Lemma 3.4.4** For any  $S \in B(H^*)$ , the mapping  $S \otimes I_X: H^* \otimes X \rightarrow H^* \otimes X$  (uniquely) extends to a bounded map

$$S \overline{\otimes} I_X: \gamma(H; X) \longrightarrow \gamma(H; X),$$

with  $\|S \overline{\otimes} I_X\| = \|S\|$ .

Now we give some particular cases for special Banach spaces  $X$ .

In the case where  $X = L$  is an Hilbert space, the class of operator  $\gamma^2(H; L)$  coincides with the class  $\mathcal{C}^2(H; L)$  of Hilbert-Schmidt operators. This means that the space  $\gamma^2(H; L)$  is the space of operators  $T: H \rightarrow L$  for which

$$\sup \sum_{k=1}^n \|T(h_k)\|^2 < \infty$$

where the supremum runs over all the orthonormal systems  $\{h_1, \dots, h_n\}$  in  $H$ ,  $n \in \mathbb{N}^*$ . Letting  $\|T\|_{\mathcal{C}^2(H; L)} = \sup \sum_{k=1}^n \|T(h_k)\|^2$ , it turns out that

$$\|T\|_{\gamma^2(H; L)} = \|T\|_{\mathcal{C}^2(H; L)}.$$

For the case of  $X = L_p(S')$  with  $p$  in  $[1, \infty)$ , the space  $\gamma(H; L_p(S'))$  is characterised in terms of square functions (see [35, Proposition 9.3.1]).

**Proposition 3.4.5** Let  $p$  in  $[1, \infty)$ . For any operator bounded operator  $T: H \rightarrow L_p(S')$ , the three following assertions are equivalent.

- (1)  $T \in \gamma^p(H; L_p(S'))$ ;
- (2)  $T \in \gamma_\infty^p(H; L_p(S'))$ ;
- (3) We have

$$\sup \left\| \left( \sum_{k=1}^N |T(h_k)|^2 \right)^{\frac{1}{2}} \right\|_{L_p(S')} < \infty,$$

where the supremum is over all finite orthonormal systems  $\{h_1, \dots, h_N\}$  in  $H$ .



In this case, we have

$$\|T\|_{\gamma^p(H;L_p(S'))} = \sup \left\| \left( \sum_{k=1}^N |T(h_k)|^2 \right)^{\frac{1}{2}} \right\|_{L_p(S')}.$$

We end this section giving the special case where  $H = L_2(\Omega)$  where  $(\Omega, dm)$  is a measure space. We identify  $L^2(\Omega)^*$  with  $L^2(\Omega)$  through the standard duality pairing

$$\langle h_1, h_2 \rangle = \int_{\Omega} h_1(t)h_2(t) dm(t), \quad h_1, h_2 \in L^2(\Omega, dm).$$

Thus we have inclusions  $L^2(\Omega) \otimes X \subset \gamma(L^2(\Omega); X) \subset B(L^2(\Omega), X)$ .

Let  $\zeta: \Omega \rightarrow X$  be a measurable function. We say that  $\zeta$  is weakly- $L^2$  provided that the function  $y \circ \zeta$  belongs to  $L^2(\Omega)$  for any  $y \in X^*$ . In this case, one can define a bounded operator  $u_{\zeta}: L^2(\Omega) \rightarrow X$  by

$$\langle y, u_{\zeta}(h) \rangle = \int_{\Omega} \langle y, \zeta(t) \rangle h(t) dm(t), \quad y \in X^*.$$

The above formula a priori defines an operator  $u_{\zeta}$  from  $L^2(\Omega)$  into  $X^{**}$ ; however it turns out that its range is included in  $X$ . We refer to [41, Section 4] and [35, Section 9.2] for details.

We let  $\gamma(\Omega; X)$  denote the space of all weakly- $L^2$  measurable functions  $\zeta: \Omega \rightarrow X$  such that  $u_{\zeta}$  belongs to  $\gamma(L^2(\Omega); X)$ . For such a function  $\zeta$ , we set

$$\|\zeta\|_{\gamma(\Omega; X)} := \|u_{\zeta}\|_{\gamma(L^2(\Omega); X)}.$$

It is plain that through the identification  $\zeta \longleftrightarrow u_{\zeta}$ ,  $\gamma(\Omega; X)$  contains  $L^2(\Omega) \otimes X$ . Thus  $\gamma(\Omega; X)$  is a dense subspace of  $\gamma(L^2(\Omega); X)$ .

**Lemma 3.4.6** *Assume that  $X$  does not contain  $c_0$  and let  $\zeta: \Omega \rightarrow X$  be a weakly- $L^2$  measurable function such that  $\zeta \cdot h \in L^1(\Omega; X)$  for any  $h \in L^2(\Omega)$ . Then  $u_{\zeta}$  belongs to  $\gamma(\Omega; X)$  if and only if there exists a constant  $C \geq 0$  such that*

$$\left\| \sum_{j \geq 1} g_j \otimes \left( \int_{\Omega} \zeta(t) e_j(t) dm(t) \right) \right\|_{\text{Gauss}_2(X)} \leq C$$

for any finite orthonormal sequence  $(e_j)_{j \geq 1}$  of  $L^2(\Omega)$ . In this case,  $\|\zeta\|_{\gamma(\Omega; X)}$  is the smallest constant  $C$  satisfying this property.

In particular, this characterisation applies to spaces  $X$  having finite cotype.

*Proof of Lemma 3.4.6.* It is clear that for any  $j \geq 1$ ,  $\int_{\Omega} \zeta(t) e_j(t) dm(t) = u_{\zeta}(e_j)$ . Hence the result follows from Definition 3.4.1 and Theorem 3.4.3.  $\square$

# Chapter 4

## Geometry of Banach spaces

### 4.1 $K$ -convexity

$K$ -convexity is a tool to understand the link between type and cotype under duality. For history and details about this, we refer the reader to [65] and [59]. All the proofs of the results of this section can be found in [35, Section 7.4] and [34, Section 4.3.b].

Let  $(r_n)_{n \geq 1}$  be a sequence of independent Rademacher variables on any probability space  $\Omega_0$ . Denote by  $R$  the orthogonal projection from  $L_2(\Omega_0)$  onto  $\text{Rad}(\mathbb{N}^*; \mathbb{C})$ , the closed subspace spanned by all the  $r_n$ .

A Banach space  $X$  is called  $K$ -convex if the operator  $R \otimes I_X$  defined a priori on the space  $L_2(\Omega_0) \otimes X$  extends to a bounded operator on  $L_2(\Omega_0; X)$  (see [70] and [59, Section 6]). Using Khintchine Kahane's inequalities, one can see that if  $X$  is a  $K$ -convex space, the space  $\text{Rad}_p(X)$  is complemented in  $L_p(\Omega_0; X)$  for any  $1 < p < \infty$ , that is  $R \otimes I_X$  extends to a bounded operator on  $L_p(\Omega_0; X)$ . We call "canonical projection" from  $L_p(\Omega_0; X)$  onto  $\text{Rad}_p(X)$  this extension.

There exists an equivalent manner to define  $K$ -convexity, which is related to the Gaussian averages.

Let  $(g_j)_{j \geq 1}$  be a independent family of complex valued standard Gaussian variables on some probability space  $(\mathcal{S}, \mathbb{P})$ . Let  $G \subset L^2(\mathcal{S})$  be the closed linear span of the  $g_j$  and let  $q: L^2(\mathcal{S}) \rightarrow L^2(\mathcal{S})$  be the orthogonal projection with range equal to  $G$ .  $X$  is  $K$ -convex if and only if  $q \otimes I_X: L^2(\mathcal{S}) \otimes X \rightarrow L^2(\mathcal{S}) \otimes X$  extends to a bounded map  $q_X$  from  $L^2(\mathcal{S}; X)$  into itself. In this case, the range of  $q_X$  is equal to  $G(X)$ , the closure of all finite sums  $\sum_j g_j \otimes x_j$ , with  $x_j \in X$ . We refer the reader to [59] for various characterizations of  $K$ -convex spaces and more information. For readers not familiar with this notion we mention that for any  $1 < p < \infty$ , either classical or noncommutative  $L^p$ -spaces are  $K$ -convex. Also we note that if  $X$  is  $K$ -convex, then its dual space  $X^*$  is  $K$ -convex as well. Further any  $K$ -convex Banach space has finite cotype.

It follows from the definition of  $\gamma$ -spaces that if  $H$  is any separable Hilbert space, then  $\gamma(H^*; X)$  is isometrically isomorphic to  $G(X)$ . In this section we will work with  $H = L^2(\mathbb{R}^d)$ , that we identify with its dual space in the usual way. It follows from above that if  $X$  is  $K$ -convex, we have a direct sum decomposition

$$L^2(\mathcal{S}; X) = \gamma(L^2(\mathbb{R}^d); X) \oplus Z_X, \tag{4.1.1}$$

for some subspace  $Z_X$  of  $L^2(\mathcal{S}; X)$ .

**Examples 4.1.1** (1) *Every Hilbert space is  $K$ -convex.*

(2) *Let  $p$  in  $(1, \infty)$  and  $\mathcal{S}$  be a measure space. If  $X$  is a  $K$ -convex space, then every Bochner space  $L_p(\mathcal{S}; X)$  is  $K$ -convex. In particular, every space  $L_p(\mathcal{S}) = L_p(\mathcal{S}; \mathbb{C})$  is a  $K$ -convex space.*

(3) *The spaces  $l^1$ ,  $c_0$  and  $l^\infty$  are not  $K$ -convex.*

The notion of  $K$ -convexity says that the tensor operator  $\Pi \otimes I_X$  from  $L_p(\Omega) \otimes X$  into  $\text{Rad}_p(\mathbb{N}^*) \otimes X$  extends to a unique operator from  $L_p(\Omega; X)$  into  $\text{Rad}_p(\mathbb{N}^*; X)$  where we denote by  $\Pi$  the Rademacher projection from  $L_p(\Omega)$  onto  $\text{Rad}_p(\mathbb{N}^*)$ .

According to [35, Proposition 7.4.5], we see that the  $K$ -convexity passes to the dual space.

**Proposition 4.1.2** *A Banach space  $X$  is  $K$ -convex if and only if its dual  $X^*$  is  $K$ -convex.*

A very important result about  $K$ -convexity is Pisier's theorem.

**Theorem 4.1.3 (Pisier)** *Let  $X$  be a Banach space. The following three assertions are equivalent.*

(1)  *$X$  is  $K$ -convex.*

(2)  *$X$  has non trivial type  $p > 1$ .*

Using duality result (Proposition 4.1.2), we deduce the following.

**Corollary 4.1.4** *A Banach space  $X$  has non-trivial type if and only if its dual  $X^*$  has non-trivial type.*

We recall that if  $X$  is a  $K$ -convex space, there is a natural isomorphism between  $\text{Rad}(X)^*$  and  $\text{Rad}(X^*)$ , that is the inclusion  $\mathcal{J}$  from  $\text{Rad}(X^*)$  into  $\text{Rad}(X)^*$  which is defined by

$$\mathcal{J} \left( \sum_{k=1}^n r_k \otimes x_k^* \right) \left[ \sum_{k=1}^n r_k \otimes x_k \right] = \sum_{k=1}^n \langle x_k, x_k^* \rangle,$$

for any families  $x_1, \dots, x_n$  of  $X$  and  $x_1^*, \dots, x_n^*$  of  $X^*$ , is a linear mapping from  $\text{Rad}(X^*)$  onto  $\text{Rad}(X)^*$ ,  $n \geq 1$ .

## 4.2 The UMD property

Most of the results we present here may be found in [34], [66].

We recall the basic definitions about the martingales. Let  $(S, \mathcal{A}, \mu)$  be a  $\sigma$ -finite measure space. We call *filtration* an increasing sequence  $(\mathcal{F}_n)_{n \geq 0}$  of sub- $\sigma$ -algebras of  $\mathcal{A}$ . It is called a  $\sigma$ -finite filtration if  $\mu$  is  $\sigma$ -finite on each  $\mathcal{F}_n$ .

A family of function  $(f_n)_{n \geq 0}$  from  $S$  into  $X$  is called *adapted* to the filtration  $(\mathcal{F}_n)_{n \geq 0}$  if for any interger  $n$ ,  $f_n$  is a measurable function from  $(S, \mathcal{F}_n, \mu)$  into  $X$ . This sequence of functions is called a *martingale* with respect to the  $\sigma$ -finite filtration if for any integers  $m \leq n$ , the function  $f_n$  is  $\sigma$ -integrable over  $\mathcal{F}_m$  and satisfies the conditional expectation condition

$$\mathbb{E}(f_n | \mathcal{F}_m) = f_m.$$

For any sequence  $(f_n)_{n \geq 0}$  of functions, we denote by  $df_n$  the functions

$$df_n = f_n - f_{n-1}, \quad n \geq 1.$$

**Definition 4.2.1 (UMD spaces)** *A Banach space  $X$  is said to have the property of unconditional martingale differences (UMD property) if for any  $p$  in  $(1, \infty)$  there exists a constant  $\beta \geq 0$  (depending on  $p$  and  $X$ ) such that the following holds.*

*Whenever  $(S, \mathcal{A}, \mu)$  is a  $\sigma$ -finite measure space,  $(\mathcal{F}_n)_{n=0}^N$  is a  $\sigma$ -finite filtration and  $(f_n)_{n=0}^N$  is a finite martingale in  $L_p(S; X)$ ,  $N \geq 1$ , then for all scalars  $\epsilon_n$ ,  $n = 1, \dots, N$  such that  $|\epsilon_n| = 1$ , we have*

$$\left\| \sum_{n=1}^N \epsilon_n df_n \right\|_{L_p(S; X)} \leq \beta \left\| \sum_{n=1}^N df_n \right\|_{L_p(S; X)}.$$

*A Banach space having the UMD property is called a UMD-space.*

It is clear that the UMD property passes to the closed subspace and the isomorphic spaces.

Non trivial properties of UMD-spaces related to duality are the important two theorems below (see [34, Proposition 4.2.17] and [34, Section 4.3]).

**Theorem 4.2.2** *If  $X$  is a UMD-space, then  $X$  is a reflexive space.*

**Theorem 4.2.3**  *$X$  is a UMD-space if and only if  $X^*$  is a UMD space.*

We now give basic examples of UMD-spaces.

**Examples 4.2.4** (1) *Every Hilbert is a UMD-space.*

(2) *Every space  $L_p(T)$  where  $(T, \mathcal{B}, \nu)$  is a UMD-space for  $p$  in  $(1, \infty)$ .*

(3) *If  $X$  is a UMD-space, then any Bochner space  $L_p(T; X)$  is a UMD-space,  $p \in (1, \infty)$ .*

- (4) Every infinite dimensional non commutative  $L_p$ -space for  $p$  in  $(1, \infty)$  is a UMD-space (in particular, the Schatten classes  $S^p$  are UMD-spaces).
- (5) The spaces  $c_0$ ,  $l^1$  and  $l^\infty$  are not UMD-spaces.

There exist an independant way from martingales to consider the UMD-property. Indeed, we define for any function  $f$  in  $L^1_{\text{loc}}(\mathbb{R}; X)$ ,  $\varepsilon > 0$  and  $R > 0$

$$H_{\varepsilon,R}f(x) = \frac{1}{\pi} \int_{\varepsilon < |x-y| < R} \frac{f(y)}{x-y} dy, \quad x \in \mathbb{R},$$

and we let in the case of existence

$$Hf(x) = \lim_{\varepsilon \downarrow 0, R \uparrow \infty} H_{\varepsilon,R}f(x), \quad x \in \mathbb{R} \quad (4.2.1)$$

where we consider that  $\varepsilon$  and  $R$  tends to their limits independently and then the limit  $Hf(x)$  is independent of this approach. In this case, we call *Hilbert transform of  $f$*  the function  $Hf$ .

A result due to Burkholder and Bourgain shows that one can take another definition of the UMD-property.

**Theorem 4.2.5** *Let  $X$  be a Banach space and  $p$  in  $(1, \infty)$ . The following assertions are equivalent*

- (1)  $X$  is a UMD-space.
- (2) For every  $f$  in  $L_p(\mathbb{R}; X)$  the limit (4.2.1) exists in  $L_p(\mathbb{R}; X)$  and there exists  $C > 0$  a constant which does not depend on  $f$  such that

$$\|Hf\|_p \leq C \|f\|_p.$$

In particular, in UMD-space, the Hilbert transform  $f \mapsto Hf$  is a bounded operator from  $L_p(\mathbb{R}; X)$  into itself for any  $p$  in  $(1, \infty)$ .

We now give links between the UMD property and the geometric properties of Banach spaces we presented in this chapter.

**Proposition 4.2.6 (UMD  $\Rightarrow$   $K$ -convex)** *Every UMD-space is  $K$ -convex.*

Using Theorem 4.1.3 and Theorem 3.2.6, we deduce the following.

**Corollary 4.2.7 (UMD  $\Rightarrow$  non trivial type and finite cotype)** *Every UMD-space has non-trivial type and finite cotype.*

Note that an ingenious counter-example due to Qiu shows that the converse properties of Proposition 4.2.6 and Corollary 4.2.7 are wrong (see [34][4.3.c]), even for reflexive space.

We will see in the sequel that UMD-property plays a major role in our subject of  $H^\infty$  functional calculus (see Section 5.3).

### 4.3 The property $(\alpha)$

We fix a sequence of Rademacher independent functions  $(r_n)_{n \geq 1}$  on some probability space  $(\Omega, \mathbb{P})$ .

As an introduction to the property  $(\alpha)$ , we recall an elementary inequality meaning that the Rademacher averages are unconditional. More precisely, for any Banach space  $X$ , for any interger  $n \geq 1$  and sequences  $(a_k)_{k \geq 1}$  and  $(x_k)_{k \geq 1}$  respectively of  $\mathbb{C}$  and  $X$ , we have

$$\left\| \sum_{k=1}^n a_k r_k \otimes x_k \right\|_{\text{Rad}(X)} \leq 2 \sup \{|a_k| : 1 \leq k \leq n\} \left\| \sum_{k=1}^n r_k \otimes x_k \right\|_{\text{Rad}(X)}. \quad (4.3.1)$$

The spaces for which unconditionality property holds true for averages in  $\text{Rad}(\text{Rad}(X))$  are said to have property  $(\alpha)$ . This property was introduced by G. Pisier in [67].

**Definition 4.3.1 (Property  $(\alpha)$ )** *Let  $X$  be a Banach space. We will say that  $X$  has the property  $(\alpha)$  if there exists a constant  $C > 0$  such that for any interger  $n \geq 1$ , for any families  $(a_{i,j})_{1 \leq i,j \leq n}$  and  $(x_{i,j})_{1 \leq i,j \leq n}$  respectively in  $\mathbb{C}$  and  $X$  we have*

$$\left\| \sum_{i,j=1}^n a_{i,j} r_i \otimes r_j \otimes x_{i,j} \right\|_{\text{Rad}(\text{Rad}(X))} \leq C \sup \{|a_{i,j}| : 1 \leq i,j \leq n\} \left\| \sum_{i,j=1}^n r_i \otimes r_j \otimes x_{i,j} \right\|_{\text{Rad}(\text{Rad}(X))}. \quad (4.3.2)$$

It is clear that the property  $(\alpha)$  passes to closed subspaces. We give classical Banach spaces which have property  $(\alpha)$  and counterexamples. This goes back to [67].

**Examples 4.3.2** (1) *Any Hilbert space has property  $(\alpha)$ .*

(2) *Any  $L_p$  (commutative) space has property  $(\alpha)$  for  $[1, \infty)$ .*

(3) *The infinite dimensional non commutative  $L_p$ -space for  $p \neq 2$  do not have property  $(\alpha)$  (in particular, the Schatten classes  $S^p$  do not have this property).*

Let  $X$  be a Banach space,  $n \geq 1$  an integer and  $(x_{i,j})_{1 \leq i,j \leq n}$  be a family of  $X$ . Let  $(r_{i,j})$  be a family of independent Rademacher function indexed by  $(\mathbb{N}^*)^2$ . Set

$$\| [x_{i,j}] \|_1 = \left\| \sum_{i,j=1}^n r_{i,j} \otimes x_{i,j} \right\|_{\text{Rad}((\mathbb{N}^*)^2; X)}$$

and

$$\| [x_{i,j}] \|_2 = \left\| \sum_{i,j=1}^n r_i \otimes r_j \otimes x_{i,j} \right\|_{\text{Rad}(\text{Rad}(X))}.$$

According to the inconditionality given by (4.3.1), we obtain the following characterisation of the property  $(\alpha)$ .

**Proposition 4.3.3** *A Banach space  $X$  has property  $(\alpha)$  if and only if there exists a constant  $K > 0$  such that for any integer  $n \geq 1$ , for any family  $(x_{i,j})_{1 \leq i,j \leq n}$  of  $X$  we have*

$$K^{-1} \|[x_{i,j}]\|_1 \leq \|[x_{i,j}]\|_2 \leq K \|[x_{i,j}]\|_1.$$

In other words, a Banach space  $X$  has property  $(\alpha)$  if and only if the linear mapping

$$\sum r_i \otimes r_j \otimes x_{i,j} \mapsto \sum r_{i,j} \otimes x_{i,j}$$

extends to an isomorphism from  $\text{Rad}(\text{Rad}(X))$  into  $\text{Rad}((\mathbb{N}^*)^2; X)$  (see [67, Remark 2.1]).

Using the fact that the property  $(\alpha)$  implies that space does not contain  $l_n^\infty$  uniformly,  $n \geq 1$ , we obtain the following.

**Proposition 4.3.4** ( $(\alpha) \implies$  **finite cotype**) *Every Banach space with property  $(\alpha)$  has finite cotype.*

The converse property does not hold true, taking the infinite dimensional non commutative  $L_p$ -spaces. There is no more link between  $(\alpha)$  and UMD because of the non commutative  $L_p$  spaces and the commutative  $L_1$  space, which have property  $(\alpha)$  but are not UMD-spaces. Moreover,  $L_1$  is not  $K$ -convex.

However, Banach lattices benefit of the following property, which is due to the two variables version of (3.2.1). In fact, in the case where the Banach lattice  $X$  has finite cotype, the Rademacher averages of type

$$\left\| \sum_{i,j=1}^n r_i \otimes r_j \otimes x_{i,j} \right\|_{\text{Rad}(\mathbb{N}^2, X)}, \quad (x_{i,j})_{1 \leq i,j \leq n} \in X$$

where  $(r_n)$  denotes a sequence of Rademacher independent functions, is equivalent to

$$\left\| \left( \sum_{i,j=1}^n |x_{i,j}|^2 \right)^{\frac{1}{2}} \right\|_X$$

defined by the Krivine's calculus. As this last expression benefits of the property of unconditionality, we conclude that  $X$  has property  $(\alpha)$ .

**Proposition 4.3.5** *Let  $X$  be a Banach lattice. Then  $X$  has finite cotype if and only if  $X$  has property  $(\alpha)$ .*

We conclude with the case of UMD lattices.

**Proposition 4.3.6** (**UMD lattices**) *Every Banach lattice  $X$  with UMD-property has property  $(\alpha)$ .*

## 4.4 The property $(\Delta)$

We present in this section a weaker property than property  $(\alpha)$ . The property  $(\Delta)$  was introduced in [40]. In particular, we will see that more classes of Banach spaces have this property. We always fix  $(r_n)_{n \geq 1}$  a sequence of independent Rademacher variables on some probability space  $(\Omega, \mathbb{P})$ .

**Definition 4.4.1 (Property  $(\Delta)$ )** *We say that a Banach space  $X$  has property  $(\Delta)$  if there exists a constant  $C > 0$  such that for any finite family  $(x_{i,j})_{1 \leq i,j \leq n}$ ,  $n \in \mathbb{N}$ , we have*

$$\left\| \sum_{i=1}^n \sum_{j=i}^n r_i \otimes r_j \otimes x_{i,j} \right\|_{\text{Rad}(\text{Rad}(X))} \leq C \left\| \sum_{i=1}^n \sum_{j=1}^n r_i \otimes r_j \otimes x_{i,j} \right\|_{\text{Rad}(\text{Rad}(X))}. \quad (4.4.1)$$

*This means that the triangular projection*

$$\sum_{i=1}^n \sum_{j=1}^n r_i \otimes r_j \otimes x_{i,j} \mapsto \sum_{i=1}^n \sum_{j=i}^n r_i \otimes r_j \otimes x_{i,j}$$

*is a bounded map of  $\text{Rad}(\text{Rad}(X))$ .*

Clearly, any Banach space with property  $(\alpha)$  has property  $(\Delta)$ , taking  $a_{i,j} = 1$  for  $j \geq i$  and  $a_{i,j} = 0$  otherwise in the inequality (4.3.2), which yields inequality (4.4.1).

**Proposition 4.4.2  $((\alpha) \implies (\Delta))$**  *Every space with property  $(\alpha)$  has property  $(\Delta)$ .*

The converse property fails. This follows from Proposition below (see [40, Proposition 3.2]).

**Proposition 4.4.3 (UMD  $\implies$  Property  $(\Delta)$ )** *Every UMD-Banach space has property  $(\Delta)$ .*

The converse property is wrong, considering the space  $L^1$ .

In particular, it turns out that any infinite dimensional non commutative  $L_p$ -space for  $p \in (1, \infty)$  has property  $(\Delta)$  (in particular, the Schatten classes  $S^p$  have this property). This is the main example of spaces having property  $(\Delta)$  but not property  $(\alpha)$ .

**Proposition 4.4.4** *Every Schatten classes  $S^p$ ,  $p \in (1, \infty)$ , has property  $(\Delta)$ .*

On the other hand, the property  $(\Delta)$  does not hold uniformly on the spaces  $l_n^\infty$ ,  $n \geq 1$ , hence we have the following.

**Proposition 4.4.5 (Property  $(\Delta) \implies$  finite cotype)** *Every Banach space with property  $(\Delta)$  has finite cotype.*

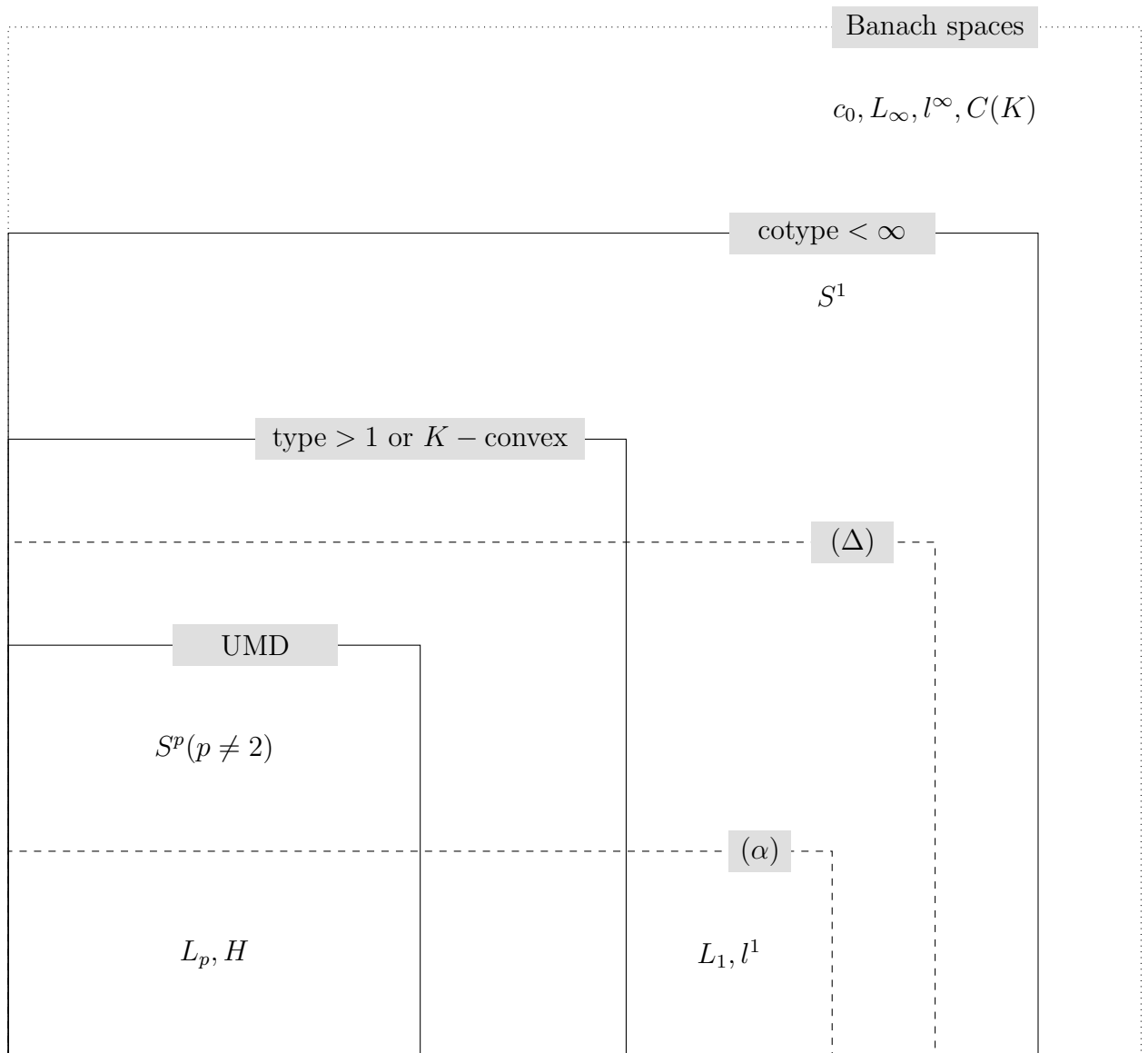
Considering the Schatten class  $S^1$  (which does not have property  $(\Delta)$ ), one sees that the converse property of the preceding proposition does not hold true. Moreover, considering space  $L_1$ , the property  $(\Delta)$  does not imply  $K$ -convexity. The issue of converse property is not resolved yet.

We finally note that the Banach space  $\bigoplus_{n \geq 1}^2 l_n^\infty$  is a reflexive space without finite cotype. Thus, this space is a counterexample to all the properties of  $K$ -convexity,  $(\alpha)$ , non trivial type and  $(\Delta)$ .

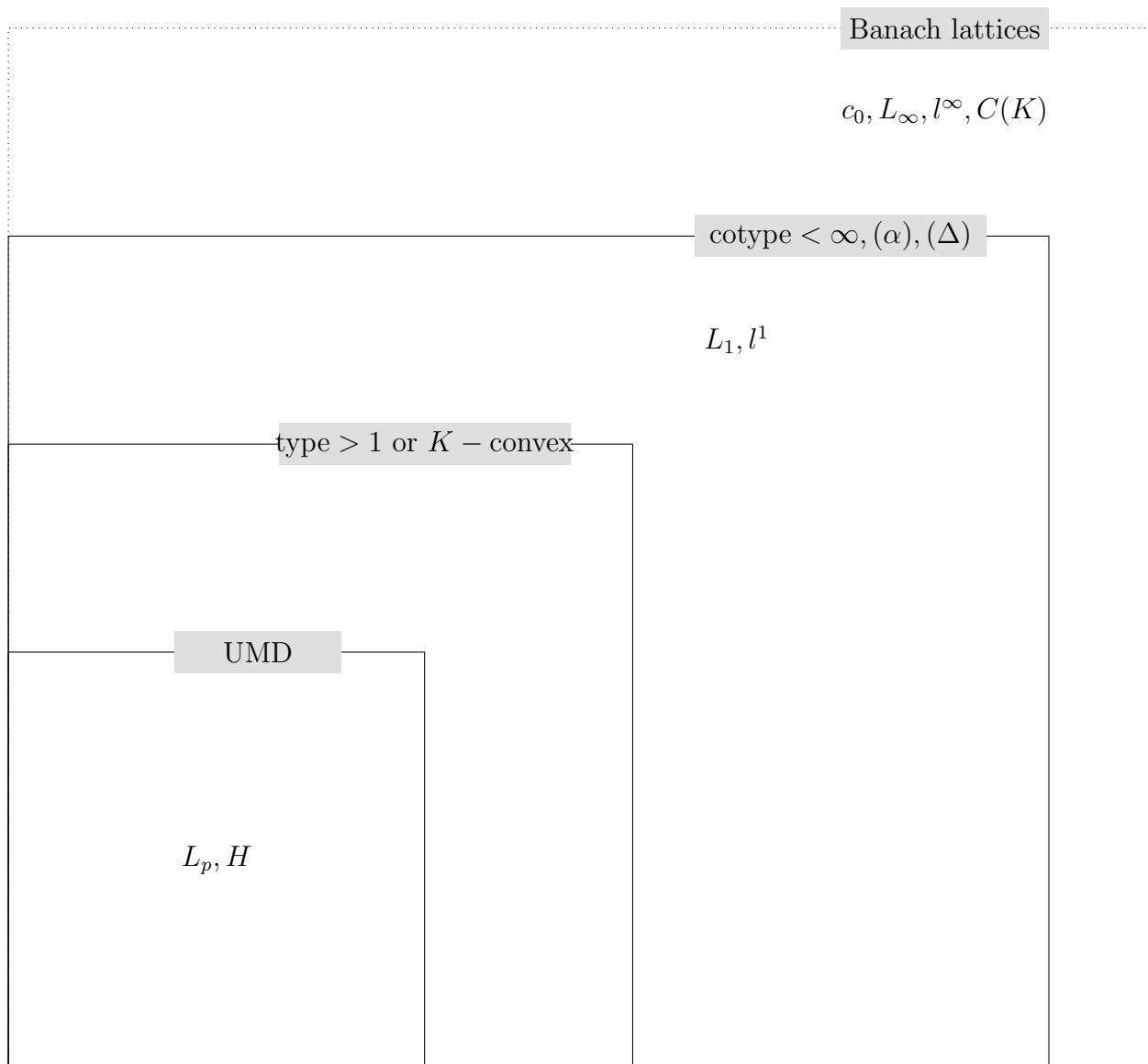


## 4.5 A summary

We summarise all the notions of geometry of Banach spaces we presented before. In all the diagram below,  $p$  belongs to  $(1, \infty)$  unless stated otherwise.  $H$  denotes an arbitrary Hilbert space and  $K$  an infinite Hausdorff compact space.



We turn now to the Banach lattices.





# Chapter 5

## Sectorial and Ritt operators

### 5.1 Definitions and basic properties

Throughout we let  $X$  be an arbitrary Banach space. For any  $\theta \in (0, \pi)$ , we let

$$\Sigma_\theta = \{z \in \mathbb{C}^* : |\operatorname{Arg}(z)| < \theta\}.$$

**Definition 5.1.1** *We say that a closed linear operator  $A: D(A) \rightarrow X$  with dense domain  $D(A) \subset X$  is sectorial of type  $\omega \in (0, \pi)$  if  $\sigma(A) \subset \overline{\Sigma_\omega}$  and for any  $\theta$  in  $(\omega, \pi)$ , there exists a constant  $C_\theta$  such that*

$$\|zR(z, A)\| \leq C_\theta, \quad z \in \mathbb{C} \setminus \overline{\Sigma_\theta}. \quad (5.1.1)$$

It follows from the Laplace formula that if  $-A$  is the generator of a bounded  $c_0$ -semigroup on  $X$ , then  $A$  is sectorial of type  $\frac{\pi}{2}$ . It is well known that  $A$  is a sectorial operator of type  $\omega < \frac{\pi}{2}$  if and only if it is the negative generator of a bounded analytic semigroup, which is  $(e^{-tA})_{t \geq 0}$  (see [64] for more on semigroup).

Let now  $A_1, \dots, A_d$  be commuting sectorial operators on  $X$ ,  $d \geq 1$ . We recall that if  $X$  is reflexive, then for any  $k = 1, \dots, d$ , we have a direct sum decomposition

$$X = \overline{\operatorname{Ran}(A_k)} \oplus \operatorname{Ker}(A_k), \quad (5.1.2)$$

see [35, Section 10.1]. We call it “ergodic decomposition” in the sequel.

For any  $\Lambda \subset \{1, \dots, d\}$ , set

$$X_\Lambda = \left( \bigcap_{k \in \Lambda} \overline{\operatorname{Ran}(A_k)} \right) \cap \left( \bigcap_{k \notin \Lambda} \operatorname{Ker}(A_k) \right). \quad (5.1.3)$$

**Lemma 5.1.2** *If  $X$  is reflexive, we have a direct sum decomposition*

$$X = \bigoplus_{\Lambda \subset \{1, \dots, d\}} X_\Lambda,$$

where the  $X_\Lambda$  are defined by (5.1.3).

*Proof of Lemma 5.1.2.* We start from the decomposition  $X = \overline{\text{Ran}(A_1)} \oplus \text{Ker}(A_1)$  given by (5.1.2). Since  $A_2$  commutes with  $A_1$ , the subspaces  $\overline{\text{Ran}(A_1)}$  and  $\text{Ker}(A_1)$  are both  $A_2$ -invariant. Thus we have restrictions

$$A_{2,1}: D(A_2) \cap \overline{\text{Ran}(A_1)} \longrightarrow \overline{\text{Ran}(A_1)} \quad \text{and} \quad A_{2,2}: D(A_2) \cap \text{Ker}(A_1) \longrightarrow \text{Ker}(A_1)$$

of  $A_2$ , which are sectorial operators. Since  $\overline{\text{Ran}(A_1)}$  and  $\text{Ker}(A_1)$  are reflexive,  $A_{2,1}$  and  $A_{2,2}$  admit ergodic decompositions. Since

$$\overline{\text{Ran}(A_{2,1})} = \overline{\text{Ran}(A_2)} \cap \overline{\text{Ran}(A_1)} \quad \text{and} \quad \text{Ker}(A_{2,1}) = \text{Ker}(A_2) \cap \overline{\text{Ran}(A_1)},$$

we obtain that

$$\overline{\text{Ran}(A_1)} = (\text{Ker}(A_2) \cap \overline{\text{Ran}(A_1)}) \oplus (\overline{\text{Ran}(A_2)} \cap \overline{\text{Ran}(A_1)}).$$

Likewise,

$$\text{Ker}(A_1) = (\text{Ker}(A_2) \cap \text{Ker}(A_1)) \oplus (\overline{\text{Ran}(A_2)} \cap \text{Ker}(A_1)).$$

Gluing these two ergodic decompositions together, we obtain the result in the case  $d = 2$ . The general case follows by induction.  $\square$

We end the basics on sectorial operators with the notion of  $R$ -sectoriality. If  $A$  is a sectorial operator on  $X$ , we say that  $A$  is  $R$ -sectorial of  $R$ -type  $\omega \in (0, \pi)$  if  $\sigma(A) \subset \overline{\Sigma_\omega}$  and for any  $\theta$  in  $(\omega, \pi)$ , the set

$$\{zR(z, A) : z \in \mathbb{C} \setminus \overline{\Sigma_\theta}\} \subset B(X)$$

is  $R$ -bounded. We refer e.g. to [35, Chapter 8] for information and references.

We now introduce Ritt operators, giving an equivalence between boundedness of certain sequences of  $B(X)$ , resolvent condition and semigroup condition.

**Proposition 5.1.3 (Ritt conditions)** *Let  $T : X \rightarrow X$  be a bounded operator. The following assertions are equivalent.*

(1) *There exists a constant  $C > 0$  such that*

$$\|T^n\| \leq C \quad \text{and} \quad \|n(T^n - T^{n-1})\| \leq C, \quad n \geq 1.$$

(2)  *$\sigma(T) \subset \overline{\mathbb{D}}$  and there exists a constant  $K > 0$  such that*

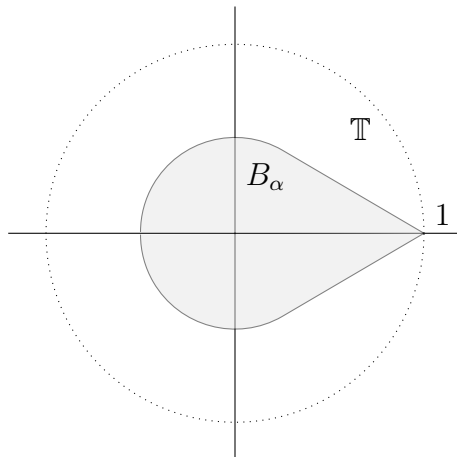
$$\|(\lambda - 1)R(\lambda, T)\| \leq K, \quad \lambda \in \mathbb{C}, \quad |\lambda| > 1.$$

(3) *The semigroup  $(e^{-t(I-T)})_{t \geq 0}$  is a bounded analytic semigroup and  $\sigma(T) \subset \mathbb{D} \cup \{1\}$ .*

**Definition 5.1.4 (Ritt operators)** *We will say that a bounded  $T : X \rightarrow X$  is a Ritt operator if it verifies one the condition (1),(2) or (3) of Proposition 5.1.3.*

In particular, we see that  $T$  is a Ritt operator if and only if  $A = I_X - T$  is a sectorial operator of type  $\omega < \frac{\pi}{2}$  and  $\sigma(T) \subset \mathbb{D} \cup \{1\}$  (see e.g [28, Theorem 4.1]).

We precise the spectrum of a Ritt operator. For any  $\alpha$  in  $(0, \frac{\pi}{2})$ , let  $B_\alpha$  denote the *Stolz domain* of angle  $\alpha$ , defined as the interior of the convex hull of 1 and the disc  $D(0, \sin(\alpha))$ .



Clearly, the Stolz domains have an increasing property, that is if  $0 < \alpha < \beta < \frac{\pi}{2}$  then  $B_\alpha \subset B_\beta$ . Moreover, we have

$$\bigcup_{0 < \alpha < \frac{\pi}{2}} B_\alpha = \mathbb{D} \cup \{1\}.$$

It turns out that if  $T$  is a Ritt operator, then  $\sigma(T) \subset \overline{B_\alpha}$  for some  $\alpha$  in  $(0, \frac{\pi}{2})$ . More precisely (see [53, Lemma 2.1]), one can find  $\alpha \in (0, \frac{\pi}{2})$  such that  $\sigma(T) \subset \overline{B_\alpha}$  and for any  $\beta \in (\alpha, \frac{\pi}{2})$ , there exists a constant  $K_\beta > 0$  such that

$$\|(\lambda - 1)R(\lambda, T)\| \leq K_\beta, \quad \lambda \in \mathbb{C} \setminus \overline{B_\beta}. \quad (5.1.4)$$

If this property holds, then we say that  $T$  is a Ritt operator of type  $\alpha$ . We refer to [58, 62, 63] for the above facts and also to [53] and the references therein for complements on the class of Ritt operators.

Now we give another useful characterisation of the Stolz domain. For every  $\sigma$  in  $(1, \infty)$ , we let

$$S_\sigma = \left\{ z \in \mathbb{D} \mid \frac{|1-z|}{1-|z|} < \sigma \right\} \cup \{1\}.$$

By [28, Section 4], we have

$$1 - \overline{S_\sigma} \subset \overline{\Sigma_\omega}, \quad \omega = \arccos\left(\frac{1}{\sigma}\right).$$

Moreover, using compactness argument, one sees that there exists  $\gamma$  in  $[\omega, \frac{\pi}{2})$  such that  $S_\sigma \subset B_\gamma$ . As a consequence, the set

$$\left\{ \frac{|1-z|}{1-|z|} : z \in B_\alpha \right\}$$

is bounded for every  $\alpha$  in  $(0, \frac{\pi}{2})$ .

Let now  $T_1, \dots, T_d$  be commuting Ritt operators on  $X$ ,  $d \geq 1$ . We recall the "mean ergodic theorem" allows us to say that if  $X$  is reflexive, then for any  $k = 1, \dots, d$ , we have a direct sum decomposition

$$X = \overline{\text{Ran}(I_X - T_k)} \oplus \text{Ker}(I_X - T_k),$$

as every  $T_k$  is a power bounded operator,  $k = 1, \dots, d$ . We call it "ergodic decomposition" in the sequel.

For any  $\Lambda \subset \{1, \dots, d\}$ , set

$$X_\Lambda = \left( \bigcap_{k \in \Lambda} \overline{\text{Ran}(I_X - T_k)} \right) \cap \left( \bigcap_{k \notin \Lambda} \text{Ker}(I_X - T_k) \right). \quad (5.1.5)$$

Mimicking the proof of Lemma 5.1.2, we obtain an analogue result for the Ritt case.

**Lemma 5.1.5** *If  $X$  is reflexive, we have a direct sum decomposition*

$$X = \bigoplus_{\Lambda \subset \{1, \dots, d\}} X_\Lambda,$$

where the  $X_\Lambda$  are defined by (5.1.5).

We end this section with  $R$ -Ritt operators. A Ritt operator  $T$  on  $X$  is called  $R$ -Ritt provided that the two sets

$$\{T^n, n \in \mathbb{N}\}, \quad \{n(T^n - T^{n-1}), n \geq 1\}$$

are  $R$ -bounded. In this case, there exists  $\delta$  in  $(0, \frac{\pi}{2})$  such that the set

$$\{(\lambda - 1)R(\lambda, T) : z \in \mathbb{C} \setminus \overline{B_\delta}\}$$

is  $R$ -bounded. We call  $R$ -type of  $T$  the infimum of all  $\delta$  for which this holds true (see [5] for the details).

## 5.2 $H^\infty$ joint functional calculus of a $d$ -tuple of sectorial operators or Ritt operators

Recall that for any  $\theta \in (0, \pi)$ , we let  $\Sigma_\theta = \{z \in \mathbb{C}^* : |\text{Arg}(z)| < \theta\}$ .

Let  $d \geq 1$  be an integer and let  $\theta_1, \dots, \theta_d$  be elements of  $(0, \pi)$ . For any subset  $\Lambda \subset \{1, \dots, d\}$ , we denote by  $H_0^\infty(\prod_{i \in \Lambda} \Sigma_{\theta_i})$  the subalgebra of  $H^\infty(\Sigma_{\theta_1} \times \dots \times \Sigma_{\theta_d})$  of all holomorphic bounded functions depending only on the variables  $(z_i)_{i \in \Lambda}$  and such that there exist positive constants  $c$  and  $(s_i)_{i \in \Lambda}$  verifying

$$|f(z_1, \dots, z_d)| \leq c \prod_{i \in \Lambda} \frac{|z_i|^{s_i}}{1 + |z_i|^{2s_i}}, \quad (z_i)_{i \in \Lambda} \in \prod_{i \in \Lambda} \Sigma_{\theta_i}. \quad (5.2.1)$$

When  $\Lambda = \emptyset$ ,  $H_0^\infty(\prod_{i \in \emptyset} \Sigma_{\theta_i})$  is the space of constant functions on  $\Sigma_{\theta_1} \times \dots \times \Sigma_{\theta_d}$ .

Let  $(A_1, \dots, A_d)$  be a family of commuting sectorial operators on  $X$ . Here the commuting property means that for any  $k, l$  in  $\{1, \dots, d\}$ , the resolvent operators  $R(z_k, A_k)$  and  $R(z_l, A_l)$  commute for any  $z_k$  in  $\mathbb{C} \setminus \sigma(A_k)$  and  $z_l$  in  $\mathbb{C} \setminus \sigma(A_l)$ . Assume that for every  $k = 1, \dots, d$ ,  $A_k$  is of type  $\omega_k \in (0, \theta_k)$  and let  $\nu_k \in (\omega_k, \theta_k)$ .

For any  $f$  in  $H_0^\infty(\prod_{i \in \Lambda} \Sigma_{\theta_i})$  with  $\Lambda \subset \{1, \dots, d\}$ ,  $\Lambda \neq \emptyset$ , we let

$$f(A_1, \dots, A_d) = \left( \frac{1}{2\pi i} \right)^{|\Lambda|} \int_{\prod_{i \in \Lambda} \partial \Sigma_{\nu_i}} f(z_1, \dots, z_d) \prod_{i \in \Lambda} R(z_i, A_i) \prod_{i \in \Lambda} dz_i, \quad (5.2.2)$$

where the boundaries  $\partial \Sigma_{\nu_i}$  are oriented counterclockwise for all  $i$  in  $\Lambda$ . By the commuting assumption on  $(A_1, \dots, A_d)$ , the product operator  $\prod_{i \in \Lambda} R(z_i, A_i)$  is well-defined. Further the conditions (5.1.1) and (5.2.1) ensure that this integral is absolutely convergent and defines an element of  $B(X)$ . By Cauchy's Theorem, this definition does not depend on the choice of the  $\nu_i$ 's. Moreover the linear mapping  $f \mapsto f(A_1, \dots, A_d)$  is an algebra homomorphism from  $H_0^\infty(\prod_{i \in \Lambda} \Sigma_{\theta_i})$  into  $B(X)$ . The proofs of these facts are similar to the ones for a single operator and are omitted.

If  $f \equiv a$  is a constant function on  $\Sigma_{\theta_1} \times \dots \times \Sigma_{\theta_d}$  (the case when  $\Lambda = \emptyset$ ), then we set  $f(A_1, \dots, A_d) = aI_X$ .

Next we let

$$H_{0,1}^\infty(\Sigma_{\theta_1} \times \dots \times \Sigma_{\theta_d}) \subset H^\infty(\Sigma_{\theta_1} \times \dots \times \Sigma_{\theta_d})$$

be the sum of all the  $H_0^\infty(\prod_{i \in \Lambda} \Sigma_{\theta_i})$ , with  $\Lambda \subset \{1, \dots, d\}$ . We claim that this sum is a direct one, so that we actually have

$$H_{0,1}^\infty(\Sigma_{\theta_1} \times \dots \times \Sigma_{\theta_d}) = \bigoplus_{\Lambda \subset \{1, \dots, d\}} H_0^\infty \left( \prod_{i \in \Lambda} \Sigma_{\theta_i} \right). \quad (5.2.3)$$

Let us prove this fact. For any  $i$  in  $\{1, \dots, d\}$ , let  $p_i$  be the operator defined on the space  $H_{0,1}^\infty(\Sigma_{\theta_1} \times \dots \times \Sigma_{\theta_d})$  by

$$[p_i(f)](z_1, \dots, z_d) = f(z_1, \dots, z_{i-1}, 0, z_{i+1}, \dots, z_d), \quad f \in H_{0,1}^\infty(\Sigma_{\theta_1} \times \dots \times \Sigma_{\theta_d}). \quad (5.2.4)$$

In this definition,  $f(z_1, \dots, z_{i-1}, 0, z_{i+1}, \dots, z_d)$  stands for the limit, when  $z \in \Sigma_{\theta_i}$  and  $z \rightarrow 0$ , of  $f(z_1, \dots, z_{i-1}, z, z_{i+1}, \dots, z_d)$ , provided that this limit exists. This is the case when  $f$  belongs to  $H_{0,1}^\infty(\Sigma_{\theta_1} \times \dots \times \Sigma_{\theta_d})$ . Note that the operators  $p_i$  commute,  $i = 1, \dots, d$ .

