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# A framework for locally structured spaces - application to geometric models of concurrency

Pierre-Yves Coursole

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Thèse de doctorat



# A framework for locally structured spaces - Application to geometric models of concurrency

Thèse de doctorat de l'Institut Polytechnique de Paris  
préparée à École polytechnique

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**PIERRE-YVES COURSOLE**

Composition du Jury :

Jean GOUBAULT-LARRECQ Full professor, ENS Paris Saclay (Laboratoire Spécification et Vérification) & CNRS	Président
Maria Manuel CLEMENTINO Professeure, Universidade de Coimbra (Departamento de Matemática)	Rapporteur
Robert VAN GLABBEEK Directeur de recherche, CISRO (Commonwealth Scientific and Industrial Research Organisation)	Rapporteur
Dirk HOFMANN Associate Professor, University of Aveiro (Department of Mathematics)	Examineur
Martin RAUSSEN Full professor, Aalborg University (Department of Mathematical Sciences)	Examineur
Emmanuel HAUCOURT Professeur, École polytechnique (Laboratoire d'Informatique de l'École polytechnique)	Directeur de thèse
Thomas SEILLER Chargé de recherche, CNRS (Laboratoire d'Informatique de Paris Nord)	Co-directeur de thèse



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# Chapter 1

## Résumé

L'usage de méthodes provenant de la Topologie Algébrique dans l'étude de la théorie de la concurrence a été explicitement initié dans [FGR06]. L'un des ingrédients clés est la *réalisation des ensembles précubiques* dans la catégorie des espaces localement ordonnés (définition 4.1.6). Toutefois, cette dernière catégorie n'est pas cocomplète, voir chapitre 4. En outre, le concept d'espace localement ordonné n'est pas totalement fixé : plusieurs définitions *non équivalentes* se retrouvent en effet dans la littérature [Law89, FGR06, BW06, Kah09]. C'est un inconvénient sérieux car les colimites d'espaces localement ordonnés sont extrêmement sensibles à des changements apparemment anodins dans les définitions, voir chapitre 4. Deux autres cadres alternatifs ayant de bien meilleures propriétés catégoriques ont été introduits : les *d-espaces* [Gra09, 1.4.0] et les *streams* [Kri09]. Comme inconvénient, tous deux autorisent des pathologies comme les vortex. On se retrouve avec une pléthore de notions similaires mais non équivalentes, toutes basées sur la topologie, cherchant à formaliser la même idée. On cherche à définir un cadre unifié pour pouvoir faire des comparaisons entre elles.

Puisque tous les modèles auxquels on s'intéresse sont basés sur la topologie, une idée naturelle est de généraliser cette dernière. Au vu des applications mentionnées dans le préambule, on aurait besoin d'étendre la notion de topologie sur un ensemble  $X$  de façon à ce que l'ensemble des parties  $\mathcal{P}(X)$  soit remplacé par un simple ensemble préordonné  $T(X)$  avec une relation supplémentaire  $\in_T^X$  remplaçant la relation d'appartenance. Typiquement, les membres de  $T(X)$  sont des sous-ensembles de  $X$  équipés d'une structure additionnelle (par exemple des ordres ou des préordres). Le préordre sur  $T(X)$  est alors donné par l'inclusion des ensembles sous-jacents et par des conditions de préservation des structures par cette inclusion. Par exemple, on peut demander à ce que les inclusions soient des plongements ou juste des morphismes : ce choix en apparence anodin a des conséquences majeures sur les catégories des 'espaces généralisés' qui en résultent, voir chapitre 4.

Le cadre que l'on a développé jusque-là permet de parler des 'sous-ensembles' ; on doit encore distinguer ceux qui sont ouverts. Contrairement à la théorie des locales, nous devons exprimer notre notion de 'topologie généralisée' sans utiliser les unions ni les intersections finies, puisqu'elles ne sont pas supposées exister dans  $T(X)$ , et même si elles existent, la distributivité est fautive en générale. Cela nous laisse avec plusieurs approches possibles en fonction de la caractérisation de la topologie que l'on choisit, parmi ces caractérisations, on retrouve

- les bases ([Kel55, Theo 11 p.47], [Eng89, 1.2.1 p.21])
- les voisinages ([Bro06, Sect 2.1, p.20])

On définit nos 'espaces généralisés' grâce aux bases et on prouve que la notion qui en résulte correspond à celle que l'on aurait obtenue avec une approche basée sur les voisinages (5.2.10). Une base de topologie classique sur  $X$  est un sous-ensemble  $\mathcal{B} \subset \mathcal{P}(X)$  vérifiant

pour toute famille finie  $(B_j)$  de  $\mathcal{B}$ , pour tout  $x \in X$  tel que  $x \in B_j$  pour tout  $j$ ,

il existe  $B \in \mathcal{B}$  tel que  $x \in B$  et  $B \subset B_j$  pour tout  $j$ .

De manière analogue, dans notre cadre, une base de  $T$ -topologie est un sous-ensemble  $\mathcal{B} \subset T(X)$  vérifiant

pour toute famille finie  $(B_j)$  de  $\mathcal{B}$ , pour tout  $x \in X$  tel que  $x \in_T^X B_j$  pour tout  $j$ ,

il existe  $B \in \mathcal{B}$  tel que  $x \in_T^X B$  et  $B \subset_T^X B_j$  pour tout  $j$ .

De là, on définit les  $T$ -ouverts associés à une base et on dit que deux bases sont équivalentes si elles induisent les mêmes  $T$ -ouverts. Pour que ces concepts aient de bonnes propriétés, on suppose que :

- la relation  $\subset_T^X$  est un préordre sur  $T(X)$ , et
- la relation  $\in_T^X$  est un module (voir 3.1.1) de  $(X, =)$  vers  $(T(X), \subset_T^X)$ , autrement dit

$$(x \in_T^X A \subset_T^X B) \Rightarrow x \in_T^X B.$$

L'étape suivante est de définir les applications  $f : X \rightarrow Y$   $T$ -continues en  $x$ . Pour adapter la définition standard de la continuité exprimée en termes de bases de topologie  $\mathcal{B}$  et  $\mathcal{B}'$  sur  $X$  et  $Y$ , c'est-à-dire

$$\forall B' \in \mathcal{B}' \text{ tel que } f(x) \in B', \exists B \in \mathcal{B} \text{ tel que } x \in B \text{ et } f(B) \subset B',$$

on a besoin d'une relation  $T(f)$  de  $T(X)$  vers  $T(Y)$  jouant le rôle de l'énoncé " $f(B) \subset B'$ ". La relation  $T(f)$  est faite pour remplir ce manque : la fonction  $f$  est  $T$ -continue en  $x$ , avec  $\mathcal{B}$  et  $\mathcal{B}'$  des bases de  $T$ -topologies sur  $X$  et  $Y$ , quand

$$\forall B' \in \mathcal{B}' \text{ tel que } f(x) \in_T^Y B', \exists B \in \mathcal{B} \text{ tel que } x \in_T^X B \text{ et } B T(f) B'.$$

Pour s'assurer que la définition ci-dessus ne soit pas dépendante du choix des bases, il suffit de supposer que

- la relation  $T(f)$  soit un module (voir 3.1.1), autrement dit

$$(A \subset_T^X A', A' T(f) B', B' \subset_T^Y B) \Rightarrow A T(f) B,$$

- et que les modules  $\in_T^X$  et  $\in_T^Y$  satisfassent la condition de naturalité lax (diagramme 5.2) :

$$x \in_T^X A \text{ et } A T(f) B \Rightarrow f(x) \in_T^Y B.$$

Par exemple, soient  $T(X)$  et  $T(Y)$  les collections des ensembles ordonnés dont les ensembles sous-jacents sont inclus dans  $X$  et  $Y$  respectivement. Alors, pour n'importe quels membres  $A$  et  $B$  de  $T(X)$  et de  $T(Y)$ , on pose  $A T(f) B$  quand  $f(A) \subset B$  et la restriction de  $f$  à  $A$  est une application croissante de  $A$  vers  $B$ .

Bien sûr, on voudrait que les espaces  $T$ -topologiques et les applications  $T$ -continues forment une catégorie concrète. C'est le cas lorsque  $T$  est un foncteur lax de la catégorie des ensembles vers celle des modules (définition 5.1.1). On appelle *théorie topologique* une paire  $T = (T, \in_T)$  qui satisfait les hypothèses précédentes, la catégorie correspondante est notée  $\mathbf{Top}_T$ .

La construction décrite jusqu'ici considère la théorie  $T$  comme un paramètre fixe. Étant donné une autre théorie  $T'$ , on appelle *transformation sémantique* (définition 5.3.1) n'importe quel foncteur concret de  $\mathbf{Top}_T$  vers  $\mathbf{Top}_{T'}$ . Un tel foncteur peut oublier trop d'informations. Pour cette raison, on introduit la notion de *données de changement de bases* (définition 5.3.3) depuis lesquelles on déduit des transformations sémantiques qui ont de bonnes propriétés appelées *foncteurs de changement de bases* (définition 5.3.14). La théorie topologique

$$X : \mathbf{Set} \mapsto ((\mathcal{P}(X), \subset), \in),$$

que l'on note  $(\mathcal{P}, \in)$ , est l'objet final de la catégorie des théories topologiques avec les données de changement de bases comme morphismes. Les espaces  $(\mathcal{P}, \in)$ -topologiques correspondant sont les espaces topologiques usuels. Par conséquent, il existe un foncteur de changement de bases

canonique de  $\mathbf{Top}_T$  dans  $\mathbf{Top}$  induit par l'unique donnée de changement de bases  $T \rightarrow (\mathcal{P}, \in)$ . La topologie sous-jacente d'un espace  $T$ -topologique est son image par ce foncteur canonique. Beaucoup de notions classiques de topologie peuvent être relevées le long de ce foncteur.

À présent, décrivons rapidement le contenu de ce manuscrit :

Dans le troisième chapitre, on rappelle certains concepts de théorie des ordres, principalement pour fixer la terminologie utilisée qui diverge sur certains points de celle standard. On introduit aussi certains concepts moins usuels comme les modules (au sens de la théorie des ordres). De nombreux exemples intéressants de théories topologiques peuvent être formulés en termes de quantales, bien que ceux-ci ne soient pas impliqués dans le développement de la théorie ; pour cette raison on leur dédie un chapitre en annexe.

Le quatrième chapitre, qui est principalement une reproduction de [CH21], se concentre sur les espaces localement ordonnés, qui sont historiquement les premiers exemples de modèles de la concurrence basés sur la topologie. Il commence avec une introduction détaillée et contient une étude approfondie de leurs colimites puisque celles-ci jouent un rôle crucial dans les applications à la théorie de la concurrence.

Le cinquième chapitre est dédié aux notions élémentaires associées à la  $T$ -topologie. On formalise les concepts esquissés précédemment. On décrit aussi la construction  $\mathbf{Loc}$ , qui associe fonctoriellement à chaque théorie topologique  $T$  une nouvelle théorie  $\mathbf{Loc}(T)$ . On voit que les espaces  $\mathbf{Loc}(T)$ -topologiques admettent une description plus concrète et possèdent de meilleures propriétés que les espaces  $T$ -topologiques généraux. De plus, la construction  $\mathbf{Loc}$  fournit de nombreux exemples utiles comme les espaces localement (pré)ordonnés.

Dans le sixième chapitre, on adapte naturellement des notions classiques de topologie, comme la convergence, la quasi-compacité et divers axiomes de séparation, aux espaces  $T$ -topologiques. Certaines de ces généralisations se résument simplement au fait que l'espace topologique sous-jacent satisfasse la notion standard.

Dans le septième chapitre, on étudie les (co)limites dans  $\mathbf{Top}_T$  via les relèvements initiaux et finaux le long du foncteur d'oubli  $\mathbf{U}_T : \mathbf{Top}_T \rightarrow \mathbf{Set}$ . Pour cela, on généralise les concepts de topologies finales (7.2), initiales (7.4) et induites (7.5) aux espaces  $T$ -topologiques. Ce faisant, des difficultés apparaissent du fait que  $T$  est seulement un foncteur *lax* ; elles sont étudiées au début du chapitre (7.1). On se concentre ensuite sur le cas de  $\mathbf{Loc}(T)$  (7.6) : si  $\mathbf{U}_T$  est une fibration alors  $\mathbf{Loc}(T)$  satisfait les hypothèses requises pour l'utilisation de la  $\mathbf{Loc}(T)$ -topologie induite, à partir de laquelle on prouve que  $\mathbf{U}_{\mathbf{Loc}(T)}$  est aussi une fibration. Avec une approche similaire, on prouve que  $\mathbf{U}_{\mathbf{Loc}(T)}$  est topologique quand  $\mathbf{U}_T$  l'est.

Dans le huitième chapitre, on se concentre sur les espaces  $T$ -topologiques  $X$  dans lesquels l'intersection de n'importe quelle famille de  $T$ -ouverts de  $X$  qui sont deux-à-deux  $\simeq_T$ -équivalents est encore un  $T$ -ouvert de  $X$  (on écrit  $A \simeq_T B$  quand  $x \in_T A \Leftrightarrow x \in_T B$  pour tout  $x \in X$ ). De tels espaces  $T$ -topologiques sont appelés des  $T$ -streams à cause de leur relation avec les streams de Krishnan [Kri09].

Dans le neuvième chapitre, suivant [GL13, 5.2-4], on étudie comment une structure monoïdale sur  $\mathbf{Top}_T$  (telle que le foncteur d'oubli dans  $\mathbf{Set}$  soit un foncteur monoïdal strict) peut être étendue à la catégorie  $\mathbf{Top}_{\mathbf{Loc}(T)}$ . Pour cette occasion, les  $T$ -streams jouent un rôle important : quand  $\mathbf{Top}_T$  est une catégorie monoïdale close, les  $\mathbf{Loc}(T)$ -streams core-compact sont exponentiables. En adaptant [GL13, 5.6], on montre que, sous certaines hypothèses sur une classe d'objets donnée  $\mathcal{C}$ , la catégorie des objets  $\mathcal{C}$ -engendrés est cartésienne close.





# Chapter 2

## Introduction

The usage of methods from Algebraic Topology in the study of Concurrency Theory was explicitly initiated in [FGR06]. One of its key ingredients is the *realization* of *precubical sets*<sup>1</sup> in the category of locally ordered spaces (Definition 4.1.6). Nevertheless, these latter do not form a co-complete category, see Chapter 4. Moreover, the concept of a locally ordered space is not firmly set: various *nonequivalent* definitions have indeed appeared in the literature [Law89, FGR06, BW06, Kah09]. This is a serious drawback because the colimits of locally ordered spaces are extremely sensitive to seemingly anodyne modification in their definition, see Chapter 4. In the meantime, two alternative frameworks enjoying much better categorical properties were introduced: the *d-spaces* [Gra09, 1.4.0] and the *streams* [Kri09]. As a drawback, both of them allow pathologies like vortex. Both are related by an adjunction whose center [PT91] was concretely described in [Hau12]. We end up with plethora of similar (yet non-equivalent) mathematical notions, all grounded on topology, intended to formalize the same idea. We aim at designing a unified framework to compare them. We have explained the goal, let us discuss the mean.

Since all the models we would like to deal with are based on topology theory, a natural idea is to generalize the latter. From here, pointless topology is certainly the first option to consider. Pointless topology extends classical topology by allowing any Heyting algebra to play the role of the lattice of open sets, thus focusing on the distributivity of the meet operator [Bor94c, 1] [Joh82, 2] [PT+03, 2]. In particular, pointless topology does not let us deal with ‘non-open subsets’. In view of the applications mentioned in the preamble, we would need to extend topology on a set  $X$  in a way that the powerset  $\mathcal{P}(X)$  is replaced by a mere preordered set  $T(X)$  together with an additional relation  $\in_T^X$  standing for the membership relation. Typically, the members of  $T(X)$  are subsets of  $X$  endowed with an additional structure (e.g. preorders or orders). The preorder on  $T(X)$  is then given by underlying set inclusion satisfying extra requirements about structure preservation. For example, one can require inclusions to induce embeddings or just morphisms: this seemingly anodyne choice has dramatic consequences on the resulting category of ‘generalized spaces’, see Chapter 4. The setting we have described so far just let us deal with ‘subsets’, so we still have to explain what the open ones are. By opposition with pointless topology, we have to express our notion of ‘generalized topology’ without using joins nor finite meets, since they are not supposed to be available in  $T(X)$ , and even if they are, distributivity may not hold. This leaves us with several approaches depending on the chosen characterization of topologies, among which

- bases ([Kel55, Theo 11 p.47], [Eng89, 1.2.1 p.21])
- neighbourhoods ([Bro06, Sect 2.1, p.20])<sup>2</sup>

We define our ‘generalized spaces’ by means of bases, and prove that the resulting notion matches the one we would have obtained from an approach based on neighbourhoods (5.2.10).

---

<sup>1</sup>Precubical sets are to higher dimensional automata as graphs are to automata. For a detailed account of the importance of higher dimensional automata in concurrency theory see [vG06].

<sup>2</sup>In our context, Axiom N3 from [Bro06] has to be read «if  $x$  belongs the neighbourhoods  $U$  and  $V$ , there exists a neighbourhood  $W$  containing  $x$ , and included in both  $U$  and  $V$ ».

A standard basis of topology on  $X$  is defined as a subset  $\mathcal{B} \subset \mathcal{P}(X)$  satisfying

- for all finite family  $(B_j)$  of  $\mathcal{B}$ , for all  $x \in X$  such that  $x \in B_j$  for every  $j$ ,
- there is  $B \in \mathcal{B}$  such that  $x \in B$  and  $B \subset B_j$  for every  $j$ .

By analogy, in our framework, a basis of  $T$ -topology is a subset  $\mathcal{B} \subset T(X)$  satisfying

- for all finite family  $(B_j)$  of  $\mathcal{B}$ , for all  $x \in X$  such that  $x \in_T^X B_j$  for every  $j$ ,
- there is  $B \in \mathcal{B}$  such that  $x \in_T^X B$  and  $B \subset_T^X B_j$  for every  $j$ .

From there we define  $T$ -open members related to the basis, and state that two such bases are equivalent if they induce the same  $T$ -open members. In order to make these concepts well-behaved, we make the following assumptions:

- the relation  $\subset_T^X$  is a preorder on  $T(X)$ , and
- the relation  $\in_T^X$  is an order-theoretic module (see 3.1.1) from  $(X, =)$  to  $(T(X), \subset_T^X)$ , i.e.

$$(x \in_T^X A \subset_T^X B) \Rightarrow x \in_T^X B.$$

The next step is to define the  $T$ -continuous functions  $f : X \rightarrow Y$  at  $x$ . In order to adapt the standard notion expressed in terms of the bases of topology  $\mathcal{B}$  and  $\mathcal{B}'$  on  $X$  and  $Y$ , i.e.

$$\forall B' \in \mathcal{B}' \text{ such that } f(x) \in B', \exists B \in \mathcal{B} \text{ such that } x \in B \text{ and } f(B) \subset B',$$

we need a relation  $T(f)$  from  $T(X)$  to  $T(Y)$  playing the role of the statement  $f(B) \subset B'$ . Indeed, the usual inverse and direct image operators associated to  $f$  may not induce morphisms between the preordered sets  $(T(X), \subset_T^X)$  and  $(T(Y), \subset_T^Y)$ . The relation  $T(f)$  is intended to fill that gap: the function  $f$  is said to be  $T$ -continuous at  $x$ , with  $\mathcal{B}$  and  $\mathcal{B}'$  bases of  $T$ -topology on  $X$  and  $Y$ , when

$$\forall B' \in \mathcal{B}' \text{ such that } f(x) \in_T^Y B', \exists B \in \mathcal{B} \text{ such that } x \in_T^X B \text{ and } B T(f) B'.$$

To ensure the above definition does not depend on the bases, it is enough to assume that

- the relation  $T(f)$  is a module (see 3.1.1), i.e.

$$(A \subset_T^X A', A' T(f) B', B' \subset_T^Y B) \Rightarrow A T(f) B,$$

- and that the modules  $\in_T^X$  and  $\in_T^Y$  satisfy the lax naturality condition (diagram 5.2):

$$x \in_T^X A \text{ and } A T(f) B \Rightarrow f(x) \in_T^Y B.$$

For example, let  $T(X)$  and  $T(Y)$  be the collections of ordered sets whose underlying sets are included in  $X$  and  $Y$ . Then, for  $A$  and  $B$  members of  $T(X)$  and  $T(Y)$ , we write  $A T(f) B$  to mean that  $f(A) \subset B$  and the restriction of  $f$  to  $A$  induces an increasing map from  $A$  to  $B$ . In this specific case, the relation  $T(f)$  is neither representable nor corepresentable in the sense of Definition 3.1.2.

Of course we expect that  $T$ -topological spaces and  $T$ -continuous maps form a concrete category. It is so when  $T$  is a lax functor from the category of sets to that of (order-theoretic) modules (Definition 5.1.1). Call *topological theory*<sup>3</sup> any pair  $T = (T, \in_T)$  that satisfies the preceding assumptions, the corresponding category is denoted by  $\mathbf{Top}_T$ .

We draw attention to the fact that being isomorphic in  $\mathbf{Top}_T$  may be a rather weak notion of ‘sameness’: some properties of  $T$ -topological spaces may not transfer along their isomorphisms. In the end, it strongly depends on the properties of  $T$ . Nevertheless, all the ‘natural’ topological theories studied in this manuscript induce a ‘strong’ notion of isomorphism.

<sup>3</sup>This notion differs from its homonym in [Hof07] even if they are driven by the same principle: from a given kind of ‘syntactic object’ we define ‘models’ which are seen as the ‘topological spaces’ associated to it.

The construction described so far takes the theory  $T$  as a fixed parameter. Given another theory  $T'$ , we call *semantic transformation* (Definition 5.3.1) any concrete functor from  $\mathbf{Top}_T$  to  $\mathbf{Top}_{T'}$ . Such a functor may forget too much information. For this reason, we introduce the notion of a *changing of bases data* (Definitions 5.3.3) from which we deduce well-behaved semantic transformations called *changing of bases functors* (Definition 5.3.14). The topological theory

$$X : \mathbf{Set} \mapsto ((\mathcal{P}(X), \subset), \in),$$

which we denote by  $(\mathcal{P}, \in)$ , is the terminal object of the category of topological theories with changing of bases data as morphisms. The corresponding  $(\mathcal{P}, \in)$ -topological spaces are the usual ones. Consequently, we have a canonical change base functor from  $\mathbf{Top}_T$  to  $\mathbf{Top}$  induced by the unique changing of bases  $T \rightarrow (\mathcal{P}, \in)$ . The underlying topology of  $T$ -topological space is its image under this canonical functor. Many notions of usual topology can be lifted along it.

We now briefly describe the content of the manuscript:

In the third chapter, we recall some concepts of order theory mainly to fix some terminology which, in some cases, slightly diverges from standard one. We also introduce some unusual concepts like (order-theoretic) modules. Many the relevant examples of topological theories can be presented in terms of quantales, though they are not involved in the development of the theory; for this reason we dedicate an appendix to them.

The fourth chapter, which is mainly a reproduction of [CH21], focuses on locally ordered spaces, which are historically the first model of concurrency based on topology. It starts with a detailed introduction, and contains a thorough study of their colimits, since the latter play a crucial role in applications to concurrency theory.

The fifth chapter is dedicated to the elementary notions related to  $T$ -topology. Basically, we formalize the concepts sketched in the above introduction. We also describe the construction  $\mathbf{Loc}$ , which functorially associates each topological theory  $T$  with a new one  $\mathbf{Loc}(T)$ . We see that  $\mathbf{Loc}(T)$ -topological spaces admit more concrete descriptions, and enjoy better properties than  $T$ -topological spaces. On the top of that, the  $\mathbf{Loc}$  construction provides a wealth of useful examples, including the locally (pre)ordered spaces.

In the sixth chapter, we naturally adapt classical notions of point-set topology like convergence, compactness, and various separation axioms, to the  $T$ -topological spaces. Some of them boil down to having the corresponding standard notion satisfied by the underlying topological space.

In the seventh chapter, we study (co)limits in  $\mathbf{Top}_T$  by means of initial and final lifting along the forgetful functor  $\mathbf{U}_T : \mathbf{Top}_T \rightarrow \mathbf{Set}$ . To this aim, we generalize the concepts of final (7.2), initial (7.4), and induced topology (7.5) to  $T$ -topological spaces. Doing so, issues arise from the fact that  $T$  is just a *lax* functor; they are early dealt with in (7.1). Then we focus on  $\mathbf{Loc}(T)$  (7.6): if  $\mathbf{U}_T$  is a fibration (i.e. it lifts morphisms ‘nicely’) then  $\mathbf{Loc}(T)$  satisfies the required assumptions for the use of induced  $\mathbf{Loc}(T)$ -topology, from which we prove that  $\mathbf{U}_{\mathbf{Loc}(T)}$  is also a fibration. By a similar approach we prove that  $\mathbf{U}_{\mathbf{Loc}(T)}$  is topological when so is  $\mathbf{U}_T$ .

In the eighth chapter, we focus on the  $T$ -topological spaces in which the meet of any family of  $T$ -open members of  $X$  that are pairwise  $\simeq_T$ -equivalent, is still a  $T$ -open member of  $X$  (we write  $A \simeq_T B$  when  $x \in_T A \Leftrightarrow x \in_T B$  for all  $x \in X$ ). Such  $T$ -topological spaces are called  $T$ -streams because of their relation to Krishnan’s streams [Kri09]. The typical kind of  $T$ -topological space that we want to rule out is the following:  $X = \mathbb{R}$ ,  $T(\mathbb{R})$  is the collection of all preorders on the set  $\mathbb{R}$ , the relation  $\in_T$  is always true,  $\subset_T$  is the mere inclusion of preorders. The  $T$ -open members are the preorders  $\preceq$  containing the partial order  $\leq_r$  for some  $r \in \mathbb{R}$ , where  $x \leq_r y$  means  $x = y$  or  $r \leq x \leq y$ . Indeed, we have an infinite strictly decreasing chain of  $T$ -open members all sharing the same underlying set, namely  $\mathbb{R}$ , whose meet is not a  $T$ -open member.

In the ninth chapter, following [GL13, 5.2-4], we study how a monoidal structure on the category  $\mathbf{Top}_T$  (with a strictly monoidal forgetful functor to  $\mathbf{Set}$ ) can be extended to the category  $\mathbf{Top}_{\mathbf{Loc}(T)}$ . On this occasion, the  $T$ -streams play an important role: provided that  $\mathbf{Top}_T$  is (monoidal) closed, the core-compact  $\mathbf{Loc}(T)$ -streams are exponentiable. Every  $T$ -topological space, seen as a  $\mathbf{Loc}(T)$ -topological space, is a core-compact  $\mathbf{Loc}(T)$ -stream. Adapting [GL13, 5.6] we show that under certain assumptions about a given class of objects  $\mathcal{C}$ , the category of

$\mathcal{C}$ -generated objects form a cartesian closed category.

# Chapter 3

## Preliminaries

In this chapter, we recall some order-theoretic notions and fix some terminology and notations that will be pervasively used in the rest of the manuscript. Many results presented here can be found, up to slightly different terminology, in [HST14, II.1].

A *relation* from a set  $M$  to a set  $N$  is a function  $R : M \times N \rightarrow \{0, 1\}$ ; such a function will be denoted by  $R : M \rightarrow N$ . Let  $A \in M$  and  $B \in N$ , we write  $A R B$  when  $R(A, B) = 1$  and we say that  $A$  is  $R$ -related to  $B$ . Two relations  $R : M \rightarrow N$  and  $S : N \rightarrow O$  are composed as usual: given  $A \in M$  and  $C \in O$ , we set  $A (S \circ R) C$  when there exists  $B \in N$  such that  $A R B$  and  $B S C$ . Moreover, two parallel relations  $R, R' : M \rightarrow N$  can be compared: we say that  $R$  is lesser than  $R'$  when, for every  $A \in M$  and  $B \in N$ ,  $A R B$  implies  $A R' B$ <sup>1</sup>. One easily checks that for every relations  $R, R' : M \rightarrow N$  and for every relations  $S, S' : N \rightarrow O$ , if  $R$  (respectively  $S$ ) is lesser than  $R'$  (respectively  $S'$ ) then  $S \circ R$  is lesser than  $S' \circ R'$ .

A *preorder* on a set  $X$  is a relation  $\leq : X \rightarrow X$  which is

- *reflexive*: for every  $x \in X$ ,  $x \leq x$ , and,
- *transitive*: for every  $x, y, z \in X$ ,  $x \leq y$  and  $y \leq z$  imply  $x \leq z$ .

An *order* on  $X$  is a preorder which is moreover

- *antisymmetric*: for every  $x, y \in X$ ,  $x \leq y$  and  $y \leq x$  imply  $x = y$ .

A *(pre)ordered set* is a set equipped with a (pre)order. As usual, we often identify a (pre)ordered set and its underlying set when there is no risk of confusion. Let  $(X, \leq_X)$  and  $(Y, \leq_Y)$  be two preordered sets, a function  $f : X \rightarrow Y$  is an *increasing map* from  $(X, \leq_X)$  to  $(Y, \leq_Y)$  when  $x \leq_X x'$  implies  $f(x) \leq_Y f(x')$  for all  $x, x' \in X$ . We denote by **Rel** the ordered category<sup>2</sup> of sets and relations, by **Ord** the category of preordered sets and increasing maps, and by **SOrd** the full subcategory of ordered sets.

### 3.1 Modules

**Definition 3.1.1** (Modules). Let  $(M, \leq_M)$  and  $(N, \leq_N)$  be two preordered sets. A *module* from  $(M, \leq_M)$  to  $(N, \leq_N)$  is a relation  $R : M \rightarrow N$  such that, for all  $A, A' \in M$  and for all  $B, B' \in N$ ,

$$(A \leq_M A', A' R B', B' \leq_N B) \Rightarrow A R B.$$

<sup>1</sup>By taking the inverse image of  $\{1\}$ , we can identify the relations  $M \rightarrow N$  and the subsets of  $M \times N$ . Hence, we also say that a relation  $R$  is included in a relation  $R'$  when the former is lesser than the latter.

<sup>2</sup>A (pre)ordered category is just a category with a (pre)order on each hom-set such that the composition is increasing in each variable (see [HST14, II.4.5]) for more details). It can be seen as a particular instance of a 2-category (see [Bor94a, 7.1]).

$$\begin{array}{ccc}
 A & \overset{R}{\dashrightarrow} & B \\
 \downarrow \leq_M & & \uparrow \leq_N \\
 A' & \xrightarrow{R} & B'
 \end{array}$$

Any module from  $M$  to  $N$  can be thought of as a preorder  $\leq_R$  on the disjoint union  $M \sqcup N$  such that:

- the preorder  $\leq_R$  matches with  $\leq_M$  on  $M$  and with  $\leq_N$  on  $N$ , and,
- no element of  $N$  is lesser than an element of  $M$ .

When a relation  $R : M \dashrightarrow N$  is a module from  $(M, \leq_M)$  to  $(N, \leq_N)$ , it will be denoted by  $R : (M, \leq_M) \dashrightarrow (N, \leq_N)$ , or, more concisely, by  $R : M \dashrightarrow N$  if there is no risk of confusion about the concerned preorders.

One readily checks that the composite of two modules (viewed as relations) is a module and that, for all preordered set  $(M, \leq_M)$ , the relation  $(\leq_M) : M \dashrightarrow M$  is a module and is a neutral element for module composition. We denote by **Mod** the category of preordered sets and modules. The category **Mod** inherits from the order of the category **Rel**.

Every increasing map  $f : (M, \leq_M) \rightarrow (N, \leq_N)$  induces a module  $f_* : (M, \leq_M) \dashrightarrow (N, \leq_N)$  and a module  $f^* : (N, \leq_N) \dashrightarrow (M, \leq_M)$ , defined, for  $A \in M$  and  $B \in N$ , by

$$A f_* B \text{ if } f(A) \leq_N B$$

$$B f^* A \text{ if } B \leq_N f(A)$$

For every preordered set  $(M, \leq_M)$ , we have

$$(\text{Id}_M)^* = (\text{Id}_M)_* = \leq_M .$$

For every increasing maps  $f : M \rightarrow N$  and  $g : N \rightarrow P$ , one has

$$g_* \circ f_* = (g \circ f)_* \quad \text{and} \quad f^* \circ g^* = (g \circ f)^* .$$

In particular, we get two functors  $(\_)_{*}$  and  $(\_)^{*}$ , respectively covariant and contravariant, from **Ord** to **Mod**. In the ordered category **Mod**, the morphisms  $f_{*}$  and  $f^{*}$  form an adjunction:

$$(\leq_M) \leq (f^* \circ f_*) \quad \text{and} \quad (f_* \circ f^*) \leq (\leq_N) .$$

Modules of the form  $f^*$  play a crucial role all over the manuscript.

**Definition 3.1.2** (Representable and corepresentable modules). Let  $(M, \leq_M)$  and  $(N, \leq_N)$  be preordered sets, and let  $R : M \dashrightarrow N$  be a module. The module  $R$  is *representable* when there is an increasing map  $f : (M, \leq_M) \rightarrow (N, \leq_N)$ , called *representation*, such that  $R = f_*$ . Likewise, the module  $R$  is *corepresentable* when there is an increasing map  $g : (N, \leq_N) \rightarrow (M, \leq_M)$ , called *corepresentation*, such that  $R = g^*$ .

In the previous definition, the assumption that  $R$  is a module makes monotonicity assumption on  $g$  (and  $f$ ) superfluous.

**Lemma 3.1.3.** Let  $(M, \leq_M)$  and  $(N, \leq_N)$  be preordered sets, let  $R : M \dashrightarrow N$  be a module, and let  $g : N \rightarrow M$  be a function such that, for all  $A \in M$  and  $B \in N$ ,  $A R B$  if, and only if,  $A \leq_M g(B)$ . Then we have  $g(B) R B$  for all  $B \in N$ .

*Proof.* Trivial since  $g(B) \leq_M g(B)$ . □

**Proposition 3.1.4.** Let  $(M, \leq_M)$  and  $(N, \leq_N)$  be preordered sets, let  $R : M \dashrightarrow N$  be a module, and let  $g : N \rightarrow M$  be a function such that, for all  $A \in M$  and  $B \in N$ ,  $A R B$  if, and only if,  $A \leq_M g(B)$ . Then the function  $g$  is increasing (and then  $R = g^*$ ).

*Proof.* Let  $B, B' \in N$  such that  $B \leq_N B'$ . By the previous lemma, we have  $g(B) R B$ . Since  $R$  is a module, we have  $g(B) R B'$ . By hypothesis on  $g$ , this implies  $g(B) \leq_M g(B')$ .  $\square$

We deduce the following proposition from the functoriality of  $(\_)^*$ :

**Proposition 3.1.5.** If  $R : M \dashrightarrow N$  is corepresented by  $g : N \rightarrow M$  and if  $S : N \dashrightarrow P$  is corepresented by  $h : P \rightarrow N$ , then the composite  $g \circ h$  is a corepresentation of  $S \circ R : M \dashrightarrow P$ .

Similar results hold for representable modules.

The functor  $(\_)^* : \mathbf{Ord} \rightarrow \mathbf{Mod}$  also allows us to define a canonical functor  $\mathbf{Eq} : \mathbf{Set} \rightarrow \mathbf{Mod}$ , where  $\mathbf{Set}$  denotes the category of sets. This functor associates every function  $f : X \rightarrow Y$  with the module  $f_* : (X, =) \dashrightarrow (Y, =)$ . If there is no risk of confusion, the module  $f_* : (X, =) \dashrightarrow (Y, =)$  is simply denoted by  $f : X \rightarrow Y$ .

## 3.2 Filters

The notion of filter defined below is weaker than the one classically used in order theory. We prefer it because it has better stability properties for non-complete preordered spaces, as we shall see.

**Definition 3.2.1** (Filter). Let  $(M, \leq)$  be a preordered set. A subset  $\mathcal{F} \subset M$  is a *filter* if, for every finite subset  $F$  of  $\mathcal{F}$ , for every  $A \in M$ , if  $\forall C, (\forall B \in F, C \leq B) \Rightarrow C \leq A$  then  $A \in \mathcal{F}$  (in other words, an element of  $M$  which is an upper bound of the set of all lower bounds of a finite part of  $\mathcal{F}$  is in  $\mathcal{F}$ ).

*Remark 3.2.1.* This notion of filter is in fact the dual notion of Frink ideals (see [Fri54] and [Nie06]).

*Remark 3.2.2.* Letting  $F$  be a singleton in the above definition, we deduce that  $\mathcal{F}$  is an *upward closed* subset: if  $A \leq B$  and if  $A \in \mathcal{F}$ , then  $B \in \mathcal{F}$ .

*Remark 3.2.3.* The empty set is a filter in  $(M, \leq)$  if, and only if, there is no greatest element in  $M$ .

*Remark 3.2.4.* When  $M$  is finitely complete, this notion of filter matches with the classical one: a non-empty subset, upward closed, and stable under binary meets.

**Definition 3.2.2.** Let  $(M, \leq)$  be a preordered set and let  $\mathcal{F}$  and  $\mathcal{F}'$  be two filters on it. The filter  $\mathcal{F}$  is *finer* than  $\mathcal{F}'$  if  $\mathcal{F}' \subset \mathcal{F}$ . Dually,  $\mathcal{F}'$  is *coarser* than  $\mathcal{F}$ .

One easily checks that the intersection of a family of filters is a filter. Thus the set  $Fl(M)$  of all filters on  $(M, \leq)$  ordered by inclusion is a complete ordered set (and the meets are classical subset intersections). This property may not hold for the classical notion of filter.

The following result, which is also generally false for the classical notion of filter, justifies the choice of the terminology.

**Proposition 3.2.3.** Let  $(M, \leq)$  be a preordered set and let  $P \subset M$ . The subset

$$\mathcal{F}(P) := \{A \in M ; \exists F \subset P \text{ finite such that } \forall C \in M, (\forall B \in F, C \leq B) \Rightarrow C \leq A\}$$

is a filter and it is the coarsest one among the filters containing  $P$ .

*Proof.* Let  $(B_j)_{j \in J}$  be a finite family of  $\mathcal{F}(P)$ . For every  $j \in J$ , by definition, there is a finite family  $(B_{j,k})_{k \in J_j}$  of  $P$  such that  $B_j$  is an upper bound of the set of all lower bounds of  $(B_{j,k})_{k \in J_j}$ . Let  $A \in M$  such that  $A$  is an upper bound of the set of all lower bounds of  $(B_j)_{j \in J}$ . Let  $C \in M$  a lower bound of  $(B_{j,k})_{j \in J, k \in J_j}$ . For every  $j \in J$ ,  $C$  is a lower bound of  $(B_{j,k})_{k \in J_j}$ , hence  $C \leq B_j$ . Therefore  $C$  is a lower bound of  $(B_j)_{j \in J}$  and so  $C \leq A$ . Thus,  $A$  is an upper bound of the set



of all lower bounds of  $(B_{j,k})_{j \in J, k \in J_j}$ , and, since  $(B_{j,k})_{j \in J, k \in J_i}$  is a finite family of  $P$ , one finally obtains  $A \in \mathcal{F}(P)$ . Consequently  $\mathcal{F}(P)$  is a filter on  $M$ .

Let  $A \in P$ . The singleton set  $\{A\}$  is a finite subset of  $P$  and  $A$  is an upper bound of the set of all its lower bounds, hence  $A \in \mathcal{F}(P)$ .

The filter  $\mathcal{F}(P)$  is clearly the coarsest one among the ones containing  $P$  because the elements of  $\mathcal{F}(P)$  are, by definition, upper bounds of the set of all lower bounds of some finite family of  $P$ .  $\square$

We say that  $\mathcal{F}(P)$  is the *filter generated by  $P$*  or that  $P$  is a *filter basis of  $\mathcal{F}(P)$* .

**Definition 3.2.4** (Filtered and cofiltered preordered set). A preordered set  $(M, \leq)$  is *filtered* when every finite subset of  $M$  has an upper bound and is *cofiltered* when every finite subset has a lower bound.

*Remark 3.2.5.* We choose to use the category theory terminology (see [Bor94a, 2.13.1] or [Mac98, 9.1]).

The classical notion of filter in order theory matches with cofiltered filters (for the induced preorder) as previously defined.

*Remark 3.2.6.* Let  $(M, \leq)$  be a preordered set and let  $P \subset M$  a cofiltered subset (for the induced preorder). Then the filter generated by  $P$  is  $\{A \in M; \exists B \in P/B \leq A\}$  and it is a cofiltered subset.

**Definition 3.2.5.** Let  $M$  and  $N$  be two preordered sets, and let  $R : M \dashrightarrow N$  be a module. The *image of a part  $P \subset M$  under  $R$*  is the set

$$P_R := \{B \in N; \exists A \in M \text{ such that } A R B\}.$$

When  $P$  is a filter, the *direct image filter* of  $P$  under  $R$  is the filter generated by  $P_R$ .

*Remark 3.2.7.* The direct image filter of a non-empty filter can be empty.

*Remark 3.2.8.* The direct image filter under a module of a cofiltered filter may not be cofiltered. However, it is when the module is representable.

Another construction working nicely with the chosen notion of filter is the following

**Definition 3.2.6** (Kowalsky sum). Let  $(M, \leq)$  a preordered set and let  $\mathfrak{F}$  be a filter in the powerset  $\mathcal{P}(Fl(M))$  ordered by inclusion. The Kowalsky sum  $\sum \mathfrak{F}$  of  $\mathfrak{F}$  is the following subset of  $M$ :

$$\sum \mathfrak{F} := \{A \in M; A^{Fl} \in \mathfrak{F}\}$$

where  $A^{Fl} := \{\mathcal{F} \in Fl(M); A \in \mathcal{F}\}$ .

