Diskoids and extensions of valuations

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Diskoids and Extensions of Valuations

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Contents

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Abstract

Résumé (Français)

Cette thèse est consacrée à l'étude des extensions d'une valuation ν sur un corps K à l'anneau des polynômes à une variable $K[X]$. Notre objectif est de mettre en lace une interprétation géométrique d'une large classe de telles extensions, dites extensions transcendantes.
Nous commençons par revoir les concepts fondamentaux de polynômes clefs abstraits et de paires minimales. Nous élargissons la correspondance entre ces derniers, établie par Novacoski en mettons en relation les valuations engendrées par chacun de ces objets. Nous appelons ce procédé descente de Galois pour les troncatures et nous en montrons quelques applications directes.
Nous pouvons alors, dans un deuxième temps, donner une interprétation géométrique des valuations données par ces paires minimales et polynômes clefs. Pour ce faire nous employons un objet appelé discoïde, qui est une généralisation du concept de boule dans un corps valué non-archimédien.

Mots clefs

Valuations, Polynômes clefs, Paires Minimales, Discoides.

Abstract (English)

This thesis is dedicated to the study of extensions of a valuation ν over K to the ring of polynomials in one variable $K[X]$. Our objective is to give a geometric interpretation to a large class of these extensions, called valuation transcendental valuations.
We start by reviewing the fundamental concepts of abstract key polynomials and minimal pairs. We build on the correspondence between them, established by Novacoski and show how to relate the valuations generated by each of these two objects. We call this process Galois desent for truncations and we illustrate it by giving some direct applications.
We can then give a geometric interpretation of valuations built from key polynomials and minimal pairs. To do so we employ an object called diskoid, which is a generalisation of the classical concept of ball in non-archimedian valued fields.

Key words

Valuations, Key polynomials, Minimal Pairs, Diskoids.

Introduction (Français)

Étendre une valuation ν définie sur corps K à un anneau de polynômes à une variable $K[X]$ peut se poursuivre selon différentes strtégies. Le *Fundamentalsatz* [44, §11, IX, p. 378] d'Ostrowski les décrit comme des limites des suites de valeurs prises par le polynôme le long d'un suite dite *suite pseudo-convergente*: si $(a_n)_n$ est une telle suite et ν est une valuation discrète de rang 1 alors l'application définie par

$$f \in K[X],\ \mu(f) = \lim_n \nu(f(a_n)),$$

est une extension de ν. À la suite de ces travaux, Mac Lane introduisit dans [32] l'idée de polynôme clef et des valuations augmentées afin d'étudier et classifier les extensions de ν quand celle-ci est discrète de rang 1. Ceci lui permit entre autres de retrouver des résultats provenant de la théorie classique des valuations [33]. Le problème vit croître un intérêt notoire, notamment dû à son application au problème de l'uniformisation locale. Les problèmes de ramification des extensions de valuations et l'étude du défaut sont profondément liées à l'uniformisation locale, comme l'ont montrées [2], [1], [5], [13], [14], [15], [29], [49], [53].
Le cas de la dimension 2 possède des aspects géométriques:

- La théorie des courbes planes et des espaces d'arcs a pu donner une description des suites génératrices minimales des valuations sur des corps de fonctions de dimension 2 ([51], [4], [36], [20]).
- Dans [50], l'auteur utilise les curvettes, méthodes généralisées ensuite aux corps non-algébriquement clos par Cutkosky et Vinh dans [16].

L'objectif de cette thèse est d'établir une correspondance entre une classe de polynômes clefs et des objets de nature ultramétrique, qui pourra donner une bijection entre les valuations induites par ces polynômes et ces objets ultramétriques. Nous précisons quelque peu les notions de polynômes clefs utilisées ici. En effet plusieurs versions des polynômes clefs ont été étudiées depuis leur conception par Mac Lane en 1936. Vaquié a généralisé son approche, où la requête pour ν d'être discrète de rang 1 tombe. Pour cela il introduit la notion de polynôme clefs limite ainsi que famille continue de valuations augmentées. Une étude indépendante donna une approche différente, menée par Spivakovsky, Olalla Acosta et Hererra Govantes dans [23], [21]. Enfin, une approche alternative des polynômes clefs, qui nous intéresse tout particulièrement est celle des polynômes clefs abstraits présentée dans [18]. Nous reprenons cette théorie dans le deuxième chapitre de la thèse. Cette approche permet de construire des valuations à travers le processus de troncature: à partir d'une valuation μ sur $K[X]$ et un polynôme clef abstrait Q, nous développons $f \in K[X]$ selon les puissances de Q

$$f = f_0 + f_1 Q + \ldots + f_n Q^n,\ \forall i,\ \deg f_i < \deg Q$$

et définissons l'application μ_Q par

$$\mu_Q(f) = \min_i \mu(f_i Q^i).$$

Le fait que le polynôme Q est un polynôme clef abstrait garantit que μ_Q soit une valuation.
Nous nous concentrerons dans ce texte autour d'une certaine classe de valuations en particulier: les valuations de type transcendant. Ce sont les extensions μ des valuations ν à $K[X]$, qui soit ont un élément transcendant dans leur extension résiduelle, soit un élément sans torsion dans le quotient des groupes des valeurs (Section 1.4). Nous sommes particulièrement intéressés par la conjecture suivante.

Conjecture. Soit μ une valuation sur $K[X]$ et $Q \in K[X]$ un polynôme clef abstrait pour μ tel que $\mu(Q) \in \mathbb{Q} \otimes \mu(K^\times)$. Fixons une extension de μ, $\overline{\mu}$ à $\overline{K}[X]$, où $\overline{K}$ est la clôture algébrique de K.

1. Il existe un sous-ensemble $\Delta(Q) \subseteq \overline{K}$, que nous appellerons *discoïde*, tel que le minimum $\min_{x\in\Delta(Q)} \overline{\mu}(f(x))$ soit atteint pour tout $f \in K[X]$, et soit égal à μ_Q:
$$\forall f \in K[X],\ \mu_Q(f) = \min_{x\in\Delta(Q)} \mu(f(x)).$$
De plus, $\Delta(Q)$ est une union fine de boules. Nous dirons que $\Delta(Q)$ *induit la valuation* μ_Q.
2. Il existe une bijection entre les valuations de type transcendent μ sur $K[X]$, qui sont donnés comme troncatures par un polynôme clef abstrait Q, et les discoïdes $\Delta = \Delta(Q)$.

Ce que nous remarquons ici est le fait que plusieurs polynômes clefs peuvent induire la même valuation (tronquée), mais donnerons également le même discoïde, tout comme une boule ouverte dans un corps non-archimédien a tous ses points comme centres. Nous démontrerons la conjecture dans les cas où K est henselien ou $\mu|_K$ est de rang 1.
La notion de discoïde peut être rapprochée dans un certain sens de la notion d'ensemble divisoriel maximal pour les valuations divisorielles (*cf.* [24]).

Une manière différente d'étendre les valuations est apparue dans les articles d'Alexandru, Popescu et Zăhărescu dans les années 90 ([6], [7], [9], [8]). Leur idée consiste dans un premier temps à étendre ν en une valuation $\overline{\nu}$ sur $\overline{K}$, la clôture algébrique de K. Le problème est nettement plus simple à traiter sur $\overline{K}[X]$ puisque les polynômes irréductibles (parmi lesquels se situent les polynômes clefs abstraits) sont de degré 1. Les auteurs ont défini les paires minimales afin de classifier les extensions de $\overline{K}$ à $\overline{K}[X]$. Soit un couple (a, δ) avec $a \in \overline{K}$ et $\delta = \overline{\mu}(X - a)$, où $\overline{\mu}$ est une extension quelconque de μ à $\overline{K}[X]$. Ce couple sera une paire minimale si $\deg_K(a)$ est minimal parmi les $\deg_K(c)$, où les $c \in \overline{K}$ satisfont $\overline{\mu}(X - c) = \delta$. Cette paire induit une valuation sur $\overline{K}[X]$ de la manière suivante
$$\overline{\mu}_{a,\delta}(f) = \min\{\overline{\mu}(a_i) + i\delta\}, \quad f = a_0 + a_1(X - a) + \ldots + a_n(X - a)^n,\ a_i \in \overline{K}.$$
Nous considérons alors la restriction de $\overline{\mu}_{a,\delta}$ à $K[X] \subseteq \overline{K}[X]$. Les paires minimales ont été employées afin d'étudier les extensions de valuations à extension résiduelle transcendante [1] et se sont montrées particulièrement utiles pour investir des invariants associés aux éléments de $\overline{K}$. Ce travail est spécialement promu par S. Khanduja (*cf.* [27]).

Dans ce texte nous montrons que les valuations induites par les paires minimales sont à rapprocher des valuations induites par des polynômes clefs. En effet, nous montrons que les valuations données par des paires minimales restreintes à $K[X]$ sont des valuations tronquées par des polynômes clefs pour μ. Une correspondance entre paires minimales et polynômes clefs se trouve déjà dans [40]. Afin de se rapprocher de notre interprétation ultramétrique, nous avons besoin de prolonger cette correspondance. Nous le faisons dans Theorem 4.3, qui peut être formulé ainsi.

Theorem. 1. Soit $Q \in K[X]$. Une racine $a \in \overline{K}$ de Q est dite racine optimisante si la valeur $\overline{\mu}(X - a)$ est minimale parmi les valeurs $\{\overline{\mu}(X - c);\ Q(c) = 0\}$. Alors Q est un polynôme clef abstrait pour μ si et seulement si $(a, \overline{\mu}(X - a))$ est une paire minimale pour $\overline{\mu}$.
2. De plus, nous avons l'égalité suivante
$$\overline{\mu}_{a,\delta}\big|_{K[X]} = \mu_Q.$$

Le premier point de ce théorème est l'objectif principal de [40], le second point est notre contribution propre. Des résultats similaires se trouvent dans [46, Theorem 5.1], néanmoins ils sont établis pour des polynômes clefs au sens de Mac Lane et Vaquié et les notres dans le contexte des polynômes clefs abstraits. De plus, leurs résultats sont démontrés pour des valuations augmentées non-limites de valuations d'extensions RT, or nous n'avons pas besoin de ces hypothèses. On pourrait imaginer que ceux-cis peuvent être étendus grâce aux travaux de Vaquié, mais nous avons toujours besoin d'une version de [46, Theorem 5.1] pour les valuations augmentées limites. Enfin, notre démonstration fait usage de l'algèbre graduée de la valuation, ce qui offre

[1] Abrégées en extensions RT.

une approche notable, dans notre opinion. Nous donnerons quelques applications que nous trouvons dignes d'intérêt en soi (*cf.* Proposition 4.12, Proposition 4.13 and Theorem 4.15).
Nous généralisons enfin [40, Proposition 3.1], en employant un certain type de polygone de Newton. Ceci provient de notre souhait d'encoder la configuration des racines d'un polynôme dans les données d'un polygone de Newton. La première pente de ce polygone est l'invariant δ qui apparaît dans les travaux concernant les paires minimales.

Nous résumons les contributions ainsi que leurs auteurs dans le tableau synoptique suivant.

$$\begin{tikzcd}
\left\{\begin{array}{c}\text{Polnômes clefs}\\ \text{Abstraits}\end{array}\right\} \arrow[r, leftrightarrow, "\text{Novacoski}"] \arrow[d, leftrightarrow] & \left\{\begin{array}{c}\text{Paires}\\ \text{Minimales}\end{array}\right\} \arrow[d, leftrightarrow] \\
\left\{\begin{array}{c}\text{Valuations}\\ \text{Tronquées}\\ \text{sur } K[X]\end{array}\right\} \arrow[r, shift left, "\text{extension}"] & \left\{\begin{array}{c}\text{Valuations}\\ \text{Tronquées}\\ \text{sur } \overline{K}[X]\end{array}\right\} \arrow[l, shift left, "\text{restriction}"]
\end{tikzcd}$$

(Mac Lane, Vaquié, Spivakovsky,...) (Alexandru, Popescu, Zăhărescu)

Notre travail concerne la partie extension-restriction et nous détaillons ceci dans Chapter 4.

Enfin nous cherchons à établir une interprétation ultramétrique des extensions RT. Pour commencer, nous revoyons les valuations induites par des paires minimales et démontrons que ce sont les valuations obtenues comme minimum sur des boules fermées ultramétriques. En effet, fixons (a, δ) une paire minimale et $f \in K[X]$. Soit alors la boule fermée

$$D(a, \delta) = \{b \in \overline{K} \mid \overline{\nu}(b - a) \geqslant \delta\}.$$

Cet ensemble n'est pas réduit à un point si $\mathrm{rk}(\mu) = \mathrm{rk}(\nu)$. En effet, si $\mathrm{rk}(\mu) = \mathrm{rk}(\nu) + 1$ et si on suppose que δ est strictement plus grand que toute valeur prise par les éléments de K, alors pour tout $x \in \overline{K}^*$, $\delta > \nu(x)$. Dans ce cas

$$D(a, \delta) = \{a\}.$$

Afin d'éviter de telles situations nous allons nous restreindre aux extensions RT. Dans un tel contexte $\mathrm{rk}(\mu) = \mathrm{rk}(\nu)$ et donc, si$\delta < \infty$, $D(a, \delta)$ n'est pas un singleton. Alors nous considérons la valuation définie comme suit

$$\overline{\nu}_{D(a,\delta)}(f) = \min_{x \in D(a,\delta)} \overline{\nu}(f(x)).$$

Nous montrerons que ce minimum est atteint et que nous pouvons même le calculer de manière explicite à partir du polynôme f (*cf.* Lemma 5.3) En effet, nous avons

$$\overline{\nu}_{D(a,\delta)} = \overline{\nu}_{a,\delta}.$$

Plus encore, ce résultat permet de montrer qu'il y a une bijection explicite entre extensions RT et boules fermées ultramétriques. En effet deux paires minimales $(a, \delta), (a', \delta')$ vont induire la même valuation si et seulement si elles donnent la même boule ultramétrique $D(a, \delta) = D(a', \delta')$. Nous souhaitons établir une correspondance similaire sur K, en trouvant un objet que nous appellerons discoïde et qui consiste en un sous-ensemble $\Delta(Q) \subseteq \overline{K}$ associé à un polynôme clef abstrait Q. Ces discoïdes doivent vérifier

$$\mu_Q = \overline{\nu}_{\Delta(Q)}, \text{ où } \overline{\nu}_{\Delta(Q)}(f) = \min_{x \in \Delta(Q)} \overline{\nu}(f(x))$$
$$\overline{\nu}_{\Delta(Q_1)} = \overline{\nu}_{\Delta(Q_2)} \implies \Delta(Q_1) = \Delta(Q_2).$$

Nous ne pouvons pas définir $\Delta(Q)$ comme de simples boules ultramétriques, vu que plusieurs boules peuvent donner la même valuation sur $K[X]$. Si par exemple μ_Q est donnée par $\overline{\nu}_{D(a,\delta)}\big|_{K[X]}$ et $\sigma \in \mathrm{Aut}_K(\overline{K})$ est tel que $\overline{\nu} \circ \sigma = \overline{\nu}$, alors μ_Q est également la restriction sur $K[X]$ de $\overline{\nu}_{\sigma(D(a,\delta))}$. Nous avons besoin de sous-ensembles de $\overline{K}$ plus grands. Nous souhaitons aussi qu'ils aient une forme facilement compréhensible. Nous donnerons un candidat raisonnable, déjà utilisé par Julian Rüth dans sa thèse de doctorat [47] dont nous retrouvons des applications dans [43].

Dans sa thèse, Rüth prend un courbe lisse C sur un corps K muni d'une valuation discrète de rang 1 ν et cherche à implémenter un algorithme pour calculer des modèles normaux de C. Nous entendons par modèle normal, un schéma plat $\mathcal{C}$ sur $\mathscr{O}(\nu)$ [2], équipé d'un isomorphisme entre C est la fibre générique $\mathcal{C}_g$. Il cherche en un premier temps à définir explicitement une injection qui prend une classe d'isomorphisme de modèles normaux de C et lui associe un sous-ensemble fini d'extensions RT de (K, ν) à $K(C)$ [3]. De plus, cette application est bijective si (K, ν) est henselien. Pour plus de détails on peut consulter [47, Ch. 3] ou [22].
La demarche exclusivement algorithmique de Rüth employe la méthode de Mac Lane des valuations aygmentées. Comme $K(C)$ est une extension séparable d'un corps de fonctions rationnelles en une variable $K(t)$, ce procédé permet d'encoder fidèlement les ensembles finis d'extensions RT. L'auteur procède alors en définissant les discoïdes et les utilise afin d'éliminer la ramification de ses extensions RT (*cf.* [47, Prop. 6.16]). Plus explicitement, pour une extension RT μ de ν à $K(t)$, on trouve une extension finie K'/K, telle que toute extension de μ à $K'(t)$ n'aie pas d'etension de groupe de valeurs (*i.e.,* elle est *faiblement ramifiée*, *cf.* [47, Definition 6.1 and 6.2]).

Nous voyons donc qu'un formalisme général pour les discoïdes a un intérêt étendu. Dans le Chapter 5 nous les étudierons en soi et montrerons comment ils se décomposent en des unions disjointes de boules ultramétriques fermées et comment le groupe de Galois absolu de K, *i.e.,* $G_K = \mathrm{Gal}(K^{sep}/K) = \mathrm{Aut}_K(\overline{K})$ agit sur ces boules. Une étude minutieuse de cette action est faite dans [58], lorsque le corps de base K est henselien. Notre travail n'emploie cette action que pour montrer comment une valuation tronquée (donnée par un polynôme clef abstrait Q) est la valuation donnée par un discoïde (associé à Q). Notre but est de montrer que cette correspondance est une bijection et c'est ce que nous appelons notre interprétation ultramétrique. Nous démontrons ces choses dans le cas particulier où (K, ν) est henselien ou de rang 1.

Nous rajoutons un chapitre final, Chapter 6, où nous résumons nos réflexions pour de nouvelles pistes de recherche concernant les problèmes soulevés ici, ainsi que des idées de possibles applications des outils établis dans cette thèse.

L'article à paraître dans *Journal of Algebra (Août 2021)* [10] a été tiré des travaux menés dans les études doctorales présentées dans ce texte.

[2] $\mathscr{O}(\nu)$ est l'anneau de valuation de ν, *cf.* Chapter 1.
[3] $K(V)$ est le corps de fonctions régulières sur la variété algébrique V définie sur le corps de base K.

Introduction

Extending valuations from a field K onto a simple polynomial ring $K[X]$ can be encoded in different ways and following different strategies. Ostrowski's *Fundamentalsatz* [44, §11, IX, p. 378] describes extensions as limits of sequences of values taken by a polynomial on what he calls *pseudo-convergent sequences*: if $(a_n)_n$ is such a sequence and ν is discretely valued of rank 1, then the map

$$f \in K[X],\ \mu(f) = \lim_n \nu(f(a_n)),$$

is a well-defined extension of ν. Mac Lane subsequently introduced key polynomials and augmented valuations in order to study extensions of discretely valued fields of rank 1 [32]. This allowed him to prove among other things results in classical valuation theory [33]. As the problem of local uniformization regained new interest, so did the study of the extension problem of valuations from K to $K[X]$. Problems of ramifications and the study of defect of extensions of valuations are deeply related to local uniformization as it has been shown in [2], [1], [5], [13], [14], [15], [29], [49], [53].

In dimension 2, the problem has a geometric flavour:

- The theory of plane curves and jet schemes has been used to give a precise description of minimal generating sequences of valuations in dimension 2 ([51], [4], [36], [20]).
- Curvettes are being used in [50] whose results have later been generalised to non-algebraically closed fields by Cutkosky and Vinh in [16].

Our goal in this thesis is to establish a correspondence between certain key polynomials and ultrametric objects that would give a faithful correspondence between the valuations induced by these key polynomials and the valuations given by these ultrametric objects. We briefly mention the various notions that are involved in our study starting with key polynomials. Different notions of key polynomials appeared since Mac Lane's first definition in 1936. Vaquié generalised this first approach in a series of articles ([55], [56], [57]) where he defines limit key polynomials and augmented limit valuations in order to extend Mac Lane's theory to not-necessarily discrete valuations of rank 1. Independently Spivakovsky, Olalla Acosta and Hererra Govantes gave a different approach to key polynomials, in [23], [21]. An alternative approach of key polynomials, which will be of interest to us, was introduced in [18], that of abstract key polynomials (see Chapter 2). These allow to construct valuations, via a process called truncation: from a valuation μ on $K[X]$ and a key polynomial Q, we consider for any $f \in K[X]$, its Q-expansion

$$f = f_0 + f_1 Q + \ldots + f_n Q^n,\ \forall i,\ \deg f_i < \deg Q$$

and define a map μ_Q, by setting

$$\mu_Q(f) = \min_i \mu(f_i Q^i).$$

The fact that Q is an abstract key polynomial ensures that μ_Q is a valuation.

Our focus in this thesis revolves around a certain type of extensions: valuation transcendental valuations. These are extensions μ over $K[X]$ of valuations ν over K that either have a transcendental element in their residual field extension, or a rationally free element in their value group extension (Section 1.4). We are concerned with the following conjecture.

Conjecture. Consider a valuation μ on $K[X]$ and an abstract key polynomial $Q \in K[X]$ for μ, such that $\mu(Q) \in \mathbb{Q} \otimes \mu(K^\times)$. Consider an extension of μ, $\overline{\mu}$ to $\overline{K}[X]$, where $\overline{K}$ is the algebraic closure of K.

1. There exists a subset $\Delta(Q) \subseteq \overline{K}$ that we shall call a *diskoid*, such that for every $f \in K[X]$, the minimum over $\Delta(Q)$, *i.e.*, $\min_{x\in\Delta(Q)} \overline{\mu}(f(x))$ exists and is equal to the the truncation of μ along Q:
$$\forall f \in K[X],\, \mu_Q(f) = \min_{x\in\Delta(Q)} \mu(f(x)).$$
Furthermore, $\Delta(Q)$ is a finite union of balls. We will say that $\Delta(Q)$ *induces the valuation* μ_Q.
2. There is a bijective correspondence between valuation transcendental valuations μ over $K[X]$, which can be given by a key polynomial Q, and diskoids $\Delta = \Delta(Q)$.

What is remarkable here is that several key polynomials may induce the same truncated valuations, but they would also induce the same diskoid Δ, just like an open ball in a non-archimedian field may have all of its points as centres. We will prove the conjecture when K is henselian or $\mu|_K$ is of rank one.
The notion of diskoids mirrors in some sense the notion of maximal divisorial set for divisorial valuations (*cf.* [24]).

A different approach to the extension problem came from a series of articles by Alexandru, Popescu and Zăhărescu in the 90s [6], [7], [9], [8]. Their idea is to extend the problem to $\overline{K}$, the algebraic closure of K. The problem is simpler to study over $\overline{K}[X]$ since all irreducible polynomials (among which, the key polynomials) are of degree 1. The authors coined the concept of minimal pairs in order to classify the extensions from $\overline{K}$ to $\overline{K}[X]$. Consider a couple (a, δ) with $a \in \overline{K}$ and $\delta = \overline{\mu}(X - a)$, where $\overline{\mu}$ stands for an arbitrary extension of μ to $\overline{K}[X]$. It will be called a minimal pair if $\deg_K(a)$ is minimal among the $\deg_K(c)$, where $c \in \overline{K}$ such that $\overline{\mu}(X - c) = \delta$. The pair induces a new valuation on $\overline{K}[X]$ as follows
$$\overline{\mu}_{a,\delta}(f) = \min\{\overline{\mu}(a_i) + i\delta\}, \quad f = a_0 + a_1(X - a) + \ldots + a_n(X - a)^n,\ a_i \in \overline{K}.$$
We then can consider the restriction of $\overline{\mu}_{a,\delta}$, to $K[X] \subseteq \overline{K}[X]$. Minimal pairs have been used to study residually transcendental extensions[4] and helped investigating invariants associated to elements of $\overline{K}$. This work is promoted especially by S. Khanduja (*cf.* [27]).

In this thesis we show that the valuations given by minimal pairs are in fact the same as the ones given by abstract key polynomials. Indeed we show that the trace of the valuation given by a minimal pair over $K[X]$, is a truncated valuation of μ. We can already find a correspondence between minimal pairs and abstract key polynomials in the work of Novacoski [40]. In order to get closer to our geometric interpretation we need to deepen this correspondence. We do this in our Theorem 4.3. The correspondence can thus be stated as follows.

Theorem. 1. Consider $Q \in K[X]$. A root $a \in \overline{K}$ of Q is said to be an optimising root if the value $\overline{\mu}(X - a)$ is minimal among the values $\{\overline{\mu}(X - c);\ Q(c) = 0\}$. Then Q is an abstract key polynomial for μ if and only if $(a, \overline{\mu}(X - a))$ is a minimal pair for $\overline{\mu}$.
2. Furthermore, one has an equality between valuations
$$\overline{\mu}_{a,\delta}\big|_{K[X]} = \mu_Q.$$

The first part of this theorem is the main goal of [40], the second part is our contribution to the theory. Similar results have been stated and proven in [46, Theorem 5.1]. However they are made in the context of key polynomials in the sense of Mac Lane and Vaquié whereas we work in the context of abstract key polynomials. Furthermore, they provide a proof only for augmented valuations of residually transcendental valuations and we do not need these assumptions. One could imagine that we could prove our statement with the help of the work of Vaquié's extension of Mac Lane's theory, however we would still need a similar result as [46, Theorem 5.1] but for limit-augmented valuations. Finally our proof will make use of the structure of the graded algebras of a valuation which in our opinion, is an approach worth noting. We will also give a couple of applications that we find noteworthy (see Proposition 4.12, Proposition 4.13 and Theorem 4.15).
We also generalise [40, Proposition 3.1], by using a type of Newton polygon. This comes from our desire to encode the root configuration of a polynomial in the Newton polygon. The first slope of this polygon will be the δ invariant in the work surrounding minimal pairs.

[4] Abbreviated RT extensions.

We sum up the contributions with their contributors in the following synoptic diagram.

$$\begin{tikzcd}
\left\{\begin{array}{c}\text{Abstract key}\\ \text{Polynomials}\end{array}\right\} \arrow[r, leftrightarrow, "\text{Novacoski}"] \arrow[d, leftrightarrow] & \left\{\begin{array}{c}\text{Minimal}\\ \text{Pairs}\end{array}\right\} \arrow[d, leftrightarrow] \\
\left\{\begin{array}{c}\text{Truncated}\\ \text{valuation}\\ \text{on } K[X]\end{array}\right\} \arrow[r, shift left, "\text{extension}"] & \left\{\begin{array}{c}\text{Truncated}\\ \text{valuation}\\ \text{on } \overline{K}[X]\end{array}\right\} \arrow[l, shift left, "\text{restriction}"]
\end{tikzcd}$$

(Mac Lane, Vaquié, Spivakovsky,...)

(Alexandru, Popescu, Zăhărescu)

Our work concerns the bottom part, the extension-restriction arrows and we make an extensive account of this contribution in Chapter 4.

Lastly we are concerned with establishing the geometric interpretation of residually transcendental valuations. To start off, valuations given by minimal pairs can be interpreted as minimal values of polynomials $f \in K[X]$ attained on balls. Indeed take a minimal pair (a, δ), so that we can consider the ball

$$D(a, \delta) = \{b \in \overline{K} \mid \overline{\nu}(b - a) \geqslant \delta\}.$$

This type of set is not reduced to a single point if $\mathrm{rk}(\mu) = \mathrm{rk}(\nu)$. Indeed if $\mathrm{rk}(\mu) = \mathrm{rk}(\nu) + 1$ and we assume δ to be strictly in the last convex subgroup of $\mu(K(X)^*)$, *i.e.,* let δ be larger than the value group of K, then for any $x \in \overline{K}^*$, $\delta > \nu(x)$. In this case

$$D(a, \delta) = \{a\}.$$

In order to avoid such situations we will restrict our focus to residually transcendental valuations. In such a context $\mathrm{rk}(\mu) = \mathrm{rk}(\nu)$ and thus, if $\delta < \infty$, $D(a, \delta)$ is not reduced to a point. Then one considers a valuation $\overline{\nu}_{D(a,\delta)}$ defined as follows

$$\overline{\nu}_{D(a,\delta)}(f) = \min_{x \in D(a,\delta)} \overline{\nu}(f(x)).$$

We can prove that the minimum is attained and we can even exhibit the way it is computed (see Lemma 5.3). Indeed we show that

$$\overline{\nu}_{D(a,\delta)} = \overline{\nu}_{a,\delta}.$$

Furthermore this allows for a clear bijection between residually transcendental valuations and balls. Two minimal pairs $(a, \delta), (a', \delta')$ may yield the same valuation, however that is only the case if they yield the same ball $D(a, \delta) = D(a', \delta')$. We wish to build a similar bijection over K, by finding what we will call diskoids $\Delta(Q) \subseteq \overline{K}$ for key polynomials Q. These will need to verify

$$\mu_Q = \overline{\nu}_{\Delta(Q)}, \text{ where } \overline{\nu}_{\Delta(Q)}(f) = \min_{x \in \Delta(Q)} \overline{\nu}(f(x))$$
$$\overline{\nu}_{\Delta(Q_1)} = \overline{\nu}_{\Delta(Q_2)} \implies \Delta(Q_1) = \Delta(Q_2).$$

We can not just take $\Delta(Q)$ to be balls, since different balls can induce the same valuation over K. If for instance μ_Q is the same as $\overline{\nu}_{D(a,\delta)}\big|_{K[X]}$ and $\sigma \in \mathrm{Aut}_K(\overline{K})$ such that $\overline{\nu} \circ \sigma = \overline{\nu}$, then μ_Q is also the same as the restriction to $K[X]$ of $\overline{\nu}_{\sigma(D(a,\delta))}$. We will use larger subsets of $\overline{K}$. We also want them to have a manageable shape and be easy to grasp. We will give a reasonable candidate, called diskoids, that was studied by Julian Rüth in his PhD thesis [47] and later used in [43].

In his thesis Rüth starts with an irreducible smooth projective curve C over a discretely valued field of rank 1 (K, ν) and looks for an explicit way of computing normal models of C. A normal model is, simply put, a flat

$\mathscr{O}(\nu)$-scheme [5] $\mathcal{C}$ equipped with an isomorphism from C to its generic fiber $\mathcal{C}_g$. He starts by detailing how to define an injective map that takes isomorphism classes of normal models of C, and assigns them finite sets of RT extensions from (K,ν) to $K(C)$ [6]. This map is furthermore bijective if the base valued field (K,ν) is henselian. For details one can consult [47, Ch. 3] or, for more general results concerned with normal models over general valuations, one can consult [22].
The explicit approach in Rüth's thesis makes extensive use of Mac Lane's augmentation process. Since $K(C)$ is a separable extension of a transcendental extension $K(t)$ of K, this process gives an algorithmic approach to finding all possible finite sets of RT extensions. The author then proceeds to construct the language of diskoids and uses it in order to give an explicit way of eliminating ramification of his RT extensions [47, Prop. 6.16]. For any RT extension μ of ν, he finds a finite field extension K'/K, such that any extension of μ to $K'(t)$ has no value group extension (*i.e.,* it is *weakly unramified*, *cf.* [47, Definition 6.1 and 6.2]).

It is thus motivating to establish a general formalism for diskoids. In Chapter 5 we will see how diskoids decompose into simple balls and how the absolute Galois group of K, *i.e.,* $G_K = \mathrm{Gal}(K^{sep}/K) = \mathrm{Aut}_K(\overline{K})$ acts on these balls. A very minute study of this action is done in [58], when the base valued field K is henselian. We only use the Galois action to show that a truncated valuation (by a fixed abstract key polynomial Q) is the valuation given by a diskoid (associated to Q). Our goal here is to show that this correspondence is in fact a bijection and this is what we wish to call our ultrametric interpretation. We prove this statement when (K,ν) is henselian or of rank 1.