For any  $\Lambda \subset \{1, \dots, d\}$ , we can therefore define

$$P_\Lambda = \prod_{i \in \Lambda} (I - p_i) \prod_{i \in \Lambda^c} p_i. \quad (5.2.5)$$

It is easy to check that  $P_\Lambda(f) = f$  if  $f$  belongs to  $H_0^\infty(\prod_{i \in \Lambda} \Sigma_{\theta_i})$  and  $P_\Lambda(f) = 0$  if  $f$  belongs to  $H_0^\infty(\prod_{i \in \Lambda'} \Sigma_{\theta_i})$  for some  $\Lambda' \neq \Lambda$ . The direct sum property (5.2.3) follows at once.

Moreover,

$$P_\Lambda: H_{0,1}^\infty(\Sigma_{\theta_1} \times \dots \times \Sigma_{\theta_d}) \longrightarrow H_{0,1}^\infty(\Sigma_{\theta_1} \times \dots \times \Sigma_{\theta_d})$$



is the projection onto  $H_0^\infty(\prod_{i \in \Lambda} \Sigma_{\theta_i})$  with kernel equal to the direct sum of the  $H_0^\infty(\prod_{i \in \Lambda'} \Sigma_{\theta_i})$ , with  $\Lambda' \neq \Lambda$ .

For any function  $f = \sum_{\Lambda \subset \{1, \dots, d\}} f_\Lambda$  in  $H_{0,1}^\infty(\Sigma_{\theta_1} \times \dots \times \Sigma_{\theta_d})$ , where each  $f_\Lambda$  belongs to  $H_0^\infty(\prod_{i \in \Lambda} \Sigma_{\theta_i})$ , we naturally set

$$f(A_1, \dots, A_d) = \sum_{\Lambda \subset \{1, \dots, d\}} f_\Lambda(A_1, \dots, A_d), \quad (5.2.6)$$

the operator  $f_\Lambda(A_1, \dots, A_d)$  being defined by (5.2.2). In the sequel,  $f \mapsto f(A_1, \dots, A_d)$  is called the functional calculus mapping associated with  $(A_1, \dots, A_d)$ .

We note that if  $f_\Lambda$  is in  $H_0^\infty(\prod_{i \in \Lambda} \Sigma_{\theta_i})$  and  $f_{\Lambda'}$  is in  $H_0^\infty(\prod_{i \in \Lambda'} \Sigma_{\theta_i})$ , then  $f_\Lambda f_{\Lambda'}$  is in  $H_0^\infty(\prod_{i \in \Lambda \cup \Lambda'} \Sigma_{\theta_i})$ . Thus  $H_{0,1}^\infty(\Sigma_{\theta_1} \times \dots \times \Sigma_{\theta_d})$  is a subalgebra of  $H^\infty(\Sigma_{\theta_1} \times \dots \times \Sigma_{\theta_d})$ .

**Lemma 5.2.1** *The functional calculus mapping  $f \mapsto f(A_1, \dots, A_d)$  is an algebra homomorphism from  $H_{0,1}^\infty(\Sigma_{\theta_1} \times \dots \times \Sigma_{\theta_d})$  into  $B(X)$ .*

*Proof of Lemma 5.2.1.* The linearity being obvious, it suffices to check that for any subsets  $\Lambda, \Lambda'$  of  $\{1, \dots, d\}$ , for any  $f_\Lambda$  in  $H_0^\infty(\prod_{i \in \Lambda} \Sigma_{\theta_i})$  and  $f_{\Lambda'}$  in  $H_0^\infty(\prod_{i \in \Lambda'} \Sigma_{\theta_i})$ , we have

$$f_\Lambda(A_1, \dots, A_d) f_{\Lambda'}(A_1, \dots, A_d) = (f_\Lambda f_{\Lambda'})(A_1, \dots, A_d). \quad (5.2.7)$$

We let  $\Lambda_0 = \Lambda \cap \Lambda'$  and we set  $\Lambda_1 = \Lambda \setminus \Lambda_0$  and  $\Lambda'_1 = \Lambda' \setminus \Lambda_0$ . For convenience we set, for any subset  $K$  of  $\{1, \dots, d\}$ ,

$$z_K = (z_i)_{i \in K}, \quad dz_K = \prod_{i \in K} dz_i, \quad R_K(z_K) = \prod_{i \in K} R(z_i, A_i) \quad \text{and} \quad \Gamma_K = \prod_{i \in K} \partial \Sigma_{\nu_i}.$$

Using Fubini's theorem, we have

$$\begin{aligned} & f_\Lambda(A_1, \dots, A_d) f_{\Lambda'}(A_1, \dots, A_d) \\ &= \left( \frac{1}{2\pi i} \right)^{|\Lambda|+|\Lambda'|} \left( \int_{\Gamma_\Lambda} f_\Lambda(z_1, \dots, z_d) R_\Lambda(z_\Lambda) dz_\Lambda \right) \left( \int_{\Gamma_{\Lambda'}} f_{\Lambda'}(z_1, \dots, z_d) R_{\Lambda'}(z_{\Lambda'}) dz_{\Lambda'} \right) \\ &= \left( \frac{1}{2\pi i} \right)^{|\Lambda|+|\Lambda'|} \int_{\Gamma_{\Lambda_1}} \left( \int_{\Gamma_{\Lambda_0}} f_\Lambda(z_1, \dots, z_d) R_{\Lambda_0}(z_{\Lambda_0}) dz_{\Lambda_0} \right) R_{\Lambda_1}(z_{\Lambda_1}) dz_{\Lambda_1} \\ &\quad \times \int_{\Gamma_{\Lambda'_1}} \left( \int_{\Gamma_{\Lambda_0}} f_{\Lambda'}(z_1, \dots, z_d) R_{\Lambda_0}(z_{\Lambda_0}) dz_{\Lambda_0} \right) R_{\Lambda'_1}(z_{\Lambda'_1}) dz_{\Lambda'_1} \\ &= \left( \frac{1}{2\pi i} \right)^{|\Lambda|+|\Lambda'|} \int_{\Gamma_{\Lambda_1} \times \Gamma_{\Lambda'_1}} \left[ \left( \int_{\Gamma_{\Lambda_0}} f_\Lambda(z_1, \dots, z_d) R_{\Lambda_0}(z_{\Lambda_0}) dz_{\Lambda_0} \right) \right. \\ &\quad \left. \times \left( \int_{\Gamma_{\Lambda_0}} f_{\Lambda'}(z_1, \dots, z_d) R_{\Lambda_0}(z_{\Lambda_0}) dz_{\Lambda_0} \right) \right] R_{\Lambda_1}(z_{\Lambda_1}) R_{\Lambda'_1}(z_{\Lambda'_1}) dz_{\Lambda_1} dz_{\Lambda'_1}. \end{aligned}$$

For fixed variables  $z_i$ , for  $i \notin \Lambda_0$ , the two functions

$$(z_i)_{i \in \Lambda_0} \mapsto f_\Lambda(z_1, \dots, z_d) \quad \text{and} \quad (z_i)_{i \in \Lambda_0} \mapsto f_{\Lambda'}(z_1, \dots, z_d)$$

both belong to  $H_0^\infty(\prod_{i \in \Lambda_0} \Sigma_{\theta_i})$ . We noticed before that the functional calculus mapping is a homomorphism from  $H_0^\infty(\prod_{i \in \Lambda_0} \Sigma_{\theta_i})$  into  $B(X)$ . Consequently,

$$\begin{aligned} & \left( \frac{1}{2\pi i} \right)^{2|\Lambda_0|} \left( \int_{\Gamma_{\Lambda_0}} f_\Lambda(z_1, \dots, z_d) R_{\Lambda_0}(z_{\Lambda_0}) dz_{\Lambda_0} \right) \left( \int_{\Gamma_{\Lambda_0}} f_{\Lambda'}(z_1, \dots, z_d) R_{\Lambda_0}(z_{\Lambda_0}) dz_{\Lambda_0} \right) \\ &= \left( \frac{1}{2\pi i} \right)^{|\Lambda_0|} \int_{\Gamma_{\Lambda_0}} f_\Lambda f_{\Lambda'}(z_1, \dots, z_d) R_{\Lambda_0}(z_{\Lambda_0}) dz_{\Lambda_0}. \end{aligned}$$

Hence the above computation leads to

$$\begin{aligned} & f_\Lambda(A_1, \dots, A_d) f_{\Lambda'}(A_1, \dots, A_d) \\ &= \left( \frac{1}{2\pi i} \right)^{|\Lambda|+|\Lambda'|-|\Lambda_0|} \int_{\Gamma_{\Lambda_1} \times \Gamma_{\Lambda'_1}} \int_{\Gamma_{\Lambda_0}} f_\Lambda f_{\Lambda'}(z_1, \dots, z_d) R_{\Lambda_0}(z_{\Lambda_0}) dz_{\Lambda_0} R_{\Lambda_1}(z_{\Lambda_1}) R_{\Lambda'_1}(z_{\Lambda'_1}) dz_{\Lambda_1} dz_{\Lambda'_1} \\ &= \left( \frac{1}{2\pi i} \right)^{|\Lambda|+|\Lambda'|-|\Lambda_0|} \int_{\Gamma_{\Lambda \cup \Lambda'}} f_\Lambda f_{\Lambda'}(z_1, \dots, z_d) R_{\Lambda \cup \Lambda'}(z_{\Lambda \cup \Lambda'}) dz_{\Lambda \cup \Lambda'} \\ &= (f_\Lambda f_{\Lambda'})(A_1, \dots, A_d), \end{aligned}$$

since  $\Lambda \cup \Lambda'$  is the disjoint union of  $\Lambda_0, \Lambda_1$  and  $\Lambda'_1$ . This proves (5.2.7).  $\square$

**Definition 5.2.2** ( $H^\infty$  Joint functional calculus of sectorial operators) *We say that  $(A_1, \dots, A_d)$  admits an  $H^\infty(\Sigma_{\theta_1} \times \dots \times \Sigma_{\theta_d})$  joint functional calculus if the functional calculus mapping associated with  $(A_1, \dots, A_d)$  is bounded, that is, there exists a constant  $K > 0$  such that for every  $f$  in  $H_{0,1}^\infty(\Sigma_{\theta_1} \times \dots \times \Sigma_{\theta_d})$ ,*

$$\|f(A_1, \dots, A_d)\| \leq K \|f\|_{\infty, \Sigma_{\theta_1} \times \dots \times \Sigma_{\theta_d}}. \quad (5.2.8)$$

It will be convenient to extend Definition 5.2.2 to the limit case when  $\theta_k$  is replaced by  $\omega_k$ , as follows.

**Definition 5.2.3** *We say that  $(A_1, \dots, A_d)$  admits an  $H^\infty(\Sigma_{\omega_1} \times \dots \times \Sigma_{\omega_d})$  joint functional calculus if it admits an  $H^\infty(\Sigma_{\theta_1} \times \dots \times \Sigma_{\theta_d})$  joint functional calculus for any  $\theta_k \in (\omega_k, \pi)$ , and there exists a constant  $K \geq 0$  such that (5.2.8) holds true for any  $\theta_k \in (\omega_k, \pi)$ ,  $k = 1, \dots, d$ .*

Each  $p_i$  from (5.2.4) is a contraction, hence each  $P_\Lambda$  from (5.2.5) is a bounded operator on  $H_{0,1}^\infty(\Sigma_{\theta_1} \times \dots \times \Sigma_{\theta_d})$ . This implies that  $(A_1, \dots, A_d)$  admits an  $H^\infty(\Sigma_{\theta_1} \times \dots \times \Sigma_{\theta_d})$  joint functional calculus if and only if  $f \mapsto f(A_1, \dots, A_d)$  is bounded on  $H_0^\infty(\prod_{i \in \Lambda} \Sigma_{\theta_i})$  for any  $\Lambda \subset \{1, \dots, d\}$ . Consequently if  $(A_1, \dots, A_d)$  admits an  $H^\infty(\Sigma_{\theta_1} \times \dots \times \Sigma_{\theta_d})$  joint functional calculus, then every subfamily  $(A_i)_{i \in \Lambda}$ , where  $\Lambda \subset \{1, \dots, d\}$ , also admits an  $H^\infty(\prod_{i \in \Lambda} \Sigma_{\theta_i})$  joint functional calculus. In particular, for every  $k = 1, \dots, d$ ,  $A_k$  admits an  $H^\infty(\Sigma_{\theta_k})$  functional calculus in the usual sense (see [31, Chapter 5]). The converse does not hold true. Indeed it follows from [48, Theorem 3.9] that for any  $1 \leq p \neq 2 \leq \infty$ , there exists a commuting couple  $(A_1, A_2)$  of sectorial operators

on the Schatten space  $S^p$  such that  $A_k$  admits an  $H^\infty(\Sigma_{\theta_k})$  functional calculus for any  $\theta_k \in (0, \pi)$ ,  $k = 1, 2$ , but  $(A_1, A_2)$  has no joint functional calculus.

For any integer  $m \geq 1$ , we set

$$\Phi_m(z) = \frac{m^2 z}{(m+z)(1+mz)}, \quad z \in \mathbb{C} \setminus \mathbb{R}_-. \quad (5.2.9)$$

A key property of this sequence is that whenever  $A$  is a sectorial operator, we have

$$\forall x \in \overline{\text{Ran}(A)}, \quad \lim_m \Phi_m(A)x = x, \quad (5.2.10)$$

see e.g. [52, 2.C]. We show below that this approximation property allows to simplify the definition of  $H^\infty$  joint functional calculus in the case when all the  $A_k$  have a dense range ( $H_{0,1}^\infty$  is replaced by  $H_0^\infty$  in the estimate (5.2.8)).

**Lemma 5.2.4** *Assume that  $\overline{\text{Ran}(A_k)} = X$  for all  $k = 1, \dots, d$ . If there exists a constant  $K > 0$  such that*

$$\|f(A_1, \dots, A_d)\| \leq K \|f\|_{\infty, \Sigma_{\theta_1} \times \dots \times \Sigma_{\theta_d}} \quad (5.2.11)$$

for any  $f \in H_0^\infty(\Sigma_{\theta_1} \times \dots \times \Sigma_{\theta_d})$ , then  $(A_1, \dots, A_d)$  admits an  $H^\infty(\Sigma_{\theta_1} \times \dots \times \Sigma_{\theta_d})$  joint functional calculus.

*Proof of Lemma 5.2.4.* Let  $f \in H_{0,1}^\infty(\Sigma_{\theta_1} \times \dots \times \Sigma_{\theta_d})$ . For any  $m \geq 1$ , let

$$\Phi_m^{\otimes d} = \Phi_m \otimes \dots \otimes \Phi_m \quad (5.2.12)$$

be the element of  $H_0^\infty(\Sigma_{\theta_1} \times \dots \times \Sigma_{\theta_d})$  defined by  $\Phi_m^{\otimes d}(z_1, \dots, z_d) = \Phi_m(z_1) \cdots \Phi_m(z_d)$ . Then the product  $f\Phi_m^{\otimes d}$  belongs to  $H_0^\infty(\Sigma_{\theta_1} \times \dots \times \Sigma_{\theta_d})$  and

$$(f\Phi_m^{\otimes d})(A_1, \dots, A_d) = f(A_1, \dots, A_d)\Phi_m(A_1) \cdots \Phi_m(A_d).$$

Applying (5.2.11) to this function we obtain that

$$\|f(A_1, \dots, A_d)\Phi_m(A_1) \cdots \Phi_m(A_d)\| \leq K \left( \prod_{k=1}^d \|\Phi_m\|_{\infty, \Sigma_{\theta_k}} \right) \|f\|_{\infty, \Sigma_{\theta_1} \times \dots \times \Sigma_{\theta_d}}.$$

The dense range assumption and (5.2.10) ensure that

$$f(A_1, \dots, A_d)\Phi_m(A_1) \cdots \Phi_m(A_d) \longrightarrow f(A_1, \dots, A_d)$$

strongly, as  $m \rightarrow \infty$ . Moreover  $(\Phi_m)_{m \geq 1}$  is uniformly bounded on all sectors  $\Sigma_\theta$ . Hence the above inequality yields an estimate

$$\|f(A_1, \dots, A_d)\| \lesssim \|f\|_{\infty, \Sigma_{\theta_1} \times \dots \times \Sigma_{\theta_d}}.$$

□

We now pass to the case of  $d$ -tuples of Ritt operators.  $H^\infty$ -functional calculus for Ritt operators was formally introduced in [53]. We now extend this definition to commuting families. We follow the same pattern as for families of sectorial operators.

Let  $d \geq 1$  be an integer and let  $\gamma_1, \dots, \gamma_d$  be elements of  $(0, \frac{\pi}{2})$ . For any subset  $\Lambda$  of  $\{1, \dots, d\}$ , we denote by  $H_0^\infty(\prod_{i \in \Lambda} B_{\gamma_i})$  the subalgebra of  $H^\infty(B_{\gamma_1} \times \dots \times B_{\gamma_d})$  of all holomorphic bounded functions  $f$  depending only on variables  $(\lambda_i)_{i \in \Lambda}$  and such that there exist positive constants  $c$  and  $(s_i)_{i \in \Lambda}$  verifying

$$|f(\lambda_1, \dots, \lambda_d)| \leq c \prod_{i \in \Lambda} |1 - \lambda_i|^{s_i}, \quad (\lambda_i)_{i \in \Lambda} \in \prod_{i \in \Lambda} B_{\gamma_i}. \quad (5.2.13)$$

When  $\Lambda = \emptyset$ ,  $H_0^\infty(\prod_{i \in \emptyset} B_{\gamma_i})$  is the space of constant functions on  $B_{\gamma_1} \times \dots \times B_{\gamma_d}$ .

Let  $(T_1, \dots, T_d)$  be a  $d$ -tuple of commuting Ritt operators. Assume that for any  $k = 1, \dots, d$ ,  $T_k$  is of type  $\alpha_k \in (0, \gamma_k)$  and let  $\beta_k \in (\alpha_k, \gamma_k)$ .

For any  $f$  in  $H_0^\infty(\prod_{i \in \Lambda} B_{\gamma_i})$  with  $\Lambda \subset \{1, \dots, d\}$ ,  $\Lambda \neq \emptyset$ , we let

$$f(T_1, \dots, T_d) = \left( \frac{1}{2\pi i} \right)^{|\Lambda|} \int_{\prod_{i \in \Lambda} \partial B_{\beta_i}} f(\lambda_1, \dots, \lambda_d) \prod_{i \in \Lambda} R(\lambda_i, T_i) \prod_{i \in \Lambda} d\lambda_i, \quad (5.2.14)$$

where the  $\partial B_{\beta_i}$  are oriented counterclockwise for all  $i \in \Lambda$ . With (5.2.13) and characterisation (2) of Proposition 5.1.3, we see that this integral is absolutely convergent, hence defines an element of  $B(X)$ , its definition does not depend on the  $\beta_i$  and the linear mapping  $f \mapsto f(T_1, \dots, T_d)$  is an algebra homomorphism from  $H_0^\infty(\prod_{i \in \Lambda} B_{\gamma_i})$  into  $B(X)$ . If  $f \equiv a$  is a constant function, then we let  $f(T_1, \dots, T_d) = aI_X$ .

Next we define

$$H_{0,1}^\infty(B_{\gamma_1} \times \dots \times B_{\gamma_d}) = \bigoplus_{\Lambda \subset \{1, \dots, d\}} H_0^\infty \left( \prod_{i \in \Lambda} B_{\gamma_i} \right).$$

As in the sectorial case, the above sum is indeed a direct one. More precisely, set

$$[q_i(f)](\lambda_1, \dots, \lambda_d) = f(\lambda_1, \dots, \lambda_{i-1}, 1, \lambda_{i+1}, \dots, \lambda_d), \quad f \in H_{0,1}^\infty(B_{\gamma_1} \times \dots \times B_{\gamma_d}),$$

for  $i = 1, \dots, d$ , and

$$Q_\Lambda = \prod_{i \in \Lambda} (I - q_i) \prod_{i \in \Lambda^c} q_i, \quad (5.2.15)$$

for  $\Lambda \subset \{1, \dots, d\}$ . These mappings are well-defined and

$$Q_\Lambda: H_{0,1}^\infty(B_{\gamma_1} \times \dots \times B_{\gamma_d}) \longrightarrow H_{0,1}^\infty(B_{\gamma_1} \times \dots \times B_{\gamma_d})$$

is the projection onto  $H_0^\infty(\prod_{i \in \Lambda} B_{\gamma_i})$  with kernel equal to the direct sum of the spaces  $H_0^\infty(\prod_{i \in \Lambda'} B_{\gamma_i})$ , with  $\Lambda' \neq \Lambda$ .

For any function  $f = \sum_{\Lambda \subset \{1, \dots, d\}} f_\Lambda$  in  $H_{0,1}^\infty(B_{\gamma_1} \times \dots \times B_{\gamma_d})$ , with  $f_\Lambda \in H_0^\infty(\prod_{i \in \Lambda} B_{\gamma_i})$ , we let  $f(T_1, \dots, T_d) = \sum_{\Lambda \subset \{1, \dots, d\}} f_\Lambda(T_1, \dots, T_d)$ , where every  $f_\Lambda(T_1, \dots, T_d)$  is defined by (5.2.14). The mapping  $f \mapsto f(T_1, \dots, T_d)$  is called the functional calculus mapping associated with  $(T_1, \dots, T_d)$ . As in the sectorial case (see Lemma 5.2.1), one shows that this is an algebra homomorphism from  $H_{0,1}^\infty(B_{\gamma_1} \times \dots \times B_{\gamma_d})$  into  $B(X)$ .

**Definition 5.2.5** We say that  $(T_1, \dots, T_d)$  admits an  $H^\infty(B_{\gamma_1} \times \dots \times B_{\gamma_d})$  joint functional calculus if the above functional calculus mapping is bounded, that is, there exists a constant  $K > 0$  such that for every  $f$  in  $H_{0,1}^\infty(B_{\gamma_1} \times \dots \times B_{\gamma_d})$ , we have

$$\|f(T_1, \dots, T_d)\| \leq K \|f\|_{\infty, B_{\gamma_1} \times \dots \times B_{\gamma_d}}. \quad (5.2.16)$$

As in the sectorial case, we observe that  $(T_1, \dots, T_d)$  admits an  $H^\infty(B_{\gamma_1} \times \dots \times B_{\gamma_d})$  joint functional calculus if and only if  $f \mapsto f(T_1, \dots, T_d)$  is bounded on  $H_0^\infty(\prod_{i \in \Lambda} B_{\gamma_i})$  for any  $\Lambda \subset \{1, \dots, d\}$ . This follows from the fact that each  $q_i$  is a contraction, hence each  $Q_\Lambda$  is bounded.

Further if  $(T_1, \dots, T_d)$  admits an  $H^\infty(B_{\gamma_1} \times \dots \times B_{\gamma_d})$  joint functional calculus, then every subfamily of this  $d$ -tuple admits an  $H^\infty$  joint functional calculus. In particular, for every  $k = 1, \dots, d$ ,  $T_k$  admits an  $H^\infty(\Sigma_{\theta_k})$  functional calculus in the sense of [53, Definition 2.4].

It is natural to consider polynomial functional calculus in this context. We let  $\mathcal{P}_d$  denote the algebra of all complex valued polynomials in  $d$  variables. Clearly  $\mathcal{P}_d$  can be regarded as a subalgebra of  $H_{0,1}^\infty(B_{\gamma_1} \times \dots \times B_{\gamma_d})$  and for  $\phi \in \mathcal{P}_d$ , the definition of  $\phi(T_1, \dots, T_d)$  given by replacing the variables  $(z_1, \dots, z_d)$  by the operators  $(T_1, \dots, T_d)$  coincides with the one given by the functional calculus mapping. This follows from the basic properties of the Dunford-Riesz functional calculus. We will show below that to obtain an  $H^\infty(B_{\gamma_1} \times \dots \times B_{\gamma_d})$  joint functional calculus for  $(T_1, \dots, T_d)$ , it suffices to consider polynomials in (5.2.16).

To prove this result, we will use the following form of Runge's Lemma.

**Lemma 5.2.6** Let  $d \geq 1$  be an integer and  $V_1, \dots, V_d$  be compact subsets of  $\mathbb{C}$  such that  $\mathbb{C} \setminus V_i$  is connected for all  $i = 1, \dots, d$ . Let  $\Omega_1, \dots, \Omega_d$  be open subsets of  $\mathbb{C}$  such that  $V_i \subset \Omega_i$  for all  $i = 1, \dots, d$ . Let  $f: \Omega_1 \times \dots \times \Omega_d \rightarrow \mathbb{C}$  be a holomorphic function. Then there exists a sequence  $(\phi_m)_{m \geq 1}$  in  $\mathcal{P}_d$  which converges uniformly to  $f$  on  $V_1 \times \dots \times V_d$ .

In the case  $d = 1$ , this statement is [71, Theorem 13.7]. The proof of the latter readily extends to the  $d$ -variable case so we omit it.

**Proposition 5.2.7** Let  $d \geq 1$  be an integer and let  $(T_1, \dots, T_d)$  be a commuting family of Ritt operators. Let  $\gamma_i \in (0, \frac{\pi}{2})$  for  $i = 1, \dots, d$ . The following assertions are equivalent.

- (i)  $(T_1, \dots, T_d)$  admits an  $H^\infty(B_{\gamma_1} \times \dots \times B_{\gamma_d})$  joint functional calculus.
- (ii) There exists a constant  $K > 0$  such that for any  $\phi \in \mathcal{P}_d$  we have

$$\|\phi(T_1, \dots, T_d)\| \leq K \|\phi\|_{\infty, B_{\gamma_1} \times \dots \times B_{\gamma_d}}. \quad (5.2.17)$$

*Proof of Proposition 5.2.7.* The implication (i)  $\Rightarrow$  (ii) is obvious. Conversely assume (ii). As noticed after (5.2.16) it suffices, to prove (i), to establish the boundedness of  $f \mapsto f(T_1, \dots, T_d)$  on  $H_0^\infty(\prod_{i \in \Lambda} B_{\gamma_i})$  for any  $\Lambda \subset \{1, \dots, d\}$ . By induction, it actually suffices to prove the estimate

$$\|f(T_1, \dots, T_d)\| \leq K \|f\|_{\infty, B_{\gamma_1} \times \dots \times B_{\gamma_d}} \quad (5.2.18)$$

for any  $f$  in  $H_0^\infty(B_{\gamma_1} \times \cdots \times B_{\gamma_d})$ .

Let  $f$  be such a function and consider  $r \in (0, 1)$  and  $r' \in (r, 1)$ . Let  $\Gamma = \partial(r'B_{\gamma_1}) \times \cdots \times \partial(r'B_{\gamma_d})$ , where all the  $\partial(r'B_{\gamma_i})$  are oriented counterclockwise. By Lemma 5.2.6 applied with  $V_i = r'\overline{B_{\gamma_i}}$  and  $\Omega_i = B_{\gamma_i}$ , there exists a sequence  $(\phi_m)_{m \geq 1}$  of  $\mathcal{P}_d$  such that  $\phi_m \rightarrow f$  uniformly on the compact set  $r'\overline{B_{\gamma_1}} \times \cdots \times r'\overline{B_{\gamma_d}}$ .

Since we have  $\sigma(rT_i) \subset r'B_{\gamma_i}$  for all  $i = 1, \dots, d$ , the Dunford-Riesz functional calculus provides

$$\phi_m(rT_1, \dots, rT_d) = \left( \frac{1}{2\pi i} \right)^d \int_{\Gamma} \phi_m(\lambda_1, \dots, \lambda_d) R(\lambda_1, rT_1) \cdots R(\lambda_d, rT_d) d\lambda_1 \cdots d\lambda_d$$

and

$$f(rT_1, \dots, rT_d) = \left( \frac{1}{2\pi i} \right)^d \int_{\Gamma} f(\lambda_1, \dots, \lambda_d) R(\lambda_1, rT_1) \cdots R(\lambda_d, rT_d) d\lambda_1 \cdots d\lambda_d.$$

The uniform convergence of  $(\phi_m)_{m \geq 1}$  to  $f$  on  $r'\overline{B_{\gamma_1}} \times \cdots \times r'\overline{B_{\gamma_d}}$  implies that

$$\phi_m(rT_1, \dots, rT_d) \xrightarrow{m \rightarrow \infty} f(rT_1, \dots, rT_d) \quad \text{and} \quad \|\phi_m\|_{\infty, r'B_{\gamma_1} \times \cdots \times r'B_{\gamma_d}} \xrightarrow{m \rightarrow \infty} \|f\|_{\infty, r'B_{\gamma_1} \times \cdots \times r'B_{\gamma_d}}$$

Using (5.2.17) we have, for any interger  $m \geq 1$ ,

$$\|\phi_m(rT_1, \dots, rT_d)\| \leq K \|\phi_m\|_{\infty, r'B_{\gamma_1} \times \cdots \times r'B_{\gamma_d}} \leq K \|\phi_m\|_{\infty, r'B_{\gamma_1} \times \cdots \times r'B_{\gamma_d}}.$$

Passing to the limit when  $m \rightarrow \infty$ , we deduce that

$$\|f(rT_1, \dots, rT_d)\| \leq K \|f\|_{\infty, r'B_{\gamma_1} \times \cdots \times r'B_{\gamma_d}} \leq K \|f\|_{\infty, B_{\gamma_1} \times \cdots \times B_{\gamma_d}}.$$

Finally, we have  $\lim_{r \rightarrow 1} f(rT_1, \dots, rT_d) = f(T_1, \dots, T_d)$  by Lebesgue's dominated convergence Theorem. We deduce (5.2.18).  $\square$

### 5.3 The role of the UMD property

We give in this section statements which explains the role of UMD-property in the  $H^\infty$  functional calculus. The first result concerns the generator of a bounded  $c_0$  group, which is a sectorial operator of type  $\frac{\pi}{2}$ .

**Theorem 5.3.1** *Let  $X$  be a UMD Banach space. Let  $(U_t)_{t \in \mathbb{R}}$  be a bounded  $c_0$  group acting on  $X$ . Let  $-A$  be its generator. Then for any  $\theta > \frac{\pi}{2}$ ,  $A$  admits an  $H^\infty(\Sigma_\theta)$  functional calculus.*

Remark that this theorem fails if we consider a pair of commuting bounded  $c_0$  group  $(U_t^1)_{t \in \mathbb{R}}$  and  $(U_t^2)_{t \in \mathbb{R}}$ . Indeed, a counterexample due to Lancien ([45]) shows the necessity of the property  $(\alpha)$  to obtain a joint functional calculus. Combining this result with the Akcoglu-Sucheston's dilation Theorem, one can show that the negative generator of a bounded  $c_0$ -semigroup of positive contraction on  $L_p$  admits an  $H^\infty$ -functional calculus ( $p \in (1, \infty)$ ) (see [32]).

The second result, which relies on Theorem 5.3.1, considers fractional power of an isomorphism. Recall that for any  $a$  in  $(0, 1)$  and  $z$  in  $\mathbb{D}$ , we have a power series expansion

$$(1 - z)^a = 1 + \sum_{k=1}^{\infty} c_{k,a} z^k$$

with  $c_{k,a}$  the complex coefficients of the series. Then we can define

$$T_a = I_X - (I_X - T)^a = - \sum_{k=1}^{\infty} c_{k,a} T^k$$

for any power bounded operator  $T$  on an arbitrary Banach space  $X$ . According to [23],  $T_a$  is a Ritt operator. [4, Theorem 3.3] provides the following result.

**Theorem 5.3.2** *Let  $X$  be a UMD Banach space. Let  $U$  be a isomorphism acting on  $X$  such that  $(U^n)_{n \in \mathbb{Z}}$  is bounded. Then for every  $a$  in  $(0, 1)$ , the operator  $U_a$  admits an  $H^\infty(B_\gamma)$  functional calculus for some  $\gamma$  in  $(0, \frac{\pi}{2})$ .*

## **Part II**

# **The Franks-McIntosh decompositions**





# Chapter 6

## Franks-McIntosh decomposition on sectors

Here we recall the original Franks-McIntosh decomposition which is done in [26, Section 3]. Furthermore we give adaptation of these results in order to have other suitable decompositions for our work. This decomposition deals with bounded holomorphic functions on some  $\Sigma_\nu$ , where

$$\Sigma_\nu = \{z \in \mathbb{C}^* : |\operatorname{Arg}(z)| < \nu\}, \quad \nu \in (0, \pi).$$

It uses sequences of holomorphic functions of  $H_0^\infty(\Sigma_\mu)$  for some  $\mu$  in  $(0, \nu)$ . The generalisation to multivariate bounded holomorphic functions on  $\Sigma_{\nu_1} \times \cdots \times \Sigma_{\nu_d}$ ,  $d \geq 2$ ,  $\nu_k \in (0, \pi)$ ,  $k = 1, \dots, d$  is sketched in [26, Section 4] and we pay a particular attention to obtain a complete proof.

In the next two lemmas, we fix two angles  $0 < \mu < \gamma < \pi$ . The following can be extracted from [26, Section 3].

**Lemma 6.0.1** *There exist constants  $\rho > 1$  and  $C > 0$ , and a family  $(\phi_{m,k,j})_{\substack{m=1,2; \\ k \in \mathbb{Z}; j \geq 0}}$  in  $H_0^\infty(\Sigma_\mu)$  such that:*

(1) *For any  $n \in \mathbb{Z}$  and for any  $z \in \Sigma_\mu$  satisfying  $\rho^n \leq |z| \leq \rho^{n+1}$ , we have*

$$|\phi_{m,k,j}(z)| \leq C 2^{-j} \rho^{-\frac{1}{2}|k-n|}, \quad m \in \{1, 2\}, k \in \mathbb{Z}, j \geq 0.$$

(2) *For any  $f \in H^\infty(\Sigma_\gamma)$ , there exists a family  $(\alpha_{m,k,j})_{\substack{m=1,2; \\ k \in \mathbb{Z}; j \geq 0}}$  of complex numbers such that*

$$|\alpha_{m,k,j}| \leq C \|f\|_{\infty, \Sigma_\gamma}, \quad m \in \{1, 2\}, k \in \mathbb{Z}, j \geq 0,$$

and

$$f(z) = \sum_{\substack{m=1,2; \\ k \in \mathbb{Z}; j \geq 0}} \alpha_{m,k,j} \phi_{m,k,j}(z), \quad z \in \Sigma_\mu.$$

(3) For any  $m \in \{1, 2\}$ ,  $k \in \mathbb{Z}$ ,  $j \geq 0$ , the function

$$z \mapsto \phi_{m,k,j}(z) \left( \frac{1+z^2}{z} \right)^{\frac{1}{2}}$$

is bounded on  $\Sigma_\mu$ .

Let  $\rho > 1$  be given by Lemma 6.0.1. For any  $k \in \mathbb{Z}$ , we define  $\sigma_k \in H^\infty(\Sigma_\mu)$  by

$$\sigma_k(z) = \frac{\rho^{\frac{k}{4}} z^{\frac{1}{4}}}{(\rho^k e^{i\gamma})^{\frac{1}{2}} - z^{\frac{1}{2}}}, \quad z \in \Sigma_\mu. \quad (6.0.1)$$

**Lemma 6.0.2** *There exist two constants  $0 < c_1 < c_2$  such that for any  $k, n \in \mathbb{Z}$  and for any  $z \in \Sigma_\mu$  satisfying  $\rho^n \leq |z| \leq \rho^{n+1}$ , we have*

$$c_1 \rho^{-\frac{1}{4}|k-n|} \leq |\sigma_k(z)| \leq c_2 \rho^{-\frac{1}{4}|k-n|}.$$

*Proof of Lemma 6.0.2.* Consider  $z \in \Sigma_\mu$  satisfying  $\rho^n \leq |z| \leq \rho^{n+1}$ . On the one hand, we have

$$\left| (\rho^k e^{i\gamma})^{\frac{1}{2}} - z^{\frac{1}{2}} \right| \leq \rho^{\frac{k}{2}} + |z|^{\frac{1}{2}} \leq (1 + \rho^{\frac{1}{2}}) \rho^{\frac{1}{2} \max\{k, n\}}.$$

On the other hand, we observe that for some  $\delta > 0$  (only depending on  $\mu, \gamma$  and  $\rho$ ), the distance  $\left| (\rho^k e^{i\gamma})^{\frac{1}{2}} - z^{\frac{1}{2}} \right|$  is bounded from below by either  $\delta \rho^{\frac{n}{2}}$  if  $k \leq n$ , or  $\delta \rho^{\frac{k}{2}}$  if  $k \geq n$ . Thus we have

$$\left| (\rho^k e^{i\gamma})^{\frac{1}{2}} - z^{\frac{1}{2}} \right| \geq \delta \rho^{\frac{1}{2} \max\{k, n\}}.$$

These estimates imply

$$|\sigma_k(z)| \approx \frac{\rho^{\frac{k}{4}} |z|^{\frac{n}{4}}}{\rho^{\frac{1}{2} \max\{k, n\}}} \approx \rho^{\frac{k}{4}} \rho^{\frac{n}{4}} \rho^{-\frac{1}{2} \max\{k, n\}} = \rho^{-\frac{1}{4}|k-n|},$$

under the condition  $\rho^n \leq |z| \leq \rho^{n+1}$ . This yields the result.  $\square$

Using Lemma 6.0.1, we obtain the following.

**Theorem 6.0.3** *Let  $d \geq 1$  be an integer, let  $\nu_k$  in  $(0, \pi)$  and  $\mu_k$  in  $(0, \nu_k)$ ,  $k = 1, \dots, d$ . There exist sequences  $(\Psi_{k, i_k})_{i_k \geq 1}$  and  $(\tilde{\Psi}_{k, i_k})_{i_k \geq 1}$  in  $H_0^\infty(\Sigma_{\mu_k})$  verifying the following properties.*

(1) For every real number  $p > 0$  and for any  $k = 1, \dots, d$ ,

$$\sup \left\{ \sum_{i_k=1}^{\infty} |\Psi_{k, i_k}(\zeta_k)|^p : \zeta_k \in \Sigma_{\mu_k} \right\} < \infty \quad \text{and} \quad \sup \left\{ \sum_{i_k=1}^{\infty} \left| \tilde{\Psi}_{k, i_k}(\zeta_k) \right|^p : \zeta_k \in \Sigma_{\mu_k} \right\} < \infty. \quad (6.0.2)$$

(2) There exists a constant  $C > 0$  such that for every  $h$  in  $H^\infty(\Sigma_{\nu_1} \times \dots \times \Sigma_{\nu_d})$ , there exists a family  $(a_{i_1, \dots, i_d})_{i_1, \dots, i_d \geq 1}$  of complex numbers such that

$$|a_{i_1, \dots, i_d}| \leq C \|h\|_{\infty, \Sigma_{\nu_1} \times \dots \times \Sigma_{\nu_d}}, \quad (i_1, \dots, i_d) \in \mathbb{N}^{*d}, \quad (6.0.3)$$

and for every  $(\zeta_1, \dots, \zeta_d)$  in  $\prod_{k=1}^d \Sigma_{\mu_k}$ ,

$$h(\zeta_1, \dots, \zeta_d) = \sum_{i_1, \dots, i_d \geq 1} a_{i_1, \dots, i_d} \Psi_{1, i_1}(\zeta_1) \tilde{\Psi}_{1, i_1}(\zeta_1) \cdots \Psi_{d, i_d}(\zeta_d) \tilde{\Psi}_{d, i_d}(\zeta_d). \quad (6.0.4)$$

*Proof of Theorem 6.0.3.* First, we prove the result for  $d = 1$  (single variable case). Let  $\nu \in (0, \pi)$  and  $\mu \in (0, \nu)$ . Considering the functions  $\phi_{m,k,j}$  given by Lemma 6.0.1, we let

$$F_{m,k,j}(z) = \left( \frac{1+z^2}{z} \right)^{\frac{1}{2}} \phi_{m,k,j}(z), \quad z \in \Sigma_\mu,$$

for  $m \in \{1, 2\}, j \geq 0, k \in \mathbb{Z}$ . By assertion (3) of Lemma 6.0.1, all the functions  $F_{m,k,j}$  belong to  $H^\infty(\Sigma_\mu)$ . Then we apply inner-outer decomposition to each function  $F_{m,k,j}$  to write  $F_{m,k,j} = G_{m,k,j} \tilde{G}_{m,k,j}$  on  $\Sigma_\mu$  and having  $|\tilde{G}_{m,k,j}| = |G_{m,k,j}| = |F_{m,k,j}|^{\frac{1}{2}}$ , a-e on  $\partial\Sigma_\mu$ . According to the maximum principle, all the functions  $G_{m,k,j}$  and  $\tilde{G}_{m,k,j}$  belong to  $H^\infty(\Sigma_\mu)$ ,  $m \in \{1, 2\}, j \geq 0, k \in \mathbb{Z}$ . Hence we deduce a factorisation of  $\phi_{m,k,j}$  as

$$\phi_{m,k,j}(z) = \left( \frac{z}{1+z^2} \right)^{\frac{1}{4}} G_{m,k,j}(z) \left( \frac{z}{1+z^2} \right)^{\frac{1}{4}} \tilde{G}_{m,k,j}(z), \quad z \in \Sigma_\mu,$$

for  $m \in \{1, 2\}, j \geq 0, k \in \mathbb{Z}$ . Now we let  $\Psi_{m,k,j}(z) = \left( \frac{z}{1+z^2} \right)^{\frac{1}{4}} G_{m,k,j}(z)$  and  $\tilde{\Psi}_{m,k,j}(z) = \left( \frac{z}{1+z^2} \right)^{\frac{1}{4}} \tilde{G}_{m,k,j}(z)$ , which are functions of  $H_0^\infty(\Sigma_\mu)$ .

Next we prove the estimations (1) of Theorem 6.0.3. Using the maximum principle on the partial sums of the series and part (1) of Lemma 6.0.1, one obtain for every real number  $p > 0$ ,

$$\sup \left\{ \sum_{m,k,j} |\Psi_{m,k,j}(\zeta)|^p : \zeta \in \Sigma_\mu \right\} < \infty \quad \text{and} \quad \sup \left\{ \sum_{m,k,j} |\tilde{\Psi}_{m,k,j}(\zeta)|^p : \zeta \in \Sigma_\mu \right\} < \infty,$$

where the summations run over all  $m \in \{1, 2\}, j \geq 0, k \in \mathbb{Z}$ . Applying part (2) of Lemma 6.0.1, for any holomorphic function  $f$  of  $H^\infty(\Sigma_\nu)$ , there exists a family  $(\alpha_{m,k,j})_{\substack{m=1,2; \\ k \in \mathbb{Z}; j \geq 0}}$  of complex numbers such that

$$|\alpha_{m,k,j}| \leq C \|f\|_{\infty, \Sigma_\nu}, \quad m \in \{1, 2\}, k \in \mathbb{Z}, j \geq 0,$$

and

$$f(z) = \sum_{\substack{m=1,2; \\ k \in \mathbb{Z}; j \geq 0}} \alpha_{m,k,j} \Psi_{m,k,j}(z) \tilde{\Psi}_{m,k,j}(z), \quad z \in \Sigma_\mu.$$

Reindexing the families  $(\alpha_{m,k,j}), (\Psi_{m,k,j}), (\tilde{\Psi}_{m,k,j})$  and  $(\phi_{m,k,j})$ , one we can write

$$f(z) = \sum_{i=1}^{\infty} \alpha_i \Psi_i(z) \tilde{\Psi}_i(z) = \sum_{i=1}^{\infty} \alpha_i \phi_i(z), \quad z \in \Sigma_\mu.$$

We now give the generalisation for any  $d \geq 2$  (multivariable case), starting with the case  $d = 2$ . Let  $\nu_1, \nu_2$  be elements of  $(0, \pi)$  and  $(\mu_1, \mu_2) \in (0, \nu_1) \times (0, \nu_2)$ .

Let now  $h$  be in  $H^\infty(\Sigma_{\nu_1} \times \Sigma_{\nu_2})$ . Fixing  $z_1$  in  $\Sigma_{\mu_1}$  and applying the preceding decomposition of  $h(z_1, \cdot)$ , we obtain that

$$h(z_1, z_2) = \sum_{i_2=1}^{\infty} \alpha_{i_2}(z_1) \phi_{2,i_2}(z_2), \quad z_2 \in \Sigma_{\mu_2}, \quad (6.0.5)$$

with a uniform estimate  $|\alpha_{i_2}(z_1)| \leq C_2 \|f\|_{\infty, \Sigma_{\nu_1} \times \Sigma_{\nu_2}}$ ,  $z_1 \in \Sigma_{\mu_1}$ , for some constant  $C_2 > 0$ . By [26, Section 4], which gives an explicit expression of the functions  $\alpha_{i_2} : \Sigma_{\mu_1} \rightarrow \mathbb{C}$ , we see that these functions are all holomorphic functions on  $\Sigma_{\mu_1}$ ,  $i_2 \geq 1$ . Moreover we have  $\|\alpha_{i_2}\|_{\infty, \Sigma_{\mu_1}} \leq C \|f\|_{\infty, \Sigma_{\nu_1} \times \Sigma_{\nu_2}}$ , which ensures that these functions belong to  $H^\infty(\Sigma_{\mu_1})$ . Thus, we apply the case  $d = 1$  to all the  $\alpha_{i_2}$ , which yields

$$\alpha_{i_2}(z_1) = \sum_{i_1=1}^{\infty} \alpha_{i_1, i_2} \phi_{1, i_1}(z_1), \quad z_1 \in \Sigma_{\mu_1}, \quad i_2 \geq 1, \quad (6.0.6)$$

where the coefficients  $\alpha_{i_1, i_2}$  verify a uniform estimate

$$|\alpha_{i_1, i_2}| \leq C_1 C_2 \|f\|_{\infty, \Sigma_{\nu_1} \times \Sigma_{\nu_2}}, \quad i_1 \geq 0, i_2 \geq 0.$$

Since the series  $\sum_{i_1 \geq 1} \phi_{1, i_1}(z_1)$  and  $\sum_{i_2 \geq 1} \phi_{2, i_2}(z_2)$  are absolutely convergent for any  $(z_1, z_2)$  in  $\Sigma_{\mu_1} \times \Sigma_{\mu_2}$ , the implementation of (6.0.6) into (6.0.5) allows us to write

$$h(z_1, z_2) = \sum_{i_1=1, i_2=2}^{\infty} \alpha_{i_1, i_2} \phi_{1, i_1}(z_1) \phi_{2, i_2}(z_2), \quad (z_1, z_2) \in \Sigma_{\mu_1} \times \Sigma_{\mu_2}.$$

It remains to use factorisations of the functions  $\phi_{1, i_1} = \Psi_{1, i_1} \tilde{\Psi}_{1, i_1}$  and  $\phi_{2, i_2} = \Psi_{2, i_2} \tilde{\Psi}_{2, i_2}$  as we describe in beginning of this proof to obtain the complete formula (6.0.4) with  $d = 2$ . The general result for  $d \geq 2$  follows by iterating this process.  $\square$

Now, the following decomposition of the unit (of independent interest) will be derived from the above two lemmas.

**Proposition 6.0.4** *Let  $\mu \in (0, \pi)$ . There exist three sequences  $(\Delta_i)_{i \geq 1}$ ,  $(\psi_i)_{i \geq 1}$  and  $(\tilde{\psi}_i)_{i \geq 1}$  in  $H_0^\infty(\Sigma_\mu)$  satisfying the following properties.*

(1) *There exists a constant  $C \geq 0$  such that*

$$\forall z \in \Sigma_\mu, \quad \sum_{i=1}^{\infty} |\psi_i(z)| \leq C \quad \text{and} \quad \sum_{i=1}^{\infty} |\tilde{\psi}_i(z)| \leq C.$$

(2) *For any  $\nu \in (0, \mu)$ , there exists a constant  $K \geq 0$  such that*

$$\forall i \geq 1, \quad \int_{\partial \Sigma_\nu} |\Delta_i(z)| \left| \frac{dz}{z} \right| \leq K.$$

(3) *There exists a constant  $C \geq 0$  such that*

$$\forall i \geq 1, \quad \forall z \in \Sigma_\mu, \quad |\Delta_i(z)| \leq C,$$

and

$$\forall z \in \Sigma_\mu, \quad 1 = \sum_{i=1}^{\infty} \Delta_i(z) \psi_i(z) \tilde{\psi}_i(z).$$

*Proof of Proposition 6.0.4.* We fix  $\gamma \in (\mu, \pi)$  and consider  $\rho > 1$  and the family  $(\phi_{m,k,j})_{\substack{m=1,2; \\ k \in \mathbb{Z}; j \geq 0}}$  given by Lemma 6.0.1. We apply part (2) of the Lemma 6.0.1 to the constant function  $f \equiv 1$ . This yields a bounded family  $(\alpha_{m,k,j})_{\substack{m=1,2; \\ k \in \mathbb{Z}; j \geq 0}}$  of  $\mathbb{C}$  such that

$$1 = \sum_{\substack{m=1,2; \\ k \in \mathbb{Z}; j \geq 0}} \alpha_{m,k,j} \phi_{m,k,j}(z), \quad z \in \Sigma_\mu. \quad (6.0.7)$$

Consider  $m \in \{1, 2\}$ ,  $k \in \mathbb{Z}$ ,  $j \geq 0$ . We define, with  $\sigma_k$  given in (6.0.1)

$$\varphi_{m,k,j} = \frac{\phi_{m,k,j}}{\sigma_k}.$$

According to part (3) of Lemma 6.0.1 and Lemma 6.0.2, this is an element of  $H_0^\infty(\Sigma_\mu)$ . As in Theorem 6.0.3, the inner-outer factorisation of functions in  $H^\infty(\Sigma_\mu)$  provides two functions  $\psi_{m,k,j}$  and  $\tilde{\psi}_{m,k,j}$  in  $H_0^\infty(\Sigma_\mu)$  such that

$$\varphi_{m,k,j}(z) = \psi_{m,k,j}(z) \tilde{\psi}_{m,k,j}(z), \quad z \in \Sigma_\mu,$$

and

$$|\psi_{m,k,j}(z)| = |\tilde{\psi}_{m,k,j}(z)| = |\varphi_{m,k,j}(z)|^{\frac{1}{2}}, \quad \text{a.e. } z \in \partial\Sigma_\mu.$$

According to part (1) of Lemma 6.0.1 and the lower bound in Lemma 6.0.2, there exists a constant  $c_0 > 0$  such that for any  $z \in \Sigma_\mu$  satisfying  $\rho^n \leq |z| \leq \rho^{n+1}$ , we have

$$|\varphi_{m,k,j}(z)| \leq c_0 2^{-j} \rho^{-\frac{1}{4}|k-n|}, \quad m \in \{1, 2\}, k \in \mathbb{Z}, j \geq 0.$$

This implies that

$$\sup_{z \in \partial\Sigma_\mu} \sum_{\substack{m=1,2; \\ k \in \mathbb{Z}; j \geq 0}} |\psi_{m,k,j}(z)| = \sup_{z \in \partial\Sigma_\mu} \sum_{\substack{m=1,2; \\ k \in \mathbb{Z}; j \geq 0}} |\varphi_{m,k,j}(z)|^{\frac{1}{2}} < \infty.$$

Using the maximum principle of holomorphic functions we deduce that

$$\sup_{z \in \Sigma_\mu} \sum_{\substack{m=1,2; \\ k \in \mathbb{Z}; j \geq 0}} |\psi_{m,k,j}(z)| < \infty. \quad (6.0.8)$$

Likewise we have

$$\sup_{z \in \Sigma_\mu} \sum_{\substack{m=1,2; \\ k \in \mathbb{Z}; j \geq 0}} |\tilde{\psi}_{m,k,j}(z)| < \infty. \quad (6.0.9)$$

Let  $\nu \in (0, \mu)$ . Applying the upper bound in Lemma 6.0.2, we obtain that for any  $k \in \mathbb{Z}$ , we have

$$\begin{aligned} \int_{\partial\Sigma_\nu} |\sigma_k(z)| \left| \frac{dz}{z} \right| &= \sum_{n \in \mathbb{Z}} \int_{\partial\Sigma_\nu \cap \{\rho^n \leq |z| \leq \rho^{n+1}\}} |\sigma_k(z)| \left| \frac{dz}{z} \right| \\ &\leq c_2 \sum_{n \in \mathbb{Z}} \rho^{-\frac{1}{4}|k-n|} \int_{\partial\Sigma_\nu \cap \{\rho^n \leq |z| \leq \rho^{n+1}\}} \left| \frac{dz}{z} \right|. \end{aligned}$$

This last term is equal to

$$2c_2 \log(\rho) \sum_{m \in \mathbb{Z}} \rho^{-\frac{m}{4}},$$

which is independent of  $k$ . Hence

$$\sup_{k \in \mathbb{Z}} \int_{\partial \Sigma_\nu} |\sigma_k(z)| \left| \frac{dz}{z} \right| < \infty. \quad (6.0.10)$$

Now define

$$\Delta_{m,k,j} = \alpha_{m,k,j} \sigma_k,$$

for any  $m \in \{1, 2\}$ ,  $k \in \mathbb{Z}$ ,  $j \geq 0$ . This is a uniformly bounded family and (6.0.7) now reads

$$1 = \sum_{\substack{m=1,2; \\ k \in \mathbb{Z}; j \geq 0}} \Delta_{m,k,j}(z) \psi_{m,k,j}(z) \tilde{\psi}_{m,k,j}(z), \quad z \in \Sigma_\mu.$$

Reindexing the families  $(\Delta_{m,k,j})_{m,k,j}$ ,  $(\psi_{m,k,j})_{m,k,j}$  and  $(\tilde{\psi}_{m,k,j})_{m,k,j}$  as sequences, the result follows from this identity and the estimates (6.0.8), (6.0.9) and (6.0.10).  $\square$

# Chapter 7

## Franks-McIntosh decomposition on Stolz domains

We saw in the preceding Chapter that Franks and McIntosh gave in [26] a decomposition of bounded holomorphic functions on sectors, using series of specific functions. In [26][Section 4], they actually considered more general domains than sectors. The case of Stolz domains  $B_\gamma$  is implicit therein.