**Proposition 3.2.7.** Let  $(M, \leq)$  a preordered set and let  $\mathfrak{F}$  be a filter in the powerset  $\mathcal{P}(Fl(M))$  ordered by inclusion. The Kowalsky sum  $\sum \mathfrak{F}$  is a filter in  $(M, \leq)$ .

*Proof.* Let  $(A_j)_{j \in J}$  be a finite family of  $\sum \mathfrak{F}$  and let  $C \in M$  be an upper bound of the set of all lower bounds of  $(A_j)_{j \in J}$ . We are to show that  $\bigcap_{j \in J} A_j^{Fl} \subset C^{Fl}$ . Since  $\mathfrak{F}$  is a filter and since  $A_j^{Fl} \in \mathfrak{F}$ , this will prove that  $C \in \mathfrak{F}$  from which we can conclude that  $C \in \sum \mathfrak{F}$ .

Let  $\mathcal{F} \in \bigcap_{j \in J} A_j^{Fl}$ . Then  $(A_j)_{j \in J}$  is a finite family of the filter  $\mathcal{F}$ , hence  $C \in \mathcal{F}$ , i.e.  $\mathcal{F} \in C^{Fl}$ .  $\square$

### 3.3 Order-theoretic compactness

**Definition 3.3.1** ((order-theoretic) compactness). Let  $(M, \leq)$  be a preordered set. Define the relation  $\ll$  on  $M$  by  $A \ll B$  when for every family  $(A_i)_{i \in I}$ <sup>3</sup> such that  $\forall C, (\forall i \in I, A_i \leq C) \Rightarrow$

<sup>3</sup>For many authors, notably in domain theory, like in [GL13, 5.1.1], the family is also assumed to be filtered and to have a join in  $M$ . Anyway the way-below relation will be mostly used in cases where the preordered set  $M$  is complete; then every family has a join and can be modifying by adding the joins of every finite subfamily, hence the two definitions match.

$B \leq C$  (i.e.  $B$  is a lower bound of the set of all upper bounds of  $(A_i)_{i \in I}$ ), there is a finite subset  $J \subset I$  such that  $\forall C, (\forall i \in J, A_i \leq C) \Rightarrow A \leq C$  (i.e.  $A$  is a lower bound of the set of all upper bounds of  $(A_i)_{i \in J}$ ).

One can easily check that, for all  $A, B$ , and  $C \in M$ , one has

- $A \ll B \Rightarrow A \leq B$ ,
- $A \leq B \ll C \Rightarrow A \ll C$ , and,
- $A \ll B \leq C \Rightarrow A \ll C$ .

The relation is called  $\ll$  the *way-below relation*. An element  $A$  of  $M$  is *compact* when  $A \ll A$ .

**Proposition 3.3.2.** Let  $(M, \leq)$  be a preordered set,  $(A_k)_{k \in K}$  be a finite family of  $M$ , and  $B \in M$ . If, for all  $k \in K$ ,  $A_k \ll B$ , and if  $(A_k)_{k \in K}$  has a join  $\bigvee_{k \in K} A_k$  in  $(M, \leq)$  then  $\bigvee_{k \in K} A_k \ll B$ .

*Proof.* Let  $(C_i)_{i \in I}$  be a family of  $M$  such that  $B$  is a lower bound of the set of all upper bounds of the family. For each  $k \in K$ , since  $A_k \ll B$ , there is a finite subset  $J_k \subset I$  such that  $A_k$  is a lower bound of the set of all upper bounds of  $(C_i)_{i \in J_k}$ . Let  $D$  be an upper bound of  $(C_i)_{i \in \bigcup_{k \in K} J_k}$ . Then, for all  $k \in K$ ,  $D$  is an upper bound of  $(C_i)_{i \in J_k}$  hence  $A_k \leq D$ . Consequently,  $\bigvee_{k \in K} A_k \leq D$ . Thus  $\bigvee_{k \in K} A_k$  is a lower bound of the set of all upper bounds of the finite family  $(C_i)_{i \in \bigcup_{k \in K} J_k}$ . It follows that  $\bigvee_{k \in K} A_k \ll B$ .  $\square$

**Definition 3.3.3** (Continuous preordered set). A preordered set  $(M, \leq)$  is *continuous* when, for all  $A \in M$ , the set

$$S_{\ll}(A) := \{B \in M ; B \ll A\}$$

is filtered and  $A$  is a join of  $S_{\ll}(A)$ .

**Theorem 3.3.4** (Interpolation lemma). Let  $(M, \leq)$  be a continuous preordered set and let  $A, C \in M$ . If  $A \ll C$  then there is  $B \in M$  such that  $A \ll B \ll C$ .

*Proof.* Write  $N := \{D \in M ; \exists B \in M \text{ such that } D \ll B \ll C\}$ .

Let  $(D_j)_{j \in J}$  be a finite family of  $N$  and let  $(B_j)_{j \in J}$  be a family such that, for every  $j \in J$ ,  $D_j \ll B_j \ll C$ . By continuity of  $M$ , the set  $S_{\ll}(C)$  is filtered so there is  $B \in M$  such that  $B \ll C$  and, for every  $j \in J$ ,  $B_j \leq B$ . Let  $j \in J$ , since  $D_j \ll B_j \leq B$ , one has  $D_j \ll B$ . Then, since  $S_{\ll}(D)$  is filtered, there is  $D \in M$  such that  $D \ll B$  and, for every  $j \in J$ ,  $D_j \leq D$ . Thus  $N$  is filtered.

Let  $E$  be an upper bound of  $N$ . Let  $B \in M$  such that  $B \ll C$ . For every  $D \in M$  such that  $D \ll B$ , one has  $D \in N$ , hence  $D \leq E$ . Therefore,  $E$  is an upper bound of  $S_{\ll}(B)$ . It follows that  $B \leq E$ , because, since  $(M, \leq)$  is continuous,  $B$  is a join of  $S_{\ll}(B)$ . Similarly, we deduce that  $E$  an upper bound of  $S_{\ll}(C)$ , then that  $C \leq E$ . Thus  $C$  is a lower bound of the set of all upper bounds of  $N$ .

Since  $A \ll C$  and since  $N$  is filtered, there is  $D \in N$  such that  $A \leq D$ . In other words, there is  $B \in M$  such that  $A \leq D \ll B \ll C$ . Consequently, one has  $A \ll B \ll C$ .  $\square$

We also use the notion of supercompactness which is a variant of the notion of compactness.

**Definition 3.3.5** (Supercompactness). Let  $(M, \leq)$  be a preordered set. We define the relation  $\lll$  on  $M$  by  $A \lll B$  when for every family  $(A_i)_{i \in I}$  such that  $\forall C, (\forall i \in I, A_i \leq C) \Rightarrow B \leq C$ , there is  $i_0 \in I$  such that  $A \leq A_{i_0}$ .

The relation  $\lll$  satisfies properties similar to those satisfied by the relation  $\ll$ . One easily checks that, for all  $A, B$ , and  $C \in M$ , one has

- $A \lll B \Rightarrow A \ll B$ ,
- $A \leq B \lll C \Rightarrow A \lll C$ , and,
- $A \lll B \leq C \Rightarrow A \lll C$ .

An element  $A$  of  $M$  is *supercompact* when  $A \lll A$ .

*Remark 3.3.1.* When  $M$  is complete and thus has all joins, the previous definitions match with the usual ones.

**Example 3.3.1.** Let  $X$  be a set. In  $(\mathcal{P}(X), \subset)$ , the compact elements are the finite subsets, and the supercompact elements are the singleton subsets.

# Chapter 4

## A motivating example

This chapter has been submitted as an article [CH21].

We provide several variants of the notion of a locally ordered space, only differing by the separation properties their underlying spaces are required to satisfy. Quite surprisingly, these variations have dramatic consequences on colimits, even on their mere existence. Yet, for the rest of the introduction, we remain vague about the variant under consideration and write  $\mathbf{LOSp}$  to denote one of them. After having provided a catalog of pathological coequalizers in  $\mathbf{LOSp}$  (and proven that some of them not even exist), our purpose is to give an insight into locally ordered spaces colimits.

No algebraic topologist would imagine working in categories that are not cocomplete<sup>1</sup>, or in which certain seemingly obvious colimits are so ill-behaved. It is thus necessary, before dwelling in the technical details, to explain why locally ordered spaces should be taken seriously. As a first argument, we cite the work by J. D. Lawson in which the equivalence between *ordered manifolds* (a certain kind of locally ordered spaces) and *conal manifolds* is established [Law89, Theorem 2.7], the relation to Lie theory of semigroups [HHL89], and also to *causal orientation* in cosmology [Seg76, pp.22–28]. Beyond that somewhat argument of authority, and adopting the computer scientist point of view, the crucial property of locally ordered spaces is that they are free of *vortices* (a vortex is a point every neighbourhood of which contains a non-trivial directed loop). Regardless of the chosen perspective, vortices are pathological. Then we have to face a dilemma. On one hand, we can require our working category to be *topological* over  $\mathbf{Top}$  [Bor94b, 7.3] so colimits be well-behaved. We thus have a convenient framework for homotopical methods, but in which vortices are pervasive. This is the case with the category of *d-spaces* [Gra09, 1.4.7]. On the other hand, we can ban vortices from our class of models, though this comes at the price of missing or poorly behaved colimits. This is the case with all the variants of the category of locally ordered spaces considered in this chapter.

A natural idea to prove that  $\mathbf{LOSp}$  is not cocomplete consists of identifying all the points visited by a directed loop to contradict the fact that a locally ordered space has no vortex. We experiment this approach on the standard directed cylinder  $S^1 \times \mathbb{R}$ , see Example 4.2.1<sup>2</sup>. More precisely, one tries to ‘create’ a vortex by identifying all the points of the form  $(s, 0)$  with  $s \in \overrightarrow{S^1}$ . Depending on the category of locally ordered spaces under consideration, this coequalizer may or may not exist, see Corollaries 4.2.9, 4.2.14, and 4.2.15 in Section 4.2.

A lucid analysis of the directed cylinder example reveals that any coequalizer in  $\mathbf{LOSp}$  (when it exists) is obtained by identifying points of a topological space. Nevertheless, the antisymmetry locally imposed by the elements of an ordered basis (Definition 4.1.1) often forces much more points to be identified than in an ordinary topological quotient. Yet, the effects of the phenomenon described above may be limited by the underlying topology. Formally, the more points are ordered, the more points are identified by quotient construction. Dually, the finer the topology is, the less extra points are identified. Sections 4.3 and 4.4 illustrate this claim:

<sup>1</sup>E.g. every model category is required to be bicomplete.

<sup>2</sup>The latter is, in particular, a conal manifold in the sense of [Law89]: the cones are induced by the canonical parallelization [BG80, Appendix 3B].

In Section 4.3, we start again from the standard cylinder. However, we equip it with a local order so that it contains a countable family of pairwise disconnected directed loops which converges, in a certain sense, to another directed loop  $\gamma$ .<sup>3</sup> Trying to identify all the points visited by  $\gamma$  results in a diagram whose coequalizer does not exist.

In Section 4.4, we identify a section of a cylinder whose basis is totally disconnected. In this case, the coequalizer exists, and its underlying topology even matches the coequalizer of the underlying topological spaces.

Despite the pathological behaviour of colimits in the category of locally ordered spaces, a wide class of precubical sets can be realized in it. Moreover, for any precubical set of this class, the underlying space of the realization in  $\mathbf{LOSp}$  matches the realization in  $\mathbf{Top}$  [FGR06]. In contrast, we observe that the above property is no longer satisfied if we consider cubical sets instead of precubical ones. Indeed, identifying a section of the directed cylinder results in a colimit that is very close to the one required to realize the following cubical set:

$$K_2 = \{s\} \quad \text{with} \quad \partial_1^+ s = \partial_1^- s \quad \sigma \partial_0^- \partial_0^- s = \partial_0^- s .$$

Concretely, this cubical set identifies the two vertical edges of the square  $s$  and reduces the lower horizontal edge to a single point.

Hence, it seems that locally ordered spaces have been especially tailored for precubical set realization. This observation, together with the fact that they naturally occur in some well established branches of mathematics and physics, have motivated the attention paid to locally ordered spaces.

## 4.1 Locally ordered spaces

For all basic definitions related to General Topology, we refer to the standard textbooks [Mun00] and [Kel55]. The order of an ordered set  $P$  is denoted by  $\leq_P$ , its underlying set by  $P$  (or  $|P|$  when we need to emphasize on the distinction).

**Definition 4.1.1** (Ordered bases). Let  $X$  be a topological space. An *ordered basis* on  $X$  is a set  $\vec{\mathfrak{B}}$  of ordered sets such that:

- the underlying sets of the elements of  $\vec{\mathfrak{B}}$  form a basis of the topology of  $X$ , and
- for all  $x \in X$  and all  $B, B' \in \vec{\mathfrak{B}}$  such that

$$x \in B \cap B'$$

there is  $B'' \in \vec{\mathfrak{B}}$  such that

$$x \in B'' \subset B \cap B'$$

and the partial order  $\leq_{B''}$  is so that  $p \leq_{B''} q$  implies  $p \leq_B q$  and  $p \leq_{B'} q$ . Since this relation between ordered subsets is pervasively used throughout the rest of this section, we give it a name: for every pair of ordered sets  $B$  and  $B'$ , we denote  $B \subset_{lax} B'$  when  $|B| \subset |B'|$ , and  $p \leq_B q$  implies  $p \leq_{B'} q$  for all  $p, q \in B$ .

The basis is said to be *strict* when, in the above definition, the order  $\leq_{B''}$  actually coincides with the restrictions of  $\leq_B$  and  $\leq_{B'}$  to  $B''$ ; this stronger relation will be denoted by  $B \subset_{str} B'$ . Most of the examples met in this section are of the latter type.

An ordered basis  $\vec{\mathfrak{B}}'$  is *coarser* than  $\vec{\mathfrak{B}}$  if, for every  $x \in X$  and for every  $B' \in \vec{\mathfrak{B}}'$  such that  $x \in B'$ , there is  $B \in \vec{\mathfrak{B}}$  such that

$$x \in B \subset_{lax} B' .$$

We say that  $\vec{\mathfrak{B}}$  and  $\vec{\mathfrak{B}}'$  are *equivalent* when, in addition,  $\vec{\mathfrak{B}}$  is coarser than  $\vec{\mathfrak{B}}'$ .

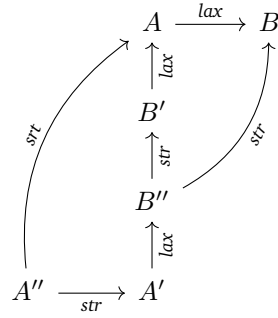
It is natural to define a notion of strict equivalence between strict bases by replacing the order  $\subset_{lax}$  in the above definition by  $\subset_{str}$  but it does not bring anything new:

<sup>3</sup>The directed loops mentioned here are to be understood as nontrivial.

**Proposition 4.1.2.** Two strict ordered bases  $\vec{\mathfrak{B}}$  and  $\vec{\mathfrak{B}}'$  on the topological space  $X$  are equivalent if, and only if, they are strictly equivalent.

*Proof.* Two strictly equivalent bases are equivalent because the relation  $\subset_{str}$  is stronger than the relation  $\subset_{lax}$ .

Conversely, assume that  $\vec{\mathfrak{B}}$  and  $\vec{\mathfrak{B}}'$  are equivalent. Let  $B \in \vec{\mathfrak{B}}'$  containing a point  $x \in X$ . There exist  $A \in \vec{\mathfrak{B}}$  such that  $x \in A \subset_{lax} B$ , and  $B' \in \vec{\mathfrak{B}}'$  such that  $x \in B' \subset_{lax} A$ . Since the basis  $\vec{\mathfrak{B}}'$  is strict, there is  $B'' \in \vec{\mathfrak{B}}'$  such that  $x \in B'' \subset_{str} B, B'$ . Once again, since  $\vec{\mathfrak{B}}$  and  $\vec{\mathfrak{B}}'$  are equivalent, there is  $A' \in \vec{\mathfrak{B}}$  such that  $x \in A' \subset_{lax} B''$ . Since  $\vec{\mathfrak{B}}$  is a strict basis, there is  $A'' \in \vec{\mathfrak{B}}$  such that  $x \in A'' \subset_{str} A, A'$ . We now check that  $A'' \subset_{str} B$ . Since  $A'' \subset_{str} A \subset_{lax} B$ , we have  $A'' \subset_{lax} B$ . Let  $x', x'' \in A''$  such that  $x' \leq_B x''$ . We have  $x' \leq_{B''} x''$  because  $B'' \subset_{str} B$ . From  $B'' \subset_{str} B'$  and  $B' \subset_{lax} A$ , we deduce that  $x' \leq_A x''$ . Finally, since  $A'' \subset_{str} A$ , we obtain  $x' \leq_{A''} x''$ . The other direction in the definition of strict equivalence is obtained by symmetry. The above reasoning is summarized in the following diagram:



□

**Definition 4.1.3.** If  $\vec{\mathfrak{B}}$  and  $\vec{\mathfrak{B}}'$  are ordered bases of topological spaces  $X$  and  $X'$  then the collection

$$\vec{\mathfrak{B}} \times \vec{\mathfrak{B}}' = \{B \times B' ; B \in \vec{\mathfrak{B}} \text{ and } B' \in \vec{\mathfrak{B}}'\}$$

is an ordered basis on  $X \times X'$ . Note that if  $\vec{\mathfrak{B}}$  and  $\vec{\mathfrak{B}}'$  are strict then so is  $\vec{\mathfrak{B}} \times \vec{\mathfrak{B}}'$ .

The equivalent class of  $\vec{\mathfrak{B}}$  admits a greatest element  $\vec{\mathcal{O}}(\vec{\mathfrak{B}})$  with respect to inclusion. Its elements are the ordered sets  $A$  such that:

- the underlying set of  $A$  is included in  $X$ , and
- for all  $x \in A$ , there exists  $B \in \vec{\mathfrak{B}}$  such that  $x \in B \subset_{lax} A$ .

One readily checks that the underlying set of any element of  $\vec{\mathcal{O}}(\vec{\mathfrak{B}})$  is open in  $X$ .

**Lemma 4.1.4.** Let  $\vec{\mathfrak{B}}$  be an ordered basis on the topological space  $X$ , and  $O$  be an element of  $\vec{\mathcal{O}}(\vec{\mathfrak{B}})$ . Every open subset  $O'$  of  $O$  equipped with the restriction of  $\leq_O$  to  $O'$ , belongs to  $\vec{\mathcal{O}}(\vec{\mathfrak{B}})$ .

*Proof.* Let  $x \in O'$ . Since  $O'$  is an open of  $X$ , there exists  $B \in \vec{\mathfrak{B}}$  such that  $x \in B \subset O'$ . Then we have  $B' \in \vec{\mathfrak{B}}$  such that  $x \in B' \subset_{lax} O, B$ . Thus, we have  $B' \subset O'$  and  $\leq_{B'}$  is included in  $\leq'$ . □

**Definition 4.1.5.** An *ordered space* is a topological space  $X$  equipped with a order. A *Nachbin ordered space* is an ordered space whose order is closed as a subspace of the product  $X \times X$ .<sup>4</sup>

<sup>4</sup>Nachbin ordered spaces should not be confused with Nachbin-Hewitt spaces, which is another name for ‘realcompact spaces’ [Joh82, p.166].

**Definition 4.1.6** (Locally ordered spaces). A *locally ordered space* is an ordered pair  $(X, \mathcal{E})$  where  $\mathcal{E}$  is an equivalence class of ordered bases on the topological space  $X$ . The greatest element of  $\mathcal{E}$ , whose elements are called the *open ordered subsets* of  $(X, \mathcal{E})$ , is denoted by  $\vec{\mathcal{O}}(X, \mathcal{E})$ . We will often use the same denotation for a locally ordered space and its underlying topological space. A locally ordered space is said to be Hausdorff when so is its underlying topological space. A *strictly locally ordered space* is an ordered space  $(X, \mathcal{E})$  such that  $\mathcal{E}$  contains a strict ordered basis.

**Definition 4.1.7.** A *locally Nachbin ordered space*  $X$  is a locally ordered space such that for every  $x \in X$  and every  $O \in \vec{\mathcal{O}}(X)$  containing  $x$ , there exists  $O' \in \vec{\mathcal{O}}(X)$  containing  $x$ , which is a Nachbin ordered space (with the topology inherited from  $X$ ) such that  $O' \subset_{lax} O$ .

In strictly locally ordered spaces, we have the following characterisation :

**Proposition 4.1.8.** Assume that  $(X, \mathcal{E})$  is a strictly locally ordered space with a chosen strict basis  $\vec{\mathfrak{B}} \in \mathcal{E}$ . Then  $(X, \mathcal{E})$  is a locally Nachbin ordered space if, and only if, for every  $x \in X$ , there is  $B \in \vec{\mathfrak{B}}$  containing  $x$  such that  $B$  is a Nachbin space.

*Proof.* Let  $x \in X$ . There is  $B \in \vec{\mathfrak{B}}$  containing  $x$ . By hypothesis, we have  $O \in \vec{\mathcal{O}}(X)$  such that  $x \in O \subset_{lax} B$  and the order  $\leq_O$  is closed. Then there is  $B' \in \vec{\mathfrak{B}}$  such that  $x \in B' \subset_{lax} O$ . Since  $\vec{\mathfrak{B}}$  is a strict ordered basis, there is  $B'' \in \vec{\mathfrak{B}}$  such that  $x \in B'' \subset_{str} B, B'$ . We check that  $\leq_{B''}$  is a closed order. Let  $x', x'' \in B''$ . If  $x' \leq_{B''} x''$ , then  $x' \leq_{B'} x''$  and  $x' \leq_O x''$ . The other way round, if  $x' \leq_O x''$  then  $x' \leq_B x''$ , and we also have  $x' \leq_{B''} x''$  because  $B'' \subset_{str} B$ . We have proven that  $\leq_{B''}$  is the restriction of the closed relation  $\leq_O$  to  $B''$ . The converse implication is obvious.  $\square$

By [Nac65], the underlying topological space of a locally Nachbin ordered space is a locally Hausdorff space.

*Remark 4.1.1.* If  $\vec{\mathfrak{B}}$  is an ordered basis of the locally ordered space  $X$ , and  $Y$  is a subspace of the underlying space of  $X$ , then  $\{Y \cap B \mid B \in \vec{\mathfrak{B}}\}$  is an ordered basis on  $Y$  (note that if  $\vec{\mathfrak{B}}$  is strict then so is this basis). All the ordered bases of  $Y$  obtained this way are equivalent, which allows us to define the (strictly) locally ordered subspace  $Y$  of  $X$ .

*Remark 4.1.2.* Let  $X$  and  $X'$  be two locally ordered spaces: an ordered basis of  $X \times X'$  is given by  $\vec{\mathfrak{B}} \times \vec{\mathfrak{B}}'$ , with  $\vec{\mathfrak{B}}$  and  $\vec{\mathfrak{B}}'$  being any ordered bases of  $X$  and  $X'$  respectively. The equivalence class of  $\vec{\mathfrak{B}} \times \vec{\mathfrak{B}}'$  only depends on the equivalence classes of  $\vec{\mathfrak{B}}$  and  $\vec{\mathfrak{B}}'$ .

*Remark 4.1.3.* Every topological space  $X$  can be seen as a strictly locally ordered space with a canonical strict ordered basis which consists of all the open subsets of  $X$  equipped with the equality. It is a locally Nachbin ordered space if, and only if, it is a locally Hausdorff space.

*Remark 4.1.4.* More generally, every ordered space  $(X, \leq)$  can be seen as a strictly locally ordered space with a canonical strict ordered basis which consists of all the open subsets of  $X$  equipped with the restriction of  $\leq$ . If  $(X, \leq)$  is moreover a Nachbin space then it is a locally Nachbin ordered space.

**Definition 4.1.9.** Let  $X$  and  $Y$  be locally ordered spaces and let  $\vec{\mathfrak{B}}$  (resp.  $\vec{\mathfrak{B}}'$ ) be an ordered basis in the equivalent class of ordered bases of  $X$  (resp.  $Y$ ). A function  $f : X \rightarrow Y$  is *locally increasing* at  $x \in X$  when, for all  $B' \in \vec{\mathfrak{B}}'$  such that  $f(x) \in B'$ , there exists  $B \in \vec{\mathfrak{B}}$  such that  $x \in B$ ,  $f(B) \subset B'$  and  $f_B : B \rightarrow B'$  is increasing. One verifies that this notion only depends on the equivalence classes of  $\vec{\mathfrak{B}}$  and  $\vec{\mathfrak{B}}'$ . The function  $f$  is a *locally increasing map* if it is locally increasing at every point of  $X$ .

One easily checks that a function locally increasing at a point  $x$  is continuous at  $x$  for the underlying topological spaces.

**Example 4.1.1.** Let  $(X, \leq_X)$  and  $(Y, \leq_Y)$  be two ordered spaces and let  $f : X \rightarrow Y$  be a function. If  $f$  is continuous and increasing then one easily checks that it is a locally increasing map with  $(X, \leq_X)$  and  $(Y, \leq_Y)$  seen as locally ordered spaces following Remark 4.1.4. The

converse is false: for example consider the ordered space  $X := ([0, 1] \cup [2, 3], \leq)$  where  $[0, 1] \cup [2, 3]$  is endowed with the topology induced by the usual one on  $\mathbb{R}$  and  $\leq$  is the restriction of the usual order of  $\mathbb{R}$ . Then the function

$$\begin{cases} x \mapsto x + 2 & \text{if } x \in [0, 1] \\ x \mapsto x - 2 & \text{if } x \in [2, 3] \end{cases}$$

from  $X$  into itself is locally increasing but not increasing. Thus we get a (non-full) concrete functor from the category of (Nachbin) ordered spaces and continuous increasing maps to the category of locally (Nachbin) ordered spaces and locally increasing maps.

In the case where the target space is a strictly locally ordered space, we have a convenient characterisation of locally increasing maps.

**Proposition 4.1.10.** Let  $X$  be a locally ordered space,  $Y$  be a strictly locally ordered space given by a strict ordered basis  $\vec{\mathfrak{B}}'$  and let  $f : X \rightarrow Y$  be a function. The map  $f$  is locally increasing at  $x \in X$  if, and only if, it is continuous at  $x$  and there exists  $O \in \vec{\mathcal{O}}(X)$  and  $B_0 \in \vec{\mathfrak{B}}'$  such that

$$x \in O, \quad f(O) \subset B_0, \quad \text{and} \quad f_O : O \rightarrow B_0 \text{ is increasing}$$

where  $f_O$  is the restriction of  $f$  to  $O$ .

*Proof.* The condition is clearly necessary. On the other hand, we suppose that there exist  $O \in \vec{\mathcal{O}}(X)$  and  $B_0 \in \vec{\mathfrak{B}}'$  such that  $x \in O$ ,  $f(O) \subset B_0$  and  $f_O : O \rightarrow B_0$  is increasing. Let  $B \in \vec{\mathfrak{B}}'$  such that  $f(x) \in B$ . There is  $B' \in \vec{\mathfrak{B}}'$  such that  $f(x) \in B' \subset_{str} B_0, B$ . Since  $f$  is continuous at  $x$ , there is an open  $O' \subset O$  such that  $x \in O'$  and  $f(O') \subset B'$ . We denote by  $\leq_{O'}$  the restriction of  $\leq_O$  to  $O'$ . By Lemma 4.1.4 the ordered subset  $(O', \leq_{O'})$  belongs to  $\vec{\mathcal{O}}(X)$ . Then we have  $f(O') \subset B' \subset B$ , and:

- the map  $f_{O'} : O' \rightarrow B_0$  is increasing since  $f_O : O \rightarrow B_0$  is increasing and  $\leq_{O'}$  is the restriction of  $\leq_O$ ,
- the map  $f_{O'} : O' \rightarrow B'$  is increasing since  $f_{O'} : O' \rightarrow B_0$  is increasing,  $f(O') \subset B'$ , and  $\leq_{B'}$  is the restriction of  $\leq_{B_0}$ , and finally

the map  $f_{O'} : O' \rightarrow B$  is increasing since  $f_{O'} : O' \rightarrow B'$  is increasing and the relation  $\leq_{B'}$  is the restriction of the relation  $\leq_B$  to  $B'$ .  $\square$

## 4.2 Cylinder

The compact unit circle (with its usual topology) is

$$S^1 := \{z \in \mathbb{C} ; |z| = 1\} = \{e^{ix} | x \in \mathbb{R}\} .$$

**Definition 4.2.1.** An *ordered arc* is a proper open arcs of  $S^1$ , i.e. a subset of the form

$$\widehat{ab} := \{e^{ix} ; x \in ]a, b[ \}$$

with  $a, b \in \mathbb{R}$  such that  $0 < b - a < 2\pi$ , equipped with the *standard order*

$$e^{ix} \leq_{a,b} e^{iy} \quad \text{if} \quad a < x < y < b .$$

We observe that if we have  $a', b'$  such that  $\widehat{ab} = \widehat{a'b'}$ , then their standard orders match. So we denote by  $\leq_\alpha$  the standard order on a proper open arc  $\alpha$ .



One readily checks that the proper open arcs with their standard order form a strict ordered basis. The resulting locally ordered space is the *directed (unit) circle*, we denote it by  $\overrightarrow{S^1}$ . The *unordered (unit) circle* is obtained the same way, replacing the standard orders on open proper arcs by the discrete ones.

The counter-examples we are about to describe are based on products of locally ordered spaces of the following form (with  $X$  denoting any locally ordered space)

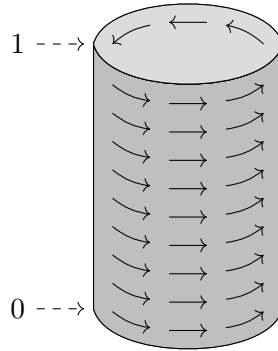
$$\overrightarrow{S^1} \times X$$

We write  $p_2 : S^1 \times X \rightarrow X$  for the second projection, and

$$i_t : S^1 \rightarrow S^1 \times X, \quad s \mapsto (s, t) \quad (t \in X)$$

for the *section* at the level  $t$ .

Figure 4.1: Cylinder with  $X = [0, 1]$



We fix a point  $*$  of  $X$  and give a criterion on the lattice of neighbourhoods of  $*$  for the coequalizer of the pair  $(i_*, c_*) : S^1 \rightarrow \overrightarrow{S^1} \times X$  (with  $c_* := s \mapsto (1, *)$ ) to exist.

Let  $f : \overrightarrow{S^1} \times X \rightarrow Y$  be a locally increasing map and  $K(f)$  be the set

$$\{ t \in X ; \forall s, s' \in S^1, f(s, t) = f(s', t) \} . \quad (4.1)$$

**Lemma 4.2.2.** The set  $K(f)$  is an open subset of  $X$ .

*Proof.* Let  $t_0 \in K(f)$  and let  $U \in \overrightarrow{\mathcal{O}}(Y)$  such that  $f(1, t_0) \in U$ .

For all  $s \in S^1$ ,  $f$  is locally increasing at  $(s, t_0)$ , so, by Definition 4.1.9, there exists an ordered arc  $\alpha_s$  containing  $s$ , an open ordered subset  $O_s$  of  $X$  containing  $t_0$  such that  $f(\alpha_s \times O_s) \subset U$  and the restriction  $f_s : \alpha_s \times O_s \rightarrow U$  is increasing. Since  $S^1$  is compact, the open covering made of the proper open arcs of the form  $\alpha_s$  admits a finite subcovering

$$\mathcal{F} = \{ \alpha_s ; s \in J \} .$$

We denote by  $O$  the finite intersection

$$\bigcap_{s \in J} O_s ,$$

which is thus an open neighbourhood of  $t_0$ . We are to show that  $O \subset K(f)$ . Let  $t \in O$ ,  $x, y \in S^1$ . There exists a finite sequence  $r_0, \dots, r_n \in S^1$  such that  $r_0 = x$ ,  $r_n = y$ , and for all  $k \in \{1, \dots, n\}$  there is  $s_k \in J$  such that  $r_k, r_{k+1} \in \alpha_{s_k}$  and  $r_k$  is less than  $r_{k+1}$  in  $\alpha_{s_k}$ . In particular  $(r_k, t)$  is less than  $(r_{k+1}, t)$  in the product ordered set  $\alpha_{s_k} \times O_{s_k}$  from which we deduce that

$$f(r_k, t) \leq_U f(r_{k+1}, t)$$

because the restriction  $f_{s_k}$  is increasing. By transitivity of  $\leq_U$ , we have  $f(x, t) \leq_U f(y, t)$ . By swapping the roles of  $x$  and  $y$  in the previous reasoning we prove that  $f(y, t) \leq_U f(x, t)$ . From the antisymmetry of  $\leq_U$ , we deduce that  $f(x, t) = f(y, t)$ , so  $t$  belongs to  $K(f)$ , which is therefore open in  $X$ .  $\square$

*Remark 4.2.1.* Lemma 4.2.2 remains valid if one replaces the directed circle by a compact locally ordered space that is *strongly connected* in the sense that for every ordered pair of points  $(a, b)$  there is a directed path from  $a$  to  $b$ .

By the above Lemma, if  $f \circ i_* = f \circ c_*$ , then  $K(f)$  is an open neighbourhood of  $*$  in  $X$ .

We will see that if  $f$  collapses the section at level  $*$  to the point  $*$ , then the collapsing spreads around the sections whose level are close to  $*$ .

Now, for every open neighbourhood  $O$  of  $*$ , we construct a locally ordered space  $X_O$  and a locally increasing map  $q_O : \overrightarrow{S^1} \times X \rightarrow X_O$  such that  $K(q_O) = O$ .

We define the set

$$X_O := O \sqcup \{ (s, t) ; s \in S_1, t \in X/O \}$$

and the (set theoretic) map  $q_O : \overrightarrow{S^1} \times X \rightarrow X_O$  by

$$q_O(s, t) = \begin{cases} t & \text{if } t \in O \\ (s, t) & \text{if } t \in X/O. \end{cases} \quad (4.2)$$

We note that  $K(q_O) = O$ .

**Lemma 4.2.3.** The final topology of  $q_O$  is generated by the subsets of the form

$$U_{\alpha, A} := O \cap A \sqcup \{ (s, t) ; s \in \alpha, t \in (X/O) \cap A \}$$

with  $\alpha$  proper open arc, and  $A \in \overrightarrow{\mathcal{O}}(X)$ .

*Proof.* Indeed, these subsets are open in the final topology because  $q_O^{-1}(U_{\alpha, A}) = S^1 \times (O \cap A) \cup \alpha \times A$  is an open subset of  $S^1 \times X$ . Conversely, let  $B$  be a subset of  $X_O$  such that  $q_O^{-1}(B)$  is an open subset of  $S^1 \times X$  and  $x \in B$ . Since  $q_O$  is a surjection, there exists  $(s, t) \in S^1 \times X$  such that  $q_O(s, t) = x$ . Then  $q_O^{-1}(B)$  is an open neighbourhood of  $(s, t)$  in  $S^1 \times X$  so there are a proper open arc  $\alpha$  and an open ordered subset  $A$  of  $X$  such that  $(s, t) \in \alpha \times A \subset q_O^{-1}(B)$ . We verify that  $U_{\alpha, A} \subset B$ .  $\square$

An element  $u \in U_{\alpha, A}$  is either an element of  $O \cap A$  or an ordered pair  $(s, t) \in \alpha \times ((X/O) \cap A)$ . Depending on the case, the *second component* of  $u$  refers to  $u$  itself or to  $t$ , we denote it by  $p_2(u)$ . We provide every set  $U_{\alpha, A}$  with the image of the order  $\leq_{\alpha \times A}$  under the mapping  $q_O$ , which we denote by  $\leq_{\alpha, A}^1$ .

**Lemma 4.2.4.** The relation  $\leq_{\alpha, A}^1$  matches with the relation  $\sqsubseteq$  defined below:

$$u \sqsubseteq u' \quad \text{if} \quad p_2(u) \leq_A p_2(u') \quad \text{and} \quad \begin{cases} \{u, u'\} \cap O \neq \emptyset \\ \text{or} \\ u = (s, t), u' = (s', t'), \text{ and } s \leq_\alpha s'. \end{cases}$$

*Proof.* Let  $u$  and  $u'$  be elements of  $U_{\alpha, A}$  such that  $u \sqsubseteq u'$ .

- If  $u = t \in O$  and  $u' = t' \in O$ , let  $s_0 \in \alpha$ , we have  $(s_0, t) \leq_{\alpha \times A} (s_0, t')$ ,  $q_O(s_0, t) = u$  and  $q_O(s_0, t') = u'$ , hence  $u \leq_{\alpha, A}^1 u'$ .
- If  $u = t \in O$  and  $u' = (s', t') \in \alpha \times ((X/O) \cap A)$ , we have  $(s', t) \leq_{\alpha \times A} (s', t')$ ,  $q_O(s', t) = u$  and  $q_O(s', t') = u'$ , hence  $u \leq_{\alpha, A}^1 u'$ .

- If  $u = (s, t) \in \alpha \times ((X/O) \cap A)$  and  $u' = t' \in O$ , we have  $(s, t) \leq_{\alpha \times A} (s, t')$ ,  $q_O(s, t) = u$  and  $q_O(s, t') = u'$ , hence  $u \leq_{\alpha, A}^1 u'$ .
- If  $u = (s, t) \in \alpha \times ((X/O) \cap A)$  and  $u' = (s', t') \in \alpha \times ((X/O) \cap A)$ , we have  $(s, t) \leq_{\alpha \times A} (s', t')$ ,  $q_O(s, t) = u$  and  $q_O(s', t') = u'$ , hence  $u \leq_{\alpha, A}^1 u'$ .

The fact that  $u \leq_{\alpha, A}^1 u'$  implies  $u \sqsubseteq u'$  readily derives from the definition of  $q_O$ .  $\square$

**Lemma 4.2.5.** The transitive closure of the relation  $\leq_{\alpha, A}^1$ , which we denote by  $\leq_{\alpha, A}$ , is antisymmetric. Moreover we have  $u \leq_{\alpha, A} u'$  if and only if

$$u \sqsubseteq u' \quad \text{or} \quad \exists u'' \in O \cap A \quad \text{such that} \quad u \sqsubseteq u'' \sqsubseteq u'.$$

We write  $u \trianglelefteq u'$  when the above condition is satisfied.

*Proof.* We check that the relation  $\leq_{\alpha, A}$  is antisymmetric. Indeed, if  $u \leq_{\alpha, A} u'$  and  $u' \leq_{\alpha, A} u$  then we have  $p_2(u) = p_2(u')$ . It follows that  $u, u'$ , and also any  $u''$  such that  $u \leq_{\alpha, A} u'' \leq_{\alpha, A} u'$  all belong to  $O$  or to its complement. The first case is obvious, in the second one we have  $s \leq_{\alpha} s' \leq_{\alpha} s$ , from which we deduce that  $s = s'$ . The relation  $\trianglelefteq$  is indeed an extension of  $\leq_{\alpha, A}^1$  and is clearly included in  $\leq_{\alpha, A}$ . In order to prove that  $\trianglelefteq$  is transitive, we first make an observation about the relation  $\sqsubseteq$ : assume that  $u_0 \sqsubseteq u_1 \sqsubseteq u \sqsubseteq u_2 \sqsubseteq u_3$  with  $u \in O \cap A$ . We have  $p_2(u_0) \leq_A p_2(u) \leq_A p_2(u_3)$ , and then  $u_0 \sqsubseteq u \sqsubseteq u_3$ . We now check that  $\trianglelefteq$  is transitive: assume that  $u_0 \trianglelefteq u_1 \trianglelefteq u_2$ . If  $u_0 = (s_0, t_0)$ ,  $u_1 = (s_1, t_1)$  and  $u_2 = (s_2, t_2)$ , and if  $u_0 \sqsubseteq u_1 \sqsubseteq u_2$  then, by transitivity of  $\leq_{\alpha}$  and of  $\leq_A$ , we have  $u_0 \sqsubseteq u_2$ . Otherwise, we meet one of the following cases:

1. one of the elements  $u_0, u_1$ , and  $u_2$  belongs to  $O \cap A$ , or
2. there exists  $\tilde{u} \in O \cap A$  such that  $u_0 \sqsubseteq \tilde{u} \sqsubseteq u_1$  or  $u_1 \sqsubseteq \tilde{u} \sqsubseteq u_2$ .

In any case the observation we made about  $\sqsubseteq$  allows us conclude that there exists  $u \in O \cap A$  such that  $u_0 \sqsubseteq u \sqsubseteq u_2$ , and therefore  $u_0 \trianglelefteq u_2$ . Finally, the relations  $\sqsubseteq$  and  $\leq_{\alpha, A}$  match.  $\square$

Consequently, if  $(X/O) \cap A$  is order-convex<sup>5</sup> in  $(A, \leq_A)$  then  $\leq_{\alpha, A} = \leq_{\alpha, A}^1$ .

