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Multivariable zeta functions, multivariable tauberian theorems and applications

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sous la direction de Driss ESSOUABRI

pour obtenir le grade de

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Chapter I

General Introduction

1 English version

This thesis consists of two parts. In the first part (Chapter II) we obtain, by two different methods, closed and explicit formulas for the values of a class of multivariable zeta functions at non-positive integers points. In the second part (Chapters III, IV and V), we establish some new results on the mean values of multivariable multiplicative functions. Let $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_n) \in \mathbb{C}^n$ and $\mathbf{b} = (b_1, \dots, b_n) \in \mathbb{C}^n$ be two vectors of complex parameters such that $\operatorname{Re}(\gamma_j) > 0$ and $\operatorname{Re}(b_j) > -\operatorname{Re}(\gamma_1)$ for all $j \in \{1, \dots, n\}$. The generalized Euler-Zagier multiple zeta function is defined formally for $\mathbf{s} = (s_1, \dots, s_n) \in \mathbb{C}^n$ by

$$\zeta_n(\mathbf{s}; \boldsymbol{\gamma}; \mathbf{b}) := \sum_{\substack{m_1 \geq 1 \\ m_2, \dots, m_n \geq 0}} \frac{1}{\prod_{j=1}^n (\gamma_j m_1 + \dots + \gamma_j m_j + b_j)^{s_j}} \quad (\text{I.1})$$

This series converges absolutely in the domain

$$\mathcal{D}_n := \{\mathbf{s} = (s_1, \dots, s_n) \in \mathbb{C}^n \mid \operatorname{Re}(s_j + \dots + s_n) > n + 1 - j \text{ for all } j \in \{1, \dots, n\}\} \quad (\text{I.2})$$

and has meromorphic continuation to the whole complex space \mathbb{C}^n whose singularities are located in the union of the hyperplanes

$$s_j + \dots + s_n = (n + 1 - j) - k_j \quad (1 \leq j \leq n, k_1, \dots, k_n \in \mathbb{N}_0).$$

Moreover it is known that for $n \geq 2$, almost all non-positive integers points lie in the singular locus above and are points of indeterminacy. The study of the directional values of these multiple zeta functions at these points was first considered by S. Akiyama, S. Egami and Y. Tanigawa [1]. These values are now well understood thanks to several works such as [20], [25], [23] and [10].

In [20], Y. Komori proved that for $\mathbf{N} = (N_1, \dots, N_n) \in \mathbb{N}_0^n$ and $\boldsymbol{\theta} = (\theta_1, \dots, \theta_n) \in \mathbb{C}^n$ such that $\theta_j + \dots + \theta_n \neq 0$ for all $j \in \{1, \dots, n\}$, the directional limit

$$\zeta_n^{\boldsymbol{\theta}}(-\mathbf{N}; \boldsymbol{\gamma}; \mathbf{b}) := \lim_{t \rightarrow 0} \zeta_n(-\mathbf{N} + t\boldsymbol{\theta}; \boldsymbol{\gamma}; \mathbf{b}) \quad (\text{I.3})$$

exists, and expressed this limit in terms of \mathbf{N} , $\boldsymbol{\theta}$ and generalized Bernoulli numbers defined implicitly as coefficients of some multiple series.

In [10], D. Essouabri and K. Matsumoto obtained, by a different method, a closed explicit formula for $\zeta_n^{\boldsymbol{\theta}}(-\mathbf{N}; \boldsymbol{\gamma}; \mathbf{b})$ in terms of \mathbf{N} , $\boldsymbol{\theta}$ and only classical Bernoulli numbers.

Recently Essouabri and Matsumoto [8] extended this result to a more general class of multivariable zeta functions defined by

$$\zeta_n(\mathbf{s}, \mathbf{P}) = \sum_{m_1, \dots, m_n \geq 1} \prod_{j=1}^n P_j(m_1, \dots, m_j)^{-s_j}, \quad (\text{I.4})$$

where for all $j \in \{1, \dots, n\}$, $P_j \in \mathbb{R}[X_1, \dots, X_j]$ is a polynomial satisfying some general assumptions. In this general case, instead of Bernoulli numbers, certain periods appear in the expression of values. Moreover, the nonlinearity of the polynomials P_j reveals values at the n -tuple of non-positive integers which can be transcendent even if all the coefficients of the polynomials P_j are algebraic.

Consider now the twisted case. Let $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$, $\boldsymbol{\mu} \in (\mathbb{T} \setminus \{1\})^k \times \{1\}^{n-k}$, $Q \in \mathbb{R}[X_1, \dots, X_n]$ and $\mathbf{P} = (P_1, \dots, P_T) \in (\mathbb{R}[X_1, \dots, X_n])^T$, we associate to the triplet $(Q; \mathbf{P}; \boldsymbol{\mu})$ the (k, n) -twisted MZF (non-twisted if $\boldsymbol{\mu} = \mathbf{1}$) defined formally for $\mathbf{s} = (s_1, \dots, s_T) \in \mathbb{C}^T$ by

$$Z(Q; \mathbf{P}; \boldsymbol{\mu}; \mathbf{s}) = \sum_{\mathbf{m}=(m_1, \dots, m_n) \in \mathbb{N}^n} \frac{Q(m_1, \dots, m_n) \mu_1^{m_1} \dots \mu_k^{m_k}}{\prod_{t=1}^T P_t(m_1, \dots, m_n)^{s_t}} \quad (\text{I.5})$$

If $P_1 \dots P_T(\mathbf{X}) \rightarrow +\infty$ as $\|\mathbf{X}\| \rightarrow +\infty$ and $\mathbf{X} \in [1, +\infty)^n$, the existence of the meromorphic continuation of these Dirichlet series to whole space \mathbb{C}^T follows from the method introduced by Essouabri ([15], [16]) if P_1, \dots, P_T verify the probably optimal hypothesis H_0S (see Definition 1 below). Moreover, the complexity of the singular locus of these Dirichlet series is maximal in the non-twisted case (i.e. $\boldsymbol{\mu} = \mathbf{1}$) and we expect to have a holomorphic extension in the completely twisted case (i.e $\boldsymbol{\mu} \in (\mathbb{T} \setminus \{1\})^n$). A counterexample gave by Crisenoy shows that this is not the case under the general assumption H_0S .

Under the slightly more restrictive assumption (HDF) (see Definition 1 below), Marc de Crisenoy ([6] Théorèmes A et B) proved in the completely twisted case (i.e $\boldsymbol{\mu} \in (\mathbb{T} \setminus \{1\})^n$) the holomorphy of these Dirichlet series and gave simple closed and explicit formulas for their values at T -tuples of non-positive integers.

In the not completely twisted case, the existence of singularities makes the study of values at non-positive integers points more difficult. In addition to the works mentioned above, Essouabri and Matsumoto [11] obtained, by combining the result of Crisenoy and the Mellin-Barnes integral representation formula (27), the values of the $(n-1, n)$ -twisted and $(n-2, n)$ -twisted MZF in the cases where the P_j 's are linear or diagonal polynomials. *In chapter 2 of this thesis, we consider a new class of $(n-1, n)$ -twisted MZF (see Definition (II.10)). More precisely, we will determine (see Theorems 2 and 3 of Chapter II) for this class of $(n-1, n)$ -twisted MZF, by two different methods, an explicit meromorphic continuation and determine explicitly their values at T -tuples of non-positive integers.*

The second part of this thesis (Chapters III, IV and V) is devoted to the study of the means values of multivariable arithmetic functions $f : \mathbb{N}^n \rightarrow \mathbb{C}$.

A typical problem that one meets in many different arithmetic or geometric contexts is to describe with reasonable precision the asymptotic behavior of a density function of the form

$$\mathbf{x} = (x_1, \dots, x_n) \mapsto \Psi(f; \mathbf{x}) := \sum_{1 \leq m_i \leq x_i \quad \forall i \in \{1, \dots, n\}} f(m_1, \dots, m_n) \quad (\text{I.6})$$

or of the form

$$x \mapsto \Psi(f; P; x) := \sum_{P(m_1, \dots, m_n) \leq x} f(m_1, \dots, m_n) \quad (\text{I.7})$$

where the function $f : \mathbb{N}^n \rightarrow \mathbb{C}$ is an arithmetic function, typically, though not always, multiplicative, and where $P \in \mathbb{R}[X_1, \dots, X_n]$ is a suitable polynomial.

For $n = 1$ there are several results as Delange's Theorem [4]. In several variables case ($n \geq 2$) there are several works that inspired this part of our thesis : Delange [3]; Lichtin [22, 21]; La Bretèche [7]; Essouabri [9, 12, 14]; Tóth and Zahai [30, 29].

In Chapter 3 of this thesis, we establish in Theorems 13 and 14 asymptotic formulas for the means values $(x_1, \dots, x_n) \mapsto \Psi(f; \mathbf{x})$ as $\inf(x_i) \rightarrow +\infty$ for a class of multivariable multiplicative functions which does not fit within the framework of the preceding works [3], [7] and [30]. We also establish (see Proposition 1 and its Corollary 3) as a consequence of the Essouabri's works [9, 12, 14] precise asymptotic formulas for the means values $x \mapsto \Psi(f; P; x)$ for a class of multivariable multiplicative functions f naturally associated to some classical arithmetic functions as the sum of divisors function, the Euler's Totient function φ , etc. A more detailed description of these results as well as their links with previous works is given in the introduction of Chapter 3.

Chapter 4 of this thesis consists of a submitted paper in collaboration with Essouabri and Tóth. This chapter is motivated by the following recent results and conjectures. Let $n \in \mathbb{N}$, for $m_1, \dots, m_n \in \mathbb{N}$ we set $c_n(m_1, \dots, m_n)$ the number of cyclic subgroups of the group $\mathbb{Z}_{m_1} \times \dots \times \mathbb{Z}_{m_n}$. Nowak and Tóth [24] proved the asymptotic formula

$$\sum_{1 \leq m_1, m_2 \leq x} c_2(m_1, m_2) = x^2 \left(\frac{12}{\pi^4} (\ln x)^3 + a_2 (\ln x)^2 + a_1 (\ln x) + a_0 \right) + O(x^{\frac{1117}{701} + \varepsilon}) \text{ as } x \rightarrow +\infty,$$

where a_0, a_1 and a_2 are explicit constants.

The case $n = 3$ was investigated by Tóth and Zhai [29] showing that

$$\sum_{1 \leq m_1, m_2, m_3 \leq x} c_3(m_1, m_2, m_3) = x^3 \sum_{j=0}^7 c_j (\ln x)^j + O(x^{8/3 + \varepsilon}) \text{ as } x \rightarrow +\infty$$

where c_j ($0 \leq j \leq 7$) are explicit constants. It is natural to conjecture that such a result holds for $n \geq 4$.

Hilberdink, Luca and Tóth [17] investigated the following three averages associated with the LCM function:

$$S_n(x) := \sum_{1 \leq m_1, \dots, m_n \leq x} \frac{1}{\text{lcm}(m_1, \dots, m_n)}, \quad (\text{I.8})$$

$$U_n(x) := \sum_{\substack{1 \leq m_1, \dots, m_n \leq x \\ \gcd(m_1, \dots, m_n) = 1}} \frac{1}{\text{lcm}(m_1, \dots, m_n)}, \quad (\text{I.9})$$

and

$$V_n(x) := \sum_{1 \leq m_1, \dots, m_n \leq x} \frac{m_1 \dots m_n}{\text{lcm}(m_1, \dots, m_n)}. \quad (\text{I.10})$$

By using the convolution method, they obtained in their paper asymptotic formulas with error terms for $S_2(x)$, $U_2(x)$ and $V_2(x)$. For $n \geq 3$, they only obtained the estimates

$$\begin{aligned} (\ln x)^{2^n - 1} &\ll S_n(x) \ll (\ln x)^{2^n - 1}, & (\ln x)^{2^n - 2} &\ll U_n(x) \ll (\ln x)^{2^n - 2}, \\ x^n &\ll V_n(x) \ll x^n (\ln x)^{2^n - 2} & \text{as } x &\rightarrow +\infty, \end{aligned}$$

and conjectured that asymptotic formulas with error terms also exist for these three averages for $n \geq 3$.

In order to prove these conjectures, we introduce a reasonably large class of multivariable multiplicative functions (see Definition 13). For a function $f : \mathbb{N}^n \rightarrow \mathbb{R}_+$ in this class, we establish in Theorem 15 the existence of the meromorphic continuation of the associated multiple zeta function

$$\mathbf{s} = (s_1, \dots, s_n) \rightarrow \mathcal{M}(f; \mathbf{s}) := \sum_{\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{N}^n} \frac{f(m_1, \dots, m_n)}{m_1^{s_1} \dots m_n^{s_n}} \quad (\text{I.11})$$

and derive several precise properties of this meromorphic continuation. By combining our Theorem 15 and La Bretèche's multivariable tauberian Theorem (i.e., Theorems 1 and 2 of [7]) we deduce in our Theorem 16 a precise asymptotic formula for the multivariable average

$$\mathcal{N}_\infty(f; x) := \sum_{\substack{\mathbf{m}=(m_1, \dots, m_n) \in \mathbb{N}^n \\ \|\mathbf{m}\|_\infty = \max_i m_i \leq x}} f(m_1, \dots, m_n) \quad \text{as } x \rightarrow +\infty,$$

and derive from it four corollaries. Our first application, namely Corollary 4, establishes the conjecture concerning the number of cyclic subgroups of the group $\mathbb{Z}_{m_1} \times \dots \times \mathbb{Z}_{m_n}$, in any dimension n . Our Corollaries 5, 6 and 7 prove the conjectures on the above sums (I.8), (I.9) and (I.10) associated with the LCM function.

Variants of Theorem 16 with other norm choices can be obtained by combining our Theorem 15 and Essouabri's multivariable tauberian Theorem (i.e., Corollary 2 of [12]). For example, for the class of Holder's norms $\|\mathbf{x}\|_d := \sqrt[d]{|x_1|^d + \dots + |x_n|^d}$ ($d \geq 1$), we obtain in Theorem 17 an asymptotic for the multivariable average

$$\mathcal{N}_d(f; x) := \sum_{\substack{\mathbf{m}=(m_1, \dots, m_n) \in \mathbb{N}^n \\ \|\mathbf{m}\|_d = \sqrt[d]{m_1^d + \dots + m_n^d} \leq x}} f(m_1, \dots, m_n) \quad \text{as } x \rightarrow +\infty.$$

As an application of Theorem 17, we derive in Corollaries 8 and 9 the analogues of Corollaries 4 and 7 for the Holder's norms $\|\cdot\|_d$.

Chapter 5 of this thesis is a complement to Chapter 4. In, Chapter 4 we have not given the analogues of the Corollaries 5 and 6 for the Holder's norms. The reason is that Essouabri's multivariable tauberian Theorem does not apply as it is to these two cases. We therefore start in Chapter 5 with a slight extension of Essouabri's Theorem on mixed zeta functions (18, Th. 3) and deduce from it a more general version of Theorem 17 of Chapter 4. As an application, we obtain in the Corollaries 10 and 11 the analogues of Corollaries 5 and 6 for Holder's norms.

2 Version en français

Cette thèse est constituée de deux parties. Dans la première partie (Chapitre II), nous établissons par deux méthodes différentes des formules closes et explicites des valeurs aux T -tuples d'entiers négatifs d'une classe de fonctions zêtas multivariables. Dans la deuxième partie (Chapitres III, IV et V), nous établissons quelques résultats nouveaux sur les moyennes des fonctions multiplicatives multivariables.

Soient $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_n) \in \mathbb{C}^n$ et $\mathbf{b} = (b_1, \dots, b_n) \in \mathbb{C}^n$ deux vecteurs de paramètres complexes tels que $\operatorname{Re}(\gamma_j) > 0$ et $\operatorname{Re}(b_j) > -\operatorname{Re}(\gamma_1)$ pour chaque $j \in \{1, \dots, n\}$. La fonction zêta multiple d'Euler-Zagier généralisée est définie formellement en $\mathbf{s} = (s_1, \dots, s_n)$ par

$$\zeta_n(\mathbf{s}; \boldsymbol{\gamma}; \mathbf{b}) := \sum_{\substack{m_1 \geq 1 \\ m_2, \dots, m_n \geq 0}} \frac{1}{\prod_{j=1}^n (\gamma_1 m_1 + \dots + \gamma_j m_j + b_j)^{s_j}} \quad (\text{I.12})$$

Cette série converge absolument dans le domaine

$$\mathcal{D}_n := \{ \mathbf{s} = (s_1, \dots, s_n) \in \mathbb{C}^n \mid \operatorname{Re}(s_j + \dots + s_n) > n + 1 - j \ \forall j \in \{1, \dots, n\} \} \quad (\text{I.13})$$

et possède un prolongement méromorphe à tout l'espace complexe \mathbb{C}^n dont les singularités possibles sont situés dans la réunion des hyperplans

$$s_j + \dots + s_n = (n + 1 - j) - k_j \quad (1 \leq j \leq n, \ k_1, \dots, k_n \in \mathbb{N}_0)$$

De plus, il est connu que pour $n \geq 2$, les points $\mathbf{s} = -\mathbf{N}$, où $\mathbf{N} = (N_1, \dots, N_n) \in \mathbb{N}^n$, se trouvent dans la plupart des cas dans le lieu singulier ci-dessus et sont des points d'indétermination. L'étude des valeurs directionnelles de ces fonctions zêta multivariables en ces points a été d'abord considérée par S. Akiyama, S. Egami and Y. Tanigawa [1]. Ces valeurs sont maintenant bien comprises grâce à plusieurs travaux comme [20], [25], [23] et [10].

Dans [20], Y. Komori a montré que pour $\mathbf{N} = (N_1, \dots, N_n) \in \mathbb{N}^n$ et $\boldsymbol{\theta} = (\theta_1, \dots, \theta_n) \in \mathbb{C}^n$ vérifiant $\theta_j + \dots + \theta_n \neq 0$ pour tout $j \in \{1, \dots, n\}$, la limite directionnelle

$$\zeta_n^{\boldsymbol{\theta}}(-\mathbf{N}; \boldsymbol{\gamma}; \mathbf{b}) := \lim_{t \rightarrow 0} \zeta_n(-\mathbf{N} + t\boldsymbol{\theta}; \boldsymbol{\gamma}; \mathbf{b}) \quad (\text{I.14})$$

existe, et a exprimé cette limite en fonction de \mathbf{N} , $\boldsymbol{\theta}$ et des nombres de Bernoulli généralisés définis implicitement comme coefficients de certaines séries multivariable.

Dans [10], D. Essouabri et K. Matsumoto ont donné finalement, par une méthode différente, des formules closes et explicites de $\zeta_n^{\boldsymbol{\theta}}(-\mathbf{N}; \boldsymbol{\gamma}; \mathbf{b})$ en fonction de \mathbf{N} , $\boldsymbol{\theta}$ et seulement les nombres de Bernoulli classiques.

Récemment dans [8], Essouabri et Matsumoto ont étendu ce résultat à une classe plus générale de fonctions zêtas multivariable définies par

$$\zeta_n(\mathbf{s}, \mathbf{P}) = \sum_{m_1, \dots, m_n \geq 1} \prod_{j=1}^n P_j(m_1, \dots, m_j)^{-s_j}, \quad (\text{I.15})$$

où $\mathbf{P} = (P_1, \dots, P_n)$ avec des polynômes assez généraux $P_j \in \mathbb{R}[X_1, \dots, X_j]$. Dans ce cas plus général, au lieu des nombres de Bernoulli, certaines périodes apparaissent dans le résultat. De plus la non linéarité des polynômes P_j fait apparaître des valeurs aux n -tuples d'entiers négatifs qui peuvent être transcendantes même si les coefficients des polynômes P_j sont algébriques.

Considérons maintenant le cas tordu. Soient $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$, $\boldsymbol{\mu} \in (\mathbb{T} \setminus \{1\})^k \times \{1\}^{n-k}$, $Q \in \mathbb{R}[X_1, \dots, X_n]$ et $\mathbf{P} = (P_1, \dots, P_T) \in (\mathbb{R}[X_1, \dots, X_n])^T$, on associe au triplet

$(Q; \mathbf{P}; \boldsymbol{\mu})$ la FZM (k, n) -tordue (non tordue si $\boldsymbol{\mu} = \mathbf{1}$) définie formellement pour $\mathbf{s} = (s_1, \dots, s_T)$ par

$$Z(Q; \mathbf{P}; \boldsymbol{\mu}; \mathbf{s}) = \sum_{\mathbf{m}=(m_1, \dots, m_n) \in (\mathbb{N}^*)^n} \frac{Q(m_1, \dots, m_n) \mu_1^{m_1} \cdots \mu_k^{m_k}}{\prod_{t=1}^T P_t(m_1, \dots, m_n)^{s_t}} \quad (\text{I.16})$$

Si $P_1 \dots P_T(\mathbf{X}) \rightarrow +\infty$ lorsque $\|\mathbf{X}\| \rightarrow +\infty$ et $\mathbf{X} \in [1, +\infty)^n$, l'existence d'un prolongement méromorphe de ses séries de Dirichlet à tout l'espace \mathbb{C}^T découle de la méthode introduite par Essouabri ([16], [15]) si P_1, \dots, P_T vérifient l'hypothèse probablement optimale H_0S (voir Définition 1 au Chapitre II). Il est connu que la complexité du lieu singulier de ses séries de Dirichlet est maximale dans le cas non tordu (i.e. $\boldsymbol{\mu} = \mathbf{1}$) et on s'attend à avoir un prolongement holomorphe dans le cas complètement tordu (i.e. $\boldsymbol{\mu} \in (\mathbb{T} \setminus \{1\})^n$). Un contre-exemple montre que ce n'est pas le cas sous l'hypothèse générale H_0S .

Sous l'hypothèse un peu plus restrictive (HDF) (voir Définition 1 du Chapitre II), Marc de Crisenoy ([6] Théorèmes A et B) a montré dans le cas complètement tordu (i.e dans le cas des FZM (n, n) -tordues) l'holomorphie de ces série de Dirichlet et a surtout donné des expressions closes et très simples de leurs valeurs aux T -tuplets d'entiers négatifs ou nuls.

Dans le cas non complètement tordue, la présence de singularités rend l'étude des valeurs aux T -tuplets d'entiers négatifs ou nuls beaucoup plus complexe. En plus des travaux cités ci-dessus, Essouabri et Matsumoto [11], ont obtenu en combinant le résultat de Crisenoy et la formule de représentation intégrale de Mellin-Barnes (27), les valeurs des FZM $(n-1, n)$ -tordues et $(n-2, n)$ -tordues dans les cas où les polynômes P_i sont linéaires ou diagonales.

Dans le Chapitre 2 de ce mémoire, nous avons étudié une classe de FZM $(n-1, n)$ -tordues (voir Définition (II.10)) qui ne rentre pas dans ces travaux. Plus précisément, nous avons déterminé (voir Théorèmes 2 et 3 du Chapitre II), par deux méthodes différentes et en utilisant aussi le résultat de Crisenoy, un prolongement méromorphe explicite ainsi que les valeurs aux T -tuples des entiers négatifs de cette classe de FZM $(n-1, n)$ -tordues.

La deuxième partie de cette thèse (Chapitres III, IV et V) est consacrée à l'étude des moyennes des fonctions arithmétiques multivariable $f : (\mathbb{N}^*)^n \rightarrow \mathbb{C}$.

Plusieurs problèmes en arithmétique et en géométrie sont liées à la détermination d'un développement asymptotique de fonctions densités de la forme

$$\mathbf{x} = (x_1, \dots, x_n) \mapsto \Psi(f; \mathbf{x}) := \sum_{1 \leq m_i \leq x_i \quad \forall i \in \{1, \dots, n\}} f(m_1, \dots, m_n) \quad (\text{I.17})$$

ou de la forme

$$x \mapsto \Psi(f; P; x) := \sum_{P(m_1, \dots, m_n) \leq x} f(m_1, \dots, m_n) \quad (\text{I.18})$$

où $f : (\mathbb{N}^*)^n \rightarrow \mathbb{C}$ est une fonction arithmétique dans beaucoup de cas multiplicative, et où $P \in \mathbb{R}[X_1, \dots, X_n]$ est un polynôme possédant des propriétés raisonnables.

Pour le cas $n = 1$ on a des résultats comme ceux de Delange [4]; pour le cas multivariable on trouve ceux de Delange [3], Lichtin [22, 21], La Bretèche [7], Essouabri [9, 12, 14], et Tóth et Zahai [30, 29] qu'on a utilisé comme source d'inspiration.

Dans le Chapitre 3 de cette thèse, nous avons établi (voir Théorèmes 13 et 14) des formules asymptotiques pour les moyennes $(x_1, \dots, x_n) \mapsto \Psi(f; \mathbf{x})$ lorsque $\inf(x_i) \rightarrow +\infty$ pour une classe de fonctions multiplicatives multivariables qui ne rentre pas dans le cadre des travaux [3], [7] et [30]. Nous avons aussi établi (Proposition 1 et Corollaire 3) comme conséquence des travaux [9, 12, 14] des formules asymptotiques précises pour les moyennes

$x \mapsto \Psi(f; P; x)$ pour une classe de fonctions multiplicatives multivariables associées naturellement aux fonctions arithmétiques classiques: somme des diviseurs, φ indicatrice d'Euler, etc. Une description plus détaillée de ses résultats ainsi que de leur articulation avec les travaux précédents se trouve dans l'introduction du Chapitre 3 de cette thèse.

Le Chapitre 4 de cette thèse est constitué d'un article soumis en collaboration avec D. Essouabri et L. Tóth. Voici une brève description du contenu de ce chapitre. Soit $n \in \mathbb{N}^*$, pour $m_1, \dots, m_n \in \mathbb{N}^*$ on définit $c_n(m_1, \dots, m_n)$ comme le nombre de sous groupes cycliques du groupe $\mathbb{Z}_{m_1} \times \dots \times \mathbb{Z}_{m_n}$. W. G. Nowak et L. Tóth [24] ont prouvé la formule asymptotique suivante

$$\sum_{1 \leq m_1, m_2 \leq x} c_2(m_1, m_2) = x^2 \left(\frac{12}{\pi^4} (\ln x)^3 + a_2 (\ln x)^2 + a_1 (\ln x) + a_0 \right) + O(x^{\frac{1117}{701} + \varepsilon}) \text{ lorsque } x \rightarrow +\infty,$$

où a_0, a_1 et a_2 sont des constantes explicites.

Le cas $n = 3$ a été étudié par L. Tóth et W. Zhai [29] qui ont montré que

$$\sum_{1 \leq m_1, m_2, m_3 \leq x} c_3(m_1, m_2, m_3) = x^3 \sum_{j=0}^7 c_j (\ln x)^j + O(x^{8/3 + \varepsilon}) \text{ lorsque } x \rightarrow +\infty,$$

où les c_j ($0 \leq j \leq 7$) sont des constantes explicites. Il est donc naturel de conjecturer qu'un tel résultat est valable pour $n \geq 4$.

T. Hilberdink, F. Luca, et L. Tóth [17] ont aussi étudié les trois moyennes multivariables suivantes associées à la fonction PPCM :

$$S_n(x) := \sum_{1 \leq m_1, \dots, m_n \leq x} \frac{1}{PPCM(m_1, \dots, m_n)}, \quad (\text{I.19})$$

$$U_n(x) := \sum_{\substack{1 \leq m_1, \dots, m_n \leq x \\ PGCD(m_1, \dots, m_n) = 1}} \frac{1}{PPCM(m_1, \dots, m_n)}, \quad (\text{I.20})$$

$$V_n(x) := \sum_{1 \leq m_1, \dots, m_n \leq x} \frac{m_1 \dots m_n}{PPCM(m_1, \dots, m_n)}. \quad (\text{I.21})$$

En utilisant la méthode de convolution, ils ont obtenu dans leur papier des formules asymptotiques avec des termes d'erreurs pour $S_2(x)$, $U_2(x)$ et $V_2(x)$.

Pour $n \geq 3$, ils n'ont obtenu que les estimations suivantes

$$(\ln x)^{2^n - 1} \ll S_n(x) \ll (\ln x)^{2^n - 1}, \quad (\ln x)^{2^n - 2} \ll U_n(x) \ll (\ln x)^{2^n - 2},$$

$$x^n \ll V_n(x) \ll x^n (\ln x)^{2^n - 2} \text{ lorsque } x \rightarrow +\infty,$$

et ils ont conjecturé que des formules asymptotiques avec des termes d'erreur existent également pour ces trois moyennes pour $n \geq 3$.

Afin de prouver ces conjectures, nous avons introduit une classe raisonnablement grande de fonctions multiplicatives multivariable (voir Définition 13). Pour une fonction $f : (\mathbb{N}^*)^n \rightarrow \mathbb{R}_+$ dans cette classe, nous avons établi (voir Théorème 15) l'existence du prolongement méromorphe de la fonction zêta multivariable associée

$$\mathbf{s} = (s_1, \dots, s_n) \rightarrow \mathcal{M}(f; \mathbf{s}) := \sum_{\mathbf{m} = (m_1, \dots, m_n) \in (\mathbb{N}^*)^n} \frac{f(m_1, \dots, m_n)}{m_1^{s_1} \dots m_n^{s_n}}$$

ainsi que plusieurs propriétés fines de ce prolongement. En combinant notre Théorème 15 et le Théorème taubérien multivariable de La Bretèche's (i.e., Théorèmes 1 et 2 de

[7]) nous avons déduit dans notre Théorème 16 une formule asymptotique précise pour la moyenne

$$\mathcal{N}_\infty(f; x) := \sum_{\substack{\mathbf{m}=(m_1, \dots, m_n) \in (\mathbb{N}^*)^n \\ \|\mathbf{m}\|_\infty = \max_i m_i \leq x}} f(m_1, \dots, m_n) \quad \text{lorsque } x \rightarrow +\infty.$$

Cette formule nous a permis d'obtenir quatre corollaires qui résolvent les quatre conjectures citées ci-dessus. Plus précisément, notre première application dans ce chapitre, à savoir le Corollaire 4, établit la conjecture concernant le nombre de sous-groupes cycliques du groupe $\mathbb{Z}_{m_1} \times \dots \times \mathbb{Z}_{m_n}$, en toute dimension n . Nos Corollaires 5, 6 et 7 prouvent les conjectures sur les trois sommes (I.19), (I.20) et (I.21) ci-dessus associées à la fonction PPCM, en toute dimension n .

Des variantes du théorème 16 avec d'autres choix de normes peuvent être obtenues en combinant notre Théorème 15 et le Théorème taubérien multivariable d'Essouabri (i.e., Corollaire 2 de [12]). Par exemple, pour la classe des normes Hölderiennes $\|\mathbf{x}\|_d := \sqrt[d]{|x_1|^d + \dots + |x_n|^d}$ ($d \geq 1$), nous obtenons dans le Théorème 17 un développement asymptotique des moyennes multivariable

$$\mathcal{N}_d(f; x) := \sum_{\substack{\mathbf{m}=(m_1, \dots, m_n) \in (\mathbb{N}^*)^n \\ \|\mathbf{m}\|_d = \sqrt[d]{m_1^d + \dots + m_n^d} \leq x}} f(m_1, \dots, m_n) \quad \text{lorsque } x \rightarrow +\infty.$$

Comme application du Théorème 17, nous obtenons dans les Corollaires 8 et 9 les analogues des Corollaires 4 et 7 pour les normes Hölderiennes $\|\cdot\|_d$.

Le Chapitre 5 de ce mémoire est un complément au Chapitre 4. Dans, le Chapitre 4 nous n'avons pas donné les analogues des Corollaires des 5 et 6 pour les normes Hölderiennes. La raison est que le Théorème taubérien multivariable d'Essouabri ne s'applique pas tel qu'il est à ces deux cas. Nous avons donc commencé dans ce Chapitre 5 par une légère extension du Théorème taubérien multivariable d'Essouabri sur les fonctions zêtas mixtes (18, Théorème 3) et nous en avons ensuite déduit une version un peu plus générale du Théorème 17 du Chapitre 4. Comme application, nous avons obtenu aussi dans les Corollaires 10 et 11 les analogues des Corollaires 5 et 6 pour les normes Hölderiennes.

3 Versión en español

La presente tesis tiene dos partes. En la primera parte (Capítulo II), utilizando dos métodos diferentes, establecemos dos fórmulas explícitas y cerradas para los valores en T -uplas de enteros negativos o nulos de una clase de funciones zeta multivariable. En la segunda parte (Capítulos III, IV et V), establecemos algunos resultados nuevos sobre los promedios de funciones multiplicativas multivariable.

Sean $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_n) \in \mathbb{C}^n$ y $\mathbf{b} = (b_1, \dots, b_n) \in \mathbb{C}^n$ dos vectores de parámetros complejos tales que $\operatorname{Re}(\gamma_j) > 0$ y $\operatorname{Re}(b_j) > -\operatorname{Re}(\gamma_1)$ para todo $j = 1, \dots, n$. La función zeta múltiple generalizada de Euler–Zagier es definida formalmente en $\mathbf{s} = (s_1, \dots, s_n)$ por

$$\zeta_n(\mathbf{s}; \boldsymbol{\gamma}; \mathbf{b}) := \sum_{\substack{m_1 \geq 1 \\ m_2, \dots, m_n \geq 0}} \frac{1}{\prod_{j=1}^n (\gamma_j m_j + b_j)^{s_j}}. \quad (\text{I.22})$$

Esta serie es absolutamente convergente en el dominio

$$\mathcal{D}_n := \{\mathbf{s} = (s_1, \dots, s_n) \in \mathbb{C}^n \mid \operatorname{Re}(s_j + \dots + s_n) > n + 1 - j \text{ for all } j \in \{1, \dots, n\}\} \quad (\text{I.23})$$

y posee una extensión meromorfa a todo el espacio complejo \mathbb{C}^n con posibles singularidades situadas en la reunión de hiperplanos

$$s_j + \dots + s_n = (n + 1 - j) - k_j \quad (1 \leq j \leq n, k_1, \dots, k_n \in \mathbb{N}_0).$$

Adicionalmente, es conocido que para $n \geq 2$ los puntos $\mathbf{s} = -\mathbf{N}$, donde $\mathbf{N} = (N_1, \dots, N_n) \in \mathbb{N}_0^n$, en la mayoría de casos se encuentran en el lugar singular citado anteriormente y son puntos de indeterminación.

El estudio de valores direccionales de estas funciones zeta multivariable en estos puntos de indeterminación fueron abordados por primera vez por S. Akiyama, S. Egami e Y. Tanigawa [1]. Actualmente estos valores son bien comprendidos gracias a múltiples trabajos como [20], [25], [23] y [10].

En [20], Y. Komori mostró que para $\mathbf{N} = (N_1, \dots, N_n) \in \mathbb{N}^n$ y $\boldsymbol{\theta} = (\theta_1, \dots, \theta_n) \in \mathbb{C}^n$ verificando $\theta_j + \dots + \theta_n \neq 0$ para todo $j \in \{1, \dots, n\}$, el límite direccional

$$\zeta_n^\theta(-\mathbf{N}; \boldsymbol{\gamma}; \mathbf{b}) := \lim_{t \rightarrow 0} \zeta_n(-\mathbf{N} + t\boldsymbol{\theta}; \boldsymbol{\gamma}; \mathbf{b}) \quad (\text{I.24})$$

existe, y expresó estos valores límites en función de \mathbf{N} , $\boldsymbol{\theta}$ y los números de Bernoulli generalizados definidos implícitamente como coeficientes de ciertas series multivariable.

En [10], D. Essouabri y K. Matsumoto dieron finalmente, gracias a un método diferente, fórmulas cerradas y explícitas para $\zeta_n^\theta(-\mathbf{N}; \boldsymbol{\gamma}; \mathbf{b})$ en función de \mathbf{N} , $\boldsymbol{\theta}$ y solamente los números clásicos de Bernoulli.

Recientemente Essouabri y Matsumoto [8] extendieron este resultado a una clase más general de funciones zeta multivariable definidas por

$$\zeta_n(\mathbf{s}, \mathbf{P}) = \sum_{\mathbf{m}=(m_1, \dots, m_n) \in \mathbb{N}^n} \prod_{j=1}^n P_j(m_1, \dots, m_j)^{-s_j}, \quad (\text{I.25})$$

donde $\mathbf{P} = (P_1, \dots, P_n)$ es un vector formado por polinomios genéricos $P_j \in \mathbb{R}[X_1, \dots, X_j]$. En este caso más general, en lugar de los números de Bernoulli, ciertos periodos aparecen en el resultado. Además la no linealidad de los polinomios P_j hacen aparecer valores de n -uplas de enteros negativos o nulos que pueden resultar trascendentes incluso si los coeficientes de los polinomios P_j son algebraicos.

Consideremos ahora el caso con torsión. Sean $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$, $\boldsymbol{\mu} \in (\mathbb{T} \setminus \{1\})^k \times \{1\}^{n-k}$, $Q \in \mathbb{R}[\mathbf{X}]$ y $\mathbf{P} = (P_1, \dots, P_T) \in (\mathbb{R}[\mathbf{X}])^T$, asociamos a la tripleta $(Q; \mathbf{P}; \boldsymbol{\mu})$ la FZM con (k, n) -torsión (sin torsión si $\boldsymbol{\mu} = \mathbf{1}$) definida por

$$Z(Q; \mathbf{P}; \boldsymbol{\mu}; \mathbf{s}) = \sum_{\mathbf{m}=(m_1, \dots, m_n) \in \mathbb{N}^n} \frac{Q(m_1, \dots, m_n) \mu_1^{m_1} \cdots \mu_k^{m_k}}{\prod_{t=1}^T P_t(m_1, \dots, m_n)^{s_t}} \quad (\text{I.26})$$

Si $P_1 \dots P_T(\mathbf{X}) \rightarrow +\infty$ cuando $\|\mathbf{X}\| \rightarrow +\infty$ y $\mathbf{X} \in [1, +\infty)^n$, la existencia de la extensión meromorfe de estas series de Dirichlet a todo el espacio \mathbb{C}^T se sigue del método introducido por Essouabri en ([16], [15]) si P_1, \dots, P_T verifican la hipótesis probablemente optimal H_0S (ver definición 1 del Capítulo II). Es conocido que la complejidad del lugar singular de estas series de Dirichlet es maximal en el caso sin torsión (i.e. $\boldsymbol{\mu} = \mathbf{1}$) y se espera tener una extensión holomorfa en el caso con torsión total (i.e. $\boldsymbol{\mu} \in (\mathbb{T} \setminus \{1\})^n$). Un contra ejemplo muestra que esto no es verdad bajo la hipótesis general H_0S . Bajo la hipótesis un poco más restrictiva (HDF) (ver definición 1 del Capítulo II), Marc de Crisenoy ([6] Teoremas A y B) mostró que en el caso de torsión total (i.e. $\boldsymbol{\mu} \in (\mathbb{T} \setminus \{1\})^n$) la holomorfía de estas series de Dirichlet y sobretodo dió expresiones cerradas y simples para los valores de T -uplas de enteros negativos o nulos.

En el caso sin torsión total, la presencia de singularidades transforman el estudio de los valores en T -uplas de enteros negativos o nulos mucho más complejo. Además de los trabajos citados anteriormente, Essouabri y Matsumoto [11] obtuvieron, combinando el resultado de Crisenoy y la fórmula de representación integral de Mellin–Barnes (27), los valores de FZM con $(n-1, n)$ -torsión y con $(n-2, n)$ -torsión en los casos donde los polinomios P_i son lineales o diagonales.

En el Capítulo 2 de esta tesis, hemos estudiado una clase de FZM con $(n-1, n)$ -torsión (ver Definición (II.10)) que no entra en los trabajos ya mencionados. De manera más precisa, hemos determinado (ver Teoremas 2 y 3 del capítulo II), de dos maneras diferentes y utilizando el resultado de Crisenoy, una extensión meromorfa explícita y hemos determinado los valores en T -uplas de enteros negativos o nulos de esta clase de FZM con $(n-1, n)$ -torsión.

La segunda parte de esta tesis es consagrada al estudio de promedios de funciones aritméticas multivariable $f : \mathbb{N}^n \rightarrow \mathbb{C}$.

Varios problemas en aritmética y en geometría son ligados a la determinación de desarrollos asintóticos de funciones densidad de la forma

$$\mathbf{x} = (x_1, \dots, x_n) \mapsto \Psi(f; \mathbf{x}) := \sum_{1 \leq m_i \leq x_i \quad \forall i=1, \dots, n} f(m_1, \dots, m_n) \quad (\text{I.27})$$

o de la forma

$$x \mapsto \Psi(f; P; x) := \sum_{P(m_1, \dots, m_n) \leq x} f(m_1, \dots, m_n) \quad (\text{I.28})$$

donde $f : \mathbb{N}^n \rightarrow \mathbb{C}$ es una función aritmética en muchos casos multiplicativa, y donde $P \in \mathbb{R}[X_1, \dots, X_n]$ es un polinomio con propiedades razonables.

En el caso $n = 1$ tenemos resultados como los de Delange [4]; para el caso multivariable tenemos resultados como los de Delange [3], Lichtin [22, 21], La Bretèche [7], Essouabri [9, 12, 14], Tóth y Zhai [30, 29] los cuales utilizamos como fuente de inspiración.

En el Capítulo 3 de esta tesis hemos establecido (ver Theoremas 13 y 14) fórmulas asintóticas para los promedios $(x_1, \dots, x_n) \mapsto \Psi(f; \mathbf{x})$ cuando $\inf(x_i) \rightarrow +\infty$ para una clase de funciones multiplicativas multivariable que no entran en las hipótesis de los trabajos [3], [7] y [30]. También hemos establecido (Proposición 1 y Corolario 3), como consecuencia de los trabajos de Essouabri [9, 12, 14], algunas formulas asintóticas precisas

para los promedios $x \mapsto \Psi(f; P; x)$ de una clase de funciones multiplicativas multivariable asociadas naturalmente a las funciones aritméticas clásicas : suma de divisores, función φ indicatriz de Euler, entre otros. Una descripción más detallada de estos resultados así como la conexión con los trabajos precedentes es dada en la introducción del Capítulo 3 de esta tesis.

El Capítulo 4 de esta tesis esta constituida por un artículo sometido en colaboración con D. Essouabri y L. Tóth. Aquí una breve descripción del contenido de este capítulo. Sea $n \in \mathbb{N}$. Definimos para $m_1, \dots, m_n \in \mathbb{N}$ $c_n(m_1, \dots, m_n)$ como el número de subgrupos cíclicos del grupo $\mathbb{Z}_{m_1} \times \dots \times \mathbb{Z}_{m_n}$. W. G. Nowak y L. Tóth [24] demostraron la siguiente fórmula asintótica

$$\sum_{1 \leq m_1, m_2 \leq x} c_2(m_1, m_2) = x^2 \left(\frac{12}{\pi^4} (\ln x)^3 + a_2 (\ln x)^2 + a_1 (\ln x) + a_0 \right) + O(x^{\frac{1117}{701} + \varepsilon}) \text{ cuando } x \rightarrow +\infty,$$

donde a_0, a_1 y a_2 son constantes explícitas.

El caso $n = 3$ fue estudiado por L. Tóth y W. Zhai en [29] y probaron que

$$\sum_{1 \leq m_1, m_2, m_3 \leq x} c_3(m_1, m_2, m_3) = x^3 \sum_{j=0}^7 c_j (\ln x)^j + O(x^{8/3 + \varepsilon}) \text{ cuando } x \rightarrow +\infty,$$

donde los c_j ($0 \leq j \leq 7$) son constantes explícitas. Es natural conjeturar un resultado análogo para $n \geq 4$.

T. Hilberdink, F. Luca, y L. Tóth [17] también estudiaron los tres promedios multivariables siguientes asociados a la función MCM :

$$S_n(x) := \sum_{1 \leq m_1, \dots, m_n \leq x} \frac{1}{MCM(m_1, \dots, m_n)}, \quad (\text{I.29})$$

$$U_n(x) := \sum_{\substack{1 \leq m_1, \dots, m_n \leq x \\ MCD(m_1, \dots, m_n) = 1}} \frac{1}{MCM(m_1, \dots, m_n)}, \quad (\text{I.30})$$

y

$$V_n(x) := \sum_{1 \leq m_1, \dots, m_n \leq x} \frac{m_1 \dots m_n}{MCM(m_1, \dots, m_n)}. \quad (\text{I.31})$$

Utilizando el método de convolución, en su artículo ellos obtuvieron fórmulas asintóticas con términos error para las funciones $S_2(x)$, $U_2(x)$ y $V_2(x)$. Para $n \geq 3$, ellos obtuvieron solamente las siguientes cotas

$$(\ln x)^{2^n - 1} \ll S_n(x) \ll (\ln x)^{2^n - 1}, \quad (\ln x)^{2^n - 2} \ll U_n(x) \ll (\ln x)^{2^n - 2},$$

$$x^n \ll V_n(x) \ll x^n (\ln x)^{2^n - 2} \quad \text{cuando } x \rightarrow +\infty,$$

y conjeturaron que fórmulas asintóticas con términos de error existen para estos tres promedios en el caso $n \geq 3$.

Para probar estas conjeturas, introducimos una clase razonablemente grande de funciones multiplicativas multivariable (ver Definición 13). Para una función $f : \mathbb{N}^n \rightarrow \mathbb{R}_+$ en esta clase, hemos establecido (ver Teorema 15) la existencia de una extensión meromorfa de la función zeta multivariable asociada

$$\mathbf{s} = (s_1, \dots, s_n) \rightarrow \mathcal{M}(f; \mathbf{s}) := \sum_{\mathbf{m}=(m_1, \dots, m_n) \in \mathbb{N}^n} \frac{f(m_1, \dots, m_n)}{m_1^{s_1} \dots m_n^{s_n}}$$

asi como varias propiedades de esta extensión. Combinando nuestro Teorema 15 y el Teorema tauberiano multivariable de La Bretèche (i.e., Teorema 1 y 2 de [7]) hemos deducido en nuestro Teorema 16 una fórmula asintótica precisa para el promedio

$$\mathcal{N}_\infty(f; x) := \sum_{\substack{\mathbf{m}=(m_1, \dots, m_n) \in \mathbb{N}^n \\ \|\mathbf{m}\|_\infty = \max_i m_i \leq x}} f(m_1, \dots, m_n) \quad \text{cuando } x \rightarrow +\infty.$$

Esta fórmula nos ha permitido obtener cuatro corolarios que responden a las cuatro conjeturas citadas anteriormente. De manera más precisa, nuestra primera aplicación en este capítulo, ver Corolario 4, establece la conjetura concerniente al número de subgrupos cíclicos del grupo $\mathbb{Z}_{m_1} \times \dots \times \mathbb{Z}_{m_n}$, para toda dimensión n . Nuestros Corolarios 5, 6 y 7 prueban las conjeturas sobre las tres sumas (I.29), (I.30) y (I.31) citadas anteriormente asociadas a la función MMC, en toda dimensión n .

Algunas variantes del Teorema 16 con otras normas pueden ser obtenidas combinando nuestro Teorema 15 y el Teorema tauberiano multivariable de Essouabri (i.e., Corolario 2 de [12]). Por ejemplo, para la clase de normas Hölderianas $\|\mathbf{x}\|_d := \sqrt[d]{|x_1|^d + \dots + |x_n|^d}$ ($d \geq 1$), obtenemos en el Teorema 17 un desarrollo asintótico de los promedios multivariables

$$\mathcal{N}_d(f; x) := \sum_{\substack{\mathbf{m}=(m_1, \dots, m_n) \in \mathbb{N}^n \\ \|\mathbf{m}\|_d = \sqrt[d]{m_1^d + \dots + m_n^d} \leq x}} f(m_1, \dots, m_n) \quad \text{cuando } x \rightarrow +\infty.$$

Como aplicación del Teorema 17, obtenemos en los Corolarios 8 y 9 los análogos de los Corolarios 4 y 7 para las normas Hölderianas $\|\cdot\|_d$.

El Capítulo 5 de esta tesis es un complemento del Capítulo 4. En el Capítulo 4 no hemos dado los análogos de los Corolarios 5 y 6 para las normas hölderianas. La razón es que el Teorema tauberiano de Essouabri no se aplica de manera directa en estos dos casos. Entonces hemos comenzado el Capítulo 5 por una ligera extensión del Teorema de Essouabri sobre las funciones zeta mixtas (18, Teorema. 3) y luego como consecuencia deducimos una versión un poco más general del Teorema 17 del Capítulo 4. Como aplicación, obtenemos también en los Corolarios 10 y 11 los análogos de los Corolarios 5 y 6 para las normas Hölderianas.

Chapter II

Zeta function in several variables and its special values

Notations

1. $\mathbb{N} = \{1, 2, \dots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, \mathcal{P} the set of prime numbers.
2. For $n \in \mathbb{N}$, we set $\mathbb{R}_+^n = [0, +\infty[^n$, $(\mathbb{R}_+^*)^n =]0, \infty[^n$, $(\mathbb{R}_+^n)^* = [0, \infty[^n \setminus \{\mathbf{0}\}$ and $J^n = [1, +\infty[^n$.
3. $\mathbb{T} = \{s \in \mathbb{C} \mid \|s\| = 1\}$
4. For $a, b \in \mathbb{R}$, we set $\llbracket a; b \rrbracket := \{k \in \mathbb{Z} \mid a \leq k \leq b\}$.
5. For I a finite set, we set $|I|$ its cardinal.
6. $E = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is the canonical basis of \mathbb{R}^n .
7. For any $\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{C}^n$ and $k \in \llbracket 1, n \rrbracket$, we denote $\mathbf{z}_k = (z_1, \dots, z_k)$ and $|\mathbf{z}_k| = z_1 + \dots + z_k$.
8. $\|\cdot\|_1, \|\cdot\|, \|\cdot\|_\infty$ are the sum, euclidean and maximum norms respectively.
9. For any $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{C}^n$ we set $\mathbf{x}^{\mathbf{s}} = x_1^{s_1} \cdots x_n^{s_n}$
10. The sequence of Bernoulli polynomials, denoted by $(B_n(X))_{n \in \mathbb{N}}$, is the unique sequence defined by the following series expansion

$$\frac{te^{Xt}}{e^t - 1} = \sum_{n \geq 0} B_n(X) \frac{t^n}{n!}$$

The Bernoulli numbers are defined by $B_n = B_n(0)$,

11. For $s \in \mathbb{C}$ and $k \in \mathbb{N}$ we set $(s)_0 = 1$ and $(s)_k = \prod_{i=0}^{k-1} (s - i)$,
12. For $\mathbf{s} \in \mathbb{C}^n$ and $\mathbf{k} \in \mathbb{N}_0^n$ we set $\binom{\mathbf{s}}{\mathbf{k}} = \prod_{i=1}^n \binom{s_i}{k_i} = \prod_{i=1}^n \frac{(s_i)_{k_i}}{k_i!}$
13. Let $P \in \mathbb{C}[X_1, \dots, X_n]$, we denote $\deg(P)$, $\deg_i(P)$ and $\text{Supp}(P)$ its absolute degree, its degree with respect to variable X_i and its support respectively.

1 Introduction

In 1735 Euler gave the value of series

$$\sum_{k \in \mathbb{N}} k^{-2} = \frac{\pi^2}{6}$$

Almost a century later, Dirichlet introduced the notion of Dirichlet's series associated to an arithmetic function $f : \mathbb{N} \rightarrow \mathbb{R}$:

$$Z(f; s) = \sum_{k \in \mathbb{N}} f(k)k^{-s}, \text{ for } s > s_0 \quad (\text{II.1})$$

Dirichlet studied the convergence of these series and realized a survey of these functions as real variable functions. In particular for f constant equal to 1 we have

$$\zeta(s) = \sum_{k \in \mathbb{N}} k^{-s}, \text{ for } s > 1 \quad (\text{II.2})$$

Dirichlet gave the explicit values of $\zeta(-k)$ for $k \in \mathbb{N}_0$, today called special values, in terms of Bernoulli numbers, more precisely he obtained

$$\zeta(-k) = (-1)^k \frac{B_{k+1}}{k+1}. \quad (\text{II.3})$$

Some years later, Riemann published a paper in 1859 where he began the study of this function as a function of complex variable. In this paper Riemann proved numerous results about the zeta function (II.2) and announced his famous Riemann hypothesis.

Riemann proved that this function, today called *Riemann zeta function*, has a meromorphic continuation to \mathbb{C} with a single simple pole at $s = 1$ of residue 1.

Today we have several techniques that allow us to prove this fact : integral representation, Euler-Maclaurin summatory formula, Taylor expansion formula, Poisson summatory formula, etc.

A first generalization of Riemann zeta function is given by

$$Z(P; \mu; s) = \sum_{k \in \mathbb{N}} \mu^k P(k)^{-s}, \text{ for } \text{Re}(s) > 1 \quad (\text{II.4})$$

where $P \in \mathbb{R}[X]$ such that $|P(X)| \rightarrow +\infty$ as $X \rightarrow +\infty$ and $\mu \in \mathbb{T}$. If $\mu \neq 1$ this series is called twisted zeta function and non-twisted zeta function otherwise. If $P(X) = X$ and $\mu \neq 1$ the series (II.4) is called Lerch zeta function and it's denoted by ζ_μ .

Classical techniques can be used to show that this series have a meromorphic continuation (holomorphic in twisted case) to \mathbb{C} , as well as to determine its special values. More precisely for $k \in \mathbb{N}_0$ and $\mu \neq 1$ we have

$$\zeta_\mu(-k) = \frac{(-1)^k \mu}{1 - \mu} \sum_{\ell=0}^k \frac{\ell! S(k, \ell)}{(\mu - 1)^\ell} \quad (\text{II.5})$$

where $S(k, \ell)$ is the number of partitions in ℓ parts of a set with k elements, it's also called Stirling number of the second kind or Stirling partition number, see [6, Lemma 5.7].

Let $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_n) \in \mathbb{C}^n$ and $\mathbf{b} = (b_1, \dots, b_n) \in \mathbb{C}^n$ be two vectors of complex parameters such that $\text{Re}(\gamma_j) > 0$ and $\text{Re}(b_j) > -\text{Re}(\gamma_1)$ for all $j \in \{1, \dots, n\}$. The generalized Euler-Zagier multiple zeta function is defined formally for $\mathbf{s} = (s_1, \dots, s_n) \in \mathbb{C}^n$ by

$$\zeta_n(\mathbf{s}; \boldsymbol{\gamma}; \mathbf{b}) := \sum_{\substack{m_1 \geq 1 \\ m_2, \dots, m_n \geq 0}} \frac{1}{\prod_{j=1}^n (\gamma_1 m_1 + \dots + \gamma_j m_j + b_j)^{s_j}}. \quad (\text{II.6})$$

This series converges absolutely in the domain

$$\mathcal{D}_n := \{\mathbf{s} = (s_1, \dots, s_n) \in \mathbb{C}^n \mid \operatorname{Re}(s_j + \dots + s_n) > n + 1 - j \text{ for all } j \in \{1, \dots, n\}\} \quad (\text{II.7})$$

and has a meromorphic continuation to the whole complex space \mathbb{C}^n whose singularities are included in the union of the hyperplanes

$$s_j + \dots + s_n = (n + 1 - j) - k_j \quad (1 \leq j \leq n, k_1, \dots, k_n \in \mathbb{N}_0).$$

Moreover it is known that for $n \geq 2$, almost all non-positive integers points lie in the singular locus above and are points of indeterminacy.

The study of the directional values of these multiple zeta functions at these points was first considered by S. Akiyama, S. Egami and Y. Tanigawa [1]. These values are now well understood thanks to several works such as [20], [25], [23] et [10].

In [20], Y. Komori proved that for $\mathbf{N} = (N_1, \dots, N_n) \in \mathbb{N}_0^n$ and $\boldsymbol{\theta} = (\theta_1, \dots, \theta_n) \in \mathbb{C}^n$ such that $\theta_j + \dots + \theta_n \neq 0$ for all $j \in \{1, \dots, n\}$, the directional limit

$$\zeta_n^\theta(-\mathbf{N}; \boldsymbol{\gamma}; \mathbf{b}) := \lim_{t \rightarrow 0} \zeta_n(-\mathbf{N} + t\boldsymbol{\theta}; \boldsymbol{\gamma}; \mathbf{b}) \quad (\text{II.8})$$

exists, and expressed this limit in terms of \mathbf{N} , $\boldsymbol{\theta}$ and generalized Bernoulli numbers defined implicitly as coefficients of some multiple series.

In [10], Essouabri and Matsumoto obtained by a different method a closed explicit formula for $\zeta_n^\theta(-\mathbf{N}; \boldsymbol{\gamma}; \mathbf{b})$ in terms of \mathbf{N} , $\boldsymbol{\theta}$ and only classical Bernoulli numbers.

Recently Essouabri and Matsumoto [8] extended this result to a more general class of multivariable zeta functions defined by

$$\zeta_n(\mathbf{s}, \mathbf{P}) = \sum_{\mathbf{m}=(m_1, \dots, m_n) \in \mathbb{N}^n} \prod_{j=1}^n P_j(m_1, \dots, m_j)^{-s_j} \quad (\text{II.9})$$

where for all $j \in \{1, \dots, n\}$, $P_j \in \mathbb{R}[X_1, \dots, X_j]$ is a polynomial satisfying some general assumptions. In this general case, instead of Bernoulli numbers, certain periods appear in the expression of values. Moreover, the nonlinearity of the polynomials P_j reveals values at the n -tuple of non-positive integers which can be transcendent even if all the coefficients of the polynomials P_j are algebraic.

Consider now the twisted case. Let $\boldsymbol{\mu} \in (\mathbb{T} \setminus \{1\})^k \times \{1\}^{n-k}$, $Q \in \mathbb{R}[X_1, \dots, X_n]$ and $\mathbf{P} = (P_1, \dots, P_T) \in (\mathbb{R}[X_1, \dots, X_n])^T$, we associate to the triple $(Q; \mathbf{P}; \boldsymbol{\mu})$ the (k, n) -twisted MZF (non-twisted if $\boldsymbol{\mu} = \mathbf{1}$) defined formally for $\mathbf{s} = (s_1, \dots, s_T) \in \mathbb{C}^T$ by

$$Z(Q; \mathbf{P}; \boldsymbol{\mu}; \mathbf{s}) = \sum_{\mathbf{m}=(m_1, \dots, m_n) \in \mathbb{N}^n} \frac{Q(m_1, \dots, m_n) \mu_1^{m_1} \dots \mu_k^{m_k}}{\prod_{t=1}^T P_t(m_1, \dots, m_n)^{s_t}} \quad (\text{II.10})$$

If $P_1 \dots P_T(\mathbf{X}) \rightarrow +\infty$ as $\|\mathbf{X}\| \rightarrow +\infty$ and $\mathbf{X} \in [1, +\infty)^n$, the existence of a meromorphic continuation of these Dirichlet series to whole space \mathbb{C}^T follows from the method introduced by Essouabri ([15], [16]) if P_1, \dots, P_T verify the probably optimal hypothesis H_0S (see Definition 1 below). Moreover, the complexity of the singular locus of these Dirichlet series is maximal in the non-twisted case (i.e. $\boldsymbol{\mu} = \mathbf{1}$) and we expect to have a holomorphic extension in the completely twisted case (i.e. $\boldsymbol{\mu} \in (\mathbb{T} \setminus \{1\})^n$). A counterexample shows that this is not the case under the general assumption H_0S .

Under the slightly more restrictive assumption (HDF) (see Definition 1 below), Marc de Crisenoy obtained in the completely twisted case (i.e. $\boldsymbol{\mu} \in (\mathbb{T} \setminus \{1\})^n$) the following general result which shows the holomorphy of these Dirichlet series and gives simple expressions of their values at T -tuples of non-positive integers:

Theorem 1 ([6] Theorems A and B). *Let $Q, P_1, \dots, P_T \in \mathbb{R}[X_1, \dots, X_n]$ and $\mathbf{N} = (N_1, \dots, N_T) \in \mathbb{N}_0^T$. We set $Q(\mathbf{X}) \prod_{t=1}^T P_t^{N_t}(\mathbf{X}) = \sum_{\mathbf{v} \in S} a_{\mathbf{v}} \mathbf{X}^{\mathbf{v}}$. If $\boldsymbol{\mu} \in (\mathbb{T} \setminus \{1\})^n$, P_1, \dots, P_T are polynomials verifying HDF condition, see Definition 1, and*

$$\prod_{t=1}^T P_t(\mathbf{X}) \xrightarrow[\substack{|\mathbf{X}| \rightarrow +\infty \\ \mathbf{X} \in [1, +\infty)^n}]{} +\infty$$

then $\mathbf{s} \mapsto Z(Q; \mathbf{P}; \boldsymbol{\mu}; \mathbf{s})$ has a holomorphic continuation to \mathbb{C} .

Moreover, $Z(Q; \mathbf{P}; \boldsymbol{\mu}; -\mathbf{N}) = \sum_{\mathbf{v} \in S} a_{\mathbf{v}} \prod_{j=1}^n \zeta_{\mu_j}(-v_j)$ where the values $\zeta_{\mu_j}(-v_j)$ were given in equation (II.5).

In the not completely twisted case, the existence of singularities makes the study of values at non-positive integers points more difficult. In addition to the works mentioned above, Essouabri and Matsumoto [11], obtained by combining the result of Crisenoy and the Mellin-Barnes integral representation formula (27), the values of the $(n-1, n)$ -twisted and $(n-2, n)$ -twisted MZF in the cases where the polynomials P_j are linear or diagonal. In this chapter, we consider a new class of $(n-1, n)$ -twisted MZF (see Definition (II.10)). More precisely, we determine (see Theorems 2 and 3 below) for this class of $(n-1, n)$ -twisted MZF, by two different methods, an explicit meromorphic continuation and determine explicitly their values at T -tuples of non-positive integers.

2 Preliminaries

We start this section giving some definitions connected with polynomials necessary to understand the results.

Definition 1. *Let $n \in \mathbb{N}$ and $P(\mathbf{X}) = \sum_{\alpha \in S} a_{\alpha} \mathbf{X}^{\alpha} \in \mathbb{C}[X_1, \dots, X_n]$ be a non-zero polynomial.*

1. *Let $\ell \leq \deg(P)$, we set and P_{ℓ} the homogeneous part of P of degree ℓ .*
2. *The polynomial P is called homogeneous of degree p if $P = P_{\deg(P)}$.*

Let P be a polynomial of degree p

- (a) *P is said to be an elliptic polynomial if $P_p \in \mathbb{R}_+[X_1, \dots, X_n]$ and $P_p(\mathbf{X}) \neq 0$ pour tout $\mathbf{X} = (X_1, \dots, X_n) \in [0, +\infty[^n \setminus \{\mathbf{0}\}$.*

We set $\mathfrak{E}_n = \{P \in \mathbb{R}[X_1, \dots, X_n] \mid P \text{ is an elliptic polynomial}\}$.

- (b) *P is said to be a non degenerate polynomial if $\deg(P) > 0$ and $P(\mathbf{X}) \asymp P^+(\mathbf{X})$ uniformly in $\mathbf{X} \in [1, +\infty)^n$, where $P^+(\mathbf{X}) = \sum_{\alpha \in S} \|a_{\alpha}\| \mathbf{X}^{\alpha}$.*

- (c) *P is said to be an hypoelliptic polynomial if*

- $\forall \mathbf{X} \in [1, +\infty)^n$, $P(\mathbf{X}) > 0$ and
- $\forall \alpha \in \mathbb{N}_0^n \setminus \{\mathbf{0}\}$, $\frac{\partial^{\alpha} P}{P}(\mathbf{X}) \xrightarrow[\substack{|\mathbf{X}| \rightarrow +\infty \\ \mathbf{X} \in [1, +\infty)^n}]{} 0$.

- (d) *P satisfies the HDF assumption, introduced by Crisenoy, if for each $\varepsilon > 0$*

- $\forall \mathbf{X} \in [1, +\infty)^n$, $P(\mathbf{X}) > 0$ and

- $\forall \alpha \in \mathbb{N}_0^n, k \in \llbracket 1, n \rrbracket; \alpha_k \geq 1 \Rightarrow \frac{\partial^\alpha P}{P}(\mathbf{X}) \ll X_k^{-\varepsilon}$ uniformly in $\mathbf{X} \in [1, +\infty)^n$.

(e) P satisfies the H_0S assumption, introduced by Essouabri, if

- $\forall \mathbf{X} \in [1, +\infty)^n, P(\mathbf{X}) > 0$ and
- $\forall \alpha \in \mathbb{N}_0^n, \frac{\partial^\alpha P}{P}(\mathbf{X}) \ll 1$ uniformly in $\mathbf{X} \in [1, +\infty)^n$.

Remark 1. We have the relations (a) \subsetneq (b) and (b), (c) \subsetneq (d) \subsetneq (e).

Example 1. $P(X_1, X_2) = X_1 X_2$ verify (b) but not (c) and $P(X_1, X_2) = (X_1 - X_2)^2 X_1 + X_2^2$ verify (c) but not (b). It follows that (b) and (c) are not comparable sets.

Example 2. $P(X_1, X_2) = (X_1 - X_2)^2 X_1 + X_1$ given by Essouabri verify the condition (e) but not (b) neither (c).

Definition 2. Let $p, q \in \mathbb{N}_0$ and $f \in \mathbb{R}$, we set

$$I(q; p; f) = \left\{ \mathbf{v} = (v_1, \dots, v_p) \in \mathbb{N}_0^p \mid \sum_{j=1}^p (p+1-j)v_j = 1+q-pf \right\}$$

Remark 2. The set $I(q; p; f)$ is a finite set. Indeed, for $\mathbf{v} \in I(q, p, f)$ we have

$$\|\mathbf{v}\|_1 \leq \sum_{j=1}^p (p+1-j)v_j \leq 1+q-pf$$

Remark 3. For $l \in \mathbb{N}_0$ we have the following identity

$$\binom{-s}{l} = \prod_{k=1}^l (-s-k+1) = \frac{\Gamma(1-s)}{\Gamma(1+l)\Gamma(1-s-l)}$$

Lemma 1. (Zeta function associated to a polynomial in one variable) Let $q \in \mathbb{N}_0, P(X) = \sum_{j=0}^p a_j X^j \in \mathbb{C}[X]$ with $a_p \neq 0$ a non-constant polynomial such that $0 \notin P(\mathbb{N})$. Then the series $Z(X^q; P; s) = \sum_{m \in \mathbb{N}} m^q P(m)^{-s}$ is absolutely and uniformly convergent in compact sets of $\left\{ \operatorname{Re}(s) > \frac{1+q}{p} \right\}$. Moreover it has a meromorphic continuation to \mathbb{C} with simple poles included in

$$\mathcal{S}'(q, p) = \{f \in \mathbb{Q} \setminus (-\mathbb{N}_0) \mid I(q; p; f) \neq \emptyset\} \subset \frac{1}{p}(1+q-\mathbb{N}_0) \setminus (-\mathbb{N}_0)$$

Furthermore, for $k \in \mathbb{N}_0$ we have

$$\begin{aligned} Z(X^q; P; -k) &= \sum_{\substack{\mathbf{v} \in \mathbb{N}_0^p \\ |\mathbf{v}| \leq k}} \binom{k}{\mathbf{v}} a_p^{k-|\mathbf{v}|} \prod_{j=1}^p a_{j-1}^{v_j} \zeta\left(-p(k-|\mathbf{v}|) - \sum_{j=1}^p (j-1)v_j - q\right) \\ &\quad + \sum_{\mathbf{v} \in I(q; p; -k)} \frac{(-1)^{|\mathbf{v}|-k} k! (|\mathbf{v}|-k-1)!}{v_1! \cdots v_p! p} a_p^{k-|\mathbf{v}|} \prod_{j=1}^p a_{j-1}^{v_j} \\ &= \sum_{v \in S_k} c_v \zeta(-v) + \sum_{\mathbf{v} \in I(q; p; -k)} \frac{(-1)^{|\mathbf{v}|-k} k! (|\mathbf{v}|-k-1)! \prod_{j=1}^p a_{j-1}^{v_j}}{v_1! \cdots v_p! p a_p^{|\mathbf{v}|-k}} \end{aligned} \quad (\text{II.11})$$

where c_v 's are defined by the relation $X^q P(X)^k = \sum_{v \in S_k} c_v X^v$ and $\zeta(-k) = (-1)^k \frac{B_{k+1}}{k+1}$.

3 Statements of theorems for $(n - 1, n)$ -twisted MZF

3.1 Hypothesis and main results of this chapter

Let $n \in \mathbb{N}$, $P_1, \dots, P_T \in \mathbb{R}[X_1, \dots, X_{n-1}]$ and $P_{T+1}(X_1, \dots, X_n) = Q(X_1, \dots, X_{n-1}) + R(X_n) \in \mathbb{R}[X_1, \dots, X_n]$. We suppose that:

A1) P_1, \dots, P_T, Q verify the HDF condition, see Definition 1-(d),

A2) $R \in \mathbb{R}_+[X_n]$ of degree r a non-constant polynomial such that $R(0) = 0$,

A3) $Q(\mathbf{X}') \xrightarrow[\substack{\|\mathbf{X}'\| \rightarrow +\infty \\ \mathbf{X}' \in [1, +\infty)^{n-1}}]{+} +\infty$, where $\mathbf{X}' = (X_1, \dots, X_{n-1})$.

We set $\mathbf{P} = (P_1, \dots, P_{T+1})$ and $\mathbf{P}' = (P_1, \dots, P_T)$.

Remark 4. 1. From A3) and Essouabri [15, Lemma 1] there exists $\delta > 0$ such that

$$Q(X_1, \dots, X_{n-1}) \gg (X_1 \cdots X_{n-1})^\delta \text{ uniformly in } \mathbf{X}' \in [1, +\infty)^{n-1} \quad (\text{II.12})$$

we fix a such $\delta > 0$.

2. There exists $D > 0$ such that

$$P_1(\mathbf{X}') \cdots P_T(\mathbf{X}') \ll (X_1 \cdots X_{n-1})^D \text{ uniformly in } \mathbf{X}' \in [1, +\infty)^{n-1} \quad (\text{II.13})$$

we fix a such $D > 0$.

By A2) we have $r \geq 1$ and there exists a_j 's real numbers such that $R(x) = \sum_{j=1}^r a_j x^j$.

Let $\boldsymbol{\mu}' = (\mu_1, \dots, \mu_{n-1}) \in (\mathbb{T} \setminus \{1\})^{n-1}$ and $\mathbf{q} \in \mathbb{N}_0^n$. For $\mathbf{s} \in \mathbb{C}^{T+1}$ we define the formal series

$$Z(\mathbf{X}^{\mathbf{q}}; \mathbf{P}; \boldsymbol{\mu}'; \mathbf{s}) = \sum_{\mathbf{m} \in \mathbb{N}^N} \frac{\mathbf{m}^{\mathbf{q}} \boldsymbol{\mu}'^{\mathbf{m}'}}{P(\mathbf{m})^{s_{T+1}} \prod_{t=1}^T P_t(\mathbf{m}')^{s_t}} \quad (\text{II.14})$$

If $\mathbf{q} = \mathbf{0}$ we write $Z(\mathbf{P}; \boldsymbol{\mu}'; \mathbf{s})$.

From A1) to A3) we have

$$P_1(\mathbf{X}') \cdots P_T(\mathbf{X}') P_{T+1}(\mathbf{X}) \xrightarrow[\substack{\|\mathbf{X}\| \rightarrow +\infty \\ \mathbf{X} \in [1, +\infty)^n}]{+} +\infty \quad \text{and} \quad P_1(\mathbf{X}') \cdots P_T(\mathbf{X}') Q(\mathbf{X}') \xrightarrow[\substack{\|\mathbf{X}'\| \rightarrow +\infty \\ \mathbf{X}' \in [1, +\infty)^{n-1}}]{+} +\infty,$$

it follows from [15, Lemma 1] that there exists $\nu > 0$ such that

$$\begin{aligned} P_1(\mathbf{X}') \cdots P_T(\mathbf{X}') P_{T+1}(\mathbf{X}) &\geq (X_1 \cdots X_n)^\nu \text{ uniformly in } \mathbf{X} \in [1, +\infty)^n \\ P_1(\mathbf{X}') \cdots P_T(\mathbf{X}') Q(\mathbf{X}') &\geq (X_1 \cdots X_{n-1})^\nu \text{ uniformly in } \mathbf{X}' \in [1, +\infty)^{n-1}. \end{aligned} \quad (\text{II.15})$$

In consequence the series $Z(\mathbf{X}^{\mathbf{q}}; \mathbf{P}; \boldsymbol{\mu}'; \mathbf{s})$ is absolutely convergent for \mathbf{s} in

$$\mathcal{D} = \left\{ \mathbf{s} \in \mathbb{C}^{T+1} \mid \forall i \in \llbracket 1, T+1 \rrbracket, \sigma_i = \text{Re}(s_i) > c \right\} \text{ where } c = \frac{1 + |\mathbf{q}|_\infty}{\nu} > 0. \quad (\text{II.16})$$

The following two theorems are the mains results of this chapter.

Theorem 2. Let $\mathbf{N} = (N_1, \dots, N_{T+1}) \in \mathbb{N}_0^{T+1}$. Under assumptions A1) to A3) we have $\mathbf{s} = -\mathbf{N}$ is a regular point of $Z(\mathbf{P}; \boldsymbol{\mu}'; \mathbf{s})$ and

$$\begin{aligned} Z(\mathbf{P}; \boldsymbol{\mu}'; -\mathbf{N}) &= \sum_{\boldsymbol{\beta} \in I(N_{T+1})} \frac{(-1)^{\mathcal{L}(\boldsymbol{\beta})} (\mathcal{L}(\boldsymbol{\beta}) - 1)! N_{T+1}! \prod_{j=1}^{r-1} a_j^{\beta_j}}{r a_r^{\mathcal{L}(\boldsymbol{\beta})} (N_{T+1} - |\boldsymbol{\beta}| + \mathcal{L}(\boldsymbol{\beta}))! \boldsymbol{\beta}!} \times \\ &\quad Z(\mathbf{P}', Q; \boldsymbol{\mu}'; -\mathbf{N}', -N_{T+1} + |\boldsymbol{\beta}| - \mathcal{L}(\boldsymbol{\beta})) \\ &+ \sum_{\substack{\boldsymbol{\alpha} \in \mathbb{N}_0^r \\ |\boldsymbol{\alpha}| \leq N_{T+1}}} \frac{(-1)^{\sum_{j=1}^r j \alpha_j} B_{1+\sum_{j=1}^r j \alpha_j} \binom{N_{T+1}}{|\boldsymbol{\alpha}|} \binom{|\boldsymbol{\alpha}|}{\boldsymbol{\alpha}} \prod_{j=1}^r a_j^{\alpha_j}}{1 + \sum_{j=1}^r j \alpha_j} \times \\ &\quad Z(\mathbf{P}', Q; \boldsymbol{\mu}'; -\mathbf{N}', -N_{T+1} + |\boldsymbol{\alpha}|) \end{aligned}$$

where for $\boldsymbol{\beta} \in \mathbb{N}_0^{r-1}$, $\mathcal{L}(\boldsymbol{\beta}) = \frac{1}{r} \left(1 + \sum_{j=1}^{r-1} j \beta_j \right)$ and

$$I(N_{T+1}) = \left\{ \boldsymbol{\beta} \in \mathbb{N}_0^{r-1} \mid |\boldsymbol{\beta}| - \mathcal{L}(\boldsymbol{\beta}) \leq N_{T+1} \text{ and } r \text{ divides } 1 + \sum_{j=1}^{r-1} j \beta_j \right\}$$

is a finite subset of $\{\boldsymbol{\beta} \in \mathbb{N}_0^{r-1} \mid |\boldsymbol{\beta}| \leq rN_{T+1} + 1\}$.

