



Interactions and incitatives : between contract theory and mean-field games

Emma Hubert

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INTERACTIONS ET INCITATIONS : entre théorie des contrats et jeux à champs moyen

Soutenue le 10 décembre 2020 par

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1. Déformation professionnelle.

2. Présence physique ou en distanciel, rien n'est encore assuré à ce jour. . .

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INTERACTIONS ET INCITATIONS : entre théorie des contrats et jeux à champs moyen

Résumé. Dans cette thèse, nous nous intéressons principalement à trois thèmes de recherche, relativement indépendants, mais néanmoins connexes au travers du fil conducteur des *interactions et incitations*, comme souligné dans l'introduction constituant le [Chapitre 1](#).

Dans la [Partie I](#), nous présentons des extensions de la théorie des contrats, permettant notamment de considérer une multitude de joueurs dans des modèles principal-agent, avec contrôle du drift et de la volatilité, en présence d'aléa moral. En particulier, le [Chapitre 2](#) présente un problème d'incitations optimales en temps continu au sein d'une hiérarchie, inspiré du modèle à une période de [Sung \[327\] \(2015\)](#), et éclairant à deux égards : d'une part, il présente un cadre où le contrôle de la volatilité intervient de manière parfaitement naturelle, et, d'autre part, il souligne l'importance de considérer des modèles en temps continu. En ce sens, cet exemple motive l'étude complète et générale des modèles hiérarchiques effectuée dans le [Chapitre 3](#), qui va de pair avec la théorie récente des équations différentielles stochastiques du second ordre (2EDSR). Enfin, dans le [Chapitre 4](#), nous proposons une extension du modèle principal-agent développé par [Aïd, Possamaï et Touzi \[11\] \(2019\)](#) à un continuum d'agents, dont les performances sont en particulier impactées par un aléa commun. Ces études nous guident notamment vers une généralisation des contrats dits *révélateurs*, proposés initialement par [Cvitanic, Possamaï et Touzi \[111\] \(2018\)](#) dans un modèle à un seul agent.

Dans la [Partie II](#), nous présentons deux applications des problèmes principal-agent au domaine de l'énergie. La première, développée dans le [Chapitre 5](#), utilise le formalisme et les résultats théoriques introduits dans le [Chapitre 4](#) pour améliorer les programmes de réponse à la demande en électricité, déjà considérés dans [\[11\]](#). En effet, en prenant en compte l'infinité de consommateurs que doit fournir en électricité un producteur, il est possible d'utiliser cette information supplémentaire pour construire les incitations optimales, afin notamment de mieux gérer le risque résiduel impliqué par les aléas climatiques. Dans un second temps, le [Chapitre 6](#) propose, à travers un modèle principal-agent avec sélection adverse, une assurance susceptible de prévenir certaines formes de précarité, en particulier la précarité énergétique.

Enfin, nous terminons cette thèse par l'étude dans la [Partie III](#) d'un second domaine d'application, celui de l'épidémiologie, et plus précisément le contrôle de la diffusion d'une maladie contagieuse au sein d'une population. Nous considérons en premier lieu dans le [Chapitre 7](#) le point de vue des individus, à travers un jeu à champs moyen : chaque individu peut choisir son taux d'interaction avec les autres, en conciliant d'un côté son besoin d'interactions sociales et de l'autre sa peur d'être à son tour contaminé, et de contribuer à la diffusion plus large de la maladie. Nous prouvons l'existence d'un équilibre de Nash entre les individus, et l'exhibons numériquement. Dans le [Chapitre 8](#), nous prenons le point de vue du gouvernement, souhaitant inciter la population, représentée maintenant dans son ensemble, à diminuer ses interactions de manière à contenir l'épidémie. Nous montrons que la mise en place de sanctions en cas de non-respect du confinement peut s'avérer efficace, mais que, pour une maîtrise totale de l'épidémie, il est nécessaire de développer une politique de dépistage consciencieuse, accompagnée d'un isolement scrupuleux des individus testés positifs.

Mots-clés. Théorie des contrats, jeux à champ moyen, EDSR du second ordre et à champ moyen, équations de McKean–Vlasov, équation de Hamilton–Jacobi–Bellman, aléa moral et sélection adverse, organisations hiérarchiques, gestion de la demande en énergie, précarité énergétique, contrôle d'épidémie, COVID-19.

INTERACTIONS AND INCENTIVES: between contract theory and mean-field games

Abstract. In this thesis, we are mainly interested in three research themes, relatively independent, but nevertheless related through the Ariadne's thread of *interactions and incentives*, as outlined in the introduction which constitutes [Chapter 1](#).

In [Part I](#), we present some extensions of contract theory, allowing in particular to consider a multitude of players in principal-agent models, with drift and volatility control, and under moral hazard. More precisely, [Chapter 2](#) presents a continuous-time problem of optimal incentives within a hierarchy, inspired by the one-period model of [Sung \[327\] \(2015\)](#), and enlightening in two respects: on the one hand, it introduces a framework where volatility control occurs in a perfectly natural way, and, on the other hand, it underlines the importance of considering models in continuous time. In this sense, this example motivates the complete and general study of hierarchical models conducted in [Chapter 3](#), which goes hand in hand with the recent theory of second-order stochastic differential equations (2BSDE). Finally, in [Chapter 4](#), we propose an extension of the principal-agent model developed by [Aïd, Possamaï, and Touzi \[11\] \(2019\)](#) to a continuum of agents, whose performances are in particular impacted by a common hazard. These studies notably guide us towards a generalisation of the so-called *revealing* contracts, proposed by [Cvitanić, Possamaï, and Touzi \[111\] \(2018\)](#) in a single-agent model.

In [Part II](#), we present two applications of principal-agent problems to the field of energy. The first one, developed in [Chapter 5](#), uses the framework and the theoretical results introduced in [Chapter 4](#), to improve electricity demand response programs, previously considered in [\[11\]](#). Indeed, by taking into account the infinite number of consumers that a producer must supply with electricity, it is possible to use this additional information to build optimal incentives, allowing in particular to better manage the residual risk implied by climatic hazards. In a second step, [Chapter 6](#) proposes, through a principal-agent model with adverse selection, an insurance which is likely to prevent certain forms of precariousness, in particular fuel poverty.

Finally, we conclude this thesis by studying in [Part III](#) a second field of application, namely epidemiology, and more precisely the control of the diffusion of a contagious disease within a population. In [Chapter 7](#), we first consider the individuals' point of view, through a mean-field game: each individual can choose his interaction rate with others, balancing on the one hand his need for social interactions and on the other hand his fear of being infected in turn, and thus of contributing to the wider spread of the disease. We prove the existence of a Nash equilibrium between individuals, and exhibit it numerically. In [Chapter 8](#), we take the point of view of the government, wishing to encourage the population, now represented as a whole, to decrease its interactions in order to contain the epidemic. We show that the implementation of penalties in the event of non-compliance with containment can be effective, but that for a better control of the epidemic, it is essential to adopt a thorough testing policy, along with a careful isolation of individuals with positive test results.

Key words. Contract theory, mean-field games, second-order BSDE, mean-field BSDE, McKean-Vlasov equations, Hamilton-Jacobi-Bellman equations, moral hazard and adverse selection, hierarchical organisations, energy demand management, fuel poverty, epidemic control, COVID-19.

PUBLICATIONS ET TRAVAUX EN COURS RELATIFS À CETTE THÈSE

1. CONTINUOUS-TIME INCENTIVES IN HIERARCHIES.
Pré-publication [arXiv:2007.10758](#).
Voir [Partie I](#), [Chapitres 2](#) et [3](#).
2. MEAN-FIELD MORAL HAZARD FOR OPTIMAL ENERGY DEMAND RESPONSE MANAGEMENT.
En collaboration avec Romuald ÉLIE, Thibaut MASTROLIA et Dylan POSSAMAÏ.
Pré-publication [arXiv:1902.10405](#) – To appear in *Mathematical Finance*, 2021.
Voir [Partie I](#), [Chapitre 4](#) et [Partie II](#), [Chapitre 5](#).
3. OPTIMAL INSURANCE CONTRACT WITH BENEFITS IN KIND UNDER ADVERSE SELECTION.
En collaboration avec Clémence ALASSEUR et Corinne CHATON.
Pré-publication [arXiv:2001.02099](#).
Voir [Partie II](#), [Chapitre 6](#).
4. CONTACT RATE EPIDEMIC CONTROL OF COVID-19: AN EQUILIBRIUM VIEW.
En collaboration avec Romuald ÉLIE et Gabriel TURINICI.
Paru dans *Mathematical Modelling of Natural Phenomena*, 15(1):35, 2020.
Voir [Partie III](#), [Chapitre 7](#).
5. INCENTIVES, LOCKDOWN, AND TESTING: FROM THUCYDIDES’S ANALYSIS TO COVID-19 PANDEMIC.
En collaboration avec Thibaut MASTROLIA, Dylan POSSAMAÏ et Xavier WARIN.
Pré-publication [arXiv:2009.00484](#).
Voir [Partie III](#), [Chapitre 8](#).

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General notations

Throughout this thesis, we will make use of the following more or less classical notations. Let $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$ be the set of positive integers, $\mathbb{R}_+ := [0, \infty)$ and $\mathbb{R}_+^* := (0, \infty)$. For every d -dimensional (column) vector b with $d \in \mathbb{N}^*$, we denote by $b^1, \dots, b^d$ its coordinates and for $(\alpha, \beta) \in \mathbb{R}^d \times \mathbb{R}^d$, we denote by $\alpha \cdot \beta$ the usual inner product, with associated norm $\|\cdot\|$, which we simplify to $|\cdot|$ when d is equal to 1. Let $\mathbf{0}_d$ and $\mathbf{1}_d$ be the vectors of size d whose coordinates are all equal to respectively 0 and 1. For any $(\ell, c) \in \mathbb{N}^* \times \mathbb{N}^*$, $\mathbb{M}^{\ell, c}$ will denote the space of $\ell \times c$ matrices with real entries. Elements of the matrix $M \in \mathbb{M}^{\ell, c}$ will be denoted by $(M^{i, j})_{1 \leq i \leq \ell, 1 \leq j \leq c}$. The transpose of $M \in \mathbb{M}^{\ell, c}$ will be denoted by $M^\top$ and the zero element in $\mathbb{M}^{\ell, c}$ is denoted by $\mathbf{0}_{\ell, c}$. When $\ell = c$, we let $\mathbb{M}^\ell := \mathbb{M}^{\ell, \ell}$. We also identify $\mathbb{M}^{\ell, 1}$ and $\mathbb{R}^\ell$. The identity matrix in $\mathbb{M}^\ell$ will be denoted by I_ℓ . We also denote by $\mathbb{S}^\ell$ (resp. $\mathbb{S}_+^\ell$) the set of symmetric positive (resp. symmetric definite positive) matrices in $\mathbb{M}^\ell$. The trace of a matrix $M \in \mathbb{M}^\ell$ will be denoted by $\text{Tr}[M]$. For any vector $x \in \mathbb{R}^d$, we denote by $\text{diag}[x]$ the diagonal matrix in $\mathbb{M}^d$ whose entries are the d elements of the vector x .

Let $k \in \mathbb{N}^*$ be any positive integer, and $T > 0$ be a final time horizon. For any $t \in [0, T]$, let $\mathcal{C}_t^k := \mathcal{C}([0, t], \mathbb{R}^k)$ denotes the space of continuous functions from $[0, t]$ into $\mathbb{R}^k$, and simplify notations when $k = 1$ by setting $\mathcal{C} := \mathcal{C}^1$. Define then on $\mathcal{C}_t^k$ the evaluation mappings π_t by $\pi_t(x) = x_t$ and the truncated supremum norms $\|\cdot\|_t$ by $\|x\|_t = \sup_{s \in [0, t]} \|x_s\|$. Unless otherwise stated, $\mathcal{C}_t^k$ is endowed with the topology associated to the uniform convergence on the compact $[0, t]$, *i.e.* with the norm $\|\cdot\|_t$. Moreover, for every finite dimensional Euclidean space E , we let $\mathcal{C}_b(E, \mathbb{R})$ be the space of bounded, continuous functions from E into $\mathbb{R}$, as well as $\mathcal{C}_b^k(E, \mathbb{R})$ the subset of $\mathcal{C}_b(E, \mathbb{R})$ of all k -times continuously differentiable functions on E , with bounded derivatives. Finally, for every $\varphi \in \mathcal{C}^2(E, \mathbb{R})$, we denote by $\nabla \varphi$ its gradient, identified as a column vector, and by $\nabla^2 \varphi$ its Hessian matrix.

For any subset A of $\mathbb{R}^n$, we will define by $\mathbb{A}$ the set of all finite and positive Borel measures on $[0, T] \times A$, whose projection on $[0, T]$ is the Lebesgue measure. In other words, every $q \in \mathbb{A}$ can be disintegrated as $q(ds, dv) = q_s(dv)ds$, for an appropriate Borel measurable kernel $(q_s)_{s \in [0, T]}$. More precisely, this means that for any $s \in [0, T]$, q_s is a finite positive Borel measure on A , and the map $[0, T] \ni s \mapsto q_s$ is Borel measurable, when the space of measures on A is endowed with the topology of weak convergence. The weak formulation for the infection models requires to consider a subset of $\mathbb{A}$, namely the set $\mathbb{A}_0$ of all $q \in \mathbb{A}$ such that the kernel q_s is of the form $\delta_{\phi_s}(dv)$ for some Borel function $\phi : [0, T] \mapsto A$, where as usual, for any $a \in A$, δ_a is the Dirac mass at a .

CHAPITRE 1

Prolégomènes

Tu voudrais qu'y ait des ascenseurs
Au fond des précipices

Mathématiques Souterraines
HUBERT FÉLIX THIÉFAINE

Interactions & incitations

Ces deux termes illustrent à eux seuls le thème de recherche que je poursuis actuellement, et qui sera détaillé tout au long de ce manuscrit. Mais selon moi, ces mots résument surtout une grande partie de nos vies, et du fonctionnement de notre société. Il s'avère que ce fait est parfaitement mis en lumière par la situation singulière que nous traversons actuellement. À l'heure de la rédaction de cette introduction, de nombreux individus subissent encore des mesures de confinement, mises en place par la plupart des États dans l'optique de lutter contre la pandémie de COVID-19. Ces mesures visent à limiter les interactions entre les individus, car c'est à travers les contacts au sein d'une population qu'une épidémie se propage. Cependant, ce confinement préconisé, voire imposé, par la plupart des États, a un *coût* pour les individus, qui doivent renoncer à leurs loisirs, à leurs habitudes, et même potentiellement à leur salaire. Afin de limiter les interactions entre individus au sein d'un pays, les décideurs peuvent fermer certains lieux de rencontre, interdire les rassemblements, ou encore proposer des incitations financières, comme un système d'amende en cas de non-respect du confinement. Cette illustration semble donc soulever deux questions principales.

- (i) Comment modéliser le comportement des individus face à une épidémie ?
- (ii) Comment les inciter de manière optimale à rester confiner, afin de limiter sa propagation ?

L'objectif de cette thèse est justement de répondre à ce type d'interrogations, dans diverses situations, en modélisant le comportement des acteurs économiques et les jeux auxquels ils participent, afin d'apporter une aide à la décision pour les autorités sur des problèmes de société majeurs. Dans le cadre de mon travail, les termes *interaction* et *incitation* prennent un sens plus économique. Le premier réfère à l'interaction entre consommateurs, travailleurs, plus généralement appelés *agents*, à travers leurs décisions, leurs actions, ou encore via un paramètre externe, comme le prix d'un bien. Lorsque nous examinons ces interactions, le but est généralement de trouver un équilibre entre les agents, de sorte que personne n'ait intérêt à en dévier. Cet équilibre dit de Nash peut être étendu dans le cas d'une infinité d'agents, et rejoint alors une théorie plus récente, celle des jeux à champ moyen. Le terme incitation a également un contenu économique et fait référence à la théorie des contrats, et plus précisément aux problèmes principal-agent. Dans ce cas, un agent (qui sera désigné par le pronom *il*) est mandaté pour agir au nom d'un principal (*elle*). L'objectif du principal est de trouver des incitations appropriées pour encourager l'agent à travailler dans son intérêt. En considérant ces questions d'un point de vue mathématique, le comportement des agents, dans un environnement incertain et en temps continu, peut être modélisé comme un problème de *contrôle stochastique*. La recherche regroupée dans ce manuscrit constitue un chemin oscillant entre problèmes principal-agent, équilibres de Nash et jeux à champ moyen, en utilisant des outils récents et novateurs de contrôle stochastique.

1.1 Théorie des contrats et aléa moral

Les problèmes principal-agent permettent de modéliser des situations où le principal cherche à encourager l'agent à agir dans son intérêt, en lui proposant un *contrat*, qui consiste généralement en un paiement – rémunération ou taxe – effectué à une date future T fixée par le contrat. Le but du principal est de trouver le contrat optimal, *i.e.*, qui maximise son utilité et qui est accepté par l'agent. Évidemment, le contrat ne peut dépendre que de ce qui est observé par le principal. Or, dans la plupart des problèmes principal-agent, excepté dans les cas dits *first-best*, nous considérons que le principal n'est pas parfaitement informée des caractéristiques de l'agent *ex ante* la signature du contrat (antisélection, ou sélection adverse, ou encore *third-best*), ou de son action pendant la durée du contrat (aléa moral, ou *second-best*), mais observe les résultats de l'agent, représentés en temps continu par un processus stochastique. Pour le moment, nous nous concentrerons particulièrement sur les problèmes avec aléa moral, ceux avec antisélection seront traités en [Section 1.3.2](#). Nous présentons dans cette section une vue d'ensemble des études liées à la théorie des contrats, depuis les premières évocations de la notion d'incitations, jusqu'aux travaux récents de Cvitanic, Possamaï et Touzi [111] (2018) et [110] (2017), sur lesquels mon travail de recherche est en partie basé.

1.1.1 Historique

La théorie des incitations, bien qu'abordée sous un angle mathématique tout au long de cette thèse, est avant tout une théorie économique. À notre connaissance, et comme mentionné par Laffont et Martimort [218] (2009), Smith [314] (1776) semble être le premier à reconnaître la nature contractuelle de l'interaction entre employés et employeurs, ainsi qu'à mentionner le caractère incitatif des salaires. Babbage [29] (1832) comprend ensuite la nécessité d'une mesure précise des performances d'un employé pour mettre en place des contrats à la pièce, à la commission, ou proposer des participations aux bénéfices. La volonté d'optimiser l'organisation du travail de manière réellement scientifique trouve ses origines au début du XX^{ème} siècle, à travers les travaux de Taylor, pionnier de la théorie du *scientific management* (organisation scientifique du travail). L'objectif principal de cette théorie est d'améliorer l'efficacité économique, en particulier la productivité du travail, et a été l'une des premières tentatives d'appliquer la science à l'ingénierie des processus de gestion. L'objectif principal du modèle de Taylor, développé dans sa monographie intitulée *The principles of scientific management*, pourrait se résumer ainsi : comment faire en sorte que les travailleurs soient performants dans l'intérêt de l'employeur, c'est-à-dire de la manière la plus rentable et avec le moins de résistance possible ? Cependant, pour répondre à cette question, Taylor promeut la supervision des travailleurs, considération très *first-best* de la relation employé/employeur, en opposition à ce qu'il appelle « the management of initiative and incentive » (Taylor [328, p. 34] (1911)), à savoir l'encadrement par l'initiative et l'incitation sur lequel nous nous concentrons.

Il faut alors attendre les travaux de Barnard [36] (1938) pour voir naître une première tentative de définition d'un cadre général concernant les incitations en matière de management. Néanmoins inspiré par Taylor, il préconise également la nécessité de créer des relations hiérarchiques au sein des organisations, ce qui nous amènera à étudier ce type de structure par la suite (voir [Section 1.2.2](#)). Bien qu'il ait souligné les issues importantes liées à l'aléa moral, ce concept même a été introduit dans la littérature sur le contrôle de gestion près de trente ans plus tard par Arrow [28] (1963). Le concept d'aléa moral a en réalité été emprunté à la théorie de l'assurance, comme le montre certaines mentions de ce concept dans les travaux de Dickerson [124] (1955) par exemple. Les modèles mathématiques sur la théorie des incitations et l'étude de l'aléa moral se sont ensuite répandus dans les années 1970, notamment grâce aux nombreux travaux de Mirrlees [249] (1971), [250] (1972), [251] (1974), [252] (1976), [253] (1976), [254] (1999). Nous pouvons citer également les modèles de Ross [296] (1973), Guesnerie et Laffont [167] (1978), Shavell [311] (1979), Harris et Raviv [173] (1979), Grossman et Hart [162] (1983). Comme mentionné ci-dessus, le concept d'aléa

moral est également très présent dans la littérature sur l'assurance, comme en témoignent les travaux de Pauly [274] (1978), Helpman et Laffont [178] (1975), Zeckhauser [350] (1970), Spence et Zeckhauser [322] (1971) et Wesson [342] (1972). Il faut préciser que la théorie des contrats et les problèmes d'aléa moral en temps discrets sont associés à une très vaste littérature, qui, dans un souci de relative concision, ne peut être entièrement mentionnée ici. Nous nous contenterons donc pour le moment de ces quelques références sélectionnées, et renvoyons le lecteur intéressé aux ouvrages fondateurs de Bolton et Dewatripont [63] (2005), Salanié [300] (2005), ou Laffont et Martimort [218] (2009) pour plus de références, avant de se lancer vers le temps continu, thème plus crucial pour cette thèse.

1.1.2 Développements en temps continu

À la fin des années 1980, la littérature sur la théorie des contrats s'est étendue pour inclure des modèles en temps continu. Le premier article, qui fait figure de référence sur les problèmes principal-agent en temps continu, a été rédigé par Holmström et Milgrom [181] (1987). Dans leur modèle, l'agent contrôle le drift d'un processus de diffusion représentant le résultat global, qui peut être interprété comme la valeur de l'entreprise. Le principal n'observe que le résultat, de sorte que les actions de l'agent sont cachées. En considérant des utilités exponentielles à la fois pour le principal et l'agent, le résultat essentiel de ce travail est que le contrat optimal s'avère être une fonction linéaire du résultat. Pour plus de précision, nous décrivons brièvement le modèle ci-dessous, dont la présentation est adaptée pour s'aligner avec la suite de cette thèse.

1.1.2.1 Le modèle d'Holmström et Milgrom [181] (1987)

Fixons tout d'abord un horizon temporel $T > 0$, correspondant à la fin de la période contractuelle, à savoir le moment où une compensation ξ sera versée par le principal à l'agent, en récompense de ses efforts sur la période $[0, T]$. Pour modéliser le processus contrôlé par l'agent, nous considérons un espace de probabilité $(\Omega, \mathcal{F}, \mathbb{P})$, supportant un mouvement brownien uni-dimensionnel W . Par la suite, nous serons amenés à considérer la filtration naturelle $\mathbb{F}$, engendrée par W et augmentée (complétée et continue à droite). Le processus représentant le résultat est alors donné par :

$$X_t := \sigma W_t, \quad t \in [0, T], \quad \sigma > 0.$$

Un effort admissible pour l'agent est représenté par un contrôle α , $\mathbb{F}$ -adapté, à valeur dans $\mathbb{R}$, satisfaisant la condition suivante :

$$\mathcal{E}\left(\int_0^\cdot \frac{\alpha_s}{\sigma} dW_s\right) := \exp\left(\int_0^\cdot \frac{\alpha_s}{\sigma} dW_s - \frac{1}{2} \int_0^\cdot \frac{|\alpha_s|^2}{|\sigma|^2} ds\right) \text{ est une } \mathbb{P}\text{-martingale uniformément intégrable.}$$

L'ensemble des efforts admissible sera noté $\mathcal{A}$. Sous la condition précédente, il est alors possible de définir, pour tout $\alpha \in \mathcal{A}$, une nouvelle probabilité $\mathbb{P}^\alpha$ sur $(\Omega, \mathcal{F})$, à travers sa dérivée de Radon-Nikodým :

$$\frac{d\mathbb{P}^\alpha}{d\mathbb{P}} \Big|_{\mathcal{F}_t} = \mathcal{E}\left(\int_0^t \frac{\alpha_s}{\sigma} dW_s\right), \quad t \in [0, T].$$

Ainsi, en appliquant le théorème de Girsanov, nous obtenons, pour tout $t \in [0, T]$:

$$X_t = \int_0^t \alpha_s ds + \sigma W_t^\alpha, \quad \text{où } W_t^\alpha := W_t - \int_0^t \frac{\alpha_s}{\sigma} ds \text{ est un mouvement brownien sous } \mathbb{P}^\alpha.$$

L'ensemble Ξ des contrats admissibles est défini comme l'ensemble des variables aléatoires $\mathcal{F}_T$ -mesurables, satisfaisant certaines conditions d'intégrabilité¹. Autrement dit, un contrat $\xi \in \Xi$ est une

1. Ces conditions simplement évoquées ici seront très largement détaillées tout au long de la [Partie I](#).

fonction mesurable des trajectoires de X jusqu'en T , que nous noterons $X_{\cdot \wedge T}$. Dans la suite, nous considérons que l'agent et le principal ont tous deux une utilité exponentielle dite CARA, avec des paramètres d'aversion au risque respectivement données par $R_A > 0$ et $R_P > 0$, *i.e.*,

$$U_A(x) = -e^{-R_A x} \quad \text{and} \quad U_P(x) = -e^{-R_P x}, \quad x \in \mathbb{R}.$$

Évidemment, effectuer un effort n'est pas anodin pour l'agent, et nous supposons donc que cela engendre un coût quadratique défini par $c(a) = \frac{1}{2k}|a|^2$, $a \in \mathbb{R}$, pour une certaine constante $k > 0$. Ainsi, étant donné un contrat $\xi \in \Xi$, il est naturel de considérer que l'agent résout le problème d'optimisation suivant :

$$V^A(\xi) := \sup_{\alpha \in \mathcal{A}} J^A(\xi, \alpha), \quad \text{où, pour tout } \alpha \in \mathcal{A}, \quad J^A(\xi, \alpha) := \mathbb{E}^{\mathbb{P}^\alpha} \left[U_A \left(\xi - \int_0^T c(\alpha_s) ds \right) \right]. \quad (1.1.1)$$

D'autre part, comme dans la plupart des problèmes principal-agent, il est supposé que l'agent a une *utilité de réservation*, notée $R_0 < 0$, *i.e.* un niveau en-dessous duquel il refuse le contrat proposé par le principal. Autrement dit, si $\xi \in \Xi$ est tel que $V^A(\xi) \geq R_0$, alors l'agent accepte de travailler pour le principal. Il est donc important pour le principal de proposer un contrat satisfaisant cette contrainte, dite de participation. L'ensemble des efforts optimaux $\alpha \in \mathcal{A}$ pour un contrat donné $\xi \in \Xi$ est alors logiquement donné par :

$$\mathcal{A}^*(\xi) := \{\alpha \in \mathcal{A}, V^A(\xi) = J^A(\xi, \alpha)\}.$$

Le problème du principal est finalement défini par :

$$V^P := \sup_{\xi \in \Xi : V^A(\xi) \geq R_0} \sup_{\alpha \in \mathcal{A}^*(\xi)} \mathbb{E}^{\mathbb{P}^\alpha} [U_P(X_T - \xi)]. \quad (1.1.2)$$

Le supremum sur un ensemble vide étant égal par convention à $-\infty$, le principal ne proposera jamais un contrat ξ tel que $\mathcal{A}^*(\xi) = \emptyset$. Ainsi, pour tout contrat $\xi \in \Xi$, il existe au moins un effort optimal $\alpha^* \in \mathcal{A}^*(\xi)$.

1.1.2.2 Sur la résolution du modèle

Le modèle considéré par [Holmström et Milgrom](#) a ensuite été largement étendu, et les principaux contributeurs à cet égard sont [Schättler et Sung](#) [305] (1993), [Sannikov](#) [303] (2008), [Biais, Mariotti, Rochet et Villeneuve](#) [56] (2007) [56] (2007), [57] (2010), ainsi que [Cvitanic et Zhang](#) [106] (2012). Il peut être distingué dans la littérature trois méthodes distinctes pour aborder ce genre de problèmes. La première, développée par [Sung](#) [325] (1995), [326] (1997) ; [Müller](#) [258] (1998), [259] (2000) ; [Hellwig et Schmidt](#) [176] (2002) et [Hellwig](#) [177] (2007), est basée sur une extension de l'approche du premier ordre, populaire pour résoudre les problèmes en temps discret. Plus récemment, [Williams](#) [345] (2009), [346] (2011), [347] (2015) et [Cvitanic, Wan et Zhang](#) [107] (2006), [108] (2008), [109] (2009) ont caractérisé la compensation optimale pour des fonctions d'utilité plus générales, en utilisant le principe du maximum stochastique et un système couplé d'équations différentielles stochastiques (EDS), directes et rétrogrades. Cette approche rigoureuse est très générale, mais aboutit à des systèmes compliqués à partir desquels il est difficile d'extraire un contrat optimal.

Finalement, la méthode la plus facilement applicable et prometteuse semble être celle développée par [Sannikov](#) [303] (2008). Du point de vue méthodologique, [Sannikov](#) montre, en temps continu, que l'approche développée par [Spear et Srivastava](#) [319] (1987) en temps discret est toujours valable, et permet d'aboutir à un problème de contrôle plus ordinaire au niveau du principal. L'idée essentielle de cette approche est de considérer l'utilité de continuation dynamique de l'agent en tant que variable d'état pour le problème du principal. Plus précisément, en utilisant le principe de la représentation martingale, il est possible d'écrire

l'utilité de continuation de l'agent comme un processus contrôlé, associée à une équation différentielle stochastique rétrograde (EDSR). Contrairement à la méthode précédente, celle-ci permet de déduire une forme de contrat particulière, qui est sans perte de généralité du point de vue du principal, et permet de ramener son problème à un problème de contrôle stochastique standard : au lieu d'optimiser sur tous les contrats possibles, le principal choisit de manière optimale uniquement deux paramètres. Le premier, $y_0 \in \mathbb{R}$, est choisi de manière à satisfaire la contrainte de participation de l'agent. Le deuxième est un processus Z , indexant le contrat sur la dynamique du résultat décrit par le processus stochastique contrôlé X . Il faut cependant noter que la description de cette méthode dans [303] comporte quelques zones d'ombre du point de vue théorique, récemment comblées par le travail rigoureux de Possamaï et Touzi [283] (2020).

En résumé, le problème auquel l'agent est confronté est un problème de contrôle non-Markovien, avec pour variable d'état X et pour contrôle α . Comme dans ce modèle, seul le drift du processus X est contrôlé, il faut mentionner que le problème est profondément lié à une équation différentielle stochastique rétrograde (EDSR). Ces équations particulières ont été introduites pour la première fois par Bismut [61] (1973) dans le cas linéaire, puis généralisées par Pardoux et Peng [272] (1990). Selon ces auteurs, la solution d'une EDSR consiste en un couple (Y, Z) de processus adaptés, satisfaisant en particulier :

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dW_s, \quad t \in [0, T], \quad \mathbb{P}\text{-p.s.},$$

où la fonction f et la variable aléatoire ξ sont respectivement appelés *générateur* et *condition terminale*. L'étude de ces EDSR a donné naissance à une littérature extrêmement vaste, que nous ne pouvons évidemment détailler ici. Nous renvoyons le lecteur intrigué à l'ouvrage édité par El Karoui et Mazliak [137] (1997), exposant notamment le lien susmentionné entre les EDSR et les problèmes de contrôle stochastique, ainsi qu'aux livres de Carmona [80] (2016) et Zhang [353] (2017), présentant chacun une extension intéressante pour la suite de cette thèse, respectivement les EDSR dites à *champ moyen* et les EDSR du second-ordre.

En gardant à l'esprit cet outil essentiel que constituent les EDSR, revenons maintenant à notre problème, et mentionnons rapidement deux chemins possibles pour le résoudre :

- (i) le premier est similaire à l'approche par vérification utilisée dans le contrôle stochastique Markovien, et se résume à écrire directement l'EDSR associée au problème, puis à utiliser un théorème de comparaison pour l'identifier à la fonction de valeur (voir El Karoui, Peng et Quenez [139] (1997)) ;
- (ii) la seconde approche repose sur le principe de programmation dynamique, et consiste à prouver directement que la fonction valeur dynamique de l'agent résout la dite EDSR.

Nous verrons par la suite qu'il existe une approche plus naturelle, consistant à conjecturer une forme particulière de contrats, permettant de résoudre facilement le problème de l'agent, et de prouver ensuite que cette classe de contrat est sans perte de généralité du point de vue du principal. En particulier, cette classe de contrat peut être obtenue de manière informelle en imitant ce qu'il se passerait dans un cadre Markovien. Cette méthode sera largement développée tout au long de cette thèse, c'est pourquoi nous nous contentons de décrire ici de manière très brève les résultats obtenus dans le cadre du modèle d'Holmström et Milgrom [181] (1987).

L'EDSR pertinente mentionnée précédemment est la suivante :

$$Y_t = \xi + \int_t^T \sup_{a \in \mathbb{R}} \{aZ_s - c(a)\} ds - \int_t^T Z_s dX_s - \frac{1}{2} R_A \int_t^T |\sigma|^2 |Z_s|^2 ds, \quad t \in [0, T].$$

La fonction valeur de l'agent satisfait alors $V^A(\xi) = U_A(Y_0)$. En particulier, l'effort optimal α^* de l'agent est unique et donné à tout instant $t \in [0, T]$ par le maximiseur de la fonction $a \mapsto \sup_{a \in \mathbb{R}} \{aZ_t - c(a)\}$, i.e.,

$\alpha_t^* = kZ_t$. Le point le plus important à remarquer ici est que l'EDSR précédente suggère qu'il est suffisant dans ce cadre d'étude de considérer des contrats $\xi^{y_0, Z}$ donnés par la valeur terminale du processus suivant :

$$\xi_t^{y_0, Z} = y_0 - \int_0^t \sup_{a \in \mathbb{R}} \{aZ_s - c(a)\} ds + \int_0^t Z_s dX_s + \frac{R_A}{2} \int_0^t |\sigma|^2 |Z_s|^2 ds, \quad t \in [0, T], \quad (1.1.3)$$

pour $y_0 \in \mathbb{R}$ et Z un processus $\mathbb{F}$ -prévisible, à valeur dans $\mathbb{R}$, et suffisamment intégrable pour que ξ soit bien défini et admissible, ce que nous résumerons en notant $Z \in \mathcal{Z}$.

Ainsi, au lieu d'optimiser sur tous les contrats admissibles possibles, *i.e.*, sur $\xi \in \Xi$, il n'y a pas de perte de généralité à considérer que le principal propose un contrat $\xi^{y_0, Z}$ de la forme (1.1.3), et choisit donc uniquement, de manière optimale, $(y_0, Z) \in \mathbb{R} \times \mathcal{Z}$. Son problème peut donc se réécrire de la façon suivante :

$$V^P = \sup_{y_0 \in \mathbb{R} : U_A(y_0) \geq R_0} \sup_{Z \in \mathcal{Z}} \mathbb{E}^{\mathbb{P}^{\alpha^*}} [U_P(X_T - \xi^{y_0, Z})],$$

et correspond maintenant à un problème de contrôle stochastique standard, à deux variables d'état X et $Y^{y_0, Z}$. En particulier, considérer l'équivalent certain de l'utilité de continuation de l'agent (*i.e.* ici $\xi^{y_0, Z}$) comme variable d'état en addition de X rend le problème du principal Markovien, comme mentionné à l'origine par Spear et Srivastava [319] (1987). Sa résolution est maintenant directe, et nous obtenons les contrôles optimaux suivants :

$$y_0^* := U_A^{-1}(R_0), \quad Z^* := \frac{k + |\sigma|^2 R_P}{k + |\sigma|^2 (R_A + R_P)}, \quad \text{et} \quad \alpha_t^* := \frac{k(k + |\sigma|^2 R_P)}{k + |\sigma|^2 (R_A + R_P)}, \quad t \in [0, T].$$

En particulier, le contrôle optimal Z^* du principal est déterministe et même constant. L'effort optimal α^* de l'agent hérite inmanquablement des mêmes propriétés. Finalement, le contrat optimal est donné par :

$$\xi^* = U_A^{-1}(R_0) + \frac{T}{2} |Z^*|^2 (|\sigma|^2 R_A - k) + Z^* X_T.$$

Ce paiement terminal n'a évidemment aucune raison d'être un positif. Cependant, certaines extensions permettent de prendre en compte la responsabilité limitée, afin d'assurer un revenu positif pour l'agent, comme les travaux de DeMarzo et Sannikov [118] (2006), [119] (2017) ; Biais, Mariotti, Rochet et Villeneuve [57] (2010) ; Décamps et Villeneuve [116] (2019) ; ou Possamaï, Réveillac et Villeneuve [285] (2020).

1.1.3 Vers le contrôle de la volatilité

Plus récemment, Cvitanić, Possamaï et Touzi [111] (2018) ont développé une théorie générale qui permet d'aborder un large spectre de problèmes d'aléa moral en temps continu, par le biais de la programmation dynamique et des équations différentielles stochastiques rétrogrades du second ordre (2EDSR en abrégé). Avec cette approche, le problème auquel est confronté le principal devient un problème de contrôle optimal standard, comme montré précédemment dans le cas du modèle de Holmström et Milgrom. Plus important encore, le cadre qu'ils considèrent permet de prendre en compte non seulement un effort de l'agent sur le drift du processus qu'il contrôle, mais aussi sur la volatilité. Nous développons brièvement ci-dessous l'approche et les résultats importants, notamment la forme pertinente des contrats.

Pour simplifier, reprenons le cadre précédemment défini dans le cas du modèle de Holmström et Milgrom, excepté que nous considérons ici que l'agent peut également contrôler la volatilité du processus résultat X . Plus précisément, l'agent contrôle le drift via un processus d'effort α , à valeur dans un ensemble A , et la volatilité par un effort β , à valeur dans B . Admettons que le processus contrôlé soit donné par :

$$X_t := \int_0^t \alpha_s ds + \int_0^t \sigma(\beta_s) \cdot dW_s, \quad t \in [0, T],$$

où maintenant, W est un mouvement brownien d -dimensionnel ($d > 1$), et $\sigma : B \rightarrow \mathbb{R}^d$ est une fonction. Ne nous attardons pas pour le moment sur la définition des ensemble d'efforts admissibles, notés respectivement $\mathcal{A}$ et $\mathcal{B}$, ils seront définis rigoureusement pour des cadres plus généraux tout au long de cette thèse, notamment en [Partie I](#). Gardons uniquement en tête que ces processus doivent naturellement être $\mathbb{F}$ -adaptés. L'ensemble des contrats admissibles noté Ξ est toujours défini comme l'ensemble des variables aléatoires $\mathcal{F}_T$ -mesurable, satisfaisant certaines conditions d'intégrabilité. Nous considérons toujours que les préférences de l'agent et du principal sont représentées par des utilités CARA, et rappelons que leurs problèmes d'optimisation respectifs sont définis par (1.1.1) et (1.1.2), même si maintenant, l'optimisation de l'agent est sur le couple $(\alpha, \beta) \in \mathcal{A} \times \mathcal{B}$. Évidemment, il est nécessaire d'ajouter un coût relatif au contrôle de la volatilité. Pour préserver une certaine simplicité, le coût est maintenant défini par $c(a, b) = \frac{1}{2k}|a|^2 + \frac{1}{2}c_v(b)$, $(a, b) \in A \times B$, pour une certaine fonction c_v .

Les étapes essentielles de l'approche prescrite dans [111] sont les suivantes :

- (i) identifier une sous-classe de contrats proposés par le principal, dits *révélateurs*, signifiant que la fonction de meilleure réponse de l'agent et son contrôle optimal peuvent être calculés directement ;
- (ii) prouver que la restriction à cette classe de contrats est sans perte de généralité ;
- (iii) résoudre le problème de contrôle optimal maintenant standard du principal.

Concernant le premier point, le meilleur moyen d'identifier une telle classe de contrats est de raisonner dans le cas Markovien. Nous regarderons dans les [Chapitres 3](#) et [4](#) deux exemples de problèmes où l'identification de la classe de contrats révélateurs se fait intuitivement de cette manière. Dans le cadre considéré ici, nous obtenons que les contrats révélateurs sont paramétrés par une constante $y_0 \in \mathbb{R}$ et un couple de processus (Z, Γ) , $\mathbb{F}$ -prévisibles et à valeurs dans $\mathbb{R}$. La forme optimale est la suivante :

$$\xi_t^{y_0, Z, \Gamma} = y_0 - \int_0^t \mathcal{H}(Z_s, \Gamma_s) ds + \int_0^t Z_s dX_s + \frac{1}{2} \int_0^t (\Gamma_s + R_A |Z_s|^2) d\langle X \rangle_s, \quad t \in [0, T], \quad (1.1.4)$$

où $\mathcal{H} : \mathbb{R}^2 \rightarrow \mathbb{R}$ correspond à l'Hamiltonien de l'agent, donné pour tout $(z, \gamma) \in \mathbb{R}^2$ par :

$$\mathcal{H}(z, \gamma) := \sup_{(a, b) \in A \times B} \left\{ az + \frac{1}{2} \gamma \|\sigma(b)\|^2 - c(a, b) \right\} = \sup_{a \in A} \left\{ az - \frac{|a|^2}{2k} \right\} + \frac{1}{2} \sup_{b \in B} \{ \gamma \|\sigma(b)\|^2 - c_v(b) \}.$$

Ainsi, dans le cas où l'agent contrôle la volatilité du processus résultat X , il est nécessaire de rajouter à la forme optimale de contrat précédemment définie par (1.1.3) un terme indexé sur la variation quadratique de X , notée $\langle X \rangle$, via un processus Γ . Dans le cas considéré, la variation quadratique vérifie :

$$d\langle X \rangle_t = \|\sigma(\beta_t)\|^2, \quad t \in [0, T].$$

Ce processus Γ devra être choisi de manière optimale par le principal, en plus du processus Z . Remarquons que si la volatilité n'est pas contrôlée par l'agent, alors le deuxième supremum dans l'Hamiltonien se simplifie avec la partie du contrat en $\frac{1}{2} \int_0^T \Gamma_t d\langle X \rangle_t$. Nous retrouvons dans ce cas la forme donnée par (1.1.3). De manière informelle, ce type de contrat est révélateur dans le sens où il est très facile de calculer les efforts optimaux de l'agent ainsi que sa fonction valeur. En particulier, les efforts optimaux sont donnés par les maximiseurs de son Hamiltonien (s'ils existent). Le deuxième point consiste à montrer que cette forme de contrat est sans perte de généralité. Cette démonstration repose sur la théorie des 2EDSR, et en particulier sur des arguments développés par [Possamaï, Tan et Zhou \[284\] \(2018\)](#). Nous verrons dans les [Chapitres 3](#) et [4](#) des exemples détaillés de preuves dans des extensions à une multitude d'agents. Pour le dernier point, le problème du principal est ramené à un problème de contrôle standard, similaire à celui obtenu dans le modèle de [Holmström et Milgrom](#), avec, pour variables d'état X et $\xi^{y_0, Z, \Gamma}$, et pour contrôles y_0 , Z et Γ .

Cette approche, évidemment bien plus générale que l'exemple simplifié que nous présentons ici, a été étendue et appliquée dans de diverses situations. Nous pouvons citer de manière non exhaustive les applications en finance par Cvitanic, Possamaï et Touzi [110] (2017) et Cvitanic et Xing [105] (2018) ; les travaux de Aïd, Possamaï et Touzi [11] (2019) (sur lequel nous nous attarderons un peu plus en [Section 1.3.1.2](#)) et Alasseur, Farhat et Saguan [16] (2019) pour des applications liées au secteur de l'énergie ; ainsi que les extensions variées de Hernández Santibáñez et Mastrolia [179] (2019), Hu, Ren et Touzi [182] (2019), Kharroubi, Lim et Mastrolia [207] (2020) et Lin, Ren, Touzi et Yang [233] (2020). Cependant, une limite importante de la théorie exposée ci-dessus est qu'elle ignore le fait qu'en général, il n'y a pas qu'un seul employé, qu'un seul client pour le principal, mais plusieurs, et qu'ils peuvent interagir les uns avec les autres. C'est ce que nous appellerons problèmes principal – multi-agents. Dans la prochaine partie, nous nous concentrerons sur les problèmes d'aléa moral considérant une multitude d'agents, voire une infinité. Un des principaux apports théorique de cette thèse réside justement en l'extension de l'approche de Cvitanic, Possamaï et Touzi [111] (2018) à ces problèmes principal-agent à plus grande échelle. Plus précisément, les deux modèles que nous introduirons dans la section suivante considèrent l'ajout du contrôle de la volatilité dans le cas, d'une part d'une multitude d'agents en [Section 1.2.2](#), et d'autre part d'une infinité d'agents en [Section 1.2.3](#).

1.2 Problèmes principal-agent à grande échelle

1.2.1 Des Nash aux jeux à champ moyen

En général, lorsque nous considérons des interactions entre agents, l'objectif est de caractériser un équilibre de Nash, à savoir une solution du jeu où chaque joueur n'a rien à gagner en changeant de manière unilatérale sa stratégie. Utilisant ce concept d'équilibre entre joueurs, les problèmes principal-agent en temps continu ont récemment été étendus à des modèles avec plusieurs *principals*, notamment grâce aux travaux de Mastrolia et Ren [238] (2018) et Hu, Ren et Yang [183] (2019). Dans notre contexte, nous sommes plus particulièrement intéressés par les extensions à plusieurs agents, comme dans les modèles de Koo, Shim et Sung [210] (2008), Goukasian et Wan [155] (2010), Élie et Possamaï [140] (2019) ou encore Baldacci, Possamaï et Rosenbaum [34] (2019).²

Cependant, il est souvent plus facile de considérer la limite lorsque le nombre d'agents tend vers l'infini, ce qui nous entraîne vers une théorie plus récente, empruntée à la physique, dite des jeux à champ moyen. Cette théorie fut introduite dans le domaine mathématique par Lasry et Lions [225, 226] (2006), [227] (2007) ; et indépendamment par Huang, Malhamé et Caines [185] (2006), [186, 187, 188] (2007). Une description complète peut être trouvée dans les livres de Carmona et Delarue [81, 82] (2018) et Cardaliaguet, Delarue, Lasry et Lions [78] (2019). Cette théorie se concentre sur l'étude du comportement d'agents en interaction au sein d'une population contenant un nombre infini d'individus, et des équilibre de Nash résultant de ces comportements. Ce point de vue asymptotique simplifie l'analyse du jeu, car il est couramment supposé que l'impact d'un individu sur l'ensemble de la population est négligeable, ce qui permet de se concentrer sur l'étude des choix d'un agent en particulier, communément appelé *agent représentatif*.

Les domaines d'application des jeux à champ moyen sont extrêmement variés, et incluent, comme listé notamment par Guéant, Lasry et Lions [166] (2011) :

2. Il convient de noter que l'extension des modèles à une multitude d'agents prend évidemment son origine en temps discret, domaine dans lequel il existe une littérature plus abondante. Parmi les travaux les plus notables dans ce sens, nous pouvons citer ceux de Holmström [180] (1982), Harris, Kriebel et Raviv [174] (1982), Green et Stokey [159] (1983), Nalebuff et Stiglitz [261] (1983), Mookherjee [255] (1984), Demski et Sappington [121] (1984), Itoh [193] (2004), Rey-Biel [290] (2008), Bartling et von Siemens [41] (2010), Grund et Sliwka [163] (2005), Demougin, Fluet et Helm [120] (2006), ainsi que Neilson et Stowe [264] (2010), ou encore Kragl [211] (2015).

(i) la finance, avec les travaux de Garnier, Papanicolaou et Yang [151] (2013) et Carmona, Fouque et Sun [90] (2015) sur le risque systémique, de Carmona, Delarue et Lacker [92] (2017), Élie, Ichiba et Laurière [142] (2020) sur la faillite de banques, Cardaliaguet et Lehalle [77] (2018) sur le trading ;

(ii) la modélisation du comportement de foule, par Lachapelle et Wolfram [213] (2011) et Achdou et Laurière [9] (2016) ;

(iii) l'énergie par Alasseur, Ben Tahar et Matoussi [15] (2017) et Aïd, Dumitrescu et Tankov [12] (2020), et plus largement les ressources naturelles comme l'eau par Bouveret, Dumitrescu et Tankov [69] (2020) ;

ou encore d'autres applications plus surprenantes comme celle de Carmona et Graves [83] (2020) sur le décalage horaire.³ Les applications qui nous intéresseront par la suite concernent particulièrement les programmes de réponse à la demande en énergie et l'épidémiologie, respectivement introduites en Sections 1.3.1 et 1.4.2, puis détaillées dans les Chapitres 5 et 7.

Néanmoins, les problèmes principal – multi-agent en temps continu développés dans la littérature sont pour le moment limités au contrôle de drift. Nous avons déjà cité précédemment ceux avec un nombre fini d'agents, mais il convient d'ajouter à cette liste ceux de Élie, Mastrolia et Possamaï [141] (2018) et Carmona et Wang [87] (2018) (qui présentent par ailleurs une application au contrôle d'une épidémie) sur un continuum d'agents avec des interactions de type champ moyen. Nous introduirons dans cette section deux approches multi-agents avec contrôle de la volatilité. D'une part, en Section 1.2.2, nous décrirons les problèmes d'incitations optimales au sein d'une hiérarchie, et donc avec une multitude d'agents, introduisant ainsi les Chapitres 2 et 3 de cette thèse. D'autre part, en Section 1.2.3, nous résumerons une autre contribution, développée dans le Chapitre 4, considérant une infinité d'agents avec des interactions de type champ moyen. Ces deux contributions constituent le principal apport théorique de cette thèse, proposant une extension relativement générale de l'approche de Cvitanić, Possamaï et Touzi [111] (2018) à ces problèmes d'incitations à plus grande échelle.

1.2.2 Le cas des organisations hiérarchisées

1.2.2.1 Contexte

Comme évoqué en Section 1.1.1, la volonté d'optimiser l'organisation du travail, de manière scientifique, trouve ses origines au début du XXème siècle, à travers les travaux de Taylor. Néanmoins, depuis, l'organisation du travail a sensiblement évolué. Ce changement de paradigme est principalement dû au fait que la nature même du travail a changé. En effet, avec l'introduction des nouvelles technologies à tous les niveaux de production, la prédominance du travail indirect exige des formes de gestion qui rompent avec le modèle classique d'organisation taylorienne. En outre, la reconnaissance de l'importance du climat psychologique, et en particulier l'idée que la reconnaissance des travailleurs stimule leur productivité, représentent également une avancée importante par rapport à l'approche de Taylor. Tant l'élimination progressive des emplois simples que le désir de responsabiliser l'employé ont rendu la supervision des travailleurs difficile et même contre-productive. Ces deux transformations impliquent que la gestion repose aujourd'hui davantage sur l'initiative des employés, et sur le développement d'incitations visant à rapprocher les intérêts de l'employé et de l'employeur, que sur la supervision des travailleurs. Néanmoins, l'organisation hiérarchique du travail à la Taylor est toujours considérée comme la structure habituelle des entreprises.

Une organisation se compose donc généralement d'une entité de pouvoir au sommet et de niveaux de pouvoir subséquents en dessous. Cette structure est le mode dominant dans notre société contemporaine. En effet, la plupart des entreprises, des gouvernements et même des organisations criminelles ont une structure hiérarchique, avec différents niveaux d'autorité. Cette structure particulière des organisations soulève de

3. Cette liste n'a en aucun cas la prétention d'être exhaustive.

nombreuses questions : sur son efficacité, son coût, sa taille optimale... Pour répondre à ces questions, une littérature abondante est apparue dans des domaines très variés, de la philosophie aux mathématiques, en passant par les sciences sociales et de gestion. Le premier modèle mathématique pour l'étude de la structure optimale d'une hiérarchie semble être l'œuvre de Williamson [348] (1967), mais, comme il le mentionne, cette question, qui présente un sérieux dilemme en théorie des organisations, a été introduite à l'origine par Knight [208] (1921). De nombreux auteurs ont suivi cette tendance, notamment Calvo et Wellisz [72, 73] (1978, 1979), et Keren et Levhari [204] (1979), ainsi que Qian [288] (1994) qui propose une généralisation pour prendre en compte la notion d'incitations.

Rappelons que la première tentative de définition d'un cadre général pour les incitations en matière de gestion est attribuée à Barnard [36] (1938), qui préconise spécifiquement la nécessité de créer des relations hiérarchiques au sein des organisations. Cependant, il faut attendre les années 1970 pour constater un véritable développement des travaux sur la théorie des contrats et incitations. Dans le cas d'une hiérarchie, nous sommes confrontés à une succession de problèmes principal-agent liés entre eux, ou en d'autres termes, à une séquence d'équilibres de Stackelberg imbriqués. Le formalisme mathématique des problèmes principal-agent permet notamment de modéliser les asymétries d'information au sein d'une hiérarchie, qu'elles soient *ex ante* (sélection adverse) ou *ex post* (aléa moral) la signature de contrats entre les entités constituant la hiérarchie. Les principaux auteurs à cet égard sont Tirole [331] (1986), Demski et Sappington [122] (1987), Baiman, Evans et Noel [33] (1987) et Kofman et Lawarree [209] (1993) sur les phénomènes de collusion et l'audit au sein d'une hiérarchie, ainsi que Melumad, Mookherjee et Reichelstein [243] (1995), McAfee et McMillan [239] (1995), Laffont et Martimort [217] (1997) et Mookherjee [256] (2006) sur la sélection adverse. Dans notre cadre, nous nous concentrerons sur les incitations optimales en présence d'aléa moral au sein d'une hiérarchie, comme dans les travaux de Laffont [216] (1990), Yang [349] (1995), Macho-Stadler et Pérez-Castrillo [235] (1998), Itoh [192] (2001) et Jost et Lammers [198] (2010), bien qu'un essai de Miller et Whitford [246] (2006) présente certaines limites à cette considération.

1.2.2.2 Vers le temps continu

Il est important de remarquer que les travaux susmentionnés sont des modèles à temps discret, constitués pour la plupart d'une seule et unique période. Le développement des modèles principal-agent à une multitude d'agents, décrit en Section 1.2.1, constitue un progrès significatif pour l'application de la théorie des contrats en temps continu aux hiérarchies. Néanmoins, ce type de modèle semble pour l'instant pouvoir être compté sur les doigts d'une seule main.

Premièrement, Miller et Yang [245] (2015) considèrent une hiérarchie de n joueurs ($n > 3$), avec une relation principal-agent entre chacun. Plus précisément, le premier joueur propose un contrat au deuxième, qui lui même propose un contrat au suivant, et ainsi de suite. En utilisant l'approche de Evans, Miller et Yang [143] (2015) basée sur la programmation dynamique, ils identifient les conditions sous lesquelles la construction d'un contrat optimal peut être réduite à un seul espace d'état et à un ensemble de contrôle uni-dimensionnel, indépendamment de la taille de la hiérarchie. Toutefois, l'approche adoptée dans Evans, Miller et Yang [143] (2015) pour caractériser les contrats optimaux en temps continu est moins générale que celle de Cvitanić, Possamaï et Touzi [111] (2018). En particulier, elle ne permet pas de contrôler la volatilité, ce qui est en réalité inévitable dans le cadre que nous étudierons par la suite.

Dans un deuxième temps, Li et Yu [231] (2018) ont étendu l'approche développée par Cvitanić, Wan et Zhang et Williams mentionnée en Section 1.1.2.2. Cette méthode est basée sur la résolution des systèmes couplés d'équations différentielles stochastiques directes et rétrogrades pour caractériser l'équilibre d'un jeu de Stackelberg généralisé à une hiérarchie à plusieurs niveaux. Néanmoins, cette approche comporte les mêmes défauts que la méthode initiale, et ne permet une résolution que dans un cadre linéaire-quadratique.

Finalement, Keppo, Touzi et Zuo [203] (2020) modélisent les relations entre un investisseur, un partenaire et un gestionnaire de fonds comme un problème d'incitations. Plus précisément, le gestionnaire est un agent pour le partenaire, le partenaire est un principal pour le gestionnaire et un agent pour l'investisseur, et enfin l'investisseur est un principal pour le partenaire. Le cadre est similaire aux deux modèles précédents, mais l'approche est différente et basée sur celle de Cvitanić, Possamaï et Touzi [111] (2018), plus précisément sur l'application à des utilités exponentielles CARA dans [110].

Néanmoins, ces trois modèles considèrent que les différentes entités de la hiérarchie contrôlent et observent le même processus résultat, tandis que l'aléa moral les empêche d'observer directement les contrôles. Le modèle qui nous intéressera ici est celui développé par Sung [327] (2015), qui contient deux niveaux d'aléa moral. Plus précisément, dans ce modèle, un principal propose un contrat à un manager désigné, afin de l'inciter à agir dans son intérêt. Ce manager propose ensuite un contrat à un nombre fini n d'agents. Chaque travailleur i (manager et agents) contrôle de manière indépendante la moyenne de sa propre variable aléatoire notée X^i , représentant son résultat personnel. Le premier niveau d'aléa moral est classique puisqu'il consiste à supposer que le manager, indexé par $i = 0$, n'observe pas directement les efforts des agents, mais uniquement leurs résultats individuels. Le deuxième niveau est plus original, car le principal n'observe pas non plus les efforts des travailleurs, mais surtout elle n'observe qu'une fonction particulière des résultats. Cette fonction est donnée par la différence entre la somme des résultats de tous les travailleurs et la somme des contrats payés aux agents, ce qui représente les bénéfices nets de la hiérarchie. Autrement dit, admettons que le manager propose un paiement ξ^i à l'agent i , alors la variable observée par le principal est :

$$\zeta := \sum_{i=0}^n X^i - \sum_{i=1}^n \xi^i. \quad (1.2.1)$$

1.2.2.3 Une étude rigoureuse du modèle de Sung

Le modèle considéré par Sung [327] (2015) est très similaire à celui de Holmström et Milgrom [181] (1987), brièvement décrit en Section 1.1.2. En effet, il suppose que chaque travailleur i contrôle uniquement la moyenne de son propre résultat X^i , et a une utilité exponentielle CARA de paramètre d'aversion au risque $R^i > 0$. En revanche, les préférences du principal sont représentées par une utilité risque neutre. Cependant, au lieu d'étudier une version à temps continu de son modèle, Sung considère que le modèle à une période est plus simple et sans perte de généralité. Étant donné le raisonnement de Holmström et Milgrom, il limite donc l'étude aux *contrats linéaires*, à savoir de la forme (1.1.3), et déclare que cette hypothèse est également sans perte de généralité, tant que les résultats sont interprétés dans le contexte de modèles à temps continu comme ceux de Holmström et Milgrom [181] (1987) ou Sung [325] (1995). Cette remarque est nécessaire, étant donné que la restriction aux contrats linéaires dans un modèle comme le sien à une période est loin d'être sans perte de généralité. En effet, Mirrlees [254] (1999) montre que, dans beaucoup de cas, incluant notamment le cadre de [327], il existe une suite de contrats, appelée *forcing contracts*, permettant d'obtenir à la limite les résultats du *first-best* (cas sans aléa moral), mais qu'il n'existe pas de contrat optimal.⁴

Cependant, nous allons remarquer, en étudiant l'équivalent du modèle [327] en temps continu dans le Chapitre 2, que les contrats linéaires ne sont tout de même pas optimaux. Le modèle en temps continu sera largement décrit dans le dit chapitre, nous nous contentons de présenter ici les intuitions et résultats principaux. Le résultat d'un travailleur i est représenté dans ce cas par un processus stochastique X^i , de dynamique identique à celle considérée dans le modèle de Holmström et Milgrom :

$$dX_t^i = \alpha_t^i dt + \sigma^i dW_t^i, \quad \sigma^i > 0, \quad t \in [0, T].$$

4. Cette dégénérescence a lieu dès que le rapport de vraisemblance monotone – défini comme le rapport entre : (i) la dérivée par rapport à l'effort de la densité du processus considéré et (ii) la densité elle-même – n'est pas borné par en-dessous. En particulier, c'est le cas dans le modèle défini par Sung car les résultats X^i sont des variables aléatoires gaussiennes.

Le travailleur i influe sur son résultat X^i entre 0 et un temps terminal $T > 0$ grâce à son effort α^i , à valeurs dans $\mathbb{R}$, et induisant un coût $\frac{1}{2k^i}|a|^2$, $a \in \mathbb{R}$. Pour simplifier, nous supposons que les W^i sont des mouvements brownien indépendants, et que l'effort d'un travailleur n'a d'impact que sur son propre résultat.

Dans un modèle principal – multi-agents classique, le principal propose des contrats aux agents sans intermédiaire. En revanche, dans le cadre d'une hiérarchie, le principal nomme un manager qui :

- (i) influe sur son propre résultat X^0 en choisissant un processus d'effort α^0 ;
- (ii) engage des agents pour effectuer les tâches restantes et conçoit leurs contrats;
- (iii) reporte au principal des bénéfices nets, en temps continu.

Nous supposons dans ce cas simple que le contrat ξ^i de l'agent i ne peut être indexé que sur sa propre performance, *i.e.* son résultat X^i . Comme l'agent ne contrôle que le drift de X^i , nous pouvons appliquer les résultats classiques mentionnés en [Section 1.1.2](#), pour obtenir la forme optimale de contrats [\(1.1.3\)](#), indexée par un paramètre Z^i choisi par le gestionnaire. Cela nous permet de calculer l'effort optimal de l'agent, et donc la dynamique optimale de X^i et ξ^i . Par conséquent, jusqu'à présent, les résultats sont conformes à ceux annoncés par [Sung](#).

Rappelons que le principal n'observe pas indépendamment les résultats X^i ou les paiements ξ^i , ni le résultat X^0 du manager. En effet, le manager ne lui communique que le bénéfice total ζ défini par [\(1.2.1\)](#), en temps continu. Par conséquent, elle ne peut indexer le contrat ξ^0 pour le manager que sur le processus ζ . Or la dynamique de ce processus, sous les effort optimaux des agents, est donnée par :

$$d\zeta_t = \left[\alpha_t^0 + \sum_{i=1}^n \left(k^i Z_t^i - \frac{1}{2} |Z_t^i|^2 (k^i + |\sigma^i|^2 R^i) \right) \right] dt + \sigma^0 dW_t^0 + \sum_{i=1}^n \sigma^i (1 - Z_t^i) dW_t^i, \quad t \in [0, T].$$

En examinant attentivement la dynamique de ζ , nous pouvons remarquer que, même si les agents ne contrôlent que le drift de leur processus, le manager contrôle le drift mais également la volatilité de ζ , à travers le choix des paramètres Z^i des contrats pour les agents. Par conséquent, nous devons envisager une classe de contrats plus étendue que celle considérée par [Sung](#). En effet, en temps continu, [Cvitanić, Possamaï et Touzi \[111\] \(2018\)](#) indiquent qu'il n'est plus suffisant de se limiter à des contrats linéaires lorsque la volatilité de la variable d'état est contrôlée. En suivant leur raisonnement, développé en [Section 1.1.3](#), la forme optimale de contrat est donnée par [\(1.1.4\)](#), mais indexée dans ce cas sur ζ au lieu de X :

$$\xi_T^0 = \xi_0^0 - \int_0^T \mathcal{H}^0(Z_s, \Gamma_s) ds + \int_0^T Z_s d\zeta_s + \frac{1}{2} \int_0^T (\Gamma_s + R^0 |Z_s|^2) d\langle \zeta \rangle_s,$$

où ξ_0 , Z et Γ sont choisis par le principal, et $\mathcal{H}^0$ est l'Hamiltonien classique du manager.

Ainsi, si la restriction aux contrats de la forme [\(1.1.3\)](#) peut être justifiée dans le cadre de [Sung](#) pour le premier équilibre de Stackelberg, entre les agents et le manager, ce n'est plus le cas pour le contrat proposé par le principal au manager. Il apparaît en effet que le type de contrats considéré par [Sung](#) pour ce second niveau est sous-optimal en temps continu, car il est nécessaire d'indexer le contrat sur la variation quadratique de la variable d'état, à travers le paramètre Γ . Cependant, dans le modèle à une période, cette variation quadratique contrôlée ne peut être estimée (contrairement au temps continu), ce qui conduit [Sung](#) à forcer $\Gamma = -R_0 |Z|^2$, alors que le Γ optimal est en réalité égal à $-R_0 Z^3$.

Cet exemple est éclairant à deux égards. Premièrement, il présente un cadre naturel où le contrôle de la volatilité intervient de manière parfaitement justifiée. D'autre part, il souligne l'importance de considérer des modèles en temps continu, qui sont pourtant souvent désavoués du fait de leur trop grande complexité, afin de prendre en compte ce contrôle de la volatilité. Il existe donc un écart fondamental entre ces deux cadres, qui motive *a priori* une étude complète et générale en temps continu des modèles hiérarchiques à la [Sung](#), ce qui va de pair avec la théorie récente des 2EDSR.

1.2.2.4 Contributions théoriques

Guidés par les intuitions précédemment évoquées, et détaillées dans le [Chapitre 2](#), nous étudions dans le [Chapitre 3](#), un problème d'incitation optimal en temps continu au sein d'une hiérarchie, généralisant ainsi le modèle de [Sung \[327\] \(2015\)](#) à de nombreux égards. Premièrement, le principal propose cette fois des contrats à un nombre m (fini) de managers, qui à leur tour suggèrent chacun une rémunération pour leurs agents respectifs. Dans ce cadre, chaque manager communique au principal les bénéfices nets de sa propre équipe. Deuxièmement, les travailleurs (managers et agents) contrôlent indépendamment en temps continu à la fois le drift et la volatilité d'un processus stochastique représentant leur résultat. Enfin, nous considérons des utilités et des fonctions de coûts les plus générales possibles pour les acteurs de la hiérarchie.

La recherche du contrat optimal à chaque étape de la hiérarchie nécessite d'utiliser la théorie des EDSR du second-ordre (2EDSR), sous réserve d'une légère extension pour tenir compte de la pluralité des travailleurs dans la hiérarchie. Étant donné l'indépendance de leurs projets, nous pourrions penser qu'il n'y a pas d'interaction entre les travailleurs. Néanmoins, il n'est en général pas possible de considérer que le contrat d'un travailleur n'est indexé que sur sa propre performance. En effet, cela conduirait à envisager un problème où certaines variables d'état ne sont pas observées par les entités de la hiérarchie, dépassant ainsi les cadres classiques, ce qui nécessiterait une étude plus approfondie. Par conséquent, les compensations des travailleurs sont également indexées sur les performances des autres, induisant ainsi un jeu à chaque étage de la hiérarchie. Finalement, le problème du manager est réduit à un espace d'état et de contrôle de dimensions respectives m et $2m$, et est donc indépendant du nombre d'étages dans la hiérarchie. Cependant, plusieurs hypothèses sont nécessaires pour compléter notre étude, notamment sur la forme de la dynamique des variables d'état, mais nous verrons que ces hypothèses sont satisfaites dans les exemples les plus courants.

Le modèle théorique que nous développons permet donc de déterminer la forme optimale d'incitation pour une structure hiérarchique particulière, qui peut être étendue de manière relativement directe à une hiérarchie à plus grande échelle. Bien que théoriques, les résultats que nous obtenons donnent des intuitions basées sur des considérations théoriques solides pour déterminer des leviers à activer pour inciter les travailleurs au sein d'une hiérarchie. En particulier, l'indexation du contrat sur la variation quadratique des bénéfices nets pour les managers plaide en faveur d'une rémunération par le biais de stock options. Ces résultats peuvent être appliqués aux problèmes d'incitation au sein d'une entreprise à structure hiérarchique, mais aussi et surtout dès lors que le travail est délégué à une entité externe. Par exemple, ces problèmes d'incitations à plusieurs niveaux peuvent être utilisés pour modéliser les relations entre une entreprise et ses filiales ou sous-traitants, ou les relations entre un investisseur, une société d'investissement et un gestionnaire de fonds, comme dans le modèle de [Keppo, Touzi et Zuo \[203\] \(2020\)](#).

1.2.3 Vers une infinité d'agents

1.2.3.1 Contexte

Comme mentionné précédemment, les modèles principal – multi-agents en temps continu restent pour le moment limités au contrôle de drift. Parmi les modèles déjà évoqués, rappelons que [Koo, Shim et Sung \[210\] \(2008\)](#) considèrent un cadre multi-agents classique, alors que [Goukasian et Wan \[155\] \(2010\)](#) et [Élie et Possamaï \[140\] \(2019\)](#) introduisent des interactions entre les agents. Plus récemment, [Élie, Mastrolia et Possamaï \[141\] \(2018\)](#) et [Carmona et Wang \[87\] \(2018\)](#) considèrent un continuum d'agents, qui contrôlent chacun de manière indépendante des processus non corrélés. Cependant, grâce à l'approche générale de [Cvitanic, Possamaï et Touzi \[111\] \(2018\)](#) sur les problèmes d'aléa moral en temps continu, il est maintenant possible d'étendre ses problèmes multi-agents au cadre dans lequel les agents contrôlent la volatilité de

leur processus résultant. Précédemment, nous nous sommes concentrés sur un modèle hiérarchique avec un nombre fini d’agents, contrôlant chacun de manière indépendante un processus. Dans la suite de cette section, nous allons nous intéresser aux modèles avec un continuum d’agents, et où les processus résultants de leurs contrôles sont corrélés.

Pour représenter la corrélation au sein de ce continuum d’agents, la méthode classique est d’ajouter un bruit, commun à tous les processus contrôlés par les agents. L’introduction de ce bruit commun induit un large éventail de difficultés mathématiques supplémentaires, qui n’ont été abordées que récemment, entre autre par Carmona, Delarue et Lacker [91] (2016), mais uniquement pour les jeux à champ moyen purs, sous-entendu sans addition d’un équilibre de Stackelberg. Rappelons que la résolution des problèmes de contrôle de drift entre multi-agents repose essentiellement sur la théorie des EDSR. Ainsi, naturellement, la résolution des problèmes considérant un continuum d’agents avec des interactions de type champ moyen est basée des EDSR dites à *champ moyen*, distinguées des EDSR classiques entre autre par la présence du bruit commun. Intuitivement, étant donné que quand les agents contrôlent la volatilité, nous sommes amenés à considérer des 2EDSR, il devrait être nécessaire d’utiliser une théorie plus générale d’EDSR du second-ordre, qui sera également qualifiée d’à *champ moyen*.

Dans la plupart des jeux à champ moyen, il est supposé que les agents sont identiques, mais surtout *petits*, en ce sens qu’ils n’ont pas d’influence sur le résultat global. Autrement dit, le choix des efforts d’un agent n’a pas d’impact sur le système commun, sur la répartition d’ensemble des agents, représentée par la loi des processus contrôlés. Cependant, certains modèles, comme ceux de Bensoussan, Chau et Yam [51] (2015), Carmona et Zhu [88] (2016), Carmona et Wang [86] (2017), [85] (2016), ou Cardaliaguet, Cirant et Porretta [79] (2020), introduisent la présence d’un agent plus important au sein de ce continuum, qui lui impacte de manière non-négligeable le système global. Une autre théorie liée aux jeux à champ moyen, le contrôle de type champ moyen (plus connu sous le nom anglais *mean-field type of control*), considère qu’un agent peut contrôler le processus mais également sa loi. Cette théorie est développée entre autre par Bensoussan, Frehse et Yam [50] (2013) et Laurière et Pironneau [228] (2014), et rejoint dans l’idée les approches économiques considérant qu’un planificateur social peut décider des actions de la population. La résolution de ces problèmes repose alors sur les équations de McKean–Vlasov, et a été récemment étudiée, notamment par Carmona, Delarue et Lachapelle [89] (2013); Bensoussan, Frehse et Yam [50] (2013), [52] (2015), [53] (2017); Lacker [214] (2017); Pham et Wei [278] (2016), [279] (2017), [280] (2018); Acciaio, Backhoff-Veraguas et Carmona [8] (2019); Basei et Pham [42] (2019); ainsi que par Bayraktar, Cosso et Pham [44] (2018); ou encore Bouchard, Djehiche et Kharroubi [68] (2020); et finalement par Djete, Possamaï et Tan [126] (2019), [127] (2020).

De manière très intuitive, un modèle principal-agent avec un continuum d’agents combine les deux cadres précédemment mentionnés. En effet, si nous considérons d’une part que les agents contrôlent leur processus mais sont trop insignifiants pour impacter le système global, nous sommes alors dans un cadre de jeu à champ moyen classique. D’autre part, le principal offre des contrats pour inciter les agents à se comporter de manière optimale de son point de vue à elle. Ce contrat impacte les efforts de chaque agent, et donc la loi globale des processus contrôlés. Ainsi, indirectement, le principal contrôle la répartition en loi des agents, à la manière d’un planificateur social, ce qui rejoint l’angle McKean–Vlasov précédemment mentionné. Cette intuition est confirmée par le travail réalisé dans le Chapitre 4, et décrit brièvement ci-dessous.

1.2.3.2 Contributions

Dans le Chapitre 4, nous choisissons d’étendre le cadre défini par Aïd, Possamaï et Touzi [11] (2019) à un modèle avec un champ moyen d’agents. Plus précisément, nous considérons que chaque agent contrôle un processus stochastique sur la période $[0, T]$, et que ces processus sont soumis à un bruit commun, témoignant

de l'environnement aléatoire dans lequel évoluent tous les agents. En supposant que tous les agents sont identiques, il est possible de restreindre l'étude à un agent dit représentatif, pouvant impacter le drift et la volatilité d'un processus X à travers des contrôles α et β , à valeurs respectives dans A et B inclus tous deux dans $\mathbb{R}^d$. Plus exactement le processus X est donné par :

$$X_t = X_0 - \int_0^t \alpha_s \cdot \mathbf{1}_d ds + \int_0^t \sigma(\beta_s) \cdot dW_s + \int_0^t \sigma^\circ dW_s^\circ, \quad t \in [0, T].$$

De manière similaire, les autres agents contrôlent un processus noté $\hat{X}$, à travers des effort $\hat{\alpha}$ et $\hat{\beta}$, impliquant une dynamique pour $\hat{X}$ équivalente à celle de X , impactée par le même bruit commun W° :

$$\hat{X}_t = X_0 - \int_0^t \hat{\alpha}_s \cdot \mathbf{1}_d ds + \int_0^t \sigma(\hat{\beta}_s) \cdot d\hat{W}_s + \int_0^t \sigma^\circ dW_s^\circ, \quad t \in [0, T].$$

En suivant l'approche de Cvitanić, Possamaï et Touzi [111] (2018), le but de ce chapitre est de déterminer la forme optimale des contrats à considérer dans ce cadre, puis de caractériser l'équilibre à champ moyen entre les agents, et enfin de résoudre le problème du principal.

Dans [11], les auteurs ne considèrent qu'un seul agent, et utilisent directement les résultats de [111] pour établir que le principal offre à l'agent un contrat de la forme (1.1.4). Dans le présent cadre, *i.e.*, avec un champ moyen d'agents, le principal observe toutes les trajectoires des processus contrôlés et peut donc tirer partie de ce supplément d'information. En particulier, elle peut proposer un contrat à l'agent représentatif qui soit fonction à la fois de son processus mais aussi des statistiques globales de la population. Il est nécessaire de souligner que, conformément au cadre champ moyen, le principal est confronté à une masse d'agents identiques et indiscernables, et ne peut donc pas indexer le contrat de l'agent représentatif sur le processus d'un autre agent en particulier, d'où la considération de statistiques globales. La statistique sur laquelle nous allons nous concentrer, notée $\hat{\mu}$, représente la loi de $\hat{X}$ conditionnellement au bruit commun. Considérer cette loi conditionnelle vient naturellement de la limite de la distribution empirique des processus $\hat{X}$ d'un cadre avec n agents. Ainsi, nous supposons dans la suite que le contrat de l'agent représentatif est une fonction de son propre processus X et de la loi conditionnelle $\hat{\mu}$.

Pour deviner la forme optimale de contrats dans ce cadre, nous développons un raisonnement très intuitif dans le cas Markovien, utilisant en particulier la *chain rule with common noise* définie par [82, Theorem 4.17], qui consiste en une extension de la formule d'Itô pour des fonctions de la loi d'un processus. Finalement, nous obtenons maintenant qu'un contrat doit être indexé par un triplet de processus $\zeta := (Z, Z^\mu, \Gamma)$, où, moralement, le couple (Z, Γ) permet de construire un contrat de la forme (1.1.4), alors que Z^μ indexe le contrat sur la loi des autres $\hat{\mu}$. Plus précisément, le contrat se présente sous la forme suivante :

$$\begin{aligned} \xi_t^{\xi_0, \zeta} := & \xi_0 - \int_0^t \mathcal{H}(X_s, \hat{\mu}_s, \zeta_s, \hat{\alpha}_s^*) ds + \int_0^t Z_s dX_s + \frac{1}{2} \int_0^t (\Gamma_s + R_A |Z_s|^2) d\langle X \rangle_s + \int_0^t \hat{\mathbb{E}}^{\hat{\mathbb{P}}_s} \left[Z_s^\mu (\hat{X}_{s \wedge \cdot}) d\hat{X}_s \right] \\ & + \frac{1}{2} R_A \int_0^t \hat{\mathbb{E}}^{\hat{\mathbb{P}}_s} \tilde{\mathbb{E}}^{\tilde{\mathbb{P}}_s} \left[Z_s^\mu (\hat{X}_{s \wedge \cdot}) Z_s^\mu (\tilde{X}_{s \wedge \cdot}) d\langle \hat{X}, \tilde{X} \rangle_s \right] + R_A \int_0^t Z_s \hat{\mathbb{E}}^{\hat{\mathbb{P}}_s} \left[Z_s^\mu (\hat{X}_{s \wedge \cdot}) d\langle X, \hat{X} \rangle_s \right], \end{aligned}$$

pour un certain $\xi_0 \in \mathbb{R}$, choisit de manière classique afin de satisfaire la contrainte de participation de l'agent considéré. Évidemment, cette forme de contrat fait intervenir des notations dont nous n'avons pas encore évoquées l'existence. Il serait trop pesant de les détailler ici, et le lecteur est prié de se référer au [Chapitre 4](#) pour une explication rigoureuse. Le point que nous voulons souligner ici est simplement que la nouvelle forme de contrat que nous obtenons dans ce cadre consiste à ajouter au contrat déterminé par Cvitanić, Possamaï et Touzi [111] (2018) des termes indexés sur la loi conditionnelle des autres $\hat{\mu}$. Il convient de noter qu'en absence de bruit commun, *i.e.*, pour $\sigma^\circ = 0$, le contrat que nous présentons ci-dessus se réduirait à la forme classique (1.1.4) considérée dans [11].

Étant donnée cette forme de contrats, et sous l'hypothèse de fonctions de coûts bien choisies, maximiser l'Hamiltonien $\mathcal{H}$ de l'agent permet de déterminer ses contrôles optimaux, et de caractériser l'équilibre à champ moyen entre les agents, qui s'avère être unique. Les efforts optimaux sont en réalité des fonctions des paramètres d'indexation Z et Γ , identiques à celle obtenues dans le cadre de [11]. L'indexation du contrat sur la loi des autres n'a donc pas d'influence directe sur les efforts de l'agent considéré. Le résultat majeur de ce chapitre consiste à montrer que considérer cette forme de contrats est sans perte de généralité, ce qui nécessite d'utiliser la théorie des EDSR du second–ordre généralisée à un cadre à champ moyen avec bruit commun. Le problème du principal, que ne détaillerons pas ici, est ainsi ramené à un problème de contrôle de type McKean–Vlasov. En effet, en choisissant les paramètres (Z, Z^μ, Γ) indexant le contrat des agents, le principal impacte leurs efforts optimaux α^* et β^* , ce qui induit une dynamique particulière pour le processus X et donc une loi conditionnelle μ^* . Le principal contrôle donc, de manière indirecte via les paramètres (Z, Z^μ, Γ) , le processus X ainsi que sa loi conditionnelle μ^* .

Il est nécessaire de préciser que nous travaillons dans ce chapitre avec un modèle spécifique, très similaire à celui de Aïd, Possamaï et Touzi [11] (2019). En effet, cette étude est motivée par la même application, à savoir l'optimisation des programmes de réponse à la demande en électricité, qui sera introduite en Section 1.3.1 et détaillée dans le Chapitre 5. Cependant, le *modus operandi* que nous développons dans cet article peut facilement être étendu à des problèmes plus généraux d'aléa moral avec un continuum d'agents ayant des interactions de type champ moyen. C'est l'une des raisons pour lesquelles nous avons fait des efforts spécifiques pour nous assurer que toutes nos déclarations et définitions soient totalement rigoureuses sur le plan mathématique, en particulier concernant la définition de la formulation faible, à la fois pour les jeux de champ moyen avec bruit commun et pour les problèmes de contrôle optimal des EDS de McKean–Vlasov, ces deux outils constituant les pierres angulaires fondamentales de notre approche. Nous pensons que cela s'avérera utile pour d'autres applications que celle étudiée dans le Chapitre 5, et pour de futures potentielles généralisations de nos techniques.

1.2.4 Perspectives

Les deux contributions précédemment évoquées, qui constituent les Chapitres 2, 3 et 4, ouvrent de nouvelles possibilités, mais soulèvent également d'autres questions. Rappelons que les problèmes de contrôle sont étroitement liés à la théorie des EDSR. Pour résumer brièvement, dans un cadre avec un seul agent contrôlant uniquement le drift de son processus, nous sommes amenés à examiner une EDSR simple, alors que dans un cadre avec contrôle de la volatilité, il est nécessaire d'introduire des 2EDSR. Pour ce qui est des jeux entre de multiples agents en interaction, la recherche d'un équilibre de Nash pour le contrôle du drift équivaut à la recherche d'une solution à un système d'EDSR couplées, comme considéré par Élie et Possamaï [140] (2019). Ainsi, naturellement, dans le cas où les agents peuvent également contrôler la volatilité, nous sommes incités à envisager un système d'EDSR du second–ordre. En effet, cette intuition sera précisée dans le Chapitre 3 : les managers contrôlant le drift et la volatilité d'un même processus ζ représentant les bénéfices nets par équipe, la résolution du Nash entre ces managers repose sur un système de 2EDSR. Cependant, dans le cadre que nous considérons, en particulier grâce aux hypothèses relatives à l'espace de contrats admissibles, ce système est naturellement bien posé et sa solution existe et est unique. En revanche, dans un cadre plus général, en particulier en absence de contrats et donc d'hypothèses satisfaisantes, il n'existe pas de résultats affirmant que le système de 2EDSR résultant est bien posé et possède une solution. En effet, ce type de systèmes n'est à notre connaissance pas encore étudié dans la littérature.

Ainsi, une perspective possible serait l'étude des EDSR du second–ordre multidimensionnelle, et en particulier leur lien avec les équilibres de Nash. En particulier, la conjecture serait qu'il existe une correspondance univoque entre (i) l'existence d'équilibres de Nash dans un jeu entre agents contrôlant le drift et

la volatilité d'un processus et (ii) les solutions à un système multidimensionnel de 2EDSR couplées. Cette conjecture est donc basée sur les résultats préliminaires que nous avons obtenus dans le [Chapitre 3](#), et renforcée par un article récent de [Possamaï, Touzi et Zhang \[286\] \(2020\)](#). Cet article lie la formulation faible des jeux différentiels à somme nulle, où les deux joueurs peuvent contrôler des processus de diffusion continus non-markoviens, à une nouvelle classe de 2EDSR, obtenue comme un infimum de 2EDSR. L'approche qu'ils développent pourrait être un premier pas vers l'application de techniques similaires au problème plus délicat des jeux à somme non nulle avec une multitude de joueurs. Cette conjecture est également basée sur les résultats obtenus dans le [Chapitre 4](#), où le lien est établi entre une EDSR du second-ordre à champ moyen et l'équilibre de Nash entre agents, dans le cadre d'un modèle spécifique. Intuitivement, ce type d'EDSR devrait être la limite d'un système d'EDSR du second-ordre dans le cas d'une infinité d'agents identiques et indiscernables. La considération de ce type d'EDSR à champ moyen a été également étudiée dans un jeu plus général, mais sans équilibre de Stackelberg, par [Barrasso et Touzi \[38\] \(2020\)](#).

Un tel résultat théorique liant un système de 2EDSR à l'équilibre de Nash d'un jeu entre agents fournirait *a priori* la première représentation probabiliste de la valeur d'un jeux différentiel stochastique général, permettant de contrôler à la fois le drift et la volatilité d'un processus de dynamique non-Markovienne, et par extension aux solutions de systèmes de d'équations aux dérivées partielles (EDP) non linéaires. L'idée serait donc d'étendre les résultats établis sur les 2EDSR uni-dimensionnelles par [Soner, Touzi et Zhang \[317\] \(2012\)](#), et plus récemment par [Possamaï, Tan et Zhou \[284\] \(2018\)](#), aux systèmes de 2EDSR. La difficulté réside dans le fait que les 2EDSR classiques sont définis comme supremum de familles d'EDSR. Comme il n'existe pas d'ordre total sur $\mathbb{R}^n$ ($n > 1$), la définition par supremum est floue. L'utilisation des extensions multidimensionnelles du théorème de comparaison (voir notamment [Hu et Peng \[184\] \(2006\)](#)) pourrait constituer une première étape pour obtenir les généralisations souhaitées.

1.3 Des incitations pour une meilleure consommation de l'énergie

Laissons pour le moment cette perspective de côté, et concentrons nous sur une application, à première vue intrigante, des deux théories principales de cette thèse au domaine de l'énergie. À travers l'application des modèles principal-agent aux questions énergétiques, nous cherchons à résoudre deux problèmes, à priori très opposés, mais finalement liés par le formalisme mathématique de la théorie des contrats.

Le premier problème concerne les programmes de *réponse à la demande*, définie comme le changement d'utilisation de l'électricité par le consommateur par rapport à son usage habituel, en réponse à des variations de prix de l'électricité, ou à de potentiels remboursements. Le but de ces programmes est de diminuer la consommation globale d'électricité lors de pics de prix sur les marchés de gros, ou lors de période d'instabilité du réseau, en cas de forte demande par exemple. En réfléchissant à ce problème en terme d'incitations, le producteur ou distributeur d'électricité doit déterminer de manière optimale les signaux de prix ou les remboursements à transmettre aux consommateurs de manière à ce que l'impact sur la consommation globale soit celle souhaitée.

Un autre problème de société lié à l'énergie concerne la précarité énergétique, définie en France dans la loi dite Grenelle 2 par : « [e]st en situation de précarité énergétique au titre de la présente loi une personne qui éprouve dans son logement des difficultés particulières à disposer de la fourniture d'énergie nécessaire à la satisfaction de ses besoins élémentaires en raison de l'inadaptation de ses ressources ou de ses conditions d'habitat. » (*Loi n° 2010-788 du 12 juillet 2010 portant engagement national pour l'environnement* [1, Article 11]). Pour lutter contre la précarité énergétique, il est nécessaire de trouver un moyen d'aider ces personnes en difficulté, ou d'éviter qu'elles le deviennent. Certaines mesures peuvent être mises en place pour remédier à cette situation. L'État français distribue par exemple aux plus précaires des chèques énergie, destinés à couvrir leur dépenses énergétiques, mais aussi d'éventuels travaux d'isolation.

À première vue, ces deux situations semblent opposées : d'un côté, les programmes de réponse à la demande procurent des incitations à la baisse de la consommation, alors que dans le cas d'un ménage souffrant de précarité énergétique, il serait à l'inverse nécessaire de fournir des incitations à l'augmentation de la consommation. En réalité, ces deux problèmes s'inscrivent dans une logique de transition énergétique et sociale, et, en particulier, dans le changement des habitudes de consommation et d'une meilleure répartition des ressources disponibles. D'autre part, comme mentionné précédemment, ces deux questions sont également reliées par le formalisme mathématique sous-jacent, étant donné qu'elles impliquent toutes deux des problèmes d'incitations optimales, que nous traduirons à travers des modèles principal-agent. Notons également que ces deux problèmes impliquent la considération d'une infinité de consommateurs.

1.3.1 Aléa moral dans les programmes de réponse à la demande

1.3.1.1 Contexte

Pour agir en faveur de la transition énergétique, il est nécessaire de repenser la production mais aussi notre consommation d'énergie. Dans cette partie, nous nous intéresserons particulièrement à l'électricité, qui est caractérisée par ses possibilités de stockage très limitées. En effet, il existe aujourd'hui très peu de solutions pour stocker l'électricité sur le long terme et en quantité substantielle, excepté les barrages hydrauliques. Cela implique une grande particularité pour le marché de l'électricité : la nécessité d'un équilibre offre-demande à tout instant. Autrement dit, la production doit être égale à la consommation, et ce à toute heure du jour et de la nuit. Pour assurer cet équilibre, les décideurs peuvent ajuster le taux de production des centrales, allumer ou éteindre des générateurs, ou encore importer de l'électricité depuis d'autres pays.

Afin de simplifier la résolution de cette équation, il est nécessaire d'apporter de la flexibilité d'un côté ou de l'autre. Malheureusement, au niveau de l'offre, plusieurs obstacles persistent. D'une part, certaines unités de production peuvent mettre beaucoup de temps à atteindre leur pleine capacité, et d'autres sont très coûteuses à exploiter. Les moyens de production les plus flexibles, comme les centrales à charbon ou gaz, demeurent les plus polluants, les plus carbonés. Or, un des objectifs majeurs de la transition énergétique est de limiter le recours à ces moyens de production. Cependant, un autre objectif est d'intégrer largement les moyens de production dits renouvelables, qui présentent le défaut d'être extrêmement intermittents et aléatoires, et donc très peu flexibles. Ainsi, pour limiter l'utilisation de moyens de productions carbonés et permettre le développement des énergies renouvelables, il semble nécessaire d'améliorer et de renforcer la flexibilité de la demande, à travers le développement des programmes de gestion de la consommation. Il faut également souligner qu'il arrive que la demande dépasse la capacité de toutes les centrales électriques disponibles réunies, renforçant la nécessité de mettre en place ce type de programmes.

La gestion de la consommation, en particulier l'efficacité énergétique et la réponse à la demande, est donc indéniablement l'un des défis les plus importants dans le secteur de l'énergie, vu qu'elle constitue une base majeure pour garantir la réalisation des objectifs de l'Union Européenne en matière d'énergie verte et de réduction des émissions. La réponse à la demande est définie comme un changement dans la consommation d'énergie d'un consommateur, en réponse à des signaux envoyés par le fournisseur, afin d'améliorer l'adéquation entre la demande et l'offre proposée. Ces signaux envoyés par le fournisseur d'énergie peuvent prendre différentes formes. Généralement, ils sont modélisés par des signaux de prix, définis à la signature du contrat : le consommateur bénéficie la plupart du temps d'un prix unitaire inférieur au tarif standard, en échange de prix nettement plus élevés à certaines périodes très courtes et prédéfinies dans le contrat. Pendant ces périodes ponctuelles, le client est alors incité à ajuster sa consommation, par exemple en différant les utilisations coûteuses d'électricité, faute de quoi il paierait un prix élevé. Il est important de noter que le fort développement des compteurs électriques intelligents favorise ce type d'initiative.

De nombreuses expériences ont été développées pour évaluer précisément les avantages des programmes de réponse à la demande sur la consommation, nous référons à l'article [11] qui compile la plupart de ces expériences. Dans cet article, les auteurs Aïd, Possamaï et Touzi se concentrent en particulier sur l'expérience de réponse à la demande du *Low Carbon London Pricing Trial*. Cette expérience a montré des résultats significatifs, néanmoins les mécanismes de réponse à la demande sont confrontés à des défis qui doivent être relevés, avant de pouvoir offrir un niveau de flexibilité comparable à celui des centrales thermiques. En particulier, ces programmes induisent une variance substantielle dans la réponse du consommateur au signal de prix : ses efforts sont peu prévisibles et aléatoires, ce qui conduit à une incertitude sur la réponse totale de la population sollicitée. Cette grande variance, appelée *effet de réactivité*, constitue une lacune importante pour une gestion efficace de la demande.

1.3.1.2 Gestion de la demande : un premier modèle d'incitation

Dans cette perspective, Aïd, Possamaï et Touzi [11] (2019) formulent le mécanisme des programmes de réponse à la demande comme un problème principal-agent en temps continu en présence d'aléa moral. L'agent est un consommateur peu enclin au risque, dont les préférences sont représentées par une utilité exponentielle CARA. Il a une consommation de base d'électricité, donnée par la somme des consommations aléatoires de chaque usage : chauffage, climatisation, éclairage, télévision, machine à laver, ordinateurs, etc. Sa consommation est adaptée en fonction du prix de l'électricité et de son mode de vie : ses préférences, la taille et l'isolation thermique de son logement... Il peut cependant, s'il y est incité, accepter de s'écarter de sa demande habituelle, en réduisant le niveau moyen de sa consommation, ainsi que la volatilité, pour chaque usage, moyennant un coût. Cette réduction de la volatilité peut s'interpréter comme un lissage de la consommation de l'agent, mais également comme une augmentation de la réactivité du consommateur face aux incitations, améliorant ainsi l'effet de réactivité précédemment mentionné.

Le principal est un producteur ou fournisseur d'énergie, dont l'aversion au risque est également modélisée par une utilité CARA, qui a l'obligation de satisfaire la demande aléatoire d'électricité du consommateur. Elle est soumise à des coûts de production ou fourniture d'énergie, mais aussi à des coûts liés à la volatilité de la consommation, qui traduisent la flexibilité limitée de la production d'électricité. Comme habituellement en théorie des contrats, le principal souhaite définir des incitations optimales pour d'encourager l'agent à réduire la moyenne et la volatilité de sa consommation, afin de minimiser les coûts auxquels elle est confrontée. Cependant, la présence d'aléa moral l'empêche d'observer les efforts de l'agent, et donc d'offrir une compensation directement liée à cet effort. Les auteurs supposent en effet que le principal n'observe que la consommation de l'agent en temps continu, hypothèse très naturelle au vu du contexte considéré.

Grâce aux travaux récents de Cvitanić, Possamaï et Touzi [111] (2018), il est possible de restreindre l'étude à des contrat de la forme (1.1.3), qui consistent donc en une somme de quatre termes. Le premier, constant, permet d'assurer la participation du consommateur au contrat. Plus précisément, dans ce modèle, les auteurs considèrent une utilité de réservation endogène pour le consommateur, déterminée par son utilité sans contrat. La deuxième partie est un paiement déterministe, qui dépend de la durée de la réponse à la demande. La troisième est une pénalisation indexée de manière linéaire sur la consommation de l'agent, incitant l'agent à diminuer la moyenne de sa consommation. Enfin, le dernier terme est une pénalisation linéaire sur la variation quadratique réalisée, incitant donc l'agent à réduire la volatilité de sa consommation. Les avantages de ce modèle de réponse à la demande sont doubles. D'une part, le contrat optimal permet au système de supporter un risque plus important, même quand la volatilité qui en résulte peut augmenter. D'autre part, le contrôle de la volatilité de la consommation par l'agent conduit à une augmentation significative de la réactivité du consommateur face au contrat proposé. En d'autres termes, l'agent est plus enclin à répondre de manière fiable aux incitations, étant donné que sa rémunération prend en compte cet effort.

L'approche proposée dans [11] présente donc des résultats significatifs, et constitue une avancée majeure dans la modélisation mathématique de la gestion de la demande. Cependant, plusieurs particularités de la consommation d'électricité ne sont pas prises en compte dans ce modèle. D'une part, la plupart des producteurs et fournisseurs d'électricité font face à une multitude de consommateurs, et doivent satisfaire la demande globale. Il serait donc plus réaliste de considérer que le principal souhaite optimiser la consommation globale, ou moyenne, d'un grand nombre d'agents, au lieu de considérer les profils de consommation des agents de manière individuelle. D'autre part, la consommation d'électricité est extrêmement dépendante des aléas climatiques, impliquant que ces profils sont très corrélés entre eux. En effet, en cas de forte chaleur ou de grand froid par exemple, tous les consommateurs de la région concernée vont réagir de la même manière, en allumant leurs climatiseurs dans un cas, ou leurs radiateurs dans l'autre. Ainsi, le but du principal serait plutôt d'inciter un groupe de consommateurs relativement similaires, dont la consommation est impactée par les mêmes conditions météorologiques. Cette considération complique évidemment le modèle initial, mais peut être bénéfique pour le principal : elle a alors accès à tous les profils de consommations, et donc à une grande quantité d'informations, qu'elle peut utiliser (de manière réglementée) pour concevoir les contrats de réponse à la demande.

1.3.1.3 Contributions

Dans le [Chapitre 5](#), nous étendons le cadre défini par [Aïd, Possamaï et Touzi \[11\] \(2019\)](#) à un modèle avec un continuum d'agent, dont les consommations sont impactées par un bruit, commun à tous, afin de représenter les particularités mentionnées ci-dessus, notamment les conséquences des variations météorologiques sur la consommation d'électricité. Le problème du principal est donc d'inciter un ensemble de consommateurs similaires, soumis aux mêmes aléas climatiques (représentés par le bruit commun), à réduire et lisser leur consommation pendant la durée du contrat, afin de limiter les effets néfastes sur le climat des pics de production d'électricité. Outre l'aspect champ moyen et la corrélation entre les consommations par le bruit commun, la formulation du problème est volontairement aussi proche que possible de celle de [Aïd, Possamaï et Touzi \[11\] \(2019\)](#), et nous utilisons les paramètres calibrés dans [11] pour l'application numérique. Néanmoins, en raison de la complexité induite par la présence d'un continuum de consommateurs en interaction, nous ne pouvons pas en général nous appuyer sur les résultats de [11], et devons utiliser les nouvelles techniques développées dans le [Chapitre 4](#), résumées précédemment en [Section 1.2.3](#).

Nous travaillons donc dans un cadre classique de jeu à champ moyen, où tous les agents sont identiques et indiscernables. Cette hypothèse est justifiée pour un grand producteur ou fournisseur d'électricité, qui gère un nombre important de consommateurs, et peut donc les regrouper en des catégories similaires suffisamment vastes. Outre les avantages techniques, ce cadre nous empêche d'indexer la rémunération d'un consommateur sur un profil de consommation particulier, à l'exception du sien. C'est en effet le cas dans le secteur de l'énergie, notamment en Europe : le *General Data Protection Regulation* [règlement général sur la protection des données] stipule que les *Energy Distribution Organising Authorities* [autorités organisatrices de la distribution d'énergie], ont uniquement le droit d'accéder à des données agrégées (et donc anonymes) sur la production et la consommation d'électricité sur un territoire donné. En France, ce règlement est précisé par deux décrets, *Décret n° 2017-976* et *Décret n° 2017-948* du 10 mai 2017. Ces décrets sont également complétés par un rapport de la Commission Nationale de l'Informatique et des Libertés (CNIL), intitulé *Pack de conformité – Les compteurs communicants*, qui spécifie qu'un fournisseur ou un bailleur social ne peut utiliser les données de consommation d'une personne pour établir des statistiques que si les données sont anonymes et agrégées, et ne permettent donc pas l'identification d'une personne physique.

Pour améliorer les résultats de [11], nous reprenons l'idée suggérée précédemment, à savoir que la prise en compte d'un continuum de consommateurs peut être profitable pour le principal. En effet, dans notre cadre,

et compte tenu du développement des compteurs intelligents, le principal a accès à une grande quantité de profils de consommation anonymes. Par conséquent, elle peut calculer des statistiques à partir de ces données, et, en particulier, approximer la loi, conditionnelle au bruit commun, de la consommation de ses clients. Ainsi, suivant le raisonnement évoqué en [Section 1.2.3](#), elle peut concevoir un nouveau contrat indexé sur cette loi, de manière à, par exemple, pénaliser un consommateur qui fait moins d'efforts, en moyenne, que le reste du groupe, ou le récompenser s'il fait plus d'efforts. Ce type de contrat est également inspiré par le développement récent d'applications qui permettent de comparer sa propre consommation avec celle de ménages similaires, voire avec celle des moins consommateurs.⁵ En outre, les résultats de l'étude de Dolan et Metcalfe [129] (2015) sur l'efficacité énergétique ont montré que la comparaison de la consommation d'énergie avec celle de ménages similaires et l'octroi d'incitations financières peuvent conduire à une réduction moyenne de 7% de la consommation d'énergie des ménages.

Cette extension permet une amélioration significative des résultats : les contrats que nous développons incitent de manière plus efficace les ménages à modifier leurs habitudes de consommation d'électricité. Pour illustrer ce fait, nous nous concentrerons sur un cas particulier, à savoir quand *l'écart de valeur énergétique* est linéaire. Cet écart est défini par la différence entre l'utilité marginal d'un consommateur pour sa consommation et le coût marginal de production associé. Succinctement, si cet écart est positif, cela signifie que l'énergie a plus de valeur pour le consommateur qu'elle n'est coûteuse pour le producteur. À l'inverse, si la différence est négative, une diminution de la consommation aura moins d'effets négatifs sur le bien-être du consommateur que d'effets positifs sur les économies du producteur. Dans le cas linéaire, il est facile de montrer que l'utilité du producteur est (strictement) accrue par l'utilisation de nos nouveaux contrats. De plus, dans la plupart des cas, ces nouveaux contrats induisent plus d'efforts de la part des consommateurs pour réduire le niveau moyen de leur consommation et avec moins de volatilité. Plus précisément, en utilisant les valeurs des paramètres calibrés dans [11], nous pouvons constater une augmentation :

- (i) de l'utilité du principal jusqu'à 50% (respectivement 15%) ;
- (ii) de l'effort des agents pour réduire leur consommation jusqu'à 50% (resp. 30%) ;
- (iii) de l'effet de réactivité (effort pour réduire la volatilité) jusqu'à 4% (resp. 3%) ;

en fonction de la corrélation avec le bruit commun, pour un principal neutre au risque (resp. pour l'aversion au risque calibrée dans [11]). Il est important de souligner que plus la variance expliquée par le bruit commun est importante, plus les résultats sont significatifs. Par conséquent, ces nouveaux contrats pourraient améliorer la réponse à la demande pendant les périodes où la consommation est fortement affectée par les conditions météorologiques, par exemple en hiver, lorsque le risque de panne d'électricité est élevé, et donc justement quand la gestion de la demande est plus que nécessaire.

Tout au long de cette application, nous considérons que le principal ne peut pas observer le bruit commun, ou du moins qu'il existe certaines réglementations l'empêchant de l'utiliser directement dans le contrat. Cette hypothèse est relativement bien établie dans le domaine de la consommation d'énergie. En effet, bien que certains fournisseurs d'électricité proposent des prix différents en fonction du jour ou de l'heure de consommation (heures creuses, heures pleines...) ⁶, ces offres tarifaires correspondent davantage à une indexation indirecte sur la météo par le biais du prix spot de l'électricité. Nos résultats montrent que l'indexation du contrat sur les autres consommations est en réalité une alternative à l'indexation directe sur la météo. En effet, elle permet au principal de diviser la consommation en deux parties : (i) la partie *réellement contrôlée* par l'agent, qui correspond à une sorte de consommation corrigée des aléas climatiques,

5. Voir par exemple l'application téléphonique *EDF & Moi*, du producteur et fournisseur d'électricité français *Électricité de France*, ou le site web de *Synergy*, producteur et fournisseur australien.

6. Il est possible de référer par exemple à la proposition du *Environmental Defense Fund* [Fond de défense de l'environnement] dans https://www.edf.org/sites/default/files/time-variant_pricing_fact_sheet_-_april_2015.pdf

et (ii) le bruit commun. Ainsi, d’une part, elle peut proposer une compensation indexée sur la consommation corrigée pour inciter l’agent à faire un effort. D’autre part, si elle est peu encline à prendre des risques, elle peut ajouter à ce contrat une partie indexée sur les autres consommations, permettant de partager le risque restant, dû au bruit commun.

Il convient de souligner que les contrats obtenus sont très théoriques et soulèvent donc une question importante, celle de leur mise en œuvre dans la pratique. Il est évident qu’il ne semble pas concevable de proposer directement de tels contrats aux consommateurs, qui en resteraient complètement perplexes. L’idée, déjà mentionnée dans [11], est plutôt de considérer le contrat optimal comme un signal de prix à envoyer aux consommateurs.

1.3.2 Précarité énergétique : diversité des risques et sélection adverse

1.3.2.1 Contexte

En France, la loi [1] du 12 juillet 2010 précédemment évoquée pose une définition de la précarité énergétique. Cependant, plusieurs méthodes peuvent être utilisées pour quantifier plus précisément le nombre de ménages en situation de précarité énergétique. Selon l’Observatoire National de la Précarité Énergétique (ONPE), en 2018,⁷

- (i) 11.7% des français consacraient plus de 8% de leurs revenus aux dépenses d’énergie dans le logement ;
- (ii) 15% des français déclaraient souffrir du froid dans leur logement.

Les nombreuses données chiffrées montrent que cette précarité touche toutes les classes d’âges, et de nombreuses situations socio-économiques. Bien qu’elle soit expliquée en partie par la mauvaise qualité thermique du parc de logements français, selon Chaton et Gouraud [98] (2020) la précarité énergétique est essentiellement liée à une perte temporaire de revenus : elle touche particulièrement les ménages vulnérable, à faible revenu, qui ont une propension à épargner insuffisante et dont une grande partie des revenus est déjà consacrée à l’énergie.

Même s’il est toujours difficile d’établir les relations de causalité, la précarité énergétique peut avoir des incidences sur la santé, comme souligné dans le rapport de Bidault, Bellois et Daude [59] (2019) de l’ONPE. Lacroix et Chaton [215] (2015) indiquent par exemple que cette situation peut conduire les ménages à adopter des comportements à risque, entraînant des problèmes de santé et une détérioration des logements. Par exemple, pour maintenir la chaleur à l’intérieur de leur logement, certains obstruent les bouches d’aération, générant ainsi de l’humidité et de la moisissure. Les ménages en situation de précarité énergétique sont donc souvent contraints de faire des choix qui ont des conséquences néfastes sur leur santé : choisir de manger ou de se chauffer, renoncer aux soins de santé ou aux interactions sociales. Ainsi, les pathologies chroniques sont plus fréquentes chez les ménages précaires, et nous constatons également plus de fragilité au niveau de la santé mentale et sociale.

Ces conséquences indirectes sont souvent négligées par les ménages, qui sont en général dans une précarité à d’autres niveaux plus primordiaux, mais sont néfastes, et de surcroît très coûteuses pour la société. Pour les éviter, des mécanismes sont mis en place pour aider les ménages vulnérables. Par exemple, des *chèques énergie* sont distribués par l’État français depuis 2018. Ce bon peut être utilisé pour payer non seulement les dépenses énergétiques telles que les factures d’énergie, mais aussi pour financer des rénovations au sein du logement. En 2018, ce chèque concernait 5,7 millions de ménages. Malheureusement, vu que ce type de précarité ne disparaît pas, les aides gouvernementales se révèlent insuffisantes, et les décideurs ne semblent pas être en mesure de s’impliquer pour aider tous les ménages en difficulté, ni généraliser l’usage des chèques à d’autres biens essentiels.

7. Voir https://onpe.org/sites/default/files/onpe_chiffres_cles_v6.pdf.

Il serait donc nécessaire de déterminer des mécanismes qui pourraient être proposés par d'autres acteurs du marché (assureur, fournisseur, producteur...), afin de compléter les aides de l'État. Pour pouvoir être implémentés, ces mécanismes doivent évidemment être profitables pour le secteur privé mais surtout avoir les effets souhaités par l'État, *i.e.*, permettre aux ménages en précarité de consommer suffisamment d'énergie, voire de leur éviter cette précarité. Nous étudierons en particulier le côté préventif : étant donné qu'une cause majeure de la précarité énergétique relève essentiellement de la perte de revenus, l'idée est de développer un mécanisme d'assurance, fourni par le secteur privé, qui pourrait garantir au ménage assuré une consommation suffisante d'énergie en cas de perte de revenu.

Deux fournisseurs d'électricité français proposent chacun des options d'assurance légèrement différentes : *Assurénergie*, proposée par Electricité de France (EDF), et *Assurance Facture*, proposée par ENGIE. Ces deux assurances mensuelles offrent un remboursement d'une partie de la facture d'électricité en cas de perte d'emploi, de congé maladie, d'hospitalisation, d'invalidité ou de décès. Avec l'assurance Assurénergie, le montant remboursé dépend du contrat choisi dans le menu proposé, tandis que l'Assurance Facture est un contrat unique. Un exemple plus général est l'*Utilities Insurance*, proposée par la société canadienne Trans Global Insurance, qui permet aux ménages d'être couverts pour certains biens de base. Cet assureur propose un double menu de contrats, en ce sens que le ménage peut choisir à la fois le niveau de couverture (3 niveaux possibles) ainsi que les différents biens à couvrir (électricité, chauffage, eau, internet...). Il est important de souligner qu'offrir des menus de contrats est très classique dans le domaine de l'assurance, puisque cela contribue à atténuer les effets de la sélection adverse.

1.3.2.2 Sélection adverse en assurance

Compte tenu de l'idée susmentionnée, il est nécessaire de s'attarder sur la notion de sélection adverse, et particulièrement en assurance. La sélection adverse est une situation de marché dans laquelle l'un des participants possède une caractéristique inconnue de l'autre participant. Cette asymétrie d'information est *ex ante* la signature du contrat, en opposition à l'aléa moral. L'exemple canonique est celui du *Market for « lemons »* d'Akerlof [13] (1970), modélisant le marché des occasions automobiles. Il montre qu'en situation d'information imparfaite, le risque de sélection adverse (ou antisélection) va conduire à un marché de l'occasion qui ne propose que des voitures avec des vices cachés. En effet, le prix moyen sera la résultante de la moyenne des prix des bonnes et mauvaises occasions. Comme le consommateur est en situation d'information imparfaite (il ne reconnaît pas les bonnes voitures des mauvaises), il ne sera prêt à payer que le prix moyen pour minimiser ses risques. Or, si les vendeurs de mauvaises occasions ne pourront que se réjouir, les vendeurs de bonnes occasions, n'ayant pas d'espoir de vendre, sortiront de ce marché : seuls resteront les vendeurs de mauvaises voitures. Finalement, les acheteurs conscients de ceci se désintéresseront à leur tour du marché de l'occasion.

Dans le cas qui nous intéresse, *i.e.*, celui d'une assurance en cas de perte de revenus, l'asymétrie d'information réside dans le risque auquel l'assuré est exposé. En effet, il semble raisonnable de supposer que l'assuré en sait plus sur son propre niveau de risque de perte de revenus que la compagnie d'assurance. Ainsi, les risques étant différents au sein de la population considérée, il sera nécessaire pour l'assureur de proposer un menu de contrats : différents types d'assurance devraient être proposés pour permettre à chaque ménage de choisir le contrat le mieux adapté à ses besoins et à son risque. Cette offre de menu de contrats, traditionnelle dans le domaine de l'assurance (voir l'enquête de Dionne, Fombaron et Doherty [125] (2013)), permettra à l'assureur de remédier partiellement à l'asymétrie d'information entre lui et l'assuré.

Les modèles d'assurance avec sélection adverse peuvent être divisés en deux catégories, selon le statut de l'assureur. Dans la première catégorie, les auteurs considèrent l'assurance comme un modèle de concurrence pure et parfaite entre les compagnies d'assurance, impliquant que le prix de l'assurance est fixé de telle

sorte que les assureurs ne réalisent aucun profit. L'un des principaux modèles de cette tendance est le modèle de Rothschild et Stiglitz [297] (1976) et ses diverses extensions, en particulier celles de Boone [65] (2015), Chassagnon et Chiappori [96] (1997), De Donder et Hindriks [114] (2009), Cook et Graham [102] (1977), Alary et Bien [14] (2008), Janssen et Karamychev [194] (2005) et l'enquête de Mimra et Wambach [248] (2014). Nous nous intéresserons plus particulièrement à la seconde catégorie, celle où l'assureur est en position de monopole. Parmi la littérature sur ces modèles avec un assureur monopolistique, nous pouvons notamment citer l'extension du modèle de Rothschild et Stiglitz à un monopole par Stiglitz [323] (1977) et, plus généralement, la catégorie des modèles avec sélection adverse (hors domaine de l'assurance) développée par Mirrlees [249] (1971), Spence [321] (1974), Guesnerie et Laffont [168] (1984), Salanié [300] (2005), Laffont et Martimort [218] (2009), ou plus récemment par Guasoni et Wang [165] (2018), et leurs extensions aux multiproduits par Armstrong [27] (1996) et Rochet et Choné [293] (1998), ou au temps continu comme par Alasseur, Ekeland, Élie, Hernández Santibáñez et Possamaï [17] (2020).

Enfin, conformément au thème global de cette thèse, nous serons également particulièrement inspiré par les modèles de théorie des contrats modélisant la sélection adverse, notamment par les travaux pionniers de Baron et Myerson [37] (1982), Guesnerie et Laffont [168] (1984), Maskin et Riley [237] (1984) et les quelques applications aux problèmes d'assurance par Stiglitz [323] (1977), ou Landsberger et Meilijson dans [222] (1994), [223] (1996) et [224] (1999), ainsi que par les travaux plus récents de Janssen et Karamychev [194] (2005), Chade et Schlee [94] (2012), Diasakos et Koufopoulos [123] (2018) et Bensalem, Hernández Santibáñez et Kazi-Tani [49] (2020).

1.3.2.3 Contribution

En utilisant un cadre principal-agent, nous souhaitons donc modéliser dans le [Chapitre 6](#) un problème d'assurance en présence de sélection adverse, afin de protéger un ménage modeste et vulnérable d'une éventuelle perte de revenus, qui conduirait à une réduction de sa consommation d'un bien essentiel particulier. Rappelons que les biens essentiels sont des produits que les consommateurs ne peuvent ou ne veulent pas supprimer de leur budget, comme la nourriture et l'eau. Ils ont tendance à consommer ces biens à un niveau relativement constant, indépendamment de leur situation financière ou du prix des marchandises. Toutefois, en cas de perte de revenus importante, les consommateurs réduiront le budget alloué à certains biens de base, tels que les aliments frais, l'énergie, les médicaments ou encore les produits d'hygiène féminine (précarité menstruelle), qui peuvent entraîner des maladies graves. En effet, ces biens de base particuliers, contrairement à la nourriture ou à l'eau, peuvent ne pas sembler essentiels aux consommateurs en situation de précarité : un ménage, confronté au choix de se chauffer, de se soigner ou de manger, fera systématiquement le choix de manger. Cette question a fait l'objet de nombreuses enquêtes et discussions sur ce type de biens qui, pour les ménages en situation très précaire, peuvent être considérés, en un sens, comme un luxe. Nous pouvons notamment citer les travaux de Milne et Molana [247] (1991), Freeman [149] (2003) sur les soins de santé et de Meier, Jamasb et Orea [242] (2013), Schulte et Heindl [308] (2017) sur l'énergie. Nous considérons dans ce document ce type particulier de biens de base, dont la consommation est fortement influencée par les pertes de revenus. Cependant, la motivation principale est d'appliquer cette assurance au bien énergie, pour proposer un mécanisme complémentaire de lutte contre la précarité énergétique.

Nous choisissons de nous concentrer sur les ménages modestes et vulnérables, en particulier ceux qui n'ont pas l'habitude d'épargner en prévision de l'avenir. Comme expliqué ci-dessus, une perte de revenus les obligerait donc à réduire leur consommation de certains produits de base particuliers, comme l'énergie, ce qui pourrait entraîner des problèmes de santé (voir Lacroix et Chaton [215] (2015)). Une solution courante consiste à offrir une aide financière, *i.e.* à distribuer de l'argent au ménage en cas de perte de revenu. Cependant, dans la situation considérée, le ménage dépensera l'argent reçu pour acheter des produits qui

sont plus essentiels de son point de vue. Ce type de mécanisme ne semble donc pas être une solution appropriée au problème auquel nous sommes confrontés. Notre proposition est donc un soutien en nature plutôt qu'une aide financière : si le ménage souffre d'une perte de revenus, il reçoit une quantité déterminée du bien particulier considéré, ce qui lui assure une consommation adéquate de ce bien. Cette approche, qui pourrait être perçue comme paternaliste, devrait protéger les individus des effets négatifs de certaines de leurs décisions et ainsi prévenir les problèmes de santé liés à l'absence de ce bien. En effet, de nombreux auteurs, comme Blackorby et Donaldson [62] (1988) ou Slesnick [313] (1996), montrent l'efficacité du soutien en nature par rapport au soutien financier dans la lutte contre la pauvreté.

Afin d'évaluer les bénéfices d'une assurance pour les ménages, nous choisissons d'étudier le scénario qui serait le moins efficace de leurs points de vue : nous supposons que l'assureur est un monopole. L'étude de ce scénario de monopole est également motivée par l'application que nous avons à l'esprit, à savoir la précarité énergétique. En effet, dans cette situation, l'assureur le mieux adapté est le fournisseur d'énergie actuel du client, qui le connaît mieux que les autres entreprises. Néanmoins, même si l'assureur est un monopole, le ménage a le choix de souscrire ou non une police d'assurance parmi celles proposées par le monopole. Par conséquent, nous supposons que le ménage refusera tout contrat si aucun ne lui fournit plus d'utilité que celle attendue sans assurance, définie comme son *utilité de réservation*, qui est donc endogène et, en particulier, dépend de son risque. Le problème de l'utilité de réservation endogène est étudié dans certains modèles de sélection adverse, tels que ceux de Lewis et Sappington [230] (1989), Biglaiser et Mezzetti [60] (1993), Maggi et Rodriguez-Clare [236] (1995), Jullien [199] (2000), ou encore Guasoni et Wang [165] (2018), et Alasseur, Ekeland, Élie, Hernández Santibáñez et Possamaï [17] (2020) discutent également de cette question pour une application proche de celle que nous avons à l'esprit. Toutefois, à notre connaissance, ce problème est rarement abordé pour un continuum de types dans les modèles d'assurance, comme dans notre cadre. D'autre part, rappelons que nous supposons ici que seul l'agent connaît son risque de perte de revenus, le principal ne connaît elle que la répartition de ces risques au sein de la population considérée. Cette hypothèse est classique dans la littérature, en particulier dans toutes les extensions de Rothschild et Stiglitz [297] (1976) et Stiglitz [323] (1977). Pour atténuer les effets de cette sélection adverse, le principal doit déterminer de manière optimale un menu de contrats : différents types d'assurance devraient être proposés pour permettre à chaque ménage de choisir le contrat le mieux adapté à son risque, et ainsi révéler ce risque.

Enfin, l'une des pierres angulaires de notre approche est de combiner un modèle de consommation (deux biens, sous contrainte budgétaire) avec un modèle d'assurance sous sélection adverse. Une autre particularité de notre méthode réside dans le choix d'un modèle d'assurance à deux périodes, comme par Schlesinger et Zhuang [306] (2014), [307] (2019). La plupart des ouvrages sur l'assurance avec sélection adverse se concentrent sur une seule période, où l'agent paie et reçoit l'assurance en même temps, ou sur des versions répétées de ce régime. L'application la plus proche dans laquelle ce type de modèle à deux périodes a été développé est sur le thème de l'auto-prévention. Dans les travaux de Eeckhoudt, Huang et Tzeng [133] (2012), Wang et Li [338] (2015), Peter [277] (2017) (modèle avec épargne) et Menegatti [244] (2009), Courbage et Rey [104] (2012) (sans épargne), des modèles à deux périodes sont considérés pour tenir compte du délai entre l'effort de prévention et son bénéfice réel. Dans notre cadre, un modèle à deux périodes est nécessaire pour modéliser le fait que l'assuré ne se trouve pas dans une situation précaire lorsqu'il souscrit au plan d'assurance destiné à le couvrir sur la période suivante. La prise en compte d'un modèle à deux périodes nous permet également de comparer, avec et sans assurance, l'évolution de la consommation de l'agent lorsqu'il subit une perte de revenus.

En résumé, le principal (une compagnie d'assurance, un fournisseur...), en situation de monopole, peut proposer à l'agent (un ménage) une assurance lui permettant de recevoir un montant fixe $e_{\min}$ d'un bien de première nécessité en cas de perte de revenus. Le principal fait face à un phénomène d'antisélection, car elle

ne connaît pas le type de l'agent, à savoir son risque de perte de revenus, mais connaît la répartition des risques au sein de la population considérée. L'utilité de réservation de l'agent est définie comme son utilité sans assurance, et dépend donc de son type. L'objectif de ce travail est de trouver la conception optimale du menu de contrats pour l'assurance considérée, et d'étudier le choix optimal de contrat de l'agent. Afin d'énoncer plus précisément les résultats que nous obtenons, il est nécessaire de décrire brièvement le modèle.

À l'instant $t \in \{0, 1\}$, l'agent dispose d'un revenu w_t qui lui permet de consommer :

- une quantité $e_t > 0$ du bien de base considéré au prix unitaire p_e ,
- une quantité $y_t > 0$ d'un autre bien, représentant tous les autres biens, au prix unitaire p_y .

Entre $t = 0$ et $t = 1$, l'agent est confronté à un risque de perte de revenus : nous définissons le revenu aléatoire d'un agent de type $\varepsilon \in [0, 1]$ au temps $t = 1$ par $w_1 := \omega w_0$ où w_0 est le revenu au temps $t = 0$ et

$$\omega := \begin{cases} \omega^h & \text{avec probabilité } 1 - \varepsilon, \\ \omega^\ell & \text{avec probabilité } \varepsilon, \end{cases}$$

où $\omega^h > \omega^\ell > 0$. Si l'agent de type ε décide de souscrire au contrat d'assurance $(e_{\min}, T)$, où T correspond au premium, alors l'assurance s'enclenche en $t = 1$ si $\omega = \omega^\ell$, et son utilité espérée est donnée par :

$$\begin{aligned} \text{EU}^Q(\varepsilon, e_{\min}, T) &= \max_{(e_0, y_0) \in \mathbb{R}_+^2} \{ \alpha \ln(e_0) + \ln(y_0) \} + \max_{(e_1, y_1) \in \mathbb{R}_+^2} \mathbb{E}^\varepsilon [\alpha \ln(e_1 + e_{\min} \mathbb{1}_{\omega \leq \omega^\ell}) + \ln(y_1)] \\ \text{s.c. } T + e_0 p_e + y_0 p_y &\leq w_0 \text{ et } e_1 p_e + y_1 p_y \leq \omega w_0. \end{aligned}$$

Traditionnellement, dans les modèles de sélection adverse (voir Salanié [300] (2005) pour la théorie générale), le contrat proposé par le principal doit être *incentive-compatible* (IC) : un agent de type ε doit avoir intérêt à souscrire le contrat qui lui correspond, et révéler ainsi son type ε , jusque là inconnu du principal. En utilisant cette contrainte, nous pouvons prouver l'un de nos principaux résultats : un menu de contrats $(e_{\min}, T)$ est IC si et seulement s'il existe une fonction continue non décroissante q (suffisamment régulière) telle que le prix T pour une quantité $e_{\min}$ est déterminé à une constante près $c_q \geq 0$. Cette constante c_q est en fait liée à la *contrainte de participation*. Rappelons que, par définition dans notre cadre, un agent de type $\varepsilon \in [0, 1]$ acceptera le contrat $(e_{\min}, T)$ s'il lui procure une utilité plus grande que sans assurance, *i.e.* si $\text{EU}^Q(\varepsilon, e_{\min}, T) \geq \text{EU}^Q(\varepsilon, 0, 0)$. Un autre résultat important que nous pouvons alors affirmer est que, en contrôlant la constante c_q , le principal peut choisir de sélectionner ou non les agents de type plus petit. Autrement dit, les types sélectionnés constituent un intervalle $[\underline{\varepsilon}, 1]$, pour un niveau $\underline{\varepsilon}$ déterminé par le principal, à travers le choix de c_q .

En conclusion, en partant d'un problème économique classique de maximisation de l'utilité, et en y ajoutant un problème d'assurance, nous pouvons explicitement écrire la classe de contrats IC et satisfaisant également la contrainte de participation, bien que l'utilité de réservation soit endogène. Enfin, nous pouvons affirmer que le problème de maximisation d'un principal neutre au risque peut être écrit sous la forme d'un problème de calcul de variation, et est donc équivalent à la résolution d'une équation différentielle ordinaire (EDO) non linéaire du second-ordre donnée par les conditions d'Euler-Lagrange. À notre connaissance, l'EDO que nous obtenons ne peut être résolue que numériquement.

En utilisant les paramètres estimés dans Chaton [97] (2020), nous pouvons appliquer ce raisonnement pour trouver un menu optimal de contrats dans le cadre d'une assurance contre la précarité énergétique, et mettre en évidence certains résultats prometteurs : ce type d'assurance pourrait permettre à un ménage de consommer une quantité d'énergie suffisante pour vivre dans des conditions convenables, malgré une perte de revenus conséquente. Ce type de mécanisme peut donc, même dans le scénario le moins efficace d'un assureur monopolistique, constituer une solution partielle pour protéger les ménages à haut risque de

la précarité énergétique. Néanmoins, nos résultats mettent en évidence une certaine forme d'irrationalité des agents, due à leur réticence à économiser de l'argent d'une période à l'autre. Nous mentionnons donc également dans le [Chapitre 6](#) un moyen de remédier à ce problème d'irrationalité, en proposant une option de prépaiement en plus de l'assurance.

1.3.3 Perspectives

Dans la continuité de l'application des problèmes principal-agent à l'énergie, nous aimerions modéliser entre autre l'aléa moral et les incitations optimales dans le cadre de la recharge des véhicules électriques. Plus précisément, nous voudrions développer un modèle proche de celui de Aïd, Possamaï et Touzi [11] (2019) et du cadre multi-agents du [Chapitre 3](#), pour modéliser un principal cherchant à encourager N sites de flottes de véhicules électriques à suivre une courbe de charge particulière. Chaque site $j \in \{1, \dots, N\}$ disposerait d'une flotte de véhicules électriques n_j , en charge entre 0 et T . À l'instant T , les batteries doivent être suffisamment chargées pour permettre l'utilisation du véhicule. Nous indiquons par $X_t^{j,i}$ l'état de la batterie du (j, i) -ième véhicule (i -ième véhicule du j -ième site) au temps $t \in [0, T]$. Nous supposons que pour tout $j \in \{1, \dots, m\}$ et $i \in \{1, \dots, n_j\}$, $X^{j,i}$ est un processus stochastique de dynamique :

$$dX_t^{j,i} = (\tilde{p}_t^{j,i} + \alpha_t^{j,i})dt + \sigma^{j,i}\sqrt{\beta_t^{j,i}}dW_t^{j,i}, \quad t \in [0, T],$$

où $\tilde{p}_t^{j,i}$ représente le profil de charge préféré du (j, i) -ième véhicule. Ainsi, le site j peut contrôler la courbe de charge de son parc de véhicules électriques en contrôlant la moyenne et la volatilité des $X^{j,i}$ par les efforts $\alpha^{j,i}$ et $\beta^{j,i}$ pour tout $i \in \{1, \dots, n_j\}$.

Nous supposerions alors que le principal n'observe pas les profils de tous les véhicules, mais uniquement les sommes des courbes de charges par site, notée $\bar{X}^j$ pour $j \in \{1, \dots, N\}$. D'autre part, elle est particulièrement intéressée par l'agrégation de la courbe de charge de tous les sites, à savoir la somme des $\bar{X}^j$. En suivant l'intuition que nous avons développée dans le cadre multi-agents d'une hiérarchie en [Section 1.2.2](#), le principal devrait proposer un contrat pour un site indexé sur sa propre courbe de charge, mais aussi sur celle des autres sites. De plus, en utilisant les résultats des [Chapitres 4 et 5](#) introduits notamment en [Section 1.3.1](#), nous pourrions également créer une interaction entre les sites en ajoutant un bruit commun, qui pourrait dans ce cas modéliser l'influence de la température sur la charge des véhicules électriques. La partie la plus conséquente serait d'ajouter les contraintes de roulement et de faisabilité : la batterie des véhicules doit être suffisamment chargée à la fin de la période considérée, afin de permettre leur utilisation, mais surtout le niveau de charge de la batterie doit évidemment rester dans un certain intervalle $[0, p_{\max}]$. Pour ajouter ces contraintes, il est nécessaire d'utiliser entre autres la théorie des *cibles stochastiques*, développée notamment par Soner and Touzi [315, 316] (2002), et utilisé dans le cadre du contrôle optimal par exemple par Bouchard, Élie et Imbert [66] (2010).

1.4 Application au contrôle d'une épidémie

En parallèle de l'énergie, la situation actuelle nous pousse à considérer dans la [Partie III](#) de cette thèse un autre domaine d'application des recherches entre jeux à champ moyen et théorie des contrats : l'épidémiologie. En effet, l'année 2020 est marquée, pour une grande partie de la population mondiale, par des mesures de confinement imposées par les gouvernements afin de limiter la propagation du virus SARS-CoV-2, médiatisée pandémie de COVID-19. Ces restrictions concernent principalement la réduction des interactions sociales. Elles sont mises en place à différentes échelles géographiques (habitations, villes, pays...) et ont déjà eu un impact économique mondial important. Tous les individus touchés par les mesures de contrôle de l'épidémie paient un coût monétaire, ou en termes d'interaction sociales, de santé (les rapports

montrent une sous-déclaration des maladies graves telles que les maladies du cœur, les accidents ischémiques cérébraux...), de pression psychologique (augmentation de la violence domestique)... Cependant, jusqu'à ce qu'une proportion suffisante de la population soit immunisée (par infection ou si un vaccin est mis au jour), et dans l'attente d'un traitement efficace contre cette maladie, le choix des mesures de contrôle ainsi qu'un suivi adéquat de la propagation du virus sont essentiels pour limiter l'épidémie.

1.4.1 Historique

Débutant vers 430 avant J.-C., et connue comme la première épidémie historiquement bien documentée, la peste d'Athènes a tué entre un quart et un tiers des Athéniens. Thucydide décrit la réaction des Athéniens et des médecins de l'époque en ces termes :

Rien n'y faisait, ni les médecins qui, soignant le mal pour la première fois, se trouvaient devant l'inconnu (et qui étaient même les plus nombreux à mourir, dans la mesure où ils approchaient le plus de malades), ni aucun autre moyen humain. De même, les supplications dans les sanctuaires, ou le recours aux oracles et autres possibilités de ce genre, tout restait inefficace : pour finir, ils renoncèrent, s'abandonnant au mal. (de Romilly [115, Volume II, Livre II, p. 34] (1962))

En analysant les conséquences de cette épidémie, Thucydide conclut que l'absence totale de tout remède utile avait suscité un bouleversement moral chez les Athéniens. En effet, le peuple a alors réalisé que les politiques traditionnelles, essentiellement religieuses, censées prévenir et guérir les tragédies, n'avaient en réalité aucun effet sur l'épidémie. En fin de compte, la maladie n'a pu être annihilée, au bout de quatre années, qu'à travers le développement d'une immunité naturelle au sein de la population. Concernant maintenant plus spécifiquement la propagation de la maladie elle-même, Thucydide écrit :

[L]a contagion, qui se communiquait au cours des soins mutuels et semait la mort comme dans un troupeau : c'est là ce qui faisait le plus de victimes. Si, par crainte, les gens refusaient de s'approcher les uns des autres, ils périssaient dans l'abandon, et bien des maisons furent ainsi vidées, faute de quelqu'un pour donner ses soins ; mais, s'ils s'approchaient, le mal les terrassait, surtout ceux qui prétendaient à quelque générosité, et qui, par respect humain, entraînaient, sans regarder à leur vie, auprès de leurs amis ; aussi bien, les proches eux-mêmes, pour finir, n'avaient seulement plus le force de pleurer ceux qui s'en allaient : l'ampleur du mal triomphait d'eux. (de Romilly [115, Volume II, Livre II, pp. 37–38] (1962))

De cette analyse de Thucydide sur la peste d'Athènes, nous pouvons extraire trois questions fondamentales qui doivent être abordées chaque fois qu'une épidémie inconnue se produit.

- (1) Comment peut-on modéliser une maladie quand nous ne disposons, au mieux, que d'informations parcellaires sur la façon dont elle se propage dans la population ?
- (2) Comment résoudre le nœud gordien associé aux interactions au sein de la population : d'une part, profiter de la présence des autres pour éviter l'isolement et la solitude, d'autre part, propager la maladie de manière potentiellement dramatique ?
- (3) Comment les gouvernements et décideurs peuvent-ils inciter la population à se comporter de telle sorte que le contrôle de la propagation de l'épidémie soit plus efficace ?

1.4.1.1 Sur la modélisation d'une épidémie

La première question est naturellement liée à plusieurs aspects de la recherche fondamentale, tant pour les mathématiciens que pour les médecins, abordant le problème du choix d'un modèle épidémique pertinent. La paternité du premier modèle mathématique destiné à décrire l'évolution d'une épidémie est souvent attribuée à Bernoulli [55] (1760) et concerne l'épidémie de variole. Cependant, le véritable développement

de la théorie, et plus précisément des modèles déterministes, n'a eu lieu qu'à partir du XX^{ème} siècle, avec les contributions notables de Hamer [171] (1906), Ross [294, 295] (1910, 1915), Soper [318] (1929); et plus tard par Kermack et McKendrick [205] (1927), McKendrick [241] (1940); et enfin Bartlett [39] (1949) qui a proposé l'une des premières études générales sur l'évolution des systèmes déterministes en interaction, ensuite appliquée à l'épidémiologie par Kendall [202] (1956). La liste précédente n'est en aucun cas exhaustive, et nous renvoyons le lecteur intéressé à la monographie de Bailey [32] (1975) pour plus de détails historiques.

Cependant, il a rapidement été constaté que les modèles déterministes étaient insuffisants pour tenir compte de l'incertitude liée à la propagation de la maladie, ainsi que des difficultés techniques habituellement rencontrées pour sa détection. Cette prise de conscience a favorisé l'émergence de modèles stochastiques, dont les premiers semblent être ceux de McKendrick [240] (1925) et Greenwood [160] (1931). Pour une comparaison précise entre les modèles déterministes et stochastiques, dans des cadres à temps discret, nous renvoyons le lecteur aux études de Bailey [32] (1975), Bartlett [40] (1956) et Allen et Burgin [19] (2000), ainsi qu'à Allen [18] (2008) pour des références plus à jour et un aperçu des modèles épidémiologiques récents. Nous allons maintenant décrire quelques modèles épidémiologiques spécifiques, appartenant à la classe générale des modèles compartimentaux, et qui seront au cœur de notre travail dans la [Partie III](#).

Le premier modèle, un des plus basiques, considère en quelque sorte le pire scénario, dans lequel une immunité n'est pas développée après l'infection. Ce modèle est particulièrement pertinent dans le cas de certaines infections sexuellement transmissibles ou pour les maladies bactériennes. Dans ce modèle, les personnes infectées, redeviennent, en guérissant, susceptibles de contracter à nouveau la maladie. Plus précisément, ces modèles appelés SIS (pour Susceptible – Infecté – Susceptible) considèrent une population divisée en deux groupes. Les individus dits infectés (de la classe I), peuvent contaminer les individus dits susceptibles (de la classe S), en interagissant avec eux. Une fois guéris, les individus retournent alors dans la classe S, et passent donc d'une classe à l'autre de manière répétée. Ce modèle peut-être étendu de manière relativement directe pour prendre en compte les dynamiques naturelles de la population, comme les naissances et morts. Ce modèle a été présenté pour la première fois par Weiss et Dishon [341] (1971), généralisant une version plus simpliste de Bailey [31] (1950), qui considérait alors uniquement des processus de naissance et mort. Elle a ensuite été étudiée plus en détail par Kryscio et Lefèvre [212] (1989), qui ont calculé le temps moyen d'extinction d'une infection. Ces modèles, à temps discret, ont ensuite été étendus par Näsell [262] (1996), [263] (1999), qui a trouvé la distribution quasi-stationnaire d'un modèle SIS stochastique à temps continu, mais sans prendre en compte les naissances et décès au sein de la population. Plus récemment, Gray, Greenhalgh, Hu, Mao et Pan [158] (2011) ont proposé de modéliser un processus SIS stochastique en temps continu, comme solution à une EDS bidimensionnelle impactée par un mouvement brownien unique.

Alternativement à ce scénario assez pessimiste, nous pouvons également supposer, ou savoir, qu'une immunité apparaîtra chez les individus ayant été contaminés. Dans ce cas, il est nécessaire de distinguer une classe additionnelle, représentant les individus rétablis, et sous-entendu immunisés. Dans ce modèle appelé SIR, les susceptibles (classe S) peuvent être contaminés, les infectés (I) sont actuellement atteints par la maladie, et enfin les personnes guéries (R) ont développé des anticorps, et sont donc maintenant immunisés. Introduit à l'origine par Kermack et McKendrick [205] (1927), ce type de modèle a été étudié en détail par Anderson et May [23] (1979) dans un cadre déterministe, et par Beretta, Kolmanovskii et Shaikhet [54] (1998) dans un cadre stochastique. La modélisation d'un processus SIR stochastique comme la solution à une EDS induite par un mouvement brownien a ensuite été proposée par Tornatore, Buccellato et Vetro [333] (2005) et Jiang, Yu, Ji et Shi [197] (2011).

Pour une modélisation plus spécifique à certaines épidémies, de nombreux auteurs ont étendu ces modèles basiques SIS et SIR. En particulier, dans le cas de la pandémie de COVID-19, pour tenir compte de la

phase de latence relativement longue de la maladie, Bacaër [30] (2020) et Dolbeault et Turinici [130] (2020) considèrent un modèle SEIR, dans lequel un individu qui contracte la maladie ne devient pas immédiatement contagieux. L'individu est alors considéré comme *exposé*, *i.e.*, infecté mais pas infectieux, et ne peut donc pas encore contaminer d'autres individus. Selon la durée de latence de la maladie, il devient ensuite infecté au sens contaminé et contagieux, et enfin rétabli. De même que pour les modèles SIS/SIR, une autre variante de ce modèle, appelé SEIRS, considère qu'il n'y a qu'une immunité partielle, et que les individus remis peuvent revenir dans la classe des susceptibles. Dans le cadre du Chapitre 8, nous considérerons les versions stochastiques en temps continu de ces modèles, et utiliserons donc celles introduites par Mummert et Otunuga [260] (2019). Il est important de souligner qu'il existe évidemment une grande variété de modèles qu'il n'est pas possible de mentionner ici dans son intégralité. En effet, ne serait-ce que pour modéliser l'épidémie de COVID-19, des modèles particulièrement exhaustifs ont été développés, afin de prendre en considération toutes les particularités de la propagation de l'épidémie, comme le SLIADRE examiné par Danchin, Ng et Turinici [113] (2020) ; mais aussi pour prendre en compte son contrôle, via l'hospitalisation, l'isolation et les tests, comme le SIDUHR+/- développé par Charpentier, Élie, Laurière et Tran [95] (2020).

1.4.1.2 Sur le contrôle d'une épidémie

Les différents modèles qui nous seront utiles dans la Partie III ayant été présentés, nous pouvons maintenant nous concentrer davantage sur la deuxième question, liée à la propagation de la maladie par les interactions au sein de la population. Dans la plupart des modèles, l'infection se propage dans la population par un taux d'incidence β , et proportionnellement au produit du nombre d'individus susceptibles et infectés. En l'absence de remède ou de vaccin, ce taux de transmission apparaît comme la seule variable de contrôle des individus ou des institutions publiques, pour endiguer la propagation d'une épidémie. En effet, bien que ce taux β soit en partie déterminé par les caractéristiques intrinsèques de la maladie (niveau de contagiosité par exemple), il peut être impacté par des actions de la population : en réduisant les interactions entre les individus, il est possible de diminuer le taux de transmission effectif de la maladie. En effet, en poussant le raisonnement à l'extrême, il est clair que si aucun individu n'interagit, la maladie ne peut se propager, quelle que soit sa contagiosité. Nous aborderons donc la deuxième question sous l'angle de la théorie du contrôle. Au cœur de cette approche se trouve l'idée simple que face à une épidémie, une population parfaitement rationnelle tentera de trouver un taux d'interaction équilibré, en conciliant la nécessité de rester en contact avec les autres et la peur naturelle de propager l'infection.

Ce point de vue n'est en rien nouveau, et les articles traitant de l'utilisation de la théorie du contrôle formel en épidémiologie remontent aux années 1970. Les premiers contributeurs à cet égard semblent être Taylor [329] (1968), Jaquette [195] (1970), Sanders [302] (1971), ou encore Gupta et Rink [169] (1971), [170] (1973). Le contrôle optimal d'une épidémie est souvent modélisé par une combinaison de stratégies d'isolement et de vaccination. En particulier, les travaux d'Abakuks dans [6] (1973) et [7] (1974) se concentrent sur le moment optimal de vaccination des individus, en combinaison avec une stratégie d'isolement immédiate et sans coût. Des impacts et contraintes plus réalistes sur les stratégies de quarantaine et d'isolement ont ensuite été pris en compte par Morton et Wickwire [257] (1974), Wickwire [343] (1975) ou encore Sethi et Staats [310] (1978). L'objectif principal de ces politiques est de contrôler le taux de reproduction de l'épidémie (voir Perasso [276] (2018)) tout en limitant l'impact social et économique. Plus récemment, et davantage en phase avec notre objectif, nous pouvons référer aux travaux de Behncke [45] (2000) et Riley et al. [291] (2003), qui ont étudié l'impact du contrôle du taux de transmission sur l'épidémie de SRAS de 2002–2004 à Hong Kong. Sur les différents moyens d'interférer avec la propagation d'une épidémie, nous pouvons également citer les travaux de Piunovskiy et Clancy [282] (2008), Hansen et Day [172] (2011), Fenichel et al. [144] (2011), Kandhway et Kuri [200] (2014), Sélley, Besenyei, Kiss et Simon [309] (2015), et plus largement à la monographie de Lenhart et Workman [229] (2007).

Une part importante de la littérature récente sur la pandémie COVID-19 a également adopté le formalisme du contrôle, pour proposer des solutions afin de ralentir la propagation de cette épidémie. Ces mesures ainsi que leurs impacts médicaux, sociétaux et économiques sont notamment examinés par Alvarez, Argente et Lippi [20] (2020), Anderson, Heesterbeek, Klinkenberg et Hollingsworth [24] (2020), Bonnans et Gianatti [64] (2020), Charpentier, Élie, Laurière et Tran [95] (2020), Colbourn [101] (2020), Del Rio et Malani [117] (2020), Djidjou-Demasse, Michalakis, Choisy, Sofonea et Alizon [128] (2020), Ferguson et al. [145] (2020), Fowler, Hill, Levin et Obradovich [148] (2020), Grigorieva, Khailov et Korobeinikov [161] (2020), Hatchimonji, Swendiman et Seamon [175] (2020), Kantner [201] (2020), Ketcheson [206] (2020), Piguillem et Shi [281] (2020), Thunström, Newbold, Finnoff, Ashworth et Shogren [330] (2020), Toda [332] (2020), ou encore par Wilder-Smith, Chiew et Lee [344] (2020). Un exemple révélateur dans la liste ci-dessus est le rapport de l'Imperial College London par Ferguson et al., qui évalue l'impact des interventions non pharmaceutiques visant à réduire le taux de contact au sein d'une population pour la pandémie de COVID-19. Ils distinguent les stratégies d'atténuation, *i.e.* la réduction des pics d'hospitalisation en protégeant les personnes les plus susceptibles d'être infectées, des stratégies de suppression, *i.e.* l'inversion de la croissance de la maladie grâce à des mesures d'isolement pour l'ensemble de la population. Leur conclusion est qu'il faut privilégier les stratégies de suppression, car une politique d'atténuation ne contribuera qu'à limiter les décès ainsi que la saturation des systèmes de santé.

1.4.1.3 Différents point de vue

Un aspect important, et légèrement irréaliste, du cadre de la plupart des articles précédemment évoqués est qu'ils considèrent le point de vue d'un planificateur social, pouvant contrôler le taux d'interaction de la population, et donc indirectement le taux de transmission de la maladie. Bien qu'il semble raisonnable de supposer qu'au moins certains individus, ayant peur de tomber malades, diminueront naturellement leur taux d'interaction, il est en réalité rare que les décideurs ou pouvoirs publics aient la possibilité de contrôler les actions de la population. Selon notre opinion, les articles précédents négligent donc deux particularités importantes. La première est que ce sont avant tout les individus qui choisissent leur taux de contact avec les autres individus potentiellement contagieux, en comparant d'une part le risque qu'ils encourent s'ils développent la maladie et, d'autre part, le coût induit par la réduction de leurs interactions. Il est donc nécessaire de regarder le problème du point de vue des individus. Ceci nous amène alors vers la deuxième particularité, qui rejoint la troisième question formulée au début de cette section, à savoir que si les pouvoirs publics souhaitent contrôler l'épidémie, il faut considérer l'aléa moral associé au problème de contrôle. Ainsi, au vu des réflexions précédentes, nous aborderons dans la [Partie III](#) la question du contrôle d'une épidémie de deux point de vue différents.

Nous serons alors inspiré par une autre partie de la littérature en épidémiologie, qui tente de prendre en compte la réponse comportementale individuelle à la politique d'isolement dans un tel contexte. Nous pouvons notamment citer à cet égard les travaux de Fenichel et al. [144] (2011), Sahneh, Chowdhury et Scoglio [299] (2012), Rizzo, Frasca et Porfiri [292] (2014), d'Onofrio, Manfredi et Salinelli [131, 132] (2007, 2008), Buonomo, d'Onofrio et Lacitignola [71] (2008) et Wang et al. [339] (2016). Cette modélisation de la réponse individuelle est bien sûr fortement influencée par les habitudes culturelles ainsi que par le besoin sociétal, économique ou religieux, d'interactions sociales. L'ajout d'un tel effet de rétroaction individuel sur la politique gouvernementale de quarantaine est essentiel dans la modélisation du contrôle optimal de la dynamique épidémique. À cet égard, nous considérerons dans le [Chapitre 7](#) le point de vue des individus, et modéliserons donc leurs interactions et processus de prise de décision face à une épidémie, ainsi que la dynamique d'épidémie en résultant. Une brève description de ce chapitre sera donnée dans la section suivante, *i.e.* [Section 1.4.2](#).

Enfin, à la lumière des questions que nous avons soulevées, une conclusion naturelle semble s'imposer : même si une approche théorique de lutte pour atténuer l'impact d'une épidémie est clairement souhaitable, il n'y a, *a priori*, aucune preuve que face à des politiques publiques claires, une population adoptera directement un comportement de distanciation sociale conduisant à un taux de transmission optimal du point de vue de la société. En outre, en l'absence d'un système permettant de suivre réellement le niveau d'interaction au sein de la population, les gouvernements sont confrontés à une situation évidente d'aléa moral. Par conséquent, une politique d'incitation devrait également être calibrée par les gouvernements. Cela nous amène, comme prévu, à la troisième question, qui est celle où notre approche s'écarte considérablement de la littérature existante. Cette approche sera brièvement décrite en [Section 1.4.3](#), et développée tout au long du [Chapitre 8](#). Il convient cependant de souligner ici que plusieurs pays dans le monde ont décidé d'utiliser des outils de recherche des contacts, tels que des applications pour téléphones portables⁸, conçues pour aider à retracer les contacts d'une personne ayant été identifiée comme infectée ultérieurement (voir par exemple les études de [Cho, Ippolito et Yu \[99\] \(2020\)](#) ou [Reichert, Brack et Scheuermann \[289\] \(2020\)](#)). Leur utilisation limiterait en principe l'aléa moral de cette situation, à condition qu'une proportion suffisante de la population utilise l'application de manière fiable, et que des tests soient organisés à grande échelle. Même en admettant que ce soit le cas, ces outils ont cependant soulevé des questions complexes de vie privée, mentionnés entre autres par [Ienca et Vayena \[191\] \(2020\)](#) et [Park, Choi et Ko \[273\] \(2020\)](#), et font donc polémique. En tout état de cause, l'approche incitative que nous proposerons peut toujours être considérée comme un complément utile à toute autre stratégie adoptée.

1.4.2 Du point de vue individuel...

Nous abordons dans le [Chapitre 7](#) la question du contrôle de l'épidémie en utilisant une approche basée sur l'individu. Pour ce faire, nous modélisons l'épidémie à travers un modèle compartimenté SIR (Susceptible – Infecté – Rétabli), dont la dynamique est donnée par :

$$\begin{cases} dS_t = -\bar{\beta}_t S_t I_t dt \\ dI_t = \bar{\beta}_t S_t I_t dt - \gamma I_t dt \\ dR_t = \gamma I_t dt, \end{cases} \quad t \in [0, T], \quad (1.4.1)$$

étant donnée une distribution initiale (S_0, I_0, R_0) des individus au temps 0, supposée connue. La variable $T > 0$ sera déterminée de manière à ce que l'épidémie soit considérée comme terminée à ce moment. Pendant la propagation de l'épidémie, chaque individu peut être soit « Susceptible », « Infecté » ou « Rétabli », et le triplet (S_t, I_t, R_t) représente la proportion de chaque catégorie au moment $t \geq 0$. La dynamique présentée ci-dessus fait intervenir deux paramètres : alors que la constante $\gamma > 0$ indique moralement la vitesse à laquelle un individu infecté est guéri de la maladie, le paramètre le plus important dans notre contexte est $\bar{\beta} > 0$, qui modélise le taux de transmission de la maladie au sein de la population considérée.

Ce taux de transmission $\bar{\beta}$ résulte, dans notre approche, des choix individuels. Plus précisément, nous allons supposer que chaque individu choisit, de manière optimale et pendant toute la durée de l'épidémie, son taux de contact β avec les autres individus. Pour le choisir, l'individu compare d'une part le coût lié à l'infection, multiplié par la probabilité d'être infecté, et d'autre part le coût associé à la réduction de ces interactions sociales. L'agrégation des taux de contact de chacun induit un taux de transmission global $\bar{\beta}$ de la maladie au sein de la population, et donc une dynamique pour l'épidémie. Ainsi, d'une part, le taux de transmission du virus $\bar{\beta}$ influe sur les choix individuels β à travers la probabilité d'infection. D'autre part, le taux de transmission global résulte de l'agrégation des choix individuels. Cela signifie entre autre que le taux

8. Nous pouvons citer dans le cas français l'application StopCovid, lancée en mai 2020.

de transmission, généralement constant et exogène dans les modèles SIR standards, est endogène et dépend du temps dans notre cadre d'étude. Dans ce contexte, chaque individu doit donc trouver un équilibre entre les résultats individuels et collectifs de son comportement, lorsqu'il choisit son niveau d'interaction sociale avec la population. Par exemple, si l'ensemble de la population exerce très peu d'effort pour limiter la propagation de la maladie, l'individu peut, par peur d'être contaminé, diminuer sensiblement ses interactions. À l'inverse, si les autres individus font des efforts conséquents, il peut être tenté d'utiliser la stratégie dite du *free lunch*, *i.e.*, profiter de la faible activité épidémique, due aux efforts des autres, tout en ne contribuant pas à l'effort lui-même. En pratique, un équilibre se forme entre les individus et nous l'analysons dans ce travail.

Chaque individu peut donc choisir de réduire son taux de contact, afin de minimiser sa probabilité d'être infecté, mais cela a un coût, en terme de bien être social, économique et même de santé. L'impact sur l'ensemble de l'épidémie d'un seul individu est négligeable, mais le comportement cumulé de tous les individus détermine l'évolution de l'épidémie. Cette évolution est formalisée par une approche de type champ moyen. Parmi les travaux qui considèrent la propagation d'une épidémie comme le résultat de l'agrégation des choix individuels, nous pouvons citer ceux de Bauch et Earn [43] (2004), Laguzet et Turinici [220] (2015), Laguzet, Turinici et Yahiaoui [221] (2016), Hubert et Turinici [190] (2018), Salvarani et Turinici [301] (2018). Cependant, ces travaux se concentrent sur l'impact des décisions individuelles de vaccination sur la dynamique de l'épidémie.

Dans le [Chapitre 7](#), nous suivons une approche similaire, mais étudions l'impact des décisions individuelles concernant la distanciation sociale et l'isolement, dans une dynamique épidémique où aucune vaccination n'est (encore) disponible. Dans ce contexte avec une infinité d'individus, cela nous amène donc à étudier l'existence d'un équilibre de Nash de ce jeu à champ moyen, *i.e.* une situation telle qu'aucun individu n'a intérêt à choisir un taux de contact différent du taux de contact sociétal global. Plus précisément, nous prouvons l'existence d'un taux de transmission β^* satisfaisant les contraintes de

- (i) *rationalité individuelle* : si la maladie se propage au taux $\bar{\beta} = \beta^*$, il est optimal pour un individu représentatif de choisir également ce taux de contact ;
- (ii) *cohérence au niveau de la population* : si tous les individus choisissent le taux de contact β^* , alors l'épidémie se propage au taux de transmission $\bar{\beta} = \beta^*$.

Nous effectuons ensuite des simulations numériques afin de trouver cet équilibre, qui semble être unique pour les cas testés. Ces expériences montrent que la stratégie d'équilibre d'auto-isolement se caractérise par une réduction rapide du taux de contact, suivie d'un retour plus lent au niveau d'interactions sociales habituel, noté β° , qui correspond de manière équivalente au taux de transmission naturel de la maladie, *i.e.* sans aucun effort de la population. Le taux de transmission induit par l'équilibre permet une nette amélioration de l'évolution de l'épidémie, par rapport à celle avec le taux de transmission initial β° . En particulier, malgré l'égoïsme des individus (qui ne cherchent qu'à minimiser leur propre coût), leurs efforts permettent de réduire de 25% le nombre de personnes touchées par la maladie. En outre, le pic d'infection est moins critique, ce qui limite, voire empêche, la saturation du système de soins de santé et implique donc indirectement une diminution du taux de mortalité de la maladie considérée.