We add a final chapter, Chapter 6, where we summarise possible directions we could explore to seek solutions to the problems touched upon here, as well as possible ideas for finding applications of the results presented in this thesis.

The article [10], to appear in *Journal of Algebra (August 2021)* has been in great part taken from the work presented here.

[5] $\mathscr{O}(\nu)$ is the valuation ring of ν, see Chapter 1.
[6] $K(V)$ is the function field of a algebraic variety V over the field K.

Chapter 1

Basics on Valuations

In these notes we adopt the convention that the set $\mathbb{N}$ contains 0: $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$. We will write $\mathbb{N}^*$ for the positive integers: $\mathbb{N}^* = \{1, 2, 3, \ldots\}$.
This first chapter is dedicated to the very basics of valuation theory. We introduce the notion of valuation, valuation rings, graded rings, composition of valuations as well as some of their numerical invariants (rank, rational rank, transcendence degree). We focus on extensions of valuations and more specifically on the simple transcendental extension case. Much of this content is standard and reference to detailed treatments include (and are not limited to) [11, Ch. 6], [53], [52], [19].

1.1 Valuations and Graded Algebras

We start by defining our most basic objects, valuations. Let R be a commutative, unitary ring and Γ a totally ordered abelian group. We write $+$ for the addition law of this group and we add an element, not in Γ, that we denote ∞. We write $\Gamma_\infty = \Gamma \cup \{\infty\}$ and extend the addition operation and order relation so that ∞ plays the role of a biggest element.

Definition 1.1. A *valuation* ν on R, is a map $\nu : R \longrightarrow \Gamma_\infty$ satisfying
(V1) $\forall a, b \in R,\ \nu(a \cdot b) = \nu(a) + \nu(b)$.
(V2) $\forall a, b \in R,\ \nu(a + b) \geqslant \min\{\nu(a), \nu(b)\}$. This is called the ultrametric inequality.
(V3) $\nu(1_R) = 0$ and $\nu(0) = \infty$.
(V4) $\nu^{-1}(\infty) = (0)$.
We often write (R, ν) for a *valuative pair*, *i.e.,* a ring equipped with a valuation.

Remark 1.2.
1. When we talk of embedding a pair (R, ν) into (S, μ), we mean we set an injective morphism $R \overset{\iota}{\hookrightarrow} S$, such that $\mu \circ \iota = \nu$.
2. (V4) implies that R is a domain. One could then extend ν to a valuation over $\mathrm{Frac}(R)$, the quotient field of R, by setting $\nu(a/b) = \nu(a) - \nu(b)$. It is easy to see it is well-defined.
3. For any integer $n \in \mathbb{Z}$, $\nu(n \cdot 1_R) \geqslant 0$. Indeed

$$\nu(n \cdot 1_R) = \nu(\underbrace{1_R + \ldots + 1_R}_{n \text{ times}}) \geqslant \nu(1_R) = 0.$$

   Furthermore, we have $\nu(-1_R) = \nu(1_R) = 0$ as $0 = \nu(1_R) = \nu((-1_R)^2) = 2\nu(-1_R)$.

The very first remarkable property of valuations is a consequence of the ultrametric inequality, *i.e.,* property (V2).

Proposition 1.3. Consider a valuative pair (R, ν) and $a, b \in R$ such that $\nu(a) < \nu(b)$. Then $\nu(a + b) = \nu(a)$.

Proof. Indeed we first have $\nu(a+b) \geqslant \min\{\nu(a), \nu(b)\} = \nu(a)$ by (V2). If the inequality is strict, then we have

$$\nu(a) = \nu((a + b) - b) \geqslant \min\{\nu(a + b), \nu(b)\} > \nu(a)$$

which is absurd. □

Example 1.4. 1. The most basic examples of valuations are the p-adic valuations on $\mathbb{Q}$ or its completions, the p-adic numbers $\mathbb{Q}_p$, where p is a prime number in $\mathbb{Z}$. By the prime factorisation theorem in $\mathbb{Z}$ any integer $n \in \mathbb{Z}$ can be written as $n = p^r m$ with $p \nmid m$. The map that assigns to each n its corresponding $\nu_p(n) := r$ is the p-adic valuation. It can then be extended to $\mathbb{Q}$ and its values lie in $\mathbb{Z}$.

2. Similarly on $k((T))$, where k is a field, one can define the T-multiplicity of a formal power series. Any such series f can be written as

$$f(T) = a_n T^n + a_{n+1} T^{n+1} + (\text{higher powers of } T), \quad a_i \in k,\ a_n \neq 0$$

so that $\operatorname{ord}_T(f) = n \in \mathbb{Z}$. This again is a valuation.

3. On $k[X]$ the map $-\deg$ is a valuation with values in $\mathbb{Z}$.
4. Consider $R = k[X, Y]$ and for any $f \in R$, write it as

$$f(X, Y) = a_n(X) Y^n + a_{n+1}(X) Y^{n+1} + \ldots + a_N(X) Y^N, \quad a_i \in k[X],\ a_n \neq 0$$

and set $\nu(f) = (n, \operatorname{ord}_X(a_n(X)))$. This is a valuation with values in $\mathbb{Z}^2$, which is equipped with the lexicographic order.

Our study of valuations requires a powerful tool, the *graded algebra of a valuation*. For $\gamma \in \nu(R \setminus \{0\})$ and define the following groups

$$\mathcal{P}_\gamma = \mathcal{P}_\gamma(R, \nu) = \{a \in R \mid \nu(a) \geqslant \gamma\}$$
$$\mathcal{P}_\gamma^+ = \mathcal{P}_\gamma^+(R, \nu) = \{a \in R \mid \nu(a) > \gamma\}.$$

The graded ring $\operatorname{gr}_\nu(R)$ is

$$\operatorname{gr}_\nu(R) = \bigoplus_{\gamma \in \Gamma} \frac{\mathcal{P}_\gamma}{\mathcal{P}_\gamma^+}.$$

This comes equipped with a map, called the *initial form*, $\operatorname{in}_\nu \ : \ R \setminus \{0\} \longrightarrow \operatorname{gr}_\nu(R)$ assigning to a its class modulo $\mathcal{P}_\gamma^+(R, \nu)$ with $\gamma = \nu(a)$. By definition, any homogeneous element [1] is thus the initial form of some element in R and any initial form of any non-zero element is a non-zero element in the graded algebra. We can assign to 0 the value 0 in the graded ring.

Remark 1.5. For general filtered modules (or rings, algebras etc.) there is also a notion of initial form which fails to be a morphism in general. It may not even be multiplicative, however when considering the graded algebra associated to a valuation, we can still state some simple rules of computation:

1. $\operatorname{in}_\nu(a \cdot b) = \operatorname{in}_\nu(a) \cdot \operatorname{in}_\nu(b),\ \forall a, b \in R$. Thus the graded ring is an integral domain.
2. if $\nu(a) > \nu(b)$ then $\operatorname{in}_\nu(a + b) = \operatorname{in}_\nu(b)$.
3. if $\nu(a) = \nu(b) < \nu(a + b)$ then $\operatorname{in}_\nu(a + b) \neq \operatorname{in}_\nu(a) + \operatorname{in}_\nu(b) = 0$.
4. if $\nu(a) = \nu(b) = \nu(a + b)$ then $\operatorname{in}_\nu(a + b) = \operatorname{in}_\nu(a) + \operatorname{in}_\nu(b)$.

1. is simply a consequence of (V1) and the rest amount to using (V2).

The structure of the graded algebra of a valuation is a central object for the algebraic study of valuations. We give here some very basic concepts relating properties of elements of a valued ring to some very simple arithmetic properties of their initial parts in the graded algebra.

Definition 1.6. Fix two elements $a, b \in R$ where (R, ν) is a valuative pair. The two following definitions are originally due to Mac Lane [32, I.2, p. 2]. We say that:

1. a and b are *ν-equivalent* elements if $\operatorname{in}_\nu(a) = \operatorname{in}_\nu(b)$ and we write it as $a \underset{\nu}{\sim} b$. This is equivalent to affirming that $\nu(a - b) > \nu(a) = \nu(b)$.

[1] *i.e.*, any element belonging to one of the direct factors $\frac{\mathcal{P}_\gamma}{\mathcal{P}_\gamma^+}$.

2. b is *ν-divisible* by a is $\mathrm{in}_\nu(a)$ divides $\mathrm{in}_\nu(b)$ in $\mathrm{gr}_\nu(R)$. Since both $\mathrm{in}_\nu(a)$ and $\mathrm{in}_\nu(b)$ are homogeneous, this means that there is a $c \in R$ such that

$$\mathrm{in}_\nu(b) = \mathrm{in}_\nu(a)\mathrm{in}_\nu(c) = \mathrm{in}_\nu(ac).$$

In another way to put it, a μ-divides b if there is a $c \in R$ such that $b \underset{\nu}{\sim} ac$, or equivalently $\nu(b - ac) > \nu(b) = \nu(ac)$. We will write this $a \underset{\nu}{|} b$.

Remark 1.7. 1. If $\gamma \in \Gamma$ is never a value of an element of R, then $\frac{\mathcal{P}_\gamma}{\mathcal{P}_\gamma^+} = 0$, so that in fact

$$\mathrm{gr}_\nu(R) = \bigoplus_{\gamma \in \nu(R \setminus \{0\})} \frac{\mathcal{P}_\gamma}{\mathcal{P}_\gamma^+}.$$

2. The condition (V4) may be relaxed and we can allow for non-zero elements to have infinite value. We call this set, its *support*, or *socle*:

$$\mathrm{supp}(\nu) = \nu^{-1}(\infty) = \{x \in R; \nu(x) = \infty\}.$$

It is a prime ideal in R. We will use the term *semivaluation* to speak of maps verifying (V1)-(V3) and valuations, the semivaluations of trivial support. This is an arbitrary choice: some authors distinguish between valuations and Krull valuations, for instance in [20], some other will use the term *pseudo-valuation* for what we call semivaluations. We can deduce a bijection between semivaluations of R of support $\mathfrak{p}$ and valuations of $R/\mathfrak{p}$: for a semivaluation ν of R of support $\mathfrak{p}$, set the valuation

$$\forall x \in R \setminus \mathfrak{p},\ \widetilde{\nu}(x \mod \mathfrak{p}) = \nu(x).$$

Semivaluations still have graded algebras and we have the following isomorphism

$$\mathrm{gr}_\nu(R) \cong \mathrm{gr}_{\widetilde{\nu}}(R/\mathrm{supp}(\nu)).$$

If we extend the definition of graded algebras and allow for the infinite values to appear in our graded components, then the above isomorphism does not hold true. Indeed, if we allow$\mathcal{P}_\infty(R,\nu) = \{a \in R \mid \nu(a) = \infty\}$, $\mathcal{P}_\infty^+(R,\nu) = \{0\}$ [2] and

$$\widetilde{\mathrm{gr}}_\nu(R) = \bigoplus_{\gamma \in \Gamma_\infty} \frac{\mathcal{P}_\gamma}{\mathcal{P}_\gamma^+}$$

then, we now have

$$\widetilde{\mathrm{gr}}_\nu(R) \cong \mathrm{gr}_{\widetilde{\nu}}(R/\mathrm{supp}(\nu)) \oplus \mathcal{P}_\infty(R,\nu).$$

For most of this text, we will only adopt the usual notion of $\mathrm{gr}_\nu(R)$.

We now wish to show how the graded algebra has a functorial behaviour. This will prove to be of key importance.

Theorem 1.8. Consider two valuative pairs (R,ν), (S,μ) with a ring morphism $\phi : R \to S$ such that $\nu \leqslant \mu \circ \phi$, *i.e.,* $\forall f \in R$, $\nu(f) \leqslant \mu(\phi(f))$ (we assume ν and μ take values inside a common ordered subgroup Γ). There is a canonical homogeneous map of graded rings (*i.e.,* sending homogeneous elements to either 0 or homogeneous of same degree)

$$\mathrm{gr}\phi : \mathrm{gr}_\nu(R) \longrightarrow \mathrm{gr}_\mu(S)$$

sending $\mathrm{in}_\nu f$, $f \in R$ to $\phi(f) \mod \mathcal{P}^+_{\nu(f)}(S,\mu)$. Its kernel is the homogeneous ideal

$$\mathrm{Ker}(\mathrm{gr}\phi) = \Big\langle \mathrm{in}_\nu(I) \Big\rangle, \text{ where } I = \{f \in R; \nu(f) < \mu(\phi(f))\}.$$

[2] Nothing is greater than ∞.

Proof. By hypothesis we have for every $\gamma \in \Gamma$

$$\phi\left(\mathcal{P}_\gamma(R,\nu)\right) \subseteq \mathcal{P}_\gamma(S,\mu)$$
$$\phi\left(\mathcal{P}^+_\gamma(R,\nu)\right) \subseteq \mathcal{P}^+_\gamma(S,\mu).$$

This induces a mapping of abelian groups

$$\frac{\mathcal{P}_\gamma(R,\nu)}{\mathcal{P}^+_\gamma(R,\nu)} \longrightarrow \frac{\mathcal{P}_\gamma(S,\mu)}{\mathcal{P}^+_\gamma(S,\mu)}, \quad f \mod \mathcal{P}^+_{\nu(f)}(R,\nu) = \mathrm{in}_\nu f \longmapsto \phi(f) \mod \mathcal{P}^+_{\nu(f)}(S,\mu).$$

These maps put together generate a global mapping of abelian groups

$$\mathrm{gr}\phi : \mathrm{gr}_\nu R \longrightarrow \mathrm{gr}_\mu S.$$

Since ϕ is a ring morphism, it sends 1_R to 1_S, so $\mathrm{gr}\phi$ clearly sends $1_{\mathrm{gr}\nu R}$ to $1_{\mathrm{gr}_\mu S}$. By (V1) and the fact that ϕ is multiplicative, $\mathrm{gr}\phi$ is also multiplicative. It is enough to show it for two homogeneous elements $\mathrm{in}_\nu f, \mathrm{in}_\nu g,\ f, g \in R$

$$\begin{aligned}
\mathrm{gr}\phi(\mathrm{in}_\nu f \times \mathrm{in}_\nu g) &= \mathrm{gr}\phi(\mathrm{in}_\nu fg) \\
&= \phi(fg) \mod \mathcal{P}^+_{\nu(fg)}(S,\mu) \\
&= \phi(f)\phi(g) \mod \mathcal{P}^+_{\nu(f)+\nu(g)}(S,\mu) \\
&= \left(\phi(f) \mod \mathcal{P}^+_{\nu(f)}(S,\mu)\right) \times \left(\phi(g) \mod \mathcal{P}^+_{\nu(g)}(S,\mu)\right) \\
&= \mathrm{gr}\phi(\mathrm{in}_\nu f)\mathrm{gr}\phi(\mathrm{in}_\nu g).
\end{aligned}$$

The kernel of $\mathrm{gr}\phi$ is in fact the abelian group generated in $\mathrm{gr}_\nu R$ by the homogeneous elements $\mathrm{in}_\nu f$ that are sent to 0 in $\mathrm{gr}_\mu S$. In other words, these homogeneous elements are the $\mathrm{in}_\nu f$ such that $\phi(f) \in \mathcal{P}^+_{\nu(f)}(S,\mu)$. This last condition is equivalent to $\nu(f) < \mu(\phi(f))$. □

Example 1.9. Here are some basic examples of graded rings

1. The graded algebra of p-adics is a polynomial algebra over a finite field

$$\mathrm{gr}_{\nu_p}\mathbb{Z} = \mathrm{gr}_{\nu_p}\mathbb{Z}_p = \mathbb{F}_p[s], \quad \mathrm{gr}_{\nu_p}\mathbb{Q} = \mathrm{gr}_{\nu_p}\mathbb{Q}_p = \mathbb{F}_p[s, s^{-1}], \text{ where } s = \mathrm{in}_{\nu_p} p.$$

2. The graded algebra over the Laurent series has a polynomial structure as well

$$\mathrm{gr}_{\mathrm{ord}_t} k[[t]] = k[\tau], \quad \mathrm{gr}_{\mathrm{ord}_t} k((t)) = k[\tau, \tau^{-1}], \text{ where } \tau = \mathrm{in}_{\mathrm{ord}_t} t.$$

3. For the $-\deg$ valuation

$$\mathrm{gr}_{-\deg} k[X] = k[t^{-1}], \text{ where } t^{-1} = \mathrm{in}_{-\deg} X.$$

4. For the last valuation of Example 1.4,

$$\mathrm{gr}_\nu R = k[x, y], \text{ where } x = \mathrm{in}_\nu X,\ y = \mathrm{in}_\nu Y.$$

For the first three examples, the gradation of the graded algebra is the same as the gradation of the polynomial structure that it is isomorphic to. This is not the case in the fourth example: indeed, the gradation is given by the group $\mathbb{Z}^2$ equipped with the lexicographic order: the element y is of degree $(1, 0)$ and x is of degree $(0, 1)$. The classic polynomial algebra $k[x, y]$ is graded by $\mathbb{Z}$ where homogeneous elements are k-linear combinations of monomials of same total degree.

1.2 Valuation Rings

Definition 1.10. The *valuation ring* of the pair (R, ν) is a local subring of K, written $\mathscr{O}(\nu)$ with maximal ideal $\mathfrak{m}(\nu)$ and residual field $\kappa(\nu)$. We define them as follows:

$$\mathscr{O}(\nu) = \{a \in K \mid \nu(a) \geqslant 0\}$$
$$\mathfrak{m}(\nu) = \{a \in K \mid \nu(a) > 0\}$$
$$\kappa(\nu) = \mathscr{O}(\nu)/\mathfrak{m}(\nu).$$

It turns out that valuation rings have a simple characterisation. We will say a local ring $(B, \mathfrak{n})$ dominates the local ring $(A, \mathfrak{m})$ if $A \subset B$ and $\mathfrak{n} \cap A = \mathfrak{m}$. The condition $\mathfrak{n} \cap A = \mathfrak{m}$ is also equivalent to $\mathfrak{m} \subset \mathfrak{n}$.

Proposition 1.11. A subring of $V \subset K$, where K is a field, is a *valuation ring* if one of the following equivalent conditions is verified:

1. $\forall x \in K$, if $x \in K \setminus V$ then $x^{-1} \in V$.
2. $\mathrm{Frac}(V) = K$ and the set of ideals of V are totally ordered by inclusion.
3. $\mathrm{Frac}(V) = K$ and the set of principal ideals of V are totally ordered by inclusion.
4. V is local and it is maximal for the domination order.

Proof. $2 \implies 3$ is clear. Let's prove $1 \implies 2$: consider $I, J \subset V$ two ideal with $I \not\subseteq J$ and take $x \in I \setminus J$. Take any $y \in J$, so that $x = y\frac{x}{y}$, thus if $\frac{x}{y} \in V$, then $x \in J$ which is contradictory, thus $\frac{x}{y}^{-1} = \frac{y}{x} \in V$, so that $y = x\frac{y}{x} \in I$. It is clear that in such a situation $\mathrm{Frac}(V) = K$. Let us prove $3 \implies 1$: consider $x \in K = \mathrm{Frac}(V)$ and write it as $x = \frac{a}{b}, a, b \in V$. Then we either have $aV \subset bV$, which is equivalent to $x = \frac{a}{b} \in V$, or $aV \supseteq bV$, which is equivalent to $x^{-1} = \frac{b}{a} \in V$.
Proving the equivalence with 4 is a bit more delicate. Let's assume that V verifies $1, 2$ and 3 and let us show 4. By 2 we already have that V is local (it has a maximal ideal by Zorn's lemma and every ideal should either contain it, or be contained in it, but by maximality, only the latter is possible). Write $\mathfrak{m}$ for its maximal ideal. Consider any other local ring $(W, \mathfrak{n})$ dominating $(V, \mathfrak{m})$. Suppose there is $x \in W \setminus V$ so that $x^{-1} \in V$. Even more so $x^{-1} \in \mathfrak{m} \subset \mathfrak{n}$ so that $x \notin W$ which contradicts $x \in W$.
Let us now prove that 4 implies 1. Consider $x \in K, x \notin V$. We show that $\mathfrak{m}V[x] = V[x]$. Indeed, if it were not the case, $\mathfrak{m}V[x]$ would be contained in a maximal ideal $\mathfrak{M}$ of $V[x]$, so $V[x]_{\mathfrak{M}}$ strictly dominates V. This implies that there are $r_i \in \mathfrak{m}$ such that

$$1 = r_0 + xr_1 + \ldots + r_n x^n$$

for some $n \in \mathbb{N}$. But since $1 - r_0 \in V^\times$, we deduce that $\frac{1}{x}$ is integral over V. Let us then show that V is integrally closed. Suppose $V \subseteq W \subseteq K$, where W is a subring, integral over V. By the lying-over (Cohen-Seidenberg theorems), there is a maximal ideal $\mathfrak{n} \subset W$ such that $\mathfrak{n} \cap V = \mathfrak{m}$. Thus $W_{\mathfrak{n}}$ dominates V, so that $W \subseteq W_{\mathfrak{n}} = V$. □

Remark 1.12.

1. For any valuation ν, we can easily show that $\mathscr{O}(\nu)$ is a valuation ring. Conversely, given a valuation ring V there should be a valuation ν such that $\mathscr{O}(\nu) = V$. We can order $K^\times / V^\times$ in the following way:

$$\forall x \in K^\times,\ x \mod V^\times \geqslant 0 \iff x \in V.$$

Since V is a valuation ring, this order is total. We can define the following map

$$\nu(x) = x \mod V^\times \in K^\times / V^\times.$$

which can be shown to be the sought after valuation.

2. We should observe that knowing a valuation boils down to knowing which elements are of positive value. This implies that a valuation is essentially given by its valuation ring V, even if Φ is not given, since we can rebuild it from V. This gives rise to the idea of identifying valuations if they have the same valuation rings:

$$\nu_1 \equiv \nu_2 \iff \mathscr{O}(\nu_1) = \mathscr{O}(\nu_2)$$

where the equality of rings is a strong equality and not an isomorphism. This is equivalent to saying there is an isomorphism of abelian groups $\lambda : \Phi(\nu_1) \to \Phi(\nu_2)$ such that $\nu_2 = \lambda \circ \nu_1$.

3. We can show that the integral closure of a domain R in a field K (not necessarily its field of fractions) written $\mathrm{Int}(R, K)$ can be built with valuation rings:

$$\mathrm{Int}(R, K) = \bigcap_{V \supseteq R} V$$

where the intersection is taken over all valuation rings of K containing R. Indeed, we can extract from the proof of the above proposition that any valuation ring is integrally closed, so the inclusion

$\mathrm{Int}(R,K) \subseteq \bigcap_{V \supseteq R} V$ is clear. Now consider $x \in K$ not integral over R, so that $\frac{1}{x} \notin R[\frac{1}{x}]^\times$ (otherwise we could build an integral equation for x with coefficients in R). Thus $\frac{1}{x}$ is contained in a maximal ideal $\mathfrak{m}$ of $R[\frac{1}{x}]$. By Zorn's lemma, there is a valuation ring V dominating $R[\frac{1}{x}]_\mathfrak{m}$, so that $\frac{1}{x} \in \mathfrak{m}(V)$ thus $x \notin V$ and $V \supseteq R$.

4. If we write $p = \mathrm{char}\kappa(\nu)$ we will have several situations.
   (a) either $\mathrm{char}R = \mathrm{char}\mathscr{O}(\nu) = p$, in which case $\nu(p) = \infty$.
   (b) or $\mathrm{char}R = 0$ and $\mathrm{char}\mathscr{O}(\nu) = p > 0$ in which case $\infty > \nu(p) > 0$.

1.3 Value Groups and Residue Fields

There are many ways in which one can measure the complexity of a valuation. We start with assigning invariants to its value group.

Definition 1.13. Consider a valuative pair (R, ν) and K the fraction field of R. The *value group of* ν written $\Phi(\nu)$ is

$$\Phi(\nu) := \nu(K^\times).$$

It is a totally ordered abelian group. Naturally, ordered abelian groups have an additional class of sub-objects, that should, in informal terms, be closed under taking convex hulls.

Definition 1.14. Given a totally ordered abelian group $(G, +, \leqslant)$ one can define its *isolated subgroups.* They are subgroups $H \subset G$ such that:

$$\forall h, k \in H,\ [h,k] \subset H$$

where $[h,k]$ is the segment with extremities h, k: $[h,k] = \{g \in G;\ h \leqslant g \leqslant k\}$.
Similarly, the isolated (sub-)groups of a valuation ν are the isolated subgroups of $\Phi(\nu)$.

If we are given an isolated subgroup $H \subset G$ it is clear it inherits the order on G, but it is probably a bit less obvious that G/H can be given an order in a canonical way.

Proposition 1.15. There exists a natural way of ordering G/H, so that the canonical projection $\pi : G \to G/H$ is an order-preserving map. If we have $g + H \in G/H$, then we will set $g + H > 0$ iff $g + H \neq H$ and $g > 0$.

Proof. We show by contradiction, that our order is well-defined. Take $g, g' \in G$, $g + H = g' + H$ such that $g > 0$. We thus wish to show that $g' > 0$. Suppose on the contrary that $g' \leqslant 0 < g$. Naturally we have $g' - g \leqslant -g < 0$. Since $g' - g \in H$ and H is an isolated subgroup, so $-g \in H$, which yields $g + H = H$, but this is excluded.
It is clear that π is order-preserving under this relation. □

Observe that we additionally proved that if $g + H > 0$ then all elements of the coset $g + H$ are positive. Furthermore, the coset H is seen as the zero element in G/H, but all elements $h \in H$, $h < 0$ will be sent to the coset H which is not negative.

Definition 1.16. We define the *rank of* ν, written $\mathrm{rk}(\nu)$

$$\mathrm{rk}(\nu) := \mathrm{ord}\{\text{strict isolated subgroups of } \Phi(\nu)\}$$

where ord denotes the ordinal type of the set. We restrict ourselves to the cases where indeed, the set of isolated subgroups does form a well-ordered set. Next we assign to ν its *rational rank*, written $r(\nu)$

$$r(\nu) := \dim_\mathbb{Q} \Phi(\nu) \otimes_\mathbb{Z} \mathbb{Q}.$$

Remark 1.17. It is quite possible that $\nu(R \setminus \{0\})$ is a lot smaller than $\Phi(\nu)$, however $\nu(R \setminus \{0\})$ generates $\Phi(\nu)$ as a subgroup, since R and $\mathscr{O}(\nu)$ have the same field of fractions.

Definition 1.18. By *extension of valued rings*, we mean an extension of rings $R \subseteq S$, each equipped with a respective valuation ν and μ, such that μ restricts to ν on R,

$$\mu\big|_R = \nu.$$

This is equivalent to saying that the natural inclusion map $R \hookrightarrow S$ defines an embedding of valued pairs $(R, \nu) \hookrightarrow (S, \mu)$.

Given such an extension one can compare value groups and residue fields. Indeed, one has $\nu(R\backslash\{0\}) \subset \mu(S\backslash\{0\})$, thus $\Phi(\nu) \subset \Phi(\mu)$. Secondly we have $\mathscr{O}(\nu) \subseteq \mathscr{O}(\mu)$ and $\mathscr{O}(\nu) \cap \mathfrak{m}(\mu) = \mathfrak{m}(\nu)$ thus giving us a natural inclusion of residual fields $\kappa(\nu) \subseteq \kappa(\mu)$. We will call them *value group extension* and *residual extension* respectively.

Proposition 1.19. We have the following inequalities given for any valuation ν on K and any valued extension (L, μ):

1. $\mathrm{rk}(\nu) \leqslant r(\nu)$.
2. $\dim_{\mathbb{Q}}(\Phi(\mu)/\Phi(\nu)) \otimes_{\mathbb{Z}} \mathbb{Q} + \mathrm{tr.deg}_{\kappa(\nu)}\kappa(\mu) \leqslant \mathrm{tr.deg}_K L$

when these quantities are well-defined and finite. The second one is called the Zariski-Abhyankar inequality (see [11, Ch. 6, §10, no. 3, Cor. 1] for a proof). Abhyankar showed how we can generalise it to valuations centred on any Noetherian local ring, *i.e.,* whose valuation ring dominates the aforementioned local ring (see [59, Appendix 2, Prop. 2] or [3, Theorem 1]).

1.4 Simple transcendental extensions

In this section we only deal with valuations μ over the ring $K[X]$ of polynomials in one variable, extending a given valuation ν over the field K. Abhyankar's inequality shows that the rational rank of a μ as its residual transcendence degree, can not jump more than once, compared to that of ν

$$r(\nu) \leqslant r(\mu) \leqslant r(\nu) + 1$$
$$\mathrm{tr.deg}_{\kappa(\nu)}\kappa(\mu) \leqslant 1.$$

We can classify them according to whether or not these invariants jump or not. We write down (r, d) the triple representing whether or not the rational rank and transcendence degree jump respectively. We will call this the *type of the extension*. Thus, according to Proposition 1.19 $(r, d) \in \{(0,0), (0,1), (1,0)\}$ (we abbreviate "transcendental" to "transc." in Table 1.1).

(r,d)	$\Phi(\mu)/\Phi(\nu)$	$\kappa(\mu)/\kappa(\nu)$	Type		Abbreviation
$(0,0)$	torsion	algebraic	Valuation algebraic		VA
$(0,1)$	torsion	transc.	Valuation transc.	Residually transc.	RT
$(1,0)$	free element	algebraic		Value transc.	VT

Table 1.1: Classification of simple Extensions

Remark 1.20.
1. Valuation transcendental extensions are also well-specified valuations (*"valuations bien-spécifiées"* in French) in Vaquié's work [54, Remarque 1.1] as they are exactly the extensions satisfying a condition relative to admissible families of valuations. Namely they are the extensions that have admissible families with a last element.
2. If μ is a semivaluation over $K[X]$ with non-trivial support, we can obtain a valuation in two different fashions. Indeed, if the support is not reduced to (0), it will be a principal prime ideal $\mathrm{supp}(\mu) = (Q)$, where $Q \in K[X]$ is a monic irreducible polynomial. Then we can obtain a valuation on the algebraic extension $L = K(\theta) = K[X]/(Q)$, where θ is simply the image of X in the quotient. An alternative way of constructing a valuation is through what we call *rank incrementation.* For any polynomial $f \in K[X]$ we factorise it as $f = Q^n g$ where $Q \nmid g$, then we set $[\mu](f) = (n, \mu(g))$. This map has values in $\mathbb{Z} \times \Phi(\mu)$ equipped with the lexicographical order. We can also determine the graded algebra of $[\mu]$
$$\mathrm{gr}_{[\mu]}K[X] = \left(\mathrm{gr}_\mu K[X]\right)\left[\mathrm{in}_{[\mu]}Q\right].$$
This construction allows for the study of valuations of simple algebraic extensions to be included in the study of simple transcendental (semi)valuations.

If we extend our valuations ν and μ to the algebraic $\overline{K}$ and $\overline{K}[X]$, then the algebraic closure of K, the type is preserved.

Proposition 1.21. Consider $(K(X), \mu)$ a valued pair extending (K, ν). We consider an extension $\overline{\mu}$ of μ to $\overline{K}(X)$ and set $\overline{\nu} = \overline{\mu}|_{\overline{K}}$. Then $\overline{\mu}/\overline{\nu}$ and μ/ν have the same type.