In the chapter, we provide a detailed proof of the Franks-McIntosh decomposition on these Stolz domains (see Section 5.1 for the definition).

Then we generalise it for holomorphic functions on the open unit disc, which is suitable when we deal with bounded polynomial calculus of the Ritt operators.

The main purpose of this chapter is the following result.

**Theorem 7.0.1** *Let  $d \geq 1$  be an integer, let  $\beta_k$  in  $(0, \frac{\pi}{2})$  and  $\alpha_k$  in  $(0, \beta_k)$ ,  $k = 1, \dots, d$ . There exist sequences  $(\Psi_{k,i_k})_{i_k \geq 1}$  and  $(\tilde{\Psi}_{k,i_k})_{i_k \geq 1}$  in  $H_0^\infty(B_{\alpha_k})$  verifying the following properties.*

(1) *For every real number  $p > 0$  and for any  $k = 1, \dots, d$ ,*

$$\sup \left\{ \sum_{i_k=1}^{\infty} |\Psi_{k,i_k}(\zeta_k)|^p : \zeta_k \in B_{\alpha_k} \right\} < \infty \quad \text{and} \quad \sup \left\{ \sum_{i_k=1}^{\infty} |\tilde{\Psi}_{k,i_k}(\zeta_k)|^p : \zeta_k \in B_{\alpha_k} \right\} < \infty. \quad (7.0.1)$$

(2) *There exists a constant  $C > 0$  such that for every  $h$  in  $H^\infty(B_{\beta_1} \times \dots \times B_{\beta_d})$ , there exists a family  $(a_{i_1, \dots, i_d})_{i_1, \dots, i_d \geq 1}$  of complex numbers such that*

$$|a_{i_1, \dots, i_d}| \leq C \|h\|_{\infty, B_{\beta_1} \times \dots \times B_{\beta_d}}, \quad (i_1, \dots, i_d) \in \mathbb{N}^{*d}, \quad (7.0.2)$$

*and for every  $(\zeta_1, \dots, \zeta_d)$  in  $\prod_{k=1}^d B_{\alpha_k}$ ,*

$$h(\zeta_1, \dots, \zeta_d) = \sum_{i_1, \dots, i_d \geq 1} a_{i_1, \dots, i_d} \Psi_{1,i_1}(\zeta_1) \tilde{\Psi}_{1,i_1}(\zeta_1) \cdots \Psi_{d,i_d}(\zeta_d) \tilde{\Psi}_{d,i_d}(\zeta_d). \quad (7.0.3)$$

The main part of the proof will consist in showing the following one-variable result.



**Proposition 7.0.2** *Let  $0 < \alpha < \beta < \frac{\pi}{2}$ . There exist a sequence  $(\Phi_i)_{i \geq 1}$  in  $H_0^\infty(B_\alpha)$  and a constant  $C > 0$  such that*

$$\sup \left\{ \sum_{i=1}^{\infty} |\Phi_i(\zeta)|^p : \zeta \in B_\alpha \right\} < \infty$$

for any  $p > 0$ , and for any  $h \in H^\infty(B_\beta)$ , there exists a sequence  $(a_i)_{i \geq 1}$  of complex numbers such that  $|a_i| \leq C \|h\|_{\infty, B_\beta}$  for any  $i \geq 1$  and

$$h(\zeta) = \sum_{i=1}^{\infty} a_i \Phi_i(\zeta), \quad \zeta \in B_\alpha. \quad (7.0.4)$$

**Remark 7.0.3** Since  $B_\alpha$  is a simply connected domain bounded by a rectifiable Jordan curve, any element of  $H^\infty(B_\alpha)$  admits boundary values. Further for any  $\Phi \in H_0^\infty(B_\alpha)$ , there exist  $\Psi, \tilde{\Psi}$  in  $H_0^\infty(B_\alpha)$  such that

$$|\Psi(\zeta)| = |\tilde{\Psi}(\zeta)| = |\Phi(\zeta)|^{\frac{1}{2}}, \quad \text{a-e } \zeta \in \partial B_\alpha. \quad (7.0.5)$$

Indeed given  $\Phi \in H_0^\infty(B_\alpha)$ , there exists  $s > 0$  and  $F \in H^\infty(B_\alpha)$  such that  $(1 - \zeta)^{-s} \Phi(\zeta) = F(\zeta)$  for any  $\zeta \in B_\alpha$ . Then using inner-outer factorisation, we may write  $F = \varphi \tilde{\varphi}$  with  $|\varphi| = |\tilde{\varphi}| = |F|^{\frac{1}{2}}$  on the boundary of  $B_\alpha$ . Then we obtain (7.0.5) by taking  $\Psi(\zeta) = (1 - \zeta)^{\frac{s}{2}} \varphi(\zeta)$  and  $\tilde{\Psi}(\zeta) = (1 - \zeta)^{\frac{s}{2}} \tilde{\varphi}(\zeta)$ .

Combining the above factorisation property with Proposition 7.0.2, we immediately obtain Theorem 7.0.1 in the case  $d = 1$ .

Before proceeding to the proof of Proposition 7.0.2, we need some preliminary constructions. We fix some  $0 < \alpha < \mu < \beta < \frac{\pi}{2}$ .

We let  $\Gamma_0$  denote the arc of the circle centered at 0 with radius  $\sin(\mu)$ , joining  $\sin(\mu)e^{i(\frac{\pi}{2}-\mu)}$  to  $\sin(\mu)e^{i(\mu-\frac{\pi}{2})}$  counterclockwise. Then we let  $\Gamma_1$  and  $\Gamma_2$  denote the segments joining 1 to  $\sin(\mu)e^{i(\frac{\pi}{2}-\mu)}$  and  $\sin(\mu)e^{i(\mu-\frac{\pi}{2})}$  to 1, respectively. Clearly  $\Gamma_0, \Gamma_1$  and  $\Gamma_2$  divide  $\partial B_\mu$ .

We divide  $\Gamma_0$  into a finite number of arcs  $\{\gamma_{0,k}\}_{k=0}^N$  with fixed length  $\delta \leq \frac{1}{2} \text{dist}(\partial B_\alpha, \Gamma_0)$ . For any  $0 \leq k \leq N$ , we denote by  $z_{0,k}$  the center of  $\gamma_{0,k}$  and we let  $D_{0,k}$  be the open ball centered at  $z_{0,k}$  with radius  $\delta$ . Thus  $D_{0,k}$  does not intersect  $\partial B_\alpha$ .

Let  $l = \cos(\mu)$ ; this is the length of the segment  $\Gamma_1$ . We introduce the sequence of points of  $\Gamma_1$

$$y_{1,k} = 1 + l\rho^{-k} e^{i(\pi-\mu)}, \quad k \geq 0 \quad (7.0.6)$$

which divide  $\Gamma_1$  into a sequence of segments  $\gamma_{1,k} = [y_{1,k}, y_{1,k+1}]$ , that is

$$\gamma_{1,k} = \{z \in \Gamma_1 : l\rho^{-k-1} \leq |1 - z| \leq l\rho^{-k}\}, \quad k \geq 0,$$

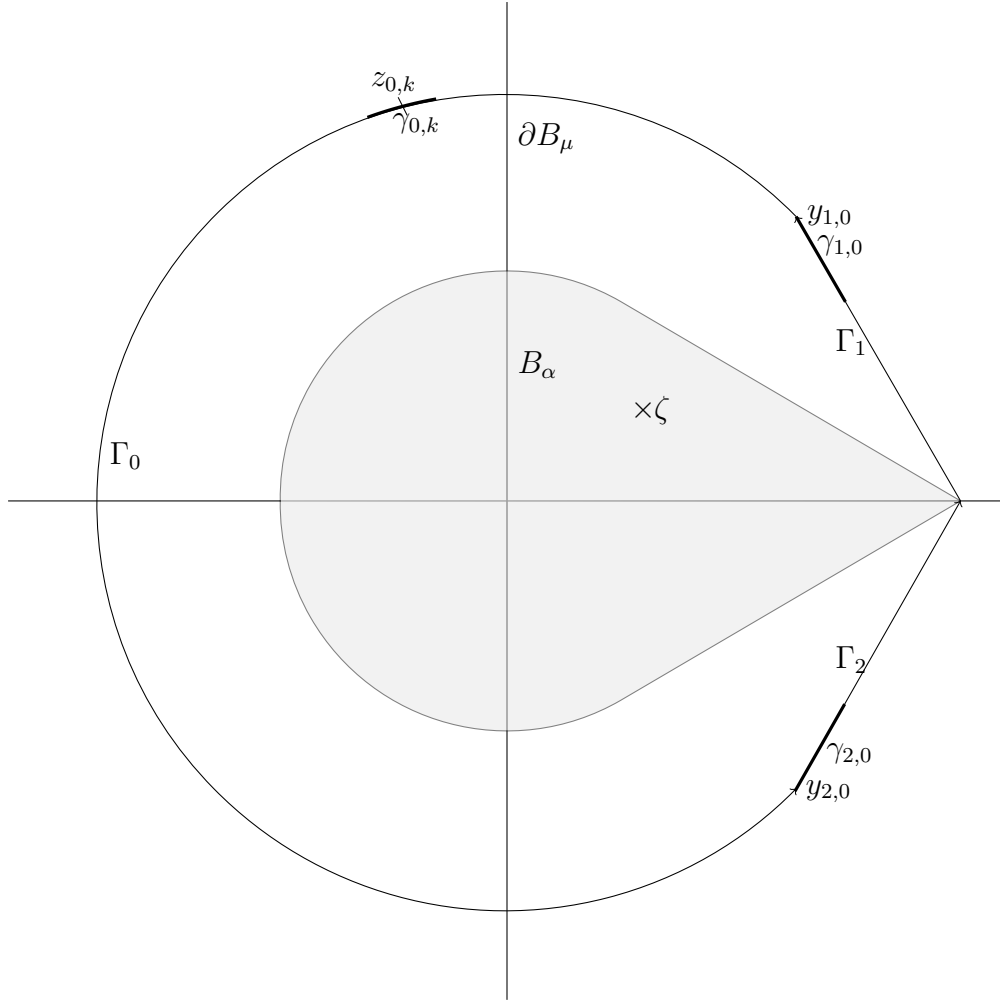
for some  $\rho > 1$  which will be chosen below. Let  $z_{1,k}$  be the center of  $\gamma_{1,k}$  and let  $D_{1,k}$  be the open ball centered at  $z_{1,k}$  with radius

$$s_k = l(\rho^{-k} - \rho^{-k-1}). \quad (7.0.7)$$

We choose  $\rho$  such that for every  $k \geq 0$ , the closure of  $D_{1,k}$  does not intersect  $\partial B_\alpha$ .

We divide  $\Gamma_2$  in the same manner by setting, for any  $k \geq 0$ ,

$$y_{2,k} = \overline{y_{1,k}}, \quad \gamma_{2,k} = \{\bar{z} : z \in \gamma_{1,k}\}, \quad z_{2,k} = \overline{z_{1,k}} \quad \text{and} \quad D_{2,k} = \{\bar{z} : z \in D_{1,k}\}. \quad (7.0.8)$$



For any  $\zeta$  in  $B_\alpha$  and any  $z$  in the union of  $\cup_{k=0}^N D_{0,k}$ ,  $\cup_{k=0}^\infty D_{1,k}$  and  $\cup_{k=0}^\infty D_{2,k}$ , we let

$$K(z, \zeta) = \frac{(1-z)^{\frac{1}{2}}(1-\zeta)^{\frac{1}{2}}}{z-\zeta}.$$

For  $z, \zeta$  as above, elementary computations yield estimates

$$|1-\zeta| \lesssim |z-\zeta| \quad \text{and} \quad |1-z| \lesssim |z-\zeta|. \quad (7.0.9)$$

We derive that for  $m = 1, 2$  and for any  $r \in \mathbb{N}$ , we have estimates

$$\sup \{|K(z, \zeta)| : z \in D_{m,k}, \zeta \in B_\alpha, l\rho^{-r-1} \leq |1-\zeta| \leq l\rho^{-r}\} \lesssim \rho^{-\frac{|k-r|}{2}}. \quad (7.0.10)$$

Indeed for  $z, \zeta$  as above, we have  $|1-z| \lesssim \rho^{-k}$ ,  $|1-\zeta| \lesssim \rho^{-r}$  and by (7.0.9), we have  $\rho^{-\min(k,r)} \lesssim |z-\zeta|$ . These three estimates yield (7.0.10).

It readily follows from the above definitions that for  $m = 1, 2$  and  $k \geq 0$ , we have

$$\int_{\gamma_{m,k}} \left| \frac{dz}{1-z} \right| = \log(\rho). \quad (7.0.11)$$

For  $m = 1, 2$  and  $k \geq 0$ , we let  $\{e_{m,k,j}\}_{j=0}^{\infty}$  be an orthonormal family of  $L^2(\gamma_{m,k}, \left| \frac{dz}{1-z} \right|)$  such that for any  $n \in \mathbb{N}$ ,  $\text{Span}\{e_{m,k,0}, \dots, e_{m,k,n}\}$  is equal to the subspace of polynomial functions with degree less than or equal to  $n$ . Likewise, for  $0 \leq k \leq N$ , we let  $\{e_{0,k,j}\}_{j=0}^{\infty}$  be an orthonormal family of  $L^2(\gamma_{0,k}, \left| \frac{dz}{z} \right|)$  such that for any  $n \in \mathbb{N}$ ,  $\text{Span}\{e_{0,k,0}, \dots, e_{0,k,n}\}$  is equal to the subspace of polynomial functions with degree less than or equal to  $n$ .

Next for any  $m \in \{0, 1, 2\}$  and any  $k \geq 0$  (with the convention that  $k \leq N$  if  $m = 0$ ), we define  $\Phi_{m,k,j}: B_{\alpha} \rightarrow \mathbb{C}$  by

$$\Phi_{m,k,j}(\zeta) = \frac{1}{2\pi i} \int_{\gamma_{m,k}} \overline{e_{m,k,j}(z)} K(z, \zeta) \frac{dz}{1-z}, \quad \zeta \in B_{\alpha}.$$

These functions are well defined holomorphic functions. Moreover, we prove that they all belong to  $H_0^{\infty}(B_{\alpha})$  with a special estimate.

**Proposition 7.0.4** *For every function  $\Phi_{m,k,j}$ ,  $m \in \{0, 1, 2\}$ ,  $k \geq 0$ ,  $j \geq 1$ , there exists a constant  $C_{m,k} > 0$  such that*

$$|\Phi_{m,k,j}(\zeta)| \leq C_{m,k} |1 - \zeta|^{\frac{1}{2}}, \quad \zeta \in B_{\alpha}. \quad (7.0.12)$$

Moreover, there exists a constant  $C > 0$  (which does not depend on  $(m, k, j)$ ) such that

$$|\Phi_{0,k,j}(\zeta)| \leq C |1 - \zeta|^{\frac{1}{2}}, \quad \zeta \in B_{\alpha}. \quad (7.0.13)$$

*Proof of Proposition 7.0.4.* Indeed according to the definition of  $K$  and the Cauchy-Schwarz inequality, we have

$$|\Phi_{m,k,j}(\zeta)| \leq \frac{1}{2\pi} \left( \int_{\gamma_{m,k}} |K(z, \zeta)|^2 \frac{|dz|}{|1-z|} \right)^{\frac{1}{2}} \leq \frac{|1 - \zeta|^{\frac{1}{2}}}{2\pi} \left( \int_{\gamma_{m,k}} \frac{|dz|}{|\zeta - z|^2} \right)^{\frac{1}{2}} \leq C_{m,k} |1 - \zeta|^{\frac{1}{2}},$$

since  $|z - \zeta| \geq \text{dist}(B_{\alpha}, \gamma_{m,k}) > 0$  and we obtain (7.0.12). The underlying constant  $C_{m,k,j}$  above only depends on  $k$  and the distance  $\text{dist}(B_{\alpha}, \gamma_{m,k})$ . As we have a finite number of indexes  $k$  for  $m = 0$  (which come from the finite division of  $\Gamma_0$ ) and that  $\text{dist}(B_{\alpha}, \gamma_{0,k}) > d > 0$  for every  $k \geq 0$ , one can choose constant  $C_{0,k}$  independent of  $k$ , which give (7.0.13).  $\square$

A more precise estimate is now given to compute series of the  $\Phi_{m,k,j}$ 's.

**Lemma 7.0.5** *There exists a constant  $c > 0$  such that if  $\zeta \in B_{\alpha}$  satisfies*

$$l\rho^{-r-1} \leq |1 - \zeta| \leq l\rho^{-r} \quad (7.0.14)$$

for some  $r \in \mathbb{N}$ , then for any  $k \geq 0$ ,  $j \geq 1$  and  $m = 1, 2$ , we have

$$|\Phi_{0,k,j}(\zeta)| \leq c 2^{-j} \quad \text{and} \quad |\Phi_{m,k,j}(\zeta)| \leq c 2^{-j} \rho^{-\frac{|k-r|}{2}}. \quad (7.0.15)$$

*Proof of Lemma 7.0.5.* We start proving the second estimate. Let  $m \in \{1, 2\}$  and  $k \geq 0$ . For any fixed  $\zeta \in B_\alpha$ , the restriction of  $K(\cdot, \zeta)$  to  $D_{m,k}$  is analytic. Recall (7.0.7) and consider the normalised power series expansion,

$$K(z, \zeta) = \sum_{n=0}^{\infty} b_{m,k,n} \left( \frac{z - z_{m,k}}{s_k} \right)^n.$$

Assume the estimate (7.0.14). Then according to (7.0.10), we have

$$\sup \{ |K(z, \zeta)| : z \in D_{m,k} \} \lesssim \rho^{-\frac{|k-r|}{2}} \quad (7.0.16)$$

Using Bessel-Parseval in  $H = L^2 \left( \partial D_{m,k}, \frac{|dz|}{2\pi s_k} \right)$  and (7.0.16), one obtains

$$\left( \sum_{n=0}^{\infty} |b_{m,k,n}|^2 \right)^{\frac{1}{2}} = \|K(\cdot, \zeta)\|_H = \left( \int_{\partial D_{m,k}} |K(z, \zeta)|^2 \frac{|dz|}{2\pi s_k} \right)^{\frac{1}{2}} \lesssim \rho^{-\frac{|k-r|}{2}}. \quad (7.0.17)$$

By construction,  $\gamma_{m,k}$  is included in the ball centered at  $z_{m,k}$  with radius  $\frac{s_k}{2}$ . Hence for any  $z \in \gamma_{m,k}$  and any integer  $N \geq 0$ , we have

$$\begin{aligned} \sum_{n=j}^{\infty} \left| b_{m,k,n} \left( \frac{z - z_{m,k}}{s_k} \right)^n \right| &\leq \left( \sum_{n=j}^{\infty} |b_{m,k,n}|^2 \right)^{\frac{1}{2}} \left( \sum_{n=j}^{\infty} \left| \frac{z - z_{m,k}}{s_k} \right|^{2n} \right)^{\frac{1}{2}} \\ &\leq \left( \sum_{n=j}^{\infty} |b_{m,k,n}|^2 \right)^{\frac{1}{2}} \left( \sum_{n=j}^{\infty} 4^{-n} \right)^{\frac{1}{2}} \lesssim \rho^{-\frac{|k-r|}{2}} 2^{-j}. \end{aligned}$$

Now recall that in  $L^2(\gamma_{m,k}, \left| \frac{dz}{1-z} \right|)$ ,  $e_{m,k,j}$  is orthogonal to every polynomial function with degree  $< j$ , hence orthogonal to  $(z - z_{m,k})^n$  for any  $n < j$ . Further  $\frac{dz}{1-z}$  is the opposite of  $\left| \frac{dz}{1-z} \right|$ . This implies that

$$\begin{aligned} |\Phi_{m,k,j}(\zeta)| &= \left| \frac{1}{2\pi i} \int_{\gamma_{m,k}} \overline{e_{m,k,j}(z)} \sum_{n=j}^{\infty} b_{m,k,n} \left( \frac{z - z_{m,k}}{s_k} \right)^n \frac{dz}{1-z} \right| \\ &\lesssim \rho^{-\frac{|k-r|}{2}} 2^{-j} \int_{\gamma_{m,k}} |e_{m,k,j}(z)| \left| \frac{dz}{1-z} \right|. \end{aligned}$$

Applying (7.0.11), we deduce the second estimate in (7.0.15).

The proof of the first estimate is similar, using the fact that on each  $\gamma_{0,k}$ ,  $\frac{dz}{z}$  is proportional to  $\left| \frac{dz}{z} \right|$ , and replacing (7.0.16) by the observation that the set

$$\left\{ \frac{zK(z, \zeta)}{1-z} : z \in \bigcup_{k=0}^N D_{0,k}, \zeta \in B_\alpha \right\}$$

is bounded.  $\square$

*Proof of Proposition 7.0.2.* Lemma 7.0.5 implies that for any  $p > 0$  and  $m \in \{0, 1, 2\}$ ,

$$\sup \left\{ \sum_{k,j=0}^{\infty} |\Phi_{m,k,j}(\zeta)|^p : \zeta \in B_\alpha \right\} < \infty. \quad (7.0.18)$$

Let  $h \in H^\infty(B_\beta)$ . By Cauchy's formula,

$$h(\zeta) = \frac{1}{2\pi i} \int_{\partial B_\mu} h(z) K(z, \zeta) \frac{dz}{1-z}, \quad \zeta \in B_\alpha. \quad (7.0.19)$$

For  $m = 1, 2$ ,  $k \geq 0$  and  $j \geq 0$ , set

$$a_{m,k,j} = \int_{\gamma_{m,k}} h(z) e_{m,k,j}(z) \left| \frac{dz}{1-z} \right|. \quad (7.0.20)$$

Likewise, for  $0 \leq k \leq N$  and  $j \geq 0$ , set

$$a_{0,k,j} = \int_{\gamma_{0,k}} h(z) e_{0,k,j}(z) \left| \frac{dz}{z} \right|. \quad (7.0.21)$$

By the Cauchy-Schwarz inequality and (7.0.11), we have a uniform estimate

$$|a_{m,k,j}| \lesssim \|h\|_{\infty, B_\beta}, \quad (7.0.22)$$

for  $m \in \{0, 1, 2\}$ ,  $k \geq 0$  and  $j \geq 0$ .

For  $m = 1, 2$  and  $k \geq 0$ , let  $H_{m,k}$  denote the subspace of all polynomial functions of  $L^2(\gamma_{m,k}, \left| \frac{dz}{1-z} \right|)$ . This is a dense subspace. Hence we have a series expansion

$$h|_{\gamma_{m,k}} = \sum_{j=0}^{\infty} a_{m,k,j} \overline{e_{m,k,j}} \quad (7.0.23)$$

in the latter space.

Likewise, for  $0 \leq k \leq N$ , let  $H_{0,k}$  denote the subspace of all polynomial functions of  $L^2(\gamma_{0,k}, \left| \frac{dz}{z} \right|)$ . This is no longer a dense subspace. However by Runge's approximation Theorem (see e.g. [71, Theorem 13.8]), every holomorphic function on an open neighborhood of  $\gamma_{0,k}$  is uniformly approximated by polynomials, hence belongs to  $\overline{H_{0,k}}^{\|\cdot\|_2}$ . This implies that the series expansion (7.0.23) holds true as well in this case.

From (7.0.19), we can write  $h(\zeta) = h_0(\zeta) + h_1(\zeta) + h_2(\zeta)$  for any  $\zeta \in B_\alpha$ , where

$$h_m(\zeta) = \frac{1}{2\pi i} \int_{\Gamma_m} h(z) K(z, \zeta) \frac{dz}{1-z}.$$

for each  $m = 0, 1, 2$ . The  $L^2$ -convergence in (7.0.23) a fortiori holds in the  $L^1$ -sense, hence

$$\begin{aligned} h_m(\zeta) &= \frac{1}{2\pi i} \sum_{k=0}^{\infty} \int_{\gamma_{m,k}} h(z) K(z, \zeta) \frac{dz}{1-z} \\ &= \frac{1}{2\pi i} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} a_{m,k,j} \int_{\gamma_{m,k}} \overline{e_{m,k,j}(z)} K(z, \zeta) \frac{dz}{1-z}, \end{aligned}$$

and hence

$$h_m(\zeta) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} a_{m,k,j} \Phi_{m,k,j}(z). \quad (7.0.24)$$

Finally, we have for any  $\zeta$  in  $B_\alpha$  the equality

$$h(\zeta) = \sum_{m \in \{0,1,2\}} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} a_{m,k,j} \Phi_{m,k,j}(z).$$

After a suitable reindexing, we obtain the result by combining (7.0.22), (7.0.24) and (7.0.18).  $\square$

*Proof of Theorem 7.0.1.* The case  $d = 1$  was settled at the end of Remark 7.0.3.

Assume that  $d = 2$ . Let  $h \in H^\infty(B_{\beta_1} \times B_{\beta_2})$ . Let  $(\Phi_{2,i})_{i \geq 1}$  be the sequence of  $H_0^\infty(B_{\alpha_2})$  obtained by applying Proposition 7.0.2 to the couple  $(\alpha_2, \beta_2)$ . For any  $\zeta_1 \in B_{\beta_1}$ , the one variable function  $h(\zeta_1, \cdot)$  belongs to  $H^\infty(B_{\beta_2})$ . Hence we have a decomposition

$$h(\zeta_1, \zeta_2) = \sum_{i=1}^{\infty} a_i(\zeta_1) \Phi_{2,i}(\zeta_2), \quad \zeta_1 \in B_{\beta_1}, \zeta_2 \in B_{\alpha_2},$$

with a uniform estimate  $|a_i(\zeta_1)| \leq C_2 \|h\|_{\infty, B_{\beta_1} \times B_{\beta_2}}$ . Recall from the proof of Proposition 7.0.2 that the complex numbers  $a_i(\zeta_1)$  are defined by (7.0.20) and (7.0.21). This implies that each  $a_i: B_{\beta_1} \rightarrow \mathbb{C}$  is a holomorphic function. Further the above estimates show that for any  $i \geq 1$ ,  $a_i \in H^\infty(B_{\beta_1})$  with  $\|a_i\|_{\infty, B_{\beta_1}} \leq C_2 \|h\|_{\infty, B_{\beta_1} \times B_{\beta_2}}$ .

Let  $(\Phi_{1,i})_{i \geq 1}$  be the sequence of  $H_0^\infty(B_{\alpha_1})$  obtained by applying Proposition 7.0.2 to the couple  $(\alpha_1, \beta_1)$ . Applying the latter to each  $a_i$ , we deduce the existence of a family  $(a_{ij})_{i,j \geq 1}$  of complex numbers such that

$$|a_{ij}| \leq C_1 C_2 \|h\|_{\infty, B_{\beta_1} \times B_{\beta_2}}, \quad i, j \geq 1,$$

for some constant  $C_1 > 0$ , and

$$a_i(\zeta_1) = \sum_{j=1}^{\infty} a_{ij} \Phi_{1,j}(\zeta_1), \quad \zeta_1 \in B_{\alpha_1}, \quad i \geq 1.$$

Since  $\sum_j |\Phi_{1,j}(\zeta_1)| < \infty$  and  $\sum_i |\Phi_{2,i}(\zeta_2)| < \infty$  for any  $(\zeta_1, \zeta_2) \in B_{\alpha_1} \times B_{\alpha_2}$ , we deduce from above that

$$h(\zeta_1, \zeta_2) = \sum_{i,j=1}^{\infty} a_{ij} \Phi_{1,j}(\zeta_1) \Phi_{2,i}(\zeta_2), \quad (\zeta_1, \zeta_2) \in B_{\alpha_1} \times B_{\alpha_2}.$$

Now using Remark 7.0.3 as in the case  $d = 1$ , we deduce the result in the case  $d = 2$ .

The general case is obtained by iterating this process.  $\square$

We end this chapter with a useful result for the sequel, which is an analogue of Proposition 6.0.4. We denote by  $H_0^\infty(\mathbb{D})$  the algebra of all holomorphic functions on  $\mathbb{D}$  such that there exists positive constant  $c > 0, s > 0$  for which

$$|h(\zeta)| \leq c |1 - \zeta|^s, \quad \zeta \in \mathbb{D}.$$

**Theorem 7.0.6** *Let  $\alpha$  in  $(0, \frac{\pi}{2})$ . There exist sequences  $(\phi_i)_{i \geq 1}$ ,  $(\varphi_i)_{i \geq 1}$ ,  $(\theta_i)_{i \geq 1}$  and  $(\psi_i)_{i \geq 1}$  of  $H_0^\infty(B_\alpha)$  such that*

(i) *For every  $i \geq 1$ , we have  $\phi_i = \theta_i \varphi_i \psi_i$ ;*

(ii) *There exists a constant  $c > 0$  such that for every  $z$  in  $B_\alpha$*

$$\sum_{i=1}^{\infty} |\varphi_i(z)| \leq c, \quad \sum_{i=1}^{\infty} |\psi_i(z)| \leq c;$$

(iii) *For every  $\gamma \in (0, \alpha)$ , there exists a constant  $e > 0$  such that for every  $i \geq 1$*

$$\int_{\partial B_\gamma} \frac{|\theta_i(z)|}{|1-z|} |dz| \leq e;$$

(iv) *For every  $z$  in  $B_\alpha$ , the series  $\sum_{i \geq 1} \phi_i(z)$  absolutely converges and there exists a constant  $c'$  such that*

$$\sup \left\{ \sum_{i=1}^{\infty} |\phi_i(z)| : z \in B_\alpha \right\} \leq c'.$$

Moreover, we have

$$\sum_{i=1}^{\infty} \phi_i(z) = 1, \quad z \in B_\alpha.$$

This theorem is a straightforward consequence of Proposition 6.0.4 taking angle  $\mu_k = \frac{\pi}{2}$  and changing variable  $z_k$  by  $1 - z_k$  to define functions  $\theta_{k,i_k}$ ,  $\varphi_{k,i_k}$ ,  $\psi_{k,i_k}$  of  $H_0^\infty(\mathbb{D})$ ,  $k = 1, \dots, d$ .

**Remark 7.0.7** All the decomposition results of this Part hold true for bounded holomorphic multivariate functions  $f$  defined on product of sectors or Stolz domains and taking their values in an arbitrary Banach space  $Z$  (vectorial valued functions). One can see that the proofs readily work in this more general case.

## Part III

### Some key properties of $H^\infty$ joint functional calculus





# Chapter 8

## Automaticity of joint functional calculus

Let  $(T_1, \dots, T_d)$  be a commuting family of Ritt operators or sectorial operators on some Banach space  $X$ . If this  $d$ -tuple admits an  $H^\infty$  joint functional calculus, then each  $T_k$  admits an  $H^\infty$  functional calculus (see Section 5.2).

In this part, we are interested with the Banach spaces  $X$  for which the following property holds true.

**(P)** : If  $(T_1, \dots, T_d)$  is a  $d$  tuple of commuting Ritt operators or sectorial operators such that each of the  $T_k$  admits an  $H^\infty$  functional calculus, then the  $d$ -tuple admits an  $H^\infty$  joint functional calculus.

Let us recall the known answers to problem **(P)**.

In [26, Theorem 4.3], the authors shows that any commutative  $L_p$  space admits a positive answer to the problem **(P)**. This uses the so-called Franks-McIntosh decomposition for holomorphic functions on sectors around the real axis.

Further, Lancien, Lancien and Le Merdy proved a deeper property for a larger class of Banach space, that is the spaces with property  $(\alpha)$  (see [48]) and the Banach lattices. Indeed, the authors shows that if  $A_1, A_2$  are sectorial operators acting on one of these spaces and each having an  $H^\infty$  functional calculus, then  $(A_1, A_2)$  has an  $H^\infty$  joint functional calculus.

Combining these approaches and appealing to the Franks-McIntosh decomposition on Stolz domains, we are ready to study the joint functional calculus of a  $d$ -tuple of Ritt operators or sectorial operators.

The purpose of this section is to show that **(P)** admits a positive answer if either  $X$  is a Banach lattice or  $X$  (or its dual space  $X^*$ ) has property  $(\alpha)$ .

We refer the reader to [57] for definitions and basic properties of Banach lattices.

The main result of this section is the following.

**Theorem 8.0.1** *Let  $X$  be a Banach space. Assume that either  $X$  is a Banach lattice, or  $X$  or  $X^*$  has property  $(\alpha)$ . Let  $d \geq 2$  be an integer. Then the following two properties hold :*

- (P1) Let  $(T_1, \dots, T_d)$  be a commuting  $d$ -tuple of Ritt operators on  $X$  and assume that for some  $0 < \gamma_1, \dots, \gamma_d < \frac{\pi}{2}$ ,  $T_k$  has an  $H^\infty(B_{\gamma_k})$  functional calculus for any  $k = 1, \dots, d$ . Then for any  $\gamma'_k \in (\gamma_k, \frac{\pi}{2})$ ,  $k = 1, \dots, d$ ,  $(T_1, \dots, T_d)$  admits an  $H^\infty(B_{\gamma'_1} \times \dots \times B_{\gamma'_d})$  joint functional calculus.
- (P2) Let  $(A_1, \dots, A_d)$  be a commuting  $d$ -tuple of sectorial operators on  $X$  and assume that for some  $0 < \theta_1, \dots, \theta_d < \pi$ ,  $A_k$  has an  $H^\infty(\Sigma_{\theta_k})$  functional calculus for any  $k = 1, \dots, d$ . Then for any  $\theta'_k \in (\theta_k, \pi)$ ,  $k = 1, \dots, d$ ,  $(A_1, \dots, A_d)$  admits an  $H^\infty(\Sigma_{\theta'_1} \times \dots \times \Sigma_{\theta'_d})$  joint functional calculus.

Proving property (P1) will require the Franks-McIntosh decomposition presented in Chapters 6 and 7.

To proceed we need more ingredients on Rademacher averages. Let  $d \geq 1$  be an integer.

We denote by  $\text{Rad}^d(X)$  the closure in  $L^2(\Omega_0^d; X)$  of the space of all elements of the form

$$\sum_{1 \leq i_1, \dots, i_d \leq n} r_{i_1} \otimes \dots \otimes r_{i_d} \otimes x_{i_1, \dots, i_d}, \quad n \in \mathbb{N}^*, \quad x_{i_1, \dots, i_d} \in X.$$

Clearly we can rewrite this space as

$$\text{Rad}^d(X) = \underbrace{\text{Rad}(\text{Rad}(\dots \text{Rad}(X) \dots))}_{d \text{ times}}. \quad (8.0.1)$$

For convenience we set

$$N_d([x_{i_1, \dots, i_d}]) = \left\| \sum_{1 \leq i_1, \dots, i_d \leq n} r_{i_1} \otimes \dots \otimes r_{i_d} \otimes x_{i_1, \dots, i_d} \right\|_{\text{Rad}^d(X)} \quad (8.0.2)$$

for any family  $(x_{i_1, \dots, i_d})_{1 \leq i_1, \dots, i_d \leq n}$  in  $X$ .

We will say that  $X$  satisfies property  $(A_d)$  if there exists a constant  $C > 0$  such that for any integer  $n \geq 1$ , for any family of complex numbers  $(a_{i_1, \dots, i_d})_{1 \leq i_1, \dots, i_d \leq n}$  and for any families  $(x_{i_1, \dots, i_d})_{1 \leq i_1, \dots, i_d \leq n}$  in  $X$  and  $(x_{i_1, \dots, i_d}^*)_{1 \leq i_1, \dots, i_d \leq n}$  in  $X^*$ , we have

$$\left| \sum_{i_1, \dots, i_d} a_{i_1, \dots, i_d} \langle x_{i_1, \dots, i_d}^*, x_{i_1, \dots, i_d} \rangle \right| \leq C \sup_{i_1, \dots, i_d} \{|a_{i_1, \dots, i_d}|\} N_d([x_{i_1, \dots, i_d}]) N_d([x_{i_1, \dots, i_d}^*]). \quad (8.0.3)$$

Theorem 8.0.1 is a straightforward consequence of the next three propositions, that will be proved in the rest of this section.

**Proposition 8.0.2** *If  $X$  satisfies property  $(A_d)$  for some integer  $d \geq 2$ , then (P1) and (P2) hold true.*

**Proposition 8.0.3** *Every Banach lattice satisfies property  $(A_d)$  for every integer  $d \geq 2$ .*

**Proposition 8.0.4** *If  $X$  or  $X^*$  has property  $(\alpha)$ , then  $X$  satisfies property  $(A_d)$  for every integer  $d \geq 2$ .*

*Proof of Proposition 8.0.2.* Assume that  $X$  satisfies property  $(A_d)$  for some  $d \geq 2$ . We only prove **(P1)**, the proof of **(P2)** being similar. We consider commuting Ritt operators  $T_1, \dots, T_d$  such that, for every  $k = 1, \dots, d$ ,  $T_k$  has a bounded  $H^\infty(B_{\gamma_k})$  functional calculus. Let  $\gamma'_k$  in  $(\gamma_k, \frac{\pi}{2})$ . By Section 5.2, and a simple induction argument, it suffices to have an estimate  $\|h(T_1, \dots, T_d)\| \lesssim \|h\|_{\infty, B_{\gamma'_1} \times \dots \times B_{\gamma'_d}}$  for functions  $h$  in  $H_0^\infty(B_{\gamma'_1} \times \dots \times B_{\gamma'_d})$ .

For  $h \in H_0^\infty(B_{\gamma'_1} \times \dots \times B_{\gamma'_d})$ , we consider the Franks-McIntosh decomposition given by Theorem 7.0.1. According to this statement we may write, for every  $(\zeta_1, \dots, \zeta_d) \in \prod_{k=1}^d B_{\gamma_k}$ ,

$$h(\zeta_1, \dots, \zeta_d) = \sum_{(i_1, \dots, i_d) \in \mathbb{N}^{*d}} a_{i_1, \dots, i_d} \Psi_{1, i_1}(\zeta_1) \tilde{\Psi}_{1, i_1}(\zeta_1) \cdots \Psi_{d, i_d}(\zeta_d) \tilde{\Psi}_{d, i_d}(\zeta_d), \quad (8.0.4)$$

where  $(a_{i_1, \dots, i_d})$  is a family of complex numbers satisfying an estimate

$$|a_{i_1, \dots, i_d}| \lesssim \|h\|_{\infty, B_{\gamma'_1} \times \dots \times B_{\gamma'_d}}, \quad (i_1, \dots, i_d) \in \mathbb{N}^{*d}, \quad (8.0.5)$$

the functions  $\Psi_{k, i_k}$  and  $\tilde{\Psi}_{k, i_k}$  belong to  $H_0^\infty(B_{\gamma_k})$  and they satisfy inequalities

$$\sup \left\{ \sum_{i_k=1}^{\infty} |\Psi_{k, i_k}(\zeta_k)| : \zeta_k \in B_{\gamma_k} \right\} \leq C \quad \text{and} \quad \sup \left\{ \sum_{i_k=1}^{\infty} |\tilde{\Psi}_{k, i_k}(\zeta_k)| : \zeta_k \in B_{\gamma_k} \right\} \leq C \quad (8.0.6)$$

for every  $k = 1, \dots, d$ , and for a constant  $C > 0$  not depending on  $h$ .

We consider the partial sums in (8.0.4), defined for every  $n \geq 1$  and every  $(\zeta_1, \dots, \zeta_d)$  in  $\prod_{k=1}^d B_{\gamma_k}$  by

$$h_n(\zeta_1, \dots, \zeta_d) = \sum_{1 \leq i_1, \dots, i_d \leq n} a_{i_1, \dots, i_d} \Psi_{1, i_1}(\zeta_1) \tilde{\Psi}_{1, i_1}(\zeta_1) \cdots \Psi_{d, i_d}(\zeta_d) \tilde{\Psi}_{d, i_d}(\zeta_d).$$

The functions  $\Psi_{k, i_k}$  and  $\tilde{\Psi}_{k, i_k}$  both belong to  $H_0^\infty(B_{\gamma_k})$  hence this implies

$$h_n(T_1, \dots, T_d) = \sum_{1 \leq i_1, \dots, i_d \leq n} a_{i_1, \dots, i_d} \Psi_{1, i_1}(T_1) \tilde{\Psi}_{1, i_1}(T_1) \cdots \Psi_{d, i_d}(T_d) \tilde{\Psi}_{d, i_d}(T_d). \quad (8.0.7)$$

Let us prove the existence of a constant  $K > 0$ , not depending either on  $n$  or  $h$ , such that

$$\|h_n(T_1, \dots, T_d)\| \leq K \|h\|_{\infty, B_{\gamma'_1} \times \dots \times B_{\gamma'_d}}. \quad (8.0.8)$$

We let  $x \in X$  and  $x^* \in X^*$ . Applying (8.0.7), we write

$$\langle x^*, h_n(T_1, \dots, T_d)x \rangle = \sum_{1 \leq i_1, \dots, i_d \leq n} a_{i_1, \dots, i_d} \langle \tilde{\Psi}_{1, i_1}(T_1)^* \cdots \tilde{\Psi}_{d, i_d}(T_d)^* x^*, \Psi_{1, i_1}(T_1) \cdots \Psi_{d, i_d}(T_d)x \rangle.$$

We let

$$x_{i_1, \dots, i_d} = \Psi_{1, i_1}(T_1) \cdots \Psi_{d, i_d}(T_d)x \quad \text{and} \quad x_{i_1, \dots, i_d}^* = \tilde{\Psi}_{1, i_1}(T_1)^* \cdots \tilde{\Psi}_{d, i_d}(T_d)^* x^*.$$

Using property  $(A_d)$  and the estimate (8.0.5), we have

$$|\langle x^*, h_n(T_1, \dots, T_d)x \rangle| \lesssim \|h\|_{\infty, B_{\gamma'_1} \times \dots \times B_{\gamma'_d}} N_d([x_{i_1, \dots, i_d}]) N_d([x_{i_1, \dots, i_d}^*])$$

Recall that  $(r_i)_{i \geq 1}$  is a sequence of Rademacher independent variables on  $\Omega_0$ . Let us momentarily fix some  $(t_1, \dots, t_d)$  in  $\Omega_0^d$ . By (8.0.6) and the  $H^\infty(B_{\gamma_k})$  functional calculus property of  $T_k$  for all  $k = 1, \dots, d$ , we have estimates

$$\begin{aligned} \left\| \sum_{1 \leq i_1, \dots, i_d \leq n} r_{i_1}(t_1) \cdots r_{i_d}(t_d) \Psi_{1, i_1}(T_1) \cdots \Psi_{d, i_d}(T_d)x \right\| &\leq \prod_{k=1}^d \left( \left\| \sum_{i_k=1}^n r_{i_k}(t_k) \Psi_{k, i_k}(T_k) \right\| \right) \|x\| \\ &\lesssim \prod_{k=1}^d \left( \left\| \sum_{i_k=1}^n r_{i_k}(t_k) \Psi_{k, i_k} \right\|_{\infty, B_{\gamma_k}} \right) \|x\| \\ &\lesssim \prod_{k=1}^d \left( \left\| \sum_{i_k=1}^n |\Psi_{k, i_k}| \right\|_{\infty, B_{\gamma_k}} \right) \|x\| \\ &\lesssim \|x\|. \end{aligned}$$

Now taking the average on  $(t_1, \dots, t_d)$ , we deduce that

$$N_d([x_{i_1, \dots, i_d}]) \lesssim \|x\|.$$

The same method yields a similar estimate  $N_d([x_{i_1, \dots, i_d}^*]) \lesssim \|x^*\|$ . We obtain that

$$|\langle x^*, h_n(T_1, \dots, T_d)x \rangle| \lesssim \|h\|_{\infty, B_{\gamma'_1} \times \dots \times B_{\gamma'_d}} \|x\| \|x^*\|.$$

Next the Hahn-Banach Theorem yields the inequality (8.0.8).

The same estimate holds true when  $(T_1, \dots, T_d)$  is replaced by  $(rT_1, \dots, rT_d)$  for any  $r \in (0, 1)$ . Further the above argument also shows that  $(h_n)_{n \geq 1}$  is a bounded sequence of the space  $H_0^\infty(B_{\gamma'_1} \times \dots \times B_{\gamma'_d})$ . Moreover, the sequence  $(h_n)_{n \geq 1}$  converges pointwise to  $h$ . Hence applying Lebesgue's dominated convergence Theorem twice we have

$$\lim_{n \rightarrow \infty} h_n(rT_1, \dots, rT_n) = h(rT_1, \dots, rT_n)$$

for any  $r \in (0, 1)$  and

$$\lim_{r \rightarrow 1^-} h(rT_1, \dots, rT_n) = h(T_1, \dots, T_n).$$

We therefore deduce from (8.0.8) that

$$\|h(T_1, \dots, T_d)\| \leq K \|h\|_{\infty, B_{\gamma'_1} \times \dots \times B_{\gamma'_d}},$$

which concludes the proof.  $\square$

*Proof of Proposition 8.0.3.* Let  $X$  be a Banach lattice and let  $d \geq 2$  be an integer. For any integer  $n \geq 1$ , for any family of complex numbers  $(a_{i_1, \dots, i_d})_{1 \leq i_1, \dots, i_d \leq n}$  and for any families  $(x_{i_1, \dots, i_d})_{1 \leq i_1, \dots, i_d \leq n}$  in  $X$  and  $(x_{i_1, \dots, i_d}^*)_{1 \leq i_1, \dots, i_d \leq n}$  in  $X^*$ , we have

$$\begin{aligned} & \left| \sum_{1 \leq i_1, \dots, i_d \leq n} a_{i_1, \dots, i_d} \langle x_{i_1, \dots, i_d}^*, x_{i_1, \dots, i_d} \rangle \right| \\ & \leq \sup_{1 \leq i_1, \dots, i_d \leq n} \{|a_{i_1, \dots, i_d}|\} \left\| \left( \sum_{1 \leq i_1, \dots, i_d \leq n} |x_{i_1, \dots, i_d}|^2 \right)^{\frac{1}{2}} \right\|_X \left\| \left( \sum_{1 \leq i_1, \dots, i_d \leq n} |x_{i_1, \dots, i_d}^*|^2 \right)^{\frac{1}{2}} \right\|_{X^*}, \end{aligned}$$

where  $\left( \sum_{1 \leq i_1, \dots, i_d \leq n} |x_{i_1, \dots, i_d}|^2 \right)^{\frac{1}{2}}$  and  $\left( \sum_{1 \leq i_1, \dots, i_d \leq n} |x_{i_1, \dots, i_d}^*|^2 \right)^{\frac{1}{2}}$  are defined in [57, Section 1.d]. This follows from basic properties of Krivine's functional calculus on Banach lattices.

By the  $d$ -variable Khintchine inequality, there exists a constant  $C > 0$  (not depending on the  $x_{i_1, \dots, i_d}$ ) such that we have an inequality

$$\left( \sum_{1 \leq i_1, \dots, i_d \leq n} |x_{i_1, \dots, i_d}|^2 \right)^{\frac{1}{2}} \leq C \int_{\Omega_0^d} \left| \sum_{1 \leq i_1, \dots, i_d \leq n} r_{i_1}(t_1) \dots r_{i_d}(t_d) x_{i_1, \dots, i_d} \right| d\mathbb{P}^d(t_1, \dots, t_d)$$

in  $X$ . By the triangle inequality and recalling (8.0.2), this implies that

$$\left\| \left( \sum_{1 \leq i_1, \dots, i_d \leq n} |x_{i_1, \dots, i_d}|^2 \right)^{\frac{1}{2}} \right\|_X \leq CN_d([x_{i_1, \dots, i_d}]).$$

Likewise, we have

$$\left\| \left( \sum_{1 \leq i_1, \dots, i_d \leq n} |x_{i_1, \dots, i_d}^*|^2 \right)^{\frac{1}{2}} \right\|_{X^*} \leq CN_d([x_{i_1, \dots, i_d}^*]).$$

Combining these three estimates we obtain that  $X$  satisfies property  $(A_d)$ .  $\square$

Before giving the proof of Proposition 8.0.4, we show that any Banach space with property  $(\alpha)$  verifies a  $d$ -variable version of inequality (4.3.2).