Cependant, l'équilibre de Nash n'est évidemment pas le meilleur qui puisse être atteint en comparaison à une situation où les individus sont altruistes et ne voient que le bien de la société dans son ensemble. Pour quantifier cette différence, nous comparons avec une situation où un planificateur global, cherchant l'optimum social, est capable de contrôler tous les taux d'interaction entre les individus. Nous observons que la divergence entre les deux stratégies arrive à la fois avant et après le pic de l'épidémie. Plus précisément, l'équilibre de Nash permet, par des efforts prématurés des individus, de diminuer le pic d'infection, alors que le contrôle centralisé de l'épidémie entraîne une diminution plus rapide de la proportion d'individus

infectés après le pic, en maintenant des efforts intenses. En conséquence, il y a un *coût de l'anarchie* : l'équilibre de Nash induit évidemment un coût plus élevé que l'optimum social. Bien que l'arrêt précoce des efforts des individus, à un moment où l'épidémie diminue mais n'est pas encore terminée, puisse s'expliquer intuitivement par la nature égoïste des individus, le fait qu'ils décident collectivement de commencer les efforts plus tôt que l'optimum sociétal est une caractéristique plus intrigante. Ce fait est en réalité conforme aux résultats expérimentaux documentés par Ghader et al. [153] (2020).

Pour tenir compte de la longue phase de latence de la maladie induite par le COVID-19, nous choisissons, par extension, de considérer un modèle SEIR. Ce choix peut cependant être discuté, car la maladie induite par le COVID-19 possède de nombreuses autres caractéristiques complexes, comme par exemple un grand nombre de porteurs asymptomatiques. Nous référons aux travaux collaboratifs de Danchin, Ng et Turinici dans [266] (2003), [335] (2006), et [113] (2020), pour les modèles alternatifs déjà utilisés dans les épidémies de coronavirus. Il serait également possible de prendre en compte d'autres moyens de contrôler une épidémie, comme la vaccination, même si aucun vaccin n'est connu à ce jour. Enfin, nous considérons que tous les individus sont rationnels, identiques, et qu'ils ont une parfaite connaissance de la dynamique de l'épidémie. Nous pourrions envisager d'ajouter une certaine hétérogénéité au sein de la population, puisque la COVID-19 n'affecte pas les individus de la même manière, et est plus coûteuse pour ceux qui sont à risque.

Néanmoins, malgré la relative simplicité de notre modèle, le fait de prendre en compte les actions individuelles peut fournir des indications sur l'impact des décisions politiques et leur réception au sein d'une population. Toutefois, l'étude du coût de l'anarchie soulève la question des incitations, introduite par la troisième et dernière question, qui peut être reformulée comme suit : quels leviers les autorités sanitaires peuvent-elles utiliser pour rapprocher l'équilibre de Nash de l'optimum social ? En effet, de nombreux pays ont introduit des amendes, voire des peines de prison, en cas de non-respect du confinement. Le modèle développé dans le Chapitre 7 peut apporter des précisions en ce sens, mais invite à une étude plus approfondie d'un tel scénario, effectuée dans le Chapitre 8 et introduite dans la section suivante.

1.4.3 ... au point de vue des pouvoirs publics

La pandémie COVID-19 a souligné qu'une politique de contrôle doit être complétée par des sanctions, si les mesures de confinement ne sont pas respectées par la population. Cependant, cette politique est soumise à deux questions principales. Premièrement, quel que soit le nombre et le type de contrôles mis en place, il est impossible pour la plupart des pays de garantir l'application de ces mesures d'isolement, et il est donc impensable d'avoir un contrôle absolu sur le comportement de tous les individus et leurs interactions. Deuxièmement, il convient de trouver un équilibre entre la sévérité des sanctions ou d'autres types d'incitations visant à réduire la propagation de la maladie et le désir naturel des citoyens d'interagir.

À notre connaissance, aucun véritable calibrage, fondé sur des critères quantitatifs, des politiques d'incitation appropriées n'a été étudié dans les modèles épidémiologiques. Il existe un certain nombre d'articles qui étudient la propagation des maladies à travers le prisme de l'aléa moral ou de la sélection adverse. Cependant, ces articles s'intéressent surtout aux maladies liées au bétail, pour lesquelles les producteurs disposent naturellement d'informations privées sur les mesures préventives qu'ils ont pu adopter, avant la contamination (risque moral *ex ante*), et peuvent ou non déclarer si leur troupeau est infecté après la contamination (sélection adverse *ex post*). Ces questions et la conception de politiques appropriées sont examinées par exemple par Valeeva et Backus [336] (2007) ou par Gramig, Horan et Wolf dans [156] (2005) et [157] (2009), mais la problématique est complètement différente de celle qui nous intéresse.

Nous proposons donc dans le Chapitre 8 une voie pour combler ce manque, en étudiant comment une politique de confinement, considérée comme une stratégie de suppression pour faire écho au rapport déjà mentionné de Ferguson et al. [145] (2020), peut contribuer à limiter le nombre de personnes infectées pendant

une épidémie, tout en modélisant les incertitudes sur le nombre réel de personnes touchées, et sur leur niveau d'adhésion à une telle politique. Plus particulièrement, nous cherchons à résoudre ce problème d'aléa moral en trouvant :

- (i) le meilleur effort de la population pour réduire l'interaction en réaction à une politique gouvernementale spécifique donnée ;
- (ii) la politique optimale, composée à la fois d'une taxe payée par la population à une échéance fixe, et d'une politique de dépistage visant à réduire l'incertitude sur le nombre estimé de personnes infectées.

Comme déjà évoqué par Carmona et Wang [87, Section 5] (2018), ce problème s'inscrit parfaitement dans un cadre principal-agent classique avec aléa moral, et se résume à déterminer un équilibre de Stackelberg entre le principal (ici le gouvernement) proposant une politique à un agent (la population) pour interagir de façon optimale, afin de réduire la propagation de la maladie. Plus précisément, nous supposons que la dynamique de l'épidémie est donnée par un modèle SIR stochastique⁹ :

$$\begin{cases} S_t = s_0 - \int_0^t \beta_s \sqrt{\alpha_s} S_s I_s ds + \sigma \int_0^t \alpha_s S_s I_s dW_s, \\ I_t = i_0 - \int_0^t (\gamma - \beta_s \sqrt{\alpha_s} S_s) I_s ds - \sigma \int_0^t \alpha_s S_s I_s dW_s, \\ R_t = r_0 + \int_0^t \gamma I_s ds, \end{cases} \quad t \in [0, T].$$

Soulignons que nous considérons ici une dynamique stochastique, ce qui permet de représenter les incertitudes auxquelles les décideurs et la population sont soumis. En effet, face à une épidémie très mal connue comme celle du COVID-19, et sans politique de dépistage extrêmement développée, il paraît impossible d'imaginer que la dynamique de l'épidémie est connue avec exactitude. L'incertitude représentée par le mouvement brownien W témoigne donc de la fiabilité relative de l'estimation des paramètres de transmission de la maladie, et de l'incapacité à déterminer le statut S, I ou R de chaque individu à tout instant.

Dans le cadre de cette étude, la population contrôle toujours le taux de transmission β , mais cette fois de manière unanime, vu que nous ne considérons qu'un seul et unique agent représentant la population dans son ensemble. Le paramètre α est lui contrôlé par le principal, à travers une politique de dépistage associée à l'isolation des individus testés positifs. En effet, le contrôle α agit d'une part sur la volatilité globale du système : augmenter le nombre de tests (*i.e.* décroître α) permet de préciser la dynamique de l'épidémie, d'en diminuer l'incertitude. D'autre part, ce contrôle agit également sur le taux de transmission effectif de la maladie, à travers l'isolement des personnes détectées comme infectées. Plus précisément, si $\alpha = 1$, nous considérons que le gouvernement ne met pas en place de politique de dépistage spécifique. L'épidémie se propage donc au taux de transmission β choisi par la population, et l'incertitude est donnée par le paramètre σ . En instaurant une politique de dépistage $\alpha \in (0, 1)$, le taux de transmission effectif et l'incertitude du système diminuent respectivement vers $\beta\sqrt{\alpha}$ et $\sigma\alpha$. Ainsi, le problème du gouvernement est de trouver une politique de test optimale α , ainsi qu'un système de taxe χ , pour inciter la population à diminuer son taux de contact β , et ainsi limiter voire endiguer la propagation de l'épidémie.

Pour résoudre cet équilibre de Stackelberg, rappelons que l'idée de base développée par Cvitanić, Pos-samaï et Touzi [111] (2018) est d'identifier une sous-classe de contrats proposés par le principal, dits révélateurs, puis de prouver que la restriction à cette classe est sans perte de généralité. Avec cette approche, le problème auquel est confronté le principal devient un problème de contrôle optimal standard. Il y a cependant deux hypothèses fondamentales pour que cette approche fonctionne, l'une d'entre elles étant une

9. Par souci de lisibilité dans cette introduction, la dynamique présentée ici est en réalité une version simplifiée de celle étudiée dans le Chapitre 8.

condition de structure spécifique sur la dynamique contrôlée, *i.e.* ici le couple (S, I) donnant le nombre d'individus susceptibles et infectés dans la population. Cette condition impose que le drift de ce processus contrôlé par l'agent soit proportionnel à la matrice de volatilité. Du fait de la présence d'un unique mouvement brownien impactant le couple (S, I) , cette hypothèse fondamentale n'est pas satisfaite dans notre modèle. Nous ne pouvons donc pas nous appuyer directement sur les résultats existants pour résoudre notre problème. Dans ces cadres dits dégénérés, la littérature s'est jusqu'à présent appuyée sur le principe du maximum stochastique de Pontryagin, voir par exemple l'approche de Hu, Ren et Touzi [182] (2019), mais cela nécessite des hypothèses strictes, comme une dynamique linéaire, qui sont automatiquement exclues pour les modèles d'épidémiologie. Nous prouvons cependant que, dans notre cadre spécifique, il est possible d'identifier toute une famille de représentation pour le contrat (au lieu d'une représentation unique dans le cadre classique), qui est différente de celle obtenue dans [111], mais qui nous permet encore de réinterpréter le problème du principal comme un problème de contrôle stochastique standard. À notre connaissance, ce modèle est le premier de la littérature qui utilise une approche de programmation dynamique pour résoudre un problème principal-agent dégénéré, et cela constitue la principale contribution mathématique de ce chapitre.

Plus précisément, le résultat théorique majeur de cette étude stipule qu'étant donné un contrat admissible, à savoir une politique de dépistage α et une taxe χ , il existe une unique constante $y_0 \in \mathbb{R}$ et un unique processus Z tel que la représentation suivante est vérifiée :

$$U(-\chi) = y_0 - \int_0^T (\gamma Z_t I_t + u(t, \beta_t^*, I_t) - \beta_t^* \sqrt{\alpha_t} S_t I_t Z_t) dt - \int_0^T Z_t dI_t,$$

où β^* est l'unique taux de contact optimal pour la population, dont les préférences sont en partie représentées par la fonction d'utilité U et la fonction de coût u . Ainsi, sous certaines hypothèses d'existence et de régularité de l'inverse de la fonction U , l'équation précédente donne une représentation générale pour la taxe χ . Au travers de cette représentation, la taxe χ est donc indexée sur les variations infinitésimales de la proportion d'individus infectés, grâce au processus Z , choisi ensuite de manière optimale par le principal. Néanmoins, en utilisant le lien entre la dynamique de I et S , nous pouvons écrire une seconde représentation équivalente :

$$U(-\chi) = y_0 - \int_0^T (u(t, \beta_t^*, I_t) - \beta_t^* \sqrt{\alpha_t} S_t I_t Z_t) dt + \int_0^T Z_t dS_t.$$

Par conséquent, étant donné le lien étroit entre la proportion de susceptibles et d'infectés, la taxe peut être indexée de manière indifférente sur les variations d'une seule de ces deux quantités.

Au vu des formes de contrats exhibées tout au long de la [Section 1.1](#), le lecteur aura remarqué que la représentation précédente pour la taxe χ n'est pas exactement celle attendue, car elle n'est pas indexée sur les variations des deux processus S et I , mais seulement sur un seul. Toutefois, la forme que nous présentons pour la taxe est en fait fortement liée à la représentation habituelle. Comme mentionné précédemment, cela est dû au lien étroit entre les dynamiques de S et I . Premièrement, jusqu'au signe, les volatilités dans la dynamique de S et de I sont exactement les mêmes. Deuxièmement, les deux processus S et I sont animés par le même mouvement brownien W . Par conséquent, intuitivement, afin d'offrir des incitations à la population, le gouvernement peut se permettre d'indexer la taxe sur un seul des deux processus. Mathématiquement, il est également simple de montrer qu'étant donné une décomposition arbitraire du processus Z en $Z =: Z^s - Z^i$, nous obtenons la représentation suivante :

$$U(-\chi) = y_0 - \int_0^T \mathcal{H}(t, S_t, I_t, Z_t^s, Z_t^i) dt + \int_0^T Z_t^s dS_t + \int_0^T Z_t^i dI_t,$$

où $\mathcal{H}$ est l'Hamiltonien usuel de la population. Nous retrouvons alors la forme générale proposée dans [111]. La principale différence est que dans [111], Z^s et Z^i sont tous deux donnés de manière unique, alors que dans notre représentation, seule leur différence compte réellement. Par conséquent, il existe un nombre infini de représentations possibles pour la taxe χ dans notre modèle dégénéré.

Malheureusement, il est très compliqué d'extirper de notre modèle d'autres résultats théoriques, notamment sur la forme des contrôles Z et α optimaux. Il est donc nécessaire d'effectuer des simulations numériques. Afin de résoudre le problème considéré, nous devons nous appuyer sur une méthode permettant de traiter les équations d'Hamilton–Jacobi–Bellman dégénérées. Nous avons choisi de mettre en œuvre des schémas semi-lagrangiens, proposés pour la première fois par Camilli et Falcone [74] (1995). Il s'agit de schémas explicites, utilisant un pas de temps fixé Δt , et nécessitant une interpolation sur la grille des points où l'équation est résolue. Cette interpolation peut être soit linéaire, comme proposé initialement dans [74], soit invoquer des interpolateurs tronqués d'ordre supérieur, comme proposé par Warin [340] (2016), impliquant une convergence de la solution numérique vers la solution de viscosité du problème. Un point clé ici, qui rend l'approche délicate, est que le domaine sur lequel les EDP sont résolues n'est pas borné. Il est donc nécessaire de définir un domaine de résolution, sur lequel la solution numérique pourra effectivement être calculée. Ce domaine doit, d'une part, être suffisamment étendu, ce qui d'autre part crée des difficultés supplémentaires dans le traitement des conditions limites nouvellement introduites. Afin de traiter ces questions, nous utilisons de nombreuses techniques numériques récemment développées, entre autre dans [340], comme la parallélisation.

Les résultats numériques sont concluants, et confirment la pertinence d'une politique de taxe et de tests pour améliorer le contrôle d'une épidémie. En premier lieu, dans le cas de référence, considéré comme le cas où le gouvernement ne met pas en place de politique spécifique (*i.e.*, mathématiquement, pour $\chi = 0$ et $\alpha = 1$), les efforts de la population ne sont pas suffisants pour contenir l'épidémie. Cela confirme la nécessité de mettre en place des incitations. En effet, si une politique de taxe est mise en place, même en absence de politique de dépistage ($\alpha = 1$), la population est alors incitée à diminuer de manière significative ses interactions, ce qui permet de contenir l'épidémie jusqu'à la fin de la période considérée. Cependant, pour une durée de confinement fixée (numériquement à 200 jours), la population relâche ses efforts à la toute fin, entraînant une reprise de l'épidémie à ce moment-là. Enfin, dans le cas où le gouvernement ajuste également sa politique de dépistage $\alpha \in (0, 1]$, moins d'effort est demandé du côté de la population, les individus peuvent donc interagir de manière quasiment habituelle, et l'épidémie s'éteint en grande partie grâce à l'isolement ciblé des personnes infectées.

Étant donné la diversité des réponses des pouvoirs publics face à l'épidémie de COVID-19, il est intéressant de mettre ces différents résultats en relation avec les politiques effectivement appliquées. Le cas de référence correspond globalement à une inaction totale des gouvernements, comme ce fut le cas dans la plupart des pays au tout début de l'épidémie, certains individus se confinant alors volontairement par peur. La deuxième stratégie s'interprète comme un confinement suggéré par les états, dont le non-respect est passible d'une amende. Cette politique a pour but principal de retarder le pic épidémique, afin d'éviter une saturation des établissements de soins de santé, et permet gagner du temps pour préparer d'autres politiques plus efficaces et de long-terme. Dès lors que la fin du confinement s'amorce, il est nécessaire de mettre en place ces politiques plus sophistiquées, pour éviter une reprise des contaminations. La politique de dépistage fait partie de ses politiques de long-terme, car elle permet de ramener le niveau d'interaction à un rythme presque habituel, et autorise donc une reprise des activités, notamment économiques, du pays considéré. Cependant, cette politique n'est que très rarement applicable dès le début d'une épidémie, surtout si la maladie associée est peu connue, car elle nécessite à la fois une recherche conséquente et une logistique importante.

1.4.4 Perspectives

En résumé, le [Chapitre 7](#) propose une modélisation basée sur les choix individuels, et exhibe un équilibre de Nash entre tous les individus, dans le cas où les gouvernements émettent tout au plus des recommandations pour limiter la propagation de l'épidémie. Cette stratégie induit un coût de l'anarchie, l'équilibre de Nash n'étant pas optimal au niveau de la société. Cela nous conduit donc, dans le [Chapitre 8](#), à étudier des mécanismes incitatifs, imposés par le gouvernement à la population, afin d'encourager des efforts plus consciencieux de la part des individus. Les résultats présentés tout au long de ce chapitre permettent d'évaluer de manière chiffrée les différentes politiques publiques qui peuvent être mises en place afin d'empêcher la propagation d'une épidémie, ainsi que leur incidence sur la dynamique de l'épidémie elle-même.

Cependant, l'approche précédente néglige les interactions dans les choix individuels. Une perspective naturelle serait de combiner ces deux modèles, à la manière du problème principal-agent avec un continuum d'agent, mentionné en [Section 1.2.3](#) et développé dans les [Chapitres 4](#) et [5](#). Un tel modèle permettrait de mettre en lumière à la fois les interactions entre les individus, cause principale de la propagation d'une épidémie, et la réaction à différentes politiques publiques. Il est nécessaire de souligner que l'idée d'utiliser un modèle principal-agent avec un continuum d'agents pour modéliser le contrôle d'une épidémie n'est pas inédite, [Carmona et Wang \[87, Section 5\] \(2018\)](#) l'ayant déjà étudiée dans le cadre d'un espace d'état fini.

Dans un tel cadre d'étude, il serait intéressant de justifier le recours aux dynamiques stochastiques que nous considérons, ou de déterminer des dynamiques plus rationnelles. Cela peut être effectué en se concentrant dans un premier temps sur le problème d'un nombre fini d'individus, et en modélisant explicitement leurs interactions. A partir de ce modèle, il est alors possible, en utilisant des théorèmes limites, de construire un modèle plus pertinent (ou de justifier la pertinence de celui que nous considérons) pour une infinité d'agents. Cette démarche est mise en lumière dans les nombreux travaux très récents de [Pang et Pardoux \[269, 270, 271\] \(2020\)](#), ainsi qu'en collaboration avec [Forien \[146, 147\] \(2020\)](#).

D'autre part, les résultats obtenus dans le [Chapitre 8](#) mettent en évidence, dans certains cas, une résurgence de l'épidémie à la fin de la période considérée. Ces résultats sont une conséquence directe du choix du modèle, en particulier de la maturité fixe et finie. Le gouvernement ne s'intéresse à la dynamique de l'épidémie que sur la période $[0, T]$, et son critère d'optimisation ne fait intervenir aucune contrainte sur l'avenir. Une première idée pour remédier à ce problème serait d'ajouter au critère du principal une pénalisation sur l'état de l'épidémie au terme de la période considérée, pour tenir compte de sa volonté que l'épidémie soit maîtrisée au temps T . Cependant, ces résultats soulèvent plus largement la question du choix de la période de confinement $[0, T]$. Toujours dans le cadre d'un modèle similaire, une autre extension pertinente consisterait à autoriser le principal à déterminer de manière optimale les dates de début et de fin du confinement. Il n'y a en effet aucune raison que cette période soit fixée de manière exogène et indépendamment de la volonté des pouvoirs publics.

Dans un registre un peu différent, au vu du fort développement des applications destinées à retracer les contacts d'une personne testée à posteriori positive à la maladie, un modèle de jeu à champ moyen pourrait être développé dans l'optique de modéliser les choix individuels d'adhérer ou non à ce traçage. En effet, si une telle application est mise en place, mais qu'elle n'est pas suffisamment utilisée par la population, ses effets sont très limités. Il est nécessaire pour son bon fonctionnement qu'une proportion suffisante de la population l'utilise. S'en suit alors un jeu entre les individus, à la manière de celui développé dans le cas du confinement en [Chapitre 7](#), ou plus généralement dans la littérature sur la vaccination.

1.5 Perspectives générales

Tout au long de cette thèse, par souci de cohérence, nous nous concentrons sur la théorie de contrats et les jeux à champ moyen, appliqués soit au domaine de l'énergie, soit au contrôle d'une épidémie. Cependant, les problèmes principal-agent et la théorie des jeux à champ moyen peuvent être utilisés pour modéliser une grande variété de situations, notamment en finance et en assurance.

En ce qui concerne les problèmes principal-agent par exemple, l'extension au contrôle de la volatilité développée par Cvitanić, Possamaï et Touzi [111] (2018) peut être appliquée à la gestion des portefeuilles : dans [110], les mêmes auteurs présentent, à titre d'exemple concret, le problème de sélection de portefeuille de Merton, avec deux actifs risqués et un actif sans risque. En outre, l'approche principal-agent apparaît depuis peu dans la littérature sur les politiques de frais de transaction, ou plus précisément sur les subventions et taxes pour l'apport et la consommation de liquidité, au sein d'un marché. Par exemple, El Euch, Mastrolia, Rosenbaum et Touzi [136] (2018) utilisent cette approche pour considérer les incitations mises en place par une bourse qui souhaite encourager un teneur de marché à augmenter la liquidité sur la plateforme. Dans cet esprit, nous pouvons également citer les travaux de Baldacci, Possamaï et Rosenbaum [34] (2019) qui étendent le modèle précédent en considérant plusieurs agents en interaction. Dans leur modèle, les teneurs de marché optimisent leurs cotations les uns par rapport aux autres et en fonction du contrat proposé par la bourse, impliquant la recherche d'un équilibre de Nash. Dans ces deux articles, le cadre contractuel avec incitations conduit à une augmentation de la liquidité et de sa fiabilité, ainsi qu'à des coûts de négociation plus faibles pour les investisseurs, et donc à un marché plus efficace.

En ce qui concerne la théorie des jeux à champ moyen, son utilité en finance a également été démontrée par une récente augmentation de la littérature sur le sujet. Parmi les nombreux articles, nous pouvons citer par exemple l'application au trading par Cardaliaguet et Lehalle [77] (2018). Dans cet article, ils formulent un problème de liquidation optimale par un jeu entre une infinité d'acteurs. L'opérateur de marché (*trader*) est confronté à l'incertitude habituelle des prix, mais pas seulement : il doit également faire face aux changements de prix générés par d'autres acteurs similaires du marché. Cette approche est novatrice car, jusqu'alors, les cadres mathématiques considéraient généralement un agent majeur, faisant face à un *bruit de fond* représentant les autres acteurs du marché, plus insignifiants. Nous pouvons également mentionner quelques applications dans le contexte de risque systémique, par Garnier, Papanicolaou et Yang [151] (2013) ou Carmona, Fouque et Sun [90] (2015). Ces derniers proposent un modèle simple d'emprunt et de prêt interbancaires où l'évolution des réserves de N banques est décrite par un système de processus de diffusion couplés. La stabilité du système dépend du taux d'emprunt et de prêt interbancaires. Chaque banque contrôle son taux d'emprunt/prêt à une banque centrale. L'optimisation reflète le désir des banques d'emprunter à la banque centrale lorsque leurs réserves monétaires tombent en dessous d'un niveau critique, ou inversement de prêter si elles dépassent un certain niveau. Ils étudient le jeu à champ moyen correspondant à la limite d'un grand nombre de banques, en présence d'un bruit commun. Dans le même contexte, certains articles étudient également le phénomène des runs bancaires à travers le jeu à champ moyen. En particulier, Carmona, Delarue et Lacker [91] (2016) introduisent un ensemble de problèmes appelés *mean-field games of timing*. Plus précisément, ils développent une théorie mathématique pour les jeux stochastiques à temps continu où les décisions stratégiques des joueurs ne sont que des choix de moments où ils quittent le jeu, et où l'interaction entre les joueurs stratégiques est de type champ moyen.

Un domaine d'application des deux théories centrales de cette thèse qui semble le plus intéressant et novateur concerne le domaine de l'assurance. Au vu du modèle introduit en Section 1.3.2 sur une assurance contre la précarité énergétique, qui sera développé dans le Chapitre 6, il paraît clair que le formalisme mathématique de la théorie des contrats peut être utile pour modéliser les problèmes d'assurance. L'extension et le développement de ce type de modèles pourraient être intéressants pour étudier d'autres situations

d'assurance. En outre, la théorie des jeux à champ moyen pourrait ouvrir la voie à un nouvel aspect de l'assurance : cette théorie pourrait être utilisée pour étudier des situations où la prime d'assurance dépend des actions de la population, et où un acteur seul n'a aucune influence sur cette prime, ce qui est généralement le cas en assurance. Dans ce cadre, chaque agent résoudrait son problème d'optimisation, en fonction de son risque, de ses préférences, mais aussi de la prime d'assurance, qui serait impactée par la distribution ou les actions des autres agents. Par conséquent, les agents interagiraient à travers le prix de l'assurance, qui est à son tour influencé par leurs décisions. Nous pourrions également rechercher un contrat d'assurance qui incite les agents à être conscients de leur impact sur la prime d'assurance. Étonnamment, il ne semble pas exister de modèle combinant les jeux à champ moyen et l'assurance, orientation qui semble pourtant pertinente.

PART I

Large-scale principal-agent problems

Abstract

In this part, we present some theoretical results on large-scale principal-agent problems. First, in [Chapter 2](#), we present a continuous-time version of [Sung's](#) model, developed in [\[327\] \(2015\)](#). This example is illuminating in two aspects. First, it presents a natural framework in which volatility control is perfectly justified. Second, it underlines the importance of considering continuous-time models, which are often disavowed because of their excessive complexity, in order to take into account this control of volatility. There is therefore a fundamental gap between the two frameworks, which motivates *a priori* a complete and general study in continuous time of hierarchical models *à la* [Sung](#), which goes hand in hand with the recent theory of second-order backward stochastic differential equations (2BSDE).

Then, guided by the intuitions detailed in [Chapter 2](#), we study in [Chapter 3](#) a general problem of optimal incentives within a hierarchy, thus generalising the model of [Sung \[327\] \(2015\)](#) in many respects. The search for the optimal contract at each stage of the hierarchy requires the use of the theory of 2BSDEs, subject to a slight extension to take into account the plurality of workers in the hierarchy. The theoretical model we are developing therefore makes it possible to determine the optimal form of incentives for a particular hierarchical structure, which can be extended relatively directly to a larger scale hierarchy.

Finally, in [Chapter 4](#), we choose to extend the framework defined by [Aïd, Possamaï et Touzi \[11\] \(2019\)](#) to a model with a continuum of agents with mean-field interactions. More precisely, we consider that each agent controls a stochastic process, and that these processes are subject to a common noise, reflecting the random environment in which all agents evolve. Technically, this leads to the consideration of a new class of 2BSDEs, coined of mean-field type, for the agents' problem. Conversely, for the principal's problem, we are led to consider at the end a control problem of McKean–Vlasov type.

Notations I

Throughout this part, the constant $T > 0$ denotes some maturity fixed in the considered contract. Recall that, for any positive integer k , and any $t \in [0, T]$, let $\mathcal{C}_t^k := \mathcal{C}([0, t], \mathbb{R}^k)$ denote the set of continuous functions from $[0, t]$ to $\mathbb{R}^k$, and $\mathcal{C}_t := \mathcal{C}_t^1$. For a probability space of the form $\Omega := \mathcal{C}_T^k \times \tilde{\Omega}$ and an associated filtration $\mathbb{F}$, we will have to consider processes $\psi : [0, T] \times \mathcal{C}_T^k \rightarrow E$, taking values in some Polish space E , which are $\mathbb{F}$ -optional, *i.e.*, $\mathcal{O}(\mathbb{F})$ -measurable where $\mathcal{O}(\mathbb{F})$ is the so-called optional σ -field generated by $\mathbb{F}$ -adapted right-continuous processes. In particular, such a process ψ is non-anticipative in the sense that $\psi(t, x) = \psi(t, x_{\cdot \wedge t})$, for all $t \in [0, T]$ and $x \in \mathcal{C}_T^k$.

For a measurable space $(\Omega, \mathcal{F})$, let $\mathcal{P}(\Omega)$ denote the set of probability measures on $(\Omega, \mathcal{F})$. In particular, for $t \in [0, T]$, we denote by $\mathcal{P}(\mathcal{C}_t^n)$ the set of all probability measures on $\mathcal{C}([0, t], \mathbb{R}^n)$. For $\mu \in \mathcal{P}(\mathcal{C}_T^n)$, let $\mu_t \in \mathcal{P}(\mathcal{C}_t^n)$ denote the image of μ under $\pi_{\cdot \wedge t}$. For $p \geq 0$ and a separable metric space (E, ℓ) , let $\mathcal{P}^p(E)$ denote the set of $\mu \in \mathcal{P}(E)$ with $\int_E \ell^p(x, x^\circ) \mu(dx) < +\infty$ for some (and thus for any) $x^\circ \in E$. For $p \geq 1$ and

$\mu, \nu \in \mathcal{P}^p(E)$, let $\ell_{E,p}$ denote the p -Wasserstein distance, given by

$$\ell_{E,p}(\mu, \nu) := \inf \left\{ \left(\int_{E \times E} \ell^p(x, y) \gamma(dx, dy) \right)^{1/p} : \gamma \in \mathcal{P}(E \times E) \text{ has marginals } \mu, \nu \right\}.$$

The space $\mathcal{P}^p(E)$ is equipped with the metric $\ell_{E,p}$, and $\mathcal{P}(E)$ has the topology of weak convergence. Both are equipped with the corresponding Borel σ -fields, which coincides with the σ -field generated by the mappings $\mu \in \mathcal{P}^p(E)$ (resp. $\mathcal{P}(E)$) $\longrightarrow \mu(F)$, F being any Borel subset of E .

Let $\mathbb{X} := (\mathcal{X}_t)_{t \in [0, T]}$ and $\mathfrak{P}$ be respectively a filtration and a set of probability measures on some space $(\Omega, \mathcal{F}_T)$. First, we will denote by $\mathbb{X}_+ := (\mathcal{X}_t^+)_{t \in [0, T]}$ the right limit of $\mathbb{X}$, i.e., $\mathcal{X}_t^+ := \bigcap_{s > t} \mathcal{X}_s$ for all $t \in [0, T]$ and $\mathcal{X}_T^+ := \mathcal{X}_T$. For any $\mathbb{P} \in \mathfrak{P}$, we denote by $\mathbb{X}^\mathbb{P} := (\mathcal{X}_t^\mathbb{P})_{t \in [0, T]}$ the completed filtration, where for all $t \in [0, T]$, $\mathcal{X}_t^\mathbb{P}$ is the completed σ -field of $\mathcal{X}_t$ under $\mathbb{P}$. Denote also by $\mathbb{X}_+^\mathbb{P}$ the right limit of $\mathbb{X}^\mathbb{P}$, so that $\mathbb{X}_+^\mathbb{P}$ satisfies the usual conditions. In addition, the filtrations $\mathbb{X}^\mathfrak{P} := (\mathcal{X}_t^\mathfrak{P})_{t \in [0, T]}$ and $\mathbb{X}^{\mathfrak{P}+} := (\mathcal{X}_t^{\mathfrak{P}+})_{t \in [0, T]}$ are defined as follows:

$$\mathcal{X}_t^\mathfrak{P} := \bigcap_{\mathbb{P} \in \mathfrak{P}} \mathcal{X}_t^\mathbb{P}, \text{ for } t \in [0, T], \quad \mathcal{X}_t^{\mathfrak{P}+} := \mathcal{X}_{t+}^\mathfrak{P}, \text{ for } t \in [0, T), \text{ and } \mathcal{X}_T^{\mathfrak{P}+} := \mathcal{X}_T^\mathfrak{P}.$$

Moreover, we will be led to consider the following probability space:

$$\mathcal{P}(t, \mathbb{P}, \mathbb{X}) := \{ \mathbb{P}' \in \mathfrak{P} \text{ s.t. } \mathbb{P}[E] = \mathbb{P}'[E] \text{ for all } E \in \mathcal{X}_t^+ \}, \text{ for any } (\mathbb{P}, t) \in \mathfrak{P} \times [0, T].$$

Finally, let Σ be an $\mathbb{S}^\ell$ -valued process, and $p > 1$. To properly define the solution of a 2BSDE in our framework, we will have to consider the following spaces with their associated norms:

(i) $\mathbb{H}_\ell^p(\mathbb{X}, \mathfrak{P}, \Sigma)$ the space of $\mathbb{X}^\mathfrak{P}$ -progressively measurable $\mathbb{R}^\ell$ -valued processes Z , satisfying:

$$\|Z\|_{\mathbb{H}_\ell^p(\mathbb{X}, \mathfrak{P}, \Sigma)}^p := \sup_{\mathbb{P} \in \mathfrak{P}} \mathbb{E}^\mathbb{P} \left[\left(\int_0^T Z_t^\top \Sigma_t Z_t dt \right)^{p/2} \right] < +\infty;$$

(ii) $\mathbb{D}^p(\mathbb{X}, \mathfrak{P})$ the space of $\mathbb{X}^{\mathfrak{P}+}$ -optional $\mathbb{R}$ -valued càdlàg processes Y , satisfying:

$$\|Y\|_{\mathbb{D}^p(\mathbb{X}, \mathfrak{P})}^p := \sup_{\mathbb{P} \in \mathfrak{P}} \mathbb{E}^\mathbb{P} \left[\sup_{0 \leq t \leq T} |Y_t|^p \right] < +\infty;$$

(iii) $\mathbb{I}^p(\mathbb{X}, \mathfrak{P})$ the space of all $\mathbb{X}^{\mathfrak{P}+}$ -optional càdlàg and non-decreasing processes K , satisfying $K_0 = 0$, and

$$\|K\|_{\mathbb{I}^p(\mathbb{X}, \mathfrak{P})}^p := \sup_{\mathbb{P} \in \mathfrak{P}} \mathbb{E}^\mathbb{P} [K_T^p] < +\infty.$$

An introduction to continuous–time and volatility control

I'm lost but I'm not stranded yet.

Lost, NOIR DÉSIR

Introduction

The present chapter is inspired by the model developed by Sung [327] (2015). In this model, a manager is hired by a principal to subcontract with n agents, for $n \in \mathbb{N}^*$. Each worker (manager and agents) controls his own output process, and all outputs are assumed to be independent. His model includes a *bi-level* moral hazard. First, the manager does not observe the effort of the agents, but only the resulting outputs. Second, the principal observes only the total benefit of the hierarchy, *i.e.*, the difference between the sum of the outputs of all workers and the sum of the contracts paid to the agents, which will be denoted by ζ . Instead of studying a continuous–time version of the model, Sung considers that the one–period model is simpler and *without loss of generality*. Extending the reasoning of Holmström and Milgrom, he therefore restricts the study to *linear contracts*, in the sense that they are linear with respect to the outcome.

In this chapter, we consider the continuous–time version of Sung's model [327] (2015). This opening example highlights the differences between the one–period model and its continuous–time equivalent, in particular concerning the volatility control and the form of the contracts. Indeed, while the restriction to linear contracts can be justified in Sung's framework for the first Stackelberg equilibrium, *i.e.*, between the manager and the agents, this is no longer the case for the contract offered by the principal to the manager. More precisely, even if the workers are only controlling the drift of their outputs, the manager controls both the drift and the volatility of the net benefit. Therefore, according to the work of Cvitanić, Possamaï, and Touzi [111] (2018), it appears that the type of contracts considered by Sung is sub–optimal. Indeed, in continuous time, it is not sufficient to limit oneself to linear contracts, in the sense of Holmström and Milgrom [181] (1987), when the volatility of the state variable is controlled. More precisely, when studying the continuous–time model, the optimal form of contracts should contain an additional part, indexed in particular on the quadratic variation of the net benefit ζ observed by the principal.

This leads to a fundamental gap between the two frameworks, which motivates a full study of Sung's model in continuous time. With this in mind, the continuous–time version of Sung's model is introduced and solved in Section 2.1. Then, we consider some extensions, (i) by looking at different types of reporting from the manager to the principal (other than the reporting of the net profit, see Section 2.3), (ii) by adding an ability parameter for the manager to justify his position in the hierarchy (see Section 2.4.1), (iii) by extending to a more general hierarchy structure (see Section 2.4.2). The details and proofs of the main results of this chapter can be found in Section 2.5. We insist on the fact that this chapter presents the results in an informal way. The rigorous study is postponed to Chapter 3, where a more general model is presented and solved, using the theory of second–order backward stochastic differential equations (2BSDEs).

2.1 An opening example: Sung's model in continuous time

We present in this section a simple hierarchical contracting problem, similar to the one considered by Sung [327] (2015). In this hierarchy, a principal contracts with a manager who in turn subcontracts with many agents. The hierarchy is illustrated in Figure 2.1. Despite its simplicity, this illuminating example shows the need to take volatility control into account, and therefore justifies the use of the 2BSDE theory, on which we will rely throughout this thesis. The reasoning will remain informal throughout this chapter, the reader is referred to the following chapter, namely Chapter 3, for a rigorous framework in continuous time.

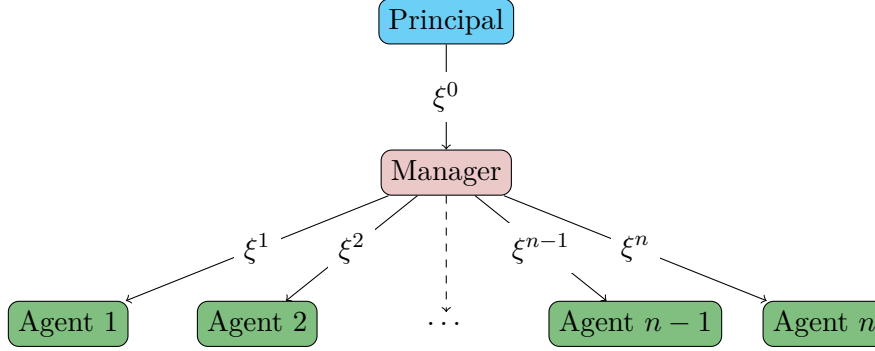


Figure 2.1 – Hierarchy in Sung's model

The only difference with Sung's model is that we consider here a continuous-time model: between 0 and some time $T > 0$, denoting the maturity fixed in the contract, the firm has $n + 1$ tasks which have to be carried out by $n + 1$ workers. The outputs of the tasks are represented by $n + 1$ stochastic processes, denoted by X^i , with dynamic

$$dX_t^i = \alpha_t^i dt + \sigma^i dW_t^i, \quad \sigma^i > 0, \quad t \in [0, T], \quad (2.1.1)$$

for $i \in \{0, \dots, n\}$. More precisely, the i -th worker carries out the task with outcome X^i by choosing a costly effort $\alpha^i \in \mathcal{A}^i$ with values in $\mathbb{R}$, where $\mathcal{A}^i$ is the set of control processes¹. For simplicity, we assume that W^i for $i \in \{0, \dots, n\}$ are independent Brownian motions and that the efforts of a worker only impact his own project, which means that all projects are independent and no workers collude. Again for the sake of simplicity in this chapter, we will consider the following quadratic cost of effort:

$$c^i(a) = \frac{1}{2} \frac{a^2}{k^i}, \quad k^i > 0, \quad \text{for } a \in \mathbb{R}, \quad i \in \{0, \dots, n\}. \quad (2.1.2)$$

Still following Sung's framework in [327], we also assume that the preference of each worker i is represented by a CARA utility function with risk aversion coefficient $R^i > 0$. The holder of the firm (the principal) is risk-neutral and seeks to maximise the expected difference between the sum of the outputs and the sum of the compensations paid to the workers. The minimum level of utility that must be guaranteed by a contract to make it acceptable to a worker, *i.e.*, his reservation utility, is defined by his utility without contract.

We define the *direct contracting case* (DC case) as the case where the principal can directly contract with the agents, without the help of a manager. In this setting, the optimal efforts of the workers are deterministic constant processes given by the results of Holmström and Milgrom [181] (1987), and summarised

1. The set of admissible control processes will be rigorously defined, in weak formulation, in Chapter 3, more precisely in Section 3.1.

by [Lemma 2.5.1](#). This case is also discussed by [Sung](#), but the main point of his paper [327], and thus of this chapter, is to study the case where the principal contracts with a manager, who in turn subcontracts with the agents.

In the *hierarchical contracting case* (HC case) considered by [Sung](#), the principal cannot directly contract with the workers. She hires a manager (the worker indexed by $i = 0$) who:

- (i) carries out his own task by choosing an effort process $\alpha^0 \in \mathcal{A}^0$;
- (ii) hires n agents to carry out the n remaining tasks: each agent i handles the outcome X^i , by choosing his effort level $\alpha^i \in \mathcal{A}^i$, for $i \in \{1, \dots, n\}$;
- (iii) reports to the principal the total benefit, that is the difference between the sum of the outputs and the sum of the compensations to be paid to the agents.

We will show that, in the continuous-time framework, we can improve [Sung's](#) results in [327] by considering a more general form of contracts.

2.1.1 A continuous-time principal-manager-agents problem

As already mentioned, we are faced with a bi-level principal-agent problem, in the sense that a principal contracts with a manager who in turn subcontracts with many agents. In this section we define the continuous-time equivalent formulation of the value functions considered in [327], as well as the admissible set of contracts.

Agent's problem. Consider first $i \in \{1, \dots, n\}$ to focus on the i -th agent's problem. He controls his own output X^i with dynamic (2.1.1) by choosing an effort $\alpha^i \in \mathcal{A}^i$. Given a contract ξ^i offered by his supervisor, namely the manager, the i -th agent's value function is simply defined by:

$$V_0^i(\xi^i) := \sup_{\alpha^i \in \mathcal{A}^i} J_0^i(\xi^i, \alpha^i), \quad \text{where } J_0^i(\xi^i, \alpha^i) := \mathbb{E}^{\mathbb{P}^i} \left[-e^{-R^i(\xi^i - \int_0^T c^i(\alpha_t^i) dt)} \right], \quad (2.1.3)$$

where $\mathbb{P}^i$ is the probability associated to the effort $\alpha^i \in \mathcal{A}^i$. We assume that the manager cannot observe the efforts of the agents, which implies a first level of moral hazard in our framework. The manager only observes the outcome processes X^i for $i \in \{1, \dots, n\}$. In order to follow [Sung's](#) model as closely as possible, we also assume that the compensation for the i -th agent can only be indexed on his performance, *i.e.*, his outcome process X^i , and denote the set of admissible contracts by Ξ^i .² Note that since the reservation utility of the i -th agent is defined as his utility without contract, it is given by $V_0^i(0) = -1$.

Manager's problem. The manager controls his own output X^0 with dynamic (2.1.1) by choosing an effort $\alpha^0 \in \mathcal{A}^0$. He also designs the compensations for the agents, namely a collection of contracts

$$\xi^A \in \Xi^A := \{(\xi^i)_{i=1}^n, \text{ s.t. } \xi^i \in \Xi^i \forall i \in \{1, \dots, n\}\}.$$

Although we consider that the manager designs the contracts for the agents, all compensations, whether for the agents or the manager, are paid by the principal. Given a contract ξ^b designed by his supervisor (the principal), the manager's value function is defined by

$$V_0^{0,b}(\xi^b) := \sup_{(\alpha^0, \xi^A) \in \mathcal{A}^0 \times \Xi^A} J_0^0(\xi^b, \alpha^0, \xi^A), \quad \text{where } J_0^0(\xi^b, \alpha^0, \xi^A) := \mathbb{E}^{\mathbb{P}^0} \left[-e^{-R^0(\xi^b - \int_0^T c^0(\alpha_t^0) dt)} \right], \quad (2.1.4)$$

where, informally, $\mathbb{P}^0$ is the probability associated to both the effort $\alpha^0 \in \mathcal{A}^0$ and the choice of the contracts $\xi^A \in \Xi^A$, under the optimal efforts of the n agents. As in [327], the second level of moral hazard is linked

2. It is worth noticing that this restriction is not admissible in a more general model, as we will see in [Remark 2.1.3](#).

to the fact that the manager only reports (in continuous time) to the principal the total benefit ζ^b , *i.e.*, the difference between the sum of the outputs and the sum of the compensations to be paid:

$$\zeta_t^b := \sum_{i=0}^n X_t^i - \sum_{i=1}^n \xi_t^i, \quad t \in [0, T]. \quad (2.1.5)$$

More precisely, in this setting, the principal cannot independently observe the agents' outputs X^i or the certainty equivalent of their continuation utility, ξ^i . Nor does she observe the manager's outcome X^0 , nor his effort α^0 . Therefore, she can only index the contract ξ^b for the manager on the total benefit, *i.e.*, the variable ζ^b . The contract ξ^b is thus a measurable function of ζ^b , and the corresponding set of admissible contract is denoted by Ξ^b .

Principal's problem. Finally, we consider a risk-neutral principal whose problem is to maximise the sum of the outputs minus the sum of the compensations to be paid to each worker, by choosing the optimal contract ξ^b for the manager. Mathematically speaking, we define her value function as follows:

$$V_0^{\text{P},b} := \sup_{\xi^b \in \Xi^b} J_0^{\text{P}}(\xi^b), \quad \text{where} \quad J_0^{\text{P}}(\xi^b) := \mathbb{E}^{\mathbb{P}^b} \left[\sum_{i=0}^n X_T^i - \sum_{i=1}^n \xi_T^i - \xi_T^b \right] = \mathbb{E}^{\mathbb{P}^b} [\zeta_T^b - \xi_T^b], \quad (2.1.6)$$

where $\mathbb{P}^b$ is the probability associated to the choice of the contract $\xi^b \in \Xi^b$, under optimal efforts of the workers.

Remark 2.1.1. *It should be noted that the three value functions defined above by (2.1.3–2.1.4–2.1.6) should be written in weak formulation. However, for the sake of simplicity in this section, we avoid this technical side for the moment and refer the reader to the general model in Chapter 3 for a rigorous writing of these three value functions.*

2.1.2 Solving the sequence of Stackelberg equilibria

In order to solve this hierarchical problem, we can follow the general theory developed by Cvitanic, Possamai, and Touzi [111] (2018) or the application in [110] to a framework with CARA utility functions. More precisely, for each Stackelberg equilibrium, starting with the manager-agent problem, we should:

- (i) identify a sub-class of contracts, offered to a considered worker by his supervisor, which are revealing in the sense that the best-reaction function of the worker and his optimal control can be computed straightforwardly;
- (ii) prove that the restriction to revealing contracts is without loss of generality;
- (iii) solve the supervisor's problem, which boils down to a standard optimal control problem.

Revealing contract for an agent. Consider $i \in \{1, \dots, n\}$ to focus on the contract for the i -th agent. Recall that his contract ξ^i is assumed to be a measurable function of his output X^i . By applying classical results of contract theory for drift control only (see, *e.g.*, the work by Sannikov [303] (2008)), the optimal form of contracts is the terminal value ξ_T^i of the certainty equivalent of the continuation utility, which is defined for all $t \in [0, T]$ as follows:

$$\xi_t^i = \xi_0^i - \int_0^t \mathcal{H}^i(Z_s^i) ds + \int_0^t Z_s^i dX_s^i + \frac{1}{2} R^i \int_0^t |Z_s^i|^2 d\langle X^i \rangle_s, \quad \xi_0^i \in \mathbb{R}, \quad (2.1.7)$$

where Z^i is a payment rate chosen by the manager, and $\mathcal{H}^i(z) := \sup_{a \in \mathbb{R}} \{az - c^i(a)\}$ for all $z \in \mathbb{R}$ is the Hamiltonian of the i -th agent. This form of contract is exactly the continuous-time equivalent of the

linear contract considered by Sung in [327, Equation (7)]. Note that this form of compensation includes in particular a fixed part ξ_0^i , which is chosen so as to satisfy the i -th agent's participation constraint. Assuming for simplicity that c^i is a standard quadratic cost function defined by (2.1.2), we can establish the following result, whose proof is straightforward³ in the light of the choice of contract's form (2.1.7).

Proposition 2.1.2. *Fix $i \in \{1, \dots, n\}$. Let $\xi_0^i \in \mathbb{R}$, Z^i be an $\mathbb{R}$ -valued process, predictable with respect to the filtration generated by X^i , satisfying appropriate integrability conditions⁴, and consider the associated contract ξ^i defined through (2.1.7). Given this contract, the optimal effort of the i -th agent is given by*

$$\alpha_t^{i,\text{HC}} = a^{i,\text{HC}}(Z_t^i), \quad t \in [0, T], \quad \text{where } a^{i,\text{HC}}(z) := k^i z, \quad z \in \mathbb{R}.$$

Moreover, under the probability $\mathbb{P}^{i,\text{HC}}$ associated to the optimal effort $\alpha^{i,\text{HC}}$, the dynamics of X^i and ξ^i satisfy

$$dX_t^i = k^i Z_t^i dt + \sigma^i dW_t^i \quad \text{and} \quad d\xi_t^i = \frac{1}{2} \tilde{R}^i |Z_t^i|^2 dt + Z_t^i \sigma^i dW_t^i, \quad t \in [0, T], \quad \text{where } \tilde{R}^i = k^i + R^i |\sigma^i|^2.$$

Simplification of the manager's problem. Recall that the manager, in addition to controlling his own output X^0 with dynamic (2.1.1) by choosing an effort $\alpha^0 \in \mathcal{A}^0$, also designs the collection of contracts $\xi^A \in \Xi^A$ for the agents. As mentioned above, instead of studying all possible contracts $\xi^i \in \Xi^i$ for the i -th agent, it has been proved by Sannikov [303] (2008) that it is sufficient to restrict the study to contracts of the form (2.1.7). Therefore, the manager's problem boils down to a standard optimal control problem: to design the compensation for the i -th agent, the manager only have to choose the payment rate Z^i . We thus define by $\mathcal{V}^0$ the collection of all processes $Z : [0, T] \times \mathcal{C}_T \rightarrow \mathbb{R}^n$, where each Z^i is predictable with respect to the filtration generated by X^i , and satisfies appropriate integrability conditions. We can now rewrite the manager's problem defined by (2.1.4) in a more standard way:

$$V_0^{0,b}(\xi^b) := \sup_{(\alpha^0, Z) \in \mathcal{A}^0 \times \mathcal{V}^0} J_0^0(\xi^b, \alpha^0, \xi^A).$$

Recall that we consider that the manager designs the contracts for the agents, but that all compensations, whether for the agents or the manager, are paid by the principal. For this reason, we assume that the principal chooses, for all $i \in \{1, \dots, n\}$, the constant ξ_0^i in the i -th agent's contract, noticing that this constant must be chosen in order to ensure that his participation constraint is satisfied.

Since the manager only reports to the principal the total benefit ζ^b in continuous time, defined by (2.1.5), his compensation ξ^b offered by the principal can only be a measurable function of ζ^b . Then, given the form of the manager's utility and the dynamic of the output X^i for $i \in \{1, \dots, n\}$, ζ^b is the only state variable of his control problem. Therefore, his optimal control, namely $(\alpha^0, Z) \in \mathcal{A}^0 \times \mathcal{V}^0$, will naturally be adapted to the filtration generated by ζ^b .

Remark 2.1.3. *In fact, the set of admissible control processes for the manager cannot be properly defined in this framework. Indeed, recall that we choose to restrict the contract for an agent i to his own X^i . Therefore, the payment rate Z^i should not depend on anything other than X^i . In particular, it cannot be predictable with respect to the filtration generated by ζ^b , since it contains information generated by the outputs of other workers. However, under this assumption, the optimal control of the manager on the i -th agent's contract,*

3. As the proof of this result is very similar to that of Proposition 2.1.5 below, providing a similar result for the manager, we have chosen not to detail it here. The reader is thus referred to Section 2.5.1 for a sketch of the proof.

4. We have to require minimal integrability on the process Z so that the stochastic integral with respect to X is well-defined. Nevertheless, since this section is informal, the conditions of integrability are ignored for the time being. The reader can refer to the general model in Chapter 3 for a rigorous definition of the admissible process Z .

which will be denoted by $Z^{i,b}$, should be a function of X^i , and thus cannot be computed by the principal, since she only observes ζ^b . Moreover, we will generically have $Z_t^{i,b} := z^{i,b}(Z_t, \Gamma_t)$, $t \in [0, T]$, where Z and Γ are two processes chosen by the principal, assumed to be predictable with respect to what she observes, i.e., with respect to the filtration generated by ζ^b . Indeed, the manager's contract is restricted to a measurable function of ζ^b , therefore the payment rates Z and Γ indexing the contract on ζ^b should also be functions of ζ^b . We thus obtain a contradiction. Nevertheless, in this particular example, every optimal efforts and controls turn out to be deterministic (and even constant), and we can thus index the contract for the i -th agent only on his own output X^i . This model has been chosen in this section to easily compare our results with those of Sung [327] (2015). However, in a more general case, we will not be able to restrict the study to such contracts. More precisely, we will be forced to consider that each agent knows the output of other workers, and that his contract can be indexed on it. Some assumptions will thus be necessary to ensure that the controls of the manager are predictable with respect to the filtration generated by ζ^b , and can thus be computed at the optimum by the principal.

Towards volatility control. Let us set aside for the moment the previous remark, it will be dealt with in the general model in Chapter 3. The most important thing to notice at this stage is that, even if the agents are only controlling the drift of their outputs, the manager controls both the drift and the volatility of ζ^b , as we can see from its dynamic under optimal efforts of the agents, which is as follows:

$$d\zeta_t^b = \left[\alpha_t^0 + \sum_{i=1}^n \left(k^i Z_t^i - \frac{1}{2} \tilde{R}^i |Z_t^i|^2 \right) \right] dt + \sigma^0 dW_t^0 + \sum_{i=1}^n \sigma^i (1 - Z_t^i) dW_t^i, \quad (2.1.8)$$

recalling that $\tilde{R}^i = k^i + R^i |\sigma^i|^2$ for all $i \in \{1, \dots, n\}$. Indeed, by choosing optimally the payment rate in each agent's contract (Z^i for all $i \in \{1, \dots, n\}$), the manager controls in a way the volatility of the certainty equivalent of agents' continuation utilities ξ^i (through the term $Z_t^i \sigma^i dW_t^i$), and thus the volatility of ζ^b . Therefore, we must consider a more extensive class of contracts than the one used by Sung in [327]. Indeed, in continuous time, it is not sufficient to limit oneself to linear contracts (in the sense of Holmström and Milgrom [181] (1987)) when the volatility of the state variable is controlled, as demonstrated by Cvitanić, Possamaï, and Touzi [111] (2018). This is where our model and Sung's will diverge. Instead of studying the model in continuous time, Sung considers that the one-period model is simpler and without loss of generality. He therefore continues to restrict the study to contracts that are linear with respect to the outcome, insisting that this restriction is “without loss of generality, as long as our results are interpreted in the context of continuous-time models.” (Sung [327] (2015)). According to our study in continuous time, it appears that the type of contracts considered by Sung is sub-optimal. We refer to Section 2.2 for the theoretical and numerical analysis of the results.

Revealing contract for the manager. Let $\mathbb{V}^b$ be the set of all $(z, \gamma) \in \mathbb{R}^2$ such that $\tilde{R}^i z - |\sigma^i|^2 \gamma > 0$ for all $i \in \{1, \dots, n\}$. We define by $\mathcal{V}^b$ the collection of all processes $(Z, \Gamma) : [0, T] \times \mathcal{C}_T \longrightarrow \mathbb{V}^b$, predictable with respect to the filtration generated by ζ^b , satisfying appropriate⁵ integrability conditions. The set $\mathcal{V}^b$ represents the admissible control processes for the principal, when she only observes the variable ζ^b . Taking into account the previous discussion, it is necessary to use recent results on optimal contracting for drift and volatility control, and therefore the theory of 2BSDEs, to state the following result. We refer to the previously mentioned works of Cvitanić, Possamaï, and Touzi [111] (2018) for the general result, [110] for an application with exponential utilities, as well as Lin, Ren, Touzi, and Yang [233] (2020) for an extension to random time horizon.

5. Similarly as noticed in Proposition 2.1.2, we have to require minimal integrability on the process Z . Once again, we refer to Chapter 3 for a rigorous definition of the set of admissible control processes $\mathcal{V}^b$.

Proposition 2.1.4. *Assuming that the principal only observes ζ^b , the optimal form of contracts offered by the principal to the manager is given by*

$$\xi_T^b = \xi_0^b - \int_0^T \mathcal{H}^b(Z_s, \Gamma_s) ds + \int_0^T Z_s d\zeta_s^b + \frac{1}{2} \int_0^T (\Gamma_s + R^0 |Z_s|^2) d\langle \zeta^b \rangle_s, \quad \xi_0^b \in \mathbb{R}, \quad (2.1.9)$$

where $\mathcal{H}^b$ is the manager's Hamiltonian, and $(Z, \Gamma) \in \mathcal{V}^b$ is a pair of processes optimally chosen by the principal. In addition, similar to the agent's contract form (2.1.7), ξ_0^b represents a fixed part of the compensation, which is chosen so as to satisfy the manager's participation constraint.

Given the dynamic of the state variable ζ^b and its quadratic variation, the manager's Hamiltonian is defined, for any $(z, \gamma) \in \mathbb{V}^b$, as follows:

$$\mathcal{H}^b(z, \gamma) = \frac{1}{2} \gamma |\sigma^0|^2 + \sup_{a \in \mathbb{R}} \{az - c^0(a)\} + \sum_{i=1}^n \sup_{z^i \in \mathbb{R}} \left\{ z \left(k^i z^i - \frac{1}{2} \tilde{R}^i |z^i|^2 \right) + \frac{1}{2} \gamma |\sigma^i|^2 |1 - z^i|^2 \right\}. \quad (2.1.10)$$

Considering any contracts of the form (2.1.9), we can easily solve the manager's problem, mainly by maximising the previous Hamiltonian. The proof of the following proposition is therefore a direct consequence of the considered form of contracts, and is detailed in Section 2.5.1.

Proposition 2.1.5. *Let $(Z, \Gamma) \in \mathcal{V}^b$ and $\xi_0^b \in \mathbb{R}$. Given the associated contract ξ_T^b defined by (2.1.9), the optimal effort and the optimal control on the i -th agent's compensation ($i \in \{1, \dots, n\}$) chosen by the manager are respectively given by $\alpha_t^b := k^0 Z_t$ and $Z_t^{i,b} := z^{i,b}(Z_t, \Gamma_t)$ for all $t \in [0, T]$, where,*

$$z^{i,b}(z, \gamma) := \frac{k^i z - |\sigma^i|^2 \gamma}{\tilde{R}^i z - |\sigma^i|^2 \gamma}, \quad (z, \gamma) \in \mathbb{V}^b. \quad (2.1.11)$$

Moreover, under the optimal probability $\mathbb{P}^b$ associated to the optimal efforts of both the agents and the manager, the dynamics of ζ^b and ξ^b are respectively given, for all $t \in [0, T]$, by:

$$d\zeta_t^b = \left[k^0 Z_t + \sum_{i=1}^n \left(k^i Z_t^{i,b} - \frac{1}{2} \tilde{R}^i |Z_t^{i,b}|^2 \right) \right] dt + \sigma^0 dW_t^0 + \sum_{i=1}^n \sigma^i (1 - Z_t^{i,b}) dW_t^i,$$

and $d\xi_t^b = \frac{1}{2} |Z_t|^2 \left(k^0 + R^0 |\sigma^0|^2 + R^0 \sum_{i=1}^n |\sigma^i|^2 |1 - Z_t^{i,b}|^2 \right) dt + Z_t \sigma^0 dW_t^0 + Z_t \sum_{i=1}^n \sigma^i (1 - Z_t^{i,b}) dW_t^i.$

Solving the principal's problem. Proposition 2.1.4 states that it is sufficient to restrict the space of contracts to those of the form (2.1.9), which simplifies the principal's problem. Recall that we assume that the principal chooses all the constants ξ_0^i in each agent's contract, as well as the constant ξ_0^b in the manager's contract. Informally, these constants have to be chosen such that each contract satisfies the participation constraint of the corresponding worker. Given the form of the manager's and agents' utility, and in particular since the manager does not pay the compensation for his agents, he is indifferent to the value of ξ_0^i , for $i \in \{1, \dots, n\}$, as long as the agents accept the contracts. This is why we consider that the principal chooses it, and we denote by $\xi_0 \in \mathbb{R}^{n+1}$ the collection of ξ_0^b and ξ_0^i , for $i \in \{1, \dots, n\}$. Moreover, recall that we have assumed as in [327] that the reservation utility level of each worker is given by his utility without any contract, thus equal to -1 , and that the initial outputs $(X_0^i)_{i=0}^n$ are equal to zero. With this in mind, the following proposition solves the principal's problem, whose proof is detailed in Section 2.5.1.

Proposition 2.1.6. *The principal's problem, initially defined by (2.1.6), is reduced to the choice of the triple $(\xi_0, Z, \Gamma) \in \mathbb{R}^{n+1} \times \mathbb{R} \times \mathcal{V}^b$ to define the contract ξ^b through (2.1.9):*

$$V_0^{P,b} = \sup_{(\xi_0, Z, \Gamma) \in \mathbb{R}^{n+1} \times \mathbb{R} \times \mathcal{V}^b} J_0^P(\xi^b).$$

By solving this standard control problem, we find her optimal controls, the optimal contracts and her utility.

- (i) The optimal payment rates for the manager are the constant processes $Z^b := z^b$ and $\Gamma^b := -R^0(z^b)^3$, where z^b is solution to the following maximisation problem:

$$\sup_{z>0} \left\{ k^0 z - \frac{1}{2} \tilde{R}^0 |z|^2 + \sum_{i=1}^n h^{i,b}(z, -R^0 z^3) \right\}, \quad (2.1.12)$$

where $\tilde{R}^0 := k^0 + R^0 |\sigma^0|^2$ and, for all $i \in \{1, \dots, n\}$ and any $(z, \gamma) \in \mathbb{V}^b$,

$$h^{i,b}(z, \gamma) := k^i z^{i,b}(z, \gamma) - \frac{1}{2} \tilde{R}^i |z^{i,b}(z, \gamma)|^2 - \frac{1}{2} R^0 |\sigma^i|^2 |z|^2 |1 - z^{i,b}(z, \gamma)|^2. \quad (2.1.13)$$

- (ii) The optimal contract offered by the principal to the manager is given by:

$$\xi_T^b = -\mathcal{H}^b(z^b, -R^0(z^b)^3)T + z^b \zeta_T^b + \frac{1}{2} R^0 |z^b|^2 (1 - z^b) \langle \zeta^b \rangle_T,$$

where $\mathcal{H}^b$ is the manager's Hamiltonian defined by (2.1.10). In particular, the optimal choice of the fixed part of the compensation ξ_0^b is the one that saturates the manager's participation constraint, i.e., such that he obtains exactly his reservation utility. In this case, since his utility reservation is equal to -1 , the optimal ξ_0^b is 0.

- (iii) For all $i \in \{1, \dots, n\}$, the optimal contract offered by the manager to the i -th agent is given by:

$$\xi_T^i = -\mathcal{H}^i(z^{i,b}(z^b, -R^0(z^b)^3))T + z^{i,b}(z^b, -R^0(z^b)^3)X_T^i + \frac{1}{2} R^i |z^{i,b}(z^b, -R^0(z^b)^3)|^2 \langle X^i \rangle_T,$$

where $\mathcal{H}^i$ is the i -th agent's Hamiltonian, and recalling that $z^{i,b}$ is defined in Proposition 2.1.5. In particular, as for the manager, the optimal choice of the fixed part of the compensation ξ_0^i is 0.

- (iv) Finally, the value function of the principal is given by:

$$V_0^b = T \left(k^0 z^b - \frac{1}{2} \tilde{R}^0 |z^b|^2 + \sum_{i=1}^n h^{i,b}(z^b, -R^0(z^b)^3) \right).$$

2.2 The benefits of continuous time

2.2.1 Non-optimality of linear contracts in continuous time

Propositions 2.1.4 and 2.1.6 require that the optimal contracts for the manager must be indexed on the quadratic variation of the net profit ζ^b through the parameter $\Gamma^b := -R^0(z^b)^3$. However, in [327], Sung restricts the analysis to linear contracts: although he remarks that decisions on middle managerial contracts are affecting the volatility of the net profit of the firm, he chooses to view them as a case similar to unobservable project choice decisions, modelled in [325] (1995). More precisely, he states the following:

As shall be seen in our hierarchical contracting problem, the top manager turns out to choose not only the mean of the outcome of his own effort but, in effect, the volatility of the total profit of the firm as he chooses middle managerial contracts. Thus, our problem turns out to be similar to the unobservable project choice problem in Sung [21]. (Sung [327, pp. 3] (2015))

In the aforementioned article [325], Sung studies a principal-agent problem in continuous time where the volatility can be controlled. He distinguishes two cases.

- (i) One where the variance is observed, but since the Brownian motion is only one dimensional, there is no moral hazard on the volatility's effort anymore. Indeed, in this case, the variance is equal to the square of the volatility effort, and since the variance is observed, the effort is easily computable by the principal. Therefore, the principal directly controls the volatility's effort of the agent and the model degenerates to the first-best case (no moral hazard) for the volatility.

- (ii) One where the variance is not observed by the principal, and therefore she cannot index the contract on the quadratic variation of the outputs, which obviously leads to consider only linear contracts.

In [327], as Sung considers that the variance is not observed (case (ii) above), the principal cannot offer a contract to the manager indexed on it, which is equivalent to forcing $\Gamma^b + R^0|Z^b|^2 = 0$ in our extended class of contracts defined by (2.1.9). Therefore, he does not optimise the utility of the principal with respect to Γ^b , since he forces $\Gamma^b := -R^0|Z^b|^2$.

Nevertheless, in continuous time, it seems natural to consider that the principal observes the quadratic variation of the total benefit, $\langle \zeta^b \rangle$, and can therefore contract on it. Indeed, she observes ζ^b in continuous time and can therefore estimate the quadratic variation through the sum of the squared increments. Moreover, a famous result of Bichteler [58] (1981) states that this quadratic variation, even controlled, can be defined independently of the probability associated to the effort. Therefore, contrary to (ii) above, the contract can be indexed on the quadratic variation. Moreover, since the process ζ^b is naturally driven by $n + 1$ independent Brownian motions, the principal does not perfectly observe the control $(Z^i)_{i=1}^n$ of the manager, but only a functional of these Z^i . This prevents the volatility control case from degenerating into the first-best case, contrary to what is mentioned in (i) above.

Therefore, Sung's argument in [327] to justify restricting the study to linear contracts, namely that his model has to be understood as a continuous-time model in which linear contracts are supposedly optimal, seems not to be valid. One way to fix this problem in the one-period model would be to propose contracts indexed on the variance of ζ^b . However, as this variance is controlled, it depends on the probability chosen by the manager, which is unknown to the principal when the efforts are not optimal. Indeed, unlike in continuous time, where the quadratic variation, even controlled, can be defined independently of the effort probability, this is not the case for the variance in the one-period model. Therefore, it is not easy to find an equivalent to the contract indexed on the quadratic variation for the one-period model.⁶ However, in any case, it is a well-known result that it is not possible to find an optimal contract in the one-period model, even without volatility control, as soon as the monotone likelihood ratio⁷ is not bounded from below, which is the case in [327] since the output processes are Gaussian. Indeed, Mirrlees [254] (1999) shows that, in this case, there is a sequence of contracts, called forcing contracts, that allows to obtain the results of the first-best case (when there is no moral hazard) at the limit, but that there is no optimal contract. Restricting oneself to linear contracts in the case of drift control only in the one-period model is justified because these are the optimal contracts in continuous time, but, unfortunately, this reasoning is no longer valid in the case of volatility control.

In conclusion, unlike the case of drift control only, in the case of volatility control it is not possible to consider the one-period model by limiting the study to linear contracts, and expect to obtain the same results as in continuous time. This result therefore justifies the full study of continuous-time models and the use of the recent theory of 2BSDEs, from a theoretical point of view. In the following, we will see through numerical results that it is obviously beneficial in a practical way for the principal to consider the problem in continuous time.

2.2.2 Numerical results

In [327, Theorem 2], Sung presents some interesting facts such as the decrease in the efforts of the manager and agents when the total number of workers increases, as well as their limits for an infinitely large company. These facts seem also be true in our framework, but we do not believe it is necessary to dwell on

6. It is worth noticing that in a discrete-time framework, but with multiple periods, one could also approximate the variance.

7. The monotone likelihood ratio is defined in this case by the ratio between the derivative of the considered process density with respect to the effort and the density itself.

proving the same results. We find it more interesting to focus on the differences between the two models, and of course on the benefits of the approach being considered in this chapter. In our opinion, the simplest way to achieve this goal is to present in this subsection some numerical simulations.

Therefore, to illustrate the benefits induced by considering contracts with the quadratic variation term, we decide to perform some numerical simulations based on the parameters chosen in [327, Section 5], in particular in the case of identical workers. More precisely, we let for all $i \in \{0, \dots, n\}$, $k^i = k$, $R^i = R$ and $\sigma^i = \sigma$, where $(k, R, \sigma) := (1000, 50, 1)$. We thus represent in the left graphs of Figures 2.2, 2.3 and 2.4, respectively the optimal *pay-for-performance sensitivities* for the agents, for the manager, as well as the value per workers for the principal, in three cases:

- (i) in the DC case (blue line), *i.e.*, without any manager (see Lemma 2.5.1 for theoretical results);
- (ii) in the case of Sung's (orange curve), *i.e.*, when the manager's contract is linear;
- (iii) in our framework (green curve), *i.e.*, when his contract is more sophisticated with an indexation on the quadratic variation.

All curves are represented with respect to the number of workers, starting from 2 (*i.e.* $n = 1$), to consider at least two agents in the DC case or one agent and one manager in the HC case. Pay-for-performance sensitivity (PPS for short) is a common proxy for the strength of incentives (see Gryglewicz, Hartman-Glaser, and Zheng [164] (2020)). In our framework, this sensitivity is directly related to the efforts of the workers. Indeed, for all $i \in \{1, \dots, n\}$, the i -th agent's optimal effort is given by $\alpha^{i, \text{HC}} := k^i Z^i$, where Z^i is precisely the PPS for the i -th agent's contract. A similar relation stands for the manager's effort and his PPS. We could also have represented workers' efforts, but we decide to use this indicator to simplify the comparison with the results of Sung given in [327, Table 1], although they have been recalculated in our case for different numbers of agents. On the right graphs of the three figures below, we represent the relative gain induced by considering sophisticated contracts versus linear ones.

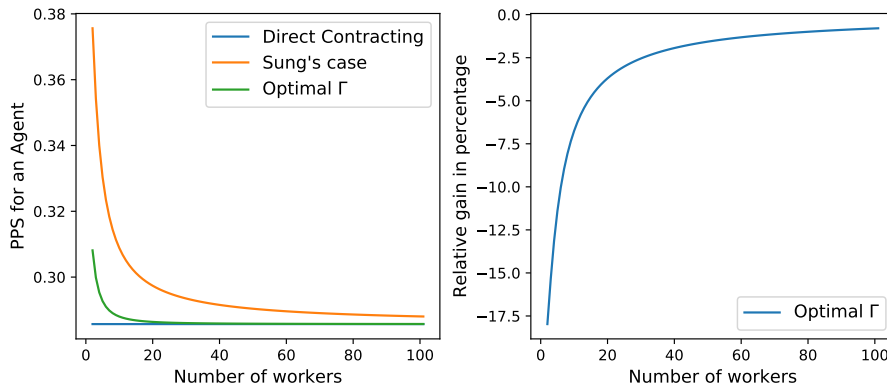


Figure 2.2 – PPS for an agent and relative gain.

Our results obviously present the same features as those outlined in [327]. Since the manager can subcontract with the agents, he can benefit, to some extent, from the results of agents' efforts and transfers his own compensation risk to them. As a consequence, agents are induced to work harder than implied in the direct contracting case (see Figure 2.2, left). To counterbalance this undesirable risk-shifting motivation of the risk-averse manager, the principal set his contract sensitivity to a level lower than that of the contract in the DC case. Consequently, the manager makes less effort than what would be required in the direct contracting situation (see Figure 2.3, left). In addition, the bigger the company, the more motivated the manager is to shift the risk onto the agents. Consequently, the larger the number of workers, the lower-powered the managerial incentive contract. Sung concludes that the results obtained with this model on

the low managerial effort can serve as an explanation of the empirical finding of Jensen and Murphy [196] (1990) that the average CEO contract sensitivity of large firms is lower than that of small firms.

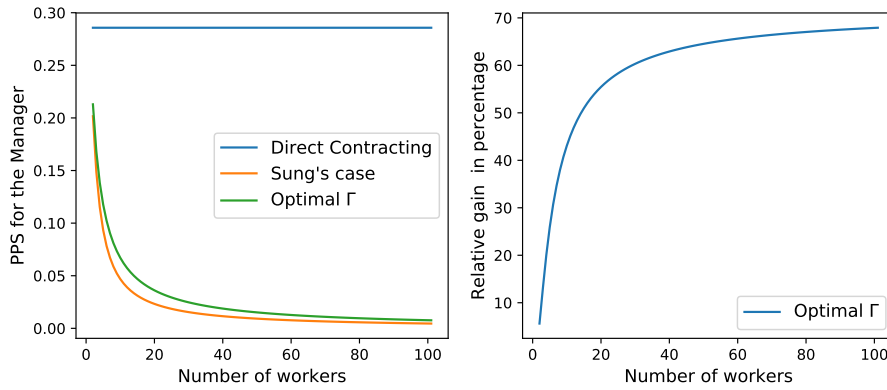


Figure 2.3 – PPS for the manager and relative gain.

Nevertheless, our sophisticated contracts allow an improvement of the results. More specifically, the PPSs we obtain, both for the manager and for the agents, are closer to the PPS in the DC case, compared to those obtained by Sung. More precisely, using a contract with the quadratic variation for the manager allows the principal to better monitor his own performance. This results in a higher PPS for the manager (see Figure 2.3, left), and therefore forces him to make more effort. The relative gain (Figure 2.3, right) is increasing with respect to the number of workers and reaches for example 60% for 30 workers in the company (29 agents in addition to the manager). The new contracts we consider therefore mitigates the undesirable risk-shifting motivation of the risk-averse manager. The manager still benefits from the results of agents' efforts and transfers a part of his own compensation risk to them, but less than with linear contracts. Consequently, agents are still induced to work harder than implied in the DC case (see Figure 2.2, left), but less than in Sung's framework.

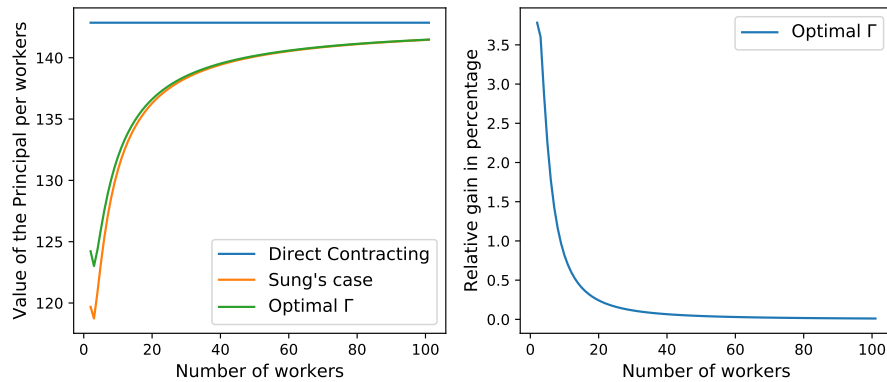


Figure 2.4 – Principal's value per workers.

Figure 2.4 (left) represents the value function of the principal per workers. With the sophisticated contracts, this value is obviously higher than with linear contracts, which confirms the interest of our study. Even if the relative gain seems small (see Figure 2.4, right), this result motivates a full study with even more sophisticated contracts, in Chapter 3. Indeed, even if this only leads to a small increase in the principal's value per worker when the number of workers is large, the gain has to be multiplied by the number of workers. Moreover, when the number of workers is small, the gain is significant nonetheless, but above all

it allows to reduce the effort gaps between the agents and the manager. It is interesting to consider the benefit of these contracts not only from the principal's point of view, but also from a global managerial perspective. Indeed, by developing this type of contracts, the principal better monitors the manager's efforts, and therefore regulates his risk-shifting motivation, which results in improved conditions for the agents and a better division of work and risk between the agents and the manager.

Remark 2.2.1. *One can notice that the principal's profit per worker is not monotonous when the number of workers is small. In particular, her profit is higher when there are two workers instead of three, while it is then increasing with the number of workers. This is actually explained by the fact that when there is only one agent supervised by the manager, there is less loss of information when going up the hierarchy. Indeed, since the principal can estimate the quadratic variation of ζ^b , given in this case by:*

$$d\langle \zeta^b \rangle_t = (|\sigma^0|^2 + |\sigma^1|^2 |1 - Z_t^1|^2) dt, \quad t \in [0, T],$$

she has access (up to a sign) to the volatility control of the manager, i.e., the indexation parameter Z^1 . Therefore, in this particular case, there is 'less' moral hazard on volatility control, and we could expect that the model degenerates towards the first-best case regarding volatility control. This fact should (at least partially) explain the higher profit of the principal.

2.3 On other types of reporting

Throughout [Section 2.1](#), we have assumed that the manager reports only the net benefit ζ^b to his supervisor, the principal. The goal of this subsection is to provide some interesting results on other types of reporting. For example, the manager can report the sum of the cost and the sum of the outputs, instead of only reporting the difference between the two sums (the benefits). We will see that more precise reporting, in particular a separate reporting of the manager's performance, will lead to a degeneracy of the model towards the direct contracting case.

2.3.1 Reporting of profits and costs

Within the same hierarchical structure as before, we consider here that the manager reports to the principal the sum of the profits and the sum of the costs, and not only the net profit ζ^b defined by (2.1.5), i.e., the difference between the two values. Therefore his contract will be indexed on the following 2-dimensional state variable

$$\zeta^{\text{pc}} = \left(\sum_{i=0}^n X^i, \sum_{i=1}^n \xi^i \right).$$

Since the agents' problem remain unchanged, the dynamic of ζ^{pc} under their optimal efforts is given by

$$d\zeta_t^{\text{pc}} = \begin{pmatrix} \alpha_t^0 + \sum_{i=1}^n k^i Z_t^i \\ \frac{1}{2} \sum_{i=1}^n |Z_t^i|^2 \tilde{R}^i \end{pmatrix} dt + \begin{pmatrix} \sigma^0 & \sigma^1 & \dots & \sigma^n \\ 0 & \sigma^1 Z_t^1 & \dots & \sigma^n Z_t^n \end{pmatrix} \begin{pmatrix} dW_t^0 \\ \vdots \\ dW_t^n \end{pmatrix}, \quad t \in [0, T]. \quad (2.3.1)$$

We consider the same criterion for the manager as before. Following the reasoning behind [Proposition 2.1.4](#), we are led to consider contracts similar to (2.1.9), but now indexed on the 2-dimensional variable ζ^{pc} :

$$\xi_T^{\text{pc}} = \xi_0^{\text{pc}} - \int_0^T \mathcal{H}^{\text{pc}}(Z_s, \Gamma_s) ds + \int_0^T Z_s \cdot d\zeta_s^{\text{pc}} + \frac{1}{2} \int_0^T \text{Tr} \left[(\Gamma_s + R^0 Z_s (Z_s)^\top) d\langle \zeta^{\text{pc}} \rangle_s \right], \quad (2.3.2)$$

where $\mathcal{H}^{\text{pc}}$ is the manager's Hamiltonian, and (Z, Γ) is a tuple of parameters optimally chosen by the principal. We thus denote by $\mathcal{V}^{\text{pc}}$ the set of admissible control processes for the principal, defined as the collection of all processes $(Z, \Gamma) : [0, T] \times \mathcal{C}_T^2 \longrightarrow \mathbb{V}^{\text{pc}}$, predictable with respect to the filtration generated by ζ^{pc} and satisfying appropriate integrability conditions, where

$$\mathbb{V}^{\text{pc}} := \left\{ (z, \gamma) \in \mathbb{R}^2 \times \mathbb{R}^{2 \times 2} \text{ s.t. } \forall i \in \{1, \dots, n\}, \tilde{R}^i z^2 - |\sigma^i|^2 \gamma^{22} > 0 \right\}.$$

Given the dynamic (2.3.1) of the state variable ζ^{pc} and its quadratic variation, we can compute the manager's Hamiltonian and thus establish a result similar to [Proposition 2.1.5](#).

Proposition 2.3.1. *Let $(Z, \Gamma) \in \mathcal{V}^{\text{pc}}$ and $\xi_0^{\text{pc}} \in \mathbb{R}$. Given the associated contract ξ_T^{b} defined by (2.3.2), the optimal effort and the optimal control on the i -th agent's compensation ($i \in \{1, \dots, n\}$) chosen by the manager are respectively given by $\alpha_t^{\text{pc}} := k^0 Z_t^1$ and $Z_t^{i, \text{pc}} := z^{i, \text{pc}}(Z_t, \Gamma_t)$, $t \in [0, T]$, where*

$$z^{i, \text{pc}}(z, \gamma) := -\frac{k^i z^1 + |\sigma^i|^2 \gamma^{12}}{\tilde{R}^i z^2 + |\sigma^i|^2 \gamma^{22}}, \text{ for } (z, \gamma) \in \mathbb{V}^{\text{pc}} \text{ and } i \in \{1, \dots, n\}.$$

Under the probability $\mathbb{P}^{\text{pc}}$ associated to the optimal controls of the workers, the dynamics of ζ^{pc} and ξ^{pc} respectively satisfy:

$$\begin{aligned} d\zeta_t^{\text{pc}} &= \left(k^0 Z_t^1 + \sum_{i=1}^n k^i z^{i, \text{pc}}(Z_t, \Gamma_t) \right) dt + \begin{pmatrix} \sigma^0 & \sigma^1 & \dots & \sigma^n \\ 0 & \sigma^1 z^{1, \text{pc}}(Z_t, \Gamma_t) & \dots & \sigma^n z^{n, \text{pc}}(Z_t, \Gamma_t) \end{pmatrix} \begin{pmatrix} dW_t^0 \\ \vdots \\ dW_t^n \end{pmatrix}, \\ d\xi_t^{\text{pc}} &= \frac{\tilde{R}^0}{2} |Z_t^1|^2 dt + \frac{R^0}{2} \sum_{i=1}^n |\sigma^i|^2 |Z_t^1 + Z_t^2 z^{i, \text{pc}}(Z_t, \Gamma_t)|^2 dt + Z_t^1 \sigma^0 dW_t^0 + \sum_{i=1}^n \sigma^i (Z_t^1 + Z_t^2 z^{i, \text{pc}}(Z_t, \Gamma_t)) dW_t^i, \end{aligned}$$

where $\tilde{R}^0 := k^0 + R^0 |\sigma^0|^2$.

The principal's problem remains essentially unchanged compared to (2.1.6), she still maximises the difference between the sum of the outputs and sum of the compensations owed to the manager and the agents. The only difference is the contract space on which she optimises her criterion. In particular, given the optimal form (2.3.2) of the contracts, her problem becomes:

$$V_0^{\text{pc}} := \sup_{(\xi_0, Z, \Gamma) \in \mathbb{R}^{n+1} \times \mathcal{V}^{\text{pc}}} \mathbb{E}^{\mathbb{P}^{\text{pc}}} \left[\sum_{i=0}^n X_T^i - \sum_{i=1}^n \xi^i - \xi^{\text{pc}} \right].$$

The previous optimisation on $(Z, \Gamma) \in \mathcal{V}^{\text{pc}}$ is in fact equivalent to the following maximisation problem:

$$\sup_{(z, \gamma) \in \mathbb{V}^{\text{pc}}} \left\{ k^0 z^1 - \frac{1}{2} \tilde{R}^0 |z^1|^2 + \sum_{i=1}^n h^{i, \text{pc}}(z, \gamma) \right\}, \quad (2.3.3)$$

where $h^{i, \text{pc}}$ is defined for all $i \in \{1, \dots, n\}$ and $(z, \gamma) \in \mathbb{V}^{\text{pc}}$ by

$$h^{i, \text{pc}}(z, \gamma) := k^i z^{i, \text{pc}}(z, \gamma) - \frac{1}{2} \tilde{R}^i |z^{i, \text{pc}}(z, \gamma)|^2 - \frac{1}{2} R^0 |\sigma^i|^2 |z^1 + z^2 z^{i, \text{pc}}(z, \gamma)|^2. \quad (2.3.4)$$

The supremum in (2.3.3) is not easily computable in the general case, but if all agents have the same characteristics, we obtain the following result, whose proof is postponed to [Section 2.5.2](#).

Proposition 2.3.2. *If all agents are identical, the principal can achieve her utility in the DC case, denoted by V^{DC} and given by (2.5.2) in Lemma 2.5.1.*

Remark 2.3.3. *In this setting, we could also consider the case where the manager directly pays the agents he manages (not only designs their contracts). His value function is thus defined by:*

$$\tilde{V}_0^{0,\text{pc}}(\tilde{\xi}^{\text{pc}}) := \sup_{(\alpha^0, Z) \in \mathcal{A}^0 \times \mathcal{V}^0} \mathbb{E}^{\mathbb{P}^{\text{pc}}} \left[-\exp \left(-R^0 \left(\tilde{\xi}^{\text{pc}} - \sum_{i=1}^n \xi^i - \int_0^T c^0(\alpha_t^0) dt \right) \right) \right],$$

and the optimal form of contract for the manager is as follows:

$$\tilde{\xi}^{\text{pc}} = \tilde{\xi}_0^{\text{pc}} + \sum_{i=1}^n \xi^i - \int_0^T \mathcal{H}^{\text{pc}}(Z_s, \Gamma_s) ds + \int_0^T Z_s \cdot d\zeta_s^{\text{pc}} + \frac{1}{2} \int_0^T \text{Tr} \left[(\Gamma_s + R^0 Z_s (Z_s)^\top) d\langle \zeta^{\text{pc}} \rangle_s \right].$$

Since the manager is directly paying the agents, the problem of the principal is only to maximise the difference between the sum of the outputs and the compensation due to the manager:

$$\tilde{V}_0^{\text{pc}} := \sup_{(Z, \Gamma) \in \mathcal{V}^{\text{pc}}} \tilde{J}_0^{\text{P}}(\tilde{\xi}^{\text{pc}}), \quad \text{where} \quad \tilde{J}_0^{\text{P}}(\tilde{\xi}^{\text{pc}}) := \mathbb{E}^{\mathbb{P}^{\text{pc}}} \left[\sum_{i=0}^n X_T^i - \tilde{\xi}^{\text{pc}} \right],$$

which leads to the exact same maximisation as before. Indeed, in this case, the compensation for the manager $\tilde{\xi}^{\text{pc}}$ is equal to the sum of the compensations for the agents and the contract ξ^{pc} of the previous case:

$$\tilde{\xi}^{\text{pc}} = \xi^{\text{pc}} + \sum_{i=1}^n \xi^i,$$

and thus the two frameworks are strictly equivalent, since

$$\tilde{J}_0^{\text{P}}(\tilde{\xi}^{\text{pc}}) = \mathbb{E}^{\mathbb{P}^{\text{pc}}} \left[\sum_{i=0}^n X_T^i - \sum_{i=1}^n \xi^i - \xi^{\text{pc}} \right].$$

2.3.2 Separate reporting of the manager's performance

For now, we focused our study on two frameworks, one where the manager reports only net benefits (Section 2.1), the other where he reports the sum of the total profit (sum of outputs) and the total cost (sum of payments) separately (Section 2.3.1). We could consider other scenarii where the manager reports more information to the principal, for example if he reports his personal outcome X^0 separately. However, with this reporting, the HC case degenerates towards the DC case, in the sense that optimal efforts of the agents and the manager, as well as the value of the principal, can be equal to those in the DC case, given by Lemma 2.5.1. The proof of the following proposition is postponed to Section 2.5.2.

Proposition 2.3.4. *If the manager reports to the principal his own outcome X^0 separately, the problem degenerates towards the DC case. More precisely:*

- (i) *if the manager reports $\zeta^{\text{b},0} = (\sum_{i=1}^n X^i - \sum_{i=1}^n \xi^i, X^0)$, it is possible to find a sequence of contracts such that, at the limit, all workers apply the optimal efforts of the DC case, and the principal receives the maximum utility possible, i.e., V^{DC} ;*
- (ii) *if the manager reports $\zeta^{\text{pc},0} = (\sum_{i=1}^n X^i, \sum_{i=1}^n \xi^i, X^0)$, the result of (i) holds and moreover, if agents are identical, we can find a contract which allows to attain the DC case.*

Since these reporting leads to a degeneration towards the DC case, they are less interesting mathematically speaking. However, from a managerial point of view, it is relevant to observe that reporting the manager's output separately makes it possible to reduce the moral hazard within the hierarchy.

2.4 Explicit extensions of Sung's model

In this section, we propose some basic extensions of Sung's model developed in [Section 2.1](#). The first possible extension is to add a coefficient of ability for the manager, allowing to highlight the interest for the principal to delegate the management of the agents. Indeed, a good manager should have a positive impact on the work of the agents below him, and improve the efficiency of the hierarchical organisation to the point that above a certain number of workers, it becomes more profitable to group them in a working team led by a manager. The second extension considers a more complicated hierarchy, with a top manager in-between the principal and m managers. In this case, we show that, although it is more complicated to obtain analytical results, the resolution tools remain the same and it is thus theoretically very simple to add a level in the hierarchy using our approach.

2.4.1 On the positive impact of hierarchical organisations

As we have seen in the previous numerical results, the hierarchical organisation considered in Sung's model is not recommended compared to the direct contracting case. Indeed, the utility obtained by the principal, when she hires a manager, is smaller compared to the case when she contracts directly with all the workers. This decrease in utility is linked to the fact that the 'severity' of moral hazard increases with the number of levels in the hierarchy, but, above all, because we do not model any specific ability of the manager. In reality, hierarchical structures appear for logistical reasons, since it would be complicated for a principal to supervise a large number of workers, but also because a manager should have more ability to manage a small group of workers than the owner (or the investors) of the firm. We therefore provide a simple extension to take into account the manager's ability to improve the productivity of his workers.

We can say that the manager's skills are defined by a pair $(m, \tilde{m})$ where m measures the help he provides to the agents under his supervision, and $\tilde{m}$ is a penalty suffered because of his management activities. Indeed, it seems natural to consider that, by helping the agents, the manager has less time for his own work. We consider that the manager's skill $m > 0$ and $\tilde{m} \in [0, 1)$ respectively affect the cost functions of the agents and the manager as follows

$$c^i(a) = \frac{1}{2} \frac{|a|^2}{k^i \left(1 + \frac{m}{n}\right)}, \text{ for } i \in \{1, \dots, n\} \text{ and } c^0(a) = \frac{1}{2} \frac{|a|^2}{k^0 (1 - \tilde{m})}.$$

In the one hand, this means that the manager's ability m decreases the cost of effort of each agent under his supervision. However, the more agents he is responsible for, the weaker the effect is. On the other hand, the parameter $\tilde{m}$ increases its own cost, representing the fact that helping agents leaves him less time for his own work. In other words, devoting time to helping the agents penalises his own result. We obtain the same form of solutions as in the previous section, more precisely by replacing k^i and k^0 respectively by

$$\tilde{k}^i := k^i \left(1 + \frac{m}{n}\right) \text{ and } \tilde{k}^0 := k^0 (1 - \tilde{m}).$$

In order to evaluate the effects of the manager's ability parameters, we present some numerical results, with the same set of parameters as in the previous simulations (see parameters in [Section 2.2.2](#)). In [Figure 2.5](#), we represent, in addition to the previous results, the PPS for an agent (on the left), for the manager (in the middle), as well as the value of the principal (on the right) when we take into account the skills of the manager (red curves). Specifically, we set $(m, \tilde{m}) = (0.6, 0.1)$. We can see that the agents are incentivised to work more than without the help of their manager, but still less than in Sung's framework. The results are the opposite for the manager. The key point to observe here is that the value of the principal is higher than in the DC case, which means that a hierarchical structure can be beneficial for the principal when the manager has an ability to supervise the agents.

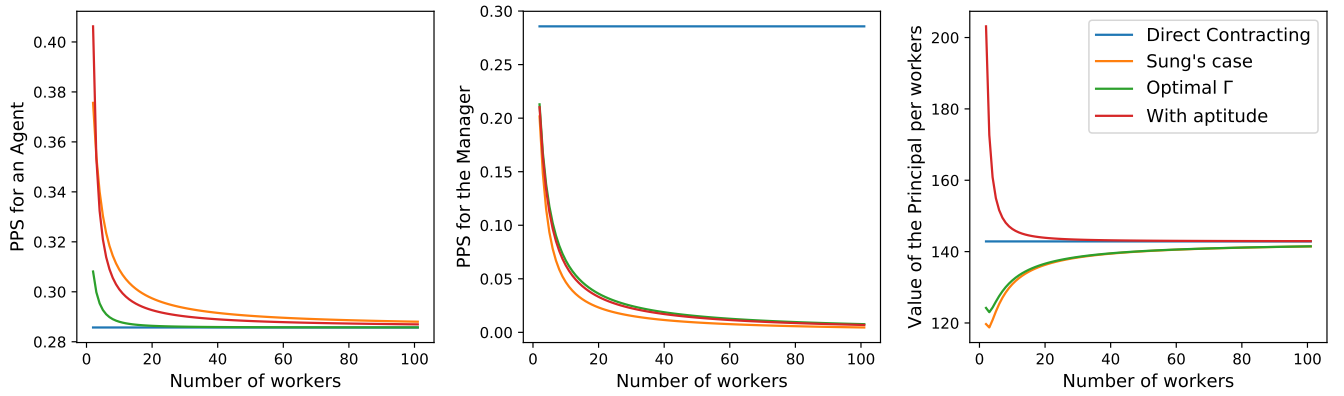
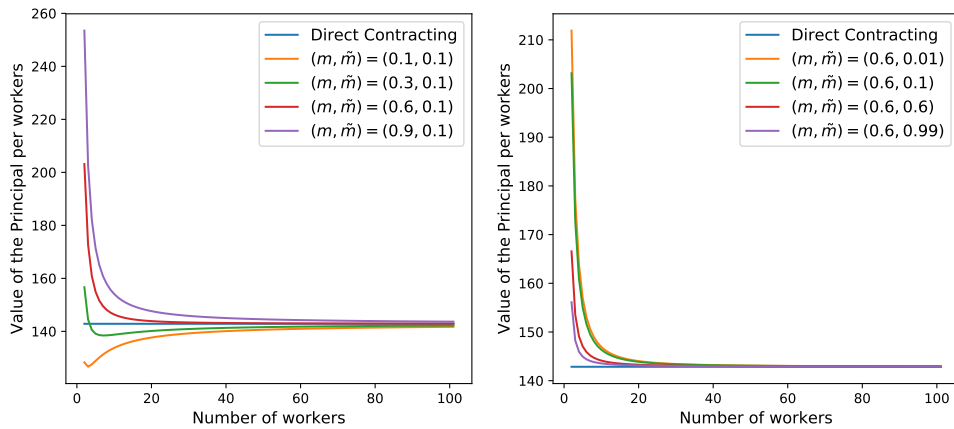


Figure 2.5 – PPS and principal's value per workers.

Therefore, this modeling highlights the interest for the principal to delegate the management of the agents. As can be seen in Figure 2.6, if the manager has good management skills, *i.e.*, m large enough, it becomes more cost-effective for the principal to form a working team headed by a manager. The right graph of Figure 2.6 shows that the influence of the parameter $\tilde{m}$ is more negligible, in the sense that if m is large enough, then the value of the principal is higher than in the DC case regardless of the value of $\tilde{m}$. This is in line with the previous results. Indeed, since under hierarchical contracting, the manager works less than in the DC case, the fact that his cost is higher matters less, as long as his ability is sufficiently beneficent to the agents.

Figure 2.6 – Principal's value per workers for different values of $(m, \tilde{m})$.

Remark 2.4.1. We could consider another case where the manager's skills m and $\tilde{m}$ respectively affect the outputs' drift of the agents and the manager in the following way:

$$dX_t^i = \left(\alpha_t^i + \frac{m}{n} \right) dt + \sigma^i dW_t^i, \text{ for } i \in \{1, \dots, n\} \text{ and } dX_t^0 = (\alpha_t^0 - \tilde{m}) dt + \sigma^0 dW_t^0, t \in [0, T].$$

This basic extension finally leads to the same problem and solution as Sung's model, only the utility of the principal is increased by $m - \tilde{m}$ (is decreased if $m - \tilde{m} < 0$). However, this model is not necessarily very realistic, because if the agents do not make any effort, the manager's ability is sufficient to increase the outputs. The previous model therefore seems more interesting, both in terms of interpretations and results.

2.4.2 On a more complex hierarchy

We consider in this section a more complex hierarchy illustrated by Figure 2.7: the principal hires a top manager, who hires m managers, and each manager j hires n_j agents. This new hierarchy requires small adjustments in notations, which will be reused in the general model in Chapter 3. First, the top manager controls his own outcome X^0 and receives the compensation ξ^0 designed by the principal. Then, the m managers, indexed by $j \in \{1, \dots, m\}$, each carry out their own outcome $X^{j,0}$ and receive the compensation $\xi^{j,0}$ designed by the top manager. For $j \in \{1, \dots, m\}$ and $i \in \{1, \dots, n_j\}$, the (j, i) -th agent is the i -th agent of the j -th manager. He controls the output $X^{j,i}$ and will receive the compensation $\xi^{j,i}$ designed by his manager. The dynamics of the output processes are given by

$$dX_t^0 = \alpha^0 dt + \sigma^0 dW_t^0 \quad \text{and} \quad dX_t^{j,i} = \alpha_t^{j,i} dt + \sigma^{j,i} dW_t^{j,i}, \quad t \in [0, T],$$

for all $j \in \{1, \dots, m\}$ and $i \in \{0, \dots, n_j\}$, where W^0 and all $W^{j,i}$ are independent Brownian motions.

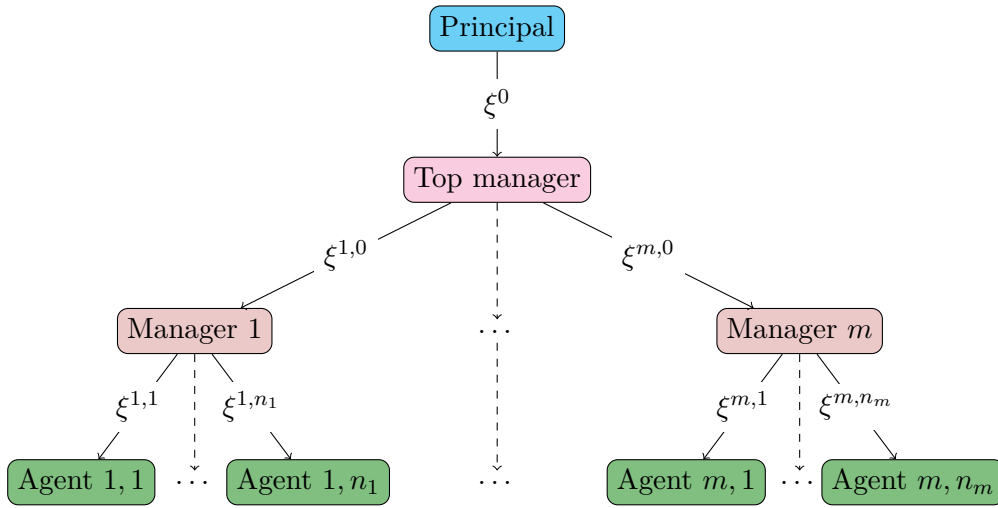


Figure 2.7 – A more complex hierarchy

Agent's problem. Apart from these notation changes, the problem for the agents remains the same. Therefore, for $j \in \{1, \dots, m\}$ and $i \in \{1, \dots, n_j\}$, the contract form for the (j, i) -th agent is similar to (2.1.7):

$$\xi_t^{j,i} = \xi_0^{j,i} - \int_0^t \mathcal{H}^{j,i}(Z_s^{j,i}) ds + \int_0^t Z_s^{j,i} dX_s^{j,i} + \frac{1}{2} R^{j,i} \int_0^t |Z_s^{j,i}|^2 d\langle X^{j,i} \rangle_s, \quad t \in [0, T],$$

where $Z^{j,i}$ is an $\mathbb{R}$ -valued process, predictable with respect to the filtration generated by $X^{j,i}$, satisfying appropriate integrability conditions, chosen by the j -th manager. This contract leads to the same optimal effort $\alpha_t^{j,i,*} = k^{j,i} Z_t^{j,i}$, $t \in [0, T]$.

Manager's problem. Let $j \in \{1, \dots, m\}$. The problem of the j -th manager is also equivalent to the manager's problem in Sung's model. In addition to choosing his effort $\alpha^{j,0}$, he designs the compensation $\xi^{j,i}$ for the i -th agent under his supervision by choosing the payment rate $Z^{j,i}$, for $i \in \{1, \dots, n_j\}$, and receives the payment $\xi^{j,0}$ offered by his supervisor, the top manager. We assume that the j -th manager only reports to his supervisor the total benefit of his working team, *i.e.*, the following variable

$$\zeta_t^j = \sum_{i=0}^{n_j} X_t^{j,i} - \sum_{i=1}^{n_j} \xi_t^{j,i}, \quad t \in [0, T].$$

Under optimal efforts of the agents, and using the notation $\tilde{R}^{j,i} := k^{j,i} + R^{j,i}|\sigma^{j,i}|^2$, the dynamic of ζ^j is given by

$$d\zeta_t^j = \left(\alpha_t^{j,0} + \sum_{i=1}^{n_j} \left(k^{j,i} Z_t^{j,i} - \frac{1}{2} \tilde{R}^{j,i} |Z_t^{j,i}|^2 \right) \right) dt + \sigma^{j,0} dW_t^{j,0} + \sum_{i=1}^{n_j} \sigma^{j,i} (1 - Z_t^{j,i}) dW_t^{j,i}, \quad t \in [0, T].$$

We assume that the contract for each manager is indexed on the total benefit of his working team, *i.e.*, each contract $\xi^{j,0}$ is a measurable function of ζ^j only. Given the form of his value function, ζ^j is the only state variable of the j -th manager's control problem. Since he controls both the drift and the volatility of ζ^j , the optimal form of contract is given, for a constant $\xi_0^{j,0} \in \mathbb{R}$, by the terminal value of:

$$\xi_t^{j,0} = \xi_0^{j,0} - \int_0^t \mathcal{H}^j(Z_s^j, \Gamma_s^j) ds + \int_0^t Z_s^j d\zeta_s^j + \frac{1}{2} \int_0^t (\Gamma_s^j + R^j |Z_s^j|^2) d\langle \zeta^j \rangle_s, \quad t \in [0, T], \quad (2.4.1)$$

where $\mathcal{H}^j$ is his Hamiltonian and $(Z^j, \Gamma^j) \in \mathcal{V}^j$ is a pair of parameters optimally chosen by the top manager. More precisely, we define by $\mathcal{V}^j$ the collection of all processes $(Z, \Gamma) : [0, T] \times \mathcal{C}_T \rightarrow \mathbb{V}^j$, predictable with respect to the filtration generated by ζ^j and satisfying appropriate integrability conditions, where

$$\mathbb{V}^j := \left\{ (z, \gamma) \in \mathbb{R} \times \mathbb{R} \text{ s.t. } \forall i \in \{1, \dots, n_j\}, \tilde{R}^{j,i} z - |\sigma^{j,i}|^2 \gamma > 0 \right\}.$$

The set $\mathcal{V}^0 := \prod_{j=1}^m \mathcal{V}^j$ thus represents the admissible control processes for the top manager. By computing and maximising the j -th manager's Hamiltonian, we obtain the following proposition.

Proposition 2.4.2. *Let $j \in \{1, \dots, m\}$ and consider a contract of the form (2.4.1) indexed by a pair $(Z^j, \Gamma^j) \in \mathcal{V}^j$ and a constant $\xi_0^{j,0} \in \mathbb{R}$. Then, the optimal effort of the j -th manager and his optimal control on the i -th agent's compensation ($i \in \{1, \dots, n_j\}$) are respectively given by the processes $\alpha^{j,0,*} := k^{j,0} Z^j$ and $Z^{j,i,*} := z^{j,i,*}(Z^j, \Gamma^j)$ where*

$$z^{j,i,*}(z, \gamma) := \frac{k^{j,i} z - |\sigma^{j,i}|^2 \gamma}{\tilde{R}^{j,i} z - |\sigma^{j,i}|^2 \gamma}, \quad \text{for all } (z, \gamma) \in \mathbb{V}^j.$$

Remark 2.4.3. *The problem highlighted in Remark 2.1.3 obviously also arises here. Indeed, to restrict the contract $\xi^{j,i}$ for the (j, i) -th agent to a measurable function of $X^{j,i}$, the payment process $Z^{j,i}$ must be predictable with respect to the filtration generated by $X^{j,i}$. Similarly, to restrict the contract $\xi^{j,0}$ for the j -th manager to a measurable function of ζ^j , the payment rate processes Z^j and Γ^j must be predictable with respect to the filtration generated by ζ^j . Since the optimal payment rate $Z^{j,i,*}$ is a function of Z^j and Γ^j , the model is consistent if and only if Z^j and Γ^j are deterministic functions of time only, which is actually the case in this example (they are even constant).*

Top manager's problem. The top manager carries out his own output X^0 , by choosing his effort level α^0 , and designs the contracts for the m managers. Like other managers, he has a CARA utility function with a risk-aversion parameter R^0 , and maximises the utility of the difference between the payment he receives from the principal, ξ^0 , and his cost of effort:

$$V_0^0(\xi^0) := \sup_{(\alpha^0, \mathcal{Z}^0) \in \mathcal{A}^0 \times \mathcal{V}^0} \mathbb{E}^{\mathbb{P}^0} \left[-e^{-R^0(\xi^0 - \int_0^T c^0(\alpha_t^0) dt)} \right],$$

where $\mathbb{P}^0$ is the probability associated to the effort $\alpha^0 \in \mathcal{A}^0$ and the choice of the process $\mathcal{Z}^0 := (Z^j, \Gamma^j)_{j=1}^m \in \mathcal{V}^0$, under the optimal efforts of the m managers and their agents. In this setting, the top manager observes in continuous time the net benefit of each working team led by a manager, that is the tuple $(\zeta^j)_{j=1}^m$. Moreover,

like every managers, the top manager reports in continuous time to the principal the benefits of his team of workers composed of all managers and agents below him, namely the following variable:

$$\zeta_t^0 = X_t^0 + \sum_{j=1}^m \zeta_t^j - \sum_{j=1}^m \xi_t^j, \quad t \in [0, T].$$

Therefore the principal can only offer to the top manager a contract indexed on ζ^0 , and, since he controls the volatility of ζ^0 through his choice of contracts for the managers, the optimal form of his compensation is the same as (2.4.1) but indexed on the variable ζ^0 :

$$\xi_T^0 := \xi_0^0 - \int_0^T \mathcal{H}^0(Z_s, \Gamma_s) ds + \int_0^T Z_s d\zeta_s^0 + \frac{1}{2} \int_0^T (\Gamma_s + R^0 |Z_s|^2) d\langle \zeta^0 \rangle_s,$$

for some constant $\xi_0^0 \in \mathbb{R}$ and where $(Z, \Gamma) \in \mathcal{V}$ is a pair of processes optimally chosen by the principal. More precisely, we define by $\mathcal{V}$ the collection of all processes $(Z, \Gamma) : [0, T] \times \mathcal{C}_T \rightarrow \mathbb{V}$, predictable with respect to the filtration generated by ζ^0 and satisfying appropriate integrability conditions, where $\mathbb{V}$ is the set of all $(z, \gamma) \in \mathbb{R}^2$ such that the top manager's Hamiltonian $\mathcal{H}^0$ defined below by (2.4.2) is well defined.

Under optimal efforts and controls of the managers and the agents (and associated probability $\mathbb{P}^0$), the dynamic of ζ^0 is given for all $t \in [0, T]$ by:

$$d\zeta_t^0 = \left(\alpha_t^0 + \sum_{j=1}^m h^{0,j}(Z_t^j, \Gamma_t^j) \right) dt + \sigma^0 dW_t^0 + \sum_{j=1}^m (1 - Z_t^j) \left(\sigma^j dW_t^j + \sum_{i=1}^{n_j} \sigma^{j,i} (1 - z^{j,i,*}(Z_t^j, \Gamma_t^j)) dW_t^{j,i} \right),$$

where, for all $j \in \{1, \dots, m\}$, $\tilde{R}^j := k^j + R^j |\sigma^j|^2$ and in addition for all $(z, \gamma) \in \mathbb{V}^j$,

$$h^{0,j}(z, \gamma) := k^j z - \frac{\tilde{R}^j}{2} |z|^2 + \sum_{i=1}^{n_j} \left(k^{j,i} z^{j,i,*}(z, \gamma) - \frac{\tilde{R}^{j,i}}{2} |z^{j,i,*}(z, \gamma)|^2 - \frac{R^j}{2} |z|^2 |\sigma^{j,i}|^2 |1 - z^{j,i,*}(z, \gamma)|^2 \right).$$

Therefore, the top manager's Hamiltonian is defined as follows:

$$\begin{aligned} \mathcal{H}^0(z, \gamma) := & \sum_{j=1}^m \sup_{(z^j, \gamma^j) \in \mathbb{V}^j} \left\{ z h^{0,j}(z^j, \gamma^j) + \frac{1}{2} \gamma |1 - z^j|^2 \left(|\sigma^j|^2 + \sum_{i=1}^{n_j} |\sigma^{j,i}|^2 |1 - z^{j,i,*}(z^j, \gamma^j)|^2 \right) \right\} \\ & + \sup_{a \in A^0} \{ a z - c^0(a) \} + \frac{1}{2} \gamma |\sigma^0|^2. \end{aligned} \quad (2.4.2)$$

The first supremum is attained for the optimal effort $\alpha_t^{0,*} = k^0 Z_t$ for $t \in [0, T]$. In addition, for all $j \in \{1, \dots, m\}$, the optimal controls of the top manager for the j -th managers' contracts are given by the processes $Z_t^{j,*} := z^{j,*}(Z_t, \Gamma_t)$ and $\Gamma_t^{j,*} := \gamma^{j,*}(Z_t, \Gamma_t)$, $t \in [0, T]$, with, for $(z, \gamma) \in \mathbb{V}$,

$$\gamma^{j,*}(z, \gamma) = -R^j (z^{j,*}(z, \gamma))^3 + \frac{\gamma}{z} z^{j,*}(z, \gamma) |1 - z^{j,*}(z, \gamma)|^2,$$

where $z^{j,*}(z, \gamma)$ can be numerically computed as the maximiser of the previous Hamiltonian $\mathcal{H}^0$.

Remark 2.4.4. In addition to the measurability issues of the manager's control, mentioned in Remark 2.4.3, we also have, in this more complex model, a measurability problem for the top manager's control. Indeed, since we restrict the space of the contracts for the j -th manager to measurable contracts with respect to his own ζ^j , the processes Z^j and Γ^j defining the contract must be adapted to the filtration generated by ζ^j . In fact, these processes, chosen by the top manager, are functions of Z and Γ , the principal's controls. Since the principal only observes ζ^0 , the processes Z and Γ should be adapted to the filtration generated by ζ^0 , contradicting the fact that Z^j and Γ^j are adapted to the filtration generated by ζ^j . Again, this question of measurability is actually not a problem in this particular case since all optimal controls are deterministic or even constant, but will be in a more general framework.

Principal's problem. We consider the same problem for the principal as before, namely that she maximises the difference between the sum of the outputs and the sum of the costs at terminal time T , which can be summarised by $\zeta_T^0 - \xi^0$. Her problem is reduced to the optimal choice of the indexation parameters in the contract ξ^0 , the pair (Z, Γ) , and thus to the following maximisation problem:

$$\sup_{(z, \gamma) \in \mathbb{V}} \left\{ k^0 z - \frac{\tilde{R}^0}{2} |z|^2 + \sum_{j=1}^m \left(h^{0,j,\star}(z, \gamma) - \frac{R^0}{2} |z|^2 |1 - z^{j,\star}(z, \gamma)|^2 \left(|\sigma^j|^2 + \sum_{i=1}^{n_j} |\sigma^{j,i}|^2 |1 - z^{j,i,\star}(z, \gamma)|^2 \right) \right) \right\},$$

where $\tilde{R}^0 := k^0 + R^0 |\sigma^0|^2$ and, abusing notations slightly, we have denoted, for all $(z, \gamma) \in \mathbb{V}$,

$$h^{0,j,\star}(z, \gamma) := h^{0,j}(z^{j,\star}(z, \gamma), \gamma^{j,\star}(z, \gamma)) \quad \text{and} \quad z^{j,i,\star}(z, \gamma) := z^{j,i}(z^{j,\star}(z, \gamma), \gamma^{j,\star}(z, \gamma)).$$

Optimal parameters $(z, \gamma) \in \mathbb{V}$ are constant over time, and their values can be obtained thanks to a simple numerical optimisation.

By comparing to the first example where there is no top manager and only one manager, we can see that the structure of the problem is the same. Adding a level in the hierarchy is no more complicated, all it takes is writing an additional control problem. With this in mind, and in order to avoid overloading the notations, we will consider only three levels in the hierarchy for the general model in [Chapter 3](#), *i.e.*, one principal, m managers, with a fixed number of agents each.

2.5 Further comments on Sung's model

Throughout this chapter, we compare our results in continuous time with those of the one-period model detailed in [327], but also with the results that would be obtained in a direct contracting framework, *i.e.*, in the case where the principal contracts directly with the agents, without the intermediary of a manager. In the latter case, we are faced with a more traditional principal–agents problem (with a finite number of agents), which can be solved in a straightforward way, by extending the results obtained by [Holmström and Milgrom \[181\] \(1987\)](#) to a multitude of agents. The following lemma reports the optimal efforts of the agents as well as the utility of the principal in this case. A similar result is mentioned in [327], but we should refer to the work of [Koo, Shim, and Sung \[210\] \(2008\)](#) for a rigorous result, or to the more general model of [Élie and Possamaï \[140\] \(2019\)](#).

Lemma 2.5.1. *With direct contracting, the optimal efforts of the workers are given by*

$$\alpha^{i,\text{DC}} := k^i Z^{i,\text{DC}}, \quad \text{where} \quad Z^{i,\text{DC}} := \frac{k^i}{\tilde{R}^i} \quad \text{and} \quad \tilde{R}^i := k^i + R^i |\sigma^i|^2. \quad (2.5.1)$$

Moreover, if $X_0 = 0$, then the value V^{DC} for the principal is equal to:

$$V^{\text{DC}} := \frac{1}{2} \sum_{i=0}^n \frac{|k^i|^2}{\tilde{R}^i}. \quad (2.5.2)$$

2.5.1 Proofs related to the initial model

This section contains the proofs of the main results related to the continuous-time version of [Sung's](#) model, established in [Section 2.1](#).

Proof of [Proposition 2.1.5](#). As in the statement of the proposition, let $(Z, \Gamma) \in \mathcal{V}^b$. According to [Proposition 2.1.4](#), the optimal form of contract is given by (2.1.9). Recall that the Hamiltonian $\mathcal{H}^b$ is defined by

(2.1.10), and that the dynamic of ζ^b under optimal effort of the agents is given by (2.1.8). Replacing in the value function of the manager, defined by (2.1.4), we obtain:

$$J_0^0(\xi^b, \alpha^0, (Z^i)_{i=1}^n) = -e^{-R^0 \xi_0^b} \exp \left(-R^0 \left(\int_0^T \left(h^b(Z_s, \Gamma_s, (Z_s^i)_{i=1}^n, \alpha_s^0) - \mathcal{H}^b(Z_s, \Gamma_s) \right) ds \right) \right. \\ \left. \times \mathbb{E}^{\mathbb{P}^0} \left[\mathcal{E} \left(-R^0 \int_0^T Z_s \left(\sigma^0 dW_s^0 + \sum_{i=1}^n \sigma^i (1 - Z_s^i) dW_s^i \right) \right) \right]_T \right],$$

where $\mathcal{E}$ denotes for the Doléans–Dade exponential and the function h^b is defined by:

$$h^b(z, \gamma, (z^i)_{i=1}^n, a) := \frac{1}{2} \gamma |\sigma^0|^2 + za - c^0(a) + \sum_{i=1}^n \left(z \left(k^i z^i - \frac{\tilde{R}^i}{2} |z^i|^2 \right) + \frac{1}{2} \gamma |\sigma^i|^2 |1 - z^i|^2 \right).$$

Under the appropriate integrability conditions on Z , the previously considered Doleans–Dade exponential is a martingale, which implies:

$$J_0^0(\xi^b, \alpha^0, (Z^i)_{i=1}^n) = -e^{-R^0 \xi_0^b} \exp \left(-R^0 \left(\int_0^T \left(h^b(Z_s, \Gamma_s, (Z_s^i)_{i=1}^n, \alpha_s^0) - \mathcal{H}^b(Z_s, \Gamma_s) \right) ds \right) \right) \leq -e^{-R^0 \xi_0^b}.$$

The inequality in the previous equation stems from the fact that $\mathcal{H}^b \geq h^b$ by definition. In particular, the equality is attained when $(Z^i)_{i=1}^n$ and α^0 maximise the function h^b . More precisely, we have:

$$k^0 z = \arg \max_{a \in \mathbb{R}} \{az - c^0(a)\}, \quad \text{and} \quad z^{i,b}(z, \gamma) = \arg \max_{z^i \in \mathbb{R}} \left\{ z \left(k^i z^i - \frac{\tilde{R}^i}{2} |z^i|^2 \right) + \frac{1}{2} \gamma |\sigma^i|^2 |1 - z^i|^2 \right\},$$

where the function $z^{i,b}$ is defined in the statement of the proposition by (2.1.11). Therefore, the optimal controls of the manager are given for all $t \in [0, T]$ by $\alpha_t^b := k^0 Z_t$ and $Z_t^{i,b} := z^{i,b}(Z_t, \Gamma_t)$.

To complete the proof, it remains to compute the dynamics of ζ^b and ξ^b under the optimal efforts of the manager. In the one hand, by plugging the optimal effort in the dynamic of ζ^b defined by (2.1.8), it is straightforward to obtain the desired result:

$$d\zeta_t^b = \left(k^0 Z_t + \sum_{i=1}^n \left(k^i z^{i,b}(Z_t, \Gamma_t) - \frac{\tilde{R}^i}{2} |z^{i,b}(Z_t, \Gamma_t)|^2 \right) \right) dt + \sigma^0 dW_t^0 + \sum_{i=1}^n \sigma^i (1 - z^{i,b}(Z_t, \Gamma_t)) dW_t^i.$$

Then, using both the previous dynamic for ζ^b and (2.1.9), we obtain:

$$d\xi_t^b = \left(h^b(Z_t, \Gamma_t, (Z_t^{i,b})_{i=1}^n, \alpha_t^b) - \mathcal{H}^b(Z_t, \Gamma_t) \right) dt + c^0(\alpha_t^b) dt \\ + \frac{1}{2} R^0 Z_t^2 \left(|\sigma^0|^2 + \sum_{i=1}^n |\sigma^i|^2 |1 - Z_t^{i,b}|^2 \right) dt + Z_t \left(\sigma^0 dW_t^0 + \sum_{i=1}^n \sigma^i (1 - Z_t^{i,b}) dW_t^i \right).$$

By definition of h^b and the optimal controls of the manager, the difference $h^b - \mathcal{H}^b$ is equal to zero. Noticing that the cost $c^0(\alpha_t^b)$ is equal to $k^0 Z_t^2/2$, we obtain the desired dynamic for ξ^b , which concludes the proof. $\square$

Proof of Proposition 2.1.6. Since the principal is risk-neutral, her reward function can be computed as follows:

$$J_0^P(\xi^b) = \zeta_0^b - \xi_0^b + \mathbb{E}^{\mathbb{P}^b} \left[\int_0^T \sigma^0 (1 - Z_t) dW_t^0 \right] + \sum_{i=1}^n \mathbb{E}^{\mathbb{P}^b} \left[\int_0^T \sigma^i (1 - Z_t) (1 - z^{i,b}(Z_t, \Gamma_t)) dW_t^i \right] \\ + \mathbb{E}^{\mathbb{P}^b} \left[\int_0^T \left(k^0 Z_t - \frac{1}{2} \tilde{R}^0 Z_t^2 + \sum_{i=1}^n h^{i,b}(Z_t, \Gamma_t) \right) dt \right],$$

for $(Z, \Gamma) \in \mathcal{V}^b$ and recalling that $\tilde{R}^0 := k^0 + R^0|\sigma^0|^2$. First, the expectation of the two stochastic integrals is equal to zero. Then, maximising the reward function is equivalent to maximising inside the last expectation and the integral with respect to time, which leads to the following optimisation problem:

$$\sup_{(z, \gamma) \in \mathbb{V}^b} \left\{ k^0 z - \frac{1}{2} \tilde{R}^0 z^2 + \sum_{i=1}^n h^{i,b}(z, \gamma) \right\}. \quad (2.5.3)$$

Since $h^{i,b}$ is strictly concave in γ , the first-order condition (FOC) is sufficient to obtain the optimal γ :

$$0 = \sum_{i=1}^n \left(k^i \partial_\gamma z^{i,b}(z, \gamma) - \tilde{R}^i z^{i,b}(z, \gamma) \partial_\gamma z^{i,b}(z, \gamma) + z^2 R^0 |\sigma^i|^2 \partial_\gamma z^{i,b}(z, \gamma) (1 - z^{i,b}(z, \gamma)) \right).$$

By computing the derivatives of $z^{i,b}(z, \gamma)$ with respect to γ for all $i \in \{1, \dots, n\}$, namely

$$\partial_\gamma z^{i,b}(z, \gamma) = \frac{-R^i |\sigma^i|^4 z}{|\tilde{R}^i z - |\sigma^i|^2 \gamma|^2},$$

one obtain the following FOC

$$0 = z(\gamma + z^3 R^0) \sum_{i=1}^n \frac{-|R^i|^2 |\sigma^i|^8}{(\tilde{R}^i z - |\sigma^i|^2 \gamma)^3},$$

and therefore it is optimal to set $\Gamma_t^b := -R^0 (Z_t^b)^3$, for all $t \in [0, T]$. Finally, since the optimal control of the principal (Z^b, Γ^b) must be in $\mathcal{V}^b$, the process Z^b must be positive. Unfortunately, we cannot obtain an explicit value for Z^b , but we can estimate it thanks to a simple numerical optimisation, as the maximiser of (2.1.12). Noticing that the optimisation problem (2.1.12) does not depend on time nor on the state variable, its maximiser z^b is a positive constant.

The point (ii) of the proposition is a simple computation of the contract under the optimal payment rates chosen by the principal:

$$\begin{aligned} \xi_T^b &= \xi_0^b - \int_0^T \mathcal{H}^b(z^b, -R^0 (z^b)^3) dt + z^b \int_0^T d\zeta_t^b + \frac{1}{2} R^0 |z^b|^2 (1 - z^b) \int_0^T d\langle \zeta^b \rangle_t \\ &= \xi_0^b - \mathcal{H}^b(z^b, -R^0 (z^b)^3) T + z^b (\zeta_T^b - \zeta_0^b) + \frac{T}{2} R^0 |z^b|^2 (1 - z^b) \langle \zeta^b \rangle_T. \end{aligned}$$

The same type of computation leads to the point (iii). Moreover, since the reservation utility of the agents and the managers are equal to -1 , we obtain that the optimal choice of ξ_0^b is zero. To prove the last point of the proposition, it is then sufficient to compute the utility of the principal for the optimal contracts. $\square$

2.5.2 Proofs related to the extensions of the initial model

This section contains the proofs of the main results established in Section 2.3, which concern the extensions to different reporting of the initial model developed in Section 2.1.

Proof of Proposition 2.3.2. As mentioned in the proposition, we assume that all agents are identical, in the sense that for all $i \in \{1, \dots, n\}$, $k^i = k$, $R^i = R$, $\sigma^i = \sigma$, for some $(k, R, \sigma) \in (\mathbb{R}_+^*)^3$. In this case, the optimisation problem (2.3.3) becomes:

$$\sup_{(z, \gamma) \in \mathbb{V}^{pc}} \left\{ k^0 z^1 - \frac{1}{2} \tilde{R}^0 |z^1|^2 + n h^{1,pc}(z, \gamma) \right\},$$

where $h^{\cdot, \text{pc}} := h^{i, \text{pc}}$, defined by (2.3.4) for all $i \in \{1, \dots, n\}$. By Lemma 2.5.1, to achieve the optimal effort of the direct contracting case for all workers, we should have:

$$z^{1, \star} = \frac{k^0}{\tilde{R}^0} \text{ and, for all } i \in \{1, \dots, n\}, z^{i, \text{pc}}(z, \gamma) = z^{\cdot, \text{pc}, \star} := \frac{k}{\tilde{R}}, \text{ where } \tilde{R} := k + R|\sigma|^2.$$

Indeed, if $z^1 = z^{1, \star}$ and $z^{\cdot, \text{pc}} = z^{\cdot, \text{pc}, \star}$, then this imply the optimal effort of the DC case for respectively the manager and the agents. Moreover, the value of supremum has to be equal to the value of the principal in the DC case, given by

$$V^{\text{DC}} = \frac{1}{2} \frac{|k^0|^2}{\tilde{R}^0} + \frac{1}{2} \frac{n|k|^2}{\tilde{R}}.$$

Using the expected values of z^1 and z^{pc} , we obtain the following value for the supremum:

$$\frac{1}{2} \frac{|k^0|^2}{\tilde{R}^0} + \frac{n}{2} \frac{|k|^2}{\tilde{R}} - \frac{n}{2} R^0 |\sigma|^2 \inf_{(z, \gamma) \in \mathbb{V}^{\text{pc}}} \left\{ \left| \frac{k^0}{\tilde{R}^0} + z^2 \frac{k}{\tilde{R}} \right|^2 \right\}.$$

The infimum in the previous equation is attained for $z^{2, \star} = -k^0 \tilde{R} / (k \tilde{R}^0)$, and we thus obtain the value of the DC case. It then remains to solve, for $(\gamma^{12}, \gamma^{22}) \in \mathbb{R}^2$ such that $(z, \gamma) \in \mathbb{V}^{\text{pc}}$,

$$-\frac{k^i z^{1, \star} + |\sigma^i|^2 \gamma^{12}}{\tilde{R}^i z^{2, \star} + |\sigma^i|^2 \gamma^{22}} = \frac{k}{\tilde{R}},$$

to ensure that $z^{\cdot, \text{pc}}$ has the requested value. This equation is equivalent to:

$$\gamma^{12} = \frac{k^0}{\tilde{R}^0} R - \frac{k}{\tilde{R}} \gamma^{22}, \text{ for } \gamma^{22} < -\frac{k^0 |\tilde{R}|^2}{k |\sigma|^2 \tilde{R}^0}.$$

In conclusion, for any γ^{22} satisfying the previous inequality, by setting

$$z^1 := \frac{k^0}{\tilde{R}^0}, z^2 := -\frac{k^0 \tilde{R}}{k \tilde{R}^0} \text{ and } \gamma^{12} := \frac{k^0}{\tilde{R}^0} R - \frac{k}{\tilde{R}} \gamma^{22},$$

the value function of the principal and the efforts of both the agents and the manager are equal to the those in the DC case. $\square$

Proof of Proposition 2.3.4. (i) To prove the first point of the proposition, we follow the reasoning of the previous subsection by considering contract for the manager of the form (2.3.2), indexed on the 2-dimensional reported variable $\zeta^{\text{b}, 0}$ by an $\mathbb{R}^2 \times \mathbb{R}^{2 \times 2}$ -valued process (Z, Γ) . We find that the optimal effort of the manager and the optimal payment rate for the i -th agents are respectively given for all $t \in [0, T]$ by $\alpha_t^{\text{b}, 0} = k^0 Z_t^2$ and $Z_t^{i, \text{b}, 0} = z^{i, \text{b}}(Z_t^1, \Gamma_t^{11})$, where $(Z^1, \Gamma^{11}) \in \mathcal{V}^{\text{b}}$. The principal's problem is then equivalent to:

$$\sup_{z^2 \in \mathbb{R}} \left\{ k^0 z^2 - \frac{1}{2} \tilde{R}^0 |z^2|^2 \right\} + \sup_{(z^1, \gamma^{11}) \in \mathbb{V}^{\text{b}}} \sum_{i=1}^n h^{i, \text{b}}(z^1, \gamma^{11}),$$

recalling that the function $h^{i, \text{b}}$ is defined for all $i \in \{1, \dots, n\}$ by (2.1.13). Therefore, by choosing the constant processes $Z_t^2 := Z^{0, \text{DC}}$, defined by (2.5.1), and $\Gamma_t^{11} := 0$, for all $t \in [0, T]$, all workers do the optimal effort of the DC case and the supremum becomes:

$$\frac{1}{2} \frac{|k^0|^2}{\tilde{R}^0} + \frac{1}{2} \sum_{i=1}^n \frac{|k^i|^2}{\tilde{R}^i} - \frac{1}{2} R^0 \inf_{z^1 > 0} \sum_{i=1}^n |\sigma^i|^2 |z^1|^2 \left| 1 - \frac{|k^i|^2}{\tilde{R}^i} \right|.$$

The infimum is equal to that in the DC case and can be achieved by considering a sequence $(z^{1,n})_{n \geq 0}$ converging to zero. Therefore, the principal can construct a sequence of contracts, namely $\xi^{b,0,n}$, with for example $Z^{1,n} := 1/n$, allowing the workers to apply optimal efforts of the DC case and such that her utility converges to V^{DC} .

(ii) To prove the second point, we consider contracts of the form (2.3.2), but indexed on the 3-dimensional variable $\zeta^{\text{pc},0}$, by an $\mathbb{R}^3 \times \mathbb{R}^{3 \times 3}$ -valued process (Z, Γ) . The fact that the principal's problem is also degenerating into the DC case can be shown in the same way as we prove the first point. Moreover, in the case of identical agents, following the same reasoning as in Section 2.3.1, the optimal contract which allow to attain the DC case exists. Indeed, in this case, the principal's problem is equivalent to:

$$\sup_{(z, \gamma, z^3) \in \mathbb{V}^{\text{pc}} \times \mathbb{R}} \left\{ k^0 z^3 - \frac{1}{2} \tilde{R}^0 |z^3|^2 + \sum_{i=1}^n h^{i, \text{pc}}(z, \gamma) \right\},$$

recalling that, for all $i \in \{1, \dots, n\}$, $h^{i, \text{pc}}$ is defined by (2.3.4). The optimal payment rate z^3 is clearly given by the ratio $k^0/\tilde{R}^0$. Moreover, if the agents are identical, in the sense that for all $i \in \{1, \dots, n\}$, $k^i = k$, $R^i = R$, $\sigma^i = \sigma$, for some $(k, R, \sigma) \in (\mathbb{R}_+^*)^3$, the previous optimisation problem is equal to

$$\frac{1}{2} \frac{|k^0|^2}{\tilde{R}^0} + n \sup_{(z, \gamma) \in \mathbb{V}^{\text{pc}}} \left\{ k z^{\cdot, \text{pc}}(z, \gamma) - \frac{1}{2} \tilde{R} |z^{\cdot, \text{pc}}(z, \gamma)|^2 - \frac{1}{2} R^0 |\sigma|^2 |z^1|^2 + z^2 z^{\cdot, \text{pc}}(z, \gamma) \right\},$$

where $\tilde{R} := k + R|\sigma|^2$ and $z^{\cdot, \text{pc}} := z^{i, \text{pc}}$ for all $i \in \{1, \dots, n\}$. By setting

$$z^1 = -\frac{k}{\tilde{R}} z^2 \quad \text{and} \quad \gamma^{12} = -\frac{k}{\tilde{R}} (R z^2 + \gamma^{22}) \quad \text{for all } (z^2, \gamma^{22}) \in \mathbb{R}^2,$$

one obtain $z^{\cdot, \text{pc}}(z, \gamma) = k/\tilde{R}$ and $z^1 + z^2 z^{\cdot, \text{pc}}(z, \gamma) = 0$, implying that the efforts of the workers and the value of the supremum are equal to those in the DC case. $\square$

A general hierarchical principal–agent model

Y'a plus de problème de hiérarchie
 Car c'est toujours moi qui commande
 C'est toujours moi qui obéit
 Faut de la discipline dans une bande

Je suis une bande de jeunes, RENAUD

Introduction

This chapter is devoted to the study of a more general hierarchical principal–agent model than the one described in the previous chapter. In particular, we assume that the workers (managers and agent) can also control the volatility of their output process, in addition to the drift. Moreover, we consider general utilities, allowing us to recover the exponential utility functions (CARA) of Sung's model [327], but also other cases such as the one of risk–neutral workers. Finally, we consider a more general hierarchy since the principal contracts with m managers, who in turns subcontract with the agents in their teams. The model is described in [Section 3.1](#).

We thus provide in this chapter, more precisely from [Sections 3.3 to 3.5](#), a systematic and rigorous method to solve any hierarchy problem of the same type described in the previous chapter. As mentioned in the previous chapter, due to volatility control, the search for the optimal form of contracts requires more recent theoretical tools, developed notably by Cvitanić, Possamaï, and Touzi [111] (2018). More precisely, the resolution is based on the theory of second–order BSDEs, subject to a slight extension to take into account the plurality of agents in the hierarchy.

The theoretical model we develop allows us to determine the optimal form of incentives for a particular hierarchical structure, which can be extended in a straightforward way to a larger scale hierarchy. The main result is that, in a general model in continuous time, the optimal form of contracts for a given worker must also include an indexation on the outputs of other workers at his level, in addition to being indexed on the quadratic variation of his own controlled process. However, several hypotheses are necessary to complete our study, notably on the shape of the dynamics of the state variables. Nevertheless, we will see that these assumptions are satisfied in the most common examples.

As mentioned above, the method we developed throughout the theoretical part of this chapter can be applied to any hierarchical structure. Moreover, we assume that the agents (at the bottom of the hierarchy) do not interact with each other, in the sense that each of them controls his own output and that these outputs are uncorrelated. We could actually assume that they interact, in the same way as the managers finally do. Furthermore, we could also consider, instead of a finite number agents, a continuum of workers with mean–field interaction. There is no reason why this issue could not be addressed, for example by extending the results of the following chapter, namely [Chapter 4](#), to this hierarchical framework.

Additional notations

Throughout this chapter, for $j \in \{1, \dots, m\}$, $i \in \{0, \dots, n_j\}$ and $x^{j,i} \in S^{j,i}$ for some set $S^{j,i}$, we will make use of the following notations for vectors:

$$x^j := (x^{j,i})_{i \in \{0, \dots, n_j\}}, \quad x := (x^j)_{j \in \{1, \dots, m\}}, \quad x^M := (x^{j,0})_{j \in \{1, \dots, m\}} \text{ and } x^{-j} := (x^\ell)_{\ell \in \{1, \dots, m\} \setminus \{j\}},$$

and their corresponding sets:

$$x^j \in S^j := \prod_{i=0}^{n_j} S^{j,i}, \quad x \in S := \prod_{j=1}^m \prod_{i=0}^{n_j} S^{j,i}, \quad x^M \in S^M := \prod_{j=1}^m S^{j,0}, \quad \text{and } x^{-j} \in S^{-j} := \prod_{\ell=1, \ell \neq j}^m S^\ell.$$

In a similar way, we also define

- $x^A \in S^A$, the vector obtained by suppressing the elements x^M of x ;
- $x^{-(j,i)} \in S^{-(j,i)}$, the vector obtained by suppressing the element $x^{j,i}$ of x^A ;
- $x^{j\setminus 0} := (x^{j,i})_{i \in \{1, \dots, n_j\}} \in S^{j\setminus 0}$ and $x^{-j\setminus 0} := (x^{\ell\setminus 0})_{\ell \in \{1, \dots, m\} \setminus \{j\}} \in S^{-j\setminus 0}$.

We will also use the following notations for sums:

$$\bar{x}^j := \sum_{i=0}^{n_j} x^{j,i} \in \bar{S}^j, \quad \text{and } \bar{x}^{-j} := (\bar{x}^\ell)_{\ell \in \{1, \dots, m\} \setminus \{j\}} \in \bar{S}^{-j}.$$

3.1 A general model: preliminaries

In this chapter, we focus on the hierarchy represented in [Figure 3.1](#): the principal contracts with m managers, and each manager j for $j \in \{1, \dots, m\}$ in turn subcontracts with n_j agents, indexed by (j, i) for $i \in \{1, \dots, n_j\}$. The (j, i) -th agent is thus the i -th agent of the j -th manager. The term *workers* will refer to the actors in the hierarchy who are in charge of managing a project, *i.e.*, both agents and managers. The total number of workers is given by $w := m + \sum_{j=1}^m n_j$.

Throughout this chapter, we fix a positive integer d , representing the dimension of the noise which affects each project.¹ We assume here for simplicity that the projects' outputs are one-dimensional² and uncorrelated, meaning that tasks to be performed have been clearly separated. In addition, we assume that each worker in the hierarchy can only impact his own project. This set up can be justified by the fact that each worker have a specific set of skills, implying that they are the only ones able to perform their tasks. In fact, we could let them interact, but this would make the Nash equilibrium hard to solve. In addition, interactions between workers will naturally occur at the level of managers, and therefore a way to handle this issue will be explained at that time.

The most important aspect that we wish to address in such a hierarchy is the loss of information by moving up the hierarchy. To model this, we assume that each manager j only reports the results of his team to the principal through a (possibly multidimensional) variable ζ^j , as in [Sung's model](#) detailed in [Chapter 2](#). Thus, the principal does not have a separate access to the results of each worker, which could lead to a degeneracy of the problem towards the DC case, as we have seen in particular in [Section 2.3.2](#).

1. We assume here that d does not depend on a specific worker. This is without loss of generality, as we can always add unused coordinates to a given project.

2. We could consider that each output is multi-dimensional, such that each coordinate represents the output generated by a task managed by a worker. Nevertheless, at some point we would be led to consider the total profit generated by a worker, which will naturally corresponds to the sum of the coordinates of his output. Therefore, to simplify, we choose to directly consider each output as the total profit generated by a worker, and thus avoid increasing the notations by considering multidimensional vectors and finally taking the sum of their coordinates.

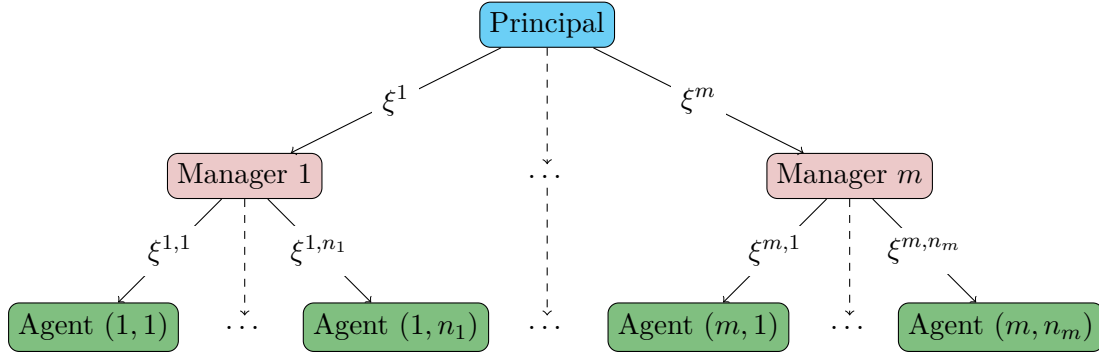


Figure 3.1 – A general hierarchy

3.1.1 Workers' effort

Fix throughout this section $j \in \{1, \dots, m\}$ and $i \in \{0, \dots, n_j\}$, to consider all the workers, *i.e.*, both the agents and the managers. Each worker (j, i) takes care of his own task by choosing a pair $\nu^{j,i} := (\alpha^{j,i}, \beta^{j,i})$, where $\alpha^{j,i}$ and $\beta^{j,i}$ are respectively $A^{j,i}$ - and $B^{j,i}$ -valued, for some subsets $A^{j,i}$ and $B^{j,i}$ of Polish spaces. More specifically, $\alpha^{j,i}$ and $\beta^{j,i}$ represent the effort of the worker (j, i) to impact respectively the expected value and the variability of his output, denoted by $X^{j,i}$.³ In addition, we will consider the following functions, assumed to be bounded:

$$\lambda^{j,i} : [0, T] \times A^{j,i} \longrightarrow \mathbb{R}^d \quad \text{and} \quad \sigma^{j,i} : [0, T] \times B^{j,i} \longrightarrow \mathbb{R}^d.$$

More precisely, the scalar product between the two functions will represent the drift of the output of the (j, i) -th worker, while $\sigma^{j,i}$ will represent its volatility. We will denote for simplicity $U^{j,i} := A^{j,i} \times B^{j,i}$, as well as U the Cartesian product of the sets $U^{j,i}$, using [Additional notations](#) (p. 70). To easily write the dynamic of the column vector X composed by the collection of all the $X^{j,i}$, we define by $\Lambda : [0, T] \times U \longrightarrow \mathbb{R}^{dw}$ and $\Sigma : [0, T] \times U \longrightarrow \mathbb{M}^{dw,w}$ the functions that will correspond respectively to the drift and the volatility of the process X . These functions Λ , Σ will be defined for all $t \in [0, T]$ and $u = (a, b) \in U$ respectively by:

$$\Lambda(t, u) := \left((\sigma^{j,i}(t, b^{j,i}) \cdot \lambda^{j,i}(t, a^{j,i}))_{j,i} \right), \quad (3.1.1)$$

in the sense that Λ is a column vector composed by the collection of all the scalar product $\sigma^{j,i}(t, b^{j,i}) \cdot \lambda^{j,i}(t, a^{j,i})$, meaning that $\Lambda^{j,i}(t, u) := \sigma^{j,i}(t, b^{j,i}) \cdot \lambda^{j,i}(t, a^{j,i})$, and

$$\Sigma(t, b) := \bigoplus_{j=1}^m \bigoplus_{i=0}^{n_j} \sigma^{j,i}(t, b^{j,i}), \quad (3.1.2)$$

where $\oplus$ symbolises direct sum⁴ of matrices (vectors in this case). To be consistent with the weak formulation of control problems, we need to define the canonical space Ω for the workers. Nevertheless, before that, we should discuss about what should be observed by the agents and their managers.

3. If each output represents the value added by a worker, then naturally α and β are some efforts to respectively increase the average and decrease the volatility. However, in more general terms, the output may represent different measures of a worker's performance, and it is possible that the worker may need to decrease the average of the output and/or increase its volatility.

4. The symbol $\oplus$ denote for direct sum of matrices, which is defined for two matrix A and B by:

$$A \oplus B := \begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & B \end{pmatrix}.$$

3.1.2 Different level of monitoring

According to the intuitions developed in [Remarks 2.1.3](#) and [2.4.4](#) in the previous chapter, we cannot assume that the (j, i) -th agent only observes his own output $X^{j,i}$, since his contract cannot be restricted to a measurable function of this output. Indeed, in the general case, the parameters $Z^{j,i}$ and $\Gamma^{j,i}$ chosen by his manager and indexing the contract on his output would not be *a priori* deterministic functions of time. Moreover, if we restrict the payment rates $Z^{j,i}$ and $\Gamma^{j,i}$ for the (j, i) -th agent to be adapted only to the filtration generated by $X^{j,i}$, the principal, who does not observe $X^{j,i}$, will not be able to compute the j -th manager's Hamiltonian. In fact, to better understand this measurability problem, we have to approach it from the top of the hierarchy.

We want to study a case of loss of information by proceeding up the hierarchy, modelled by the fact that each manager j reports only the variable ζ^j to the principal, representing the performance of his work team. Indeed, if we consider for example that the principal represents the company's shareholders as in [\[327\]](#), it is logical to assume that she is not aware of the precise results of each team led by a manager, and that she is probably only interested in the profits and costs, or even the net profits/benefit of each team, represented by the vector ζ . This assumption is particularly relevant if we consider, for example, that each team is a department of the company (or a subsidiary of the parent company), and that shareholders can only compare the benefits of the different departments to optimise their investments and the importance given to each department.

Therefore, the principal only observes the collection of the ζ^j , for $j \in \{1, \dots, m\}$. Under some restrictive conditions⁵, she may offer a contract for the j -th manager which only depends on the result ζ^j of his working team. However, more generally, her controls would be adapted to any information available to her, *i.e.*, the filtration generated by $\zeta := (\zeta^j)_{j=1}^m$, and therefore it makes little sense to restrict the space of contracts for the j -th manager to measurable functions of ζ^j . We are thus led to study a more general space of contracts for the managers, measurable with respect to the filtration generated by ζ .

Therefore, each manager receives a contract indexed on ζ , and we should make some assumptions ensuring that it is the only state variable of his control problem, in order to avoid the more challenging case where the manager's problem depend on another process, unobservable by the principal. Under these assumptions (see in particular [Assumption 3.4.1](#) in the following), even if a manager observes independently the outputs of his agents, his optimal controls will be adapted to the filtration generated by ζ , and thus computable at the optimum by the principal. Nevertheless, the managers can use the detailed information they have to index the contract for their agents on it. Indeed, if a manager indexes his agents' contracts only on ζ , he does not benefit from the information he knows over that known by the principal, and there is then no loss of information between the manager and the principal. Therefore, it must be in the interest of the j -th manager to index the compensation for his workers on all the information he has. In particular, we will assume that the j -th manager observes in continuous time:

- (i) the output produced by the workers of his team, including his own, *i.e.*, the $(n_j + 1)$ -dimensional process X^j ;
- (ii) the sum of the results of each of the other teams, *i.e.*, the $(m - 1)$ -dimensional process $\bar{X}^{-j}$;

using [Additional notations](#) (p. 70). Indeed, in the context of a hierarchy in a company, it seems quite logical to assume that a manager is well informed about the results of his agents, and that during meetings

5. These conditions could be that the principal's problem is separable in each ζ^j , with ζ^j independent and independently controlled by each manager. To illustrate a separable problem for the principal, one can consider Sung's model developed in [Chapter 2, Section 2.1](#) and adding other working teams, led by managers, also reporting the net benefit of their working team to the principal. In this case, if each manager controls only his own ζ^j and if all ζ^j are independent, since the net benefit is just a difference of sums, and the principal is risk-neutral, her problem is completely separable in each ζ^j .

between managers, everyone communicates only the overall result of his team. The canonical space of each agent should contain every processes observable by his manager. Indeed, we want to focus on the loss of information by going up the hierarchy, between the agents and the managers, and then between the managers and the principal. Considering in addition a loss of information by going down the hierarchy would seriously complicate the problem, and would require further study.

We could also assume that the j -th manager observes in particular the output produced by the workers of the other teams, *i.e.*, the process X^{-j} , instead of the sum process $\bar{X}^{-j}$, and so that the agents of his team have access to it as well. In practice, it seems difficult to imagine that a manager would have access to the individual results of the agents from another team. Nevertheless, for the sake of clarity and simplicity, we will consider that an agent observes all the outputs of all workers. Indeed, since each agent can only impact its own project, and its objective function depends only on its project and remuneration, the state variables of his problem will only be the one on which the contract is indexed. This consideration is therefore without loss of generality and makes it possible to define a single canonical space for all agents, regardless of the team they are in.

In summary, the following framework is considered: the agents, at the bottom of the hierarchy, observe all the workers' output, *i.e.*, the global output process X taking values in $\mathbb{R}^w$. As explained above, they will not use all the information contained in X , the important thing is that they have access to the information held by their manager. Each manager perfectly observes the outputs of his agents, as well as his own. However, he does not have access to the detailed results of the other teams, but only to the sum of the outputs produced per team. Finally, the principal only observes m variables, namely $\zeta := (\zeta^j)_{j=1}^m$, each representing the aggregation of a team's results.

3.1.3 Canonical space

Following the previous reasoning, we are thus led to consider the canonical space $\Omega := \mathcal{C}_T^w \times \mathcal{C}_T^{dw} \times \mathbb{U}$, where $\mathbb{U}$ is defined from U as defined in [General notations \(p. xv\)](#). This space is supporting a canonical process (X, W, Π) , where for any $(t, x, \omega, q) \in [0, T] \times \Omega$,

$$X_t(x, \omega, q) := x(t), \quad W_t(x, \omega, q) := \omega(t), \quad \Pi(x, \omega, q) := q.$$

Less formally, X represents the collection of the w one-dimensional outputs $X^{j,i}$ controlled by the workers. Each $X^{j,i}$ is affected by a d -dimensional idiosyncratic noise $W^{j,i}$, and W is the collection of the w noises $W^{j,i}$, using [Additional notations \(p. 70\)](#). Then, the canonical filtration $\mathbb{F} := (\mathcal{F}_t)_{t \in [0, T]}$ is defined by

$$\mathcal{F}_t := \sigma\left((X_s, W_s, \Delta_s(\varphi)) \text{ s.t. } (s, \varphi) \in [0, t] \times \mathcal{C}_b([0, T] \times U, \mathbb{R})\right), \quad t \in [0, T],$$

and for any $(s, \varphi) \in [0, T] \times \mathcal{C}_b([0, T] \times U, \mathbb{R})$,

$$\Delta_s(\varphi) := \int_0^s \int_U \varphi(r, u) \Pi(dr, du).$$

For any $(t, \psi) \in [0, T] \times \mathcal{C}_b^2(\mathbb{R}^w \times \mathbb{R}^{dw}, \mathbb{R})$, we set

$$M_t^A(\psi) := \psi(X_t, W_t) - \int_0^t \int_U \left(\tilde{\Lambda}(s, u) \cdot \nabla \psi(X_s, W_s) + \frac{1}{2} \text{Tr} \left[\nabla^2 \psi(X_s, W_s) \tilde{\Sigma}(s, u) \tilde{\Sigma}(s, u)^\top \right] \right) \Pi(ds, du),$$

$\tilde{\Lambda}, \tilde{\Sigma}$ are respectively the drift vector and the diffusion matrix of the $(w + dw)$ -dimensional vector process $(X, W)^\top$, defined by:

$$\tilde{\Lambda}(s, u) := \begin{pmatrix} \Lambda(s, u) \\ \mathbf{0}_{wd} \end{pmatrix}, \quad \tilde{\Sigma}(s, u) := \begin{pmatrix} \mathbf{0}_{w,w} & \Sigma(s, u)^\top \\ \mathbf{0}_{wd,w} & \mathbf{I}_{wd} \end{pmatrix}, \quad s \in [0, T], \quad u := (a, b) \in U,$$

where Λ, Σ are defined by (3.1.1–3.1.2), so that $\tilde{\Lambda}$ and $\tilde{\Sigma}$ take values respectively in $\mathbb{R}^{w+wd}$ and $\mathbb{M}^{w+wd}$.

We fix some initial conditions, namely $x_0 \in \mathbb{R}^w$ representing the initial value of X . Recall that, as mentionned in [General notations](#) (p. xv), the weak formulation requires to consider a subset of $\mathbb{U}$, namely the set $\mathbb{U}_0$ of all $q \in \mathbb{U}$ such that $q(dt, du) = \delta_{\phi_t}(du)dt$, for $t \in [0, T]$ and some Borel function ϕ .

Definition 3.1.1. *Let $\mathcal{P}$ be the set of all probability measures $\mathbb{P}$ on $(\Omega, \mathcal{F}_T)$ such that:*

- (i) $M^A(\psi)$ is a $(\mathbb{F}, \mathbb{P})$ –local martingale on $[0, T]$ for all $\psi \in \mathcal{C}_b^2(\mathbb{R}^w \times \mathbb{R}^{dw}, \mathbb{R})$;
- (ii) there exists some $w_0 \in \mathbb{R}^{wd}$ such that $\mathbb{P}[(X_0, W_0) = (x_0, w_0)] = 1$;
- (iii) $\mathbb{P}[\Pi \in \mathbb{U}_0] = 1$.

The previous definition does not give us access directly to the dynamic of X . It is however a classical result that, enlarging the canonical space if necessary, one can construct Brownian motions allowing to write rigorously the dynamic of X , see for instance [Stroock and Varadhan](#) [324, Theorem 4.5.2] (1997). It turns out here that, since we enlarged the canonical space right from the start to account for the idiosyncratic noises, any further enlargement is not required. Indeed, arguing as in the proof of [Lin, Ren, Touzi, and Yang](#) [234, Lemma 2.2] (2020), we can prove the following.

Lemma 3.1.2. *For all $\mathbb{P} \in \mathcal{P}$, we have $\Pi(ds, du) = \delta_{\nu_s^\mathbb{P}}(du)ds$ $\mathbb{P}$ –a.s. for some $\mathbb{F}$ –predictable control process $\nu^\mathbb{P} := (\alpha^{j,i,\mathbb{P}}, \beta^{j,i,\mathbb{P}})_{j,i}$, and we obtain the following representation for X :*

$$X_t = x_0 + \int_0^t \Lambda(s, \nu_s^\mathbb{P}) ds + \int_0^t \Sigma(s, \beta_s^\mathbb{P})^\top dW_s, \quad t \in [0, T], \quad \mathbb{P}\text{--a.s.} \quad (3.1.3)$$

More precisely, for any $j \in \{1, \dots, m\}$, $i \in \{0, \dots, n_j\}$,

$$X_t^{j,i} = x_0^{j,i} + \int_0^t \sigma^{j,i}(s, \beta_s^{j,i,\mathbb{P}}) \cdot \left[\lambda^{j,i}(s, \alpha_s^{j,i,\mathbb{P}}) ds + dW_s^{j,i} \right], \quad t \in [0, T], \quad \mathbb{P}\text{--a.s.} \quad (3.1.4)$$

Remark 3.1.3. *By construction, the collection of outputs $X^{j,i}$ are not defined as solutions of stochastic differential equations (SDE for short), as in more general framework of principal–agent models (see [Cvitanic, Possamai, and Touzi](#) [111] (2018) for example), they are just standard Itô processes. The reader is referred to [Remark 3.3.1](#) for the motivations of this assumption.*

3.2 The principal–managers–agents hierarchy

The hierarchy is modeled by a series of interlinked principal–agent problems: the principal contracts with m managers, and each manager j , for $j \in \{1, \dots, m\}$, hires n_j agents, indexed by (j, i) for $i \in \{1, \dots, n_j\}$. More precisely, for $j \in \{1, \dots, m\}$, the j –th manager receives a compensation ξ^j from the principal and must design what will be the remuneration $\xi^{j,i}$ of each agent (j, i) , for $i \in \{1, \dots, n_j\}$. Finally, as in the model developed in the previous sections, we assume that the principal chooses the continuation utilities of the agents at time $t = 0$ (in addition to those of the managers), *i.e.*, the expected value finally obtained by the agents at time T , denoted by $Y_0^{j,i} \in \mathbb{R}$ for the (j, i) –th agent.