Proof. We can show (see for instance [53, Prop. 1.16], [52, Prop. 5.2]) or [19, Theorem 3.2.4]) that the two quotient groups $\Phi(\overline{\mu})/\Phi(\mu)$ and $\Phi(\overline{\nu})/\Phi(\nu)$ are torsion, and that the two extensions $\kappa(\overline{\mu})/\kappa(\mu)$ and $\kappa(\overline{\nu})/\kappa(\nu)$ are algebraic. Thus, by the transitive properties of the torsion groups, $\Phi(\overline{\mu})/\Phi(\overline{\nu})$ is torsion if and only if $\Phi(\mu)/\Phi(\nu)$ is torsion. By the transitive properties of algebraic extensions $\kappa(\overline{\mu})/\kappa(\overline{\nu})$ is algebraic if and only if $\kappa(\mu)/\kappa(\nu)$ is algebraic. This concludes the proof. □

Remark 1.22. Adopting the same notations as the ones in the above proposition, we see that if μ/ν is VA, then $\overline{\mu}/\overline{\nu}$ is in fact an *immediate extension*: the value groups and residual fields of the extension coincide. Indeed, since the quotient group $\Phi(\mu)/\Phi(\nu)$ is torsion, we then have

$$\Phi(\overline{\mu})/\Phi(\overline{\nu}) = (\mathbb{Q}\otimes\Phi(\mu))/(\mathbb{Q}\otimes\Phi(\nu)) = \mathbb{Q}\otimes\big(\Phi(\mu)/\Phi(\nu)\big) = 0.$$

Furthermore, since $\kappa(\overline{\mu})/\kappa(\overline{\nu})$ is algebraic and $\kappa(\overline{\nu})$ algebraically closed, we see that $\kappa(\overline{\mu}) = \kappa(\overline{\nu})$.

Example 1.23. 1. *Depth zero valuations*: consider an ordered embedding of ordered groups $\Phi(\nu) \hookrightarrow \Gamma$, set $a \in K$ and $\gamma \in \Gamma$. We construct then the depth zero valuation over $K[X]$,defined by a and γ, written $\nu_{a,\gamma}$, as follows[39, §2.2]:

$$\nu_{a,\gamma}\left(\sum_{i=0}^{n} a_i(X-a)^i\right) = \min_{0\leqslant i\leqslant n}\{\nu(a_i)+i\gamma\}.$$

To see it is a valuation, we can refer the reader to Chapter 2. Depending on whether or not $\gamma \in \mathbb{Q}\otimes\Phi(\nu)$, $\nu_{a,\gamma}$ is RT or VT, respectively.

2. Consider the formal power series $\omega(t) = \exp(t) - 1 = t + \frac{t^2}{2} + \frac{t^3}{6} + \ldots \in \mathbb{C}((t))$. We define the valuation μ over $\mathbb{C}(t)[X]$ in the following way:

$$f \in \mathbb{C}(t)[X], \quad \mu(f) = \mathrm{ord}_t f(\omega(t)).$$

Since $\omega(t)$ is transcendental over $\mathbb{C}(t)$, it is clear that μ respects (V4). The other axioms are easily seen to be verified. It is a valuation that clearly extends the valuation ord_t restricted to $\mathbb{C}(t)$. Furthermore it defines an immediate extension: indeed the value group is the same (*i.e.*, $\mathbb{Z}$) and the residue field of $\mathbb{C}(t)$ is $\mathbb{C}$. The residue field of $\mathbb{C}(t)(X) = \mathrm{Frac}(\mathbb{C}(t)[X])$ is also $\mathbb{C}$: indeed take $f(X) \in \mathbb{C}(t)(X)$ such that $\mu(f) = 0$; since $\omega(t) \in t\mathbb{C}((t))$, this implies that $f(0) = a_0(t) \in \mathbb{C}(t)$ with $a_0(0)$ being well-defined and not 0. It is then easy to show that $f(X) \longmapsto a_0(0)$ is the canonical projection of the valuation ring onto the residue field.

1.5 Composition

When we are given a valuative pair (K,ν) and an isolated subgroup $\Delta \subset \Phi(\nu)$ then the composition of maps

$$K^\times \xrightarrow{\nu} \Phi(\nu) \xrightarrow{\pi_\Delta} \Phi(\nu)/\Delta$$

is another valuation of K. We will write it either ν' or ν_Δ. Then ν will induce a valuation on $\kappa(\nu')$, that we will write $\overline{\nu}$, as follows:

$$\overline{\nu}(x \mod \mathfrak{m}(\nu')) = \nu(x), \forall x \in K^\times.$$

We say that ν is the *composition of* ν' *and* $\overline{\nu}$, and write $\nu = \nu' \circ \overline{\nu}$. We can then prove that $\Phi(\overline{\nu}) = \Delta$ and that we have a short exact sequence of ordered abelian groups:

$$0 \longrightarrow \Phi(\overline{\nu}) \longrightarrow \Phi(\nu) \longrightarrow \Phi(\nu') \longrightarrow 0$$

We can prove that $\kappa(\overline{\nu}) = \kappa(\nu)$ and show the following inclusions:

$$\mathfrak{m}(\nu') \subset \mathfrak{m}(\nu) \subset \mathscr{O}(\nu) \subset \mathscr{O}(\nu') \subset K$$

and $\mathfrak{m}(\nu')$ is a prime ideal of $\mathscr{O}(\nu)$. In fact, one can prove that $\mathscr{O}(\overline{\nu})$ and $\mathfrak{m}(\overline{\nu})$ are the respective images of $\mathscr{O}(\nu)$ and $\mathfrak{m}(\nu)$ under the canonical projection $\mathscr{O}(\nu') \to \kappa(\nu')$ and

$$\mathscr{O}(\nu') = \mathscr{O}(\nu)_{\mathfrak{m}(\nu')}.$$

In some sense one could say that ν gives a finer information about the elements of K, than ν'. We propose to say that ν' is *coarser* than ν. This defines an order among valuations over K, that we will write $\nu' \preceq \nu$.

We can iterate compositions. If $\nu = \nu_1 \circ \ldots \circ \nu_r$ and write $\nu_{(i)} = \nu_1 \circ \ldots \circ \nu_i$.
Finally we can evoke the correspondence between the isolated subgroups of ν and the prime ideals of $\mathscr{O}(\nu)$: Δ will correspond to $\mathfrak{m}(\nu')$. It can be made explicit (*cf.* [11, 53]). This shows that

$$\mathrm{rk}(\nu) = \dim \mathscr{O}(\nu)$$

and

$$\mathrm{rk}(\nu' \circ \overline{\nu}) = \mathrm{rk}(\nu') + \mathrm{rk}(\overline{\nu}).$$

Chapter 2

Key Polynomials

In the following chapter we want to talk about a series of methods of extending a valuation from K to $K[X]$. It is the simplest form of the extension problem and this situation could be seen as an inductive step to understanding valuations on function fields. The method of interest employs key polynomials. Several theories of such objects have appeared:

1. Ostrowski (1935) showed that for any discretely valued field of rank 1 (K, ν), *i.e.,* $\Phi(\nu) = \mathbb{Z}$, every valuation (of rank 1) on $K(X)$ is obtained as a limit defined by a pseudo-convergent sequence [44, §11, IX, p. 378]. More precisely his *Fundamentalsatz* states that any extension μ of ν to $K(X)$ is obtained as follows: there exists a sequence $(a_n)_{n\in\mathbb{N}}$, with values in $\overline{K}$, which is pseudo-convergent with respect to an extension $\overline{\nu}$ of ν to $\overline{K}$, *i.e.,* for all sufficiently large enough n, either $a_{n+1} = a_n$ or $\overline{\nu}(a_{n+1} - a_n) > \overline{\nu}(a_n - a_{n-1})$, such that for all $f \in K(X)$
$$\mu(f) = \lim_{n\to+\infty} \overline{\nu}(f(a_n)).$$
His work on pseudo-convergent sequences opened up the path to the study of general valuations of which Kaplansky's work on general. pseudo-convergent sequences [25], [26] is a first instance.
2. Mac Lane (1936, [32], [33]) solved the problem of extending ν to μ on $K(X)$, when $\Phi(\nu) = \mathbb{Z}$, without having to extend to an extension of ν to $\overline{K}$. He uses key polynomials in order to augment valuations and thus obtains a complete classifications of all the possible valuations μ extending ν to $K(X)$.
3. Vaquié (2007, [55],[57], [56], [54]) prolonged and completed Mac Lane's strategy through the use of limit key polynomials.
4. Roughly at the same time as Vaquié, Spivakovsky and his collaborators (*cf.* [21], [23]) elaborated a new type of key polynomials, which allow to truncate a valuation.

Roughly speaking augmenting and truncating valuations are dual:

- *Augmenting* a valuation μ on $K(X)$, extending ν defined on K, amounts to finding a valuation μ' extending ν as well, such that $\mu \leqslant \mu'$, *i.e.,* $\forall f \in K[X], \mu(f) \leqslant \mu'(f)$.
- *Truncating* a valuation μ consists in finding a valuation μ'' such that $\mu'' \leqslant \mu$, both extending ν.

We will present both techniques in the first two sections (giving a more exhaustive presentation of the truncation process as it will serve the rest of this work). In the third and last section we will briefly explain how these two approaches amount to the same objects.

Most of our presentation is taken from [32], [55], [37] and [39] for key polynomials and from [18], [42], [17] for abstract key polynomials and their link to classical Mac Lane-Vaquié polynomials.

All along this chapter, we will fix a couple of notations: we suppose we are given a valued field (K, ν) and we wish to study the set of extensions of ν to $K[X]$. Given such a valuation, we will write it with a Greek letter μ with various indices or exponents (*e.g.,* $\mu', \mu_G, \mu^*, \ldots$).

2.1 Key Polynomials

The goal is to obtain every valuation on $K[X]$ extending ν, as a limit of simple augmentations, starting off with a very simple valuation: the Gauss valuation μ_G, *i.e.,* $\mu_G = \nu_{0,0}$ according to the notations on Example 1.23: for any $\sum_i a_i X^i \in K[X]$

$$\mu_G\left(\sum_i a_i X^i\right) = \min_i \nu(a_i).$$

This allows for an algorithmic computation of values of polynomials. We define simple augmentations, then describe the necessity to also define limit augmentations.

2.1.1 Simple augmentations

To augment a valuation in an elementary way, we need to single out polynomials that enjoy certain arithmetic properties in $\mathrm{gr}_\nu K[X]$. We give these concepts here:

Definition 2.1. [32, Definition 4.1] [55, Def. par. 1.1] Consider a polynomial $Q \in K[X]$ and a valuation μ on $K[X]$. We say that Q is a *Mac Lane-Vaquié key polynomial for* μ (we will abbreviate this by MVKP), if it verifies the following properties:

(M1) Q is μ-irreducible, that is, $\mathrm{in}_\mu(Q)$ is a prime element in $\mathrm{gr}_\mu K[X]$. In other words

$$\forall f, g \in K[X],\ Q \underset{\mu}{|} fg \implies Q \underset{\mu}{|} f \text{ or } Q \underset{\mu}{|} g.$$

(M2) Q is μ-minimal, that is, any polynomial μ-divisible by Q is of degree not less that $\deg Q$:

$$\forall f \in K[x],\ Q \underset{\mu}{|} f \implies \deg Q \leqslant \deg f.$$

(M3) Q is monic.

Remark 2.2. If μ is a semivaluation with non-trivial support, then it does not have any key polynomial. Indeed, if $\mathrm{supp}(\mu) \neq (0)$, then it is a maximal ideal. Since $\mathrm{gr}_\mu K[X] \cong \mathrm{gr}_{\widetilde{\mu}} K[X]/\mathrm{supp}(\mu)$, $K[X]/\mathrm{supp}(\mu)$ being a field, any non-zero homogeneous element of $\mathrm{gr}_\mu K[X]$ is invertible. The graded algebra does not have any homogeneous prime element.

μ-minimal polynomials enjoy a very useful characterisation.

Lemma 2.3. [37, Prop. 2.3] Fix a polynomial $Q \in K[X]$. The following are equivalent:

1. Q is μ-minimal
2. For any $f \in K[X]$ with expansion $f = f_0 + f_1 Q + \ldots + f_n Q^n$, $\quad \forall i,\ Q \underset{\mu}{\nmid} f_i$ we have

$$\mu(f) = \min_i \mu(f_i Q^i).$$

3. For any non-zero $f \in K[X]$ with expansion $f = f_0 + f_1 Q + \ldots + f_n Q^n$, $\quad \forall i,\ Q \underset{\mu}{\nmid} f_i$ we have

$$Q \underset{\mu}{\nmid} f \iff \mu(f) = \mu(f_0).$$

As a corollary of this lemma, if Q is μ-minimal, then the standard Q-expansion of a polynomial, *i.e.,* $f = f_0 + f_1 Q + \ldots + f_n Q^n$, $\deg f_i < \deg Q$, gives a simple Q-expansion for which the properties of the lemma are verified, since $\forall i,\ Q \nmid f_i$. From now on, we will choose this as *the* Q-expansion.

Remark 2.4. The set of key polynomials has properties that can be proved algebraically. For instance Nart proved that the set of key polynomials for a given valuation μ, up to μ-equivalence, is isomorphic to the maximal spectrum of the 0-component of $\mathrm{gr}_\mu K[X]$. The use of key polynomials can also prove a factorisation theorem for homogeneous elements in $\mathrm{gr}_\mu K[X]$. For details, see [37].

To augment our valuations we may take the characterisation of μ-minimality and force the value of Q to be larger. We do this formally and thus we will suppose that $\Phi(\mu)$ is contained as an ordered group, in a larger ordered abelian group Γ.

Theorem 2.5. [55, Theorem 1.2, Prop. 1.5], [37, Prop. 7.2]
Consider a key polynomial Q for μ and $\gamma \in \Gamma$, $\gamma > \mu(Q)$. For any Q-expansion $f = f_0 + f_1Q + \ldots + f_nQ^n$, $\deg f_i < \deg Q$ where $f \in K[X]$ consider the map μ' defined by

$$\mu'(f) = \min_i \mu(f_i) + i\gamma.$$

We write this map $\mu' = [\mu; Q, \gamma]$. We then have:

1. The map μ' is a valuation such that $\mu \leqslant \mu'$. We have equality if and only if $Q \nmid_{\mu} f$ or $f = 0$.
2. If $Q \nmid_{\mu} f$, then $\mathrm{in}_{\mu'} f$ is a unit in $\mathrm{gr}_{\mu'} K[X]$.
3. Q is a key polynomial for μ'.

Under this set of properties and thanks to Theorem 1.8, we can elucidate the structure of $\mathrm{gr}_{\mu'} K[x]$.

Proposition 2.6. [55, Theorem 1.7] There is a canonical map $g : \mathrm{gr}_{\mu} K[X] \to \mathrm{gr}_{\mu'} K[X]$, with kernel the ideal $(\mathrm{in}_{\mu} Q)$ generated by the initial form of Q. This induces an isomorphism of rings

$$G : \frac{\mathrm{gr}_{\mu} K[X]}{(\mathrm{in}_{\mu} Q)}[T] \xrightarrow{\cong} \mathrm{gr}_{\mu'} K[X] \qquad T \longmapsto \mathrm{in}_{\mu'} Q.$$

We are now interested in finding key polynomials. Let us start with a simple remark

Corollary 2.7. If a valuation μ on $K[X]$ is maximal, *i.e.,* there is no valuation μ^* on $K[X]$ (extending ν) such that $\mu < \mu^*$, then μ has no key polynomials.

We can actually prove the converse. Suppose on the contrary that μ is not maximal and take μ^* strictly larger than μ. Then we can set the following

$$\begin{aligned} d(\mu, \mu^*) &:= \min\{\deg(f);\ \mu(f) < \mu^*(f)\} \\ \mathbf{\Phi}(\mu, \mu^*) &:= \{Q \in K[X];\ \mu(Q) < \mu^*(Q),\ Q \text{ is monic and } \deg(Q) = d(\mu, \mu^*).\} \end{aligned}$$

In other words, $\mathbf{\Phi}(\mu, \mu^*)$ is made up of polynomials on which μ and μ^* disagree and are of minimal degree. Our goal is to find ways of finding valuations μ' such that, on the one hand, are "close enough" to μ, ideally we want them to be simple augmented valuations, and on the other hand, verify $d(\mu', \mu^*) > d(\mu, \mu^*)$. This last point amounts to saying that the set of polynomials on which μ^* agrees with μ' is increased relative to the values it agrees with μ. Informally μ' is a better approximation of μ^* than μ.

Theorem 2.8. [55, Théorème 1.15][37, Cor. 2.6]

1. $\mathbf{\Phi}(\mu, \mu^*)$ is made up of key polynomials for μ. Furthermore, for any $Q \in \mathbf{\Phi}(\mu, \mu^*)$ we have
$$\mu < [\mu; Q, \mu^*(Q)] \leqslant \mu^*.$$
2. Any two $Q_1, Q_2 \in \mathbf{\Phi}(\mu, \mu^*)$ are μ-equivalent.
3. $\mathbf{\Phi}(\mu, \mu^*)$ is unchanged by further augmentation of μ^*:
$$\mu < \mu^* < \mu^{**} \implies \mathbf{\Phi}(\mu, \mu^*) = \mathbf{\Phi}(\mu, \mu^{**}).$$

We now set
$$\Lambda(\mu, \mu^*) := \{\mu^*(Q);\ Q \in \mathbf{\Phi}(\mu, \mu^*)\}.$$

Two situations may arise: either $\Lambda(\mu, \mu^*)$ has a maximal element or it has no maximal element. For the rest of this subsection, we suppose $\Lambda(\mu, \mu^*)$ has a maximal element. We address the other case in the next subsection. We take $Q \in \mathbf{\Phi}(\mu, \mu^*)$ such that $\mu^*(Q) = \max \Lambda(\mu, \mu^*)$. By Theorem 2.8 this Q will be a key polynomial (of minimal degree) and $\mu' := [\mu; Q, \mu^*(Q)]$ will be a new valuation such that $\mu < \mu' \leqslant \mu^*$, with $d(\mu, \mu^*) < d(\mu', \mu^*)$ (*cf.* [55, p.3453].

2.1.2 Limit augmentations

If we have no assumptions on ν, there is no reason for $\Lambda(\mu,\mu^*)$ to have a maximal element. In this situation we need to establish a new type of key polynomial. We now need to define a key polynomial for families of valuations. We start by choosing a totally ordered set A that can index $\Lambda(\mu,\mu^*)$, *i.e.,*

$$\Lambda(\mu,\mu^*)=\{\gamma_a;\ a\in A\}$$

where $a<a'$ if and only if $\gamma_a<\gamma_{a'}$.

Remark 2.9. More generally one can simply choose a cofinal set A' inside A, *i.e.,* such that $\forall a\in A, \exists a'\in A',\ a'>a$.
Observe that in this situation, A' will not contain a maximal element, since A does not have one either.

For every $a\in A$ we choose $Q_a\in\mathbf{\Phi}(\mu,\mu^*)$ such that $\mu^*(Q_a)=\gamma_a$. All these polynomials are of the same degree $d(\mu,\mu^*)$. We can also prove that $\forall a\in A,\ \gamma_a\in\Phi(\nu)$ (*cf.* [55, Lemme 1.17]). The family of key polynomials $(Q_a)_{a\in A}$ define successive augmentations $\mu_a:=[\mu;Q_a,\gamma_a]$ that form what is called a continuous family of iterative augmented valuations, or a continuous Mac Lane-Vaquié chain of valuations.

Definition 2.10. [55, §1.4][37, §7.2] A family of valuations $\mathcal{F}=(\mu_a)_{a\in A}$ forms a *continuous family of iterative augmented valuations* or a *continuous Mac Lane-Vaquié chain of valuations* based on μ if

$$\forall a\in A,\ \mu_a=[\mu;Q_a,\gamma_a],\ \text{where } Q_a \text{ is a key polynomial for } \mu \text{ and } \mu(Q_a)<\gamma_a\in\Phi(\mu)$$

and if they additionally verify the following:

1. $\deg(Q_a)$ is independent of $a\in A$. Hence we can write d_A.
2. The mapping $A\longrightarrow\Phi(\mu), a\longmapsto\gamma_a$ is an order-preserving embedding of A into $\Phi(\mu)$ and A has no maximal element.
3. For all $a<a'$ in A, $Q_{a'}$ is a key polynomial for μ_a, $Q_a \underset{\mu_a}{\nsim} Q_{a'}$ and $\mu_{a'}=[\mu_a;Q_{a'},\gamma_{a'}]$.

For any family of such valuations, we will call a polynomial $f\in K[X]$, *$\mathcal{F}$-stable* if there is $a_0\in A$ such that $\forall a\geqslant a_0,\ \mu_a(f)=\mu_{a_0}(f)$. We will denote this asymptotic value by $\mu_{\mathcal{F}}(f)$ exists.

Lemma 2.11. For any non $\mathcal{F}$-stable polynomial f we have $\mu_a(f)<\mu_{a'}(f)$, $\forall a<a'$ in A.

We can now define limit key polynomials associated to a continuous family of iterated augmented valuations. We will say that a polynomial g is $\mathcal{F}$-divisible by f if there is $a_0\in A$ such that $\forall a\geqslant a_0,\ f\underset{\mu_a}{\mid} g$. We then write $f\underset{\mathcal{F}}{\mid} g$.

Definition 2.12. Let $Q\in K[X]$ be a polynomial and $\mathcal{F}=(\mu_a)_{a\in A}$ be a continuous family of iterated augmented valuations. We say that Q is a *limit key polynomial* for $\mathcal{F}=(\mu_a)_{a\in A}$ if it satisfies the following conditions:

(LM1) Q is *$\mathcal{F}$-irreducible*, *i.e.,*

$$\forall f,g\in K[X], Q\underset{\mathcal{F}}{\mid} fg\implies Q\underset{\mathcal{F}}{\mid} f \text{ or } Q\underset{\mathcal{F}}{\mid} g.$$

(LM2) Q is *$\mathcal{F}$-minimal*, *i.e.,*

$$\forall f\in K[X],\ Q\underset{\mathcal{F}}{\mid} f\implies \deg(Q)\leqslant\deg(f).$$

(LM3) Q is monic.

We can obtain limit key polynomials in a similar way to ordinary key polynomials. We write down

$$d(\mathcal{F}):=\min\{\deg(f);\ \mu_a(f)<\mu_{a'}(f),\ \forall a<a' \text{ in } A\}$$
$$\mathbf{\Phi}(\mathcal{F}):=\{f\in K[X];\ \mu_a(f)<\mu_{a'}(f),\ \forall a<a' \text{ in } A \text{ and } \deg(f)=d(\mathcal{F})\}.$$

Proposition 2.13. Every unitary polynomial $Q\in\mathbf{\Phi}(\mathcal{F})$ is a limit key polynomial for $\mathcal{F}=(\mu_a)_{a\in A}$.

We can furthermore augment the continuous Mac Lane-Vaquié chain in a similar way to the ordinary augmentation process.

Definition 2.14. Consider $\mathcal{F} = (\mu_a)_{a\in A}$ a continuous Mac Lane-Vaquié chain based in μ, Q a limit key polynomial for this chain, of minimal degree (*i.e.,* it is in $\mathbf{\Phi}(\mathcal{F})$). Suppose $\Phi(\mu) \subseteq \Gamma$ is order-preserving embedded in another ordered group Γ and $\gamma \in \Gamma$ such that $\gamma > \mu_a(Q)$, $\forall a \in A$. The limit augmented valuation of a continuous chain with respect to Q and γ is the mapping

$$\mu' : K[X] \to \Gamma_\infty$$

assigning to any $f \in K[X]$, with Q-expansion $f = f_0 + f_1 Q + \ldots + f_n Q^n$, $\deg f_i < \deg Q$ the value

$$\mu'(f) = \min_i \mu_{\mathcal{F}}(g_i) + i\gamma.$$

We denote it by $\mu' = [\mu_{\mathcal{F}}; Q, \gamma]$.

Note that this definition makes sense as any polynomial g with degree $\deg(g) < \deg(Q)$ is $\mathcal{F}$-stable and thus $\mu_{\mathcal{F}}(g)$ is well-defined.

Proposition 2.15. Consider the same notations as in the previous definition.

1. The mapping $\mu' = [\mu_{\mathcal{F}}; Q, \gamma]$ is a valuation such that $\mu_a \leqslant \mu'$, $\forall a \in A$. We have
$$\mu_a(f) = \mu'(f) \iff Q \underset{\mathcal{F}}{\nmid} f.$$
2. $\forall f \in K[X]$, $\deg(f) < \deg(Q)$, $\mathrm{in}_{\mu'} f$ is a unit in $\mathrm{gr}_{\mu'} K[X]$.
3. Q is a key polynomial for μ' of minimal degree.

We thus obtained a valuation μ' such that $d(\mu', \mu^*) > d(\mu, \mu^*)$ [55, Prop. 1.27].

2.2 Abstract Key Polynomials and Truncation

We now wish to present an alternative presentation of the idea of key polynomials that has been initiated by M. Spivakovsky and his collaborators in [23], [21]. Subsequent works include [42], [40], [35] and [18]. This is in fact a dual approach to the Mac Lane-Vaquié theory as we now try and find truncated valuations, instead of augmented ones. One apparent advantage this other strategy has over MVKPs is the fact that key polynomials and limit key polynomials do not need separate treatment. We will detail the link between the two approaches in the third and last section of this chapter.

We will suppose from now on that we want to approximate a valuation μ by truncating it. We will set two polynomials $f, Q \in K[X]$ with $\deg Q > 0$, compute the Q-expansion of f

$$f = f_0 + f_1 Q + \ldots + f_n Q^n, \quad \forall i, \deg f_i < \deg Q$$

and define the mapping

$$\mu_Q(f) = \min_i \mu(f_i Q^i)$$

called the *truncated map of μ, with respect to Q* . The Q-expansion is unique and we denote by $\deg_Q(f)$ the largest i such that $f_i \neq 0$. For $f \in K[X]$ and Q, μ as above, we will write

$$S_{Q,\mu}(f) = \{i \in \mathbb{N};\ \mu(f_i Q^i) = \mu_Q(f)\}$$

or just $S_Q(f)$ for short, when the μ is fixed once and for all. We call this the *Q-support of f*. We also define $d_{Q,\mu}(f) = d_Q(f) := \max S_{Q,\mu}(f)$, the *$Q$-degree of f*.

The truncated map is not always a valuation. It still is a map extending the valuation $\nu = \mu\big|_K$ and it still verifies (V2) and (V3) (we assume that our valuation has trivial support and thus, μ_Q also satisfies (V4)). It may however fail to verify (V1).

Example 2.16. [42, Example 2.5] Suppose we have a valuation μ on $K[X]$ and $a \in K$ such that

$$\nu(a), \mu(X) > 0.$$

If we set $Q = X^2 + 1$, then we can compute $\mu_Q(X^2 - a^2)$:

$$\mu_Q(X^2 - a^2) = \mu_Q((X^2 + 1) - (a^2 + 1)) = \min\{\mu(X^2 + 1), \mu(a^2 + 1)\} = 0,$$

however

$$\mu_Q(X \pm a) = \mu(X \pm a) \geqslant \min\{\mu(X), \mu(a)\} > 0,$$

so

$$\mu_Q((X - a)(X + a)) = 0 < \mu_Q(X - a) + \mu_Q(X + a).$$

The truncated map μ_Q does not verify (V1).

There is a natural condition for which it is a valuation, that is if Q is an abstract key polynomial. In this case we will talk about *truncated valuation in* Q or even more succinctly, a *truncation*. We will need to define the ϵ level or ϵ factor of a polynomial f.

Definition 2.17. For any valuation μ on $K[X]$ and polynomial $f \in K[X]$, we define

$$\epsilon_\mu(f) = \max_{i \geqslant 1} \frac{\mu(f) - \mu(\partial_i f)}{i}.$$

We will call it the ϵ factor of f . For our fixed μ we will simply write $\epsilon(f) = \epsilon_\mu(f)$.

Here ∂_i represents the ith Hasse-Schmidt derivative on $K[X]$. This is an operator that can be defined by means of the Taylor expansion of polynomials in two variables

$$f(X + Y) = \sum_{i \geqslant 0} \partial_i f(X) Y^i.$$

By multiplying together the Taylor expansions of two polynomials $f, g \in K[X]$, we see that these Hasse-Schmidt derivatives satisfy the Leibniz rule

$$\partial_i(f \cdot g) = \sum_{i=j+k} \partial_j f \cdot \partial_k g.$$

Furthermore we can compose Hasse-Schmidt derivatives. By expanding $f(X + Y + Z)$ in two different ways, we can show that

$$\partial_i \circ \partial_j = \binom{i+j}{j} \partial_{i+j}.$$

We now define abstract key polynomials.

Definition 2.18. [18, Definition 11]
Let $Q \in K[X]$ be a monic polynomial. We say that Q is an *abstract key polynomial for* μ(that we abbreviate ABKP) if, for any $f \in K[X]$

$$\deg f < \deg Q \implies \epsilon_\mu(f) < \epsilon_\mu(Q).$$

As basic examples, any degree one polynomials are abstract key polynomials according to this definition. Less obvious examples are the key polynomials given by Mac Lane-Vaquié's key polynomials. See [18, Section 3] or the next section of this chapter. Now let us go through some basic properties the ABKPs verify.