Theorem 3. Let $\mathbf{q} \in \mathbb{N}_0^n$, $\boldsymbol{\mu}' \in (\mathbb{T} \setminus \{1\})^{n-1}$ and $\mathbf{N} = (\mathbf{N}', N_{T+1}) \in \mathbb{N}_0^{T+1}$. Under assumptions A1) to A3) we have $\mathbf{s} = -\mathbf{N}$ is a regular point of $Z(\mathbf{X}^{\mathbf{q}}; \mathbf{P}; \boldsymbol{\mu}'; \mathbf{s})$ and

$$Z(\mathbf{X}^{\mathbf{q}}; \mathbf{P}; \boldsymbol{\mu}'; -\mathbf{N}) = \sum_{k=0}^{N_{T+1}} \binom{N_{T+1}}{k} Z(X^{\mathbf{q}_n}; R; -k) Z(\mathbf{X}'^{\mathbf{q}'}; \mathbf{P}', Q; \boldsymbol{\mu}'; -\mathbf{N}', -N_{T+1} + k)$$

Remark 5. Since the series $Z(\mathbf{X}'^{\mathbf{q}'}; \mathbf{P}', Q; \boldsymbol{\mu}'; \mathbf{s}_{T+1})$ is a $(n-1, n-1)$ -twisted MZF and $-N_{T+1} + k \in -\mathbb{N}_0$ for $k \in \llbracket 0, N_{T+1} \rrbracket$, we can use Crisenoy's theorem [6, Theorem 2] to determinate explicitly its values. The value of $Z(X^{\mathbf{q}_n}; R; -k)$ was given in Lemma 1.

Corollary 1. Let $P_1(x_1)$, $P(\mathbf{x}) = Q(x_1) + R(x_2)$, $\boldsymbol{\mu}' \in \mathbb{T} \setminus \{1\}$ and $\mathbf{N} = (N_1, N_2) \in \mathbb{N}_0^2$. Then

$$Z(\mathbf{P}; \boldsymbol{\mu}'; -\mathbf{N}) = \sum_{k=0}^{N_2} \binom{N_2}{k} Z(R; -k) Z(P_1, Q; \boldsymbol{\mu}'; -N_1, -N_2 + k).$$

Example 3. We can apply above theorems to $\mathbf{P} = (P_1, \dots, P_N)$, where :

1. $P_t(x_1, \dots, x_t) = \sum_{j=1}^t x_j$ for $t \in \llbracket 1, N \rrbracket$.
2. $P_t(x_1, \dots, x_t) = \sum_{j=1}^t x_j^{\ell_{t,j}}$ and $\ell_{t,j} \in \mathbb{N}$ for $t \in \llbracket 1, N \rrbracket$.

4 Proofs of Lemma 1, Theorems 2 and 3

We start by a proof of Lemma 1:

Proof of Lemma 1. Let $q \in \mathbb{N}_0$, $K \subset \left\{ s \in \mathbb{C}; \operatorname{Re}(s) > \frac{1+q}{p} \right\}$ a compact set, $s \in K$ and

$M \in \mathbb{N}$ such that $M \geq \frac{2}{\|a_p\|} \sum_{j=1}^p \|a_{j-1}\|$. We set

$$Z_M(X^q; P; s) = Z(X^q; P; s) - \sum_{m \leq M} m^q P(m)^{-s} \quad \text{and} \quad u(x) = \frac{\sum_{j=0}^{p-1} a_j x^j}{a_p x^p}$$

So we have $|u(x)| \leq 1/2$ for $x > M$. By applying the Taylor formula with integral remain to the function $u \mapsto (1+u)^s$, we obtain that for $|u| \leq 1/2$ and $r \in \mathbb{N}$

$$(1+u)^s = \sum_{k=0}^r \binom{-s}{k} u^k + (r+1) \binom{-s}{r+1} u^{r+1} \int_0^1 (1-t)^r (1+tu)^{-s-r-1} dt \quad (\text{II.17})$$

It follows that for $\text{Re}(s) \gg 1$,

$$\begin{aligned} Z_M(X^q; P; s) &= \sum_{m>M} m^q P(m)^{-s} = \sum_{m>M} m^q (a_p m^p)^{-s} (1+u(m))^{-s} \\ &= \sum_{k=0}^r \frac{1}{a_p^s} \binom{-s}{k} \sum_{m>M} m^{q-ps} u(m)^k + \frac{r+1}{a_p^s} \binom{-s}{r+1} \times \\ &\quad \sum_{m>M} m^{q-ps} u(m)^{r+1} \int_0^1 (1-t)^r (1+tu(m))^{-s-r-1} dt \end{aligned} \quad (\text{II.18})$$

Using the identity $(X_1 + \dots + X_p)^k = \sum_{\substack{\mathbf{v}=(v_1, \dots, v_p) \in \mathbb{N}_0^p \\ |\mathbf{v}|=k}} \binom{k}{\mathbf{v}} X_1^{v_1} \dots X_p^{v_p}$, we have for $\text{Re}(s) \gg 1$,

$$\begin{aligned} Z_M(X^q; P; s) &= \sum_{\substack{\mathbf{v} \in \mathbb{N}_0^{p+1} \\ |\mathbf{v}| \leq r}} \binom{-s}{|\mathbf{v}|} \binom{|\mathbf{v}|}{\mathbf{v}} \frac{\prod_{j=1}^p a_{j-1}^{v_j}}{a_p^{s+|\mathbf{v}|}} \zeta\left(ps + \sum_{j=1}^p (p+1-j)v_j - q\right) \\ &\quad - \sum_{\substack{\mathbf{v} \in \mathbb{N}_0^p \\ |\mathbf{v}| \leq r}} \binom{-s}{|\mathbf{v}|} \binom{|\mathbf{v}|}{\mathbf{v}} \frac{\prod_{j=1}^p a_{j-1}^{v_j}}{a_p^{s+|\mathbf{v}|}} \sum_{m \leq M} m^{q-ps - \sum_{j=1}^p (p+1-j)v_j} \\ &\quad + (r+1) \sum_{\substack{\mathbf{v} \in \mathbb{N}_0^p \\ |\mathbf{v}|=r+1}} \binom{-s}{r+1} \binom{r+1}{\mathbf{v}} \frac{\prod_{j=1}^p a_{j-1}^{v_j}}{a_p^{s+r+1}} \sum_{m>M} m^{q-ps - \sum_{j=1}^p (p+1-j)v_j} \times \\ &\quad \int_0^1 (1-u)^r \left(1 + u \sum_{j=1}^p \frac{a_{j-1}}{a_p} m^{j-1-p}\right)^{-s-r-1} du \end{aligned} \quad (\text{II.19})$$

On the right hand side of (II.19) we have

- First term has a meromorphic continuation to \mathbb{C} with simple poles included in

$$\mathcal{S}(q; p) = \left\{ \frac{1+q}{p} - \sum_{j=1}^p \frac{p+1-j}{p} v_j \mid \mathbf{v} = (v_1, \dots, v_p) \in \mathbb{N}^p \right\} \subset \frac{1+q - \mathbb{N}_0}{p}$$

- Second term has a holomorphic continuation to \mathbb{C} .
- Third term has a holomorphic continuation in $\left\{ \text{Re}(s) > \frac{1+q-r}{p} \right\}$.

It follows that equation (II.19) defines a meromorphic continuation of $Z_M(X^q; P; s)$ to $s \in \left\{ \text{Re}(s) > \frac{1+q-r}{p} \right\}$. More precisely, the function $H_{\mathbf{v}} : \mathbb{C} \setminus \mathcal{S}(q; p) \rightarrow \mathbb{C}$ defined by

$$H_{\mathbf{v}}(s) = \binom{-s}{|\mathbf{v}|} \binom{|\mathbf{v}|}{\mathbf{v}} \zeta\left(ps + \sum_{j=1}^p (p+1-j)v_j - q\right)$$

has a meromorphic continuation that is regular at all points of $-\mathbb{N}_0$ and we have for each $k \in \mathbb{N}_0$ that

$$H_{\mathbf{v}}(-k) = \begin{cases} \binom{k}{|\mathbf{v}|} \binom{|\mathbf{v}|}{\mathbf{v}} \zeta\left(-p(k - |\mathbf{v}|) - \sum_{j=1}^p (j-1)v_j - q\right) & , \text{ if } k \geq |\mathbf{v}| \\ \frac{(-1)^{|\mathbf{v}|-k} k! (|\mathbf{v}| - k - 1)!}{\mathbf{v}! p} & , \text{ if } \mathbf{v} \in I(q; p; -k) \\ 0 & , \text{ otherwise.} \end{cases}$$

It follows that all poles of $Z(X^q; P; s)$ are included in $\mathcal{S}'(q; p) = \mathcal{S}(q; p) \setminus -\mathbb{N}_0$. To evaluate at $s = k$ we take $r = 3 + pk$ and use two following identities

$$\sum_{\substack{\mathbf{v} \in \mathbb{N}_0^p \\ |\mathbf{v}| \leq r-1}} \binom{k}{|\mathbf{v}|} \binom{|\mathbf{v}|}{\mathbf{v}} \frac{\prod_{j=1}^p a_{j-1}^{v_j}}{a_p^{|\mathbf{v}|-k}} \sum_{\mathbf{m} \leq M} m^{pk - \sum_{j=1}^p (p+1-j)v_j} = \sum_{\mathbf{m} \leq M} P(\mathbf{m})^k$$

and

$$\begin{aligned} X^q P(X)^k &= \sum_{\substack{\mathbf{v} \in \mathbb{N}_0^p \\ |\mathbf{v}| \leq k}} \binom{k}{|\mathbf{v}|} \binom{|\mathbf{v}|}{\mathbf{v}} X^q (a_p X^p)^{k-|\mathbf{v}|} \prod_{j=1}^p (a_{j-1} X^{j-1})^{v_j} \\ &= \sum_{\substack{\mathbf{v} \in \mathbb{N}_0^{p+1} \\ |\mathbf{v}|=k}} \binom{k}{|\mathbf{v}|} \binom{|\mathbf{v}|}{\mathbf{v}} X^q (a_p X^p)^{v_{p+1}} \prod_{j=1}^p (a_{j-1} X^{j-1})^{v_j} \end{aligned}$$

This finish the proof of Lemma 1. \square

For the proof of Theorems 2 and 3 the main idea is to obtain a connection between the $(n-1, n)$ -twisted MZF

$$Z(\mathbf{X}^q; \mathbf{P}; \boldsymbol{\mu}'; \mathbf{s}) = \sum_{\mathbf{m} \in \mathbb{N}^n} \frac{\mathbf{m}^q \boldsymbol{\mu}'^{\mathbf{m}'}}{P(\mathbf{m})^{s_{T+1}} \prod_{t=1}^T P_t(\mathbf{m}')^{s_t}}$$

and the $(n-1, n-1)$ -twisted MZF

$$Z(\mathbf{X}'^q; \mathbf{P}', Q; \boldsymbol{\mu}'; \mathbf{s}) = \sum_{\mathbf{m}' \in \mathbb{N}^{n-1}} \frac{\mathbf{m}'^q \boldsymbol{\mu}'^{\mathbf{m}'}}{Q(\mathbf{m}')^{s_{T+1}} \prod_{t=1}^T P_t(\mathbf{m}')^{s_t}}$$

in order to use Crisenoy's Theorem.

We divide the proof into seven preliminary lemmas below, see Lemmas 2 to 8.

Lemma 2. *Let $a \in \mathbb{R}_+^*$, $R(X) \in \mathbb{R}_+[X]$ of degree r such that $R(0) = 0$ and $s \in \mathbb{C}$. Then there exists $\eta > 0$ such that the function*

$$h_a :] - \eta, +\infty[\rightarrow \mathbb{C} \\ x \mapsto (a + R(x))^{-s}$$

is \mathcal{C}^∞ in $] - \eta, +\infty[$ and for any $k \in \mathbb{N}_0$ and $x \in] - \eta, +\infty[$ we have

$$h_a^{(k)}(x) = k! \sum_{\substack{\boldsymbol{\alpha} \in \mathbb{N}_0^r \\ \sum_{j=1}^r j\alpha_j = k}} \binom{-s}{|\boldsymbol{\alpha}|} \binom{|\boldsymbol{\alpha}|}{\boldsymbol{\alpha}} (a + R(x))^{-s-|\boldsymbol{\alpha}|} \prod_{j=1}^r \left(\frac{R^{(j)}(x)}{j!} \right)^{\alpha_j} \quad (\text{II.20})$$

In particular,

$$h_a^{(k)}(0) = k! \sum_{\substack{\boldsymbol{\alpha} \in \mathbb{N}_0^r \\ \sum_{j=1}^r j\alpha_j = k}} \binom{-s}{|\boldsymbol{\alpha}|} \binom{|\boldsymbol{\alpha}|}{\boldsymbol{\alpha}} a^{-s-|\boldsymbol{\alpha}|} \prod_{j=1}^r \left(\frac{R^{(j)}(0)}{j!} \right)^{\alpha_j} \quad (\text{II.21})$$

Proof. It is enough to apply a recurrence on the parameter k . \square

Let $M \in \mathbb{N}_0$, $\mathbf{m}' = (m_1, \dots, m_{n-1}) \in \mathbb{N}^{n-1}$ and $\mathbf{s} = (s_1, \dots, s_{T+1}) \in \mathbb{C}^{T+1}$ such that $\sigma_i = \operatorname{Re}(s_i) > \frac{1}{\deg(R)}$ for each $i \in \llbracket 1, T+1 \rrbracket$. The Euler-Maclaurin formula applied to the function $\ell \mapsto P^{-s_{T+1}}(\mathbf{m}', \ell)$ gives

$$\begin{aligned} \sum_{\ell \in \mathbb{N}} P^{-s_{T+1}}(\mathbf{m}', \ell) &= \sum_{\ell \in \mathbb{N}} (Q(\mathbf{m}') + R(\ell))^{-s_{T+1}} \\ &= \int_{\mathbb{R}_+} (Q(\mathbf{m}') + R(x))^{-s_{T+1}} dx \\ &\quad + \sum_{k=0}^M (-1)^k \frac{B_{k+1}}{(k+1)!} \left(\frac{\partial}{\partial x} \right)^k (Q(\mathbf{m}') + R(x))^{-s_{T+1}} \Big|_{x=0} \\ &\quad + \frac{(-1)^M}{(M+1)!} \int_{\mathbb{R}_+} \left[\left(\frac{\partial}{\partial x} \right)^{M+1} (Q(\mathbf{m}') + R(x))^{-s_{T+1}} \right] B_{M+1}(x) dx \end{aligned}$$

Lemma 2 ,with $a = Q(\mathbf{m}') > 0$ by assumption A1), implies that

$$\begin{aligned} \sum_{\ell \in \mathbb{N}} P^{-s_{T+1}}(\mathbf{m}', \ell) &= \int_{\mathbb{R}_+} (Q(\mathbf{m}') + R(x))^{-s_{T+1}} dx + \sum_{k=0}^M \frac{(-1)^k B_{k+1}}{k+1} \times \\ &\quad \sum_{\substack{\boldsymbol{\alpha} \in \mathbb{N}_0^r \\ \sum_{j=1}^r j \alpha_j = k}} \binom{-s_{T+1}}{|\boldsymbol{\alpha}|} \binom{|\boldsymbol{\alpha}|}{\boldsymbol{\alpha}} Q(\mathbf{m}')^{-s_{T+1}-|\boldsymbol{\alpha}|} \prod_{j=1}^r \left(\frac{R^{(j)}(0)}{j!} \right)^{\alpha_j} \\ &\quad + (-1)^M \sum_{\substack{\boldsymbol{\alpha} \in \mathbb{N}_0^r \\ \sum_{j=1}^r j \alpha_j = M+1}} \binom{-s_{T+1}}{|\boldsymbol{\alpha}|} \binom{|\boldsymbol{\alpha}|}{\boldsymbol{\alpha}} \times \\ &\quad \int_{\mathbb{R}_+} B_{M+1}(x) (Q(\mathbf{m}') + R(x))^{-s_{T+1}-|\boldsymbol{\alpha}|} \prod_{j=1}^r \left(\frac{R^{(j)}(x)}{j!} \right)^{\alpha_j} dx \end{aligned}$$

which implies the following identity: for $\mathbf{s} \in \mathbb{C}^{T+1}$ such that $\sigma_i \gg 1$ for each $i \in \llbracket 1, T+1 \rrbracket$

$$\begin{aligned} Z(\mathbf{P}; \boldsymbol{\mu}'; \mathbf{s}) &= \sum_{\mathbf{m}' \in \mathbb{N}^{n-1}} \frac{\boldsymbol{\mu}'^{\mathbf{m}'}}{\prod_{j=1}^T P_j(\mathbf{m}')^{s_j}} \int_{\mathbb{R}_+} (Q(\mathbf{m}') + R(x))^{-s_{T+1}} dx \\ &\quad + \sum_{k=0}^M \frac{(-1)^k B_{k+1}}{k+1} \sum_{\substack{\boldsymbol{\alpha} \in \mathbb{N}_0^r \\ \sum_{j=1}^r j \alpha_j = k}} \binom{-s_{T+1}}{|\boldsymbol{\alpha}|} \binom{|\boldsymbol{\alpha}|}{\boldsymbol{\alpha}} \times \\ &\quad \prod_{j=1}^r \left(\frac{R^{(j)}(0)}{j!} \right)^{\alpha_j} Z(\mathbf{P}', Q; \boldsymbol{\mu}'; \mathbf{s}', s_{T+1} + |\boldsymbol{\alpha}|) \\ &\quad + (-1)^M \sum_{\substack{\boldsymbol{\alpha} \in \mathbb{N}_0^r \\ \sum_{j=1}^r j \alpha_j = M+1}} \binom{-s_{T+1}}{|\boldsymbol{\alpha}|} \binom{|\boldsymbol{\alpha}|}{\boldsymbol{\alpha}} R_M(\mathbf{P}', Q, R; \boldsymbol{\alpha}; \mathbf{s}) \boldsymbol{\mu}' \end{aligned} \tag{II.22}$$

where

$$\begin{aligned} R_M(\mathbf{P}', Q, R; \boldsymbol{\alpha}; \mathbf{s}) &= \sum_{\mathbf{m}' \in \mathbb{N}^{n-1}} \frac{\boldsymbol{\mu}'^{\mathbf{m}'}}{\prod_{j=1}^T P_j(\mathbf{m}')^{s_j}} \times \\ &\quad \int_{\mathbb{R}_+} B_{M+1}(x) (Q(\mathbf{m}') + R(x))^{-s_{T+1}-|\boldsymbol{\alpha}|} \prod_{j=1}^r \left(\frac{R^{(j)}(x)}{j!} \right)^{\alpha_j} dx \end{aligned} \tag{II.23}$$

To use this identity we need first to prove the following three lemmas.

Lemma 3. *Let $M \in \mathbb{N}_0$. The function $G_M : \mathbb{C} \times]-1, +\infty[\rightarrow \mathbb{C}$ defined by*

$$(1+y)^{-s} = \sum_{k=0}^M \binom{-s}{k} y^k + \binom{-s}{M+1} G_M(s; y)$$

verifies that:

1. for each $y \in]-1, +\infty[$, the function $s \mapsto G_M(s; y)$ is holomorphic in \mathbb{C} ;
2. for each $\delta, \gamma \in \mathbb{R}$ verifying $-1 < \delta \leq \gamma$ and each compact set K of \mathbb{R} we have

$$G_M(s; y) \ll_{\delta, \gamma, M, K} |y|^{M+1}, \text{ uniformly in } y \in [\delta, \gamma], \tau \in \mathbb{R} \text{ and } \sigma \in K$$

Proof. This is a particular case of Lemma 3.1 of [13]. □

Lemma 4. *Let $a, b \in \mathbb{C}$ such that $\operatorname{Re}(a), \operatorname{Re}(b) > 0$, $\alpha \in \mathbb{R}_+$ and $\beta \in \mathbb{R}_+^*$. Then, for each $s \in \mathbb{C}$ such that $\operatorname{Re}(s) > \frac{1+\alpha}{\beta}$ we have*

$$\begin{aligned} V(a, b; \alpha, \beta; s) &:= \int_{\mathbb{R}_+} \frac{x^\alpha}{(a + bx^\beta)^s} dx \\ &= \frac{1}{\beta a^{s - \frac{1+\alpha}{\beta}} b^{\frac{1+\alpha}{\beta}}} \frac{\Gamma\left(\frac{1+\alpha}{\beta}\right) \Gamma\left(s - \frac{1+\alpha}{\beta}\right)}{\Gamma(s)} \end{aligned} \quad (\text{II.24})$$

Proof. It's enough to proceed with the case $a > 0$ and $b > 0$. The general case is a consequence for this case by analytic continuation. The change of variable $t = bx^\beta$, implies that $x = b^{-1/\beta} t^{1/\beta}$, $dx = \frac{1}{\beta} b^{-1/\beta} t^{1/\beta - 1} dt$ and

$$V(a, b; \alpha, \beta; s) = \int_{\mathbb{R}_+} \frac{1}{(a+t)^s} b^{-\frac{\alpha}{\beta}} t^{\frac{\alpha}{\beta}} \frac{1}{\beta} b^{-\frac{1}{\beta}} t^{\frac{1}{\beta} - 1} dt = \frac{b^{-\frac{1+\alpha}{\beta}}}{\beta} \int_{\mathbb{R}_+} \frac{t^{\frac{1+\alpha}{\beta} - 1}}{(a+t)^s} dt$$

Set $t = ay$, then $dt = a dy$ and

$$V(a, b; \alpha, \beta; s) = \frac{b^{-\frac{1+\alpha}{\beta}}}{\beta} \int_{\mathbb{R}_+} \frac{a^{\frac{1+\alpha}{\beta} - 1} y^{\frac{1+\alpha}{\beta} - 1}}{a^s (1+y)^s} a dy = \frac{a^{-s + \frac{1+\alpha}{\beta}} b^{-\frac{1+\alpha}{\beta}}}{\beta} \int_{\mathbb{R}_+} \frac{y^{\frac{1+\alpha}{\beta} - 1}}{(1+y)^s} dy$$

Set $z = \frac{1}{1+y}$, then $y = \frac{1-z}{z}$, $dy = -\frac{dz}{z^2}$ and

$$\begin{aligned} V(a, b; \alpha, \beta; s) &= \frac{a^{-s + \frac{1+\alpha}{\beta}} b^{-\frac{1+\alpha}{\beta}}}{\beta} \int_0^1 \left(\frac{1-z}{z}\right)^{\frac{1+\alpha}{\beta} - 1} z^s \frac{dz}{z^2} \\ &= \frac{a^{-s + \frac{1+\alpha}{\beta}} b^{-\frac{1+\alpha}{\beta}}}{\beta} \int_0^1 (1-z)^{\frac{1+\alpha}{\beta} - 1} z^{s - \frac{1+\alpha}{\beta} - 1} dz \\ &= \frac{a^{-s + \frac{1+\alpha}{\beta}} b^{-\frac{1+\alpha}{\beta}}}{\beta} \frac{\Gamma\left(\frac{1+\alpha}{\beta}\right) \Gamma\left(s - \frac{1+\alpha}{\beta}\right)}{\Gamma(s)} \end{aligned}$$

□

Lemma 5. Let $a \in \mathbb{R}_+^*$, $M \in \mathbb{N}_0$, $R(X) \in \mathbb{R}_+[X]$ a polynomial of degree $r \geq 1$ verifying $R(0) = 0$. We write R as $R(X) = \sum_{j=1}^r a_j X^j$ where $a_j \geq 0$ and $a_r > 0$. Then for all $s \in \mathbb{C}$ verifying $\operatorname{Re}(s) > 1/r$ we have

$$\begin{aligned} Y(a; R; s) &:= \int_{\mathbb{R}_+} (a + R(x))^{-s} dx \\ &= \sum_{\substack{\beta \in \mathbb{N}_0^{r-1} \\ |\beta| \leq M}} \binom{-s}{|\beta|} \binom{|\beta|}{\beta} \prod_{j=1}^{r-1} a_j^{\beta_j} \frac{\Gamma(\mathcal{L}(\beta)) \Gamma(s + |\beta| - \mathcal{L}(\beta))}{r a_r^{\mathcal{L}(\beta)} a^{s+|\beta| - \mathcal{L}(\beta)} \Gamma(s + |\beta|)} \\ &\quad + \binom{-s}{M+1} \int_{\mathbb{R}_+} (a + a_r x^r)^{-s} G_M \left(s; \frac{\sum_{j=1}^{r-1} a_j x^j}{a + a_r x^r} \right) dx \end{aligned} \quad (\text{II.25})$$

where for $\beta \in \mathbb{N}_0^{r-1}$ we set $\mathcal{L}(\beta) = \frac{1 + \sum_{j=1}^{r-1} j \beta_j}{r}$.

Proof. Since $\frac{\sum_{j=1}^{r-1} a_j x^j}{a + a_r x^r} \geq 0$ for all $x \in \mathbb{R}_+$, Lemma 3 imply that for each $x \in \mathbb{R}_+$, we have

$$\begin{aligned} (a + R(x))^{-s} &= (a + a_r x^r + x \sum_{j=1}^{r-1} a_j x^j)^{-s} = (a + a_r x^r)^{-s} \left(1 + \frac{\sum_{j=1}^{r-1} a_j x^j}{a + a_r x^r} \right)^{-s} \\ &= (a + a_r x^r)^{-s} \left[\sum_{k=0}^M \binom{-s}{k} \frac{(\sum_{j=1}^{r-1} a_j x^j)^k}{(a + a_r x^r)^k} + \binom{-s}{M+1} G_M \left(s; \frac{\sum_{j=1}^{r-1} a_j x^j}{a + a_r x^r} \right) \right] \end{aligned}$$

Using relation $(b_1 + \dots + b_{p-1})^k = \sum_{\substack{\beta \in \mathbb{N}_0^{p-1} \\ |\beta|=k}} \binom{k}{\beta} \prod_{j=1}^{p-1} b_j^{\beta_j}$, it follows that for each $x \geq 0$, we have

$$\begin{aligned} (a + R(x))^{-s} &= \sum_{k=0}^M \binom{-s}{k} (a + a_r x^r)^{-s-k} \sum_{\substack{\beta \in \mathbb{N}_0^{r-1} \\ |\beta|=k}} \binom{k}{\beta} \prod_{j=1}^{r-1} a_j^{\beta_j} x^{\sum_{j=1}^{r-1} j \beta_j} \\ &\quad + (a + a_r x^r)^{-s} \binom{-s}{M+1} G_M \left(s; \frac{\sum_{j=1}^{r-1} a_j x^j}{a + a_r x^r} \right) \\ &= \sum_{\substack{\beta \in \mathbb{N}_0^{r-1} \\ |\beta| \leq M}} \binom{-s}{|\beta|} \binom{|\beta|}{\beta} \prod_{j=1}^{r-1} r_j^{\beta_j} (a + a_r x^r)^{-s-|\beta|} x^{\sum_{j=1}^{r-1} j \beta_j} \end{aligned} \quad (\text{II.26})$$

$$+ (a + a_r x^r)^{-s} \binom{-s}{M+1} G_M \left(s; \frac{\sum_{j=1}^{r-1} a_j x^j}{a + a_r x^r} \right) \quad (\text{II.27})$$

For determine the integral of (II.26) we use Lemma 3 to $\frac{x^{\sum_{j=1}^{r-1} j \beta_j}}{(a + a_r x^r)^{s+|\beta|}}$, and for the convergence of integral of (II.27) we use Lemma 4. In consequence for all $s \in \mathbb{C}$ verifying $\operatorname{Re}(s) > \frac{1}{r}$, in particular for $\operatorname{Re}(s + |\beta|) > \frac{1 + \sum_{j=1}^{r-1} j \beta_j}{r}$, we have

$$\begin{aligned}
Y(a; R; s) &= \sum_{\substack{\beta \in \mathbb{N}_0^{r-1} \\ |\beta| \leq M}} \binom{-s}{|\beta|} \binom{|\beta|}{\beta} \prod_{j=1}^{r-1} a_j^{\beta_j} V \left(a, a_r; \sum_{j=1}^{r-1} j\beta_j, r; s + |\beta| \right) \\
&\quad + \binom{-s}{M+1} \int_{\mathbb{R}_+} (a + a_r x^r)^{-s} G_M \left(s; \frac{\sum_{j=1}^{r-1} a_j x^j}{a + a_r x^r} \right) dx \\
&= \sum_{\substack{\beta \in \mathbb{N}_0^{r-1} \\ |\beta| \leq M}} \binom{-s}{|\beta|} \binom{|\beta|}{\beta} \prod_{j=1}^{r-1} a_j^{\beta_j} \frac{\Gamma \left(\frac{1 + \sum_{j=1}^{r-1} j\beta_j}{r} \right) \Gamma \left(s + |\beta| - \frac{1 + \sum_{j=1}^{r-1} j\beta_j}{r} \right)}{r a_r^{\frac{1 + \sum_{j=1}^{r-1} j\beta_j}{r}} a^{s + |\beta| - \frac{1 + \sum_{j=1}^{r-1} j\beta_j}{r}} \Gamma(s + |\beta|)} \\
&\quad + \binom{-s}{M+1} \int_{\mathbb{R}_+} (a + a_r x^r)^{-s} G_M \left(s; \frac{\sum_{j=1}^{r-1} a_j x^j}{a + a_r x^r} \right) dx.
\end{aligned}$$

This finishes the proof of Lemma 5. \square

Applying Lemmas 5 and 2 to the equation (II.22) we have for $M \in \mathbb{N}_0$ and $\mathbf{s} \in \mathbb{C}^{T+1}$ verifying $\sigma_i \gg 1$ for each $i \in \llbracket 1, T+1 \rrbracket$ that

$$\begin{aligned}
Z(\mathbf{P}; \boldsymbol{\mu}'; \mathbf{s}) &= \sum_{\substack{\beta \in \mathbb{N}_0^{r-1} \\ |\beta| \leq M}} \binom{-s_{T+1}}{|\beta|} \binom{|\beta|}{\beta} \frac{\prod_{j=1}^{r-1} a_j^{\beta_j} \Gamma(\mathcal{L}(\beta)) \Gamma(s_{T+1} + |\beta| - \mathcal{L}(\beta))}{r a_r^{\mathcal{L}(\beta)} \Gamma(s_{T+1} + |\beta|)} \\
&\quad \times Z(\mathbf{P}', Q; \boldsymbol{\mu}'; \mathbf{s}', s_{T+1} + |\beta| - \mathcal{L}(\beta)) \\
&\quad + \sum_{k=0}^M \frac{(-1)^k B_{k+1}}{k+1} \sum_{\substack{\alpha \in \mathbb{N}_0^r \\ \sum_{j=1}^r j\alpha_j = k}} \binom{-s_{T+1}}{|\alpha|} \binom{|\alpha|}{\alpha} \prod_{j=1}^r \left(\frac{R^{(j)}(0)}{j!} \right)^{\alpha_j} \quad (\text{II.28}) \\
&\quad \times Z(\mathbf{P}', Q; \boldsymbol{\mu}'; \mathbf{s}', s_{T+1} + |\alpha|) + \binom{-s_{T+1}}{M+1} H_M(\mathbf{s}) + R_M(\mathbf{s})
\end{aligned}$$

where $R_M(\mathbf{s}) = (-1)^M \sum_{\substack{\alpha \in \mathbb{N}_0^r \\ \sum_{j=1}^r j\alpha_j = M+1}} \binom{-s_{T+1}}{|\alpha|} \binom{|\alpha|}{\alpha} R_M(\mathbf{P}', Q, R; \alpha; \mathbf{s})$ and

$$H_M(\mathbf{s}) = \sum_{\mathbf{m}' \in \mathbb{N}^{n-1}} \frac{\boldsymbol{\mu}'^{\mathbf{m}'}}{\prod_{j=1}^T P_j(\mathbf{m}')^{s_j}} \int_{\mathbb{R}_+} (Q(\mathbf{m}') + a_r x^r)^{-s_{T+1}} G_M \left(s_{T+1}; \frac{\sum_{j=1}^{r-1} a_j x^j}{Q(\mathbf{m}') + a_r x^r} \right) dx.$$

Now we will study in the three following lemmas the terms which appear on the right side of the previous identity.

Lemma 6. *Let $M \in \mathbb{N}_0$ and $B > 0$. Then*

$$\mathbf{s} \mapsto H_M(\mathbf{s}) = \sum_{\mathbf{m}' \in \mathbb{N}^{n-1}} \frac{\boldsymbol{\mu}'^{\mathbf{m}'}}{\prod_{j=1}^T P_j(\mathbf{m}')^{s_j}} \int_{\mathbb{R}_+} (Q(\mathbf{m}') + a_r x^r)^{-s_{T+1}} G_M \left(s_{T+1}; \frac{\sum_{j=1}^{r-1} a_j x^j}{Q(\mathbf{m}') + a_r x^r} \right) dx$$

is a holomorphic function in the domain

$$\mathcal{D}_{B,M} = \left\{ \mathbf{s} \in \mathbb{C}^{T+1} \mid \operatorname{Re}(s_j) > -B \text{ for each } j \in \llbracket 1, T \rrbracket \text{ and } \operatorname{Re}(s_{T+1}) > \frac{1 + DB}{\delta} - \frac{M}{r} \right\}$$

where $D, \delta > 0$ verify

$$\prod_{j=1}^T P_j(\mathbf{X}') \ll (X_1 \cdots X_{n-1})^D \quad \text{and} \quad Q(\mathbf{X}') \gg (X_1 \cdots X_{n-1})^\delta \quad \text{uniformly in } \mathbf{X}' \in [1, +\infty)^{n-1}$$

Proof. Let K a compact set of $\mathcal{D}_{B,M}$ and the canonical projection

$$\begin{aligned} \pi' : \quad \mathbb{C}^{T+1} &\longrightarrow \mathbb{C}^T \\ \mathbf{s} = (s_1, \dots, s_{T+1}) &\mapsto \mathbf{s}' = (s_1, \dots, s_T) \end{aligned}$$

Then $K' = \pi'(K) \subset \{\mathbf{s}' \in \mathbb{C}^T \mid \operatorname{Re}(s_j) > -B \text{ for each } j \in \llbracket 1, T \rrbracket\}$ and K' is a compact set.

Let π'' the canonical projection

$$\begin{aligned} \pi'' : \quad \mathbb{C}^{T+1} &\longrightarrow \mathbb{C} \\ \mathbf{s} = (s_1, \dots, s_{T+1}) &\mapsto s_{T+1} \end{aligned}$$

Then $K'' = \pi''(K) \subset \{s_{T+1} \in \mathbb{C} \mid \operatorname{Re}(s_{T+1}) > \frac{1+DB}{\delta} - \frac{M}{r} \text{ for each } j \in \llbracket 1, T \rrbracket\}$ and K'' is a compact set. It follows that there exists $\varepsilon > 0$ such that

$$\forall s_{T+1} \in K'', \operatorname{Re}(s_{T+1}) \geq \frac{1+DB}{\delta} - \frac{M}{r} + \frac{\varepsilon}{\delta}$$

and

$$\left(\operatorname{Re}(s_{T+1}) + \frac{M}{r} \right) \delta - DB \geq 1 + \varepsilon \quad (\text{II.29})$$

From Lemma 3 we have uniformly in $\mathbf{m}' \in \mathbb{N}_0^{n-1}$, $x \in [0, +\infty[$ and $s_{T+1} \in K''$

$$\begin{aligned} &\left| (Q(\mathbf{m}') + a_r x^r)^{-s_{T+1}} G_M \left(s_{T+1}; \frac{\sum_{j=1}^{r-1} a_j x^j}{Q(\mathbf{m}') + a_r x^r} \right) \right| \\ &\ll_{M,K} (Q(\mathbf{m}') + a_r x^r)^{-\sigma_{T+1} - M - 1} \left(\sum_{j=1}^{r-1} a_j x^j \right)^{(M+1)} \\ &\ll_{M,K} (Q(\mathbf{m}') + a_r x^r)^{-\sigma_{T+1} - M - 1 + (M+1)\frac{r-1}{r}} \\ &= (Q(\mathbf{m}') + a_r x^r)^{-\sigma_{T+1} - \frac{M+1}{r}} \end{aligned}$$

and Lemma 4 implies that

$$s_{T+1} \mapsto \int_{\mathbb{R}_+} (Q(\mathbf{m}') + a_r x^r)^{-s_{T+1}} G_M \left(s_{T+1}; \frac{\sum_{j=1}^{r-1} a_j x^j}{Q(\mathbf{m}') + a_r x^r} \right) dx := W_M(s_{T+1}; \mathbf{m}')$$

is absolutely convergent and holomorphic function in

$$\left\{ \operatorname{Re}(s_{T+1}) + \frac{M+1}{r} > \frac{1}{r} \right\} = \left\{ \operatorname{Re}(s_{T+1}) > -\frac{M}{r} \right\} \supset K''.$$

Furthermore, from Lemma 4 we have

$$\begin{aligned} W_M(s_{T+1}; \mathbf{m}') &\ll_K \frac{1}{Q(\mathbf{m}')^{\sigma_{T+1} + \frac{M}{r}}} \frac{\Gamma(s_{T+1} + \frac{M}{r})}{\Gamma(s_{T+1} + \frac{M+1}{r})} \\ &\ll_K \frac{1}{Q(\mathbf{m}')^{\sigma_{T+1} + \frac{M}{r}}} \end{aligned}$$

uniformly in $s_{T+1} \in K''$ and $\mathbf{m}' \in \mathbb{N}^{n-1}$.

It follows that we have uniformly in $\mathbf{s} \in K$ and $\mathbf{m}' \in \mathbb{N}^{n-1}$

$$\begin{aligned}
& \left| \frac{\boldsymbol{\mu}'^{\mathbf{m}'}}{\prod_{j=1}^T P_j(\mathbf{m}')^{s_j}} \int_{\mathbb{R}_+} (Q(\mathbf{m}') + a_r x^r)^{-s_{T+1}} G_M \left(s_{T+1}; \frac{\sum_{j=1}^{r-1} a_j x^j}{Q(\mathbf{m}') + a_r x^r} \right) dx \right| \\
& \ll_K \frac{1}{\prod_{j=1}^T P_j(\mathbf{m}')^{\sigma_j}} \frac{1}{Q(\mathbf{m}')^{\sigma_{T+1} + \frac{M}{r}}} \\
& \ll_K \frac{1}{\prod_{j=1}^T P_j(\mathbf{m}')^{-B} Q(\mathbf{m}')^{\sigma_{T+1} + \frac{M}{r}}} = \frac{\prod_{j=1}^T P_j(\mathbf{m}')^B}{Q(\mathbf{m}')^{\sigma_{T+1} + \frac{M}{r}}} \\
& \ll_K \frac{(m_1 \cdots m_{N-1})^{BD}}{(m_1 \cdots m_{n-1})^{\delta(\sigma_{T+1} + \frac{M}{r})}} = \prod_{j=1}^{n-1} m_j^{BD - \delta(\sigma_{T+1} + \frac{M}{r})} \\
& \ll_K \prod_{j=1}^{n-1} m_j^{-1-\varepsilon}
\end{aligned}$$

since $\sum_{\mathbf{m}' \in \mathbb{N}^{n-1}} \prod_{j=1}^{n-1} m_j^{-1-\varepsilon}$ is convergent, then we conclude by the theorem of holomorphy under the sum sign. This completes the proof of the Lemma 6. \square

Lemma 7. Let $M \in \mathbb{N}$, $B > 0$, $\mathcal{D}_{B,M}$ the domain of \mathbb{C}^{T+1} defined in Lemma 6, $\boldsymbol{\alpha} \in \mathbb{N}^r$ such that $\sum_{j=1}^r j\alpha_j = M + 1$ and $R_M(\mathbf{P}', Q, R; \boldsymbol{\alpha}; \mathbf{s})$ the function defined in (II.23). The function $\mathbf{s} \mapsto R_M(\mathbf{P}', Q, R; \boldsymbol{\alpha}; \mathbf{s})$ is holomorphic in the domain $\mathcal{D}_{B,M}$.

Proof. Let $K \subset \mathcal{D}_{B,M}$ a compact set and K', K'' as in the proof of 6. We have uniformly in $s_{T+1} \in K''$ and $x \in \mathbb{R}_+$

$$\begin{aligned}
& \left| B_{M+1}(x) (Q(\mathbf{m}') + R(x))^{-s_{T+1} - |\boldsymbol{\alpha}|} \prod_{j=1}^p \left(\frac{R^{(j)}(x)}{j!} \right)^{\alpha_j} \right| \\
& \ll (Q(\mathbf{m}') + R(x))^{-\sigma_{T+1} - |\boldsymbol{\alpha}|} \prod_{j=1}^r |R^{(j)}(x)|^{\alpha_j} \\
& \ll (Q(\mathbf{m}') + R(x))^{-\sigma_{T+1} - |\boldsymbol{\alpha}|} x^{\sum_{j=1}^r \alpha_j (p-j)} \\
& \ll (Q(\mathbf{m}') + a_r x^r)^{-\sigma_{T+1} - |\boldsymbol{\alpha}|} x^{r|\boldsymbol{\alpha}| - M - 1}
\end{aligned}$$

last inequality because $|\boldsymbol{\alpha}| \geq \frac{M+1}{r}$ and it follows that $\sigma_{T+1} + |\boldsymbol{\alpha}| > \frac{1}{r} > 0$.

Lemma 4 imply that

$$s_{T+1} \mapsto \int_{\mathbb{R}_+} B_{M+1}(x) (Q(\mathbf{m}') + R(x))^{-s_{T+1} - |\boldsymbol{\alpha}|} \prod_{j=1}^r \left(\frac{R^{(j)}(x)}{j!} \right)^{\alpha_j} dx := L(\mathbf{m}'; s_{T+1})$$

is absolutely convergent and it's a holomorphic function in

$$\left\{ \operatorname{Re}(s_{T+1}) + |\boldsymbol{\alpha}| > \frac{1 + r|\boldsymbol{\alpha}| - M - 1}{r} \right\} = \left\{ \operatorname{Re}(s_{T+1}) > -\frac{M}{r} \right\}$$

and since $\sigma_{T+1} + |\boldsymbol{\alpha}| \geq \sigma_{T+1} + \frac{M+1}{r} > \frac{1}{r}$ it follows that

$$\begin{aligned}
L(\mathbf{m}'; s_{T+1}) & \ll_K \frac{1}{r Q(\mathbf{m}')^{\sigma_{T+1} + \frac{M}{r}}} \frac{|\Gamma(s_{T+1} + \frac{M}{r})|}{|\Gamma(s_{T+1} + |\boldsymbol{\alpha}|)|} \\
& \ll_K \frac{1}{Q(\mathbf{m}')^{\sigma_{T+1} + \frac{M}{r}}}
\end{aligned} \tag{II.30}$$

uniformly in $s_{T+1} \in K''$ and $\mathbf{m}' \in \mathbb{N}^{n-1}$. We conclude as in the proof of Lemma 6. \square

Lemma 8. Let $r \in \mathbb{N}$, $\boldsymbol{\beta} = (\beta_1, \dots, \beta_{r-1}) \in \mathbb{N}_0^{r-1}$, $\mathcal{L}(\boldsymbol{\beta}) = \frac{1 + \sum_{j=1}^{r-1} j\beta_j}{r}$. We define for $s \in \mathbb{C} \setminus \{\mathcal{L}(\boldsymbol{\beta}) - |\boldsymbol{\beta}| - j \mid j \in \mathbb{N}_0\}$ the function

$$G_{\boldsymbol{\beta}}(s) = \binom{-s}{|\boldsymbol{\beta}|} \binom{|\boldsymbol{\beta}|}{\boldsymbol{\beta}} \frac{\Gamma(s + |\boldsymbol{\beta}| - \mathcal{L}(\boldsymbol{\beta}))\Gamma(\mathcal{L}(\boldsymbol{\beta}))}{\Gamma(s + |\boldsymbol{\beta}|)}$$

Then, $s \mapsto G_{\boldsymbol{\beta}}(s)$ is a meromorphic function in \mathbb{C} with simple poles in $\mathcal{S} = \{\frac{1-j}{r} \mid j \in \mathbb{N}\} \setminus (-\mathbb{N}_0)$. In particular, non positive integer numbers are regular points of $G_{\boldsymbol{\beta}}$. Furthermore,

1. if $\mathcal{L}(\boldsymbol{\beta}) \notin \mathbb{N}_0$, then for each $k \in \mathbb{N}_0$ we have $G_{\boldsymbol{\beta}}(-k) = 0$;
2. if $\mathcal{L}(\boldsymbol{\beta}) \in \mathbb{N}_0$, then we have two possibilities

(a) $G_{\boldsymbol{\beta}}(-k) = 0$, if $k \leq |\boldsymbol{\beta}| - \mathcal{L}(\boldsymbol{\beta}) - 1$;

$$(b) \quad G_{\boldsymbol{\beta}}(-k) = \frac{(-1)^{\mathcal{L}(\boldsymbol{\beta})} \binom{|\boldsymbol{\beta}|}{\boldsymbol{\beta}} (\mathcal{L}(\boldsymbol{\beta}) - 1)! k!}{(k - |\boldsymbol{\beta}| + \mathcal{L}(\boldsymbol{\beta}))! |\boldsymbol{\beta}|!} = \frac{(-1)^{\mathcal{L}(\boldsymbol{\beta})} (\mathcal{L}(\boldsymbol{\beta}) - 1)! k!}{(k - |\boldsymbol{\beta}| + \mathcal{L}(\boldsymbol{\beta}))! |\boldsymbol{\beta}|!},$$

if $k \geq |\boldsymbol{\beta}| - \mathcal{L}(\boldsymbol{\beta})$

Proof. For $s \in \mathbb{C}$, $k \in \mathbb{N}_0$ we have

$$\binom{s}{k} = \frac{s(s-1)\cdots(s-(k-1))}{k!} = \frac{\Gamma(s+1)}{k!\Gamma(s-k+1)}$$

and $\binom{-s}{k} = \frac{\Gamma(1-s)}{k!\Gamma(-s-k+1)}$. It follows that

$$\begin{aligned} G_{\boldsymbol{\beta}}(s) &= \binom{|\boldsymbol{\beta}|}{\boldsymbol{\beta}} \Gamma(\mathcal{L}(\boldsymbol{\beta})) \frac{\Gamma(s + |\boldsymbol{\beta}| - \mathcal{L}(\boldsymbol{\beta}))}{\Gamma(s + |\boldsymbol{\beta}|)} \frac{\Gamma(1-s)}{|\boldsymbol{\beta}|! \Gamma(-s - |\boldsymbol{\beta}| + 1)} \\ &= \frac{\binom{|\boldsymbol{\beta}|}{\boldsymbol{\beta}} \Gamma(\mathcal{L}(\boldsymbol{\beta}))}{|\boldsymbol{\beta}|! \pi} \Gamma(s + |\boldsymbol{\beta}| - \mathcal{L}(\boldsymbol{\beta})) \Gamma(1-s) \sin(\pi s + \pi |\boldsymbol{\beta}|) \\ &= \frac{(-1)^{|\boldsymbol{\beta}|} \binom{|\boldsymbol{\beta}|}{\boldsymbol{\beta}} \Gamma(\mathcal{L}(\boldsymbol{\beta}))}{|\boldsymbol{\beta}|! \pi} \Gamma(1-s) \Gamma(s + |\boldsymbol{\beta}| - \mathcal{L}(\boldsymbol{\beta})) \sin(\pi s) \end{aligned}$$

Let $k \in \mathbb{N}_0$.

Case 1: $\mathcal{L}(\boldsymbol{\beta}) \notin \mathbb{N}_0$, then $\mathcal{L}(\boldsymbol{\beta}) \notin \mathbb{Z}$. In consequence $G_{\boldsymbol{\beta}}(-k) = 0$.

Case 2: $\mathcal{L}(\boldsymbol{\beta}) \in \mathbb{N}_0$,

i) if $-k + |\boldsymbol{\beta}| - \mathcal{L}(\boldsymbol{\beta}) > 0$, then $G_{\boldsymbol{\beta}}(-k) = 0$.

ii) if $-k + |\boldsymbol{\beta}| - \mathcal{L}(\boldsymbol{\beta}) \leq 0$, we set $m = k - |\boldsymbol{\beta}| + \mathcal{L}(\boldsymbol{\beta}) \in \mathbb{N}$ and $s = -k + a$ with $a \rightarrow 0$. Then $s + |\boldsymbol{\beta}| - \mathcal{L}(\boldsymbol{\beta}) = -k + |\boldsymbol{\beta}| - \mathcal{L}(\boldsymbol{\beta}) + a = -m + a$ and

$$\begin{aligned} \Gamma(s + |\boldsymbol{\beta}| - \mathcal{L}(\boldsymbol{\beta})) &= \Gamma(-m + a) \underset{a \rightarrow 0}{\sim} \frac{\text{Res}(\Gamma; -m)}{a} \\ \sin(\pi s) &= \sin(-\pi k + \pi a) = (-1)^k \sin(\pi a) \underset{a \rightarrow 0}{\sim} \pi (-1)^k a \end{aligned}$$

so

$$\begin{aligned} \Gamma(s + |\boldsymbol{\beta}| - \mathcal{L}(\boldsymbol{\beta})) \sin(\pi s) &\underset{s \rightarrow -k}{\sim} \pi (-1)^k \text{Res}(\Gamma; -m) = \pi (-1)^k \frac{(-1)^m}{m!} \\ &\underset{s \rightarrow -k}{\sim} \pi \frac{(-1)^{k+k-|\boldsymbol{\beta}|+\mathcal{L}(\boldsymbol{\beta})}}{(k - |\boldsymbol{\beta}| + \mathcal{L}(\boldsymbol{\beta}))!} = \frac{\pi (-1)^{|\boldsymbol{\beta}| - \mathcal{L}(\boldsymbol{\beta})}}{(k - |\boldsymbol{\beta}| + \mathcal{L}(\boldsymbol{\beta}))!} \end{aligned}$$

which implies that

$$\begin{aligned} G_{\boldsymbol{\beta}}(-k) &= \frac{(-1)^{|\boldsymbol{\beta}|} \binom{|\boldsymbol{\beta}|}{\boldsymbol{\beta}} (\mathcal{L}(\boldsymbol{\beta}) - 1)! \Gamma(1+k)}{|\boldsymbol{\beta}|! \pi} \frac{\pi (-1)^{|\boldsymbol{\beta}| - \mathcal{L}(\boldsymbol{\beta})}}{(k - |\boldsymbol{\beta}| + \mathcal{L}(\boldsymbol{\beta}))!} \\ &= \frac{(-1)^{\mathcal{L}(\boldsymbol{\beta})} (\mathcal{L}(\boldsymbol{\beta}) - 1)! k!}{\boldsymbol{\beta}! (k - |\boldsymbol{\beta}| + \mathcal{L}(\boldsymbol{\beta}))!} \end{aligned}$$

this finishes the proof of Lemma 8. □

4.1 Proof of Theorem 2

Let $M \in \mathbb{N}_0$, equation (II.28) implies that for $\mathbf{s} \in \mathbb{C}^{T+1}$ and $\sigma_i \gg 1$ for each $j \in \llbracket 1, T+1 \rrbracket$, we have

$$\begin{aligned} Z(\mathbf{P}; \boldsymbol{\mu}'; \mathbf{s}) &= \sum_{\substack{\boldsymbol{\beta} \in \mathbb{N}_0^{r-1} \\ |\boldsymbol{\beta}| \leq M}} \frac{\prod_{j=1}^{r-1} a_j^{\beta_j}}{r a_r^{\mathcal{L}(\boldsymbol{\beta})}} G_{\boldsymbol{\beta}}(s_{T+1}) Z(\mathbf{P}', Q; \boldsymbol{\mu}'; \mathbf{s}', s_{T+1} + |\boldsymbol{\beta}| - \mathcal{L}(\boldsymbol{\beta})) \\ &+ \sum_{k=0}^M \frac{(-1)^k B_{k+1}}{k+1} \sum_{\substack{\boldsymbol{\alpha} \in \mathbb{N}_0^r \\ \sum_{j=1}^r j \alpha_j = k}} \binom{-s_{T+1}}{|\boldsymbol{\alpha}|} \binom{|\boldsymbol{\alpha}|}{\boldsymbol{\alpha}} \prod_{j=0}^r \left(\frac{R^{(j)}(0)}{j!} \right)^{\alpha_j} \times \\ &Z(\mathbf{P}', Q; \boldsymbol{\mu}'; \mathbf{s}', s_{T+1} + |\boldsymbol{\alpha}|) + \binom{-s_{T+1}}{M+1} H_M(\mathbf{s}) \\ &+ (-1)^M \sum_{\substack{\boldsymbol{\alpha} \in \mathbb{N}_0^r \\ \sum_{j=1}^r j \alpha_j = M+1}} \binom{-s_{T+1}}{|\boldsymbol{\alpha}|} \binom{|\boldsymbol{\alpha}|}{\boldsymbol{\alpha}} R_M(\mathbf{P}', Q, R; \boldsymbol{\alpha}; \mathbf{s}) \end{aligned} \quad (\text{II.31})$$

Crisenoy's Theorem and Lemmas 6, 7, 8 imply that for all $B > 0$ the function $\mathbf{s} \mapsto Z(\mathbf{P}; \boldsymbol{\mu}'; \mathbf{s})$ is meromorphic in

$$\mathcal{D}_{B,M} = \left\{ \mathbf{s} \in \mathbb{C}^{T+1} \mid \operatorname{Re}(s_j) > -B \forall j \in \llbracket 1, T \rrbracket \text{ and } \operatorname{Re}(s_{T+1}) > \frac{1+DB}{\delta} - \frac{M}{r} \right\}$$

with singularities included in the reunion of hyperplanes $s_{T+1} = \frac{1}{r}$ and $s_{T+1} = -\frac{j}{r}$ ($j \in \mathbb{N}_0, r \nmid j$).

Since $\mathbb{C}^{T+1} = \bigcup_{B>0, M \in \mathbb{N}} \mathcal{D}_{B,M}$ it follows that $\mathbf{s} \mapsto Z(\mathbf{P}; \boldsymbol{\mu}'; \mathbf{s})$ has a meromorphic continuation to \mathbb{C}^{T+1} with singularities included in the reunion of hyperplanes

$$s_{T+1} = \frac{1}{p} \text{ and } s_{T+1} = -\frac{j}{p} \quad (j \in \mathbb{N}_0, p \nmid j).$$

Let $\mathbf{k} = (k_1, \dots, k_{T+1}) \in \mathbb{N}_0^{T+1}$. We choose $B > \max\{k_j \mid j \in \llbracket 1, T \rrbracket\}$; for example $B = \max\{k_j \mid j \in \llbracket 1, T \rrbracket\} + \varepsilon$ with $\varepsilon \in]0, 1[$.

Let $M \in \mathbb{N}_0$ such that $-k_{T+1} > \frac{1+DB}{\delta} - \frac{M}{r}$; that is $M > r k_{T+1} + \frac{r}{\delta}(1+DB)$. Then we have $\mathbf{s} = -\mathbf{k} \in \mathcal{D}_{B,M}$.

In particular H_M and $R_M(\mathbf{P}', Q, R; \boldsymbol{\alpha}; -)$ with $\sum_{j=1}^r j \alpha_j = M+1$ are holomorphic in $\mathbf{s} = -\mathbf{k}$.

For $\mathbf{s} = -\mathbf{k}$ we have $\binom{-s_{T+1}}{M+1} = \binom{k_{T+1}}{M+1} = 0$, because $M > rk_{T+1} + \frac{r}{\delta} > k_{T+1}$ and if $\boldsymbol{\alpha} \in \mathbb{N}_0^p$ verifies $\sum_{j=1}^r j\alpha_j = M + 1$, then $r|\boldsymbol{\alpha}| \geq M + 1 > M > rk_{T+1}$. Which implies that $|\boldsymbol{\alpha}| > k_{T+1}$ and therefore that

$$\binom{-s_{T+1}}{|\boldsymbol{\alpha}|} \Big|_{s_{T+1}=-k_{T+1}} = \binom{k_{T+1}}{|\boldsymbol{\alpha}|} = 0.$$

Relation (II.31) implies

$$\begin{aligned} Z(\mathbf{P}; \boldsymbol{\mu}'; -\mathbf{k}) &= \sum_{\substack{\boldsymbol{\beta} \in \mathbb{N}_0^{r-1} \\ |\boldsymbol{\beta}| \leq M}} \frac{\prod_{j=1}^{r-1} a_j^{\beta_j}}{r a_r^{\mathcal{L}(\boldsymbol{\beta})}} G_{\boldsymbol{\beta}}(-k_{T+1}) Z(\mathbf{P}', Q; \boldsymbol{\mu}'; -\mathbf{k}', -k_{T+1} + |\boldsymbol{\beta}| - \mathcal{L}(\boldsymbol{\beta})) \\ &+ \sum_{k=0}^M \frac{(-1)^k B_{k+1}}{k+1} \sum_{\substack{\boldsymbol{\alpha} \in \mathbb{N}_0^r \\ \sum_{j=1}^r j\alpha_j = k}} \binom{k_{T+1}}{|\boldsymbol{\alpha}|} \binom{|\boldsymbol{\alpha}|}{\boldsymbol{\alpha}} \prod_{j=0}^r \left(\frac{R^{(j)}(0)}{j!} \right)^{\alpha_j} \times \\ &Z(\mathbf{P}', Q; \boldsymbol{\mu}'; -\mathbf{k}', -k_{T+1} + |\boldsymbol{\alpha}|) \end{aligned} \quad (\text{II.32})$$

Let $I(k_{T+1}) = \left\{ \boldsymbol{\beta} \in \mathbb{N}_0^{r-1} \mid \mathcal{L}(\boldsymbol{\beta}) = \frac{1 + \sum_{j=1}^{r-1} j\beta_j}{r} \in \mathbb{N}_0 \text{ and } |\boldsymbol{\beta}| - \mathcal{L}(\boldsymbol{\beta}) \leq k_{T+1} \right\}$, we remark that $I(k_{T+1})$ is a finite set. Indeed $\boldsymbol{\beta} \in I(k_{T+1})$ implies that

$$k_{T+1} \geq |\boldsymbol{\beta}| - \frac{1}{r} \left(1 + \sum_{j=1}^{r-1} j\beta_j \right) = \frac{1}{r} \left(\sum_{j=1}^{r-1} (r-j)\beta_j \right) - \frac{1}{r} \geq \frac{1}{r} |\boldsymbol{\beta}| - \frac{1}{r}$$

and therefore that $|\boldsymbol{\beta}| \leq rk_{T+1} + 1$.

Lemma 8 implies that

$$\begin{aligned} Z(\mathbf{P}; \boldsymbol{\mu}'; -\mathbf{k}) &= \sum_{\boldsymbol{\beta} \in I(k_{T+1})} \frac{\prod_{j=1}^{r-1} a_j^{\beta_j}}{r a_r^{\mathcal{L}(\boldsymbol{\beta})}} \frac{(-1)^{\mathcal{L}(\boldsymbol{\beta})} (\mathcal{L}(\boldsymbol{\beta}) - 1)! k_{T+1}!}{(k_{T+1} - |\boldsymbol{\beta}| + \mathcal{L}(\boldsymbol{\beta}))! \boldsymbol{\beta}!} \times \\ &Z(\mathbf{P}', Q; \boldsymbol{\mu}'; -\mathbf{k}', -k_{T+1} + |\boldsymbol{\beta}| - \mathcal{L}(\boldsymbol{\beta})) \\ &+ \sum_{\substack{\boldsymbol{\alpha} \in \mathbb{N}_0^r \\ \sum_{j=1}^r j\alpha_j \leq M \\ |\boldsymbol{\alpha}| \leq k_{T+1}}} \frac{(-1)^{\sum_{j=1}^r j\alpha_j} B_{1+\sum_{j=1}^r j\alpha_j}}{1 + \sum_{j=1}^r j\alpha_j} \binom{k_{T+1}}{|\boldsymbol{\alpha}|} \binom{|\boldsymbol{\alpha}|}{\boldsymbol{\alpha}} \times \\ &\prod_{j=0}^r \left(\frac{R^{(j)}(0)}{j!} \right)^{\alpha_j} Z(\mathbf{P}', Q; \boldsymbol{\mu}'; -\mathbf{k}', -k_{T+1} + |\boldsymbol{\alpha}|) \end{aligned} \quad (\text{II.33})$$

On the other side, since $|\boldsymbol{\alpha}| \leq k_{T+1}$ implies that $\sum_{j=1}^r j\alpha_j \leq r|\boldsymbol{\alpha}| \leq rk_{T+1} < M$, see equation (II.29), we can erase the condition $\sum_{j=1}^r j\alpha_j \leq M$ in the second sum. This ends the proof of Theorem 2. \square

4.2 Proof of Theorem 3

Lemma 9. *Let $\sigma_0 \in \mathbb{R}_-$ and $\mathbf{q} \in \mathbb{N}_0^n$. Under assumptions A1) to A3) we have that the $(n-1, n)$ -twisted MZF $Z(\mathbf{X}^{\mathbf{q}}; \mathbf{P}; \boldsymbol{\mu}'; \mathbf{s})$ has a meromorphic continuation $\text{Re}(\mathbf{s}) > \sigma_0 \mathbf{1}$*

given by

$$\begin{aligned}
Z(\mathbf{X}^{\mathbf{q}}; \mathbf{P}; \boldsymbol{\mu}'; \mathbf{s}) &= \sum_{k=0}^{\lfloor -\rho' \rfloor} \frac{(-1)^k \Gamma(s_{T+1} + k)}{k! \Gamma(s_{T+1})} Z(X^{q_n}; R; -k) Z(\mathbf{X}'^{\mathbf{q}'}; \mathbf{P}', Q; \boldsymbol{\mu}'; \mathbf{s}', s_{T+1} + k) \\
&+ \sum_{f \in \mathcal{S}'} \frac{\Gamma(s_{T+1} - f) \Gamma(f)}{\Gamma(s_{T+1})} Z(\mathbf{X}'^{\mathbf{q}'}; \mathbf{P}', Q; \boldsymbol{\mu}'; \mathbf{s}', s_{T+1} - f) \operatorname{Res}_{z=f} Z(X^{q_n}; R; z) \\
&+ \frac{1}{2i\pi} \int_{(\rho')} \frac{\Gamma(s_{T+1} - z) \Gamma(z)}{\Gamma(s_{T+1})} Z(X^{q_n}; R; z) Z(\mathbf{X}'^{\mathbf{q}'}; \mathbf{P}', Q; \boldsymbol{\mu}'; \mathbf{s}', s_{T+1} - z) dz
\end{aligned}$$

where $\rho' \in \left[-2c + \sigma_0 - \frac{1}{2 \deg(R)}, -2c + \sigma_0\right]$ and c is defined in equation (II.16). Moreover, its singularities are included in $\{\mathbf{s} \in \mathbb{C}^{T+1} \mid s_{T+1} \in \mathcal{S}' = \mathcal{S}'(q_N; \deg(R)) \cap [\rho', +\infty[\}$.

Proof. This proof is based in the Mellin–Barnes integral representation formula, see Lemma 27.

Let $\sigma_0 \in \mathbb{R}^-$, $\boldsymbol{\mu}' = (\mu_1, \dots, \mu_{n-1}) \in (\mathbb{T} \setminus \{1\})^{n-1}$, $\mathbf{q} \in \mathbb{N}_0^n$ and c as in equation (II.16). We set $w_0 = Q(\mathbf{m}')$, $w_1 = R(m_n)$ and $\rho = 2c$.

Then for $\mathbf{s} \in \mathbb{C}^{T+1}$ such that $\operatorname{Re}(s_i) > 4c$ for each $i \in \llbracket 1, T+1 \rrbracket$ we have $\operatorname{Re}(z) = 2c$, $\operatorname{Re}(s_{T+1} - z) > 4c - 2c = 2c$ and the Mellin-Barnes formula gives us

$$\begin{aligned}
\sum_{\mathbf{m} \in \mathbb{N}^n} \frac{\mathbf{m}^{\mathbf{q}} \prod_{j=1}^{n-1} \mu_j^{m_j}}{P_{T+1}(\mathbf{m}', m_n)^{s_{T+1}} \prod_{t=1}^T P_t(\mathbf{m}')^{s_t}} &= \frac{1}{2i\pi} \int_{(\rho)} \frac{\Gamma(s_{T+1} - z) \Gamma(z)}{\Gamma(s_{T+1})} \times \\
&Z(X^{q_n}; R; z) Z(\mathbf{X}'^{\mathbf{q}'}; \mathbf{P}', Q; \boldsymbol{\mu}'; \mathbf{s}_T, s_{T+1} - z) dz
\end{aligned} \tag{II.34}$$

To give a meromorphic continuation of this series the main idea is to shift the integration line to the left until $\operatorname{Re}(z_j) = \rho' \in \left[-2c + \sigma_0 - \frac{1}{2 \deg(R)}, -2c + \sigma_0\right]$ such that $\rho' \notin -\frac{1}{2 \deg(R)} \mathbb{N}_0$ to guarantee the non-existence of poles on this new integration line. To verify that we can shift to the left and to determine the set of possible poles of the integrant we use the following three facts:

1. Let $z = x + iy$ with $x, y \in \mathbb{R}$, then there exists $A > 0$ such that for all $\varepsilon, \delta > 0$ we have uniformly in $\operatorname{Re}(z) > c - N$ and $|y| \geq \delta$

$$Z(X^{q_n}; R; z) \ll_{\mathbf{N}, \varepsilon, \delta} (1 + |y|)^{A \max\{c-x; 0\} + \varepsilon} \tag{II.35}$$

This estimate follows from the classical estimates of the Riemann zeta function.

2. By Crisenoy's theorem we have the function $(\mathbf{s}_T, z) \mapsto Z(X^q; \mathbf{P}', Q; \boldsymbol{\mu}; \mathbf{s}_T, z)$ has a holomorphic continuation to \mathbb{C}^{T+1} .
3. Let $z_i = x_j + iy_j$ with $x_j, y_j \in \mathbb{R}$, then there exists $A > 0$ such that for all $\varepsilon > 0$ we have uniformly in $\operatorname{Re}(\mathbf{z}) > \mathbf{c} - \mathbf{N}$

$$Z(X^q; \mathbf{P}', Q; \boldsymbol{\mu}; z_1, \dots, z_T) \ll_{\mathbf{N}, \varepsilon} (1 + |(y_1, \dots, y_T)|_1)^{A \sum_{t=1}^T \max\{c-x_t; 0\} + \varepsilon} \tag{II.36}$$

This estimate follows from the proof of holomorphic of $\mathbf{z} \mapsto Z(X^q; \mathbf{P}', Q; \boldsymbol{\mu}; \mathbf{z})$ continuation given by Crisenoy.

For $\operatorname{Re}(\mathbf{s}) > \sigma_0 \mathbf{1}$ and $\operatorname{Re}(z) \in [\rho', \rho]$ we have $\operatorname{Re}(s_{T+1} - z) > 2c$, it follows that all integrant's singularities come from singularities of $\Gamma(z)$ factor in $\llbracket [-\rho'], 0 \rrbracket$ and from singularities of $Z(X^{q_n}; R; z)$ factor in $\mathcal{S}' = \mathcal{S}'(q_n, \deg(R)) \cap]\rho', +\infty[$.

Therefore for $\operatorname{Re}(\mathbf{s}) > 2c \mathbf{1}$ the integral on the right hand side of (II.34) is equal to

$$\begin{aligned} & \sum_{k=0}^{\lfloor -\rho' \rfloor} \frac{\Gamma(s_{T+1} + k)}{\Gamma(s_{T+1})} Z(X^{q_n}; R; -k) Z(\mathbf{X}'^{\mathbf{q}'}; \mathbf{P}', Q; \boldsymbol{\mu}'; \mathbf{s}_T, s_{T+1} + k) \operatorname{Res}_{z=-k} \Gamma(z) \\ & + \sum_{f \in \mathcal{S}'} \frac{\Gamma(s_{T+1} - f) \Gamma(f)}{\Gamma(s_{T+1})} Z(\mathbf{X}'^{\mathbf{q}'}; \mathbf{P}', Q; \boldsymbol{\mu}'; \mathbf{s}_T, s_{T+1} - f) \operatorname{Res}_{z=f} Z(X^{q_n}; R; z) \\ & + \frac{1}{2i\pi} \int_{(\rho')} \frac{\Gamma(s_{T+1} - z) \Gamma(z)}{\Gamma(s_{T+1})} Z(X^{q_n}; R; z) Z(\mathbf{X}'^{\mathbf{q}'}; \mathbf{P}', Q; \boldsymbol{\mu}'; \mathbf{s}_T, s_{T+1} - z) dz \end{aligned} \quad (\text{II.37})$$

since estimate (II.35) and Stirling complex formula imply that we have uniformly in $\operatorname{Re}(z) \in [A, B]$

$$z^k \Gamma(z) Z(X^{q_n}; R; z) \rightarrow 0 \text{ as } |\operatorname{Im}(z)| \rightarrow +\infty$$

and $Z(\mathbf{X}'^{\mathbf{q}'}; \mathbf{P}', Q; \boldsymbol{\mu}'; \mathbf{s}_T, s_{T+1} - z) = O(1)$.

To verify that the integral in equation (II.37), the third term on the right hand side of (II.37), define a holomorphic function in $\mathbb{C}^T \times \{s_{T+1} > \rho'\}$ it is enough to apply the estimates (II.35), (II.36) and the Stirling complex formula.

The first two terms on the right hand side of (II.37) have a meromorphic continuation to \mathbb{C}^{T+1} with singularities included in the reunion of hyperplanes

$$\left\{ \mathbf{s} \in \mathbb{C}^{T+1} \mid s_{T+1} \in \frac{1}{\deg(R)} (1 + q_n - \mathbb{N}_0) \setminus (-\mathbb{N}_0) \right\}$$

Let $f \in \mathcal{S}'(q, \deg(R))$ the function $H_f : s \mapsto \frac{\Gamma(s - f)}{\Gamma(s)}$ has a meromorphic continuation to \mathbb{C} with poles included in $f - \mathbb{N}_0 \subset \frac{1}{\deg(R)} (1 + q_n - \mathbb{N}_0) \setminus (-\mathbb{N}_0)$, moreover $H_f(-N) = 0$. \square

The previous theorem proves that $Z(\mathbf{X}^{\mathbf{q}}; \mathbf{P}; \boldsymbol{\mu}'; \mathbf{s})$ has a meromorphic continuation to \mathbb{C}^{T+1} with singularities outside of $-\mathbb{N}_0^{T+1}$, in consequence it is possible to determine its values at $\mathbf{s} = (-N_1, \dots, -N_{T+1})$.

End of proof of Theorem 3

Let $\mathbf{N} \in \mathbb{N}_0^{T+1}$ and $-\rho' > 1 + 2c + |\mathbf{N}|_\infty$.