Once again, it seems more appropriate to think about this hierarchy from top to bottom. First, and as mentioned above, the principal chooses the initial values of the agents’ continuation utility. This will lead us to fix, until the principal’s problem, $Y_0^{j,i} \in \mathbb{R}$ for all $j \in \{1, \dots, m\}$ and $i \in \{1, \dots, n_j\}$, denoted by $Y_0^A \in \mathbb{R}^{w-m}$. The principal also offers to each manager a contract indexed on the variable ζ she observes, which leads to the first Stackelberg game. Given his contract and the choices of other managers, the j –th manager will choose:

- (i) an optimal effort $\alpha^{j,0}$ to improve the expected value of his own output, and the associated probability;
- (ii) a compensation scheme for the agents he manages, *i.e.*, $(\xi^{j,i})_{i=1}^{n_j}$.

Then, given the choices of the managers, and in particular the agents' contracts, the agents will determine their optimal efforts. To write their problems, it is therefore necessary to fix a contract and an effort probability chosen by the managers. In addition, since an agent receives a contract which depends on his colleagues' output, his optimal response must therefore also be defined in relation to the efforts of the other agents. The intuition is thus to define the optimal response of an agent given:

- (i) a contract;
- (ii) the efforts of the managers;
- (iii) the efforts of the other agents.

In a classical way, the two Stackelberg games are solved from the bottom to the top. First we look for the optimal response of an agent to arbitrary choices of others. Next, the Nash equilibrium between the agents can be solved, under fixed contracts and efforts of the managers. Then, knowing the optimal response of the agents, each manager will then be able to choose his efforts and the remuneration for his agents, in order to optimise his criterion, given the choices of other managers, and the contracts designed by the principal. Since the manager's choices depend on those of other managers, it is also necessary to find a Nash equilibrium between managers. Finally, given the optimal response of each manager to a contract, the principal will be able to design the optimal contract for each manager in order to optimise her own criterion.

3.2.1 A Nash equilibrium between the agents

To consider a particular agent, we fix throughout this section $j \in \{1, \dots, m\}$ and $i \in \{1, \dots, n_j\}$. Following the previous reasoning, we have to define two subsets of the canonical space Ω , in order to look for the optimal response of the (j, i) -th agent, given the choices of other workers (managers and agents). Using [Additional notations](#) (p. 70), we define the sets U^M and $U^{-(j,i)}$ from the collection of sets $U^{j,i}$. In the same manner we have defined $\mathbb{U}$ and the corresponding set $\mathbb{U}_0$ for the canonical space in [Section 3.1.3](#), we define:

- (i) $\mathbb{U}^{-(j,i)}$, the collection of all finite and positive Borel measures on $[0, T] \times U^{-(j,i)}$, whose projection on $[0, T]$ is the Lebesgue measure, and the associated subset $\mathbb{U}_0^{-(j,i)}$;
- (ii) $\mathbb{U}^M$, the collection of all finite and positive Borel measures on $[0, T] \times U^M$, whose projection on $[0, T]$ is the Lebesgue measure, and the associated subset $\mathbb{U}_0^M$.

Informally, the set $\mathbb{U}_0$ will allow us to define the set of admissible efforts of all the workers, while the set $\mathbb{U}_0^{-(j,i)}$ will be used to define the efforts of other agents, apart from the (j, i) -th agent, and the set $\mathbb{U}_0^M$ will be used to define the efforts of the managers. In the same way we have defined Ω , with its canonical process, its canonical filtration and the appropriate subset of probability $\mathcal{P}$ by [Definition 3.1.1](#), we can define the two following canonical spaces:

- (i) $(\Omega^{-(j,i)}, \mathcal{F}_T^{-(j,i)})$ the canonical space of other agents (apart from the agent (j, i)), where

$$\Omega^{-(j,i)} := \mathcal{C}_T^{w-m-1} \times \mathcal{C}_T^{d(w-m-1)} \times \mathbb{U}^{-(j,i)},$$

with canonical process $(X^{-(j,i)}, W^{-(j,i)}, \Pi^{-(j,i)})$, and canonical filtration $\mathbb{F}^{-(j,i)} := (\mathcal{F}_t^{-(j,i)})_{t \in [0, T]}$, on which we define the subset $\mathcal{P}^{-(j,i)}$ of probability measures satisfying the appropriate properties;

(ii) $(\Omega^M, \mathcal{F}_T^M)$ the canonical space of managers, where

$$\Omega^M := \mathcal{C}_T^m \times \mathcal{C}_T^{dm} \times \mathbb{U}^M,$$

with canonical process (X^M, W^M, Π^M) , and canonical filtration $\mathbb{F}^M := (\mathcal{F}_t^M)_{t \in [0, T]}$, on which we define the subset $\mathcal{P}^M$ of probability measures satisfying the appropriate properties.

Informally, the canonical space $\Omega^{-(j,i)}$ contains the same information as Ω , except that the components concerning the (j, i) -th agent and the managers are removed, and thus, $\Omega^{-(j,i)}$ is a subset of Ω . Similarly, the space Ω^M contains the information of Ω concerning the managers. We can now define, for the (j, i) -th agent, the set of his admissible response to others.

Definition 3.2.1 (Admissible Response). *Consider two probability measures, $(\mathbb{P}^M, \mathbb{P}^{-(j,i)}) \in \mathcal{P}^M \times \mathcal{P}^{-(j,i)}$, respectively chosen by the managers and the other agents. The set of admissible response of the (j, i) -th agent, denoted by $\mathcal{P}^{j,i}(\mathbb{P}^{-(j,i)}, \mathbb{P}^M)$, is given by all probability measures $\mathbb{P} \in \mathcal{P}$ on $(\Omega, \mathcal{F}_T)$ satisfying:*

- (i) *the restriction of $\mathbb{P}$ to $(\Omega^{-(j,i)}, \mathcal{F}_T^{-(j,i)})$ is $\mathbb{P}^{-(j,i)}$;*
- (ii) *the restriction of $\mathbb{P}$ to $(\Omega^M, \mathcal{F}_T^M)$ is $\mathbb{P}^M$.*

Following the discussion in [Section 3.1.2](#), we assume that the j -th manager observes the output of the agents under his supervision, but also the sum of the outputs of each of the other teams. Formally, this means that the j -th manager proposes to the (j, i) -th agent a contract $\xi^{j,i}$, which is a random variable measurable with respect to the natural filtration generated by X^j and $\bar{X}^{-j}$, denoted $\mathbb{G}^j$. In other words, $\xi^{j,i}$ must be a measurable functional of the paths of X^j and $\bar{X}^{-j}$:

$$\begin{aligned} \xi^{j,i} : \mathcal{C}_T^{n_j+1} \times \mathcal{C}_T^{m-1} &\longrightarrow \mathbb{R}, \\ (X^j, \bar{X}^{-j}) &\longmapsto \xi^{j,i}(X^j, \bar{X}^{-j}). \end{aligned} \quad (3.2.1)$$

One can notice that the contracts for the agents of the team j are measurable with respect to $\mathbb{G}^j$, *i.e.*, all agents in a team will receive a contract indexed on the same results. The set of admissible contracts for the (j, i) -th agent will be denoted by $\Xi^{j,i}$, and we refer to [Definition 3.2.5](#) below for a rigorous description.

Given this contract and two probability measures $(\mathbb{P}^{-(j,i)}, \mathbb{P}^M) \in \mathcal{P}^{-(j,i)} \times \mathcal{P}^M$ chosen by others, we introduce the (j, i) -th agent's objective function

$$J^{j,i}(\mathbb{P}, \xi^{j,i}) := \mathbb{E}^{\mathbb{P}} \left[\mathcal{K}_{0,T}^{j,i,\mathbb{P}} g^{j,i}(X_{\cdot \wedge T}^{j,i}, \xi^{j,i}) - \int_0^T \mathcal{K}_{0,s}^{j,i,\mathbb{P}} c^{j,i}(s, X_s^{j,i}, \nu_s^{j,i,\mathbb{P}}) ds \right], \text{ for } \mathbb{P} \in \mathcal{P}^{j,i}(\mathbb{P}^{-(j,i)}, \mathbb{P}^M), \quad (3.2.2)$$

where

- (i) $g^{j,i} : \mathcal{C}_T \times \mathbb{R} \longrightarrow \mathbb{R}$ is a utility function assumed to be Borel measurable in each arguments, such that for any $x \in \mathcal{C}_T$, the map $\xi \longmapsto g^{j,i}(x, \xi)$ is invertible and we will denote by $\bar{g}^{j,i}$ its inverse.
- (ii) $c^{j,i} : [0, T] \times \mathcal{C}_T \times U^{j,i} \longrightarrow \mathbb{R}$ is a cost function assumed to be Borel measurable in each arguments, satisfying for any $u \in U^{j,i}$, the map $(t, x) \longmapsto c^{j,i}(t, x, u)$ is $\mathbb{F}$ -optional, and there exists some $p > 1$ such that

$$\sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}} \left[\int_0^T |c^{j,i}(t, X_t^{j,i}, \nu_t^{j,i,\mathbb{P}})|^p dt \right] < +\infty; \quad (3.2.3)$$

(iii) the discount factor

$$\mathcal{K}_{t,s}^{j,i,\mathbb{P}} := \exp \left(- \int_t^s \int_U k^{j,i}(v, X_v^{j,i}, u^{j,i}) \Pi(dv, du) \right), \text{ for } 0 \leq t \leq s \leq T \text{ and } \mathbb{P} \in \mathcal{P}^{j,i}(\mathbb{P}^{-(j,i)}, \mathbb{P}^M),$$

is defined by means of the function $k^{j,i} : [0, T] \times \mathcal{C}_T \times U^{j,i} \longrightarrow \mathbb{R}$ assumed to be bounded and Borel measurable in each arguments, and such that for any $u \in U^{j,i}$, the map $(t, x) \longmapsto k^{j,i}(t, x, u)$ is $\mathbb{F}$ -optional.

The previous setting and assumptions are relatively standard in control theory. The map $g^{j,i}$ is assumed to be invertible so that one can recover the contract $\xi^{j,i}$ from the continuation utility of the agent. More precisely, our first goal is to obtain the dynamic of the continuation utility of the (j, i) -th agent, which will be denoted $(Y_t^{j,i})_{t \in [0, T]}$, satisfying at the end of the contracting period the following equality: $Y_T^{j,i} = g^{j,i}(X^{j,i}, \xi^{j,i})$. The contract will thus be given by $\xi^{j,i} = \bar{g}^{j,i}(X^{j,i}, Y_T^{j,i})$, recalling that $\bar{g}^{j,i} : \mathcal{C}_T \times \mathbb{R} \rightarrow \mathbb{R}$ is the inverse of $g^{j,i}$. However, we are forced to make an additional hypothesis, and the reader is referred to [Remark 3.3.2](#) for the motivation.

Assumption 3.2.2. *There exists $c_x^{j,i}, k_x^{j,i} : [0, T] \times \mathcal{C}_T \rightarrow \mathbb{R}$ and $c_u^{j,i}, k_u^{j,i} : [0, T] \times U^{j,i} \rightarrow \mathbb{R}$ such that for any $(t, x, u) \in [0, T] \times \mathcal{C}_T \times U^{j,i}$, we can write:*

$$c^{j,i}(t, x, u) = c_x^{j,i}(t, x) + c_u^{j,i}(t, u) \quad \text{and} \quad k^{j,i}(t, x, u) = k_x^{j,i}(t, x) + k_u^{j,i}(t, u).$$

Despite this unusual assumption, the framework under consideration can at least accommodate the two most standard cases for the agent's utility function, *i.e.*, both the risk-averse case from [327] and the risk-neutral case. Indeed, to recover for example [Sung's](#) model developed in [Chapter 2](#), it suffices to set, for all $(t, x, y) \in [0, T] \times \mathbb{R}^2$, $u := (a, b) \in U^{j,i}$, $c^{j,i}(t, x, u) = k_x^{j,i}(t, x) \equiv 0$, $g^{j,i}(x, y) := -e^{-R^{j,i}y}$ and $k_u^{j,i}(t, u) := -R^{j,i}a^2/2k^{j,i}$, for some $(R^{j,i}, k^{j,i}) \in (\mathbb{R}_+^*)^2$.

Given a compensation $\xi^{j,i}$ promised by his manager, as well as a pair of probabilities $(\mathbb{P}^{-(j,i)}, \mathbb{P}^M) \in \mathcal{P}^{-(j,i)} \times \mathcal{P}^M$ chosen by other workers, the optimisation problem faced by the (j, i) -th agent is defined by

$$V_0^{j,i}(\mathbb{P}^{-(j,i)}, \mathbb{P}^M, \xi^{j,i}) := \sup_{\mathbb{P} \in \mathcal{P}^{j,i}(\mathbb{P}^{-(j,i)}, \mathbb{P}^M)} J^{j,i}(\mathbb{P}, \xi^{j,i}). \quad (3.2.4)$$

For $V_0^{j,i}(\mathbb{P}^{-(j,i)}, \mathbb{P}^M, \xi^{j,i})$ to make sense, we require minimal integrability on the contracts, and thus impose that there is some $p > 1$ such that

$$\sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}} \left[|g^{j,i}(X^{j,i}, \xi^{j,i})|^p \right] < +\infty. \quad (\mathcal{I}_A^p)$$

Definition 3.2.3 (Optimal Response). *A probability measure $\mathbb{P} \in \mathcal{P}$ is an optimal response to the probabilities $\mathbb{P}^{-(j,i)}$ and $\mathbb{P}^M$ chosen by others and to a contract $\xi^{j,i} \in \Xi^{j,i}$ if (i) $\mathbb{P}$ is admissible, *i.e.*, $\mathbb{P} \in \mathcal{P}^{j,i}(\mathbb{P}^{-(j,i)}, \mathbb{P}^M)$; and (ii) $V_0^{j,i}(\mathbb{P}^{-(j,i)}, \mathbb{P}^M, \xi) = J^{j,i}(\mathbb{P}, \mathbb{P}^{-(j,i)}, \mathbb{P}^M, \xi)$. We denote by $\mathcal{P}^{j,i,*}(\mathbb{P}^{-(j,i)}, \mathbb{P}^M, \xi)$ the collection of all such optimal probability measures.*

We define by ξ^A the collection of contracts for all agents, *i.e.*, the collection of $\xi^{j,i} \in \Xi^{j,i}$ for $j \in \{1, \dots, m\}$ and $i \in \{1, \dots, n_j\}$, and denote the corresponding set by Ξ^A . A Nash equilibrium between the agents can thus be defined as an optimal response to the manager's choices for any agents:

Definition 3.2.4 (Nash Equilibrium). *Fix a probability measure $\mathbb{P}^M \in \mathcal{P}^M$ chosen by the managers and a collection $\xi^A \in \Xi^A$ of contracts for the agents. We denote by $\mathcal{P}^{A,*}(\mathbb{P}^M, \xi^A)$ the set of Nash equilibria between agents, *i.e.* the set of probability measure $\mathbb{P} \in \mathcal{P}$ such that for any $j \in \{1, \dots, m\}$ and any $i \in \{1, \dots, n_j\}$, $\mathbb{P} \in \mathcal{P}^{j,i,*}(\mathbb{P}^{-(j,i)}, \mathbb{P}^M, \xi^{j,i})$, where $\mathbb{P}^{-(j,i)}$ is the restriction of $\mathbb{P}$ on $\Omega^{-(j,i)}$.*

To simplify the scope of our study and avoid unnecessary complexities, we will subsequently require that all eligible contracts for agents are those that induce a unique Nash equilibrium between agents, and we thus define the set of admissible contract as follows:

Definition 3.2.5 (Admissible contracts). *Fix $Y_0^A \in \mathbb{R}^{w-m}$ the collection of $Y_0^{j,i} \in \mathbb{R}$ for all $j \in \{1, \dots, m\}$, $i \in \{1, \dots, n_j\}$ and $\mathbb{P}^M \in \mathcal{P}^M$. A contract of the form (3.2.1) satisfying [Condition \(I_A^p\)](#) is called admissible. The corresponding class is denoted by $\Xi^{j,i}$. Moreover, a collection ξ^A of contracts for the agents is admissible*

if (i) the induced Nash equilibrium $\mathbb{P}^* \in \mathcal{P}^{A,*}$ between the agents is unique; and (ii) for all $j \in \{1, \dots, m\}$, $i \in \{1, \dots, n_j\}$, $\xi^{j,i} \in \Xi^{j,i}$ and the (j, i) -th agent's value function at equilibrium is equal to $Y_0^{j,i}$. In this case, we will write $\xi^A \in \Xi^A$.⁶

In addition, similarly to classical principal-agent problems, we assume that the (j, i) -th agent has a reservation utility level $\rho^{j,i} \in \mathbb{R}$ below which he will refuse to work, and thus decline the contract offered by his manager. Mathematically speaking, a contract should thus satisfy the following inequality:

$$V_0^{j,i}(\mathbb{P}^{-(j,i)}, \mathbb{P}^M, \xi^{j,i}) \geq \rho^{j,i}. \quad (\text{PC}_A)$$

Note that, if the collection ξ^A of contracts for the agents is admissible in the sense of the previous definition, then there exists a unique Nash equilibrium $\mathbb{P}^*$ between agents, which satisfies in addition $V_0^{j,i}(\mathbb{P}^{-(j,i)*}, \mathbb{P}^M, \xi^{j,i}) = Y_0^{j,i}$. Therefore, in order to ensure that the participation constraint (PC_A) of the (j, i) -th agent is satisfied, the principal only has to choose $Y_0^{j,i} \geq \rho^{j,i}$ in the end.

3.2.2 A Nash equilibrium between the managers

Throughout the following, we fix $j \in \{1, \dots, m\}$ to informally define the j -th manager's optimisation problem. The weak formulation of this problem will be rigorously defined in [Section 3.4](#), since it will be necessary to consider a new canonical space taking into account the optimal response of the agents. Let us just recall for the moment that the j -th manager is in charge of a task that generates an output $X^{j,0}$. His effort to improve his output $X^{j,0}$ is defined in an informal way by a pair $\nu^{j,0} := (\alpha^{j,0}, \beta^{j,0}) \in \mathcal{U}^{j,0}$ taking values in $U^{j,0} := A^{j,0} \times B^{j,0}$.

We suppose that the j -th manager reports in continuous time to the principal the variable ζ^j , assumed to be h -dimensional for some $h > 0$, and measuring the global result of his entire working team (including himself). Therefore, ζ^j will depend on the outputs of his team, namely X^j (the outputs of the agents he manages and his own) and the collection $\xi^{j,0} := (\xi^{j,i})_{i=1}^{n_j}$ of compensations to be paid to the agents. Therefore, the principal only knows the result of each team, *i.e.*, the process $\zeta := (\zeta^j)_{j=1}^m$, and thus the contract designed by the principal for the j -th manager depends exclusively on the path of ζ :

$$\xi^{j,0} : \zeta \in \mathcal{C}_T^{hm} \mapsto \xi^{j,0}(\zeta) \in \mathbb{R}. \quad (3.2.5)$$

In other words, we can only consider contracts for managers which are $\mathcal{G}_T$ -measurable, where $\mathbb{G} := (\mathcal{G}_t)_{t \in [0, T]}$ is the natural filtration of ζ . The set of admissible contracts for the j -th manager will be denoted by $\Xi^{j,0}$, and we refer to [Definition 3.2.9](#) below for a rigorous description.

We define the characteristics of the j -th manager as follows:

- (i) a utility function $g^{j,0} : \mathcal{C}_T^h \times \mathbb{R} \rightarrow \mathbb{R}$, assumed to be Borel measurable in each arguments, such that for any $x \in \mathcal{C}_T^h$, the map $y \mapsto g^{j,0}(x, y)$ is invertible and we will denote by $\bar{g}^{j,0}$ its inverse;
- (ii) a cost function $c^{j,0} : [0, T] \times \mathcal{C}_T^h \times U^{j,0} \rightarrow \mathbb{R}$, assumed to be Borel measurable in each arguments, such that for any $u \in U^{j,0}$, the map $(t, x) \mapsto c^{j,0}(t, x, u)$ is $\mathbb{G}$ -optional, and satisfying for some $p > 1$

$$\sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}} \left[\int_0^T |c^{j,0}(t, \zeta^j, \nu_t^{j,0})|^p dt \right] < +\infty; \quad (3.2.6)$$

6. Note that the space of admissible contracts Ξ^A depends on the collection of $Y_0^A \in \mathbb{R}^{w-m}$. Nevertheless, in order to lighten the notations, we omit this dependency.

- (iii) a discount factor $k^{j,0} : [0, T] \times \mathcal{C}_T^h \times U^{j,0} \rightarrow \mathbb{R}$, assumed to be bounded and Borel measurable in each arguments, and such that for any $u \in U^{j,0}$, the map $(t, x) \mapsto k^{j,0}(t, x, u)$ is $\mathbb{G}$ -optional, and its associated quantity

$$\mathcal{K}_{t,s}^{j,0,\mathbb{P}} := \exp \left(- \int_t^s \int_U k^{j,0}(v, \zeta^j, u^{j,0}) \Pi(dv, du) \right), \quad \text{for } 0 \leq t \leq s \leq T \quad \text{and } \mathbb{P} \in \mathcal{P};$$

- (iv) a reservation utility level $\rho^{j,0} \in \mathbb{R}$ below which he will refuse to work.

Given a probability $\mathbb{P} \in \mathcal{P}$ on the canonical space, as well as a contract $\xi^{j,0}$ designed for him by the principal, the criterion of the j -th manager is defined as follows:

$$J^{j,0}(\mathbb{P}, \xi^{j,0}) := \mathbb{E}^{\mathbb{P}} \left[\mathcal{K}_{0,T}^{j,0,\mathbb{P}} g^{j,0}(\zeta^j, \xi^{j,0}) - \int_0^T \mathcal{K}_{0,t}^{j,0,\mathbb{P}} c^{j,0}(t, \zeta^j, \nu_t^{j,0,\mathbb{P}}) dt \right]. \quad (3.2.7)$$

Remark 3.2.6. *The attentive reader will have noticed that the characteristics of the manager, i.e., the functions $g^{j,0}$, $c^{j,0}$ and $k^{j,0}$, are assumed to be functions of ζ^j , and not of the collection X of outputs. Indeed, these functions can only depend on the variable observed by the principal. Otherwise, if these functions depend on some variable which is not observed by the principal, she cannot compute the managers' Hamiltonians, even for their optimal efforts. This would raise major issues, not yet addressed in the literature on continuous time (to our knowledge), which would require a full study before it can be considered in our case. Nevertheless, it is worth noticing that some works attempt to address similar problems, such as the paper by Huang, Ju, and Xing [189] (2017). Nevertheless, for the sake of generality, we could let them depend on the collection of the ζ^j , since the manager's contract will be a measurable function of $\zeta := (\zeta^j)_{j=1}^m$, and ζ will thus be the natural state variable of each manager's problem.*

The j -th manager must optimise the specific criterion defined by (3.2.7), given the contract he receives, but also given the choices of the other managers. More precisely, let us fix a contract $\xi^{j,0}$ of the form (3.2.5), as well as the decision of other managers, namely, for all $\ell \in \{1, \dots, m\} \setminus \{j\}$,

- (i) the effort $\nu^{\ell,0} \in \mathcal{U}^{\ell,0}$ of the ℓ -th manager;

- (ii) the collection of contracts $\xi^{\ell,0} \in \Xi^{\ell,0}$ offered by the ℓ -th manager to his n_ℓ agents.

Given this, the j -th manager must thus choose an optimal control $\nu^{j,0} \in \mathcal{U}^{j,0}$, as well as a contract $\xi^{j,i} \in \Xi^{j,i}$ for each agent (j, i) under his supervision. We summarise the controls of the managers by a tuple $\tilde{\chi} \in \tilde{\mathcal{X}}$, such that

$$\tilde{\chi}^j := (\nu^{j,0}, (\xi^{j,i})_{i=1}^{n_j}) \in \tilde{\mathcal{X}}^j := \mathcal{U}^{j,0} \times \prod_{i=1}^{n_j} \Xi^{j,i},$$

is the control of the j -th manager, for all $j \in \{1, \dots, m\}$. Informally, the probability $\mathbb{P}^M \in \mathcal{P}^M$ results from the effort choice of all managers, namely $\tilde{\chi} \in \tilde{\mathcal{X}}$. Therefore, the collection of all contracts chosen by the managers for their agents, denoted ξ^A , depends on $\tilde{\chi}$. As mentioned in Definition 3.2.5, we require that all eligible contracts for agents are those that induce a unique Nash equilibrium between agents, i.e., $\mathcal{P}^{A,*}(\mathbb{P}^M, \xi^A) = \{\mathbb{P}^*(\tilde{\chi})\}$, where the notation is made to highlight the dependence of the probability $\mathbb{P}^*$ on the control $\tilde{\chi}$.

Remark 3.2.7. *It is well-known that the uniqueness of a Nash equilibrium is more the exception than the rule. However, if we do not want to assume uniqueness, this leads to considerations that can become complex. Indeed, assume that there is no uniqueness of the Nash equilibrium. In usual principal–agent problems⁷, the*

7. We refer here to *usual* principal–agent problems in the sense that there is no limited liability, contrary to Sannikov [303] (2008) or DeMarzo and Sannikov [119] (2017), and the agent does not control the discount factor.

agents actually receive exactly their reservation utility. If this is true in our case, each equilibrium should therefore give the same utility to the agents. In this case, it would therefore be reasonable to assume that each manager chooses the most advantageous equilibrium between his agents from his point of view, and that the principal then chooses the best equilibrium between the managers. Unfortunately, to do this, it would first be necessary to define a Nash equilibrium between the agents per team (i.e. for all $j \in \{1, \dots, m\}$), and then look at the Nash equilibrium between the teams, before we address the Nash equilibrium between the managers. Moreover, at this stage of the study, it is also possible that the various equilibria give distinct utilities to the agents. In this case, if one equilibrium Pareto–dominates the others from the agents’ point of view, then it would make sense for the agents to choose it, and the manager would not be able to optimise over all possible equilibria. In our opinion, introducing the possibility of multiple Nash equilibria would therefore only overload the framework, while it is not the major focus of this study.

The j -th manager’s optimisation problem can then be informally written as follows:

$$V_0^{j,0}(\xi^{j,0}, \tilde{\chi}^{-j}) := \sup_{\tilde{\chi}^j \in \tilde{\mathcal{X}}^j} J^{j,0}(\mathbb{P}^*(\tilde{\chi}), \xi^{j,0}). \quad (3.2.8)$$

Similar to the agents problem, we require minimal integrability on the contracts:

$$\sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}} \left[|g^{j,0}(\zeta^j, \xi^{j,0})|^p \right] < +\infty, \text{ for some } p > 1. \quad (\mathbb{I}_M^p)$$

Definition 3.2.8 (Nash equilibrium between managers). *Given a collection $\xi^M := (\xi^{j,0})_{j=1}^m \in \Xi^M$ of contracts, a Nash equilibrium between the managers is a tuple of control $\tilde{\chi} \in \tilde{\mathcal{X}}$, such that for all $j \in \{1, \dots, m\}$,*

$$V_0^{j,0}(\xi^{j,0}, \tilde{\chi}^{-j}) = J^{j,0}(\mathbb{P}^*(\tilde{\chi}), \xi^{j,0}).$$

We denote by $\mathcal{P}^{M,*}(\xi^M)$ the set of Nash equilibria given $\xi^M \in \Xi^M$.

Similar to the previous Stackelberg game, to simplify the scope of our study, we will require that all eligible contracts for managers are those that induce a unique Nash equilibrium between them, i.e., $\mathcal{P}^{M,*}(\xi^M) = \{\mathbb{P}^*(\xi^M)\}$.

Definition 3.2.9. *A contract of the form (3.2.5) satisfying Condition $(\mathbb{I}_M^p)$ is called admissible. The corresponding class is denoted by $\Xi^{j,0}$. The product of the sets $\Xi^{j,0}$ for $j \in \{1, \dots, m\}$, such that the resulting collection ξ^M of contracts for the managers induces a unique Nash equilibrium between them, will be denoted by Ξ^M .*

In addition, similarly to the agent’s problems, we assume that the j -th manager has a reservation utility level $\rho^{j,0} \in \mathbb{R}$ below which he will refuse to work, and thus decline the contract offered by the principal. Mathematically speaking, a contract should also satisfy the following inequality:

$$V_0^{j,0}(\xi^{j,0}) \geq \rho^{j,0}. \quad (\text{PCM})$$

As with the agents’ admissible contracts, we could have assumed that the collection of admissible contracts for the managers is such that for any $j \in \{1, \dots, m\}$, the j -th manager’s value function equals a particular $Y_0^{j,0} \in \mathbb{R}$ at equilibrium. Then, the principal would optimise these $Y_0^{j,0} \in \mathbb{R}$ in order to satisfy the manager’s participation constraint. Nevertheless, contrary to the agents’ problem, here the principal chooses both the value of the manager’s continuation utility and his contract. Therefore, there is no need to first consider a subset of admissible contracts for a given level of continuation utility and then optimise this level.

3.2.3 A principal at the top

It remains to define, still informally at this point, the principal's problem. Contrary to the managers, her problem is more classical, as she does not directly control any process, she just designs the collection of contracts ξ^M for the m managers. Her criterion is defined by:

$$J^P(\xi^M) := \mathbb{E}^{\mathbb{P}(\xi^M)} \left[\mathcal{K}_{0,T}^P g^P(\zeta, \xi^M) \right], \quad (3.2.9)$$

where $\mathbb{P}(\xi^M)$ can be seen informally as the probability resulting from the optimal controls of all managers and agents under her supervision given the contracts ξ^M , and

- (i) $g^P : \mathbb{R}^{hm} \times \mathbb{R}^m \rightarrow \mathbb{R}$ is a given utility function, non-increasing and concave in the second argument;
- (ii) the discount factor

$$\mathcal{K}_{s,t}^P := \exp \left(\int_s^t k^P(r, \zeta) dr \right), \quad 0 \leq s \leq t \leq T,$$

is defined by means of a bounded discount rate function $k^P : [0, T] \times \mathcal{C}_T^h \rightarrow \mathbb{R}$, $\mathbb{G}$ -optional, recalling that $\mathbb{G} := (\mathcal{G}_t)_{t \in [0, T]}$ is the natural filtration of ζ .

The principal must therefore optimise the specific criterion defined by (3.2.9) by choosing the collection of contracts $\xi^M \in \Xi^M$, as well as the initial value of the agents' continuation utility $Y_0^A \in \mathbb{R}^{w-m}$. Since we have assumed uniqueness of the Nash equilibrium between managers given a collection of admissible contracts, the principal's problem is simply given by:

$$V_0^P := \sup_{Y_0^A \in \mathbb{R}^{w-m}} \sup_{\xi^M \in \Xi^M} J^P(\xi^M). \quad (3.2.10)$$

Note that the choice Y_0^A does not directly appear in the principal's optimisation problem. Nevertheless, this choice does have an impact on the optimal controls of both the managers and agents, as well as on the value of ζ . However, to lighten the notations, we choose not to emphasise this dependency.

3.2.4 Main result: reduction to a standard stochastic control problem

Applying the recent results of Cvitanić, Possamaï, and Touzi [111] (2018), it is relatively simple to solve the (j, i) -th agent's problem. Indeed, by limiting our study to contracts indexed on the dynamics of the outputs X^j and $\bar{X}^{-j}$ and their quadratic variations, the optimal efforts are given by the maximisers of his Hamiltonian. It is then sufficient to show that the restriction to contracts of this form for the manager is without loss of generality.

The main result of this chapter is that this reasoning can be extended to the Stackelberg game between the managers and the principal. Therefore, following the same intuition, we will limit our study to contracts for the managers indexed on a well-chosen state variable, namely ζ , and its quadratic variation, through a tuple of parameters $\mathcal{Z} \in \mathcal{V}$ chosen by the principal. We will prove that the problem of the managers is relatively simple to solve for this particular class of contract, and that this restriction is in fact without loss of generality for the principal. More precisely, we establish in Theorem 3.4.9 that, at the end of the day, the principal's problem defined by (3.2.10) boils down to the following standard control problem:

$$V_0^P = \sup_{Y_0 \geq \rho} \sup_{\mathcal{Z} \in \mathcal{V}} \mathbb{E}^{\mathbb{P}^*(\mathcal{Z})} \left[\mathcal{K}_{0,T}^P g^P(\zeta, \xi^M) \right],$$

where

- (i) the inequality $Y_0 \geq \rho$ has to be understood componentwise, *i.e.*, for all $j \in \{1, \dots, m\}$ and $i \in \{0, \dots, m\}$, $Y_0^{j,i} \geq \rho^{j,i}$, and ensure that the participation constraint of all workers is satisfied;
- (ii) $\mathbb{P}^*(\mathcal{Z})$ is the unique Nash equilibrium between the workers, given the control $\mathcal{Z} \in \mathcal{V}$ chosen by the principal;
- (iii) ξ^M is the collection of *revealing contracts* for the managers, thoroughly characterised by the choice of $Y_0^M \in \mathbb{R}^m$ and $\mathcal{Z} \in \mathcal{V}$.

3.3 Contracting with the agents

This section is devoted to solving the problem of a particular agent. With this in mind, we fix throughout the following $j \in \{1, \dots, m\}$ and $i \in \{1, \dots, n_j\}$, $Y_0^{j,i} \in \mathbb{R}$, as well as the probabilities $\mathbb{P}^{-(j,i)}$ and $\mathbb{P}^M$ chosen by other workers, and thus the associated efforts $\hat{\nu} := (\nu^{-(j,i)}, \nu^M) \in \mathcal{U}^{-(j,i)} \times \mathcal{U}^M$. We have assumed previously that, given his manager's observation, an admissible contract $\xi^{j,i} \in \Xi^{j,i}$ for the (j, i) -th agent is restricted to functions of the form (3.2.1), and more precisely satisfies Definition 3.2.5. Therefore, in view of his objective function (3.2.2), we can already point out that the state variables of his optimisation problem (3.2.4) are X^j and $\bar{X}^{-j}$. By considering the dynamic version $Y_t^{j,i}$ of his value function, we should have:

$$Y_0^{j,i} = V_0^{j,i}(\mathbb{P}^{-(j,i)}, \mathbb{P}^M, \xi^{j,i}), \text{ and } Y_T^{j,i} = g^{j,i}(X_{\cdot \wedge T}^{j,i}, \xi^{j,i}). \quad (3.3.1)$$

From this definition, we notice that we have an explicit relationship between the compensation $\xi^{j,i}$ and the terminal value function $Y_T^{j,i}$. Given the probabilities chosen by other workers, and the associated efforts, we first write the Hamiltonian of the considered agent (j, i) in Section 3.3.1. Intuitively, this Hamiltonian appears by simply applying Itô's formula to the dynamic function of the consumer and by considering the associated Hamilton–Jacobi–Bellman (HJB for short) equation. The next step is to derive a class of so-called revealing contracts, thus extending to a many-agents framework the results of Cvitanić, Possamaï, and Touzi [111] (2018), which considered general moral hazard problems with one agent, or similarly extending to volatility control the results of Élie and Possamaï [140] (2019), where the agents controlled only the drift of the output process X . Similarly, the class of revealing contracts can be obtained by applying Itô's formula, still in an informal way. For more details about this intuition, the reader may refer to Section 3.6.1, where the reasoning is rigorously written in the Markovian framework. Finally, we will see in Theorem 3.3.7 that the restriction to revealing contracts is in fact without loss of generality.

3.3.1 Agent's Hamiltonian

Recalling that the agent's problem is defined by (3.2.4), and since his contract is restricted to functions of the form (3.2.1), the state variables of his problem are X^j and $\bar{X}^{-j}$. Nevertheless, the agent only controls the process $X^{j,i}$, while the dynamic of the other state variables are fixed through the probabilities $\mathbb{P}^{-(j,i)}$ and $\mathbb{P}^M$. Following the reasoning in [111], as well as the intuition in the Markovian case developed in Section 3.6.1, the agent's Hamiltonian is a sum of two components:

- (i) one is the classical Hamiltonian part as in [111], given by the supremum on the agent's effort $u := (a, b) \in U^{j,i}$ of the following function:

$$h^{j,i}(t, x, y, z, \gamma, u) := -c^{j,i}(t, x, u) - k^{j,i}(t, x, u)y + \Lambda^{j,i}(t, u)z + \frac{1}{2}\|\sigma^{j,i}(t, b)\|^2\gamma, \quad (3.3.2)$$

for $(t, x) \in [0, T] \times \mathcal{C}_T$, $(y, z, \gamma) \in \mathbb{R}^3$ and recalling that $\Lambda^{j,i}(t, u) := \sigma^{j,i}(t, b) \cdot \lambda^{j,i}(t, a)$;

- (ii) the second part is related to the indexation of the contract on the outputs of the other workers, and thus indexed on their effort $\hat{\nu} := (\nu^{-(j,i)}, \nu^M) \in \mathcal{U}^{-(j,i)} \times \mathcal{U}^M$, fixed by $\mathbb{P}^{-(j,i)}$ and $\mathbb{P}^M$:

$$H^{j,i}(t, z, \tilde{z}, \hat{\nu}) := z \cdot \left(\Lambda^{j,\ell}(t, \hat{\nu}_t^{j,\ell}) \right)_{\ell=0, \ell \neq i}^{n_j} + \tilde{z} \cdot \left(\sum_{\ell=0}^{n_k} \Lambda^{k,\ell}(t, \hat{\nu}_t^{k,\ell}) \right)_{k=1, k \neq j}^m, \quad (3.3.3)$$

for $t \in [0, T]$ and $(z, \tilde{z}) \in \mathbb{R}^{n_j} \times \mathbb{R}^{m-1}$.

Remark 3.3.1. Without the assumption on the independence of the drift and the volatility with respect to the outputs X , the second part of $H^{j,i}$, defined by (3.3.3), would depend on the outputs of the other teams, which are not supposed to be observable by the manager of the j -th team. Unfortunately, the manager would not be able to compute the Hamiltonian of his agents in this case. This would lead to the more challenging case, already mentioned in Remark 3.2.6, where the agent's problem depends on another process, unobservable by the manager. Again, to our knowledge, this problem is not yet addressed in the literature in continuous time (see the aforementioned paper [189] for a particular example), and would require a full study before it could be considered in our case.

The Hamiltonian $\mathcal{H}^{j,i}$ of the (j, i) -th agent is the supremum on his effort of the sum of the two previous terms:

$$\mathcal{H}^{j,i}(t, x, y, z, \tilde{z}, \gamma, \hat{\nu}) := \sup_{u \in U^{j,i}} h^{j,i}(t, x, y, z^i, \gamma, u) + H^{j,i}(t, z^{-i}, \tilde{z}, \hat{\nu}), \quad (3.3.4)$$

defined for any $(t, x, y, z, \tilde{z}, \gamma) \in [0, T] \times \mathcal{C}_T \times \mathbb{R} \times \mathbb{R}^{n_j+1} \times \mathbb{R}^{m-1} \times \mathbb{R}$ and along $\hat{\nu} \in \mathcal{U}^{-(j,i)} \times \mathcal{U}^M$. By definition of $h^{j,i}$ in (3.3.2), and using Assumption 3.2.2, we can already notice that a maximiser of the Hamiltonian, if it exists, can be written as a function $u^{j,i,*} : [0, T] \times \mathbb{R}^3 \longrightarrow A^{j,i} \times B^{j,i}$:

$$u^{j,i,*}(t, y, z^i, \gamma) := (a^{j,i,*}, b^{j,i,*})(t, y, z^i, \gamma). \quad (3.3.5)$$

This maximiser, which will be proved later to be the optimal effort of the (j, i) -th agent, only depends on the time, the variable y (which will be the agent's continuation utility), and the parameters z^i and γ (which will represent the indexation of the contract on respectively his output $X^{j,i}$ and its quadratic variation). In particular, this maximiser does not depend on the effort of the other workers $\hat{\nu}$, nor even on the indexation of the agent's contract on the outputs of the others, represented by the parameter $\tilde{z}^{-j}$. Therefore, an agent will in fact optimise his efforts independently of others.

Remark 3.3.2. Without Assumption 3.2.2, the maximiser of the Hamiltonian would also be a function of $X^{j,i}$. In this case, even if the agent's supervisor could still compute his optimal effort, the manager of another team could not, which would be an issue, similar to the one mentioned in Remark 3.3.1.

Similarly, we can define the Hamiltonian of any agents in the same way $\mathcal{H}^{j,i}$ is defined for the (j, i) -th agent. Therefore, the function (3.3.5) is defined for any $j \in \{1, \dots, m\}$ and $i \in \{1, \dots, n_j\}$. Apart from the (j, i) -th agent, we will denote by $\nu^{-(j,i),*}$ the collection of the effort of other agents such that their Hamiltonians are maximised. To simplify the reasoning from now on, we will make the following assumption.

Assumption 3.3.3. For all $j \in \{1, \dots, m\}$ and $i \in \{1, \dots, n_j\}$, there exists a unique Borel-measurable map $u^{j,i,*} : [0, T] \times \mathbb{R}^3 \longrightarrow U^{j,i}$, defined by (3.3.5), maximising the Hamiltonian $\mathcal{H}^{j,i}$ given by (3.3.4).

3.3.2 A relevant form of contracts leading to a Nash equilibrium

We define the relevant subset of contracts, similarly as Cvitanić, Possamaï, and Touzi [111, Definition 3.2] (2018), but extended to a multi-agents framework, in the spirit of Élie and Possamaï [140] (2019), although with volatility control in addition. As for the definition of the Hamiltonian, this relevant form of contract is intuited from the Markovian framework, developed in Section 3.6.1.

Let $\mathbb{V}^j := \mathbb{R}^{n_j+1} \times \mathbb{R}^{m-1} \times \mathbb{R}$. For any $\mathbb{V}^j$ -valued $\mathbb{G}^j$ -predictable processes $\mathcal{Z} := (Z, \tilde{Z}, \Gamma)$, and any $Y_0^{j,i} \in \mathbb{R}$, let us define $\mathbb{P}$ -a.s. for all $\mathbb{P} \in \mathcal{P}^{j,i}(\mathbb{P}^{-(j,i)}, \mathbb{P}^M)$ the process $Y^{j,i}$ by:

$$Y_t^{j,i} := Y_0^{j,i} - \int_0^t \mathcal{H}^{j,i}(r, X^{j,i}, Y_r^{j,i}, \mathcal{Z}_r, \hat{\nu}_r^*) dr + \int_0^t Z_r \cdot dX_r^j + \int_0^t \tilde{Z}_r \cdot d\bar{X}_r^{-j} + \frac{1}{2} \int_0^t \Gamma_r d\langle X^{j,i} \rangle_r, \quad (3.3.6)$$

for all $t \in [0, T]$, where $\mathcal{H}^{j,i}$ is defined by (3.3.4) and $\hat{\nu}^* := (\nu^{-(j,i),*}, \nu^M)$ is the collection of optimal efforts of the other agents, given by their Hamiltonian maximisers through (3.3.5), and fixed effort of the managers. This process $Y^{j,i}$ will represent the continuation utility of the (j, i) -th agent, given the action of others.

Remark 3.3.4. Note that the process $Y^{j,i}$ is defined by (3.3.6) as a solution of an ODE with random coefficients. It is therefore necessary to mention that this ODE is well defined so that the solution exists and is unique. This is indeed the case because the Hamiltonian $\mathcal{H}^{j,i}$ defined by (3.3.4) is Lipschitz continuous in the variable y (due to the discount factor $k^{j,i}$ being bounded), thus guaranteeing that $Y^{j,i}$ is well-defined as the unique solution of (3.3.6).

Definition 3.3.5. Let $Y_0^{j,i} \in \mathbb{R}$. We denote by $\mathcal{V}^{j,i}$ the set of $\mathbb{V}^j$ -valued $\mathbb{G}^j$ -predictable process $\mathcal{Z}$, such that the process $Y^{j,i}$ defined by (3.3.6) satisfies the following integrability condition:

$$\sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}} \left[\sup_{t \in [0, T]} |Y_t^{j,i}|^p \right] < +\infty, \quad (\mathbf{J}_A^p)$$

for some $p > 1$. For any $\mathcal{Z} \in \mathcal{V}^{j,i}$, we call random variables of the form $\xi^{j,i} = \bar{g}^{j,i}(X^{j,i}, Y_T^{j,i})$ **revealing contracts** for the (j, i) -th agent, and denote the corresponding set by $\Xi_R^{j,i}$.

By considering revealing contracts, we are able to compute the optimal efforts of each agent, which were given informally by (3.3.5): intuitively, maximising each agent's Hamiltonian is sufficient to obtain his optimal efforts. Since the agent's optimal efforts do not depend on the efforts of the others, it simplifies the task of characterising a Nash equilibrium, in the sense of Definition 3.2.4. In other words, each agent controls his output independently of others. Still informally, Assumption 3.3.3 is in force to ensure existence and uniqueness of the Nash equilibrium, thus avoiding technical considerations at the level of the managers' problem, which, in our opinion, are not relevant for this analysis. These results are rigorously presented in the following proposition.

Proposition 3.3.6. Fix $\mathbb{P}^M$ the probability chosen by the managers. For all $j \in \{1, \dots, m\}$ and $i \in \{1, \dots, n_j\}$, let $Y_0^{j,i} \in \mathbb{R}$ and $\mathcal{Z}^{j,i} := (Z^{j,i}, \tilde{Z}^{j,i}, \Gamma^{j,i}) \in \mathcal{V}^{j,i}$. Define $Y^{j,i}$ and the associated contract $\xi^{j,i} \in \Xi_R^{j,i}$ as in Definition 3.3.5. Denote by $\xi^A \in \Xi_R^A$ the subsequent collection of agents' revealing contracts. Then, $\xi^A \in \Xi^A$ in the sense of Definition 3.2.5, and the unique Nash equilibrium $\mathbb{P}^* \in \mathcal{P}^{A,*}(\mathbb{P}^M, \xi^A)$ is characterised by, for all $j \in \{1, \dots, m\}$ and $i \in \{1, \dots, n_j\}$,

- (i) the optimal effort of the (j, i) -th agent is given by the unique maximiser of his Hamiltonian, defined by (3.3.5):

$$\nu_t^{j,i,*} := u^{j,i,*}(t, Y_t^{j,i}, (Z_t^{j,i})^i, \Gamma_t^{j,i}), \quad dt \otimes \mathbb{P}^* \text{-a.s. for all } t \in [0, T];$$

$$(ii) \quad Y_0^{j,i} = V_0^{j,i}(\mathbb{P}^{-(j,i),*}, \mathbb{P}^M, \xi^{j,i}).$$

While the result given by the previous proposition is relatively intuitive, its formal proof is based on the 2BSDE theory, and is thus reported to [Section 3.8.1](#). The crux of the argument is to show that we can construct directly a solution to a 2BSDE (more precisely, to [2BSDE \$\(j, i\)\$](#) defined in [Section 3.7.1.3](#)) whenever $\xi^{j,i} \in \Xi_R^{j,i}$, and this for all $j \in \{1, \dots, m\}$ and $i \in \{1, \dots, n_j\}$. Finally, the main point is to prove that the restriction of our study to revealing contracts is not restrictive from the managers' point of view. This is precisely the purpose of the following section.

3.3.3 Optimality of the revealing contracts

Let us fix throughout this section $j \in \{1, \dots, m\}$, in order to focus on the j -th managers' problem. Given an admissible contract $\xi^{j,0} \in \Xi^{j,0}$, in the sense of [Definition 3.2.9](#), as well as the decision of other managers summarised by the control $\tilde{\chi}^{-j}$, we recall that the j -th manager's optimisation problem is defined by [\(3.2.8\)](#). Following the general approach by [Cvitanic, Possamaï, and Touzi \[111\] \(2018\)](#), we can prove that there is no loss of generality for the manager to restrict to contracts in $\Xi_R^{j,0}$, in the sense of [Definition 3.3.5](#), instead of contracts in $\Xi^{j,0}$. More precisely, to incentivise the (j, i) -th agent under his supervision, it is sufficient to offer him a revealing contract $\xi^{j,i} \in \Xi_R^{j,i}$, parametrised by a process $\mathcal{Z}^{j,i} \in \mathcal{V}^{j,i}$, instead of considering all admissible contracts in $\Xi^{j,i}$. Therefore, the j -th manager has to choose in an optimal way:

(i) his own effort $\nu^{j,0} \in \mathcal{U}^{j,0}$;

(ii) the triple of payment rates for each agent under his supervision, *i.e.*, $\mathcal{Z}^{j,i} \in \mathcal{V}^{j,i}$ for all $i \in \{1, \dots, n_j\}$.

The control of the j -th manager can be summarised by a process $\chi^j := (\nu^{j,0}, (\mathcal{Z}^{j,i})_{i=1}^{n_j}) \in \mathcal{X}^j$ and, extending this reasoning to the other managers, the controls of all managers will be denoted by χ , defined as follows:

$$\chi := (\chi^j)_{j=1}^m \in \mathcal{X}, \quad \text{where } \mathcal{X} := \prod_{j=1}^m \mathcal{X}^j, \quad \text{and } \mathcal{X}^j := \mathcal{U}^{j,0} \times \prod_{i=1}^{n_j} \mathcal{V}^{j,i}.$$

Note that for all $j \in \{1, \dots, m\}$, the process χ^j takes values in $\mathfrak{X}^j := U^{j,0} \times (\mathbb{V}^j)^{n_j}$. The process χ thus takes values in $\mathfrak{X}$, naturally defined as the Cartesian product of all $\mathfrak{X}^j$. With this in hand, we can now turn to the main theorem of this first Stackelberg game, whose proof is postponed to [Section 3.8.1](#).

Theorem 3.3.7. *Consider a collection of admissible contracts $\xi^M := (\xi^{j,0})_{j=1}^m \in \Xi^M$ for the managers, in the sense of [Definition 3.2.9](#), then the following equality holds*

$$V_0^{j,0}(\xi^{j,0}, \chi^{-j}) = \sup_{\chi^j \in \mathcal{X}^j} J^{j,0}(\mathbb{P}^*(\chi), \xi^{j,0}), \quad \text{for all } j \in \{1, \dots, m\}, \quad (3.3.7)$$

where $\mathbb{P}^*(\chi)$ is the unique Nash equilibrium between the agents, given the control $\chi \in \mathcal{X}$ of the managers.

To summarise the agents' problem, first, [Proposition 3.3.6](#) solves the Nash equilibrium for a probability $\mathbb{P}^M$ and a collection of revealing contracts $\xi^A \in \Xi_R^A$ chosen by the managers. Then, [Theorem 3.3.7](#) states that the restriction to revealing contracts is without loss of generality. Using the previous results and notations, we can write the value function of each agent at equilibrium as follows:

$$V_0^{j,i,*}(\chi) := V_0^{j,i}(\mathbb{P}^{-(j,i),*}, \mathbb{P}^M, \xi^{j,i}), \quad \text{for all } j \in \{1, \dots, m\}, \quad i \in \{1, \dots, n_j\}. \quad (3.3.8)$$

3.4 Contracting with the managers

Throughout this section, we fix $j \in \{1, \dots, m\}$ and focus our attention on the j -th manager's problem. Recall that the j -th manager controls his own project with output $X^{j,0}$ as well as the compensations for his n_j agents. [Theorem 3.3.7](#) ensures that it is sufficient to restrict the admissible contract space for the (j, i) -th agent to $\Xi_R^{j,i}$, and thus limit the j -th manager's optimisation problem to choosing an optimal effort $\nu^{j,0} \in \mathcal{U}^{j,0}$, as well as n_j triples $\mathcal{Z}^{j,i} := (Z^{j,i}, \tilde{Z}^{j,i}, \Gamma^{j,i}) \in \mathcal{V}^{j,i}$, for $i \in \{1, \dots, n_j\}$, to set up the contracts of the agents under his supervision. Therefore, the manager's goal is to choose his control process $\chi^j \in \mathcal{X}^j$ optimally, given the results of his working team ζ^j and his compensation $\xi^{j,0}$ chosen by the principal.

Through the equality (3.3.7), the manager's value function is similar to that of an agent in a classical principal-agent problem, since given a contract $\xi^{j,0}$, the manager chooses his optimal controls. However, the state variable ζ^j , as a function of $\xi^{j,0}$, X^j and $\bar{X}^{-j}$, seems to be considered partially in strong and weak formulation. Indeed, $\xi^{j,0}$ is considered in strong formulation (indexed by the control $\chi^j \in \mathcal{X}^j$), while the vector of outputs X is considered in weak formulation (the control $\chi^j \in \mathcal{X}^j$ only impacts the distribution of X through $\mathbb{P}^*$). It makes little sense to consider a control problem of this form directly, and we should adopt the weak formulation to state the problem of each manager, since this is the one which makes sense for the agents' problem.

However, before turning to the weak formulation, it is necessary to determine which are the state variables of the manager problem. Recall that the contract $\xi^{j,0}$ for the manager can only be indexed on $\zeta := (\zeta^j)_{j=1}^m$, as defined in (3.2.5), since it is the only variable observable by the principal. Nevertheless, each ζ^j measures the global result of the entire j -th working team (including the manager) and therefore depends on the outputs of the team, namely X^j , and the collection of payments $\xi^{j,i}$, for $i \in \{1, \dots, n_j\}$, to be made to the agents, denoted by $\xi^{j,0}$. More precisely, we can state that, for all $t \in [0, T]$, $\zeta_t^j = f^j(t, X^j, \xi^{j,0})$, for some function $f^j : [0, T] \times \mathbb{R}^{n_j+1} \times \mathbb{R}^{n_j} \rightarrow \mathbb{R}^h$.

In all generality and without any particular assumptions on the function f^j , there is no reason the dynamic of ζ^j should not depend on the collections of all outputs and continuation utilities, which would therefore constitute the state variables of j -th manager. Unfortunately, this would lead us, once again, to the case where the principal does not observe all the state variables of the managers' problem. As already explained in [Remarks 3.2.6](#) and [3.3.1](#), this would raise major issues requiring a comprehensive study before being addressed in our framework. We are therefore forced to make the following major assumption on the shape of the induced dynamic for ζ .

Assumption 3.4.1. *The dynamic of ζ depends only on time, on ζ itself, and on the managers' controls summarised by the process $\chi \in \mathcal{X}$. More precisely, there exist two bounded functions:*

$$\Lambda_M : [0, T] \times \mathcal{C}_T^{mh} \times \mathfrak{X} \rightarrow \mathbb{R}^{dw} \text{ and } \Sigma_M : [0, T] \times \mathcal{C}_T^{mh} \times \mathfrak{X} \rightarrow \mathbb{M}^{hm,dw},$$

satisfying $\Lambda_M(\cdot, z)$ and $\Sigma_M(\cdot, z)$ $\mathbb{G}$ -optional for any $z \in \mathfrak{X}$, such that ζ is solution to the following SDE:

$$d\zeta_t := \Sigma_M(t, \zeta, \chi_t)(\Lambda_M(t, \zeta, \chi_t)dt + dW_t), \text{ for all } t \in [0, T]. \quad (3.4.1)$$

Although restrictive, this assumption nevertheless allows the study of interesting frameworks that we may have in mind, including the context described by [Sung \[327\] \(2015\)](#). The reader is referred to [Section 3.6.3](#), where interesting cases in which this hypothesis is satisfied are presented (see [Lemmas 3.6.1](#) and [3.6.2](#) for the linear and exponential utility functions respectively). Throughout the following, we will assume that [Assumption 3.4.1](#) holds.

Even under [Assumption 3.4.1](#), the dynamic of ζ is controlled by all managers in a non-trivial way. Indeed, while the agents control uniquely their own output, in the sense that the (j, i) -th agent only impacts the

dynamic of $X^{j,i}$, the j -th manager does not only control the variable ζ^j , but also the other components of the vector ζ . This is due to the fact that the component ζ^k (for $k \neq j$) depends on the collection of contracts $\xi^{k,0}$, which are indexed in particular on $\bar{X}^{-k}$, and thus depend in particular on the optimal effort of the (j,i) -th agent given by

$$\nu_t^{j,i,*} := u^{j,i,*}(t, Y_t^{j,i}, (Z_t^{j,i})^i, \Gamma_t^{j,i}), \text{ dt} \otimes \mathbb{P}^* \text{-a.s. for all } t \in [0, T].$$

Since the pair $(Z^{j,i}, \Gamma^{j,i})$ is chosen by the j -th manager, he somehow controls the volatility of ζ^k , for all $k \neq j$. Note that he also controls the volatility through his own effort $\nu^{j,0}$. This leads us to a control problem with interacting agents, as for example, in the work of [Élie and Possamaï \[140\] \(2019\)](#) (see also [Élie, Mastrolia, and Possamaï \[141\] \(2018\)](#) for the case of an infinite number of interacting agents), but with volatility control in addition. The following weak formulation is inspired by the work of [Possamaï, Touzi, and Zhang \[286, Section 6.1\] \(2020\)](#), where a zero-sum game is considered between two players, controlling both the drift and the volatility of the same output process. Therefore, it only requires to extend their formulation to a nonzero-sum game with m interacting agents.

3.4.1 Canonical space for the managers

Following the previous reasoning, in particular under [Assumption 3.4.1](#), and given the form (3.2.5) for the managers' contract, ζ is clearly the only state variable of the managers' problems. We are thus led to consider the following canonical space, $\Omega^M := \mathcal{C}_T^{hm} \times \mathcal{C}_T^{wd} \times \mathbb{X}$, where, similarly as for the initial canonical space defined in [Section 3.1.3](#), $\mathbb{X}$ is the collection of all finite and positive Borel measures on $[0, T] \times \mathfrak{X}$, whose projection on $[0, T]$ is the Lebesgue measure. The weak formulation requires to consider a subset of $\mathbb{X}$, namely the set $\mathbb{X}_0$ of all $q \in \mathbb{X}$ such that $q(ds, du) = \delta_{\phi_s}(du)ds$, for some Borel function ϕ . The canonical process is denoted by (ζ, W, Π^M) where, for any $(t, \omega, \varpi, q) \in [0, T] \times \Omega$,

$$\zeta_t(\omega, \varpi, q) := \omega_t, \quad W_t(\omega, \varpi, q) := \varpi_t, \quad \text{and} \quad \Pi^M(\omega, \varpi, q) := q.$$

The associated canonical filtration is defined by $\mathbb{F}^M := (\mathcal{F}_t^M)_{t \in [0, T]}$ with

$$\mathcal{F}_t^M := \sigma \left(\left(\zeta_s, \int_0^s \int_{\mathfrak{X}} \varphi(r, u) \Pi^M(dr, du) \right) \text{ s.t. } (s, \varphi) \in [0, t] \times \mathcal{C}_b([0, T] \times \mathfrak{X}, \mathbb{R}) \right), \quad t \in [0, T].$$

Then, for any $(t, \psi) \in [0, T] \times \mathcal{C}_b^2(\mathbb{R}^{hm} \times \mathbb{R}^{dw}, \mathbb{R})$, we set

$$M_t^M(\psi) := \psi(\zeta_t, W_t) - \int_0^t \int_{\mathfrak{X}} \left(\tilde{\Lambda}_M(s, \zeta, u) \cdot \nabla \psi(\zeta_s, W_s) + \frac{1}{2} \text{Tr}[\nabla^2 \psi(\zeta_s, W_s) (\tilde{\Sigma}_M \tilde{\Sigma}_M^\top)(s, \zeta, u)] \right) \Pi^M(ds, du).$$

where $\tilde{\Lambda}_M$ and $\tilde{\Sigma}_M$ are respectively the drift vector and the diffusion matrix of the $(hm + dw)$ -dimensional vector process $(\zeta, W)^\top$, defined for all $s \in [0, T]$, $x \in \mathcal{C}_T^{hm}$ and $u \in \mathfrak{X}$ by:

$$\tilde{\Lambda}_M(s, x, u) := \begin{pmatrix} (\Sigma_M \Lambda_M)(s, x, u) \\ \mathbf{0}_{wd} \end{pmatrix}, \quad \tilde{\Sigma}_M(s, x, u) := \begin{pmatrix} \mathbf{0}_{hm, hm} & \Sigma_M(s, x, u) \\ \mathbf{0}_{wd, hm} & \mathbf{I}_{wd} \end{pmatrix},$$

where Λ_M and Σ_M are defined in [Assumption 3.4.1](#).

We fix an initial condition for the process ζ , namely $\varrho_0 \in \mathbb{R}^{hm}$. Similarly as in [Definition 3.1.1](#) for the initial control problem, we define the subset $\mathcal{P}^M$ of probability measures $\mathbb{P}$ on $(\Omega^M, \mathcal{F}_T^M)$ satisfying the following conditions:

- (i) $M^M(\psi)$ is a $(\mathbb{F}^M, \mathbb{P})$ -local martingale on $[0, T]$ for all $\psi \in \mathcal{C}_b^2(\mathbb{R}^{hm} \times \mathbb{R}^{dw}, \mathbb{R})$;

$$(ii) \mathbb{P}[(\zeta_0, W_0) = (\varrho_0, w_0)] = 1;$$

$$(iii) \mathbb{P}[\Pi^M \in \mathbb{X}_0] = 1.$$

Similarly to Lemma 3.1.2, we know that for all $\mathbb{P} \in \mathcal{P}^M$, we have $\Pi^M(ds, du) = \delta_{\chi_s^{\mathbb{P}}}(du)ds$ $\mathbb{P}$ -a.s. for some $\mathbb{F}^M$ -predictable control process $\chi^{\mathbb{P}}$, and we thus obtain the representation (3.4.1) for the dynamic of ζ , but controlled by $\chi := \chi^{\mathbb{P}}$. However, this representation only gives access to an admissible set of controls in terms of probability measures for all managers, namely $\mathcal{P}^M$.

3.4.2 Weak formulation of a manager's problem

To properly define the choice of a particular manager, in response to the choices of others, we need to define its own canonical space. With this in mind, we fix throughout the following $j \in \{1, \dots, m\}$, as well as the controls $\chi^{-j} \in \mathcal{X}^{-j}$ chosen by other managers. The canonical space for the j -th manager is defined by $\Omega^j := \mathcal{C}_T^{hm} \times \mathcal{C}_T^{wd} \times \mathbb{X}^j$, where $\mathbb{X}^j$ is the collection of all finite and positive Borel measures on $[0, T] \times \mathfrak{X}^j$, whose projection on $[0, T]$ is the Lebesgue measure. Similarly as before, we define the corresponding set $\mathbb{X}_0^j$ of all $q \in \mathbb{X}^j$ such that $q(ds, du) = \delta_{\phi_s}(du)ds$ for some Borel function ϕ . The canonical process is denoted by (ζ, W, Π^j) where,

$$\zeta_t(\omega, \varpi, q) := \omega_t, \quad W_t(\omega, \varpi, q) := \varpi_t, \quad \text{and } \Pi^j(\omega, \varpi, q) := q, \quad \text{for any } (t, \omega, \varpi, q) \in [0, T] \times \Omega^j.$$

The associated canonical filtration is defined by $\mathbb{F}^j := (\mathcal{F}_t^j)_{t \in [0, T]}$ with

$$\mathcal{F}_t^j := \sigma\left(\left(\zeta_s, \int_0^s \int_{\mathfrak{X}^j} \varphi(r, u) \Pi^j(dr, du)\right) \text{ s.t. } (s, \varphi) \in [0, t] \times \mathcal{C}_b([0, T] \times \mathfrak{X}^j, \mathbb{R})\right), \quad t \in [0, T].$$

Then, for any $(t, \psi) \in [0, T] \times \mathcal{C}_b^2(\mathbb{R}^{hm} \times \mathbb{R}^{dw}, \mathbb{R})$, we set

$$\begin{aligned} M_t^j(\psi) := & \psi(\zeta_t, W_t) - \int_0^t \int_{\mathfrak{X}^j} \left(\tilde{\Lambda}_M(s, \zeta, u \otimes_j \chi_s^{-j}) \cdot \nabla \psi(\zeta_s, W_s) \right. \\ & \left. + \frac{1}{2} \text{Tr}[\nabla^2 \psi(\zeta_s, W_s) (\tilde{\Sigma}_M \tilde{\Sigma}_M^\top)(s, \zeta, u \otimes_j \chi_s^{-j})] \right) \Pi^j(ds, du). \end{aligned}$$

where $(u \otimes_j \chi_t^{-j})^j = u$ and $(u \otimes_j \chi_t^{-j})^k = \chi_t^k$ for $k \neq j$, recalling that χ^{-j} is fixed throughout this section.

We can then define the subset $\mathcal{P}^j(\chi^{-j})$ of probability measures $\mathbb{P}$ on $(\Omega^j, \mathcal{F}_T^j)$ satisfying the following conditions:

- (i) $M^j(\psi)$ is a $(\mathbb{F}^j, \mathbb{P})$ -local martingale on $[0, T]$ for all $\psi \in \mathcal{C}_b^2(\mathbb{R}^{hm} \times \mathbb{R}^{dw}, \mathbb{R})$;
- (ii) there exists some $\iota \in \mathbb{R}^{wd}$ such that $\mathbb{P} \circ (\zeta_0, W_0)^{-1} = \delta_{(\varrho, \iota)}$;
- (iii) $\mathbb{P}[\Pi^j \in \mathbb{X}_0^j] = 1$.

We know that for all $\mathbb{P} \in \mathcal{P}^j(\chi^{-j})$, we have $\Pi^j(ds, du) = \delta_{\chi_s^{j, \mathbb{P}}}(du)ds$ $\mathbb{P}$ -a.s. for some $\mathbb{F}^j$ -predictable control process $\chi^{j, \mathbb{P}}$, and we obtain the representation (3.4.1) for the dynamic of ζ , but controlled by $\chi := \chi^{j, \mathbb{P}} \otimes_j \chi^{-j}$.

Therefore, given $\chi^{-j} \in \mathcal{X}^{-j}$ chosen by other managers, the j -th manager must choose an optimal probability measure $\mathbb{P} \in \mathcal{P}^j(\chi^{-j})$, which leads to consider the following weak formulation for his optimisation problem (3.2.8):

$$V_0^{j,0}(\xi^{j,0}, \chi^{-j}) := \sup_{\mathbb{P} \in \mathcal{P}^j(\chi^{-j})} J^{j,0}(\mathbb{P}, \xi^{j,0}), \quad (3.4.2)$$

recalling that $J^{j,0}$ is defined by (3.2.7). We can then adapt Definition 3.2.8 to define a Nash equilibrium between the managers in weak formulation.

Definition 3.4.2 (Nash equilibrium between managers, in weak formulation). *Fix a collection of contracts designed by the principal for the managers, namely $\xi^M := (\xi^{j,0})_{j=1}^m \in \Xi^M$. A Nash equilibrium between the managers is a control $\chi \in \mathcal{X}$, such that there exists a probability measure $\mathbb{P}^* \in \mathcal{P}^M$ satisfying, for all $j \in \{1, \dots, m\}$, $\mathbb{P}^* \in \mathcal{P}^j(\chi^{-j})$ and such that the supremum in (3.4.2) is attained for this $\mathbb{P}^*$. We denote by $\mathcal{P}^{M,*}(\xi^M)$ the set of Nash equilibria.*

3.4.3 Relevant form of contracts for the managers

Recall that [Assumption 3.4.1](#) is enforced to ensure that ζ is the only state variable of the j -th manager's optimisation problem. Following the line developed in [Section 3.3.1](#) for the agents (based on [111]), the Hamiltonian of the j -th manager is defined by:

$$\mathcal{H}^j(t, x, y, z, \gamma, \chi^{-j}) := \sup_{u \in \mathfrak{X}^j} h^j(t, x, y, z, \gamma, \chi^{-j}, u), \quad (3.4.3)$$

for $(t, x, y) \in [0, T] \times \mathcal{C}_T^{hm} \times \mathbb{R}$, $(z, \gamma) \in \mathbb{R}^{hm} \times \mathbb{M}^{hm}$ and $\chi^{-j} \in \mathcal{X}^{-j}$ chosen by other managers, where in addition, for $u \in \mathfrak{X}^j$,

$$\begin{aligned} h^j(t, x, y, z, \gamma, \chi^{-j}, u) := & - (c^{j,0} + yk^{j,0})(t, x^j, u) + (\Sigma_M \Lambda_M)(t, x, u \otimes \chi^{-j}) \cdot z \\ & + \frac{1}{2} \text{Tr}[(\Sigma_M \Sigma_M^\top)(t, x, u \otimes \chi^{-j})\gamma]. \end{aligned}$$

We can define the Hamiltonian of any manager in the same way $\mathcal{H}^j$ is defined for the j -th manager.

We can already notice that a maximiser of the j -th manager's Hamiltonian, if it exists, depends on the time, the paths of the state variable ζ , the parameter y (which will be the manager's continuation utility), the parameters z and γ (which will index the contract on respectively the reporting ζ and its quadratic variation). But more importantly, it depends on the efforts of other managers, namely χ^{-j} .

Assumption 3.4.3. *For all $j \in \{1, \dots, m\}$, there exists at least a Borel-measurable map $u^j : [0, T] \times \mathcal{C}_T^{hm} \times \mathbb{R} \times \mathbb{R}^{hm} \times \mathbb{M}^{hm} \times \mathcal{X}^{-j} \rightarrow \mathfrak{X}^j$, satisfying:*

$$\mathcal{H}^j(t, x, y, z, \gamma, \chi^{-j}) = h^j(t, x, y, z, \gamma, \chi^{-j}, u^j(t, x, y, z, \gamma, \chi^{-j})),$$

for all $(t, x, y, z, \gamma) \in [0, T] \times \mathcal{C}_T^{hm} \times \mathbb{R} \times \mathbb{R}^{hm} \times \mathbb{M}^{hm}$ and given the actions of other managers $\chi^{-j} \in \mathcal{X}^{-j}$.

Contrary to the optimal efforts of agents, who were independent of the efforts of other agents, the maximiser of the Hamiltonian therefore depends on the efforts of other managers, *i.e.*, χ^{-j} . Similarly, these efforts χ^{-j} will be defined through the maximiser of other managers' Hamiltonian, and will thus depends on $(t, x, y^{-j}, z^{-j}, \gamma^{-j})$, but also on χ^j . This leads us to consider in some way a fixed point of a multidimensional Hamiltonian, where each component represents the Hamiltonian of a manager.

Assumption 3.4.4. *There exists a unique Borel-measurable map $u^* : [0, T] \times \mathcal{C}_T^{hm} \times \mathbb{R}^m \times \mathbb{M}^{hm,m} \times (\mathbb{M}^{hm})^m \rightarrow \prod_{j=1}^m \mathfrak{X}^j$, such that for all $j \in \{1, \dots, m\}$, the j -th component $u^{j,*}$ takes values in $\mathfrak{X}^j$ and satisfies:*

$$\mathcal{H}^j(t, x, y^j, z^j, \gamma^j, u^{-j,*}(t, x, y, z, \gamma)) = h^j(t, x, y^j, z^j, \gamma^j, u^{-j,*}(t, x, y, z, \gamma), u^{j,*}(t, x, y, z, \gamma)),$$

for all $(t, x) \in [0, T] \times \mathcal{C}_T^{hm}$, $y := (y^j)_{j=1}^m \in \mathbb{R}^m$, and $(z, \gamma) \in \mathbb{M}^{hm,m} \times (\mathbb{M}^{hm})^m$ where for all $j \in \{1, \dots, m\}$, $z^j \in \mathbb{R}^{hm}$ is the j -th column of z and $\gamma^j \in \mathbb{M}^{hm}$.

Remark 3.4.5. *The previous assumption, in particular on the existence of such a function, is classical in multi-agents problems to ensure existence of an equilibrium for the managers (see for example Élie and Possamaï [140, Assumption 4.1] (2019) for a multi-agents problem with drift control only). Uniqueness is assumed to simplify the study, as in the similar hypothesis for the agents (see Assumption 3.3.3). Indeed, in the absence of uniqueness, it would be possible to have several Nash equilibria, and it would then be necessary to represent the preferences of the managers and the principal between these different Nash equilibria (see also [140, Section 4.1.1] for an example).*

Thanks to Assumption 3.4.4, we can define the following function, which corresponds to the Hamiltonian of the j -th manager under optimal efforts of all managers:

$$\mathcal{H}^{j,\star}(t, x, y, z, \gamma) := \mathcal{H}^j(t, x, y^j, z^j, \gamma^j, u^{-j,\star}(t, x, y, z, \gamma)), \quad (3.4.4)$$

for $(t, x, y, z, \gamma) \in [0, T] \times \mathcal{C}_T^{hm} \times \mathbb{R}^m \times \mathbb{M}^{hm,m} \times (\mathbb{M}^{hm})^m$, and taking values in $\mathbb{R}$.

We can now define the relevant subset of contracts, similarly as for the agents, except that the contract has to be indexed on ζ , and that the managers' Hamiltonians are coupled. Let $\mathbb{V} := \mathbb{M}^{hm,m} \times (\mathbb{M}^{hm})^m$. For any $\mathbb{V}$ -valued $\mathbb{G}$ -predictable processes (Z, Γ) , and any $Y_0^M := (Y_0^j)_{j=1}^m \in \mathbb{R}^m$, let us define the multidimensional process $\mathcal{Y}^M$ such that each component $\mathcal{Y}^j$, for $j \in \{1, \dots, m\}$, satisfies:

$$\mathcal{Y}_t^j := Y_0^j - \int_0^t \mathcal{H}^{j,\star}(r, \zeta, \mathcal{Y}_r^M, Z_r, \Gamma_r) dr + \int_0^t Z_r^j \cdot d\zeta_r + \frac{1}{2} \int_0^t \text{Tr}[\Gamma_r^j d\langle \zeta \rangle_r], \quad t \in [0, T], \quad (3.4.5)$$

where $\mathcal{H}^{j,\star}$ is defined by (3.4.4). Each component $\mathcal{Y}^j$ will thus represent the continuation utility of the j -th manager, given the action of others.

Similarly as for the agents' problem (see Remark 3.3.4), each component $\mathcal{Y}^j$ of the process $\mathcal{Y}^M$ is defined by (3.4.5) as a solution to an ODE with random coefficients. The following assumption is made to ensure that this ODE is well defined, so that the solution exists and is unique.

Assumption 3.4.6. *The multidimensional Hamiltonian $\mathcal{H}^\star$, whose components are defined by (3.4.4), is uniformly Lipschitz continuous with respect to the variable $y \in \mathbb{R}^m$.*

One can notice that if, for all $j \in \{1, \dots, m\}$, the discount factor $k^{j,0}$ is not controlled, meaning that $k^{j,0} : [0, T] \times \mathcal{C}_T^h \rightarrow \mathbb{R}$, then the previous assumption is not necessary. Indeed, in this case, the optimal effort of the managers, defined through the function $u^\star$ will be independent of the variable $y \in \mathbb{R}^m$, thus also implying the independence of the Hamiltonian $\mathcal{H}^\star$. Similarly, in the case of exponential (CARA) utility functions for the managers, a change of variable allows to suppress the dependence of the control on the variable y . Therefore, in these two cases, Assumption 3.4.6 is trivially satisfied, ensuring the well-definition of the process $\mathcal{Y}^M$.

Definition 3.4.7. *Let $Y_0^M \in \mathbb{R}^m$. We denote by $\mathcal{V}$ the set of $\mathbb{V}$ -valued $\mathbb{G}$ -predictable process $\mathcal{Z}$, such that for all $j \in \{1, \dots, m\}$, each component $\mathcal{Y}^j$ of the m -dimensional process $\mathcal{Y}^M$ defined by (3.4.5) satisfies the following integrability condition, for some $p > 1$:*

$$\sup_{\mathbb{P} \in \mathcal{P}^M} \mathbb{E}^{\mathbb{P}} \left[\sup_{t \in [0, T]} |\mathcal{Y}_t^j|^p \right] < +\infty, \quad (J_M^p)$$

Consider the function $\bar{g}^M : \mathcal{C}_T^{hm} \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ such that for all $j \in \{1, \dots, m\}$, the j -th component is given by the map $\bar{g}^{j,0} : \mathcal{C}_T^h \times \mathbb{R} \rightarrow \mathbb{R}$ defined in the managers' problem. For $\mathcal{Z} \in \mathcal{V}$ and $Y_0^M := (Y_0^j)_{j=1}^m \in \mathbb{R}^m$, we consider the m -dimensional random variable $\xi^M := \bar{g}^M(\zeta, \mathcal{Y}_T^M)$, and denote the corresponding set by Ξ_R^M . We will say that $\xi^M \in \Xi_R^M$ is a collection of **revealing contracts** for the managers. In particular, for all $j \in \{1, \dots, m\}$, its j -th component satisfies $\xi^j = \bar{g}^{j,0}(\zeta^j, \mathcal{Y}_T^j)$, and the corresponding set is denoted by Ξ_R^j .

3.4.4 Nash and optimality of revealing contracts

By considering revealing contracts, we are able to compute the optimal efforts of each manager, which were given informally by the maximiser of their Hamiltonian. Contrary to the agents' problem, the manager's optimal efforts depend on the efforts of other managers. Informally, [Assumption 3.4.4](#) is in force to ensure existence and uniqueness of the Nash equilibrium, thus avoiding technical considerations on the preferences between the different Nash equilibria, which, in our opinion, are not relevant for this analysis. These results are rigorously presented in the following proposition.

Proposition 3.4.8. *Let $Y_0^M := (Y_0^j)_{j=1}^m \in \mathbb{R}^m$ and $\mathcal{Z} := (Z, \Gamma) \in \mathcal{V}$. By [Definition 3.4.7](#), consider the m -dimensional process $\mathcal{Y}^M$ and the associated collection of contracts $\xi^M := (\xi^j)_{j=1}^m \in \Xi_R^M$. Then, $\xi^M \in \Xi^M$ in the sense of [Definition 3.2.9](#) and there exists a unique Nash equilibrium in the sense of [Definition 3.4.2](#), i.e., a control $\chi^* \in \mathcal{X}$ associated to a probability measure $\mathbb{P}^*$. This Nash equilibrium is characterised by:*

- (i) *for all $j \in \{1, \dots, m\}$, the optimal effort of the j -th manager is given by the j -th component of the unique fixed point of the multidimensional Hamiltonian, defined through [\(3.4.4\)](#), i.e.:*

$$\chi_t^{j,*} := w^{j,*}(t, \zeta, \mathcal{Y}_t^M, Z_t, \Gamma_t), \text{ dt } \otimes \mathbb{P}^* \text{-a.s. for all } t \in [0, T]; \quad (3.4.6)$$

- (ii) $Y_0^j = V_0^{j,0}(\xi^j, \chi^{-j,*})$.

The formal proof of the previous result is based on the 2BSDE theory, and is thus reported to [Section 3.8.2](#). It follows the same reasoning as the one developed in the proof of [Proposition 3.3.6](#) (see [Section 3.8.1](#)). [Proposition 3.4.8](#) solves the Nash equilibrium for a collection of revealing contracts $\xi^M \in \Xi_R^M$ chosen by the principal. At equilibrium, we can write the value function of the j -th manager as follows:

$$V^{j,0,*}(\xi^M) := V_0^{j,0}(\xi^j, \chi^{-j,*}), \text{ for all } j \in \{1, \dots, m\}. \quad (3.4.7)$$

Finally, it remains to prove that the specialisation of our study to revealing contracts is not restrictive from the principal's point of view. This result is given by the following theorem, which echoes [Theorem 3.3.7](#) for the manager-agent problem. Its formal proof is also postponed to [Section 3.8.2](#).

Theorem 3.4.9. *Recalling that the principal's problem is defined by [\(3.2.10\)](#), the following equality holds*

$$V_0^P = \sup_{Y_0 \geq \rho} \bar{V}^P(Y_0) \text{ where } \bar{V}^P(Y_0) := \sup_{\mathcal{Z} \in \mathcal{V}} \mathbb{E}^{\mathbb{P}^*(\mathcal{Z})} \left[\mathcal{K}_{0,T}^P g^P(\zeta, \xi^M) \right], \quad (3.4.8)$$

where

- (i) *the inequality $Y_0 \geq \rho$ has to be understood componentwise, i.e., for all $j \in \{1, \dots, m\}$ and $i \in \{0, \dots, m\}$, $Y_0^{j,i} \geq \rho^{j,i}$, and ensure that the participation constraint of all workers is satisfied;*
- (ii) *$\mathbb{P}^*(\mathcal{Z})$ is the unique Nash equilibrium between the workers, given the control $\mathcal{Z} \in \mathcal{V}$ chosen by the principal;*
- (iii) *ξ^M is the collection of revealing contracts for the managers, thoroughly characterised by the choice of $Y_0^M \in \mathbb{R}^m$ and $\mathcal{Z} \in \mathcal{V}$.*

Note that the choices of the workers' initial continuation utility, namely $Y_0 := (Y_0^A, Y_0^M)$, directly impact the initial value of ζ . Moreover, they also have an impact on the Nash equilibrium between the managers. However, again with a view to lighten the notations, this dependency is not explicitly mentioned.

3.5 Principal's problem

Following the previous reasoning, in particular under [Assumption 3.4.1](#), ζ , which is the only variable observable by the principal, is also clearly the only state variable of her problem together the continuation utilities of the managers, namely $\mathcal{Y}^M$, since $\xi^M \in \Xi_R^M$ satisfies $\xi^M = \bar{g}^M(\zeta, \mathcal{Y}_T^M)$.

3.5.1 Canonical space for the principal

First, we should write the dynamics of ζ and $\mathcal{Y}^M$ under managers' optimal efforts. With this in mind, and recalling the definition of the map u^* in [Assumption 3.4.4](#), we define two functions $\Lambda_M^*, \Sigma_M^* : [0, T] \times \mathcal{C}_T^{hm} \times \mathbb{R}^m \times \mathbb{M}^{hm,m} \times (\mathbb{M}^{hm})^m \longrightarrow \mathbb{R}^{dw}, \mathbb{M}^{hm,dw}$, satisfying:

$$\Lambda_M^*(t, x, y, v) := \Lambda_M(t, x, u^*(t, x, y, z, \gamma)) \quad \text{and} \quad \Sigma_M^*(t, x, y, v) := \Sigma_M(t, \zeta, u^*(t, x, y, z, \gamma)), \quad (3.5.1)$$

for all $(t, x, y) \in [0, T] \times \mathcal{C}_T^{hm} \times \mathbb{R}^m$ and $v := (z, \gamma) \in \mathbb{M}^{hm,m} \times (\mathbb{M}^{hm})^m$. Similarly, we define the functions $c^{M,*}, k^{M,*} : [0, T] \times \mathcal{C}_T^{hm} \times \mathbb{R}^m \times \mathbb{M}^{hm,m} \times (\mathbb{M}^{hm})^m \longrightarrow \mathbb{R}^m$ such that for all $j \in \{1, \dots, m\}$, the j -th components $c^{j,*}$ and $k^{j,*}$ respectively satisfy:

$$c^{j,*}(t, x, y, v) := c^{j,0}(t, x^j, u^{j,*}(t, x, y, z, \gamma)) \quad \text{and} \quad k^{j,*}(t, x, y, v) := k^{j,0}(t, x^j, u^{j,*}(t, x, y, z, \gamma)),$$

for $(t, x, y, v) \in [0, T] \times \mathcal{C}_T^{hm} \times \mathbb{R}^m \times \mathbb{V}$ and more precisely $x := (x^j)_{j=1}^m$, where $x^j \in \mathcal{C}_T^h$.

With these notations, ζ is solution to the following SDE:

$$\zeta_t = \zeta_0 + \int_0^t \Sigma_M^*(s, \zeta, \mathcal{Y}_s^M, \mathcal{Z}_s) (\Lambda_M^*(s, \zeta, \mathcal{Y}_s^M, \mathcal{Z}_s) ds + dW_s), \quad t \in [0, T], \quad (3.5.2)$$

under $\mathbb{P}^*(\mathcal{Z})$, for some $\mathcal{Z} := (Z, \Gamma) \in \mathcal{V}$. Then, we can compute the value at equilibrium of the multidimensional Hamiltonian $\mathcal{H}^*$ defined by (3.4.4), and use the dynamic (3.5.2) of ζ , to write the SDE satisfied by $\mathcal{Y}^M$. More precisely, starting from (3.4.5), we can state that $\mathcal{Y}^M$ is such that each component $\mathcal{Y}^j$, for all $j \in \{1, \dots, m\}$, satisfies:

$$\mathcal{Y}_t^j = Y_0^j + \int_0^t (c^{j,*} + \mathcal{Y}_s^j k^{j,*})(s, \zeta, \mathcal{Y}_s^M, \mathcal{Z}_s) ds + \int_0^t (Z_s^j)^\top \Sigma_M^*(s, \zeta, \mathcal{Y}_s^M, \mathcal{Z}_s) dW_s, \quad t \in [0, T]. \quad (3.5.3)$$

Note that the column vector process $\mathcal{Y}^M$ taking values in $\mathbb{R}^m$ satisfies the following multidimensional SDE:

$$d\mathcal{Y}_t^M = (c^{M,*} + \mathcal{Y}_t^M \cdot k^{M,*})(t, \zeta, \mathcal{Y}_t^M, \mathcal{Z}_t) dt + Z_t^\top \Sigma_M^*(t, \zeta, \mathcal{Y}_t^M, \mathcal{Z}_t) dW_t, \quad t \in [0, T]. \quad (3.5.4)$$

We are thus led to consider the following canonical space for the principal, $\Omega^P := \mathcal{C}_T^{hm} \times \mathcal{C}_T^m \times \mathcal{C}_T^{wd} \times \mathbb{V}^P$, where $\mathbb{V}^P$ is the collection of all finite and positive Borel measures on $[0, T] \times \mathbb{V}$, whose projection on $[0, T]$ is the Lebesgue measure, and we consider the subset $\mathbb{V}_0^P$ of all $q \in \mathbb{V}^P$ such that $q(ds, du) = \delta_{\phi_s}(du) ds$, for some Borel function ϕ . The canonical process is denoted by $(\zeta, \mathcal{Y}^M, W, \Pi^P)$, and associated canonical filtration is defined by $\mathbb{F}^P := (\mathcal{F}_t^P)_{t \in [0, T]}$ as usual. Then, for any $(t, \psi) \in [0, T] \times \mathcal{C}_b^2(\mathbb{R}^{hm} \times \mathbb{R}^m \times \mathbb{R}^{dw}, \mathbb{R})$, we set

$$\begin{aligned} M_t^P(\psi) := & \psi(\zeta_t, \mathcal{Y}_t^M, W_t) - \int_0^t \int_{\mathbb{V}} \left(\tilde{\Lambda}_P(s, \zeta, \mathcal{Y}_s^M, v) \cdot \nabla \psi(\zeta_s, \mathcal{Y}_s^M, W_s) \right. \\ & \left. + \frac{1}{2} \text{Tr} \left[\nabla^2 \psi(\zeta_s, \mathcal{Y}_s^M, W_s) (\tilde{\Sigma}_P \tilde{\Sigma}_P^\top)(s, \zeta, \mathcal{Y}_s^M, v) \right] \right) \Pi^P(ds, dv), \end{aligned}$$

where $\tilde{\Lambda}_P$ and $\tilde{\Sigma}_P$ are respectively the drift vector and the diffusion matrix of the $(hm + m + dw)$ -dimensional vector process $(\zeta, \mathcal{Y}^M, W)$, defined for all $(s, x, y) \in [0, T] \times \mathcal{C}_T^{hm} \times \mathbb{R}^m$ and $v := (z, \gamma) \in \mathbb{V}$, by:

$$\tilde{\Lambda}_P(s, x, y, v) := \begin{pmatrix} (\Sigma_M^* \Lambda_M^*)(s, x, y, v) \\ (c^{M,*} + y \cdot k^{M,*})(s, x, y, v) \\ \mathbf{0}_{wd} \end{pmatrix}, \quad \tilde{\Sigma}_P(s, x, y, v) := \begin{pmatrix} \mathbf{0}_{hm, hm} & \mathbf{0}_{hm, m} & \Sigma_M^*(s, x, v) \\ \mathbf{0}_{m, hm} & \mathbf{0}_{m, m} & z^\top \Sigma_M^*(s, x, y, v) \\ \mathbf{0}_{wd, hm} & \mathbf{0}_{wd, m} & I_{wd} \end{pmatrix},$$

recalling that Λ_M and Σ_M are defined in [Assumption 3.4.1](#).

Similarly as in [Definition 3.1.1](#) for the initial control problem of the agents, or in [Section 3.4.1](#) for the managers' problem, we fix an initial condition for $\mathcal{Y}^M$, namely $Y_0^M \in \mathbb{R}^m$, and we define the subset $\mathcal{P}^P$ of probability measures $\mathbb{P}$ on $(\Omega^P, \mathcal{F}_T^P)$ satisfying the following conditions:

- (i) $M^P(\psi)$ is a $(\mathbb{F}^P, \mathbb{P})$ -local martingale on $[0, T]$ for all $\psi \in \mathcal{C}_b^2(\mathbb{R}^{hm} \times \mathbb{R}^m \times \mathbb{R}^{dw}, \mathbb{R})$;
- (ii) $\mathbb{P}[(\zeta_0, \mathcal{Y}_0^M, W_0) = (\varrho_0, Y_0^M, w_0)] = 1$;
- (iii) $\mathbb{P}[\Pi^P \in \mathbb{V}_0^P] = 1$.

As usual, we know that for all $\mathbb{P} \in \mathcal{P}^P$, $\Pi^P(ds, du) = \delta_{\mathcal{Z}_s^P}(du)ds$ $\mathbb{P}$ -a.s. for some $\mathbb{F}^P$ -predictable control process $\mathcal{Z}^P \in \mathcal{V}$, and the representations [\(3.5.2\)](#) and [\(3.5.3\)](#) holds respectively for ζ and $\mathcal{Y}^M$, driven by the control $\mathcal{Z}^P$.

3.5.2 On solving the principal's problem

Recall that the principal's problem is initially defined by [\(3.2.10\)](#), and then simplified by [Theorem 3.4.9](#) into a standard control problem. Thanks to the previous section, we can finally rigorously write her problem in weak formulation:

$$\bar{V}^P(Y_0) = \sup_{\mathbb{P} \in \mathcal{P}^P} \mathbb{E}^{\mathbb{P}} \left[\mathcal{K}_{0,T}^P g^P(\zeta, \xi^M) \right], \text{ and thus } V_0^P = \sup_{Y_0 \geq \rho} \bar{V}^P(Y_0). \quad (3.5.5)$$

Let $(t, x, y, y^P) \in [0, T] \times \mathcal{C}_T^{hm} \times \mathbb{R}^m \times \mathbb{R}$, $\nabla := (\nabla^\zeta, \nabla^\mathcal{Y}) \in \mathbb{R}^{hm} \times \mathbb{R}^m$, and $\Delta \in \mathbb{M}^{hm+m}$ the following symmetric block matrix

$$\Delta := \begin{pmatrix} \Delta^\zeta & (\Delta^{\zeta, \mathcal{Y}})^\top \\ \Delta^{\zeta, \mathcal{Y}} & \Delta^\mathcal{Y} \end{pmatrix}, \text{ where } \Delta^\zeta \in \mathbb{M}^{hm}, \Delta^\mathcal{Y} \in \mathbb{M}^m, \Delta^{\zeta, \mathcal{Y}} \in \mathbb{M}^{m, hm}.$$

Given the dynamics of the state variables ζ and $\mathcal{Y}^M$, we can define the principal's Hamiltonian as follows:

$$\mathcal{H}^P(t, x, y, y^P, \nabla, \Delta) := \sup_{v \in \mathbb{V}} h^P(t, x, y, y^P, \nabla, \Delta, v) \quad (3.5.6)$$

where, in addition for $v := (z, \gamma) \in \mathbb{V}$,

$$\begin{aligned} h^P(t, x, y, y^P, \nabla, \Delta, v) := & -y^P k^P(t, x) + (\Sigma_M^* \Lambda_M^*)(t, x, y, v) \cdot \nabla^\zeta + (c^{M,*} + y \cdot k^{M,*})(t, x, y, v) \cdot \nabla^\mathcal{Y} \\ & + \frac{1}{2} \text{Tr}[(\Sigma_M^* (\Sigma_M^*)^\top)(t, x, y, v) \Delta^\zeta] + \frac{1}{2} \text{Tr}[z^\top (\Sigma_M^* (\Sigma_M^*)^\top)(t, x, y, v) z \Delta^\mathcal{Y}] \\ & + \text{Tr}[(\Sigma_M^* (\Sigma_M^*)^\top)(t, x, y, v) z \Delta^{\zeta, \mathcal{Y}}]. \end{aligned}$$

We are then led to consider the following HJB equation for all $(t, x, y) \in [0, T] \times \mathcal{C}_T^{hm} \times \mathbb{R}^m$,

$$-\partial_t V(t, x, y) - \mathcal{H}^P(t, x, y, V(t, x, y), \nabla V(t, x, y), \nabla^2 V(t, x, y)) = 0, \quad (3.5.7)$$

with terminal condition $V(T, x, y) = g^P(x, \bar{g}^M(x, y))$.

Given the previous HJB equation, it is clear that the principal's problem $\bar{V}^P$ boils down to a more standard control problem. Nevertheless, it should be noticed that the previous HJB equation is path-dependent, since the Hamiltonian at time $t \in [0, T]$ depends on the paths of the variable ζ up to t . Therefore, in this general case, solving the principal's problem $\bar{V}^P(Y_0)$ is equivalent to solving a path-dependent partial differential equation (path-dependent PDE for short) under appropriate conditions for the solution. We refer to the works of Ekren, Touzi et Zhang (2016) [134, 135] for more details on the resolution of this type of problems through the notion of viscosity solutions. Intuitively, the optimal control $\mathcal{Z} \in \mathcal{V}$ will correspond to the maximiser of the Hamiltonian. The final step is then to find the optimal $Y_0 \geq \rho$ in order to maximise the previously obtain value function.

If we consider a Markovian framework, in the sense that the function g^P only depends on the terminal value of ζ (*i.e.* ζ_T) and that the Hamiltonian at time $t \in [0, T]$ only depends on the current value ζ_t , then solving the principal's problem $\bar{V}^P$ boils down to solving a more standard PDE. In this case, following the line of Cvitanić, Possamaï, and Touzi [111, Theorem 3.9] (2018) we could write a verification result for the problem $\bar{V}^P$. In particular, assume that there exists a function $V : [0, T] \times \mathbb{R}^{hm} \times \mathbb{R}^m$, *smooth enough*, solution to HJB equation (3.5.7), and a function $v^* : [0, T] \times \mathbb{R}^{hm} \times \mathbb{R}^m \rightarrow \mathbb{V}$ satisfying, for all $(t, x, y) \in [0, T] \times \mathbb{R}^{hm} \times \mathbb{R}^m$,

$$\mathcal{H}^P(t, x, y, V(t, x, y), \nabla V(t, x, y), \nabla^2 V(t, x, y)) = h^P(t, x, y, V(t, x, y), \nabla V(t, x, y), \nabla^2 V(t, x, y), v^*(t, x, y)).$$

Then, intuitively and under *additional appropriate condition* on this two functions, we should obtain that $\bar{V}^P(Y_0) = V(0, \zeta_0, Y_0^M)$, and that the process $\mathcal{Z}^*$ defined for all $t \in [0, T]$ by $\mathcal{Z}_t := v^*(t, \zeta, \mathcal{Y}_t^M)$ is an optimal control for the principal. As mentioned above, the final step is to optimise on the initial value of the workers' continuation utility.

Finally, the main aspect to notice concerning the principal's problem is that, thanks to the optimal form of contracts for managers, and in particular by Theorem 3.4.9, the dimension of this problem does not explode. More precisely, if the principal supervises m managers, then her problem has $2m$ state variables, potentially multidimensional but of dimension independent of the number of managers. Indeed, on the one hand, each manager j communicates his results through a variable ζ^j , of fixed dimension h , which constitutes a state variable for the principal. On the other hand, thanks to the elegant reasoning of Sannikov [303] (2008), later developed by Cvitanić, Possamaï, and Touzi [111] (2018), considering in addition the one-dimensional continuation utility of the said manager is sufficient to solve the principal's problem. Therefore, throughout this chapter, we have shown that the method used to solve a contracting problem between one principal and one agent can be extended to a hierarchical structure and preserves the same main features, namely that the principal's problem boils down to a more classical control problem with two state variables per agent under her direct supervision.

3.6 Intuition and further comments

3.6.1 Intuition in the Markovian framework

One of the cornerstones of the approach to continuous-time moral hazard problems, pioneered by Sannikov [303] (2008), and studied in full generality by Cvitanić, Possamaï, and Touzi [111] (2018), is to obtain an appropriate probabilistic representation for incentive-compatible contracts. Intuitively, we expect that the continuation utility $Y^{j,i}$ of the (j, i) -th agent, given a contract $\xi \in \Xi^{j,i}$, in particular $\mathbb{G}^j$ -measurable, and actions of other workers subsumed by the collections of outputs X^j and $\bar{X}^{-j}$, may be written as follows:

$$Y_t^{j,i} = v^{j,i}(t, X_{\cdot \wedge t}^j, \bar{X}_{\cdot \wedge t}^{-j}),$$

that is, the process $Y^{j,i}$ at time t depends on time t and on the path history of X^j and $\bar{X}^{-j}$. Recall that the effort of other agents, $\nu^{-(j,i)}$, are fixed through the probability $\mathbb{P}^{-(j,i)}$, as well as the efforts ν^M of the managers, fixed through $\mathbb{P}^M$. We thus consider $\mathbb{P} \in \mathcal{P}^{j,i}(\mathbb{P}^{-(j,i)}, \mathbb{P}^M)$.

To intuit the form of contracts used in [Section 3.3.2](#), the focus here is on the Markovian case: we assume that the continuation utility $Y^{j,i}$ can be written at each time $t \in [0, T]$ as a function of X_t^j and $\bar{X}_t^{-j}$, and thus not on their paths up to t . In particular, if this value function is smooth enough, we can apply Itô's formula to the process $(\mathcal{K}_{0,t}^{j,i,\mathbb{P}} Y_t^{j,i})_{t \in [0,T]}$ under $\mathbb{P}$:

$$\begin{aligned} \mathcal{K}_{0,t}^{j,i,\mathbb{P}} Y_t^{j,i} - \mathcal{K}_{0,s}^{j,i,\mathbb{P}} Y_s^{j,i} &= \int_s^t \mathcal{K}_{0,r}^{j,i,\mathbb{P}} \left((\partial_t v^{j,i} - k^{j,i}(r, X_r^{j,i}, \nu_r^{j,i}) v^{j,i}) dr + \nabla_x v^{j,i} \cdot dX_r^j + \nabla_{\bar{x}} v^{j,i} \cdot d\bar{X}_r^{-j} \right. \\ &\quad \left. + \frac{1}{2} \text{Tr} \left[\nabla_x^2 v^{j,i} d\langle X^j \rangle_r + \nabla_{\bar{x}}^2 v^{j,i} d\langle \bar{X}^{-j} \rangle_r + 2 \nabla_{x\bar{x}}^2 v^{j,i} d\langle X^j, \bar{X}^{-j} \rangle_r \right] \right). \end{aligned}$$

Noticing in particular that the outputs are not correlated, we can rewrite the previous form as follows:

$$\begin{aligned} \mathcal{K}_{0,t}^{j,i,\mathbb{P}} Y_t^{j,i} - \mathcal{K}_{0,s}^{j,i,\mathbb{P}} Y_s^{j,i} &= \int_s^t \mathcal{K}_{0,r}^{j,i,\mathbb{P}} \left((\partial_t v^{j,i} - k^{j,i}(r, X_r^{j,i}, \nu_r^{j,i}) v^{j,i}) dr + \sum_{\ell=0}^{n_j} \left(\partial_{x^{j,\ell}} v^{j,i} dX_r^{j,\ell} + \frac{1}{2} \partial_{x^{j,\ell}}^2 v^{j,i} d\langle X^{j,\ell} \rangle_r \right) \right. \\ &\quad \left. + \sum_{k=0, k \neq j}^m \left(\partial_{\bar{x}^k} v^{j,i} d\bar{X}_r^k + \frac{1}{2} \partial_{\bar{x}^k}^2 v^{j,i} d\langle \bar{X}^k \rangle_r \right) \right). \end{aligned}$$

Recall that, for any $t \in [0, T]$, the output $X_t^{j,i}$ of the (j, i) -th worker is given by [\(3.1.4\)](#), which implies

$$d\bar{X}_t^k = \sum_{\ell=0}^{n_k} \Lambda^{k,\ell}(t, \nu_t^{k,\ell}) dt + \sum_{\ell=0}^{n_k} \sigma^{k,\ell}(t, \beta_t^{k,\ell}) \cdot dW_t^{k,\ell}, \text{ for all } k \in \{0, \dots, m\}.$$

Moreover, the relevant quadratic variations are given by:

$$d\langle X^{j,\ell} \rangle_t = \|\sigma^{j,\ell}(t, \beta_t^{j,\ell})\|^2 dt, \ell \in \{0, \dots, n_j\}, \text{ and } d\langle \bar{X}^k \rangle_t = \sum_{\ell=0}^{n_k} \|\sigma^{k,\ell}(t, \beta_t^{k,\ell})\|^2 dt, k \in \{0, \dots, m\},$$

which implies the following:

$$\begin{aligned} \mathcal{K}_{0,t}^{j,i,\mathbb{P}} Y_t^{j,i} - \mathcal{K}_{0,s}^{j,i,\mathbb{P}} Y_s^{j,i} &= \int_s^t \mathcal{K}_{0,r}^{j,i,\mathbb{P}} \left(\partial_t v^{j,i} + \tilde{h}^{j,i}(r, X_r^{j,i}, v^{j,i}, \nabla v^{j,i}, \nabla^2 v^{j,i}, \hat{\nu}_r, \nu_r^{j,i}) + c^{j,i}(r, X_r^{j,i}, \nu_r^{j,i}) \right) dr \\ &\quad + \int_s^t \mathcal{K}_{0,r}^{j,i,\mathbb{P}} \left(\sum_{\ell=0}^{n_j} \partial_{x^{j,\ell}} v^{j,i} \sigma^{j,\ell}(r, \beta_r^{j,\ell}) \cdot dW_r^{j,\ell} + \sum_{k=0, k \neq j}^m \partial_{\bar{x}^k} v^{j,i} \sum_{\ell=0}^{n_k} \sigma^{k,\ell}(r, \beta_r^{k,\ell}) \cdot dW_r^{k,\ell} \right), \end{aligned}$$

where for $t \in [0, T]$, $(x, y) \in \mathbb{R}^2$, $(z, \tilde{z}) \in \mathbb{R}^{n_j+1} \times \mathbb{R}^{m-1}$, $(\gamma, \tilde{\gamma}) \in \mathbb{R}^{n_j+1} \times \mathbb{R}^{m-1}$, $u \in U^{j,i}$,

$$\begin{aligned} \tilde{h}^{j,i}(t, x, y, (z, \tilde{z}), (\gamma, \tilde{\gamma}), \hat{\nu}_t, u) &:= - (c^{j,i} + y k^{j,i})(t, x, u) + z^i \Lambda^{j,i}(t, u) + \frac{1}{2} \gamma^i \|\sigma^{j,i}(t, b)\|^2 \\ &\quad + \sum_{\ell=0, \ell \neq i}^{n_j} z^\ell \Lambda^{j,\ell}(t, \nu_t^{j,\ell}) + \frac{1}{2} \sum_{\ell=0, \ell \neq i}^{n_j} \gamma^\ell \|\sigma^{j,\ell}(t, \beta_t^{j,\ell})\|^2 \\ &\quad + \tilde{z} \cdot \left(\sum_{\ell=0}^{n_k} \Lambda^{k,\ell}(t, \nu_t^{k,\ell}) \right)_{k=1, k \neq j}^m + \frac{1}{2} \tilde{\gamma} \cdot \left(\sum_{\ell=0}^{n_k} \|\sigma^{k,\ell}(t, \beta_t^{k,\ell})\|^2 \right)_{k=0, k \neq j}^m. \end{aligned}$$

recalling that $\hat{\nu} := (\nu^{-(j,i)}, \nu^M) \in \mathcal{U}^{-(j,i)} \times \mathcal{U}^M$ is fixed by $\mathbb{P}^{-(j,i)}$ and $\mathbb{P}^M$, and $\Lambda^{j,i}(t, u) := \sigma^{j,i}(t, b) \cdot \lambda^{j,i}(t, a)$.

Then, under fairly general conditions, the value of the agent's problem is given by $V_0^{j,i} = v^{j,i}(0, X_0^j, \bar{X}_0^{-j})$, where the function $v^{j,i} : [0, T] \times \mathbb{R}^{n_j+1} \times \mathbb{R}^{m-1} \rightarrow \mathbb{R}$ can be characterised as the unique viscosity solution (with appropriate growth at infinity) of the following Hamilton–Jacobi–Bellman (HJB) equation:

$$-\partial_t v^{j,i}(t, x^j, \bar{x}^{-j}) - \sup_{u^{j,i} \in U^{j,i}} \tilde{h}^{j,i}(t, X_t^{j,i}, v^{j,i}, \nabla v^{j,i}, \nabla^2 v^{j,i}, \nu_t^{-(j,i)}, \nu_t^M, u^{j,i}) = 0.$$

This implies, for $s = 0$ and $t = T$, and by taking expectation under $\mathbb{P}$,

$$V_0^{j,i}(\mathbb{P}^{j,i}, \mathbb{P}^M, \xi^{j,i}) \geq \mathbb{E}^{\mathbb{P}} \left[\mathcal{K}_{0,T}^{j,i,\mathbb{P}} g^{j,i}(X_T^{j,i}, \xi^{j,i}) - \int_0^T \mathcal{K}_{0,r}^{j,i,\mathbb{P}} c^{j,i}(r, X_r^{j,i}, \nu_r^{j,i}) dr \right].$$

In particular, the equality in the previous inequality is attained for the maximiser of the Hamiltonian.

The same reasoning allows to obtain, still in the Markovian case, the form of the continuation utility $Y^{j,i}$. In particular, by Itô's formula on $Y^{j,i}$ and using the HJB, we have:

$$\begin{aligned} Y_t^{j,i} &= Y_0^{j,i} - \int_0^t \sup_{u^{j,i} \in U^{j,i}} \tilde{h}^{j,i}(r, X_r^{j,i}, v^{j,i}, \nabla v^{j,i}, \nabla^2 v^{j,i}, \nu_r^{-(j,i)}, \nu_r^M, \nu_r^{j,i}) dr + \int_0^t \nabla_{x^j} v^{j,i} \cdot dX_r^j \\ &\quad + \int_0^t \nabla_{\bar{x}^{-j}} v^{j,i} \cdot d\bar{X}_r^{-j} + \frac{1}{2} \sum_{\ell=0}^{n_j} \int_0^t \partial_{x^j, \ell}^2 v^{j,i} d\langle X^{j,\ell} \rangle_r + \frac{1}{2} \sum_{k=0, k \neq j}^m \int_0^t \partial_{\bar{x}^k}^2 v^{j,i} d\langle \bar{X}^k \rangle_r. \end{aligned}$$

Replacing the quadratic variations and $\tilde{h}^{j,i}$ by their values, several simplifications are possible, especially between the terms related to second order derivatives, and we obtain:

$$\begin{aligned} Y_t^{j,i} &= Y_0^{j,i} - \int_0^t \left(\sup_{u^{j,i} \in U^{j,i}} h^{j,i}(r, X_r^{j,i}, v^{j,i}, \partial_{x^j, i} v^{j,i}, \partial_{x^j, i}^2 v^{j,i}, u^{j,i}) + H^{j,i}(r, (\partial_{x^j, \ell} v^{j,i})_{\ell \neq i}, \partial_{\bar{x}^k} v^{j,i}, \nu^{-(j,i)}, \nu^M) \right) dr \\ &\quad + \int_0^t \nabla_{x^j} v^{j,i} \cdot dX_r^j + \int_0^t \nabla_{\bar{x}^{-j}} v^{j,i} \cdot d\bar{X}_r^{-j} + \frac{1}{2} \int_0^t \partial_{x^j, i}^2 v^{j,i} d\langle X^{j,i} \rangle_r, \end{aligned}$$

where $h^{j,i}$ and $H^{j,i}$ are respectively defined by (3.3.2) and (3.3.3).

On the one hand, the previous reasoning explains the particular form of the (j, i) -th agent's Hamiltonian, denoted by $\mathcal{H}^{j,i}$ and defined by (3.3.4). In particular, this Hamiltonian is the supremum on the (j, i) -th agent's effort of the sum of the two previous terms $h^{j,i}$ and $H^{j,i}$. On the other hand, we also obtain that the (j, i) -th agent's continuation utility $Y^{j,i}$ should be parameterised by a triple $\mathcal{Z} := (Z, \tilde{Z}, \Gamma)$, where

$$Z := \nabla_x v^{j,i}, \quad \tilde{Z} := \nabla_{\bar{x}} v^{j,i}, \quad \text{and} \quad \Gamma := \nabla_{x^j, i}^2 v^{j,i},$$

and satisfies for $t \in [0, T]$:

$$Y_t^{j,i} = Y_0^{j,i} - \int_0^t \mathcal{H}^{j,i}(r, X_r^{j,i}, Y_r^{j,i}, \mathcal{Z}_r, \hat{\nu}^*) dr + \int_0^t Z_r \cdot dX_r^j + \int_0^t \tilde{Z}_r \cdot d\bar{X}_r^{-j} + \frac{1}{2} \int_0^t \Gamma_r d\langle X^{j,i} \rangle_r,$$

where $\hat{\nu}^* := (\nu^{-(j,i),*}, \nu^M)$ and $\mathcal{H}^{j,i}$ is defined by (3.3.4).

Therefore, the process $Y^{j,i}$ is the continuation utility of the (j, i) -th agent in the Markovian case, and the associated contract $\xi^{j,i}$ should be such that $Y_T^{j,i} = g^{j,i}(X_T^{j,i}, \xi^{j,i})$. However, this form of contract cannot be used directly in the context of a principal–agent problem with moral hazard. Indeed, this form depends explicitly on the efforts of other agents, namely $\nu^{-(j,i)}$, through the Hamiltonian $\mathcal{H}^{j,i}$, and these efforts are not supposed to be observable, nor contractible upon, for the manager. Nevertheless, we can overcome this difficulty by replacing $\nu^{-(j,i)}$ by the optimal efforts process of other agents, which has to be formally

computed as the maximiser in the Hamiltonian denoted by $\nu^{-(j,i),*}$ and defined by (3.3.5). Indeed, at equilibrium, each agent should apply their optimal efforts. Moreover, one can notice that the Hamiltonian also depends on the effort of the managers. Indeed, for now, the j -th manager offers a contract to his agents, given any efforts made by other managers, and thus communicates these efforts to his agents. We will see later, when solving the Nash equilibrium between managers, that the Hamiltonian will naturally also be computed with the optimal efforts of other managers.

3.6.2 Extending the dynamics

In this subsection, we show the limits when considering each output $X^{j,i}$, for $j \in \{1, \dots, m\}$ and $i \in \{0, \dots, n_j\}$, as a solution to the following SDE

$$dX_t^{j,i} = \sigma^{j,i}(t, X_t^{j,i}, \beta_t^{j,i}) \cdot \left[\lambda^{j,i}(t, X_t^{j,i}, \alpha_t^{j,i}) dt + dW_t^{j,i} \right], \quad t \in [0, T], \quad \mathbb{P}\text{-a.s.} \quad (3.6.1)$$

Following the reasoning developed in Section 3.3, the Hamiltonian of the (j, i) -th agent is defined by

$$\mathcal{H}^{j,i}(t, x, y, z, \tilde{z}, \gamma, \hat{\nu}) := \sup_{u \in U^{j,i}} h^{j,i}(t, x^{j,i}, y, z^i, \gamma, u) + H^{j,i}(t, x, z^{-i}, \tilde{z}, \hat{\nu}), \quad (3.6.2)$$

for any $(t, x, y, z, \tilde{z}, \gamma) \in [0, T] \times \mathcal{C}_T^w \times \mathbb{R} \times \mathbb{R}^{n_j+1} \times \mathbb{R}^{m-1} \times \mathbb{R}$ and $\hat{\nu} \in \mathbb{U}_0^{-(j,i)} \times \mathbb{U}_0^M$, where

(i) for $(t, x) \in [0, T] \times \mathcal{C}_T$, $(y, z, \gamma) \in \mathbb{R}^3$, $u := (a, b) \in U^{j,i}$,

$$h^{j,i}(t, x, y, z, \gamma, u) := -c^{j,i}(t, x, u) - k^{j,i}(t, x, u)y + \Lambda^{j,i}(t, x, u)z + \frac{1}{2} \|\sigma^{j,i}(t, x, b)\|^2 \gamma; \quad (3.6.3)$$

(ii) for $(t, x) \in [0, T] \times \mathcal{C}_T^w$, $(z, \tilde{z}) \in \mathbb{R}^{n_j} \times \mathbb{R}^{m-1}$ and $\hat{\nu} := (\nu^{-(j,i)}, \nu^M) \in \mathcal{U}^{-(j,i)} \times \mathcal{U}^M$,

$$H^{j,i}(t, x, z, \tilde{z}, \hat{\nu}) := z \cdot \left(\Lambda^{j,\ell}(t, x^{j,\ell}, \hat{\nu}_t^{j,\ell}) \right)_{\ell=0, \ell \neq i}^{n_j} + \tilde{z} \cdot \left(\sum_{\ell=0}^{n_k} \Lambda^{k,\ell}(t, x^{k,\ell}, \hat{\nu}_t^{k,\ell}) \right)_{k=1, k \neq j}^m. \quad (3.6.4)$$

First, remark that the (j, i) -th agent's Hamiltonian depends on every components of X throughout the drift and volatility functions. Limiting the study to the agent, this is not a problem since we have assumed that the agents observes X . However, plugging this Hamiltonian in the contract is not possible, since the j -th manager only observes X^j and $\bar{X}^{-j}$. Therefore, the manager cannot compute the Hamiltonian part $H^{j,i}$ of his (j, i) -th agent. This would lead to an additional assumption, similar to Assumption 3.4.1, on the shape of the dynamic of $\bar{X}^{-j}$ for all $j \in \{1, \dots, m\}$.

Nevertheless, even with this type of assumption, we are faced with a much more serious problem. The part of the Hamiltonian optimised by the agent, *i.e.*, the part given by (3.6.3), depends on the output of the agent. Thus, his optimal effort will a priori be a functional of his output. Considering only the relation between the manager and his agent, it is not an issue since both observes the output. However, moving to the problem of another team, since the contract of an agent is written with his Hamiltonian on the optimal efforts of others, it will in fact depend on the output of the agents of another team through their optimal efforts. Since these outputs are not observed by the manager, we are not allowed to write the contract in this way either.

3.6.3 On the reporting of the managers

The goal of this section is to find interesting cases where Assumption 3.4.1 holds. With this in mind, we fix $j \in \{1, \dots, m\}$ and we recall that $\zeta_t^j := f^j(t, X^j, \xi^{j,0})$, for some function f^j . Assuming that the function

f^j is smooth enough, we can apply Itô's formula to write explicitly the dynamics of ζ^j with respect to X^j and $\xi^{j,0}$:

$$d\zeta_t^j = \partial_t f^j dt + \nabla_x f^j \cdot dX_t^j + \nabla_y f^j \cdot d\xi_t^{j,0} + \frac{1}{2} \text{Tr}[\nabla_x^2 f^j d\langle X^j \rangle_t + \nabla_y^2 f^j d\langle \xi^{j,0} \rangle_t + 2\nabla_{xy}^2 f^j d\langle X^j, \xi^{j,0} \rangle_t].$$

Nevertheless, to obtain a dynamics of the form (3.4.1) for ζ , it is necessary to develop the previous equation using the dynamic of X^j and $\xi^{j,0}$ under the optimal effort of the agents.

Given a probability $\mathbb{P}^M$ and a collection ξ^A of contracts for the agents, both chosen by the managers, the Nash equilibrium between the agents is represented by the probability $\mathbb{P}^*(\mathbb{P}^M, \xi^A)$, which will be denoted by $\mathbb{P}^*$ for simplicity. Under this probability and for all $j \in \{1, \dots, m\}$ and $i \in \{0, \dots, n_j\}$, we have the following dynamics for $X^{j,i}$, for all $t \in [0, T]$, $\mathbb{P}^*$ -a.s.

$$dX_t^{j,0} = \sigma^{j,0}(t, \beta_t^{j,0}) \cdot [\lambda^{j,0}(t, \alpha_t^{j,0})dt + dW_t^{j,0}], \text{ and } dX_t^{j,i} = \sigma^{j,i}(t, \beta_t^{j,i,*}) \cdot [\lambda^{j,i}(t, \alpha_t^{j,i,*})dt + dW_t^{j,i}],$$

for $i \in \{1, \dots, n_j\}$, where

$$(\alpha_t^{j,i,*}, \beta_t^{j,i,*}) = u^{j,i,*}(t, Y_t^{j,i}, (Z_t^{j,i})^i, \Gamma_t^{j,i}), \text{ dt } \otimes \mathbb{P}^*\text{-a.s. for all } t \in [0, T],$$

while the effort $\nu^{j,0}$ for all j are fixed through $\mathbb{P}^M$.

On the other hand, for all $i \in \{1, \dots, n_j\}$, we have $\xi_t^{j,i} = \bar{g}^{j,i}(X_{\wedge t}^{j,i}, Y_t^{j,i})$, for $t \in [0, T]$, where $Y^{j,i}$ is defined by (3.3.6). Under the optimal effort of the agents, some parts of the Hamiltonian simplifies with the drift parts of the stochastic integrals, and we obtain in particular that the dynamic of $Y^{j,i}$ is given by:

$$\begin{aligned} dY_t^{j,i} &= (c^{j,i} + Y_t^{j,i} k^{j,i})(t, X_t^{j,i}, \nu_t^{j,i,*})dt + (Z_t^{j,i})^0 \sigma^{j,0}(t, \beta_t^{j,0}) \cdot dW_t^{j,0} + \sum_{\ell=1}^{n_j} (Z_t^{j,i})^\ell \sigma^{j,\ell}(t, \beta_t^{j,\ell,*}) \cdot dW_t^{j,\ell} \\ &\quad + \tilde{Z}_t^{j,i} \cdot \left(\sigma^{k,0}(t, \beta_t^{k,0}) \cdot dW_t^{k,0} + \sum_{\ell=1}^{n_k} \sigma^{k,\ell}(t, \beta_t^{k,\ell,*}) \cdot dW_t^{k,\ell} \right)_{k=1, k \neq j}^m. \end{aligned} \quad (3.6.5)$$

More precisely for the reporting, let us keep in mind that the most consistent forms are those considered in the examples detailed in Chapter 2, see in particular Sections 2.1 and 2.4. We can assume that the j -th manager reports in continuous time to the principal the sum of all the outputs of his working team including him, and the sum of compensations paid to the agents under his supervision, *i.e.*, the 2-dimensional variable

$$\zeta_t^j := \left(\sum_{i=0}^{n_j} X_t^{j,i}, \sum_{i=1}^{n_j} \xi_t^{j,i} \right), \text{ for } t \in [0, T], \quad (3.6.6)$$

as considered in the example in Section 2.3.1. Based on this form of reporting, it will be relatively simple to also consider the case where the j -th manager only reports in continuous time the net benefits of his working team, as in Sung's model developed in Section 2.1, *i.e.*, the one-dimensional variable

$$\zeta_t^j := \sum_{i=0}^{n_j} X_t^{j,i} - \sum_{i=1}^{n_j} \xi_t^{j,i}, \text{ for } t \in [0, T]. \quad (3.6.7)$$

These two potential choices of reporting are based on the results of Section 2.3.2, which shows that a more accurate reporting of agents' results leads to a degeneracy of the principal's problem into the direct contracting case.

Looking at the two reporting choices mentioned above, it is clear that if ζ has an independent dynamic in the second case, then the same is true in the first case. We thus focus on a reporting ζ given by (3.6.7). In the following, we detail two interesting cases:

- (i) the linear case, in the sense that $g^{j,i}(x, y) := g_x^{j,i}x + y$, for some $g_x^{j,i} \in \mathbb{R}$;
- (ii) the exponential case, which corresponds to $g^{j,i}(x, y) := -e^{-R^{j,i}(x+y)}$ for some $R^{j,i} > 0$, to cover the examples provided in the first two sections.

Finally, we will make some comments on other reporting.

Linear case. We assume that $g^{j,i}(x, y) := g_x^{j,i}x + y$ for all $(x, y) \in \mathbb{R}^2$ and some $g_x^{j,i} \in \mathbb{R}$. In this case, remark that $\xi_t^{j,i} = Y_t^{j,i} - g_x^{j,i}X_t^{j,i}$. Therefore, the reporting ζ^j given by (3.6.7) admits the following dynamics:

$$d\zeta_t^j = dX_t^{j,0} + \sum_{i=1}^{n_j} (1 + g_x^{j,i})dX_t^{j,i} - \sum_{i=1}^{n_j} dY_t^{j,i}, \quad \text{for } t \in [0, T].$$

There are two problems in attempting to obtain an independent dynamic:

- (i) the optimal effort of each agent is a function of his continuation utility, and there is no reason that, finally, the dynamics will only make the sum of $Y^{j,i}$ appear;
- (ii) the drift parts of the continuation utilities are particular functions of the outputs and the continuation utilities themselves, and, in the same way as for the first point, there is no apparent reason to get the sum at the end.

The easiest way to tackle the first problem is to assume that the discount rate $k^{j,i}$ is not controlled, i.e., $k^{j,i}(t, x, u) = k_x^{j,i}(t, x)$, recalling that the function $k_x^{j,i}$ is defined by Assumption 3.2.2. Under this specification, and given the form (3.3.2) of the Hamiltonian part to be maximised by the agent, we can state that the optimal efforts of the agents do not depend on their continuation utilities anymore. It remains to deal with the second problem. Since the drift part of $Y^{j,i}$ can only depends on $X^{j,i}$ and $Y^{j,i}$, by summing for $i = 1$ to n_j , there is no way we can obtain something that depends on $X^{j,0}$. It is therefore impossible to obtain a function of ζ^j in the drift. We are thus led to assume that $c^{j,i}(t, x, u)$ is independent of x and that $k^{j,i}$ is in fact equal to 0. Under these strong assumptions, it is now clear that the dynamics of ζ is of the desired form, since it does not depend on the outputs and continuation utilities anymore. One can note that if we had considered the reporting form (3.6.6) with $g_x^{j,i} = 0$, we could have let $k^{j,i}$ depends on time t . Indeed, in this case, by summing the dynamics of the continuation utilities, we obtain a drift depending only on the sum, which corresponds to the second component of ζ^j .

Lemma 3.6.1. *Consider the linear case, i.e., when $g^{j,i}(x, y) := g_x^{j,i}x + y$ for some $g_x^{j,i} \in \mathbb{R}$. If the reporting ζ is defined by (3.6.6), then Assumption 3.4.1 is satisfied if, for all $j \in \{1, \dots, m\}$, $i \in \{1, \dots, n_j\}$, $c^{j,i}(t, x, u) = c_u^{j,i}(t, u)$ and $k^{j,i}(t, x, u) = k(t)$ for $(t, x, u) \in [0, T] \times \mathbb{R} \times U^{j,i}$. Moreover, if ζ is defined by (3.6.7), then we have to assume in addition that $k(t) = 0$ for all $t \in [0, T]$.*

Exponential case. We assume that $g^{j,i}(x, y) := -e^{-R^{j,i}(x+y)}$ for $(x, y) \in \mathbb{R}^2$, where $R^{j,i}$ is a positive constant representing the risk-aversion of the (j, i) -th agent, and $c^{j,i} \equiv 0$, in order to recover the classical exponential utility case. In this case, we remark that

$$\xi_t^{j,i} = -\frac{1}{R^{j,i}} \ln(-Y_t^{j,i}) - X_t^{j,i}.$$

Therefore, by applying Itô's formula and using the dynamics of $Y^{j,i}$ given by (3.6.5), we obtain the following dynamics for $\xi^{j,i}$, for all $t \in [0, T]$:

$$\begin{aligned} d\xi_t^{j,i} = & -\frac{1}{R^{j,i}} k^{j,i}(t, X_t^{j,i}, \nu_t^{j,i,*}) dt + \frac{1}{2} R^{j,i} |(\hat{Z}_t^{j,i})^0|^2 \|\sigma^{j,0}(t, \beta_t^{j,0})\|^2 dt + \frac{1}{2} R^{j,i} \sum_{\ell=1}^{n_j} |(\hat{Z}_t^{j,i})^\ell|^2 \|\sigma^{j,\ell}(t, \beta_t^{j,\ell,*})\|^2 dt \\ & + \frac{1}{2} R^{j,i} \sum_{k=1, k \neq j}^m |(\check{Z}_t^{j,i})^k|^2 \left(\|\sigma^{k,0}(t, \beta_t^{k,0})\|^2 + \sum_{\ell=1}^{n_k} \|\sigma^{k,\ell}(t, \beta_t^{k,\ell,*})\|^2 \right) dt - g_x^{j,i} dX_t^{j,i} \\ & + (\hat{Z}_t^{j,i})^0 \sigma^{j,0}(t, \beta_t^{j,0}) \cdot dW_t^{j,0} + \sum_{\ell=1}^{n_j} (\hat{Z}_t^{j,i})^\ell \sigma^{j,\ell}(t, \beta_t^{j,\ell,*}) \cdot dW_t^{j,\ell} \\ & + \check{Z}_t^{j,i} \cdot \left(\sigma^{k,0}(t, \beta_t^{k,0}) \cdot dW_t^{k,0} + \sum_{\ell=1}^{n_k} \sigma^{k,\ell}(t, \beta_t^{k,\ell,*}) \cdot dW_t^{k,\ell} \right)_{k=1, k \neq j}^m, \end{aligned}$$

where for all $t \in [0, T]$,

$$\hat{Z}_t^{j,i} := -\frac{Z_t^{j,i}}{R^{j,i} Y_t^{j,i}}, \quad \check{Z}_t^{j,i} := -\frac{\tilde{Z}_t^{j,i}}{R^{j,i} Y_t^{j,i}}, \quad \text{and} \quad \hat{\Gamma}_t^{j,i} := -\frac{\Gamma_t^{j,i}}{R^{j,i} Y_t^{j,i}}.$$

The previous change of variable is classical when considering exponential utilities, and implies that the optimal control is in fact independent of the continuation utility. Indeed, the Hamiltonian's part to maximise, defined by (3.3.2), is now given by:

$$h^{j,i}(t, X_t^{j,i}, Y_t^{j,i}, (Z_t^{j,i})^i, \Gamma_t^{j,i}, u) = -Y_t^{j,i} \left(k^{j,i}(t, X_t^{j,i}, u) + R^{j,i} \Lambda^{j,i}(t, u) (\hat{Z}_t^{j,i})^i + \frac{1}{2} \|\sigma^{j,i}(t, b)\|^2 R^{j,i} \hat{\Gamma}_t^{j,i} \right),$$

and its maximiser $\nu_t^{j,i,*}$ is thus independent of y . Therefore, the first issue in the linear case does not arise in the exponential case. However, in order to obtain an independent dynamic for ζ , we are still led to assume that $k^{j,i}$ is in fact independent of $X^{j,i}$.

Lemma 3.6.2. *Consider the exponential case, i.e., when $g^{j,i}(x, y) := -e^{-R^{j,i}(x+y)}$ for some $R^{j,i} > 0$. If the reporting ζ is defined by (3.6.6) or (3.6.7), then Assumption 3.4.1 is satisfied if, for all $j \in \{1, \dots, m\}$, $i \in \{1, \dots, n_j\}$, $c^{j,i}(t, x, u) = 0$ and $k^{j,i}(t, x, u) = k_u^{j,i}(t, u)$ for $(t, x, u) \in [0, T] \times \mathbb{R} \times U^{j,i}$.*

Remark 3.6.3. *One may note that the assumption on $k^{j,i}$, mainly that it is bounded (see Assumption 3.2.2), is made to ensure that the agent's Hamiltonian, defined in (3.3.2), is Lipschitz in y (the continuation utility), and is not necessary if we only consider CARA utility functions as in this section.*

Other kinds of reporting. Finally, we could imagine alternative types of reporting than those mentioned above. For example, we can assume that the j -th manager reports to the principal the sum of the outputs and, separately, the sum of the discounted continuation utilities. In this case, if we arbitrary assume that the agent's optimal efforts are independent of their continuation utility, the dynamic of ζ^j is also independent of $Y^{j,0}$. Indeed, using (3.6.5), we have:

$$\begin{aligned} d(\mathcal{K}_{0,t}^{j,i,\mathbb{P}^*} Y_t^{j,i}) = & \mathcal{K}_{0,t}^{j,i,\mathbb{P}^*} \left(c^{j,i}(t, X_t^{j,i}, \nu_t^{j,i,*}) dt + (Z_t^{j,i})^0 \sigma^{j,0}(t, \beta_t^{j,0}) \cdot dW_t^{j,0} + \sum_{\ell=1}^{n_j} (Z_t^{j,i})^\ell \sigma^{j,\ell}(t, \beta_t^{j,\ell,*}) \cdot dW_t^{j,\ell} \right. \\ & \left. + \check{Z}_t^{j,i} \cdot \left(\sigma^{k,0}(t, \beta_t^{k,0}) \cdot dW_t^{k,0} + \sum_{\ell=1}^{n_k} \sigma^{k,\ell}(t, \beta_t^{k,\ell,*}) \cdot dW_t^{k,\ell} \right)_{k=1, k \neq j}^m \right), \end{aligned}$$

for all $i \in \{1, \dots, n_j\}$. Nevertheless, we still have an issue with the dependency in the output X .

To prevent this issue, we can also imagine that the j -th manager reports the following:

$$\zeta_t^j := \left(\sum_{i=0}^{n_j} X_t^{j,i}, \sum_{i=1}^{n_j} \kappa_{0,t}^{j,i,\mathbb{P}^*} Y_t^{j,i} - \int_0^t c^{j,i}(t, X_t^{j,i}, \nu_t^{j,i,*}) dt \right), \text{ for } t \in [0, T].$$

In this case, under the same assumption as below, namely that the agents' optimal efforts are independent of their continuation utility, then the dynamic of ζ^j is both independent of X^j and $Y^{j \setminus 0}$.

In short, there appear to be many cases in which [Assumption 3.4.1](#) is satisfied. Unfortunately, it seems complicated to define a general framework with weak assumptions on the form of the reporting ζ and on the characteristic functions of the agents ensuring that this hypothesis is satisfied, though the assumption itself can easily be checked on a case-by-case basis.

3.7 2BSDE representation...

The theoretical framework developed throughout this chapter strongly relies on the recent theory of 2BSDEs, which is thus presented in this section, using [Notations I](#) (p. 43).

3.7.1 ... for an agent

This section clarifies the link between the Nash for the agents and the theory of 2BSDEs, while the link between the Nash for managers and 2BSDEs is postponed to the next section (see [Section 3.7.2](#)).

3.7.1.1 Another representation for the set of measures

Recall the set of probability measures $\mathcal{P}$, specified by [Definition 3.1.1](#). The general approach to moral hazard problems by Cvitanić, Possamaï, and Touzi [111] (2018) requires to distinguish between the efforts of the agents which give rise to absolutely continuous probability measures in $\mathcal{P}$, namely the ones for which only the drift changes, or for which the volatility control changes, while keeping fixed the quadratic variation of X . The goal of this subsection is to provide the appropriate formulation in our setting.

For simplicity, recalling that $\Sigma(t, b) \in \mathbb{M}^{dw,w}$ is given by [\(3.1.2\)](#), we define the following diagonal matrix:

$$\Sigma^2(t, b) := \Sigma(t, b)^\top \Sigma(t, b) = \text{diag} \left[\left(\|\sigma^{j,i}(t, b^{j,i})\|^2 \right)_{j,i} \right], \text{ for } t \in [0, T] \text{ and } b \in B.$$

Definition 3.7.1. We define by $\bar{\mathcal{P}}$ the set of probability measures $\bar{\mathbb{P}}$ on $(\Omega, \mathcal{F}_T)$ such that

- (i) the canonical vector process $(X, W)^\top$ is an $(\mathbb{F}, \bar{\mathbb{P}})$ -local martingale for which there exists an $\mathbb{F}$ -predictable and B -valued process $\beta^{\bar{\mathbb{P}}}$ such that the $\bar{\mathbb{P}}$ -quadratic variation of $(X, W)^\top$ is equal to

$$\begin{pmatrix} \Sigma^2(t, \beta_t^{\bar{\mathbb{P}}}) & \Sigma(t, \beta_t^{\bar{\mathbb{P}}})^\top \\ \Sigma(t, \beta_t^{\bar{\mathbb{P}}}) & I_{wd} \end{pmatrix}, \quad t \in [0, T], \quad \bar{\mathbb{P}}\text{-a.s.}$$

- (ii) $\bar{\mathbb{P}}[\Pi \in \mathbb{U}_0] = 1$.

Similarly to [Lemma 3.1.2](#), we know that for all $\bar{\mathbb{P}} \in \bar{\mathcal{P}}$, we have the following representation for X :

$$X_t = x_0 + \int_0^t \Sigma(s, \beta_s^{\bar{\mathbb{P}}})^\top dW_s, \quad t \in [0, T], \quad \bar{\mathbb{P}}\text{-a.s.}$$

More precisely, for any $j \in \{1, \dots, m\}$ and $i \in \{0, \dots, n_j\}$,

$$X_t^{j,i} = x_0^{j,i} + \int_0^t \sigma^{j,i}(s, \beta_s^{\bar{\mathbb{P}}, j, i}) \cdot dW_s^{j,i}, \quad t \in [0, T], \quad \bar{\mathbb{P}} - \text{a.s.}$$

Recall that using classical results of Bichteler [58] (1981) or Neufeld and Nutz [265, Proposition 6.6] (2014), we can define a pathwise version of the $\mathbb{F}$ -predictable quadratic variation $\langle X \rangle$, allowing us to define the $w \times w$ non-negative symmetric matrix $\hat{\sigma}_t$ for all $t \in [0, T]$ such that

$$\hat{\sigma}_t^2 := \limsup_{n \rightarrow +\infty} n(\langle X \rangle_t - \langle X \rangle_{t-1/n}).$$

Since $\hat{\sigma}_t^2$ takes values in $\mathbb{S}^w$, we can naturally define its square root $\hat{\sigma}_t$. In particular, we will denote for all $j \in \{1, \dots, m\}$ and $i \in \{0, \dots, n_j\}$, the process $S^{j,i}$, taking values in $\mathbb{R}$, defined as follows:

$$S_t^{j,i} := \limsup_{n \rightarrow +\infty} n(\langle X^{j,i} \rangle_t - \langle X^{j,i} \rangle_{t-1/n}), \quad \text{for all } t \in [0, T]. \quad (3.7.1)$$

Definition 3.7.2. Let $\bar{\mathbb{P}} \in \bar{\mathcal{P}}$ and consider the process $\beta^{\bar{\mathbb{P}}}$ associated to $\bar{\mathbb{P}}$ in the sense of Definition 3.7.1 (i). For any $A \times B$ -valued and $\mathbb{F}$ -predictable processes $\nu := (\alpha, \beta)$ ⁸ such that, for all $t \in [0, T]$, $\Sigma^2(t, \beta) = \Sigma^2(t, \beta^{\bar{\mathbb{P}}})$ $\bar{\mathbb{P}}$ -a.s., we define the equivalent measures $\bar{\mathbb{P}}^\nu$ by their Radon-Nikodym density on $\mathcal{F}_T$,

$$\frac{d\bar{\mathbb{P}}^\nu}{d\bar{\mathbb{P}}} := \exp \left(\int_0^T \lambda(s, \alpha_s) \cdot dW_s - \frac{1}{2} \int_0^T \|\lambda(s, \alpha_s)\|^2 ds \right),$$

where λ is defined as the column vector composed of all the functions $\lambda^{j,i} : [0, T] \times A \mapsto \mathbb{R}^d$, implying that λ takes values in $\mathbb{R}^{dw}$.

Notice that such a measure is well-defined since, for all $j \in \{1, \dots, m\}$ and $i \in \{0, \dots, n_j\}$, $\lambda^{j,i}$ is bounded. It is then immediate to check that the set $\mathcal{P}$ coincides exactly with the set of all probability measures of the form $\bar{\mathbb{P}}^\nu$, which satisfy in addition that there exists $w_0 \in \mathbb{R}^{dw}$ such that $\bar{\mathbb{P}}^\nu \circ (X_0, W_0)^{-1} = \delta_{(x_0, w_0)}$. For any $\bar{\mathbb{P}} \in \bar{\mathcal{P}}$, we denote by $\bar{\mathcal{U}}(\bar{\mathbb{P}})$ the set of controls $\nu \in \mathcal{U}$ such that $\bar{\mathbb{P}}^\nu \in \mathcal{P}$.

Following the reasoning developed in Section 3.2, it is necessary to characterise the space and the actions of other workers to properly define the admissible response of a considered agent. In particular, this leads to the definitions of $\mathcal{P}^{-(j,i)}$ and $\mathcal{P}^M$ in Section 3.2.1, in addition to the definition of $\mathcal{P}$ on the whole canonical space in Section 3.1.3. We are therefore led to consider the sets $\bar{\mathcal{P}}^{-(j,i)}$ and $\bar{\mathcal{P}}^M$ corresponding respectively to $\mathcal{P}^{-(j,i)}$ and $\mathcal{P}^M$, in the same way that we just constructed $\bar{\mathcal{P}}$ corresponding to $\mathcal{P}$ in Definition 3.7.2. Similarly to Definition 3.2.1, we can then define the set $\bar{\mathcal{P}}^{j,i}(\bar{\mathbb{P}}^{-(j,i)}, \bar{\mathbb{P}}^M)$ of admissible responses $\bar{\mathbb{P}} \in \bar{\mathcal{P}}$ of the (j, i) -th agent to some probabilities $\bar{\mathbb{P}}^{-(j,i)} \in \bar{\mathcal{P}}^{-(j,i)}$ and $\bar{\mathbb{P}}^M \in \bar{\mathcal{P}}^M$ respectively chosen by the other agents and the managers.

3.7.1.2 Semilinear Hamiltonian

In the following, in order to focus on the (j, i) -th agent, let us consider $j \in \{1, \dots, m\}$ and $i \in \{1, \dots, n_j\}$. We also fix the probabilities $\bar{\mathbb{P}}^{-(j,i)} \in \bar{\mathcal{P}}^{-(j,i)}$ and $\bar{\mathbb{P}}^M \in \bar{\mathcal{P}}^M$, and consider the associated efforts $\hat{\nu} := (\nu^{-(j,i)}, \nu^M) \in \mathcal{U}^{-(j,i)} \times \mathcal{U}^M$ of the other workers. In order to lighten the notations, we will consider $\mathfrak{P} := \bar{\mathcal{P}}^{j,i}(\bar{\mathbb{P}}^{-(j,i)}, \bar{\mathbb{P}}^M)$.

For any $t \in [0, T]$, we denote by $\mathcal{S}_t^{j,i}$ the image of $B^{j,i}$ by the map $b \in B^{j,i} \mapsto \|\sigma^{j,i}(t, b)\|^2 \in \mathbb{R}_+$, i.e., $\mathcal{S}_t^{j,i} := \{\|\sigma^{j,i}(t, b)\|^2, \text{ for } b \in B^{j,i}\}$. Conversely, for any $S \in \mathcal{S}_t^{j,i}$, we define $\tilde{U}_t^{j,i}(S) := \{(a, b) \in A^{j,i} \times$

8. Strictly speaking, the process β should be indexed by the measure $\bar{\mathbb{P}}$, but we chose to not do so in order to alleviate notations.

$B^{j,i}$, s.t. $\|\sigma^{j,i}(t, b)\|^2 = S\}$. Thanks to these notations, we can isolate the partial maximisation with respect to the squared diffusion in the Hamiltonian of the (j, i) -th agent. Indeed, we can define a map $F^{j,i} : [0, T] \times \mathcal{C}_T \times \mathbb{R} \times \mathbb{R}^{n_j+1} \times \mathbb{R}^{m-1} \times \mathcal{U}^{-(j,i)} \times \mathcal{U}^M \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ as follows:

$$F^{j,i}(t, x, y, z, \tilde{z}, \hat{v}, S) := \sup_{u \in \tilde{\mathcal{U}}_t^{j,i}(S)} \tilde{h}^{j,i}(t, x, y, z^i, u) + H^{j,i}(t, z^{-i}, \tilde{z}, \hat{v}), \quad (3.7.2)$$

for all $(t, x, y, z, \tilde{z}, \hat{v}, S) \in [0, T] \times \mathcal{C}_T \times \mathbb{R} \times \mathbb{R}^{n_j+1} \times \mathbb{R}^{m-1} \times \mathcal{U}^{-(j,i)} \times \mathcal{U}^M \times \mathbb{R}_+$, where in addition for $z^i \in \mathbb{R}$ and $u \in \mathcal{U}^{j,i}$, $\tilde{h}^{j,i}(t, x, y, z^i, u) := -c^{j,i}(t, x, u) - k^{j,i}(t, x, u)y + \sigma^{j,i}(t, b) \cdot \lambda^{j,i}(t, a)z^i$. We thus obtain that the Hamiltonian $\mathcal{H}^{j,i}$ of the (j, i) -th agent, defined by (3.3.4), satisfies:

$$\mathcal{H}^{j,i}(t, x, y, z, \tilde{z}, \gamma, \hat{v}) = \sup_{S \in \mathcal{S}_t^{j,i}} \left\{ F^{j,i}(t, x, y, z, \tilde{z}, \hat{v}, S) + \frac{1}{2} S \gamma \right\},$$

for all $(t, x, y, z, \tilde{z}, \gamma, \hat{v}) \in [0, T] \times \mathcal{C}_T \times \mathbb{R} \times \mathbb{R}^{n_j+1} \times \mathbb{R}^{m-1} \times \mathbb{R} \times \mathcal{U}^{-(j,i)} \times \mathcal{U}^M$.

3.7.1.3 Best-reaction functions of an agent

In this subsection, for a given admissible contract $\xi^{j,i} \in \Xi^{j,i}$, in the sense of Definition 3.2.5, and a pair of probability measures $(\bar{\mathbb{P}}^{-(j,i)}, \bar{\mathbb{P}}^M)$ chosen by the other workers, we wish to relate the best-reaction function $V_0^{j,i}(\bar{\mathbb{P}}^{-(j,i)}, \bar{\mathbb{P}}^M, \xi)$ of the (j, i) -th agent to an appropriate 2BSDE.

Recalling that $S^{k,\ell}$ is well defined for all $k \in \{1, \dots, m\}$ and all $\ell \in \{0, \dots, n_k\}$ by (3.7.1), we denote by $\tilde{\Sigma}^j$ the following process:

$$\tilde{\Sigma}_t^j := \text{diag} \left[(S_t^{j,\ell})_{\ell \in \{0, \dots, n_j\}} \right] \oplus \text{diag} \left[\left(\sum_{\ell=1}^{n_k} S_t^{k,\ell} \right)_{k \in \{1, \dots, m\} \setminus \{j\}} \right], \quad t \in [0, T],$$

taking values in the set of diagonal positive $(n_j + m - 1)$ -dimensional matrices. This process $\tilde{\Sigma}^j$ represents the (pathwise) quadratic variation of the vector $(X^j, \bar{X}^{-j})$, which corresponds to the state variables of any agents under the supervision of the j -th manager.

Given an admissible contract $\xi^{j,i} \in \Xi^{j,i}$, we are led to consider the following 2BSDE, indexed by (j, i) :

$$Y_t = g^{j,i}(X^{j,i}, \xi^{j,i}) + \int_t^T F^{j,i}(s, X^{j,i}, Y_s, Z_s, \tilde{Z}_s, \hat{v}_s, S_s^{j,i}) ds - \int_t^T Z_s \cdot dX_s^j - \int_t^T \tilde{Z}_s \cdot d\bar{X}_s^{-j} + \int_t^T dK_s, \quad \text{2BSDE } (j, i)$$

where $F^{j,i}$ is defined by (3.7.2). Recall that we consider here $\mathfrak{P} := \bar{\mathcal{P}}^{j,i}(\bar{\mathbb{P}}^{-(j,i)}, \bar{\mathbb{P}}^M)$, and that $\mathbb{G}^j$ is the natural filtration generated by X^j and $\bar{X}^{-j}$. The following definition adapts the classic notion of 2BSDE to our framework, using Notations I (p. 43).

Definition 3.7.3. *We say that $(Y, (Z, \tilde{Z}), K)$ is a solution to 2BSDE (j, i) if 2BSDE (j, i) holds $\mathfrak{P}$ -q.s., and if for some $k > 1$, $Y \in \mathbb{D}^k(\mathbb{G}^j, \mathfrak{P})$, $(Z, \tilde{Z}) \in \mathbb{H}_{n_j+m-1}^k(\mathbb{G}^j, \mathfrak{P}, \tilde{\Sigma}^j)$, $K \in \mathbb{I}^k(\mathbb{G}^j, \mathfrak{P})$, where K satisfies in addition the following minimality condition*

$$0 = \text{ess inf}_{\bar{\mathbb{P}} \in \bar{\mathcal{P}}(t, \mathbb{P}, \mathbb{G}^j)}^{\bar{\mathbb{P}}} \mathbb{E}^{\bar{\mathbb{P}}} \left[K_T - K_t \middle| (\mathcal{G}_t^j)^{\bar{\mathbb{P}}} \right], \quad 0 \leq t \leq T, \quad \bar{\mathbb{P}}\text{-a.s. for all } \bar{\mathbb{P}} \in \bar{\mathfrak{P}},$$

recalling that $(\mathbb{G}^j)^{\bar{\mathbb{P}}+}$ is the right limit of the completion of $\mathbb{G}^j$ under $\bar{\mathbb{P}}$ and that $\bar{\mathcal{P}}(t, \bar{\mathbb{P}}, \mathbb{G}^j)$ is defined in Notations I (p. 43).

The following result relates the solution to the above **2BSDE** (j, i) to the best-reaction function of the (j, i) -th agent.

Proposition 3.7.4. *Fix $(\bar{\mathbb{P}}^{-(j,i)}, \bar{\mathbb{P}}^M) \in \bar{\mathcal{P}}^{-(j,i)} \times \bar{\mathcal{P}}^M$, as well as $\xi^{j,i} \in \Xi^{j,i}$. Let $(Y, (Z, \tilde{Z}), K)$ be a solution to **2BSDE** (j, i) . We have*

$$V_0^{j,i}(\bar{\mathbb{P}}^{-(j,i)}, \bar{\mathbb{P}}^M, \xi^{j,i}) = \sup_{\bar{\mathbb{P}} \in \bar{\mathcal{P}}^{j,i}(\bar{\mathbb{P}}^{-(j,i)}, \bar{\mathbb{P}}^M)} \mathbb{E}^{\bar{\mathbb{P}}}[Y_0].$$

Conversely, the (dynamic) value function $V_t^{j,i}(\bar{\mathbb{P}}^{-(j,i)}, \bar{\mathbb{P}}^M, \xi^{j,i})$ always provides the first component "Y" of a solution to **2BSDE** (j, i) . Moreover, any optimal effort $\nu^{j,i,*}$, and the optimal measure $\bar{\mathbb{P}}$ must be such that

$$K = 0, \bar{\mathbb{P}}^\nu\text{-a.s.}, \text{ and } \nu^{j,i,*} \in \arg \max_{u \in \tilde{U}_t^{j,i}(S_t^{j,i})} \tilde{h}^{j,i}(t, X_t^{j,i}, Y_t, Z_t^i, u), \bar{\mathbb{P}}^\nu\text{-a.s.}, \quad (3.7.3)$$

where $\bar{\mathbb{P}}^\nu$ is defined from $\bar{\mathbb{P}}$ and ν by **Definition 3.7.2**, where ν results from the collection of all workers' efforts, and is thus composed by the optimal effort $\nu^{j,i,*}$ of the (j, i) -th agent, and by the efforts of other workers $(\nu^{-(j,i)}, \nu^M)$ fixed through the pair of probability measures $(\bar{\mathbb{P}}^{-(j,i)}, \bar{\mathbb{P}}^M)$.

Proof. The proof is classical and follows the lines of Cvitanić, Possamaï, and Touzi [111, Proof of Propositions 4.5 and 4.6] (2018). We thus only mention here why the assumptions required to apply the results of Possamaï, Tan, and Zhou [284] (2018) are satisfied in our framework.

First of all, recall that $k^{j,i}$, $\sigma^{j,i}$, and $\lambda^{j,i}$ are bounded for all $j \in \{1, \dots, m\}$ and $i \in \{1, \dots, n_j\}$. As in [111, Proof of Proposition 4.5], it follows from the definition of admissible controls that $F^{j,i}$ satisfies the Lipschitz continuity assumptions required in [284, Assumption 2.1 (i)]. Indeed, for all $t \in [0, T]$, in the one hand we have, for $(x, y, z) \in \mathcal{C}_T \times \mathbb{R} \times \mathbb{R}$, $S^{j,i} \in \mathcal{S}_t^{j,i}$, and $(y', z') \in \mathbb{R}^2$,

$$\left| \sup_{u \in \tilde{U}_t^{j,i}(S^{j,i})} \tilde{h}^{j,i}(t, x, y, z, u) - \sup_{u \in \tilde{U}_t^{j,i}(S^{j,i})} \tilde{h}^{j,i}(t, x, y', z', u) \right| \leq |k^{j,i}|_\infty |y - y'| + |\lambda^{j,i}|_\infty |S^{j,i}|^{1/2} |z - z'|.$$

On the other hand, for $(z, \tilde{z}) \in \mathbb{R}^{n_j+1} \times \mathbb{R}^m$, $(z', \tilde{z}') \in \mathbb{R}^{n_j+1} \times \mathbb{R}^m$ and $\hat{\nu} \in \mathcal{U}^{-(j,i)} \times \mathcal{U}^M$, we have

$$\begin{aligned} |H^{j,i}(t, z, \tilde{z}, \hat{\nu}) - H^{j,i}(t, z', \tilde{z}', \hat{\nu})| &\leq \sum_{\ell=0, \ell \neq i}^{n_j} |\lambda^{j,\ell}|_\infty |S^{j,\ell}|^{1/2} |z^\ell - (z')^\ell| \\ &\quad + \sum_{k=1, k \neq j}^m \left| \sum_{\ell=0}^{n_k} \lambda^{k,\ell} \right|_\infty \left| \sum_{\ell=0}^{n_k} S^{k,\ell} \right|^{1/2} |\tilde{z}^k - (\tilde{z}')^k|. \end{aligned}$$

Combining the two inequalities, we obtain that $F^{j,i}$ is Lipschitz in y and in $(\tilde{\Sigma}^j)^{1/2}(z, \tilde{z})$, as requested in [284, Assumption 2.1 (i)].

Moreover, by **Definition 3.2.5** of the set of admissible contracts $\Xi^{j,i}$, the terminal condition $g^{j,i}(X^{j,i}, \xi)$ satisfies (I_A^P) . Using in addition the integrability condition (3.2.3) for $c^{j,i}$, it then follows that the terminal condition $g^{j,i}(X^{j,i}, \xi)$ and $F^{j,i}$ satisfy the integrability properties in [284, Assumption 1.1 (ii)]. In addition, [284, Assumption 3.1] is also satisfied thanks to the integrability condition (3.2.3) for $c^{j,i}$, as explained in [111, Proof of Proposition 4.5]. Next, [284, Assumption 1.1 (iii)-(v)] are also satisfied by the set of measures $\mathfrak{P}$, see for instance [268]. Finally, the set $\mathfrak{P}$ is saturated in the sense of [284, Definition 5.1], see [284, Remark 5.1]. $\square$

3.7.1.4 Characterisation of the Nash equilibrium between agents

With [Proposition 3.7.4](#) in hand, we can now characterise a Nash equilibria between the agents, thanks to a collection of decoupled 2BSDEs, reminiscent of the multidimensional BSDE obtained in the setting of [Élie and Possamaï \[140\] \(2019\)](#) where only the drift of the canonical process was controlled.

Theorem 3.7.5. *Let $\bar{\mathbb{P}}^M \in \bar{\mathcal{P}}^M$, as well as a collection $\xi^A \in \Xi^A$ of contracts for the agents. A probability measure $\mathbb{P}^*$ belongs to $\mathcal{P}^{A,*}(\bar{\mathbb{P}}^M, \xi^A)$, in the sense of [Definition 3.2.4](#), if and only if $\mathbb{P}^* = \bar{\mathbb{P}}^{\nu^*}$ where $\nu^* := (\alpha^{\mathbb{P}^*}, \beta^{\mathbb{P}^*})$ is such that for any $j \in \{1, \dots, m\}$ and any $i \in \{1, \dots, n_j\}$,*

$$K^{j,i} = 0, \bar{\mathbb{P}}^{\nu^*}\text{-a.s.}, \text{ and } \nu^{j,i,*} = \arg \max_{u \in \tilde{U}_t^{j,i}(S_t^{j,i})} \tilde{h}^{j,i}(t, X_t^{j,i}, Y_t^{j,i}, (Z_t^{j,i})^i, u), \bar{\mathbb{P}}^{\nu^*}\text{-a.s.},$$

where $(Y^{j,i}, (Z^{j,i}, \tilde{Z}^{j,i}), K^{j,i})$ is a solution to [2BSDE \$\(j, i\)\$](#) , in the sense of [Definition 3.7.3](#).