**Lemma 4.2.6.** The family of ordered sets  $(U_{\alpha, A}, \leq_{\alpha, A})$  is an ordered basis on  $X_O$ .

*Proof.* Let  $u \in X_O$  such that  $u \in U_{\alpha_0, A_0} \cap U_{\alpha_1, A_1}$  with  $\alpha_0, \alpha_1$  proper open arcs and  $A_0, A_1 \in \vec{\mathcal{O}}(X)$ . If  $u \in O$ , there exists  $A_2 \in \vec{\mathcal{O}}(X)$  such that  $A_2 \subset O$  and  $u \in A_2 \subset_{\text{Iax}} A_0, A_1$ . Then  $U_{\alpha_0, A_2}$  is such that  $u \in U_{\alpha_0, A_2} \subset_{\text{Iax}} U_{\alpha_0, A_0}, U_{\alpha_1, A_1}$ . We note that if  $A_2 \subset_{\text{str}} A_0, A_1$ , then  $U_{\alpha_0, A_2} \subset_{\text{str}} U_{\alpha_0, A_0}, U_{\alpha_1, A_1}$ . If  $u = (s, t) \in S^1 \times (X/O)$ , there exists a proper open arc  $\alpha_2$  and  $A_2 \in \vec{\mathcal{O}}(X)$  such that  $s \in \alpha_2 \subset \alpha_0 \cap \alpha_1$ , and  $t \in A_2 \subset_{\text{Iax}} A_0, A_1$ . Then  $U_{\alpha_2, A_2}$  is such that  $u \in U_{\alpha_2, A_2} \subset_{\text{Iax}} U_{\alpha_0, A_0}, U_{\alpha_1, A_1}$ . We note that if  $A_2 \subset_{\text{str}} A_0, A_1$  and if the subsets  $(X/O) \cap A_i$  are order-convex in the ordered sets  $(A_i, \leq_{A_i})$ , then  $U_{\alpha_2, A_2} \subset_{\text{str}} U_{\alpha_0, A_0}, U_{\alpha_1, A_1}$ . Finally,  $X_O$  is a locally ordered space.  $\square$

*Remark 4.2.2.* Moreover, if there is a *strict* ordered basis  $\vec{\mathfrak{B}}$  such that for all  $B \in \vec{\mathfrak{B}}$ ,  $(X/O) \cap B$  is order-convex in  $B$ , then  $X_O$  is a *strictly* locally ordered space. In particular, if  $X$  a strictly locally ordered space coming from a topological space (4.1.3), then the canonical ordered basis satisfies the latter order-convexity condition. Besides, when  $O$  is a clopen subset, any strict basis  $\vec{\mathfrak{B}}$  can be turned into a strict basis satisfying the order-convexity condition by keeping only those elements  $B \in \vec{\mathfrak{B}}$  such that  $B \subset O$  or  $B \subset X/O$ .

**Lemma 4.2.7.** The map  $q_O$  is locally increasing.

*Proof.* Let  $x = (s, t) \in S^1 \times X$  and let  $U_{\alpha, A}$  such that  $q_O(x) \in U_{\alpha, A}$ , with  $\alpha$  proper open arc, and  $A \in \vec{\mathcal{O}}(X)$ . If  $q_O(x) \in O$ , we can assume that  $A \subset O$ . Let  $\alpha'$  be a proper open arc containing  $s$ , then  $x \in \alpha' \times A \in \vec{\mathcal{O}}(S^1 \times X)$ ,  $q_O(\alpha' \times A) \subset U_{\alpha, A}$  and the restriction of  $q_O$  to  $\alpha' \times A$  is increasing from  $\alpha' \times A$  to  $U_{\alpha, A}$ . If  $q_O(x) = (s, t) \in S^1 \times (X/O)$ , then  $x \in \alpha \times A \in \vec{\mathcal{O}}(S^1 \times X)$ ,  $q_O(\alpha \times A) \subset U_{\alpha, A}$  and the restriction of  $q_O$  to  $\alpha \times A$  is increasing from  $\alpha \times A$  to  $U_{\alpha, A}$ .  $\square$

<sup>5</sup>Recall that a subset  $P$  of a preordered set  $(M, \leq_M)$  is order-convex when, for every  $x, y, z \in A$  such that  $x \leq_M y \leq_M z$ , if  $x$  and  $z$  belong to  $P$  then  $y \in P$  too.

**Lemma 4.2.8.** Any locally increasing map  $f : \overrightarrow{S^1} \times X \rightarrow Y$  such that  $O \subset K(f)$  factorizes through the map  $q_O$  in a unique way.

*Proof.* The map  $h : X_O \rightarrow Y$  soundly defined by  $q_O(s, t) = f(s, t)$  is the only one satisfying  $f = h \circ q_O$ . Since the underlying topology on  $X_O$  is the final topology associated to  $q_O$ , the map  $h$  is continuous. Let  $u \in X_O$  and  $W \in \overrightarrow{\mathcal{O}}(Y)$  such that  $h(u) \in W$ . Let  $(s, t) \in \overrightarrow{S^1} \times X$  such that  $u = q_O(s, t)$ . Since  $f$  is locally increasing, we have a proper open arc  $\alpha$  and an open ordered subset  $A$  of  $\overrightarrow{\mathcal{O}}(X)$  such that  $(s, t) \in \alpha \times A$ ,  $f(\alpha \times A) \subset W$  and the restriction of  $f$  to  $\alpha \times A$  is increasing from  $\alpha \times A$  to  $W$ . Then  $u \in U_{\alpha, A}$  and  $h(U_{\alpha, A}) = h(q_O(\alpha \times A)) = f(\alpha \times A) \subset W$ . It remains to show that the restriction of  $h$  to  $U_{\alpha, A}$  is an increasing map from  $U_{\alpha, A}$  to  $W$ . Let  $u'$  and  $u''$  be elements of  $U_{\alpha, A}$  such that  $u' \leq_{\alpha, A}^1 u''$ . There exist two elements  $x'$  and  $x''$  of  $\alpha \times A$  such that  $x' \leq_{\alpha \times A} x''$ ,  $q_O(x') = u'$  and  $q_O(x'') = u''$ . Hence  $h(u') = f(x') \leq_W f(x'') = h(u'')$ .  $\square$

**Corollary 4.2.9.** In the category of locally ordered spaces, there exists a coequalizer of  $i_*$  and  $c_*$  if and only if the family of open neighbourhoods of  $*$  has a smallest element. If  $O$  is such a neighbourhood, then  $q_O : \overrightarrow{S^1} \times X \rightarrow X_O$  is the coequalizer.

*Proof.* Assume that there is a coequalizer  $f : \overrightarrow{S^1} \times X \rightarrow Y$  of  $i_*$  and  $c_*$ . Let  $O$  be an open neighbourhood of  $*$ . The map  $q_O : \overrightarrow{S^1} \times X \rightarrow X_O$  coequalizes  $i_*$  and  $c_*$  so there is a map  $h : Y \rightarrow X_O$  such that  $q_O = h \circ f$ , hence  $K(f) \subset K(q_O) = O$ . Moreover we know from Lemma 4.2.2 that  $K(f)$  is an open neighbourhood of  $*$  in  $X$ .

Conversely, let  $O$  be the least element among the open neighbourhoods of  $*$  in  $X$ . Given  $f : \overrightarrow{S^1} \times X \rightarrow Y$  that coequalizes  $i_*$  and  $c_*$ , the subset  $K(f)$  is an open neighbourhood of  $*$  (Lemma 4.2.2) so  $O \subset K(f)$ . Thus, by Lemma 4.2.8, there exists a unique factorization of  $f$  through  $q_O$ .  $\square$

**Corollary 4.2.10.** Let  $X$  be a strictly locally ordered space. Assume that  $\mathcal{V}$  is an open neighbourhood basis of  $*$  satisfying the following property: for every  $O \in \mathcal{V}$  there is a strict ordered basis  $\overrightarrow{\mathcal{B}}$  of  $X$  such that  $(X/O) \cap B$  is order-convex in each  $B \in \overrightarrow{\mathcal{B}}$ . Then, in the category of strictly locally ordered spaces, the coequalizer of  $i_*$  and  $c_*$  exists if and only if  $\mathcal{V}$  has a least element. If  $O$  is the least element of  $\mathcal{V}$ , then  $q_O : \overrightarrow{S^1} \times X \rightarrow X_O$  is the coequalizer of  $i_*$  and  $c_*$ .

*Proof.* The proof of Corollary 4.2.9 still holds taking Remark 4.2.2 into account.  $\square$

**Lemma 4.2.11.** For every continuous map  $f : S^1 \times X \rightarrow Y$  with  $Y$  locally Hausdorff, the set  $K(f)$  is a closed subset of  $X$ .

*Proof.* Let  $t \in \overline{K(f)}$ , we are to show that the continuous map  $f \circ i_t : S^1 \rightarrow Y$  is locally constant, which is sufficient to prove that  $f \circ i_t$  is constant (i.e.  $t \in K(f)$ ) because  $S^1$  is a connected space.

Let  $s \in S^1$  and let  $U \in \mathcal{O}(Y)$  such that  $f(s, t) \in U$  and  $U$  Hausdorff.

Since  $f$  is continuous at  $(s, t)$ , there exists an open neighbourhood  $\alpha \times A$  of  $(s, t)$  such that  $f(\alpha \times A) \subset U$  with  $\alpha$  denoting a proper open arc and  $A$  an open of  $X$ . From  $t \in \overline{K(f)}$ , we deduce that there is a net  $(t_i)_{i \in I}$  of  $A \cap K(f)$  that converges to  $t$ . Let  $s' \in \alpha$ . The nets  $(s, t_i)_{i \in I}$  and  $(s', t_i)_{i \in I}$  converge respectively to  $(s, t)$  and  $(s', t)$ . Each  $t_i$  belongs to  $K(f)$  hence  $f(s, t_i) = f(s', t_i)$ . The images of the nets  $(s, t_i)_{i \in I}$  and  $(s', t_i)_{i \in I}$  under  $f$  are thus equal and converge, by continuity of  $f$ , to  $f(s, t)$  and  $f(s', t)$ . We deduce that  $f(s, t) = f(s', t)$  because  $U$  is Hausdorff.  $\square$

*Remark 4.2.3.* Connectedness is the only property of  $S^1$  that is really used in the above proof. Under the stronger assumption that  $Y$  is Hausdorff, the above lemma is valid for any topological space instead of  $S^1$ .

**Proposition 4.2.12.** If  $X$  is a locally ordered space whose underlying topology is Hausdorff, then the two following statements are equivalent:

1. The underlying topology of the locally ordered space  $X_O$  is Hausdorff.

2. The open subset  $O$  of  $X$  is closed.

*Proof.* The first statement imply the second one by Lemma 4.2.11 (take  $f = q_O$ ). Conversely, assume that  $O$  is closed. Let  $u, u' \in X_O$  with  $u \neq u'$ . We have two situations to consider:

- There are  $s, s' \in S^1$  and  $t, t' \in X$  with  $t \neq t'$  such that  $u = q_O(s, t)$  and  $u' = q_O(s', t')$ . Since  $X$  is Hausdorff, there are disjoint open ordered subsets  $A$  and  $A'$  containing  $t$  and  $t'$  respectively. For any proper open arc  $\alpha$  containing  $s$  and  $s'$ , the subsets  $U_{\alpha, A}$  and  $U_{\alpha, A'}$  are disjoint open neighbourhood of  $u$  and  $u'$  respectively.
- There are  $t \in X$  and  $s, s' \in S^1$  with  $s \neq s'$  and  $u = q_O(s, t)$  and  $u' = q_O(s', t)$ . Let  $\alpha$  and  $\alpha'$  be disjoint proper open arcs containing  $s$  and  $s'$  respectively and let  $A$  be open ordered subset containing  $t$  and included in  $X/O$  (which is an open subset by hypothesis). Then the subsets  $U_{\alpha, A}$  and  $U_{\alpha', A}$  are disjoint open neighbourhood of  $u$  and  $u'$  respectively.

□

**Proposition 4.2.13.** Assume that  $X$  is a locally Nachbin ordered space. The two following statements are equivalent:

1. The space  $X_O$  is locally Nachbin ordered.
2. The open subset  $O$  of  $X$  is closed.

*Proof.* The first statement imply the second one by Lemma 4.2.11. Conversely, assume that  $O$  is closed. Let  $A, \alpha$  and  $u$  be an element of  $\overrightarrow{\mathcal{O}}(X)$ , an ordered arc, and an element of  $U_{\alpha, A}$  respectively. We have two cases to deal with. On one hand, if  $u = t \in O \cap A$ , there is  $A' \in \overrightarrow{\mathcal{O}}(X)$  included in  $O$  such that  $t \in A' \subset_{lax} A$ , and the order  $\leq_{A'}$  is closed for the topology induced by  $X$ . So the sets  $U_{\alpha, A'}$  and  $A'$  are equal, the topologies induced on them by  $X_O$  and  $X$  are the same, and the orders  $\leq_{\alpha, A'}$  and  $\leq_{A'}$  coincide. Moreover, we have  $U_{\alpha, A'} \subset_{lax} U_{\alpha, A}$ . On the other hand, if  $x = (s, t) \in \alpha \times (A \cap (X/O))$  with  $X/O$  open, there is  $A' \in \overrightarrow{\mathcal{O}}(X)$  included in  $X/O$  such that  $t \in A' \subset_{lax} A$  and the order  $\leq_{A'}$  is closed for the topology induced by  $X$ . So the sets  $U_{\alpha, A'}$  and  $\alpha \times A'$  are equal, the topologies induced on them by  $X_O$  and  $S^1 \times X$  are the same, and the orders  $\leq_{\alpha, A'}$  and  $\leq_{\alpha} \times \leq_{A'}$  coincide. Moreover, we have  $U_{\alpha, A'} \subset_{lax} U_{\alpha, A}$ . □

**Corollary 4.2.14.** In the category of (strictly) locally Nachbin ordered spaces, there exists a coequalizer of  $i_*$  and  $c_*$  if and only if the family of clopen neighbourhoods of  $*$  has a smallest element. If  $O$  is such a neighbourhood, then  $q_O : S^1 \times X \rightarrow X_O$  is the coequalizer.

*Proof.* Assume that there is a coequalizer  $f : S^1 \times X \rightarrow Y$  of  $i_*$  and  $c_*$  with  $Y$  a (strictly) locally Nachbin ordered space. Then, by Lemma 4.2.2 and Lemma 4.2.11,  $K(f)$  is a clopen neighbourhood of  $*$  in  $X$ . Let  $O$  be a clopen neighbourhood of  $*$ . The map  $q_O : S^1 \times X \rightarrow X_O$  coequalizes  $i_*$  and  $c_*$  and, by Remark 4.2.2 and by Proposition 4.2.13,  $X_O$  is a (strictly) locally Nachbin ordered space, so there is a map  $h : Y \rightarrow X_O$  such that  $q_O = h \circ f$ , hence

$$K(f) \subset K(q_O) = O$$

Conversely, if  $O$  is the least element among the clopen neighbourhoods of  $*$  in  $X$ . Still by Remark 4.2.2 and by Proposition 4.2.13, the space  $X_O$  is a (strictly) locally Nachbin ordered space. For every  $f : S^1 \times X \rightarrow Y$  that coequalizes  $i_*$  and  $c_*$  with  $Y$  (strictly) locally Nachbin ordered, the subset  $K(f)$  is a clopen neighbourhood of  $*$  (see Lemmas 4.2.2 and 4.2.11). Therefore  $O$  is included in  $K(f)$ , and there exists a unique factorization of  $f$  through  $q_O$ . □

**Corollary 4.2.15.** Assume that the underlying topology of  $X$  is Hausdorff. In the category of (strictly) locally (Nachbin) ordered Hausdorff spaces, there exists a coequalizer of  $i_*$  and  $c_*$  if and only if the family of clopen neighbourhoods of  $*$  has a smallest element  $O$ . In that case,  $q_O : S^1 \times X \rightarrow X_O$  is the coequalizer.

*Proof.* Assume that there is a coequalizer  $f : S^1 \times X \rightarrow Y$  of  $i_*$  and  $c_*$  with  $Y$  a (strictly) locally (Nachbin) ordered Hausdorff space. Then, by Lemmas 4.2.2 and 4.2.11,  $K(f)$  is a clopen neighbourhood of  $*$  in  $X$ . Let  $O$  be a clopen neighbourhood of  $*$ . The map  $q_O : S^1 \times X \rightarrow X_O$  coequalizes  $i_*$  and  $c_*$  and, by Remark 4.2.2 and by Propositions 4.2.12 and 4.2.13,  $X_O$  is a (strictly) locally (Nachbin) ordered Hausdorff space, so there is a map  $h : Y \rightarrow X_O$  such that  $q_O = h \circ f$ , hence

$$K(f) \subset K(q_O) = O$$

Conversely, let  $O$  be the least clopen neighbourhood of  $*$ . Still by Remark 4.2.2 and by Propositions 4.2.12 and 4.2.13, the space  $X_O$  is a (strictly) locally (Nachbin) ordered space. For every  $f : S^1 \times X \rightarrow Y$  that coequalizes  $i_*$  and  $c_*$  with  $Y$  a (strictly) locally (Nachbin) ordered space, the subset  $K(f)$  is an clopen neighbourhood of  $*$  (see Lemmas 4.2.2 and 4.2.11) so  $O$  is included in  $K(f)$ . So there exists a unique factorization of  $f$  through  $q_O$ .  $\square$

It is now time to provide some examples:

**Example 4.2.1.** The spaces  $\mathbb{R}$  and  $\mathbb{Q}$  are the real line and the space of rational numbers. The coarsest refinement of the topology of  $\mathbb{R}$  in which every singleton  $\{x\}$  with  $x \neq 0$  is open induces a topological space that is denoted by  $\mathbb{R}_*$ .

The collection of open subsets of  $\mathbb{R}$ , each equipped with the standard order, forms a strictly locally Nachbin ordered space which we denote by  $\overrightarrow{\mathbb{R}}$ . Let  $S$  be an infinite set with a distinguished element  $\bar{s}$ . We denote by  $\mathcal{U}$  the topological space on  $S$  in which a subset is open when it contains  $\bar{s}$ <sup>6</sup>. The table here below summarizes the cases where the coequalizer of  $i_*$  and  $c_*$  exists (see 4.1.3). The distinguished elements of  $\mathbb{R}_*$  and  $\mathcal{U}$  are respectively 0 and  $\bar{s}$ . In all the other cases, the distinguished element can be any point  $*$  of the space.

category \ X	$\mathbb{R}$	$\overrightarrow{\mathbb{R}}$	$\overrightarrow{S^1}$	$\mathbb{R}_*$	$\mathbb{Q}$	$\mathcal{U}$
(strictly) locally ordered spaces			$\times$			$\checkmark$
(strictly) locally ordered Hausdorff spaces		$\checkmark$		$\times$		$\checkmark$
(strictly) locally Nachbin ordered spaces		$\checkmark$		$\times$		n/a
(strictly) locally Nachbin ordered Hausdorff spaces		$\checkmark$		$\times$		n/a

In the first three columns, the coequalizer, when it exists, is the second projection. It is also the coequalizer when the space under consideration is  $\mathcal{U}$  and the ambient category is that of locally ordered Hausdorff spaces. In the latter case, if we drop the Hausdorffness assumption, the coequalizer is the quotient map  $q_O$  where  $O = \{\bar{s}\}$ , see (Eq. 4.2). All the spaces appearing in the above table are strict and satisfy the hypotheses of Corollary 4.2.10. Consequently, the results summarized in the table are valid regardless of the fact that the coequalizers are taken in categories of strictly or laxly locally ordered spaces.

### 4.3 Zebra cylinder

We emphasize that any point of  $[0, 1]$  admits the whole space as its smallest clopen neighbourhood, so the according to the results from Section 4.2 the coequalizer of  $(i_0, c_0) : S^1 \rightarrow S^1 \times [0, 1]$  with the product ordered basis on  $S^1 \times [0, 1]$  exists in the category of locally ordered Hausdorff spaces (it is actually  $[0, 1]$ ).

In this Section, we describe a strict ordered basis  $\overrightarrow{\mathfrak{B}}$  on  $S^1 \times [0, 1]$  so that the coequalizer of the morphisms  $(i_0, c_0)$  no longer exists in the category of (strictly) locally (Nachbin) ordered (Hausdorff) spaces. As before, the strategy consists of setting the ordered basis in a way that:

- all the sections in a chosen neighbourhood  $V$  of the section  $i_0$  are collapsed, and
- the neighbourhood  $V$  can be made arbitrarily small.

<sup>6</sup>The open subsets of  $\mathcal{U}$  are the elements of the principal ultrafilter on  $S$  generated by  $\bar{s}$  [DP02, p.233]

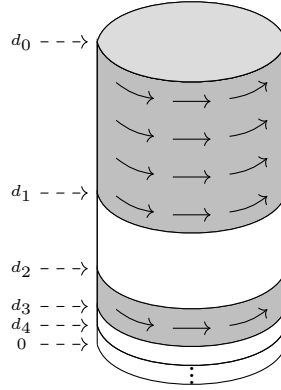
Let  $\overline{\mathbb{N}}$  be the set of extended natural numbers  $\mathbb{N} \cup \{+\infty\}$ , and  $(d_n)_{n \in \mathbb{N}}$  be a strictly decreasing sequence with values in the compact unit interval  $I := [0, 1]$ . Assume that  $\inf \{d_n ; n \in \mathbb{N}\} = 0$ . For all  $n \in \mathbb{N}$ ,  $I(n)$  denotes the interval  $[d_{2n+1}, d_{2n}]$  and  $I(+\infty)$  the degenerated interval  $\{0\}$ .

The elements of  $\overrightarrow{\mathfrak{B}}$  are of the form  $\alpha \times O$  where  $\alpha$  is an ordered arc and  $O$  is an open subset of  $I$ , ordered as follows:

$$(s, u) \preceq_{\alpha}^O (s', u') \quad \text{if} \quad \begin{cases} s \leq_{\alpha} s' \text{ and } u = u' \in I(n) \text{ for some } n \in \overline{\mathbb{N}}, \\ \text{or } s = s' \text{ and } u = u'. \end{cases} \quad (4.3)$$

We denote by  $Z$  the resulting *strictly locally Nachbin ordered Hausdorff space*.

Figure 4.2: Zebra cylinder



Let  $f : Z \rightarrow X$  be a locally increasing map such that

$$f \circ i_0 = f \circ c_0$$

and recall that  $K(f)$  is the set

$$\{ t \in I ; \forall s, s' \in S^1, f(s, t) = f(s', t) \}.$$

We prove a result similar to lemma 4.2.2.

**Lemma 4.3.1.** There exists  $n \in \mathbb{N}$  such that, for all  $m \geq n$ ,  $I(m) \subset K(f)$ .

*Proof.* The arguments are mostly the same as in the proof of Lemma 4.2.2. By hypothesis 0 belongs to  $K(f)$ . Let  $U \in \overrightarrow{\mathcal{O}}(X)$  such that  $f(1, 0) \in U$ . For all  $s \in S^1$ , the function  $f$  is locally increasing at  $(s, 0)$ , so there exists an open neighbourhood  $\alpha_s \times O_s$  of  $(s, t_0)$  such that  $f(\alpha_s \times O_s) \subset U$  and the restriction  $f_s : (\alpha_s \times O_s, \preceq_s) \rightarrow U$  is increasing with  $\alpha_s$  denoting a proper open arc and  $\preceq_s$  the order  $\preceq_{\alpha_s}^{O_s}$  on  $\alpha_s \times O_s$ .

Since  $S^1$  is compact, the open covering made of the proper open arcs of the form  $\alpha_s$  admits a finite subcovering

$$\mathcal{F} = \{ \alpha_s ; s \in J \} .$$

We denote by  $O$  the finite intersection

$$\bigcap_{s \in J} O_s ,$$

which is thus an open neighbourhood of 0.

Since the sequence  $(d_n)_{n \in \mathbb{N}}$  tends to 0, there exists  $n \in \mathbb{N}$  such that the intervals  $I(m)$  are included in  $O$  for every  $m \geq n$ . Let  $m$  be such a natural number. We are to show that  $I(m) \subset K(f)$ . Given  $t \in I(m)$ ,  $x, y \in S^1$ , there exists a finite sequence  $r_0, \dots, r_l \in S^1$  such that  $r_0 = x$ ,  $r_l = y$ , and for all  $k \in \{1, \dots, l\}$  there is  $s_k \in J$  such that  $r_k, r_{k+1} \in \alpha_{s_k}$  and  $r_k \leq_k r_{k+1}$

with  $\leq_k$  denoting the standard order on the proper open arc  $\alpha_{s_k}$ . In particular, since  $t \in I(m)$ , we have

$$(r_k, t) \preceq_{s_k} (r_{k+1}, t)$$

from which we deduce that

$$f(r_k, t) \leq_U f(r_{k+1}, t)$$

because the restriction  $f_{s_k}$  is increasing. By transitivity of  $\leq_U$ , we have  $f(x, t) \leq_U f(y, t)$ . By swapping the roles of  $x$  and  $y$  in the previous reasoning we prove that  $f(y, t) \leq_U f(x, t)$ . From the antisymmetry of  $\leq_U$ , we deduce that  $f(x, t) = f(y, t)$ , so  $t$  belongs to  $K(f)$ , which therefore contains  $I(m)$ .  $\square$

**Proposition 4.3.2.** The pair of morphisms  $(i_0, c_0 : S^1 \rightarrow Z)$  does not have any coequalizer in the category of (strictly) locally (Nachbin) ordered (Hausdorff) spaces.

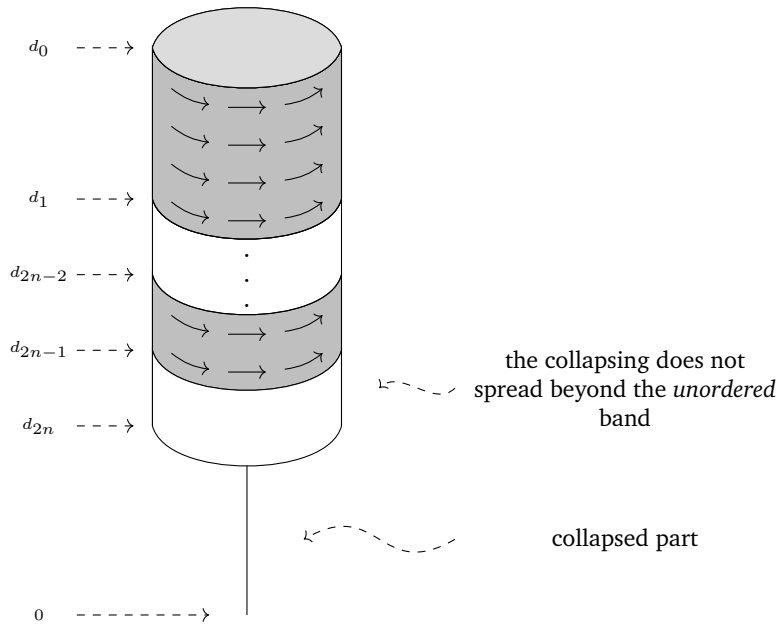
*Proof.* Let  $f : Z \rightarrow X, g : Z \rightarrow Y$  be two locally increasing maps which coequalize  $i_0$  and  $c_0$ . If there is  $h : Y \rightarrow X$  such that  $f = h \circ g$ , then  $K(g) \subset K(f)$ .

We construct, for each  $n \in \mathbb{N}$ , a locally ordered space  $X_n$  (which is actually strict, Nachbin, and Hausdorff) and a locally increasing map  $f_n : Z \rightarrow X_n$  such that  $f_n \circ i_0 = f \circ c_0$  and  $K(f_n) = [0, d_{2n}]$ . If  $g$  was the coequalizer of  $(i_0, c_0)$ , we would have

$$0 \in K(g) \subset \bigcap_{n \in \mathbb{N}} K(f_n) = \bigcap_{n \in \mathbb{N}} [0, d_{2n}] = \{0\}$$

but this is in contradiction with Lemma 4.3.1. For  $n \in \mathbb{N}$  we define the set

Figure 4.3: The locally ordered space  $X_n$



$$X_n := [0, d_{2n}] \sqcup \{ (s, t) ; s \in S_1, t \in ]d_{2n}, 1] \}$$

and the (set theoretic) map  $f_n : Z \rightarrow X_n$  by

$$f_n(s, t) = \begin{cases} (s, t) & \text{if } t > d_{2n} \\ t & \text{if } t \leq d_{2n} . \end{cases}$$



We note that  $f_n \circ i_0 = f_n \circ c_0$  and  $K(f_n) = [0, d_{2n}]$ . The final topology of  $f_n$  is generated by the subsets of the form

$$O_{\alpha, A} := \begin{cases} B & \text{if } A \subseteq [0, d_{2n}[ \\ B \sqcup (S_1 \times C) & \text{if } d_{2n} \in A \\ \alpha \times C & \text{if } A \subseteq ]d_{2n}, 1] \end{cases}$$

with  $\alpha$  proper open arc,  $A$  an open interval of  $I$ ,  $B := A \cap [0, d_{2n}[$  and  $C := A \cap ]d_{2n}, 1]$ . In order to define a (strict) ordered basis on  $X_n$  that makes the map  $f_n$  a morphism of locally ordered spaces, we only consider the intervals  $A$  whose length is (strictly) less than  $d_{2n-1} - d_{2n}$ . The order  $\leq_{\alpha, A}$  on  $O_{\alpha, A}$  is then defined as the equality in the first and second cases, and matches the order described at (4.3) in the third case. In particular, the condition on the length of  $A$  guarantees that the order on  $B \sqcup (S_1 \times C)$  on one hand, and the order on  $\alpha \times A$  from the ordered basis of  $Z$  on the other hand (for any proper open arc  $\alpha$ ), both match on  $\alpha \times C$ : this key observation ensures that  $f_n$  is indeed a morphism of locally ordered spaces.  $\square$

## 4.4 Rational based cylinder

Previously, we saw that coequalizers in the category of locally ordered spaces may behave differently than in the category of topological space because of the collapsing spreading described in Lemmas 4.2.2 and 4.3.1. These latter rely on the fact that the directed loops are continuous (i.e. they are indexed by  $\overrightarrow{S^1}$ ). In this section we replace  $\overrightarrow{S^1}$  by some of its dense totally disconnected subspace. Then we exhibit a pair of morphisms whose coequalizer exists, and whose underlying space matches with the topological coequalizer.

**Definition 4.4.1.** The subspace  $\{e^{ix} ; x \in \mathbb{Q}\}$  of  $S^1$  is denoted by  $S_{\mathbb{Q}}^1$ . The *directed rational unit circle*  $\overrightarrow{S}_{\mathbb{Q}}^1$  is the subspace  $S_{\mathbb{Q}}^1$  with the locally ordered space structure inherited from the directed unit circle, see Remark 4.1.1.

We overload the denotations  $i_0$  and  $c_0$  which now designate the mappings

$$s \in S_{\mathbb{Q}}^1 \mapsto (s, 0) \in \overrightarrow{S}_{\mathbb{Q}}^1 \times I \quad \text{and} \quad s \in S_{\mathbb{Q}}^1 \mapsto (1, 0) \in \overrightarrow{S}_{\mathbb{Q}}^1 \times I$$

As before we identify all the points of the section  $S_{\mathbb{Q}}^1 \times \{0\}$ . We now describe the resulting coequalizer in the category of topological spaces. The underlying set is the disjoint union

$$\{0\} \sqcup S_{\mathbb{Q}}^1 \times ]0, 1] \quad ,$$

and the quotient map is denoted by

$$q : S_{\mathbb{Q}}^1 \times [0, 1] \rightarrow \{0\} \sqcup (S_{\mathbb{Q}}^1 \times ]0, 1]) \quad .$$

A basis of open neighbourhoods of  $(s, t)$  with  $t > 0$  is given by the traces of the products  $\alpha \times ]a, b[$  with  $s \in \alpha$  proper open arc and  $0 < a < t < b$ . The associated order is given by the restriction of the product order  $\leq_{\alpha \times ]a, b[}$ .

The neighbourhoods of 0 are a bit harder to describe. We provide a basis of open neighbourhoods whose elements will be the supports of the orders around 0. To this aim, we consider the set  $\mathcal{H}$  of all functions  $h : S_{\mathbb{Q}}^1 \rightarrow [0, 1]$  which are continuous, strictly positive, and such that

$$\inf h = 0 \quad .$$

For every function  $h \in \mathcal{H}$ , we define the set

$$O_h = \{0\} \sqcup \{ (s, t) \in S_{\mathbb{Q}}^1 \times ]0, 1] ; t < h(s) \} \quad .$$

We note that  $\mathcal{H}$  is a inf-semilattice<sup>7</sup> with the minimum being computed pointwise. Moreover, the map  $h \mapsto O_h$  is a morphism of inf-semilattices. We are going to prove that the set of all  $O_h$  is a basis of open neighbourhoods of 0.

<sup>7</sup>Any pair of elements has a meet.

Firstly, in order to prove that  $O_h$  is an open subset, we show that  $q^{-1}(O_h) = \{(s, t) \in S_{\mathbb{Q}}^1 \times I ; t < h(s)\}$  is an open subset of  $S_{\mathbb{Q}}^1 \times I$ . Let  $(s, t) \in q^{-1}(O_h)$ , since  $t < h(s)$ , there exists two disjoint open intervals  $A$  and  $A'$  of  $I$  such that  $A < A'$ <sup>8</sup>,  $t \in A$  and  $h(s) \in A'$ . By continuity of  $h$ , there is an open subset  $O$  of  $S_{\mathbb{Q}}^1$  containing  $s$  such that  $h(O) \subset A'$ . Therefore,  $O \times A$  is an open neighbourhood of  $(s, t)$  included in  $q^{-1}(O_h)$ .

Secondly, in order to prove that any open neighbourhood of 0 contains some  $O_h$ , we use the following lemma:

**Lemma 4.4.2.** Let  $O$  be an open neighbourhood of  $i_0(S_{\mathbb{Q}}^1)$ . Let  $(s_n)_{n \in \mathbb{N}}$  be an enumeration of  $S_{\mathbb{Q}}^1$ . We inductively define a family  $(A_j, t_j)_{j \in J}$  (with  $J \subseteq \mathbb{N}$ ) such that

- for all  $j \in J$  :
  - the set  $A_j$  is open,
  - the real number  $t_j$  belongs to  $]0, 1]$ , and
  - the product  $A_j \times [0, t_j[$  is included in  $O$ ,
- the sets  $A_j$  form a partition of  $S_{\mathbb{Q}}^1$ , and
- the greatest lower bound of the set  $\{t_j ; j \in J\}$  is 0.

Then, for any open neighbourhood  $U$  of 0, the set  $q^{-1}(U)$  is an open neighbourhood of  $i_0(S_{\mathbb{Q}}^1)$ . Therefore, by applying the lemma, we get a family  $(A_j, t_j)$  from which we define the map  $h : S_{\mathbb{Q}}^1 \rightarrow [0, 1]$  which sends  $s \in A_j$  to  $t_j$ . We observe that it belongs to  $\mathcal{H}$ , and that  $q^{-1}(O_h) \subset q^{-1}(U)$ . Consequently, we get  $O_h \subset U$ . This concludes the proof that the set of all  $O_h$  is a basis of open neighbourhoods of 0.

*Proof of Lemma 4.4.2.* Since  $O$  is open and contains  $(s_0, 0)$ , there exists an open neighbourhood of  $(s_0, 0)$  of the form  $A_0 \times [0, t_0[$ . Taking  $A_0$  to be the trace of a proper open arc whose extremities are  $e^{ia}$  and  $e^{ib}$  with  $a$  and  $b$  in  $\mathbb{R}/\mathbb{Q}$ , we obtain a clopen subset of  $S_{\mathbb{Q}}^1$ . Moreover, we choose  $A_0$  so that  $A_0 \neq S_{\mathbb{Q}}^1$ . Define  $J_0 = \{0\}$ .

Suppose that we have already defined  $A_j$  and  $t_j$  for  $j \in J_N$ , with  $N + 1$  denoting the cardinality of  $J_N$ . We actually suppose that the following stronger hypotheses are satisfied:

- each  $A_j$  is a clopen, we have  $t_j < \frac{1}{j+1}$ , and
- the family of sets  $A_j$ , with  $j \in J_N$ , does not cover  $S_{\mathbb{Q}}^1$  though it contains  $\{s_0, \dots, s_N\}$ .

Let  $n$  be the smallest integer such that  $s_n$  does not belong to the union  $U_N$  of the sets  $A_j$  for  $j \in J_N$ . We have  $n > N$  and we define  $J_{N+1} = J_N \cup \{n\}$ . We can find a clopen  $A_n$  which contains  $s_n$  and a number  $t_n < \frac{1}{n+1}$  so that  $A_n \times [0, t_n[$  is included in  $O$ . The union  $U_N$  is closed because so is each  $A_j$ , so we can suppose that  $A_n$  does not meet  $U_N$ . Of course we can also restrict  $A_n$  so that  $U_N \cup A_n \neq S_{\mathbb{Q}}^1$ .  $\square$

We equip the sets  $O_h$  with orders  $\leq_h$  so that they become the elements of the expected ordered basis containing 0.

By definition, we have  $(s, t) \leq_h (s', t')$  when  $t = t'$  and there exists a proper open arc  $\alpha$  such that  $s \leq_{\alpha} s'$  and  $(\alpha \cap S_{\mathbb{Q}}^1) \times \{t\} \subset O_h$  (and of course  $0 \leq_h 0$ ).

Let  $h, h' \in \mathcal{H}$ . Since  $O_{\min(h, h')}$  is the intersection of  $O_h$  and  $O_{h'}$ , the order  $\leq_{\min(h, h')}$  matches the restrictions of both  $\leq_h$  and  $\leq_{h'}$ .

The collection of ordered sets  $\alpha \times ]a, b[$  (with  $s \in \alpha$  proper open arc and  $0 < a < b$ ) and  $O_h$  (with  $h \in \mathcal{H}$ ) thus forms a (strict) ordered basis. We denote by  $W$  the corresponding locally order space on  $\{0\} \sqcup S_{\mathbb{Q}}^1 \times ]0, 1]$ .

**Proposition 4.4.3.** The quotient map  $q$  induces the coequalizer of  $i_0$  and  $c_0$ .

<sup>8</sup>i.e.  $a < a'$  holds for all  $a \in A$  and  $a' \in A'$ .

*Proof.* One easily checks that the map  $q$  is locally increasing. Let  $f : \overrightarrow{S_{\mathbb{Q}}^1} \times I \rightarrow X$  be a locally increasing map such that  $f \circ i_0 = f \circ c_0$ . The underlying topology of  $W$  is the final one so we have a unique continuous map  $g$  from the underlying space of  $W$  to that of  $X$  such that  $f = g \circ q$ . The only point of  $W$  around which  $g$  is not trivially increasing is 0. This latter case has to be treated carefully. Let  $U \in \overrightarrow{\mathcal{O}}(X)$  such that  $g(0) \in U$ . Let  $(s_n)_{n \in \mathbb{N}}$  be an enumeration of  $S_{\mathbb{Q}}^1$ . We construct by induction a family  $(\alpha_j, t_j)_{j \in J}$  (with  $J \subseteq \mathbb{N}$ ) such that

- for all  $j \in J$  :
  - $\alpha_j$  is a proper open arc of the form  $\widehat{a_j b_j}$  with  $a_j, b_j \in \mathbb{R}/\mathbb{Q}$ .
  - the number  $t_j$  belongs to  $]0, 1]$ ,
  - $f((\alpha \cap S_{\mathbb{Q}}^1) \times [0, t_j]) \subset U$ , and
  - one has  $f(s, t) \leq_U f(s', t)$  when  $s \leq_{\alpha_j} s'$  for  $s, s' \in \alpha_j \cap S_{\mathbb{Q}}^1$  and  $t \in [0, t_j[$
- the sets  $\alpha_j \cap S_{\mathbb{Q}}^1$  form a partition of  $S_{\mathbb{Q}}^1$ .

Since  $f$  is locally increasing at  $(s_0, 0)$ , there exists an open neighbourhood of  $(s_0, 0)$  of the form  $A_0 \times [0, t_0[$  such that the restriction of  $f$  to  $A_0 \times [0, t_0[$  with values in  $U$  is increasing. Taking  $A_0$  to be the trace of a proper open arc  $\alpha_0 = \widehat{a_0 b_0}$  with  $a_0$  and  $b_0$  in  $\mathbb{R}/\mathbb{Q}$ , we obtain a clopen subset of  $S_{\mathbb{Q}}^1$ . Define  $J_0 = \{0\}$ . Suppose that we have already defined  $\alpha_j$  and  $t_j$  for  $j \in J_N$ , with  $N + 1$  denoting the cardinality of  $J_N$ . If the union  $U_N$  of the sets  $\alpha_j \cap S_{\mathbb{Q}}^1$  for  $j \in J_N$  is  $S_{\mathbb{Q}}^1$ , then  $J := J_n$  and the construction is over. Otherwise, let  $n$  be the smallest integer such that  $s_n$  does not belong to  $U_N$ . We have  $n > N$  and we define  $J_{N+1} = J_N \cup \{n\}$ . We can find a clopen  $A_n$  which contains  $s_n$  and a number  $0 < t_n \leq 1$  so that the restriction of  $f$  to  $A_n \times [0, t_n[$  with values in  $U$  is increasing.

The union  $U_N$  is closed because so is each  $\alpha_j \cap S_{\mathbb{Q}}^1$ , so we can suppose that  $A_n$  does not meet  $U_N$ . Finally, we can assume that  $A_n$  is the trace of a proper open arc  $\alpha_n = \widehat{a_n b_n}$  with  $a_n$  and  $b_n$  in  $\mathbb{R}/\mathbb{Q}$ .