For $\mathbf{s} \in \left\{ \operatorname{Re}(s_j) > -N_j - \frac{1}{2 \deg(R)} \forall j \in \llbracket 1, T+1 \rrbracket \right\}$ the first term of (II.37) is equal to

$$\begin{aligned} & \sum_{k=0}^{\lfloor -\rho' \rfloor} \frac{(-1)^k \Gamma(s_{T+1} + k)}{k! \Gamma(s_{T+1})} Z(X^{q_n}; R; -k) Z(\mathbf{X}'^{\mathbf{q}'}; \mathbf{P}', Q; \boldsymbol{\mu}'; \mathbf{s}', s_{T+1} + k) \\ & = \sum_{k=0}^{\lfloor -\rho' \rfloor} \frac{(-1)^k}{k!} Z(X^{q_n}; R; -k) Z(\mathbf{X}'^{\mathbf{q}'}; \mathbf{P}', Q; \boldsymbol{\mu}'; \mathbf{s}', s_{T+1} + k) \prod_{j=1}^k (s_{T+1} + j - 1) \end{aligned}$$

Then the value of equation (II.37) at $\mathbf{s} = -\mathbf{N}$ is equal to

$$\begin{aligned}
&= \sum_{k=0}^{N_{T+1}} \frac{N_{T+1}!}{k!(N_{T+1} - k)!} Z(X^{q_{n+1}}; R; -k) Z(\mathbf{X}'^q; \mathbf{P}', Q; \boldsymbol{\mu}'; -\mathbf{N}_{\mathbf{T}}, -N_{T+1} + k) \\
&= \sum_{k=0}^{N_{T+1}} \binom{N_{T+1}}{k} Z(X^{q_n}; R; -k) Z(\mathbf{X}'^q; \mathbf{P}', Q; \boldsymbol{\mu}'; -\mathbf{N}_{\mathbf{T}}, -N_{T+1} + k)
\end{aligned}$$

Applying the Crisenoy's Theorem to the above terms, implies that they can be expressed in terms of $\boldsymbol{\mu}'$, Bernoulli numbers, \mathbf{N} and coefficients of polynomials Q, R, P_i 's. This ends the proof of Theorem 3. \square

Chapter III

Means values of multivariable arithmetic functions

Notations

1. Let $m \in \mathbb{N}$ and $p \in \mathcal{P}$, $\nu_p(m)$ is the p -adic valuation of m .
2. For $p \in \mathcal{P}$ and $\mathbf{w} = (w_1, \dots, w_n) \in \mathbb{N}_0^n$ we write $p^{\mathbf{w}} = (p^{w_1}, \dots, p^{w_n})$.
3. Let $\mathbf{x} = (x_1, \dots, x_n), \mathbf{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$, we denote
 - $x_i < y_i$ for all $i \in \llbracket 1, n \rrbracket$ by $\mathbf{x} < \mathbf{y}$;
 - $x_i \leq y_i$ for all $i \in \llbracket 1, n \rrbracket$ by $\mathbf{x} \leq \mathbf{y}$.
4. For $\mathbf{x} \in [1, +\infty[^n, \boldsymbol{\gamma} \in \mathbb{R}^n$ and $f : \mathbb{N}^n \rightarrow \mathbb{C}$, we set
 - $\Psi(f; \mathbf{x}) = \sum_{\substack{\mathbf{m}=(m_1, \dots, m_n) \in \mathbb{N}^n \\ \mathbf{m} \leq \mathbf{x}}} f(m_1, \dots, m_n)$
 - $f_{\boldsymbol{\gamma}} : \mathbb{N}^n \rightarrow \mathbb{C}$ is defined by $f_{\boldsymbol{\gamma}}(m_1, \dots, m_n) = f(m_1, \dots, m_n) m_1^{\gamma_1} \cdots m_n^{\gamma_n}$
5. Let $h : \mathbb{C} \rightarrow \mathbb{C}$, we denote $\int_{s \in (c)} h(s) ds$ the integral defined by $i \int_{-\infty}^{+\infty} h(c + it) dt$
6. Let $P(\mathbf{X}) = \sum_{j=0}^p a_j \mathbf{X}^{\alpha_j} \in \mathbb{C}[\mathbf{X}]$ such that $\text{Supp}(P) \subset \mathbb{R}_+^n$ and $G \subset \text{Supp}(P)$, we set $P_G(\mathbf{X}) = \sum_{\alpha_j \in G} a_j \mathbf{X}^{\alpha_j}$
7. $\mu : \mathbb{N} \rightarrow \mathbb{R}$ is the Möbius function.
8. Let $f : \mathbb{N}^{n_1} \rightarrow \mathbb{C}$ and $g : \mathbb{N}^{n_2} \rightarrow \mathbb{C}$, we define $f \otimes g : \mathbb{N}^{n_1+n_2} \rightarrow \mathbb{C}$ by $f \otimes g(m_1, \dots, m_{n_1+n_2}) = f(m_1, \dots, m_{n_1})g(m_{n_1+1}, \dots, m_{n_1+n_2})$.
9. Let $\mathbf{g} : \{\mathbf{k}_1, \dots, \mathbf{k}_I\} \subset \mathbb{N}_0^n \setminus \{\mathbf{0}\} \rightarrow \mathbb{N}$, we set $g_i = \mathbf{g}(\mathbf{k}_i)$ for each $i \in \llbracket 1, I \rrbracket$. We define the multiplicative function, see Definition 5, $\tau_{\mathbf{g}} : \mathbb{N}^n \rightarrow \mathbb{C}$ by

$$\tau_{\mathbf{g}}(p^{\mathbf{w}}) = \sum_{\substack{\boldsymbol{\ell}=(\ell_1, \dots, \ell_I) \in \mathbb{N}_0^I \\ \sum_{i=1}^I \ell_i \mathbf{k}_i = \mathbf{w}}} \prod_{i=1}^I \binom{g_i + \ell_i - 1}{\ell_i} \quad (\text{III.1})$$

for all $p \in \mathcal{P}$ and $\mathbf{w} \in \mathbb{N}_0^n$.

1 Preliminaries

1.1 Some standard constructions

Let $p, n \in \mathbb{N}$ and $I = \{\alpha_0, \dots, \alpha_p\} \subset \mathbb{R}^n$

1. The convex hull of I is

$$\text{conv}(I) := \left\{ \sum_{i=0}^p \lambda_i \alpha_i \mid \lambda_i \geq 0 \forall i \in \llbracket 0, p \rrbracket \quad \text{and} \quad |\boldsymbol{\lambda}| = \lambda_0 + \dots + \lambda_p = 1 \right\}.$$

2. The relative interior of the convex hull of I is

$$\text{conv}^*(I) := \left\{ \sum_{i=0}^p \lambda_i \alpha_i \mid \lambda_i > 0 \forall i \in \llbracket 0, p \rrbracket \quad \text{and} \quad |\boldsymbol{\lambda}| = 1 \right\}.$$

3. The convex cone of I is $\text{con}(I) := \left\{ \sum_{i=0}^p \lambda_i \alpha_i \mid \lambda_i \geq 0 \forall i \in \llbracket 0, p \rrbracket \right\}$.

4. The relative interior of the convex cone of I is

$$\text{con}^*(I) := \left\{ \sum_{i=0}^p \lambda_i \alpha_i \mid \lambda_i > 0 \forall i \in \llbracket 0, p \rrbracket \right\}.$$

5. The Newton polyhedron of I is defined by $\mathcal{E}(I) := \text{conv}(I + \mathbb{R}_+^n)$.

6. The Newton polyhedron at infinity of I is defined by $\mathcal{E}^\infty(I) := \text{conv}(I - \mathbb{R}_+^n)$.

Let $I \subset \mathbb{R}_+^n \setminus \{\mathbf{0}\}$ be a finite set and $\mathbf{a} \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}$ we set

1. $\Sigma(I) := \{\mathbf{x} \in \mathbb{R}_+^n \mid \boldsymbol{\beta} \cdot \mathbf{x} \geq 1 \text{ for all } \boldsymbol{\beta} \in I\}$,
2. $m(\mathbf{a}) := \min\{\mathbf{a} \cdot \mathbf{x} \mid \mathbf{x} \in \Sigma(I)\}$
3. The faces of $\Sigma(I)$ are the sets $\mathcal{F}_{\Sigma(I)}(\mathbf{a}) := \{\mathbf{x} \in \Sigma(I) \mid \mathbf{a} \cdot \mathbf{x} = m(\mathbf{a})\}$. A facet of $\Sigma(I)$ is a face of maximal dimension.
4. The index of $\Sigma(I)$ is $\iota(\Sigma(I)) := \min\{|\boldsymbol{\alpha}|_1 \mid \boldsymbol{\alpha} \in \Sigma(I)\}$. In particular

$$\mathcal{F}_{\Sigma(I)}(\mathbf{1}) := \{\mathbf{x} \in \Sigma(I) \mid |\mathbf{x}|_1 = \iota(\Sigma(I))\}$$

5. Let F be a face of $\Sigma(I)$, $\text{Pol}(F) := \{\mathbf{a} \in \mathbb{R}_+^n \setminus \{\mathbf{0}\} \mid F = \mathcal{F}_{\Sigma(I)}(\mathbf{a})\}$ is the polar cone of F and its elements are called polar vectors of F . A polar vector $\mathbf{a} \in \text{Pol}(F)$ is called normalized polar vector of F if $m(\mathbf{a}) = 1$ and we set $\text{Pol}_0(F)$ the set of normalized polar vectors of F .

1.2 Newton polyhedron of multiplicative functions

Let $f : \mathbb{N}^n \rightarrow \mathbb{C}$ be a multiplicative function. In [9] section 3.1.1 Essouabri defined the following objects:

1. $S(f) := \{\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{N}_0^n \setminus \{\mathbf{0}\} \mid \text{there exists } p \in \mathcal{P} \text{ such that } f(p^{k_1}, \dots, p^{k_n}) \neq 0\}$,
2. $S_\infty(f) := \{\mathbf{k} \in \mathbb{N}_0^n \setminus \{\mathbf{0}\} \mid \text{there exists an infinity of } p \in \mathcal{P} \text{ such that } f(p^{k_1}, \dots, p^{k_n}) \neq 0\}$,
3. $\mathcal{E}(f) := \mathcal{E}(S_\infty(f)) = \text{conv}(S_\infty(f) + \mathbb{R}_+^n)$,
4. $I(f) := \{\mathbf{k} \in S_\infty(f) \mid \mathbf{k} \text{ is in a compact face of } \mathcal{E}(f)\}$,
5. $\Sigma(f) := \{\mathbf{x} \in \mathbb{R}_+^n \mid \forall \mathbf{k} \in I(f) \text{ we have } \mathbf{k} \cdot \mathbf{x} \geq 1\}$ the Newton polyhedron of f ,

6. $\iota(f) := \min\{|\mathbf{x}| \mid \mathbf{x} \in \Sigma(f)\}$ the index of $\Sigma(f)$,
7. $\mathcal{F}_{\Sigma(f)}(\mathbf{1}) := \{\mathbf{x} \in \Sigma(f) \mid |\mathbf{x}| = \iota(f)\}$ the face of $\Sigma(f)$ of polar vector $\mathbf{1}$,
8. $I_0(f) := \{\mathbf{k} \in I(f) \mid \mathcal{F}_{\Sigma(f)}(\mathbf{1}) \subset \{\mathbf{x} \in \Sigma(f) \mid \mathbf{k} \cdot \mathbf{x} = 1\}\}$.

Remark 6. a) By construction $I(f)$ is a finite set.

b) The sets $\mathcal{E}(f)$ and $I(f)$ are connected by identity $\mathcal{E}(f) = \text{conv}(I(f) + \mathbb{R}_+^n)$.

In particular for $f : \mathbb{N} \rightarrow \mathbb{C}$ be a multiplicative function of a single variable and $\mathbf{d} \in \mathbb{N}^n$. We set $f_{\mathbf{d}} : \mathbb{N}^n \rightarrow \mathbb{C}$ the function defined by $f_{\mathbf{d}}(m_1, \dots, m_n) = f(m_1^{d_1} \cdots m_n^{d_n})$, so $f_{\mathbf{d}}$ is also multiplicative and

1. $S(f_{\mathbf{d}}) = \{\mathbf{b} \in \mathbb{N}_0^n \setminus \{\mathbf{0}\} \mid \mathbf{b} \cdot \mathbf{d} \in S(f)\}$,
2. $S_{\infty}(f_{\mathbf{d}}) = \{\mathbf{b} \in \mathbb{N}_0^n \setminus \{\mathbf{0}\} \mid \mathbf{b} \cdot \mathbf{d} \in S_{\infty}(f)\}$,
3. $\mathcal{E}(f_{\mathbf{d}}) = \text{conv}\{S_{\infty}(f_{\mathbf{d}}) + \mathbb{R}^n\}$,
4. $I(f_{\mathbf{d}}) = \{\mathbf{b} \in S_{\infty}(f_{\mathbf{d}}) \mid \mathbf{b}$ is in a compact face of $\mathcal{E}(f_{\mathbf{d}})\}$,
5. $\Sigma(f_{\mathbf{d}}) = \{\mathbf{x} \in \mathbb{R}_+^n \mid \mathbf{x} \cdot \mathbf{b} \geq 1 \text{ for all } \mathbf{b} \in I(f_{\mathbf{d}})\}$,
6. $\iota(f_{\mathbf{d}}) = \min\{|\mathbf{x}| \mid \mathbf{x} \in \Sigma(f_{\mathbf{d}})\}$,
7. $\mathcal{F}_{\Sigma(f_{\mathbf{d}})}(\mathbf{1}) = \{\mathbf{x} \in \Sigma(f_{\mathbf{d}}) \mid |\mathbf{x}|_1 = \iota(f_{\mathbf{d}})\}$,
8. $I_0(f_{\mathbf{d}}) = \{\mathbf{b} \in I(f_{\mathbf{d}}) \mid \mathcal{F}_{\Sigma(f_{\mathbf{d}})}(\mathbf{1}) \subset \{\mathbf{x} \in \mathbb{R}_+^n \mid \mathbf{b} \cdot \mathbf{x} = 1\}\}$.

1.3 The Sargos volume constant $A(\boldsymbol{\eta}; P)$

Let $\boldsymbol{\eta} \in (\mathbb{R}_+^*)^n$ and $P(\mathbf{X}) = \sum_{j=0}^p a_j \mathbf{X}^{\alpha_j}$ be a generalized polynomial of n variables, dependent in all its variables, with real coefficients and not degenerate with respect to its Newton polyhedron at infinity such that $\text{Supp}(P) = \{\boldsymbol{\alpha}_0, \dots, \boldsymbol{\alpha}_p\} \subset \mathbb{R}_+^n$. Sargos, cf. [27, Chap. 3, §1.3 and §4.1], was interested in asymptotic behavior in neighbor of the first real pole of integrals and series like:

$$Y(\boldsymbol{\eta}; P; s) = \int_{\mathbf{x} \in [1, +\infty[^n} \mathbf{x}^{\boldsymbol{\eta}-1} P(\mathbf{x})^{-s} d\mathbf{x} \quad (\text{III.2})$$

$$Z(\boldsymbol{\eta}; P; s) = \sum_{\mathbf{m} \in \mathbb{N}^n} \mathbf{m}^{\boldsymbol{\eta}-1} P(\mathbf{m})^{-s} \quad (\text{III.3})$$

For this purpose Sargos used the following notations and constructions, see [27]:

1. We set G_0 the intersection of all faces of $\mathcal{E}^{\infty}(\text{Supp}(P))$ crossed by the half-line $\mathbb{R}_+ \boldsymbol{\eta}$. Equivalently the smallest face crossed by the half-line $\mathbb{R}_+ \boldsymbol{\eta}$.
2. $\sigma_0 = \max\{\lambda_F \cdot \boldsymbol{\eta} \mid F \text{ face of } \mathcal{E}^{\infty}(\text{Supp}(P)) \text{ and } \lambda_F \text{ its normalized polar vector}\} \in \mathbb{R}_+^*$. Geometrically it can be visualized as the inverse of the smallest $t \in \mathbb{R}_+$ such that $t \cdot \boldsymbol{\eta}$ intersects $\mathcal{E}^{\infty}(\text{Supp}(P))$.
3. $\theta = \text{codim}(G_0)$ et $\widetilde{G}_0 = \text{Span}\{G_0 - g \mid g \in G_0\}$.

4. By permutation of coordinates, we can suppose that:

$$\mathbb{R}^n = \widetilde{G}_0 \oplus \sum_{k=1}^{\theta} \mathbb{R}\mathbf{e}_k = \widetilde{G}_0^\perp \oplus \sum_{k=\theta+1}^n \mathbb{R}\mathbf{e}_k.$$

5. G_0 is parallel to $\mathbf{e}_{q+1}, \dots, \mathbf{e}_n$ and not parallel to another \mathbf{e}_k .

6. Let $\boldsymbol{\varepsilon}_1, \dots, \boldsymbol{\varepsilon}_\ell$ be the normalized polar vectors of facets $\mathcal{E}^\infty(\text{Supp}(P))$ crossed by $\mathbb{R}_+\boldsymbol{\eta}$.

7. $K = \text{conv}\{\mathbf{0}, \boldsymbol{\varepsilon}_1, \dots, \boldsymbol{\varepsilon}_\ell, \mathbf{e}_{\theta+1}, \dots, \mathbf{e}_n\}$ is a n -dimensional polytope.

Definition 3. *The Sargos constants associated to a pair $(\boldsymbol{\eta}; P)$ are:*

$$A(\boldsymbol{\eta}; P) = n! \text{Vol}(K) \int_{[1, +\infty[^{n-q}} \left(\prod_{l=q+1}^n y_l^{\eta_l-1} \right) \left[\int_{\mathbb{R}_+^{q-\theta}} \left(\prod_{k=\theta+1}^q x_k^{\eta_k-1} \right) P_{G_0}(\mathbf{1}, \mathbf{x}, \mathbf{y})^{-\sigma_0} \right] d\mathbf{x} d\mathbf{y},$$

$$B(\boldsymbol{\eta}; P) = n! \text{Vol}(K) \sum_{\boldsymbol{\nu} \in \mathbb{N}^{n-q}} \left(\prod_{l=q+1}^n \nu_l^{\eta_l-1} \right) \left[\int_{\mathbb{R}_+^{q-\theta}} \left(\prod_{k=\theta+1}^q x_k^{\eta_k-1} \right) P_{G_0}(\mathbf{1}, \mathbf{x}, \boldsymbol{\nu})^{-\sigma_0} d\mathbf{x} \right]$$

where $\mathbf{x} = (x_{\theta+1}, \dots, x_q)$, $\mathbf{y} = (y_{q+1}, \dots, y_n)$. Moreover, if P is a polynomial with real coefficients the constants $A(\boldsymbol{\eta}; P)$ and $B(\boldsymbol{\eta}; P)$ are strictly positives.

For $\boldsymbol{\eta} = \mathbf{1}$ we set $A_0(P) := A(\mathbf{1}; P)$ and $B_0(P) := B(\mathbf{1}; P)$.

Theorem 4 ([27], Th. 4.1, Sargos). *Let $\boldsymbol{\eta} \in (\mathbb{R}_+^*)^n$ and P a generalized polynomial of n variables with real positive coefficients and non degenerated with respect to its Newton polyhedron at infinity. Then we have*

1. $Y(P; \boldsymbol{\eta}; s)$ is absolutely convergent in $\{s \in \mathbb{C} \mid \text{Re}(s) > \sigma_0\}$,
2. there exists $\varepsilon > 0$ such that $s \mapsto Y(P; \boldsymbol{\eta}; s)$ has a meromorphic continuation into the region $\{s \in \mathbb{C} \mid \text{Re}(s) > \sigma_0 - \varepsilon\}$ with a single pole at $s = \sigma_0$ of order $\theta = \text{codim}(G_0)$,
3. $A(\boldsymbol{\eta}; P) > 0$ and

$$Y(P; \boldsymbol{\eta}; s) \sim A(\boldsymbol{\eta}; P)(s - \sigma_0)^{-\theta} \text{ as } s \rightarrow \sigma_0$$

1.4 Essouabri's mixed volume constant $A_0((I; \mathbf{u}); P)$

These constants were introduced by Essouabri, see [14, §2.3.3], to describe the asymptotic behavior near to the first real pole of mixed zeta functions.

The volume constant $A_0(I; \mathbf{g}; \mathbf{a})$

Let $I = \{\mathbf{v}_1, \dots, \mathbf{v}_{|I|}\} \subset \mathbb{R}_+^p \setminus \{\mathbf{0}\}$, $\mathbf{g} : I \rightarrow \mathbb{N}$ and $\mathbf{a} = (a_1, \dots, a_p) \in (\mathbb{R}_+^*)^p$. We set $|\mathbf{g}| = \sum_{\mathbf{v} \in I} \mathbf{g}(\mathbf{v})$, $\mathbf{v}_i = (v_{i,1}, \dots, v_{i,p})$, $\mathbf{w}_j = \underbrace{(v_{1,j}, \dots, v_{1,j})}_{\mathbf{g}(\mathbf{v}_1) \text{ times}}, \dots, \underbrace{(v_{|I|,j}, \dots, v_{|I|,j})}_{\mathbf{g}(\mathbf{v}_{|I|}) \text{ times}} \in \mathbb{R}_+^{|\mathbf{g}|}$ and

$P_{(I; \mathbf{g}; \mathbf{a})}(\mathbf{Y}) = \sum_{j=1}^p a_j \mathbf{Y}^{\mathbf{w}_j}$. We define $A_0(I; \mathbf{g}; \mathbf{a}) := A_0(P_{(I; \mathbf{g}; \mathbf{a})})$ as the Sargos volume constant associated with the generalized polynomial $P_{(I; \mathbf{g}; \mathbf{a})}$.

The mixed volume constant $A_0((I; \mathbf{u}); P)$

Definition 4. Let $P(\mathbf{X}) = \sum_{j=1}^p a_j \mathbf{X}^{\alpha_j} \in \mathbb{R}_+[X_1, \dots, X_n]$ be a generalized polynomial which depends on all its variables, $I \subset \mathbb{R}_+^n \setminus \{\mathbf{0}\}$ a finite set and $\mathbf{u} : I \rightarrow \mathbb{N}$. We set

1. $\mathbf{a} = (a_1, \dots, a_p)$,
2. $\Gamma = (\alpha_1; \dots; \alpha_p) \in \mathcal{M}_{p,n}(\mathbb{R})$ the matrix with rows $\alpha_1, \dots, \alpha_p$,
3. $\tilde{I} = \{\Gamma \cdot \mathbf{v} \mid \mathbf{v} = (v_1, \dots, v_n) \in I\} \subset \mathbb{R}_+^p \setminus \{\mathbf{0}\}$,
4. $\tilde{\mathbf{u}} : \tilde{I} \rightarrow \mathbb{N}$ defined by $\tilde{\mathbf{u}}(\mathbf{w}) = \sum_{\substack{\mathbf{v} \in I \\ \Gamma \cdot \mathbf{v} = \mathbf{w}}} \mathbf{u}(\mathbf{v})$ for $\mathbf{w} = (w_1, \dots, w_p)$,
5. $P_{(\tilde{I}; \tilde{\mathbf{u}}; \mathbf{a})} : \mathbb{R}^{|\tilde{\mathbf{u}}|} = \mathbb{R}^{|\mathbf{u}|} \rightarrow \mathbb{R}$ for $\mathbf{y} = (y_{\mathbf{w},j})_{\mathbf{w} \in \tilde{I}, j \in [1, \tilde{\mathbf{u}}(\mathbf{w})]} \in \mathbb{R}^{|\tilde{\mathbf{u}}|}$ is defined by

$$P_{(\tilde{I}; \tilde{\mathbf{u}}; \mathbf{a})}(\mathbf{y}) = \sum_{k=1}^p a_k \prod_{\mathbf{w} \in \tilde{I}} \prod_{j=1}^{\tilde{\mathbf{u}}(\mathbf{w})} y_{\mathbf{w},j}^{w_k}$$

Also see construction (III.4) below. The mixed volume constant is defined by

$$A_0((I; \mathbf{u}); P) := A_0(\tilde{I}; \tilde{\mathbf{u}}; \mathbf{a}) = A_0(P_{(\tilde{I}; \tilde{\mathbf{u}}; \mathbf{a})})$$

We set $\tilde{I} = \{\mathbf{w}_1, \dots, \mathbf{w}_r\}$, $r = |\tilde{I}|$, $q_{-1} = 0$, $q_i = q_{i-1} + \tilde{\mathbf{u}}(\mathbf{w}_i)$ for each $i \in [1, r]$ and $q = q_r = \sum_{i=1}^r \tilde{\mathbf{u}}(\mathbf{w}_i)$ where $\mathbf{w}_i = (w_{i,1}, \dots, w_{i,p})$. Let $\mathbf{y} = (y_1, \dots, y_q) \in \mathbb{R}^q$ we have

$$\begin{aligned} P_{\tilde{I}; \tilde{\mathbf{u}}; \mathbf{a}}(\mathbf{y}) &= \sum_{k=1}^p a_k \prod_{i=1}^r \prod_{j=1}^{\tilde{\mathbf{u}}(\mathbf{w}_i)} y_{q_{i-1}+j}^{w_{i,k}} = \sum_{k=1}^p a_k \prod_{i=1}^r \left(\prod_{j=1}^{\tilde{\mathbf{u}}(\mathbf{w}_i)} y_{q_{i-1}+j} \right)^{w_{i,k}} \\ &= Q \left(\prod_{j=1}^{\tilde{\mathbf{u}}(\mathbf{w}_1)} y_j, \dots, \prod_{j=1}^{\tilde{\mathbf{u}}(\mathbf{w}_r)} y_{q_{r-1}+j} \right) \end{aligned} \quad (\text{III.4})$$

where $Q(z_1, \dots, z_r) = \sum_{k=1}^p a_k \prod_{i=1}^r z_i^{w_{i,k}}$

Example 4. Let $P(\mathbf{X}) = a_1 X_1^3 + a_2 X_1^2 X_2 + a_3 X_1 X_2^2 + a_4 X_2^3 \in \mathbb{R}_+[X_1, X_2]$, $I = \{(2, 0); (1, 1); (0, 2)\}$, $\mathbf{u}(2, 0) = 3$, $\mathbf{u}(0, 2) = 2$ and $\mathbf{u}(1, 1) = 1$. Then we have

$$\Gamma = \begin{pmatrix} 3 & 0 \\ 2 & 1 \\ 1 & 2 \\ 0 & 3 \end{pmatrix}, \quad \tilde{I} = \{(6, 4, 2, 0); (3, 3, 3, 3); (0, 2, 4, 6)\},$$

$\tilde{\mathbf{u}}(6, 4, 2, 0) = 3$, $\tilde{\mathbf{u}}(3, 3, 3, 3) = 2$, $\tilde{\mathbf{u}}(0, 2, 4, 6) = 1$ and

$$P_{(\tilde{I}; \tilde{\mathbf{u}}; \mathbf{a})}(y_1, \dots, y_6) = a_1 y_1^6 y_2^6 y_3^6 y_4^3 y_5^3 + a_2 y_1^4 y_2^4 y_3^4 y_4^3 y_5^3 y_6^2 + a_3 y_1^2 y_2^2 y_3^2 y_4^3 y_5^3 y_6^4 + a_4 y_4^3 y_5^3 y_6^6$$

2 Introduction

A *multivariable arithmetic function* $f : \mathbb{N}^n \rightarrow \mathbb{C}$ is a function defined on a set of vectors with positive integer coordinates and with values in the set of complex numbers \mathbb{C} . We denote by \mathcal{A}^n the set of arithmetic functions in n variables and by \mathcal{A}_1^n the set of arithmetic function $f \in \mathcal{A}_1^n$ such that $|f| \leq 1$. It's easy to see that $(\mathcal{A}^n, +, \times)$ is a ring and that $(\mathcal{A}_1^n, \times)$ is a semigroup.

For $f \in \mathcal{A}^n$, we define the following three Dirichlet (formal) series:

$$\mathcal{M}(f; \mathbf{s}) = \sum_{\mathbf{m} \in \mathbb{N}^n} \frac{f(\mathbf{m})}{\mathbf{m}^{\mathbf{s}}} = \sum_{\mathbf{m}=(m_1, \dots, m_n) \in \mathbb{N}^n} \frac{f(m_1, \dots, m_n)}{m_1^{s_1} \cdots m_n^{s_n}}, \quad (\text{III.5})$$

$$\mathcal{M}(|f|; \mathbf{s}) = \sum_{\mathbf{m} \in \mathbb{N}^n} \frac{|f(\mathbf{m})|}{\mathbf{m}^{\mathbf{s}}} = \sum_{\mathbf{m}=(m_1, \dots, m_n) \in \mathbb{N}^n} \frac{|f(m_1, \dots, m_n)|}{m_1^{s_1} \cdots m_n^{s_n}} \quad (\text{III.6})$$

and for an elliptic polynomial $P \in \mathfrak{E}_n$ of degree $d > 0$ (see Chapter 2 Definition 1),

$$Z(f; P; \mathbf{s}) = \sum_{\mathbf{m} \in \mathbb{N}^n} \frac{f(\mathbf{m})}{P(\mathbf{m})^{s/d}} = \sum_{\mathbf{m}=(m_1, \dots, m_n) \in \mathbb{N}^n} \frac{f(m_1, \dots, m_n)}{P(m_1, \dots, m_n)^{s/d}} \quad (\text{III.7})$$

Remark 7. 1. Let $f \in \mathcal{A}_1^n$ the two series $\mathcal{M}(f; \mathbf{s})$ and $\mathcal{M}(|f|; \mathbf{s})$, see (III.5) and (III.6), are absolutely and uniformly convergent on compact subsets of $\{\text{Re}(\mathbf{s}) > \mathbf{1}\}$. Therefore they define a holomorphic on $\{\text{Re}(\mathbf{s}) > \mathbf{1}\}$.

2. There exists $c > 0$ such that $Z(f; P; \mathbf{s})$, see (III.7), is absolutely and uniformly convergent on compact subsets of $\{\text{Re}(s) > c\}$.

For $\mathbf{g} : \{\mathbf{k}_1, \dots, \mathbf{k}_I\} \subset \mathbb{N}^n \setminus \{\mathbf{0}\} \rightarrow \mathbb{N}$, we set

$$Z_{\mathbf{g}}(\mathbf{s}) := \prod_{i=1}^I \zeta(1 + \mathbf{k}_i \cdot \mathbf{s})^{g_i} \quad \text{and} \quad \Theta_{\mathbf{g}}(\mathbf{s}) := \prod_{i=1}^I (\mathbf{k}_i \cdot \mathbf{s})^{g_i} \zeta(1 + \mathbf{k}_i \cdot \mathbf{s})^{g_i} \quad (\text{III.8})$$

where $g_i = \mathbf{g}(\mathbf{k}_i)$ for each $i \in \llbracket 1, I \rrbracket$.

Remark 8. $Z_{\mathbf{g}}$ defines a holomorphic function on the half-space $\{\text{Re}(\mathbf{s}) > \mathbf{0}\}$ and has meromorphic continuation to the whole complex space \mathbb{C}^n . The function $\Theta_{\mathbf{g}}$ is holomorphic in \mathbb{C}^n . These two functions are connected by the following relation:

$$Z_{\mathbf{g}}(\mathbf{s}) = \frac{\Theta_{\mathbf{g}}(\mathbf{s})}{\prod_{i=1}^I (\mathbf{k}_i \cdot \mathbf{s})^{g_i}} \quad (\text{III.9})$$

Definition 5. An arithmetic function $f : \mathbb{N}^n \rightarrow \mathbb{C}$ is said to be multiplicative if for any $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_n)$ in \mathbb{N}^n such that $\text{gcd}(a_1 \cdots a_n; b_1 \cdots b_n) = 1$, we have

$$f(a_1 b_1, \dots, a_n b_n) = f(\mathbf{a}) f(\mathbf{b}).$$

We denote by \mathcal{M}^n the set of multiplicative functions in n variables and by $\mathcal{M}_1^n = \mathcal{M}^n \cap \mathcal{A}_1^n$. For $f \in \mathcal{M}^n$ we have the following factorization

$$f(\mathbf{m}) = \prod_{p \in \mathcal{P}} f(p^{\nu_p(m_1)}, \dots, p^{\nu_p(m_n)}) \quad \text{for all } \mathbf{m} = (m_1, \dots, m_n) \in \mathbb{N}^n,$$

where ν_p is the p -adic valuation.

Let $\mathbf{d} = (d_1, \dots, d_n) \in \mathbb{N}^n$ and $f : \mathbb{N} \rightarrow \mathbb{C}$ be a one variable multiplicative function, for example $f = \sigma_k$ where $k \in \mathbb{N}_0$ and $\sigma_k(m) = \sum_{d|m} d^k$ or $f = \varphi$ the Euler's indicator function. We associate to f and \mathbf{d} the multivariable arithmetic function $f_{\mathbf{d}} : \mathbb{N}^n \rightarrow \mathbb{C}$ defined by

$$f_{\mathbf{d}}(m_1, \dots, m_n) = f\left(\prod_{j=1}^n m_j^{d_j}\right). \quad (\text{III.10})$$

It is clear that $f_{\mathbf{d}}$ is also a multivariable multiplicative function.

As in the one variable case, we can define on \mathcal{A}^n the *Dirichlet convolution*. For $f, g \in \mathcal{A}^n$, the convolution of f and g is the function $f * g : \mathbb{N}^n \rightarrow \mathbb{C}$ defined for $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{N}^n$ by

$$(f * g)(\mathbf{m}) = \sum_{\substack{\mathbf{d}_i = (d_{i,1}, \dots, d_{i,n}) \in \mathbb{N}^n, i \in \{1,2\} \\ d_{1,k} d_{2,k} = m_k, k \in \{1, \dots, n\}}} f(\mathbf{d}_1) g(\mathbf{d}_2). \quad (\text{III.11})$$

Moreover, we have the following formal identities

$$\mathcal{M}(f * g; \mathbf{s}) = \mathcal{M}(f; \mathbf{s}) \mathcal{M}(g; \mathbf{s}) \quad \text{and} \quad \mathcal{M}(f; \mathbf{s}) = \prod_{p \in \mathcal{P}} \sum_{\mathbf{k} \in \mathbb{N}_0^n} \frac{f(p^{k_1}, \dots, p^{k_n})}{p^{\mathbf{k} \cdot \mathbf{s}}} \quad (\text{III.12})$$

More precisely, the product which defines $\mathcal{M}(f; \mathbf{s})$ is absolutely convergent if and only if

$$\sum_{p \in \mathcal{P}} \sum_{\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{N}_0^n \setminus \{\mathbf{0}\}} \frac{|f(p^{k_1}, \dots, p^{k_n})|}{p^{\mathbf{k} \cdot \mathbf{s}}} < +\infty \quad (\text{III.13})$$

and the identities (III.12) hold if $\mathcal{M}(f; \mathbf{s})$ and $\mathcal{M}(g; \mathbf{s})$ are absolutely convergent. We define the p -factor of $\mathcal{M}(f; \mathbf{s})$ by

$$\mathcal{M}_p(f; \mathbf{s}) = \sum_{\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{N}_0^n} \frac{f(p^{k_1}, \dots, p^{k_n})}{p^{\mathbf{k} \cdot \mathbf{s}}} \quad (\text{III.14})$$

A typical problem that one meets in many different arithmetic or geometric contexts is to describe with reasonable precision the asymptotic behavior of a density function of the form

$$\mathbf{x} = (x_1, \dots, x_n) \mapsto \Psi(f; \mathbf{x}) := \sum_{\substack{\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{N}^n \\ 1 \leq m_i \leq x_i, \forall i \in \{1, \dots, n\}}} f(m_1, \dots, m_n) \quad (\text{III.15})$$

or of the form

$$x \mapsto \Psi(f; P; x) := \sum_{\substack{\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{N}^n \\ P(m_1, \dots, m_n)^{1/d} \leq x}} f(m_1, \dots, m_n) \quad (\text{III.16})$$

where the function $f : \mathbb{N}^n \rightarrow \mathbb{C}$ is an arithmetic function typically, though not always, multiplicative and where $P \in \mathbb{R}[X_1, \dots, X_n]$ is a proper polynomial of degree d .

For $n = 1$ there are several results as Delange's theorem [4]. In several variables case ($n \geq 2$) there are several works that inspired this part of our thesis : Delange [3]; Lichtin [22, 21]; La Bretèche [7]; Essouabri [9, 12, 14]; Tóth and Zahai [30, 29].

Definition 6. Let $f \in \mathcal{A}_1^1$, we will said that f has a mean value $M(f) \in \mathbb{C}$ if the limit

$$\lim_{m \rightarrow +\infty} \frac{1}{m} \sum_{k \leq m} f(k) \text{ exists and is equal to } M(f).$$

In 1961, Delange proved the following result

Theorem 5 ([4], Theo. 2). *If $f \in \mathcal{M}_1^1$ and the series $\sum_{p \in \mathcal{P}} \frac{1-f(p)}{p}$ is convergent, then $M(f)$ exists and is given by*

$$M(f) = \prod_{p \in \mathcal{P}} \left(1 - \frac{1}{p}\right) \left[1 + \sum_{j=1}^{+\infty} \frac{f(p^j)}{p^j}\right]$$

Moreover, $M(f) = 0$ if and only if $f(2^r) = -1$ for all $r \geq 1$.

Delange [4, Theo. 1] proved also the following converse of above theorem.

Theorem 6 ([4], Theo. 1). *Assume that $f \in \mathcal{M}_1^1$, $M(f)$ exists and $M(f) \neq 0$. Then,*

1. *the series $\sum_{p \in \mathcal{P}} \frac{1-f(p)}{p}$ is convergent, and*
2. *there exists $r \in \mathbb{N}$ such that $f(2^r) \neq -1$.*

Delange also proved in [3] the following multivariable extension of his theorem:

Theorem 7 ([3]). *Let $f : \mathbb{N}^n \rightarrow \mathbb{C}$ be a multiplicative function such that $|f| \leq 1$. Assume that for any vector $\mathbf{e}_i = (e_{i,1}, \dots, e_{i,n})$ of the canonical basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ of \mathbb{R}^n , the series*

$$\sum_{p \in \mathcal{P}} \frac{1-f(p^{e_{i,1}}, \dots, p^{e_{i,n}})}{p} \text{ converges.} \quad (\text{III.17})$$

Then, as $(x_1, \dots, x_n) \rightarrow (+\infty, \dots, +\infty)$,

$$\sum_{\substack{\mathbf{m}=(m_1, \dots, m_n) \in \mathbb{N}^n \\ m_i \leq x_i, \forall i \in [1, n]}} f(\mathbf{m}) = M(f) x_1 \dots x_n + o(x_1 \dots x_n),$$

where $M(f) := \prod_{p \in \mathcal{P}} \left(1 - \frac{1}{p}\right)^n \left(\sum_{\nu=(\nu_1, \dots, \nu_n) \in \mathbb{N}_0^n} \frac{f(p^{\nu_1}, \dots, p^{\nu_n})}{p^{\nu_1 + \dots + \nu_n}} \right)$.

In 2001, La Bretèche [7] established the following general result:

Theorem 8 ([7], Theo. 1). *Let $f \in \mathcal{A}^n$ be a non-negative multivariable arithmetic function. Assume that there exists $\mathbf{c} \in \mathbb{R}_+^n$ such that the following three assumptions hold:*

1. *The series $\mathcal{M}(f; \mathbf{s})$ is absolutely convergent if $\text{Re}(\mathbf{s}) \in \mathbf{c} + (\mathbb{R}_+^*)^n$,*
2. *there exists two sets $W = \{\mathbf{k}_1, \dots, \mathbf{k}_I\}$, $W' = \{\mathbf{k}'_1, \dots, \mathbf{k}'_r\} \subset \mathbb{R}_+^n \setminus \{\mathbf{0}\}$ such that the function $\mathbf{s} \mapsto \Theta(f; \mathbf{s}) : \mathbb{C}^n \rightarrow \mathbb{C}$ defined by*

$$\Theta(f; \mathbf{s}) = \mathcal{M}(f; \mathbf{s} + \mathbf{c}) \prod_{i=1}^I \mathbf{k}_i \cdot \mathbf{s}$$

has holomorphic continuation to the domain

$$\mathcal{D}(\delta_1, \delta_3) = \{\mathbf{s} \in \mathbb{C}^n \mid \forall i \in [1, I], \mathbf{k}_i \cdot \boldsymbol{\sigma} > -\delta_1 \text{ and } \forall j \in [1, r], \mathbf{k}'_j \cdot \boldsymbol{\sigma} > -\delta_3\}$$

where δ_1, δ_3 are positive real numbers.

3. there exists $\delta_2 > 0$ such that for all $\varepsilon > 0$, $\varepsilon' > 0$ the estimate

$$|\Theta(f; \mathbf{s})| \ll (1 + |\boldsymbol{\tau}|_1^\varepsilon) \prod_{i=1}^I (1 + |\mathbf{k}_i \cdot \boldsymbol{\tau}|)^{1 - \delta_2 \min\{0, \mathbf{k}_i \cdot \boldsymbol{\sigma}\}}$$

holds uniformly in the domain $\mathcal{D}(\delta_1 - \varepsilon', \delta_3 - \varepsilon')$.

Let $J(\mathbf{c}) = \{j \in \llbracket 1, n \rrbracket \mid c_j = 0\}$. Set $r = \#J(\mathbf{c})$ and $\mathbf{k}_{I+1}, \dots, \mathbf{k}_{I+r}$ the r vectors \mathbf{e}_j where $j \in J(\mathbf{c})$. Then, for any $\boldsymbol{\beta} \in (\mathbb{R}_+^*)^n$ there exists a polynomial $Q_\beta \in \mathbb{R}[X]$ of degree at most $I + r - \text{Rank}(\{\mathbf{k}_i\}_{i=1}^{I+r})$ and $\theta = \theta(W, W', \delta_1, \delta_2, \delta_3, \mathbf{c}, \boldsymbol{\beta}) > 0$ such that

$$\Psi(f; t^{\beta_1}, \dots, t^{\beta_n}) = t^{\boldsymbol{\beta} \cdot \mathbf{c}} (Q_\beta(\log t) + O(t^{-\theta})) \quad \text{as } t \rightarrow +\infty.$$

Moreover, La Bretèche determine in the following result, under fairly general geometric assumptions, the exact degree and the main term of the polynomial Q_β :

Theorem 9 ([7], Theo. 2). *Let $f \in \mathcal{A}^n$ satisfying all assumption of Theorem 8. Let $J(\mathbf{c}) = \{j \in \llbracket 1, n \rrbracket \mid c_j = 0\}$. Set $r = \#J(\mathbf{c})$ and $\mathbf{k}_{I+1}, \dots, \mathbf{k}_{I+r}$ the r vecteurs \mathbf{e}_j where $j \in J(\mathbf{c})$.*

1. *If $\boldsymbol{\beta} \notin \text{Span}(\{\mathbf{k}_i\}_{i=1}^{I+r})$, then $Q_\beta = 0$.*

2. *If $\mathbf{s} \mapsto \mathcal{M}(f; \mathbf{s})$ satisfies the following two additional assumptions*

(a) *the exists a function G such that $\Theta(f; \mathbf{s}) = G(\mathbf{k}_1 \cdot \mathbf{s}, \dots, \mathbf{k}_{I+r} \cdot \mathbf{s})$ and*

(b) *$\boldsymbol{\beta} \in \text{Span}(\{\mathbf{k}_i\}_{i=1}^{I+r})$ and there is no proper subset \widetilde{W} of $\{\mathbf{k}_i\}_{i=1}^{I+r}$ such that $\boldsymbol{\beta} \in \text{Span}(\widetilde{W})$ and*

$$\#\widetilde{W} - \text{Rank}(\widetilde{W}) = \#(\{\mathbf{k}_i\}_{i=1}^{I+r}) - \text{Rank}(\{\mathbf{k}_i\}_{i=1}^{I+r})$$

Then, the polynomial Q_β verifies

$$Q_\beta(\log t) = C_0 t^{-\boldsymbol{\beta} \cdot \mathbf{c}} I(t^\beta) + O((\log t)^{\rho-1}) \quad (t \geq 3)$$

where $C_0 = \Theta(f; \mathbf{0})$, $\rho = I + r - \text{Rank}(\{\mathbf{k}_i\}_{i=1}^{I+r})$

$$I(t^\beta) = \int_{A(t^\beta)} \frac{dy_1 \dots dy_n}{\prod_{i=1}^I y_i^{1 - \mathbf{k}_i \cdot \mathbf{c}}},$$

and

$$A(t^\beta) = \left\{ \mathbf{y} = (y_1, \dots, y_n) \in [1, +\infty[^I \mid \forall j \in \llbracket 1, N \rrbracket, \prod_{i=1}^I y_i^{\mathbf{k}_{i,j}} \leq t^{\beta_j} \right\}$$

3. *If $C_0 \neq 0$ and $I + r = n = \text{Rank}(\{\mathbf{k}_i\}_{i=1}^{I+r})$. Then, there exists a family of real numbers $(\{a_i\}_{i=1}^{I+r})$ such that $\boldsymbol{\beta} = \sum_{i=1}^{I+r} a_i \mathbf{k}_i$.*

If for all $i \in \{1, \dots, I + r\}$, $a_i > 0$, then $\deg(Q_\beta) = 0$.

If there exists $i_0 \in \{1, \dots, I + r\}$ such that $a_{i_0} < 0$, then $Q_\beta \equiv 0$.

4. *If $C_0 \neq 0$, $\text{Rank}(\{\mathbf{k}_i\}_{i=1}^{I+r}) = n$ and $\boldsymbol{\beta} \in \text{conv}^*(\{\mathbf{k}_i\}_{i=1}^{I+r})$, then*

$$\deg(Q_\beta) = \rho = I + r - n.$$

Example 5. Let $\{\mathbf{k}_1, \dots, \mathbf{k}_I\} \subset \mathbb{R}_+^n \setminus \{\mathbf{0}\}$, $\mathbf{c} \in (\mathbb{R}_+^*)^n$ and $\mathbf{g} : \{\mathbf{k}_1, \dots, \mathbf{k}_I\} \rightarrow \mathbb{N}$ such that for all $i \in \llbracket 1, I \rrbracket$ on a $\mathbf{k}_i \cdot \mathbf{c} = 1$. Set $g_i = \mathbf{g}(\mathbf{k}_i)$ for all $i \in \llbracket 1, I \rrbracket$. Then, the function f defined by identity $\mathcal{M}(f; \mathbf{s}) = \prod_{i=1}^I \zeta^{g_i}(\mathbf{k}_i \cdot \mathbf{s})$ satisfies the assumptions of Theorems 8, 9 with $W = \{\mathbf{k}_1, \dots, \mathbf{k}_I\}$ and $W' = J = \emptyset$.

La Bretèche's Theorems give us, under fairly general assumption on the function f , an asymptotic of $\Psi(f; \mathbf{x})$ when the variables x_1, \dots, x_n are dependent; i.e. the case where $x_1 = t^{a_1}, \dots, x_n = t^{a_n}$ ($t \geq 1$) with a_1, \dots, a_n fixed positive numbers. Unfortunately, his method does not seem to apply easily if the x_i are independent.

Later Essouabri obtains, under fairly general hypotheses on the arithmetic function f and the polynomial P , the asymptotic expansion of the function $x \mapsto \Psi(f; P; x)$, see theorems 10, 11 and 12 below. To obtain these results, he introduced the notions: function of finite type, Newton's polyhedron of a multiplicative function, mixed volume constants [12, 14], moderate growth and growth exponent, see definitions below.

Definition 7 ([12], §2.1). (*Moderate growth*). A meromorphic function $F : \Omega \subset \mathbb{C}^n \rightarrow \mathbb{C}$ with singular locus \mathcal{D} is said to have moderate growth if there exist a and $b > 0$ such that: for any compact subset K of \mathbb{R}^n and any $\varepsilon > 0$,

$$F(\mathbf{s}) \ll_{K, \varepsilon} 1 + |\tau|_1^{a|\sigma|^{1+b}} \text{ uniformly in } \mathbf{s} = \sigma + i\tau \in \Omega \text{ such that } \sigma \in K \text{ and } \text{dist}(\mathbf{s}; \mathcal{D}) \geq \varepsilon.$$

Definition 8 ([12], §3.2.1). (*Finite type*) A function $f : \mathbb{N}^n \rightarrow \mathbb{C}$ is said to be of **finite type** if there exists a point $\mathbf{c} = (c_1, \dots, c_n) \in (\mathbb{R}_+^*)^n$ such that $\mathcal{M}(f; \mathbf{s})$ converges absolutely if $\sigma_i = \text{Re}(s_i) > c_i$ for all $i \in \llbracket 1, n \rrbracket$ and it can be continued as a meromorphic function to a neighborhood of \mathbf{c} , as follows:

There exists a pair $\mathcal{T}_{\mathbf{c}} = (I_{\mathbf{c}}, \mathbf{u})$, where $I_{\mathbf{c}}$ is a finite non empty subset of $\mathbb{R}_+^n \setminus \{\mathbf{0}\}$ and $\mathbf{u} = (u(\beta))_{\beta \in I_{\mathbf{c}}}$ is a vector of positive integers, such that

$$\mathbf{s} \mapsto H(f; \mathcal{T}_{\mathbf{c}}; \mathbf{s}) := \left(\prod_{\beta \in I_{\mathbf{c}}} \langle \mathbf{s}, \beta \rangle^{u(\beta)} \right) \mathcal{M}(f; \mathbf{c} + \mathbf{s}) \quad (\text{III.18})$$

has a holomorphic continuation with moderate growth (see Definition 7) to the set $\{\mathbf{s} \in \mathbb{C}^n \mid \sigma_i > -\varepsilon_0 \text{ for all } i \in \llbracket 1, n \rrbracket\}$ for some $\varepsilon_0 > 0$.

We can assume¹ that $\langle \beta, \mathbf{c} \rangle = 1$ for each $\beta \in I_{\mathbf{c}}$. In this case we call $\mathcal{T}_{\mathbf{c}} = (I_{\mathbf{c}}, \mathbf{u})$ a “regularizing pair” of $\mathcal{M}(f; \mathbf{s})$ at \mathbf{c} .

For non negative functions f , this is essentially the class of arithmetic functions studied by La Bretèche.

For this class of function, Essouabri proved the following general result:

Theorem 10 ([12], Th. 3). Let $f \in \mathcal{A}^n$ be a multivariable arithmetic function of finite type. Let $\mathbf{c} \in (\mathbb{R}_+^*)^n$ and $\mathcal{T}_{\mathbf{c}} = (I_{\mathbf{c}}; \mathbf{u})$ a regularizing pair of $\mathcal{M}(f; \mathbf{s})$ at \mathbf{c} . Let $P \in \mathfrak{E}_n$ be a homogeneous elliptic polynomial of degree $D > 0$. Then, $s \mapsto Z(f; P; s)$ is holomorphic in the half-plane $\{\text{Re}(s) > |\mathbf{c}| := \sum_{i=1}^n c_i\}$ and there exists $\eta > 0$ such that $s \mapsto Z(f; P; s)$ has a meromorphic continuation with moderate growth to the half-plane $\{\sigma > |\mathbf{c}| - \eta\}$ with at most one pole at $s = |\mathbf{c}|$ of order at most $\rho_0(\mathcal{T}_{\mathbf{c}}) := \sum_{\beta \in I_{\mathbf{c}}} u(\beta) - \text{rank}(I_{\mathbf{c}}) + 1$.

Assume in addition that the following two properties are satisfied:

1. $\mathbf{1} := (1, \dots, 1) \in \text{con}^*(I_{\mathbf{c}})$ (see §1.1);

¹by replacing each vector β by the vector $\frac{1}{\langle \beta, \mathbf{c} \rangle} \beta$

2. there exists a function K holomorphic in a tubular neighborhood² of $\mathbf{0}$ such that $H(f; \mathcal{T}_c; \mathbf{s}) = K(\langle \beta, \mathbf{s} \rangle)_{\beta \in I_c}$, where $H(f; \mathcal{T}_c; \mathbf{s})$ is the function defined above in (III.18).

Then,

$$Z(f; P; s) = \frac{H(f; \mathcal{T}_c; \mathbf{0}) d^{\rho_0(\mathcal{T}_c)} A_0(\mathcal{T}_c, P)}{(s - |\mathbf{c}|)^{\rho_0(\mathcal{T}_c)}} + O\left(\frac{1}{(s - |\mathbf{c}|)^{\rho_0(\mathcal{T}_c) - 1}}\right) \text{ as } s \rightarrow |\mathbf{c}|,$$

where $A_0(\mathcal{T}_c, P) > 0$ is the mixed volume constant associated to P and \mathcal{T}_c (see §1.4). In particular, $s = |\mathbf{c}|$ is a pole of order $\rho_0(\mathcal{T}_c)$ if and only if $H(f; \mathcal{T}_c; \mathbf{0}) \neq 0$.

A standard consequence of this theorem is a description of the main contribution to the asymptotic of $x \mapsto \Psi(f; P; x)$ as follows:

Theorem 11 ([12], Cor. 2). *Let $f : \mathbb{N}^n \mapsto \mathbb{R}_+$ be a non-negative arithmetic function of finite type and let $P \in \mathfrak{E}_n$ be a homogeneous elliptic polynomial of degree $d > 0$. Let $\mathbf{c} \in (\mathbb{R}_+^*)^n$ and $\mathcal{T}_c = (I_c, \mathbf{u})$ a regularizing pair of $\mathcal{M}(f; -)$ at \mathbf{c} . Then there exist a polynomial Q of degree at most $\rho_0(\mathcal{T}_c) - 1$ and $\delta > 0$ such that:*

$$\Psi(f; P; x) := \sum_{\substack{\mathbf{m}=(m_1, \dots, m_n) \in \mathbb{N}^n \\ P(\mathbf{m})^{1/d} \leq x}} f(m_1, \dots, m_n) = x^{|\mathbf{c}|} Q(\log x) + O(x^{|\mathbf{c}| - \delta}) \text{ as } x \rightarrow +\infty.$$

In particular, there exists a non-negative constant $C_0(f; P)$ such that

$$\Psi(f; P; x) = C_0(f; P) x^{|\mathbf{c}|} (\log x)^{\rho_0(\mathcal{T}_c) - 1} + O(x^{|\mathbf{c}|} (\log x)^{\rho_0(\mathcal{T}_c) - 2}) \text{ as } t \rightarrow +\infty.$$

Assume in addition that the assumptions 1 and 2 of Theorem 10 are satisfied. Then,

$$C_0(f; P) := \frac{H(f; \mathcal{T}_c; \mathbf{0}) d^{\rho_0(\mathcal{T}_c)} A_0(\mathcal{T}_c, P)}{|\mathbf{c}| (\rho_0(\mathcal{T}_c) - 1)!} \geq 0.$$

Thus, $C_0(f; P) > 0$ if and only if $H(f; \mathcal{T}_c; \mathbf{0}) \neq 0$.

Essouabri also introduced in [9] the notion of Newton's polyhedron $\Sigma(f)$ (see §1.2 at the beginning of this chapter) of a multiplicative function $f \in \mathcal{M}^n$ and Delange type conditions (see Definition 10 below) which allowed him to obtain, in the multiplicative case, more explicit versions of Theorems 10 and 11.

Definition 9 ([9], Def. 3). *(Finite growth) A function $f \in \mathcal{M}^n$ is said of finite growth if for all $i \in \llbracket 1, n \rrbracket$, we have*

$$\gamma_i := \max \left\{ \limsup_{m_i \rightarrow \infty} \frac{\log(|f(1, \dots, 1, m_i, 1, \dots, 1)|)}{\log m_i}; 0 \right\} \in \mathbb{R}_+ \quad (\text{III.19})$$

and for all $\varepsilon > 0$

$$|f(m_1, \dots, m_n)| \ll_{\varepsilon} m_1^{\gamma_1 + \varepsilon} \dots m_n^{\gamma_n + \varepsilon} \text{ uniformly in } \mathbf{m} \in \mathbb{N}^n \quad (\text{III.20})$$

The vector $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_n)$ is called growth exponent of f .

²By tubular neighborhood of $\mathbf{0}$, we mean a neighborhood of the form $\{\mathbf{s} \in \mathbb{C}^n; |\operatorname{Re}(s_i)| < \varepsilon \forall i\}$ ($\varepsilon > 0$).

Definition 10 (Delange type conditions, [9]). Let I be a finite subset of $\mathbb{N}_0^n \setminus \{\mathbf{0}\}$, $\theta \leq 1$, $\gamma \in \mathbb{R}_+^n$, and $\mathbf{g} : I \rightarrow \mathbb{N}$. Let $f \in \mathcal{M}^n$ be a multivariable multiplicative function. Let $\Sigma(f)$ its Newton Polyhedron, $\mathcal{F}_{\Sigma(f)}(\mathbf{1})$ the smallest face of $\Sigma(f)$ in the diagonal direction and $I_0(f)$ the finite set of polar vectors of the faces containing $\mathcal{F}_{\Sigma(f)}(\mathbf{1})$ (see §1.2). The function $f \in \mathcal{M}^n$ belongs to the set $\mathfrak{D}(\gamma; \mathbf{g}; \theta)$ if

- f has a growth exponent γ ,
- $I_0(f) = I$,
- for all $\mathbf{k} \in I$, the series

$$\sum_{p \in \mathcal{P}} \frac{1}{p^\theta} \left| \frac{f(p^{\mathbf{k}})}{p^{\mathbf{k} \cdot \gamma}} - \mathbf{g}(\mathbf{k}) \right| \quad \text{converges.} \quad (\text{III.21})$$

Definition 11 ([9], Def. 4). Let $f \in \mathcal{M}^n$ be a multivariable multiplicative function. Assume that f belongs to the class $\mathfrak{D}(\gamma; \mathbf{g}; \theta)$ as in Definition 10 above. Let $\mathbf{c}' \in \mathcal{F}_{\Sigma(f)}(\mathbf{1})$. Set $\mathbf{c} := \mathbf{c}' + \gamma$,

$$I_{\mathbf{c}} := \left\{ \frac{1}{1 + \mathbf{k} \cdot \gamma} \mathbf{k} \mid \mathbf{k} \in I \right\} \quad \text{and} \quad \mathbf{u}(\mathbf{v}) := \mathbf{g}(\mathbf{k}) \quad \text{for all} \quad \mathbf{v} = \frac{1}{1 + \mathbf{k} \cdot \gamma} \mathbf{k} \in I_{\mathbf{c}}.$$

The **combinatorial type** of f at \mathbf{c} is the pair $\mathcal{T}_{\mathbf{c}}(f) = (I_{\mathbf{c}}; \mathbf{u})$. For all $p \in \mathcal{P}$ we set

$$\mathcal{H}_p(f; \mathbf{g}; \mathbf{c}'; \gamma; \mathbf{s}) = \left(\sum_{\mathbf{v} \in \mathbb{N}_0^n} \frac{f(p^{\mathbf{v}})}{p^{\mathbf{v} \cdot (\mathbf{s} + \mathbf{c}' + \gamma)}} \right) \prod_{\mathbf{k} \in I} \left(1 - \frac{1}{p^{\mathbf{k} \cdot (\mathbf{s} + \mathbf{c}')}} \right)^{\mathbf{g}(\mathbf{k})} \quad (\text{III.22})$$

and

$$\mathcal{H}(f; \mathbf{g}; \mathbf{c}'; \gamma; \mathbf{s}) = \prod_{p \in \mathcal{P}} \mathcal{H}_p(f; \mathbf{g}; \mathbf{c}'; \gamma; \mathbf{s}) \quad (\text{III.23})$$

Remark. In Proposition 2 we will prove that the product defining $\mathcal{H}(f; \mathbf{c}'; \gamma; \mathbf{s})$ is absolutely convergent in the closed half-space $\{\text{Re}(\mathbf{s}) \geq \mathbf{0}\}$.

Remark 9. Let $I = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be the canonical basis of \mathbb{R}^n and $g : I \rightarrow \mathbb{N}$ defined by $g(\mathbf{e}_i) = 1$ for all $i \in \llbracket 1, n \rrbracket$. If $f \in \mathfrak{D}(\mathbf{0}; g; 1) \cap \mathcal{M}^n$ and $|f| \leq 1$. Then, f satisfies all assumptions of Delange's multivariable Theorem (see Theorem 3 above) and this theorem implies that there exists $M(f) \in \mathbb{R}$ such that $\Psi(f; x_1, \dots, x_n) = x_1 \dots x_n (M(f) + o(1))$ as $(x_1, \dots, x_n) \rightarrow (+\infty, \dots, +\infty)$.

Remark 10. The function $\mathbf{u} : I_{\mathbf{c}} \rightarrow \mathbb{N}$ in Definition 11 is well defined. Indeed, if there exists $\mathbf{k}_1, \mathbf{k}_2 \in I$ such that $\frac{1}{1 + \mathbf{k}_1 \cdot \gamma} \mathbf{k}_1 = \frac{1}{1 + \mathbf{k}_2 \cdot \gamma} \mathbf{k}_2 \in I_{\mathbf{c}}$, then there exists $t \in \mathbb{R}_+$ such that $\mathbf{k}_1 = t \mathbf{k}_2$. Since $\mathbf{c}' \in \mathcal{F}_{\Sigma(f)}(\mathbf{1}) \subset \{\mathbf{x}; \mathbf{k}_1 \cdot \mathbf{x} = 1\} \cap \{\mathbf{x}; \mathbf{k}_2 \cdot \mathbf{x} = 1\}$, it follows that $1 = \mathbf{k}_1 \cdot \mathbf{c}' = t \mathbf{k}_2 \cdot \mathbf{c}' = t$ and therefore that $\mathbf{k}_1 = \mathbf{k}_2$.

For a multiplicative function $f : \mathbb{N}^n \rightarrow \mathbb{R}_+$ and P a non constant homogeneous and elliptic polynomial. Essouabri showed, in the following theorem, that $\Psi(f; P; t)$ has an asymptotic expansion when f satisfies Delange type conditions, see Definition 10, with $\theta < 1$.

Theorem 12 ([9], Theo. 3). *Let $f \in \mathfrak{D}(\gamma; \mathbf{g}; \theta)$ as in Definition 10. Let $P \in \mathfrak{E}_n$ homogeneous of degree $d > 0$. Assume that $\theta < 1$, $\mathcal{F}_{\Sigma(f)}(\mathbf{1}) \not\subset \bigcup_{i=1}^n \{\mathbf{x} \in \mathbb{R}^n; x_i = 0\}$ and $\text{Rank}(S(f)) = \text{Rank}(I_0(f))$. Set $\rho_0(f) = 1 + \sum_{\mathbf{k} \in I_0(f)} \mathbf{g}(\mathbf{k}) - \text{Rank}(I_0(f))$ and $\mathbf{c} = \mathbf{c}' + \gamma$.*

Then, there exists a polynomial Q of degree at most $\rho_0(f) - 1$ and $\delta > 0$ such that

$$\Psi(f; P; t) = t^{\iota(f)+|\gamma|} Q(\log t) + O(t^{\iota(f)+|\gamma|-\delta}) \quad \text{as } t \rightarrow +\infty. \quad (\text{III.24})$$

In particular, there exists a constant $M(f; P)$ such that

$$\Psi(f; P; t) = M(f; P) t^{\iota(f)+|\gamma|} (\log t)^{\rho_0(f)-1} + O(t^{\iota(f)+|\gamma|} (\log t)^{\rho_0(f)-2}) \quad \text{as } t \rightarrow +\infty.$$

Moreover, for any $\mathbf{c}' \in \text{int}(\mathcal{F}_{\Sigma(f)}(\mathbf{1}))$, we have

$$M(f; P) = \left(\prod_{\mathbf{k} \in I_0(f)} (1 + \mathbf{k} \cdot \gamma)^{-g(\mathbf{k})} \right) \frac{d^{\rho_0(f)} A_0(\mathcal{T}_{\mathbf{c}}; P)}{(\rho_0(f) - 1)! (\iota(f) + |\gamma|)} \times \prod_{p \in \mathcal{P}} \left[(1 - p^{-1})^{\sum_{\mathbf{k} \in I_0(f)} g(\mathbf{k})} \left(\sum_{\mathbf{v} \in \mathbb{N}_0^n} \frac{f(p^{\mathbf{v}})}{p^{\mathbf{v} \cdot \mathbf{c}}} \right) \right]$$

where $\mathcal{T}_{\mathbf{c}}$ is the combinatorial type of f at \mathbf{c} (see Definition 11) and $A_0(\mathcal{T}_{\mathbf{c}}; P) > 0$ the mixed volume constant associated to $\mathcal{T}_{\mathbf{c}}$ and P .

As the example given by Essouabri in [9, Theo. 7] illustrates, the multiple zeta function $\mathcal{M}(f; \mathbf{s})$ does not necessarily have a meromorphic extension to a small ball centered on \mathbf{c} if we only suppose that $\theta = 1$.

Consequently, to treat the case $\theta = 1$, neither the method of La Bretèche nor that of Essouabri can be used. Moreover, the multivariable Delange's theorem only applies if $I_0(f)$ is reduced to the canonical basis and $|f| \leq 1$.

Our main results in this chapter (i.e. Theorems 13 and 14 below) study some special class of multiplicative multivariable functions in the case $\theta = 1$. The methods of La Bretèche and Essouabri use analytic continuation and multivariable complex integration. Another method for studying the means values of arithmetic functions is the convolution method. This method has been extended to the multivariable case in the work of Delange [3] and more recently in those of Tóth and Zahai [30, 29]. To obtain our main results in this chapter, we use a method which is a mixture of the multivariable complex integration method and the multivariable convolution method.

3 Statements of the main results

As a first main result, see Theorem 13 below, we are going to treat a particular case, which does not fit into the La Bretèche framework since $\theta = 1$ and we do not suppose the variables x_1 and x_2 to be linked. Neither does it fit within the scope of Delange since $I_0(f)$ is not reduced to the canonical basis $\{\mathbf{e}_1, \mathbf{e}_2\}$.

Under the notations of the Definition 10 and the equation (III.23) we consider

$$\gamma \in \mathbb{R}_+^2, \quad \mathbf{g} : \{\mathbf{a}\mathbf{e}_1, \alpha\mathbf{e}_1 + \beta\mathbf{e}_2, \mathbf{b}\mathbf{e}_2\} \rightarrow \{1\} \quad \text{and} \quad f \in \mathfrak{D}(\gamma; \mathbf{g}; 1),$$

where $\mathbf{a}, \mathbf{b}, \alpha, \beta \in \mathbb{N}^*$ and $(\alpha, \beta) \in \text{conv}^*\{\mathbf{a}\mathbf{e}_1; \mathbf{b}\mathbf{e}_2\}$.

Theorem 13. Let $a, b, \alpha, \beta \in \mathbb{N}$, $\gamma \in \mathbb{R}_+^2$ and $\mathbf{g} : \{(a, 0), (\alpha, \beta), (0, b)\} \rightarrow \{1\}$ such that $(\alpha, \beta) \in \text{conv}^*\{\mathbf{a}\mathbf{e}_1, \mathbf{b}\mathbf{e}_2\}$. Set $\mathbf{c}' = (c'_1, c'_2) = (a^{-1}, b^{-1})$ et $\mathbf{c} = (c_1, c_2) = \mathbf{c}' + \gamma$. Let $f : \mathbb{N}^2 \rightarrow \mathbb{C}$ be a multiplicative function. Assume that $f \in \mathfrak{D}(\gamma; \mathbf{g}; 1)$, then

$$\Psi(f; x_1, x_2) = \sum_{\substack{m_1 \leq x_1 \\ m_2 \leq x_2}} f(m_1, m_2) = \left(\frac{\mathcal{H}(f; \mathbf{g}; \mathbf{c}'; \gamma; \mathbf{0})}{ab\alpha\beta c_1 c_2} + o(1) \right) x_1^{c_1} x_2^{c_2} \log\left(\min\{x_1^\beta; x_2^\alpha\}\right)$$

as $\min\{x_1; x_2\} \rightarrow +\infty$, where

$$\mathcal{H}(f; \mathbf{g}; \mathbf{c}'; \gamma; \mathbf{0}) = \prod_{p \in \mathcal{P}} \left(\sum_{\mathbf{v} \in \mathbb{N}_0^2} \frac{f(p^{\mathbf{v}})}{p^{\mathbf{v} \cdot (\mathbf{c}' + \gamma)}} \right) \prod_{\mathbf{k} \in I} \left(1 - \frac{1}{p^{\mathbf{k} \cdot \mathbf{c}'}} \right)^{\mathbf{g}(\mathbf{k})}.$$

(This infinite product is convergent).

Two cases in which we can apply this result are given by the examples 6 and 7 below.

Example 6. The multiplicative functions $f_1, f_2 : \mathbb{N}^2 \rightarrow \mathbb{R}$ defined by

$$f_1(p^{\nu_1}, p^{\nu_2}) = \begin{cases} 1, & \text{if } |\nu|_1 \text{ is even,} \\ 0, & \text{otherwise.} \end{cases} \quad \text{and} \quad f_2(p^{\nu_1}, p^{\nu_2}) = \begin{cases} 1 + |\nu|_1, & \text{if } |\nu|_1 \text{ is even,} \\ 0, & \text{otherwise.} \end{cases}$$

satisfy all the assumptions of Theorem 13.

Example 7. Let $a, b, \alpha, \beta \in \mathbb{N}$, $I_0 = \{(a, 0), (\alpha, \beta), (0, b)\}$, $\mathbf{g} : I_0 \rightarrow \{1\}$ and $\gamma = \mathbf{0}$ such that $a^{-1}\alpha + b^{-1}\beta = 1$. The function $f = \tau_{\mathbf{g}}$, see (III.1), satisfies all the assumptions of Theorem 13 avec $I_0(f) = I_0$ et $\mathbf{c} = (a^{-1}, b^{-1})$.

For examples with $\gamma \in \mathbb{R}_+^2 \setminus \{\mathbf{0}\}$, it suffices to consider the function $f(\mathbf{m}) = \mathbf{m}^\gamma \tau_{\mathbf{g}}(\mathbf{m})$.

Our second main result, see Theorem 14 below, also deal with a particular case in which $\theta = 1$ and which therefore does not fit into the La Bretèche's framework nor in that of Essouabri's framework.

Theorem 14. Let $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{N}^n$, $f : \mathbb{N}^n \rightarrow \mathbb{C}$ a multiplicative function, $\mathbf{g} : \{a_1 \mathbf{e}_1, \dots, a_n \mathbf{e}_n\} \rightarrow \mathbb{N}$ and $\gamma \in \mathbb{R}_+^n$. Set $\mathbf{c}' = (c'_1, \dots, c'_n) = (a_1^{-1}, \dots, a_n^{-1})$, $\mathbf{c} = (c_1, \dots, c_n) = \mathbf{c}' + \gamma$ and $g_i = \mathbf{g}(a_i \mathbf{e}_i)$ (see Definition 10). If $f \in \mathfrak{D}(\gamma; \mathbf{g}; 1)$, then

$$\Psi(f; \mathbf{x}) = \sum_{m_i \leq x_i \quad \forall i \in [1, n]} f(m_1, \dots, m_n) = (M(f) + o(1)) x_1^{c_1} \dots x_n^{c_n} \prod_{i=1}^n \log^{g_i-1}(x_i) \quad (\text{III.25})$$

as $\min\{x_1, \dots, x_n\} \rightarrow \infty$, where

$$M(f) = \left(\prod_{i=1}^n \frac{a_i^{-g_i}}{(g_i - 1)! c_i} \right) \prod_{p \in \mathcal{P}} \left[\left(\sum_{\mathbf{v} \in \mathbb{N}_0^n} \frac{f(p^{\mathbf{v}})}{p^{\mathbf{v} \cdot \mathbf{c}}} \right) \left(1 - \frac{1}{p} \right)^{|\mathbf{g}|} \right] \quad (\text{III.26})$$

In the particular case where $a_1 = \dots = a_n = 1$, this result implies Delange's Theorem . Above theorem can be applied to the following family of multiplicative functions:

Example 8. Let $\phi : \mathbb{N} \rightarrow \mathbb{N}$ be a multiplicative function and $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{N}^n$ such that $\text{gcd}(a_i, a_j) = 1$ for $i \neq j$. Set $\widehat{a}_i = \prod_{j=1, j \neq i}^n a_j$, then the function

$$f(\mathbf{m}) = \begin{cases} \phi(m_1^{\widehat{a}_1} \dots m_n^{\widehat{a}_n}), & \text{if } m_1^{\widehat{a}_1} \dots m_n^{\widehat{a}_n} \text{ is } \prod_{i=1}^n a_i \text{ powerfull,} \\ 0, & \text{otherwise} \end{cases},$$

verifies all assumptions of Theorem 14 with $\mathbf{g} : I = \{a_1 \mathbf{e}_1, \dots, a_n \mathbf{e}_n\} \rightarrow \{1\}$.

Let $P \in \mathfrak{E}_n$ of degree $d > 0$, $\mathbf{d} \in \mathbb{N}^n$ and $f : \mathbb{N} \rightarrow \mathbb{C}$ a one variable multiplicative function satisfying the assumptions:

(H1) there exist $\gamma, L \in \mathbb{R}^+$, $\delta > 0$ and $g : \mathbb{N} \rightarrow \mathbb{N}$ such that for all $\varepsilon > 0$ we have $g(\ell) \ll e^{\varepsilon \ell}$ and

$$f(p^\ell) = g(\ell)p^{\gamma \ell} + O(\ell^L p^{\gamma \ell - \delta}) \text{ uniformly in } p \text{ and } \ell,$$

(H2) $C = (\mathbb{R}_+^*)^n \cap \text{int } \mathcal{F}_{\Sigma(f_{\mathbf{d}})}(\mathbf{1}) \neq \emptyset$, we choose \mathbf{c}' an element of C .

As a third result of this chapter, we give (see Proposition 1) as application of Essouabri's Theorem 12 the asymptotic expansion of the density function

$$\Psi(f_{\mathbf{d}}; P; x) = \sum_{\substack{\mathbf{m}=(m_1, \dots, m_n) \in \mathbb{N}^n \\ P(\mathbf{m})^{1/d} \leq x}} f_{\mathbf{d}}(\mathbf{m}) = \sum_{\substack{\mathbf{m} \in \mathbb{N}^n \\ P(\mathbf{m})^{1/d} \leq x}} f(m^{d_1} \dots m_n^{d_n})$$

We set

$$\mathcal{H}(f; g; \mathbf{c}'; \gamma; \mathbf{d}; \mathbf{s}) = \prod_{p \in \mathcal{P}} \left[\left(\sum_{\nu \in \mathbb{N}_0^n} \frac{f(p^{\mathbf{d} \cdot \nu})}{p^{\nu \cdot (\mathbf{s} + \mathbf{c}' + \gamma \mathbf{d})}} \right) \prod_{\mathbf{k} \in I_0(f_{\mathbf{d}})} \left(1 - \frac{1}{p^{1 + \mathbf{k} \cdot \mathbf{s}}} \right)^{g(\mathbf{d} \cdot \mathbf{k})} \right] \quad (\text{III.27})$$

Proposition 1. *Let $P \in \mathfrak{E}_n$ of degree $d > 0$ and $\mathbf{c}' \in (\mathbb{R}_+^*)^n \cap \text{int } \mathcal{F}_{\Sigma(f_{\mathbf{d}})}(\mathbf{1})$. Assume that the assumptions (H1) et (H2) hold. Then, there exists $\delta' > 0$ such that the series $Z(f_{\mathbf{d}}; P; s)$ is holomorphic in the domain $\{\text{Re}(s) > |\mathbf{c}'|_1 + \gamma|\mathbf{d}|_1\}$ and has a meromorphic continuation with moderate growth to the domain $\{\text{Re}(s) > |\mathbf{c}'|_1 + \gamma|\mathbf{d}|_1 - \delta'\}$ with at most one pole in $\sigma_0 = |\mathbf{c}'| + \gamma|\mathbf{d}|$ of order at most $\rho = \sum_{\mathbf{k} \in I_0(f_{\mathbf{d}})} g(\mathbf{d} \cdot \mathbf{k}) - \text{Rank}(I_0(f_{\mathbf{d}})) + 1$. If in addition $\mathcal{H}(f; g; \mathbf{c}'; \gamma; \mathbf{d}; \mathbf{0}) \neq 0$, see (III.27), then*

$$Z(f_{\mathbf{d}}; P; s) = \frac{\mathcal{H}(f; g; \mathbf{c}'; \gamma; \mathbf{d}; \mathbf{0}) A_0((I_{\mathbf{c}'}; g); P) d^\rho}{(s - \sigma_0)^\rho \prod_{\mathbf{v} \in I_0(f_{\mathbf{d}})} (1 + \gamma \mathbf{v} \cdot \mathbf{d})^{g(\mathbf{v} \cdot \mathbf{d})}} + O((s - \sigma_0)^{1-\rho}), \text{ as } s \rightarrow \sigma_0$$

and

$$\Psi(f_{\mathbf{d}}; P; x) = \frac{\mathcal{H}(f; g; \mathbf{c}'; \gamma; \mathbf{d}; \mathbf{0}) A_0((I_{\mathbf{c}'}; g); P) d^\rho x^{\sigma_0} (\log x)^{\rho-1} \left(1 + O((\log x)^{-1}) \right)}{\sigma_0(\rho-1)! \prod_{\mathbf{v} \in I_0(f_{\mathbf{d}})} (1 + \gamma \mathbf{v} \cdot \mathbf{d})^{g(\mathbf{v} \cdot \mathbf{d})}}$$

where $A_0((I_{\mathbf{c}'}; g); P)$ is the mixed volume constant associated to the pair $\mathcal{T}_{\mathbf{c}'} = (I_{\mathbf{c}'}; g)$ and the polynomial P .

