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Sur quelques équations aux dérivées partielles fractionnaires, théorie et applications

Ali Tfayli

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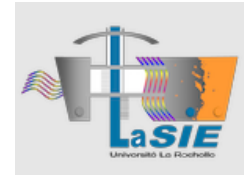
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pour l'obtention du grade de docteur
de l'université de La Rochelle
École doctorale : **EUCLIDE**
Laboratoire : **LaSIE**
Mention : **Mécanique des fluides**

présentée par
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**Sur quelques équations aux dérivées partielles
fractionnaires, théorie et applications**

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Chapitre 1

Introduction

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Dans cette thèse on s'intéresse à la résolution des problèmes liés à des équations aux **dérivées fractionnaires**. On a traité trois problèmes :

- Problèmes d'estimation de paramètre (**Problème d'identification**),
- **Problème inverse**,
- Problèmes variationnels et méthode **PGD**.

1.1 Outils mathématiques

On rappelle dans cette section certaines définitions et propriétés nécessaires qu'on utilisera dans la suite. Soit $f : [a, b] \rightarrow \mathbb{R}$ une fonction intégrable.

Définition 1.1.1 (Intégrale fractionnaire de Riemann-Liouville). On appelle intégrale de Riemann-Liouville à gauche d'ordre $0 < \alpha < 1$ de f l'expression :

$$I_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x-t)^{1-\alpha}} dt, \quad (1.1)$$

et on appelle intégrale de Riemann-Liouville à droite d'ordre $\alpha > 0$ de f l'expression :

$$I_b^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{f(t)}{(t-x)^{1-\alpha}} dt, \quad (1.2)$$

où Γ est la fonction Gamma d'Euler.

Définition 1.1.2 (Dérivée fractionnaire de Riemann-Liouville). On considère que $f \in AC^n([a, b])$. La dérivée fractionnaire de Riemann-Liouville à gauche d'ordre $\alpha > 0$ est définie par

$$D_x^\alpha f(x) = \frac{d^n}{dx^n} I_x^{n-\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^x (x-t)^{n-1-\alpha} f(t) dt, \quad n = [\alpha] + 1, \quad (1.3)$$

où $[\alpha] + 1$ désigne la partie entière de α .

La dérivée fractionnaire de Riemann-Liouville à droite d'ordre $\alpha > 0$ est définie par

$$D_b^\alpha f(x) = \frac{d^n}{dx^n} I_b^{n-\alpha} f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_x^b (t-x)^{n-1-\alpha} f(t) dt, \quad n = [\alpha] + 1. \quad (1.4)$$

En particulier, pour $0 < \alpha < 1$ on peut écrire les définitions de la dérivée :

1. La dérivée de Riemann-Liouville à gauche s'écrit

$${}_a D_x^\alpha u(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \left(\int_a^x (x-t)^{-\alpha} u(t) dt \right). \quad (1.5)$$

2. La dérivée de Riemann-Liouville à droite s'écrit

$${}_x D_b^\alpha u(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \left(\int_x^b (t-x)^{-\alpha} u(t) dt \right). \quad (1.6)$$

Il existe une autre définition de la dérivée fractionnaire, qui est attribuée à Caputo :

Définition 1.1.3 (Dérivée fractionnaire de Caputo). On considère que $f \in AC^n([a, b])$. La dérivée fractionnaire de Caputo à gauche d'ordre $\alpha > 0$ est définie par

$$\mathcal{D}_x^\alpha f(x) = I_x^{n-\alpha} f^{(n)}(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-1-\alpha} f^{(n)}(t) dt, \quad n = [\alpha] + 1, \quad (1.7)$$

où $f^{(n)}$ désigne la n -ème dérivée classique de f . La dérivée fractionnaire de Caputo à droite d'ordre $\alpha > 0$ est définie par

$$\mathcal{D}_b^\alpha f(x) = I_b^{n-\alpha} f^{(n)}(x) = \frac{1}{\Gamma(n-\alpha)} \int_x^b (t-x)^{n-1-\alpha} f^{(n)}(t) dt, \quad n = [\alpha] + 1. \quad (1.8)$$

Lors de la résolution des problèmes de valeurs propres avec une dérivée d'ordre fractionnaire, une famille de fonctions, dite de Mittag-Leffler apparait naturellement.

Définition 1.1.4 (Fonction de Mittag-Leffler). La fonction de Mittag-Leffler à un paramètre $\alpha > 0$ est définie par

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \text{ pour tout } z \in \mathbb{C}, \quad (1.9)$$

et la fonction de Mittag-Leffler à deux paramètres $\alpha > 0$ et $\beta > 0$ est définie par

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \text{ pour tout } z \in \mathbb{C}. \quad (1.10)$$

Il est clair que $E_{\alpha,1} = E_\alpha$. Dans [45], nous avons la propriété suivante

$$\int_0^1 x^{\alpha-1} E_{\alpha,\alpha}(\lambda x^\alpha) dx = E_{\alpha,\alpha+1}(\lambda), \quad (\alpha > 0 \text{ \& } \lambda \in \mathbb{R}). \quad (1.11)$$

De plus, dans [44] nous avons le théorème suivant :

Théorème 1.1.1. Soient $\alpha \in]0, 2[$, $\beta \in \mathbb{R}$ et μ un réel vérifiant $\frac{\alpha\pi}{2} < \mu < \min\{\pi, \alpha\pi\}$.

Alors il existe une constante $C \geq 0$ telle que pour tout $z \in \mathbb{C}$ vérifiant $\mu \leq |\arg(z)| \leq \pi$, nous avons

$$|E_{\alpha,\beta}(z)| \leq \frac{C}{1 + |z|}. \quad (1.12)$$

Nous aurons besoin du lemme suivant [47].

lemme 1.2. Soit $(f_i)_{i \in \mathbb{N}}$, une suite de fonctions définies sur l'intervalle $]a, b[$. Supposons que les conditions suivantes soient remplies :

1. Pour un $\alpha > 0$ donné, la dérivée fractionnaire $D_t^\alpha f_i(t)$ existe pour tous $i \in \mathbb{N}$, $t \in]a, b[$;
2. Les deux séries $\sum_{i=1}^{\infty} f_i(t)$ et $\sum_{i=1}^{\infty} D_t^\alpha f_i(t)$ convergent uniformément sur l'intervalle $[a + \epsilon, b]$ pour tout $\epsilon > 0$.

Alors la fonction définie par la série $\sum_{i=1}^{\infty} f_i(t)$ est α -différentiable, de plus on a

$$D_t^\alpha \left(\sum_{i=1}^{\infty} f_i(t) \right) = \sum_{i=1}^{\infty} D_t^\alpha f_i(t). \quad (1.13)$$

1.3 Identification de l'ordre de la dérivée dans des équations différentielles fractionnaires

Des phénomènes de diffusion anormale dans les milieux poreux, rhéologiques, viscoélastiques et fractals ont attiré une grande attention au cours des dernières années.

Ces processus de diffusion anormaux sont décrits par des opérateurs différentiels d'ordre fractionnaires. En effet, dans le cadre du mouvement aléatoire des particules, la loi de Fick considère que les particules ne peuvent sauter qu'aux emplacements adjacents pendant un pas de temps donné. Cependant, dans de nombreuses observations expérimentales dans des milieux hétérogènes, les particules peuvent sauter non seulement vers les emplacements adjacents, mais aussi vers des emplacements plus éloignés avec une probabilité plus faible. Cette propriété de non-localisation peut être modélisée à l'aide d'une dérivée fractionnaire.

Dans un récent article de synthèse [50], les auteurs présentent un grand nombre de champs d'application des dérivées fractionnaires. En physique, les auteurs présentent l'équation fractionnaire de Langevin pour les milieux complexes viscoélastiques et la diffusion anormale sous-jacente. En spectroscopie électrique, les résultats expérimentaux montrent que la diffusion manifestée par les ions peut ne pas toujours dépendre seulement de la gamme de fréquences considérée mais peut aussi dépendre des effets de surface. En acoustique, les modèles d'ondes fractionnaires sont en accord avec les résultats expérimentaux obtenus lors de la propagation des ondes acoustiques dépendant de la fréquence. D'autres exemples concernant la thermoélasticité et la classification des impuretés dans des solutions solides sont décrits. En théorie du contrôle, les systèmes à mémoire d'ordre fractionnaire sont plus pertinents pour prendre en compte la dépendance de l'état courant en fonction des états précédents. Dans le traitement du signal et de l'image, les modèles de variation totale d'ordre fractionnaire (FOTV) peuvent combiner à la fois l'élimination de l'effet d'escalier et la préservation des contours en sélectionnant l'ordre de la dérivée de manière adéquate. D'autres modèles intéressants sont décrits en mécanique et en systèmes dynamiques, biologie, économie, etc.

Dans [57], les propriétés de transport des matériaux granulaires semblables au béton sont étudiées. Pour de tels matériaux, l'anisotropie et l'hétérogénéité inhérentes à l'échelle microscopique ne conviennent pas à la loi de Fick classique. Les auteurs soulignent l'importance de comprendre la relation entre l'indice fractionnaire et l'hétérogénéité de la structure du matériau.

Pour plus de détails sur le calcul fractionnaire et les applications, nous renvoyons le lecteur intéressé à [39, 41, 59, 29, 20] et les références qui y figurent.

Dans les applications où les matériaux présentent des propriétés de sous-diffusion ou de super-diffusion, les ordres fractionnaires sous-jacents des opérateurs différentiels associés ne sont pas vraiment connus et seules des mesures sont disponibles. Ainsi, la détermination de l'ordre des dérivées fractionnaires via des techniques de problèmes inverses est une question d'un grand intérêt. En effet, une connaissance précise de cet ordre fractionnaire peut aider à mieux comprendre la signification physique des modèles de diffusion fractionnaire et à résoudre davantage les problèmes

pratiques d'ingénierie.

Dans [11], les auteurs ont appliqué des schémas numériques pour identifier les ordres des dérivées fractionnaires dans un modèle de diffusion anormale simple, à partir de données réelles. Considérant une équation de diffusion fractionnaire unidimensionnelle à la fois en espace et en temps, les auteurs ajustent les données en choisissant plusieurs valeurs des ordres fractionnaires par minimisation numérique des «erreurs» de norme uniforme, représentant l'écart entre la solution numérique de l'équation modèle et les données expérimentales. Notons qu'aucune équation satisfaite par la méthode de descente du gradient n'est fournie dans ce processus de minimisation, seule une approximation numérique de la direction de descente est proposée.

Dans [8], une équation parabolique à une dimension spatiale est considérée, avec une dérivée d'ordre fractionnaire dans le temps ($0 < \alpha < 1$) et une diffusion classique à diffusivité variable $\lambda(x)$, où $x \in [0, \ell]$ et $t \in [0, T]$. Les auteurs ont montré un résultat d'existence et d'unicité pour les inconnues (α, λ) si la solution u est connue au bord gauche $x = 0$, pour tout temps $t \in [0, T]$, pour un $T > 0$ donné.

D'autre part, d'autres travaux se sont intéressés à la partie numérique de l'identification des paramètres dans les systèmes différentiels fractionnaires. On peut renvoyer à [62, 58, 28, 56, 37], où des algorithmes génétiques ont été appliqués pour effectuer de telles identifications.

Dans le présent travail, la question centrale est : si nous avons une équation différentielle partielle avec une dérivée d'ordre fractionnaire α , dont la solution est notée $u(\alpha)$, peut-on obtenir l'équation différentielle partielle satisfaite par $\frac{d u(\alpha)}{d \alpha}$, la dérivée de $u(\alpha)$ par rapport à α ? Dans les références citées précédemment, seule une approximation de cette dérivée est donnée par

$$\frac{u(\alpha + \varepsilon) - u(\alpha)}{\varepsilon},$$

avec ε assez petit. La nouveauté de notre travail est la détermination du problème exact satisfait par $\frac{d u(\alpha)}{d \alpha}$. Nous montrons que cette dérivée satisfait le même problème, mais avec un autre terme source. Notre nouveau résultat permet d'effectuer un calcul exact de la dérivée de la fonction coût par rapport à α pour avoir des algorithmes de descente du gradient efficaces et précis. De plus, la dérivée de la fonction coût par rapport à α ne nécessite pas de nouvelles inversions matricielles, puisque le problème satisfait par $\frac{d u}{d \alpha}$ est identique à celui satisfait par $u(\alpha)$, avec un autre terme source. Ce résultat peut être appliqué dans d'autres cadres mathématiques comme les problèmes de contrôle optimal ou l'étude de la sensibilité par rapport à l'ordre des dérivées.

Dans ce qui suit, nous considérerons le cadre de Riemann-Liouville concernant les

opérateurs différentiels fractionnaires sur un intervalle $[a, b]$. Nous rappelons pour chaque $\alpha \in]0, 1[$ et pour chaque fonction continue u sur $[a, b]$, on définit la dérivée fractionnaire mixte, de paramètre $\theta \in [0, 1]$, de Riemann-Liouville de u en $x \in [a, b]$ par :

$$D_x^{\alpha, \theta} u = \theta {}_a D_x^\alpha u + (1 - \theta) {}_x D_b^\alpha u.$$

Maintenant, considérons le problème aux limites suivant

$$\begin{cases} -\frac{d}{dx} \left[k(x) D_x^{\alpha, \theta} u(x) \right] = f(x), & x \in \Omega = (a, b), \\ u(a) = u(b) = 0, \end{cases} \quad (1.14)$$

où $0 < \alpha < 1$, $0 \leq \theta \leq 1$, le terme source $f \in L^2(\Omega)$ et le coefficient de diffusion $k \in C^1(\overline{\Omega})$ est (strictement) positif. Ici, $D_x^{\alpha, \theta} u(x)$ désigne la dérivée fractionnaire mixte de Riemann-Liouville décrite précédemment. Notons que si la fonction de diffusivité k est constante, alors l'opérateur $\frac{d}{dx} \left[k D_x^{\alpha, \theta} \right]$ n'est autre que la dérivée fractionnaire de Riemann-Liouville $D_x^{1+\alpha, \theta}$, à une constante multiplicative près k .

On se limite au cas où le coefficient θ , la fonction de diffusivité k et le terme source f sont connus, mais l'ordre de dérivation fractionnaire α est inconnu. Pour déterminer le paramètre inconnu α , nous supposons que nous avons une observation z , sur Ω , de la solution u_α du problème (1.14). Par conséquent, nous introduisons la fonction de coût quadratique

$$J(\alpha) = \frac{1}{2} \int_{\Omega} (u_\alpha - z)^2 dx. \quad (1.15)$$

Le problème d'identification de paramètre pour (1.14) est de trouver le paramètre optimal $\hat{\alpha}$ satisfaisant $J(\hat{\alpha}) = \inf \{ J(\alpha); \alpha \in]0, 1[\}$. La condition nécessaire d'optimalité associée est $J'(\hat{\alpha}) = 0$.

Theorem 1.3.1. *Soit u_α la solution de (1.14), alors $\frac{du_\alpha}{d\alpha}$ est la solution du problème aux limites suivant*

$$\begin{cases} -\frac{d}{dx} \left[k(x) D_x^{\alpha, \theta} w \right] = \frac{d}{dx} \left[k(x) \widehat{D}_x^{\alpha, \theta} u_\alpha \right], & x \in \Omega = (a, b), \\ w(a) = w(b) = 0, \end{cases} \quad (1.16)$$

avec

$$\widehat{D}_x^{\alpha, \theta} u_\alpha = \theta {}_a \widehat{D}_x^\alpha u_\alpha + (1 - \theta) {}_x \widehat{D}_b^\alpha u_\alpha,$$

où

$$\begin{cases} {}_a \widehat{D}_x^\alpha u_\alpha = \frac{d}{dx} \left[\int_a^x \frac{(x-t)^{-\alpha}}{\Gamma(1-\alpha)} \ln \left(\frac{1}{x-t} \right) u_\alpha(t) dt \right] + \frac{\Gamma'(1-\alpha)}{\Gamma(1-\alpha)} {}_a D_x^\alpha u_\alpha, \\ {}_x \widehat{D}_b^\alpha u_\alpha = \frac{d}{dx} \left[\int_x^b \frac{(t-x)^{-\alpha}}{\Gamma(1-\alpha)} \ln \left(\frac{1}{t-x} \right) u_\alpha(t) dt \right] + \frac{\Gamma'(1-\alpha)}{\Gamma(1-\alpha)} {}_x D_b^\alpha u_\alpha. \end{cases}$$

En outre,

$$J'(\alpha) = \int_{\Omega} w_{\alpha}(x)[u_{\alpha}(x) - z(x)] dx$$

où w_{α} est la solution de (1.16).

Pour trouver le minimum de la fonction coût J , nous utilisons l'algorithme de descente du gradient à pas fixe. Ainsi, pour trouver le paramètre inconnu $\hat{\alpha} \in]0, 1[$, nous introduisons la suite itérative

1. *Initialisation* : Choix de $\alpha_0 \in]0, 1[$.
2. *Résoudre les problèmes (1.14) et (1.16) avec α_n pour obtenir u_{α_n} et w_{α_n} .*
3. *Direction de descente* : $-J'(\alpha_n) = - \int_{\Omega} w_{\alpha_n}(x)[u_{\alpha_n}(x) - z(x)] dx$.
4. *Mise à jour* : $\alpha_{n+1} = \alpha_n - \rho J'(\alpha_n)$, avec un pas suffisamment petit $\rho > 0$.

Pour résoudre les problèmes (1.14) et (1.16), nous adaptons la méthode numérique développée par Mustapha et al [42] basée sur les différences finies. Notons que dans [42], les auteurs montrent la convergence théorique et numérique de la méthode proposée.

On a considéré une application à l'écoulement annulaire coaxial dans lequel un fluide est confiné entre deux cylindres de rayons R_{in} et R_{out} . Ce type d'écoulement de Taylor-Couette a une importance particulière pour permettre la caractérisation rhéologique des matériaux en chargeant le matériau inconnu dans l'espace annulaire, en faisant tourner un cylindre et en mesurant le couple résultant sur la deuxième surface cylindrique [16]. Plus précisément, considérons le domaine cylindrique spatial

$$O =]R_{in}, R_{out}[\times]0, 2\pi[\times]z_{min}, z_{max}[$$

et un intervalle de temps $[0, T]$.

Pour $\theta \in [0, 2\pi]$ et $z \in]z_{min}, z_{max}[$ fixes, la composante selon la direction θ de la vitesse du fluide $\vec{u}(r, \theta, z, t)$ sera désignée par $u_{\theta}(r, t)$. Le domaine spatial est donc $\Omega =]R_{in}, R_{out}[$, pour $(\theta, z) \in]0, 2\pi[\times]z_{min}, z_{max}[$ fixés, et l'intervalle de temps est $[0, T]$. Nous nous intéressons au problème satisfait par la composante selon θ de la vitesse du fluide $u_{\theta}(r, t)$. Dans [24], les auteurs ont proposé un modèle d'ordre fractionnaire gouvernant la vitesse du fluide $u_{\theta}(r, t)$:

$$\begin{cases} \frac{\rho}{\mathbb{V}} \frac{\partial u_{\theta}}{\partial t} + \frac{\rho}{\mathbb{G}} D_t^{2-\beta} u_{\theta} = \frac{\partial^2 u_{\theta}}{\partial r^2} + \frac{1}{r} \frac{\partial u_{\theta}}{\partial r} - \frac{u_{\theta}}{r^2}, & R_{in} < r < R_{out}, 0 < t \leq T \\ u_{\theta}(R_{in}, t) = \phi_i(t), u_{\theta}(R_{out}, t) = \phi_o(t), & 0 \leq t \leq T, \\ u_{\theta}(r, 0) = \frac{\partial u_{\theta}(r, 0)}{\partial t} = 0, & R_{in} < r < R_{out}. \end{cases} \quad (1.17)$$

et notre objectif ici est de trouver l'ordre fractionnaire de la dérivée temporelle $\beta \in]0, 1[$, en utilisant une mesure de $u_{\theta}(r, t)$ sur $]R_{in}, R_{out}[\times]0, T[$. Les constantes ρ , \mathbb{V} et \mathbb{G} sont des paramètres physiques positifs donnés.

1.4 Problème inverse

1.4.1 Définition générale

Supposons que nous ayons un modèle mathématique d'un processus physique. On suppose que le modèle décrit systématiquement le processus et ses conditions de fonctionnement, et explique les grandeurs principales du modèle (voir Figure 1.1) : entrées, paramètres du système et sorties. Dans la plupart des cas, la description du système est donnée sur la base d'un ensemble d'équations (équations différentielles ordinaires ou partielles, équations intégrales, etc.) avec certains paramètres.

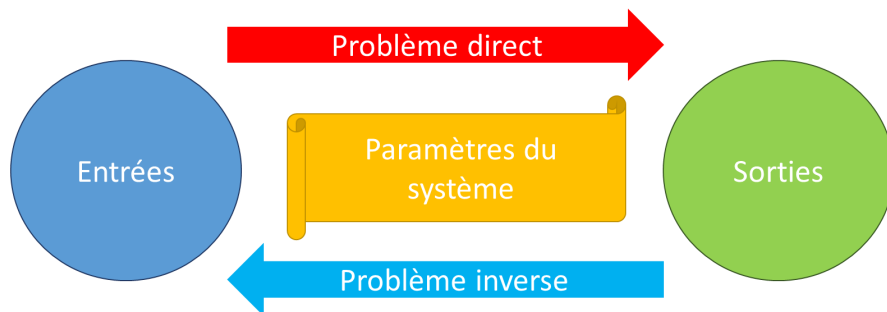


FIGURE 1.1 : un modèle mathématique d'un processus physique

L'analyse d'un processus physique donné à travers un modèle mathématique peut être divisée en trois types de problèmes différents :

1. Problème direct : A partir de l'entrée (input) et des paramètres du système, on doit déterminer la sortie (output) du modèle.
2. Problème de "reconstruction" : A partir des paramètres du système et de la sortie, on doit déterminer l'entrée.
3. Problème d'identification (où estimation des paramètres) : A partir de l'entrée et de la sortie, déterminer les paramètres du système qui sont cohérents avec la relation entre l'entrée et le sortie.

Nous appelons un problème de type (1) un problème direct car il est basé sur une série de relations causales. En ce sens, les problèmes de types (2) et (3) sont appelés problèmes inverses.

1.4.2 Description des résultats

On considère le problème aux limites d'évolution fractionnaire non homogène

$$\begin{cases} \mathcal{D}_t^\gamma u(x, y, t) - D_x^\alpha u(x, y, t) - D_y^\beta u(x, y, t) = p(x, y) f(t), & (x, y, t) \in \Omega \times (0, T] \end{cases} \quad (1.18)$$

$$\begin{cases} u(0, y, t) = u(1, y, t) = 0, & (y, t) \in [0, 1] \times [0, T], \end{cases} \quad (1.19)$$

$$\begin{cases} u(x, 0, t) = u(x, 1, t) = 0, & (x, t) \in [0, 1] \times [0, T], \end{cases} \quad (1.20)$$

$$\begin{cases} u(x, y, 0) = \varphi(x, y), & (x, y) \in \overline{\Omega}, \end{cases} \quad (1.21)$$

où $\mathcal{D}_t^\gamma u(x, y, t)$ désigne la dérivée fractionnaire au sens de Caputo, avec $0 < \gamma < 1$, et $D_x^\alpha u(x, y, t)$, $D_y^\beta u(x, y, t)$ sont les dérivées fractionnaires au sens de Riemann-Liouville, avec $1 < \alpha, \beta < 2$. Le domaine spatial est le carré ouvert $\Omega =]0, 1[\times]0, 1[$ et $T > 0$ désigne la longueur de l'intervalle de temps.

On précise la nature des problèmes qui seront étudiés par la suite : lorsque les données initiales φ et le terme source (p, f) est connu, le problème direct consiste à chercher la fonction inconnue u , dans un espace fonctionnel adéquat. D'autre part, lorsque les données initiales φ et seulement la partie spatiale p sont données, le problème inverse consiste à chercher les fonctions inconnues (u, f) via la donnée supplémentaire

$$\iint_{\Omega} u(x, y, t) dx dy = \psi(t), \quad t \in [0, T], \quad (1.22)$$

où ψ est une fonction donnée sur $[0, T]$.

Notez que les opérateurs fractionnaires D_x^α , D_y^β sont des opérateurs non autoadjoints admettant des bases complètes dénombrables de fonctions propres, dans le sens $L^2(0, 1)$. Cela nous permettra de rechercher des solutions classiques par l'intermédiaire des représentations tensorielles biorthogonales (en x et y). En effet, la convergence uniforme des séries résultantes en $\overline{\Omega} \times [0, T]$ (et de leurs dérivées) et l'unicité des limites correspondantes, pour les normes $\|\cdot\|_2$ et $\|\cdot\|_\infty$, conduiront à l'existence de solutions classiques du problème direct. Pour le problème inverse, nous utiliserons le théorème du point fixe de Schauder dans un sous-ensemble convexe borné avec la norme de Bielecki.

On énonce brièvement les résultats obtenus dans ce second travail :

Theorem 1.4.1. *Soit $p(x, y) \in \mathcal{C}(\overline{\Omega}, \mathbb{R})$, $f(t) \in \mathcal{C}([0, T], \mathbb{R})$ et $\varphi(x, y) \in \mathcal{C}(\overline{\Omega}, \mathbb{R})$ tels que $D_x^\alpha \varphi(x, y) \in \mathcal{C}(\overline{\Omega}, \mathbb{R})$, $D_y^\beta \varphi(x, y) \in \mathcal{C}(\overline{\Omega}, \mathbb{R})$ et $\varphi(0, y) = \varphi(1, y) = \varphi(x, 0) = \varphi(x, 1) = 0$. Alors, il existe une solution classique unique au problème direct.*

Pour montrer ce théorème, on utilise la méthode de séparation de variables en utilisant la fonction de Mittag-Leffler.

Theorem 1.4.2. *Dans les mêmes conditions que dans le théorème 1.4.1, la solution du problème direct dépend continûment des données.*

Theorem 1.4.3. *Soit $p(x, y) \in \mathcal{C}(\overline{\Omega}, \mathbb{R})$, $\psi(t) \in \mathcal{C}([0, T], \mathbb{R})$, tels que $\mathcal{D}_t^\gamma \psi(t) \in \mathcal{C}([0, T], \mathbb{R})$, et $\varphi(x, y) \in \mathcal{C}(\overline{\Omega}, \mathbb{R})$ tel que $D_x^\alpha \varphi(x, y) \in \mathcal{C}(\Omega, \mathbb{R})$, $D_y^\beta \varphi(x, y) \in \mathcal{C}(\overline{\Omega}, \mathbb{R})$ et $\varphi(0, y) = \varphi(1, y) = \varphi(x, 0) = \varphi(x, 1) = 0$. Alors, il existe une solution unique au problème inverse.*

Pour montrer ce théorème on utilise le théorème du point fixe de Schauder. On termine ce paragraphe par le résultat de continuité suivant :

Theorem 1.4.4. *La solution du problème inverse dépend continûment des données $\varphi, p \in L^\infty(\Omega)$, avec p non identiquement nulle sur Ω , et ψ définie sur $[0, T]$, vérifiant $\mathcal{D}_t^\gamma \psi \in L^\infty(0, T)$.*

1.5 Décomposition progressive généralisée (PGD)

La PGD est une nouvelle famille de méthodes. Le premier article de recherche sur la PGD a été publié par Ammar et al. en 2006 [3], le premier livre sur cette méthode a été écrit par Chinesta et al. 2013- 2014 [10].

Dans cette thèse, nous étendons cette méthode aux équations aux dérivées partielles fractionnaires. Nous démontrerons des résultats de convergence théorique, que nous illustrons ensuite par des simulations numériques.

La représentation tensorielle (séparation de variables) de la solution n'est pas spécifique à la PGD. Cette décomposition a été proposée auparavant par Pierre Ladevèze dans sa méthode LATIN (Large Time Incremental method) [34, 35], il a considéré une séparation en espace-temps comme suit :

$$u(x, t) \approx \sum_{i=1}^m X_i(x) \times T_i(t) = \sum_{i=1}^m X_i \otimes T_i(x, t),$$

où \otimes désigne le produit tensoriel de deux fonctions. Cela fait de la PGD une généralisation de la méthode LATIN introduite par Ladevèze. Cette représentation a également été largement utilisée en chimie quantique. D'autre part, les méthodes Hartree-Fock et Post-Hartree-Fock utilisent des approximations de la forme séparée de la fonction d'onde pour l'équation de Schrödinger [6].

La méthode PGD appartient à la grande famille des approximations des tenseurs d'ordre (ou rang) élevé par des tenseurs d'ordre petit. Dans [61], Zhang et Golub ont étudié certaines propriétés de l'approximation des tenseurs d'ordre élevé par des tenseur de rang 1 (SVD multidimensionnel) ainsi que des algorithmes numériques proposés.

Bien que des progrès significatifs aient été réalisés dans la compréhension théorique des algorithmes de la PGD, seuls quelques résultats de convergence pour des

problèmes variationnels sont disponibles de nos jours dans la littérature. La question de la convergence de la méthode PGD a d'abord été étudiée par Le Bris et *al.* [5] en exploitant son lien avec les algorithmes dit "glouton" de la théorie de l'approximation non linéaire. Les auteurs ont prouvé la convergence de la PGD dans le cas de l'équation de Poisson. Dans Cances et *al.* [7], les auteurs ont étudié une méthode pour les problèmes variationnels d'obstacles, toujours dans le cadre des espaces de Hilbert tensoriels, et ont démontré un résultat de convergence analogue. Pour la PGD variationnelle dans les espaces de tenseur de Banach, la convergence a d'abord été établie par Falco et Nouy [22]. Notons que dans un ouvrage antérieur [21], les mêmes auteurs ont également fourni une autre preuve de la convergence des problèmes variationnels elliptiques via le célèbre théorème d'Eckart et Young [17]. Récemment, A. El Hamidi et *al.* ont démontré la convergence de l'étape cruciale de la PGD, celle de l'enrichissement de l'approximation PGD en utilisant la méthode des directions alternées [18].