**Lemma 8.0.5** *Let  $X$  be a Banach space with property  $(\alpha)$ . For any integer  $d \geq 2$ , there exists a constant  $C > 0$  such that for any integer  $n \geq 1$ , any family  $(a_{i_1, \dots, i_d})_{1 \leq i_1, \dots, i_d \leq n}$  of complex numbers and any family  $(x_{i_1, \dots, i_d})_{1 \leq i_1, \dots, i_d \leq n}$  in  $X$ ,*

$$N_d([a_{i_1, \dots, i_d} x_{i_1, \dots, i_d}]) \leq C \sup_{1 \leq i_1, \dots, i_d \leq n} \{|a_{i_1, \dots, i_d}|\} N_d([x_{i_1, \dots, i_d}]). \quad (8.0.9)$$

*Proof of Lemma 8.0.5.* According to [67, Remark 2.1] and results we cite in Proposition 4.3.3, property  $(\alpha)$  is equivalent to the fact that the linear mapping

$$\sum_{i,j} r_{i,j} \otimes x_{i,j} \mapsto \sum_{i,j} r_i \otimes r_j \otimes x_{i,j}$$

induces an isomorphism from  $\text{Rad}(\mathbb{N}^{*2}; X)$  onto  $\text{Rad}(\text{Rad}(X)) = \text{Rad}^2(X)$ . This readily implies that for any countable sets  $I_1, I_2$ , we have a natural isomorphism

$$\text{Rad}(I_1 \times I_2; X) \approx \text{Rad}(I_1; \text{Rad}(I_2; X))$$

when  $X$  has property  $(\alpha)$ .

Under this assumption, we thus have

$$\text{Rad}(\text{Rad}(\mathbb{N}^{*2}; X)) \approx \text{Rad}(\mathbb{N}^* \times \mathbb{N}^{*2}; X) = \text{Rad}(\mathbb{N}^{*3}; X)$$

and

$$\text{Rad}(\text{Rad}(\mathbb{N}^{*2}; X)) \approx \text{Rad}(\text{Rad}(\text{Rad}(X))) = \text{Rad}^3(X),$$

whence a natural isomorphism

$$\text{Rad}(\mathbb{N}^{*3}; X) \approx \text{Rad}^3(X).$$

Proceeding by induction, we obtain that

$$\text{Rad}(\mathbb{N}^{*d}; X) \approx \text{Rad}^d(X).$$

This means that for finite families  $(x_{i_1, \dots, i_d})$  of  $X$ ,  $N_d([x_{i_1, \dots, i_d}])$  and  $\|\sum r_{i_1, \dots, i_d} \otimes x_{i_1, \dots, i_d}\|$  are equivalent. Now recall that by (4.3.1), we have

$$\begin{aligned} & \left\| \sum_{1 \leq i_1, \dots, i_d \leq n} a_{i_1, \dots, i_d} r_{i_1, \dots, i_d} \otimes x_{i_1, \dots, i_d} \right\|_{\text{Rad}(\mathbb{N}^{*d}; X)} \\ & \leq 2 \sup_{1 \leq i_1, \dots, i_d \leq n} \{|a_{i_1, \dots, i_d}|\} \left\| \sum_{1 \leq i_1, \dots, i_d \leq n} r_{i_1, \dots, i_d} \otimes x_{i_1, \dots, i_d} \right\|_{\text{Rad}(\mathbb{N}^{*d}; X)} \end{aligned}$$

for every finite family  $(a_{i_1, \dots, i_d})$  of complex numbers. The inequality (8.0.9) follows at once.  $\square$

*Proof of Proposition 8.0.4.* Assume that  $X$  has property  $(\alpha)$ . Let  $(x_{i_1, \dots, i_d})$ ,  $(x_{i_1, \dots, i_d}^*)$  and  $(a_{i_1, \dots, i_d})$  be finite families of  $X$ ,  $X^*$  and  $\mathbb{C}$ , respectively, indexed by  $(i_1, \dots, i_d) \in \mathbb{N}^{*d}$ .

By the independence of Rademacher variables, we have

$$\begin{aligned} & \sum_{i_1, \dots, i_d} a_{i_1, \dots, i_d} \langle x_{i_1, \dots, i_d}^*, x_{i_1, \dots, i_d} \rangle = \\ & \int_{\Omega_0^d} \left\langle \sum_{i_1, \dots, i_d} r_{i_1}(t_1) \cdots r_{i_d}(t_d) x_{i_1, \dots, i_d}^*, \sum_{i_1, \dots, i_d} a_{i_1, \dots, i_d} r_{i_1}(t_1) \cdots r_{i_d}(t_d) x_{i_1, \dots, i_d} \right\rangle d\mathbb{P}^d(t_1, \dots, t_d). \end{aligned}$$

By the Cauchy-Schwarz inequality, this implies that

$$\begin{aligned} & \left| \sum_{i_1, \dots, i_d} a_{i_1, \dots, i_d} \langle x_{i_1, \dots, i_d}^*, x_{i_1, \dots, i_d} \rangle \right| \leq \left\| \sum_{i_1, \dots, i_d} r_{i_1} \otimes \cdots \otimes r_{i_d} \otimes x_{i_1, \dots, i_d}^* \right\|_{\text{Rad}^d(X^*)} \\ & \quad \times \left\| \sum_{i_1, \dots, i_d} a_{i_1, \dots, i_d} r_{i_1} \otimes \cdots \otimes r_{i_d} \otimes x_{i_1, \dots, i_d} \right\|_{\text{Rad}^d(X)}. \end{aligned}$$

By Lemma 8.0.5, we deduce an estimate

$$\left| \sum_{i_1, \dots, i_d} a_{i_1, \dots, i_d} \langle x_{i_1, \dots, i_d}^*, x_{i_1, \dots, i_d} \rangle \right| \leq C \sup \{ |a_{i_1, \dots, i_d}| \}$$

$$\times \left\| \sum_{i_1, \dots, i_d} r_{i_1} \otimes \cdots \otimes r_{i_d} \otimes x_{i_1, \dots, i_d}^* \right\|_{\text{Rad}^d(X^*)} \left\| \sum_{i_1, \dots, i_d} r_{i_1} \otimes \cdots \otimes r_{i_d} \otimes x_{i_1, \dots, i_d} \right\|_{\text{Rad}^d(X)},$$

which proves  $(A_d)$ .

The same proof holds true if  $X^*$  verifies the property  $(\alpha)$ .  $\square$

*Proof of Theorem 8.0.1.* Let  $(T_1, \dots, T_d)$  be a  $d$ -tuple of commuting Ritt operators acting on  $X$ , a Banach space. If  $X$  is a Banach lattice, then it has property  $(A_d)$  by Proposition 8.0.3. If  $X$  or  $X^*$  has property  $(\alpha)$ , then  $X$  has property  $(A_d)$  by Proposition 8.0.4. In one or the other case, the properties **(P1)** or **(P2)** hold true on  $X$ . This concludes the proof.  $\square$





# Chapter 9

## Convex combinations of Ritt operators

In this chapter, we show that any convex combination of commuting Ritt operators is a Ritt operator too. Then we give a consequence on the spaces with property  $(\Delta)$  (see Section 4.4).

### 9.1 Stability under convex combinations

**Proposition 9.1.1** *Let  $X$  be a Banach space. Let  $d \geq 2$  be an integer. Let  $T_1, \dots, T_d$  be commuting Ritt operators acting on  $X$ . Let  $c_1, \dots, c_d$  be positive real numbers such that  $\sum_{k=1}^d c_k = 1$ . Then  $\sum_{k=1}^d c_k T_k$  is a Ritt operator.*

We note that we may proceed by induction on  $d$ . Thus, the case where  $d = 2$  is the key result. It is possible to show the result obtaining one or the other characterisation (1),(2) or (3) of Proposition 5.1.3. We will give two of the three proofs.

*Proof of Proposition 9.1.1 by (1) of Proposition 5.1.3.* Let  $d = 2$  and let  $T_1 = T$ ,  $T_2 = S$  be two commuting Ritt operators. Let  $C_0 > 0, C_1 > 0$  such that

$$\|T^n\| \leq C_0, \quad \|S^n\| \leq C_0, \quad n \geq 0 \quad (9.1.1)$$

and

$$\|nT^{n-1}(I_X - T)\| \leq C_1, \quad \|nS^{n-1}(I_X - S)\| \leq C_1, \quad n \geq 1. \quad (9.1.2)$$

Let  $c$  in  $(0, 1)$  and  $n \geq 0$  an integer. We prove that  $U = cT + (1 - c)S$  is a Ritt operator.

We prove that  $(U^n)_{n \geq 0}$  is bounded in  $B(X)$ . Indeed, as  $T$  and  $S$  commute, we have

$$U^n = \sum_{k=0}^n \binom{n}{k} c^k (1 - c)^{n-k} T^k S^{n-k}. \quad (9.1.3)$$

Using (9.1.1) et (9.1.2), we have

$$\|U^n\| \leq \sum_{k=0}^n \binom{n}{k} c^k (1-c)^{n-k} C_0^2 = (1-c+c)^n C_0^2 = C_0^2,$$

which shows that  $(U^n)$  is bounded.

Next we study the sequence  $(n(I-U)U^{n-1})_{n \geq 1}$ . Writing

$$n(I_X - U)U^{n-1} = cn(I_X - T)U^{n-1} + (1-c)n(I_X - S)U^{n-1},$$

we begin by  $cn(I-T)U^{n-1}$ ,  $n \geq 1$ . Using again (9.1.3), we have

$$cn(I_X - T)U^{n-1} = cn \sum_{k=0}^{n-1} \binom{n-1}{k} c^k (1-c)^{n-1-k} (I_X - T)T^k S^{n-1-k}.$$

Thus, noting that  $\|(I_X - T)T^k\| \leq \frac{C_1}{k+1}$  we obtain the inequality

$$\|cn(I - T)U^{n-1}\| \leq cn \sum_{k=0}^{n-1} \binom{n-1}{k} c^k (1-c)^{n-1-k} \frac{C_0 C_1}{k+1}.$$

Recall that  $\frac{n}{k+1} \binom{n-1}{k} = \binom{n}{k+1}$  for  $0 \leq k \leq n-1$ . Hence we obtain

$$\begin{aligned} cn \sum_{k=0}^{n-1} \binom{n-1}{k} c^k (1-c)^{n-1-k} \frac{C_0 C_1}{k+1} &= C_0 C_1 \sum_{j=1}^n \binom{n}{j} c^j (1-c)^{n-j} \\ &= C_0 C_1 (1 - (1-c)^n) \leq C_0 C_1. \end{aligned}$$

Then we obtain that  $\|cn(I_X - T)U^{n-1}\| \leq C_0 C_1$  and similary  $\|(1-c)n(I_X - S)U^{n-1}\| \leq C_0 C_1$ , which allows us to write

$$\|n(I_X - U)U^{n-1}\| \leq 2C_0 C_1.$$

The proof is complete.  $\square$

A way to use condition (3) in Proposition 5.1.3 make use of the following Lemma (see e.g [18, Theorem 3.3]).

**Lemma 9.1.2** *Let  $A$  and  $B$  be closed and densely defined sectorial operators with respective types  $\omega_A$  and  $\omega_B$  (and domains  $D(A)$  and  $D(B)$ ). Suppose that  $A$  and  $B$  are commuting operators (which means that their resolvent commute) and that  $\omega_A + \omega_B < \pi$ . Then we have the three following properties*

- (i)  $D(A) \cap D(B)$  is dense in  $X$ .
- (ii)  $A + B : D(A) \cap D(B) \rightarrow X$  is closable in  $\overline{A + B}$ .
- (iii)  $\overline{A + B}$  is sectorial of type  $\max(\omega_A, \omega_B)$ .

*Proof of Proposition 9.1.1 by (3) of Proposition 5.1.3.* We always take  $T$  and  $S$  two commuting Ritt operators acting on some Banach space  $X$ . By [72, Theorem 11.23], we see that  $\sigma(T + S) \subseteq \sigma(T) + \sigma(S)$  where  $\sigma(T) + \sigma(S) = \{\lambda + \mu : \lambda \in \sigma(T), \mu \in \sigma(S)\}$ . Let  $c$  in  $(0, 1)$  and  $U = cT + (1 - c)S$ .

We prove that  $\sigma(cT + (1 - c)S) \subset \mathbb{D} \cup \{1\}$  and that  $I - U$  is a sectorial operator of type  $\omega < \frac{\pi}{2}$ .

Firstly, we have

$$\sigma(cT + (1 - c)S) \subset \sigma(cT) + \sigma((1 - c)S) \subset c(\mathbb{D} \cup \{1\}) + (1 - c)(\mathbb{D} \cup \{1\}) \subset \mathbb{D} \cup \{1\}$$

where we used that  $\mathbb{D} \cup \{1\}$  is a convex set.

Secondly, let us write  $I_X - (cT + (1 - c)S) = c(I_X - T) + (1 - c)(I_X - S)$ . By hypothesis, operators  $I_X - T$  et  $I_X - S$  are sectorial operators with type  $\omega_{I_X - T} < \frac{\pi}{2}$  and  $\omega_{I_X - S} < \frac{\pi}{2}$ . This holds for  $c(I_X - T)$  and  $(1 - c)(I_X - S)$ . According to Lemma 9.1.2, we know that  $c(I_X - T) + (1 - c)(I_X - S)$  is sectorial with type  $\max(\omega_{I_X - T}, \omega_{I_X - S}) < \frac{\pi}{2}$ . By (3) of Proposition 5.1.3, we conclude that  $cT + (1 - c)S$  is a Ritt operator.  $\square$

## 9.2 The special case of the spaces with property $(\Delta)$

Once we know that any convex combination of commuting Ritt operators  $T_1, \dots, T_d$  is also a Ritt operator, a natural question is whether such a convex combination admits an  $H^\infty$  functional calculus if all the  $T_k$ 's admit one. The answer is positive for Ritt operators acting on a Banach space  $X$  with property  $(\Delta)$  (see Section 4.4).

We recall first a known result for the case of sectorial operators ([51, Theorem 1.1]).

**Theorem 9.2.1** *Let  $X$  be a Banach space with property  $(\Delta)$ . Let  $A$  and  $B$  be two commuting sectorial operators acting on  $X$  and let  $\theta_1, \theta_2$  be angles of  $(0, \pi)$  such that  $\theta_1 + \theta_2 < \pi$ . Assume that  $A$  has an  $H^\infty(\Sigma_{\theta_1})$  bounded functional calculus and  $B$  has an  $H^\infty(\Sigma_{\theta_2})$  bounded functional calculus. Then  $A + B$  is closed and for any  $\pi > \theta > \max(\theta_1, \theta_2)$ ,  $A + B$  has an  $H^\infty(\Sigma_\theta)$  bounded functional calculus.*

Using the relationship between sectorial operators and Ritt operators, we deduce the following.

**Theorem 9.2.2** *Let  $X$  be a Banach space with property  $(\Delta)$  and  $d \geq 2$  be an integer. Let  $T_1, \dots, T_d$  be commuting Ritt operators acting on  $X$ . Let  $c_1, \dots, c_d$  be positive real numbers such that  $\sum_{k=1}^d c_k = 1$ . Suppose that each  $T_k$  admits an  $H^\infty$  functional calculus,  $k = 1, \dots, d$ . Then  $T = \sum_{k=1}^d c_k T_k$  admits an  $H^\infty$  functional calculus.*

*Proof of Theorem 9.2.2.* First, using Proposition 9.1.1, we know that  $T = \sum_{k=1}^d c_k T_k$  is a Ritt operator. By induction, one sees that it suffices to prove the result for  $d = 2$ . Let  $c_1, c_2 \geq 0$  such that  $c_1 + c_2 = 1$  and  $\gamma_1, \gamma_2$  be elements of  $(0, \frac{\pi}{2})$  such that  $T_k$  admits an  $H^\infty(B_{\gamma_k})$  functional calculus for  $k = 1, 2$ .

We let  $A_1 = I_X - T_1$  and  $A_2 = I_X - T_2$ , which are bounded sectorial operators of type  $\omega < \frac{\pi}{2}$  on  $X$  according to Section 5.1. As the  $T_k$ 's admit  $H^\infty(B_{\gamma_k})$  functional calculus,

$k = 1, 2$ , we can use a transfer principle ([53, Proposition 4.1]) to say that there exists  $\theta_1, \theta_2$  in  $(0, \frac{\pi}{2})$  such that  $A_k$  admits an  $H^\infty(\Sigma_{\gamma_k})$  functional calculus for  $k = 1, 2$ . Then  $c_k T_k$  admits an  $H^\infty(\Sigma_{\gamma_k})$  functional calculus for  $k = 1, 2$  and as we have  $\theta_1 + \theta_2 < \pi$ ,  $A = c_1 A_1 + c_2 A_2$  admits an  $H^\infty(\Sigma_\theta)$  functional calculus for any  $\frac{\pi}{2} > \theta > \max(\theta_1, \theta_2)$  by Theorem 9.2.2.

Using the converse property of the transfer principle ([53, Proposition 4.1]), we obtain that  $T = I_X - A = c_1 T_1 + c_2 T_2$  admits an  $H^\infty(B_\gamma)$  functional calculus for some  $\gamma$  in  $(0, \frac{\pi}{2})$ .  $\square$

**Remark 9.2.3** One can prove Theorem 9.2.2 for  $X$  having property  $(\alpha)$  or  $X$  being a Banach lattice not using Theorem 9.2.1 but appealing to Theorem 8.0.1. Indeed, taking an holomorphic function  $\varphi$  on some  $B_\gamma$ ,  $\gamma \in (0, \frac{\pi}{2})$ , one can consider function  $\phi$  on  $B_{\gamma_1} \times \cdots \times B_{\gamma_d}$ , defined by  $\phi(z_1, \dots, z_d) = \varphi(c_1 z_1 + \cdots + c_d z_d)$  with  $\gamma_k < \gamma$ ,  $k = 1, \dots, d$ .

**Remark 9.2.4** Theorem 9.2.2 applies to the noncommutative  $L_p$ -spaces, although they do not have property  $(\alpha)$ .

# Chapter 10

## Angle reduction of $H^\infty$ joint functional calculus

Let  $(A_1, \dots, A_d)$  be a  $d$ -tuple of commuting sectorial operators on some Banach space  $X$ . It is plain that if  $(A_1, \dots, A_d)$  admits an  $H^\infty(\Sigma_{\theta_1} \times \dots \times \Sigma_{\theta_d})$  joint functional calculus, then it also admits an  $H^\infty(\Sigma_{\theta'_1} \times \dots \times \Sigma_{\theta'_d})$  joint functional calculus whenever  $\theta'_k \geq \theta_k$ ,  $k = 1, \dots, d$ . Likewise, if  $(T_1, \dots, T_d)$  is a  $d$ -tuple of Ritt operators having an  $H^\infty(B_{\gamma_1} \times \dots \times B_{\gamma_d})$  joint functional calculus, then it also admits an  $H^\infty(B_{\gamma'_1} \times \dots \times B_{\gamma'_d})$  for any  $\gamma'_k \geq \gamma_k$ ,  $k = 1, \dots, d$ .

The purpose of this section is to prove the following theorem, which allows to consider the case when  $\theta'_k < \theta_k$  or  $\gamma'_k < \gamma_k$ .

**Theorem 10.0.1** *Let  $X$  be a Banach space. Let  $d \geq 2$  be an integer. Then the following two properties hold :*

- (1) *Let  $(A_1, \dots, A_d)$  be a  $d$ -tuple of commuting  $R$ -sectorial operators on  $X$ , of respective  $R$ -types  $\omega_1, \dots, \omega_d$ . Let  $\theta_k \in (\omega_k, \pi)$ , for  $k = 1, \dots, d$ , and assume that  $(A_1, \dots, A_d)$  admits an  $H^\infty(\Sigma_{\theta_1} \times \dots \times \Sigma_{\theta_d})$  joint functional calculus. Then for any  $\theta'_k \in (\omega_k, \pi)$ ,  $k = 1, \dots, d$ , the family  $(A_1, \dots, A_d)$  admits an  $H^\infty(\Sigma_{\theta'_1} \times \dots \times \Sigma_{\theta'_d})$  joint functional calculus.*
- (2) *Let  $(T_1, \dots, T_d)$  be a  $d$ -tuple of commuting  $R$ -Ritt operators on  $X$ , of respective  $R$ -types  $a_1, \dots, a_d$ . Let  $\gamma_k \in (a_k, \frac{\pi}{2})$ , for  $k = 1, \dots, d$ , and assume that  $(T_1, \dots, T_d)$  admits an  $H^\infty(B_{\gamma_1} \times \dots \times B_{\gamma_d})$  joint functional calculus. Then for any  $\gamma'_k \in (a_k, \frac{\pi}{2})$ ,  $k = 1, \dots, d$ , the family  $(T_1, \dots, T_d)$  admits an  $H^\infty(B_{\gamma'_1} \times \dots \times B_{\gamma'_d})$  joint functional calculus.*
- (3) *Let  $(T_1, \dots, T_d)$  be a  $d$ -tuple of commuting operators acting on  $X$  such that every  $T_k$  is a  $R$ -Ritt operator,  $k = 1, \dots, d$ . Suppose that  $(T_1, \dots, T_d)$  is polynomially bounded, that is there exists a constant  $C \geq 1$  such that for any polynomial function  $h$  of  $d$  variables we have*

$$\|h(T_1, \dots, T_d)\| \leq C \sup \{ |h(z_1, \dots, z_d)| : (z_1, \dots, z_d) \in \mathbb{T}^d \}. \quad (10.0.1)$$

*Then  $(T_1, \dots, T_d)$  admits a  $H^\infty$  joint functional calculus.*

Part (1) of this theorem is a multivariable version of a widely used result due to Kalton-Weis [40, Proposition 5.1]. The proof below is a new one the last reference.

*Proof of Theorem 10.0.1.* We prove (1). We assume that  $(A_1, \dots, A_d)$  admits an  $H^\infty(\Sigma_{\theta_1} \times \dots \times \Sigma_{\theta_d})$  joint functional calculus. We consider  $\theta'_k > \omega_k$ ,  $k = 1, \dots, d$ , and we fix  $\nu_k, \mu \in (0, \pi)$  such that

$$\omega_k < \nu_k < \theta'_k < \mu \quad \text{and} \quad \theta_k < \mu, \quad k = 1, \dots, d.$$

We consider the three sequences  $(\Delta_i)_{i \geq 1}$ ,  $(\psi_i)_{i \geq 1}$  and  $(\tilde{\psi}_i)_{i \geq 1}$  from Proposition 6.0.4. For any  $i_1, \dots, i_d \geq 1$ , we let

$$\Delta_{i_1, \dots, i_d} = \Delta_{i_1} \otimes \dots \otimes \Delta_{i_d} \in H_0^\infty(\Sigma_\mu \times \dots \times \Sigma_\mu)$$

denote the function taking any  $(z_1, \dots, z_d)$  in  $\Sigma_\mu^d$  to  $\Delta_{i_1}(z_1) \cdots \Delta_{i_d}(z_d)$ . We similarly define  $\psi_{i_1, \dots, i_d} = \psi_{i_1} \otimes \dots \otimes \psi_{i_d}$  and  $\tilde{\psi}_{i_1, \dots, i_d} = \tilde{\psi}_{i_1} \otimes \dots \otimes \tilde{\psi}_{i_d}$ .

According to Proposition 6.0.4,

$$1 = \sum_{i_1, \dots, i_d=1}^{\infty} \Delta_{i_1, \dots, i_d} \psi_{i_1, \dots, i_d} \tilde{\psi}_{i_1, \dots, i_d} \quad (10.0.2)$$

on  $\Sigma_\mu^d$  and

$$\sup_{(z_1, \dots, z_d) \in \Sigma_\mu^d} \sum_{i_1, \dots, i_d=1}^{\infty} |(\Delta_{i_1, \dots, i_d} \psi_{i_1, \dots, i_d} \tilde{\psi}_{i_1, \dots, i_d})(z_1, \dots, z_d)| < \infty. \quad (10.0.3)$$

Let  $f \in H_0^\infty(\Sigma_{\theta'_1} \times \dots \times \Sigma_{\theta'_d})$ . For convenience we set  $\mathcal{A} = (A_1, \dots, A_d)$  and in this proof, we write  $f(\mathcal{A})$  instead of  $f(A_1, \dots, A_d)$ . It is easy to deduce from (10.0.2), (10.0.3) and Fubini's theorem that

$$\sum_{i_1, \dots, i_d=1}^{\infty} \|f(\mathcal{A}) \Delta_{i_1, \dots, i_d}(\mathcal{A}) \psi_{i_1, \dots, i_d}(\mathcal{A}) \tilde{\psi}_{i_1, \dots, i_d}(\mathcal{A})\| < \infty$$

and

$$f(\mathcal{A}) = \sum_{i_1, \dots, i_d=1}^{\infty} f(\mathcal{A}) \Delta_{i_1, \dots, i_d}(\mathcal{A}) \psi_{i_1, \dots, i_d}(\mathcal{A}) \tilde{\psi}_{i_1, \dots, i_d}(\mathcal{A}).$$

We consider two arbitrary elements  $x \in X$  and  $y \in X^*$ . For any integer  $N \geq 1$ , we set

$$f_N = \sum_{i_1, \dots, i_d=1}^N f \Delta_{i_1, \dots, i_d} \psi_{i_1, \dots, i_d} \tilde{\psi}_{i_1, \dots, i_d}.$$

Let now  $(r_{i_1, \dots, i_d})_{i_1 \geq 1, \dots, i_d \geq 1}$  be a family of independent Rademacher function indexed by  $(\mathbb{N}^*)^d$ . By Cauchy-Schwarz, we have

$$\begin{aligned} |\langle f_N(\mathcal{A})x, y \rangle| &= \left| \sum_{i_1, \dots, i_d=1}^N \langle f(\mathcal{A})\Delta_{i_1, \dots, i_d}(\mathcal{A})\psi_{i_1, \dots, i_d}(\mathcal{A})x, \tilde{\psi}_{i_1, \dots, i_d}(\mathcal{A})^*y \rangle \right| \\ &\leq \left\| \sum_{i_1, \dots, i_d=1}^N r_{i_1, \dots, i_d} \otimes f(\mathcal{A})\Delta_{i_1, \dots, i_d}(\mathcal{A})\psi_{i_1, \dots, i_d}(\mathcal{A})x \right\|_{\text{Rad}(X)} \\ &\quad \times \left\| \sum_{i_1, \dots, i_d=1}^N r_{i_1, \dots, i_d} \otimes \tilde{\psi}_{i_1, \dots, i_d}(\mathcal{A})^*y \right\|_{\text{Rad}(X^*)}. \end{aligned}$$

We shall now estimate each of the two factors in the upper bound.

According to the  $R$ -sectoriality assumption, each set  $\{zR(z, A_k) : z \in \partial\Sigma_{\nu_k}\}$  is  $R$ -bounded, hence by [35, Proposition 8.1.19], the product set

$$\left\{ \prod_{k=1}^d (z_k R(z_k, A_k)) : z_k \in \partial\Sigma_{\nu_k}, k = 1, \dots, d \right\}$$

is  $R$ -bounded. Further it follows from part (2) of Proposition 6.0.4 that

$$\int_{\prod_{k=1}^d \partial\Sigma_{\nu_k}} |(f\Delta_{i_1, \dots, i_d})(z_1, \dots, z_d)| \prod_{k=1}^d \left| \frac{dz_k}{z_k} \right| \lesssim \|f\|_{\infty, \Sigma_{\theta'_1} \times \dots \times \Sigma_{\theta'_d}}.$$

By (5.2.2), we have

$$f(\mathcal{A})\Delta_{i_1, \dots, i_d}(\mathcal{A}) = \left( \frac{1}{2\pi i} \right)^d \int_{\prod_{k=1}^d \partial\Sigma_{\nu_k}} (f\Delta_{i_1, \dots, i_d})(z_1, \dots, z_d) \prod_{k=1}^d (z_k R(z_k, A_k)) \prod_{k=1}^d \frac{dz_k}{z_k}.$$

Applying [35, Theorem 8.5.2], we deduce from the above two results that the set

$$\mathcal{F}_f := \{f(\mathcal{A})\Delta_{i_1, \dots, i_d}(\mathcal{A}) : i_1, \dots, i_d \geq 1\}$$

is  $R$ -bounded, with an estimate

$$\mathcal{R}(\mathcal{F}_f) \lesssim \|f\|_{\infty, \Sigma_{\theta'_1} \times \dots \times \Sigma_{\theta'_d}}.$$

Consequently,

$$\begin{aligned} \left\| \sum_{i_1, \dots, i_d=1}^N r_{i_1, \dots, i_d} \otimes f(\mathcal{A})\Delta_{i_1, \dots, i_d}(\mathcal{A})\psi_{i_1, \dots, i_d}(\mathcal{A})x \right\|_{\text{Rad}(X)} \\ \lesssim \|f\|_{\infty, \Sigma_{\theta'_1} \times \dots \times \Sigma_{\theta'_d}} \left\| \sum_{i_1, \dots, i_d=1}^N r_{i_1, \dots, i_d} \otimes \psi_{i_1, \dots, i_d}(\mathcal{A})x \right\|_{\text{Rad}(X)}. \end{aligned}$$



By part (1) of Proposition 6.0.4, there exists a constant  $C \geq 0$  such that

$$\left\| \sum_{i_1, \dots, i_d=1}^N \eta_{i_1, \dots, i_d} \psi_{i_1, \dots, i_d} \right\|_{\infty, \Sigma_\mu^d} \leq C$$

for any  $\eta_{i_1, \dots, i_d} = \pm 1$ . Since  $(A_1, \dots, A_d)$  has an  $H^\infty(\Sigma_{\theta_1} \times \dots \times \Sigma_{\theta_d})$  joint functional calculus, this implies an estimate

$$\left\| \sum_{i_1, \dots, i_d=1}^N \eta_{i_1, \dots, i_d} \psi_{i_1, \dots, i_d}(\mathcal{A}) \right\| \leq C', \quad \eta_{i_1, \dots, i_d} = \pm 1.$$

Consequently, we have an estimate

$$\left\| \sum_{i_1, \dots, i_d=1}^N r_{i_1, \dots, i_d} \otimes \psi_{i_1, \dots, i_d}(\mathcal{A})x \right\|_{\text{Rad}(X)} \lesssim \|x\|,$$

and hence an estimate

$$\left\| \sum_{i_1, \dots, i_d=1}^N r_{i_1, \dots, i_d} \otimes f(\mathcal{A})\Delta_{i_1, \dots, i_d}(\mathcal{A})\psi_{i_1, \dots, i_d}(\mathcal{A})x \right\|_{\text{Rad}(X)} \lesssim \|f\|_{\infty, \Sigma_{\theta'_1} \times \dots \times \Sigma_{\theta'_d}} \|x\|,$$

not depending on  $N$ .

The second factor in the majorisation of  $|\langle f_N(\mathcal{A})x, y \rangle|$  can be treated similarly and we obtain an estimate

$$\left\| \sum_{i_1, \dots, i_d=1}^N \varepsilon_{i_1, \dots, i_d} \otimes \tilde{\psi}_{i_1, \dots, i_d}(\mathcal{A})^* y \right\|_{\text{Rad}(X^*)} \lesssim \|y\|,$$

not depending on  $N$ .

Altogether we thus have

$$|\langle f_N(\mathcal{A})x, y \rangle| \lesssim \|f\|_{\infty, \Sigma_{\theta'_1} \times \dots \times \Sigma_{\theta'_d}} \|x\| \|y\|.$$

Passing to the limit and taking the supremum over all  $x, y$  of norms less than or equal to 1, we obtain

$$\|f(A_1, \dots, A_d)\| \lesssim \|f\|_{\infty, \Sigma_{\theta'_1} \times \dots \times \Sigma_{\theta'_d}}.$$

This estimate can be proved as well for any  $f \in H_{0,1}^\infty(\Sigma_{\theta'_1} \times \dots \times \Sigma_{\theta'_d})$ , which yields the result. Appealing to Theorem 7.0.6 in place of Proposition 6.0.4, one can obtain assertion (2) by mimicking what precedes. (3) is a straightforward consequence of (2). We omit the details.  $\square$

Combining Theorem 10.0.1 and Theorem 8.0.1, we easily have obtain this last result.

**Theorem 10.0.2** *Let  $X$  be a Banach space. Assume that either  $X$  is a Banach lattice, or  $X$  or  $X^*$  has property  $(\alpha)$ . Let  $d \geq 2$  be an integer. Then the following two properties hold :*

- (1) Let  $(A_1, \dots, A_d)$  be a  $d$ -tuple of commuting  $R$ -sectorial operators of types  $\omega_k \in (0, \pi)$ ,  $k = 1, \dots, d$ . Suppose that  $A_k$  admits an  $H^\infty(\Sigma_{\theta_k})$  functional calculus for some  $\theta_k \in (\omega_k, \pi)$ ,  $k = 1, \dots, d$ . Then  $(A_1, \dots, A_d)$  admits an  $H^\infty(\Sigma_{\theta'_1} \times \dots \times \Sigma_{\theta'_d})$  joint functional calculus for any  $\theta'_k \in (\omega_k, \pi)$ ,  $k = 1, \dots, d$ .
- (2) Let  $(T_1, \dots, T_d)$  be a  $d$ -tuple of commuting  $R$ -Ritt operators of types  $a_k \in (0, \frac{\pi}{2})$ ,  $k = 1, \dots, d$ . Suppose that  $T_k$  admits an  $H^\infty(B_{\gamma_k})$  functional calculus for some  $\gamma_k \in (a_k, \frac{\pi}{2})$ ,  $k = 1, \dots, d$ . Then  $(T_1, \dots, T_d)$  admits an  $H^\infty(B_{\gamma'_1} \times \dots \times B_{\gamma'_d})$  joint functional calculus for any  $\gamma'_k \in (a_k, \frac{\pi}{2})$ ,  $k = 1, \dots, d$ .



## Part IV

# Characterisations by dilations and contraction properties



# Chapter 11

## The case of UMD spaces with property $(\alpha)$

In this chapter, we give characterisations of  $H^\infty$  joint functional calculus for commuting families of either Ritt or sectorial operators acting on a UMD Banach space  $X$  with property  $(\alpha)$ . We pay a special attention to the case when  $X$  is an  $L^p$ -space, for  $p \in (1, \infty)$ . These characterisations generalise some of the main results of [4].

We refer the reader to [12], [66, Chapter 5], Section 4.2 and Section 5.3 for information on the UMD property

### 11.1 Characterisations for commuting Ritt operators

We first establish a general result about combining dilations of commuting operators through Bochner spaces. Given any  $p \in [1, \infty)$ , any measure space  $\Omega$ , any Banach space  $X$ , and any bounded operators  $T: L^p(\Omega) \rightarrow L^p(\Omega)$  and  $S: X \rightarrow X$ , consider the operator  $T \otimes S$  acting on  $L^p(\Omega) \otimes X$ . If this operator extends to a bounded operator on  $L^p(\Omega; X)$ , we denote this extension by

$$T \overline{\otimes} S: L^p(\Omega; X) \longrightarrow L^p(\Omega; X).$$

By the density of  $L^p(\Omega) \otimes X$  in  $L^p(\Omega; X)$ , this extension is necessarily unique. We recall that if  $T$  is a positive operator (meaning that  $T(x) \geq 0$  for every  $x \geq 0$ ), then  $T \otimes S$  has a bounded extension as described above.

**Lemma 11.1.1** *Let  $d \geq 2$  be an integer, let  $T_1, \dots, T_d$  be commuting operators on a Banach space  $X$  and let  $p \in [1, \infty)$ . Let  $1 \leq m \leq d$ . Assume that:*

- (1) *For every  $k = 1, \dots, m$ , there exist a positive operator  $V_k$  on some  $L^p(\Omega)$  and two bounded operators  $J_k: X \rightarrow L^p(\Omega; X)$  and  $Q_k: L^p(\Omega; X) \rightarrow X$  such that*

$$T_k^{n_k} = Q_k(V_k \overline{\otimes} I_X)^{n_k} J_k, \quad n_k \in \mathbb{N}. \quad (11.1.1)$$

(2) If  $m < d$ , there exist a Banach space  $Y$ , two bounded operators  $J_{m+1}: X \rightarrow Y$  and  $Q_{m+1}: Y \rightarrow X$  as well as commuting bounded operators  $V_{m+1}, \dots, V_d$  on  $Y$  such that

$$T_{m+1}^{n_{m+1}} \cdots T_d^{n_d} = Q_{m+1} V_{m+1}^{n_{m+1}} \cdots V_d^{n_d} J_{m+1}, \quad (n_{m+1}, \dots, n_d) \in \mathbb{N}^{d-m}. \quad (11.1.2)$$

(3) For every  $i = 1, \dots, m$  and  $j = 1, \dots, d$ , we have

$$J_i T_j = (I_{L^p(\Omega)} \overline{\otimes} T_j) J_i. \quad (11.1.3)$$

Then there exist two bounded operators  $J: X \rightarrow L^p(\Omega^m; Y)$  and  $Q: L^p(\Omega^m; Y) \rightarrow X$  such that

$$T_1^{n_1} \cdots T_d^{n_d} = Q U_1^{n_1} \cdots U_d^{n_d} J, \quad (n_1, \dots, n_d) \in \mathbb{N}^d, \quad (11.1.4)$$

where the operators  $U_1, \dots, U_d: L^p(\Omega^m; Y) \rightarrow L^p(\Omega^m; Y)$  are given by

$$U_k = I^{\otimes k-1} \overline{\otimes} V_k \overline{\otimes} I^{\otimes m-k} \overline{\otimes} I_Y, \quad k = 1, \dots, m; \quad (11.1.5)$$

$$U_k = I^{\otimes m} \overline{\otimes} V_k, \quad k = m+1, \dots, d. \quad (11.1.6)$$

Here  $I = I_{L^p(\Omega)}$  and  $I^{\otimes l} = \underbrace{I \overline{\otimes} \cdots \overline{\otimes} I}_{l \text{ factors}}$  for every integer  $l \geq 1$ .

*Proof of Lemma 11.1.1.* We define  $\widetilde{Q}_m: L^p(\Omega^m; X) \rightarrow X$  and  $\widetilde{J}_m: X \rightarrow L^p(\Omega^m; X)$  by letting

$$\widetilde{Q}_m = Q_1 (I \overline{\otimes} Q_2) (I^{\otimes 2} \overline{\otimes} Q_3) \cdots (I^{\otimes m-1} \overline{\otimes} Q_m) \quad (11.1.7)$$

and

$$\widetilde{J}_m = (I^{\otimes m-1} \overline{\otimes} J_m) \cdots (I^{\otimes 2} \overline{\otimes} J_3) (I \overline{\otimes} J_2) J_1. \quad (11.1.8)$$

Then we define  $S_{k,m}: L^p(\Omega^m; X) \rightarrow L^p(\Omega^m; X)$  by

$$S_{k,m} = I^{\otimes k-1} \overline{\otimes} V_k \overline{\otimes} I^{\otimes m-k} \overline{\otimes} I_X, \quad 1 \leq k \leq m. \quad (11.1.9)$$

Our first aim is to prove by induction on  $m$  that we have the following dilation property,

$$T_1^{n_1} \cdots T_m^{n_m} = \widetilde{Q}_m S_{1,m}^{n_1} \cdots S_{m,m}^{n_m} \widetilde{J}_m, \quad (n_1, \dots, n_m) \in \mathbb{N}^m. \quad (11.1.10)$$

We will see that this property only depends on the assumptions (11.1.1) and (11.1.3).

The case  $m = 1$  is trivial. Let  $m \geq 2$ , suppose that (11.1.7), (11.1.8), (11.1.9) and (11.1.10) hold true for  $m-1$ , and let us prove the latter dilation property for  $m$ . For every  $(n_1, \dots, n_m) \in \mathbb{N}^m$ , we write

$$T_1^{n_1} \cdots T_{m-1}^{n_{m-1}} T_m^{n_m} = \widetilde{Q}_{m-1} S_{1,m-1}^{n_1} \cdots S_{m-1,m-1}^{n_{m-1}} \widetilde{J}_{m-1} T_m^{n_m}. \quad (11.1.11)$$

We compute the last term  $\widetilde{J}_{m-1} T_m^{n_m}$ . First by (11.1.3), we have

$$J_1 T_m^{n_m} = (I \overline{\otimes} T_m^{n_m}) J_1.$$

Applying (11.1.3) again, we then have

$$(I\overline{\otimes}J_2)(I\overline{\otimes}T_m^{n_m})J_1 = (I^{\otimes 2}\overline{\otimes}T_m^{n_m})(I\overline{\otimes}J_2)J_1.$$

Repeating this process with each factor of  $\widetilde{J_{m-1}}$ , we obtain

$$\widetilde{J_{m-1}}T_m^{n_m} = (I^{\otimes m-1}\overline{\otimes}T_m^{n_m})\widetilde{J_{m-1}}. \quad (11.1.12)$$

Using (11.1.1) for  $T_m$ , we see that

$$I^{\otimes m-1}\overline{\otimes}T_m^{n_m} = (I^{\otimes m-1}\overline{\otimes}Q_m)(I^{\otimes m-1}\overline{\otimes}V_m\overline{\otimes}I_X)^{n_m}(I^{\otimes m-1}\overline{\otimes}J_m).$$

Combining with (11.1.11) and (11.1.12), and using the fact that  $I^{\otimes m-1}\overline{\otimes}V_m\overline{\otimes}I_X = S_{m,m}$ , we deduce that

$$T_1^{n_1} \cdots T_m^{n_m} = \widetilde{Q_{m-1}}S_{1,m-1}^{n_1} \cdots S_{m-1,m-1}^{n_{m-1}}(I^{\otimes m-1}\overline{\otimes}Q_m)S_{m,m}^{n_m}(I^{\otimes m-1}\overline{\otimes}J_m)\widetilde{J_{m-1}}.$$

A thorough look at (11.1.9) reveals that for any  $k = 1, \dots, m-1$ ,

$$S_{k,m-1}(I^{\otimes m-1}\overline{\otimes}Q_m) = (I^{\otimes m-1}\overline{\otimes}Q_m)S_{k,m}.$$

Consequently

$$T_1^{n_1} \cdots T_m^{n_m} = \widetilde{Q_{m-1}}(I^{\otimes m-1}\overline{\otimes}Q_m)S_{1,m}^{n_1} \cdots S_{m-1,m}^{n_{m-1}}S_{m,m}^{n_m}(I^{\otimes m-1}\overline{\otimes}J_m)\widetilde{J_{m-1}}.$$

Since

$$\widetilde{Q_m} = \widetilde{Q_{m-1}}(I^{\otimes m-1}\overline{\otimes}Q_m) \quad \text{and} \quad \widetilde{J_m} = (I^{m-1}\overline{\otimes}J_m)\widetilde{J_{m-1}},$$

this yields property (11.1.10).

If  $m = d$ , the preceding computation proves the lemma. Assume now that  $m \leq d-1$ . It follows from (11.1.10) that for any  $(n_1, \dots, n_d) \in \mathbb{N}^d$ , we have

$$T_1^{n_1} \cdots T_d^{n_d} = \widetilde{Q_m}S_{1,m}^{n_1} \cdots S_{m,m}^{n_m}\widetilde{J_m}T_{m+1}^{n_{m+1}} \cdots T_d^{n_d}.$$

Using (11.1.3) we obtain that for any  $k = m+1, \dots, d$ ,

$$\widetilde{J_m}T_k^{n_k} = (I^{\otimes m}\overline{\otimes}T_k)^{n_k}\widetilde{J_m}. \quad (11.1.13)$$

Applying (11.1.2), we therefore obtain that

$$\begin{aligned} T_1^{n_1} \cdots T_d^{n_d} &= \widetilde{Q_m}S_{1,m}^{n_1} \cdots S_{m,m}^{n_m} \\ &\quad \times (I^{\otimes m}\overline{\otimes}Q_{m+1})(I^{\otimes m}\overline{\otimes}V_{m+1})^{n_{m+1}} \cdots (I^{\otimes m}\overline{\otimes}V_d)^{n_d}(I^{\otimes m}\overline{\otimes}J_{m+1})\widetilde{J_m}. \end{aligned}$$

Using (11.1.6), this yields

$$T_1^{n_1} \cdots T_d^{n_d} = \widetilde{Q_m}S_{1,m}^{n_1} \cdots S_{m,m}^{n_m}(I^{\otimes m}\overline{\otimes}Q_{m+1})U_{m+1}^{n_{m+1}} \cdots U_d^{n_d}(I^{\otimes m}\overline{\otimes}J_{m+1})\widetilde{J_m}. \quad (11.1.14)$$

Now it follows from (11.1.9) that for any  $k = 1, \dots, m$ ,

$$S_{k,m}^{n_k}(I^{\otimes m}\overline{\otimes}Q_{m+1}) = (I^{\otimes m}\overline{\otimes}Q_{m+1})U_k, \quad (11.1.15)$$

where the  $U_k$  are given by (11.1.5). Set

$$Q = \widetilde{Q_m}(I^{\otimes m}\overline{\otimes}Q_{m+1}) \quad \text{and} \quad J = (I^{\otimes m}\overline{\otimes}J_{m+1})\widetilde{J_m}.$$

Then (11.1.4) follows from the factorisation (11.1.14) and the relation (11.1.15).  $\square$

The following result is a  $d$ -variable version of [4, Theorem 4.1].



**Theorem 11.1.2** *Let  $X$  be a reflexive Banach space such that  $X$  and  $X^*$  have finite cotype. Let  $T_1, \dots, T_d$  be commuting Ritt operators on  $X$  such that every  $T_k$  has an  $H^\infty(B_{\gamma_k})$  functional calculus for some  $\gamma_k \in (0, \frac{\pi}{2})$ . Let  $p \in (1, \infty)$ . Then there exist a measure space  $\Omega$ , commuting isometric isomorphisms  $U_1, \dots, U_d$  on  $L^p(\Omega^d; X)$ , and two bounded operators  $J: X \rightarrow L^p(\Omega^d; X)$  and  $Q: L^p(\Omega^d; X) \rightarrow X$  such that*

$$T_1^{n_1} \cdots T_d^{n_d} = QU_1^{n_1} \cdots U_d^{n_d} J, \quad (n_1, \dots, n_d) \in \mathbb{N}^d. \quad (11.1.16)$$

*Proof of Theorem 11.1.2.* We shall apply Lemma 11.1.1 in the case  $m = d$ , using the construction devised in the proof of [4, Theorem 4.1].

We recall this construction. We let  $(r_n)_{n \in \mathbb{Z}}$  be an independent sequence of Rademacher variables on some probability space  $\Omega_0$ .

For any  $k = 1, \dots, d$ , recall the ergodic decomposition  $X = \text{Ker}(I - T_k) \oplus \overline{\text{Ran}(I - T_k)}$ . It is shown in [4] that the operator

$$J_k: \begin{array}{l} X = \text{Ker}(I - T_k) \oplus \overline{\text{Ran}(I - T_k)} \rightarrow X \oplus_p L^p(\Omega_0; X) \\ x_0 + x_1 \mapsto (x_0, \sum_{n=1}^{\infty} r_n \otimes T_k^n (I - T_k)^{\frac{1}{2}} (I + T_k)(x_1)) \end{array}$$

is well-defined and bounded, under the assumption that  $T_k$  has an  $H^\infty(B_{\gamma_k})$  functional calculus for some  $\gamma_k \in (0, \frac{\pi}{2})$ . More precisely, the series

$$\sum_{n=1}^{\infty} r_n \otimes T_k^n (I - T_k)^{\frac{1}{2}} (I + T_k)(x_1)$$

converges in  $L^p(\Omega_0; X)$  for any  $x_1 \in X$  and the norm of the resulting sum is  $\lesssim \|x_1\|$ .

Define  $\Omega$  as the disjoint union of  $\Omega_0$  and a singleton, so that

$$X \oplus_p L^p(\Omega_0; X) \simeq L^p(\Omega; X).$$

It also follows from the proof of [4, Theorem 4.1] that there exist an isometric isomorphism  $U: L^p(\Omega) \rightarrow L^p(\Omega)$  (which does not depend on  $T_k$ ,  $k = 1, \dots, d$ ) and operators  $Q_k: L^p(\Omega; X) \rightarrow X$  such that

$$T_k^{n_k} = Q_k(U \otimes I_X)^{n_k} J_k, \quad n_k \in \mathbb{N}.$$

We set  $V_k = U$  for any  $k = 1, \dots, d$ , so that  $T_1, \dots, T_d$  satisfy (11.1.1).

Let us show that  $T_1, \dots, T_d$  also satisfy (11.1.3). Consider arbitrary  $i, j$  in  $\{1, \dots, d\}$ ,  $i \neq j$  and an element  $x_0 + x_1 \in X = \text{Ker}(I - T_i) \oplus \overline{\text{Ran}(I - T_i)}$ . Since  $T_i$  and  $T_j$  commute,  $T_j(x_0)$  belongs to  $\text{Ker}(T_i)$ . Consequently,

$$\begin{aligned} J_i(T_j(x_0 + x_1)) &= (T_j(x_0), \sum_{n=1}^{\infty} r_n \otimes T_i^n (I - T_i)^{\frac{1}{2}} (I + T_i) T_j(x_1)) \\ &= (T_j(x_0), \sum_{n=1}^{\infty} r_n \otimes T_j T_i^n (I - T_i)^{\frac{1}{2}} (I + T_i)(x_1)) \\ &= \left( T_j(x_0), (I_{L^p(\Omega_0)} \otimes T_j) \left( \sum_{n=1}^{\infty} r_n \otimes (T_i^n (I - T_i)^{\frac{1}{2}} (I + T_i)(x_1)) \right) \right) \\ &= (I_{L^p(\Omega)} \otimes T_j) J_i(x_0 + x_1). \end{aligned}$$

This proves (11.1.3).

Applying Lemma 11.1.1, we deduce the existence of two bounded operators  $Q: L^p(\Omega^d; X) \rightarrow X$  and  $J: X \rightarrow L^p(\Omega^d; X)$  such that

$$T_1^{n_1} \cdots T_d^{n_d} = QU_1^{n_1} \cdots U_d^{n_d} J, \quad (n_1, \dots, n_d) \in \mathbb{N}^d,$$

where  $U_1, \dots, U_d$  are given by

$$U_k = (I_{L^p(\Omega)})^{\otimes k-1} \bar{\otimes} U \bar{\otimes} (I_{L^p(\Omega)})^{\otimes d-k-1} \bar{\otimes} I_X, \quad k = 1, \dots, d.$$

Since  $U$  is an isometric isomorphism of  $L^p(\Omega)$ , it is clear that each  $U_k$  is an isometric isomorphism of  $L^p(\Omega^d; X)$  as well.  $\square$

We are now in position to extend [4, Theorem 5.1] to  $d$ -tuples of Ritt operators.