Proof. As in the statement of the theorem, we fix $\bar{\mathbb{P}}^M \in \bar{\mathcal{P}}^M$, as well as a collection $\xi^A \in \Xi^A$ of contracts for the agents. Recall that by definition of the set Ξ^A , the collection of contracts ξ^A leads to a unique Nash equilibrium between the agents (see [Definition 3.2.9](#)). By [Proposition 3.7.4](#), we have a characterisation of the best-reaction function of the (j, i) -th agent to an arbitrary pair of probability measures $(\bar{\mathbb{P}}^{-(j,i)}, \bar{\mathbb{P}}^M)$ chosen by the managers and the other agents. A Nash equilibrium $\mathbb{P}^*$ then necessitates only that for each $j \in \{1, \dots, m\}$ and $i \in \{1, \dots, n_j\}$, $\mathbb{P}^*$ is the best-reaction function of the (j, i) -th agent to $(\bar{\mathbb{P}}^{-(j,i)}, \bar{\mathbb{P}}^M)$, that are respectively defined as the restrictions of $\mathbb{P}^*$ to $\Omega^{-(j,i)}$ and Ω^M . In other words, the probability $\mathbb{P}^*$ and the associated effort $\nu^* = (\nu^{j,i,*})_{j,i}$ have to satisfy [\(3.7.3\)](#) for all $j \in \{1, \dots, m\}$ and $i \in \{1, \dots, n_j\}$, which is exactly what is written in the statement of the theorem. $\square$

3.7.2 ... for a manager

This section provides the slight adaptation of the 2BSDE theory needed to study and solve the managers problem.

3.7.2.1 Another representation for the set of measures

Following the reasoning developed in [Section 3.7.1.1](#) for the agents' problem, we need to distinguish between the efforts of the managers which give rise to absolutely continuous probability measures, namely the ones for which only the drift changes, or for which the volatility control changes, while keeping fixed the quadratic variation of ζ .

Definition 3.7.6. *We define by $\bar{\mathcal{P}}^M$ the set of probability measures $\bar{\mathbb{P}}$ on $(\Omega^M, \mathcal{F}_T^M)$ such that*

- (i) *the canonical vector process $(\zeta, W)^\top$ is an $(\mathbb{F}^M, \bar{\mathbb{P}})$ -local martingale for which there exists an $\mathbb{F}^M$ -predictable and $\mathfrak{X}$ -valued process $\chi^\mathbb{P}$ such that the $\bar{\mathbb{P}}$ -quadratic variation of $(\zeta, W)^\top$ is equal to*

$$\begin{pmatrix} \Sigma_M \Sigma_M^\top(t, \zeta, \chi_t^\mathbb{P}) & \Sigma_M(t, \zeta, \chi_t^\mathbb{P}) \\ \Sigma_M^\top(t, \zeta, \chi_t^\mathbb{P}) & I_{wd} \end{pmatrix}, t \in [0, T], \bar{\mathbb{P}}\text{-a.s.},$$

recalling that Σ_M is defined in [Assumption 3.4.1](#);

- (ii) $\bar{\mathbb{P}}[\Pi \in \mathbb{X}_0] = 1$.

We thus know that for all $\bar{\mathbb{P}} \in \bar{\mathcal{P}}^M$, we have the following representation for ζ :

$$\zeta_t = \zeta_0 + \int_0^t \Sigma_M(s, \zeta, \chi_s^\mathbb{P}) dW_s, t \in [0, T], \bar{\mathbb{P}}\text{-a.s.}$$

We can also define a pathwise version of the $\mathbb{F}^M$ -predictable quadratic variation $\langle \zeta \rangle$, allowing us to define the process $\hat{\Sigma}$, taking values in the set of hm -dimensional non-negative symmetric matrices, by

$$\hat{\Sigma}_t := \limsup_{n \rightarrow +\infty} n(\langle \zeta \rangle_t - \langle \zeta \rangle_{t-1/n}), \quad t \in [0, T]. \quad (3.7.4)$$

Since $\hat{\Sigma}$ takes values in $\mathbb{S}^{hm}$, we can naturally define its square root $\hat{\Sigma}_t^{1/2}$.

Definition 3.7.7. Let $\bar{\mathbb{P}} \in \bar{\mathcal{P}}^M$ and consider the process $\chi^{\bar{\mathbb{P}}}$ associated to $\bar{\mathbb{P}}$ in the sense of Definition 3.7.6 (i). For any $\mathfrak{X}$ -valued and $\mathbb{F}^M$ -predictable processes χ such that

$$\Sigma_M \Sigma_M^\top(t, \zeta, \chi_t) = \Sigma_M \Sigma_M^\top(t, \zeta, \chi_t^{\bar{\mathbb{P}}}), \quad t \in [0, T], \quad \bar{\mathbb{P}} - \text{a.s.},$$

we define the equivalent measures $\bar{\mathbb{P}}^\chi$ by their Radon-Nikodym density on $\mathcal{F}_T^M$,

$$\frac{d\bar{\mathbb{P}}^\chi}{d\bar{\mathbb{P}}} := \exp \left(\int_0^T \Lambda_M(s, \zeta, \chi_s) \cdot dW_s - \frac{1}{2} \int_0^T \|\Lambda_M(s, \zeta, \chi_s)\|^2 ds \right).$$

Notice that such a measure is well-defined since Λ_M is bounded. It is then immediate to check that the set $\mathcal{P}^M$ coincides exactly with the set of all probability measures of the form $\bar{\mathbb{P}}^\chi$, which satisfy in addition that there exists $\iota \in \mathbb{R}^{wd}$ such that $\bar{\mathbb{P}}^\chi \circ (\zeta_0, W_0)^{-1} = \delta_{(\varrho, \iota)}$. For any $\bar{\mathbb{P}} \in \bar{\mathcal{P}}^M$, we denote by $\bar{\mathcal{X}}(\bar{\mathbb{P}})$ the set of controls $\chi \in \mathcal{X}$ such that $\bar{\mathbb{P}}^\chi \in \mathcal{P}^M$.

Following the reasoning developed in Section 3.4, it is necessary to characterise the space and the actions of a considered manager in response of other managers' choices. In particular, this leads to the definition of $\mathcal{P}^j(\chi^{-j})$ in Section 3.4.2, in addition to the definition of $\mathcal{P}$ on the whole canonical space in Section 3.4.1, when actions of other managers are fixed through $\chi^{-j} \in \mathcal{X}^{-j}$. We are therefore led to consider the set $\bar{\mathcal{P}}^j(\chi^{-j})$ corresponding to $\mathcal{P}^j(\chi^{-j})$, in the same way that we just constructed $\bar{\mathcal{P}}^M$ corresponding to $\mathcal{P}^M$ in Definition 3.7.7.

3.7.2.2 Best-reaction function of a manager

In this subsection, for a given admissible contract $\xi^j \in \Xi^{j,0}$, in the sense of Definition 3.2.9, and the choices of other managers, namely χ^{-j} , we wish to relate the best-reaction function $V_0^{j,0}(\xi^{j,0}, \chi^{-j})$ of the j -th manager, defined by (3.4.2), to an appropriate 2BSDE. With this in mind, we fix throughout the following $j \in \{1, \dots, m\}$ in order to focus on the j -th manager, as well as the effort of other managers summarised by $\chi^{-j} \in \mathcal{X}^{-j}$. For simplicity, we will denote $\mathfrak{P} := \bar{\mathcal{P}}^j(\chi^{-j})$.

First, we should adapt the notations defined in Section 3.7.1.2 for the agents' problem to the managers' problem, by defining for any $(t, x) \in [0, T] \times \mathcal{C}_T^{hm}$,

$$\begin{aligned} \mathcal{S}_t^j(x, \chi^{-j}) &:= \left\{ \Sigma_M \Sigma_M^\top(t, x, u \otimes_j \chi_t^{-j}) \in \mathbb{S}_+^{hm}, \text{ for } u \in \mathfrak{X}^j \right\}, \\ \text{and } \tilde{\mathfrak{X}}_t^j(x, \chi^{-j}, S) &:= \{u \in \mathfrak{X}^j, \text{ s.t. } \Sigma_M \Sigma_M^\top(t, x, u \otimes_j \chi_t^{-j}) = S\}, \text{ for } S \in \mathcal{S}_t^j(x, \chi^{-j}). \end{aligned}$$

Thanks to these notations, we can isolate the partial maximisation with respect to the squared diffusion in the Hamiltonian of the j -th manager. Indeed, we can define a map F^j as follows:

$$F^j(t, x, y, z, \chi^{-j}, S) := \sup_{u \in \tilde{\mathfrak{X}}_t^j(x, \chi^{-j}, S)} \tilde{h}^j(t, x, y, z, \chi^{-j}, u), \quad (3.7.5)$$

for all $(t, x, y, z, \chi^{-j}, S) \in [0, T] \times \mathcal{C}_T^{hm} \times \mathbb{R} \times \mathbb{R}^{hm} \times \mathcal{X}^{-j} \times \mathbb{S}_+^{hm}$ and where, in addition,

$$\tilde{h}^j(t, x, y, z, \chi^{-j}, u) := -(\mathfrak{c}^{j,0} + yk^{j,0})(t, x^j, u) + (\Sigma_M \Lambda_M)(t, x, u \otimes_j \chi_t^{-j}) \cdot z, \text{ for } u \in \tilde{\mathfrak{X}}_t^j(x, \chi^{-j}, S).$$

We thus obtain that the Hamiltonian $\mathcal{H}^j$ of the j -th manager, defined by (3.4.3), satisfies:

$$\mathcal{H}^j(t, x, y, z, \gamma, \chi^{-j}) = \sup_{S \in \mathcal{S}_t^j(x, \chi^{-j})} \left\{ F^j(t, x, y, z, \chi^{-j}, S) + \frac{1}{2} \text{Tr}[S\gamma] \right\},$$

for all $(t, x, y, z, \gamma, \chi^{-j}) \in [0, T] \times \mathcal{C}_T^{hm} \times \mathbb{R} \times \mathbb{R}^{hm} \times \mathbb{M}^{hm} \times \mathcal{X}^{-j}$.

Given an admissible contract $\xi^j \in \Xi^{j,0}$, we are led to consider the following 2BSDE, indexed by j :

$$\mathcal{Y}_t = g^{j,0}(\zeta^j, \xi^j) + \int_t^T F^j(s, \zeta, \mathcal{Y}_s, Z_s, \chi^{-j}, \hat{\Sigma}_s) ds - \int_t^T Z_s \cdot d\zeta_s + \int_t^T dK_s, \quad \text{2BSDE } j$$

where F^j is defined by (3.7.5). Recall that we consider here $\mathfrak{P} := \bar{\mathcal{P}}^j(\chi^{-j})$, and that $\mathbb{G}$ is the natural filtration generated by ζ . The following definition, which echoes Definition 3.7.3, adapts the classic notion of 2BSDE to our framework, using Notations I (p. 43).

Definition 3.7.8. *We say that $(\mathcal{Y}, Z, K)$ is a solution to 2BSDE j if 2BSDE j holds $\mathfrak{P}$ -q.s., and if for some $k > 1$, $\mathcal{Y} \in \mathbb{D}^k(\mathbb{G}, \mathfrak{P})$, $Z \in \mathbb{H}_{hm}^k(\mathbb{G}, \mathfrak{P}, \hat{\Sigma})$, $K \in \mathbb{I}^k(\mathbb{G}, \mathfrak{P})$, where K satisfies in addition the following minimality condition*

$$0 = \underset{\bar{\mathbb{P}} \in \bar{\mathcal{P}}(t, \mathbb{P}, \mathbb{G})}{\text{ess inf}} \mathbb{E}^{\bar{\mathbb{P}}} \left[K_T - K_t \middle| \mathcal{G}_t^{\mathbb{P}^+} \right], \quad 0 \leq t \leq T, \quad \bar{\mathbb{P}} - \text{a.s. for all } \bar{\mathbb{P}} \in \mathfrak{P},$$

recalling that $\mathbb{G}^{\mathbb{P}^+}$ is the right limit of the completion of $\mathbb{G}$ under $\bar{\mathbb{P}}$ and that $\bar{\mathcal{P}}(t, \bar{\mathbb{P}}, \mathbb{G})$ is defined in Notations I (p. 43).

The following result relates the solution to the above 2BSDE to the best-reaction function of the j -th manager.

Proposition 3.7.9. *Fix $\chi^{-j} \in \mathcal{X}^{-j}$ and $\xi^j \in \Xi^{j,0}$. Let $(\mathcal{Y}, Z, K)$ be a solution to 2BSDE j . We have*

$$V_0^{j,0}(\xi^j, \chi^{-j}) = \sup_{\bar{\mathbb{P}} \in \bar{\mathcal{P}}^j(\chi^{-j})} \mathbb{E}^{\bar{\mathbb{P}}}[\mathcal{Y}_0].$$

Conversely, the (dynamic) value function $V_t^{j,0}(\xi^j, \chi^{-j})$ always provides the first component " $\mathcal{Y}$ " of a solution to 2BSDE j . Moreover, any optimal effort $\tilde{\chi}^{j,*}$, and the optimal measure $\bar{\mathbb{P}}$ must be such that

$$K = 0, \quad \bar{\mathbb{P}}^\chi\text{-a.s.}, \quad \text{and} \quad \tilde{\chi}_t^{j,*} \in \arg \max_{u \in \mathfrak{X}_t^j(\zeta, \chi^{-j}, \hat{\Sigma}_t)} \tilde{h}^j(t, \zeta, \mathcal{Y}_t, Z_t, \chi_t^{-j}, u), \quad t \in [0, T], \quad \bar{\mathbb{P}}^\chi\text{-a.s.}, \quad (3.7.6)$$

where $\bar{\mathbb{P}}^\chi$ is defined from $\bar{\mathbb{P}}$ and χ by Definition 3.7.7, where χ results from the collection of all managers' efforts, and is thus composed by the optimal effort $\tilde{\chi}^{j,*}$ of the j -th manager, and by the arbitrary efforts of other workers $\chi^{-j} \in \mathcal{X}^{-j}$. More precisely, $\chi := \tilde{\chi}^{j,*} \otimes_j \chi^{-j}$.

Proof. As in the proof of Proposition 3.7.4, it suffices to mention why the assumptions required to apply the results of [284] are satisfied within this framework. First of all, recall that by definition of $k^{j,0}$ in Section 3.2.2, and Σ_M, Λ_M in Assumption 3.4.1, these functions are assumed to be bounded. As in [111, Proof of Proposition 4.5], it follows that F^j satisfies the Lipschitz continuity assumptions required in [284, Assumption 2.1 (i)]. Indeed, we have

$$|F^j(t, x, y, z, \chi^{-j}, S) - F^j(t, x, y', z', \chi^{-j}, S)| \leq |k^{j,0}|_\infty |y - y'| + |\Lambda_M|_\infty |S^{1/2}(z - z')|,$$

for all $(t, x, y, z) \in [0, T] \times \mathcal{C}_T^{hm} \times \mathbb{R} \times \mathbb{R}^{hm}$, $S \in \mathcal{S}_t^j(x, \chi^{-j})$, and $(y', z') \in \mathbb{R} \times \mathbb{R}^{hm}$, ensuring that F^j is Lipschitz continuous in y and in $S^{1/2}z$, as requested in [284, Assumption 2.1 (i)]. Moreover, by Definition 3.2.9 of the set of admissible contracts $\Xi^{j,0}$, the terminal condition $g^{j,0}(\zeta^j, \xi^j)$ satisfies (I_M^p) . Using in addition the integrability condition (3.2.6) for $c^{j,0}$, it then follows that the terminal condition $g^{j,0}(\zeta^j, \xi^j)$ and F^j satisfy the integrability properties in [284, Assumption 1.1 (ii)]. Indeed, we have, for some $p > 1$,

$$\sup_{\mathbb{P} \in \mathfrak{P}} \mathbb{E}^{\mathbb{P}} \left[|g^{j,0}(\zeta^j, \xi^j)|^p \right] < +\infty, \text{ by Condition } (I_M^p)$$

$$\text{and } \sup_{\mathbb{P} \in \mathfrak{P}} \mathbb{E}^{\mathbb{P}} \left[\int_t^T |F^j(s, \zeta, 0, 0, \chi^{-j}, \hat{\Sigma}_s)|^p ds \right] \leq \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}} \left[\int_0^T |c^{j,0}(t, \zeta^j, \chi^{j,\mathbb{P}})|^p ds \right] < +\infty, \text{ by (3.2.6), } t \in [0, T].$$

In addition, [284, Assumption 3.1] is also satisfied thanks to the integrability condition (3.2.6) for $c^{j,0}$, as explained in [111, Proof of Proposition 4.5]. Next, [284, Assumption 1.1 (iii)–(v)] are also satisfied by the set of measures $\mathfrak{P} := \bar{\mathcal{P}}^j(\chi^{-j})$, see for instance [268]. Finally, the set $\mathfrak{P}$ is saturated in the sense of [284, Definition 5.1], see [284, Remark 5.1]. $\square$

3.7.2.3 Characterisation of the Nash equilibrium between managers

With Proposition 3.7.9 in hand, we can now characterise a Nash equilibria between the managers, thanks to a collection of coupled 2BSDEs, reminiscent of the multidimensional BSDE obtained in the setting of Élie and Possamaï [140] (2019) where only the drift of the canonical process was controlled.

Theorem 3.7.10. *Let $\xi^M \in \Xi^M$ be the collection of contracts for the managers, meaning that the j -th manager receives a contract $\xi^j \in \Xi^{j,0}$. The unique Nash equilibrium $\mathbb{P} \in \mathcal{P}^{M,*}(\xi^M)$, in the sense of Definition 3.4.2, is characterised by $\mathbb{P} = \bar{\mathbb{P}}^{\tilde{\chi}^*}$ where $\tilde{\chi}^*$ is such that for any $j \in \{1, \dots, m\}$,*

$$K_t^j = 0 \text{ and } \tilde{\chi}^{j,*} = \arg \max_{u \in \tilde{\mathfrak{X}}_t^j(\zeta, \tilde{\chi}^{-j,*}, \hat{\Sigma}_t)} \tilde{h}^j(t, \zeta, \mathcal{Y}_t^j, Z_t^j, \tilde{\chi}_t^{-j,*}, u), \text{ for all } t \in [0, T], \bar{\mathbb{P}}^{\tilde{\chi}^*}\text{-a.s.,}$$

where $(\mathcal{Y}^j, Z^j, K^j)$ is a solution to 2BSDE j , in the sense of Definition 3.7.8, on $\bar{\mathcal{P}}^j(\tilde{\chi}^{-j,*})$.

Proof. As in the statement of the theorem, we fix a collection $\xi^M \in \Xi^M$ of contracts for the managers. Recall that by definition of the set Ξ^M , the collection of contracts ξ^M leads to a unique Nash equilibrium between the managers (see Definition 3.2.9). By Proposition 3.7.9, we have a characterisation of the best-reaction function of the j -th manager to an arbitrary tuple of controls χ^{-j} chosen by the other managers. A Nash equilibrium $\mathbb{P}^*$ associated to an optimal effort $\chi^* := (\chi^{j,*})_{j=1}^m$ then necessitates only that for all $j \in \{1, \dots, m\}$, $\mathbb{P}^*$ is the best-reaction function of the j -th managers to $\chi^{-j,*}$. In other words, the probability $\mathbb{P}^*$ and the associated effort χ^* have to satisfy (3.7.6) for all $j \in \{1, \dots, m\}$, which is exactly what is written in the statement of the theorem. $\square$

The result of the previous theorem leads us to consider, if it exists, a map $\tilde{u}$ taking values in $\mathbb{R}^m$, such that each component satisfies, for all $j \in \{1, \dots, m\}$,

$$\tilde{u}^j(t, x, y, z, S) = \arg \max_{u \in \tilde{\mathfrak{X}}_t^j(x, \tilde{u}^{-j}(t, x, y, z, S), S)} \tilde{h}^j(t, x, y^j, z^j, \tilde{u}^j(t, x, y, z, S), u), \quad (3.7.7)$$

for $(t, x) \in [0, T] \times \mathcal{C}_T^{hm}$, $y = (y^j)_{j=1}^m \in \mathbb{R}^m$, $z = (z^j)_{j=1}^m \in (\mathbb{R}^{hm})^m$ and $S \in \mathbb{S}_+^{hm}$. Let then define $F^* : [0, T] \times \mathcal{C}_T^{hm} \times \mathbb{R}^m \times (\mathbb{R}^{hm})^m \times \mathbb{S}_+^{hm} \rightarrow \mathbb{R}^m$ such that each component satisfies

$$F^{j,*}(t, x, y, z, S) := F^j(t, x, y^j, z^j, \tilde{u}^j(t, x, y, z, S)), \quad j \in \{1, \dots, m\}.$$

We can then consider a triple $(\mathcal{Y}, Z, K)$, solution of a multidimensional 2BSDE, in the sense that each component $(\mathcal{Y}^j, Z^j, K^j)$ is solution to the following 2BSDE for $t \in [0, T]$,

$$\mathcal{Y}_t^j = g^{j,0}(\zeta^j, \xi^j) + \int_t^T F^{j,\star}(s, \zeta, \mathcal{Y}_s, Z_s, \hat{\Sigma}_s) ds - \int_t^T Z_s^j \cdot d\zeta_s + \int_t^T dK_s^j, \quad \mathcal{P}^j(\tilde{\chi}^{-j,\star})\text{-q.s.}, \quad (3.7.8)$$

where $\tilde{\chi}^\star$ is defined component by component by $\tilde{\chi}_t^{j,\star} := \tilde{u}^j(t, \zeta, \mathcal{Y}_t, Z_t, \hat{\Sigma}_t)$, $j \in \{1, \dots, m\}$.

This multidimensional 2BSDE is an extension of the pair of 2BSDEs considered by [Possamaï, Touzi, and Zhang \[286, Equations \(3.22–3.23\)\] \(2020\)](#) in their framework of a zero-sum game with two interacting players. One can also relate this multidimensional 2BSDE to the Mean-Field and Mc-Kean Vlasov 2BSDEs obtained by [Barrasso and Touzi \[38\] \(2020\)](#) or in [Chapter 4](#), for a framework with a continuum of agents with Mean-Field interactions.

3.8 Technical proofs...

This section regroups the proofs of the propositions and theorems established in the chapter, for the managers-agents and the principal-managers problems, using the results established respectively in [Sections 3.7.1 and 3.7.2](#).

3.8.1 ... for the agents' problem

Proof of Proposition 3.3.6. As in the statement of the proposition, we fix $\mathbb{P}^M$ a probability chosen by the managers. For all $j \in \{1, \dots, m\}$ and $i \in \{0, \dots, n_j\}$, let $Y_0^{j,i} \in \mathbb{R}$ and $\mathcal{Z}^{j,i} \in \mathcal{V}^{j,i}$, and consider the continuation utility $Y^{j,i}$ as well as the associated contract $\xi^{j,i}$ defined through [Definition 3.3.5](#). Note that each $\xi^{j,i}$ naturally satisfies the properties in order to be admissible in the sense of [Definition 3.2.5](#), and that it suffices to prove uniqueness of the Nash equilibrium to ensure that $\xi^A \in \Xi^A$. We will first show that the equilibrium suggested in the proposition is indeed a Nash equilibrium, and then that it is unique.

Existence of a Nash equilibrium. We first fix $j \in \{1, \dots, m\}$ and $i \in \{0, \dots, n_j\}$ to focus on the (j, i) -th agent. We assume for now that other agents are playing according to $\nu^{-(j,i),\star}$, i.e., $\mathbb{P}^{-(j,i)} = \mathbb{P}^{-(j,i),\star}$, and resume by $\hat{\nu}^\star := (\nu^{-(j,i),\star}, \nu^M) \in \mathcal{U}^{-(j,i)} \times \mathcal{U}^M$ the effort of other workers. We look for the best admissible response of the (j, i) -th agent to this probability $\mathbb{P}^{-(j,i),\star}$, and to the probability $\mathbb{P}^M$ chosen by the managers, i.e., a probability $\mathbb{P} \in \mathcal{P}^{j,i}(\mathbb{P}^{-(j,i),\star}, \mathbb{P}^M)$ that maximises his utility. First, by assumption, the contract $\xi^{j,i}$ belongs to the set $\Xi_R^{j,i}$, thus the continuation utility $Y^{j,i}$ satisfies the formula in [Definition 3.3.5](#), for $Y_0^{j,i} \in \mathbb{R}$ and $\mathcal{Z}^{j,i} = (Z^{j,i}, \tilde{Z}^{j,i}, \Gamma^{j,i}) \in \mathcal{V}^{j,i}$, i.e., for all $t \in [0, T]$,

$$Y_t^{j,i} = Y_0^{j,i} - \int_0^t \mathcal{H}^{j,i}(r, X^{j,i}, Y_r^{j,i}, Z_r^{j,i}, \hat{\nu}_r) dr + \int_0^t Z_r^{j,i} \cdot dX_r^j + \int_0^t \tilde{Z}_r^{j,i} \cdot d\bar{X}_r^{-j} + \frac{1}{2} \int_0^t \Gamma_r^{j,i} d\langle X^{j,i} \rangle_r.$$

Define then, for $t \in [0, T]$,

$$K_t^{j,i} := \int_0^t \left(\mathcal{H}^{j,i}(r, X^{j,i}, Y_r^{j,i}, Z_r^{j,i}, \hat{\nu}_r) - \frac{1}{2} \Gamma_r^{j,i} S_r^{j,i} - F^{j,i}(r, X^{j,i}, Y_r^{j,i}, Z_r^{j,i}, \tilde{Z}_r^{j,i}, \hat{\nu}_r, S_r^{j,i}) \right) dr,$$

where $F^{j,i}$ is given by [\(3.7.2\)](#). Replacing in the previous form of the continuation utility, we thus obtain:

$$Y_t^{j,i} = Y_0^{j,i} - \int_0^t F^{j,i}(r, X^{j,i}, Y_r^{j,i}, Z_r^{j,i}, \tilde{Z}_r^{j,i}, \hat{\nu}_r, S_r^{j,i}) dr + \int_0^t Z_r^{j,i} \cdot dX_r^j + \int_0^t \tilde{Z}_r^{j,i} \cdot d\bar{X}_r^{-j} - \int_0^t dK_r^{j,i},$$

for $t \in [0, T]$. Finally, recalling the contract satisfies $\xi^{j,i} = \bar{g}^{j,i}(X^{j,i}, Y_T^{j,i})$, we have $Y_T^{j,i} = g^{j,i}(X^{j,i}, \xi^{j,i})$ and we can rewrite the previous equation in a backward form as follows:

$$Y_t^{j,i} = Y_T^{j,i} + \int_t^T F^{j,i}(r, X^{j,i}, Y_r^{j,i}, Z_r^{j,i}, \tilde{Z}_r^{j,i}, \hat{\nu}_r, S_r^{j,i}) dr - \int_t^T Z_r^{j,i} \cdot dX_r^j - \int_t^T \tilde{Z}_r^{j,i} \cdot d\bar{X}_r^{-j} + \int_t^T dK_r^{j,i},$$

which exactly corresponds to **2BSDE** (j, i) under $\mathfrak{P} := \bar{\mathcal{P}}^{j,i}(\bar{\mathbb{P}}^{-(j,i),*}, \bar{\mathbb{P}}^M)$. By definition of $F^{j,i}$, we can directly check that $K^{j,i}$ is always a non-decreasing process, which vanishes on the support of any probability measure corresponding to the efforts $\nu^{j,i,*}$ defined in the statement of the proposition. To ensure that $(Y^{j,i}, (Z^{j,i}, \tilde{Z}^{j,i}), K^{j,i})$ solves **2BSDE** (j, i) , it therefore remains to check that all the integrability requirements in **Definition 3.7.3** are satisfied. The one for $Y^{j,i}$ is immediate by definition of the set $\mathcal{V}^{j,i}$. The required integrability on $(Z^{j,i}, \tilde{Z}^{j,i}, K^{j,i})$ then follows from Bouchard, Possamaï, Tan, and Zhou [67, Theorem 2.1 and Proposition 2.1] (2018).

Uniqueness. We have therefore obtained that the candidate provided in the statement of the proposition was indeed an equilibrium. Let us now prove uniqueness. Let $\nu^{-(j,i)}$ be the arbitrary efforts of other agents, and the associated probability measure $\mathbb{P}^{-(j,i)}$. In this case, the continuation utility of the (j, i) -th agent, given a contract $\xi^{j,i} \in \Xi_R^{j,i}$, does not satisfy **2BSDE** (j, i) , since other agents' efforts are not necessarily optimal anymore. Nevertheless, $\Xi_R^{j,i} \subset \Xi^{j,i}$ and by **Proposition 3.7.4**, the optimal effort $\nu^{j,i,*}$ is the maximiser of the map $F^{j,i}$, which coincides with the maximiser (3.3.5) of his Hamiltonian $\mathcal{H}^{j,i}$ given by (3.3.4). By **Assumption 3.3.3**, this optimal effort is unique, and in particular does not depend on $\nu^{-(j,i)}$ (nor on ν^M). To sum up, given a contract in $\Xi_R^{j,i}$ and for arbitrary efforts $\nu^{-(j,i)}$ of others, the agent (j, i) has a unique optimal effort $\nu^{j,i,*}$, independent of others actions. We can therefore conclude that the optimal effort of each agent is given by the maximiser of his Hamiltonian. This induces a unique equilibrium in terms of efforts, given by $\nu^* = (\alpha^*, \beta^*)$, and inducing the law $\mathbb{P}^*$. It is therefore the unique equilibrium. $\square$

Proof of Theorem 3.3.7. Before explaining how to prove the aforementioned result, notice that if we can prove that the restriction to revealing contracts in $\Xi_R^{j,i}$ is without loss of generality, then the equality (3.3.7) is trivial. Indeed, as mentioned before the theorem and by **Definition 3.3.5**, given a constant $Y_0^{j,i} \in \mathbb{R}$, choosing a contract $\xi^{j,i} \in \Xi_R^{j,i}$ is strictly equivalent of choosing a triple of payment rates $\mathcal{Z}^{j,i} := (Z^{j,i}, \tilde{Z}^{j,i}, \Gamma^{j,i}) \in \mathcal{V}^{j,i}$.

The fact that the restriction to revealing contracts in the sense of **Definition 3.3.5** is without loss of generality relies on arguments similar to the ones developed in the aforementioned works [111, 141, 11, 140]. We thus consider an arbitrary collection $\xi^A \in \Xi^A$ of contracts, such that each agent (j, i) receives the contract $\xi^{j,i} \in \Xi^{j,i}$. Starting from the admissible collection of contracts ξ^A , the goal is to show that for all agents, we can define an approximation ξ^ε of his contract $\xi^{j,i}$, leading to the same Nash equilibrium, such that the associated continuation utility Y^ε has the required dynamics (3.3.6), and moreover, $\xi^\varepsilon = \xi^{j,i}$ at the Nash equilibrium. This will thus ensure that the manager receives the same value when considering revealing contracts in the sense of **Definition 3.3.5** instead of arbitrary admissible contracts.

First, using **Proposition 3.7.4** and **Theorem 3.7.5**, we know that for a collection of contracts $\xi^A \in \Xi^A$ and a probability $\mathbb{P}^M$ chosen by the managers, there exists a unique Nash equilibrium $\mathbb{P}^* \in \mathcal{P}^{A,*}(\mathbb{P}^M, \xi^A)$, associated to an optimal effort $\nu^* \in \mathcal{U}$, satisfying for any $j \in \{1, \dots, m\}$ and $i \in \{1, \dots, n_j\}$,

$$K^{j,i} = 0, \mathbb{P}^*\text{-a.s.}, \text{ and } \nu^{j,i,*} \in \arg \max_{u \in \tilde{\mathcal{U}}_t^{j,i}(S_t^{j,i})} \tilde{h}^{j,i}(t, X^{j,i}, Y_t^{j,i}, (Z_t^{j,i})^i, u), \mathbb{P}^*\text{-a.s.},$$

where $(Y^{j,i}, (Z^{j,i}, \tilde{Z}^{j,i}), K^{j,i})$ is a solution to **2BSDE** (j, i) , in the sense of **Definition 3.7.3**.

Given an arbitrary but admissible collection $\xi^A \in \Xi^A$ of contracts, the idea is to use the aforementioned solution $(Y^{j,i}, (Z^{j,i}, \tilde{Z}^{j,i}), K^{j,i})$ to **2BSDE** (j, i) to construct an approximation of the contract $\xi^{j,i}$. Since the

reasoning is similar for all agents, we fix $j \in \{1, \dots, m\}$, $i \in \{1, \dots, n_j\}$, and focus on the approximation of the (j, i) -th agent's contract $\xi^{j,i}$. The main difference between contracts in $\Xi^{j,i}$ and $\Xi_R^{j,i}$ comes from whether the process $K^{j,i}$ above is absolutely continuous with respect to Lebesgue measure or not. Since it is not in general, we will approximate it by a sequence of absolutely continuous ones. With this in mind, fix some $\varepsilon > 0$, and define the absolutely continuous approximation of $K^{j,i}$:

$$K_t^\varepsilon := \frac{1}{\varepsilon} \int_{(t-\varepsilon)^+}^t K_s^{j,i} ds, \quad t \in [0, T].$$

Recalling the notation $\mathfrak{P} := \bar{\mathcal{P}}^{j,i}(\bar{\mathbb{P}}^{-(j,i)}, \bar{\mathbb{P}}^M)$, we have that K^ε is $(\mathbb{G}^j)^\mathfrak{P}$ -predictable, non-decreasing $\mathfrak{P}$ -q.s. and satisfies

$$K^\varepsilon = 0, \quad \mathbb{P}^\star\text{-a.s. for all } \mathbb{P}^\star \in \mathcal{P}^{A,\star}(\bar{\mathbb{P}}^M, \xi^A). \quad (3.8.1)$$

We next define for any $t \in [0, T]$ the process

$$Y_t^\varepsilon := Y_0^{j,i} - \int_0^t F^{j,i}(s, X_s^{j,i}, Y_s^\varepsilon, Z_s^{j,i}, \tilde{Z}_s^{j,i}, \hat{\nu}_s^\star, S_s^{j,i}) ds + \int_0^t Z_s^{j,i} \cdot dX_s^j + \int_0^t \tilde{Z}_s^{j,i} \cdot d\bar{X}_s^{-j} - \int_0^t dK_s^\varepsilon, \quad (3.8.2)$$

where $\hat{\nu}^\star := (\nu^{-(j,i),\star}, \nu^M) \in \mathcal{U}^{-(j,i)} \times \mathcal{U}^M$ denotes for the effort of others under $\mathbb{P}^\star \in \mathcal{P}^{A,\star}(\bar{\mathbb{P}}^M, \xi^A)$, as defined in [Proposition 3.3.6](#). We first verify that $(Y^\varepsilon, (Z^{j,i}, \tilde{Z}^{j,i}), K^\varepsilon)$ solves [2BSDE \(j, i\)](#), with terminal condition Y_T^ε and generator $F^{j,i}$. First, by (3.8.1), K^ε clearly satisfies the required minimality condition. Then, noticing that $0 \leq K^\varepsilon \leq K^{j,i}$, K^ε inherits the integrability of $K^{j,i}$. By the integrability of $Y^{j,i}$, $(Z^{j,i}, \tilde{Z}^{j,i})$ and $K^{j,i}$, the stability of solutions to SDEs with Lipschitz generator implies that $\sup_{\mathbb{P} \in \bar{\mathcal{P}}} \mathbb{E}^\mathbb{P}[|Y_T^\varepsilon|^p] < \infty$. Therefore, $(Y^\varepsilon, (Z^{j,i}, \tilde{Z}^{j,i}), K^\varepsilon)$ is solution to [2BSDE \(j, i\)](#) in the sense of [Definition 3.7.3](#), which implies by [\[284, Theorem 4.4\]](#) the following estimates:

$$\|Y^\varepsilon\|_{\mathbb{S}^{\bar{p}}(\mathbb{G}^j, \mathfrak{P})} + \|(Z^{j,i}, \tilde{Z}^{j,i})\|_{\mathbb{H}_{n_j+m-1}^{\bar{p}}(\mathbb{G}^j, \mathfrak{P}, \tilde{\Sigma}^j)} < \infty, \quad \text{for } \bar{p} \in (1, p). \quad (3.8.3)$$

We finally observe that a probability measure $\mathbb{P}$ satisfies $K^{j,i} = 0$, $\mathbb{P}$ -a.s. if and only if it satisfies $K^\varepsilon = 0$, $\mathbb{P}$ -a.s. An approximation ξ^ε of the admissible contract $\xi^{j,i}$ can thus be defined as a particular function of the terminal value of Y^ε , more precisely by $\xi^\varepsilon := \bar{g}^{j,i}(X^{j,i}, Y_T^\varepsilon)$, recalling that $\bar{g}^{j,i}$ corresponds to the inverse of $g^{j,i}$ with respect to the second variable. In other words, the approximation ξ^ε satisfies $Y_T^\varepsilon = g^{j,i}(X^{j,i}, \xi^\varepsilon)$.

To prove that the previously defined contract ξ^ε is a revealing contract, meaning that it belongs to the set $\Xi_R^{j,i}$, we should in particular make the parameter Γ appear. With this in mind, notice that for any $(t, \omega, x, y, z, \tilde{z}) \in [0, T] \times \Omega \times \mathcal{C}_T \times \mathbb{R} \times \mathbb{R}^{n_j+1} \times \mathbb{R}^{m-1}$, the map

$$\gamma \longmapsto \mathcal{H}^{j,i}(t, x, y, z, \tilde{z}, \gamma, \hat{\nu}^\star) - \frac{1}{2} \gamma S(\omega) - F^{j,i}(t, x, y, z, \tilde{z}, \hat{\nu}^\star, S(\omega)), \quad (3.8.4)$$

is surjective on $(0, +\infty)$. Indeed, it is non-negative, by definition of $\mathcal{H}^{j,i}$ and $F^{j,i}$, convex, continuous on the interior of its domain, and is coercive by the boundedness of the functions $\lambda^{j,i}$, $\sigma^{j,i}$, $k^{j,i}$ and $c^{j,i}$. Let $\dot{K}^\varepsilon$ denote the density of the absolutely continuous process K^ε with respect to the Lebesgue measure. Applying a classical measurable selection argument (the maps appearing here are continuous, and we can use the results from [\[47, 48\]](#)), we may deduce the existence of a $\mathbb{G}^j$ -predictable process Γ^ε such that

$$\dot{K}_s^\varepsilon = \mathcal{H}^{j,i}(s, X_s^{j,i}, Y_s^\varepsilon, Z_s^{j,i}, \tilde{Z}_s^{j,i}, \Gamma_s^\varepsilon, \hat{\nu}_s^\star) - \frac{1}{2} \Gamma_s^\varepsilon S_s^{j,i} - F^{j,i}(s, X_s^{j,i}, Y_s^\varepsilon, Z_s^{j,i}, \tilde{Z}_s^{j,i}, \hat{\nu}_s^\star, S_s^{j,i}), \quad s \in [0, T].$$

Indeed, if $\dot{K}_s^\varepsilon > 0$, the existence of Γ_s^ε is clear from (3.8.4), and if $\dot{K}_s^\varepsilon = 0$, Γ_s^ε can be chosen arbitrarily. Substituting in (3.8.2), it follows that the following representation for Y^ε holds

$$Y_t^\varepsilon := Y_0^{j,i} - \int_0^t \mathcal{H}^{j,i}(r, X_r^{j,i}, Y_r^\varepsilon, Z_r^{j,i}, \tilde{Z}_r^{j,i}, \Gamma_r^\varepsilon, \hat{\nu}_r^\star) dr + \int_0^t Z_s^{j,i} \cdot dX_s^j + \int_0^t \tilde{Z}_s^{j,i} \cdot d\bar{X}_s^{-j} + \frac{1}{2} \int_0^t \Gamma_s^\varepsilon d\langle X^{j,i} \rangle_s.$$

This shows that the continuation utility Y^ε has the required dynamics (3.3.6). The fact that the contract ξ^ε induced by Y^ε belongs to $\Xi_R^{j,i}$ then stems from (3.8.3). Moreover, notice that the admissible contract $\xi^{j,i}$ and its approximation ξ^ε coincides at the equilibrium, in the sense that $\xi^\varepsilon = \xi^{j,i}$, $\mathbb{P}^*$ -a.s. This reasoning is true for all $j \in \{1, \dots, m\}$ and $i \in \{1, \dots, n_j\}$, and we have therefore constructed a well-suited approximation of the collection ξ^A of contract belonging to Ξ_R^A . Using Propositions 3.3.6 and 3.7.4, we can then conclude as in the proof of [111, Theorem 3.6], since both collection of contracts lead to the same unique Nash equilibrium. $\square$

3.8.2 ... for the managers' problem

Proof of Proposition 3.4.8. As in the statement of the proposition, let $Y_0^M := (Y_0^j)_{j=1}^m$ and $\mathcal{Z} := (Z, \Gamma) \in \mathcal{V}$. By Definition 3.4.7, consider the m -dimensional process $\mathcal{Y}^M := (\mathcal{Y}^j)_{j=1}^m$ as well as the associated collection of contracts $\xi^M := (\xi^j)_{j=1}^m \in \Xi_R^M$. Note that each ξ^j naturally satisfies the properties in order to be admissible in the sense of Definition 3.2.9, and that it suffices to prove uniqueness of the Nash equilibrium to ensure that $\xi^M \in \Xi^M$.

Existence of a Nash equilibrium. We first fix $j \in \{1, \dots, m\}$, and we assume that other managers apart from the j -th are playing according to $\chi^{-j,*}$, defined by the first point of the proposition, *i.e.*,

$$\chi^{-j,*} = (\chi_t^{\ell,*})_{\ell=1, \ell \neq j}^m, \quad \text{where } \chi_t^{\ell,*} = u^{\ell,*}(t, \zeta, \mathcal{Y}, Z, \Gamma), \quad t \in [0, T], \quad \ell \in \{1, \dots, m\} \setminus \{j\}. \quad (3.8.5)$$

We look for the best admissible response of the j -th manager with respect to the effort of others, *i.e.*, a probability $\mathbb{P} \in \mathcal{P}^j(\chi^{-j,*})$ that maximises his utility. More precisely, we want to prove that $\chi^{j,*}$ is also given by (3.8.5). By assumption on the contract, namely that $\xi^j \in \Xi_R^j$, we have in particular that the continuation utility $\mathcal{Y}^j$ of the j -th manager satisfies the formula (3.4.5). Recalling the definition of $\mathcal{H}^{j,*}$ in (3.4.4), *i.e.*, for $(t, x, y, z, \gamma) \in [0, T] \times \mathcal{C}_T^{hm} \times \mathbb{R}^m \times (\mathbb{R}^{hm})^m \times (\mathbb{M}^{hm})^m$,

$$\mathcal{H}^{j,*}(t, x, y, z, \gamma) := \mathcal{H}^j(t, x, y^j, z^j, \gamma^j, u^{-j,*}(t, x, y, z, \gamma)),$$

it is easy to see that $\mathcal{Y}^j$ satisfies the following:

$$\mathcal{Y}_t^j = Y_0^j - \int_0^t \mathcal{H}^j(r, \zeta, \mathcal{Y}_r^j, Z_r^j, \Gamma_r^j, \chi_r^{-j,*}) dr + \int_0^t Z_r^j \cdot d\zeta_r + \frac{1}{2} \int_0^t \text{Tr}[\Gamma_r^j d\langle \zeta \rangle_r], \quad t \in [0, T].$$

Recalling that $\hat{\Sigma}$ denotes the pathwise version of the quadratic variation $\langle \zeta \rangle$ (see (3.7.4)), we define

$$K_t^j := \int_0^t \left(\mathcal{H}^j(r, \zeta, \mathcal{Y}_r^j, Z_r^j, \Gamma_r^j, \chi_r^{-j,*}) - \frac{1}{2} \text{Tr}[\Gamma_r^j \hat{\Sigma}_r] - F^j(r, \zeta, \mathcal{Y}_r^j, Z_r^j, \chi_r^{-j,*}, \hat{\Sigma}_r) \right) dr,$$

for $t \in [0, T]$, where F^j is defined by (3.7.5). Replacing in the previous form of the continuation utility, we obtain:

$$\mathcal{Y}_t^j = Y_0^j - \int_0^t F^j(r, \zeta, \mathcal{Y}_r^j, Z_r^j, \chi_r^{-j,*}, \hat{\Sigma}_r) dr + \int_0^t Z_r^j \cdot d\zeta_r - \int_0^t dK_r^j.$$

Finally, recalling the contract satisfies $\xi^j = \bar{g}^j(\zeta^j, \mathcal{Y}_T^j)$, we have $\mathcal{Y}_T^j = g^j(\zeta^j, \xi^j)$ and we can rewrite the previous equation in a backward form as follows:

$$\mathcal{Y}_t^j = g^j(\zeta^j, \xi^j) + \int_t^T F^j(r, \zeta, \mathcal{Y}_r^j, Z_r^j, \chi_r^{-j,*}, \hat{\Sigma}_r) dr - \int_t^T Z_r^j \cdot d\zeta_r + \int_t^T dK_r^j,$$

which exactly corresponds to **2BSDE j** under $\mathfrak{P} := \bar{\mathcal{P}}^j(\chi^{-j,*})$. By definition of F^j , we can directly check that K^j is always a non-decreasing process, which vanishes on the support of any probability measure corresponding to the efforts $\chi^{j,*}$ defined in the statement of the proposition. To ensure that $(\mathcal{Y}^j, Z^j, K^j)$ solves **2BSDE j** , it therefore remains to check that all the integrability requirements in **Definition 3.7.8** are satisfied. The one for $\mathcal{Y}^j$ is immediate by definition of the set $\mathcal{V}$. The required integrability on (Z^j, K^j) then follows from Bouchard, Possamaï, Tan, and Zhou [67, Theorem 2.1 and Proposition 2.1] (2018).

Uniqueness. We have therefore obtained that the candidate provided in the statement of the proposition was indeed an equilibrium. Let us now prove uniqueness. Let χ^{-j} be the arbitrary efforts of other managers. In this case, the continuation utility of the j -th manager, given a contract $\xi^j \in \Xi_R^j$, does not satisfies **2BSDE j** , since other agents' efforts are not necessarily optimal anymore. Nevertheless, $\Xi_R^j \subset \Xi^{j,0}$ and by **Proposition 3.7.9**, an optimal effort $\chi^{j,*}$ is a maximiser of the map F^j , which coincides with a maximiser of his Hamiltonian $\mathcal{H}^j$ given by (3.4.3). Recall that the existence of such a maximiser is ensured by **Assumption 3.4.3**, but uniqueness is not assumed. Nevertheless, this reasoning is valid for all managers, implying that χ^* should satisfies for all $j \in \{1, \dots, m\}$, $\chi_t^{j,*} = u^j(t, \zeta, \mathcal{Y}^j, Z^j, \Gamma^j, \chi^{-j,*})$. This condition is equivalent to the definition of the map u^* in **Assumption 3.4.4**. Moreover, since by **Assumption 3.4.4** there is a unique map u^* guaranteeing the maximization of each manager's Hamiltonian simultaneously, this induces a unique equilibrium in terms of efforts, given by χ^* . $\square$

Proof of Theorem 3.4.9. The main point is to prove that the restriction to revealing contracts in the sense of **Definition 3.4.7** is without loss of generality. This proof relies on arguments similar to the ones developed in the proof of **Theorem 3.3.7** and initially in [111, Proof of Theorem 3.6]. We thus consider an arbitrary collection $\xi^M \in \Xi^M$ of contracts, in the sense that the j -th manager receives the contract $\xi^j \in \Xi^{j,0}$. Starting from this admissible collection of contracts, the goal is to prove that, for all managers, we can define an approximation $\xi^{j,\varepsilon}$ of his contract ξ^j , such that the new collection of contracts ξ^ε is admissible and gives the same Nash equilibrium. Moreover, we should verify that for all $j \in \{1, \dots, m\}$, $\xi^{j,\varepsilon} = \xi^j$ at the Nash equilibrium, and that the associated continuation utility $\mathcal{Y}^{M,\varepsilon}$ satisfies the representation (3.4.5), in addition to required integrability conditions, to ensure that ξ^ε is a collection of revealing contracts, in the sense of **Definition 3.4.7**.

First, using **Proposition 3.7.9** and **Theorem 3.7.10**, we know that for a collection of contracts $\xi^M \in \Xi^M$, there exists a unique equilibrium $\mathbb{P}^* \in \mathcal{P}^{M,*}(\xi^M)$, associated to an optimal effort $\tilde{\chi}^* \in \mathcal{X}$, satisfying for any $j \in \{1, \dots, m\}$,

$$K^j = 0, \mathbb{P}^*\text{-a.s.}, \text{ and } \tilde{\chi}_t^{j,*} = \arg \max_{u \in \tilde{\mathcal{X}}_t^j(\zeta, \tilde{\chi}^{-j,*}, \hat{\Sigma}_t)} \tilde{h}^j(t, \zeta, \mathcal{Y}_t^j, Z_t^j, \tilde{\chi}_t^{-j,*}, u), \mathbb{P}^*\text{-a.s.}, \quad (3.8.6)$$

where $(\mathcal{Y}^j, Z^j, K^j)$ is a solution to **2BSDE j** , in the sense of **Definition 3.7.8**.

Given this arbitrary but admissible collection ξ^M of contracts, the idea is to use the aforementioned solution $(\mathcal{Y}^j, Z^j, K^j)$ to **2BSDE j** to construct an approximation $\xi^{j,\varepsilon}$ of the contract ξ^j . Similar to the agents' problem, let us fix some $\varepsilon > 0$, and define, for all $j \in \{1, \dots, m\}$, the absolutely continuous approximation of K^j :

$$K_t^{j,\varepsilon} := \frac{1}{\varepsilon} \int_{(t-\varepsilon)^+}^t K_s^j ds, \quad t \in [0, T].$$

The process $K^{j,\varepsilon}$ naturally inherits some properties of K^j . More precisely, given the effort of other managers χ^{-j} , and recalling the notation $\mathfrak{P} := \bar{\mathcal{P}}^j(\chi^{-j})$, we have that $K^{j,\varepsilon}$ is $\mathbb{G}^{\mathfrak{P}}$ -predictable, non-decreasing $\mathfrak{P}$ -q.s. and satisfies

$$K^{j,\varepsilon} = 0, \mathbb{P}^*\text{-a.s. for all } \mathbb{P}^* \in \mathcal{P}^{M,*}(\xi^M). \quad (3.8.7)$$

We next define the m -dimensional process $\mathcal{Y}^\varepsilon$ such that each component $\mathcal{Y}^{j,\varepsilon}$ satisfies, for any $t \in [0, T]$,

$$\mathcal{Y}_t^{j,\varepsilon} := Y_0^j - \int_0^t F^j(s, \zeta, \mathcal{Y}_s^{j,\varepsilon}, Z_s^j, \tilde{\chi}_s^{-j,\star}, \hat{\Sigma}_s) ds + \int_0^t Z_s^j \cdot d\zeta_s - \int_0^t dK_s^{j,\varepsilon}, \quad \mathcal{P}^j(\tilde{\chi}^{-j,\star})\text{-q.s.}, \quad (3.8.8)$$

recalling that the optimal efforts $\tilde{\chi}^{-j,\star}$ of other managers are defined omega per omega through (3.8.6).

We first verify that for all $j \in \{1, \dots, m\}$, $(\mathcal{Y}^{j,\varepsilon}, Z^j, K^{j,\varepsilon})$ solves 2BSDE j under $\mathfrak{P} := \bar{\mathcal{P}}^j(\chi^{-j,\star})$, with terminal condition $\mathcal{Y}_T^{j,\varepsilon}$ and generator F^j . With this in mind, let us fix $j \in \{1, \dots, m\}$ as well the other components, namely $(\mathcal{Y}^{-j,\varepsilon}, Z^{-j}, K^{-j,\varepsilon})$. First, by (3.8.7), $K^{j,\varepsilon}$ clearly satisfies the required minimality condition. Then, noticing that $K^{j,\varepsilon} \leq K^j$, $K^{j,\varepsilon}$ inherits the integrability of K^j , and moreover we can verify that $\sup_{\mathbb{P} \in \bar{\mathcal{P}}^M} \mathbb{E}^\mathbb{P}[\|\mathcal{Y}_T^{j,\varepsilon}\|^p] < \infty$, similarly to the equivalent proof for the manager-agents problem. Therefore, by [284, Theorem 4.4], we have the following estimates

$$\|\mathcal{Y}^{j,\varepsilon}\|_{\mathbb{S}^{\bar{p}}(\mathbb{G}, \mathfrak{P})} + \|Z^j\|_{\mathbb{H}_{hm}^{\bar{p}}(\mathbb{G}, \mathfrak{P}, \hat{\Sigma})} < \infty, \quad \text{for } \bar{p} \in (1, p). \quad (3.8.9)$$

We finally observe that a probability measure $\mathbb{P}$ satisfies $K^j = 0$, $\mathbb{P}$ -a.s. if and only if it satisfies $K^\varepsilon = 0$, $\mathbb{P}$ -a.s. An approximation $\xi^{j,\varepsilon}$ of the admissible contract ξ^j can thus be defined omega per omega from the terminal value of $\mathcal{Y}^{j,\varepsilon}$ by $\xi^{j,\varepsilon} := \bar{g}^{j,0}(\zeta^j, \mathcal{Y}_T^{j,\varepsilon})$, recalling that $\bar{g}^{j,0}$ corresponds to the inverse of $g^{j,0}$ with respect to the second variable. In other words, the approximation $\xi^{j,\varepsilon}$ satisfies $\mathcal{Y}_T^{j,\varepsilon} := g^{j,0}(\zeta^j, \xi^{j,\varepsilon})$.

To prove that the previously defined contract $\xi^{j,\varepsilon}$ is a revealing contract, meaning that it belongs to the set $\Xi_R^{j,0}$, we should in particular make the parameter Γ appears in the representation (3.8.8). With this in mind, notice that for any $(t, \omega, x, y, z) \in [0, T] \times \Omega^M \times \mathcal{C}_T^{hm} \times \mathbb{R}^{hm} \times \mathbb{R}^{n_j+1}$, the map

$$\gamma \longmapsto \mathcal{H}^j(t, x, y, z, \gamma, \tilde{\chi}^{-j,\star}) - \frac{1}{2} \text{Tr}[\gamma S(\omega)] - F^j(t, x, y, z, \tilde{\chi}^{-j,\star}, S(\omega)), \quad (3.8.10)$$

is surjective on $(0, +\infty)$. Indeed, it is non-negative, by definition of $\mathcal{H}^j$ and F^j , convex, continuous on the interior of its domain, and is coercive by the boundedness of the functions Λ_M , Σ_M , $k^{j,0}$ and $c^{j,0}$. Let $\dot{K}^{j,\varepsilon}$ denote the density of the absolutely continuous process $K^{j,\varepsilon}$ with respect to the Lebesgue measure. Applying a classical measurable selection argument (the maps appearing here are continuous, and we can use the results from [47, 48]), we can deduce the existence of a $\mathbb{G}$ -predictable process $\Gamma^{j,\varepsilon}$ such that

$$\dot{K}_s^{j,\varepsilon} = \mathcal{H}^j(r, \zeta, \mathcal{Y}_s^{j,\varepsilon}, Z_s^j, \Gamma_s^{j,\varepsilon}, \tilde{\chi}_s^{-j,\star}) - \frac{1}{2} \text{Tr}[\Gamma_s^{j,\varepsilon} \hat{\Sigma}_s] - F^j(r, \zeta, \mathcal{Y}_s^{j,\varepsilon}, Z_s^j, \tilde{\chi}_s^{-j,\star}, \hat{\Sigma}_s), \quad s \in [0, T].$$

Indeed, if $\dot{K}_s^{j,\varepsilon} > 0$, the existence of $\Gamma_s^{j,\varepsilon}$ is clear from (3.8.10), and if $\dot{K}_s^{j,\varepsilon} = 0$, $\Gamma_s^{j,\varepsilon}$ can be chosen arbitrarily. Substituting in (3.8.8), it follows that the following representation for $\mathcal{Y}^{j,\varepsilon}$ holds

$$\mathcal{Y}_t^{j,\varepsilon} := Y_0^j - \int_0^t \mathcal{H}^j(r, \zeta, \mathcal{Y}_r^{j,\varepsilon}, Z_r^j, \Gamma_r^{j,\varepsilon}, \tilde{\chi}_r^{-j,\star}) dr + \int_0^t Z_r^j \cdot d\zeta_r + \frac{1}{2} \int_0^t \text{Tr}[\Gamma_r^{j,\varepsilon} d\langle \zeta \rangle_r].$$

This shows that the continuation utility $\mathcal{Y}^\varepsilon$ has the required dynamics (3.4.5), since, at equilibrium, the effort $\tilde{\chi}^\star = \chi^\star$ and is unique. The fact that the contract $\xi^{j,\varepsilon}$ induced by $\mathcal{Y}^{j,\varepsilon}$ belongs to Ξ_R^j then stems from (3.8.9). Moreover, notice that the admissible contract ξ^j and its approximation $\xi^{j,\varepsilon}$ coincides at the equilibrium, in the sense that $\xi^{j,\varepsilon} = \xi^j$, $\mathbb{P}^\star$ -a.s. This reasoning is true for all $j \in \{1, \dots, m\}$, and we have therefore constructed a well-suited approximation of the collection ξ^M of contracts, belonging to Ξ_R^M . Using Propositions 3.4.8 and 3.7.9, we can then conclude as in the proof of [111, Theorem 3.6] since both collection of contracts lead to the same unique Nash equilibrium.

Finally, the equality (3.4.8) is now trivial. Indeed, by Definition 3.4.7, choosing a collection $\xi^M \in \Xi^M$ of contracts is strictly equivalent of choosing both a pair of payment rates $\mathcal{Z} := (Z, \Gamma) \in \mathcal{V}$ to index the

contract of each manager respectively on $d\zeta$ and $d\langle\zeta\rangle$, and a constant $Y_0^M := (Y_0^j)_{j=1}^m \in \mathbb{R}^m$. Nevertheless, for all $j \in \{1, \dots, m\}$, the constant $Y_0^j \in \mathbb{R}$ has to be chosen so that the participation constraint for the j -th manager is satisfied. Moreover, the principal also chooses the initial value $Y_0^A \in \mathbb{R}$ of the agents' continuation utility, such that their participation constraints, *i.e.*, [Condition \(PC_A\)](#), are satisfied. Using [Propositions 3.3.6](#) and [3.4.8](#) respectively for the agents and the managers, these conditions are satisfied if and only if:

$$Y_0^{j,i} = V_0^{j,i,*}(\chi^*) \geq \rho^{j,i} \quad \text{and} \quad Y_0^j = V_0^{j,0,*}(\xi^M) \geq \rho^{j,0}, \quad \text{for all } j \in \{1, \dots, m\}, i \in \{0, \dots, n_j\},$$

recalling that $V^{j,i,*}$ and $V_0^{j,0,*}$ are respectively defined by [\(3.3.8\)](#) and [\(3.3.8\)](#). This justifies the equality [\(3.4.8\)](#) and ends the proof. $\square$

Towards a mean–field of agents

Où sont tes ailes pour le futur ?
Si tu choisis d'enfanter la démesure.

N'appartiens jamais à personne
BERNARD LAVILLIERS

Introduction

In this chapter, we explore another path to broaden principal–agent problems to multi–agents models, by considering a continuum of agents with mean–field interaction, whose output process denoted by X is impacted by a common noise, thus extending the framework considered by Aïd, Possamaï, and Touzi [11] (2019). More precisely, we work under a classical mean–field framework, where all agents are identical: they all have the same characteristics, the same costs, the same risk–aversion parameter. First, this simplification allows us to restrict the analysis to a representative agent, who is a single agent identical to others, and too small to have an impact on the global result from the population’s point of view. Therefore, we can focus on the moral hazard problem between this representative agent and the principal, who offers him a contract corresponding to a compensation paid at a future specified time. Moreover, this framework with indistinguishable agents prevents the principal from indexing the compensation for an agent on a particular output, except his own. The rigorous formulation of the problem is developed in [Section 4.1](#).

The goal of this chapter is to find a way for the principal to benefit from dealing with a continuum of agents. In usual principal–agent models with drift and volatility control, following the general approach by Cvitanić, Possamaï, and Touzi [111] (2018) mentioned in the previous chapters, we are led to restrict the study to *revealing contracts*, indexed on the output process X and its quadratic variation. The idea we develop in this chapter is to add a component to this revealing contract. Indeed, in our framework, the principal has access to a large quantity of anonymous output profiles. Therefore, she can approximate the conditional law, with respect to the common noise, of the output process of the pool of agents she manages. We are thus led to study a new class of contracts, adding to the relevant contract for drift and volatility control, a component indexing the contract on the law of the output of other agents.

First, [Section 4.2](#) is devoted to the intuition leading to this new type of contracts. Then, [Section 4.3](#) properly define the relevant form of contracts and provide the main results on the representative agent’s optimal response and on the resulting mean–field equilibrium between agents. Finally, the principal’s problem is solved in [Section 4.4](#). We first present the main result of this chapter, namely [Theorem 4.4.1](#), which states that our form of revealing contract, given by [Definition 4.3.1](#), is without loss of generality. Therefore, the principal’s problem is reduced to a McKean–Vlasov problem, because she impacts through the contract, and thus somehow indirectly controls, the efforts of the agents as well as the conditional law of the outputs. We then provide some theoretical results for solving the principal’s problem, in both cases of risk–neutral and CARA utility functions. [Section 4.5](#) presents the first–best case as well as an extension to contractible common noise. All technical proofs are postponed to [Section 4.6](#).

The theoretical approach developed throughout this chapter is motivated by the application of the results to energy demand response programs, in order to extend the model in [11] to an infinite number of consumers. The application itself is addressed in the following chapter, *i.e.*, Chapter 5. This motivation justifies why, apart from the mean-field aspect and the correlation between agents through the common noise, the framework is voluntarily as close as possible to that of [11]. Nevertheless, due to the complexity induced by the presence of a continuum of agents in interaction, we do not rely on the results of [11], besides some straightforward algebra, and need to develop new techniques, based on the general approach in [111] and on the theory of mean-field games with common noise, developed in particular by Carmona and Delarue [82] (2018). Moreover, it should be emphasised that although the model is very specific, the *modus operandi* we develop in this chapter can readily be extended to a more general moral hazard problem between a continuum of agents with mean-field interaction and common noise. Finally, even if this chapter is strongly linked to Chapter 5, we have made a specific effort to ensure that these two chapters can be read independently, even if this can lead to unavoidable redundancies. Nevertheless, we provide here the theoretical results and proofs, to which the reader can refer if necessary when reading the next chapter.

4.1 Problem formulation

We work under a classical mean-field framework, where all agents are identical, which allows us to restrict our study to a representative agent, who is identical to others, and too small to impact the global result. This agent controls his own output process X , driven by a d -dimensional idiosyncratic noise, for $d \in \mathbb{N}^*$ fixed, and an uni-dimensional common noise, by choosing a pair $\nu := (\alpha, \beta)$, where α and β are respectively A - and B -valued. For technical reasons, we need to consider bounded efforts, we then set

$$A := [0, \rho^1 A_{\max}] \times \cdots \times [0, \rho^d A_{\max}] \quad \text{and} \quad B := [B_{\min}, 1]^d, \quad (4.1.1)$$

for some constants $A_{\max} > 0$, $B_{\min} \in (0, 1)$ and a vector $\rho \in (0, +\infty)^d$. We denote for simplicity $U := A \times B$. More specifically, α represents the effort of the agent to impact the nominal level of his output and β is the effort to reduce the variability. We also define a vector $\sigma \in (0, +\infty)^d$ representing the variability of the agent's output when he does not make any effort, and a constant $\sigma^\circ \in \mathbb{R}_+$ representing the correlation with the common noise. Finally, we define a map $\sigma : B \rightarrow \mathbb{R}^d$ as follows:

$$\sigma(b) := (\sigma^1 \sqrt{b^1}, \dots, \sigma^d \sqrt{b^d})^\top, \quad \text{for } b \in B.$$

Therefore, each agent controls exclusively his own output process, but all outputs are correlated with each other through the common noise. We will see in the following that the principal wants to implement contracts to incentivise them to reduce the mean and the volatility of their output. But first, in order to properly define the mathematical set up of our problem, we should define a canonical space Ω for the representative agent (see Section 4.1.1), as well as a canonical space $\hat{\Omega}$ for others (see Section 4.1.2).

4.1.1 Canonical space for the representative agent

As introduced in *General notations* (p. xv), we define $\mathbb{U}$ and $\mathbb{U}_0$ from the subset $U := A \times B$, to consider the weak formulation. In order to combine the theoretical formulations of mean-field games and McKean-Vlasov problems, we are led to consider the following canonical space

$$\Omega := \Omega^\circ \times \Omega^1 \times \Omega^2 \times \mathbb{U}, \quad \text{where} \quad \Omega^\circ := \mathcal{C}_T, \quad \Omega^1 := \mathcal{C}_T \times \mathcal{C}_T^d \quad \text{and} \quad \Omega^2 := \mathcal{P}(\mathcal{C}_T),$$

recalling from *General notations* (p. xv) that $\mathcal{C}_T^d := \mathcal{C}([0, T], \mathbb{R}^d)$ and $\mathcal{C}_T := \mathcal{C}([0, T], \mathbb{R})$. This space is supporting a canonical process $(W^\circ, X, W, \mu, \Pi)$, where for any $t \in [0, T]$ and $\omega := (w^\circ, x, w, u, q) \in \Omega$,

$$W_t^\circ(\omega) := w^\circ(t), \quad X_t(\omega) := x(t), \quad W_t(\omega) := w(t), \quad \mu_t(\omega) := u(t), \quad \text{and} \quad \Pi_t(\omega) := q.$$

In words, X represents the output controlled by the representative agent, affected by an idiosyncratic noise W and a common noise W° . The process μ allows to keep track of the law of X conditionally to the common noise and the space $\mathbb{U}$ corresponds to the controls. The canonical filtration $\mathbb{F} := (\mathcal{F}_t)_{t \in [0, T]}$ is defined as

$$\mathcal{F}_t := \sigma\left((W_s^\circ, X_s, W_s, \mu_s, \Delta_s(\varphi)) : (s, \varphi) \in [0, t] \times \mathcal{C}_b([0, T] \times U, \mathbb{R})\right), \quad t \in [0, T],$$

recalling that $\mathcal{C}_b([0, T] \times U, \mathbb{R})$ denotes the set of all bounded continuous functions from $[0, T] \times U$ to $\mathbb{R}$, and for any $(s, \varphi) \in [0, T] \times \mathcal{C}_b([0, T] \times U, \mathbb{R})$, $\Delta_s(\varphi) := \int_0^s \int_U \varphi(r, v) \Pi(dr, dv)$. We will also need a smaller filtration containing only the information generated by the common noise and the conditional law of X . Namely, we define $\mathbb{F}^\circ := (\mathcal{F}_t^\circ)_{t \in [0, T]}$ by $\mathcal{F}_t^\circ := \sigma((W_s^\circ, \mu_s) : s \in [0, t])$, $t \in [0, T]$.

Remark 4.1.1. *The restriction to the filtration $\mathbb{F}^\circ$ stems from the presence of common noise in our model. As pointed out by Carmona, Delarue, and Lacker [91] (2016), it is commonplace for control problems in weak formulation, and actually already for weak solutions to SDEs, that the underlying driving noise is not rich enough to obtain a solution which is only adapted to it. In our context, this translates into the fact that in general the law of X may fail to be measurable only with respect to the information generated by the common noise W° , which justifies the enlargement in the definition of $\mathbb{F}^\circ$. This is linked to the so-called compatibility condition in the MFG theory with common noise, which intuitively means that a given player in the game does have access to the full information generated by the idiosyncratic and common noises W and W° , as well as the distribution of all other players' states, and his controls are allowed to be randomised externally to these observations, but such a randomisation must be conditionally independent of future information given current information.*

Recall that $\mathcal{C}_b^2(\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}, \mathbb{R})$ denotes the set of $\mathcal{C}^2$ bounded functions from $\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}$ to $\mathbb{R}$, with bounded derivatives. For any $(s, \varphi) \in [0, T] \times \mathcal{C}_b^2(\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}, \mathbb{R})$, we set

$$M_s(\varphi) := \varphi(X_s, W_s, W_s^\circ) - \int_0^s \int_U \left(A(v) \cdot \nabla \varphi(X_r, W_r, W_r^\circ) + \frac{1}{2} \text{Tr}[\nabla^2 \varphi(X_r, W_r, W_r^\circ)(BB^\top)(v)] \right) \Pi(dr, dv),$$

where A and B are respectively the drift vector and the diffusion matrix of the vector process $(X, W, W^\circ)^\top$:

$$A(v) := \begin{pmatrix} -a \cdot \mathbf{1}_d \\ \mathbf{0}_d \\ 0 \end{pmatrix}, \quad B(v) := \begin{pmatrix} 0 & \sigma^\top(b) & \sigma^\circ \\ \mathbf{0}_d & \text{Id} & \mathbf{0}_d \\ 0 & \mathbf{0}_d^\top & 1 \end{pmatrix}, \quad v := (a, b) \in U.$$

Therefore, the covariation matrix of the vector process $(X, W, W^\circ)^\top$ is defined for all $v \in U$ by

$$(BB^\top)(v) = \begin{pmatrix} \Sigma(b) + |\sigma^\circ|^2 & \sigma^\top(b) & \sigma^\circ \\ \sigma(b) & \text{Id} & \mathbf{0}_d \\ \sigma^\circ & \mathbf{0}_d^\top & 1 \end{pmatrix},$$

where $\Sigma(b) := \sigma^\top(b)\sigma(b)$ for all $b \in B$. We now fix some initial conditions, namely a probability measures ϱ on $\mathbb{R}$ representing the law at 0 of X .

Definition 4.1.2. *Let $\mathcal{P}$ be the set of probability measures $\mathbb{P}$ on $(\Omega, \mathcal{F}_T)$ satisfying*

- (i) $M(\varphi)$ is a $(\mathbb{P}, \mathbb{F})$ -local martingale on $[0, T]$ for all $\varphi \in \mathcal{C}_b^2(\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}, \mathbb{R})$;
- (ii) $\mathbb{P} \circ (X_0)^{-1} = \varrho$ and there exists a measure ι on $\mathbb{R}^d \times \mathbb{R}$ such that $\mathbb{P} \circ ((W_0, W_0^\circ))^{-1} = \iota$;
- (iii) $\mathbb{P}[\Pi \in \mathbb{U}_0] = 1$;

- (iv) for $\mathbb{P}$ -a.e. $\omega \in \Omega$ and for every $t \in [0, T]$, we have $\mu_t(\omega) = \mathbb{P}_t^\omega \circ (X_{t \wedge \cdot})^{-1}$, where $(\mathbb{P}_t^\omega)_{\omega \in \Omega}$ is a family of regular conditional probability distribution¹ (r.c.p.d for short) for $\mathbb{P}$ given $\mathcal{F}_t^\circ$. We will denote by $\mathbb{E}^{\mathbb{P}_t^\omega}$ the expectation under the distribution $\mathbb{P}_t^\omega$. For ease of notation, we will often omit the ω in the notation for the expectation;
- (v) (W°, μ) is $\mathbb{P}$ -independent of W .

Roughly speaking, the set $\mathcal{P}$ represents the set of admissible controls in the weak formulation. Nevertheless, the previous formulation does not give us access directly to the dynamic of the output process X . It is however a classical result that, enlarging the canonical space if necessary, one can construct Brownian motions allowing to write rigorously the dynamic of X (see for instance Stroock and Varadhan [324, Theorem 4.5.2] (1997)). It turns out here that since we have enlarged the canonical space right from the start to account for the idiosyncratic and common noises, any further enlargement is not required.

Lemma 4.1.3. *For all $\mathbb{P} \in \mathcal{P}$, $\Pi(ds, dv) = \delta_{\nu_s^\mathbb{P}}(dv)ds$ $\mathbb{P}$ -a.s., for some $\mathbb{F}$ -predictable control process $\nu^\mathbb{P} := (\alpha^\mathbb{P}, \beta^\mathbb{P})$ and*

$$X_t = X_0 - \int_0^t \alpha_s^\mathbb{P} \cdot \mathbf{1}_d ds + \int_0^t \sigma(\beta_s^\mathbb{P}) \cdot dW_s + \int_0^t \sigma^\circ dW_s^\circ, \quad t \in [0, T], \quad \mathbb{P}\text{-a.s.} \quad (4.1.2)$$

Notice that the previous lemma allows to properly define the set $\mathcal{U} = \mathcal{A} \times \mathcal{B}$ of admissible efforts for the representative agent. Its proof is postponed to Section 4.6.

4.1.2 Canonical space of other agents and copies

To model the output of other agents (*i.e.*, apart from the representative agent), affected by the same common noise W° , we need to define an alternative canonical probability space $\hat{\Omega} := \Omega^\circ \times \hat{\Omega}^1 \times \hat{\Omega}^2 \times \hat{\mathbb{U}}$. This space is supporting a canonical process $(W^\circ, \hat{X}, \hat{W}, \hat{\mu}, \hat{\Pi})$, and the canonical filtration $\hat{\mathbb{F}} := (\hat{\mathcal{F}}_t)_{t \in [0, T]}$ is defined in the same way as $\mathbb{F}$ on Ω . We can then consider the set $\hat{\mathcal{P}}$ of all probability measures on $(\hat{\Omega}, \hat{\mathcal{F}}_T)$, in the same way we have defined $\mathcal{P}$ through Definition 4.1.2. In particular, the notation $\hat{\mathbb{E}}^{\hat{\mathbb{P}}_t}$ will stand for the expectation under the r.c.p.d. $\hat{\mathbb{P}}_t$ of $\hat{\mathbb{P}} \in \hat{\mathcal{P}}$ given $\hat{\mathcal{F}}_t^\circ$, on the space $(\hat{\Omega}, \hat{\mathcal{F}})$. Still applying Lemma 4.1.3, for any $\hat{\mathbb{P}} \in \hat{\mathcal{P}}$, we can write

$$\hat{X}_t = \hat{X}_0 - \int_0^t \alpha_s^{\hat{\mathbb{P}}} \cdot \mathbf{1}_d ds + \int_0^t \sigma(\beta_s^{\hat{\mathbb{P}}}) \cdot d\hat{W}_s + \int_0^t \sigma^\circ dW_s^\circ, \quad t \in [0, T], \quad \hat{\mathbb{P}}\text{-a.s.}, \quad (4.1.3)$$

where $\nu^{\hat{\mathbb{P}}} := (\alpha^{\hat{\mathbb{P}}}, \beta^{\hat{\mathbb{P}}})$ is some $\hat{\mathbb{F}}$ -predictable control process, chosen by other agents, satisfying $\hat{\Pi}(ds, dv) = \delta_{\nu_s^{\hat{\mathbb{P}}}}(dv)ds$ $\hat{\mathbb{P}}$ -a.s. This allows us to properly define the set $\hat{\mathcal{U}} = \hat{\mathcal{A}} \times \hat{\mathcal{B}}$ of admissible efforts of others.

In order to apply the chain rule with common noise, as defined by Carmona and Delarue [82, Theorem 4.17] (2018), we will need a copy of the process X , denoted by $\tilde{X}$, driven by the same common noise W° , and with the same conditional law μ . For this purpose, we define a copy of the initial canonical space.

Definition 4.1.4 (Copy of a space). *Let Ω be a canonical space of the form $\Omega := \Omega^\circ \times \Omega^1 \times \Omega^2 \times \mathbb{U}$. A copy of Ω is defined by $\hat{\Omega} := \Omega^\circ \times \hat{\Omega}^1 \times \Omega^2 \times \hat{\mathbb{U}}$ where $\hat{\Omega}^1$ and $\hat{\mathbb{U}}$ are respectively standard copies of the spaces Ω^1 and $\mathbb{U}$.*

1. We recall that these objects are such that for any $\omega \in \Omega$, $\mathbb{P}_t^\omega$ is a probability measure on $(\Omega, \mathcal{F})$, such that for any $A \in \mathcal{F}$, the map $\omega \mapsto \mathbb{P}_t^\omega[A]$ is $\mathcal{F}_t^\circ$ -measurable, and such that for any $\mathbb{P}$ -integrable random variable ξ on $(\Omega, \mathcal{F})$, we have

$$\mathbb{E}^\mathbb{P}[\xi | \mathcal{F}_t^\circ](\omega) = \mathbb{E}^{\mathbb{P}_t^\omega}[\xi], \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

Notice that since $(\Omega, \mathcal{F})$ is a Polish space and $\mathcal{F}_t^\circ$ is countably generated, the existence of these r.c.p.d. is guaranteed for instance by Cohen and Elliott [100, Theorems 2.6.5 and 2.6.7] (2015).

This canonical space $\tilde{\Omega}$ is supporting a canonical process $(W^\circ, \tilde{X}, \tilde{W}, \mu, \tilde{\Pi})$, and the canonical filtration $\tilde{\mathbb{F}} := (\tilde{\mathcal{F}}_t)_{t \in [0, T]}$ is defined exactly as $\mathbb{F}$. For a given probability $\mathbb{P} \in \mathcal{P}$, we can associate in a unique way a probability $\tilde{\mathbb{P}}$ satisfying [Definition 4.1.2](#) on $(\tilde{\Omega}, \tilde{\mathcal{F}}_T)$, where, in particular, $\tilde{\mathbb{P}}$ and $\mathbb{P}$ have the same r.c.p.d. Therefore, abusing notations slightly, $\tilde{\mathbb{E}}^{\mathbb{P}_t}$ will stand for the expectation on $(\tilde{\Omega}, \tilde{\mathcal{F}}_T)$ under the r.c.p.d. $\mathbb{P}_t$ for $\tilde{\mathbb{P}} \in \mathcal{P}$ given $\mathcal{F}_t^\circ$. Then, by [Lemma 4.1.3](#), for any $\tilde{\mathbb{P}}$, we have

$$\tilde{X}_t = \tilde{X}_0 - \int_0^t \alpha_s^{\tilde{\mathbb{P}}} \cdot \mathbf{1}_d ds + \int_0^t \sigma(\beta_s^{\tilde{\mathbb{P}}}) \cdot d\tilde{W}_s + \int_0^t \sigma^\circ dW_s^\circ, \quad t \in [0, T], \quad \tilde{\mathbb{P}}\text{-a.s.}, \quad (4.1.4)$$

where $\nu^{\tilde{\mathbb{P}}} := (\alpha^{\tilde{\mathbb{P}}}, \beta^{\tilde{\mathbb{P}}})$ is some $\tilde{\mathbb{F}}$ -predictable process, satisfying $\tilde{\Pi}(ds, dv) = \delta_{\nu_s^{\tilde{\mathbb{P}}}}(dv)ds$ $\tilde{\mathbb{P}}$ -a.s.

Definition 4.1.5 (Copy of a process). *The process $\tilde{X}$ defined above by (4.1.4) is called a copy of X .*

Copies of the output of others, denoted by $\tilde{X}$, are defined in the same way as copies $\tilde{X}$ of X by [Definition 4.1.5](#), on the space $\tilde{\Omega} := \Omega^\circ \times \tilde{\Omega}^1 \times \hat{\Omega}^2 \times \tilde{\mathcal{U}}$, itself a copy of $\hat{\Omega}$ in the sense of [Definition 4.1.4](#). This space is supporting a canonical process and the associated canonical filtration $\tilde{\mathbb{F}} := (\tilde{\mathcal{F}}_t)_{t \in [0, T]}$. The notation $\tilde{\mathbb{E}}^{\hat{\mathbb{P}}_t}$ stands for the expectation under the r.c.p.d. $\hat{\mathbb{P}}_t$ on the space $(\tilde{\Omega}, \tilde{\mathcal{F}})$.

In words, the canonical process $(W^\circ, X, W, \mu, \Pi)$ defined on the space Ω represents the choices of the representative agent regarding his output. Similarly, the canonical process $(W^\circ, \tilde{X}, \tilde{W}, \hat{\mu}, \hat{\Pi})$ defined on $\hat{\Omega}$ allows us to represent the choices of other agents, which may be different from the representative agent, but are affected by the same common noise W° . Then, notations involving $\tilde{\cdot}$ will refer to copies of the initial space Ω , while notations involving $\hat{\cdot}$ will refer to copies of the canonical space of others $\hat{\Omega}$. In order to compute his terminal payment ξ , the representative agent is going to assume that the others have played some distribution $\hat{\mu}$ (namely the conditional law of $\tilde{X}$ under some $\hat{\mathbb{P}} \in \hat{\mathcal{P}}$ given $\hat{\mathbb{F}}^\circ$ on $(\hat{\Omega}, \hat{\mathcal{F}}_T)$), and he is going to compute ξ along his own deviation X and $\hat{\mu}$.

4.1.3 Definition of a contract

In the general work by [Cvitanic, Possamaï, and Touzi \[111\] \(2018\)](#), the principal offers a contract to an agent indexed on his output. In our investigation, the principal faces a continuum of agents with mean-field interaction and can therefore benefit from this additional information. However, in accordance with the mean-field framework, the principal is facing a mass of identical and indistinguishable agents, and thus cannot choose the output of a particular agent to index the contract for the representative agent on it. Nevertheless, we assume that the principal can perform some statistics at the population level, and can therefore offer contracts depending on both the output of a given agent, and the aggregate statistics of the output of other agents.² Formally, we assume that the principal proposes to the representative agent a contract ξ which is a random variable measurable with respect to the natural filtration generated by $(X, \hat{\mu})$, denoted $\mathbb{F}^{\text{obs}}$, recalling that X is the output of the representative agent and $\hat{\mu}$ is the law of the output of other agents, conditionally to the common noise. In other words, ξ must be a measurable functional of the paths of X and $\hat{\mu}$:

$$\xi : (X, \hat{\mu}) \in \mathcal{C}_T \times \mathcal{P}(\mathcal{C}_T) \longmapsto \xi(X, \hat{\mu}) \in \mathbb{R}. \quad (4.1.5)$$

2. Note that, when considering the application we have in mind for the following chapter, namely energy demand response programs, the electricity provider is not allowed to reveal the consumption of a particular consumer to another consumer. Hence, she cannot directly index the remuneration for a consumer on the output of another dedicated one, which justify the consideration of aggregated statistics.

Remark 4.1.6. *Considering the conditional law $\hat{\mu}$ naturally comes from the limit of the n -agents case. Indeed, if the principal monitors n agents, she wants to index the contract for the i -th agent on his own deviation X^i and on the empirical distribution of other agents, $\bar{\mu}^{-i}$, defined as:*

$$\bar{\mu}^{-i} := \left(\frac{1}{n-1} \sum_{j=1, j \neq i}^n \delta_{X_t^j} \right)_{t \geq 0}$$

where the deviations X^j are not independent since all agents are suffering from the common noise W° . Hence, in the mean-field framework, we may wonder about the convergence of the empirical measure $\bar{\mu}_t^{-i}$ as n tends to ∞ . In the absence of the common noise, the standard theory of propagation of chaos applies: asymptotically, particles become independent and the $\bar{\mu}_t^{-i}$ converges to their common asymptotic distribution. By contrast, with a common noise, even in the limit $n \rightarrow \infty$, the particles must still keep track of the common noise W° , so they cannot become independent. Nevertheless, Carmona and Delarue [82] (2018) underline that particles become asymptotically independent conditionally on the common noise, and that the empirical distribution converges towards the common conditional distribution of each particle given the common noise ($\hat{\mu}$ in our case).

Given a contract ξ of the form (4.1.5) and the conditional law of the output of other agents $\hat{\mu}$, the representative agent solves the following optimisation problem

$$V_0^A(\xi, \hat{\mu}) := \sup_{\mathbb{P} \in \mathcal{P}} J_0^A(\xi, \hat{\mu}, \mathbb{P}), \text{ where } J_0^A(\xi, \hat{\mu}, \mathbb{P}) := \mathbb{E}^\mathbb{P} \left[U^A \left(\xi(X, \hat{\mu}) - \int_0^T (c(\nu_t^\mathbb{P}) - f(X_t)) dt \right) \right], \quad (4.1.6)$$

where $c : \mathbb{R}_+^d \times (0, 1]^d \rightarrow \mathbb{R}_+$ is the cost function associated with the effort made by the agent, and the function $f : \mathbb{R} \rightarrow \mathbb{R}$ denotes the preference of the agent toward his output. The function f is required to be concave, increasing, and centred at the origin. The function U^A is an exponential utility function, with risk aversion parameter of the representative agent $R_A > 0$, defined by $U^A(x) = -e^{-R_A x}$. For simplicity, we assume that an effort $\nu \in \mathcal{U}$ induces a separable cost, meaning that for all $v := (a, b)$,

$$c(v) := \frac{1}{2} c_\alpha(a) + \frac{1}{2} c_\beta(b), \text{ where } c_\alpha(a) := \sum_{k=1}^d \frac{|a^k|^2}{\rho^k} \text{ and } c_\beta(b) := \sum_{k=1}^d \frac{|\sigma^k|^2}{\lambda^k \eta^k} \left((b^k)^{-\eta^k} - 1 \right),$$

for fixed $(\lambda, \eta) \in (0, +\infty)^d \times (1, +\infty)^d$, and recalling that $\rho \in (0, +\infty)^d$. In particular, the cost of the effort in the drift term of X , denoted by c_α , is a classical quadratic cost function, meaning that no effort for the agent induces no intrinsic cost, and such that he has no interest to provide negative efforts. The cost associated with the effort in the volatility of X prohibits the agent from removing the volatility ($b^k > 0$, for all $k \in \{1, \dots, d\}$) and is equal to zero if the agent makes no effort (case $b^k = 1$, $k \in \{1, \dots, d\}$).

For $V_0^A(\xi, \hat{\mu})$ to make sense, we require minimal integrability on the contracts, by imposing that

$$\sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^\mathbb{P} \left[e^{p R_A |\xi|} \right] < +\infty, \text{ for some } p > 1. \quad (\text{I}^p)$$

In addition and for technical reasons, we restrict our attention to contracts ξ satisfying also the technical assumption (CARA) given below, only necessary in order to solve the problem of a CARA risk-averse principal (and useless to solve the agent problem). Moreover, as usual in contract theory, we assume that agents have an exogenous reservation utility $R_0 < 0$, below which they refuse the contract offered by the principal.³ Contracts of the form (4.1.5), satisfying Condition (I^p) and Condition (CARA) (p. 137), as well as the participation constraint, are called admissible, and the corresponding class is denoted by Ξ .

3. We will see in the following chapter that we can in fact consider an endogenous reservation utility. More precisely, we will assume that R_0 is given by the agent's utility without contract, i.e., $R_0 := V_0^A(0, \hat{\mu})$.

Finally, notice here that the contracts offered by the principal have been assumed to not be indexed on the common noise W° . This can either mean that the principal cannot observe it perfectly, or that there are regulatory reasons preventing the principal from using it directly in the contract. Nevertheless, our formulation allows to incorporate the case where this becomes possible. Thus, in [Section 4.5.2](#), we will study the case where the principal is allowed to use the common noise directly in the contract. In this particular case, she can offer to the representative agent a contract ξ , measurable with respect to the natural filtration generated by X , W° and $\hat{\mu}$, denoted by $\mathbb{F}^{\text{obs},\circ}$. In other words, ξ must be in this case a measurable functional of the paths of X , W° and $\hat{\mu}$:

$$\xi : (X, W^\circ, \hat{\mu}) \in \mathcal{C}_T \times \mathcal{C}_T \times \mathcal{P}(\mathcal{C}_T) \longmapsto \xi(X, W^\circ, \hat{\mu}) \in \mathbb{R}. \quad (4.1.7)$$

The corresponding class of admissible contracts will be denoted by Ξ° .

4.1.4 Definition of a mean-field equilibrium

Recall that we work under a classical mean-field framework, where all agents are identical. Hence, similarly to [Carmona, Delarue, and Lacker \[91\] \(2016\)](#), we define a mean-field equilibrium as follows.

Definition 4.1.7 (Mean-field equilibrium). *Let $\xi \in \Xi$ be an admissible contract. We denote by $\mathcal{M}^*(\xi)$ the collection of all mean-field equilibria, i.e., pairs $(\mathbb{P}^*, \mu^*) \in \mathcal{P} \times \mathcal{P}(\mathcal{C}_T)$ such that*

(i) *given $\mu^* \in \mathcal{P}(\mathcal{C}_T)$, the probability $\mathbb{P}^* \in \mathcal{P}$ is optimal for (4.1.6), i.e.,*

$$V_0^A(\xi, \mu^*) = \mathbb{E}^{\mathbb{P}^*} \left[U^A \left(\xi(X, \mu^*) - \int_0^T (c(\nu_t^{\mathbb{P}^*}) - f(X_t)) dt \right) \right];$$

(ii) *for $\mathbb{P}^*$ -a.e. $\omega \in \Omega$ and for every $t \in [0, T]$, we have*

$$\mu_t^*(\omega) = \mathbb{P}_t^\omega \circ (X_{t \wedge \cdot})^{-1},$$

where $(\mathbb{P}_t^\omega)_{\omega \in \Omega}$ is a family of regular conditional probability distribution for $\mathbb{P}^$ given $\mathcal{F}_t^\circ$.*

We extend readily this definition to contracts in Ξ° , and denote by $\mathcal{M}^{,\circ}(\xi)$ the associated set of mean-field equilibria.*

Remark 4.1.8. *The attentive reader may note that this definition of a mean-field equilibrium, in particular the fixed point constraint, involves the probability measure on the path space, i.e., $\mu^* \in \mathcal{P}(\mathcal{C}_T)$. Although this is not standard in the MFG theory in a Markovian setting, this makes sense in our non-Markovian framework, and a similar condition is considered by [Carmona and Lacker \[84\] \(2015\)](#).*

4.1.5 The principal's problem

We now turn to the definition of the principal's problem. In the one-agent framework defined in [\[11\]](#), the principal has an exponential utility function, with risk-aversion parameter $R_P > 0$, defined by $U^P(x) = -e^{-R_P x}$ and wants to minimise:

- (i) the compensation paid to the agent, i.e., ξ ;
- (ii) a cost induced by the output, denoted by $g(X_t)$, where g is concave and increasing.
- (iii) a penalisation term indexed on the quadratic variation $\langle X \rangle_t$, which allows the principal to take into account the variations of the output over time.

The intuition for the principal problem in the mean-field case comes from the n -agents case. Formally, if we consider a n -players model, the principal would minimise the (utility of the) sum of the previous costs. To ensure stability of these sums as n grows, and therefore obtain a mean-field limit of the n -agents problem, we can follow the line by [Élie, Mastrolia, and Possamaï \[141\] \(2018\)](#) by assuming that each individual output is scaled by the total number of agents n . In their framework, *i.e.*, with exponential utility functions but without common noise, the principal becomes risk-neutral in the limit when $n \rightarrow +\infty$, by classical propagation of chaos arguments. Another interpretation of the risk-neutrality of the principal in this case is that the principal is diversifying the risk by considering a large number of agents: the random average penalised output in the n -players' game converges to a deterministic quantity. In our framework, *with common noise*, as explained in [Remark 4.1.6](#), the outputs of the agents become asymptotically independent only conditionally to the common noise. Therefore, a risk-averse principal does not become risk-neutral in our case, and remains impacted by the residual risk arising from the common noise. Nevertheless, we will consider both cases of a risk-averse and a risk-neutral principal.

As a consequence, combining [\[141\]](#) and [\[11\]](#) the criterion of the principal is defined for any $\xi \in \Xi$ by

$$J_0^P(\xi, \mathbb{P}) := \mathbb{E}^\mathbb{P} \left[U^P \left(- \mathbb{E}^\mathbb{P} \left[\xi + \int_0^T g(X_s) ds + \frac{\theta}{2} \int_0^T d\langle X \rangle_s \middle| \mathcal{F}_T^\circ \right] \right) \right], \text{ for any } \mathbb{P} \in \mathcal{P}, \quad (4.1.8)$$

where the function $U^P : \mathbb{R} \rightarrow \mathbb{R}$ is the principal's utility function and θ is a positive constant representing the costs induced by the quadratic variation of the output.⁴

Anticipating the results we obtain in [Section 4.3](#), for a contract $\xi \in \Xi$, any mean-field equilibrium $(\mathbb{P}^*, \mu^*) \in \mathcal{M}^*(\xi)$ will give the same utility to the agents, since they all have the same characteristics. Furthermore, in the absence of limited liability in our model (compensations ξ need not be non-negative), the participation constraints of the agents is going to be saturated, meaning that any optimal contract will provide them exactly their reservation utility level. Therefore, in the case where an optimal contract would lead to several possible equilibria, the agents will be indifferent to the specific one chosen, implying that we can reasonably assume, as in the standard moral hazard literature, that the principal can maximise her utility by choosing the optimal equilibrium for her. This leads to the following maximisation problem

$$V_0^P := \sup_{\xi \in \Xi} \sup_{(\mathbb{P}, \mu) \in \mathcal{M}^*(\xi)} J_0^P(\xi, \mathbb{P}), \quad (4.1.9)$$

with the usual convention $\sup_\emptyset = -\infty$. Notice that for the contracts in the class we will end up considering, there is only one possible equilibrium, which makes the above issue not really central to our analysis.

In the moral hazard contracting problem considered here, the principal has an interest in giving a contract for which there is at least a mean-field equilibrium. Otherwise, by convention, her utility is equal to $-\infty$. Thus, contracts $\xi \in \Xi$ such that $\mathcal{M}^*(\xi) = \emptyset$ will never be offered by the principal, meaning that one can implicitly assume that there will always be an optimal response from the agents to a contract proposed by the principal. Moreover, since the contract has to satisfy the participation constraint, there exists $(\mathbb{P}^*, \mu^*) \in \mathcal{M}^*(\xi)$ such that $V_0^A(\xi, \mu^*) \geq R_0$. The set Ξ of eligible contracts is now formally defined. We define similarly Ξ° and $V_0^{P,\circ}$.

The study of the first-best contracting problem, *i.e.*, when the principal can observe in continuous-time the efforts of the agents and thus can index the contract on the latter, is postponed to [Section 4.5.1](#).

One of the cornerstones of the approach to continuous-time moral hazard problems pioneered by [Sanikov \[303\] \(2008\)](#), and studied in full generality by [Cvitanić, Possamaï, and Touzi \[111\] \(2018\)](#), is to obtain

4. In the application we will study in [Chapter 5](#), this parameter will allow to account for the limited flexibility of production.

an appropriate probabilistic representation for incentive-compatible contracts. The goal of the following section is to use informal dynamic programming type arguments to deduce such a representation in a context where a continuum of agents with mean-field interactions is involved, each of them being able to control the volatility of the output process, and to then prove that mean-field equilibria are easily accessible for this class of contracts.

4.2 Intuition for a relevant form of contracts

Recall that we consider, for now, that the principal can only offer to the representative agent a contract ξ of the form (4.1.5), *i.e.*, $\mathbb{F}^{\text{obs}}$ -measurable. For a given conditional distribution $\hat{\mu}$ of other players and this contract ξ , we consider the dynamic version of the value function of the representative agent, namely V_t^A , which satisfies $V_0^A = V_0^A(\xi, \hat{\mu})$, and $V_T^A = U^A(\xi(X, \hat{\mu}))$. From this definition, we notice that the following explicit relationship between the payoff and the terminal value function holds

$$\xi(X, \hat{\mu}) = -\frac{1}{R_A} \ln(-V_T^A). \quad (4.2.1)$$

In this section, we will start, for a given contract and given efforts chosen by other agents, by introducing the appropriate Hamiltonian functional, which will allow to first compute formally the optimal response of the agent. Intuitively, this Hamiltonian appears by applying the chain rule with common noise defined by Carmona and Delarue [82, Theorem 4.17] (2018) to the dynamic value function of the agent and considering the associated Master equation. Our next step is then to derive a class of so-called *revealing* contracts, thus extending to a general mean-field game framework the main arguments of Cvitanić, Possamaï, and Touzi [111] (2018), who consider general moral hazard problems with one agent, and Élie, Mastrolia, and Possamaï [141] (2018), who consider mean-field game moral hazard problems where the agents controlled only the drift of the output process X . Informally, the class of revealing contract is obtained by still using the chain rule with common noise, but applying it to a transformed function of the agent's dynamic value function, defined by (4.2.1).

We insist on the fact that the analysis we make in this section, to find the agent's Hamiltonian as well as the relevant form of contracts, is informal. Indeed, we consider for now the Markovian framework, *i.e.*, we suppose that the agent's dynamic value function at time t only depends on X_t and $\hat{\mu}_t$, where here, $\hat{\mu}_t$ is the conditional law of $\hat{X}_t$ (and not of the paths of $\hat{X}$ up to time t) knowing the common noise. This framework allows us to apply the chain rule with common noise, defined in [82, Theorem 4.17]. Nevertheless, the analysis we make, though informal at this point, relies strongly on recent progresses on the dynamic programming approach to the control of McKean–Vlasov SDEs and will rigorously be justified later in the chapter, mainly in Section 4.6. Moreover, the important results in the non-Markovian framework are rigorously established in the following section, *i.e.*, Section 4.3.

With this in mind, we fix throughout this section a probability $\hat{\mathbb{P}} \in \hat{\mathcal{P}}$, chosen by other agents on their own canonical space $\hat{\Omega}$. Using Definition 4.1.2, we denote by $\hat{\mathbb{P}}_t^\omega$ the r.c.p.d. for $\mathbb{P}$ given $\mathcal{F}_t^\circ$, and $\hat{\mu}_t$ the associated conditional law of $\hat{X}_{t \wedge \cdot}$. Almost by definition of $\hat{\mathbb{P}} \in \hat{\mathcal{P}}$, more precisely by Lemma 4.1.3, there exists a control process representing the effort of other agents, denoted for simplicity $\hat{\nu} := (\hat{\alpha}, \hat{\beta}) \in \hat{\mathcal{U}}$, such that the dynamic of their output $\hat{X}$ is:

$$d\hat{X}_t = -\hat{\alpha}_t \cdot \mathbf{1}_d dt + \sigma(\hat{\beta}_t) \cdot d\hat{W}_t + \sigma^\circ dW_t^\circ, \quad t \in [0, T].$$

Therefore, the efforts $\hat{\nu} \in \hat{\mathcal{U}}$ of other agents as well as the conditional law $\hat{\mu} \in \mathcal{P}(\mathcal{C}_T)$ are fixed through the probability $\hat{\mathbb{P}} \in \hat{\mathcal{P}}$.

Intuitively, in view of (4.2.1), we expect that a contract $\mathbb{F}^{\text{obs}}$ -measurable is any terminal value of the following process, as a function of t , the path of X up to t , and $\hat{\mu}_t$, the conditional law of $\hat{X}_{t\wedge\cdot}$ with respect to the common noise:

$$\xi_t := -\frac{1}{R_A} \ln(-V_t^A) = u^A(t, X_{t\wedge\cdot}, \hat{\mu}_t), \quad t \in [0, T], \quad (4.2.2)$$

where $u^A : [0, T] \times \mathcal{C}_T \times \mathcal{P}(\mathcal{C}_T) \rightarrow \mathbb{R}$. Therefore, given a contract $\xi \in \Xi$ and efforts of other agents subsumed by the distribution $\hat{\mu}$, the agent's (dynamic) continuation utility V_t^A , for all $t \in [0, T]$, may be written as

$$V_t^A := v^A(t, X_{t\wedge\cdot}, \hat{\mu}_t), \quad \text{where } v^A := -e^{-R_A \cdot} \circ u^A,$$

i.e., the process V^A at time t depends on t , on the path history of X , and on the conditional law $\hat{\mu}$ of output $\hat{X}$ of others. Indeed, the contract being only indexed on X and $\hat{\mu}$, the continuation utility should not depend on all the information contained in $\mathbb{F}$ and $\hat{\mathbb{F}}$.

To find the relevant form of contracts, the intuition is to focus on the Markovian framework, *i.e.* when the function v^A , and thus the function u^A , only depend on t , X_t , and $\hat{\mu}_t$, where here, $\hat{\mu}_t$ is the conditional law of $\hat{X}_t$ (and not of the paths of $\hat{X}$ up to time t) with respect to the common noise. Note that in this particular case, both v^A and u^A are functions from $[0, T] \times \mathbb{R} \times \mathcal{P}(\mathbb{R})$ with values in $\mathbb{R}$.

4.2.1 Agent's Hamiltonian

In the Markovian framework, *i.e.*, when V_t^A only depends on X_t and $\hat{\mu}_t$ where here, $\hat{\mu}_t$ is the conditional law of $\hat{X}_t$ (and not of the paths of $\hat{X}$ up to time t) knowing the common noise, and if v^A is smooth enough in the sense of Carmona and Delarue [82, Section 4.3.4] (2018), we can apply the chain rule with common noise (for function of both the state and the measure, see [82, Theorem 4.17]) to $v^A : [0, T] \times \mathbb{R} \times \mathcal{P}(\mathbb{R}) \rightarrow \mathbb{R}$:

$$\begin{aligned} V_t^A &= V_0^A + \int_0^t \partial_s v^A(s, X_s, \hat{\mu}_s) ds + \int_0^t \partial_x v^A(s, X_s, \hat{\mu}_s) dX_s + \int_0^t \hat{\mathbb{E}}^{\hat{\mathbb{P}}_s} \left[\partial_\mu v^A(s, X_s, \hat{\mu}_s)(\hat{X}_s) d\hat{X}_s \right] \\ &\quad + \frac{1}{2} \int_0^t \partial_{xx}^2 v^A(s, X_s, \hat{\mu}_s) d\langle X \rangle_s + \int_0^t \hat{\mathbb{E}}^{\hat{\mathbb{P}}_s} \left[\partial_x \partial_\mu v^A(s, X_s, \hat{\mu}_s)(\hat{X}_s) d\langle X, \hat{X} \rangle_s \right] \\ &\quad + \frac{1}{2} \int_0^t \hat{\mathbb{E}}^{\hat{\mathbb{P}}_s} \left[\partial_v \partial_\mu v^A(s, X_s, \hat{\mu}_s)(\hat{X}_s) d\langle \hat{X} \rangle_s \right] + \frac{1}{2} \int_0^t \hat{\mathbb{E}}^{\hat{\mathbb{P}}_s} \tilde{\mathbb{E}}^{\tilde{\mathbb{P}}_s} \left[\partial_\mu^2 v^A(s, X_s, \hat{\mu}_s)(\hat{X}_s, \tilde{X}_s) d\langle \hat{X}, \tilde{X} \rangle_s \right], \end{aligned} \quad (4.2.3)$$

for $t \in [0, T]$ and where $\tilde{X}$ is a copy of $\hat{X}$ in the sense of Definition 4.1.5. One may note that this particular Itô's formula involves derivatives with respect to a measure. We refer to [82, Section 4.3.4] for a rigorous definition of these types of derivatives, and we denote by $\tilde{\mathcal{L}}$ (resp. $\tilde{\mathcal{L}}^2$) the set of Borel measurable functions from $\mathbb{R}$ (resp. $\mathbb{R}^2$) into $\mathbb{R}$, to consider the latter.

Intuitively, as in classical control theory, the Hamiltonian of the representative agent should be composed of all the drift terms appearing in the previous Itô's expansion. First, we can compute the quadratic variations and covariations between the output of the representative agent and the others, for all $t \in [0, T]$:

$$d\langle X \rangle_t = (\Sigma(\beta_t^{\mathbb{P}}) + |\sigma^\circ|^2) dt, \quad d\langle \hat{X} \rangle_t = (\Sigma(\hat{\beta}_t) + |\sigma^\circ|^2) dt, \quad d\langle X, \hat{X} \rangle_t = d\langle \hat{X}, \tilde{X} \rangle_t = |\sigma^\circ|^2 dt.$$

Using in addition the dynamics of X and $\hat{X}$, one obtain the following form for the hamiltonian,

$$H(t, x, y, p, \hat{\mu}_t, \hat{\nu}_t) := \sup_{v \in U} h(t, x, y, p, \hat{\mu}_t, \hat{\nu}_t, v), \quad (4.2.4)$$

for $(t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}$, $p := (z, z^\mu, \gamma, \gamma^\mu, \gamma^{\mu,1}, \gamma^{\mu,2}) \in \mathbb{R} \times \tilde{\mathcal{L}} \times \mathbb{R} \times \tilde{\mathcal{L}} \times \tilde{\mathcal{L}}^2 \times \tilde{\mathcal{L}}$, and computed along $(\hat{\mu}, \hat{\nu}) \in \mathcal{P}(\mathcal{C}_T) \times \tilde{\mathcal{U}}$, where in addition for $v \in U$,

$$\begin{aligned} h(t, x, y, p, \hat{\mu}_t, \hat{\nu}_t, v) := & -R_A(c(v) + f(x))y - za \cdot \mathbf{1}_d - \hat{\mathbb{E}}^{\hat{\mathbb{P}}_t}[z^\mu(\hat{X}_t)\hat{\alpha}_t \cdot \mathbf{1}_d] + \frac{1}{2}\gamma(\Sigma(b) + |\sigma^\circ|^2) \\ & + |\sigma^\circ|^2 \hat{\mathbb{E}}^{\hat{\mathbb{P}}_t}[\gamma^\mu(\hat{X}_t)] + \frac{1}{2}\hat{\mathbb{E}}^{\hat{\mathbb{P}}_t}[\gamma^{\mu,1}(\hat{X}_t)(\Sigma(\hat{\beta}_t) + |\sigma^\circ|^2)] + \frac{1}{2}|\sigma^\circ|^2 \hat{\mathbb{E}}^{\hat{\mathbb{P}}_t} \check{\mathbb{E}}^{\check{\mathbb{P}}_t}[\gamma^{\mu,2}(\hat{X}_t, \check{X}_t)]. \end{aligned}$$

Moreover, following the classical reasoning in control theory, the value function should satisfy the following Hamilton–Jacobi–Bellman (HJB) equation, for all $(t, x, \hat{\mu}) \in [0, T] \times \mathbb{R} \times \mathcal{P}(\mathcal{C}_T)$,

$$-\partial_t v^A(t, x, \hat{\mu}_t) - H(t, x, \hat{\mu}_t, v^A(t, x, \nabla v^A(t, x, \hat{\mu}_t), \nabla^2 v^A(t, x, \hat{\mu}_t), \hat{\mu}_t, \hat{\nu}_t) = 0, \quad (4.2.5)$$

where

$$\begin{aligned} \nabla v^A(t, x, \hat{\mu}_t) &:= (\partial_x v^A(t, x, \hat{\mu}_t), \partial_\mu v^A(t, x, \hat{\mu}_t)), \\ \text{and } \nabla^2 v^A(t, x, \hat{\mu}_t) &:= (\partial_{xx}^2 v^A(t, x, \hat{\mu}_t), \partial_x \partial_\mu v^A(t, x, \hat{\mu}_t), \partial_v \partial_\mu v^A(t, x, \hat{\mu}_t), \partial_\mu^2 v^A(t, x, \hat{\mu}_t)). \end{aligned}$$

4.2.2 Toward a relevant form of contract

By still considering the Markovian framework, and assuming that we can apply to the function u^A defined by (4.2.2) the chain rule with common noise under $\mathcal{C}^{1,2,2}$ -regularity, defined in [82, Theorem 4.17], we can obtain a formula similar to (4.2.3) for u^A . By computing⁵ the partial derivatives of u^A in terms of the partial derivatives of v^A , we obtain after some tedious but simple computations:

$$\begin{aligned} \xi_t = & -\frac{\ln(-V_0^A)}{R_A} + \int_0^t \partial_s u^A(s, X_s, \hat{\mu}_s) ds + \int_0^t Z_s dX_s + \int_0^t \hat{\mathbb{E}}^{\hat{\mathbb{P}}_s}[Z_s^\mu(\hat{X}_s) d\hat{X}_s] + \frac{1}{2} \int_0^t (\Gamma_s + R_A |Z_s|^2) d\langle X \rangle_s \\ & + \frac{1}{2} \int_0^t \hat{\mathbb{E}}^{\hat{\mathbb{P}}_s}[\Gamma_s^{\mu,1}(\hat{X}_s) d\langle \hat{X} \rangle_s] + \frac{1}{2} \int_0^t \hat{\mathbb{E}}^{\hat{\mathbb{P}}_s} \check{\mathbb{E}}^{\check{\mathbb{P}}_s}[(\Gamma_s^{\mu,2}(\hat{X}_s, \check{X}_s) + R_A Z_s^\mu(\hat{X}_s) Z_s^\mu(\check{X}_s)) d\langle \hat{X}, \check{X} \rangle_s] \\ & + \int_0^t \hat{\mathbb{E}}^{\hat{\mathbb{P}}_s}[(\Gamma_s^\mu(\hat{X}_s) + R_A Z_s Z_s^\mu(\hat{X}_s)) d\langle X, \hat{X} \rangle_s], \end{aligned} \quad (4.2.6)$$

where the process $(Z, Z^\mu, \Gamma, \Gamma^\mu, \Gamma^{\mu,1}, \Gamma^{\mu,2})$ takes values in $\mathbb{R} \times \tilde{\mathcal{L}} \times \mathbb{R} \times \tilde{\mathcal{L}} \times \tilde{\mathcal{L}}^2 \times \tilde{\mathcal{L}}$ and is defined by:

$$(Z_t, Z_t^\mu, \Gamma_t, \Gamma_t^\mu, \Gamma_t^{\mu,1}, \Gamma_t^{\mu,2}) := -\frac{1}{R_A V_t^A} (\partial_x v^A, \partial_\mu v^A, \partial_{xx}^2 v^A, \partial_x \partial_\mu v^A, \partial_v \partial_\mu v^A, \partial_\mu^2 v^A)(t, X_t, \hat{\mu}_t), \quad t \in [0, T].$$

Using the HJB equation (4.2.5) satisfied by v^A , we can state the HJB equation satisfied by u^A :

$$-\partial_t u^A(t, x, \hat{\mu}_t) - \tilde{H}(t, x, Z_t, Z_t^\mu, \Gamma_t, \Gamma_t^\mu, \Gamma_t^{\mu,1}, \Gamma_t^{\mu,2}, \hat{\mu}_t, \hat{\nu}_t) = 0, \quad (4.2.7)$$

where $\tilde{H}$ is a slightly modified version of the initial Hamiltonian H , see (4.2.4), more convenient when dealing with CARA utility functions, and satisfying

$$\tilde{H}(t, x, z, z^\mu, \gamma, \gamma^{\mu,1}, \gamma^{\mu,2}, \gamma^\mu, \hat{\mu}_t, \hat{\nu}_t) = \frac{1}{2} H_d(z) + \frac{1}{2} H_v(\gamma) + H_c(x, \gamma) + H_o(z^\mu, \gamma^{\mu,1}, \gamma^{\mu,2}, \gamma^\mu, \hat{\mu}_t, \hat{\nu}_t), \quad (4.2.8)$$

5. Once again, we refer to [82, Section 4.3.4] for computations rules of derivatives, but to give an idea, we have:

$$\partial_\mu u^A(t, x, \hat{\mu}_t)(\hat{x}) = -\frac{1}{R_A V_t^A} \partial_\mu v^A(t, x, \hat{\mu}_t)(\hat{x}), \text{ and } \partial_v \partial_\mu u^A(t, x, \hat{\mu}_t)(\hat{x}) = -\frac{1}{R_A V_t^A} \partial_v \partial_\mu v^A(t, x, \hat{\mu}_t)(\hat{x}).$$

for $(\hat{\mu}, \hat{\nu}) \in \mathcal{P}(\mathcal{C}_T) \times \hat{\mathcal{U}}$, $(t, x) \in [0, T] \times \mathbb{R}$, and $(z, z^\mu, \gamma, \gamma^\mu, \gamma^{\mu,1}, \gamma^{\mu,2}) \in \mathbb{R} \times \tilde{\mathcal{L}} \times \mathbb{R} \times \tilde{\mathcal{L}} \times \tilde{\mathcal{L}}^2 \times \tilde{\mathcal{L}}$, where

$$H_d(z) := -\inf_{a \in A} \{2za \cdot \mathbf{1}_d + c_\alpha(a)\}, \quad H_v(\gamma) := -\inf_{b \in B} \{c_\beta(b) - \gamma \Sigma(b)\}, \quad H_c(x, \gamma) := \frac{|\sigma^\circ|^2}{2} \gamma + f(x),$$

$$\text{and } H_o(z^\mu, \gamma^{\mu,1}, \gamma^{\mu,2}, \gamma^\mu, \hat{\mu}_t, \hat{\nu}_t) := -\hat{\mathbb{E}}^{\hat{\mathbb{P}}_t} [z^\mu(\hat{X}_t) \hat{\alpha}_t \cdot \mathbf{1}_d] + |\sigma^\circ|^2 \hat{\mathbb{E}}^{\hat{\mathbb{P}}_t} [\gamma^\mu(\hat{X}_t)] \\ + \frac{1}{2} \hat{\mathbb{E}}^{\hat{\mathbb{P}}_t} [\gamma^{\mu,1}(\hat{X}_t) (\Sigma(\hat{\beta}_t) + |\sigma^\circ|^2)] + \frac{|\sigma^\circ|^2}{2} \hat{\mathbb{E}}^{\hat{\mathbb{P}}_t} \check{\mathbb{E}}^{\check{\mathbb{P}}_t} [\gamma^{\mu,2}(\hat{X}_t, \check{X}_t)].$$

Therefore, the Hamiltonian of the representative agent in this case consists of four parts. The first three, H_d , H_v and H_c , are the classical parts for drift and volatility control, which do not depend on the efforts and the distribution of other players' states. The last part, H_o , does depend on the law and the efforts of others, and act as a constant part for the representative agent, since he cannot control it. Note that the optimisers are given by

$$a^{k,\star}(z) := \rho^k(z^- \wedge A_{\max}) \quad \text{and} \quad b^{k,\star}(\gamma) := 1 \wedge (\lambda^k \gamma^-)^{\frac{-1}{\eta^k+1}} \vee B_{\min}, \quad \text{for } k \in \{1, \dots, d\}. \quad (4.2.9)$$

We thus claim that, in our framework, the Hamiltonian of the representative agent should be somehow a path dependent version of (4.2.8), and that the relevant contracts should be of the form (4.2.6), parametrised by a process $(Z, Z^\mu, \Gamma, \Gamma^\mu, \Gamma^{\mu,1}, \Gamma^{\mu,2})$. Nevertheless, some modifications are necessary, mainly by considering a path dependent version, but also some simplifications are possible. In particular, by writing explicitly the quadratic variations of last three integrals of (4.2.6), we can show that the terms indexed by Γ^μ , $\Gamma^{\mu,1}$ and $\Gamma^{\mu,2}$ can be simplified with some part of the Hamiltonian $\tilde{H}$, and are therefore unnecessary. We provide details in the next section.

4.3 Solving the mean-field thanks to simple contracts

The intuition in the Markovian framework developed in the previous section allows us to intuit the form of revealing contracts in our framework, given below in Definition 4.3.1. In particular, the relevant form of contract is inspired by (4.2.6), adapted for a non-Markovian framework, and noticing that some simplification are possible. Therefore, starting from a contract indexed by a tuple of processes $(Z, Z^\mu, \Gamma, \Gamma^\mu, \Gamma^{\mu,1}, \Gamma^{\mu,2})$, we finally obtain that the tuple of process $\zeta := (Z, Z^\mu, \Gamma)$ should be sufficient to set up the relevant contract. This type of contracts, indexed only by ζ , then allows us to easily compute the optimal efforts of the representative agent and the associated mean-field equilibrium (see Theorem 4.3.4) whose proof is based on the theory of 2BSDEs. The main result of this chapter, mainly that the restriction to this type of so-called revealing contracts is in fact without loss of generality, is postponed to the next section (see Theorem 4.4.1).

Throughout the following, we denote for simplicity, for any positive integer n , by $\mathcal{L}^n$ the set of Borel measurable functionals from $\mathcal{C}([0, T], \mathbb{R}^n)$ into $\mathbb{R}$, and $\mathcal{L} := \mathcal{L}^1$. Indeed, instead of considering that the indexation parameter Z^μ takes values in $\tilde{\mathcal{L}}$ as in the previous section, we should now consider, in our non-Markovian framework, that for all $t \in [0, T]$, the function Z_t^μ can be applied to the paths of $\hat{X}$ until t .

4.3.1 Simple contracts

Recall that in view of (4.2.1), we expect that a contract $\mathbb{F}^{\text{obs}}$ -measurable is defined by (4.2.2), as a function of t , the path of X up to t , and $\hat{\mu}_t$, the conditional law of $\hat{X}_{t \wedge \cdot}$. Thanks to the reasoning developed in the previous section, and noticing moreover that, replacing the Hamiltonian $\tilde{H}$ by its value (see (4.2.8)) in the form of the contract (4.2.6), some simplifications are possible between parts of the Hamiltonian and terms related to the quadratic variations. In particular, the simplified Hamiltonian is given by:

$$\mathcal{H}(x, \hat{\mu}_t, z, z^\mu, \gamma, \hat{\alpha}_t) := \frac{1}{2} H_d(z) + \frac{1}{2} H_v(\gamma) + H_c(x, \gamma) - \hat{\mathbb{E}}^{\hat{\mathbb{P}}_t} [z^\mu(\hat{X}_{t \wedge \cdot}) \hat{\alpha}_t \cdot \mathbf{1}_d], \quad (4.3.1)$$

for $(x, z, z^\mu, \gamma) \in \mathbb{R} \times \mathbb{R} \times \mathcal{L} \times \mathbb{R}$ and $(\hat{\mu}, \hat{\alpha}) \in \mathcal{P}(\mathcal{C}_T) \times \hat{\mathcal{A}}$. We then obtain that the contract should only be indexed by a process $\zeta := (Z, Z^\mu, \Gamma)$, taking values in $\mathbb{R} \times \mathcal{L} \times \mathbb{R}$, and should satisfy for some $\xi_0 \in \mathbb{R}$,

$$\begin{aligned} \xi_t = & \xi_0 - \int_0^t \mathcal{H}(X_s, \hat{\mu}_s, \zeta_s, \hat{\alpha}_s) ds + \int_0^t Z_s dX_s + \int_0^t \hat{\mathbb{E}}^{\hat{\mathbb{P}}_s} \left[Z_s^\mu (\hat{X}_{s \wedge \cdot}) d\hat{X}_s \right] + \frac{1}{2} \int_0^t (\Gamma_s + R_A |Z_s|^2) d\langle X \rangle_s \\ & + \frac{1}{2} R_A \int_0^t \hat{\mathbb{E}}^{\hat{\mathbb{P}}_s} \check{\mathbb{E}}^{\check{\mathbb{P}}_s} \left[Z_s^\mu (\hat{X}_{s \wedge \cdot}) Z_s^\mu (\check{X}_{s \wedge \cdot}) d\langle \hat{X}, \check{X} \rangle_s \right] + R_A \int_0^t Z_s \hat{\mathbb{E}}^{\hat{\mathbb{P}}_s} \left[Z_s^\mu (\hat{X}_{s \wedge \cdot}) d\langle X, \hat{X} \rangle_s \right]. \end{aligned} \quad (4.3.2)$$

Therefore, $\mathcal{H}$ defined by (4.3.1) is the relevant Hamiltonian in our non-Markovian framework. One may note that $\mathcal{H}$ is a simplified version of the Hamiltonian $\tilde{H}$ defined by the equation (4.2.8). Indeed, some parts of the Hamiltonian which are not controlled by the agent simplify with some parts of the contract: the triple $(\Gamma^{\mu,1}, \Gamma^{\mu,2}, \Gamma^\mu)$ no longer appears. We thus obtain a simplified form for the contract, only indexed by $\zeta := (Z, Z^\mu, \Gamma) \in \mathbb{R} \times \mathcal{L} \times \mathbb{R}$. This control ζ will be called the triple of payment rates.

Moreover, one may note that the Hamiltonian as well as the contract for the representative agent do not depend anymore on the other agents' effort on the volatility, namely $\hat{\beta}$, but still depends on the effort on the drift of other agents, *i.e.* $\hat{\alpha}$. Therefore, the form of contracts given by (4.3.2) depends explicitly on $\hat{\alpha}$ through the Hamiltonian, while this effort is not supposed to be observable, nor contractible upon, for the principal. Hence, though the expression (4.3.2) is appealing to be the generic contract form, it is not possible to use it directly in the context of a principal-agent problem with moral hazard.

Nevertheless, we can overcome this difficulty by replacing $\hat{\alpha}$ by the optimal drift process of other agents, which has to be formally computed as the maximiser in the Hamiltonian denoted by $\hat{\alpha}^*$ and defined by (4.2.9) so that $\hat{\alpha}^{k,*}(\hat{z}) := \rho^k(\hat{z}^- \wedge A_{\max})$, $k \in \{1, \dots, d\}$, where $\hat{z}$ is the payment rate for the other agents' drift effort. Indeed, at equilibrium, each agent should consume optimally. Moreover, in our mean-field framework, the agents are identical and indistinguishable. Therefore, the principal will offer the same contract for all agents, that is the payment rate for drift effort will be the same for all the agents, since no discrimination is allowed. Hence, the optimal drift process of other agents will be $\hat{\alpha}^*(Z)$. We are thus led to consider a particular type of revealing contracts, precisely described in the definition below.

Definition 4.3.1 (Simple contracts). *For any $\mathbb{R} \times \mathcal{L} \times \mathbb{R}$ -valued $\mathbb{F}^{\text{obs}}$ -predictable process $\zeta := (Z, Z^\mu, \Gamma)$, and any $\xi_0 \in \mathbb{R}$, let us define the following process $\xi^{\xi_0, \zeta}$ for all $t \in [0, T]$ by*

$$\begin{aligned} \xi_t^{\xi_0, \zeta} := & \xi_0 - \int_0^t \mathcal{H}(X_s, \hat{\mu}_s, \zeta_s, \hat{\alpha}_s^*) ds + \int_0^t Z_s dX_s + \frac{1}{2} \int_0^t (\Gamma_s + R_A |Z_s|^2) d\langle X \rangle_s + \int_0^t \hat{\mathbb{E}}^{\hat{\mathbb{P}}_s} \left[Z_s^\mu (\hat{X}_{s \wedge \cdot}) d\hat{X}_s \right] \\ & + \frac{R_A}{2} \int_0^t \hat{\mathbb{E}}^{\hat{\mathbb{P}}_s} \check{\mathbb{E}}^{\check{\mathbb{P}}_s} \left[Z_s^\mu (\hat{X}_{s \wedge \cdot}) Z_s^\mu (\check{X}_{s \wedge \cdot}) d\langle \hat{X}, \check{X} \rangle_s \right] + R_A \int_0^t Z_s \hat{\mathbb{E}}^{\hat{\mathbb{P}}_s} \left[Z_s^\mu (\hat{X}_{s \wedge \cdot}) d\langle X, \hat{X} \rangle_s \right], \end{aligned} \quad (4.3.3)$$

where $\mathcal{H}$ is defined by (4.3.1). We let $\mathcal{V}$ the set of $\mathbb{R} \times \mathcal{L} \times \mathbb{R}$ -valued $\mathbb{F}^{\text{obs}}$ -predictable process ζ such that

$$\sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}} \left[\sup_{0 \leq t \leq T} \exp(p R_A |\xi_t^{\xi_0, \zeta}|) \right] < +\infty, \quad (\text{J}^p) \quad (4.3.4)$$

where p is the same as in Condition (IP). We call random variables of the form $(\xi_T^{\xi_0, \zeta})$, for $(\xi_0, \zeta) \in \mathbb{R} \times \mathcal{V}$, simple contracts, and denote the corresponding set by Ξ_S . Moreover, for any process $\zeta := (Z, Z^\mu, \Gamma) \in \mathcal{V}$, we denote $\bar{\zeta} = (Z, \bar{Z}^\mu, \Gamma)$, where $\bar{Z}_t^\mu = \hat{\mathbb{E}}^{\hat{\mathbb{P}}_t} [Z_t^\mu (\hat{X}_{t \wedge \cdot})]$, for $t \in [0, T]$, and we define by $\bar{\mathcal{V}}$ the corresponding set.

Remark 4.3.2. Notice that the integrability requirement in the definition of the set $\mathcal{V}$ is rather implicit. It is however clear that $\mathcal{V}$ is not empty as it contains trivially constant processes, since the drift and the volatility of X are always bounded. Besides, this is exactly the integrability we need to be able to solve the MFG for the agents given a contract in Ξ_S , as the proof of Theorem 4.3.4 below will make clear.

4.3.2 Interpretation of the form of contracts

The form of contracts given by [Definition 4.3.1](#) is mainly composed of two parts. One part is an indexation on the process controlled by the considered agent. In particular, similarly to the results in [111], the contract has a linear part in the output X and the corresponding quadratic variation $\langle X \rangle$, with linearity coefficients Z and Γ . This part corresponds to the classical contract for drift and volatility control, while the other one is an indexation on other agents' results, through the law $\hat{\mu}$.

The constant part is slightly different from the usual one in moral hazard framework. In fact, remark that we can divide the Hamiltonian as follows:

$$\mathcal{H}(X_s, \hat{\mu}_s, \zeta_s, \hat{\alpha}_s^*) = \underbrace{\frac{1}{2}(H_d(Z_s) + H_v(\Gamma_s) + 2f(X_s))}_{(i)} - \underbrace{\hat{\mathbb{E}}^{\hat{\mathbb{P}}_s}[Z_s^\mu(\hat{X}_{s\wedge\cdot})\hat{\alpha}_s^* \cdot \mathbf{1}_d]}_{(ii)} + \underbrace{\frac{1}{2}|\sigma^\circ|^2\Gamma_s}_{(iii)}.$$

The integral of (i) represents the certainty equivalent of the utility gain of the agent that can be achieved by an optimal response to the contract. This part is thus subtracted from the compensation, in agreement with the standard contract's form in usual moral hazard problems, see [11] for example. Moreover, due to the risk-aversion of the agent, the infinitesimal payment $Z_t dX_t$ must be compensated by the additional payment $\frac{1}{2}R_A|Z_t|^2 d\langle X \rangle_t$. Following the same reasoning, the integral of (ii) and the additional payment

$$\frac{1}{2}R_A|\sigma^\circ|^2 \int_0^T \left| \hat{\mathbb{E}}^{\hat{\mathbb{P}}_s}[Z_s^\mu(\hat{X}_{s\wedge\cdot})] \right|^2 ds,$$

in (4.3.3) are compensations for the infinitesimal payment $\hat{\mathbb{E}}^{\hat{\mathbb{P}}_s}[Z_s^\mu(\hat{X}_{s\wedge\cdot})d\hat{X}_s]$ indexed on others. Finally, the integral of the last term (iii) is a compensation for the covariation induced by the two infinitesimal payments.

In summary, the principal will choose the triple of controls $\zeta = (Z, Z^\mu, \Gamma)$, where the payment rates (Z, Γ) index the contract on the output of the considered agent, in agreement with standard contracts in usual principal-agent problems with drift and volatility control, while the payment rate Z^μ indexes the contract on the behaviour of other agent, represented by the conditional law $\hat{\mu}$.

In addition, one can notice that if other agents are playing according to the optimal drift effort $\hat{\alpha}^*$, some simplifications appears between the Hamiltonian and the indexation on others in the contract $\xi_T^{\xi_0, \zeta} \in \Xi_S$. In particular, recalling the notation $\bar{Z}_s^\mu = \hat{\mathbb{E}}^{\hat{\mathbb{P}}_s}[Z_s^\mu(\hat{X}_{s\wedge\cdot})]$, for $s \in [0, T]$, we can write:

$$\begin{aligned} \xi_T^{\xi_0, \zeta} &= \xi_0 - \int_0^T \mathcal{H}^\circ(X_s, Z_s, \Gamma_s) ds + \int_0^T Z_s dX_s + \frac{1}{2} \int_0^T (\Gamma_s + R_A|Z_s|^2) d\langle X \rangle_s + \sigma^\circ \int_0^T \bar{Z}_s^\mu dW_s^\circ \\ &\quad + \frac{R_A}{2} \int_0^T \hat{\mathbb{E}}^{\hat{\mathbb{P}}_s} \tilde{\mathbb{E}}^{\tilde{\mathbb{P}}_s} [Z_s^\mu(\hat{X}_{s\wedge\cdot}) Z_s^\mu(\tilde{X}_{s\wedge\cdot}) d\langle \hat{X}, \tilde{X} \rangle_s] + R_A \int_0^T Z_s \hat{\mathbb{E}}^{\hat{\mathbb{P}}_s} [Z_s^\mu(\hat{X}_{s\wedge\cdot}) d\langle X, \hat{X} \rangle_s], \end{aligned}$$

where for $(x, z, \gamma) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}$,

$$\mathcal{H}^\circ(x, z, \gamma) := \frac{1}{2}H_d(z) + \frac{1}{2}H_v(\gamma) + H_c(x, \gamma). \quad (4.3.4)$$

We can even go further in the simplifications by noticing that, given the common noise, $\hat{X}$ and $\tilde{X}$ are independent. Hence, we have, for all $s \in [0, T]$,

$$\begin{aligned} \hat{\mathbb{E}}^{\hat{\mathbb{P}}_s} \tilde{\mathbb{E}}^{\tilde{\mathbb{P}}_s} [Z_s^\mu(\hat{X}_{s\wedge\cdot}) Z_s^\mu(\tilde{X}_{s\wedge\cdot}) d\langle \hat{X}, \tilde{X} \rangle_s] &= |\sigma^\circ|^2 \hat{\mathbb{E}}^{\hat{\mathbb{P}}_s} [Z_s^\mu(\hat{X}_{s\wedge\cdot})] \tilde{\mathbb{E}}^{\tilde{\mathbb{P}}_s} [Z_s^\mu(\tilde{X}_{s\wedge\cdot})] ds = |\sigma^\circ|^2 |\bar{Z}_s^\mu|^2 ds, \\ \text{and } \hat{\mathbb{E}}^{\hat{\mathbb{P}}_s} [Z_s Z_s^\mu(\hat{X}_{s\wedge\cdot}) d\langle X, \hat{X} \rangle_s] &= |\sigma^\circ|^2 Z_s \hat{\mathbb{E}}^{\hat{\mathbb{P}}_s} [Z_s^\mu(\hat{X}_{s\wedge\cdot})] ds = |\sigma^\circ|^2 Z_s \bar{Z}_s^\mu ds. \end{aligned}$$

Therefore, we can finally write the contract as follows:

$$\begin{aligned} \xi_T^{\xi_0, \bar{\zeta}} = & \xi_0 - \int_0^T \mathcal{H}^\circ(X_s, Z_s, \Gamma_s) ds + \int_0^T Z_s dX_s + \sigma^\circ \int_0^T \bar{Z}_s^\mu dW_s^\circ + \frac{1}{2} \int_0^T (\Gamma_s + R_A |Z_s|^2) d\langle X \rangle_s \\ & + \frac{R_A}{2} |\sigma^\circ|^2 \int_0^T \bar{Z}_s^\mu (\bar{Z}_s^\mu + 2Z_s) ds. \end{aligned} \quad (4.3.5)$$

In particular, this form shows that indexing on the conditional law is actually a hidden indexing on the common noise (see [Remark 4.3.3](#) below). Moreover, it underlines the fact that if $\sigma^\circ = 0$, a simple contract in the sense of [Definition 4.3.1](#) is exactly a standard contract for drift and volatility control (see [\[11, 110, 111\]](#)). Therefore, in a similar framework but without common noise, it is straightforward to conclude that considering contracts indexed on the output of the population will not improve the results given in [\[11\]](#). We refer to the next chapter, and more precisely to [Section 5.4](#), for more details on this result.

However, the previous form of contracts cannot be offered by the Principal in the actual case, since she does not observe the common noise. Nevertheless, the previous computations will be useful in the proofs of the main results established in this section, as well as in the remark below.

Remark 4.3.3. *Until now, we have assumed that the principal cannot observe the common noise, or at least was not allowed to directly index compensations on it. Whenever she can do so, using the previous computations, we notice that the initial contract defined by (4.3.2) also satisfies the formula (4.3.5), and thus does not depend on the others' effort $\hat{\alpha}$ anymore. Moreover, the actual compensation parameter of the contract in this case is the triple $\bar{\zeta} := (Z, \bar{Z}^\mu, \Gamma) \in \bar{\mathcal{V}}$. The form (4.3.5) of contract is nothing more than a rewriting of the one given by (4.3.2). This shows that indexing on the conditional law is actually a hidden indexing on the common noise: the compensation term in the contract depending on others is rewritten as a term depending only on the common noise. Therefore, in the case where the producer is allowed to use W° , she can directly offer this type of contracts. Otherwise, if there are some regulatory reasons preventing her from using it directly in the contract, she has to offer the contract (4.3.3) in [Definition 4.3.1](#).*

This being said, when the principal uses contracts in Ξ° , we nonetheless need to add the common noise as a state variable in the value function of the representative agent. Nevertheless, using a similar reasoning, we can show that the contract is indexed on the common noise through a parameter Z° . The payment rates chosen by the principal then have to be predictable with respect to the natural filtration generated by X , W° and $\hat{\mu}$. Hence, if the principal observes the common noise, we simply have to extend the space of controls chosen by the principal. In fact, the form of optimal contracts will be the same, leading to the same effort of the agent and the same utility for the principal. We refer to [Section 4.5.2](#) for the detailed contract and the resolution of optimal contracting in this particular case.

4.3.3 Solving the mean-field game

By considering simple contracts, we are able to compute the optimal efforts of the representative agent, which were given informally by (4.2.9). Intuitively, maximising the Hamiltonian given by (4.3.1) is sufficient to obtain optimal efforts, but the formal proof relies on the theory of 2BSDEs. We will note that the agent's optimal efforts do not depend on the efforts of the others, which simplifies the task of obtaining the unique mean-field equilibrium given by [Theorem 4.3.4](#). In other words, each agent optimises his output independently of everyone else. Therefore, there is a unique mean-field equilibrium, given by the following theorem.

Theorem 4.3.4. *Given a contract $\xi_T^{\xi_0, \bar{\zeta}} \in \Xi_S$ indexed by the triple of parameters $\bar{\zeta} := (Z, \bar{Z}^\mu, \Gamma) \in \bar{\mathcal{V}}$, in the sense of [Definition 4.3.1](#), there exists a unique mean-field equilibrium in the sense of [Definition 4.1.7](#) denoted by $(\mathbb{P}^*, \mu^*)$ where*

- (i) the optimal efforts of the agent on the drift and the volatility are respectively given by the processes $\alpha^* := a^*(Z)$ and $\beta^* := b^*(\Gamma)$, where the functions $a : \mathbb{R} \rightarrow A$ and $b : \mathbb{R} \rightarrow B$ are defined component by component (i.e., for all $k \in \{1, \dots, d\}$) by (4.2.9);
- (ii) $\mathbb{P}^*$ is the law of X driven by optimal controls, i.e., with the following dynamic:

$$dX_t = -\bar{\rho}(Z_t^- \wedge A_{\max})dt + \sigma^*(\Gamma_t) \cdot dW_t + \sigma^\circ dW_t^\circ, \quad t \in [0, T], \quad \text{with } \bar{\rho} := \rho \cdot \mathbf{1}_d; \quad (4.3.6)$$

- (iii) μ^* is the conditional law of X given $\mathbb{F}^\circ$.

The proof of the previous theorem, summarised below, relies on the theory of 2BSDEs, mostly postponed to Section 4.6.1: the crux of the argument here is to use the general result of Proposition 4.6.3 and Theorem 4.6.4, and show that we can construct directly a solution to 2BSDE (4.6.2) whenever $\xi^{\xi_0, \zeta} \in \Xi_S$. Indeed, contrary to the intuition presented in the previous section, the chain rule with common noise cannot be applied in our non-Markovian framework.

Proof. We first assume that other agents are playing according $\mathbb{P}^*$, meaning in particular that their efforts are $\nu^* = (\alpha^*, \beta^*)$ and that $\hat{\mu} = \mu^*$. In this case, one can notice that the simplification (4.3.5) of the contract holds, since other agents are playing according α^* . Keeping this in mind, consider for any $S \geq 0$, the pre-image $\Sigma^{-1}(S)$ of the singleton $\{S\}$ by the map $\Sigma : B \rightarrow \mathbb{R}_+$, as well as $\Sigma(B)$ the image of B by Σ . Then, we define an $\mathbb{R}$ -valued map F for all $(x, y, z, S) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \Sigma(B)$ by

$$F(x, y, z, S) := \sup_{(a, b) \in A \times \Sigma^{-1}(S)} \left\{ -a \cdot \mathbf{1}_d z + R_A y (c(a, b) - f(x)) \right\}. \quad (4.3.7)$$

We can now define, for $t \in [0, T]$,

$$Y_t := -e^{-R_A \xi_t^{\xi_0, \zeta}}, \quad Z_t^1 := -R_A Y_t Z_t, \quad Z_t^2 := -R_A Y_t \hat{\mathbb{E}}^{\mathbb{P}^*}_t [Z_t^\mu (\hat{X}_{t \wedge \cdot})], \quad \Gamma_t := -R_A Y_t \Gamma_t,$$

and $K_t := \int_0^t \left(-R_A Y_s \mathcal{H}^\circ(X_s, \zeta_s) - \frac{1}{2} \Gamma_s (S_s + |\sigma^\circ|^2) - F(X_s, Y_s, Z_s^1, S_s) \right) ds,$

recalling that $\mathcal{H}^\circ$ is given by (4.3.4). A simple application of the usual Itô's formula on the process Y between $t \in [0, T]$ and T then leads to the following equation

$$Y_t = -e^{-R_A \xi_T} + \int_t^T F(X_s, Y_s, Z_s^1, S_s) ds - \int_t^T Z_s^1 dX_s - \sigma^\circ \int_t^T Z_s^2 dW_s^\circ + \int_t^T dK_s.$$

The goal is to show that $(Y, (Z^1, Z^2)^\top, K)$ solves 2BSDE (4.6.2), defined in Section 4.6.1, in which the link between the agent's problem and the theory of 2BSDE is detailed. First, by definition of $\mathcal{H}^\circ$ and F , we directly check that K is always a non-decreasing process, which vanishes on the support of any probability measure corresponding to the efforts β^* defined in the statement of the proposition. Indeed, the dynamic of K satisfies, for all $s \in [0, T]$,

$$dK_s = \frac{1}{2} R_A Y_s \left(\inf_{b \in B} \{c_\beta(b) - \Gamma_s \Sigma(b)\} - \inf_{b \in \Sigma^{-1}(S_s)} c_\beta(b) + \Gamma_s S_s \right) ds.$$

To ensure that $(Y, (Z^1, Z^2)^\top, K)$ solves 2BSDE (4.6.2), it therefore remains to check that all the integrability requirements in Definition 4.6.2 are satisfied, since the fixed-point constraint is satisfied by definition. The one for Y is immediate by definition of the set $\mathcal{V}$. The required integrability on $((Z^1, Z^2)^\top, K)$ then follows from Bouchard, Possamaï, Tan, and Zhou [67, Theorem 2.1 and Proposition 2.1] (2018). Therefore, starting from a contract $\xi^{\xi_0, \zeta}$, we have constructed a solution $(Y, (Z^1, Z^2)^\top, K)$ to 2BSDE (4.6.2). We can then

conclude by [Proposition 4.6.3](#) that the effort provided in the statement of the proposition is an optimal response for the representative agent, which implies that $(\mathbb{P}^*, \nu^*)$ is indeed an equilibrium.

We have therefore obtained that the candidate provided in the statement of the proposition was indeed an equilibrium. Let us now prove uniqueness. Let $\hat{\nu} = (\hat{\alpha}, \hat{\beta})$ be the arbitrary effort of other agents, and the associated conditional distribution $\hat{\mu}$. In this case, a contract $\xi^{\xi_0, \zeta} \in \Xi_S$ no longer admits the decomposition (4.3.5). Nevertheless, $\Xi_S \subset \Xi$ and by [Proposition 4.6.3](#), the optimal effort ν^* is the maximiser of the map F , and does not depend on $\hat{\nu}$. Although there is no uniqueness in general of the probability $\mathbb{P}^*$, and therefore of the effort β^* , the effort α^* is unique and is the one defined in the statement of the theorem. To sum up, given a contract in Ξ_S and for arbitrary efforts $\hat{\nu}$ of others, each agent has a unique optimal drift effort α^* , independent of $\hat{\nu}$. We can therefore already conclude that the optimal effort α^* is the same for all agents. Using the dynamic of $\hat{X}$ with α^* , the contract $\xi^{\xi_0, \zeta}$ admits the decomposition (4.3.5). Therefore, we can apply the reasoning above to construct a solution to 2BSDE (4.6.2), and we have now uniqueness of the probability $\mathbb{P}^*$, and therefore on the volatility effort β^* , given by point (ii) of the theorem. To conclude, given an arbitrary law $\hat{\mu}$ and a contract $\xi^{\xi_0, \zeta} \in \Xi_S$, the optimal effort is $\nu^* = (\alpha^*, \beta^*)$, inducing the law $\mathbb{P}^*$ and the conditional law μ^* . It is therefore the unique equilibrium. $\square$

Throughout this chapter, we will denote by $v^*(z, \gamma) := (a^*(z), b^*(\gamma))$ the optimal response of a given agent to a payment rate $(z, \gamma) \in \mathbb{R}^2$, and the cost associated to this effort will be denoted by $c^*(z, \gamma)$. To summarise this section, we provide a new form of contracts in [Definition 4.3.1](#), called simple contracts, allowing us, by [Theorem 4.3.4](#), to compute the optimal efforts of the agents and the associated unique mean-field equilibrium. The aim of the following section is to prove that there is no loss of generality to consider only simple contracts (see [Theorem 4.4.1](#)), and to solve the principal's problem thus restricted to these simple contracts.

4.4 Solving the principal's problem

We recall that the optimisation problem of the principal is given by (4.1.9), using the criterion defined by (4.1.8). First, to lighten the notations, we define for any $(\xi_0, \zeta) \in \mathbb{R} \times \mathcal{V}$ the following process

$$L_t^{\xi_0, \zeta} := \xi_t^{\xi_0, \zeta} + \int_0^t g(X_s) ds + \frac{\theta}{2} \int_0^t d\langle X \rangle_s, \quad t \in [0, T].$$

Following the general approach of [111], we expect that there is no loss of generality for the principal to restrict to contracts in Ξ_S , in the sense of [Definition 4.3.1](#), instead of Ξ . This property had been obtained before in [Élie, Mastrolia, and Possamaï \[141\] \(2018\)](#) for general moral hazard problems with a continuum of agents with mean-field interaction, but who were constrained to simply control the drift of the diffusion X . We show here that this general result also extends to cases where volatility can be controlled as well, using 2BSDEs theory. This constitutes the key result of this chapter, rigorously stated in [Theorem 4.4.1](#) below, whose proof is postponed to [Section 4.6.1](#).

Theorem 4.4.1. *The following equality holds*

$$V_0^P = \sup_{(\xi_0, \zeta) \in [U_A^{-1}(R_0), +\infty) \times \mathcal{V}} \mathbb{E}^{\mathbb{P}^*} \left[U^P \left(- \mathbb{E}^{\mathbb{P}^*} \left[L_T^{\xi_0, \zeta} \middle| \mathcal{F}_T^\circ \right] \right) \right] = \sup_{\zeta \in \mathcal{V}} \mathbb{E}^{\mathbb{P}^*} \left[U^P \left(- \mathbb{E}^{\mathbb{P}^*} \left[L_T^{U_A^{-1}(R_0), \zeta} \middle| \mathcal{F}_T^\circ \right] \right) \right].$$

In addition to establishing that the restriction to contracts in Ξ_S is without loss of generality, the previous theorem also ensures that the agents' participation constraint is saturated. Indeed, the constant ξ_0 is chosen equal to $U_A^{-1}(R_0)$, so that $V_0^A(\xi^{\xi_0, \zeta}, \mu^*) = R_0$. To lighten the notations, the exponent $U_A^{-1}(R_0)$

will be omitted, since we consider that $R_0 < 0$ is fixed. Notice that from the principal's point of view, when considering contracts in Ξ_S , and when the agents are at the unique equilibrium, the deviation of other agents $\hat{X}$ is nothing but a copy of X , in the sense of [Definition 4.1.5](#), which we denoted by $\tilde{X}$. Besides, noticing then that the distributions $\hat{\mathbb{P}}$ and $\mathbb{P}^*$ coincide, in particular that $\hat{\mu} = \mu^*$, and imposing $\xi_0 := U_A^{-1}(R_0)$ from now on, one obtain:

$$\begin{aligned} \xi_t^\zeta = & \xi_0 - \int_0^t \mathcal{H}(X_s, \mu_s^*, \zeta_s, \alpha_s^*) ds + \int_0^t Z_s dX_s + \int_0^t \tilde{\mathbb{E}}^{\mathbb{P}^*}_s [Z_s^\mu(\tilde{X}_s) d\tilde{X}_s] + \frac{1}{2} \int_0^t (\Gamma_s + R_A |Z_s|^2) d\langle X \rangle_s \\ & + \frac{1}{2} R_A |\sigma^\circ|^2 \int_0^t \left| \tilde{\mathbb{E}}^{\mathbb{P}^*}_s [Z_s^\mu(\tilde{X}_s)] \right|^2 ds + R_A |\sigma^\circ|^2 \int_0^t Z_s \tilde{\mathbb{E}}^{\mathbb{P}^*}_s [Z_s^\mu(\tilde{X}_s)] ds. \end{aligned}$$

Given the form of the principal's control problem and of the contract, the intuition is that her value function should depend only on time and on the conditional law μ^Y of the state variable $Y = (X, L)^\top$.

4.4.1 The general case

The attentive reader have noticed that the right-hand side of the equation in [Theorem 4.4.1](#) looks like the value function of a stochastic control problem of a McKean–Vlasov SDE with common noise. However, one of the two state variables, mainly L , seems to be considered in the strong formulation (it is indexed by the control ζ), while the other state variable X is considered in weak formulation (the control ζ only impacts the distribution of X through $\mathbb{P}^*$). As highlighted by [Cvitanic and Zhang \[106, Remark 5.1.3\] \(2012\)](#), it makes little sense to consider a control problem of this form directly. Therefore, from our point of view, there is no reason why we should adopt anything but the weak formulation to state the problem of the principal, contrary to what is usually done in principal–agent problems (see, e.g., [\[111\]](#)), since this is the one which makes sense for the problem of the agent.⁶ We thus formulate it below.

To detail the weak formulation, we consider the sets $\mathbb{V}$ and $\mathbb{V}_0$, defined from the set $V := \mathbb{R}^3$ thanks to [General notations \(p. xv\)](#). The intuition is that the principal's problem depends only on time and on the conditional law μ^Y of the state variable $Y = (X, L)^\top$. Following the same methodology used for the agent's problem, we are led to consider the following canonical space:

$$\Omega^P := \Omega^\circ \times \Omega^{P,1} \times \Omega^{P,2} \times \mathbb{V}, \text{ where } \Omega^{P,1} := \mathcal{C}([0, T], \mathbb{R}^2 \times \mathbb{R}^d), \text{ and } \Omega^{P,2} := \mathcal{P}_T(\mathbb{R}^2),$$

with canonical process $(W^\circ, Y, W, \mu^Y, \Pi^P)$, where for any $t \in [0, T]$ and $\omega := (t, w^\circ, y, w, u, q) \in \Omega^P$

$$W_t^\circ(\omega) := w^\circ(t), Y_t(\omega) := y(t), W_t(\omega) := w(t), \mu_t^Y(\omega) := u(t), \Pi_t^P(\omega) := q.$$

Less formally, for all $t \in [0, T]$, $\mu_t^Y \in \mathcal{P}(\mathbb{R}^2)$ will be the conditional distribution of $Y_t = (X_t, L_t)$, and we will denote by μ^X and μ^L the marginal distributions of μ^Y . When no confusion is possible, in order to lighten the notations, we will often omit the space for the integrals with respect to the conditional distribution, by denoting for example:

$$\int \phi(x, \ell) \mu^Y(dx, d\ell) := \int_{\mathbb{R}^2} \phi(x, \ell) \mu^Y(dx, d\ell), \text{ for any } \phi : \mathbb{R}^2 \longrightarrow \mathbb{R}.$$

The canonical filtration $\mathbb{F}^P := (\mathcal{F}_t^P)_{t \in [0, T]}$ is defined as

$$\mathcal{F}_t^P := \sigma\left((W_s^\circ, Y_s, W_s, \mu_s^Y, \Delta_s(\varphi)) : (s, \varphi) \in [0, t] \times \mathcal{C}_b([0, T] \times \mathbb{R}^3, \mathbb{R})\right), t \in [0, T],$$

6. Notice that at the end of the day, this is not really an issue. Indeed, provided that the problem has enough regularity (typically some semi-continuity of the terminal and running reward with respect to state), one can expect the strong and weak formulations to coincide. This has recently been proven in a setting with common noise by [Djete, Possamaï et Tan](#) in [\[126, 127\]](#).

where for any $(s, \varphi) \in [0, T] \times \mathcal{C}_b([0, T] \times \mathbb{R}^3, \mathbb{R})$, $\Delta_s(\varphi) := \int_0^s \int_{\mathbb{R}^3} \varphi(r, \bar{v}) \Pi^P(dr, d\bar{v})$. We also define by $\mathbb{F}^{P, \circ} := (\mathcal{F}_t^{P, \circ})_{t \in [0, T]}$, a smaller filtration containing only the information generated by the common noise and the conditional law of Y , i.e., $\mathcal{F}_t^{P, \circ} := \sigma((W_s^\circ, \mu_s^Y) : s \in [0, t])$, $t \in [0, T]$. Then, for any $(s, \varphi) \in [0, T] \times \mathcal{C}_b^2(\mathbb{R}^2 \times \mathbb{R}^d \times \mathbb{R}, \mathbb{R})$, we set

$$M_s^P(\varphi) := \varphi(Y_s, W_s, W_s^\circ) - \int_0^s \int_U \left(A_P^\top(\bar{v}) \nabla \varphi(Y_r, W_r, W_r^\circ) + \frac{1}{2} \text{Tr}[\nabla^2 \varphi(Y_r, W_r, W_r^\circ) (B_P B_P^\top)(\bar{v})] \right) \Pi^P(dr, d\bar{v}),$$

where A_P and B_P are respectively the drift vector and the diffusion matrix of the vector process $(Y, W, W^\circ)^\top$:

$$A_P(\bar{v}) := \begin{pmatrix} -\bar{\rho}(z^- \wedge A_{\max}) \\ b(X_t, \bar{v}) \\ \mathbf{0}_d \\ 0 \end{pmatrix}, \quad B_P(\bar{v}) := \begin{pmatrix} 0 & 0 & \sigma^\star(\gamma)^\top & \sigma^\circ \\ 0 & 0 & z\sigma^\star(\gamma)^\top & (z + \bar{z}^\mu)\sigma^\circ \\ \mathbf{0}_d & \mathbf{0}_d & \text{Id} & \mathbf{0}_d \\ 0 & 0 & \mathbf{0}_d^\top & 1 \end{pmatrix}, \quad (4.4.1)$$

for $\bar{v} := (z, \bar{z}^\mu, \gamma) \in \mathbb{R}^3$ and recalling that $\bar{\rho} := \rho \cdot \mathbf{1}_d$, where in addition for $x \in \mathbb{R}$,

$$b(x, \bar{v}) := c^\star(z, \gamma) + g(x) - f(x) + \frac{1}{2} R_A |z|^2 \Sigma^\star(\gamma) + \frac{1}{2} R_A |\sigma^\circ|^2 |z + \bar{z}^\mu|^2 + \frac{\theta}{2} (\Sigma^\star(\gamma) + |\sigma^\circ|^2).$$

In addition, we fix some initial condition, namely a probability measure ϱ^P on $\mathbb{R}^2$ for the law of Y at 0.

Definition 4.4.2. Let $\mathbb{M}^P$ be the set of all probability measures on $(\Omega^P, \mathcal{F}_T^P)$. The subset $\mathcal{Q} \subset \mathbb{M}^P$ is composed of all $\mathbb{P}$ such that

- (i) $M^P(\varphi)$ is a $(\mathbb{P}, \mathbb{F}^P)$ -local martingale on $[0, T]$ for all $\varphi \in \mathcal{C}_b^2(\mathbb{R}^2 \times \mathbb{R}^d \times \mathbb{R}, \mathbb{R})$;
- (ii) $\mathbb{P} \circ (Y_0)^{-1} = \varrho^P$, and $\mathbb{P} \circ ((W_0, W_0^\circ))^{-1} = \nu$;
- (iii) $\mathbb{P}[\Pi^P \in \mathbb{V}_0] = 1$;
- (iv) for $\mathbb{P}$ -a.e. $\omega \in \Omega^P$ and for every $t \in [0, T]$, we have $\mu_t^Y(\omega) = \mathbb{P}_t^\omega \circ (Y_t)^{-1}$, where $(\mathbb{P}_t^\omega)_{\omega \in \Omega}$ is a family of r.c.p.d. for $\mathbb{P}$ given $\mathcal{F}_t^{P, \circ}$. Similarly as before, we will denote by $\mathbb{E}^{\mathbb{P}_t}$ the conditional expectation under the r.c.p.d. $\mathbb{P}_t^\omega$;
- (v) (W°, μ^Y) is $\mathbb{P}$ -independent of W .

Remark 4.4.3. One may notice that the previous definition, in particular the point (iv), does not involve a probability measure on the path space anymore, contrary to [Definition 4.1.2](#), and as noticed in [Remark 4.1.8](#). Indeed, the form of contracts we are considering makes the principal's problem Markovian in that sense.

Following the reasoning developed in [Section 4.1.2](#), we can construct a copy of the canonical space Ω^P and a copy of Y in the sense of [Definitions 4.1.4](#) and [4.1.5](#). Thanks to the previous formulation, we can write the weak formulation of the principal's problem as follows

$$V_0^P = \sup_{\mathbb{P} \in \mathcal{Q}} \mathbb{E}^{\mathbb{P}} \left[U^P \left(-\mathbb{E}^{\mathbb{P}_T} \left[L_T^{\bar{\mathbb{P}}} \right] \right) \right], \quad (4.4.2)$$

where, given some $\mathbb{P} \in \mathcal{Q}$, the notation $\mathbb{E}^{\mathbb{P}_t}$ will refer to the conditional expectation under the r.c.p.d. $\mathbb{P}_t$ of some $\mathbb{P} \in \mathcal{Q}$ given $\mathcal{F}_t^{P, \circ}$ for all $t \in [0, T]$, in the sense of [Definition 4.4.2](#) (iv).