Pour illustrer cette méthode, nous considérons l'exemple classique suivant [10] :

Exemple 1.5.1 (Solution PGD de l'équation de Poisson). Considérons l'équation de Poisson

$$\Delta u(x, y) = f(x, y) \quad (1.23)$$

dans un domaine rectangulaire à deux dimensions $\Omega = \Omega_x \times \Omega_y = (0, L) \times (0, H)$. Nous spécifions des conditions aux limites de Dirichlet homogènes pour la fonction inconnue $u(x, y)$, c'est-à-dire $u(x, y)$ s'annule à la frontière Γ du domaine Ω . De plus, nous supposons que le terme source f est constant dans le domaine. Pour toutes les fonctions test appropriées u^* , on doit avoir

$$\int_{\Omega_x \times \Omega_y} u^* \cdot \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - f \right) dx \cdot dy = 0. \quad (1.24)$$

Notre objectif est d'obtenir une solution PGD approximative de 1.23 sous la forme séparée

$$u(x, y) = \sum_{i=1}^{\infty} X_i(x) \cdot Y_i(y). \quad (1.25)$$

Nous le ferons en calculant chaque terme de la série, un par un, de manière itérative, ainsi enrichir l'approximation PGD jusqu'à ce qu'un critère de convergence approprié soit satisfait.

Construction progressive de la représentation séparée. A chaque étape d'enrichissement n , ($n \geq 1$), nous avons déjà calculé les $n-1$ premiers termes de l'approximation PGD (1.25) :

$$u^{n-1}(x, y) = \sum_{i=1}^{n-1} X_i(x) \cdot Y_i(y) \quad (1.26)$$

Nous souhaitons maintenant calculer le terme suivant $X_n(x) \cdot Y_n(y)$ pour obtenir la solution PGD enrichie à l'itération n

$$\begin{aligned}
u_n(x, y) &= u_{n-1}(x, y) + X_n(x) \cdot Y_n(y) \\
&= \sum_{i=1}^{n-1} X_i(x) \cdot Y_i(y) + X_n(x) \cdot Y_n(y).
\end{aligned} \tag{1.27}$$

Les deux fonctions $X_n(x)$ et $Y_n(y)$ sont inconnues à l'étape d'enrichissement actuelle n , et elles apparaissent sous la forme d'un produit. Le problème résultant est donc non linéaire et un schéma itératif de point fixe approprié est requis. Nous utiliserons l'indice p pour désigner une itération particulière.

A l'étape d'enrichissement n , l'approximation PGD $u_{n,p}$ obtenue à l'itération p se lit ainsi

$$u_{n,p}(x, y) = u_{n-1}(x, y) + X_n^p(x) \cdot Y_n^p(y). \tag{1.28}$$

Le schéma itératif le plus simple est la méthode des directions alternées qui calcule $X_n^p(x)$ de $Y_n^{p-1}(y)$, puis $Y_n^p(y)$ de $X_n^p(x)$. Une initialisation arbitraire $Y_n^0(y)$ est requise pour démarrer le processus itératif. Les itérations non linéaires se poursuivent jusqu'à atteindre un point fixe à une certaine tolérance ϵ spécifiée par l'utilisateur, c'est-à-dire

$$\frac{\|X_n^p \otimes Y_n^p - X_n^{p-1} \otimes Y_n^{p-1}\|}{\|X_n^{p-1} \otimes Y_n^{p-1}\|} < \epsilon, \tag{1.29}$$

où $\|\cdot\|$ est une norme appropriée. L'étape d'enrichissement n se termine donc par les affectations $X_n(x) \leftarrow X_n^p(x)$ et $Y_n(y) \leftarrow Y_n^p(y)$. Le processus d'enrichissement s'arrête lui-même lorsqu'une mesure d'erreur appropriée $\epsilon(n)$ devient suffisamment petit, c'est-à-dire $\epsilon(n) < \tilde{\epsilon}$. Plusieurs critères d'arrêt sont disponibles, comme nous le verrons plus loin. Nous décrivons maintenant plus en détail une itération de direction alternée à une étape d'enrichissement donnée.

Stratégie de direction alternative Chaque itération du schéma de direction alternée comprend les deux étapes suivantes :

- Calcul de $X_n^p(x)$ à partir de $Y_n^{p-1}(y)$. Dans ce cas, l'approximation indiquée

$$u_n^p(x, y) = \sum_{i=1}^{n-1} X_i(x) \cdot Y_i(y) + X_n^p(x) \cdot Y_n^{p-1}(y), \tag{1.30}$$

où toutes les fonctions sont connues sauf $X_n^p(x)$. Le choix le plus simple pour la fonction test u^* dans la formulation résiduelle (1.24) est

$$u^*(x, y) = X_n^*(x) \cdot Y_n^{p-1}(y), \tag{1.31}$$

ce qui revient à sélectionner la forme résiduelle pondérée de Galerkin de l'équa-

tion de Poisson. En injectant (1.30) et (1.31) dans (1.24), on obtient

$$\begin{aligned}
 & \int_{\Omega_x \times \Omega_y} X_n^* \cdot Y_n^{p-1} \cdot \left(\frac{d^2 X_n^p}{dx^2} \cdot Y_n^{p-1} + X_n^p \cdot \frac{d^2 Y_n^{p-1}}{dy^2} \right) dx \cdot dy \\
 &= - \int_{\Omega_x \times \Omega_y} X_n^* \cdot Y_n^{p-1} \cdot \sum_{i=1}^{n-1} \left(\frac{d^2 X_i}{dx^2} \cdot Y_i + X_i \cdot \frac{d^2 Y_i}{dy^2} \right) dx \cdot dy \\
 & \quad + \int_{\Omega_x \times \Omega_y} X_n^* \cdot Y_n^{p-1} \cdot f dx \cdot dy.
 \end{aligned} \tag{1.32}$$

Puisque toutes les fonctions de y sont connues dans l'expression ci-dessus, nous pouvons calculer les intégrales unidimensionnelles suivantes sur Ω_y :

$$\begin{cases}
 \alpha^x = \int_{\Omega_y} \left(Y_n^{p-1}(y) \right)^2 dy, \\
 \beta^x = \int_{\Omega_y} Y_n^{p-1}(y) \cdot \frac{d^2 Y_n^{p-1}}{dy^2} dy, \\
 \gamma_i^x = \int_{\Omega_y} Y_n^{p-1}(y) \cdot Y_i(y) dy, \\
 \delta_i^x = \int_{\Omega_y} Y_n^{p-1}(y) \cdot \frac{d^2 Y_i}{dy^2} dy, \\
 \xi^x = \int_{\Omega_y} Y_n^{p-1}(y) \cdot f dy.
 \end{cases} \tag{1.33}$$

L'équation (1.32) devient

$$\begin{aligned}
 & \int_{\Omega_x} X_n^* \cdot \left(\alpha^x \cdot \frac{d^2 X_n^p}{dx^2} + \beta^x \cdot X_n^p \right) dx \\
 &= - \int_{\Omega_x} X_n^* \cdot \sum_{i=1}^{n-1} \left(\gamma_i^x \cdot \frac{d^2 X_i}{dx^2} + \delta_i^x \cdot X_i \right) dx + \int_{\Omega_x} X_n^* \cdot \xi^x dx.
 \end{aligned} \tag{1.34}$$

Nous avons ainsi obtenu la forme résiduelle d'un problème unidimensionnel défini sur Ω_x qui peut être résolu (par exemple par la méthode des éléments finis) pour obtenir la fonction X_n^p que nous recherchons. Alternativement, nous pouvons revenir à la formulation forte correspondante

$$\alpha^x \cdot \frac{d^2 X_n^p}{dx^2} + \beta^x \cdot X_n^p = - \sum_{i=1}^{n-1} \left(\gamma_i^x \cdot \frac{d^2 X_i}{dx^2} + \delta_i^x \cdot X_i \right) + \xi^x, \tag{1.35}$$

et le résoudre numériquement au moyen de toute méthode numérique appropriée (par exemple, différences finies, techniques pseudo-spectrales, ...). La forme forte (1.35) est une équation différentielle ordinaire de second ordre pour X_n^p . Avec les formulations résiduelles ou fortes, les conditions aux limites de Dirichlet homogènes $X_n^p(x=0) = X_n^p(x=L) = 0$ sont facilement spécifiées. Ayant ainsi calculé $X_n^p(x)$, nous sommes maintenant en mesure à passer à la deuxième étape de l'itération p .

- Calcul de $Y_n^p(y)$ à partir du $X_n^p(x)$ juste calculé. Pour cela, il suffit simplement d'échanger les rôles joués par toutes les fonctions pertinentes de x et y . L'approximation PGD actuelle se lit comme suit :

$$u_n^p(x, y) = \sum_{i=1}^{n-1} X_i(x) \cdot Y_i(y) + X_n^p(x) \cdot Y_n^p(y), \quad (1.36)$$

où toutes les fonctions sont connues sauf $Y_n^p(y)$. La formulation de Galerkin de (1.24) est obtenue avec le choix particulier

$$u^*(x, y) = X_n^p(x) \cdot Y_n^*(y), \quad (1.37)$$

Ensuite, en introduisant (1.36) et (1.37) dans (1.24), nous obtenons

$$\begin{aligned} & \int_{\Omega_x \times \Omega_y} X_n^p \cdot Y_n^* \cdot \left(\frac{d^2 X_n^p}{dx^2} \cdot Y_n^p + X_n^p \cdot \frac{d^2 Y_n^p}{dy^2} \right) dx \cdot dy \\ &= - \int_{\Omega_x \times \Omega_y} X_n^p \cdot Y_n^* \cdot \sum_{i=1}^{n-1} \left(\frac{d^2 X_i}{dx^2} \cdot Y_i + X_i \cdot \frac{d^2 Y_i}{dy^2} \right) dx \cdot dy \\ & \quad + \int_{\Omega_x \times \Omega_y} X_n^p \cdot Y_n^* \cdot f dx \cdot dy. \end{aligned} \quad (1.38)$$

Comme toutes les fonctions de x sont connues, les intégrales sur Ω_x peuvent être calculées pour obtenir

$$\left\{ \begin{array}{l} \alpha^y = \int_{\Omega_x} (X_n^p(x))^2 dx, \\ \beta^y = \int_{\Omega_x} X_n^p(x) \cdot \frac{d^2 X_n^p}{dx^2} dx, \\ \gamma_i^y = \int_{\Omega_x} X_n^p(x) \cdot X_i(x) dx, \\ \delta_i^y = \int_{\Omega_x} X_n^p(x) \cdot \frac{d^2 X_i}{dx^2} dx, \\ \xi^y = \int_{\Omega_x} X_n^p(x) \cdot f dx. \end{array} \right. \quad (1.39)$$

L'équation (1.38) devient

$$\begin{aligned} & \int_{\Omega_y} Y_n^* \cdot \left(\alpha^y \cdot \frac{d^2 Y_n^p}{dy^2} + \beta^y \cdot Y_n^p \right) dy \\ &= - \int_{\Omega_y} Y_n^* \cdot \sum_{i=1}^{n-1} \left(\gamma_i^y \cdot \frac{d^2 Y_i}{dy^2} + \delta_i^y \cdot Y_i \right) dy + \int_{\Omega_y} Y_n^* \cdot \xi^y dy. \end{aligned} \quad (1.40)$$

Comme précédemment, nous avons ainsi obtenu la forme résiduelle d'un problème elliptique défini sur Ω_y dont la solution est la fonction $Y_n^p(y)$. Alternativement, la formulation forte correspondante de ce problème unidimensionnel s'écrit :

$$\alpha^y \cdot \frac{d^2 Y_n^p}{dy^2} + \beta^y \cdot Y_n^p = - \sum_{i=1}^{n-1} \left(\gamma_i^y \cdot \frac{d^2 Y_i}{dy^2} + \delta_i^y \cdot Y_i \right) + \xi^y. \quad (1.41)$$

Avec les formulations résiduelles et fortes, les conditions aux limites homogènes de Dirichlet $Y_n^p(y=0) = Y_n^p(y=L) = 0$ sont facilement spécifiées.

Nous avons ainsi terminé l'itération p de l'étape d'enrichissement n . Il est important de réaliser que l'équation de Poisson bidimensionnelle définie sur $\Omega = \Omega_x \times \Omega_y$ a été transformée dans le cadre du PGD en un série de problèmes unidimensionnels découplés en x et y .

Comme nous le détaillerons plus loin, si nous considérons l'équation de Poisson définie sur un domaine de dimension D , c'est-à-dire $\Omega_1 \times \Omega_2 \times \dots \times \Omega_D$, alors sa solution PGD entraîne également une série de problèmes unidimensionnels découplés formulés dans chaque Ω_i . Ceci explique bien sûr pourquoi la solution PGD des problèmes de grande dimension d'une grande utilité.

Critère d'arrêt du processus d'enrichissement Le processus d'enrichissement se termine quand une mesure appropriée de l'erreur $\varepsilon(n)$ devient suffisamment petite, c'est-à-dire $\varepsilon(n) < \tilde{\varepsilon}$. Nous avons opté pour le critère d'arrêt suivant :

$$\varepsilon(n) = \frac{\|X_n \otimes Y_n\|}{\|u^n\|} = \frac{\|X_n \otimes Y_n\|}{\left\| \sum_{i=1}^n X_i \otimes Y_i \right\|}. \quad (1.42)$$

Pour préciser le cadre théorique de la méthode PGD, nous présentons dans le paragraphe suivant les grandes lignes de l'analyse fonctionnelle sous jacente pour les problèmes variationnels.

1.5.1 Espaces de Hilbert Tensoriel et PGD variationnelle

Cette section traite des problèmes de minimisation de fonctionnelles convexes définies sur des espaces de Hilbert tensoriels. La structure tensorielle des espaces de Hilbert permet d'effectuer la minimisation d'une manière progressive via les méthodes de réduction dimensionnelle. Pour la commodité du lecteur, nous reprenons brièvement les grandes lignes de la PGD variationnelle.

On considère une famille finie d'espaces de Hilbert réels $(V_k, \|\cdot\|_k)_{1 \leq k \leq d}$, où $d \geq 2$ est un entier. L'espace tensoriel algébrique de Hilbert engendré par la famille $(V_k, \|\cdot\|_k)_{1 \leq k \leq d}$, noté par

$$V := {}_a \bigotimes_{k=1}^d V_k,$$

est l'ensemble de toutes les combinaisons linéaires des tenseurs élémentaires (d'ordre 1 ou de rang 1) $v = \bigotimes_{k=1}^d v_k$, avec $v_k \in V_k$. Le suffixe " a " dans " ${}_a \bigotimes$ " indique la nature

"algébrique" du produit tensoriel, i.e,

$$V = \left\{ \sum_{j=1}^m v_1^{(j)} \otimes v_2^{(j)} \otimes \cdots \otimes v_d^{(j)} : m \in \mathbb{N}^* \text{ et } v_k^{(j)} \in V_k \text{ pour tout } k = 1, 2, \dots, d \right\}. \quad (1.43)$$

On dit que $V_{\|\cdot\|}$ est un espace de Hilbert tensoriel s'il existe un espace tensoriel algébrique V et une norme $\|\cdot\|$ sur V tels que $V_{\|\cdot\|}$ est le complété de V pour la norme $\|\cdot\|$, i.e.

$$V_{\|\cdot\|} = \overline{\sum_{k=1}^d V_k}^{\|\cdot\|}.$$

Par conséquent, chaque élément de $v \in V_{\|\cdot\|}$ peut être écrit comme

$$v = \sum_{j=1}^{+\infty} v_1^{(j)} \otimes v_2^{(j)} \otimes \cdots \otimes v_d^{(j)},$$

dans le sens

$$\lim_{m \rightarrow +\infty} \left\| v - \sum_{j=1}^m v_1^{(j)} \otimes v_2^{(j)} \otimes \cdots \otimes v_d^{(j)} \right\| = 0.$$

Dans la suite, le cône des tenseurs élémentaires dans V sera noté par $\mathcal{R}_1(V)$, i.e.

$$\mathcal{R}_1(V) = \{v_1 \otimes v_2 \otimes \cdots \otimes v_d : v_k \in V_k \text{ for } k = 1, 2, \dots, d\}. \quad (1.44)$$

Alors l'espace engendré par $\mathcal{R}_1(V)$ est V , qui est à son tour une partie dense dans $V_{\|\cdot\|}$.

Maintenant, on considère une fonctionnelle $J : V_{\|\cdot\|} \rightarrow \mathbb{R}$, où $V_{\|\cdot\|}$ est un espace tensoriel de Hilbert et on s'intéresse au problème de minimisation suivant

$$\inf_{u \in V_{\|\cdot\|}} J(u), \quad (1.45)$$

où J satisfait les hypothèses

- (H₁) J is $\mathcal{C}^1(V_{\|\cdot\|}, \mathbb{R})$,
- (H₂) J est coercive,
- (H₃) J est strictement convexe.

En appliquant des arguments classiques de calcul variationnel, on montre que le problème (4.3) admet une solution unique $\hat{u} \in V_{\|\cdot\|}$, sous les hypothèses (H₁)–(H₃) et cette solution est un point critique de J , c'est-à-dire $J'(\hat{u}) = 0$. Pour définir la méthode PGD, nous supposons en outre

$$(H_4) \quad \mathcal{R}_1(V) \text{ est faiblement fermé dans } V_{\|\cdot\|},$$

où $V := \sum_{k=1}^d V_k$ est l'espace tensoriel algébrique associé à $V_{\|\cdot\|}$.

Notons que pour toute $u \in V$, le problème de minimisation

$$\inf_{z \in \mathcal{R}_1(V)} J(u + z) \quad (1.46)$$

admet au moins une solution, sous les hypothèses $(H_1) - (H_4)$. La multiplicité éventuelle de solutions du problème (1.46) est due au fait que $\mathcal{R}_1(V)$ n'est pas convexe.

La méthode PGD progressive associée au problème (4.3) consiste à construire une suite $(u_m)_{m \in \mathbb{N}} \subset V$ comme suit

(i) Initialisation : $u_0 := 0$.

(ii) Direction de descente : choisir $\hat{z}_m \in \operatorname{argmin}_{z \in \mathcal{R}_1(V)} J(u_{m-1} + z)$.

(iii) Progression

$$u_m := u_{m-1} + \hat{z}_m.$$

Nous référons le lecteur intéressé à [22] pour des méthodes d'approximation plus générales. Cette méthode PGD est dite convergente si $(u_m)_{m \in \mathbb{N}} \subset V$ converge vers la solution unique \hat{u} dans $V_{\|\cdot\|}$.

Chapter 2

Identification of the derivative order in fractional differential equations

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2.1 Introduction

Anomalous diffusion phenomena in porous, rheological, viscoelastic and fractal media have attracted great attention during the present decade. Such anomalous diffusion processes are described by nonlocal differential operators with fractional orders. Indeed, in the framework of particles random motion, the Fick's law considers that particles can only jump to the adjacent locations during a unit time step. However, in many experimental observations in heterogenous media, particles can jump not only to the adjacent locations but also to further locations with a lower probability. This nonlocality feature can be captured by using fractional derivative.

In a recent review article [50], the authors present short summaries in the field of fractional calculus. In physics, the authors present the fractional Langevin equation for viscoelastic complex media and the underlying anomalous diffusion. In the electrical spectroscopy impedance, experimental results show that the diffusion manifested by the ions may not always depend on the frequency range considered and related to the surface effects. In acoustics, fractional wave models agreed well with the obtained experimental results in the frequency-dependent acoustic wave propagation. Other examples concerning thermoelasticity and clusterization of impurities in solid solutions are described. In control theory, fractional order memory systems are more relevant to take into account the dependence of the current state of the whole history. An interesting application of fractional order calculus in active damping of flexible structures by reducing spillover effect due to the uncontrolled modes. In signal and image processing, the fractional order total variation (FOTV) models can ease the contention between staircase effect elimination and edge preservation by selecting the adequate order of derivative. Other interesting models are described in mechanics and dynamical systems, biology, economics, etc.

In [57], transport properties of concrete-like granular materials are studied. For such materials, the inherent anisotropy and heterogeneity at the microscopic scale is unsuitable with the classical Fick's law. The authors point out the importance of understanding the relation between the fractional index and the heterogeneity of the material structure.

A magnetohydrodynamic model with fractional order derivative is studied in [51]. The author used the Atangana and Baleanu's fractional derivative, in time. The Atangana and Baleanu's fractional derivative provides a fractional nonlocal differential operator with a nonsingular kernel (the Mittag-Leffler function). The author showed that the obtained results are more realistic, in particular the fluid velocity decreases with respect to the fractional derivative order.

For more details on fractional calculus and applications, we refer the interested reader to [39, 41, 59, 29, 20] and the references therein.

In applications where materials present sub-diffusion or super-diffusion properties, the underlying fractional orders of the associated differential operators are not really known and only measurements are available. Thus, the determination of the order of fractional derivatives via inverse problems techniques is a question of great interest. Indeed, an accurate knowledge of this fractional order can help to better understand the physical meaning of the fractional diffusion models and further solve the practical engineering problems.

In [11], the authors applied numerical schemes to identify the orders of the fractional derivatives in a simple anomalous diffusion model, starting from real data. Considering a one-dimensional fractional diffusion equation in both space and time, the authors fit the data by choosing several values of the fractional orders by numerical minimization of the uniform-norm "errors", representing the discrepancy between the numerical solution to the model equation and the experimental data. Notice that no exact problem giving the descent direction is provided in this minimization process,

only numerical approximation of the descent direction is studied.

In [8], a one space-dimensional parabolic equation is considered, with a fractional order derivative in time ($0 < \alpha < 1$) and a classical diffusion with variable diffusivity $\lambda(x)$, where $x \in [0, \ell]$ and $t \in [0, T]$. The authors showed an existence and uniqueness result for the unknowns (α, λ) if the solution u is known at the left boundary $x = 0$, for every time $t \in [0, T]$, for some given $T > 0$. The proof is based on the expansion of the weak solution to the initial boundary value problem and the Gel'fand-Levitan theory.

In [30], the authors extended the work of [8] and bring two essential results: the first one concerns the case $1 < \alpha < 2$. In this situation, the authors proved that the differential order α and the diffusivity function λ can be uniquely determined if the time integral of the solution u is known at the left boundary $x = 0$ for each time $t \in [0, T]$, that is, if $\int_0^t u(0, t) dt$ is given for each $t \in [0, T]$. The second result is an extension to the two dimensional space with a source term $f(x, y)$. The authors proved that for $0 < \alpha < 1$, the source term $f(x, y)$ can be uniquely determined if the solution u is known at the point $(x, y) = (0, 0)$, for every time $t \in [0, T]$. Notice that in [8, 30], only theoretical results are presented and neither algorithms nor numerical analysis are performed.

Based on the Tikhonov regularization scheme and the quasi-optimality criterion, the authors construct in [33] the computational algorithm to find the fractional order in quasilinear subdiffusion from noisy discrete measurements. Several numerical tests illustrating the algorithm are presented.

On the other hand, other works were interested in the numerical part of the parameter identification in fractional differential systems. We can refer the interested reader to [62, 58, 28, 56, 37], where genetic algorithms were applied to perform such identifications.

In the present work, the central question is: if we have a partial differential equation with a fractional order derivative α , whose solution is denoted by $u(\alpha)$, can we derive the partial differential equation satisfied by $\frac{du(\alpha)}{d\alpha}$, the derivative of $u(\alpha)$ with respect to α ? In the references cited before, only an approximation of this derivative are given by

$$\frac{u(\alpha + \varepsilon) - u(\alpha)}{\varepsilon},$$

with ε small enough. The novelty of our work is the determination of the exact problem satisfied by $\frac{du(\alpha)}{d\alpha}$, we show that this amount satisfies the same problem, but with another source term. Our new result allows to carry out an exact computation of the cost functional derivative with respect to α to perform more accurate and efficient steepest descent minimization algorithms. Moreover, the cost functional derivative with respect to α does not require new matrix inversions, since the problem satisfied by $\frac{du}{d\alpha}$ is the same as the one satisfied by $u(\alpha)$, with another source term. This result can be applied in other mathematical frameworks as optimal control problems or the sensitivity with respect to derivative order.

The manuscript is organized as follows: Section 2 deals with the identification problem of the fractional derivative order. The proposed algorithm and numerical method are presented in Section 3. Numerical simulations are performed in Section 4 in the case of a one dimensional elliptic problem with variable diffusivity coefficient. In Section 5, we end our work with an application to the fractional order derivative in time for the Taylor-Couette flow and its numerical simulation. A discussion-conclusion is presented in Section 6, where interesting theoretical aspects are suggested and limitations of our work are highlighted.

2.2 Identification of the fractional order derivative problem

In what follows, we will consider the Riemann-Liouville framework concerning the fractional differential operators on an interval $[a, b]$. We recall for every $\alpha \in]0, 1[$ and for every continuous function u on $[a, b]$, we have :

1. The left-sided Riemann-Liouville fractional derivative of u at $x \in [a, b]$

$${}_a D_x^\alpha u(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \left(\int_a^x (x-t)^{-\alpha} u(t) dt \right).$$

2. The right-sided Riemann-Liouville fractional derivative of u at $x \in [a, b]$

$${}_x D_b^\alpha u(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \left(\int_x^b (t-x)^{-\alpha} u(t) dt \right).$$

3. The two-sided Riemann-Liouville fractional derivative of u at $x \in [a, b]$

$$D_x^{\alpha, \theta} u = \theta {}_a D_x^\alpha u + (1-\theta) {}_x D_b^\alpha u,$$

with the skew coefficient $0 \leq \theta \leq 1$.

Now, let us consider the following elliptic boundary value problem

$$\begin{cases} -\frac{d}{dx} \left[k(x) D_x^{\alpha, \theta} u(x) \right] = f(x), & x \in \Omega = (a, b), \\ u(a) = u(b) = 0, \end{cases} \quad (2.1)$$

where $0 < \alpha < 1$, $0 \leq \theta \leq 1$, the source term $f \in L^2(\Omega)$ and the diffusivity function $k \in C^1(\overline{\Omega})$ is positive. Here, $D_x^{\alpha, \theta} u(x)$ denotes the two-sided Riemann-Liouville fractional derivative described before. Notice that if the diffusivity function k is constant, then the operator $\frac{d}{dx} \left[k D_x^{\alpha, \theta} \right]$ is not other than the Riemann-Liouville fractional derivative $D_x^{1+\alpha, \theta}$, up to the multiplicative constant k .

We confine ourselves to the case where the skew coefficient θ , the diffusivity function k and the source term f are known, but the fractional differential order α is unknown. To determine the unknown parameter α , we assume that we have an observation z , on Ω , of the solution u_α to Problem (2.1). Therefore, we introduce the quadratic cost function

$$J(\alpha) = \frac{1}{2} \int_{\Omega} (u(\alpha) - z)^2 dx. \quad (2.2)$$

The parameter identification problem for (2.1) is to find the optimal parameter $\hat{\alpha}$ satisfying $J(\hat{\alpha}) = \inf \{J(\alpha); \alpha \in]0, 1[\}$. As it is well known the necessary optimality condition for the cost J is given by $J'(\hat{\alpha}) = 0$.

Theorem 2.2.1. *Let u_α be the solution of (2.1), then $\frac{du_\alpha}{d\alpha}$ is the solution to the following boundary value problem*

$$\begin{cases} -\frac{d}{dx} [k(x)D_x^{\alpha,\theta} w] = \frac{d}{dx} [k(x)\hat{D}_x^{\alpha,\theta} u_\alpha], & x \in \Omega = (a, b), \\ w(a) = w(b) = 0, \end{cases} \quad (2.3)$$

with

$$\hat{D}_x^{\alpha,\theta} u_\alpha = \theta {}_a\hat{D}_x^\alpha u_\alpha + (1-\theta) {}_x\hat{D}_b^\alpha u_\alpha,$$

where

$$\begin{cases} {}_a\hat{D}_x^\alpha u_\alpha = \frac{d}{dx} \left[\int_a^x \frac{(x-t)^{-\alpha}}{\Gamma(1-\alpha)} \ln\left(\frac{1}{x-t}\right) u_\alpha(t) dt \right] + \frac{\Gamma'(1-\alpha)}{\Gamma(1-\alpha)} {}_aD_x^\alpha u_\alpha, \\ {}_x\hat{D}_b^\alpha u_\alpha = \frac{d}{dx} \left[\int_x^b \frac{(t-x)^{-\alpha}}{\Gamma(1-\alpha)} \ln\left(\frac{1}{t-x}\right) u_\alpha(t) dt \right] + \frac{\Gamma'(1-\alpha)}{\Gamma(1-\alpha)} {}_xD_b^\alpha u_\alpha. \end{cases}$$

Moreover,

$$J'(\alpha) = \int_{\Omega} w_\alpha(x) [u_\alpha(x) - z(x)] dx$$

where w_α is the solution of (2.3).