Proposition 2.19. [18, Proposition 13]
Let $P_1, \ldots, P_t \in K[X]$ be polynomials of degree $< \deg Q$ (assume $t \geqslant 2$). If we set the following euclidean division $\prod_{k=1}^t P_k = qQ + r$ in $K[X]$, with $\deg r < \deg Q$, then

$$\mu\left(\prod_{k=1}^{t} P_k\right) = \mu(r) < \mu(qQ).$$

Proof. We prove the statement by induction on t. For $t = 1$, the result is a consequence of the definition of ABKPs (as $\deg P_1$, $r = P_1$ and $q = 0$). We prove it for $t = 2$: we wish to show that

$$\mu(P_1P_2) = \mu(r) < \mu(qQ)$$

where $P_1P_2 = qQ + r$ is the euclidean division of P_1P_2 by Q. We assume on the contrary that $\mu(r), \mu(P_1P_2) \geqslant \mu(qQ)$. Since Q is an ABKP and the polynomials P_1, P_2, q, r are of degree strictly less than $\deg Q$ we have $\forall i \in \mathbb{N}^*$

$$\begin{aligned}
\mu(\partial_i P_1) &> \mu(P_1) - i\epsilon \\
\mu(\partial_i P_2) &> \mu(P_2) - i\epsilon \\
\mu(\partial_i q) &> \mu(q) - i\epsilon \\
\mu(\partial_i r) &> \mu(r) - i\epsilon
\end{aligned}$$

where $\epsilon = \epsilon_\mu(Q)$. We now set

$$b = \min\left\{ i \in \mathbb{N}^*;\ \frac{\mu(Q) - \mu(\partial_i Q)}{i} = \epsilon \right\}.$$

Thus for any $i = 1, \ldots, b$

$$\begin{aligned}
\mu(q\partial_b Q) &= \mu(q) + \mu(\partial_b Q) \\
&= \mu(q) + \mu(Q) - b\epsilon \\
&= \Big(\mu(q) - i\epsilon\Big) + \Big(\mu(Q) - (b-i)\epsilon\Big) \\
&< \mu(\partial_i q) + \mu(\partial_{b-i} Q).
\end{aligned}$$

Hence

$$\mu(\partial_b(qQ)) = \mu\left(\sum_{i=0}^{b} \partial_i q \partial_{b-i} Q\right) = \mu(q\partial_b Q) = \mu(qQ) - b\epsilon. \tag{E}$$

On the other hand

$$\begin{aligned}
\mu(\partial_b(qQ)) &= \mu\Big(\partial_b(P_1P_2) - \partial_b r\Big) \\
&\geqslant \min\{\mu(\partial_b(P_1P_2)), \mu(\partial_b r)\} \\
&\geqslant \min\left\{ \mu\left(\sum_{i=0}^{b} \partial_i P_1 \partial_{b-i} P_2\right), \mu(\partial_b r) \right\} \\
&\geqslant \min_{0 \leqslant i \leqslant b} \{\mu(\partial_i P_1) + \mu(\partial_{b-i} P_2), \mu(\partial_b r)\} \\
&> \min_{0 \leqslant i \leqslant b} \left\{ \Big(\mu(P_1) - i\epsilon\Big) + \Big(\mu(P_2) - (b-i)\epsilon\Big), \mu(r) - b\epsilon \right\} \\
&\geqslant \mu(qQ) - b\epsilon
\end{aligned}$$

which contradicts (E). This proves the case $t = 2$.
We now prove the proposition for $t > 2$, provided we've proven it for $t - 1$. Let $P = \prod_{k=1}^{t-1} P_k$, then we set the euclidean divisions:

$$\begin{aligned}
P &= q_1 Q + r_1 \\
r_1 P_t &= q_2 Q + r.
\end{aligned}$$

Indeed, the euclidean division of $r_1 P_t$ by Q has the same remainder as the euclidean division of $\prod_{k=1}^{t} P_k$ by Q. Furthermore the euclidean division of $\prod_{k=1}^{t} P_k$ by Q is

$$\prod_{k=1}^{t} P_k = qQ + r = (q_1 P_t + q_2)Q + r.$$

By the induction hypothesis we have $\mu(P) = \mu(r_1) < \mu(q_1 Q)$, hence

$$\mu\left(\prod_{k=1}^{t} P_k\right) = \mu(r_1 P_t) < \mu(q_1 P_t Q).$$

By the $t = 2$ case we have $\mu(r_1 P_t) = \mu(r) < \mu(q_2 Q)$, so combining the two, we get

$$\begin{aligned}\mu(qQ) &= \mu(q_1 P_t Q + q_2 Q)\\ &\geqslant \min\{\mu(q_1 P_t Q), \mu(q_2 Q)\}\\ &> \mu(r_1 P_t)\\ &= \mu\left(\prod_{k=1}^{t} P_k\right) = \mu(r).\end{aligned}$$

□

Corollary 2.20. [18, Prop. 15],[42, Prop. 2.6]
If Q is an ABKP, then μ_Q is a valuation. We will call these valuations *truncations*.

Proof. We first show (V3) and (V4). Take any $f \in K[X]$, $f \neq 0$ so that among the components $f_0, f_1, \ldots, f_n$, $\deg f_i < \deg Q$ of its Q-expansion, there are some that are non-zero. Thus there are finite values among $\mu(f_0), \mu(f_1 Q), \ldots, \mu(f_n Q^n)$.
We show that μ_Q verifies (V2). Take $f, g \in K[X]$ and write their Q-expansions

$$\begin{aligned}f &= f_0 + f_1 Q + \ldots + f_n Q^n\\ g &= g_0 + g_1 Q + \ldots + g_n Q^n\end{aligned}$$

so that

$$f + g = (f_0 + g_0) + (f_1 + g_1)Q + \ldots + (f_n + g_n)Q^n$$

is the Q-expansion of $f + g$. Thus

$$\begin{aligned}\mu_Q(f+g) &= \min_i \mu\Big((f_i + g_i)Q^i\Big)\\ &\geqslant \min_i \min\left\{\mu(f_i Q^i), \mu(g_i Q^i)\right\}\\ &\geqslant \min\left\{\min_i \mu(f_i Q^i), \min_i \mu(g_i Q^i)\right\}\\ &= \min\{\mu_Q(f), \mu_Q(g)\}.\end{aligned}$$

We now prove that μ_Q verifies (V1): set $f, g \in K[X]$, our objective being to prove

$$\mu_Q(fg) = \mu_Q(f) + \mu_Q(g). \tag{A}$$

Case 1: If $\deg f, \deg g < \deg Q$, then $\mu_Q(f) = \mu(f)$ and $\mu_Q(g) = \mu(g)$. By setting the euclidean division of fg by Q, $fg = qQ + r$, by the case $t = 2$ of Proposition 2.19 we get $\mu(fg) = \mu(r) < \mu(qQ)$. Thus $\mu_Q(fg) = \mu(fg)$ and (A) holds, since μ verifies (V1).
Case 2: we want to show (A) for $f = aQ^i$, $g = bQ^j$ with $\deg a, \deg b < \deg Q$. We set the euclidean division of ab by Q, $ab = qQ + r$ so that $\deg q, \deg r < \deg Q$ and $\mu(ab) = \mu(r) < \mu(qQ)$. This gives the Q-expansion of fg

$$fg = abQ^{i+j} = qQ^{i+j+1} + rQ^{i+j},$$

so by definition

$$\begin{aligned}\mu_Q(fg) &= \min\{\mu(qQ^{i+j+1}), \mu(rQ^{i+j})\} = \mu(rQ^{i+j}) = \mu(r) + \mu(Q^{i+j})\\ &= \mu(ab) + \mu(Q^{i+j}) = \mu(aQ^i bQ^j) = \mu(aQ^i) + \mu(bQ^j)\\ &= \mu_Q(aQ^i) + \mu_Q(bQ^j).\end{aligned}$$

Case 3: Suppose now that f, g are arbitrary polynomials and write their Q-expansions

$$f = f_0 + f_1 Q + \ldots + f_n Q^n$$
$$g = g_0 + g_1 Q + \ldots + g_m Q^m.$$

By the ultrametric property of μ_Q and Case 2, we have

$$\begin{aligned}\mu_Q(fg) &\geqslant \min_{i,j}\{\mu_Q(f_i g_j Q^{i+j})\} = \min_{i,j}\{\mu_Q(f_i Q^i) + \mu_Q(g_j Q^j)\} = \min_i\{\mu_Q(f_i Q^i)\} + \min_j\{\mu_Q(g_j Q^j)\} \\ &= \mu_Q(f) + \mu_Q(g). \end{aligned} \tag{B1}$$

For every i, j we set the euclidean division of $f_i g_j$ by Q

$$f_i g_j = q_{i,j} Q + r_{i,j}, \quad \deg q_{i,j},\, \deg r_{i,j} < \deg Q.$$

so that, by Proposition 2.19

$$\mu(f_i Q^i) + \mu(g_j Q^j) = \mu(f_i g_j) + \mu(Q^{i+j}) = \mu(r_{i,j} Q^{i+j}) < \mu(q_{i,j} Q^{i+j+1}). \tag{B2}$$

Also set

$$\begin{aligned} i_0 &= \min S_{Q,\mu}(f) := \min\{i;\ \mu_Q(f) = \mu(f_i Q^i)\} \\ j_0 &= \min S_{Q,\mu}(g) := \min\{j;\ \mu_Q(g) = \mu(g_j Q^j)\} \\ k_0 &= i_0 + j_0, \end{aligned}$$

so that for any $i < i_0$ or for any $j < j_0$, by (B2),

$$\begin{aligned} \min\{\mu(q_{i,j} Q^{i+j+1}), \mu(r_{i,j} Q^{i+j})\} &= \mu(r_{i,j} Q^{i+j}) \\ &= \mu(f_i Q^i) + \mu(g_j Q^j) \\ &> \mu(f_{i_0} Q^{i_0}) + \mu(g_{j_0} Q^{j_0}) \\ &= \mu(r_{i_0,j_0} Q^{k_0}). \end{aligned} \tag{B3}$$

If we write the Q-expansion $fg = a_0 + a_1 Q + \ldots + a_p Q^p$, then

$$a_{k_0} = \sum_{i+j+1=k_0} q_{i,j} + \sum_{i+j=k_0} r_{i,j}.$$

For couple $(i, j) \in \mathbb{N}^2$ such that $i + j = k_0$, but $(i, j) \neq (i_0, j_0)$, we either have $i < i_0$ or $j < j_0$, so by (B3)

$$\mu(r_{i,j} Q^{k_0}) > \mu(r_{i_0,j_0} Q^{k_0}).$$

In a similar fashion, for a couple $(i, j) \in \mathbb{N}^2$ such that $i + j + 1 = k_0$, either $i < i_0$ or $j < j_0$, so by (B3) again

$$\mu(q_{i,j} Q^{k_0}) > \mu(r_{i_0,j_0} Q^{k_0}).$$

By the ultrametric property we conclude that

$$\mu(a_{k_0} Q^{k_0}) = \mu(r_{i_0,j_0} Q^{k_0}).$$

Combined with (B2), we obtain

$$\mu(a_{k_0} Q^{k_0}) = \mu(r_{i_0,j_0} Q^{k_0}) = \mu(f_{i_0} Q^{i_0}) + \mu(g_{j_0} Q^{j_0}) = \mu_Q(f) + \mu_Q(g).$$

Therefore

$$\mu_Q(fg) = \min_k\{\mu(a_k Q^k)\} \leqslant \mu(a_{k_0} Q^{k_0}) = \mu_Q(f) + \mu_Q(g)$$

and, considering (B1), the proof is complete. □

We now wish to understand the structure of the graded algebra $\mathrm{gr}_{\mu_Q} K[X]$. We first need the following.

Corollary 2.21. [18, Remark 16] Let $\alpha = \deg Q$ and define

$$G_{<\alpha} = \sum_{\deg f < \deg Q} \mathrm{gr}_\nu(K) \cdot \mathrm{in}_{\mu_Q}(f) \subseteq \mathrm{gr}_{\mu_Q}(K[X])$$

then this $\mathrm{gr}_\nu(K)$-module is stable under multiplication and thus is an algebra.

Proof. We wish to show that if $f, g \in K[X]$, $\deg f, \deg g < \deg Q$ then $\mathrm{in}_{\mu_Q}(f)\mathrm{in}_{\mu_Q}(g) \in G_{<\alpha}$. This is a direct application of the $t = 2$ case of Proposition 2.19: set $fg = qQ + r$ the euclidean division of fg by Q. Since $\deg q, \deg r < \deg Q$ this is the Q-expansion of fg and $\mu(fg) = \mu(r) < \mu(qQ)$. This amounts to saying that

$$\mu_Q(fg - r) = \mu_Q(qQ) = \mu(qQ) > \mu(r) = \mu_Q(r),$$

in other words

$$\mathrm{in}_{\mu_Q}(f)\mathrm{in}_{\mu_Q}(g) = \mathrm{in}_{\mu_Q}(fg) = \mathrm{in}_{\mu_Q}(r) \in G_{<\alpha}.$$

□

Remark 2.22. We will see later (see Proposition 4.13) that the condition $\deg f < \deg Q$ under the sum can be replaced by $\epsilon_\mu(f) < \epsilon(Q)$.

This subring $G_{<\alpha}$ allows to give a simple presentation of the graded algebra $\mathrm{gr}_{\mu_Q} K[X]$.

Proposition 2.23. [18, Remark 16][17, Remarque 2.2.11]
The graded ring of μ_Q on $K[X]$ has a simple polynomial structure. More precisely

$$\mathrm{gr}_{\mu_Q}(K[X]) = G_{<\alpha}\left[\mathrm{in}_{\mu_Q} Q\right]$$

with $\mathrm{in}_{\mu_Q} Q$ being transcendental over $G_{<\alpha}$. We thus have

$$\deg_{\mathrm{gr}_{\mu_Q}}\left(\mathrm{in}_{\mu_Q}(f)\right) = \max S_{Q,\mu}(f).$$

Proof. Let us describe the action of the initial form here. Consider a polynomial $f \in K[X]$ and its Q-expansion

$$f = f_0 + f_1 Q + \ldots + f_n Q^n,\ \forall i,\ \deg f_i < \deg Q$$

so that

$$\mathrm{in}_{\mu_Q}(f) = \sum_{i \in S_Q(f)} \mathrm{in}_{\mu_Q}(f_i)\mathrm{in}_{\mu_Q}(Q)^i$$

thanks to the rules of computation with initial forms. Thus $\forall i$, $\mathrm{in}_{\mu_Q}(f_i) \in G_{<\alpha}$ and $\mathrm{in}_{\mu_Q}(f) \in G_{<\alpha}\left[\mathrm{in}_{\mu_Q} Q\right]$. It remains to prove that indeed, $\mathrm{in}_{\mu_Q} Q$ is transcendental over $G_{<\alpha}$. Suppose on the contrary that there is a non-trivial algebraic relation

$$\sum_{i=0}^{n} A_i \mathrm{in}_{\mu_Q} Q^i = 0 \tag{S}$$

with $A_i = \sum_{j=1}^{N_i} \mathrm{in}_\nu(a_{i,j})\mathrm{in}_{\mu_Q}(f_{i,j})$, $a_{i,j} \in K$, $\deg f_{i,j} < \deg Q$ and some of the A_i are non-zero. Since (S) is a sum of homogeneous terms we can regroup them according to their degree, so we will now assume that (S) has everyone of its terms of same degree. Thus, considering the computing properties of initial forms (see Remark 1.5), we can assume

$$A_i = \mathrm{in}_{\mu_Q} f_i, \quad \deg f_i < \deg Q,$$

so that

$$0 = \sum_{i=0}^{n} A_i \mathrm{in}_{\mu_Q} Q^i = \sum_{i=0}^{n} \mathrm{in}_{\mu_Q} f_i \mathrm{in}_{\mu_Q} Q^i = \mathrm{in}_{\mu_Q}\left(\sum_{i=0}^{n} f_i Q^i\right).$$

Since we have by definition

$$\mu_Q\left(\sum_{i=0}^{n} f_i Q^i\right) = \min_i \mu(f_i Q^i)$$

and considering some of the f_i are non-zero, we arrive at a contradiction. □

Truncation now gives us a valuation. Furthermore, the set of ϵ levels associated to the truncation has a maximal finite element. We state the main result concerning these numerical properties, but we will delay its proof to Section 4.4 as we would have by then introduced the tools to prove it in a fairly straightforward way.

Proposition 2.24. [18, Lemma 17] For any polynomial $f \in K[X]$

$$\epsilon_{\mu_Q}(f) \leqslant \epsilon_{\mu_Q}(Q) = \epsilon_{\mu}(Q).$$

Furthermore we can even state the case of equality:

$$\epsilon_{\mu_Q}(f) = \epsilon_{\mu_Q}(Q) \iff S_{Q,\mu}(f) \neq \{0\}.$$

2.3 Links between the two strategies

We present in this section the achievements of [18] and [41], which establish the links between the two theories of key polynomials.

2.3.1 From (non-limit) MVKPs to ABKPs

We first show that any key polynomial in the sense of Mac Lane-Vaquié can be seen as an ABKP for any augmented valuation.

Proposition 2.25. [18, Theorem 27]
Fix a valuation μ on $K[X]$ and let μ' be a valuation on $K[X]$ and Q a polynomial such that
1. $\forall f \in K[X], \deg f < \deg Q \implies \mu(f) = \mu'(f)$.
2. $\mu'(Q) > \mu(Q)$

Then Q is an ABKP for μ'.

For instance, a valuation satisfying the above properties is the augmented valuation $\mu^* = [\mu; Q, \gamma]$, for some $\gamma > \mu(Q)$ with Q a MVKP for μ.

2.3.2 From ABKPs to (non-limit) MVKPs

We wish to obtain non-limit key polynomials in the sense of Mac Lane-Vaquié starting with ABKPs. We first define successor polynomials

Definition 2.26. Let Q and Q^* be two ABKPs for a given valuation μ, such that $\epsilon_\mu(Q) < \epsilon_\mu(Q^*)$. We say that Q^* is an *immediate successor* of Q, and we write $Q < Q^*$, if

$$\deg(Q^*) = \min\{\deg(P);\ P \text{ is an ABKP for } \mu \text{ and } \epsilon_\mu(Q) < \epsilon_\mu(P)\}.$$

This property is actually characterised by the following proposition

Proposition 2.27. Let Q, Q^* be two ABKPs for μ. The following are equivalent:
- Q^* immediate successor for Q.
- $\mu_Q(Q^*) < \mu(Q^*)$ and $\deg Q^*$ is minimal with respect to this property, in other words

$$Q^* \in \mathbf{\Phi}(\mu_Q, \mu).$$

Theorem 2.28. [18, Theorem 26],[41, Theorem 6.1]
Let Q be an ABKP for μ and $Q^* \in \mathbf{\Phi}(\mu_Q, \mu)$, in other words, $Q < Q^*$. Then Q and Q^* are MVKPs for μ_Q and

$$\mu_{Q^*} = [\mu_Q; Q^*, \mu(Q^*)].$$

2.3.3 Limit MVKPs from ABKPs

In [42, Prop. 2.12], the authors give a characterisation of ABKPs and classify them into two categories.

Theorem 2.29. Take a valuative pair $(K[X], \mu)$, extending a valuative pair (K, ν) and fix $Q \in K[X]$. Then Q is an ABKP for μ if there is an ABKP Q_- for which, either

1. $Q \in \mathbf{\Phi}(\mu_{Q_-}, \mu)$, or
2. the following are verified:
   - (K1) $\deg Q_- = d(\mu_{Q_-}, \mu)$.
   - (K2) $\Lambda(\mu_{Q_-}, \mu)$ does not have a maximal element.
   - (K3) $\mu_{Q^*}(Q) < \mu(Q)$ for every $Q^* \in \mathbf{\Phi}(\mu_{Q_-}, \mu)$.
   - (K4) Q has the smallest degree among the polynomials verifying (K3).

An ABKP verifying (K1)-(K4) is called a *limit (abstract) key polynomial* based at Q_-.

Remark 2.30. Let us set $\Lambda(\mu_{Q_-}, \mu) = \{\gamma_a; a \in A\}$ such that $\gamma_a < \gamma_{a'}$ for any $a < a'$ in A and $Q_a \in \mathbf{\Phi}(\mu_{Q_-}, \mu)$, such that $\mu(Q_a) = \gamma_a$. Then (K3) and (K4) amount to saying that $Q \in \mathbf{\Phi}(A)$.

In [41], the link between MVKPs and ABKPs has been extended to limit key polynomials.

Theorem 2.31. [41, Theorem 6.2] Assume Q is a limit abstract key polynomial for μ, based at Q_-. Then

$$\mathcal{F} = (\mu_{Q^*})_{Q^* \in \mathbf{\Phi}(\mu_{Q_-}, \mu)}$$

is a continuous family of iterated augmented valuations ordered by the values $\epsilon_\mu(Q^*)$. Q is a limit key polynomial for $\mathcal{F}$ in the Mac Lane-Vaquié sense and

$$\mu_Q = [\mu_{\mathcal{F}}; Q, \mu(Q)].$$

Chapter 3

Passing to the algebraic closure

In this chapter, we wish to build valuations over $K[X]$ by restricting one defined on $\overline{K}[X]$. We start with $(K(X), \mu)$ and extend the valuation to $(\overline{K}(X), \overline{\mu})$. This can be represented in a simple diagram

$$
\begin{tikzcd}
(\overline{K}(X), \overline{\mu}) \arrow[dd, dash] \arrow[dr, dash] & \\
& (K(X), \mu) \arrow[dd, dash] \\
(\overline{K}, \overline{\nu}) \arrow[dr, dash] & \\
& (K, \nu)
\end{tikzcd}
\tag{A}
$$

Remark 3.1. One can show that $\mathrm{Aut}_K(L) \cong \mathrm{Aut}_{K(X)}(L(X))$ for any algebraic field extension L/K. Indeed the isomorphism is given by simply taking a $K(X)$-automorphism φ of $L(X)$ and restricting it to L.

$$
\begin{tikzcd}
& L(X) \arrow[rr, "\varphi"] \arrow[dr, dash] & & L(X) \\
L \arrow[ur, hook] \arrow[dr, dash] & & K(X) \arrow[ur, dash] & \\
& K \arrow[ur, hook] & &
\end{tikzcd}
$$

In order to see that this mapping is well-defined, it is enough to show that if $\varphi \in \mathrm{Aut}_{K(X)}(L(X))$ and $\alpha \in L$ then $\varphi(\alpha) \in L$. We know that α is algebraic over K, thus $\varphi(\alpha)$ is algebraic as well: there is a $P \in K[X]$ such that $P(\alpha) = 0$. If $\varphi(\alpha) \in L(X) \setminus L$, then $\varphi(\alpha)$ is transcendental over L and thus, transcendental over K, which is a contradiction. To see how to extend an element $\sigma \in \mathrm{Aut}_K(L)$ to an element $\varphi \in \mathrm{Aut}_{K(X)}(L(X))$, take any $f = \sum_k a_k X^k \in L[X]$ and set

$$\varphi(f) = \sum_k \sigma(a_k) X^k$$

and this can then be extended to $L(X)$ by setting $\varphi(f/g) = \varphi(f)/\varphi(g)$. This defines the inverse mapping to the one defined above. From now on we will identify elements from the two groups.

If the extension L/K is not algebraic, then the result fails. Indeed consider $L = K(t)$ with t a transcendental element over K, hence $L(X) = K(t, X)$. A $K(X)$-endomorphism φ of $L(X)$ is uniquely defined once we assign $\varphi(t) \in L(X)$. We here define $\varphi(t) = tX \in L(X) \setminus L$, giving us an automorphism whose inverse simply sends t to $\frac{t}{X}$. Thus $\varphi(L) \nsubseteq L$.

Lemma 3.2. For any situation where we extend ν to $\overline{\nu}$ over $\overline{K}$ and ν to μ over $K[X]$, there is a common extension $\overline{\mu}$ of both μ and $\overline{\nu}$.

Proof. We begin by taking any extension μ' of μ to $\overline{K}(X)$. Since $\overline{K}/K$ is normal, $\mathrm{Aut}_K(\overline{K})$ acts transitively on the valuations of $\overline{K}$ extending ν, so that there is $\sigma \in \mathrm{Aut}_{K(X)}(\overline{K}(X))$ such that $\mu' \circ \sigma\big|_{\overline{K}} = \overline{\nu}$. □

In this chapter we will introduce in the first section Newton polygons, that will elucidate the way in which the roots of a polynomial behave. This will extend a result by Novacoski [41, Prop. 3.1] that proves to be paramount for us.
In the second part of this chapter we study minimal pairs and more specifically minimal pairs of definition. They will serve to characterise valuation transcendental extensions $\overline{\mu}/\overline{\nu}$ and prepare the groundwork for Chapter 4.

3.1 On Newton polygons

In this section we will present a result concerning the values $\overline{\mu}(X-\lambda)$ where λ runs through the roots of a polynomial f. We will call this data the root configuration of f. We will use Newton polygons. The first slope of the polygon will be the ϵ factor. It is our hope that the whole slope data will lead us to a better understanding of the diskoids decomposition and the action of the absolute Galois group of K on them.

3.1.1 Defining our Newton polygon

Classical Newton polygons take a field F equipped with an ultrametric absolute value $|.|$ (or equivalently a valuation of rank 1, so that one can suppose $V(F^\times) \subseteq \mathbb{R}$). If however we want to work with fields of higher ranks, one needs to work out what convex sets are in $\mathbb{R} \times \Phi$ for any ordered abelian group Φ. Our presentation here takes many components from Vaquié's own work in [57]. We will consider the following groups $\mathbb{Q}\Phi = \Phi_{\mathbb{Q}} := \Phi \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\mathbb{R}\Phi = \Phi \otimes_{\mathbb{Z}} \mathbb{R}$. Since Φ has no torsion, the canonical maps

$$\begin{tikzcd} \Phi \arrow[r] & \Phi_{\mathbb{Q}} \arrow[d, hook] & \gamma \arrow[r, mapsto] & \gamma \otimes 1 \\ \Phi \arrow[r] & \Phi_{\mathbb{R}} & \gamma \arrow[r, mapsto] & \gamma \otimes 1 \end{tikzcd}$$

are all injective, thus we can consider Φ as a subgroup of $\Phi_{\mathbb{Q}}$ or $\Phi_{\mathbb{R}}$.
We define a *line* to be a subset $L \subseteq \mathbb{R} \times \Phi_{\mathbb{R}}$ defined by a linear or affine equation

$$L = L_{q,\alpha,\beta} = \{(x,\gamma) \in \mathbb{R} \times \Phi_{\mathbb{R}};\ q\gamma + \alpha x + \beta = 0\}$$

for some fixed values $q \in \mathbb{R}_{\geqslant 0},\ \alpha, \beta \in \Phi_{\mathbb{R}}$. The slope of L is given by $s(L) = s(L_{q,\alpha,\beta}) = \alpha/q$ whenever $q \neq 0$. This definition of slope is not classical and corresponds to the negative of the natural slope of a line.
There will always be a single line passing through two fixed and distinct points P_1, P_2, that we will denote (P_1, P_2). Any line defines two *half-spaces*

$$H^L_{\geq} = \{(x,\gamma) \in \mathbb{R} \times \Phi_{\mathbb{R}};\ q\gamma + \alpha x + \beta \geq 0\}$$
$$H^L_{\leq} = \{(x,\gamma) \in \mathbb{R} \times \Phi_{\mathbb{R}};\ q\gamma + \alpha x + \beta \leq 0\}.$$

They are respectively the *half-upper space* and *half-lower space*. For any subset $A \subseteq \mathbb{R} \times \Phi_{\mathbb{R}}$, we define its *convex hull* by

$$\mathrm{Conv}(A) = \bigcap_{\substack{H \text{ half-space} \\ A \subseteq H}} H$$

i.e., the intersection of half-spaces containing A. A set A is considered to be convex if $\mathrm{Conv}(A) = A$. For any set A, we define its *faces* to be subsets $F = \mathrm{Conv}(A) \cap L$ where L is a line in $\mathbb{R} \times \Phi_{\mathbb{R}}$, satisfying

- $\mathrm{Conv}(A)$ is contained in one of the half-spaces $H^L_{\geq}$ or $H^L_{\leq}$.
- $F = \mathrm{Conv}(A) \cap L$ contains at least two points.

We will say that L *supports the face* F. The slope $s(F)$ of a face F of A will simply be $s(L)$ where L supports F. Usually a Newton polygon is constructed for finite sets $X = \{(k, \gamma_k),\, 0 \leqslant k \leqslant m\}$. We will write its *Newton polygon* as

$$PN(X) = \mathrm{Conv}(\{(x,\delta);\ \exists (x,\gamma) \in X,\ \delta \geq \gamma\}) = \mathrm{Conv}\left((\{0\} \times \Phi_{\geq 0}) + X\right)$$

where $\Phi_{\geq 0} = \{\gamma \in \Phi_{\mathbb{R}};\ \gamma \geq 0\}$ and $A + B = \{a + b;\ a \in A,\ b \in B\}$ is the Minkowski sum of two subsets of $\mathbb{R} \times \Phi_{\mathbb{R}}$. The bottom boundary of $PN(X)$ is then a finite polygonal line, thus describing it will be equivalent to giving the following

- a finite sequence of non-negative integers $0 = a_0 < a_1 < \ldots < a_r = m$ (the abscissa or x-coordinates of the vertices of the polygonal line),
- a finite set of values in $\Phi_{\mathbb{R}}$: $\epsilon_1 > \ldots > \epsilon_r$ (the slopes of the segments forming the polygonal line)

verifying $\forall k, t,\ 0 \leqslant k \leqslant m, 1 \leqslant t \leqslant r$ one has

$$\gamma_k + k\epsilon_t \geqslant \gamma_{a_{t-1}} + a_{t-1}\epsilon_t = \gamma_{a_t} + a_t\epsilon_t. \tag{P}$$

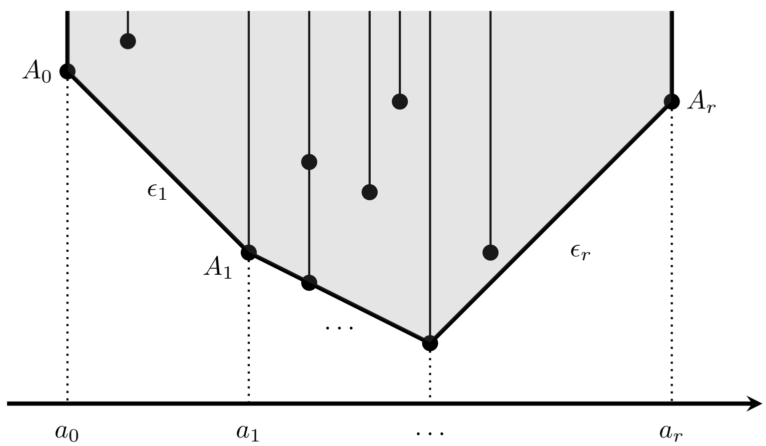

Figure 3.1: Example of a Newton polygon of a finite subset of $\mathbb{R} \times \mathbb{R}$. We have indicated by ϵ_k, the slopes of the bottom segments of the polygon and a_k the abscissa of the extremities of these segments.

Remark that if $k < a_{t-1}$ then

$$\begin{aligned}(\gamma_k + k\epsilon_t) - (\gamma_{a_{t-1}} + a_{t-1}\epsilon_{t-1}) &\geqslant (\gamma_k + k\epsilon_t) - (\gamma_k + k\epsilon_{t-1})\\ &= k(\epsilon_t - \epsilon_{t-1})\\ &> a_{t-1}(\epsilon_t - \epsilon_{t-1})\\ &= (\gamma_{a_{t-1}} + a_{t-1}\epsilon_t) - (\gamma_{a_{t-1}} + a_{t-1}\epsilon_{t-1})\end{aligned}$$

hence $\gamma_k + k\epsilon_t > \gamma_{a_{t-1}} + a_{t-1}\epsilon_t$ and a similar argument shows that if $a_t < k$, then $\gamma_k + k\epsilon_t > \gamma_{a_t} + a_t\epsilon_t$. This allows for the following interpretations: the points $A_t = (a_t, \gamma_{a_t})$ are the *vertices* of the polygon and the *faces* are simply the segments $[A_{t-1}, A_t]$ of slope ϵ_t, as by property (P) all other points (k, γ_k) lie above the line (A_{t-1}, A_t) and the A_t are points where the polygonal line of $PN(X)$ turns. We will also call $a_t - a_{t-1}$ the *length* of this face.

Let us now fix a valuation μ over $K[X]$ with values in Φ. For any $f \in K[X]$ define

$$PN(f,\mu) = PN(X(f,\mu)) \text{ where } X(f,\mu) = \{(i, \mu(\partial_i f));\ i = 0, \ldots, \deg(f)\}.$$

Definition 3.3. For any valuation μ on $K[X]$ and polynomial $f \in K[X]$, we define the *slope data* f, as the sequence of vertices of $PN(f, \mu)$
$$\big((a_1, \epsilon_1), \ldots, (a_r, \epsilon_r)\big)$$
as defined above.

In the next subsection, we relate the finite data given by our version of the Newton polygon of f to some information given by the configuration of roots of f.

3.1.2 Root configurations

In order to define what we mean by configuration of roots, we need to extend our initial valuation μ to $\overline{K}[X]$ where $\overline{K}$ is the algebraic closure of K. Decompose then f into linear factors
$$f = \alpha \prod_{i=1}^{m} (X - \lambda_i).$$
What will be of interest to us will be the values $\overline{\mu}(X - \lambda_i)$ and how many times one such value is repeated. We can fix an indexation of the a_i so that we have the following
$$\begin{aligned} \overline{\mu}(X - \lambda_1) = \ldots = \overline{\mu}(X - \lambda_{l_1}) &> \overline{\mu}(X - \lambda_{l_1+1}) = \ldots = \overline{\mu}(X - \lambda_{l_1+l_2}) \\ &\vdots \\ &> \overline{\mu}(X - \lambda_{l_1+\ldots+l_{s-1}+1}) = \ldots = \overline{\mu}(X - \lambda_{l_1+\ldots+l_s}). \end{aligned}$$
We will write $\overline{\mu}(X - \lambda_{l_1+\ldots+l_t}) = \delta_t$.