Remark 4.4.4. First, recall that simple contracts were defined as random variables of the form $\xi_T^{\xi_0, \zeta}$, for $(\xi_0, \zeta) \in \mathbb{R} \times \mathcal{V}$. By [\(4.4.1\)](#), we notice that the drift vector and the diffusion matrix of the process $(Y, W, W^\circ)^\top$ are defined as function of $\bar{v} = (z, \bar{z}^\mu, \gamma)$. This is why we consider that the principal controls, through a

probability $\mathbb{P} \in \mathcal{Q}$, the triple of controls $\zeta^{\mathbb{P}} = (Z, \bar{Z}^\mu, \Gamma) \in \bar{\mathcal{V}}$, where for all $t \in [0, T]$, $\bar{Z}_t^\mu = \tilde{\mathbb{E}}^{\mathbb{P}_t}[Z_t^\mu(\tilde{X}_t)]$, instead of $\zeta^{\mathbb{P}} \in \mathcal{V}$. Moreover, the conditional laws of X and L do not impact their dynamics, therefore the principal's problem does not seem to be a McKean–Vlasov control problem, but only a standard control problem. Nevertheless, the principal's criterion reveals the conditional law of L , which transforms the problem into a McKean–Vlasov one.

In order to apply the chain rule with common noise to functions depending on time and conditional distribution, we define the regularity assumption needed.

Definition 4.4.5 ($\mathcal{C}^{1,2}$ -regularity). *A function $u : [0, T] \times \mathcal{P}(\mathbb{R}^d) \rightarrow \mathbb{R}$, $(t, \mu) \mapsto u(t, \mu)$ is smooth enough in the sense of Chain Rule under $\mathcal{C}^{1,2}$ -regularity if*

- (i) *u is differentiable with respect to t , and the partial derivative $\partial_t u : [0, T] \times \mathcal{P}(\mathbb{R}^d) \rightarrow \mathbb{R}$ is continuous;*
- (ii) *for all $t \in [0, T]$, the mapping $\mu \in \mathcal{P}(\mathbb{R}^d) \mapsto u(t, \mu)$ is simply $\mathcal{C}^2$, in the sense defined in [82, Section 4.3.2], and satisfies the assumptions of [82, Theorem 4.14].*

The previous definition allows us to consider the natural extension for time dependent functions of the chain rule under $\mathcal{C}^2$ -regularity defined in [82, Theorem 4.14]. Thus, for any function $v : [0, T] \times \mathcal{P}(\mathbb{R}^2) \rightarrow \mathbb{R}$ smooth enough in the sense of Definition 4.4.5, the chain rule under $\mathcal{C}^{1,2}$ -regularity can be written as

$$\begin{aligned} dv(t, \mu_t^Y) &= \partial_t v(t, \mu_t^Y) dt + \mathbb{E}^{\mathbb{P}_t}[\partial_\mu v(t, \mu_t^Y)(Y_t) \cdot dY_t] + \frac{1}{2} \mathbb{E}^{\mathbb{P}_t} \tilde{\mathbb{E}}^{\mathbb{P}_t} \left[\text{Tr}[\partial_\mu^2 v(t, \mu_t^Y)(Y_t, \tilde{Y}_t) d\langle Y, \tilde{Y} \rangle_t] \right] \\ &\quad + \frac{1}{2} \mathbb{E}^{\mathbb{P}_t} \left[\text{Tr}[\partial_y \partial_\mu v(t, \mu_t^Y)(Y_t) d\langle Y \rangle_t] \right], \end{aligned}$$

where $\tilde{Y}$ is a copy of Y in the sense of Definition 4.1.5 and $\mathbb{E}^{\mathbb{P}_t}$ must be understood according to Definition 4.4.2 (iv).

Remark 4.4.6. *Notice that another way to obtain this chain rule is to simplify the chain rule in [82, Theorem 4.17] by considering a function which does not depend on the state process.*

We are thus led to consider the following HJB equation, written on the space of measures:

$$\begin{aligned} 0 &= \frac{|\sigma^\circ|^2}{2} \iint \partial_{\mu^x} v(t, \mu_t^Y)(y, \tilde{y}) \mu_t^Y(dy) \mu_t^Y(d\tilde{y}) + \frac{1}{2} \sup_{\bar{v} \in \mathbb{R}^3} h(\mu_t^Y, \partial_\mu v(t, \mu_t^Y), \partial_y \partial_\mu v(t, \mu_t^Y), \partial_\mu^2 v(t, \mu_t^Y), \bar{v}) \\ &\quad + \partial_t v(t, \mu_t^Y) + \int \left(\partial_{\mu^L} v(t, \mu_t^Y)(y) \left(\frac{\theta}{2} |\sigma^\circ|^2 + (g - f)(x) \right) + \frac{1}{2} |\sigma^\circ|^2 \partial_x \partial_{\mu^x} v(t, \mu_t^Y)(y) \right) \mu_t^Y(dy), \end{aligned} \quad (4.4.3)$$

with terminal condition $v(T, \mu_T^Y) = U^P(-\mathbb{E}^{\mu_T^L}[L_T])$ and where, for $(v_\mu, v_{y,\mu}, v_{\mu,\mu}) \in \mathcal{L}^{1 \times 2} \times \mathcal{L}^{2 \times 2} \times (\mathcal{L}^2)^{2 \times 2}$,

$$\begin{aligned} h(\mu, v_\mu, v_{y,\mu}, v_{\mu,\mu}, \bar{v}) &:= -2\bar{\rho}(z^- \wedge A_{\max}) \int v_\mu^1(y) \mu(dy) + 2c^*(z, \gamma) \int v_\mu^2(y) \mu(dy) \\ &\quad + \Sigma^*(\gamma) \int \left((\theta + R_A |z|^2) v_\mu^2(y) + v_{y,\mu}^{1,1}(y) + |z|^2 v_{y,\mu}^{2,2}(y) + 2z v_{y,\mu}^{1,2}(y) \right) (y) \mu(dy) \\ &\quad + |\sigma^\circ|^2 |z + \bar{z}^\mu|^2 \int \left(R_A v_\mu^2(y) + v_{y,\mu}^{2,2}(y) + \int v_{\mu,\mu}^{2,2}(y, \tilde{y}) \mu(d\tilde{y}) \right) \mu(dy) \\ &\quad + 2|\sigma^\circ|^2 (z + \bar{z}^\mu) \int (v_{\mu,\mu}^{1,2}(y, \tilde{y}) \mu(d\tilde{y}) + v_{y,\mu}^{1,2}(y)) \mu(dy), \end{aligned} \quad (4.4.4)$$

Theorem 4.4.7. *Assume that there is a solution v to PDE (4.4.3), smooth enough in the sense of Definition 4.4.5, with partial derivatives satisfying for each $\mathbb{P} \in \mathcal{Q}$*

$$\mathbb{E}^{\mathbb{P}} \left[\left(\int_0^T \left| \int \partial_{\mu^x} v(t, \mu_t^Y)(y) \mu_t^Y(dy) \right|^2 dt \right)^{1/2} + \sup_{0 \leq t \leq T} \left| \int \partial_{\mu^L} v(t, \mu_t^Y)(y) \mu_t^Y(dy) \right|^{\frac{p'}{p'-1}} \right] < +\infty, \quad (4.4.5)$$

where $p' > 1$ is the exponent appearing in Lemma 4.6.5, and a function $\bar{v}^* : [0, T] \times \mathcal{P}(\mathbb{R}^2) \rightarrow \mathbb{R}^3$ satisfying

$$h(\mu_t^Y, \partial_\mu v(t, \mu_t^Y), \partial_y \partial_\mu v(t, \mu_t^Y), \partial_\mu^2 v(t, \mu_t^Y), \bar{v}^*(t, \mu_t^Y)) = \sup_{\bar{v} \in \mathbb{R}^3} h(\mu_t^Y, \partial_\mu v(t, \mu_t^Y), \partial_y \partial_\mu v(t, \mu_t^Y), \partial_\mu^2 v(t, \mu_t^Y), \bar{v}).$$

Then, $v(0, \mu_0^Y) = V_0^P$ and the process $\bar{\zeta}^*$ defined for all $t \in [0, T]$ by $\bar{\zeta}_t^* := \bar{v}^*(t, \mu_t^Y)$ is an optimal triple of parameters for the contract.

The result is quite similar to those found in the literature on McKean–Vlasov problems, see, e.g., the works by Bensoussan, Frehse et Yam in [50] (2013), [52] (2015), and [53] (2017); by Pham et Wei in [279] (2017) and [280] (2018); as well as by Bayraktar, Cosso, and Pham [44] (2018), or finally by Djete, Possamaï, and Tan [126] (2019) for the most recent results. We refer to Section 4.6 for the proof of this proposition.

In the following two sections, we will specify the utility function U^P to consider two different cases:

- (i) a risk-averse principal, with a CARA utility function and a risk-aversion parameter $R_P > 0$ in Section 4.4.2;
- (ii) a risk-neutral principal, with $U^P(x) = x$, in Section 4.4.3.

All the functions defined in the first case will be indexed by "P", to make the dependency explicit on R_P : for example, V^P will denote the value of the principal's problem. For the sake of consistency, they will be indexed by 0 in the second case, meaning that, informally, it is sufficient to set $R_P = 0$ in a function defined in the risk-averse case to obtain the function in risk-neutral case.

4.4.2 Principal with CARA utility

Under a constant relative risk aversion specification of the utility function of the principal, that is $U^P(x) := -e^{-R_P x}$, for some risk-aversion $R_P > 0$, we are looking for a solution v^P of (4.4.3) with the form

$$v^P(t, \mu_t^Y) = -e^{R_P(\mathbb{E}^{\mathbb{P}^t}[L_t] - u^P(t, \mu_t^X))}, \text{ with } u^P(T, \mu_T^X) = 0. \quad (4.4.6)$$

To lighten to notations, we will denote, for all functions $u : [0, T] \times \mathcal{P}(\mathbb{R}) \rightarrow \mathbb{R}$,

$$\begin{aligned} \bar{u}_{\mu^X}(t, \mu) &:= \int \partial_{\mu^X} u(t, \mu)(x) \mu(dx), \quad \bar{u}_{x, \mu^X}(t, \mu) := \int \partial_x \partial_{\mu^X} u(t, \mu)(x) \mu(dx), \\ \text{and } \bar{u}_{\mu^X, \mu^X}(t, \mu) &:= \iint \partial_{\mu^X}^2 u^P(t, \mu)(x, \tilde{x}) \mu(dx) \mu(d\tilde{x}), \quad \text{for } (t, \mu) \in [0, T] \times \mathcal{P}(\mathbb{R}). \end{aligned}$$

We also define the following risk-aversion ratio $\bar{R}$ as $1/\bar{R} := 1/R_A + 1/R_P$, with the convention $\bar{R} = 0$ if R_A or R_P is equal to zero. In this case, we are led to solve a simplified HJB equation for the function u^P :

$$\begin{aligned} 0 = & -\partial_t u^P(t, \mu_t^X) + \int (g - f)(x) \mu_t^X(dx) + \frac{\theta}{2} |\sigma^\circ|^2 - \frac{1}{2} |\sigma^\circ|^2 \bar{u}_{x, \mu^X}^P(t, \mu_t^X) + \frac{1}{2} (|\sigma^\circ|^2 \bar{R} - \bar{\rho}) |\bar{u}_{\mu^X}^P(t, \mu_t^X)|^2 \\ & - \frac{1}{2} |\sigma^\circ|^2 \bar{u}_{\mu^X, \mu^X}^P(t, \mu_t^X) + \frac{1}{2} \inf_{z \in \mathbb{R}} \left\{ F_0(q(z, \bar{u}_{x, \mu^X}^P(t, \mu_t^X))) + \bar{\rho} |z^- \wedge A_{\max}| + \bar{u}_{\mu^X}^P(t, \mu_t^X) \right\}, \end{aligned} \quad (4.4.7)$$

with terminal condition $u^P(t, \mu_t^X) = 0$, and where

$$q(z, u) := \theta + R_A |z|^2 - u \quad \text{and} \quad F_0(q) := \inf_{\gamma < 0} \{ \Sigma^*(\gamma) q + c_\beta^*(\gamma) \}.$$

In order to apply Theorem 4.4.7, we however need to ensure that Condition (4.4.5) holds for the function v^P . Therefore, we assume that there exists some $\bar{p} > \frac{p'}{p'-1}$ such that the following technical condition on the integrability of the contracts is enforced

$$\mathbb{E}^{\mathbb{P}} \left[\sup_{0 \leq t \leq T} e^{\bar{p} R_P \mathbb{E}^{\mathbb{P}} [\xi_t^\zeta | \mathcal{F}_t^\circ]} \right] < +\infty. \quad (\text{CARA})$$

We will make use of the notation $\varepsilon := \sqrt{\bar{p}(p' - 1)/p'}$. Under this assumption on contracts, we derive from [Theorem 4.4.7](#) the following verification result, whose proof is postponed to [Section 4.6](#).

Proposition 4.4.8. *Let u be a solution to PDE (4.4.7), smooth enough in the sense of [Definition 4.4.5](#) and satisfying*

$$\mathbb{E}^{\mathbb{P}} \left[\left(\int_0^T |\bar{u}_{\mu^X}(t, \mu_t^X)|^2 dt \right)^{\frac{p'}{2}} + \sup_{0 \leq t \leq T} \exp \left(- \frac{q' p'}{p' - 1} R_P u(t, \mu_t^X) \right) \right] < +\infty, \quad (4.4.8)$$

for $q' = \varepsilon/(\varepsilon - 1)$. Moreover, let $\bar{v}^* : [0, T] \times \mathcal{P}(\mathbb{R}) \rightarrow \mathbb{R}^3$ be a function satisfying

$$h^P(\bar{u}_{\mu^X}(t, \mu_t^X), \bar{u}_{x, \mu^X}(t, \mu_t^X), \bar{v}^*(t, \mu_t^X)) = \inf_{\bar{v} \in \mathbb{R}^3} h^P(\bar{u}_{\mu^X}(t, \mu_t^X), \bar{u}_{x, \mu^X}(t, \mu_t^X), \bar{v}),$$

where, for any $(u_1, u_2, v) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^3$,

$$\begin{aligned} h^P(u_1, u_2, \bar{v}) := & \Sigma^*(\gamma)(h + R_A |z|^2 - u_2) + c_\beta^*(\gamma) + c_\alpha^*(z) + 2\bar{\rho}(z^- \wedge A_{\max})u_1 \\ & + |\sigma^\circ|^2(R_A + R_P)|z + \bar{z}^\mu|^2 - 2R_P|\sigma^\circ|^2(z + \bar{z}^\mu)u_1. \end{aligned} \quad (4.4.9)$$

Then, the following holds:

- (i) $V_0^P = -e^{R_P(\xi_0 - u(0, \mu_0^X))}$;
- (ii) the optimal payment rate Z^* is defined by $Z_t^* = z^*(t, \mu_t^X)$, $t \in [0, T]$, where the function z^* is solution of the minimisation problem in (4.4.7) and satisfies for $(t, \mu) \in [0, T] \times \mathcal{P}(\mathbb{R})$,

$$\begin{cases} z^*(t, \mu) = 0, & \text{when } \bar{u}_{\mu^X}(t, \mu) \geq 0, \\ z^*(t, \mu) \in [\bar{u}_{\mu^X}(t, \mu) \vee -A_{\max}, 0], & \text{when } \bar{u}_{\mu^X}(t, \mu) \leq 0; \end{cases} \quad (4.4.10)$$

- (iii) the optimal payment rate $\bar{Z}^{\mu, *}$ is defined for all $t \in [0, T]$ by $\bar{Z}_t^{\mu, *} = \bar{z}^{\mu, *}(t, \mu_t^X)$ where

$$\bar{z}^{\mu, *}(t, \mu) := -z^*(t, \mu) + \frac{R_P}{R_A + R_P} \bar{u}_{\mu^X}(t, \mu), \quad (t, \mu) \in [0, T] \times \mathcal{P}(\mathbb{R}),$$

and since $\bar{Z}_s^{\mu, *} = \tilde{\mathbb{E}}^{\mathbb{P}_s}[Z_s^{\mu, *}(\tilde{X}_s)]$, we can arbitrary set that $z^{\mu, *}(t, \mu)(x) \equiv \bar{z}^{\mu, *}(t, \mu)$ for all $x \in \mathbb{R}$;

- (iv) the optimal payment rate Γ^* is defined by $\Gamma_t^* = \gamma^*(t, \mu_t^X)$, $t \in [0, T]$, where, in addition for $\mu \in \mathcal{P}(\mathbb{R})$,

$$\gamma^*(t, \mu) := -\max \left\{ \theta - \bar{u}_{x, \mu^X}(t, \mu) + R_A |z^*(t, \mu)|^2, 1/\bar{\lambda} \right\}, \quad \text{where } \bar{\lambda} = \max_{k \in \{1, \dots, d\}} \lambda^k; \quad (4.4.11)$$

- (v) the second-best optimal contract is equal to

$$\begin{aligned} \xi = & \xi_0 - \int_0^T \mathcal{H}(X_s, \mu_s^X, \bar{\zeta}_s^*, \alpha_s^*) ds + \int_0^T Z_s^* (dX_s - \tilde{\mathbb{E}}^{\mathbb{P}_s}[d\tilde{X}_s]) + \frac{R_P}{R_A + R_P} \int_0^T \bar{u}_{\mu^X}(s, \mu_s^X) \tilde{\mathbb{E}}^{\mathbb{P}_s}[d\tilde{X}_s] \\ & + \frac{1}{2} \int_0^T (\Gamma_s^* + R_A |Z_s^*|^2) d\langle X \rangle_s + \frac{|\bar{R}|^2}{2R_A} |\sigma^\circ|^2 \int_0^T |\bar{u}_{\mu^X}(s, \mu_s^X)|^2 ds - \frac{R_A}{2} |\sigma^\circ|^2 \int_0^T |Z_s^*|^2 ds, \end{aligned}$$

where $\mathcal{H}$ was defined above in (4.3.1).

Remark 4.4.9. In the case where the agents are risk-neutral ($R_A = 0$), PDE (4.4.7) reduces to

$$\begin{aligned} 0 = & -\partial_t u^P(t, \mu_t^X) + \int (g - f)(x) \mu_t^X(dx) + \frac{\theta}{2} |\sigma^\circ|^2 - \frac{|\sigma^\circ|^2}{2} \bar{u}_{x, \mu^X}^P(t, \mu_t^X) - \frac{\bar{\rho}}{2} |\bar{u}_{\mu^X}^P(t, \mu_t^X)|^2 \\ & - \frac{|\sigma^\circ|^2}{2} \bar{u}_{\mu^X, \mu^X}^P(t, \mu_t^X) + \frac{1}{2} F_0(\theta - \bar{u}_{x, \mu^X}^P(t, \mu_t^X)) + \frac{1}{2} \inf_{z \in \mathbb{R}} \left\{ \bar{\rho} |z^- \wedge A_{\max}| + \bar{u}_{\mu^X}^P(t, \mu_t^X) \right\}^2. \end{aligned}$$

Noticing that the infimum is attained for all $t \in [0, T]$ at $Z_t^* = \bar{u}_{\mu^X}^P(t, \mu_t^X)$, the optimal payment rate Γ^* to induce a reduction of the volatility of the output is

$$\Gamma_t^* = -\max \left\{ \theta - \bar{u}_{x, \mu^X}^P(t, \mu_t^X), 1/\bar{\lambda} \right\},$$

and the optimal payment rate $\bar{Z}^{\mu, \star}$ is given by

$$\bar{Z}_t^{\mu, \star} = -Z_t^* + \bar{u}_{\mu^X}^P(t, \mu_t^X) = 0 \text{ for all } t \in [0, T].$$

Hence, the certainty equivalent of the principal's utility and the resulting optimal contract does not depend on the principal's risk aversion R_P . Moreover, since $\bar{Z}^{\mu, \star} = 0$, the contract is not indexed on the law of other agents. This result is particularly interesting since the contract has the classical form (in the sense of [111]), but extended to a model with common noise. Furthermore, note that unlike the literature on principal-agent problems with only one agent, the case $R_A = 0$ does not coincide with the first-best problem (see Section 4.5.1).

4.4.3 Risk-neutral principal

The value function of a risk-neutral principal is defined as $V_0^0 = \sup_{\mathbb{P} \in \mathcal{Q}} \mathbb{E}^{\mathbb{P}}[-\mathbb{E}^{\mathbb{P}_T}[L_T^{\bar{\zeta}^{\mathbb{P}}}]]$. Under this specification for U^P , we are looking for a solution v^0 of (4.4.3) with the form

$$v^0(t, \mu_t^Y) = -\mathbb{E}^{\mathbb{P}_t}[L_t^{\bar{\zeta}}] + u^0(t, \mu_t^X), \text{ with } u^0(T, \mu_T^X) = 0. \quad (4.4.12)$$

By adapting the reasoning of Proposition 4.4.8, we obtain the following HJB equation associated to u^0 :

$$\begin{aligned} 0 = & -\partial_t u^0(t, \mu_t^X) + \int (g - f)(x) \mu_t^X(dx) + \frac{\theta}{2} |\sigma^\circ|^2 - \frac{1}{2} |\sigma^\circ|^2 \bar{u}_{x, \mu^X}^0(t, \mu_t^X) - \frac{1}{2} \bar{\rho} |\bar{u}_{\mu^X}^0(t, \mu_t^X)|^2 \\ & - \frac{1}{2} |\sigma^\circ|^2 \bar{u}_{\mu^X, \mu^X}^0(t, \mu_t^X) + \frac{1}{2} \inf_{z \in \mathbb{R}} \left\{ F_0(q(z, \bar{u}_{x, \mu^X}^0(t, \mu_t^X))) + \bar{\rho} |(z^- \wedge A_{\max}) + \bar{u}_{\mu^X}^0(t, \mu_t^X)|^2 \right\}, \end{aligned} \quad (4.4.13)$$

with terminal condition $u^0(T, \mu_T^X) = 0$. Intuitively, this result can be found by setting $R_P = 0$ in (4.4.7). Similarly, we can deduce the following result from Proposition 4.4.8 with $R_P = 0$.

Proposition 4.4.10. *Assume that there exists a solution u to the PDE (4.4.13), smooth enough in the sense of Definition 4.4.5, such that the following condition is satisfied*

$$\mathbb{E}^{\mathbb{P}} \left[\left(\int_0^T |\bar{u}_{\mu^X}(t, \mu_t^X)|^2 dt \right)^{1/2} \right] < +\infty, \quad (4.4.14)$$

and a function $\bar{v}^* : [0, T] \times \mathcal{P}(\mathbb{R}) \longrightarrow \mathbb{R}^3$ satisfying

$$h^0(\bar{u}_{\mu^X}(t, \mu_t^X), \bar{u}_{x, \mu^X}(t, \mu_t^X), \bar{v}^*(t, \mu_t^X)) = \inf_{\bar{v} \in \mathbb{R}^3} h^0(\bar{u}_{\mu^X}(t, \mu_t^X), \bar{u}_{x, \mu^X}(t, \mu_t^X), \bar{v}).$$

Then, $V_0^0 = -\xi_0 + u(0, \mu_0^X)$ and the optimal payment rate $\bar{\zeta}^* := (Z^*, \bar{Z}^{\mu, \star}, \Gamma^*)$ is given for all $t \in [0, T]$ by $\bar{\zeta}_t^* := (Z_t^*, \bar{Z}_t^{\mu, \star}, \Gamma_t^*) = (z^*(t, \mu_t^X), -z^*(t, \mu_t^X), \gamma^*(t, \mu_t^X))$, where

- (i) the map z^* is the optimiser of the minimisation problem in (4.4.13), and satisfies (4.4.10);
- (ii) the map γ^* is defined by (4.4.11).

Moreover, the second-best optimal contract is equal to

$$\xi_0 - \int_0^T \left(\mathcal{H}(X_s, \mu_s^X, \bar{\zeta}_s^*, \alpha_s^*) - \frac{R_A}{2} |\sigma^\circ|^2 |Z_s^*|^2 \right) ds + \int_0^T Z_s^* (dX_s - \tilde{\mathbb{E}}^{\mathbb{P}_s}[d\tilde{X}_s]) + \frac{1}{2} \int_0^T (\Gamma_s^* + R_A |Z_s^*|^2) d\langle X \rangle_s.$$

Proof. The proof simply consists in a slight adaptation of the proof of [Proposition 4.4.8](#) postponed to [Section 4.6](#), by showing that the function v^0 , defined by [\(4.4.12\)](#), satisfies the required assumptions to apply [Theorem 4.4.7](#). More precisely, noticing that

$$\int \partial_{\mu^X} v(t, \mu_t^Y)(y) \mu_t^Y(dy) = \bar{u}_{\mu^X}(t, \mu_t^X) \quad \text{and} \quad \int \partial_{\mu^L} v(t, \mu_t^Y)(y) \mu_t^Y(dy) = -1,$$

and since u satisfies the condition [\(4.4.14\)](#), we directly deduce that v satisfies [\(4.4.5\)](#). $\square$

Remark 4.4.11. *In fact, in the risk-neutral case, the principal's value, defined by [\(4.4.2\)](#), satisfies*

$$V_0^0 := \sup_{\mathbb{P} \in \mathcal{Q}} \mathbb{E}^{\mathbb{P}} \left[-\mathbb{E}^{\mathbb{P}^T} \left[L_T^{\bar{\zeta}^{\mathbb{P}}} \right] \right] = \sup_{\mathbb{P} \in \mathcal{Q}} \mathbb{E}^{\mathbb{P}} \left[-L_T^{\bar{\zeta}^{\mathbb{P}}} \right].$$

It is therefore a standard stochastic control problem that can be solved in the classical way. Nevertheless, for the sake of consistency throughout this chapter, we choose to solve it using [Theorem 4.4.7](#).

4.5 Benchmark and extension

4.5.1 First-best problem

In this section, we focus our attention on the so-called first-best framework, *i.e.*, without moral hazard. In this case, the principal can actually choose both the contract ξ and the representative agent's efforts. Given the reservation utility level, R_0 , the principal's problem is defined by:

$$V_0^{\text{P,FB}} := \sup_{(\mathbb{P}, \mu^X) \in \mathcal{P} \times \mathcal{P}(\mathcal{C}_T)} \sup_{\xi \in \Xi^{\text{P,FB}}} J_0^{\text{P}}(\xi, \mathbb{P}), \text{ such that } J_0^{\text{A}}(\xi, \mu^X, \mathbb{P}) \geq R_0, \quad (4.5.1)$$

where $\Xi^{\text{P,FB}}$ is rigorously defined later by [\(4.6.10\)](#). Recall that the representative agent is risk-averse, with a risk-aversion parameter R_A . Using the same tools as in [Section 4.4](#), we obtain the main results below, for both the case of a risk-averse (see [Proposition 4.5.1](#)) and risk-neutral principal (see [Proposition 4.5.2](#)), while the proofs are postponed to [Section 4.6.3](#).

Proposition 4.5.1. *Consider the first-best case, and a CARA utility function $U^{\text{P}}(x) = -e^{-R_P x}$, $x \in \mathbb{R}$. Assume that there exists a smooth solution $u^{\text{P,FB}}$ to the following HJB equation, for $t \in [0, T]$,*

$$\begin{aligned} 0 = & -\partial_t u + \int (g - f)(x) \mu^X(dx) + \frac{\theta}{2} |\sigma^\circ|^2 - \frac{1}{2} |\sigma^\circ|^2 \bar{u}_{x, \mu^X} + \frac{1}{2} |\sigma^\circ|^2 \bar{R} |\bar{u}_{\mu^X}|^2 - \frac{1}{2} |\sigma^\circ|^2 \bar{u}_{\mu^X \mu^X} \\ & + \bar{\rho} \bar{u}_{\mu^X} ((\bar{u}_{\mu^X})^- \wedge A_{\max}) + \frac{1}{2} c_\alpha^*(\bar{u}_{\mu^X}) - \frac{1}{2} (\bar{u}_{x, \mu^X} - \theta) \Sigma^*(\bar{u}_{x, \mu^X} - \theta) + \frac{1}{2} c_\beta^*(\bar{u}_{x, \mu^X} - \theta), \end{aligned}$$

with terminal condition $u(T, \cdot) = -1$. Then:

- (i) *the principal's value is given by $V_0^{\text{P,FB}} = R_0 (V_0^{\bar{R}} / R_0)^{1+R_P/R_A}$ where $V_0^{\bar{R}} = -e^{-\bar{R} u^{\text{P,FB}}(0, \mu_0^X)}$;*
- (ii) *the optimal effort $\nu^{\text{FB},*} := (\alpha^{\text{FB},*}, \beta^{\text{FB},*})$ is defined for all $t \in [0, T]$ by $\alpha_t^{\text{FB},*} := a^{\text{FB},*}(t, \mu_t^X)$ and $\beta_t^{\text{FB},*} := b^{\text{FB},*}(t, \mu_t^X)$ where, for $k \in \{1, \dots, d\}$ and $(t, \mu) \in [0, T] \times \mathcal{P}(\mathbb{R})$,*

$$a^{k, \text{FB},*}(t, \mu) := \rho^k \left((\bar{u}_{\mu^X}^{\text{P,FB}}(t, \mu))^- \wedge A_{\max} \right) \quad \text{and} \quad b^{k, \text{FB},*} := 1 \wedge \left(\lambda^k (\bar{u}_{x, \mu^X}^{\text{P,FB}}(t, \mu) - \theta) \right)^{\frac{-1}{\eta^k + 1}} \vee B_{\min};$$

- (iii) *the optimal contract is given by*

$$\xi^* = -\frac{1}{R_A} \ln(-R_0) + \int_0^T (c(\nu_s^{\text{FB},*}) - f(X_s)) ds.$$

The risk-neutral principal's problem is very similar to the risk-averse one: informally the following proposition is obtained by setting $R_P = 0$ (and thus $\bar{R} = 0$) in [Proposition 4.5.1](#).

Proposition 4.5.2. *Consider the first-best case and a utility function $U^P(x) = x$, $x \in \mathbb{R}$. Assume that there exists a smooth solution $u^{0,FB}$ to the following HJB equation, for $t \in [0, T]$,*

$$\begin{aligned} 0 = & -\partial_t u + \int (g - f)(x) \mu^X(dx) + \frac{\theta}{2} |\sigma^\circ|^2 - \frac{1}{2} |\sigma^\circ|^2 \bar{u}_{x, \mu^X} - \frac{1}{2} |\sigma^\circ|^2 \bar{u}_{\mu^X, \mu^X} \\ & + \bar{\rho} \bar{u}_{\mu^X} ((\bar{u}_{\mu^X})^- \wedge A_{\max}) + \frac{1}{2} c_\alpha^\star(\bar{u}_{\mu^X}) - \frac{1}{2} (\bar{u}_{x, \mu^X} - \theta) \Sigma^\star(\bar{u}_{x, \mu^X} - \theta) + \frac{1}{2} c_\beta^\star(\bar{u}_{x, \mu^X} - \theta). \end{aligned}$$

Then the principal's value is given by $V_0^{0,FB} = \ln(-R_0)/R_A + u^{0,FB}(0, \mu_0^X)$ and the optimal efforts are the same than those in [Proposition 4.5.1](#) but applied in $u^{0,FB}$.

4.5.2 Contractible common noise

For the moment, we have studied optimal contracting in the case where the principal cannot offer a contract directly indexed on the common noise. Nevertheless, [Remark 4.3.3](#) leads us to compare with the results we would obtain in the case of a contractible common noise.

From [\(4.1.7\)](#), we recall that, in this case, the principal is offering a contract $\mathbb{F}^{\text{obs}, \circ}$ -measurable. Therefore, we expect that the value function of the agent now depends on three state variables: X , $\hat{\mu}$ and W° . We consider the dynamic version of the value function of the representative agent, V_t^A , which may be written through a function $v^A : [0, T] \times \mathcal{C}_T^2 \times \mathcal{P}(\mathcal{C}_T)$ as follows:

$$V_t^A = v^A(t, X_{t \wedge \cdot}, W_{t \wedge \cdot}^\circ, \hat{\mu}_t), \quad t \in [0, T].$$

Following the same reasoning as the one developed in [Section 4.2](#), and using the computations in [Section 4.3.2](#), we obtain a contract of the form [\(4.3.5\)](#), indexed by a triple ζ° . Therefore, if the principal observes the common noise, it is equivalent, in terms of the contract form, to index on the law of others or on the common noise, since the optimal form boils down to the same expression in both cases. Indeed, the optimal contract form in this case is a simplification of the original one, given by [\(4.3.3\)](#), and is obtained just by making the common noise explicitly appear. Therefore, the only difference with regard to contract form is whether or not we are allowed to explicitly write the common noise. Then, for a fixed triple of indexation parameter ζ , both contracts lead to the same agent's effort, and to the unique mean-field equilibrium previously exhibited. Of course, the form of contract is more explicit in this case, and we do not need to consider the unusual indexation on the law of others.

In fact, one difference that may matter is the measurability of the indexation parameters. Indeed, recall that if the principal does not observe the common noise, $\mathbb{F}^{\text{obs}}$ is the natural filtration generated by X and $\hat{\mu}$, thus ζ is $\mathbb{F}^{\text{obs}}$ -measurable. If the principal observes the common noise, ζ is $\mathbb{F}^{\text{obs}, \circ}$ -measurable, where $\mathbb{F}^{\text{obs}, \circ}$ is a bigger filtration, since it is generated by X , $\hat{\mu}$ and W° . Therefore, if the principal observes the common noise, it may be in her interest to choose the indexation parameters according to its values, in order to benefit from this additional information. To address this question, the same principal's optimisation problem must be solved, except that the supremum is taken over contracts $\xi \in \Xi^\circ$. In fact, following the same reasoning as in [Section 4.4](#), [Theorem 4.4.7](#) as well as [Propositions 4.4.8](#) and [4.4.10](#) can also be established in this case. Therefore, there is no benefice for indexing the contract on the common noise, if there is already an indexation on the law of others. This means that the original contract form [\(4.3.3\)](#) allows the principal to index the compensation on the common noise in a hidden way.

4.6 Technical proofs

This section provides a formal description of the 2BSDE theory in our framework, as well as the technical proofs omitted along this chapter. We first start with the proof of [Lemma 4.1.3](#).

Proof of Lemma 4.1.3. Let $\mathbb{P} \in \mathcal{P}$. First of all, by definition of $\mathcal{P}$ we have $\mathbb{P}[\Pi \in \mathbb{U}_0] = 1$, thus $\Pi(ds, dv) = \delta_{\nu_s^\mathbb{P}}(dv)ds$ $\mathbb{P}$ -a.s. for some $\mathbb{F}$ -predictable control process $\nu^\mathbb{P} := (\alpha^\mathbb{P}, \beta^\mathbb{P})$. Therefore, (X, W, W°) is an Itô process with drift $A(\nu^\mathbb{P})$ and quadratic variation $B(\nu_t^\mathbb{P})B^\top(\nu_t^\mathbb{P})$ under $\mathbb{P}$, where, $dt \otimes \mathbb{P}(d\omega)$ -a.e.,

$$B(\nu_t^\mathbb{P})B^\top(\nu_t^\mathbb{P}) = \begin{pmatrix} \Sigma(\beta_t^\mathbb{P}) + |\sigma^\circ|^2 & \sigma^\top(\beta_t^\mathbb{P}) & \sigma^\circ \\ \sigma(\beta_t^\mathbb{P}) & \mathbf{I}_d & \mathbf{0}_d \\ \sigma^\circ & \mathbf{0}_d^\top & 1 \end{pmatrix} = \begin{pmatrix} 0 & \sigma^\top(\beta_t^\mathbb{P}) & \sigma^\circ \\ \mathbf{0}_d & \mathbf{I}_d & \mathbf{0}_d \\ 0 & \mathbf{0}_d^\top & 1 \end{pmatrix} \times \begin{pmatrix} 0 & \mathbf{0}_d^\top & 0 \\ \sigma(\beta_t^\mathbb{P}) & \mathbf{I}_d & \mathbf{0}_d^\top \\ \sigma^\circ & \mathbf{0}_d^\top & 1 \end{pmatrix}.$$

Furthermore, following the line of [Lin, Ren, Touzi, and Yang \[234\] \(2020\)](#), we consider the extended space $\Omega^e = \Omega \times \Omega'$ where $\Omega' = \mathcal{C}([0, T], \mathbb{R}^{d+2})$. Ω' is equipped with the filtration $(\mathcal{F}'_t)_{t \geq 0}$, generated by the canonical process, and $\mathbb{P}'_0$ is the Wiener measure on Ω' . We define $\mathcal{F}_t^e := \mathcal{F}_t \otimes \mathcal{F}'_t$, $\mathbb{F}^e := \mathbb{F} \otimes \mathbb{F}'$ and $\mathbb{P}^e := \mathbb{P} \otimes \mathbb{P}'_0$. We denote X^e , W^e and $W^{\circ, e}$ the natural extensions of X and W° from Ω to Ω^e . By [Stroock and Varadhan \[324, Theorem 4.5.2\] \(1997\)](#), there is a $d+2$ -dimensional Brownian motion B^e on $(\Omega^e, \mathbb{F}^e, \mathbb{P}^e)$, such that

$$d \begin{pmatrix} X_t^e \\ W_t^e \\ W_t^{\circ, e} \end{pmatrix} = \begin{pmatrix} -\alpha_t^\mathbb{P} \cdot \mathbf{1}_d \\ \mathbf{0}_d \\ 0 \end{pmatrix} dt + \begin{pmatrix} 0 & \sigma^\top(\beta_t^\mathbb{P}) & \sigma^\circ \\ \mathbf{0}_d & \mathbf{I}_d & \mathbf{0}_d \\ 0 & \mathbf{0}_d^\top & 1 \end{pmatrix} dB_t^e.$$

Therefore, we have $dW_t^e = d(B_t^{e,2}, \dots, B_t^{e,d+1})^\top$ and $dW_t^{\circ, e} = dB_t^{e,d+2}$. Then,

$$X_t^e = - \int_0^t \alpha_s^\mathbb{P} \cdot \mathbf{1}_d ds + \int_0^t (0, \sigma^\top(\beta_s^\mathbb{P}), \sigma^\circ) dB_s^e = - \int_0^t \alpha_s^\mathbb{P} \cdot \mathbf{1}_d ds + \int_0^t \sigma(\beta_s^\mathbb{P}) \cdot dW_s^e + \int_0^t \sigma^\circ dW_s^{\circ, e},$$

for $t \geq 0$, $\mathbb{P}^e$ -a.s., which implies the desired result. $\square$

4.6.1 Link with 2BSDEs

4.6.1.1 Another representation for the set of measures

As already mentioned in the previous chapter, the general approach to moral hazard problems in [Cvitanović, Possamaï, and Touzi \[111\] \(2018\)](#) requires to distinguish between the efforts of the agent which give rise to absolutely continuous probability measures in $\mathcal{P}$, or for which the volatility control changes while keeping fixed the quadratic variation of X . With this in mind, we let $\bar{\mathcal{P}}$ be the set of probability measures $\bar{\mathbb{P}}$ on $(\Omega, \mathcal{F}_T)$ such that

- (i) the canonical vector process $(X, W, W^\circ)^\top$ is an $(\mathbb{F}, \bar{\mathbb{P}})$ -local martingale for which there exists an $\mathbb{F}$ -predictable and B -valued process $\beta^\bar{\mathbb{P}}$ such that the $\bar{\mathbb{P}}$ -quadratic variation of $(X, W, W^\circ)^\top$ is $\bar{\mathbb{P}}$ -a.s. equal to

$$\begin{pmatrix} \Sigma(\beta_s^\bar{\mathbb{P}}) + |\sigma^\circ|^2 & \sigma^\top(\beta_s^\bar{\mathbb{P}}) & \sigma^\circ \\ \sigma(\beta_s^\bar{\mathbb{P}}) & \mathbf{I}_d & \mathbf{0}_d \\ \sigma^\circ & \mathbf{0}_d^\top & 1 \end{pmatrix}, \quad s \in [0, T];$$

- (ii) $\bar{\mathbb{P}}[\Pi \in \mathbb{U}_0] = 1$.

Arguing as in [Lemma 4.1.3](#), we know that for all $\bar{\mathbb{P}} \in \bar{\mathcal{P}}$

$$X_t = X_0 + \int_0^t \sigma(\beta_s^{\bar{\mathbb{P}}}) \cdot dW_s + \int_0^t \sigma^\circ dW_s^\circ, \quad t \in [0, T], \quad \bar{\mathbb{P}}\text{-a.s.}$$

In addition, recall that using classical results of [Bichteler \[58\] \(1981\)](#) (see [Neufeld and Nutz \[265, Proposition 6.6\] \(2014\)](#) for a modern presentation), we can define a pathwise version of the quadratic variation of $\langle X \rangle$, being $\mathbb{F}$ -predictable, allowing us to define the following $\mathbb{R}_+$ -valued process

$$S_t := \limsup_{n \rightarrow +\infty} n(\langle X \rangle_t - \langle X \rangle_{t-1/n}). \quad (4.6.1)$$

Definition 4.6.1. *For any A -valued and $\mathbb{F}$ -predictable process α , any $\bar{\mathbb{P}} \in \bar{\mathcal{P}}$, and $\mathbb{F}$ -predictable and B -valued process β ⁷ such that, $\bar{\mathbb{P}}$ -a.s., $\Sigma(\beta^{\bar{\mathbb{P}}}) = \Sigma(\beta)$, we define the equivalent measures $\bar{\mathbb{P}}^\nu$, for $\nu := (\alpha, \beta)$, by their Radon–Nykodym density on $\mathcal{F}_T$:*

$$\frac{d\bar{\mathbb{P}}^\nu}{d\bar{\mathbb{P}}} := \exp \left(- \int_0^T \frac{\alpha_s \cdot \mathbf{1}_d}{\Sigma(\beta_s)} \sigma(\beta_s) \cdot dW_s - \frac{1}{2} \int_0^T |\alpha_s \cdot \mathbf{1}_d|^2 ds \right).$$

Notice that such a measure is well-defined since A is a compact set, and the B -valued processes are automatically bounded and bounded away from 0. It is then immediate to check that the $\mathcal{P}$ coincides exactly with the set of all probability measures of the form $\bar{\mathbb{P}}^\nu$, which satisfy in addition

- (i) $\bar{\mathbb{P}}^\nu \circ (X_0)^{-1} = \varrho$, and there exists a measure ι on $\mathbb{R}^d \times \mathbb{R}$ such that $\bar{\mathbb{P}}^\nu \circ ((W_0, W_0^\circ))^{-1} = \iota$;
- (ii) for $\bar{\mathbb{P}}$ -a.e. $\omega \in \Omega$ and for every $t \in [0, T]$, we have $\mu_t(\omega) = (\bar{\mathbb{P}}^\nu)_t^\omega \circ (X_{t \wedge \cdot})^{-1}$;
- (iii) (W°, μ) is $\bar{\mathbb{P}}^\nu$ -independent of W .

For any $\bar{\mathbb{P}} \in \bar{\mathcal{P}}$, we denote by $\bar{\mathcal{U}}(\bar{\mathbb{P}})$ the set of controls $\nu \in \mathcal{U}$ such that $\bar{\mathbb{P}}^\nu \in \mathcal{P}$.

4.6.1.2 Best-reaction functions of the agents and equilibrium

In this subsection, we wish to relate the best-reaction function of an agent to a given contract $\xi \in \Xi$, and a given measure $\hat{\mu}$ played by the mean-field of the other agents, in other words $V_0^A(\xi, \hat{\mu})$, to an appropriate 2BSDE. With this in mind, we define a process S^0 by

$$S_t^0 := \begin{pmatrix} S_t + |\sigma^\circ|^2 & \sigma^\circ \\ \sigma^\circ & 1 \end{pmatrix}, \quad t \in [0, T],$$

taking values in the set of symmetric positive 2×2 matrices. We then consider the following 2BSDE

$$Y_t = -e^{-R_A \xi} + \int_t^T F(X_s, Y_s, Z_s^1, S_s) ds - \int_t^T Z_s^1 dX_s - \sigma^\circ \int_t^T Z_s^2 dW_s^\circ + \int_t^T dK_s. \quad \text{2BSDE (4.6.2)}$$

We can now adapt the notion of 2BSDE to our mean-field setting, using the [Notations I \(p. 43\)](#).

Definition 4.6.2. *We say that (Y, Z, K) is a solution to [2BSDE \(4.6.2\)](#) if [2BSDE \(4.6.2\)](#) holds $\bar{\mathcal{P}}$ -q.s. and if for some $k > 1$, $Y \in \mathbb{D}^p(\mathbb{F}^{\text{obs}}, \bar{\mathbb{P}})$, $Z \in \mathbb{H}_2^p(\mathbb{F}^{\text{obs}}, \bar{\mathbb{P}}, S^0)$, and $K \in \mathbb{I}^p(\mathbb{F}^{\text{obs}}, \bar{\mathbb{P}})$ satisfying the following minimality condition:*

$$K_t = \operatorname{ess\,inf}_{\bar{\mathbb{P}}' \in \mathcal{P}(t, \bar{\mathbb{P}}, \mathbb{F}^{\text{obs}})} \mathbb{E}^{\bar{\mathbb{P}}'} \left[K_T \middle| (\mathcal{F}_t^{\text{obs}})^{\bar{\mathbb{P}}+} \right], \quad 0 \leq t \leq T, \quad \bar{\mathbb{P}}\text{-a.s. for all } \bar{\mathbb{P}} \in \bar{\mathcal{P}}. \quad (4.6.3)$$

7. Strictly speaking, the process β should be indexed by the measure $\bar{\mathbb{P}}$, but we chose to not do so in order to alleviate notations.

The following proposition links the solution to the above 2BSDE to the agent's best-reaction function.

Proposition 4.6.3. *Fix $(\xi, \hat{\mu}) \in \Xi \times \mathcal{P}(\mathcal{C}_T)$. Let $(Y, (Z^1, Z^2)^\top, K)$ be a solution to 2BSDE (4.6.2). We have*

$$V_0^A(\xi, \hat{\mu}) = \sup_{\mathbb{P} \in \bar{\mathcal{P}}} \mathbb{E}^{\mathbb{P}}[Y_0].$$

Conversely, the (dynamic) value function $V_t^A(\xi, \hat{\mu})$ always provides the first component of a solution to 2BSDE (4.6.2). Moreover, any optimal effort $\nu^ := (\alpha^{\mathbb{P}^*}, \beta^{\mathbb{P}^*})$, and the optimal measure $\bar{\mathbb{P}}^* \in \bar{\mathcal{P}}$ must be such that*

$$K = 0, (\bar{\mathbb{P}}^*)^{\nu^*} - \text{a.s.}, (\alpha_t^*, \beta_t^*) \in \underset{(a,b) \in A \times \Sigma^{-1}(S_t)}{\operatorname{argmax}} F(X_{t \wedge \cdot}, Y_t, Z_t^1, S_t), (\bar{\mathbb{P}}^*)^{\nu^*} - \text{a.s.},$$

where $(\bar{\mathbb{P}}^*)^{\nu^*}$ is defined from $\bar{\mathbb{P}}^*$ by Definition 4.6.1.

Proof. The proof is highly similar to the proofs of Propositions 3.7.4 and 3.7.9 in Chapter 3, and follows the lines of Cvitanić, Possamaï, and Touzi [111, Proposition 4.6] (2018). We thus only mention here why the required assumptions are satisfied. First of all, the map F can readily be rewritten as a function $\tilde{F}(x, y, (S^0)^{1/2}(z^1, z^2)^\top, S^0)$ as done by Possamaï, Tan, and Zhou [284] (2018), since the matrix S^0 is always invertible in our setting. Moreover, we can easily get rid off the linear term $-R_A f(x)y$ in F by considering the 2BSDE satisfied by $e^{-\int_0^t f(X_{s \wedge \cdot}) ds} Y_t$ instead. We therefore assume without loss of generality that f is 0 here. Then, $\tilde{F}$ and the terminal condition $-e^{-R_A \xi}$ satisfy the Lipschitz and integrability properties in [284, Assumption 1.1 (i) and (ii)], because controls are bounded and by definition of the set of contracts, recall (IP) (and that the densities from probabilities in $\mathcal{P}$ to probabilities in $\bar{\mathcal{P}}$ have moments of any order, uniformly on the measures). [284, Assumption 4.1] is also automatically satisfied as $\tilde{F}$ is 0 for $y = z^1 = z^2 = 0$. Next, [284, Assumption 1.1 (iii)–(v)] are also satisfied by the set of measures $\bar{\mathcal{P}}$, see for instance Nutz and van Handel [268] (2013). Finally, the set $\bar{\mathcal{P}}$ is saturated in the sense of [284, Definition 5.1], see [284, Remark 5.1]. $\square$

With Proposition 4.6.3 in hand, we can now characterise a mean-field equilibrium, thanks to a 2BSDE of mean-field type, reminiscent of the mean-field BSDE obtained in the setting by Élie, Mastrolia, and Possamaï [141] (2018) where only the drift of X was controlled.

Theorem 4.6.4. *The pair $(\mathbb{P}^*, \mu^*)$ belongs to $\mathcal{M}^*(\xi)$ if and only if there exists $\bar{\mathbb{P}}^* \in \bar{\mathcal{P}}$ and $\nu^* \in \mathcal{U}$ such that $\mathbb{P}^* = (\bar{\mathbb{P}}^*)^{\nu^*}$, where $(\bar{\mathbb{P}}^*)^{\nu^*}$ is defined from $\bar{\mathbb{P}}^*$ in the sense of Definition 4.6.1, satisfying*

$$K^* = 0, \mathbb{P}^* \text{-a.s.}, \nu_t^* \in \underset{(a,b) \in A \times \Sigma^{-1}(S_t)}{\operatorname{argmax}} F(X_{t \wedge \cdot}, Y_t^*, Z_t^{1*}, S_t), \bar{\mathbb{P}}^* \text{-a.s.},$$

where $(Y^*, (Z^{1*}, Z^{2*})^\top, K^*)$ is a solution to the (mean-field) 2BSDE (4.6.2), in the sense of Definition 4.6.2 and the fixed-point constraint

$$\mu_t(\omega) = (\mathbb{P}^*)_t^\omega \circ (X_{t \wedge \cdot})^{-1}.$$

Proof. By Proposition 4.6.3, we have a characterisation of the best-reaction function of the agent to an arbitrary $\hat{\mu}$. An equilibrium then necessitates only that $\hat{\mu}$ coincides with the conditional distribution of X under $\mathbb{P}^*$, which is exactly what is given by the mean-field 2BSDE (4.6.2). $\square$

4.6.1.3 Proof of the main result

We start this section with the following lemma which provides us with explicit integrability properties for the processes Z and Z^μ associated to a contract in Ξ_S , in the sense of [Definition 4.3.1](#). This will prove useful for us in the proof of [Theorem 4.4.1](#) below, and when analysing the problem of the principal.

Lemma 4.6.5. *For any $(Z, \bar{Z}^\mu, \Gamma) \in \bar{\mathcal{V}}$, there exists some $p' \in (1, p)$ such that*

$$\sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}} \left[\left(\int_0^T |Z_s|^2 ds \right)^{p'/2} \right] + \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}} \left[\left(\int_0^T |\bar{Z}_s^\mu|^2 ds \right)^{p'/2} \right] < +\infty.$$

Proof. First, we know by [Theorem 4.3.4](#) and its proof that if we define

$$\begin{aligned} Y_t &:= -e^{-R_A \xi_t^{\xi_0, \zeta}}, \quad Z_t^1 := -R_A Y_t Z_t, \quad Z_t^2 := -R_A Y_t \bar{Z}_t^\mu, \quad \Gamma_t := -R_A Y_t \Gamma_t, \\ K_t &:= \int_0^t \left(-R_A Y_s \mathcal{H}^\circ(X_s, \zeta_s) - \frac{1}{2} \Gamma_s (S_s + |\sigma^\circ|^2) - F(X_s, Y_s, Z_s^1, S_s) \right) ds, \quad t \in [0, T], \end{aligned}$$

then $(Y, (Z^1, Z^2)^\top, K)$ solves [2BSDE \(4.6.2\)](#) and in particular, there is some $\bar{p} \in (1, p)$ such that $(Z^1, Z^2)^\top \in \mathbb{H}_2^{\bar{p}}(\mathbb{F}^{\text{obs}}, \bar{\mathbb{P}}, S^0)$. Furthermore, we also have that $|1/Y_t| = e^{R_A \xi_t^{\xi_0, \zeta}}$, so that we deduce using Hölder's inequality, the definition of Ξ_S , and the fact that densities between measures in $\mathcal{P}$ and $\bar{\mathcal{P}}$ have moments of any order, that

$$\sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}} \left[\sup_{0 \leq t \leq T} |1/Y_t|^p \right] < +\infty.$$

Then, we have for any $\tilde{p} \in (1, \bar{p})$

$$\begin{aligned} \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}} \left[\left(\int_0^T |Z_s|^2 ds \right)^{\tilde{p}/2} \right] &= \frac{1}{R_A^{\tilde{p}}} \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}} \left[\left(\int_0^T |Z_s^1/Y_s|^2 ds \right)^{\tilde{p}/2} \right] \\ &\leq \frac{1}{R_A^{\tilde{p}}} \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}} \left[\sup_{0 \leq t \leq T} |1/Y_t|^{\tilde{p}} \left(\int_0^T |Z_s^1|^2 ds \right)^{\tilde{p}/2} \right] \\ &\leq \frac{1}{R_A^{\tilde{p}}} \sup_{\mathbb{P} \in \mathcal{P}} \left(\mathbb{E}^{\mathbb{P}} \left[\sup_{0 \leq t \leq T} |1/Y_t|^p \right] \right)^{\frac{\tilde{p}}{p}} \sup_{\mathbb{P} \in \mathcal{P}} \left(\mathbb{E}^{\mathbb{P}} \left[\left(\int_0^T |Z_s^1|^2 ds \right)^{\frac{p\tilde{p}}{2(p-\tilde{p})}} \right] \right)^{1-\tilde{p}/p}. \end{aligned}$$

To conclude, we want to make sure that we can choose $\tilde{p} \in (1, \bar{p})$ such that $p\tilde{p}/(p-\tilde{p}) \in (1, \bar{p}]$. This is equivalent to the following

$$\begin{cases} \tilde{p} > \frac{p}{1+p} \\ \tilde{p} \leq \frac{p\bar{p}}{p+\bar{p}}, \end{cases}$$

which is always possible since $p\bar{p}/(p+\bar{p}) > p/(1+p) \iff \bar{p}(p-1) > 0$. The same reasoning gives us the required result for $\bar{Z}^\mu$. $\square$

Proof of Theorem 4.4.1. Before explaining how to prove the aforementioned result, notice that the second equality in [Theorem 4.4.1](#) is trivial. Indeed, in absence of limited liability, the value of the principal is a non-increasing function of the utility obtained by the agent. Mathematically, this translates into the fact that the dynamics of both state variables in the principal's problem actually do not depend on L , so that the dependence on the associated initial value is straightforward.

The proof of the first equality relies on arguments similar to the ones developed by [Cvitanović, Possamai, and Touzi \[111\] \(2018\)](#), and already detailed in [Chapter 3](#), mainly in the proofs of [Theorems 3.3.7](#) and [3.4.9](#).

Using [Proposition 4.6.3](#) and [Theorem 4.6.4](#), we know that for $\xi \in \Xi$, there exists an equilibrium $(\mathbb{P}^*, \mu^*) \in \mathcal{M}^*(\xi)$, where $\mathbb{P}^*$ is such that for $\mathbb{P}^*$ -a.e. $\omega \in \Omega$ and for every $t \in [0, T]$, we have

$$\mu_t^*(\omega) = (\mathbb{P}^*)_t^\omega \circ (X_{t \wedge \cdot})^{-1},$$

and

$$K = 0, \mathbb{P}^* - \text{a.s.}, (\alpha_t^*, \beta_t^*) \in \operatorname{argmax}_{(a,b) \in A \times \Sigma^{-1}(S_t)} F(X_{t \wedge \cdot}, Y_t, Z_t^1, S_t), \mathbb{P}^* - \text{a.s.},$$

where K is the last component of the solution $(Y, (Z^1, Z^2)^\top, K)$ of the 2BSDE

$$Y_t = -e^{-R_A \xi} + \int_t^T F(X_s, Y_s, Z_s^1, S_s) ds - \int_t^T Z_s^1 dX_s - \sigma^\circ \int_t^T Z_s^2 dW_s^\circ + \int_t^T dK_s.$$

The main difference between contracts in Ξ and Ξ_S comes from whether the process K above is absolutely continuous with respect to Lebesgue measure or not. Since it is not in general, we will approximate it by a sequence of absolutely continuous ones. Fix thus some $\varepsilon > 0$, and define the absolutely continuous approximation of K

$$K_t^\varepsilon := \frac{1}{\varepsilon} \int_{(t-\varepsilon)^+}^t K_s ds, \quad t \in [0, T].$$

Clearly, K^ε is $\mathbb{F}^{\bar{\mathcal{P}}}$ -predictable, non-decreasing $\bar{\mathcal{P}}$ -q.s. and

$$K^\varepsilon = 0, \mathbb{P}^* - \text{a.s. for all } (\mathbb{P}^*, \mu^*) \in \mathcal{M}^*(\xi). \quad (4.6.4)$$

We next define for any $t \in [0, T]$ the process

$$Y_t^\varepsilon := Y_0 - \int_0^t F(X_s, Y_s^\varepsilon, Z_s^1, S_s) ds + \int_0^t Z_s^1 dX_s + \sigma^\circ \int_0^t Z_s^2 dW_s^\circ - \int_0^t dK_s^\varepsilon, \quad (4.6.5)$$

and verify that $(Y^\varepsilon, (Z^1, Z^2), K^\varepsilon)$ solves the 2BSDE with terminal condition $-e^{-R_A \xi^\varepsilon} := Y_T^\varepsilon$ and generator F . Indeed, since $K^\varepsilon \leq K$, K^ε does satisfy the required minimality condition, which is obvious by [\(4.6.4\)](#). We also verify that $\sup_{\bar{\mathbb{P}} \in \bar{\mathcal{P}}} \mathbb{E}^{\bar{\mathbb{P}}} [e^{-R_A \xi^\varepsilon} | p] < \infty$. Thus, by [\[284, Theorem 4.4\]](#), we have the estimates

$$\|Y^\varepsilon\|_{\mathbb{D}^{\bar{\mathbb{P}}}(\mathbb{F}^{\text{obs}}, \bar{\mathbb{P}})} + \|(Z^1, Z^2)^\top\|_{\mathbb{H}_2^{\bar{\mathbb{P}}}(\mathbb{F}^{\text{obs}}, \bar{\mathbb{P}}, S^0)} < \infty, \text{ for } \bar{p} \in (1, p). \quad (4.6.6)$$

We finally observe that a probability measure $\mathbb{P}$ satisfies $K = 0, \mathbb{P}$ -a.s. if and only if it satisfies $K^\varepsilon = 0, \mathbb{P}$ -a.s.

Notice then that for any $(t, \omega, x, y, z^1, z^2) \in [0, T] \times \Omega \times \mathbb{R}^4$, the map

$$\gamma \longmapsto -R_A y \mathcal{H}^\circ \left(x, -\frac{1}{R_A y} (z^1, z^2, \gamma) \right) - \frac{1}{2} \gamma (S_s(\omega) + |\sigma^\circ|^2) - F(x, y, z^1, S_s(\omega)), \quad (4.6.7)$$

is surjective on $(0, \infty)$. Indeed, it is non-negative, by definition of $\mathcal{H}^\circ$ and F , convex, continuous on the interior of its domain, and is coercive by the boundedness of the controls.

Let $\dot{K}^\varepsilon$ denote the density of the absolutely continuous process K^ε with respect to the Lebesgue measure. Applying a classical measurable selection argument (the maps appearing here are continuous, and we can use the results by Beneš in [\[47\] \(1970\)](#) and [\[48\] \(1971\)](#)), we may deduce the existence of an $\mathbb{F}^{\text{obs}}$ -predictable process Γ^ε such that

$$\dot{K}_s^\varepsilon = -R_A Y_s^\varepsilon \mathcal{H}^\circ \left(X_s, -\frac{1}{R_A Y_s^\varepsilon} (Z_s^1, Z_s^2, \Gamma_s^\varepsilon) \right) - \frac{1}{2} \Gamma_s^\varepsilon (S_s + |\sigma^\circ|^2) - F(X_s, Y_s^\varepsilon, Z_s^1, S_s).$$

Indeed, for $\dot{K}_s^\varepsilon > 0$, this is clear from (4.6.7), and if $\dot{K}_s^\varepsilon = 0$, Γ_s^ε can be chosen arbitrarily. Substituting in (4.6.5), it follows that the following representation for Y^ε holds

$$Y_t^\varepsilon = Y_0 + \int_0^t R_A Y_s^\varepsilon \mathcal{H}^\circ \left(X_s, -\frac{1}{R_A Y_s^\varepsilon} (Z_s^1, Z_s^2, \Gamma_s^\varepsilon) \right) ds + \int_0^t Z_s^1 dX_s + \sigma^\circ \int_0^t Z_s^2 dW_s^\circ + \frac{1}{2} \int_0^t \Gamma_s^\varepsilon d\langle X \rangle_s.$$

Then, applying Itô's formula to $-1/R_A \log(-Y_t^\varepsilon)$, we obtain:

$$\begin{aligned} \xi_t^\varepsilon &= -\frac{1}{R_A} \log(-Y_0^\varepsilon) - \int_0^t \mathcal{H}^\circ(X_s, \bar{\zeta}_s) ds + \int_0^t Z_s dX_s + \sigma^\circ \int_0^t \bar{Z}_s^\mu dW_s^\circ + \frac{1}{2} \int_0^t (\Gamma_s + R_A |Z_s|^2) d\langle X \rangle_s \\ &\quad + \frac{1}{2} R_A |\sigma^\circ|^2 \int_0^t \bar{Z}_s^\mu (\bar{Z}_s^\mu + 2Z_s) ds. \end{aligned}$$

where

$$\bar{\zeta} := (Z, \bar{Z}^\mu, \Gamma) = -\frac{1}{R_A Y^\varepsilon} (Z^1, Z^2, \Gamma^\varepsilon).$$

Finally, define Z^μ as any $\mathbb{F}^{\text{obs}}$ -predictable process, taking values in $\mathcal{L}$, such that $\widehat{\mathbb{E}}^{\mathbb{P}^*}_s[Z_s^\mu(\hat{X}_{s\wedge\cdot})] = \bar{Z}_s^\mu$. Using the dynamic of $\hat{X}$, we deduce that

$$\begin{aligned} \xi_t^\varepsilon &:= -\frac{\ln(-Y_0^\varepsilon)}{R_A} - \int_0^t \mathcal{H}(X_s, \hat{\mu}_s, \zeta_s, \alpha_s^*) ds + \int_0^t Z_s dX_s + \frac{1}{2} \int_0^t (\Gamma_s + R_A |Z_s|^2) d\langle X \rangle_s + \int_0^t \widehat{\mathbb{E}}^{\mathbb{P}^*}_s[Z_s^\mu(\hat{X}_{s\wedge\cdot}) d\hat{X}_s] \\ &\quad + \frac{1}{2} R_A \int_0^t \widehat{\mathbb{E}}^{\mathbb{P}^*}_s[\check{\mathbb{E}}^{\mathbb{P}^*}_s[Z_s^\mu(\hat{X}_{s\wedge\cdot}) Z_s^\mu(\check{X}_{s\wedge\cdot}) d\langle \hat{X}, \check{X} \rangle_s] + R_A \int_0^t Z_s \widehat{\mathbb{E}}^{\mathbb{P}^*}_s[Z_s^\mu(\hat{X}_{s\wedge\cdot}) d\langle X, \hat{X} \rangle_s], \quad t \in [0, T], \end{aligned}$$

where $\zeta := (Z, Z^\mu, \Gamma)$. This shows that the contract ξ^ε has the required dynamics (4.3.3), since, at equilibrium, $\mu^* = \hat{\mu}$ and the effort $\alpha^* = \hat{\alpha}$ and is unique. The fact that it belongs to Ξ_S then stems from (4.6.6) and arguments similar to those in Lemma 4.6.5. We can then conclude, in view of Section 4.6.1.1 and as in the proof of [111, Theorem 3.6] by noting that $\xi^\varepsilon = \xi$, $\mathbb{P}^*$ -a.s. $\square$

4.6.2 Proofs for the principal's problem

Proof of Theorem 4.4.7. Let v be a solution to the PDE (4.4.3), smooth enough in the sense of Definition 4.4.5, such that Condition (4.4.5) is satisfied. Moreover, assume that the supremum in the PDE (4.4.3) is attained for a function $\bar{v}^*$, from $[0, T] \times \mathcal{P}(\mathbb{R}^2)$ to $\mathbb{R}^3$. We need to prove that $V_0^P = v(0, \mu_0^Y)$.

On the one hand, applying the Chain Rule under $\mathcal{C}^{1,2}$ -regularity on v , as a function depending on time and on the conditional law μ^Y of $Y^\zeta = (X, L^\zeta)^\top$, we obtain:

$$\begin{aligned} dv(t, \mu_t^Y) &= \left(\partial_t v(t, \mu_t^Y) + \int \partial_{\mu^L} v(t, \mu_t^Y)(y)(g - f)(x) \mu_t^Y(dy) + \frac{1}{2} |\sigma^\circ|^2 \int \int \partial_{\mu^X}^2 v(t, \mu_t^Y)(y, \tilde{y}) \mu_t^Y(dy) \mu_t^Y(d\tilde{y}) \right. \\ &\quad + \frac{\theta}{2} |\sigma^\circ|^2 \int \partial_{\mu^L} v(t, \mu_t^Y)(y) \mu_t^Y(dy) + \frac{1}{2} |\sigma^\circ|^2 \int \partial_x \partial_{\mu^X} v(t, \mu_t^Y)(y) \mu_t^Y(dy) \\ &\quad + \frac{1}{2} h(\mu_t^Y, \partial_{\mu} v(t, \mu_t^Y), \partial_y \partial_{\mu} v(t, \mu_t^Y), \partial_{\mu}^2 v(t, \mu_t^Y), \bar{\zeta}) \Big) dt \\ &\quad + \sigma^\circ \int (-\partial_{\mu^X} v(t, \mu_t^Y)(y) + \partial_{\mu^L} v(t, \mu_t^Y)(y)(Z_t + \bar{Z}_t^\mu)) \mu_t^Y(dy) dW_t^\circ. \end{aligned}$$

Recalling that, by assumption, v is solution to the HJB equation (4.4.3), and therefore satisfies

$$\begin{aligned} v(T, \mu_T^Y) &= v(0, \mu_0^Y) + \frac{1}{2} \int_0^T \left(h(\mu_t^Y, \partial_{\mu} v, \partial_y \partial_{\mu} v, \partial_{\mu}^2 v, \bar{\zeta}) - \sup_{\bar{\zeta} \in \mathbb{R}^3} h(\mu_t^Y, \partial_{\mu} v, \partial_y \partial_{\mu} v, \partial_{\mu}^2 v, \bar{v}) \right) dt \\ &\quad + \sigma^\circ \int_0^T \int (-\partial_{\mu^X} v(t, \mu_t^Y)(y) + \partial_{\mu^L} v(t, \mu_t^Y)(y)(Z_t + \bar{Z}_t^\mu)) \mu_t^Y(dy) dW_t^\circ. \end{aligned} \quad (4.6.8)$$

On the other hand, under Assumption (4.4.5) on the partial derivatives of v , the process

$$\sigma^\circ \int_0^\cdot \int (-\partial_{\mu^X} v(t, \mu_t^Y)(y) + \partial_{\mu^L} v(t, \mu_t^Y)(y)(Z_t + \bar{Z}_t^\mu)) \mu_t^Y(dy) dW_t^\circ.$$

is a $\mathbb{P}$ -martingale, since the following quantities

$$A := \mathbb{E}^\mathbb{P} \left[\sup_{0 \leq t \leq T} \left| \int_0^t \int \partial_{\mu^X} v(s, \mu_s^Y)(y) \mu_s^Y(dy) dW_s^\circ \right| \right],$$

and $B := \mathbb{E}^\mathbb{P} \left[\sup_{0 \leq t \leq T} \left| \int_0^t \int \partial_{\mu^L} v(s, \mu_s^Y)(y)(Z_s + \bar{Z}_s^\mu) \mu_s^Y(dy) dW_s^\circ \right| \right],$

are finite. Indeed, on the one hand, using Burkholder–Davis–Gundy inequality, we have for some constant $C > 0$, independent of $\mathbb{P}$,

$$A \leq C \mathbb{E}^\mathbb{P} \left[\left(\int_0^T \left(\int \partial_{\mu^X} v(t, \mu_t^Y)(y) \mu_t^Y(dy) \right)^2 ds \right)^{1/2} \right] < +\infty,$$

by the first part of Assumption (4.4.5). On the other hand, using in addition Hölder's inequality, we have, for some constant $\tilde{C} > 0$ independent of $\mathbb{P}$, and for $p' > 1$ given by Lemma 4.6.5

$$\begin{aligned} B &\leq \tilde{C} \mathbb{E}^\mathbb{P} \left[\left(\int_0^T \left| \int \partial_{\mu^L} v(s, \mu_s^Y)(y)(Z_s + \bar{Z}_s^\mu) \mu_s^Y(dy) \right|^2 ds \right)^{1/2} \right] \\ &\leq 2\tilde{C} \mathbb{E}^\mathbb{P} \left[\sup_{0 \leq t \leq T} \left| \int \partial_{\mu^L} v(t, \mu_t^Y)(y) \mu_t^Y(dy) \right| \left(\int_0^T (|Z_s|^2 + |\bar{Z}_s^\mu|^2) ds \right)^{1/2} \right] \\ &\leq 2\tilde{C} \left(\mathbb{E}^\mathbb{P} \left[\sup_{0 \leq t \leq T} \left| \int \partial_{\mu^L} v(t, \mu_t^Y)(y) \mu_t^Y(dy) \right|^{\frac{p'}{p'-1}} \right] \right)^{1-1/p'} \left(\mathbb{E}^\mathbb{P} \left[\left(\int_0^T (|Z_s|^2 + |\bar{Z}_s^\mu|^2) ds \right)^{p'/2} \right] \right)^{1/p'} < +\infty. \end{aligned}$$

Indeed, the first term is finite by the second part of Assumption (4.4.5), and the second term is also finite since $\bar{\zeta} \in \bar{\mathcal{V}}$ implies that Z and $\bar{Z}^\mu$ are in $\mathbb{H}^{p'}$ by Lemma 4.6.5.

Therefore, by taking the expectation under $\mathbb{P}$ in (4.6.8), we obtain

$$\begin{aligned} \mathbb{E}^\mathbb{P} [v(T, \mu_T^Y)] &= v(0, \mu_0^Y) + \int_0^T \mathbb{E}^\mathbb{P} \left[h(\mu_t^Y, \partial_\mu v(t, \mu_t^Y), \partial_y \partial_\mu v(t, \mu_t^Y), \partial_\mu^2 v(t, \mu_t^Y), \bar{\zeta}_t) \right. \\ &\quad \left. - \sup_{\bar{v} \in \mathbb{R}^3} \theta(\mu_t^Y, \partial_\mu v(t, \mu_t^Y), \partial_y \partial_\mu v(t, \mu_t^Y), \partial_\mu^2 v(t, \mu_t^Y), \bar{v}) \right] dt. \end{aligned}$$

In addition, using the terminal condition $v^P(T, \mu_T^Y) = U^P(-\mathbb{E}^{\mathbb{P}^T}[L_T^{\bar{\zeta}}])$ and noticing that for all $t \in [0, T]$,

$$h(\mu_t^Y, \partial_\mu v(t, \mu_t^Y), \partial_y \partial_\mu v(t, \mu_t^Y), \partial_\mu^2 v(t, \mu_t^Y), \bar{\zeta}_t) - \sup_{\bar{v} \in \mathbb{R}^3} h(\mu_t^Y, \partial_\mu v(t, \mu_t^Y), \partial_y \partial_\mu v(t, \mu_t^Y), \partial_\mu^2 v(t, \mu_t^Y), \bar{v}) \leq 0,$$

with equality for $\bar{\zeta}_t^* := \bar{v}^*(t, \mu_t^Y)$ by assumption, we conclude that

$$\mathbb{E}^\mathbb{P} \left[U^P \left(-\mathbb{E}^{\mathbb{P}^T} [L_T^{\bar{\zeta}}] \right) \right] \leq v(0, \mu_0^Y),$$

with equality for $\bar{\zeta}^*$. Therefore, $v(0, \mu_0^Y) = V^P$. □

Proof of Proposition 4.4.8. Let u be a solution to the PDE (4.4.7), smooth enough in the sense of Definition 4.4.5, satisfying Condition (4.4.8). Let $\bar{\zeta}^* = (Z^*, \bar{Z}^{\mu,*}, \Gamma^*)$ be a process in $\bar{\mathcal{V}}$ such that for all $t \in [0, T]$, $\bar{\zeta}_t^* := \bar{v}^*(t, \mu_t^X)$ is the maximiser of h^P defined by (4.4.9). We define the function v as:

$$v(t, \mu_t^Y) = -\exp\left(R_P\left(\mathbb{E}^{\mathbb{P}^t}[L_t^{\bar{\zeta}^*}] - u(t, \mu_t^X)\right)\right).$$

To prove the first point of the proposition, it is sufficient to show that the function v satisfies the assumptions of Theorem 4.4.7. Indeed, we will have, by the first point of the theorem,

$$-e^{R_P(\xi_0 - u(0, \mu_0^X))} = v(0, \mu_0^Y) = V_0^P.$$

First of all, v has the same regularity as u , and is therefore smooth enough in the sense of Definition 4.4.5. Moreover, we can prove that v satisfies the condition (4.4.5). Indeed, since

$$\int \partial_{\mu^X} v(t, \mu_t^Y)(y) \mu_t^Y(dy) = -R_P v(t, \mu_t^Y) \bar{u}_{\mu^X}(t, \mu_t^X) \quad \text{and} \quad \int \partial_{\mu^L} v(t, \mu_t^Y)(y) \mu_t^Y(dy) = R_P v(t, \mu_t^Y),$$

Condition (4.4.5) is equivalent here to

$$\mathbb{E}^{\mathbb{P}}\left[\left(\int_0^T |\bar{u}_{\mu^X}(t, \mu_t^X) v(t, \mu_t^Y)|^2 dt\right)^{1/2}\right] + \mathbb{E}^{\mathbb{P}}\left[\sup_{0 \leq t \leq T} |v(t, \mu_t^Y)|^{p'/(p'-1)}\right] < +\infty.$$

By applying Hölder's inequality twice, we have the following upper bound for the first expectation

$$\left(\mathbb{E}^{\mathbb{P}}\left[\left(\int_0^T |\bar{u}_{\mu^X}(t, \mu_t^X)|^2 dt\right)^{p'/2}\right]\right)^{1/p'} \left(\mathbb{E}^{\mathbb{P}}\left[\sup_{0 \leq t \leq T} |v(t, \mu_t^Y)|^{p'/(p'-1)}\right]\right)^{1-1/p'},$$

where the first expectation is finite since u satisfies Condition (4.4.8). It remains to prove that the second one is finite. By applying Hölder's inequality,

$$\begin{aligned} \mathbb{E}^{\mathbb{P}}\left[\sup_{0 \leq t \leq T} |v(t, \mu_t^Y)|^{p'/(p'-1)}\right] &= \mathbb{E}^{\mathbb{P}}\left[\sup_{0 \leq t \leq T} e^{\frac{p' R_P}{p'-1}(\mathbb{E}^{\mathbb{P}^t}[L_t^{\bar{\zeta}^*}] - u(t, \mu_t^X))}\right] \\ &\leq \left(\mathbb{E}^{\mathbb{P}}\left[\sup_{0 \leq t \leq T} e^{\frac{\varepsilon p' R_P}{p'-1} \mathbb{E}^{\mathbb{P}^t}[L_t^{\bar{\zeta}^*}]}\right]\right)^{1/\varepsilon} \left(\mathbb{E}^{\mathbb{P}}\left[\sup_{0 \leq t \leq T} e^{-\frac{q' p' R_P}{p'-1} u(t, \mu_t^X)} dt\right]\right)^{1/q'}, \end{aligned}$$

for $q' = \varepsilon/(\varepsilon - 1)$ and recalling that $\varepsilon = \sqrt{\bar{p}(p' - 1)/p'}$. Thus, the second expectation is finite since u satisfies Condition (4.4.8). Moreover, since $\langle X \rangle$ is bounded and applying again Hölder's inequality with ε and q' , there exists some positive constant C such that the first expectation has the following upper bound

$$C \left(\mathbb{E}^{\mathbb{P}}\left[\sup_{0 \leq t \leq T} e^{\frac{\varepsilon^2 p' R_P}{p'-1} \mathbb{E}^{\mathbb{P}^t}[\xi_t^{\bar{\zeta}^*} | \mathcal{F}_t^{\circ}]}\right]\right)^{1/\varepsilon^2} \left(\mathbb{E}^{\mathbb{P}}\left[\sup_{0 \leq t \leq T} e^{\frac{\bar{p} R_P}{\varepsilon-1} \mathbb{E}^{\mathbb{P}^t}[\int_0^t g(X_s) ds | \mathcal{F}_t^{\circ}]}\right]\right)^{1/(\varepsilon-1)}.$$

By noting that $\frac{\varepsilon^2 p'}{p'-1} = \bar{p}$, we deduce from (CARA) that the first term is finite. Since g has linear growth and X has bounded drift and volatility, the second term is also finite.

To apply Theorem 4.4.7, it remains to prove that v is a solution to the PDE (4.4.3) and that $\bar{\zeta}^*$ satisfies the optimality condition on h . By computing the partial derivatives of v in term of u , the function h in the PDE (4.4.3) can be rewritten as:

$$\begin{aligned} h(\mu_t, \partial_{\mu} v, \partial_y \partial_{\mu} v, \partial_{\mu}^2 v, \bar{v}) &= R_P v \left(\Sigma^*(\gamma) (\theta + R_A |z|^2 - \bar{u}_{x, \mu^X}) + c_{\beta}^*(\gamma) + c_{\alpha}^*(z) + 2\bar{\rho}(z^- \wedge A_{\max}) \bar{u}_{\mu^X} \right. \\ &\quad \left. + |\sigma^{\circ}|^2 (R_A + R_P) (z + \bar{z}^{\mu})^2 - 2R_P |\sigma^{\circ}|^2 (z + \bar{z}^{\mu}) \bar{u}_{\mu^X} \right). \end{aligned}$$

Noticing that $v < 0$, we have

$$\sup_{\bar{v} \in \mathbb{R}^3} h(\mu_t^Y, \partial_\mu v(t, \mu_t^Y), \partial_y \partial_\mu v(t, \mu_t^Y), \partial_\mu^2 v(t, \mu_t^Y), \bar{v}) = R_P v(t, \mu_t^Y) \inf_{\bar{v} \in \mathbb{R}^3} h^P(\mu_t^X, \bar{u}_{\mu^X}(t, \mu_t^X), \bar{u}_{x, \mu^X}(t, \mu_t^X), \bar{v}),$$

recalling that h^P is defined by (4.4.9). Since the infimum of h^P on $\bar{z}^\mu$ is attained in

$$\bar{z}^{\mu, \star}(t, \mu_t^X) = -z + \frac{R_P}{R_A + R_P} \bar{u}_{\mu^X}(t, \mu_t^X),$$

point (iii) of the proposition has been proven and point (v) is an easy computation for simple contracts in Definition 4.3.1 with the optimal payment rate $\bar{Z}_t^{\mu, \star} = \bar{z}^{\mu, \star}(t, \mu_t^X)$. Moreover, we obtain

$$\inf_{\bar{v} \in \mathbb{R}^3} h^P(\mu_t^X, \bar{u}_{\mu^X}, \bar{u}_{x, \mu^X}, \bar{v}) = -\frac{|\sigma^\circ|^2 R_P^2}{R_A + R_P} |\bar{u}_{\mu^X}|^2 - \bar{\rho} |\bar{u}_{\mu^X}|^2 + \inf_{z \in \mathbb{R}} \left\{ F_0(q(z, \bar{u}_{x, \mu^X})) + \bar{\rho} |(z^- \wedge A_{\max}) + \bar{u}_{\mu^X}|^2 \right\}.$$

Therefore, the function v is solution to PDE (4.4.3) if

$$0 = v \left(-2\partial_t u(t, \mu_t^X) + 2 \int (g - f)(x) \mu_t^X(dx) + \theta |\sigma^\circ|^2 + (|\sigma^\circ|^2 \bar{R} - \bar{\rho}) |\bar{u}_{\mu^X}(t, \mu_t^X)|^2 \right. \\ \left. - |\sigma^\circ|^2 (\bar{u}_{x, \mu^X} + \bar{u}_{\mu^X, x})(t, \mu_t^X) + \inf_{z \in \mathbb{R}} \left\{ F_0(q(z, \bar{u}_{x, \mu^X}(t, \mu_t^X))) + \bar{\rho} |(z^- \wedge A_{\max}) + \bar{u}_{\mu^X}(t, \mu_t^X)|^2 \right\} \right),$$

and this equality is true since u is solution to the PDE (4.4.7).

Consider now the following minimisation problem:

$$\inf_{z \in \mathbb{R}} \left\{ F_0(q(z, \bar{u}_{x, \mu^X}(t, \mu_t^X))) + \bar{\rho} ((z^- \wedge A_{\max}) + \bar{u}_{\mu^X}(t, \mu_t^X))^2 \right\}. \quad (4.6.9)$$

As noticed in [11, Lemma 4.1], the function F_0 is non-decreasing, and the infimum is reached for $\gamma^\star = -q$, which proves the point (iv) of the Proposition, and we have

$$F_0(q) = q \Sigma^\star(-q) + c_\beta^\star(-q) = \sum_{k=1}^d \frac{(\sigma^k)^2}{\lambda^k} \left[\lambda^k \mathbb{1}_{\lambda^k q \leq 1} + \frac{1}{\eta^k} \left((1 + \eta^k) (\lambda^k q)^{\frac{\eta^k}{1+\eta^k}} - 1 \right) \mathbb{1}_{\lambda^k q > 1} \right].$$

Then, on the one hand, for $z \geq 0$, the minimisation problem (4.6.9) is equal to

$$\bar{\rho} |\bar{u}_{\mu^X}(t, \mu_t^X)|^2 + \inf_{z \geq 0} \left\{ F_0(q(z, \bar{u}_{x, \mu^X}(t, \mu_t^X))) \right\},$$

and since F_0 is non-decreasing, its minimum is attained on the minimum of $q(z, \bar{u}_{x, \mu^X}(t, \mu_t^X))$, i.e., for $z = 0$. Therefore, (4.6.9) is equal to $\bar{\rho} |\bar{u}_{\mu^X}^P(t, \mu_t^X)|^2 + F_0(\theta - \bar{u}_{x, \mu^X}(t, \mu_t^X))$. On the other hand, for $z \leq 0$, the minimisation problem (4.6.9) is equal to

$$\inf_{z \leq 0} \left\{ F_0(q(z, \bar{u}_{x, \mu^X}(t, \mu_t^X))) + \bar{\rho} ((-z \wedge A_{\max}) + \bar{u}_{\mu^X}(t, \mu_t^X))^2 \right\}.$$

Thus, if $\bar{u}_{\mu^X}(t, \mu_t^X) \geq 0$, the infimum is reached for $z = 0$. Otherwise, the infimum lie between $\bar{u}_{\mu^X}(t, \mu_t^X) \vee -A_{\max}$ and 0.

To summarise, the optimal process $Z^\star$ is defined for $t \in [0, T]$ by $Z_t^\star = z^\star(t, \mu_t^X)$ where the function $z^\star$ is the minimiser of (4.6.9) and satisfies (4.4.10), thus the point (ii) has been proven.

To conclude the proof, it is sufficient to notice that the triple $\bar{\zeta}^\star := (Z^\star, Z^{\mu, \star}, \Gamma^\star)$ satisfies the optimality condition and apply Theorem 4.4.7 to the function v . $\square$

4.6.3 Details on the first-best case

Adapting the reasoning in [140], we are led to introduce a slight modification of the so-called Morse–Transue space on the initial canonical space Ω , defined here for any $\mathbb{P} \in \mathcal{P}$ by

$$M^{\mathbb{P}}(\mathbb{R}) := \{\vartheta : \Omega \longrightarrow \mathbb{R}, \text{ measurable, } \mathbb{E}^{\mathbb{P}}[\phi(a\vartheta)] < +\infty, \text{ for all } a \geq 0\},$$

where $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ is the following Young function $\phi : x \longmapsto \exp(|x|) - 1$. Then, $M^{\mathbb{P}}(\mathbb{R})$ endowed with the norm $\|\vartheta\|_{\phi} := \inf \{k > 0 : \mathbb{E}^{\mathbb{P}}[\phi(\vartheta/k)] \leq 1\}$ is a Banach space. In this case, the set of admissible contracts Ξ^{FB} is defined as

$$\Xi^{\text{FB}} := \{\xi \in M^{\mathbb{P}}(\mathbb{R}), \mathcal{F}_T\text{-measurable, s.t. } \mathbb{E}^{\mathbb{P}}[\xi | \mathcal{F}_T^{\circ}] \in M^{\mathbb{P}}(\mathbb{R}), \forall \mathbb{P} \in \mathcal{P}\}. \quad (4.6.10)$$

Thus, for any $(\xi, \mu^X, \mathbb{P}) \in \Xi^{\text{FB}} \times \mathcal{P}(\mathcal{C}_T) \times \mathcal{P}$, the quantities $J_0^{\text{A}}(\xi, \mu^X, \mathbb{P})$ and $J_0^{\text{P}}(\xi, \mathbb{P})$ are well-defined.

To take into account the inequality constraint in the definition of $V_0^{\text{P,FB}}$ in (4.5.1), one has to introduce the associated Lagrangian:

$$V_0^{\text{P,FB}} = \inf_{\varpi > 0} \left\{ -\varpi R_0 + \sup_{(\mathbb{P}, \mu^X) \in \mathcal{P} \times \mathcal{P}(\mathcal{C}_T)} \sup_{\xi \in \Xi^{\text{P,FB}}} \{J_0^{\text{P}}(\xi, \mathbb{P}) + \varpi J_0^{\text{A}}(\xi, \mu^X, \mathbb{P})\} \right\},$$

where $\varpi > 0$ is the Lagrange multiplier associated to the participation constraint. We first maximise this utility with respect to ξ . Let us consider, for any probability $(\mathbb{P}, \mu^X) \in \mathcal{P} \times \mathcal{P}(\mathcal{C}_T)$, the following map $V^{\mathbb{P}} : \Xi^{\text{FB}} \longrightarrow \mathbb{R}$ defined by

$$V^{\mathbb{P}}(\xi) := \mathbb{E}^{\mathbb{P}} \left[U^{\text{P}} \left(-\mathbb{E}^{\mathbb{P}} \left[\xi + \int_0^T g(X_s) ds + \frac{\theta}{2} \int_0^T d\langle X \rangle_s \middle| \mathcal{F}_T^{\circ} \right] \right) + \varpi U^{\text{A}} \left(\xi - \int_0^T (c(\nu_s^{\mathbb{P}}) - f(X_s)) ds \right) \right].$$

Recall that the representative agent is risk-averse, with a risk-aversion parameter R_{A} . We can consider both the cases of a risk-averse or risk-neutral principal. To simplify the notations, we define

$$K_T^{\text{P}} := \int_0^T g(X_s) ds + \frac{\theta}{2} \int_0^T d\langle X \rangle_s, \quad K_T^{\text{A}, \mathbb{P}} := \int_0^T (c(\nu_s^{\mathbb{P}}) - f(X_s)) ds \quad \text{and} \quad K_T := K_T^{\text{A}, \mathbb{P}} + K_T^{\text{P}}. \quad (4.6.11)$$

We now focus on the proof of Proposition 4.5.1 by considering a CARA utility function $U^{\text{P}}(x) = -e^{-R_{\text{P}}x}$, since the risk-neutral principal case is deduced by taking $R_{\text{P}} = 0$. In this case, we obtain

$$V^{\mathbb{P}}(\xi) = \mathbb{E}^{\mathbb{P}} \left[-\exp(R_{\text{P}} \mathbb{E}^{\mathbb{P}}[\xi + K_T^{\text{P}} | \mathcal{F}_T^{\circ}]) - \varpi \exp(-R_{\text{A}} \xi + R_{\text{A}} K_T^{\text{A}, \mathbb{P}}) \right].$$

Notice that, for any $\vartheta \in \Xi^{\text{FB}}$ and $\varepsilon \geq 0$,

$$\begin{aligned} \frac{1}{\varepsilon} (V^{\mathbb{P}}(\xi + \varepsilon \vartheta) - V^{\mathbb{P}}(\xi)) &= \frac{1}{\varepsilon} \mathbb{E}^{\mathbb{P}} \left[-\exp(R_{\text{P}} \mathbb{E}^{\mathbb{P}}[\xi + \varepsilon \vartheta + K_T^{\text{P}} | \mathcal{F}_T^{\circ}]) + \exp(R_{\text{P}} \mathbb{E}^{\mathbb{P}}[\xi + K_T^{\text{P}} | \mathcal{F}_T^{\circ}]) \right] \\ &\quad + \frac{1}{\varepsilon} \mathbb{E}^{\mathbb{P}} \left[-\varpi \exp(-R_{\text{A}}(\xi + \varepsilon \vartheta) + R_{\text{A}} K_T^{\text{A}, \mathbb{P}}) + \varpi \exp(-R_{\text{A}} \xi + R_{\text{A}} K_T^{\text{A}, \mathbb{P}}) \right] \\ &= \frac{1}{\varepsilon} \mathbb{E}^{\mathbb{P}} \left[e^{R_{\text{P}} \mathbb{E}^{\mathbb{P}}[\xi + K_T^{\text{P}} | \mathcal{F}_T^{\circ}]} (1 - e^{R_{\text{P}} \varepsilon \mathbb{E}^{\mathbb{P}}[\vartheta | \mathcal{F}_T^{\circ}]}) \right] + \frac{1}{\varepsilon} \mathbb{E}^{\mathbb{P}} \left[\varpi e^{-R_{\text{A}} \xi + R_{\text{A}} K_T^{\text{A}, \mathbb{P}}} (1 - e^{-R_{\text{A}} \varepsilon \vartheta}) \right]. \end{aligned}$$

Therefore, letting $\varepsilon \rightarrow 0$, the Gâteaux derivative of $V^{\mathbb{P}}$ in $\xi \in \Xi^{\text{FB}}$ in the direction $\vartheta \in \Xi^{\text{FB}}$ is given by

$$\begin{aligned} DV^{\mathbb{P}}(\xi)[\vartheta] &= \mathbb{E}^{\mathbb{P}} \left[-R_{\text{P}} \mathbb{E}^{\mathbb{P}}[\vartheta | \mathcal{F}_T^{\circ}] e^{R_{\text{P}} \mathbb{E}^{\mathbb{P}}[\xi + K_T^{\text{P}} | \mathcal{F}_T^{\circ}]} + \varpi \vartheta R_{\text{A}} e^{-R_{\text{A}} \xi + R_{\text{A}} K_T^{\text{A}, \mathbb{P}}} \right] \\ &= \mathbb{E}^{\mathbb{P}} \left[\mathbb{E}^{\mathbb{P}} \left[-R_{\text{P}} \vartheta e^{R_{\text{P}} \mathbb{E}^{\mathbb{P}}[\xi + K_T^{\text{P}} | \mathcal{F}_T^{\circ}]} \middle| \mathcal{F}_T^{\circ} \right] + \varpi \vartheta R_{\text{A}} e^{-R_{\text{A}} \xi + R_{\text{A}} K_T^{\text{A}, \mathbb{P}}} \right]. \end{aligned}$$

Conditioning by $\mathcal{F}_T^\circ$, we obtain:

$$\begin{aligned} DV^\mathbb{P}(\xi)[\vartheta] &= \mathbb{E}^\mathbb{P} \left[\mathbb{E}^\mathbb{P} \left[-R_P \vartheta e^{R_P \mathbb{E}^\mathbb{P}[\xi + K_T^P | \mathcal{F}_T^\circ]} \middle| \mathcal{F}_T^\circ \right] + \mathbb{E}^\mathbb{P} \left[\varpi h R_A e^{-R_A \xi + R_A K_T^{A,\mathbb{P}}} \middle| \mathcal{F}_T^\circ \right] \right] \\ &= \mathbb{E}^\mathbb{P} \left[-R_P \vartheta e^{R_P \mathbb{E}^\mathbb{P}[\xi + K_T^P | \mathcal{F}_T^\circ]} + \varpi \vartheta R_A e^{-R_A \xi + R_A K_T^{A,\mathbb{P}}} \right]. \end{aligned}$$

Then, for any $\mathbb{P} \in \mathcal{P}$, let us introduce the terminal compensation $\xi^\star(\mathbb{P})$ defined by

$$\xi^\star(\mathbb{P}) := -\frac{1}{R_A + R_P} \ln \left(\frac{R_P}{\rho R_A} \right) + K_T^A - \frac{R_P}{R_A + R_P} \mathbb{E}^\mathbb{P}[K_T | \mathcal{F}_T^\circ]. \quad (4.6.12)$$

so that

$$\begin{aligned} \mathbb{E}^\mathbb{P}[\xi^\star(\mathbb{P}) | \mathcal{F}_T^\circ] &:= -\frac{1}{R_A + R_P} \ln \left(\frac{R_P}{\rho R_A} \right) + \frac{R_A}{R_A + R_P} \mathbb{E}^\mathbb{P}[K_T^A | \mathcal{F}_T^\circ] - \frac{R_P}{R_A + R_P} \mathbb{E}^\mathbb{P}[K_T^P | \mathcal{F}_T^\circ], \\ \text{and } \xi^\star(\mathbb{P}) &:= -\frac{1}{R_A} \ln \left(\frac{R_P}{\rho R_A} \right) + K_T^A - \frac{R_P}{R_A} (\mathbb{E}^\mathbb{P}[\xi^\star(\mathbb{P}) | \mathcal{F}_T^\circ] + \mathbb{E}^\mathbb{P}[K_T^P | \mathcal{F}_T^\circ]). \end{aligned}$$

Therefore, for any $\vartheta \in \Xi^{\text{FB}}$, we have $DV^\mathbb{P}(\xi^\star(\mathbb{P}))[\vartheta] = 0$ and since $V^\mathbb{P}$ is a strictly concave function, $\xi^\star(\mathbb{P})$ attains the minimum of $V^\mathbb{P}$ and is therefore optimal. Plugging these expressions back into the principal's problem and recalling that $\bar{R}$ is defined by $1/\bar{R} := 1/R_A + 1/R_P$, the value function of the principal in the first-best case rewrites:

$$\begin{aligned} V_0^{\text{P,FB}} &= \inf_{\rho > 0} \left\{ \rho \left(-R_0 + \frac{R_A + R_P}{R_P} \exp \left(\frac{R_A}{R_A + R_P} \ln \left(\frac{R_P}{\rho R_A} \right) \right) V_0^{\bar{R}} \right) \right\}, \\ \text{where } V_0^{\bar{R}} &:= \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^\mathbb{P} \left[-\exp \left(\bar{R} \mathbb{E}^\mathbb{P}[K_T | \mathcal{F}_T^\circ] \right) \right]. \end{aligned}$$

Notice that $V_0^{\bar{R}}$ does not depend on ρ . Then direct calculations lead to the following values for respectively the optimal Lagrange multiplier and first-best value function:

$$\rho^\star = \frac{R_P}{R_A} (V_0^{\bar{R}}/R_0)^{1+R_P/R_A} \text{ and } V_0^{\text{P,FB}} = R_0 (V_0^{\bar{R}}/R_0)^{1+R_P/R_A}.$$

Finally, using the same tools as in [Section 4.4](#), we can easily prove the points (i) and (ii) of [Proposition 4.5.1](#). The point (iii) is a straightforward computation of the contract defined by [\(4.6.12\)](#).

PART II

Electricity consumption: between
moral hazard and adverse selection

Abstract

In this part, we focus on a seemingly intriguing application of the two main theories of this thesis: the field of energy. Through the application of principal–agent models to energy issues, we seek to solve two problems, at first glance very different, but ultimately linked by the mathematical formalism of contract theory.

The first issue, treated in [Chapter 5](#), concerns *demand response* programs. The primary purpose of these programs is to reduce overall electricity consumption during price peak, or periods of grid instability, such as high demand periods. When considering this question in terms of incentives, the electricity producer or distributor needs to optimally determine the price signals or rebates to be transmitted to consumers so that the impact on overall consumption is as desired.

Another energy–related societal problem concerns *fuel poverty*. In order to fight against this type of precariousness, it is necessary to find a way to help households in difficulty, or to prevent them from becoming so. Some measures can be put in place to remedy this situation. With this in mind, we will study in [Chapter 6](#), an insurance with benefit in kind, targeted for vulnerable households, for which a loss of income would lead to fuel poverty.

At first glance, these two situations seem opposite: on the one hand, demand–response programmes provide incentives to reduce consumption, whereas in the case of a household suffering from fuel poverty, it would be necessary to provide incentives to increase consumption. In reality, these two problems are part of a logic of energy and social transition and, in particular, of changing consumption habits and better distribute available resources. Moreover, these two issues are also linked by the underlying mathematical formalism, since they both involve problems of optimal incentives, which will be translated through principal–agent models. Note also that these two problems involve an infinite number of consumers, and thus share some similarities with the ones developed in [Part I](#).

Incentive-based demand response management

Ils mettent des rubans autour de l'alphabet
Et sortent dans la rue leurs mots pour prendre l'air

Les Poètes, LÉO FERRÉ

Introduction

Using the theoretical results obtained in [Chapter 4](#), we extend in this chapter the problem of demand response contracts in electricity markets, introduced by [Aïd, Possamaï, and Touzi \[11\] \(2019\)](#), by considering a continuum of consumers with mean-field interaction. In particular, we consider that each consumer can deviate from his baseline consumption, and each consumption is subject to a common noise, which accounts for the common random environment where all the consumers evolve. In regard to energy, this common random environment can be interpreted as consequences of meteorological variations. The first motivation behind this extension is that it seems more realistic to consider that the producer is interested in the overall consumption of its consumers, rather than in individual consumption. We thus assume here that the problem of the producer is to manage a pool of similar consumers, whose consumption is subject to the same climate hazards. The second motivation is that considering a continuum of consumers with mean-field interaction can be profitable for the producer, as she has access to more information by observing all consumption profiles. Apart from the mean-field aspect and the correlation between agents through the common noise, the problem formulation is voluntarily as close as possible to that of [\[11\]](#), and is developed in [Section 5.1](#).

We thus work under a classical mean-field framework, where all agents are identical. This assumption is justified for a major electricity producer or provider, who has a sufficiently large number of similar consumers. Moreover, this framework prevents us from indexing the compensation of a consumer on a particular consumption profile, except his own. This is indeed the case in the energy sector: the General Data Protection Regulation specifies that the Energy Distribution Organising Authorities (EDOA), the licensing authorities, have only the right to access aggregated (and therefore anonymous) data on electricity production and consumption in a given territory ¹. Moreover, in France, a report ² stipulates that a provider or social landlord may use an individual's consumption data to compile statistics, but only if the data are anonymous or aggregated and therefore do not allow the identification of a physical person. Therefore, we can apply the results developed in [Chapter 4](#), and compare the new contracts, indexed on the law of others' consumption, to the traditional ones, within this demand-response management problem.

1. Décret n° 2017-976 of 10 May 2017 on the procedures for making electricity and gas consumption data available to consumers and Décret n° 2017-948 of 10 May 2017 on the procedures for consumers to access electricity or natural gas consumption data and for suppliers to make such data available.

2. Pack de conformité – Les compteurs communicants [Compliance package for communicating meters], Commission Nationale de l'Informatique et des Libertés [National Commission for Information Technology and Liberties].

Section 5.3 is devoted to this comparison, by implementing classical contracts in our framework. In order to have closed-form solutions, we focus here on a particular case: the linear energy value discrepancy case. The energy value discrepancy is the difference between a consumer's preference toward his deviation consumption and the production costs of this deviation. According to the results of the empirical study by Dolan and Metcalfe [129] (2015) on energy efficiency, which shows that comparing energy usage with similar households or providing financial incentives can lead to an average reduction in household energy usage of 7%, we prove mathematically that new contracts are more profitable than the traditional ones. Moreover, in most cases, these contracts induce more efforts from the consumers to reduce the average level of their consumption and with less volatility. Finally, our approach provides better management of the risk associated with common noise. Indeed, the greater the variance explained by the common noise, the more significant the results are. Therefore, these new contracts could improve demand response during periods where consumption is strongly affected by weather conditions, for example in winter, when the risk of electricity blackouts is high, and thus demand response is more than needed.

Throughout this work, we consider a principal who cannot observe the common noise, or at least assume that there exist some regulatory rules preventing her from using the common noise directly in the contract. This hypothesis is relatively well established in the field of energy consumption. Although some electricity suppliers offer different prices depending on the day or time of consumption (peak-period, off-peak period...), called *time-variant pricing*³, these tariff offers correspond more to an indirect indexation on the weather through the spot price of electricity. In fact, the optimal contract is indexed in a hidden way on the consumption adjusted for climate hazards. This allows the principal to offer a compensation indexed on the process which is really controlled by the consumer, to encourage him for making effort on the drift and the volatility of his deviation. Moreover, if the principal is risk-averse, she can add to this contract a part indexed on others, which is in fact an indexation on the common noise, to better choose the remaining risk she wants to bear. Therefore, in the case where she is authorised to index the contract directly to the common noise, we obtain the same form of contracts, as shown in Chapter 4, Section 4.5.2 (p. 141). Nevertheless, if the principal cannot observe the common noise, or if there exist some regulatory rules preventing her from using it explicitly, indexing the contract on others is a way to overcome this.

It should be emphasised that the contracts obtained are very theoretical, and thus raise an important question, that of their implementation in practice. Clearly, it does not seem conceivable to offer such contracts directly to consumers, who would be completely puzzled. The idea, already mentioned in [11], is rather to see the optimal contract as a price signal to be sent to consumers.

5.1 Problem formulation

In this section, we provide an informal formulation of the contracting problem between the principal (the electricity provider or producer) and the continuum of agents with mean-field interaction, already described in Chapter 4, Section 4.1 (p. 118). Hence, we quickly recall the framework under consideration, and the reader is referred to the previous chapter for a rigorous mathematical formulation. Moreover, apart from the mean-field aspect, the framework is voluntarily the same as in [11].

5.1.1 A mean-field of consumers

We work under a classical mean-field framework, where all agents are identical. We thus restrict our study to a representative agent, who is a single consumer, identical to a pool of others, and too small to

3. See for example the proposal of the Environmental Defense Fund in https://www.edf.org/sites/default/files/time-variant_pricing_fact_sheet_-_april_2015.pdf

impact the global consumption. His deviation consumption is driven by an idiosyncratic noise for each usage (d -dimensional) and a common noise (uni-dimensional). The agent controls this process by choosing a pair $\nu := (\alpha, \beta)$, taking values in $U := A \times B$, where $A := \prod_{k=1}^d [0, \rho^k A_{\max}]$ and $B := [B_{\min}, 1]^d$, for some constants $A_{\max} > 0$ and $B_{\min} \in (0, 1)$.⁴ More specifically, α represents the effort of the consumer to reduce the nominal level of consumption and β is the effort to reduce the variability of his consumption for each usage of electricity. We emphasise that α and β are d -dimensional vectors, thus capturing the differentiation between different usages, e.g. refrigerator, heating or air conditioning, lightning, television, washing machine, computers... The set of admissible efforts is denoted by $\mathcal{U} = \mathcal{A} \times \mathcal{B}$. Thanks to the rigorous weak formulation developed in [Section 4.1](#), we know that the set $\mathcal{U}$ of admissible control processes can be associated to a probability space $\mathcal{P}$, so that we can equivalently consider that the agent chooses a probability $\mathbb{P} \in \mathcal{P}$ or a control $\nu \in \mathcal{U}$.

We assume that an effort ν induces a separable cost c , *i.e.*, for $u := (a, b) \in U$,

$$c(u) := \frac{1}{2}c_\alpha(a) + \frac{1}{2}c_\beta(b), \text{ where } c_\alpha(a) := \sum_{k=1}^d \frac{(a^k)^2}{\rho^k}, \text{ and } c_\beta(b) := \sum_{k=1}^d \frac{(\sigma^k)^2}{\lambda^k \eta^k} \left((b^k)^{-\eta^k} - 1 \right),$$

for fixed $(\sigma, \rho, \lambda, \eta) \in (0, +\infty)^d \times (0, +\infty)^d \times (0, +\infty)^d \times (1, +\infty)^d$. In particular, the cost related to the drift term of X , denoted by c_α , is a classical quadratic cost function, meaning that no effort for the agent induces no intrinsic cost, and such that he has no interest to provide negative efforts. The cost associated with the effort β prohibits the agent from removing the volatility ($b^k > 0$) and is equal to zero if the agent makes no effort ($b^k = 1$).

To write the deviation consumption of the representative agent, we consider a map $\sigma : b \in B \mapsto (\sigma^1 \sqrt{b^1}, \dots, \sigma^d \sqrt{b^d})^\top \in \mathbb{R}^d$, as well as a constant $\sigma^\circ \in \mathbb{R}_+$ representing the correlation with the common noise. We will denote by $\Sigma(b) := \sigma^\top(b)\sigma(b)$ for all $b \in B$. Using these notations and for a chosen control $\nu = (\alpha, \beta) \in \mathcal{U}$, the agent's control process X is defined by:

$$X_t := x_0 - \int_0^t \alpha_s \cdot \mathbf{1}_d ds + \int_0^t \sigma(\beta_s) \cdot dW_s + \int_0^t \sigma^\circ dW_s^\circ, \quad t \in [0, T]. \quad (5.1.1)$$

Hence, the previous variable X represents the consumer's deviation from the deterministic profile of his consumption, and we will denote by $\mu \in \mathcal{P}(\mathcal{C}_T)$ the law of X conditionally to the common noise. Using the formulation developed in [Section 4.1.1](#) (p. 118), we will consider the canonical filtration $\mathbb{F} := (\mathcal{F}_t)_{t \in [0, T]}$ as well as the filtration containing only the information generated by the common noise and the conditional law of X , namely $\mathbb{F}^\circ := (\mathcal{F}_t^\circ)_{t \in [0, T]}$.

Similarly, we can define the deviation consumption of other consumers (that is the ones different from the representative agent), denoted by $\hat{X}$, affected by the same common noise W° :

$$\hat{X}_t = \hat{X}_0 - \int_0^t \hat{\alpha}_s \cdot \mathbf{1}_d ds + \int_0^t \sigma(\hat{\beta}_s) \cdot d\hat{W}_s + \int_0^t \sigma^\circ dW_s^\circ, \quad t \in [0, T],$$

where $\hat{\nu} := (\hat{\alpha}, \hat{\beta}) \in \hat{\mathcal{U}}$, taking values in U , is the control process chosen by others. We will denote by $\hat{\mu}$ the law of $\hat{X}$ conditionally to the common noise.

Finally, we will be led to consider *copies* of X and $\hat{X}$, in the sense of [Definition 4.1.5](#) (p. 121), denoted respectively by $\tilde{X}$ and $\tilde{\hat{X}}$ and satisfying:

$$d\tilde{X}_t = -\tilde{\alpha}_s \cdot \mathbf{1}_d ds + \sigma(\tilde{\beta}_s) \cdot d\tilde{W}_s + \sigma^\circ dW_s^\circ, \text{ and } d\tilde{\hat{X}}_t = -\tilde{\alpha}_s \cdot \mathbf{1}_d ds + \sigma(\tilde{\beta}_s) \cdot d\tilde{W}_s + \sigma^\circ dW_s^\circ, \quad t \in [0, T].$$

4. The admissible spaces are assumed to be bounded for technical reasons. Nevertheless, note that since X is a deviation from a baseline consumption, the upper bound for the drift control is natural, since the agent cannot consume a negative amount of electricity.

5.1.2 Definition of a contract

In the paper by Aïd, Possamaï, and Touzi [11] (2019), the principal – *an energy producer* – offers a contract to an agent – *a consumer* – indexed on his deviation consumption. In our investigation, the principal faces a continuum of agents with mean–field interaction and can therefore benefit from this additional information. However, in the framework we are interested in, the electricity producer is not allowed to reveal the consumption of a particular consumer to another consumer. Hence, she cannot directly design a remuneration for a consumer with respect to the deviation consumption of another dedicated one. Nevertheless, she can offer contracts depending on both the deviation of a given consumer, and the aggregate statistics of the deviation of other consumers. We insist on the fact that it is a legal requirement for electricity producers to respect the privacy of their consumers, even when they have access to their consumption profile through modern smart–meters. Moreover, in accordance with the mean–field framework, the principal is facing a mass of identical and indistinguishable consumers, and thus cannot choose a deviation consumption of another consumer to contract on it.

Formally, we assume that the principal proposes to the representative agent a contract ξ which is a random variable measurable with respect to the natural filtration generated by both X and $\hat{\mu}$, denoted $\mathbb{F}^{\text{obs}}$:

$$\xi : (X, \hat{\mu}) \in \mathcal{C}_T \times \mathcal{P}(\mathcal{C}_T) \longrightarrow \xi(X, \hat{\mu}) \in \mathbb{R}, \quad (5.1.2)$$

recalling that X is the deviation consumption of the representative agent and $\hat{\mu}$ is the law of the deviation consumption of other agents, conditionally to the common noise, which naturally comes from the limit of the n –Agents case with common noise, as already mentioned in Remark 4.1.6 (p. 121).

Given this contract ξ , and the conditional law of the deviation consumption of other consumers $\hat{\mu}$, we consider that the representative consumer solves the same optimisation problem as in Chapter 4, namely:

$$V_0^A(\xi, \hat{\mu}) := \sup_{\mathbb{P} \in \mathcal{P}} J_0^A(\xi, \hat{\mu}, \mathbb{P}), \text{ where } J_0^A(\xi, \hat{\mu}, \mathbb{P}) := \mathbb{E}^{\mathbb{P}} \left[U^A \left(\xi(X, \hat{\mu}) - \int_0^T (c(\nu_t^{\mathbb{P}}) - f(X_t)) dt \right) \right], \quad (5.1.3)$$

where the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ denotes the preference of the agent toward his deviation consumption. This function is required to be concave, increasing, and centred at the origin. This means that the reduction of the agent's consumption causes him discomfort, and conversely, if the consumption deviation is non–negative, the agent consumes more electricity, which gives him satisfaction. Closed–form solutions will be obtained for linear f . The function U^A is an exponential utility function, with risk aversion parameter $R_A > 0$, defined by $U^A(x) = -e^{-R_A x}$, $x \in \mathbb{R}$. One may notice that, apart from the contract dependency, the agent's problem is identical to the one defined in [11].

As usual in contract theory, we assume that consumers have a reservation utility R_0 , below which they refuse the contract offered by the principal. Here, we will assume that this reservation utility is endogenous. The underlying idea is that without compensation (*i.e.*, for $\xi = 0$), a consumer could already exert efforts and modify his consumption, and thus receive utility $R_0 < 0$. He will thus refuse any contract which does not provide him with at least what he could get by himself. We refer to Section 5.2.3 for more details on how to compute the value of R_0 . Moreover, as mentioned in Chapter 4, we have to require minimal integrability on the contracts, namely Condition (P) (p. 122) and Condition (CARA) (p. 137). The corresponding class of contracts of the form (5.1.2), satisfying the previous integrability conditions and the participation constraint, is denoted by Ξ .

Finally, given a contract $\xi \in \Xi$, we will denote by $\mathcal{M}^*(\xi)$ the collection of all mean–field equilibria in the sense of Definition 4.1.7 (p. 123). Less formally, a mean–field equilibrium is characterised by

- (i) a probability law $\mathbb{P}^*$ of a process X^* , which is the optimal deviation consumption of each agent;
- (ii) the conditional law μ^* of X^* with respect to the common noise.

5.1.3 The producer's problem

We now turn to the problem of the principal. In the one agent framework defined in [11], the principal has an exponential utility function, with risk-aversion parameter $R_P > 0$, defined by $U^P(x) = -e^{-R_P x}$ and wants to minimise the sum of the following.

- (i) The compensation paid to the agent: ξ .
- (ii) The cost of production, corresponding to additional costs induced by the deviation consumption: $g(X_t)$, where g is concave and increasing. It means that if the agent's deviation consumption is positive, the consumption has increased, hence the principal has an additive cost of production. Conversely, a negative deviation consumption means that the consumption is decreasing, hence the principal benefits from a decrease of production costs.
- (iii) The quadratic variation of the deviation consumption: $\langle X \rangle_t$. This penalisation term allows the principal to take into account the variations of consumption over time. This additional cost is particularly relevant in electricity markets, since the Producer has to follow the load curve, and the higher the volatility of the consumption, the more costly it is.

As already explained in Chapter 4, more precisely in Section 4.1.5 (p. 123), the intuition for the principal problem in the mean-field case comes from the n -Agents case. We only recall here that, in our framework, *with common noise*, the consumption deviations of the agents become asymptotically independent, but only conditionally to the common noise. Therefore, a risk-averse principal does not become risk-neutral in our case, and remains impacted by the residual risk arising from the common noise. Nevertheless, we can consider both cases of a risk-averse and a risk-neutral principal. As a consequence, similarly to [141] and [11], and as in Chapter 4, the principal's value is defined by

$$J_0^P(\xi, \mathbb{P}) := \sup_{\xi \in \Xi} \sup_{(\mathbb{P}, \mu) \in \mathcal{M}^*(\xi)} J_0^P(\xi, \mathbb{P}), \text{ with } \mathbb{E}^{\mathbb{P}} \left[U^P \left(-\mathbb{E}^{\mathbb{P}} \left[\xi + \int_0^T g(X_s) ds + \frac{\theta}{2} \int_0^T d\langle X \rangle_s \middle| \mathcal{F}_T^{\circ} \right] \right) \right], \quad (5.1.4)$$

where $U^P : \mathbb{R} \rightarrow \mathbb{R}$ is the principal's utility function and θ is a positive constant representing the costs induced by the quadratic variation of the consumption, accounting for the limited flexibility of production.

5.2 Contracting with the agents

The framework developed in the previous section is exactly the same as in Chapter 4. Therefore, all results apply here. For the sake of clarity, we briefly summarise in this section the main results, in particular the optimal form of contracts (see Section 5.2.1) and the mean-field equilibrium (Section 5.2.2). Furthermore, as we consider here an endogenous reservation utility, we determine this level in Section 5.2.3. Then, Section 5.2.4 quickly recall the major results concerning the resolution of the principal's problem in the general case, and provide additional interpretations. Finally, we provide in Section 5.2.5 more explicit results, in the particular case of a linear energy value discrepancy.

5.2.1 Simple contracts

We fix for now a probability $\hat{\mathbb{P}} \in \hat{\mathcal{P}}$, chosen by other consumers. Recall that there exists a control process representing the effort of other consumers, denoted $\hat{\nu} := (\hat{\alpha}, \hat{\beta}) \in \hat{\mathcal{U}}$ and associated to $\hat{\mathbb{P}} \in \hat{\mathcal{P}}$ such that the dynamic of their deviation consumption $\hat{X}$ satisfies:

$$d\hat{X}_t = -\hat{\alpha}_t \cdot \mathbf{1}_d dt + \sigma(\hat{\beta}_t) \cdot d\hat{W}_t + \sigma^\circ dW_t^\circ, \quad t \in [0, T].$$

In an ultimate attempt to simplify the notations used in [Chapter 4](#), we informally denote by $\widehat{\mathbb{E}}_t$ the conditional expectation under $\widehat{\mathbb{P}}_t$ given $\mathcal{F}_t^\circ$, and by $\widetilde{\mathbb{E}}_t$ the corresponding notations for copies $\widetilde{X}$ of $\widehat{X}$. The controls of other consumers and the conditional law of the path of $\widehat{X}$, namely $(\widehat{\mu}, \widehat{\nu}) \in \mathcal{P}(\mathcal{C}_T) \times \widehat{\mathcal{U}}$, are thus fixed throughout the choice of $\widehat{\mathbb{P}} \in \widehat{\mathcal{P}}$. Since the principal only observes X and $\widehat{\mu}$, she can only offer an $\mathbb{F}^{\text{obs}}$ -measurable contract as in (5.1.2), implying that the state variables of an agent are $(X, \widehat{\mu})$.

For any positive integer n , we denote by $\mathcal{L}^n$ the set of Borel measurable functionals from $\mathcal{C}_T^n$ into $\mathbb{R}$, and $\mathcal{L} := \mathcal{L}^1$. Following the reasoning developed in [Chapter 4](#), more precisely in [Sections 4.2](#) and [4.3](#) (pp. 125 and 128), the relevant Hamiltonian of the representative agent in our non-Markovian framework is defined by:

$$\mathcal{H}(x, \widehat{\mu}_t, z, z^\mu, \gamma, \widehat{\alpha}_t) := \frac{1}{2}H_d(z) + \frac{1}{2}H_v(\gamma) + H_c(x, \gamma) - \widehat{\mathbb{E}}_t[z^\mu(\widehat{X}_{t\wedge\cdot})\widehat{\alpha}_t \cdot \mathbf{1}_d], \quad (5.2.1)$$

$$\text{where } H_d(z) := -\inf_{a \in A} \{2za \cdot \mathbf{1}_d + c_\alpha(a)\}, \quad H_v(\gamma) := -\inf_{b \in B} \{c_\beta(b) - \gamma \Sigma(b)\}, \quad H_c(x, \gamma) := \frac{1}{2}\gamma|\sigma^\circ|^2 + f(x),$$

for any $(x, z, z^\mu, \gamma) \in \mathbb{R} \times \mathbb{R} \times \mathcal{L} \times \mathbb{R}$ and $(\widehat{\mu}, \widehat{\alpha}) \in \mathcal{P}(\mathcal{C}_T) \times \widehat{\mathcal{A}}$. The optimisers are given, for all $(z, \gamma) \in \mathbb{R}^2$, by

$$a^{k,\star}(z) := \rho^k(z^- \wedge A_{\max}) \quad \text{and} \quad b^{k,\star}(\gamma) := 1 \wedge (\lambda^k \gamma^-)^{\frac{-1}{\eta^k+1}} \vee B_{\min}, \quad \text{for } k \in \{1, \dots, d\}. \quad (5.2.2)$$

Still following the line of [Chapter 4](#), we can define the notion of revealing contracts within this framework. In particular, a contract should be parametrised by a process $\zeta := (Z, Z^\mu, \Gamma)$, taking values in $\mathbb{R} \times \mathcal{L} \times \mathbb{R}$, and by a constant part $\xi_0 \in \mathbb{R}$. First, the constant $\xi_0 \in \mathbb{R}$ should be chosen in order to satisfy the participation constraint of the consumer. The payment rates (Z, Γ) index the contract on the deviation consumption of the considered consumer, in agreement with usual principal-agent problems with drift and volatility control, while the payment rate Z^μ indexes the contract on the behaviour of other consumers, represented by the conditional law $\widehat{\mu}$. Moreover, as described in [Section 4.3.1](#) (p. 128), we have to compute the Hamiltonian in the contract all along the optimal effort of other consumers, *i.e.*, the process $\widehat{\alpha}^\star := a^\star(Z)$, where $a^\star$ is defined component by component in (5.2.2) (for $k \in \{1, \dots, d\}$). We are thus led to consider a particular type of revealing contracts, precisely described in the definition below, similar to [Definition 4.3.1](#) (p. 129).

Definition 5.2.1 (Simple contracts). *For any $\mathbb{R} \times \mathcal{L} \times \mathbb{R}$ -valued $\mathbb{F}^{\text{obs}}$ -predictable process $\zeta := (Z, Z^\mu, \Gamma)$, and any $\xi_0 \in \mathbb{R}$, let us define the following process $\xi^{\xi_0, \zeta}$ for all $t \in [0, T]$ by*

$$\begin{aligned} \xi_t^{\xi_0, \zeta} := & \xi_0 - \int_0^t \mathcal{H}(X_s, \widehat{\mu}_s, \zeta_s, \widehat{\alpha}_s^\star) ds + \int_0^t Z_s dX_s + \frac{1}{2} \int_0^t (\Gamma_s + R_A Z_s^2) d\langle X \rangle_s + \int_0^t \widehat{\mathbb{E}}_s[Z_s^\mu(\widehat{X}_{s\wedge\cdot}) d\widehat{X}_s] \\ & + \frac{1}{2} R_A \int_0^t \widehat{\mathbb{E}}_s \widetilde{\mathbb{E}}_s[Z_s^\mu(\widehat{X}_{s\wedge\cdot}) Z_s^\mu(\widetilde{X}_{s\wedge\cdot}) d\langle \widehat{X}, \widetilde{X} \rangle_s] + R_A \int_0^t Z_s \widehat{\mathbb{E}}_s[Z_s^\mu(\widehat{X}_{s\wedge\cdot}) d\langle X, \widehat{X} \rangle_s]. \end{aligned} \quad (5.2.3)$$

We let $\mathcal{V}$ the set of $\mathbb{R} \times \mathcal{L} \times \mathbb{R}$ -valued $\mathbb{F}^{\text{obs}}$ -predictable process ζ such that [Condition \(J^p\)](#) (p. 129) is satisfied. For $(\xi_0, \zeta) \in \mathbb{R} \times \mathcal{V}$, variables $\xi_T^{\xi_0, \zeta}$ are called simple contracts, and we denote the corresponding set by Ξ_S .

As mentioned in [Section 4.3.2](#) (p. 130), one can notice that if other consumers are playing according to the optimal drift effort $\widehat{\alpha}^\star$, some simplifications appears between the Hamiltonian and the indexation on others in the contract $\xi_T^{\xi_0, \zeta} \in \Xi_S$. In particular, for any process $\zeta := (Z, Z^\mu, \Gamma) \in \mathcal{V}$, let $\bar{Z}_t^\mu = \widehat{\mathbb{E}}_t[Z_t^\mu(\widehat{X}_{t\wedge\cdot})]$, for $t \in [0, T]$ and consider the process $\bar{\zeta} = (Z, \bar{Z}^\mu, \Gamma)$, as well as $\bar{\mathcal{V}}$ the corresponding set. We then obtain:

$$\begin{aligned} \xi_T^{\xi_0, \bar{\zeta}} = & \xi_0 - \int_0^T \mathcal{H}^\circ(X_s, Z_s, \Gamma_s) ds + \int_0^T Z_s dX_s + \sigma^\circ \int_0^T \bar{Z}_s^\mu dW_s^\circ + \frac{1}{2} \int_0^T (\Gamma_s + R_A |Z_s|^2) d\langle X \rangle_s \\ & + \frac{R_A}{2} |\sigma^\circ|^2 \int_0^T \bar{Z}_s^\mu (\bar{Z}_s^\mu + 2Z_s) ds. \end{aligned} \quad (5.2.4)$$

where for $(x, z, \gamma) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}$,

$$\mathcal{H}^\circ(x, z, \gamma) := \frac{1}{2}H_d(z) + \frac{1}{2}H_v(\gamma) + H_c(x, \gamma). \quad (5.2.5)$$

This simplification also applies if the principal can observe the common noise and is legally entitled to index the contract on it. This form therefore shows that indexing on the conditional law is actually a hidden indexing on the common noise. Moreover, it underlines the fact that if $\sigma^\circ = 0$, a simple contract in the sense of [Definition 5.2.1](#) is exactly a standard contract for drift and volatility control (see [11, 110, 111]). Therefore, in a similar framework but without common noise, it is straightforward to conclude that considering contracts indexed on the output of the population will not improve the results given in [11]. We refer to [Section 5.3](#) for more details on this result.

5.2.2 Mean-field equilibrium and optimal efforts

By considering simple contracts, we are able to compute the optimal efforts of the representative agent, which were given informally by (5.2.2). Intuitively, maximising the Hamiltonian given by (5.2.1) is sufficient to obtain optimal efforts, but the formal proof relies on the theory of 2BSDEs, developed in [Chapter 4](#). The consumer's optimal efforts do not depend on the efforts of the others, which simplifies the task of obtaining the unique mean-field equilibrium in the sense of [Definition 4.1.7](#) (p. 123). Therefore, [Theorem 4.3.4](#) holds: given a contract $\xi_T^{\xi_0, \zeta} \in \Xi_S$ for $\xi_0 \in \mathbb{R}$ and $\zeta \in \mathcal{V}$, there exists a unique mean-field equilibrium $(\mathbb{P}^*, \mu^*)$.

In this case, the optimal effort of the consumers are given by the processes $\alpha^* := a^*(Z)$ and $\beta^* := b^*(\Gamma)$, where the functions a^* and b^* are defined component by component (*i.e.*, for $k \in \{1, \dots, d\}$) by (5.2.2). In the following, we denote by $v^*(z, \gamma) := (a^*(z), b^*(\gamma))$ the optimal response of a given agent, and the cost associated to this effort by $c^*(z, \gamma)$, for all $(z, \gamma) \in \mathbb{R}^2$. Then, $\mathbb{P}^*$ (respectively μ^*) is the law (resp. conditional law given $\mathbb{F}^\circ$) of X driven by optimal controls, whose dynamic satisfies:

$$dX_t = -\bar{\rho}(Z_t^- \wedge A_{\max})dt + \sigma^*(\Gamma_t) \cdot dW_t + \sigma^\circ dW_t^\circ, \quad t \in [0, T]. \quad (5.2.6)$$

One can notice that the optimal efforts of the consumer are defined through the same function as the one in [11], although the contracts we consider have more components. The additional component, namely the ones indexed by Z^μ , do not affect the optimal effort of the consumer. These results are consistent with classical results on drift and volatility control (see [110, 111]): contracts indexed by (Z, Γ) are sufficient to incentivise the agent for making effort on drift and volatility. Therefore it is quite natural that another parameter in the contract will not directly affect the effort, but may increase the value function of the principal. More precisely, given a tuple of payment rates $\zeta := (Z, Z^\mu, \Gamma) \in \mathbb{R} \times \mathcal{L} \times \mathbb{R}$, we can make the following interpretations on the optimal efforts.

- (i) The payment rate Z induces an effort of the consumer on all usages to reduce his consumption in average. The effort on the k -th usage is proportional to its cost, $1/\rho^k$, non increasing with Z , positive if Z is negative and zero otherwise.
- (ii) Γ induces an effort only on the usages whose cost $1/\lambda^k$ is lower than the payment. If for $t \in [0, T]$, Γ_t is non-negative, then $b^{k,*}(\Gamma_t) = 1$ for all usages, implying that the consumer makes no effort at time t on the volatility of his consumption deviation. The more Γ is negative, the more β^* will be close to zero, *i.e.*, the more the agent will reduce the volatility of his consumption deviation;
- (iii) Finally, the payment Z^μ has no influence on the consumer's efforts: although his payment is indexed on the deviation consumption of others, the consumer will not take it into account to optimise his deviation consumption. In fact, it seems rather logical in the sense that a consumer optimises his consumption independently of what his neighbours do, even if the price of electricity depends on the global demand.

5.2.3 Reservation utility of the consumer

An admissible contract ξ , offered by the principal to the representative consumer, has to satisfy the agent's participation constraint, namely $V^A(\xi) \geq R_0$. In this chapter, we assume that R_0 is defined as the expected utility of the consumer without contract, *i.e.*,

$$R_0 := \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}} \left[-\exp \left(-R_A \int_0^T (f(X_s) - c(\nu_s^{\mathbb{P}})) ds \right) \right].$$

The underlying idea is that consumers will refuse the contract if it provides them with a utility level lower than the one they could attain by themselves without any contract. The corresponding value can then be obtained through a standard HJB equation, which can be solved explicitly when f is linear. Apart from the terms depending on the common noise, the same results as in [11] are obtained. Nevertheless, we choose to detail the proofs of the various results in Section 5.5, to highlight the impact of the common noise.

Proposition 5.2.2 (Reservation utility). *Assume that f has linear growth. Then the following holds.*

- (i) *The consumer's reservation utility is given by $R_0 = -e^{-R_A \psi(0, X_0)}$, where the corresponding certainty equivalent ψ is a solution (in the viscosity sense) of the HJB equation, for $(t, x) \in [0, T] \times \mathbb{R}$,*

$$\begin{cases} 0 = \partial_t \psi(t, x) + f(x) - \frac{1}{2} \left(c_{\beta}^*(\gamma^0(t, x)) - \gamma^0(t, x) \Sigma^*(\gamma^0(t, x)) - \gamma^0(t, x) |\sigma^\circ|^2 \right), \\ \psi(T, x) = 0, \end{cases} \quad (5.2.7)$$

where $\gamma^0(t, x) := \partial_{xx}^2 \psi(t, x) - R_A |\partial_x \psi(t, x)|^2$, for $(t, x) \in [0, T] \times \mathbb{R}$.

- (ii) *Assume that PDE (5.2.7) has a $C^{1,2}$ solution ψ such that the following condition is satisfied*

$$\mathbb{E}^{\mathbb{P}} \left[\int_0^T e^{-2R_A \psi(t, X_t)} |\partial_x \psi(t, X_t)|^2 dt \right] < +\infty, \text{ for any } \mathbb{P} \in \mathcal{P}. \quad (5.2.8)$$

Then, the consumer's optimal effort $\nu^0 := (\alpha^0, \beta^0)$ corresponds to the following feedback controls:

$$\alpha_t^0 := 0 \quad \text{and} \quad \beta_t^{0,k} := b^{k,*}(\gamma^0(t, X_t)), \quad k \in \{1, \dots, d\}, \quad t \in [0, T]. \quad (5.2.9)$$

The previous result shows that, even without contracting, the consumer's optimal behaviour exhibits a positive effort to reduce the volatility of the consumption deviation process. Of course, this is naturally due to the fact that consumers in our model are assumed to be risk-averse, and are therefore negatively impacted by the variance of their deviation. We conclude this section by providing a closed-form expression for the reservation utility when f is linear.

Proposition 5.2.3 (f linear). *Let $f(x) = \kappa x$, $x \in \mathbb{R}$, with $\kappa \geq 0$. Then, the function ψ introduced in Proposition 5.2.2 satisfies in particular $\psi(0, X_0) = \kappa T X_0 + \psi_0$, where for $(t, x) \in [0, T] \times \mathbb{R}$,*

$$\psi_0 := - \int_0^T H_0(\gamma^0(t)) dt, \quad \text{with} \quad \gamma^0(t) := -R_A |\kappa|^2 |T - t|^2 \quad \text{and} \quad H_0(\gamma) := \frac{1}{2} (c_{\beta}^*(\gamma) - \gamma \Sigma^*(\gamma) - \gamma |\sigma^\circ|^2).$$

In addition, the consumer's optimal effort on the drift and on each volatility usage are respectively given by

$$\alpha^0 := 0 \quad \text{and} \quad \beta_t^{0,k} := 1 \wedge \left(\lambda^k R_A |\kappa|^2 |T - t|^2 \right)^{\frac{-1}{\eta^{k+1}}} \vee B_{\min}, \quad k \in \{1, \dots, d\}.$$

thus inducing an optimal distribution $\mathbb{P}^0$ under which the deviation process X satisfies the following:

$$dX_t = \sigma^*(\gamma^0(t)) \cdot dW_t + \sigma^\circ dW_t^\circ.$$

Remark 5.2.4. *One can notice that the certainty equivalent ψ , introduced in Proposition 5.2.2, is a decreasing function of the correlation σ° with the common noise, since*

$$\psi(t, x) = \kappa x(T - t) - \frac{1}{2} \int_t^T \left(c^*(-R_A |A(s)|^2) + R_A |A(s)|^2 \Sigma^*(-R_A |A(s)|^2) \right) ds - \frac{1}{2} R_A |\sigma^\circ|^2 \int_t^T |A(s)|^2 ds,$$

for a particular function A specified in the proof. Therefore, the reservation utility of the consumer is negatively impacted by the presence of common noise.

Until now, we have considered the new form of contracts introduced in Chapter 4, called simple contracts, to compute the optimal efforts of the consumers and the associated unique mean-field equilibrium in this demand-response framework, using Theorem 4.3.4. In parallel, we have determined the value of the consumer's reservation utility R_0 . The aim of the following is to recall that there is no loss of generality to consider only simple contracts, and to solve the principal's problem thus restricted to these simple contracts.

5.2.4 Principal's problem

We recall that the optimisation problem of the principal is defined by (5.1.4). Following the reasoning developed in Chapter 4, there is no loss of generality for the principal to restrict the study to contracts in Ξ_S , in the sense of Definition 5.2.1, instead of Ξ . Therefore, instead of maximising her criterion on all contracts $\xi \in \Xi$, the principal only have to choose a constant $\xi_0 \in \mathbb{R}$ and a triple of payment rates $\zeta \in \mathcal{V}$. Recall that $\xi_0 \in \mathbb{R}$ should be chosen in order to ensure the participation of the consumer, *i.e.*, such that $J_0^A(\xi, \mu^*, \mathbb{P}^*) \geq R_0$, where $(\mathbb{P}^*, \mu^*) \in \mathcal{M}^*(\xi)$ is the unique mean-field equilibrium associated to $\xi \in \Xi_S$. As stated in Theorem 4.4.1 (p. 133), the optimal constant is thus given by $\xi_0 = U_A^{-1}(R_0)$, and the principal's problem boils down to the following:

$$V_0^P = \sup_{\zeta \in \mathcal{V}} \mathbb{E}^{\mathbb{P}^*} \left[U^P \left(-\mathbb{E}^{\mathbb{P}^*} [L_T^\zeta | \mathcal{F}_T^\circ] \right) \right], \text{ where } L_t^\zeta := \xi_t^\zeta + \int_0^t g(X_s) ds + \frac{\theta}{2} \int_0^t d\langle X \rangle_s, \quad t \in [0, T].$$

From now on, it is assumed that f and g have linear growths. Thus, the result of Proposition 5.2.2 holds. With this in mind, the theoretical results described in Chapter 4 hold, namely Theorem 4.4.7, Propositions 4.4.8 and 4.4.10. Nevertheless, we have chosen not to recall the results here, as we will present more explicit results in the linear energy discrepancy case in the next section, namely Section 5.2.5. We only provide here additional interpretations in relation to our study on demand-response programme.

Risk-averse principal. Under a constant relative risk aversion specification of the utility function of the producer, *i.e.*, $U^P(x) := -e^{-R_P x}$, for some risk-aversion $R_P > 0$, the optimal contract is given by Proposition 4.4.8 (p. 138) (v). The most interesting part of the contract concerns the infinitesimal payment $Z_s^*(dX_s - \tilde{\mathbb{E}}^{\mathbb{P}^*}[d\tilde{X}_s])$. Indeed, since $Z^* \leq 0$ by Proposition 4.4.8 (ii), this payment is positive if the consumer's deviation consumption is below the mean of others' deviation. Conversely, if the consumer makes less effort than the rest of the pool, this part of the compensation will be negative. Therefore, as expected, a part of the compensation is based on the comparison between the deviation consumption of a consumer and the mean of others' deviation consumption.

In addition to that, we can derive a more illuminating form of contract, by denoting by X° the deviation consumption without common noise, whose dynamic under optimal efforts is given by

$$dX_t^\circ = -\bar{\rho}((Z_t^*)^- \wedge A_{\max}) dt + \sigma^*(\Gamma_t^*) \cdot dW_t, \quad t \in [0, T].$$

The optimal contract can thus be written in terms of the common noise as follows

$$\begin{aligned}\xi_T^* &= U_A^{-1}(R_0) - \int_0^T \mathcal{H}(X_s^\circ + \sigma^\circ W_s^\circ, Z_s^*, \Gamma_s^*) ds + \int_0^T Z_s^* dX_s^\circ + \frac{1}{2} \int_0^T (\Gamma_s^* + R_A |Z_s^*|^2) d\langle X^\circ \rangle_s \\ &\quad + \frac{R_P \sigma^\circ}{R_A + R_P} \int_0^T \bar{u}_{\mu^X}^P(s, \mu_s^X) dW_s^\circ + \frac{1}{2} \frac{R_A |R_P|^2}{|R_A + R_P|^2} |\sigma^\circ|^2 \int_0^T |\bar{u}_{\mu^X}^P(s, \mu_s^X)|^2 ds,\end{aligned}$$

where $\mathcal{H}^*(x, z, \gamma) := \frac{1}{2}H_d(z) + \frac{1}{2}H_v(\gamma) + f(x)$, for $(x, z, \gamma) \in \mathbb{R}^3$. We can then divide the study of the optimal contract ξ^* given by the previous equation in two parts.

- (i) The first equation line is in fact the classical contract form for drift and volatility control, indexed on the process X° , which is the part of the deviation consumption really controlled by the agent:
 - (a) the contract is linear in the level of X° and its quadratic variation $\langle X^\circ \rangle$;
 - (b) by responding to the contract with optimal efforts, the consumer gains a certain amount of utility. Therefore, the principal can subtract the certainty equivalent of this utility gain from the contract, *i.e.*, the constant part $\int_0^T \mathcal{H}(X_s^\circ + \sigma^\circ W_s^\circ, \bar{\zeta}_s^*) ds$;
 - (c) due to the risk-aversion of the consumer, the additional payment $\frac{1}{2}R_A |Z_t^*|^2 d\langle X^\circ \rangle_t$ is needed to compensate the infinitesimal payment $Z_t^* dX_t^\circ$.

Since the common noise W° represents in our framework the climate hazards, the process X° can be interpreted as the deviation consumption adjusted for climate hazards. Therefore, this part of the contract is a fixed compensation, independent of weather conditions.

- (ii) The other part of the contract is an indexation on the common noise, that is the remaining risk the principal wants to give to the agent. As for the infinitesimal payment $Z_t dX_t$, due to the risk aversion of the consumer, the term $\frac{R_P}{R_A + R_P} \sigma^\circ \bar{u}_{\mu^X}^P(s, \mu_s^X) dW_s^\circ$, must be compensated by

$$\frac{1}{2} R_A \left| \frac{R_P}{R_A + R_P} \sigma^\circ \bar{u}_{\mu^X}^P(s, \mu_s^X) \right|^2 d\langle W^\circ \rangle_s = \frac{1}{2} \frac{R_A |R_P|^2}{|R_A + R_P|^2} |\sigma^\circ|^2 |\bar{u}_{\mu^X}^P(s, \mu_s^X)|^2 ds.$$

We can already notice that, if the principal is risk-neutral, she will not use at all the common noise to provide incentives to the agent. Indeed, if she is risk-neutral and since the consumers are risk-averse, it is too costly to share the risk, and she can bear it alone.

The conclusion of this interpretation is that the indexation of the contract on others allows the principal to divide the deviation consumption of an agent into two parts: the part which is really controlled by the agent, X° , and the common noise. Hence, she offers a compensation indexed on the controlled deviation X° to encourage the agent for making effort on the drift and the volatility. Moreover, if she is risk-averse, she adds to this contract a part indexed on the common noise, to share the remaining risk, even if regulatory rules prevent her from using the common noise directly in the contract.

Risk-neutral principal. To consider a risk-averse principal, let $U^P(x) = x$, for $x \in \mathbb{R}$. As explained in Chapter 4, by letting $R_P = 0$ in Proposition 4.4.8, we can deduce Proposition 4.4.10 (p. 139), which establishes the optimal contract in this case. As for the risk-averse case, to better see the compensation the consumer will receive, we can denote by X° his deviation consumption without common noise, and write the contract in term of common noise as follows:

$$\xi_t^* = U_A^{-1}(R_0) - \int_0^T \mathcal{H}^*(X_s^\circ + \sigma^\circ W_s^\circ, Z_s^*, \Gamma_s^*) ds + \int_0^T Z_s^* dX_s^\circ + \frac{1}{2} \int_0^T (\Gamma_s^* + R_A |Z_s^*|^2) d\langle X^\circ \rangle_s,$$

recalling that $\mathcal{H}^*(x, z, \gamma) := \frac{1}{2}H_d(z) + \frac{1}{2}H_v(\gamma) + f(x)$, for $(x, z, \gamma) \in \mathbb{R}^3$.

Therefore, in the risk-neutral case, the optimal contract is the classical contract form for drift and volatility control, indexed on the process X° , representing the deviation consumption adjusted for climatic hazards. This result comes from the fact that the principal uses the indexation of the contract on the deviation of others, *i.e.*, $\bar{Z}^\mu$, to isolate the common noise. Hence, she has a clear view of what is the deviation consumption of the agents, without the common noise (X°). Therefore, she can offer a payment which is only indexed on the part of the consumption which is really optimised by the agents. In this particular case of risk-averse agents and risk-neutral principal, the principal can bear the risk alone and offers a contract which does not depend on the common noise.

5.2.5 Application to the linear energy value discrepancy (EVD) case

The energy value discrepancy is the difference between a consumer's preference toward his deviation consumption (represented by the function f) and the production costs of this deviation (represented by g). Following the line of [11], to obtain closed-form solutions, we consider in this section the case where

$$(f - g)(x) = \delta x, \quad x \in \mathbb{R}.$$

Intuitively, if δ is positive, this means that the energy is more valuable for the consumer than it is costly for the producer. Therefore, a reduction of the consumption has a negative effect more important on the consumers' utilities than the positive effect on the producer's utility. Similarly, δ negative implies that an increase of consumption induces more cost for the producer than the benefit generated for the consumer. Therefore, intuitively, if δ is negative, it will be easier for the principal to incentivise the consumers to reduce their consumption.

Under this assumption, we derive a closed-form solution in both cases of a risk-averse and risk-neutral principal and more explicit optimal payment rates. To lighten the notation, we define

$$\bar{h}(t, z) := F_0(\theta + R_A |z|^2) + \bar{\rho} |(z^- \wedge A_{\max}) + \delta(T - t)|^2, \quad (t, z) \in [0, T] \times \mathbb{R}.$$

We sum up the results in the following proposition, whose proof is postponed to [Section 5.5](#).

Proposition 5.2.5. *Let the energy value discrepancy be linear, *i.e.* $(f - g)(x) = \delta x$, $x \in \mathbb{R}$. Define the certainty equivalent function u^P as*

$$u^P(t, \mu_t^X) := \delta(T - t) \int x \mu_t^X(dx) - \int_t^T m^P(s) ds,$$

$$\text{where } m^P(t) := \frac{\theta}{2} |\sigma^\circ|^2 + \frac{1}{2} \left(|\sigma^\circ|^2 \bar{R} - \bar{\rho} \right) |\delta|^2 |T - t|^2 + \frac{1}{2} \inf_{z \in \mathbb{R}} \bar{h}(t, z), \quad t \in [0, T],$$

and where μ^X denotes for the conditional law of X given the common noise. Then, the following holds:

- (i) *the optimal payment rate process $\bar{\zeta}^\star = (Z^\star, \bar{Z}^{\mu^\star}, \Gamma^\star)$ is a deterministic function of time, independent of σ° and is defined for all $t \in [0, T]$ by*

$$Z_t^\star := \arg \min_{z \in \mathbb{R}} \bar{h}(t, z), \quad \bar{Z}_t^{\mu^\star} := -Z_t^\star + \frac{R_P}{R_A + R_P} \delta(T - t) \text{ and } \Gamma_t^\star := -\max \{ \theta + R_A |Z_t^\star|^2, 1/\bar{\lambda} \};$$

- (ii) *the value function of a producer with a CARA utility function and a risk-aversion parameter R_P is given by $V_0^P = -e^{R_P(\xi_0 - u^P(0, \mu_0^X))}$;*
- (iii) *the value function of a risk-neutral producer is given by $V_0^0 = -\xi_0 + u^0(0, \mu_0^X)$.*

The above proposition underlines the fact that the optimal payment rates Z^* and Γ^* are the same, in both cases of a risk-averse or risk-neutral principal. Hence, the efforts of the consumers on their deviation consumptions will be the same, whatever the risk aversion of the principal. The principal controls the risk she wants to bear thanks to the control $\bar{Z}^\mu$. Indeed, in the risk-neutral case, the principal does not care about the risk, hence $\bar{Z}^\mu$ is such that the contract does not depend on the common noise. On the other hand, in the risk-averse case, the agent is remunerated for a part of the common noise: the risk induced by the common noise is shared between the agent and the principal.

Remark 5.2.6. *For all $t \in [0, T]$, the optimal payment rate Z_t^* is equal to zero if δ is non-negative and lies between $\delta(T - t) \vee -A_{\max}$ and 0 if $\delta < 0$.*

Moreover, one can notice that in this particular case, the only information that the principal uses from the conditional law μ^X is actually the conditional mean. Indeed, the only term with μ^X appears in the certainty equivalent function u^P of the producer, with the form $\int x \mu_t^X(dx)$ which is equal to $\mathbb{E}^P[X_t | \mathcal{F}_t^\circ]$.

To resume this section, we used the results established in [Chapter 4](#) to develop a new form of contract in the context of demand-response programme for electricity consumption. This new form of contract allows to easily compute the mean-field equilibrium between the consumers and is without loss of generality. Even with an endogenous reservation utility, we obtain almost closed-form solutions for the principal's problem, in particular in the case of a linear energy value discrepancy, and provide additional interpretations on the optimal contract.

5.3 Comparison with classical contracts

The aim of this section is to study the benefits of indexing the contracts on the distribution of the deviation of other consumers. We thus consider, as a benchmark case, the producer's problem when incentives are limited to payments for efforts of the considered consumer. More precisely, in the benchmark case, we consider that the principal can only offer contracts controlled by $\zeta^0 := (Z, 0, \Gamma)$ instead of $\zeta := (Z, Z^\mu, \Gamma)$. We denote by $\mathcal{V}^0$ the corresponding restriction of $\mathcal{V}$. The theoretical results in this benchmark case are postponed to [Section 5.4](#).

In this section, we focus on the linear energy value discrepancy (EVD) case to compare the results, in terms of consumers' effort and producer's utility, between the two types of contracts, in both cases of a risk-averse (see [Section 5.3.1](#)) or a risk-neutral producer ([Section 5.3.2](#)). Although some comparisons can be obtained analytically, through the formula developed all along this chapter for optimal controls and value functions, it may be required to numerically compute the optimal payment rate Z^* . Indeed, this is the only quantity for which we cannot get a closed-form expression, even in the linear EVD case. For numerical computations, the calibration of the parameters detailed in [\[11\]](#) is used. Indeed, even if they considered in [\[11\]](#) the consumption, by fixing the initial condition of the consumption to be zero, the consumption process X in their work becomes directly the observed reduction of consumption, which is our canonical process X .

More precisely, unless otherwise specified, we will consider the parameters given in [Table 5.1](#), where 'p' stands for pence. We will explicitly indicate different values for specific parameters, for instance when we investigate the influence of R_P on the utility of the principal. Concerning the volatility calibrated in [\[11\]](#) (namely the nominal volatility, with value 0.085 kW/h^{1/2}), we will suppose that it is equal to the total volatility without effort in our model, i.e. $\sqrt{\Sigma(1) + |\sigma^\circ|^2}$. Hence, we will consider different values for σ° but such that the total volatility without effort remains constant, equal to 0.085. Notice that the numerical computations will be performed for $d = 1$.

κ (p/kWh)	R_A (p ⁻¹)	ρ (kW ² h ⁻¹ p ⁻¹)	η	λ (kW ² h/p)
11.76	$5,7 \times 10^{-3}$	9.3×10^{-5}	1	2.8×10^{-2}

(a) Consumers' characteristics

θ (p/kW ² h)	R_P (p ⁻¹)
4×10^{-3}	6×10^{-3}

(b) Producer's characteristics

T (h)	δ (p/kWh)	$\sqrt{\Sigma(1) + \sigma^\circ ^2}$ (kW/h ^{1/2})
5.5	-55.44	0.085

(c) Other characteristics

Table 5.1 – Set of parameters

5.3.1 Comparison for a risk-averse principal

Recall that we focus here on the linear energy value discrepancy case. Under a CARA specification of the utility function of the producer, her utility, the optimal effort of the consumers, as well as the optimal contract are given by [Proposition 5.2.5](#), for the new type of contracts indexed by $\zeta := (Z, Z^\mu, \Gamma)$, and by [Proposition 5.4.3](#) (p. 177) for the benchmark case, *i.e.*, with classical contracts indexed by $(Z, 0, \Gamma)$. The main point is to study the consumers' effort and the producer's utility when she offers *new contracts* compared to *classical ones*, through a comparative analysis of the two aforementioned propositions.

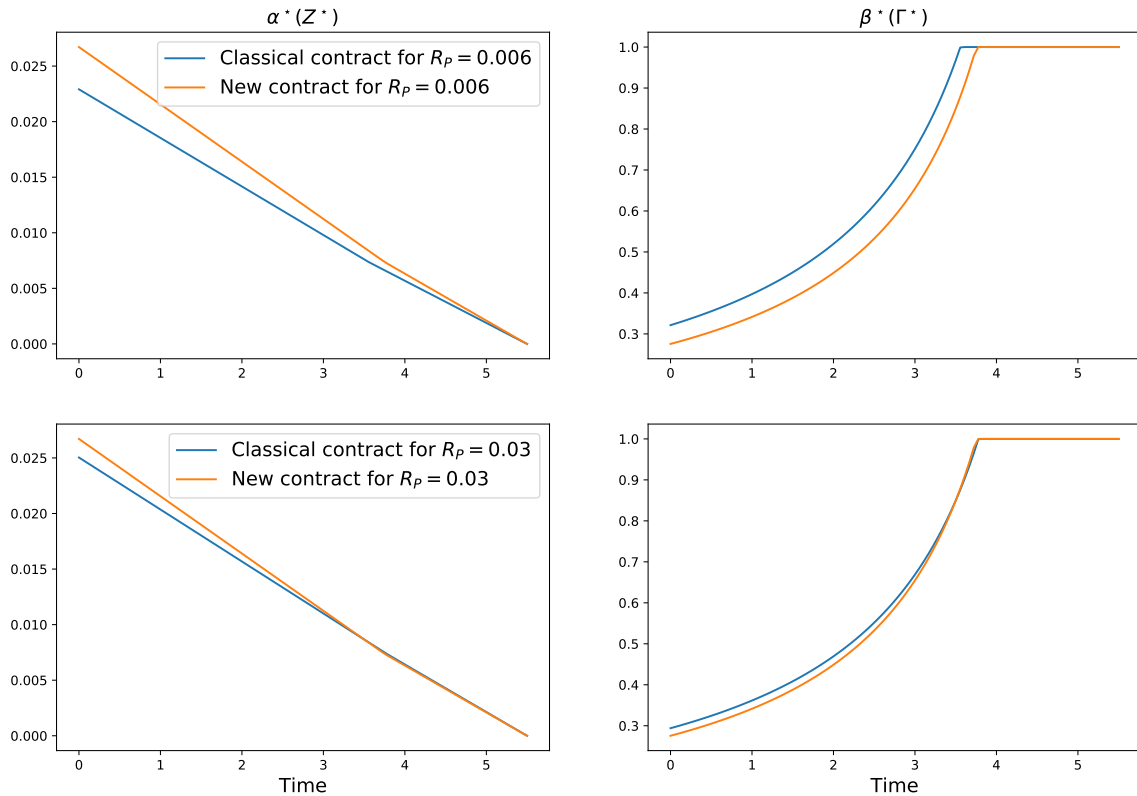


Figure 5.1 – Comparison of efforts in the linear EVD case

Parameters: $\sigma^\circ = \sigma = 0.085/\sqrt{2}$ and $R_P = 0.006$ (upper graphs) or $R_P = 0.03$ (lower graphs).

Comparison of efforts. To compare the efforts of consumers on the average level of their consumption and on its volatility, we need to compare Z^* with $Z^{0,*}$, as well as Γ^* with $\Gamma^{0,*}$, since the optimal efforts in both cases are given by the functions a^* and b^* of these payment rates. For this purpose, we must distinguish two cases, according to the sign of the linear EVD parameter δ .

(i) We first focus on the most representative case, *i.e.*, when $\delta \leq 0$, since empirical results in [11] provide $\delta = -55.44$. However, analytic comparison of efforts in this case is not clear. Figure 5.1 represents the optimal efforts of the consumers on the drift and the volatility for two values of R_P close to the one calibrated in [11], and when 50% of the variance is explained by the common noise (*i.e.* $\sigma^\circ = \sigma = 0.085/\sqrt{2} \approx 0.0601$). The blue lines represent the optimal efforts in the case of classical contracts, while the orange lines are dedicated to our new contracts. Recall that, with contracts indexed on others, the effort of the agents on the drift and on the volatility are independent of the risk aversion of the principal. Therefore,

- (a) for $R_P = 0.006$ (upper graphs), thanks to the contract indexed on others, the principal can incentivise the consumers to make more effort on the drift and on the volatility (orange curves compared to blue ones), for the entire duration of the contract;
- (b) if the risk aversion of the principal increases ($R_P = 0.03$, lower graphs), with the classical contract, she is asking the consumer to make more efforts, whereas with the contract indexed on others, she requires the same effort regardless of her risk aversion. Thus, we obtain that with the new contract, the consumer makes still more efforts in the beginning of the contract, but less after.

First, the fact that the efforts are decreasing in time can be explained as follows. At every moment $t \in [0, T]$ of the contract, what matters to the principal is rather the integral between 0 and t of the efforts than the instantaneous efforts. Indeed, she is interesting in the deviation X_t for all $t \in [0, T]$ and in the quadratic variation of X_T , informally given as integrals of the effort. Therefore, it is in the principal's interest to ask for more effort at the beginning of the contract.