Proof. Let u_α be the solution of (2.1), then

$$-\frac{d}{dx} [k(x)D_x^{\alpha,\theta} u_\alpha(x)] = f(x), \quad x \in \Omega = (a, b). \quad (2.4)$$

We will detail the derivative of the equation (2.4) with respect to α :

$$-\frac{d}{d\alpha} \left(\frac{d}{dx} [k(x)D_x^{\alpha,\theta} u_\alpha(x)] \right) = -\frac{d}{dx} \left(k(x) \frac{d}{d\alpha} [D_x^{\alpha,\theta} u_\alpha(x)] \right) = 0, \quad x \in \Omega = (a, b). \quad (2.5)$$

It is clear that

$$\frac{d}{d\alpha} [D_x^{\alpha,\theta} u_\alpha(x)] = \theta \frac{d}{d\alpha} [{}_aD_x^\alpha u_\alpha(x)] + (1-\theta) \frac{d}{d\alpha} [{}_xD_b^\alpha u_\alpha(x)].$$

Let us start by the first derivative $\frac{d}{d\alpha} [{}_a D_x^\alpha u_\alpha(x)]$. A direct computation gives

$$\begin{aligned} \frac{d}{d\alpha} [{}_a D_x^\alpha u_\alpha(x)] &= \frac{d}{d\alpha} \left[\frac{d}{dx} \left(\int_a^x \frac{(x-t)^{-\alpha}}{\Gamma(1-\alpha)} u_\alpha(t) dt \right) \right] \\ &= \frac{d}{dx} \left(\int_a^x \frac{(x-t)^{-\alpha}}{\Gamma(1-\alpha)} \ln \left(\frac{1}{x-t} \right) u_\alpha(t) dt \right) + \frac{\Gamma'(1-\alpha)}{\Gamma(1-\alpha)} {}_a D_x^\alpha u_\alpha(x) \\ &\quad + {}_a D_x^\alpha \frac{du_\alpha}{d\alpha}(x), \end{aligned}$$

obtained by differentiation under the integral sign. We mention here that the first integral

$$\int_a^x (x-t)^{-\alpha} \ln \left(\frac{1}{x-t} \right) u(t) dt$$

is of Bertrand type and is consequently convergent if u is continuous on $[a, b]$. That is,

$$\frac{d}{d\alpha} [{}_a D_x^\alpha u_\alpha(x)] = {}_a \widehat{D}_x^\alpha u_\alpha(x) + {}_a D_x^\alpha \frac{du_\alpha}{d\alpha}(x), \quad (2.6)$$

The same computations with ${}_x D_b^\alpha$ lead to

$$\frac{d}{d\alpha} [D_x^{\alpha, \theta} u_\alpha(x)] = \widehat{D}_x^{\alpha, \theta} u_\alpha(x) + {}_a D_x^{\alpha, \theta} \frac{du_\alpha}{d\alpha}(x). \quad (2.7)$$

Using the fact that

$$\frac{d}{d\alpha} [D_x^{\alpha, \theta} u_\alpha(x)] = \frac{d}{d\alpha} f(x) = 0 \quad \text{and} \quad \frac{d}{d\alpha} u_\alpha(a) = \frac{d}{d\alpha} u_\alpha(b) = 0$$

we obtain the result. □

2.3 Steepest Descent Method and Finite Difference Scheme

To find the minimum of the cost function J , we use the classical method of steepest descent. Hence, to find the unknown parameter $\alpha \in]0, 1[$, we define the iterative sequence

1. *Initialization* : Choose $\alpha_0 \in]0, 1[$.
2. *Solve Problems (2.1) and (2.3) with α_n to obtain u_{α_n} and w_{α_n} .*
3. *Descent direction* : $-J'(\alpha_n) = - \int_{\Omega} w_{\alpha_n}(x) [u_{\alpha_n}(x) - z(x)] dx$.
4. *Update* : $\alpha_{n+1} = \alpha_n - \rho J'(\alpha_n)$, with a sufficiently small fixed parameter $\rho > 0$.

To solve Problems (2.1) and (2.3), we adapt the numerical method developed by Mustapha et al [42] with the finite difference method.

2.3.1 Finite difference scheme for (2.1)

Hereafter, we present the computation of the matrices B_L and B_R associated to the left-sided and right-sided fractional derivatives, respectively. We refer the interested reader to [42] for more details about the convergence of the method. To this end, we consider first a uniform discretization $(x_n)_{0 \leq n \leq N}$ of the interval (a, b) such that $x_0 = a$ and $x_N = b$ and for every $n \in \{1, 2, \dots, N\}$, the interval $[x_{n-1}, x_n]$ will be denoted by I_n . Then, the step size $h = x_n - x_{n-1} = (b - a)/N$, we set $x_{n+\frac{1}{2}} = (x_n + x_{n+1})/2$ the center of interval I_{n+1} and use the notation $v^n = v(x_n)$. With such notations, given a function v defined on $[a, b]$, for every $n \in \{1, 2, \dots, N\}$ and $x \in I_n$, we define the backward difference at x by:

$$\delta v(x) = \delta v^n := v^n - v^{n-1}.$$

Left-sided fractional derivative

Let $\theta = 1$, then (2.1) reduces to

$$-\frac{d}{dx} [k(x) {}_a D_x^\alpha u](x) = f(x). \quad (2.8)$$

Using first a forward type difference treatment of the operator $\frac{d}{dx}$, we propose the following approximation:

$$\frac{d}{dx} [k(x) {}_a D_x^\alpha u](x_n) \simeq h^{-1} \left[k^{n+\frac{1}{2}} {}_a D_x^\alpha u(x_{n+1}) - k^{n-\frac{1}{2}} {}_a D_x^\alpha u(x_n) \right], \quad (2.9)$$

where $k^{n+\frac{1}{2}} = k(x_{n+\frac{1}{2}})$. Since $u(a) = 0$, then ${}_a D_x^\alpha u = {}_a I_x^{1-\alpha} u'$ and the equation (2.9) can be rewritten as

$$\frac{d}{dx} [k(x) {}_a D_x^\alpha u](x_n) \simeq h^{-1} \left[k^{n+\frac{1}{2}} {}_a I_x^{1-\alpha} u'(x_{n+1}) - k^{n-\frac{1}{2}} {}_a I_x^{1-\alpha} u'(x_n) \right],$$

where ${}_a I_x^\beta$ denotes the left-sided Riemann-Liouville fractional integral defined by

$${}_a I_x^\beta u(x) = \frac{1}{\Gamma(\beta)} \int_a^x (x-t)^{\beta-1} u(t) dt.$$

Now, we apply again the difference approximation to get

$$\frac{d}{dx} [k(x) {}_a D_x^\alpha u](x_n) \simeq h^{-2} \left[k^{n+\frac{1}{2}} {}_a I_x^{1-\alpha} \delta u(x_{n+1}) - k^{n-\frac{1}{2}} {}_a I_x^{1-\alpha} \delta u(x_n) \right],$$

for $n = 1, \dots, N-1$. Furthermore,

$$\begin{aligned}
{}_a I_x^{1-\alpha} \delta u(x_n) &= \sum_{j=1}^n \int_{I_j} \frac{(x_n - s)^{-\alpha}}{\Gamma(1-\alpha)} \delta u^j ds \\
&= \sum_{j=1}^n \frac{\delta u^j}{\Gamma(1-\alpha)} \int_{I_j} (x_n - s)^{-\alpha} ds \\
&= \sum_{j=1}^n \frac{\delta u^j}{\Gamma(1-\alpha)} \int_{x_{j-1}}^{x_j} (x_n - s)^{-\alpha} ds \\
&= \sum_{j=1}^n \frac{w_{n,j} h^{1-\alpha}}{(1-\alpha)\Gamma(1-\alpha)} \delta u^j \\
&= \frac{h^{1-\alpha}}{\Gamma(2-\alpha)} \sum_{j=1}^n w_{n,j} \delta u^j \\
&= \frac{h^{1-\alpha}}{\Gamma(2-\alpha)} \left(\sum_{j=1}^{n-1} [w_{n,j} - w_{n,j+1}] u^j + u^n \right),
\end{aligned} \tag{2.10}$$

with

$$w_{n,j} = (n+1-j)^{1-\alpha} - (n-j)^{1-\alpha} \quad \text{for } n \geq j \geq 1,$$

and

$$\omega(h) = \frac{h^{1-\alpha}}{\Gamma(2-\alpha)}.$$

We return now to Problem (2.8), let $(U^n)_{1 \leq n \leq N-1} \simeq (u^n)_{1 \leq n \leq N-1}$ be the finite difference solution of the problem

$$\frac{1}{h^2} \left[k^{n-\frac{1}{2}} ({}_a I_x^{1-\alpha} \delta U)(x_n) - k^{n+\frac{1}{2}} ({}_a I_x^{1-\alpha} \delta U)(x_{n+1}) \right] = f^n, \quad 1 \leq n \leq N-1. \tag{2.11}$$

Therefore, it follows

$$\left\{ \begin{array}{l} \frac{\omega(h)}{h^2} \left[k^{n-\frac{1}{2}} \sum_{j=1}^n w_{n,j} \delta U^j - k^{n+\frac{1}{2}} \sum_{j=1}^{n+1} w_{n+1,j} \delta U^j \right] = f^n, \quad 1 \leq n \leq N-1, \\ U^0 = U^N = 0. \end{array} \right.$$

Hence, we can express the finite difference scheme as

$$\begin{aligned}
&\frac{\omega(h)}{h^2} \left(\sum_{j=1}^{n-1} \left(k^{n-\frac{1}{2}} [w^{n,j} - w^{n,j+1}] - k^{n+\frac{1}{2}} [w^{n+1,j} - w^{n+1,j+1}] \right) U^j \right. \\
&\left. + \left(k^{n-\frac{1}{2}} - k^{n+\frac{1}{2}} (2^{1-\alpha} - 2) \right) U^n - k^{n+\frac{1}{2}} U^{n+1} \right) = f^n.
\end{aligned}$$

The corresponding matrix of left-sided fractional derivative is then given by

$$B_L = \frac{\omega(h)}{h^2} \begin{pmatrix} c_{1,1} & -k^{3/2} & 0 & \cdots & \cdots & 0 \\ c_{2,1} & c_{2,2} & -k^{5/2} & 0 & \cdots & \vdots \\ c_{3,1} & c_{3,2} & c_{3,3} & -k^{7/2} & \ddots & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \\ \vdots & \vdots & & & & 0 \\ c_{N-2,1} & c_{N-2,2} & \cdots & \cdots & c_{N-2,N-2} & -k^{N-3/2} \\ c_{N-1,1} & c_{N-1,2} & \cdots & \cdots & c_{N-1,N-2} & c_{N-1,N-1} \end{pmatrix} \quad (2.12)$$

where

$$c_{n,j} = \begin{cases} k^{n-\frac{1}{2}} - k^{n+\frac{1}{2}}[2^{1-\alpha} - 2] & j = n, \\ a_{n,j} - a_{n+1,j} & j < n, \end{cases}$$

and

$$a_{n,j} = \begin{cases} k^{n-\frac{1}{2}} & j = n, \\ k^{n-\frac{1}{2}}[w_{n,j} - w_{n-1,j}] & j < n. \end{cases}$$

The finite difference solution is obtained by solving the linear system $B_L U = F$, where $U = [U^1, U^2, \dots, U^{N-1}]^T$ and $F = [f^1, f^2, \dots, f^{N-1}]^T$.

Right-sided fractional derivative

Let $\theta = 0$, then (2.1) reduces to

$$-\frac{d}{dx} [k(x) {}_x D_b^\alpha u](x) = f(x). \quad (2.13)$$

In this case, we use the following approximation

$$\frac{d}{dx} [k(x) {}_x D_b^\alpha u](x_n) \simeq h^{-1} \left[k^{n+\frac{1}{2}} {}_x D_b^\alpha u(x_n) - k^{n-\frac{1}{2}} {}_x D_b^\alpha u(x_{n-1}) \right]. \quad (2.14)$$

As before, since $u(b) = 0$, then ${}_x D_b^\alpha u = {}_x I_b^{1-\alpha} u'$ and the equation (2.14) can be rewritten as

$$\frac{d}{dx} [k(x) {}_x D_b^\alpha u](x_n) \simeq h^{-1} \left[k^{n+\frac{1}{2}} {}_x I_b^{1-\alpha} u'(x_n) - k^{n-\frac{1}{2}} {}_x I_b^{1-\alpha} u'(x_{n-1}) \right],$$

with ${}_x I_b^\alpha$ is the right-sided Riemann-Liouville fractional integral defined by

$${}_x I_b^\beta u(x) = \frac{1}{\Gamma(\beta)} \int_x^b (t-x)^{\beta-1} u(t) dt.$$

At this step, we apply again the difference approximation, we get

$$\frac{d}{dx} [k(x) {}_x D_b^\alpha u](x_n) \simeq h^{-2} \left[k^{n+\frac{1}{2}} {}_x I_b^{1-\alpha} \delta u(x_n) - k_x^{n-\frac{1}{2}} I_b^{1-\alpha} \delta u(x_{n-1}) \right],$$

for $n = 1, \dots, P-1$. The finite solution $(U^n)_{1 \leq n \leq N-1} \simeq (u^n)_{1 \leq n \leq N-1}$, Problem (2.13) satisfies the linear system

$$\begin{cases} \frac{1}{h^2} \left(k^{n-\frac{1}{2}} {}_x I_b^{1-\alpha} \delta U(x_{n-1}) - k^{n+\frac{1}{2}} {}_x I_b^{1-\alpha} \delta U(x_n) \right) = f^n, & 1 \leq n \leq N-1, \\ U^0 = U^N = 0. \end{cases} \quad (2.15)$$

Apply the Riemann-Liouville fractional integral to the finite difference δv to obtain

$$\begin{aligned} {}_x I_b^{1-\alpha} \delta v(x_{n-1}) &= \sum_{j=n}^N \int_{I_j} \frac{(s-x_n)^{-\alpha}}{\Gamma(1-\alpha)} \delta v^j ds \\ &= \omega(h) \sum_{j=n}^N w_{j,n} \delta v^j \\ &= \omega(h) \left(\sum_{j=n}^{N-1} [w_{j,n} - w_{j+1,n}] v^j - w_{n,n} v^{n-1} \right). \end{aligned}$$

The finite difference scheme of (2.15) is then given by

$$\frac{\omega(h)}{h^2} \left(k^{n-\frac{1}{2}} \sum_{j=n}^N w_{j,n} \delta U^j - k^{n+\frac{1}{2}} \sum_{j=n+1}^N w_{j,n+1} \delta U^j \right) = f^n.$$

We can rewrite (2.15) as

$$\frac{\omega(h)}{h^2} \left(\sum_{j=n}^{P-1} (b_{j,n} - b_{j,n+1}) U^j - k^{n-\frac{1}{2}} U^{n-1} \right) = f^n, \quad n = 1, \dots, N-1,$$

where

$$b_{j,n} = \begin{cases} -k^{n+\frac{1}{2}} & j = n-1, \\ k^{n-\frac{1}{2}} [w_{j,n} - w_{j,n-1}] & j \geq n. \end{cases}$$

The matrix of right-sided fractional derivative is

$$B_R = \frac{\omega(h)}{h^2} \begin{pmatrix} d_{1,1} & d_{1,2} & d_{1,3} & d_{1,4} & d_{1,5} & \cdots & d_{1,N-1} \\ -k^{3/2} & d_{2,2} & d_{2,3} & d_{2,4} & d_{2,5} & \cdots & d_{2,N-1} \\ 0 & -k^{5/2} & d_{3,3} & d_{3,4} & d_{3,5} & \cdots & d_{3,N-1} \\ 0 & 0 & -k^{7/2} & d_{4,4} & d_{4,5} & \cdots & d_{4,N-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -k^{N-3/2} & d_{N-1,N-1} \end{pmatrix} \quad (2.16)$$

with

$$d_{n,j} = \begin{cases} k^{n-\frac{1}{2}} (w_{j,n} - w_{j,n-1}) - k^{n+\frac{1}{2}} (w_{j,n+1} - w_{j,n}) & j > n, \\ k^{n+\frac{1}{2}} - k^{n-\frac{1}{2}} [2^{1-\alpha} - 2], & j = n. \end{cases}$$

The finite difference solution is obtained by solving the linear system $B_R U = F$.

Two-sided fractional derivative

We return to problem (2.1). To get a finite difference approximation, we combine the difference schemes introduced in (2.11) and (2.15). The finite solution $(U^n)_n \simeq (u^n)_n$ of the fractional problem is given by

$$\begin{aligned} & k^{n-\frac{1}{2}} [\theta {}_a I_x^{1-\alpha} \delta U(x_n) + (1-\theta) {}_x I_b^{1-\alpha} \delta U(x_{n-1})] \\ & - k^{n+\frac{1}{2}} [\theta {}_a I_x^{1-\alpha} \delta U(x_{n+1}) + (1-\theta) {}_x I_b^{1-\alpha} \delta U(x_n)] = h^2 f^n, \end{aligned}$$

for $n = 1, \dots, N-1$, and $U^0 = U^N = 0$. The finite difference solution is obtained by solving the linear system $BU = F$, where $B = \theta B_L + (1-\theta)B_R$.

2.3.2 Finite difference scheme for (2.3)

Hereafter, we will use the notation:

$${}_a \tilde{D}_x^\alpha u_\alpha = \frac{d}{dx} \left(\int_a^x \frac{(x-t)^{-\alpha}}{\Gamma(1-\alpha)} \ln \left(\frac{1}{x-t} \right) u_\alpha(t) dt \right),$$

and

$${}_x \tilde{D}_b^\alpha u_\alpha = \frac{d}{dx} \left(\int_x^b \frac{(t-x)^{-\alpha}}{\Gamma(1-\alpha)} \ln \left(\frac{1}{t-x} \right) u_\alpha(t) dt \right).$$

In the left side of the equation (2.3), we can proceed as before to build the left-sided and right-sided fractional differential matrices. On the other hand, for the right side we have

$$\begin{cases} {}_a \hat{D}_x^\alpha v = {}_a \tilde{D}_x^\alpha v + \frac{\Gamma'(1-\alpha)}{\Gamma(1-\alpha)} {}_a D_x^\alpha v, \\ {}_x \hat{D}_b^\alpha v = {}_x \tilde{D}_b^\alpha v + \frac{\Gamma'(1-\alpha)}{\Gamma(1-\alpha)} {}_x D_b^\alpha v. \end{cases}$$

It is clear that we can associate the matrices $-B_L$ to the operator ${}_a D_x^\alpha$ and $-B_R$ to ${}_x D_b^\alpha$. It remains to compute the matrix of fractional derivatives corresponding to ${}_a \tilde{D}_x^\alpha$ and ${}_x \tilde{D}_b^\alpha$. To obtain these matrices, it suffices to apply the same schemes obtained in the previous subsection to

$$\begin{cases} -\frac{d}{dx} [k(x) {}_a \tilde{D}_x^\alpha v] = g, & a < x < b, \\ v(a) = v(b) = 0, \end{cases} \quad (2.17)$$

and

$$\begin{cases} -\frac{d}{dx} [k(x) {}_x \tilde{D}_b^\alpha v] = g, & a < x < b, \\ v(a) = v(b) = 0, \end{cases} \quad (2.18)$$

with appropriate source functions g .

Left-sided fractional derivative, finite difference scheme for (2.17)

We propose the following approximation

$$\frac{d}{dx} [k(x) {}_a\tilde{D}_x^\alpha v](x_n) \simeq h^{-1} \left[k^{n+\frac{1}{2}} {}_a\tilde{D}_x^\alpha v(x_{n+1}) - k^{n-\frac{1}{2}} {}_a\tilde{D}_x^\alpha v(x_n) \right]. \quad (2.19)$$

Since $u(a) = 0$, then ${}_a\tilde{D}_x^\alpha v = {}_a\tilde{I}_x^{1-\alpha} v'$ and Equation (2.19) can be rewritten as

$$\frac{d}{dx} [k(x) {}_a\tilde{D}_x^\alpha v](x_n) \simeq h^{-1} \left[k^{n+\frac{1}{2}} {}_a\tilde{I}_x^{1-\alpha} v'(x_{n+1}) - k^{n-\frac{1}{2}} {}_a\tilde{I}_x^{1-\alpha} v'(x_n) \right],$$

with ${}_a\tilde{I}_x^{1-\alpha}$ defined by

$${}_a\tilde{I}_x^{1-\alpha} v(x) = \frac{1}{\Gamma(1-\alpha)} \int_a^x (x-t)^{-\alpha} \ln\left(\frac{1}{x-t}\right) v(t) dt.$$

If we apply again the difference approximation, we get

$$\frac{d}{dx} [k(x) {}_a\tilde{D}_x^\alpha v](x_n) \approx h^{-2} \left[k^{n+\frac{1}{2}} {}_a\tilde{I}_x^{1-\alpha} \delta v(x_{n+1}) - k^{n-\frac{1}{2}} {}_a\tilde{I}_x^{1-\alpha} \delta v(x_n) \right],$$

for $n = 1, \dots, N-1$. Furthermore,

$$\begin{aligned} {}_a\tilde{I}_x^{1-\alpha} \delta v(x_n) &= \sum_{j=1}^n \int_{I_j} \frac{(x_n-s)^{-\alpha}}{\Gamma(1-\alpha)} \ln\left(\frac{1}{x_n-s}\right) \delta v^j ds \\ &= \sum_{j=1}^n \frac{\delta v^j}{\Gamma(1-\alpha)} \int_{I_j} (x_n-s)^{-\alpha} \ln\left(\frac{1}{x_n-s}\right) ds \\ &= \sum_{j=1}^n \frac{\delta v^j}{\Gamma(1-\alpha)} \int_{x_{j-1}}^{x_j} (x_n-s)^{-\alpha} \ln\left(\frac{1}{x_n-s}\right) ds \\ &= \sum_{j=1}^n \frac{\delta v^j}{\Gamma(1-\alpha)} \left[-\ln\left(\frac{1}{x_n-s}\right) \frac{(x_n-s)^{1-\alpha}}{1-\alpha} \Big|_{x_{j-1}}^{x_j} + \int_{x_{j-1}}^{x_j} \frac{(x_n-s)^{-\alpha}}{1-\alpha} ds \right] \\ &= \sum_{j=1}^n \frac{\delta v^j}{\Gamma(1-\alpha)} \left[-\ln\left(\frac{1}{x_n-s}\right) \frac{(x_n-s)^{1-\alpha}}{1-\alpha} \Big|_{x_{j-1}}^{x_j} - \frac{(x_n-s)^{1-\alpha}}{(1-\alpha)^2} \Big|_{x_{j-1}}^{x_j} \right] \\ &= \sum_{j=1}^n \frac{h^{1-\alpha}}{\Gamma(2-\alpha)} \delta v^j \left[\left(-\ln((n-j+1)h) + \frac{1}{1-\alpha} \right) (n-j+1)^{1-\alpha} \right. \\ &\quad \left. - \left(-\ln((n-j)h) + \frac{1}{1-\alpha} \right) (n-j)^{1-\alpha} \right], \end{aligned} \quad (2.20)$$

where h is the spatial step. Now, we define

$$\tilde{w}_{n,j} = \left(-\ln((n-j+1)h) + \frac{1}{1-\alpha} \right) (n-j+1)^{1-\alpha} - \left(-\ln((n-j)h) + \frac{1}{1-\alpha} \right) (n-j)^{1-\alpha}, \quad (2.21)$$

for $n \geq j \geq 1$. It follows that

$${}_a\tilde{I}_x^{1-\alpha} \delta v(x_n) = \frac{h^{1-\alpha}}{\Gamma(2-\alpha)} \sum_{j=1}^n \tilde{w}_{n,j} \delta v^j. \quad (2.22)$$

At this step, we set

$$\omega(h) = \frac{h^{1-\alpha}}{\Gamma(2-\alpha)},$$

and denote by $(V^n)_n \simeq (v^n)_n$ the finite difference solution. Then,

$$\frac{1}{h^2} \left(k^{n-\frac{1}{2}} ({}_a \tilde{I}_x^{1-\alpha} \delta V)(x_n) - k^{n+\frac{1}{2}} ({}_a \tilde{I}_x^{1-\alpha} \delta V)(x_{n+1}) \right) = g^n. \quad (2.23)$$

Therefore, we have

$$\frac{\omega(h)}{h^2} \left(k^{n-\frac{1}{2}} \sum_{j=1}^n \tilde{w}_{n,j} \delta V^j - k^{n+\frac{1}{2}} \sum_{j=1}^{n+1} \tilde{w}_{n+1,j} \delta V^j \right) = g^n,$$

for $n = 1, \dots, N-1$, with $V^0 = V^N = 0$. Moreover, the difference scheme can be expressed by

$$\begin{aligned} & \frac{\omega(h)}{h^2} \left(\sum_{j=1}^{n-1} \left(k^{n-\frac{1}{2}} [\tilde{w}^{n,j} - \tilde{w}^{n,j+1}] - k^{n+\frac{1}{2}} [\tilde{w}^{n+1,j} - \tilde{w}^{n+1,j+1}] \right) V^j \right. \\ & \left. + \left(k^{n-\frac{1}{2}} - k^{n+\frac{1}{2}} (2^{1-\alpha} - 2) \right) V^n - k^{n+1} V^{n+\frac{1}{2}} \right) = g^n. \end{aligned}$$

Hence, the matrix of left-sided fractional derivative associated to (2.17) is given by

$$\tilde{B}_L = \frac{\omega(h)}{h^2} \begin{pmatrix} \tilde{c}_{1,1} & -k^{3/2} & 0 & \cdots & \cdots & 0 \\ \tilde{c}_{2,1} & \tilde{c}_{2,2} & -k^{5/2} & 0 & \cdots & \vdots \\ \tilde{c}_{3,1} & \tilde{c}_{3,2} & \tilde{c}_{3,3} & -k^{7/2} & \ddots & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \\ \vdots & \vdots & & & & 0 \\ \tilde{c}_{N-2,1} & \tilde{c}_{N-2,2} & \cdots & \cdots & \tilde{c}_{N-2,N-2} & -k^{N-3/2} \\ \tilde{c}_{N-1,1} & \tilde{c}_{N-1,2} & \cdots & \cdots & \tilde{c}_{N-1,N-2} & \tilde{c}_{N-1,N-1} \end{pmatrix} \quad (2.24)$$

where

$$\tilde{c}_{n,j} = \begin{cases} k^{n-\frac{1}{2}} - k^{n+\frac{1}{2}} [2^{1-\alpha} - 2] & j = n, \\ \tilde{a}_{n,j} - \tilde{a}_{n+1,j} & j < n. \end{cases}$$

with

$$\tilde{a}_{n,j} = \begin{cases} k^{n-\frac{1}{2}} & j = n, \\ k^{n-\frac{1}{2}} [\tilde{w}_{n,j} - \tilde{w}_{n-1,j}] & j < n. \end{cases}$$

The finite difference solution is obtained by solving the linear system $\tilde{B}_\ell V = G$, where

$$\begin{cases} V = [V^1, V^2, \dots, V^{N-1}]^T, \\ G = [g^1, g^2, \dots, g^{N-1}]^T. \end{cases}$$

Right-sided fractional derivative, finite difference scheme for (2.18)

In this case, we propose the following approximation

$$\frac{d}{dx} [k(x)_x \tilde{D}_b^\alpha v] (x_n) \simeq h^{-1} \left[k^{n+\frac{1}{2}} {}_x \tilde{D}_b^\alpha v(x_n) - k^{n-\frac{1}{2}} {}_x \tilde{D}_b^\alpha v(x_{n-1}) \right]. \quad (2.25)$$

As previously, since $v(b) = 0$ then ${}_x \tilde{D}_b^\alpha v = {}_x \tilde{I}_b^{1-\alpha} v'$, Equation (2.25) can be written as

$$\frac{d}{dx} [k(x)_x \tilde{D}_b^\alpha v] (x_n) \approx h^{-1} \left[k^{n+\frac{1}{2}} {}_x \tilde{I}_b^{1-\alpha} v'(x_n) - k^{n-\frac{1}{2}} {}_x \tilde{I}_b^{1-\alpha} v'(x_{n-1}) \right],$$

with ${}_x \tilde{I}_b^\alpha$ defined by

$${}_x \tilde{I}_b^\alpha v(x) = \frac{1}{\Gamma(1-\alpha)} \int_x^b (t-x)^{1-\alpha} \ln\left(\frac{1}{t-x}\right) v(t) dt.$$

Now, we apply again the difference approximation and get

$$\frac{d}{dx} [k(x)_x \tilde{D}_b^\alpha v] (x_n) \simeq h^{-2} \left[k^{n+\frac{1}{2}} {}_x \tilde{I}_b^{1-\alpha} \delta v(x_n) - k^{n-\frac{1}{2}} {}_x \tilde{I}_b^{1-\alpha} \delta v(x_{n-1}) \right],$$

for $n = 1, \dots, N-1$. For the finite solution $(V^n)_n \simeq (v^n)_n$, Problem (2.18) satisfies the system

$$\frac{1}{h^2} \left(k^{n-\frac{1}{2}} ({}_x \tilde{I}_b^{1-\alpha} \delta V)(x_{n-1}) - k^{n+\frac{1}{2}} ({}_x \tilde{I}_b^{1-\alpha} \delta V)(x_n) \right) = g^n, \quad (2.26)$$

for $n = 1, \dots, N-1$, with $V^0 = V^N = 0$. We apply the Riemann-Liouville fractional derivative to the finite difference δv and obtain

$$\begin{aligned} {}_x I_b^{1-\alpha} \delta v(x_{n-1}) &= \sum_{j=n}^N \int_{\tilde{I}_j} \frac{(s-x_n)^{-\alpha}}{\Gamma(1-\alpha)} \delta v^j ds \\ &= \omega(h) \sum_{j=n}^N \tilde{w}_{j,n} \delta v^j \\ &= \omega(h) \left(\sum_{j=n}^{N-1} [\tilde{w}_{j,n} - \tilde{w}_{j+1,n}] v^j - \tilde{w}_{n,n} v^{n-1} \right). \end{aligned}$$

We can express the finite difference scheme of (2.26) as

$$\frac{\omega(h)}{h^2} \left(k^{n-\frac{1}{2}} \sum_{j=n}^N \tilde{w}_{j,n} \delta V^j - k^{n+\frac{1}{2}} \sum_{j=n+1}^N \tilde{w}_{j,n+1} \delta V^j \right) = g^n$$

and rewrite (2.23) as

$$\frac{\omega(h)}{h^2} \left(\sum_{j=n}^{N-1} (b_{j,n} - b_{j,n+1}) V^j - k^{n-\frac{1}{2}} V^{n-1} \right) = g^n, \quad n = 1, \dots, N-1,$$

where

$$b_{j,n} = \begin{cases} -k^{n+\frac{1}{2}} & j = n-1, \\ k^{n-\frac{1}{2}} [\tilde{w}_{j,n} - \tilde{w}_{j,n-1}] & j \geq n. \end{cases}$$

Whence, the corresponding right-sided fractional derivative matrix is

$$\tilde{B}_R = \frac{\omega(h)}{h^2} \begin{pmatrix} \tilde{d}_{1,1} & \tilde{d}_{1,2} & \tilde{d}_{1,3} & \tilde{d}_{1,4} & \tilde{d}_{1,5} & \cdots & \tilde{d}_{1,N-1} \\ -k^{3/2} & \tilde{d}_{2,2} & \tilde{d}_{2,3} & \tilde{d}_{2,4} & \tilde{d}_{2,5} & \cdots & \tilde{d}_{2,N-1} \\ 0 & -k^{5/2} & \tilde{d}_{3,3} & \tilde{d}_{3,4} & \tilde{d}_{3,5} & \cdots & \tilde{d}_{3,N-1} \\ 0 & 0 & -k^{7/2} & d_{4,4} & d_{4,5} & \cdots & \tilde{d}_{4,N-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -k^{N-3/2} & \tilde{d}_{N-1,N-1} \end{pmatrix} \quad (2.27)$$

with

$$\tilde{d}_{n,j} = \begin{cases} k^{n-\frac{1}{2}}(\tilde{w}_{j,n} - \tilde{w}_{j,n-1}) - k^{n+\frac{1}{2}}(\tilde{w}_{j,n+1} - \tilde{w}_{j,n}) & j > n, \\ k^{n+\frac{1}{2}} - k^{n-\frac{1}{2}}[2^{1-\alpha} - 2], & j = n. \end{cases}$$

So, the finite difference solution is obtained by solving the linear system $\tilde{B}_R V = G$.