For each  $j \in J$ , consider a continuous map  $\phi_j : \alpha_j \rightarrow [0, t_j]$  that is strictly positive, and tends to 0 on  $a_j$  and  $b_j$  (basically a *bump function* would be more than enough). Then let  $h$  be the map whose restriction to  $\alpha_j$  is

$$x \mapsto \phi_j(x) \quad \text{with} \quad x \in \alpha_j$$

One readily deduces from the definition of  $h$  that it belongs to  $\mathcal{H}$  and satisfies  $g(O_h) \subset U$ . We now check that the restriction of  $g : W \rightarrow X$  to  $O_h$  is order-preserving from  $\leq_h$  to  $\leq_U$ . Suppose that we have  $(s, t) \leq_h (s', t)$  in  $O_h$ . By definition of the order  $\leq_h$  there exists an open proper arc  $\alpha$  such that  $s \leq_{\alpha} s'$  and  $(\alpha \cap S_{\mathbb{Q}}^1) \times \{t\} \subseteq O_h$ . Let  $j \in J$  be such that  $s \in \alpha_j$ . Recall that  $\alpha$  and  $\alpha_j$  are the images of  $]a, b[$  and  $]a_j, b_j[$  under the complex exponential map  $t \in \mathbb{R} \mapsto e^{it} \in S^1$  with  $b - a < 2\pi$ ,  $b_j - a_j < 2\pi$ . Moreover, once  $a$  and  $b$  are fixed, one can choose  $a_j$  and  $b_j$  such that  $s = e^{ix}$  for some  $x \in ]a, b[ \cap ]a_j, b_j[$ . We also have  $x'$  in  $]a, b[$  such that  $x \leq x'$  and  $e^{ix'} = s'$ . We cannot have  $b_j < x'$  otherwise the interval  $]x, b_j[$  would be included in  $]a, b[$  so we would have  $x'' \in ]a, b[ \cap \mathbb{Q}$  with  $h(e^{ix''})$  arbitrarily small. In particular  $(e^{ix''}, t)$  would not belong to  $O_h$ . Moreover we have  $b_j \neq x'$  because one is rational while the other is not. Since the standard order on  $\mathbb{R}$  is total, we have  $x' < b_j$ . Hence both  $x$  and  $x'$  belong to  $]a_j, b_j[$ , and we have  $x \leq x'$  so  $s \leq_{\alpha_j} s'$ . Moreover  $t < h(s) \leq t_j$ . It follows that  $f(s, t) \leq_U f(s', t)$ .  $\square$

# Chapter 5

## $T$ -topological spaces

In this chapter we introduce all the basic notions that are used throughout the rest of the manuscript. Inspired by the commonalities between the definitions of the bases of topology and of the (strictly) ordered bases (4.1.1), we provide an unified framework: the *topological theories*. From such a theory  $T$ , we naturally derive the basic notions of *bases of  $T$ -topology*, of  *$T$ -topological spaces*, of  *$T$ -openness*, of  *$T$ -neighbourhoods*, and of (pointwise)  *$T$ -continuity*. With a well chosen theory, these notions match with their classical analogues or with the (strictly) locally ordered version developed in the previous chapter.

Two notions of morphism between topological theories arise: one that can be considered as ‘semantic’ since it only involves the  $T$ -topological spaces, and the other that can be considered as ‘syntactic’ since it directly lives at the level of the topological theories. We describe the fundamental operator **Sem** that associates each morphism of the second kind with a well-behaved morphism of the first kind without modifying the domain and the codomain.

We finally provide another operator **Loc** that associates each morphism of the first kind with a morphism of the second kind. This operator will be a major source of instances of well-behaved topological theories which are more thoroughly study in later chapters.

### 5.1 Topological theories

**Definition 5.1.1** (Lax functors). A *lax functor*<sup>1</sup> from **Set** to the ordered category **Mod** is an application  $T$ , which associates each set  $X$  with a preordered set  $(T(X), \subset_{T(X)})$  and each function  $f : X \rightarrow Y$  with a module  $T(f) : (T(X), \subset_{T(X)}) \rightarrow (T(Y), \subset_{T(Y)})$ , such that:

- for every set  $X$ ,  $(\subset_{T(X)}) \leq T(\text{Id}_X)$ , and
- for every functions  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$ ,  $T(g) \circ T(f) \leq T(g \circ f)$ .

Any functor from **Set** to **Mod** is in particular a lax functor.

For any lax functor  $T$  and any set  $X$ , we identify  $T(X)$  and its underlying set, and we denote by  $\subset_{T(X)}$  (or by  $\subset_T$  if there is no risk of confusion) its preorder, to emphasize the fact that  $(T(X), \subset_T)$  is thought as an abstraction of the ordered set  $(\mathcal{P}(X), \subset)$ . However we avoid using the symbols  $\cup$  and  $\cap$  to denote, when they exist, joins and meets in  $T(X)$  since in some examples the members of  $T(X)$  have underlying sets which are not preserved by joins or meets.

**Definition 5.1.2** (Topological theories). A *topological theory* is a pair  $(T, \in_T)$  where  $T$  is a lax functor from **Set** to **Mod** and  $\in_T = (\in_T^X : (X, =) \rightarrow T(X))_{X \in \mathbf{Set}}$  is a family of modules such that, for every function  $f : X \rightarrow Y$ , for every  $x \in X$ , and for every  $B \in T(Y)$ ,

$$\text{if } \exists A \in T(X) \text{ such that } x \in_T^X A \text{ and } A T(f) B, \text{ then } f(x) \in_T^Y B \quad (5.1)$$

<sup>1</sup>It is a particular instance of [Bor94a, 7.5.1] considering **Set** and **Mod** as particular 2-categories. See also [HST14, II.4.6].

*Remark 5.1.1.* The previous condition is equivalent to the inequality relation in the following diagram in the ordered category  $\mathbf{Mod}$ , where the sets  $X$  and  $Y$  are identified with the preordered sets  $(X, =)$  and  $(Y, =)$ , and the function  $f$  with the module  $f_*$  :

$$\begin{array}{ccc}
 & T(X) & \\
 \in_T^X \nearrow & & \searrow T(f) \\
 X & \leq & T(Y) \\
 \searrow f & & \nearrow \in_T^Y
 \end{array} \tag{5.2}$$

*Remark 5.1.2.* We often identify the topological theory  $(T, \in_T)$  with the lax functor  $T$  and  $\in_T^X$  with  $\in_T$  if there is no risk of confusion.

**Definition 5.1.3** (Topological theories with strong membership). A *topological theory with strong membership* is a topological theory  $(T, \in_T)$  such that, for every sets  $X$  and  $Y$ , every function  $f : X \rightarrow Y$ , every  $x \in X$ , and every  $B \in T(Y)$ , the converse implication of (5.1) is true. In other words, the diagram in the above remark is commutative.

*Remark 5.1.3.* The family  $\in_T$  is in fact just a lax transformation in the sense of [HST14, II.4.6] or of [Bor94a, 7.5.2] from the canonical functor  $\mathbf{Eq}$  introduced at the very end of Section 3.1 to the lax functor  $T$ . The theory is with strong membership precisely when  $\in_T$  is a natural transformation.

**Definition 5.1.4** (Topological theories with representable membership). A *topological theory with representable membership* is a topological theory  $(T, \in_T)$  such that, for every set  $X$ , there is a map  $\sigma_X : X \rightarrow T(X)$  such that  $\in_T^X = (\sigma_X)_*$  (see 3.1.2); the map  $\sigma_X$  is then unique modulo equivalence.

In a topological theory with representable membership, the assumption (5.1) is equivalent to

$$\text{if } \sigma_X(x) T(f) B \text{ then } f(x) \in_T^Y B.$$

### Examples 5.1.1.

- Let  $\mathbb{V}$  be a quantale (see Appendix A). For every set  $X$ , denote by  $T_{\mathbb{V}}(X)$  the set of all  $\mathbb{V}$ -preordered sets (Definition A.0.7) whose underlying set is  $X$ . Endow  $T_{\mathbb{V}}(X)$  with equality to obtain an ordered set. We define  $\in_T^X$  as the classical set membership relation, i.e. it is the relation which is always true in our case. Let  $f : X \rightarrow Y$  be a function, let  $(X, R) \in T_{\mathbb{V}}(X)$  and let  $(Y, S) \in T_{\mathbb{V}}(Y)$ , define the relation  $T_{\mathbb{V}}(f)$  by  $(X, R) T_{\mathbb{V}}(f) (Y, S)$  when  $f$  is a  $\mathbb{V}$ -increasing map (Definition A.0.8) from  $(X, R)$  to  $(Y, S)$ . One readily checks that one gets a topological theory. The membership relation is strong and is representable only if  $\mathbb{V} = \mathbb{1}$ ; in that case, for every set  $X$ ,  $T(X)$  is a singleton set, and, for every function  $f$ ,  $T(f)$  is the relation which is always true.
- We also consider a variant  $T_{\mathbb{V}}^l$  of the previous topological theory, where, for every set  $X$ , the equality on  $T_{\mathbb{V}}(X)$  is substituted by the  $\mathbb{V}$ -monotonicity of the identity map, and a variant  $T_{\mathbb{V}}^S$  where, for every set  $X$ ,  $T_{\mathbb{V}}^S(X)$  is the subset of  $\mathbb{V}$ -ordered sets.
- For every set  $X$ , endow the powerset  $\mathcal{P}(X)$  with the inclusion relation. For every function  $f : X \rightarrow Y$ , denote by  $\mathcal{P}(f) : \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$  the module defined, for all  $A \subset X$  and  $B \subset Y$ , by  $A \mathcal{P}(f) B$  when  $f(A) \subset B$ . The pair  $(\mathcal{P}, \in)$  is then a topological theory whose membership is strong and representable by  $x \mapsto \{x\}$ . We also denote by  $\mathbf{Loc}(T_{\mathbb{1}})$  this theory for a reason explained in Section 5.4.

- For every set  $X$ , denote by  $\mathcal{P}_{fin}(X)$  the set of all finite subsets of  $X$ , ordered by inclusion. For every function  $f : X \rightarrow Y$ , define the module  $\mathcal{P}_{fin}(f) : \mathcal{P}_{fin}(X) \rightarrow \mathcal{P}_{fin}(Y)$  as the restriction of  $\mathcal{P}(f)$ . The pair  $(\mathcal{P}_{fin}, \in)$  is then a topological theory with strong and representable membership.
- For every set  $X$ , denote by  $\mathcal{P}_{Rel}(X)$  the set of all subsets  $P$  of  $X$  equipped with a relation  $P \rightarrow P$ . The set  $\mathcal{P}_{Rel}(X)$  is ordered by  $(P, R) \subset_{\mathcal{P}_{Rel}(X)} (P', R')$  when  $P \subset P'$  and  $R \subset R'$ . For every function  $f : X \rightarrow Y$ , the module  $\mathcal{P}_{Rel}(f) : \mathcal{P}_{Rel}(X) \rightarrow \mathcal{P}_{Rel}(Y)$  is defined, for every  $(P, R) \in \mathcal{P}_{Rel}(X)$  and  $(Q, S) \in \mathcal{P}_{Rel}(Y)$ , by  $(P, R) \mathcal{P}_{Rel}(f) (Q, S)$  when  $f(P) \subset Q$  and when, for all  $x, x' \in P$ ,  $R(x, x')$  implies  $S(f(x), f(x'))$ . The pair  $(\mathcal{P}_{Rel}, \in)$  is then a topological theory whose membership relation is strong and is represented by  $x \mapsto (\{x\}, \emptyset)$ .
- Similarly denote by  $(\mathcal{P}_{rRel}, \in)$  the variant of the previous theory defined in the same way except that all the relations are assumed to be reflexive. The theory is with strong and representable membership.
- For every set  $X$ , define  $T_\emptyset(X) := \emptyset$ , and, for every function  $f$ , define  $T_\emptyset(f)$  as the sole possible relation. The pair  $(T_\emptyset, \in)$  is a topological theory with strong membership but not with representable membership.
- Let  $\mathcal{C}$  be a category and  $\mathbf{U} : \mathcal{C} \rightarrow \mathbf{Set}$  be a faithful functor whose fibres<sup>2</sup> are small. Let  $X$  be a set, define  $T_{\mathbf{U}}(X)$  as the fibre of  $\mathbf{U}$  on  $X$  and define  $\in_X^{T_{\mathbf{U}}}$  as the relation which is always true. Let  $f : X \rightarrow Y$  be a function, define a relation  $T_{\mathbf{U}}(f)$  by, for every  $A \in T_{\mathbf{U}}(X)$  and  $B \in T_{\mathbf{U}}(Y)$ ,  $A T_{\mathbf{U}}(f) B$  when there exists a (unique) morphism  $\bar{f} : A \rightarrow B$  such that  $\mathbf{U}(\bar{f}) = f$ . On  $T_{\mathbf{U}}(X)$ , we consider two preorders: the equality and  $T_{\mathbf{U}}(Id_X)$ . We easily verify that we get then two topological theories  $T_{\mathbf{U}}$  and  $T_{\mathbf{U}}^l$ . We recognize the first two examples by taking for  $\mathbf{U}$  the forgetful functor  $\mathbf{U}_{\mathbb{V}} : \mathbb{V}\text{-Ord} \rightarrow \mathbf{Set}$ .

**Definition 5.1.5.** Let  $T = (T, \in_T)$  be a topological theory and let  $X$  be a set. Given  $x \in X$  and  $A \in T(X)$ , when  $x \in_T A$ , we say that  $x$  is a  $T$ -element of  $A$ . On  $T(X)$ , we define an equivalent relation "having the same  $T$ -elements"

$$A \simeq_T A' \text{ when, for all } x \in X, (x \in_T A \Leftrightarrow x \in_T A')$$

Conversely, given  $x, x' \in X$ , we say that  $x$  and  $x'$  are  $\in_T$ -indistinguishable when, for all  $A \in T(X)$ ,  $x \in_T A \Leftrightarrow x' \in_T A$ .

If  $T$  is with a representable membership,  $x$  and  $x'$  are  $\in_T$ -indistinguishable if and only if  $\sigma(x)$  and  $\sigma(x')$  are equivalent in  $T(X)$  for the preorder  $\subset_T$ .

## 5.2 Basic definitions and properties

Fix a topological theory  $T = (T, \in_T)$ .

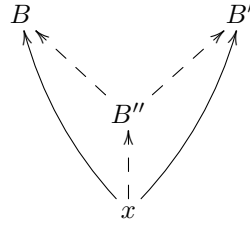
In order to avoid confusion, from now on, we will call "member" an element of a  $T(X)$ , "point" an element of a  $T$ -topological space once they have been defined, and reserve the term "element" for the element of any sets.

**Definition 5.2.1** ( $T$ -topologies). Let  $X$  be a set. A *basis of  $T$ -topology* on  $X$  is a subset  $\mathcal{B}$  of  $T(X)$  such that, for every element  $x \in X$  :

- there is  $B \in \mathcal{B}$  such that  $x \in_T B$ , and
- for all  $B, B' \in \mathcal{B}$  such that  $x \in_T B$  and  $x \in_T B'$ , there is  $B'' \in \mathcal{B}$  such that  $x \in_T B''$ ,  $B'' \subset_T B$ , and  $B'' \subset_T B'$ .

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<sup>2</sup>Recall that the fibre of a set  $X$  along  $\mathbf{U}$  is the collection of all objects  $A$  of  $\mathcal{C}$  satisfying  $\mathbf{U}(A) = X$ .



A member  $O \in T(X)$  is  $T$ -open for the basis  $\mathcal{B}$  when, for all  $x \in X$ , if  $x \in_T O$ , then there is  $B \in \mathcal{B}$  such that  $x \in_T B \subset_T O$ .

We denote by  $\mathcal{O}_T(\mathcal{B})$  the set of all members of  $T(X)$  which are  $T$ -open for  $\mathcal{B}$ . Two bases  $\mathcal{B}$  and  $\mathcal{B}'$  on  $X$  are equivalent when  $\mathcal{O}_T(\mathcal{B}) = \mathcal{O}_T(\mathcal{B}')$ .

A  $T$ -topology on  $X$  is given by an equivalent class of bases of  $T$ -topology on  $X$ . A  $T$ -topological space is a pair  $(X, \mathcal{E})$  with  $X$  a set and  $\mathcal{E}$  a  $T$ -topology on  $X$ .

As usual, when there is no risk of confusion, we identify a  $T$ -topological space and its underlying set. In particular, we denote by  $\mathcal{O}_T(X)$  the set of all members of  $T(X)$  which are  $T$ -open for any basis among the  $T$ -topology of a  $T$ -topological space  $X$ .

Let  $(X, \mathcal{E})$  be a  $T$ -topological space, and let  $\mathcal{B}$  be a basis of  $T$ -topology on  $X$ , we say that  $\mathcal{B}$  generates the  $T$ -topology of  $X$  when  $\mathcal{B} \in \mathcal{E}$ .

*Remark 5.2.1.* More concisely, let  $X$  be a set and let  $\mathcal{B} \subset T(X)$ . The set  $\mathcal{B}$  is a basis of  $T$ -topology on  $X$  if, for every  $x \in X$ , the preordered subset  $x/\mathcal{B} := \{B \in \mathcal{B} ; x \in_T B\}$  is cofiltered.

*Remark 5.2.2.* Let  $X$  be a set. A member  $A \in T(X)$  which has no  $T$ -element is  $T$ -open for any  $T$ -topology on  $X$ . In particular, on the empty set, there is only one  $T$ -topology. For the latter, all members of  $T(\emptyset)$  are  $T$ -open.

Since we assume that  $T(X)$  is a preordered set and that  $\in_T^X$  is a module, we can generalize classical results about bases of topology.

**Lemma 5.2.2.** Let  $\mathcal{B}$  be a basis of  $T$ -topology on a set  $X$ . One has  $\mathcal{B} \subset \mathcal{O}_T(\mathcal{B})$ .

*Proof.* Let  $B \in \mathcal{B}$  and  $x \in X$  such that  $x \in_T B$ . Since  $\subset_T$  is reflexive, one has  $x \in_T B \subset_T B$  therefore  $B$  is  $T$ -open.  $\square$

**Lemma 5.2.3.** Let  $\mathcal{B}_1$  and  $\mathcal{B}_2$  be two bases of  $T$ -topologies on a set  $X$ . If  $\mathcal{B}_1 \subset \mathcal{O}_T(\mathcal{B}_2)$  then  $\mathcal{O}_T(\mathcal{B}_1) \subset \mathcal{O}_T(\mathcal{B}_2)$ .

*Proof.* Assume that  $\mathcal{B}_1 \subset \mathcal{O}_T(\mathcal{B}_2)$ , and let  $O_1 \in \mathcal{O}_T(\mathcal{B}_1)$  and  $x \in X$  such that  $x \in_T O_1$ .

There is  $B_1 \in \mathcal{B}_1$  such that  $x \in_T B_1 \subset_T O_1$ . Since  $B_1$  is  $T$ -open for  $\mathcal{B}_2$ , there is  $B_2 \in \mathcal{B}_2$  such that  $x \in_T B_2 \subset_T B_1$ . We deduce that  $x \in_T B_2 \subset_T O_1$ , so that  $O_1$  is  $T$ -open for  $\mathcal{B}_2$ , and finally that  $\mathcal{O}_T(\mathcal{B}_1) \subset \mathcal{O}_T(\mathcal{B}_2)$ .  $\square$

**Corollary 5.2.4.** Let  $\mathcal{B}_1$  and  $\mathcal{B}_2$  be two bases of  $T$ -topologies on a set  $X$ . The bases  $\mathcal{B}_1$  and  $\mathcal{B}_2$  are equivalent if and only if  $\mathcal{B}_1 \subset \mathcal{O}_T(\mathcal{B}_2)$  and  $\mathcal{B}_2 \subset \mathcal{O}_T(\mathcal{B}_1)$ .

*Proof.* The direct implication comes from Lemma 5.2.2. One deduces the converse implication from the previous lemma.  $\square$

**Lemma 5.2.5.** Let  $\mathcal{B}$  be a basis of  $T$ -topology on a set  $X$ . Then  $\mathcal{O}_T(\mathcal{B})$  is a basis of  $T$ -topology and  $\mathcal{O}_T(\mathcal{O}_T(\mathcal{B})) = \mathcal{O}_T(\mathcal{B})$ .

*Proof.* Let  $x \in X$ . There is  $B \in \mathcal{B}$  such that  $x \in_T B$  because  $\mathcal{B}$  is a basis. By Lemma 5.2.2,  $B \in \mathcal{O}_T(\mathcal{B})$ . Let  $O, O' \in \mathcal{O}_T(\mathcal{B})$  such that  $x \in_T O, O'$ . Since  $O$  and  $O'$  are  $T$ -open for  $\mathcal{B}$ , there is  $B, B' \in \mathcal{B}$  such that  $x \in_T B \subset_T O$  and  $x \in_T B' \subset_T O'$ . Since  $\mathcal{B}$  is a basis, there is  $B'' \in \mathcal{B}$  such that  $x \in_T B'' \subset_T B$  et  $x \in_T B'' \subset_T B'$ . We deduce that  $x \in_T B'' \subset_T O$  and  $x \in_T B'' \subset_T O'$ . We conclude as before by Lemma 5.2.2. Thus  $\mathcal{O}_T(\mathcal{B})$  is a basis of  $T$ -topology and then, by applying the result at  $\mathcal{O}_T(\mathcal{B})$ ,  $\mathcal{O}_T(\mathcal{O}_T(\mathcal{B}))$  is also a basis.

It remains to show that  $\mathcal{O}_T(\mathcal{O}_T(\mathcal{B})) = \mathcal{O}_T(\mathcal{B})$ . By Lemma 5.2.2, one has  $\mathcal{O}_T(\mathcal{B}) \subset \mathcal{O}_T(\mathcal{O}_T(\mathcal{B}))$ . Conversely, by the previous lemma, one has  $\mathcal{O}_T(\mathcal{O}_T(\mathcal{B})) \subset \mathcal{O}_T(\mathcal{B})$ .  $\square$

**Corollary 5.2.6.** Let  $X$  be a  $T$ -topological space. The set  $\mathcal{O}_T(X)$  is the greatest basis for inclusion in the  $T$ -topology of  $X$ .

In other words, for every  $T$ -topological space, there is a canonical basis of  $T$ -topology that generates it.

**Examples 5.2.1.**

- Let  $\mathbb{V}$  be a quantale and let  $X$  be a set. The bases of  $T_{\mathbb{V}}$ -topology are the singleton subsets of  $T_{\mathbb{V}}(X)$  and two bases are equivalent if and only if they are equal. In other words, one can identify the  $T_{\mathbb{V}}$ -topologies on  $X$  and the  $\mathbb{V}$ -preordered sets whose underlying set is  $X$ . Similarly, one can identify the  $T_{\mathbb{V}}^S$ -topologies on  $X$  and the  $\mathbb{V}$ -ordered sets whose underlying set is  $X$ .
- More generally, let  $\mathbf{U} : \mathcal{C} \rightarrow \mathbf{Set}$  be a faithful functor whose fibres are small. Let  $X$  be a set. If  $X$  is empty, there is only one  $T_{\mathbf{U}}$ -topology on  $X$  and if  $X$  is not empty, one can identify the  $T_{\mathbf{U}}$ -topologies on  $X$  and the members of the fibre of  $X$  along  $\mathbf{U}$ .
- For the topological theory  $(\mathcal{P}, \in)$ , the notions of basis of  $(\mathcal{P}, \in)$ -topology, of  $(\mathcal{P}, \in)$ -openness and of  $(\mathcal{P}, \in)$ -topological spaces coincide with the classical notions of basis of topology, of open subset and of topological space.
- For the topological theory  $(\mathcal{P}_{fin}, \in)$ , the  $\mathcal{P}_{fin}$ -topological spaces are the topological spaces which have a basis of topology whose all members are finite subsets.
- For the topological theory  $(T_{\emptyset}, \in)$ , the only  $(T_{\emptyset}, \in)$ -topological space is the empty set with the unique  $T_{\emptyset}$ -topology on it.

### 5.2.1 $T$ -neighbourhoods

In this part, we introduce the  $T$ -neighbourhoods, we prove some of their basic properties, and we give a characterization of  $T$ -topological spaces via the  $T$ -neighbourhoods.

**Definition 5.2.7** ( $T$ -neighbourhoods). Let  $X$  be a  $T$ -topological space and let  $x \in X$ . A member  $V$  of  $T(X)$  is a  $T$ -neighbourhood of  $x$  in  $X$  when there is  $O \in \mathcal{O}_T(X)$  such that  $x \in_T O \subset_T V$ . The set of all  $T$ -neighbourhoods of  $x$  is denoted by  $\mathcal{V}_T^X(x)$ , or more simply by  $\mathcal{V}_T(x)$ .

**Lemma 5.2.8.** Let  $X$  be a  $T$ -topological space. A member  $O$  of  $T(X)$  is  $T$ -open if and only if, for every  $T$ -element  $x$  of  $O$ ,  $O$  is a  $T$ -neighbourhood of  $x$ .

*Proof.* The condition is clearly necessary by definition of the  $T$ -neighbourhoods and it is sufficient because the set  $\mathcal{O}_T(X)$  is a basis which generates the  $T$ -topology of  $X$ .  $\square$

**Proposition 5.2.9.** Let  $X$  be a  $T$ -topological space and let  $x \in X$ . The subset  $\mathcal{V}_T(x)$  is a filter of  $T(X)$  such that:

- $\mathcal{V}_T(x)$  is cofiltered,
- for all  $V \in \mathcal{V}_T(x)$ ,  $x \in_T V$ , and
- for all  $V \in \mathcal{V}_T(x)$ , there is  $W \in \mathcal{V}_T(x)$  such that  $W \subset_T V$  and such that, for every  $y \in X$  satisfying  $y \in_T W$ , one has  $W \in \mathcal{V}_T(y)$ .

*Proof.* Let  $(V_j)_{j \in J}$  a finite family of  $T$ -neighbourhoods of  $x$ . By definition of  $T$ -neighbourhoods, for all  $j \in J$ , there is  $O_j \in \mathcal{O}_T(X)$  such that  $x \in_T O_j \subset_T V_j$ . Since  $\mathcal{O}_T(X)$  is a basis of  $T$ -topology, there is  $O \in \mathcal{O}_T(X)$  such that, for every  $j \in J$ ,  $x \in_T O \subset_T O_j$ . Thus,  $O$  is a lower bound of  $(V_j)_{j \in J}$  and, by the previous lemma,  $O \in \mathcal{V}_T(x)$ . Therefore  $\mathcal{V}_T(x)$  is cofiltered.

Let  $V \in \mathcal{V}_T(x)$  and let  $A \in T(X)$  such that  $V \subset_T A$ . One readily deduces that  $A \in \mathcal{V}_T(x)$ , hence  $\mathcal{V}_T(x)$  is a filter because  $\mathcal{V}_T(x)$  is cofiltered.

For every  $V \in \mathcal{V}_T(x)$ , there is  $O \in \mathcal{O}_T(X)$  such that  $x \in_T O \subset_T V$ . Since  $\in_T$  is a module, we deduce that  $x \in_T V$ . From the previous lemma, we deduce that  $O \in \mathcal{V}_T(x)$  and that, for every  $y \in X$  satisfying  $y \in_T O$ , one has  $O \in \mathcal{V}_T(y)$ .  $\square$



Conversely, the  $T$ -topologies on a set  $X$  can be expressed via  $T$ -neighbourhoods:

**Proposition 5.2.10** ( *$T$ -topologies via  $T$ -neighbourhoods*). Let  $X$  be a set, and, for each  $x \in X$ , let  $\mathcal{F}(x)$  be a filter on  $T(X)$  satisfying the three assumptions of the preceding proposition. Then there exists a unique  $T$ -topology such that, for every  $x \in X$ , the filter  $\mathcal{F}(x)$  is the filter of  $T$ -neighbourhoods of  $x$ .

*Proof.*

- Unicity: if there is a  $T$ -topology on  $X$  such that, for every  $x \in X$ ,  $\mathcal{F}(x)$  is the filter of  $T$ -neighbourhoods of  $x$ . A member  $O$  of  $T(X)$  is  $T$ -open if and only if, for every  $x \in X$  such that  $x \in_T O$ ,  $O \in \mathcal{V}_T(x)$ . This is equivalent to, for every  $x \in X$  such that  $x \in_T O$ ,  $O \in \mathcal{F}(x)$ .
- Existence: write  $\mathcal{B} := \{A \in T(X); \forall x \in X, x \in_T A \Rightarrow A \in \mathcal{F}(x)\}$ .

Let  $x \in X$  and let  $(B_j)_{j \in J}$  be a finite family of  $\mathcal{B}$  such that, for all  $j \in J$ ,  $x \in_T B_j$ . For each  $j \in J$ , one has a fortiori  $B_j \in \mathcal{F}(x)$ . Since  $\mathcal{F}(x)$  is cofiltered, there is  $A \in \mathcal{F}(x)$  such that, for all  $j \in J$ ,  $A \subset_T B_j$ . By hypothesis, there exists  $B \in \mathcal{F}(x)$  such that  $B \subset_{T(X)} A$  and such that, for every  $y \in B$ ,  $y \in_T B$  implies  $B \in \mathcal{F}(y)$ . Furthermore, from  $B \in \mathcal{F}(x)$ , we deduce  $x \in_T B$ . Hence  $B \in \mathcal{B}$  is a lower bound of the  $B_j$  whose  $x$  is a  $T$ -element. Consequently,  $\mathcal{B}$  is a basis of  $T$ -topology on  $X$ .

One easily checks from the definition of  $\mathcal{B}$ , from the last hypothesis on the  $\mathcal{F}(x)$ , and from there upward closedness, that they are the filters of  $T$ -neighbourhoods for  $T$ -topology defined by  $\mathcal{B}$ .

□

## 5.2.2 $T$ -openness properties

In this part, we prove some stability properties of  $T$ -openness and we give a characterisation of  $T$ -topological spaces via the  $T$ -open members when  $T$  and  $\in_T$  have sufficiently good properties.

**Proposition 5.2.11** (*Stability under finite meets of  $T$ -openness*). Let  $X$  be a  $T$ -topological space and let  $(O_j)_{j \in J}$  be a finite family of  $\mathcal{O}_T(X)$ . Every meet  $\bigwedge_{j \in J} O_j$  of  $(O_j)_{j \in J}$  in  $T(X)$  is  $T$ -open in  $X$ .

*Proof.* Let  $x \in X$  such that  $x \in_T \bigwedge_{j \in J} O_j$ . For each  $j \in J$ , one has  $\bigwedge_{j \in J} O_j \subset_T O_j$ , hence  $x \in_T O_j$  because  $\in_T$  is a module. Since  $\mathcal{O}_T(B)$  is a basis of  $X$  and since  $J$  is a finite set, there is  $O \in \mathcal{O}_T(X)$  such that  $x \in_T O$ , and, for all  $j \in J$ ,  $O \subset_T O_j$ . We finally deduce that  $x \in_T O \subset_T \bigwedge_{j \in J} O_j$ , thus  $\bigwedge_{j \in J} O_j$  is  $T$ -open in  $X$ . □

**Definition 5.2.12** (*Supercompact  $T$ -membership*). Let  $X$  be a set, the membership module  $\in_T^X$  is *supercompact* when, for every  $x \in X$ , for every family  $(A_i)_{i \in I}$ , if  $x$  is a  $T$ -element of every upper bound of  $(A_i)_{i \in I}$ , then there exists  $i \in I$  such that  $x \in_T^X A_i$ .

When  $T$  is a topological theory with representable membership (see 5.1.4), the module  $\in_T^X$  is supercompact when, for every  $x \in X$ ,  $\sigma(x)$  is supercompact (see 3.3.5).

**Proposition 5.2.13** (*Stability under joins of  $T$ -openness*). Let  $X$  be a  $T$ -topological space and let  $(O_i)_{i \in I}$  be a family of  $\mathcal{O}_T(X)$ . Assume that  $\in_X^T$  is supercompact. Every join  $\bigvee_{i \in I} O_i$  of  $(O_i)_{i \in I}$  in  $T(X)$  is  $T$ -open in  $X$ .

*Proof.* Let  $x \in X$  such that  $x \in_T \bigvee_{i \in I} O_i$ . Since  $\in_T$  is supercompact, there is  $O_i$  such that  $x \in_T O_i$ . Thus, one has  $x \in_T O_i \subset_{T(X)} \bigvee_{i \in I} O_i$  hence  $\bigvee_{i \in I} O_i$  is  $T$ -open. □

*Remark 5.2.3.* In the previous proposition, it is enough to assume that the set of all  $T$ -elements of  $\bigvee_{i \in I} O_i$  is equal to the union of the sets of all  $T$ -elements of the  $O_i$  to conclude, even if  $\in_X^T$  is not supercompact.

**Proposition 5.2.14** (Upward stability for members with the same  $T$ -elements). Let  $X$  be a  $T$ -topological space,  $O \in \mathcal{O}_T(X)$  and  $A \in T(X)$ . If  $O \subset_T A$  and if  $O \simeq_T A$ , then  $A$  is  $T$ -open.

*Proof.* Let  $x \in X$  such that  $x \in_T A$ . Since  $O \simeq_T A$ , one has  $x \in_T O$ . Hence  $x \in_T O \subset_T A$  and thus we conclude that  $A$  is  $T$ -open because  $\mathcal{O}_T(X)$  is a basis which generate the  $T$ -topology of  $X$ .  $\square$

Conversely, if  $T(X)$  is complete and if the module  $\in_T^X$  preserves finite meets in the sense of the following definition, the three stability properties above are sufficient to characterize the  $T$ -open members of any  $T$ -topology.

**Definition 5.2.15.** Let  $X$  be a set. The module  $\in_T^X$  *preserves (finite) meets* when, for every  $x \in X$ , for every (finite) family  $(A_i)_{i \in I}$  of  $T(X)$  which has a meet,  $x \in_T^X \bigwedge_{i \in I} A_i$  if and only if, for all  $i \in I$ ,  $x \in_T^X A_i$ .

When  $\in_T^X$  is representable,  $\in_T^X$  preserves meets.

**Proposition 5.2.16** (Definition of  $T$ -topologies via  $T$ -openness). Let  $X$  be a set. Assume that  $T(X)$  is complete and that  $\in_T^X$  preserves finite meets. Every subset  $\mathcal{U}$  of  $T(X)$  which satisfies the three preceding stability properties is a basis of  $T$ -topology and  $\mathcal{U} = \mathcal{O}_T(\mathcal{U})$ .

*Proof.* Let  $x \in X$  and let  $(U_j)_{j \in J}$  be a finite family of  $\mathcal{U}$  such that  $x \in_T U_j$  for all  $j \in J$ . Since  $\mathcal{U}$  is stable under finite meets and since  $\in_T^X$  preserves the finite meets, the meet  $\bigwedge_{j \in J} U_j$  belongs to  $\mathcal{U}$ ,  $x \in_T \bigwedge_{j \in J} U_j$ , and, for every  $j \in J$ ,  $\bigwedge_{j \in J} U_j \subset_T U_j$ . Thus  $\mathcal{U}$  is a basis of  $T$ -topology, and moreover, by Lemma 5.2.2, one has the inclusion  $\mathcal{U} \subset \mathcal{O}_T(\mathcal{U})$ .

Conversely, let  $O \in \mathcal{O}_T(\mathcal{U})$ . Consider

$$U_O := \bigvee \{ U \subset_T O ; U \in \mathcal{U} \} .$$

Then one has  $U_O \in \mathcal{U}$  by stability of  $\mathcal{U}$  under joins, and  $U_O \subset_T O$ . It remains to show that  $U_O \simeq_T O$  in order to be able to conclude by upward stability of  $\mathcal{U}$  for  $\simeq_T$ -equivalent members of  $T(X)$ .

Let  $x \in X$ . If  $x \in_T U_O$  then  $x \in_T O$  because  $U_O \subset_T O$ . If  $x \in_T O$ , since  $O$  is  $T$ -open for the basis  $\mathcal{U}$ , there is  $U \in \mathcal{U}$  such that  $x \in_T U \subset_T O$ . By definition of  $U_O$ , one has  $U \subset_T(X) U_O$  and finally  $x \in_T U_O$ .  $\square$

**Proposition 5.2.17.** Let  $X$  be a set such that  $T(X)$  is complete and  $\in_T^X$  preserves meets. Then  $\in_T^X$  is representable.

*Proof.* Let  $x \in X$ . Define  $\sigma(x) = \bigwedge \{ A \in T(X) ; x \in_T A \}$ . By definition of  $\sigma(x)$ , for every  $A \in T(X)$ , if  $x \in_T A$  then  $\sigma(x) \subset_T A$ . Since  $\in_T^X$  preserves meets, one has  $x \in_T^X \sigma(x)$ . Hence, for every  $A \in T(X)$ , if  $\sigma(x) \subset_T A$  then  $x \in_T A$ .  $\square$

### 5.2.3 $T$ -continuous maps

**Definition 5.2.18** ( $T$ -continuity). Let  $X$  and  $Y$  be two  $T$ -topological spaces. A function  $f : X \rightarrow Y$  is  $T$ -continuous at  $x \in X$  when, for every  $V_Y \in \mathcal{V}_T^Y(f(x))$ , there exists  $V_X \in \mathcal{V}_T^X(x)$  such that  $V_X T(f) V_Y$ . The function  $f$  is  $T$ -continuous when  $f$  is  $T$ -continuous at every  $x \in X$ .

**Proposition 5.2.19** ( $T$ -continuity via bases). Let  $X$  and  $Y$  be two  $T$ -topological spaces and let  $\mathcal{B}_X$  and  $\mathcal{B}_Y$  be bases belonging to the topologies of  $X$  and  $Y$  respectively. A function  $f : X \rightarrow Y$  is  $T$ -continuous at  $x \in X$  if and only if, for every  $B_Y \in \mathcal{B}_Y$  such that  $f(x) \in_T^Y B_Y$ , there exists  $B_X \in \mathcal{B}_X$  such that  $x \in_T^X B_X$  and  $B_X T(f) B_Y$ .

*Proof.* Assume that  $f$  is  $T$ -continuous at  $x$ . Let  $B_Y \in \mathcal{B}_Y$  such that  $f(x) \in_T^Y B_Y$ . By Lemma 5.2.2,  $B_Y$  is  $T$ -open in  $Y$  and a fortiori is a  $T$ -neighbourhood of  $f(x)$ . Thus, by  $T$ -continuity of  $f$  at  $x$ , there is a  $T$ -neighbourhood  $V_X$  of  $x$  such that  $V_X T(f) B_Y$ . Since  $V_X$  is a  $T$ -neighbourhood of  $x$ , there is  $O_X \in \mathcal{O}_T(X)$  such that  $x \in_T^X O_X \subset_T V_X$ . Since  $\mathcal{B}_X$  is a basis of  $X$ , there is  $B_X \in \mathcal{B}_X$  such that  $x \in_T^X B_X \subset_T O_X$ . Hence one has  $x \in_T^X B_X \subset_T V_X$ . Finally, since  $T(f)$  is a module, given that one has  $B_X \subset_T V_X$  and  $V_X T(f) B_Y$ , we obtain  $B_X T(f) B_Y$ .

Conversely, assume that, for every  $B_Y \in \mathcal{B}_Y$  such that  $f(x) \in_T^Y B_Y$ , there is  $B_X \in \mathcal{B}_X$  such that  $x \in_T^X B_X$  and  $B_X T(f) B_Y$ . Let  $V_Y$  be a  $T$ -neighbourhood of  $f(x)$ . There is  $O_Y \in \mathcal{O}_T(Y)$  such that  $f(x) \in_T^Y O_Y \subset_T V_Y$ . Since  $\mathcal{B}_Y$  is a basis of  $Y$ , there is  $B_Y \in \mathcal{B}_Y$  such that  $f(x) \in_T^Y B_Y \subset_T O_Y$ . Then one has  $f(x) \in_T^Y B_Y \subset_T V_Y$ . By hypothesis, there is  $B_X \in \mathcal{B}_X$  such that  $x \in_T^X B_X$  and  $B_X T(f) B_Y$ . Since  $T(f)$  is a module, given that one has  $B_X T(f) B_Y$  and  $B_Y \subset_T V_Y$ , we obtain  $B_X T(f) V_Y$ . We can conclude because  $B_X$ , as a member of a basis of  $T$ -topology of  $X$ , is  $T$ -open and  $x$  is a  $T$ -element of it (by Lemma 5.2.2) hence is a  $T$ -neighbourhood of  $x$ .  $\square$

Since the set of all  $T$ -open members is a basis, we also obtain, thanks to the previous proposition, a characterisation of the  $T$ -continuity in terms of  $T$ -openness.

**Definition 5.2.20** (Inverse image). Let  $X$  and  $Y$  be sets and let  $f : X \rightarrow Y$  be a function. A map  $f^{-1} : T(Y) \rightarrow T(X)$  is an *inverse image* of  $T(f)$  if the module  $T(f)$  is corepresented by  $f^{-1}$  (see 3.1.2) and if, for every  $x \in X$  and every  $B \in T(Y)$ ,

$$x \in_T f^{-1}(B) \quad \text{if and only if} \quad f(x) \in_T B \quad .$$

When  $T$  is a topological theory with strong membership (5.1.3), the last condition is redundant:

**Lemma 5.2.21.** Assume that  $T$  is a topological theory with strong membership. Let  $X$  and  $Y$  be two sets and let  $f : X \rightarrow Y$  be a function. If the module  $T(f)$  is corepresented by  $f^{-1} : T(Y) \rightarrow T(X)$  then  $f^{-1}$  is an inverse image of  $T(f)$ .

*Proof.* Let  $x \in X$  and let  $B \in T(Y)$ . Assume that  $x \in_T^X f^{-1}(B)$ . Since  $f^{-1}(B) T(f) B$ , we deduce, by hypothesis on  $\in_T$ , that  $f(x) \in_T^Y B$ . Conversely, assume that  $f(x) \in_T^Y B$ . Since  $T$  is with strong membership, there is  $A \in T(X)$  such that  $x \in_T^X A$  and  $A T(f) B$ . Since  $f^{-1}$  corepresents  $T(f)$ , we deduce from  $A T(f) B$  that  $A \subset_T f^{-1}(B)$ . Then we deduce from  $x \in_T^X A$  that  $x \in_T^X f^{-1}(B)$ .  $\square$

Conversely

**Lemma 5.2.22.** Assume that, for every function  $f : X \rightarrow Y$ ,  $T(f)$  has an inverse image  $f^{-1} : T(Y) \rightarrow T(X)$ . Then  $T$  is a topological theory with strong membership.