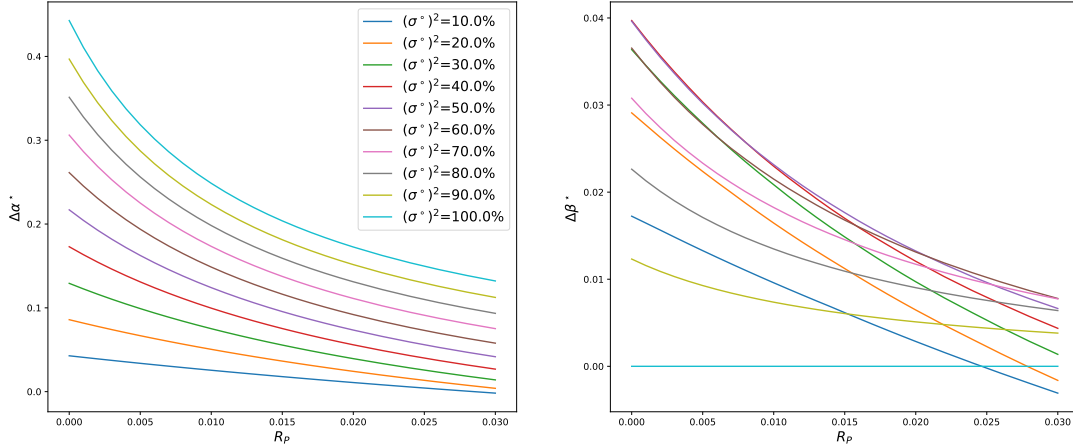


Figure 5.2 – Relative gain in mean (left) and in volatility (right) of the consumption.

Then, in order to assess the benefit of our new contracts on the efforts of the consumers, we compute the following two quantities:

$$\Delta\alpha^* := \frac{\int_0^T (a^*(Z_t^*) - a^*(Z_t^{0,*}))dt}{\int_0^T a^*(Z_t^{0,*})dt} \quad \text{and} \quad \Delta\beta^* := -\frac{\sigma^2 \int_0^T (b^*(\Gamma_t^*) - b^*(\Gamma_t^{0,*}))dt}{\sigma^2 \int_0^T b^*(\Gamma_t^{0,*})dt + T|\sigma^\circ|^2},$$

representing the relative gain respectively in mean and in volatility of the consumption between new and classical contracts. More precisely, if $\Delta\alpha^* \geq 0$ (resp. $\Delta\beta^* \geq 0$), the new contracts incentivise the consumers to decrease more the mean (resp. the volatility) of their total consumption at the end of the contract (at time T). The results are presented in Figure 5.2. We choose values for R_P of the same order than the estimation in [11] given in Table 5.1. For σ° , we set values such that $|\sigma^\circ|^2 = x\%$ of the total variance without effort (nominal variance), i.e. $\Sigma(1) + |\sigma^\circ|^2 = 0.085^2$. For example, when $\sigma^\circ = 0.085$, then $|\sigma^\circ|^2 = 100\%$ of the nominal variance: this means that the volatility in the deviation consumption is entirely related to climate hazards ($\sigma = 0$).

Figure 5.2 show that, in most cases, our new contracts lead to a significant decrease in consumption on average and on volatility compared to classical contracts. However, when the risk-aversion of the principal increases, this gain becomes negative. Nevertheless, it can be stressed that, first, for the parameters calibrated in [11], there is a significant gain regardless of the correlation with the common noise. Second, even if the principal has a relatively high risk-aversion, our new contracts allow a reduction on average and on volatility of the consumption if it is strongly impacted by weather conditions. Therefore, our contracts can help to better manage consumption cut-off during peak demand due to climate hazard.

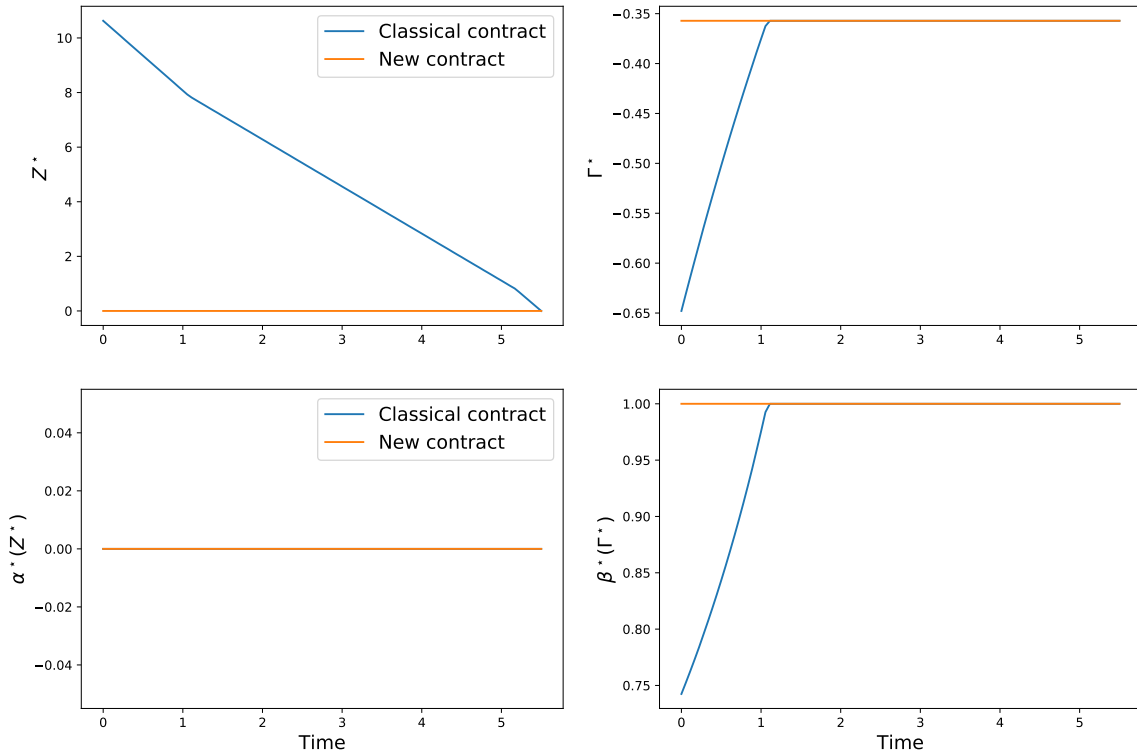


Figure 5.3 – Comparison of payment rates and efforts with $\delta \geq 0$.

Parameters: $\delta = 5$, $\lambda = 2.8$ and $\sigma^\circ = \sigma = 0.085/\sqrt{2}$.

(ii) Out of curiosity, we can also investigate the positive δ case. Using Remark 5.2.6, we can prove, analytically in this case, that $Z^{0,*} \geq Z^* = 0$. On the one hand, since the payment rate on the drift is positive in both cases, the effort of the consumer to reduce his consumption in average is zero. On the other hand, this inequality leads to $\Gamma^{0,*} \leq \Gamma^* \leq 0$. Therefore, when the principal offers classical contracts, she incentivises more the consumers to make effort on the volatility. Nevertheless, since $\lambda = 2.8 \times 10^{-2}$ and

$\theta = 4 \times 10^{-3}$, we can remark that the consumer will in fact make no effort on the volatility, since

$$\Gamma_t^{0,*} = -\max\{\theta + R_A|Z_t^{0,*}|^2, 1/\lambda\} = -1/\lambda,$$

and thus $\beta_t^*(\Gamma_t^{0,*}) = 1 \wedge (\lambda(\Gamma_t^{0,*})^-)^{-1/(\eta^k+1)} \vee B_{\min} = 1$.

Therefore, for small λ , both contracts lead to zero effort on the drift and on the volatility. To find a case where our contracts induce less effort on the volatility than classical ones, we need to increase drastically λ for example.

Figure 5.3 illustrates this particular case, with $\delta = 5$, $\lambda = 2.8$ and when 50% of the variance is explained by the common noise ($\sigma^\circ = \sigma = 0.085/\sqrt{2}$). Although the results obtained do not seem intuitive, the explanation is as follows. When δ is positive, a reduction of the consumption strongly decreases the consumers' utility in comparison to the marginal gain on the producer's utility. Therefore, it is too costly for the principal to incentivise the consumers to make efforts, in particular on the drift. Thus, the principal sets a non-negative payment rate Z (upper left graph). However, since she is risk-averse, she wants to share some risk with the consumers through the contract. In the classical contracts case, the only way to share risk is through the infinitesimal payment $Z_t^{0,*}dX_t$. This explain why $Z^{0,*}$ is positive (upper left graph, blue curve), while, in the new contracts case, the principal can share the risk through the indexation on others by $\bar{Z}^{\mu,*}$, and can therefore set $Z^* = 0$ (upper left graph, orange curve). Moreover, when the principal offers a positive payment Z , she needs to compensate it by giving a negative payment Γ (upper right graph, blue curve), in order to minimise the payment indexed on the quadratic variation $(\Gamma_t + R_A|Z_t|^2)d\langle X \rangle_t$. Otherwise, in the new contracts case, the principal manages the risk through the payment rate $\bar{Z}^\mu$, and does not need to compensate with a small (negative) Γ (upper right graph, orange curve).

We may think that the previous results are disappointing, because consumers make less effort to reduce the volatility of their consumption (lower right graph). But as mentioned above, we insist on the fact that this case is not supposed to happen, since, according to calibration in [11], $\lambda = 2.8 \times 10^{-2}$ and $\delta = -55.44 \leq 0$.

Comparison of utility. We can prove analytically that the utility of the principal is bigger in the case she can indexed contracts on others' deviation consumption. Indeed, we have

$$\begin{aligned} m^{0,P}(t) &= \frac{\theta}{2}|\sigma^\circ|^2 - \frac{1}{2}\bar{\rho}|\delta|^2|T-t|^2 + \frac{1}{2}\inf_{z \in \mathbb{R}} \bar{h}^P(t, z) \\ &\geq \frac{\theta}{2}|\sigma^\circ|^2 - \frac{1}{2}\bar{\rho}|\delta|^2|T-t|^2 + \frac{1}{2}\inf_{z \in \mathbb{R}} \bar{h}(t, z) + \frac{1}{2}|\sigma^\circ|^2 \inf_{z \in \mathbb{R}} \left\{ R_A|z|^2 + R_P|-z + \delta(T-t)|^2 \right\}. \end{aligned}$$

The second infimum is attained on $z = R_P\delta(T-t)/(R_A + R_P)$, which leads to

$$m^{0,P}(t) \geq \frac{\theta}{2}|\sigma^\circ|^2 + \frac{1}{2}\left(|\sigma^\circ|^2\bar{R} - \bar{\rho}\right)|\delta|^2|T-t|^2 + \frac{1}{2}\inf_{z \in \mathbb{R}} \bar{h}(t, z) = m^P(t).$$

Hence, for all $t \in [0, T]$ we have $m^{0,P}(t) \geq m^P(t)$, and thus $V_0^P \geq V_0^{0,P}$. This result is very intuitive since in the case of new contracts, the principal has more controls to maximise her utility (three instead of two).

In order to assure convergence when R_P tends to zero, we draw in Figure 5.4a the (absolute) utility difference defined as follow

$$\begin{aligned} \Delta V &:= (1 + V_0^P)/R_P - (1 + V_0^{0,P})/R_P = (V_0^P - V_0^{0,P})/R_P, \\ \text{since } \lim_{R_P \rightarrow 0} (1 + V_0^P)/R_P &= \lim_{R_P \rightarrow 0} \left(1 - e^{R_P(\xi_0 - u^P(0, \mu_0^X))}\right)/R_P = -\xi_0 + u^0(0, \mu_0^X) = V_0^0. \end{aligned}$$

We study the effect of the risk-aversion parameter R_P on the producer's utility, and also the impact of the percentage of variance related to common noise.

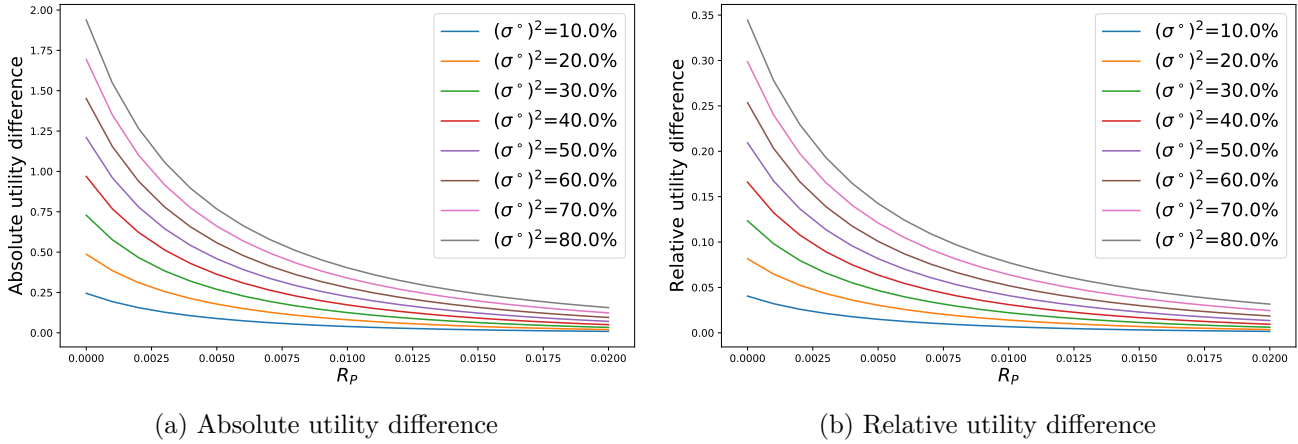


Figure 5.4 – Comparison of utility in the linear EVD case

Variation with respect to the risk aversion parameter R_P and the correlation with the common noise σ° .

First, notice that the utility difference is of order 1, but as the utility itself. Therefore, we have a significant gain by implementing this type of contracts, which is confirmed by Figure 5.4b, representing the relative utility difference, computed as:

$$\bar{\Delta}V = \frac{(1 + V_0^P)/R_P - (1 + V_0^{0,P})/R_P}{(1 + V_0^{0,P})R_P} = \frac{V_0^P - V_0^{0,P}}{1 + V_0^{0,P}}.$$

In addition, it is clear that the more significant the correlation with the common noise is, the more important the utility difference is. This is an expected result, since our type of contracts allows the principal to better choose the remaining risk she wants to bear, by indexing the contract on others. We have already noticed that, in absence of common noise ($\sigma^\circ = 0$), our contracts are reduced to classical contracts for drift and volatility control.

Finally, Figure 5.4 also shows that the gain in utility is decreasing with the risk-aversion of the principal. Indeed, with classical contracts, she is forced to give to the consumers some risk through the linear payment $Z_t dX_t$. Since the consumers are risk-averse too, this payment must be compensated by a deterministic one. Our new contracts allows her to better choose the risk she wants to bear. On the one hand, if her risk-aversion is low, she can keep all the risk to herself. Thus, she does not need to compensate the risk with a deterministic payment, and therefore she make a significant gain. On the other hand, if she is risk-averse, she adds a random part to the contract, indexed on the common noise:

$$\frac{R_P}{R_A + R_P} \sigma^\circ \delta \int_0^t (T - s) dW_s^\circ.$$

But, since the consumers are risk-averse too, this random payment must be compensated by its quadratic variation:

$$\frac{1}{2} \frac{R_A |R_P|^2}{|R_A + R_P|^2} |\sigma^\circ|^2 |\delta|^2 \int_0^t |T - s|^2 ds,$$

Therefore, the more risk-averse the principal is, the more costly it is for her to spread the risk.

5.3.2 Comparison for a risk-neutral principal

To establish the same type of results in the case of a risk-neutral principal, we compare the consumers' optimal efforts and the principal's value obtained in [Proposition 5.4.4](#) (for sophisticated contracts) and in [Proposition 5.2.5](#) (for classical contracts).

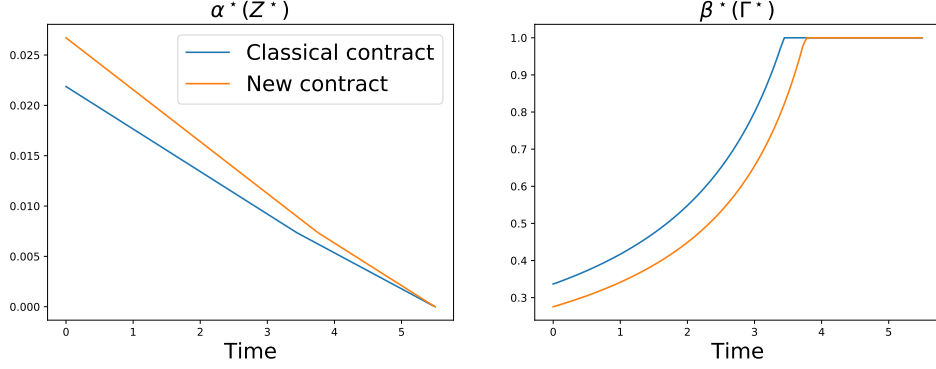


Figure 5.5 – Comparison of efforts for a risk-neutral principal.

Comparison of efforts. As previously, we need to distinguish cases according to the sign of δ .

(i) On the one hand, in the meaningful case, when $\delta \leq 0$, the efforts of the consumers are higher when the principal can index contracts on others' deviation. Indeed, let us recall that the optimal payment rates $Z_t^{0,*}$ and Z_t^* respectively satisfy, for $t \in [0, T]$,

$$Z_t^{0,*} = \arg \min_{z \in \mathbb{R}} \left\{ F_0(h + R_A|z|^2) + R_A|\sigma^\circ|^2|z|^2 + \bar{\rho}|(z^- \wedge A_{\max}) + \delta(T-t)|^2 \right\} \leq 0,$$

and $Z_t^* = \arg \min_{z \in \mathbb{R}} \left\{ F_0(h + R_A|z|^2) + \bar{\rho}|(z^- \wedge A_{\max}) + \delta(T-t)|^2 \right\} \leq 0.$

By definition of the minimum, we have:

$$\begin{aligned} & F_0(\theta + R_A|Z_t^{0,*}|^2) + R_A|\sigma^\circ|^2|Z_t^{0,*}|^2 + \bar{\rho}|((Z_t^{0,*})^- \wedge A_{\max}) + \delta(T-t)|^2 \\ & \leq F_0(\theta + R_A|Z_t^*|^2) + \bar{\rho}|(Z_t^*)^- \wedge A_{\max}) + \delta(T-t)|^2, \end{aligned}$$

for all $z \in \mathbb{R}$, and in particular for $z = Z_t^*$. In the same way,

$$F_0(\theta + R_A|Z_t^*|^2) + \bar{\rho}|((Z_t^*)^- \wedge A_{\max}) + \delta(T-t)|^2 \leq F_0(h + R_A|z|^2) + \bar{\rho}|(z^- \wedge A_{\max}) + \delta(T-t)|^2,$$

for all $z \in \mathbb{R}$ and in particular for $z = Z_t^{0,*}$. Hence, using the first inequality, and then the second, we have

$$\begin{aligned} R_A|\sigma^\circ|^2(|Z_t^{0,*}|^2 - |Z_t^*|^2) & \leq F_0(\theta + R_A|Z_t^*|^2) - F_0(\theta + R_A|Z_t^{0,*}|^2) \\ & \quad + \bar{\rho}|((Z_t^*)^- \wedge A_{\max}) + \delta(T-t)|^2 - \bar{\rho}|((Z_t^{0,*})^- \wedge A_{\max}) + \delta(T-t)|^2 \leq 0. \end{aligned}$$

Hence $R_A|\sigma^\circ|^2(Z_t^{0,*} - Z_t^*)(Z_t^{0,*} + Z_t^*) \leq 0$, and since $Z_t^{0,*}, Z_t^* \leq 0$, we obtain $0 \geq Z_t^{0,*} \geq Z_t^*$. Therefore, in the case where the principal can index contracts on the deviation consumption of others, the efforts of the consumer to reduce his deviation consumption in average is more important. Moreover, the inequality on the optimal payment rate Z implies that $0 \geq \Gamma_t^{0,*} \geq \Gamma^*$: the effort on the volatility is also more important. These results are presented in [Figure 5.5](#) for $|\sigma^\circ|^2 = 50\%$, *i.e.* 50% of the variance is explained by the common noise.

(ii) On the other hand, if $\delta \geq 0$, we obtain $Z^* = 0$ in both cases, which leads to zero effort from the consumers on their deviation consumption in average. Moreover, the optimal payment rate Γ^* is also the same in both cases and leads to the same effort on the volatility. Therefore, in this particular case, the choice of new or classical contracts does not affect the consumers efforts.

Comparison of utility. Naturally, the utility of a risk-neutral principal is higher when she can index contracts on others' deviation consumption. Indeed,

$$\begin{aligned} m^{0,0}(t) &= \frac{\theta}{2}|\sigma^\circ|^2 - \frac{\bar{\rho}}{2}|\delta|^2|T-t|^2 + \frac{1}{2} \inf_{z \in \mathbb{R}} \left\{ F_0(h + R_A|z|^2) + R_A|\sigma^\circ|^2|z|^2 + \bar{\rho}|(z^- \wedge A_{\max}) + \delta(T-t)|^2 \right\} \\ &\geq \frac{\theta}{2}|\sigma^\circ|^2 - \frac{\bar{\rho}}{2}|\delta|^2|T-t|^2 + \frac{1}{2} \inf_{z \in \mathbb{R}} \left\{ F_0(h + R_A|z|^2) + \bar{\rho}|(z^- \wedge A_{\max}) + \delta(T-t)|^2 \right\} \\ &= m^0(t). \end{aligned}$$

Hence, for all $t \in [0, T]$ we have $m^{0,0}(t) \geq m^0(t)$, which leads to

$$-\int_t^T m^{0,0}(s)ds \leq -\int_t^T m^0(s)ds, \quad t \in [0, T],$$

and we conclude that $V_0^0 \geq V_0^{0,0}$. This result is shown in Figure 5.6. The absolute utility difference and the relative utility difference are increasing with respect to the correlation with the common noise, which is consistent with the intuition that the sophisticated contracts allow the principal to manage the remaining risk. The more important this risk is, the more gain in utility the principal receives.

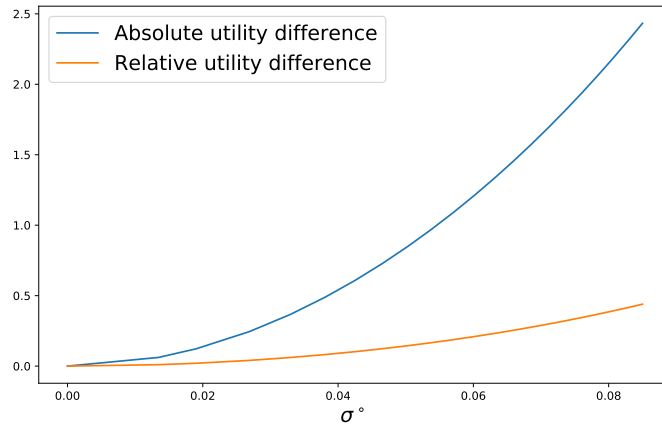


Figure 5.6 – Absolute and relative utility difference for a risk-neutral principal
Variation with respect to the correlation with the common noise σ° .

To conclude this section, even if we focused on the linear energy value discrepancy case, it seems that there is a net benefice from implementing contracts with indexation on others' deviation consumption. In addition to the substantial gain in utility for the principal, this type of contracts induces, in general, more efforts of the consumers to reduce their consumption in average and with less volatility. The best results are obtained for a risk-neutral producer. In this case, if the variance of the deviation is only explained by the common noise, the utility with new contracts is up to 1.5 the utility with classical contracts (see Figure 5.6). Moreover, Figure 5.2 (left) shows that consumers reduce their consumption by 1.5 times more on average. The best results on volatility is when the volatility is half explained by the common noise, *i.e.* for $\sigma^\circ = 0.085/\sqrt{2}$. The consumers thus increase their efforts on the volatility by almost 4%.

5.4 Theoretical results for classical contracts

In this section, we provide some theoretical results when the principal can only consider contracts controlled by $\zeta^0 := (Z, 0, \Gamma)$ instead of $\zeta := (Z, Z^\mu, \Gamma)$. We denote by $\mathcal{V}^0$ the corresponding restriction of $\mathcal{V}$. We provide the optimal contract ξ^0 and effort of the consumers in [Section 5.4.1](#). We focus on the linear energy value discrepancy (EVD) case to compare the results, in terms of consumers' effort and producer's utility, between the two types of contracts, in both cases of a risk-averse (see [Section 5.4.2](#)) or a risk-neutral producer ([Section 5.4.3](#)). These results are used below in [Section 5.3](#), to demonstrate the benefits of considering the new type of contracts, indexed on the consumption's aggregate statistics.

5.4.1 Classical contracts

We consider the same model, with the same dynamic of the deviation consumption X , but we restrict our study to contracts offered by the principal to a consumer depending only his deviation. This type of contracts has the classical form of contracts for drift and volatility control (see [\[11\]](#), [\[110\]](#), [\[111\]](#)):

$$\xi_T^0 = \xi_0 - \int_0^T \mathcal{H}^\circ(X_t, \zeta_t^0) dt + \int_0^T Z_t dX_t + \frac{1}{2} \int_0^T (\Gamma_t + R_A Z_t^2) d\langle X \rangle_t,$$

where $\zeta^0 := (Z, 0, \Gamma)$, an $\mathbb{R} \times \{0\} \times \mathbb{R}$ -valued $\mathbb{F}$ -predictable process, is the set of parameters chosen by the principal and with $\xi_0 = U_A^{-1}(R_0)$.

We have seen before that the consumer's optimal response to a contract indexed by $\zeta = (Z, Z^\mu, \Gamma)$ is independent of Z^μ , and in particular that $\nu^*(\zeta^0) = (\alpha^*(Z), \beta^*(\Gamma))$. Therefore, intuitively, in the case of classical contracts, mainly with $Z^\mu = 0$, the latter should be the same function of (Z, Γ) . This result coincides with those in [\[11\]](#), and is consistent with the interpretations detailed in [Section 5.2.2](#). Thus, these contracts lead obviously to the same unique mean-field equilibrium as the one defined in [Theorem 4.3.4](#), in terms of functional of efforts.

The principal has to choose optimally the indexation parameters (Z, Γ) to maximise her utility. Her value function is the same as before, but restricted to the controls $\zeta^0 \in \mathcal{V}^0$:

$$V_0^{0,P} := \sup_{\zeta^0 \in \mathcal{V}^0} \mathbb{E}^{\mathbb{P}} \left[U^P \left(-\mathbb{E}^{\mathbb{P}} \left[L_T^{\zeta^0} \middle| \mathcal{F}_T^\circ \right] \right) \right]. \quad (5.4.1)$$

Following the lines of [Section 4.4](#), we obtain an HJB equation similar to [\(4.4.3\)](#), but with the supremum on ζ^0 (intuitively by considering $Z^\mu = 0$ in [\(4.4.3\)](#)). Therefore, we can establish a result analogous to [Theorem 4.4.7](#) (p. 136), but for simplicity, we focus here on the risk-averse and risk-neutral cases.

5.4.2 Principal with CARA utility

Under a CARA specification of the utility function of the producer, by mimicking the reasoning in [Section 4.4.2](#), we expect a solution $v^{0,P}$ to the HJB equation associated with [\(5.4.1\)](#) to be given by

$$v^{0,P}(t, \mu_t^Y) = -e^{R_P \left(\mathbb{E}^{\mathbb{P}_t} \left[L_t^{\zeta^0} \right] - u^{0,P}(t, \mu_t^X) \right)}, \quad (5.4.2)$$

where $u^{0,P}$ satisfies the following PDE (similar to [\(4.4.7\)](#))

$$\begin{aligned} 0 = & -\partial_t u^{0,P} + \int (g - f)(x) \mu_t^X(dx) + \frac{\theta}{2} |\sigma^\circ|^2 - \frac{1}{2} |\sigma^\circ|^2 \bar{u}_{x, \mu^X}^{0,P} - \frac{1}{2} |\sigma^\circ|^2 \bar{u}_{\mu^X, \mu^X}^{0,P} - \frac{1}{2} \bar{\rho} (\bar{u}_{\mu^X}^{0,P})^2 \\ & + \frac{1}{2} \inf_{z \in \mathbb{R}} \left\{ F_0(q(z, \bar{u}_{x, \mu^X}^{0,P})) + R_A |\sigma^\circ|^2 |z|^2 + \bar{\rho} ((z^- \wedge A_{\max}) + \bar{u}_{\mu^X}^{0,P})^2 + R_P |\sigma^\circ|^2 (-z + \bar{u}_{\mu^X}^{0,P})^2 \right\}, \end{aligned} \quad (5.4.3)$$

with terminal condition $u^{0,P}(T, \mu_T^X) = 0$. We can state the equivalent of [Proposition 4.4.8 \(p. 138\)](#) in our benchmark case with classical contracts.

Proposition 5.4.1. *Assume that there is a solution u to PDE (5.4.3), smooth enough in the sense of [Definition 4.4.5 \(p. 136\)](#), satisfying [Condition \(4.4.8\) \(p. 138\)](#), as well as a function $\bar{v}^{0,*} : [0, T] \times \mathcal{P}(\mathcal{C}_T) \rightarrow \mathbb{R} \times \{0\} \times \mathbb{R}$ that reaches the infimum of h^P defined by (4.4.9). Then, $V_0^{0,P} = -e^{R_P(\xi_0 - u(0, \mu_0^X))}$ and the optimal payment rates $Z^{0,*}$ and $\Gamma^{0,*}$ are respectively given by:*

- (i) $Z_t^{0,*} := z^{0,*}(t, \mu_t^X)$, $t \in [0, T]$, where the function $z^{0,*}$ is the minimiser of the minimisation problem in (5.4.3);
- (ii) $\Gamma_t^{0,*} := \gamma^{0,*}(t, \mu_t^X)$, $t \in [0, T]$, where for $(t, \mu) \in [0, T] \times \mathcal{P}(\mathbb{R})$,

$$\gamma^{0,*}(t, \mu) := -\max \left\{ \theta - \bar{u}_{x, \mu^X}(t, \mu) + R_A |z^{0,*}(t, \mu)|^2, 1/\bar{\lambda} \right\}.$$

Remark 5.4.2. *In the case where the consumers are risk-neutral ($R_A = 0$), the PDE reduces to*

$$\begin{aligned} 0 = & -\partial_t u^{0,P} + \int (g - f)(x) \mu^X(dx) + \frac{\theta}{2} |\sigma^\circ|^2 - \frac{1}{2} |\sigma^\circ|^2 \bar{u}_{x, \mu^X}^{0,P} - \frac{1}{2} |\sigma^\circ|^2 \bar{u}_{\mu^X, \mu^X}^{0,P} - \frac{1}{2} \bar{\rho} (\bar{u}_{\mu^X}^{0,P})^2 \\ & + \frac{1}{2} F_0(\theta - \bar{u}_{x, \mu^X}^{0,P}) + \frac{1}{2} \inf_{z \in \mathbb{R}} \left\{ \bar{\rho} ((z^- \wedge A_{\max}) + \bar{u}_{\mu^X}^{0,P})^2 + R_P |\sigma^\circ|^2 (-z + \bar{u}_{\mu^X}^{0,P})^2 \right\}. \end{aligned}$$

Hence, the infimum is attained for $z^* := \bar{u}_{\mu^X}^{0,P}$ and, similarly to [Remark 4.4.9 \(p. 138\)](#), the resulting optimal contract do depend on the producer's risk aversion R_P .

In order to obtain closed-form solutions, we can study the case of the linear EVD, similarly to [Proposition 5.2.5](#). To lighten the notations, we denote for $(t, z) \in [0, T] \times \mathbb{R}$,

$$\bar{h}^P(t, z) := F_0(\theta + R_A |z|^2) + R_A |\sigma^\circ|^2 |z|^2 + \bar{\rho} (z^- \wedge A_{\max}) + \delta(T - t)^2 + R_P |\sigma^\circ|^2 (-z + \delta(T - t))^2.$$

Proposition 5.4.3. *Let the energy value discrepancy be linear, i.e. $(f - g)(x) = \delta x$, $x \in \mathbb{R}$. Then,*

- (i) *the producer's value function is given by $V_0^{0,P} = -e^{R_P(\xi_0 - u^{0,P}(0, \mu_0^X))}$ where the certainty equivalent function $u^{0,P}$ is characterised by*

$$\begin{aligned} u^{0,P}(t, \mu_t^X) &= \delta(T - t) \int x \mu_t^X(dx) - \int_t^T m^{0,P}(s) ds, \\ \text{where } m^{0,P}(t) &:= \frac{\theta}{2} |\sigma^\circ|^2 - \frac{1}{2} \bar{\rho} |\delta|^2 |T - t|^2 + \frac{1}{2} \inf_{z \in \mathbb{R}} \bar{h}^P(t, z); \end{aligned}$$

- (ii) *the optimal payment rate process $\zeta^{0,*} = (Z^{0,*}, 0, \Gamma^{0,*})$ is a deterministic function of time given by*

$$Z_t^{0,*} = \arg \min_{z \in \mathbb{R}} \bar{h}^P(t, z) \quad \text{and} \quad \Gamma_t^{0,*} = -\max \left\{ \theta + R_A |Z_t^{0,*}|^2, 1/\bar{\lambda} \right\} \quad \text{for } t \in [0, T].$$

5.4.3 Risk-neutral principal

If the principal is risk-neutral, we expect a solution of the form

$$v^{0,0}(t, \mu_t^Y) = -\mathbb{E}^{\mathbb{P}_t} [L_t^{\zeta^0}] + u^{0,0}(t, \mu_t^X). \quad (5.4.4)$$

It can be formally proved that the results of the previous section hold with $R_P = 0$. In particular, $u^{0,0}$ is solution to the PDE (5.4.3) with $R_P = 0$, and we can also state the equivalent of [Proposition 5.4.1](#) with $R_P = 0$ in this case. But, in order to obtain closed-form solution, we focus on the case of the linear energy value discrepancy.

Proposition 5.4.4. *Let the energy value discrepancy be linear, i.e. $(f - g)(x) = \delta x$, $x \in \mathbb{R}$. Then*

(i) *the producer's value function is given by $V_0^{0,0} = -\xi_0 + u^{0,0}(0, \mu_0^X)$ where for all $t \in [0, T]$,*

$$u^{0,0}(t, \mu_t^X) := \delta(T - t) \int x \mu_t^X(dx) - \int_t^T m^{0,0}(s) ds,$$

$$\text{with } m^{0,0}(t) := \frac{\theta}{2} |\sigma^\circ|^2 - \frac{1}{2} \bar{\rho} |\delta|^2 |T - t|^2 + \frac{1}{2} \inf_{z \in \mathbb{R}} \bar{h}^0(t, z);$$

(ii) *the optimal payment rate process $\zeta^{0,*} = (Z^{0,*}, \Gamma^{0,*})$ is a deterministic function of time given by*

$$Z_t^{0,*} = \arg \min_{z \in \mathbb{R}} \bar{h}^0(t, z) \quad \text{and} \quad \Gamma_t^{0,*} = -\max \{ \theta + R_A (Z_t^{0,*})^2, 1/\bar{\lambda} \}.$$

5.5 Technical proofs

Proof of Proposition 5.2.2. (i) Since the function f is non-decreasing, the consumer has no reason to make an effort on the drift of his consumption deviation, as no compensation is offered for this costly effort. More rigorously, the comparison theorem for SDEs with the same diffusion coefficient, see for instance Peng and Zhu [275, Corollary 3.1] (2006), as well as the fact that f is non-decreasing imply immediately that the supremum in the definition of R_0 can only be attained for efforts of the form $\nu^\mathbb{P} = (0, \beta^\mathbb{P}) \in \mathbb{U}$. Notice however that, due to his risk aversion, he might have interest to make effort on the volatility. Denoting by $R(t, X_t)$ the dynamic version of the reservation utility, satisfying

$$R(0, X_0) = R_0 \quad \text{and} \quad R(T, X_T) = -1;$$

and using standard stochastic control theory, we obtain the following HJB equation for the function R :

$$0 = \partial_t R(t, x) - R_A R(t, x) f(x) + \frac{1}{2} |\sigma^\circ|^2 \partial_{xx}^2 R(t, x) + \frac{1}{2} \sup_{b \in B} \{ R(t, x) R_A c_\beta(b) + \partial_{xx}^2 R(t, x) \Sigma(b) \}.$$

The optimal effort on the volatility without contract is thus given for all $t \in [0, T]$ and $k \in \{1, \dots, d\}$ by $\beta_t^{0,k} := b^{k,*}(\gamma^0(t, X_t))$ where

$$\gamma^0(t, x) := -\frac{\partial_{xx}^2 R(t, x)}{R(t, x) R_A}, \quad (t, x) \in [0, T] \times \mathbb{R}.$$

Moreover, using the same notations as in the previous subsection, we obtain, for $(t, x) \in [0, T] \times \mathbb{R}$,

$$0 = \partial_t R(t, x) - R_A R(t, x) f(x) + \frac{1}{2} R(t, x) R_A (c_\beta^*(\gamma^0(t, x)) - \gamma^0(t, x) \Sigma^*(\gamma^0(t, x)) - \gamma^0(t, x) |\sigma^\circ|^2).$$

The solution to the previous PDE is non-positive and can thus be written as $R(t, x) = -e^{-R_A \psi(t, x)}$ for all $(t, x) \in [0, T] \times \mathbb{R}$, where the function ψ is the certainty equivalent function, satisfying the PDE (5.2.7).

(ii) Let ψ be a $\mathcal{C}^{1,2}$ -solution to the PDE (5.2.7). We then can apply Itô's formula to the function $R_0(t, x) := -e^{-R_A \psi(t, x)}$ under an arbitrary $\mathbb{P} \in \mathcal{P}$

$$dR_0(s, X_s) = \partial_t R_0(s, X_s) ds - \alpha_s^\mathbb{P} \cdot \mathbf{1}_d \partial_x R_0(s, X_s) ds + \partial_x R_0(s, X_s) (\sigma(\beta_s^\mathbb{P}) \cdot dW_s + \sigma^\circ dW_s^\circ) \\ + \frac{1}{2} \partial_{xx}^2 R_0(s, X_s) (\Sigma(\beta_s^\mathbb{P}) + |\sigma^\circ|^2) ds.$$

Denoting by $M^\mathbb{P}$ the process defined by $M_t^\mathbb{P} := \mathcal{E}_t^\mathbb{P} R_0(t, X_t)$ where $\mathcal{E}_t^\mathbb{P} := e^{R_A \int_0^t (\frac{1}{2} c_\beta(\beta_s^\mathbb{P}) - f(X_s)) ds}$, for all $t \in [0, T]$, we obtain, again by Itô's formula,

$$\begin{aligned} M_t^\mathbb{P} = & M_0^\mathbb{P} + \int_0^t \mathcal{E}_s^\mathbb{P} \left(\partial_t R_0(s, X_s) - \alpha_s^\mathbb{P} \cdot \mathbf{1}_d \partial_x R_0(s, X_s) + \frac{1}{2} \partial_{xx}^2 R_0(s, X_s) (\Sigma(\beta_s^\mathbb{P}) + |\sigma^\circ|^2) \right) ds \\ & + \int_0^t M_s^\mathbb{P} R_A (c(\nu_s^\mathbb{P}) - f(X_s)) ds + \int_0^t \mathcal{E}_s^\mathbb{P} \partial_x R_0(s, X_s) (\sigma(\beta_s^\mathbb{P}) \cdot dW_s + \sigma^\circ dW_s^\circ). \end{aligned}$$

Replacing by the derivatives of ψ , we obtain

$$M_t^\mathbb{P} = M_0^\mathbb{P} + \int_0^t R_A M_s^\mathbb{P} h^\psi(X_s, \partial_x \psi, \partial_{xx}^2 \psi, \alpha_s^\mathbb{P}, \beta_s^\mathbb{P}) ds - \int_0^t R_A M_s^\mathbb{P} \partial_x \psi (\sigma(\beta_s) \cdot dW_s + \sigma^\circ dW_s^\circ),$$

$$\text{where } h^\psi(x, \partial_x \psi, \partial_{xx}^2 \psi, a, b) = -\partial_t \psi + c(\nu_s) - f(x) + a \cdot \mathbf{1}_d \partial_x \psi + \frac{1}{2} (R_A |\partial_x \psi|^2 - \partial_{xx}^2 \psi) (\Sigma(b) + |\sigma^\circ|^2).$$

Under [Condition \(5.2.8\)](#), the term $\int_0^t R_A M_s^\mathbb{P} \partial_x \psi (\sigma(\beta_s^\mathbb{P}) \cdot dW_s + \sigma^\circ dW_s^\circ)$, is a $\mathbb{P}$ -martingale. Indeed, recall that under any $\mathbb{P} \in \mathcal{P}$, the drift and the volatility of X are bounded, c_β and Σ are continuous functions on the compact set B , and f has linear growth. Hence, using in particular Cauchy–Schwarz inequality, [\(5.2.8\)](#) ensures that the above stochastic integral is in $\mathbb{H}^1(\mathbb{P})$ and is thus a $\mathbb{P}$ -martingale. We deduce

$$\mathbb{E}^\mathbb{P}[M_T^\mathbb{P}] = M_0^\mathbb{P} + \mathbb{E}^\mathbb{P} \left[\int_0^t R_A M_s^\mathbb{P} h^\psi(X_s, \partial_x \psi, \partial_{xx}^2 \psi, \alpha_s^\mathbb{P}, \beta_s^\mathbb{P}) ds \right].$$

Using the boundary condition for ψ , and replacing by their respective values of $M_0^\mathbb{P}$ and $M_T^\mathbb{P}$, we obtain

$$R_0(0, X_0) = \mathbb{E}^\mathbb{P} \left[-e^{R_A \int_0^T (\frac{1}{2} c_\beta(\beta_s^\mathbb{P}) - f(X_s)) ds} \right] - \mathbb{E}^\mathbb{P} \left[\int_0^t R_A M_s^\mathbb{P} h^\psi(X_s, \partial_x \psi, \partial_{xx}^2 \psi, \alpha_s^\mathbb{P}, \beta_s^\mathbb{P}) ds \right].$$

Finally, using the HJB equation satisfied by ψ , we have

$$h^\psi(x, \partial_x \psi, \partial_{xx}^2 \psi, a, b) = \frac{1}{2} (c_\alpha(a) + 2a \cdot \mathbf{1}_d \partial_x \psi) + \frac{1}{2} (c_\beta(b) - \gamma^0 \Sigma(b)) - \frac{1}{2} c_\beta^*(\gamma^0) + \frac{1}{2} \gamma^0 \Sigma^*(\gamma^0),$$

and by simple computations

$$\begin{aligned} c_\beta(b) - \gamma^0 \Sigma(b) &\geq \inf_{b' \in B} \{c_\beta(b') - \gamma^0 \Sigma(b')\} = -H_v(\gamma^0) = -\gamma^0 \Sigma^*(\gamma^0) + c_\beta^*(\gamma^0), \\ c_\alpha(a) + 2a \cdot \mathbf{1}_d \partial_x \psi &\geq \inf_{a' \in A} \{c_\alpha(a') + 2a' \cdot \mathbf{1}_d \partial_x \psi\} = -H_d(\partial_x \psi) = 0, \end{aligned}$$

since the function $\partial_x \psi$ is non-negative. Therefore, $h^\psi(x, \partial_x \psi, \partial_{xx}^2 \psi, a, b) \geq 0$, with equality for the optimal controls (α^0, β^0) defined by [\(5.2.9\)](#), which leads to

$$R_0(0, X_0) \geq \mathbb{E}^\mathbb{P} \left[-e^{R_A \int_0^T (\frac{1}{2} c_\beta(\beta_s^\mathbb{P}) - f(X_s)) ds} \right],$$

with equality for the optimal controls. □

Proof of [Proposition 5.2.3](#). By directly plugging the guess $\psi(t, x) = A(t)x + \psi_0(t)$, for $(t, x) \in [0, T] \times \mathbb{R}$, in the PDE [\(5.2.7\)](#), we obtain

$$0 = A'(t)x + \psi_0'(t) + \kappa x - \frac{1}{2} \left(c_\beta^*(-R_A |A(t)|^2) + R_A |A(t)|^2 \Sigma^*(-R_A |A(t)|^2) + R_A |A(t)|^2 |\sigma^\circ|^2 \right),$$

with terminal condition $A(T) = \psi_0(T) = 0$. This provides, for all $t \in [0, T]$, $A(t) = \kappa(T - t)$ and

$$\psi_0(t) = - \int_t^T H_0(-R_A |A(s)|^2) ds, \text{ with } H_0(\gamma) := \frac{1}{2}(c_\beta^*(\gamma) - \gamma \Sigma^*(\gamma) - \gamma |\sigma^\circ|^2), \gamma \in \mathbb{R}.$$

Finally the expression of the maximiser $\beta^{0,k}$ follows from [Proposition 5.2.2](#). Moreover, this smooth solution to the PDE satisfies the condition (5.2.8). Indeed, this condition is equivalent to having, for any $\mathbb{P} \in \mathcal{P}$

$$\mathbb{E}^\mathbb{P} \left[\int_0^T e^{-2R_A(\kappa(T-t)X_t + \psi_0(t))} |T - t|^2 dt \right] < +\infty,$$

which is true since X is an Itô process with bounded drift and volatility. We thus conclude with [Proposition 5.2.2 \(ii\)](#) that it is indeed the value function inducing the reservation utility. $\square$

Proof of Proposition 5.2.5. The proof of this proposition is a straightforward application of [Propositions 4.4.8](#) and [4.4.10](#) with the specification $(f - g)(x) = \delta x$, since the functions u^P and u^0 respectively satisfy [Conditions \(4.4.8\)](#) and [\(4.4.14\)](#). Indeed, noticing that $\bar{u}_{\mu^X}^P(t, \mu_t^X) = \bar{u}_{\mu^X}^0(t, \mu_t^X) = \delta(T - t)$, we clearly have

$$\mathbb{E}^\mathbb{P} \left[\left(\int_0^T |\bar{u}_{\mu^X}^P(t, \mu_t^X)|^2 dt \right)^{\frac{p'}{2}} \right] = \delta^{p'} \frac{T^{3p'/2}}{3^{p'/2}} < +\infty \quad \text{and} \quad \mathbb{E}^\mathbb{P} \left[\left(\int_0^T |\bar{u}_{\mu^X}^0(t, \mu_t^X)|^2 dt \right)^{1/2} \right] = \delta \frac{T^{3/2}}{\sqrt{3}} < +\infty.$$

Therefore, u^0 satisfies the condition (4.4.14). To show that u^P satisfies (4.4.8), it remains to prove that

$$\mathbb{E}^\mathbb{P} \left[\sup_{0 \leq t \leq T} e^{-\frac{q'p'}{p'-1} R_P (\delta(T-t) \mathbb{E}^\mathbb{P}[X_t | \mathcal{F}_t^\circ] + \int_t^T m^P(s) ds)} \right] < +\infty,$$

which is true since m^P is continuous and X has bounded drift and volatility. $\square$

An insurance model to prevent fuel poverty

Il nous fallut bien du talent
Pour être vieux sans être adultes

Les vieux amants, JACQUES BREL

Introduction

To pursue the application of principal–agent problems to the field of energy, this chapter examines a solution to protect vulnerable households from the risk of temporary poverty in a particular staple good, as electricity. Staple goods are essential products that consumers are unable or unwilling to cut out of their budgets, such as food and water. However, in the event of a substantial loss of income, consumers will reduce the budget allocated to *some* staple goods, such as fresh food, energy, medicines and feminine hygiene products, which can lead to serious illnesses. Indeed, these particular staple goods, unlike food or water, may not seem essential to poor consumers. This decrease in consumption can lead to health issues and, consequently, generate significant costs for society. We suggest that consumers can, to prevent these negative consequences, buy insurance to secure sufficient consumption of a staple good.

We thus develop in this chapter a two–period/two–good principal–agent problem with adverse selection and endogenous reservation utility to model an insurance plan with in–kind benefits. This model allows us to obtain semi–explicit solutions. Applied to the energy sector, this in–kind support can be a tool to prevent fuel poverty among households. Indeed, when a household suffers from a loss of income, if it has subscribed to an insurance contract of the sort we propose, it consumes more energy than it would without insurance. In this application, providing support in kind therefore forces the household to consume more energy and thus prevents risky behaviour that can lead to serious health problems. Therefore, even in the least efficient scenario from the households point of view, *i.e.*, when the insurance is provided by a monopoly, this in–kind mechanism improves significantly the living conditions of the riskiest households by ensuring them a sufficient consumption of energy. An insurance mechanism also makes it possible to cover agents’ fuel poverty risks and to pool costs between agents (of a high risk type).

Following the same approach as developed in this chapter, it can be shown that the conclusions on consumption would not be the same in the case of income insurance: the household would not increase its energy consumption sufficiently. This result is inline with Slesnick’s results, who underlines that “[w]ith the exception of energy and consumer goods, in–kind transfers have roughly the same impact on the poverty rate as cash” (Slesnick [313, pp. 1544] (1996)). The insurance we propose is also different from the plans provided by the two French energy suppliers EDF and ENGIE: these insurance plans offer a partial reimbursement of the electricity bill, which means the household has to pay the bill first. However, if it suffers a loss of income, the household tends to reduce its consumption for fear of not being able to pay the bill, even if it is reimbursed afterwards. An in–kind support should help to prevent this bias.

6.1 The model

6.1.1 A principal–agent model with adverse selection

We consider a two–period/two–good principal–agent model with adverse selection. The agent represents a household consuming the essential good considered and another representative good. More precisely, at each time $t \in \{0, 1\}$, the agent has an income w_t , which allows him to consume a quantity e_t of the considered staple good and a quantity y_t of another good, with respective unitary constant positive price p_e and p_y . However, between the two periods, the agent is likely to suffer from a loss of income, which will put him in a precarious situation at time $t = 1$: he will be obliged to reduce his consumption. However, if he does not consume a sufficient quantity of the staple good, this can lead to serious issues of which the agent is not necessarily aware. To prevent him from *staple good poverty*, a risk–neutral principal, who may be the good producer or supplier, an insurance company, or even the government, can offer an insurance. This insurance ensures that the agent receives a specified quantity of the staple good, denoted $e_{\min}$, in the event of an income loss. At time $t = 0$, the agent thus chooses if he wants to subscribe to the insurance plan, and if so, he pays the insurance premium T associated with a contractible quantity $e_{\min}$. At time $t = 1$, if he has purchased the insurance and if his income has decreased sufficiently, the agent receives the quantity $e_{\min}$ of the staple good.

We define the random income of an *agent of type* ε at time $t = 1$ by $w_1 := \omega w_0$, where w_0 is the income at time $t = 0$ and ω is a random variable, defined on the probability space $(\Omega, \mathcal{A}, \mathbb{P}^\varepsilon)$, where Ω is a subset in $\mathbb{R}_+$ and $\mathcal{A}$ is its natural σ –algebra. We assume that the insurance is activated when $\omega \leq \tilde{\omega}$, where the income loss barrier $\tilde{\omega}$ is set in an exogenous way. To obtain closed–form solutions, we make the following assumption:

Assumption 6.1.1. *The random variable ω takes two values, $\underline{\omega}$ with probability ε and $\bar{\omega}$ with probability $1 - \varepsilon$, where $\varepsilon \in [0, 1]$ and $\bar{\omega} > \tilde{\omega} \geq \underline{\omega} > 0$.*

We assume that the constants $\underline{\omega}$ and $\bar{\omega}$ are common knowledge. The inequality $\bar{\omega} > \tilde{\omega} \geq \underline{\omega}$ means that the insurance is only activated when $\omega = \underline{\omega}$. This model for the distribution of losses is traditional in insurance models based on the pioneering works of Rothschild and Stiglitz [297] (1976) and Stiglitz [323] (1977). We consider that the agent is better informed than the principal about the risk of income loss he is facing, which depends on his work quality, his job insecurity, the relation he has with his supervisor, etc. The principal only has access to an overview of risks among the population, which leads to an adverse selection problem.

Assumption 6.1.2 (Adverse selection – Third–best case). *The principal cannot observe the type of an agent but knows the distribution of the types of her potential clients.*

As usual in adverse selection problems, the principal has an incentive to offer a menu of contracts, *i.e.*, various quantity $e_{\min}$ with the associated premia T . The agent then chooses the contract that best suits him among all contracts offered by the principal, depending on his risk type. In our study, we look for the best continuous menu of contracts that the insurer can offer.

6.1.2 Agent’s problem

Utility. In most insurance models, the agent’s utility function is not specified. With the aim of obtaining the most explicit results possible, we choose here to represent the preferences of the agent over the goods’ consumption at time t by using a separable utility function based on logarithmic felicities:

$$U(e_t, y_t) := \alpha \ln(e_t) + \ln(y_t), \text{ for } e_t, y_t > 0, \quad (6.1.1)$$

where α parametrises the *longview* elasticity of substitution between the staple good and the composite good.

Our model does not take into account the possibility for the agent to save between the two periods (contrary to the model by Schlesinger and Zhuang [306] (2014)). This hypothesis may seem restrictive but is consistent with the literature on two-period models, in particular with the works of Menegatti [244] (2009) and Courbage and Rey [104] (2012) on prevention, and is justified in our framework in view of the particular households on which we want to focus in our study. Indeed, one can assume that a household already used to saving has built up sufficient funds to pay its bills in the event of a loss of income. This household should thus not be concerned with the insurance we develop throughout this chapter unless it has inadequate savings.

Households that do not save but want to are widespread, as evidenced by the many mobile applications or services to help them.¹ In our opinion, this highlights that there is a need to encourage households to save money and that they are willing to pay for these types of services. In fact, our insurance plan can precisely be interpreted as a form of incentive to save: it is a way for households that have no savings to obtain a quantity of staple good in case of an income loss. The concept of insurance is very effective in this type of situation and allows risks to be shared among the population. Moreover, this choice of model is also based on the willingness to keep a tractable model with (relatively) explicit solutions and to focus our study on the design of insurance contracts. In parallel with Menegatti [244] (2009) for prevention, the interaction between insurance and savings in a two-period model is a different problem but could represent a potentially interesting extension for future work.

Reservation utility. Without insurance, the agent maximises, independently in each period t , the previously defined utility, under his budget constraint:

$$V^\varnothing(w_t) := \max_{(e_t, y_t) \in \mathbb{R}_+^2} U(e_t, y_t), \text{ u.c. } e_t p_e + y_t p_y \leq w_t. \quad (6.1.2)$$

Given a discount factor $\beta \in [0, 1]$, we define the intertemporal expected utility without insurance of an agent of type ε as follows:

$$EU^\varnothing(\varepsilon) := V^\varnothing(w_0) + \beta \mathbb{E}^{\mathbb{P}^\varepsilon} [V^\varnothing(\omega w_0)]. \quad (6.1.3)$$

In our framework, the agent is likely to accept the insurance contract only if it provides him with a level of utility at least equal to his utility without it. Therefore, the *reservation utility* of an agent of type ε is defined by (6.1.3).²

Expected utility with insurance. Let us now fix an insurance contract $(e_{\min}, T)$. If the agent decides to subscribe to this contract, we assume that the payment of the insurance premium T only impacts his budget constraint at time $t = 0$, and his maximum utility is thus naturally given by:

$$V_0(w_0, T) := V^\varnothing(w_0 - T). \quad (6.1.4)$$

As described in Section 6.1.1, the insurance we consider is an in-kind support: it ensures the agent a fixed non-negative amount $e_{\min} \geq 0$ of a determined staple good at time $t = 1$ if he suffers a sufficient loss of

1. For example, the mobile application *Birdycent* rounds up each payment made by the consumer to feed a piggy bank, with a zero interest rate, which is equivalent to losing money in relation to inflation. A second application, called *Yeeld*, offers 4% in cash back on Amazon instead of an interest rate. The bank *Crédit Mutuel* proposes the service *Budget +*, which is subject to a fee, to automatically save from a current account to a savings account.

2. With the aim of simplifying the notation, we only highlight the dependency of reservation utility on the type ε .

income, *i.e.*, if $\omega = \underline{\omega}$. Therefore, his maximisation problem is:

$$V_1(\omega w_0, e_{\min}) := \max_{(e_1, y_1) \in \mathbb{R}_+^2} U(e_1 + e_{\min} \mathbb{1}_{\omega=\underline{\omega}}, y_1), \quad \text{u.c.} \quad e_1 p_e + y_1 p_y \leq \omega w_0. \quad (6.1.5)$$

Similar to the case without insurance, we define the intertemporal expected utility of an agent of type ε with an insurance contract $(e_{\min}, T)$ by:

$$\text{EU}^Q(\varepsilon, e_{\min}, T) := V_0(w_0, T) + \beta \mathbb{E}^{\mathbb{P}^\varepsilon} [V_1(\omega w_0, e_{\min})]. \quad (6.1.6)$$

6.1.3 Principal's problem

We assume that the principal is risk-neutral³ and wants to maximise her profit: she receives at time 0 the earnings from the sale of insurance plans to agents of type $\varepsilon \in [0, 1]$ who agree to subscribe but needs to provide them the quantity $e_{\min}$ they have chosen if they suffer from an income loss in the next period. We consider in this model that insurers are not in perfect competition, so that the price of insurance is not determined by the actuarial price. Therefore, in this monopoly situation, the insurer can choose the range of $e_{\min}$ she wants to offer as well as the price associated with each quantity. We properly define the notion of admissible contracts and menu in our framework:

Definition 6.1.3. *An admissible contract $(e_{\min}, T)$ is a quantity $e_{\min} \geq 0$ with an associated premium $T < w_0$. An admissible menu is then defined as a continuum of admissible contracts $(e_{\min}, T)$, *i.e.*, a continuum of non-negative quantities and a continuous price function T defined for all quantities offered. Under [Assumption 6.1.2](#), the function price T is required to be independent of the agent's type.*

The first-best case. The principal knows the type ε of the agent and can thus offer him a particular contract. Since she has to pay with probability ε the quantity $e_{\min}$ at the unitary price p_e , her optimisation problem is:

$$\pi_\varepsilon := \sup_{e_{\min}, T} (T - \varepsilon p_e e_{\min}), \quad (6.1.7)$$

under the constraint that $(e_{\min}, T)$ is an admissible contract and provides the agent of type ε with at least his reservation utility.

The third-best case. With adverse selection, we consider a menu of revealing contracts, in the sense that an agent of type ε will subscribe to the insurance contract designed for him, *i.e.*, $(e_{\min}(\varepsilon), T(\varepsilon))$. Assuming that the distribution of the type ε in the population considered by the principal has a density function f , the principal's problem is defined as follows:

$$\sup_{e_{\min}, T} \int_0^1 (T(\varepsilon) - \varepsilon p_e e_{\min}(\varepsilon)) f(\varepsilon) d\varepsilon, \quad (6.1.8)$$

under the participation constraint, and where, in this case, $e_{\min}$ and T are appropriate functions of ε . To simplify, we will assume that the distribution of the type ε in the population considered by the principal is uniform on $[0, 1]$, *i.e.* $f = \mathbb{1}_{[0,1]}$. It is equivalent to considering that, from the principal's point of view, an agent has a probability of one-half of experiencing a significant loss of income. This assumption is actually not necessary, as computations could easily be made for another distribution, but it allows for simplification of the principal's problem. This distribution models the problem of a principal who does not actually have

3. The principal's risk-neutrality is justified by the fact that shareholders of insurance companies generally have a diversified portfolio.

data on the agent's income loss. This is the case in the application in question, where the insurer is an electricity supplier, who is not intended to have insight into the distribution of the risks of income loss of its customers. Moreover, even if a probability of one-half seems high for the population, the agents likely to subscribe to our insurance are people of a rather high risk type, and the probability in the population considered by the insurer is necessarily higher than in the global population. This is particularly true given that our study focuses on middle-class households without savings, which naturally have a higher probability of losing income. Nevertheless, theoretical results can be extended to include other probability distributions.

6.1.4 Application to fuel poverty

We apply this model for a particular staple good, electricity, to develop insurance against fuel poverty. According to Chaton and Gouraud [98] (2020), fuel poverty is essentially linked to a temporary loss of income. It particularly affects low-income and vulnerable households that have a low propensity to save and that already spend a large part of their income on energy. This situation can lead households to adopt risky behaviours, causing health problems and housing deterioration (see Lacroix and Chaton [215] (2015)). For example, to keep heat inside their homes, some obstruct vents, thereby generating moisture and mould. Households in fuel poverty are often forced to make choices with harmful consequences for their health: choosing eating or heating, or giving up health care or social interactions. These consequences are often neglected by households but are harmful, and moreover highly expensive for society. To avoid them, mechanisms are being developed to help vulnerable households. For example, in France, energy vouchers have been distributed by the state since 2018. This voucher can be used to pay not only for energy expenses such as electricity, gas, wood and fuel oil bills but also for energy renovation. In 2019, it targeted 5.8 million households with modest incomes.

The motivation of this chapter is to act on the prevention side by proposing a complementary tool to keep the number of households in fuel poverty from increasing. The idea is to develop an insurance policy that is activated if the household becomes energy constrained. Two French electricity suppliers offer two slightly different insurance options: *Assurénergie*, offered by Electricité de France (EDF), and *Assurance Facture*, offered by ENGIE. These two monthly insurance plans offer a refund of part of the electricity bill in the event of job loss, sick leave, hospitalisation, disability or death. In the first insurance plan, the amount refunded depends on the contract chosen from the proposed menu, while the second insurance plan is a unique contract. A more general example is the *Utilities Insurance* provided by the Canadian company Trans Global Insurance, which allows household to be covered for some basic utilities. This insurer provides a menu of contracts in two ways: the household can choose both the level of coverage (3 possible) as well as the different basic utilities to be covered (power, heat, water, internet...). Our goal is to compute the optimal menu of contracts using contract theory with adverse selection to study the structure of the contracts obtained and to uncover what types of agents are likely to subscribe to the insurance plans. One may note that the monopolistic framework under consideration makes sense in this situation, since the fuel provider of a household has more inside information than other fuel providers or traditional insurance companies.

6.2 Solving the agent's problem

In this section, we solve the optimal consumption problem of an agent of type ε : given an insurance contract $(e_{\min}, T)$ and the utility function specified in (6.1.1), we compute the agent's optimal consumption of both goods at each period. As a result, we can compute the maximum utility the agent can achieve

for a given contract. Comparing this utility with the reservation utility, we can determine the maximum price the agent is willing to pay for insurance. This section allows us to properly define in our context the *participation constraint* mentioned in the definition of the principal's problem in [Section 6.1.3](#).

6.2.1 Optimal consumption without insurance

We first solve the consumption problem of an agent who has not subscribed to an insurance contract. Let us define the following constant:

$$C_{\alpha, p_e, p_y} := \alpha \ln(\alpha) - (1 + \alpha) \ln(1 + \alpha) - \alpha \ln(p_e) - \ln(p_y). \quad (6.2.1)$$

Since our framework does not allow the agent to transfer income from one period to another, the agent maximises his utility in each period independently by solving (6.1.2), which leads to the following result.

Lemma 6.2.1 (Without insurance). *The optimal consumptions of each good at time $t \in \{0, 1\}$ of an agent with income w_t are given by:*

$$y_t^\varnothing := \frac{1}{1 + \alpha} \frac{w_t}{p_y} \quad \text{and} \quad e_t^\varnothing := \frac{\alpha}{1 + \alpha} \frac{w_t}{p_e},$$

and induce the maximum utility $V^\varnothing(w_t) = (1 + \alpha) \ln(w_t) + C_{\alpha, p_e, p_y}$.

Reservation utility. Then, by a simple computation of the expected utility defined by (6.1.3), we can explicitly write the reservation utility:

Proposition 6.2.2. *Under [Assumption 6.1.1](#), the expected utility without insurance of an agent of type ε is given by*

$$EU^\varnothing(\varepsilon) = (1 + \alpha) \ln(\underline{\omega}^{\beta\varepsilon} \bar{\omega}^{\beta(1-\varepsilon)} w_0^2) + (1 + \beta) C_{\alpha, p_e, p_y}. \quad (6.2.2)$$

Usually, in insurance models, the reservation utility is taken to be independent of the agent's type. In our framework, we reason that an agent does not subscribe to an insurance contract if his utility without is higher. Therefore, the reservation utility we consider is endogenous and depends on the probability ε . In particular, $EU^\varnothing$ is a decreasing function of ε . This problem is addressed in some adverse selection models, for example, by Lewis and Sappington [230] (1989), Biglaiser and Mezzetti [60] (1993), Maggi and Rodríguez-Clare [236] (1995), Jullien [199] (2000), as well as by Alasseur et al. [17] (2020); but is rarely considered in insurance problems. As Laffont and Martimort [218] (2009) explain, in this case, determining which participation and incentive constraints are binding becomes a more difficult task. Nevertheless, [Proposition 6.4.3](#) establishes that only agents of a sufficiently risky type are selected by the principal, and this feature only appears in models with countervailing incentives. The principal excludes the types with lower risk, those with a low probability of losing their income, because the price they are willing to pay is very low, while the agents of a higher risk type are more profitable, since they are easily satisfied and are willing to pay much more.

6.2.2 Optimal consumption with insurance

Similarly, we can solve the consumption problem of an agent who subscribes to a given admissible contract $(e_{\min}, T)$. Recall that, with the insurance, the agent pays a price T at time $t = 0$, depending on the amount of staple good $e_{\min}$ that he wants to receive at time $t = 1$.

Lemma 6.2.3 (With insurance, at time 0). *Given an insurance premium T , the optimal agent's consumption of each good at time $t = 0$ are given by*

$$y_0^Q := \frac{1}{1+\alpha} \frac{w_0 - T}{p_y} \quad \text{and} \quad e_0^Q := \frac{\alpha}{1+\alpha} \frac{w_0 - T}{p_e};$$

and his corresponding maximum utility is $V_0(T) = (1+\alpha) \ln(w_0 - T) + C_{\alpha, p_e, p_y}$.

By setting $\bar{e}_{\min} := \alpha \underline{\omega} w_0 / p_e$, we obtain the following result on optimal consumption at time $t = 1$.

Lemma 6.2.4 (With insurance, at time 1). *Given an insurance contract $(e_{\min}, T)$, the optimal agent's consumption of each good at time $t = 1$ are given by*

$$e_1^Q := \left(\frac{\alpha}{1+\alpha} \frac{\omega w_0}{p_e} - \frac{1}{1+\alpha} e_{\min} \mathbb{1}_{\omega=\underline{\omega}} \right)^+ \quad \text{and} \quad y_1^Q := \frac{\omega w_0 - p_e e_1^Q}{p_y},$$

where $x^+ := \max\{x, 0\}$ for all $x \in \mathbb{R}$, and provide the following maximum utility to the agent:

$$V_1(\omega w_0, e_{\min}) = \begin{cases} (1+\alpha) \ln(\omega w_0 + p_e e_{\min} \mathbb{1}_{\omega=\underline{\omega}}) + C_{\alpha, p_e, p_y} & \text{if } e_{\min} \mathbb{1}_{\omega=\underline{\omega}} < \bar{e}_{\min}, \\ \ln(\omega w_0) + \alpha \ln(e_{\min}) - \ln(p_y) & \text{if } e_{\min} \mathbb{1}_{\omega=\underline{\omega}} \geq \bar{e}_{\min}. \end{cases}$$

Remark 6.2.5. *The case separation in the previous lemma is needed to ensure that the consumption e_1^Q at time $t = 1$ is non-negative. Indeed, the consumer should not be allowed to sell back part of the insured quantity $e_{\min}$, i.e., the quantity of the staple good provided under the insurance plan. First, we ignore any reselling possibility because one can notice that, if the agent could resell part of the quantity, the insurance would be strictly equivalent to income insurance. Therefore, under this no resale assumption, we are faced with two cases. In the first case, i.e., when $e_{\min} \mathbb{1}_{\omega=\underline{\omega}} < \bar{e}_{\min}$, the agent's utility at time $t = 1$ depends only on his effective income $\omega w_0 + p_e e_{\min} \mathbb{1}_{\omega=\underline{\omega}}$. Assuming that the choice of $e_{\min}$ is restricted to the interval $[0, \bar{e}_{\min}]$ is equivalent to assuming that the quantity offered is smaller than the optimal quantity consumed in the event of an income loss. Therefore, in this case, the insurance acts as an earmarked fund or a liquid asset: the agent would have spent at least the quantity $e_{\min}$, so he reacts as if his income were increased by this value. In contrast, in the second case, the agent consumes only the amount $e_{\min}$ of the staple good: the optimal e_1^Q is equal to zero. In this case, the insurance is not interpreted as a liquid asset, and the agent consumes the full quantity $e_{\min}$ offered to him, leading to a utility of $\alpha \ln(e_{\min})$, and does not consider it an increase in income. He then spends all his income $\underline{\omega} w_0$ on the other good. However, in this case, it could have been better from his point of view to resell part of the insured quantity. Nevertheless, it is precisely the purpose of this insurance model to ensure that the household consumes more of this particular good to avoid the consequences induced by a decrease in consumption, of which the household is not aware.*

Expected utility with insurance. To simplify the notations, we can assume without loss of generality that any admissible contract $(e_{\min}, T)$, in the sense of [Definition 6.1.3](#), is of the following form:

$$e_{\min} := q \alpha \underline{\omega} w_0 / p_e, \quad \text{for } q \in \mathbb{R}_+ \quad \text{and} \quad T := t_0 w_0, \quad \text{for } t_0 \in [0, 1]. \quad (6.2.3)$$

The pair (q, t_0) is referred to as an *admissible normalised contract*. Therefore, by slightly abusing the notations as in [Lemma 6.2.1](#), we denote by $V_0(t_0)$ the maximum utility the agent can achieve by optimally choosing his consumption at time $t = 0$, which can be written as

$$V_0(t_0) = (1+\alpha) \ln(w_0) + (1+\alpha) \ln(1-t_0) + C_{\alpha, p_e, p_y}. \quad (6.2.4)$$

We then denote by $\bar{U}$ the following function for $q \in \mathbb{R}_+$:

$$\bar{U}(q) := \begin{cases} (1 + \alpha) \ln(1 + q\alpha) & \text{if } q < 1, \\ \alpha \ln(q) + (1 + \alpha) \ln(1 + \alpha) & \text{if } q \geq 1. \end{cases} \quad (6.2.5)$$

The previous distinction of cases between $q < 1$ and $q \geq 1$ is related to the *no resale assumption*, as mentioned in [Remark 6.2.5](#). Recall that if $q < 1$, this quantity insured is not sufficient from the agent's point of view. He therefore supplements it by purchasing additional energy at $t = 1$. In contrast, if $q \geq 1$, the agent consumes only the corresponding amount $e_{\min}$ of the staple good, and his optimal complementary consumption is equal to zero.

Thanks to the previous notations, the maximum utility obtained by the agent at time $t = 1$ can be written as a function of ω and q as follows:

$$V_1(\omega, q) = (1 + \alpha) \ln(\omega w_0) + \bar{U}(q) \mathbf{1}_{\omega=\underline{\omega}} + C_{\alpha, p_e, p_y}. \quad (6.2.6)$$

Combining (6.2.4) and (6.2.6), we can provide an explicit form for the expected utility of an agent subscribing to an insurance contract.

Proposition 6.2.6. *Given an admissible normalised contract (q, t_0) and under [Assumption 6.1.1](#), the expected utility of an insured agent of type ε is given by:*

$$\text{EU}^Q(\varepsilon, q, t_0) = \text{EU}^\varnothing(\varepsilon) + (1 + \alpha) \ln(1 - t_0) + \beta \varepsilon \bar{U}(q). \quad (6.2.7)$$

6.2.3 Participation constraint and maximum price

It remains to determine when the agent of type ε will subscribe to the insurance plan, *i.e.*, when his expected utility with insurance is greater than his reservation utility. With this in mind, by computing the difference between (6.1.6) and (6.1.3), we can state the following proposition.

Proposition 6.2.7 (Participation constraint). *An admissible normalised contract (q, t_0) satisfies the participation constraint of the agent of type ε if and only if $t_0 \leq t_{\max}(\varepsilon, q)$, where $t_{\max}$ is defined for any $(\varepsilon, q) \in [0, 1] \times \mathbb{R}_+$ by:*

$$t_{\max}(\varepsilon, q) := 1 - \begin{cases} (1 + q\alpha)^{-\beta\varepsilon} & \text{if } q < 1, \\ q^{-\beta\varepsilon \frac{\alpha}{1+\alpha}} (1 + \alpha)^{-\beta\varepsilon} & \text{if } q \geq 1. \end{cases} \quad (6.2.8)$$

Therefore, $w_0 t_{\max}(\varepsilon, q)$ is the maximum price the agent of type ε is willing to pay for a quantity $e_{\min} = \alpha q \underline{\omega} w_0 / p_e$. In other words, if the premium T associated with a quantity $e_{\min} = \alpha q \underline{\omega} w_0 / p_e$ is below $w_0 t_{\max}(\varepsilon, q)$, the agent of type ε is willing to purchase the insurance contract. We say, in this case, that the admissible contract $(e_{\min}, T)$ satisfies the participation constraint for ε types. On the contrary, if the premium T is above $w_0 t_{\max}(\varepsilon, q)$, the agent will not purchase the insurance, meaning that the participation constraint is not satisfied.

Comparative statics. Through (6.2.8), the maximum price seems to be independent of the price p_e . In reality, this is only due to the fact that q represents a quantity normalised (among other parameters) by price. More precisely, the maximum price $w_0 t_{\max}(\varepsilon, q)$ can be reformulated as a function of $e_{\min}$:

$$T_{\max}(\varepsilon, e_{\min}) := w_0 - \begin{cases} w_0 \left(1 + \frac{p_e e_{\min}}{\underline{\omega} w_0} \right)^{-\beta\varepsilon} & \text{if } e_{\min} < \alpha \underline{\omega} w_0 / p_e, \\ w_0 \left(\frac{p_e e_{\min}}{\alpha \underline{\omega} w_0} \right)^{-\beta\varepsilon \frac{\alpha}{1+\alpha}} (1 + \alpha)^{-\beta\varepsilon} & \text{if } e_{\min} \geq \alpha \underline{\omega} w_0 / p_e. \end{cases} \quad (6.2.9)$$

Thanks to this formula, it is straightforward to make the following comparative statics.

Corollary 6.2.8. *The function $T_{\max}(\varepsilon, e_{\min})$ defined by (6.2.9) is increasing with respect to $e_{\min}$, w_0 , p_e , ε , and β , but decreasing with respect to $\underline{\omega}$. Moreover, when $e_{\min} < \alpha \underline{\omega} w_0 / p_e$, $T_{\max}$ is independent of α .*

Interpretations. The results presented in the previous corollary (see Section 6.6.2 for the proof) are, for the most part, very intuitive. First, the agent is naturally willing to pay more for a higher insured quantity. In addition, the higher the agent's initial income w_0 , the more he is willing to pay a high price. Conversely, the higher his income w_1 in $t = 1$ ($\underline{\omega}$ close to 1), the less he is willing to pay a high price. Indeed, if the agent is not likely to lose significant income between $t = 0$ and $t = 1$, this kind of insurance is of limited value to him. Moreover, if the staple good is expensive (p_e high), the agent is willing to pay more for the insurance. Similarly, the more the agent values the future (β large), the more he is ready to spend a large part of his income in $t = 0$ by buying the insurance in order to benefit from it in $t = 1$.

All the aforementioned points seem to be in line with reality and therefore validate the choice of our model. Nevertheless, the main feature to observe is that the maximum price increases with the type ε of the agents, *i.e.*, the probability of income loss. Therefore, the riskier the agent is, the more willing he is to pay a high price for the same insured quantity. Finally, the fact that for a small insured quantity ($e_{\min} < \alpha \underline{\omega} w_0 / p_e$), the maximum price is independent of α , is related to Remark 6.2.5. Indeed, in this case, the insurance acts as an income insurance, since the agent would have consumed the amount $e_{\min}$ anyway, even without insurance. The maximum price is therefore independent of his preference between the essential good and the composite good. On the other hand, when the insured quantity is large, the agent would prefer to resell the surplus. This would allow him an additional income that he could then optimally distribute between the essential good and the composite good, according to his preference represented by the parameter α . Following the reasoning developed in the proof of Corollary 6.2.8, we can state that the maximum price is increasing with respect to α for α sufficiently small and then decreasing. This result is also expected as α small means that the agent has a strong preference for the essential good, and is thus very inclined to buy insurance. On the contrary, when α gets close to 1, the agent becomes indifferent between the two goods, and is therefore less inclined to buy the insurance.

Remark 6.2.9 (Maximum price without uncertainty). *Given the form of the maximum price, it can already be noted that some agents show a certain form of irrationality due to their unwillingness to save money from one period to the next. Indeed, in our framework, the maximal price the consumer is willing to pay in the case without uncertainty is not equal to the actuarial price, $p_e e_{\min}$. Indeed, assuming that $q < 1$ and setting $\beta = 1$ for simplicity, we obtain $w_0 t_{\max}(1, q) > p_e e_{\min}$ as soon as $w_0 > \underline{\omega} w_0 + p_e e_{\min}$. Therefore, if the income of the consumer at time $t = 0$ is larger than the effective money he has at time $t = 1$ with the insurance, he is willing to pay a certain amount of money at time $t = 0$ to obtain less at time $t = 1$. Conversely, if his income w_0 is lower than the money he will receive at time $t = 1$, he is not willing to pay at time $t = 0$ the real price of the energy he will get at time $t = 1$. This result is slightly counter–intuitive but is totally explained by the choice of concave utilities in a two–period model and the absence of saving. This problem does not occur in single–period models. However, a one–period model would not allow us to model a household willing to insure against a possible loss of future income. One solution could be to offer the agent the opportunity to have savings, but this is not consistent with the type of household being considered. Therefore, an alternative approach to address this issue is initiated in Section 6.5.*

6.3 Benchmark case: the first–best problem

Thanks to the study developed in the previous section, we know the maximum price an agent is willing to pay for an insurance policy (q, t) . With this in mind, we can now solve the problem in the first–best case, *i.e.*, without adverse selection. In particular, in this case, since the principal knows the type of the agent,

she can provide him a specific contract, with which his participation constraint is binding. In other words, she may charge the insurance at the highest price the agent is willing to pay, to the point that he is, in fact, indifferent between subscribing or not subscribing to the contract.

6.3.1 Solving the principal's problem

As detailed in [Section 6.1.3](#), the principal's problem in the first-best case is defined by (6.1.7), under the participation constraint of the agent. Thanks to the reasoning developed in the previous section, and denoting by $\Xi_\varepsilon := \{(q, t_0) \in \mathbb{R}_+^2, \text{ s.t. } t_0 \leq t_{\max}(\varepsilon, q)\}$, her problem is equivalent to:

$$\pi_\varepsilon := w_0 \sup_{(q, t_0) \in \Xi_\varepsilon} (t_0 - \varepsilon \alpha q \underline{\omega}). \quad (6.3.1)$$

Proposition 6.3.1. *If $\beta > \underline{\omega}$, the optimal contract $(e_{\min}, T)$ for an agent of type $\varepsilon \in [0, 1]$ is given by $e_{\min}^{\text{FB}} := \alpha q_\varepsilon \underline{\omega} w_0 / p_e$ and $T^{\text{FB}} := w_0 t_{\max}(\varepsilon, q_\varepsilon)$ where*

$$q_\varepsilon := \begin{cases} \left(\beta(1 + \alpha)^{-\beta\varepsilon - 1} / \underline{\omega} \right)^{(1+\alpha)/(1+\alpha+\alpha\beta\varepsilon)} & \text{if } \varepsilon \leq \varepsilon_1^{\text{FB}}, \\ \frac{1}{\alpha} \left((\beta/\underline{\omega})^{1/(1+\beta\varepsilon)} - 1 \right) & \text{if } \varepsilon > \varepsilon_1^{\text{FB}}, \end{cases} \quad (6.3.2)$$

$$\text{for } \varepsilon_1^{\text{FB}} := \frac{1}{\beta} \left(\frac{\ln(\beta) - \ln(\underline{\omega})}{\ln(1 + \alpha)} - 1 \right).$$

The previous result solves the principal's optimisation problem in the first-best case, and its proof is reported in [Section 6.6.1](#). The assumption $\beta > \underline{\omega}$ is made to simplify the result and makes perfect sense in this framework (see [Remark 6.6.2](#)).

Comparative statics. We first study the optimal insured quantity as well as the optimal insurance premium in the first-best case, using (6.3.2).

Corollary 6.3.2. (i) *The quantity $e_{\min}^{\text{FB}}$ is increasing with respect to w_0 and decreasing with respect to p_e and ε .*

(ii) *The price T^{FB} is increasing with respect to w_0 , ε and β , but decreasing with respect to $\underline{\omega}$. Moreover, T^{FB} is independent of p_e .*

(iii) *When $\varepsilon > \varepsilon_1^{\text{FB}}$, both $e_{\min}^{\text{FB}}$ and T^{FB} are independent of α .*

The proof of this result is highly similar to that of [Corollary 6.2.8](#), postponed to [Section 6.6.1](#), and is therefore omitted. In our opinion, the main feature to notice is that the quantity $e_{\min}^{\text{FB}}$ is decreasing with respect to ε , while the price is increasing. This means that in the first-best case, the insurer can force high-risk households to subscribe a small insured quantity at a higher price. Indeed, by knowing the type of the agents, the principal can design very specific contract for each type of agents. In particular, since the reservation utility $\text{EU}^\varnothing$ is decreasing with respect to ε , a high-risk type agent is willing to pay more than a low-risk type agent for the same quantity, and the principal therefore exploits this fact.

Given the optimal contract $(e_{\min}^{\text{FB}}, T^{\text{FB}})$, the maximum profit obtained by the principal for each risk type ε of agent is computed explicitly in [Corollary 6.6.1](#). In particular, this profit is surprisingly independent of the price p_e . In other words, the profit of the insurer is not impacted by the variations of the staple good's price. The most important fact is that the profit is increasing with respect to ε . In other words, a risky household is much more profitable for the insurer, which is consistent with the previous remarks. Indeed, in the first-best case, the riskier the household, the smaller the insured quantity but the higher the price. One may note that the profit is also increasing in w_0 .

6.3.2 First–best insurance against fuel poverty

With the motivation described in Section 6.1.4, we apply the results of the first–best case in the context of insurance against fuel poverty in France. Many French households that belong to the poorest 30% are in fuel poverty. Indeed, in 2018, in mainland France, 78% of households in the first income decile were in fuel poverty. This percentage falls to 54% (26%) for the 2nd (3rd) income decile. These households are already struggling to have the necessary energy to avoid fuel poverty and thus do not have the financial means to subscribe to fuel poverty insurance plans.

To perform our numerical simulations, we therefore consider a middle–class household with an annual disposable income, after taxes and social benefits, of $w_0 = \text{€}35,000$. We assume that this household lives in an all–electric house (electric heating and hot water), with an annual electricity consumption of $e_0^\varnothing = 14,403$ kWh.⁴ In France, the average price p_e per kilowatt hour is around $\text{€}0.18$, and the share of household income spent on household energy expenditure was 7.41%. We deduce from the expression of $e_0^\varnothing$ the value $\alpha = 8\%$. Moreover, we set $\varpi = 0.4$: the household thus has a probability ε of having an annual disposable income equal to $\text{€}0.4 \times 35,000 = \text{€}14,000$ at time $t = 1$. To simplify, we assume $\beta = 1$.

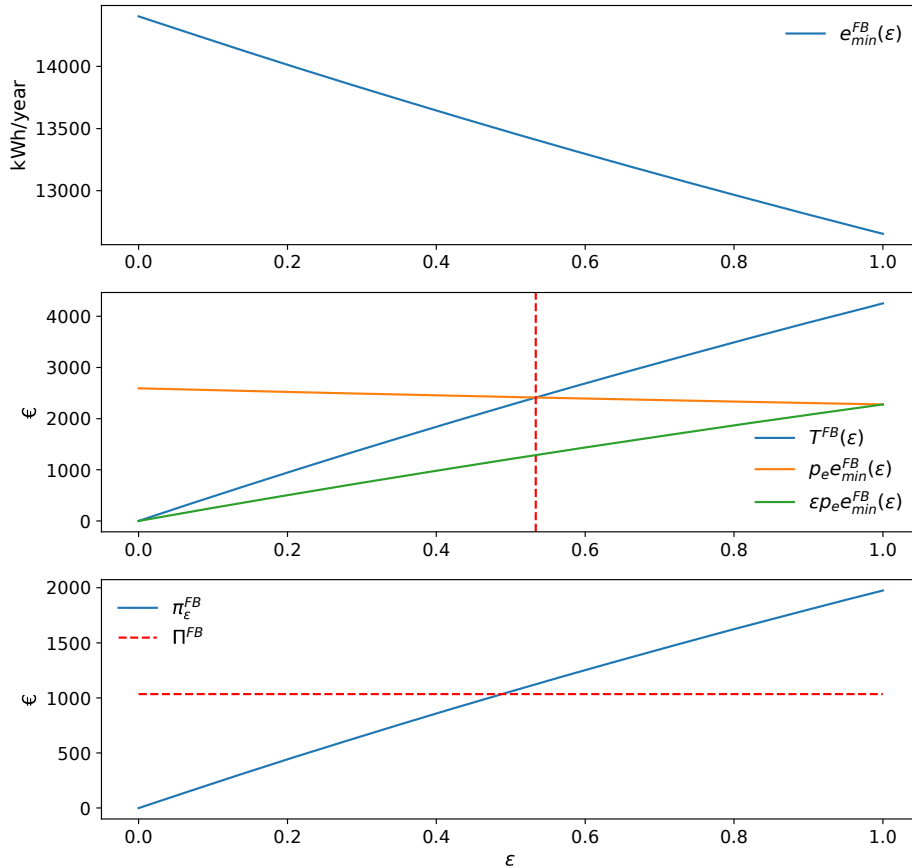


Figure 6.1 – Optimal insurance in the first–best case.

Blue lines, from top to bottom: insured quantity, insurance premium and principal's profit, with respect to the risk type ε . Middle graph: insurance price is compared to the future price of the quantity (orange line) and to its actuarial price (green line). Red dotted line, bottom graph: principal's average profit.

4. Note that this consumption is approximately the average electricity consumption of a French household, which is equal to 14,527 kWh, according to Belaïd [46] (2016).

Results. Thanks to [Proposition 6.3.1](#), we can compute the optimal insured quantity and the insurance premium, as well as the principal's profit (blue lines on [Figure 6.1](#)). In this case, an agent of a higher risk type pays a higher insurance premium for a smaller insured quantity than does an agent of a lower risk type, as we can see by combining the two upper graphs of [Figure 6.1](#). More precisely, the insured quantity varies from approximately $e_{\min}^{\text{FB}} = 14,403$ to $12,653$ kWh, while the price, T^{FB} increases from €0 to €4,252. Moreover, the middle graph of [Figure 6.1](#) shows that the riskier the agent's type, the greater the difference between the insurance premium and the actuarial price is. Since the actuarial price also corresponds to the principal's cost, the greater this difference, the greater her profit is. Therefore, from the principal's point of view, the more efficient agents are those who are willing to pay more than the actuarial price (bottom graph of [Figure 6.1](#)). Her average profit $\Pi^{\text{FB}} = \text{€}1,035$ is given by the integral of her profit per agent, assuming that the distribution of risk type is uniform.

Interpretations. Therefore, in our framework, the efficient agents are those who are at risk of losing their income, which may seem counter-intuitive. Nevertheless, this result can be explained by the reservation utility we have chosen. More precisely, one can compute the information rent, which is the difference between $\text{EU}(\varepsilon, q, t_0)$ and $\text{EU}^{\varnothing}(\varepsilon)$ for an agent of type ε . This information rent increases with ε for every q and t_0 , which means exactly that the riskier the agent's type, the more interesting it is for him to buy the insurance. In the first-best case, the principal knows the agent's type and can thus reduce his information rent to zero. This explains why an agent of a higher risk type is ready to pay more for a lower insured quantity.

The intuition for the third-best case, *i.e.*, when the principal cannot differentiate among the agents by their type since they are unknown to her, is the following: if she offers the optimal first-best contract, an agent with a positive probability of losing income may lie and pretend to be of a lower risk type to pay less for a higher insured quantity. Agents of the lowest risk type, on the other hand, have no incentive to pretend they are of a higher risk type, since that would cause them to pay more for a smaller insured quantity. As in classical adverse selection problems, here the *efficient* agents, which are those who have an incentive to lie, receive an *information rent*, generated by the informational advantage they have over the principal.

However, one may already note one limit of our model: agents of type $\varepsilon \geq 0.53$ (red dotted line, middle graph) are willing to pay more than the future price of the quantity, *i.e.*, $p_{\text{e}}e_{\min}(\varepsilon)$. This fact highlights a significant inconsistency of these agents, who are willing to pay a high price for insurance, when it would be more efficient for them to save money instead. One way to address this inconsistency would be to offer another option, in parallel with the insurance. Indeed, if insurance is the only option for households, the principal abuses her monopoly position. In our opinion, this result highlights the importance of regulating this type of market. If the state's interest is in combating fuel poverty, insurance seems to be a good option, but at the same time, alternative solutions must also be developed, such as a prepayment option (see [Section 6.5](#)).

6.4 Third-best case: under adverse selection

In this section, we focus on finding the optimal menu of insurance contracts in the presence of adverse selection. As explained above in [Section 6.3.2](#), the intuition is that the first-best contract given by [Proposition 6.3.1](#) is no longer optimal if the principal cannot observe the agents' type. Indeed, with this contract, an agent with a positive probability of losing income may lie and pretend to be of a less risky type to pay less for a higher insured quantity. Therefore, the agents of a higher risk type would receive an information rent generated by the informational advantage they have over the principal. Conversely, the agent of the lower risk type should have no information rent since he has no incentive to lie: if he pretends he is of a higher risk type, he pays a higher premium for a lower insured quantity.

In line with this reasoning, the principal has to find a new optimal menu of contracts. The classical scheme is to use the *revelation principle*: the principal has to design a menu of contracts indexed by ε , such that the agent of type ε chooses the contract designed for him. As usual in adverse selection problems, the more efficient agents, *i.e.*, in our framework, those of a high risk type, receive an information rent generated by the informational advantage they have over the principal. Moreover, the agent of the higher risk type is insured for the optimal quantity of the good computed in the first-best case. For other types ($\varepsilon < 1$), in contrast, there is a distortion in the optimal quantity: they are insured for a smaller quantity than that determined in the first-best case.

The revelation principle detailed in [Section 6.4.1](#) allows us to write the insurance premium as a function of the insured quantity and the type to within a constant c_q . In fact, [Section 6.4.2](#) explains that the value of this constant is related to the participation constraint of the agents: the principal can choose the constant depending on the type of agents she wants to select. [Section 6.4.3](#) is dedicated to solving principal's problem, and [Section 6.4.4](#) addresses the application to fuel poverty.

6.4.1 Revelation principle

Traditionally, in adverse selection models (see [Salanié \[300\] \(2005\)](#) for the general theory on adverse selection), the contract offered by the principal has to satisfy the *incentive compatibility (IC) constraint*: the contract has to be such that an agent of type ε would subscribe the contract corresponding to him, and thus reveal his type ε , which is previously unknown to the principal. Indeed, the well-known *revelation principle* implies that we can restrict the study to incentive compatible mechanisms. More precisely, the revelation principle stated by [Salanié \[300\] \(2005\)](#) can be adapted to our framework as follows: *If the optimal quantity $e_{\min}$ chosen by an agent of type ε can be implemented through some mechanism, then it can also be implemented through a direct and truthful mechanism where the agent reveals his risk ε .*

First, we can show that the Spence–Mirrlees condition, also called the *constant sign assumption* by [Guesnerie and Laffont \[168\] \(1984\)](#), as defined by [Laffont and Martimort \[218\] \(2009\)](#), is automatically satisfied in our framework (see [Lemma 6.6.3](#)). This property makes the incentive problem well behaved in the sense that only local incentive constraints need to be considered. This condition was introduced by [Spence \[320\] \(1973\)](#) in his theory of signaling on the labour market and, similarly, by [Mirrlees \[249\] \(1971\)](#) in his theory of optimal income taxation, as the *single-crossing assumption*: the condition implies that the indifference lines of two different types of agents can only cross once. This condition also has economic content, which implies in our framework that agents with a higher probability of income loss are willing to pay more for a given increase in $e_{\min}$ than agents of a less risky type. This condition ensures that it is possible to separate the agents of a high risk type from those of a low risk type by offering them better coverage in exchange for a higher premium.

To find the risk-revealing contracts, we need to associate to an admissible menu of contracts $(e_{\min}, T)$, an pair (q, t_0) of functions of $\varepsilon \in [0, 1]$. With this in mind, we define $\varepsilon_1 := \min\{\varepsilon \in [0, 1], \text{ s.t., } q(\varepsilon) = 1\}$, with the convention that $\varepsilon_1 = 0$ if $q(\varepsilon) \geq 1$ for all $\varepsilon \in [0, 1]$.

Definition 6.4.1. A mechanism (q, t_0) is said to be admissible if

- (i) q and t_0 are continuous functions on $[0, 1]$, taking values in an interval contained in $\mathbb{R}_+$ and $[0, 1]$;
- (ii) q and t_0 have continuous first and second derivatives on $(0, 1)$, except at ε_1 .

More precisely, an admissible menu of contracts $(e_{\min}, T)$ is associated with an admissible mechanism (q, t_0) if, for all quantities $e_{\min}$ available at price T , there is an $\varepsilon \in [0, 1]$ such that $e_{\min} = \alpha q(\varepsilon) \omega w_0 / p_e$ and $T = w_0 t_0(\varepsilon)$.

The first point of the previous definition is entirely based on assumptions made about an admissible menu of contracts in [Definition 6.1.3](#). The second point on the regularity of q and t_0 is more a technical assumption made to simplify the reasoning: it allows us to use the first- and second-order conditions to define an incentive compatible contract. Unfortunately, the case separation between $q < 1$ and $q \geq 1$ subsequently implies a loss of $\mathcal{C}^1$ -continuity of the quantity and price at this point.

We thus limit our study to mechanisms that are smooth enough in the sense of [Definition 6.4.1](#). According to the reasoning of [Guesnerie and Laffont \[168\] \(1984\)](#), our results could be easily extended to piece-wise continuously differentiable mechanisms of class $\mathcal{C}^1$, and some could even be generalised to all mechanisms. Nevertheless, significant additional difficulties can be avoided with this smoothness assumption. Moreover, one may note that the optimal mechanism in the first-best case is smooth in the sense of [Definition 6.4.1](#) and that it therefore makes sense to restrict our study in this way.

The IC constraint says that the utility of an agent of type $\varepsilon \in [0, 1]$ has to be maximal for the choice of the contract $(q(\varepsilon), t_0(\varepsilon))$, *i.e.*,

$$\text{EU}^Q(\varepsilon, q(\varepsilon), t_0(\varepsilon)) = \max_{\varepsilon' \in [0, 1]} \text{EU}^Q(\varepsilon, q(\varepsilon'), t_0(\varepsilon')).$$

In other words, if a menu of contracts satisfies the IC constraint, then the agent has an interest in revealing his type by choosing the contract made for him. We denote by $\mathcal{C}^Q$ the set of admissible mechanisms satisfying this constraint. With the aim of lightening the equations, we denote throughout this section:

$$Q_0(\varepsilon) := \begin{cases} \int_0^\varepsilon \ln(1 + \alpha q(\varepsilon)) d\varepsilon & \text{if } \varepsilon \in [0, \varepsilon_1] \\ \int_0^{\varepsilon_1} \ln(1 + \alpha q(\varepsilon)) d\varepsilon + \frac{\alpha}{1 + \alpha} \int_{\varepsilon_1}^\varepsilon \ln(q(\varepsilon)) d\varepsilon & \text{if } \varepsilon \in [\varepsilon_1, 1]. \end{cases} \quad (6.4.1)$$

Theorem 6.4.2. *An admissible mechanism (q, t_0) satisfies the IC constraint for all $\varepsilon \in [0, 1]$ if and only if the function q is non-decreasing on $[0, 1]$ and there exists $c_q \geq 0$ such that the price t_0 satisfies for all $\varepsilon \in [0, 1]$,*

$$t_0(\varepsilon) = 1 - c_q e^{\beta Q_0(\varepsilon)} \times \begin{cases} (1 + \alpha q(\varepsilon))^{-\beta \varepsilon} & \text{if } \varepsilon \in [0, \varepsilon_1], \\ (1 + \alpha)^{-\beta \varepsilon_1} (q(\varepsilon))^{-\beta \varepsilon \alpha / (1 + \alpha)} & \text{if } \varepsilon \in [\varepsilon_1, 1]. \end{cases} \quad (6.4.2)$$

The previous proposition provides a characterisation of an admissible mechanism (q, t_0) satisfying the IC constraint for all types of agents; its proof is postponed to [Section 6.6.3](#). Nevertheless, the concrete menu of contracts proposed by the principal must be composed of quantities $e_{\min}$ and a price T associated with each quantity, regardless of the type of agent, as specified in [Definition 6.1.3](#). The form of the practical menu of contracts associated with an admissible mechanism is a consequence of the previous theorem and is given by [Corollary 6.6.5](#). We can summarise this result by saying that considering a sufficiently smooth admissible menu of revealing contracts $(e_{\min}, T)$ is equivalent to considering an admissible mechanism (q, t_0) , where q is non-decreasing and the price t_0 is given by (6.4.2). It is now necessary to establish conditions implying that such a mechanism satisfies the agent's participation constraint.

6.4.2 Adding the participation constraint

Recall that an agent of type $\varepsilon \in [0, 1]$ accepts the contract if his utility from it is larger than his reservation utility, defined in our framework as his utility without insurance. To establish a precise result,

we define the function $\underline{c}$ for all $\varepsilon \in [0, 1]$ by:

$$\underline{c}(\varepsilon) := \begin{cases} e^{-\beta Q_0(\varepsilon)} & \text{if } \varepsilon < \varepsilon_1, \\ (1 + \alpha)^{-\beta(\varepsilon - \varepsilon_1)} e^{-\beta Q_0(\varepsilon)} & \text{if } \varepsilon \geq \varepsilon_1. \end{cases} \quad (6.4.3)$$

The following proposition states that by controlling the constant c_q in the insurance premium t_0 given by (6.4.2), the principal can choose whether to select agents of lower risk types. As a result, only the agents of a sufficiently high risk type are selected by the principal. Indeed, agents with a high probability of losing their income are easily satisfied and willing to pay much more than those of a less risky type. This result is entirely implied by the fact that the reservation utility of an agent depends on his type and only occurs in principal-agent problems with countervailing incentives. Additional information including the proof of the proposition can be found in Section 6.6.4. In particular, Remark 6.6.6 shows that if a constant reservation utility had been chosen, the selected agents would have been those of the less risky type.

Proposition 6.4.3. *If the mechanism (q, t_0) is admissible and incentive compatible, an agent of type $\varepsilon \in [0, 1]$ subscribes to the insurance plan if and only if $c_q \geq \underline{c}(\varepsilon)$. Moreover, by defining $\underline{\varepsilon} := \min\{\varepsilon \in [0, 1], \text{ s.t. }, c_q = \underline{c}(\varepsilon)\}$, the participation constraint is satisfied only for agents of type $\varepsilon \in [\underline{\varepsilon}, 1]$.*

Now that the subset of risk-revealing contracts satisfying the agents' participation constraint is well defined, we can study the principal's problem.

6.4.3 The optimal menu of contracts

In the third-best case, the principal's goal is to find an optimal admissible menu of contracts $(e_{\min}, T)$ to maximise her profit, as defined by (6.1.8), without knowing the agent's type. In fact, instead of maximising the principal's utility over all possible contracts, we restrict the study to the menus of contracts associated with an admissible mechanism (q, t_0) in the sense of Definition 6.4.1. Thus, in line with the revelation principle, it is sufficient to only consider admissible mechanisms that are risk revealing. Recalling that an agent subscribes to a contract only if it satisfies his participation constraint, her problem becomes:

$$\sup_{(q, t_0) \in \mathcal{C}^Q} \int_{\varepsilon \in \Xi(q, t_0)} \pi(\varepsilon) d\varepsilon, \quad \text{for } \pi(\varepsilon) := w_0 t_0(\varepsilon) - \varepsilon \alpha q(\varepsilon) \underline{\omega} w_0, \quad (6.4.4)$$

where $\Xi(q, t_0)$ denotes the set of $\varepsilon \in [0, 1]$ such that the participation constraint $t_0(\varepsilon) \leq t_{\max}(\varepsilon, q(\varepsilon))$ is satisfied, where $t_{\max}$ is defined by (6.2.8).

From Theorem 6.4.2, we know that an admissible mechanism (q, t_0) satisfies the IC constraint if and only if q is increasing and the price t_0 is given by (6.4.2). Moreover, we know by Proposition 6.4.3 that the participation constraint is satisfied only for agents of type $\varepsilon \in [\underline{\varepsilon}, 1]$ if and only if $c_q = \underline{c}(\underline{\varepsilon})$, where $\underline{c}$ is defined by (6.4.3). For $\underline{\varepsilon} \in [0, 1]$, we thus denote by $\mathcal{C}^Q(\underline{\varepsilon})$ the set of admissible and revealing mechanisms such that the participation constraint is satisfied for all $\varepsilon \in [\underline{\varepsilon}, 1]$ only. In line with the previous reasoning, the principal's problem is equivalent to:

$$\sup_{\underline{\varepsilon} \in [0, 1]} \Pi(\underline{\varepsilon}), \quad \text{where } \Pi(\underline{\varepsilon}) := \sup_{(q, t_0) \in \mathcal{C}^Q(\underline{\varepsilon})} \int_{\underline{\varepsilon}}^1 \pi(\varepsilon) d\varepsilon.$$

To solve the principal's problem, we first fix $\underline{\varepsilon} \in [0, 1]$. We denote by $\mathcal{Q}(\underline{\varepsilon})$ the space of functions Q defined on $[\underline{\varepsilon}, 1]$, continuous and piecewise continuously differentiable of class $\mathcal{C}^3$, satisfying:

- (i) Q is continuous on $[\underline{\varepsilon}, 1]$ and such that $Q(\underline{\varepsilon}) = 0$;
- (ii) Q' is positive and continuous except at ε_1 , where $Q'(\varepsilon_1^-) = \ln(1 + \alpha)$ and $Q'(\varepsilon_1^+) = 0$;

(iii) Q'' is positive and continuous except at ε_1 .

We consider the following second-order non-linear ordinary differential equation (ODE for short):

$$\frac{\beta}{\omega}(\beta|\varepsilon|^2 Q''(\varepsilon) - 2)e^{\beta(Q(\varepsilon) - \varepsilon Q'(\varepsilon))} + G(\varepsilon, Q) = 0, \quad (\text{ODE})$$

with initial conditions $Q(\underline{\varepsilon}) = 0$ and $Q'(\underline{\varepsilon}) = \underline{q}$, for $\underline{q} \in \mathbb{R}_+$, and where the function G is defined for any $(\varepsilon, Q) \in [\underline{\varepsilon}, 1] \times \mathcal{Q}(\underline{\varepsilon})$ by:

$$G(\varepsilon, Q) := \begin{cases} (1 + \varepsilon Q''(\varepsilon))e^{Q'(\varepsilon)}, & \text{for } \varepsilon \in [\underline{\varepsilon}, \varepsilon_1 \vee \underline{\varepsilon}), \\ (1 + \alpha)^{\beta(\varepsilon_1 \vee \underline{\varepsilon}) + 1} \left(1 + \varepsilon \frac{1 + \alpha}{\alpha} Q''(\varepsilon)\right) e^{\frac{1 + \alpha}{\alpha} Q'(\varepsilon)}, & \text{for } \varepsilon \in [\varepsilon_1 \vee \underline{\varepsilon}, 1]. \end{cases}$$

This ODE is at the heart of the resolution of the principal's problem, since it characterises the optimal admissible mechanism, for $\underline{\varepsilon}$ and $\underline{q}$ fixed.

Theorem 6.4.4. *Given $\underline{\varepsilon} \in [0, 1]$ and $\underline{q} \in \mathbb{R}_+$, if there exists $Q \in \mathcal{Q}(\underline{\varepsilon})$ solution to (ODE), then the optimal admissible mechanism (q^*, t_0^*) for the principal is given by:*

$$\begin{aligned} & \left(\frac{1}{\alpha} (e^{Q'(\varepsilon)} - 1), 1 - e^{\beta(Q(\varepsilon) - \varepsilon Q'(\varepsilon))} \right) && \text{for } \varepsilon \in [\underline{\varepsilon}, \varepsilon_1 \vee \underline{\varepsilon}), \\ & \left(e^{\frac{1 + \alpha}{\alpha} Q'(\varepsilon)}, 1 - (1 + \alpha)^{-\beta(\varepsilon_1 \vee \underline{\varepsilon})} e^{\beta(Q(\varepsilon) - \varepsilon Q'(\varepsilon))} \right) && \text{for } \varepsilon \in [\varepsilon_1 \vee \underline{\varepsilon}, 1]. \end{aligned}$$

Remark 6.4.5. *Theorem 6.4.4 only gives a sufficient condition for the principal's optimisation problem. In fact, it would be possible to obtain a necessary condition. Nevertheless, in the numerical example we are interested in (detailed in the following subsection), as the solution to (ODE) naturally satisfies the constraint of being in $\mathcal{Q}(\underline{\varepsilon})$, we decide to simplify the result by presenting it in this way. For more details, the reader is referred to Remark 6.6.8.*

For the sake of clarity, the proof of the theorem is reported in Section 6.6.5. The main point to observe is that, contrary to the first-best case, the optimal insured quantity, $e_{\min}^*(\varepsilon) = \alpha q^*(\varepsilon) \omega w_0 / p_e$, is now increasing with respect to ε . Since the premium is also increasing with respect to ε , no agent should have an interest in hiding his risk anymore.

However, (ODE) cannot be solved other than numerically. Therefore, to solve the principal's problem, one must first fix $\underline{\varepsilon} \in [0, 1]$ and an arbitrary initial value $\underline{q} \in \mathbb{R}_+$ for $Q'(\underline{\varepsilon})$. Then, the solution of the previous ODE can be computed. With this solution, one can compute the principal's profit in this case, using Corollary 6.6.7. This profit can then be maximised by choosing an optimal initial condition $\underline{q}$ and an optimal $\underline{\varepsilon} \in [0, 1]$. For the numerical results, readers are referred to the next subsection, which discusses the application of this model to a particular framework: fuel poverty.

6.4.4 Third-best insurance against fuel poverty

We consider the same household as the one studied in Section 6.3.2. From the recursive scheme explained in the previous subsection, we obtain the optimal $\underline{\varepsilon}^* \approx 0.63$ with $Q'(\underline{\varepsilon}^*) \approx 0$, and thus, agents of type $\varepsilon < 0.63$ are not insured. Solving (ODE) for these parameters, we obtain the optimal function $Q \in \mathcal{Q}(\underline{\varepsilon}^*, 0)$, and in particular $\varepsilon_1^* \approx 0.66$. Thanks to Theorem 6.4.4, we can compute the optimal revealing mechanism (q^*, t_0^*) , and thus the optimal insured quantity for an agent of type $\varepsilon \in [\underline{\varepsilon}^*, 1]$, which is $e_{\min}^*(\varepsilon) = \alpha q^*(\varepsilon) \omega w_0 / p_e$, and its corresponding price $T^*(\varepsilon) = w_0 t_0^*(\varepsilon)$.

Optimal quantity insurance. The optimal quantities and prices are represented by blue lines in Figure 6.2. The left graph shows that the insured quantity in the third-best case (blue line) is smaller than that in the first-best case (green line). Only agents of type $\varepsilon = 1$ are insured for the same quantity as they would be in the first-best case, modulo numerical errors, *i.e.*, $e_{\min}^*(1) \approx 12,665$ kWh/year. In the right graph, the green line represents the maximum price an agent of type ε is willing to pay for the quantity $e_{\min}^*(\varepsilon)$, given by $T_{\max}(\varepsilon) := w_0 t_{\max}(\varepsilon, q^*(\varepsilon))$. We thus observe that, as stated before, an agent of type ε^* pays his maximum price, and his information rent is thus reduced to zero. In contrast, an agent of type $\varepsilon > \varepsilon^*$ obtains an information rent. In particular, the information rent is increasing with ε . Finally, as was already noted in the first-best case, the insurance premium is higher than the future price of the quantity, $p_e e_{\min}^*$ (orange line). This result highlights a form of inconsistency of the agents and reflects the need to put in place, in addition to insurance, other options to encourage agents of higher risk types to save.

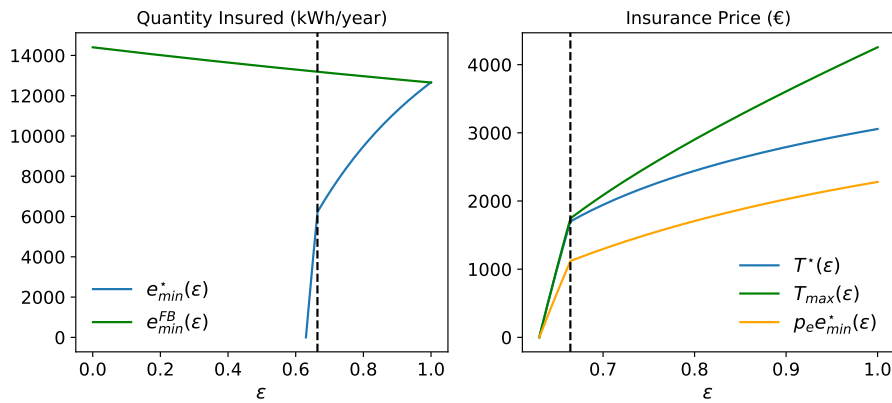


Figure 6.2 – Optimal insurance contract in the third-best case.

Left: optimal insured quantity (blue), compared to the quantity in first-best case (green). Right: optimal premium (blue), compared to the maximum price (green) and to the future price (orange). Dotted black axes: $\varepsilon = \varepsilon_1^*$.

Improvement of energy consumption. Given the menu of contracts $(e_{\min}^*, T^*)$ associated with the optimal admissible revealing mechanism (q^*, t_0^*) , we can compute the optimal consumption of the agents who subscribe to an insurance contract (see Lemmas 6.2.1, 6.2.3 and 6.2.4 for the formulas). In order to highlight the positive effects of quantity insurance, we have also computed the quantities consumed by an agent in the case of a financial insurance. In this case, the agent's problem is modified, in the sense that at time $t = 1$, he no longer receives a quantity $e_{\min}$, but sees his income $\underline{\omega}w_0$ increases by a monetary amount r . This problem is highly similar (and even simpler) to the one dealt with in this chapter, which is why the theoretical results are omitted.

The left graph of Figure 6.3 represents the optimal consumption of energy at time $t = 0$, e_0^Q with in-kind insurance (blue curve), e_0^R with classical revenue insurance (green curve) and $e_0^\emptyset$ without insurance (orange curve). Obviously, with insurance, an agent consumes less than he would without, since paying for insurance decreases the effective income he splits between the two goods. Nevertheless, the in-kind insurance allows for a lower decrease in energy consumption compared to more traditional insurance, which already partly argues in favour of this type of insurance to ensure sufficient energy consumption by households. Note that at time $t = 1$, if the agent does not suffer a loss of income, the insurance is not activated, and he will thus consume the same quantity of energy in both cases, with and without insurance.

In contrast, as we can see in Figure 6.3 (right graph), if the household suffers an income loss, he consumes more energy with insurance (blue and green curves) than without insurance (orange curve). More precisely,

without insurance, his optimal consumption is $e_1^\emptyset \approx 5,800$ KWh, which is around the level of consumption of a household of four people without electric heating. Therefore, one can conclude that such a household would renounce home heating, for example. Otherwise, if the agent of type ε is insured, he receives either the quantity $e_{\min}^*(\varepsilon)$ or a monetary compensation, depending on the insurance type. In the case of the in-kind insurance, we can compute by Lemma 6.2.4 his optimal consumption $e_1^Q(\varepsilon)$ of energy. His effective consumption $e_1^{eff}(\varepsilon)$ (blue curve) is given by the sum of $e_1^Q(\varepsilon)$ and $e_{\min}^*(\varepsilon)$. It is interesting to note that the higher the agent's risk type, the higher the insured quantity is, and it tends towards the quantity $e_0^\emptyset$ consumed with the initial income w_0 . The results for a financial insurance are similar, but the increase in consumption for the riskiest households is smaller. This second fact also argues in favour of an in-kind insurance to increase the energy consumption of households most at risk in terms of fuel poverty.

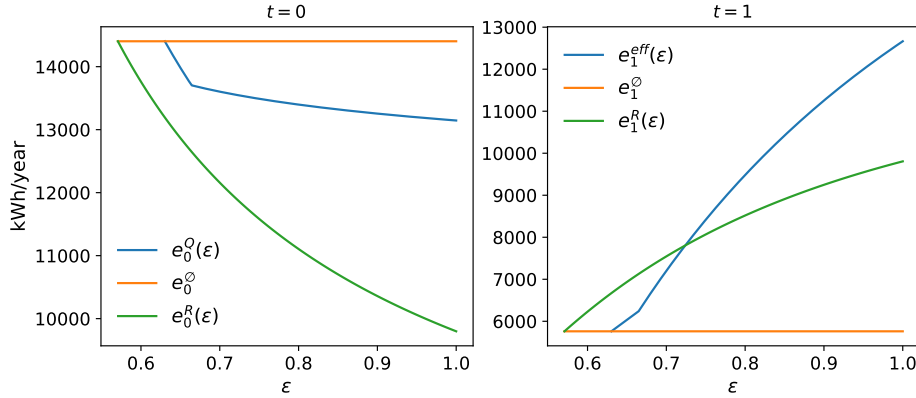


Figure 6.3 – Energy consumption in the third-best case.

Consumption with insurance (blue for in-kind, green for financial) and without (orange), at time $t = 0$, on the left, and at time $t = 1$ in the case of an income loss, on the right.

This latter fact can be explained as follows. By definition of ε_1 , an agent of type $\varepsilon \in [\underline{\varepsilon}^*, \varepsilon_1)$ receives a quantity $e_{\min}^*(\varepsilon) < \bar{e}_{\min}$. In this case, the in-kind insurance acts as an earmarked fund, and is thus equivalent to the financial insurance: the agent behaves exactly as if his income had been increased by $p_e e_{\min}^*(\varepsilon)$. More precisely, a part $\alpha/(1 + \alpha)$ of this supplementary income is dedicated to electricity consumption, and the other part, $1/(1 + \alpha)$, is dedicated to the other good, in the same way that his income w_t at time t is distributed between the two goods. More precisely, the agent has to decrease his consumption e_1^Q and increase his consumption y_1^Q to perfectly split this fictive supplementary income between the two goods. In contrast, when the insured quantity is higher, precisely if $e_{\min}^*(\varepsilon) > \bar{e}_{\min}$, the agent's consumption e_1^Q is reduced to 0, and he cannot decrease it anymore. In this case, his effective consumption is given by $e_{\min}^*(\varepsilon)$, and the agent cannot properly split this fictive supplementary income between the two goods. For these types of agents, *i.e.*, $\varepsilon \geq \varepsilon_1^*$, the insurance no longer acts as a liquid asset, thus ensuring higher electricity consumption.

These results on the energy consumption are very interesting in the situation we considered. Indeed, a household that falls into fuel poverty due to a loss of income tends to consume less electricity, which can lead to health problems and housing damage. This is exactly the kind of problem we are seeking to avoid by proposing insurance. Nevertheless, traditional income insurance allows the agent to receive money in the event of loss of income. However, an agent in fuel poverty has other needs to satisfy that he considers more important. He would therefore use the insurance money largely for these expenses, ignoring the significance for his health of heating his home sufficiently, for example. The insurance we model prevents this bias; it manages to oblige the agent to consume enough electricity to live decently in his home.

Comparison with financial insurance in terms of utility. Given the optimal contracts for both the quantity and financial insurance, we are able to compute numerically the expected utility of an household of any type $\varepsilon \in [0, 1]$ (see Figure 6.4, left). Of course, less risky households have no interest in purchasing such policies, and therefore have a utility equal to their reservation utility. For riskier households, both insurances allow an increase in utility, compared to the utility without insurance, but unfortunately, the utility obtained with quantity insurance is less than that obtained with income insurance. This is entirely due to the fact that, with in-kind insurance, the insured is forced to consume more energy than he would like for an equivalent income, and the benefit of this over-consumption is not taken into account in household utility. Indeed, the model assumes that households are not aware that under-consumption of energy can lead to long-term health problems, whose associated costs are paid by both the households and the society.

To account for the benefit of this increase in the energy consumption, one could take the point of view of a central planner, who is aware of the effects of fuel poverty. However, this would imply knowing how to measure the long-term costs of fuel poverty incurred by society as a whole, which is not a simple task. Indeed, first of all, it is difficult to directly link health problems to their intrinsic cause, and therefore to estimate the future cost of a precarious household. Moreover, these costs are difficult to quantify for each type of agent, as they do not only depend on the types considered here but on many other parameters (for *e.g.*, age, initial state of health, labour market status). Nevertheless, the direct and indirect costs incurred by fuel poverty are far from negligible since, as mentioned by Chaton [97] (2020), they represent more than one billion euros per (see Roys et al. [298] (2010) or Ahrendt et al. [10] (2016)).

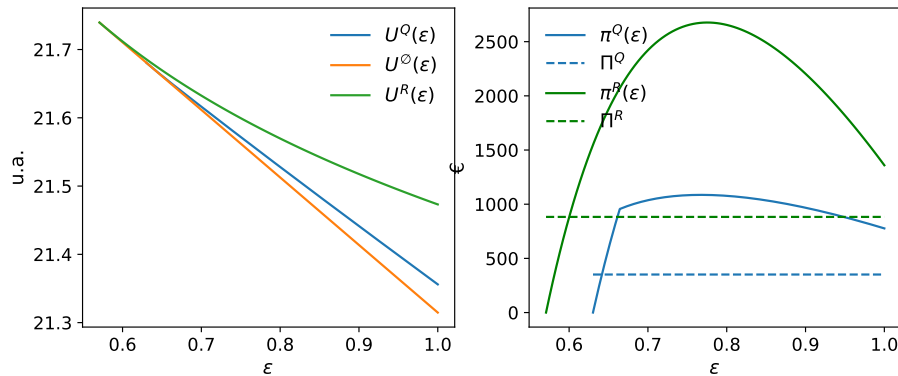


Figure 6.4 – Agent's utility and principal's profit in the third-best case.

Left: agent's utility with in-kind (blue), financial (green) and without (orange) insurance. Right: principal's profit (plain curves) and expected profit (dotted lines) in both cases of an in-kind (blue) or financial (green) insurance.

Finally, we can compute the profit π^Q of the principal for every type of agent, represented in Figure 6.4 (right – blue curve). In particular, her expected profit Π^Q from the considered population, given by the integral of $\pi^Q(\varepsilon)$ between $\underline{\varepsilon}^*$ and 1, is equal to €351. To show the benefit of considering a menu of contracts, we can compare our results to the profit induced by a unique contract. In this case, the optimal quantity and price can easily be computed: $e_{\min} \approx 10,538$ kWh/year and $T \approx \text{€}2,549$; and this contract induces an average profit of €342 for the principal. Therefore, there is a positive gain for the principal in implementing a menu instead of a unique contract (2.7% on each contract offered), and more agents are insured. Similarly, in the case of a financial insurance, we can compute (numerically) the principal's profit π^R for every type of agent (see Figure 6.4, green curve). In particular, her expected profit Π^R from the considered population is equal to €883. Therefore, proposing an in-kind insurance instead of a more classical revenue insurance implies a loss of utility from the point of view of both the household and the insurer.

Extensions. The previous result raises a new question: how could a central planner (*e.g.* the state) encourage insurers to offer such in-kind insurance, which enables households to significantly improve their energy consumption? Recall that, according to the study by Lacroix and Chaton [215] (2015), the effects of fuel poverty on the physical and mental health of individuals are not questionable. Therefore, a central planner, aware of the harmful effects of under-consumption of energy and wishing to reduce the risk of fuel poverty among the population, would be well advised to generalise the use of this type of insurance. In this case, we will be faced with a three-level principal-agent problem, where the state delegates to an insurance company the implementation of an in-kind insurance and remunerates it for this. Unfortunately, the problem would be more complicated and beyond the scope of this chapter, especially since it would be necessary to estimate the long-term costs of fuel poverty, which, as mentioned above, are difficult to quantify. Consideration could also be given to making this type of insurance mandatory, either by law or through a contract between a landlord and a tenant, for example. Indeed, on the one hand, to reduce health expenses, the state could have an interest in ensuring that agents do not experience fuel poverty. On the other hand, a landlord, or even a social landlord, could have the same interest in order to prevent the tenant from causing deterioration of the housing.

The simplicity of our model makes it also easy to extend into a multi-period model, keeping in mind that only agents who are not in precarious situations are entitled to subscribe to insurance plans. Thus, an agent who has not suffered a loss of income can again pay the premium to re-insure himself for the next period. A simple repetition of the model is sufficient to address this case. Moreover, our model could be extended to random prices, which would allow the household to have a guaranteed quantity of electricity even in case of a price increase. Finally, while we have chosen to apply it for insurance against fuel poverty, this type of model could be used for other staples with consumption that is affected by a loss of income. More generally, this type of insurance could also be purchased by manufacturing firms to ensure that they have sufficient input in the event of a temporary downturn in revenues.

6.5 Adding the prepayment option

Limits. Before considering some extensions, it would be necessary to address the issue of the high price. Indeed, the results we obtain highlight a certain form of irrationality of the agents due to their unwillingness to save money from one period to the next. More precisely, since the households considered do not think about saving to ensure a sufficient quantity of essential goods regardless of their future income, their choice is limited to subscribing to the insurance plan or doing nothing. Due to the shape of the chosen utility function, in particular its concavity, households who anticipate a loss of income with a high probability are willing to pay a very high price for the insurance we offer. They prefer to significantly reduce their disposable income by subscribing to the insurance, which only slightly reduces their utility, to ensure sufficient consumption the following year, which significantly increases their utility. This results in a very high premium, higher than the future price of the quantity. Therefore, insurance can be a tool to protect at-risk households, but it cannot be set up alone if securing the welfare of the agents is the desired goal.

Some suggestions could be considered to make the insurance premium more realistic. First, if a regulator required insurance to be offered to all types of agents, *i.e.*, $\varepsilon = 0$, the insurance price would be lower. In addition, the regulator could also better control the monopoly position of electricity suppliers in the case of energy insurance. By introducing competition in the market for this type of insurance, the price should fall towards the marginal cost of insurance. Another solution that seems, in our opinion, easy to implement would be to increase the options of agents by offering them an opportunity to engage in prepayment. This alternative can be a way to encourage agents of a high risk type to save money when they do not do so voluntarily.

No prepayment option. Therefore, we consider in this section that a regulator offers (or forces the insurer to offer) another form of contract: the prepayment option. In this situation, the agent can (i) subscribe at time $t = 0$ to an insurance contract; (ii) prepay a quantity e^P , i.e., pay at time $t = 0$ the price $p_e e^P$ to receive e^P in $t = 1$; or (iii) do nothing. In fact, adding this option only changes the agent's participation constraint: if his utility with prepayment is higher than with insurance, he does not subscribe to the insurance plan. Although some theoretical results could be obtained, we choose in this section to only present numerical results. Indeed, the theoretical formulations are in the same spirit as those developed throughout the chapter but more complicated, so it seems more relevant and meaningful in our opinion to discuss only the results obtained numerically, with the parameters defined for the application to fuel poverty. In this situation, we can see in [Section 6.5.2](#) that, even in the first-best case, this option allows a large decrease in the price of the insurance. Unfortunately, this addition implies that the third-best case detailed in [Section 6.5.3](#) is more complicated to solve, although the techniques developed throughout this chapter are a step towards resolution. In our opinion, this case would require further study, since it appears to be a very good way to lower the insurance premium and to insure agents of a medium risk type.

6.5.1 A new reservation utility

If the agent decides to prepay at time $t = 0$ a quantity e^P , his utility function is defined by (6.1.4), where $T = p_e e^P$ is the price of the chosen quantity. As previously, we assume without loss of generality that $e^P = \alpha q \underline{\omega} w_0 / p_e$ for some $q \in \mathbb{R}_+$. Through easy optimisation techniques, we obtain the following result:

Lemma 6.5.1. *If the agent subscribes to the prepayment option for a quantity $q < 1/\alpha \underline{\omega}$, his optimal consumption of each good at time $t = 0$ is given by*

$$y_0^P := \frac{w_0}{1 + \alpha} \frac{1 - \alpha q \underline{\omega}}{p_y} \quad \text{and} \quad e_0^P := \frac{\alpha w_0}{1 + \alpha} \frac{1 - \alpha q \underline{\omega}}{p_e},$$

and gives him the following utility: $V_0^P(q) = (1 + \alpha) \ln(w_0) + (1 + \alpha) \ln(1 - \alpha q \underline{\omega}) + C_{\alpha, p_e, p_y}$.

At time $t = 1$, he receives the prepaid quantity e^P in any case, not only in the case of an income loss. His utility to maximise at time $t = 1$ is thus naturally defined by:

$$V_1^P(\omega, q) = \max_{(e_1, y_1) \in \mathbb{R}_+^2} \alpha \ln(e_1 + \alpha q \underline{\omega} w_0 / p_e) + \ln(y_1), \quad \text{u.c.} \quad e_1 p_e + y_1 p_y \leq \omega w_0.$$

By maximising this utility with respect to e_1 and y_1 , we can state the following:

Lemma 6.5.2. *The optimal quantities consumed in each good at time $t = 1$ are given by:*

$$e_1^P := \frac{\alpha}{1 + \alpha} \frac{w_0}{p_e} (\omega - q \underline{\omega})^+ \quad \text{and} \quad y_1^P := \frac{\omega w_0 - p_e e_1^P}{p_y},$$

and provide the following utility to the agent: $V_1^P(\omega, q) = (1 + \alpha) \ln(\omega w_0) + C_{\alpha, p_e, p_y} + \bar{U}(q \underline{\omega} / \omega)$.

Therefore, by [Lemmas 6.5.1](#) and [6.5.2](#), the expected utility of an agent of type ε who chooses to prepay the quantity $e^P = \alpha q \underline{\omega} w_0 / p_e$ for $q \in [0, 1/\alpha \underline{\omega}]$ is given by:

$$\text{EU}^P(\varepsilon, q) = \text{EU}^\varnothing(\varepsilon) + (1 + \alpha) \ln(1 - \alpha q \underline{\omega}) + \beta \varepsilon \bar{U}(q) + \beta(1 - \varepsilon) \bar{U}(q \underline{\omega} / \bar{\omega}).$$

The agent then chooses the optimal amount he wants to prepay by maximising his expected utility over an admissible q . The easiest way to solve this optimisation problem is to perform a simple numerical optimisation to find the optimal quantity $q^P(\varepsilon)$ that an agent of type ε should prepay and his associated expected

utility, denoted $EU^{P,*}(\varepsilon)$, for every $\varepsilon \in [0, 1]$. For the parameters previously defined in Section 6.3.2, the results are presented in Figure 6.5.

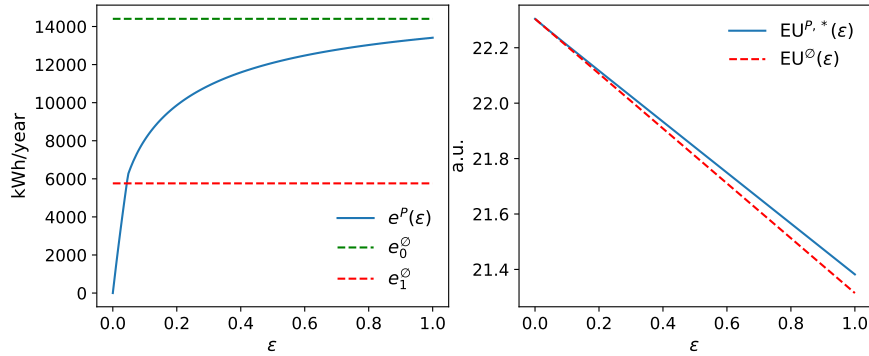


Figure 6.5 – Optimal quantity prepaid and associated expected utility.

Top: optimal quantity prepaid (blue), compared to the quantity consumed with the initial income (dotted green) and with an income loss (dotted red). Bottom: maximum expected utility with prepayment (blue), compared to the previous reservation utility (dotted orange).

Facing this new option, an agent of type ε subscribes to an insurance contract $(e_{\min}, T)$, with $e_{\min} = \alpha q \underline{w}_0 / p_e$ and $T = w_0 t_0$, if and only if the two following conditions hold:

$$EU^Q(\varepsilon, q, t_0) \geq EU^{P,*}(\varepsilon) \quad \text{and} \quad EU^Q(\varepsilon, q, t_0) \geq EU^\varnothing(\varepsilon). \quad (6.5.1)$$

By definition of $EU^{P,*}(\varepsilon)$, for every $\varepsilon \in [0, 1]$, we have $EU^{P,*}(\varepsilon) \geq EU^P(\varepsilon, 0) = EU^\varnothing(\varepsilon)$. Therefore, the second inequality in (6.5.1) is implied by the first and is thus not necessary. In this framework, the reservation utility of an agent of type ε is therefore defined by the utility he obtains thanks to the prepayment option. To simplify the notation, we denote by $\Delta EU^{P,*}$ the difference between the expected utility with prepayment of an agent and his utility without:

$$\Delta EU^{P,*}(\varepsilon) := EU^{P,*}(\varepsilon) - EU^\varnothing(\varepsilon), \quad (6.5.2)$$

which corresponds to the information rent in this framework.

6.5.2 First-best case

As detailed in Section 6.3.1, the problem of the principal in this case is defined by (6.1.7), under a new participation constraint of the agent, since his reservation utility is now given by $EU^{P,*}(\varepsilon)$: an agent accepts the contract if it provides him with at least as much utility as the prepayment. Similar to the reasoning developed in Section 6.3.1, when the principal knows the type of the agent, she may charge him the highest price he is willing to pay for the insurance. In this case, the agents' informational rents are then reduced to zero for any type $\varepsilon \in [0, 1]$.

Using the notation (6.5.2), the participation constraint of an agent of type ε is equivalent to:

$$t_0 \leq T_{\max}^P(\varepsilon, q) := 1 - \exp\left(\frac{\Delta EU^{P,*}(\varepsilon)}{1 + \alpha}\right) \times \begin{cases} (1 + q\alpha)^{-\beta\varepsilon} & \text{if } q < 1, \\ q^{-\beta\varepsilon \frac{\alpha}{1+\alpha}} (1 + \alpha)^{-\beta\varepsilon} & \text{if } q \geq 1, \end{cases}$$

However, contrary to Section 6.3.1, since it is relatively complicated to explicitly obtain the expected utility with prepayment, we cannot give a more detailed formula for the maximum price that an agent of type ε is willing to pay for insurance. Nevertheless, all the results can easily be computed numerically.

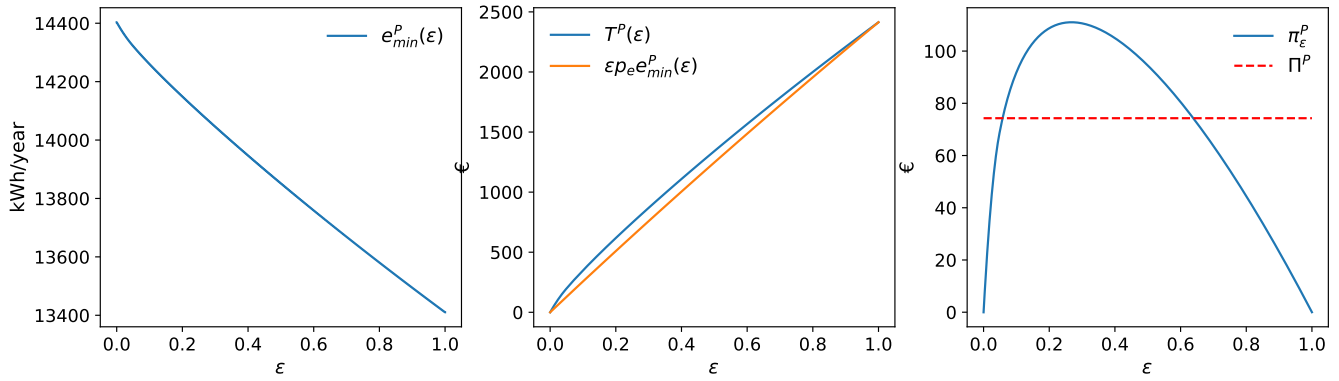


Figure 6.6 – Optimal insurance in the first-best case with prepayment.

Blue lines represent, from left to right, the insured quantity, the premium and the principal's profit, with respect to the probability ε . In the middle graph, the premium is compared to the actuarial price (orange line), which also corresponds to the principal's cost. The red dotted line on the right graph is her average profit.

Figure 6.6 presents, from left to right, the insured quantity, the price paid by the agents, and the principal's profit, in the case of insurance against fuel poverty, *i.e.*, with the parameters defined in Section 6.3.2. We can compare these graphs with those of Figure 6.1. The most interesting point is that the price of insurance is significantly lower in this new situation. Indeed, the price paid by the agents is now barely higher than the actuarial price, whereas without prepayment, it was sometimes even higher than $p_e e_{\min}$, which is actually precisely the price of the prepayment.

6.5.3 Third-best case

Since we are only changing the reservation utility, the results of Section 6.4.1 remain true. In particular, the principal offers a menu of revealing contracts $(e_{\min}, T)$ defined by Corollary 6.6.5, such that an agent of type ε chooses the quantity $e_{\min} = \alpha q(\varepsilon) \omega w_0 / p_e$ and pays the price $w_0 t_0(\varepsilon)$, given by (6.4.2), for a particular function q , which will be optimised by the principal. The only thing that changes is the participation constraint, which is now given by $EU^Q(\varepsilon, q, t_0) \geq EU^{P,\star}(\varepsilon)$. Using the form of the price given by (6.4.2), this constraint is equivalent to

$$c_q \geq \underline{c}^P(\varepsilon) := \exp\left(\frac{\Delta EU^{P,\star}(\varepsilon)}{1 + \alpha}\right) \underline{c}(\varepsilon).$$

The function $\underline{c}$, defined by (6.4.3), is decreasing, implying in Section 6.4.2 that the participation constraint was satisfied for agents of type ε above a specific level. Unfortunately, in this case, it is not possible to precisely determine the variations of the function $\underline{c}^P$, since we do not have an explicit form and, as we can see in Figure 6.5, the difference defined in (6.5.2) is increasing.

In the case with prepayment, it is therefore difficult to determine a monotonicity or even the variations of the information rent. With the help of the first-best case, we can still intuit that the information rent in the third-best case is increasing up to a specific $\varepsilon \in [0, 1]$ and then decreasing. More precisely, as seen on the graph with the principal's profit, she earns money on the agents of medium risk type, since agents of type $\varepsilon = 0$ and $\varepsilon = 1$ have no interest in subscribing to insurance plans. Indeed, on the one hand, the problem of agents of the lowest risk type remains the same as in the case without prepayment: the optimal quantity they would like to prepay is zero, which implies that their utility with prepayment is equal to their utility without insurance. Their reservation utility is therefore unchanged from the case studied throughout this paper, and these agents are not of interest from the insurer's point of view. On the other hand, the agents

of type $\varepsilon = 1$ are now indifferent between prepayment and insurance, the two options providing them with a sufficient quantity of energy for the future. The agents of the highest risk type, who were highly courted by the insurer, are now hard to satisfy and instead turn to prepayment. Hence, the most ‘interesting’ agents in this case for the Principal seem to be the intermediate ones.

Unfortunately, to address the third-best case in a rigorous manner, further study is required. Indeed, the non-monotonicity of the information rent makes the problem more difficult. Nevertheless, following the intuition previously developed, we can make the reasonable assumption that the optimal contract should only select types ε in an interval $[\underline{\varepsilon}, \bar{\varepsilon}] \subset [0, 1]$. We can then apply the reasoning developed in [Section 6.4](#) to find the ODE satisfied by the optimal contract under this assumption, which is highly similar to (ODE), and thus not detailed here. We only present in [Figure 6.7](#) the optimal quantity insured (left graph) as well as the corresponding premium (right), obtained with the usual parameters.

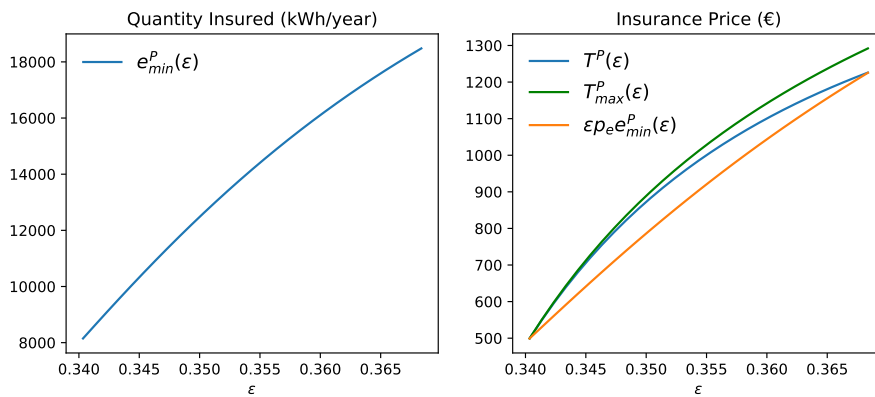


Figure 6.7 – Insurance contract in the third-best case, when the prepayment option is available.

Left: insured quantity. Right: premium (blue), compared to maximum price (green) and to actuarial price (orange).

Comparing these results with those in [Figure 6.2](#), it is clear that, when the prepayment option is available:

- the insured quantity is higher, thus clearly ensuring a sufficient level of energy consumption at time $t = 1$, which should contribute in reducing the risk of fuel poverty among the households;
- the premium of the insurance is just slightly above the actuarial price, given by $\varepsilon p_e e_{\min}^P$ for an agent of type ε , and therefore way below the future price of the quantity.

Therefore, the addition of the prepayment option seems to (i) allow agents to consume sufficient energy in case of loss of income, either by insurance or prepayment; (ii) decrease the insurance premium compared to the case without prepayment; and (iii) allow agents of a medium risk type to be insured.

However, it can be seen that the contract we obtain is designed only for a small proportion of types, approximately 2%. From this point of view, the value of implementing such insurance is largely debatable, and the prepayment option seems to be a more relevant and sufficient tool to limit fuel poverty. Nevertheless, it should be noticed that, due to the assumption we have made, a contract with more than one interval could be more optimal than this one, and would thus select more types of agents. Secondly, it is worth noticing that the interval depends on the parameters considered. For example, if we focus on slightly wealthier households ($w_0 = 50,000$ instead of 35,000, and $\alpha = 0.06$) with better expected income at time $t = 1$ ($\bar{w} = 1.5$ instead of 1), the proportion of selected types increases up to almost 10%. Finally, these results still underline the need for regulation of this market. More precisely, in this case, a regulator could force or try to incentivise the insurance company to offer more different contracts, in order to cover a larger proportion of households. In this perspective, our work could help quantify the loss of income for the insurer if she has to offer contracts for a wider range of risk types.

6.6 Technical results and proofs...

This section provides additional technical results as well as all proofs omitted throughout this chapter.

6.6.1 ... for the first-best case

The profit of the principal in the first-best case, induced by the optimal contract detailed in [Proposition 6.3.1](#), is given by the following result.

Corollary 6.6.1. *Let us assume $\beta > \underline{\omega}$. The optimal contract $(e_{\min}^{\text{FB}}, T^{\text{FB}})$ for an agent of type $\varepsilon \in [0, 1]$ induced the following profit for the principal:*

$$\pi_{\varepsilon}^{\text{FB}} := \begin{cases} w_0 \left(1 - (\beta(1 + \alpha)/\underline{\omega})^{-\beta\varepsilon/(1+\alpha+\alpha\beta\varepsilon)} - \varepsilon\underline{\omega}(\beta(1 + \alpha)^{-\beta\varepsilon}/\underline{\omega})^{(1+\alpha)/(1+\alpha+\alpha\beta\varepsilon)} \right) & \text{if } \varepsilon \leq \varepsilon_1^{\text{FB}}, \\ w_0 \left(1 + \varepsilon\underline{\omega} - (\beta/\underline{\omega})^{-\beta\varepsilon/(1+\beta\varepsilon)} - \varepsilon\underline{\omega}(\beta/\underline{\omega})^{1/(1+\beta\varepsilon)} \right) & \text{if } \varepsilon > \varepsilon_1^{\text{FB}}. \end{cases}$$

The proof of the previous corollary results from the proof of the associated proposition, detailed below.

Proof of Proposition 6.3.1. We fix the agent's type $\varepsilon \in [0, 1]$. Since the profit of the principal is increasing in T , she has an incentive to set the price of the insurance equal to the maximum price the agent is willing to pay, i.e., $t_0 = t_{\max}(\varepsilon, q)$, for $q \in \mathbb{R}_+$. The participation constraint of the agent is thus binding, and the maximisation problem of the principal (6.3.1) becomes:

$$\pi_{\varepsilon} = w_0 \max \left\{ \sup_{q \in [0, 1]} \left\{ 1 - (1 + \alpha q)^{-\beta\varepsilon} - \varepsilon \alpha q \underline{\omega} \right\}, \sup_{q \geq 1} \left\{ 1 - q^{-\beta\varepsilon \frac{\alpha}{1+\alpha}} (1 + \alpha)^{-\beta\varepsilon} - \varepsilon \alpha q \underline{\omega} \right\} \right\}.$$

Computing the first- and second-order conditions (FOC and SOC) for each supremum, and since $\beta > \underline{\omega}$, we obtain that the two suprema are attained for q_1 and q_2 , respectively, where:

$$q_1 = \min \left\{ \frac{1}{\alpha} \left((\beta/\underline{\omega})^{1/(1+\beta\varepsilon)} - 1 \right), 1 \right\}, \quad \text{and} \quad q_2 = \max \left\{ (\beta(1 + \alpha)^{-\beta\varepsilon}/\underline{\omega})^{(1+\alpha)/(1+\alpha+\alpha\beta\varepsilon)}, 1 \right\}.$$

If $\varepsilon > \varepsilon_1^{\text{FB}}$, then $q_1 < 1$ and $q_2 = 1$ and, conversely, if $\varepsilon < \varepsilon_1^{\text{FB}}$, then $q_1 = 1$ and $q_2 > 1$. Since the two suprema have the same value for $q = 1$, we conclude that q_{ε} defined by (6.3.2) is optimal. $\square$

Remark 6.6.2. *We assume in Proposition 6.3.1 and Corollary 6.6.1 that $\beta > \underline{\omega}$ because it is the most interesting case. Otherwise, we would have $\varepsilon_1^{\text{FB}} < 0$, and the maximum would be reached for q_1 defined in the previous proof. However, in this particular case, q_1 is negative for all $\varepsilon \in [0, 1]$. Therefore, the optimal q_{ε} is zero in this case for all $\varepsilon \in [0, 1]$. This means that the principal has no interest in offering insurance. Indeed, when β is too small, the agent has very little concern for his future, so he is not willing to pay for insurance to protect himself.*

6.6.2 ... to obtain comparative statics

Proof of Corollary 6.2.8. We recall that the maximum price the agent of type ε is willing to pay for a quantity $e_{\min}$ is given by (6.2.9). To establish the comparative statics detailed in the corollary, it suffices to compute the derivative of $T_{\max}$ with respect to the parameters under consideration.

On the one hand, when $e_{\min} < \alpha\underline{\omega}w_0/p_e$, we have:

$$\begin{aligned} \partial_{e_{\min}} T_{\max} &= \frac{\beta \varepsilon p_e (w_0 - T_{\max})}{\underline{\omega}w_0 + p_e e_{\min}}, \quad \partial_{p_e} T_{\max} = \frac{\beta \varepsilon e_{\min} (w_0 - T_{\max})}{\underline{\omega}w_0 + p_e e_{\min}}, \quad \partial_{\varepsilon} T_{\max} = \beta \ln \left(1 + \frac{p_e e_{\min}}{\underline{\omega}w_0} \right) (w_0 - T_{\max}), \\ \partial_{\beta} T_{\max} &= \varepsilon \ln \left(1 + \frac{p_e e_{\min}}{\underline{\omega}w_0} \right) (w_0 - T_{\max}), \quad \text{and} \quad \partial_{\underline{\omega}} T_{\max} = \frac{-\beta \varepsilon p_e e_{\min} (w_0 - T_{\max})}{\underline{\omega}(\underline{\omega}w_0 + p_e e_{\min})}. \end{aligned}$$

It is easy to see that these derivatives are non-negative, except the last one which is non-positive, since $T_{\max}$ is bounded by w_0 , and all other parameters are non-negative. Furthermore, we have

$$\partial_{w_0} T_{\max}(\varepsilon, e_{\min}) = 1 - \exp\left(-\beta\varepsilon \ln\left(1 + \frac{p_e e_{\min}}{\underline{\omega} w_0}\right)\right) \left(1 + \frac{\beta\varepsilon p_e e_{\min}}{\underline{\omega} w_0 + p_e e_{\min}}\right).$$

Further computations are needed to ensure that this derivative is non-negative. Let us introduce the following function, for all $x \in [0, 1)$:

$$f(x) = 1 - e^{-\beta\varepsilon \ln(1+x)} \left(1 + \frac{\beta\varepsilon x}{1+x}\right).$$

Noticing that the following equality holds:

$$\partial_{w_0} T_{\max}(\varepsilon, e_{\min}) = f\left(\frac{p_e e_{\min}}{\underline{\omega} w_0}\right),$$

it is only necessary to prove that f is non-negative for all $x \in [0, 1)$ in order to obtain the desired result, namely that $T_{\max}$ is increasing with respect to w_0 . First, remark that $f(0) = 0$ and that the derivative of f satisfies:

$$f'(x) = \frac{\beta\varepsilon x(1 + \beta\varepsilon)}{|1+x|^2} e^{-\beta\varepsilon \ln(1+x)} \geq 0 \text{ for all } x \in [0, 1).$$

Therefore, the function f starts at 0 for $x = 0$ and is then increasing for $x \in [0, 1)$, which implies that f is non-negative on $[0, 1)$. We thus conclude that $T_{\max}$ is increasing with respect to w_0 .

In a similar way, it is easy to see that, when $e_{\min} \geq \alpha \underline{\omega} w_0 / p_e$, the derivatives have the same sign as their counterparts on the previous interval, confirming that $T_{\max}$ is increasing with respect to w_0 , $e_{\min}$, p_e , ε and β , but decreasing in $\underline{\omega}$. □

6.6.3 ... to find a risk-revealing menu of contracts

Before seeking a risk-revealing menu of contracts, we first prove that the Spence-Mirrlees condition is satisfied (see [Lemma 6.6.3](#)). This condition is important since it makes the incentive problem well behaved in the sense that only local incentive constraints need to be considered. Together with [Lemma 6.6.4](#), this allows us to establish [Theorem 6.4.2](#), whose proof is reported below, after the two lemmas.

Lemma 6.6.3. *The marginal rates of substitution between the in-kind support and the insurance price can be ranked in a monotonic way. More precisely,*

$$\frac{\partial}{\partial \varepsilon} \left(\frac{\partial_q \text{EU}^Q(\varepsilon, q, t_0)}{\partial_{t_0} \text{EU}^Q(\varepsilon, q, t_0)} \right) \leq 0.$$

Proof. Indeed, recalling that the expected utility of an agent of type ε is given by (6.2.7), we have:

$$\partial_q \text{EU}^Q(\varepsilon, q, t_0) = \begin{cases} \beta\varepsilon\alpha(1+\alpha)/(1+q\alpha) & \text{if } q < 1, \\ \beta\varepsilon\alpha/q & \text{if } q \geq 1, \end{cases} \text{ and } \partial_{t_0} \text{EU}^Q(\varepsilon, q, t_0) = -\frac{1+\alpha}{1-t_0},$$

which leads to

$$\frac{\partial}{\partial \varepsilon} \left(\frac{\partial_q \text{EU}^Q(\varepsilon, q, t_0)}{\partial_{t_0} \text{EU}^Q(\varepsilon, q, t_0)} \right) = \begin{cases} -\frac{\beta\alpha(1-t_0)}{1+q\alpha} & \text{if } q < 1, \\ -\frac{\beta\alpha(1-t_0)}{q(1+\alpha)} & \text{if } q \geq 1. \end{cases}$$

Both quotients are indeed non-positive since the price of the insurance should be at least smaller than the agents' income, which implies $t_0 < 1$, and all other quantities and prices are positive. $\square$

Lemma 6.6.4. *Let (q, t_0) be an admissible mechanism such that q is non-decreasing and t_0 is given by (6.4.2) for some $c_q \geq 0$. If the function q is constant on some interval contained in $[0, 1]$, then the price t_0 is also constant on this interval.*

Proof. Let us first assume that q is constant on some interval $[x, y]$, where $y < \varepsilon_1$. For all ε in this interval, we have in particular $q(\varepsilon) = q(x)$ and thus:

$$\begin{aligned} t_0(\varepsilon) &= 1 - c_q(1 + \alpha q(x))^{-\beta x - \beta(\varepsilon - x)} \exp \left(\beta \int_0^x \ln(1 + \alpha q(\varepsilon)) d\varepsilon + \beta \int_x^\varepsilon \ln(1 + \alpha q(x)) d\varepsilon \right) \\ &= 1 - c_q(1 + \alpha q(x))^{-\beta x} \exp \left(\beta \int_0^x \ln(1 + \alpha q(\varepsilon)) d\varepsilon \right). \end{aligned}$$

Therefore, for all $\varepsilon \in [x, y]$, $t_0(\varepsilon) = t_0(x)$, i.e., t_0 is constant on this interval. The proof is highly similar for an interval $[x, y]$ such that $x \geq \varepsilon_1$. Finally, if the interval $[x, y]$ contains ε_1 , we necessarily have $q(\varepsilon) = 1$ for all ε in the interval. By definition of ε_1 , we actually have $x \geq \varepsilon_1$, and the problem is reduced to the previous case. $\square$

Proof of Theorem 6.4.2. (i) We first prove that q being non-decreasing with respect to $\varepsilon \in [0, 1]$ and t_0 satisfying (6.4.2) are necessary conditions for the admissible mechanism (q, t_0) to satisfy the IC constraint on $[0, 1]$. To prove this, we first fix $\varepsilon \in (0, 1)$, such that $q(\varepsilon) \neq 1$, and focus the study on an agent of type ε . From (6.2.7), his expected utility if he chooses a contract $(q(\varepsilon'), t_0(\varepsilon'))$, for some $\varepsilon' \in [0, 1]$, is as follows:

$$\text{EU}^Q(\varepsilon, q(\varepsilon'), t_0(\varepsilon')) = \text{EU}^\varnothing(\varepsilon) + (1 + \alpha) \ln(1 - t_0(\varepsilon')) + \beta \varepsilon \bar{U}(q(\varepsilon')). \quad (6.6.1)$$

The mechanism (q, t_0) is incentive compatible if the agent chooses the contract designed for him to maximise his utility. Therefore, the utility computed above must attain its maximum on $\varepsilon' = \varepsilon$. Since the mechanism is assumed to be regular enough, we can compute the first and second derivatives of the previous utility with respect to ε' . The first-order condition (FOC) states that the first derivative has to be equal to zero for $\varepsilon' = \varepsilon$. Since the derivative with respect to $\varepsilon' \in (0, 1)$ of $\text{EU}^Q(\varepsilon, q(\varepsilon'), t_0(\varepsilon'))$ is given by:

$$\partial_{\varepsilon'} \text{EU}^Q(\varepsilon, q(\varepsilon'), t_0(\varepsilon')) = -(1 + \alpha) \frac{\partial_{\varepsilon'} t_0(\varepsilon')}{1 - t_0(\varepsilon')} + (1 + \alpha) \beta \varepsilon \times \begin{cases} \frac{\alpha \partial_{\varepsilon'} q(\varepsilon')}{1 + \alpha q(\varepsilon')} & \text{if } q(\varepsilon') < 1, \\ \frac{\alpha \partial_{\varepsilon'} q(\varepsilon')}{(1 + \alpha) q(\varepsilon')} & \text{if } q(\varepsilon') > 1, \end{cases} \quad (6.6.2)$$

the FOC for the agent of type ε is as follows:

$$\partial_{\varepsilon} t_0(\varepsilon) = \begin{cases} \beta \varepsilon \frac{\alpha \partial_{\varepsilon} q(\varepsilon)}{1 + \alpha q(\varepsilon)} (1 - t_0(\varepsilon)) & \text{if } q(\varepsilon) < 1, \\ \beta \varepsilon \frac{\alpha \partial_{\varepsilon} q(\varepsilon)}{(1 + \alpha) q(\varepsilon)} (1 - t_0(\varepsilon)) & \text{if } q(\varepsilon) > 1. \end{cases} \quad (6.6.3)$$

Moreover, to check that $\varepsilon' = \varepsilon$ attains a local maximum, the second-order derivative has to be negative for $\varepsilon' = \varepsilon$ (second-order condition – SOC), which gives:

$$0 \geq -\frac{\partial_\varepsilon^2 t_0(\varepsilon)}{1 - t_0(\varepsilon)} - \left| \frac{\partial_\varepsilon t_0(\varepsilon)}{1 - t_0(\varepsilon)} \right|^2 + \beta\varepsilon \times \begin{cases} \frac{\alpha \partial_\varepsilon^2 q(\varepsilon)}{1 + \alpha q(\varepsilon)} - \frac{|\alpha|^2 |\partial_\varepsilon q(\varepsilon)|^2}{|1 + \alpha q(\varepsilon)|^2} & \text{if } q(\varepsilon) < 1, \\ \frac{\alpha \partial_\varepsilon^2 q(\varepsilon)}{(1 + \alpha)q(\varepsilon)} - \frac{\alpha |\partial_\varepsilon q(\varepsilon)|^2}{(1 + \alpha)|q(\varepsilon)|^2} & \text{if } q(\varepsilon) > 1. \end{cases} \quad (6.6.4)$$

The mechanism (q, t_0) must be revealing for every type of agent, which implies that the previous FOC and SOC have to be true at least for all $\varepsilon \in (0, 1)$ such that $q(\varepsilon) \neq 1$.