Two-sided fractional derivative, finite difference scheme for (2.3)

Let us return now to Problem (2.3). By the previous computations, we can associate $-\tilde{B}_L$ to ${}_a\tilde{D}_x^\alpha$ and $-\tilde{B}_R$ to ${}_x\tilde{D}_b^\alpha$. It follows that the matrix \hat{B}_L associated to the operator ${}_a\hat{D}_x^\alpha$ is given by

$$\hat{B}_L = -\tilde{B}_L - \frac{\Gamma'(1-\alpha)}{\Gamma(1-\alpha)} B_L,$$

and the matrix \hat{B}_R associated to ${}_x\hat{D}_b^\alpha$ is

$$\hat{B}_R = -\tilde{B}_R - \frac{\Gamma'(1-\alpha)}{\Gamma(1-\alpha)} B_R.$$

Finally, let us set $\hat{B} = \theta\hat{B}_L + (1-\theta)\hat{B}_R$, then the finite difference solution of (2.3) is obtained by solving the linear system $BV = \hat{B}U$, where U corresponds to u_α .

2.4 Numerical simulations

In this section, we confine ourselves to show how our method can be implemented in order to identify the fractional differential order in the case of stationary or evolution partial differential equations. Since, we carry out the *exact problem* satisfied by the optimal descent direction, which is the same as the original direct problem with another source term, our method is consequently more accurate and pertinent in time computation. The efficiency of our method depends essentially on the chosen algorithm to solve fractional differential equations. We favor the method developed in [42], where rigorous mathematical convergence results are proved. All the numerical tests have been performed using the *Matlab* software for numerical computation on a computer equipped with a processor Intel Core i5 and 8 Gb of RAM.

Hereafter, we choose $\theta = 1/2$ and consider the measure z of u in $\Omega =]0, 1[$ defined by

$$z(x) = 10^4 x^{3.8} (1-x)^{3.8}, \quad (2.28)$$

for the the source term f defined by

$$f(x) = D_x^{\hat{\alpha}, \theta} z(x), \quad (2.29)$$

where $\hat{\alpha} = 0.6$. Our aim is to recover the value $\hat{\alpha}$ from the data z , considered as an observation of the solution $u_{\hat{\alpha}}$. To this end, we apply the steepest descent method, described in Section 3, to the cost function $J(\alpha)$. In Figure 2.1, the curves of the cost function $J(\alpha)$ and the relative error with respect to the iteration number are presented. The convergence of the sequence $(\alpha_n)_n$ toward $\hat{\alpha}$ is illustrated, with a good accuracy.

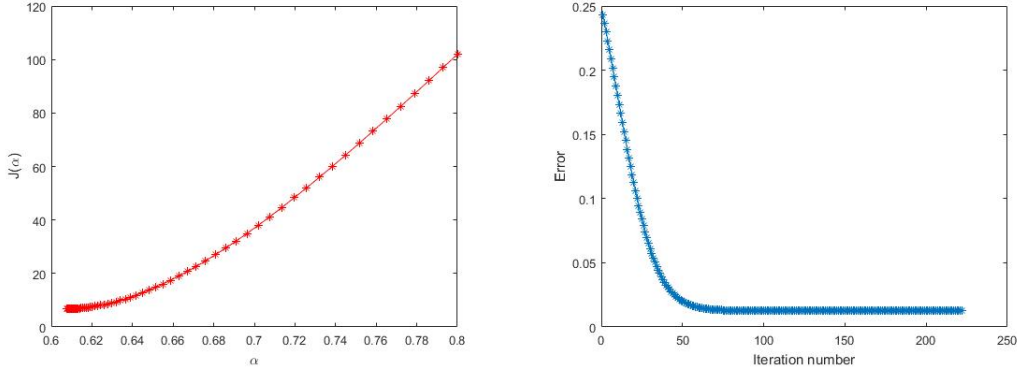


Figure 2.1: Left: The cost function $J(\alpha)$, with initialization $\alpha_0 = 0.8$. Right: Relative error for α in the steepest descent scheme.

To illustrate the convergence of our method for noisy data, the measure z is vitiated by an additive white gaussian noise with a given *signal-to-noise ratio* (SNR) as in Figure 2.2. Using this noisy measure, we can identify the fractional order $\hat{\alpha}$, as it is shown in Figure 2.3.

2.5 Application to the Taylor-Couette flow

We consider an application to coaxial annular flow in which a fluid is confined between two cylinders of radii R_{in} and R_{out} . This type of Taylor-Couette flow has the particular importance of allowing the rheological characterization of materials by loading the unknown material into the annular gap, rotating one cylinder (in a steady or unsteady motion) and measuring the resulting torque on the second cylindrical surface [16]. To be more precise, consider the spatial cylindrical domain $O =]R_{in}, R_{out}[\times]0, 2\pi[\times]z_{min}, z_{max}[$ and a time-interval $[0, T]$. For fixed $\theta \in [0, 2\pi]$ and

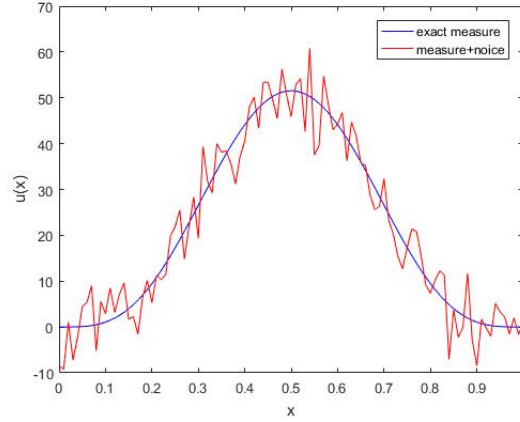


Figure 2.2: The exact observation (blue) and a noisy observation (red) with an additive white gaussian noise of SNR = 20.

$z \in]z_{min}, z_{max}[$, the θ -component of the fluid velocity $\vec{u}(r, \theta, z, t)$ will be denoted by $u_\theta(r, t)$, when the two cylinders are rotating. So the spatial domain is $\Omega =]R_{in}, R_{out}[$, for fixed $(\theta, z) \in]0, 2\pi[\times]z_{min}, z_{max}[$, and the time-interval is $[0, T]$. We are interested by the problem governing the θ -component of the fluid velocity $u_\theta(r, t)$. In [24], the authors proposed a fractional order model governing the fluid velocity $u_\theta(r, t)$:

$$\begin{cases} \frac{\rho}{\mathbb{V}} \frac{\partial u_\theta}{\partial t} + \frac{\rho}{\mathbb{G}} D_t^{2-\beta} u_\theta = \frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r} \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r^2}, & R_{in} < r < R_{out}, 0 < t \leq T \\ u_\theta(R_{in}, t) = \phi_i(t), u_\theta(R_{out}, t) = \phi_o(t), & 0 \leq t \leq T, \\ u_\theta(r, 0) = \frac{\partial u_\theta(r, 0)}{\partial t} = 0, & R_{in} < r < R_{out}. \end{cases} \quad (2.30)$$

and our aim here is to find the fractional order of the time derivative $\beta \in]0, 1[$, using a measure data z of $u_\theta(r, t)$. The constants ρ , \mathbb{V} and \mathbb{G} are given positive physical parameters.

2.5.1 Identification problem

Let us consider the following quadratic cost function

$$J(\beta) = \int_0^T \int_{R_{in}}^{R_{out}} (u_\theta(\beta) - z)^2 dr dt, \quad (2.31)$$

with z is a measure of solution on $]R_{in}, R_{out}[\times]0, T[$. The optimal parameter $\hat{\beta}$ satisfies $J(\hat{\beta}) = \inf_{\beta \in]0, 1[} J(\beta)$. Let $u_\theta(\beta)$ be the solution of (2.30). To perform a descent gradient method, we need to compute $u'_\theta(\beta)$ for the descent direction $-J'(\beta)$.

Proceeding as in Theorem 2.2.1, we obtain that $u'_\theta(\beta)$ is the solution of the following

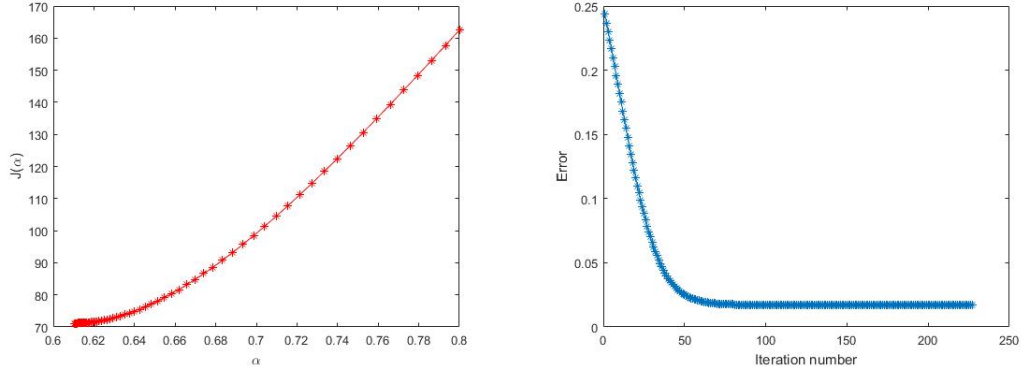


Figure 2.3: The case of noisy observation data of SNR = 20. Left: The cost function $J(\alpha)$, with initialization $\alpha_0 = 0.8$. Right: Relative error for α in the steepest descent scheme.

problem

$$\begin{cases} \frac{\rho}{\mathbb{V}} \frac{\partial w}{\partial t} + \frac{\rho}{\mathbb{G}} D_t^{2-\beta} w = \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} - \frac{w}{r^2} - \widehat{D}^{2-\beta} u_\theta(\beta), & R_{in} < r < R_{out}, 0 < t \leq T, \\ w(R_{in}, t) = 0, w(R_{out}, t) = 0, & 0 < t < T, \\ w(r, 0) = \frac{\partial w(r, 0)}{\partial t} = 0, & R_{in} < r < R_{out}. \end{cases} \quad (2.32)$$

and

$$J'(\beta) = 2 \int_0^T \int_{R_{in}}^{R_{out}} u'_\theta(\beta) (u_\theta(\beta) - z) dr dt. \quad (2.33)$$

2.5.2 Numerical scheme

Following [24], we consider the following difference scheme

$$\begin{aligned} \frac{\rho}{\mathbb{V}} \delta_t u_i^{s-1/2} + \frac{\rho}{\Gamma(\beta) \mathbb{G}} \frac{1}{\Delta t} \left[a_0 \delta_t u_i^{s-1/2} - \sum_{j=1}^{s-1} (a_{n-j-1} - a_{n-j}) \delta_t u_i^{j-1/2} \right] \\ = \delta_r^2 u_i^{s-1/2} + \frac{1}{2r_i} \left(\frac{\delta_r u_i^s + \delta_r u_i^{s-1}}{2} \right) - \frac{u_i^{s-1/2}}{r_i^2}, \end{aligned} \quad (2.34)$$

with u_i^s is an approximation of $u_\theta(r_i, t_s)$, $i = 1, \dots, N-1$, $s = 1, \dots, S$. With the same arguments, we define

$$\begin{aligned} u_i^{s-1/2} &\equiv \frac{1}{2}(u_i^s + u_i^{s-1}) \\ \delta_t u_i^{s-1/2} &\equiv \frac{1}{\Delta t}(u_i^s - u_i^{s-1}) \\ \delta_r u_{i-1/2}^s &\equiv \frac{1}{\Delta r}(u_i^s - u_{i-1}^s) \\ \delta_r^2 u_i^s &\equiv \frac{1}{\Delta r}(\delta_r u_{i+1/2}^s - \delta_r u_{i-1/2}^s), \end{aligned} \quad (2.35)$$

where $\Delta r = \frac{R_{out} - R_{in}}{N}$ and $\Delta t = T/S$ and

$$a_s = \int_{t_s}^{t_{s+1}} \frac{1}{t^{\varepsilon-1}} dt = \frac{(\Delta t)^{2-\varepsilon}}{2-\varepsilon} [(l+1)^{2+\varepsilon} - l^{2-\varepsilon}], \quad 0 \leq s \leq S-1 \text{ and } \varepsilon = 2 - \beta. \quad (2.36)$$

For $s = 1, 2, \dots, S$ and $i = 1, 2, \dots, N-1$, we may rewrite equation (2.34) as:

$$\begin{aligned} &\frac{\rho}{\mathbb{V}} \frac{1}{\Delta t} (u_i^s - u_i^{s-1}) + \frac{\rho}{\Gamma(\beta)\mathbb{G}} \frac{1}{\Delta t} \left[a_0 \frac{1}{\Delta t} (u_i^s - u_i^{s-1}) - \sum_{j=1}^{s-1} (a_{s-j-1} - a_{s-j}) \frac{1}{\Delta t} (u_i^j - u_i^{j-1}) \right] \\ &= -\frac{(u_i^s + u_i^{s-1})}{2r_i^2} + \frac{1}{(\Delta r)^2} \left[\left(\frac{u_{i+1}^s + u_{i-1}^s}{2} \right) - u_i^s + \left(\frac{u_{i+1}^{s-1} + u_{i-1}^{s-1}}{2} \right) - u_i^{s-1} \right] \\ &+ \frac{1}{2r_i \Delta r} \left(\frac{u_{i+1}^s - u_{i-1}^s + u_{i+1}^{s-1} - u_{i-1}^{s-1}}{2} \right) \end{aligned} \quad (2.37)$$

or in matrix form:

$$AU = B \quad (2.38)$$

where

$$U = \begin{bmatrix} u_1^s \\ u_2^s \\ \vdots \\ u_{N-1}^s \end{bmatrix} \quad (2.39)$$

and

$$A = \begin{pmatrix} d_1 & \frac{-1}{4r_1 \Delta r} - \frac{1}{2(\delta r)^2} & & & & & \\ \frac{1}{4r_2 \Delta r} - \frac{1}{2(\delta r)^2} & d_2 & \frac{-1}{4r_2 \Delta r} - \frac{1}{2(\delta r)^2} & & & & \\ \ddots & \ddots & \ddots & \ddots & & & \\ & & & & \ddots & \ddots & \\ & & & & & \frac{1}{4r_{N-1} \Delta r} - \frac{1}{2(\delta r)^2} & d_{N-1} \end{pmatrix} \quad (2.40)$$

with $d_i = \left(\frac{\rho}{\mathbb{V}} \frac{1}{\Delta t} + \frac{\rho}{\Gamma(\beta)\mathbb{G}} \frac{a_0}{(\Delta t)^2} + \frac{1}{2(r_i)^2} + \frac{1}{(\Delta r)^2} \right)$ for $1 \leq i \leq N-1$ and B is the column vector containing all the terms coming from the previous time iteration.

2.5.3 Taylor-Couette numerical simulations

We choose $\hat{\beta} = 0.7$ which will be supposed to be unknown. We are interested to find $\hat{\beta}$ by considering a measure z of the solution $u_\theta(\hat{\beta})$ on $]R_{in}, R_{out}[\times]0, T[$. Using the difference scheme and the result of the previous subsections, we can identify $\hat{\beta}$ as it is shown in Figure 2.4.

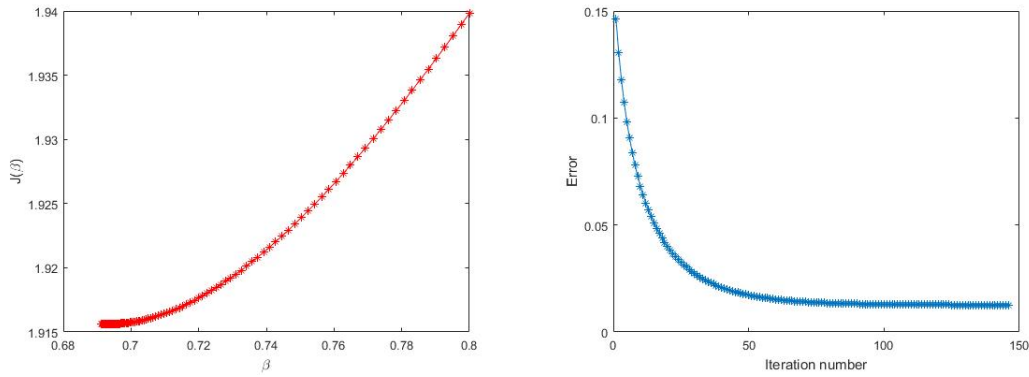


Figure 2.4: Left: The cost function $J(\beta)$, with initialization $\beta_0 = 0.8$. Right: Relative error for β in the steepest descent scheme.

2.6 Discussion and conclusion

As mentioned above, it is established that for general heterogeneous media, the classical Fick's law must be replaced by a nonlocal fractional differential operator. In such situations where anomalous diffusion occurs, an important fact is the determination of the correct orders of fractional differential operators. Indeed, an accurate computation of these fractional orders can help to better understand the physical meaning of the fractional diffusion models.

In this work, for elliptic and parabolic fractional order differential equations, we give the exact problem satisfied by the derivative of the solution with respect to the differential order. More precisely, if the solution of our boundary value problem, with a fractional differential order α , is denoted by $u(\alpha)$, we show that $\frac{d u(\alpha)}{d \alpha}$ satisfies the same problem, but with other source terms. In the literature cited before, only an approximation of this derivative are given by $\frac{u(\alpha + \varepsilon) - u(\alpha)}{\varepsilon}$, with ε small enough.

Our new result allows to carry out an exact computation of the cost functional derivative with respect to α and consequently an accurate computation of the descent direction for optimization algorithms. Notice that our result can also be applied in optimal control theory and for sensitivity analysis frameworks. We give two illustrations

of our result to identify the derivative order for the fractional Poisson's equation and Taylor-Couette flow problem with fractional time-derivative. This method can be extended without any difficulty to fractional differential systems. An interesting question is how to adapt this method for nonlinear systems of fractional differential equations.

From the theoretical point of view, it is interesting to obtain sharp regularity results on the solution of fractional differential equations with respect to the derivative order. In the best of our knowledge, there is no relevant results in this theoretical topic and one needs to imagine a new concept of differentiability. Indeed, for the regularity with respect to the fractional order, the ambient space depends on the parameter and the concept of Frechet or Gateaux differentiability is then not relevant. Such results will provide a rigorous justification of the computations carried out in the proof of Theorem 1, this will be the subject of a future research project.

Chapter 3

An inverse problem for a non-homogeneous time-space fractional equation

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3.1 Introduction

Consider the non-homogeneous fractional evolution boundary value problem

$$\begin{cases} \mathcal{D}_t^\gamma u(x, t) - \sum_{k=1}^d D_{x_k}^{\alpha_k} u(x, t) = f(t)p(x), & (x, t) \in \Omega \times]0, T[, \\ u(x, t) = 0, & (x, t) \in \partial\Omega \times [0, T], \\ u(x, 0) = \varphi(x), & x \in \overline{\Omega}, \end{cases}$$

where $\mathcal{D}_t^\gamma u(x, t)$ denotes (time) fractional derivative in the sense of Caputo, with $0 < \gamma < 1$, and $D_{x_k}^{\alpha_k} u(x, t)$ is the space fractional derivative, in the direction x_k , in the sense of Riemann-Liouville, with $1 < \alpha_k < 2$, for any $k \in \{1, 2, \dots, d\}$, $d \geq 1$. The spatial domain is the open set $\Omega =]0, 1[^d$ and $T > 0$ denotes the length of the time interval.

Let us fix the framework : When the initial data φ and the source terms (p, f) are given, the direct problem consists in looking for the unknown function u , in an adequate functional space. On the other hand, when the initial data φ and only the spatial part p are given, the

inverse problem consists in looking for the unknown functions (u, f) via the supplementary data

$$\int_{\Omega} u(x, t) dx = \psi(t), \quad t \in [0, T], \quad (3.1)$$

where ψ is a given function on $[0, T]$.

Notice that the fractional operator $D_x^{\alpha_k}$ is unbounded non self adjoint with compact resolvent admitting a countable complete basis of eigenfunctions, in the $L^2(\Omega, \mathbb{C})$ sense. This will allow us to look for classical solutions via the biorthogonal tensorial representation in the direction x . Indeed, the uniform convergence of the resulting series in $\bar{\Omega} \times [0, T]$ (and of their derivatives) and the uniqueness of the corresponding limits, with respect to the norms $\|\cdot\|_2$ and $\|\cdot\|_{\infty}$, will lead to the existence of classical solutions of the direct problem. For the inverse problems, we will also make use of the Schauder fixed point theorem in an adequate convex and bounded subset with the Bielecki's norm.

Before we present and prove our results, let us dwell on existing literature. Analytical and numerical study of inverse problems for fractional diffusion equations is undergoing an intensive development during the present decade. Kirane and Malik [31] considered at some later date a one-dimensional fractional diffusion equation with a non-local non-self adjoint boundary condition. They developed a solution and source function using a bi-orthogonal method, and obtained the uniqueness. In addition, they have proved an existence result when both the initial and the final conditions are smooth enough. In [26] the problem in [31] is generalized. Furati et al. use the same bi-orthogonal spatial system to obtain a separable formal solution and source function, and to show their uniqueness. They then use the asymptotic expansion of the generalized Mittag-Leffler function to achieve a result of existence under certain smoothness criteria on the initial and final conditions. Aleroev et al. [46] consider a linear heat equation, with a non-local boundary condition, involving a fractional derivative in time. They establish a space independent source term and the distribution of temperature for a problem with integral type over determining condition. They prove the solution's existence and uniqueness, and its continuous dependence on the data.

Other papers are devoted to determining unknown source terms [53, 52, 32].

The manuscript is organized as follows: In Section 3.2 we recall fundamental concepts of fractional calculus. In Section 3.4 we obtain results for the direct problem using Fourier's method; we determine the existence, uniqueness and continuous dependence on the data. In Section 3.5 we consider the inverse problem associated with the direct problem, we show the results of the existence, uniqueness and continuous dependence of the solution on the data.

3.2 Preliminaries

In this section, we recall basic definitions and notations from fractional calculus [47, 14]. Let $f : [a, b] \rightarrow \mathbb{R}$, be an integrable function. The left sided Riemann-Liouville fractional integral of order $\alpha > 0$ is defined by

$$I_x^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x-t)^{1-\alpha}} dt, \quad (3.2)$$

where $\Gamma(\alpha)$ is the Euler's Gamma function.

For arbitrary values of $\alpha > 0$, the left sided Riemann-Liouville fractional derivative of order α is defined by

$$D_x^\alpha f(x) = \frac{d^n}{dx^n} I_x^{n-\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^x (x-t)^{n-1-\alpha} f(t) dt, \quad n = [\alpha] + 1, \quad (3.3)$$

where $[\alpha]$ denotes the integer part of α . The left sided Caputo fractional derivative of order $\alpha > 0$ is defined by

$$\mathcal{D}_x^\alpha f(x) = I_x^{n-\alpha} f^{(n)}(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-1-\alpha} f^{(n)}(t) dt, \quad n = [\alpha] + 1. \quad (3.4)$$

where $f^{(n)}$ denotes the n -th classical derivative of f .

The Mittag-Leffler with two parameters α and β is defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \text{for any } z \in \mathbb{C}. \quad (3.5)$$

In the case $\beta = 1$, the function $E_{\alpha,1}$ is the classical Mittag-Leffler function denoted by E_α . From [45] we have the following property

$$\int_0^z t^{\beta-1} E_{\alpha,\beta}(\lambda t^\alpha) dt = z^\beta E_{\alpha,\beta+1}(\lambda z^\alpha), \quad (\alpha > 0, \beta > 0, \lambda \in \mathbb{R}). \quad (3.6)$$

We also need the following Lemma[44] due to Montel.

Theorem 3.2.1. *For every $\alpha \in]1, 2[$ and $\beta > 0$, there is a constant $c_{\alpha,\beta} > 0$ such that*

$$\forall z \in \mathbb{S}_\alpha, \quad |E_{\alpha,\beta}(z)| \leq \frac{c_{\alpha,\beta}}{1+|z|}, \quad (3.7)$$

where

$$\mathbb{S}_\alpha = \left\{ z \in \mathbb{C}, \quad \frac{\alpha\pi}{2} < |\arg(z)| \leq \pi \right\}. \quad (3.8)$$

Furthermore, from [47], we have :

Lemma 3.3. *Let $(f_i)_{i \in \mathbb{N}}$, be a sequence of functions defined on the interval $(a, b]$. Suppose the following conditions are fulfilled:*

1. *for a given $\alpha > 0$, the fractional derivative $D_z^\alpha f_i$ exists for all $i \in \mathbb{N}$ and $z \in (a, b]$;*
2. *both series $\sum_{i=1}^{\infty} f_i(z)$ and $\sum_{i=1}^{\infty} D_z^\alpha f_i(z)$ are uniformly convergent on the interval $[a + \varepsilon, b]$ for any $\varepsilon > 0$.*

Then the function defined by the series $\sum_{i=1}^{\infty} f_i(z)$ is α -differentiable on $(a, b]$ and satisfies

$$D_z^\alpha \left(\sum_{i=1}^{\infty} f_i(z) \right) = \sum_{i=1}^{\infty} D_z^\alpha f_i(z). \quad (3.9)$$

3.4 Direct problem

In this section, we will study the forward problem

$$\begin{cases} \mathcal{D}_t^\gamma u(x, t) - \sum_{k=1}^d D_{x_k}^{\alpha_k} u(x, t) = f(t)p(x), & (x, t) \in \Omega \times]0, T[, \\ u(x, t) = 0, & (x, t) \in \partial\Omega \times [0, T], \\ u(x, 0) = \varphi(x), & x \in \overline{\Omega}, \end{cases} \quad (3.10)$$

where p , f and φ are given functions such that $p, \varphi \in \mathcal{C}(\overline{\Omega}, \mathbb{R})$, $f \in \mathcal{C}([0, T], \mathbb{R})$. Since the problem (3.10) is linear, we can split its solution as

$$u(x, t) = v(x, t) + w(x, t), \quad (3.11)$$

where $v(x, t)$ is the solution of the problem

$$\begin{cases} \mathcal{D}_t^\gamma v(x, t) - \sum_{k=1}^d D_{x_k}^{\alpha_k} v(x, t) = 0, & (x, t) \in \Omega \times]0, T[, \\ v(x, t) = 0, & (x, t) \in \partial\Omega \times [0, T], \\ v(x, 0) = \varphi(x), & x \in \overline{\Omega}, \end{cases} \quad (3.12)$$

and $w(x, t)$ is the solution of the problem

$$\begin{cases} \mathcal{D}_t^\gamma w(x, t) - \sum_{k=1}^d D_{x_k}^{\alpha_k} w(x, t) = f(t)p(x), & (x, t) \in \Omega \times]0, T[, \\ w(x, t) = 0, & (x, t) \in \partial\Omega \times [0, T], \\ w(x, 0) = 0, & x \in \overline{\Omega}, \end{cases} \quad (3.13)$$

The spectral problems associated to Problem (3.12) is

$$D_{x_k}^{\alpha_k} X^k(x) = \lambda_k X^k(x), \quad 0 \leq x_k \leq 1, \quad \text{with } X^k(0) = X^k(1) = 0, \quad k \in \{1, 2, \dots, d\}. \quad (3.14)$$

The eigenfunctions of (3.14) are given by:

$$X_n^k(x) = x_k^{\alpha_k - 1} E_{\alpha_k, \alpha_k}(\lambda_n^k x^{\alpha_k}), \quad (3.15)$$

where λ_n^k are the associated eigenvalues respectively, $k \in \{1, 2, \dots, d\}$. Notice that these eigenvalues form a countable set and will be denoted by $(\lambda_n^k)_{n \geq 1}$. Moreover, they satisfy the following

1. There are only finitely many real eigenvalues and the rest appears as complex conjugate pairs.
2. $|\lambda_n^k| \sim (2\pi n)^{\alpha_k}$ as $n \rightarrow +\infty$.
3. $\frac{\alpha_k \pi}{2} < \arg(\lambda_n^k) \leq \pi$ for n sufficiently large and $\arg(\lambda_n^k) \sim \frac{\alpha_k \pi}{2}$ as $n \rightarrow +\infty$.

According to (3.7), for every $\alpha_k \in]1, 2[$:

$$\exists c_{\alpha_k} \geq 0, \quad \exists N_{\alpha_k} \in \mathbb{N}^*, \quad \forall n \geq N_{\alpha_k}, \quad \left| E_{\alpha_k, \alpha_k}(\lambda_n^k) \right| \leq \frac{c_{\alpha_k}}{1 + |\lambda_n^k|}. \quad (3.16)$$

To lighten the notations, we will present the case $d = 1$, but the all computations and results can be performed for $d \geq 1$ with straihforward changes. Hence the subscript and superscript k will be omitted in the sequel, and Ω reduces to the interval $]0, 1[$.

Based on [1], the family $(X_n)_{n \geq 1}$ is a basis of $L^2([0, 1], \mathbb{C})$, which is not orthogonal, because of the non auto-adjointness of the operator D_x^α . Consequently, for each $n \geq 1$, we introduce the family

$$X_n^*(x) = (1-x)^{\alpha-1} E_{\alpha, \alpha}(\lambda_n(1-x)^\alpha), \quad (3.17)$$

corresponding to the eigenfunctions of the adjoint operator of D_x^α , with respect to the inner product in $L^2([0, 1], \mathbb{C})$:

$$\langle f, g \rangle_{L^2([0, 1], \mathbb{C})} = \int_0^1 f(x) \bar{g}(x) dx, \quad (3.18)$$

where $\bar{g}(x)$ denotes the conjugate of the complex number $g(x)$. In the sequel, the inner products in $L^2([0, 1], \mathbb{C})$ and $L^2([0, T], \mathbb{C})$ will be simply denoted by $\langle \cdot, \cdot \rangle_{L^2(0, 1)}$ and $\langle \cdot, \cdot \rangle_{L^2(0, T)}$, respectively.