*Proof.* Let  $f : X \rightarrow Y$  be a function, let  $x \in A$  and  $B \in T(Y)$  such that  $f(x) \in_T B$ . Then one has  $x \in_T f^{-1}(B)$  and  $f^{-1}(B) T(f) B$  because  $f^{-1}(B) \subset_T f^{-1}(B)$ .  $\square$

When  $T(f)$  has an inverse image,  $T$ -continuity can be characterised thanks to it:

**Proposition 5.2.23.** Let  $X$  and  $Y$  be  $T$ -topological spaces and let  $f : X \rightarrow Y$  be a function. Assume that  $T(f)$  has an inverse image  $f^{-1} : T(Y) \rightarrow T(X)$ . Let  $\mathcal{B}$  be a basis of that generates the  $T$ -topology of  $Y$ . Then  $f$  is  $T$ -continuous if and only if, for all  $B \in \mathcal{B}$ ,  $f^{-1}(B)$  is  $T$ -open in  $X$ .

*Proof.* Assume that  $f$  is  $T$ -continuous. Let  $B \in \mathcal{B}$  and let  $x \in X$  such that  $x \in_T^X f^{-1}(B)$ . Then one has  $f(x) \in_T^Y B$ . By  $T$ -continuity of  $f$  at  $x$  and by the previous proposition, there is  $O \in \mathcal{O}_T(X)$  such that  $x \in_T^X O$  and  $O T(f) B$ . The latter assertion implies  $O \subset_T f^{-1}(B)$ .

Conversely, let  $x \in X$  and let  $B \in \mathcal{B}$  such that  $f(x) \in_T^Y B$ . Then  $f^{-1}(B)$  is  $T$ -open in  $X$ ,  $x \in_T^X f^{-1}(B)$  and  $f^{-1}(B) T(f) B$ . Finally we can conclude by the previous proposition.  $\square$

$T$ -continuity is stable under composition:

**Proposition 5.2.24.** Let  $X, Y$  and  $Z$  be  $T$ -topological spaces, and let  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  be two functions. If  $f$  is  $T$ -continuous at some  $x \in X$  and if  $g$  is  $T$ -continuous at  $f(x)$  then the composite  $g \circ f$  is  $T$ -continuous at  $x$ .

*Proof.* Let  $V_Z$  be a  $T$ -neighbourhood of  $g(f(x))$  in  $Z$ . By  $T$ -continuity of  $g$ , there is a  $T$ -neighbourhood  $V_Y$  of  $f(x)$  such that  $V_Y T(f) V_Z$ . By  $T$ -continuity of  $f$ , there is a  $T$ -neighbourhood  $V_X$  of  $x$  such that  $V_X T(f) V_Y$ . Then one has  $V_X (T(f) \circ T(f)) V_Z$ , hence  $V_X T(g \circ f) V_Z$  because  $T(g) \circ T(f) \subset T(g \circ f)$ .  $\square$

Moreover, the identity maps are  $T$ -continuous:

**Proposition 5.2.25.** Let  $X$  be a  $T$ -topological space. The map  $\text{Id}_X : X \rightarrow X$  is  $T$ -continuous.

*Proof.* Let  $x \in X$ . Let  $V$  be a  $T$ -neighbourhood of  $x$  in  $X$ . One has  $V \subset_{T(X)} V$ , hence  $VT(\text{Id}_X)V$  because  $(\subset_{T(X)}) \subset T(\text{Id}_X)$ .  $\square$

Hence one has a category  $\mathbf{Top}_T$  of  $T$ -topological spaces and  $T$ -continuous maps. The category  $\mathbf{Top}_T$  is a concrete category: one has a trivial forgetful functor  $\mathbf{U}_T$  from  $\mathbf{Top}_T$  in the category of sets  $\mathbf{Set}$  which consists in taking the underlying sets of  $T$ -topological spaces.

**Examples 5.2.2** (Continuation of Examples 5.1.1).

- Let  $\mathbb{V}$  be a quantale. The category  $\mathbf{Top}_{T_{\mathbb{V}}}$  is the category  $\mathbb{V}\text{-Ord}$  of  $\mathbb{V}$ -preordered sets and the forgetful functor  $\mathbf{U}_{T_{\mathbb{V}}}$  matches with the one given in Appendix A. In particular,  $\mathbf{Top}_{T_{\mathbb{1}}}$  is the category  $\mathbf{Set}$ ,  $\mathbf{Top}_{T_{\mathbb{B}}}$  is the category  $\mathbf{Ord}$  of preordered sets and increasing maps, and  $\mathbf{Top}_{T_{\mathbb{B}}^s}$  is the full subcategory  $\mathbf{SOrd}$  of ordered sets.
- More generally, let  $\mathbf{U} : \mathcal{C} \rightarrow \mathbf{Set}$  be a faithful functor whose fibres are small, whose fibre of the empty set  $\emptyset$  is a singleton set and whose unique member of this fibre is an initial object in the category  $\mathcal{C}$ . By identifying, for every set  $X$ , the  $T_{\mathbf{U}}$ -topological spaces on  $X$  and the members of the fibre of  $X$ , the  $T_{\mathbf{U}}$ -continuity of a function corresponds to the existence of a (unique) lifting along  $\mathbf{U}$ .
- The category  $\mathbf{Top}_{\mathcal{P}}$  is the category of standard topological spaces and continuous maps  $\mathbf{Top}$ .
- The category  $\mathbf{Top}_{T_{\emptyset}}$  has a unique object  $(\emptyset, \{\emptyset\})$  and a unique morphism  $\text{Id}_{\emptyset}$ .

## 5.3 Morphisms between topological theories

Two notions of morphisms between topological theories naturally arise. The first one derives from the idea that  $T$ -topological spaces are ‘models’ of a topological theory  $T$ .

Let  $T = (T, \in_T)$ ,  $T' = (T', \in_{T'})$ , and  $T'' = (T'', \in_{T''})$  be three topological theories.

**Definition 5.3.1** (Semantic transformations). A *semantic transformation* from  $T$  to  $T'$  is a functor from  $\mathbf{Top}_T$  to  $\mathbf{Top}_{T'}$  which commutes with the forgetful functors<sup>3</sup>. We denote by  $\mathbf{STopTh}$  the category of topological theories and semantic transformations; it comes along with a canonical functor from  $\mathbf{STopTh}$  to the slice-category  $\mathbf{Cat}/\mathbf{Set}$ .

The category  $\mathbf{STopTh}$  is a preordered category with, for every pair of semantic transformations  $F, G : T \rightarrow T'$ ,  $F \leq G$  when, for every  $T$ -topological space  $(X, \mathcal{O}_T(X))$ , the map  $\text{Id}_X$  is  $T'$ -continuous from  $F(X, \mathcal{O}_T(X))$  to  $G(X, \mathcal{O}_T(X))$ . In other words,  $F \leq G$  when there is a natural transformation from  $F$  to  $G$  whose image under the forgetful functor  $\mathbf{U}_{T'}$  is the identity natural transformation  $U_T$  (since  $\mathbf{U}_{T'}$  is faithful, the natural transformation is unique when it exists).

**Example 5.3.1.** The functor that associates every set  $X$  with the topological space  $X$  endowed with the discrete topology and the one that associates every set  $X$  with the topological space  $X$  endowed with the coarsest topology are semantic transformations from  $T_{\mathbb{1}}$  to  $\mathcal{P}$ . In  $\mathbf{STopTh}$ , the former is lesser than the latter.

Let  $(T_i = (T_i, \in_{T_i}))_{i \in I}$  be a family of topological theories. For every set  $X$ , denote by  $(\prod_{i \in I} T_i)(X)$  the set  $\prod_{i \in I} T_i(X)$  equipped by the preorder  $\subset_{(\prod_{i \in I} T_i)(X)}$  defined by

$$(A_i)_{i \in I} \subset_{(\prod_{i \in I} T_i)(X)} (B_i)_{i \in I} \text{ when, for every } i \in I, A_i \subset_{T_i(X)} B_i,$$

<sup>3</sup>In other words, it is a concrete functor.

and denote by  $\in_{\prod_{i \in I} T_i}^X$  the module defined by

$$x \in_{\prod_{i \in I} T_i}^X (A_i)_{i \in I} \text{ when, for every } i \in I, x \in_{T_i}^X A_i .$$

For every function  $f : X \rightarrow Y$ , define the module  $(\prod_{i \in I} T_i)(f)$  by

$$(A_i)_{i \in I} \left( \prod_{i \in I} T_i \right) (f) (B_i)_{i \in I} \text{ when, for every } i \in I, A_i T_i(f) B_i .$$

One easily checks that  $\prod_{i \in I} T_i = (\prod_{i \in I} T_i, \in_{\prod_{i \in I} T_i})$  is a topological theory and that the pair  $(\mathbf{Top}_{\prod_{i \in I} T_i}, \mathbf{U}_{\prod_{i \in I} T_i})$  is the product in the slice-category  $\mathbf{Cat}/\mathbf{Set}$  of the family  $(\mathbf{Top}_{T_i}, \mathbf{U}_{T_i})$ . Therefore

**Proposition 5.3.2.** The category  $\mathbf{STopTh}$  has all products.

A particular kind of products that will be useful later (in 7.6) are the one of the form  $\mathcal{P} \times T$ . In particular, the category of  $(\mathcal{P} \times T_{\mathbb{B}}^S)$ -topological spaces matches with the category of ordered spaces (Definition 4.1.5) and continuous and increasing maps.

There is a second notion of morphism between topological theories that we shall consider. It is directly defined at the topological theories level, unlike semantic transformations that are defined using the spaces associated to the corresponding topological theories.

**Definition 5.3.3** (Changing of bases data). A *changing of bases datum* from  $T$  to  $T'$  is a family of increasing maps  $v = (v_X : T(X) \rightarrow T'(X))$  such that:

- For every sets  $X$  and  $Y$ , for every function  $f : X \rightarrow Y$ , for every  $A \in T(X)$ , and for every  $B \in T(Y)$ ,

$$\text{if } A T(f) B \text{ then } v(A) T'(f) v(B) .$$

- For every set  $X$ , for every  $x \in X$ , and for every  $A \in T(X)$ ,

$$x \in_T A \text{ if and only if } x \in_{T'} v(A) .$$

**Examples 5.3.1.**

- Let  $\mathbb{V}$  and  $\mathbb{W}$  be two quantales and let  $v : \mathbb{V} \rightarrow \mathbb{W}$  be a quantales morphism. By propositions A.0.12 and A.0.13, the map  $(X, R) \mapsto (X, v(R))$ , for all  $X$  and all  $(X, R) \in T_{\mathbb{V}}(X)$ , is a changing of bases datum, still denoted by  $v$ , from  $T_{\mathbb{V}}$  to  $T_{\mathbb{W}}$ , and from  $T_{\mathbb{V}}^l$  to  $T_{\mathbb{W}}^l$ .
- Let  $\mathbb{V}$  be a quantale. The identity maps allow us to define a changing of bases datum from  $T_{\mathbb{V}}$  to  $T_{\mathbb{V}}^l$ .
- More generally, let  $T = (T, \in_T)$  be a topological theory. We define a new lax functor  $T^l$  as follow:
  - For every set  $X$ ,  $T^l(X)$  has the same members as  $T(X)$  and the preorder  $\subset_{T^l}$  is defined by  $A \subset_{T^l} B$  when  $A T(\text{Id}_X) B$ .
  - For every function  $f : X \rightarrow Y$ ,  $T^l(f) := T(f)$ .

One easily checks that  $(T^l, \in_T)$  is a topological theory and that the identity maps allow us to define a changing of bases datum from  $(T, \in_T)$  to  $(T^l, \in_T)$ .

- For every set  $X$  and for every  $A \in T(X)$ , define the subset  $!_X^T(A)$  as  $\{x \in X ; x \in_T^X A\}$ . Obviously, for all  $x \in X$ ,  $x \in !_X^T(A)$  if and only if  $x \in_T A$ . The function  $!_X^T$  is an increasing map. For every function  $f : X \rightarrow Y$ , for every  $A \in T(X)$  and  $B \in T(Y)$ , if  $A T(f) B$ , then, for all  $x \in X$ ,  $x \in_T^X A$  implies  $f(x) \in_T^Y B$ ; hence  $!_X^T(A) \mathcal{P}(f) !_Y^T(B)$ . Thus  $!^T$  is a changing of bases from  $T$  to  $(\mathcal{P}, \in)$ .

- Since every finite set is a fortiori a set, one has an obvious changing of bases from  $(\mathcal{P}_{fin}, \in)$  to  $(\mathcal{P}, \in)$ .
- For every topological theory  $T$ , there is a unique changing of bases datum  $\emptyset_T : T_\emptyset \rightarrow T$ .

**Definition 5.3.4** (Composition of changing of bases data ). Let  $v : T \rightarrow T'$  and  $\nu : T' \rightarrow T''$  be two changing of bases data. The composite  $\nu \circ v$  is defined by  $(\nu_X \circ v_X)_X$ . One readily checks that it is a changing of bases datum from  $T$  to  $T''$ .

Moreover, let  $v$  and  $v'$  be two changing of bases data from  $T$  to  $T'$ . Define  $v \leq v'$  when, for every set  $X$  and every member  $A \in T(X)$ ,  $v(A) \subset_{T'} v'(A)$ .

**Lemma 5.3.5.** Let  $v, v' : T \rightarrow T'$  and  $\nu, \nu' : T' \rightarrow T''$  be changing of bases data . If  $v \leq v'$  and  $\nu \leq \nu'$  then  $\nu \circ v \leq \nu' \circ v'$ .

*Proof.* Let  $X$  be a set and let  $A \in T(X)$ . Since  $v \leq v'$ , one has  $v(A) \subset_{T'} v'(A)$ . Then  $\nu'(v(A)) \subset_{T''(X)} \nu'(v'(A))$  because  $\nu'_X$  is increasing. Since  $\nu \leq \nu'$ , one has  $\nu(v(A)) \subset_{T''(X)} \nu'(v(A))$ . Finally, one obtains  $\nu(v(A)) \subset_{T''(X)} \nu'(v'(A))$ .  $\square$

Thus we define

**Definition 5.3.6** (Preordered category of changing of bases data ). The preordered category of changing of bases data **TopTh** is the category whose objects are the topological theories and morphisms are the changing of bases data .

**Example 5.3.2.** The operator  $(\_ )^!$  is a 2-functor from the category **TopTh** into itself.

We construct now a 2-functor **Sem** : **TopTh**  $\rightarrow$  **STopTh** such that, for every topological theory  $T = (T, \in_T)$ , **Sem**( $T$ ) =  $T$ . Let  $v : T \rightarrow T'$  be a changing of bases datum .

**Lemma 5.3.7.** Let  $X$  be a set and let  $\mathcal{B}$  be a basis of  $T$ -topology on  $X$ . Then

$$v(\mathcal{B}) := \{v(B) ; B \in \mathcal{B}\}$$

is a basis of  $T'$ -topology on  $X$ .

*Proof.* Let  $x \in X$  and let  $(B_j)_{j \in J}$  be a finite family of  $\mathcal{B}$  such that, for all  $j \in J$ ,  $x \in_{T'} v(B_j)$ . Then one has, for every  $j \in J$ ,  $x \in_T B_j$ . Since  $\mathcal{B}$  is a basis, we deduce that there is  $B \in \mathcal{B}$  such that  $x \in_T B$  and such that, for all  $j \in J$ ,  $B \subset_T B_j$ . From the assumptions on  $v$ , we easily deduce that  $x \in_{T'} v(B)$  and that, for all  $j \in J$ ,  $v(B) \subset_{T'} v(B_j)$ .  $\square$

**Lemma 5.3.8.** Let  $X$  be a set, let  $\mathcal{B}$  be a basis of  $T$ -topology on  $X$  and let  $O \in \mathcal{O}_T(\mathcal{B})$ . Then  $v(O)$  is  $T'$ -open for  $v(\mathcal{B})$ .

*Proof.* Let  $x \in X$  such that  $x \in_{T'} v(O)$ . Then one has  $x \in_T O$  and, since  $O$  belongs to  $\mathcal{O}_T(\mathcal{B})$ , there is  $B \in \mathcal{B}$  such that  $x \in_T B \subset_T O$ . We deduce that  $x \in_{T'} v(B) \subset_{T'} v(O)$  and then that  $v(O)$  is  $T'$ -open for  $v(\mathcal{B})$ .  $\square$

**Corollary 5.3.9.** Let  $X$  be a set. The  $T'$ -topology generated by  $v(\mathcal{B})$  only depends on the topology generated by  $\mathcal{B}$ .

*Proof.* Clear consequence of the previous lemma and of Corollary 5.2.4.  $\square$

Let  $X$  be a  $T$ -topological space, we denote by **Sem**( $v$ )( $X$ ), or more simply by  $v(X)$  if there is no risk of confusion, the set  $X$  endowed with the  $T'$ -topology generated by  $v(\mathcal{B})$ , for any basis of  $T$ -topology  $\mathcal{B}$  that generated the  $T$ -topology of  $X$ .

**Corollary 5.3.10.** Let  $X$  be a  $T$ -topological space, let  $x \in X$  and let  $V \in T(X)$ . If  $V$  is a  $T$ -neighbourhood of  $x$  in  $X$  then  $v(V)$  is a  $T'$ -neighbourhood of  $x$  in  $v(X)$ .

The converse of the previous lemma and of the previous corollary hold when  $v_X$  is fully faithful.

**Proposition 5.3.11.** Let  $X$  be a  $T$ -topological space, let  $x \in X$  and let  $V \in T(X)$ . Assume that  $\nu_X$  is fully faithful. If  $\nu(V)$  is a  $T'$ -neighbourhood of  $x$  in  $\nu(X)$  then  $X$  is a  $T$ -neighbourhood of  $x$  in  $X$ .

*Proof.* If  $\nu(V)$  is a  $T'$ -neighbourhood of  $x$  in  $\nu(X)$  then there is  $O \in \mathcal{O}_T(X)$  such that  $x \in_{T'} \nu(O) \subset_{T'} \nu(V)$ . Thus we conclude that  $x \in_T O \subset_T V$  because  $\nu_X$  is fully faithful.  $\square$

Since a member  $O$  of  $T(X)$  is  $T$ -open if and only if it is a  $T$ -neighbourhood of each of its  $T$ -elements by Lemma 5.2.8, we deduce:

**Corollary 5.3.12.** Let  $X$  be a  $T$ -topological space and let  $O \in T(X)$ . Assume that  $\nu_X$  is fully faithful. If  $\nu(O)$  is  $T'$ -open in  $\nu(X)$ , then  $O$  is  $T$ -open in  $X$ .

Concerning  $T$ -continuity, we have:

**Proposition 5.3.13.** Let  $X$  and  $Y$  be two  $T$ -topological spaces and let  $f : X \rightarrow Y$  be a function. If  $f$  is  $T$ -continuous at  $x \in X$ , then  $f : \nu(X) \rightarrow \nu(Y)$  is  $T'$ -continuous at  $x$ .

*Proof.* Let  $W'$  be a  $T'$ -neighbourhood of  $f(x)$  in  $\nu(Y)$ . Then there is a  $T$ -neighbourhood  $W$  of  $f(x)$  in  $Y$  such that  $\nu(W) \subset_{T'} W'$ . Since the function  $f$  is  $T$ -continuous at  $x$ , there is a  $T$ -neighbourhood  $V$  of  $x$  in  $X$  such that  $V T(f) W$ . We deduce that  $\nu(V) T'(f) \nu(W)$ . Then  $\nu(V) T'(f) W'$  because  $\nu(W) \subset_{T'} W'$ . Finally, by Corollary 5.3.10,  $\nu(V)$  is a  $T$ -neighbourhood of  $x$  in  $\nu(X)$ .  $\square$

Thus we can set the following

**Definition 5.3.14.** The *changing of bases functor* associated to  $\nu$  is the semantic transformation (5.3.1)

$$\mathbf{Sem}(\nu) : (X \in \mathbf{Top}_T) \mapsto (\nu(X) \in \mathbf{Top}_{T'})$$

from the theory  $T$  to the theory  $T'$ .

The changing of bases functors are better behaved than mere semantic transformations in the sense that they often preserve (or reflect depending on the case) the  $T$ -topological properties as we will show in the following chapters.

**Proposition 5.3.15.** The map  $\mathbf{Sem}$  is a 2-functor from the preordered category  $\mathbf{TopTh}$  to the preordered category  $\mathbf{STopTh}$ .

*Proof.* Let  $\nu : T \rightarrow T'$  and  $\nu' : T' \rightarrow T''$  be two changing of bases data. Let  $X$  be a  $T$ -topological space. The set  $\{\nu(O) ; O \in \mathcal{O}_T(X)\}$  is a basis of  $T'$ -topology that generates the  $T'$ -topology of  $\nu(X)$ , and consequently  $\{\nu'(\nu(O)) ; O \in \mathcal{O}_T(X)\}$  is a basis of  $T''$ -topology that generates the  $T''$ -topology of  $\nu'(\nu(X))$ . However, since  $\nu'(\nu(O)) = (\nu' \circ \nu)(O)$ , for all  $O \in \mathcal{O}_T(X)$ , it is also a basis of  $T''$ -topology that generates the  $T''$ -topology of  $(\nu' \circ \nu)(X)$ .

Let  $\nu, \nu' : T \rightarrow T'$  be two changing of bases data such that  $\nu \leq \nu'$ . Let  $X$  be a  $T$ -topological space. We are to show that the identity map  $\text{Id}_X$  is  $T'$ -continuous from  $\nu(X)$  to  $\nu'(X)$ . Let  $x \in X$  and let  $W$  be a  $T'$ -neighbourhood of  $x$  in  $\nu'(X)$ . There exists a  $T$ -neighbourhood  $V$  of  $x$  in  $X$  such that  $\nu'(V) \subset_{T'} W$ . By Corollary 5.3.10,  $\nu(V)$  is a  $T'$ -neighbourhood of  $x$  in  $\nu(X)$ . Since  $\nu \leq \nu'$ , one has  $\nu(V) \subset_{T'} \nu'(V)$ , hence  $\nu(V) \subset_{T'} W$ . Finally, since  $(\subset_{T'} \nu(V)) \subset T'(\text{Id}_X)$ , we deduce that  $\nu(V) T'(\text{Id}_X) W$  and that  $\text{Id}_X$  is  $T'$ -continuous.  $\square$

Let  $\nu : T \rightarrow T'$  be a changing of bases datum. Like any semantic transformation, the change base functor  $\mathbf{Sem}(\nu)$  is faithful. Concerning its fullness, one has the following:

**Proposition 5.3.16.** Let  $\nu : T \rightarrow T'$  be a changing of bases datum. Let  $X$  and  $Y$  be two  $T$ -topological spaces and let  $f : X \rightarrow Y$  be a function. Assume that, for all  $A \in T(X)$  and all  $B \in T(Y)$ ,  $A T(f) B$  if and only if  $\nu(A) T'(f) \nu(B)$ . Then the function  $f$  is  $T$ -continuous at  $x \in X$  if and only if it is  $T'$ -continuous at  $x$ .

*Proof.* The implication corresponds to Proposition 5.3.13. Conversely, if  $f$  is  $T'$ -continuous at  $x$ . Let  $W$  be a  $T$ -neighbourhood of  $f(x)$  in  $Y$ . By Corollary 5.3.10,  $v(W)$  is a  $T'$ -neighbourhood of  $f(x)$  in  $v(Y)$ . Hence, by  $T'$ -continuity, there is a  $T'$ -neighbourhood  $V'$  of  $x$  in  $v(X)$  such that  $V' T'(f) v(W)$ . There is a  $T$ -neighbourhood  $V$  of  $x$  in  $X$  such that  $v(V) \subset_{T'} V'$ . Therefore, one has  $v(V) T'(f) v(W)$ , and finally, by hypothesis,  $V T(f) W$ .  $\square$

**Corollary 5.3.17.** Assume that, for every set  $X$  and  $Y$ , for every function  $f : X \rightarrow Y$ , for every  $A \in T(X)$ , for every  $B \in T(Y)$ , one has  $A T(f) B$  if and only if  $v(A) T'(f) v(B)$ . Then the changing of bases functor  $\mathbf{Sem}(v)$  is fully faithful.

The changing of bases datum  $!^T$  defined at the beginning of the section (5.3.1) plays a special role:

**Theorem 5.3.18.** *The theory  $(\mathcal{P}, \in)$  is a final object in the category  $\mathbf{TopTh}^4$ .*

*Proof.* Let  $T$  be a topological theory. Then  $!^T$  is a changing of bases datum from  $T$  to  $(\mathcal{P}, \in)$ . Conversely, let  $v : T \rightarrow (\mathcal{P}, \in)$  be another changing of bases datum. Let  $X$  be a set and let  $A \in T(X)$ . Given  $x \in X$ ,  $x \in v(A)$  is equivalent to  $x \in_T A$ , then to  $x \in !^T_X(A)$ . Hence  $v(A) = !^T_X(A)$ .  $\square$

Hence there is a canonical forgetful functor from  $\mathbf{Top}_T$  into  $\mathbf{Top}$ ; given a  $T$ -topological space  $X$ , the topology of  $!^T(X)$  is called the *underlying topology of  $X$* . The changing of bases datum  $!^T$  will allow us to compare the classical notions of topology and their generalization to  $T$ -topological spaces.

The changing of bases datum  $\rho : T' \rightarrow T$  is a right adjoint of  $\lambda : T \rightarrow T'$  in the preordered category  $\mathbf{TopTh}$  when, for every set  $X$ , for every  $A \in T(X)$ ,

$$A \subset_T \rho(\lambda(A)) ,$$

and, for every  $A' \in T'(X)$ ,

$$\lambda(\rho(A')) \subset_{T'} A' .$$

In other words,  $\rho$  is a right adjoint of  $\lambda$  in the preordered category  $\mathbf{TopTh}$  if and only if, for every set  $X$ , the increasing map  $\rho_X$  is a right adjoint of  $\lambda_X$ .

Similarly,

**Proposition 5.3.19.** Let  $\rho : T' \rightarrow T$  and  $\lambda : T \rightarrow T'$  be two changing of bases data such that, for every set  $X$  and  $Y$ , for every  $A \in T(X)$ , for every  $B' \in T'(Y)$ , for every function  $f : X \rightarrow Y$ ,

$$\lambda(A) T'(f) B' \quad \text{if and only if} \quad A T(f) \rho(B') .$$

Then the changing of bases functor  $\mathbf{Sem}(\rho)$  is a right adjoint of the changing of bases functor  $\mathbf{Sem}(\lambda)$  in the preordered category  $\mathbf{STopTh}$ .

*Proof.* Let  $X$  be a  $T$ -topological space, let  $Y$  be a  $T'$ -topological space, let  $f : X \rightarrow Y$  be a function, and let  $x \in X$ .

Since, for every  $T'$ -neighbourhood  $V'$  of  $x$  in  $\lambda(X)$ , there is a  $T$ -neighbourhood  $V$  of  $x$  in  $X$  such that  $\lambda(V) \subset_{T'(X)} V'$ , the function  $f$  is  $T'$ -continuous at  $x$  from  $\lambda(X)$  to  $Y$  if and only if, for every  $W' \in \mathcal{V}_{T'}^{Y'}(f(x))$ , there is  $V \in \mathcal{V}_T^X(x)$  such that  $\lambda(V) T'(f) W'$ .

Since, for every  $T$ -neighbourhood  $W$  of  $f(x)$  in  $\rho(Y)$ , there is a  $T'$ -neighbourhood  $W'$  of  $f(x)$  in  $Y$  such that  $\rho(W') \subset_{T(X)} W$ , the function  $f$  is  $T$ -continuous at  $x$  from  $X$  to  $\rho(Y)$  if and only if, for every  $W' \in \mathcal{V}_{T'}^Y(f(x))$ , there is  $V \in \mathcal{V}_T^X(x)$  such that  $V T(f) \rho(W')$ .

By the hypothesis, we deduce that  $f$  is  $T'$ -continuous at  $x$  from  $\lambda(X)$  to  $Y$  if and only if it is  $T$ -continuous at  $x$  from  $X$  to  $\rho(Y)$   $\square$

*Remark 5.3.1.* One easily checks that if  $\rho$  is a right adjoint of  $\lambda$  in the preordered category  $\mathbf{TopTh}$  then  $\rho$  and  $\lambda$  satisfies the assumption of the previous proposition.

The following lemma will be useful later:

<sup>4</sup>Therefore the functor  $\mathbf{Sem}$  does not preserve products.



**Lemma 5.3.20.** Let  $X$  be a set. For every  $x, y \in X$ , if  $x$  and  $y$  are  $\in_{T'}$ -indistinguishable (Definition 5.1.5) then  $x$  and  $y$  are  $\in_T$ -indistinguishable. The converse is true when  $v_X$  is essentially surjective.

*Proof.* Assume that  $x$  and  $y$  are  $\in_{T'}$ -indistinguishable. Let  $A \in T(X)$ . If  $x \in_T A$ , then  $x \in_{T'} v(A)$ , hence  $y \in_{T'} v(A)$ , and finally  $y \in_T A$ . Similarly, one shows that  $y \in_T A$  implies  $x \in_T A$ . Conversely, assume that  $v_X$  is essentially surjective and that  $x$  and  $y$  are  $\in_T$ -indistinguishable. Let  $A' \in T'(X)$ . Since  $v_X$  is essentially surjective, there is  $A \in T(X)$  such that  $v_X(A)$  and  $A'$  are  $\subset_{T'}^X$ -equivalent, a fortiori  $v_X(A)$  and  $A'$  have the same  $T'$ -elements. If  $x \in_{T'} A'$ , then  $x \in_T A$ , hence  $y \in_T A$ , and finally  $y \in_{T'} A'$ . Similarly, one shows that  $y \in_{T'} A'$  implies  $x \in_{T'} A'$ .  $\square$

## 5.4 $\mathbf{Loc}(T)$ -spaces

Let  $T = (T, \in_T)$  be a topological theory.

In this section, we define a functor  $\mathbf{Loc} : \mathbf{STopTh} \rightarrow \mathbf{TopTh}$  in such a way that  $\mathbf{Sem} \circ \mathbf{Loc}$  is a monad on  $\mathbf{STopTh}$ . The functor  $\mathbf{Loc}$  allows us to define many useful and well-behaved examples of topological theories; indeed, we will see in following chapters that various structures and properties of  $\mathbf{Top}_T$  (and of the forgetful functor) can be lifted to  $\mathbf{Top}_{\mathbf{Loc}(T)}$ .

In general, the topological theories  $T$  and  $\mathbf{Loc}(T)$  differ.

For every set  $X$ , we consider the preordered set whose underlying set is

$$\mathbf{Loc}(T)(X) := \{(A, \mathcal{E}) \text{ } T\text{-topological space ; } A \subset X\}$$

endowed with the preorder defined by  $(A, \mathcal{E}) \subset_{\mathbf{Loc}(T)(X)} (A', \mathcal{E}')$  if  $A \subset A'$  and if the inclusion is  $T$ -continuous.

For every sets  $X$  and  $Y$ , and every function  $f : X \rightarrow Y$ , define the module

$$\mathbf{Loc}(T)(f) : \mathbf{Loc}(T)(X) \rightarrow \mathbf{Loc}(T)(Y)$$

by  $A \mathbf{Loc}(T)(f) B$  if  $f(A) \subset B$  and if the function  $f$  can be restricted to a  $T$ -continuous map  $f_A : A \rightarrow B$ , for every  $A \in \mathbf{Loc}(T)(X)$  and  $B \in \mathbf{Loc}(T)(Y)$ .

Clearly  $\mathbf{Loc}(T)$  is a lax functor from  $\mathbf{Set}$  to  $\mathbf{Mod}$  such that, for every set  $X$ ,

$$\mathbf{Loc}(T)(\text{Id}_X) = \subset_{\mathbf{Loc}(T)(X)} ,$$

and  $(\mathbf{Loc}(T), \in)$  is a topological theory.

We apply the  $\mathbf{Loc}$  construction to some of the topological theories introduced in Examples 5.1.1, 5.2.1, and 5.2.2:

### Examples 5.4.1.

- Since  $\mathbf{Top}_{\mathbf{T}_1} = \mathbf{Set}$ , one readily checks that  $\mathbf{Loc}(T_1) = \mathcal{P}$ .
- The  $\mathbf{Loc}(T_{\mathbb{B}})$ -topological spaces are similar to the locally ordered spaces defined in 4.1.6 except that preorders are used instead of orders; therefore we call *locally preordered spaces* the  $\mathbf{Loc}(T_{\mathbb{B}})$ -topological spaces. More generally, we call *locally  $\mathbb{V}$ -preordered spaces* the  $\mathbf{Loc}(T_{\mathbb{V}})$ -topological spaces, for any quantale  $\mathbb{V}$ .
- The category of  $\mathbf{Loc}(T_{\mathbb{B}}^S)$ -topological spaces matches with the category of locally ordered spaces (Definition 4.1.6) and locally increasing maps.
- For every set  $X$ ,  $\mathbf{Loc}(T_{\emptyset})(X)$  is a singleton set whose unique member is the empty set endowed with the unique  $T_{\emptyset}$ -topology on it. So there exists a  $\mathbf{Loc}(T_{\emptyset})$ -topology on  $X$  if and only if  $X$  is the empty set. Thus, the topological theories  $T_{\emptyset}$  and  $\mathbf{Loc}(T_{\emptyset})$  are isomorphic in the category  $\mathbf{STopTh}$  but not in the category  $\mathbf{TopTh}$ .

Let  $T' = (T', \in_{T'})$  be another topological theory and let  $F : T \rightarrow T'$  be a semantic transformation. For every set  $X$ , define  $\mathbf{Loc}(F)_X : \mathbf{Loc}(T)(X) \rightarrow \mathbf{Loc}(T')(X)$  by

$$\mathbf{Loc}(F)_X(A) := F(A) \quad .$$

Since  $F$  is a semantic transformation, for every set  $X$ ,  $\mathbf{Loc}(F)_X$  is an increasing map, and, for every  $x \in X$  and every  $A \in \mathbf{Loc}(T)(X)$ ,  $x \in A$  if and only if  $x \in F(A)$ . Moreover, let  $f : X \rightarrow Y$  be a function and let  $A \in \mathbf{Loc}(T)(X)$  and  $B \in \mathbf{Loc}(T)(Y)$  such that  $A \mathbf{Loc}(T)(f) B$ , i.e.  $f$  can be restricted to a  $T$ -continuous map from  $A$  to  $B$ . Since  $F$  is a semantic transformation, we deduce that  $f$  can be restricted to a  $T'$ -continuous map from  $F(A)$  to  $F(B)$ , i.e. that  $F(A) \mathbf{Loc}(T')(f) F(B)$ .

Thus  $\mathbf{Loc}(F)$  is a changing of bases datum from  $\mathbf{Loc}(T)$  to  $\mathbf{Loc}(T')$ . One readily checks that  $\mathbf{Loc}(F)$  is a functor.

Let  $F, G : T \rightarrow T'$  be two parallel semantic transformations such that  $F \leq G$ . Let  $X$  be a set and let  $A \in \mathbf{Loc}(T)(X)$ . Then  $\text{Id}_A$  is a  $T'$ -continuous map from  $F(A)$  to  $G(A)$ , so  $\mathbf{Loc}(F)_X(A) \subset_{\mathbf{Loc}(T')} \mathbf{Loc}(G)_X(A)$ . Hence  $\mathbf{Loc}(F) \leq \mathbf{Loc}(G)$  in  $\mathbf{TopTh}$ .

Therefore, we defined a 2-functor  $\mathbf{Loc}$  from the preordered category  $\mathbf{TopTh}$  to the preordered category  $\mathbf{STopTh}$ .

Let  $(X, \mathcal{E})$  be a  $T$ -topological space, denote by  $\eta_T(X, \mathcal{E})$  the  $\mathbf{Loc}(T)$ -topological space  $X$  endowed with the  $\mathbf{Loc}(T)$ -topology generated by the basis  $\{(X, \mathcal{E})\}$ .

**Proposition 5.4.1.**  $\eta$  is a natural transformation from  $\text{Id}_{\mathbf{STopTh}}$  to  $\mathbf{Sem} \circ \mathbf{Loc}$ .

*Proof.* Let  $X$  and  $Y$  be  $T$ -topological spaces and let  $f : X \rightarrow Y$  be a  $T$ -continuous map. Then  $f$  is a  $\mathbf{Loc}(T)$ -continuous map from  $\eta_T(X)$  to  $\eta_T(Y)$  by the characterization of the  $\mathbf{Loc}(T)$ -continuity via the bases (Proposition 5.2.19) and by definition of  $\mathbf{Loc}(T)(f)$ . Thus  $\eta_T$  is a semantic transformation from  $T$  to  $\mathbf{Loc}(T)$ .

Let  $F : T \rightarrow T'$  be a semantic transformation and let  $X$  be a  $T$ -topological space.

On the one hand,  $\{F(X)\}$  is a basis that generates the  $\mathbf{Loc}(T')$ -topology of  $\eta_{T'}(F(X))$ .

On the other hand, since  $\{X\}$  is a basis that generates the  $\mathbf{Loc}(T)$ -topology of  $\eta_T(X)$  and, since  $\mathbf{Loc}(F)$  is a changing of bases datum,  $\{\mathbf{Loc}(F)_X(X)\} = \{F(X)\}$  is a basis that generates the  $\mathbf{Loc}(T')$ -topology of  $\mathbf{Loc}(F)(\eta_T(X))$ . Thus

$$\eta_{T'}(F(X)) = \mathbf{Loc}(F)(\eta_T(X)).$$

Therefore  $\eta$  is a natural transformation from  $\text{Id}_{\mathbf{STopTh}}$  to  $\mathbf{Sem} \circ \mathbf{Loc}$ .  $\square$

**Proposition 5.4.2.** For every topological theory  $T$ ,  $\eta_T : \mathbf{Top}_T \rightarrow \mathbf{Top}_{\mathbf{Loc}(T)}$  is a fully faithful functor.

*Proof.* Immediate consequence of the definition of  $T_{\mathbf{Loc}(T)}$ , of  $\eta_T$ , and of the characterization of the  $\mathbf{Loc}(T)$ -continuity via the bases (Proposition 5.2.19).  $\square$

Thus, for every topological theory  $T$ , one can see  $\mathbf{Top}_T$  as a full subcategory of  $\mathbf{Top}_{\mathbf{Loc}(T)}$ . We now describe the product of the monad  $\mathbf{Sem} \circ \mathbf{Loc}$ .

Let  $T = (T, \in_T)$  be a topological theory and let  $n \in \mathbb{N}^*$ , write

$$\mathbf{Loc}^n(T) := \underbrace{(\mathbf{Loc} \circ \dots \circ \mathbf{Loc})(T)}_{n\text{-times}}.$$

In the same way, write

$$(\mathbf{Sem} \circ \mathbf{Loc})^n := \underbrace{(\mathbf{Sem} \circ \mathbf{Loc}) \circ \dots \circ (\mathbf{Sem} \circ \mathbf{Loc})}_{n\text{-times}}.$$

Let  $(X, \mathcal{O}_{\mathbf{Loc}^2(T)}(X))$  be a  $\mathbf{Loc}^2(T)$ -topological space and let  $\mathcal{B}$  be a basis that generates this space. Define

$$\mu_T(\mathcal{B}) := \{A \in \mathbf{Loc}(T)(X) ; \exists A' \in \mathcal{B} \text{ such that } A \in \mathcal{O}_{\mathbf{Loc}(T)}(A')\}.$$

**Lemma 5.4.3.** The set  $\mu_T(\mathcal{B})$  is a basis of  $\mathbf{Loc}(T)$ -topology on  $X$ .

*Proof.* Let  $x \in X$  and let  $(A_j)_{j \in J}$  be a finite family of  $\mu_T(\mathcal{B})$  such that, for all  $j \in J$ ,  $x \in A_j$ . By definition of  $\mu_T(\mathcal{B})$ , for each  $j \in J$ , there is  $A'_j \in \mathcal{B}$  such that  $A_j \in \mathcal{O}_{\mathbf{Loc}(T)}(A'_j)$ . Then one has, for every  $j \in J$ ,  $x \in A'_j$ , so, since  $\mathcal{B}$  is a basis, there is  $A' \in \mathcal{B}$  such that  $x \in A'$ , and, for every  $j \in J$ ,  $A' \subset_{\mathbf{Loc}^2(T)(X)} A'_j$ . So, for each  $j \in J$ , the inclusion  $i_j$  of  $A'$  into  $A'_j$  is  $\mathbf{Loc}(T)$ -continuous at  $x$  hence there is  $B_j \in \mathcal{O}_{\mathbf{Loc}(T)}(A')$  such that  $x \in B_j$  and  $B_j \mathbf{Loc}(T)(i_j) A'_j$ . Since  $\mathcal{O}_{\mathbf{Loc}(T)}(A')$  is a basis, there is  $A \in \mathcal{O}_{\mathbf{Loc}(T)}(A')$  such that  $x \in A$  and  $A \subset_{\mathbf{Loc}(T)(A')} B_j$ . We deduce that  $A \in \mu_T(\mathcal{B})$ , that, for all  $j \in J$ ,  $A \subset B_j \subset A_j$ , and that the inclusions are  $T$ -continuous, hence  $A \subset_{\mathbf{Loc}(T)(X)} A_j$  for any  $j$ .  $\square$

**Lemma 5.4.4.** Let  $O' \in \mathcal{O}_{\mathbf{Loc}^2(T)}(\mathcal{B})$  and let  $O \in \mathcal{O}_{\mathbf{Loc}(T)}(O')$  then  $O \in \mathcal{O}_{\mathbf{Loc}(T)}(\mu_T(\mathcal{B}))$ .