Definition 3.4. Choose $\overline{\mu}$ and f as above. Adopting the previous notations, we define the *root configuration of f* as the finite sequence
$$\big((l_1, \delta_1), \ldots, (l_s, \delta_s)\big).$$
Our main result in this section says that for any polynomial f, its root configuration and slope data are equivalent.

Theorem 3.5. *Fix $\overline{\mu}$ a valuation over $\overline{K}[X]$ and $f \in K[X]$. Then the root configuration of f, $\big((l_t, \delta_t), t = 1, \ldots, s\big)$ is encoded in its slope data $\big((a_t, \epsilon_t), t = 1, \ldots, r\big)$ as follows:*
$$\begin{aligned} s &= r \\ a_t - a_{t-1} &= l_t,\ t = 1, \ldots, r \\ \epsilon_t &= \delta_t,\ t = 1, \ldots, r \end{aligned}$$
where we have fixed $a_0 = 0$.

Remark 3.6. One can be surprised by the fact that even though the definition of the δ_t needs us to choose an extension $\overline{\mu}$ of μ to $\overline{K}$, their value is ultimately independent of this choice.

To prove Theorem 3.5 one can use the following classic lemma concerning Newton polygons.

Lemma 3.7. *Let (F, V) be any valued field and $p(T) \in F[T]$ any polynomial, whose roots are in F. We write*
$$p(T) = c \prod_{k=1}^{N} (T - c_k) = \sum_{l=0}^{N} b_l T^l \quad c, c_k, b_l \in F$$
and consider the points $\{(l, V(b_l)), l = 0, \ldots, N\}$. If ζ is a slope of its Newton polygon of length ℓ, it follows that precisely ℓ of the roots c_k have value ζ.

A proof of Lemma 3.7 can be found in [28, Ch. IV, § 3] in the rank 1 case, but it also applies for V of arbitrary rank as well.

Proof. First observe that the Newton polygon of $T^k p(T)$ is the Newton polygon of $p(T)$ translated horizontally by k units. Since this is equivalent to adding a root 0 with multiplicity k, we can assume from now on that $p(T)$ has no zero roots, so that $b_0 \neq 0$. We will write

$$\zeta_1 > \ldots > \zeta_r$$

for the different values of the roots and

$$\ell_1, \ldots, \ell_r$$

the multiplicities of these values among the roots: the value ζ_i is taken by exactly ℓ_i roots. We also write

$$m_j = \sum_{i=1}^{j} \ell_i,\ 1 \leqslant j \leqslant r$$

and set $m_0 = 0$. Up to re-ordering, we can also assume that the roots of value ζ_1 are $c_1, \ldots, c_{m_1}$, the roots of value ζ_2 are $c_{m_1+1}, \ldots, c_{m_2}$ and son on and so forth. We thus need to show that the slopes (and abscissa) of $p(T)$'s Newton polygon are the ζ_i (and m_i resp.). Given how they are characterised (inequality (P) in Section 3.1.1), we need to show that

$$\forall l = 0, \ldots, N,\ \forall j = 1, \ldots, r, \quad V(b_l) + l\zeta_j \geqslant V(b_{m_{j-1}}) + m_{j-1}\zeta_j = V(b_{m_j}) + m_j\zeta_j.$$

This is equivalent to showing that for any l and j as above,

$$V(b_l) - V(b_{m_{j-1}}) \geqslant (m_{j-1} - l)\zeta_j$$

and that we have equality for $l = m_j$. We will use the root-coefficient relations. We will fix the notation $[d] = \{1, \ldots, d\}$ for any $d \in \mathbb{N}$, so that $[0] = \emptyset$. We clearly have

$$b_l = \sum_{\substack{I \subseteq [N] \\ \#I = l}} \prod_{k \in [N] \setminus I} c_k.$$

Fix i such that $m_i \leqslant l < m_{i+1}$, so that by the ultrametric property

$$\begin{aligned} b_l &\geqslant \min_{\substack{I \subseteq [N] \\ \#I = l}} \sum_{k \in [N] \setminus I} V(c_k) \\ &= V(b_0) - \sum_{k=1}^{l} V(c_k) \\ &= V(b_0) - \Big(l_1\zeta_1 + \ldots + l_i\zeta_i + (l - m_i)\zeta_{i+1}\Big). \end{aligned} \tag{I}$$

Notice how the above inequality is actually an equality as soon as $l = m_i$. Indeed, in this case, among the products $\prod_{k \in [N] \setminus I} c_k$, the product $\prod_{k \in [N-l]} c_k$ is of least value and any other product is of strictly greater value. Thus

$$V(b_l) - V(b_{m_{j-1}}) \geqslant V(b_0) - \Big(l_1\zeta_1 + \ldots + l_i\zeta_i + (l - m_i)\zeta_{i+1}\Big) - V(b_0) + \Big(l_1\zeta_1 + \ldots + l_{j-1}\zeta_{j-1} + \Big).$$

If $i \geqslant j - 1$

$$\begin{aligned} V(b_l) - V(b_{m_{j-1}}) &\geqslant -l_j\zeta_j - \ldots - l_i\zeta_i - (l - m_i)\zeta_{i+1} \\ &\geqslant -l_j\zeta_j - \ldots - l_i\zeta_j - (l - m_i)\zeta_j \\ &= (m_i - l_i - \ldots - l_j - l)\zeta_j \\ &= (m_{j-1} - l)\zeta_j \end{aligned}$$

and similarly, if $i < j-1$

$$\begin{aligned} V(b_l) - V(b_{m_{j-1}}) &\geqslant -(l-m_i)\zeta_{i+1} + l_{i+1}\zeta_{i+1} + \ldots + l_{j-1}\zeta_{j-1} \\ &= (m_{i+1}-l)\zeta_{i+1} + l_{i+2}\zeta_{i+2} + \ldots + l_{j-1}\zeta_{j-1} \\ &> (m_{i+1}-l)\zeta_j + l_{i+2}\zeta_j + \ldots + l_{j-1}\zeta_j \\ &= (m_{i+1} + l_{i+2} + \ldots + l_{j-1} - l)\zeta_j \\ &= (m_{j-1}-l)\zeta_j. \end{aligned}$$

Furthermore, we clearly have equality as soon as $l = m_j$. Thus we conclude our proof. □

We can apply this lemma to prove Theorem 3.5, by setting $(F, V) = (\overline{K}(X), \overline{\mu})$ and

$$p(T) = f(X+T) = \sum_{i=0}^{m} \partial_i f(X) T^i \in \overline{K}(X)[T].$$

Indeed, the roots of $p(T)$, when considered as a polynomial of coefficients in $\overline{K}(X)$ are $\lambda_i - X, i = 1, \ldots, m$ [1].

3.1.3 The δ invariant

In [40], another quantity is defined in parallel to the ϵ factor.

Definition 3.8. For any valuation $\overline{\mu}$ of $\overline{K}[X]$ and any polynomial $f \in \overline{K}[X]$, we define its δ factor as follows

$$\delta_{\overline{\mu}}(f) := \max\left\{\overline{\mu}(X-b);\ b \text{ root of } f\right\}.$$

We will often abbreviate this by $\delta(f)$.

Considering the root configuration $\big((l_t, \delta_t), t = 1, \ldots, r\big)$ of f, we obviously have by definition

$$\delta(f) = \delta_1.$$

On the other hand, if $\big((l_t, \delta_t), t = 1, \ldots, r\big)$ is the slope data of f, then we can write down

$$\epsilon(f) = \epsilon_1.$$

Theorem 3.5 establishes among other things that $\epsilon_1 = \delta_1$, so we obtain the following theorem.

Theorem 3.9. [40, Prop. 3.1] For any polynomial $f \in K[X]$

$$\delta(f) = \epsilon(f).$$

Remark 3.10. We should observe that even though the choice of $\overline{\mu}$ is arbitrary, if $f \in K[X]$ then $\delta_{\overline{\mu}}(f)$ will only depend on μ.

3.2 Minimal Pairs

If we wish to construct a valuation on $\overline{K}[X]$, we can consider $a \in \overline{K}$, $\delta \in \Phi$, where Φ is a group containing $\Phi(\overline{\nu})$, and define

$$\overline{\nu}_{a,\delta}\left(\sum_i a_i (X-a)^i\right) = \min_i\{\overline{\nu}(a_i) + i\delta\}. \tag{B}$$

We say that (a, δ) is a *pair of definition* for a valuation $\overline{\mu}$ if $\overline{\mu} = \overline{\nu}_{a,\delta}$.

Remark 3.11. 1. One observes that $\forall f \in \overline{K}[X],\ a_i = \partial_i f(a)$.

[1] This short proof was suggested by Franz-Viktor Kuhlmann in a private communication.

2. If $\delta = \overline{\mu}(X-a)$ then $\overline{\nu}_{a,\delta}$ is a truncation of $\overline{\mu}$: $\overline{\nu}_{a,\delta} = \overline{\mu}_{X-a}$, so the map defined in (B) is indeed a valuation.

Several pairs of definition can yield the same valuation. We characterise these situations in the lemma below.

Lemma 3.12. Two pairs, (a,δ) and (a',δ') define the same valuation if and only if the following two conditions hold
1. $\delta = \delta'$
2. $\overline{\nu}(a-a') \geqslant \delta$

There is a proof of this fact in the case of residually transcendental extensions in [6], but we give a proof in the general case.

Proof. Consider two pairs $(a,\delta),(a',\delta')$ that define the same valuation. Then we have by definition

$$\delta' = \overline{\nu}_{a',\delta'}(X-a') = \overline{\nu}_{a,\delta}(X-a') = \min\{\delta, \overline{\nu}(a-a')\}$$

and by a symmetric argument we obtain

$$\delta = \min\{\delta', \overline{\nu}(a-a')\}$$

thus $\delta' = \delta$ and $\overline{\nu}(a-a') \geqslant \delta$.
Conversely, consider pairs that verify the two conditions of the lemma and let us show that the valuations they define are equal. It is clear that they agree on $\overline{K}$ and it is sufficient to show they agree on polynomials of type $X-b$, $b \in \overline{K}$. Indeed, each polynomial factors as products of such simple degree 1 polynomials, since $\overline{K}$ is algebraically closed. We have

$$\overline{\nu}_{a,\delta}(X-b) = \min\{\delta, \overline{\nu}(a-b)\}$$
$$\overline{\nu}_{a',\delta}(X-b) = \min\{\delta', \overline{\nu}(a'-b)\}.$$

If $\overline{\nu}(a-b) \geqslant \delta$ then $\overline{\nu}_{a,\delta}(X-b) = \delta$, but $\overline{\nu}(a'-b) = \overline{\nu}(a'-a+a-b) \geqslant \min\{\overline{\nu}(a'-a), \overline{\nu}(a-b)\} \geqslant \delta$ so that

$$\overline{\nu}_{a',\delta}(X-b) = \delta = \overline{\nu}_{a,\delta}(X-b).$$

If $\overline{\nu}(a-b) < \delta$ then $\overline{\nu}(a'-b) = \overline{\nu}(a'-a+a-b) = \overline{\nu}(a-b)$ thus

$$\overline{\nu}_{a,\delta}(X-b) = \overline{\nu}(a-b) = \overline{\nu}(a'-b) = \overline{\nu}_{a',\delta}(X-b).$$

□

One can choose among the elements of $\{a' \in \overline{K};\, \overline{\nu}(a'-a) \geqslant \delta\}$ one such that $\deg_K(a')$ is minimal. This leads us to the natural definition of minimal pairs of definition.

Definition 3.13. We will say that (a,δ) is a *minimal pair of definition for a valuation* $\overline{\mu}$, if $\overline{\mu} = \overline{\nu}_{a,\delta}$, so that (a,δ) is a pair of definition for $\overline{\mu}$ and $\deg_K(a)$ is minimal among the pairs that define it.

If we are given an arbitrary $\overline{\mu}$ one can try to approximate it by setting $\delta = \overline{\mu}(X-a)$ so that $\overline{\nu}_{(a,\delta)} \leqslant \overline{\mu}$. The definition of minimal pair consists of a couple formed by an element a and the value $\delta = \overline{\mu}(X-a)$. However, several elements a may give the same δ. We impose furthermore that the degree of a over K is minimal. We have the precise definition below.

Definition 3.14. We say that (a,δ) is a *minimal pair* for $\overline{\mu}$ when
1. $\overline{\mu}(X-a) = \delta$.
2. for any $b \in \overline{K}$, $\deg_K(b) < \deg_K(a) \implies \overline{\nu}(a-b) < \delta$.

Remark 3.15. 1. Condition 2 of Definition 3.14 above is also equivalent to the contrapositive:

$$2'. \quad b \in \overline{K},\ \overline{\nu}(a-b) \geqslant \delta \implies \deg_K(b) \geqslant \deg_K(a).$$

2. A minimal pair of definition characterizes the valuation we are studying and allows for a direct way of computing it, however a minimal pair does not necessarily characterize $\overline{\mu}$: a valuation $\overline{\mu}$ may have a minimal pair (a, δ), however it may not be a minimal pair of definition for $\overline{\mu}$ as $\overline{\nu}_{a,\delta}$ may be different from $\overline{\mu}$. Nevertheless, (a, δ) is a minimal pair of definition for $\overline{\nu}_{a,\delta}$.

Valuation-transcendental extensions are characterized in the theorem below.

Theorem 3.16. [6, Proposition 1, 2 and 3] [30, Theorem 3.11]
The following are equivalent
1. μ/ν is a valuation-transcendental extension.
2. $\overline{\mu}/\overline{\nu}$ is a valuation-transcendental extension.
3. $\exists a \in \overline{K}, \delta \in \Phi(\overline{\mu})$ such that $\overline{\mu} = \overline{\nu}_{a,\delta}$.
4. $\overline{\mu}(X - \overline{K}) := \{\overline{\mu}(X - b),\ b \in \overline{K}\}$ has a maximal element.

Proof. The equivalence between 1 and 2 follows from Proposition 1.21.

Let us prove that 3 implies 2: assume that $\overline{\mu} = \overline{\nu}_{a,\delta}$ for some $a \in \overline{K}$ and $\delta \in \Phi(\overline{\mu})$.
If $\delta \in \Phi(\overline{\mu}) \setminus \Phi(\overline{\nu})$, then it will be torsion free, since $\Phi(\overline{\nu})$ is divisible. The extension $\overline{\mu}/\overline{\nu}$ is thus VT.
Assume now $\delta \in \Phi(\overline{\nu})$ and take $c \in \overline{K}$ such that $\overline{\nu}(c) = \delta$. Then x, the residual image of $\frac{X-a}{c}$ in $\kappa(\overline{\mu})$ is transcendental over $\kappa(\overline{\nu})$: suppose there is a polynomial $f = \sum_k \alpha_k T^k \in \kappa(\overline{\nu})[T]$ such that $f(x) = 0$. If we set $a_k \in \overline{K}$ such that $a_k \mod \overline{\nu} = \alpha_k$, then we have that

$$\overline{\mu}\left(\sum_{k=0}^{n} a_k \left(\frac{X-a}{c}\right)^k\right) = \overline{\mu}\left(\sum_k \frac{a_k}{c^k}(X-a)^k\right) > 0.$$

However, since $\overline{\mu} = \overline{\nu}_{a,\delta}$, we thus get

$$\forall k = 0, \dots, n,\ \overline{\nu}\left(\frac{a_k}{c^k}\right) + k\delta = \overline{\nu}(a_k) > 0.$$

We conclude that $\alpha_k = 0$, so $f = 0$ and x is transcendental over $\kappa(\overline{\nu})$.

Let us now show how 2 implies 3.
If the extension is VT then there is an element $\overline{\mu}(f) \in \Phi(\overline{\mu})$, torsion free over $\Phi(\overline{\nu})$. We can decompose f into linear factors:

$$f = c\prod_{i=1}^{n}(X-a), \quad c, a_i \in \overline{K},$$

thus

$$\overline{\mu}(f) = \overline{\nu}(c) + \sum_{i=1}^{n} \overline{\mu}(X - a_i)$$

so we can conclude that one of the values $\overline{\mu}(X - a_i),\ i = 1, \dots, n$ is torsion free over $\Phi(\overline{\nu})$. Suppose that $\delta = \overline{\mu}(X - a)$ is such a value, so that for any $k < l \in \mathbb{N}$ and any $u, v \in \overline{K}$ we have

$$\overline{\mu}(u(X-a)^k) \neq \overline{\mu}(v(X-a)^l)$$

so that, for any $f \in \overline{K}[X] = \overline{K}[X-a],\ f = \sum_{k=0}^{n} u_k(X-a)^k$ we have

$$\overline{\mu}(f) = \min_{0 \leqslant k \leqslant n}\{\overline{\mu}(u_k(X-a)^k)\} = \min_{0 \leqslant k \leqslant n}\{\overline{\nu}(u_k) + k\delta\}$$

thus $\overline{\mu} = \overline{\nu}_{a,\delta}$.
Suppose now that $\overline{\mu}/\overline{\nu}$ is an RT extension, so that by Abhyankar's inequality $\mathrm{rat.rk}(\Phi(\overline{\mu})/\Phi(\overline{\nu})) = 0$. Since $\Phi(\overline{\nu})$ is divisible and has no torsion, we automatically have $\Phi(\overline{\mu}) = \Phi(\overline{\nu})$. Consider now f/g the lift of a residually transcendental element of $\kappa(\overline{\mu})$, where $f, g \in \overline{K}[X]$. We can then factor each of them:

$$f = \alpha\prod_{i=1}^{m}(X - a_i)$$
$$g = \beta\prod_{j=1}^{n}(X - b_j).$$

We chose for each $i = 1, \dots, m$ and $j = 1, \dots, n$ elements c_i, $d_j \in \overline{K}$ verifying:

$$\overline{\nu}(c_i) = \overline{\mu}(X - a_i)$$
$$\overline{\nu}(d_j) = \overline{\mu}(X - b_j).$$

We can replace f/g with:

$$\frac{\prod_{i=1}^{m}(X - a_i)/c_i}{\prod_{j=1}^{n}(X - b_j)/d_j} \tag{F}$$

as the transcendental element we are considering. Since all values of the factors $\frac{X-a_i}{c_i}$, $\frac{X-b_j}{d_j}$ are equal to 0, the above factorisation (F) can be brought into $\kappa(\overline{\mu})$, so there is one of these factors that is residually transcendental over $\kappa(\overline{\nu})$, let's write it $\frac{X-a}{c}$. Set $\delta = \overline{\mu}(X-a) = \overline{\nu}(c)$. Take f and write it as $(X-a)$-expansion

$$f = \sum_{i \geqslant 0} a_i (X-a)^i, \; a_i \in \overline{K}. \tag{G}$$

We divide by some $d \in \overline{K}$ so that $\forall i$, $\overline{\nu}(a_i c^i/d) \geqslant 0$ and $\overline{\nu}(a_i c^i/d) = 0$ for some i. Thus the residual expression of

$$\frac{f}{d} = \sum_i \frac{a_i c^i}{d} \left(\frac{X-a}{c}\right)^i$$

transforms into a non-trivial polynomial expression of the residual image of $\frac{X-a}{c}$. This expression is not zero, *i.e.*, $\overline{\mu}(f/d) = 0$, so that

$$\overline{\mu}(f) = \overline{\nu}(d) = \min_i \overline{\nu}(a_i c^i) = \min_i \overline{\nu}(a_i) + i\delta = \overline{\nu}_{a,\delta}(f).$$

However, since any polynomial in $\overline{K}[X]$ can be written as in (G), one simply has

$$\overline{\mu} = \overline{\nu}_{a,\delta}.$$

We now prove that 3 implies 4: if $\overline{\mu} = \overline{\nu}_{a,\delta}$ where $\delta \in \Phi(\overline{\nu})$, then clearly $\Phi(\overline{\mu}) \subseteq \Phi(\overline{\nu})$, and for any $b \in \overline{K}$:

$$\overline{\mu}(X-b) = \overline{\nu}_{a,\delta}(X-a+a-b) = \min\{\delta, \overline{\nu}(a-b)\} \leqslant \delta = \overline{\mu}(X-a).$$

Finally we show that 4 implies 3: suppose now that $\overline{\mu}(X - \overline{K})$ is bounded and contains its upper bound δ. By hypothesis we have $a \in \overline{K}$ such that

$$\delta = \overline{\mu}(X-a) = \max \overline{\mu}(X - \overline{K}).$$

$\overline{\mu}$ and $\overline{\nu}_{a,\delta}$ are both extensions of $\overline{\nu}$, so they coincide over $\overline{K}$. It is thus sufficient to show that $\overline{\mu}$ and $\overline{\nu}_{a,\delta}$ coincide over polynomials of form $X - b, b \in \overline{K}$, since any polynomial is a product of such factors and scalars. We have

$$\overline{\nu}_{a,\delta}(X-b) = \min\{\delta, \overline{\nu}(a-b)\}.$$

However, by hypothesis, we have $\overline{\mu}(X-b) \leqslant \delta$. If $\overline{\mu}(X-b) = \delta$, then $\overline{\nu}(a-b) \geqslant \delta$, so that $\overline{\nu}_{a,\delta}(X-b) = \delta$. If on the other hand $\overline{\mu}(X-b) < \delta$, then $\overline{\nu}(a-b) = \overline{\mu}(X-b-(X-a)) = \overline{\mu}(X-b)$, by the ultrametric property. In both cases we've shown that $\overline{\mu}(X-b) = \overline{\nu}_{a,\delta}(X-b)$. □

Remark 3.17.
1. Item 3 of Theorem 3.16 will be generalised to Proposition 4.12, once we have established Theorem 4.3.
2. In the foundational literature (*e.g.*, [6, 7, 9]), minimal pairs of definition are associated to RT extensions, nevertheless they characterise all valuation transcendental valuations.

The following corollary follows immediately from the above proof.

Corollary 3.18. If one of the equivalent conditions of Theorem 3.16 is verified, and $\overline{\mu} = \overline{\nu}_{a,\delta}$, then

$$\max \overline{\mu}(X - \overline{K}) = \delta.$$

Furthermore, μ/ν is RT if and only if $\delta \in \Phi(\overline{\nu})$, otherwise it is VT.

We have the following converse result.

Proposition 3.19. Assume $\overline{\mu}(X - \overline{K})$ has a maximal element δ. Then we have seen that $\overline{\mu}/\overline{\nu}$ is valuation transcendental, more precisely for any $a \in \overline{K}$ such that $\delta = \overline{\mu}(X - a)$ we have $\overline{\mu} = \overline{\nu}_{a,\delta}$. Furthermore,

1. if $\delta \in \Phi(\overline{\nu})$, then for any $b \in \overline{K}$ such that $\overline{\nu}(b) = \delta$, the residue class of $\frac{X-a}{b}$ is transcendental over $\kappa(\overline{\nu})$ and $\overline{\mu}/\overline{\nu}$ is RT.
2. if $\delta \notin \Phi(\overline{\nu})$ then the extension is VT as δ is torsion free over $\Phi(\overline{\nu})$.

Proof. We need only show that for any $b \in \overline{K}$, $\overline{\mu}(X - b) = \overline{\nu}_{a,\delta}(X - b)$ as all polynomials are products of scalars (elements of $\overline{K}$) and monic linear factors (polynomials of the form $X - b$, $b \in \overline{K}$). We have the following (in)equalities, which stem from the basic properties and definitions:

$$\overline{\mu}(X - b) = \overline{\mu}(X - a + (a - b)) \geqslant \min\{\overline{\nu}(a - b), \delta\}$$
$$\overline{\nu}_{a,\delta}(X - b) = \overline{\nu}_{a,\delta}(X - a + (a - b)) = \min\{\overline{\nu}(a - b), \delta\}.$$

If $\overline{\nu}(a - b) < \delta$ then by the ultrametric property

$$\overline{\mu}(X - b) = \overline{\mu}(a - b) = \overline{\nu}(X - b),$$

and $\overline{\nu}_{a,\delta}(X - b) = \overline{\nu}(X - b)$.
If $\overline{\nu}(a - b) \geqslant \delta$ then, since δ is the maximal element of $\overline{\mu}(X - \overline{K})$, we get

$$\delta \geqslant \overline{\mu}(X - b) \geqslant \min\{\overline{\nu}(a - b), \delta\} = \delta,$$

hence it is an equality. Furthermore $\overline{\nu}_{a,\delta}(X - b) = \delta$, so in both cases, we have $\overline{\mu}(X - b) = \overline{\nu}_{a,\delta}(X - b)$. $\square$

Pairs of definition that are minimal enjoy certain specific properties, that allow to state the value of certain polynomials plainly.

Lemma 3.20. For any $f \in K[X]$, and (a, δ) a minimal pair for $\overline{\mu}$, such that $\deg f < \deg_K(a)$, then $\delta_{\overline{\mu}}(f) < \delta$.

Proof. Consider any root of f, $b \in \overline{K}$. Then $\deg_K(b) \leqslant \deg f < \deg_K(a)$, thus by the definition of minimal pairs, we have

$$\overline{\mu}(a - b) = \overline{\nu}(a - b) < \delta.$$

Now suppose that b is such that $\delta_{\overline{\mu}}(f) = \overline{\mu}(X - b)$. By the ultrametric inequality

$$\overline{\mu}(X - b) = \overline{\mu}(X - a + (a - b)) = \overline{\mu}(a - b) < \delta.$$

$\square$

When given a minimal pair (a, δ), one can compute $\mu(f)$ for polynomials such that $\deg f < \deg_K(a)$. This can be stated more precisely in the lemma below.

Lemma 3.21. [7, Theorem 2.1(a)] Fix (a, δ) a minimal pair for $\overline{\mu}$. For any $f \in \overline{K}[X]$ with $\delta_{\overline{\mu}}(f) < \delta$ we have

$$\mathrm{in}_{\overline{\nu}_{a,\delta}}(f) = \mathrm{in}_{\overline{\nu}_{a,\delta}}(f(a)).$$

Furthermore $\overline{\nu}_{a,\delta}(f) = \overline{\mu}(f)$.

We give a presentation of the proof of Lemma 3.21 that differs from that of [7], as it shall serve the proof of Theorem 4.3.

Proof. Consider the roots of f and write it as

$$f = c\prod_{i=1}^{d}(X - c_i), \text{ with } c, c_i \in \overline{K}.$$

By hypothesis

$$\forall i,\ \overline{\mu}(X - c_i) \leqslant \delta(f) < \delta = \overline{\mu}(X - a).$$

By the ultrametric property

$$\begin{aligned}\overline{\nu}(a-c_i) &= \overline{\mu}((X-c_i)-(X-a))\\ &= \overline{\mu}(X-c_i)\\ &< \overline{\mu}(X-a),\end{aligned}$$

so we naturally have

$$\mathrm{in}_{\overline{\nu}_{a,\delta}}(X-c_i) = \mathrm{in}_{\overline{\nu}_{a,\delta}}(X-a+a-c_i) = \mathrm{in}_{\overline{\nu}_{a,\delta}}(a-c_i),$$

thus, by multiplicativity of $\mathrm{in}_{\overline{\nu}_{a,\delta}}$

$$\begin{aligned}\mathrm{in}_{\overline{\nu}_{a,\delta}}(f) &= \mathrm{in}_{\overline{\nu}_{a,\delta}}(c)\prod_{i=1}^{d}\mathrm{in}_{\overline{\nu}_{a,\delta}}(X-c_i)\\ &= \mathrm{in}_{\overline{\nu}_{a,\delta}}(c)\prod_{i=1}^{d}\mathrm{in}_{\overline{\nu}_{a,\delta}}(a-c_i)\\ &= \mathrm{in}_{\overline{\nu}_{a,\delta}}\left(c\prod_{i=1}^{d}a-c_i\right)\\ &= \mathrm{in}_{\overline{\nu}_{a,\delta}}(f(a)).\end{aligned}$$

Furthermore we can deduce that

$$\begin{aligned}\overline{\nu}_{a,\delta}(X-c_i) &= \min\left\{\overline{\mu}(X-a), \overline{\nu}(a-c_i)\right\}\\ &= \overline{\nu}(a-c_i)\\ &= \overline{\mu}(X-c_i).\end{aligned}$$

$\square$

Chapter 4

Extending truncations

In this chapter we wish to relate the extensions of valuations given by minimal pairs and those given by truncation by ABKPs. One such stride has been made in the work of Novacoski. We cite his result in the following theorem.

Theorem 4.1. [40, Proposition 3.2] Let $a \in \overline{K}$ be a root of an irreducible polynomial $Q \in K[X]$ verifying $\delta = \delta(Q) = \overline{\mu}(X - a)$. Then

$$Q \text{ is an ABKP for } \mu \iff (a, \delta) \text{ is a minimal pair for } \overline{\mu}.$$

Proof. Assume that Q is an ABKP for μ. Consider $b \in \overline{K}$ such that $\overline{\nu}(a - b) \geqslant \delta = \overline{\mu}(X - a)$. This implies that $\overline{\mu}(X - b) \geqslant \delta$. If we Take f the minimal polynomial of b over K, then, by Theorem 3.9

$$\epsilon(f) = \delta(f) \geqslant \delta = \delta(Q) = \epsilon(Q).$$

Since Q is an ABKP we get $\deg(f) \geqslant \deg(Q)$.
Assume now that (a, δ) is a minimal pair for $\overline{\mu}$. We want to show that if $f \in K[X]$, $\deg(f) < \deg(Q)$, then $\epsilon_\mu(f) < \epsilon_\mu(Q)$. By our assumption and by Theorem 3.9

$$\epsilon_\mu(f) = \delta(f) < \delta = \delta(Q) = \epsilon_\mu(Q).$$

□

Definition 4.2. For any polynomial $f \in \overline{K}[X]$ we shall call $a \in \overline{K}$ an *optimizing root of f according to $\overline{\mu}$*, if a is a root of f and $\overline{\mu}(X - a) = \delta(f)$.

We will prove that the truncated valuation μ_Q, with Q an ABKP, comes as a restriction to $K[X]$ of truncated valuations on $\overline{K}[X]$, *i.e.,* defined via a minimal pair.

Theorem 4.3. Take $Q \in K[X]$, let $a \in \overline{K}$ be an optimizing root of Q and write $\delta = \overline{\mu}(X - a) = \delta(Q)$. Then Q is an ABKP, if and only if (a, δ) is a minimal pair. In this case $\overline{\nu}_{a,\delta}$ is an extension of μ_Q

$$\overline{\nu}_{a,\delta}\big|_{K[X]} = \mu_Q$$

thus inducing a natural injective map of graded algebras

$$\theta \ : \ \mathrm{gr}_{\mu_Q}(K[X]) \hookrightarrow \mathrm{gr}_{\overline{\nu}_{a,\delta}}(\overline{K}[X]).$$

It sends $\mathrm{in}_{\mu_Q}(f)$ to $\mathrm{in}_{\overline{\nu}_{a,\delta}}(f)$, for any $f \in K[X]$.

This completes the correspondence between the situation over K and $\overline{K}$. This chapter is dedicated to the proof of this theorem and three direct applications of this descent type result. In Chapter 5 we will see how we can apply this result to bring forth a geometric interpretation for RT extensions.

4.1 Proof of Theorem 4.3

4.1.1 Step 1: building θ

Throughout this subsection and the following one, we assume Q is an ABKP for μ and a an optimising root of Q, according to an extension $\overline{\mu}$ of μ over to $\overline{K}[X]$. We set $\delta = \overline{\mu}(X-a)$. We first show the following lemma.