Notice that the adjoint operator of D_x^α is the right sided Riemann-Liouville fractional derivative of order $\alpha > 0$ defined by

$${}_x D^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_x^1 (t-x)^{n-1-\alpha} f(t) dt, \quad n = [\alpha] + 1. \quad (3.19)$$

Moreover, the family $(X_n^*)_{n \geq 1}$ is biorthogonal to $(X_n)_{n \geq 1}$ in $L^2([0, 1], \mathbb{C})$ and satisfy

$$\langle X_n, X_m^* \rangle_{L^2(0, 1)} = 0 \text{ for } n \neq m \quad \text{and} \quad \langle X_n, X_n^* \rangle_{L^2(0, 1)} = C_n > 0. \quad (3.20)$$

On the other hand, for every pair of integers $m \geq 1$, the resulting problem in the variable t is

$$\mathcal{D}_t^Y \mathcal{F}(t) = \lambda_n \mathcal{F}(t), \quad 0 < t < T, \quad \text{with } \mathcal{F}(0) = \varphi_n,$$

whose solution is given by

$$\mathcal{F}_n(t) = \varphi_n E_{\gamma, 1}(\lambda_n t^\gamma),$$

where,

$$\varphi_n = \frac{\langle \varphi, X_n^* \rangle_{L^2(0, 1)}}{\langle X_n, X_n^* \rangle_{L^2(0, 1)}}, \quad (3.21)$$

since $\varphi \in L^2(]0, 1[)$. The solution of Problem (3.12) is then formally given by:

$$v(x, t) = \sum_{n=1}^{\infty} \varphi_n E_{\gamma, 1}(\lambda_n t^\gamma) X_n(x). \quad (3.22)$$

In the same way, we can look for the classical solution of the problem (3.13) as

$$w(x, t) = \sum_{n=1}^{\infty} w_n(t) X_n(x), \quad (3.23)$$

where the coefficients $w_n(t)$ are to be found. To this end, if $p \in L^2(]0, 1[)$, then

$$p(x) = \sum_{n=1}^{\infty} p_n X_n(x), \quad (3.24)$$

with

$$p_n = \frac{\langle p, X_n^* \rangle_{L^2(0,1)}}{\langle X_n, X_n^* \rangle_{L^2(0,1)}}. \quad (3.25)$$

Substituting the solution w given by (3.23) in Equation (3.13), we find formally

$$\sum_{n=1}^{\infty} (\mathcal{D}_t^\gamma w_n(t) - \lambda_n w_n(t)) X_n(x) = \sum_{n=1}^{\infty} f(t) p_n X_n(x).$$

Therefore, for each $n \geq 1$, the coefficient w_n satisfies the equation

$$\mathcal{D}_t^\gamma w_n(t) - \lambda_n w_n(t) = p_n f(t). \quad (3.26)$$

Equation (3.26) is supplemented with the initial condition

$$w_n(0) = \frac{\langle w(x, 0), X_n^* \rangle_{L^2(0,1)}}{\langle X_n, X_n^* \rangle_{L^2(0,1)}} = 0. \quad (3.27)$$

Following [38], the solution of (3.26) is

$$w_n(t) = p_n \int_0^t s^{\gamma-1} E_{\gamma, \gamma}(\lambda_n s^\gamma) f(t-s) ds. \quad (3.28)$$

Therefore, the solution of problem (3.13) is written as

$$w(x, t) = \sum_{n=1}^{\infty} p_n X_n(x) \int_0^t s^{\gamma-1} E_{\gamma, \gamma}(\lambda_n s^\gamma) f(t-s) ds. \quad (3.29)$$

In what follows, c is a constant that might change from line to line.

Theorem 3.4.1. *Let $p \in \mathcal{C}([0, 1], \mathbb{R})$, $f \in \mathcal{C}([0, T], \mathbb{R})$ and $\varphi \in \mathcal{C}([0, 1], \mathbb{R})$ such that $D_x^\alpha \varphi \in \mathcal{C}([0, 1], \mathbb{R})$ and $\varphi(0) = \varphi(1) = 0$. Then there exists a unique classical solution of Problem (3.10).*

Proof.

– **Existence of a solution:** We need to prove the convergence of series (3.22), (3.29) and the series associated to its fractional derivatives in space and time. We set $\Phi := D_x^\alpha \varphi \in L^2(0, 1)$. Then

$$\Phi = \sum_{n \geq 1} \Phi_n X_n \quad (3.30)$$

where

$$\begin{aligned} \Phi_n &= \frac{\langle \Phi, X_n^* \rangle_{L^2(0,1)}}{\langle X_n, X_n^* \rangle_{L^2(0,1)}} \\ &= \frac{\langle D_x^\alpha \varphi, X_n^* \rangle_{L^2(0,1)}}{\langle X_n, X_n^* \rangle_{L^2(0,1)}} \\ &= \frac{\langle \varphi, {}_x D^\alpha X_n^* \rangle_{L^2(0,1)}}{\langle X_n, X_n^* \rangle_{L^2(0,1)}} \\ &= \frac{\langle \varphi, X_n^* \rangle_{L^2(0,1)}}{\lambda_n \langle X_n, X_n^* \rangle_{L^2(0,1)}} \\ &= \overline{\lambda_n} \varphi_n \end{aligned}$$

Then for every $x \in [0, 1]$ and $t \in [0, T]$, there is a constant $c \geq 0$ such that for all n sufficiently large

$$\begin{aligned} |\varphi_n X_n(x) E_{\gamma,1}(\lambda_n t^\gamma)| &= \left| \frac{\Phi_n X_n(x)}{\lambda_n} E_{\gamma,1}(\lambda_n t^\gamma) \right|, \\ &\leq \frac{c}{|\lambda_n|}, \\ &\leq \frac{c}{n^\alpha}. \end{aligned}$$

The series of general terms $a_n = \frac{c}{n^\alpha}$ is convergent since $\alpha > 1$. Using the M-test of Weierstrass, the series (3.22) is normally convergent on $[0, 1] \times [0, T]$. Consequently the function $v(x, t)$ defined by (3.22) is continuous on $[0, 1] \times [0, T]$. The same idea can be used for the series (3.29), by setting $P = D_x^\alpha p \in L^2(0, 1)$. Hence, the functions v and w verify the initial and the boundary conditions of (3.12) and (3.13), respectively.

Finally, to prove the convergence of the series associated to the fractional derivatives of v and w , it suffices to prove the convergence of series associated to $\frac{\partial v}{\partial t}$, $\frac{\partial w}{\partial t}$, $D_x^\alpha v$ and $D_x^\alpha w$. To this end, let

$$v_n(x, t) = \varphi_n X_n(x) E_{\gamma,1}(\lambda_n t^\gamma),$$

and

$$\tilde{w}_n(x, t) = p_n X_n(x) \int_0^t s^{\gamma-1} E_{\gamma,\gamma}(\lambda_n s^\gamma) f(t-s) dt;$$

so $v(x, t) = \sum_{n=1}^{\infty} v_n(x, y, t)$ and $w(x, t) = \sum_{n=1}^{\infty} \tilde{w}_n(x, t)$. Let us recall that for any $z \in \mathbb{C}$, it holds :

$$\frac{dE_{\alpha,\beta}(z)}{dz} = \frac{1}{\alpha z} (E_{\alpha,\beta-1}(z) - (\beta-1)E_{\alpha,\beta}(z)).$$

Consequently, there are constants c independent of n such that for every $x \in]0, 1[$ and $t \in]0, T[$ one has

$$\begin{aligned} \left| \frac{\partial v_n(x, t)}{\partial t} \right| &= \left| \varphi_n X_n(x) \frac{E_{\gamma,0}(\lambda_n t^\gamma)}{t} \right| \\ &\leq \frac{c}{n^\alpha t}. \end{aligned} \tag{3.31}$$

Then the series $\sum_{n \geq 1} \frac{\partial v_n(x, t)}{\partial t}$ converges normally on $[0, 1] \times [\varepsilon, T]$, for every $\varepsilon \in]0, T[$.

For the spatial derivative,

$$\begin{aligned} |D_x^\alpha v_n(x, t)| &= |\lambda_n \varphi_n X_n(x) E_{\gamma,1}(\lambda_n t^\gamma)|, \\ &= \left| \lambda_n \frac{\Phi_n}{\lambda_n} X_n(x) E_{\gamma,1}(\lambda_n t^\gamma) \right|, \\ &\leq c |E_{\gamma,1}(\lambda_n t^\gamma)|, \\ &\leq \frac{c}{1 + n^\alpha t^\gamma}. \end{aligned}$$

Then the series $\sum_{n \geq 1} D_x^\alpha v_n(x, t)$ converges normally on $[0, 1] \times [\varepsilon, T]$, for every $\varepsilon \in]0, T[$.

Applying the same ideas, we get the normal convergence of the series $\sum_{n \geq 1} \frac{\partial w_n(x, t)}{\partial t}$ and $\sum_{n \geq 1} D_x^\alpha w_n(x, t)$ on $[0, 1] \times [\varepsilon, T]$, for every $\varepsilon \in]0, T[$. The Lemma 3.3 achieves claim.

– **Uniqueness of the solution:** Let u_1, u_2 be two classical solutions of Problem (3.10) and set $u^* \equiv u_1 - u_2$. Then u^* satisfies the following problem

$$\begin{cases} \mathcal{D}_t^\gamma u^*(x, t) - D_x^\alpha u^*(x, t) - D_y^\beta u^*(x, t) = 0, & (x, t) \in]0, T[\times]0, T[\\ u^*(0, t) = u^*(1, t) = 0, & t \in [0, T], \\ u^*(x, 0) = 0, & x \in [0, 1]. \end{cases}$$

Proceeding as in before with $\varphi \equiv 0$, we obtain that $\varphi_n = 0$, for every $n \geq 1$, which implies that $u^* \equiv 0$. \square

Theorem 3.4.2. *Under the same conditions as in Theorem 2, the solution of the direct problem (3.10) depends continuously on the given data.*

Proof. Let u and \hat{u} be the solutions to the direct problem, corresponding to the data $\{p, f, \varphi\}$ and $\{\hat{p}, \hat{f}, \hat{\varphi}\}$, respectively. Using (3.11), we find

$$|u(x, t) - \hat{u}(x, t)| \leq |v(x, t) - \hat{v}(x, t)| + |w(x, t) - \hat{w}(x, t)|,$$

where $\{v, w\}$ and $\{\hat{v}, \hat{w}\}$ correspond the data $\{p, f, \varphi\}$ and $\{\hat{p}, \hat{f}, \hat{\varphi}\}$, respectively. The same arguments as before applied to (3.22) lead to

$$|v(x, t) - \hat{v}(x, t)| \leq c \|\varphi - \hat{\varphi}\|_{L^\infty(\Omega)},$$

where c is a positive constant. On the other hand, a direct computation gives

$$|w_n - \hat{w}_n| \leq \int_0^t |(t-s)^\gamma E_{\gamma, \gamma}(\lambda_n (t-s)^\gamma) (f(s) p_n - \hat{f}(s) \hat{p}_n)| ds.$$

It holds

$$\begin{aligned} |f(s) p_n - \hat{f}(s) \hat{p}_n| &= |f(s) p_n - f(s) \hat{p}_n + f(s) \hat{p}_n - \hat{f}(s) \hat{p}_n| \\ &\leq \|f\|_{L^\infty(0, T)} |p_n - \hat{p}_n| + |\hat{p}_n| \|f - \hat{f}\|_{L^\infty(0, T)}. \end{aligned}$$

Therefore, there is a constant $c > 0$ such that

$$|w(x, t) - \hat{w}(x, t)| \leq c (\|f\|_{L^\infty(\Omega)} \|p - \hat{p}\|_{L^\infty(\Omega)} + \|\hat{p}\|_{L^\infty(\Omega)} \|f - \hat{f}\|_{L^\infty(0, T)}),$$

which achieves the proof. \square

3.5 The inverse problem

Let us consider the problem

$$\begin{cases} \mathcal{D}_t^\gamma u(x, t) - D_x^\alpha u(x, t) = p(x) f(t), & (x, t) \in]0, 1[\times]0, T[, \\ u(0, t) = u(1, t) = 0, & t \in [0, T], \\ u(x, 0) = \varphi(x), & x \in [0, 1], \\ \int_0^1 u(x, t) dx = \psi(t), & t \in [0, T]. \end{cases} \quad (3.32)$$

where the functions p , φ and ψ are given and the functions u and f are unknown.

To solve Problem (3.32), we start by expanding $u(x, t)$ and $p(x)$, using systems (3.15) and get

$$u(x, t) = \sum_{n=1}^{\infty} c_n(t) X_n(x), \quad (3.33)$$

with

$$c_n(t) = \frac{\langle u(\cdot, t), X_n^* \rangle_{L^2(0,1)}}{\langle X_n, X_n^* \rangle_{L^2(0,1)}} \quad (3.34)$$

and

$$p(x) = \sum_{n=1}^{\infty} p_n X_n(x), \quad (3.35)$$

with

$$p_n = \frac{\langle p, X_n^* \rangle_{L^2(0,1)}}{\langle X_n, X_n^* \rangle_{L^2(0,1)}} \quad (3.36)$$

By substitution in (3.32), we obtain formally

$$\sum_{n=1}^{\infty} (\mathcal{D}_t^\gamma c_n(t) - \lambda_n c_n(t)) X_n(x) = f(t) \sum_{n=1}^{\infty} p_n X_n(x). \quad (3.37)$$

It follows that

$$\mathcal{D}_t^\gamma c_n(t) - \lambda_n c_n(t) = p_n f(t), \quad (3.38)$$

with the initial condition $c_n(0) = \varphi_n$ on $[0, 1]$. Using (3.33) and (3.6) we get

$$\begin{aligned} \psi(t) = \int_0^1 u(x, t) dx &= \sum_{n=1}^{\infty} c_n(t) \int_0^1 x^{\alpha-1} E_{\alpha, \alpha}(\lambda_n x^\alpha) dx, \\ &= \sum_{n=1}^{\infty} c_n(t) E_{\alpha, \alpha+1}(\lambda_n). \end{aligned}$$

Using (3.38), we find formally

$$\mathcal{D}_t^\gamma \psi(t) - \sum_{n=1}^{\infty} \lambda_n c_n(t) E_{\alpha, \alpha+1}(\lambda_n) = p_n f(t) \sum_{n=1}^{\infty} E_{\alpha, \alpha+1}(\lambda_n), \quad (3.39)$$

then

$$f(t) = \frac{\mathcal{D}_t^\gamma \psi(t) - \sum_{n=1}^{\infty} \lambda_n c_n(t) E_{\alpha, \alpha+1}(\lambda_n)}{\sum_{n=1}^{\infty} p_n E_{\alpha, \alpha+1}(\lambda_n)}. \quad (3.40)$$

According to [38], the solution of (3.38) is given by

$$c_n(t) = \varphi_n E_{\gamma, 1}(\lambda_n t^\gamma) + p_n \int_0^t s^{\gamma-1} E_{\gamma, \gamma}(\lambda_n s^\gamma) f(t-s) ds. \quad (3.41)$$

Replacing $c_n(t)$ as in (3.41) in (3.40), we then find the equation for $f(t)$:

$$\begin{aligned} f(t) &= \frac{1}{a} \left(\mathcal{D}_t^\gamma \psi(t) - \phi(t) - \int_0^t s^{\gamma-1} f(t-s) \rho(s) ds \right) \\ &= \frac{1}{a} \left(\mathcal{D}_t^\gamma \psi(t) - \phi(t) - \int_0^t (t-s)^{\gamma-1} f(s) \rho(t-s) ds \right), \end{aligned} \quad (3.42)$$

where

$$a = \sum_{n=1}^{\infty} p_n E_{\alpha, \alpha+1}(\lambda_n), \quad (3.43)$$

$$\phi(t) = \sum_{n=1}^{\infty} \lambda_n \varphi_n E_{\gamma, 1}(\lambda_n t^\gamma) E_{\alpha, \alpha+1}(\lambda_n), \quad (3.44)$$

and

$$\rho(t) = \sum_{n=1}^{\infty} \lambda_n p_n E_{\gamma, \gamma}(\lambda_n t^\gamma) E_{\alpha, \alpha+1}(\lambda_n). \quad (3.45)$$

Theorem 3.5.1. *Let $p \in \mathcal{C}([0, 1], \mathbb{R})$ such that $\int_0^1 p(x) dx \neq 0$, $\psi \in \mathcal{C}([0, T], \mathbb{R})$ such that $\mathcal{D}_t^\gamma \psi \in \mathcal{C}([0, T], \mathbb{R})$ and $\varphi \in \mathcal{C}([0, 1], \mathbb{R})$ such that $D_x^\alpha \varphi \in \mathcal{C}([0, 1], \mathbb{R})$, with $\varphi(0) = \varphi(1) = 0$. Then there exists at least a solution to Problem (3.32).*

Proof. – **Existence of a solution:**

Applying the same arguments as before, we can show that the functions ρ and ϕ defined by (3.45) and (3.44) are γ -differentiable on $]0, T[$. Moreover, the condition $\int_0^1 p(x) dx \neq 0$ implies that the real number $a \neq 0$.

At this stage, we will establish the existence of at least a solution to the inverse problem (3.32) and that this solution depends continuously on the data. First of all, we will use the Schauder fixed point theorem in Banach spaces with the Arzela-Ascoli compactness result. To this end, we start by defining the following operator :

$$B : C([0, T], \mathbb{R}) \longrightarrow C([0, T], \mathbb{R})$$

$$f \longmapsto B(f) : t \mapsto \frac{1}{a} \left(\mathcal{D}_t^\gamma \psi(t) - \phi(t) - \int_0^t (t-s)^{\gamma-1} f(s) \rho(t-s) ds \right). \quad (3.46)$$

To prove that the operator B admits a fixed point, start by showing that B maps a certain closed convex set into itself, in the space $C([0, T], \mathbb{R})$ equipped with the Bielecki norm. For every $\delta > 0$, we introduce the Bielecki norm

$$\|u\|_\delta = \sup_{t \in [0, T]} \left(e^{-\delta t} |u(t)| \right). \quad (3.47)$$

The space $(C([0, T], \mathbb{R}), \|\cdot\|_\delta)$ is a Banach space.

Lemma 3.6. *Let $1 < \alpha < 2$, then there is a positive constant $\delta_* > 0$ such that for any $\delta > \delta_*$ there is a radius $R > 0$ such that the closed convex ball*

$$K = \{f \in C([0, T], \mathbb{R}) : \|f\|_\delta \leq R\}$$

is stable by the operator B , that is, $B(K) \subset K$.

Proof. For any $f \in C([0, T], \mathbb{R})$, $t \in [0, T]$ and $\delta > 0$, we have

$$\begin{aligned} e^{-\delta t} |B(f)(t)| &= \frac{e^{-\delta t}}{|a|} \left| D_t^\gamma \psi(t) - \phi(t) - \int_0^t (t-s)^{\gamma-1} \rho(t-s) f(s) ds \right|, \\ &= \frac{1}{|a|} \left| e^{-\delta t} (D_t^\gamma \psi(t) - \phi(t)) - \int_0^t e^{-\delta(t-s)} (t-s)^{\gamma-1} \rho(t-s) e^{-\delta s} f(s) ds \right|, \\ &\leq \frac{1}{|a|} \left(\|D_t^\gamma \psi\|_\delta + \|\phi\|_\delta + \|f\|_\delta \int_0^T e^{-\delta s} s^{\gamma-1} |\rho(s)| ds \right). \end{aligned}$$

At this stage, recall that for any $\gamma > 0$, the functions $\mathcal{E}_\delta : s \mapsto e^{-\delta s} s^{\gamma-1} \rho(s)$ is integrable on $[0, T]$. Moreover, it holds that $\lim_{\delta \rightarrow +\infty} \|\mathcal{E}_\delta\|_{L^1([0, T])} = 0$; consequently,

$$\exists \delta_* > 0, \forall \delta \geq \delta_*, \quad \int_0^T e^{-\delta s} s^{\gamma-1} |\rho(s)| ds \leq \int_0^T \left| e^{-\delta_* s} s^{\gamma-1} \rho(s) \right| ds < 1.$$

That is, there are two real constants $c_1(\delta) := \frac{1}{|a|} (\|D_t^\gamma \psi\|_\delta + \|\phi\|_\delta)$ and $c_2 := \int_0^T \left| e^{-\delta_* s} s^{\gamma-1} \rho(s) \right| ds$ satisfying $c_1(\delta) \geq 0$ and $0 \leq c_2 < 1$ such that :

$$\forall \delta \geq \delta_*, \quad \|B(f)\|_\delta \leq c_1(\delta) + c_2 \|f\|_\delta.$$

Whence, for any $\delta \geq \delta_*$, there exists $R > 0$ such that the closed ball K of radius R in $(C([0, T], \mathbb{R}), \|\cdot\|_\delta)$ is stable by the operator B ; this achieves the proof. \square

Remarque 1. Using the Bielecki norm allows to have no constraint on the maximum value that T can take. On the other hand, if we use the classical infinite norm, then T must be less than a finite quantity depending on the data of the problem.

Lemma 3.7. *The family $(B(f))_{f \in K}$ is equicontinuous, that is*

$$\forall t \in [0, T], \forall \varepsilon > 0, \exists r > 0 : \forall t' \in [0, T] \cap]t-r, t+r[, \forall f \in K, |B(f)(t) - B(f)(t')| \leq \varepsilon.$$

Proof. It suffices to prove that the family $(B_1(f))_{f \in K}$ is equicontinuous, where

$$\begin{aligned} B_1 : C([0, T], \mathbb{R}) &\longrightarrow C([0, T], \mathbb{R}) \\ f &\longmapsto B_1(f) : t \mapsto \int_0^t (t-s)^{\gamma-1} \rho(t-s) f(s) ds. \end{aligned} \tag{3.48}$$

We introduce the functions $W_1 : s \mapsto s^{\gamma-1} \rho(s)$ and $W_2 : s \mapsto e^{-\delta s} W_1(s)$, which are clearly integrable on $[0, T]$. Let t and t' in $]0, T]$. We can assume, without loss of generality, that $0 < t \leq t' \leq T$.

$$\begin{aligned} |B_1(f)(t') - B_1(f)(t)| &= \left| \int_0^{t'} (W_1(t'-s) - W_1(t-s)) f(s) ds + e^{\delta t} \int_t^{t'} W_2(t-s) e^{-\delta s} f(s) ds \right| \\ &\leq e^{\delta T} \|f\|_\delta \left(\int_0^t |W_1(t'-s) - W_1(t-s)| ds + \int_0^{t'-t} |W_2(s)| ds \right). \end{aligned}$$

First, for every $s \in]0, t]$, we have

$$\begin{aligned} |W_1(t'-s) - W_1(t-s)| &\leq (t-s)^{\gamma-1} |\rho(t'-s)| + (t-s)^{\gamma-1} |\rho(t-s)|, \\ &\leq 2 \|\rho\|_\infty (t-s)^{\gamma-1}. \end{aligned}$$

Using the fact that the map $s \mapsto (t-s)^{\gamma-1}$ is in $L^1(0, t)$ and applying the dominated convergence theorem of Lebesgue, we deduce that

$$\lim_{t' \rightarrow t} \int_0^t |W_1(t'-s) - W_1(t-s)| ds = 0.$$

Fix now an $\varepsilon > 0$ and $t \in]0, T[$, it follows that

$$\exists r_1 > 0, \forall t' \in [0, T] \cap]t - r_1, t + r_1[, \int_0^t |W_1(t' - s) - W_1(t - s)| ds \leq \frac{e^{-\delta T}}{4R \|\rho\|_\infty} \varepsilon.$$

On the other hand, using the fact that the map W_2 is in $L^1(0, T)$ and applying the dominated convergence theorem of Lebesgue, we deduce that

$$\lim_{t' \rightarrow t} \int_0^{t'-t} |W_2(s)| ds = 0.$$

That is

$$\exists r_2 > 0, \forall t' \in [0, T] \cap]t - r_2, t + r_2[, \int_0^{t'-t} |W_2(s)| ds \leq \frac{e^{-\delta T}}{2R} \varepsilon.$$

Let us point out that radii r_1 and r_2 depend only on t and ε , but are independent of f . Whence, for any $t \in [0, T]$ and $\varepsilon > 0$, there exists $r := \min\{r_1, r_2\} > 0$ such that

$$\forall t' \in [0, T] \cap]t - r, t + r[, \forall f \in K, |B_1(f)(t) - B_1(f)(t')| \leq \varepsilon,$$

which achieves the proof. \square

By Arzela-Ascoli theorem, $B(K)$ is relatively compact. Now, we are able to use the Schauder fixed point theorem [60].

Theorem 3.7.1 (Schauder fixed point theorem). *Let $(E, \|\cdot\|)$ be a Banach space, let $K \subset E$ convex and closed. Let $T : K \rightarrow K$ be a continuous operator such that $T(K)$ is relatively compact. Then T has a fixed point in K .*

We know that B is a continuous operator and $B(K)$ is relatively compact, with K convex and closed. By Theorem 3.7.1 B admits a fixed point in K , so the integral equation (3.42) has a solution in $[0, T]$.

Once the existence of the source term f is established, the existence of u follows in a similar way with the arguments developed above for the direct problem.

Theorem 3.7.2. *The solution of the inverse problem (3.32) depends continuously on the given data $\varphi, p \in L^\infty(\Omega)$, such that p is not identically zero in Ω , and ψ defined on $[0, T]$ such that $\mathcal{D}_t^\gamma \psi \in L^\infty(0, T)$.*

Proof. Let $\{u(x, y, t), f(t)\}, \{\tilde{u}(x, y, t), \tilde{f}(t)\}$ be solution sets of the inverse problem, corresponding to the data $\{p, \varphi, \psi\}, \{\tilde{p}, \tilde{\varphi}, \tilde{\psi}\}$ respectively. Using (3.42) we have

$$f(t) = \frac{\mathcal{D}_t^\gamma \psi(t)}{a} - \frac{\phi(t)}{a} - \int_0^t (t-s)^{\gamma-1} f(s) \frac{\rho(t-s)}{a} ds,$$

and

$$\tilde{f}(t) = \frac{\mathcal{D}_t^\gamma \tilde{\psi}(t)}{\tilde{a}} - \frac{\tilde{\phi}(t)}{\tilde{a}} - \int_0^t (t-s)^{\gamma-1} \tilde{f}(s) \frac{\tilde{\rho}(t-s)}{\tilde{a}} ds,$$

where $\{a, \phi, \rho\}, \{\tilde{a}, \tilde{\phi}, \tilde{\rho}\}$ correspond to the data $\{p, \varphi, \psi\}, \{\tilde{p}, \tilde{\varphi}, \tilde{\psi}\}$ respectively.