*Proof.* Let  $x \in X$  such that  $x \in O$ . One has  $x \in O'$ , so, since  $O' \in \mathcal{O}_{\mathbf{Loc}^2(T)}(\mathcal{B})$ , there is  $A' \in \mathcal{B}$  such that  $x \in A'$  and  $A' \subset_{\mathbf{Loc}^2(T)(X)} O'$ . Thus  $A' \subset O'$  and the inclusion is  $\mathbf{Loc}(T)$ -continuous at  $x$ . Hence there is  $A \in \mathcal{O}_{\mathbf{Loc}(T)}(A')$  such that  $x \in A$  and  $A \mathbf{Loc}(i) O$ , where  $i$  denotes the inclusion of  $A'$  into  $O'$ . Therefore, one has  $A \subset O$  and the inclusion is  $T$ -continuous. In other words,  $A \subset_{\mathbf{Loc}(T)(X)} O$ .  $\square$

**Corollary 5.4.5.** Let  $\mathcal{B}$  and  $\mathcal{B}'$  be two bases that generate the  $\mathbf{Loc}^2(T)$ -topology of the space  $(X, \mathcal{O}_{\mathbf{Loc}^2(T)}(X))$ . The bases  $\mu_T(\mathcal{B})$  and  $\mu_T(\mathcal{B}')$  are equivalent.

*Proof.* It is a clear consequence of the previous lemma and of Corollary 5.2.4.  $\square$

We denote by  $\mu_T(X, \mathcal{O}_{\mathbf{Loc}^2(T)}(X))$  (or more simply by  $\mu_T(X)$ ) the set  $X$  endowed with the  $\mathbf{Loc}(T)$ -topology generated by the basis  $\mu_T(\mathcal{O}_{\mathbf{Loc}^2(T)}(X))$ .

**Proposition 5.4.6.**  $\mu$  is a natural transformation from  $(\mathbf{Sem} \circ \mathbf{Loc})^2$  to  $\mathbf{Sem} \circ \mathbf{Loc}$ .

*Proof.* Let  $X$  and  $Y$  be two  $\mathbf{Loc}^2(T)$ -topological spaces and let  $f : X \rightarrow Y$  be a  $\mathbf{Loc}^2(T)$ -continuous map. Let  $x \in X$  and let  $O_Y \in \mu_T(\mathcal{O}_{\mathbf{Loc}^2(T)}(Y))$  such that  $f(x) \in O_Y$ . There is  $O'_Y \in \mathcal{O}_{\mathbf{Loc}^2(T)}(Y)$  such that  $O_Y \in \mathcal{O}_{\mathbf{Loc}(T)}(O'_Y)$ . So one has  $f(x) \in O'_Y$ , then, by  $\mathbf{Loc}^2(T)$ -continuity of  $f$  at  $x$ , there exists  $O'_X \in \mathcal{O}_{\mathbf{Loc}^2(T)}(X)$  such that  $x \in O'_X$  and  $O'_X \mathbf{Loc}^2(T) O'_Y$ . Thus  $f(O'_X) \subset O'_Y$  and  $f$  can be restricted to a  $\mathbf{Loc}(T)$ -continuous map  $f|_{O'_X}$  from  $O'_X$  to  $O'_Y$ . By  $\mathbf{Loc}(T)$ -continuity of the map at  $x$ , there exists  $O_X \in \mathcal{O}_{\mathbf{Loc}(T)}(O'_X)$  such that  $x \in O_X$  and  $O_X \mathbf{Loc}(T)(f|_{O'_X}) O'_Y$ . Thus  $f|_{O'_X}(O_X) \subset O'_Y$ , and a fortiori  $f(O_X) \subset O'_Y$ , and  $f|_{O'_X}$ , and a fortiori  $f$ , can be restricted to a  $T$ -continuous map from  $O_X$  to  $O'_Y$ . Therefore  $O_X \mathbf{Loc}(T)(f) O'_Y$ , and then  $f$  is  $\mathbf{Loc}(T)$ -continuous at  $x$  from  $\mu_T(X)$  to  $\mu_T(Y)$ .

The map  $\mu_T$  is then a semantic transformation. It remains to prove the naturality of  $\mu$  in  $T$ . Let  $T' = (T', \in_{T'})$  be another topological theory and let  $F : T \rightarrow T'$  be a semantic transformation. Let  $X$  be a  $\mathbf{Loc}^2(T)$ -topological space.

By Corollary 5.4.5 and since  $(\mathbf{Sem} \circ \mathbf{Loc})(F)$  is a changing of bases functor,

$$\mathcal{B}_1 := \{F(O) ; O \in \mathbf{Loc}(T)(X) \text{ such that } \exists O' \in \mathcal{O}_{\mathbf{Loc}^2(T)}(X) \text{ verifying } O \in \mathcal{O}_{\mathbf{Loc}(T)}(O')\}$$

is a basis that generates  $(\mathbf{Sem} \circ \mathbf{Loc})(\mu_T(X))$ .

Similarly, again by Corollary 5.4.5 and since  $(\mathbf{Sem} \circ \mathbf{Loc})^2(F)$  is a changing of bases functor,

$$\mathcal{B}_2 := \{U \in \mathbf{Loc}(T')(X) ; \exists O' \in \mathcal{O}_{\mathbf{Loc}^2(T)}(X) \text{ such that } U \in \mathcal{O}_{\mathbf{Loc}(T')}(\mathbf{Loc}(F)(O'))\}$$

is a basis that generates  $\mu_{T'}((\mathbf{Sem} \circ \mathbf{Loc})^2(X))$ .

It is enough to show that the two bases are equivalent to conclude.

Let  $O' \in \mathcal{O}_{\mathbf{Loc}^2(T)}(X)$  and let  $O \in \mathcal{O}_{\mathbf{Loc}(T)}(O')$ . Since  $(\mathbf{Sem} \circ \mathbf{Loc})(F)$  is a changing of bases datum, by Lemma 5.3.8 and by definition of  $\mathbf{Loc}(F)$ ,  $F(O)$  is  $\mathbf{Loc}(T')$ -open in  $\mathbf{Loc}(F)(O')$ . Hence the inclusion  $\mathcal{B}_1 \subset \mathcal{B}_2$  holds.

Conversely, let  $O' \in \mathcal{O}_{\mathbf{Loc}^2(T)}(X)$ , let  $U \in \mathcal{O}_{\mathbf{Loc}(T')}(\mathbf{Loc}(F)(O'))$ , and let  $x \in X$  such that  $x \in U$ . By definition of  $\mathbf{Loc}(F)(O')$ , there is  $O \in \mathcal{O}_{\mathbf{Loc}(T)}(O')$  such that  $x \in O$  and  $F(O) \subset_{\mathbf{Loc}(T')} U$ . Hence one has  $U \in \mathcal{O}_{\mathbf{Loc}(T')}(\mathcal{B}_1)$  and thus  $\mathcal{B}_2 \subset \mathcal{O}_{\mathbf{Loc}(T')}(\mathcal{B}_1)$ .

We conclude by Lemma 5.2.2 and Corollary 5.2.4.  $\square$

**Theorem 5.4.7.** The triple  $((\mathbf{Sem} \circ \mathbf{Loc}), \mu, \eta)$  is a monad on  $\mathbf{STopTh}$ .

*Proof.* Let  $T = (T, \in_T)$  be a topological theory.

Firstly, we are to show that  $\mu_T \circ \mu_{\mathbf{Loc}(T)} = \mu_T \circ \mathbf{Loc}(\mu_T)$ .

Let  $X$  be a  $\mathbf{Loc}^3(T)$ -topological space and  $\mathcal{B}$  be a basis that generates  $X$ .

On the one hand, by the previous corollary,

$$\mu_{\mathbf{Loc}(T)}(\mathcal{B}) = \{A' \in T_{\mathbf{Loc}^2(T)}(X) ; \exists A'' \in \mathcal{B} \text{ such that } A' \in \mathcal{O}_{\mathbf{Loc}^2(T)}(A'')\}$$

is a basis that generates  $\mu_{\mathbf{Loc}(T)}(X)$ , then

$$\begin{aligned} \mu_T(\mu_{\mathbf{Loc}(T)}(\mathcal{B})) &= \{A \in T_{\mathbf{Loc}(T)}(X) ; \exists A' \in \mu_{\mathbf{Loc}(T)}(\mathcal{B}) \text{ such that } A \in \mathcal{O}_{\mathbf{Loc}(T)}(A')\} \\ &= \{A \in T_{\mathbf{Loc}(T)}(X) ; \exists A'' \in \mathcal{B}, \exists A' \in \mathcal{O}_{\mathbf{Loc}^2(T)}(A'') \text{ such that } A \in \mathcal{O}_{\mathbf{Loc}(T)}(A')\} \end{aligned}$$

is a basis that generates  $\mu_T(\mu_{\mathbf{Loc}(T)}(X))$ .

On the other hand,

$$\mathbf{Loc}(\mu_T)(\mathcal{B}) = \{\mathbf{Loc}(\mu_T)_X(A'') ; A'' \in \mathcal{B}\} = \{\mu_T(A'') ; A'' \in \mathcal{B}\}$$

is a basis of  $\mathbf{Loc}(\mu_T)(X)$ , then, by the previous corollary,

$$\begin{aligned} \mu_T(\mathbf{Loc}(\mu_T)(\mathcal{B})) &= \{A \in T_{\mathbf{Loc}(T)}(X) ; \exists A' \in \mathbf{Loc}(\mu_T)(\mathcal{B}) \text{ such that } A \in \mathcal{O}_{\mathbf{Loc}(T)}(A')\} \\ &= \{A \in T_{\mathbf{Loc}(T)}(X) ; \exists A'' \in \mathcal{B} \text{ such that } A \in \mathcal{O}_{\mathbf{Loc}(T)}(\mu_T(A''))\} \end{aligned}$$

is a basis that generates  $\mu_T(\mathbf{Loc}(\mu_T)(X))$ .

Since  $\mu_T(\mathcal{O}_{\mathbf{Loc}^2(T)}(A'')) \subset \mathcal{O}_{\mathbf{Loc}(T)}(\mu_T(A''))$ , one has  $\mu_T(\mu_{\mathbf{Loc}(T)}(\mathcal{B})) \subset \mu_T(\mathbf{Loc}(\mu_T)(\mathcal{B}))$ . Conversely, let  $A \in T_{\mathbf{Loc}(T)}(X)$ ,  $A'' \in \mathcal{B}$  such that  $A \in \mathcal{O}_{\mathbf{Loc}(T)}(\mu_T(A''))$ , and  $x \in A$ . Since  $\mu_T(\mathcal{O}_{\mathbf{Loc}^2(T)}(A''))$  is a basis of  $\mu_T(A'')$ , there is  $A' \in \mathcal{O}_{\mathbf{Loc}^2(T)}(A'')$  and  $B \in \mathcal{O}_{\mathbf{Loc}(T)}(A')$  such that  $x \in B \subset_{\mathbf{Loc}(T)} A$ , hence  $\mu_T(\mathbf{Loc}(\mu_T)(\mathcal{B})) \subset \mu_T(\mu_{\mathbf{Loc}(T)}(\mathcal{B}))$ . Therefore, by Lemma 5.2.2 and Corollary 5.2.4

$$\mu_T(\mu_{\mathbf{Loc}(T)}(X)) = \mu_T(\mathbf{Loc}(\mu_T)(X)) .$$

Secondly, we are to prove that  $\mu_T \circ \eta_{\mathbf{Loc}(T)} = \text{Id}_{\mathbf{Loc}(T)}$ .

Let  $X$  be a  $\mathbf{Loc}(T)$ -topological space. The set  $\{X\}$  is a basis that generates  $\eta_{\mathbf{Loc}(T)}(X)$ , then, by the previous corollary,

$$\begin{aligned} \mu_T(\{X\}) &= \{A \in T_{\mathbf{Loc}(T)}(X) ; \exists A' \in \{X\} \text{ such that } A \in \mathcal{O}_{\mathbf{Loc}(T)}(A')\} \\ &= \{A \in T_{\mathbf{Loc}(T)}(X) ; A \in \mathcal{O}_{\mathbf{Loc}(T)}(X)\} \\ &= \mathcal{O}_{\mathbf{Loc}(T)}(X) \end{aligned}$$

is a basis that generates  $\mu_T(\eta_{\mathbf{Loc}(T)}(X))$  hence  $\mu_T(\eta_{\mathbf{Loc}(T)}(X)) = X$ . Thus

$$\mu_T \circ \eta_{\mathbf{Loc}(T)} = \text{Id}_{\mathbf{Loc}(T)} .$$

Finally, we are to show that  $\mu_T \circ \mathbf{Loc}(\eta_T) = \text{Id}_{\mathbf{Loc}(T)}$ .

Let  $X$  be a  $\mathbf{Loc}(T)$ -topological space. The set

$$\begin{aligned} \mathbf{Loc}(\eta_T)(\mathcal{O}_{\mathbf{Loc}(T)}(X)) &= \{\mathbf{Loc}(\eta_T)_X(O) ; O \in \mathcal{O}_{\mathbf{Loc}(T)}(X)\} \\ &= \{\eta_T(O) ; O \in \mathcal{O}_{\mathbf{Loc}(T)}(X)\} \end{aligned}$$

is a basis that generates  $\mathbf{Loc}(\eta_T)(X)$ . Then, by the previous corollary,

$$\begin{aligned} \mu_T(\mathbf{Loc}(\eta_T)(\mathcal{O}_{\mathbf{Loc}(T)}(X))) &= \{A \in T_{\mathbf{Loc}(T)}(X) ; \exists A' \in \mathbf{Loc}(\eta_T)(\mathcal{O}_{\mathbf{Loc}(T)}(X)) \text{ such that } A \in \mathcal{O}_{\mathbf{Loc}(T)}(A')\} \\ &= \{A \in T_{\mathbf{Loc}(T)}(X) ; \exists O \in \mathcal{O}_{\mathbf{Loc}(T)}(X) \text{ such that } A \in \mathcal{O}_{\mathbf{Loc}(T)}(\eta_T(O))\} \end{aligned}$$

is a basis that generates  $\mu_T(\mathbf{Loc}(\eta_T)(X))$ .

Let  $O \in \mathcal{O}_{\mathbf{Loc}(T)}(X)$ . On the one hand,  $O$  is  $\mathbf{Loc}(T)$ -open in  $\eta_T(O)$ . On the other hand, let  $A \in T_{\mathbf{Loc}(T)}(X)$  such that  $A$  is  $\mathbf{Loc}(T)$ -open in  $\eta_T(O)$ . So  $A$  and  $O$  have the same underlying sets and  $O \subset_{\mathbf{Loc}(T)(O)} A$  then  $O \subset_{\mathbf{Loc}(T)(X)} A$ , hence, by the stability property 5.2.14,  $A$  is  $\mathbf{Loc}(T)$ -open in  $X$ . Therefore, one has  $X = \mu_T(\mathbf{Loc}(\eta_T)(X))$ , and then

$$\mu_T \circ \mathbf{Loc}(\eta_T) = \text{Id}_{\mathbf{Loc}(T)} .$$

□



# Chapter 6

## Generalization of notions of point-set topology

For this whole chapter, set two topological theories  $T = (T, \in_T)$  and  $T' = (T', \in_{T'})$  and a changing of bases datum  $v : T \rightarrow T'$ .

We generalize some classical notions of point-set topology to  $T$ -topological spaces and we study their preservation (or reflection) by changing of bases functors.

For the sake of concision, we assume classical topological properties to prove their  $T$ -topological counterparts, though direct proofs are possible. This approach is mainly based on the final changing of bases datum  $v$ .

### 6.1 Closedness

Until now, we generalized the notions of open subset and of neighbourhood. In this section, we generalize the closed subsets. An issue is that a member of  $(T(X), \subset_{T(X)})$  does not have a complement in general (unlike members of  $(\mathcal{P}(X), \subset)$ ).

We fix two  $T$ -topological spaces  $X$  and  $Y$ .

**Definition 6.1.1** (Adherent points). A point  $x \in X$  is an *adherent  $T$ -point* of  $A \in T(X)$  when, for every  $O \in \mathcal{O}_T(X)$  such that  $x \in_T O$ , there is  $x_O \in X$  such that  $x_O \in_T O$  and  $x_O \in_T A$ .

*Remark 6.1.1.* A  $T$ -element of  $A \in T(X)$  is a fortiori an adherent  $T$ -point of  $A$ .

We can then give a natural definition of

**Definition 6.1.2** ( $T$ -closedness). A member  $A \in T(X)$  is  *$T$ -closed* when every adherent  $T$ -point of  $A$  is a  $T$ -element of  $A$ .

Concerning the preservation of these notions by the change base functors, one has

**Proposition 6.1.3.** Let  $x \in X$  and let  $A \in T(X)$ . The point  $x$  is an adherent  $T$ -point of the member  $A$  in  $X$  if and only if it is an adherent  $T'$ -point of  $v(A)$  in  $v(X)$ .

*Proof.* Assume that  $x$  is an adherent  $T$ -point of  $A$  in  $X$ . Let  $O' \in \mathcal{O}_{T'}(v(X))$  such that  $x \in_{T'} O'$ . So there is  $O \in \mathcal{O}_T(X)$  such that  $x \in_T O$  and  $v(O) \subset_{T'} O'$ . Then, there is  $y \in X$  such that  $y \in_T O$  and  $y \in_T A$ . Hence  $y \in_{T'} v(O) \subset_{T'} O'$  and  $y \in_{T'} v(A)$ .

Conversely, assume that  $x$  is an adherent  $T'$ -point of  $v(A)$  in  $v(X)$ . Let  $O \in \mathcal{O}_T(X)$  such that  $x \in_T O$ . Then  $v(O)$  is  $T'$ -open in  $v(X)$  and  $x$  is a  $T'$ -element of it by Lemma 5.3.8. Hence, there is  $y \in X$  such that  $y \in_{T'} v(O)$  and  $y \in_{T'} v(A)$ . Therefore  $y \in_T O$  and  $y \in_T A$ .  $\square$

**Corollary 6.1.4.** Let  $A \in T(X)$ . Then  $A$  is  $T$ -closed in  $X$  if and only if  $v(A)$  is  $T'$ -closed in  $v(X)$ .

In particular, the two results can be applied to the final changing of bases datum  $!^T : T \rightarrow \mathcal{P}$  (see Examples 5.3.1). Thus, a point  $x \in X$  is an adherent  $T$ -point of a member  $A \in T(X)$  if and only if  $x$  is an adherent point of the set of all  $T$ -elements of  $A$  for the underlying topology of  $X$ , and  $A$  is  $T$ -closed if and only if the set of all its  $T$ -elements is closed for the underlying topology of  $X$ . Therefore, there is no real gap between these notions and the classical ones. Hence, when there is no risk of confusion, we also say that a member of  $T(X)$  is closed when it is  $T$ -closed.

This allows us to deduce the following properties of  $T$ -adherent points and of  $T$ -closeness from the properties of adherent points and closed subsets.

**Lemma 6.1.5.** Let  $A, A' \in T(X)$  such that  $A \subset_T A'$  and let  $x \in X$ . If  $x$  is an adherent  $T$ -point of  $A$ , then  $x$  is an adherent  $T$ -point of  $A'$ .

*Proof.* Assume that  $x$  is an adherent  $T$ -point of  $A$ . Then  $x$  is an adherent point of  $!^T_X(A)$  for the underlying topology of  $X$  by Proposition 6.1.3. Since  $A \subset_T A'$ , one has  $!^T_X(A) \subset !^T_X(A')$ , hence  $x$  also an adherent point of  $!^T_X(A')$ . Again by Proposition 6.1.3, we conclude that  $x$  is an adherent  $T$ -point of  $A'$ .  $\square$

**Lemma 6.1.6.** Let  $A, A' \in T(X)$  such that  $A \simeq_T A'$ , and let  $x \in X$ . If  $x$  is an adherent  $T$ -point of  $A$ , then  $x$  is an adherent  $T$ -point of  $A'$ .

*Proof.* Clear by Proposition 6.1.3 because  $!^T_X(A) = !^T_X(A')$ .  $\square$

**Proposition 6.1.7** (Stability of  $T$ -closedness under meets). Let  $(A_i)_{i \in I}$  be a family of  $T$ -closed members of  $T(X)$  which has a meet  $\bigwedge_{i \in I} A_i$  in the preordered set  $T(X)$ . Assume that  $\in_T^X$  preserves meets. Then  $\bigwedge_{i \in I} A_i$  is  $T$ -closed.

*Proof.* The assumption implies that  $!^T_X(\bigwedge_{i \in I} A_i) = \bigcap_{i \in I} !^T_X(A_i)$ . Then we conclude by Corollary 6.1.4.  $\square$

**Proposition 6.1.8** (Stability of  $T$ -closedness under finite joins). Let  $(A_j)_{j \in J}$  be a finite family of  $T$ -closed members of  $T(X)$  which has a join  $\bigvee_{j \in J} A_j$  in  $T(X)$ . Assume that the  $\in_T^X$  is supercompact (see 5.2.12). Then  $\bigvee_{j \in J} A_j$  is  $T$ -closed.

*Proof.* The assumption implies that  $!^T_X(\bigvee_{j \in J} A_j) = \bigcup_{j \in J} !^T_X(A_j)$ . We conclude again by Corollary 6.1.4.  $\square$

**Proposition 6.1.9** (Stability of  $T$ -closedness under  $\simeq_T$ -equivalence). Let  $A, A' \in T(X)$  such that  $A' \simeq_T A$ . If  $A$  is  $T$ -closed then so is  $A'$ .

*Proof.* Immediate consequence of Corollary 6.1.4 given that  $!^T_X(A) = !^T_X(A')$ .  $\square$

Unlike the case of topological spaces, given a set  $X$ , the datum of  $T$ -closed members of  $T(X)$  is generally not enough to specify a  $T$ -topology on  $X$ . This is notably due to the stability under  $T$ -equivalence, which is true for the  $T$ -closed members of  $T(X)$  but not in general for the  $T$ -open members.

When a  $T$ -open member has a ‘complement’, the latter is  $T$ -closed:

**Proposition 6.1.10.** Let  $O \in \mathcal{O}_T(X)$  and let  $A \in T(X)$  such that, for every  $x \in X$ ,  $x \in_T A$  is equivalent to  $x \notin_T O$ . Then  $A$  is  $T$ -closed in  $X$ .

*Proof.* By Lemma 5.3.8, the subset  $!^T_X(O)$  is open for the underlying topology of  $X$ . Hence its complement, which matches with  $!^T_X(A)$  by hypothesis, is closed. We conclude again by Corollary 6.1.4.  $\square$

**Proposition 6.1.11.** Let  $A \in T(X)$  and let  $\bar{A} \in T(X)$  such that the  $T$ -elements of  $\bar{A}$  match with the adherent  $T$ -points of  $A$  in  $X$ . Then  $\bar{A}$  is  $T$ -closed in  $X$ .

*Proof.* By Proposition 6.1.3, the subset  $!^T_X(\bar{A})$  is the closure of  $!^T_X(A)$  for the underlying topology of  $X$ , hence it is closed. We conclude by Corollary 6.1.4.  $\square$

The two previous propositions are particularly interesting when  $!^T_X$  is surjective. Concerning  $T$ -continuity, one has

**Proposition 6.1.12.** Let  $f : X \rightarrow Y$  be a  $T$ -continuous map, let  $A \in T(X)$ , and let  $x \in X$ . If  $x$  is an adherent  $T$ -point of  $A$ , then, for every  $B \in T(Y)$  such that  $A T(f) B$ ,  $f(x)$  is an adherent  $T$ -point of  $B$ .

*Proof.* Let  $B \in T(Y)$  such that  $A T(f) B$  and let  $O_Y \in \mathcal{O}_T(Y)$  such that  $f(x) \in_T O_Y$ . Since  $f$  is  $T$ -continuous at  $x$ , there is  $O_X \in \mathcal{O}_T(X)$  whose  $x$  is a  $T$ -element such that  $O_X T(f) O_Y$ . Since  $x$  is an adherent  $T$ -point of  $A$ , there is  $x' \in X$  such that  $x' \in_T O_X$  and  $x' \in_T A$ . Since  $A T(f) B$  and  $O_X T(f) O_Y$ , we finally deduce that  $f(x') \in_T O_Y$  and  $f(x') \in_T B$ .  $\square$

**Corollary 6.1.13.** Let  $f : X \rightarrow Y$  be a  $T$ -continuous map such that  $T(f)$  has an inverse image  $f^{-1}$ . Then, for every  $T$ -closed member  $B \in T(Y)$ ,  $f^{-1}(B)$  is  $T$ -closed in  $X$ .

*Proof.* Let  $B \in T(Y)$   $T$ -closed and let  $x \in X$  be an adherent  $T$ -point of  $f^{-1}(B)$ . Since  $f^{-1}(B) T(f) B$ , by the previous proposition, we deduce that  $f(x)$  is an adherent  $T$ -point of  $B$ . Therefore  $f(x) \in_T B$  because  $B$  is  $T$ -closed. Since  $f^{-1}$  is an inverse image, we deduce that  $x \in_T f^{-1}(B)$ .  $\square$

## 6.2 Convergence and adherent $T$ -points of parts of $T(X)$

In this section, we define the notion of convergence in a  $T$ -topological space. A difficulty comes from the fact that, in the case of topological spaces, we generally exclude the filters that contain the empty set because they would converge to any point. Therefore we need a good notion of "set of subsets that does not contain the empty set" in our framework. This notion leads us to another issue: it is not generally preserved by taking the generated filter. Thus, we cannot restrict ourself to the sole consideration of filters on  $T(X)$ : we have to provide a notion of convergence which makes sense for any part of  $T(X)$ .

**Definition 6.2.1.** Let  $X$  be set. A part  $P$  of  $T(X)$  is *finitely  $T$ -pointable* when, for every finite family  $(A_j)_{j \in J}$  of  $P$ , there is  $x \in X$  such that, for every  $j \in J$ ,  $x \in_T A_j$ .

*Remark 6.2.1.* Let  $X$  be a set. A cofiltered subset  $P$  of  $T(X)$  is finitely  $T$ -pointable if and only if, for every  $A \in P$ , there is  $x \in X$  such that  $x \in_T A$ .

**Example 6.2.1.** In a  $T$ -topological space, the  $T$ -neighbourhoods filters are finitely  $T$ -pointable.

**Lemma 6.2.2.** Let  $X$  be a set and let  $P$  and  $P'$  be two parts of  $T(X)$  such that  $P' \subset P$ . If  $P$  is finitely  $T$ -pointable then so is  $P'$ .

*Proof.* Obvious.  $\square$

*Remark 6.2.2.* Let  $X$  be a set. If  $X$  is empty,  $\emptyset \subset T(\emptyset)$  is not finitely  $T$ -pointable but if  $X \neq \emptyset$ ,  $\emptyset \subset T(X)$  is finitely  $T$ -pointable.

Under some conditions, a part of  $T(X)$  is finitely  $T$ -pointable if and only if the filter that it generates is finitely  $T$ -pointable.

**Lemma 6.2.3.** Let  $X$  be a set and let  $P \subset T(X)$ . Assume that one of the two following statement is true:

- The module  $\in_T^X$  is representable.
- The preordered set  $T(X)$  is finitely complete and  $\in_T^X$  preserves finite meets.

Then  $P$  is finitely  $T$ -pointable if and only if the filter generated by  $P$  is finitely  $T$ -pointable.

*Proof.* Write  $\mathcal{F}$  the filter generated by  $P$ . Since  $P \subset \mathcal{F}$ , by the previous lemma, if  $\mathcal{F}$  is finitely  $T$ -pointable then so is  $P$ .

Conversely, assume that  $P$  is finitely  $T$ -pointable. Let  $(A_j)_{j \in J}$  be a finite family of  $\mathcal{F}$ . For each  $j \in J$ , there is a finite family  $(A_{j,k})_{k \in J_j}$  of  $P$  such that  $A_j$  is an upper bound of the set of all lower bounds of  $(A_{j,k})_{k \in J_j}$ . Since  $(A_{j,k})_{j \in J, k \in J_j}$  is a finite family of  $P$ , there is  $x \in X$  such that, for all  $j \in J$  and for all  $k \in J_j$ ,  $x \in_T A_{j,k}$ .



If the membership module is represented by  $\sigma : X \rightarrow T(X)$ , we deduce that for all  $j \in J$  and all  $k \in J_j$ ,  $\sigma(x) \subset_T A_{j,k}$ . For each  $j \in J$ ,  $\sigma(x)$  is then a lower bound of  $(A_{j,k})_{k \in J_j}$ , hence  $\sigma(x) \subset_{T(X)} A_j$ , i.e.  $x \in_T A_j$ .

If  $T(X)$  is finitely complete and if  $\in_T^X$  preserves finite meets. Since  $x \in_T A_{j,k}$  for all  $j \in J$  and  $k \in J_j$ ,  $x$  is a  $T$ -element of a meet  $\in_T \bigwedge_{j \in J, k \in J_j} A_{j,k}$ . Given  $j \in J$ , since, for every  $k \in J_j$ ,  $\bigwedge_{j \in J, k \in J_j} A_{j,k} \subset_T A_{j,k}$ , one has  $\bigwedge_{j \in J, k \in J_j} A_{j,k} \subset_T A_j$ , hence  $x \in_T A_j$ .  $\square$

This lemma allows us, under some assumptions, to restrict ourself to the case of filters. In the general case, we have to define the following notions for any parts of  $T(X)$  rather than for the filters.

**Proposition 6.2.4.** Let  $X$  and  $Y$  be two sets and let  $f : X \rightarrow Y$  be a function. Let  $P$  be a finitely  $T$ -pointable part of  $T(X)$ . Then  $P_{T(f)}$ , the image of  $P$  under the module  $T(f)$  (see 3.2.5), is finitely  $T$ -pointable.

*Proof.* Let  $(B_j)_{j \in J}$  be a finite family of  $T(Y)$  such that, for every  $j \in J$ , there is  $A_j \in P$  such that  $A_j T(f) B_j$ . Since  $P$  is finitely  $T$ -pointable, there is  $x \in X$  such that, for every  $j \in J$ ,  $x \in_T A_j$ . Given  $j \in J$ , since  $A_j T(f) B_j$ , we deduce that  $f(x) \in_T B_j$ .  $\square$

**Corollary 6.2.5.** Let  $X$  and  $Y$  be two sets and let  $f : X \rightarrow Y$  be a function. Let  $\mathcal{F}$  be a filter on  $T(X)$ . Assume that  $Y$  satisfies one of the two statements of Lemma 6.2.3. If  $\mathcal{F}$  is finitely  $T$ -pointable, then the direct image filter of  $\mathcal{F}$  under  $T(f)$  is finitely  $T$ -pointable.

Concerning changing of bases functors, given a set  $X$ , define the following operators:

- For every  $P \subset T(X)$ , write

$$v(P) := \{A' \in T'(X); \exists A \in P \text{ such that } v(A) \subset_{T'} A'\} \quad .$$

- For every  $P' \subset T'(X)$ , write

$$v^{-1}(P') := \{A \in T(X); \exists A' \in P' \text{ such that } A' \subset_{T'} v(A)\} \quad .$$

**Proposition 6.2.6.** Let  $X$  be a set and let  $P \subset T(X)$ . Then  $P$  is finitely  $T$ -pointable if and only if  $v(P)$  is finitely  $T'$ -pointable.

*Proof.* Assume that  $P$  is finitely  $T$ -pointable. Let  $(A'_j)_{j \in J}$  be a finite family of  $T'(X)$  such that, for every  $j \in J$ , there is  $A_j \in P$  such that  $v(A_j) \subset_{T'} A'_j$ . Since  $P$  is finitely  $T$ -pointable, there is  $x \in X$  such that, for every  $j \in J$ ,  $x \in_T A_j$ . Then one has, for every  $j \in J$ ,  $x \in_{T'} v(A_j) \subset_{T'} A'_j$ , hence  $v(P)$  is finitely  $T'$ -pointable.

Conversely, assume that  $v(P)$  is finitely  $T'$ -pointable. Let  $(A_j)_{j \in J}$  be a finite family of  $T(X)$ . Then  $(v(A_j))_{j \in J}$  is a finite family of  $v(P)$ , so there is  $x \in X$  such that, for every  $j \in J$ ,  $x \in_{T'} v(A_j)$ . Thus, one has, for every  $j \in J$ ,  $x \in_T A_j$ .  $\square$

In particular, a part  $P$  of  $T(X)$  is finitely  $T$ -pointable if and only if the filter of  $\mathcal{P}(X)$  generated by  $!^T(P)$  does not contains the empty set. However the image under  $!^T$  of the filter generated by  $P$  does not necessary match with the filter generated by  $!^T(P)$ .

**Proposition 6.2.7.** Let  $X$  be a set and let  $P' \subset T'(X)$ . If  $P'$  is finitely  $T'$ -pointable then  $v^{-1}(P')$  is finitely  $T$ -pointable. Conversely, assume that  $v_X : T(X) \rightarrow T'(X)$  is essentially surjective, if  $v^{-1}(P')$  finitely  $T$ -pointable then  $P'$  finitely  $T'$ -pointable.

*Proof.* Assume that  $P'$  is finitely  $T'$ -pointable. Let  $(A_j)_{j \in J}$  be a finite family of  $T(X)$  such that, for every  $j \in J$ , there is  $A'_j \in P'$  such that  $A'_j \subset_{T'} v(A_j)$ . Since  $P'$  is finitely  $T'$ -pointable, there is  $x \in X$  such that, for every  $j \in J$ ,  $x \in_{T'} A'_j \subset_{T'} v(A_j)$ . Hence one has, for every  $j \in J$ ,  $x \in_T A_j$ .

Conversely, assume that  $v_X$  is essentially surjective and that  $v^{-1}(P')$  is finitely  $T$ -pointable. Let  $(A'_j)_{j \in J}$  be a finite family of  $P'$ . Since  $v$  is essentially surjective, there exists a finite family  $(A_j)_{j \in J}$  of  $T(X)$  such that, for every  $j \in J$ ,  $v(A_j)$  is  $\subset_{T'(X)}$ -equivalent to  $A'_j$ . The family  $(A_j)_{j \in J}$  is then a finite family of  $v^{-1}(P')$ , so there is  $x \in X$  such that, for every  $j \in J$ ,  $x \in_T A_j$ . Therefore, one has, for every  $j \in J$ ,  $x \in_{T'} v(A_j)$ , hence  $x \in_{T'} A'_j$ , since  $v(A_j)$  is  $\subset_{T'(X)}$ -equivalent to  $A'_j$ .  $\square$

**Definition 6.2.8** (Adherent  $T$ -points of a part). Let  $X$  be a  $T$ -topological space. A point  $x \in X$  is an adherent  $T$ -point of a part  $P \subset T(X)$  when, for every finite family  $(A_j)_{j \in J}$  of  $P$ , for every  $O \in \mathcal{O}_T(X)$  such that  $x \in_T O$ , there is  $y \in X$  such that  $y \in_T O$  and  $y \in_T A_j$ , for every  $j \in J$ .

*Remark 6.2.3.* In other words, since the  $T$ -neighbourhoods filter of  $x$  is cofiltered,  $x$  is an adherent  $T$ -point of  $P$  when  $\mathcal{V}_T(x) \cup P$  is finitely  $T$ -pointable. In particular, a part of  $T(X)$  which has an adherent  $T$ -point is finitely  $T$ -pointable.

*Remark 6.2.4.* Let  $X$  be a  $T$ -topological space. Let  $P$  be a part of  $T(X)$  and let  $x \in X$ . If  $P$  is cofiltered, then  $x$  is an adherent  $T$ -point of  $P$  if and only if  $x$  is an adherent  $T$ -point of every member of  $P$ .

**Proposition 6.2.9.** Let  $X$  be a  $T$ -topological space and let  $P$  and  $P'$  be two parts of  $T(X)$  such that  $P' \subset P$ . Let  $x \in X$ . If  $x$  is an adherent  $T$ -point of  $P$ , then  $x$  is an adherent  $T$ -point of  $P'$ .

*Proof.* Trivial. □

As for the finitely  $T$ -pointable parts, when one of the hypothesis of Lemma 6.2.3 are satisfied, we can restrict ourself to the case of filters:

**Lemma 6.2.10.** Let  $X$  be a  $T$ -topological space, let  $x \in X$  and let  $P$  be a part of  $T(X)$ . Assume that one of the two following statements is true:

- The module  $\in_T^X$  is representable.
- The preordered set  $T(X)$  is finitely complete and  $\in_T^X$  preserves finite meets.

Then  $x$  is an adherent  $T$ -point of  $P$  if and only if  $x$  is an adherent  $T$ -point of the filter generated by  $P$ .

*Proof.* Denote by  $\mathcal{F}$  the filter generated by  $P$  and by  $\mathcal{F}'$  the filter generated by  $\mathcal{V}_T(x) \cup P$ . Assume that  $x$  is an adherent  $T$ -point of  $\mathcal{F}$ . Since  $P \subset \mathcal{F}$ , by the previous proposition, we deduce that  $x$  is an adherent  $T$ -point of  $P$ . Conversely, assume that  $x$  is an adherent  $T$ -point of  $P$ . Then  $\mathcal{V}_T(x) \cup P$  is finitely  $T$ -pointable. By Lemma 6.2.3, we deduce that  $\mathcal{F}'$  is finitely  $T$ -pointable. Since  $\mathcal{V}_T(x) \cup \mathcal{F} \subset \mathcal{F}'$ ,  $\mathcal{V}_T(x) \cup \mathcal{F}$  is finitely  $T$ -pointable, hence  $x$  is an adherent  $T$ -point of  $\mathcal{F}$ . □

Concerning changing of bases functors, one has:

**Proposition 6.2.11.** Let  $X$  be a  $T$ -topological space and let  $P \subset T(X)$ . Let  $x \in X$ , then  $x$  is an adherent  $T$ -point of  $P$  if and only if  $x$  is an adherent  $T'$ -point of  $v(P)$ .

*Proof.* Assume that  $x$  is an adherent  $T$ -point of  $P$ . Let  $(A'_j)_{j \in J}$  be a finite family of  $T'(X)$  such that, for every  $j \in J$ , there is  $A_j \in P$  such that  $v(A_j) \subset_{T'} A'_j$ , and let  $O' \in \mathcal{O}_{T'}(v(X))$  such that  $x \in_{T'} O'$ . There is  $O \in \mathcal{O}_T(X)$  whose  $x$  is a  $T$ -element such that  $v(O) \subset_{T'} O'$ . Then there is  $y \in X$  such that  $y \in_T O$  and, for every  $j \in J$ ,  $y \in_T A_j$ . Therefore, one has  $y \in_{T'} v(O) \subset_{T'} O'$  and, for every  $j \in J$ ,  $y \in_{T'} v(A_j) \subset_{T'} A'_j$ . We conclude that  $x$  is an adherent  $T'$ -point of  $v(P)$ .

Conversely, assume that  $x$  is an adherent  $T'$ -point of  $v(P)$ . Let  $(A_j)_{j \in J}$  be a finite family of  $P$  and let  $O \in \mathcal{O}_T(X)$  such that  $x \in_T O$ . Then  $v(O)$  is  $T'$ -open in  $v(X)$ ,  $x$  is a  $T'$ -element of it, and  $(v(A_j))_{j \in J}$  is a finite family of  $v(P)$ , hence there is  $y \in X$  such that  $y \in_{T'} v(O)$  and, for every  $j \in J$ ,  $y \in_{T'} v(A_j)$ . Hence  $y \in_T O$  and, for every  $j \in J$ ,  $y \in_T A_j$ . □

**Proposition 6.2.12.** Let  $X$  be a  $T$ -topological space, let  $P' \subset T'(X)$  and let  $x \in X$ . If  $x$  is an adherent  $T'$ -point of  $P'$  then  $x$  is an adherent  $T$ -point of  $v^{-1}(P')$ . Conversely, assume that  $v_X : T(X) \rightarrow T'(X)$  is essentially surjective, if  $x$  is an adherent  $T$ -point of  $v^{-1}(P')$  then  $x$  is an adherent  $T'$ -point of  $P'$ .

*Proof.* Assume that  $x$  is an adherent  $T'$ -point of  $P'$ . Let  $(A_j)_{j \in J}$  be a finite family of  $T(X)$  such that, for every  $j \in J$ , there is  $A'_j \in P'$  such that  $A'_j \subset_{T'} v(A_j)$  and let  $O \in \mathcal{O}_T(X)$  such that  $x \in_T O$ . Then  $v(O)$  is  $T'$ -open in  $v(X)$ ,  $x$  is a  $T'$ -element of it, and  $(v(A_j))_{j \in J}$  is a finite family of  $P'$ . Therefore, there is  $y \in X$  such that  $y \in_{T'} v(O)$  and, for every  $j \in J$ ,  $y \in_{T'} v(A_j)$ . Thus  $y \in_T O$  and, for every  $j \in J$ ,  $y \in_T A_j$ .

Conversely, assume that  $v_X$  is essentially surjective and that  $x$  is an adherent  $T$ -point of  $v^{-1}(P')$ . Let  $(A'_j)_{j \in J}$  be a finite family of  $P'$  and let  $O' \in \mathcal{O}_{T'}(v(X))$  such that  $x \in_{T'} O'$ . Since  $v$  is essentially surjective, there is a finite family  $(A_j)_{j \in J}$  such that, for every  $j \in J$ ,  $v(A_j)$  is  $\subset_{T'(X)}$ -equivalent to  $A'_j$ . Besides, there is  $O \in \mathcal{O}_T(X)$  whose  $x$  is a  $T$ -element such that  $v(O) \subset_{T'} O'$ . Since  $x$  is an adherent  $T$ -point of  $v^{-1}(P')$ , we deduce that there exists  $y \in X$  such that  $y \in_T O$  and, for every  $j \in J$ ,  $y \in_T A_j$ . Then  $y \in_{T'} v(O) \subset_{T'} O'$ , so  $y \in_{T'} O'$ , and, for every  $j \in J$ ,  $y \in_{T'} v(A_j)$ , so  $y \in_T A'_j$ , because  $v(A_j)$  is  $\subset_{T'(X)}$ -equivalent to  $A'_j$ .  $\square$

**Definition 6.2.13** (Parts convergence). Let  $X$  be a  $T$ -topological space. A part  $P \subset T(X)$  converges to a point  $x \in X$  when  $\mathcal{V}_T(x) \subset P$ .<sup>1</sup>

*Remark 6.2.5.* Only the convergence of finitely  $T$ -pointable parts is interesting because, for any topological space  $X$ ,  $\mathcal{P}(X)$  converges to all the points of  $X$ .