Lemma 4.4. $\mu(Q) = \mu_Q(Q) = \overline{\nu}_{a,\delta}(Q)$ so that $\theta\left(\mathrm{in}_{\mu_Q} Q\right) = \mathrm{in}_{\overline{\nu}_{a,\delta}} Q$.

Proof. First we have by Theorem 3.5 that $\delta = \epsilon_\mu(Q) = \max_{i\geqslant 1} \frac{\mu(Q)-\mu(\partial_i Q)}{i}$, thus

$$\mu(Q) = \min_{i\geqslant 1}\{\mu(\partial_i Q) + i\delta\}.$$

Secondly $0 \notin S_{X-a}(Q)$ (because $Q(a) = 0$ so $\overline{\nu}(Q(a)) = \infty$), thus

$$\overline{\mu}_{X-a}(Q) = \min_{i\geqslant 1}\left\{\overline{\nu}(\partial_i Q(a)) + i\delta\right\}.$$

Thirdly, for any $i \geqslant 1$, $\deg \partial_i Q < \deg Q = \deg_K(a)$, so that $\delta_{\overline{\mu}}(\partial_i Q) < \delta$ by Lemma 3.20, hence by Lemma 3.21

$$\mu(\partial_i Q) = \overline{\nu}(\partial_i Q(a)).$$

Putting it all together

$$\mu(Q) = \min_{i\geqslant 1}\{\mu(\partial_i Q) + i\delta\} = \min_{i\geqslant 1}\{\overline{\nu}(\partial_i Q(a)) + i\delta\} = \overline{\mu}_{X-a}(Q).$$

$\square$

This allows us to show that $\overline{\nu}_{a,\delta}$ is greater than μ_Q.

Lemma 4.5.

$$\overline{\nu}_{a,\delta}\big|_{K[X]} \geqslant \mu_Q.$$

Proof. Fix $f \in K[X]$ and write its Q-standard decomposition

$$f = \sum_i f_i Q^i,$$

thus by ultrametric inequality we have

$$\overline{\nu}_{a,\delta}(f) \geqslant \min_i \overline{\nu}_{a,\delta}(f_i Q^i).$$

Now by Lemma 3.21 $\forall i$, $\overline{\nu}_{a,\delta}(f_i) = \overline{\mu}(f_i) = \mu(f_i)$ and by Lemma 4.4 $\overline{\nu}_{a,\delta}(Q) = \overline{\mu}(Q) = \mu(Q)$. Thus

$$\overline{\nu}_{a,\delta}(f) \geqslant \min_i \overline{\nu}_{a,\delta}(f_i Q^i) = \min_i \mu(f_i Q^i) = \mu_Q(f).$$

$\square$

We build our morphism θ, based on the previous inequality.

Corollary 4.6. Lemma 4.5 induces a map

$$\mathrm{gr}_{\mu_Q}(K[X]) \xrightarrow{\theta} \mathrm{gr}_{\overline{\nu}_{a,\delta}}(\overline{K}[X])$$

This is a simple consequence of Theorem 1.8 and Lemma 4.5. Its kernel is

$$\mathrm{Ker}(\theta) = \left\langle \mathrm{in}_{\mu_Q}(I) \right\rangle, \text{ where } I = \{f \in K[X];\, \overline{\nu}_{a,\delta}(f) > \mu_Q(f)\}.$$

Proving Theorem 4.3 amounts to showing that θ is injective.

4.1.2 Step 2: Proving that θ is injective

The main ingredient here is the structure of the graded algebra. According to Section 2.2 these have a polynomial presentation

$$\mathrm{gr}_{\mu_Q}(K[X]) = G_{<\alpha}[T]$$

where $T = \mathrm{in}_{\mu_Q}(Q)$, $\alpha = \deg Q$ and

$$G_{<\alpha} = \sum_{\deg f < \deg Q} \mathrm{gr}_\nu(K)\mathrm{in}_{\mu_Q}(f).$$

We can make the same statement for $\overline{\nu}_{a,\delta}$

$$\mathrm{gr}_{\overline{\nu}_{a,\delta}}(\overline{K}[X]) = (\mathrm{gr}_{\overline{\nu}}(\overline{K}))[\overline{T}]$$

where $\overline{T} = \mathrm{in}_{\overline{\nu}_{a,\delta}}(X-a)$. Indeed, by definition $G_{<\deg X - a}$ is generated by initial forms of polynomials of degree $< 1 = \deg(X-a)$, *i.e.,* coefficients in $\overline{K}$. This remark turns out to be crucial in many ways. One can use Lemma 3.21 to prove the following proposition (which can be seen as a reformulation of Lemma 3.21).

Proposition 4.7. The map θ restricted to $G_{<\alpha}$ induces an injective map between $G_{<\alpha}$ and $\mathrm{gr}_{\overline{\nu}}(\overline{K}) \subseteq \mathrm{gr}_{\overline{\nu}}(\overline{K})[\overline{T}]$.

The θ map will take $\mathrm{in}_{\mu_Q}(f)$ with $\deg(f) < \deg(Q)$, and send it to $\mathrm{in}_{\overline{\nu}}(f(a))$. We can display our different mappings in the following commutative diagram

$$\begin{tikzcd}
G_{<\alpha} \arrow[r, hook, "\theta"] \arrow[d, hook] & \mathrm{gr}_{\overline{\nu}}(\overline{K}) \arrow[d, hook] \\
\mathrm{gr}_{\mu_Q}(K[X]) \arrow[r, "\theta"] & \mathrm{gr}_{\overline{\nu}_{a,\delta}}(\overline{K}[X])
\end{tikzcd}$$

Again by Lemma 3.21, we have that $\theta(\mathrm{in}_{\mu_Q} Q) = \mathrm{in}_{\overline{\nu}_{a,\delta}} Q \neq 0$, because $\overline{\nu}_{a,\delta}(Q) = \mu(Q) = \mu_Q(Q)$. The initial form $\mathrm{in}_{\overline{\nu}_{a,\delta}}$ is multiplicative, so $X-a$ dividing Q in $\overline{K}[X]$ implies that $\mathrm{in}_{\overline{\nu}_{a,\delta}}(X-a)$ divides $\mathrm{in}_{\overline{\nu}_{a,\delta}}(Q)$, so that $d_{X-a,\overline{\mu}}(Q) > 0$. Hence $\mathrm{in}_{\overline{\nu}_{a,\delta}}(Q) \notin \mathrm{gr}_{\overline{\nu}}(\overline{K})$.
We will conclude by using the following basic result on polynomial rings.

Lemma 4.8. Let $\phi : R \longrightarrow S$ be an injective integral domain map, that we extend to a map $\tilde{\phi} : R[X] \longrightarrow S[Y]$ which assigns to X a non-constant polynomial $p(Y) \in S[Y] \setminus S$. Then $\tilde{\phi}$ is again injective.

Proof. Consider a non-zero polynomial $q(X) \in R[X]$. If $q \in R$ then by assumption $\tilde{\phi}(q) = \phi(q) \neq 0$. Otherwise suppose $\deg q(X) \geqslant 1$. Since all the rings we are considering are integral domains, we get $\deg \tilde{\phi}(q(X)) = \deg q \deg p \geqslant 0 \implies \tilde{\phi}(q(X)) \neq 1$. $\square$

Thus θ is injective. Furthermore, taking into account its construction, θ is homogeneous. This concludes our proof.

4.1.3 Alternative

We adopt all the notations of the previous sections. We wish to present in this last subsection another way of proving a slightly weaker version of Theorem 4.3.

Proposition 4.9. Consider a valuation μ and Q an ABKP for μ. Then there is a root $a \in \overline{K}$ of Q such that

$$\overline{\nu}_{a,\delta(Q)}\big|_{K[X]} = \mu_Q.$$

This result is slightly weaker as we don't know if the root a is an optimising root of Q or not. Our approach here is purely qualitative and relies on an astute use of the invariants defined for valuation-transcendental valuations. We start by giving a simple result, a natural consequence of our discussion around Theorem 3.5. This first lemma is already present in [58] (for instance, see Proposition 3.1 and Theorem 3.4) and is interesting for its own merit.

Lemma 4.10. Define $E(\mu) = \{\epsilon_\mu(f);\ f \in K[X]\}$. Then μ/ν is valuation-transcendental if and only if $\overline{\mu}/\overline{\nu}$ is valuation-transcendental. This is also equivalent to both $E(\mu)$ and $\overline{\mu}(X - \overline{K})$ having a maximal element, in which case

$$\max E(\mu) = \max \overline{\mu}(X - \overline{K}).$$

Proof. We only need to show that $E(\mu)$ has a maximal element if and only if $\overline{\mu}(X - \overline{K})$ has a maximal element and that they are equal if they exist. The other statements have already been proven in Theorem 3.16. So let us assume that $\overline{\mu} = \overline{\nu}_{a,\delta}$, for a minimal pair of definition (a, δ) and write $Q \in K[X]$ the minimal polynomial of a over K. We already know that Q is an ABKP according to Theorem 4.1 and we already know that

$$\overline{\mu}(X - a) = \delta = \max \overline{\mu}(X - \overline{K})$$

by Theorem 3.16 and Proposition 3.19. We now show that $E(\mu)$ has a maximal element and that this element is δ. By Theorem 3.5 we simply need to show that $\{\overline{\mu}(X - b);\, b \text{ root of a polynomial } f \in K[X]\}$ has a maximal element, but we know that $\forall f \in K[X]$ and for any b root of f we already have $\overline{\mu}(X - b) \leqslant \delta = \overline{\mu}(X - a)$. Furthermore we have $\epsilon_\mu(Q) = \overline{\mu}(X - a) = \delta \in E(\mu)$. Thus $\max E(\mu) = \delta$.
Conversely suppose that μ/ν is valuation algebraic. We wish to show that $E(\mu)$ does not have a maximal element. By Theorem 3.16 we know that $\overline{\mu}(X - \overline{K})$ does not have a maximal element. Take any $f \in K[X]$ and fix any $a \in \overline{K}$ such that $\delta(f) = \overline{\mu}(X - a)$. Then, by assumption, we can find $b \in \overline{K}$ such that $\overline{\mu}(X - b) > \overline{\mu}(X - a)$, so that, if g is the minimal polynomial of b over K, then $\epsilon_\mu(g) \geqslant \overline{\mu}(x - b) > \epsilon_\mu(f)$. ☐

We now prove our proposition. We extend μ_Q to $\overline{\mu}'$ over $\overline{K}$. By Proposition 2.24 we know that $E(\mu_Q)$ has a maximal element, namely $\epsilon_{\mu_Q}(Q) = \epsilon_\mu(Q)$. Thus μ_Q/ν is valuation-transcendental and so is $\overline{\mu}'/\overline{\nu}$. Furthermore, we know that

$$\overline{\mu}' = \overline{\nu}_{b,\delta}, \quad \delta = \max \overline{\mu}'(X - \overline{K})$$

and b can be any element of $\overline{K}$ such that $\overline{\mu}'(X - b) = \delta$. However, we know that $\delta = \max E(\mu_Q) = \epsilon_{\mu_Q}(Q)$. By Theorem 3.5 we know there is a root a of Q (an optimising root for $\overline{\mu}'$) such that $\overline{\mu}'(X - a) = \epsilon_{\mu_Q}(Q)$. Thus by Proposition 3.19

$$\overline{\mu}' = \overline{\nu}_{a,\epsilon\mu(Q)}.$$

Remark 4.11. We have just shown that the root a of Q can be chosen to be an optimising root for $\overline{\mu}'$, but not necessarily $\overline{\mu}$. Also, we have used Proposition 2.24 that we have not proved yet, since we wish to show it with the help of our Theorem 4.3. It can however be proven without the help of our descent result.

4.2 Application to valuation transcendental extensions

We start by refining Theorem 3.16. It is equivalent to [54, Prop. 1.4] insofar as it is done in the theory of MVKPs.

Proposition 4.12. [40, Theorem 1.3] Our extension $(K[X], \mu)/(K, \nu)$ is valuation transcendental if and only if there is a polynomial Q such that $\mu = \mu_Q$. Furthermore Q can be taken to be an ABKP. The extension μ/ν will be RT if and only if $\epsilon_\mu(Q) \in \mathbb{Q} \otimes \Phi(\nu) = \Phi(\overline{\nu})$.

Proof. Choose an extension $\overline{\mu}$ of μ to $\overline{K}[X]$, which restricts to $\overline{\nu}$ over $\overline{K}$ ($\overline{\nu}$ is thus an extension of ν). By Theorem 3.16, our extension is valuation transcendental if and only if we are given a pair of definition $(a, \delta) \in \overline{K} \times \Phi_{\overline{\mu}}$ for $\overline{\mu}$

$$\overline{\mu} = \overline{\nu}_{a,\delta}.$$

We can suppose that this pair is minimal, simply by choosing a of minimal degree over K, among the pairs of definition of $\overline{\mu}$. Thus by our Theorem 4.3 the minimal polynomial of Q of a is an ABKP and

$$\mu = \overline{\mu}\big|_{K[X]} = \overline{\nu}_{a,\delta}\big|_{K[X]} = \overline{\nu}_{a,\delta}\big|_{K[X]} = \mu_Q.$$

By Proposition 3.19 the extension $(K[X], \mu)/(K, \nu)$ is residually transcendental if and only if $\epsilon_\mu(Q) = \delta(Q) = \max \overline{\mu}(X - \overline{K})$ is in $\Phi(\overline{\nu})$. ☐

4.3 Application to the graded algebra structure

Now recall Remark 2.22. We can now extend the condition under the sum, defining $G_{<\alpha}$ in the proposition below.

Proposition 4.13. Take $f \in K[X]$ with $\epsilon(f) < \epsilon(Q)$, where Q is an ABKP for a valuation μ over $K[X]$. There exists $g \in K[X]$ such that $\deg g < \deg Q$ and $\mathrm{in}_\mu(f) = \mathrm{in}_\mu(g)$. Thus one can re-write the definition of $G_{<\alpha}$ where $\alpha = \deg Q$:

$$G_{<\alpha} = \sum_{\epsilon(f)<\epsilon(Q)} \mathrm{gr}_\nu(K)\mathrm{in}_{\mu_Q}(f).$$

Proof. We already have

$$G_{<\alpha} \subseteq \sum_{\epsilon(f)<\epsilon(Q)} \mathrm{gr}_\nu(K)\mathrm{in}_{\mu_Q}(f)$$

since $\deg f < \deg Q$ implies $\epsilon(f) < \epsilon(Q)$ by definition of ABKPs. Consider f with $\epsilon(f) < \epsilon(Q)$ and its Q-expansion

$$f = \sum_i f_i Q^i,\ \deg f_i < \deg Q.$$

Consider an optimising root of Q, $a \in \overline{K}$, thus providing us with a minimal pair (a, δ), $\delta = \delta_{\overline{\mu}}(Q) = \epsilon_\mu(Q)$. We thus have the following equalities

$$\begin{aligned}
\theta\left(\mathrm{in}_{\mu_Q}(f)\right) &= \mathrm{in}_{\overline{\nu}_{a,\delta}}(f)\\
&\overset{(*)}{=} \mathrm{in}_{\overline{\nu}_{a,\delta}}(f(a))\\
&= \mathrm{in}_{\overline{\nu}_{a,\delta}}(f_0(a))\\
&\overset{(**)}{=} \mathrm{in}_{\overline{\nu}_{a,\delta}}(f_0)\\
&= \theta\left(\mathrm{in}_{\mu_Q}(f_0)\right).
\end{aligned}$$

$(*)$ is a direct application of Lemma 3.21 and so is $(**)$, since having $\deg f_0 < \deg Q$, implies $\epsilon(f_0) < \epsilon(Q)$. Thus, since θ is injective, $\mathrm{in}_{\mu_Q}(f) = \mathrm{in}_{\mu_Q}(f_0) \in G_{<\alpha}$. □

Remark 4.14. 1. This result should then be compared to [32, Theorem 5.2].

2. It should be noted that a similar result, concerning the detailed structure of the graded algebra is established in [54, Prop. 2.3].

4.4 A numerical result

We now prove Proposition 2.24, that for any extension μ of a valuation ν from K to $K[x]$, any ABKP Q for μ and any polynomial$f \in K[X]$, we have $\epsilon_{\mu_Q}(f) \leqslant \epsilon_{\mu_Q}(Q) = \epsilon_\mu(Q)$. From the proof of Proposition 4.13, one can extract the fact that $S_Q(f) = \{0\}$ whenever $\epsilon(f) < \epsilon(Q)$. This is proven in [18, Prop. 17, Prop. 18] and the authors of the article also announce the proof of its converse in an upcoming article. We now give a proof of these facts and also prove the converse of [18, Prop. 18].

Theorem 4.15. For any polynomial $f \in K[X]$ and any ABKP Q for a valuation μ over $K[X]$ we have

$$\epsilon_{\mu_Q}(f) \leqslant \epsilon_{\mu_Q}(Q) = \epsilon_\mu(Q).$$

Furthermore we have equality, in other words there is a $b \geqslant 1$ such that $\frac{\mu_Q(f)-\mu_Q(\partial_b f)}{b} = \epsilon_\mu(Q)$, if and only if $S_{Q,\mu}(f) \neq \{0\}$:

$$S_{Q,\mu}(f) \neq \{0\} \iff \epsilon_{\mu_Q}(f) = \epsilon_{\mu_Q}(Q) = \epsilon_\mu(Q).$$

Proof. We will fix an extension $\overline{\mu}$ of μ to $\overline{K}[X]$ and take an optimising root a of Q. By Theorem 4.3 we know that $\overline{\nu}_{a,\delta}$ is an extension of μ_Q and by Theorem 3.5 we can write

$$\epsilon_{\mu_Q}(f) = \delta_{\overline{\nu}_{a,\delta}}(f) = \max\{\overline{\nu}_{a,\delta}(X-c);\ c \text{ root of } f\}$$

and $\epsilon_\mu(Q) = \epsilon_{\mu_Q}(Q) = \overline{\mu}(X-a)$. Thus we simply need to show that for any $c \in \overline{K}$ $\overline{\nu}_{a,\delta}(X-c) \leqslant \overline{\mu}(X-a)$, which is straightforward

$$\overline{\nu}_{a,\delta}(X-c) = \min\{\overline{\mu}(X-a), \overline{\nu}(a-c)\} \leqslant \overline{\mu}(X-a).$$

Let us henceforth write $\epsilon_\mu(Q) = \epsilon$. Since $\overline{\nu}_{a,\delta}$ is an extension of μ_Q, one can write

$$\epsilon_{\mu_Q}(f) = \epsilon_{\overline{\nu}_{a,\delta}}(f).$$

Let us abbreviate the following initial forms

$$\begin{aligned} T &= \mathrm{in}_{\mu_Q}(Q) \\ \overline{T} &= \mathrm{in}_{\overline{\nu}_{a,\delta}}(X-a). \end{aligned}$$

We can now express the following equivalences

$$\begin{aligned} S_{Q,\mu}(f) \neq \{0\} &\iff d_Q(f) = \deg_T \mathrm{in}_{\mu_Q}(f) \geqslant 1 \\ &\iff d_{X-a}(f) = \deg_{\overline{T}} \mathrm{in}_{\overline{\nu}_{a,\delta}}(f) \geqslant 1 \\ &\iff S_{X-a,\overline{\mu}}(f) \neq \{0\}. \end{aligned}$$

Indeed, recall the properties of the map θ in the proof of Theorem 4.3. It sends initial forms of polynomials f with $\deg_T \mathrm{in}_{\mu_Q} f = 0$ to an initial forms with $\deg_{\overline{T}} \mathrm{in}_{\overline{\nu}_{a,\delta}} f = 0$ and initial forms of polynomials f with $\deg_T \mathrm{in}_{\mu_Q} \geqslant 1$, to initial forms with $\deg_{\overline{T}} \mathrm{in}_{\overline{\nu}_{a,\delta}} f \geqslant 1$.
Thus one can simply work in $\overline{K}[X]$. Let $f \in \overline{K}[X]$ be such that $S_{X-a,\overline{\mu}} \neq \{0\}$ so $d = d_{X-a,\overline{\mu}}(f) \geqslant 1$, *i.e.*,

$$\overline{\nu}_{a,\delta}(f) = \overline{\mu}(\partial_d f(a)(X-a)^d).$$

By definition of truncation, we have the inequality

$$\overline{\nu}_{a,\delta}(\partial_d f) = \min_{k \geqslant 0}\{\overline{\nu}(\partial_k(\partial_d f)(a)) + k\delta\} \leqslant \overline{\nu}(\partial_d f(a))$$

which implies

$$\overline{\nu}_{a,\delta}(f) = \overline{\mu}(\partial_d f(a)) + d\epsilon \geqslant \overline{\nu}_{a,\delta}(\partial_d f) + d\epsilon.$$

Thus we have

$$\epsilon = \epsilon_{\mu_Q}(Q) \overset{(*)}{\geqslant} \epsilon_{\mu_Q}(f) = \epsilon_{\overline{\nu}_{a,\delta}}(f) \geqslant \frac{\overline{\nu}_{a,\delta}(f) - \overline{\nu}_{a,\delta}(\partial_d f)}{d} \geqslant \epsilon$$

so everything is an equality (the inequality $(*)$ is just due to the first part of the theorem, the inequality part).
Alternatively, one can start with $\epsilon_{\mu_Q}(f) = \epsilon_{\overline{\nu}_{a,\delta}}(f) < \epsilon$, so that

$$\max_{i \geqslant 1} \frac{\overline{\nu}_{a,\delta}(f) - \overline{\nu}_{a,\delta}(\partial_i f)}{i} < \epsilon;$$

in other words,

$$\forall i \geqslant 1,\ \overline{\nu}_{a,\delta}(f) < \overline{\nu}_{a,\delta}(\partial_i f) + i\epsilon \leqslant \overline{\nu}(\partial_i f(a)) + i\epsilon$$

hence $S_{X-a,\overline{\mu}}(f) = \{0\}$.

Let us now show the converse. We suppose, ad absurdum, that we have a polynomial $f \in \overline{K}[X]$ verifying the two following statements:

1. $\epsilon_{\overline{\nu}_{a,\delta}}(f) = \epsilon$
2. $S_{X-a,\overline{\mu}}(f) = \{0\}$.

The first statement says that
$$\max_{j\geqslant 1}\frac{\overline{\nu}_{a,\delta}(f)-\overline{\nu}_{a,\delta}(\partial_j f)}{j}=\epsilon,$$
in other words,
$$\forall j\geqslant 1,\ \overline{\nu}_{a,\delta}(f)\leqslant\overline{\nu}_{a,\delta}(\partial_j f)+j\epsilon \tag{1}$$
and the inequality is an equality for some $j\geqslant 1$. We write d the maximal positive integer with this property, so that
$$\overline{\nu}_{a,\delta}(f)=\overline{\nu}_{a,\delta}(\partial_d f)+d\epsilon. \tag{1'}$$
The second statement says that
$$\forall i\geqslant 1,\ \overline{\nu}_{a,\delta}(f)=\overline{\nu}(f(a))<\overline{\nu}(\partial_i f(a))+i\epsilon.$$
Taking $i=d$ we can use (1) and show that
$$\overline{\nu}_{a,\delta}(\partial_d f)+d\epsilon=\overline{\nu}_{a,\delta}(f)<\overline{\nu}(\partial_d f(a))+d\epsilon$$
or, by simplifying
$$\overline{\nu}_{a,\delta}(\partial_d f)<\overline{\nu}(\partial_d f(a)). \tag{2}$$
This implies that $S_{X-a,\overline{\mu}}(\partial_d f)\neq\{0\}$. Indeed, $S_{X-a,\overline{\mu}}(\partial_d f)=\{0\}$ means that $\overline{\nu}_{a,\delta}(\partial_d f)=\overline{\nu}(\partial_d(f(a)))$, contradicting (2). By the first direction of the statement proven above, we have that $\epsilon_{\overline{\nu}_{a,\delta}}(\partial_d f)=\epsilon$, meaning that we have a positive integer $b\geqslant 1$, such that
$$\epsilon=\frac{\overline{\nu}_{a,\delta}(\partial_d f)-\overline{\nu}_{a,\delta}(\partial_b\partial_d f)}{b}$$
i.e.,
$$\overline{\nu}_{a,\delta}(\partial_d f)=\overline{\nu}_{a,\delta}(\partial_b\partial_d f)+b\epsilon.$$
We know that $\partial_b\partial_d=\binom{b+d}{b}\partial_{b+d}$, so by combining with $(1')$ we have
$$\overline{\nu}_{a,\delta}(f)=\nu\left(\binom{b+d}{b}\right)+\overline{\nu}_{a,\delta}(\partial_{b+d}f)+(b+d)\epsilon.$$
$\binom{b+d}{b}$ is an integer so $\nu\left(\binom{b+d}{b}\right)\geqslant 0$. But by the inequality in (1) (taken for $j=b+d$) we also know that
$$\overline{\nu}_{a,\delta}(f)\leqslant\overline{\nu}_{a,\delta}(\partial_{b+d}f)+(b+d)\epsilon$$
thus we also have the reverse inequality $\nu\left(\binom{b+d}{b}\right)\leqslant 0$. In conclusion
$$\overline{\nu}_{a,\delta}(f)=\overline{\nu}_{a,\delta}(\partial_{b+d}f)+(b+d)\epsilon$$
which contradicts the maximality of d. □

Chapter 5

Diskoids, towards a geometric interpretation of truncations

In this chapter, our goal twofold. the primary objective is to introduce a geometric formalism that can allow us to give a purely geometric encoding of RT extensions. This object is called a diskoid and has already been used in [47] and [43]. When we start with a valued field (K, ν) and we extend ν to the algebraic closure $(\overline{K}, \overline{\nu})$, diskoids can be defined as

$$D(f, \rho) = \{x \in \overline{K};\ \overline{\nu}(f(x)) \geqslant \rho\}.$$

where $f \in \overline{K}[X]$ and $\rho \in \Phi(\overline{\nu})$. They can be decomposed into a disjoint union of balls. Our secondary objective is to show that if Q is an ABKP for an extension μ of ν to $K[X]$, then the common extensions of $\overline{\nu}$ and μ_Q to $\overline{K}[X]$ correspond to the set of balls the diskoid $D(Q, \mu(Q))$ is made of. The proof of the first goal will allow us to establish this second part, as we will study the action of the absolute Galois group on the balls forming the diskoid.

This chapter is divided into two parts. In the first part we build our formalism of balls and diskoids as generally as possible and in the second we see how it is possible to build our two correspondences.

5.1 The Diskoid Formalism

We will establish a relative point a view. We will assume we are given a fixed valued field (K, ν) and we will consider an extension to a valued field (F, v). We will assume that F is "big enough" in the sense that the residue field of $(F, v,\ \kappa(F)$ is an infinite field. Additionally we suppose that the value group of (F, v), $\Phi(F)$ is contained in a bigger ordered abelian group Γ and we will assume it is divisible, *i.e.,* $\mathbb{Q}\Gamma = \Gamma$. We will adopt these notations throughout this section.

5.1.1 Balls

We start by studying the simplest possible construction: balls.

Definition 5.1. Fix $a \in F$ and $\delta \in \Gamma \cup \{\infty\}$. Then we set

$$B(a, \delta) = B_F(a, \delta) = \{x \in F;\ v(x - a) \geqslant \delta\}$$

to be the *closed ball centred around a and of radius δ* [1].

Closed (ultrametric) balls have a very interesting set-theoretical properties.

Proposition 5.2. 1. Let $a \in F$, $\delta \in \Gamma \cup \{\infty\}$. For any $b \in B = B(a, \delta)$, b is a centre of B: $B(a, \delta) = B(b, \delta)$.

[1] We will often drop the F in B_F as long as there is only one field F considered in this section.

2. Two balls $B_1 = B(a_1, \delta_1)$, $B_2 = B(a_2, \delta_2)$ are either disjoint ($B_1 \cap B_2 = \emptyset$) or one is included in the other ($B_1 \subseteq B_2$ or $B_1 \supseteq B_2$).

Proof. 1. We prove that if $b \in B(a, \delta)$ then $B(a, \delta) \subseteq B(b, \delta)$. That way, $a \in B(b, \delta)$ and by a symmetrical argument $B(b, \delta) \subseteq B(a, \delta)$. So let $x \in B(a, \delta)$, *i.e.,* $v(x - a) \geqslant \delta$. Then by the ultrametric inequality

$$v(x - b) = v(x - a + a - b) \geqslant \min\{v(x - a), v(a - b)\} \geqslant \delta$$

so that $x \in B(b, \delta)$.
2. Suppose that $B_1 \cap B_2 \neq \emptyset$ and take any a in the intersection. By the first part of the proof we now know that $B_i = B(a, \delta_i)$, $i = 1, 2$, thus $B_1 \subseteq B_2$ or $B_1 \supseteq B_2$, according to whether $\delta_1 \geqslant \delta_2$ or $\delta_1 \leqslant \delta_2$. □

Lemma 5.3. Let $B = B(a, \delta)$ be a closed ball, with $\delta \in \Phi(v) \cup \{\infty\}$. Then for any polynomial $f \in F[X]$, the following minimum is well defined and can even be explicitly written

$$\min_{x \in B} v(f(x)) = \min_{i \in \mathbb{N}} \{v(\partial_i f(a)) + i\delta\}.$$

Thus the map $f \longmapsto \min_{x \in B} v(f(x))$, that we denote by v_B, is a valuation, which coincides with the depth zero valuation $v_{a,\delta}$ given by the (not necessarily minimal) defining pair (a, δ).

Proof. If $\delta = \infty$ then $B = \{a\}$ and clearly

$$\min_{x \in B} v(f(x)) = v(f(a)) = \min_{i \in \mathbb{N}} \{v(\partial_i f(a)) + i\delta\}.$$

We henceforth assume that $\delta < \infty$. Write f as

$$f(X) = \sum_{i=0}^{n} a_i \left(\frac{X - a}{b}\right)^i, \quad a_i \in F \tag{1}$$

where $b \in F$ is an element such that $v(b) = \delta$. Since $x \in B$ if and only if $v\left(\frac{x-a}{b}\right) \geqslant 0$, for such x we have by the ultrametric property

$$v(f(x)) \geqslant \min_{0 \leqslant i \leqslant n} v\left(a_i \left(\frac{X - a}{b}\right)^i\right) = \min_{0 \leqslant i \leqslant n} v(a_i) + iv\left(\frac{X - a}{b}\right) \geqslant \min_{0 \leqslant i \leqslant n} v(a_i).$$

Let us now show that in fact this last value is assumed by f inside B. We first normalise the expression of f above, *i.e.,* we divide by the a_i of minimal value and we can assume $\min_{0 \leqslant i \leqslant n} v(a_i) = 0$. Reducing the coefficients of the polynomial $g(T) = \sum_i a_i T^i$ to the residue field $\kappa(F)$ we get a non-zero polynomial $\overline{g} \in \kappa(F)[X]$. $\kappa(F)$ being infinite, we can find a nonzero $\alpha \in \kappa(F)$ such that $\overline{g}(\alpha) \neq 0$. We fix $x_0 \in F$ such that $\frac{x_0 - a}{b} \in \mathscr{O}(F)$ and its residual image is α, so that

$$v(f(x_0)) \mod \mathfrak{m}(F) = \overline{g}(\alpha) \neq 0,$$

thus $v(f(x_0)) = 0$. Since $v\left(\frac{x_0 - a}{b}\right) \geqslant 0$, necessarily $x_0 \in B$. Thus we have shown that

$$\min_{x \in B} v(f(x)) = 0 = v(f(x_0)) = \min_i v(a_i) = \min_{i \in \mathbb{N}} \{v(\partial_i f(a)) + i\delta\},$$

since $a_i = \partial_i f(a) b^i$, from the way we have written f in (1). □

Remark 5.4. 1. Thus we just have proven that v_B is a valuation over $F[X]$ and according to Theorem 3.16, it defines an RT extension of v.