**Theorem 11.1.3** *Let  $X$  be a UMD Banach space with property  $(\alpha)$  and let  $d \geq 1$  be an integer. Let  $T_1, \dots, T_d$  be commuting Ritt operators on  $X$  and let  $p \in (1, \infty)$ . The following two conditions are equivalent.*

- (1)  $(T_1, \dots, T_d)$  admits an  $H^\infty(B_{\gamma_1} \times \cdots \times B_{\gamma_d})$  joint functional calculus for some  $\gamma_k \in (0, \frac{\pi}{2})$ ,  $k = 1, \dots, d$ .
- (2) There exist a measure space  $\Omega$ , commuting contractive Ritt operators  $R_1, \dots, R_d$  on  $L^p(\Omega; X)$  such that every  $R_k$  admits an  $H^\infty(B_{\gamma'_k})$  functional calculus for some  $\gamma'_k \in (0, \frac{\pi}{2})$ ,  $k = 1, \dots, d$ , as well as two bounded operators  $J: X \rightarrow L^p(\Omega; X)$  and  $Q: L^p(\Omega; X) \rightarrow X$  such that

$$T_1^{n_1} \cdots T_d^{n_d} = QR_1^{n_1} \cdots R_d^{n_d} J, \quad (n_1, \dots, n_d) \in \mathbb{N}^d. \quad (11.1.17)$$

*Proof of Theorem 11.1.3.* The implication “(2)  $\Rightarrow$  (1)” is easy. Indeed (11.1.17) implies that for any  $\phi \in \mathcal{P}_d$  (the algebra of complex polynomials in  $d$  variables), we have

$$\phi(T_1, \dots, T_d) = Q\phi(R_1, \dots, R_d)J,$$

and hence

$$\|\phi(T_1, \dots, T_d)\| \leq \|Q\| \|J\| \|\phi(R_1, \dots, R_d)\|.$$

By assumption each  $R_k$  has an  $H^\infty(B_{\gamma'_k})$  functional calculus, with  $\gamma'_k \in (0, \frac{\pi}{2})$ . Since  $X$  has property  $(\alpha)$ , the Bochner space  $L^p(\Omega; X)$  has property  $(\alpha)$  as well. It therefore follows from Theorem 8.0.1 that the  $d$ -tuple  $(R_1, \dots, R_d)$  has an  $H^\infty(B_{\gamma_1} \times \cdots \times B_{\gamma_d})$  joint functional calculus for some  $\gamma_k \in (0, \frac{\pi}{2})$ . Applying Proposition 5.2.7, we deduce that  $(T_1, \dots, T_d)$  also has an  $H^\infty(B_{\gamma_1} \times \cdots \times B_{\gamma_d})$  joint functional calculus.

To prove the converse (and main) implication “(1)  $\Rightarrow$  (2)”, we assume (1). Every UMD Banach space is reflexive and has finite cotype, so we can apply Theorem 11.1.2 on  $X$ .

As in [4, Section 3], set

$$(T_k)_a = I_X - (I_X - T_k)^a, \quad a > 0,$$

where we recall the definition of Section 5.3. Since  $(T_1, \dots, T_d)$  has an  $H^\infty$  joint functional calculus, every  $T_k$  has an  $H^\infty$  functional calculus. Hence according to [4, Proposition 3.2], there exists  $a > 1$  such that every  $(T_k)_a$  has an  $H^\infty$  functional calculus. Applying Theorem 11.1.2, we deduce a dilation property

$$((T_1)_a)^{n_1} \cdots ((T_d)_a)^{n_d} = QU_1^{n_1} \cdots U_d^{n_d} J, \quad (n_1, \dots, n_d) \in \mathbb{N}^d,$$

where  $J: X \rightarrow L^p(\Omega; X)$  and  $Q: L^p(\Omega; X) \rightarrow X$  are bounded operators and  $U_1, \dots, U_d$  are isometric isomorphisms on  $L^p(\Omega; X)$ .

Let  $b = \frac{1}{a}$ , so that  $0 < b < 1$ . Arguing as in the proof of [4, Theorem 5.1] (see also [24], where this argument appeared for the first time), we derive that

$$T_1^{n_1} \cdots T_d^{n_d} = Q((U_1)_b)^{n_1} \cdots ((U_d)_b)^{n_d} J, \quad (n_1, \dots, n_d) \in \mathbb{N}^d.$$

We let  $R_k = (U_k)_b$  for every  $k = 1, \dots, d$ . By [4, Theorem 3.1 and 3.3] (see also Theorem 5.3.2), and the assumption that  $X$  is a UMD Banach space, every  $R_k$  is a contractive Ritt operator having an  $H^\infty(B_{\gamma'_k})$  functional calculus for some  $\gamma'_k \in (0, \frac{\pi}{2})$ , which proves (2).  $\square$

**Remark 11.1.4** It follows from the proof of [4, Theorem 4.1] that the isometric isomorphism  $U: L^p(\Omega) \rightarrow L^p(\Omega)$  appearing in the proof of Theorem 11.1.2 is positive. This implies that if  $X$  is an ordered Banach space, then the isometric isomorphisms  $U_1, \dots, U_d: L^p(\Omega; X) \rightarrow L^p(\Omega; X)$  in the latter theorem are positive operators. It therefore follows from [4, Theorem 3.1 (c)] that if  $X$  is an ordered Banach space in Theorem 11.1.3, then the contractive Ritt operators  $R_1, \dots, R_d: L^p(\Omega; X) \rightarrow L^p(\Omega; X)$  in this theorem are positive operators.

We note that any UMD Banach lattice has property  $(\alpha)$ . Hence any UMD Banach lattice satisfies Theorem 11.1.3.

We also observe that thanks to Theorem 8.0.1, assumption (1) of Theorem 11.1.3 is equivalent to the property that each  $T_k$  admits an  $H^\infty(B_{\gamma_k})$  functional calculus for some  $\gamma_k \in (0, \frac{\pi}{2})$ .

We now give a specific result on  $L^p$ -spaces. This is a  $d$ -variable version of [4, Theorem 5.2].

**Theorem 11.1.5** *Let  $\Sigma$  be a measure space and let  $p \in (1, \infty)$ . Let  $T_1, \dots, T_d$  be commuting Ritt operators on  $L^p(\Sigma)$ . The following two conditions are equivalent.*

- (1)  $(T_1, \dots, T_d)$  admits an  $H^\infty(B_{\gamma_1} \times \cdots \times B_{\gamma_d})$  joint functional calculus for some  $\gamma_k \in (0, \frac{\pi}{2})$ ,  $k = 1, \dots, d$ .
- (2) There exist a measure space  $\Omega$ , commuting positive contractive Ritt operators  $R_1, \dots, R_d$  on  $L^p(\Omega)$ , and two bounded operators  $J: L^p(\Sigma) \rightarrow L^p(\Omega)$  and  $Q: L^p(\Omega) \rightarrow L^p(\Sigma)$  such that

$$T_1^{n_1} \cdots T_d^{n_d} = QR_1^{n_1} \cdots R_d^{n_d} J, \quad (n_1, \dots, n_d) \in \mathbb{N}^d.$$

*Proof of Theorem 11.1.5.* We apply Theorem 11.1.3 above with  $X = L^p(\Sigma)$ , which is a UMD Banach space with property  $(\alpha)$ . We note that for any measure space  $\Omega$ ,  $L^p(\Omega; L^p(\Sigma))$  is an  $L^p$ -space. Further conditions (1) in Theorem 11.1.3 and Theorem 11.1.5 are identical.

Assuming (1) and applying Theorem 11.1.3 together with Remark 11.1.4, we obtain condition (2) in Theorem 11.1.5.

The converse implication follows from Theorem 11.1.3 and the fact that any positive contractive Ritt operator on an  $L^p$ -space has an  $H^\infty(B_\gamma)$  functional calculus for some  $\gamma \in (0, \frac{\pi}{2})$ . This result is proved in [54, Theorem 3.3].  $\square$

A celebrated theorem of Akcoglu and Sucheston (see [1]) asserts that if  $T: L^p(\Sigma) \rightarrow L^p(\Sigma)$  is a positive contraction, with  $p \in (1, \infty)$ , then there exist a measure space  $\Sigma'$ , an isometric isomorphism  $V: L^p(\Sigma') \rightarrow L^p(\Sigma')$  and two contractions  $J: L^p(\Sigma) \rightarrow L^p(\Sigma')$  and  $Q: L^p(\Sigma') \rightarrow L^p(\Sigma)$  such that  $T^n = QV^nJ$  for any  $n \in \mathbb{N}$ . It is an open problem whether the Akcoglu-Sucheston Theorem extends to pairs. The question reads as follows.

Consider a commuting pair  $(T_1, T_2)$  of positive contractions on  $L^p(\Sigma)$ . Does there exist a commuting pair  $(V_1, V_2)$  of isometric isomorphisms acting on some  $L^p(\Sigma')$ , as well as bounded (or even contractive) operators  $J: L^p(\Sigma) \rightarrow L^p(\Sigma')$  and  $Q: L^p(\Sigma') \rightarrow L^p(\Sigma)$  such that  $T_1^{n_1}T_2^{n_2} = QV_1^{n_1}V_2^{n_2}J$  for any  $(n_1, n_2) \in \mathbb{N}^2$ ?

The next result shows that the answer is positive if either  $T_1$  or  $T_2$  is a Ritt operator. More generally we have the following.

**Theorem 11.1.6** *Let  $\Sigma$  be a measure space and let  $p \in (1, \infty)$ . Let  $T_1, \dots, T_d$  be commuting positive contractions on  $L^p(\Sigma)$ . Assume further that  $T_1, \dots, T_{d-1}$  are Ritt operators.*

*Then there exist a measure space  $\Omega$ , two bounded operators  $J: L^p(\Sigma) \rightarrow L^p(\Omega)$  and  $Q: L^p(\Omega) \rightarrow L^p(\Sigma)$ , as well as commuting isometric isomorphisms  $U_1, \dots, U_d: L^p(\Omega) \rightarrow L^p(\Omega)$  such that*

$$T_1^{n_1} \cdots T_d^{n_d} = QU_1^{n_1} \cdots U_d^{n_d}J, \quad (n_1, \dots, n_d) \in \mathbb{N}^d.$$

*Proof of Theorem 11.1.6.* We aim at applying Lemma 11.1.1 with  $m = d - 1$  and  $X = L^p(\Sigma)$ . For any  $k = 1, \dots, d - 1$ ,  $T_k$  is a positive Ritt contraction on  $L^p(\Sigma)$ . According to [54, Theorem 3.3], this implies that it has an  $H^\infty(B_{\gamma_k})$  functional calculus for some  $\gamma_k \in (0, \frac{\pi}{2})$ . By [4, Theorem 4.1] and its proof, this implies that  $T_1, \dots, T_{d-1}$  satisfy the assumption (1) of Lemma 11.1.1.

According to the Akcoglu-Sucheston Theorem quoted above,  $T_d$  satisfies the assumption (2) of Lemma 11.1.1, with  $Y = L^p(\Sigma')$ .

Moreover the argument in the proof of Theorem 11.1.2 shows that  $(T_1, \dots, T_d)$  verifies the assumption (3) of Lemma 11.1.1.

The result now follows from this lemma and the fact that  $L^p(\Omega^m; Y) = L^p(\Omega^{d-1}; L^p(\Sigma'))$  is an  $L^p$ -space. Details are left to the reader.  $\square$

## 11.2 Characterisations for commuting sectorial operators

In this section, we give analogues of our previous results for sectorial operators and semigroups.

We refer the reader to e.g. [64] for definitions and basic properties of  $C_0$ -semigroups and bounded analytic semigroups. We recall that if  $(T_t)_{t \geq 0}$  is a  $C_0$ -semigroup on  $X$ , with generator  $-A$ , then  $A$  is sectorial of type  $< \frac{\pi}{2}$  if and only if  $(T_t)_{t \geq 0}$  is a bounded analytic semigroup.

We say that two  $C_0$ -semigroups  $(T_{1,t})_{t \geq 0}$  and  $(T_{2,t})_{t \geq 0}$  on  $X$  commute provided that

$$T_{1,t_1} T_{2,t_2} = T_{2,t_2} T_{1,t_1}, \quad t_1 \geq 0, t_2 \geq 0. \quad (11.2.1)$$

Assume that  $(T_{1,t})_{t \geq 0}$  and  $(T_{2,t})_{t \geq 0}$  are bounded analytic semigroups with respective generators  $-A_1$  and  $-A_2$ . Then (11.2.1) holds true if and only if the sectorial operators  $A_1, A_2$  commute (in the resolvent sense, see Section 5.2).

It is easy to adapt the proof of Lemma 11.1.1 to semigroups to obtain the following result. We skip the proof.

**Lemma 11.2.1** *Let  $d \geq 2$  be an integer, let  $(T_{1,t})_{t \geq 0}, \dots, (T_{d,t})_{t \geq 0}$  be commuting  $C_0$ -semigroups on a Banach space  $X$  and let  $p \in [1, \infty)$ . Let  $1 \leq m \leq d$ . Assume that:*

- (1) *For every  $k = 1, \dots, m$ , there exist a  $C_0$ -semigroup  $(V_{k,t})_{t \geq 0}$  of positive operators on some  $L^p(\Omega)$  and two bounded operators  $J_k: X \rightarrow L^p(\Omega; X)$  and  $Q_k: L^p(\Omega; X) \rightarrow X$  such that*

$$T_{k,t} = Q_k(V_{k,t} \overline{\otimes} I_X) J_k, \quad t \geq 0. \quad (11.2.2)$$

- (2) *If  $m < d$ , there exist a Banach space  $Y$ , two bounded operators  $J_{m+1}: X \rightarrow Y$  and  $Q_{m+1}: Y \rightarrow X$  as well as commuting  $C_0$ -semigroups  $(V_{m+1,t})_{t \geq 0}, \dots, (V_{d,t})_{t \geq 0}$  on  $Y$  such that*

$$T_{m+1,t_{m+1}} \cdots T_{d,t_d} = Q_{m+1} V_{m+1,t_{m+1}} \cdots V_{d,t_d} J_{m+1}, \quad t_{m+1} \geq 0, \dots, t_d \geq 0.$$

- (3) *For every  $i = 1, \dots, m$  and  $j = 1, \dots, d$ , and for any  $t \geq 0$ , we have*

$$J_i T_{j,t} = (I_{L^p(\Omega)} \overline{\otimes} T_{j,t}) J_i. \quad (11.2.3)$$

*Then there exist two bounded operators  $J: X \rightarrow L^p(\Omega^m; Y)$  and  $Q: L^p(\Omega^m; Y) \rightarrow X$  such that*

$$T_{1,t_1} \cdots T_{d,t_d} = Q U_{1,t_1} \cdots U_{d,t_d} J, \quad t_1 \geq 0, \dots, t_d \geq 0, \quad (11.2.4)$$

*where  $(U_{1,t})_{t \geq 0}, \dots, (U_{d,t})_{t \geq 0}$  are  $C_0$ -semigroups on  $L^p(\Omega^m; Y)$  given by*

$$U_{k,t} = I^{\otimes k-1} \overline{\otimes} V_{k,t} \overline{\otimes} I^{\otimes m-k} \overline{\otimes} I_Y, \quad k = 1, \dots, m;$$

$$U_{k,t} = I^{\otimes m} \overline{\otimes} V_{k,t}, \quad k = m+1, \dots, d.$$

*Here  $I = I_{L^p(\Omega)}$  and  $I^{\otimes l} = \underbrace{I \overline{\otimes} \cdots \overline{\otimes} I}_{l \text{ factors}}$  for every integer  $l \geq 1$ .*

The construction in the proof of [4, Theorem 4.5] is an analogue of the construction in the proof of [4, Theorem 4.1] where discrete square functions based on Rademacher averages are replaced by continuous square functions provided by Brownian motion. Using this construction and using Lemma 11.2.1 instead of Lemma 11.1.1, we obtain the following sectorial version of Theorem 11.1.2.

**Theorem 11.2.2** *Let  $X$  be a reflexive Banach space such that  $X$  and  $X^*$  have finite cotype. Let  $A_1, \dots, A_d$  be commuting sectorial operators on  $X$  such that every  $A_k$  has an  $H^\infty(\Sigma_{\theta_k})$  functional calculus for some  $\theta_k$  in  $(0, \frac{\pi}{2})$ . Let  $p \in (1, \infty)$ . Then there exist a measure space  $\Omega$ , commuting  $C_0$ -groups of isometries  $(U_{1,t})_{t \in \mathbb{R}}, \dots, (U_{d,t})_{t \in \mathbb{R}}$  on  $L^p(\Omega; X)$ , and two bounded operators  $J: X \rightarrow L^p(\Omega; X)$  and  $Q: L^p(\Omega; X) \rightarrow X$  such that*

$$e^{-t_1 A_1} \dots e^{-t_d A_d} = QU_{1,t_1} \dots U_{d,t_d} J, \quad t_1 \geq 0, \dots, t_d \geq 0.$$

*Proof of Theorem 11.2.2.* We shall apply Lemma 11.2.1 in the case  $m = d$ , using the construction devised in the proof of [4, Theorem 4.5] and we refer the reader to [4, Section 4] for the details. We let  $(T_{k,t_k})_{t_k \geq 0} = (e^{-t_k A_k})_{t_k \geq 0}$  be the semigroups generated by the  $A_k$ ,  $k = 1, \dots, d$ .

Let  $\widehat{\Omega}$  be a probability space. We note first that the construction in [4] uses a Gaussian random process  $W_p: L_2(\mathbb{R}) \rightarrow L_p(\widehat{\Omega})$  such that the operator  $W_p \otimes I_X: L_2(\mathbb{R}) \otimes X \rightarrow L_p(\widehat{\Omega}) \otimes X$  extends to an isometry  $W_{p,X}: \gamma^p(L_2(\mathbb{R}); X) \rightarrow L_p(\widehat{\Omega}; X)$ .

An important fact for the computations below is the following. Let  $T$  be a bounded operator acting on  $X$ . We have

$$(I_{L_p(\widehat{\Omega})} \otimes T) \circ (W_p \otimes I_X) = (W_p \otimes I_X) \circ (I_{L_2(\mathbb{R})} \otimes T)$$

and passing to the extended operators we obtain

$$(I_{L_p(\widehat{\Omega})} \overline{\otimes} T) \circ W_{p,X} = W_{p,X} \circ (I_{L_2(\mathbb{R})} \overline{\otimes} T). \quad (11.2.5)$$

Since  $X$  is reflexive, we have  $X = \text{Ker}(A_k) \oplus \overline{\text{Ran}(A_k)}$  for  $k = 1, \dots, d$ . Let  $\chi = \chi_{(0,\infty)}$ . It is shown in [4] that the operator

$$J_k: \begin{array}{l} X = \text{Ker}(A_k) \oplus \overline{\text{Ran}(A_k)} \rightarrow X \oplus_p L^p(\widehat{\Omega}; X) \\ x_0 + x_1 \quad \mapsto (x_0, W_{p,X}(s \mapsto \chi(s) A_k^{\frac{1}{2}} T_{k,s} x_1)) \end{array}$$

is well-defined and bounded, under the assumptions that  $A_k$  has an  $H^\infty(\Sigma_{\theta_k})$  functional calculus and  $X$  has finite cotype. This follows from the more general fact (see [41]). Let  $x$  in  $X$  and let  $\varphi$  be a function of  $H_0^\infty(\Sigma_{\theta'_k})$ ,  $\theta'_k > \theta_k$ . Then the function  $s \mapsto \varphi(sA)x$  defines a bounded operator  $u_\varphi$  of  $\gamma^p(L_2(\mathbb{R}_+^*, dt/t); X)$ , such that

$$\|s \mapsto \varphi(sA)x\|_{\gamma^p(\mathbb{R}_+^*, dt/t; X)} = \|u_\varphi\|_{\gamma^p(L_2(\mathbb{R}_+^*, dt/t); X)} \lesssim \|x\|$$

(see Section 3.4 for the precise definition of  $\gamma^p(L_2(\mathbb{R}_+^*, dt/t); X)$ ).

Define  $\Omega$  as the disjoint union of  $\widehat{\Omega}$  and a singleton, so that

$$X \oplus_p L^p(\widehat{\Omega}; X) \simeq L^p(\Omega; X).$$

It also follows from the proof of [4, Theorem 4.4] that there exist a  $c_0$ -group of positive isometric isomorphisms  $(V_t)_{t \in \mathbb{R}}: L^p(\Omega) \rightarrow L^p(\Omega)$  (which does not depend on the  $A_k$ ,  $k = 1, \dots, d$ ) and operators  $Q_k: L^p(\Omega; X) \rightarrow X$  such that

$$T_{k,t_k} = Q_k(V_{t_k} \overline{\otimes} I_X) J_k, \quad t_k \geq 0.$$

We set  $V_{k,t_k} = V_{t_k}$  for any  $k = 1, \dots, d$ ,  $t_k \geq 0$ , so that  $T_{1,t_1}, \dots, T_{d,t_d}$  satisfy (11.2.2) of Lemma 11.2.1.

Let us show that  $T_{1,t_1}, \dots, T_{d,t_d}$  also satisfy (11.2.3). Consider arbitrary  $i, j$  in  $\{1, \dots, d\}$ ,  $i \neq j$  and an element  $x_0 + x_1 \in X = \text{Ker}(A_i) \oplus \overline{\text{Ran}(A_i)}$ . Since  $T_{i,t_i}$  and  $T_{j,t_j}$  commute for any  $t_i, t_j \geq 0$ ,  $T_{j,t_j}(x_0)$  belongs to  $\text{Ker}(A_i)$ .

Consequently, taking in account relation (11.2.5), we have for any  $t_j \geq 0$

$$\begin{aligned} J_i(T_{j,t_j}(x_0 + x_1)) &= (T_{j,t_j}(x_0), W_{p,X}(s \mapsto \chi(s) A_i^{\frac{1}{2}} T_{s,i} T_{j,t_j}(x_1))) \\ &= (T_{j,t_j}(x_0), W_{p,X}(s \mapsto \chi(s) T_{j,t_j} A_i^{\frac{1}{2}} T_{s,i}(x_1))) \\ &= (T_{j,t_j}(x_0), W_{p,X} \circ (I_{L_2(\mathbb{R})} \otimes T_{j,t_j})(s \mapsto \chi(s) A_i^{\frac{1}{2}} T_{s,i}(x_1))) \\ &= (T_{j,t_j}(x_0), (I_{L_p(\widehat{\Omega})} \overline{\otimes} T_{j,t_j}) \circ W_{p,X}(s \mapsto \chi(s) A_i^{\frac{1}{2}} T_{s,i}(x_1))) \\ &= (I_{L_p(\Omega)} \overline{\otimes} T_{j,t_j})(x_0, W_{p,X}(s \mapsto \chi(s) A_i^{\frac{1}{2}} T_{s,i}(x_1))) \\ &= (I_{L_p(\Omega)} \overline{\otimes} T_{j,t_j}) J_i(x_0 + x_1). \end{aligned}$$

This proves (11.2.3).

Applying Lemma 11.2.1, we deduce the existence of two bounded operators  $J: X \rightarrow L^p(\Omega^d; Y)$  and  $Q: L^p(\Omega^d; Y) \rightarrow X$  such that

$$T_{1,t_1} \cdots T_{d,t_d} = Q U_{1,t_1} \cdots U_{d,t_d} J, \quad t_1 \geq 0, \dots, t_d \geq 0,$$

where  $(U_{1,t})_{t \geq 0}, \dots, (U_{d,t})_{t \geq 0}$  are  $C_0$ -semigroups on  $L^p(\Omega^d; Y)$  given by

$$U_{k,t} = (I_{L^p(\Omega)})^{\otimes k-1} \overline{\otimes} V_{k,t} \overline{\otimes} (I_{L^p(\Omega)})^{\otimes d-k} \overline{\otimes} I_X, \quad k = 1, \dots, d, \quad t \geq 0.$$

Since  $(V_{k,t_k})_{t_k \geq 0}$  is a semigroup of isometric isomorphisms of  $L^p(\Omega)$ , it is clear that each  $(U_{k,t_k})_{t_k \geq 0}$  is an isometric isomorphism of  $L^p(\Omega^d; X)$  as well.  $\square$

Using the previous result and adapting the proof of [4, Theorem 5.6] to the  $d$ -variable case, we obtain the following sectorial version of Theorem 11.1.3.

**Theorem 11.2.3** *Let  $X$  be a UMD Banach space with property  $(\alpha)$  and let  $d \geq 1$  be an integer. Let  $A_1, \dots, A_d$  be commuting sectorial operators and let  $p \in (1, \infty)$ . The following two conditions are equivalent.*

- (1)  $(A_1, \dots, A_d)$  admits an  $H^\infty(\Sigma_{\theta_1} \times \cdots \times \Sigma_{\theta_d})$  joint functional calculus for some  $\theta_k \in (0, \frac{\pi}{2})$ ,  $k = 1, \dots, d$ .
- (2) There exist a measure space  $\Omega$ , commuting sectorial operators  $B_1, \dots, B_d$  on  $L^p(\Omega; X)$  such that every  $B_k$  admits an  $H^\infty(\Sigma_{\theta'_k})$  functional calculus for some  $\theta'_k \in (0, \frac{\pi}{2})$ ,  $k = 1, \dots, d$ , as well as two bounded operators  $J: X \rightarrow L^p(\Omega; X)$  and  $Q: L^p(\Omega; X) \rightarrow X$  such that

$$e^{-t_1 A_1} \cdots e^{-t_d A_d} = Q e^{-t_1 B_1} \cdots e^{-t_d B_d} J, \quad t_1 \geq 0, \dots, t_d \geq 0, \quad (11.2.6)$$

and all the  $(e^{-t B_k})_{t \geq 0}$  are semigroups of contractions.

*Proof of Theorem 11.2.3.* The proof of implication "(2)  $\implies$  (1)" is similar to the one for Theorem 11.1.3. Indeed, relation (11.2.6) implies using the Laplace formula for semigroups that

$$R(\lambda_1, A_1) \cdots R(\lambda_d, A_d) = QR(\lambda_1, A_1) \cdots R(\lambda_d, A_d)J,$$

for any  $\lambda_k \in \mathbb{C} \setminus (\sigma(A_k) \cup \sigma(B_k))$ ,  $k = 1, \dots, d$ . Now we use definition 5.2.2 to obtain that for any function  $f$  in  $H_{0,1}^\infty(\Sigma_{\nu_1} \times \cdots \times \Sigma_{\nu_d})$  (with  $\nu_k > \theta'_k$ ) for which both of terms are well defined, we have

$$f(A_1, \dots, A_d) = Qf(B_1, \dots, B_d)J.$$

Hence we deduce  $\|f(A_1, \dots, A_d)\| \leq \|Q\| \|J\| \|f(B_1, \dots, B_d)\| \lesssim \|f\|_{\infty, \Sigma_{\theta'_1} \times \cdots \times \Sigma_{\theta'_d}}$ , which proves that  $(A_1, \dots, A_d)$  admits an  $H^\infty(\Sigma_{\theta'_1} \times \cdots \times \Sigma_{\theta'_d})$  joint functional calculus.

To prove the converse implication "(1)  $\implies$  (2)", we assume (1). Every UMD Banach space is reflexive and has finite cotype, so we can apply Theorem 11.2.2 on  $X$ .

Since  $(A_1, \dots, A_d)$  has an  $H^\infty$  joint functional calculus, every  $A_k$  has an  $H^\infty$  functional calculus. Hence according to [4, Lemma 2.1], there exists  $\alpha > 1$  such that every  $A_k^\alpha$  has an  $H^\infty(\Sigma_\nu)$  functional calculus with  $\nu < \frac{\pi}{2}$ . Applying Theorem 11.2.2, we deduce a dilation property

$$e^{-t_1 A_1^\alpha} \cdots e^{-t_d A_d^\alpha} = QU_{1,t_1} \cdots U_{d,t_d} J, \quad t_1 \geq 0, \dots, t_d \geq 0.$$

where  $J: X \rightarrow L^p(\Omega; X)$  and  $Q: L^p(\Omega; X) \rightarrow X$  are bounded operators and  $(U_{1,t_1}), \dots, (U_{d,t_d})$  are commuting  $C_0$ -groups of isometries on  $L^p(\Omega; X)$ . We denote by  $C_1, \dots, C_d$  the respective negative generators of the  $C_0$ -groups  $(U_{1,t_1}), \dots, (U_{d,t_d})$ . As  $X$  is a UMD-space, every  $C_k$  admits an  $H^\infty(\Sigma_{\omega_k})$  functional calculus for every  $\omega_k > \frac{\pi}{2}$ ,  $k = 1, \dots, d$  (see Theorem 5.3.1).

Let  $\beta = \frac{1}{\alpha}$ , so that  $0 < \beta < 1$ . Choose  $\omega_k > \frac{\pi}{2}$  sufficiently close to  $\frac{\pi}{2}$  such that  $\theta'_k = \beta\omega_k < \frac{\pi}{2}$ ,  $k = 1, \dots, d$ . As we see in the proof of [4, Theorem 5.6], every  $C_k^\beta$  admits an  $H^\infty(\theta'_k)$  functional calculus for some  $\theta'_k$  in  $(0, \frac{\pi}{2})$ . Using subordination (see [80, XI 11]), we know that for any  $t_k > 0$ , there exists a nonnegative function  $f_{t_k, \beta}$  in  $L_1(\mathbb{R}_+)$  with  $\int_0^\infty f_{t_k, \beta}(s) ds = 1$ , such that we have in the strong sense

$$e^{-t_k C_k^\beta} = \int_0^\infty f_{t_k, \beta}(s) U_{k,s} ds$$

which shows that every semigroup  $(e^{-t_k C_k^\beta})_{t_k \geq 0}$  is a semigroups of contractions. Likewise

$$e^{-t_k A_k} = e^{-t_k (A_k^\alpha)^\beta} = \int_0^\infty f_{t_k, \beta}(s) e^{-s A_k^\alpha} ds.$$

Using now the commutation of the  $C_0$ -groups and the Fubini's Theorem, we have



for any  $t_1 > 0, \dots, t_d > 0$

$$\begin{aligned}
e^{-t_1 A_1} \dots e^{-t_d A_d} &= \left( \int_0^\infty f_{t_1, \beta}(s_1) e^{-s_1 A_1^\alpha} ds_1 \right) \cdots \left( \int_0^\infty f_{t_d, \beta}(s_d) e^{-s_d A_d^\alpha} ds_d \right) \\
&= \int_{(0, \infty)^d} f_{t_1, \beta}(s_1) \cdots f_{t_d, \beta}(s_d) e^{-s_1 A_1^\alpha} \cdots e^{-s_d A_d^\alpha} ds_1 \cdots ds_d \\
&= \int_{(0, \infty)^d} f_{t_1, \beta}(s_1) \cdots f_{t_d, \beta}(s_d) Q U_{1, t_1} \cdots U_{d, t_d} J ds_1 \cdots ds_d \\
&= Q \left( \int_{(0, \infty)^d} f_{t_1, \beta}(s_1) \cdots f_{t_d, \beta}(s_d) U_{1, t_1} \cdots U_{d, t_d} ds_1 \cdots ds_d \right) J \\
&= Q e^{-t_1 C_1^\beta} \cdots e^{-t_d C_d^\beta} J.
\end{aligned}$$

We conclude setting  $B_k = C_k^\beta$ , which ends the proof.  $\square$

We now give the sectorial version of Theorem 11.1.5.

**Theorem 11.2.4** *Let  $\Sigma$  be a measure space and let  $p \in (1, \infty)$ . Let  $A_1, \dots, A_d$  be commuting sectorial operators on  $L^p(\Sigma)$ . The following conditions are equivalent.*

- (1)  $(A_1, \dots, A_d)$  admits an  $H^\infty(\Sigma_{\theta_1} \times \cdots \times \Sigma_{\theta_d})$  joint functional calculus for some  $\theta_k \in (0, \frac{\pi}{2})$ ,  $k = 1, \dots, d$ .
- (2) There exist a measure space  $\Omega$ , commuting sectorial operators  $B_1, \dots, B_d$  on  $L^p(\Omega)$  of type  $< \frac{\pi}{2}$ , and two bounded operators  $J: L^p(\Sigma) \rightarrow L^p(\Omega)$  and  $Q: L^p(\Omega) \rightarrow L^p(\Sigma)$  such that

$$e^{-t_1 A_1} \dots e^{-t_d A_d} = Q e^{-t_1 B_1} \dots e^{-t_d B_d} J, \quad t_1 \geq 0, \dots, t_d \geq 0,$$

and all the  $(e^{-t B_k})_{t \geq 0}$  are semigroups of positive contractions.

*Proof of Theorem 11.2.4.* If  $B$  is a sectorial operator of type  $< \frac{\pi}{2}$  on  $L^p(\Omega)$  such that  $e^{-tB}$  is a positive contraction for any  $t \geq 0$ , then  $B$  has an  $H^\infty(\Sigma_\theta)$  functional calculus for some  $\theta < \frac{\pi}{2}$ . This result is due to Weis, see [78, 40]. Using this and arguing as in the proof of Theorem 11.1.5, the result follows at once.  $\square$

We conclude with a semigroup version of Theorem 11.1.6. We first recall that Fendler [25] proved the following semigroup version of the Akcoglu-Sucheston Theorem: Let  $(T_t)_{t \geq 0}$  be a  $C_0$ -semigroups of positive contractions on  $L^p(\Sigma)$ , with  $p \in (1, \infty)$ . Then there exist a measure space  $\Sigma'$ , a  $C_0$ -group  $(V_t)_{t \geq 0}$  of isometric isomorphisms on  $L^p(\Sigma')$  and two contractions  $J: L^p(\Sigma) \rightarrow L^p(\Sigma')$  and  $Q: L^p(\Sigma') \rightarrow L^p(\Sigma)$  such that  $T_t = QV_tJ$  for any  $t \geq 0$ .

Using this result and Lemma 11.2.1, and arguing as in the proof of Theorem 11.1.6, we obtain the following.

**Theorem 11.2.5** *Let  $\Sigma$  be a measure space and let  $p \in (1, \infty)$ . Let  $(T_{1,t})_{t \geq 0}, \dots, (T_{d,t})_{t \geq 0}$  be  $C_0$ -semigroups of positive contractions on  $L^p(\Sigma)$ . Assume further that  $(T_{1,t})_{t \geq 0}, \dots, (T_{d-1,t})_{t \geq 0}$  are commuting bounded analytic semigroups.*

Then there exist a measure space  $\Omega$ , two bounded operators  $J: L^p(\Sigma) \rightarrow L^p(\Omega)$  and  $Q: L^p(\Omega) \rightarrow L^p(\Sigma)$ , as well as commuting  $C_0$ -groups  $(U_{1,t})_{t \geq 0}, \dots, (U_{d,t})_{t \geq 0}$  of isometric isomorphisms on  $L^p(\Omega)$  such that

$$T_{1,t_1} \cdots T_{d,t_d} = QU_{1,t_1} \cdots U_{d,t_d}J, \quad t_1 \geq 0, \dots, t_d \geq 0.$$



# Chapter 12

## The case of Hilbert spaces

### 12.1 Characterisations for commuting Ritt operators

This chapter is devoted to commuting operators on Hilbert space  $H$ . We will be interested in the following two issues.

First recall that if  $T: H \rightarrow H$  is a Ritt operator, then  $T$  has an  $H^\infty(B_\gamma)$  functional calculus for some  $\gamma < \frac{\pi}{2}$  if and only if  $T$  is similar to a contraction, that is, there exists a bounded invertible operator  $S: H \rightarrow H$  such that  $S^{-1}TS$  is a contraction on  $H$ . This is proved in [53, Theorem 8.1]. We will extend this characterisation to  $d$ -tuples of Ritt operators, see Corollary 12.1.2 below.

Second let  $(T_1, \dots, T_d)$  be a  $d$ -tuple of commuting contractions on  $H$ . Recall that  $\mathcal{P}_d$  denotes the set of polynomial functions on  $d$  variables,  $d \geq 1$ . If  $d = 2$ , Ando's Theorem [9] (see also [69, Theorem 1.2]) asserts that  $\|\phi(T_1, T_2)\| \leq \|\phi\|_{\infty, \mathbb{D}^2}$  for any polynomial  $\phi \in \mathcal{P}_2$ . This result does not extend to  $d \geq 3$  and it is unknown whether there exists a universal constant  $C \geq 1$  such that

$$\|\phi(T_1, \dots, T_d)\| \leq C\|\phi\|_{\infty, \mathbb{D}^d} \quad (12.1.1)$$

for any  $\phi \in \mathcal{P}_d$  (see [69, Chapter 1] for more on this problem). Theorem 12.1.1 below shows that an estimate (12.1.1) holds true when at least  $d - 2$  of these contractions are Ritt operators.

**Theorem 12.1.1** *Let  $d \geq 3$  be an integer and let  $H$  be a Hilbert space. Let  $T_1, \dots, T_d$  be commuting operators on  $H$  such that:*

- (i) *For every  $j$  in  $\{1, \dots, d - 2\}$ ,  $T_j$  is a Ritt operator which is similar to a contraction.*
- (ii) *There exists a bounded invertible operator  $S: H \rightarrow H$  such that  $S^{-1}T_{d-1}S$  and  $S^{-1}T_dS$  are both contractions.*

*Then we have the following three properties:*

- (1) There exist a Hilbert space  $K$ , two bounded operators  $J: H \rightarrow K$  and  $Q: K \rightarrow H$  and commuting unitary operators  $U_1, \dots, U_d$  on  $K$  such that

$$T_1^{n_1} \dots T_d^{n_d} = QU_1^{n_1} \dots U_d^{n_d} J, \quad (n_1, \dots, n_d) \in \mathbb{N}^d. \quad (12.1.2)$$

- (2) There exists  $C \geq 1$  such that for any polynomial  $\phi$  in  $\mathcal{P}_d$ ,

$$\|\phi(T_1, \dots, T_d)\| \leq C \|\phi\|_{\infty, \mathbb{D}^d}. \quad (12.1.3)$$

- (3) There exists a bounded invertible operator  $S: H \rightarrow H$  such that for any  $j = 1, \dots, d$ ,  $S^{-1}T_jS$  is a contraction.

*Proof of Theorem 12.1.1.* The proof of (1) will rely on Lemma 11.1.1. The Ritt operators  $T_1, \dots, T_{d-2}$  are similar to contractions hence according to [53, Theorem 8.1],  $T_k$  has an  $H^\infty(B_{\gamma_k})$  functional calculus for some  $\gamma_k$  in  $(0, \frac{\pi}{2})$ , for all  $k = 1, \dots, d-2$ . The argument in the proof of Theorem 11.1.2 shows that there exist a measure space  $\Omega$ , unitaries  $V_1, \dots, V_{d-2}$  on  $L^2(\Omega)$  and bounded operators

$$J_1, \dots, J_{d-2}: H \longrightarrow L^2(\Omega; H) \quad \text{and} \quad Q_1, \dots, Q_{d-2}: L^2(\Omega; H) \longrightarrow H,$$

such that for any  $k = 1, \dots, d-2$ ,

$$T_k^{n_k} = Q_k(V_k \overline{\otimes} I_H)^{n_k} J_k, \quad n_k \in \mathbb{N},$$

and

$$J_k R = (I_{L^2(\Omega)} \overline{\otimes} R) J_k$$

for any  $R: H \rightarrow H$  commuting with  $T_k$ .

By assumption there exists an invertible  $W: H \rightarrow H$  such that  $W^{-1}T_{d-1}W$  and  $W^{-1}T_dW$  are contractions. By Ando's Theorem [9], there exist a Hilbert space  $L$  containing  $H$  as a closed subspace and two unitaries  $V_{d-1}, V_d: L \rightarrow L$  such that

$$(W^{-1}T_{d-1}W)^{n_{d-1}} (W^{-1}T_dW)^{n_d} = P_H V_{d-1}^{d-1} V_d^{n_d} J_H, \quad (n_{d-1}, n_d) \in \mathbb{N}^2,$$

where  $J_H: H \rightarrow L$  and  $P_H = J_H^*: L \rightarrow H$  denote the inclusion map and the orthogonal projection, respectively. This can be written as

$$T_{d-1}^{n_{d-1}} T_d^{n_d} = Q_{d-1} V_{d-1}^{d-1} V_d^{n_d} J_{d-1}, \quad (n_{d-1}, n_d) \in \mathbb{N}^2,$$

with  $Q_{d-1} = W P_H$  and  $J_{d-1} = H_H W^{-1}$ .

We can therefore apply Lemma 11.1.1 to  $(T_1, \dots, T_d)$  with  $m = d-2$  and  $Y = L$ . Thus there exist two bounded operators  $J: H \rightarrow L_2(\Omega^{d-2}; L)$  and  $Q: L_2(\Omega^{d-2}; L) \rightarrow H$ , as well as operators  $U_1, \dots, U_d$  on  $L_2(\Omega^{d-2}; L)$  such that

$$T_1^{n_1} \dots T_d^{n_d} = QU_1^{n_1} \dots U_d^{n_d} J, \quad (n_1, \dots, n_d) \in \mathbb{N}^d \quad (12.1.4)$$

and the operators  $U_k$  are given by

$$\begin{aligned} U_k &= I^{\otimes k-1} \overline{\otimes} V_k \overline{\otimes} I^{\otimes d-2-k} \overline{\otimes} I_L, & k &= 1, \dots, d-2; \\ U_k &= I^{\otimes d-2} \overline{\otimes} U_k, & k &= d-1, d. \end{aligned}$$

Clearly  $K = L_2(\Omega^{d-2}; L)$  is a Hilbert space and  $U_1, \dots, U_d$  are commuting unitaries operators. This shows (1).

(2) is a direct consequence of (1). Indeed for any  $\phi \in \mathcal{P}_d$ , (1) implies

$$\|\phi(T_1, \dots, T_d)\| \leq \|Q\| \|J\| \|\phi(U_1, \dots, U_d)\|,$$

and by the functional calculus of unitary operators which asserts that  $\|\phi(U_1, \dots, U_d)\| \leq \|\phi\|_{\infty, \mathbb{D}^d}$ , we obtain

$$\|\phi(T_1, \dots, T_d)\| \lesssim \|\phi\|_{\infty, \mathbb{D}^d}.$$

We turn now to the proof of (3). We appeal to [4, Proposition 2.4]. Consider the algebraic semigroup  $\mathcal{G} = (\mathbb{N}^d, +)$  and its representations

$$\pi: \begin{array}{ccc} \mathcal{G} & \rightarrow & B(H) \\ (n_1, \dots, n_d) & \mapsto & T_1^{n_1} \dots T_d^{n_d} \end{array} \quad \text{and} \quad \rho: \begin{array}{ccc} \mathcal{G} & \rightarrow & B(K) \\ (n_1, \dots, n_d) & \mapsto & U_1^{n_1} \dots U_d^{n_d}, \end{array}$$

where  $K$  and  $U_1, \dots, U_d$  are provided by (1).

According to (12.1.4), we have two bounded operators  $J: H \rightarrow K$  and  $Q: K \rightarrow H$  such that

$$\pi(n_1, \dots, n_d) = Q\rho(n_1, \dots, n_d)J, \quad (n_1, \dots, n_d) \in \mathcal{G}.$$

Hence by [4, Proposition 2.4], there exist two  $\rho$ -invariant closed subspaces  $M \subset N \subset K$ , as well as an isomorphism  $S: H \rightarrow N/M$  such that the compressed representation  $\tilde{\rho}: \mathcal{G} \rightarrow B(N/M)$  satisfies

$$\pi(n_1, \dots, n_d) = S^{-1}\tilde{\rho}(n_1, \dots, n_d)S, \quad (n_1, \dots, n_d) \in \mathcal{G}. \quad (12.1.5)$$

For any  $k = 1, \dots, d$ , define  $R_k: N/M \rightarrow N/M$  by  $R_k(\dot{x}) = \overbrace{U_k}^{\bullet}(x)$  for any  $x \in N$ , where  $\dot{x}$  denotes its class modulo  $M$ . Then  $R_1, \dots, R_d$  are contractions and (12.1.5) can be equivalently written as

$$T_1^{n_1} \dots T_d^{n_d} = S^{-1}R_1^{n_1} \dots R_d^{n_d}S, \quad (n_1, \dots, n_d) \in \mathcal{G}.$$

This implies that

$$T_k = S^{-1}R_kS$$

for any  $k = 1, \dots, d$ . By construction,  $N/M$  is a Hilbert space. Since it is isomorphic to  $H$ , through  $S$ , it is isometrically isomorphic to  $H$ . In other words, there exists a unitary  $V: N/M \rightarrow H$ . The above identity can be written as

$$T_k = S^{-1}V^*VR_kV^*VS$$

for any  $k = 1, \dots, d$ . Now changing  $S$  into  $VS$  and  $R_k$  into  $VR_kV^*$ , property (3) follows at once.  $\square$

The next corollary is a straightforward consequence of the previous theorem.

Before stating it, we recall that Pisier showed in [68] the existence of a pair  $(T_1, T_2)$  of commuting operators on Hilbert space  $H$  such that  $T_1$  and  $T_2$  are both similar to contractions (that is, there exist bounded invertible operators  $S_1, S_2: H \rightarrow H$  such that  $S_1^{-1}T_1S_1$  and  $S_2^{-1}T_2S_2$  are contractions) but there is no common bounded invertible  $S: H \rightarrow H$  such that  $S^{-1}T_1S$  and  $S^{-1}T_2S$  are contractions.

**Corollary 12.1.2** *Let  $d \geq 2$  be an integer and let  $(T_1, \dots, T_d)$  be a commuting family of Ritt operators on Hilbert space  $H$ . The following assertions are equivalent.*

- (1)  $(T_1, \dots, T_d)$  admits an  $H^\infty(B_{\gamma_1} \times \dots \times B_{\gamma_d})$  functional calculus for some  $\gamma_k \in (0, \frac{\pi}{2})$ ,  $k = 1, \dots, d$ .
- (2) There exists a bounded invertible operator  $S: H \rightarrow H$  such that for any  $k = 1, \dots, d$ ,  $S^{-1}T_k S$  is a contraction.

## 12.2 Characterisations for commuting sectorial operators

We finally mention that Theorem 12.1.1 and Corollary 12.1.2 have semigroup versions, that can be obtained by adapting the previous arguments.

The following results come from [49, Theorem 3.1].

**Theorem 12.2.1** *Let  $H$  be an Hilbert space and  $d \geq 2$  be an integer. Let  $(T_{1,t})_{t \geq 0}, \dots, (T_{d,t})_{t \geq 0}$  be a family of commuting  $c_0$ -semigroups, such that*

- (i) *For any  $1 \leq k \leq d - 2$ ,  $(T_{k,t})_{t \geq 0}$  is an analytic semigroups and there exists a bounded invertible operator  $S_k: H \rightarrow H$  such that  $(S_k T_{k,t} S_k^{-1})_{t \geq 0}$  is a semigroup of contractions.*
- (ii) *There exists a bounded invertible operator  $S: H \rightarrow H$  such that  $(S T_{d-1,t} S^{-1})_{t \geq 0}$  and  $(S T_{d,t} S^{-1})_{t \geq 0}$  are semigroups of contractions.*

*Then the two following properties hold.*

- (1) *There exist an Hilbert space  $K$ , two bounded operators  $J: H \rightarrow K$  and  $Q: K \rightarrow H$  and commuting  $c_0$ -semigroups of unitary operators  $(U_{1,t})_{t \geq 0}, \dots, (U_{d,t})_{t \geq 0}$  such that*

$$T_{1,t_1} \cdots T_{d,t_d} = Q U_{1,t_1} \cdots U_{d,t_d} J, \quad t_1 \geq 0, \dots, t_d \geq 0.$$

- (2) *There exists a bounded invertible operator  $W: H \rightarrow H$  such that for any  $1 \leq k \leq d$ ,  $(W T_{k,t} W^{-1})_{t \geq 0}$  is a semigroup of contractions.*

**Corollary 12.2.2** *Let  $H$  be an Hilbert space and  $d \geq 2$  be an integer. Let  $(T_{1,t})_{t \geq 0}, \dots, (T_{d,t})_{t \geq 0}$  be a family of commuting bounded analytic  $c_0$ -semigroups. Let  $-A_k$  be the negative generator of  $(T_{k,t})_{t \geq 0}$ ,  $k = 1, \dots, d$ . The two following assertions are equivalent.*

- (1)  $(A_1, \dots, A_d)$  admits an  $H^\infty(\Sigma_{\theta_1} \times \dots \times \Sigma_{\theta_d})$  joint functional calculus for some  $\theta_k$  in  $(0, \frac{\pi}{2})$ ,  $k = 1, \dots, d$ .
- (2) There exists a bounded invertible operator  $W: H \rightarrow H$  such that for any  $1 \leq k \leq d$ ,  $(W T_{k,t} W^{-1})_{t \geq 0}$  is a semigroup of contractions.

## Part V

# Square functions for $d$ -tuples





# Chapter 13

## Square functions on general Banach spaces

### 13.1 Square functions for $d$ -tuple of Ritt operators

In this section, we introduce the square functions related to a commuting family of Ritt operators on general Banach spaces.

We let  $(r_{k_1, \dots, k_d})$  be an independent Rademacher variables indexed by  $(\mathbb{N}^*)^d$ .

Theorem 13.1.2 below gives a criterion of summability, which is a generalisation of Kwapien's Theorem saying that if  $X$  is a Banach space which does not contain  $c_0$ , a series  $\sum r_n \otimes x_n$  converges in  $\text{Rad}(X)$  if and only if its partial sums are uniformly bounded.

This generalisation uses the following lemma (see [35, Prop. 6.1.5]).