**Proposition 6.2.14.** Let  $X$  be a  $T$ -topological space and let  $P$  and  $P'$  be two parts of  $T(X)$  such that  $P \subset P'$ . Let  $x \in X$ . If  $P$  converges to  $x$  then  $P'$  converges to  $x$ .

*Proof.* Obvious.  $\square$

Thus, if  $P \subset T(X)$  converges to  $x$  in a  $T$ -topological space  $X$ , then the filter generated by  $P$  also converges to  $x$ .

**Proposition 6.2.15.** Let  $X$  be a  $T$ -topological space and let  $P, P' \subset T(X)$  such that  $P' \subset P$ . Let  $x \in X$ . If  $P$  is finitely  $T$ -pointable and converges to  $x$  then  $x$  is an adherent  $T$ -point of  $P'$ .

*Proof.* Since  $P' \subset P$  and  $\mathcal{V}_T(x) \subset P$ , one has  $P' \cup \mathcal{V}_T(x) \subset P$ . Since  $P$  is finitely  $T$ -pointable, we deduce that  $P' \cup \mathcal{V}_T(x)$  is finitely  $T$ -pointable then that  $x$  is an adherent  $T$ -element of  $P'$ .  $\square$

Conversely

**Proposition 6.2.16.** Let  $X$  be a  $T$ -topological space and let  $P' \subset T(X)$ . Let  $x \in X$ . If  $x$  is an adherent  $T$ -element of  $P'$  then there is  $P \subset T(X)$  finitely  $T$ -pointable such that  $P' \subset P$  and  $P$  converges to  $x$ .

*Proof.* Write  $P := P' \cup \mathcal{V}_T(x)$ . Since  $x$  is an adherent  $T$ -element of  $P'$ ,  $P$  is finitely  $T$ -pointable. The fact that  $P' \subset P$  and that  $P$  converges to  $x$  is clear.  $\square$

Like in the case of topological spaces, we can characterize the  $T$ -closedness by the convergence of some parts.

**Proposition 6.2.17.** Let  $X$  be a  $T$ -topological space. A member  $F$  of  $T(X)$  is  $T$ -closed if and only if, for every  $P \subset T(X)$  such that there is a  $T$ -element  $x_A$  of  $F$  and of  $A$  for all  $A \in P$ , and for every  $x \in X$ , if  $P$  converges to  $x$  then  $x \in_T F$ .

*Proof.* Assume that  $F$  is  $T$ -closed. Let  $P \subset T(X)$  such that, for all  $A \in P$ , there is a  $T$ -element  $x_A$  of  $F$  and of  $A$ . Let  $x \in X$  such that  $P$  converges to  $x$ . Let  $O \in \mathcal{O}_T(X)$  such that  $x \in_T O$ . Since  $P$  converges to  $x$ , one has  $O \in P$ , hence, by assumption, there is a  $T$ -element  $x_O$  of  $O$  and of  $F$ . Thus  $x$  is an adherent  $T$ -point of  $F$ . Consequently  $x \in_T F$  because  $F$  is  $T$ -closed.

Conversely, assume that for every  $P \subset T(X)$  such that, for all  $A \in P$ , there is a  $T$ -element  $x_A$  of  $F$  and of  $A$ , for every  $x \in X$ , if  $P$  converges to  $x$  then  $x \in_T F$ . Let  $x \in X$  such that  $x$  is an adherent  $T$ -point of  $F$ . Consider the filter  $\mathcal{V}_T(x)$ . For all  $V \in \mathcal{V}_T(x)$ , since  $x$  is an adherent  $T$ -point of  $F$ , there is a  $T$ -element  $x_V$  of  $F$  and of  $V$ . Since  $\mathcal{V}_T(x)$  converges to  $x$ , we conclude that, par hypothesis,  $x \in F$ .  $\square$

*Remark 6.2.6.* By the previous proposition, it is enough to verify the hypothesis for the cofiltered and finitely  $T$ -pointable filters because the  $T$ -neighbourhoods filters satisfy those statements, and they are the ones that are used in the preceding proof of the converse implication.

Concerning  $T$ -continuous maps,

<sup>1</sup>We could have defined the convergence of part  $P$  to a point  $x$  by requiring that  $\mathcal{V}_T(x)$  is included in the filter generated by  $P$  but then Proposition 6.2.15 does not hold without additional assumptions.

**Proposition 6.2.18.** Let  $X$  and  $Y$  be  $T$ -topological spaces, let  $f : X \rightarrow Y$  be a function  $T$ -continuous at  $x \in X$ , and let  $P \subset T(X)$ . If  $x$  is an adherent  $T$ -point of  $P$ , then  $f(x)$  is an adherent  $T$ -point of  $P_{T(f)}$ , where  $P_{T(f)}$  is the image of  $P$  under the module  $T(f)$  (Definition 3.2.5).

*Proof.* Let  $(B_j)_{j \in J}$  be a finite family of  $T(Y)$  such that, for every  $j \in J$ , there is  $A_j \in \mathcal{F}$  such that  $A_j T(f) B_j$ . Let  $V_Y \in \mathcal{V}_T(f(x))$ . Since  $f$  is  $T$ -continuous at  $x$ , there is  $V_X \in \mathcal{V}_T(x)$  such that  $V_X T(f) V_Y$ . Since  $x$  is an adherent  $T$ -point of  $P$ , there is  $y \in X$  such that  $y \in_T V_X$  and, for every  $j \in J$ ,  $y \in_T A_j$ . Since  $V_X T(f) V_Y$  and, for every  $j \in J$ ,  $A_j T(f) B_j$ , one deduces that  $f(y) \in_T V_Y$  and, for every  $j \in J$ ,  $f(y) \in_T B_j$ .  $\square$

**Proposition 6.2.19.** Let  $X$  and  $Y$  be two  $T$ -topological spaces and let  $f : X \rightarrow Y$  be a function. The function  $f$  is  $T$ -continuous at  $x \in X$  if and only if, for every  $P \subset T(X)$  converging to  $x$ ,  $P_{T(f)}$  converges to  $f(x)$ .

*Proof.* Assume that  $f$  is  $T$ -continuous at  $x \in X$  and let  $P \subset T(X)$  converging to  $x$ . Let  $V_Y \in \mathcal{V}_T(f(x))$ . Since  $f$  is  $T$ -continuous at  $x$ , there is  $V_X \in \mathcal{V}_T(x)$  such that  $V_X T(f) V_Y$ . Since  $P$  converges to  $x$ , one has  $V_X \in P$ , then  $V_Y \in P_{T(f)}$ .

Conversely, assume that for every  $P \subset T(X)$  converging to  $x$ ,  $P_{T(f)}$  converges to  $f(x)$ . Let  $V_Y \in \mathcal{V}_T(f(x))$ . Since  $\mathcal{V}_T(x)$  converges to  $x$ , by hypothesis,  $V_Y$  belongs to the image of  $\mathcal{V}_T(x)$  under the module  $T(f)$ , i.e. there is  $V_X \in \mathcal{V}_T(x)$  such that  $V_X T(f) V_Y$ .  $\square$

Concerning changing of bases functors:

**Proposition 6.2.20.** Let  $X$  be a  $T$ -topological space, let  $P \subset T(X)$  and let  $x \in X$ . If  $P$  converges to  $x$  in  $X$  then  $v(P)$  converges to  $x$  in  $v(X)$ . Conversely, if  $v_X : T(X) \rightarrow T'(X)$  is fully faithful and if  $P$  is upward closed, the convergence of  $v(P)$  to  $x$  in  $v(X)$  implies the convergence of  $P$  to  $x$  in  $X$ .

*Proof.* Assume that  $P$  converges to  $x$  in  $X$ . Let  $V' \in \mathcal{V}_{T'}(x)$ . There is  $V \in \mathcal{V}_T(x)$  such that  $v(V) \subset_{T'} V'$ . Since  $P$  converges to  $x$  in  $X$ , one has  $V \in P$ , and then, by definition of  $v(P)$ ,  $V' \in v(P)$ .

Conversely, assume that  $v_X$  is fully faithful, that  $P$  is upward closed, and that  $v(P)$  converges to  $x$  in  $v(X)$ . Let  $V \in \mathcal{V}_T(x)$ . Then  $v(V) \in \mathcal{V}_{T'}(x)$  by Corollary 5.3.10, hence  $v(V) \in vP$  because  $v(P)$  converges to  $x$  in  $v(X)$ . Therefore, there is  $A \in P$  such that  $v(A) \subset_{T'} v(V)$ . Since  $v_X$  is fully faithful, we deduce that  $A \subset_T V$ . Finally  $V \in P$  because  $P$  is upward closed.  $\square$

**Proposition 6.2.21.** Let  $X$  be a  $T$ -topological space, let  $P' \subset T'(X)$  and let  $x \in X$ . If  $P'$  converges to  $x$  in  $v(X)$  then  $v^{-1}(P')$  converges to  $x$  in  $X$ . Conversely, if  $v_X : T(X) \rightarrow T'(X)$  is fully faithful and essentially surjective, and if  $P'$  is upward closed, then the convergence of  $v^{-1}(P')$  to  $x$  in  $X$  implies the convergence of  $P'$  to  $x$  in  $v(X)$ .

*Proof.* Assume that  $P'$  converges to  $x$  in  $v(X)$ . Let  $V \in \mathcal{V}_T(x)$ . Then  $v(V) \in \mathcal{V}_{T'}(x)$ , hence  $v(V) \in P'$  since  $P'$  converges to  $x$  in  $v(X)$ . By definition of  $v^{-1}(P')$ , we deduce that  $V \in v^{-1}(P')$ .

Conversely, assume that  $v_X : T(X) \rightarrow T'(X)$  is fully faithful and essentially surjective, that  $P'$  is upward closed, and that  $v^{-1}(P')$  converges to  $x$  in  $X$ . Let  $V' \in \mathcal{V}_{T'}(x)$ . Since  $v_X$  is essentially surjective, there is  $V \in T(X)$  such that  $v(V)$  and  $V'$  are  $\subset_{T'(X)}$ -equivalent. Since  $V'$  is a  $T'$ -neighbourhood of  $x$  in  $v(X)$ , so is  $v(V)$ . By proposition 5.3.11, since  $v_X$  is fully faithful, one has  $V \in \mathcal{V}_T(x)$ . Since  $v^{-1}(P')$  converges to  $x$ , one has  $V \in v^{-1}(P')$ . Consequently, there is  $A \in P'$  such that  $A \subset_{T'} v(V)$ . Since  $v(V)$  and  $V'$  are  $\subset_{T'(X)}$ -equivalent, we deduce that  $A \subset_{T'} V'$ . Finally, since  $P'$  is upward closed,  $V' \in P'$ .  $\square$

## 6.3 Separation properties

We study the generalization of some separation properties to  $T$ -topological spaces.

### 6.3.1 Specialization preorder and associated separation properties

The specialization preorder naturally generalizes to  $T$ -topological spaces:

**Definition 6.3.1** (Specialization preorder). Let  $X$  be a  $T$ -topological space. The  $T$ -specialization preorder is a preorder  $\preceq_T^X$  (or more simply  $\preceq_T$ ) on  $X$  defined by  $x \preceq_T y$  if  $\mathcal{V}_T(x) \subset \mathcal{V}_T(y)$ .

*Remark 6.3.1.* Let  $X$  be a  $T$ -topological space and let  $x, y \in X$ . If, for every  $A \in T(X)$ ,  $x \in_T A$  implies  $y \in_T A$ , then  $x \preceq_T y$ . For every  $A \in T(X)$ , if  $x \preceq_T y$  and if  $y$  is an adherence  $T$ -point of  $A$  then  $x$  is an adherence  $T$ -point of  $A$ .

From this generalization of the specialization preorder, one easily derives generalizations of the separation axioms  $T0$ ,  $R0$ , and  $T1$ .

**Definition 6.3.2** (Indistinguishable elements). Let  $X$  be a  $T$ -topological space. Two elements  $x$  and  $y$  of  $X$  are  $T$ -indistinguishable in the  $T$ -topological space  $X$  when they are equivalent for the  $T$ -specialization preorder, i.e. if  $\mathcal{V}_T(x) = \mathcal{V}_T(y)$ .

**Definition 6.3.3** ( $T$ - $T0$  spaces). Let  $X$  be a  $T$ -topological space. The space  $X$  is  $T$ - $T0$  when, for every elements  $x$  and  $y$  of  $X$ , if  $x$  and  $y$  are  $T$ -indistinguishable in  $X$  then  $x = y$ . In other words,  $X$  is  $T$ - $T0$  when the  $T$ -specialization preorder is an order.

Equality between points of a  $T$ -topological space may be a too strong notion; this justifies the following definition.

**Definition 6.3.4** (Weakly  $T$ - $T0$  spaces). Let  $X$  be a  $T$ -topological space. The space  $X$  is weakly  $T$ - $T0$  when, for every elements  $x$  and  $y$  of  $X$ , if  $x$  and  $y$  are  $T$ -indistinguishable in  $X$  then  $x$  and  $y$  are  $\in_T$ -indistinguishable (Definition 5.1.5). In other words, the space  $X$  is weakly  $T$ - $T0$  when the  $T$ -specialization preorder is an order relatively to the equivalence relation of  $\in_T$ -indistinguishability.

*Remark 6.3.2.* A  $T$ -topological space  $X$  is  $T$ - $T0$  if and only if  $X$  is weakly  $T$ - $T0$  and the relation of  $\in_T$ -indistinguishability is the equality relation.

**Example 6.3.1.** With the topological theory  $\mathcal{P}$ , the two notions matches with the classical separation axiom  $T0$  in point-set topology.

**Definition 6.3.5** ( $T$ - $R0$  spaces). A  $T$ -topological space  $X$  is  $T$ - $R0$  when the  $T$ -specialization preorder is an equivalence relation, i.e. when for every  $x, y \in X$ , if  $\mathcal{V}_T(x) \subset \mathcal{V}_T(y)$  then  $\mathcal{V}_T(x) = \mathcal{V}_T(y)$ .

*Remark 6.3.3.* Let  $X$  be a  $T$ -topological space  $T$ - $R0$ . Given  $x, y \in X$ , if, for every  $A \in T(X)$ ,  $x \in_T A$  implies  $y \in_T A$ , then  $x$  and  $y$  are  $T$ -indistinguishable in  $X$ .

**Definition 6.3.6** ((weakly)  $T$ - $T1$  spaces). A  $T$ -topological space  $X$  is (weakly)  $T$ - $T1$  when it is  $T$ - $R0$  and (weakly)  $T$ - $T0$ .

**Example 6.3.2.** Let  $\mathbb{V}$  be a quantale. All  $T_{\mathbb{V}}$ -topological spaces are weakly  $T_{\mathbb{V}}$ - $T1$  but only the ones whose underlying set is the empty set or a singleton set are  $T_{\mathbb{V}}$ - $T1$ .

**Lemma 6.3.7.** Let  $X$  be a  $T$ - $R0$  space. Assume that  $\in_T^X$  is represented by  $\sigma : X \rightarrow T(X)$ . Then, for every  $x$  and  $y \in X$ ,  $x$  is an adherent  $T$ -point of  $\sigma(y)$  if and only if  $x$  and  $y$  are  $T$ -indistinguishable in  $X$ .

*Proof.* Assume that  $x$  is an adherent  $T$ -point of  $\sigma(y)$ . Let  $O \in \mathcal{O}_T(X)$  such that  $x \in_T O$ . Then there is  $z \in X$  such that  $z \in_T O$  and  $z \in_T \sigma(y)$ . For every  $A \in T(X)$  such that  $y \in_T A$ . So one has  $\sigma(y) \subset_T A$  then  $z \in_T A$ . Thus  $y \preceq_T z$ , hence  $z \preceq_T y$  because  $X$  is  $T$ - $R0$ . Since  $z \in_T O$  and since  $O$  is  $T$ -open, we deduce that  $y \in_T O$ . Hence  $x \preceq_T y$ , and then  $x$  and  $y$  are  $T$ -indistinguishable in  $X$  since  $X$  are  $T$ - $R0$ . The converse implications are obvious because since  $y \in_T \sigma(y)$ .  $\square$

**Proposition 6.3.8.** Let  $X$  be a  $T$ - $R0$  space. Assume that  $\in_T^X$  is represented by  $\sigma : X \rightarrow T(X)$ . The space  $X$  is  $T$ - $T1$  if and only if, for every  $x, y \in X$ , if  $x$  is an adherent  $T$ -point of  $\sigma(y)$  in  $X$  then  $x = y$ . The space  $X$  is weakly  $T$ - $T1$  if and only if, for every  $x, y \in X$ , if  $x$  is an adherent  $T$ -point of  $\sigma(y)$  in  $X$  then  $x$  and  $y$  are  $\in_T$ -equivalent.

**Corollary 6.3.9.** Let  $X$  be a weakly  $T$ - $T1$  space such that  $\in_T^X$  is represented by  $\sigma$ . For every  $x \in X$ ,  $\sigma(x)$  is closed.

Concerning changing of bases functors, one has

**Proposition 6.3.10.** Let  $X$  be a  $T$ -topological space and let  $x, y \in X$ . One has  $x \preceq_T y$  in  $X$  if and only if  $x \preceq_{T'} y$  in  $v(X)$ .

*Proof.* Assume that  $x \preceq_T y$  in  $X$ . Let  $O' \in \mathcal{O}_{T'}(v(X))$  such that  $x \in_{T'} O'$ . Hence there is  $O \in \mathcal{O}_T(X)$  such that  $x \in_T O$  and  $v(O) \subset_{T'} O'$ . Since  $x \preceq_T y$ , one has  $y \in_T O$ , then  $y \in_{T'} v(O)$ , and finally  $y \in_{T'} O'$ .

Assume that  $x \preceq_{T'} y$  in  $v(X)$ . Let  $O \in \mathcal{O}_T(X)$  such that  $x \in_T O$ . Then  $v(O)$  is  $T'$ -open in  $v(X)$  and  $x$  is a  $T'$ -element of it. Since  $x \preceq_{T'} y$  in  $v(X)$ , one has  $y \in_{T'} v(O_y)$ , then finally  $y \in_T O_y$ .  $\square$

**Corollary 6.3.11.** Let  $X$  be a  $T$ -topological space. The space  $X$  is  $T$ - $T0$  (resp.  $T$ - $R0$ ,  $T$ - $T1$ ) if and only if  $v(X)$  is  $T'$ - $T0$  (resp.  $T'$ - $R0$ ,  $T'$ - $T1$ ).

**Corollary 6.3.12.** Let  $X$  and  $Y$  be  $T$ -topological spaces and let  $f : X \rightarrow Y$  be an injective  $T$ -continuous map. If  $Y$  is  $T$ - $T0$  (resp.  $T$ - $T1$ ), then  $X$  is  $T$ - $T0$  (resp.  $T$ - $T1$ ).

*Remark 6.3.4.* A priori this result does not hold in general for the weak versions since we do not have interesting map between  $T(X)$  and  $T(Y)$ , just a module. Even when the spaces  $X$  and  $Y$  are isomorphic in the concrete category  $\mathbf{Top}_T$ , it may be possible that  $X$  is weakly  $T0$  (respectively weakly  $T1$ ) whereas  $Y$  is not. It illustrates a particular phenomenon which is due to the fact that  $T$  is only a *lax* functor: two isomorphic  $T$ -topological spaces may be quite different, even when the concerned isomorphism is an identity function<sup>2</sup>. This will be a major source of complexity in the following chapters.

In particular, with the final changing of bases datum  $!^T$ , we deduce that a  $T$ -topological space  $X$  is  $T$ - $T0$  (resp.  $T$ - $R0$ ,  $T$ - $T1$ ) if and only if the underlying topological space  $!^T(X)$  is  $T0$  (resp.  $R0$ ,  $T1$ ). Therefore, only weak versions of the separation axioms really differ from their classical counterparts.

**Corollary 6.3.13.** Let  $X$  be a  $T$ -topological space. If  $v(X)$  is weakly  $T'$ - $T0$  (resp. weakly  $T'$ - $T1$ ) then  $X$  is weakly  $T$ - $T0$  (resp. weakly  $T$ - $T1$ ). Assume that, for every  $x, y \in X$ , the  $\in_T$ -indistinguishability of  $x$  and  $y$  implies the  $\in_{T'}$ -indistinguishability of  $x$  and  $y$ . Then the converse implication holds.

*Proof.* It clearly follows from the previous proposition and from Lemma 5.3.20.  $\square$

### 6.3.2 $T$ -Hausdorff and $T$ -compact spaces

Now we have generalized the notion of convergence and adherent points of filters, we can naturally generalize the notion of Hausdorff and compact spaces.

**Definition 6.3.14.** Let  $X$  be a  $T$ -topological space. The space  $X$  is  $T$ -Hausdorff (or  $T$ - $T2$ ) when, for every finitely  $T$ -pointable part  $P \subset T(X)$ , for every  $x, y \in X$ , if  $P$  converges to  $x$  and to  $y$  in  $X$  then  $x = y$ . Similarly  $X$  is weakly  $T$ -Hausdorff when, for every finitely  $T$ -pointable part  $P \subset T(X)$ , for every  $x, y \in X$ , if  $P$  converges to  $x$  and to  $y$  in  $X$  then  $x$  and  $y$  are  $\in_T$ -indistinguishable.

The classical characterizations of Hausdorff spaces holds.

<sup>2</sup>In univalent foundations, it seems to correspond to the distinction between *precategories* and *categories* (see [Uni13, 9.1]).

**Proposition 6.3.15.** Let  $X$  be a  $T$ -topological space. The following statements are equivalent:

1. The space  $X$  is (weakly)  $T$ -Hausdorff.
2. For every  $x, y \in X$ , if for every  $V_x \in \mathcal{V}_T(x)$  and  $V_y \in \mathcal{V}_T(y)$ , there is  $z \in X$  satisfying  $z \in_T V_x$  and  $z \in_T V_y$ , then  $x = y$  (respectively,  $x$  and  $y$  are  $\in_T$ -indistinguishable).
3. For every part  $P \subset T(X)$ , for every  $x, y \in X$ , if  $P$  converges to  $x$  and if  $y$  is an adherent  $T$ -point of  $P$ , then  $x = y$  (respectively,  $x$  and  $y$  are  $\in_T$ -indistinguishable).

*Proof.*

- 1.  $\Rightarrow$  2.: Let  $x$  and  $y$  be two elements of  $X$  such that, for every  $V_x \in \mathcal{V}_T(x)$  and  $V_y \in \mathcal{V}_T(y)$ , there is  $z \in X$  satisfying  $z \in_T V_x$  and  $z \in_T V_y$ . Write  $P := \mathcal{V}_T(x) \cup \mathcal{V}_T(y)$ . Then  $P$  converges to  $x$  and  $y$  in  $X$ . By 1., it is enough to show that  $P$  is finitely  $T$ -pointable to conclude.

Let  $(A_j)_{j \in J_x \cup J_y}$  be a finite family of  $T(X)$  such that, for every  $j \in J_x$ ,  $A_j \in \mathcal{V}_T(x)$ , and that, for every  $j \in J_y$ ,  $A_j \in \mathcal{V}_T(y)$ . Since the filter  $\mathcal{V}_T(x)$  is cofiltered, there is  $V_x \in \mathcal{V}_T(x)$  such that, for every  $j \in J_x$ ,  $V_x \subset_T A_j$ . In the same way, since  $\mathcal{V}_T(y)$  is cofiltered, there is  $V_y \in \mathcal{V}_T(y)$  such that, for every  $j \in J_y$ ,  $V_y \subset_T A_j$ . Par hypothesis, there is  $z \in X$  satisfying  $z \in_T V_x$  and  $z \in_T V_y$ . We finally deduce that, for every  $j \in J_x \cup J_y$ ,  $z \in_T A_j$ .

- 2.  $\Rightarrow$  3.: Let  $V_x \in \mathcal{V}_T(x)$  and  $V_y \in \mathcal{V}_T(y)$ . Since  $P$  converges to  $x$ , one has  $V_x \in P$ . Since  $y$  is an adherent  $T$ -point of  $P$  and since  $V_x \in P$ , there is  $z \in X$  such that  $z \in_T V_x$  and  $z \in_T V_y$ . By 2. allows us to conclude.
- 3.  $\Rightarrow$  1.: It is a clear consequence of Proposition 6.2.16 and of Proposition 6.2.14.

□

*Remark 6.3.5.* When one of the two statements of Lemma 6.2.3 is satisfied, we can restrict ourself to the filters.

Concerning changing of bases functors, one has

**Proposition 6.3.16.** Let  $X$  be a  $T$ -topological space. The space  $X$  is  $T$ -Hausdorff if and only if the space  $v(X)$  is  $T'$ -Hausdorff.

*Proof.* Assume that  $X$  is  $T$ -Hausdorff. Let  $x, y \in X$ . Let  $P' \subset T'(X)$  be a finitely  $T'$ -pointable part that converges to  $x$  and to  $y$  in  $v(X)$ . Then, by Propositions 6.2.7 and 6.2.21,  $v^{-1}(P')$  is finitely  $T$ -pointable and converges to  $x$  and to  $y$  in  $X$ . Hence  $x = y$  because  $X$  is  $T$ -Hausdorff.

Assume that  $v(X)$  is  $T'$ -Hausdorff. Let  $x, y \in X$ . Let  $P \subset T(X)$  be a finitely  $T$ -pointable part that converges to  $x$  and to  $y$  in  $X$ . Then, by Propositions 6.2.6 and 6.2.20,  $v(P)$  is finitely  $T'$ -pointable and converges to  $x$  and to  $y$  in  $v(X)$ . Hence  $x = y$  because  $v(X)$  is  $T'$ -Hausdorff. □

In particular, it applies to the final changing of bases datum  $!^T$ : A  $T$ -topological space is  $T$ -Hausdorff if and only if its underlying topological space is Hausdorff.

**Corollary 6.3.17.** Let  $X$  and  $Y$  be  $T$ -topological spaces and let  $f : X \rightarrow Y$  be an injective  $T$ -continuous map. If  $Y$  is  $T$ -Hausdorff, then so is  $X$ .

*Remark 6.3.6.* A priori, as Corollary 6.3.12, this result does not hold in general for weak  $T$ -Hausdorff spaces.

**Corollary 6.3.18.** Let  $X$  be a  $T$ -topological space. If the space  $X$  is  $T$ -Hausdorff, then it is  $T$ - $T1$ .

This result remains true for the weak versions:

**Proposition 6.3.19.** Let  $X$  be a  $T$ -topological space. If the spaces  $X$  is weakly  $T$ -Hausdorff, then it is weakly  $T$ - $T1$ .

*Proof.* Let  $x, y \in X$  such that  $x \preceq_T y$ . Let  $V_x$  be a  $T$ -neighbourhood of  $x$  and  $V_y$  be a  $T$ -neighbourhood of  $y$ . Since  $x \preceq_T y$ ,  $V_x$  is a  $T$ -neighbourhood of  $y$ . Then  $y \in_T V_x, V_y$ . Hence  $x$  and  $y$  are  $\in_T$ -indistinguishable by Proposition 6.3.15 because  $X$  is weakly  $T$ -Hausdorff.  $\square$

**Proposition 6.3.20.** Let  $X$  be a  $T$ -topological space. If the space  $v(X)$  is  $T'$ -Hausdorff then the space  $X$  is  $T$ -Hausdorff. The converse implication holds when, for every  $x, y \in X$ , the  $\in_T$ -indistinguishability of  $x$  and  $y$  implies the  $\in_{T'}$ -indistinguishability of  $x$  and  $y$ .

*Proof.* The proof is similar to the one of Proposition 6.3.16 using Lemma 5.3.20.  $\square$

Now that we defined the Hausdorffness as the unicity of elements to which a finitely  $T$ -pointable part converges, we define compactness.

**Definition 6.3.21** ( $T$ -compact spaces). Let  $X$  be a  $T$ -topological space. The space  $X$  is  $T$ -compact when every finitely  $T$ -pointable part of  $T(X)$  has an adherent  $T$ -point.

*Remark 6.3.7.* When any of the statements from Lemma 6.2.3 is satisfied, it is enough to verify the hypothesis for filters.

Like in classical point-set topology, modulo the choice axiom,  $T$ -compactness is equivalent to the convergence of some parts.

**Definition 6.3.22** ( $T$ -ultraparts). Let  $X$  be a set. A  $T$ -ultrapart of  $T(X)$  is a maximum in the set of finitely  $T$ -pointable parts of  $T(X)$  ordered by inclusion.

In other words, a finitely  $T$ -pointable part of  $T(X)$  is a  $T$ -ultrapart when every part that strictly contains it is not finitely  $T$ -pointable.

*Remark 6.3.8.* Let  $X$  be a set. A finitely  $T$ -pointable part  $P \subset T(X)$  is a  $T$ -ultrapart if and only if, for every  $A \in T(X)$ , if  $P \cup \{A\}$  is finitely  $T$ -pointable, then  $A \in P$ .

**Lemma 6.3.23.** Let  $X$  be a set and let  $P_0 \subset T(X)$  be a finitely  $T$ -pointable part. The set of all finitely  $T$ -pointable parts of  $T(X)$  containing  $P_0$ , ordered by inclusion, is an inductive preordered set.

*Proof.* Let  $S$  be a chain of the set of all finitely  $T$ -pointable parts of  $T(X)$  containing  $P_0$  ordered by inclusion. Let  $S' := S \cup \{P_0\}$  and write  $P_{S'} := \bigcup_{P \in S'} P$ . One has, for every  $P \in S'$ ,  $P \subset P_{S'}$ . To conclude, it is enough to show that  $P_{S'}$  is finitely  $T$ -pointable.

Let  $(A_j)_{j \in J}$  be a finite family of  $P_{S'}$ . Since  $S'$  is clearly a non empty chain, there is  $P \in S'$  such that, for every  $j \in J$ ,  $A_j \in P$ . Since  $P$  is finitely  $T$ -pointable, there is  $x \in X$  such that, for every  $j \in J$ ,  $x \in_T A_j$ .  $\square$

We deduce that:

**Corollary 6.3.24.** [Assuming the choice axiom] Let  $X$  be a set. For every finitely  $T$ -pointable part of  $P \subset T(X)$ , there is a  $T$ -ultrapart  $\mathcal{U}$  of  $X$  such that  $P \subset \mathcal{U}$ .

*Remark 6.3.9.* By Lemma 6.2.3, when one of its two statements is true, every  $T$ -ultrapart is a filter (we call it then a  $T$ -ultrafilter).

**Lemma 6.3.25.** Let  $X$  be a set, let  $\mathcal{U}$  be a  $T$ -ultrapart on  $T(X)$  and let  $A, A' \in T(X)$ . If, for every  $x \in X$ ,  $x \in_T A$  or  $x \in_T A'$ , then  $A \in \mathcal{U}$  or  $A' \in \mathcal{U}$ .

*Proof.* If  $A \notin \mathcal{U}$ , there is a finite family  $(U_j)_{j \in J}$  of  $\mathcal{U}$  such that, for every  $x \in X$ , if, for every  $j \in J$ ,  $x \in_T U_j$  then  $x \notin_T A$ . In the same way, if  $A' \notin \mathcal{U}$ , there is a finite family  $(U_j)_{j \in J'}$  of  $\mathcal{U}$  such that, for every  $x \in X$ , if, for every  $j \in J'$ ,  $x \in_T U_j$  then  $x \notin_T A'$ . Hence, if  $A, A' \notin \mathcal{U}$ , the family  $(U_j)_{j \in J \sqcup J'}$  is a finite family of  $\mathcal{U}$  and there does not exist  $x \in X$  such that, for every  $j \in J \sqcup J'$ ,  $x \in_T U_j$ ; this contradicts the assumption that  $\mathcal{U}$  is finitely  $T$ -pointable.  $\square$

**Proposition 6.3.26.** Let  $X$  be a set and let  $\mathcal{U}$  be a  $T$ -ultrapart. Then  $\mathcal{U}$  is upward closed.



*Proof.* Let  $A, A' \in T(X)$  such that  $A \subset_T A'$  and  $A \in \mathcal{U}$ . Since  $\mathcal{U}$  is a  $T$ -ultrapart, it is enough to show that  $\mathcal{U} \cup \{A'\}$  is finitely  $T$ -pointable. Let  $(A_j)_{j \in J}$  be a finite family of  $\mathcal{U}$ . Define  $J' := J \sqcup \{*\}$  and  $(B_j)_{j \in J'}$  with,  $B_* := A$  and, for every  $j \in J$ ,  $B_j := A_j$ . Since  $\mathcal{U}$  is finitely  $T$ -pointable, there is  $x \in X$  such that, for every  $j \in J'$ ,  $x \in_T B_j$ . Since  $A \subset_T A'$ , one has  $x \in_T A_j$  and  $x \in_T A'$  for every  $j \in J$ .  $\square$

**Proposition 6.3.27.** Let  $X$  be a  $T$ -topological space. If the space  $X$  is  $T$ -compact then every  $T$ -ultrapart of  $T(X)$  converges to some point of  $X$ .

*Proof.* Let  $\mathcal{U}$  be a  $T$ -ultrapart of  $T(X)$ . A fortiori  $\mathcal{U}$  is finitely  $T$ -pointable, so, since  $X$  is  $T$ -compact, there is  $x \in X$  such that  $x$  an adherent  $T$ -point of  $\mathcal{U}$ . By Proposition 6.2.16, there is a finitely  $T$ -pointable part  $P \subset T(X)$  such that  $\mathcal{U} \subset P$  converging to  $x$  in  $X$ . However, since  $\mathcal{U}$  is a  $T$ -ultrapart, one has  $P = \mathcal{U}$ , hence  $\mathcal{U}$  converges to  $x$  in  $X$ .  $\square$

If we assume the choice axiom, the converse implication is true:

**Proposition 6.3.28.** [Assuming the choice axiom] Let  $X$  be a  $T$ -topological space. The space  $X$  is  $T$ -compact if and only if every  $T$ -ultrapart of  $T(X)$  converges to some point of  $X$ .

*Proof.* The direct implication corresponds to the previous proposition. The converse implication is a trivial consequence of Proposition 6.2.15 and of Corollary 6.3.24.  $\square$

Basic properties of compact spaces can be adapted to our context:

**Proposition 6.3.29.** Let  $X$  be a  $T$ -compact space. Let  $P$  be a finitely  $T$ -pointable part of  $T(X)$  whose members are  $T$ -closed. Then, there is  $x \in X$  such that, for every  $A \in P$ ,  $x \in_T A$ .

*Proof.* Since  $X$  is  $T$ -compact, there is  $x \in X$  such that  $x$  is an adherent  $T$ -element of  $P$ . A fortiori, for every  $A \in P$ ,  $x$  is an adherent  $T$ -element of  $A$ , hence  $x \in_T A$  because  $A$  is  $T$ -closed.  $\square$

**Proposition 6.3.30.** Let  $X$  be a  $T$ -compact space. Let  $(O_i)_{i \in I}$  be a family of  $\mathcal{O}_T(X)$ . Assume that, for every  $i \in I$ , there is  $A_i \in T(X)$  such that, for every  $x \in X$ ,  $x \in_T A_i$  if and only if  $x \notin_T O_i$ . If, for every  $x \in X$ , there is  $i \in I$  such that  $x \in_T O_i$ , then there exists a finite subset  $J \subset I$  such that, for every  $x \in X$ , there is  $i \in J$  such that  $x \in_T O_i$ .

*Proof.* By Proposition 6.1.10, for every  $i \in I$ ,  $A_i$  is  $T$ -closed. For every  $x \in X$ , by hypothesis, there is  $i \in I$  such that  $x \in_T O_i$ , hence  $x \notin_T A_i$ . By the contraposé of the previous proposition, there is a finite subset  $J \subset I$  such that, for every  $x \in X$ , there is  $i \in J$  such that  $x \notin_T A_i$ . In other words, for every  $x \in X$ , there is  $i \in J$  such that  $x \in_T O_i$ .  $\square$

The previous proposition is similar to a kind of Borel-Lebesgue axiom. We can define more generally:

**Definition 6.3.31** ( $T$ -Borel-Lebesgue spaces). Let  $X$  be a  $T$ -topological space. The space  $X$  is a  $T$ -Borel-Lebesgue space when, for every family  $(O_i)_{i \in I}$  of  $\mathcal{O}_T(X)$  satisfying

$$\forall x \in X, \exists i \in I \text{ such that } x \in_T O_i ,$$

there is a finite subset  $J \subset I$  satisfying

$$\forall x \in X, \exists i \in J \text{ such that } x \in_T O_i .$$

Concerning changing of bases functors, one has the following results:

**Proposition 6.3.32.** Let  $X$  be a  $T$ -topological space. If  $\nu(X)$  is  $T'$ -compact then  $X$  is  $T$ -compact. If  $\nu_X$  is essentially surjective, then the converse is true.

*Proof.* Assume that  $v(X)$  is  $T'$ -compact. Let  $P$  be a finitely  $T$ -pointable part of  $T(X)$ . Then, by Proposition 6.2.6,  $v(P)$  is a finitely  $T'$ -pointable part of  $T'(X)$ . Since  $v(X)$  is  $T'$ -compact, there is  $x \in X$  an adherent  $T'$ -point of  $v(P)$ . Thus,  $x$  is an adherent  $T$ -point of  $P$  by Proposition 6.2.11.

Assume that  $v_X$  is essentially surjective and that  $X$  is  $T$ -compact. Let  $P'$  be a finitely  $T'$ -pointable part of  $T'(X)$ . Then, by Proposition 6.2.7,  $v^{-1}(P')$  is a finitely  $T$ -pointable part of  $T(X)$ . Since  $X$  is  $T$ -compact, there is  $x \in X$  an adherent  $T$ -point of  $v^{-1}(P')$ . Since  $v_X$  is essentially surjective, by Proposition 6.2.12,  $x$  is an adherent  $T'$ -point of  $P'$ .  $\square$

**Proposition 6.3.33.** Let  $X$  be a  $T$ -topological space. Then  $X$  is a  $T$ -Borel-Lebesgue space if and only if  $v(X)$  is a  $T'$ -Borel-Lebesgue space.

*Proof.* Assume that  $X$  is a  $T$ -Borel-Lebesgue space. Let  $(O'_i)_{i \in I}$  be a family of  $\mathcal{O}_{T'}(v(X))$  such that, for every  $x \in X$ , there is  $i \in I$  such that  $x \in_{T'} O'_i$ . For every  $i \in I$ , for every  $x \in_{T'} O'_i$ , there is  $O_{i,x} \in \mathcal{O}_T(X)$  such that  $x \in_{T'} v(O_{i,x}) \subset_{T'} O'_i$ . Hence, for every  $x \in X$ , there is  $i \in I$  such that  $x \in_T O_{i,x}$ . Since  $X$  is a  $(T)$ -Borel-Lebesgue space, there is a finite subset  $J \subset \bigsqcup_{i \in I} \{x \in X; x \in_{T'} O'_i\}$  such that, for every  $x \in X$ , there is  $(i, y) \in J$  such that  $x \in_T O_{i,y}$ . Write  $J' := \{i \in I; \exists y \in X \text{ such that } (i, y) \in J\}$ . Since  $J$  is finite,  $J'$  is also finite. Given  $x \in X$ , there is  $(i, y) \in J$  such that  $x \in_T O_{i,y}$ , then one has  $x \in_{T'} O'_i$  and  $i \in J'$ .

Conversely, assume that  $v(X)$  is a  $T'$ -Borel-Lebesgue space. Let  $(O_i)_{i \in I}$  be a family of  $\mathcal{O}_T(X)$  such that, for every  $x \in X$ , there is  $i \in I$  such that  $x \in_T O_i$ . Then, by Lemma 5.3.8,  $(v(O_i))_{i \in I}$  is a family of  $\mathcal{O}_{T'}(v(X))$  such that, for every  $x \in X$ , there is  $i \in I$  such that  $x \in_{T'} v(O_i)$ . Since  $v(X)$  is a  $T'$ -Borel-Lebesgue space, there is a finite subset  $J \subset I$  such that, for every  $x \in X$ , there is  $i \in J$  such that  $x \in_{T'} v(O_i)$ , so such that  $x \in_T O_i$ .  $\square$

**Corollary 6.3.34.** Let  $X$  be a  $T$ -topological space. If  $X$  is a  $T$ -Borel-Lebesgue space then  $X$  is  $T$ -compact. The converse is true when  $!^T_X$  is essentially surjective.

*Proof.* Just apply the two previous propositions to the final changing of bases datum  $!^T$ , given the fact that for the classical topological spaces, i.e. the  $(\mathcal{P}, \in)$ -topological space, compactness is equivalent to the Borel-Lebesgue axiom.  $\square$

### Examples 6.3.1.

- Let  $\mathbb{V}$  be a quantale. Any  $T_{\mathbb{V}}$ -topological space has a unique  $T_{\mathbb{V}}$ -open member, therefore it is a  $T_{\mathbb{V}}$ -Borel-Lebesgue space, and thus a  $T_{\mathbb{V}}$ -compact space.
- For the topological theory  $(\mathcal{P}, \in)$ , the notions of  $\mathcal{P}$ -Borel-Lebesgue spaces and of  $\mathcal{P}$ -compact spaces match with the classical notion of compact spaces.
- Let  $X$  be a  $\mathcal{P}_{fin}$ -topological space and let  $P \subset \mathcal{P}_{fin}(X)$  finitely  $\mathcal{P}_{fin}$ -pointable. If  $P$  is empty, then  $X \neq \emptyset$  and all elements of  $X$  are adherent  $\mathcal{P}_{fin}$ -points of  $P$ . If  $P \neq \emptyset$ , there exists  $A_0 \in P$  with the smallest cardinal among the members of  $P$ . Since  $P \subset \mathcal{P}_{fin}$  and  $P$  is  $\mathcal{P}_{fin}$ -pointable, the cardinal of  $A_0$  is a positive integer. Since the membership module is representable, we can assume that  $P$  is a filter. For every  $A \in P$ , one has  $A \cap A_0 \in P$  and its cardinal is smaller than the one of  $A_0$ , consequently the two cardinals are equal. Since  $A_0$  is finite, it follows that  $A_0 = A \cap A_0$ . Hence every element of  $A_0$  is an adherent  $\mathcal{P}_{fin}$ -element of  $P$ . Thus, every  $\mathcal{P}_{fin}$ -topological space is  $\mathcal{P}_{fin}$ -compact.