On the one hand, by differentiating (6.6.3), we prove that t_0 should satisfy:

$$\partial_\varepsilon^2 t_0(\varepsilon) = \begin{cases} \frac{\beta\alpha(1 - t_0(\varepsilon))}{|1 + \alpha q(\varepsilon)|^2} \left((\partial_\varepsilon q(\varepsilon) + \varepsilon \partial_\varepsilon^2 q(\varepsilon))(1 + \alpha q(\varepsilon)) - \alpha \varepsilon |\partial_\varepsilon q(\varepsilon)|^2 (1 + \beta\varepsilon) \right) & \text{if } q(\varepsilon) < 1, \\ \frac{\beta\alpha(1 - t_0(\varepsilon))}{(1 + \alpha)|q(\varepsilon)|^2} \left((\partial_\varepsilon q(\varepsilon) + \varepsilon \partial_\varepsilon^2 q(\varepsilon))q(\varepsilon) - \varepsilon |\partial_\varepsilon q(\varepsilon)|^2 \left(1 + \frac{\alpha}{1 + \alpha} \beta\varepsilon \right) \right) & \text{if } q(\varepsilon) > 1. \end{cases}$$

Replacing the first and second derivatives of t_0 with their values computed above, we obtain that the SOC (6.6.4) is equivalent in both cases to $\partial_\varepsilon q(\varepsilon) \geq 0$. Therefore, q is non-decreasing before attaining 1 and non-decreasing thereafter. By continuity of the function q , it can either cross the constant line equal to 1 only once or be equal to 1 over an interval. In both cases, it implies that the function q is non-decreasing on $(0, 1)$. We can thus denote by $[\varepsilon_1, \varepsilon_2]$ the interval on which q is constant equal to 1, with the convention that this interval is reduced to $\{\varepsilon_1\}$ if there exists only one point where q is equal to 1, $\varepsilon_1 = \varepsilon_2 = 1$ if q is always strictly less than 1 and $\varepsilon_1 = \varepsilon_2 = 0$ if q is always strictly greater than 1.

On the other hand, by solving (6.6.3) when $q(\varepsilon) < 1$, *i.e.*, $\varepsilon \in (0, \varepsilon_1)$, we obtain

$$t_0(\varepsilon) = 1 - c_q (1 + \alpha q(\varepsilon))^{-\beta\varepsilon} e^{\beta Q_0(\varepsilon)},$$

for some constant $c_q \in \mathbb{R}$, using the notation defined in (6.4.1). This proves the first form in (6.4.2). Moreover, solving the second part of (6.6.3), *i.e.*, for $\varepsilon \in (\varepsilon_2, 1)$, leads to:

$$t_0(\varepsilon) = 1 - \tilde{c}_q (q(\varepsilon))^{-\beta\varepsilon/(1+\alpha)} \exp \left(\frac{\beta\alpha}{1 + \alpha} \int_{\varepsilon_2}^{\varepsilon} \ln(q(\varepsilon)) d\varepsilon \right),$$

for some $\tilde{c}_q \in \mathbb{R}$. The two previous forms are valid on $[0, \varepsilon_1]$ and $[\varepsilon_2, 1]$ by the assumed continuity of t_0 . If $\varepsilon_1 = \varepsilon_2 \in (0, 1)$, the price is continuous at this point if and only if

$$\tilde{c}_q = c_q (1 + \alpha)^{-\beta\varepsilon_1} e^{\beta Q_0(\varepsilon_1)}, \quad (6.6.5)$$

and we thus obtain the second form in (6.4.2). With this setting, if $\varepsilon_1 = \varepsilon_2 = 0$, we obtain $\tilde{c}_q = c_q$, and the price is given by (6.4.2) for all $\varepsilon \in (0, 1)$, to within the constant c_q . Similar reasoning applies if $\varepsilon_1 = \varepsilon_2 = 1$. It remains to deal with the case where q is constant on the interval $[\varepsilon_1, \varepsilon_2]$, not reduced to a singleton. To address this case, let us consider an agent of type $\varepsilon \in (\varepsilon_1, \varepsilon_2)$. For the mechanism to be revealing for him, his expected utility $\text{EU}^Q(\varepsilon, q(\varepsilon'), t_0(\varepsilon'))$ must at least reach a local maximum in $\varepsilon' = \varepsilon$. His utility for any $\varepsilon' \in (\varepsilon_1, \varepsilon_2)$ is as follows, since q is constant equal to 1 on this interval:

$$\text{EU}^Q(\varepsilon, q(\varepsilon'), t_0(\varepsilon')) = \text{EU}^\varnothing(\varepsilon) + (1 + \alpha) \ln(1 - t_0(\varepsilon')) + \beta\varepsilon(1 + \alpha) \ln(1 + \alpha).$$

Therefore, ε is a maximum point on $(\varepsilon_1, \varepsilon_2)$ if for any ε' on this interval, $t_0(\varepsilon) \leq t_0(\varepsilon')$. Conversely, the mechanism is revealing for the agent of type ε' at least if $t_0(\varepsilon) \geq t_0(\varepsilon')$. This naturally implies that t_0 is also constant on $(\varepsilon_1, \varepsilon_2)$. In particular, by continuity of the price, we should have $t_0(\varepsilon_1) = t_0(\varepsilon_2)$, which also implies (6.6.5). Finally, we obtain for all $\varepsilon \in [\varepsilon_2, 1]$:

$$\begin{aligned} t_0(\varepsilon) &= 1 - c_q(1 + \alpha)^{-\beta\varepsilon_1}(q(\varepsilon))^{-\beta\varepsilon\alpha/(1+\alpha)}e^{\beta Q_0(\varepsilon_1)} \exp\left(\frac{\beta\alpha}{1+\alpha} \int_{\varepsilon_2}^{\varepsilon} \ln(q(\varepsilon))d\varepsilon\right) \\ &= 1 - c_q(1 + \alpha)^{-\beta\varepsilon_1}(q(\varepsilon))^{-\beta\varepsilon\alpha/(1+\alpha)}e^{\beta Q_0(\varepsilon)}, \end{aligned}$$

where the second equality is implied by the fact that q is constant equal to 1 on $[\varepsilon_1, \varepsilon_2]$. Therefore, the form (6.4.2) is proven to be true in any case. Finally, since the mechanism has to be admissible in the sense of Definition 6.4.1, we should have $t_0(\varepsilon) < 1$ for all $\varepsilon \in [0, 1]$, which implies $c_q > 0$. We therefore have shown that q being non-decreasing with respect to $\varepsilon \in [0, 1]$ and t_0 satisfying (6.4.2) are **necessary conditions** for the menu of contracts to satisfy the IC constraint on $[0, 1]$.

(ii) It remains to us to prove that these conditions are **sufficient**. To this end, we recall that the expected utility of an agent of type ε who chooses a contract $(q(\varepsilon'), t_0(\varepsilon'))$ is given by (6.6.1). In particular, its derivative with respect to ε' for $\varepsilon' \in (0, 1)$ such that $q(\varepsilon') \neq 1$ is given by (6.6.2). Since t_0 satisfies (6.6.3) in particular in ε' , we obtain

$$\partial_{\varepsilon'} \text{EU}^Q(\varepsilon, q(\varepsilon'), t_0(\varepsilon')) = \begin{cases} (1 + \alpha) \frac{\beta\alpha \partial_{\varepsilon'} q(\varepsilon')}{1 + \alpha q(\varepsilon')} (\varepsilon - \varepsilon') & \text{if } q(\varepsilon') < 1, \\ \beta \frac{\alpha \partial_{\varepsilon'} q(\varepsilon')}{q(\varepsilon')} (\varepsilon - \varepsilon') & \text{if } q(\varepsilon') > 1. \end{cases}$$

Moreover, if we consider without loss of generality that q is constant equal to 1 on some interval $[\varepsilon_1, \varepsilon_2]$ and take $\varepsilon \in (\varepsilon_1, \varepsilon_2)$, then, by Lemma 6.6.4, for any neighbourhood of ε contained in $[\varepsilon_1, \varepsilon_2]$, the price given by (6.4.2) is also constant. Therefore, the expected utility $\text{EU}^Q(\varepsilon, q(\varepsilon'), t_0(\varepsilon'))$ is, in fact, also differentiable on this neighbourhood, and its derivative is equal to zero. In summary, the following values are obtained for the derivative of the expected utility:

$$\partial_{\varepsilon'} \text{EU}^Q(\varepsilon, q(\varepsilon'), t_0(\varepsilon')) = \begin{cases} \beta(1 + \alpha) \frac{\alpha \partial_{\varepsilon'} q(\varepsilon')}{1 + \alpha q(\varepsilon')} (\varepsilon - \varepsilon') & \text{if } 0 < \varepsilon' < \varepsilon_1, \\ 0 & \text{if } \varepsilon' \in (\varepsilon_1, \varepsilon_2), \\ \beta\alpha \frac{\partial_{\varepsilon'} q(\varepsilon')}{q(\varepsilon')} (\varepsilon - \varepsilon') & \text{if } \varepsilon_2 < \varepsilon' < 1. \end{cases}$$

We first check that the contract is revealing for the interior types of agents, *i.e.*, where the previous derivative is defined. It suffices to remark that the expected utility of an agent of type ε is non-decreasing for $\varepsilon' \leq \varepsilon$ and non-increasing after, which proves, by continuity of the utility, that $\varepsilon' = \varepsilon$ is a maximiser. If the agent's type is $\varepsilon = 0$, his continuous utility is non-increasing with $\varepsilon' \in (0, 1)$, and a maximum is attained for $\varepsilon' = 0$. Similar reasoning can be applied if the agent's type is $\varepsilon = 1$, and therefore, the contract is revealing for the extreme types. For $\varepsilon = \varepsilon_1$ (resp. $\varepsilon = \varepsilon_2$), the utility is non-decreasing before ε , constant on $(\varepsilon_1, \varepsilon_2)$, and non-increasing after. Therefore, the (continuous) utility is constant and maximal on the interval $[\varepsilon_1, \varepsilon_2]$; in particular, the maximum is also attained at ε_1 (resp. ε_2). Therefore, the conditions stated in the proposition are **sufficient** for the mechanism to satisfy the IC constraint for all $\varepsilon \in [0, 1]$. $\square$

Theorem 6.4.2 thus provides a characterisation of a mechanism (q, t_0) satisfying the IC constraint for all types of agents. However, the real menu of contracts offered by the principal must be composed of quantities $e_{\min}$ and a price T associated with each quantity, independent of the type of agent, which is not observed by

the principal. Therefore, in the end, we have to obtain a price T that is only a function of $e_{\min}$, not also a function of ε . Nevertheless, [Lemma 6.6.4](#) states that when the function q is constant, the associated price t_0 is necessarily constant as well. Together with the fact that the function q is non-decreasing, this naturally implies that if two different types of agents choose the same quantity, they pay the same price. This result therefore prevents the contract resulting from a revealing mechanism from depending on the type of agent.

To precisely define the menu of contracts associated with an admissible revealing mechanism, let us fix an interval $I \subset \mathbb{R}_+$ and define $\tilde{I} := \{k \in \mathbb{R}_+ \text{ s.t. } \alpha k \underline{\omega} w_0 / p_e \in I\}$. For a function f that is non decreasing on $[0, 1]$, taking values in $\tilde{I}$, its generalised inverse for all $k \in \tilde{I}$ is defined by:

$$f^{-1}(k) = \inf\{\varepsilon \in [0, 1] \text{ such that } f(\varepsilon) = k\}. \quad (6.6.6)$$

The following corollary allows us to characterise a sufficiently smooth admissible menu of revealing contracts. The proof of this result is highly similar to that of [Theorem 6.4.2](#).

Corollary 6.6.5. *An admissible menu of contracts $(e_{\min}, T)$ for $e_{\min} \in I$ is associated with an admissible revealing mechanism if and only if there exists a non-decreasing continuous function q with values in $\tilde{I}$ and continuous second derivatives except where it is equal to 1, such that the price T for a quantity $e_{\min} = \alpha k \underline{\omega} w_0 / p_e$ is given for some $c_q \geq 0$ by:*

$$T(k) = w_0 - c_q w_0 e^{\beta Q_0(q^{-1}(k))} \times \begin{cases} (1 + \alpha k)^{-\beta q^{-1}(k)}, & \text{if } k < 1, \\ (1 + \alpha)^{-\beta q^{-1}(1)} k^{-\beta q^{-1}(k)\alpha/(1+\alpha)}, & \text{if } k \geq 1 \end{cases} \quad (6.6.7)$$

for $k \in \tilde{I}$ and where q^{-1} is the generalised inverse of q , as defined in (6.6.6).

Proof. (i) To prove that it is a necessary condition, let us fix an admissible menu of contracts $(e_{\min}, T)$ and an associated admissible revealing mechanism (q, t_0) . Since the mechanism (q, t_0) is admissible and satisfies the IC constraint, by [Theorem 6.4.2](#), we obtain that q is non-decreasing, and the price function t_0 is given by (6.4.2) with a constant $c_q > 0$. Moreover, by the previous discussion on admissible contracts, T should be independent of ε and thus constant when q is constant, which is true by [Lemma 6.6.4](#). Hence, we can write the price t_0 given by (6.4.2) in $\varepsilon = q^{-1}(k)$, where $k := p_e e_{\min} / (\alpha \underline{\omega} w_0)$ and q^{-1} is the generalised inverse of q . Moreover, noticing that $\varepsilon < \varepsilon_1$ is equivalent to $k < q^{-1}(\varepsilon_1) = 1$, and conversely, if $\varepsilon \geq \varepsilon_1$ then $k \geq 1$, we obtain that the price of a quantity $e_{\min} := \alpha k \underline{\omega} w_0$ is given by $T(k) = w_0 t_0(q^{-1}(k))$, which is (6.6.7).

(ii) To prove the equivalence, let us consider an admissible menu of contracts $(e_{\min}, T)$, where T is given by (6.6.7), and assume that the function q has the right properties. First, we can show that given this menu of contracts, the optimal quantity chosen by an agent of type ε is $e_{\min} = \alpha k \underline{\omega} w_0 / p_e$, where $k = q(\varepsilon)$. Indeed, by computing the derivative of his utility given by (6.2.7) with respect to the normalised quantity k , we obtain the following FOC for the optimal k :

$$0 = -\frac{\partial_k t_0(k)}{1 - t_0(k)} + \beta \varepsilon \times \begin{cases} \frac{\alpha}{1 + \alpha k} & \text{if } k < 1, \\ \frac{\alpha}{(1 + \alpha)k} & \text{if } k > 1. \end{cases}$$

Since the derivative of t_0 with respect to k satisfies:

$$\partial_k t_0(k) = \begin{cases} \frac{\beta \alpha q^{-1}(k)}{1 + \alpha k} (1 - t_0(k)) & \text{if } k < 1, \\ \frac{\beta \alpha q^{-1}(k)}{(1 + \alpha)k} (1 - t_0(k)) & \text{if } k > 1, \end{cases}$$

the previous FOC is equivalent to $k = q(\varepsilon)$. By continuity of q , the result is extendable to $k = 1$. It remains to us to check the following SOC:

$$0 \geq -\frac{\partial_k^2 t_0(k)(1 - t_0(k)) + |\partial_k t_0(k)|^2}{|1 - t_0(k)|^2} - \beta\varepsilon \times \begin{cases} \frac{|\alpha|^2}{|1 + \alpha k|^2} & \text{if } k < 1, \\ \frac{\alpha}{(1 + \alpha)|k|^2} & \text{if } k > 1. \end{cases}$$

The second-order derivative of T satisfies:

$$\partial_k^2 t_0(k) = \begin{cases} \frac{\beta\alpha(1 - t_0(k))}{|1 + \alpha k|^2} \left(\partial_k q^{-1}(k)(1 + \alpha k) - \alpha q^{-1}(k) - \beta\alpha|q^{-1}(k)|^2 \right) & \text{if } k < 1, \\ \frac{\beta\alpha(1 - t_0(k))}{|1 + \alpha|^2 k^2} \left((\partial_k q^{-1}(k)k - q^{-1}(k))(1 + \alpha) - \beta\alpha|q^{-1}(k)|^2 \right) & \text{if } k > 1. \end{cases}$$

Thus, the SOC is equivalent to:

$$\begin{cases} \partial_k q^{-1}(k)(1 + \alpha k) + \alpha(\varepsilon - q^{-1}(k)) \geq 0 & \text{if } k < 1, \\ \partial_k q^{-1}(k)k - q^{-1}(k) + \varepsilon \geq 0 & \text{if } k > 1. \end{cases}$$

In $k = q(\varepsilon) \neq 1$, the SOC becomes in both cases $\partial_k q^{-1}(k) \geq 0$, which is true since q is non-decreasing. By continuity of the utility, this result is also true for $k = 1$. Therefore, an agent of type ε chooses the quantity $e_{\min} = \alpha q(\varepsilon) \omega w_0 / p_e$, which is an available quantity because q takes values in $\tilde{I}$ and thus $e_{\min} \in I$. By computing the function $T(k)$ for $k = q(\varepsilon)$ and dividing it by w_0 , we recover the function t_0 defined by (6.4.2), which associates with any ε the price $w_0 t_0(\varepsilon)$ of the normalised quantity $k = q(\varepsilon)$. Moreover, since the mechanism (q, t_0) satisfies the assumptions to be admissible in the sense of Definition 6.4.1, by Theorem 6.4.2, the mechanism associated with the menu $(e_{\min}, T)$ is admissible and satisfies the IC constraint. $\square$

6.6.4 ... to select the agents

Given an admissible revealing mechanism (q, t_0) , we can write the informational rent of an agent of type ε as a function of ε :

$$\Delta \text{EU}^Q(\varepsilon) = (1 + \alpha) \ln(1 - t_0(\varepsilon)) + \beta\varepsilon \bar{U}(q(\varepsilon)).$$

Since (q, t_0) is a menu of revealing contracts, we can use the FOC (6.6.3) to compute its derivative:

$$\partial_\varepsilon \Delta \text{EU}^Q(\varepsilon) = \beta \bar{U}(q(\varepsilon)) = \begin{cases} \beta(1 + \alpha) \ln(1 + \alpha q(\varepsilon)) & \text{if } q(\varepsilon) < 1, \\ \beta\alpha \ln(q(\varepsilon)) + \beta(1 + \alpha) \ln(1 + \alpha) & \text{if } q(\varepsilon) > 1. \end{cases}$$

This derivative is non-negative in both cases and implies that the information rent is non-decreasing. Therefore, if there exists $\underline{\varepsilon} \in [0, 1]$ such that $\Delta \text{EU}^Q(\underline{\varepsilon}) \geq 0$, then for all $\varepsilon \in [\underline{\varepsilon}, 1]$, $\Delta \text{EU}^Q(\varepsilon) \geq 0$, which means that the participation constraint of agents of type $\varepsilon \in [\underline{\varepsilon}, 1]$ is satisfied. A more precise result is established in Proposition 6.4.3, and its proof is reported below.

Proof of Proposition 6.4.3. We consider an admissible and incentive compatible mechanism (q, t_0) . Applying Theorem 6.4.2, the price t_0 satisfies (6.4.2). In the one hand, if $\varepsilon \in [0, 1]$ is such that $q(\varepsilon) < 1$, i.e., $\varepsilon < \varepsilon_1$, the participation constraint of the agent of type ε becomes:

$$c_q \geq \exp \left(-\beta \int_0^\varepsilon \ln(1 + \alpha q(\varepsilon)) d\varepsilon \right) = \underline{c}(\varepsilon).$$

On the other hand, if $\varepsilon \geq \varepsilon_1$, the participation constraint is equivalent to:

$$c_q \geq (1 + \alpha)^{-\beta(\varepsilon - \varepsilon_1)} e^{\beta Q_0(\varepsilon)} = \underline{c}(\varepsilon).$$

The participation constraint for an agent of type $\varepsilon \in [0, 1]$ is thus equivalent in both cases to $c_q \geq \underline{c}(\varepsilon)$. We can then compute the derivative of $\underline{c}$ with respect to ε :

$$\underline{c}'(\varepsilon) = -\beta \underline{c}(\varepsilon) \times \begin{cases} \ln(1 + \alpha q(\varepsilon)) & \text{if } \varepsilon < \varepsilon_1, \\ \ln(1 + \alpha) + \frac{\alpha}{1 + \alpha} \ln(q(\varepsilon)) & \text{if } \varepsilon > \varepsilon_1. \end{cases}$$

Since $q(\varepsilon) \geq 0$ for all $\varepsilon \in [0, \varepsilon_1)$ and $q(\varepsilon) \geq 1$ for all $\varepsilon \in (\varepsilon_1, 1]$, we obtain that the derivative of $\underline{c}$ is negative in both cases. Since the function $\underline{c}$ is continuous on $[0, 1]$ (in particular in ε_1), the function is non-increasing on $[0, 1]$. Moreover, by definition of $\underline{\varepsilon}$ and continuity of $\underline{c}$, $\underline{c}(\underline{\varepsilon}) = c_q$. Thus, for any $\varepsilon \in [\underline{\varepsilon}, 1]$, we have $\underline{c}(\varepsilon) \leq \underline{c}(\underline{\varepsilon}) = c_q$, and thus the participation constraint of the agent of type ε is satisfied. Conversely, for any $\varepsilon \in [0, \underline{\varepsilon})$, we have $\underline{c}(\varepsilon) > \underline{c}(\underline{\varepsilon}) = c_q$, which means that the participation constraint is not satisfied. $\square$

Proposition 6.4.3 thus states that only the agents of a sufficiently risky type are selected by the principal. This result is entirely implied by the fact that the reservation utility of an agent depends on his type and only occurs in principal-agent problems with countervailing incentives. Indeed, the following remark shows that if a constant reservation utility had been chosen, the selected agents would have been those of a sufficiently low risk type.

Remark 6.6.6. *If the agents' reservation utility is assumed to be a constant R_0 , the participation constraint for an agent of type ε becomes $\text{EU}^Q(\varepsilon) \geq R_0$, where $\text{EU}^Q(\varepsilon)$ is defined by (6.2.7) for a revealing contract $(q(\varepsilon), t_0(\varepsilon))$. By computing the derivative of $\text{EU}^Q(\varepsilon)$ with respect to ε for a menu of revealing contracts, using FOC (6.6.3), we obtain:*

$$\partial_\varepsilon \text{EU}^Q(\varepsilon) = (1 + \alpha)\beta \ln(\underline{\omega}/\bar{\omega}) + \beta \bar{U}(q(\varepsilon)).$$

Under the assumption⁵ that $(1 + \alpha)\underline{\omega} \leq \bar{\omega}$, the information rent $\text{EU}^Q(\varepsilon) - R_0$ is decreasing for all $\varepsilon \in [0, 1]$ such that $q(\varepsilon) \in (1, (\bar{\omega}/(\underline{\omega}(1 + \alpha)))^{(1 + \alpha)/\alpha}]$. Thus, in this case, if there exists $\bar{\varepsilon} \in [0, 1]$ such that $\text{EU}^Q(\bar{\varepsilon}) \geq R_0$, then the participation constraint of agents of type $\varepsilon \in [0, \bar{\varepsilon}]$ is satisfied.

Nevertheless, in our opinion, it makes little sense to consider in our framework that the reservation utility is constant for any agents, regardless of their type.

6.6.5 ... to solve the principal's problem

Corollary 6.6.7. *Let $\underline{\varepsilon} \in [0, 1]$ and $\underline{q} \in \mathbb{R}_+$. If there is a solution $Q \in \mathcal{Q}(\underline{\varepsilon})$ to (ODE), the principal's profit given by (6.4.4) is equal to:*

$$\Pi(\underline{\varepsilon}) = w_0(1 - \underline{\varepsilon}) + \frac{1}{2} \underline{\omega} w_0 (|\varepsilon_1 \vee \underline{\varepsilon}|^2 - |\underline{\varepsilon}|^2) - w_0 F_1(Q) - w_0 F_2(Q),$$

where F_1 and F_2 are, respectively, defined as follows:

$$F_1(Q) := \int_{\underline{\varepsilon}}^{\varepsilon_1 \vee \underline{\varepsilon}} \left(e^{\beta(Q(\varepsilon) - \varepsilon Q'(\varepsilon))} + \varepsilon \underline{\omega} e^{Q'(\varepsilon)} \right) d\varepsilon, \quad (6.6.8a)$$

$$F_2(Q) := \int_{\varepsilon_1 \vee \underline{\varepsilon}}^1 \left((1 + \alpha)^{-\beta(\varepsilon_1 \vee \underline{\varepsilon})} e^{\beta(Q(\varepsilon) - \varepsilon Q'(\varepsilon))} + \varepsilon \alpha e^{\frac{1 + \alpha}{\alpha} Q'(\varepsilon)} \underline{\omega} \right) d\varepsilon. \quad (6.6.8b)$$

5. This is the case in the application considered throughout this chapter, since $\alpha = 0.08$, $\underline{\omega} = 0.4$ and $\bar{\omega} = 1$.

Proof of Theorem 6.4.4 and Corollary 6.6.7. Let us fix a mechanism $(q, t_0) \in \mathcal{C}^Q(\underline{\varepsilon})$. This mechanism satisfies the assumption of Theorem 6.4.2, and the price t_0 is therefore given by (6.4.2). Moreover, since this mechanism is assumed to be in $\mathcal{C}^Q(\underline{\varepsilon})$, the participation constraint has to be satisfied only for all $\varepsilon \in [\underline{\varepsilon}, 1]$, which implies by Proposition 6.4.3 that the constant c_q in the price is given by $c_q = \underline{c}(\underline{\varepsilon})$. We thus obtain that, if $\underline{\varepsilon} \in [0, \varepsilon_1)$, the price for all $\varepsilon \in [\underline{\varepsilon}, 1]$ is given by:

$$t_0(\varepsilon) = 1 - e^{\beta(Q_0(\varepsilon) - Q_0(\underline{\varepsilon}))} \times \begin{cases} (1 + \alpha q(\varepsilon))^{-\beta \varepsilon}, & \text{if } \varepsilon \in [\underline{\varepsilon}, \varepsilon_1), \\ (1 + \alpha)^{-\beta \varepsilon_1} (q(\varepsilon))^{-\beta \varepsilon \alpha / (1 + \alpha)}, & \text{if } \varepsilon \in [\varepsilon_1, 1]. \end{cases}$$

Similarly, if $\underline{\varepsilon} \in [\varepsilon_1, 1]$, the price for all $\varepsilon \in [\underline{\varepsilon}, 1]$ is given by:

$$t_0(\varepsilon) = 1 - e^{\beta(Q_0(\varepsilon) - Q_0(\underline{\varepsilon}))} (1 + \alpha)^{-\beta \underline{\varepsilon}} (q(\varepsilon))^{-\beta \varepsilon \alpha / (1 + \alpha)}.$$

To reconcile the two cases, we denote by Q the following function for all $\varepsilon \in [\underline{\varepsilon}, 1]$:

$$Q(\varepsilon) := \begin{cases} \int_{\underline{\varepsilon}}^{\varepsilon} \ln(1 + \alpha q(\varepsilon)) d\varepsilon & \text{if } \varepsilon \in [\underline{\varepsilon}, \varepsilon_1 \vee \underline{\varepsilon}), \\ \int_{\underline{\varepsilon}}^{\varepsilon_1 \vee \underline{\varepsilon}} \ln(1 + \alpha q(\varepsilon)) d\varepsilon + \frac{\alpha}{1 + \alpha} \int_{\varepsilon_1 \vee \underline{\varepsilon}}^{\varepsilon} \ln(q(\varepsilon)) d\varepsilon & \text{if } \varepsilon \in [\varepsilon_1 \vee \underline{\varepsilon}, 1]. \end{cases} \quad (6.6.9)$$

Since (q, t_0) is an admissible revealing mechanism, q is continuous on $[0, 1]$ and $\mathcal{C}^2$ on $(0, 1)$ except where it is equal to 1, and by Theorem 6.4.2, q is a non-decreasing function. This naturally implies that the function Q satisfies the right properties to be in $\mathcal{Q}(\underline{\varepsilon})$.

Thanks to the definition of the function Q , we can write q as a function of Q' for all $\varepsilon \in [\underline{\varepsilon}, 1]$:

$$q(\varepsilon) = \begin{cases} \frac{1}{\alpha} (e^{Q'(\varepsilon)} - 1) & \text{if } \varepsilon \in [\underline{\varepsilon}, \varepsilon_1 \vee \underline{\varepsilon}), \\ e^{\frac{1+\alpha}{\alpha} Q'(\varepsilon)} & \text{if } \varepsilon \in [\varepsilon_1 \vee \underline{\varepsilon}, 1]. \end{cases}$$

Therefore, the price t_0 can be written as follows for all $\varepsilon \in [\underline{\varepsilon}, 1]$:

$$t_0(\varepsilon) = \begin{cases} 1 - e^{\beta(Q(\varepsilon) - \varepsilon Q'(\varepsilon))} & \text{if } \varepsilon \in [\underline{\varepsilon}, \varepsilon_1 \vee \underline{\varepsilon}), \\ 1 - (1 + \alpha)^{-\beta(\varepsilon_1 \vee \underline{\varepsilon})} e^{\beta(Q(\varepsilon) - \varepsilon Q'(\varepsilon))} & \text{if } \varepsilon \in [\varepsilon_1 \vee \underline{\varepsilon}, 1]. \end{cases}$$

Moreover, optimising on admissible revealing mechanisms $(q, t_0) \in \mathcal{C}^Q(\underline{\varepsilon})$ is thus equivalent to optimising on $Q \in \mathcal{Q}(\underline{\varepsilon})$, and the principal's problem for $\underline{\varepsilon} \in [0, 1]$ fixed is thus given by:

$$\begin{aligned} \Pi(\underline{\varepsilon}) = & w_0(1 - \underline{\varepsilon}) + \frac{1}{2} \underline{\omega} w_0(|\varepsilon_1 \vee \underline{\varepsilon}|^2 - |\underline{\varepsilon}|^2) - w_0 \inf_{Q \in \mathcal{Q}(\underline{\varepsilon})} \left\{ \int_{\underline{\varepsilon}}^{\varepsilon_1 \vee \underline{\varepsilon}} \left(e^{\beta(Q(\varepsilon) - \varepsilon Q'(\varepsilon))} + \varepsilon \underline{\omega} e^{Q'(\varepsilon)} \right) d\varepsilon \right. \\ & \left. + w_0 \int_{\varepsilon_1 \vee \underline{\varepsilon}}^1 \left((1 + \alpha)^{-\beta(\varepsilon_1 \vee \underline{\varepsilon})} e^{\beta(Q(\varepsilon) - \varepsilon Q'(\varepsilon))} + \varepsilon \alpha e^{\frac{1+\alpha}{\alpha} Q'(\varepsilon)} \underline{\omega} \right) d\varepsilon \right\}. \end{aligned} \quad (6.6.10)$$

This problem is relatively standard in the field of calculus of variation. Given the form of the previous optimisation, we study the optimal function Q separately on $(\underline{\varepsilon}, \varepsilon_1 \vee \underline{\varepsilon})$ and on $(\varepsilon_1 \vee \underline{\varepsilon}, 1)$.

With this in mind, we first study the problem on $(\underline{\varepsilon}, \varepsilon_1 \vee \underline{\varepsilon})$. Let R be an arbitrary function that has at least one derivative and vanishes at the endpoints $\underline{\varepsilon}$ and $\varepsilon_1 \vee \underline{\varepsilon}$. For any $\eta \in \mathbb{R}$, we denote $g_1(\eta) := F_1(Q + \eta R)$, where F_1 is defined by (6.6.8a). We can compute the derivative of g_1 with respect to η :

$$g'_1(\eta) = \int_{\underline{\varepsilon}}^{\varepsilon_1 \vee \underline{\varepsilon}} \left(\beta(R(\varepsilon) - \varepsilon R'(\varepsilon)) e^{\beta(Q(\varepsilon) + \eta R(\varepsilon) - \varepsilon Q'(\varepsilon) - \varepsilon \eta R'(\varepsilon))} + \varepsilon \underline{\omega} R'(\varepsilon) e^{Q'(\varepsilon) + \eta R'(\varepsilon)} \right) d\varepsilon.$$

The Gâteaux differential of F_1 with respect to Q in the direction R denoted by $DF_1(Q)(R)$ is given by $g'_1(0)$:

$$DF_1(Q)(R) = \int_{\underline{\varepsilon}}^{\varepsilon_1 \vee \underline{\varepsilon}} \left(\beta(R(\varepsilon) - \varepsilon R'(\varepsilon)) e^{\beta(Q(\varepsilon) - \varepsilon Q'(\varepsilon))} + \varepsilon \underline{\omega} R'(\varepsilon) e^{Q'(\varepsilon)} \right) d\varepsilon.$$

By computing an integration by parts and since $R(\underline{\varepsilon}) = R(\varepsilon_1 \vee \underline{\varepsilon}) = 0$ by assumption, we obtain:

$$DF_1(Q)(R) = - \int_{\underline{\varepsilon}}^{\varepsilon_1 \vee \underline{\varepsilon}} R(\varepsilon) \left(\beta(|\beta| \varepsilon^2 Q''(\varepsilon) - 2) e^{\beta(Q(\varepsilon) - \varepsilon Q'(\varepsilon))} + \underline{\omega} (1 + \varepsilon Q''(\varepsilon)) e^{Q'(\varepsilon)} \right) d\varepsilon.$$

Therefore, the Euler–Lagrange equation associated with the optimisation problem on $[\underline{\varepsilon}, \varepsilon_1 \vee \underline{\varepsilon}]$ is equivalent to the following non-linear second-order ODE:

$$0 = \underline{\omega} (1 + \varepsilon Q''(\varepsilon)) e^{Q'(\varepsilon)} + \beta(|\beta| \varepsilon^2 Q''(\varepsilon) - 2) e^{\beta(Q(\varepsilon) - \varepsilon Q'(\varepsilon))}. \quad (6.6.11)$$

Moreover, we can compute the second derivative of g_1 with respect to η :

$$g''_1(\eta) = \int_{\underline{\varepsilon}}^{\varepsilon_1 \vee \underline{\varepsilon}} \left(|\beta|^2 |R(\varepsilon) - \varepsilon R'(\varepsilon)|^2 e^{\beta(Q(\varepsilon) + \eta R(\varepsilon) - \varepsilon Q'(\varepsilon) - \varepsilon \eta R'(\varepsilon))} + \varepsilon \underline{\omega} |R'(\varepsilon)|^2 e^{Q'(\varepsilon) + \eta R'(\varepsilon)} \right) d\varepsilon.$$

This second derivative is therefore positive for any $\eta \in \mathbb{R}$ and implies that F_1 attains a minimum for Q the solution on $[\underline{\varepsilon}, \varepsilon_1 \vee \underline{\varepsilon}]$ of ODE (6.6.11), if it exists.

Similarly, to study the problem on $(\varepsilon_1 \vee \underline{\varepsilon}, 1)$, we consider F_2 defined by (6.6.8b). Applying the same reasoning than for F_1 , we obtain that the Euler–Lagrange equation associated with the optimisation problem on $[\underline{\varepsilon}, \varepsilon_1 \vee \underline{\varepsilon}]$ is equivalent to the following non-linear first-order ODE:

$$0 = \underline{\omega} (1 + \alpha)^{\beta(\varepsilon_1 \vee \underline{\varepsilon}) + 1} \left(1 + \varepsilon \frac{1 + \alpha}{\alpha} Q''(\varepsilon) \right) e^{\frac{1 + \alpha}{\alpha} Q'(\varepsilon)} + \beta(|\beta| \varepsilon^2 Q''(\varepsilon) - 2) e^{\beta(Q(\varepsilon) - \varepsilon Q'(\varepsilon))}. \quad (6.6.12)$$

Moreover, the second derivative of g_2 with respect to η is positive for any $\eta \in \mathbb{R}$, which implies that F_2 attains a minimum for Q , the solution on $[\varepsilon_1 \vee \underline{\varepsilon}, 1]$ of ODE (6.6.12), if it exists.

We can thus conclude that if there is a function $Q \in \mathcal{Q}(\underline{\varepsilon})$ solution to ODE (6.6.11) on $[\underline{\varepsilon}, \varepsilon_1 \vee \underline{\varepsilon}]$ and to ODE (6.6.12) on $[\varepsilon_1 \vee \underline{\varepsilon}, 1]$, it maximises the principal's profit for some $\underline{\varepsilon} \in [0, 1]$ and q fixed. Combining both ODEs leads to (ODE), which proves the theorem. Moreover, using (6.6.10), we obtain the form of the principal's profit given in Corollary 6.6.7. $\square$

Remark 6.6.8. As explained in Remark 6.4.5, Theorem 6.4.4 only gives a sufficient condition for the principal's optimisation problem. To obtain a necessary condition, one should adapt the previous proof by writing the Euler–Lagrange equation for the problem with constraints. A new ODE would then be obtained, and the existence of a solution to this ODE would be equivalent to the existence of an optimal contract. Nevertheless, in the application developed in Section 6.4.4, solving the (ODE) is sufficient since its solution has the required regularity. Moreover, one can prove that the ODE has a unique solution for $\underline{\varepsilon}$ bounded away from 0, which confirms that the numerical scheme converges to the solution of the principal's problem in our application. More precisely, on $[\underline{\varepsilon}, \varepsilon_1 \vee \underline{\varepsilon}]$, (ODE) can be written as a system of two first-order ODEs as follows:

$$\begin{cases} Q'(\varepsilon) = \frac{\beta Q(\varepsilon) - R(\varepsilon)}{1 + \beta \varepsilon}, \\ R'(\varepsilon) = \frac{1 + \beta \varepsilon}{\varepsilon} \frac{1 - 2\beta e^{R(\varepsilon)}/\underline{\omega}}{1 + |\beta|^2 \varepsilon e^{R(\varepsilon)}/\underline{\omega}}. \end{cases}$$

By the Cauchy–Lipschitz theorem, the second ODE has a unique solution if $\underline{\varepsilon} \geq c > 0$. This solution is, in particular, bounded with bounded derivatives on the interval considered, which implies that the first ODE also has a unique solution. The same reasoning can be applied on the interval $[\varepsilon_1 \vee \underline{\varepsilon}, 1]$.

PART III

On the control of epidemics

Abstract

The current worldwide situation leads us to consider in this part another field of application of research between mean-field games and contract theory: epidemiology. Indeed, the year 2020 is marked, for a large proportion of the world's population, by containment measures imposed by governments in order to limit the spread of the SARS-CoV-2 virus, referred to as the COVID-19 pandemic. These restrictions mainly concern the reduction of social interactions. They are implemented at different geographical scales and already have a significant global economic impact. However, until a sufficient proportion of the population is immunised (by infection or if a vaccine is discovered), and pending effective treatment against this disease, the choice of control measures and adequate monitoring of the spread of the virus are essential to limit the epidemic.

In [Chapter 7](#), we consider the individual perspective on the control of an epidemic. The epidemic is represented by an SIR dynamic, which can be extended in a straightforward way to an SEIR, to take into account the latency phase of the COVID-19 disease. We model the interactions between individuals, responsible for the spread of the virus, and their decision-making processes, in a mean-field fashion. We thus study a Nash equilibrium among the individual, leading to a weaker epidemic due to the efforts of individuals to limit their contact with each other. However, a Nash equilibrium is obviously not the best that can be achieved when compared to a situation where individuals are altruistic and see only the good of society as a whole. More precisely, the study of the cost of anarchy raises the question of incentives: what levers can health authorities use to bring the Nash equilibrium closer to the optimum for the society as a whole?

With the previous remark in mind, we develop in [Chapter 8](#) a principal-agent problem with moral hazard to model the control of the epidemic in a country. More precisely, the government (the principal) wants to incentivise the population to reduce its contact rate, in order to limit the spread of the disease. However, the government cannot force individuals to follow a specific rate of contact with each other, which naturally leads to a moral hazard situation. The goal is therefore to calibrate an optimal incentive policy as well as a testing policy, chosen by the government, to better control the spread of the disease. In this model, we consider a stochastic SIR model, to account for uncertainty in the spreading of the COVID-19 epidemic.

The individual's point of view: a mean-field equilibrium

Tu me manques, je sais même plus combien
J'ai jamais su compter si loin

Tous les jours, MANO SOLO

Introduction

In this chapter, we consider, within both a SIR and SEIR models, the question of the COVID-19 control measures as seen from the point of view of the individuals. We assume that each individual can choose to decrease his/her social interactions in order to slow down the spread of the virus. This means that the transmission rate, usually constant and exogenous in standard SIR models, is endogenous and time-dependent. Individuals can choose to lower their contact rate, which will allow them to minimise the likeliness of being infected, but this comes at a cost. The impact on the overall epidemic unfold from a single individual is negligible but the aggregating behaviour of all individuals determine the epidemic evolution. This is formalised through a Mean-Field approach.

We prove the existence of a Nash/Mean-Field equilibrium (Nash-MF for short), *i.e.*, a contact rate such that no individual has an interest in choosing a contact rate different from the overall societal contact rate, thanks to the Schauder fixed point theorem. We then perform numerical simulations in order to find this equilibrium, which seems to be unique for the tested cases. The transmission rate of the disease induced by the equilibrium allows a clear improvement in the evolution of the epidemic, compared to the evolution with the initial transmission rate β° , which corresponds to the transmission rate of the disease without any effort of the population. In particular, despite the selfishness of individuals (who only seek to minimise their own cost), their efforts make possible to reduce the number of people affected by the disease by 25%. In addition, the infection peak is less critical, which limits, and may even prevent, the saturation of the health care system and a corresponding decrease in the mortality rate of the virus.

However, this equilibrium is not the best that can be achieved if compared with a situation where individuals are 100% altruistic and only see the good of the society as a whole. To quantify this difference, we compute the societal-wide optimum to compare the two strategies. We observe that the divergence between the two strategies arrives both before and after the peak of the epidemic. More precisely, the Nash-MF equilibrium allows, through premature efforts, to decrease the peak of infected, but the centralised epidemic control implies a more rapid decrease in the proportion infected after the peak, by maintaining intense efforts. As a consequence, there is a *cost of anarchy*, meaning that the Nash-MF equilibrium induces a higher cost than the societal optimum. This cost will motivate the study of incentives in the next chapter, in order to determine the possible levers that health authorities can use to bring the interests of the individual and society closer together.

7.1 Individual based modelling of the epidemic dynamics

7.1.1 The SIR model

The dynamics of the epidemic is modelled by the standard SIR (Susceptible – Infectious – Recovered) compartment model represented in Figure 7.1. We refer to the works of Anderson, Anderson, and May [25] (1992), Capasso [75] (1993) and Capasso and Serio [76] (1978) for additional details on this model, as well as the description of many other mathematical epidemic propagation models. In addition, for models dedicated to coronavirus, we can refer to the works of Ng, Turinici, and Danchin [266] (2003) and Turinici and Danchin [335] (2006); as well as to the works of Danchin, Ng, and Turinici [113] (2020) and Volpert et al. [337] (2020) for models specific to the COVID-19 epidemic. As a first step, we choose to restrict our analysis to a simple SIR model rather than considering more complex (and realistic) models, in order to focus our analysis on the role of individual decisions aggregation on the epidemic dynamics. Nevertheless, our study can be extended to more complicated epidemic models. In particular, an extension to the SEIR compartment model is presented in Section 7.3.4.

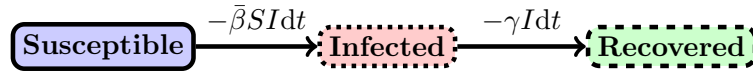


Figure 7.1 – Graphical illustration of the SIR model.

During the epidemic propagation, each individual can be either *Susceptible*, *Infectious* or *Recovered*, and (S_t, I_t, R_t) denotes the proportion of each category at time $t \geq 0$. More precisely, at each time $t \geq 0$, while S_t represents the proportion of the population that have not yet been infected by the virus at time t , I_t is the proportion of individuals currently infected. Finally, the *Recovered* class represents the collection of individuals whose behaviour does not impact the transmission of the virus anymore. In this chapter, we assume that this compartment includes both cured individuals and those who did not survive the epidemic, as well as individuals tested positive and isolated in perfect quarantine. Besides, we hereby implicitly assume that the immunity acquired by the recovered lasts for ever. The (continuous-time) evolution of the disease is described by the following equations:

$$\begin{cases} dS_t = -\bar{\beta}_t S_t I_t dt \\ dI_t = \bar{\beta}_t S_t I_t dt - \gamma I_t dt \\ dR_t = \gamma I_t dt, \end{cases} \quad (7.1.1)$$

with a given initial compartmental distribution of individuals at time 0 denoted (S_0, I_0, R_0) , which is supposed to be known. We assume that initially $S_0 + I_0 + R_0 = 1$, and observe that the system (7.1.1) possesses a conservation law, *i.e.*, for all $t \geq 0$, $S_t + I_t + R_t = 1$.¹ It is also worth noticing that, as the dynamics of the epidemic is not impacted by the size of the class R , we can restrict our focus on the evolution of the proportion (S, I) of both susceptible and infected.

7.1.2 The lockdown strategy

The model described by (7.1.1) involves two parameters, γ and $\bar{\beta}$. The constant parameter $\gamma > 0$ is exogenous and identifies the recovery rate, *i.e.*, it represents the inverse of the length (in days) of the

1. The SIR model can be extended to include births and deaths, see Chapter 8 for an example. Nevertheless, given that the duration of the COVID-19 pandemic is assumed to be relatively short with respect to the life expectancy at birth in most of the concerned countries, the demographic dynamics is not relevant and will not be taken into account in this chapter.

contagious period (or time before strict isolation). In our framework, the key parameter is the transmission rate of the disease, denoted $(\bar{\beta}_t)_{t \geq 0}$, which is considered to be endogenous and time-dependent, contrary to more classical SIR models. To focus our study on this transmission rate, we assume that the parameter γ is fixed and known. The transmission rate $\bar{\beta}$ depends essentially on two factors: the disease characteristics and the contact rate within the population. We will denote by β° the constant initial transmission rate of the disease, *i.e.*, without any control measures or effort from the population. Although the society cannot modify the disease characteristics, it can produce (possibly strong) incentives to each individual to reduce his/her contact rate with other individuals in the population.

This mitigation strategy (lockdown) has been chosen by many countries during the COVID-19 epidemic propagation in 2020, some countries starting strict measures in January (China) while other countries in February, March and so on; this is implemented in order to slow down the epidemic propagation. However, for each individual, reducing the contact rate with others comes at a cost, such as health hazards, psychological pressures, loss of social relationships, income uncertainty; and it may even expose the individual to yet unknown risks. The latter is especially true for long epidemics that require substantial efforts from the individuals. On the other hand, if everybody else lowers his/her own contact rate then the epidemic diffusion will be vanishing and a given individual, having this knowledge, can act as a free-rider and effectively increase his/her contact rate without too much risk.

Nevertheless, in most countries, governments only provide recommendations, and therefore individuals are generally free to choose whether or not to follow these recommendations. Moreover, even if governments issue injunctions, the actions of all individuals cannot generally be monitored. They may therefore not comply with the injunctions, depending on the risk of being sanctioned. Therefore, we assume that each individual is free to choose his/her own contact rate β , roughly by comparing the cost of lockdown with the cost related to the infection, and thus chooses the risk of infection he/she wants to take. Finally, the global epidemics propagation rate $\bar{\beta}$ of the society follows from the aggregation of all individuals contact rates, so that an equilibrium is reached between all contact rates β chosen by each individual and the overall epidemic transmission rate $\bar{\beta}$ in the society.

As we are interested in the unfold of the COVID-19 pandemic (as opposed to an endemic disease such as measles, mumps, rubella, pertussis...) we will consider a finite time horizon $T > 0$, large enough so that the epidemic is deemed to be over at time T . Considering a finite time horizon also allows some technical simplifications of the mathematical analysis, mainly for the proof of the existence of a Nash-MF equilibrium, without sacrificing qualitative conclusions. Nevertheless, the results can be extended to an infinite horizon, with relevant technical adaptations. Moreover, since the numerical simulations can only tackle finite times, this choice has the advantage to use an unified framework for both technical and numerical contributions.

7.1.3 Individual viewpoint: contact rate optimisation

We focus in this section on the individual perspective for the epidemic control. Consistent with the literature on Mean-Field Games, we suppose that the population can be partitioned into several collections of identical individuals (here the *Susceptible*, *Infectious* and *Recovered* classes), so that each individual in a given collection takes the same decisions as other individuals in the same class. This implies that we can consider a *representative* individual, who corresponds to an arbitrary individual in a given class. In the following, as in traditional MFG frameworks, we assumed that this representative individual is too small to impact the dynamics of the epidemic, when choosing his/her own contact rate β .

An important variable in the decision making process of the representative individual is his/her probability of being infected during the epidemic. This probability depends on both his/her own contact rate β with the population, together with the proportion I_t of infectious individuals in the population. More

precisely, for all $t \in [0, +\infty)$, we denote by $\varphi_t^{\bar{\beta}}(\beta)$ the probability that infection occurs before time t for an individual choosing his/her own contact rate β , while the epidemic evolves according the population's transmission rate $\bar{\beta}$. In other words, if τ denotes the random infection time of the representative individual, then for any time $t \geq 0$, we have $\varphi_t^{\bar{\beta}}(\beta) = \mathbb{P}(\tau \leq t)$, where the probability $\mathbb{P}$ obviously depends on the choices of β and $\bar{\beta}$. In particular, we can show that for any time $t \in [0, T]$, $\varphi_t^{\bar{\beta}}(\beta)$ has the following dynamics:

$$d\varphi_t^{\bar{\beta}}(\beta) = \beta_t I_t (1 - \varphi_t^{\bar{\beta}}(\beta)) dt, \quad (7.1.2)$$

where I_t is the proportion of (contagious) infected at time t , whose dynamics is driven by the population contact rate $\bar{\beta}$, as described in (7.1.1). The previous dynamic is consistent with the statements of Laguzet and Turinici [220] (2015) and Laguzet, Turinici, and Yahiaoui [221] (2016), but for the sake of completeness, a detailed proof is given below in Section 7.4.1.

We assume that, while in the Susceptible class, *i.e.*, before the (possible) infection time τ , the representative individual can choose his/her contact rate $\beta_t \in [\beta_{\min}, \beta^\circ]$ (for all $t \leq \tau$). Recall that β° represents the transmission rate of the disease without any control measures, while $\beta_{\min} > 0$ is the lowest possible contact rate value that can be attained by the representative individual. The efforts of an individual for decreasing his/her social interactions, in order to lower the transmission rate of the virus from β° to some $\beta \in [\beta_{\min}, \beta^\circ]$, induce an instantaneous cost, represented by a decreasing function $c : [\beta_{\min}, \beta^\circ] \rightarrow \mathbb{R}^+$. In addition to this cost related to the lockdown, we assume that if the individual is infected during the considered period $[0, T]$, *i.e.* if $\tau \leq T$, then a cost incurred at the infection time τ . This cost is defined by $r_1 \mathbb{1}_{\tau \leq T}$, where r_1 denotes the unitary cost of infection as quantified by the representative individual, together with any other costs that the presence in the *Infectious* class implies. Note that r_1 is taken here as a constant which does not depend on the pandemic evolution neither on the individual control measures that are specific to the *Infectious* class. Nevertheless, we could consider that the unitary cost r_1 depends, for example, on the proportion of infected people, to model, among other things, the individual's fear of hospital congestion.

Therefore, and similarly to the framework defined by Laguzet and Turinici [220] (2015), we assume that the global cost (seen from time $t = 0$) of an individual, while infected at time τ and choosing a dynamic contact rate β , is defined by:

$$C(\beta, \tau) := \int_0^{\tau \wedge T} c(\beta_s) ds + r_1 \mathbb{1}_{\tau \leq T}, \quad (7.1.3)$$

where $\tau \wedge T$ denotes for the minimum between τ and T . We do not specify here the nature of these previous costs, both have health components but may also include other types of costs. We refer the reader to the literature on the QALY/DALY scales for further details, see for example the works of Zeckhauser and Shepard [351] (1976), Anand and Hanson [22] (1997) and Sassi [304] (2006).

Note that given the form of the cost related to the reduction of social interactions until the time of infection, represented by the integral term in (7.1.3), if the individual is not infected during the period $[0, T]$, then his/her efforts are costly up to the horizon time T . On the contrary, if the individual is infected before T , his/her effort are only costly before τ . Indeed, we assume here for simplicity that only susceptible individuals choose their contact rate. This means in particular that, once an individual is infected, he/she is supposed to follow the societal transmission rate, which thus results from the aggregation of Susceptible choices. This questionable assumption is made to simplify the reasoning, but can be justified as follows: an infected individual has no more incentives to make an effort, since his/her effort has no influence on the societal contact rate, by the Mean-Field hypothesis. Indeed, the individual is assumed to be too small to influence the transmission rate of the disease. Therefore, since the effort is costly, the infected individual will

choose the maximal contact rate β° . Nevertheless, an infected individual can be hospitalised, and is thus forced to stop social interaction, or may not be aware that he/she is infected, and thus still limit his/her interaction. This leads to many cases to consider, and will make the analysis unnecessarily burdensome, since in reality, an infected individual will still take care not to transmit the disease anyway. Therefore, in our opinion, it seems more convenient to assume that only susceptible individuals choose their contact rate in order to minimise their cost, and infected individuals inexpensively follow the overall epidemic.

The expected cost of the representative individual, seen from time $t = 0$, when using individual contact rate β while the population contact rate is $\bar{\beta}$, is given by the expectation of the cost defined above in (7.1.3), namely:

$$\mathfrak{C}(\beta, \bar{\beta}) := \mathbb{E}[C(\beta, \tau)] = \mathbb{E}\left[\int_0^{\tau \wedge T} c(\beta_s) ds + r_1 \mathbf{1}_{\tau \leq T}\right], \quad (7.1.4)$$

where the distribution of τ is given by (7.1.2) and thus indirectly driven both by β and $\bar{\beta}$.

Finally, in order to determine his/her optimal contact rate $\beta^\star$ while the population rate driving I is $\bar{\beta}$, the representative individual faces the following minimisation problem:

$$\min_{\beta \in \mathcal{B}} \mathfrak{C}(\beta, \bar{\beta}), \quad \text{for } \mathcal{B} := \{\beta : [0, T] \longrightarrow [\beta_{\min}, \beta^\circ], \beta \text{ measurable}\}, \quad (7.1.5)$$

where $\mathcal{B}$ represents the set of admissible contact rate strategies.

7.1.4 Societal viewpoint: induced epidemic dynamics

In the previous subsection, we described the situation from the point of view of a representative individual. We assumed that each individual chooses his/her own contact rate β in order to minimise the associated cost $\mathfrak{C}(\beta, \bar{\beta})$, while $\bar{\beta}$ represents the population's contact rate. In our (Mean-Field games) framework, one individual taken alone has no influence on the pandemic dynamics when choosing his/her contact rate. Nevertheless, the global epidemic propagation rate $\bar{\beta}$ of the society is induced by the aggregation of all individuals contact rates. Besides, the behaviour of the representative individual described in the previous section, is the one used by all individuals assumed to be identical.

Hence, following the prescriptions of the Mean-Field games framework, an equilibrium is sought between the contact rate β , chosen by each individual, and the overall epidemic transmission rate $\bar{\beta}$, observed at the society level. More precisely, we look for an equilibrium among all individuals in the population, in the sense of the following definition.

Definition 7.1.1 (Nash/Mean-Field equilibrium). *The contact rate $\beta^\star \in \mathcal{B}$ together with the epidemic dynamics $(S^\star, I^\star, R^\star)$ is a Nash-MF equilibrium if the following relations are satisfied.*

- (i) **Individual rationality.** *It is optimal for the representative individual to choose the contact rate $\beta = \beta^\star$ when the epidemic dynamics is $(S^\star, I^\star, R^\star)$;*
- (ii) **Population consistency.** *Whenever the population contact rate is $\bar{\beta} = \beta^\star$, the induced epidemic dynamics is given by $(S^\star, I^\star, R^\star)$.*

Identifying such an equilibrium boils down to a fixed point property of the optimal best response function of the representative individual as described in the following section.

7.2 Theoretical results and numerical approximation

7.2.1 Existence of a Nash/Mean-Field equilibrium

As emphasised in the previous section, the contact rate chosen by each individual is impacted by the epidemic dynamics through the proportion of infected I , while this proportion is a direct consequence of the aggregation of all individuals' behaviour. We thus look for an equilibrium in such context, *i.e.*, in the sense of Definition 7.1.1. In fact, finding a Nash-MF equilibrium is equivalent to identifying a fixed point of the so-called best response function, which provides an optimal individual contact rate $\beta^* \in \mathcal{B}$ (or a set of optimal strategies if non-uniqueness of the optimal strategy) in response to a population contact rate $\bar{\beta}$.

We therefore introduce the best response function $\mathcal{T}$, which associates to any societal transmission rate $\bar{\beta} \in \mathcal{B}$, the optimal individual contact rate β^* :

$$\mathcal{T} : \bar{\beta} \in \mathcal{B} \longrightarrow \beta^* \in \left\{ \arg \min_{\beta \in \mathcal{B}} \mathfrak{C}(\beta, \bar{\beta}) \right\}. \quad (7.2.1)$$

As mentioned earlier, although we expect to find at least one fixed point to this application, in general $\mathcal{T}$ is a multi-valued mapping.

Assumption 7.2.1. *The cost function c is decreasing, two times differentiable with continuous second derivative (*i.e.*, of $\mathcal{C}^2$ -class) and such that:*

$$\inf_{\beta \in [\beta_{\min}, \beta^\circ]} c''(\beta) > 0. \quad (7.2.2)$$

Throughout the following, we will assume that the hypothesis above holds. Nevertheless, it is worth mentioning that the following results should also be valid under much less demanding hypotheses, but at the cost of increasing technicalities, deemed superfluous in the context of our study.

We first state the theoretical result concerning the existence of a unique best response strategy in $\mathcal{T}(\bar{\beta})$ for a given $\bar{\beta} \in \mathcal{B}$.

Lemma 7.2.2. *Suppose $\bar{\beta} \in \mathcal{B}$ is fixed. Under Assumption 7.2.1, there exists a unique $\beta^{*, \bar{\beta}} \in \mathcal{B}$ satisfying*

$$\mathfrak{C}(\beta^{*, \bar{\beta}}, \bar{\beta}) \leq \mathfrak{C}(\beta, \bar{\beta}), \text{ for all } \beta \in \mathcal{B}.$$

We are now in position to state the main mathematical result of this chapter, concerning the existence of a Nash-MF equilibrium.

Theorem 7.2.3. *Under Assumption 7.2.1, the model (7.1.1) admits a Nash/Mean-Field equilibrium $\beta^* \in \mathcal{B}$ together with $(S^{\beta^*}, I^{\beta^*}, R^{\beta^*})$, in the sense of Definition 7.1.1, *i.e.*, for any $\beta \in \mathcal{B}$,*

$$\mathfrak{C}(\beta^*, \beta^*) \leq \mathfrak{C}(\beta, \beta^*).$$

For sake of clarity, the proofs of both Lemma 7.2.2 and Theorem 7.2.3 are postponed to Section 7.4.4. Note that the result of Theorem 7.2.3 only informs on the existence of an equilibrium, and not on its possible uniqueness. The uniqueness of a Mean-Field equilibrium has been recognised as a difficult problem from the very beginning of the MFG theory (see counter-examples by Lasry and Lions [225, Remark below Theorem 2.2] (2006)) and very little theoretical guidance exists to support such a claim. Among the methods that can allow to obtain uniqueness one can list the monotony assumptions (see the aforementioned work of Lasry and Lions [225, 226, 227]) or the evolution dynamics (see Turinici [334] (2017)) but none seems to directly apply here. Moreover, with few exceptions, the epidemic Nash-MF equilibrium has rarely been proved to possess systematic uniqueness properties.

7.2.2 Numerical approach

We describe below the methodology used in the numerical experiments in order to approximate the Nash–MF equilibrium β^* . Recall that the cost for the representative individual when choosing a contact rate β while the societal transmission rate is $\bar{\beta}$, is given by $\mathfrak{C}(\beta, \bar{\beta})$.

The numerical approach we follow to find an equilibrium is based on the *metric equilibrium flow* approach, introduced by Turinici [334] (2017). The reader may refer to [334] for a rigorous mathematical transcription of the objects below, and also to the work of Salvarani and Turinici [301] (2018) for an application to epidemic. This approach prescribes the following iterative procedure in order to reach an equilibrium: choose a pseudo–time step $h > 0$ and define iteratively β^{n+1} as a minimiser of the following functional

$$\beta \mapsto \frac{|d(\beta, \beta^n)|^2}{h} + \mathfrak{C}(\beta, \beta^n). \quad (7.2.3)$$

Here, the function $d : \mathcal{B} \times \mathcal{B} \rightarrow \mathbb{R}$ denotes a geodesic distance on the space $\mathcal{B}$ of all possible individual choices β . Note that if $\mathcal{B}$ is a Hilbert space, the cost function $\mathfrak{C}$ is *smooth enough*, and d is the distance induced by the canonical norm, then the fact that β^{n+1} is a minimiser of the functional in (7.2.3) implies:

$$\beta^{n+1} = \beta^n - h \nabla_1 \mathfrak{C}(\beta^{n+1}, \beta^n), \quad (7.2.4)$$

where ∇_1 denotes for the derivative with respect to the first argument.

Relation (7.2.4) is similar to the JKO scheme used in one variable gradient flows, as by Ambrosio, Gigli, and Savare [21] (2008). The problem with (7.2.4) is that it is implicit and thus not directly compatible, in this form, with numerical computation. In practice, when the vectorial structure on $\mathcal{B}$ is compatible with the topological structure, one can propose an iterative procedure to find β^{n+1} solution of (7.2.4).

(i) *Initialisation*: start with $\beta^{n+1,0} = \beta^n$, for some $\beta^n \in \mathcal{B}$.

(ii) *Iteration*: for all $\ell \geq 0$, let $\beta^{n+1,\ell+1} = \beta^n - h \nabla_1 \mathfrak{C}(\beta^{n+1,\ell}, \beta^n)$.

It is standard to see that for h small enough and in presence of (e.g. Lipschitz) regularity of $\nabla_1 \mathfrak{C}$ with respect to its first argument, the iterations are guaranteed to converge, by a Picard fixed point argument, to the (unique) solution of (7.2.4). However, for numerical convenience, in practice only L iterations are performed, and for our numerical simulations, $L = 1$ worked just fine. Therefore, we implement the following proxy for the minimisation in (7.2.3):

$$\beta^{n+1} = \beta^n - h \nabla_1 \mathfrak{C}(\beta^n, \beta^n). \quad (7.2.5)$$

When $\mathcal{B}$ is a Hilbert space and forgetting any possible regularity issues, we obtain an Explicit Euler discretisation of the equilibrium flow defined in [334]. Thus, we can expect that $\lim_{n \rightarrow \infty} \beta^n$ will be the equilibrium we want to compute.

7.2.3 Choice of parameters

The following numerical experiments are done using a contact rate reduction cost function defined by:

$$c(b) := \frac{\beta^\circ}{b} - 1, \quad b \in [\beta_{\min}, \beta^\circ]. \quad (7.2.6)$$

Recall that the parameter $\beta_{\min}$ represents the minimal achievable contact rate by a representative individual, while β° denotes the usual contact rate used before the beginning of the lockdown measures. In other words, β° represents the transmission rate of the disease without any isolation effort of the population. The shape

of the cost function encompasses the increasing difficulty to bring the contact rate closer to zero. Conversely, without effort of the individual at time t , meaning that $\beta_t = \beta^\circ$, the associated cost $c(\beta_t)$ is equal to zero. In addition, note that this function c satisfies [Assumption 7.2.1](#).

The set of parameters used in the experiments are provided in [Table 7.1](#). The associated reproduction number $\mathcal{R}_0$ without isolation measure, commonly defined by $\mathcal{R}_0 := \beta^\circ/\gamma$ in the literature on epidemic models, is equal to 2.0 in our framework, and is thus in the confidence interval of available data, see for example the study by [Li et al. \[232\] \(2020\)](#). The parameter γ corresponds to the inverse of the virus contagious period, as explained by [Anderson, Hollingsworth, and Nokes \[26\] \(2009\)](#). We assume that the initial proportion of infected I_0 in the overall population is around 1%, when the contact rate optimisation starts. We set the cost r_I incurred by an infected individual to $r_I = 300$. We recall that this cost is not necessarily expressed in terms of money, but can also be medical side effects or general morbidity (see the aforementioned works [\[351, 22, 304\]](#) for an introduction on QALY/DALY) and is relative to the definition of the cost function c .

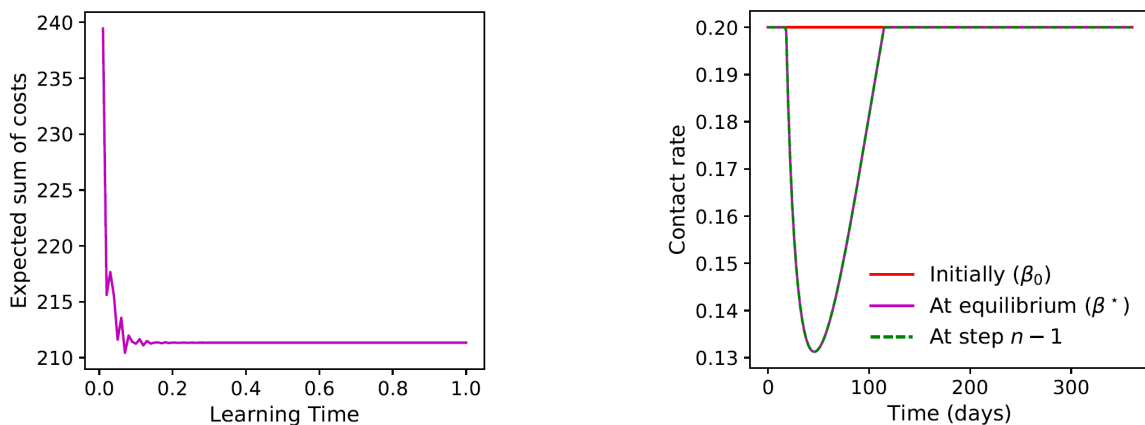
(S_0, I_0, R_0)	γ	β°	$\beta_{\min}$	$\underline{\beta}$	T	r_I
(0.99, 0.01, 0.00)	1/10	0.20	0.05	0.14	360 days	300

Table 7.1 – Set of parameters for the numerical experiments

Unless explicitly stated otherwise, numerical simulations will be performed with the parameters described in [Table 7.1](#). Finally, since there is considerable uncertainty in the medical literature on the choice of all parameters described above, the sensibility to the values chosen has been tested in [Section 7.3.3](#).

7.2.4 Equilibrium and resulting epidemic dynamics

The numerical approximation of the Nash–MF equilibrium is obtained using the algorithm described in [Section 7.2.2](#). The initial guess β^0 is taken constant $\beta^0(t) = \beta^\circ, \forall t \geq 0$. It is worth noticing that we tested other initial guesses and all the convergence rate was the same. Moreover, in all cases, the same equilibrium is obtained.



(a) Global cost $\mathcal{C}$ in terms of the learning time.

(b) Contact rate β^* at equilibrium (magenta curve), identical to the contact rate at the penultimate step of the algorithm (green dashed) and compared to β° (red).

Figure 7.2 – Convergence of the algorithm to the Nash/Mean-Field equilibrium of the system [\(7.1.1\)](#).

We plot in Figure 7.2 the convergence towards the minimal cost; the decay of the cost is very fast in terms of the learning time h . The contact rate at equilibrium is represented in the right plot, while the empirical convergence of the algorithm is confirmed by its superposition with the approximate contact rate computed at the previous step. At this numerical Nash/Mean-Field equilibrium, the cost function $\mathfrak{C}(\beta^*, \beta^*)$ is 211.33 and provides for each individual a relative gain of 12% in comparison to the zero effort strategy β° . In addition, only 85% of this cost is explained in terms of the wealth impact (instead of 100% for the strategy β°), with the remaining 15% being related to the cost of social effort; the probability of being infected over the interval $[0, T]$ decreases from 80% in the 0-effort benchmark scenario, to 62%, see Figure 7.3. Note that in this case, the minimum attainable infection probability is 50% (as calculated from [219, Lemma A.1] or from equation (7.1.1)).

The equilibrium contact rate β^* is characterised by three major phases, in response to the Nash/Mean-Field equilibrium epidemic dynamics presented in Figure 7.3. First, at the beginning of the epidemic, the number I of infected people is relatively low. Individuals make therefore no effort in order to reduce their social interactions and the virus is transmitted at the normal rate β° . This leads to a large augmentation in the proportion of Infectious, implying a significant increase of the individual's probability of contracting the virus. In response to this, individuals begin to reduce significantly their social interactions, implying a strong reduction of the transmission rate of the disease. Finally, after the epidemic peak, all individuals slowly reduce their effort until the number of infected people is relatively close to 0.

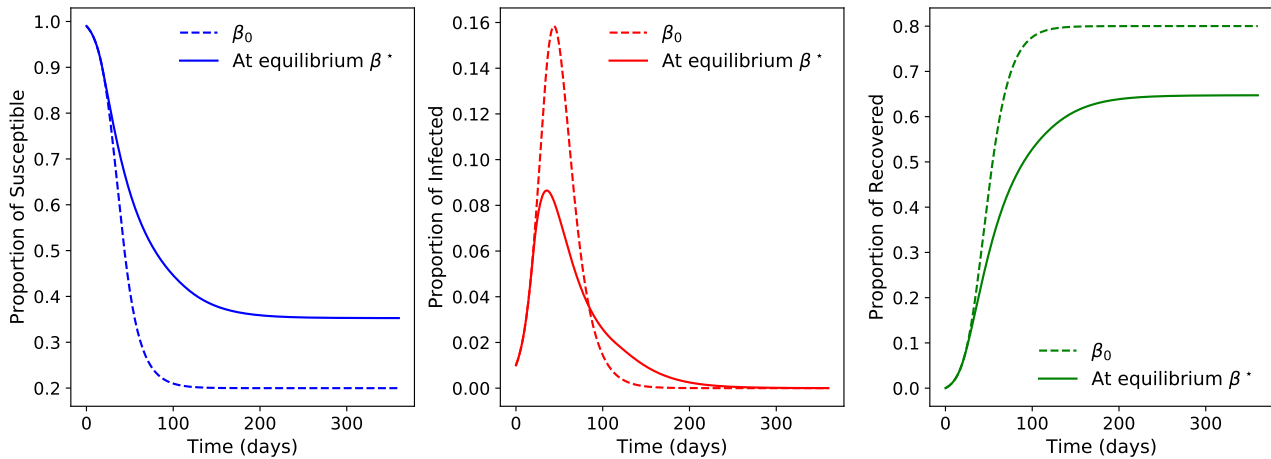


Figure 7.3 – SIR dynamics at the Nash–MF equilibrium.

Evolution of *Susceptible* (left), *Infectious* (middle) and *Recovered* (right) proportions for the contact rate $\beta^* \in \mathcal{B}$ (solid lines), comparison with the initial transmission rate β° (dashed lines).

Figure 7.3 provides a comparison between the SIR dynamics of the Nash/Mean-Field equilibrium (solid lines), *i.e.* with transmission rate β^* , and the one generated by the no-effort strategy β° (dashed lines). Even if driven by self-interest, individual efforts do reduce social interactions population-wide. This can be explained as follows. First, we observe that the proportion R_T of recovered at terminal date drops from 80% to 60% when the population is applying the Nash/Mean-Field equilibrium strategy. This means that the proportion of the population spared by the virus goes from 20% in the case without effort to 40% at equilibrium. Second, one can observe that the infection peak occurring around $t = 50$ days is twice less critical at equilibrium, but, as a counterpart, the epidemic lasts longer as the number of infected decreases more slowly after the epidemic peak. As the infection peak is less critical, it limits, and may even prevent, the saturation of the healthcare facilities. Although not represented in our model, this necessarily implies a decrease in the mortality rate of the virus.

7.3 Comparison and extensions

7.3.1 Impulse control equilibrium

In this section, we focus on the situation where the set of admissible strategies is restricted to a subset of $\mathcal{B}$, denoted by $\bar{\mathcal{B}}$, of particular piece-wise constant strategies:

$$\bar{\mathcal{B}} := \{\beta \in \mathcal{B}, \beta(t) = \underline{\beta} \cdot \mathbf{1}_{[t_1, t_2]}(t) + \beta^\circ \cdot \mathbf{1}_{[0, T] \setminus [t_1, t_2]}(t); 0 \leq t_1 \leq t_2 \leq T\}, \quad (7.3.1)$$

where we took $\underline{\beta} = 0.14$ as described in Table 7.1. This framework encompasses the realistic situation where the instantaneous contact rate of each individual can not be, in practice, chosen within the whole set $\mathcal{B}$ and has to be restricted to a unique control period. The representative individual will optimally select the times t_1 and t_2 , representing respectively the beginning and the end of his/her lockdown period.

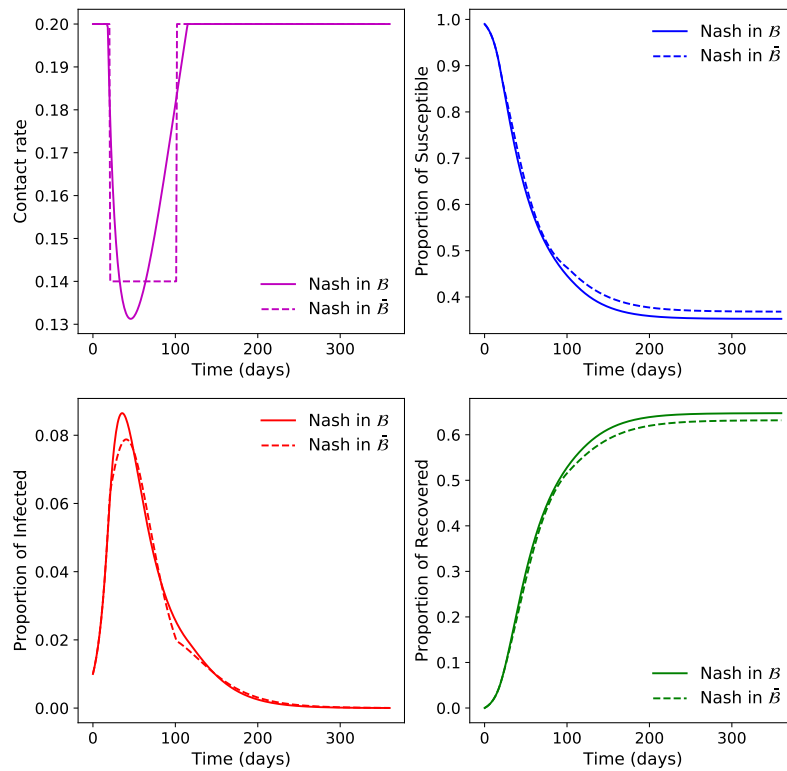


Figure 7.4 – Comparison between the Nash–MF equilibria in $\mathcal{B}$ and in $\bar{\mathcal{B}}$.

Upper left: contact rate; upper right: proportion of *Susceptible*; bottom left: proportion of *Infectious*; and bottom right: proportion of *Recovered*; for the Nash–MF contact rate in $\mathcal{B}$ (solid lines) and in $\bar{\mathcal{B}}$ (dashed lines).

Figure 7.4 compares the Nash–MF equilibria obtained over both sets $\bar{\mathcal{B}}$ and $\mathcal{B}$. The equilibrium strategy over $\bar{\mathcal{B}}$ starts isolation measures at time $t_1 = 20$ by decreasing the contact rate from $\beta^\circ = 0.20$ to $\underline{\beta} = 0.14$. The duration of the lockdown is 81 days, after which the contact rate is immediately returning to normal, *i.e.*, β° . The cost induced for each individual is around 210.78, which is lower than the one around 211.33 associated to the equilibrium over $\mathcal{B}$. Moreover, we observe that the induced SIR dynamics provides a lower epidemic size R_T together with a lower proportion of infected at the epidemic peak, hereby reducing the potential mortality rate induced by the virus. This observation enlightens as well how the Nash/Mean-Field equilibrium β^* in $\mathcal{B}$ is not optimal for the society as a whole, as it can be improved by restricting the set of admissible strategies to $\bar{\mathcal{B}}$. This observation leads us to look towards the optimal societal contact rate for the population as a whole.

7.3.2 Cost of anarchy

In our previous equilibrium analysis, each individual is considered to be too small in order to impact the epidemic dynamics of the society and can hence act in a selfish manner: each individual minimises his own cost $\mathfrak{C}(\cdot, \bar{\beta})$ in response to the transmission rate $\bar{\beta}$ of the epidemic. On the other hand, a global planner, *e.g.*, a government with full empowerment, will optimise the global cost of the entire society with respect to the choice of the transmission rate in the society. Namely, the global planner will solve:

$$\min_{\bar{\beta} \in \mathcal{B}} \mathfrak{C}(\bar{\beta}, \bar{\beta}).$$

This leads to a different optimisation problem, which is well documented in the literature on epidemics control, see for example the works by Sethi and Staats [310] (1978), Behncke [45] (2000), Hansen and Day [172] (2011) and more recently, the application to the COVID-19 disease by Djidjou-Demasse et al. [128] (2020). This type of optimisation problem is even more widely discussed concerning the topic of vaccination, by Abakuks [7] (1974), Morton and Wickwire [257] (1974), Piunovskiy and Clancy [282] (2008) and Laguzet and Turinici [219] (2015), among others.

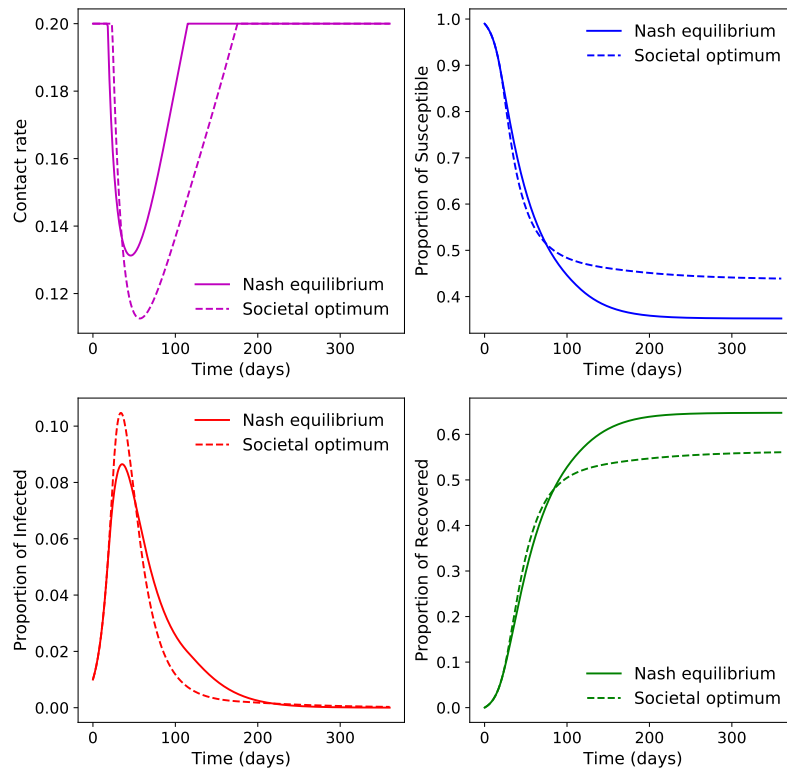


Figure 7.5 – Comparison between Nash–MF equilibrium and societal optimum.

Upper left: contact rate; upper right: proportion of *Susceptible*; bottom left: proportion of *Infectious*; and bottom right: proportion of *Recovered*; for $\beta^* \in \mathcal{B}$ (solid lines) and the societal optimum contact rate (dashed lines).

In our framework, we can compute, through classical optimisation procedures (*e.g.*, the Pontryagin principle), the optimal control of the transmission rate from the society point of view. We refer to the literature previously cited for the details on the techniques allowing to find this optimal control and we only present here the numerical results. Note that in such context there are several additional, classical, procedures available to compute the optimal control (see for instance the forward–backward sweep method developed by Lenhart and Workman [229, Chapter 4] (2007)). Nevertheless, in our framework, a slight modification of the previously implemented gradient descent detailed in Section 7.2.2 works just fine.

First, observe in Figure 7.5 that the societal optimal transmission rate imposes larger effort at the beginning of the control period, and relieves these constraints much more slowly. The total control duration period for the societal optimum is 151 days in comparison to 96 days for the Nash–MF equilibrium, although societal control begins later than in the case of the Nash. Secondly, observe that the optimal transmission rate chosen by the global planner accentuates the already encouraging results obtained with the Nash–MF equilibrium on the epidemic dynamics: the epidemic size R_T represents only 55% of the total population.

The societal optimum allows to reach an individual cost around 200.25, while the Nash–MF equilibrium provides a cost valued around 211.33. This phenomenon allows to mathematically quantify the so-called *cost of anarchy*, induced when letting each individual decide on his/her own, instead of letting a global planner take decisions for the population as a whole. Of course, the societal optimal strategy is not a Nash equilibrium: given the optimal transition rate for the society, an individual is tempted to make less effort in reducing his/her own contact rate, which will drive away the global rate towards the Nash–MF equilibrium.

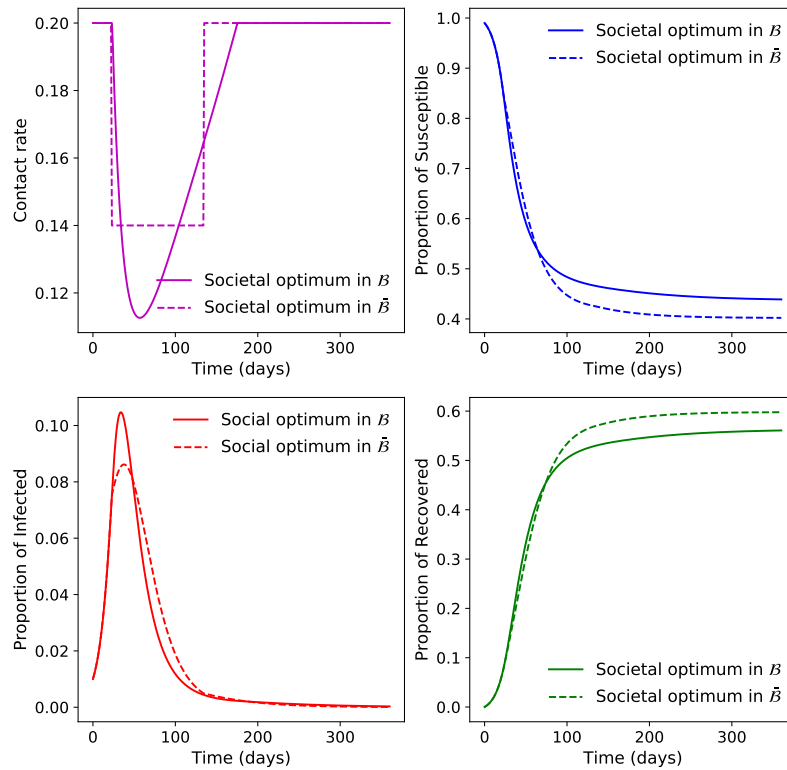


Figure 7.6 – Comparison between the societal optimum in $\mathcal{B}$ and in $\bar{\mathcal{B}}$.

Upper left: contact rate; upper right: proportion of *Susceptible*; bottom left: proportion of *Infectious*; and bottom right: proportion of *Recovered*; for the societal optimum contact rate in $\mathcal{B}$ (solid lines) and in $\bar{\mathcal{B}}$ (dashed lines).

We also compare in Figure 7.6 the societal optimal transmission rate in $\mathcal{B}$ to the optimal one in $\bar{\mathcal{B}}$, where only one strong effort period is allowed. The societal-wide optimal control period in $\bar{\mathcal{B}}$ starts at time $t_1 = 23$ and lasts 111 days. The induced cost is around 2% higher than the one induced by the optimal social diffusion rate. This may allow to quantify the importance of going through a progressive lockdown strategy.

Finally, Figure 7.7 provides a closer look on the four aforementioned strategies, focusing on the time interval $[10, 150]$, in order to highlight the control (lockdown) and control-less (lockout) properties. One can observe that for both Nash/Mean-Field equilibrium strategies, the control period starts and ends earlier in comparison to the societal optimum situation. Indeed, on one hand, individuals engage in preventive

measures by decreasing their interactions earlier than a global planner would recommend, due to fear of the infection spreading. On the other hand, they release their efforts just after the peak of infection, whereas a global planner would recommend maintaining a relatively high level of effort in order to avoid further spreading of the virus. Therefore, while the Nash equilibrium of individuals allows, through premature efforts, to decrease the peak of infection, the socially optimal strategy allows a more rapid decrease in the proportion of infected after the peak, by maintaining intense efforts. While stopping early, at a point where the epidemic decreases but is not yet over, may be intuitively explained by the selfish nature of individuals, the fact that they decide collectively to begin efforts early than the societal optimum is a more intriguing feature. Nevertheless, the latter fact is consistent with the experimental results documented in the remarkable work by Ghader et al. [153] (2020).

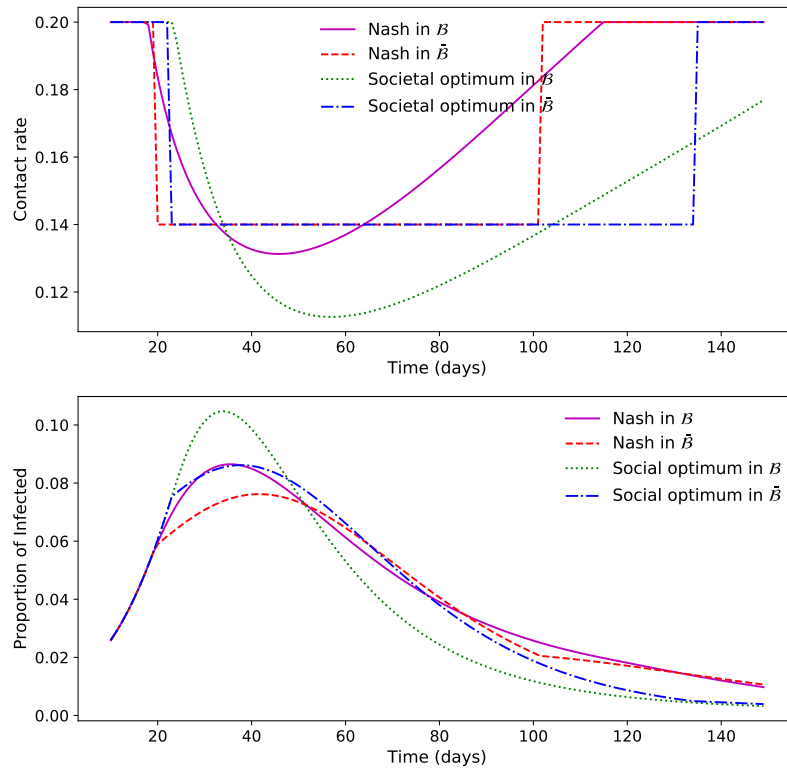


Figure 7.7 – Contact rate (top) and proportion of infected (bottom) for four strategies.

Comparison on the time interval $[10, 150]$ of the Nash–MF equilibrium in $\mathcal{B}$ (solid magenta lines) and in $\bar{\mathcal{B}}$ (dashed red), and of the societal optimum in $\mathcal{B}$ (dotted green) and in $\bar{\mathcal{B}}$ (dash–dot blue).

However, remember that we are considering the same cost for the individual and the global planner, whereas this is not necessarily the case. In particular, the cost of a global planner could also take into account the possible saturation of the healthcare system, contrary to our model, which could lead to a lower epidemic peak than the one induced by the Nash–MF equilibrium. Moreover, the study of the cost of anarchy raises the question of incentives, to bring the Nash–MF equilibrium closer to the societal optimal. Our model can give details in this direction, and invites a finer assessment of the costs related to the epidemic, but also to the lockdown.

Of course, any model is but an imperfect description of the reality and thus there are some limitations of the model presented in this chapter. First of all, as there is no consensus in the medical literature on the parameters of the epidemic provided in Table 7.1 above, this impacts our model too.

7.3.3 Sensitivity to parameters

The recent literature on the COVID-19 epidemic is abundant but discordant on the dynamics of the epidemic, in particular on the parameters γ and $\mathcal{R}_0 := \beta^\circ/\gamma$. Therefore, we tested the sensitivity of our results with respect to the choice of the main parameters of the model. The parameters are those described in Table 7.1, except when other values are explicitly mentioned.

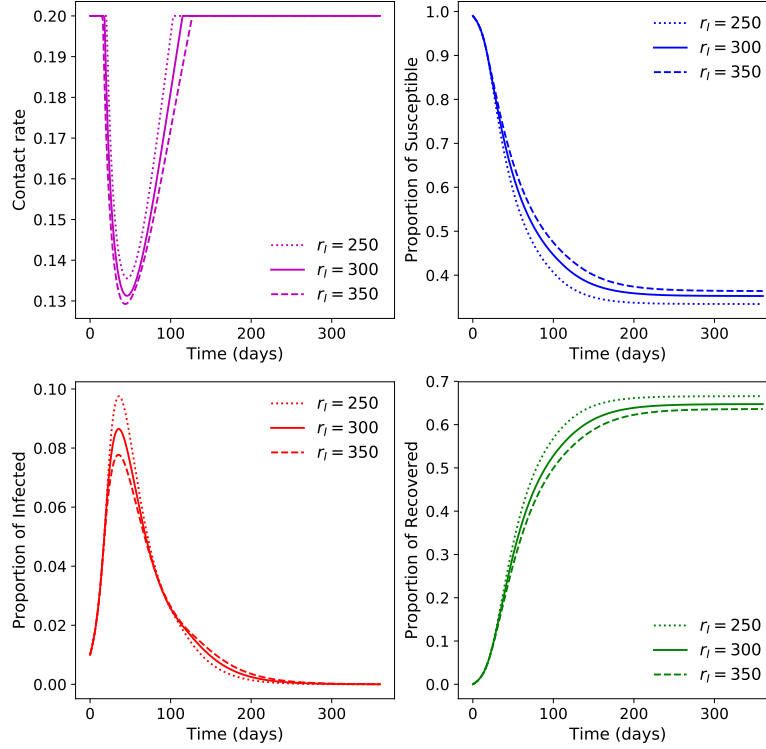


Figure 7.8 – Nash-MF contact rate and SIR dynamics for three different values for r_I .
Solid lines: $r_I = 300$; dotted line: $r_I = 250$; dashed lines: $r_I = 350$.

First, Figure 7.8 provides the Nash-MF equilibrium and the epidemic dynamics for three different values for r_I , describing the relative effects of the two parts of the costs. As expected, the more costly is the infection for an individual, the more efforts he/she will do in order to limit his social interactions, in order to decrease his/her probability of infection. In particular, reducing the sanitary cost r_I from 350 to 250 implies a 20% decrease on the level of the epidemic peak.

Then, Figure 7.9 provides a similar study for three different values of the reproducing number $\mathcal{R}_0$. For higher $\mathcal{R}_0$, individuals make more effort in order to reduce their social interactions. Bringing $\mathcal{R}_0$ from 2.5 to 1.8 reduces the epidemic peak by 40% together with reducing the size R_T of the epidemic from 70% to around 60%. This result is perfectly understandable since the higher $\mathcal{R}_0$ is, the higher the probability of being infected without effort is. Each individual limits his social interaction in order to decrease the probability of being infected, and thus diminishes the wealth impact of the epidemic.

Finally, Figure 7.10 studies the sensibility of our findings with respect to the initial proportion I_0 of infected at time 0. A higher I_0 induces an earlier beginning of the control period, together with a stronger efficacy of it. At terminal date, the total proportion of susceptible remains similar. This implies that the long term effects of a late detection of the epidemic can be compensated by a stronger isolation equilibrium policy. Once again, we omit here the possible negative outcomes induced by the saturation of the health care system.

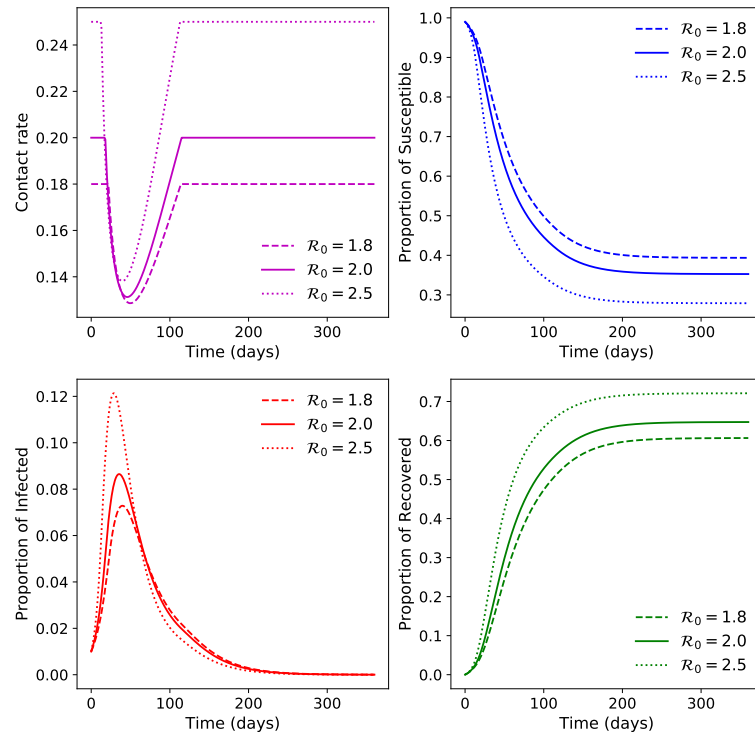


Figure 7.9 – Nash–MF contact rate and SIR dynamics for three different values for $\mathcal{R}_0$.
Dashed lines: $\mathcal{R}_0 = 1.8$; solid lines: $\mathcal{R}_0 = 2.0$; dotted line: $\mathcal{R}_0 = 2.5$.

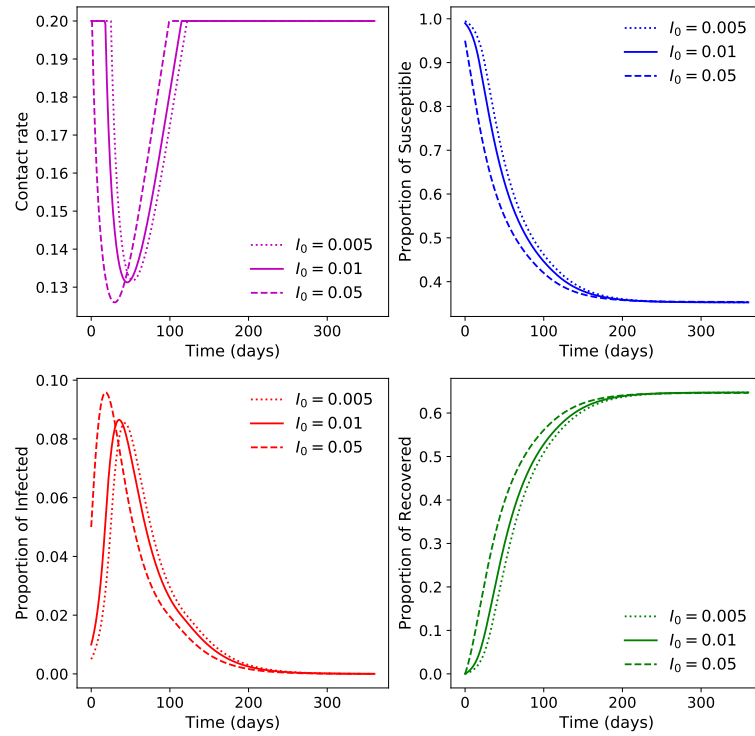


Figure 7.10 – Nash–MF contact rate and SIR dynamics for three different values for I_0 .
Solid lines: $I_0 = 0.01$; dotted line: $I_0 = 0.005$; dashed lines: $I_0 = 0.05$.

7.3.4 The SEIR model and application to COVID-19

In addition to the incertitude on the parameters, the appropriate estimation of the cost of effort c and its counterpart r_I are still object of research, as they are not necessarily monetary or economic costs. Indeed, these costs are often costs related to health, disease and social interactions. To actually estimate these parameters would require an extensive sociological and economic study, that we do not presume to be able to perform at this stage, linking, for example, the individual costs to the DALY/QALY concepts already mentioned.

Nevertheless, our model has the advantage that it can be easily extended to epidemiological models other than the SIR dynamics. Therefore, in order to take into account one of the major characteristic of the COVID-19 disease, we extend the study in this section to a SEIR epidemic model, and present some numerical results. Indeed, it should be noted that the COVID-19 disease is characterised by a relatively long latency phase (as well as many other complex dynamics). To account for this latent phase, where infected individuals are not yet contagious, we extend our reasoning to an SEIR model, where the class E represents the individuals infected but not yet infectious. The dynamics of the SEIR model are described by the following system:

$$\begin{cases} dS_t = -\bar{\beta}_t S_t I_t dt, \\ dE_t = \bar{\beta}_t S_t I_t dt - \alpha E_t dt, \\ dI_t = \alpha E_t dt - \gamma I_t dt, \\ dR_t = \gamma I_t dt, \end{cases} \quad (7.3.2)$$

where $\bar{\beta}$ still represents the societal transmission rate, and $\alpha > 0$ is a parameter specific to the SEIR model, representing the rate at which an exposed person becomes infectious. The average incubation period is therefore given by $1/\alpha$.

The computations provided in [Section 7.4](#) still hold for the SEIR model. Therefore, the application of the numerical scheme described in [Section 7.2.2](#) is straightforward, and only the numerical results are presented here. The parameters used for the numerical simulations are those provided by [Bacaër \[30\] \(2020\)](#), also used by [Dolbeault and Turinici \[130\] \(2020\)](#), and are described in the following table. Other parameters given in [Table 7.1](#) remain unchanged.

(S_0, E_0, I_0, R_0)	γ	α	β°
$(0.99984, 8.81 \times 10^{-5}, 1.88 \times 10^{-5}, 5.35 \times 10^{-5})$	1	0.25	2.33

Table 7.2 – Set of parameters for the SEIR model

The numerical results are provided in [Figure 7.11](#) and have the same features as for the SIR model. More precisely, the proportion of the population spared by the virus goes from less than 20% in the case without effort to 40% at equilibrium. Moreover, the infection peak occurring around $t = 50$ days is three times less critical at equilibrium. This result would limit, and may even prevent, the saturation of the healthcare facilities, and thus implies a decrease in the mortality rate of the virus. However, as a counterpart, the epidemic lasts longer as the proportion of exposed and infected individuals decreases more slowly after the epidemic peak. Finally, and similarly as for the SIR model, the cost of anarchy exists. In particular, the societal optimal transmission rate requires a greater and longer-lasting effort, even if it starts a little later. The optimal transmission rate at the societal level thus improves the rate of recovery from 60% to 55%. However, since the effort begins later, the infection peak is higher than for the Nash equilibrium.

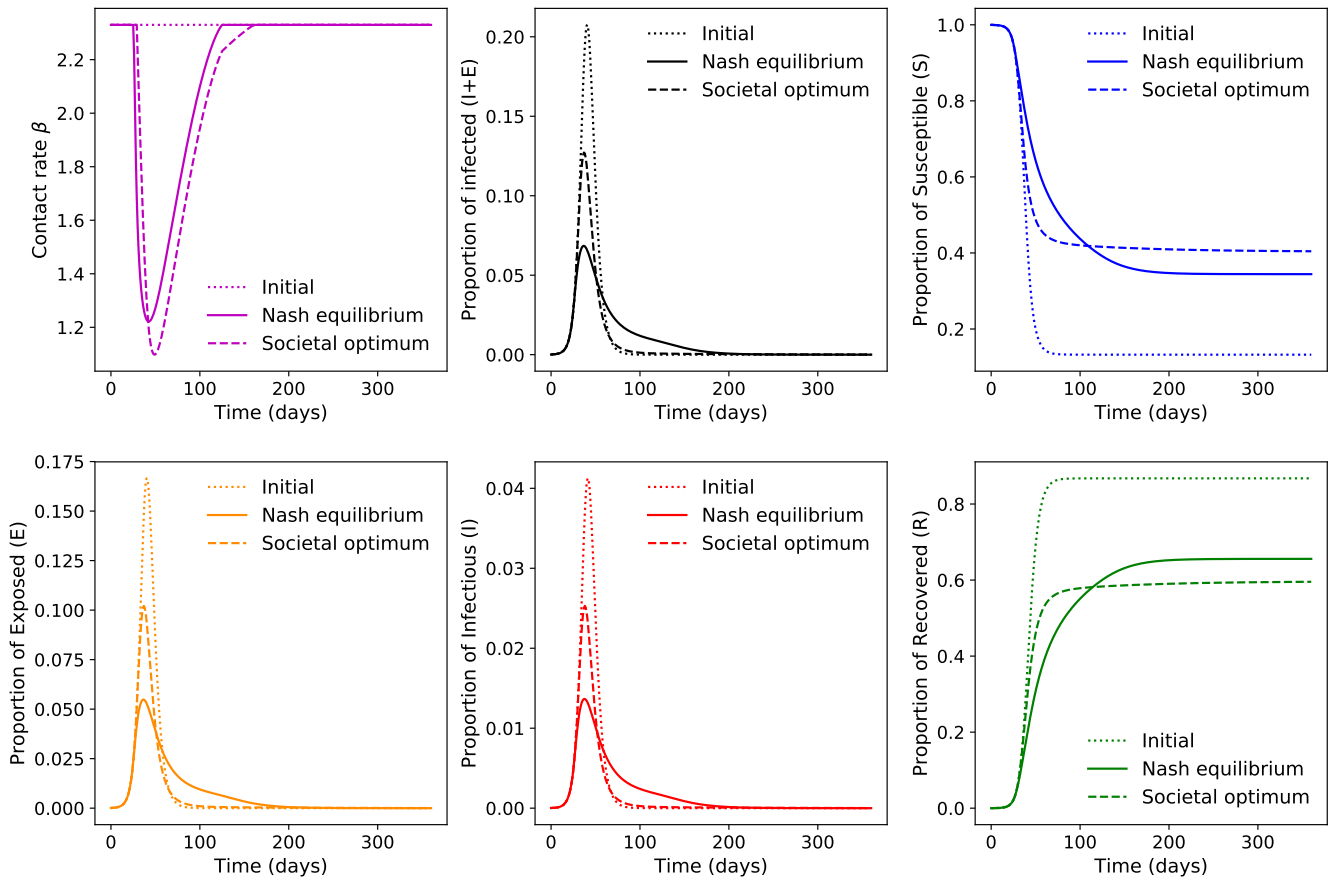


Figure 7.11 – Comparison of SEIR dynamics for different contact rate.

Transmission rate (upper left) and proportions of *Susceptible* (upper right), *Exposed* (bottom left), *Infectious* (bottom middle) and *Recovered* (bottom right), at the equilibrium (plain lines), at the societal optimum (dashed) and with the transmission rate β° (dotted). Upper middle: total proportion of infected, *i.e.*, $E + I$.

Of course, the choice of an SEIR model can also be discussed, since the disease induced by the COVID-19 has many other complex dynamics, for example the large number of asymptomatic carriers. We refer to the aforementioned works by Danchin, Ng et Turinici, in particular [266] (2003), [335] (2006) and [113] (2020) for alternatives, already used in coronavirus epidemics.

Moreover, our model could also be extended to take into account other ways to control an epidemic, such as vaccination for example, even if no vaccine is known to date. Moreover we assume in this chapter that all individuals are rational, identical, and that they have a perfect knowledge of the epidemic dynamics. A possible extension could be to add some heterogeneity between individuals, since the epidemic does not affect individuals in the same way, and is more costly for those at risk.

However, all the aforementioned extensions remains for future works. To conclude this chapter, we provide in the following section some mathematical details. More precisely, we present the computation of the expected cost of an individual, given by (7.1.4). In particular, these computations provide us with an explicit formula for the gradient of the cost, necessary to perform the numerical approach presented in Section 7.2.2. More importantly, this section reports the proof of the main theoretical results of this chapter, namely Lemma 7.2.2 and Theorem 7.2.3.

7.4 Mathematical details

In this section, we detail the computations and proofs when the epidemic is modelled by a SIR. Nevertheless, these computations still hold if we consider instead an SEIR model, as mentioned in [Section 7.3.4](#). More precisely, since the individual is considered infected as soon as he/she enters class E , the probability of being infected in the considered time interval $[0, T]$ remains unchanged. In particular, the cost satisfies the same formula (7.4.5) as for the SIR model, as well as the gradient formula (7.4.7).

7.4.1 Probability of infection

We fix throughout the following the population's contact rate $\bar{\beta} \in \mathcal{B}$. We introduce $(S_t^{\bar{\beta}}, I_t^{\bar{\beta}}, R_t^{\bar{\beta}})$ in order to denote, at time $t \geq 0$, the solution of the system (7.1.1) with contact rate $\bar{\beta}$. For all $t \in [0, +\infty)$, we denote by $\varphi_t^{\bar{\beta}}(\beta)$ the probability that infection occurs before time t for an individual choosing his/her own contact rate β , while the epidemic evolves according the population's contact rate $\bar{\beta}$.

Lemma 7.4.1. *The probability of being infected before time $t \geq 0$, for an individual choosing a contact rate $\beta \in \mathcal{B}$, while the epidemic evolves according the population's contact rate $\bar{\beta}$, is equal to:*

$$\varphi_t^{\bar{\beta}}(\beta) = 1 - \exp\left(-\int_0^t \beta_s I_s^{\bar{\beta}} ds\right). \quad (7.4.1)$$

Proof. To compute this probability, we follow the reasoning developed by [Laguzet and Turinici \[220\] \(2015\)](#) as well as by [Laguzet, Turinici, and Yahiaoui \[221\] \(2016\)](#). The Markov chain of an individual, who chooses a contact rate $\beta \in \mathcal{B}$ with infected individuals, and whose state at time $t \in [0, T]$ is denoted by M_t , is described in terms of the following passage probabilities:

$$\begin{aligned} \mathbb{P}[M_{t+\Delta t} = S | M_t = S] &= 1 - \beta_t I_t^{\bar{\beta}} \Delta t + o(\Delta t), \\ \mathbb{P}[M_{t+\Delta t} = I | M_t = S] &= \beta_t I_t^{\bar{\beta}} \Delta t + o(\Delta t), \\ \mathbb{P}[M_{t+\Delta t} = I | M_t = I] &= 1 - \gamma I_t^{\bar{\beta}} \Delta t + o(\Delta t), \\ \mathbb{P}[M_{t+\Delta t} = R | M_t = I] &= \gamma I_t^{\bar{\beta}} \Delta t + o(\Delta t), \\ \mathbb{P}[M_{t+\Delta t} = R | M_t = R] &= 1 + o(\Delta t). \end{aligned}$$

The probability of being infected before time $t + \Delta t$ can be written as follows:

$$\begin{aligned} \varphi_{t+\Delta t}^{\bar{\beta}}(\beta) &= \mathbb{P}[\exists \tau \in [0, t + \Delta t], M_\tau = I | M_0 = S] \\ &= \mathbb{P}[M_{t+\Delta t} \in \{I \cup R\} | M_0 = S] \\ &= \mathbb{P}[M_{t+\Delta t} \in \{I \cup R\} | M_t = S] \times \mathbb{P}[M_t = S | M_0 = S] \\ &\quad + \mathbb{P}[M_{t+\Delta t} \in \{I \cup R\} | M_t \in \{I \cup R\}] \times \mathbb{P}[M_t \in \{I \cup R\} | M_0 = S]. \end{aligned} \quad (7.4.2)$$

Noting that

$$\begin{aligned} \mathbb{P}[M_{t+\Delta t} \in \{I \cup R\} | M_t = S] &= \mathbb{P}[M_{t+\Delta t} = I | M_t = S] = \beta_t I_t^{\bar{\beta}} \Delta t + o(\Delta t), \\ \text{and } \mathbb{P}[M_{t+\Delta t} \in \{I \cup R\} | M_t \in \{I \cup R\}] &= 1 + o(\Delta t), \end{aligned}$$

and replacing in (7.4.2), we obtain:

$$\varphi_{t+\Delta t}^{\bar{\beta}}(\beta) = \beta_t I_t^{\bar{\beta}} \Delta t \times (1 - \varphi_t^{\bar{\beta}}(\beta)) + 1 \times \varphi_t^{\bar{\beta}}(\beta) + o(\Delta t).$$

Using the fact that $\varphi_0^{\bar{\beta}}(\beta) = 0$, and letting $\Delta t \rightarrow 0$, the previous equation naturally implies the required formula (7.4.1). $\square$

7.4.2 Computation of the cost

Recall that given a finite time horizon T , the expected cost of an individual is defined by (7.1.4). In other words, we have:

$$\mathfrak{C}(\beta, \bar{\beta}) = \mathbb{E} \left[\mathbb{1}_{\tau \leq T} \int_0^\tau c(\beta_s) ds + \mathbb{1}_{\tau > T} \int_0^T c(\beta_s) ds + r_I \mathbb{1}_{\tau \leq T} \right]. \quad (7.4.3)$$

Since the cumulative distribution function of the random variable τ at time t corresponds to the individual's probability of being infected before time t , which is denoted by $\varphi_t^{\bar{\beta}}(\beta)$, we obtain:

$$\begin{aligned} \mathfrak{C}(\beta, \bar{\beta}) &= \int_0^T \int_0^t c(\beta_s) ds d\varphi_t^{\bar{\beta}}(\beta) + (1 - \varphi_T^{\bar{\beta}}(\beta)) \int_0^T c(\beta_s) ds + r_I \varphi_T^{\bar{\beta}}(\beta) \\ &= \int_0^T c(\beta_s) (1 - \varphi_s^{\bar{\beta}}(\beta)) ds + r_I \varphi_T^{\bar{\beta}}(\beta). \end{aligned} \quad (7.4.4)$$

Moreover, using (7.4.1), we can also write:

$$\mathfrak{C}(\beta, \bar{\beta}) = \int_0^T \left(c(\beta_t) + r_I \beta_t I_t^{\bar{\beta}} \right) \exp \left(- \int_0^t \beta_s I_s^{\bar{\beta}} ds \right) dt. \quad (7.4.5)$$

7.4.3 Gradient of the cost

As described in Section 7.2.2, in order to obtain preform numerical simulations, we need to compute the gradient of this cost with respect to the first variable. With this in mind, we compute the Gateau derivative $D_h \mathfrak{C}$ of $\mathfrak{C}$ with respect to the first variable β in the direction h . Using Equation (7.4.4), we obtain:

$$\begin{aligned} D_h \mathfrak{C}(\beta, \bar{\beta}) &:= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} (\mathfrak{C}(\beta + \varepsilon h, \bar{\beta}) - \mathfrak{C}(\beta, \bar{\beta})) \\ &= r_I D_h \varphi_T^{\bar{\beta}}(\beta) + \int_0^T \left((1 - \varphi_s^{\bar{\beta}}(\beta)) D_h c(\beta_s) - c(\beta_s) D_h \varphi_s^{\bar{\beta}}(\beta) \right) ds. \end{aligned} \quad (7.4.6)$$

Some preliminary computations of the Gateaux derivatives allow to write:

$$\begin{aligned} D_h \varphi_s^{\bar{\beta}}(\beta) &:= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} (\varphi_s^{\bar{\beta}}(\beta + \varepsilon h) - \varphi_s^{\bar{\beta}}(\beta)) = (1 - \varphi_s^{\bar{\beta}}(\beta)) \int_0^s h_u I_u^{\bar{\beta}} du, \\ \text{and } D_h c(\beta_s) &:= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} (c(\beta_s + \varepsilon h_s) - c(\beta_s)) = c'(\beta_s) h_s. \end{aligned}$$

Replacing in (7.4.6), we therefore obtain the following relation:

$$D_h \mathfrak{C}(\beta, \bar{\beta}) = \langle h, \nabla_1 \mathfrak{C}(\beta, \bar{\beta}) \rangle_T,$$

where

$$\nabla_1 \mathfrak{C}(\beta, \bar{\beta}) = I^{\bar{\beta}} \left(r_I (1 - \varphi_T^{\bar{\beta}}(\beta)) - L_*(\beta, \bar{\beta}) \right) + (1 - \varphi_T^{\bar{\beta}}(\beta)) c'(\beta), \quad (7.4.7)$$

and

$$L_s(\beta, \bar{\beta}) := \int_s^T c(\beta_u) (1 - \varphi_u^{\bar{\beta}}(\beta)) du.$$

Equation (7.4.7) is used for numerical simulation, in particular in the equilibrium flow descent described in Section 7.2.2.

7.4.4 Proof of the existence of an equilibrium

Before addressing the proof of [Lemma 7.2.2](#) and [Theorem 7.2.3](#), we should recall that the system (7.1.1) has a unique solution for any $\bar{\beta} \in \mathcal{B}$, as demonstrated for example by [Bressan and Rampazzo \[70\] \(1991\)](#), [Dal Maso and Rampazzo \[112\] \(1991\)](#) and [Silva and Vinter \[312\] \(1997\)](#). In particular one can prove that if $\bar{\beta}_n$ is a sequence of functions in $\mathcal{B}$ converging in L^1 (thus also in L^2) to some $\bar{\beta}_\infty$ then the corresponding solution $(S_t^{\bar{\beta}_n}, I_t^{\bar{\beta}_n}, R_t^{\bar{\beta}_n})$ of (7.1.1) converges (for any given t) to $(S_t^{\bar{\beta}_\infty}, I_t^{\bar{\beta}_\infty}, R_t^{\bar{\beta}_\infty})$. Moreover for any $\beta \in \mathcal{B}$ the solution $(S_t^\beta, I_t^\beta, R_t^\beta)$ is a Lipschitz function of time with Lipschitz constant $L_S \leq \beta^\circ + \gamma$.

Proof of Lemma 7.2.2. As in the statement of the Lemma, we consider that $\bar{\beta}$ is fixed. We define the dynamic value function Π , such that its initial value coincides with the value defined by (7.4.5). More precisely, we have at any time $t \in [0, T]$,

$$\Pi_t := \inf_{\beta \in \mathcal{B}} \int_t^T c(\beta_s)(1 - \varphi_s^\beta(\beta))ds + r_I \frac{1 - \varphi_T^\beta(\beta)}{1 - \varphi_t^\beta(\beta)} = \inf_{\beta \in \mathcal{B}} \int_t^T \left(c(\beta_s) + r_I \beta_s I_s^\beta \right) \exp \left(- \int_t^s \beta_u I_u^\beta du \right) ds.$$

This corresponds to the optimal cost of an individual starting at time t in the susceptible class. By invoking standard arguments (see [Bardi and Capuzzo–Dolcetta \[35\] \(1997\)](#)) one can show that Π_t is the (unique) solution of the following Hamilton–Jacobi–Bellman equation:

$$\frac{d\Pi_t}{dt} = H(t, \Pi_t), \quad \text{where } H(t, x) = \min_{y \in [\beta_{\min}, \beta^\circ]} \left\{ c(y) + (r_I - x)y I_t^\beta \right\} \quad (7.4.8)$$

with terminal value $\Pi_T = 0$. Under [Assumption 7.2.1](#), the minimisation in (7.4.8) is straightforward to analyse (and in general is related to the Fenchel transform of the function c). Moreover there exists a unique value y^* such that the minimum is attained, and the following mapping

$$x \mapsto \beta_{\text{opt}}(x, I_t^\beta) = y^* = \arg \min_{y \in [\beta_{\min}, \beta^\circ]} \left\{ c(y) + (r_I - x) \cdot y \cdot I_t^\beta \right\},$$

is Lipschitz continuous in both arguments, with a constant L_H valid for all $x \in [0, r_I]$ and $I_t^\beta \in [0, 1]$. Since I_t^β is Lipschitz, and in particular continuous, we obtain that $\beta_{\text{opt}}(\Pi_t, I_t^\beta)$ is continuous with respect to time and thus Π_t is a classical solution of (7.4.8). It is a Lipschitz function of time with Lipschitz constant $L_\Pi \leq c(\beta^\circ) + r_I \beta^\circ$. $\square$

Proof of Theorem 7.2.3. The proof consists in applying Schauder Theorem, whose rigorous enunciation can be found in the books of [Zeidler \[352, Theorem 1.C\] \(2012\)](#) or [Poznyak \[287, Theorem 18.20\] \(2008\)](#), to the mapping $\mathcal{T}$, in order to prove that it has a fixed point. To this end, we define a subset $\mathcal{D} \subset \mathcal{B}$ consisting of Lipschitz functions with constant $L_H(L_\Pi + \beta^\circ + \gamma)$. Obviously $\mathcal{D}$ is a compact set in $L^2(0, T)$. Previous considerations show that for any $\bar{\beta}$, the corresponding optimal individual choice $\beta_{\text{opt}}(\Pi_t, I_t^\beta)$ is in $\mathcal{D}$. Moreover $\mathcal{D}$ is a compact subset of $L^2(0, T)$. For any sequence $\bar{\beta}^n$ in $\mathcal{B}$ converging in L^2 to some $\bar{\beta}^\infty$, we have that, on the one hand $I_t^{\bar{\beta}^n} \rightarrow I_t^{\bar{\beta}^\infty}$ when $n \rightarrow +\infty$ for any t . On the other hand, the corresponding value functions Π_t^n , which depend on $\bar{\beta}^n$ through $I_t^{\bar{\beta}^n}$, will also converge pointwise. This means that, finally, the optimal values $\beta_{\text{opt}}(\Pi_t^n, I_t^{\bar{\beta}^n})$ will converge to the corresponding limits $\beta_{\text{opt}}(\Pi_t^\infty, I_t^{\bar{\beta}^\infty})$. By the dominated convergence theorem we obtain the L^2 -convergence of $\mathcal{T}(\bar{\beta}^n)$ to $\mathcal{T}(\bar{\beta}^\infty)$ which ends the proof, by application of the Schauder fixed point theorem. $\square$

The government's point of view: between lockdown and testing policy

Il y a rien faire par moments
 Regarder le monde à l'envers
 Croire en tout, en l'éphémère
 Décider d'aller de l'avant

La traversée du désert
 LOUISE ATTAQUE

Introduction

The study of the cost of anarchy in the previous chapter raises the question of incentives: what levers can healthcare authorities use to bring the Nash equilibrium closer to the societal optimal? However, to the best of our knowledge, no real calibration, founded on quantitative criteria, of appropriate incentive policies has been investigated in epidemiological models. There are a certain number of papers studying disease spreading through the lens of either moral hazard or adverse selection. However, these papers are mostly interested in livestock related diseases, where producers naturally have private information on preventive measures they may have adopted, prior to contamination (*ex ante* moral hazard), and may or may not declare whether their herd is infected after contamination (*ex post* adverse selection). Such issues and the design of appropriate policies are considered for instance by Valeeva and Backus [336] (2007), and by Gramig, Horan, and Wolf in [156] (2005), [157] (2009), but the problematic is completely different from the one we are interested in. The present chapter proposes to fulfil this gap by studying how a lockdown policy, seen as a suppression strategy to echo the report by Ferguson et al. [145] (2020), can limit the number of infected people during an epidemic, with uncertainties on the actual number of affected individuals, and on their level of adherence to such a policy.

We consider the control of the COVID-19 pandemic via incentives, through either a stochastic SIS or SIR compartmental models, on the time interval $[0, T]$, where the time horizon $T > 0$ is fixed. When the epidemic is ongoing, the population can reduce interactions between individuals in order to decrease the rate of transmission of the disease, and thus limit its spread. However, this effort comes at a cost for the population. Therefore, the government can put in place a policy of incentives to encourage the lockdown of the population. On the other hand, the government may also choose a testing policy in order to know more precisely the spread of the epidemic within the country, and to isolate infected individuals. This problem perfectly fits with a classical principal-agent problem with moral hazard, as already underlined by Carmona and Wang [87, Section 5] (2018), and boils down to finding a Stackelberg equilibrium between the government proposing a policy to the population, to interact optimally in order to reduce the spread of the disease. We provide numerical examples, as well as a straightforward extension to a stochastic SEIR compartmental model to account for the relatively long latency period of the COVID-19 disease.

Unfortunately, it is very complicated to extract from our model explicit results, especially on the shape of optimal controls. It is therefore necessary to perform numerical simulations, by implementing semi-Lagrangian schemes, proposed for the first time by Camilli and Falcone [74] (1995), using some truncated high-order interpolators, as proposed by Warin [340] (2016). The numerical results for both SIS and SIR models are conclusive, and confirm the relevance of a tax and testing policy to improve the control of an epidemic. First, in the benchmark case, considered as the case where the government does not put in place a specific policy, the efforts of the population are not sufficient to contain the epidemic. In our opinion, this supports the need for incentives. Indeed, if a tax policy is put in place, even in the absence of a specific testing policy, the population is then encouraged to significantly reduce its interactions, thus containing the epidemic until the end of the period under consideration. However, for a fixed containment period, the population relaxes its effort at the very end, leading to a resumption of the epidemic at that point. Finally, if the government also adjusts its testing policy, less effort is required on the population side, so individuals can interact in a way almost usual, and the epidemic is largely contained by the targeted isolation of positively-tested individuals.

8.1 Informal pandemic models and main results

In this section, in order to highlight the results we obtained throughout this chapter, we present our model in an informal way. We thus detail the compartmental epidemic models we consider to represent the spreading of the virus, *i.e.*, either a SIS or a SIR model. Indeed, at the beginning of an epidemic, it is unlikely that decision-makers, let alone the population, will have sufficient data to conclude that infected individuals become immune to the virus in question once they have recovered. This is particularly the case when the virus is new, as in the case of the COVID-19.

With this in mind, we concentrate our attention to two classical models in epidemiology: the SIS model, for the case where infected individuals do not develop an immunity to the disease, and can therefore re-contract it, and the SIR model in the opposite case. Our study is therefore able to deal with both models, and one of the important points will be to compare the results obtained for each of them. We insist on the fact that this entire section is informal, and the reader is referred to [Section 8.3](#) for the rigorous mathematical study.

8.1.1 Deterministic SIS/SIR dynamics with controlled transmission rate

Some parameters will be common in the considered models. In particular, they both involve four non-negative parameters, λ , μ , β and ρ . The parameters λ and μ represent respectively the birth and (natural) death rates among the population, and therefore reflect the demographic dynamics unrelated to the epidemic¹, while ρ represents the death rate associated to the disease. All these parameters are assumed to be constant and exogenous. In most epidemic models, the parameter β , representing the transmission rate of the disease, is also assumed to be constant and exogenous. Nevertheless, in our framework, and similarly to [Chapter 7](#), we will consider that β is endogenous and time-dependent, in order to model the influence that the population can have on this transmission rate.

More precisely, the transmission rate β depends essentially on two factors: the disease characteristics and the contact rate within the population. Although the population cannot modify the disease characteristics,

1. It should be noted that if the length of the epidemic is relatively short in relation to the life expectancy at birth in the concerned country, the demographic dynamics become less relevant and may be dismissed altogether, by setting $\lambda = \mu = 0$. Nevertheless, for the sake of generality, we chose to take these dynamics into account, in order to allow for a straightforward application of our study to other types of epidemics.

each individual can choose (or be incentivised) to reduce his/her contact rate with other individuals in the population. We will thus assume that the population can control the transmission rate β of the disease, by reducing social interactions. With this in mind, we will denote by $\beta^\circ > 0$ the constant initial transmission rate of the disease, *i.e.*, without any control measures or effort from the population. Unfortunately, reducing social interactions is costly for the population. This cost takes into account both the obvious social cost, due to accrued isolation during the lockdown period, and an economic cost (loss of employment due to the lockdown,...). From now on, β will thus denote the time-dependent transmission rate of the disease, controlled by the population. More precisely, we fix some constant $\beta^{\max} \geq \beta^\circ$ representing the maximum rate of interaction that can be considered, and we define $B := [0, \beta^{\max}]$. The process β will be assumed to be B -valued, and we will denote by $\mathcal{B}$ the corresponding set of processes.²

8.1.1.1 No known immunity: the SIS model

One of the two epidemic models we will study is inspired by the well-known SIS (Susceptible–Infected–Susceptible) compartment model, which mainly considers two classes S and I within the population: the class S represents the ‘Susceptible’, while the class I represents the ‘Infected’. In this model, during the epidemic, each individual can be either ‘Susceptible’ or ‘Infected’, and (S_t, I_t) denotes the proportion of each category at time $t \geq 0$. More precisely, as in classical SIS models, we assume that an infected individual returns, after recovery, to the class of susceptible individuals, and can therefore re-contract the disease. We denote by ν the associated rate, which is assumed to be a non-negative constant. We also take into account the demographic dynamics of the population, *i.e.*, births and deaths (related to the considered disease or not), through the previously mentioned parameters λ , μ and ρ . To sum up, the model is represented in Figure 8.1 below, and the (continuous-time) evolution of the disease is described by the following system

$$\begin{cases} dS_t = (\lambda - \mu S_t + \nu I_t - \beta_t S_t I_t) dt, \\ dI_t = -((\mu + \nu + \rho) I_t - \beta_t S_t I_t) dt, \end{cases} \quad \text{for } t \in [0, T], \quad (8.1.1)$$

for an initial compartmental distribution of individuals at time 0, denoted by $(s_0, i_0) \in \mathbb{R}_+^2$, supposed to be known.

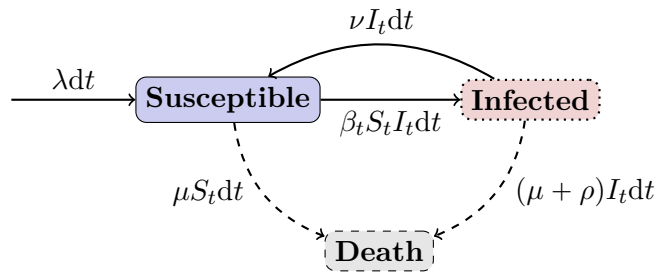


Figure 8.1 – SIS model with demographic dynamics

8.1.1.2 Immunity after infection: the SIR model

The second epidemic model we will focus on is the classical SIR (Susceptible–Infected–Recovered) compartment model. As in the SIS model, the class S represents the ‘Susceptible’ and the class I represents the ‘Infected’. The SIR model is used to describe epidemics in which infected individuals develop immunity

2. We refer to Section 8.3.1.2 for a more precise definition of the set $\mathcal{B}$, taking into account the information flow in the model.

to the virus. This therefore involves a third class, namely R , representing the ‘Recovered’, *i.e.*, individuals who have contracted the disease, are now cured, and therefore immune to the virus under consideration. We denote by γ the recovery rate, which is assumed to be a fixed non-negative constant. Therefore, during the epidemic, each individual can be either ‘Susceptible’ or ‘Infected’ or in ‘Recovery’, and (S_t, I_t, R_t) denotes the proportion of each category at time $t \geq 0$. As in the previously described SIS model, we also take into account the demographic dynamics of the population, through the parameters λ , μ and ρ . To sum up, the epidemic scheme is represented in Figure 8.2 (similar to Figure 7.1 but with birth and death rates), and the (continuous-time) evolution of the disease is described by the following system

$$\begin{cases} dS_t = (\lambda - \mu S_t - \beta_t S_t I_t)dt, \\ dI_t = -((\mu + \rho + \gamma)I_t - \beta_t S_t I_t)dt, \quad \text{for } t \in [0, T], \\ dR_t = (\gamma I_t - \mu R_t)dt, \end{cases} \quad (8.1.2)$$

for a given initial distribution of individuals at time 0, denoted by (s_0, i_0, r_0) .

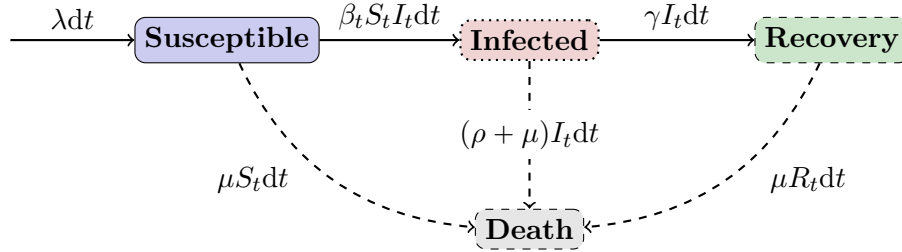


Figure 8.2 – SIR model with demographic dynamics

8.1.2 Uncertainty and testing policy

The use of a deterministic model is widespread and generally justified for most epidemics. However, in our case study, and given what is currently happening in many countries, it appears that the number of infected individuals is not so simple to quantify and estimate. Indeed, without a large testing campaign, it seems complicated to know precisely the proportion of infected in the population. This is particularly true in the case of the COVID-19 epidemic: the absence of symptoms for a significant proportion of infected individuals leads to uncertainty about the actual number of susceptible and infected.

As a consequence, it seems more realistic in our study to turn both the SIS and SIR deterministic controlled models previously described, into stochastic controlled models. Concerning the deterministic part, the dynamics written in the previous systems remain identical. The volatility is partly represented by a fixed and deterministic parameter $\sigma > 0$, and by a time-dependent process α , representing the actions of the government in terms of testing policy. More precisely, in our model, an increase of the number of tests in the population, represented by a decrease of the parameter α , leads to a decrease in the volatility of the processes S and I . Hence, both the population and the government have a clearer view of the number of susceptible and infected, and thus on the epidemic. However, this strategy comes at an economic cost for the government. We then assume that, without any specific effort of the government, α is equal to 1. We also fix a small parameter $\varepsilon \in (0, 1)$ to consider the subset $A := [\varepsilon, 1]$.³ The control α of the government is assumed to be A -valued, and we denote by $\mathcal{A}$ the corresponding set of processes.⁴

3. The lower bound ε is here to insist on the fact that it is not possible, or prohibitively expensive, to cancel completely the uncertainty linked to the disease's dynamics, by taking α to be 0.

4. We refer to Section 8.3.1 for the rigorous definition of the set $\mathcal{A}$.

In addition, the testing policy allows the government to isolate individuals with positive test results. Therefore, the control α also has an impact on the effective transmission rate of the disease. More precisely, without any testing policy, *i.e.* $\alpha = 1$, the government cannot isolate contaminated individuals efficiently. In this case, all infected people spread the disease, and the transmission rate of the virus is given by β . Conversely, if a testing policy is put into place by the government, *i.e.* when $\alpha < 1$, we consider that individuals with positive test results can be isolated, and as a consequence less infected people spread the disease. In this case, the effective transmission rate is lower.

Inspired by the continuous-time SIS process considered by Gray et al. [158] (2011), as a solution to a bi-dimensional SDE driven by a Brownian motion, we define a stochastic version of the SIS model previously defined by (8.1.1): the number of infected, and therefore the number of susceptible, are impacted at each time t by a Brownian motion W_t . More precisely, the dynamic of the epidemic is now given by the following system

$$\begin{cases} S_t = s_0 + \int_0^t (\lambda - \mu S_s + \nu I_s - \beta_s \sqrt{\alpha_s} S_s I_s) ds + \int_0^t \sigma \alpha_s S_s I_s dW_s, \\ I_t = i_0 - \int_0^t ((\mu + \nu + \rho) I_s - \beta_s \sqrt{\alpha_s} S_s I_s) ds - \int_0^t \sigma \alpha_s S_s I_s dW_s, \end{cases} \quad t \in [0, T]. \quad (8.1.3)$$

Similarly to the SIS model, we consider that the deterministic model SIR described by (8.1.2) is also subject to a noise in the estimation of the proportion of susceptible and infected individuals. Inspired by the stochastic SIR model in Tornatore, Buccellato, and Vetro [333] (2005), the dynamic of the epidemic is now given by the following system

$$\begin{cases} S_t = s_0 + \int_0^t (\lambda - \mu S_s - \beta_s \sqrt{\alpha_s} S_s I_s) ds + \int_0^t \sigma \alpha_s S_s I_s dW_s, \\ I_t = i_0 - \int_0^t ((\mu + \gamma + \rho) I_s - \beta_s \sqrt{\alpha_s} S_s I_s) ds - \int_0^t \sigma \alpha_s S_s I_s dW_s, \\ R_t = r_0 + \int_0^t (\gamma I_s - \mu R_s) ds, \end{cases} \quad t \in [0, T]. \quad (8.1.4)$$

Note that the proportion R of individuals in recovery is also uncertain, but only through its dependency with respect to I . More precisely, we assume that there is no uncertainty on the recovery rate γ , implying that if the proportion of infected individual is perfectly known, the proportion of recovered is also known without uncertainty. This modelling choice is consistent with most stochastic SIR models, and emphasises that the major uncertainty in the current epidemic is related to the non-negligible proportion of (nearly) asymptomatic individuals. Indeed, an asymptomatic individual may be mis-classified as susceptible. This is also the case for an individual in recovery, who has been asymptomatic, but the uncertainty is solely related to the fact that he was not classified as infected when he actually was.

In order to provide a unified framework for both the SIS and SIR models, and simplify the presentation, we will consider the following dynamic for the epidemic

$$\begin{cases} S_t = s_0 + \int_0^t (\lambda - \mu S_s + \nu I_s - \beta_s \sqrt{\alpha_s} S_s I_s) ds + \int_0^t \sigma \alpha_s S_s I_s dW_s, \\ I_t = i_0 - \int_0^t ((\mu + \nu + \rho + \gamma) I_s - \beta_s \sqrt{\alpha_s} S_s I_s) ds - \int_0^t \sigma \alpha_s S_s I_s dW_s, \\ R_t = r_0 + \int_0^t (\gamma I_s - \mu R_s) ds, \end{cases} \quad t \in [0, T]. \quad (8.1.5)$$

Notice that to recover the SIS model, one has to set $\gamma = 0$, and conversely, $\nu = 0$ to obtain the SIR model.

Remark that in the three previous systems, we do not assume that the impact of the testing policy on the volatility of S and I , and on the transmission rate has the same magnitude. Indeed, we expect a lower reduction of the effective transmission rate, compared to the volatility reduction for a given policy α . This should be understood as a manifestation of the fact that it is easier to reduce the uncertainty on the number of infected people, compared to actually isolate individuals who have been identified as infected. We thus assume a linear dependency with respect to α for the volatility of both S and I , while the effective transmission rate is chosen equal to $\beta\sqrt{\alpha}$, so that the number of infected people spreading the disease at time t is given by $\sqrt{\alpha}I_t$.

8.1.3 Incentive policies

In addition to the choice of a testing policy, the government can also incentivise the population to limit their social interactions, in order to decrease the transmission rate of the disease, by introducing financial penalties. More precisely, at time 0, the government informs the population about its testing policy $\alpha \in \mathcal{A}$, as well as its fine policy $\chi \in \mathfrak{C}$ ⁵, for the lockdown period $[0, T]$. Knowing this, the population will choose an interacting behaviour according to the following rules

- (i) an increase in the tax lowers its utility;
- (ii) an increase in the level of interaction (up to a specific threshold, namely β°) improves its well-being;
- (iii) the population is scared of having a large number of infected people.

8.1.3.1 Population optimisation problem

We stylise the previous facts by considering that the population solves the following optimal control problem, for a given pair $(\alpha, \chi) \in \mathcal{A} \times \mathfrak{C}$

$$V_0^A(\alpha, \chi) := \sup_{\beta \in \mathcal{B}} \mathbb{E} \left[\int_0^T u(t, \beta_t, I_t) dt + U(-\chi) \right], \quad (8.1.6)$$

where $u : [0, T] \times B \times \mathbb{R}_+ \rightarrow \mathbb{R}$ and $U : \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions in all their arguments, and U is a bijection from $\mathbb{R}$ to $\mathbb{R}$. Given a pair (α, χ) , the set of optimal contact rates β will be denoted $\mathcal{B}^*(\alpha, \chi)$.⁶

The functions u and U should be interpreted as functions translating respectively the actual value of interaction from the point of view of the population, and the disutility associated to the fine. More precisely, the function U is assumed to be an increasing function, according to (i) above. Concerning the function u , it should be non-decreasing in the second variable up to β° , and then non-increasing, modelling (ii) above. On the other hand, the function u is assumed to be non-increasing with respect to the proportion of infected individual in the population. In particular, this allows to take into account both the fear of the infection (as mentioned in (iii) above) and the cost that is incurred if an individual is infected. From the population's point of view, this cost is not actually expressed in terms of money, but mainly corresponds to medical side effects or general morbidity. We refer to Anand and Hanson [22] (1997), Zeckhauser and Shepard [351] (1976) and Sassi [304] (2006), for an introduction to QALY/DALY (Quality- and Disability-Adjusted Life-Year), the generic measures of disease burden used in economic evaluation to assess the value of medical interventions.

We choose to normalise the utility of the population to zero when there is no epidemic. In other words, if $i_0 = 0$, then $I_t = 0$ for all $t \in [0, T]$, and thus the utility of the population should be equal to 0. With

5. See Section 8.3.1.3 for a rigorous definition of the set $\mathfrak{C}$ of admissible fine policies.

6. The reader is referred to Section 8.3.1.3, and more precisely to (8.3.8), for a rigorous definition of $\mathcal{B}^*$.

this in mind, we assume that $U(0) = 0$, which means that without a fine, the population does not suffer any disutility. Moreover, when there is no epidemic, the population should not reduce its social interaction, meaning that for all $t \in [0, T]$, $\beta_t = \beta^\circ$. This leads us to assume that

$$u(t, \beta^\circ, 0) = 0, \text{ for all } t \in [0, T].$$

Example 8.1.1 (Utility functions for the population). *As previously mentioned, the function $u : [0, T] \times B \times \mathbb{R}_+ \rightarrow \mathbb{R}$ represents the social cost of lockdown policy, and thus should capture the two rules (ii) and (iii), as well as satisfy $u(t, \beta^\circ, 0) = 0$ for all $t \in [0, T]$. In particular, we could consider a separable utility function u of the form*

$$u(t, b, i) := -u_I(t, i) - u_\beta(t, b), \quad (t, b, i) \in [0, T] \times B \times \mathbb{R}_+, \quad (8.1.7)$$

where the function $u_I : \mathbb{R}_+ \rightarrow \mathbb{R}$ represents the fear of the infection for the population. In order to choose this function, we would like model the fact that when the proportion of infected is close to 0, the population underestimates the epidemic, while when this proportion becomes large, the population becomes irrationally afraid. Therefore, we can consider a function independent of t , and take

$$u_I(t, i) = c_p i^3, \quad (t, i) \in [0, T] \times \mathbb{R}_+, \text{ for some } c_p \geq 0. \quad (8.1.8)$$

Next, the function u_β represents the sensitivity of the population with respect to the initial transmission rate β° of the disease, i.e., without any lockdown measure. During the lockdown period, the social cost of distancing measures becomes more and more important for the population, and we thus expect the cost u_β to also reflect this sensitivity with respect to time. More precisely, we can consider two particular functions, for some $\eta_p > 0$, to model these stylised facts:

- (i) either $u_\beta(t, b) := \eta_p \psi(t) |\beta^\circ - b|^2 / 2$, $(t, b) \in [0, T] \times B$ to insist on the fact that it is costly for the population to deviate from its usual contact rate, i.e. its level of interactions in an epidemic-free environment, inducing the natural transmission rate of the disease β° ;
- (ii) or $u_\beta(t, b) := \eta_p \psi(t) ((\beta^\circ)^{\eta_p} b^{-\eta_p} - 1)$, $(t, b) \in [0, T] \times B$, to emphasise that it is very costly, if not impossible, to reduce the level of interaction between the population to zero, and thus to prevent the transmission of the disease.

In the two previous forms, ψ is a non-decreasing and convex $\mathbb{R}^+$ -valued function, to represent the increasing aversion to the lockdown for the population as time passes. Indeed, the longer the lockdown period, the more sensitive the population is to the social cost of distancing measures. In other words, deviating from its usual level of interaction entails a social cost to the population that is greater as the duration increases. More precisely, we can imagine that the function ψ takes the form

$$\psi(t) := e^{\tau_p t}, \quad t \in [0, T], \text{ for some } \tau_p > 0.$$

Finally, concerning the utility of the Agent with respect to the tax χ , we choose a mixed CARA-risk-neutral utility function

$$U(x) := \frac{1 - e^{-\theta_p x}}{\theta_p} + \phi_p x, \quad x \in \mathbb{R},$$

where $\theta_p > 0$ is the risk-aversion of the population, and $\phi_p > 0$, so that $U(0) = 0$, and U is an increasing and strictly concave bijection from $\mathbb{R}$ to $\mathbb{R}$. For later use, we record that the inverse of U , denoted by $U^{(-1)}$, has an explicit form (see Corless et al. [103] (1996) for more details on the LambertW function) :

$$U^{(-1)}(y) := \frac{1}{\theta_p} \text{LambertW}\left(\phi_p^{-1} e^{\frac{1-\theta_p y}{\phi_p}}\right) + \frac{\theta_p y - 1}{\theta_p \phi_p}, \quad y \in \mathbb{R}.$$

8.1.3.2 A benchmark case: optimal interaction without taxation and testing policies

Before turning to the principal–agent problem itself, we aim at solving (8.3.7) for $\alpha = 1$ fixed, and $\chi = 0$, *i.e.* without tax and testing policy. Similar problems have been studied for instance by [Kandhway and Kuri \[200\] \(2014\)](#). Mathematically speaking, the optimisation problem faced by the population without contract is informally given by

$$V_0^A(1, 0) = \sup_{\beta \in \mathcal{B}} \mathbb{E} \left[\int_0^T u(t, \beta_t, I_t) dt \right], \quad (8.1.9)$$

since we assumed $U(0) = 0$. Notice that by assumption on the function u , in the no–epidemic case, *i.e.*, if $i_0 = 0$, the population should not make any effort, and therefore the optimal contact rate β over the period $[0, T]$ is equal to β° . We thus consider in the following a fixed initial condition $(s_0, i_0) \in (\mathbb{R}_+^*)^2$, which implies that for all $t \in [0, T]$, both S_t and I_t are (strictly) positive.

Without tax, the population's problem boils down to a standard control problem, with two state variables S and I . We will give the associated PDE in [Section 8.1.4.1](#) below.

8.1.3.3 Relevant form of tax policy

One of the main theoretical result of our study is given by [Theorem 8.3.7](#). Informally, this theorem states that given an admissible contract, namely a testing policy $\alpha \in \mathcal{A}$ and a tax $\chi \in \mathfrak{C}$, there exist a unique Y_0 and Z such that the following representation holds

$$U(-\chi) = Y_0 - \int_0^T \left(Z_t(\mu + \nu + \rho + \gamma)I_t + u(t, \beta_t^*, I_t) - \beta_t^* \sqrt{\alpha_t} S_t I_t Z_t \right) dt - \int_0^T Z_t dI_t, \quad (8.1.10)$$

where β^* is the unique optimal contact rate for the population. More precisely, we can state that for (Lebesgue–almost every) $t \in [0, T]$, $\beta_t^* := b^*(t, S_t, I_t, Z_t)$ is the maximiser of the function $b \in B \mapsto u(t, b, I_t) - b S_t I_t Z_t$. Under some assumptions for existence and smoothness of the inverse of the function U , the previous equation gives a representation for the tax χ .

Based on (8.1.10), the tax χ will be indexed on the variation of the proportion of infected I , through the stochastic integral $\int_0^\cdot Z_s dI_s$, and not on the variation of susceptible S (though it is indexed on S through the dt integral). Nevertheless, using the link between the dynamics of I and S , we can write a representation equivalent to (8.1.10)

$$U(-\chi) = Y_0 - \int_0^T \left(u(t, \beta_t^*, I_t) - \beta_t^* \sqrt{\alpha_t} S_t I_t Z_t - Z_t(\lambda - \mu S_t + \nu I_t) \right) dt + \int_0^T Z_t dS_t. \quad (8.1.11)$$

Through this equation, we can state that the tax can be indexed on S instead of I . Therefore, given the strong link between the number of Susceptible and the number of Infected, it is sufficient to index the tax on only one of these two quantities, and one can therefore choose indifferently to index the tax χ on the variations of I or S .

The reader familiar with contract theory in continuous–time will have noticed that the previous representation for the tax χ is not exactly the expected one. Indeed, referring for instance to [Cvitanić, Possamaï, and Touzi \[111\] \(2018\)](#) the contract is usually the sum of three components:

- (i) a constant similar to Y_0 , chosen by the Principal in order to satisfy the participation constraint of the Agent;
- (ii) an integral with respect to time $t \in [0, T]$ of the Agent's Hamiltonian;
- (iii) a stochastic integral with respect to the controlled process, *i.e.*, in our framework, (S, I) .

Neither the representation (8.1.10) nor (8.1.11) are, *a priori*, of this form. This difference is due to the fact that the dynamics of (S, I) is degenerated. More precisely, there is a fundamental structure condition in [111] requiring that the drift of the output process belongs to the range of its volatility. In words, defining for $(s, i) \in \mathbb{R}_+^2$ and $(a, b) \in A \times B$,

$$\sigma(s, i, a) := \sigma a s i \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \text{ and } \lambda(s, i, b, a) := \begin{pmatrix} \lambda - \mu s + \nu i + b\sqrt{a} s i \\ -(\mu + \nu + \rho + \gamma)i + b\sqrt{a} s i \end{pmatrix},$$

the condition assumed in [111, Equation (2.1)] implies that

$$\lambda(s, i, b, a) \propto \sigma(s, i, a), \text{ for any } (s, i, a, b) \in \mathbb{R}_+^2 \times A \times B,$$

which is obviously impossible here. Therefore, we cannot use directly any existing result in the literature, and we should not expect, *a priori*, to be able to obtain a contract representation similar to the one in [111], nor that the so-called dynamic programming approach will prove effective in our case. Indeed, as far as we know, such degenerate models have only been tackled using the stochastic maximum principle, see Hu, Ren, and Touzi [182] (2019).

However, and somewhat surprisingly, the form we exhibit for the tax is actually strongly related to the usual representation. The reason for this is twofold. First, up to the sign, the volatilities in the dynamics of both S and I are exactly the same. Second, both the processes S and I are driven by the same Brownian motion W . Therefore, intuitively, in order to provide incentives to the population, the government can afford to index the tax on only one of the two processes. Mathematically, it is also straightforward to show that given an arbitrary decomposition of the process Z in (8.1.10) of the form $Z =: Z^s - Z^i$, we have

$$U(-\chi) = Y_0 - \int_0^T H(t, S_t, I_t, Z_t^s, Z_t^i) dt + \int_0^T Z_t^s dS_t + \int_0^T Z_t^i dI_t,$$

where H is the Hamiltonian of the population, and this is exactly the general form provided in [111]. The main difference is that in [111], Z^s and Z^i are both uniquely given, while in our representation, only their difference actually matters. Hence, there is an infinite number of possible representations for the tax χ in our degenerate model.

8.1.3.4 Government optimisation problem

As already explained, the government can choose the tax $\chi \in \mathfrak{C}$ paid by the population together with the testing policy $\alpha \in \mathcal{A}$. It aims at minimising the number of infected people until the end of the quarantine period, and we informally write its minimisation problem as

$$V_0^P := \sup_{(\alpha, \chi) \in \Xi} \sup_{\beta \in \mathcal{B}^*(\alpha, \chi)} \mathbb{E} \left[\chi - \int_0^T (c(I_t) + k(t, S_t, I_t, \alpha_t)) dt \right], \quad (8.1.12)$$

where $c : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and $k : [0, T] \times \mathbb{R}_+ \times \mathbb{R}_+ \times A \rightarrow \mathbb{R}$ are continuous functions. The function c denotes the instantaneous cost implied by the proportion of infected people during the quarantine period, and is thus assumed to be non-decreasing, while the function k represents the cost of the testing policy.

In addition, the set Ξ takes into account the so-called participation constraint for the population. This means that the government is benevolent, which translates into the fact that it has committed to ensure that the living conditions of the population do not fall below a minimal level. Mathematically, the government can only implement policies $(\alpha, \chi) \in \mathcal{A} \times \mathfrak{C}$ such that $V_0^A(\alpha, \chi) \geq \underline{v}$, where the minimal utility $\underline{v} \in \mathbb{R}$ is given. This is what is encoded in the set Ξ .

Example 8.1.2 (Cost functions for the government). *The function c can be linear to represent the cost per unit of infected people, or quadratic to highlight the cost induced by the saturation of the health-care system when the number of infected is too high. Typically, we would take $c(i) := c_g(i + i^2)$, $i \in \mathbb{R}_+$, for some $c_g > c_p$, to take into account that the marginal cost linked to the proportion of infected people in the population is higher for the government than for the population itself. We also point out that we choose a linear-quadratic cost in i for the government, while we took a purely cubic one for the population. This choice emphasises that, on the one hand, even for a small number of infected, the marginal cost faced by the government is not close to 0 (hence the linear term). On the other hand, the population is more likely to incur very high and lasting costs (loss of revenues, employment, life,...) when the disease spreads uncontrollably, when compared to the government which mostly faces pecuniary costs.*

Concerning the cost function k associated with the testing policy, we recall that $\alpha = 1$ means no testing policy, so no cost for the government. As soon as α is different from 1, the cost has to be higher. We may consider the following function for the testing policy k , for some $\eta_g > 0$ and $\kappa_g > 0$,

$$k(t, s, i, a) := \frac{\kappa_g}{\eta_g} (a^{-\eta_g} - 1), \quad (t, s, i, a) \in [0, T] \times \mathbb{R}_+^2 \times A. \quad (8.1.13)$$

This function highlights the fact that it is very costly, if not impossible, to eliminate the uncertainty associated with the epidemic. Indeed, in a relatively populous country, it seems impossible to develop a testing policy sufficient to know exactly the proportion of susceptible and infected people.

8.1.3.5 A second benchmark case: the first-best

Another interesting case to compare our results with, corresponds to the so-called first-best case. This is the best-possible scenario where the government can enforce whichever interaction rate $\beta \in \mathcal{B}$ it desires, and simply has to satisfy the participation constraint of the population. From the practical point of view, this could correspond to a situation where the government would be able to track every individual and force them to stop interacting. The problem faced by the government is then

$$V_0^{\text{P,FB}} := \sup_{(\alpha, \chi, \beta) \in \mathcal{A} \times \mathcal{C} \times \mathcal{B}} \mathbb{E} \left[\chi - \int_0^T (c(I_t) + k(t, S_t, I_t, \alpha_t)) dt \right], \quad (8.1.14)$$

such that $\mathbb{E} \left[\int_0^T u(t, \beta_t, I_t) dt + U(-\chi) \right] \geq \underline{v}$.

8.1.4 Main results and comparison

In this section, we present the main theoretical results obtained when the dynamic of the epidemic is given by (8.1.5). Recall that, in order to consider the SIS or the SIR model, one has to set respectively $\gamma = 0$ or $\nu = 0$.

8.1.4.1 The benchmark case

As mentioned in Section 8.1.3.2, the benchmark problem is a standard Markovian stochastic control problem, whose Hamiltonian is defined, for $t \in [0, T]$, $(s, i) \in (\mathbb{R}_+^*)^2$, $p := (p_1, p_2) \in \mathbb{R}^2$ and $M \in \mathbb{S}^2$ by

$$H^A(t, s, i, p, M) := \sup_{b \in B} \{ -bsi(p_1 - p_2) + u(t, b, i) \} + (\lambda - \mu s + \nu i)p_1 - (\mu + \nu + \rho + \gamma)ip_2 + \frac{1}{2} |\sigma|^2 |si|^2 (M_{11} - 2M_{12} + M_{22}). \quad (8.1.15)$$

We then have the natural identification $V_0^A(1, 0) = v(0, s_0, i_0)$, where v solves the associated Hamilton–Jacobi–Bellman (HJB for short) equation

$$-\partial_t v(t, s, i) - H^A(t, s, i, \nabla v, \nabla^2 v) = 0, \quad (t, s, i) \in \mathcal{D}, \quad v(T, s, i) = 0, \quad (s, i) \in \mathcal{D}_T, \quad (8.1.16)$$

where

$$\mathcal{D} := \{(t, s, i) \in [0, T) \times (\mathbb{R}_+^*)^2 : 0 < s + i \leq F(t, s_0, i_0)\}, \quad \mathcal{D}_T := \{(s, i) \in \mathbb{R}_+^2 : 0 < s + i < F(T, s_0, i_0)\},$$

for a particular function F defined by (8.3.4) in Section 8.3.1. Note that if we consider separable utilities with the form

$$u(t, b, i) := -u_I(t, i) - \frac{\eta_P \psi(t)}{2} |\beta^\circ - b|^2, \quad (8.1.17a)$$

$$\text{or } u(t, b, i) := -u_I(t, i) - \eta_P \psi(t) ((\beta^\circ)^{\eta_P} b^{-\eta_P} - 1), \quad (8.1.17b)$$

the maximiser of the Hamiltonian is given by $b^\circ(s, i, p_1 - p_2)$ where the map $b^\circ : (\mathbb{R}_+^*)^2 \times \mathbb{R} \rightarrow B$ is defined for all $(s, i, z) \in (\mathbb{R}_+^*)^2 \times \mathbb{R}$ by

$$b^\circ(s, i, z) := \begin{cases} \beta^{\max}, & \text{if } z < -\frac{\eta_P \psi(t)}{si} (\beta^{\max} - \beta^\circ), \\ \beta^\circ - \frac{siz}{\eta_P \psi(t)}, & \text{if } z \in \left[-\frac{\eta_P \psi(t) (\beta^{\max} - \beta^\circ)}{si}, \frac{\beta^\circ \eta_P \psi(t)}{si} \right], \\ 0, & \text{if } z > \frac{\beta^\circ \eta_P \psi(t)}{si}, \end{cases} \quad (8.1.18a)$$

$$\text{or } b^\circ(s, i, z) := \begin{cases} \beta^{\max}, & \text{if } z < \frac{\eta_P \psi(t) (\beta^\circ)^{\eta_P}}{si (\beta^{\max})^{1+\eta_P}}, \\ \left(\frac{\eta_P \psi(t) (\beta^\circ)^\eta}{siz} \right)^{\frac{1}{1+\eta}}, & \text{otherwise.} \end{cases} \quad (8.1.18b)$$

In particular, the optimal interaction rate is given by $\beta_t^B = b^\circ(S_t, I_t, (\partial_s v - \partial_i v)(t, S_t, I_t))$, $t \in [0, T]$.

8.1.4.2 The first–best case

To find the optimal interaction rate $\beta \in \mathcal{B}$, as well as the optimal contract $(\alpha, \chi) \in \mathcal{A} \times \mathfrak{C}$, in the first–best case, one has to solve the government’s problem defined by (8.1.14). Mathematical details are postponed to Section 8.3.3.3, but we present here an overview of the main results.