Applying Proposition 1 to the functions $\sigma_{k, \mathbf{d}}$ and $\varphi_{\mathbf{d}}$, gives the following results:

Corollary 2. *Let $\mathbf{d} \in \mathbb{N}^n$ and $f \in \{\varphi, \sigma_k; k \in \mathbb{N}\}$. Then,*

With notation of §1.2 we have $S_\infty(f_{\mathbf{d}}) = \mathbb{N}_0^n \setminus \{\mathbf{0}\}$ and $I(f_{\mathbf{d}}) = I_0(f_{\mathbf{d}}) = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$.

Moreover, the assumptions (H1) and (H2) hold and invariants g, \mathbf{c}', γ and ρ which appear in them and in Proposition 1 verify

1. for $f = \sigma_{0; \mathbf{d}}$: $\mathbf{c}' = \mathbf{1}$, $\gamma = 0$, $\rho = 1 + |\mathbf{d}|_1$ and $g(\ell) = \ell + 1 \forall \ell \in \mathbb{N}_0$;
2. for $f = \sigma_{k; \mathbf{d}}$ with $k \in \mathbb{N}$: $\mathbf{c}' = \mathbf{1}$, $\gamma = k$, $\rho = 1$ and $g(\ell) = 1 \forall \ell \in \mathbb{N}_0$;
3. for $f = \varphi_{\mathbf{d}}$: $\mathbf{c}' = \mathbf{1}$, $\gamma = 1$, $\rho = 1$ and $g(\ell) = 1 \forall \ell \in \mathbb{N}_0$.

In particular, for $P \in \mathfrak{E}_n$ of degree p , there exists $\varepsilon > 0$ such that

$$Z(f_{\mathbf{d}}; P; s) = \sum_{\mathbf{m} \in \mathbb{N}^n} \frac{f_{\mathbf{d}}(\mathbf{m})}{P(\mathbf{m})^{s/p}} = \sum_{\mathbf{m} \in \mathbb{N}^n} \frac{f(m_1^{d_1} \cdots m_n^{d_n})}{P(\mathbf{m})^{s/p}}$$

is holomorphic in $\{\operatorname{Re}(s) > |\mathbf{c}'|_1 + \gamma|\mathbf{d}|_1\}$ and has a meromorphic continuation with moderate growth to $\{\operatorname{Re}(s) > |\mathbf{c}'|_1 + \gamma|\mathbf{d}|_1 - \varepsilon\}$ with one pole at $s = \sigma_0 := |\mathbf{c}'|_1 + \gamma|\mathbf{d}|_1$ of order ρ . Moreover,

$$Z(f_{\mathbf{d}}; P; s) = \mathcal{H}(f; g; \mathbf{c}'; \gamma; \mathbf{d}; \mathbf{0}) A_0(P^\sharp) p^\rho (s - \sigma_0)^{-\rho} (1 + O(s - \sigma_0)), \text{ as } s \rightarrow \sigma_0,$$

where $A_0(P^\sharp)$ is the Sargos's volume constant (see §1.3) associated to the polynomial P^\sharp with

1. $P^\sharp((X_j^{(k)})_{j \in \llbracket 1, 1+d_k \rrbracket}^{k \in \llbracket 1, n \rrbracket}) = P(\prod_{j=1}^{1+d_1} X_j^{(1)}, \dots, \prod_{j=1}^{1+d_n} X_j^{(n)})$ if $f = \sigma_0$,
2. $P^\sharp(\mathbf{X}) = P(X_1^{\frac{1}{1+k d_1}}, \dots, X_n^{\frac{1}{1+k d_n}})$ if $f = \sigma_k$ et $k \in \mathbb{N}$,
3. $P^\sharp(\mathbf{X}) = P(X_1^{\frac{1}{1+d_1}}, \dots, X_n^{\frac{1}{1+d_n}})$ if $f = \varphi$.

Corollary 3. Under assumption of Corollary 2 we have

$$\sum_{\substack{\mathbf{m} \in \mathbb{N}^n \\ P(\mathbf{m})^{1/p} \leq t}} \sigma_0(m_1^{d_1} \cdots m_n^{d_n}) = p^{1+|\mathbf{d}|_1} \mathcal{H}(\sigma_0; g; \mathbf{1}; 0; \mathbf{d}; \mathbf{0}) A_0(P^\sharp) \frac{t^n (\log t)^{|\mathbf{d}|_1}}{|\mathbf{d}|_1! n} (1 + O((\log t)^{-1})) \quad (\text{III.28})$$

$$\sum_{\substack{\mathbf{m} \in \mathbb{N}^n \\ P(\mathbf{m})^{1/p} \leq t}} \sigma_k(m_1^{d_1} \cdots m_n^{d_n}) = \frac{p \mathcal{H}(\sigma_k; g; \mathbf{1}; k; \mathbf{d}; \mathbf{0}) A_0(P^\sharp)}{\prod_{\ell=1}^n (1 + k d_\ell)} \frac{t^{\frac{n+k|\mathbf{d}|_1}{p}}}{n + k|\mathbf{d}|_1} (1 + O(t^{-\delta})) \text{ si } k \in \mathbb{N} \quad (\text{III.29})$$

$$\sum_{\substack{\mathbf{m} \in \mathbb{N}^n \\ P(\mathbf{m})^{1/p} \leq t}} \varphi(m_1^{d_1} \cdots m_n^{d_n}) = \frac{p \mathcal{H}(\varphi; g; \mathbf{1}; 1; \mathbf{d}; \mathbf{0}) A_0(P^\sharp)}{\prod_{\ell=1}^n (1 + d_\ell)} \frac{t^{\frac{n+|\mathbf{d}|_1}{p}}}{n + |\mathbf{d}|_1} (1 + O(t^{-\delta})). \quad (\text{III.30})$$

where $\delta > 0$ is a positive constant.

4 Auxiliary results

4.1 About La Bretèche's method

Let $f : \mathbb{N}^n \rightarrow \mathbb{R}_+$ be an arithmetic function with non-negative values. In this paragraph we give a brief sketch of La Bretèche's method to study the mean values

$$\Psi(f; \mathbf{x}) = \sum_{\mathbf{m} \leq \mathbf{x}} f(\mathbf{m}) = \sum_{m_i \leq x_i \forall i \in \llbracket 1, n \rrbracket} f(m_1, \dots, m_n).$$

For $\xi \in]0, 1]$, we define the function $\eta_\xi : [0, +\infty[\rightarrow \mathbb{R}_+$ by

$$\eta_\xi(t) = \begin{cases} 1, & \text{si } t \in [0, 1] \\ (1 + \xi - t)/\xi, & \text{if } t \in [1, 1 + \xi] \\ 0, & \text{si } t > 1 + \xi, \end{cases}$$

and its Fourier transform $\widehat{\eta}_\xi : \mathbb{C} \rightarrow \mathbb{C}$ is defined by

$$\widehat{\eta}_\xi(s) := - \int_{0+}^{+\infty} t^s d\eta_\xi(t) = \frac{(1+\xi)^{s+1} - 1}{\xi(s+1)}, \text{ if } \operatorname{Re}(s) > -1, \quad (\text{III.31})$$

we also denote by $\widehat{\eta}_\xi$ its holomorphic continuation to \mathbb{C} . In particular we have

$$\widehat{\eta}_\xi(0) = 1 \quad , \quad \widehat{\eta}_\xi(-1) = \frac{\log(1+\xi)}{\xi}$$

and the following uniform estimates in $\xi \in]0, 1]$ and $\operatorname{Re}(s)$ in a compact set

$$\widehat{\eta}_\xi(s) \ll \min \left\{ 1; \frac{1}{\xi|1+s|} \right\} \quad , \quad \widehat{\eta}_\xi(s) = 1 + O(\xi|s|) \quad , \quad \widehat{\eta}_\xi^{(k)}(s) \ll \xi^k \quad (\text{III.32})$$

Let $\boldsymbol{\xi} = (\xi_1, \dots, \xi_n) \in]0, 1]^n$. We set

$$\Psi(f; \mathbf{x}; \boldsymbol{\xi}) = \sum_{\mathbf{m} \leq \mathbf{x}} f(\mathbf{m}) \prod_{j=1}^n \eta_{\xi_j} \left(\frac{m_j}{x_j} \right) \quad (\text{III.33})$$

Then, since f is non-negative, we have the following inequalities

$$\Psi(f; (1+\xi_1)^{-1}x_1, \dots, (1+\xi_n)^{-1}x_n; \boldsymbol{\xi}) \leq \Psi(f; \mathbf{x}) \leq \Psi(f; \mathbf{x}; \boldsymbol{\xi})$$

$$0 \leq \Psi(f; \mathbf{x}; \boldsymbol{\xi}) - \Psi(f; \mathbf{x}) \leq \Psi(f; \mathbf{x}; \boldsymbol{\xi}) - \Psi(f; (1+\xi_1)^{-1}x_1, \dots, (1+\xi_n)^{-1}x_n; \boldsymbol{\xi}),$$

which implies that

$$\Psi(f; \mathbf{x}) = \Psi(f; \mathbf{x}; \boldsymbol{\xi}) + O \left(\Psi(f; \mathbf{x}; \boldsymbol{\xi}) - \Psi(f; (1+\xi_1)^{-1}x_1, \dots, (1+\xi_n)^{-1}x_n; \boldsymbol{\xi}) \right).$$

This last equation reduces the study of $\Psi(f; \mathbf{x})$ to that of the more regular function $\Psi(f; \mathbf{x}; \boldsymbol{\xi})$. In order to study $\Psi(f; \mathbf{x}; \boldsymbol{\xi})$, we choose $\mathbf{D} \in (\mathbb{R}_+^*)^n$ such that the multiple series $\mathcal{M}(f; \mathbf{s})$ converge absolutely in $\mathbf{D} + (\mathbb{R}_+^*)^n$, and $\boldsymbol{\kappa} = (\kappa_1, \dots, \kappa_n)$ such that $\kappa_i \asymp \frac{1}{\log(3+x_i)}$. By using Perron's formula we get the following integral representation

$$\Psi(f; \mathbf{x}; \boldsymbol{\xi}) = \frac{1}{(2\pi i)^n} \int_{s_1 \in (D_1 + \kappa_1)} \cdots \int_{s_n \in (D_n + \kappa_n)} \mathcal{M}(f; \mathbf{s}) x_1^{s_1} \cdots x_n^{s_n} \prod_{j=1}^n \frac{\widehat{\eta}_{\xi_j}(s_j)}{s_j} ds \quad (\text{III.34})$$

We split the integral III.34 into two pieces. More precisely, for $T > 1$, we set

$$E(f; \mathbf{x}; \boldsymbol{\xi}; T) = \frac{1}{(2\pi i)^n} \int_{\substack{\mathbf{s} \in (\mathbf{D} + \boldsymbol{\kappa}) \\ \|\boldsymbol{\tau}\|_\infty > T}} \mathcal{M}(f; \mathbf{s}) x_1^{s_1} \cdots x_n^{s_n} \prod_{j=1}^n \frac{\widehat{\eta}_{\xi_j}(s_j)}{s_j} ds \quad (\text{III.35})$$

as the first piece and

$$\Psi(f; \mathbf{x}; \boldsymbol{\xi}; T) = \frac{1}{(2\pi i)^n} \int_{\substack{\mathbf{s} \in (\mathbf{D} + \boldsymbol{\kappa}) \\ \|\boldsymbol{\tau}\|_\infty \leq T}} \mathcal{M}(f; \mathbf{s}) x_1^{s_1} \cdots x_n^{s_n} \prod_{j=1}^n \frac{\widehat{\eta}_{\xi_j}(s_j)}{s_j} ds \quad (\text{III.36})$$

as second piece. We estimate each integral in terms of T and $\boldsymbol{\xi}$ and finally we take particular optimal values of T and $\boldsymbol{\xi}$ to obtain a main term and an error term. La Bretèche uses this method to obtain the general result we mentioned above in the case where the variables x_i are dependent.

4.2 A key proposition and first preliminaries

Let $\gamma \in \mathbb{R}_+^n$, $f \in \mathcal{M}^n$, $\mathbf{g} : \{\mathbf{k}_1, \dots, \mathbf{k}_I\} \rightarrow \mathbb{N}$ and $\mathbf{c}' \in \text{int}(\mathcal{F}_{\Sigma(f)}(\mathbf{1})) \cap (\mathbb{R}_+^*)^n$ such that $f \in \mathcal{D}(\gamma; \mathbf{g}; 1)$. Keeping notations as in Theorem 14 we define $h_{f,\mathbf{g}} \in \mathcal{M}^n$ by

$$\left(\sum_{\mathbf{v} \in \mathbb{N}_0^n} f(p^{\mathbf{v}}) p^{-\mathbf{v} \cdot (\mathbf{s} + \gamma)} \right) \prod_{i=1}^I (1 - p^{-\mathbf{k}_i \cdot \mathbf{s}})^{g_i} = \sum_{\mathbf{v} \in \mathbb{N}_0^n} \frac{h_{f,\mathbf{g}}(p^{\mathbf{v}})}{p^{\mathbf{v} \cdot \mathbf{s}}} \quad (\text{III.37})$$

extended to \mathbb{N}^n by $h_{f,\mathbf{g}}(\mathbf{m}) = \prod_{p \in \mathcal{P}} h_{f,\mathbf{g}}(p^{\nu_p(m_1)}, \dots, p^{\nu_p(m_n)})$.

Remark 11. *We have the relation*

$$\mathcal{M}(h_{f,\mathbf{g}}; \mathbf{s} + \mathbf{c}') = \sum_{m_1, \dots, m_n \geq 1} \frac{h_{f,\mathbf{g}}(m_1, \dots, m_n)}{m_1^{c'_1 + s_1} \dots m_n^{c'_n + s_n}} = \mathcal{H}(f; \mathbf{g}; \mathbf{c}'; \gamma; \mathbf{s}), \quad (\text{III.38})$$

where $\mathcal{H}(f; \mathbf{g}; \mathbf{c}'; \gamma; \mathbf{s})$ is defined by (III.23).

In our proofs of Theorems 13 and 14 we will use the following key proposition:

Proposition 2. *Let $\gamma \in \mathbb{R}_+^n$, $f \in \mathcal{M}^n$ and $\mathbf{g} : \{\mathbf{k}_1, \dots, \mathbf{k}_I\} \rightarrow \mathbb{N}$. If $f \in \mathcal{D}(\gamma; \mathbf{g}; 1)$ and $\mathbf{c}' \in \text{int}(\mathcal{F}_{\Sigma(f)}(\mathbf{1})) \cap (\mathbb{R}_+^*)^n$, then $\mathbf{s} \mapsto Z(|h_{f,\mathbf{g}}|; \mathbf{s})$ and $\mathbf{s} \mapsto Z(h_{f,\mathbf{g}}; \mathbf{s})$ are uniformly convergent in the closed set $\{\text{Re}(\mathbf{s}) \in \mathbf{c}' + \gamma + \mathbb{R}_+^n\}$ and are holomorphic in its interior.*

We will apply the above proposition in the following two cases:

1. In proof of theorem 13 with $\mathbf{k}_1 = (a, 0)$; $\mathbf{k}_2 = (0, b)$, $\mathbf{k}_3 = (\alpha, \beta)$ and $\mathbf{g} = 1$.
2. In proof of theorem 14 with $\mathbf{k}_i = a_i \mathbf{e}_i$ for each $i \in \llbracket 1, n \rrbracket$.

We also need the next lemma:

Lemma 10. *Let $\mathbf{g} : W = \{\mathbf{k}_1, \dots, \mathbf{k}_I\} \subset \mathbb{R}^n \rightarrow \mathbb{N}$. We set $g_i = \mathbf{g}(\mathbf{k}_i)$ and*

$$\Theta_{\mathbf{g}}(\mathbf{s}) = \prod_{i=1}^I [(\mathbf{k}_i \cdot \mathbf{s}) \zeta(1 + \mathbf{k}_i \cdot \mathbf{s})]^{g_i}.$$

Then, for any $\varepsilon > 0$ and any compact subset K of \mathbb{R}^n , we have uniformly in \mathbf{s} such that $\text{Re}(\mathbf{s}) \in K$,

$$\Theta_{\mathbf{g}}(\mathbf{s}) \ll_{\varepsilon, K} (1 + |\tau|_1)^\varepsilon \prod_{i=1}^I (1 + |\mathbf{k}_i \cdot \tau|_1)^{g_i - \frac{g_i}{2}(0 \wedge \mathbf{k}_i \cdot \sigma)} \quad (\text{III.39})$$

Proof. This estimation follows directly from the classic estimation of Riemann zeta function: for any $\varepsilon > 0$ and any compact subset K of \mathbb{R} , we have uniformly in s such that $\text{Re}(s) \in K$,

$$|s \zeta(1 + s)| \ll_{\varepsilon, K} (1 + |\tau|)^{1 + \varepsilon - \frac{1}{2}(0 \wedge \sigma)}$$

□

In the proof of theorem 13; that is in the case $I = 3$, $W = \{(a, 0); (\alpha, \beta); (0, b)\}$ and $\mathbf{g} : W \rightarrow \{1\}$ lemma 10 implies that for any $\varepsilon > 0$ and any compact subset K of \mathbb{R}^2 , we have uniformly in \mathbf{s} such that $\text{Re}(\mathbf{s}) \in K$,

$$\begin{aligned} \Theta_{\mathbf{g}}(\mathbf{s}) \ll_{\varepsilon, K} (1 + |\tau_1|)^{1 + \varepsilon - \frac{3}{2}(0 \wedge \sigma_1)} (1 + |\tau_2|)^{1 + \varepsilon - \frac{1}{2}(0 \wedge \sigma_2)} \times \\ (1 + |\alpha \tau_1 + \beta \tau_2|)^{1 + \varepsilon - \frac{1}{2}(0 \wedge (\alpha \sigma_1 + \beta \sigma_2))} \end{aligned} \quad (\text{III.40})$$

Additionally, in proof of theorem 13 we also need the following values:

$$\Theta_{\mathbf{g}}(\mathbf{0}) = 1 \quad , \quad \frac{d}{ds}\Theta_{\mathbf{g}}(s, 0)\Big|_{s=0} = (\alpha + a)\gamma \quad , \quad \frac{d}{ds}\Theta_{\mathbf{g}}(0, s)\Big|_{s=0} = (\beta + b)\gamma \quad (\text{III.41})$$

$$\frac{d}{ds}\Theta_{\mathbf{g}}\left(s, -\frac{\alpha}{\beta}s\right)\Big|_{s=0} = (a\beta - \alpha b)\frac{\gamma}{\beta} \quad , \quad \frac{d}{ds}\Theta_{\mathbf{g}}\left(-\frac{\beta}{\alpha}s, s\right)\Big|_{s=0} = (\alpha b - a\beta)\frac{\gamma}{\alpha} \quad (\text{III.42})$$

These values follow from the Laurent series expansion $\zeta(1+s) = \frac{1}{s} + \gamma + O(s)$ at $s = 0$.

5 Proof of Proposition 2

This proof be will divided into three lemmas:

Lemma 11. *Let $f \in \mathcal{M}^n$ and $S_{\infty}(f)$, $I(f)$, $I_0(f)$, Σ_f , $\mathcal{F}_{\Sigma_f}(\mathbf{1})$ as in the §1.2. If $\mathbf{c}' \in \text{int}(\mathcal{F}_{\Sigma_f}(\mathbf{1})) \cap (\mathbb{R}_+^*)^n$, then there exists $\delta > 0$ such that*

$$(1 - \delta)\mathbf{v} \cdot \mathbf{c}' > 1 \text{ for each } \mathbf{v} \in S_{\infty}(f) \setminus I_0(f)$$

Proof. We set $E = \text{Ext}(\mathcal{F}_{\Sigma_f}(\mathbf{1}))$ the set of extremal points of face $\mathcal{F}_{\Sigma_f}(\mathbf{1})$, $S_1 = \{\mathbf{v} \in S_{\infty} \setminus I_0(f), \mathbf{v} \cdot \mathbf{c}' \leq 2\}$ and S_2 such that $S_{\infty}(f) \setminus I_0(f) = S_1 \sqcup S_2$.

For each $\mathbf{v} \in S_2$, it's enough to take $\delta = 1/2$.

Since S_1 is a bounded closed discrete set, more precisely it's a subset of $(\mathbb{N}_0^n) \cap \frac{2}{\min\{c_1, \dots, c_n\}} B(0; 1)$, it follows that S_1 is a finite compact set. Consequently the function $C : S_1 \rightarrow \mathbb{R}_+$ defined by $\mathbf{v} \mapsto \mathbf{v} \cdot \mathbf{c}'$ has a minimum that it reaches at a point \mathbf{v}_0 of S_1 .

Fact $\mathbf{v}_0 \cdot \mathbf{c}' > 1$: We will prove this fact by contradiction.

Assume, for sake a contradiction, that $\mathbf{v}_0 \cdot \mathbf{c}' \leq 1$.

1. For $\mathbf{y} \in E \subset \Sigma_f$ and $\mathbf{v}_i \in I(f)$, we have $\mathbf{v}_i \cdot \mathbf{y} \geq 1$.
2. Since $\mathbf{c}' \in \text{int}(\mathcal{F}_{\Sigma_f}(\mathbf{1})) \subset \Sigma(f)$ it follows that there exists $\{t_{\mathbf{y}}\}_{\mathbf{y} \in E} \subset]0, 1]$ such that $\sum_{\mathbf{y} \in E} t_{\mathbf{y}} = 1$ and $\mathbf{c}' = \sum_{\mathbf{y} \in E} t_{\mathbf{y}} \mathbf{y}$.
3. Since $\mathbf{v}_0 \in S_1 \subset S_{\infty}(f) \setminus I_0(f) \subset \text{conv}(I(f) + \mathbb{R}_+^n)$ it follows that there exists $\{\mathbf{v}_1, \dots, \mathbf{v}_r\} \subset I(f)$ and $\{t_i : i \in \llbracket 1, r \rrbracket\} \subset]0, 1]$ such that $\sum_{i=1}^r t_i = 1$ and $\sum_{i=1}^r t_i \mathbf{v}_i \leq \mathbf{v}_0$

The vectors \mathbf{v}_i are elements of $I_0(f)$. Indeed, we have

$$1 \geq \mathbf{v}_0 \cdot \mathbf{c}' \geq \sum_{i=1}^r t_i \mathbf{v}_i \cdot \mathbf{c}' \geq \sum_{i=1}^r \sum_{\mathbf{y} \in E} t_i t_{\mathbf{y}} \mathbf{v}_i \cdot \mathbf{y} \geq \sum_{i=1}^r \sum_{\mathbf{y} \in E} t_i t_{\mathbf{y}} = 1,$$

it follows that for each $i \in \llbracket 1, r \rrbracket$ and each $\mathbf{y} \in E$ we have $\mathbf{v}_i \cdot \mathbf{c}' = \mathbf{v}_i \cdot \mathbf{y} = 1$, consequently $\mathbf{v}_i \in I_0(f)$.

We also have $\sum_{i=1}^r t_i \mathbf{v}_i = \mathbf{v}_0$. Indeed, if $\sum_{i=1}^r t_i \mathbf{v}_i < \mathbf{v}_0$ then there exists \mathbf{e}_j vector of canonical basis of \mathbb{R}^n and $\delta > 0$ such that $\sum_{i=1}^r t_i \mathbf{v}_i + \delta \mathbf{e}_j < \mathbf{v}_0$, and therefore

$$1 = \mathbf{v}_0 \cdot \mathbf{c}' > \delta c'_j + \sum_{i=1}^r t_i \mathbf{v}_i \cdot \mathbf{c}' \geq 1 + \delta c'_j > 1$$

which is an absurdity. Therefore the fact $\mathbf{v}_0 \cdot \mathbf{c}' > 1$ is true. Since $\mathbf{v}_0 = \sum_{i=1}^r t_i \mathbf{v}_i$ it follows that $\mathbf{v}_0 \in I_0(f)$ which is also an absurdity because $S_1 \cap I_0(f) = \emptyset$.

This finishes the proof of lemma 11. □

Lemma 12. Let $\gamma, f, \mathbf{g} : \{\mathbf{k}_1, \dots, \mathbf{k}_I\} \rightarrow \mathbb{N}$ as in the Proposition 2. An explicit version of $h_{f,\mathbf{g}} : \mathbb{N}^n \rightarrow \mathbb{C}$ defined in equation (III.37) is given by $h_{f,\mathbf{g}}(\mathbf{1}) = 1$,

$$h_{f,\mathbf{g}}(p^{\mathbf{k}_i}) = f(p^{\mathbf{k}_i})p^{-\mathbf{k}_i \cdot \gamma} - g_i \quad \text{if } i \in \llbracket 1, I \rrbracket \quad (\text{III.43})$$

$$\begin{aligned} h_{f,\mathbf{g}}(p^{\mathbf{v}}) &= -f(p^{\mathbf{v}})p^{-\mathbf{v} \cdot \gamma} \mathbf{1}_{S(f) \setminus I_0(f)}(\mathbf{v}) + \sum_{\substack{\mathbf{j} \leq \mathbf{g} \\ |\mathbf{j}|_1 \geq 2}} (-1)^{|\mathbf{j}|_1} \binom{\mathbf{g}}{\mathbf{j}} \mathbf{1}_{\{\mathbf{v}\}} \left(\sum_{i=1}^I j_i \mathbf{k}_i \right) \\ &+ \sum_{\substack{\mathbf{j} \leq \mathbf{g} \\ |\mathbf{j}|_1 \geq 1}} \sum_{\mathbf{u} \in S(f)} (-1)^{|\mathbf{j}|_1} \binom{\mathbf{g}}{\mathbf{j}} f(p^{\mathbf{u}}) p^{-\mathbf{u} \cdot \gamma} \mathbf{1}_{\{\mathbf{v}\}} \left(\mathbf{u} + \sum_{i=1}^I j_i \mathbf{k}_i \right), \text{ otherwise.} \end{aligned} \quad (\text{III.44})$$

Proof. Let $\mathbf{j} = (j_1, \dots, j_I) \in \mathbb{N}_0^I$, we set $\mathbf{g} = (g_1, \dots, g_I)$ and $\binom{\mathbf{g}}{\mathbf{j}} = \prod_{i=1}^I \binom{g_i}{j_i}$.

Developing the product defining the function $\mathbf{s} \mapsto \sum_{\mathbf{v} \in \mathbb{N}_0^n} \frac{h_{f,\mathbf{g}}(p^{\mathbf{v}})}{p^{\mathbf{v} \cdot \mathbf{s}}}$ for $\sigma \gg \mathbf{1}$ implies that

$$\begin{aligned} \sum_{\mathbf{v} \in \mathbb{N}_0^n} \frac{h_{f,\mathbf{g}}(p^{\mathbf{v}})}{p^{\mathbf{v} \cdot \mathbf{s}}} &= \left(1 + \sum_{i=1}^I \frac{f(p^{\mathbf{k}_i})p^{-\mathbf{k}_i \cdot \gamma}}{p^{\mathbf{k}_i \cdot \mathbf{s}}} + \sum_{\mathbf{v} \in S(f) \setminus I_0(f)} \frac{f(p^{\mathbf{v}})p^{-\mathbf{v} \cdot \gamma}}{p^{\mathbf{v} \cdot \mathbf{s}}} \right) \times \\ &\left(1 - \sum_{i=1}^I \frac{g_i}{p^{\mathbf{k}_i \cdot \mathbf{s}}} + \sum_{\substack{\mathbf{j} \leq \mathbf{g} \\ |\mathbf{j}|_1 \geq 2}} \binom{\mathbf{g}}{\mathbf{j}} \frac{(-1)^{|\mathbf{j}|_1}}{p^{(\sum_{i=1}^I j_i \mathbf{k}_i) \cdot \mathbf{s}}} \right) \\ &= 1 + \sum_{i=1}^I \frac{f(p^{\mathbf{k}_i})p^{-\mathbf{k}_i \cdot \gamma} - g_i}{p^{\mathbf{k}_i \cdot \mathbf{s}}} + \sum_{\mathbf{v} \in S(f) \setminus I_0(f)} \frac{f(\mathbf{v})p^{-\mathbf{v} \cdot \gamma}}{p^{\mathbf{v} \cdot \mathbf{s}}} \\ &+ \sum_{\substack{\mathbf{j} \leq \mathbf{g} \\ |\mathbf{j}|_1 \geq 2}} \binom{\mathbf{g}}{\mathbf{j}} \frac{(-1)^{|\mathbf{j}|_1}}{p^{(\sum_{i=1}^I j_i \mathbf{k}_i) \cdot \mathbf{s}}} + \sum_{\substack{\mathbf{j} \leq \mathbf{g} \\ |\mathbf{j}|_1 \geq 1}} \sum_{\mathbf{v} \in S(f)} \binom{\mathbf{g}}{\mathbf{j}} \frac{(-1)^{|\mathbf{j}|_1} f(p^{\mathbf{v}})p^{-\mathbf{v} \cdot \gamma}}{p^{(\mathbf{v} + \sum_{i=1}^I j_i \mathbf{k}_i) \cdot \mathbf{s}}} \end{aligned}$$

It follows that $h_{f,\mathbf{g}}(\mathbf{1}) = 1$, $h_{f,\mathbf{g}}(p^{\mathbf{k}_i}) = f(p^{\mathbf{k}_i})p^{-\mathbf{k}_i \cdot \gamma} - g_i$ for $\mathbf{k}_i \in I_0(f)$ and

$$\begin{aligned} h_{f,\mathbf{g}}(p^{\mathbf{v}}) &= -f(p^{\mathbf{v}})p^{-\mathbf{v} \cdot \gamma} \mathbf{1}_{S(f) \setminus I_0(f)}(\mathbf{v}) + \sum_{\substack{\mathbf{j} \leq \mathbf{g} \\ |\mathbf{j}|_1 \geq 2}} (-1)^{|\mathbf{j}|_1} \binom{\mathbf{g}}{\mathbf{j}} \mathbf{1}_{\{\mathbf{v}\}} \left(\sum_{i=1}^I j_i \mathbf{k}_i \right) \\ &+ \sum_{\substack{\mathbf{j} \leq \mathbf{g} \\ |\mathbf{j}|_1 \geq 1}} \sum_{\mathbf{u} \in S(f)} (-1)^{|\mathbf{j}|_1} \binom{\mathbf{g}}{\mathbf{j}} f(p^{\mathbf{u}}) p^{-\mathbf{u} \cdot \gamma} \mathbf{1}_{\{\mathbf{v}\}} \left(\mathbf{u} + \sum_{i=1}^I j_i \mathbf{k}_i \right) \end{aligned} \quad (\text{III.45})$$

for $\mathbf{v} \notin I_0(f) \cup \{\mathbf{0}\}$. □

Lemma 13. We suppose that assumptions of lemma 12 hold. Let $\tilde{h}_{f,\mathbf{g}} : \mathbb{N}^n \rightarrow \mathbb{R}_+$ be the multiplicative function defined by $\tilde{h}_{f,\mathbf{g}}(\mathbf{1}) = 1$,

$$\tilde{h}_{f,\mathbf{g}}(p^{\mathbf{k}_i}) = |f(p^{\mathbf{k}_i})p^{-\mathbf{k}_i \cdot \gamma} - g_i|, \text{ for } i \in \llbracket 1, I \rrbracket \quad (\text{on } I_0(f)); \quad (\text{III.46})$$

$$\begin{aligned} \tilde{h}_{f,\mathbf{g}}(p^{\mathbf{v}}) &= |f(p^{\mathbf{v}})|p^{-\mathbf{v} \cdot \gamma} \mathbf{1}_{S(f) \setminus I_0(f)}(\mathbf{v}) + \sum_{\substack{\mathbf{j} \leq \mathbf{g} \\ |\mathbf{j}|_1 \geq 2}} \binom{\mathbf{g}}{\mathbf{j}} \mathbf{1}_{\{\mathbf{v}\}} \left(\sum_{i=1}^I j_i \mathbf{k}_i \right) \\ &+ \sum_{\substack{\mathbf{j} \leq \mathbf{g} \\ |\mathbf{j}|_1 \geq 1}} \sum_{\mathbf{u} \in S(f)} \binom{\mathbf{g}}{\mathbf{j}} |f(p^{\mathbf{u}})|p^{-\mathbf{u} \cdot \gamma} \mathbf{1}_{\{\mathbf{v}\}} \left(\mathbf{u} + \sum_{i=1}^I j_i \mathbf{k}_i \right), \text{ otherwise.} \end{aligned} \quad (\text{III.47})$$

Then, $|h_{f,\mathbf{g}}| \leq \tilde{h}_{f,\mathbf{g}}$ and $\mathcal{M}(\tilde{h}_{f,\mathbf{g}}; \mathbf{s})$ is convergent in the closed set $\{\operatorname{Re}(\mathbf{s}) \in \mathbf{c}' + \mathbb{R}_+^n\}$. Consequently, $\mathcal{M}(|h_{f,\mathbf{g}}|; \mathbf{s})$ and $\mathcal{M}(h_{f,\mathbf{g}}; \mathbf{s})$ are also convergent.

Proof. It's enough to prove that the series $\sum_{p \in \mathcal{P}} \sum_{\mathbf{v} \in (\mathbb{N}_0^n)^*} \frac{\tilde{h}_{f,\mathbf{g}}(p^\mathbf{v})}{p^{\mathbf{v} \cdot \boldsymbol{\sigma}}}$ is convergent for $\boldsymbol{\sigma} \in \mathbf{c}' + \mathbb{R}_+^n$.

Let $\boldsymbol{\sigma} \in \mathbf{c}' + \mathbb{R}_+^n$. We split the series into two terms:

- the first term is the sum running on $\mathbf{v} \in I_0(f)$,
- the second term is the sum running on $\mathbf{v} \in S(\tilde{h}) \setminus I_0(f)$, it will be dominated by three series as follows:
 - the first series is the sum running on $\mathbf{v} \in \{\sum_{i=1}^I j_i \mathbf{k}_i; \mathbf{j} \leq \mathbf{g}, |\mathbf{j}|_1 \geq 2\}$,
 - the second series is the sum running on $\mathbf{v} \in S(f) \setminus I_0(f)$,
 - the third series is the sum running on $\mathbf{v} \in \{\mathbf{u} + \sum_{i=1}^I j_i \mathbf{k}_i; \mathbf{u} \in S(f), \mathbf{j} \leq \mathbf{g}, |\mathbf{j}|_1 \geq 1\}$.

1. The sum running on \mathbf{v} in $I_0(f) = \{\mathbf{k}_1, \dots, \mathbf{k}_I\}$: we have

$$\sum_{p \in \mathcal{P}} \sum_{i=1}^I \frac{\tilde{h}_{f,\mathbf{g}}(p^{\mathbf{k}_i})}{p^{\mathbf{k}_i \cdot \boldsymbol{\sigma}}} \leq \sum_{i=1}^I \sum_{p \in \mathcal{P}} \frac{|f(p^{\mathbf{k}_i}) p^{-\mathbf{k}_i \cdot \boldsymbol{\gamma}} - g_i|}{p^{\mathbf{k}_i \cdot \mathbf{c}'}} \leq \sum_{i=1}^I \sum_{p \in \mathcal{P}} \frac{|f(p^{\mathbf{k}_i}) p^{-\mathbf{k}_i \cdot \boldsymbol{\gamma}} - g_i|}{p}$$

and the last sum is $O(1)$ since by hypothesis each series over \mathcal{P} is convergent.

2. The sum running on \mathbf{v} in $S(\tilde{h}_{f,\mathbf{g}}) \setminus I_0(f)$: In this case $\tilde{h}_{f,\mathbf{g}}(p^\mathbf{v})$ is splitting into three terms which leads to the three following sub-sums.

(a) For the first sub-sum, we have

$$\sum_{\substack{p \in \mathcal{P} \\ \mathbf{v} \in (\mathbb{N}^n)^*}} \sum_{\substack{\mathbf{j} \leq \mathbf{g} \\ |\mathbf{j}|_1 \geq 2}} \binom{\mathbf{g}}{\mathbf{j}} \frac{\mathbf{1}_{\{\mathbf{v}\}} \left(\sum_{i=1}^I j_i \mathbf{k}_i \right)}{p^{\mathbf{v} \cdot \boldsymbol{\sigma}}} \leq \sum_{\substack{\mathbf{j} \leq \mathbf{g} \\ |\mathbf{j}|_1 \geq 2}} \binom{\mathbf{g}}{\mathbf{j}} \sum_{p \in \mathcal{P}} \frac{1}{p^{(\sum_{i=1}^I j_i \mathbf{k}_i) \cdot \mathbf{c}'}} \leq \sum_{\substack{\mathbf{j} \leq \mathbf{g} \\ |\mathbf{j}|_1 \geq 2}} \binom{\mathbf{g}}{\mathbf{j}} \sum_{p \in \mathcal{P}} \frac{1}{p^2}$$

and the last series is $O(1)$.

(b) For the second sub-sum, we have

$$\sum_{p \in \mathcal{P}} \sum_{\mathbf{v} \in (\mathbb{N}_0^n)^*} \frac{|f(p^\mathbf{v})| p^{-\mathbf{v} \cdot \boldsymbol{\gamma}} \mathbf{1}_{S(f) \setminus I_0(f)}(\mathbf{v})}{p^{\mathbf{v} \cdot \boldsymbol{\sigma}}} = \sum_{p \in \mathcal{P}} \sum_{\mathbf{v} \in S(f) \setminus I_0(f)} \frac{|f(p^\mathbf{v})| p^{-\mathbf{v} \cdot \boldsymbol{\gamma}}}{p^{\mathbf{v} \cdot \boldsymbol{\sigma}}}$$

Set

$$S(\mathbf{v}) = \{p \in \mathcal{P}; f(p^\mathbf{v}) \neq 0\}, \quad \varepsilon = \frac{1}{4} \min \left\{ \min_{i \in [1, I]} \frac{1}{|\mathbf{k}_i|_1}; \min_{i \in [1, n]} c'_i; \min_{\mathbf{v} \in I(f) \setminus I_0(f)} (\mathbf{v} \cdot \mathbf{c}' - 1) \right\},$$

$$p(M) = \sup \{S(\mathbf{v}); \mathbf{v} \in \mathbb{N}_0^n \setminus \{\mathbf{0}\}, |\mathbf{v}|_1 \leq M\} \quad \text{and} \quad M = \mathbf{1} + \left\lfloor \frac{\mathbf{8}}{\varepsilon} + \max_{i \in [1, I]} |\mathbf{k}_i|_1 \right\rfloor.$$

We remark that $S(\mathbf{v})$ is a finite set if and only if $\mathbf{v} \notin S_\infty(f) \cup \{\mathbf{0}\}$.

i. For $\mathbf{v} \in S(f)$ and $|\mathbf{v}|_1 \geq M$ we have:

- $|f(p^\mathbf{v})| p^{-\mathbf{v} \cdot \boldsymbol{\gamma}} \ll p^{\varepsilon |\mathbf{v}|_1}$, because f has growth exponent $\boldsymbol{\gamma}$ (see definition 9);

• $p^{-\mathbf{v}\cdot\boldsymbol{\sigma}} \ll p^{-2\varepsilon M - 2\varepsilon|\mathbf{v}|_1}$, because $\mathbf{v}\cdot\boldsymbol{\sigma} \geq \mathbf{v}\cdot\mathbf{c}' \geq |\mathbf{v}|_1 \inf c'_i \geq 2\varepsilon M + 2\varepsilon|\mathbf{v}|_1$;
it follows

$$\begin{aligned} \sum_{\substack{\mathbf{v} \in S(f) \\ |\mathbf{v}|_1 \geq M}} \sum_{p \in \mathcal{P}} \frac{|f(p^{\mathbf{v}})| p^{-\mathbf{v}\cdot\boldsymbol{\gamma}}}{p^{\mathbf{v}\cdot\boldsymbol{\sigma}}} &= \sum_{p \in \mathcal{P}} \sum_{k \geq M} \sum_{\substack{\mathbf{v} \in S(f) \\ |\mathbf{v}|_1 = k}} \frac{|f(p^{\mathbf{v}})| p^{-\mathbf{v}\cdot\boldsymbol{\gamma}}}{p^{\mathbf{v}\cdot\boldsymbol{\sigma}}} \\ &\ll \sum_{p \in \mathcal{P}} \sum_{k \geq M} \frac{\binom{k+n-1}{n-1} p^{\varepsilon k}}{p^{2\varepsilon M + 2\varepsilon k}} \\ &\leq \sum_{p \in \mathcal{P}} \frac{1}{p^8} \sum_{k \geq M} \frac{\binom{k+n-1}{n-1}}{p^{\varepsilon k}} \end{aligned} \quad (\text{III.48})$$

and the last series is $O_\varepsilon(1)$.

ii. For $\mathbf{v} \in S(f) \setminus S_\infty(f)$ and $|\mathbf{v}|_1 \leq M$ we have $|f(p^{\mathbf{v}})| p^{-\mathbf{v}\cdot\boldsymbol{\gamma}} \ll p^{\varepsilon|\mathbf{v}|_1}$ and $p^{-\mathbf{v}\cdot\boldsymbol{\sigma}} \leq p^{-\mathbf{v}\cdot\mathbf{c}'} \ll p^{-2\varepsilon|\mathbf{v}|_1}$,

$$\sum_{\substack{\mathbf{v} \in S(f) \setminus S_\infty(f) \\ |\mathbf{v}|_1 \leq M}} \sum_{p \in \mathcal{P}} \frac{|f(p^{\mathbf{v}})| p^{-\mathbf{v}\cdot\boldsymbol{\gamma}}}{p^{\mathbf{v}\cdot\boldsymbol{\sigma}}} \ll \sum_{\substack{\mathbf{v} \in S(f) \setminus S_\infty(f) \\ |\mathbf{v}|_1 \leq M}} \sum_{p \in S(\mathbf{v})} \frac{1}{p^{\varepsilon|\mathbf{v}|_1}} \leq \sum_{\substack{p \in \mathcal{P} \\ p \leq p(M)}} \sum_{k \leq M} \frac{\binom{k+n-1}{n-1}}{p^{\varepsilon k}}$$

and the last series is $O_\varepsilon(1)$.

iii. For $\mathbf{v} \in S_\infty(f) \setminus I_0(f)$ and $|\mathbf{v}|_1 \leq M$. By lemma 11 there exists $\delta > 0$ such that $(1 - \delta)\mathbf{v}\cdot\mathbf{c}' > 1 + \delta$, ainsi $|f(p^{\mathbf{v}})| p^{-\mathbf{v}\cdot\boldsymbol{\gamma}} \ll p^{\varepsilon\delta|\mathbf{v}|_1}$ and $\mathbf{v}\cdot\boldsymbol{\sigma} \geq \mathbf{v}\cdot\mathbf{c}' = (1 - \delta)\mathbf{v}\cdot\mathbf{c}' + \delta\mathbf{v}\cdot\mathbf{c}' \geq 1 + \delta + 2\varepsilon\delta|\mathbf{v}|_1$,

$$\sum_{\substack{\mathbf{v} \in S_\infty(f) \setminus I_0(f) \\ |\mathbf{v}|_1 \leq M}} \sum_{p \in \mathcal{P}} \frac{|f(p^{\mathbf{v}})| p^{-\mathbf{v}\cdot\boldsymbol{\gamma}}}{p^{\mathbf{v}\cdot\boldsymbol{\sigma}}} \ll \sum_{p \in \mathcal{P}} \sum_{k \leq M} \frac{\binom{k+n-1}{n-1}}{p^{1+\delta+\varepsilon\delta k}} \ll \sum_{p \in \mathcal{P}} \frac{1}{p^{1+\delta}} \sum_{k \leq M} \frac{\binom{k+n-1}{n-1}}{p^{\varepsilon\delta k}}$$

and the last term is $O_{\varepsilon,\delta}(1)$.

The three points (i),(ii) and (iii) imply that $\sum_{p \in \mathcal{P}} \sum_{\mathbf{v} \in S(f) \setminus I_0(f)} \frac{|f(p^{\mathbf{v}})| p^{-\mathbf{v}\cdot\boldsymbol{\gamma}}}{p^{\mathbf{v}\cdot\boldsymbol{\sigma}}} = O_{\varepsilon,\delta}(1)$.

(c) For the third sub-sum, we have

$$\begin{aligned} \sum_{\substack{p \in \mathcal{P} \\ \mathbf{v} \in (\mathbb{N}_0^n)^*}} \sum_{\substack{\mathbf{j} \leq \mathbf{g} \\ |\mathbf{j}|_1 \geq 1}} \sum_{\mathbf{u} \in S(f)} \binom{\mathbf{g}}{\mathbf{j}} \frac{|f(p^{\mathbf{u}})| p^{-\mathbf{u}\cdot\boldsymbol{\gamma}} \mathbb{1}_{\{\mathbf{v}\}} \left(\mathbf{u} + \sum_{i=1}^I j_i \mathbf{k}_i \right)}{p^{\mathbf{v}\cdot\boldsymbol{\sigma}}} &\leq \sum_{\substack{\mathbf{j} \leq \mathbf{g} \\ |\mathbf{j}|_1 \geq 1}} \binom{\mathbf{g}}{\mathbf{j}} \sum_{\substack{\mathbf{u} \in S(f) \\ p \in \mathcal{P}}} \frac{|f(p^{\mathbf{u}})| p^{-\mathbf{u}\cdot\boldsymbol{\gamma}}}{p^{(\mathbf{u} + \sum_{i=1}^I j_i \mathbf{k}_i) \cdot \mathbf{c}'}} \\ &\leq \sum_{\substack{\mathbf{j} \leq \mathbf{g} \\ |\mathbf{j}|_1 \geq 1}} \binom{\mathbf{g}}{\mathbf{j}} \sum_{p \in \mathcal{P}} \sum_{\mathbf{u} \in S(f) \setminus I_0(f)} \frac{|f(p^{\mathbf{u}})| p^{-\mathbf{u}\cdot\boldsymbol{\gamma}}}{p^{1+\mathbf{u}\cdot\mathbf{c}}} + \sum_{\substack{\mathbf{j} \leq \mathbf{g} \\ |\mathbf{j}|_1 \geq 1}} \binom{\mathbf{g}}{\mathbf{j}} \sum_{i=1}^I \sum_{p \in \mathcal{P}} \frac{|f(p^{\mathbf{k}_i})| p^{-\mathbf{k}_i \cdot \boldsymbol{\gamma}}}{p^{1+\mathbf{k}_i \cdot \mathbf{c}'}} \end{aligned}$$

Moreover, we have the two following points:

i. it follows from b) that $\sum_{p \in \mathcal{P}} \sum_{\mathbf{v} \in S(f) \setminus I_0(f)} \frac{|f(p^{\mathbf{v}})| p^{-\mathbf{v}\cdot\boldsymbol{\gamma}}}{p^{\mathbf{v}\cdot\boldsymbol{\sigma}}} = O_{\varepsilon,\delta}(1)$,

ii. Since $|f(p^{\mathbf{k}_i})| p^{-\boldsymbol{\gamma}\cdot\mathbf{k}_i} \ll p^{\varepsilon|\mathbf{k}_i|} \ll p^{1/4}$, it follows that $\sum_{i=1}^I \sum_{p \in \mathcal{P}} \frac{|f(p^{\mathbf{k}_i})| p^{-\boldsymbol{\gamma}\cdot\mathbf{k}_i}}{p^{1+\mathbf{k}_i \cdot \mathbf{c}'}} \ll \sum_{p \in \mathcal{P}} \frac{1}{p^{7/4}}$ and the last series is $O(1)$.

and these two points imply that $\sum_{p \in \mathcal{P}} \sum_{\mathbf{u} \in S(f)} \frac{|f(p^{\mathbf{u}})| p^{-\mathbf{u} \cdot \boldsymbol{\gamma}}}{p^{(\mathbf{u} + \sum_{i=1}^I j_i \mathbf{k}_i) \cdot \mathbf{c}'}} = O_{\varepsilon, \delta}(1)$.

Combining the points a), b) and c) implies that $\sum_{p \in \mathcal{P}} \sum_{\mathbf{v} \in S(\tilde{h}_{f; \mathbf{g}}) \setminus I_0(f)} \frac{\tilde{h}_{f; \mathbf{g}}(p^{\mathbf{v}})}{p^{\mathbf{v} \cdot \boldsymbol{\sigma}}} = O_{\varepsilon, \delta}(1)$.

The proof of lemma 13 follows from the estimates obtained in points 1 and 2. \square

End of proof of proposition 2:

Lemma 12 give us explicit values of $h_{f; \mathbf{g}}$, Lemma 13 give us an upper bound function $\tilde{h}_{f; \mathbf{g}}$ for $|h_{f; \mathbf{g}}|$ such that the Dirichlet's series $Z(\tilde{h}_{f; \mathbf{g}}; \mathbf{s})$ converges in the closed set $\{\operatorname{Re}(\mathbf{s}) \in \mathbf{c}' + \mathbb{R}_+^n\}$. We deduce that $Z(h_{f; \mathbf{g}}; \mathbf{s})$ is also convergent in the same closed set $\{\operatorname{Re}(\mathbf{s}) \in \mathbf{c}' + \mathbb{R}_+^n\}$. This finishes the proof of proposition 2. \square

In the following sections we use the notations as in the above section, see (III.34), (III.35) and (III.36).

At first we give a lemma connecting $Z(\tau_{\mathbf{g}}; \mathbf{s})$ and the Riemann zeta function.

Lemma 14. *Let $\mathbf{g} : \{\mathbf{k}_1, \dots, \mathbf{k}_I\} \subset \mathbb{N}^n \setminus \{\mathbf{0}\} \rightarrow \mathbb{N}$, we set $g_i = \mathbf{g}(\mathbf{k}_i)$ for each $i \in \llbracket 1, I \rrbracket$. The function $\tau_{\mathbf{g}}$, see (III.1), verify*

$$Z(\tau_{\mathbf{g}}; \mathbf{s}) = \sum_{\mathbf{m} \in \mathbb{N}^n} \frac{\tau_{\mathbf{g}}(\mathbf{m})}{\mathbf{m}^{\mathbf{s}}} = \prod_{i=1}^I \zeta^{g_i}(\mathbf{k}_i \cdot \mathbf{s}) \text{ for } \operatorname{Re}(\mathbf{s}) \gg 1$$

Proof. For $\operatorname{Re}(\mathbf{s}) \gg 1$ we have

$$\begin{aligned} \prod_{i=1}^I \zeta^{g_i}(\mathbf{k}_i \cdot \mathbf{s}) &= \prod_{p \in \mathcal{P}} \prod_{i=1}^I \left(1 - \frac{1}{p^{\mathbf{k}_i \cdot \mathbf{s}}}\right)^{-g_i} = \prod_{p \in \mathcal{P}} \prod_{i=1}^I \left(\sum_{j_i \geq 0} \binom{-g_i}{j_i} \frac{(-1)^{j_i}}{p^{j_i \mathbf{k}_i \cdot \mathbf{s}}}\right) \\ &= \prod_{p \in \mathcal{P}} \left(\sum_{\mathbf{j} \in \mathbb{N}^I} \frac{(-1)^{|\mathbf{j}|} \prod_{i=1}^I \binom{-g_i}{j_i}}{p^{(\sum_{i=1}^I j_i \mathbf{k}_i) \cdot \mathbf{s}}}\right) = \prod_{p \in \mathcal{P}} \left(\sum_{\mathbf{j} \in \mathbb{N}^I} \frac{\prod_{i=1}^I \binom{g_i + j_i - 1}{j_i}}{p^{(\sum_{i=1}^I j_i \mathbf{k}_i) \cdot \mathbf{s}}}\right) \\ &= \prod_{p \in \mathcal{P}} \left(\sum_{\mathbf{w} \in \mathbb{N}_0^n} p^{-\mathbf{w} \cdot \mathbf{s}} \sum_{\substack{\mathbf{j} \in \mathbb{N}^I \\ \sum_{i=1}^I j_i \mathbf{k}_i = \mathbf{w}}} \prod_{i=1}^I \binom{g_i + j_i - 1}{j_i}\right) \\ &= \prod_{p \in \mathcal{P}} \left(\sum_{\mathbf{w} \in \mathbb{N}_0^n} \tau_{\mathbf{g}}(p^{\mathbf{w}}) p^{-\mathbf{w} \cdot \mathbf{s}}\right) = \sum_{\mathbf{m} \in \mathbb{N}^n} \frac{\tau_{\mathbf{g}}(\mathbf{m})}{\mathbf{m}^{\mathbf{s}}} \end{aligned}$$

\square

We also need the following lemma:

Lemma 15. *Let $f \in \mathcal{D}(\boldsymbol{\gamma}; \mathbf{g}; 1)$ and $h_{f; \mathbf{g}} : \mathbb{N}^n \rightarrow \mathbb{C}$ the function defined in Lemma 12. Then, for all $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{N}^n$*

$$f(\mathbf{m}) = \mathbf{m}^{\boldsymbol{\gamma}} (h_{f; \mathbf{g}} * \tau_{\mathbf{g}})(\mathbf{m})$$

Proof. For $\operatorname{Re}(\mathbf{s}) \gg 1$, we have

$$\begin{aligned} Z(f_{-\boldsymbol{\gamma}}; \mathbf{s}) &= Z(f; \mathbf{s} + \boldsymbol{\gamma}) = \prod_{p \in \mathcal{P}} \left[\left(1 + \sum_{\nu \in (\mathbb{N}_0^n)^*} \frac{f(p^{\nu}) p^{-\nu \cdot \boldsymbol{\gamma}}}{p^{\nu \cdot \mathbf{s}}}\right) \prod_{i=1}^I \left(1 - \frac{1}{p^{\mathbf{k}_i \cdot \mathbf{s}}}\right)^{g_i} \right] \\ &\quad \times \prod_{i=1}^I \zeta(\mathbf{k}_i \cdot \mathbf{s})^{g_i} \\ &= Z(h_{f; \mathbf{g}}; \mathbf{s}) Z(\tau_{\mathbf{g}}; \mathbf{s}) = Z(h_{f; \mathbf{g}} * \tau_{\mathbf{g}}; \mathbf{s}) \end{aligned}$$

it follows that $f(\mathbf{m})\mathbf{m}^{-\gamma} = (h_{f;\mathbf{g}} * \tau_{\mathbf{g}})(\mathbf{m})$ consequently $f(\mathbf{m}) = \mathbf{m}^{\gamma}(h_{f;\mathbf{g}} * \tau_{\mathbf{g}})(\mathbf{m})$. \square

Remark 12. For $\operatorname{Re}(\mathbf{s}) > \mathbf{c}'$ we have

$$Z(h_{f;\mathbf{g}}; \mathbf{s} + \mathbf{c}') = \mathcal{H}(f; \mathbf{c}'; \gamma; \mathbf{s}) \quad (\text{III.49})$$

where $\mathbf{s} \mapsto \mathcal{H}(f; \mathbf{c}'; \gamma; \mathbf{s})$ was defined by equation (III.23).

6 Proof of Theorem 13

We also need the next proposition:

Proposition 3. Let $(\alpha, \beta) \in \operatorname{conv}^*\{\mathbf{a}\mathbf{e}_1, \mathbf{b}\mathbf{e}_2\}$. Then, we have

$$\sum_{\mathbf{m} \leq \mathbf{x}} \mathbf{m}^{\gamma} \tau_{\mathbf{g}}(\mathbf{m}) = \mathbf{x}^{\mathbf{c}} \left(\frac{\log(\min\{x_1^{\beta}; x_2^{\alpha}\})}{\mathbf{a}\mathbf{b}\alpha\beta c_1 c_2} + O(1) \right) \quad (\text{III.50})$$

In particular, $\sup_{\mathbf{x} \in [1, +\infty]^2} \left| \frac{\sum_{\mathbf{m} \leq \mathbf{x}} \mathbf{m}^{\gamma} \tau_{\mathbf{g}}(\mathbf{m})}{\mathbf{x}^{\mathbf{c}} \log(3 + \min\{x_1^{\beta}; x_2^{\alpha}\})} \right|$ is bounded.

Proof of proposition 3:

Let $\kappa_j > 0$ such that $x_j^{\kappa_j} = O(1)$ for all $j \in \llbracket 1, 2 \rrbracket$. It follows from equation (III.34) that

$$\begin{aligned} \Psi((\tau_{\mathbf{g}})_{\gamma}; \mathbf{x}; \boldsymbol{\xi}) &= \frac{1}{(2\pi i)^2} \int_{\mathbf{s} \in (\mathbf{c}' + \gamma + \boldsymbol{\kappa})} \left(\sum_{\mathbf{m} \in \mathbb{N}^2} \frac{(\tau_{\mathbf{g}})_{\gamma}(\mathbf{m})}{\mathbf{m}^{\mathbf{s}}} \right) \prod_{j=1}^2 x_j^{s_j} \frac{\widehat{\eta}_{\boldsymbol{\xi}_j}(s_j)}{s_j} ds \\ &= \frac{1}{(2\pi i)^2} \int_{\mathbf{s} \in (\mathbf{c}' + \gamma + \boldsymbol{\kappa})} \left(\sum_{\mathbf{m} \in \mathbb{N}^2} \frac{\tau_{\mathbf{g}}(\mathbf{m})}{\mathbf{m}^{\mathbf{s} - \gamma}} \right) \prod_{j=1}^2 x_j^{s_j} \frac{\widehat{\eta}_{\boldsymbol{\xi}_j}(s_j)}{s_j} ds \\ &= \frac{1}{(2\pi i)^2} \int_{\mathbf{s} \in (\mathbf{c}' + \boldsymbol{\kappa})} Z_{\mathbf{g}}(\mathbf{s}) \prod_{j=1}^2 x_j^{s_j + \gamma_j} \frac{\widehat{\eta}_{\boldsymbol{\xi}_j}(s_j + \gamma_j)}{s_j + \gamma_j} ds \\ &= \frac{\mathbf{x}^{\mathbf{c}' + \gamma}}{(2\pi i)^2} \int_{\mathbf{s} \in (\boldsymbol{\kappa})} Z_{\mathbf{g}}(\mathbf{c}' + \mathbf{s}) \prod_{j=1}^2 x_j^{s_j} \frac{\widehat{\eta}_{\boldsymbol{\xi}_j}(s_j + \mathbf{c}'_j + \gamma_j)}{s_j + \mathbf{c}'_j + \gamma_j} ds \end{aligned}$$

We set $\mathbf{c} = \mathbf{c}' + \gamma$. We split last integral into two terms

$$\Psi((\tau_{\mathbf{g}})_{\gamma}; \mathbf{x}; \boldsymbol{\xi}) = E((\tau_{\mathbf{g}})_{\gamma}; \mathbf{x}; \boldsymbol{\xi}; T) + \Psi((\tau_{\mathbf{g}})_{\gamma}; \mathbf{x}; \boldsymbol{\xi}; T)$$

the first term is called an error term and the second is called principal term, where

$$E((\tau_{\mathbf{g}})_{\gamma}; \mathbf{x}; \boldsymbol{\xi}; T) = \frac{\mathbf{x}^{\mathbf{c}}}{(2\pi i)^2} \int_{\substack{\mathbf{s} \in (\boldsymbol{\kappa}) \\ \|\boldsymbol{\tau}\|_{\infty} > T}} Z_{\mathbf{g}}(\mathbf{c}' + \mathbf{s}) \prod_{j=1}^2 x_j^{s_j} \frac{\widehat{\eta}_{\boldsymbol{\xi}_j}(s_j + \mathbf{c}_j)}{s_j + \mathbf{c}_j} ds \quad (\text{III.51})$$

$$\Psi((\tau_{\mathbf{g}})_{\gamma}; \mathbf{x}; \boldsymbol{\xi}; T) = \frac{\mathbf{x}^{\mathbf{c}}}{(2\pi i)^2} \int_{\substack{\mathbf{s} \in (\boldsymbol{\kappa}) \\ \|\boldsymbol{\tau}\|_{\infty} \leq T}} Z_{\mathbf{g}}(\mathbf{c}' + \mathbf{s}) \prod_{j=1}^2 x_j^{s_j} \frac{\widehat{\eta}_{\boldsymbol{\xi}_j}(s_j + \mathbf{c}_j)}{s_j + \mathbf{c}_j} ds \quad (\text{III.52})$$

Bounding the error term $E((\tau_{\mathbf{g}})_{\gamma}; \mathbf{x}; \boldsymbol{\xi}; T)$

Let $T > 1$. Set

$$\begin{aligned} F(\boldsymbol{\sigma}; \boldsymbol{\tau}) &:= Z_{\mathbf{g}}(\mathbf{c}' + \mathbf{s}) \prod_{j=1}^2 \frac{\widehat{\eta}_{\xi_j}(s_j + c_j)}{s_j + c_j} \\ &= \frac{\Theta_{\mathbf{g}}(\boldsymbol{\sigma} + i\boldsymbol{\tau})}{\prod_{i=1}^3 [\mathbf{k}_i \cdot \boldsymbol{\sigma} + i\mathbf{k}_i \cdot \boldsymbol{\tau}]} \prod_{j=1}^2 \frac{\widehat{\eta}_{\xi_j}(\sigma_j + c_j + i\tau_j)}{\sigma_j + c_j + i\tau_j} \end{aligned} \quad (\text{III.53})$$

where $\mathbf{k}_1 = a\mathbf{e}_1$, $\mathbf{k}_2 = b\mathbf{e}_2$ and $\mathbf{k}_3 = \alpha\mathbf{e}_1 + \beta\mathbf{e}_2$.

We have the uniform estimate for $\text{Re}(\mathbf{s})$ in compact sets

$$\frac{1}{|\mathbf{k}_i \cdot \mathbf{s}|} \leq \frac{\sqrt{2} \max\{1, |\mathbf{k}_i \cdot \boldsymbol{\sigma}|^{-1}\}}{1 + |\mathbf{k}_i \cdot \boldsymbol{\tau}|}, \quad \text{if } \mathbf{k}_i \cdot \boldsymbol{\sigma} > 0 \quad (\text{III.54})$$

From estimates (III.32) and (III.40), we have for $\mathbf{s} \in (\boldsymbol{\kappa})$ and $|\boldsymbol{\tau}|_1 \geq T$ that:

$$\begin{aligned} F(\boldsymbol{\sigma}; \boldsymbol{\tau}) &\ll_{\varepsilon} \frac{(1 + |\boldsymbol{\tau}|_1)^{\varepsilon} \prod_{i=1}^3 \max\{1, |\mathbf{k}_i \cdot \boldsymbol{\sigma}|^{-1}\}}{\prod_{j=1}^2 (1 + |\tau_j|) \max\{1, \xi_j(1 + |\tau_j|)\}} \\ &\ll_{\varepsilon} \frac{(1 + |\boldsymbol{\tau}|_1)^{\varepsilon} \max\{1, \frac{1}{a\sigma_1}\} \max\{1, \frac{1}{b\sigma_2}\} \max\{\frac{1}{\alpha\sigma_1 + \beta\sigma}\}}{\prod_{j=1}^2 (1 + |\tau_j|) \max\{1, \xi_j(1 + |\tau_j|)\}} \end{aligned}$$

uniformly in $\boldsymbol{\sigma}$ in compact sets.

Particularly for $|\tau_2| \leq |\tau_1|$ we have:

$$\begin{aligned} F(\boldsymbol{\sigma}; \boldsymbol{\tau}) &\ll_{\varepsilon} \frac{(1 + |\boldsymbol{\tau}|_1)^{\varepsilon} \kappa_1^{-1} \kappa_2^{-1} (\kappa_1 + \kappa_2)^{-1}}{\xi_1 (1 + |\tau_1|)^2 (1 + |\tau_2|)} \\ &\ll_{\varepsilon} \frac{\kappa_1^{-1} \kappa_2^{-1} (\kappa_1 + \kappa_2)^{-1}}{\xi_1 (1 + |\tau_1|)^{2-2\varepsilon} (1 + |\tau_2|)^{1+\varepsilon}} \end{aligned}$$

and for $|\tau_1| \leq |\tau_2|$ we have:

$$\begin{aligned} F(\boldsymbol{\sigma}; \boldsymbol{\tau}) &\ll_{\varepsilon} \frac{(1 + |\boldsymbol{\tau}|_1)^{\varepsilon} \kappa_1^{-1} \kappa_2^{-1} (\kappa_1 + \kappa_2)^{-1}}{\xi_2 (1 + |\tau_1|) (1 + |\tau_2|)^2} \\ &\ll_{\varepsilon} \frac{\kappa_1^{-1} \kappa_2^{-1} (\kappa_1 + \kappa_2)^{-1}}{\xi_2 (1 + |\tau_1|)^{1+\varepsilon} (1 + |\tau_2|)^{2-2\varepsilon}} \end{aligned}$$

We deduce the following estimate of the error term:

$$\begin{aligned} \frac{E((\tau_{\mathbf{g}})_{\gamma}; \mathbf{x}; \boldsymbol{\xi}; T)}{\mathbf{x}^{\mathbf{c}}} &= \frac{x_1^{\kappa_1} x_2^{\kappa_2}}{(2\pi)^2} \int_{\tau_1=T}^{+\infty} \int_{\tau_2=-\tau_1}^{\tau_1} [F(\boldsymbol{\kappa}; \tau_1, \tau_2) x^{i\tau_1} x^{i\tau_2} + F(\boldsymbol{\kappa}; -\tau_1, \tau_2) x^{-i\tau_1} x^{i\tau_2}] d\tau_2 d\tau_1 \\ &\quad + \frac{x_1^{\kappa_1} x_2^{\kappa_2}}{(2\pi)^2} \int_{\tau_2=T}^{+\infty} \int_{\tau_1=-\tau_2}^{\tau_2} [F(\boldsymbol{\kappa}; \tau_1, \tau_2) x^{i\tau_1} x^{i\tau_2} + F(\boldsymbol{\kappa}; \tau_1, -\tau_2) x^{i\tau_1} x^{-i\tau_2}] d\tau_1 d\tau_2 \\ &\ll_{\varepsilon} \frac{\kappa_1^{-1} \kappa_2^{-1} (\kappa_1 + \kappa_2)^{-1}}{\xi_1} \int_{\tau_1=T}^{+\infty} \int_{\tau_2=-\tau_1}^{\tau_1} \frac{d\boldsymbol{\tau}}{(1 + |\tau_1|)^{2-2\varepsilon} (1 + |\tau_2|)^{1+\varepsilon}} \\ &\quad + \frac{\kappa_1^{-1} \kappa_2^{-1} (\kappa_1 + \kappa_2)^{-1}}{\xi_2} \int_{\tau_2=T}^{+\infty} \int_{\tau_1=-\tau_2}^{\tau_2} \frac{d\boldsymbol{\tau}}{(1 + |\tau_1|)^{1+\varepsilon} (1 + |\tau_2|)^{2-2\varepsilon}} \\ &\ll_{\varepsilon} \frac{\kappa_1^{-1} \kappa_2^{-1} (\kappa_1 + \kappa_2)^{-1}}{(1 + T)^{1-2\varepsilon} \min\{\xi_1, \xi_2\}} \end{aligned} \quad (\text{III.55})$$

Study of the main term $\Psi((\tau_{\mathbf{g}})_{\gamma}; \mathbf{x}; \boldsymbol{\xi}; T)$

We are looking for the main term of $\Psi((\tau_{\mathbf{g}})_{\gamma}; \mathbf{x}; \boldsymbol{\xi}; T)$ for this we will use the residue theorem. To do this, we will integrate with respect to one of the variables and then with respect to the other.

We assume first that $x_2^\alpha \leq x_1^\beta$. We will integrate first with respect to the variable s_1 . Let $\varepsilon \in]0; 1/16[$, $N_1 = 2 \max \left\{ 1; \frac{\beta}{\alpha} \right\}$,

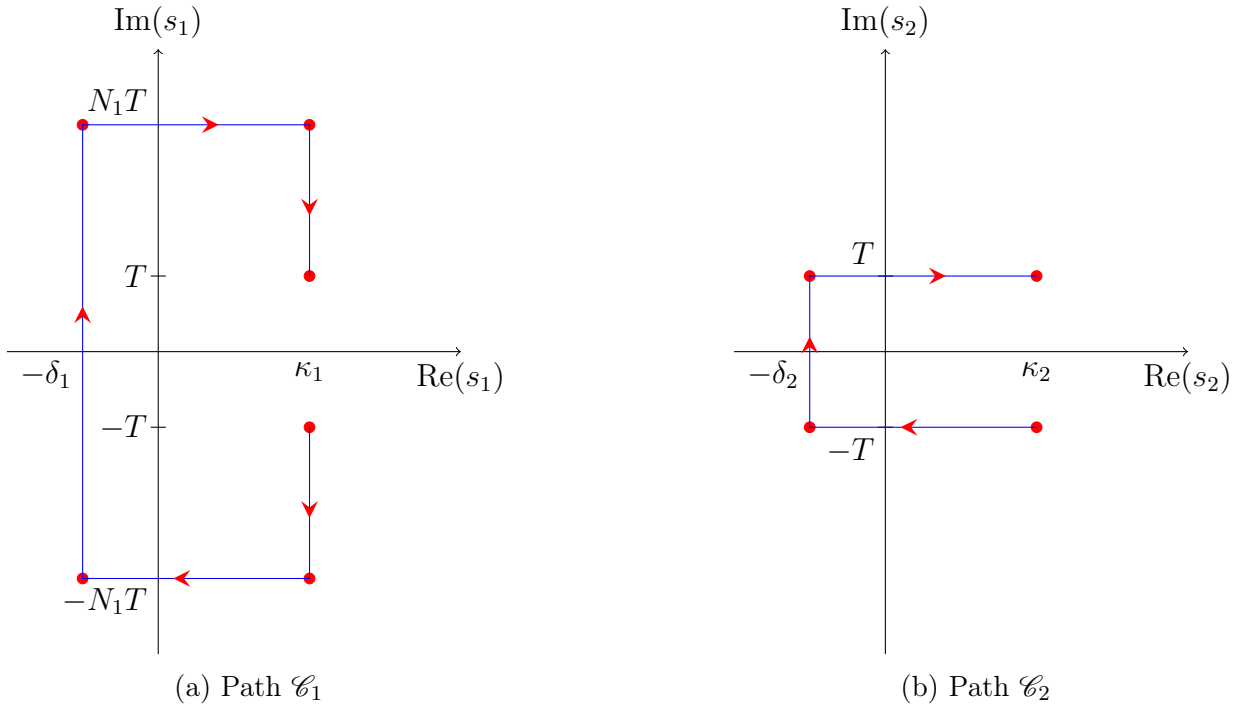
$$\delta_1 = \min \left\{ \frac{1}{4(a + \alpha)}; \frac{\beta}{2ab}; \frac{c'_1 + \gamma_1}{2}; \frac{\beta}{\alpha} \frac{c'_2 + \gamma_2}{2} \right\} \text{ and}$$

$$\delta_2 = \min \left\{ \frac{1}{4(b + \beta)}; \frac{\alpha}{2a\beta}; \frac{c'_2 + \gamma_2}{2}; \frac{\alpha}{\beta} \frac{c'_1 + \gamma_1}{2} \right\},$$

we consider the two polygonal paths \mathcal{C}_1 and \mathcal{C}_2 , see figures (a) and (b) below, defined by

$$\mathcal{C}_1 := \kappa_1 - iT \rightarrow \kappa_1 - iN_1T \rightarrow -\delta_1 - iN_1T \rightarrow -\delta_1 + iN_1T \rightarrow \kappa_1 + iN_1T \rightarrow \kappa_1 + iT$$

$$\mathcal{C}_2 := \kappa_2 - iT \rightarrow -\delta_2 - iT \rightarrow -\delta_2 + iT \rightarrow \kappa_2 + iT.$$



Applying the residue theorem (in the variable s_1) implies that

$$\begin{aligned} \frac{\Psi((\tau_{\mathbf{g}})_{\gamma}; \mathbf{x}; \boldsymbol{\xi}; T)}{\mathbf{x}^{\mathbf{c}}} &= \frac{1}{2\pi i} \int_{\kappa_2 - iT}^{\kappa_2 + iT} \operatorname{Res}_{s_1=0} \left(\frac{\Theta_{\mathbf{g}}(\mathbf{s}) x_1^{s_1} x_2^{s_2}}{\operatorname{abs}_1 s_2 (\alpha s_1 + \beta s_2)} \prod_{j=1}^2 \frac{\widehat{\eta}_{\xi_j}(s_j + c_j)}{s_j + c_j} \right) ds_2 \\ &+ \frac{1}{2\pi i} \int_{\kappa_2 - iT}^{\kappa_2 + iT} \operatorname{Res}_{s_1 = -\frac{\beta}{\alpha} s_2} \left(\frac{\Theta_{\mathbf{g}}(\mathbf{s}) x_1^{s_1} x_2^{s_2}}{\operatorname{abs}_1 s_2 (\alpha s_1 + \beta s_2)} \prod_{j=1}^2 \frac{\widehat{\eta}_{\xi_j}(s_j + c_j)}{s_j + c_j} \right) ds_2 \\ &+ \frac{1}{(2\pi i)^2} \int_{\kappa_2 - iT}^{\kappa_2 + iT} \int_{s_1 \in \mathcal{C}_1} \frac{\Theta_{\mathbf{g}}(\mathbf{s}) x_1^{s_1} x_2^{s_2}}{\operatorname{abs}_1 s_2 (\alpha s_1 + \beta s_2)} \prod_{j=1}^2 \frac{\widehat{\eta}_{\xi_j}(s_j + c_j)}{s_j + c_j} ds_2 ds_1 \end{aligned}$$

The computation of the residues implies then that

$$\frac{\Psi((\tau_{\mathbf{g}})_{\gamma}; \mathbf{x}; \boldsymbol{\xi}; T)}{\mathbf{x}^{\mathbf{c}}} = \frac{1}{2\pi i a b \beta} \int_{\kappa_2 - iT}^{\kappa_2 + iT} \frac{\Theta_{\mathbf{g}}(0, s_2)}{s_2^2} \frac{\widehat{\eta}_{\xi_1}(c_1)}{c_1} \frac{\widehat{\eta}_{\xi_2}(s_2 + c_2)}{s_2 + c_2} x_2^{s_2} ds_2 \quad (\text{III.56})$$

$$- \frac{1}{2\pi i a b \beta} \int_{\kappa_2 - iT}^{\kappa_2 + iT} \frac{\Theta_{\mathbf{g}}(-\frac{\beta}{\alpha} s_2, s_2)}{s_2^2} \frac{\widehat{\eta}_{\xi_1}(-\frac{\beta}{\alpha} s_2 + c_1)}{-\frac{\beta}{\alpha} s_2 + c_1} \frac{\widehat{\eta}_{\xi_2}(s_2 + c_2)}{s_2 + c_2} \left(\frac{x_2^{\alpha}}{x_1^{\beta}}\right)^{\frac{s_2}{\alpha}} ds_2 \quad (\text{III.57})$$

$$+ \frac{1}{(2\pi i)^2} \int_{s_1 \in \mathcal{C}_1} \int_{\kappa_2 - iT}^{\kappa_2 + iT} \frac{\Theta_{\mathbf{g}}(\mathbf{s}) x_1^{s_1} x_2^{s_2}}{a b s_1 s_2 (\alpha s_1 + \beta s_2)} \prod_{j=1}^2 \frac{\widehat{\eta}_{\xi_j}(s_j + c_j)}{s_j + c_j} ds \quad (\text{III.58})$$

We will now study the three terms (III.56), (III.57) and (III.58).