2. We can extract from the proof the fact that the value $\min_{x \in B} v(f(x))$ is attained for $x \in \partial B := \{x \in F;\ v(x - a) = \delta\}$. Indeed, if we consider the notations inside the proof, α is non-zero inside $\kappa(F)$, so that, if $\frac{x_0 - a}{b}$ reduces to α, then $v\left(\frac{x_0 - a}{b}\right) = 0$, in other words $v(x_0 - a) = \delta$. So for $x_0 \in \partial B$ we get $v(f(x_0)) = v_B(f)$.

5.1.2 Diskoids

We now build a more general object called diskoid.

Definition 5.5. For any polynomial $f \in F[X]$ and any value $\rho \in \Gamma \cup \{\infty\}$ we define the *diskoid centred at f of radius ρ*

$$D(f,\rho) = D_F(f,\rho) = \{x \in F;\ v(f(x)) \geqslant \rho\}.$$

Observe that we then have $D(X-a,\delta) = B(a,\delta)$, so diskoids are a generalisation of balls. Just like in the previous section, we wish to show that for a polynomial $f \in F[X]$, the mapping

$$f \in K[X] \longmapsto \min_{x \in D(f,\rho)} v(f(x))$$

is well-defined (at least when $\delta \in \Phi(v) \cup \{\infty\}$). This can easily be proven thanks to the next lemma, which shows that diskoids are simply disjoint unions of closed balls.

Lemma 5.6. Set $f \in F[X]$, $a \in F$ a root of f and $\rho \in \Phi(v)$. Then the quantity

$$\epsilon(a; f,\rho) := \min\{\lambda \in \Gamma;\ B(a,\lambda) \subseteq D(f,\rho)\}$$

is well-defined and can even be explicitly written

$$\epsilon(a; f,\rho) = \max_{i \in \mathbb{N}^*} \frac{\rho - v(\partial_i f(a))}{i}.$$

Proof. Let us write f as

$$f(X) = \sum_{i=0}^{n} a_i (X-a)^i, \quad a_i \in \overline{K}$$

so that $a_i = \partial_i f(a)$. We remark that $a_0 = 0$ since $f(a) = 0$, so considering Lemma 5.3, for any $\lambda \in \Phi(\overline{v})$, $\min_{x \in B(a,\lambda)} v(f(x)) = \min_{i \geqslant 1}\{v(a_i) + i\lambda\}$. Thus we can write the following equivalences

$$\begin{aligned} B(a,\lambda) \subseteq D(f,\rho) &\iff \forall x \in B(a,\lambda),\ v(f(x)) \geqslant \rho \\ &\iff \min_{x \in B(a,\lambda)} v(f(x)) \geqslant \rho \\ &\iff \min_{1 \leqslant i \leqslant n} \{v(a_i) + i\lambda\} \geqslant \rho \\ &\iff \lambda \geqslant \max_{1 \leqslant i \leqslant n} \frac{\rho - v(a_i)}{i}. \end{aligned}$$

Thus we can state the following expression

$$\epsilon(a; f,\rho) = \max_{1 \leqslant i \leqslant n} \frac{\rho - v(a_i)}{i}.$$

□

We can now show how diskoids decompose.

Lemma 5.7. Suppose that a monic polynomial $f \in F[X]$ has all of its roots in F, and fix $\rho \in \Gamma \cup \{\infty\}$. The diskoid $D(f,\rho)$ is the union of some balls centred around the roots of f

$$D(f,\rho) = \bigcup_{f(c)=0} B(c, \epsilon(c; f,\rho)).$$

Proof. Let $c_1, \ldots, c_n$ be the possibly repeated roots of f in F. By Lemma 5.6 we can assign to each c_i the value $\epsilon_i = \epsilon(c_i; f,\rho) \in \Phi(v)$ such that $B(c_i,\epsilon_i) \subseteq D(f,\rho)$, with ϵ_i minimal for this property. We now show $D(f,\rho) \subseteq \cup_i B(c_i,\epsilon_i)$, the other inclusion being clear. Let $x \in D(f,\rho)$

$$\sum_{i=1}^{n} v(x - c_i) = v(f(x)) \geqslant \rho.$$

We can rearrange the c_i so that

$$v(x-c_1) \geqslant \ldots \geqslant v(x-c_n).$$

In other words we have relabelled the roots to make c_1 the closest root to x. Now we will show that $B(c_1, v(x-c_1)) \subseteq D(f,\rho)$ so that $\epsilon_1 \leqslant v(x-c_1)$. For $y \in B(c_1, v(x-c_1))$ we have

$$\begin{aligned} v(f(y)) &= \sum_{i=1}^{n} v(y-c_i) \\ &\geqslant v(x-c_1) + \sum_{i=2}^{n} v(\underbrace{y-c_1}_{\geqslant v(c_1-x)} + \underbrace{c_1-x}_{\geqslant v(x-c_i)} + x - c_i) \\ &\geqslant v(x-c_1) + \sum_{i=2}^{n} v(x-c_i) \\ &\geqslant \rho. \end{aligned}$$

□

For the rest of this section, we will fix a unitary polynomial $\mathbf{f} \in \mathbf{F}[\mathbf{X}]$ *that has all of its roots in* $\mathbf{F}$.

Definition 5.8. For any polynomial $g \in F[X]$ and value $\rho \in \Phi(v) \cup \{\infty\}$, the following

$$\min_{x \in D(f,\rho)} v(g(x))$$

is well-defined, according to Lemma 5.7 and Lemma 5.3. We can thus define the map

$$\begin{aligned} v_{D(f,\rho)} : K[X] &\longrightarrow \Phi(v) \cup \{\infty\} \\ g &\longmapsto \min_{x \in D(f,\rho)} v(g(x)). \end{aligned}$$

Proposition 5.9. The map $\overline{\nu}_{D(f,\rho)}$ is ultrametric (verifies (V2)).

Proof. Take $g, h \in \overline{K}[X]$ and set $x \in D(f,\rho)$ such that $v_{D(f,\rho)}(g+h) = v(g(x)+h(x))$. We have the following

$$\begin{aligned} v_{D(f,\rho)}(g+h) &= v(g(x)+h(x)) \\ &\geqslant \min\{v(g(x)), v(h(x))\} \\ &\geqslant \min\{v_{D(f,\rho)}(g), v_{D(f,\rho)}(h)\}. \end{aligned}$$

□

Remark 5.10. A similar argument shows that

$$\forall g, h \in F[X],\ v_{D(f,\rho)}(gh) \geqslant v_{D(f,\rho)}(g) + v_{D(f,\rho)}(h).$$

However $v_{D(f,\rho)}$ may fail to be multiplicative (the condition (V1) for valuations). We give a simple counter-example: set $a \in F$ such that $v(a) < 0$. Define $B_a = B(a,0)$, $B_0 = B(0,0)$ and $D = B_a \cup B_0$. It can be realised as a diskoid of $f = X(X-a)$

$$D(X(X-a), \nu(a)) = B(a,0) \cup B(0,0) = D$$

since

$$\epsilon(0; f, v(a)) = \epsilon(a; f, v(a)) = \max\left\{\frac{v(a)}{2}, v(a) - v(a)\right\} = 0.$$

Indeed, $\partial_1 X(X-a) = 2X - a$ and $\partial_2 X(X-a) = 1$. Then we have

$$\begin{array}{ll} v_{B_0}(X) = 0 & v_{B_a}(X) = v(a) \\ v_{B_0}(X-a) = \nu(a) & v_{B_a}(X-a) = 0 \end{array}$$

so clearly

$$v_{D(f,v(a))}(X) = v_{D(f,v(a))}(X-a) = v(a).$$

Furthermore $v_{D(f,v(a))}(X(X-a)) = v(a)$ but

$$v_{D(f,v(a))}(X) + v_{D(f,v(a))}(X-a) = 2v(a) < v(a) = v_{D(f,v(a))}(X(X-a))$$

hence, $v_{D(f,v(a))}$ is not a valuation.

5.2 Correspondence for RT extensions and diskoids

In this section we try and build a correspondence between RT extension and diskoids. We will set here $F = \overline{K}$ and $v = \overline{\nu}$. Thus $\kappa(v)$ is algebraically closed, hence infinite, and any polynomial $f \in F[X]$ will have all of its roots in F, so that all maps v_D are well-defined.

5.2.1 Balls and RT extensions over $\overline{K}$

In this subsection, we study in particular RT extensions of $\overline{\nu}$, *i.e.,* a valuation over $\overline{K}$, the algebraic closure of a field K. If we fix a centre $a \in \overline{K}$ and a radius $\delta \in \Phi(\overline{\nu})$, we saw that the map $\overline{\nu}_{B(a,\delta)}$ is exactly the map $\overline{\nu}_{a,\delta}$, which is trivially a valuation. According to Theorem 3.16, it is an RT extension of $\overline{\nu}$.

Remark 5.11. If $\delta = \infty$, then $B = B(a,\delta) = \{a\}$ so we have

$$\min_{x \in B} \overline{\nu}(f(x))) = \overline{\nu}(f(a)).$$

This is not a valuation anymore, but a semivaluation, as $\overline{\nu}_B(X-a) = \infty$.

We can restate and extend Theorem 3.16 in the following way.

Theorem 5.12. There is a one-to-one correspondence between residually transcendental extensions and balls as we have just defined. More precisely, we can define the following maps, which form a commutative diagram

$$\begin{tikzcd}
 & \left\{ \begin{array}{c} \text{Minimal Pairs } (a,\delta) \\ \text{with } \delta \in \Phi(\overline{\nu}) \end{array} \right\} \arrow[dl, "\text{onto}"'] \arrow[dr, "\text{onto}"] & \\
\{\text{RT Extensions}\} & & \{\text{balls}\} \arrow[ll, "\text{one-to-one}"']
\end{tikzcd}$$

$$\begin{tikzcd}
 & (a,\delta) \arrow[dl, mapsto] \arrow[dr, mapsto] & \\
\overline{\nu}_B = \overline{\nu}_{a,\delta} & & B = B(a,\delta) \arrow[ll, mapsto]
\end{tikzcd}$$

Proof. We already know that the above maps are well-defined. We also know that they are onto. Indeed, on the one hand, any residually transcendental extension is given by a minimal pair as we have established in Theorem 3.16. On the other hand, by the ultrametric property we know that any point inside a ball $B(b,\delta)$ is a centre of the ball, thus we can choose a centre a such that $\deg_K(a)$ is minimal, which yields a minimal pair. Finally the horizontal map is injective. Indeed consider two balls $B = B(a,\delta)$, $B' = B(a',\delta')$, yielding the same valuations $\overline{\nu}_B = \overline{\nu}_{B'}$. We can furthermore suppose that $(a,\delta), (a',\delta')$ are minimal pairs, thus we have $\overline{\nu}_{a,\delta} = \overline{\nu}_{a',\delta'}$ and by Lemma 3.12, we have $\delta = \delta'$ and $\overline{\nu}(a-a') \geqslant \delta$, which in turn is equivalent to $B = B'$. $\square$

This correspondence is decreasing in the following sense.

Proposition 5.13. Consider two RT extensions μ_1, μ_2 and their corresponding balls B_1, B_2. Then we have

$$\mu_1 \leqslant \mu_2 \iff B_1 \supseteq B_2.$$

Proof. Suppose $B_1 \supseteq B_2$ and take $f \in \overline{K}[X]$. Choose $x \in B_2$ such that $\mu_2(f) = \overline{\nu}(f(x))$, so that, since $x \in B_1$, $\mu_1(f) = \min_{y \in B_1} \overline{\nu}(f(y)) \leqslant \overline{\nu}(f(x))$.
Set now $B_i = B(a_i, \delta_i)$, $i = 1, 2$. If we suppose that $\mu_1 \leqslant \mu_2$, then consider any $x \in B_2$, *i.e.*, $\overline{\nu}(x - a_2) \geqslant \delta_2$. Then we have the following series of inequalities:

$$\delta_1 = \min_{\overline{\nu}(y-a_1)\geqslant\delta_1} \overline{\nu}(y - a_1) = \mu_1(X - a_1) \leqslant \mu_2(X - a_1) = \min_{\overline{\nu}(y-a_2)\geqslant\delta_2} \overline{\nu}(y - a_1) \leqslant \overline{\nu}(x - a_1)$$

thus $\overline{\nu}(x - a_1) \geqslant \delta_1$ and $x \in B_1$. □

5.2.2 Diskoids and RT extensions (general case)

Our wish is to generalise Theorem 5.12 to any RT extension. We hence suppose that $\mu = \mu_Q$ with Q an ABKP such that $\epsilon_\mu(Q) \in \mathbb{Q} \otimes \Phi(\nu) = \Phi(\overline{\nu})$. More specifically we wish to find subsets $D \subseteq \overline{K}$ that can verify the following:

(D1) $\forall f \in K[X]$, $\overline{\nu}_D := \min_{x \in D} \overline{\nu}\big(f(x)\big)$ is well-defined.
(D2) $\overline{\nu}_D$ is a valuation.
(D3) There is a way of associating to Q an ABKP for μ, with $\epsilon_\mu(Q) \in \mathbb{Q} \otimes \Phi(\nu)$, a subset D_Q, satisfying (D1) and (D2), such that $\overline{\nu}_{D_Q} = \mu_Q$.
(D4) The mapping $D \longmapsto \overline{\nu}_D$ between such subsets and RT extensions over $K[X]$ is bijective and decreasing.

Our natural candidates here for D_Q are diskoids. More precisely we want to show that in certain specific cases $D_Q = D(Q, \mu(Q))$ will serve as the natural replacement for balls in the correspondence of Theorem 5.12. We already know that the map $f \in K[X] \longmapsto \overline{\nu}_{D(Q,\mu(Q))}(f)$ is well defined, let us study its properties more closely.

Remark 5.14. If (a, δ) is a minimal pair associated to an abstract key polynomial Q, then we can give a clear interpretation for the quantity $\epsilon(a; Q, \mu(Q))$. Indeed, since for any $i \geqslant 1$, $\deg \partial_i Q < \deg Q$, we get by Theorem 4.15 $\mu(\partial_i Q) = \mu_Q(\partial_i Q) = \overline{\nu}_{a,\delta}(\partial_i Q) = \overline{\nu}(\partial_i Q(a))$, since $\epsilon_{\mu_Q}(\partial_i Q) = \epsilon_\mu(\partial_i Q) < \epsilon_\mu(Q)$. By Lemma 5.6

$$\epsilon(a; Q, \mu(Q)) = \max_{i \geqslant 1} \frac{\mu(Q) - \overline{\nu}(\partial_i Q(a))}{i} = \frac{\mu(Q) - \mu(\partial_i Q(a))}{i} = \epsilon_\mu(Q) = \delta_{\overline{\mu}}(Q).$$

One can hope that if we restrict to studying only abstract key polynomials Q, with $\epsilon_\mu(Q) \in \mathbb{Q} \otimes \Phi(\overline{\nu})$, the diskoids $D(Q, \mu(Q))$ yield true valuations. Let us state the following conjectures.

Conjecture 5.15.
(C1) If Q is an ABKP for μ with $\epsilon_\mu(Q) \in \Phi(\overline{\nu})$, then $\overline{\nu}_{D(Q,\mu(Q))}$ is a valuation.
(C2) If μ is an RT extension, there is a unique diskoid D such that $\mu = \overline{\nu}_D$, *i.e.*, if there are two diskoids $D_i = D(Q_i, \rho_i)$, such that $\overline{\nu}_{D_1} = \mu = \overline{\nu}_{D_2}$, then $D_1 = D_2$.

Our approach needs some basic facts about henselizations.

5.2.3 Henselian valued fields and henselizations

In this section we recall some facts about the interaction between extensions of valued fields and the Galois theory of the corresponding field extension. We start with the following crucial notion.

Definition 5.16. A valued field (K, ν) is called *henselian* if there is only one extension of ν to the algebraic closure of K.

Several ways to characterize henselian fields exist and relate more or less with Hensel's lemma (one can consult [29] for a deeper understanding).

Example 5.17. Henselian fields include discrete valued fields of rank 1 which are complete, such as the field of p-adic numbers $\mathbb{Q}_p$ equipped with its natural p-adic absolute value, or the formal power series $k((t))$ where k is any field, which we equip with the natural t-adic valuation.

Not every field is henselian, but one can find a smallest henselian extension of a valued pair (K,ν) which is henselian. It is called a henselization of (K,ν).

Definition 5.18. An extension $(\widetilde{K},\widetilde{\nu})$ of (K,ν) is called a *henselization* of (K,ν) if it is henselian and if for every henselian valued field (E,ζ) and every embedding $\lambda:(K,\nu)\hookrightarrow(E,\zeta)$ there exists a unique embedding $\widetilde{\lambda}:(\widetilde{K},\widetilde{\nu})\hookrightarrow(E,\zeta)$ extending λ, *i.e.,* making the following diagram

$$\begin{tikzcd}
(K,\nu) \arrow[r, "\lambda"] \arrow[d] & (E,\zeta) \\
(\widetilde{K},\widetilde{\nu}) \arrow[ur, dashed, "\widetilde{\lambda}"'] &
\end{tikzcd}$$

commutative.

Henselizations exist and can be constructed in the following way: choose a separable closure K^s of K and an extension ν^s of ν to K^s (or $\overline{K}$). Write $G_K=\mathrm{Gal}(K^s/K)=\mathrm{Aut}_K(\overline{K})$ and

$$G^h=\{\sigma\in G_K;\ \nu^s\circ\sigma=\nu^s\}$$

the decomposition group of ν^s. The decomposition field of ν^s, which is by definition

$$K^{h(\nu^s)}=\{x\in K^s;\ \sigma(x)=x,\ \forall\sigma\in G^h\},$$

is a henselization of (K,ν). Any other choice for the extension ν^s will give another henselization. All henselizations are isomorphic up to unique isomorphism fixing K so we will talk about *the* henselization of (K,ν) and write it (K^h,ν^h), as we make a choice of one henselization throughout this text once and for all. We will also assume that $\overline{\nu}$ is an extension of ν^s and ν^h.

Remark 5.19. Since the extension $\overline{K}/K^s$ is purely inseparable, the extension from ν^s to $\overline{K}$ is unique [19, Cor. 3.2.10]. This is equivalent to saying that K^s is henselian. Thus we will know that for any valuations $\overline{\nu}_1,\overline{\nu}_2$ over $\overline{K}$, we have

$$\overline{\nu}_1=\overline{\nu}_2 \iff \overline{\nu}_1\big|_{K^s}=\overline{\nu}_2\big|_{K^s}$$

In our situation we are interested in certain types of extensions. Once we have our truncated valuation μ_Q, we wish to enumerate the valuations that extend both μ_Q and $\overline{\nu}$, to $\overline{K}[X]$. We use the following result concerning the permutation of valuations by automorphisms of extensions. Consider (K,ν) a valued field and L/K a field extension. We will write $\mathcal{E}(L,\nu)$ the set of valuations on L extending ν.

Proposition 5.20. [19, Conjugation Theorem 3.2.15] Let L/K be a normal field extension and ν a valuation of K. $\mathrm{Aut}_K(L)$ acts *transitively* on the set of extensions of ν to L as follows

$$\begin{aligned}
\mathrm{Aut}_K(L)\times\mathcal{E}(L,\nu) &\longrightarrow \mathcal{E}(L,\nu)\\
(\sigma,\mu) &\longmapsto \mu\circ\sigma^{-1}=\mu^{\sigma}.
\end{aligned}$$

Once we have a common extension $\overline{\mu}$ of our μ_Q and $\overline{\nu}$, so for instance $\overline{\nu}_{a,\delta}$ where (a,δ) is a minimal pair associated to Q, then we can let $G_K=\mathrm{Aut}_K(\overline{K})$ act on $\overline{\mu}$ so that we obtain all the extensions of μ_Q. Indeed $\overline{K}(X)/K(X)$ is a normal extension with same Galois group as $\overline{K}/K$. So, if $\sigma\in G_K$, we have the following equivalences

$$\begin{aligned}
\overline{\mu}\circ\sigma^{-1}\in\mathcal{E}(\overline{K}[X],\overline{\nu}) &\iff \overline{\mu}\circ\sigma^{-1}\big|_{\overline{K}}=\overline{\nu}\\
&\iff \overline{\mu}\big|_{\overline{K}}\circ\sigma^{-1}\big|_{\overline{K}}=\overline{\nu} \qquad \text{since } \sigma^{-1}(\overline{K})\subset\overline{K}\\
&\iff \overline{\nu}\circ\sigma^{-1}\big|_{\overline{K}}=\overline{\nu}\\
&\iff \sigma^{-1}\big|_{\overline{K}}\in G^h.
\end{aligned}$$

Thus, if we identify the automorphism groups of $\overline{K}(X)/K(X)$ and $\overline{K}/K$, we can state the proposition below.

Proposition 5.21. Take an abstract key polynomial Q and (a, δ) an associated minimal pair. We thus have

$$\begin{aligned}\mathcal{E}(\overline{K}[X], \overline{\nu}) \cap \mathcal{E}(\overline{K}[X], \mu_Q) &= \{\overline{\nu}_{a,\delta} \circ \sigma^{-1};\ \sigma \in G^h\}\\ &= \{\overline{\nu}_{\sigma(a),\delta};\ \sigma \in G^h\}.\end{aligned}$$

Proof. It remains to prove the following identity

$$\overline{\nu}_{a,\delta} \circ \sigma^{-1} = \overline{\nu}_{\sigma(a),\delta}$$

for any $\sigma \in G^h$. If $f \in \overline{K}[X]$ decomposes as $f = \sum_{i \geqslant 0} c_i (X - \sigma(a))^i$

$$\begin{aligned}\overline{\nu}_{a,\delta} \circ \sigma^{-1}(f) &= \overline{\nu}_{a,\delta}\left(\sum_{i \geqslant 0} \sigma^{-1}(c_i)(X-a)^i\right)\\ &= \min_{i \geqslant 0}\{\overline{\nu}(\sigma^{-1}(c_i)) + i\delta\}\\ &= \min_{i \geqslant 0}\{\overline{\nu}(c_i) + i\delta\}\\ &= \overline{\nu}_{\sigma(a),\delta}(f).\end{aligned}$$

□

Remark 5.22. Just as G^h acts on valuations, it also acts on balls. Set $\sigma \in G^h$ and $(a, \delta) \in \overline{K} \times \Phi(\overline{\nu})$. Then

$$\sigma\left(B(a,\delta)\right) = B(\sigma(a), \delta).$$

Indeed, for any $x \in \overline{K}$, one has $\overline{\nu}(x - a) = \overline{\nu}(\sigma(x) - \sigma(a))$.

5.2.4 Return to diskoids.

We can use the action of G^h to make the structure of diskoids easier to grasp and make more sense of the maps $\overline{\nu}_{D(f,\mu(f))}$.

Proposition 5.23. Consider a polynomial $f \in K^h[X]$, such that G^h acts transitively on its roots and $\epsilon_{\overline{\mu}}(f) \in \Phi(\overline{\nu})$. If we set $a \in \overline{K}$ an optimising root of f (*i.e.*, $\overline{\mu}(X - a) = \delta_{\overline{\mu}}(f) = \epsilon_{\overline{\mu}}(f)$), then we have

$$D(f, \overline{\mu}(f)) = \bigcup_{\sigma \in G^{h(\overline{\nu})}} B(\sigma(a), \delta_{\overline{\mu}}(f)).$$

Thus $\overline{\nu}_{D(f,\overline{\mu}(f))}$ is in fact a valuation on $K[X]$ (or even $K^h[X]$) and its extensions to $\overline{K}[X]$ are $\overline{\nu}_{B(\sigma(a),\delta(f))}$, $\sigma \in G^h$.

Proof. Considering Lemma 5.7, the union is well-indexed, as the roots of f will be $\sigma(a)$, $\sigma \in G^h$. We already know $\overline{\nu}_{B(\sigma(a),\delta(f))}$ is well-defined valuation and equal to the valuation given by the defining pair $(a, \delta(f))$, by Lemma 5.3. Consider any $g \in K^h[X]$, we will show that in fact $\overline{\nu}_{B(\sigma(a),\delta(f))}(g) = \overline{\nu}_{B(a,\delta(f))}(g)$ this value thus being also equal to $\overline{\nu}_{D(f,\overline{\mu}(f))}(g)$. Indeed, for any $\sigma \in G^h$, σ will not change the coefficients of g, thus we can safely write

$$\begin{aligned}\overline{\nu}_{B(\sigma(a),\delta(f))}(g) &= \overline{\nu}_{\sigma(a),\delta(f)}(g)\\ &= \overline{\nu}_{a,\delta(f)} \circ \sigma^{-1}(g)\\ &= \overline{\nu}_{a,\delta(f)}(g)\\ &= \overline{\nu}_{B(a,\delta(f))}(g).\end{aligned}$$

Since $\overline{\nu}_{B(a,\delta(f))}$ is a valuation over $\overline{K}[X]$, extending $\overline{\nu}_{D(f,\overline{\mu}(f))}$, by Proposition 5.20, we know that all the other extensions are $\overline{\nu}_{a,\delta(f)} \circ \sigma^{-1} = \overline{\nu}_{B(\sigma(a),\delta(f))}$. □

For instance if the polynomial f in the preceding theorem is irreducible, then it has its roots permuted transitively.

Corollary 5.24. Let Q be an abstract key polynomial with $\mu(Q) \in \Phi(\overline{\nu})$. If Q is irreducible over K^h, then $\overline{\nu}_{D(Q,\mu(Q))}$ is a valuation and in fact

$$\mu_Q = \overline{\nu}_{D(Q,\mu(Q))}.$$

Proof. By Theorem 4.3, if we assign to Q one of its minimal pairs (a, δ)

$$\begin{aligned} \mu_Q &= \overline{\nu}_{a,\delta}\big|_{K[X]} \\ &= \overline{\nu}_{D(a,\delta)}\big|_{K[X]} \\ &= \overline{\nu}_{D(Q,\mu(Q))}. \end{aligned}$$

□

We are thus led to the following concept, which is classical.

Definition 5.25. We say that a polynomial $f \in K[X]$ is *analytically irreducible* if it is irreducible over K^h.

If all ABKPs Q are analytically irreducible, we can then extend the correspondence of Theorem 5.12, into the one below.

$$\begin{tikzcd}
 & \left\{ \begin{array}{c} \text{ABKPs } Q \\ \text{with } \epsilon_\mu(Q) \in \mathbb{Q} \otimes \Phi(\nu) \end{array} \right\} \arrow[dl, "\text{onto}"'] \arrow[dr, "\text{onto}"] & \\
\{\text{RT Extensions}\} & & \{\text{Diskoids}\} \arrow[ll, "\text{one-to-one}"']
\end{tikzcd}$$

$$\begin{tikzcd}
 & Q \arrow[dl, mapsto] \arrow[dr, mapsto] & \\
\mu_Q = \overline{\nu}_D & & D = D(Q, \mu(Q)) \arrow[ll, mapsto]
\end{tikzcd}$$

If the base field (K, ν) is itself henselian, Q is obviously analytically irreducible (since ABKPs are irreducible over K to begin with), thus the correspondence is true in this case. In the next subsection, we give a case where this is always the case.

5.2.5 The rank one case.

This section is dedicated to showing the following theorem.

Theorem 5.26. Suppose ν is of rank 1 and Q is an ABKP for a valuation μ of $K[X]$ such that μ_Q/ν is an RT extension,*i.e.,* $\epsilon_\mu(Q) \in \mathbb{Q} \otimes \Phi(\nu)$. Then Q is analytically irreducible.

The reasoning can be done by reductio ad absurdum. Suppose on the contrary that Q is reducible in $K^h[X]$. We can write $Q = PR,\ P, R \in K^h[X]$ with P irreducible over K^h and $\epsilon_\mu(P) = \epsilon_\mu(Q)$. Indeed consider a an optimising root of Q and suppose P is its minimal polynomial over K^h. Then $P|Q$ and since Q is not irreducible, we get that $P \neq Q$.
In rank one, the completion satisfies Hensel's lemma (*cf.* [19, Theorem 1.3.1]), thus it is henselian. This shows that the henselization can be embedded in the completion, hence K is dense in K^h, since K is dense in its completion. This means that

$$\forall z \in K^h,\ \forall \epsilon \in \Phi(\nu),\ \exists z^* \in K,\ \nu^h(z - z^*) > \epsilon.$$

We now use the continuity of roots of a monic polynomial of constant degree relative to its coefficients. We use as a reference [12, Theorem 2], but more specifically the remark that follows Theorem 2 of the aforementioned paper.

Theorem 5.27. Let f and f^* be monic polynomials of common degree $n > 1$ with coefficients in an algebraically closed valued field (F, v). We suppose that these coefficients are of positive value. We may then write

$$f = \prod_{k=1}^{n} (X - \alpha_k)$$
$$f^* = \prod_{k=1}^{n} (X - \alpha_k^*)$$

such that $\forall k,\ \nu(\alpha_k - \alpha_k^*) \geqslant V(f - f^*)/n$, where

$$V\left(\sum_l a_l X^l\right) = \min_l v(a_l).$$

Let us come back to our proof. We can arbitrarily approximate each coefficient of P by elements in K, thus one can find a monic $P^* \in K[X]$ of same degree, such that $V(P - P^*) > n\epsilon_\mu(P)$. We decompose them in linear factors

$$P = \prod_{k=1}^{r} (X - a_k)$$
$$P^* = \prod_{k=1}^{r} (X - a_k^*)$$

so that $\forall k,\ \overline{\nu}(a_k - a_k^*) > \epsilon_\mu(P)$. Thus $\forall k,\ \overline{\mu}(X - \alpha_k) = \overline{\mu}(X - \alpha_k^*)$, so the polynomial $P^* \in K^h$, is such that $\epsilon(P^*) = \epsilon(P) = \epsilon(Q)$ but $\deg(P^*) = \deg(P) < \deg(Q)$. This contradicts the fact that Q is an ABKP.

In higher rank cases, this proof breaks down, since K^h is no longer in the completion of K. In a private communication with F.-V. Kuhlmann, he expressed his pessimism about analytic irreducibility of ABKPs for valuations of ranks higher than 1.

Chapter 6

Open Problems

In this last chapter we wish to touch upon the conjectures evoked in this text. We discuss the difficulty we encountered and make suggestions as to how we would begin to tackle them. The very last section concerns a simple suggestion that might prove to be worthwhile even though it does not directly touch upon our concerns.

6.1 Diskoids and common extensions

Our central result Theorem 4.3 starts with an optimising root a of an ABKP Q, and we show that

$$\overline{\nu}_{a,\delta}\big|_{K[X]} = \mu_Q, \quad \delta = \epsilon_\mu(Q).$$

In [34], the authors ask whether or not this kind of result applies for other roots of Q. First of all, they start by proving an extension result, similar to our own Theorem 4.3. In this whole section, we will set μ a valuation over $K[X]$, extending a valuation ν over K. Fix a valuation $\overline{\nu}$ extending ν over $\overline{K}$, the algebraic closure of K. For any $f \in \overline{K}[X]$, we write

$$\mathcal{R}(f) := \{x \in \overline{K};\ f(x) = 0\}$$

the set of roots of f in $\overline{K}$.