**Lemma 13.1.1** *Let  $X$  be a Banach space and  $(\Omega, \mathbb{P})$  be a measure space. Let  $\xi$  and  $\eta$  be random variables from  $\Omega$  to  $X$ . If  $\eta$  is real-symmetric (i.e  $\eta$  and  $-\eta$  are identically distributed) and independent of  $\xi$ , then for all  $1 \leq p \leq \infty$  we have*

$$\|\xi\|_{L^p(\Omega; X)} \leq \|\xi + \eta\|_{L^p(\Omega; X)}. \quad (13.1.1)$$

**Theorem 13.1.2** *Let  $X$  be a Banach space which does not contain  $c_0$ . Let  $d \geq 2$  be an integer and  $(x_{k_1, \dots, k_d})_{(k_1, \dots, k_d) \in (\mathbb{N}^*)^d}$  be a family of  $X$ . Suppose that there exists a constant  $K \geq 0$  such that*

$$\left\| \sum_{1 \leq k_1, \dots, k_d \leq N} r_{k_1, \dots, k_d} \otimes x_{k_1, \dots, k_d} \right\|_{\text{Rad}((\mathbb{N}^*)^d; X)} \leq K, \quad N \in \mathbb{N}^*. \quad (13.1.2)$$

*Then the family  $(r_{k_1, \dots, k_d} \otimes x_{k_1, \dots, k_d})_{(k_1, \dots, k_d) \in (\mathbb{N}^*)^d}$  is summable in  $\text{Rad}((\mathbb{N}^*)^d; X)$ .*

*Proof of Theorem 13.1.2 :* Let  $\psi : \mathbb{N}^* \rightarrow (\mathbb{N}^*)^d$  be an arbitrary bijection. We prove that  $\sum r_{\psi(n)} \otimes x_{\psi(n)}$  converges in  $\text{Rad}(X)$ . By Kwapien's Theorem, it suffices to prove that the partial sums of this series are uniformly bounded. Let  $M \geq 1$  be an integer. There exists  $N \geq 1$  such that

$$\{\psi(1), \dots, \psi(M)\} \subset \{1, \dots, N\}^d.$$

Then by Lemma 13.1.1 and assumption (13.1.2), we have

$$\left\| \sum_{n=1}^M r_{\psi(n)} \otimes x_{\psi(n)} \right\|_{\text{Rad}(X)} \leq \left\| \sum_{1 \leq k_1, \dots, k_d \leq N} r_{k_1, \dots, k_d} \otimes x_{k_1, \dots, k_d} \right\|_{\text{Rad}((\mathbb{N}^*)^d; X)} \leq K.$$

This yields the result.  $\square$

For  $\alpha > 0$ , define  $p_\alpha : z \mapsto (1 - z)^\alpha$  which is an element of  $H_0^\infty(B_\gamma)$  for every  $\gamma$  in  $(0, \frac{\pi}{2})$ . Then we let  $(I_X - T)^\alpha = p_\alpha(T)$  in the sense of the functional calculus defined in (5.2.14).

We now define the square functions for commuting Ritt operators. Let  $T = (T_1, \dots, T_d)$  be a  $d$ -tuple of commuting Ritt operators on  $X$  and  $\alpha = (\alpha_1, \dots, \alpha_d)$  in  $(\mathbb{R}_+^*)^d$ . Let  $\Lambda$  be a subset of  $\{1, \dots, d\}$ . Define

$$\alpha_\Lambda = (\alpha_i)_{i \in \Lambda} \in (\mathbb{R}_+^*)^\Lambda.$$

Then we let  $(r_{(k_i)_{i \in \Lambda}})$  be a family of independent Rademacher variables indexed by  $(\mathbb{N}^*)^\Lambda$ . Define now for any  $(k_i)_{i \in \Lambda} \in (\mathbb{N}^*)^\Lambda$  and  $x$  in  $X$

$$x_{(k_i)} = \left[ \prod_{i \in \Lambda} k_i^{\alpha_i - \frac{1}{2}} T_i^{k_i - 1} (I_X - T_i)^{\alpha_i} \right] x.$$

We let for any  $x$  of  $X$

$$\|x\|_{T, \alpha_\Lambda} = \left\| \sum_{(k_i)_{i \in \Lambda} \in (\mathbb{N}^*)^\Lambda} r_{(k_i)} \otimes x_{(k_i)} \right\|_{\text{Rad}((\mathbb{N}^*)^\Lambda; X)}, \quad (13.1.3)$$

if the family  $(r_{(k_i)} \otimes x_{(k_i)})_{(k_i) \in (\mathbb{N}^*)^\Lambda}$  is summable and we let  $\|x\|_{T, \alpha_\Lambda} = \infty$  otherwise.

If  $\Lambda = \{1, \dots, d\}$ , we will simply use the notation

$$\|x\|_{T, \alpha} = \left\| \sum_{k_1, \dots, k_d \geq 1} \left( \prod_{i=1}^d k_i^{\alpha_i - \frac{1}{2}} \right) r_{k_1, \dots, k_d} \otimes \left( \prod_{i=1}^d T_i^{k_i - 1} (I_X - T_i)^{\alpha_i} x \right) \right\|_{\text{Rad}((\mathbb{N}^*)^d; X)}. \quad (13.1.4)$$

In the case where  $X = E(S)$  is a Banach lattice of functions with finite cotype, the Khintchine-Maurey inequalities imply that the family  $(r_{(k_i)} \otimes x_{(k_i)})_{(k_i) \in (\mathbb{N}^*)^\Lambda}$  is summable if and only if  $\left\| \left( \sum_{(k_i) \in (\mathbb{N}^*)^\Lambda} |x_{(k_i)}|^2 \right)^{\frac{1}{2}} \right\|_{E(S)}$  is finite and in this case, we have

$$\|x\|_{T, \alpha_\Lambda} \simeq \left\| \left( \sum_{(k_i) \in (\mathbb{N}^*)^\Lambda} |x_{(k_i)}|^2 \right)^{\frac{1}{2}} \right\|_{E(S)} ;$$

(see [35]).

We are interested in the  $d$ -tuples  $(T_1, \dots, T_d)$  for which there exists a constant  $K > 0$  such that for any subset  $\Lambda$  of  $\{1, \dots, d\}$  and for any  $x$  in  $X$ , we have

$$\|x\|_{T, \alpha_\Lambda} \leq K \|x\|, \quad (13.1.5)$$

for some  $d$ -tuple  $(\alpha_1, \dots, \alpha_d)$ . If such an inequality (13.1.5) happens, we will say that the  $d$ -tuple  $(T_1, \dots, T_d)$  admits a *square function estimate*.

Note that this square function estimate depends a priori on the  $d$ -tuple  $\alpha$ . We will get back to this problem with Theorem 14.2.1 below.

## 13.2 Square functions for sectorial operators

This section is devoted to the definition of square functions for a  $d$ -tuple of sectorial operators. We use here all the background given in Section 3.4. In particular, we keep its notations letting  $X$  a Banach space,  $H$  an Hilbert space and  $(g_n)$  a sequence of complex valued independant Gaussian variables on some probability space.

For any measure space  $(\Omega, \mu)$ , we let  $\gamma(\Omega, X)$  be the space defined at the end of Section 3.4.

In the sequel we will work with the measure space  $\Omega_0 = (\mathbb{R}_+^*, \frac{dt}{t})$  and its powers  $\Omega_0^d$  equipped with

$$dM(t) := \frac{dt_1}{t_1} \cdots \frac{dt_d}{t_d}, \quad t = (t_1, \dots, t_d) \in \Omega_0^d. \quad (13.2.1)$$

Throughout the rest of this section, we let  $(A_1, \dots, A_d)$  be a commuting family of sectorial operators on  $X$ , with respective types  $\omega_1, \dots, \omega_d$ . Let  $F \in H_{0,1}^\infty(\Sigma_{\nu_1} \times \cdots \times \Sigma_{\nu_d})$ , with  $\nu_k \in (\omega_k, \pi)$  for all  $k = 1, \dots, d$ . By Lebesgue's Theorem and (5.2.2), the function

$$\varphi : t = (t_1, \dots, t_d) \mapsto F(t_1 A_1, \dots, t_d A_d) \quad (13.2.2)$$

is continuous from  $\Omega_0^d$  into  $B(X)$ . Let  $x \in X$ . If  $t \mapsto F(t_1 A_1, \dots, t_d A_d)x$  belongs to  $\gamma(\Omega_0^d; X)$ , then we set

$$\|x\|_F = \left\| t \mapsto F(t_1 A_1, \dots, t_d A_d)x \right\|_{\gamma(\Omega_0^d; X)}.$$

We set  $\|x\|_F = \infty$  otherwise.

Extending the usual terminology for single sectorial operators (see [41, 30, 35]), we introduce the following.

**Definition 13.2.1** *We say that  $(A_1, \dots, A_d)$  admits a square function estimate with respect to  $F \in H_{0,1}^\infty(\Sigma_{\nu_1} \times \cdots \times \Sigma_{\nu_d})$  if there exists a constant  $K \geq 0$  such that*

$$\|x\|_F \leq K \|x\|, \quad x \in X. \quad (13.2.3)$$

We end this section telling a word on Banach lattices. Let  $(S, \mathcal{T}, \nu)$  be a measure space. If  $X = E(S)$  is a Banach lattice of functions on  $S$  with a finite cotype, [35,

Theorem 9.3.8] says that  $\|x\|_{A,F} < \infty$  if and only if the function  $\varphi$  defined by (13.2.2) belongs to  $E(S; L_2(\Omega_d))$ , that is

$$\left\| \left( \int_{\Omega_d} |\varphi(t_1, \dots, t_d)|^2 \frac{dt_1}{t_1} \cdots \frac{dt_d}{t_d} \right)^{\frac{1}{2}} \right\|_{E(S)} < \infty$$

and in this case, we have

$$\|x\|_{A,F} \simeq \left\| \int_{\Omega_d} |\varphi(t_1, \dots, t_d)|^2 \frac{dt_1}{t_1} \cdots \frac{dt_d}{t_d} \right\|_{E(S)}^{\frac{1}{2}}. \quad (13.2.4)$$

# Chapter 14

## $H^\infty$ joint functional calculus and square functions

### 14.1 From $H^\infty$ joint functional calculus to square function estimates

In this section, we aim to show the following theorem.

**Theorem 14.1.1** *Let  $X$  be a Banach space with a finite cotype. Suppose that  $T = (T_1, \dots, T_d)$  is a  $d$ -tuple of commuting Ritt operators on  $X$  which has an  $H^\infty(B_{\gamma_1} \times \dots \times B_{\gamma_d})$  joint functional calculus for some  $\gamma_1, \dots, \gamma_d$  in  $(0, \frac{\pi}{2})$ . Let  $\alpha = (\alpha_1, \dots, \alpha_d)$  in  $(\mathbb{R}_+^*)^d$ . Then  $T$  verifies the following square function estimate : there exists a constant  $K > 0$  such that for any  $x$  in  $X$  we have*

$$\|x\|_{T,\alpha} \leq K \|x\|. \quad (14.1.1)$$

**Remark 14.1.2** *We recall that every subfamily of a  $d$ -tuple  $(T_1, \dots, T_d)$  as in Theorem 14.1.1 has an  $H^\infty$  joint functional calculus too. Thus, one can obtain the inequality (14.1.1) replacing  $\alpha$  by  $\alpha_\Lambda$ ,  $\Lambda \subset \{1, \dots, d\}$ , where we refer to (13.1.3) for the definition of square function.*

This result appeals to the notion of quadratic functional calculus which is defined as follows.

**Definition 14.1.3** *Let  $X$  be a Banach space. Let  $T = (T_1, \dots, T_d)$  be a  $d$ -tuple of commuting Ritt operators on  $X$  such that  $T_k$  is of type  $a_k$  for  $k = 1, \dots, d$ . Let  $\gamma_k \in (a_k, \frac{\pi}{2})$  for  $k = 1, \dots, d$ . We say that  $T$  admits a quadratic  $H^\infty(B_{\gamma_1} \times \dots \times B_{\gamma_d})$  functional calculus if there exists a constant  $C > 0$  such that for any finite family  $(\varphi_i)_{i \in \mathcal{I}}$  in  $H_{0,1}^\infty(B_{\gamma_1} \times \dots \times B_{\gamma_d})$  and  $x$  in  $X$ ,*

$$\left\| \sum_{i \in \mathcal{I}} r_i \otimes \varphi_i(T_1, \dots, T_d)(x) \right\|_{\text{Rad}(\mathcal{I}; X)} \leq C \|x\| \left\| \left( \sum_{i \in \mathcal{I}} |\varphi_i|^2 \right)^{\frac{1}{2}} \right\|_{\infty, B_{\gamma_1} \times \dots \times B_{\gamma_d}}, \quad (14.1.2)$$

with  $(r_i)_{i \in \mathcal{I}}$  a family of independent Rademacher variables.

Theorem 14.1.1 is obtained by combining the following two propositions.

**Proposition 14.1.4** *Let  $X$  be a Banach space with a finite cotype. Let  $T = (T_1, \dots, T_d)$  be a  $d$ -tuple of commuting Ritt operators on  $X$ . Suppose that  $T$  has an  $H^\infty(B_{b_1} \times \dots \times B_{b_d})$  joint functional calculus for some  $b_1, \dots, b_d$  in  $(0, \frac{\pi}{2})$ . Then for any  $\gamma_1, \dots, \gamma_d$  such that  $\frac{\pi}{2} > \gamma_k > b_k$  for  $k = 1, \dots, d$ ,  $T$  has a quadratic  $H^\infty(B_{\gamma_1} \times \dots \times B_{\gamma_d})$  functional calculus.*

**Proposition 14.1.5** *Let  $X$  be a Banach which does not contain  $c_0$ . Let  $T = (T_1, \dots, T_d)$  be a  $d$ -tuple of commuting Ritt operators on  $X$ . Suppose that  $T$  has a quadratic  $H^\infty(B_{\gamma_1} \times \dots \times B_{\gamma_d})$  functional calculus for some  $\gamma_1, \dots, \gamma_d$  in  $(0, \frac{\pi}{2})$ . Let  $\alpha = (\alpha_1, \dots, \alpha_d)$  in  $(\mathbb{R}_+^*)^d$ . Then  $T$  satisfies a square function estimate, that is there exists a constant  $K > 0$  such that*

$$\|x\|_{T, \alpha} \leq K \|x\|. \quad (14.1.3)$$

*Proof of Proposition 14.1.4 :* The first step of the proof relies on a decomposition principle for holomorphic functions of several variables. The original idea of such a decomposition is due to Franks and McIntosh (see [26]). We have a decomposition for Stolz domain (see [6, Section 6]) which is useful for  $d$ -tuple of Ritt operators.

We let  $0 < b_k < \gamma_k < \frac{\pi}{2}$  be angles for  $k = 1, \dots, d$ .

The decomposition in [6, Section 6] provides sequences of holomorphic functions  $(\Psi_{k, i_k})_{i_k \in \mathbb{N}}$  and  $(\tilde{\Psi}_{k, i_k})_{i_k \in \mathbb{N}}$  in  $H_0^\infty(B_{b_k})$  for  $k = 1, \dots, d$  such that the two following properties hold.

(i) For every  $p > 0$ , there exists a constant  $C_p > 0$  such that

$$\sup \left\{ \sum_{i_k=1}^{\infty} |\Psi_{k, i_k}(\zeta_k)|^p : \zeta_k \in B_{b_k} \right\} \leq C_p, \quad k = 1, \dots, d, \quad (14.1.4)$$

$$\sup \left\{ \sum_{i_k=1}^{\infty} |\tilde{\Psi}_{k, i_k}(\zeta_k)|^p : \zeta_k \in B_{b_k} \right\} \leq C_p, \quad k = 1, \dots, d; \quad (14.1.5)$$

(ii) For any Banach space  $Z$  and a function  $h$  in  $H^\infty(B_{\gamma_1} \times \dots \times B_{\gamma_d}; Z)$ , there exists a family  $(a_{i_1, \dots, i_d})$  in  $Z$  indexed by  $\mathbb{N}^d$  such that for every  $(\zeta_1, \dots, \zeta_d)$  in  $B_{b_1} \times \dots \times B_{b_d}$  we have

$$h(\zeta_1, \dots, \zeta_d) = \sum_{i_1, \dots, i_d} a_{i_1, \dots, i_d} \Psi_{1, i_1}(\zeta_1) \tilde{\Psi}_{1, i_1}(\zeta_1) \cdots \Psi_{d, i_d}(\zeta_d) \tilde{\Psi}_{d, i_d}(\zeta_d), \quad (14.1.6)$$

and there exists a constant  $C > 0$  (independent of  $h$ ) such that for every  $(i_1, \dots, i_d)$  in  $(\mathbb{N}^*)^d$  and  $k = 1, \dots, d$ ,

$$\|a_{i_1, \dots, i_d}\| \leq C \|h\|_{\infty, B_{b_1} \times \dots \times B_{b_d}}. \quad (14.1.7)$$

The original proof is done for  $Z = \mathbb{C}$  in [6, Section 6]. However, This proof works as well for any Banach space  $Z$  and for any function  $h$  in  $H^\infty(B_{\gamma_1} \times \cdots \times B_{\gamma_d}; Z)$ .

Let now  $(\eta_{i_1, \dots, i_d})$  be a finite family of complex numbers and let  $m \geq 1$  be an integer. As  $(T_1, \dots, T_d)$  has an  $H^\infty(B_{b_1} \times \cdots \times B_{b_d})$  joint functional calculus, we have the estimate

$$\begin{aligned} \left\| \sum_{i_1, \dots, i_d=1}^m \eta_{i_1, \dots, i_d} \Psi_{1, i_1}(T_1) \cdots \Psi_{d, i_d}(T_d) \right\| &\lesssim \sup_{(z_1, \dots, z_d) \in B_{b_1} \times \cdots \times B_{b_d}} \left| \sum_{i_1, \dots, i_d=1}^m \eta_{i_1, \dots, i_d} \Psi_{1, i_1}(z_1) \cdots \Psi_{d, i_d}(z_d) \right| \\ &\leq \sup |\eta_{i_1, \dots, i_d}| \sup_{B_{b_1} \times \cdots \times B_{b_d}} \sum_{i_1, \dots, i_d=1}^m |\Psi_{1, i_1}(z_1) \cdots \Psi_{d, i_d}(z_d)|. \end{aligned}$$

Hence, using (14.1.4), we have

$$\sup_{m \geq 1, \eta_{i_1, \dots, i_d} = \pm 1} \left\| \sum_{i_1, \dots, i_d=1}^m \eta_{i_1, \dots, i_d} \Psi_{1, i_1}(T_1) \cdots \Psi_{d, i_d}(T_d) \right\| < \infty. \quad (14.1.8)$$

Likewise, by (14.1.5), we also have

$$\sup_{m \geq 1, \eta_{i_1, \dots, i_d} = \pm 1} \left\| \sum_{i_1, \dots, i_d=1}^m \eta_{i_1, \dots, i_d} \tilde{\Psi}_{1, i_1}(T_1) \cdots \tilde{\Psi}_{d, i_d}(T_d) \right\| < \infty. \quad (14.1.9)$$

We now prove the quadratic  $H^\infty(B_{\gamma_1} \times \cdots \times B_{\gamma_d})$  functional calculus property. Let  $(\varphi_j)_{j \in \mathcal{I}}$  be a finite family of functions of  $H_{0,1}^\infty(B_{\gamma_1} \times \cdots \times B_{\gamma_d})$ . Considering  $Z = l_2^{\mathcal{I}}$  with its canonical basis denoted by  $(e_j)_{j \in \mathcal{I}}$ , we define

$$h = \sum_{j \in \mathcal{I}} \varphi_j \otimes e_j \quad (14.1.10)$$

regarded as an element of  $H^\infty(B_{\gamma_1} \times \cdots \times B_{\gamma_d}; Z)$ .

Let now  $(a_{i_1, \dots, i_d})$  be the family of  $Z$  provided by (14.1.6) for  $h$  defined in (14.1.10). We decompose every  $a_{i_1, \dots, i_d}$  on basis  $(e_j)$  as

$$a_{i_1, \dots, i_d} = \sum_{j \in \mathcal{I}} c_{(i_1, \dots, i_d); j} e_j, \quad (i_1, \dots, i_d) \in (\mathbb{N}^*)^d.$$

For convenience, we will use notations  $(i) = (i_1, \dots, i_d)$ . Then we can write for every  $j$  in  $\mathcal{I}$  and  $(\zeta_1, \dots, \zeta_d) \in B_{b_1} \times \cdots \times B_{b_d}$

$$\varphi_j(\zeta_1, \dots, \zeta_d) = \sum_{(i) \in \mathbb{N}^d} c_{(i); j} \Psi_{1, i_1}(\zeta_1) \tilde{\Psi}_{1, i_1}(\zeta_1) \cdots \Psi_{d, i_d}(\zeta_d) \tilde{\Psi}_{d, i_d}(\zeta_d). \quad (14.1.11)$$

We now use that

$$\|h\|_{\infty, B_{\gamma_1} \times \cdots \times B_{\gamma_d}} = \left\| \left( \sum_{j \in \mathcal{I}} |\varphi_j|^2 \right)^{\frac{1}{2}} \right\|_{\infty, B_{\gamma_1} \times \cdots \times B_{\gamma_d}}$$



together with estimation (14.1.7) to say that

$$\sup_{(i_1, \dots, i_d) \in (\mathbb{N}^*)^d} \left( \sum_{j \in \mathcal{I}} |c_{(i);j}|^2 \right)^{\frac{1}{2}} \lesssim \left\| \left( \sum_{j \in \mathcal{I}} |\varphi_j|^2 \right)^{\frac{1}{2}} \right\|_{\infty, B_{\gamma_1} \times \dots \times B_{\gamma_d}}. \quad (14.1.12)$$

For any  $j$  in  $\mathcal{I}$  and any integer  $m \geq 1$ , we consider the function

$$h_{j;m} = \sum_{1 \leq i_1, \dots, i_d \leq m} c_{(i);j} \Psi_{1,i_1} \tilde{\Psi}_{1,i_1} \otimes \dots \otimes \Psi_{d,i_d} \tilde{\Psi}_{d,i_d},$$

which belongs to  $H_0^\infty(B_{\gamma_1} \times \dots \times B_{\gamma_d})$  and pointwise converges to  $\varphi_j$  when  $m \rightarrow \infty$ , according to (14.1.11).

Fix now  $(r_j)_{j \in \mathcal{I}}$  and  $(r_{i_1, \dots, i_d})_{i_1, \dots, i_d \geq 1}$  two families of Rademacher independent variables on some probability space  $(\Omega_0, \mathbb{P})$ . Let  $x$  in  $X$ . By the Khintchine-Kahane inequality, we have

$$\begin{aligned} & \left\| \sum_{j \in \mathcal{I}} r_j \otimes h_{j;m}(T_1, \dots, T_d)x \right\|_{\text{Rad}(X)} \\ &= \left( \int_{\Omega_0} \left\| \sum_{(i);j} c_{(i);j} r_j(\omega) \Psi_{1,i_1}(T_1) \tilde{\Psi}_{1,i_1}(T_d) \dots \Psi_{d,i_d}(T_d) \tilde{\Psi}_{d,i_d}(T_d)x \right\|^2 d\mathbb{P}(\omega) \right)^{\frac{1}{2}} \\ &\lesssim \int_{\Omega_0} \left\| \sum_{(i)} \Psi_{1,i_1}(T_1) \dots \Psi_{d,i_d}(T_d) \left( \sum_j c_{(i);j} r_j(\omega) \tilde{\Psi}_{1,i_1}(T_d) \dots \tilde{\Psi}_{d,i_d}(T_d)x \right) \right\| d\mathbb{P}(\omega), \end{aligned} \quad (14.1.13)$$

where indexes are such that  $1 \leq i_1, \dots, i_d \leq m$  and  $j$  in  $\mathcal{I}$ .

For any  $(x_{i_1, \dots, i_d})$ , we note that

$$\begin{aligned} & \sum_{1 \leq i_1, \dots, i_d \leq m} \Psi_{1,i_1}(T_1) \dots \Psi_{d,i_d}(T_d) x_{i_1, \dots, i_d} \\ &= \int_{\Omega_0} \left( \sum_{1 \leq i_1, \dots, i_d \leq m} r_{i_1, \dots, i_d}(\omega) \Psi_{1,i_1}(T_1) \dots \Psi_{d,i_d}(T_d) \right) \left( \sum_{1 \leq i_1, \dots, i_d \leq m} r_{i_1, \dots, i_d}(\omega) x_{i_1, \dots, i_d} \right) d\mathbb{P}(\omega), \end{aligned}$$

using that the  $r_{i_1, \dots, i_d}$ 's are independent.

Hence, by (14.1.8), we obtain

$$\left\| \sum_{1 \leq i_1, \dots, i_d \leq m} \Psi_{1,i_1}(T_1) \dots \Psi_{d,i_d}(T_d) x_{i_1, \dots, i_d} \right\| \lesssim \int_{\Omega_0} \left\| \sum_{1 \leq i_1, \dots, i_d \leq m} r_{i_1, \dots, i_d}(\omega) x_{i_1, \dots, i_d} \right\| d\mathbb{P}(\omega).$$

We apply it in (14.1.13) with

$$x_{i_1, \dots, i_d} = \sum_{j \in \mathcal{I}} c_{(i_1, \dots, i_d);j} r_j(\omega) \tilde{\Psi}_{1,i_1}(T_d) \dots \tilde{\Psi}_{d,i_d}(T_d)x$$

to have

$$\left\| \sum_{j \in \mathcal{I}} r_j \otimes h_{j;m}(T_1, \dots, T_d)x \right\|_{\text{Rad}(X)} \lesssim \left\| \sum_{(i);j} c_{(i);j} r_j \otimes r_{(i)} \tilde{\Psi}_{1,i_1}(T_d) \cdots \tilde{\Psi}_{d,i_d}(T_d)x \right\|_{\text{Rad}(\text{Rad}(X))}.$$

We recall now that from [36], we have

$$\left\| \sum_{(i);j} z_{(i);j} r_{(i)} \otimes r_j \otimes y_{(i)} \right\|_{\text{Rad}(\text{Rad}(X))} \lesssim \sup_{(i)} \left( \sum_j |z_{(i);j}|^2 \right)^{\frac{1}{2}} \left\| \sum_{(i)} r_{(i)} \otimes y_{(i)} \right\|_{\text{Rad}(X)}$$

for any finite family of complex numbers  $(z_{(i);j})$  and elements  $(y_{(i)})$  of  $X$ .

We apply this last inequality with  $y_{(i)} = y_{i_1, \dots, i_d} = \tilde{\Psi}_{1,i_1}(T_d) \cdots \tilde{\Psi}_{d,i_d}(T_d)x$  together with (14.1.9), (14.1.12) and (14.1.5) to obtain

$$\left\| \sum_{j \in \mathcal{I}} r_j \otimes h_{j;m}(T_1, \dots, T_d)x \right\|_{\text{Rad}(X)} \lesssim \|x\| \left\| \left( \sum_{j \in \mathcal{I}} |\varphi_j|^2 \right)^{\frac{1}{2}} \right\|_{\infty, B_{\gamma_1} \times \cdots \times B_{\gamma_d}}. \quad (14.1.14)$$

The final step of the proof uses approximation arguments. We use the same argument as in the end of the proof of [6, Proposition 3.2]. We reproduce it here for convenience.

The inequality (14.1.14) holds true when  $(T_1, \dots, T_d)$  is replaced by  $(rT_1, \dots, rT_d)$  for any  $r \in (0, 1)$ . Further, we know that  $(h_{j;m})_{m \geq 1}$  is a bounded sequence of the space  $H_0^\infty(B_{\gamma_1} \times \cdots \times B_{\gamma_d})$ . Moreover, the sequence  $(h_{j;m})_{m \geq 1}$  converges pointwise to  $\varphi_j$ . Hence applying Lebesgue's dominated convergence Theorem twice we have for any  $j$

$$\lim_{m \rightarrow \infty} h_{j;m}(rT_1, \dots, rT_d) = \varphi_j(rT_1, \dots, rT_d)$$

for any  $r \in (0, 1)$ , and

$$\lim_{r \rightarrow 1^-} \varphi_j(rT_1, \dots, rT_d) = \varphi_j(T_1, \dots, T_d).$$

We therefore deduce from (14.1.14) that

$$\left\| \sum_{j \in \mathcal{I}} r_j \otimes \varphi_j(T_1, \dots, T_d)x \right\|_{\text{Rad}(X)} \lesssim \|x\| \left\| \left( \sum_{j \in \mathcal{I}} |\varphi_j|^2 \right)^{\frac{1}{2}} \right\|_{\infty, B_{\gamma_1} \times \cdots \times B_{\gamma_d}},$$

which concludes the proof.  $\square$

The proof of Proposition 14.1.5 will require the following lemma.

**Lemma 14.1.6** *Let  $\alpha > 0$  and  $\gamma \in (0, \frac{\pi}{2})$ . There exists a constant  $C > 0$  such that for any  $z$  in  $B_\gamma$ , we have*

$$|1 - z|^{2\alpha} \sum_{k=1}^{+\infty} k^{2\alpha-1} (|z|^2)^{k-1} \leq C. \quad (14.1.15)$$

*Proof of Lemma 14.1.6 :* We fix  $x$  in  $[0, 1)$  and  $\alpha > 0$ . Let  $f : t \mapsto t^{2\alpha-1}x^{t-1}$  from  $(0, \infty)$  into  $\mathbb{R}$ . If  $2\alpha - 1 > 0$ ,  $f$  is increasing on  $(0, \frac{2\alpha-1}{-\log(x)})$  and  $f$  is decreasing on  $(\frac{2\alpha-1}{-\log(x)}, \infty)$ .

Using a comparison test, we obtain an estimate

$$\sum_{k=1}^{\infty} k^{2\alpha-1}x^{k-1} \leq C \int_0^{\infty} t^{2\alpha-1}x^{t-1}dt, \quad (14.1.16)$$

where the constant  $C > 0$  does not depend on  $x$ .

Changing variable  $u = -\log(x)t$  in (14.1.16), we obtain

$$\sum_{k=1}^{\infty} k^{2\alpha-1}x^{k-1} \lesssim \frac{\Gamma(2\alpha)}{x(-\log(x))^{2\alpha}}, \quad (14.1.17)$$

where  $\Gamma(y) = \int_0^{\infty} u^{y-1}e^{-u}du, y > 0$ .

If we now take  $z$  in a Stolz domain  $B_\gamma$ , with  $z$  in a neighbourhood of 1, we can apply (14.1.17) to have

$$|1 - z|^{2\alpha} \sum_{k=1}^n k^{2\alpha-1}(|z|^2)^{k-1} \lesssim \frac{\Gamma(2\alpha) |1 - z|^{2\alpha}}{|z|^2 (-\log(|z|^2))^{2\alpha}} \lesssim \frac{\Gamma(2\alpha) |1 - z|^{2\alpha}}{2^{2\alpha} |z|^2 (1 - |z|)^{2\alpha}} \lesssim \left( \frac{|1 - z|}{1 - |z|} \right)^{2\alpha}.$$

We know that there exists a constant  $A > 0$  such that for every  $\omega \in B_\gamma \setminus \{1\}$ , we have  $\frac{|1-\omega|}{1-|\omega|} \leq A$ . Thus,  $|1 - z|^{2\alpha} \sum_{k=1}^n k^{2\alpha-1}(|z|^2)^{k-1}$  is uniformly bounded for  $z \in B_\gamma$  and  $n \in \mathbb{N}^*$ .

If  $2\alpha - 1 < 0$ , the function  $f : t \mapsto t^{2\alpha-1}x^{t-1}$  is decreasing on  $\mathbb{R}^{+*}$  and one can proceed in the same way to obtain (14.1.16) and (14.1.17).  $\square$

*Proof of Proposition 14.1.5 :* Let  $T = (T_1, \dots, T_d)$  be a  $d$ -tuple of commuting Ritt operators and  $\alpha = (\alpha_1, \dots, \alpha_d)$  in  $(\mathbb{R}_+^*)^d$ . In all this proof, we will use notation

$$x_{k_1, \dots, k_d} = \prod_{j=1}^d k_j^{\alpha_j - \frac{1}{2}} T_j^{k_j - 1} (I_X - T_j)^{\alpha_j} (x),$$

in order to rewrite (13.1.4) as

$$\|x\|_{T, \alpha} = \left\| \sum_{k_1, \dots, k_d \geq 1} r_{k_1, \dots, k_d} \otimes x_{k_1, \dots, k_d} \right\|_{\text{Rad}((\mathbb{N}^*)^d; X)}. \quad (14.1.18)$$

As  $X$  does not contain  $c_0$ , Theorem 13.1.2 ensures that we only need to prove the existence of a constant  $K > 0$  such that for any integer  $n \geq 1$  we have

$$\left\| \sum_{1 \leq k_1, \dots, k_d \leq n} r_{k_1, \dots, k_d} \otimes x_{k_1, \dots, k_d} \right\|_{\text{Rad}((\mathbb{N}^*); X)} \leq K \|x\|. \quad (14.1.19)$$

To have inequality (14.1.19), we apply (14.1.2) with the functions

$$\varphi_{k_1, \dots, k_d}(z_1, \dots, z_d) = k_1^{\alpha_1 - \frac{1}{2}} \dots k_d^{\alpha_d - \frac{1}{2}} z_1^{k_1 - 1} (1 - z_1)^{\alpha_1} \dots z_d^{k_d - 1} (1 - z_d)^{\alpha_d}.$$

It is clear that all these functions belong to  $H_{0,1}^\infty(B_{\gamma_1} \times \cdots \times B_{\gamma_d})$  for any  $\gamma_1, \dots, \gamma_d$  in  $(0, \frac{\pi}{2})$ . Moreover, definition of fractional power calculus says that

$$\varphi_{k_1, \dots, k_d}(T_1, \dots, T_d) = k_1^{\alpha_1 - \frac{1}{2}} \cdots k_d^{\alpha_d - \frac{1}{2}} T_1^{k_1 - 1} (I_X - T_1)^{\alpha_1} \cdots z_d^{k_d - 1} (I_X - T_d)^{\alpha_d}.$$

As  $(T_1, \dots, T_d)$  has a quadratic  $H^\infty(B_{\gamma_1} \times \cdots \times B_{\gamma_d})$  functional calculus by Proposition 14.1.4 we have an estimate

$$\left\| \sum_{1 \leq k_1, \dots, k_d \leq n} r_{k_1, \dots, k_d} \otimes \varphi_{k_1, \dots, k_d}(T_1, \dots, T_d)(x) \right\|_{\text{Rad}(X)} \lesssim \|x\| \left\| \left( \sum_{1 \leq k_1, \dots, k_d \leq n} |\varphi_{k_1, \dots, k_d}|^2 \right)^{\frac{1}{2}} \right\|_{\infty, B_{\gamma_1} \times \cdots \times B_{\gamma_d}}.$$

Next we study the right hand side. Let  $(z_1, \dots, z_d) \in B_{\gamma_1} \times \cdots \times B_{\gamma_d}$ . We write

$$\sum_{1 \leq k_1, \dots, k_d \leq n} |\varphi_{k_1, \dots, k_d}(z_1, \dots, z_d)|^2 = \prod_{j=1}^d \left[ |1 - z_j|^{2\alpha_j} \left( \sum_{k_j=1}^n k_j^{2\alpha_j - 1} (|z_j|^2)^{k_j - 1} \right) \right],$$

According to Lemma 14.1.6, there exists constants  $C_1, \dots, C_d > 0$  such that for any  $j$  in  $\{1, \dots, d\}$ ,  $z_j$  in  $B_{\gamma_j}$  and  $n$  in  $\mathbb{N}^*$ , we have

$$|1 - z_j|^{2\alpha_j} \left( \sum_{k_j=1}^n k_j^{2\alpha_j - 1} (|z_j|^2)^{k_j - 1} \right) \leq C_j.$$

Combining these inequalities, we see that  $\prod_{j=1}^d \left[ |1 - z_j|^{2\alpha_j} \left( \sum_{k_j=1}^n k_j^{2\alpha_j - 1} (|z_j|^2)^{k_j - 1} \right) \right]$  is uniformly bounded on  $z_j \in B_{\gamma_j}$ ,  $j = 1, \dots, d$  and  $n \in \mathbb{N}^*$ , that is there exists a constant  $K > 0$  such that

$$\left\| \left( \sum_{1 \leq k_1, \dots, k_d \leq n} |\varphi_{k_1, \dots, k_d}|^2 \right)^{\frac{1}{2}} \right\|_{\infty, B_{\gamma_1} \times \cdots \times B_{\gamma_d}} \leq K,$$

with  $K$  not depending on  $n$ . We finally obtain (14.1.19) and this suffices to prove the result.  $\square$

We now pass to the sectorial operators. For this case, the main result of this section is the following theorem, which generalises [35, Theorem 10.4.16] (see also [41, Proposition 7.7] and [15, Corollary 6.7] and [56, Theorem 7.6]) from the single to the multivariable case.

**Theorem 14.1.7** *Assume that  $X$  has finite cotype and that  $(A_1, \dots, A_d)$  admits an  $H^\infty(\Sigma_{\theta_1} \times \cdots \times \Sigma_{\theta_d})$  joint functional calculus. Then for any  $F \in H_{0,1}^\infty(\Sigma_{\nu_1} \times \cdots \times \Sigma_{\nu_d})$ , with  $\nu_k \in (\theta_k, \pi)$ ,  $(A_1, \dots, A_d)$  admits a square function estimate with respect to  $F$ .*

The proof of this theorem will use the following proposition, which extends [53, Theorem 6.3]. In the case of a single sectorial operator, a similar approach was developed in [30] (see also [35, 10.4.c]).

**Proposition 14.1.8** *Assume that  $X$  has finite cotype and that  $(A_1, \dots, A_d)$  admits an  $H^\infty(\Sigma_{\theta_1} \times \dots \times \Sigma_{\theta_d})$  joint functional calculus. Then for any  $\nu_k \in (\theta_k, \pi)$ ,  $k = 1, \dots, d$ , there exists a constant  $K \geq 0$  such that*

$$\left\| \sum_{j=1}^n r_j \otimes F_j(A_1, \dots, A_d)x \right\|_{\text{Rad}(X)} \leq K \|x\| \left\| \left( \sum_{j=1}^n |F_j|^2 \right)^{\frac{1}{2}} \right\|_{\infty, \Sigma_{\nu_1} \times \dots \times \Sigma_{\nu_d}},$$

for any  $n \geq 1$ , for any  $F_1, \dots, F_n$  in  $H_0^\infty(\Sigma_{\nu_1} \times \dots \times \Sigma_{\nu_d})$  and for any  $x \in X$ .

The proof of [53, Theorem 6.3] (which corresponds to Proposition 14.1.8 in the case  $d = 1$ ) relies on the Franks-McIntosh decomposition and more precisely on Theorem 6.0.3 and Remark 7.0.7 in place of Theorem 7.0.1.

The proof of Proposition 14.1.8 is a repetition of the proof of [53, Theorem 6.3], so we skip it.

*Proof of Theorem 14.1.7.* We may and do assume that  $F \in H_0^\infty(\Sigma_{\nu_1} \times \dots \times \Sigma_{\nu_d})$ . If  $x \in N(A_k)$  for some  $k \in \{1, \dots, d\}$ , then  $F(t_1 A_1, \dots, t_d A_d)x = 0$  for any  $t = (t_1, \dots, t_d) \in \Omega_0^d$ , hence  $\|x\|_F = 0$ . It therefore follows from Lemma 5.1.2 that to prove an estimate (13.2.3), we may assume that

$$x \in \overline{\text{Ran}(A_1)} \cap \dots \cap \overline{\text{Ran}(A_d)}. \quad (14.1.20)$$

Recall (13.2.1). Let  $(e_j)_{j \geq 1}$  be an orthonormal sequence of  $L^2(\Omega_0^d)$ . For any  $j \geq 1$ , define a function  $F_j: \Sigma_{\nu_1} \times \dots \times \Sigma_{\nu_d} \rightarrow \mathbb{C}$  by

$$F_j(z_1, \dots, z_d) = \int_{\Omega_0^d} F(t_1 z_1, \dots, t_d z_d) e_j(t) dM(t).$$

It is clear that  $F_j \in H^\infty(\Sigma_{\nu_1} \times \dots \times \Sigma_{\nu_d})$ . (Note that  $F_j$  does not necessarily belong to  $H_0^\infty(\Sigma_{\nu_1} \times \dots \times \Sigma_{\nu_d})$ ; this is why we need the approximation process below.)

For any integer  $m \geq 1$ , consider  $\Phi_m$  and  $\Phi_m^{\otimes d}$  defined by (5.2.9) and (5.2.12). Then  $F_j \Phi_m^{\otimes d}$  belongs to  $H_0^\infty(\Sigma_{\nu_1} \times \dots \times \Sigma_{\nu_d})$  for any  $j \geq 1$ .

The argument in the proof of [56, Lemma 6.5] shows that the function

$$t \mapsto F(t_1 A_1, \dots, t_d A_d) \Phi_m(A_1) \cdots \Phi_m(A_d)$$

belongs to  $L^2(\Omega_0^d; B(X))$ . Then for any  $j \geq 1$ , the function

$$t \mapsto F(t_1 A_1, \dots, t_d A_d) \Phi_m(A_1) \cdots \Phi_m(A_d) e_j(t)$$

belongs to  $L^1(\Omega_0^d; B(X))$  and by Fubini's theorem, we obtain that

$$(F_j \Phi_m^{\otimes d})(A_1, \dots, A_d) = \int_{\Omega_0^d} F(t_1 A_1, \dots, t_d A_d) \Phi_m(A_1) \cdots \Phi_m(A_d) e_j(t) dM(t). \quad (14.1.21)$$

According to Proposition 14.1.8 and to the finite cotype assumption, there is a constant  $K$  such that

$$\left\| \sum_{j=1}^n g_j \otimes (F_j \Phi_m^{\otimes d})(A_1, \dots, A_d)x \right\|_{G(X)} \leq K \|x\| \left\| \left( \sum_{j=1}^n |F_j \Phi_m^{\otimes d}|^2 \right)^{\frac{1}{2}} \right\|_{\infty, \Sigma_{\nu_1} \times \dots \times \Sigma_{\nu_d}}, \quad (14.1.22)$$

for any  $n \geq 1$ ,  $m \geq 1$  and  $x \in X$ . Let  $(z_1, \dots, z_d) \in \Sigma_{\nu_1} \times \dots \times \Sigma_{\nu_d}$ . Since  $(e_j)_{j \geq 1}$  is an orthonormal family of  $L^2(\Omega_0^d)$ , we have

$$\left( \sum_{j=1}^n |F_j(z_1, \dots, z_d)|^2 \right)^{\frac{1}{2}} \leq \left( \int_{\Omega_0^d} |F(t_1 z_1, \dots, t_d z_d)|^2 dM(t) \right)^{\frac{1}{2}}.$$

By assumption,  $F \in H_0^\infty(\Sigma_{\nu_1} \times \dots \times \Sigma_{\nu_d})$  hence by (5.2.1) there exist real numbers  $C \geq 0$  and  $s_1, \dots, s_d > 0$  (not depending on  $z_1, \dots, z_d$ ) such that

$$\begin{aligned} \left( \sum_{j=1}^n |F_j(z_1, \dots, z_d)|^2 \right)^{\frac{1}{2}} &\leq C \left( \int_{\Omega_0^d} \prod_{k=1}^d \frac{|t_k z_k|^{s_k}}{(1 + |t_k z_k|^{s_k})^2} dM(t) \right)^{\frac{1}{2}} \\ &= C \prod_{k=1}^d \left( \int_0^\infty \frac{|t_k z_k|^{s_k}}{(1 + |t_k z_k|^{s_k})^2} \frac{dt_k}{t_k} \right)^{\frac{1}{2}}. \end{aligned}$$

A standard calculation shows that that the right-hand side of the above inequality is bounded on  $\Sigma_{\nu_1} \times \dots \times \Sigma_{\nu_d}$ . Moreover the sequence  $(\phi_m^{\otimes d})_{m \geq 1}$  is uniformly bounded on  $\Sigma_{\nu_1} \times \dots \times \Sigma_{\nu_d}$ . Implementing these estimates in (14.1.22) we obtain a constant  $K' > 0$  such that

$$\left\| \sum_{j=1}^n g_j \otimes (F_j \Phi_m^{\otimes d})(A_1, \dots, A_d)x \right\|_{G(X)} \leq K' \|x\|, \quad x \in X,$$

for all  $n, m \geq 1$ .

Since  $X$  has finite cotype, it does not contain  $c_0$ . Hence according to (14.1.22) and Lemma 3.4.6, the above estimate shows that the function

$$(t_1, \dots, t_d) \mapsto F(t_1 A_1, \dots, t_d A_d) \Phi_m(A_1) \cdots \Phi_m(A_d)x$$

belongs to  $\gamma(\Omega_0^d; X)$ , and that its norm in the space  $\gamma(L^2(\Omega_0^d); X)$  is  $\leq K' \|x\|$ .

Now assume that  $x$  satisfies (14.1.20). Applying (5.2.10) to  $A_k$  for any  $k = 1, \dots, d$ , we obtain that  $\Phi_m(A_1) \cdots \Phi_m(A_d)x \rightarrow x$  when  $m \rightarrow \infty$ . It therefore follows from [41, Lemma 4.10 (b)] that the function  $(t_1, \dots, t_d) \mapsto F(t_1 A_1, \dots, t_d A_d)x$  belongs to  $\gamma(\Omega_0^d; X)$  and that its norm in the space  $\gamma(L^2(\Omega_0^d); X)$  is  $\leq K' \|x\|$ . This proves the square function estimate.  $\square$

## 14.2 From square functions to $H^\infty$ joint functional calculus

In this section, we show that square function estimates imply the  $H^\infty$  joint functional calculus property supposing we have  $R$ -Ritt commuting operators or  $R$ -sectorial operators.

Let  $T_1, \dots, T_d$  be commuting Ritt operators or let  $A_1, \dots, A_d$  be commuting sectorial operators acting on reflexive Banach space  $X$ . We recall by Lemma 5.1.2 and Lemma 5.1.5 that we have an "ergodic decomposition"

$$X = \bigoplus_{\Lambda \subset \{1, \dots, d\}} X_\Lambda, \quad (14.2.1)$$

where we let

$$X_\Lambda = \left[ \bigcap_{i \in \Lambda} \overline{\text{Ran}(I_X - T_i)} \right] \cap \left[ \bigcap_{i \notin \Lambda} \text{Ker}(I_X - T_i) \right], \quad (14.2.2)$$

or

$$X_\Lambda = \left[ \bigcap_{i \in \Lambda} \overline{\text{Ran}(A_i)} \right] \cap \left[ \bigcap_{i \notin \Lambda} \text{Ker}(A_i) \right], \quad (14.2.3)$$

for any subset  $\Lambda$  of  $\{1, \dots, d\}$ .

**Theorem 14.2.1** *Let  $X$  be a reflexive Banach space such that  $X$  and  $X^*$  have finite cotype. Let  $T = (T_1, \dots, T_d)$  be commuting Ritt operators on  $X$  such that every  $T_k$  is  $R$ -Ritt of  $R$ -type  $\delta_k \in (0, \frac{\pi}{2})$  for  $k = 1, \dots, d$ . Suppose that there exists a constant  $C > 0$  such that for any subset  $\Lambda$  of  $\{1, \dots, d\}$ , there exist  $\alpha_\Lambda = (\alpha_k)_{k \in \Lambda}$  and  $\beta_\Lambda = (\beta_k)_{k \in \Lambda}$  in  $(\mathbb{R}_+^*)^\Lambda$  such that*

$$\|x\|_{T, \alpha_\Lambda} \leq C \|x\|, \quad x \in X, \quad (14.2.4)$$

$$\|y\|_{T^*, \beta_\Lambda} \leq C \|y\|, \quad y \in X^*. \quad (14.2.5)$$

*Then  $(T_1, \dots, T_d)$  admits a  $H^\infty(B_{\gamma_1} \times \dots \times B_{\gamma_d})$  joint functional calculus for every  $\gamma_k \in (\delta_k, \frac{\pi}{2})$ .*

**Remark 14.2.2** *We note that Theorem 14.2.1 and Theorem 14.1.1 are new results, even for a single operator  $T$ , when  $\alpha \notin \mathbb{N}^*$ .*

Firstly, we prove the equivalence between square functions on reflexive spaces.

**Theorem 14.2.3** *Let  $X$  be a reflexive Banach space with finite cotype. Let  $T = (T_1, \dots, T_d)$  be a  $d$ -tuple of Ritt operators such that every  $T_k$  is an  $R$ -Ritt operator,  $k = 1, \dots, d$ . Let  $\alpha = (\alpha_1, \dots, \alpha_d)$  and  $\beta = (\beta_1, \dots, \beta_d)$  be  $d$ -tuples of  $(\mathbb{R}_+^*)^d$ . Then  $\|\cdot\|_{T, \alpha}$  and  $\|\cdot\|_{T, \beta}$  are equivalent, that is there exists a constant  $C > 0$  such that*

$$C^{-1} \|x\|_{T, \alpha} \leq \|x\|_{T, \beta} \leq C \|x\|_{T, \alpha}. \quad (14.2.6)$$

*Proof of Theorem 14.2.3.* This proof uses many ideas of the one of [5, Theorem 3.3]. Let  $\gamma_1, \dots, \gamma_d$  be positive numbers such that every  $N_j = \alpha_j + \gamma_j$  is a positive integer,  $j = 1, \dots, d$ . For every integers  $k \geq 1$  and  $j = 1, \dots, d$ , we let

$$c_{k,j} = \frac{k(k+1) \cdots (k+N_j-2)}{k^{\alpha_j - \frac{1}{2}}}$$

if  $N_j \geq 2$  and  $c_{k,j} = \frac{1}{k^{\alpha_j - \frac{1}{2}}}$  if  $N_j = 1$ . By [5, Theorem 3.3], we have for  $j = 1, \dots, d$  and  $r$  in  $(0, 1)$

$$\sum_{k=1}^{\infty} c_{k,j} k^{\alpha_j - \frac{1}{2}} (rT_j)^{2k-2} (I_X - (rT_j)^2)^{N_j} = (N_j - 1)! I_X$$

the series being absolutely convergent and as every  $(I_X + rT_j)^{N_j}$  is invertible

$$\sum_{k=1}^{\infty} c_{k,j} (rT_j)^{k-1} (I_X - rT_j)^{\gamma_j} k^{\alpha_j - \frac{1}{2}} (rT_j)^{k-1} (I_X - rT_j)^{\alpha_j} = (N_j - 1)! (I_X + rT_j)^{-N_j}. \quad (14.2.7)$$

Now let for any intergers  $k_1, \dots, k_d, m_1, \dots, m_d \geq 1$ ,  $r$  in  $(0, 1)$  and  $j = 1, \dots, d$ ,

$$S_{j,m_j,k_j} = m_j^{\beta_j - \frac{1}{2}} (rT_j)^{m_j+k_j-2} (I_X - rT_j)^{\beta_j + \gamma_j},$$

$$R_{j,k_j} = k_j^{\alpha_j - \frac{1}{2}} (rT_j)^{k_j-1} (I_X - rT_j)^{\alpha_j}.$$

Then we define for any intergers  $m_1, \dots, m_d \geq 1$ ,  $r$  in  $(0, 1)$  and  $x$  in  $X$

$$y_{(m_1, \dots, m_d)}(r) = \left[ \prod_{j=1}^d (N_j - 1)! (I_X + rT_j)^{-N_j} m_j^{\beta_j - \frac{1}{2}} (rT_j)^{m_j-1} (I - rT_j)^{\beta_j} \right] x.$$

As every series in (14.2.7) is absolutely convergent, one can see that

$$y_{(m_1, \dots, m_d)}(r) = \sum_{k_1, \dots, k_d=1}^{\infty} c_{k_1,1} \cdots c_{k_d,d} S_{1,m_1,k_1} \cdots S_{d,m_d,k_d} R_{1,k_1} \cdots R_{d,k_d} x.$$

For any integer  $n \geq 1$ , define the partial sum

$$y_{(m_1, \dots, m_d); n}(r) = \sum_{k_1, \dots, k_d=1}^n c_{k_1,1} \cdots c_{k_d,d} S_{1,m_1,k_1} \cdots S_{d,m_d,k_d} R_{1,k_1} \cdots R_{d,k_d} x.$$

Next we consider the square functions as follows. For any integers  $k_1, \dots, k_d \geq 1$  we let

$$x_{(k_1, \dots, k_d); (\alpha_1, \dots, \alpha_d)} = k_1^{\alpha_1 - \frac{1}{2}} \cdots k_d^{\alpha_d - \frac{1}{2}} (rT_1)^{k_1-1} \cdots (rT_d)^{k_d-1} (I_X - rT_1)^{\alpha_1} \cdots (I_X - rT_d)^{\alpha_d} x$$

and similary for other integers  $m_1, \dots, m_d \geq 1$

$$x_{(m_1, \dots, m_d); (\beta_1, \dots, \beta_d)} = m_1^{\beta_1 - \frac{1}{2}} \cdots m_d^{\beta_d - \frac{1}{2}} (rT_1)^{m_1-1} \cdots (rT_d)^{m_d-1} (I_X - rT_1)^{\beta_1} \cdots (I_X - rT_d)^{\beta_d} x.$$

By this way, square functions (13.1.4) may be written as

$$\|x\|_{(rT_1, \dots, rT_d), (\alpha_1, \dots, \alpha_d)} = \left\| \sum_{k_1, \dots, k_d=1}^{\infty} r_{k_1, \dots, k_d} \otimes x_{(k_1, \dots, k_d); (\alpha_1, \dots, \alpha_d)} \right\|_{\text{Rad}(\mathbb{N}^d, X)}$$



and

$$\|x\|_{(rT_1, \dots, rT_d), (\beta_1, \dots, \beta_d)} = \left\| \sum_{m_1, \dots, m_d=1}^{\infty} r_{m_1, \dots, m_d} \otimes x_{(m_1, \dots, m_d); (\beta_1, \dots, \beta_d)} \right\|_{\text{Rad}(\mathbb{N}^d, X)}. \quad (14.2.8)$$

The aim is to have an estimate

$$\|x\|_{(rT_1, \dots, rT_d), (\beta_1, \dots, \beta_d)} \lesssim \|x\|_{(rT_1, \dots, rT_d), (\alpha_1, \dots, \alpha_d)}. \quad (14.2.9)$$

Let us study  $\|x\|_{(rT_1, \dots, rT_d), (\beta_1, \dots, \beta_d)}$ . One can remark that

$$y_{(m_1, \dots, m_d)}(r) = \left[ \prod_{j=1}^d (N_j - 1)! (I_X + rT_j)^{-N_j} \right] x_{(m_1, \dots, m_d); (\beta_1, \dots, \beta_d)}.$$

As every set of operators  $\{(N_j - 1)^{-1} (I_X + rT_j)^{N_j} : r \in (0, 1)\}$  is bounded for  $j = 1, \dots, d$ , we can consider  $\sum_{m_1, \dots, m_d=1}^{\infty} r_{m_1, \dots, m_d} \otimes y_{(m_1, \dots, m_d)}(r)$  in place of  $\sum_{m_1, \dots, m_d=1}^{\infty} r_{m_1, \dots, m_d} \otimes x_{(m_1, \dots, m_d); (\beta_1, \dots, \beta_d)}$  in (14.2.8).