1. **First term III.56:** We will prove the following estimate:

$$\begin{aligned} \frac{1}{2\pi i a b \beta} \int_{\kappa_2 - iT}^{\kappa_2 + iT} \frac{\Theta_{\mathbf{g}}(0, s_2)}{s_2^2} \frac{\widehat{\eta}_{\xi_1}(c_1)}{c_1} \frac{\widehat{\eta}_{\xi_2}(s_2 + c_2)}{s_2 + c_2} x_2^{s_2} ds_2 &= \frac{\log(x_2)}{a b \beta c_1 c_2} \\ &+ O\left(1 + |\boldsymbol{\xi}|_1 \log(3 + x_2) + \xi_2^{-1} x_2^{-\delta_2}\right) \end{aligned} \quad (\text{III.59})$$

We set $G_{\boldsymbol{\xi}}(s) = \Theta_{\mathbf{g}}(0, s) \frac{\widehat{\eta}_{\xi_1}(c_1)}{c_1} \frac{\widehat{\eta}_{\xi_2}(s + c_2)}{s + c_2}$, it follows from the residue theorem that

$$\begin{aligned} \int_{\kappa_2 - iT}^{\kappa_2 + iT} \frac{G_{\boldsymbol{\xi}}(s_2)}{s_2^2} x_2^{s_2} ds_2 &= \int_{s_2 \in \mathcal{C}_2} \frac{G_{\boldsymbol{\xi}}(s_2)}{s_2^2} x_2^{s_2} ds_2 \\ &+ 2\pi i \left[G_{\boldsymbol{\xi}}(0) \log x_2 + G_{\boldsymbol{\xi}}^{(1)}(0) \right] \end{aligned} \quad (\text{III.60})$$

We continue with an estimate of each term on right side of (III.60). From estimate $\widehat{\eta}_{\xi}(s) = 1 + O(\xi|s|)$, see equation (III.32), we have

$$G_{\boldsymbol{\xi}}(0) = \Theta_{\mathbf{g}}(\mathbf{0}) \frac{\widehat{\eta}_{\xi_1}(c_1)}{c_1} \frac{\widehat{\eta}_{\xi_2}(c_2)}{c_2} = \frac{\widehat{\eta}_{\xi_1}(c_1) \widehat{\eta}_{\xi_2}(c_2)}{c_1 c_2} = \frac{1}{c_1 c_2} + O(|\boldsymbol{\xi}|_1). \quad (\text{III.61})$$

The values of $\Theta_{\mathbf{g}}(\mathbf{0})$ and $\frac{d}{ds} \Theta_{\mathbf{g}}(0, s)|_{s=0}$ given in equation (III.41) imply that

$$\begin{aligned} G_{\boldsymbol{\xi}}^{(1)}(0) &= \frac{d}{ds} \Theta_{\mathbf{g}}(0, s)|_{s=0} \frac{\widehat{\eta}_{\xi_1}(c_1)}{c_1} \frac{\widehat{\eta}_{\xi_2}(c_2)}{c_2} + \Theta_{\mathbf{g}}(\mathbf{0}) \frac{\widehat{\eta}_{\xi_1}(c_1)}{c_1} \frac{\widehat{\eta}_{\xi_2}^{(1)}(c_2)}{c_2} - \Theta_{\mathbf{g}}(\mathbf{0}) \frac{\widehat{\eta}_{\xi_1}(c_1)}{c_1} \frac{\widehat{\eta}_{\xi_2}(c_2)}{c_2^2} \\ &= (b + \beta)\gamma \frac{\widehat{\eta}_{\xi_1}(c_1)}{c_1} \frac{\widehat{\eta}_{\xi_2}(c_2)}{c_2} + \frac{\widehat{\eta}_{\xi_1}(c_1)}{c_1} \frac{\widehat{\eta}_{\xi_2}^{(1)}(c_2)}{c_2} - \frac{\widehat{\eta}_{\xi_1}(c_1)}{c_1} \frac{\widehat{\eta}_{\xi_2}(c_2)}{c_2^2} \end{aligned}$$

by using in addition the estimates $\widehat{\eta}_{\xi}^{(k)}(s) \ll \xi^k$ and $\widehat{\eta}_{\xi} = 1 + O(\xi s)$, we obtain

$$G_{\boldsymbol{\xi}}^{(1)}(0) = \frac{(b + \beta)\gamma}{c_1 c_2} - \frac{1}{c_1 c_2^2} + O(|\boldsymbol{\xi}|_1) = O(1), \quad (\text{III.62})$$

The estimate $\Theta_{\mathbf{g}}(0, s) \ll_{\varepsilon} (1 + |\tau|)^{2+\varepsilon - \frac{b+\beta}{2}(0 \wedge \sigma)}$, see equation (III.40), implies that

(a) in the horizontal line segments of \mathcal{C}_2 , we have:

$$\begin{aligned} \frac{G_{\boldsymbol{\xi}}(s_2)}{s_2^2} x_2^{s_2} &= \Theta_{\mathbf{g}}(0, s_2) \frac{\widehat{\eta}_{\xi_1}(c_1)}{c_1} \frac{\widehat{\eta}_{\xi_2}(s_2 + c_2)}{(s_2 + c_2) s_2^2} x_2^{s_2} \\ &= O\left((1 + T)^{-1+\varepsilon + \frac{b+\beta}{2}\delta_2}\right) \end{aligned} \quad (\text{III.63})$$

because $x_2^{\kappa_2} = O(1)$ and $\widehat{\eta}_{\xi_2}(s_2 + c_2) = O(1)$.

(b) in the vertical line segments of \mathcal{C}_2 , by using in addition the estimate $\widehat{\eta}_\xi(s) \ll 1 \wedge \frac{1}{\xi^{|1+s|}}$, we obtain that

$$\begin{aligned} \frac{G_\xi(s_2)}{s_2^2} x_2^{s_2} &= \Theta_{\mathbf{g}}(0, s_2) \frac{\widehat{\eta}_{\xi_1}(c_1)}{c_1} \frac{\widehat{\eta}_{\xi_2}(s_2 + c_2) x_2^{s_2}}{(s_2 + c_2) s_2^2} \\ &= O\left(\xi_2^{-1} x_2^{-\delta_2} (1 + |\tau_2|)^{-2+\varepsilon+\frac{b+\beta}{2}\delta_2}\right) \end{aligned} \quad (\text{III.64})$$

because $\text{Re}(s + c_2) \in [c_2/2, 2c_2]$.

Consequently we have the following estimate for the integral on the right hand side of (III.60)

$$\int_{s_2 \in \mathcal{C}_2} \frac{G_{\xi_2}(s_2)}{s_2^2} x_2^{s_2} ds_2 = O\left(\xi_2^{-1} x_2^{-\delta_2} + (1 + T)^{-1+\varepsilon+\frac{b+\beta}{2}\delta_2}\right) \quad (\text{III.65})$$

Using estimates (III.61), (III.62) and (III.65) in equation (III.60) gives the estimate (III.59).

2. **Second term III.57:** We will prove the following estimate:

$$\int_{\kappa_2 - iT}^{\kappa_2 + iT} \frac{\Theta_{\mathbf{g}}\left(-\frac{\beta}{\alpha} s_2, s_2\right) \widehat{\eta}_{\xi_1}\left(-\frac{\beta}{\alpha} s_2 + c_1\right) \widehat{\eta}_{\xi_2}(s_2 + c_2)}{s_2^2 \left(-\frac{\beta}{\alpha} s_2 + c_1\right) (s_2 + c_2)} \left(\frac{x_2^\alpha}{x_1^\beta}\right)^{\frac{s_2}{\alpha}} ds_2 = O(1) \quad (\text{III.66})$$

$$\text{Set } H_\xi(s) = \Theta_{\mathbf{g}}\left(-\frac{\beta}{\alpha} s, s\right) \frac{\widehat{\eta}_{\xi_1}\left(-\frac{\beta}{\alpha} s + c_1\right) \widehat{\eta}_{\xi_2}(s + c_2)}{-\frac{\beta}{\alpha} s + c_1} \frac{1}{s + c_2}.$$

We have the following three estimates

- (a) $\Theta_{\mathbf{g}}\left(-\frac{\beta}{\alpha} s, s\right) \ll (1 + |\tau|)^{2+\varepsilon+\frac{a\beta\sigma}{2\alpha}}$ for $\text{Re}(s) \geq 0$, see equation (III.40),
- (b) $H_\xi(s) \ll (1 + T)^{\varepsilon+\frac{a\beta\sigma}{2\alpha}}$ for $\text{Re}(s) \geq 0$ and $|\text{Im}(s)| = T$, because $\widehat{\eta}_\xi(s) = O(1)$,
- (c) $H_\xi(s) \ll (1 + |\tau|)^{\varepsilon+\frac{a\beta\sigma}{2\alpha}}$ for $\text{Re}(s) = \delta_2$ because $\text{Re}\left(-\frac{\beta}{\alpha} \delta_2 + c_1\right) \in \left[\frac{c_1}{2}, c_1\right]$.

This being the case, it follows from the residue theorem, the estimate (b) and the fact $x_2^\alpha \leq x_1^\beta$ that

$$\int_{\kappa_2 - iT}^{\kappa_2 + iT} \frac{H_\xi(s)}{s^2} \left(\frac{x_2^\alpha}{x_1^\beta}\right)^{\frac{s}{\alpha}} ds = \int_{\delta_2 - iT}^{\delta_2 + iT} \frac{H_\xi(s)}{s^2} \left(\frac{x_2^\alpha}{x_1^\beta}\right)^{\frac{s}{\alpha}} ds + O\left(T^{-2+\varepsilon+\frac{a\beta\delta_2}{2\alpha}}\right),$$

Since from the estimate (c) we have

$$\int_{\delta_2 - iT}^{\delta_2 + iT} \frac{H_\xi(s)}{s^2} \left(\frac{x_2^\alpha}{x_1^\beta}\right)^{\frac{s}{\alpha}} ds \ll \int_0^T (1 + |\tau|)^{-2+\varepsilon+\frac{a\beta\delta_2}{2\alpha}} = O(1)$$

and $\delta_2 \leq \frac{\alpha}{a\beta}$, we conclude that (III.66) holds.

3. **Third term III.58:** We will prove the following estimate:

$$\begin{aligned} &\int_{s_1 \in \mathcal{C}_1} \int_{\kappa_2 - iT}^{\kappa_2 + iT} \frac{\Theta_{\mathbf{g}}(\mathbf{s}) x_1^{s_1} x_2^{s_2}}{\text{abs}_1 s_2 (\alpha s_1 + \beta s_2)} \prod_{j=1}^2 \frac{\widehat{\eta}_{\xi_j}(s_j + c_j)}{s_j + c_j} ds \\ &\ll \frac{\kappa_2^{-1} \xi_2^{-1}}{(1 + T)^{1/2}} + \frac{\xi_1^{-1} \xi_2^{-1} \kappa_2^{-1} (\kappa_1 + \kappa_2)^{-1}}{(1 + T)^{1/2}} + \frac{\xi_1^{-1} \xi_2^{-1} \kappa_2^{-1}}{x_1^{\delta_1}} \end{aligned} \quad (\text{III.67})$$

We set $J_{\xi}(\mathbf{s}) = \Theta_{\mathbf{g}}(\mathbf{s}) \prod_{j=1}^2 \frac{\widehat{\eta}_{\xi_j}(s_j + c_j)}{s_j + c_j}$. We have following estimates:

(a) We set Δ_1 the polygonal path $s_1 \in \kappa_1 - iN_1T \rightarrow -\delta_1 - iN_1T$, $s_2 \in \kappa_2 - iT \rightarrow \kappa_2 + iT$ and Δ_2 the polygonal path $-\delta_1 + iN_1T \rightarrow \kappa_1 + iN_1T$, $s_2 \in \kappa_2 - iT \rightarrow \kappa_2 + iT$. For $\mathbf{s} \in \Delta_1 \cup \Delta_2$ we have

- i. $\frac{1}{|s_1|} \ll \frac{1}{1+T}$, $\frac{1}{|s_2|} \ll \frac{1}{\kappa_2(1+|\tau_2|)}$ see equation (III.54),
- ii. $\frac{1}{|\alpha s_1 + \beta s_2|} \ll \frac{1}{1+T}$,
- iii. $x_1^{s_1} x_2^{s_2} \ll 1$,
- iv. $\frac{\widehat{\eta}_{\xi_1}(s_1 + c_1)}{s_1 + c_1} \ll \frac{1}{1+|\tau_1|}$ because $\text{Re}(s_1 + c_1) \in [\frac{c_1}{2}, c_1]$,
- v. $\frac{\widehat{\eta}_{\xi_2}(s_2 + c_2)}{s_2 + c_2} \ll \frac{1}{\xi_2(1+|\tau_2|)^2}$ because $\text{Re}(s_2 + c_2) \in [\frac{c_2}{2}, c_2]$,
- vi. $\Theta_{\mathbf{g}}(\mathbf{s}) \ll (1+|\tau_1|)^{1+\varepsilon+\frac{\alpha}{2}\delta_1} (1+|\tau_2|)^{1+\varepsilon} (1+|\alpha\tau_1 + \beta\tau_2|)^{1+\varepsilon+\frac{\alpha}{2}\delta_1}$

it follows that

$$\frac{J_{\xi}(\mathbf{s})x_1^{s_1}x_2^{s_2}}{\text{abs}_1s_2(\alpha s_1 + \beta s_2)} \ll \frac{(1+|\tau_1|)^{1+\varepsilon+\frac{\alpha}{2}\delta_1}(1+|\tau_2|)^{1+\varepsilon}(1+|\alpha\tau_1 + \beta\tau_2|)^{1+\varepsilon+\frac{\alpha}{2}\delta_1}}{\xi_2\kappa_2(1+T)^2(1+|\tau_2|)^3(1+|\alpha\tau_1 + \beta\tau_2|)}$$

since $|\tau_1| \geq |\tau_2|$ we have $1+|\alpha\tau_1 + \beta\tau_2| \ll 1+|\tau_1|$ and therefore

$$\frac{J_{\xi}(\mathbf{s})x_1^{s_1}x_2^{s_2}}{\text{abs}_1s_2(\alpha s_1 + \beta s_2)} \ll \frac{\kappa_2^{-1}\xi_2^{-1}}{(1+T)^{1-2\varepsilon-\frac{\alpha+\alpha}{2}\delta_1}(1+|\tau_2|)^{2-\varepsilon}} \quad (\text{III.68})$$

Integrating over $\Delta_1 \cup \Delta_2$ and using the estimate (III.68) imply that

$$\iint_{\Delta_1 \cup \Delta_2} \frac{J_{\xi}(\mathbf{s})x_1^{s_1}x_2^{s_2}}{\text{abs}_1s_2(\alpha s_1 + \beta s_2)} \ll \frac{\kappa_2^{-1}\xi_2^{-1}}{(1+T)^{1/2}}$$

(b) We set Δ_3 the polygonal path $s_1 \in \kappa_1 - iT \rightarrow \kappa_1 - iN_1T$, $s_2 \in \kappa_2 - iT \rightarrow \kappa_2 + iT$ and Δ_4 the polygonal path $\kappa_1 + iT \rightarrow \kappa_1 + iN_1T$, $s_2 \in \kappa_2 - iT \rightarrow \kappa_2 + iT$.

For $\mathbf{s} \in \Delta_3 \cup \Delta_4$ we have

- i. $x_1^{s_1} x_2^{s_2} \ll 1$,
- ii. $\frac{1}{|s_2|} \ll \frac{1}{\kappa_2(1+|\tau_2|)}$, $\frac{1}{|s_1|} \ll \frac{1}{1+T}$,
- iii. $\frac{1}{|\alpha s_1 + \beta s_2|} \ll \frac{1}{(\kappa_1 + \kappa_2)(1+|\alpha\tau_1 + \beta\tau_2|)}$,
- iv. $\frac{\widehat{\eta}_{\xi_j}(s_j + c_j)}{s_j + c_j} \ll \frac{1}{\xi_j(1+|\tau_j|)^2}$ because $\text{Re}(s_j + c_j) \in [\frac{c_j}{2}, c_j]$,
- v. $\Theta_{\mathbf{g}}(\mathbf{s}) \ll (1+|\tau_1|)^{1+\varepsilon}(1+|\tau_2|)^{1+\varepsilon}(1+|\alpha\tau_1 + \beta\tau_2|)^{1+\varepsilon}$

consequently

$$\frac{J_{\xi}(\mathbf{s})x_1^{s_1}x_2^{s_2}}{\text{abs}_1s_2(\alpha s_1 + \beta s_2)} \ll \frac{(1+|\tau_1|)^{1+\varepsilon}(1+|\tau_2|)^{1+\varepsilon}(1+|\alpha\tau_1 + \beta\tau_2|)^{1+\varepsilon}}{\xi_1\xi_2\kappa_2(\kappa_1 + \kappa_2)(1+|\tau_1|)^3(1+|\tau_2|)^3(1+|\alpha\tau_1 + \beta\tau_2|)}$$

since $|\tau_1|_1 \geq |\tau_2|_1$ we have $1+|\alpha\tau_1 + \beta\tau_2| \ll 1+|\tau_1|$ and consequently

$$\frac{J_{\xi}(\mathbf{s})x_1^{s_1}x_2^{s_2}}{\text{abs}_1s_2(\alpha s_1 + \beta s_2)} \ll \frac{\xi_1^{-1}\xi_2^{-1}\kappa_2^{-1}(\kappa_1 + \kappa_2)^{-1}}{(1+|\tau_1|)^{2-3\varepsilon}(1+|\tau_2|)^2} \quad (\text{III.69})$$

Integrating over $\Delta_3 \cup \Delta_4$ and using the estimate (III.69) imply that

$$\int \int_{\Delta_3 \cup \Delta_4} \frac{J_{\xi}(\mathbf{s}) x_1^{s_1} x_2^{s_2}}{\text{ab} s_1 s_2 (\alpha s_1 + \beta s_2)} \ll \frac{\xi_1^{-1} \xi_2^{-1} \kappa_2^{-1} (\kappa_1 + \kappa_2)^{-1}}{(1+T)^{1-3\varepsilon}}$$

(c) We set Δ_5 the polygonal path $s_1 \in -\delta_1 - iN_1T \rightarrow -\delta_1 + iN_1T$, $s_2 \in \kappa_2 - iT \rightarrow \kappa_2 + iT$. For $\mathbf{s} \in \Delta_5$ we have

- i. $x_1^{s_1} x_2^{s_2} \ll x_1^{-\delta_1}$,
- ii. $\frac{1}{|s_2|} \ll \frac{1}{\kappa_2(1+|\tau_2|)}$, $\frac{1}{|s_1|} \ll \frac{1}{1+|\tau_1|}$,
- iii. $\frac{1}{|\alpha s_1 + \beta s_2|} \ll \frac{1}{1+|\alpha\tau_1 + \beta\tau_2|}$,
- iv. $\frac{\widehat{\eta}_{\xi_j}(s_j + c_j)}{s_j + c_j} \ll \frac{1}{\xi_j(1+|\tau_j|)^2}$ because $\text{Re}(s_j + c_j) \in [\frac{c_j}{2}, c_j]$,
- v. $\Theta_{\mathbf{g}}(\mathbf{s}) \ll (1+|\tau_1|)^{1+\varepsilon+\frac{3}{2}\delta_1} (1+|\tau_2|)^{1+\varepsilon} (1+|\alpha\tau_1 + \beta\tau_2|)^{1+\varepsilon+\frac{\alpha}{2}\delta_1}$

it follows that

$$\begin{aligned} \frac{J_{\xi}(\mathbf{s}) x_1^{s_1} x_2^{s_2}}{\text{ab} s_1 s_2 (\alpha s_1 + \beta s_2)} &\ll \frac{x_1^{-\delta_1} (1+|\tau_1|)^{1+\varepsilon+\frac{3}{2}\delta_1} (1+|\tau_2|)^{1+\varepsilon} (1+|\alpha\tau_1 + \beta\tau_2|)^{1+\varepsilon+\frac{\alpha}{2}\delta_1}}{\kappa_2 \xi_1 \xi_2 (1+|\tau_1|)^3 (1+|\tau_2|)^3 (1+|\alpha\tau_1 + \beta\tau_2|)} \\ &\ll \frac{\kappa_2^{-1} \xi_1^{-1} \xi_2^{-1} x_1^{-\delta_1}}{(1+|\tau_1|)^{2-2\varepsilon-\frac{3+\alpha}{2}\delta_1} (1+|\tau_2|)^{2-2\varepsilon-\frac{\alpha}{2}\delta_1}} \end{aligned} \quad (\text{III.70})$$

Integrating over Δ_5 and using the estimate (III.70) imply that

$$\int \int_{\Delta_5} \frac{J_{\xi}(\mathbf{s}) x_1^{s_1} x_2^{s_2}}{\text{ab} s_1 s_2 (\alpha s_1 + \beta s_2)} \ll \kappa_2^{-1} \xi_1^{-1} \xi_2^{-1} x_1^{-\delta_1}$$

The estimate (III.67) follows from the estimates obtained in (a), (b) and (c) above.

The estimates 1, 2 and 3 above imply that

$$\begin{aligned} \frac{\Psi((\tau_{\mathbf{g}})_{\gamma}; \mathbf{x}; \xi; T)}{\mathbf{x}^{\mathbf{c}}} &= \frac{\log(x_2)}{\text{ab}\beta c_1 c_2} + O(1 + |\xi|_1 \log(3 + x_2)) \\ + O\left(\frac{\xi_2^{-1}}{x_2^{\delta_2}} + \frac{\xi_1^{-1} \xi_2^{-1} \kappa_2^{-1}}{x_1^{\delta_1}} + \frac{\xi_1^{-1} \xi_2^{-1} \kappa_2^{-1} (\kappa_1 + \kappa_2)^{-1}}{(1+T)^{1/2}} + \frac{\kappa_2^{-1} \xi_2^{-1}}{(1+T)^{1/2}}\right) \end{aligned} \quad (\text{III.71})$$

With this in mind, if $x_2^{\alpha} \leq x_1^{\beta}$, we choose $\kappa_1^{-1} = \log(3 + x_1^{\beta})$, $\kappa_2^{-1} = \log(3 + x_2^{\alpha})$, $\xi_1 = \kappa_1^2$, $\xi_2 = \kappa_2^2$ and $T = \log^{20}(3 + x_1^{\beta} + x_2^{\alpha})$. Then the error term equation (III.55) verifies:

$$\begin{aligned} E((\tau_{\mathbf{g}})_{\gamma}; \mathbf{x}; \xi; T) &\ll \mathbf{x}^{\mathbf{c}} \frac{\kappa_1^{-1} \kappa_2^{-1} (\kappa_1 + \kappa_2)^{-1}}{(1+T)^{1-2\varepsilon} \min\{\xi_1; \xi_2\}} \\ &= \mathbf{x}^{\mathbf{c}} O\left(\frac{\log^2(3 + x_2^{\alpha}) \log^3(3 + x_1^{\beta})}{(1+T)^{1-2\varepsilon}}\right) = o(\mathbf{x}^{\mathbf{c}}) \end{aligned}$$

and equation (III.71) becomes

$$\Psi((\tau_{\mathbf{g}})_{\gamma}; \mathbf{x}; \xi; T) = \mathbf{x}^{\mathbf{c}} \frac{\log(x_2^{\alpha})}{\text{ab}\alpha\beta c_1 c_2} + O(\mathbf{x}^{\mathbf{c}})$$

If $x_2^\alpha \geq x_1^\beta$, we proceed in a similar way, at first we integrate with respect to s_2 after which we integrate with respect to s_1 and we obtain estimates $E((\tau_{\mathbf{g}})_\gamma; \mathbf{x}; \boldsymbol{\xi}; T) = o(\mathbf{x}^c)$ and

$$\Psi((\tau_{\mathbf{g}})_\gamma; \mathbf{x}; \boldsymbol{\xi}; T) = \mathbf{x}^c \frac{\log(x_1^\beta)}{ab\alpha\beta c_1 c_2} + O(\mathbf{x}^c)$$

Finally putting both estimates together gives

$$\begin{aligned} \Psi((\tau_{\mathbf{g}})_\gamma; \mathbf{x}; \boldsymbol{\xi}) &= \Psi((\tau_{\mathbf{g}})_\gamma; \mathbf{x}; \boldsymbol{\xi}; T) + E((\tau_{\mathbf{g}})_\gamma; \mathbf{x}; \boldsymbol{\xi}; T) \\ &= \mathbf{x}^c \frac{\log(\min\{x_1^\beta; x_2^\alpha\})}{ab\alpha\beta c_1 c_2} + O(\mathbf{x}^c) \end{aligned}$$

Consequently

$$\begin{aligned} &\Psi((\tau_{\mathbf{g}})_\gamma; x_1, x_2; \boldsymbol{\xi}) - \Psi\left((\tau_{\mathbf{g}})_\gamma; \frac{x_1}{1+\xi_1}, \frac{x_2}{1+\xi_2}; \boldsymbol{\xi}\right) \\ &= \frac{x_1^{c_1} x_2^{c_2}}{ab\alpha\beta c_1 c_2} \left(\log(\min\{x_1^\beta; x_2^\alpha\}) - \frac{\log\left(\min\left\{\left(\frac{x_1}{1+\xi_1}\right)^\beta; \left(\frac{x_2}{1+\xi_2}\right)^\alpha\right\}\right)}{(1+\xi_1)^{c_1} (1+\xi_2)^{c_2}} \right) + O(\mathbf{x}^c) \\ &= \frac{x_1^{c_1} x_2^{c_2}}{ab\alpha\beta c_1 c_2} \left(\log(\min\{x_1^\beta; x_2^\alpha\}) - \frac{\log\left(\min\{x_1^\beta; x_2^\alpha\}\right)}{(1+\xi_1)^{c_1} (1+\xi_2)^{c_2}} \right. \\ &\quad \left. - \frac{\log\left(\min\left\{\left(\frac{x_1}{1+\xi_1}\right)^\beta; \left(\frac{x_2}{1+\xi_2}\right)^\alpha\right\}\right) - \log\left(\min\{x_1^\beta; x_2^\alpha\}\right)}{(1+\xi_1)^{c_1} (1+\xi_2)^{c_2}} \right) + O(\mathbf{x}^c) \\ &= \frac{x_1^{c_1} x_2^{c_2}}{ab\alpha\beta c_1 c_2} \left(O((\xi_1 \vee \xi_2) \log(\min\{x_1^\beta; x_2^\alpha\})) - \frac{\log\left(\min\left\{\frac{\max\{1; \frac{x_1^\beta}{x_2^\alpha}\}; \max\{1; \frac{x_2^\alpha}{x_1^\beta}\}\right\}}{(1+\xi_1)^\beta; (1+\xi_2)^\alpha}\right)}{(1+\xi_1)^{c_1} (1+\xi_2)^{c_2}} \right) + O(\mathbf{x}^c) \\ &\ll \mathbf{x}^c \left(O\left(\frac{1}{\log(\min\{x_1^\beta; x_2^\alpha\})}\right) + \frac{\log\left(\max\left\{\frac{(1+\xi_1)^\beta}{\max\{1; \frac{x_1^\beta}{x_2^\alpha}\}}; \frac{(1+\xi_2)^\alpha}{\max\{1; \frac{x_2^\alpha}{x_1^\beta}\}}\right\}\right)}{(1+\xi_1)^{c_1} (1+\xi_2)^{c_2}} \right) + \mathbf{x}^c \\ &\ll \mathbf{x}^c O(1 + \log(\max\{1 + \xi_1; 1 + \xi_2\})) + \mathbf{x}^c = O(\mathbf{x}^c) \end{aligned}$$

From inequality $\Psi((\tau_{\mathbf{g}})_\gamma; \frac{x_1}{1+\xi_1}, \frac{x_2}{1+\xi_2}; \boldsymbol{\xi}) \leq \Psi((\tau_{\mathbf{g}})_\gamma; \mathbf{x}) \leq \Psi((\tau_{\mathbf{g}})_\gamma; x_1, x_2; \boldsymbol{\xi})$ it follows then that

$$\Psi((\tau_{\mathbf{g}})_\gamma; \mathbf{x}) = \Psi((\tau_{\mathbf{g}})_\gamma; \mathbf{x}; \boldsymbol{\xi}) + O(\mathbf{x}^c) = \frac{\mathbf{x}^c \log(\min\{x_1^\beta; x_2^\alpha\})}{ab\alpha\beta c_1 c_2} + O(\mathbf{x}^c) \quad (\text{III.72})$$

This completes the proof of the proposition 13. \square

Now that we have obtained the asymptotic (III.72) we will deduce by the convolution method an asymptotic for $\Psi(f; \mathbf{x})$.

Lemma 15 allows us to describe the sum $\Psi(f; \mathbf{x}) = \sum_{\mathbf{m} \leq \mathbf{x}} f(\mathbf{m})$ using sums associated with h and $\tau_{\mathbf{g}}$. More precisely, Lemma 15 implies that

$$\begin{aligned} \Psi(f; \mathbf{x}) &= \sum_{\mathbf{m} \leq \mathbf{x}} \mathbf{m}^\gamma (h_{f; \mathbf{g}} * \tau_{\mathbf{g}})(\mathbf{m}) = \sum_{\substack{(u_1 v_1, \dots, u_n v_n) \leq \mathbf{x} \\ \mathbf{u}, \mathbf{v} \in \mathbb{N}^n}} \mathbf{u}^\gamma \mathbf{v}^\gamma h_{f; \mathbf{g}}(\mathbf{u}) \tau_{\mathbf{g}}(\mathbf{v}) \\ &= \sum_{\mathbf{u} \leq \mathbf{x}} \mathbf{u}^\gamma h_{f; \mathbf{g}}(\mathbf{u}) \sum_{\substack{\mathbf{v}_j \leq x_j / u_j \\ j \in \{1, 2\}}} \mathbf{v}^\gamma \tau_{\mathbf{g}}(\mathbf{v}) \end{aligned} \quad (\text{III.73})$$

We can now finish the proof of theorem 13 as follows:

Let $\varepsilon > 0$, proposition 2 implies the convergence of the series $\mathcal{M}(|h_{f;g}|; \mathbf{c}') = \sum_{\mathbf{d} \in \mathbb{N}^n} \frac{|h_{f;g}(\mathbf{d})|}{\mathbf{d}^{\mathbf{c}'}}$.

It follows that there exists $t_0 \geq 3$ such that

$$t \geq t_0 \Rightarrow \sum_{\substack{\mathbf{d} \in \mathbb{N}^2 \\ |\mathbf{d}|_\infty \geq t}} \frac{|h_{f;g}(\mathbf{d})|}{d_1^{c'_1} d_2^{c'_2}} < \varepsilon \quad (\text{III.74})$$

We set $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$. Let $t \in]t_0^2, +\infty[$, $\delta(t) = \frac{\log(t_0)}{\log(t)}$ and $\mathbf{x} \geq t\mathbf{1}$, then $x_i^{\delta(t)} \geq t_0 \geq 3$ and $x_i^{1-\delta(t)} \geq t_0$. With this in mind, we have the next split

$$\Psi(f; \mathbf{x}) = \sum_{\substack{u_j \leq x_j^{\delta(t)} \\ j \in \{1;2\}}} \mathbf{u}^\gamma h_{f;g}(\mathbf{u}) \sum_{\substack{v_j \leq x_j/u_j \\ j \in \{1;2\}}} \mathbf{v}^\gamma \tau_g(\mathbf{v}) + \sum_{\substack{x_1 \geq u_1 > x_1^{\delta(t)} \text{ or} \\ x_2 \geq u_2 > x_2^{\delta(t)}}} \mathbf{u}^\gamma h_{f;g}(\mathbf{u}) \sum_{\substack{v_j \leq x_j/u_j \\ j \in \{1;2\}}} \mathbf{v}^\gamma \tau_g(\mathbf{v}) \quad (\text{III.75})$$

since $\mathbf{c} = \mathbf{c}' + \gamma$ we have the estimate

$$\begin{aligned} \left| \sum_{\substack{x_1 \geq u_1 > x_1^{\delta(t)} \text{ or} \\ x_2 \geq u_2 > x_2^{\delta(t)}}} \mathbf{u}^\gamma h_{f;g}(\mathbf{u}) \sum_{\substack{v_j \leq x_j/u_j \\ j \in \{1;2\}}} \mathbf{v}^\gamma \tau_g(\mathbf{v}) \right| &\leq \sum_{\substack{|\mathbf{u}|_\infty > t_0 \\ \mathbf{u} \leq \mathbf{x}}} \mathbf{u}^\gamma |h_{f;g}|(\mathbf{u}) \sum_{\substack{v_j \leq x_j/u_j \\ j \in \{1;2\}}} \mathbf{v}^\gamma \tau_g(\mathbf{v}) \\ &= \mathbf{x}^{\mathbf{c}} \sum_{\substack{|\mathbf{u}|_\infty > t_0 \\ \mathbf{u} \leq \mathbf{x}}} \frac{|h_{f;g}|(\mathbf{u})}{\mathbf{u}^{\mathbf{c}'}} \sum_{\substack{v_j \leq x_j/u_j \\ j \in \{1;2\}}} \mathbf{v}^\gamma \tau_g(\mathbf{v}) \prod_{j=1}^2 \left(\frac{x_j}{u_j} \right)^{-c_j} \end{aligned}$$

Using proposition 3 gives

$$\left| \sum_{\substack{x_1 \geq u_1 > x_1^{\delta(t)} \text{ or} \\ x_2 \geq u_2 > x_2^{\delta(t)}}} \mathbf{u}^\gamma h_{f;g}(\mathbf{u}) \sum_{\substack{v_j \leq x_j/u_j \\ j \in \{1;2\}}} \mathbf{v}^\gamma \tau_g(\mathbf{v}) \right| \ll \mathbf{x}^{\mathbf{c}} \sum_{\substack{|\mathbf{u}|_\infty > t_0 \\ \mathbf{u} \leq \mathbf{x}}} \frac{|h_{f;g}|(\mathbf{u})}{\mathbf{u}^{\mathbf{c}'}} \left(\frac{\log(\min\{(\frac{x_1}{u_1})^\beta; (\frac{x_2}{u_2})^\alpha\})}{ab\alpha\beta c_1 c_2} + O(1) \right)$$

which leads to the following estimate

$$\begin{aligned} \left| \sum_{\substack{x_1 \geq u_1 > x_1^{\delta(t)} \text{ or} \\ x_2 \geq u_2 > x_2^{\delta(t)}}} \mathbf{u}^\gamma h_{f;g}(\mathbf{u}) \sum_{\substack{v_j \leq x_j/u_j \\ j \in \{1;2\}}} \mathbf{v}^\gamma \tau_g(\mathbf{v}) \right| &\ll \mathbf{x}^{\mathbf{c}} \left(\log(\min\{x_1^\beta; x_2^\alpha\}) + O(1) \right) \sum_{|\mathbf{u}|_\infty > t_0} \frac{|h_{f;g}|(\mathbf{u})}{\mathbf{u}^{\mathbf{c}'}} \\ &= O\left(\varepsilon \mathbf{x}^{\mathbf{c}} \log(3 + \min\{x_1^\beta; x_2^\alpha\}) \right) \end{aligned}$$

We deduce that for $t \geq t_0$ and $\mathbf{x} \geq t\mathbf{1}$, we have

$$\Psi(f; \mathbf{x}) = \sum_{\substack{u_j \leq x_j^{\delta(t)} \\ j \in \{1;2\}}} \mathbf{u}^\gamma h_{f;g}(\mathbf{u}) \sum_{\substack{v_j \leq x_j/u_j \\ j \in \{1;2\}}} \mathbf{v}^\gamma \tau_g(\mathbf{v}) + O\left(\varepsilon \mathbf{x}^{\mathbf{c}} \log(3 + \min\{x_1^\beta; x_2^\alpha\}) \right). \quad (\text{III.76})$$

The term $\sum_{\mathbf{v} \leq \mathbf{x}} \mathbf{v}^\gamma \tau_{\mathbf{g}}(\mathbf{v})$ was estimated in proposition 3. This proposition 3 imply that:

$$\begin{aligned}
\sum_{\substack{\mathbf{u}_j \leq x_j^{\delta(t)} \\ j \in \{1,2\}}} \mathbf{u}^\gamma h_{f;\mathbf{g}}(\mathbf{u}) \sum_{\substack{\mathbf{v}_j \leq x_j/\mathbf{u}_j \\ j \in \{1,2\}}} \mathbf{v}^\gamma \tau_{\mathbf{g}}(\mathbf{v}) &= \mathbf{x}^{\mathbf{c}} \sum_{\substack{\mathbf{u}_j \leq x_j^{\delta(t)} \\ j \in \{1,2\}}} \frac{h_{f;\mathbf{g}}(\mathbf{u})}{\mathbf{u}^{\mathbf{c}'}} \sum_{\substack{\mathbf{v}_j \leq x_j/\mathbf{u}_j \\ j \in \{1,2\}}} \mathbf{v}^\gamma \tau_{\mathbf{g}}(\mathbf{v}) \prod_{j=1}^2 \left(\frac{x_j}{\mathbf{u}_j} \right)^{-c_j} \\
&= \mathbf{x}^{\mathbf{c}} \sum_{\substack{\mathbf{u}_j \leq x_j^{\delta(t)} \\ j \in \{1,2\}}} \frac{h_{f;\mathbf{g}}(\mathbf{u})}{\mathbf{u}^{\mathbf{c}'}} \left(\frac{\log(\min\{(\frac{x_1}{\mathbf{u}_1})^\beta; (\frac{x_2}{\mathbf{u}_2})^\alpha\})}{ab\alpha\beta c_1 c_2} + O(1) \right) \\
&= \mathbf{x}^{\mathbf{c}} \sum_{\substack{\mathbf{u}_j \leq x_j^{\delta(t)} \\ j \in \{1,2\}}} \frac{h_{f;\mathbf{g}}(\mathbf{u})}{\mathbf{u}^{\mathbf{c}'}} \left(\frac{\log(\min\{(\frac{x_1}{\mathbf{u}_1})^\beta; (\frac{x_2}{\mathbf{u}_2})^\alpha\})}{ab\alpha\beta c_1 c_2} \right) + O\left(\mathbf{x}^{\mathbf{c}} \sum_{\substack{\mathbf{u}_j \leq x_j^{\delta(t)} \\ j \in \{1,2\}}} \frac{|h_{f;\mathbf{g}}(\mathbf{u})|}{\mathbf{u}^{\mathbf{c}'}} \right)
\end{aligned}$$

By using in addition the estimate

$$\sum_{\substack{\mathbf{u}_j \leq x_j^{\delta(t)} \\ j \in \{1,2\}}} \frac{|h_{f;\mathbf{g}}(\mathbf{u})|}{\mathbf{u}^{\mathbf{c}'}} \leq \sum_{\mathbf{u} \in (\mathbb{N}^*)^2} \frac{|h_{f;\mathbf{g}}(\mathbf{u})|}{\mathbf{u}^{\mathbf{c}'}} = O(1),$$

we deduce that

$$\begin{aligned}
\sum_{\substack{\mathbf{u}_j \leq x_j^{\delta(t)} \\ j \in \{1,2\}}} \mathbf{u}^\gamma h_{f;\mathbf{g}}(\mathbf{u}) \sum_{\substack{\mathbf{v}_j \leq x_j/\mathbf{u}_j \\ j \in \{1,2\}}} \mathbf{v}^\gamma \tau_{\mathbf{g}}(\mathbf{v}) &= \mathbf{x}^{\mathbf{c}} \sum_{\substack{\mathbf{u}_j \leq x_j^{\delta(t)} \\ j \in \{1,2\}}} \frac{h_{f;\mathbf{g}}(\mathbf{u})}{\mathbf{u}^{\mathbf{c}'}} \left(\frac{\log(\min\{(\frac{x_1}{\mathbf{u}_1})^\beta; (\frac{x_2}{\mathbf{u}_2})^\alpha\})}{ab\alpha\beta c_1 c_2} \right) \\
&\quad + O\left(\mathbf{x}^{\mathbf{c}} \right) \tag{III.77}
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
&\left| \sum_{\substack{\mathbf{u}_j \leq x_j^{\delta(t)} \\ j \in \{1,2\}}} \frac{h_{f;\mathbf{g}}(\mathbf{u})}{\mathbf{u}^{\mathbf{c}}} \log\left(\min\left\{\left(\frac{x_1}{\mathbf{u}_1}\right)^\beta, \left(\frac{x_2}{\mathbf{u}_2}\right)^\alpha\right\}\right) - \sum_{\substack{\mathbf{u}_j \leq x_j^{\delta(t)} \\ j \in \{1,2\}}} \frac{h_{f;\mathbf{g}}(\mathbf{u})}{\mathbf{u}^{\mathbf{c}'}} \log\left(\min\{x_1^\beta; x_2^\alpha\}\right) \right| \\
&= \sum_{\substack{\mathbf{u}_j \leq x_j^{\delta(t)} \\ j \in \{1,2\}}} \frac{h_{f;\mathbf{g}}(\mathbf{u})}{\mathbf{u}^{\mathbf{c}'}} \log\left(\min\{x_1^\beta, x_2^\alpha\} \cdot \min\left\{\left(\frac{x_1}{\mathbf{u}_1}\right)^\beta; \left(\frac{x_2}{\mathbf{u}_2}\right)^\alpha\right\}^{-1}\right) \\
&\leq \sum_{\substack{\mathbf{u}_j \leq x_j^{\delta(t)} \\ j \in \{1,2\}}} \frac{|h_{f;\mathbf{g}}(\mathbf{u})|}{\mathbf{u}^{\mathbf{c}'}} \log\left(\min\{x_1^{\beta\delta(t)}; x_2^{\alpha\delta(t)}\}\right) = \delta(t) \log\left(\min\{x_1^\beta; x_2^\alpha\}\right) \sum_{\substack{\mathbf{u}_j \leq x_j^{\delta(t)} \\ j \in \{1,2\}}} \frac{|h_{f;\mathbf{g}}(\mathbf{u})|}{\mathbf{u}^{\mathbf{c}'}} \\
&= O\left(\delta(t) \log\left(\min\{x_1^\beta; x_2^\alpha\}\right)\right).
\end{aligned}$$

It follows then from (III.77) and (III.76) that for $t \geq t_0$ and $\mathbf{x} \geq t\mathbf{1}$, we have

$$\begin{aligned}
\Psi(f; \mathbf{x}) &= \frac{\mathbf{x}^{\mathbf{c}} \log\left(\min\{x_1^\beta; x_2^\alpha\}\right)}{ab\alpha\beta c_1 c_2} \sum_{\substack{\mathbf{u}_j \leq x_j^{\delta(t)} \\ j \in \{1,2\}}} \frac{h_{f;\mathbf{g}}(\mathbf{u})}{\mathbf{u}^{\mathbf{c}'}} \\
&\quad + O\left(\delta(t) \mathbf{x}^{\mathbf{c}} \log\left(3 + \min\{x_1^\beta; x_2^\alpha\}\right)\right) + O\left(\varepsilon \mathbf{x}^{\mathbf{c}} \log\left(3 + \min\{x_1^\beta; x_2^\alpha\}\right)\right) + O(\mathbf{x}^{\mathbf{c}}).
\end{aligned}$$

By using in addition (III.74) and the fact that

$$\sum_{(\mathbf{u}_1, \mathbf{u}_1) \in \mathbb{N}^2} \frac{|h_{f;\mathbf{g}}(\mathbf{u})|}{\mathbf{u}^{\mathbf{c}'}} = \mathcal{M}(h_{f;\mathbf{g}}; \mathbf{c}') = \mathcal{H}(f; \mathbf{c}'; \gamma; \mathbf{0}),$$

we deduce that for any $\varepsilon > 0$, there exist $t \geq t_0$ such that if $\mathbf{x} \geq t\mathbf{1}$ we have

$$\begin{aligned} \Psi(f; \mathbf{x}) &= \frac{\mathbf{x}^c \log\left(\min\{x_1^\beta; x_2^\alpha\}\right)}{ab\alpha\beta c_1 c_2} \mathcal{H}(f; \mathbf{c}'; \gamma; \mathbf{0}) \\ &\quad + O\left(\delta(t)\mathbf{x}^c \log\left(3 + \min\{x_1^\beta; x_2^\alpha\}\right)\right) + O\left(\varepsilon\mathbf{x}^c \log\left(3 + \min\{x_1^\beta; x_2^\alpha\}\right)\right) + O(\mathbf{x}^c). \end{aligned}$$

Since $\delta(t) \rightarrow 0$ as $t \rightarrow \infty$, we deduce from the previous identity that

$$\Psi(f; \mathbf{x}) \sim \mathbf{x}^c \log\left(\min\{x_1^\beta; x_2^\alpha\}\right) \frac{\mathcal{H}(f; \mathbf{c}'; \gamma; \mathbf{0})}{ab\alpha\beta c_1 c_2}$$

as $\mathbf{x} \geq t\mathbf{1}$ and $t \rightarrow \infty$. This completes the proof of Theorem 13. \square

7 Proof of Theorem 14

In this case we have $\mathbf{g} : \{a_1\mathbf{e}_1, \dots, a_n\mathbf{e}_n\} \rightarrow \mathbb{N}$, we set $g_i = \mathbf{g}(a_i\mathbf{e}_i)$ for each $i \in \llbracket 1, n \rrbracket$. It follows from definition of $\tau_{\mathbf{g}}$, see (III.1), and Lemma 14 that

$$Z(\tau_{\mathbf{g}}; \mathbf{s}) = \sum_{\mathbf{m} \in \mathbb{N}^n} \frac{\tau_{\mathbf{g}}(\mathbf{m})}{\mathbf{m}^{\mathbf{s}}} = \prod_{i=1}^n \zeta^{g_i}(a_i s_i) = \prod_{i=1}^n \left(\sum_{m \in \mathbb{N}} \frac{\tau_{g_i}(m)}{m^s} \right) \text{ for } \operatorname{Re}(\mathbf{s}) \gg \mathbf{1}$$

where $\zeta^{g_i}(a_i s) = \sum_{m \in \mathbb{N}} \frac{\tau_{g_i}(m)}{m^s}$ and consequently $\tau_{\mathbf{g}} = \otimes_{i=1}^n \tau_{g_i}$.

From definition of function $h_{f; \mathbf{g}} \in \mathcal{M}^n$, see (III.37) or its explicit version given in lemma 12 we have

$$\sum_{\mathbf{v} \in \mathbb{N}_0^n} \frac{h_{f; \mathbf{g}}(p^{\mathbf{v}})}{p^{\mathbf{v} \cdot \mathbf{s}}} = \left(\sum_{\mathbf{v} \in \mathbb{N}_0^n} f(p^{\mathbf{v}}) p^{-\mathbf{v} \cdot (\mathbf{s} + \gamma)} \right) \prod_{i=1}^I (1 - p^{-\mathbf{k}_i \cdot \mathbf{s}})^{g_i}$$

It follows that

$$f(\mathbf{m}) = \mathbf{m}^\gamma (h_{f; \mathbf{g}} * \tau_{\mathbf{g}})(\mathbf{m}) = (h_{f; \mathbf{g}})_\gamma * (\tau_{\mathbf{g}})_\gamma(\mathbf{m}) \quad (\text{III.78})$$

where the first equality was proved in lemma 15 and the second equality follows from definition of convolution product.

Additionally to proposition 2 we need the next lemma:

Lemma 16. *Let $\gamma \in \mathbb{R}_+$ and $a, g \in \mathbb{N}$. We set $c = a^{-1} + \gamma$ and $\zeta^g(as) = \sum_{m \in \mathbb{N}} \tau_g(m) m^{-s}$, then we have*

$$\Psi((\tau_g)_\gamma; x) = \sum_{m \leq x} (\tau_g)_\gamma(m) = \frac{a^{-g} (1 + o(1))}{(g-1)!c} x^c \log^{g-1}(x) \text{ as } x \rightarrow +\infty$$

where $(\tau_g)_\gamma(m) = m^\gamma \tau_g(m)$.

Proof. We know that the function $\zeta(as - a\gamma)$ has a meromorphic extension to \mathbb{C} with only a single simple pole at $s = c$ of residue a^{-1} .

It follows that $s \mapsto T(s) := \sum_{m \in \mathbb{N}} \frac{(\tau_g)_\gamma(m)}{m^s} = \zeta^g(as - a\gamma)$ has a meromorphic extension

with only one pole at $s = c$ of order g . Moreover, $T(s) \sim \frac{a^{-g}}{(s-c)^g}$ as $s \rightarrow c$. We conclude by using Delange's tauberian theorem, see appendix theorem 20. \square

End of the proof of Theorem 14:

Let $\mathbf{x} \geq \mathbf{1}$, from equation (III.78) we have

$$\begin{aligned} \Psi(f; \mathbf{x}) &= \sum_{\mathbf{m}=(m_1, \dots, m_n) \leq \mathbf{x}} f(\mathbf{m}) = \sum_{\mathbf{m}=(m_1, \dots, m_n) \leq \mathbf{x}} (h_{f; \mathbf{g}})_\gamma * (\tau_\gamma)_\gamma(\mathbf{m}) \\ &= \sum_{\mathbf{m}=(m_1, \dots, m_n) \leq \mathbf{x}} \sum_{\substack{\mathbf{d}_1=(d_{1,1}, \dots, d_{1,n}) \in \mathbb{N}^n \\ \mathbf{d}_2=(d_{2,1}, \dots, d_{2,n}) \in \mathbb{N}^n \\ d_{1,j}d_{2,j}=m_j, j \in \llbracket 1, n \rrbracket}} (h_{f; \mathbf{g}})_\gamma(\mathbf{d}_1) (\tau_\gamma)_\gamma(\mathbf{d}_2) \\ &= \sum_{\mathbf{d}_1 \leq \mathbf{x}} (h_{f; \mathbf{g}})_\gamma(\mathbf{d}_1) \sum_{\substack{d_{2,j} \leq x_j/d_{1,j} \\ j \in \llbracket 1, n \rrbracket}} (\tau_\gamma)_\gamma(\mathbf{d}_2) = \sum_{\mathbf{d} \leq \mathbf{x}} (h_{f; \mathbf{g}})_\gamma(\mathbf{d}) \prod_{j=1}^n \Psi \left((\tau_{g_j})_{\gamma_j}; \frac{x_j}{d_j} \right). \end{aligned}$$

For each $j \in \llbracket 1, n \rrbracket$, we set $r_j(t) := \frac{\sum_{m \leq t} m^{\gamma_j} \tau_{g_j}(m) - a_j^{-g_j} \frac{t^{c_j} \max\{2; \log(t)\}^{g_j-1}}{(g_j-1)!c_j}}{a_j^{-g_j} \frac{t^{c_j} \max\{2; \log(t)\}^{g_j-1}}{(g_j-1)!c_j}}$. Lemma 16

implies that all $j = 1, \dots, n$ the function r_j is bounded in $[1, +\infty[$ and verifies $r_j(t) \rightarrow 0$ as $t \rightarrow +\infty$. By using this notation, we obtain that for $\mathbf{x} \geq \mathbf{1}$, we have

$$\Psi(f; \mathbf{x}) = \mathbf{x}^{\mathbf{c}} \sum_{\mathbf{d} \leq \mathbf{x}} \frac{h_{f; \mathbf{g}}(\mathbf{d})}{\mathbf{d}^{\mathbf{c}'}} \prod_{j=1}^n a_j^{-g_j} \frac{\max\{2; \log(x_j/d_j)\}^{g_j-1}}{(g_j-1)!c_j} [1 + r_j(x_j/d_j)] \quad (\text{III.79})$$

Let $\varepsilon > 0$. From lemma 13 we know that the series $Z(|h_{f; \mathbf{g}}|; \mathbf{c}')$ is convergent. It follows that there exists $t_0 = t_0(\varepsilon) \geq 4$ such that

$$t \geq t_0 \Rightarrow \sum_{\substack{\mathbf{d} \in \mathbb{N}^n \\ |\mathbf{d}|_\infty \geq t}} \frac{|h_{f; \mathbf{g}}(\mathbf{d})|}{d_1^{c'_1} \dots d_n^{c'_n}} < \varepsilon.$$

Let $t \geq t_0^2$ and $\delta(t) = \log(t_0)/\log(t) \leq 1/2$, we split (III.79) into two terms as follows:

$$\Psi(f; \mathbf{x}) = \mathbf{x}^{\mathbf{c}} \sum_{\mathbf{d} \leq \mathbf{x}^{\delta(t)}} \frac{h_{f; \mathbf{g}}(\mathbf{d})}{\mathbf{d}^{\mathbf{c}'}} \prod_{j=1}^n a_j^{-g_j} \frac{\max\{2; \log(x_j/d_j)\}^{g_j-1}}{(g_j-1)!c_j} [1 + r_j(x_j/d_j)] \quad (\text{III.80})$$

$$+ \mathbf{x}^{\mathbf{c}} \sum_{\substack{\mathbf{d} \not\leq \mathbf{x}^{\delta(t)} \\ \mathbf{d} \leq \mathbf{x}}} \frac{h_{f; \mathbf{g}}(\mathbf{d})}{\mathbf{d}^{\mathbf{c}'}} \prod_{j=1}^n a_j^{-g_j} \frac{\max\{2; \log(x_j/d_j)\}^{g_j-1}}{(g_j-1)!c_j} [1 + r_j(x_j/d_j)] \quad (\text{III.81})$$

and we remarks that for $\mathbf{x} \geq e^t \mathbf{1}$ we have the following two points:

- $x_j^{1-\delta(t)} \geq e^{t-\delta(t)} \geq e^{t/2}$ and therefore $2 \leq t_0/2 \leq \log(x_j/d_j)$ for $\mathbf{d} \leq \mathbf{x}^{\delta(t)}$;
- $|\mathbf{d}|_\infty \geq t_0$, for $\mathbf{d} \not\leq \mathbf{x}^{\delta(t)}$.

The first sum (III.80) is equal to

$$\begin{aligned}
& \mathbf{x}^{\mathbf{c}} \sum_{\mathbf{d} \leq \mathbf{x}^{\delta(t)}} \frac{h_{f;\mathbf{g}}(\mathbf{d})}{\mathbf{d}^{\mathbf{c}'}} \prod_{j=1}^n a_j^{-g_j} \frac{(\log x_j - \log d_j)^{g_j-1}}{(g_j-1)!c_j} \sum_{\mathbf{u} \in \{0,1\}^n} \prod_{j=1}^n r_j \left(\frac{x_j}{d_j} \right)^{u_j} \\
&= \mathbf{x}^{\mathbf{c}} \sum_{\mathbf{d} \leq \mathbf{x}^{\delta(t)}} \frac{h_{f;\mathbf{g}}(\mathbf{d})}{\mathbf{d}^{\mathbf{c}'}} \sum_{\substack{\mathbf{u} \in \{0,1\}^n \\ \ell \leq \mathbf{g}-1}} (-1)^{|\ell|} \prod_{j=1}^n \binom{g_j-1}{\ell_j} a_j^{-g_j} \frac{\log^{g_j-\ell_j-1}(x_j) \log^{\ell_j} d_j}{(g_j-1)!c_j} r_j \left(\frac{x_j}{d_j} \right)^{u_j} \\
&= \sum_{\substack{\mathbf{u}=\mathbf{0} \\ \ell=0}} \sum_{\mathbf{d} \leq \mathbf{x}^{\delta(t)}} + \sum_{\substack{\mathbf{u}=\mathbf{0} \\ 0 \neq \ell \leq \mathbf{g}-1}} \sum_{\mathbf{d} \leq \mathbf{x}^{\delta(t)}} + \sum_{\substack{\mathbf{u} \in \{0,1\}^n \setminus \{\mathbf{0}\} \\ \ell \leq \mathbf{g}-1}} \sum_{\mathbf{d} \leq \mathbf{x}^{\delta(t)}} \\
&= \mathcal{M}(h_{f;\mathbf{g}}; \mathbf{c}') \mathbf{x}^{\mathbf{c}} \prod_{j=1}^n a_j^{-g_j} \frac{\log^{g_j-1}(x_j)}{(g_j-1)!c_j} + O\left(\left(\sum_{|\mathbf{d}|_{\infty} \geq t_0} \frac{|h_{f;\mathbf{g}}(\mathbf{d})|}{\mathbf{d}^{\mathbf{c}'}} \right) \mathbf{x}^{\mathbf{c}} \prod_{j=1}^n \log^{g_j-1}(x_j) \right) \\
&\quad + O\left((\delta(t) + \sup_{\substack{x_j^{1-\delta(t)} \leq v_j \leq x_j \\ j \in [1,n]}} r_j(v_j)) \mathbf{x}^{\mathbf{c}} \mathcal{M}(|h_{f;\mathbf{g}}|; \mathbf{c}') \prod_{j=1}^n \log^{g_j-1}(x_j) \right) \\
&= \left[\mathcal{M}(h_{f;\mathbf{g}}; \mathbf{c}') + (\varepsilon + \delta(t)) O(1) \right] \mathbf{x}^{\mathbf{c}} \prod_{j=1}^n a_j^{-g_j} \frac{\log^{g_j-1}(x_j)}{(g_j-1)!c_j}
\end{aligned}$$

We upper bound the second sum (III.81) as follows

$$\ll \mathbf{x}^{\mathbf{c}} \sum_{|\mathbf{d}|_{\infty} \geq t_0} \frac{|h_{f;\mathbf{g}}(\mathbf{d})|}{\mathbf{d}^{\mathbf{c}'}} \prod_{j=1}^n \log^{g_j-1}(x_j) \leq \varepsilon \mathcal{M}(|h_{f;\mathbf{g}}|; \mathbf{c}') \mathbf{x}^{\mathbf{c}'} \prod_{j=1}^n \log^{g_j-1}(x_j)$$

Combining these two facts implies that

$$\Psi(f; \mathbf{x}) = \left[\mathcal{M}(h_{f;\mathbf{g}}; \mathbf{c}') + o(1) \right] \mathbf{x}^{\mathbf{c}} \prod_{j=1}^n \frac{a_j^{-g_j} \log^{g_j-1}(x_j)}{(g_j-1)!c_j}$$

as $\mathbf{x} \geq e^{t\mathbf{1}}$ and $t \rightarrow \infty$. This completes the proof of Theorem 14. \square

8 Proofs of Proposition 1 and Corollaries 2 and 3

We use the notation introduced in section 2.

Let $P \in \mathfrak{E}_n$ of degree $d > 0$ and $f : \mathbb{N} \rightarrow \mathbb{C}$ a one variable multiplicative function satisfying the assumptions (H1) and (H2) of proposition 1 Let $\mathbf{d} = (d_1, \dots, d_n) \in \mathbb{N}^n$.

Recall that the multivariable function $f_{\mathbf{d}} : \mathbb{N}^n \rightarrow \mathbb{C}$ is defined by

$$f_{\mathbf{d}}(\mathbf{m}) = f(m^{d_1} \cdots m^{d_n}) \quad \forall \mathbf{m} = (m_1, \dots, m_n) \in \mathbb{N}^n.$$

Let $\mathbf{c}' \in (\mathbb{R}_+^*)^n \cap \text{int } \mathcal{F}_{\Sigma(f_{\mathbf{d}})}(\mathbf{1})$. Set $\boldsymbol{\gamma} := \boldsymbol{\gamma} \mathbf{d} = (\gamma d_1, \dots, \gamma d_n)$, $\mathbf{c} = \mathbf{c}' + |\boldsymbol{\gamma}| = \mathbf{c}' + \boldsymbol{\gamma} \mathbf{d}$.

Assumption (H1) implies that the multiplicative function $f_{\mathbf{d}}$ is of finite growth (see Definition 9) with $\boldsymbol{\gamma}$ as growth exponent.

Assumption (H2) implies that for all $\mathbf{k} \in I_0(f_{\mathbf{d}})$,

$$\sum_{p \in \mathcal{P}} \frac{1}{p^{1-\delta/2}} \left| \frac{f_{\mathbf{d}}(p^{\mathbf{k}})}{p^{\mathbf{k} \cdot \boldsymbol{\gamma}}} - g(\mathbf{k}, \mathbf{d}) \right| = \sum_{p \in \mathcal{P}} \frac{1}{p^{1-\delta/2}} \left| \frac{f(p^{\mathbf{k}, \mathbf{d}})}{p^{\mathbf{k} \cdot \boldsymbol{\gamma}}} - g(\mathbf{k}, \mathbf{d}) \right| \ll \sum_{p \in \mathcal{P}} \frac{1}{p^{1+\delta/2}} < +\infty$$

It follows that the multiplicative function $f_{\mathbf{d}}$ belongs to the class $\mathfrak{D}(\boldsymbol{\gamma}; \mathbf{g}; 1 - \delta/2)$ (see Definition 10), where $I = I_0(f_{\mathbf{d}})$ and $\mathbf{g} : I \rightarrow \mathbb{N}$ is defined by $\mathbf{g}(\mathbf{k}) = g(\mathbf{k}, \mathbf{d})$ for all $\mathbf{k} \in I$. Proposition 1 follows then from Essouabri's Theorem [[9], Theo. 3] (see Theorem 12 above).

Proofs of Corollaries 2 and 3:

It is enough to verify that functions $f \in \{\varphi, \sigma_k; k \in \mathbb{N}\}$ satisfy all conditions of proposition 1.

Case $f = \sigma_0$: $\sigma_{0;\mathbf{d}} : \mathbb{N}^n \rightarrow \mathbb{R}_+$ is defined by $\sigma_{0;\mathbf{d}}(\mathbf{m}) = \sigma_0(\prod_{j=1}^n m_j^{d_j})$.

Particularly, for each $\boldsymbol{\nu} = (\nu_1, \dots, \nu_n) \in \mathbb{N}^n$ we have

$$\sigma_{0;\mathbf{d}}(p^{\boldsymbol{\nu}}) = \sigma_{0;\mathbf{d}}(p^{\nu_1}, \dots, p^{\nu_n}) = \sigma_0(p^{\boldsymbol{\nu} \cdot \mathbf{d}}) = 1 + \boldsymbol{\nu} \cdot \mathbf{d}$$

It follows that $I(f_{\mathbf{d}}) = I_0(f_{\mathbf{d}}) = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$, $\gamma = 0$, $\mathbf{g}(\mathbf{e}_i) = 1 + d_i$, $\mathbf{c}' = \mathbf{1}$ and $\rho = 1 + |\mathbf{d}|_1$. The result follows from application of proposition 1.

Case $f = \sigma_k$ and $k \in \mathbb{N}_0$: $\sigma_{k;\mathbf{d}} : \mathbb{N}^n \rightarrow \mathbb{R}_+$ is defined by $\sigma_{k;\mathbf{d}}(\mathbf{m}) = \sigma_k(\prod_{\ell=1}^n m_{\ell}^{d_{\ell}})$.

Particularly, for each $\boldsymbol{\nu} \in \mathbb{N}_0^n$ we have

$$\sigma_{k;\mathbf{d}}(p^{\nu_1}, \dots, p^{\nu_n}) = \sigma_k(p^{\boldsymbol{\nu} \cdot \mathbf{d}}) = \frac{p^{k(1+\boldsymbol{\nu} \cdot \mathbf{d})} - 1}{p^k - 1} = p^{k\boldsymbol{\nu} \cdot \mathbf{d}} - \frac{p^{k(\boldsymbol{\nu} \cdot \mathbf{d})} - 1}{p^k - 1} = p^{k\boldsymbol{\nu} \cdot \mathbf{d}} + O(p^{k\boldsymbol{\nu} \cdot \mathbf{d} - \delta})$$

It follows that $I(f_{\mathbf{d}}) = I_0(f_{\mathbf{d}}) = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$, $\gamma = k$, $\mathbf{g}(\mathbf{e}_i) = 1$, $\mathbf{c}' = \mathbf{1}$ and $\rho = 1$.

The result follows from application of proposition 1.

Case $f = \varphi$: $\varphi_{\mathbf{d}} : \mathbb{N}^n \rightarrow \mathbb{R}_+$ is defined by $\varphi_{\mathbf{d}}(\mathbf{m}) = \varphi(\prod_{j=1}^n m_j^{d_j})$.

Particularly, for $\boldsymbol{\nu} \in \mathbb{N}_0^n$ we have

$$\varphi_{\mathbf{d}}(p^{\nu_1}, \dots, p^{\nu_n}) = \varphi(p^{\boldsymbol{\nu} \cdot \mathbf{d}}) = \begin{cases} p^{\boldsymbol{\nu} \cdot \mathbf{d}} - p^{\boldsymbol{\nu} \cdot \mathbf{d} - 1} = p^{\boldsymbol{\nu} \cdot \mathbf{d}} + O(p^{\boldsymbol{\nu} \cdot \mathbf{d} - \delta}), & \text{si } \boldsymbol{\nu} \neq \mathbf{0} \\ 1, & \text{si } \boldsymbol{\nu} = \mathbf{0} \end{cases}$$

It follows that $I(f_{\mathbf{d}}) = I_0(f_{\mathbf{d}}) = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$, $\gamma = 1$, $\mathbf{g}(\mathbf{e}_i) = 1$, $\mathbf{c}' = \mathbf{1}$ and $\rho = 1$.

The result follows from application of proposition 1.

□

Chapter IV

Mean values of multivariable multiplicative functions and applications to the average number of cyclic subgroups and multivariable averages associated with the LCM function.

This chapter consists of a submitted joint paper with Driss Essouabri and László Tóth.

Abstract. *We use multiple zeta functions to prove, under suitable assumptions, precise asymptotic formulas for the averages of multivariable multiplicative functions. As applications, we prove some conjectures on the average number of cyclic subgroups of the group $\mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_n}$ and multivariable averages associated with the LCM function.*

Mathematics Subject Classifications: 11N37, 11M32, 11M41, 11M45.

Key words: Mean values of multivariable arithmetic functions, multiplicative functions, Zeta functions, meromorphic continuation, tauberian theorems, subgroups averages, LCM multivariable averages.

1 Introduction

Our paper is motivated by the following recent results and conjectures. Let $n \in \mathbb{N}$ and for $m_1, \dots, m_n \in \mathbb{N}$ let $c_n(m_1, \dots, m_n)$ denote the number of cyclic subgroups of the group $\mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_n}$. W. G. Nowak and L. Tóth [24] proved the asymptotic formula

$$\sum_{1 \leq m_1, m_2 \leq x} c_2(m_1, m_2) = x^2 \left(\frac{12}{\pi^4} (\ln x)^3 + a_2 (\ln x)^2 + a_1 (\ln x) + a_0 \right) + O(x^{\frac{1117}{701} + \varepsilon}) \quad \text{as } x \rightarrow +\infty,$$

where a_0, a_1 and a_2 are explicit constants. This error term was improved by L. Tóth and W. Zhai [31] into $O(x^{3/2} (\ln x)^{13/2})$. The case $n = 3$ was investigated by L. Tóth and W. Zhai [29] showing that

$$\sum_{1 \leq m_1, m_2, m_3 \leq x} c_3(m_1, m_2, m_3) = x^3 \sum_{j=0}^7 c_j (\ln x)^j + O(x^{8/3 + \varepsilon}),$$

where c_j ($0 \leq j \leq 7$) are explicit constants. For the proof they used a multidimensional Perron formula and the complex integration method. It is natural to conjecture that such a result holds for $n \geq 4$.

T. Hilberdink, F. Luca, and L. Tóth [17] investigated the following three averages associated with the LCM function:

$$S_n(x) := \sum_{1 \leq m_1, \dots, m_n \leq x} \frac{1}{\text{lcm}(m_1, \dots, m_n)}, \quad (\text{IV.1})$$

$$U_n(x) := \sum_{\substack{1 \leq m_1, \dots, m_n \leq x \\ \gcd(m_1, \dots, m_n) = 1}} \frac{1}{\text{lcm}(m_1, \dots, m_n)}, \quad (\text{IV.2})$$

and

$$V_n(x) := \sum_{1 \leq m_1, \dots, m_n \leq x} \frac{m_1 \dots m_n}{\text{lcm}(m_1, \dots, m_n)}. \quad (\text{IV.3})$$

By using the convolution method, they obtained in their paper asymptotic formulas with error terms for $S_2(x)$, $U_2(x)$ and $V_2(x)$. For $n \geq 3$, they only obtained the estimates

$$(\ln x)^{2^n - 1} \ll S_n(x) \ll (\ln x)^{2^n - 1}, \quad (\ln x)^{2^n - 2} \ll U_n(x) \ll (\ln x)^{2^n - 2},$$

$$x^n \ll V_n(x) \ll x^n (\ln x)^{2^n - 2} \quad \text{as } x \rightarrow +\infty,$$

and conjectured that asymptotic formulas with error terms also exist for these three averages for $n \geq 3$.

In order to prove these conjectures, we introduce a reasonably large class of multivariable multiplicative functions (see Definition 13). For a function $f : \mathbb{N}^n \rightarrow \mathbb{R}_+$ in this class, we establish in Theorem 15 the existence of the meromorphic continuation of the associated multiple zeta function

$$\mathbf{s} = (s_1, \dots, s_n) \rightarrow \mathcal{M}(f; \mathbf{s}) := \sum_{m_1 \geq 1, \dots, m_n \geq 1} \frac{f(m_1, \dots, m_n)}{m_1^{s_1} \dots m_n^{s_n}}$$

and derive several precise properties of this meromorphic continuation. By combining our Theorem 15 and La Bretèche's multivariable Tauberian Theorem (i.e., Theorems 1 and 2 of [7]) we deduce in our Theorem 2 a precise asymptotic formula for the multivariable average

$$\mathcal{N}_\infty(f; x) := \sum_{\substack{\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{N}^n \\ \|\mathbf{m}\|_\infty = \max_i m_i \leq x}} f(m_1, \dots, m_n) \quad \text{as } x \rightarrow +\infty,$$

and derive from it four corollaries.

Our first application, namely Corollary 4, establishes the conjecture concerning the number of cyclic subgroups of the group $\mathbb{Z}_{m_1} \times \dots \times \mathbb{Z}_{m_n}$, in any dimension n . Our Corollaries 5, 6 and 7 prove the conjectures on the above sums associated with the LCM function.

Variants of Theorem 16 with other norm choices can be obtained by combining our Theorem 15 and the first author's multivariable tauberian theorem (i.e., Corollary 2 of [12]). For example, for the class of Holder's norms $\|\mathbf{x}\|_d := \sqrt[d]{|x_1|^d + \dots + |x_n|^d}$ ($d \geq 1$), we obtain in Theorem 17 an asymptotic for the multivariable average

$$\mathcal{N}_d(f; x) := \sum_{\substack{\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{N}^n \\ \|\mathbf{m}\|_d = \sqrt[d]{m_1^d + \dots + m_n^d} \leq x}} f(m_1, \dots, m_n) \quad \text{as } x \rightarrow +\infty.$$

As an application of Theorem 17, we derive in Corollaries 8 and 9 the analogues of Corollaries 4 and 7 for the Holder's norms $\|\cdot\|_d$.