Remark first that ρ , a , and ϕ depend continuously on p and φ . Indeed, for the function ρ , using (3.36) and (3.45), we have the estimate :

$$\|\rho - \tilde{\rho}\|_{L^\infty([0,T])} \leq c \|p - \tilde{p}\|_{L^\infty(\bar{\Omega})}. \quad (3.49)$$

For the real constant a , using (3.43) and (3.36), we have the estimate :

$$|a - \tilde{a}| \leq c \|p - \tilde{p}\|_{L^\infty(\bar{\Omega})}. \quad (3.50)$$

For the function ϕ , using (3.44) and (3.21), we have the estimate :

$$|\phi(t) - \tilde{\phi}(t)| \leq c \|\varphi - \tilde{\varphi}\|_{L^\infty(\bar{\Omega})}. \quad (3.51)$$

Let $g(t)$ and $\tilde{g}(t)$ be such that

$$g(t) = \int_0^t (t-s)^{\gamma-1} f(s) \frac{\rho(t-s)}{a} ds, \quad \tilde{g}(t) = \int_0^t (t-s)^{\gamma-1} \tilde{f}(s) \frac{\tilde{\rho}(t-s)}{\tilde{a}} ds.$$

then, for any $\delta > 0$

$$e^{-\delta t} |f(t) - \tilde{f}(t)| \leq e^{-\delta t} \left(\left| \frac{\mathcal{D}_t^\gamma \psi(t)}{a} - \frac{\mathcal{D}_t^\gamma \tilde{\psi}(t)}{\tilde{a}} \right| + \left| \frac{\phi(t)}{a} - \frac{\tilde{\phi}(t)}{\tilde{a}} \right| + |g(t) - \tilde{g}(t)| \right). \quad (3.52)$$

Now

$$\begin{aligned} e^{-\delta t} \left| \frac{\mathcal{D}_t^\gamma \psi(t)}{a} - \frac{\mathcal{D}_t^\gamma \tilde{\psi}(t)}{\tilde{a}} \right| &\leq \left| \frac{\mathcal{D}_t^\gamma \psi(t)}{a} - \frac{\mathcal{D}_t^\gamma \psi(t)}{\tilde{a}} + \frac{\mathcal{D}_t^\gamma \psi(t)}{\tilde{a}} - \frac{\mathcal{D}_t^\gamma \tilde{\psi}(t)}{\tilde{a}} \right| \\ &\leq \|\mathcal{D}_t^\gamma \psi\|_{L^\infty(0,T)} \left| \frac{1}{a} - \frac{1}{\tilde{a}} \right| + \frac{1}{|\tilde{a}|} \|\mathcal{D}_t^\gamma \psi - \mathcal{D}_t^\gamma \tilde{\psi}\|_{L^\infty(0,T)}, \end{aligned}$$

and consequently

$$e^{-\delta t} \left| \frac{\mathcal{D}_t^\gamma \psi(t)}{a} - \frac{\mathcal{D}_t^\gamma \tilde{\psi}(t)}{\tilde{a}} \right| \leq \frac{c M_1}{|a \tilde{a}|} \|p - \tilde{p}\|_{L^\infty(\bar{\Omega})} + \frac{1}{|\tilde{a}|} \|\mathcal{D}_t^\gamma \psi - \mathcal{D}_t^\gamma \tilde{\psi}\|_\delta,$$

where M_1 is a positive constant such that $\|\mathcal{D}_t^\gamma \psi\|_{L^\infty(0,T)} \leq M_1$. Similarly, we get :

$$e^{-\delta t} \left| \frac{\phi(t)}{a} - \frac{\tilde{\phi}(t)}{\tilde{a}} \right| \leq \frac{c \|\phi\|_{L^\infty(0,T)}}{|a \tilde{a}|} \|p - \tilde{p}\|_{L^\infty(\bar{\Omega})} + \frac{1}{|\tilde{a}|} \|\phi - \tilde{\phi}\|_{L^\infty(0,T)}.$$

It follows that

$$e^{-\delta t} \left| \frac{\phi(t)}{a} - \frac{\tilde{\phi}(t)}{\tilde{a}} \right| \leq \frac{c M_2}{|a \tilde{a}|} \|p - \tilde{p}\|_{L^\infty(\bar{\Omega})} + \frac{c}{|\tilde{a}|} \|\varphi - \tilde{\varphi}\|_{L^\infty(\bar{\Omega})},$$

where M_2 is a positive constant such that $\|\phi\|_{L^\infty(0,T)} \leq M_2$. Now, we look at

$$\begin{aligned} e^{-\delta t} |g(t) - \tilde{g}(t)| &= e^{-\delta t} \int_0^t (t-s)^{\gamma-1} \left| f(t) \frac{\rho(t-s)}{a} - \tilde{f}(t) \frac{\tilde{\rho}(t-s)}{\tilde{a}} \right| ds \\ &\leq e^{-\delta t} \int_0^t (t-s)^{\gamma-1} \left| f(t) \frac{\rho(t-s)}{a} - \tilde{f}(t) \frac{\rho(t-s)}{a} \right| ds + \\ &\quad e^{-\delta t} \int_0^t (t-s)^{\gamma-1} \left| \tilde{f}(t) \frac{\rho(t-s)}{a} - \tilde{f}(t) \frac{\tilde{\rho}(t-s)}{\tilde{a}} \right| ds \\ &\leq \|\tilde{f}\|_{L^\infty([0,T])} \int_0^t (t-s)^{\gamma-1} \left| \frac{\rho(t-s)}{a} - \frac{\tilde{\rho}(t-s)}{\tilde{a}} \right| ds + \frac{M_3}{|a|} \|f - \tilde{f}\|_\delta \mathbb{I}(\delta), \end{aligned}$$

where M_3 is a positive constant such that $\|\rho\|_{L^\infty([0,T])} \leq M_3$ and

$$\mathbb{I}(\delta) = \int_0^T (t-s)^{\gamma-1} e^{-\delta(t-s)} ds. \quad (3.53)$$

Similarly, we get

$$\begin{aligned} \left| \frac{\rho(t-s)}{a} - \frac{\tilde{\rho}(t-s)}{\tilde{a}} \right| &\leq \frac{cM_3}{|a\tilde{a}|} \|p - \tilde{p}\|_{L^\infty(\bar{\Omega})} + \frac{1}{|\tilde{a}|} \|\rho - \tilde{\rho}\|_{L^\infty([0,T])}, \\ &\leq \frac{c(M_3+1)}{|a\tilde{a}|} \|p - \tilde{p}\|_{L^\infty(\bar{\Omega})}. \end{aligned}$$

Then

$$e^{-\delta t} \int_0^t (t-s)^{\gamma-1} \left| \frac{\rho(t-s)}{a} - \frac{\tilde{\rho}(t-s)}{\tilde{a}} \right| ds \leq \frac{c(M_3+1)}{|a\tilde{a}|} \|p - \tilde{p}\|_{L^\infty(\bar{\Omega})} \frac{T^\gamma}{\gamma}.$$

Therefore, we obtain the estimate

$$|g(t) - \tilde{g}(t)| \leq \frac{T^\gamma}{\gamma} \frac{c(M_3+1)}{|a\tilde{a}|} \|p - \tilde{p}\|_{L^\infty(\bar{\Omega})} + \frac{M_3 \mathbb{I}(\delta)}{|a|} \|f - \tilde{f}\|_{L^\infty([0,T])}.$$

Combing the last estimate and (3.52), we deduce the estimate on the source term :

$$\left(1 - \frac{M_3 \mathbb{I}(\delta)}{|a|}\right) \|f - \tilde{f}\|_\delta \leq \tilde{c} \left(\|\mathcal{D}_t^\gamma \psi - \mathcal{D}_t^\gamma \tilde{\psi}\|_{L^\infty([0,T])} + \|\varphi - \tilde{\varphi}\|_{L^\infty(\bar{\Omega})} + \|p - \tilde{p}\|_{L^\infty(\bar{\Omega})} \right), \quad (3.54)$$

where \tilde{c} is a positive constant depending on a , \tilde{a} , T , M_1 , M_2 and M_3 .

Since $\lim_{\delta \rightarrow +\infty} \mathbb{I}(\delta) = 0$, there is $\delta^* > 0$ such that

$$1 - \frac{M_3 \mathbb{I}(\delta)}{|a|} > 0, \quad \text{for any } \delta \geq \delta^*$$

holds true. Whence, there is a positive constant $C > 0$, independent of f and \tilde{f} such that

$$\|f - \tilde{f}\|_\delta \leq C \left(\|\mathcal{D}_t^\gamma \psi - \mathcal{D}_t^\gamma \tilde{\psi}\|_{L^\infty([0,T])} + \|\varphi - \tilde{\varphi}\|_{L^\infty(\bar{\Omega})} + \|p - \tilde{p}\|_{L^\infty(\bar{\Omega})} \right),$$

that is, the solution f depends continuously on the data.

The continuity of the part u of the solution follows with classical arguments for linear problems with the norm $\|u\|_{\infty,\delta} = \sup_{x \in [0,1]} \sup_{t \in [0,T]} |e^{-\delta t} u(x,t)|$. \square

Chapter 4

Proper Generalized Decomposition (PGD) for Fractional variational PDE's

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4.1 Introduction

Proper Generalized Decompositions are a class of recent reduction order methods, they belong to the large family of low-rank approximation of high order tensors. In [61], Zhang and Golub investigated certain properties of rank-one approximation of high order tensors (multi-dimensional SVD) as well as numerical algorithms. For a complete study of tensor spaces and numerical tensor calculus, we refer to the recent book by Huckbusch [27].

The first paper in the PGD framework is due to Ammar et al. [4], even if similar ideas figure in the previous work of Ladeveze [36] on the LATIN method. The first books on this subject have been recently published by Chinesta et al. [10, 9]. Although significant advances have been made in theoretical understanding of PGD algorithms, only few convergence results for special variational problems are available nowadays in the literature. The question of PGD convergence was first studied by Le Bris et al. [5] by exploiting its connection to greedy algorithms from nonlinear approximation theory. The authors proved the PGD convergence in the case

of Poisson equation. In Cances et al. [7], the authors studied a greedy method for variational obstacle problems, always in the framework of Hilbert tensor spaces, and carried out its convergence. For variational PGD in Banach tensor spaces, the convergence was first established by Falco and Nouy [22]. Notice that in an earlier work [21], the same authors also provided an alternate proof for the convergence of linear elliptic variational problems via the celebrated Theorem of Eckart and Young [17]. Notice that in [48], Schmidt introduced long before the SVD and the corresponding truncation for Hilbert-Schmidt (compact) operators.

We mention finally other convergence results for greedy algorithms dedicated to a class of linear systems, by Ammar et al. [2], and for the Fokker-Planck equation by Figueroa and Suli [25]. It is important to mention here that in the cited previous works, the question of the optimal rank-one tensor computation is not considered.

The PGD provides an approximate d -dimensional separated representation of the form:

$$u(x_1, x_2, \dots, x_d) \simeq \sum_{j=1}^m u_1^{(j)}(x_1) \times u_2^{(j)}(x_2) \times \dots \times u_d^{(j)}(x_d),$$

where x_k are of some moderate dimensions, $k \in \{1, 2, \dots, d\}$. This approximation is carried out without any *a priori* knowledge of the solution u , unlike Proper Orthogonal Decompositions (POD) (see for example [54, 13] and the references therein). Different classes of PGD algorithms exist in the literature, we refer the interested reader to Nouy [43] for a detailed description. In the present work, we will focus on the simplest definition of the PGD: the *rank-one PGD*. This version of the PGD seeks to find iteratively an optimal rank-one separated representation (or rank-one tensor) $u_1^{(j)}(x_1) \times u_2^{(j)}(x_2) \times \dots \times u_d^{(j)}(x_d)$ for each $j \in \{1, 2, \dots, m\}$. Notice that in general, there is no best rank- r approximation, $r \geq 2$, for tensors of rank- m with $m > r$. In the special case $m = 3$ and $r = 2$, such a result was proved by De Silva and Lim [49].

The *rank-one PGD* iteration can be described as the following: if the rank- m approximation is previously computed, it is simply moved to the right hand side of the PDE and a next optimal rank-*one* tensor is sought. This rank-*one* tensor will serve to update the rank- m approximation and obtain the rank- $(m + 1)$ approximation.

4.2 Hilbert Tensor Spaces and Variational PGDs

This section deals with minimization problems with convex functionals defined on Hilbert tensor spaces. Tensorial structure of Hilbert spaces allows optimization via dimensionality reduction methods in a progressive way. For the reader's convenience, we describe briefly the main lines of variational PGDs. Consider a finite family of real Hilbert spaces $(V_k, \|\cdot\|_k)_{1 \leq k \leq d}$, where $d \geq 2$ is an integer. The algebraic tensor Hilbert space spanned by the family $(V_k, \|\cdot\|_k)_{1 \leq k \leq d}$, denoted by

$$V := \bigotimes_{k=1}^d V_k,$$

is the set of all finite linear combinations of rank-one tensors $v = \bigotimes_{k=1}^d v_k$, with $v_k \in V_k$. The suffix "a" in " $a \otimes$ " refers to the "algebraic" nature of the tensor product. That is,

$$V = \left\{ \sum_{j=1}^m v_1^{(j)} \otimes v_2^{(j)} \otimes \cdots \otimes v_d^{(j)} : m \in \mathbb{N}^* \text{ and } v_k^{(j)} \in V_k \text{ for } k = 1, 2, \dots, d \right\}. \quad (4.1)$$

We say that $V_{\|\cdot\|}$ is a Hilbert tensor space if there exist an algebraic tensor Hilbert space V and a norm $\|\cdot\|$ on V such that $V_{\|\cdot\|}$ is the completion of V with respect to the norm $\|\cdot\|$, *i.e.*

$$V_{\|\cdot\|} = \overline{a \bigotimes_{k=1}^d V_k}^{\|\cdot\|}.$$

Therefore, every element of $v \in V_{\|\cdot\|}$ can be written as

$$v = \sum_{j=1}^{+\infty} v_1^{(j)} \otimes v_2^{(j)} \otimes \cdots \otimes v_d^{(j)},$$

in the sense

$$\lim_{m \rightarrow +\infty} \left\| v - \sum_{j=1}^m v_1^{(j)} \otimes v_2^{(j)} \otimes \cdots \otimes v_d^{(j)} \right\| = 0.$$

In the sequel, the cone of all rank-one tensors in V will be denoted by $\mathcal{R}_1(V)$, *i.e.*

$$\mathcal{R}_1(V) = \{v_1 \otimes v_2 \otimes \cdots \otimes v_d : v_k \in V_k \text{ for } k = 1, 2, \dots, d\}. \quad (4.2)$$

Thus the space spanned by $\mathcal{R}_1(V)$ is V which is in turn a dense subset of $V_{\|\cdot\|}$.

Now, consider a functional $J : V_{\|\cdot\|} \rightarrow \mathbb{R}$, where $V_{\|\cdot\|}$ is a (reflexive) tensor Hilbert space. We are interested in the minimization problem

$$\inf_{u \in V_{\|\cdot\|}} J(u), \quad (4.3)$$

where J satisfies the hypotheses

- (H₁) J is $\mathcal{C}^1(V_{\|\cdot\|}, \mathbb{R})$,
- (H₂) J is coercive,
- (H₃) J is strictly convex.

Applying standard arguments of variational calculus, one shows that problem (4.3) admits a unique solution $\hat{u} \in V_{\|\cdot\|}$, under hypotheses (H₁) – (H₃) and this solution is a critical point of J , that is $J'(\hat{u}) = 0$. To define PGD method, we assume furthermore

$$(H_4) \quad \mathcal{R}_1(V) \text{ is weakly closed in } V_{\|\cdot\|},$$

where $V := a \bigotimes_{k=1}^d V_k$ is the algebraic tensor space associated to $V_{\|\cdot\|}$.

Notice that for any $u \in V$, the minimization problem

$$\inf_{z \in \mathcal{R}_1(V)} J(u + z) \quad (4.4)$$

has at least one solution, under hypotheses $(H_1) - (H_4)$. The possible non-uniqueness of the solution to problem (4.4) is due to the fact that $\mathcal{R}_1(V)$ is not convex.

The rank-one PGD method associated to problem (4.3) consists in the construction of a sequence $(u_m)_{m \in \mathbb{N}} \subset V$ as follows

- (i) Initialization: $u_0 := 0$.
- (ii) Descent direction: choose $\hat{z}_m \in \underset{z \in \mathcal{R}_1(V)}{\operatorname{argmin}} J(u_{m-1} + z)$.
- (iii) Update strategy

$$u_m := u_{m-1} + \hat{z}_m.$$

We refer the interested reader to [22] for more general update strategies. This PGD method is said to be convergent if $(u_m)_{m \in \mathbb{N}} \subset V$ converges to the unique solution \hat{u} in $V_{\|\cdot\|}$.

4.3 Fractional Sobolev spaces and Riemann-Liouville partial derivatives

4.3.1 Fractional Sobolev spaces

For the convenience of the reader, we recall in what follows some basic definitions and results on fractional-order function spaces and operators. Let Ω be a non-empty subset of \mathbb{R}^d , where d is an integer satisfying $d \geq 2$, if $u : \Omega \rightarrow \mathbb{R}$ is sufficiently smooth for them to exist, then we denote the partial derivatives of u by

$$\partial^s u(x) = \left(\frac{\partial}{\partial x_1} \right)^{s_1} \cdots \left(\frac{\partial}{\partial x_d} \right)^{s_d} u(x),$$

where $s = (s_1, \dots, s_d)$ is a *multi-index*, i.e., a d -tuple of nonnegative integers. The *order* of the partial derivative $\partial^s u$ is the integer $|s| = s_1 + \dots + s_d$. To define Sobolev spaces of fractional order $\mu \in]0, 1[$ on Ω , we denote the Slobodeckii semi-norm by

$$|u|_{\mu, \Omega} := \left(\iint_{\Omega \times \Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{2\mu + d}} dx dy \right)^{1/2}.$$

For $s = r + \mu > 0$, where r is the *integer part* of s , we define

$$W^{s,2}(\Omega) := \{u \in W^{r,2}(\Omega) : |\partial^j u|_{\mu, \Omega} < +\infty \text{ for } |j| = r\},$$

where $W^{r,2}(\Omega)$ is the classical Sobolev space defined by

$$W^{r,2}(\Omega) := \{u \in L^2(\Omega) : \partial^j u \in L^2(\Omega) \text{ for } |j| \leq r\}.$$

Equipped with the norm

$$\|u\|_{W^{s,2}(\Omega)} := \left(\|u\|_{W^{r,2}(\Omega)}^2 + \sum_{|j|=r} |\partial^j u|_{\mu,\Omega}^2 \right)^{1/2},$$

the space $W^{s,2}(\Omega)$ has Hilbert structure with the inner product

$$\langle u, v \rangle_{W^{s,2}(\Omega)} := \langle u, v \rangle_{W^{r,2}(\Omega)} + \sum_{|j|=r} \iint_{\Omega \times \Omega} \frac{[\partial^j u(x) - \partial^j u(y)] [\partial^j v(x) - \partial^j v(y)]}{|x - y|^{\mu + \frac{d}{2}}} dx dy.$$

We introduce a second family of Sobolev spaces, more adapted as functional framework, to the Riemann-Liouville differential operators, which later will turn out to be equivalent to the one given before.

Let us denote by

$$\mathcal{S}(\mathbb{R}^d) := \{\varphi \in \mathcal{C}^\infty(\mathbb{R}^d) : \sup_{x \in \mathbb{R}^d} |x^s \partial^\beta \varphi(x)| < +\infty, \text{ for all multi-indices } s \text{ and } \beta\}$$

the Schwartz space of rapidly decreasing \mathcal{C}^∞ functions on \mathbb{R}^d . It is well-known that the Fourier transform

$$\begin{aligned} \mathcal{F} : \mathcal{S}(\mathbb{R}^d) &\longrightarrow \mathcal{S}(\mathbb{R}^d) \\ u &\longmapsto \mathcal{F}[u] : \xi \longmapsto \mathcal{F}[u](\xi) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} u(x) e^{-i\xi \cdot x} dx \end{aligned}$$

is a linear and (sequentially) continuous whose inverse \mathcal{F}^{-1} is not other than its adjoint Fourier transform \mathcal{F}^* defined by

$$\mathcal{F}^*[u] : \xi \longmapsto \mathcal{F}^*[u](\xi) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} u(x) e^{i\xi \cdot x} dx.$$

By density of $\mathcal{S}(\mathbb{R}^d)$ in $L^2(\mathbb{R}^d)$, the Fourier transform can be uniquely extended to a linear continuous of $L^2(\mathbb{R}^d)$ preserving L^2 -norm: $\|u\|_{L^2(\mathbb{R}^d)} = \|\mathcal{F}[u]\|_{L^2(\mathbb{R}^d)} = \|\mathcal{F}^*[u]\|_{L^2(\mathbb{R}^d)}$.

For $s \in]0, +\infty[$, let the linear and continuous operator $\mathcal{I}^s : \mathcal{S}(\mathbb{R}^d) \longrightarrow \mathcal{S}(\mathbb{R}^d)$, called the *Bessel potential* of order s , by

$$\mathcal{I}^s u(x) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} (1 + |\xi|^2)^{s/2} \widehat{u}(\xi) e^{i\xi \cdot x} d\xi, \quad \text{for } x \in \mathbb{R}^d,$$

where \widehat{u} is the Fourier transform of u . In this way, $\mathcal{F}[\mathcal{I}^s u](\xi) = (1 + |\xi|^2)^{s/2} \widehat{u}(\xi)$.

We define $H^s(\mathbb{R}^d)$, the Sobolev space of order s on \mathbb{R}^d , by

$$H^s(\mathbb{R}^d) := \{u \in \mathcal{S}^*(\mathbb{R}^d) : \mathcal{I}^s u \in L^2(\mathbb{R}^d)\}$$

and equip this space with the inner product

$$\langle u, v \rangle_{H^s(\mathbb{R}^d)} := \langle \mathcal{I}^s u, \mathcal{I}^s v \rangle_{L^2(\mathbb{R}^d)}$$

and the induced norm

$$\|u\|_{H^s(\mathbb{R}^d)} := \sqrt{\langle u, u \rangle_{H^s(\mathbb{R}^d)}} = \|\mathcal{I}^s u\|_{L^2(\mathbb{R}^d)}.$$

Notice that the Bessel potential $\mathcal{J}^s : H^s(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^d)$ is a unitary isomorphism and

$$\|u\|_{H^s(\mathbb{R}^d)}^2 = \int_{\mathbb{R}^d} (1 + |\xi|^2)^s |\widehat{u}(\xi)|^2 d\xi.$$

Now, let Ω be any non-empty open set in \mathbb{R}^d , we define

$$H^s(\Omega) := \{U|_{\Omega} : U \in H^s(\mathbb{R}^d)\}.$$

The space $H^s(\Omega)$ is equipped with the (quotient) norm, ,

$$\|u\|_{H^s(\Omega)} := \inf\{\|U\|_{H^s(\mathbb{R}^d)} : U|_{\Omega} = u\}, \text{ for any } s > 0.$$

It is well-known that if Ω is bounded Lipschitz domain then

$$H^s(\Omega) = W^{s,2}(\Omega),$$

where $W^{s,2}(\Omega)$ is defined above.

We also define the Sobolev space on Ω ,

$$H_0^s(\Omega) := \overline{\mathcal{D}(\Omega)}^{H^s(\Omega)}, \text{ the closure of } \mathcal{D}(\Omega) \text{ in } H^s(\Omega).$$

Applying classical results concerning trace operators on fractional Sobolev spaces, and a refined result in [12, 15], we have the following characterization of $H_0^s(\Omega)$ for bounded Lipschitz domain Ω :

$$H_0^s(\Omega) = \begin{cases} H^s(\Omega) & \text{if } 0 < s \leq 1/2, \\ \{u \in H^s(\Omega) : u|_{\partial\Omega} = 0\} & \text{if } 1/2 < s < 3/2, \end{cases}$$

where $\partial\Omega$ denotes the boundary of Ω . Another important characterization of $H_0^s(\Omega)$, which allows extension by zero of functions in $H_0^s(\Omega)$ to get functions in $H^s(\mathbb{R}^d)$ is:

$$\text{if } 1/2 < s < 3/2 \text{ then } H_0^s(\Omega) = \{u \in H^s(\Omega) : \tilde{u} \in H^s(\mathbb{R}^d)\},$$

where \tilde{u} is the extension by zero of u on $\mathbb{R}^d \setminus \Omega$. This last characterization allows consequently the use of the Fourier transform tools. Therefore, we get

$$\|u\|_{H_0^s(\Omega)} = \|\tilde{u}\|_{H^s(\mathbb{R}^d)}.$$

If no confusion is likely to arise in what follows, the function \tilde{u} will be denoted simply by u . The norm $\|\cdot\|_{H_0^s(\Omega)}$ will be denoted simply by $\|\cdot\|$.

4.3.2 Riemann-Liouville partial derivatives

In the sequel, the canonical basis of \mathbb{R}^d will be denoted by $(\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_d)$ and points of Ω will be denoted by $x = (x_1, x_2, \dots, x_d)$. Motivated by the PGD framework, we limit ourselves to Lipschitz bounded domains of the form

$$\Omega = \Omega_1 \times \Omega_2 \times \dots \times \Omega_d,$$

where $\Omega_i =]a_k, b_k[$ is a non-empty bounded and open real interval and $a = (a_1, a_2, \dots, a_d)$, $b = (b_1, b_2, \dots, b_d)$ are given points in \mathbb{R}^d . Moreover, to deal with variational boundary value

problems with an adequate functional framework (trace results and good characterizations of fractional Sobolev spaces) we will assume henceforth $1/2 < s < 3/2$.

The *Gamma* function, which will be involved in fractional derivatives, is defined by:

$$\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} dt.$$

Définition 4.3.1. Let $u \in C_0^\infty(\Omega)$ and a real parameter $s > 0$. The s -th Riemann-Liouville integral of u in the directions \mathbf{e}_k and $-\mathbf{e}_k$, for $k = 1, 2, \dots, d$, are defined by

$$\begin{aligned} \mathcal{R}_{\mathbf{e}_k}^{-s}[u](x) &:= \frac{1}{\Gamma(s)} \int_{a_k}^{x_k} (x_k - t)^{s-1} u(x_1, \dots, x_{k-1}, t, x_{k+1}, \dots, x_d) dt, \\ \mathcal{R}_{-\mathbf{e}_k}^{-s}[u](x) &:= \frac{1}{\Gamma(s)} \int_{x_k}^{b_k} (t - x_k)^{s-1} u(x_1, \dots, x_{k-1}, t, x_{k+1}, \dots, x_d) dt. \end{aligned}$$

The s -th Riemann-Liouville derivatives of u in the directions \mathbf{e}_k and $-\mathbf{e}_k$ are defined by

$$\begin{aligned} \mathcal{R}_{\mathbf{e}_k}^s[u](x) &:= \frac{\partial^\ell}{\partial x_k^\ell} \left(\mathcal{R}_{\mathbf{e}_k}^{s-\ell}[u](x) \right), \\ \mathcal{R}_{-\mathbf{e}_k}^s[u](x) &:= \frac{\partial^\ell}{\partial x_k^\ell} \left(\mathcal{R}_{-\mathbf{e}_k}^{s-\ell}[u](x) \right), \end{aligned}$$

where ℓ is the unique integer satisfying $\ell - 1 < s \leq \ell$.

If $u \in C_0^\infty(\mathbb{R}^d)$, we define similarly

$$\begin{aligned} \mathcal{R}_{\mathbf{e}_k}^{-s}[u](x) &:= \frac{1}{\Gamma(s)} \int_{-\infty}^{x_k} (x_k - t)^{s-1} u(x_1, \dots, x_{k-1}, t, x_{k+1}, \dots, x_d) dt, \\ \mathcal{R}_{-\mathbf{e}_k}^{-s}[u](x) &:= \frac{1}{\Gamma(s)} \int_{x_k}^{+\infty} (t - x_k)^{s-1} u(x_1, \dots, x_{k-1}, t, x_{k+1}, \dots, x_d) dt, \end{aligned}$$

and $\mathcal{R}_{\mathbf{e}_k}^s[u]$, $\mathcal{R}_{-\mathbf{e}_k}^s[u]$ are defined in the same way.

These integrals are well defined, since u is bounded and with compact support in Ω (resp. in \mathbb{R}^d).

Lemma 4.4. *The Riemann-Liouville partial derivatives can be uniquely extended to $H^s(\mathbb{R}^d)$ and $H^s(\Omega)$. Then every $u \in H^s(\mathbb{R}^d)$ and $s \in]\frac{1}{2}, \frac{3}{2}[$, the Fourier transform of the s -th Riemann-Liouville derivative of u in the direction \mathbf{e}_k satisfies:*

$$\mathcal{F}(\mathcal{R}_{\mathbf{e}_k}^s[u])(\xi) = (i\xi_k)^s \mathcal{F}(u)(\xi). \quad (4.5)$$

Moreover, it holds for every $u \in H^s(\Omega)$:

$$\langle \mathcal{R}_{\mathbf{e}_k}^s[u], \mathcal{R}_{-\mathbf{e}_k}^s[u] \rangle_{L^2(\Omega)} = \cos(\pi s) \| \mathcal{R}_{\mathbf{e}_k}^s[u] \|_{L^2(\Omega)}^2. \quad (4.6)$$

Proof. We recall the following standard result about the Gamma function:

$$\forall s \in]0, 1[, \text{ one has : } \lim_{R \rightarrow +\infty} \int_0^R t^{s-1} e^{-it} dt = i^{-s} \Gamma(s) \quad (4.7)$$

Let us prove (4.5) at first for $u \in C_0^\infty(\mathbb{R}^d)$. By the simple shifting ($t' = x - t$), we can rewrite:

$$\mathcal{R}_{\mathbf{e}_k}^{-s}[u](x) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} u(x - t\mathbf{e}_k) dt.$$

We distinguish two cases:

Case 1: $\frac{1}{2} < s < 1$.

$$\begin{aligned} \mathcal{F}(\mathcal{R}_{\mathbf{e}_k}^{-s}[u])(\xi) &= \frac{1}{\Gamma(s)} \int_{\mathbb{R}^d} e^{-ix \cdot \xi} \left(\lim_{R \rightarrow \infty} \int_0^R t^{s-1} u(x - t\mathbf{e}_k) dt \right) dx, \\ &= \frac{1}{\Gamma(s)} \lim_{R \rightarrow \infty} \int_{\mathbb{R}^d} \int_0^R e^{-ix \cdot \xi} t^{s-1} u(x - t\mathbf{e}_k) dt dx, \end{aligned}$$

where the last inequality follows thanks to the Dominated Convergence Theorem. If $\text{supp}(u) \subset B(0, \rho)$, for some $\rho > 0$, then we have that $\text{supp}(u(\cdot - t\mathbf{e}_k)) \subset B(0, \rho + R)$ for any $t \in [0, R]$, so we have

$$\begin{aligned} \int_{\mathbb{R}^d} \int_0^R |e^{-ix \cdot \xi} t^{s-1} u(x - t\mathbf{e}_k)| dt dx &\leq \|u\|_{\infty} \text{meas}(B(0, \rho + R)) \frac{R^s}{s} \\ &< \infty, \end{aligned}$$

Then, by the Fubini Theorem, we get:

$$\begin{aligned} \mathcal{F}(\mathcal{R}_{\mathbf{e}_k}^{-s}[u])(\xi) &= \frac{1}{\Gamma(s)} \lim_{R \rightarrow \infty} \int_0^R t^{s-1} \left(\int_{\mathbb{R}^d} e^{-ix \cdot \xi} u(x - t\mathbf{e}_k) dx \right) dt \\ &= \frac{\hat{u}(\xi)}{\Gamma(s)} \lim_{R \rightarrow \infty} \int_0^R t^{s-1} e^{-i(\mathbf{e}_k \cdot \xi)t} dt \\ &= \frac{\hat{u}(\xi)}{\Gamma(s)} \lim_{R \rightarrow \infty} \int_0^R t^{s-1} e^{-i\xi_k t} dt \\ &= \frac{\xi_k^{-s} \hat{u}(\xi)}{\Gamma(s)} \lim_{R \rightarrow \infty} \int_0^R t^{s-1} e^{-it} dt \\ &= (i\xi_k)^{-s} \mathcal{F}(u)(\xi), \end{aligned}$$

where the last equality follows thanks to (4.7). Then,

$$\begin{aligned} \mathcal{F}(\mathcal{R}_{\mathbf{e}_k}^s[u])(\xi) &= \mathcal{F} \left[\frac{\partial}{\partial x_k} (\mathcal{R}_{\mathbf{e}_k}^{s-1}[u]) \right] (\xi) \\ &= (i\xi_k) \mathcal{F}[\mathcal{R}_{\mathbf{e}_k}^{s-1}[u]](\xi) \\ &= (i\xi_k)(i\xi_k)^{s-1} \mathcal{F}(u)(\xi) \\ &= (i\xi_k)^s \mathcal{F}(u)(\xi). \end{aligned}$$

Case 2: $1 < s < \frac{3}{2}$. We use the semi-group property, [19]

$$\mathcal{R}_{\mathbf{e}_k}^{-\alpha} \mathcal{R}_{\mathbf{e}_k}^{-\beta} = \mathcal{R}_{\mathbf{e}_k}^{-(\alpha+\beta)}, \text{ for any } \alpha > 0 \text{ and } \beta > 0,$$

with $\alpha = 1$ and $\beta = s - 1$, which achieves the claim.