Fix now integers  $n, M \geq 1$ . Let  $(m) = (m_1, \dots, m_d)$  and  $(k) = (k_1, \dots, k_d)$ . Considering that the summations run over all indexes  $1 \leq m_1, \dots, m_d \leq M$  and  $1 \leq k_1, \dots, k_d \leq n$ , we have

$$\begin{aligned} & \sum_{m_1, \dots, m_d} r_{(m)} \otimes y_{(m); n}(r) \\ &= \sum_{m_1, \dots, m_d} r_{(m)} \otimes \left( \sum_{k_1, \dots, k_d} c_{k_1, 1} \cdots c_{k_d, d} S_{1, m_1, k_1} \cdots S_{d, m_d, k_d} R_{1, k_1} \cdots R_{d, k_d} x \right) \\ &= \sum_{m_1, \dots, m_d} r_{(m)} \otimes \left( \sum_{k_1, \dots, k_d} \delta_{k_1, m_1} \cdots \delta_{k_d, m_d} W_{1, m_1, k_1} \cdots W_{d, m_d, k_d} R_{1, k_1} \cdots R_{d, k_d} x \right) \end{aligned}$$

where we write  $\delta_{k_j, m_j} = \frac{c_{k_j, j} m_j^{\beta_j - \frac{1}{2}}}{(m_j + k_j - 1)^{\gamma_j + \beta_j}}$  in order to have  $c_{k_j, j} S_{j, m_j, k_j} = \delta_{k_j, m_j} W_{j, m_j, k_j}$ , with

$$W_{j, m_j, k_j} = (m_j + k_j - 1)^{\gamma_j + \beta_j} (rT_j)^{m_j + k_j - 2} (I_X - rT_j)^{\beta_j + \gamma_j}, \quad j = 1, \dots, d.$$

All the sets  $F_j = \{W_{j, m_j, k_j} : m_j, k_j \geq 1, r \in (0, 1)\}$  are  $R$ -bounded according to [5, Proposition 2.8] for  $j = 1, \dots, d$  and using Lemma 3.3.2, the set  $F_1 \cdots F_d$  is also  $R$ -bounded (using notation (3.3.3)). Since  $X$  has finite cotype, this last set is then  $\gamma$ -bounded.

Moreover, the infinite matrices  $[\delta_{k_j, m_j}]_{k_j, m_j \geq 1}$  represent an element of  $B(l^2)$  denoted by  $h_j$ , for  $j = 1, \dots, d$ . Then the operator  $h_1 \otimes \cdots \otimes h_d$ , whose matrix has coefficients  $\delta_{k_1, m_1} \cdots \delta_{k_d, m_d}$  represents an element of  $B\left(l^2 \otimes^2 \cdots \otimes^2 l^2\right)$  and we have  $\|h_1 \otimes \cdots \otimes h_d\| = \|h_1\| \cdots \|h_d\|$ . Moreover, the coefficients of  $h_1 \otimes \cdots \otimes h_d$  belong to  $[0, \infty)$ .

With all these properties in hand and considering Gaussian averages in place of Rademacher averages, we can use [5, Proposition 2.6] to say that we have an inequality of type

$$\begin{aligned} & \left\| \sum_{1 \leq m_1, \dots, m_d \leq M} r_{m_1, \dots, m_d} \otimes y_{(m_1, \dots, m_d); n}(r) \right\|_{\text{Rad}((\mathbb{N}^*)^d; X)} \\ & \lesssim \left\| \sum_{k_1, \dots, k_d=1}^{\infty} r_{k_1, \dots, k_d} \otimes x_{(k_1, \dots, k_d); (\alpha_1, \dots, \alpha_d)} \right\|_{\text{Rad}((\mathbb{N}^*)^d; X)} \end{aligned}$$

which suffices to have the finiteness of square functions and to have inequality (14.2.9).

It remains to study the case where  $r \rightarrow 1^-$  to obtain the result. Choose  $\nu$  an integer such that  $\nu \geq \alpha_j + 1$  and  $\nu \geq \beta_j + 1$ ,  $j = 1, \dots, d$ . Using [5, Lemma 3.2 (3)], the limit  $r \rightarrow 1^-$  exists for  $x$  in  $\bigcap_{j=1}^d \text{Ran}((I - T_j)^\nu)$  and we have for such an element  $x$

$$\|x\|_{(T_1, \dots, T_d), (\beta_1, \dots, \beta_d)} \lesssim \|x\|_{(T_1, \dots, T_d), (\alpha_1, \dots, \alpha_d)}.$$

Let now for some integers  $p_1, \dots, p_d$  the operators

$$\theta_{p_k} = \frac{1}{p_k + 1} \sum_{l_k=0}^{p_k} (I - T_k^{l_k}), \quad k = 1, \dots, d$$

and

$$\Theta_{p_1, \dots, p_d} = \theta_{p_1} \cdots \theta_{p_d}.$$

It is clear that  $\Theta_{p_1, \dots, p_d}^\nu$  maps  $X$  into  $\bigcap_{j=1}^d \text{Ran}((I - T_j)^\nu)$ . Thus, we have the uniform estimate

$$\|\Theta_{p_1, \dots, p_d}^\nu(x)\|_{(T_1, \dots, T_d), (\beta_1, \dots, \beta_d)} \lesssim \|\Theta_{p_1, \dots, p_d}^\nu(x)\|_{(T_1, \dots, T_d), (\alpha_1, \dots, \alpha_d)}$$

for any  $x$  in  $X$  and  $p_1, \dots, p_d \geq 0$ .

Since all the  $T_k$  are power bounded, the sequences  $(\theta_{p_k})_{p_k \geq 0}$  are bounded. Using [5, Lemma 3.2 (1)], we obtain

$$\|\Theta_{p_1, \dots, p_d}^\nu(x)\|_{(T_1, \dots, T_d), (\beta_1, \dots, \beta_d)} \lesssim \|x\|_{(T_1, \dots, T_d), (\alpha_1, \dots, \alpha_d)}$$

for any  $x$  in  $X$  and  $p_1, \dots, p_d \geq 0$ .

Further, we know that for any  $x \in \bigcap_{j=1}^d \text{Ran}(I - T_j)$  we have  $\Theta_{p_1, \dots, p_m}(x) \rightarrow x$  and  $\Theta_{p_1, \dots, p_m}^\nu(x) \rightarrow x$  for  $p_1, \dots, p_d \rightarrow \infty$ . Considering finite sums with  $q \geq 1$  in square functions with estimates

$$\begin{aligned} & \left\| \sum_{1 \leq m_1, \dots, m_d \leq q} \prod_{j=1}^d m_j^{\beta_j - \frac{1}{2}} r_{m_1, \dots, m_d} \otimes \left( \prod_{j=1}^d T_j^{m_j} (I_X - T_j)^{\beta_j} \Theta_{p_1, \dots, p_m}^\nu(x) \right) \right\|_{\text{Rad}((\mathbb{N}^*)^d; X)} \\ & \lesssim \|x\|_{(T_1, \dots, T_d), (\alpha_1, \dots, \alpha_d)} \end{aligned}$$

and passing to the limit  $p_1, \dots, p_d \rightarrow \infty$  in the left hand sum then letting  $q \rightarrow \infty$  yields estimate  $\|x\|_{(T_1, \dots, T_d), (\alpha_1, \dots, \alpha_d)} \lesssim \|x\|_{(T_1, \dots, T_d), (\beta_1, \dots, \beta_d)}$  for every  $x$  in  $\bigcap_{j=1}^d \text{Ran}(I - T_j)$ .

To conclude the proof, take ergodic decomposition given by (??) and (??).

We remark that square functions  $\|x\|_{(T_1, \dots, T_d), (\alpha_1, \dots, \alpha_d)}$  and  $\|x\|_{(T_1, \dots, T_d), (\beta_1, \dots, \beta_d)}$  vanish on every subspace  $\text{Ker}(I - T_k)$ ,  $k = 1, \dots, d$ , and then on every subspace  $X_\Lambda$  where  $\Lambda \neq \{1, \dots, d\}$ . This means that the estimate  $\|x\|_{(T_1, \dots, T_d), (\beta_1, \dots, \beta_d)} \lesssim \|x\|_{(T_1, \dots, T_d), (\alpha_1, \dots, \alpha_d)}$  on  $\bigcap_{j=1}^d \text{Ran}(I - T_j)$  suffice to have this one in all the space  $X$ , which ends the proof.  $\square$

*Proof of Theorem 14.2.1.* Taking into account Theorem 14.2.3, we may and do assume that  $\alpha = \beta = (1, \dots, 1)$ . Let  $\gamma_k \in (\delta_k, \frac{\pi}{2})$  for  $k = 1, \dots, d$ . Let  $x \in X$ ,  $y \in X^*$  and  $\varphi \in H_{0,1}(B_{\gamma_1} \times \dots \times B_{\gamma_d})$ . The aim is to have an estimate

$$|\langle \varphi(T_1, \dots, T_d)x, y \rangle| \lesssim \|x\| \|y\| \|\varphi\|_{\infty, B_{\gamma_1} \times \dots \times B_{\gamma_d}}. \quad (14.2.10)$$

By [6, Propostion 2.5], it suffices to prove (14.2.10) when  $\varphi$  is a polynomial function on  $d$  variables.

We suppose first that  $\varphi$  is such a polynomial function of the form  $\varphi(z_1, \dots, z_d) = (1 - z_1) \cdots (1 - z_d) \varphi_1(z_1, \dots, z_d)$  where  $\varphi_1$  is another polynomial function.

Then we see that  $\varphi(T_1, \dots, T_d)x \in \bigcap_{k=1}^d \text{Ran}(I - T_k)$  for every  $x$  in  $X$ .

According to [53, Lemma 7.2], we have for every  $k = 1, \dots, d$  and  $y$  in  $\text{Ran}(I - T_k)$

$$\sum_{i_k=1}^{\infty} i_k(i_k + 1) T_k^{i_k-1} (I - T_k)^3 y = 2y.$$

We deduce that

$$\begin{aligned} \sum_{i_d=1}^{\infty} \cdots \sum_{i_1=1}^{\infty} i_d(i_d + 1) \cdots i_1(i_1 + 1) T_d^{i_d-1} \cdots T_1^{i_1-1} (I - T_d)^3 \cdots (I - T_1)^3 \varphi(T_1, \dots, T_d)x \\ = 2^d \varphi(T_1, \dots, T_d)x, \quad x \in X. \end{aligned} \quad (14.2.11)$$

Consider the polynomial function  $\psi$  defined by  $\psi(z) = \frac{1}{2}(1 + z + z^2)^3$ . Also for any integers  $i_1, \dots, i_d \geq 1$ , we set

$$f(i_1, \dots, i_d) = \prod_{k=1}^d (i_k + 1) \prod_{k=1}^d T_k^{i_k-1} (I - T_k) \varphi(T_1, \dots, T_d),$$

$$g(i_1, \dots, i_d) = \prod_{k=1}^d i_k^{\frac{1}{2}} \prod_{k=1}^d T_k^{i_k-1} (I - T_k),$$

$$h(i_1, \dots, i_d) = \prod_{k=1}^d i_k^{\frac{1}{2}} \prod_{k=1}^d (T_k^*)^{i_k-1} (I - T_k^*) \psi(T_k^*).$$

For convenience, we will write only  $\sum_{i_1, \dots, i_d}$  in place of  $\sum_{i_d=1}^{\infty} \cdots \sum_{i_1=1}^{\infty}$ , keeping the order of summation. Then it follows from (14.2.11) that for any  $x$  in  $X$  and  $y$  in  $X^*$  we have

$$\langle \varphi(T_1, \dots, T_d)x, y \rangle = \sum_{i_1, \dots, i_d} \langle f(i_1, \dots, i_d)g(i_1, \dots, i_d)x, h(i_1, \dots, i_d)y \rangle. \quad (14.2.12)$$

Let us now consider independent Rademacher variables  $(r_{i_1, \dots, i_d})_{(i_1, \dots, i_d) \in (\mathbb{N}^*)^d}$  and families  $(x_{i_1, \dots, i_d})_{(i_1, \dots, i_d) \in (\mathbb{N}^*)^d}$  and  $(y_{i_1, \dots, i_d})_{(i_1, \dots, i_d) \in (\mathbb{N}^*)^d}$  of  $X$  and  $X^*$  respectively. For any integers  $N_1, \dots, N_d \geq 1$ , writing  $\sum_{N_1, \dots, N_d}$  in place of  $\sum_{i_1=1}^{N_1} \cdots \sum_{i_d=1}^{N_d}$ , the independence of the  $r_{i_1, \dots, i_d}$  yields

$$\sum_{N_1, \dots, N_d} \langle x_{i_1, \dots, i_d}, y_{i_1, \dots, i_d} \rangle = \int_{\Omega} \left\langle \sum_{N_1, \dots, N_d} r_{i_1, \dots, i_d}(u) x_{i_1, \dots, i_d}, \sum_{N_1, \dots, N_d} r_{i_1, \dots, i_d}(u) y_{i_1, \dots, i_d} \right\rangle d\mathbb{P}(u). \quad (14.2.13)$$

We now let

$$S_{N_1, \dots, N_d} = \sum_{N_1, \dots, N_d} \langle f(i_1, \dots, i_d) g(i_1, \dots, i_d) x, h(i_1, \dots, i_d) y \rangle,$$

the partial sums of (14.2.12) for any integers  $N_1, \dots, N_d \geq 1$ . Letting

$$x_{i_1, \dots, i_d} = f(i_1, \dots, i_d) g(i_1, \dots, i_d) x, \quad y_{i_1, \dots, i_d} = h(i_1, \dots, i_d) y,$$

for any integers  $i_1, \dots, i_d \geq 1$  and using the Cauchy-Schwarz inequality in (14.2.13), we obtain

$$|S_{N_1, \dots, N_d}| \leq \left\| \sum_{N_1, \dots, N_d} r_{i_1, \dots, i_d} \otimes f(i_1, \dots, i_d) g(i_1, \dots, i_d) x \right\|_{\text{Rad}(X)} \times \left\| \sum_{N_1, \dots, N_d} r_{i_1, \dots, i_d} \otimes h(i_1, \dots, i_d) y \right\|_{\text{Rad}(X^*)}. \quad (14.2.14)$$

We now prove that the family of operators  $(f(i_1, \dots, i_d))$  is  $R$ -bounded. First, as  $\varphi$  is a function of  $H_0^\infty(B_{\gamma_1} \times \cdots \times B_{\gamma_d})$ , operators  $f(i_1, \dots, i_d)$  are defined by

$$f(i_1, \dots, i_d) = \left( \frac{1}{2\pi i} \right)^d \int_{\partial B_{(\gamma)}} \prod_{k=1}^d (i_k + 1) \varphi(\lambda_1, \dots, \lambda_d) \prod_{k=1}^d \lambda_k^{i_k} (\lambda_k - 1) \prod_{k=1}^d R(\lambda_k, T_k) d(\lambda),$$

where  $\partial B_{(\gamma)} = \partial B_{\gamma_1} \times \cdots \times \partial B_{\gamma_d}$  and  $d(\lambda) = d\lambda_1 \cdots d\lambda_d$ .

We consider the subset of  $B(X)$

$$E = \{(\lambda_1 - 1)R(\lambda_1, T_1) \cdots (\lambda_1 - 1)R(\lambda_1, T_1) : (\lambda_1, \dots, \lambda_d) \in \partial B_{\gamma_1} \times \cdots \times \partial B_{\gamma_d}\}. \quad (14.2.15)$$

Every  $E_k = \{(\lambda_k - 1)R(\lambda_k, T_k) : \lambda_k \in \partial B_{\gamma_k}\}$  is  $R$ -bounded as  $T_k$  is  $R$ -Ritt of  $R$ -type  $\delta_k$  and  $\delta_k < \gamma_k$ . Using Lemma 3.3.2, we know that  $E$  is  $R$ -bounded.

Let us now use Lemma 3.3.3. Consider the function  $F : (\lambda_1, \dots, \lambda_d) \mapsto (\lambda_1 - 1)R(\lambda_1, T_1) \cdots (\lambda_1 - 1)R(\lambda_1, T_1)$  from  $\partial B_{(\gamma)}$  to  $E$ . We prove that  $\chi_{i_1, \dots, i_d} : (\lambda_1, \dots, \lambda_d) \mapsto \prod_{k=1}^d (i_k + 1) \lambda_k^{i_k} \varphi(\lambda_1, \dots, \lambda_d)$  is uniformly bounded in  $L^1(\partial B_{(\gamma)}, |d(\lambda)|)$  when  $(i_1, \dots, i_d) \in (\mathbb{N}^*)^d$ . Recall that for any angle  $\theta \in (0, \frac{\pi}{2})$ , one can check that

$$\sup \left\{ \int_{\partial B_\theta} (m+1) |z|^m |dz|, m \in \mathbb{N} \right\} < \infty.$$

$\chi_{i_1, \dots, i_d}$  is therefore uniformly bounded in  $L^1(\partial B_{(\gamma)}, |d(\lambda)|)$  with  $\|h_{i_1, \dots, i_d}\|_{L^1(\partial B_{(\gamma)}, |d(\lambda)|)} \lesssim \|\varphi\|_{\infty, \partial B_{(\gamma)}}$ . We apply Lemma 3.3.3 to obtain that the family  $(f(i_1, \dots, i_d))$  is  $R$ -bounded with

$$\mathcal{R} \{f(i_1, \dots, i_d) : (i_1, \dots, i_d) \in (\mathbb{N}^*)^d\} \lesssim \|\varphi\|_{\infty, \partial B_{(\gamma)}}. \quad (14.2.16)$$

We now use (14.2.16) together with (14.2.14) to have

$$|S_{n_1, \dots, n_d}| \lesssim \|\varphi\|_{\infty, \partial B_{(\gamma)}} \left\| \sum_{i_1, \dots, i_d}^{N_1, \dots, N_d} r_{i_1, \dots, i_d} \otimes g(i_1, \dots, i_d)x \right\|_{\text{Rad}(X)} \times \left\| \sum_{i_1, \dots, i_d}^{N_1, \dots, N_d} r_{i_1, \dots, i_d} \otimes h(i_1, \dots, i_d)y \right\|_{\text{Rad}(X^*)}.$$

Finally, we let successively  $N_1 \rightarrow \infty, \dots, N_d \rightarrow \infty$  to obtain

$$|\langle \varphi(T_1, \dots, T_d)x, y \rangle| \lesssim \|\varphi\|_{\infty, \partial B_{(\gamma)}} \|x\|_{(T_1, \dots, T_d), (1, \dots, 1)} \|y\|_{(T_1^*, \dots, T_d^*), (1, \dots, 1)}.$$

Then it suffices to use (14.2.4) and (14.2.5) with  $\alpha = \beta = (1, \dots, 1)$  to deduce that

$$|\langle \varphi(T_1, \dots, T_d)x, y \rangle| \lesssim \|\varphi\|_{\infty, B_{(\gamma)}} \|x\| \|y\|$$

which provides  $\|\varphi(T_1, \dots, T_d)\| \lesssim \|\varphi\|_{\infty, B_{(\gamma)}}$ .

To conclude the proof, one sees that we have the estimation (14.2.10) for any function polynomial function  $\varphi$  of type  $\varphi(z_1, \dots, z_d) = \prod_{i \in \Lambda} (1 - z_i) \varphi_1(z_1, \dots, z_d)$  where  $\varphi_1$  is another polynomial function depending on variables  $(z_i)_{i \in \Lambda}$  with  $\Lambda$  a subset of  $\{1, \dots, d\}$ . Indeed, arguments above holds verbatim using square functions  $\|\cdot\|_{T, \alpha_\Lambda}$  and  $\|\cdot\|_{T^*, \beta_\Lambda}$ . Hence, (14.2.10) is verified for any polynomial function.  $\square$

**Remark 14.2.4** Looking at the preceding proof, one sees that we do not need to suppose that  $X$  is a reflexive space or  $X$  has finite cotype in Theorem 14.2.1 if we only consider square functions estimates  $\|x\|_{T, (1, \dots, 1)} \lesssim \|x\|$  and  $\|y\|_{T^*, (1, \dots, 1)} \lesssim \|y\|$  for  $x \in X$  and  $y \in X^*$ .

Recall that for  $p \in (1, \infty)$ ,  $p \neq 2$ , the noncommutative  $L_p$ -spaces do not have property  $(\alpha)$  (see e.g [6, Section 3] for a definition of this property). Thus, the results of [6, Section 3] do not apply to this class of Banach space and in particular, noncommutative  $L_p$ -spaces do not have the joint functional calculus property (see [48]). The following result, which generalises [53, Corollary 7.5], gives a characterisation of the joint functional calculus of a  $d$ -tuple of Ritt operators on spaces having property  $(\Delta)$  (see [40] for the definition). This result applies to the noncommutative  $L_p$ -spaces. Note that we must appeal to Theorem 14.1.1 and Remark 14.1.2 to obtain the next corollary.

**Corollary 14.2.5** *Let  $X$  be a Banach space with property  $(\Delta)$ . Let  $T_1, \dots, T_d$  be commuting Ritt operators on  $X$ . The following two assertions are equivalent.*

- i)  $T = (T_1, \dots, T_d)$  admits a  $H^\infty(B_{\gamma_1} \times \dots \times B_{\gamma_d})$  joint functional calculus for some  $\gamma_k \in (0, \frac{\pi}{2})$ ,  $k = 1, \dots, d$ .*

ii) Every  $T_k$  is  $R$ -Ritt and for every  $\Lambda$  subset of  $\{1, \dots, d\}$  and for any  $\alpha_\Lambda$  and  $\beta_\Lambda$  in  $(\mathbb{R}_+^*)^\Lambda$ , there exists a constant  $C > 0$  such that we have

$$\begin{aligned} \|x\|_{T, \alpha_\Lambda} &\leq C \|x\|, & x \in X, \\ \|y\|_{T^*, \beta_\Lambda} &\leq C \|y\|, & y \in X^*. \end{aligned}$$

We now pass to the case of sectorial operators. It is well-known that if  $X$  is reflexive, then each  $A_k^*$  is a sectorial operator on  $X^*$ . It is plain that if  $(A_1, \dots, A_d)$  admits an  $H^\infty(\Sigma_{\theta_1} \times \dots \times \Sigma_{\theta_d})$  joint functional calculus, then  $(A_1^*, \dots, A_d^*)$  admits an  $H^\infty(\Sigma_{\theta_1} \times \dots \times \Sigma_{\theta_d})$  joint functional calculus as well. Indeed for any  $f \in H_{0,1}^\infty(\Sigma_{\theta_1} \times \dots \times \Sigma_{\theta_d})$ ,

$$f(A_1^*, \dots, A_d^*) = \widetilde{f}(A_1, \dots, A_d)^*, \quad (14.2.17)$$

where  $\widetilde{f} \in H_{0,1}^\infty(\Sigma_{\theta_1} \times \dots \times \Sigma_{\theta_d})$  is defined by  $\widetilde{f}(z_1, \dots, z_d) = \overline{f(\overline{z_1}, \dots, \overline{z_d})}$ .

It therefore follows from Theorem 14.1.7 that when  $X$  is reflexive and  $X$  and  $X^*$  both have finite cotype, we have the following result: if  $(A_1, \dots, A_d)$  admits an  $H^\infty(\Sigma_{\theta_1} \times \dots \times \Sigma_{\theta_d})$  joint functional calculus, then both  $(A_1, \dots, A_d)$  and  $(A_1^*, \dots, A_d^*)$  admit square function estimates with respect to any  $F \in H_{0,1}^\infty(\Sigma_{\nu_1} \times \dots \times \Sigma_{\nu_d})$ , with  $\nu_k \in (\theta_k, \pi)$ .

In the single case, Theorem 14.1.7 has various converse statements. These statements all assert that under certain assumptions, some square function estimates for  $A$  and  $A^*$  imply that  $A$  has an  $H^\infty$ -functional calculus. The next statement, involving  $R$ -sectoriality, is essentially a multivariable version of [35, Theorem 10.4.9].

**Theorem 14.2.6** *Assume that  $X$  is reflexive with finite cotype, and that for any  $k = 1, \dots, d$ ,  $A_k$  is  $R$ -sectorial of  $R$ -type  $\omega_k \in (0, \pi)$  and has dense range. Assume further that there exist non zero functions  $F_1, F_2 \in H_0^\infty(\Sigma_{\nu_1} \times \dots \times \Sigma_{\nu_d})$ , for some  $\nu_k \in (\omega_k, \pi)$ , such that  $(A_1, \dots, A_d)$  admits a square function estimate with respect to  $F_1$  and  $(A_1^*, \dots, A_d^*)$  admits a square function estimate with respect to  $F_2$ . Then for any  $\theta_k \in (\omega_k, \pi)$ ,  $k = 1, \dots, d$ ,  $(A_1, \dots, A_d)$  admits an  $H^\infty(\Sigma_{\theta_1} \times \dots \times \Sigma_{\theta_d})$  joint functional calculus.*

*Proof of Theorem 14.2.6.* We use classical arguments so we will be deliberately brief. We set  $\mathcal{A} = (A_1, \dots, A_d)$  and we write  $F(t\mathcal{A})$  instead of  $F(t_1A_1, \dots, t_dA_d)$  for any  $t = (t_1, \dots, t_d) \in \mathbb{R}_+^{*d}$ , whenever  $F \in H_0^\infty(\Sigma_{\theta_1} \times \dots \times \Sigma_{\theta_d})$ , with  $\theta_k \in (\omega_k, \pi)$ .

We may assume that  $\theta_k \leq \nu_k$  for any  $k = 1, \dots, d$ . Define  $\widetilde{F}_2 \in H_0^\infty(\Sigma_{\nu_1} \times \dots \times \Sigma_{\nu_d})$  so that

$$\widetilde{F}_2(A_1, \dots, A_d)^* = F_2(A_1^*, \dots, A_d^*), \quad (14.2.18)$$

see (14.2.17). Since  $F_1, F_2$  are non zero, we can find  $\Psi \in H_0^\infty(\Sigma_{\nu_1} \times \dots \times \Sigma_{\nu_d})$  such that

$$\int_{\Omega_0^d} \Psi(t) F_1(t) \widetilde{F}_2(t) dM(t) = 1.$$

By analytic continuation, this implies that for any  $(z_1, \dots, z_d) \in \Sigma_{\theta_1} \times \dots \times \Sigma_{\theta_d}$ , we have

$$\int \Psi(t_1z_1, \dots, t_dz_d) F_1(t_1z_1, \dots, t_dz_d) \widetilde{F}_2(t_1z_1, \dots, t_dz_d) dM(t) = 1.$$

Fix  $f \in H_0^\infty(\Sigma_{\theta_1} \times \cdots \times \Sigma_{\theta_d})$ . The argument in the proof of [56, Lemma 6.5 (1)] shows that the function  $t \mapsto f(\mathcal{A})\Psi(t\mathcal{A})F_1(t\mathcal{A})\widetilde{F}_2(t\mathcal{A})$  belongs to  $L^1(\Omega_0^d; B(X))$  and by Fubini's theorem,

$$\int_{\Omega_0^d} f(\mathcal{A})\Psi(t\mathcal{A})F_1(t\mathcal{A})\widetilde{F}_2(t\mathcal{A}) dM(t) = f(\mathcal{A}). \quad (14.2.19)$$

Arguing as in the proof of Theorem 10.0.1, we may deduce from the assumption that  $A_k$  is  $R$ -sectorial of  $R$ -type  $\omega_k$  that the set

$$\mathcal{F}_f := \{f(\mathcal{A})\Psi(t\mathcal{A}) : t \in \mathbb{R}_+^{*d}\}$$

is  $R$ -bounded, with an estimate

$$\mathcal{R}(\mathcal{F}_f) \lesssim \|f\|_{\infty, \Sigma_{\theta_1} \times \cdots \times \Sigma_{\theta_d}}. \quad (14.2.20)$$

We assumed that  $(A_1, \dots, A_d)$  admits a square function estimate with respect to  $F_1$ . Thus for any  $x \in X$ ,  $t \mapsto F_1(t\mathcal{A})x$  belongs to  $\gamma(\Omega_0^d; X)$ , with an estimate

$$\|x\|_{F_1} = \|t \mapsto F_1(t\mathcal{A})x\|_{\gamma(\Omega_0^d; X)} \lesssim \|x\|.$$

Since  $X$  has finite cotype, it therefore follows from (14.2.20) and the so-called  $\gamma$ -multiplier theorem (see [35, Theorem 9.5.1]) that the function  $t \mapsto f(\mathcal{A})\Psi(t\mathcal{A})F_1(t\mathcal{A})x$  belongs to  $\gamma(\Omega_0^d; X)$  for any  $x \in X$ , with an estimate

$$\|t \mapsto f(\mathcal{A})\Psi(t\mathcal{A})F_1(t\mathcal{A})x\|_{\gamma(\Omega_0^d; X)} \lesssim \|f\|_{\infty, \Sigma_{\theta_1} \times \cdots \times \Sigma_{\theta_d}} \|x\|. \quad (14.2.21)$$

Furthermore  $(A_1^*, \dots, A_d^*)$  admits a square function estimate with respect to  $F_2$  hence for any  $y \in X^*$ , the function  $t \mapsto F_2(t\mathcal{A}^*)y$  belongs to  $\gamma(\Omega_0^d; X^*)$ , with an estimate

$$\|y\|_{F_2} = \|t \mapsto F_2(t\mathcal{A}^*)y\|_{\gamma(\Omega_0^d; X^*)} \lesssim \|y\|. \quad (14.2.22)$$

Let us now use the identities (14.2.19) and (14.2.18). Applying the so-called trace duality theorem on  $\gamma$ -spaces (see [35, Theorem 9.2.14]), the estimates (14.2.21) and (14.2.22) imply that

$$\begin{aligned} |\langle f(\mathcal{A})x, y \rangle| &= \left| \int_{\Omega_0^d} \langle f(\mathcal{A})\Psi(t\mathcal{A})F_1(t\mathcal{A})x, F_2(t\mathcal{A}^*)y \rangle dM(t) \right| \\ &\leq \|t \mapsto \Psi(t\mathcal{A})F_1(t\mathcal{A})x\|_{\gamma(\Omega_0^d; X)} \|t \mapsto F_2(t\mathcal{A}^*)y\|_{\gamma(\Omega_0^d; X^*)} \\ &\lesssim \|f\|_{\infty, \Sigma_{\theta_1} \times \cdots \times \Sigma_{\theta_d}} \|x\| \|y\|, \end{aligned}$$

for  $x \in X, y \in X^*$ . This yields an estimate

$$\|f(\mathcal{A})\| \lesssim \|f\|_{\infty, \Sigma_{\theta_1} \times \cdots \times \Sigma_{\theta_d}}$$

for  $f \in H_0^\infty(\Sigma_{\theta_1} \times \cdots \times \Sigma_{\theta_d})$ . By Lemma 5.2.4 and the dense range assumption, this shows that  $(A_1, \dots, A_d)$  admits an  $H^\infty(\Sigma_{\theta_1} \times \cdots \times \Sigma_{\theta_d})$  joint functional calculus.  $\square$

**Remark 14.2.7**

(1) The following variant of Theorem 14.2.6 is easy to deduce from the above proof: Assume that  $X$  is reflexive with finite cotype, and that for any  $k = 1, \dots, d$ ,  $A_k$  is  $R$ -sectorial of  $R$ -type  $\omega_k \in (0, \pi)$ . Let  $\nu_k \in (\omega_k, \pi)$  for any  $k = 1, \dots, d$ , and assume that for any  $F \in H_{0,1}^\infty(\Sigma_{\nu_1} \times \dots \times \Sigma_{\nu_d})$ ,  $(A_1, \dots, A_d)$  and  $(A_1^*, \dots, A_d^*)$  both admit a square function estimate with respect to  $F$ . Then for any  $\theta_k \in (\omega_k, \pi)$ ,  $k = 1, \dots, d$ ,  $(A_1, \dots, A_d)$  admits an  $H^\infty(\Sigma_{\theta_1} \times \dots \times \Sigma_{\theta_d})$  joint functional calculus.

(2) Combining the above part (1) and Theorem 14.1.7, we obtain a new proof of Theorem 10.0.1 in the case when  $X$  is reflexive and both  $X$  and  $X^*$  have finite cotype.

We refer the reader to Section 4.4 for definitions of property  $(\Delta)$ . It is proved in [40, Theorem 5.3] that if  $A$  is a sectorial operator with an  $H^\infty(\Sigma_\theta)$  functional calculus on  $X$  with property  $(\Delta)$ , then  $A$  is  $R$ -sectorial of  $R$ -type  $\theta$ . Combining that result with Theorem 14.1.7 and Remark 14.2.7 (1), we obtain the following.

**Corollary 14.2.8** *Assume that  $X$  is a reflexive Banach space with property  $(\Delta)$  and that  $X^*$  has finite cotype. The following are equivalent.*

- (i) *For any  $\theta_k \in (\omega_k, \pi)$ ,  $k = 1, \dots, d$ ,  $(A_1, \dots, A_d)$  admits an  $H^\infty(\Sigma_{\theta_1} \times \dots \times \Sigma_{\theta_d})$  joint functional calculus.*
- (ii) *For any  $\theta_k \in (\omega_k, \pi)$ ,  $k = 1, \dots, d$ , and for any  $F \in H_{0,1}^\infty(\Sigma_{\theta_1} \times \dots \times \Sigma_{\theta_d})$ ,  $(A_1, \dots, A_d)$  and  $(A_1^*, \dots, A_d^*)$  both admit a square function estimate with respect to  $F$ .*

In the rest of this section, we discuss the following natural problem :

**(P)** *Let  $F_k \in H_0^\infty(\Sigma_{\nu_k})$ , with  $\nu_k \in (\omega_k, \pi)$  for all  $k = 1, \dots, d$ . Consider  $F_1 \otimes \dots \otimes F_d$  in  $H_0^\infty(\Sigma_{\nu_1} \times \dots \times \Sigma_{\nu_d})$  given by  $(F_1 \otimes \dots \otimes F_d)(z_1, \dots, z_d) = F_1(z_1) \dots F_d(z_d)$ . Assume that for each  $k = 1, \dots, d$ ,  $A_k$  admits a square function estimate with respect to  $F_k$ . Does it imply that the  $d$ -tuple  $(A_1, \dots, A_d)$  admits a square function estimate with respect to  $F_1 \otimes \dots \otimes F_d$ ?*

Assume first that  $d = 2$ . Observe that if  $A_1$  (resp.  $A_2$ ) admits a square function estimate with respect to  $F_1$  (resp.  $F_2$ ), then for any  $x \in X$ , the function  $(t_1, t_2) \mapsto F_1(t_1 A_1) F_2(t_2 A_2) x$  belongs to  $\gamma(\Omega_0; \gamma(L^2(\Omega_0); X))$  and we have an estimate

$$\|(t_1, t_2) \mapsto F_1(t_1 A_1) F_2(t_2 A_2) x\|_{\gamma(\Omega_0; \gamma(L^2(\Omega_0); X))} \lesssim \|x\|. \quad (14.2.23)$$

Indeed for  $i = 1, 2$ , let  $\Gamma_i: X \rightarrow \gamma(L^2(\Omega_0); X)$  be the operator taking any  $x \in X$  to the function  $t \mapsto F_i(t A_i) x$ . Let  $\widetilde{\Gamma}_2: \gamma(L^2(\Omega_0); X) \rightarrow \gamma(L^2(\Omega_0); \gamma(L^2(\Omega_0); X))$  be the tensor extension of  $\Gamma_2$  defined by  $\widetilde{\Gamma}_2(u) = \Gamma_2 \circ u$  for any  $u \in \gamma(L^2(\Omega_0); X)$ . Then the operator

$$\widetilde{\Gamma}_2 \circ \Gamma_1: X \longrightarrow \gamma(L^2(\Omega_0); \gamma(L^2(\Omega_0); X))$$

takes any  $x \in X$  to the function  $(t_1, t_2) \mapsto F_1(t_1 A_1) F_2(t_2 A_2) x$ . This yields the estimate (14.2.23).



It follows from this observation that  $(A_1, A_2)$  admits a square function estimate with respect to  $F_1 \otimes F_2$  provided that the natural inclusion  $L^2(\Omega_0) \otimes L^2(\Omega_0) \otimes X \rightarrow L^2(\Omega_0^2) \otimes X$  extends to a bounded map

$$\gamma(L^2(\Omega_0); \gamma(L^2(\Omega_0); X)) \longrightarrow \gamma(L^2(\Omega_0^2); X). \quad (14.2.24)$$

By a simple reiteration argument, the above result extends to any  $d \geq 2$ . That is, if  $X$  satisfies (14.2.24), if  $A_k$  admits a square function estimate with respect to  $F_k$  for any  $k = 1, \dots, d$ , then  $(A_1, \dots, A_d)$  admits a square function estimate with respect to  $F_1 \otimes \dots \otimes F_d$ .

We note that property (14.2.24) is satisfied if either  $X$  has property  $(\alpha)$  or  $X$  has type 2 [74, Proposition 2.7]. Consequently, the above problem **(P)** has an affirmative answer if either  $X$  has property  $(\alpha)$  or  $X$  has type 2.

**Example 14.2.9** We now turn to a counterexample. For any  $1 < p < \infty$ , let  $S^p$  denote the  $p$ -Schatten class on  $\ell^2$ . Let  $(e_k)_{k \geq 1}$  denote the standard basis of  $\ell^2$  and let  $c \in B(\ell^2)$  be the positive selfadjoint operator satisfying  $c(e_k) = 2^{-k}e_k$  for any  $k \geq 1$ . Let  $A_1, A_2: S^p \rightarrow S^p$  be the left and right multiplication operators defined by  $A_1(x) = cx$  and  $A_2(x) = xc$ , respectively, for  $x \in S^p$ . It is well-known that  $A_1, A_2$  are sectorial operators with dense range, and that they admit an  $H^\infty(\Sigma_\theta)$  functional calculus for any  $\theta \in (0, \pi)$  (see e.g. [56, Section 8.A] for details). Further  $A_1$  and  $A_2$  commute. It follows from the proof of [48, Theorem 3.9] that if  $p \neq 2$ , then for any  $\theta_1, \theta_2 \in (0, \pi)$ , the couple  $(A_1, A_2)$  does not have an  $H^\infty(\Sigma_{\theta_1} \times \Sigma_{\theta_2})$  joint functional calculus.

Assume that  $1 < p < 2$  and let  $p' = \frac{p}{p-1}$  be its conjugate number. Let  $F_1, F_2$  be non zero functions in  $H_0^\infty(\Sigma_\nu)$  for some  $\nu \in (0, \pi)$ , and let  $F = F_1 \otimes F_2$ . Since  $A_1$  admits an  $H^\infty(\Sigma_\theta)$  functional calculus for any  $\theta \in (0, \pi)$ , both  $A_1$  and  $A_1^*$  admit square function estimates with respect to  $F_1$ . Likewise  $A_2$  and  $A_2^*$  admit square function estimates with respect to  $F_2$ . The Banach space  $S^{p^*} = S^{p'}$  has type 2, hence it follows from the discussion preceding this example that  $(A_1^*, A_2^*)$  admits a square function estimates with respect to  $F$ . If  $(A_1, A_2)$  admitted a square function estimates with respect to  $F$  as well, it would follow from Theorem 14.2.6 that  $(A_1, A_2)$  admits an  $H^\infty(\Sigma_\theta \times \Sigma_\theta)$  joint functional calculus for some  $\theta$ , which is false. Consequently,  $(A_1, A_2)$  does not admit a square function estimate with respect to  $F = F_1 \otimes F_2$ .

We refer the reader to [56, Chapters 8-10] for more examples of sectorial operators on non-commutative  $L^p$ -spaces with an  $H^\infty$  functional calculus.

## Part VI

# Characterisations by isomorphic dilations



# Chapter 15

## Dilation from $H^\infty$ joint functional calculus

### 15.1 The case of commuting Ritt operators

In this section, we give an application of square functions in terms of dilation of a  $d$ -tuple of Ritt operators. The framework of this part is  $K$ -convex spaces. We refer to Section 4.1 for background on these spaces.

Let now  $Y$  be another Banach space and  $U_1, \dots, U_d$  be commuting isomorphisms on  $Y$ . We say that  $U = (U_1, \dots, U_d)$  admits a  $C(\mathbb{T}^d)$  bounded functional calculus if there exists a constant  $C \geq 1$  such that for any trigonometric polynomial of  $d$  variables

$$\phi(s_1, \dots, s_d) = \sum_{n_1, \dots, n_d \in \mathbb{Z}^d} a_{n_1, \dots, n_d} e^{i(n_1 s_1 + \dots + n_d s_d)},$$

where  $(a_{n_1, \dots, n_d})$  is a finite family of complex numbers, we have

$$\|\phi(U_1, \dots, U_d)\| \leq C \sup \{ |\phi(s_1, \dots, s_d)| : (s_1, \dots, s_d) \in \mathbb{R}^d \}, \quad (15.1.1)$$

where we let  $\phi(U_1, \dots, U_d) = \sum a_{n_1, \dots, n_d} U_1^{n_1} \dots U_d^{n_d}$ .

As trigonometric polynomial functions of  $d$  variables are dense in the space  $C(\mathbb{T}^d)$  of all continuous functions on  $\mathbb{T}^d$ ,  $C(\mathbb{T}^d)$  bounded functional calculus property is equivalent to the existence of a unique unital bounded homomorphism  $\omega : C(\mathbb{T}^d) \rightarrow B(X)$  such that  $\omega(e(j)) = U_j$  where  $e(j) : (s_1, \dots, s_d) \mapsto e^{is_j}$ .

**Theorem 15.1.1** *Let  $X$  be a reflexive  $K$ -convex Banach space and  $p$  in  $(1, \infty)$ . Let  $T = (T_1, \dots, T_d)$  be a  $d$ -tuple of commuting Ritt operators on  $X$ . Suppose moreover that  $T$  admits a  $H^\infty(B_{\gamma_1} \times \dots \times B_{\gamma_d})$  joint functional calculus for some  $\gamma_1, \dots, \gamma_d$  in  $(0, \frac{\pi}{2})$ .*

*Then there exists a measure space  $\Sigma$ , a  $d$ -tuple of commuting isomorphisms  $(U_1, \dots, U_d)$  on  $L_p(\Sigma; X)$  admitting a  $C(\mathbb{T}^d)$  bounded calculus and two bounded operators  $J : X \rightarrow L_p(\Sigma; X)$  and  $Q : L_p(\Sigma; X) \rightarrow X$  such that*

$$T_1^{n_1} \dots T_d^{n_d} = Q U_1^{n_1} \dots U_d^{n_d} J, \quad (n_1, \dots, n_d) \in \mathbb{N}^d. \quad (15.1.2)$$

*Proof of Theorem 15.1.1 :* Throughout the proof, we let  $\Lambda$  be a subset of  $\{1, \dots, d\}$ , denoted by

$$\Lambda = \{i_1, \dots, i_k\}, \quad i_1 < \dots < i_k, \quad (15.1.3)$$

where  $|\Lambda| = k$ . Let  $(r_{(l_{i_1}, \dots, l_{i_k})})$  be a family of independent Rademacher variables indexed by  $\mathbb{Z}^\Lambda$  for  $\Lambda \neq \emptyset$  on some probability space  $\Omega_\Lambda$ . Consider then space  $\text{Rad}_p(\mathbb{Z}^\Lambda; \mathbb{C})$ , letting  $\text{Rad}_p(\mathbb{Z}^\emptyset; \mathbb{C}) = \mathbb{C}$ .

For any  $j$  in  $\{1, \dots, d\}$  and  $\Lambda \neq \emptyset$ , we define the operator  $v_{j,\Lambda}$  on the space  $\text{Rad}_p(\mathbb{Z}^\Lambda; \mathbb{C})$  by  $v_{j,\Lambda}(r_{(l_{i_1}, \dots, l_{i_k})}) = r_{(l_{i_1}, \dots, l_{i_k})}$  if  $j \notin \Lambda$  and if  $j = i_m \in \Lambda$ , we let  $v_{j,\Lambda}(r_{(l_{i_1}, \dots, l_{i_m}, \dots, l_{i_k})}) = r_{(l_{i_1}, \dots, l_{i_m-1}, \dots, l_{i_k})}$ . If  $\Lambda = \emptyset$ , let  $v_{j,\emptyset} = I_{\mathbb{C}}$  for  $j = 1, \dots, d$ .

Then  $v_{j,\Lambda}$  is an isometric isomorphism of  $\text{Rad}_p(\mathbb{Z}^\Lambda; \mathbb{C})$ . Further, every  $v_{j,\Lambda} \otimes I_X$  extends to a unique operator  $V_{j,\Lambda}$  from  $\text{Rad}_p(\mathbb{Z}^\Lambda; X)$  into itself, which is an isometric isomorphism too.

Next we define

$$V_j = \bigoplus_{\Lambda \subset \{1, \dots, d\}} V_{j,\Lambda},$$

the direct sum of operators  $V_{j,\Lambda}$ , from  $\bigoplus_{\Lambda \subset \{1, \dots, d\}}^p \text{Rad}_p(\mathbb{Z}^\Lambda; X)$  into  $\bigoplus_{\Lambda \subset \{1, \dots, d\}}^p \text{Rad}_p(\mathbb{Z}^\Lambda; X)$  defined by  $V_j(\sum_{\Lambda \subset \{1, \dots, d\}} x_\Lambda) = \sum_{\Lambda \subset \{1, \dots, d\}} V_{j,\Lambda}(x_\Lambda)$ .

Recall now (??) and similary  $(X^*)_\Lambda = \left[ \bigcap_{i \in \Lambda} \overline{\text{Ran}(I - T_i^*)} \right] \cap \left[ \bigcap_{i \notin \Lambda} \text{Ker}(I - T_i^*) \right]$ . As  $X$  is reflexive, by (??), we have

$$X = \bigoplus_{\Lambda \subset \{1, \dots, d\}} X_\Lambda \quad \text{and} \quad X^* = \bigoplus_{\Lambda \subset \{1, \dots, d\}} (X^*)_\Lambda.$$

For any  $\Lambda \subset \{1, \dots, d\}$ , taking notation of (15.1.3), we know that  $T_\Lambda = (T_{i_1}, \dots, T_{i_k})$  has a joint functional calculus as a subfamily of  $T$ . Using Theorem 14.1.1 with the  $|\Lambda|$ -tuple  $\alpha = (\frac{1}{2}, \dots, \frac{1}{2})$ , we can define a bounded operator  $J_{\Lambda,1}$  using square functions. We let

$$\begin{aligned} X_\Lambda &\rightarrow \text{Rad}_p(\mathbb{Z}^\Lambda; X) \\ J_{\Lambda,1} : x &\mapsto \sum_{l_{i_1}, \dots, l_{i_k} \geq 1} r_{(l_{i_1}, \dots, l_{i_k})} \otimes \prod_{s=1}^k T_{i_s}^{l_{i_s}-1} (I_X - T_{i_s})^{\frac{1}{2}} x. \end{aligned}$$

Since  $H^\infty$  functional calculus passes to the adjoint, we have square function estimates for  $(T_1^*, \dots, T_d^*)$  and we can define in the same way

$$\begin{aligned} (X^*)_\Lambda &\rightarrow \text{Rad}_{p'}(\mathbb{Z}^\Lambda; X^*) \\ J_{\Lambda,2} : y &\mapsto \sum_{l_{i_1}, \dots, l_{i_k} \geq 1} r_{(l_{i_1}, \dots, l_{i_k})} \otimes \prod_{s=1}^k ((T_{i_s}^*)^{l_{i_s}-1} (I_X - (T_{i_s}^*)^*)^{\frac{1}{2}} y), \end{aligned}$$

where  $p'$  is the conjugate exponent of  $p$  (verifying  $\frac{1}{p} + \frac{1}{p'} = 1$ ).