1.1 Notations

1. $\mathbb{N} = \{1, 2, \dots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$; $\mathbb{R}_+ = [0, +\infty)$.
2. The expression: $f(\lambda, \mathbf{y}, \mathbf{x}) \ll_{\mathbf{y}} g(\mathbf{x})$ uniformly in $\mathbf{x} \in X$ and $\lambda \in \Lambda$ means there exists $A = A(\mathbf{y}) > 0$, such that, $\forall \mathbf{x} \in X$ and $\forall \lambda \in \Lambda$ $|f(\lambda, \mathbf{y}, \mathbf{x})| \leq Ag(\mathbf{x})$;
3. For any $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$, we set $\|\mathbf{x}\| = \|\mathbf{x}\|_2 = \sqrt{x_1^2 + \dots + x_n^2}$, $|\mathbf{x}| = |\mathbf{x}|_1 = |x_1| + \dots + |x_n|$ and $|\mathbf{x}|_\infty = \max_{i=1, \dots, n} |x_i|$. We denote the canonical basis of \mathbb{R}^n by $(\mathbf{e}_1, \dots, \mathbf{e}_n)$ (i.e. $\mathbf{e}_{i,j} = 1$ if $i = j$ and $\mathbf{e}_{i,j} = 0$ if $i \neq j$). The standard inner product on \mathbb{R}^n is denoted by $\langle \cdot, \cdot \rangle$. We set also $\mathbf{0} = (0, \dots, 0)$ and $\mathbf{1} = (1, \dots, 1)$;
4. We denote a vector in \mathbb{C}^n by $\mathbf{s} = (s_1, \dots, s_n)$, and write $\mathbf{s} = \boldsymbol{\sigma} + i\boldsymbol{\tau}$, where $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_n)$ and $\boldsymbol{\tau} = (\tau_1, \dots, \tau_n)$ are the real resp. imaginary components of \mathbf{s} (i.e. $\sigma_i = \operatorname{Re}(s_i)$ and $\tau_i = \operatorname{Im}(s_i)$ for all i). We also write $\langle \mathbf{x}, \mathbf{s} \rangle$ for $\sum_i x_i s_i$ if $\mathbf{x} \in \mathbb{R}^n, \mathbf{s} \in \mathbb{C}^n$;
5. A function $f : \mathbb{N}^n \rightarrow \mathbb{C}$ is said to be multiplicative (resp. additive) if for all $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{N}^n$ and $\mathbf{m}' = (m'_1, \dots, m'_n) \in \mathbb{N}^n$ satisfying $\gcd(\operatorname{lcm}(m_i), \operatorname{lcm}(m'_i)) = 1$ we have $f(m_1 m'_1, \dots, m_n m'_n) = f(\mathbf{m}) \cdot f(\mathbf{m}')$ (resp. $= f(\mathbf{m}) + f(\mathbf{m}')$);
6. Let F be a meromorphic function on a domain \mathcal{D} of \mathbb{C}^n and let \mathcal{S} be the support of its polar divisor. F is said to be of moderate growth if $\exists a, b > 0$ such that $\forall \delta > 0$, $F(\mathbf{s}) \ll_{\sigma, \delta} 1 + |\boldsymbol{\tau}|^{a|\boldsymbol{\sigma}|+b}$ uniformly in $\mathbf{s} = \boldsymbol{\sigma} + i\boldsymbol{\tau} \in \mathcal{D}$ verifying $d(\mathbf{s}, \mathcal{S}) \geq \delta$;

2 A class of multivariable multiplicative functions and statement of the main results

2.1 A class of multivariable multiplicative functions

To simplify the exposition, we introduce first the following three definitions.

Definition 12. A quadruple $(g, \kappa, \mathbf{c}, \delta)$ is said to be a data if

1. $g : \mathbb{N}_0^n \rightarrow \mathbb{N}_0$ is a function of subexponential growth; that is g verifies for any $\varepsilon > 0$ $g(\boldsymbol{\nu}) \ll_{\varepsilon} e^{\varepsilon|\boldsymbol{\nu}|}$ uniformly in $\boldsymbol{\nu} \in \mathbb{N}_0^n$;
2. $\kappa : \mathbb{N}_0^n \rightarrow [1, +\infty) \cup \{0\}$ is a function verifying $\kappa(\mathbf{0}) = 0$ and $\inf_{\boldsymbol{\nu} \in \mathbb{N}_0^n \setminus \{\mathbf{0}\}} \frac{\kappa(\boldsymbol{\nu})}{|\boldsymbol{\nu}|} > 0$;
3. $\mathbf{c} = (c_1, \dots, c_n) \in [0, +\infty)^n$ and $\delta \in (0, +\infty)$.

We now introduce the class of multivariable multiplicative functions on which we will focus in this paper.

Definition 13. Let $(g, \kappa, \mathbf{c}, \delta)$ be a data as in Definition 12.

A multivariable multiplicative function $f : \mathbb{N}^n \rightarrow \mathbb{R}$ is said to be in the class $\mathcal{C}(g, \kappa, \mathbf{c}, \delta)$ if for any $\varepsilon > 0$,

$$f(p^{\nu_1}, \dots, p^{\nu_n}) - g(\boldsymbol{\nu}) p^{\langle \mathbf{c}, \boldsymbol{\nu} \rangle - \kappa(\boldsymbol{\nu})} \ll_{\varepsilon} e^{\varepsilon|\boldsymbol{\nu}|} p^{\langle \mathbf{c}, \boldsymbol{\nu} \rangle - \kappa(\boldsymbol{\nu}) - \delta}, \quad (\text{IV.4})$$

uniformly in $\boldsymbol{\nu} \in \mathbb{N}_0^n$ and p prime number.

We will need also the following integral definition.

Definition 14. Let I be a finite subset of $\mathbb{N}_0^n \setminus \{\mathbf{0}\}$, $\mathbf{u} = (u(\boldsymbol{\nu}))_{\boldsymbol{\nu} \in I}$ be a finite sequence of positive integers and $\mathbf{c} = (c_1, \dots, c_n) \in [0, +\infty)^n$. We denote by $\boldsymbol{\nu}^1, \dots, \boldsymbol{\nu}^r$ the elements of I where $r = \#I$, and define the finite sequence q_k ($0 \leq k \leq r$) by

$$q_0 = 0 \quad \text{and} \quad q_k = \sum_{j=1}^k u(\boldsymbol{\nu}^j) \quad \forall k \in \{1, \dots, r\}.$$

We define then for $x > 0$ the integral

$$\mathcal{I}_n(I, \mathbf{u}, \mathbf{c}; x) := \int_{\mathcal{A}(I, \mathbf{u}; x)} \frac{dy_1 \dots dy_r}{\prod_{k=1}^r \prod_{\ell=q_{k-1}+1}^{q_k} y_\ell^{1-\langle \boldsymbol{\nu}^k, \mathbf{c} \rangle}},$$

$$\text{where } \mathcal{A}(I, \mathbf{u}; x) := \left\{ \mathbf{y} \in [1, +\infty)^{q_r}; \prod_{k=1}^r \prod_{\ell=q_{k-1}+1}^{q_k} y_\ell^{\langle \boldsymbol{\nu}^k, \mathbf{e}_j \rangle} \leq x \quad \forall j \in \{1, \dots, n\} \right\}.$$

2.2 Statement of the main results

Let $f : \mathbb{N}^n \rightarrow \mathbb{R}$ be a multivariable multiplicative function. We assume that f belongs to the class $\mathcal{C}(g, \kappa, \mathbf{c}, \delta)$ associated to the data $(g, \kappa, \mathbf{c}, \delta)$ (see Definitions 1 and 2 above). We assume also that the finite set

$$I = I(\kappa, g) := \{\boldsymbol{\nu} \in \mathbb{N}_0^n \mid \kappa(\boldsymbol{\nu}) = 1 \text{ and } g(\boldsymbol{\nu}) \neq 0\} \quad \text{is nonempty.} \quad (\text{IV.5})$$

The following theorem is *the main analytic ingredient* of this paper:

Theorem 15. 1. *the multiple zeta function*

$$\mathbf{s} = (s_1, \dots, s_n) \rightarrow \mathcal{M}(f; \mathbf{s}) := \sum_{m_1 \geq 1, \dots, m_n \geq 1} \frac{f(m_1, \dots, m_n)}{m_1^{s_1} \dots m_n^{s_n}}$$

converges absolutely in the domain $\{\mathbf{s} \in \mathbb{C}^n \mid \text{Re}(s_i) > c_i \quad \forall i \in \{1, \dots, n\}\}$;

2. *there exists $\varepsilon_0 > 0$ such that the function*

$$\mathbf{s} = (s_1, \dots, s_n) \rightarrow \mathcal{H}(f, \mathbf{c}; \mathbf{s}) := \left(\prod_{\boldsymbol{\nu} \in I} \langle \boldsymbol{\nu}, \mathbf{s} \rangle^{g(\boldsymbol{\nu})} \right) \mathcal{M}(f; \mathbf{c} + \mathbf{s})$$

has **holomorphic** continuation to the domain $\{\mathbf{s} \in \mathbb{C}^n \mid \text{Re}(s_i) > -\varepsilon_0 \quad \forall i \in \{1, \dots, n\}\}$ and verifies in it the following estimate: for all $\varepsilon > 0$,

$$\mathcal{H}(f, \mathbf{c}; \mathbf{s}) \ll_\varepsilon \prod_{\boldsymbol{\nu} \in I} (|\langle \boldsymbol{\nu}, \mathbf{s} \rangle| + 1)^{g(\boldsymbol{\nu})(1 - \frac{1}{2} \min(0, \text{Re}(\langle \boldsymbol{\nu}, \mathbf{s} \rangle))) + \varepsilon};$$

3. $\mathcal{H}(f, \mathbf{c}; \mathbf{0})$ is given by the following convergent Euler product:

$$\mathcal{H}(f, \mathbf{c}; \mathbf{0}) = \prod_p \left(1 - \frac{1}{p} \right)^{\sum_{\boldsymbol{\nu} \in I} g(\boldsymbol{\nu})} \left(\sum_{\boldsymbol{\nu} \in \mathbb{N}_0^n} \frac{f(p^{\nu_1}, \dots, p^{\nu_n})}{p^{\langle \boldsymbol{\nu}, \mathbf{c} \rangle}} \right). \quad (\text{IV.6})$$

Combining our Theorem 15 and La Bretèche's multivariable Tauberian Theorem (i.e Theorems 1 and 2 of [7]) yields to the following multivariable mean value theorem:

Theorem 16. Let $f : \mathbb{N}^n \rightarrow \mathbb{R}_+$ be a nonnegative multivariable multiplicative function satisfying assumptions of Theorem 15.

Set $J := \{\mathbf{e}_i \mid c_i = 0\}$ where $(\mathbf{e}_1, \dots, \mathbf{e}_n)$ is the canonical basis of \mathbb{R}^n .

Set also $\rho := (\sum_{\nu \in I} g(\nu)) + \#J - \text{Rank}(I \cup J)$.

Then,

there exist a polynomial Q_∞ of degree at most ρ and a positive constant $\mu_\infty > 0$ such that

$$\mathcal{N}_\infty(f; x) := \sum_{\substack{\mathbf{m}=(m_1, \dots, m_n) \in \mathbb{N}^n \\ \|\mathbf{m}\|_\infty = \max_i m_i \leq x}} f(m_1, \dots, m_n) = x^{|\mathbf{c}|} Q_\infty(\ln x) + O(x^{|\mathbf{c}| - \mu_\infty}) \quad \text{as } x \rightarrow +\infty.$$

Furthermore, if we assume in addition that the two following assumptions hold:

1. $\text{Rank}(I \cup J) = n$;
2. $\mathbf{1} = (1, \dots, 1)$ is in the interior of the cone generated by $I \cup J$; that is $\mathbf{1} \in \text{con}^*(I \cup J) := \{\sum_{\nu \in I \cup J} \lambda_\nu \nu \mid \lambda_\nu \in (0, +\infty) \forall \nu \in I \cup J\}$,

Then, the degree of the polynomial Q_∞ is equal to $\rho = (\sum_{\nu \in I} g(\nu)) + \#J - n$ and the main term of $\mathcal{N}_\infty(f; x)$ is given by

$$\mathcal{N}_\infty(f; x) = C_n(f) K_n(f, \|\cdot\|_\infty) x^{|\mathbf{c}|} (\ln x)^\rho + O((\ln x)^{\rho-1}) \quad \text{as } x \rightarrow +\infty,$$

where $C_n(f) := \mathcal{H}(f, \mathbf{c}; \mathbf{0}) > 0$ is defined by the Euler product (IV.6) and

$$K_n(f, \|\cdot\|_\infty) := \lim_{x \rightarrow +\infty} \mathcal{I}_n(I, \mathbf{u}, \mathbf{c}; x) x^{-|\mathbf{c}|} (\ln x)^{-\rho} > 0, \quad \text{where}$$

$\mathcal{I}_n(I, \mathbf{u}, \mathbf{c}; x)$ is the integral (see Definition 14) associated to the finite set I , the finite sequence $\mathbf{u} = (g(\nu))_{\nu \in I}$ and to the vector \mathbf{c} .

Remark 13. If $\{\mathbf{e}_1, \dots, \mathbf{e}_n\} \subset I \cup J$, then the two assumptions $\text{Rank}(I \cup J) = n$ and $\mathbf{1} \in \text{con}^*(I \cup J)$ clearly hold.

Combining our Theorem 15 and the first author's multivariable tauberian theorem (i.e Corollary 2 of [12]) yields to the following multivariable mean value theorem for Holder's norms $\|\mathbf{x}\|_d := \sqrt[d]{|x_1|^d + \dots + |x_n|^d}$ ($d \geq 1$):

Theorem 17. Let $f : \mathbb{N}^n \rightarrow \mathbb{R}_+$ be a nonnegative multivariable multiplicative function satisfying assumptions of Theorem 15.

Assume that $\mathbf{c} = (c_1, \dots, c_n) \in (0, +\infty)^n$. Set

1. $\rho := (\sum_{\nu \in I} g(\nu)) - \text{Rank}(I)$;
2. $I_{\mathbf{c}} := \{\langle \mathbf{c}, \nu \rangle^{-1} \nu \mid \nu \in I\}$ and $\mathbf{u} := (u(\beta))_{\beta \in I_{\mathbf{c}}}$ where $u(\beta) = \sum_{\nu \in I, \langle \mathbf{c}, \nu \rangle^{-1} \nu = \beta} g(\nu)$.

Then,

there exist a polynomial Q of degree at most ρ and a positive constant $\mu > 0$ such that

$$\mathcal{N}_d(f; x) := \sum_{\substack{\mathbf{m}=(m_1, \dots, m_n) \in \mathbb{N}^n \\ \|\mathbf{m}\|_d = \sqrt[d]{m_1^d + \dots + m_n^d} \leq x}} f(m_1, \dots, m_n) = x^{|\mathbf{c}|} Q(\ln x) + O(x^{|\mathbf{c}| - \mu}) \quad \text{as } x \rightarrow +\infty.$$

Furthermore, if we assume in addition that $\text{Rank}(I) = n$ and $\mathbf{1} \in \text{con}^*(I)$, then, the degree of the polynomial Q is equal to $\rho = (\sum_{\nu \in I} g(\nu)) - n$ and the main term of $\mathcal{N}_d(f; x)$ is given by

$$\mathcal{N}_d(f; x) = C_n(f) K_n(f, \|\cdot\|_d) x^{|\mathbf{c}|} (\ln x)^\rho + O((\ln x)^{\rho-1}) \quad \text{as } x \rightarrow +\infty,$$

where $C_n(f) := \mathcal{H}(f, \mathbf{c}; \mathbf{0}) > 0$ is defined by the Euler product (IV.6) above and

$$K_n(f, \|\!\|_d) := \left(\prod_{\nu \in I} \langle \nu, \mathbf{c} \rangle^{-g(\nu)} \right) \frac{d^{\rho+1} A_0(\mathcal{T}_{\mathbf{c}}, P_d)}{|\mathbf{c}| \rho!} > 0.$$

where $A_0(\mathcal{T}_{\mathbf{c}}, P_d) > 0$ is the mixed volume constant (see §2.3.3 of [12]) associated to the pair $\mathcal{T}_{\mathbf{c}} := (I_{\mathbf{c}}, \mathbf{u})$ and the polynomial $P_d = X_1^d + \dots + X_n^d$.

2.3 Applications

We will now give the applications that motivated our general results of section §2.2.

On the average number of cyclic subgroups of the group $\mathbb{Z}_{m_1} \times \dots \times \mathbb{Z}_{m_n}$

Let $n \in \mathbb{N}$. For $m_1, \dots, m_n \in \mathbb{N}$ denote by $c_n(m_1, \dots, m_n)$ the number of cyclic subgroups of the group $\mathbb{Z}_{m_1} \times \dots \times \mathbb{Z}_{m_n}$. Set

$$G_n(x) := \sum_{1 \leq m_1, \dots, m_n \leq x} c_n(m_1, \dots, m_n).$$

As we mentioned in the introduction, precise asymptotic for $G_2(x)$ was obtained by W. G. Nowak and L. Tóth in [24] and improved by L. Tóth and W. Zhai in [31]. The case $n = 3$ was also investigated by L. Tóth and W. Zhai in [29]. It is natural to conjecture that such a result holds for $n \geq 4$. The following result establish this conjecture in any dimension n .

Corollary 4. *Let $n \in \mathbb{N}$. There exists a polynomial Q_1 of degree $2^n - 1$ and $\mu_1 > 0$ such that*

$$G_n(x) := \sum_{1 \leq m_1, \dots, m_n \leq x} c_n(m_1, \dots, m_n) = x^n Q_1(\ln x) + O(x^{n-\mu_1}) \quad \text{as } x \rightarrow +\infty.$$

In particular, we have

$$G_n(x) = C_n(c_n) K_n(c_n, \|\!\|_{\infty}) x^n (\ln x)^{2^n-1} + O(x^n (\ln x)^{2^n-2}) \quad \text{as } x \rightarrow +\infty,$$

where

$$C_n(c_n) := \prod_p \left(1 - \frac{1}{p} \right)^{2^n+n-1} \left(\sum_{\nu \in \mathbb{N}_0^n} \frac{c_n(p^{\nu_1}, \dots, p^{\nu_n})}{p^{|\nu|}} \right) > 0 \quad (\text{IV.7})$$

and

$$K_n(c_n, \|\!\|_{\infty}) := \lim_{x \rightarrow +\infty} \mathcal{I}_n(I, \mathbf{u}; x) x^{-n} (\ln x)^{-2^n+1} > 0, \quad \text{where}$$

$\mathcal{I}_n(I, \mathbf{u}, \mathbf{c}; x)$ is the integral (see Definition 14) associated to $I = \{0, 1\}^n \setminus \{\mathbf{0}\}$, to the sequence $\mathbf{u} = (u(\nu))_{\nu \in I}$ defined by $u(\mathbf{e}_i) = 2 \forall i = 1, \dots, n$ and $u(\nu) = 1 \forall \nu \in I \setminus \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ and to the vector $\mathbf{c} = \mathbf{1}$.

Remark 14. *We will compute more explicitly in §7 below the constants $C_n(c_n)$ and $K_n(c_n, \|\!\|_{\infty})$ for $n = 2$ and $n = 3$. In particular, we will prove in §7.1 and §7.3 that $C_2(c_2) = \frac{36}{\pi^4}$ and $K_2(c_2, \|\!\|_{\infty}) = \frac{1}{3}$. Thus, our mains term in the asymptotic of $G_2(x)$ agree with the main term obtained by the convolution method in [24] by W. G. Nowak and L. Tóth.*

Some multivariable averages associated to the LCM function

As we mentioned in the introduction, T. Hilberdink, F. Luca, and L. Tóth introduced in [17] the three averages (IV.1), (IV.2) and (IV.3) associated to the LCM function and obtained in this paper asymptotic formulas for $S_2(x)$, $U_2(x)$ and $V_2(x)$. For $n \geq 3$, they only obtained the following estimates

$$(\ln x)^{2^n-1} \ll S_n(x) \ll (\ln x)^{2^n-1}, \quad (\ln x)^{2^n-2} \ll U_n(x) \ll (\ln x)^{2^n-2},$$

$$x^n \ll V_n(x) \ll x^n (\ln x)^{2^n-2},$$

and conjectured that asymptotic formulas also exist for these three averages for $n \geq 3$. The following three corollaries prove these conjectures.

Corollary 5. *Let $n \in \mathbb{N}$. There exists a polynomial Q_2 of degree $2^n - 1$ and $\mu_2 > 0$ such that*

$$S_n(x) := \sum_{1 \leq m_1, \dots, m_n \leq x} \frac{1}{\text{lcm}(m_1, \dots, m_n)} = Q_2(\ln x) + O(x^{-\mu_2}) \quad \text{as } x \rightarrow +\infty.$$

In particular, we have

$$S_n(x) = C_n(s_n) K_n(s_n, \|\|\|_\infty) (\ln x)^{2^n-1} + O((\ln x)^{2^n-2}) \quad \text{as } x \rightarrow +\infty,$$

where

$$C_n(s_n) := \prod_p \left(1 - \frac{1}{p}\right)^{2^n-1} \left(\sum_{k=0}^{\infty} \frac{(k+1)^n - k^n}{p^k}\right) > 0, \quad (\text{IV.8})$$

and

$$K_n(s_n, \|\|\|_\infty) := \lim_{x \rightarrow +\infty} \mathcal{I}_n(I, \mathbf{u}, \mathbf{c}; x) (\ln x)^{-2^n+1} > 0, \quad \text{where}$$

$\mathcal{I}_n(I, \mathbf{u}, \mathbf{c}; x)$ is the integral (see Definition 14) associated to $I = \{0, 1\}^n \setminus \{\mathbf{0}\}$, to the sequence $\mathbf{u} = (u(\boldsymbol{\nu}))_{\boldsymbol{\nu} \in I}$ defined by $u(\boldsymbol{\nu}) = 1 \forall \boldsymbol{\nu} \in I$ and to the vector $\mathbf{c} = \mathbf{0}$.

Corollary 6. *Let $n \in \mathbb{N} \setminus \{1\}$. There exists a polynomial Q_3 of degree $2^n - 2$ and $\mu_3 > 0$ such that*

$$U_n(x) := \sum_{\substack{1 \leq m_1, \dots, m_n \leq x \\ \text{gcd}(m_1, \dots, m_n) = 1}} \frac{1}{\text{lcm}(m_1, \dots, m_n)} = Q_3(\ln x) + O(x^{-\mu_3}) \quad \text{as } x \rightarrow +\infty.$$

In particular, we have

$$U_n(x) = C_n(u_n) K_n(u_n, \|\|\|_\infty) (\ln x)^{2^n-2} + O((\ln x)^{2^n-3}) \quad \text{as } x \rightarrow +\infty,$$

where

$$C_n(u_n) := \prod_p \left(1 - \frac{1}{p}\right)^{2^n-2} \left(\sum_{k=0}^{\infty} \frac{(k+1)^n - k^n - 1}{p^k}\right) > 0, \quad (\text{IV.9})$$

and

$$K_n(u_n, \|\|\|_\infty) := \lim_{x \rightarrow +\infty} \mathcal{I}_n(I, \mathbf{u}, \mathbf{c}; x) (\ln x)^{-2^n+2} > 0, \quad \text{where}$$

$\mathcal{I}_n(I, \mathbf{u}, \mathbf{c}; x)$ is the integral (see Definition 14) associated to $I = \{0, 1\}^n \setminus \{\mathbf{0}, \mathbf{1}\}$, to the sequence $\mathbf{u} = (u(\boldsymbol{\nu}))_{\boldsymbol{\nu} \in I}$ defined by $u(\boldsymbol{\nu}) = 1 \forall \boldsymbol{\nu} \in I$ and to the vector $\mathbf{c} = \mathbf{0}$.

Corollary 7. Let $n \in \mathbb{N}$. There exists a polynomial Q_4 of degree $2^n - n - 1$ and $\mu_4 > 0$ such that

$$V_n(x) := \sum_{1 \leq m_1, \dots, m_n \leq x} \frac{m_1 \dots m_n}{\text{lcm}(m_1, \dots, m_n)} = x^n Q_4(\ln x) + O(x^{n-\mu_4}) \quad \text{as } x \rightarrow +\infty.$$

In particular, we have

$$V_n(x) = C_n(v_n) K_n(v_n, \|\|\|\infty) x^n (\ln x)^{2^n - n - 1} + O(x^n (\ln x)^{2^n - n - 2}) \quad \text{as } x \rightarrow +\infty,$$

where

$$C_n(v_n) := \prod_p \left(1 - \frac{1}{p}\right)^{2^n - 1} \left(\sum_{k=0}^{\infty} \frac{(k+1)^n - k^n}{p^k}\right) > 0, \quad (\text{IV.10})$$

and

$$K_n(v_n, \|\|\|\infty) := \lim_{x \rightarrow +\infty} \mathcal{I}_n(I, \mathbf{u}, \mathbf{c}; x) x^{-n} (\ln x)^{-2^n + n + 1} > 0, \quad \text{where}$$

$\mathcal{I}_n(I, \mathbf{u}, \mathbf{c}; x)$ is the integral (see Definition 14) associated to $I = \{0, 1\}^n \setminus \{\mathbf{0}\}$, to the sequence $\mathbf{u} = (u(\boldsymbol{\nu}))_{\boldsymbol{\nu} \in I}$ defined by $u(\boldsymbol{\nu}) = 1 \forall \boldsymbol{\nu} \in I$ and to the vector $\mathbf{c} = \mathbf{1}$.

Remark 15. The constants $C_n(s_n)$, $C_n(u_n)$ and $C_n(v_n)$ are equal. We will compute more explicitly in sections §7.2, §7.4, §7.5 and §7.6 below the constants $C_n(\cdot)$ and $K_n(\cdot, \|\|\|\infty)$ for $n = 2$ and $n = 3$. More precisely, we will prove that

1. $C_2(s_2) = C_2(u_2) = C_2(v_2) = \frac{6}{\pi^2}$ and $C_3(s_3) = C_3(u_3) = C_3(v_3) = \prod_p \left(1 - \frac{9}{p^2} + \frac{16}{p^3} - \frac{9}{p^4} + \frac{1}{p^6}\right)$;
2. $K_2(s_2, \|\|\|\infty) = \frac{1}{3}$ and $K_2(u_2, \|\|\|\infty) = K_2(v_2, \|\|\|\infty) = 1$;
3. $K_3(s_3, \|\|\|\infty) = \frac{11}{3366}$, $K_3(u_3, \|\|\|\infty) = \frac{11}{480}$ and $K_3(v_3, \|\|\|\infty) = \frac{1}{16}$.

In particular, our main term in the asymptotic of $S_2(x)$, $U_2(x)$ and $V_2(x)$ agree with those obtained by the convolution method in [17] by T. Hilberdink, F. Luca, and L. Tóth.

Multivariable averages with other norms

The following two results give analogues of Corollaries 4 and 7 for some other choices of norms.

Corollary 8. Let $n \in \mathbb{N}$ and $d \geq 1$. There exists a polynomial Q_5 of degree $2^n - 1$ and $\mu_5 > 0$ such that

$$G_{n,d}(x) := \sum_{\substack{\mathbf{m}=(m_1, \dots, m_n) \in \mathbb{N}^n \\ \|\mathbf{m}\|_d = \sqrt[d]{m_1^d + \dots + m_n^d} \leq x}} c_n(m_1, \dots, m_n) = x^n Q_5(\ln x) + O(x^{n-\mu_5}) \quad \text{as } x \rightarrow +\infty.$$

Moreover, if we set $\tilde{I} := \{|\boldsymbol{\nu}|^{-1} \boldsymbol{\nu} \mid \boldsymbol{\nu} \in \{0, 1\}^n \setminus \{\mathbf{0}\}\}$ and $\mathbf{u} = (u(\boldsymbol{\beta}))_{\boldsymbol{\beta} \in \tilde{I}}$ where $u(\boldsymbol{\beta}) = 2$ if $\boldsymbol{\beta} \in \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ and $u(\boldsymbol{\beta}) = 1$ otherwise, then

$$G_{n,d}(x) = C_n(c_n) K_n(c_n, \|\|\|_d) x^n (\ln x)^{2^n - 1} + O(x^n (\ln x)^{2^n - 2}) \quad \text{as } x \rightarrow +\infty,$$

where $C_n(c_n) > 0$ is given by (IV.7) and

$$K_n(c_n, \|\|\|_d) = \left(\prod_{k=2}^n k^{-\binom{n}{k}}\right) \frac{d^{2^n} A_0(\mathcal{T}, P_d)}{n (2^n - 1)!} > 0,$$

where $A_0(\mathcal{T}, P_d)$ is the mixed volume constant (see §2.3.3 of [12]) associated to the pair $\mathcal{T} = (\tilde{I}, \mathbf{u})$ and the polynomial $P_d = X_1^d + \dots + X_n^d$.

Corollary 9. *Let $n \in \mathbb{N}$ and $d \geq 1$. There exists a polynomial Q_6 of degree $2^n - n - 1$ and $\mu_6 > 0$ such that*

$$V_{n,d}(x) := \sum_{\substack{\mathbf{m}=(m_1,\dots,m_n) \in \mathbb{N}^n \\ \|\mathbf{m}\|_d = \sqrt[d]{m_1^d + \dots + m_n^d} \leq x}} \frac{m_1 \dots m_n}{\text{lcm}(m_1, \dots, m_n)} = x^n Q_6(\ln x) + O(x^{n-\mu_6}) \quad \text{as } x \rightarrow +\infty.$$

Moreover, if we set $\tilde{I} := \{|\boldsymbol{\nu}|^{-1} \boldsymbol{\nu} \mid \boldsymbol{\nu} \in \{0, 1\}^n \setminus \{\mathbf{0}\}\}$ and $\mathbf{u} = (u(\boldsymbol{\beta}))_{\boldsymbol{\beta} \in \tilde{I}}$ where $u(\boldsymbol{\beta}) = 1 \forall \boldsymbol{\beta} \in \tilde{I}$, then

$$V_{n,d}(x) = C_n(v_n) K_n(v_n, \|\cdot\|_d) x^n (\ln x)^{2^n - n - 1} + O(x^n (\ln x)^{2^n - n - 2}) \quad \text{as } x \rightarrow +\infty,$$

where $C_n(v_n) > 0$ is given by (IV.10) and

$$K_n(v_n, \|\cdot\|_d) = \left(\prod_{k=2}^n k^{-\binom{n}{k}} \right) \frac{d^{2^n - n} A_0(\mathcal{T}, P_d)}{n (2^n - n - 1)!} > 0,$$

where $A_0(\mathcal{T}, P_d)$ is the mixed volume constant (see §2.3.3 of [12]) associated to the pair $\mathcal{T} = (\tilde{I}, \mathbf{u})$ and the polynomial $P_d = X_1^d + \dots + X_n^d$.

Remark 16. *The constants $C_n(\cdot)$, are independent on the choice of the norm. The constants $K_n(\cdot)$ depend on the choice of the norm. We will compute more explicitly in §7.7 below the constants $K_n(c_n, \|\cdot\|_d)$ and $K_n(v_n, \|\cdot\|_d)$ for $n = 2$ and $n = 3$. More precisely, we will prove that*

1. $K_2(c_2, \|\cdot\|_d) = \frac{1}{6d} \frac{\Gamma(\frac{1}{d})^2}{\Gamma(\frac{2}{d})}$ and $K_2(v_2, \|\cdot\|_d) = \frac{1}{2d} \frac{\Gamma(1/d)^2}{\Gamma(2/d)}$;
2. $K_3(c_3, \|\cdot\|_d) = \frac{31}{30240} \frac{\Gamma(\frac{1}{d})^3}{d^2 \Gamma(\frac{3}{d})}$ and $K_3(v_3, \|\cdot\|_d) = \frac{\Gamma(\frac{1}{d})^3}{2 d^2 \Gamma(\frac{3}{d})}$.

3 Proof of Theorem 15

Let $f : \mathbb{N}^n \rightarrow \mathbb{R}$ be a multiplicative function in the class $\mathcal{C}(g, \kappa, \mathbf{c}, \delta)$. Define for $\boldsymbol{\nu} \in \mathbb{N}_0^n$ and p prime $V(p, \boldsymbol{\nu})$ by the formula

$$f(p^{\nu_1}, \dots, p^{\nu_n}) = (g(\boldsymbol{\nu}) + V(p, \boldsymbol{\nu})) p^{(\mathbf{c}, \boldsymbol{\nu}) - \kappa(\boldsymbol{\nu})} \quad (\text{IV.11})$$

Since $(g, \kappa, \mathbf{c}, \delta)$ is a data, point 1 of Definition 12 and assumption (IV.4) can then be written in the following more convenient equivalent form:

$$\forall \varepsilon > 0, \quad g(\boldsymbol{\nu}) \ll_{\varepsilon} e^{\varepsilon|\boldsymbol{\nu}|} \quad \text{and} \quad V(p, \boldsymbol{\nu}) \ll_{\varepsilon} e^{\varepsilon|\boldsymbol{\nu}|} p^{-\delta}, \quad (\text{IV.12})$$

uniformly in $\boldsymbol{\nu} \in \mathbb{N}_0^n$ and in p prime number.

Moreover, point 2 of Definition 12 implies that there exists $\beta > 0$ such that

$$\kappa(\boldsymbol{\nu}) \geq \max(1, \beta|\boldsymbol{\nu}|) \quad \forall \boldsymbol{\nu} \in \mathbb{N}_0^n \setminus \{\mathbf{0}\}. \quad (\text{IV.13})$$

3.1 Proof of point 1 of Theorem 15

Let $\mathbf{s} = (s_1, \dots, s_n) \in \mathbb{C}^n$ be such that $\sigma_i = \operatorname{Re}(s_i) > c_i \forall i = 1, \dots, n$.

Set $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_n)$ and $\eta = \frac{1}{2} \min_{i=1, \dots, n} (\sigma_i - c_i) > 0$.

So, we have $\sigma_i \geq c_i + 2\eta \forall i$ and $\langle \boldsymbol{\sigma}, \boldsymbol{\nu} \rangle \geq \langle \mathbf{c}, \boldsymbol{\nu} \rangle + 2\eta |\boldsymbol{\nu}| \forall \boldsymbol{\nu} \in \mathbb{N}_0^n$. Choose $\varepsilon > 0$ small enough such that $e^\varepsilon < 2^\eta$. It follows then from (IV.11) and (IV.12) that we have for any prime number p ,

$$\begin{aligned} \sum_p \sum_{|\boldsymbol{\nu}| \geq 1} \left| \frac{f(p^{\nu_1}, \dots, p^{\nu_n})}{p^{\langle \mathbf{s}, \boldsymbol{\nu} \rangle}} \right| &= \sum_p \sum_{|\boldsymbol{\nu}| \geq 1} \frac{|f(p^{\nu_1}, \dots, p^{\nu_n})|}{p^{\langle \boldsymbol{\sigma}, \boldsymbol{\nu} \rangle}} \ll_\varepsilon \sum_p \sum_{|\boldsymbol{\nu}| \geq 1} \frac{e^{\varepsilon |\boldsymbol{\nu}|} p^{\langle \mathbf{c}, \boldsymbol{\nu} \rangle - \kappa(\boldsymbol{\nu})}}{p^{\langle \mathbf{c}, \boldsymbol{\nu} \rangle + 2\eta |\boldsymbol{\nu}|}} \\ &\ll_\varepsilon \sum_p \sum_{|\boldsymbol{\nu}| \geq 1} \frac{e^{\varepsilon |\boldsymbol{\nu}|}}{p^{\kappa(\boldsymbol{\nu}) + 2\eta |\boldsymbol{\nu}|}} \ll_\varepsilon \sum_p \frac{1}{p^{1+\eta}} \sum_{|\boldsymbol{\nu}| \geq 1} \frac{e^{\varepsilon |\boldsymbol{\nu}|}}{p^{\eta |\boldsymbol{\nu}|}} \\ &\ll_\varepsilon \sum_p \frac{1}{p^{1+\eta}} \sum_{|\boldsymbol{\nu}| \geq 1} \left(\frac{e^\varepsilon}{2^\eta} \right)^{|\boldsymbol{\nu}|} \ll_\varepsilon \sum_p \frac{1}{p^{1+\eta}} < \infty. \end{aligned}$$

The multiplicativity of f implies then that $\mathbf{s} \rightarrow \mathcal{M}(f; \mathbf{s})$ converges absolutely and that

$$\mathcal{M}(f; \mathbf{s}) = \sum_{\mathbf{m} \in \mathbb{N}^n} \frac{f(m_1, \dots, m_n)}{m^{s_1} \dots m^{s_n}} = \prod_p \left(\sum_{\boldsymbol{\nu} \in \mathbb{N}_0^n} \frac{f(p^{\nu_1}, \dots, p^{\nu_n})}{p^{\langle \mathbf{s}, \boldsymbol{\nu} \rangle}} \right). \quad (\text{IV.14})$$

This ends the proof of point 1 of Theorem 15. \square

3.2 Two useful lemmas

Recall that $I = I(\kappa, g) := \{\boldsymbol{\nu} \in \mathbb{N}_0^n \mid \kappa(\boldsymbol{\nu}) = 1 \text{ and } g(\boldsymbol{\nu}) \neq 0\}$ is a nonempty set.

For all $t \in \mathbb{R}$, set $U_t := \{\mathbf{s} \in \mathbb{C}^n \mid \sigma_i = \operatorname{Re}(s_i) > t \forall i = 1, \dots, n\}$.

We need the following two lemmas:

Lemma 17. *There exists $\varepsilon_1, \eta_1 > 0$ such that for any prime number p , the function*

$$\mathbf{s} \mapsto R_p(\mathbf{s}) := \left(\sum_{|\boldsymbol{\nu}| \geq 1} \frac{f(p^{\nu_1}, \dots, p^{\nu_n})}{p^{\langle \mathbf{c} + \mathbf{s}, \boldsymbol{\nu} \rangle}} \right) - \left(\sum_{\boldsymbol{\nu} \in I} \frac{g(\boldsymbol{\nu})}{p^{1 + \langle \boldsymbol{\nu}, \mathbf{s} \rangle}} \right)$$

is holomorphic in the domain $U_{-\varepsilon_1}$ and verifies in it the estimate

$$R_p(\mathbf{s}) \ll p^{-1-\eta_1} \quad \text{uniformly in } p.$$

Lemma 18. *Set $\varepsilon_2 = \inf_{\boldsymbol{\nu} \in I} \frac{1}{4|\boldsymbol{\nu}|}$ and $\eta_2 = \frac{1}{2}$. Then, for any prime number p , the function*

$$\mathbf{s} \mapsto L_p(\mathbf{s}) := \left(\prod_{\boldsymbol{\nu} \in I} \left(1 - \frac{1}{p^{1 + \langle \boldsymbol{\nu}, \mathbf{s} \rangle}} \right)^{g(\boldsymbol{\nu})} \right) - 1 + \left(\sum_{\boldsymbol{\nu} \in I} \frac{g(\boldsymbol{\nu})}{p^{1 + \langle \boldsymbol{\nu}, \mathbf{s} \rangle}} \right)$$

is holomorphic in the domain $U_{-\varepsilon_2}$ and verifies in it the estimate

$$L_p(\mathbf{s}) \ll p^{-1-\eta_2} \quad \text{uniformly in } p.$$

Proof of Lemma 17

Fix $\beta > 0$ such that (IV.13) holds. Fix also a positive integer N verifying $N \geq \max\left(4\beta^{-1}, \max_{\nu \in I} |\nu|\right)$. Identity (IV.11) implies that for p prime number and $\mathbf{s} \in U_0 = \{\mathbf{s} \in \mathbb{C}^n \mid \sigma_i > 0 \forall i\}$, we have

$$R_p(\mathbf{s}) = R_p^1(\mathbf{s}) + R_p^2(\mathbf{s}), \quad \text{where} \quad (\text{IV.15})$$

$$R_p^1(\mathbf{s}) = \sum_{1 \leq |\nu| \leq N} \frac{V(p, \nu)}{p^{\kappa(\nu) + \langle \nu, \mathbf{s} \rangle}} + \sum_{\substack{\nu \notin I \\ 1 \leq |\nu| \leq N}} \frac{g(\nu)}{p^{\kappa(\nu) + \langle \nu, \mathbf{s} \rangle}} \quad \text{and} \quad R_p^2(\mathbf{s}) = \sum_{|\nu| > N} \frac{g(\nu) + V(p, \nu)}{p^{\kappa(\nu) + \langle \nu, \mathbf{s} \rangle}}.$$

To prove Lemma 17 it suffices to verify that both $\mathbf{s} \mapsto R_p^1(\mathbf{s})$ and $\mathbf{s} \mapsto R_p^2(\mathbf{s})$ satisfy the conclusions of Lemma 17.

CLAIM 1: $\mathbf{s} \mapsto R_p^1(\mathbf{s})$ satisfies the conclusions of Lemma 17.

Proof of CLAIM 1: It's clear that $\mathbf{s} \mapsto R_p^1(\mathbf{s})$ is holomorphic in the whole space \mathbb{C}^n .

Let $\varepsilon > 0$. It follows from (IV.12) and (IV.13) that for p prime number and $\mathbf{s} \in U_{-\varepsilon} = \{\mathbf{s} \in \mathbb{C}^n \mid \sigma_i > -\varepsilon \forall i\}$, we have

$$|R_p^1(\mathbf{s})| \leq \sum_{1 \leq |\nu| \leq N} \frac{|V(p, \nu)|}{p^{1 + \langle \nu, \sigma \rangle}} + \sum_{\substack{\nu \notin I \\ 1 \leq |\nu| \leq N}} \frac{g(\nu)}{p^{\kappa(\nu) + \langle \nu, \sigma \rangle}} \ll \sum_{\nu \in I} \frac{p^{-\delta}}{p^{1 - \varepsilon |\nu|}} + \sum_{\substack{\nu \notin I, g(\nu) \neq 0 \\ 1 \leq |\nu| \leq N}} \frac{1}{p^{\kappa(\nu) - \varepsilon |\nu|}} \quad (\text{IV.16})$$

Since $\kappa(\nu) > 1$ if $\nu \notin I \cup \{0\}$ and $g(\nu) \neq 0$, it is clear that we can choose $\varepsilon > 0$ small enough such that

$$\mu_1 := \min_{\nu \in I} (\delta - \varepsilon |\nu|) > 0 \quad \text{and} \quad \mu_2 := \min\{\kappa(\nu) - \varepsilon |\nu| - 1 \mid 1 \leq |\nu| \leq N, \nu \notin I \text{ and } g(\nu) \neq 0\} > 0.$$

Set $\mu = \min(\mu_1, \mu_2) > 0$. It follows then from (IV.16) that we have $R_p^1(\mathbf{s}) \ll p^{-1-\mu}$ uniformly in p prime number and in $\mathbf{s} \in U_{-\varepsilon}$. This ends the proof of CLAIM 1. \square

CLAIM 2: $\mathbf{s} \mapsto R_p^2(\mathbf{s})$ satisfies the conclusions of Lemma 17.

Proof of CLAIM 2: Fix $\varepsilon > 0$ such that $e^\varepsilon < 2^{\beta/4}$. Assumptions (IV.12) and (IV.13) imply that we have uniformly in p prime number and in $\mathbf{s} \in U_{-\beta/2}$,

$$\begin{aligned} \sum_{|\nu| > N} \left| \frac{g(\nu) + V(p, \nu)}{p^{\kappa(\nu) + \langle \nu, \mathbf{s} \rangle}} \right| &\ll_\varepsilon \sum_{|\nu| > N} \frac{e^{\varepsilon |\nu|}}{p^{\beta |\nu| + \langle \nu, \sigma \rangle}} \leq \sum_{|\nu| > N} \frac{e^{\varepsilon |\nu|}}{p^{\frac{\beta}{2} |\nu|}} \ll_\varepsilon \frac{1}{p^{\frac{\beta}{4} N}} \sum_{|\nu| > N} \frac{e^{\varepsilon |\nu|}}{2^{\frac{\beta}{4} |\nu|}} \\ &\ll_\varepsilon \frac{1}{p^{\frac{\beta}{4} N}} \sum_{|\nu| > N} \left(\frac{e^\varepsilon}{2^{\frac{\beta}{4}}} \right)^{|\nu|} \ll_\varepsilon \frac{1}{p^{\frac{\beta}{4} N}} \leq \frac{1}{p^2}. \end{aligned}$$

We deduce that $\mathbf{s} \mapsto R_p^2(\mathbf{s})$ is holomorphic in $U_{-\beta/2}$ and verifies the estimates $R_p^2(\mathbf{s}) \ll p^{-2}$ uniformly in p prime number and $\mathbf{s} \in U_{-\beta/2}$. This ends the proof of CLAIM 2 and also ends the proof of Lemma 17. \square

3.3 Proof of Lemma 18

It is clear that $\mathbf{s} \mapsto L_p(\mathbf{s})$ is holomorphic in \mathbb{C}^n for any p .

Set now $\varepsilon_2 = \inf_{\nu \in I} \frac{1}{4|\nu|}$. It follows that for $\mathbf{s} \in U_{-\varepsilon_2}$ and $\nu \in I$, $1 + \langle \nu, \sigma \rangle \geq 1 - \varepsilon_2 |\nu| \geq 3/4$.

Newton Binomial theorem implies then that we have uniformly in $\mathbf{s} \in U_{-\varepsilon_2}$ and in p prime

number,

$$\begin{aligned}
|L_p(\mathbf{s})| &= \left| \left(\prod_{\nu \in I} \left(1 - \frac{1}{p^{1+\langle \nu, \mathbf{s} \rangle}} \right)^{g(\nu)} \right) - 1 + \left(\sum_{\nu \in I} \frac{g(\nu)}{p^{1+\langle \nu, \mathbf{s} \rangle}} \right) \right| \\
&= \left| \sum_{\substack{0 \leq k_\nu \leq g(\nu) \ \forall \nu \in I, \\ \sum_{\nu \in I} k_\nu \geq 2}} \frac{\prod_{\nu \in I} (-1)^{k_\nu} \binom{g(\nu)}{k_\nu}}{p^{\sum_{\nu \in I} k_\nu (1+\langle \nu, \mathbf{s} \rangle)}} \right| \ll \sum_{\substack{0 \leq k_\nu \leq g(\nu) \ \forall \nu \in I, \\ \sum_{\nu \in I} k_\nu \geq 2}} \frac{1}{p^{\sum_{\nu \in I} k_\nu (1+\langle \nu, \mathbf{s} \rangle)}} \\
&\ll \sum_{\substack{0 \leq k_\nu \leq g(\nu) \ \forall \nu \in I, \\ \sum_{\nu \in I} k_\nu \geq 2}} \frac{1}{p^{\frac{3}{4} \sum_{\nu \in I} k_\nu}} \ll \frac{1}{p^{3/2}}.
\end{aligned}$$

This ends the proof of Lemma 18.

Proof of parts 2 and 3 of Theorem 15

Define the function $\mathbf{s} = (s_1, \dots, s_n) \mapsto \mathcal{E}(f; \mathbf{s})$ by

$$\mathcal{E}(f; \mathbf{s}) := \left(\prod_{\nu \in I} \zeta(1 + \langle \nu, \mathbf{s} \rangle)^{-g(\nu)} \right) \mathcal{M}(f; \mathbf{c} + \mathbf{s}). \quad (\text{IV.17})$$

Part 1 of Theorem 15 implies then that $\mathbf{s} \mapsto \mathcal{E}(f; \mathbf{s})$ converges absolutely in the domain $U_0 = \{\mathbf{s} \in \mathbb{C}^n \mid \sigma_i > 0 \ \forall i\}$. Moreover, The multiplicativity of f imply that for all $\mathbf{s} \in U_0$:

$$\mathcal{E}(f; \mathbf{s}) = \prod_p \mathcal{E}_p(f; \mathbf{s}), \quad \text{where} \quad (\text{IV.18})$$

$$\mathcal{E}_p(f; \mathbf{s}) := \prod_{\nu \in I} \left(1 - \frac{1}{p^{1+\langle \nu, \mathbf{s} \rangle}} \right)^{g(\nu)} \left(\sum_{\nu \in \mathbb{N}_0^n} \frac{f(p^{\nu_1}, \dots, p^{\nu_n})}{p^{\langle \nu, \mathbf{c} + \mathbf{s} \rangle}} \right).$$

We will now prove the following needed lemma:

Lemma 19. *There exists $\varepsilon_0 > 0$ such that the Euler product $\mathbf{s} \mapsto \mathcal{E}(f; \mathbf{s}) = \prod_p \mathcal{E}_p(f; \mathbf{s})$ converges absolutely and defines a bounded holomorphic function in $U_{-\varepsilon_0} = \{\mathbf{s} \in \mathbb{C}^n \mid \sigma_i > -\varepsilon_0 \ \forall i = 1, \dots, n\}$.*

Proof of Lemma 19:

We will use in the sequel of this proof notation of Lemmas 1 and 2. Lemmas 1 and 2 imply that for any prime p and any $\mathbf{s} \in U_0$,

$$\begin{aligned}
\mathcal{E}_p(f; \mathbf{s}) &= \left(1 - \left(\sum_{\nu \in I} \frac{g(\nu)}{p^{1+\langle \nu, \mathbf{s} \rangle}} \right) + L_p(\mathbf{s}) \right) \left(1 + \left(\sum_{\nu \in I} \frac{g(\nu)}{p^{1+\langle \nu, \mathbf{s} \rangle}} \right) + R_p(\mathbf{s}) \right) \\
&= 1 - A_p(\mathbf{s})^2 + B_p(\mathbf{s}) + C_p(\mathbf{s}), \quad \text{where} \quad (\text{IV.19})
\end{aligned}$$

$$A_p(\mathbf{s}) := \sum_{\nu \in I} \frac{g(\nu)}{p^{1+\langle \nu, \mathbf{s} \rangle}}, \quad B_p(\mathbf{s}) := (1 - A_p(\mathbf{s})) R_p(\mathbf{s}) \quad \text{and} \quad C_p(\mathbf{s}) := L_p(\mathbf{s}) (1 + A_p(\mathbf{s}) + R_p(\mathbf{s})).$$

Let $\varepsilon_1, \varepsilon_2, \eta_1, \eta_2 > 0$ the positive constants defined in Lemmas 1 and 2.

Set $\varepsilon_0 = \min(\varepsilon_1, \varepsilon_2) > 0$ and $\eta_0 = \min(\eta_1, \eta_2) = \min(\eta_1, 1/2) > 0$. Lemmas 1 and 2 imply that the three function A_p , B_p and C_p are holomorphic in $U_{-\varepsilon_0}$ and that we have uniformly in p prime number and $\mathbf{s} \in U_{-\varepsilon_0}$ the following estimates:

1. $A_p(\mathbf{s}) \ll \sum_{\nu \in I} \frac{1}{p^{1+\langle \nu, \sigma \rangle}} \leq \sum_{\nu \in I} \frac{1}{p^{1-\varepsilon_0|\nu|}} \ll \frac{1}{p^{3/4}};$
2. $A_p(\mathbf{s})^2 \ll \frac{1}{p^{3/2}} \ll \frac{1}{p^{1+\eta_0}};$
3. $B_p(\mathbf{s}) \ll \left(1 + \frac{1}{p^{3/4}}\right) \frac{1}{p^{1+\eta_1}} \ll \frac{1}{p^{1+\eta_0}};$
4. $C_p(\mathbf{s}) \ll \frac{1}{p^{1+\eta_2}} \left(1 + \frac{1}{p^{3/4}} + \frac{1}{p^{1+\eta_1}}\right) \ll \frac{1}{p^{1+\eta_0}}.$

It follows that for any prime number p , the function $\mathbf{s} \mapsto \mathcal{E}_p(f; \mathbf{s}) - 1$ is holomorphic in $U_{-\varepsilon_0}$ and verifies $\mathcal{E}_p(f; \mathbf{s}) - 1 \ll \frac{1}{p^{1+\eta_0}}$ uniformly in $\mathbf{s} \in U_{-\varepsilon_0}$ and in the prime number p .

We deduce that the Euler product $\mathbf{s} \mapsto \mathcal{E}(f; \mathbf{s}) = \prod_p \mathcal{E}_p(f; \mathbf{s})$ converges absolutely and defines a bounded holomorphic function in $U_{-\varepsilon_0}$. This ends the proof of Lemma 19. \square
This being done we are now ready to prove points 2 and 3 of Theorem 15. Combining part 1 of Theorem 1, (IV.17) and (IV.18) implies that for $\mathbf{s} \in U_0$,

$$\mathcal{H}(f, \mathbf{c}; \mathbf{s}) := \left(\prod_{\nu \in I} \langle \nu, \mathbf{s} \rangle^{g(\nu)} \right) \mathcal{M}(f; \mathbf{c} + \mathbf{s}) = \left(\prod_{\beta \in I} (\langle \nu, \mathbf{s} \rangle \zeta(1 + \langle \nu, \mathbf{s} \rangle))^{g(\nu)} \right) \mathcal{E}(f; \mathbf{s}). \quad (\text{IV.20})$$

Part 2 of Theorem 15 follows then from Lemma 19 and the following two classical properties of Riemann zeta function: $s \mapsto \zeta(1 + s)$ is holomorphic in \mathbb{C} and verifies in the half-plane $\{\text{Re}(s) > -1\}$ the estimate $s \zeta(1 + s) \ll_\varepsilon (1 + |s|)^{1 - \frac{1}{2} \min(0, \text{Re}(s)) + \varepsilon}$, $\forall \varepsilon > 0$. Moreover, since $s \zeta(1 + s)|_{s=0} = 1$, we deduce from (IV.20) and (IV.18) that

$$\mathcal{H}(f, \mathbf{c}; \mathbf{0}) = \mathcal{E}(f; \mathbf{0}) = \prod_p \left(1 - \frac{1}{p}\right)^{\sum_{\nu \in I} g(\nu)} \left(\sum_{\nu \in \mathbb{N}_0^n} \frac{f(p^{\nu_1}, \dots, p^{\nu_n})}{p^{\langle \nu, \mathbf{c} \rangle}} \right).$$

This ends the proof of point 3 and also the proof of Theorem 15. \square

4 Proofs of Theorems 16 and 17

4.1 Proof of Theorem 16

We will now explain how the combination of our Theorem 15 and La Bretèche's multivariable Tauberian Theorem (i.e Theorems 1 and 2 of [7]) yields to our Theorem 16. Our notations are different from La Bretèche's notations. To simplify the exposition, we will first recall La Bretèche's Tauberian Theorem 1 and the part we use of his Tauberian Theorem 2 by using our notations:

Theorem A: (Theorem 1 of [7]):

Let $f : \mathbb{N}^n \rightarrow \mathbb{R}_+$ be a nonnegative function and F the associated Dirichlet's series defined by

$$F(\mathbf{s}) = F(s_1, \dots, s_n) = \sum_{m_1, \dots, m_n \geq 1} \frac{f(m_1, \dots, m_n)}{m_1^{s_1} \dots m_n^{s_n}}.$$

Denote by $\mathcal{LR}_n^+(\mathbb{C})$ the set of \mathbb{C} -linear forms from \mathbb{C}^n to \mathbb{C} that are nonnegative on $(\mathbb{R}_+)^n$.

We assume that there exists $\mathbf{c} = (c_1, \dots, c_n) \in (\mathbb{R}_+)^n$ such that:

1. $F(\mathbf{s})$ converges absolutely for $\mathbf{s} \in \mathbb{C}^n$ such that $\operatorname{Re}(s_i) > c_i \forall i \in \{1, \dots, n\}$;
2. There exist a finite family $\mathcal{L} = (\ell^{(i)})_{1 \leq i \leq q}$ of nonzero elements of $\mathcal{L}\mathcal{R}_n^+(\mathbb{C})$, a finite family $(h^{(i)})_{1 \leq i \leq q'}$ of elements of $\mathcal{L}\mathcal{R}_n^+(\mathbb{C})$ and $\delta_1, \delta_2, \delta_3 > 0$ such that the function H defined by

$$H(\mathbf{s}) = F(\mathbf{c} + \mathbf{s}) \prod_{i=1}^q \ell^{(i)}(\mathbf{s})$$

has **holomorphic** continuation to the domain

$$\mathcal{D}(\delta_1, \delta_3) := \{\mathbf{s} \in \mathbb{C}^n \mid \operatorname{Re}(\ell^{(i)}(\mathbf{s})) > -\delta_1 \forall i \in \llbracket 1, q \rrbracket \text{ and } \operatorname{Re}(h^{(i)}(\mathbf{s})) > -\delta_3 \forall i \in \llbracket 1, q' \rrbracket\}$$

and verifies the estimate: for $\varepsilon, \varepsilon' > 0$ we have uniformly in $\mathbf{s} \in \mathcal{D}(\delta_1 - \varepsilon', \delta_3 - \varepsilon')$

$$H(\mathbf{s}) \ll \prod_{i=1}^q (|\operatorname{Im}(\ell^{(i)}(\mathbf{s}))| + 1)^{1 - \delta_2 \min(0, \operatorname{Re}(\ell^{(i)}(\mathbf{s})))} (1 + (|\operatorname{Im}(s_1)| + \dots + |\operatorname{Im}(s_n)|)^\varepsilon).$$

Set $J = J(\mathbf{c}) = \{j \in \{1, \dots, n\} \mid c_j = 0\}$. Denote by $w = \#J$ the cardinality of the set J and by $j_1 < \dots < j_w$ its elements in increasing order. Define the w linear forms $\ell^{(q+i)}$ ($1 \leq i \leq w$) by $\ell^{(q+i)}(\mathbf{s}) = \mathbf{e}_{j_i}^*(\mathbf{s}) = s_{j_i}$.

Then, for any $\boldsymbol{\beta} = (\beta_1, \dots, \beta_n) \in (0, \infty)^n$, there exist a polynomial $Q_{\boldsymbol{\beta}} \in \mathbb{R}[X]$ of degree at most $q + w - \operatorname{Rank}\{\ell^{(1)}, \dots, \ell^{(q+w)}\}$ and $\theta > 0$ such that

$$\sum_{1 \leq m_1 \leq x^{\beta_1}} \dots \sum_{1 \leq m_n \leq x^{\beta_n}} f(m_1, \dots, m_n) = x^{(\mathbf{c}, \boldsymbol{\beta})} Q_{\boldsymbol{\beta}}(\log x) + O(x^{(\mathbf{c}, \boldsymbol{\beta}) - \theta}) \text{ as } x \rightarrow +\infty.$$

Theorem B: (parts (ii) and (iv) from Theorem 2 of [7]):

Let $f : \mathbb{N}^n \rightarrow \mathbb{R}_+$ be a function satisfying assumptions of Theorem A.

Let $\boldsymbol{\beta} = (\beta_1, \dots, \beta_n) \in (0, \infty)^n$. Set $\mathcal{B} = \sum_{i=1}^n \beta_i \mathbf{e}_i^* \in \mathcal{L}\mathcal{R}_n^+(\mathbb{C})$.

- **(ii)** If the Dirichlet's series F satisfies the additional two assumptions:
 - (C1) There exists a function G such $H(\mathbf{s}) = G(\ell^{(1)}(\mathbf{s}), \dots, \ell^{(q+w)}(\mathbf{s}))$.
 - (C2) $\mathcal{B} \in \operatorname{Vect}(\{\ell^{(k)} \mid k = 1, \dots, q + w\})$ and there is no subfamily \mathcal{L}' of $\mathcal{L}_0 := (\ell^{(k)})_{1 \leq k \leq q+w}$ such that $\mathcal{L}' \neq \mathcal{L}_0$, $\mathcal{B} \in \operatorname{Vect}(\mathcal{L}')$ and $\#\mathcal{L}' - \operatorname{Rank}(\mathcal{L}') = \#\mathcal{L}_0 - \operatorname{Rank}(\mathcal{L}_0)$.

Then, the polynomial $Q_{\boldsymbol{\beta}}$ satisfies the relation

$$Q_{\boldsymbol{\beta}}(\log x) = H(\mathbf{0})x^{-(\mathbf{c}, \boldsymbol{\beta})} \mathcal{J}_{\boldsymbol{\beta}}(x) + O((\log x)^{\rho-1}),$$

where $\rho := q + w - \operatorname{Rank}\{\ell^{(1)}, \dots, \ell^{(q+w)}\}$ and

$$\mathcal{J}_{\boldsymbol{\beta}}(x) := \int_{\mathcal{A}_{\boldsymbol{\beta}}(x)} \frac{dy_1 \dots dy_q}{\prod_{i=1}^q y_i^{1 - \ell^{(i)}(\mathbf{c})}},$$

with

$$\mathcal{A}_{\boldsymbol{\beta}}(x) := \{\mathbf{y} \in [1, \infty)^q \mid \prod_{i=1}^q y_i^{\ell^{(i)}(\mathbf{e}_j)} \leq x^{\beta_j} \quad \forall j = 1, \dots, n\}.$$

- **(iv)** If $\operatorname{Rank}\{\ell^{(1)}, \dots, \ell^{(q+w)}\} = n$, $H(\mathbf{0}) \neq 0$ and $\mathcal{B} \in \operatorname{conv}^*(\{\ell^{(1)}, \dots, \ell^{(q+w)}\})$, then $\deg(Q_{\boldsymbol{\beta}}) = \rho = q + w - n$.

Remark : If assumptions of point (iv) hold, then assumptions of the point (ii) also clearly hold.

Proof of Theorem 16:

Let $f : \mathbb{N}^n \rightarrow \mathbb{R}_+$ be a multivariable multiplicative function. We assume that f belongs to the class $\mathcal{C}(g, \kappa, \mathbf{c}, \delta)$ associated to the data $(g, \kappa, \mathbf{c}, \delta)$ (see definitions 1 and 2).

We assume also that the finite set

$$I = I(\kappa, g) := \{\boldsymbol{\nu} \in \mathbb{N}_0^n \mid \kappa(\boldsymbol{\nu}) = 1 \text{ and } g(\boldsymbol{\nu}) \neq 0\} \text{ is nonempty.}$$

We denote by $\boldsymbol{\nu}^1, \dots, \boldsymbol{\nu}^r$ the elements of I where $r = \#I$, and define the finite sequence q_k ($0 \leq k \leq r$) by

$$q_0 = 0 \quad \text{and} \quad q_k = \sum_{j=1}^k g(\boldsymbol{\nu}^j) \quad \forall k \in \{1, \dots, r\}.$$

We define the linear forms $\ell^{(i)}$ ($1 \leq i \leq q_r$) by

$$\ell^{(i)}(\mathbf{s}) = \langle \boldsymbol{\nu}^k, \mathbf{s} \rangle \quad \text{if } q_{k-1} < i \leq q_k \text{ and } 1 \leq k \leq r.$$

We define also the set $J = J(\mathbf{c}) = \{j \in \{1, \dots, n\} \mid c_j = 0\}$. we denote by $w = \#J$ the cardinality of the set J and by $j_1 < \dots < j_w$ its elements in increasing order.

We define also the w linear forms $\ell^{(q+i)}$ ($1 \leq i \leq w$) by

$$\ell^{(q+i)}(\mathbf{s}) = \mathbf{e}_{j_i}^*(\mathbf{s}) = s_{j_i} \quad (1 \leq i \leq w).$$

By using notation of our Theorem 15 it's easy to see that the Dirichlet's series associated to f is

$$F(\mathbf{s}) = \sum_{m_1, \dots, m_n \geq 1} \frac{f(m_1, \dots, m_n)}{m_1^{s_1} \dots m_n^{s_n}} = \mathcal{M}(f; \mathbf{s})$$

and

$$H(\mathbf{s}) = \left(\prod_{i=1}^{q_r} \ell^{(i)}(\mathbf{s}) \right) F(\mathbf{c} + \mathbf{s}) = \left(\prod_{\boldsymbol{\nu} \in I} \langle \boldsymbol{\nu}, \mathbf{s} \rangle^{g(\boldsymbol{\nu})} \right) \mathcal{M}(f; \mathbf{c} + \mathbf{s}) = \mathcal{H}(f, \mathbf{c}; \mathbf{s}).$$

Our Theorem 15 implies then that the assumptions of Theorem A above hold. By applying Theorem A with $\boldsymbol{\beta} = \mathbf{1} = (1, \dots, 1)$, we deduce that there exist a polynomial Q_1 of degree at most

$$\rho = q_r + w - \text{Rank} \{ \ell^{(1)}, \dots, \ell^{(q+w)} \} = \left(\sum_{\boldsymbol{\nu} \in I} g(\boldsymbol{\nu}) \right) + \#J - \text{Rank}(I \cup J)$$

and a positive constant $\eta > 0$ such that

$$\mathcal{N}_\infty(f; x) := \sum_{\substack{\mathbf{m}=(m_1, \dots, m_n) \in \mathbb{N}^n \\ \|\mathbf{m}\|_\infty = \max_i m_i \leq x}} f(m_1, \dots, m_n) = x^{|\mathbf{c}|} Q_1(\ln x) + O(x^{|\mathbf{c}|-\eta}) \quad \text{as } x \rightarrow +\infty.$$

This ends the proof of the first part of our Theorem 16.

Assume now in addition that the two following assumptions hold:

1. $\text{Rank}(I \cup J) = n$;

2. $\mathbf{1} = (1, \dots, 1)$ is in the interior of the cone generated by $I \cup J$; that is $\mathbf{1} \in \text{conv}^*(I \cup J) := \{\sum_{\nu \in I \cup J} \lambda_\nu \nu \mid \lambda_\nu \in (0, \infty) \forall \nu \in I \cup J\}$,

By duality, we deduce that $\text{Rank} \{\ell^{(1)}, \dots, \ell^{(q+w)}\} = n$ and $\mathbf{1}^* \in \text{conv}^*(\{\ell^{(1)}, \dots, \ell^{(q+w)}\})$. Moreover since f is nonnegative, our Theorem 15 implies that

$$H(\mathbf{0}) = \mathcal{H}(f, \mathbf{c}; \mathbf{0}) = \prod_p \left(1 - \frac{1}{p}\right)^{\sum_{\nu \in I} g(\nu)} \left(\sum_{\nu \in \mathbb{N}_0^n} \frac{f(p^{\nu_1}, \dots, p^{\nu_n})}{p^{(\nu, \mathbf{c})}} \right) > 0.$$

It follows that assumptions of point (iv) (and therefore assumptions of point (ii)) of Theorem B above hold. Theorem B implies then that

$$\text{deg}(Q_1) = \rho = \left(\sum_{\nu \in I} g(\nu) \right) + \#J - n$$

and

$$Q_1(\log x) = H(\mathbf{0})x^{-|\mathbf{c}|} \mathcal{J}_1(x) + O((\log x)^{\rho-1}), \quad (\text{IV.21})$$

where

$$\mathcal{J}_1(x) = \int_{\mathcal{A}_1(x)} \frac{dy_1 \dots dy_{q_r}}{\prod_{i=1}^{q_r} y_i^{1-\ell^{(i)}(\mathbf{c})}},$$

with

$$\mathcal{A}_1(x) := \{y \in [1, +\infty)^{q_r} \mid \prod_{i=1}^{q_r} y_i^{\ell^{(i)}(\mathbf{e}_j)} \leq x \quad \forall j \in \{1, \dots, n\}\}.$$

By using notations of Definition 14, it's easy to see that

$$\mathcal{J}_1(x) = \mathcal{J}_n(I, \mathbf{u}, \mathbf{c}; x) \quad \text{and} \quad \mathcal{A}_1(x) = \mathcal{A}(I, \mathbf{u}; x),$$

where \mathbf{u} is the sequence $\mathbf{u} = (g(\nu))_{\nu \in I}$.

Since the degree of the polynomial Q_1 is equal to $\rho = (\sum_{\nu \in I} g(\nu)) + \#J - n$, there exists a positive constant $C > 0$ such that $Q_1(x) = Cx^\rho + O(x^{\rho-1})$ as $x \rightarrow +\infty$ and (IV.21) implies that

$$H(\mathbf{0})x^{-|\mathbf{c}|} \mathcal{J}_1(x) = C(\log x)^\rho + O((\log x)^{\rho-1}).$$

It follows that

$$C = H(\mathbf{0}) \lim_{x \rightarrow \infty} x^{-|\mathbf{c}|} (\log x)^{-\rho} \mathcal{J}_1(x) = H(\mathbf{0}) \lim_{x \rightarrow \infty} x^{-|\mathbf{c}|} (\log x)^{-\rho} \mathcal{J}_n(I, \mathbf{u}, \mathbf{c}; x).$$

We deduce that the main term of $\mathcal{N}_\infty(f; x)$ is given by

$$\mathcal{N}_\infty(f; x) = C_n(f) K_n(f, \|\|\|_\infty) x^{|\mathbf{c}|} (\ln x)^\rho + O((\ln x)^{\rho-1}) \quad \text{as } x \rightarrow +\infty,$$

where $C_n(f) := H(\mathbf{0}) = \mathcal{H}(f, \mathbf{c}; \mathbf{0}) > 0$ is defined by the Euler product (IV.6) and

$$K_n(f, \|\|\|_\infty) := \lim_{x \rightarrow \infty} \mathcal{J}_n(I, \mathbf{u}, \mathbf{c}; x) x^{-|\mathbf{c}|} (\ln x)^{-\rho} > 0.$$

This ends the proof of Theorem 16. □

4.2 Proof of Theorem 17

Let $f : \mathbb{N}^n \rightarrow \mathbb{R}$ be a multivariable multiplicative function. We assume that f belongs to the class $\mathcal{C}(g, \kappa, \mathbf{c}, \delta)$ associated to the data $(g, \kappa, \mathbf{c}, \delta)$ (see definitions 1 and 2).

We assume also that the finite set

$$I = I(\kappa, g) := \{\boldsymbol{\nu} \in \mathbb{N}_0^n \mid \kappa(\boldsymbol{\nu}) = 1 \text{ and } g(\boldsymbol{\nu}) \neq 0\} \text{ is nonempty.}$$

We define the set $I_{\mathbf{c}} := \{\frac{1}{\langle \boldsymbol{\nu}, \mathbf{c} \rangle} \boldsymbol{\nu} \mid \boldsymbol{\nu} \in I\}$ and the sequence $\mathbf{u} := (u(\boldsymbol{\beta}))_{\boldsymbol{\beta} \in I_{\mathbf{c}}}$ where

$$u(\boldsymbol{\beta}) = \sum_{\boldsymbol{\nu} \in I; \frac{1}{\langle \boldsymbol{\nu}, \mathbf{c} \rangle} \boldsymbol{\nu} = \boldsymbol{\beta}} g(\boldsymbol{\nu}) \quad \text{for all } \boldsymbol{\beta} \in I_{\mathbf{c}}.$$

We Define also the pair $\mathcal{T}_{\mathbf{c}} := (I_{\mathbf{c}}, \mathbf{u})$.

Theorem 15 implies that

$$\mathbf{s} \rightarrow \mathcal{M}(f; \mathbf{s}) := \sum_{m_1 \geq 1, \dots, m_n \geq 1} \frac{f(m_1, \dots, m_n)}{m_1^{s_1} \dots m_n^{s_n}}$$

converges absolutely in the domain $\{\mathbf{s} \in \mathbb{C}^n \mid \operatorname{Re}(s_i) > c_i \ \forall i = 1, \dots, n\}$; and that there exists $\varepsilon_0 > 0$ such that the function

$$\begin{aligned} \mathbf{s} \rightarrow H(f; \mathcal{T}_{\mathbf{c}}; \mathbf{s}) &:= \left(\prod_{\boldsymbol{\beta} \in I_{\mathbf{c}}} \langle \boldsymbol{\beta}, \mathbf{s} \rangle^{u(\boldsymbol{\beta})} \right) \mathcal{M}(f; \mathbf{c} + \mathbf{s}) \\ &= \left(\prod_{\boldsymbol{\nu} \in I} \langle \boldsymbol{\nu}, \mathbf{c} \rangle^{-g(\boldsymbol{\nu})} \right) \left(\prod_{\boldsymbol{\nu} \in I} \langle \boldsymbol{\nu}, \mathbf{s} \rangle^{g(\boldsymbol{\nu})} \right) \mathcal{M}(f; \mathbf{c} + \mathbf{s}) \\ &= \left(\prod_{\boldsymbol{\nu} \in I} \langle \boldsymbol{\nu}, \mathbf{c} \rangle^{-g(\boldsymbol{\nu})} \right) \mathcal{H}(f, \mathbf{c}; \mathbf{s}) \end{aligned} \tag{IV.22}$$

has *holomorphic* continuation to the domain $\{\mathbf{s} \in \mathbb{C}^n \mid \operatorname{Re}(s_i) > -\varepsilon_0 \ \forall i \in \{1, \dots, n\}\}$ and verifies in it the following estimate: for all $\varepsilon > 0$,

$$H(f, \mathcal{T}_{\mathbf{c}}; \mathbf{s}) \ll_{\varepsilon} \prod_{\boldsymbol{\nu} \in I} (|\langle \boldsymbol{\nu}, \mathbf{s} \rangle| + 1)^{g(\boldsymbol{\nu})(1 - \frac{1}{2} \min(0, \operatorname{Re}(\langle \boldsymbol{\nu}, \mathbf{s} \rangle))) + \varepsilon};$$

We deduce that f is of *finite type* with $\mathcal{T}_{\mathbf{c}} := (I_{\mathbf{c}}, \mathbf{u})$ as a *regularizing pair* (see Definition 2 of [12]). It follows then from Corollary 2 of [12] that there exist a polynomial Q of degree at most

$$\rho := \left(\sum_{\boldsymbol{\beta} \in I_{\mathbf{c}}} u(\boldsymbol{\beta}) \right) - \operatorname{Rank}(I_{\mathbf{c}}) = \left(\sum_{\boldsymbol{\nu} \in I} g(\boldsymbol{\nu}) \right) - \operatorname{Rank}(I)$$

and a positive constant $\mu > 0$ such that

$$\mathcal{N}_d(f; x) := \sum_{\substack{\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{N}^n \\ \|\mathbf{m}\|_d = \sqrt[d]{m_1^d + \dots + m_n^d} \leq x}} f(m_1, \dots, m_n) = x^{|\mathbf{c}|} Q(\ln x) + O(x^{|\mathbf{c}| - \mu}) \quad \text{as } x \rightarrow +\infty.$$

This ends the proof of part 1 of Theorem 17.

Assume now in addition that $\operatorname{Rank}(I) = n$ and $\mathbf{1} \in \operatorname{con}^*(I)$. It follows that

1. $\operatorname{Rank}(I_{\mathbf{c}}) = n$ and it's clear then that there exists a function holomorphic in a tubular neighborhood of $\mathbf{0}$ such that $H(f, \mathcal{T}_{\mathbf{c}}; \mathbf{s}) = K(\langle \boldsymbol{\beta}, \mathbf{s} \rangle)_{\boldsymbol{\beta} \in I_{\mathbf{c}}}$;

2. $\mathbf{1} \in \text{con}^*(I_{\mathbf{c}})$.

Therefore, the additional assumptions 1 and 2 of Theorem 3 of [12] are satisfied and the second part of Corollary 2 of [12] implies then that

$$\mathcal{N}_d(f; x) = C_0(f, P_d) x^{|\mathbf{c}|} (\ln x)^\rho + O((\ln x)^{\rho-1}) \quad \text{as } x \rightarrow +\infty,$$

where $C_0(f, P_d) := \frac{H(f, \mathcal{T}_{\mathbf{c}}; \mathbf{0}) d^{\rho+1} A_0(\mathcal{T}_{\mathbf{c}}, P_d)}{|\mathbf{c}| \rho!}$ and $A_0(\mathcal{T}_{\mathbf{c}}, P_d) > 0$ is the mixed volume constant (see §2.3.3 of [12]) associated to the pair $\mathcal{T}_{\mathbf{c}} := (I_{\mathbf{c}}, \mathbf{u})$ and the polynomial $P_d = X_1^d + \dots + X_n^d$.