The (linear) operator $\mathcal{R}_{\mathbf{e}_k}^s$ is then continuous from $(C_0^\infty(\mathbb{R}^d), \|\cdot\|_{H^s(\mathbb{R}^d)})$ to

$$(C_0^\infty(\mathbb{R}^d), \|\cdot\|_{L^2(\mathbb{R}^d)}).$$

Indeed:

$$\begin{aligned} \|\mathcal{R}_{\mathbf{e}_k}^s[u]\|_{L^2(\mathbb{R}^d)} &= \|\mathcal{F}(\mathcal{R}_{\mathbf{e}_k}^s[u])\|_{L^2(\mathbb{R}^d)} \\ &= \|(i\xi_k)^s \hat{u}\|_{L^2(\mathbb{R}^d)} \\ &\leq \|u\|_{H^s(\mathbb{R}^d)}. \end{aligned}$$

By density, the s -th Riemann-Liouville differential operator $\mathcal{R}_{\mathbf{e}_k}^s$ can be uniquely extended to a linear continuous operator from $H^s(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$, for each $k = 1, 2, \dots, d$. Now, let $u \in H^s(\mathbb{R}^d)$ then there is a sequence $(\varphi_n)_n \subset C_0^\infty(\mathbb{R}^d)$ such that $\varphi_n \rightarrow u$ in $H^s(\mathbb{R}^d)$. Hence,

$$\begin{aligned} \|(i\xi_k)^s \mathcal{F}(u) - \mathcal{F}(\mathcal{R}_{\mathbf{e}_k}^s[u])\|_{L^2(\mathbb{R}^d)} &\leq \|(i\xi_k)^s (\mathcal{F}(u) - \mathcal{F}(\varphi_n))\|_{L^2(\mathbb{R}^d)} \\ &\quad + \|\mathcal{F}(\mathcal{R}_{\mathbf{e}_k}^s[\varphi_n]) - \mathcal{F}(\mathcal{R}_{\mathbf{e}_k}^s[u])\|_{L^2(\mathbb{R}^d)}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \|(i\xi_k)^s (\mathcal{F}(u) - \mathcal{F}(\varphi_n))\|_{L^2(\mathbb{R}^d)} &= \|(i\xi_k)^s \mathcal{F}(u - \varphi_n)\|_{L^2(\mathbb{R}^d)} \\ &\leq \|(1 + |\xi|^2)^{s/2} \mathcal{F}(u - \varphi_n)\|_{L^2(\mathbb{R}^d)} \\ &= \|u - \varphi_n\|_{H^s(\mathbb{R}^d)} \rightarrow 0 \text{ as } n \rightarrow +\infty \end{aligned}$$

and

$$\begin{aligned} \|\mathcal{F}(\mathcal{R}_{\mathbf{e}_k}^s[\varphi_n]) - \mathcal{F}(\mathcal{R}_{\mathbf{e}_k}^s[u])\|_{L^2(\mathbb{R}^d)} &= \|\mathcal{F}(\mathcal{R}_{\mathbf{e}_k}^s[\varphi_n] - \mathcal{R}_{\mathbf{e}_k}^s[u])\|_{L^2(\mathbb{R}^d)} \\ &= \|\mathcal{R}_{\mathbf{e}_k}^s[\varphi_n] - \mathcal{R}_{\mathbf{e}_k}^s[u]\|_{L^2(\mathbb{R}^d)} \\ &\leq \|\varphi_n - u\|_{H^s(\mathbb{R}^d)} \rightarrow 0 \text{ as } n \rightarrow +\infty \end{aligned}$$

We conclude that (4.5) holds true for every $u \in H^s(\mathbb{R}^d)$ and $s \in]\frac{1}{2}, \frac{3}{2}[$.

To prove the last claim, we recall the following observation:

$$\overline{(it)^s} = \begin{cases} e^{-i\pi s} \overline{(-it)^s} & \text{if } t \geq 0, \\ e^{i\pi s} \overline{(-it)^s} & \text{if } t < 0, \end{cases} \quad \forall t \in \mathbb{R}, \quad (4.8)$$

and the Fourier transform property:

$$\int_{\mathbb{R}^d} v \overline{w} = \int_{\mathbb{R}^d} \widehat{v} \overline{\widehat{w}}. \quad (4.9)$$

Thus, let $s \in]\frac{1}{2}, \frac{3}{2}[$ and $u \in H^s(\Omega)$, then

$$\begin{aligned} \langle \mathcal{R}_{\mathbf{e}_k}^s[u], \mathcal{R}_{-\mathbf{e}_k}^s[u] \rangle_{L^2(\Omega)} &\stackrel{(4.9)}{=} \int_{\mathbb{R}^d} (i\xi_k)^s \mathcal{F}(\widehat{u})(\xi) \times \overline{(-i\xi_k)^s \mathcal{F}(\widehat{u})(\xi)} d\xi \\ &= \int_{\{\xi_k \geq 0\}} (i\xi_k)^s \mathcal{F}(\widehat{u})(\xi) \times \overline{(-i\xi_k)^s \mathcal{F}(\widehat{u})(\xi)} d\xi \\ &\quad + \int_{\{\xi_k \leq 0\}} (i\xi_k)^s \mathcal{F}(\widehat{u})(\xi) \times \overline{(-i\xi_k)^s \mathcal{F}(\widehat{u})(\xi)} d\xi, \end{aligned}$$

where the extension by zero of u outside Ω is denoted by \widehat{u} . Using (4.8), we can rewrite

$$\begin{aligned} \langle \mathcal{R}_{\mathbf{e}_k}^s[u], \mathcal{R}_{-\mathbf{e}_k}^s[u] \rangle_{L^2(\Omega)} &= e^{i\pi s} \int_{\{\xi_k \geq 0\}} |(i\xi_k)^s \mathcal{F}(\widehat{u})(\xi)|^2 d\xi \\ &\quad + e^{-i\pi s} \int_{\{\xi_k \leq 0\}} |(i\xi_k)^s \mathcal{F}(\widehat{u})(\xi)|^2 d\xi \\ &= i \sin(\pi s) \left(\int_{\{\xi_k \geq 0\}} |(i\xi_k)^s \mathcal{F}(\widehat{u})(\xi)|^2 d\xi \right. \\ &\quad \left. - \int_{\{\xi_k \leq 0\}} |(i\xi_k)^s \mathcal{F}(\widehat{u})(\xi)|^2 d\xi \right) \\ &\quad + \cos(\pi s) \int_{\mathbb{R}^d} |(i\xi_k)^s \mathcal{F}(\widehat{u})(\xi)|^2 d\xi. \end{aligned}$$

Finally, since $\overline{\mathcal{F}(\tilde{u})(-\xi)} = \mathcal{F}(\tilde{u})(\xi)$, we get

$$\int_{\{\xi_k \leq 0\}} |(i\xi_k)^s \mathcal{F}(\tilde{u})(\xi)|^2 d\xi = \int_{\{\xi_k \geq 0\}} |(i\xi_k)^s \mathcal{F}(\tilde{u})(\xi)|^2 d\xi,$$

and therefore,

$$\begin{aligned} \langle \mathcal{R}_{\mathbf{e}_k}^s [u], \mathcal{R}_{-\mathbf{e}_k}^s [u] \rangle_{L^2(\Omega)} &= \langle \mathcal{R}_{\mathbf{e}_k}^s [\tilde{u}], \mathcal{R}_{-\mathbf{e}_k}^s [\tilde{u}] \rangle_{L^2(\mathbb{R}^d)} \\ &= \cos(\pi s) \int_{\mathbb{R}^d} |\mathcal{F}(\mathcal{R}_{\mathbf{e}_k}^s [\tilde{u}])(\xi)|^2 d\xi \\ &= \cos(\pi s) \int_{\mathbb{R}^d} |\mathcal{R}_{\mathbf{e}_k}^s [\tilde{u}](\xi)|^2 d\xi \\ &= \cos(\pi s) \int_{\Omega} |\mathcal{R}_{\mathbf{e}_k}^s [u](\xi)|^2 d\xi \end{aligned}$$

which achieves the result. \square

Lemma 4.5. For any $u, v \in H^s(\Omega)$ and $s \in]\frac{1}{2}, \frac{3}{2}[$, the identity

$$\langle \mathcal{R}_{\mathbf{e}_k}^s [u], v \rangle_{L^2(\Omega)} = \langle u, \mathcal{R}_{-\mathbf{e}_k}^s [v] \rangle_{L^2(\Omega)},$$

holds true, for every $k = 1, 2, \dots, d$.

Proof. Let $\varphi, \psi \in C_0^\infty(\Omega)$. If we denote by $\tilde{\varphi}$ and $\tilde{\psi}$ the extensions of φ and ψ resp. by zero outside Ω , then for every $\mu > 0$, one has

$$\begin{aligned} \langle \mathcal{R}_{\mathbf{e}_k}^{-\mu} [\varphi], \psi \rangle_{L^2(\Omega)} &= \langle \mathcal{R}_{\mathbf{e}_k}^{-\mu} [\tilde{\varphi}], \tilde{\psi} \rangle_{L^2(\mathbb{R}^d)} \\ &= \int_{\mathbb{R}^d} \left(\int_{-\infty}^{x_k} (x_k - t)^{\mu-1} \tilde{\varphi}(x_1, \dots, x_{k-1}, t, x_{k+1}, \dots, x_n) dt \right) \tilde{\psi}(x) dx \\ &= \frac{1}{\Gamma(\mu)} \int_{\mathbb{R}^d} \left(\int_0^{+\infty} t^{\mu-1} \tilde{\varphi}(x_1, \dots, x_{k-1}, x_k - t, x_{k+1}, \dots, x_n) dt \right) \tilde{\psi}(x) dx \\ &= \frac{1}{\Gamma(\mu)} \int_{\mathbb{R}^d} \int_0^{+\infty} t^{\mu-1} \tilde{\varphi}(x_1, \dots, x_{k-1}, x_k - t, x_{k+1}, \dots, x_n) \tilde{\psi}(x) dt dx \\ &= \frac{1}{\Gamma(\mu)} \int_0^{+\infty} \int_{\mathbb{R}^d} t^{\mu-1} \tilde{\varphi}(x) \tilde{\psi}(x_1, \dots, x_{k-1}, x_k + t, x_{k+1}, \dots, x_n) dx dt \\ &= \langle \tilde{\varphi}, \mathcal{R}_{-\mathbf{e}_k}^{-\mu} [\tilde{\psi}] \rangle_{L^2(\mathbb{R}^d)} \\ &= \langle \varphi, \mathcal{R}_{-\mathbf{e}_k}^{-\mu} [\psi] \rangle_{L^2(\Omega)}, \end{aligned}$$

where we have used two simple changes of variable and Fubini's Theorem to change the order of integration. Therefore,

$$\begin{aligned} \langle \mathcal{R}_{\mathbf{e}_k}^s [\varphi], \psi \rangle_{L^2(\Omega)} &= \langle \mathcal{R}_{\mathbf{e}_k}^s [\varphi], \mathcal{R}_{-\mathbf{e}_k}^{-s} \mathcal{R}_{-\mathbf{e}_k}^s [\psi] \rangle_{L^2(\Omega)} \\ &= \langle \mathcal{R}_{\mathbf{e}_k}^{-s} \mathcal{R}_{\mathbf{e}_k}^s [\varphi], \mathcal{R}_{-\mathbf{e}_k}^s [\psi] \rangle_{L^2(\Omega)} \\ &= \langle \varphi, \mathcal{R}_{-\mathbf{e}_k}^s [\psi] \rangle_{L^2(\Omega)}. \end{aligned}$$

Now for $u, v \in H^s(\Omega)$ there exists $(\varphi_n)_{n \in \mathbb{N}}, (\psi_n)_{n \in \mathbb{N}} \in C_0^\infty(\Omega)$ such that

$$\varphi_n \xrightarrow[n \rightarrow \infty]{} u \quad \text{et} \quad \psi_n \xrightarrow[n \rightarrow \infty]{} v \quad \text{dans} \quad H_0^s(\Omega).$$

Since $\mathcal{R}_{\mathbf{e}_k}^s$ and $\mathcal{R}_{-\mathbf{e}_k}^s$ are linear continuous from $H^s(\Omega)$ to $L^2(\Omega)$, then

$$\begin{aligned} \langle \mathcal{R}_{\mathbf{e}_k}^s[u], v \rangle_{L^2(\Omega)} &= \lim_{n \rightarrow \infty} \langle \mathcal{R}_{\mathbf{e}_k}^s[\varphi_n], \psi_n \rangle_{L^2(\Omega)} \\ &= \lim_{n \rightarrow \infty} \langle \varphi_n, \mathcal{R}_{-\mathbf{e}_k}^s[\psi_n] \rangle_{L^2(\Omega)} \\ &= \langle u, \mathcal{R}_{-\mathbf{e}_k}^s[v] \rangle_{L^2(\Omega)}. \end{aligned}$$

□

At this stage, we prove that the space $H_0^s(\Omega)$ has a tensorial structure, that is:

Proposition 4.5.1. *For every $s \in]\frac{1}{2}, \frac{3}{2}[$, one has*

$$H_0^s(\Omega) = \overline{a \bigotimes_{k=1}^d H_0^s(\Omega_k)}^{H^s(\Omega)}.$$

Proof. Let $u \in H_0^s(\Omega) := \overline{\mathcal{D}(\Omega)}^{H^s(\Omega)}$, then there is a sequence $(u_n)_n \subset \mathcal{D}(\Omega)$ such that

$$u_n \xrightarrow[n \rightarrow \infty]{} u \quad \text{in} \quad H^s(\Omega) \quad \text{and thus} \quad u_n \xrightarrow[n \rightarrow \infty]{} u \quad \text{in} \quad H_0^s(\Omega). \quad (4.10)$$

On the other hand, it is well-known that

$$\mathcal{D}(\Omega) = \overline{a \bigotimes_{k=1}^d \mathcal{D}(\Omega_k)}^{\mathcal{D}(\Omega)},$$

it follows that for any integer n , there is a sequence $(\varphi_m^n)_m \subset a \bigotimes_{k=1}^d \mathcal{D}(\Omega_k) \subset a \bigotimes_{k=1}^d H_0^s(\Omega_k)$ such that

$$\varphi_m^n \xrightarrow[m \rightarrow \infty]{} u_n \quad \text{in} \quad \mathcal{D}(\Omega) \quad \text{and consequently} \quad \varphi_m^n \xrightarrow[m \rightarrow \infty]{} u_n \quad \text{in} \quad H_0^{[s]+1}(\Omega), \quad (4.11)$$

where $[s]$ denotes the integer part of s .

Now, let $n \in \mathbb{N}$, then

$$\exists N_1(n) \in \mathbb{N} \quad \text{such that} \quad \forall p \geq N_1(n) \quad \text{one has} \quad \|u_p - u\|_{H_0^s(\Omega)} \leq 1/2^{n+1},$$

and in particular

$$\|u_{N_1(n)} - u\|_{H_0^s(\Omega)} \leq 1/2^{n+1}.$$

In the same way $\exists N_2(n) \in \mathbb{N}$ such that $\forall m \geq N_2(n)$ one has

$$\|\varphi_m^{N_1(n)} - u_{N_1(n)}\|_{H_0^{[s]+1}(\Omega)} \leq 1/2^{n+1}.$$

Therefore, for $v_n := \varphi_{N_2(n)}^{N_1(n)}$, we get

$$\begin{aligned} \|v_n - u\|_{H_0^s(\Omega)} &\leq \|v_n - u_{N_1(n)}\|_{H_0^s(\Omega)} + \|u_{N_1(n)} - u\|_{H_0^s(\Omega)} \\ &\leq c_1 \|v_n - u_{N_1(n)}\|_{H_0^{[s]+1}(\Omega)} + \|u_{N_1(n)} - u\|_{H_0^s(\Omega)} \quad [40] \\ &\leq \max(1, c_1) / 2^n \xrightarrow[n \rightarrow \infty]{} 0. \end{aligned}$$

which ends the proof. □

4.6 A variational Problem and convergence of the PGD method

As in the last section, we limit ourselves to Lipschitz bounded domains of the form

$$\Omega = \Omega_1 \times \Omega_2 \times \cdots \times \Omega_d,$$

where $\Omega_i =]a_k, b_k[$ is a non-empty bounded and open real interval and $a = (a_1, a_2, \dots, a_d)$, $b = (b_1, b_2, \dots, b_d)$ are given points in \mathbb{R}^d .

Consider the variational partial fractional equation with homogeneous boundary problem

$$\begin{cases} \sum_{k=1}^d (\mathcal{R}_{-\mathbf{e}_k}^s \mathcal{R}_{\mathbf{e}_k}^s + \mathcal{R}_{\mathbf{e}_k}^s \mathcal{R}_{-\mathbf{e}_k}^s)[u] = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (4.12)$$

where $(\mathbf{e}_k)_{k=1, \dots, d}$ is the canonical basis of \mathbb{R}^d and $\frac{1}{2} < s < \frac{3}{2}$, with f is a given function in $L^2(\Omega)$.

Définition 4.6.1. A function $u \in H_0^s(\Omega)$ is said to be a weak solution of Problem (4.12) if for any $v \in H_0^s(\Omega)$,

$$\sum_{k=1}^d \int_{\Omega} (\mathcal{R}_{\mathbf{e}_k}^s [u](x) \mathcal{R}_{\mathbf{e}_k}^s [v](x) + \mathcal{R}_{-\mathbf{e}_k}^s [u](x) \mathcal{R}_{-\mathbf{e}_k}^s [v](x)) dx = \int_{\Omega} f(x) v(x) dx,$$

holds true.

The Euler-Lagrange functional corresponding to Problem (4.12) on $H_0^s(\Omega)$ is given by:

$$\begin{aligned} J: H_0^s(\Omega) &\longrightarrow \mathbb{R} \\ u &\longrightarrow \frac{1}{2} \sum_{k=1}^d \int_{\Omega} [(\mathcal{R}_{\mathbf{e}_k}^s [u])^2 + (\mathcal{R}_{-\mathbf{e}_k}^s [u])^2] - \int_{\Omega} f u, \end{aligned} \quad (4.13)$$

In this way, weak solutions to (4.12) are simply critical points of the functional J .

Lemma 4.7. *The functional J is $\mathcal{C}^2(H_0^s(\Omega), \mathbb{R})$ and 2-elliptic, that is*

$$\exists c > 0, \forall u, v \in H_0^s(\Omega) : \langle J'(u) - J'(v), u - v \rangle_{H^{-s}(\Omega), H_0^s(\Omega)} \geq c \|u - v\|^2,$$

where

$$J'(u) \cdot v = \sum_{k=1}^d \int_{\Omega} (\mathcal{R}_{\mathbf{e}_k}^s [u](x) \mathcal{R}_{\mathbf{e}_k}^s [v](x) + \mathcal{R}_{-\mathbf{e}_k}^s [u](x) \mathcal{R}_{-\mathbf{e}_k}^s [v](x)) dx - \int_{\Omega} f v.$$

Therefore, J is strictly convex, bounded below and coercive.

Proof. To prove that J is $\mathcal{C}^2(H_0^s(\Omega), \mathbb{R})$, we will show that J admits a continuous 2-order Gâteaux derivative on $H_0^s(\Omega)$. For all $u, v \in H_0^s(\Omega)$ and $t > 0$, we have

$$\begin{aligned} J(u + tv) &= \frac{1}{2} \sum_{k=1}^d \int_{\Omega} [(\mathcal{R}_{\mathbf{e}_k}^s [u + tv])^2 + (\mathcal{R}_{-\mathbf{e}_k}^s [u + tv])^2] - \int_{\Omega} f(u + tv) \\ &= \frac{1}{2} \sum_{k=1}^d \int_{\Omega} [(\mathcal{R}_{\mathbf{e}_k}^s [u])^2 + (\mathcal{R}_{-\mathbf{e}_k}^s [u])^2 + t^2 [(\mathcal{R}_{\mathbf{e}_k}^s [v])^2 + (\mathcal{R}_{-\mathbf{e}_k}^s [v])^2] \\ &\quad + 2t [\mathcal{R}_{\mathbf{e}_k}^s [u] \mathcal{R}_{\mathbf{e}_k}^s [v] + \mathcal{R}_{-\mathbf{e}_k}^s [u] \mathcal{R}_{-\mathbf{e}_k}^s [v]]] - \int_{\Omega} f(u + tv) \\ &= J(u) + t \sum_{k=1}^d \int_{\Omega} [\mathcal{R}_{\mathbf{e}_k}^s [u] \mathcal{R}_{\mathbf{e}_k}^s [v] + \mathcal{R}_{-\mathbf{e}_k}^s [u] \mathcal{R}_{-\mathbf{e}_k}^s [v]] - t \int_{\Omega} f v + t \varepsilon(t) \end{aligned}$$

where $\lim_{t \rightarrow 0} \varepsilon(t) = 0$. Then the Gâteaux derivative of J exists and verifies

$$J'(u) \cdot v = \sum_{k=1}^d \int_{\Omega} (\mathcal{R}_{\mathbf{e}_k}^s[u](x) \mathcal{R}_{\mathbf{e}_k}^s[v](x) + \mathcal{R}_{-\mathbf{e}_k}^s[u](x) \mathcal{R}_{-\mathbf{e}_k}^s[v](x)) dx - \int_{\Omega} f v.$$

It's easy to verify that $J'(u)$ is linear and continuous, thus J is Fréchet-differentiable. Moreover, J' is continuous on $H_0^s(\Omega)$. Indeed, if $z_n \rightarrow z$ in $H_0^s(\Omega)$ then we have

$$\begin{aligned} \|J'(z_n) - J'(z)\|_{H^{-s}(\Omega)} &= \sup_{\|v\|=1} |J'(z_n) \cdot v - J'(z) \cdot v| \\ &\leq \sup_{\|v\|=1} \sum_{k=1}^d \int_{\Omega} |\mathcal{R}_{\mathbf{e}_k}^s[z_n - z](x) \mathcal{R}_{\mathbf{e}_k}^s[v](x)| dx \\ &\quad + \int_{\Omega} |\mathcal{R}_{-\mathbf{e}_k}^s[z_n - z](x) \mathcal{R}_{-\mathbf{e}_k}^s[v](x)| dx \\ &\leq \sup_{\|v\|=1} \sum_{k=1}^d \|\mathcal{R}_{\mathbf{e}_k}^s[z_n - z](x)\|_{L^2(\Omega)} \cdot \|\mathcal{R}_{\mathbf{e}_k}^s[v](x)\|_{L^2(\Omega)} \\ &\quad + \|\mathcal{R}_{-\mathbf{e}_k}^s[z_n - z](x)\|_{L^2(\Omega)} \cdot \|\mathcal{R}_{-\mathbf{e}_k}^s[v](x)\|_{L^2(\Omega)} \\ &= \sup_{\|v\|=1} \sum_{k=1}^d \|(i\xi_k)^s \mathcal{F}[z_n - z]\|_{L^2(\mathbb{R}^d)} \cdot \|(i\xi_k)^s \mathcal{F}[v]\|_{L^2(\mathbb{R}^d)} \\ &\quad + \|(-i\xi_k)^s \mathcal{F}[z_n - z]\|_{L^2(\mathbb{R}^d)} \cdot \|(-i\xi_k)^s \mathcal{F}[v]\|_{L^2(\mathbb{R}^d)} \\ &= 2 \sup_{\|v\|=1} \sum_{k=1}^d \|\xi_k\|^s \mathcal{F}[z_n - z]\|_{L^2(\mathbb{R}^d)} \cdot \|\xi_k\|^s \mathcal{F}[v]\|_{L^2(\mathbb{R}^d)} \\ &\leq 2d \sup_{\|v\|=1} \|z_n - z\|_{H_0^s(\Omega)} \cdot \|v\|_{H_0^s(\Omega)} \xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

Thus $J \in \mathcal{C}^1(H_0^s(\Omega))$. Moreover, we have

$$\begin{aligned} &\langle J'(u) - J'(v), u - v \rangle_{H^{-s}(\Omega), H_0^s(\Omega)} \\ &= \sum_{k=1}^d \int_{\Omega} (\mathcal{R}_{\mathbf{e}_k}^s[v - u](x))^2 + (\mathcal{R}_{-\mathbf{e}_k}^s[v - u](x))^2 dx \\ &= \sum_{k=1}^d \int_{\mathbb{R}^d} |\mathcal{F}(\mathcal{R}_{\mathbf{e}_k}^s[v - u](x))(\xi)|^2 + |\mathcal{F}(\mathcal{R}_{-\mathbf{e}_k}^s[v - u](x))(\xi)|^2 d\xi \\ &= 2 \sum_{k=1}^d \int_{\mathbb{R}^d} |\xi_k|^{2s} |\mathcal{F}(v - u)(x)(\xi)|^2 d\xi \\ &\geq c \int_{\mathbb{R}^d} |\xi|^{2s} |\mathcal{F}(v - u)(x)(w)|^2 d\xi \\ &\geq c \|v - u\|_{H_0^s(\Omega)}^2, \end{aligned}$$

where we have used in the two last line the equivalence between the l_2 and l_{α} norms in \mathbb{R}^d ($\alpha \geq 1$), and one of the Fractional Poincaré-Friderichs inequalities [55].

Now, for all $u, v, w \in H_0^s(\Omega)$ we have

$$\begin{aligned} J'(u + tv) \cdot w &= \sum_{k=1}^d \int_{\Omega} (\mathcal{R}_{\mathbf{e}_k}^s[u + tv](x) \mathcal{R}_{\mathbf{e}_k}^s[w](x) \\ &\quad + \mathcal{R}_{-\mathbf{e}_k}^s[u + tv](x) \mathcal{R}_{-\mathbf{e}_k}^s[w](x)) dx - \int_{\Omega} f w dx \\ &= J'(u) \cdot w + t \sum_{k=1}^d \int_{\Omega} (\mathcal{R}_{\mathbf{e}_k}^s[v](x) \mathcal{R}_{\mathbf{e}_k}^s[w](x) \\ &\quad + \mathcal{R}_{-\mathbf{e}_k}^s[v](x) \mathcal{R}_{-\mathbf{e}_k}^s[w](x)) dx, \end{aligned}$$

then

$$\begin{aligned} J''(u) \cdot (v, w) &:= \lim_{t \rightarrow 0} \frac{J'(u + tv) \cdot w - J'(u) \cdot w}{t} \\ &= \sum_{k=1}^d \int_{\Omega} (\mathcal{R}_{\mathbf{e}_k}^s[v](x) \mathcal{R}_{\mathbf{e}_k}^s[w](x) + \mathcal{R}_{-\mathbf{e}_k}^s[v](x) \mathcal{R}_{-\mathbf{e}_k}^s[w](x)) dx. \end{aligned}$$

It's clear that J'' is continuous. Therefore $J \in \mathcal{C}^2(H_0^s(\Omega))$.

Following [23, Lemma 3], we conclude that J is strictly convex, bounded below and coercive. \square

Applying standard direct methods in the calculus of variations, the minimization problem

$$\inf_{v \in H_0^s(\Omega)} J(v) \tag{4.14}$$

has a unique solution.

Lemma 4.8. *For every family $u_k : \Omega_k \rightarrow \mathbb{R}$, $1 \leq k \leq d$, of measurable functions such that*

$\bigotimes_{k=1}^d u_k \neq 0$, *one has*

$$\bigotimes_{k=1}^d u_k \in H_0^s(\Omega) \iff u_k \in H_0^s(\Omega_k), \forall k = 1, \dots, d.$$

Proof. Let $u_k \in H_0^s(\Omega_k)$, $1 \leq k \leq d$, be a given family of measurable functions. It is clear that $\bigotimes_{k=1}^d u_k|_{\partial\Omega} = 0$ and $\bigotimes_{k=1}^d u_k \in L^2(\Omega)$. Moreover, up to an extension by zero of u_k outside Ω_k , $1 \leq k \leq d$, we get

$$\begin{aligned} \int_{\mathbb{R}^d} |\xi|^{2s} |\mathcal{F}(\bigotimes_{k=1}^d u_k)(\xi)|^2 d\xi &= \int_{\mathbb{R}^n} |\xi|^{2s} |\mathcal{F}(\bigotimes_{k=1}^d u_k)(\xi)|^2 d\xi \\ &\leq c_1 \int_{\mathbb{R}^d} \left(\sum_{k=1}^d |\xi_k|^{2s} \right) \prod_{k=1}^d |\mathcal{F}(u_k)(\xi_k)|^2 d\xi_1 \cdots d\xi_d \\ &= c_1 \sum_{k=1}^d \left(\int_{\mathbb{R}} |\xi_k|^{2s} |\mathcal{F}(u_k)(\xi_k)|^2 d\xi_k \times \prod_{i=1, i \neq k}^d \int_{\mathbb{R}} |\mathcal{F}(u_i)(\xi_i)|^2 d\xi_i \right) \\ &< +\infty. \end{aligned}$$

Hence, $\bigotimes_{k=1}^d u_k \in H_0^s(\Omega)$.