For any  $x \in X_\Lambda$ ,  $y \in (X^*)_\Lambda$  and  $n_1, \dots, n_k$  integers, we compute

$$\begin{aligned}
\langle V_{i_1}^{n_1} \cdots V_{i_k}^{n_k} J_{\Lambda,1}x, J_{\Lambda,2}y \rangle &= \sum_{l_1, \dots, l_k \geq 1} \left\langle \prod_{s=1}^k T_{i_s}^{l_s + n_s - 1} (I_X - T_{i_s})^{\frac{1}{2}} x, \prod_{s=1}^k ((T_{i_s})^*)^{l_s - 1} (I_{X^*} - (T_{i_s})^*)^{\frac{1}{2}} y \right\rangle \\
&= \sum_{l_1, \dots, l_k \geq 1} \left\langle \prod_{s=1}^k T_{i_s}^{n_s + 2(l_s - 1)} (I_X - T_{i_s}) x, y \right\rangle \\
&= \sum_{l_1, \dots, l_k \geq 1} \left\langle \prod_{s=1}^k T_{i_s}^{n_s + 2(l_s - 1)} (I_X - T_{i_s}^2) (I_X + T_{i_s})^{-1} x, y \right\rangle.
\end{aligned} \tag{15.1.4}$$

Now recall that for any  $x$  in  $\overline{\text{Ran}(I_X - T_i)}$ ,  $i = 1, \dots, d$  we have

$$\sum_{k=1}^{\infty} T_i^{2(k-1)} (I_X - T_i^2)(x) = x.$$

Indeed, one has  $\sum_{k=1}^P T_i^{2(k-1)} (I_X - T_i^2)(x) = x - T_i^{2P}(x)$  for any  $x$  and  $P \geq 1$ . If  $x = (I_X - T_i)z$ ,  $z \in X$ , we have  $\|T_i^{2P}(I_X - T_i)z\| \lesssim \frac{\|z\|}{2^P}$  using Ritt condition and this term tends to 0 as  $P$  tends to  $\infty$ .

Hence, letting  $S_\Lambda = \prod_{s=1}^k (I + T_{i_s})^{-1}$  and developping the last sum in (15.1.4), we finally obtain

$$\langle V_{i_1}^{n_1} \cdots V_{i_k}^{n_k} J_{\Lambda,1}x, J_{\Lambda,2}y \rangle = \langle T_{i_1}^{n_1} \cdots T_{i_k}^{n_k} S_\Lambda x, y \rangle.$$

Next, for any  $j \notin \Lambda$ , we see that  $T_j(x) = x, (T_j)^*(y) = y$  for  $x \in X_\Lambda$  and  $y \in (X^*)_\Lambda$ . Moreover,  $V_j$  acts as identity operator on  $\text{Rad}(\mathbb{Z}^\Lambda; X)$ , so that for any integers  $n_1, \dots, n_d$

$$\langle V_1^{n_1} \cdots V_d^{n_d} J_{\Lambda,1}x, J_{\Lambda,2}y \rangle = \langle T_1^{n_1} \cdots T_d^{n_d} S_\Lambda x, y \rangle, \quad x \in X_\Lambda, \quad y \in (X^*)_\Lambda. \tag{15.1.5}$$

Note that for any  $a$  in  $\text{Ker}(I_X - T_i)$  and  $b$  in  $\overline{\text{Ran}(I_X - T_i^*)}$ , we have clearly  $\langle a, b \rangle = 0$ ,  $i = 1, \dots, d$ . Thus, for any  $a$  in  $X_\Lambda$  and  $b$  in  $(X^*)_{\Lambda'}$  with  $\Lambda \neq \Lambda'$ , one has  $\langle a, b \rangle = 0$ . Since each  $T_i$  maps  $X_\Lambda$  into itself for any  $i$  in  $\{1, \dots, d\}$  and  $\Lambda \subset \{1, \dots, d\}$ , one obtains that for subsets  $\Lambda \neq \Lambda'$ , we have

$$\langle V_1^{n_1} \cdots V_d^{n_d} J_{\Lambda,1}x, J_{\Lambda',2}y \rangle = 0, \quad x \in X_\Lambda, \quad y \in (X^*)_{\Lambda'}. \tag{15.1.6}$$

$X$  being a  $K$ -convex space, every space  $\text{Rad}_p(\mathbb{Z}^\Lambda; X)$  is complemented in  $L_p(\Omega_\Lambda; X)$ . Thus, the space  $W = \bigoplus_{\Lambda \subset \{1, \dots, d\}} \text{Rad}_p(\mathbb{Z}^\Lambda; X)$  is complemented in  $\bigoplus_{\Lambda \subset \{1, \dots, d\}} L_p(\Omega_\Lambda; X)$ , which is identified to  $L_p(\Sigma; X)$  where  $\Sigma = \sqcup_{\Lambda \subset \{1, \dots, d\}} \Omega_\Lambda$  equipped with the sum measure. Then we write  $L_p(\Sigma; X) = W \oplus E$ . It is the same for

$$W' = \bigoplus_{\Lambda \subset \{1, \dots, d\}} \text{Rad}_{p'}(\mathbb{Z}^\Lambda; X^*),$$

which is complemented in  $L_{p'}(\Sigma; X^*)$ .

We can define operators  $J_0$  and  $Q_0$  as follows. We let

$$J_0 = \oplus_{\Lambda \subset \{1, \dots, d\}} J_{\Lambda, 1} (S_\Lambda)^{-1}$$

from  $X$  into  $W$ , then we let  $J_2 = \oplus_{\Lambda \subset \{1, \dots, d\}} J_{\Lambda, 2}$  from  $X^*$  into  $W'$  and finally  $Q_0 = J_2^*$ , which maps  $(W')^*$  into  $X$ . Using that  $W$  acts by duality on  $W'$  and the natural complementation of  $W$  in  $L_p(\Sigma; X)$  as well as complementation of  $W'$  in  $L_{p'}(\Sigma; X^*)$  (which relies on the canonical projection), we identify  $(W')^*$  to  $W$ , so that  $Q_0$  maps  $W$  into  $X$ . Identities (15.1.5) and (15.1.6) guarantee that we have

$$T_1^{n_1} \cdots T_d^{n_d} = Q_0 V_1^{n_1} \cdots V_d^{n_d} J_0, \quad n_1, \dots, n_d \in \mathbb{N}. \quad (15.1.7)$$

We let  $J = \mathcal{J} \circ J_0$  where  $\mathcal{J} : W \hookrightarrow L_p(\Sigma; X)$  is the inclusion,  $Q = Q_0 \circ \Pi$  where  $\Pi$  is the canonical projection from  $L_p(\Sigma; X)$  onto  $W$  and finally let  $U_j = V_j \oplus I_E$ ,  $j = 1, \dots, d$ . Then every  $U_j$  is an isomorphism of  $L_p(\Sigma; X)$  for  $j = 1, \dots, d$  having with (15.1.7)

$$T_1^{n_1} \cdots T_d^{n_d} = Q U_1^{n_1} \cdots U_d^{n_d} J \quad n_1, \dots, n_d \in \mathbb{N}.$$

It is clear that  $U_1, \dots, U_d$  are commuting operators and we obtain (15.1.2).

It remains to prove that  $U = (U_1, \dots, U_d)$  admits a  $C(\mathbb{T}^d)$  bounded calculus. It suffices to prove this fact for  $V = (V_1, \dots, V_d)$  and hence for any  $(V_{1, \Lambda}, \dots, V_{d, \Lambda})$  for any  $\Lambda \subset \{1, \dots, d\}$ . Let  $\phi = \sum a_{n_1, \dots, n_d} e^{in_1} \cdots e^{in_d}$  be a trigonometric polynomial function. Noting that  $V_{j, \Lambda} = v_{j, \Lambda} \overline{\otimes} I_X$ , we have  $\phi(V_{1, \Lambda}, \dots, V_{d, \Lambda}) = \phi(v_{1, \Lambda}, \dots, v_{d, \Lambda}) \overline{\otimes} I_X$ .

We take  $(g_{l_{i_1}, \dots, l_{i_k}})$  a sequence of independant Gaussian variable on some probability space  $\Xi$  and we can define the space  $G_2(\mathbb{Z}^\Lambda)$  as the closed subspace of  $L_2(\Xi)$  spanned by  $(g_{l_{i_1}, \dots, l_{i_k}})$ . Next we define  $v_{j, \Lambda}$  on  $G_2(\mathbb{Z}^\Lambda)$  in the same manner as  $v_{j, \Lambda}$  on  $\text{Rad}_p(\mathbb{Z}^\Lambda)$ . It is clear that  $v_{j, \Lambda}$  is a unitary operator of the Hilbert space  $G_2(\mathbb{Z}^\Lambda)$ . According to the Spectral Theorem, we obtain

$$\|\phi(v_{1, \Lambda}, \dots, v_{d, \Lambda})\|_{G_2(\mathbb{Z}^\Lambda) \rightarrow G_2(\mathbb{Z}^\Lambda)} = \sup \{ |\phi(s_1, \dots, s_d)|, (s_1, \dots, s_d) \in \mathbb{R}^d \}. \quad (15.1.8)$$

Recall that for any bounded operator  $\psi : G_2(\mathbb{Z}^\Lambda) \rightarrow G_2(\mathbb{Z}^\Lambda)$ , the operator  $\psi \otimes I_X : G_2(\mathbb{Z}^\Lambda) \otimes X \rightarrow G_2(\mathbb{Z}^\Lambda) \otimes X$  extends to a unique bounded operator  $\psi \overline{\otimes} I_X : G_2(\mathbb{Z}^\Lambda; X) \rightarrow G_2(\mathbb{Z}^\Lambda; X)$  with  $\|\psi \overline{\otimes} I_X\| = \|\psi\|$  (see [35, Prop. 6.1.23]).

As  $X$  is a  $K$ -convex space, it has finite cotype. Thus, the Rademacher averages and Gaussian averages on  $X$  are equivalent. More precisely, the spaces  $\text{Rad}_p(\mathbb{Z}^\Lambda; X)$  and  $G_2(\mathbb{Z}^\Lambda; X)$  are naturally isomorphic and we have therefore an equivalence

$$\|\phi(V_{1, \Lambda}, \dots, V_{d, \Lambda})\|_{\text{Rad}_p(\mathbb{Z}^\Lambda; X) \rightarrow \text{Rad}_p(\mathbb{Z}^\Lambda; X)} \simeq \|\phi(v_{1, \Lambda}, \dots, v_{d, \Lambda}) \overline{\otimes} I_X\|_{G_2(\mathbb{Z}^\Lambda; X) \rightarrow G_2(\mathbb{Z}^\Lambda; X)}$$

and then

$$\|\phi(V_{1, \Lambda}, \dots, V_{d, \Lambda})\|_{\text{Rad}_p(\mathbb{Z}^\Lambda; X) \rightarrow \text{Rad}_p(\mathbb{Z}^\Lambda; X)} \simeq \|\phi(v_{1, \Lambda}, \dots, v_{d, \Lambda})\|_{G_2(\mathbb{Z}^\Lambda) \rightarrow G_2(\mathbb{Z}^\Lambda)}. \quad (15.1.9)$$

Combining (15.1.8) and (15.1.9), we obtain an inequality of type

$$\|\phi(V_{1, \Lambda}, \dots, V_{d, \Lambda})\|_{\text{Rad}(\mathbb{Z}^\Lambda; X)} \lesssim \sup \{ |\phi(s_1, \dots, s_d)|, (s_1, \dots, s_d) \in \mathbb{R}^d \},$$

which is exactly the property of  $C(\mathbb{T}^d)$  bounded calculus.  $\square$

## 15.2 The case of commuting semigroups

This section is devoted to dilation properties of commuting families of bounded analytic semigroups and their connections with joint functional calculus. We refer to Section 4.1 for background on  $K$ -convexity and state two elementary lemmas.

Let  $X$  be a Banach space.

Assume that  $X$  is a reflexive Banach space and let  $(A_1, \dots, A_d)$  be a commuting  $d$ -tuple of sectorial operators on  $X$ . For any  $\Lambda \subset \{1, \dots, d\}$ , let  $X_\Lambda$  be defined by (5.1.3). Recall that  $(A_1^*, \dots, A_d^*)$  is a commuting  $d$ -tuple of sectorial operators on  $X^*$ . Then we similarly define  $X_\Lambda^* \subset X^*$  as the intersection of  $\bigcap_{k \in \Lambda} \overline{R(A_k^*)}$  and  $\bigcap_{k \notin \Lambda} N(A_k^*)$ . Let us use the notation  $(X_\Lambda)^*$  for the dual space of  $X_\Lambda$ . The closeness of the notations  $(X_\Lambda)^*$  and  $X_\Lambda^*$  is justified by the fact they are isomorphic, as follows.

**Lemma 15.2.1** *The mapping  $X_\Lambda^* \rightarrow (X_\Lambda)^*$  which takes any  $\eta \in X_\Lambda^*$  to its restriction  $\eta|_{X_\Lambda}$ , is an isomorphism.*

*Proof of Lemma 15.2.1.* For any  $k = 1, \dots, d$ ,  $N(A_k)^\perp = \overline{R(A_k^*)}$  and  $\overline{R(A_k)}^\perp = N(A_k^*)$ . Hence we have

$$X_\Lambda^\perp = \bigoplus_{\substack{\Lambda' \subset \{1, \dots, d\} \\ \Lambda' \neq \Lambda}} X_{\Lambda'}^*.$$

Then the result follows from standard duality principles.  $\square$

Recall the sequence  $(\Phi_m)_{m \geq 1}$  given by (5.2.9). According to [56, Lemma 6.5], which holds true on any Banach space, we have the following fact.

**Lemma 15.2.2** *Let  $(T_t)_{t \geq 0}$  be a bounded analytic semigroup on  $X$ , and let  $A$  be its negative generator. For any  $m \geq 1$ , the function  $s \mapsto AT_s \Phi_m(A)$  belongs to  $L^1(\mathbb{R}_+^*; B(X))$ , and*

$$\int_0^\infty AT_s \Phi_m(A) ds = \Phi_m(A).$$

Let  $((T_t^1)_{t \geq 0}, \dots, (T_t^d)_{t \geq 0})$  be a  $d$ -tuple of bounded  $C_0$ -semigroups on  $X$  and for any  $k = 1, \dots, d$ , let  $A_k$  be the negative generator of  $(T_t^k)_{t \geq 0}$ . It is plain that the sectorial operators  $A_1, \dots, A_d$  commute if and only if the semigroups  $(T_t^1)_{t \geq 0}, \dots, (T_t^d)_{t \geq 0}$  commute, that is,  $T_t^k T_{t'}^{k'} = T_{t'}^{k'} T_t^k$  for any  $1 \leq k, k' \leq d$  and any  $t, t' \geq 0$ .

The following theorem should be regarded as a multivariable version of [27, Theorem 5.1] on  $K$ -convex reflexive spaces. It is also related albeit different from [6, Theorem 4.8]. Note that Definition 5.2.2 is used in the statement.

**Theorem 15.2.3** *Assume that  $X$  is reflexive and  $K$ -convex. Let  $((T_t^1)_{t \geq 0}, \dots, (T_t^d)_{t \geq 0})$  be a  $d$ -tuple of commuting bounded analytic semigroups, and let  $A_1, \dots, A_d$  denote their negative generators. Assume that  $(A_1, \dots, A_d)$  admits an  $H^\infty(\Sigma_{\theta_1} \times \dots \times \Sigma_{\theta_d})$  joint functional calculus, for some  $\theta_k \in (0, \frac{\pi}{2})$ ,  $k = 1, \dots, d$ . Then there exist a measure space  $(\Omega, dm)$ , two bounded operators  $J: X \rightarrow L^2(\Omega; X)$  and  $Q: L^2(\Omega; X) \rightarrow X$ , as well as a  $d$ -tuple  $((U_t^1)_{t \in \mathbb{R}}, \dots, (U_t^d)_{t \in \mathbb{R}})$  of commuting bounded  $C_0$ -groups on  $L^2(\Omega; X)$  such that:*



- (a) The  $d$ -tuple  $(B_1, \dots, B_d)$ , where  $B_k$  is the negative generator of  $(U_t^k)_t$  for all  $k = 1, \dots, d$ , admits an  $H^\infty(\Sigma_{\frac{\pi}{2}} \times \dots \times \Sigma_{\frac{\pi}{2}})$  joint functional calculus;
- (b) For any  $t_1, \dots, t_d$  in  $\mathbb{R}_+$ , we have

$$T_{t_1}^1 \cdots T_{t_d}^d = QU_{t_1}^1 \cdots U_{t_d}^d J.$$

*Proof of Theorem 15.2.3.* We set  $X_1 = \overline{R(A_1)} \cap \dots \cap \overline{R(A_d)}$  and  $X_1^* = \overline{R(A_1^*)} \cap \dots \cap \overline{R(A_d^*)}$ .

For any  $x \in X_1$ , let  $\varphi_x: \mathbb{R}^d \rightarrow X$  be defined by  $\varphi_x(s_1, \dots, s_d) = A_1^{\frac{1}{2}} T_{s_1}^1 \cdots A_d^{\frac{1}{2}} T_{s_d}^d x$  if  $(s_1, \dots, s_d) \in \mathbb{R}_+^{*d}$ , and  $\varphi_x(s_1, \dots, s_d) = 0$  otherwise.

Consider  $F \in H_0^\infty(\Sigma_{\theta_1} \times \dots \times \Sigma_{\theta_d})$  defined by

$$F(z_1, \dots, z_d) = (z_1^{\frac{1}{2}} e^{-z_1}) (z_2^{\frac{1}{2}} e^{-z_2}) \cdots (z_d^{\frac{1}{2}} e^{-z_d}). \quad (15.2.1)$$

By the joint functional calculus assumption and Theorem 14.1.7,  $(A_1, \dots, A_d)$  admits a square function estimate with respect to  $F$ . Since

$$F(s_1 A_1, \dots, s_d A_d) = s_1^{\frac{1}{2}} A_1^{\frac{1}{2}} T_{s_1}^1 \cdots s_d^{\frac{1}{2}} A_d^{\frac{1}{2}} T_{s_d}^d$$

for any  $s_1, \dots, s_d > 0$ , this implies that  $\varphi_x$  belongs to  $\gamma(\mathbb{R}^d; X)$  for any  $x$  and that we have a uniform estimate

$$\|x\|_F = \|\varphi_x\|_{\gamma(\mathbb{R}^d; X)} \lesssim \|x\|, \quad x \in X_1.$$

Using the embedding of  $\gamma(L^2(\mathbb{R}^d); X)$  into  $L^2(\mathcal{S}; X)$ , this yields a bounded operator

$$J_1: X_1 \longrightarrow L^2(\mathcal{S}; X)$$

given by  $J_1(x) = \varphi_x$ , for any  $x \in X_1$ .

Likewise we have a bounded operator

$$\tilde{J}_1: X_1^* \longrightarrow L^2(\mathcal{S}; X^*)$$

given by  $\tilde{J}_1(y) = \tilde{\varphi}_y$  for any  $y \in X_1^*$ , where  $\tilde{\varphi}_y(s_1, \dots, s_d) = A_1^{*\frac{1}{2}} T_{s_1}^{1*} \cdots A_d^{*\frac{1}{2}} T_{s_d}^{d*} y$  if  $(s_1, \dots, s_d) \in \mathbb{R}_+^{*d}$ , and  $\tilde{\varphi}_y(s_1, \dots, s_d) = 0$  otherwise.

For any  $k = 1, \dots, d$ , and any  $t \in \mathbb{R}$ , let  $\tau_t^k: L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$  be the translation operator defined by setting

$$[\tau_t^k(h)](s_1, \dots, s_d) = h(s_1, \dots, s_{k-1}, s_k + t, s_{k+1}, \dots, s_d), \quad (s_1, \dots, s_d) \in \mathbb{R}^d, \quad (15.2.2)$$

for any  $h \in L^2(\mathbb{R}^d)$ .

According to Lemma 3.4.4,  $\tau_t^k \otimes I_X$  extends to an isometric isomorphism

$$V_t^k: \gamma(L^2(\mathbb{R}^d); X) \longrightarrow \gamma(L^2(\mathbb{R}^d); X).$$

A thorough look at this tensor extension process shows that  $V_t^k(u) = u \circ \tau_{-t}^k$  for any  $u \in \gamma(L^2(\mathbb{R}^d); X)$ . We deduce that for any  $\zeta \in \gamma(\mathbb{R}^d; X)$ , we have

$$[V_t^k(u_\zeta)] = u_{\zeta_t^k}, \quad (15.2.3)$$

where  $\zeta_t^k \in \gamma(\mathbb{R}^d; X)$  is given by

$$\zeta_t^k(s_1, \dots, s_d) = \zeta(s_1, \dots, s_{k-1}, s_k + t, s_{k+1}, \dots, s_d), \quad (s_1, \dots, s_d) \in \mathbb{R}^d. \quad (15.2.4)$$

Moreover for any  $k = 1, \dots, d$ ,  $(V_t^k)_{t \in \mathbb{R}}$  is an isometric  $C_0$ -group on  $\gamma(L^2(\mathbb{R}^d); X)$ . Using (4.1.1), we now define

$$U_t^k: L^2(\mathcal{S}; X) \longrightarrow L^2(\mathcal{S}; X)$$

by setting  $U_t^k(u, w) = (V_t^k(u), w)$  for any  $u \in \gamma(L^2(\mathbb{R}^d); X)$  and any  $w \in Z_X$ . It follows from above that  $((U_t^1)_{t \in \mathbb{R}}, \dots, (U_t^d)_{t \in \mathbb{R}})$  is a commuting family of bounded  $C_0$ -groups on  $L^2(\mathcal{S}; X)$ .

Let  $x \in X_1$ , let  $y \in X_1^*$  and let  $t_1, \dots, t_d \geq 0$ . Since  $X$  is reflexive, we have

$$L^2(\mathcal{S}; X^*) = L^2(\mathcal{S}; X)^* \quad (15.2.5)$$

isometrically, see e.g. [20, Section IV.1]. Then according to (15.2.3), (15.2.4) and trace duality on  $\gamma$ -spaces (see [35, Section 9.1.f]), we have

$$\langle U_{t_1}^1 \cdots U_{t_d}^d J_1(x), \tilde{J}_1(y) \rangle = \int_{\mathbb{R}^d} \langle \varphi_x(s_1 + t_1, \dots, s_d + t_d), \tilde{\varphi}_y(s_1, \dots, s_d) \rangle ds_1 \cdots ds_d.$$

Let  $x_m = \Phi_m(A_1) \cdots \Phi_m(A_d)x$  for any  $m \geq 1$ . Using the definitions of  $\varphi_x, \tilde{\varphi}_y$  and the above identity, we obtain that

$$\begin{aligned} \langle U_{t_1}^1 \cdots U_{t_d}^d J_1(x_m), \tilde{J}_1(y) \rangle &= \int_0^\infty \cdots \int_0^\infty \langle (A_1 T_{2s_1+t_1}^1 \Phi_m(A_1)) \cdots (A_d T_{2s_d+t_d}^d \Phi_m(A_d))(x), y \rangle ds_1 \cdots ds_d \\ &= \int_0^\infty \cdots \int_0^\infty \langle (A_1 T_{2s_1}^1 \Phi_m(A_1)) \cdots (A_d T_{2s_d}^d \Phi_m(A_d))(T_{t_1}^1 \cdots T_{t_d}^d x), y \rangle ds_1 \cdots ds_d. \end{aligned}$$

Applying Lemma 15.2.2 for each  $A_k$ , we deduce that

$$\langle U_{t_1}^1 \cdots U_{t_d}^d J_1(x_m), \tilde{J}_1(y) \rangle = 2^{-d} \langle T_{t_1}^1 \cdots T_{t_d}^d x_m, y \rangle.$$

Since  $x \in X_1$ , we know that  $x_m \rightarrow x$ , by (5.2.10). We infer that

$$\langle U_{t_1}^1 \cdots U_{t_d}^d J_1(x), \tilde{J}_1(y) \rangle = 2^{-d} \langle T_{t_1}^1 \cdots T_{t_d}^d x, y \rangle.$$

Using Lemma 15.2.1, let us identify  $X_1^*$  with  $(X_1)^*$  and note that  $L^2(\mathcal{S}; X^*)^* = L^2(\mathcal{S}; X)$ , by (15.2.5). Set  $Q_1 = 2^d \tilde{J}_1^*: L^2(\mathcal{S}; X) \rightarrow X_1$ . Then the above identity shows that

$$Q_1 U_{t_1}^1 \cdots U_{t_d}^d J_1 = T_{t_1}^1 \cdots T_{t_d}^d|_{X_1}.$$

For any  $k = 1, \dots, d$ , let  $B_k$  (resp.  $C_k$ ) be the negative generator of the  $C_0$ -group  $(U_t^k)_t$  (resp.  $(V_t^k)_t$ ). Then let  $c_k$  be the negative generator of the  $C_0$ -group  $(\tau_t^k)_t$ . It is easy to check that for any  $\theta_1, \dots, \theta_d$  in  $(\frac{\pi}{2}, \pi)$  and any  $f \in H_{0,1}^\infty(\Sigma_{\theta_1} \times \cdots \times \Sigma_{\theta_d})$ , the operator  $f(C_1, \dots, C_d): \gamma(L^2(\mathbb{R}^d); X) \rightarrow \gamma(L^2(\mathbb{R}^d); X)$  coincides with  $f(c_1, \dots, c_d) \otimes I_X$  on  $L^2(\mathbb{R}^d) \otimes X$ . According to Lemma 3.4.4, this implies that

$$\|f(C_1, \dots, C_d)\| = \|f(c_1, \dots, c_d)\|.$$

Now since  $(\tau_t^1)_t, \dots, (\tau_t^d)_t$  are commuting unitary groups, it follows from spectral theory that  $\|f(c_1, \dots, c_d)\| \leq \|f\|_{\infty, \Sigma_{\frac{\pi}{2}}^d}$ . This proves that  $(C_1, \dots, C_d)$  admits an  $H^\infty(\Sigma_{\frac{\pi}{2}} \times \dots \times \Sigma_{\frac{\pi}{2}})$  joint functional calculus. It readily implies that the same holds true for  $(B_1, \dots, B_d)$ .

What we have proved so far is that the theorem holds true on  $X_\Lambda$  instead of  $X$ . Now it is easy to adapt the argument to show that more generally, the theorem holds true on  $X_\Lambda$  for any  $\Lambda \subset \{1, \dots, d\}$ , using

$$F_\Lambda(z_1, \dots, z_d) = \prod_{k \in \Lambda} (z_k^{\frac{1}{2}} e^{-z_k})$$

instead of (15.2.1).

Finally the result holds true on  $X$  using Lemma 5.1.2 and a simple direct sum argument.  $\square$

**Remark 15.2.4** Using the  $\gamma^p$ -spaces (see [35, Section 9.1]) in place of the  $\gamma$ -spaces considered in the present paper, we easily obtain that for any  $1 < p < \infty$ , Theorem 15.2.3 holds as well with  $L^p(\Omega; X)$  in place of  $L^2(\Omega; X)$ .

# Chapter 16

## $H^\infty$ joint functional calculus from dilation

### 16.1 The case of commuting Ritt operators

For the purpose of having a converse property of Theorem 15.1.1, we give Theorem 16.1.1 below whose proof relies on (3) of Theorem 10.0.2.

**Theorem 16.1.1** *Let  $X$  be a Banach space and  $p \in (1, \infty)$ . Let  $(T_1, \dots, T_d)$  a  $d$ -tuple of commuting operators acting on  $X$  such that every  $T_k$  is a  $R$ -Ritt operator,  $k = 1, \dots, d$ . Suppose that there exist a measure space  $\Sigma$ , a  $d$ -tuple of isomorphisms  $(U_1, \dots, U_d)$  acting on  $L_p(\Sigma; X)$  having a  $C(\mathbb{T}^d)$  bounded calculus and two bounded operator  $J : X \rightarrow L_p(\Sigma; X)$ ,  $Q : L_p(\Sigma; X) \rightarrow X$  such that (15.1.2) is verified.*

*Then there exist  $\gamma_1, \dots, \gamma_d$  in  $(0, \frac{\pi}{2})$  such that  $(T_1, \dots, T_d)$  admits a  $H^\infty(B_{\gamma_1} \times \dots \times B_{\gamma_d})$  joint functional calculus.*

*Proof of Theorem 16.1.1.* If (15.1.2) is verified, then for any polynomial function  $P$  of  $d$  variables we have  $P(T_1, \dots, T_d) = QP(U_1, \dots, U_d)J$  and then

$$\|P(T_1, \dots, T_d)\| \leq \|Q\| \|J\| \|P(U_1, \dots, U_d)\|.$$

Now, if  $(U_1, \dots, U_d)$  admits a  $C(\mathbb{T}^d)$  bounded calculus, then for any polynomial function  $P$  as above we have

$$\|P(U_1, \dots, U_d)\| \lesssim \sup \{ |P(z_1, \dots, z_d)| : (z_1, \dots, z_d) \in \mathbb{T}^d \}.$$

Thus,  $(T_1, \dots, T_d)$  is polynomially bounded and as every  $T_k$  is a  $R$ -Ritt operator, (3) of Theorem 10.0.2 implies that  $(T_1, \dots, T_d)$  admits a  $H^\infty$  joint functional calculus.  $\square$

### 16.2 The case of commuting semigroups

The following result provides a converse to Theorem 15.2.3, up to an  $R$ -sectoriality assumption on the  $A_k$ .

**Theorem 16.2.1** *Let  $((T_t^1)_{t \geq 0}, \dots, (T_t^d)_{t \geq 0})$  be a  $d$ -tuple of commuting bounded analytic semigroups on some Banach space  $X$ , and let  $A_1, \dots, A_d$  denote their negative generators. Assume that each  $A_k$  is  $R$ -sectorial of  $R$ -type  $\omega_k < \frac{\pi}{2}$ . Assume further that there exist a Banach space  $Y$ , two bounded operators  $J: X \rightarrow Y$  and  $Q: Y \rightarrow X$ , as well as a  $d$ -tuple  $((U_t^1)_{t \in \mathbb{R}}, \dots, (U_t^d)_{t \in \mathbb{R}})$  of commuting bounded  $C_0$ -groups on  $Y$  such that:*

- (a) *The  $d$ -tuple  $(B_1, \dots, B_d)$ , where  $B_k$  is the negative generator of  $(U_t^k)_t$  for all  $k = 1, \dots, d$ , admits an  $H^\infty(\Sigma_{\theta_1} \times \dots \times \Sigma_{\theta_d})$  joint functional calculus, for some  $\theta_k \in [\frac{\pi}{2}, \pi)$ ;*
- (b) *For any  $t_1, \dots, t_d$  in  $\mathbb{R}_+$ , we have*

$$T_{t_1}^1 \cdots T_{t_d}^d = QU_{t_1}^1 \cdots U_{t_d}^d J.$$

*Then  $(A_1, \dots, A_d)$  admits an  $H^\infty(\Sigma_{\theta'_1} \times \dots \times \Sigma_{\theta'_d})$  joint functional calculus, for all  $\theta'_k \in (\omega_k, \frac{\pi}{2})$ ,  $k = 1, \dots, d$ .*

*Proof of Theorem 16.2.1.* By condition (b) and the Laplace formula, we have

$$R(z_1, A_1) \cdots R(z_d, A_d) = QR(z_1, A_1) \cdots R(z_d, A_d)J.$$

for any complex numbers  $z_1, \dots, z_d$  with negative real parts. This implies, by (5.2.2), that

$$f(A_1, \dots, A_d) = Qf(B_1, \dots, B_d)J,$$

and hence

$$\|f(A_1, \dots, A_d)\| \leq \|Q\| \|J\| \|f(B_1, \dots, B_d)\|, \quad (16.2.1)$$

for any  $\theta_1, \dots, \theta_d \in (\frac{\pi}{2}, \pi)$  and any  $f \in H_{0,1}^\infty(\Sigma_{\theta_1} \times \dots \times \Sigma_{\theta_d})$ .

By condition (a), we may choose  $\theta_1, \dots, \theta_d$  in such a way that  $(B_1, \dots, B_d)$  admits an  $H^\infty(\Sigma_{\theta_1} \times \dots \times \Sigma_{\theta_d})$  joint functional calculus. Then by (16.2.1),  $(A_1, \dots, A_d)$  admits an  $H^\infty(\Sigma_{\theta_1} \times \dots \times \Sigma_{\theta_d})$  joint functional calculus as well. Now applying Theorem 10.0.1, we obtain the result.  $\square$

Recall that if  $X$  has property  $(\Delta)$ , then any sectorial operator  $A$  on  $X$  with an  $H^\infty(\Sigma_\theta)$  functional calculus is automatically  $R$ -sectorial of  $R$ -type  $\theta$  [40, Theorem 5.3]. Combining this result with Theorems 15.2.3 and 16.2.1, we obtain the following corollary. We note that any UMD Banach space is both reflexive,  $K$ -convex (see e.g. [35, Section 4.3]) and has property  $(\Delta)$  (see [40, Proposition 3.2]); hence the next result applies to such spaces.

**Corollary 16.2.2** *Let  $X$  be a reflexive,  $K$ -convex Banach space, with property  $(\Delta)$ . Let  $((T_t^1)_{t \geq 0}, \dots, (T_t^d)_{t \geq 0})$  be a  $d$ -tuple of commuting bounded analytic semigroups on  $X$ , and let  $A_1, \dots, A_d$  denote their negative generators. The following assertions are equivalent.*

- (i) *The  $d$ -tuple  $(A_1, \dots, A_d)$  admits an  $H^\infty(\Sigma_{\theta_1} \times \dots \times \Sigma_{\theta_d})$  joint functional calculus, for some  $\theta_k \in (0, \frac{\pi}{2})$ ,  $k = 1, \dots, d$ .*

(ii) Each  $A_k$  is  $R$ -sectorial of  $R$ -type  $< \frac{\pi}{2}$  and there exist a measure space  $(\Omega, dm)$ , two bounded operators  $J: X \rightarrow L^2(\Omega; X)$  and  $Q: L^2(\Omega; X) \rightarrow X$ , as well as a  $d$ -tuple  $((U_t^1)_{t \in \mathbb{R}}, \dots, (U_t^d)_{t \in \mathbb{R}})$  of commuting bounded  $C_0$ -groups on  $L^2(\Omega; X)$  such that:

- (a) The  $d$ -tuple  $(B_1, \dots, B_d)$ , where  $B_k$  is the negative generator of  $(U_t^k)_t$  for all  $k = 1, \dots, d$ , admits an  $H^\infty(\Sigma_{\frac{\pi}{2}} \times \dots \times \Sigma_{\frac{\pi}{2}})$  joint functional calculus;
- (b) For any  $t_1, \dots, t_d$  in  $\mathbb{R}_+$ , we have

$$T_{t_1}^1 \cdots T_{t_d}^d = QU_{t_1}^1 \cdots U_{t_d}^d J.$$

To conclude, we wish to emphasize that for a commuting family of generators of bounded  $C_0$ -groups, to have an  $H^\infty(\Sigma_{\frac{\pi}{2}} \times \dots \times \Sigma_{\frac{\pi}{2}})$  joint functional calculus is a very strong property.

**Proposition 16.2.3** *Let  $((U_t^1)_{t \in \mathbb{R}}, \dots, (U_t^d)_{t \in \mathbb{R}})$  be a commuting family of bounded  $C_0$ -groups on some Banach space  $Y$ . The following are equivalent.*

(i) *The  $d$ -tuple  $(B_1, \dots, B_d)$ , where  $B_k$  is the negative generator of  $(U_t^k)_t$  for all  $k = 1, \dots, d$ , admits an  $H^\infty(\Sigma_{\frac{\pi}{2}} \times \dots \times \Sigma_{\frac{\pi}{2}})$  joint functional calculus.*

(ii) *There exists a constant  $K \geq 0$  such that for any  $b \in L^1(\mathbb{R}^d)$ ,*

$$\left\| \int_{\mathbb{R}^d} b(t_1, \dots, t_d) U_{t_1}^1 \cdots U_{t_d}^d dt_1 \cdots dt_d \right\| \leq K \|\widehat{b}\|_{\infty, \mathbb{R}^d}. \quad (16.2.2)$$

In this statement,  $\widehat{b}: \mathbb{R}^d \rightarrow \mathbb{C}$  denotes the Fourier transform of  $b \in L^1(\mathbb{R}^d)$  and the integral in the left-hand side of (16.2.2) is defined in the strong sense. Property (ii) means that the mapping  $b \mapsto \int_{\mathbb{R}^d} b(t_1, \dots, t_d) U_{t_1}^1 \cdots U_{t_d}^d dt_1 \cdots dt_d$  “extends” to a bounded homomorphism  $C_0(\mathbb{R}^d) \rightarrow B(Y)$ . More precisely, there exists a (necessarily unique) bounded homomorphism  $\rho: C_0(\mathbb{R}^d) \rightarrow B(Y)$  such that

$$\rho(\widehat{b}) = \int_{\mathbb{R}^d} b(t_1, \dots, t_d) U_{t_1}^1 \cdots U_{t_d}^d dt_1 \cdots dt_d, \quad b \in L^1(\mathbb{R}^d).$$

This property is optimal since by Fourier-Plancherel,  $\|\widehat{b}\|_{\infty, \mathbb{R}^d}$  coincides with the norm of the operator  $\int_{\mathbb{R}^d} b(t_1, \dots, t_d) U_{t_1}^1 \cdots U_{t_d}^d dt_1 \cdots dt_d$  in the case when  $Y = L^2(\mathbb{R}^d)$  and  $U_t^k = \tau_t^k$  is given by (15.2.2).

*Proof of Proposition 16.2.3.* For any non empty  $\Lambda \subset \{1, \dots, d\}$  and any  $\beta \in L^1(\mathbb{R}_+^\Lambda)$ , we let  $L_\beta: \overline{\Sigma_{\frac{\pi}{2}}^d} \rightarrow \mathbb{C}$  be the Laplace transform of  $\beta$  defined by

$$L_\beta(z_1, \dots, z_d) = \int_{\mathbb{R}_+^\Lambda} \beta((t_k)_{k \in \Lambda}) \prod_{k \in \Lambda} e^{-z_k t_k} \prod_{k \in \Lambda} dt_k, \quad (z_1, \dots, z_d) \in \overline{\Sigma_{\frac{\pi}{2}}^d}.$$

This function only depends on the variables  $(z_k)_{k \in \Lambda}$ .

(ii)  $\Rightarrow$  (i): We fix  $\theta \in (\frac{\pi}{2}, \pi)$ . Let  $f \in H_0^\infty(\Sigma_\theta^\Lambda)$ , for some  $\Lambda \subset \{1, \dots, d\}$ ,  $\Lambda \neq \emptyset$ . Arguing as in [31, Section 3.3], we find that there exists  $\beta \in L^1(\mathbb{R}_+^\Lambda)$  such that  $L_\beta$  coincides with  $f$  on  $\Sigma_{\frac{\pi}{2}}^d$  and

$$f(B_1, \dots, B_d) = \int_{\mathbb{R}^\Lambda} \beta((t_k)_{k \in \Lambda}) \prod_{k \in \Lambda} U_{t_k}^k \prod_{k \in \Lambda} dt_k. \quad (16.2.3)$$

For any integer  $n \geq 1$ , define

$$b_n(t_1, \dots, t_d) = \beta((t_k)_{k \in \Lambda}) \prod_{j \notin \Lambda} (ne^{-nt_j}), \quad t_1, \dots, t_d \in \mathbb{R}_+.$$

Then  $b_n \in L^1(\mathbb{R}_+^d)$  and

$$\widehat{b}_n(s_1, \dots, s_d) = \widehat{\beta}((s_k)_{k \in \Lambda}) \prod_{j \notin \Lambda} \left( \frac{n}{n + is_j} \right), \quad s_1, \dots, s_d \in \mathbb{R}.$$

Consequently,

$$\|\widehat{b}_n\|_{\infty, \mathbb{R}^d} = \|f\|_{\infty, \Sigma_{\frac{\pi}{2}}^d}, \quad n \geq 1.$$

Moreover for any  $n \geq 1$ , we have

$$\int_{\mathbb{R}^d} b_n(t_1, \dots, t_d) U_{t_1}^1 \cdots U_{t_d}^d dt_1 \cdots dt_d = (-1)^{d-|\Lambda|} f(B_1, \dots, B_d) \prod_{j \notin \Lambda} (n(nI_Y + B_j)^{-1})$$

by (16.2.3) and the Laplace formula. Hence (ii) implies that for any  $n \geq 1$ ,

$$\left\| f(B_1, \dots, B_d) \prod_{j \notin \Lambda} (n(nI_Y + B_j)^{-1}) \right\| \leq K \|f\|_{\infty, \Sigma_{\frac{\pi}{2}}^d}$$

Since  $n(nI_Y + B_j)^{-1} \rightarrow I_Y$  strongly, the above inequality yields  $\|f(B_1, \dots, B_d)\| \leq K \|f\|_{\infty, \Sigma_{\frac{\pi}{2}}^d}$ . According to the comment after [6, Definition 2.2], this implies (i).

(i)  $\Rightarrow$  (ii): Assume (i) and set  $K_k = \sup\{\|U_t^k\| : t \in \mathbb{R}\}$  for any  $k = 1, \dots, d$ . To prove (ii), it clearly suffices to prove an estimate (16.2.2) for  $b$  lying in a dense subset of  $L^1(\mathbb{R}^d)$ . Let  $b \in L^1([-s, \infty)) \otimes \cdots \otimes L^1([-s, \infty))$  for some  $s > 0$ . Let  $b_s \in L^1(\mathbb{R}^d)$  be defined by  $b_s(t_1, \dots, t_d) = b(t_1 - s, \dots, t_d - s)$  for any  $(t_1, \dots, t_d) \in \mathbb{R}^d$ . Then

$$\begin{aligned} \int_{\mathbb{R}^d} b(t_1, \dots, t_d) U_{t_1}^1 \cdots U_{t_d}^d dt_1 \cdots dt_d &= \int_{\mathbb{R}^d} b_s(t_1, \dots, t_d) U_{t_1-s}^1 \cdots U_{t_d-s}^d dt_1 \cdots dt_d \\ &= U_{-s}^1 \cdots U_{-s}^d \int_{\mathbb{R}^d} b_s(t_1, \dots, t_d) U_{t_1}^1 \cdots U_{t_d}^d dt_1 \cdots dt_d. \end{aligned}$$

Hence

$$\left\| \int_{\mathbb{R}^d} b(t_1, \dots, t_d) U_{t_1}^1 \cdots U_{t_d}^d dt_1 \cdots dt_d \right\| \leq K_1 \cdots K_d \left\| \int_{\mathbb{R}^d} b_s(t_1, \dots, t_d) U_{t_1}^1 \cdots U_{t_d}^d dt_1 \cdots dt_d \right\|$$

Moreover  $\|\widehat{b}\|_{\infty, \mathbb{R}^d} = \|\widehat{b}_s\|_{\infty, \mathbb{R}^d}$  and  $b_s$  has support in  $[0, \infty)^d$ , that is,  $b_s \in L^1(\mathbb{R}_+^d)$ . Hence to prove (ii), it suffices to establish an estimate (16.2.2) for

$$b \in L^1(\mathbb{R}_+) \otimes \cdots \otimes L^1(\mathbb{R}_+). \quad (16.2.4)$$

If, for any  $b$  as above,  $L_b$  had an extension to an element of  $H_{0,1}^\infty(\Sigma_\theta^d)$  for some  $\theta > \frac{\pi}{2}$ , then the argument in the proof of “(ii)  $\Rightarrow$  (i)” would provide an estimate (16.2.2). This does not hold true in general and the rest of the proof is just a way to circumvent that.

Following [31, Section 2.2] we let

$$\mathcal{E}(\Sigma_\theta) = H_0^\infty(\Sigma_\theta) \oplus \text{Span}\{1, R\} \subset H^\infty(\Sigma_\theta),$$

where  $R$  is defined by  $R(z) = (1+z)^{-1}$ . Then for any  $\theta \in (\frac{\pi}{2}, \pi)$ , holomorphic functional calculus associated with  $(B_1, \dots, B_d)$  can be naturally defined on

$$\mathcal{E}(\Sigma_\theta) \otimes \cdots \otimes \mathcal{E}(\Sigma_\theta) \subset H^\infty(\Sigma_\theta^d).$$

It is easy to check (left to the reader) that the assumption (i) implies the existence of a constant  $K \geq 1$ , not depending on  $\theta$ , such that

$$\|F(B_1, \dots, B_d)\| \leq K \|F\|_{\infty, \Sigma_{\frac{\pi}{2}}^d}, \quad F \in \mathcal{E}(\Sigma_\theta) \otimes \cdots \otimes \mathcal{E}(\Sigma_\theta). \quad (16.2.5)$$

For any  $\varepsilon > 0$ , we set  $h_\varepsilon(z) = \varepsilon + z(1 + \varepsilon z)^{-1}$  and we define

$$B_{k,\varepsilon} = h_\varepsilon(B_k) = \varepsilon I_Y + B_k(I_Y + \varepsilon B_k)^{-1}, \quad k = 1, \dots, d.$$

Each of these operators is bounded and invertible, with spectrum included in  $\Sigma_{\frac{\pi}{2}}$ . For convenience we set

$$H_\varepsilon(z_1, \dots, z_d) = (h_\varepsilon(z_1), \dots, h_\varepsilon(z_d)), \quad (z_1, \dots, z_d) \in \Sigma_{\frac{\pi}{2}}^d.$$

For any  $a \in L^1(\mathbb{R}_+)$ ,  $L_a \circ h_\varepsilon$  extends to an element of  $H^\infty(\Sigma_\theta)$  for some  $\theta \in (\frac{\pi}{2}, \pi)$  depending on  $\varepsilon$ . This extension is holomorphic at 0 and  $\infty$ , hence belongs to  $\mathcal{E}(\Sigma_\theta)$ . Further by the composition rule (see [31, Section 2.4]),  $L_a \circ h_\varepsilon(B_k) = L_a(B_{k,\varepsilon})$  for any  $k = 1, \dots, d$ .

This readily implies that for any  $b$  as in (16.2.4), we have

$$\|L_b \circ H_\varepsilon(B_1, \dots, B_d)\| \leq K \|L_b \circ H_\varepsilon\|_{\infty, \Sigma_{\frac{\pi}{2}}^d} \leq K \|\widehat{b}\|_{\infty, \mathbb{R}^d},$$

by (16.2.5), and

$$L_b(B_{1,\varepsilon}, \dots, B_{d,\varepsilon}) = L_b \circ H_\varepsilon(B_1, \dots, B_d).$$

Now observe that

$$L_b(B_{1,\varepsilon}, \dots, B_{d,\varepsilon}) = \int_{\mathbb{R}^d} b(t_1, \dots, t_d) e^{-t_1 B_{1,\varepsilon}} \cdots e^{-t_d B_{d,\varepsilon}} dt_1 \cdots dt_d.$$

Consequently,

$$\left\| \int_{\mathbb{R}^d} b(t_1, \dots, t_d) e^{-t_1 B_{1,\varepsilon}} \cdots e^{-t_d B_{d,\varepsilon}} dt_1 \cdots dt_d \right\| \leq K \|\widehat{b}\|_{\infty, \mathbb{R}^d}$$

for any  $\varepsilon > 0$ .

For any  $k = 1, \dots, d$ , the semigroups  $(e^{-tB_{k,\varepsilon}})_{t \geq 0}$  are uniformly bounded and for any  $t \geq 0$ ,  $e^{-tB_{k,\varepsilon}} \rightarrow U_t^k$  strongly, when  $\varepsilon \rightarrow 0$ . This follows from the classical Yosida approximation principle. Hence using Lebesgue's dominated convergence Theorem, the above inequality implies an estimate (16.2.2) for  $b$  satisfying (16.2.4).  $\square$





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**Résumé :** Ce travail de recherche est consacré au  $H^\infty$  calcul fonctionnel joint de  $d$ -uplets commutants d'opérateurs sectoriels ou opérateurs de Ritt. La première partie donne les outils nécessaires, notamment les moyennes de Rademacher et gaussiennes ainsi que la notion d'opérateurs  $R$ -bornés et  $\gamma$ -bornés. On décrit ensuite les propriétés géométriques des espaces de Banach qui interviennent dans les résultats obtenus. Ensuite, on étend une décomposition de fonctions holomorphes, originellement due à Franks et McIntosh, au cas des fonctions de plusieurs variables. Les premiers résultats traitent du caractère automatique du calcul joint dans le sens suivant : si chaque élément du  $d$ -uplets admet un  $H^\infty$  calcul fonctionnel, sous quelles conditions ce  $d$ -uplets admet-il un  $H^\infty$  calcul fonctionnel joint ? On donne après des caractérisations en terme de dilatation de  $d$ -uplets sur des espaces de Bochner, sous certaines conditions géométriques sur l'espace de Banach sous-jacent. Le cas des espaces de Hilbert donne lieu à d'autres caractérisations en terme de similarités à des contractions. Les autres résultats traitent du  $H^\infty$  calcul fonctionnel joint relié aux fonctions carré associées à de tels  $d$ -uplets. Enfin, on propose des résultats de dilations nouveaux qui ne font pas directement appels au  $H^\infty$  calcul fonctionnel de chaque opérateurs mais aux fonctions carré ou au  $H^\infty$  calcul fonctionnel joint du  $d$ -uplet.

**Mots-clés :** Ritt et sectoriels ; Calcul fonctionnel  $H^\infty$  ; Fonctions carrés ; Dilations ; Espaces de Banach .

**Abstract :** This research work is dedicated to the  $H^\infty$  joint functional calculus of  $d$ -tuples of commuting sectorial operators or Ritt operators. The first part of this work gives the necessary tools, particularly the Rademacher averages and Gaussian averages and the notions of  $R$ -boundedness and  $\gamma$ -boundedness. We describe next the geometric properties of Banach spaces which intervene in our results. Next, we extend a decomposition result of holomorphic functions, due to Franks and McIntosh, to the case of multivariate functions. The first results deal with the automaticity of the joint calculus in the following sense : if any element of the  $d$ -tuple admits an  $H^\infty$  functional calculus, when does the  $d$ -tuple admit an  $H^\infty$  joint functional calculus? We give in the sequel some characterisations in terms of dilation of  $d$ -tuples on Bochner space provided certain geometric conditions on the underlying Banach space. The case of Hilbert spaces lead to other characterisations in term of similarity to contractions. The other results deal with  $H^\infty$  joint functional calculus related to square functions associated to such  $d$ -tuples. Finally, we offer new dilation results which does not appeal to  $H^\infty$  functional calculus of each of the operators but to square functions or  $H^\infty$  joint functional calculus of the  $d$ -tuple.

**Keywords :** Ritt and sectorial ;  $H^\infty$  functional calculus ; Square functions ; Dilations ; Banach spaces.