Combining (IV.22) and the expression of $\mathcal{H}(f, \mathbf{c}; \mathbf{0})$ given by Theorem 15 implies that

$$H(f, \mathcal{T}_{\mathbf{c}}; \mathbf{0}) = \left(\prod_{\nu \in I} \langle \nu, \mathbf{c} \rangle^{-g(\nu)} \right) C_n(f),$$

where $C_n(f) := \mathcal{H}(f, \mathbf{c}; \mathbf{0}) > 0$ is defined by the Euler product (IV.6). Moreover, if we set

$$K_n(f, \|\cdot\|_d) := \left(\prod_{\nu \in I} \langle \nu, \mathbf{c} \rangle^{-g(\nu)} \right) \frac{d^{\rho+1} A_0(\mathcal{T}_{\mathbf{c}}, P_d)}{|\mathbf{c}| \rho!} > 0,$$

then the the constant $C_0(f, P_d)$ is positive and is given by

$$C_0(f, P_d) = C_n(f) K_n(f, \|\cdot\|_d) > 0.$$

In particular, the degree of the polynomial Q is equal to $\rho = (\sum_{\nu \in I} g(\nu)) - n$. This ends the proof of Theorem 17. \square

5 Proof of Corollary 4

Define the function $g_1 : \mathbb{N}_0^n \rightarrow \mathbb{N}_0$ by

$$g_1(\nu) = \begin{cases} 1 & \text{if } \exists i \neq j \in \{1, \dots, n\} \text{ such that } \nu_i = \nu_j = |\nu|_\infty; \\ |\nu|_\infty - \max(\{\nu_i \mid i \in \{1, \dots, n\}\} \setminus \{|\nu|_\infty\}) + 1 & \text{otherwise,} \end{cases}$$

where $|\nu|_\infty = \max_{i \in \{1, \dots, n\}} \nu_i$.

We will first prove the following needed lemma.

Lemma 20. *We have*

$$c_n(p^{\nu_1}, \dots, p^{\nu_n}) = g_1(\nu) p^{|\nu|_1 - |\nu|_\infty} + O((1 + |\nu|_1) p^{|\nu|_1 - |\nu|_\infty - 1})$$

uniformly in $\nu = (\nu_1, \dots, \nu_n) \in \mathbb{N}_0^n$ and p prime number.

Proof of Lemma 20:

In the proof of this lemma we will use the notations: $a \wedge b = \min(a, b)$ and $a \vee b = \max(a, b)$.

First we recall the following formula by Tóth [32]:

$$c_n(p^{\nu_1}, \dots, p^{\nu_n}) = \sum_{\substack{0 \leq \ell_i \leq \nu_i \\ i \in [1, n]}} \frac{\varphi(p^{\ell_1}) \dots \varphi(p^{\ell_n})}{\varphi(p^{\max\{\ell_1, \dots, \ell_n\}}}. \quad (\text{IV.23})$$

If $n = 1$, then $c_1(\nu_1) = 1 + \nu_1$ and the lemma holds.

Let $n \geq 2$. Set $k = n - 1 \in \mathbb{N}$.

Let p be a prime number and $\boldsymbol{\nu} = (\nu_1, \dots, \nu_n) \in \mathbb{N}_0^n$. Without loss of generality we can assume that

$$\nu_1 \leq \nu_2 \leq \dots \leq \nu_n.$$

It follows that

$$\begin{aligned} c_n(p^{\nu_1}, \dots, p^{\nu_n}) &= \sum_{\substack{0 \leq \ell_i \leq \nu_i \\ i \in \llbracket 1, k+1 \rrbracket \\ \ell_1 \vee \dots \vee \ell_k \leq \ell_{k+1}}} \varphi(p^{\ell_1}) \cdots \varphi(p^{\ell_k}) + \sum_{\substack{0 \leq \ell_i \leq \nu_i \\ i \in \llbracket 1, k+1 \rrbracket \\ \ell_1 \vee \dots \vee \ell_k > \ell_{k+1}}} \varphi(p^{\ell_{k+1}}) \frac{\varphi(p^{\ell_1}) \cdots \varphi(p^{\ell_k})}{\varphi(p^{\max\{\ell_1, \dots, \ell_k\}})} \\ &= \sum_{\ell=0}^{\nu_{k+1}} p^{\nu_1 \wedge \ell + \dots + \nu_k \wedge \ell} + \sum_{\substack{0 \leq \ell_i \leq \nu_i \\ i \in \llbracket 1, k \rrbracket \\ \ell_1 \vee \dots \vee \ell_k \geq 1}} \frac{\varphi(p^{\ell_1}) \cdots \varphi(p^{\ell_k})}{\varphi(p^{\max\{\ell_1, \dots, \ell_k\}})} \sum_{\ell_{k+1}=0}^{\ell_1 \vee \dots \vee \ell_k - 1} \varphi(p^{\ell_{k+1}}) \\ &= (\nu_{k+1} - \nu_k + 1) p^{\nu_1 + \dots + \nu_k} + \sum_{\ell=0}^{\nu_k - 1} p^{\nu_1 \wedge \ell + \dots + \nu_k \wedge \ell} + \sum_{\substack{0 \leq \ell_i \leq \nu_i \\ i \in \llbracket 1, k \rrbracket \\ \ell_1 \vee \dots \vee \ell_k \geq 1}} \frac{\varphi(p^{\ell_1}) \cdots \varphi(p^{\ell_k})}{\varphi(p^{\max\{\ell_1, \dots, \ell_k\}})} p^{\ell_1 \vee \dots \vee \ell_k - 1} \\ &= (\nu_{k+1} - \nu_k + 1) p^{\nu_1 + \dots + \nu_k} + \sum_{\ell=0}^{\nu_k - 1} p^{\nu_1 \wedge \ell + \dots + \nu_k \wedge \ell} + \sum_{\substack{0 \leq \ell_i \leq \nu_i \\ i \in \llbracket 1, k \rrbracket}} \frac{\varphi(p^{\ell_1}) \cdots \varphi(p^{\ell_k})}{p-1} - \frac{1}{p-1} \\ &= (\nu_{k+1} - \nu_k + 1) p^{\nu_1 + \dots + \nu_k} + \sum_{\ell=0}^{\nu_k - 1} p^{\nu_1 \wedge \ell + \dots + \nu_k \wedge \ell} + \frac{p^{\nu_1 + \dots + \nu_k} - 1}{p-1} \\ &= (\nu_{k+1} - \nu_k + 1) p^{\nu_1 + \dots + \nu_k} + \sum_{\ell=0}^{\nu_k - 1} p^{\nu_1 \wedge \ell + \dots + \nu_{k-1} \wedge \ell} p^\ell + \sum_{\ell=0}^{\nu_1 + \dots + \nu_k - 1} p^\ell \end{aligned}$$

Thus, for $\boldsymbol{\nu} = (\nu_1, \dots, \nu_n) \in \mathbb{N}_0^n$ such that $\nu_1 \leq \nu_2 \leq \dots \leq \nu_n$, we have

$$c_n(p^{\nu_1}, \dots, p^{\nu_n}) = (\nu_n - \nu_{n-1} + 1) p^{\nu_1 + \dots + \nu_{n-1}} + \sum_{\ell=0}^{\nu_{n-1} - 1} p^{\nu_1 \wedge \ell + \dots + \nu_{n-2} \wedge \ell} p^\ell + \sum_{\ell=0}^{\nu_1 + \dots + \nu_{n-1} - 1} p^\ell \quad (\text{IV.24})$$

We deduce that

$$\begin{aligned} 0 &\leq c_n(p^{\nu_1}, \dots, p^{\nu_n}) - (\nu_n - \nu_{n-1} + 1) p^{|\boldsymbol{\nu}|_1 - |\boldsymbol{\nu}|_\infty} \\ &\leq p^{\nu_1 + \dots + \nu_{k-1}} \sum_{\ell=0}^{\nu_k - 1} p^\ell + (\nu_1 + \dots + \nu_k) p^{\nu_1 + \dots + \nu_k - 1} \\ &\leq (\nu_1 + \dots + \nu_{k-1} + 2\nu_k) p^{\nu_1 + \dots + \nu_k - 1} \leq 2|\boldsymbol{\nu}| p^{|\boldsymbol{\nu}|_1 - |\boldsymbol{\nu}|_\infty - 1}. \end{aligned}$$

This ends the proof of Lemma 20. \square

This being done we are now ready to prove Corollary 4.

It's clear that $c_n : (m_1, \dots, m_n) \rightarrow c_n(m_1, \dots, m_n)$ is a multiplicative function. Moreover, Lemma 20 implies that c_n belongs to the class $\mathcal{C}(g, \kappa, \mathbf{c}, \delta)$ (see Definition 13), where $g = g_1$, $\mathbf{c} = \mathbf{1} = (1, \dots, 1)$, $\delta = 1$ and κ is the function defined by $\kappa(\boldsymbol{\nu}) = \max_{i \in \{1, \dots, n\}} \nu_i \forall \boldsymbol{\nu} \in \mathbb{N}_0^n$. Furthermore, if we denote by $(\mathbf{e}_1, \dots, \mathbf{e}_n)$ the canonical basis of \mathbb{R}^n , then

$$I = I(\kappa, g) := \{\boldsymbol{\nu} \in \mathbb{N}_0^n \mid \kappa(\boldsymbol{\nu}) = 1 \text{ and } g(\boldsymbol{\nu}) \neq 0\} = \{0, 1\}^n \setminus \{\mathbf{0}\}.$$

Since $J = \{\mathbf{e}_i \mid c_i = 0\} = \emptyset$ and $\mathbf{e}_1, \dots, \mathbf{e}_n \in I = I \cup J$, it follows that the two assumptions $\text{Rank}(I \cup J) = n$ and $\mathbf{1} \in \text{con}^*(I \cup J)$ hold. Set

$$\rho := \left(\sum_{\boldsymbol{\nu} \in I} g(\boldsymbol{\nu}) \right) + \#J - n = \left(\sum_{\boldsymbol{\nu} \in I} g(\boldsymbol{\nu}) \right) - n.$$

Since $g(\mathbf{e}_i) = 2 \forall i = 1, \dots, n$ and $g(\boldsymbol{\nu}) = 1 \forall \boldsymbol{\nu} \in I \setminus \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$, we have

$$\rho = 2n + (\#I - n) - n = \#I = 2^n - 1.$$

Theorem 16 implies then that there exist a polynomial Q_1 of degree ρ and a positive constant $\mu_1 > 0$ such that

$$\begin{aligned} G_n(x) &:= \sum_{1 \leq m_1, \dots, m_n \leq x} c_n(m_1, \dots, m_n) = x^n Q_1(\ln x) + O(x^{n-\mu_1}) \quad \text{as } x \rightarrow +\infty, \\ &= C_n(c_n) K_n(c_n, \|\!\| \!\|_\infty) x^n (\ln x)^{2^n-1} + O(x^n (\ln x)^{2^n-2}) \quad \text{as } x \rightarrow +\infty, \end{aligned}$$

where

$$C_n(c_n) := \mathcal{H}(c_n, \mathbf{c}; \mathbf{0}) = \prod_p \left(1 - \frac{1}{p}\right)^{2^n+n-1} \left(\sum_{\boldsymbol{\nu} \in \mathbb{N}_0^n} \frac{c_n(p^{\nu_1}, \dots, p^{\nu_n})}{p^{|\boldsymbol{\nu}|}} \right) > 0$$

and

$$K_n(c_n, \|\!\| \!\|_\infty) := \lim_{x \rightarrow \infty} \mathcal{J}_n(I, \mathbf{u}; x) x^{-n} (\ln x)^{-2^n+1} > 0, \quad \text{where}$$

$\mathcal{J}_n(I, \mathbf{u}, \mathbf{c}; x)$ is the integral (see Definition 14) associated to the set $I = \{0, 1\}^n \setminus \{\mathbf{0}\}$ and to the sequence $\mathbf{u} = (u(\boldsymbol{\nu}))_{\boldsymbol{\nu} \in I}$ defined by $u(\mathbf{e}_i) = 2 \forall i \in \{1, \dots, n\}$ and $u(\boldsymbol{\nu}) = 1 \forall \boldsymbol{\nu} \in I \setminus \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ and to the vector $\mathbf{c} = \mathbf{1}$. This ends the proof of Corollary 4. \square

6 Proof of Corollaries 5, 6, 7, 8 and 9

6.1 Proof of Corollary 5

Let $s_n : \mathbb{N}^n \rightarrow \mathbb{R}_+$ the function defined by

$$s_n(m_1, \dots, m_n) = \frac{1}{\text{lcm}(m_1, \dots, m_n)} \quad \forall (m_1, \dots, m_n) \in \mathbb{N}^n.$$

It clear that the function s_n is multiplicative and that for $\boldsymbol{\nu} = (\nu_1, \dots, \nu_n) \in \mathbb{N}_0^n$ and p prime number, we have

$$s_n(p^{\nu_1}, \dots, p^{\nu_n}) = p^{-\max_{i=1, \dots, n} \nu_i}.$$

Thus, s_n belongs to the class $\mathcal{C}(g, \kappa, \mathbf{c}, \delta)$ (see Definition 13), where $g \equiv 1$, $\mathbf{c} = \mathbf{0} = (0, \dots, 0)$, $\delta = 1$ and κ is the function defined by $\kappa(\boldsymbol{\nu}) = \max_{i=1, \dots, n} \nu_i \forall \boldsymbol{\nu} \in \mathbb{N}_0^n$.

Moreover, we have

$$I = I(\kappa, g) := \{\boldsymbol{\nu} \in \mathbb{N}_0^n \mid \kappa(\boldsymbol{\nu}) = 1 \text{ and } g(\boldsymbol{\nu}) \neq 0\} = \{0, 1\}^n \setminus \{\mathbf{0}\}$$

and $J = \{\mathbf{e}_i \mid c_i = 0\} = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$. It follows that the two assumptions $\text{Rank}(I \cup J) = n$ and $\mathbf{1} \in \text{con}^*(I \cup J)$ hold. Moreover, $\rho := (\sum_{\boldsymbol{\nu} \in I} g(\boldsymbol{\nu})) + \#J - n = (\sum_{\boldsymbol{\nu} \in I} g(\boldsymbol{\nu})) = 2^n - 1$. Theorem 16 implies then that there exist a polynomial Q_2 of degree $\rho = 2^n - 1$ and a positive constant $\mu_2 > 0$ such that

$$\begin{aligned} S_n(x) &:= \sum_{1 \leq m_1, \dots, m_n \leq x} \frac{1}{\text{lcm}(m_1, \dots, m_n)} = Q_2(\ln x) + O(x^{-\mu_2}) \quad \text{as } x \rightarrow +\infty, \\ &= C_n(s_n) K_n(s_n, \|\!\| \!\|_\infty) (\ln x)^{2^n-1} + O((\ln x)^{2^n-2}) \quad \text{as } x \rightarrow +\infty, \end{aligned}$$

where

$$\begin{aligned} C_n(s_n) &:= \mathcal{H}(c_n, \mathbf{c}; \mathbf{0}) = \prod_p \left(1 - \frac{1}{p}\right)^{2^n - 1} \left(\sum_{\nu \in \mathbb{N}_0^n} \frac{1}{p^{|\nu|_\infty}}\right) \\ &= \prod_p \left(1 - \frac{1}{p}\right)^{2^n - 1} \left(\sum_{k=0}^{\infty} \frac{(k+1)^n - k^n}{p^k}\right) > 0, \end{aligned}$$

and

$$K_n(s_n, \|\|\|_\infty) := \lim_{x \rightarrow \infty} \mathcal{I}_n(I, \mathbf{u}, \mathbf{c}; x) (\ln x)^{-2^n + 1} > 0, \quad \text{where}$$

$\mathcal{I}_n(I, \mathbf{u}, \mathbf{c}; x)$ is the integral (see Definition 14) associated to $I = \{0, 1\}^n \setminus \{\mathbf{0}\}$, to the sequence $\mathbf{u} = (u(\boldsymbol{\nu}))_{\boldsymbol{\nu} \in I}$ defined by $u(\boldsymbol{\nu}) = 1 \forall \boldsymbol{\nu} \in I$ and to the vector $\mathbf{c} = \mathbf{0}$. This ends the proof of Corollary 5. \square

6.2 Proof of Corollary 6

Let $n \in \mathbb{N} \setminus \{1\}$. Let $u_n : \mathbb{N}^n \rightarrow \mathbb{R}_+$ the function defined by

$$u_n(m_1, \dots, m_n) = \frac{1}{\text{lcm}(m_1, \dots, m_n)} \text{ if } \gcd(m_1, \dots, m_n) = 1 \text{ and } u_n(m_1, \dots, m_n) = 0 \text{ otherwise.}$$

It clear that the function u_n is multiplicative and that for $\boldsymbol{\nu} = (\nu_1, \dots, \nu_n) \in \mathbb{N}_0^n$ and p prime number, we have

$$u_n(p^{\nu_1}, \dots, p^{\nu_n}) = p^{-\max_{i=1, \dots, n} \nu_i} \text{ if } \min_{i \in \{1, \dots, n\}} \nu_i = 0 \text{ and } u_n(p^{\nu_1}, \dots, p^{\nu_n}) = 0 \text{ otherwise.}$$

Thus, u_n belongs to the class $\mathcal{C}(g, \kappa, \mathbf{c}, \delta)$ (see Definition 13), where $\mathbf{c} = \mathbf{0} = (0, \dots, 0)$, $\delta = 1$, κ is the function defined by $\kappa(\boldsymbol{\nu}) = \max_{i \in \{1, \dots, n\}} \nu_i \forall \boldsymbol{\nu} \in \mathbb{N}_0^n$ and g is the function defined by

$$g(\boldsymbol{\nu}) = 1 \text{ if } \min_{i=1, \dots, n} \nu_i = 0 \text{ and } g(\boldsymbol{\nu}) = 0 \text{ otherwise.}$$

Thus, we have $I = I(\kappa, g) := \{\boldsymbol{\nu} \in \mathbb{N}_0^n \mid \kappa(\boldsymbol{\nu}) = 1 \text{ and } g(\boldsymbol{\nu}) \neq 0\} = \{0, 1\}^n \setminus \{\mathbf{0}, \mathbf{1}\}$ and $J = \{\mathbf{e}_i \mid c_i = 0\} = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$. It follows that the two assumptions $\text{Rank}(I \cup J) = n$ and $\mathbf{1} \in \text{con}^*(I \cup J)$ hold. Moreover, $\rho := (\sum_{\boldsymbol{\nu} \in I} g(\boldsymbol{\nu})) + \#J - n = (\sum_{\boldsymbol{\nu} \in I} g(\boldsymbol{\nu})) = 2^n - 2$. Theorem 16 implies then that there exist a polynomial Q_3 of degree $2^n - 2$ and $\mu_3 > 0$ such that

$$\begin{aligned} U_n(x) &:= \sum_{\substack{1 \leq m_1, \dots, m_n \leq x \\ \gcd(m_1, \dots, m_n) = 1}} \frac{1}{\text{lcm}(m_1, \dots, m_n)} = Q_3(\ln x) + O(x^{-\mu_3}) \quad \text{as } x \rightarrow +\infty \\ &= C_n(u_n) K_n(u_n, \|\|\|_\infty) (\ln x)^{2^n - 2} + O((\ln x)^{2^n - 3}) \quad \text{as } x \rightarrow +\infty, \end{aligned}$$

where

$$\begin{aligned} C_n(u_n) &:= \mathcal{H}(c_n, \mathbf{c}; \mathbf{0}) = \prod_p \left(1 - \frac{1}{p}\right)^{2^n - 2} \left(\sum_{\substack{\boldsymbol{\nu} \in \mathbb{N}_0^n \\ \min_{i=1, \dots, n} \nu_i = 0}} \frac{1}{p^{|\boldsymbol{\nu}|_\infty}}\right) \\ &= \prod_p \left(1 - \frac{1}{p}\right)^{2^n - 2} \left(1 + \sum_{k=1}^{\infty} \frac{(k+1)^n + (k-1)^n - 2k^n}{p^k}\right) \\ &= \prod_p \left(1 - \frac{1}{p}\right)^{2^n - 1} \left(\sum_{k=0}^{\infty} \frac{(k+1)^n - k^n}{p^k}\right) > 0, \end{aligned}$$

and

$$K_n(u_n, \|\|\|\infty) := \lim_{x \rightarrow \infty} \mathcal{I}_n(I, \mathbf{u}, \mathbf{c}; x) (\ln x)^{-2^n+2} > 0, \quad \text{where}$$

$\mathcal{I}_n(I, \mathbf{u}, \mathbf{c}; x)$ is the integral (see Definition 14) associated to $I = \{0, 1\}^n \setminus \{\mathbf{0}, \mathbf{1}\}$, to the sequence $\mathbf{u} = (u(\boldsymbol{\nu}))_{\boldsymbol{\nu} \in I}$ defined by $u(\boldsymbol{\nu}) = 1 \forall \boldsymbol{\nu} \in I$ and to the vector $\mathbf{c} = \mathbf{0}$. This ends the proof of Corollary 6. \square

6.3 Proof of Corollary 7

Let $v_n : \mathbb{N}^n \rightarrow \mathbb{R}_+$ the function defined by

$$v_n(m_1, \dots, m_n) = \frac{m_1 \dots m_n}{\text{lcm}(m_1, \dots, m_n)} \quad \forall (m_1, \dots, m_n) \in \mathbb{N}^n.$$

It clear that the function f is multiplicative and that for $\boldsymbol{\nu} = (\nu_1, \dots, \nu_n) \in \mathbb{N}_0^n$ and p prime number, we have

$$v_n(p^{\nu_1}, \dots, p^{\nu_n}) = p^{|\boldsymbol{\nu}| - \max_{i \in \{1, \dots, n\}} \nu_i}.$$

Thus, v_n belongs to the class $\mathcal{C}(g, \kappa, \mathbf{c}, \delta)$ (see Definition 13), where $g \equiv 1$, $\mathbf{c} = \mathbf{1} = (1, \dots, 1)$, $\delta = 1$ and κ is the function defined by $\kappa(\boldsymbol{\nu}) = \max_{i \in \{1, \dots, n\}} \nu_i \forall \boldsymbol{\nu} \in \mathbb{N}_0^n$.

Moreover, we have

$$I = I(\kappa, g) := \{\boldsymbol{\nu} \in \mathbb{N}_0^n \mid \kappa(\boldsymbol{\nu}) = 1 \text{ and } g(\boldsymbol{\nu}) \neq 0\} = \{0, 1\}^n \setminus \{\mathbf{0}\}$$

and $J = \{\mathbf{e}_i \mid c_i = 0\} = \emptyset$. It follows that the two assumptions $\text{Rank}(I \cup J) = n$ and $\mathbf{1} \in \text{con}^*(I \cup J)$ hold. Moreover, $\rho := (\sum_{\boldsymbol{\nu} \in I} g(\boldsymbol{\nu})) + \#J - n = 2^n - 1 - n$.

Theorem 16 implies then that there exist a polynomial Q_4 of degree $\rho = 2^n - n - 1$ and a positive constant $\mu_4 > 0$ such that

$$\begin{aligned} V_n(x) &:= \sum_{1 \leq m_1, \dots, m_n \leq x} \frac{m_1 \dots m_n}{\text{lcm}(m_1, \dots, m_n)} = x^n Q_4(\ln x) + O(x^{n-\mu_4}) \quad \text{as } x \rightarrow +\infty \\ &= C_n(v_n) K_n(v_n, \|\|\|\infty) x^n (\ln x)^{2^n-n-1} + O(x^n (\ln x)^{2^n-n-2}) \quad \text{as } x \rightarrow +\infty, \end{aligned}$$

where

$$C_n(v_n) := \mathcal{H}(c_n, \mathbf{c}; \mathbf{0}) = \prod_p \left(1 - \frac{1}{p}\right)^{2^n-1} \left(\sum_{\boldsymbol{\nu} \in \mathbb{N}_0^n} \frac{1}{p^{|\boldsymbol{\nu}|_\infty}}\right) = \prod_p \left(1 - \frac{1}{p}\right)^{2^n-1} \left(\sum_{k=0}^{\infty} \frac{(k+1)^n - k^n}{p^k}\right) > 0,$$

and

$$K_n(v_n, \|\|\|\infty) := \lim_{x \rightarrow \infty} \mathcal{I}_n(I, \mathbf{u}, \mathbf{c}; x) x^{-n} (\ln x)^{-2^n+n+1} > 0, \quad \text{where}$$

$\mathcal{I}_n(I, \mathbf{u}, \mathbf{c}; x)$ is the integral (see Definition 14) associated to $I = \{0, 1\}^n \setminus \{\mathbf{0}\}$, to the sequence $\mathbf{u} = (u(\boldsymbol{\nu}))_{\boldsymbol{\nu} \in I}$ defined by $u(\boldsymbol{\nu}) = 1 \forall \boldsymbol{\nu} \in I$ and to the vector $\mathbf{c} = \mathbf{1}$. This ends the proof of Corollary 7. \square

6.4 Proof of Corollaries 8 and 9

Proof of Corollary 8 (resp. Corollary 9) is similar to the proof of Corollary 4 (resp. Corollary 7) by using Theorem 17 instead of Theorem 16 and the identity $\prod_{\boldsymbol{\nu} \in \{0, 1\}^n, |\boldsymbol{\nu}| \geq 2} |\boldsymbol{\nu}| = \prod_{k=2}^n k \binom{n}{k}$.

6.5 Computation of $C_2(c_2)$ in Corollaries 4 and 8

The identity (IV.24) implies that for $\nu \in \mathbb{N}_0^2$ such that $0 \leq \nu_1 \leq \nu_2$ we have

$$c_2(p^{\nu_1}, p^{\nu_2}) = (\nu_2 - \nu_1 - 1)p^{\nu_1} + 2 \frac{p^{\nu_1+1} - 1}{p - 1}.$$

We deduce by symmetry that

$$\begin{aligned} C_2(c_2) &= \prod_p \left(1 - \frac{1}{p}\right)^5 \left[2 \sum_{\nu_2 \geq 0} \sum_{\nu_1=0}^{\nu_2} \frac{c_2(p^{\nu_1}, p^{\nu_2})}{p^{\nu_1+\nu_2}} - \sum_{\nu_1 \geq 0} \frac{c_2(p^{\nu_1}, p^{\nu_1})}{p^{2\nu_1}} \right] \\ &= \prod_p \left(\frac{p-1}{p}\right)^5 \left[2 \frac{p^2(p^2+p+2)}{(p-1)^3(p+1)} - \frac{p(p^2+1)}{(p-1)^3} \right] \\ &= \prod_p \left(1 - \frac{1}{p^2}\right)^2 = \frac{1}{\zeta^2(2)} = \frac{36}{\pi^4}. \end{aligned}$$

6.6 Computation of $C_n(\cdot)$ ($n = 2, 3, 4$) in Corollaries 5, 6, 7 and 9

Constants $C_n(\cdot)$ in Corollaries 5, 6, 7 and 9 are equal. We will denote them by C_n in this subsection.

- In dimension $n = 2$, we have

$$\begin{aligned} C_2 &= \prod_p \left(1 - \frac{1}{p}\right)^3 \left(\sum_{k \geq 0} \frac{2k+1}{p^k}\right) = \prod_p \frac{(p-1)^3}{p^3} \left(\frac{2p}{(p-1)^2} + \frac{p}{p-1}\right) = \prod_p \frac{p^2-1}{p^2} \\ &= \zeta(2)^{-1} = \frac{6}{\pi^2} \end{aligned}$$

- In dimension $n = 3$, we have

$$\begin{aligned} C_3 &= \prod_p \left(1 - \frac{1}{p}\right)^7 \left(\sum_{k \geq 0} \frac{3k^2+3k+1}{p^k}\right) = \prod_p \frac{(p-1)^7}{p^7} \left(\frac{3p(p+1)}{(p-1)^3} + \frac{3p}{(p-1)^2} + \frac{p}{p-1}\right) \\ &= \prod_p \frac{(p-1)^4(p^2+4p+1)}{p^6} \\ &= \prod_p \left(1 - \frac{9}{p^2} + \frac{16}{p^3} - \frac{9}{p^4} + \frac{1}{p^6}\right) \end{aligned}$$

- In dimension $n = 4$, we have

$$\begin{aligned} C_4 &= \prod_p \left(1 - \frac{1}{p}\right)^{15} \left(\sum_{k \geq 0} \frac{4k^3+6k^2+4k+1}{p^k}\right) \\ &= \prod_p \frac{(p-1)^{15}}{p^{15}} \left(\frac{4p^3+16p^2+4p}{(p-1)^4} + \frac{6p^2+6p}{(p-1)^3} + \frac{4p}{(p-1)^2} + \frac{p}{p-1}\right) \\ &= \prod_p \frac{(p-1)^{11}(p^3+11p^2+11p+1)}{p^{14}} \\ &= \prod_p \left(1 - \frac{55}{p^2} + \frac{320}{p^3} - \frac{891}{p^4} + \frac{1408}{p^5} - \frac{1155}{p^6} + \frac{1155}{p^8} - \frac{1408}{p^9} + \frac{891}{p^{10}} - \frac{320}{p^{11}} + \frac{55}{p^{12}} - \frac{1}{p^{14}}\right) \end{aligned}$$

6.7 Computation of $K_n(c_n, \|\cdot\|_\infty)$ ($n = 2, 3$) in Corollary 4

• In dimension $n = 2$, we have: $I = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2\}$, $\mathbf{u} = (2, 2, 1)$ and $\mathbf{c} = \mathbf{1}$. It follows from definition 14 that

$$\begin{aligned} \mathcal{J}_2(I, \mathbf{u}, \mathbf{c}; x) &= \int_{\substack{\mathbf{y} \in [1, +\infty[^5 \\ y_1 y_2 y_5 \leq x \\ y_3 y_4 y_5 \leq x}} y_5 d\mathbf{y} = \int_{y_5 \in [1, x[} y_5 \left(\int_{\substack{y_1, y_2 \in [1, +\infty[\\ y_1 y_2 \leq x/y_5}} dy_1 dy_2 \right)^2 dy_5 \\ &= \int_{y_5 \in [1, x[} y_5 \left(\frac{x}{y_5} \ln \left(\frac{x}{y_5} \right) - \frac{x}{y_5} + 1 \right)^2 dy_5 \\ &= \frac{1}{3} x^2 \ln^3(x) - x^2 \ln^2(x) + x^2 \ln(x) - 2x \ln(x) + \frac{x^2}{2} - \frac{1}{2} \end{aligned}$$

We deduce that

$$K_2(c_2, \|\cdot\|_\infty) := \lim_{x \rightarrow \infty} \mathcal{J}_2(I, \mathbf{u}, \mathbf{c}; x) x^{-2} (\ln x)^{-3} = \frac{1}{3}.$$

• In dimension $n = 3$: We have $I = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_3, \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3\}$, $\mathbf{u} = (2, 2, 2, 1, 1, 1, 1)$ and $\mathbf{c} = \mathbf{1}$. It follows from definition 14 that

$$\begin{aligned} \mathcal{J}_3(I, \mathbf{u}, \mathbf{c}; x) &= \int_{\substack{\mathbf{y} \in [1, +\infty[^{10} \\ y_1 y_2 y_7 y_8 y_{10} \leq x \\ y_3 y_4 y_7 y_9 y_{10} \leq x \\ y_5 y_6 y_8 y_9 y_{10} \leq x}} y_7 y_8 y_9 y_{10}^2 d\mathbf{y} = \int_{\substack{y_8, y_9, y_{10} \in [1, +\infty[\\ y_8 y_9 y_{10} \leq x}} y_8 y_9 y_{10}^2 \int_{\substack{y_1, \dots, y_7 \in [1, +\infty[\\ y_1 y_2 y_7 \leq x / y_8 y_{10} \\ y_3 y_4 y_7 \leq x / y_9 y_{10} \\ y_5 y_6 \leq x / y_8 y_9 y_{10}}} y_7 d\mathbf{y} \\ &= \int_{\substack{y_8, y_9, y_{10} \in [1, +\infty[\\ y_8 y_9 y_{10} \leq x}} y_8 y_9 y_{10}^2 h \left(\frac{x}{y_8 y_9 y_{10}} \right) \int_{\substack{y_1, y_2, y_3, y_4, y_7 \in [1, +\infty[\\ y_1 y_2 y_7 \leq x / y_8 y_{10} \\ y_3 y_4 y_7 \leq x / y_9 y_{10}}} y_7 dy_{1,2,3,4,7} dy_{8,9,10}, \end{aligned}$$

where $h(u) = u \ln(u) - u + 1$. We deduce that

$$\begin{aligned} \mathcal{J}_3(I, \mathbf{u}, \mathbf{c}; x) &= \int_{\substack{y_8, y_9, y_{10} \in [1, +\infty[\\ y_8 y_9 y_{10} \leq x}} y_8 y_9 y_{10}^2 h \left(\frac{x}{y_8 y_9 y_{10}} \right) \int_{y_7=1}^{\frac{x}{y_{10}(y_8 \vee y_9)}} y_7 \int_{\substack{y_1, \dots, y_4 \in [1, +\infty[\\ y_1 y_2 \leq x / y_7 y_8 y_{10} \\ y_3 y_4 \leq x / y_7 y_9 y_{10}}} dy_{1,2,3,4} dy_7 dy_{8,9,10} \\ &= \int_{\substack{y_8, y_9, y_{10} \in [1, +\infty[\\ y_8 y_9 y_{10} \leq x}} y_8 y_9 y_{10}^2 h \left(\frac{x}{y_8 y_9 y_{10}} \right) \int_{y_7=1}^{\frac{x}{y_{10}(y_8 \vee y_9)}} y_7 h \left(\frac{x}{y_7 y_8 y_{10}} \right) h \left(\frac{x}{y_7 y_9 y_{10}} \right) dy_7 dy_{8,9,10} \\ &= \int_{y_{10} \in [1, x[} y_{10}^2 \int_{\substack{y_8, y_9 \in [1, +\infty[\\ y_8 y_9 \leq x / y_{10}}} y_8 y_9 h \left(\frac{x}{y_8 y_9 y_{10}} \right) \int_{y_7=1}^{\frac{x}{y_{10}(y_8 \vee y_9)}} y_7 h \left(\frac{x}{y_7 y_8 y_{10}} \right) h \left(\frac{x}{y_7 y_9 y_{10}} \right) dy_7 dy_{8,9,10} \end{aligned}$$

The symmetry on the variables y_8 and y_9 implies then that

$$\begin{aligned} \mathcal{I}_3(I, \mathbf{u}, \mathbf{c}; x) &= 2 \int_{y_{10} \in [1, x]} y_{10}^2 \int_{y_9=1}^{\sqrt{\frac{x}{y_{10}}}} \int_{y_8=1}^{y_9} \int_{y_7=1}^{\frac{x}{y_8 y_{10}}} y_7 y_8 y_9 h\left(\frac{x}{y_8 y_9 y_{10}}\right) h\left(\frac{x}{y_7 y_8 y_{10}}\right) h\left(\frac{x}{y_7 y_9 y_{10}}\right) \\ &\quad dy_7 dy_8 dy_9 dy_{10} \\ &+ 2 \int_{y_{10} \in [1, x]} y_{10}^2 \int_{y_9=\sqrt{\frac{x}{y_{10}}}}^{\frac{x}{y_{10}}} \int_{y_8=1}^{\frac{x}{y_9 y_{10}}} \int_{y_7=1}^{\frac{x}{y_8 y_{10}}} y_7 y_8 y_9 h\left(\frac{x}{y_8 y_9 y_{10}}\right) h\left(\frac{x}{y_7 y_8 y_{10}}\right) h\left(\frac{x}{y_7 y_9 y_{10}}\right) \\ &\quad dy_7 dy_8 dy_9 dy_{10} \end{aligned}$$

The change of variables $z_i = \ln(y_i)/\ln(x)$ implies then that

$$\begin{aligned} \mathcal{I}_3(I, \mathbf{u}, \mathbf{c}; x) &= 2 \ln^4(x) \int_{z_{10}=0}^1 \int_{z_9=0}^{(1-z_{10})/2} \int_{z_8=0}^{z_9} \int_{z_7=0}^{1-z_8-z_{10}} x^{2z_7+2z_8+2z_9+3z_{10}} h(x^{1-z_8-z_9-z_{10}}) h(x^{1-z_7-z_8-z_{10}}) \times \\ &\quad h(x^{1-z_7-z_9-z_{10}}) dz_7 dz_8 dz_9 dz_{10} \\ &+ 2 \ln^4(x) \int_{z_{10}=0}^1 \int_{z_9=(1-z_{10})/2}^{1-z_{10}} \int_{z_8=0}^{1-z_9-z_{10}} \int_{z_7=0}^{1-z_8-z_{10}} x^{2z_7+2z_8+2z_9+3z_{10}} h(x^{1-z_8-z_9-z_{10}}) \times \\ &\quad h(x^{1-z_7-z_8-z_{10}}) h(x^{1-z_7-z_9-z_{10}}) dz_7 dz_8 dz_9 dz_{10} \end{aligned}$$

$$= x^3 \left(\frac{47}{16128} \ln^7(x) - \frac{217}{11520} \ln^6(x) + \frac{11}{240} \ln^5(x) - \frac{1}{32} \ln^4(x) + \frac{4}{3} \ln^3(x) - \frac{1}{4} \ln(x) - \frac{973}{36} + O(x^{\varepsilon-1}) \right).$$

It follows that

$$K_3(c_3, \|\|\|_{\infty}) := \lim_{x \rightarrow \infty} \mathcal{I}_3(I, \mathbf{u}, \mathbf{c}; x) x^{-3} (\ln x)^{-7} = \frac{47}{16128}.$$

6.8 Computation of $K_n(s_n, \|\|\|_{\infty})$ ($n = 2, 3$) in Corollary 5

• In dimension $n = 2$: We have $I = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2\}$, $\mathbf{u} = (1, 1, 1)$ and $\mathbf{c} = \mathbf{0}$. It follows that

$$\mathcal{I}_2(I, \mathbf{u}, \mathbf{c}; x) = \int_{\substack{y_1, y_2, y_3 \in [1, +\infty[\\ y_1 y_3 \leq x \\ y_2 y_3 \leq x}} \frac{d\mathbf{y}}{y_1 y_2 y_3} = \frac{1}{3} \ln^3(x).$$

We deduce that

$$K_2(s_2, \|\|\|_{\infty}) := \lim_{x \rightarrow \infty} \mathcal{I}_2(I, \mathbf{u}, \mathbf{c}; x) (\ln x)^{-3} = \frac{1}{3}.$$

• In dimension $n = 3$: We have $I = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_3, \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3\}$, $\mathbf{u} = \mathbf{1}$

and $\mathbf{c} = \mathbf{0}$. It follows that

$$\begin{aligned}
\mathcal{I}_3(I, \mathbf{u}, \mathbf{c}; x) &= \int_{\substack{y_1, \dots, y_7 \in [1, +\infty[\\ y_1 y_4 y_5 y_7 \leq x \\ y_2 y_4 y_6 y_7 \leq x \\ y_3 y_5 y_6 y_7 \leq x}} \frac{d\mathbf{y}}{y_1 y_2 y_3 y_4 y_5 y_6 y_7} = \ln^7(x) \int_{\substack{z_1, \dots, z_7 \in [0, +\infty[\\ z_1 + z_4 + z_5 + z_7 \leq 1 \\ z_2 + z_4 + z_6 + z_7 \leq 1 \\ z_3 + z_5 + z_6 + z_7 \leq 1}} dz \\
&= \ln^7(x) \int_{\substack{z_5, z_6, z_7 \in [0, +\infty[\\ z_5 + z_6 + z_7 \leq 1}} \int_{\substack{z_1, z_2, z_4 \in [0, +\infty[\\ z_1 + z_4 \leq 1 - z_5 - z_7 \\ z_2 + z_4 \leq 1 - z_6 - z_7}} (1 - z_5 - z_6 - z_7) dz_{1,2,4} dz_{5,6,7} \\
&= \ln^7(x) \int_{\substack{z_5, z_6, z_7 \in [0, +\infty[\\ z_5 + z_6 + z_7 \leq 1}} \int_{z_4=0}^{1-z_7-z_5 \vee z_6} \int_{\substack{z_1, z_2 \in [0, +\infty[\\ z_1 \leq 1 - z_4 - z_5 - z_7 \\ z_2 \leq 1 - z_4 - z_6 - z_7}} (1 - z_5 - z_6 - z_7) dz_{1,2,4} dz_{5,6,7} \\
&= \ln^7(x) \int_{\substack{z_5, z_6, z_7 \in [0, +\infty[\\ z_5 + z_6 + z_7 \leq 1}} \int_{z_4=0}^{1-z_7-z_5 \vee z_6} (1 - z_4 - z_5 - z_7)(1 - z_4 - z_6 - z_7)(1 - z_5 - z_6 - z_7) dz_4 dz_{5,6,7}
\end{aligned}$$

The symmetry in z_5 and z_6 implies that

$$\begin{aligned}
\mathcal{I}_3(I, \mathbf{u}, \mathbf{c}; x) &= 2 \ln^7(x) \left(\int_{z_7=0}^1 \int_{z_6=0}^{(1-z_7)/2} \int_{z_5=0}^{z_6} \int_{z_4=0}^{1-z_6-z_7} + \int_{z_7=0}^1 \int_{z_6=(1-z_7)/2}^{1-z_7} \int_{z_5=0}^{1-z_6-z_7} \int_{z_4=0}^{1-z_6-z_7} \right) \\
&= \frac{11}{3360} \ln^7(x)
\end{aligned}$$

We deduce that

$$K_3(s_3, \|\|\|\infty) := \lim_{x \rightarrow \infty} \mathcal{I}_3(I, \mathbf{u}, \mathbf{c}; x) (\ln x)^{-7} = \frac{11}{3360}.$$

6.9 Computation of $K_n(u_n, \|\|\|\infty)$ ($n = 2, 3$) in Corollary 6

• In dimension $n = 2$: We have $I = \{\mathbf{e}_1, \mathbf{e}_2\}$, $\mathbf{u} = (1, 1)$ and $\mathbf{c} = \mathbf{0}$. It follows that

$$\mathcal{I}_2(I, \mathbf{u}, \mathbf{c}; x) = \int_{\substack{y_1, y_2 \in [1, +\infty[\\ y_1 \leq x \\ y_2 \leq x}} \frac{d\mathbf{y}}{y_1 y_2} = \ln^2(x)$$

We deduce that

$$K_2(u_2, \|\|\|\infty) := \lim_{x \rightarrow \infty} \mathcal{I}_2(I, \mathbf{u}, \mathbf{c}; x) (\ln x)^{-2} = 1.$$

• In dimension $n = 3$: We have $I = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_3, \mathbf{e}_2 + \mathbf{e}_3\}$, $\mathbf{u} = \mathbf{1}$ and $\mathbf{c} = \mathbf{0}$.

It follows that

$$\begin{aligned}
\mathcal{I}_3(I, \mathbf{u}, \mathbf{c}; x) &= \int_{\substack{y_1, \dots, y_6 \in [1, +\infty[\\ y_1 y_4 y_5 \leq x \\ y_2 y_4 y_6 \leq x \\ y_3 y_5 y_6 \leq x}} \frac{d\mathbf{y}}{y_1 y_2 y_3 y_4 y_5 y_6} = \ln^6(x) \int_{\substack{z_1, \dots, z_6 \in [0, +\infty[\\ z_1 + z_4 + z_5 \leq 1 \\ z_2 + z_4 + z_6 \leq 1 \\ z_3 + z_5 + z_6 \leq 1}} dz \\
&= \ln^6(x) \int_{\substack{z_5, z_6 \in [0, +\infty[\\ z_5 + z_6 \leq 1}} \int_{\substack{z_1, z_2, z_4 \in [0, +\infty[\\ z_1 + z_4 \leq 1 - z_5 \\ z_2 + z_4 \leq 1 - z_6}} (1 - z_5 - z_6) dz_{1,2,4} dz_{5,6} \\
&= \ln^6(x) \int_{\substack{z_5, z_6 \in [0, +\infty[\\ z_5 + z_6 \leq 1}} \int_{z_4=0}^{1 - z_5 \vee z_6} \int_{\substack{z_1, z_2 \in [0, +\infty[\\ z_1 \leq 1 - z_4 - z_5 \\ z_2 \leq 1 - z_4 - z_6}} (1 - z_5 - z_6) dz_{1,2} dz_4 dz_{5,6} \\
&= \ln^6(x) \int_{\substack{z_5, z_6 \in [0, +\infty[\\ z_5 + z_6 \leq 1}} \int_{z_4=0}^{1 - z_5 \vee z_6} (1 - z_4 - z_5)(1 - z_4 - z_6)(1 - z_5 - z_6) dz_4 dz_{5,6}
\end{aligned}$$

Using the symmetry of the integrant in z_5 and z_6 implies that

$$\begin{aligned}
\mathcal{I}_3(I, \mathbf{u}, \mathbf{c}; x) &= 2 \ln^6(x) \left(\int_{z_6=0}^{1/2} \int_{z_5=0}^{z_6} \int_{z_4=0}^{1-z_6} + \int_{z_6=1/2}^1 \int_{z_5=0}^{1-z_6} \int_{z_4=0}^{1-z_6} \right) \\
&= \frac{11}{480} \ln^6(x)
\end{aligned}$$

We deduce that

$$K_3(u_3, \|\|\|_\infty) := \lim_{x \rightarrow \infty} \mathcal{I}_3(I, \mathbf{u}, \mathbf{c}; x) (\ln x)^{-6} = \frac{11}{480}.$$

6.10 Computation of $K_n(v_n, \|\|\|_\infty)$ ($n = 2, 3$) in Corollary 7

- In dimension $n = 2$: We have $I = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2\}$, $\mathbf{u} = (1, 1, 1)$ and $\mathbf{c} = (1, 1)$.

$$\mathcal{I}_2(I, \mathbf{u}, \mathbf{c}; x) = \int_{\substack{y_1, y_2, y_3 \in [1, +\infty[\\ y_1 y_3 \leq x \\ y_2 y_3 \leq x}} y_3 d\mathbf{y} = x^2 \ln(x) - \frac{3}{2}x^2 + 2x - \frac{1}{2}.$$

We deduce that

$$K_2(v_2, \|\|\|_\infty) := \lim_{x \rightarrow \infty} \mathcal{I}_2(I, \mathbf{u}, \mathbf{c}; x) x^{-2} (\ln x)^{-1} = 1.$$

- In dimension $n = 3$: We have $I = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_3, \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3\}$, $\mathbf{u} = \mathbf{1}$

and $\mathbf{c} = (1, 1, 1)$. It follows that

$$\begin{aligned}
\mathcal{I}_3(I, \mathbf{u}, \mathbf{c}; x) &= \int_{\substack{y_1, \dots, y_7 \in [1, +\infty[\\ y_1 y_4 y_5 y_7 \leq x \\ y_2 y_4 y_6 y_7 \leq x \\ y_3 y_5 y_6 y_7 \leq x}} y_4 y_5 y_6 y_7^2 d\mathbf{y} = \ln^7(x) \int_{\substack{z_1, \dots, z_7 \in [0, +\infty[\\ z_1 + z_4 + z_5 + z_7 \leq 1 \\ z_2 + z_4 + z_6 + z_7 \leq 1 \\ z_3 + z_5 + z_6 + z_7 \leq 1}} x^{z_1 + \dots + z_3 + 2z_4 + \dots + 2z_6 + 3z_7} dz \\
&= \ln^7(x) \int_{\substack{z_5, z_6, z_7 \in [0, +\infty[\\ z_5 + z_6 + z_7 \leq 1}} \int_{\substack{z_1, z_2, z_4 \in [0, +\infty[\\ z_1 + z_4 \leq 1 - z_5 - z_7 \\ z_2 + z_4 \leq 1 - z_6 - z_7}} x^{z_1 + z_2 + 2z_4 + \dots + 2z_6 + 3z_7} \frac{x^{1 - z_5 - z_6 - z_7} - 1}{\ln(x)} dz_{1,2,4} dz_{5,6,7} \\
&= \ln^4(x) \int_{\substack{z_5, z_6, z_7 \in [0, +\infty[\\ z_5 + z_6 + z_7 \leq 1}} \int_{\substack{z_4 = 0 \\ 1 - z_7 - z_5 \vee z_6}} x^{2z_4 + \dots + 2z_6 + 3z_7} (x^{1 - z_5 - z_6 - z_7} - 1)(x^{1 - z_4 - z_5 - z_7} - 1) \times \\
&\quad \times (x^{1 - z_4 - z_6 - z_7} - 1) dz_4 dz_{5,6,7}
\end{aligned}$$

By symmetry in z_5 and z_6 we get

$$\begin{aligned}
\mathcal{I}_3(I, \mathbf{u}, \mathbf{c}; x) &= 2 \ln^4(x) \left(\int_{z_7=0}^1 \int_{z_6=0}^{(1-z_7)/2} \int_{z_5=0}^{z_6} \int_{z_4=0}^{1-z_6-z_7} + \int_{z_7=0}^1 \int_{z_6=(1-z_7)/2}^{1-z_7} \int_{z_5=0}^{1-z_6-z_7} \int_{z_4=0}^{1-z_6-z_7} \right) \\
&= \frac{1}{16} x^3 \ln^4(x) - \frac{1}{4} x^3 \ln^3(x) + \frac{5}{2} x^3 \ln(x) - \frac{67}{12} x^3 + O(x^{2+\varepsilon}).
\end{aligned}$$

We deduce that

$$K_3(v_3, \|\|\|_\infty) := \lim_{x \rightarrow \infty} \mathcal{I}_3(I, \mathbf{u}, \mathbf{c}; x) x^{-3} (\ln x)^{-4} = \frac{1}{16}.$$

6.11 Computation of $K_n(\cdot, \|\|\|_d)$ ($n = 2, 3$) in Corollaries 8 and 9

Sargos's volume constant

First we will recall some notations from §2.3.1 of [12]. Let $Q(\mathbf{X}) = \sum_{\alpha \in \text{supp}(Q)} a_\alpha \mathbf{X}^\alpha$ be a generalized polynomial with positive coefficients that depends upon all the variables X_1, \dots, X_n . We apply the discussion in [27] (see also [26]) to define a ‘‘volume constant’’ for Q .

By definition, the Newton polyhedron of Q (at infinity) is the set $\mathcal{E}^\infty(Q) := (\text{conv}(\text{supp}(Q)) - \mathbb{R}_+^n)$.

Let G_0 be the smallest face of $\mathcal{E}^\infty(Q)$ which meets the diagonal $\Delta = \mathbb{R}_+ \mathbf{1}$. We denote by σ_0 the unique positive real number t that satisfies $t^{-1} \mathbf{1} \in G_0$. Thus, there exists a unique vector subspace \vec{G}_0 of largest codimension ρ_0 such that $G_0 \subset \sigma_0^{-1} \mathbf{1} + \vec{G}_0$. Both ρ_0, σ_0 evidently depend upon Q , but it is not necessary to indicate this in the notation. We also set $Q_{G_0}(X) = \sum_{\alpha \in G_0} a_\alpha \mathbf{X}^\alpha$.

There exist finitely many facets of $\mathcal{E}^\infty(Q)$ that intersect in G_0 . We denote their normalized polar vectors by $\lambda_1, \dots, \lambda_N$.

By a permutation of the coordinates X_i one can suppose that $\bigoplus_{i=1}^{\rho_0} \mathbb{R} \mathbf{e}_i \oplus \vec{G}_0 = \mathbb{R}^n$, and that $\{\mathbf{e}_{m+1}, \dots, \mathbf{e}_n\}$ is the set of vectors to which G_0 is parallel (i.e. for which $G_0 = G_0 - \mathbb{R}_+ \mathbf{e}_i$). If G_0 is compact then $m = n$.

Set $\Lambda = \text{Conv}\{\mathbf{0}, \lambda_1, \dots, \lambda_N, \mathbf{e}_{\rho_0+1}, \dots, \mathbf{e}_n\}$. It follows that $\dim \Lambda = n$.

Definition 1. *The volume constant associated to Q is:*

$$A_0(Q) := n! \text{Vol}(\Lambda) \int_{[1, +\infty)^{n-m}} \left(\int_{\mathbb{R}_+^{m-\rho_0}} P_{G_0}^{-\sigma_0}(\mathbf{1}, \mathbf{x}, \mathbf{y}) d\mathbf{x} \right) d\mathbf{y}.$$

In ([27], chap 3, th. 1.6) (also see [26]), P. Sargos proved the following important result about the function

$$s \rightarrow Y(Q; s) := \int_{[1, +\infty)^n} Q(\mathbf{x})^{-s} d\mathbf{x},$$

Theorem (P. Sargos): *Let Q be a generalized polynomial with positive coefficients. Then $Y(Q; s)$ converges absolutely in $\{\operatorname{Re} s > \sigma_0\}$, and has a meromorphic continuation to \mathbb{C} with largest pole at $s = \sigma_0$ of order ρ_0 . In addition, the volume constant of Q is given by*

$$A_0(Q) = \lim_{s \rightarrow \sigma_0} (s - \sigma_0)^{\rho_0} Y(Q; s) > 0. \quad (\text{IV.25})$$

Mellin's Formula

We will also use the following classical Mellin's formula:

Let $w_1, \dots, w_r \in \mathbb{C}$ such that $\operatorname{Re}(w_i) > 0 \forall i = 0, \dots, r$, Let $\rho_1, \dots, \rho_r > 0$. Then, for $s \in \mathbb{C}$ verifying $\operatorname{Re}(s) > \rho_1 + \dots + \rho_r$, we have :

$$\frac{\Gamma(s)}{(\sum_{k=0}^r w_k)^s} = \frac{1}{(2\pi i)^r} \int_{(\rho_1)} \dots \int_{(\rho_r)} \frac{\Gamma(s - |\mathbf{z}|) \prod_{i=1}^r \Gamma(z_i) dz}{w_0^{s-|\mathbf{z}|} (\prod_{k=1}^r w_k^{z_k})}, \quad (\text{IV.26})$$

where the notation $\int_{(\rho)}$ denote the integral on the vertical line $\operatorname{Re}(s) = \rho$.

Computation of $K_n(c_n, \|\|\|_d)$ ($n = 2, 3$) in Corollary 8

- In dimension $n = 2$: Corollary 8 implies that

$$K_2(c_2, \|\|\|_d) = \frac{d^4}{24} A_0(\mathcal{S}, P), \quad (\text{IV.27})$$

where $A_0(\mathcal{S}, P)$ is the mixed volume constant (see §2.3.3 of [12]) associated to the polynomial $P = X_1^d + X_2^d$ and the pair $\mathcal{S} = (\tilde{I}, \mathbf{u} = (u(\beta))_{\beta \in \tilde{I}})$, where $\tilde{I} = \{\frac{1}{2}(\mathbf{e}_1 + \mathbf{e}_2), \mathbf{e}_1, \mathbf{e}_2\}$, and

$$u\left(\frac{1}{2}(\mathbf{e}_1 + \mathbf{e}_2)\right) = 1 \text{ and } u(\mathbf{e}_1) = u(\mathbf{e}_2) = 2.$$

It follows then from the construction given in §2.3.3 of [12] that

$$A_0(\mathcal{S}, P) = A_0(Q), \quad (\text{IV.28})$$

where $A_0(Q)$ is the volume constant associated to the polynomial

$$Q(X_1, X_2, X_3, X_4, X_5) = X_1^{d/2} X_2^d X_3^d + X_1^{d/2} X_4^d X_5^d.$$

By using notations of §7.7.1, we have

$$\mathcal{E}^\infty(Q) := \operatorname{conv}(\operatorname{supp}(Q) - \mathbb{R}_+^5) = \operatorname{conv}(\{(d/2, d, d, 0, 0), (d/2, 0, 0, d, d)\} - \mathbb{R}_+^5),$$

$$G_0 = \operatorname{conv}\{(d/2, d, d, 0, 0), (d/2, 0, 0, d, d)\}, \quad \sigma_0 = 2/d \quad \text{and} \quad \rho_0 = 4.$$

Sargos's Theorem above implies then that

$$A_0(Q) = \lim_{s \rightarrow 2/d} \left(s - \frac{2}{d}\right)^4 Y(Q; s). \quad (\text{IV.29})$$

This being done we will now compute the principal part of the integral $Y(Q; s)$. First we remark that for $\operatorname{Re}(s) > 2/d$, we have

$$Y(Q; s) = \int_{[1, \infty)^5} (x_1^{d/2} x_2^d x_3^d + x_1^{d/2} x_4^d x_5^d)^{-s} dx_{1,2,3,4,5} = \frac{2}{d} \frac{1}{s - \frac{2}{d}} \int_{[1, \infty)^4} (x_2^d x_3^d + x_4^d x_5^d)^{-s} dx_{2,3,4,5} \quad (\text{IV.30})$$

Mellin's formula (IV.26) implies that for $\operatorname{Re}(s) > 2/d$,

$$\begin{aligned} \int_{[1, \infty)^4} (x_2^d x_3^d + x_4^d x_5^d)^{-s} dx_{2,3,4,5} &= \frac{1}{2\pi i} \int_{[1, \infty)^4} \int_{(2/d)} \frac{\Gamma(s-z)\Gamma(z)}{\Gamma(s)} (x_2 x_3)^{-d(s-z)} (x_4 x_5)^{-dz} dx_{2,3,4,5} dz \\ &= \frac{1}{2\pi i} \int_{(2/d)} \frac{\Gamma(s-z)\Gamma(z)}{d^4 \Gamma(s)} \frac{1}{[(s-z) - \frac{1}{d}]^2} \frac{1}{[z - \frac{1}{d}]^2} dz \end{aligned}$$

Moving the integration line to left until $\frac{1}{2d}$ and using residues theorem imply that

$$\begin{aligned} \int_{[1, \infty)^4} (x_2^d x_3^d + x_4^d x_5^d)^{-s} dx_{2,3,4,5} &= \frac{1}{2\pi i} \int_{(\frac{1}{2d})} \frac{\Gamma(s-z)\Gamma(z)}{d^4 \Gamma(s)} \frac{1}{[(s-z) - \frac{1}{d}]^2} \frac{1}{[z - \frac{1}{d}]^2} dz \quad (\text{IV.31}) \\ &+ \frac{\Gamma^{(1)}(\frac{1}{d}) \Gamma(s - \frac{1}{d})}{d^4 \Gamma(s) [s - \frac{2}{d}]^2} - \frac{\Gamma(\frac{1}{d}) \Gamma^{(1)}(s - \frac{1}{d})}{d^4 \Gamma(s) [s - \frac{2}{d}]^2} + \frac{2\Gamma(\frac{1}{d}) \Gamma(s - \frac{1}{d})}{d^4 \Gamma(s) [s - \frac{2}{d}]^3} \end{aligned}$$

Since the integral in the right side of (IV.31) define a holomorphic function in $\operatorname{Re}(s) > \frac{3}{2d}$, we deduce by using in addition (IV.30) that

$$A_0(Q) = \lim_{s \rightarrow 2/d} \left(s - \frac{2}{d} \right)^4 Y(Q; s) = \frac{4\Gamma(\frac{1}{d})^2}{d^5 \Gamma(s)}. \quad (\text{IV.32})$$

Combining (IV.27), (IV.28) and (IV.32) implies that

$$K_2(c_2, \|\|\|_d) = \frac{1}{6d} \frac{\Gamma(\frac{1}{d})^2}{\Gamma(\frac{2}{d})}. \quad (\text{IV.33})$$

• In dimension $n = 3$: Corollary 8 implies that

$$K_3(c_3, \|\|\|_d) = \frac{d^8}{72 \times 7!} A_0(\mathcal{I}, P), \quad (\text{IV.34})$$

where $A_0(\mathcal{I}, P)$ is the mixed volume constant (see §2.3.3 of [12]) associated to the polynomial $P = X_1^d + X_2^d + X_3^3$ and the pair $\mathcal{I} = \left(\tilde{I}, \mathbf{u} = (u(\boldsymbol{\beta}))_{\boldsymbol{\beta} \in \tilde{I}} \right)$, where

$$\tilde{I} = \left\{ \frac{1}{3}(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3), \frac{1}{2}(\mathbf{e}_1 + \mathbf{e}_2), \frac{1}{2}(\mathbf{e}_1 + \mathbf{e}_3), \frac{1}{2}(\mathbf{e}_2 + \mathbf{e}_3), \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \right\}$$

and $\mathbf{u} = (1, 1, 1, 1, 2, 2, 2)$. It follows then from the construction given in §2.3.3 of [12] that

$$A_0(\mathcal{I}, P) = A_0(Q), \quad (\text{IV.35})$$

where $A_0(Q)$ is the volume constant associated to the polynomial

$$Q(x_1, \dots, x_{10}) = x_1^{d/3} x_2^{d/2} x_3^{d/2} x_5^d x_8^d + x_1^{d/3} x_2^{d/2} x_4^{d/2} x_6^d x_9^d + x_1^{d/3} x_3^{d/2} x_4^{d/2} x_7^d x_{10}^d.$$

By using notations of §7.7.1, we have

$$G_0 = \operatorname{conv}\{(d/3, d/2, d/2, 0, d, d, 0, 0, 0, 0), (d/3, d/2, 0, d/2, 0, 0, d, d, 0, 0), (d/3, 0, d/2, d/2, 0, 0, 0, 0, d, d)\},$$

$$\sigma_0 = 3/d \quad \text{and} \quad \rho_0 = 8.$$

Sargos's Theorem implies then that

$$A_0(Q) = \lim_{s \rightarrow 3/d} \left(s - \frac{3}{d} \right)^8 Y(Q; s). \quad (\text{IV.36})$$

This being done we will now compute the principal part of the integral $Y(Q; s)$ at $s = 3/d$. More precisely, we will show that for $\text{Re}(s) > 3/d$,

$$Y(Q; s) = \frac{372\Gamma\left(\frac{1}{d}\right)^2 \Gamma\left(s - \frac{2}{d}\right)}{d^{10}\Gamma(s) \left[s - \frac{3}{d}\right]^8} + \frac{H(s)}{\left[s - \frac{3}{d}\right]^7}, \quad (\text{IV.37})$$

where H is a holomorphic function in the bigger domain $\Omega = \{\text{Re}(s) > \frac{5}{2d}\}$.

In the next computation we will denote by H a generic holomorphic function in Ω .

By using Mellin's formula (IV.26), we obtain that for $\text{Re}(s) > 3/d$,

$$\begin{aligned} Y(Q; s) &= \frac{24}{d^{10} \left[s - \frac{3}{d}\right]} \frac{1}{(2\pi i)^2} \int_{\substack{(\rho_1)=\frac{3}{2d} \\ (\rho_2)=\frac{3}{2d}}} \frac{\Gamma(s - z_1 - z_2)\Gamma(z_1)\Gamma(z_2)\Gamma(s)^{-1} dz_{1,2}}{\left(s - z_1 - z_2 - \frac{1}{d}\right)^2 \left(z_1 + z_2 - \frac{2}{d}\right) \prod_{j=1}^2 \left[\left(z_j - \frac{1}{d}\right)^2 \left(s - z_j - \frac{2}{d}\right)\right]} \\ &= \frac{24}{d^{10} \left[s - \frac{3}{d}\right]} \frac{1}{(2\pi i)^2} \int_{\substack{(\rho_1)=\frac{1}{2d} \\ (\rho_2)=\frac{1}{2d}}} \frac{\Gamma(s - z_1 - z_2)\Gamma(z_1)\Gamma(z_2)\Gamma(s)^{-1} dz_{1,2}}{\left(s - z_1 - z_2 - \frac{1}{d}\right)^2 \left(z_1 + z_2 - \frac{2}{d}\right) \prod_{j=1}^2 \left[\left(z_j - \frac{1}{d}\right)^2 \left(s - z_j - \frac{2}{d}\right)\right]} \\ &\quad + \frac{24}{d^{10} \left[s - \frac{3}{d}\right]} \frac{1}{2\pi i} \int_{(\rho_1)=\frac{3}{2d}} \frac{\frac{\partial}{\partial z_2} \left[\frac{\Gamma(s - z_1 - z_2)\Gamma(z_1)\Gamma(z_2)}{\left(s - z_1 - z_2 - \frac{1}{d}\right)^2 \left(z_1 + z_2 - \frac{2}{d}\right) \left(s - z_2 - \frac{2}{d}\right)} \right]_{z_2=\frac{1}{d}} dz_1}{\Gamma(s) \left(z_1 - \frac{1}{d}\right)^2 \left(s - z_1 - \frac{2}{d}\right)} \\ &\quad + \frac{24}{d^{10} \left[s - \frac{3}{d}\right]} \frac{1}{2\pi i} \int_{(\rho_2)=\frac{1}{2d}} \frac{\frac{\partial}{\partial z_1} \left[\frac{\Gamma(s - z_1 - z_2)\Gamma(z_1)\Gamma(z_2)}{\left(s - z_1 - z_2 - \frac{1}{d}\right)^2 \left(z_1 + z_2 - \frac{2}{d}\right) \left(s - z_1 - \frac{2}{d}\right)} \right]_{z_1=\frac{1}{d}} dz_2}{\Gamma(s) \left(z_2 - \frac{1}{d}\right)^2 \left(s - z_2 - \frac{2}{d}\right)} \\ &\quad - \frac{12}{d^{10} \left[s - \frac{3}{d}\right]} \frac{1}{2\pi i} \int_{(\rho_2)=\frac{1}{2d}} \frac{\Gamma\left(s - \frac{2}{d}\right)\Gamma(z_2)\Gamma\left(\frac{2}{d} - z_2\right)}{\Gamma(s) \left(s - \frac{3}{d}\right)^3 \left(z_2 - \frac{1}{d}\right)^4} \left[\frac{1}{s + z_2 - \frac{4}{d}} + \frac{1}{s - z_2 - \frac{2}{d}} \right] dz_2 \\ &= \frac{H(s)}{\left(s - \frac{3}{d}\right)^7} - \frac{12}{d^{10} \left(s - \frac{3}{d}\right)} \frac{1}{2\pi i} \int_{(\rho_2)=\frac{1}{2d}} \frac{\Gamma\left(s - \frac{2}{d}\right)\Gamma(z_2)\Gamma\left(\frac{2}{d} - z_2\right)}{\Gamma(s) \left(s - \frac{3}{d}\right)^3 \left(z_2 - \frac{1}{d}\right)^4 \left(s + z_2 - \frac{4}{d}\right)} dz_2 \\ &\quad + \frac{24}{d^{10} \left(s - \frac{3}{d}\right)} \frac{1}{2\pi i} \int_{(\rho_1)=\frac{3}{2d}} \frac{\frac{\partial}{\partial z_2} \left[\frac{\Gamma(s - z_1 - z_2)\Gamma(z_1)\Gamma(z_2)}{\left(s - z_1 - z_2 - \frac{1}{d}\right)^2 \left(z_1 + z_2 - \frac{2}{d}\right) \left(s - z_2 - \frac{2}{d}\right)} \right]_{z_2=\frac{1}{d}} dz_1}{\Gamma(s) \left(z_1 - \frac{1}{d}\right)^2 \left(s - z_1 - \frac{2}{d}\right)} \\ &= \frac{H(s)}{\left(s - \frac{3}{d}\right)^7} + \frac{2}{d^{10} \left(s - \frac{3}{d}\right)^4} \frac{\partial^3}{\partial z_2^3} \left[\frac{\Gamma\left(s - \frac{2}{d}\right)\Gamma(z_2)\Gamma\left(\frac{2}{d} - z_2\right)}{\Gamma(s) \left(s + z_2 - \frac{4}{d}\right)} \right]_{z_2=\frac{1}{d}} \\ &\quad + \frac{24}{d^{10}\Gamma(s)} \left[\frac{-\Gamma\left(\frac{1}{d}\right)}{\left(s - \frac{3}{d}\right)^2} \text{Res}_{z_1=\frac{1}{d}} \frac{\Gamma(z_1)\Gamma^{(1)}\left(s - z_1 - \frac{1}{d}\right)}{\left(s - z_1 - \frac{2}{d}\right)^3 \left(z_1 - \frac{1}{d}\right)^3} + \frac{\Gamma^{(1)}\left(\frac{1}{d}\right)}{\left(s - \frac{3}{d}\right)^2} \text{Res}_{z_1=\frac{1}{d}} \frac{\Gamma(z_1)\Gamma\left(s - z_1 - \frac{1}{d}\right)}{\left(s - z_1 - \frac{2}{d}\right)^3 \left(z_1 - \frac{1}{d}\right)^3} \right] \\ &\quad + \frac{2\Gamma\left(\frac{1}{d}\right)}{\left(s - \frac{3}{d}\right)^2} \text{Res}_{z_1=\frac{1}{d}} \frac{\Gamma(z_1)\Gamma\left(s - z_1 - \frac{1}{d}\right)}{\left(s - z_1 - \frac{2}{d}\right)^4 \left(z_1 - \frac{1}{d}\right)^3} - \frac{\Gamma\left(\frac{1}{d}\right)}{\left(s - \frac{3}{d}\right)^2} \text{Res}_{z_1=\frac{1}{d}} \frac{\Gamma(z_1)\Gamma\left(s - z_1 - \frac{1}{d}\right)}{\left(s - z_1 - \frac{2}{d}\right)^3 \left(z_1 - \frac{1}{d}\right)^4} \end{aligned}$$

$$\begin{aligned}
& + \frac{\Gamma\left(\frac{1}{d}\right)}{\left(s - \frac{3}{d}\right)^3} \operatorname{Res}_{z_1 = \frac{1}{d}} \left[\frac{\Gamma(z_1) \Gamma\left(s - z_1 - \frac{1}{d}\right)}{\left(s - z_1 - \frac{2}{d}\right)^3 \left(z_1 - \frac{1}{d}\right)^3} \right] \\
& = \frac{H(s)}{\left(s - \frac{3}{d}\right)^7} - \frac{12}{d^{10} \left(s - \frac{3}{d}\right)^8} \frac{\Gamma\left(s - \frac{2}{d}\right) \Gamma\left(\frac{1}{d}\right)^2}{\Gamma(s)} + \frac{24 * 20}{d^{10} \left(s - \frac{3}{d}\right)^8} \frac{\Gamma\left(s - \frac{2}{d}\right) \Gamma\left(\frac{1}{d}\right)^2}{\Gamma(s)} \\
& - \frac{24 * 10}{d^{10} \left(s - \frac{3}{d}\right)^8} \frac{\Gamma\left(s - \frac{2}{d}\right) \Gamma\left(\frac{1}{d}\right)^2}{\Gamma(s)} + \frac{24 * 6}{d^{10} \left(s - \frac{3}{d}\right)^8} \frac{\Gamma\left(s - \frac{2}{d}\right) \Gamma\left(\frac{1}{d}\right)^2}{\Gamma(s)} \\
& = \frac{H(s)}{\left(s - \frac{3}{d}\right)^7} + \frac{12 * 31}{d^{10} \left(s - \frac{3}{d}\right)^8} \frac{\Gamma\left(s - \frac{2}{d}\right) \Gamma\left(\frac{1}{d}\right)^2}{\Gamma(s)}
\end{aligned}$$

This ends the proof of (IV.37). Combining (IV.34), (IV.35), (IV.36) and (IV.37) implies that

$$K_3(c_3, \|\!\|_d) = \frac{31 \Gamma\left(\frac{1}{d}\right)^3}{30240 d^2 \Gamma\left(\frac{3}{d}\right)}. \quad (\text{IV.38})$$

Computation of $K_n(v_n, \|\!\|_d)$ ($n = 2, 3$) in Corollary 9

• In dimension $n = 2$: Corollary 9 implies that

$$K_2(v_2, \|\!\|_d) = \frac{d^2}{4} A_0(\mathcal{S}, P), \quad (\text{IV.39})$$

where $A_0(\mathcal{S}, P)$ is the mixed volume constant (see §2.3.3 of [12]) associated to the polynomial $P = X_1^d + X_2^d$ and the pair $\mathcal{S} = \left(\tilde{I}, \mathbf{u} = (u(\boldsymbol{\beta}))_{\boldsymbol{\beta} \in \tilde{I}}\right)$, where $\tilde{I} = \left\{\frac{1}{2}(\mathbf{e}_1 + \mathbf{e}_2), \mathbf{e}_1, \mathbf{e}_2\right\}$, and

$$u\left(\frac{1}{2}(\mathbf{e}_1 + \mathbf{e}_2)\right) = u(\mathbf{e}_1) = u(\mathbf{e}_2) = 1.$$

It follows then from the construction given in §2.3.3 of [12] that

$$A_0(\mathcal{S}, P) = A_0(Q), \quad (\text{IV.40})$$

where $A_0(Q)$ is the volume constant associated to the polynomial

$$Q(X_1, X_2, X_3) = X_1^{d/2} X_2^d + X_1^{d/2} X_3^d.$$

By using notations of §7.7.1, we have

$$G_0 = \operatorname{conv}\{(d/2, d, 0), (d/2, 0, d)\}, \quad \sigma_0 = 2/d, \quad \rho_0 = 2 \quad \text{and} \quad \Lambda = \operatorname{conv}\left\{\mathbf{0}, \frac{1}{d}(2, 0, 0), \frac{1}{d}(0, 1, 1), \mathbf{e}_3\right\}.$$

It follows then from Definition 1 above that

$$A_0(\mathcal{S}, P) = A_0(Q) = 3! \operatorname{Vol}(\Lambda) \int_{\mathbb{R}_+} P(\mathbf{1}, x_3)^{-2/d} dx_3 = \frac{2}{d^3} \frac{\Gamma(1/d)^2}{\Gamma(2/d)}.$$

By using in addition (IV.39) we obtain that

$$K_2(v_2, \|\!\|_d) = \frac{1}{2d} \frac{\Gamma(1/d)^2}{\Gamma(2/d)}. \quad (\text{IV.41})$$

• In dimension $n = 3$: Corollary 9 implies that

$$K_3(v_3, \|\!\|_d) = \frac{d^5}{3 \times 4!} A_0(\mathcal{S}, P), \quad (\text{IV.42})$$

where $A_0(\mathcal{I}, P)$ is the mixed volume constant (see §2.3.3 of [12]) associated to the polynomial $P = X_1^d + X_2^d + X_3^3$ and the pair $\mathcal{I} = \left(\tilde{I}, \mathbf{u} = (u(\beta))_{\beta \in \tilde{I}} \right)$, where

$$\tilde{I} = \left\{ \frac{1}{3}(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3), \frac{1}{2}(\mathbf{e}_1 + \mathbf{e}_2), \frac{1}{2}(\mathbf{e}_1 + \mathbf{e}_3), \frac{1}{2}(\mathbf{e}_2 + \mathbf{e}_3), \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \right\}$$

and $\mathbf{u} = (1, 1, 1, 1, 1, 1, 1)$. It follows then from the construction given in §2.3.3 of [12] that

$$A_0(\mathcal{I}, P) = A_0(Q), \quad (\text{IV.43})$$

where $A_0(Q)$ is the volume constant associated to the polynomial

$$Q(x_1, \dots, x_7) = x_1^{d/3} x_2^{d/2} x_3^{d/2} x_5^d + x_1^{d/3} x_2^{d/2} x_4^{d/2} x_6^d + x_1^{d/3} x_3^{d/2} x_4^{d/2} x_7^d.$$

By using notations of §7.7.1, we have

$$G_0 = \text{conv}\{(d/3, d/2, d/2, 0, d, 0, 0), (d/3, d/2, 0, d/2, 0, d, 0), (d/3, 0, d/2, d/2, 0, 0, d)\},$$

$$\sigma_0 = 3/d \quad \text{and} \quad \rho_0 = 5.$$

Sargos's Theorem above implies then that

$$A_0(Q) = \lim_{s \rightarrow 3/d} \left(s - \frac{3}{d} \right)^5 Y(Q; s). \quad (\text{IV.44})$$

This being done we will now compute the principal part of the integral $Y(Q; s)$ at $s = 3/d$. Using Mellin's formula (IV.26) as in the proof of (IV.37) imply that for $\text{Re}(s) > 3/d$, we have

$$\begin{aligned} Y(Q; s) &= \int_{[1, \infty)^7} (x_1^{d/3} x_2^{d/2} x_3^{d/2} x_5^d + x_1^{d/3} x_2^{d/2} x_4^{d/2} x_6^d + x_1^{d/3} x_3^{d/2} x_4^{d/2} x_7^d)^{-s} dx_{1,2,3,4,5,6,7} \\ &= \frac{36\Gamma\left(s - \frac{2}{d}\right) \Gamma\left(\frac{1}{d}\right)^2}{d^7 \Gamma(s) \left[s - \frac{3}{d}\right]^5} + \frac{H(s)}{\left[s - \frac{3}{d}\right]^4}, \end{aligned}$$

where H is a holomorphic function in the domain $\Omega = \{\text{Re}(s) > \frac{5}{2d}\}$.