Conversely, let $(u_k)_{1 \leq k \leq d}$ be a given family of measurable functions such that $\bigotimes_{k=1}^d u_k \in H_0^s(\Omega)$ and $\bigotimes_{k=1}^d u_k \neq 0$. It is easy to verify that $u_k|_{\partial\Omega_k} = 0$, for every $k \in \{1, \dots, d\}$. Moreover, since

$$0 < \left\| \bigotimes_{k=1}^d u_k \right\|_{L^2(\Omega)} = \prod_{k=1}^d \|u_k\|_{L^2(\Omega_k)} < +\infty$$

it follows that $u_k \in L^2(\Omega_k)$, for any $k \in \{1, \dots, d\}$. On the other hand, up once again to an exten-

sion by zero of u_k outside Ω_k , $1 \leq k \leq d$, we get

$$\begin{aligned}
& \int_{\mathbb{R}^d} |\xi|^{2s} |\mathcal{F}(\otimes_{k=1}^d u_k)(\xi)|^2 d\xi \\
& \geq c_2 \int_{\mathbb{R}^d} \left(\sum_{k=1}^d |\xi_k|^{2s} \right) \prod_{k=1}^d |\mathcal{F}(u_k)(\xi_k)|^2 d\xi_1 \cdots d\xi_d \\
& = c_2 \sum_{k=1}^d \left(\int_{\mathbb{R}} |\xi_k|^{2s} |\mathcal{F}(u_k)(\xi_k)|^2 d\xi_k \times \prod_{i=1, i \neq k}^d \int_{\mathbb{R}} |\mathcal{F}(u_i)(\xi_i)|^2 d\xi_i \right) \\
& \geq c_2 \int_{\mathbb{R}} |\xi_k|^{2s} |\mathcal{F}(u_k)(\xi_k)|^2 d\xi_k \times \prod_{i=1, i \neq k}^d \|u_k\|_{L^2(\Omega_k)}^2, \quad \forall k \in \{1, \dots, d\},
\end{aligned}$$

which achieves the proof, thanks to $0 < \prod_{i=1, i \neq k}^d \|u_k\|_{L^2(\Omega_k)}^2 < +\infty$. \square

Proposition 4.8.1. $\mathcal{M}_1 := \mathcal{M}_1\left(\underset{k=1}{\overset{d}{\otimes}} H_0^s(\Omega_k)\right)$ is weakly closed in $H_0^s(\Omega)$.

Proof. Let $(z_n)_n$ be a sequence in \mathcal{M}_1 which converges weakly to z in $H_0^s(\Omega)$. On one hand, it is clear that

$$\mathcal{M}_1\left(\underset{k=1}{\overset{d}{\otimes}} H_0^s(\Omega_k)\right) \subset \mathcal{M}_1\left(\underset{k=1}{\overset{d}{\otimes}} L^2(\Omega_k)\right)$$

On the other hand, it is well-known that $\mathcal{M}_1\left(\underset{k=1}{\overset{d}{\otimes}} L^2(\Omega_k)\right)$ is weakly closed in $L^2(\Omega)$ (see [23]), which implies that $z \in \mathcal{M}_1\left(\underset{k=1}{\overset{d}{\otimes}} L^2(\Omega_k)\right)$, that is $z = \underset{k=1}{\overset{d}{\otimes}} z_k$ with $z_k \in L^2(\Omega_k)$, $k = 1, \dots, d$. Therefore, Lemma 4.8 leads to $z \in \mathcal{M}_1$. \square

As a consequence of this Proposition is the following:

Corollary 4.8.0.1. *The minimization problem :*

$$\inf_{v \in \mathcal{M}_1\left(\underset{k=1}{\overset{d}{\otimes}} H_0^s(\Omega_k)\right)} J(v) \tag{4.15}$$

admits a least one solution.

4.9 PGD and its convergence

Lemma 4.10. *If $u \in H_0^s(\Omega)$ satisfies*

$$J(u) \leq J(u + z), \quad \forall z \in \mathcal{M}_1\left(\underset{k=1}{\overset{d}{\otimes}} H_0^s(\Omega_k)\right)$$

then u is the unique weak solution of (4.12).

Proof. Let $z \in \mathcal{M}_1 \left(\bigotimes_{k=1}^d H_0^s(\Omega_k) \right)$ and let the function $g : \mathbb{R} \rightarrow \mathbb{R}$ defined by $g(t) = J(u + tz)$. It is clear that the function g is differentiable on \mathbb{R} and realizes its global minimum at 0. Consequently $g'(0) = J'(u) \cdot (z) = 0$ and the linearity of $J'(u)$ implies that $J'(u) = 0$ on $\bigotimes_{k=1}^d H_0^s(\Omega_k)$. Finally the density of $\bigotimes_{k=1}^d H_0^s(\Omega_k)$ in $H_0^s(\Omega)$ achieves the proof. \square

We will consider the method PGD with the strategy (c) for the progressive approximation of the solution to Problem (4.12):

$$(M) \left\{ \begin{array}{l} (i) \text{ Initialization: } u_0 := 0 \text{ in } H_0^s(\Omega). \\ (ii) \text{ Descent direction: } z_m := \underset{z \in \mathcal{M}_1 \left(\bigotimes_{k=1}^d H_0^s(\Omega_k) \right)}{\operatorname{argmin}} J(u_{m-1} + z), \\ \forall m \geq 1. \\ (iii) \text{ Update: } u_m = u_{m-1} + z_m, \forall m \geq 1. \end{array} \right.$$

This is the most used PGD method thanks to its implementation simplicity. We prove in the sequel that every PGD sequence (M) is convergent in $H_0^s(\Omega)$.

Theorem 4.10.1. *The sequence $(u_m)_{m \in \mathbb{N}} \subset \bigotimes_{k=1}^d H_0^s(\Omega_k)$, defined by (M), converges weakly in $H_0^s(\Omega)$ toward the unique weak solution of (4.12).*

Proof. Let $(u_m)_{m \in \mathbb{N}} \subset H_0^s(\Omega)$ be a PGD sequence defined by (M). It is clear that the real sequence $(J(u_m))_{m \in \mathbb{N}}$ is decreasing, bounded below and consequently convergent in \mathbb{R} , we denote its limit by ℓ . The coerciveness of J implies that $(u_m)_{m \in \mathbb{N}}$ is bounded in $H_0^s(\Omega)$.

To show that $(u_m)_m$ converges weakly in $H_0^s(\Omega)$, we show that $(u_m)_m$ has a unique adherence value for the weak topology of $H_0^s(\Omega)$.

Let $u \in H_0^s(\Omega)$ be an adherence value of $(u_m)_m$ for the weak topology of $H_0^s(\Omega)$ and let $(u_{m_n})_n$ a subsequence of $(u_m)_m$ converging weakly to u .

As $u_{m_n} \rightharpoonup u$ in $H_0^s(\Omega)$ then $u_{m_n} + z \rightharpoonup u + z$, for every $z \in H_0^s(\Omega)$. Consequently, we get $|\xi_k|^s \mathcal{F}(u_{m_n} + z) \rightharpoonup |\xi_k|^s \mathcal{F}(u + z)$ in $L^2(\Omega)$, for any $z \in H_0^s(\Omega)$ and $k \in \{1, 2, \dots, d\}$. On the other hand, if the extension of u and u_{m_n} by zero outside Ω are still denoted by u and u_{m_n} , we get

$$\begin{aligned} \||\xi_k|^s \mathcal{F}(u_{m_n} - u)\|_{L^2(\Omega)}^2 &= \||\xi_k|^s \mathcal{F}(u_{m_n} + z) - |\xi_k|^s \mathcal{F}(u + z)\|_{L^2(\Omega)}^2 \\ &= \||\xi_k|^s \mathcal{F}(u_{m_n} + z)\|_{L^2(\Omega)}^2 + \||\xi_k|^s \mathcal{F}(u + z)\|_{L^2(\Omega)}^2 \\ &\quad - 2 \langle |\xi_k|^s \mathcal{F}(u_{m_n} + z), |\xi_k|^s \mathcal{F}(u + z) \rangle_{L^2(\Omega)} \end{aligned}$$

and using that $|\xi_k|^s \mathcal{F}(u_{m_n} + z) \rightharpoonup |\xi_k|^s \mathcal{F}(u + z)$ in $L^2(\Omega)$, we get

$$\||\xi_k|^s \mathcal{F}(u_{m_n} - u)\|_{L^2(\Omega)}^2 = \||\xi_k|^s \mathcal{F}(u_{m_n} + z)\|_{L^2(\Omega)}^2 - \||\xi_k|^s \mathcal{F}(u + z)\|_{L^2(\Omega)}^2 + o_n(1), \quad (4.16)$$

and consequently

$$\|\mathcal{R}_{\mathbf{e}_k}^s [u_{m_n} - u]\|_{L^2(\mathbb{R}^d)}^2 = \|\mathcal{R}_{\mathbf{e}_k}^s [u_{m_n} + z]\|_{L^2(\mathbb{R}^d)}^2 - \|\mathcal{R}_{\mathbf{e}_k}^s [u + z]\|_{L^2(\mathbb{R}^d)}^2 + o_n(1). \quad (4.17)$$

In the same way, we obtain

$$\|\mathcal{R}_{-\mathbf{e}_k}^s [u_{m_n} - u]\|_{L^2(\mathbb{R}^d)}^2 = \|\mathcal{R}_{-\mathbf{e}_k}^s [u_{m_n} + z]\|_{L^2(\mathbb{R}^d)}^2 - \|\mathcal{R}_{-\mathbf{e}_k}^s [u + z]\|_{L^2(\mathbb{R}^d)}^2 + o_n(1). \quad (4.18)$$

We conclude the following decomposition

$$J(u_{m_n} - u) = J(u_{m_n} + z) - J(u + z) + o_n(1), \quad \forall z \in H_0^s(\Omega). \quad (4.19)$$

On the other hand,

$$J(u_{m_{n+1}}) \leq J(u_{m_n+1}) \leq J(u_{m_n} + z), \quad \forall z \in \mathcal{M}_1 \left(a \bigotimes_{k=1}^d H_0^s(\Omega_k) \right). \quad (4.20)$$

Finally, we obtain

$$\begin{aligned} J(u_{m_{n+1}} - u) + J(u) &= J(u_{m_{n+1}}) + o_n(1), \quad \text{by setting } z = 0 \text{ in (4.19)} \\ &\leq J(u_{m_n} + z) + o_n(1), \quad \forall z \in \mathcal{M}_1 \left(a \bigotimes_{k=1}^d H_0^s(\Omega_k) \right) \\ &\quad \text{by (4.20),} \\ &= J(u_{m_n} - u) + J(u + z) + o_n(1), \quad \text{by (4.19).} \end{aligned}$$

Using the fact that $J(u_{m_{n+1}} - u)$ and $J(u_{m_n} - u)$ have the same limit $\ell - J(u)$ as $n \rightarrow +\infty$, we obtain:

$$J(u) \leq J(u + z), \quad \forall z \in \mathcal{M}_1 \left(a \bigotimes_{k=1}^d H_0^s(\Omega_k) \right).$$

Lemma 4.10 implies that u is thus the unique solution (4.12). Hence, the sequence $(u_m)_{m \in \mathbb{N}}$ has a unique adherence value for the weak topology of $H_0^s(\Omega)$, then it converges weakly to the unique solution of Problem (4.12). \square

Corollary 4.10.1.1. *Under the same hypotheses as Theorem 4.10.1, we have:*

$$\lim_{m \rightarrow +\infty} J'(u_m) \cdot z = 0, \quad \forall z \in H_0^s(\Omega).$$

Preuve. Simply write the expression of $J'(u_m) \cdot z$ and use the weak convergence of $(u_m)_m$ to the unique solution of (4.12). \square

We give now some technical lemmas in order to show the strong convergence of PGD sequences given by the scheme (M).

Lemma 4.11. *The sequence of successive descent directions z_m defined in (M) for the PGD sequence $(u_m)_m$ satisfies:*

$$\sum_{k=1}^{+\infty} \|z_k\|^2 < +\infty,$$

in particular, $\lim_{m \rightarrow +\infty} z_m = 0$ in $H_0^s(\Omega)$.

Proof. Remark first that

$$J'(u_m) \cdot z_m = 0, \quad \forall m \in \mathbb{N}. \quad (4.21)$$

Indeed, for any $t \in \mathbb{R}$, one has $J(u_m + t z_m) = J(u_{m-1} + (1+t)z_m) \geq J(u_m)$, which gives the claim. Now, introduce the quadratic part of J defined on $H_0^s(\Omega)$ by:

$$\tilde{J}(u) := J(u) - J'(0) \cdot u.$$

Then,

$$J'(u) \cdot w = \tilde{J}'(u) \cdot w + J'(0) \cdot w,$$

where the form defined on $H_0^s(\Omega) \times H_0^s(\Omega)$ by: $(u, w) \mapsto \tilde{J}'(u) \cdot w$ is bilinear and symmetric. We get thus

$$\begin{aligned} J'(u_m) \cdot u_m &= J'(u_m) \cdot u_{m-1} \quad \text{by (4.21)} \\ &= \tilde{J}'(u_m) \cdot u_{m-1} + J'(0) \cdot u_{m-1} \\ &= \tilde{J}'(u_{m-1}) \cdot u_{m-1} + J'(0) \cdot u_{m-1} + \tilde{J}'(z_m) \cdot u_{m-1} \\ &= J'(u_{m-1}) \cdot u_{m-1} + \tilde{J}'(u_{m-1}) \cdot z_m \\ &\vdots \\ &= \sum_{j=1}^{m-1} \tilde{J}'(u_j) \cdot z_{j+1}, \quad \text{since } u_0 = 0 \text{ and } \tilde{J}'(0) = 0 \\ &= \sum_{1 \leq i < j \leq m} \tilde{J}'(z_i) \cdot z_j, \quad \text{by the bilinearity of } (u, w) \rightarrow \tilde{J}'(u) \cdot w \\ &= \frac{1}{2} \left(\tilde{J}'(u_m) \cdot u_m - \sum_{k=1}^m \tilde{J}'(z_k) \cdot z_k \right) \\ &= \frac{1}{2} \left(J'(u_m) \cdot u_m - J'(0) \cdot u_m - \sum_{k=1}^m \tilde{J}'(z_k) \cdot z_k \right). \end{aligned}$$

Whence, one obtains

$$J'(u_m) \cdot u_m = -J'(0) \cdot u_m - \sum_{k=1}^m \tilde{J}'(z_k) \cdot z_k. \quad (4.22)$$

Since $\tilde{J}'(z_k) \cdot z_k \geq c_1 \|z_k\|^2$, for any k and $J'(u_m) \cdot u_m \leq c_2 \|u_m\|^2$, for any m , we deduce that

$$\sum_{k=1}^m \|z_k\|^2 < +\infty,$$

in particular, $\lim_{m \rightarrow +\infty} \|z_m\| = 0$. □

Lemma 4.12. *Let $(u_m)_{m \in \mathbb{N}}$ be a PGD sequence for the Problem (4.12), then there exists $C > 0$ such that*

$$|J'(u_{m-1}) \cdot z| \leq C \|z_m\| \|z\|,$$

for any $z \in \mathcal{M}_1 \left(\underset{k=1}{\overset{d}{\otimes}} H_0^s(\Omega_k) \right)$, where z_m is the descent direction at the iteration $m \geq 1$, defined in (M).

Proof. First, remark that $J' : H_0^s(\Omega) \rightarrow H^{-s}(\Omega)$ is Lipschitz continuous. On the other hand, the convexity of J leads to

$$J'(u_{m-1} + z) \cdot (z_m - z) \leq J(u_{m-1} + z_m) - J(u_{m-1} + z) \leq 0,$$

thus

$$J'(u_{m-1} + z) \cdot z_m \leq J'(u_{m-1} + z) \cdot z. \quad (4.23)$$

$$\begin{aligned}
-J'(u_{m-1}) \cdot z &= (J'(u_{m-1} + z) - J'(u_{m-1})) \cdot z - J'(u_{m-1} + z) \cdot z \\
&\leq C\|z\|^2 - J'(u_{m-1} + z) \cdot z, \\
&\quad \text{since } J \text{ is Lipschitz continuous} \\
&\leq C\|z\|^2 - J'(u_{m-1} + z) \cdot z_m, \quad (\text{by 4.23}) \\
&= C\|z\|^2 - (J'(u_{m-1} + z) - J'(u_{m-1} + z_m)) \cdot z_m \\
&\leq C(\|z\|^2 + \|z - z_m\| \|z_m\|) \\
&\leq C(\|z\|^2 + \|z\| \|z_m\| + \|z_m\|^2).
\end{aligned}$$

Hence, replacing z by $\pm z \frac{\|z_m\|}{\|z\|}$ in the previous inequalities, we obtain

$$|J'(u_{m-1}) \cdot z| \leq 3C\|z_m\| \|z\|. \quad \square$$

Lemma 4.13. *With the same notations as Theorem 4.10.1, one has :*

$$\lim_{m \rightarrow +\infty} J'(u_m) \cdot u_m = 0$$

and the sequence PGD $(u_m)_m$ given by the scheme (M) converges to the solution of (4.12).

Proof. Using (4.22), we get

$$J'(u_m) \cdot u_m = -J'(0) \cdot u_m - \sum_{k=1}^m \tilde{J}'(z_k) \cdot z_k.$$

Since $\sum_{k=1}^{+\infty} \tilde{J}'(z_k) \cdot z_k < +\infty$ and $\lim_{m \rightarrow +\infty} J'(0) \cdot u_m = J'(0) \cdot \hat{u}$, where \hat{u} is the weak limit of $(u_m)_{m \in \mathbb{N}}$ (and at the same time the unique solution of (4.12) from Theorem 4.10.1), then the real sequence $(J'(u_m) \cdot u_m)_m$ is convergent. On the other hand, one has

$$\begin{aligned}
|J'(u_m) \cdot u_m| &\leq \sum_{k=1}^m |J'(u_m) \cdot z_k| \\
&\leq C \sum_{k=1}^m \|z_{m+1}\| \|z_k\| \quad (\text{Lemma 4.12}) \\
&\leq C(m \|z_{m+1}\|^2)^{1/2} \left(\sum_{k=1}^m \|z_k\|^2 \right)^{1/2} \quad (\text{Hölder's inequality}) \\
&\leq C((m+1) \|z_{m+1}\|^2)^{1/2} \left(\sum_{k=1}^{+\infty} \|z_k\|^2 \right)^{1/2}
\end{aligned}$$

Finally, as $\sum_{k=1}^{+\infty} \|z_k\|^2 < +\infty$ then there is a subsequence $(z_{m_k})_k$ de $(z_m)_m$ such that

$\lim_{k \rightarrow +\infty} m_k \|z_{m_k}\|^2 = 0$. Therefore,

$$\lim_{m \rightarrow +\infty} J'(u_m) \cdot u_m = \lim_{k \rightarrow +\infty} J'(u_{m_k}) \cdot u_{m_k} = 0.$$

Applying the fact that

$$\begin{cases} u_m & \rightarrow u \text{ dans } W_{\|\cdot\|} \\ J'(u_m) \cdot u_m & \rightarrow 0 \text{ quand } m \rightarrow +\infty \end{cases}$$

and the decomposition formula (4.17) and (4.18), we deduce that

$$J'(u_m - u) \cdot (u_m - u) \rightarrow 0 \quad \text{quand } m \rightarrow +\infty.$$

Consequently, we obtain

$$J'(0) \cdot (u_m - u) + \tilde{J}'(u_m - u) \cdot (u_m - u) \rightarrow 0 \quad \text{quand } m \rightarrow +\infty.$$

Which ends the result. □

4.14 Numerical results

In this section, we limit ourselves to showing the efficiency of the PGD method compared to the classical method. We propose in this section to use the difference numerical schemes defined in [42] for both methods. All digital tests have been performed using *Matlab* software for numerical calculation on a computer equipped with an with a processor Intel Core i5 and 8 GB of RAM.

Let $\Omega_1 = \Omega_2 =]0, 1[$ and $\Omega = \Omega_1 \times \Omega_2$, the finite difference discretization of Ω_1 and Ω_2 uses uniform spatial meshes with $N_x = 2^{\ell_1}$, $N_y = 2^{\ell_2}$ subintervals, with $\ell_1, \ell_2 > 1$ respectively, such that $h_x = \frac{1}{N_x}$, $h_y = \frac{1}{N_y}$. The solution error Er is measured using the discrete L^∞ -norm $\|u\|_\infty = \max_{0 \leq i \leq N_x} \max_{0 \leq j \leq N_y} |u(x_i, y_j)|$. Now, consider the following problem:

$$\begin{cases} \sum_{k=1}^2 (\mathcal{R}_{-\mathbf{e}_k}^s \mathcal{R}_{\mathbf{e}_k}^s + \mathcal{R}_{\mathbf{e}_k}^s \mathcal{R}_{-\mathbf{e}_k}^s)[u] = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (4.24)$$

We first consider the source term f leading to the exact solution

$$u_{ex} = 10^8 x^5 (1-x)^5 y^5 (1-y)^5. \quad (4.25)$$

We first fix $N_x = N_y = 128$, $s = 0.7$ and represent in Figure 4.1 the estimates errors Er_{PGD} and Er for the PGD method and the classical method respectively. Then, we take $N_x = N_y = 2048$, $s = 0.7$ and represent in Figure 4.2 the estimates errors Er_{PGD} and Er for the PGD method and the classical method respectively. We fix $N_x = N_y = n$, Table 4.1 reports the L^∞ -norm of the error, the elapsed time and the iterations number for different values of the number of points of discretization n for the two algorithms for $s = 0.7$. Figure 4.3 shows the elapsed time for both algorithms with respect to the number of subintervals n for $s = 0.7$ Figure 4.4 shows the elapsed time for both algorithms w.r.t the order of derivation s for $n = 256$. Figure 4.5 shows the elapsed time for both algorithms w.r.t the order of derivation s and the number of subintervals n .

Classical algorithm			
n	Er	Elapsed time	Iterations number
128	9.8778e-07	0.3959 s	416
256	9.8016e-07	3.9842 s	819
512	9.9182e-07	37.8177 s	1618
1024	9.9678e-07	6 min 3 s	3209
2048	9.9831e-07	1 h 24 min 40 s	6382
PGD algorithm			
n	Er_{PGD}	Elapsed time	Iterations number
128	9.1927e-07	0.2437 s	25
256	9.7413e-07	0.6854 s	31
512	7.0378e-07	2.8003 s	31
1024	9.2544e-07	12.3756 s	35
2048	9.0840e-07	1 min 5 s	30

Table 4.1: L^∞ -norm of the error, the elapsed time and the iterations number.

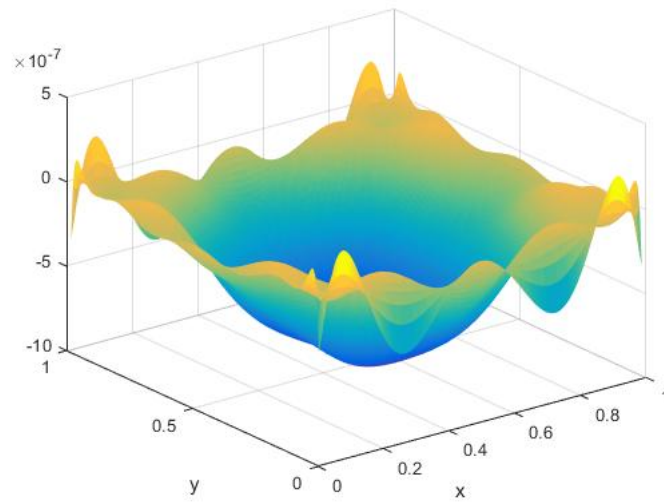
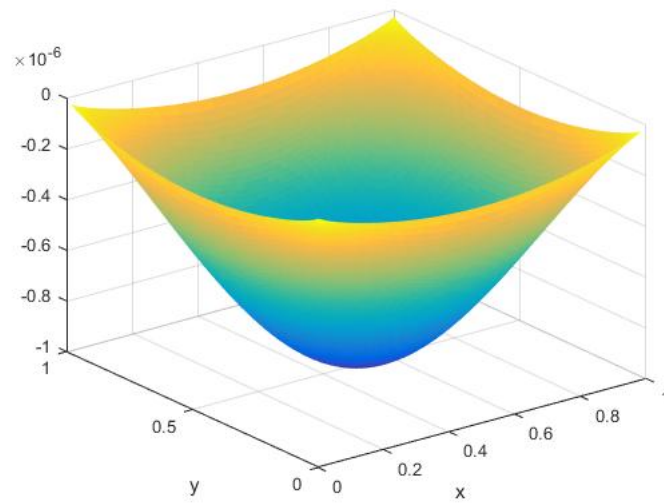
(a) Er_{pgd} (b) Er

Figure 4.1: 4.1(a): PGD method error. 4.1(b): Classical method error.

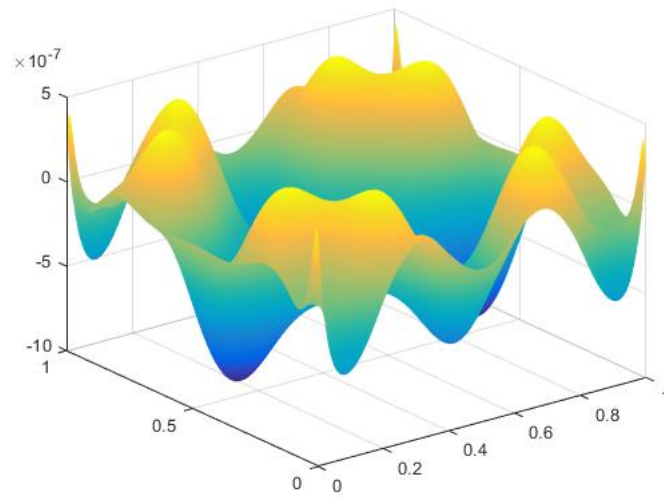
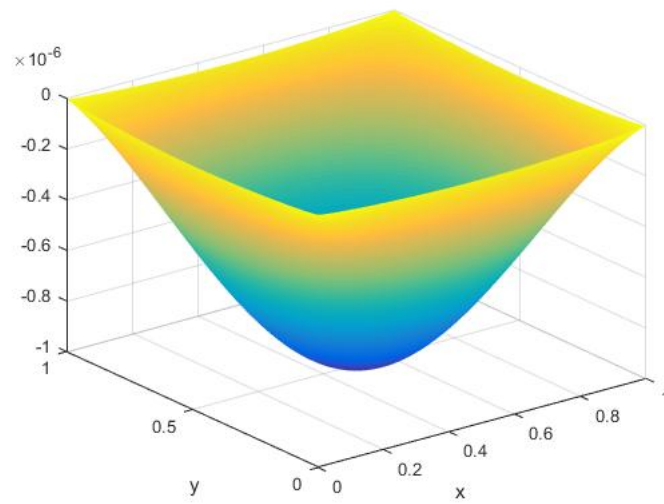
(a) $E_{r_{pgd}}$ (b) E_r

Figure 4.2: 4.2(a): PGD method error. 4.2(b): Classical method error.

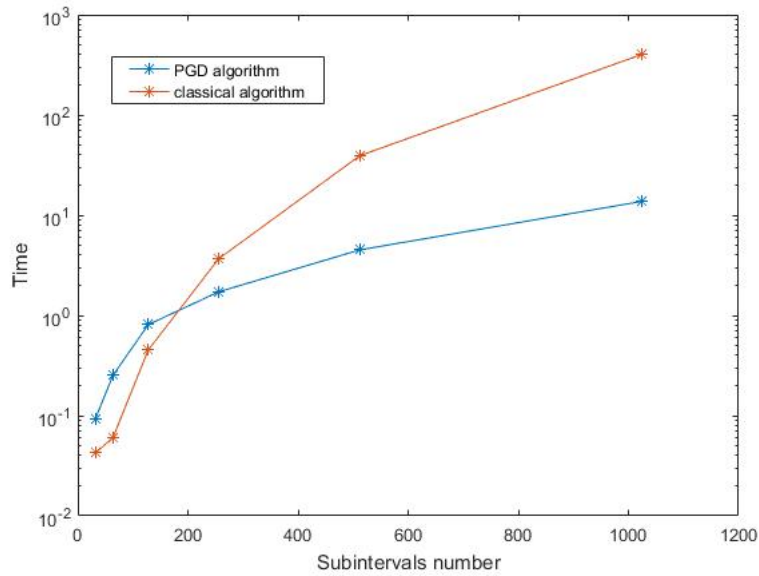


Figure 4.3: Elapsed time for both algorithms w.r.t the number of subintervals n for $s = 0.7$.

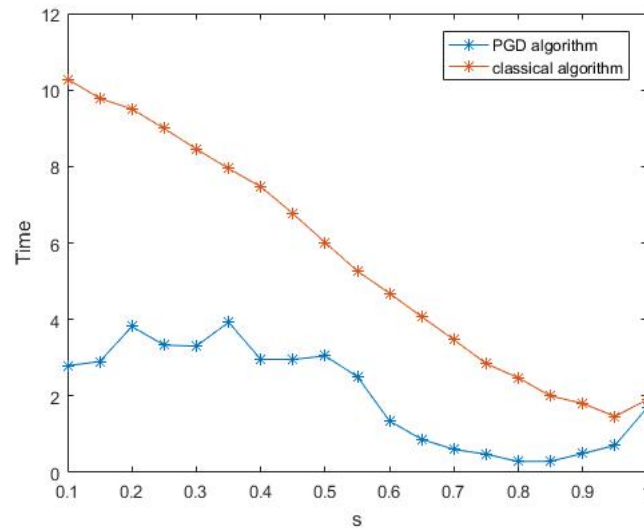


Figure 4.4: Elapsed time for both algorithms w.r.t the order of derivation s for $n = 256$.

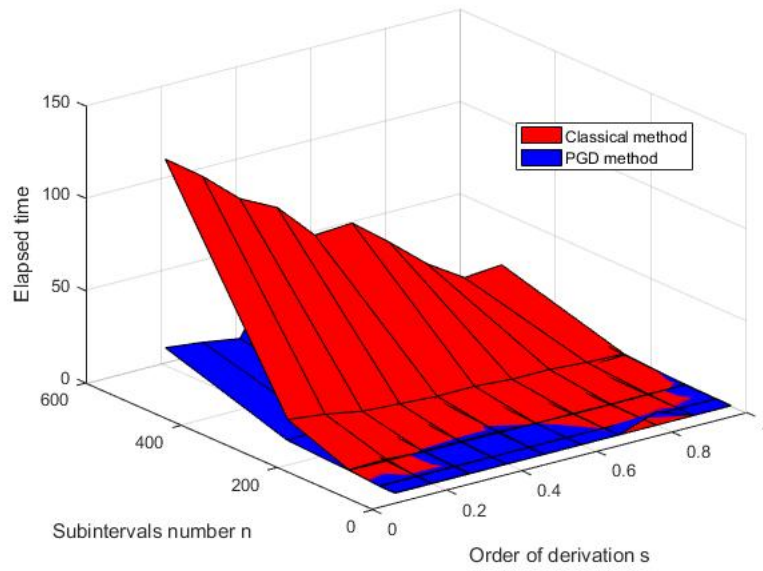


Figure 4.5: Elapsed time for both algorithms w.r.t the order of derivation s and the number of subintervals n .

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