Theorem 6.1. [34, Theorem 5.6]
The valuation μ is valuation transcendental if and only if it is a truncated valuation, *i.e.*, there is an ABKP Q such that $\mu = \mu_Q$.
Furthermore, for any common extension $\overline{\mu}$ of μ_Q and $\overline{\nu}$, there is a root $a \in \mathcal{R}(Q)$ such that

$$\overline{\mu} = \overline{\nu}_{a,\delta}$$

where

$$\delta = \max \overline{\mu}(X - \overline{K}) := \{\overline{\mu}(X - c);\ c \in \overline{K}\}.$$

This value δ is also equal to $\epsilon_\mu(Q)$ as we have seen in Chapter 3.

We feel it is relevant to introduce a new notion concerning ABKPs.

Definition 6.2. Let Q be an ABKP for μ. Then we will say that Q is *homoradicial* if for all roots $a \in \mathcal{R}(Q)$, $\overline{\nu}_{a,\delta}$ is a common extension of μ_Q and $\overline{\nu}$.

The results of [34] prove the following theorem.

Theorem 6.3. [34, Theorem 6.6] If Q is homoradicial, then any of its immediate successors Q^* is also homoradicial.

We believe that studying whether or not an ABKP is homoradicial or not is linked to our question concerning diskoids. Indeed we have the following lemma, which is a weaker result than [34, Lemma 6.1].

Lemma 6.4. By Theorem 4.3, there is a root a of our ABKP Q, such that $\overline{\nu}_{a,\delta}$ is a common extension of $\overline{\nu}$ and μ_Q. Consider $b \in \mathcal{R}(Q)$. If $\overline{\nu}_{b,\delta}$ is a common extension of μ_Q and $\overline{\nu}$, then for all $g \in K[X]$, $\deg g < \deg Q$.

Proof. Consider $g \in K[X]$, $\deg g < \deg Q$, so that by Theorem 3.5

$$\max_{g(c)=0} \overline{\nu}_{b,\delta}(X-c) = \delta_{\overline{\nu}_{b,\delta}}(g) = \epsilon_{\mu_Q}(g) = \epsilon_\mu(g) < \epsilon(Q) = \delta,$$

thus for all $c \in \overline{K},\ g(c) = 0$,

$$\min\{\delta, \overline{\nu}(b-c)\} = \overline{\nu}_{b,\delta}(X-c) < \delta,$$

in other words,

$$\forall c \in \overline{K},\ g(c) = 0, \quad \overline{\nu}_{b,\delta}(X-c) = \overline{\nu}(b-c).$$

If we expand g into linear factors

$$g(X) = c\prod_{i=1}^{n}(X-c_i),\ c, c_i \in \overline{K},$$

then

$$\overline{\nu}_{b,\delta}(g) = \overline{\nu}(c) + \sum_{i=1}^{n} \overline{\nu}_{b,\delta}(X-c_i) = \overline{\nu}(c) + \sum_{i=1}^{n} \overline{\nu}(b-c_i) = \overline{\nu}(g).$$

□

Assume now that $b \in \mathcal{R}(Q)$ is a root of an ABKP Q such that $\overline{\nu}_{b,\delta}$ is a common extension to μ_Q and $\overline{\nu}$. Recall the decomposition of diskoids, $D(Q, \mu(Q))$ is the union of balls centred around the roots a of Q and of radius $\epsilon(a; Q, \mu(Q))$. By the Lemma 6.4 above

$$\forall i \geqslant 1,\ \overline{\nu}(\partial_i Q(b)) = \overline{\nu}_{b,\delta}(\partial_i Q) = \mu_Q(\partial_i Q),$$

so by Lemma 5.3

$$\epsilon(b; Q, \mu(Q)) = \max_{i\geqslant 1} \frac{\mu(Q) - \overline{\nu}(\partial_i Q(b))}{i} = \max_{i \geqslant 1} \frac{\mu(Q) - \mu(\partial_i Q)}{i} = \epsilon_\mu(Q).$$

If Q is homoradicial (and $\epsilon_\mu(Q) \in \Phi(\overline{\nu})$), then for all $b \in \mathcal{R}$, the valuation $\overline{\nu}_{b,\delta} = \overline{\nu}_{B(b,\epsilon(b;Q,\mu(Q))}$ is an extension of μ_Q to $\overline{K}[X]$ thus the mapping $\overline{\nu}_{D(Q,\mu(Q))}$ is indeed the valuation μ_Q.

It is our opinion that proof of the main result [34, Theorem 6.6] owes its success to the following structure theorem for key polynomials which are immediate successors, *i.e.,* non-limit key polynomials.

Proposition 6.5. [34, Prop. 4.10], [32, Theorem 9.4], [55, Theorem 1.11]
Consider two ABKPs Q and Q^* for μ, such that $\deg Q \leqslant \deg Q^*$ and Q^* is not μ_Q-equivalent to Q. If we write $Q^* = q_n Q^n + \ldots + q_t Q^t + \ldots + q_0$ the Q-expansion of Q^*, then

- $q_n = 1$.
- $\mu_Q(Q^*) = n\mu(Q) = \mu(q_t) + t\mu(Q)$.

Remark 6.6. The hypotheses of this theorem are satisfied if Q^* is an immediate successor of Q.

6.2 Extending the correspondence and small extension closure

We come back to our correspondence results in Chapter 5. We have established such a geometric interpretation for RT extensions and we wish to discuss possible ways to extend this interpretation to other valuations, namely VT extensions.

6.2.1 A final criterion

We start off by giving another characterization of the type of μ/ν, according to the nature of the set $\overline{\mu}(X-\overline{K})$.

Lemma 6.7. $\overline{\mu}(X-\overline{K})\cap\Phi(\overline{\nu})$ is an initial segment of $\Phi(\overline{K})$. If $\overline{\mu}(X-\overline{K})$ has a maximal element δ, then the extension $\overline{\mu}/\overline{\nu}$ is valuation transcendental and

1. if $\delta\in\Phi(\overline{\nu})$, then μ/ν is RT and $\overline{\mu}(X-\overline{K})\subseteq\Phi(\overline{\nu})$.
2. if $\delta\notin\Phi(\overline{\nu})$, then
$$\overline{\mu}(X-\overline{K})\setminus\{\delta\}\subseteq\Phi(\overline{\nu}).$$

If $\overline{\mu}(X-\overline{K})$ does not have a maximal element, then $\overline{\mu}(X-\overline{K})\subseteq\Phi(\overline{\nu})$ and the extension μ/ν is valuation algebraic ($\overline{\mu}/\overline{\nu}$ is thus an immediate extension).

Proof. Consider $\gamma=\overline{\mu}(X-a)\in\overline{\mu}(X-\overline{K})\cap\Phi(\overline{\nu})$ and $\eta=\overline{\nu}(x)\in\Phi(\overline{\nu})$, where $a,x\in\overline{K}$ and assume $\eta<\gamma$. Then we have by the ultrametric property

$$\overline{\mu}(X-a+x)=\min\{\overline{\mu}(X-a),\overline{\nu}(x)\}=\eta$$

so $\eta\in\overline{\mu}(X-\overline{K})\cap\Phi(\overline{\nu})$. The set $\overline{\mu}(X-\overline{K})\cap\Phi(\overline{\nu})$ is an initial segment of $\Phi(\overline{\nu})$.
Suppose that $\overline{\mu}(X-\overline{K})$ has a maximal element $\delta=\overline{\mu}(X-a)$, $a\in\overline{K}$. By Theorem 3.16 $\overline{\mu}=\overline{\nu}_{a,\delta}$, so that if $\delta\in\Phi(\overline{\nu})$, then $\overline{\mu}(X-\overline{K})\subseteq\Phi(\overline{\nu})$. Assume $\delta\notin\Phi(\overline{\nu})$ and take any $\eta\in\overline{\mu}(X-\overline{K})$, $\eta<\delta$. Set $b\in\overline{K}$ such that $\eta=\overline{\mu}(X-b)$, so that

$$\eta=\overline{\mu}(X-b)=\overline{\mu}(X-a+a-b)=\min\{\delta,\overline{\nu}(a-b)\}=\overline{\nu}(a-b)\in\Phi(\overline{\nu}),$$

which ends the argument.
Assume now that $\overline{\mu}(X-\overline{K})$ does not have a maximal element and let us show that $\overline{\mu}(X-\overline{K})\subseteq\Phi(\overline{\nu})$. Take $\eta\in\overline{\mu}(X-\overline{K})$ and $b\in\overline{K}$ such that $\eta=\overline{\mu}(X-b)$. Since η is not maximal, there is a $\delta=\overline{\mu}(X-a)$, $a\in\overline{K}$ such that $\delta>\eta$. By the ultrametric property

$$\eta=\min\{\delta,\eta\}=\overline{\mu}((X-b)-(X-a))=\overline{\nu}(a-b)\in\Phi(\overline{\nu}).$$

$\square$

We will now assume that $\overline{\mu}=\overline{\nu}_{a,\delta}$ with $a\in\overline{K}$. For VT extensions the values of $\overline{\mu}$ are not in $\Phi(\overline{\nu})$ anymore, so we need to generalize the valuations $\overline{\nu}_D$, with $D\subset\overline{K}$, since they take values inside $\Phi(\overline{\nu})$. A possible solution consists of taking values not in $\Phi(\overline{\nu})$ but taking values in a complete ordered abelian group Γ in which we can embed $\Phi(\overline{\nu})$.

Definition 6.8. An ordered abelian group $(\Gamma,\leqslant)$ is *complete* if every non-empty set of Γ admits a least upper bound (supremum) and a greatest lower bound (infinum).

Thus if Γ is complete then the values $\sup\overline{\mu}(X-\overline{K})$ and $\inf_{x\in B(a,\delta)}\overline{\nu}(f(x))$ are always well-defined, even if they may not be defined in $\Phi(\overline{\nu})$. If we do this operation then we are on our way to defining our valuations. We give a quick presentation of a candidate that enjoys this property and more.

6.2.2 Small extensions

Let $\Gamma\overset{i}{\hookrightarrow}\Lambda$ be an embedding of ordered abelian groups and define

$$\Gamma^{\Lambda}:=\{\lambda\in\Lambda;\ \exists n\in\mathbb{N}^*,\ n\lambda\in i(\Gamma)\}.$$

Definition 6.9. We say that an embedding $\Gamma\overset{i}{\hookrightarrow}\Lambda$ is a *small extension* if $\Lambda/\Gamma^{\mathrm{com}}$ is a cyclic group, *i.e.*, isomorphic to $\mathbb{Z}$.

Recall our situation having two valuations μ over $K[X]$ extending ν over K. Passing to the algebraic closure $\overline{K}$ we have a valuative pair $(\overline{K}[X],\overline{\mu})$ extending $(\overline{K},\overline{\nu})$. The canonical inclusions $\Phi(\nu)\hookrightarrow\Phi(\mu)$ and $\Phi(\overline{\nu})\hookrightarrow\Phi(\overline{\mu})$ are small extensions [30] and $\Phi(\overline{\nu})=\mathbb{Q}\otimes\Phi(\nu)$. In [38], Nart constructs ordered sets

$$\Gamma\subset\Gamma_{\mathbb{Q}}\subset\Gamma_{\mathbb{R}}\subset\Gamma_{\mathrm{sme}}$$

that classify small extensions in the following way.

Theorem 6.10. [38, Prop. 2.17] Let $\Gamma \xrightarrow{i} \Lambda$ be a small extension of Γ and let $\lambda \in \Lambda$ be such that $\Lambda = \big\langle \Gamma^{\mathrm{com}}, \lambda \big\rangle$. Let $\Gamma^{\mathrm{com}} \xrightarrow{\simeq} \Delta \subset \Gamma_{\mathbb{Q}}$ be the canonical embedding of Γ^{com} into $\Gamma_{\mathbb{Q}}$. Then there is a unique $\beta \in \Gamma_{\mathrm{sme}}$ for which there exists a unique isomorphism of ordered groups

$$\Lambda \xrightarrow{\simeq} \big\langle \Delta, \beta \big\rangle,$$

sending λ to β and and extending the canonical isomorphism $\Gamma^{\mathrm{com}} \xrightarrow{\simeq} \Delta$. The set Γ_{sme} is called the *small extension closure of* Γ .

Remark 6.11. 1. The group $\Gamma_{\mathbb{Q}}$ is simply $\Gamma \otimes_{\mathbb{Z}} \mathbb{Q}$, the divisible hull of Γ. It enjoys the following universal property: for any divisible group G and any morphism of ordered abelian groups $\Gamma \xrightarrow{f} G$, there exists a unique morphism of ordered abelian groups $\Gamma_{\mathbb{Q}} \xrightarrow{\widetilde{f}} G$, extending f, *i.e.,* making the diagram

$$\begin{tikzcd}
\Gamma \arrow[r, "f"] \arrow[d] & G \\
\Gamma_{\mathbb{Q}} \arrow[ur, dashed, "\widetilde{f}"'] &
\end{tikzcd}$$

commutative, where the vertical map is simply the canonical $\gamma \in \Gamma \longmapsto \gamma \otimes 1 \in \Gamma_{\mathbb{Q}}$.

2. Nart defines the *equal-rank closure* $\Gamma_{\mathbb{R}}$ [38, Definition p.15], which is not to be confused with $\Gamma \otimes_{\mathbb{Z}} \mathbb{R}$, that allows to classify the equal rank extensions. Indeed, if we adopt the notations of Theorem 6.10, then $\mathrm{rk}(\Lambda) = \mathrm{rk}(\Gamma) \iff \beta \in \Gamma_{\mathbb{R}}$. Otherwise $\mathrm{rk}(\Lambda) = \mathrm{rk}(\Gamma) + 1$.

Having defined these sets, Nart proceeds to prove that Γ_{sme} is complete [38, Theorem 3.7] and he applies it to extending our result Lemma 5.3. We establish once and for all an embedding $\Phi(\overline{\nu}) \overset{\ell_\nu}{\hookrightarrow} \Phi(\nu)_{\mathrm{sme}}$ and we can define our balls and diskoids as usual as subsets of $\overline{K}$. Our correspondence between RT extensions of $\overline{K}$ and balls is thus generalised to valuation transcendental valuations as a result of the following theorem.

Theorem 6.12. [38, Prop. 4.5]
Suppose that μ/ν is a valuation transcendental extension. Then $\overline{\mu} = \overline{\nu}_{a,\delta}$ for some $a \in \overline{K}$ and $\delta \in \Phi(\overline{\nu})_{\mathrm{sme}}$. If we set

$$B = B(a,\delta) = \{x \in \overline{K};\, \overline{\nu}(x-a) \geqslant \delta\}$$

then the map

$$\begin{aligned} \overline{\nu}_B :\ \overline{K}[X] &\longrightarrow \Phi(\overline{\nu})_{\mathrm{sme}} \\ f &\longmapsto \inf\Big\{\ell_\nu\big(\overline{\nu}(f(x))\big);\ x \in B\Big\} \end{aligned}$$

is well-defined and is equivalent to the valuation $\overline{\mu}$, in the following sense:

1. if $\delta \in \mathbb{Q}\Phi(\nu) = \Phi(\overline{\nu})$, then this is just the result of Lemma 5.3. This is the case where μ/ν is RT.
2. if $\delta \notin \mathbb{Q}\Phi(\nu)$ and $\delta \in \Phi(\overline{\nu})_{\mathbb{R}}$, then μ and ν have the same rank, but not the same rational rank.
3. if $\delta \notin \Phi(\overline{\nu})_{\mathbb{R}}$, then $\delta > \Phi(\overline{\nu})$, μ/ν is VT and $\overline{\mu}$ is equivalent to the rank incrementation $[\overline{\nu}_{a,\infty}]$, *i.e.,* if $g = \sum_k a_k(X-a)^k$, then
$$[\overline{\nu}_{a,\infty}](g) = \min_k(k, \overline{\nu}(a_k)),$$
where the $(k, \overline{\nu}(a_k))$ are ordered lexicographically.

We can furthermore prove that $\overline{\nu}_{a,\delta} = \overline{\nu}_{a',\delta'}$ if and only if $B(a,\delta) = B(a',\delta')$ (see Lemma 3.12 and [38, Cor. 4.4]).

Observe that the valuation $\overline{\nu}_B$ is defined for any polynomial in $\overline{K}[X]$ whereas it is restricted to $K[X]$ in [38], but the proof works just as well.

6.3 From valuations to quasi-valuations

In this section we touch upon the axioms of valuations themselves. If we relax the condition (V1) to

$$\text{(V1')}\ \forall a, b \in R,\ \nu(ab) \geqslant \nu(a) + \nu(b),$$

we arrive at the notion of *quasi-valuation* or *order functions* (see [31]). We can prove that an order function still defines a filtration and thus a graded algebra which will be an integral domain if and only if ν verifies the stronger axiom (V1). The rules of Remark 1.5 remain the same, except for the first one, which now becomes

$$\begin{array}{l} 1'.\ \forall a, b \in R,\ \mathrm{in}_\nu(a \cdot b) = \mathrm{in}_\nu(a) \cdot \mathrm{in}_\nu(b) \text{ if } \nu(ab) = \nu(a) + \nu(b), \\ \ \text{otherwise } \nu(ab) > \nu(a) = \nu(b) \text{ and } \mathrm{in}_\nu(a \cdot b) \neq 0 = \mathrm{in}_\nu(a) \cdot \mathrm{in}_\nu(b). \end{array}$$

This allows us to deal with elements that are not necessarily valuations, stricto sensu, but are still interesting objects of inquiry. For instance, we can postpone the proof of Corollary 2.20, which shows that the truncation μ_Q is a valuation. Our Theorem 4.3 still holds nonetheless, so we can show that $\overline{\nu}_{a,\delta}\big|_{K(X)} = \mu_Q$ and since it is easier to show that $\overline{\nu}_{a,\delta}$ is valuations, we get an alternative proof of μ_Q being a valuation.
Fialy let us observe that the maps $\overline{\nu}_D$, where D is a diskoid, are not always valuations, as we have shown in a counter-example (see Remark 5.10), however they are quasi-valuations.

6.4 Kaplansky embeddings

As a final idea for future developments, we endeavor to potentially link Kaplansky embeddings with sequences of ABKPs (or MVKPs for that matter). In [25], Kaplansky showed that any valued field (K, ν) can be seen as a sub-valued field of a generalized power series field, equipped with its order valuation. More precisely write Γ for the divisible hull of $\Phi(\nu)$, $\overline{\kappa(\nu)}$ for the algebraic closure of the residue field of ν, and

$$A := \begin{cases} \overline{\kappa(\nu)} & \text{if } \mathrm{char}(K) = \mathrm{char}(\kappa(\nu)) \text{ (equicharacteristic case)} \\ W(\overline{\kappa(\nu)}) & \text{if } \mathrm{char}(K) = 0 \text{ and } \mathrm{char}(\kappa(\nu)) = p > 0 \text{ (mixed characteristic case),} \end{cases}$$

where $W(R)$ represents the Witt vector ring of R. One then constructs the ring $A((\varpi^\Gamma))$ of Hahn power series (also known as Malcev-Neumann power series or simply generalized power series) with coefficients in A and and exponents in the value group Γ. Poonen also invented a mixed characteristic version in [45], thus extending Kaplansky's result. Later still, San Saturnino gave an effective embedding of any local complete regular valued ring of rank 1, by using abstract key polynomials in [48].

Kaplansky's embedding theorem states that we can find an embedding of valued fields

$$(K, \nu) \xhookrightarrow{\vartheta} \big(A((\varpi^\Gamma)), \mathrm{ord}_\varpi\big),$$

where ord_ϖ is the order of multiplicity in ϖ. If we apply this result to $(K(X), \mu)$, an extension of (K, ν), we can write

$$\forall f \in K[X],\ \mu(f) = \mathrm{ord}_\varpi(f^\vartheta(\theta)).$$

Here $\theta = \theta(\varpi) \in A((\varpi^\Gamma))$ is the image via ϑ of X and if $f = \sum_i a_i X^i$, then $f^\vartheta = \sum_i \vartheta(a_i) X^i$. Our goal would then be to find an explicit link between sequences of key polynomials and the series of truncations of the "arc" $\theta(\varpi)$.

Bibliography

[1] S. Abhyankar, *On the ramification of algebraic functions*, American Journal of Mathematics **77** (1955), no. 3, 575–592.

[2] ______, *Local uniformization on algebraic surfaces over ground fields of characteristic $p \neq 0^1$*, Annals of Mathematics **63** (1956), no. 3, 491–526.

[3] ______, *On the valuations centered in a local domain*, American Journal of Mathematics **78** (1956), no. 2, 321–348.

[4] ______, *Lectures on expansion techniques in algebraic geometry*, Tata Institute of Fundamental Research Lectures on Mathematics and Physics, Tata Institute of Fundamental Research, Bombay, 1977.

[5] ______, *Ramification theoretic methods in algebraic geometry (am-43)*, vol. 43, Princeton University Press, 2016.

[6] V. Alexandru and N. Popescu, *Sur une classe de prolongements à K(X) d'une valuation sur un corps K*, Revue Roumaine de Mathématiques Pures et Appliquées **5** (1988), 393–400.

[7] V. Alexandru, N. Popescu, and A. Zaharescu, *A theorem of characterisation of residual transcendental extension*, Journal of Mathematics of Kyoto University **28** (1988), 579–592.

[8] ______, *All valuations on K(X)*, Journal of Mathematics of Kyoto University **30** (1990), 281–296.

[9] ______, *Minimal pairs of definition of a residually transcendental extension of a valuation*, Journal of Mathematics of Kyoto University **30** (1990), 207–225.

[10] A. Benguş-Lasnier, *Minimal pairs, truncations and diskoids*, Journal of Algebra **579** (1 August 2021), 388–427.

[11] N. Bourbaki, *Algèbre commutative, chapitres 5 à 7*, Éléments de mathématique, Springer-Verlag, Berlin Heidelberg, 2006.

[12] D. Brink, *New light on Hensel's lemma*, Expositiones Mathematicae **24** (2006), 291–306.

[13] V. Cossart and O. Piltant, *Resolution of singularities of threefolds in positive characteristic. I.: Reduction to local uniformization on Artin–Schreier and purely inseparable coverings*, Journal of Algebra **320** (2008), no. 3, 1051–1082.

[14] S. D. Cutkosky and H. Mourtada, *Defect and local uniformization*, Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas **113** (2019), 4211–4226.

[15] S. D. Cutkosky and O. Piltant, *Ramification of valuations*, Advances in Mathematics **183** (2004), no. 1, 1–79.

[16] S. D. Cutkosky and P. A. Vinh, *Valuation semigroups of two dimensional local rings*, Proceedings of the London Mathematical Society **108** (2014), no. 2, 350–384.

[17] J. Decaup, *Uniformisation locale simultanée par monomialisation d'éléments clefs*, Ph.D. thesis, École doctorale Mathématiques, informatique et télécommunications (Toulouse), 2018.

[18] J. Decaup, W. Mahboub, and M. Spivakovsky, *Abstract key polynomials and comparison theorems with the key polynomials of Mac Lane - Vaquie*, Illinois J. Math. **62** (2018), no. 1-4, 253–270.

[19] A. J. Engler and A. Prestel, *Valued fields*, Springer Monographs in Mathematics (SMM), Springer-Verlag, Berlin Heidelberg, 2005.

[20] C. Favre and M. Jonsson, *The valuative tree*, Lecture Notes in Mathematics, Springer-Verlag, Berlin-Heidelberg, 2008.

[21] F. J. H. Govantes, W. Mahboub, M. Acosta, and M. Spivakovsky, *Key polynomials for simple extensions of valued fields*, https://arxiv.org/abs/1406.0657, 2014 (last updated 2018), preprint.

[22] B. Green, *On curves over valuation rings and morphisms to* $\mathbb{P}^1$, Journal of Number Theory **59** (1996), 262–290.

[23] F. J Herrera Govantes, M. A. Olalla Acosta, and M. Spivakovsky, *Valuations in algebraic field extensions*, Journal of Algebra **312** (2007), 1033–1074.

[24] S. Ishii, *Maximal divisorial sets in arc spaces*, Algebraic geometry in East Asia—Hanoi 2005 (Tokyo), Math. Soc. Japan, 2008, pp. 237–249.

[25] I. Kaplanski, *Maximal fields with valuations*, Duke Math. J. **9** (1942), 303–321.

[26] ______, *Maximal fields with valuations, ii*, Duke Math. J. **12** (1945), 243–248.

[27] S. K. Khanduja, N. Popescu, and K. W. Roggenkamp, *On minimal pairs and residually transcendental extensions of valuations*, Mathematika **49** (2002), 93–106.

[28] N. Koblitz, *p-adic numbers, p-adic analysis and zeta functions*, Graduate Texts in Mathematics, Springer-Verlag, New York, 1984.

[29] F.-V. Kuhlmann, *Valuation theoretic and model theoretic aspects of local uniformization*, Resolution of Singularities: A research textbook in tribute to Oscar Zariski Based on the courses given at the Working Week in Obergurgl, Austria, September 7-14, 1997 (Basel), Birkhäuser Basel, 2000, pp. 381–456.

[30] ______, *Value groups, residue fields, and bad places of rational function fields*, Transactions of the American Mathematical Society **356** (2004), no. 11, 4559–4600.

[31] M. Lejeune-Jalabert and B. Teissier, *Cloture intégrale des idéaux et équisingularité, avec 7 compléments*, Annales de la Faculté des Sciences de Toulouse **Vol 17** (2008), 781–859.

[32] S. Mac Lane, *A construction for absolute values in polynomial rings*, Trans. Amer. Math. Soc. **40** (1936), 363–395.

[33] ______, *A construction for prime ideals as absolute values of an algebraic field*, Duke Mathematical Journal **2** (1936), 492–510.

[34] W. Mahboub, A. Mansour, and M. Spivakovsky, *On common extensions of valued fields*, https://arxiv.org/abs/2007.13578, 2020.

[35] W. Mahboub and Mark Spivakovsky, *Key polynomials in dimension 2*, Singularities - Kagoshima 2017, Proceedings of the 5th Franco-Japanese-Vietnamese Symposium on Singularities (Masaharu Ishikawa, ed.), World Scientific, 2020.

[36] H. Mourtada, *Jet schemes and generating sequences of divisorial valuations in dimension two*, Michigan Math. J. **66** (2017), no. 1, 155–174.

[37] E. Nart, *Key polynomials over valued fields*, Publicacions Matemàtiques de la Universitat Autònoma de Barcelona **64** (2020), no. 1, 195–232.

[38] ———, *Small extensions of abelian ordered groups*, https://arxiv.org/abs/2011.05975, 2020, preprint.

[39] ———, *Mac Lane-Vaquié chains of valuations on a polynomial ring*, Pacific Journal of Mathematics **311** (2021), no. 1, 165–195.

[40] J. Novacoski, *Key polynomials and minimal pairs*, Journal of Algebra **523** (2019), 1–14.

[41] ———, *On Mac Lane-Vaquié key polynomials*, Journal of Pure and Applied Algebra **225** (2021), 106644.

[42] J. Novacoski and M. Spivakovsky, *Key polynomials and pseudo-convergent sequences*, Journal of Algebra **495** (2018), 199–219.

[43] A. Obus and S. Wewers, *Explicit resolution of weak wild quotient singularities on arithmetic surfaces*, Journal of Algebraic Geom. (2019), no. 29, 691–728.

[44] A. Ostrowski, *Untersuchungen zur arithmetischen theorie der körper*, Mathematische Zeitschrift **39** (1935), 269–404.

[45] B. Poonen, *Maximally complete fields*, L'Enseignement Mathématique **39** (1993), 87–106.

[46] L. Popescu and N. Popescu, *On the residual transcendental extensions of a valuation. key polynomials and augmented valuation*, Tsukuba Journal of Mathematics **15** (1991), no. 1, 57–78.

[47] J. P. Rüth, *Models of curves and valuations*, Ph.D. thesis, Universität Ulm, 2014.

[48] J.-C. San Saturnino, *Théorème de Kaplansky effectif pour des valuations de rang 1 centrées sur des anneaux locaux reguliers et complets*, Annales des l'Institut Fourier, Grenoble **64** (2014), no. 3, 1177–1202.

[49] ———, *Defect of an extension, key polynomials and local uniformization*, Journal of Algebra **481** (2017), 91–119.

[50] M. Spivakovsky, *Valuations in function fields of surfaces*, American Journal of Mathematics **112** (1990), no. 1, 107–156.

[51] B. Teissier, *Appendix to Oscar Zariski's article: The moduli problem for plane branches*, University Lecture Series, no. 39, pp. 111–143, American Mathematical Society, Providence, RI, 2006.

[52] M. Vaquié, *Valuations*, pp. 539–590, Birkhäuser Basel, Basel, 2000.

[53] ———, *Valuations and local uniformisation*, Advanced Studies in Pure Mathematics **43** (2006), 477–527.

[54] ———, *Algebre graduée associee a une valuation de $K[x]$*, Advanced Studies in Pure Mathematics **46** (2007), no. 2, 259–271.

[55] ———, *Extension d'une valuation*, Transactions of the American Mathematical Society **359** (2007), 3439–3481.

[56] ———, *Famille admissible de valuations et défaut d'une extension*, Journal of Algebra **311** (2007), no. 2, 859–876.

[57] ———, *Extension de valuation et polygone de newton*, Annales de l'Institut Fourier **58** (2008), no. 7, 2503–2541.

[58] ———, *Augmented valuation and minimal pair*, https://arxiv.org/abs/2005.03298, 2020, preprint.

[59] O. Zariski and P. Samuel, *Commutative algebra, Volume II*, Graduate Texts in Mathematics, Springer-Verlag, Berlin Heidelberg, 1960.

List of Symbols and Notations

Chapter 1 - Basics on Valuation

$\mathbb{N}$	the set of natural numbers, *i.e.,* the non-negative integers, 13
$\mathbb{N}^*$	the set of positive integers, 13
Γ_∞	disjoint union of an ordered abelian group with a maximal (infinite) element, 13
(R, ν)	a valuative pair, 13
$\mathbb{Q}_p$	the ring of p-adic numbers, 14
$k((T))$	the field of (Laurent) formal power series over the field k, 14
ord_T	order valuation along the variable T, 14
$\mathrm{gr}_\nu(R)$	graded ring of the valuative pair (R, ν), 14
$\underset{\nu}{\sim}$	ν-equivalence relation, 14
$\underset{\nu}{\mid}$	ν-divisibility relation, 15
$\mathrm{supp}(\nu)$	support of a valuation ν, 15
$\widetilde{\mathrm{gr}}_\nu(R)$	extended graded algebra of a valuative pair (R, ν), 15
$\mathrm{gr}\phi$	graded morphism associated to a morphism ϕ of valued pairs, 15
$\mathscr{O}(\nu)$	valuation ring of a valuation ν, 16
$\mathfrak{m}(\nu)$	maximal ideal of a valuation ν, 16
$\kappa(\nu)$	residue field of a valuation ν, 16
$\mathrm{Int}(R, K)$	integral closure of a ring R in a field K, 17
$\Phi(\nu)$	value group of a valuation ν, 18
$\mathrm{rk}(\nu)$	rank or height of a valuation ν, 18
$r(\nu)$	rational rank of a valuation ν, 18
$\nu_{a,\gamma}$	(depth zero) valuation defined by a and γ, 20
$\nu' \circ \overline{\nu}$	composition of the valuations ν' and $\overline{\nu}$, 20
$\mathrm{in}_\nu f$	initial form of an element f of a ring with respect to a valuation ν, 14

Chapter 2 - Key Polynomials

Chapter 3 - Passing to the algebraic closure

Chapter 4 - Galois Descent for truncations of valuations

Chapter 5 - Diskoids, towards a geometric interpretation of truncations

Index