We deduce that

$$A_0(\mathcal{I}, P) = A_0(Q) = \frac{36\Gamma\left(\frac{1}{d}\right)^3}{d^7 \Gamma\left(\frac{3}{d}\right)}.$$

It follows then from (IV.42) that

$$K_3(v_3, \|\|\|_d) = \frac{\Gamma\left(\frac{1}{d}\right)^3}{2 d^2 \Gamma\left(\frac{3}{d}\right)}. \quad (\text{IV.45})$$

Chapter V

Complements to the Chapter IV

This Chapter 5 is a complement to Chapter 4. In, Chapter 4 we have not given the analogues of the corollaries of 5 and 6 for the Holder's norms. The reason is that Essouabri's Tauberian theorem does not apply as it is to these two cases. We therefore start in this chapter with a slight extension of Essouabri's theorem on mixed zetas functions (18, Th. 3) and deduce from it a more general version of Theorem 17 of Chapter 4. As an application, we will obtain in the Corollaries 10 and 11 the analogues of Corollaries 5 and 6 for Holder's norms.

1 Preliminaries

Let $P \in \mathbb{R}_+[X_1, \dots, X_n]$ be an elliptic homogeneous polynomial of degree $d > 0$ with non-negative coefficients and let $f : \mathbb{N}^n \rightarrow \mathbb{C}$ be a multiplicative function.

We suppose that there exist $\mathbf{c} = (c_1, \dots, c_n) \in \mathbb{R}_+^n$, $I = \{\beta_1, \dots, \beta_r\} \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}$, $\mathbf{u} : I \rightarrow \mathbb{N}$ and $\varepsilon \in \mathbb{R}_+^*$ such that

1. $\mathbf{s} \mapsto \mathcal{M}(f; \mathbf{s}) := \sum_{\mathbf{m} \in \mathbb{N}^n} \frac{f(m_1, \dots, m_n)}{m_1^{s_1} \dots m_n^{s_n}}$ is absolutely convergent in $\{\sigma_i > c_i \ \forall i \in \llbracket 1, n \rrbracket\}$,
2. $\mathbf{s} \mapsto \mathcal{H}(f, \mathbf{c}; \mathbf{s}) := \prod_{i=1}^r (\beta_i \cdot \mathbf{s})^{\mathbf{u}(\beta_i)} \mathcal{M}(f; \mathbf{c} + \mathbf{s})$ has a holomorphic continuation with moderate growth to $\{\mathbf{s}_i > c_i - \varepsilon \ \forall i \in \llbracket 1, n \rrbracket\}$.

We will introduce in this section several objects attached to P , \mathbf{c} , I and \mathbf{u} that we will use in the statements and the proofs of the main results of this chapter.

First we write P in the form

$$P(\mathbf{X}) = \sum_{k=1}^p a_k \mathbf{X}^{\alpha_k},$$

where $\alpha_1, \dots, \alpha_p \in \mathbb{N}_0^n$ verify $|\alpha_1| = \dots = |\alpha_p| = d$ and $a_1, \dots, a_p \in \mathbb{R}_+^*$.

In particular, we have $\text{Supp}(P) = \{\alpha_1, \dots, \alpha_p\}$.

We also set $(\mathbf{e}_1, \dots, \mathbf{e}_n)$ the canonical basis of \mathbb{R}^n .

Since P is elliptic, we have $\{d\mathbf{e}_i \mid i \in \llbracket 1, n \rrbracket\} \subset \text{Supp}(P)$ and therefore the cone generated by $\text{Supp}(P)$ is equal to \mathbb{R}_+^n . It follows that $\mathbf{c} \in \text{con}(\text{Supp}(P)) = \text{con}(\{\alpha_1, \dots, \alpha_p\})$.

We define a subset J of $\llbracket 1, p \rrbracket$ and a vector $\delta^{(0)}$ of \mathbb{R}_+^p as follows:

1. If $\mathbf{c} \neq \mathbf{0}$, we define $J \subset \llbracket 1, p \rrbracket$ by $\mathbf{c} \in \text{con}^* \{\boldsymbol{\alpha}_j; j \notin J\}$. Thus, there exists $\{\delta_j^{(0)}\}_{j \notin J} \in \mathbb{R}_+^*$ such that $\mathbf{c} = \sum_{\substack{j \in \llbracket 1, p \rrbracket \\ j \notin J}} \delta_j^{(0)} \boldsymbol{\alpha}_j$. We also set in this case

$$\boldsymbol{\delta}^{(0)} = \sum_{\substack{j \in \llbracket 1, p \rrbracket \\ j \notin J}} \delta_j^{(0)} \mathbf{e}_j \in \mathbb{R}_+^p. \quad (\text{V.1})$$

In particular,

$$\mathbf{c} = \sum_{j=1}^p \delta_j^{(0)} \boldsymbol{\alpha}_j \quad (\text{V.2})$$

and

$$\frac{|\mathbf{c}|}{d} = \sum_{j \notin J} \delta_j^{(0)} = |\boldsymbol{\delta}^{(0)}| \text{ since } |\boldsymbol{\alpha}_j| = d \ \forall j \in \llbracket 1, p \rrbracket.$$

2. If $\mathbf{c} = \mathbf{0}$ we set $J = \llbracket 1, p \rrbracket$ and $\boldsymbol{\delta}^{(0)} = \mathbf{0}$.

We also set in both cases

$$E_J = \{\mathbf{e}_j \mid j \in J\}, \quad \boldsymbol{\delta}^{(1)} = \sum_{j \in J} \mathbf{e}_j \quad \text{and} \quad \boldsymbol{\delta} = (\delta_1, \dots, \delta_p) = \boldsymbol{\delta}^{(0)} + \boldsymbol{\delta}^{(1)} \in (\mathbb{R}_+^*)^p \quad (\text{V.3})$$

In particular we have $|\boldsymbol{\delta}| = \frac{|\mathbf{c}|}{d} + |J|$.

Let $I = \{\boldsymbol{\beta}_1, \dots, \boldsymbol{\beta}_r\} \subset \mathbb{R}_+^n \setminus \{\mathbf{0}\}$ and $\mathbf{u} : I \rightarrow \mathbb{N}$ as in the beginning of this section.

We define $T'_P : I \rightarrow \mathbb{R}_+^p$, I' and $\mathbf{u}'_P : I' \rightarrow \mathbb{N}$ by

$$\forall \boldsymbol{\beta}_i = (\beta_{i,1}, \dots, \beta_{i,n}) \in I, \quad T'_P(\boldsymbol{\beta}_i) = (\beta_{i,1}, \dots, \beta_{i,n}) \begin{pmatrix} \alpha_{1,1} & \cdots & \alpha_{p,1} \\ & \ddots & \\ \alpha_{1,n} & \cdots & \alpha_{p,n} \end{pmatrix} = (\boldsymbol{\beta}_i \cdot \boldsymbol{\alpha}_1, \dots, \boldsymbol{\beta}_i \cdot \boldsymbol{\alpha}_p)$$

$$I' = T'_P(I) \cup E_J \quad \text{and} \quad \forall \mathbf{v} \in I', \quad \mathbf{u}'_P(\mathbf{v}) = \mathbf{1}_{\mathbf{v} \in E_J}(\mathbf{v}) + \sum_{\substack{i \in \llbracket 1, r \rrbracket \\ T'_P(\boldsymbol{\beta}_i) = \mathbf{v}}} \mathbf{u}(\boldsymbol{\beta}_i) \quad (\text{V.4})$$

Remark 17. We have $\mathbf{v} \cdot \boldsymbol{\delta} > 0 \ \forall \mathbf{v} \in I'$. Indeed,

If $\mathbf{v} \in E_J$ then $\mathbf{v} \cdot \boldsymbol{\delta} = 1$.

If $\mathbf{v} \in I' \setminus E_J$ then there exists $l \in \llbracket 1, r \rrbracket$ such that $\mathbf{v} = T'_P(\boldsymbol{\beta}_l)$ and we have

$$\begin{aligned} \mathbf{v} \cdot \boldsymbol{\delta} &= \sum_{i=1}^p \left(\sum_{k=1}^n \beta_{l,k} \alpha_{i,k} \right) \delta_i = \sum_{k=1}^n \beta_{l,k} \left(\sum_{i=1}^p \alpha_{i,k} \delta_i \right) \\ &\geq \sum_{k=1}^n \beta_{l,k} \left(\min_{i \in \llbracket 1, p \rrbracket} \delta_i \right) \left(\sum_{i=1}^p \alpha_{i,k} \right) \geq 0, \end{aligned} \quad (\text{V.5})$$

as $\min_{i \in \llbracket 1, p \rrbracket} \delta_i > 0$ it follows that $\mathbf{v} \cdot \boldsymbol{\delta} = 0$ imply that for each $k \in \llbracket 1, n \rrbracket$ such that $\beta_{l,k} > 0$ we have $\sum_{i=1}^p \alpha_{i,k} = 0$. Consequently for k as above we have $\alpha_{i,k} = 0$ for each $i \in \llbracket 1, p \rrbracket$, it follows that P is independent of the variable X_k . Hence au contradiction. It follows that $\mathbf{v} \cdot \boldsymbol{\delta} > 0$. This ends the proof of the remark.

Remark 17 allows us to define $T_P : I' \rightarrow \mathbb{R}_+^p$, \tilde{I} and $\mathbf{u}_P : \tilde{I} \rightarrow \mathbb{N}$ by

$$\forall \mathbf{v} \in I' \quad T_P(\mathbf{v}) = (\mathbf{v} \cdot \boldsymbol{\delta})^{-1} \mathbf{v}, \quad \tilde{I} = T_P(I') \quad \text{and} \quad \forall \mathbf{w} \in \tilde{I} \quad \mathbf{u}_P(\mathbf{w}) = \sum_{\substack{\mathbf{v} \in I' \\ T_P(\mathbf{v}) = \mathbf{w}}} \mathbf{u}'_P(\mathbf{v}) \quad (\text{V.6})$$

Moreover, we have the identity

$$\sum_{\mathbf{w} \in \tilde{I}} \mathbf{u}_P(\mathbf{w}) = \sum_{\mathbf{w} \in \tilde{I}} \sum_{\substack{\mathbf{v} \in I' \\ T_P(\mathbf{v})=\mathbf{w}}} \mathbf{u}'_P(\mathbf{v}) = \sum_{\mathbf{v} \in I'} \mathbf{u}'_P(\mathbf{v}) = |J| + \sum_{\beta \in I} \mathbf{u}(\beta) \quad (\text{V.7})$$

and by using definitions of T'_P , T_P and relations (V.1), (V.3),(V.4) we also have

$$\mathbf{v} \cdot \boldsymbol{\delta} = \begin{cases} \beta \cdot \mathbf{c} + \sum_{j \in J} \beta \cdot \alpha_j & \text{if } \mathbf{v} = T'_P(\beta) \\ 1 & \text{if } \mathbf{v} \in E_J \end{cases} \quad (\text{V.8})$$

$$\tilde{I} = \left\{ \frac{1}{\beta \cdot \mathbf{c} + \sum_{j \in J} \beta \cdot \alpha_j} (\beta \cdot \alpha_1, \dots, \beta \cdot \alpha_p); \beta \in I \right\} \cup J \quad (\text{V.9})$$

Furthermore

$$\begin{aligned} \prod_{\mathbf{v} \in I'} (\mathbf{v} \cdot \boldsymbol{\delta})^{\mathbf{u}'_P(\mathbf{v})} &= \prod_{\mathbf{v} \in E_J} (\mathbf{v} \cdot \boldsymbol{\delta})^{\mathbf{u}'_P(\mathbf{v})} \prod_{\mathbf{v} \in I' \setminus E_J} (\mathbf{v} \cdot \boldsymbol{\delta})^{\mathbf{u}'_P(\mathbf{v})} \\ &= \prod_{\substack{\beta \in I \\ T'_P(\beta) \notin E_J}} (T'_P(\beta) \cdot \boldsymbol{\delta})^{\mathbf{u}(\beta)} \end{aligned}$$

because $\mathbf{v} \cdot \boldsymbol{\delta} = \delta_j = 1$ for each $\mathbf{v} \in E_J$.

From definition of T'_P and relation (V.3) we get

$$\begin{aligned} \prod_{\mathbf{v} \in I'} (\mathbf{v} \cdot \boldsymbol{\delta})^{\mathbf{u}'_P(\mathbf{v})} &= \prod_{\substack{\beta \in I \\ T'_P(\beta) \notin E_J}} \left(\sum_{i=1}^n \beta_i \sum_{j=1}^p \alpha_{j,i} \delta_j \right)^{\mathbf{u}(\beta)} \\ &= \prod_{\substack{\beta \in I \\ T'_P(\beta) \notin E_J}} \left(\sum_{i=1}^n \beta_i \sum_{j=1}^p \alpha_{j,i} \delta_j^{(0)} + \sum_{i=1}^n \beta_i \sum_{j=1}^p \alpha_{j,i} \delta_j^{(1)} \right)^{\mathbf{u}(\beta)} \\ &= \prod_{\substack{\beta \in I \\ T'_P(\beta) \notin E_J}} (\beta \cdot \mathbf{c} + \sum_{j \in J} \beta \cdot \alpha_j)^{\mathbf{u}(\beta)} \\ &= \prod_{\beta \in I} (\beta \cdot \mathbf{c} + \sum_{j \in J} \beta \cdot \alpha_j)^{\mathbf{u}(\beta)} \end{aligned} \quad (\text{V.10})$$

2 Main results of this chapter

The following theorem is an extension of Essouabri's result on mixed zeta functions, see [12, Theo. 3]:

Theorem 18. *Let $P \in \mathbb{R}_+[X_1, \dots, X_n]$ be an elliptic homogeneous polynomial of degree $d > 0$ and with non-negative coefficients. We write it as $P(\mathbf{X}) = \sum_{k=1}^p a_k \mathbf{X}^{\alpha_k}$ with $a_j > 0$ for each $j \in \llbracket 1, p \rrbracket$.*

Let $f : \mathbb{N}^n \rightarrow \mathbb{C}$ be a multiplicative function. We suppose that there exists $\mathbf{c} \in [0, +\infty[^n$, $I = \{\beta_1, \dots, \beta_r\} \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}$, $\mathbf{u} : I \rightarrow \mathbb{N}$ and $\varepsilon \in \mathbb{R}_+^$ such that*

1. $\mathbf{s} \mapsto \mathcal{M}(f; \mathbf{s}) = \sum_{\mathbf{m} \in \mathbb{N}^n} \frac{f(m_1, \dots, m_n)}{m_1^{s_1} \dots m_n^{s_n}}$ is absolutely convergent in $\{\sigma_i > c_i \ \forall i \in \llbracket 1, n \rrbracket\}$,
2. $\mathbf{s} \mapsto \mathcal{H}(f, \mathbf{c}; \mathbf{s}) = \prod_{i=1}^r (\beta_i \cdot \mathbf{s})^{\mathbf{u}(\beta_i)} \mathcal{M}(f; \mathbf{c} + \mathbf{s})$ has a holomorphic continuation with moderate growth in $\{s_i > c_i - \varepsilon \ \forall i \in \llbracket 1, n \rrbracket\}$.

Let $J \subset \llbracket 1, p \rrbracket$ the set defined by $\mathbf{c} \in \text{con}^* \{\boldsymbol{\alpha}_j; j \notin J\}$ if $\mathbf{c} \neq \mathbf{0}$ and $J = \llbracket 1, p \rrbracket$ if $\mathbf{c} = \mathbf{0}$. We set $E_J = \{\mathbf{e}_j; j \in J\}$ and $\rho = \sum_{\boldsymbol{\beta} \in I} \mathbf{u}(\boldsymbol{\beta}) + |J| - \text{Rank}(I \cup E_J) + 1 - \mathbf{1}_{\{\mathbf{0}\}}(\mathbf{c})$. Then,

$$\mathbf{s} \mapsto Z(f; P; \mathbf{s}) = \sum_{\mathbf{m} \in \mathbb{N}^n} \frac{f(m_1, \dots, m_n)}{P(m_1, \dots, m_n)^{s/d}} \quad (\text{V.11})$$

is holomorphic in $\{\text{Re}(s) > |\mathbf{c}|\}$ and there exists $\eta > 0$ such that $\mathbf{s} \mapsto Z(f; P; \mathbf{s})$ has a meromorphic extension to $\{\sigma > |\mathbf{c}| - \eta\}$ with one possible pole at $s = |\mathbf{c}|$ of order at most ρ .

If we assume in addition that $\mathbf{1} \in \text{con}^*(I \cup E_J)$ and there exists K a holomorphic function in a tubular neighbor of $\mathbf{0}$ such that $\mathcal{H}(f, \mathbf{c}; \mathbf{s}) = K((\boldsymbol{\beta} \cdot \mathbf{s})_{\boldsymbol{\beta} \in I \cup E_J})$. Then, by using notation $\boldsymbol{\delta}, \tilde{I}, \mathbf{u}_P$ defined in section §1, we have

$$Z(f; P; s) = \frac{T(d, \boldsymbol{\delta}, \mathbf{a}; \mathbf{c}) \mathcal{H}(f, \mathbf{c}; \mathbf{0}) d^{\rho-1} A_0(\tilde{I}; \mathbf{u}_P; \mathbf{a})}{\prod_{\boldsymbol{\beta} \in I} (\boldsymbol{\beta} \cdot \mathbf{c} + \sum_{j \in J} \boldsymbol{\beta} \cdot \boldsymbol{\alpha}_j)^{\mathbf{u}(\boldsymbol{\beta})}} \frac{1}{(s - |\mathbf{c}|)^\rho} + O((s - |\mathbf{c}|)^{-(\rho-1)}) \quad \text{as } s \rightarrow |\mathbf{c}|. \quad (\text{V.12})$$

where $T(d, \boldsymbol{\delta}, \mathbf{a}; \mathbf{c}) = \frac{d\Gamma(|\boldsymbol{\delta}|) \prod_{j \in J} a_j}{\Gamma\left(\frac{|\mathbf{c}|}{d}\right)}$ if $\mathbf{c} \neq \mathbf{0}$ and $T(d, \boldsymbol{\delta}, \mathbf{a}; \mathbf{c}) = \Gamma(p) \prod_{j=1}^p a_j$ if $\mathbf{c} = \mathbf{0}$,

and where $A_0(\tilde{I}; \mathbf{u}_P; \mathbf{a})$ is the volume constant, see Chapter II section 1.4, associated to triplet $(\tilde{I}, \mathbf{u}_P, \mathbf{a})$ with $\mathbf{a} = (a_1, \dots, a_p)$.

Remark 18. Essouabri established above theorem in the case where $\mathbf{c} \in (\mathbb{R}_+^*)^n$. With a slight extension of his argument, we will prove it in the case of $\mathbf{c} \in \mathbb{R}_+^n$.

The combination of this extension and Theorem 15 of Chapter IV implies the following extension of Theorem 17 of Chapter IV:

Theorem 19. Let $d \geq 1$. Let $f : \mathbb{N}^n \rightarrow \mathbb{R}_+$ be a multiplicative non-negative function. We suppose that f belongs to the class $\mathcal{C}(g, \kappa, \mathbf{c}, \delta)$ associated to the data $(g, \kappa, \mathbf{c}, \delta)$, see Definition 12. We set $I = I(\kappa, g) = \{\boldsymbol{\nu} \in \mathbb{N}_0 \setminus \{\mathbf{0}\} \mid \kappa(\boldsymbol{\nu}) = 1 \text{ and } g(\boldsymbol{\nu}) \neq 0\}$. Let $J = \{j \in \llbracket 1, n \rrbracket \mid c_j = 0\}$ and $E_J = \{\mathbf{e}_j; j \in J\}$. Set

$$\rho_0 = \sum_{\boldsymbol{\nu} \in I} g(\boldsymbol{\nu}) + |J| - \text{Rank}(I \cup E_J).$$

Then, there exist a polynomial Q of degree at most ρ_0 and a positive constant $\eta > 0$ such that

$$\mathcal{N}_d(f; x) := \sum_{\substack{\mathbf{m}=(m_1, \dots, m_n) \in \mathbb{N}^n \\ \|\mathbf{m}\|_d = \sqrt[d]{m_1^d + \dots + m_n^d} \leq x}} f(m_1, \dots, m_n) = x^{|\mathbf{c}|} Q(\ln x) + O(x^{|\mathbf{c}|-\eta}) \quad \text{as } x \rightarrow +\infty.$$

Furthermore, if we assume in addition that $\text{Rank}(I \cup E_J) = n$ and $\mathbf{1} \in \text{con}^*(I \cup E_J)$, then, the degree of the polynomial Q is equal to $\rho_0 = (\sum_{\boldsymbol{\nu} \in I} g(\boldsymbol{\nu})) + |J| - n$ and the main term of $\mathcal{N}_d(f; x)$ is given by

$$\mathcal{N}_d(f; x) = C_n(f) K_n(f, \|\cdot\|_d) x^{|\mathbf{c}|} (\ln x)^{\rho_0} + O((\ln x)^{\rho_0-1}) \quad \text{as } x \rightarrow +\infty,$$

where $C_n(f) := \mathcal{H}(f, \mathbf{c}; \mathbf{0}) > 0$ is defined by the Euler product (IV.6) and

$$K_n(f, \|\cdot\|_d) := \frac{d^{\rho_0} \Gamma(|\boldsymbol{\delta}|) A_0(\tilde{I}; \mathbf{u}_P; \mathbf{1})}{\Gamma\left(\frac{|\mathbf{c}|}{d}\right) |\mathbf{c}| \rho_0! \prod_{\boldsymbol{\nu} \in I} (\boldsymbol{\beta} \cdot \mathbf{c} + d \sum_{j \in J} \nu_j)^{g(\boldsymbol{\nu})}} \quad \text{if } \mathbf{c} \neq \mathbf{0}$$

and

$$K_n(f, \|\cdot\|_d) := \frac{\Gamma(n)d^{-1} A_0(\tilde{I}; \mathbf{u}_P; \mathbf{1})}{\rho_0! \prod_{\nu \in I} |\nu|^{g(\nu)}} \quad \text{if } \mathbf{c} = \mathbf{0}.$$

where δ , \tilde{I} and \mathbf{u}_P are defined from $I := I(\kappa, g)$ and $\mathbf{u} := g$ as in section §1, and where $A_0(\tilde{I}; \mathbf{u}_P; \mathbf{1}) > 0$ is the volume constant (see Chapter II section 1.4) associated to \tilde{I} , \mathbf{u}_P and $\mathbf{1}$.

As a consequence of theorem 19 we obtain the following two corollaries:

Corollary 10. *Let $n \in \mathbb{N}$ and $d \geq 1$. There exists a polynomial Q_7 of degree $2^n - 1$ and $\mu_7 > 0$ such that*

$$S_{n,d}(x) := \sum_{\substack{\mathbf{m}=(m_1, \dots, m_n) \in \mathbb{N}^n \\ \|\mathbf{m}\|_d = \sqrt[d]{m_1^d + \dots + m_n^d} \leq x}} \frac{1}{\text{lcm}(m_1, \dots, m_n)} = Q_7(\ln x) + O(x^{-\mu_7}) \quad \text{as } x \rightarrow +\infty.$$

In particular, we have

$$S_{n,d}(x) = C_n(s_n) K_n(s_n, \|\cdot\|_d) (\ln x)^{2^n - 1} + O((\ln x)^{2^n - 2}) \quad \text{as } x \rightarrow +\infty,$$

where

$$C_n(s_n) := \prod_p \left(1 - \frac{1}{p}\right)^{2^n - 1} \left(\sum_{k=0}^{\infty} \frac{(k+1)^n - k^n}{p^k}\right) > 0, \quad (\text{V.13})$$

and

$$K_n(s_n, \|\cdot\|_d) := \frac{(n-1)! \prod_{j=1}^n j^{-\binom{n}{j}}}{d (2^n - 1)!} A_0(\tilde{I}; \mathbf{u}_P; \mathbf{1}) > 0,$$

where \tilde{I} and \mathbf{u}_P are defined as in section §1 from $\mathbf{c} = \mathbf{0}$, $I = \{0, 1\}^n \setminus \{\mathbf{0}\}$ and $\mathbf{u} : I \rightarrow \mathbb{N}$ defined by $\mathbf{u}(\nu) = 1 \forall \nu \in I$ and where $A_0(\tilde{I}; \mathbf{u}_P; \mathbf{1}) > 0$ is the volume constant associated to \tilde{I} , \mathbf{u}_P and $\mathbf{1}$.

Corollary 11. *Let $n \in \mathbb{N} \setminus \{1\}$ and $d \geq 1$. There exists a polynomial Q_8 of degree $2^n - 2$ and $\mu_8 > 0$ such that*

$$U_{n,d}(x) := \sum_{\substack{\mathbf{m}=(m_1, \dots, m_n) \in \mathbb{N}^n \\ \text{gcd}(m_1, \dots, m_n) = 1 \\ \|\mathbf{m}\|_d = \sqrt[d]{m_1^d + \dots + m_n^d} \leq x}} \frac{1}{\text{lcm}(m_1, \dots, m_n)} = Q_8(\ln x) + O(x^{-\mu_8}) \quad \text{as } x \rightarrow +\infty.$$

In particular, we have

$$U_{n,d}(x) = C_n(u_n) K_n(u_n, \|\cdot\|_\infty) (\ln x)^{2^n - 2} + O((\ln x)^{2^n - 3}) \quad \text{as } x \rightarrow +\infty,$$

where

$$C_n(u_n) := \prod_p \left(1 - \frac{1}{p}\right)^{2^n - 2} \left(\sum_{k=0}^{\infty} \frac{(k+1)^n - k^n - 1}{p^k}\right) > 0, \quad (\text{V.14})$$

and

$$K_n(u_n, \|\cdot\|_\infty) := \frac{(n-1)! \prod_{j=1}^{n-1} j^{-\binom{n}{j}}}{d (2^n - 2)!} A_0(\tilde{I}; \mathbf{u}_P; \mathbf{1}) > 0,$$

where \tilde{I} and \mathbf{u}_P are defined as in section §1 from $\mathbf{c} = \mathbf{0}$, $I = \{0, 1\}^n \setminus \{\mathbf{0}, \mathbf{1}\}$ and $\mathbf{u} : I \rightarrow \mathbb{N}$ defined by $\mathbf{u}(\nu) = 1 \forall \nu \in I$ and where $A_0(\tilde{I}; \mathbf{u}_P; \mathbf{1}) > 0$ is the volume constant associated to \tilde{I} , \mathbf{u}_P and $\mathbf{1}$.

3 Proof of Theorems 18 and 19

We will use the notations introduced in §1.

First we will prove the following lemma:

Lemma 21. *Let $\delta \in (\mathbb{R}_+^*)^p$ defined by (V.3) in §1. Let $\delta' \in (\mathbb{R}_+^*)^p$ and $\mathbf{z} = (z_1, \dots, z_p) \in \delta' + \delta + i\mathbb{R}^p$, then for $\mathbf{w} \in \tilde{I}$ we have $\operatorname{Re}(\mathbf{w} \cdot \mathbf{z}) > 1$.*

Proof. Let $\mathbf{w} \in \tilde{I}$, then there exists $\mathbf{v} \in I'$ such that $\mathbf{w} = (\mathbf{v} \cdot \delta)^{-1} \mathbf{v}$. It follows that $\operatorname{Re}(\mathbf{w} \cdot \mathbf{z}) = \mathbf{w} \cdot (\delta' + \delta) = \frac{\mathbf{v} \cdot \delta'}{\mathbf{v} \cdot \delta} + 1$. So to conclude it's enough to verify that $\mathbf{v} \cdot \delta' > 0$. As in (17), if $\mathbf{v} = T_P(\beta)$ with $\beta = (\beta_1, \dots, \beta_n) \in I$ then, we have

$$\mathbf{v} \cdot \delta' \geq \sum_{k=1}^n \beta_k \left(\min_{i \in \llbracket 1, p \rrbracket} \delta'_i \right) \left(\sum_{i=1}^p \alpha_{i,k} \right) > 0$$

The term on the right side is positive since P depends on all the variables X_1, \dots, X_n . \square

We need also the next tauberian lemma for Dirichlet series with abscissa of convergence equal to 0:

Lemma 22 (Tauberian). *Let $D(s) = \sum_{n \geq 1} \frac{a_n}{\lambda_n^s}$ be a Dirichlet series with non-negative coefficients with $0 < \lambda_1 < \lambda_2 < \dots$. We suppose that $s \mapsto D(s)$ is absolutely convergent in $\{\operatorname{Re}(s) > 0\}$ and that it has a meromorphic extension with moderate growth to a half-plane $\{\operatorname{Re}(s) > -\eta\}$ with $\eta > 0$. We also assume that D has a single pole in this half-plane at $s = 0$ and $D(s) \sim_{s \rightarrow 0} \frac{c_\rho}{s^\rho}$ with $c_\rho > 0$ and $\rho \in \mathbb{N}$. Then, there exists a polynomial of degree ρ and $\delta \in]0, 1[$ such that*

$$A(x) = \sum_{\lambda_n \leq x} a_n = Q(\ln x) + O(x^{-\delta}) \quad (\text{V.15})$$

Moreover,

$$A(x) = \frac{c_\rho}{\rho!} (\ln x)^\rho + O((\ln x)^{\rho-1}) \quad \text{as } x \rightarrow +\infty.$$

Proof. We set $b_n = \lambda_n a_n$ for each $n \in \mathbb{N}$, $B(x) = \sum_{\lambda_n \leq x} b_n$ and $G(s) = \sum_{n \geq 1} \frac{b_n}{\lambda_n^s}$.

We have for $\operatorname{Re}(s) > 1$,

$$G(s) = \sum_{n \geq 1} \frac{b_n}{\lambda_n^s} = \sum_{n \geq 1} \frac{a_n}{\lambda_n^{s-1}} = D(s-1)$$

It follows that $G(s)$ is absolutely convergent in $\{\operatorname{Re}(s) > 1\}$ and has a meromorphic extension with moderate growth to $\{\operatorname{Re}(s) > 1 - \eta\}$ with only one pole at $s = 1$ of order ρ and

$$G(s) \sim \frac{c_\rho}{(s-1)^\rho} \quad \text{as } s \rightarrow 1$$

Landau's tauberian theorem (see Theorem 21 in the Appendix), implies then that there exists $H \in \mathbb{C}[x]$ polynomial of degree $\rho - 1$ and $\delta > 0$ such that

$$B(x) = \sum_{\lambda_n \leq x} b_n = xH(\ln x) + O(x^{1-\delta}) \quad \text{as } x \rightarrow +\infty \quad (\text{V.16})$$

Landau taberian theorem implies also that the coefficient of the dominant term of $H(x)$ is equal to $\frac{c_\rho}{(\rho-1)!}$.

Writing the error term as $x^{1-\delta}u(x)$ with $u(x) = O(1)$ and using the Riemann-Stieljes integral we deduce that

$$\begin{aligned}
A(x) &= \sum_{\lambda_n \leq x} a_n = \sum_{\lambda_n \leq x} \frac{1}{\lambda_n} b_n = \int_{\lambda_1^-}^{x^+} \frac{1}{t} dB(t) \\
&= \frac{B(t)}{t} \Big|_{\lambda_1^-}^{x^+} + \int_{\lambda_1^-}^{x^+} \frac{B(t)}{t^2} dt \\
&= \frac{B(x^+)}{x^+} - \frac{B(\lambda_1^-)}{\lambda_1^-} + \int_{\lambda_1^-}^{x^+} \frac{B(t)}{t^2} dt \\
&= \frac{B(x^+)}{x} + \int_{\lambda_1^-}^{x^+} \frac{B(t)}{t^2} dt
\end{aligned} \tag{V.17}$$

By using in addition equation (V.16), we deduce that

$$\begin{aligned}
A(x) &= H(\ln x) + x^{-\delta}u(x^+) + \int_{\lambda_1^-}^{x^+} \frac{H(\ln t)}{t} dt + \int_{\lambda_1^-}^{x^+} \frac{u(t)}{t^{1+\delta}} dt \\
&= H(\ln x) + x^{-\delta}u(x^+) + \int_{\lambda_1^-}^{x^+} \frac{H(\ln t)}{t} dt + \int_{\lambda_1^-}^{+\infty} \frac{u(t)}{t^{1+\delta}} dt - \int_{x^+}^{+\infty} \frac{u(t)}{t^{1+\delta}} dt
\end{aligned} \tag{V.18}$$

since $u(x) = O(1)$ for $x \in \mathbb{R}_+$, we have

- $x^{-\delta}u(x^+) = O(x^{-\delta})$,
- $\int_{x^+}^{+\infty} \frac{u(t)}{t^{1+\delta}} dt = O(x^{-\delta})$,
- $\int_{\lambda_1^-}^{+\infty} \frac{u(t)}{t^{1+\delta}} dt$ is convergent and we set $K = \int_{\lambda_1^-}^{+\infty} \frac{u(t)}{t^{1+\delta}} dt$

Let $G \in \mathbb{R}[x]$ such that $G' = H$ and $G(\ln(\lambda_1)) = 0$. Equation (V.18) implies then that

$$\begin{aligned}
A(x) &= H(\ln x) + G(\ln x) - G(\ln(\lambda_1)) + K + O(x^{-\delta}) \\
&= G(\ln x) + H(\ln x) + K + O(x^{-\delta})
\end{aligned}$$

The function $G(x) + H(x) + K \in \mathbb{R}[x]$ is a polynomial of degree ρ , moreover since $G' = H$ we deduce that the coefficient of the dominant term of G is $\frac{c_\rho}{(\rho-1)!} = \frac{c_\rho}{\rho!}$. \square

3.1 End of proof of Theorem 18

First we remark that it suffices to prove the assertion of the theorem for $Z(f, 1 + P; s)$. Indeed, if we set σ_0 the abscissa of convergence of $Z(f; P; s)$, then Taylor formula implies that for any $K \in \mathbb{N}$, and $s \in \mathbb{C}$ such that $\sigma = \text{Re}(s) > \sigma_0$, we have

$$\begin{aligned}
Z(f; P; s) &= \sum_{\mathbf{m} \in \mathbb{N}^n} \frac{f(\mathbf{m})}{(1 + P(\mathbf{m}))^{s/d}} \left[1 - \frac{1}{1 + P(\mathbf{m})} \right]^{-s/d} \\
&= \sum_{k=0}^K \binom{-s/d}{k} (-1)^k Z(f, 1 + P, s + dk) \\
&\quad + O\left((1 + |s|)^{K+1} Z(|f|, 1 + P, s + d(K+1))\right).
\end{aligned} \tag{V.19}$$

Let $\boldsymbol{\delta}^{(0)} = (\delta_1^{(0)}, \dots, \delta_p^{(0)}) \in \mathbb{R}_+^p$ defined by (V.1). Let $\boldsymbol{\delta}' = (\delta'_1, \dots, \delta'_p) \in \mathbb{R}_+^{*p}$. Mellin-Barnes integral formula (see Lemma 27 in the Appendix), implies that for $s \in \mathbb{C}$ such that $\text{Re}(s) \gg 1$, we have

$$\begin{aligned}
Z(f; 1 + P; s) &= \sum_{\mathbf{m} \in \mathbb{N}^n} \frac{f(m_1, \dots, m_n)}{(1 + \sum_{j=1}^p a_j \mathbf{m}^{\alpha_j})^{s/d}} \\
&= \sum_{\mathbf{m} \in \mathbb{N}^n} \frac{1}{(2\pi i)^p} \int_{(\delta_1^{(0)} + \delta'_1)} \cdots \int_{(\delta_p^{(0)} + \delta'_p)} \frac{\Gamma(\frac{s}{d} - |\mathbf{z}|) \prod_{j=1}^p \Gamma(z_j)}{\Gamma(\frac{s}{d}) \prod_{j=1}^p a_j^{z_j}} \frac{f(m_1, \dots, m_n)}{\prod_{j=1}^p \mathbf{m}^{z_j \alpha_j}} dz_1 \dots dz_p \\
&= \frac{1}{(2\pi i)^p} \int_{(\delta_1^{(0)} + \delta'_1)} \cdots \int_{(\delta_p^{(0)} + \delta'_p)} \frac{\Gamma(\frac{s}{d} - |\mathbf{z}|) \prod_{j=1}^p \Gamma(z_j)}{\Gamma(\frac{s}{d}) \prod_{j=1}^p a_j^{z_j}} \times \\
&\quad \mathcal{M}(f; (\sum_{j=1}^p z_j \alpha_{j,1}, \dots, \sum_{j=1}^p z_j \alpha_{j,n})) dz_1 \dots dz_p \\
&= \frac{1}{(2\pi i)^p} \int_{(\delta'_1)} \cdots \int_{(\delta'_p)} \frac{\Gamma(\frac{s}{d} - |\mathbf{z}| - |\boldsymbol{\delta}^{(0)}|) \prod_{j=1}^p \Gamma(z_j + \delta_j^{(0)})}{\Gamma(\frac{s}{d}) \prod_{j=1}^p a_j^{z_j + \delta_j^{(0)}}} \times \\
&\quad \mathcal{M}\left(f; \left(\sum_{j=1}^p \delta_j^{(0)} \alpha_{j,1} + \sum_{j=1}^p z_j \alpha_{j,1}, \dots, \sum_{j=1}^p \delta_j^{(0)} \alpha_{j,n} + \sum_{j=1}^p z_j \alpha_{j,n}\right)\right) dz_1 \dots dz_p \\
&= \frac{1}{(2\pi i)^p} \int_{(\delta'_1)} \cdots \int_{(\delta'_p)} \frac{\Gamma(\frac{s}{d} - |\mathbf{z}| - |\boldsymbol{\delta}^{(0)}|) \prod_{j=1}^p \Gamma(z_j + \delta_j^{(0)})}{\Gamma(\frac{s}{d}) \prod_{j=1}^p a_j^{z_j + \delta_j^{(0)}}} \times \\
&\quad \mathcal{M}\left(f; \left(c_1 + \sum_{j=1}^p z_j \alpha_{j,1}, \dots, c_n + \sum_{j=1}^p z_j \alpha_{j,n}\right)\right) dz_1 \dots dz_p
\end{aligned}$$

the last equality follows from (V.2).

By using the function $\mathbf{s} \mapsto \mathcal{H}(f, \mathbf{c}; \mathbf{s}) = \prod_{i=1}^r (\beta_i \cdot \mathbf{s})^{\mathbf{u}(\beta_i)} \mathcal{M}(f; \mathbf{c} + \mathbf{s})$ and also the function T'_p defined in (V.4), we obtain that for $s \in \mathbb{C}$ such that $\text{Re}(s) \gg 1$, we have

$$\begin{aligned}
Z(f; 1 + P; s) &= \frac{1}{(2\pi i)^p} \int_{(\delta'_1)} \cdots \int_{(\delta'_p)} \frac{\Gamma(\frac{s}{d} - |\mathbf{z}| - |\boldsymbol{\delta}^{(0)}|) \prod_{j=1}^p \Gamma(z_j + \delta_j^{(0)})}{\Gamma(\frac{s}{d}) \prod_{j=1}^p a_j^{z_j + \delta_j^{(0)}}} \times \\
&\quad \frac{\mathcal{H}(f, \mathbf{c}; (\sum_{j=1}^p z_j \alpha_{j,1}, \dots, \sum_{j=1}^p z_j \alpha_{j,n}))}{\prod_{\beta \in I} (T'_p(\beta) \cdot \mathbf{z})^{\mathbf{u}(\beta)}} dz_1 \dots dz_p
\end{aligned}$$

Let $\boldsymbol{\delta} = (\delta_1, \dots, \delta_p) = \boldsymbol{\delta}^{(0)} + \boldsymbol{\delta}^{(1)} \in (\mathbb{R}_+^*)^p$ defined by (V.3). Using the relations (V.3) and (V.4) implies that for $\text{Re}(s) \gg 1$, we have

$$\begin{aligned}
Z(f; 1 + P; s) &= \frac{\prod_{j \in J} a_j}{(2\pi i)^p} \int_{(\delta'_1)} \cdots \int_{(\delta'_p)} \frac{\Gamma(\frac{s}{d} + |J| - |\mathbf{z}| - |\boldsymbol{\delta}|) \prod_{j=1}^p \Gamma(z_j + \delta_j)}{\Gamma(\frac{s}{d}) \prod_{j=1}^p a_j^{z_j + \delta_j}} \times \\
&\quad \frac{\mathcal{H}(f, \mathbf{c}; (\sum_{j=1}^p z_j \alpha_{j,1}, \dots, \sum_{j=1}^p z_j \alpha_{j,n}))}{\prod_{j \in J} z_j \prod_{\beta \in I} (T'_p(\beta) \cdot \mathbf{z})^{\mathbf{u}(\beta)}} dz_1 \dots dz_p \\
&= \frac{\Gamma(\frac{s}{d} + |J|) \prod_{j \in J} a_j}{(2\pi i)^p \Gamma(\frac{s}{d})} \int_{(\delta'_1)} \cdots \int_{(\delta'_p)} \frac{\Gamma(\frac{s}{d} + |J| - |\mathbf{z}| - |\boldsymbol{\delta}|) \prod_{j=1}^p \Gamma(z_j + \delta_j)}{\Gamma(\frac{s}{d} + |J|) \prod_{j=1}^p a_j^{z_j + \delta_j}} \times \\
&\quad \frac{\mathcal{H}(f, \mathbf{c}; (\sum_{j=1}^p z_j \alpha_{j,1}, \dots, \sum_{j=1}^p z_j \alpha_{j,n}))}{\prod_{\mathbf{v} \in I'} (\mathbf{v} \cdot \mathbf{z})^{\mathbf{u}'_P(\mathbf{v})}} dz_1 \dots dz_p.
\end{aligned}$$

It follows that for $\boldsymbol{\delta}' = (\delta'_1, \dots, \delta'_p) \in \mathbb{R}_+^{*p}$, we have for $\operatorname{Re}(s) \gg 1$:

$$Z(f; 1 + P; s - d|J|) = G(s) (F_1(s) + F_2(s)) \quad (\text{V.20})$$

where

$$G(s) = \frac{d\Gamma\left(\frac{s}{d}\right) \prod_{j \in J} a_j}{\Gamma\left(\frac{s}{d} - |J|\right)}, \quad (\text{V.21})$$

and for $i \in \{1, 2\}$,

$$F_i(s) = \frac{1}{(2\pi i)^p} \int_{(\delta'_1)} \dots \int_{(\delta'_p)} \frac{L_i(\mathcal{H}; s, \mathbf{z}) dz_1 \dots dz_p}{(s - d|\boldsymbol{\delta}| - \langle d\mathbf{1}, \mathbf{z} \rangle) \prod_{\mathbf{v} \in I'} (\mathbf{v} \cdot \mathbf{z})^{\mathbf{u}'_P(\mathbf{v})}}, \quad (\text{V.22})$$

with

$$L_1(\mathcal{H}; s, \mathbf{z}) = \frac{\Gamma\left(\frac{s}{d} - |\mathbf{z}| - |\boldsymbol{\delta}| + 1\right) \prod_{j=1}^p \Gamma(z_j + \delta_j)}{\Gamma\left(\frac{s}{d}\right) \prod_{j=1}^p a_j^{z_j + \delta_j}} \times \mathcal{H}(f, \mathbf{c}; \mathbf{0}) \quad (\text{V.23})$$

and

$$L_2(\mathcal{H}; s, \mathbf{z}) = \frac{\Gamma\left(\frac{s}{d} - |\mathbf{z}| - |\boldsymbol{\delta}| + 1\right) \prod_{j=1}^p \Gamma(z_j + \delta_j)}{\Gamma\left(\frac{s}{d}\right) \prod_{j=1}^p a_j^{z_j + \delta_j}} \times \left(\mathcal{H}(f, \mathbf{c}; \left(\sum_{j=1}^p z_j \alpha_{j,1}, \dots, \sum_{j=1}^p z_j \alpha_{j,n}\right)) - \mathcal{H}(f, \mathbf{c}; \mathbf{0}) \right) \quad (\text{V.24})$$

Since there exists $\varepsilon > 0$ such that $\mathbf{s} \mapsto \mathcal{H}(f, \mathbf{c}; \mathbf{s})$ has a holomorphic continuation with moderate growth to $\{\operatorname{Re}(s_i) > c_i - \varepsilon \forall i \in \llbracket 1, n \rrbracket\}$, we deduce from [12, Lemme 2] that there exist $\eta, \varepsilon > 0$ and $A, B > 0$ such that for $i \in \{1, 2\}$, the function $(s, \mathbf{z}) \mapsto L_i(\mathcal{H}; s, \mathbf{z})$ is holomorphic in the domain

$$\mathcal{D}(\eta, \varepsilon) := \{(s, \mathbf{z}) \in \mathbb{C} \times \mathbb{C}^p \mid \operatorname{Re}(s) > d|\boldsymbol{\delta}| - \eta \text{ and } |\operatorname{Re}(z_j)| < \varepsilon \forall j \in \llbracket 1, p \rrbracket\}$$

and verifies in it the uniform estimate

$$L_i(\mathcal{H}; s, \mathbf{z}) \ll_{\sigma, \eta, \varepsilon} (1 + |\tau|)^{2\frac{|\sigma|}{d} + A} \prod_{j=1}^p (1 + |y_j|)^{\frac{|\sigma|}{d} + B} e^{\frac{\pi}{2}(|\frac{\tau}{d}| - \sum_{j=1}^p |y_j| - |\frac{\tau}{d} - \sum_{j=1}^p y_j|)} \quad (\text{V.25})$$

where $\sigma = \operatorname{Re}(s)$, $\tau = \operatorname{Im}(s)$ and $y_j = \operatorname{Im}(z_j) \forall j \in \llbracket 1, p \rrbracket$.

It follows then from Essouabri's Lemma [12, lemme 3] that for $i \in \{1, 2\}$, there exist $\mu > 0$ such that the function $s \mapsto F_i(s)$ has a meromorphic continuation with moderate growth to the domain $\{\operatorname{Re}(s) > d|\boldsymbol{\delta}| - \mu\}$ with only one possible pole at $s = d|\boldsymbol{\delta}|$ of order at most

$$\rho_i = 1 + \sum_{\mathbf{v} \in I'} \mathbf{u}'_P(\mathbf{v}) - \operatorname{Ran}(I') - \varepsilon(L_i) = 1 + \sum_{\boldsymbol{\beta} \in I} \mathbf{u}(\boldsymbol{\beta}) + |J| - \operatorname{Ran}(I \cup E_J) - \varepsilon(L_i), \quad (\text{V.26})$$

Where

$$\varepsilon(L_i) = \begin{cases} 1 & , \text{ if } \mathbf{1} \in \operatorname{con}^*(I') \text{ and there exists two functions } H, U \text{ holomorphic in } \mathcal{D}(\eta; \varepsilon) \\ & \text{ such that } L_i(\mathcal{H}; s, \mathbf{z}) = U(s; \mathbf{z})H(s; (\mathbf{v} \cdot \mathbf{z})_{\mathbf{v} \in I'}) \text{ and } H(s; \mathbf{0}) \equiv 0 \\ 0 & , \text{ otherwise} \end{cases} \quad (\text{V.27})$$

Moreover, since $|\boldsymbol{\delta}| = \frac{|\mathbf{c}|}{d} + |J| > 0$ (see (V.3)), there exists $\varepsilon > 0$ such that the function

$$s \mapsto G(s) := \frac{d\Gamma\left(\frac{s}{d}\right) \prod_{j \in J} a_j}{\Gamma\left(\frac{s}{d} - |J|\right)}$$

is holomorphic in the domain $\{\operatorname{Re}(s) > d|\boldsymbol{\delta}| - \varepsilon\}$ and verifies

$$G(d|\boldsymbol{\delta}|) = \left(d\Gamma(|\boldsymbol{\delta}|) \prod_{j \in J} a_j \right) \frac{1}{\Gamma(t)} \Big|_{t=\frac{|\mathbf{c}|}{d}}$$

In particular,

$$G(d|\boldsymbol{\delta}|) = \frac{d\Gamma(|\boldsymbol{\delta}|) \prod_{j \in J} a_j}{\Gamma\left(\frac{|\mathbf{c}|}{d}\right)} > 0 \quad \text{if } \mathbf{c} \neq \mathbf{0}, \quad (\text{V.28})$$

and if $\mathbf{c} = \mathbf{0}$ we have $|J| = |\boldsymbol{\delta}| = p$,

$$G \text{ has a zero of order one at } s = d|\boldsymbol{\delta}| = dp \text{ and } G(s) \sim \Gamma(p) \prod_{j=1}^p a_j (s - d|\boldsymbol{\delta}|). \quad (\text{V.29})$$

Combining (V.20), (V.26), (V.28) and (V.29) implies that there exist $\mu > 0$ such that

$$s \mapsto Z(f; 1 + P; s - d|J|) = G(s) (F_1(s) + F_2(s))$$

has a meromorphic continuation with moderate growth to the domain $\{\operatorname{Re}(s) > d|\boldsymbol{\delta}| - \mu\}$ with only one possible pole at $s = d|\boldsymbol{\delta}|$ of order at most

$$\rho = 1 + \sum_{\mathbf{v} \in I'} \mathbf{u}'_P(\mathbf{v}) - \operatorname{Ran}(I') - \mathbf{1}_{\{\mathbf{0}\}}(\mathbf{c}) = 1 + \sum_{\boldsymbol{\beta} \in I} \mathbf{u}(\boldsymbol{\beta}) + |J| - \operatorname{Ran}(I \cup E_J) - \mathbf{1}_{\{\mathbf{0}\}}(\mathbf{c}). \quad (\text{V.30})$$

By using $|\boldsymbol{\delta}| = \frac{|\mathbf{c}|}{d} + |J|$, we deduce that there exist $\mu > 0$ such that $s \mapsto Z(f; 1 + P; s)$ has a meromorphic continuation with moderate growth to the domain $\{\operatorname{Re}(s) > |\mathbf{c}| - \mu\}$ with only one possible pole at $s = |\mathbf{c}|$ of order at most ρ given by (V.30). This ends the proof of the first part of Theorem 18.

Now we will assume in addition that $\mathbf{1} \in \operatorname{con}^*(I \cup E_J)$ and there exists K a holomorphic function in a tubular neighbor of $\mathbf{0}$ such that $\mathcal{H}(f, \mathbf{c}; \mathbf{s}) = K((\boldsymbol{\beta} \cdot \mathbf{s})_{\boldsymbol{\beta} \in I \cup E_J})$. Under these additional assumptions we will determine the principal part of $s \mapsto Z(f; 1 + P; s)$ near $s = |\mathbf{c}|$.

The second part of Theorem 18 follows from (V.20) and the following lemma:

Lemma 23. *Let ρ defined by (V.30). Then,*

1. $s \mapsto G(s)F_2(s)$ has at most a pole of order $\rho - 1$ at $s = d|\boldsymbol{\delta}|$;
2. $s \mapsto G(s)F_1(s)$ has at most a pole of order ρ at $s = d|\boldsymbol{\delta}|$ and

$$G(s)F_1(s) = \frac{T(d, \boldsymbol{\delta}, \mathbf{a}; \mathbf{c}) \mathcal{H}(f, \mathbf{c}; \mathbf{0}) d^{\rho-1} A_0(\tilde{I}; \mathbf{u}_P; \mathbf{a})}{\prod_{\mathbf{v} \in I'} (\mathbf{v} \cdot \boldsymbol{\delta})^{\mathbf{u}'_P(\mathbf{v})}} \frac{1}{(s - d|\boldsymbol{\delta}|)^\rho} + O((s - d|\boldsymbol{\delta}|)^{-(\rho-1)}) \quad \text{as } s \rightarrow d|\boldsymbol{\delta}|,$$

$$\text{where } T(d, \boldsymbol{\delta}, \mathbf{a}; \mathbf{c}) = \frac{d\Gamma(|\boldsymbol{\delta}|) \prod_{j \in J} a_j}{\Gamma\left(\frac{|\mathbf{c}|}{d}\right)} \text{ if } \mathbf{c} \neq \mathbf{0} \text{ and } T(d, \boldsymbol{\delta}, \mathbf{a}; \mathbf{c}) = \Gamma(p) \prod_{j=1}^p a_j \text{ if } \mathbf{c} = \mathbf{0}$$

and where $A_0(\tilde{I}; \mathbf{u}_P; \mathbf{a})$ is the volume constant, see Chapter II section 1.4, associated to triplet $(\tilde{I}, \mathbf{u}_P, \mathbf{a})$. (the notations $I', \tilde{I}, \mathbf{u}_P, \mathbf{u}'_P$ were defined in section §1).

Proof of Lemma 23:

• Proof of point 1 of Lemma 23:

Since we assume in addition that $\mathbf{1} \in \text{con}^*(I \cup E_J) = \text{con}^*(I')$ and there exists K a holomorphic function in a tubular neighbor of $\mathbf{0}$ such that $\mathcal{H}(f, \mathbf{c}; \mathbf{s}) = K((\boldsymbol{\beta} \cdot \mathbf{s})_{\boldsymbol{\beta} \in I \cup E_J})$, it follows from (V.27) that $s \mapsto F_2(s)$ has a meromorphic continuation with moderate growth to the domain $\{\text{Re}(s) > d|\boldsymbol{\delta}| - \mu\}$ with only one possible pole at $s = d|\boldsymbol{\delta}|$ of order at most

$$\rho_2 = \sum_{\mathbf{v} \in I'} \mathbf{u}'_P(\mathbf{v}) - \text{Ran}(I') = \sum_{\boldsymbol{\beta} \in I} \mathbf{u}(\boldsymbol{\beta}) + |J| - \text{Ran}(I \cup E_J).$$

Point 1 of Lemma 23 follows then from the combination of this fact with (V.28) and (V.29).

• Proof of point 2 of Lemma 23:

The identities (V.22) and (V.23) imply that for $\text{Re}(s) \gg 1$, we have

$$\begin{aligned} F_1(s) &= \frac{\mathcal{H}(f, \mathbf{c}; \mathbf{0})}{(2\pi i)^p d} \int_{(\delta'_1)} \dots \int_{(\delta'_p)} \frac{\Gamma(\frac{s}{d} - |\mathbf{z}| - |\boldsymbol{\delta}|) \prod_{j=1}^p \Gamma(z_j + \delta_j)}{\Gamma(\frac{s}{d}) \prod_{j=1}^p a_j^{z_j + \delta_j} \prod_{\mathbf{v} \in I'} (\mathbf{v} \cdot \mathbf{z})^{\mathbf{u}'_P(\mathbf{v})}} dz_1 \dots dz_p \\ &= \frac{\mathcal{H}(f, \mathbf{c}; \mathbf{0})}{(2\pi i)^p d \prod_{\mathbf{v} \in I'} (\mathbf{v} \cdot \boldsymbol{\delta})^{\mathbf{u}'_P(\mathbf{v})}} \int_{(\delta'_1)} \dots \int_{(\delta'_p)} \frac{\Gamma(\frac{s}{d} - |\mathbf{z}| - |\boldsymbol{\delta}|) \prod_{j=1}^p \Gamma(z_j + \delta_j)}{\Gamma(\frac{s}{d}) \prod_{j=1}^p a_j^{z_j + \delta_j} \prod_{\mathbf{w} \in \tilde{I}} (\mathbf{w} \cdot \mathbf{z})^{\mathbf{u}_P(\mathbf{w})}} dz_1 \dots dz_p, \end{aligned}$$

where \tilde{I} and \mathbf{u}_P are defined in (V.6).

Since $\mathbf{1} \in \text{con}(\tilde{I}) = \text{con}(I')$ and $\mathbf{w} \cdot \boldsymbol{\delta} = 1 \forall \mathbf{w} \in \tilde{I}$, we deduce from Essouabri's Lemma [12, Lemma 4] that there exists $\eta > 0$ such that $s \mapsto F_1(s)$ has a meromorphic continuation to $\{\text{Re}(s) > d|\boldsymbol{\delta}| - \eta\}$ with at most one pole at $s = d|\boldsymbol{\delta}|$ of order

$$\begin{aligned} \rho' &:= 1 + \sum_{\mathbf{w} \in \tilde{I}} \mathbf{u}_P(\mathbf{w}) - \text{Ran}(\tilde{I}) \\ &= 1 + \sum_{\mathbf{v} \in I'} \mathbf{u}'_P(\mathbf{v}) - \text{Ran}(I') = 1 + \sum_{\boldsymbol{\beta} \in I} \mathbf{u}(\boldsymbol{\beta}) + |J| - \text{Ran}(I \cup E_J). \end{aligned}$$

Lemma [12, Lemma 4] also implies that

$$F_1(s) = \frac{\mathcal{H}(f, \mathbf{c}; \mathbf{0}) d^{\rho'-1} A_0(\tilde{I}; \mathbf{u}_P; \mathbf{a})}{\prod_{\mathbf{v} \in I'} (\mathbf{v} \cdot \boldsymbol{\delta})^{\mathbf{u}'_P(\mathbf{v})} (s - d|\boldsymbol{\delta}|)^{\rho'}} + O\left((s - d|\boldsymbol{\delta}|)^{-(\rho'-1)}\right) \quad \text{as } s \rightarrow d|\boldsymbol{\delta}|.$$

where $A_0(\tilde{I}; \mathbf{u}_P; \mathbf{a})$ is the volume constant, see chapter II section 1.4, associated to triplet $(\tilde{I}, \mathbf{u}_P, \mathbf{a})$.

We conclude by using in addition (V.28) and (V.29). This ends the proof of Lemma 23. \square

The identity (V.20) and Lemma 23 imply that

$$Z(f; 1+P; s-d|J|) = \frac{T(d, \boldsymbol{\delta}, \mathbf{a}; \mathbf{c}) \mathcal{H}(f, \mathbf{c}; \mathbf{0}) d^{\rho-1} A_0(\tilde{I}; \mathbf{u}_P; \mathbf{a})}{\prod_{\mathbf{v} \in I'} (\mathbf{v} \cdot \boldsymbol{\delta})^{\mathbf{u}'_P(\mathbf{v})} (s - d|\boldsymbol{\delta}|)^\rho} + O\left((s - d|\boldsymbol{\delta}|)^{-(\rho-1)}\right) \quad \text{as } s \rightarrow d|\boldsymbol{\delta}|,$$

By using in addition the equality $|\boldsymbol{\delta}| = \frac{|\mathbf{c}|}{d} + |J|$ (see (V.3)), we deduce that

$$Z(f; 1+P; s) = \frac{T(d, \boldsymbol{\delta}, \mathbf{a}; \mathbf{c}) \mathcal{H}(f, \mathbf{c}; \mathbf{0}) d^{\rho-1} A_0(\tilde{I}; \mathbf{u}_P; \mathbf{a})}{\prod_{\mathbf{v} \in I'} (\mathbf{v} \cdot \boldsymbol{\delta})^{\mathbf{u}'_P(\mathbf{v})} (s - |\mathbf{c}|)^\rho} + O\left((s - |\mathbf{c}|)^{-(\rho-1)}\right) \quad \text{as } s \rightarrow |\mathbf{c}|.$$

It follows then from (V.19) and (V.10) that

$$Z(f; P; s) = \frac{T(d, \boldsymbol{\delta}, \mathbf{a}; \mathbf{c}) \mathcal{H}(f, \mathbf{c}; \mathbf{0}) d^{\rho-1} A_0(\tilde{I}; \mathbf{u}_P; \mathbf{a})}{\prod_{\boldsymbol{\beta} \in I} (\boldsymbol{\beta} \cdot \mathbf{c} + \sum_{j \in J} \boldsymbol{\beta} \cdot \boldsymbol{\alpha}_j)^{\mathbf{u}(\boldsymbol{\beta})} (s - |\mathbf{c}|)^\rho} + O\left((s - |\mathbf{c}|)^{-(\rho-1)}\right) \quad \text{as } s \rightarrow |\mathbf{c}|.$$

This ends the proof of Theorem 18. □

Proof of Theorem 19. .

Theorem 15 of Chapter IV implies that f verifies the assumptions of Theorem 18 with $I = I(\kappa, g) = \{\boldsymbol{\nu} \in \mathbb{N}_0 \setminus \{\mathbf{0}\} \mid \kappa(\boldsymbol{\nu}) = \mathbf{1} \text{ and } \mathbf{g}(\boldsymbol{\nu}) \neq \mathbf{0}\}$ and $\mathbf{u} : I \rightarrow \mathbb{N}$ defined by $\mathbf{u}(\boldsymbol{\nu}) = g(\boldsymbol{\nu})$ for all $\boldsymbol{\nu} \in I$.

So to conclude, it suffices to use Theorem 18 and the classical Landau tauberian theorem (see Theorem 21 in the Appendix) if $\mathbf{c} \neq \mathbf{0}$ or the Tauberian Lemma 22 if $\mathbf{c} = \mathbf{0}$. □

Proof of Corollaries 10 and 11. .

Proof of Corollary 10 (resp. Corollary 11) is similar to the proof of Corollary 5 (resp. Corollary 6) by using Theorem 19 instead of Theorem 16. □

Chapter VI

Appendix

We gather in this appendix some classic results that we have used in this thesis.

Theorem 20 (Delange's tauberian theorem). *Let $A : (0, +\infty) \rightarrow [0, +\infty[$ a non-decreasing function such that $\int_0^{+\infty} e^{-\sigma y} A(y) dy < +\infty$ for $\sigma > \sigma_0 \geq 0$. We suppose that the function $s \mapsto \int_0^\infty e^{-sy} A(y) dy - \sum_{k=1}^{\rho} \frac{a_k}{(s - \sigma_0)^k}$ has a continuous extension to the closed half-plane $\{\operatorname{Re}(s) \geq \sigma_0\}$ with $a_\rho \neq 0$, then*

$$\lim_{y \rightarrow \infty} \frac{A(y)}{y^{\rho-1}} e^{-y\sigma_0} = \frac{a_\rho}{(\rho-1)!} \quad (\text{VI.1})$$

Corollary 12. *Let $D(s) = \sum_{n \geq 1} a_n \lambda_n^{-s}$ be a Dirichlet series with non-negative coefficients. Let $\sigma_0 > 0$. Assume that $D(s)$ converges in the open half-plane $\{\sigma = \operatorname{Re}(s) > \sigma_0\}$ and that the function $s \mapsto D(s) - \sum_{k=1}^{\rho} \frac{c_k}{(s - \sigma_0)^k}$ has a continuous extension to the closed half-plane $\{\sigma = \operatorname{Re}(s) \geq \sigma_0\}$ with $c_\rho \neq 0$, then*

$$\sum_{\lambda_k \leq y} a_k \sim \frac{c_\rho}{\sigma_0 (\rho-1)!} y^{\sigma_0} (\log y)^{\rho-1} \quad \text{as } y \rightarrow +\infty. \quad (\text{VI.2})$$

Theorem 21 (Landau's tauberian theorem). *Let $D(s) = \sum_{n \geq 1} \frac{a_n}{\lambda_n^s}$ be a Dirichlet series with non-negative coefficients. Let $\sigma_0 > 0$. We suppose that $s \mapsto D(s)$ is absolutely convergent in $\{\operatorname{Re}(s) > \sigma_0\}$ and that it has a meromorphic continuation with moderate growth to a half-plane $\{\operatorname{Re}(s) > \sigma_0 - \eta\}$ with $\eta > 0$. We also assume that D has a single pole in this half-plane at $s = \sigma_0$ and $D(s) \sim_{s \rightarrow \sigma_0} \frac{c_\rho}{(s - \sigma_0)^\rho}$ with $c_\rho > 0$ and $\rho \in \mathbb{N}$. Then, there exists a polynomial of degree $\rho - 1$ and $\delta > 0$ such that*

$$A(x) = \sum_{\lambda_n \leq x} a_n = x^{\sigma_0} Q(\ln x) + O(x^{\sigma_0 - \delta}) \quad \text{as } x \rightarrow +\infty. \quad (\text{VI.3})$$

Moreover,

$$A(x) = C x^{\sigma_0} (\ln x)^{\rho-1} + O((\ln x)^{\rho-2}) \quad \text{as } x \rightarrow +\infty,$$

where $C = \frac{c_\rho}{\sigma_0 (\rho-1)!}$.

Lemma 24 (Euler-Maclaurin formula). *Let $p, q, r \in \mathbb{N}$ with $p < q$ and $f \in C^{r+1}([p, q], \mathbb{C})$. Then*

$$\begin{aligned} \sum_{m=p+1}^q f(m) &= \int_p^q f(x) dx + \sum_{j=0}^r \frac{(-1)^{j+1} B_{j+1}}{(j+1)!} \{f^{(j)}(q) - f^{(j)}(p)\} \\ &\quad + \frac{(-1)^r}{(r+1)!} \int_p^q f^{(r+1)}(x) B_{r+1}(\{x\}) dx \end{aligned} \quad (\text{VI.4})$$

$$\begin{aligned} \sum_{m=p}^q f(m) &= \int_p^q f(x) dx + \sum_{j=1}^k \frac{B_{2j}}{(2j)!} \{f^{(2j-1)}(q) - f^{(2j-1)}(p)\} \\ &\quad + \frac{1}{2} [f(q) + f(p)] + \frac{-1}{(2k)!} \int_p^q f^{(2k)}(x) B_{2k}(\{x\}) dx \end{aligned} \quad (\text{VI.5})$$

$$\begin{aligned} \sum_{m=p}^{q-1} f(m) &= \int_p^q f(x) dx + \sum_{j=0}^r \frac{B_{j+1}}{(j+1)!} \{f^{(j)}(q) - f^{(j)}(p)\} \\ &\quad + \frac{(-1)^r}{(r+1)!} \int_p^q f^{(r+1)}(x) B_{r+1}(\{x\}) dx, \end{aligned} \quad (\text{VI.6})$$

where the $B_j(x)$ are the Bernoulli polynomials and the $B_j := B_j(0)$ are the Bernoulli numbers.

Lemma 25 (Taylor formula with integral remainder). *Let $r \in \mathbb{N}$, I an open interval containing 0 and $g \in C^{r+1}(I)$. Then for $t \in I$ we have*

$$g(t) = g(0) + \sum_{k=1}^r \frac{g^{(k)}(0)}{k!} t^k + \frac{t^{r+1}}{r!} \int_0^1 (1-u)^r g^{(r+1)}(tu) du \quad (\text{VI.7})$$

Lemma 26 ([13], Lem. 1). *Let $N \in \mathbb{N}$, $L \in \mathbb{N}_0$. We define the function $R_{N,L} : \mathbb{C}^N \times]-1, +\infty[\rightarrow \mathbb{C}$ by*

$$R_{N,L}(\mathbf{s}, \mathbf{x}) := \prod_{i=1}^N (1+x_i)^{-s_i} - \sum_{\substack{\mathbf{k} \in \mathbb{N}^N \\ |\mathbf{k}| \leq L}} \binom{-\mathbf{s}}{\mathbf{k}} \mathbf{x}^{\mathbf{k}},$$

Then we have the identity

$$R_{N,L}(\mathbf{s}, \mathbf{x}) = (L+1) \sum_{\substack{\mathbf{k} \in \mathbb{N}_0^N \\ |\mathbf{k}|=L+1}} \binom{-\mathbf{s}}{\mathbf{k}} \mathbf{x}^{\mathbf{k}} \int_0^1 (1-t)^L \prod_{i=1}^N (1+tx_i)^{-s_i-k_i} dt \quad (\text{VI.8})$$

Furthermore, for each $\delta, \gamma \in \mathbb{R}$ such that $-1 < \delta \leq \gamma$, we have $|R_{N,L}(\mathbf{s}, \mathbf{x})| \ll_{\delta, \gamma, L, N, \sigma} (1+|\boldsymbol{\tau}|_1)^{L+1} |\mathbf{x}|_1^{L+1}$ uniformly in $\mathbf{x} \in [\delta, \gamma]^N$ and $\boldsymbol{\tau} \in \mathbb{R}^N$.

Lemma 27 (Mellin-Barnes integral formula). *Let $\mathbf{w} = (w_1, \dots, w_p) \in \mathbb{C}^p$ such that $\text{Re}(w_j) > 0 \forall j \in \llbracket 0, p \rrbracket$, and let $\boldsymbol{\rho} = (\rho_1, \dots, \rho_p) \in (\mathbb{R}_+^*)^n$. Then for $s \in \mathbb{C}$ such that $\text{Re}(s) > |\boldsymbol{\rho}| = \rho_1 + \dots + \rho_p$ (i.e. $\text{Re}(s) > |\boldsymbol{\rho}|$), we have*

$$\frac{\Gamma(s)}{\left(\sum_{j=0}^p w_j\right)^s} = \frac{1}{(2i\pi)^p} \int_{\rho_1-i\infty}^{\rho_1+i\infty} \dots \int_{\rho_p-i\infty}^{\rho_p+i\infty} \frac{\Gamma(s-|\mathbf{z}|) \prod_{j=1}^p \Gamma(z_j) dz}{w_0^{s-|\mathbf{z}|} \left(\prod_{j=1}^p w_j^{z_j}\right)} \quad (\text{VI.9})$$

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FONCTION ZÊTAS MULTIVARIABLES, THÉORÈMES TAUBERIENS MULTIVARIABLES ET APPLICATIONS

Résumé: Cette thèse porte deux axes de recherche:

Le premier vise la détermination de valeurs spéciales d'une famille de séries de Dirichlet multivariable partiellement tordues. Pour réaliser cet étude on donne un prolongement méromorphe explicite à l'aide de deux techniques: formule de sommation d'Euler-Maclaurin et la représentation intégrale. Ensuite nous montrons la régularité sur les points de coordonnées entières non positives, pour enfin utiliser un résultat du à Crisenoy lié à la détermination des valeurs spéciales des séries totalement tordues.

Le deuxième axe de recherche est lié au problème de moyennes de fonctions arithmétiques multivariable et les théorèmes taubériens multivariable. Dans un premier temps nous obtenons une généralisation du théorème taubérien multivariable de Delange, pour ce fait on utilise un mélange de deux méthodes: un venant de l'analyse complexe et l'autre de la convolution.

Ensuite on aborde l'étude d'une famille de fonctions arithmétiques multiplicatives multivariable. On établit certaines propriétés analytiques liées aux séries de Dirichlet multivariable, comme conséquence nous obtenons un théorème qui joint au théorème taubérien multivariable soit celui de La Bretèche soit celui d'Essouabri nous permet de donner une réponse affirmative à quelques conjectures liées aux moyennes de nombre de sous-groupes cycliques de $\mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_n}$ et aux moyennes de certaines fonctions liées au PPCM.

Mots-clés. Valeurs spéciales, fonction zêta tordue, représentation intégrale, valeurs moyennes, moyennes multi-variables de PPCM, moyennes de sous-groupes, théorème taubérien multivariable, fonction multiplicative.

MULTIVARIABLE ZETA FUNCTIONS, MULTIVARIABLE TAUBERIAN THEOREMS AND APPLICATIONS

Abstrac: This thesis is concerned with two lines of research:

The first, is the determination of special values of a partially twisted multivariable Dirichlet series family. To carry out this study we give an explicit meromorphic extension using two techniques: Euler-Maclaurin summation formula and integral representation. Then, we show the regularity on the non-positive integer coordinate points, to finally use a Crisenoy's result related to the determination of the special values of the totally twisted series.

The second, is related to the problem of means of multivariable arithmetic functions and multivariable tauberian theorems. We obtain a generalization of Delange's multivariable tauberian theorem, for this we use a mixture of two methods: one coming from complex analysis and the other from convolution.

Then we approached the study of a family of multivariable multiplicative arithmetic functions. We obtain some analytical properties related to the multivariable Dirichlet series, as a consequence we obtain a theorem which joins to the La Bretèche's or Essouabri's multivariable tauberian theorem allows us to give an affirmative answer to some conjectures related to the means of number of cyclic subgroups of $\mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_n}$ and averages of some functions related to LCM.

Keywords. Special values, twisted zeta functions, integral representation, mean values, LCM multivariable averages, subgroups averages, multivariable tauberian theorems, multiplicative function.