To take into account the inequality constraint in the definition of $V_0^{P,FB}$, one has to introduce the associated Lagrangian. Given a Lagrange multiplier $\varpi > 0$, we first remark that the optimal tax is constant and given by

$$\chi^*(\varpi) := -(U')^{(-1)}\left(\frac{1}{\varpi}\right). \quad (8.1.19)$$

Then, defining for any $\varpi > 0$

$$\bar{V}_0(\varpi) := \sup_{(\alpha, \beta) \in \mathcal{A} \times \mathcal{B}} \mathbb{E} \left[\int_0^T (\varpi u(t, \beta_t, I_t) - c(I_t) - k(t, S_t, I_t, \alpha_t)) dt \right], \quad (8.1.20)$$

we have

$$V_0^{P,FB} = \inf_{\varpi > 0} \left\{ \chi^*(\varpi) + \varpi (U(-\chi^*(\varpi)) - \underline{v}) + \bar{V}_0(\varpi) \right\}. \quad (8.1.21)$$

Note that $\bar{V}_0(\varpi)$ is the value function of a standard stochastic control problem, and therefore we expect to have $\bar{V}_0(\varpi) = v^\varpi(0, s_0, i_0)$, for a function $v^\varpi : [0, T] \times \mathbb{R}_+^2 \rightarrow \mathbb{R}$ solution to the following HJB PDE

$$-\partial_t v^\varpi(t, s, i) + c(i) - (\lambda - \mu s + \nu i) \partial_s v^\varpi + (\mu + \nu + \rho + \gamma) i \partial_i v^\varpi - \mathcal{H}^\varpi(t, s, i, \nabla v^\varpi, \nabla^2 v^\varpi) = 0,$$

for $(t, s, i) \in \mathcal{D}$, and with terminal condition $v^\varpi(T, s, i) = 0$ for $(s, i) \in \mathcal{D}_T$, where the Hamiltonian is defined, for $t \in [0, T]$, $(s, i) \in (\mathbb{R}_+^*)^2$, $p := (p_1, p_2) \in \mathbb{R}^2$ and $M \in \mathbb{S}^2$ by

$$\mathcal{H}^\varpi(t, s, i, p, M) := \sup_{(a, b) \in A \times B} \left\{ \varpi u(t, b, i) - b s i \sqrt{a} (p_1 - p_2) - k(t, s, i, a) + \frac{1}{2} |\sigma s i a|^2 (M_{11} - 2M_{12} + M_{22}) \right\}.$$

In particular, if we consider separable utilities with the forms (8.1.17), for a given testing policy $\alpha \in \mathcal{A}$ and a Lagrange multiplier $\varpi > 0$, the optimal interaction rate is given for all $t \in [0, T]$ by $\beta_t^\varpi = b^\varpi(S_t, I_t, \partial v^\varpi(t, S_t, I_t), \alpha_t)$, where, recalling that b° is defined by (8.1.18),

$$b^\varpi(s, i, p, a) := b^\circ(s, i, \sqrt{a}(p_1 - p_2)/\varpi), \quad \text{for all } (s, i, p, a) \in (\mathbb{R}_+^*)^2 \times \mathbb{R}^2 \times A.$$

8.1.4.3 The general case

Thanks to the reasoning developed in Section 8.3, we are able to determine the optimal design of the fine policy, the optimal testing policy, as well as the optimal effort of the population.

First, as informally explained in Section 8.1.3.3, to implement a tax policy $\chi \in \mathfrak{C}$, the government only needs to choose a constant Y_0 and a process Z . Given these two parameters, we can state that the optimal contact rate for the population is defined by $\beta_t^* := b^*(t, S_t, I_t, Z_t, \alpha_t)$, such that the function $b \in B \mapsto u(t, b, I_t) - b \sqrt{\alpha_t} S_t I_t Z_t$ is maximised for (Lebesgue-almost every) $t \in [0, T]$.

Remark 8.1.3. Note that if we consider separable utilities with the forms (8.1.17), the maximiser b^* is defined for all $(t, s, i, z, a) \in [0, T] \times (\mathbb{R}_+^*)^2 \times \mathbb{R} \times A$ by $b^*(t, s, i, z, a) := b^\circ(s, i, z \sqrt{a})$, recalling that b° is defined by (8.1.18).

It thus remains to solve the government's problem in order to determine the optimal choice of Y_0 and Z . The reader is referred to Section 8.3.3 for the rigorous government's problem, but, to summarise the results, the optimal process Z as well as the optimal testing policy α are determined so as to maximise the government's Hamiltonian, given by

$$\begin{aligned} H^P(t, s, i, p, M) = & \sup_{z \in \mathbb{R}, a \in A} \left\{ b^*(t, s, i, z, a) \sqrt{a} s i (p_2 - p_1) + \frac{1}{2} |\sigma a s i|^2 f(z, M) - k(t, a, s, i) - u^*(t, s, i, z, a) p_3 \right\} \\ & + (\lambda - \mu s + \nu i) p_1 - (\mu + \nu + \rho + \gamma) i p_2 - c(i), \end{aligned}$$

for $(t, s, i, p, M) \in [0, T] \times \mathbb{R}_+^2 \times \mathbb{R}^3 \times \mathbb{S}^3$, where, in addition for $z \in \mathbb{R}$,

$$f(z, M) := M_{11} - 2M_{12} + M_{22} - 2z(M_{23} - M_{13}) + |z|^2 M_{33}, \quad \text{and } u^*(t, s, i, z, a) := u(t, b^*(t, s, i, z, a), i).$$

Finally, it remains to solve numerically the following HJB equation, for all $t \in [0, T]$ and $x := (s, i, y) \in \mathbb{R}^3$

$$-\partial_t v(t, x) - H^P(t, x, \nabla_x v, \nabla_x^2 v) = 0, \quad (t, x) \in \mathcal{O}, \quad v(T, x) = -U^{(-1)}(y), \quad x \in \mathcal{O}_T, \quad (8.1.22)$$

where the natural domain over which the above PDE must be solved is

$$\begin{aligned} \mathcal{O} &:= \{(t, s, i, y) \in [0, T] \times \mathbb{R}_+^2 \times \mathbb{R} : 0 < s + i < F(t, s_0, i_0)\}, \\ \text{and } \mathcal{O}_T &:= \{(s, i, y) \in \mathbb{R}_+^2 \times \mathbb{R} : 0 < s + i < F(T, s_0, i_0)\}. \end{aligned}$$

8.2 Numerical experiments

The results presented in [Section 8.1.4](#) are quite theoretical: except for the optimal transmission rate, it is complicated to obtain explicit formulae for the other variables sought, in particular for the optimal testing policy α , even if we consider separable utility functions as in [\(8.1.17\)](#). It is therefore necessary to perform numerical simulations to evaluate the optimal efforts of the population and the government, as well as the optimal tax policy. Given the similarities in the results between the SIS and SIR models, only those related to the SIR model are presented in this section, while the results for the SIS model are postponed to [Section 8.5](#).

8.2.1 Choice of parameters

The following numerical experiments are implemented using the utility and cost functions respectively mentioned in [Example 8.1.1](#) for the population and in [Example 8.1.2](#) for the government. To summarise, we choose for the population [\(8.2.1a\)](#) and for the government [\(8.2.1b\)](#):

$$u(t, b, i) := -c_p i^3 - \frac{\eta_p \psi(t)}{2} |\beta^\circ - b|^2, \text{ with } \psi(t) := e^{\tau_p t}, \text{ and } U(x) := \frac{1 - e^{-\theta_p x}}{\theta_p} + \phi_p x; \quad (8.2.1a)$$

$$k(t, s, i, a) := \kappa_g (a^{-\eta_g} - 1), \text{ and } c(i) := c_g (i + |i|^2), \quad (8.2.1b)$$

for all $(t, x, s, i) \in [0, T] \times \mathbb{R} \times \mathbb{R}_+^2$, $a \in A := [\varepsilon, 1]$ and $b \in B := [0, \beta^{\max}]$. These functions require to specify several parameters, provided in [Table 8.1](#).

Parameters	c_p	η_p	θ_p	τ_p	ϕ_p	$\beta^{\max}$
Values	0.5	1	4	0	0.5	0.2

(a) Characteristics of the population

Parameters	κ_g	c_g	η_g	ε
Values	0.001	1	0.01	0.01

(b) Characteristics of the government

Table 8.1 – Set of parameters for cost and utility functions

	T	(s_0, i_0, r_0)	(λ, μ)	ρ	ν	γ	σ	β°
SIR model (8.1.4)	200	$(0.99984, 1.07 \times 10^{-4}, 5.3 \times 10^{-5})$	$(0, 0)$	0.01	0	0.1	0.1	0.2

Table 8.2 – Set of parameters for the simulations of SIR model

In addition, the set of parameters used for the simulations of the epidemic dynamics given by [\(8.1.5\)](#) are provided in [Table 8.2](#) and are inspired by those chosen in [Chapter 7, Table 7.1](#) (p. 226). Recall that the parameter β° denotes the usual contact rate within the population, before the beginning of the lockdown. In other words, β° represents the initial and effective transmission rate of the disease, without any specific effort of the population. The associated reproduction number $\mathcal{R}_0$, commonly defined by $\mathcal{R}_0 := \beta^\circ / (\nu + \gamma)$ in the literature on epidemic models, is equal to 2.0, and is thus in the confidence interval of available data, see for example [Li et al. \[232\] \(2020\)](#). Recall that the parameters λ and μ represent respectively the birth and (natural) death rates among the population, and therefore reflect the demographic dynamics unrelated to the epidemic, while ρ represents the death rate associated to the disease. To simplify, and since the duration of the COVID-19 epidemic should be relatively short in comparison to the life expectancy at birth, we choose to disregard the demographic dynamics by setting $\lambda = \mu = 0$. In contrast, we set $\rho = 1\%$, since the mortality associated with the disease appears to be significant. Finally, recall that the parameters

ν and γ correspond respectively to the recovery rates in the SIS and SIR models, *i.e.*, the inverse of the virus contagious period. Since we want to consider here a SIR dynamic, we let $\nu = 0$ and $\gamma = 0.1$, to account for the average 10-day duration of COVID-19 disease.

When not explicitly specified, the simulations presented in this section are performed with the sets of parameters described in Tables 8.1 and 8.2. However, the parameters used to describe in particular the utility and cost functions of the population and government are set in a relatively arbitrary way. To actually estimate these parameters would require an extensive sociological and economic study, that we do not presume to be able to perform at this stage, and linking, for example, the population's costs to the DALY/QALY concepts already mentioned, and the government's costs to those of the health care system and its possible congestion. Moreover, there is considerable uncertainty in the medical literature on the choice of all parameters used to describe the dynamics of the epidemic, in particular because the COVID-19 is a new type of virus and therefore we do not have sufficient hindsight to reliably estimate its characteristics. It will therefore be necessary to study the sensitivity of the results obtained with respect to the selected parameters.

Finally, it should be remembered that, in contrast to usual principal-agent problems, the government implements a mandatory tax, which the population cannot refuse. Nevertheless, we consider that the government is benevolent, in the sense that it still wishes to ensure that the utility of the population remains above a certain level, denoted by $\underline{v}$. To fix this level, we assume that the government wants to ensure at the very least to the population the same living conditions it would have had in the event of an uncontrolled epidemic, *i.e.*, without any effort on the part of neither the population nor the government, meaning $\beta = \beta^\circ$, $\alpha = 1$ and $\chi = 0$. Mathematically, this is equivalent to the following, since u is separable of the form (8.1.7), such that for all $t \in [0, T]$, $u_\beta(t, \beta^\circ) = 0$ and u_I satisfies (8.1.8)

$$\underline{v} := \mathbb{E}^{\mathbb{P}^{1, \beta^\circ}} \left[\int_0^T u(t, \beta^\circ, I_t) dt + U(0) \right] = -c_p \mathbb{E}^{\mathbb{P}^{1, \beta^\circ}} \left[\int_0^T I_t^3 dt \right], \quad (8.2.2)$$

Notice that the reservation utility $\underline{v}$ is given by the worst case scenario, without any sanitary precaution neither from the population nor from the government. This level may be judged too severe, and one could consider a model where the government is more benevolent. In particular, one could set $\underline{v}$ closer to the value that the population achieves in the benchmark case, *i.e.*, when it makes optimal efforts in the absence of government policy. Nevertheless, the value of $\underline{v}$ should not be of major importance, since it should only impact the initial value Y_0 .

8.2.2 Numerical approach

In order to solve (8.1.16) corresponding to the population's problem in the benchmark case, as well as (8.1.22) for the government's problem, we need a method permitting to deal with degenerate HJB equations. We choose to implement semi-Lagrangian schemes, first introduced by Camilli and Falcone [74] (1995). These are explicit schemes using a given time-step Δt , and requiring interpolation on the grid of points where the equation is solved. This interpolation can be either linear, as in [74], or using some truncated higher-order interpolators, as proposed by Warin [340] (2016), leading to convergence of the numerical solution to the viscosity solution of the problem. A key point here, which makes the approach delicate, is that the domain over which the PDEs are solved is unbounded. It is therefore necessary to define a so-called resolution domain, over which the numerical solution will be actually computed, which on the one hand must be large enough, and which on the other hand creates additional difficulties in the treatment of newly introduced boundary conditions. In order to treat these issues, we use two special tricks:

- (i) picking randomly the control in (8.1.5) for the benchmark case, and in (8.3.16) for the general case, and using the forward SDE with an Euler scheme, a Monte–Carlo method allows us to get an envelop of the reachable domain with a high probability at each time–step. Then, given a discretisation step fixed once and for all, the grid of points used by the semi–Lagrangian scheme is defined at each time–step with bounds set by the reachable domain estimated by Monte–Carlo. Therefore, at time step 0, the grid is only represented by a single mesh, while the number of meshes can reach millions near T ;
- (ii) since the scheme is explicit, starting at a given point at date t , it requires to use only some discretisation points at date $t + \Delta t$, and a modification of the general scheme is implemented to use only points inside the grid at date $t + \Delta t$, as shown in [340].

Lastly, in dimension 3 or above, parallelisation techniques defined in [340] have to be used in order to accelerate the resolution of the problems. The numerical results below are obtained using the StOpt library, see Gevret et al. [152] (2018).

8.2.3 The benchmark case

We first focus on the benchmark case, when the government does not implement any particular policy to tackle the epidemic, *i.e.*, $\alpha = 1$ and $\chi = 0$. Recall that in this case, the population’s problem is given by (8.1.9), and is then equivalent to solving the HJB equation (8.1.16).

For our simulations, we choose a number of time–steps equal to 200, and a discretisation step equal to 0.0025. The interpolator is chosen linear, and the optimal command b° used to maximise the Hamiltonian is discretised with 200 points given a step discretisation of 0.005. Once the PDE is solved, a forward Euler scheme is used to obtain trajectories of the optimally controlled S and I , meaning with the optimal transmission rate b° . In order to check the accuracy of the method described in Section 8.2.2, we implement two versions of the resolution

- (i) the first version is a direct resolution of (8.1.16) with the Hamiltonian (8.1.15);
- (ii) the second one relies on a change of variable. More precisely, we consider $(s, x := (s + i))$ as state variables, instead of (s, i) , and then solve the problem (8.1.16), but with a slightly modified Hamiltonian to take into account this change of variable

$$\begin{aligned} \tilde{H}^A(t, s, x, p, M) := & \sup_{b \in B} \{ -bs(x - s)p_1 + u(t, b, x - s) \} + (\lambda - (\mu + \nu)s + \nu x)p_1 \\ & + (\lambda - (\mu + \rho + \gamma)x + (\rho + \gamma)s)p_2 + \frac{1}{2}|\sigma|^2 |s(x - s)|^2 M_{11}. \end{aligned}$$

The advantage of the second representation is that the dispersion of $I_t + S_t$ is zero and thus smaller than the one of I_t , leading to the use of grids with a smaller number of points.

First, to give an overview of the overall trend, we plot, on Figure 8.3, 100 trajectories of the optimal interaction rate β^* , and the associated proportions S_t and I_t of susceptible and infected, using the resolution method (i) mentioned above, *i.e.*, with state variables (S, I) . For more accurate trajectories, we compare on Figure 8.4 two different trajectories of the optimal interaction rate β^* , together with the corresponding dynamic of the proportion I of infected. For these two simulations, we compare the results given by the two aforementioned methods. More precisely, while the blue curve is obtained through the direct resolution, the orange one results from the second method, *i.e.*, with state variables $(S, S - I)$. Finally, on Figures 8.5 and 8.6, we test the influence of the parameter τ_p by setting $\tau_p = 0.01$, instead of 0.

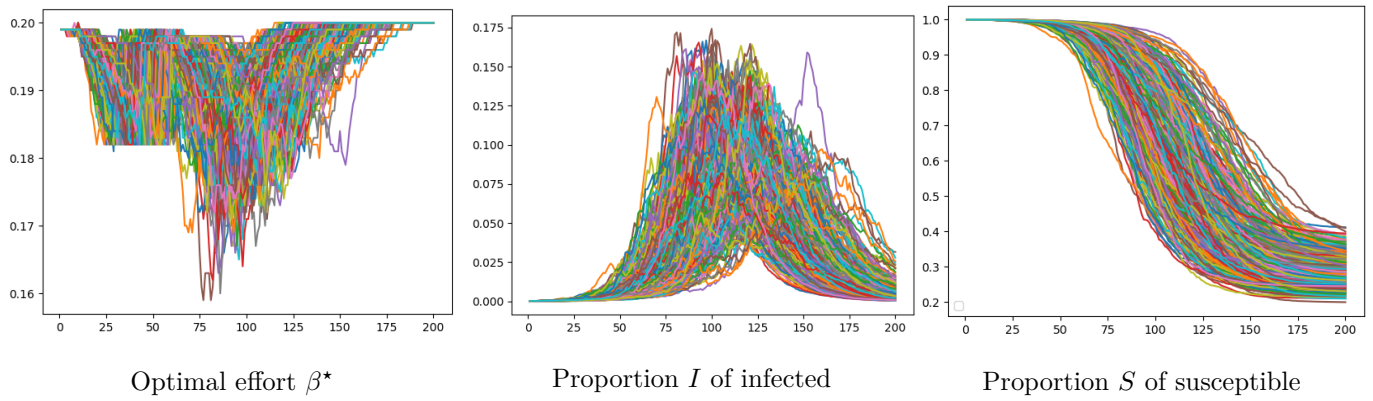
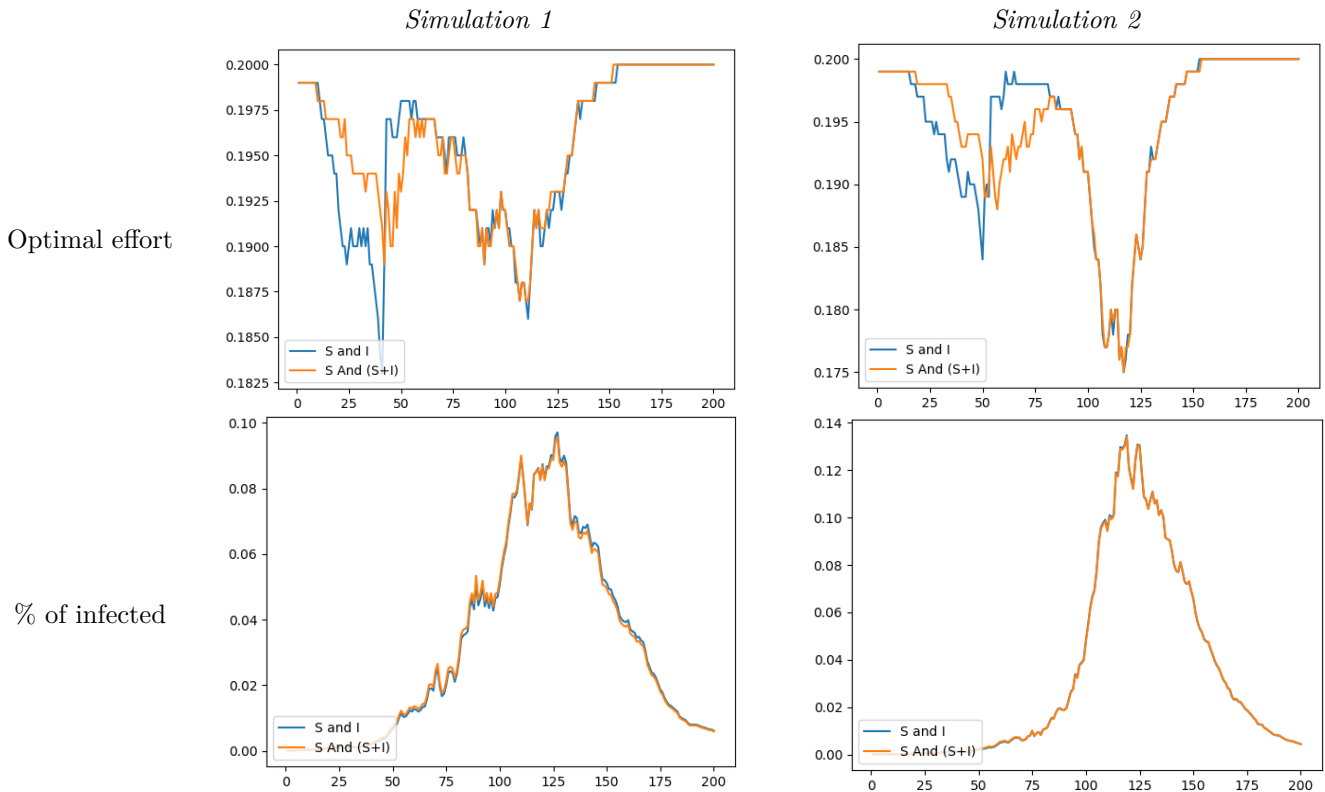


Figure 8.3 – 100 simulations with respect to time of the SIR model in the benchmark case.

Voluntary lockdown of the population. As expected, the optimal behaviour is to start at $\beta^* = \beta^\circ$, then we note that β^* decreases as the disease spreads in the population. More specifically, two waves of effort can be observed: the first one delays the acceleration of the epidemic, and the second, generally more significant, takes place during the peak of the epidemic. Approaching the fixed maturity, individuals come back to their usual behaviour β° . However, even if the population chooses to decrease the interaction rate among individuals, the range of β^* stays quite small with minimum 0.16 and maximum $\beta^\circ = 0.2$.

Figure 8.4 – The optimal transmission rate β and the resulting proportion I in the benchmark case. Comparison between of the two methods aforementioned on two simulations.

Sensitivity with respect to the method. As we can notice in Figure 8.4 (top), the optimal effort obtained for these two simulations exhibits the same features as those previously described. Moreover, the blue curve and the orange curve, representing respectively the results of the two aforementioned methods, are very close, except at the beginning of the time interval, probably because of the very small initial value i_0 . Nevertheless, we can see on the bottom graphs that the two methods lead to the same dynamic for the proportion of infected, since the two curves, blue and orange, are almost superposed. Therefore, a small error on the computation of the optimal effort at the beginning does not impact the optimally controlled trajectories of I . The resolution with respect to $(s, s+i)$ seems to be more regular, and may give a command closer to the analytical one.

The fear of the infection is not enough. Without a proper government policy to encourage the lockdown, we observe that the natural reduction of the interaction rate among individuals is not sufficient to contain the disease, so that it spreads with a high infection peak, up to 0.175. As a result, even if at the end of the time interval under consideration, the epidemic appears to be over, between 60 and 80% of the population has been contaminated by the virus, since the proportion S at time $T = 200$ lies between 0.2 and 0.4. In conclusion, without some governmental measures, the fear of the epidemic is not sufficient to encourage the population to make sufficient effort, in order to significantly reduce the rate of transmission of the disease. The introduction by the government of an effective lockdown policy together with an active testing policy should improve the results of the benchmark case, in particular by reducing the peak of infection and the total number of infected people over the considered period.

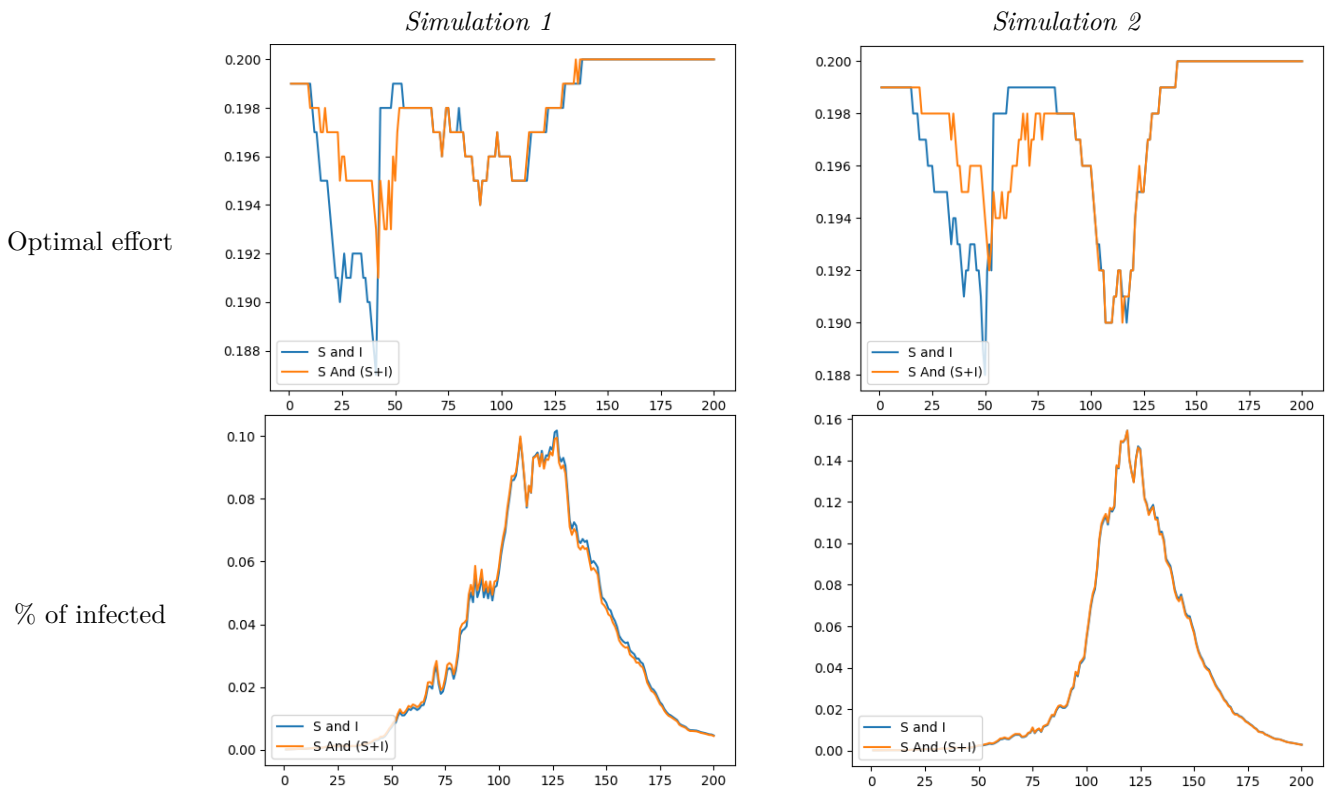


Figure 8.5 – The optimal transmission rate β and the resulting proportion I with $\tau_p = 0.01$
Comparison between of the two methods on two simulations.

The lockdown fatigue. By setting $\tau_p = 0.01$ instead of 0, the cost of the lockdown from the population's point of view is now increasing with time. This allows to take into account the possible fatigue the population may suffer if the lockdown continues for too long. As expected, by comparing Figures 8.3 and 8.6, the impatience of the population, gives higher values of optimal interaction rate β . Moreover, comparing Figures 8.4 and 8.5, we can see that in both simulations, the second wave of effort is of course more impacted (*i.e.*, the contact rate is less reduced) by the impatience of the population than the first one.

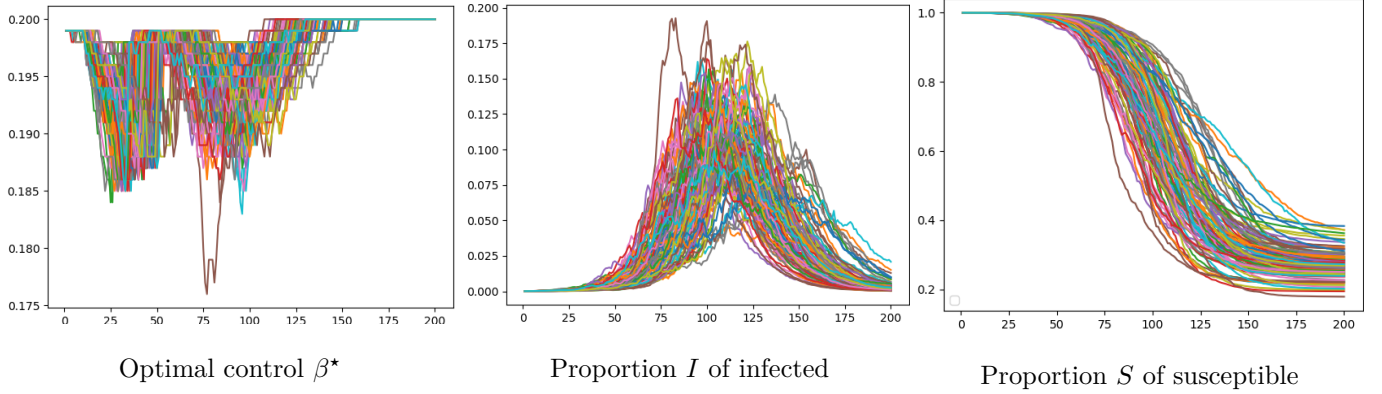


Figure 8.6 – 100 simulations of the SIR model in the benchmark case with $\tau_p = 0.01$

8.2.4 Lockdown policy, without testing

We focus in this section on the tax policy, by assuming that $A = \{1\}$. In words, we assume that the government does not implement a specific testing policy, which means $\alpha = 1$ as in the benchmark case, but only encourages the population to lockdown through the tax policy χ . In such a situation, *i.e.*, without a proper testing policy, the detection and hence the isolation of ill people becomes very intricate. The only possibility to regain control of the epidemic was to reduce the interaction rate of the population.

This case is interesting, as it corresponds to the lockdown policy that most of western countries have implemented in 2020, when faced with the COVID-19 disease, while a very small number of tests was available. Indeed, most countries put in place systems of fines, or even prison sentences, to incentivise people to lockdown. Although the penalties for non-compliance are not as sophisticated as in our model, most governments did adapt the level of penalties according to the stage of the epidemic: higher fines during periods of strict lockdown (hence at the peak of the epidemic), or in case of recidivism, for example. This reflects the adjustment of sanctions in many countries according to the health situation, and therefore a notion of dynamic adaptation to circumstances, which is exactly what is suggested by our tax system. Though it is clear that our model is different from reality, since we consider a fine/compensation, paid at some terminal time T , and equal for each individual, whereas in most countries, the fine is paid by a particular individual who has not complied with the injunctions, we still believe it allows to highlight sensible guidelines.

The numerical approach is highly similar to the method used to solve the benchmark case. One difference is that we have to estimate the reservation utility of the population, namely $\underline{v}$, given by (8.2.2). Using a Monte-Carlo method and an Euler scheme with 200 time-steps and 10^6 trajectories, we obtain an approximated value $\underline{v} = -0.02937$. Then, we can solve (8.1.22) through the aforementioned semi-Lagrangian scheme, with 200 time steps, as well as a step discretisation for the grid in (s, i, y) corresponding to $(0.0025, 0.0025, 0.005)$, leading to a number of meshes at maturity equal to $250 \times 70 \times 800$.

A technical point concerns the domain of the control Z . Although this control of the government, used

to index the tax on the proportion of infected, can take high values, we have to bound its domain in order to perform the numerical simulations. We choose to restrict its domain to an interval $[-Z_{\max}, Z_{\max}]$, and we consider a discretisation step equal to 0.5. One naturally expect that a larger choice would lead to better solutions. However, this neglects a fundamental numerical issue: large values of Z increase the numerical cost, as they enlarge the volatility of the process Y (given by σZIS). As such, since the volatility cone becomes larger, it is necessary to sample a much larger grid in order to be able to cover the region where Y will most likely take its values. Too large values of $Z_{\max}$ therefore become numerically intractable, unless one is willing to sacrifice accuracy. A balance need to be struck, which is why we capped $Z_{\max}$ at 30. A sensitivity analysis with respect to variations of $Z_{\max}$ is provided in Figure 8.8. Though the trajectories of the optimal Z are somewhat impacted, Figure 8.7 confirms that this is minimal impact on the trajectories of I itself. Indeed, for different values for $Z_{\max}$, the shape of the parameter Z remains the same. More importantly, we will see that the paths of the optimal transmission rate, namely β^* , associated to different $Z_{\max}$, are almost superposed. As a consequence, the dynamic of I also follows almost the same paths independently of $Z_{\max}$.

First, we present in Figure 8.7 different trajectories of the proportion I of infected when the government implements the optimal tax policy, and compare it to the trajectories obtained in the benchmark case. As mentioned before, we also want to study the sensibility with respect to the arbitrary bound $Z_{\max}$, and we thus represent the paths of I in three cases, in addition to the benchmark case: for $Z_{\max} = 10$ (orange curves), $Z_{\max} = 20$ (green), and $Z_{\max} = 30$ (red). Then, the corresponding simulations of the optimal control Z of the government, used to index the tax on the proportion of infected, is given in Figure 8.8. We compare optimal controls β and Z for the tax policy with different lockdown time period in Figure 8.9. Finally, Figure 8.10 regroups the simulations of the optimal transmission rate β^* obtained with the tax policy, and compare it to β^B obtained in the benchmark case.

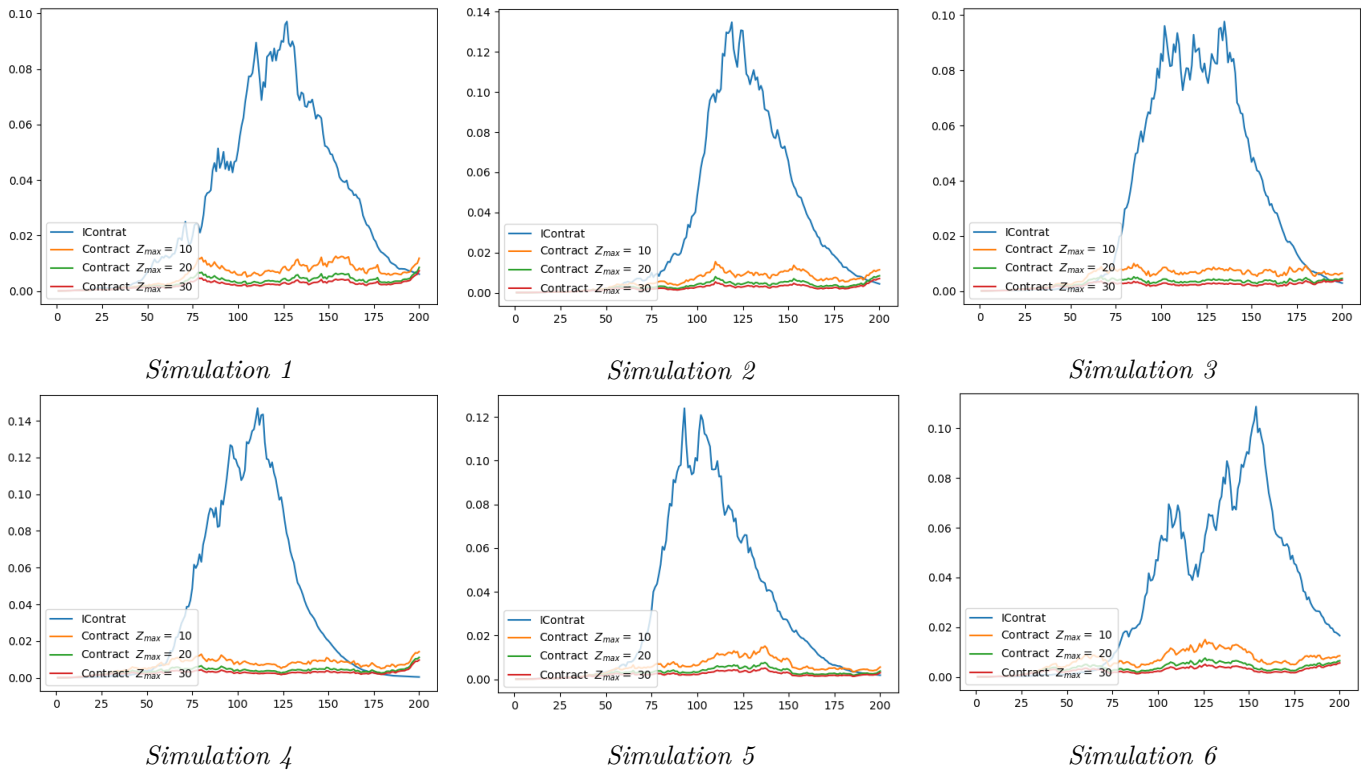


Figure 8.7 – Optimal trajectories of I without testing
Comparison for different values of $Z_{\max}$ and in the benchmark case.

The epidemic is at best contained, and at worst delayed. Compared to the benchmark case, we observe in Figure 8.7 that the optimal lockdown policy prevents the epidemic peak in most cases by maintaining the epidemic to low levels of infection during the lockdown period. Therefore, the government has more time to prepare for a possible infection peak after the lockdown, specifically to increase hospital capacity and provide safety equipment (surgical masks, hydro-alcoholic gel, respirators...). The government can also use this time to fund the development of tests to detect the virus, as well as the research on a vaccine or a remedy for the related disease.

Nevertheless, we can see that at the end of the lockdown period, in many cases the virus is not exterminated and the epidemic may even restart. This is particularly well illustrated by Figure 8.11, representing 500 trajectories of I , obtained with the optimal control. Such a phenomenon can be understood as follows: as the lockdown slows down the epidemic, a very small proportion of the population has been infected and is therefore immune. We thus cannot thus rely on herd immunity, which is reached if at least 50% of the population has been contaminated, to prevent a resurgence of the epidemic. Consequently, this lockdown policy is a powerful leverage to control an epidemic, but this tool needs to be supplemented by alternative policies, such as those mentioned above, in order to be fully effective. If the time saved through lockdown is not exploited, it will have no impact on the final consequences of the epidemic, measured by the economic and social cost associated with the total number of people infected and deceased during the total duration of the epidemic.

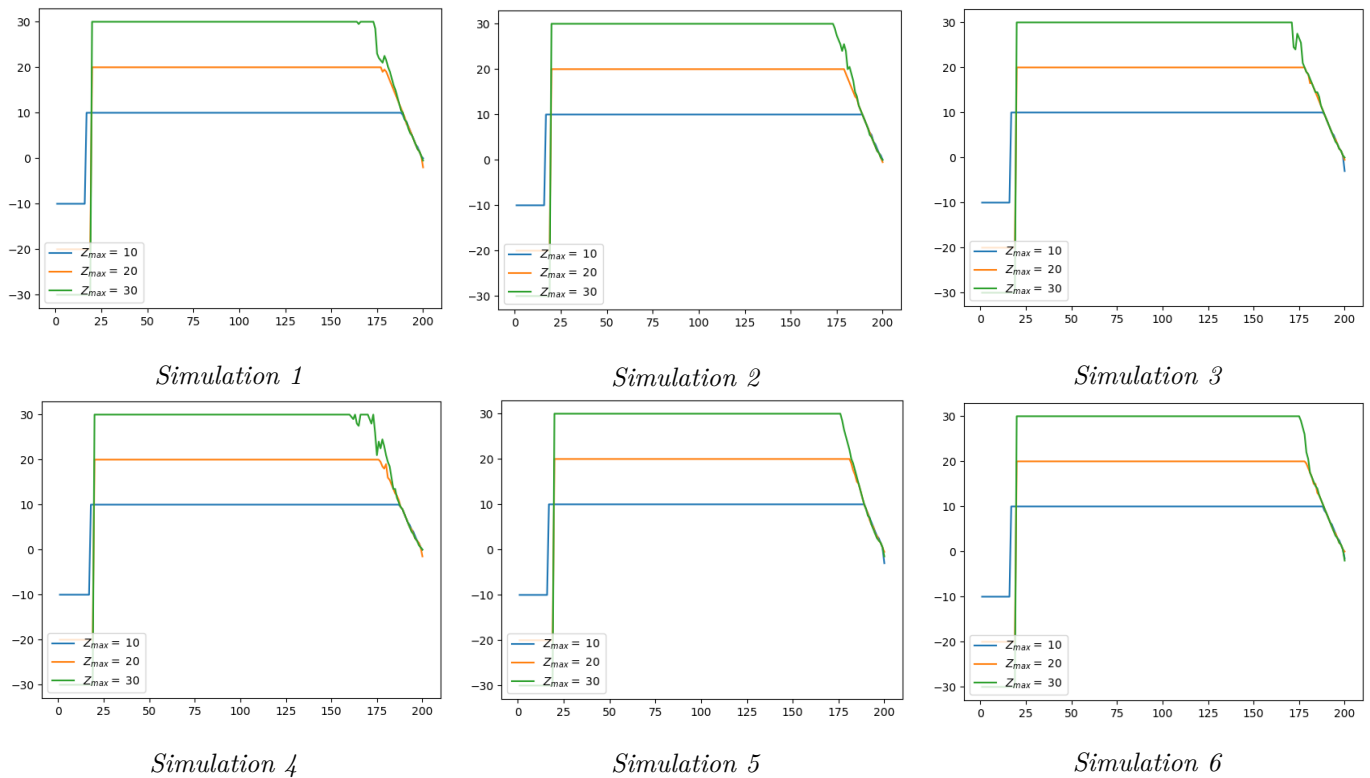


Figure 8.8 – Optimal trajectories of the control Z without testing.

Comparison for different values of $Z_{\max}$, with $A = \{1\}$.

Policy implications By comparing the graphs in Figure 8.8, we first remark that the shape of the optimal indexation parameter rate Z remains the same, regardless of the simulation and the value of $Z_{\max}$. The control takes the most negative value possible ($-Z_{\max}$) for about 20 days, then increases almost

instantaneously to reach the maximum value $Z_{\max}$, before slowly decreasing to 0. Therefore, the optimal tax scheme set by the government is as follows. First, at the beginning of the epidemic, it seems optimal to give to the population a compensation (corresponding to a negative tax) as maximal as possible, by setting $Z = -Z_{\max}$. Though this may be a numerical artefact, given that the initial values of I and its variations are extremely low, the fact that the same phenomenon appeared in virtually all our simulations tends to show that it is actually significant. We interpret this as the government anticipating the negative consequences of the lockdown policy by immediately providing monetary relief to the population. This is exactly what happened in several countries, for instance in the USA with stimulus checks sent to every citizen, and our model endogenously reproduces this aspect. Policy-wise, it also shows that maximum efficiency for such stimulus packages is attained when they are provided to the population as early as possible. After this initial phase, when the epidemic spreads among the population, the government suddenly increases Z , so that the tax becomes positive and is in fact maximum, in order to deter people from interacting.

Approaching the maturity, the government eases the lockdown little by little. However, this end of lockdown may be premature, since we have observed in the previous figures that the epidemic may restart at the end of the considered period. Indeed, considering a final time horizon is equivalent to assuming that 'the world' stops at that time: all potential costs generated by the epidemic after T are not taken into account in the model. The government thus has no interest in implementing costly measures, whose subsequent impact on the epidemic will not be measured. Nevertheless, we remark in the numerical results that if we consider a more distant time T , the lockdown certainly lasts longer, but follows the exact same paths during most of the lockdown period, and its release occurs around the same time before maturity. Moreover, the lockdown period should still end at some time, which is why a finite terminal time is assumed. This time may correspond to an estimate of the time needed to implement other more sustainable policies than lockdown, such as the implementation of an active testing policy, or to hope for the discovery of a vaccine or cure, as mentioned above.

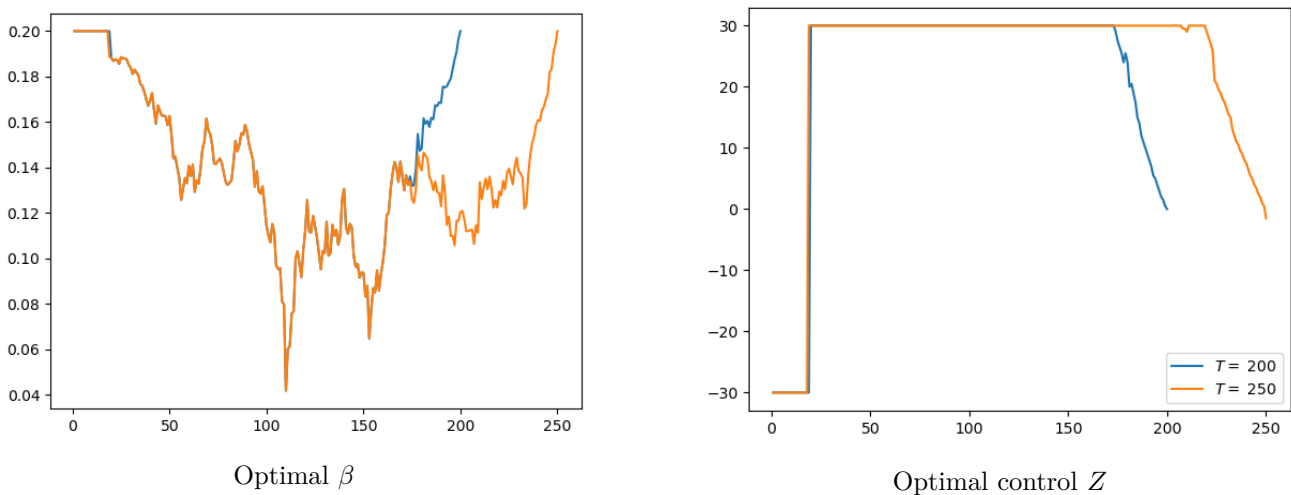


Figure 8.9 – Maturity effect for the tax policy in the SIR model

Comparison of the optimal trajectories of Z for $T = 200$ and $T = 250$, with $Z_{\max} = 30$.

Optimal tax sensitivity with respect to the lockdown duration. On Figure 8.9, we give two trajectories of the optimal contact rate β (on the left) and the optimal indexation parameter Z (on the right) for two different maturities. It is clear that both trajectories follow the same paths until some point. Regardless of the maturity, the contact rate β and the parameter Z have the same characteristics as those shown respectively in Figures 8.8 and 8.10. As one approaches the shortest maturity, *i.e.* $T = 200$,

the parameter Z decreases towards 0 for the contract of this maturity, while the other remains at the maximum, and decreases later, as its maturity approaches. Therefore, the fact that Z decreases at maturity, as mentioned in the paragraph 'Policy implications' above appears to be a boundary effect since it is not sensitive with respect to the maturity.

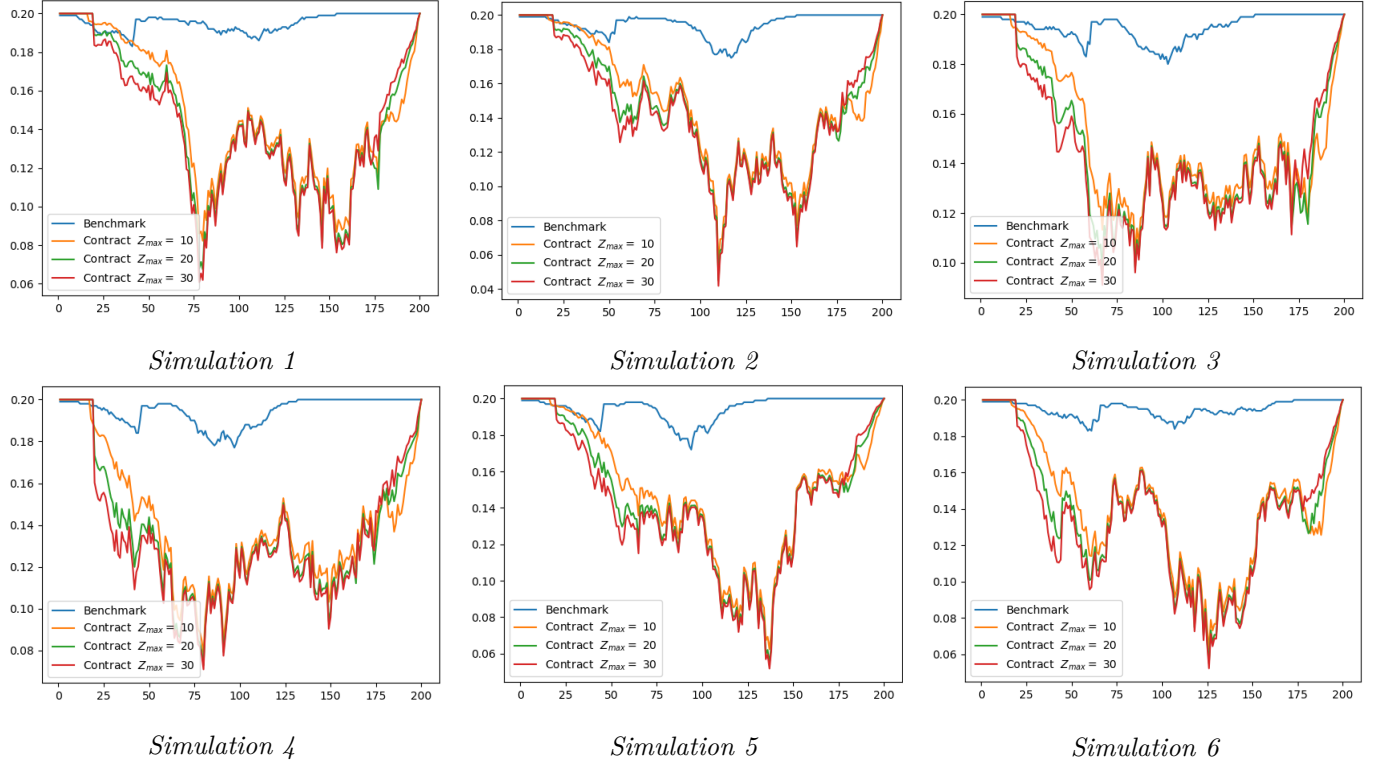


Figure 8.10 – Optimal transmission rate β without testing
Comparison for different $Z_{\max}$ and with the benchmark case, in the case $A = \{1\}$.

Optimal interaction rate and comparison with the benchmark case. We now explain the general trend of the optimal interaction rate, impacted by the optimal indexation parameter Z chosen optimally by the government, and represented in Figure 8.8 above. At the beginning, remember that the parameter Z , indexing the tax on the dynamic of I , is negative, meaning that the tax is negatively indexed on the variation of I . In other words, since I is globally (but very slightly) increasing at the beginning of the epidemic, the compensation increases with I , which means that the population is not incentivised at all to decrease their contact rate, and thus the transmission rate of the virus, which remains equal to the initial level $\bar{\beta}$. Then, as the epidemic spreads, Z becomes very high, which now incentivises the population to reduce the transmission rate below $\bar{\beta}$. Finally, near the end of the lockdown period, the simulations of Z decreases to zero, which naturally implies that the optimal contact rate β^* goes back to its usual level $\bar{\beta}$. By comparing with the benchmark case, we see that the tax policy succeeds in reducing significantly the interaction rate. As a consequence, and as we have seen in Figure 8.7, the tax policy contains the spread of the disease during the considered time period, unlike in the case without intervention of the government.

We have observed in Figure 8.7 that the optimal tax policy seems to limit the high values of I . This is generally confirmed with the print in Figure 8.11 of 100 trajectories of I , obtained with the optimal control (left) and in the benchmark case (right). Nevertheless, it can be noted that, surprisingly, between 10 and 15% of the simulations show no improvement in the control of the epidemic.

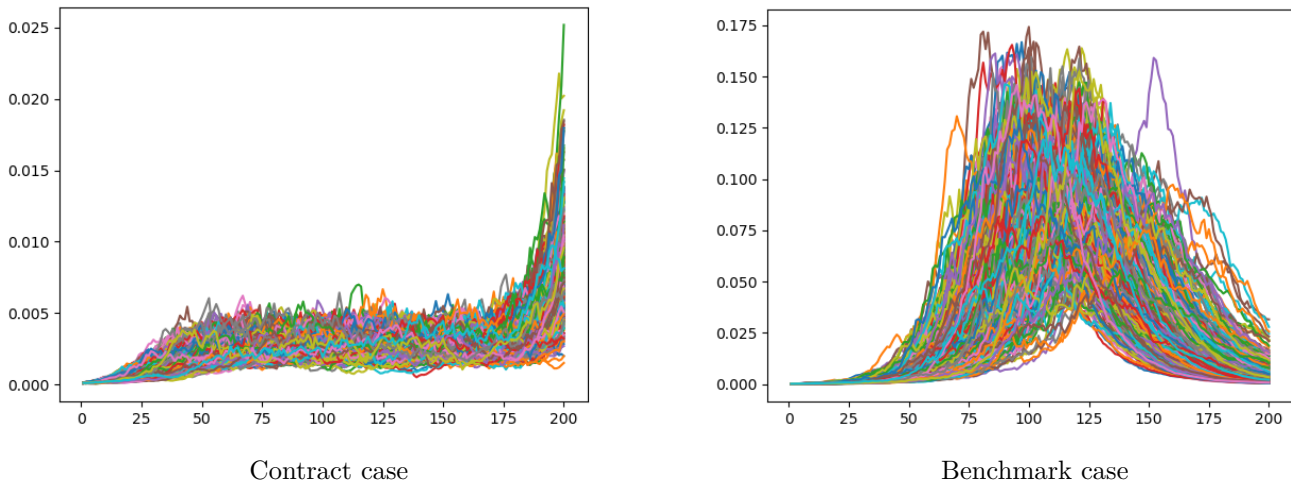


Figure 8.11 – 500 simulations of the proportion I of infected in the SIR model

Comparison between the case with tax policy (but without testing) on the left and the benchmark case on the right.

8.2.5 Tax policy with testing

In this section, we now study the case where the government can implement an active testing policy, in addition to the incentive policy for lockdown, to contain the spread of the epidemic. This policy is similar to the one adopted by most European governments in June 2020, after relatively strict containment periods and at a time when the COVID-19 epidemic seemed to be under control. Indeed, the lockdown periods in Europe have generally made it possible to delay the epidemic, and thus to give public authorities time to prepare a meaningful testing policy by developing and increasing the number of available tests.

This testing policy has two major interests. First, it allows the identification of clusters, and therefore provides a more precise knowledge of the dynamics of the epidemic in real time on the different territories. Second, by identifying infected people, we can force them to remain isolated, in order to avoid the contamination of their relatives. This policy therefore constitutes another leverage, in addition to containment, to reduce the contact rate within the population. Thus, by developing a robust testing policy, public authorities can in fact relax the lockdown while keeping the rate of disease transmission at a sufficiently low level. Therefore, comparing with the no-testing policy case, we expect that

- (i) the government will be able to control the epidemic at least as well as with just the lockdown policy;
- (ii) it will allow the population to regain a contact rate closer to the desired and initial level β° .

To study the optimal testing policy α^* , taking values in $A := [\varepsilon, 1]$, we consider the cost of effort k given by (8.2.1b). This cost function emphasises the fact that testing the entire population every day is inconceivable, and therefore results in an explosion of cost when α takes values close to 0. Recall that the parameters for the function k , namely κ_g and η_g are given in Table 8.1b. Finally, A is discretised with a step equal to 0.05 and we consider $Z_{\max} = 30$.

As we can see from the three selected simulations below, the control Z is very regular (see Figure 8.12), while the control α is less regular and concentrated at the heart of the epidemic (see Figure 8.13). Figure 8.15 gives a global overview of the 500 simulations, which confirms the intuition given by the three selected ones.

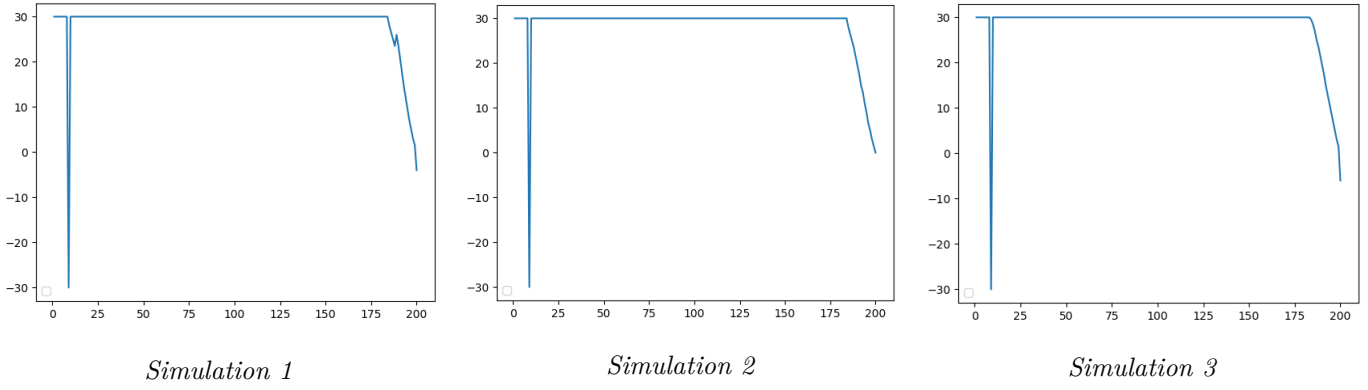


Figure 8.12 – Optimal trajectories of Z with testing policy.

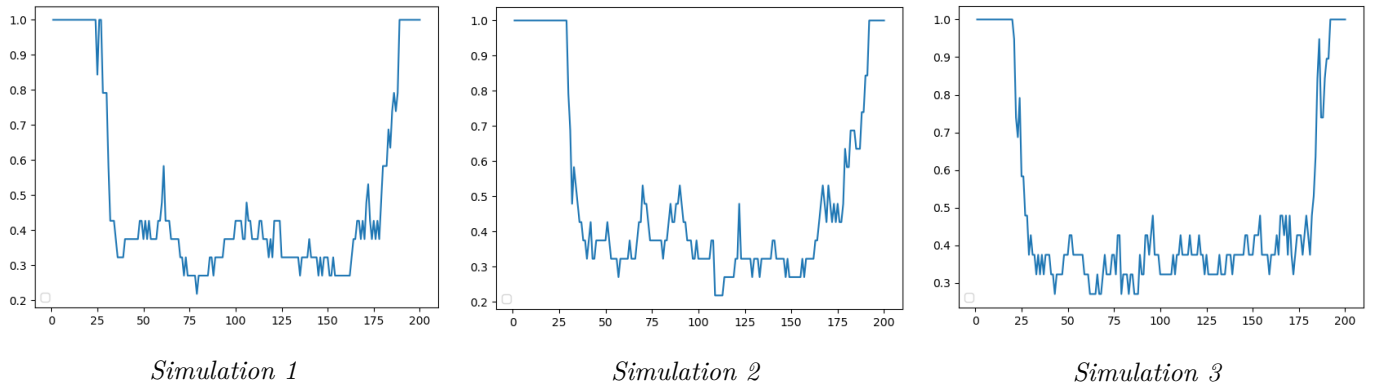


Figure 8.13 – Optimal trajectories of the testing policy α

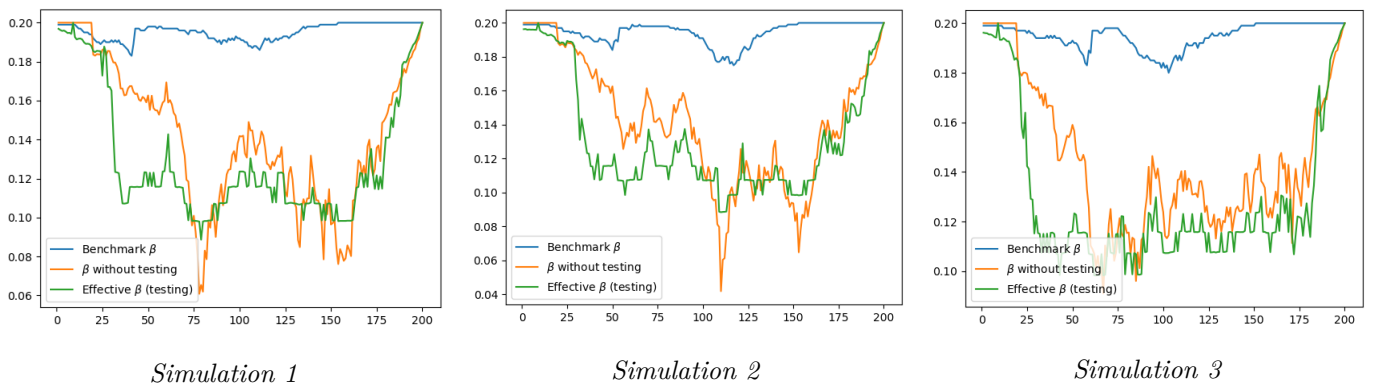
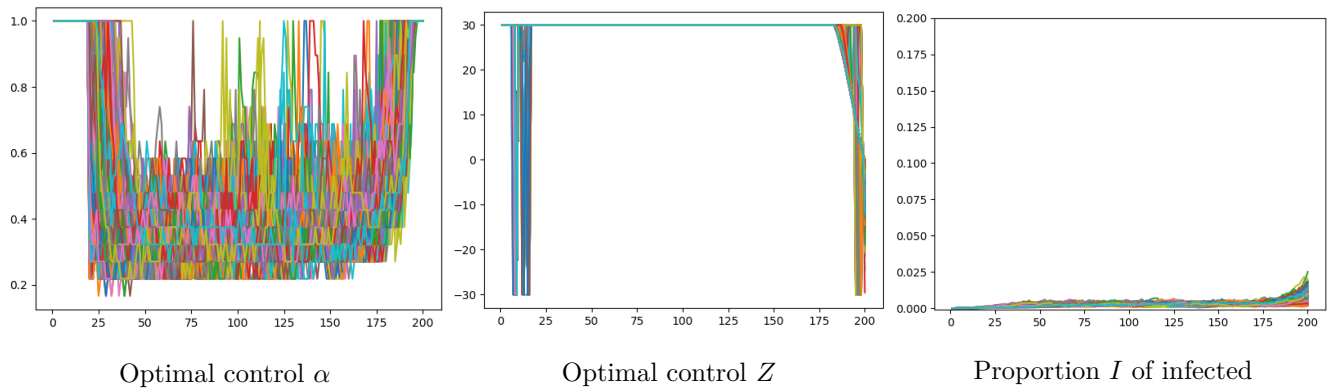


Figure 8.14 – Optimal effective transmission rate $\beta\sqrt{\alpha}$ with testing policy
Comparison between the three cases, the benchmark, with, and without testing.

Figure 8.15 – 500 simulations of optimal government's controls, with the resulting trajectories of I .

Relaxed lockdown but lower effective transmission rate. First, comparing Figures 8.8 and 8.12, the optimal control Z presents the same shape in both cases, except at the beginning, since now Z is not negative initially. In fact, in this case, we observe that the government is asking for less effort from the population, and therefore the *initial stimulus* mentioned in the paragraph 'Policy implications' still happens, but later and for a much shorter length. Figure 8.16 (left) also shows that the optimal contact rate is closer to the initial level $\bar{\beta}$, which should induce a more violent spread of the disease. Nevertheless, the control α , representing the testing policy and given by Figure 8.13, balances this effect. Indeed, the testing allows an isolation of targeted infected individual, and therefore contribute to the decrease of the effective transmission rate of the disease, represented in Figure 8.16 (right). Therefore, comparing Figure 8.15 with Figure 8.11, we notice that the control of the epidemic is more efficient than in the case $A = \{1\}$, since the proportion of infected is globally decreased.

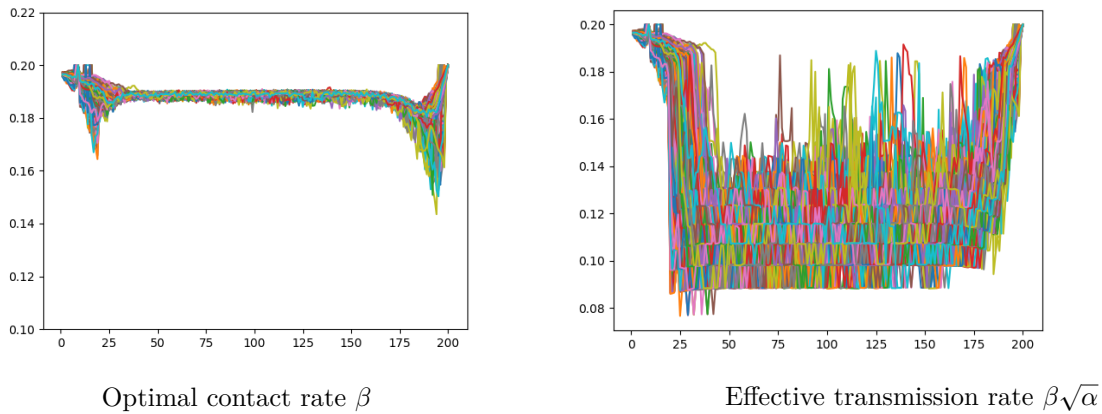


Figure 8.16 – 500 simulations of the transmission rate with testing policy

8.2.6 The first-best case

First, remark that, with the particular choice of utility functions, we have :

$$\chi^*(\varpi) = \frac{1}{\theta_p} \ln \left(\frac{1}{\varpi} - \phi_p \right), \text{ if } 0 < \varpi < \frac{1}{\phi_p} = 2.$$

Otherwise, if $\varpi \geq 2$, the optimal tax policy is equal to $-\infty$, which cannot be optimal from the government's point of view, since it leads to an infimum on ϖ equal to $+\infty$ (see (8.1.21)).

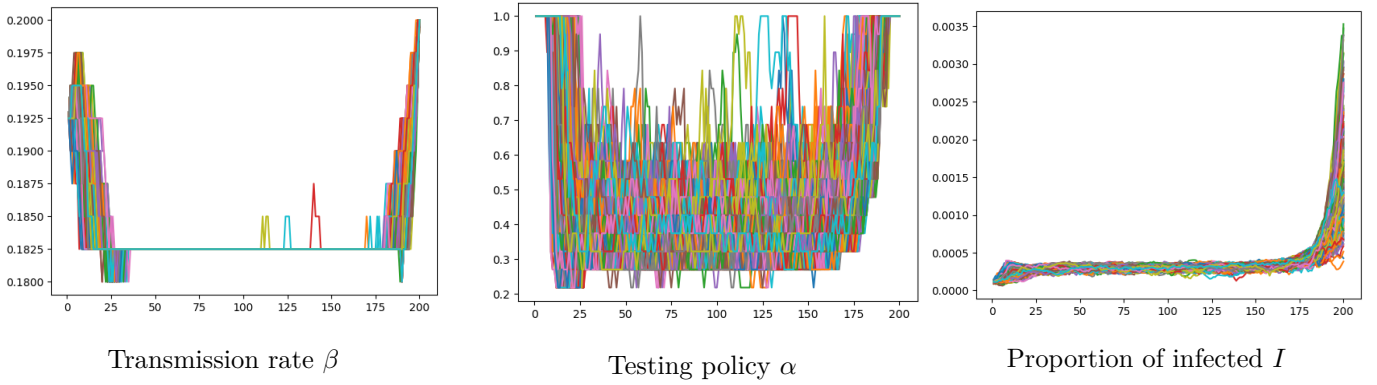


Figure 8.17 – 500 trajectories obtained in the first-best case.

For each value of the Lagrange parameter, a two dimensional PDE with a two-dimensional control (α, β) is achieved. A step discretisation for the grid in (s, i) is taken equal to $(0.001, 0.001)$. $A = [\varepsilon, 1]$ is discretised with 20 values and the values of β are discretised with 80 equally spaced values (to reduce the cost of optimization). We then search for the optimal ϖ parameter with a step of 0.01 within the interval $(0, 2)$. We obtain in this case an optimal value equal to 0.64 and we give on Figure 8.17 the results, which shows that the epidemic is controlled in a similar way as in the second-best case, with incentives and testing policy.

The shape of the optimal controls β and α , as well as the trajectories for the proportion I of infected, are highly similar to those obtained in the previous case. The only clear difference is the principal's value. Indeed, we can compare the optimal value V_0^P for the government in the moral hazard case, to the first best value $V_0^{P,FB}$. Using $1e4$ trajectories and the previously optimal control computed, we estimate $V_0^{P,FB} = -0.249$ while $V_0^P = -0.287$. The first best value appears to be a good approximation of the second-best value. This is partly explained by the fact that the testing is profitable both for the government and for the population as it allows the optimal control β to take values very close to its usual level $\bar{\beta}$, as shown on Figure 8.17.

8.3 Incentive policy for epidemic stochastic models

8.3.1 The stochastic model

8.3.1.1 Initial canonical space

We fix a small parameter $\varepsilon \in (0, 1)$ to consider the subset $A := [\varepsilon, 1]$. We then define $\mathbb{A}$ and $\mathbb{A}_0$ from the subset A , using [General notations](#) (p. xv). We are then led to consider the following canonical space

$$\Omega := \mathcal{C}_T^2 \times \mathbb{A},$$

whose canonical process is denoted by (S, I, Λ) , in the sense that

$$S_t(\mathbf{s}, \iota, q) := \mathbf{s}(t), \quad I_t(\mathbf{s}, \iota, q) := \iota(t), \quad \Lambda(\mathbf{s}, \iota, q) := q, \quad \forall (t, \mathbf{s}, \iota, q) \in [0, T] \times \Omega.$$

We let $\mathcal{F}$ be the Borel σ -algebra on Ω , and $\mathbb{F} := (\mathcal{F}_t)_{t \in [0, T]}$ be the natural filtration of the canonical process

$$\mathcal{F}_t := \sigma\left((S_s, I_s, \Delta_s(\varphi)) : (s, \varphi) \in [0, t] \times \mathcal{C}_b([0, T] \times A, \mathbb{R})\right), \quad t \in [0, T],$$

where for any $(s, \varphi) \in [0, T] \times \mathcal{C}_b([0, T] \times A, \mathbb{R})$, $\Delta_s(\varphi) := \iint_{[0, s] \times A} \varphi(r, a) \Lambda(dr, da)$. Recall that in this framework $\mathcal{F} = \mathcal{F}_T$. Let $\mathcal{M}$ be the set of probability measures on $(\Omega, \mathcal{F}_T)$. For any $\mathbb{P} \in \mathcal{M}$, we let $\mathcal{N}^\mathbb{P}$ be the collection of all $\mathbb{P}$ -null sets, that is to say

$$\mathcal{N}^\mathbb{P} := \{N \in 2^\Omega : \exists N' \in \mathcal{F}_T, N \subset N', \mathbb{P}[N'] = 0\},$$

and we let $\mathbb{F}^\mathbb{P} := (\mathcal{F}_t^\mathbb{P})_{t \in [0, T]}$ be the $\mathbb{P}$ -augmentation of $\mathbb{F}$, where $\mathcal{F}_t^\mathbb{P} := \mathcal{F}_t \vee \sigma(\mathcal{N}^\mathbb{P})$. We let $\mathbb{F}^{\mathbb{P}^+} := (\mathcal{F}_t^{\mathbb{P}^+})_{t \in [0, T]}$ the corresponding right limit. Similarly, for any subset $\Pi \subset \mathcal{M}$, we let $\mathbb{F}^\Pi := (\mathcal{F}_t^\Pi)_{t \in [0, T]}$ be the Π -universal completion of $\mathbb{F}$.

Let us now introduce the drift and volatility functions for our controlled model, namely $B : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ and $\Sigma : \mathbb{R}^2 \times A \longrightarrow \mathbb{R}^2$, defined by

$$B(x, y) := \begin{pmatrix} \lambda - \mu x + \nu y \\ -(\mu + \nu + \rho + \gamma)y \end{pmatrix}, \quad \Sigma(x, y, a) := \begin{pmatrix} \sigma a x y \\ -\sigma a x y \end{pmatrix}, \quad (x, y, a) \in \mathbb{R}^2 \times A,$$

where the parameters $(\lambda, \mu, \nu, \rho, \sigma) \in [0, \infty)^4 \times \mathbb{R}_+^*$ are given. For any $(s, \varphi) \in [0, T] \times \mathcal{C}_b^2(\mathbb{R}^2, \mathbb{R})$, we set

$$M_s(\varphi) := \varphi(S_s, I_s) - \iint_{[0, s] \times A} \left(B(S_r, I_r) \cdot \nabla \varphi(S_r, I_r) + \frac{1}{2} \text{Tr}[\nabla^2 \varphi(S_r, I_r) (\Sigma \Sigma^\top)(S_r, I_r, a)] \right) \Lambda(dr, da),$$

and we give ourselves some initial values $(s_0, i_0) \in \mathbb{R}_+^2$.⁷

Definition 8.3.1. We define the subset $\mathcal{P} \subset \mathcal{M}$ as the one composed of all $\mathbb{P} \in \mathcal{M}$ such that

- (i) $M(\varphi)$ is an $(\mathbb{F}, \mathbb{P})$ -local martingale on $[0, T]$ for all $\varphi \in \mathcal{C}_b^2(\mathbb{R}^2, \mathbb{R})$;
- (ii) $\mathbb{P}[(S_0, I_0) = (s_0, i_0)] = 1$;
- (iii) $\mathbb{P}[\Lambda \in \mathbb{A}_0] = 1$.

We can follow Bichteler [58] (1981), or Neufeld and Nutz [265, Proposition 6.6] (2014) to define a pathwise version of the density of the quadratic variation of S , denoted by $\hat{\sigma} : [0, T] \times \Omega \longrightarrow \mathbb{R}$, by

$$\hat{\sigma}_t^2(\omega) := \limsup_{n \rightarrow \infty} n(\langle S \rangle_t(\omega) - \langle S \rangle_{t-1/n}(\omega)), \quad (t, \omega) \in [0, T] \times \Omega.$$

Lévy's characterisation of Brownian motion ensures that the process⁸

$$W_t := \int_0^t \hat{\sigma}_s^{-1/2} \mathbf{1}_{\hat{\sigma}_s \neq 0} dS_s, \quad t \in [0, T], \quad (8.3.1)$$

7. Notice that the initial value of r_0 of R , which appears in the SIR version of the model, is irrelevant at this stage.

8. More precisely, one should first use the result of Stroock and Varadhan [324, Theorem 4.5.2] (1997) to obtain that on an enlargement of $(\Omega, \mathcal{F}_T)$, there is for any $\mathbb{P} \in \mathcal{P}$, a Brownian motion $W^\mathbb{P}$, and an $\mathbb{F}$ -predictable process, A -valued process $\alpha^\mathbb{P}$ such that

$$S_t = s_0 + \int_0^t (\lambda - \mu S_s + \nu I_s) ds + \int_0^t \sigma \alpha_s^\mathbb{P} S_s I_s dW_s^\mathbb{P}, \quad t \in [0, T], \quad \mathbb{P}\text{-a.s.}$$

The result for W is then immediate. Notice in addition that since W is defined as a stochastic integral, it should also depend on explicitly on $\mathbb{P}$. We can however use Nutz [267, Theorem 2.2] (2012) to define W universally, as an $\mathbb{F}^\mathcal{P}$ -adapted and continuous process. This requires some set-theoretic assumptions which we implicitly consider here, see Possamaï, Tan, and Zhou [284, Footnote 7] (2018) for details.

is an $(\mathbb{F}^{\mathcal{P}}, \mathbb{P})$ -Brownian motion for any $\mathbb{P} \in \mathcal{P}$. For any $\mathbb{P} \in \mathcal{P}$, we denote by $\mathcal{A}_o(\mathbb{P})$ the set of $\mathbb{F}$ -predictable and A -valued process $\alpha := (\alpha_s)_{s \in [0, T]}$ such that, $\mathbb{P}$ -a.s.

$$\begin{cases} S_t = s_0 + \int_0^t (\lambda - \mu S_s + \nu I_s) ds + \int_0^t \sigma \alpha_s S_s I_s dW_s, & t \in [0, T], \\ I_t = i_0 - \int_0^t (\mu + \rho + \nu + \gamma) I_s ds - \int_0^t \sigma \alpha_s S_s I_s dW_s, & t \in [0, T]. \end{cases} \quad (8.3.2)$$

Once again, it is a classical result (see for instance Stroock and Varadhan [324, Theorem 4.5.2] (1997), Lin, Ren, Touzi, and Yang [234, Lemma 2.2] (2020), or Chapter 3, Lemma 3.1.2) that $\mathcal{A}_o(\mathbb{P})$ is not empty.

We recall that the term $\lambda \geq 0$ denotes the birth rate, the parameter $\mu \geq 0$ is the natural death rate in the population (susceptible and infected), $\rho \geq 0$ is the death rate related to the disease. The parameters ν and γ correspond to recovery rates, depending on whether we are considering a SIS or a SIR model, see the remark below for more details.

Remark 8.3.2. *From now on, one should have in mind that*

- (i) *if $\gamma = 0$, the constant $\nu > 0$ is the rate of recovery for infected people, going back in the class of susceptible. This case corresponds to the classical SIS model whose dynamics are described by the system (8.1.3);*
- (ii) *if $\nu = 0$, the constant $\gamma > 0$ is the recovery rate for infected individual, going into a class of recovered people, whose proportion is denoted by R . This case corresponds to the classical SIR model described by (8.1.4).*

It can be noted that our model, which results from a mixing of the SIS and SIR models, can be interpreted as a SIR model with partial immunisation, in the sense that only a part of the population develops antibodies for the disease after being infected. Thus, a proportion γ of the infected moves to the class R, and can no longer be infected. Conversely, the proportion of the infected who do not develop antibodies reverts to the class S, and can therefore contract the disease again. This resulting model is similar to the one developed by Zhang et al. [354] (2018) and called SISRS. This type of model seems in fact well suited to model epidemics related to new viruses, such as the COVID-19, when the immunity of infected persons has not yet been proved.

Before pursuing, we need a bit more notations, and will consider the following sets

$$\mathcal{A}_o := \bigcup_{\mathbb{P} \in \mathcal{P}} \mathcal{A}_o(\mathbb{P}),$$

as well as, for any $\alpha \in \mathcal{A}_o$, $\mathcal{P}(\alpha) := \{\mathbb{P} \in \mathcal{P} : \alpha \in \mathcal{A}_o(\mathbb{P})\}$. We will require that the controls chosen by the government lead to only one weak solution to (8.3.2), and are such that the processes S and I remain non-negative. We will therefore concentrate our attention to the set $\mathcal{A}$ of admissible controls defined by

$$\mathcal{A} := \{\alpha \in \mathcal{A}_o : \mathcal{P}(\alpha) \text{ is a singleton } \{\mathbb{P}^\alpha\}, \text{ and } (S, I) \text{ is } \mathbb{R}_+^2\text{-valued, } \mathbb{P}^\alpha\text{-a.s.}\}.$$

Notice that the set $\mathcal{A}$ is not empty. Indeed, any constant A -valued process automatically belongs to $\mathcal{A}$, as a direct consequence of Gray et al. [158, Section 3] (2011) or Gao, Song, Wang, and Liu [150, Lemma 2.3] (2019). Notice that for any $\alpha \in \mathcal{A}$, we have

$$\hat{\sigma}_t = \sigma S_t I_t \alpha_t, \quad d\mathbb{P}^\alpha \otimes dt\text{-a.e.}$$

Notice finally that for any $\alpha \in \mathcal{A}$, we have

$$S_t + I_t = s_0 + i_0 + \int_0^t (\lambda - \mu(S_s + I_s) - (\rho + \gamma)I_s) ds, \quad t \in [0, T], \quad \mathbb{P}^\alpha\text{-a.s.}$$

We thus deduce, using the positivity of S and I , that

$$0 \leq S_t + I_t = e^{-\mu t}(s_0 + i_0) + \int_0^t e^{-\mu(t-s)}(\lambda - (\rho + \gamma)I_s) ds \leq F(t, s_0, i_0), \quad t \in [0, T], \quad \mathbb{P}^\alpha\text{-a.s.}, \quad (8.3.3)$$

where for all $(t, s, i) \in [0, T] \times \mathbb{R}_+^2$

$$F(t, s, i) := e^{-\mu t}(s + i) + \lambda \left(\frac{1 - e^{-\mu t}}{\mu} \mathbf{1}_{\{\mu > 0\}} + t \mathbf{1}_{\{\mu = 0\}} \right). \quad (8.3.4)$$

This result proves in particular that S and I are actually $\mathbb{P}^\alpha$ -almost surely bounded, for any $\alpha \in \mathcal{A}$. Moreover, if $(s_0, i_0) \in (\mathbb{R}_+^*)^2$, then for all $t \in [0, T]$, both S_t and I_t are (strictly) positive.

Remark 8.3.3. Note that in the SIR model described by the system (8.1.4), we have, for all $t \in [0, T]$,

$$R_t = r_0 e^{-\mu t} + \gamma \int_0^t I_s e^{-\mu(t-s)} ds,$$

so that R_t depends only on the observation of I_s for $s \leq t$. In addition to that

$$0 \leq S_t + I_t + R_t \leq e^{-\mu t}(s_0 + i_0 + r_0) + \int_0^t e^{-\mu(t-s)}(\lambda - \rho I_s) ds \leq F(t, s_0, i_0) + r_0 e^{-\mu t}.$$

8.3.1.2 Impact of the interaction

The basic model from (8.3.2) takes into account the testing policy put into place by the government, but ignores so far the interacting behaviour of the population. We model this through an additional control process chosen by the population. More precisely, we fix some constant $\beta^{\max} > 0$ representing the maximum rate of interaction that can be considered, and we define $B := [0, \beta^{\max}]$. Let $\mathcal{B}$ be the set of all $\mathbb{F}$ -predictable and B -valued processes. Given a testing policy $\alpha \in \mathcal{A}$ implemented by the government, notice that the following stochastic exponential

$$\left(\exp \left(- \int_0^t \frac{\beta_s}{\sigma \sqrt{\alpha_s}} dW_s - \frac{1}{2} \int_0^t \frac{|\beta_s|^2}{|\sigma|^2 \alpha_s} ds \right) \right)_{t \in [0, T]},$$

is an $(\mathbb{F}, \mathbb{P}^\alpha)$ -martingale, given that the process $\beta/(\sigma\sqrt{\alpha})$ takes values in $[0, \beta^{\max}/(\sigma\sqrt{\varepsilon})]$, $\mathbb{P}^\alpha$ -a.s. Therefore, for any $(\alpha, \beta) \in \mathcal{A} \times \mathcal{B}$, we can define a probability measure $\mathbb{P}^{\alpha, \beta}$ on $(\Omega, \mathcal{F})$, equivalent to $\mathbb{P}^\alpha$, by

$$\frac{d\mathbb{P}^{\alpha, \beta}}{d\mathbb{P}^\alpha} := \exp \left(- \int_0^t \frac{\beta_s}{\sigma \sqrt{\alpha_s}} dW_s - \frac{1}{2} \int_0^t \frac{|\beta_s|^2}{|\sigma|^2 \alpha_s} ds \right).$$

Using Girsanov's theorem, we know that the process

$$W_t^\beta := W_t + \int_0^t \frac{\beta_s}{\sigma \sqrt{\alpha_s}} ds, \quad t \in [0, T], \quad (8.3.5)$$

is an $(\mathbb{F}, \mathbb{P}^{\alpha, \beta})$ -Brownian motion, and we have

$$\begin{cases} S_t = s_0 + \int_0^t (\lambda - \mu S_s + \nu I_s - \beta_s \sqrt{\alpha_s} S_s I_s) ds + \int_0^t \sigma \alpha_s S_s I_s dW_s^\beta, & t \in [0, T], \\ I_t = i_0 - \int_0^t ((\mu + \nu + \rho + \gamma) I_s - \beta_s \sqrt{\alpha_s} S_s I_s) ds - \int_0^t \sigma \alpha_s S_s I_s dW_s^\beta, & t \in [0, T]. \end{cases} \quad (8.3.6)$$

8.3.1.3 Optimisation problems

At time 0, the government informs the population about its testing policy $\alpha \in \mathcal{A}$, as well as its fine policy χ , which for now will be an $\mathcal{F}_T$ -measurable and $\mathbb{R}$ -valued random variable (a set we denote by $\mathfrak{C}$). The population solves the following optimal control problem

$$V_0^A(\alpha, \chi) := \sup_{\beta \in \mathcal{B}} \mathbb{E}^{\mathbb{P}^{\alpha, \beta}} \left[\int_0^T u(t, \beta_t, I_t) dt + U(-\chi) \right], \quad (8.3.7)$$

The interpretation of the functions u and U is detailed in [Section 8.1.3.1](#), where the population's problem was informally defined.

For any $(\alpha, \chi) \in \mathcal{A} \times \mathfrak{C}$, we recall that we denoted by $\mathcal{B}^*(\alpha, \chi)$ the set of optimal controls for $V_0^A(\alpha, \chi)$, that is to say

$$\mathcal{B}^*(\alpha, \chi) := \left\{ \beta \in \mathcal{B} : V_0^A(\alpha, \chi) = \mathbb{E}^{\mathbb{P}^{\alpha, \beta}} \left[\int_0^T u(t, \beta_t, I_t) dt + U(-\chi) \right] \right\}. \quad (8.3.8)$$

We require minimal integrability assumptions at this stage, and insist that there exists some $p > 1$ such that

$$\mathbb{E}^{\mathbb{P}^\alpha} [|U(-\chi)|^p] < \infty, \text{ for any } \alpha \in \mathcal{A}. \quad (8.3.9)$$

Remark 8.3.4. Notice that since for any $\alpha \in \mathcal{A}$ the Radon–Nykodým density $d\mathbb{P}^{\alpha, \beta}/d\mathbb{P}^\alpha$ has moments of any order under $\mathbb{P}^\alpha$ (since any $\beta \in \mathcal{B}$ is bounded and any $\alpha \in \mathcal{A}$ is bounded and bounded away from 0), a simple application of Hölder's inequality ensures that (8.3.9) implies that for any $p' \in (1, p)$ and any $\beta \in \mathcal{B}$

$$\mathbb{E}^{\mathbb{P}^{\alpha, \beta}} [|U(-\chi)|^{p'}] < \infty.$$

Recall that the government can only implement policies $(\alpha, \chi) \in \mathcal{A} \times \mathfrak{C}$ such that $V_0^A(\alpha, \chi) \geq \underline{v}$, where the minimal utility $\underline{v} \in \mathbb{R}$ is given. We denote the subset of $\mathcal{A} \times \mathfrak{C}$ satisfying this constraint and (8.3.9) by Ξ .

In line with the informal reasoning developed in [Section 8.1.3.4](#), the government aims at minimising the number of infected people until the end of the lockdown period, and we write rigorously its minimisation problem as

$$V_0^P := \sup_{(\alpha, \chi) \in \Xi} \sup_{\beta \in \mathcal{B}^*(\alpha, \chi)} \mathbb{E}^{\mathbb{P}^{\alpha, \beta}} \left[\chi - \int_0^T (c(I_t) + k(t, S_t, I_t, \alpha_t)) dt \right], \quad (8.3.10)$$

where the functions $c : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and $k : [0, T] \times \mathbb{R}_+ \times \mathbb{R}_+ \times A \rightarrow \mathbb{R}$ were introduced in [Section 8.1.3.4](#).

8.3.2 Optimal interaction of the population given tax and test policies

8.3.2.1 A relevant contract form

Since the fine policy χ is an $\mathcal{F}_T$ -measurable random variable, where $\mathbb{F}$ is the filtration generated by the process (S, I) , we should expect that in general $V_0^A(\alpha, \chi) = v(0, s_0, i_0)$, where the map $v : [0, T] \times \mathcal{C}_T^2 \rightarrow \mathbb{R}$ satisfies an informal Hamilton Jacobi Bellman (HJB for short) equation, and as such has the dynamic

$$dv(t, S_t, I_t) = -H(S_t, I_t, Z_t^s, Z_t^i, \alpha_t) dt + Z_t^s dS_t + Z_t^i dI_t,$$

where the population's Hamiltonian $H : [0, T] \times (\mathbb{R}_+^*)^2 \times \mathbb{R}^2 \times A \longrightarrow \mathbb{R}$ is defined, for $(t, s, i, z, z', a) \in [0, T] \times (\mathbb{R}_+^*)^2 \times \mathbb{R}^2 \times A$, by

$$H(t, s, i, z, z', a) := \sup_{b \in B} h(t, s, i, z, z', a, b),$$

where $h(t, s, i, z, z', a, b) := (\lambda - \mu s + \nu i - b\sqrt{a}si)z - ((\mu + \nu + \rho + \gamma)i - b\sqrt{a}si)z' + u(t, b, i)$, for $b \in B$.

In particular, defining $Z := Z^s - Z^i$, we should have

$$\begin{aligned} U(-\chi) &= V_0^A(\alpha, \chi) - \int_0^T H(t, S_t, I_t, Z_t^s, Z_t^i, \alpha_t) dt + \int_0^T Z_t^s dS_t + \int_0^T Z_t^i dI_t \\ &= V_0^A(\alpha, \chi) - \int_0^T \sup_{b \in B} \{u(t, b, I_t) - b\sqrt{\alpha_t} S_t I_t (Z_t^s - Z_t^i)\} dt + \int_0^T \sigma \alpha_t S_t I_t (Z_t^s - Z_t^i) dW_t \\ &= V_0^A(\alpha, \chi) - \int_0^T \left((\mu + \nu + \rho + \gamma) I_t Z_t + \sup_{b \in B} \{u(t, b, I_t) - b\sqrt{\alpha_t} S_t I_t Z_t\} \right) dt - \int_0^T Z_t dI_t. \end{aligned} \quad (8.3.11)$$

Given the supremum appearing above, the following assumption will be useful for us.

Assumption 8.3.5. *There exists a unique Borel-measurable map $b^* : [0, T] \times \mathbb{R}_+^* \times \mathbb{R}_+^* \times \mathbb{R} \times A \longrightarrow B$ such that*

$$b^*(t, s, i, z, a) \in \operatorname{argmax}_{b \in B} \{u(t, b, i) - b\sqrt{a}si\}, \quad \forall (t, s, i, z, a) \in [0, T] \times (\mathbb{R}_+^*)^2 \times \mathbb{R} \times A. \quad (8.3.12)$$

Remark 8.3.6. *We would like to insist on the fact that, for the SIR model, and in view of Remark 8.3.3, it is not necessary to consider that the process R is a state variable. Indeed, its value at time t can be deduced from the paths of I until time t . More precisely, following the previous reasoning to find the relevant form of contracts, one could consider*

$$dv(t, S_t, I_t) = -\tilde{H}(S_t, I_t, R_t, Z_t^s, Z_t^i, \alpha_t) dt + Z_t^s dS_t + Z_t^i dI_t + Z_t^r dR_t,$$

where, in this case, the population's Hamiltonian $\tilde{H} : [0, T] \times (\mathbb{R}_+^*)^2 \times \mathbb{R}^2 \times A$ is defined by

$$\tilde{H}(t, s, i, r, z, z', \tilde{z}, a) := \sup_{b \in B} \{h(t, s, i, z, z', a, b)\} + (\gamma i - \mu r)\tilde{z},$$

for any $(t, s, i, z, z', \tilde{z}, a) \in [0, T] \times (\mathbb{R}_+^*)^2 \times \mathbb{R}^3 \times A$. Since the dynamics of R is deterministic and not controlled, a simplification occurs between the additional part of the Hamiltonian $(\gamma i - \mu r)\tilde{z}$ and the integral with respect to dR , which leads to the same form for the utility function as previously mentioned, i.e., (8.3.11).

8.3.2.2 The general analysis

Let us start this section by defining two useful spaces. For any $\alpha \in \mathcal{A}$, and any $m \in \mathbb{N}^*$, we define $\mathcal{S}^m(\mathbb{P}^\alpha)$ and $\mathbb{H}^m(\mathbb{P}^\alpha)$ as respectively the sets of $\mathbb{R}$ -valued, $\mathbb{F}^{\mathbb{P}^\alpha}$ -adapted continuous processes Y such that $\|Y\|_{\mathcal{S}^m(\mathbb{P}^\alpha)} < \infty$, and the set of $\mathbb{F}^{\mathbb{P}^\alpha}$ -predictable, $\mathbb{R}$ -valued processes Z with $\|Z\|_{\mathbb{H}^m(\mathbb{P}^\alpha)} < \infty$, where

$$\|Y\|_{\mathcal{S}^m(\mathbb{P}^\alpha)}^m := \mathbb{E}^{\mathbb{P}^\alpha} \left[\sup_{t \in [0, T]} |Y_t|^m \right], \quad \|Z\|_{\mathbb{H}^m(\mathbb{P}^\alpha)}^m := \mathbb{E}^{\mathbb{P}^\alpha} \left[\left(\int_0^T |\hat{\sigma}_s Z_s|^2 ds \right)^{m/2} \right], \quad (Y, Z) \in \mathcal{S}^m(\mathbb{P}^\alpha) \times \mathbb{H}^m(\mathbb{P}^\alpha).$$

Theorem 8.3.7. *Let $(\alpha, \chi) \in \Xi$. There exists a unique $\mathcal{F}_0^{\mathbb{P}^\alpha}$ -measurable random variable Y_0 and a unique $Z \in \mathbb{H}^p(\mathbb{P}^\alpha)$ such that*

$$U(-\chi) = Y_0 - \int_0^T \left(Z_t(\mu + \nu + \rho + \gamma)I_t + u(t, \beta_t^*, I_t) - \beta_t^* \sqrt{\alpha_t} S_t I_t Z_t \right) dt - \int_0^T Z_t dI_t, \quad \mathbb{P}^\alpha\text{-a.s.}, \quad (8.3.13)$$

with $\beta_t^* := b^*(t, S_t, I_t, Z_t, \alpha_t)$ for all $t \in [0, T]$. Moreover, $\mathcal{B}^*(\alpha, \chi) = \{\beta^*\}$ and $V_0^A(\alpha, \chi) = \mathbb{E}^{\mathbb{P}^\alpha}[Y_0]$.

Proof. Fix $(\alpha, \chi) \in \Xi$ as in the statement of the theorem. Let us consider the solution (Y, Z) of the following BSDE

$$Y_t = U(-\chi) + \int_t^T \sup_{b \in B} \{u(r, b, I_r) - Z_r b \sqrt{\alpha_r} S_r I_r\} dr - \int_t^T Z_r \sigma \alpha_r S_r I_r dW_r, \quad t \in [0, T]. \quad (8.3.14)$$

Since $\chi \in \mathfrak{C}$, u is continuous, I and S are bounded, and B is a compact set, it is immediate this BSDE is well-posed and admits a unique solution $(Y, Z) \in \mathcal{S}^p(\mathbb{P}^\alpha) \times \mathbb{H}^p(\mathbb{P}^\alpha)$ (in a more general context, one may refer for instance to Bouchard, Possamaï, Tan, and Zhou [67, Theorem 4.1] (2018)). Therefore, using the dynamic of I under $\mathbb{P}^\alpha$, given by (8.3.2), as well as the definition of β^* , and letting $t = 0$, we obtain that (8.3.13) is satisfied. Next, using this representation for $U(\chi)$, notice that for any $\beta \in \mathcal{B}$, we have

$$\begin{aligned} & \mathbb{E}^{\mathbb{P}^{\alpha, \beta}} \left[\int_0^T u(t, \beta_t, I_t) dt + U(-\chi) \right] \\ &= \mathbb{E}^{\mathbb{P}^{\alpha, \beta}} \left[Y_0 + \int_0^T \left(u(t, \beta_t, I_t) - Z_t(\mu + \nu + \rho + \gamma)I_t - u(t, \beta_t^*, I_t) + \beta_t^* \sqrt{\alpha_t} S_t I_t Z_t \right) dt - \int_0^T Z_t dI_t \right] \\ &= \mathbb{E}^{\mathbb{P}^\alpha} [Y_0] + \sup_{\beta \in \mathcal{B}} \mathbb{E}^{\mathbb{P}^{\alpha, \beta}} \left[\int_0^T \left(u(t, \beta_t, I_t) - \beta_t S_t I_t Z_t - u(t, \beta_t^*, I_t) + \beta_t^* \sqrt{\alpha_t} S_t I_t Z_t \right) dt \right] \\ &\leq \mathbb{E}^{\mathbb{P}^\alpha} [Y_0], \end{aligned}$$

where we used the fact that $Z \in \mathbb{H}^p(\mathbb{P}^\alpha)$, and that the process

$$\mathcal{E}^\beta := \exp \left(- \int_0^\cdot \frac{\beta_s}{\sigma \sqrt{\alpha_s}} dW_s - \frac{1}{2} \int_0^\cdot \frac{|\beta_s|^2}{|\sigma|^2 \alpha_s} ds \right),$$

is continuous, and both an $(\mathbb{F}^{\mathbb{P}^\alpha}, \mathbb{P}^\alpha)$ - and an $(\mathbb{F}^{\mathbb{P}^{\alpha+}}, \mathbb{P}^\alpha)$ -martingale (see for instance Neufeld and Nutz [265, Proposition 2.2] (2014)), so that for any $\beta \in \mathcal{B}$

$$\mathbb{E}^{\mathbb{P}^{\alpha, \beta}} [Y_0] = \mathbb{E}^{\mathbb{P}^\alpha} [\mathcal{E}_T^\beta Y_0] = \mathbb{E}^{\mathbb{P}^\alpha} [\mathcal{E}_0^\beta Y_0] = \mathbb{E}^{\mathbb{P}^\alpha} [Y_0].$$

The previous inequality implies that

$$V_0^A(\alpha, \chi) \leq \mathbb{E}^{\mathbb{P}^\alpha} [Y_0].$$

Moreover, thanks to Assumption 8.3.5, equality is achieved if and only if we choose the control β^* . This shows that

$$V_0^A(\alpha, \chi) = \mathbb{E}^{\mathbb{P}^\alpha} [Y_0], \text{ and } \mathcal{B}^*(\alpha, \chi) = \{\beta^*\}.$$

□

In the previous result, the fact that (8.3.13) holds with an $\mathcal{F}_0^{\mathbb{P}^{\alpha+}}$ -measurable random variable and not a constant is somewhat annoying. The next lemma shows that we can actually have the representation with a constant without loss of generality.

Lemma 8.3.8. *Let $\alpha \in \mathcal{A}$, and fix an $\mathcal{F}_0^{\mathbb{P}^{\alpha+}}$ -measurable random variable Y_0 and some $Z \in \mathbb{H}^p(\mathbb{P}^\alpha)$. Define the following contracts*

$$\begin{aligned} \chi &:= -U^{(-1)} \left(Y_0 - \int_0^T \left(Z_t(\mu + \nu + \rho + \gamma)I_t + u(t, \beta_t^*, I_t) - \beta_t^* \sqrt{\alpha_t} S_t I_t Z_t \right) dt - \int_0^T Z_t dI_t \right), \\ \chi' &:= -U^{(-1)} \left(\mathbb{E}^{\mathbb{P}^\alpha} [Y_0] - \int_0^T \left(Z_t(\mu + \nu + \rho + \gamma)I_t + u(t, \beta_t^*, I_t) - \beta_t^* \sqrt{\alpha_t} S_t I_t Z_t \right) dt - \int_0^T Z_t dI_t \right). \end{aligned}$$

Then

$$V_0^A(\alpha, \chi) = V_0^A(\alpha, \chi') = \mathbb{E}^{\mathbb{P}^\alpha} [Y_0], \quad \mathcal{B}^*(\alpha, \chi) = \mathcal{B}^*(\alpha, \chi') = \{\beta^*\}.$$

Proof. The equalities for (α, χ) are immediate from [Theorem 8.3.7](#). For (α, χ') , we have, using the fact that $Z \in \mathbb{H}^p(\mathbb{P}^\alpha)$, and thus $Z \in \mathbb{H}^q(\mathbb{P}^{\alpha, \beta})$ for any $\beta \in \mathcal{B}$ and any $q \in (1, p)$

$$\begin{aligned} V_0^A(\alpha, \chi') &= \mathbb{E}^{\mathbb{P}^\alpha}[Y_0] + \sup_{\beta \in \mathcal{B}} \mathbb{E}^{\mathbb{P}^{\alpha, \beta}} \left[\int_0^T \left(u(t, \beta_t, I_t) - \beta_t \sqrt{\alpha_t} S_t I_t Z_t - u(t, \beta_t^*, I_t) + \beta_t^* \sqrt{\alpha_t} S_t I_t Z_t \right) dt \right] \\ &\leq \mathbb{E}^{\mathbb{P}^\alpha}[Y_0]. \end{aligned}$$

Since the equality is attained if and only if we choose $\beta = \beta^*$, this ends the proof. $\square$

8.3.2.3 Characterisation of the class of admissible contracts

We introduce the class $\bar{\Xi}$ of contracts defined by all pairs $(\alpha, U^{(-1)}(-Y_T^{y_0, Z}))$ with $\alpha \in \mathcal{A}$, and $Y^{y_0, Z}$ a process given, $\mathbb{P}^\alpha$ -a.s., for all $t \in [0, T]$ by

$$Y_t^{y_0, Z} = y_0 - \int_0^t \left(Z_r(\mu + \nu + \rho + \gamma)I_r + u^*(r, S_r, I_r, Z_r, \alpha_r) - b^*(r, S_r, I_r, Z_r, \alpha_r)\sqrt{\alpha_r}S_rI_rZ_r \right) dr - \int_0^t Z_r dI_r,$$

with $Z \in \mathbb{H}^p(\mathbb{P}^\alpha)$ and $y_0 \in [\underline{v}, \infty)$, recalling that $u^*(t, s, i, z, a) := u(t, b^*(t, s, i, z, a), i)$ for all $(t, s, i, z, a) \in [0, T] \times \mathbb{R}_+^2 \times \mathbb{R} \times A$. We also denote for simplicity $\mathbb{P}^{*, \alpha, Z} := \mathbb{P}^{\alpha, b^*(S, I, Z)}$.

Lemma 8.3.9. *The problem of the government given by (8.3.10) can be rewritten*

$$V_0^P = \sup_{(\alpha, Z) \in \mathcal{A} \times \mathbb{H}^p(\mathbb{P}^\alpha)} \mathbb{E}^{\mathbb{P}^{*, \alpha, Z}} \left[-U^{(-1)}(Y_T^{\underline{v}, Z}) - \int_0^T (c(I_s) + k(s, S_s, I_s, \alpha_s)) ds \right]. \quad (8.3.15)$$

Proof. From [Theorem 8.3.7](#) and [Lemma 8.3.8](#), we know that $\Xi \subset \bar{\Xi}$. To prover the reverse inclusion, let us now consider a pair $(\alpha, -U^{(-1)}(Y_T^{y_0, Z})) \in \bar{\Xi}$. We simply need to ensure that $-U^{(-1)}(Y_T^{y_0, Z}) \in \mathfrak{C}$. We have, using the fact that u is continuous, B is compact, α is bounded below by ε , and S and I are bounded, that there exists some constant $C > 0$, which may change value from ligne to ligne, such that

$$\begin{aligned} \mathbb{E}^{\mathbb{P}^\alpha} [|Y_T^{y_0, Z}|^p] &\leq C \left(1 + \mathbb{E}^{\mathbb{P}^\alpha} \left[\left(\int_0^T |S_r I_r Z_r| dr \right)^p + \left| \int_0^T \hat{\sigma}_r Z_r dW_r \right|^p \right] \right) \\ &\leq C \left(1 + \mathbb{E}^{\mathbb{P}^\alpha} \left[\left(\int_0^T \sigma \alpha_r |S_r I_r Z_r| dr \right)^p \right] + \|Z\|_{\mathbb{H}^p(\mathbb{P}^\alpha)}^p \right) \\ &\leq C(1 + \|Z\|_{\mathbb{H}^p(\mathbb{P}^\alpha)}^p) < \infty, \end{aligned}$$

where we used Burkholder–Davis–Gundy’s inequality and Cauchy–Schwarz’s inequality. This proves the reverse inclusion and thus that $\Xi = \bar{\Xi}$.

Next, we use [Lemma 8.3.8](#) to realise that :

$$\mathcal{B}^*(\alpha, -U^{(-1)}(Y_T^{y_0, Z})) = \{b^*(\cdot, S, I, Z, \alpha)\}, \text{ and } V_0^A(\alpha, -U^{(-1)}(Y_T^{y_0, Z})) = y_0,$$

which implies

$$V_0^P = \sup_{y_0 \geq \underline{v}} \sup_{(\alpha, Z) \in \mathcal{A} \times \mathbb{H}^p(\mathbb{P}^\alpha)} \mathbb{E}^{\mathbb{P}^{*, \alpha, Z}} \left[-U^{(-1)}(Y_T^{y_0, Z}) - \int_0^T (c(I_s) + k(s, S_s, I_s, \alpha_s)) ds \right].$$

To conclude, it is enough to notice that the following map

$$[\underline{v}, \infty) \ni y_0 \longmapsto \mathbb{E}^{\mathbb{P}^{*, \alpha, Z}} \left[-U^{(-1)}(Y_T^{y_0, Z}) - \int_0^T (c(I_s) + k(s, S_s, I_s, \alpha_s)) ds \right] \in \mathbb{R},$$

is non-increasing. $\square$

8.3.3 Optimal tax and test policies under moral hazard for epidemic models

8.3.3.1 Weak formulation for the government's problem

[Lemma 8.3.9](#) states that the problem of the government can be reduced to a more standard stochastic control problem. However, in the current formulation, one of the three state variables, namely Y , is considered in the strong formulation, while the other state variables S and I are considered in weak formulation. Indeed, the variable Y is indexed by the control Z , while the control (α, Z) only impacts the distribution of S and I through $\mathbb{P}^{\star, \alpha, Z}$. As highlighted by [Cvitanic and Zhang \[106, Remark 5.1.3\] \(2012\)](#), it makes little sense to consider a control problem of this form directly. Therefore, contrary to what is usually done in principal-agent problems (see, *e.g.*, [Cvitanic, Possamaï, and Touzi \[111\] \(2018\)](#)), we decided to adopt the weak formulation to rigorously write the problem of the principal, since this is the formulation which makes sense for the agent's problem. We will thus formulate it below, for the sake of thoroughness.⁹

Let $V := \mathbb{R} \times A$ and consider the sets $\mathbb{V}$ and $\mathcal{V}_o$ defined using V , as we defined $\mathbb{A}$ and $\mathcal{A}_o$ from A . The intuition is that the principal's problem depends only on time and on the state variable $X = (S, I, Y)$. Following the same methodology used for the agent's problem, to properly define the weak formulation of the principal's problem, we are led to consider the following canonical space

$$\Omega^P := \mathcal{C}^3 \times \mathbb{V},$$

with canonical process (S, I, Y, Λ^P) , where for any $(t, \mathfrak{s}, \iota, y, q) \in [0, T] \times \Omega^P$

$$S_t(\mathfrak{s}, \iota, y, q) := \mathfrak{s}(t), \quad I_t(\mathfrak{s}, \iota, y, q) := \iota(t), \quad Y_t(\mathfrak{s}, \iota, y, q) := y(t), \quad \Lambda^P(\mathfrak{s}, \iota, y, q) := q.$$

We let $\mathcal{G}$ be the Borel σ -algebra on Ω^P , and $\mathbb{G} := (\mathcal{G}_t)_{t \in [0, T]}$ the natural filtration of (S, I, Y, Λ^P) , defined in the same way as $\mathbb{F}$ in the previous canonical space Ω (see [Section 8.3.1](#)). Let then $\mathcal{M}^P$ be the set of probability measures on $(\Omega^P, \mathcal{G}_T)$. For any $\mathbb{P} \in \mathcal{M}^P$, we can define $\mathbb{G}^{\mathbb{P}}$ the $\mathbb{P}$ -augmentation of $\mathbb{G}$, its right limit $\mathbb{G}^{\mathbb{P}+}$, as well as $\mathbb{F}^{\Pi} := (\mathcal{F}_t)_{t \in [0, T]}^{\Pi}$ the Π -universal completion of $\mathbb{F}$ for any subset $\Pi \subset \mathcal{M}^P$.

The drift and volatility functions for the process X are now defined by

$$B^P(t, s, i, z, a) := \begin{pmatrix} \lambda - \mu s + \nu i - b^{\star}(t, s, i, z, a) \sqrt{a} s i \\ -(\mu + \nu + \rho + \gamma) i + b^{\star}(t, s, i, z, a) \sqrt{a} s i \\ -u^{\star}(t, s, i, z, a) \end{pmatrix}, \quad \Sigma^P(s, i, z, a) := \sigma a s i \begin{pmatrix} 1 \\ -1 \\ z \end{pmatrix}, \quad (8.3.16)$$

for any $(t, s, i, z, a) \in [0, T] \times (\mathbb{R}_+^{\star})^2 \times V$, where in addition where $u^{\star}(t, s, i, z, a) := u(t, b^{\star}(t, s, i, z, a), i)$. For any $(t, \varphi) \in [0, T] \times \mathcal{C}_b^2(\mathbb{R}^3, \mathbb{R})$, we define

$$M_t^P(\varphi) := \varphi(X_t) - \iint_{[0, t] \times V} \left(B^P(r, S_r, I_r, v) \cdot \nabla \varphi(X_r) + \frac{1}{2} \text{Tr}[\nabla^2 \varphi(X_r) (\Sigma^P(\Sigma^P)^{\top})(r, S_r, I_r, v)] \right) \Lambda^P(dr, dv).$$

In the spirit of [Definition 8.3.1](#) for $\mathcal{P} \subset \mathcal{M}$, we define the subset $\mathcal{Q} \subset \mathcal{M}^P$ as the one consisting of all $\mathbb{P} \in \mathcal{M}^P$ such that

- (i) $M^P(\varphi)$ is a $(\mathbb{G}, \mathbb{P})$ -local martingale on $[0, T]$ for all $\varphi \in \mathcal{C}_b^2(\mathbb{R}^3, \mathbb{R})$;
- (ii) $\mathbb{P}[X_0 = x_0] = 1$, where $x_0 := (s_0, i_0, \underline{v})$;

9. Notice that at the end of the day, this is not really an issue. Indeed, provided that the problem has enough regularity (typically some semi-continuity of the terminal and running reward with respect to state), one can expect the strong and weak formulations to coincide. See for instance [El Karoui and Tan \[138, Theorem 4.5\] \(2013\)](#)

(iii) $\mathbb{P}[\Lambda^P \in \mathcal{V}_o] = 1$.

Still following the line of [Section 8.3.1](#), we know that for any $\mathbb{P} \in \mathcal{Q}$, we can define a $(\mathbb{G}^{\mathcal{Q}}, \mathbb{P})$ -Brownian motion W^P . We then denote by $\mathcal{V}_o(\mathbb{P})$ the set of $\mathbb{G}$ -predictable and V -valued process (Z, α) such that, $\mathbb{P}$ -a.s. and for all $t \in [0, T]$,

$$\begin{cases} S_t = s_0 + \int_0^t (\lambda - \mu S_r + \nu I_r - b^*(r, S_r, I_r, Z_r, \alpha_r) \sqrt{\alpha_r} S_r I_r) dr + \int_0^t \sigma \alpha_r S_r I_r dW_r^P, \\ I_t = i_0 - \int_0^t ((\mu + \nu + \rho + \gamma) I_r - b^*(r, S_r, I_r, Z_r, \alpha_r) \sqrt{\alpha_r} S_r I_r) dr - \int_0^t \sigma \alpha_r S_r I_r dW_r^P, \\ Y_t = \underline{v} - \int_0^t u^*(r, S_r, I_r, Z_r, \alpha_r) dr + \int_0^t Z_r \sigma \alpha_r S_r I_r dW_r^P. \end{cases} \quad (8.3.17)$$

8.3.3.2 Solving the government's problem

Thank to the analysis conducted in the previous subsection, the problem of the government given by [\(8.3.10\)](#) can now be written rigorously in weak formulation

$$V_0^P = \sup_{\mathbb{P} \in \mathcal{Q}} \mathbb{E}^{\mathbb{P}} \left[-U^{(-1)}(Y_T) - \int_0^T (c(I_s) + k(s, S_s, I_s, \alpha_s)) ds \right]. \quad (8.3.18)$$

We then define the Hamiltonian of the government, for all $t \in [0, T]$, $x := (s, i, y) \in \mathbb{R}^3$ and $(p, M) \in \mathbb{R}^3 \times \mathbb{S}^3$, by

$$H^P(t, x, p, M) := \sup_{(z, a) \in V} \left\{ B^P(t, s, i, z, a) \cdot p + \frac{1}{2} \text{Tr} \left[M (\Sigma^P (\Sigma^P)^\top)(t, s, i, z, a) \right] - k(t, s, i, a) \right\} - c(i),$$

where $\mathbb{S}^3$ represents the set of 3×3 symmetric positive matrices with real entries. More explicitly, the Hamiltonian can be written as follows

$$\begin{aligned} H^P(t, x, p, M) = & \sup_{z \in \mathbb{R}, a \in A} \left\{ b^*(t, s, i, z, a) \sqrt{a} s i (p_2 - p_1) - u^*(t, s, i, z, a) p_3 + \frac{1}{2} |\sigma a s i|^2 f(z, M) - k(t, s, i, a) \right\} \\ & + (\lambda - \mu s + \nu i) p_1 - (\mu + \nu + \rho + \gamma) i p_2 - c(i), \end{aligned}$$

where $f(z, M) := M_{11} - 2M_{12} + M_{22} - 2z(M_{23} - M_{13}) + z^2 M_{33}$, for all $(z, M) \in \mathbb{R} \times \mathbb{S}^3$.

We are then led to consider the following HJB equation, for all $t \in [0, T]$ and $x = (s, i, y) \in \mathbb{R}^3$:

$$-\partial_t v(t, x) - H^P(t, x, \nabla_x v, \nabla_x^2 v) = 0, \quad (t, x) \in \mathcal{O}, \quad (8.3.19)$$

with terminal condition $v(T, x) := -U^{(-1)}(y)$, and where the natural domain over which the above PDE must be solved is ¹⁰

$$\mathcal{O} := \{(t, s, i, y) \in [0, T) \times \mathbb{R}_+^2 \times \mathbb{R} : 0 < s + i < F(t, s_0, i_0)\},$$

recalling that F is defined by [\(8.3.4\)](#).

10. The boundary of the domain cannot be reached by the processes S and I , which is why it is not necessary to specify a boundary condition there. Notice though that the upper bound can formally only be attained when I is constantly 0, in which case S becomes deterministic, and the government's best choice for α is clearly 1, and its choice of Z becomes irrelevant. In such a situation, we would immediately have $V_0^P = \underline{v}$.

Remark 8.3.10. *Standard arguments from viscosity solution theory allow to prove that $V_0^P = v^P(0, x_0)$ (recalling that $x_0 := (s_0, i_0, \underline{v})$) where v^P should be understood as the unique viscosity solution, in an appropriate class of functions, of the PDE (8.3.19). Obtaining further regularity results is by far more challenging. Indeed, it is a second-order, fully non-linear, parabolic PDE, which is clearly not uniformly elliptic, the corresponding diffusion matrix being degenerate. This makes the question of proving the existence of an optimal contract a very complicated one, which is clearly outside the scope of our study. As a sanity check though, we recall that ε -optimal contracts always exist, and can be indeed approximated numerically. See for instance Kharroubi, Lim, and Mastrolia [207] (2020) for an explicit construction of such ε -optimal contracts in a particular case dealing with the stochastic logistic equation.*

8.3.3.3 Comparison with the first-best case

As already mentioned, the first-best case corresponds to the case where the government can enforce whichever interaction rate $\beta \in \mathcal{B}$ it desires (in addition to a contract $(\alpha, \chi) \in \mathcal{A} \times \mathfrak{C}$), and simply has to satisfy the participation constraint of the population. In order to find the optimal interaction rate in this scenario, as well as the optimal contract, one has to solve the government's problem defined by (8.1.14).

The simplest way to take into account the inequality constraint in the definition of $V_0^{P,FB}$ is to introduce the associated Lagrangian. By strong duality, we then have

$$V_0^{P,FB} = \inf_{\varpi > 0} \sup_{(\alpha, \chi, \beta) \in \mathcal{A} \times \mathfrak{C} \times \mathcal{B}} \left\{ \mathbb{E} \left[\chi - \int_0^T (c(I_t) + k(t, S_t, I_t, \alpha_t)) dt \right] + \varpi \left(\mathbb{E} \left[\int_0^T u(t, \beta_t, I_t) dt + U(-\chi) \right] - \underline{v} \right) \right\}.$$

First, by concavity of U , it is immediate that for any given Lagrange multiplier $\varpi > 0$, the optimal tax is constant and given by (8.1.19). Then, using the definition of $\bar{V}_0(\varpi)$ for any $\varpi > 0$ in (8.1.20), we have:

$$V_0^{P,FB} = \inf_{\varpi > 0} \left\{ \chi^*(\varpi) + \varpi (U(-\chi^*(\varpi)) - \underline{v}) + \bar{V}_0(\varpi) \right\}.$$

Note that $\bar{V}_0(\varpi)$ is the value function of a standard stochastic control problem. Therefore, we expect to have $\bar{V}_0(\varpi) = v^\varpi(0, s_0, i_0)$, where the function $v^\varpi : [0, T] \times \mathbb{R}_+^2 \rightarrow \mathbb{R}$ solves the following HJB PDE

$$-\partial_t v^\varpi(t, s, i) + c(i) - (\lambda - \mu s + \nu i) \partial_s v^\varpi + (\mu + \nu + \rho + \gamma) i \partial_i v^\varpi - \mathcal{H}^\varpi(t, s, i, \partial v^\varpi, \nabla^2 v^\varpi) = 0,$$

for $(t, s, i) \in \mathcal{D}$, with terminal condition $v^\varpi(T, s, i) = 0$, $(s, i) \in \mathcal{D}_T$, where the Hamiltonian is defined, for $t \in [0, T]$, $(s, i) \in (\mathbb{R}_+^*)^2$, $p := (p_1, p_2) \in \mathbb{R}^2$ and $M \in \mathbb{S}^2$ by

$$\mathcal{H}^\varpi(t, s, i, p, M) := \sup_{(a, b) \in A \times B} \left\{ \varpi u(t, b, i) - b s i \sqrt{a} (p_1 - p_2) \right\} - k(t, s, i, a) + \frac{|\sigma s i a|^2}{2} |M| \Bigg\},$$

where $|M| := M_{11} - 2M_{12} + M_{22}$.

To simplify, let us consider separable utilities with the forms (8.1.17). We focus on the maximisation of the Hamiltonian $\mathcal{H}^\varpi$ with respect to $b \in B$, to obtain the optimal interaction rate β^ϖ . The maximiser b^ϖ is defined by

$$b^\varpi(s, i, p, a) := b^\circ(s, i, \sqrt{a}(p_1 - p_2)/\varpi), \text{ for all } (s, i, p, a) \in (\mathbb{R}_+^*)^2 \times \mathbb{R}^2 \times A,$$

recalling that b° is defined by (8.1.18). In particular, for a given testing policy $\alpha \in \mathcal{A}$ and a Lagrange multiplier $\varpi > 0$, the optimal interaction rate in this case is given for all $t \in [0, T]$ by $\beta_t^\varpi = b^\varpi(S_t, I_t, \partial v^\varpi(t, S_t, I_t), \alpha_t)$. We thus obtain, for all $(t, s, i, p, M) \in [0, T] \times (\mathbb{R}_+^*)^2 \times \mathbb{R}^2 \times \mathbb{S}^2$,

$$\mathcal{H}^\varpi(t, s, i, p, M) = \sup_{a \in A} \left\{ \varpi u^\varpi(t, s, i, p, a) - b^\varpi(s, i, p, a) s i \sqrt{a} (p_1 - p_2) - k(t, s, i, a) + \frac{|\sigma s i a|^2}{2} |M| \right\},$$

where in addition for $a \in A$, $u^\varpi(t, s, i, p, a) := u(t, b^\varpi(s, i, p, a), i)$. Then, the optimal testing policy is given for all $t \in [0, T]$ by $\alpha_t^\varpi := a^\varpi(t, S_t, I_t, \partial v^\varpi(t, S_t, I_t), \nabla^2 v^\varpi(t, S_t, I_t))$, where $a^\varpi : [0, T] \times (\mathbb{R}_+^*)^2 \times \mathbb{R}^2 \times \mathbb{S}^2 \rightarrow A$ is the maximiser of the previous Hamiltonian on $a \in A$, if it exists.

8.4 Extensions and generalisations

8.4.1 Diseases with latency periods: SEIS, SEIR

We now focus on the SEIR/S (Susceptible – Exposed – Infected – Recovered or Susceptible) compartment model. Again, the class S represents the ‘Susceptible’ and the class I represents the ‘Infected’ and infectious. The SEIR and SEIS models are used to describe epidemics in which individuals are not directly contagious after contracting the disease. This therefore involves a fourth class, namely E , representing the ‘Exposed’, *i.e.*, individuals who have contracted the disease but are not yet infectious. With this in mind, we denote by ι the rate at which an exposed person becomes infectious, which is assumed to be a fixed non-negative constant. Therefore, during the epidemic, each individual can be either ‘Susceptible’ or ‘Exposed’ or ‘Infected’ or in ‘Recovery’, and (S_t, E_t, I_t, R_t) denotes the proportion of each category at time $t \geq 0$. The difference between SEIS and SEIR models is embedded into the immunity toward the disease: for SEIR models, it is assumed that the immunity is permanent, *i.e.*, after being infected, an individual goes and stays in the class R , whereas for SEIS models, there is no immunity, *i.e.*, infected individual come back in the susceptible class at rate $\nu \geq 0$, similarly to SIS models.

As in the previously described SIR model, we also take into account the demographic dynamics of the population, through the parameters λ , μ and ρ . To sum up, the epidemic dynamics is represented in Figure 8.18, and the (continuous-time) evolution of the disease is described by the following system (see for instance Mummert and Otunuga [260] (2019)):

$$\begin{cases} dS_t = (\lambda - \mu S_t - \beta_t S_t I_t + \nu I_t) dt, \\ dE_t = -((\mu + \iota) E_t - \beta_t S_t I_t) dt, \\ dI_t = -((\mu + \nu + \rho + \gamma) I_t - \iota E_t) dt, \\ dR_t = (\gamma I_t - \mu R_t) dt, \end{cases} \quad \text{for } t \in [0, T], \quad (8.4.1)$$

for a given initial distribution of individuals at time 0, denoted by $(s_0, e_0, i_0, r_0) \in \mathbb{R}_+^4$ and assumed to be known.

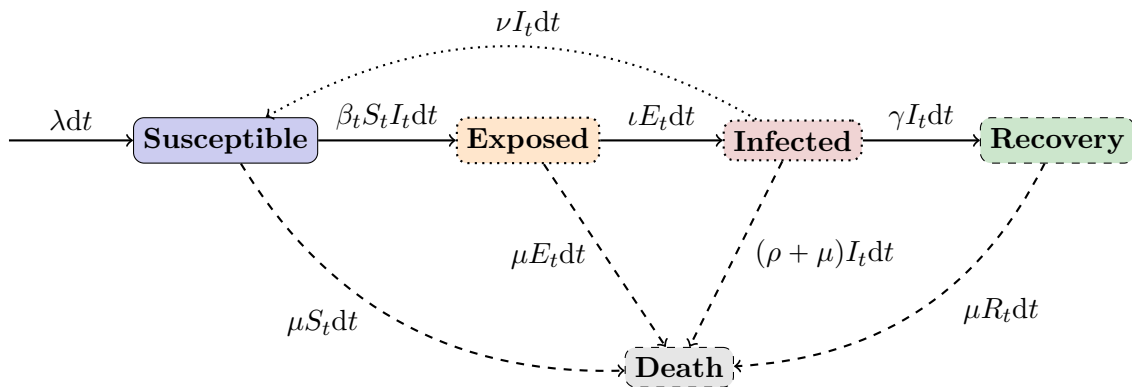


Figure 8.18 – SEIR/SEIS model with demographic dynamics

8.4.1.1 SEIS/SEIR controlled model

Similarly to the previous models, we consider that the deterministic model described by (8.4.1) is also subject to a noise in the estimation of the proportion of susceptible and infected individuals. Inspired by the stochastic model in Mummert and Otunuga [260, Equation (3)] (2019), the dynamics of the epidemic is now given by the following system

$$\begin{cases} S_t = s_0 + \int_0^t (\lambda - \mu S_s - \beta_s \sqrt{\alpha_s} S_s I_s + \nu I_s) ds + \int_0^t \sigma \alpha_s S_s I_s dW_s, \\ E_t = e_0 - \int_0^t ((\mu + \iota) E_s - \beta_s \sqrt{\alpha_s} S_s I_s) ds - \int_0^t \sigma \alpha_s S_s I_s dW_s, \\ I_t = i_0 - \int_0^t ((\mu + \nu + \rho + \gamma) I_s - \iota E_s) ds, \\ R_t = r_0 + \int_0^t (\gamma I_s - \mu R_s) ds, \end{cases} \quad \text{for } t \in [0, T]. \quad (8.4.2)$$

Note that the proportion I of infected and infectious is also uncertain, but only through its dependence on E and the proportion R of recovery is uncertain only through its dependence on I . More precisely, we assume that there is no uncertainty on both the recovery rate γ , the rate ι at which infected people becomes infectious and the (potentially) rate ν at which an individual loses immunity, implying that if the proportion of exposed individual is perfectly known, the proportion of infected is also known without uncertainty and consequently the proportion of recovery is also certainly known. Again this modelling choice is consistent with most stochastic SEIRS models, and emphasises that the major uncertainty in the current epidemic is related to the non-negligible proportion of (nearly) asymptomatic individuals. Indeed, an asymptomatic individual may be misclassified as susceptible or exposed.

8.4.1.2 The contracting problem

We will now give (informally) the optimisation problems faced by both the population and the government, the rigorous treatment can be done following the lines of Section 8.3. The most important change compared to SIS/SIR models is that the criteria should now depend on the sum $E + I$, representing the proportion of the population having contracted the disease, rather than just the proportion I of infectious people. Unless otherwise stated, the notations are those of Section 8.3.

The problem of the population is now

$$V_0^A(\alpha, \chi) := \sup_{\beta \in \mathcal{B}} \mathbb{E} \left[\int_0^T u(t, \beta_t, E_t + I_t) dt + U(-\chi) \right], \quad (8.4.3)$$

while that of the government becomes

$$V_0^P := \sup_{(\alpha, \chi) \in \Xi} \sup_{\beta \in \mathcal{B}^*(\alpha, \chi)} \mathbb{E} \left[\chi - \int_0^T (c(E_t + I_t) + k(t, S_t, I_t, \alpha_t)) dt \right]. \quad (8.4.4)$$

Notice that in the cost function k , we did not replace I by $I + E$. This is due to the fact that this cost should scale with the volatility of $I + E$ (see the discussion in Example 8.1.2), which is still $|\sigma \alpha \cdot S \cdot I|^2$ in the model (8.4.2).

The population's Hamiltonian is given by $H : [0, T] \times (\mathbb{R}_+^*)^3 \times \mathbb{R}_+ \times \mathbb{R}^4 \times A$ and defined, for any $(t, s, e, i, z^s, z^e, z^i, z^r, a) \in [0, T] \times (\mathbb{R}_+^*)^2 \times \mathbb{R}^4 \times A$, by

$$H(t, s, e, i, r, z^s, z^e, z^i, z^r, a) := \sup_{b \in B} \{h(t, s, e, i, z^s, z^e, a, b)\} + (\gamma i - \mu r) z^r - ((\mu + \nu + \rho + \gamma) i - \iota e) z^i,$$

$$\text{where } h(t, s, e, i, z^s, z^e, a, b) := (\lambda - \mu s + \nu i - b \sqrt{a s i}) z^s - ((\mu + \iota) e - b s i) z^e + u(t, b, i + e),$$

for $b \in B$. Given the supremum appearing above, and similarly to [Assumption 8.3.5](#), we make the following assumption.

Assumption 8.4.1. *There exists a unique Borel-measurable map $b^* : [0, T] \times (\mathbb{R}_+^*)^3 \times \mathbb{R} \times a \longrightarrow B$ such that*

$$b^*(t, s, e, i, z, a) \in \operatorname{argmax}_{b \in B} \{u(t, b, i + e) - b\sqrt{asiz}\}, \quad \forall (t, s, e, i, z, a) \in [0, T] \times (\mathbb{R}_+^*)^3 \times \mathbb{R} \times A. \quad (8.4.5)$$

Therefore, a straightforward adaption of our earlier arguments will show that every admissible contract will take the form $\chi := -U^{(-1)}(Y_T)$ where

$$Y_t := Y_0 - \int_0^t \left(Z_t(\mu + \iota)E_t + u(t, \beta_t^*, E_t + I_t) - \beta_t^* \sqrt{\alpha_t} S_t I_t Z_t \right) dt - \int_0^t Z_t dE_t, \quad (8.4.6)$$

where $\beta_t^* := b^*(t, S_t, E_t, I_t, Z_t, \alpha_t)$ for all $t \in [0, T]$ is the optimal control of the population. It thus remain to solve the government's problem. Unlike in the previous SIS/SIR models, there are now four state variables for the government's problem, namely (S, E, I, Y) , whose dynamic under the optimal effort of the population is as follows

$$\begin{cases} S_t = s_0 + \int_0^t (\lambda - \mu S_s - b^*(s, S_s, E_s, I_s, Z_s) \sqrt{\alpha_s} S_s I_s + \nu I_s) ds + \int_0^t \sigma \alpha_s S_s I_s dW_s, \\ E_t = e_0 - \int_0^t ((\mu + \iota)E_s - b^*(s, S_s, E_s, I_s, Z_s) \sqrt{\alpha_s} S_s I_s) ds - \int_0^t \sigma \alpha_s S_s I_s dW_s, \\ I_t = i_0 - \int_0^t ((\mu + \nu + \rho + \gamma)I_s - \iota E_s) ds, \\ Y_t = Y_0 - \int_0^t u^*(s, S_s, E_s, I_s, Z_s, \alpha_s) ds + \int_0^t Z_s \sigma \sqrt{\alpha_s} S_s I_s dW_s, \end{cases} \quad \text{for } t \in [0, T], \quad (8.4.7)$$

where $u^*(t, s, e, i, z, a) = u^*(t, b^*(t, s, e, i, z, a), i + e, a)$, for all $(t, s, e, i, z, a) \in [0, T] \times (\mathbb{R}_+^*)^3 \times \mathbb{R} \times A$.

We then define the Hamiltonian of the government, for all $t \in [0, T]$, $x := (s, e, y, i) \in \mathbb{R}^4$ and $(p, M) \in \mathbb{R}^4 \times \mathbb{S}^4$, by

$$\begin{aligned} H^P(t, x, p, M) = \sup_{(z, a) \in \mathbb{R} \times A} & \left\{ [\sqrt{asiz}(p_2 - p_1)b^* - p_4 u^*](t, s, e, i, z, a) + \frac{|\sigma asiz|^2}{2} f(z, M) - k(t, s, i, a) \right\} \\ & + (\lambda - \mu s + \nu i)p_1 - (\mu + \iota)ep_2 - ((\mu + \nu + \rho + \gamma)i - \iota e)p_3 - c(e + i), \end{aligned}$$

where $f(z, M) := M_{11} - 2M_{12} + M_{22} - 2z(M_{24} - M_{14}) + |z|^2 M_{44}$, for all $(z, M) \in \mathbb{R} \times \mathbb{S}^4$.

We are then led to consider the following HJB equation

$$-\partial_t v(t, x) - H^P(t, x, \nabla v, \nabla^2 v) = 0, \quad (t, x) \in \tilde{\mathcal{O}}, \quad (8.4.8)$$

with terminal condition $v(T, x) := -U^{(-1)}(y)$, and where the natural domain over which the above PDE must be solved is

$$\tilde{\mathcal{O}} := \{(t, s, e, i, y) \in [0, T] \times \mathbb{R}_+^3 \times \mathbb{R} : 0 < s + i + e < F(t, s_0, i_0 + e_0)\},$$

recalling that F is defined by [\(8.3.4\)](#).

Solving numerically [\(8.4.8\)](#) is really more challenging since it increases the dimension of the problem. A numerical investigation seems to be complicated as far as we now, and we left these numerical issues for future researches.

8.4.2 Beyond SEIS/SEIR models: a theoretically tractable method

There are of course plethora of generalisations of the models we have considered so far. For instance, in SEIRS (or also SIRS) models, the immunity is temporary, *i.e.* people in the class R may come back into the class S at rate ν . Using a similar stochastic extension of this model, it is straightforward that all our results extend, *mutatis mutandis*, to this case as well, albeit with one important difference: the control problem faced by the government now has 5 states variables, namely (S, E, I, R, Y) . Even more generally, our approach can readily be adapted to compartmental models considering additional classes: for instance the SIDARTHE (susceptible (S), infected (I), diagnosed (D), ailing (A), recognized (R), threatened (T), healed (H) and extinct (E)) model investigated by Giordano et al. [154] (2020) for COVID-19. Of course the price to pay is that the number of state variables in the government's problem will increase with the number of compartments, and numerical procedures to solve the HJB equation will become more delicate to implement, and should in general be based on neural networks.

8.5 Simulations for the SIS model

Similar to Section 8.2, we present in this appendix the numerical results obtained when considering a SIS compartmental model, whose dynamic is given by (8.1.3), or equivalently by (8.1.5) with $\gamma = 0$.

Choice of parameters. We take the same parameters as for the SIR case to model the preferences of the government and the population, *i.e.* the parameters given in Table 8.1, except for $\beta^{\max} = 0.5$. To model the SIS dynamic, we consider a different set of parameters, in order to obtain the same shape for the proportion of infected at the beginning of the epidemic in both cases of an SIR and SIS dynamics. This choice is made to model the fact that, at the beginning of a relatively unknown epidemic such as that of COVID-19, the proportion of infected people is observed (using noise), but the authorities do not necessarily know whether this disease allows immunity to be acquired. Therefore, the parameters used to model an SIS dynamic similar to the beginning of the SIR dynamic previously considered are given in Table 8.3.

	T	(s_0, i_0, r_0)	(λ, μ)	ρ	ν	γ	σ	β°
SIS model (8.1.3)	600	$(0.99984, 1.07 \times 10^{-4}, 0)$	$(0, 0)$	0.01	0.04	0	0.1	0.5

Table 8.3 – Set of parameters for the simulation of SIS dynamics

8.5.1 The benchmark case

To solve the benchmark case, we follow the method described in Section 8.2.3 when considering the SIR dynamics, although we choose here a number of time steps equal to 600. In particular, we choose a time step discretisation is equal to 0.0025, a linear interpolator, and the optimal command β used to maximise the Hamiltonian is discretised with 200 points given a step discretisation of 0.005. Once the PDE is solved, a simulator is used in forward using the optimal command and giving the dynamic of the proportion (S, I) . As for the numerical resolution of the benchmark case for the SIR model in Section 8.2.3, in order to check the accuracy of the method, we propose to implement two versions of the resolution. The first one is the direct resolution, while the second one consists in a slight change of variable, by considering $(S, S + I)$ as the couple of state variables.

As the numerical results obtained in the benchmark case when the epidemic dynamic is given by a SIS model have the same characteristics as with the SIR dynamic, we describe the graphs only briefly below.

Figure 8.19. We plot 100 trajectories of the optimal interaction rate β^* , the proportion of susceptible S , as well as the proportion of infected I , using the resolution with initial state variables, *i.e.*, (S, I) . The population behaviour is similar to the behaviour obtained by the SIR model: fits the population does as usual, then begin the react limiting it natural trend by reducing β and finally goes back as usual as the epidemic disappears. We observe once again that the population's fear of infection is not sufficient to prevent the epidemic.

Figure 8.20. As in the SIR case, the trajectories of β obtained through the two aforementioned resolutions are rather close, and the corresponding trajectories for I coincide.

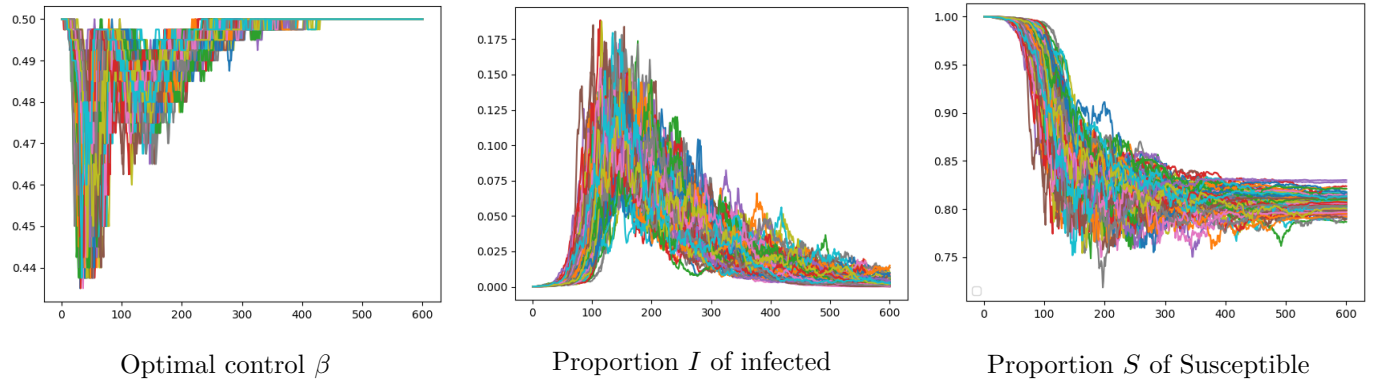


Figure 8.19 – Optimal trajectories in the benchmark case, for an SIS dynamic.

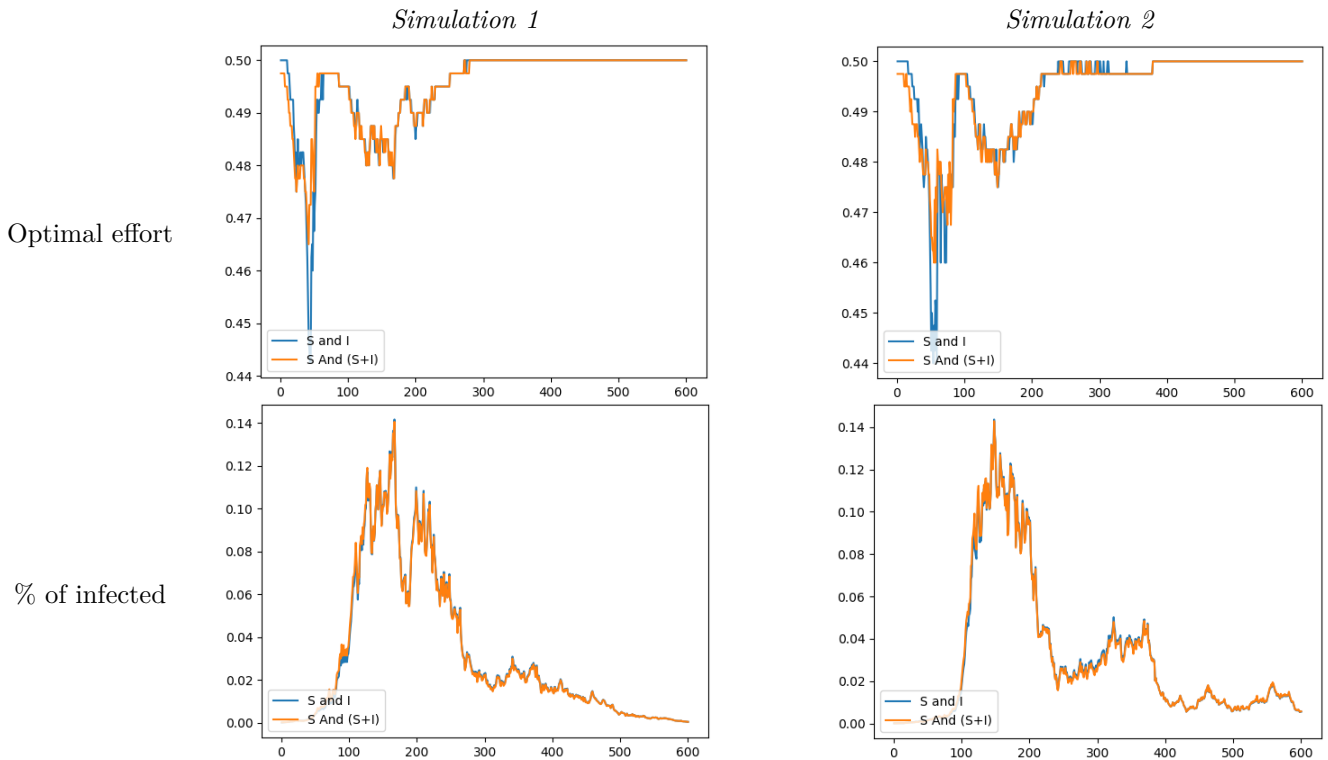


Figure 8.20 – Two simulations of the SIS in the benchmark case
Comparison between the two methods.

8.5.2 The lockdown policy without testing

As for the benchmark case, the numerical method to obtain the optimal lockdown policy is similar to the one used in the case of an SIR dynamics. We only recall here the key points of the method. We first solve (8.1.22) with the semi-Lagrangian scheme, taking $\underline{v}$ given by (8.2.2) and estimated with a Monte Carlo method, and by using an Euler scheme with a time-discretisation of 600 time steps and 10^6 trajectories. The estimated value for $\underline{v}$ is -0.0878063 . We then take a step discretisation for the grid in (s, i, y) corresponding to $(0.0025, 0.0025, 0.005)$, leading to a number of meshes at maturity equal to $150 \times 120 \times 1200$. We consider the bounded set of values $[-10, 10]$ for the control Z , and a step discretisation equal to 0.5.

The graphs obtained are briefly described below.

Figure 8.21. We present some trajectories of the optimal controls β and Z , as well as the resulting proportion I of infected individuals.

Figure 8.22. We compare on some simulations the optimal transmission rate obtained with the contract to the one obtained in the benchmark case. We see that the tax succeeds in reducing significantly the interaction rate compared to the no-tax policy case.

Figure 8.23. As a consequence, we see through different simulations of the trajectory of the proportion of infected I that a tax policy contains the spreading of the disease along the considered time period, contrary to the benchmark case. The optimal control thus allows to limit the high values of I .

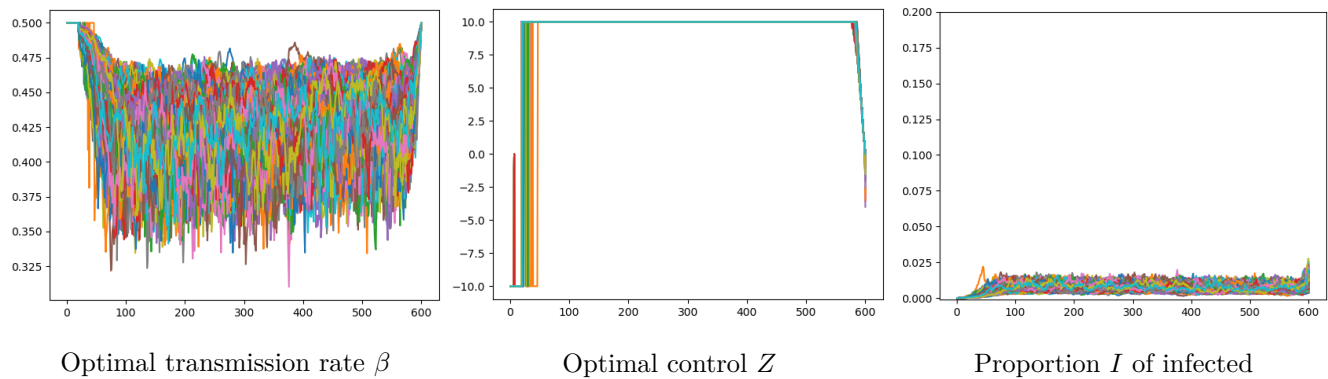


Figure 8.21 – 500 trajectories of optimal β , Z and I , without testing policy

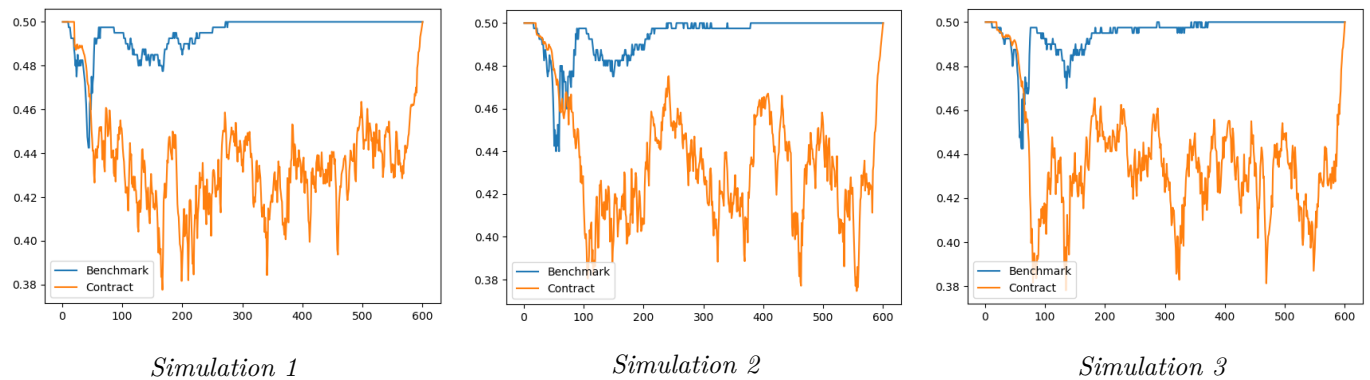


Figure 8.22 – Trajectories of the optimal transmission rate without testing policy
Comparison with the benchmark case.

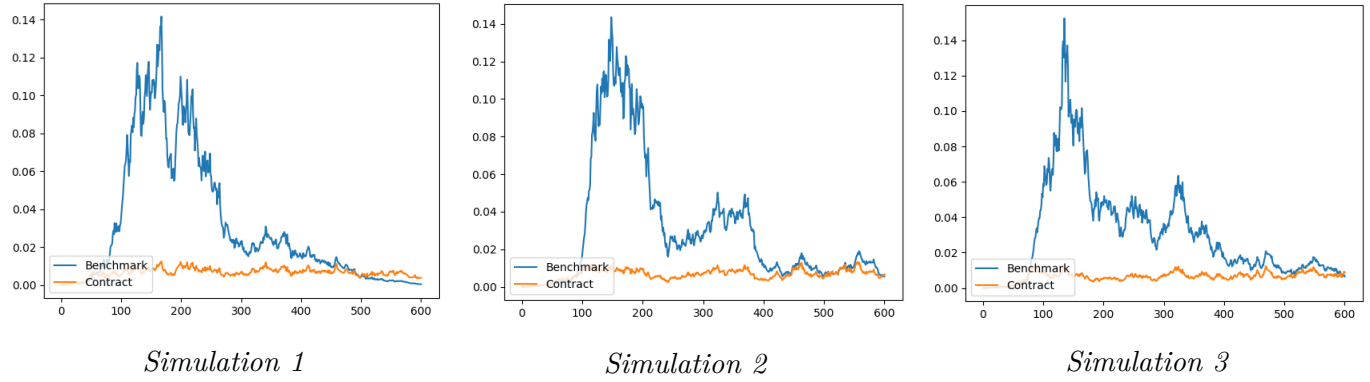


Figure 8.23 – Trajectories of I resulting from optimal lockdown without testing policy
Comparison with the benchmark case.

8.5.3 The testing policy

Due to the larger terminal time horizon, the computation time is particularly significant. To reduce it, the discretisation used to find the optimal control Z is reduced to 1. The resulting graphs are briefly described below.

Figure 8.24. We present trajectories of the optimal controls β , α and Z , and the resulting proportion I of infected.

Figure 8.25. We compare on simulations the optimal proportion of infected with the two previous cases (benchmark case and only tax policy): with testing, the epidemic is now totally under control.

Figure 8.26. We present simulations of the optimal effective transmission rate in this case, and compare it to the optimal β obtained in the benchmark case and without testing policy.

Figure 8.27. We present simulations of the optimal α : its quick variations explain the swift changes in the effective β .

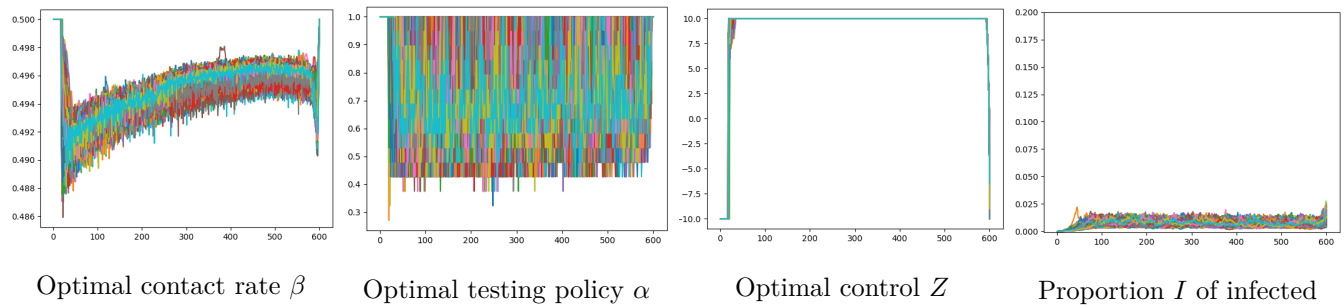


Figure 8.24 – 500 optimal trajectories of β , α , Z , and I , with testing policy

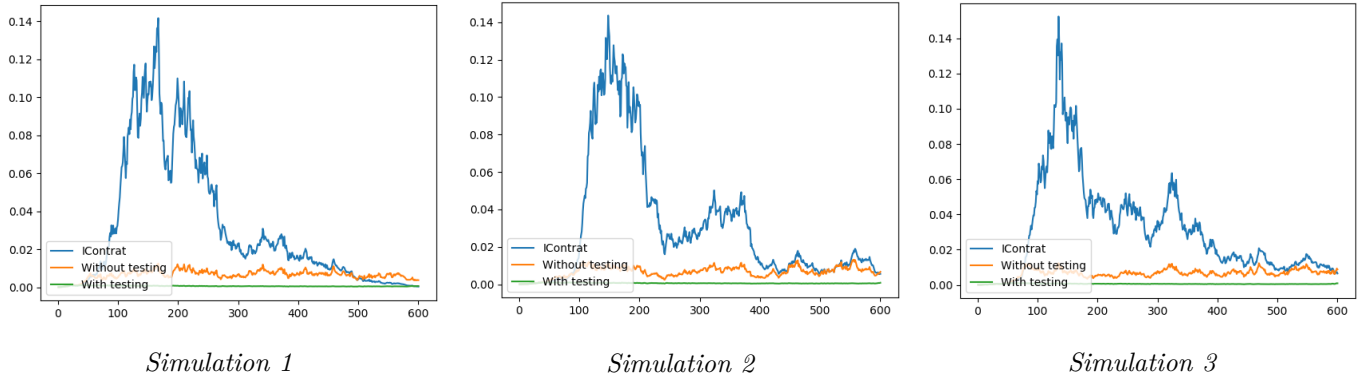


Figure 8.25 – Optimal trajectories of I with and without testing, as well as in the benchmark case.

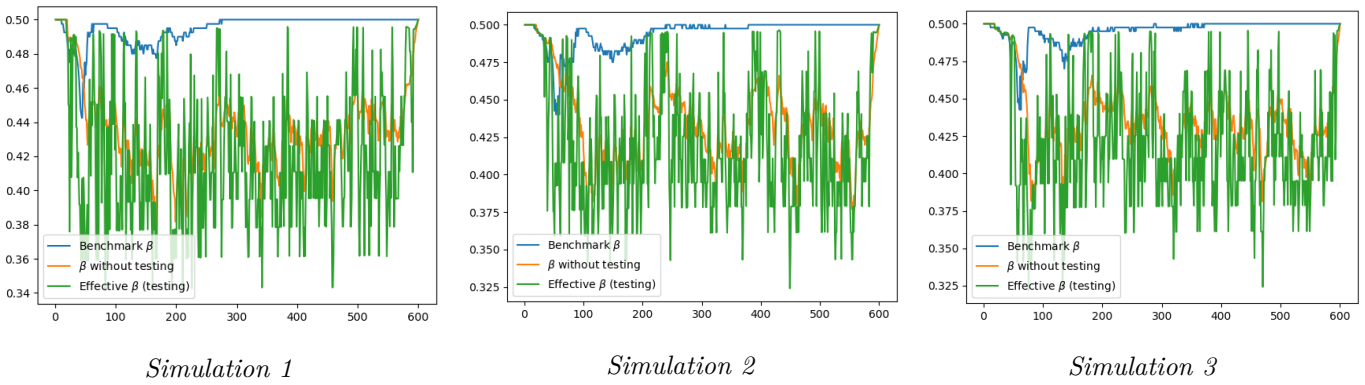


Figure 8.26 – Optimal effective transmission rate $\beta\sqrt{\alpha}$, compared to the previous cases.

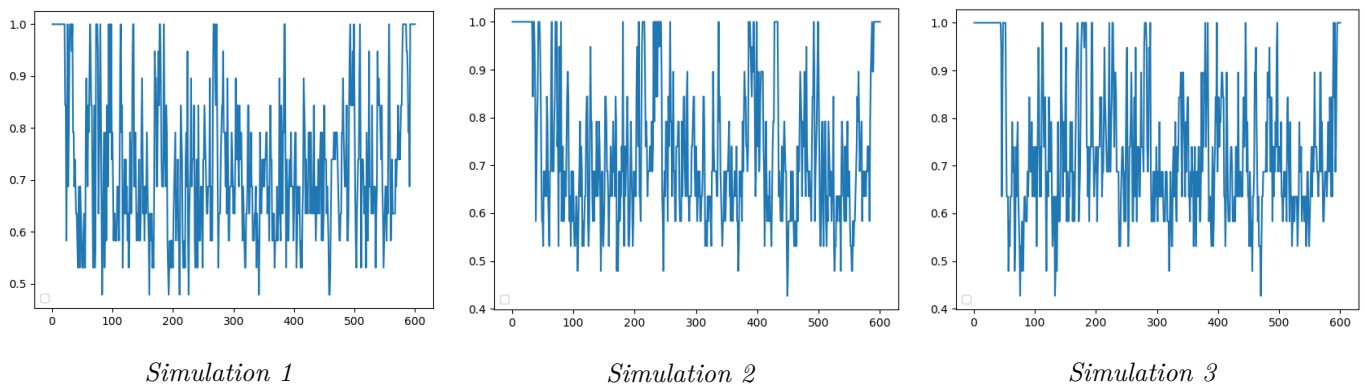


Figure 8.27 – Optimal trajectories of the testing policy α

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