The dual of Corollary 6.3.17 is harder.

**Proposition 6.3.35.** Let  $X$  and  $Y$  be two  $T$ -topological spaces and let  $f : X \rightarrow Y$  be a surjective  $T$ -continuous map. Assume that  $T(f)$  has an inverse image  $f^{-1} : T(Y) \rightarrow T(X)$ . If  $X$  is  $T$ -compact then so is  $Y$ .

*Proof.* Let  $P^Y$  be a finitely  $T$ -pointable part of  $T(Y)$ . Write

$$P^X := \{f^{-1}(B); B \in P_T\}.$$

Let  $(B_j)_{j \in J}$  be a finite family of  $P^Y$ . Given that  $P^Y$  is finitely  $T$ -pointable, there is  $y \in Y$  such that, for every  $j \in J$ ,  $y \in_T B_j$ . Since  $f$  is surjective, there is  $x \in X$  such that  $f(x) = y$ . Since  $f^{-1}$  has an inverse image, it follows that, for every  $j \in J$ ,  $x \in_T f^{-1}(B_j)$ .

Thus  $P^X$  is finitely  $T$ -pointable. Given that  $X$  is  $T$ -compact, there is  $x' \in X$  such that  $x'$  is an adherent  $T$ -point of  $P^X$ . By Proposition 6.2.18, since  $f$  is  $T$ -continuous,  $f(x')$  is an adherent  $T$ -point of  $P_{T(f)}^X$ , image of  $P^X$  under the module  $T(f)$ .

Let  $B \in P^Y$ , one has  $f^{-1}(B) \in P^X$  and  $f^{-1}(B) T(f) B$  because  $f^{-1}(B) \subset_T f^{-1}(B)$ , hence  $B \in P_{T(f)}^X$ . Thus  $P^Y \subset P_{T(f)}^X$ , so, by Lemma 6.2.9,  $f(x')$  is an adherent  $T$ -point of  $P^Y$ .  $\square$

The result of the previous proposition still holds with other assumptions:

**Proposition 6.3.36.** Let  $X$  and  $Y$  be two  $T$ -topological spaces and let  $f : X \rightarrow Y$  be a  $T$ -continuous function which has a section  $s : Y \rightarrow X$  such that  $T(f \circ s) = T(f) \circ T(s)$ . If  $X$  is  $T$ -compact then so is  $Y$ .

*Proof.* Let  $P$  be a finitely  $T$ -pointable part of  $T(Y)$ . Then the image  $P_{T(s)}$  of  $P$  under  $T(s)$  is finitely  $T$ -pointable by Proposition 6.2.4. Therefore, given that  $X$  is  $T$ -compact, there is  $x \in X$  which is an adherent  $T$ -point of  $P_{T(s)}$  in  $X$ . By Proposition 6.2.18, since  $f$  is  $T$ -continuous,  $f(x)$  is an adherent  $T$ -element of  $(P_{T(s)})_{T(f)}$ , image of  $P_{T(s)}$  under  $T(f)$ . By Lemma 6.2.9, it is enough to show that  $P \subset (P_{T(s)})_{T(f)}$ .

Let  $B \in P^Y$ . One has  $B \subset_{T(Y)} B$ , hence

$$B T(Id_Y) B .$$

In other words, since  $f \circ s = Id_Y$ ,

$$B T(f \circ s) B .$$

Given that

$$T(f \circ s) = T(f) \circ T(s) ,$$

there is  $A \in T(X)$  such that  $B T(s) A$  and  $A T(f) B$ . From  $B T(s) A$ , we deduce that  $A \in P_{T(s)}$ , then, from  $A T(f) B$ , we deduce that  $B \in (P_{T(s)})_{T(f)}$ .  $\square$

### 6.3.3 $T$ -Alexandroff spaces

**Definition 6.3.37** ( $T$ -Alexandroff spaces). Let  $X$  be a  $T$ -topological space. The space  $X$  is a  $T$ -Alexandroff space when, for every  $x \in X$ , there is  $V_x^0 \in \mathcal{V}_T(x)$  such that, for every  $V \in \mathcal{V}_T(x)$ , one has  $V_x^0 \subset_T V$ .

*Remark 6.3.10.* Since every  $T$ -neighbourhood contains a member of  $\mathcal{O}_T(X)$ ,  $V_x^0$  is  $T$ -open in  $X$ .

**Proposition 6.3.38.** Let  $X$  be a  $T$ -Alexandroff space, let  $(O_i)_{i \in I}$  be a family of  $\mathcal{O}_T(X)$  which has a meet  $\bigwedge_{i \in I} O_i$  in  $T(X)$ . Then  $\bigwedge_{i \in I} O_i$  is  $T$ -open in  $X$ .

*Proof.* Let  $x \in X$  such that  $x \in_T \bigwedge_{i \in I} O_i$ . For every  $i \in I$ , one has  $x \in_T O_i$ , hence  $x \in_T V_x^0 \subset_T O_i$ . It follows that  $x \in_T V_x^0 \subset_T \bigwedge_{i \in I} O_i$ .  $\square$

**Proposition 6.3.39.** Let  $X$  and  $Y$  be two  $T$ -topological spaces and let  $f : X \rightarrow Y$  be a  $T$ -continuous map which has a  $T$ -continuous section  $g$ . Assume that  $T(Id_Y) = (\subset_{T(Y)})$ . If  $X$  is a  $T$ -Alexandroff space, then so is  $Y$ .

*Proof.* Let  $y \in Y$  and let  $W \in \mathcal{V}_T^Y(y)$ . Denote by  $V^0$  the smallest member of  $\mathcal{V}_T^X(g(y))$ . Since  $g$  is  $T$ -continuous at  $y$ , there is  $W^0 \in \mathcal{V}_T^Y(y)$  such that  $W^0 T(g) V^0$ . Since  $f$  is  $T$ -continuous at  $g(y)$ , one has  $V^0 T(f) W$ . Then one has  $W^0 T(f \circ g) W$ , hence  $W^0 T(Id_Y) W$ . Given that  $T(Id_Y) = (\subset_{T(Y)})$ , we finally deduce that  $W^0 \subset_T W$ .  $\square$

**Proposition 6.3.40.** Let  $X$  be a  $T$ -topological space. If  $X$  is a  $T$ -Alexandroff space, then  $\nu(X)$  is a  $T'$ -Alexandroff space. If  $\nu_X$  is fully faithful, then the converse implication holds.

*Proof.* Assume that  $X$  is a  $T$ -Alexandroff space. Let  $x \in X$  and let  $V_x^0$  be a smallest member of  $\mathcal{V}_T(x)$ . Then  $v(V_x^0)$  is a  $T'$ -neighbourhood of  $x$  in  $v(X)$ . Let  $V'$  be a  $T'$ -neighbourhood of  $x$  in  $v(X)$ . Then there is a  $T$ -neighbourhood of  $x$  in  $X$  such that  $v(V) \subset_{T'} V'$ . By hypothesis,  $V_x^0 \subset_T V$ , hence  $v(V_x^0) \subset_{T'} v(V)$ , and finally  $v(V_x^0) \subset_{T'} V'$ .

Conversely, assume that  $v_X$  is fully faithful and that  $v(X)$  is a  $T'$ -Alexandroff space. Let  $x \in X$  and let  $V_x'^0$  be a smallest member of  $\mathcal{V}_{T'}(x)$ . Then there is a  $T$ -neighbourhood  $V_x^0$  of  $x$  in  $X$  such that  $v(V_x^0) \subset_{T'} V_x'^0$ . Let  $V$  be a  $T$ -neighbourhood of  $x$  in  $X$ . Then  $v(V)$  is a  $T'$ -neighbourhood of  $x$  in  $v(X)$ , hence, by assumption,  $V_x'^0 \subset_{T'} v(V)$ , then  $v(V_x^0) \subset_{T'} v(V)$ . Finally, given that  $v_X$  is fully faithful,  $V_x^0 \subset_T V$ .  $\square$



## Chapter 7

# Limits and colimits in the category of $T$ -topological spaces

For this whole chapter, set a topological theory  $T = (T, \in_T)$ .

Initial and final topologies play a major role: the induced topology and the product topology are special cases of initial topology, while the quotient topology is a special case of final topology. They are also used to prove that the forgetful functor into **Set** is a topological functor. In this chapter, we intent to generalize this notions to the  $T$ -topological spaces. In order to fix the terminology, we first recall some definitions, with  $\mathbf{U} : \mathcal{C} \rightarrow \mathcal{D}$  a functor:

**Definition 7.0.1** (Initial lifting). Let  $D \in \mathcal{D}$ , and let  $(d_i : D \rightarrow \mathbf{U}(C_i))$  be a family of morphisms. An  $\mathbf{U}$ -initial lifting of the cone  $(D, (d_i))$  is a pair  $(C, (c_i : C \rightarrow C_i))$  such that  $\mathbf{U}(C) = D$ ,  $\mathbf{U}(c_i) = d_i$  for every  $i$ , and, for every pair  $(C', (c'_i : C' \rightarrow C_i))$  and every  $f : \mathbf{U}(C') \rightarrow \mathbf{U}(C)$  verifying  $\mathbf{U}(c_i) \circ f = \mathbf{U}(c'_i)$ , there is a unique  $h : C' \rightarrow C$  such that  $\mathbf{U}(h) = f$  and  $c_i \circ h = c'_i$  for every  $i$ .

Similarly,

**Definition 7.0.2** (Final lifting). Let  $D \in \mathcal{D}$ , and let  $(d_i : \mathbf{U}(C_i) \rightarrow D)$  be a family of morphisms. An  $\mathbf{U}$ -final lifting of the cocone  $(D, (d_i))$  is a pair  $(C, (c_i : C_i \rightarrow C))$  such that  $\mathbf{U}(C) = D$ ,  $\mathbf{U}(c_i) = d_i$  for every  $i$ , and, for every pair  $(C', (c'_i : C_i \rightarrow C'))$  and every  $f : \mathbf{U}(C) \rightarrow \mathbf{U}(C')$  verifying  $f \circ \mathbf{U}(c_i) = \mathbf{U}(c'_i)$ , there is a unique  $h : C \rightarrow C'$  such that  $\mathbf{U}(h) = f$  and  $h \circ c_i = c'_i$  for every  $i$ .

There is three special cases that will interest us:

**Definition 7.0.3** (Topological functors). The functor  $\mathbf{U}$  is a *topological functor* when every cone has an  $\mathbf{U}$ -initial lifting or, equivalently, when every cocone has an  $\mathbf{U}$ -final lifting.

**Definition 7.0.4** ( $\mathbf{U}$ -discrete and  $\mathbf{U}$ -indiscrete objects). Let  $C$  be an object of  $\mathcal{C}$  and let  $D$  be an object of  $\mathcal{D}$ .  $C$  is  $\mathbf{U}$ -indiscrete (respectively  $\mathbf{U}$ -discrete) over  $D$  when  $(C, \emptyset)$  is an  $\mathbf{U}$ -initial (respectively  $\mathbf{U}$ -final) lifting of  $(D, \emptyset)$ .

**Definition 7.0.5** (Fibration). The functor  $\mathbf{U}$  is a *fibration* when every cone of the form  $(D, f : D \rightarrow \mathbf{U}(C))$  has an  $\mathbf{U}$ -initial lifting. When  $(C', h : C' \rightarrow C)$  is an  $\mathbf{U}$ -initial lifting of  $(\mathbf{U}(C'), \mathbf{U}(h))$  with  $\mathbf{U}(h)$  a monomorphism, we say that  $h$  (or  $C'$  when there is no risk of confusion) is a *restriction* of  $C$ .

For results about this notions, we refer to [Bor94b, 7.3 and 8] or to [HST14, II.5.6]. We avoid using the terminology ‘initial structure’ as in [HST14, II.5.6] because of the weakness of the property of being isomorphic in the category  $\mathbf{Top}_T$ , see Remark 6.3.4. For the same reason, we do not use the notion of ‘topological functor’ developed in [AHS06, V.21] which is too strong. The fact that  $T$  is only a *lax* functor is a source of difficulties; the first paragraph is dedicated to this concern.

## 7.1 Inverse images relative to $T$ and saturation

Let  $X$  and  $Y$  be two sets and let  $f : X \rightarrow Y$  be a function. We defined in 5.2.3 the notion of inverse image. In some sense, this concept is pointwise because it refers to the sole module  $T(f)$ , not to the other modules of the form  $T(g)$ . In the following, we need a more global variant of this notion.

**Definition 7.1.1** (Inverse images relative to  $T$ ). Let  $X$  and  $Y$  be two sets and let  $f : X \rightarrow Y$  be a function. An *inverse image of  $T(f)$  relative to  $T$*  is an increasing map  $f^{-1} : T(Y) \rightarrow T(X)$  such that

- for every function  $g : W \rightarrow X$ , for every  $A \in T(W)$ , for every  $B \in T(Y)$ ,

$$A T(g) f^{-1}(B) \quad \text{if and only if} \quad A T(f \circ g) B$$

- for every  $x \in X$ , for every  $B \in T(Y)$ ,

$$x \in_T f^{-1}(B) \quad \text{if and only if} \quad f(x) \in_T B$$

As with inverse images, when the membership is strong (5.1.3), the first hypothesis implies the second one

**Lemma 7.1.2.** Let  $X$  and  $Y$  be two sets and let  $f : X \rightarrow Y$  be a function. Assume that the theory  $T$  is a theory with strong membership. Let  $f^{-1} : T(Y) \rightarrow T(X)$  be an increasing map that satisfies the first hypothesis of the previous definition. For every  $B \in T(Y)$ , one has

$$f^{-1}(B) T(f) B .$$

*Proof.* Let  $B \in T(Y)$ . One has  $f^{-1}(B) \subset_{T(X)} f^{-1}(B)$ , hence  $f^{-1}(B) T(\text{Id}_X) f^{-1}(B)$ , and finally  $f^{-1}(B) T(f \circ \text{Id}_X) B$ .  $\square$

**Lemma 7.1.3.** Let  $X$  and  $Y$  be two sets and let  $f : X \rightarrow Y$  be a function. Assume that  $T$  is a theory with strong membership. Let  $f^{-1} : T(Y) \rightarrow T(X)$  be an increasing map that satisfies the first hypothesis of the previous definition. Then  $f^{-1} : T(Y) \rightarrow T(X)$  is an inverse image  $T(f)$  relative to  $T$ .

*Proof.* Let  $x \in X$  and let  $B \in T(Y)$ . Assume that  $x \in_T f^{-1}(B)$ . Since  $f^{-1}(B) T(f) B$  by the previous lemma, it follows that  $f(x) \in_T B$ .

Conversely, assume that  $f(x) \in_T B$ . Given that  $T$  is with strong membership, there is  $A \in T(X)$  such that  $x \in_T A$  and  $A T(f) B$ . Consequently, one has  $A T(\text{Id}_X) f^{-1}(B)$ . Finally, because  $x \in_T A$ , we get  $x = \text{Id}_X(x) \in_T f^{-1}(B)$ .  $\square$

Conversely,

**Lemma 7.1.4.** Assume that, for every function  $f : X \rightarrow Y$ ,  $T(f)$  has an inverse image  $f^{-1} : T(Y) \rightarrow T(X)$  relative to  $T$ . Then  $T$  is a topological theory with strong membership.

*Proof.* Let  $f : X \rightarrow Y$  be a function, let  $x \in X$ , and let  $B \in T(Y)$  such that  $f(x) \in_T B$ . Then one has  $x \in_T f^{-1}(B)$  and  $f^{-1}(B) T(f) B$  by Lemma 7.1.2.  $\square$

*Remark 7.1.1.* Let  $X$  and  $Y$  be two sets and let  $f : X \rightarrow Y$  be a function. Let  $f^{-1} : T(Y) \rightarrow T(X)$  be an increasing map.

- Assume that  $T(\text{Id}_X) = (\subset_{T(X)})$ . If  $f^{-1}$  is an inverse image of  $T(f)$  relative to  $T$ , then it is also an inverse image of  $T(f)$ .
- Assume that, for every function  $g : W \rightarrow X$ ,  $T(f \circ g) = T(f) \circ T(g)$ . If  $f^{-1}$  is an inverse image of  $T(f)$  then it is also an inverse image of  $T(f)$  relative to  $T$ .
- If  $T(f)$  has an inverse image relative to  $T$ , then, for every  $g : W \rightarrow X$ ,  $T(f \circ g) = T(f) \circ T(g)$ .

In particular, when  $T$  is a functor, the two notions match.

**Examples 7.1.1.**

- For the topological theory  $(\mathcal{P}, \in)$ , the inverse image of subsets, in classical sense, allows us to define an inverse image  $f^{-1} : \mathcal{P}(Y) \rightarrow \mathcal{P}(X)$  in the previous sense for every function  $f : X \rightarrow Y$ . Since  $\mathcal{P}$  is a functor, the inverse images are also inverse images relative to  $\mathcal{P}$ .
- Let  $\mathbb{V}$  be a quantale. For the theory  $(T_{\mathbb{V}}, \in)$  (Examples 5.1.1, 5.2.1, 5.2.2), the inverse image  $\mathbb{V}$ -preorder (A.0.9) allows us to define, for every function  $f : X \rightarrow Y$ , an inverse image of  $T_{\mathbb{V}}(f)$  relative to  $T_{\mathbb{V}}$ . The inverse images *relative* to  $T_{\mathbb{V}}$  are not in general inverse images. However the distinction vanishes when  $T_{\mathbb{V}}$  is replaced by  $T_{\mathbb{V}}^l$ , and the inverse images relative to  $T_{\mathbb{V}}$  are also inverse images relative to  $T_{\mathbb{V}}^l$ .

Inverse images relative to  $T$  are functorial (while inverse images may not be):

**Proposition 7.1.5.** Let  $X, Y$  and  $Z$  be three sets and let  $f : X \rightarrow Y$  and  $e : Y \rightarrow Z$  be two functions. If  $f^{-1}$  and  $e^{-1}$  are inverse images of respectively  $T(f)$  and  $T(e)$  relative to  $T$ , then  $f^{-1} \circ e^{-1}$  is an inverse image of  $T(e \circ f)$  relative to  $T$ .

*Proof.* Let  $g : W \rightarrow X$  be a function, let  $A \in T(W)$  and let  $B \in T(Z)$ . The statement  $A T(g) f^{-1}(e^{-1}(B))$  is equivalent to  $A T(f \circ g) e^{-1}(B)$  then to  $A T(e \circ f \circ g) B$ .  $\square$

In order to use the notion of inverse image relative to  $T$ , we need an analogue of Proposition 5.2.23. For this, we use the following notion:

**Definition 7.1.6** (Pseudo- $T$ -openness). Let  $X$  be a  $T$ -topological space. A member  $A \in T(X)$  is *pseudo- $T$ -open* in  $X$  when

$$\forall x \in X, (x \in_T A) \Rightarrow (\exists O \in \mathcal{O}_T(X) \text{ such that } x \in_T O T(\text{Id}_X) A)$$

Denote by  $\mathcal{O}_T^{\text{sat}}(X)$  the set of all members of  $T(X)$  that are pseudo- $T$ -open in  $X$ .

**Examples 7.1.2.**

- For the topological theory  $(\mathcal{P}, \in)$ , since  $\mathcal{P}$  is a functor, openness and pseudo-openness match.
- Let  $\mathbb{V}$  be a quantale. For the theory  $(T_{\mathbb{V}}^l, \in)$ , since  $T_{\mathbb{V}}^l$  is a functor,  $T_{\mathbb{V}}^l$ -openness and pseudo- $T_{\mathbb{V}}^l$ -openness match. Let  $X = (X, \{(X, R)\})$  be a  $T_{\mathbb{V}}$ -topological space.  $(X, R)$  is the only member which is  $T_{\mathbb{V}}$ -open in  $X$  and the  $(X, R') \in T_{\mathbb{V}}(X)$  satisfying  $R \leq R'$  are the only members which are pseudo- $T_{\mathbb{V}}$ -open.
- More generally, in a  $T^l$ -topological space (Examples 5.3.1),  $T^l$ -openness and pseudo- $T^l$ -openness match. In a  $T$ -topological space  $X$ , the pseudo- $T$ -open members in  $X$  are the  $T^l$ -open members in  $X$  seen as a  $T^l$ -topological spaces; they form a basis of  $T^l$ -topology but not always of  $T$ -topology.

*Remark 7.1.2.* Since  $(\subset_{T(X)}) \subset T(\text{Id}_X)$ , every  $A \in T(X)$  which is  $T$ -open is a fortiori pseudo- $T$ -open, i.e.  $\mathcal{O}_T(X) \subset \mathcal{O}_T^{\text{sat}}(X)$ . The converse inclusion is true when  $(\subset_{T(X)}) = T(\text{Id}_X)$ .

We get then the following characterization of  $T$ -continuity:

**Proposition 7.1.7.** Let  $X$  and  $Y$  be two  $T$ -topological spaces and let  $f : X \rightarrow Y$  be a function. Assume that  $T(f)$  has an inverse image  $f^{-1}$  relative to  $T$ . Let  $\mathcal{B}$  be a basis of  $T$ -topology that generates  $Y$ . If, for every  $B \in \mathcal{B}$ ,  $f^{-1}(B)$  is  $T$ -open in  $X$ , then  $f$  is  $T$ -continuous. Conversely, if  $f$  is  $T$ -continuous, then, for every  $B \in \mathcal{B}$ ,  $f^{-1}(B)$  is pseudo- $T$ -open in  $X$ .

*Proof.* Assume that, for every  $B \in \mathcal{B}$ ,  $f^{-1}(B)$  is  $T$ -open in  $X$ . Let  $x \in X$  and let  $O \in \mathcal{O}_T(Y)$  such that  $f(x) \in_T O$ . There is  $B \in \mathcal{B}$  such that  $f(x) \in_T B \subset_T O$ . Then  $f^{-1}(B)$  is  $T$ -open in  $X$ ,  $x$  is a  $T$ -element of it, and  $f^{-1}(B) T(f) B$  by Lemma 7.1.2, therefore  $f^{-1}(B) T(f) O$ .

Conversely, assume that  $f$  is  $T$ -continuous. Let  $O \in \mathcal{O}_T(Y)$  and let  $x \in X$  such that  $x \in_T f^{-1}(O)$  because  $f$  is  $T$ -continuous in  $x$ . One has  $f(x) \in_T O$ , hence there is  $O' \in \mathcal{O}_T(X)$  such that  $x \in_T O' T(f) O$ . Consequently, since  $f^{-1}$  is an inverse image of  $T(f)$  relative to  $T$ ,  $x \in_T O' T(\text{Id}_X) f^{-1}(O)$ .  $\square$



There is an analogue of Proposition 6.3.35.

**Proposition 7.1.8.** Let  $X$  and  $Y$  be two  $T$ -topological spaces and let  $f : X \rightarrow Y$  be a surjective  $T$ -continuous map. Assume that  $T(f)$  has an inverse image  $f^{-1}$  relative to  $T$ . If  $X$  is  $T$ -compact then so is  $Y$ .

*Proof.* Let  $P^Y$  be a finitely  $T$ -pointable part of  $T(Y)$ . Write

$$P^X := \{f^{-1}(B) ; B \in \mathcal{F}\}$$

Let  $(B_j)_{j \in J}$  be a finite family of  $P^Y$ . Since  $P^Y$  is finitely  $T$ -pointable, there is  $y \in Y$  such that, for every  $j \in J$ ,  $y \in_T B_j$ . Given that  $f$  is surjective, there is  $x \in X$  such that  $f(x) = y$ . Consequently, one has, for every  $j \in J$ ,  $x \in_T f^{-1}(B_j)$ . Thus  $P^X$  is finitely  $T$ -pointable.

Since  $X$  is  $T$ -compact, there is  $x' \in X$  such that  $x'$  is an adherent  $T$ -point of  $P^X$ . By Proposition 6.2.18, given that  $f$  is  $T$ -continuous,  $f(x')$  is an adherent  $T$ -point of  $P_{T(f)}^X$ , image of  $P^X$  under  $T(f)$ . Let  $B \in P^Y$ , one has  $f^{-1}(B) \in P^X$  and, by Lemma 7.1.2,  $f^{-1}(B) T(f) B$ , hence  $B \in P_{T(f)}^X$ . Consequently  $P^Y \subset P_{T(f)}^X$ , and then, by Lemma 6.2.9,  $f(x')$  is an adherent  $T$ -point of  $P^Y$ .  $\square$

Many of the following statements require that the pseudo- $T$ -open members form a basis of  $T$ -topology.

**Definition 7.1.9** (Saturated and saturable  $T$ -topological spaces). Let  $X$  be a  $T$ -topological space. The space  $X$  is *saturated* when  $\mathcal{O}_T(X) = \mathcal{O}_T^{sat}(X)$  and is *saturable* when  $\mathcal{O}_T^{sat}(X)$  is a basis of  $T$ -topology on  $X$ . When  $X$  is saturable, write  $X^{sat} := (X, \mathcal{O}_T^{sat}(X))$  and call this space *the saturated of  $X$* .

*Remark 7.1.3.* Since  $\mathcal{O}_T(X) \subset \mathcal{O}_T^{sat}(X)$ ,  $X$  is saturable if and only if, for every  $x \in X$ , for every  $A, A' \in \mathcal{O}_T^{sat}(X)$  such that  $x \in_T A, A'$ , there is  $A'' \in \mathcal{O}_T^{sat}(X)$  such that  $x \in_T A'' \subset_T A, A'$ .

**Example 7.1.1.** Let  $\mathbb{V}$  be a quantale and let  $X = (X, \{(X, R)\})$  be a  $T_{\mathbb{V}}$ -topological space. The space  $X$  is saturable if and only if  $R$  is a maximal  $\mathbb{V}$ -preorder on  $X$ . In this case  $X^{sat} = X$ .

**Lemma 7.1.10.** Let  $X$  be a  $T$ -topological space. If  $X$  is saturable then  $\mathcal{O}_T(\mathcal{O}_T^{sat}(X)) = \mathcal{O}_T^{sat}(X)$ .

*Proof.* Let  $O' \in \mathcal{O}_T(\mathcal{O}_T^{sat}(X))$  and let  $x \in X$  such that  $x \in_T O'$ . There is then  $A \in \mathcal{O}_T^{sat}(X)$  such that  $x \in_T A \subset_T O'$ . It follows that, since  $A$  is pseudo- $T$ -open in  $X$ , there is  $O \in \mathcal{O}_T(X)$  such that  $x \in_T O T(\text{Id}_X) A$ . Since  $T(\text{Id}_X)$  is a module, we deduce that  $x \in_T O T(\text{Id}_X) O'$ . Thus  $O'$  is pseudo- $T$ -open in  $X$ , hence  $O' \in \mathcal{O}_T^{sat}(X)$ .  $\square$

The following proposition justifies the name given to  $X^{sat}$ .

**Proposition 7.1.11.** Let  $X$  be a  $T$ -topological space. If  $X$  is saturable then the saturated  $X^{sat}$  of  $X$  is a saturated  $T$ -topological space.

*Proof.* Let  $A' \in \mathcal{O}_T^{sat}(X^{sat})$  and let  $x \in X$  such that  $x \in_T A$ . Then, there is  $A \in \mathcal{O}_T(\mathcal{O}_T^{sat}(X))$  such that  $x \in_T A \subset_T A'$ . By the previous lemma,  $A$  is pseudo- $T$ -open in  $X$ . Thus, there is  $O \in \mathcal{O}_T(X)$  such that  $x \in_T O T(\text{Id}_X) A$ . Consequently  $x \in_T O T(\text{Id}_X) A'$  and then  $A'$  is pseudo- $T$ -open in  $X$ .  $\square$

**Proposition 7.1.12.** Let  $X$  be a  $T$ -topological space. If  $X$  is saturable, then the map  $\text{Id}_X$  is an isomorphism in  $\mathbf{Top}_T$  between  $X$  and  $X^{sat}$ .

*Proof.* Given that  $\mathcal{O}_T(X) \subset \mathcal{O}_T^{sat}(X)$ , the map  $\text{Id}_X$  is  $T$ -continuous from  $X^{sat}$  to  $X$ . Conversely, let  $x \in X$  and let  $A \in \mathcal{O}_T^{sat}(X)$  such that  $x \in_T A$ . By definition of pseudo- $T$ -openness, there is  $O \in \mathcal{O}_T(X)$  such that  $x \in_T O T(\text{Id}_X) A$ . Thus the map  $\text{Id}_X$  is  $T$ -continuous from  $X$  to  $X^{sat}$ .  $\square$

Thus, concerning  $T$ -continuity, and in particular for Proposition 7.1.7, if a  $T$ -topological space  $X$  is saturable, we can assume that the pseudo- $T$ -open members in  $X$  match with the  $T$ -open members.

**Definition 7.1.13** (Saturable and saturated topological theories). A topological theory  $T$  is *saturable* (respectively *saturated*) when every  $T$ -topological space is saturable (respectively saturated).

We have the following sufficient condition for saturability:

**Lemma 7.1.14.** Let  $X$  be a  $T$ -topological space. Assume that  $T(\text{Id}_X)$  has an inverse image  $i^{-1}$ . Let  $A \in T(X)$ . Then  $A$  is pseudo- $T$ -open if and only if  $i^{-1}(A)$  is  $T$ -open.

*Proof.* Indeed, given  $x \in X$  and  $O$  a  $T$ -open member of  $X$ , one has  $x \in_T O \cap T(i) A$  if and only if  $x \in_T O \subset_T i^{-1}(A)$ .  $\square$

**Proposition 7.1.15.** Let  $X$  be a  $T$ -topological space. Assume that  $T(X)$  is finitely complete and that  $T(\text{Id}_X)$  has an inverse image  $i^{-1}$  preserving finite meets, then  $X$  is saturable.

*Proof.* Let  $(A_j)_{j \in J}$  be a finite family of pseudo- $T$ -open members of  $X$ . Denote by  $\bigwedge_{j \in J} A_j$  a meet of the  $A_j$  in  $T(X)$ . Since every  $i^{-1}(A_j)$  is  $T$ -open by the previous lemma and since  $i^{-1}$  preserves finite meets, by Proposition 5.2.11,  $i^{-1}(\bigwedge_{j \in J} A_j) = \bigwedge_{j \in J} i^{-1}(A_j)$  is  $T$ -open, hence  $\bigwedge_{j \in J} A_j$  is pseudo- $T$ -open.  $\square$

Pseudo- $T$ -openness is preserved by changing of bases:

**Proposition 7.1.16.** Let  $T = (T, \in_T)$  and  $T' = (T', \in_{T'})$  be two topological theories and let  $v : T \rightarrow T'$  be a changing of bases datum. Let  $X$  be a  $T$ -topological space and let  $A \in T(X)$ . If  $A$  is pseudo- $T$ -open in  $X$  then  $v_X(A)$  is pseudo- $T'$ -open in  $v(X)$ .

*Proof.* Let  $x \in X$  such that  $x \in_{T'} v_X(A)$ . Then  $x \in_T A$ , hence there is  $O \in \mathcal{O}_T(X)$  such that  $x \in_T O \cap T(\text{Id}_X) A$ . It follows that  $x \in_{T'} v_X(O) \cap T'(\text{Id}_X) v_X(A)$ .  $\square$

## 7.2 Final $T$ -topology

Assume that, for every function  $f : X \rightarrow Y$ , the module  $T(f)$  has an inverse image  $f^{-1}$  relative to  $T$ .

Let  $(X_i)_{i \in I}$  be a family of  $T$ -topological spaces, let  $X$  be a set, and let  $(f_i : X_i \rightarrow X)_{i \in I}$  be a family of functions. Also assume that  $T(X)$  is finitely complete, that every  $f_i^{-1}$  preserves finite meets, that  $x_T^X$  preserves finite meets (Definition 5.2.15), and that every space  $X_i$  is saturable.

Write

$$\mathcal{B} := \{B \in T(X) ; \forall i \in I, f_i^{-1}(B) \in \mathcal{O}_T(X_i^{\text{Sat}})\}.$$

**Lemma 7.2.1.** The set  $\mathcal{B}$  is a basis of  $T$ -topology on  $X$ .

*Proof.* Let  $x \in X$  and let  $(B_j)_{j \in J}$  be a finite family of  $\mathcal{B}$  such that, for every  $j \in J$ ,  $x \in_T B_j$ . Given that  $\in_T^X$  preserves finite meets, one has  $x \in_T \bigwedge_{j \in J} B_j$ , where  $\bigwedge_{j \in J} B_j$  is a finite meet of  $(B_j)_{j \in J}$  in the finitely complete preordered set  $T(X)$ . It is enough to show that  $\bigwedge_{j \in J} B_j \in \mathcal{B}$  to conclude.

Let  $i \in I$ . Since  $f_i^{-1}$  preserves finite meets, one has  $f_i^{-1}(\bigwedge_{j \in J} B_j) = \bigwedge_{j \in J} f_i^{-1}(B_j)$ . By definition, the  $f_i^{-1}(B_j)$  are  $T$ -open in  $X_i^{\text{Sat}}$ , hence, by stability of  $T$ -openness under finite meets (Proposition 5.2.11),  $\bigwedge_{j \in J} f_i^{-1}(B_j) \in \mathcal{O}_T(X_i^{\text{Sat}})$ .  $\square$

Call this  $T$ -topology the *final  $T$ -topology* on  $X$  associated to the  $f_i$  and call  $X$  endowed with it the *final  $T$ -topological space* associated to the  $(f_i)$ . From now on, we assume that  $X$  is endowed with this topology.

**Lemma 7.2.2.** The functions  $f_i : X_i \rightarrow X$  are  $T$ -continuous.

*Proof.* Let  $i \in I$ . Since  $X_i$  is saturable, the map  $\text{Id}_{X_i}$  is an isomorphism in  $\mathbf{Top}_T$  between  $X_i$  and  $X_i^{\text{Sat}}$ . It follows that each  $f_i : X_i \rightarrow X$  is  $T$ -continuous if and only if  $f_i : X_i^{\text{Sat}} \rightarrow X$  is  $T$ -continuous. The map  $f_i$  is  $T$ -continuous from  $X_i^{\text{Sat}}$  to  $X$  by Proposition 7.1.7.  $\square$

**Corollary 7.2.3.** The subset  $\mathcal{B}$  is equal to the subset  $\mathcal{O}_T(\mathcal{B})$ .

*Proof.* Indeed, given  $O \in \mathcal{O}_T(\mathcal{B})$ , by Proposition 7.1.7, since the  $f_i$  are  $T$ -continuous, the  $f_i^{-1}(O)$  are pseudo- $T$ -open, hence  $O \in \mathcal{B}$ .  $\square$

Let  $Y$  be a  $T$ -topological space, and, for every  $i \in I$ , let  $g_i : X_i \rightarrow Y$  be a  $T$ -continuous map. Let  $h : X \rightarrow Y$  be a function such that, for every  $i \in I$ ,  $g_i = h \circ f_i$ . Then  $h$  is  $T$ -continuous. Indeed, given  $O \in \mathcal{O}_T(Y)$ , by Proposition 7.1.7, it is enough to show that  $h^{-1}(O)$  is  $T$ -open in  $X$ . Let  $i \in I$ , since  $f_i^{-1} \circ h^{-1}$  is an inverse image of  $T(h \circ f_i)$  relative to  $T$  by Proposition 7.1.5 and since  $g_i = h \circ f_i$  is  $T$ -continuous from  $X_i$  to  $Y$ , by Proposition 7.1.7,  $f_i^{-1}(h^{-1}(O))$  is pseudo- $T$ -open in  $X_i$ . Then, by definition of the final  $T$ -topology of  $X$ ,  $h^{-1}(O)$  is  $T$ -open in  $X$ .

Thus  $X$  endowed with the final  $T$ -topology and the  $f_i$  seen as  $T$ -continuous maps are a  $\mathbf{U}_T$ -final lifting of the cocone  $(X, (f_i))$  in the sense of Definition 7.0.1. Therefore, we have the following theorem:

**Theorem 7.2.4.** Assume that the theory  $T$  is saturable, that, for every set  $X$ ,  $T(X)$  is finitely complete and  $\in_T^X$  preserves finite meets, and that, for every function  $f : X \rightarrow Y$ ,  $T(f)$  has an inverse image relative to  $T$  which preserves finite meets. Then the forgetful functor  $\mathbf{U}_T : \mathbf{Top}_T \rightarrow \mathbf{Set}$  is a topological functor.

*Remark 7.2.1.* The assumptions of the previous theorem are in particular satisfied when, for every set  $X$ ,  $T(X)$  is finitely complete and  $\in_T^X$  preserves finite meets, for every function  $f : X \rightarrow Y$ ,  $T(f)$  has an inverse image which preserves finite meets, and  $T$  commutes with  $\circ$ .

**Example 7.2.1.** Let  $\mathbb{V}$  a be quantale. One easily checks, since  $T_{\mathbb{V}}^l$  is a functor and thanks to results of Section A, that  $T_{\mathbb{V}}^l$  satisfies the assumptions as the previous theorem and thus that  $\mathbf{U}_{T_{\mathbb{V}}^l}$  is a topological functor.

Hence, by Propositions [Bor94b, 7.3.7 and 7.3.8], we deduce the

**Corollary 7.2.5.** With the same assumptions as the previous theorem, the category  $\mathbf{Top}_T$  is complete and cocomplete, the forgetful functor preserves limits and colimits, and has fully faithful left and right adjoints.

## 7.3 Discrete and coarsest $T$ -topologies

We can give a more concrete description of the adjoints of the forgetful functor, which holds with weaker assumptions.

**Definition 7.3.1** (Discrete  $T$ -topology). A set  $X$  has a discrete  $T$ -topology when  $T(X)$  is a basis of  $T$ -topology on  $X$ . The  $T$ -topology  $T(X)$  is then called *discrete  $T$ -topology* and the space  $(X, T(X))$  is called the *discrete  $T$ -topological space* on  $X$ .

**Proposition 7.3.2.** Let  $X$  be a set. Assume that one of the two following assumptions is true:

- The preordered set  $T(X)$  is finitely complete and  $\in_T^X$  preserves finite meets.
- The membership module  $\in_T^X$  est representable.

Then  $X$  has a discrete  $T$ -topology.

*Proof.* Trivial.  $\square$

**Proposition 7.3.3.** Let  $X$  and  $Y$  be two  $T$ -topological spaces with  $X$  endowed with the discrete  $T$ -topology and let  $f : X \rightarrow Y$  be a function. Moreover, assume that one of the following statements is true:

- The topological theory  $T$  is with strong membership.
- The module  $T(f)$  has an inverse image.

- The module  $T(f)$  has an inverse image relative to  $T$ .

Then  $f$  is  $T$ -continuous.

*Proof.* Let  $x \in X$  and let  $O \in \mathcal{O}_T(Y)$  such that  $f(x) \in_T O$ .

Assume that the topological theory  $T$  is with strong membership. Given that  $f(x) \in_T O$ , there is  $A \in T(X)$  such that  $x \in_T A$  and  $T(f)O$ , and moreover, by definition of the discrete  $T$ -topology,  $A \in \mathcal{O}_T(X)$ .

Now, assume that  $f^{-1}$  an inverse image or has an inverse image relative to  $T$   $f^{-1}$  of  $T(f)$ . In both cases, one has  $x \in_T f^{-1}(O)$  and  $f^{-1}(O) \in \mathcal{O}_T(X)$  by definition of the discrete  $T$ -topology.  $\square$

Hence, when one of the three statements of the previous proposition is satisfied,  $X$  endowed with the discrete  $T$ -topology (when it exists) is a  $\mathbf{U}_T$ -discrete object as defined in 7.0.4.

**Corollary 7.3.4.** Assume that every set has a discrete  $T$ -topology and that one of the two following statements is true:

- The topological theory  $T$  is with strong membership.
- For every function  $f$ ,  $T(f)$  has an inverse image or has an inverse image relative to  $T$ .

Then the forgetful functor  $\mathbf{U}_T : \mathbf{Top}_T \rightarrow \mathbf{Set}$  has a left adjoint given by the discrete  $T$ -topological spaces.

**Corollary 7.3.5.** With the same hypothesis as in the previous corollary, the forgetful functor  $\mathbf{U}_T : \mathbf{Top}_T \rightarrow \mathbf{Set}$  preserves limits.

**Definition 7.3.6** (Indiscrete  $T$ -topology). Let  $X$  be a set such that the preordered set  $T(X)$  has greatest members. Then the set of all greatest members of  $T(X)$  is a basis of  $T$ -topology. This  $T$ -topology is called the *coarsest* or *indiscrete  $T$ -topology* on  $X$  and the set  $X$  equipped with this  $T$ -topology is called the *coarsest* or *indiscrete  $T$ -topological space* on  $X$ .

**Proposition 7.3.7.** Let  $X$  and  $Y$  be two  $T$ -topological spaces with  $Y$  endowed with the indiscrete  $T$ -topology and let  $f : X \rightarrow Y$  be a function. If, for every  $x \in X$ , there is  $O \in \mathcal{O}_T(X)$  and  $B \in T(Y)$  such that  $x \in_T O$  and  $O T(f) B$ , then  $f$  is  $T$ -continuous.

*Proof.* Let  $x \in X$  and let  $W \in \mathcal{V}_T(f(x))$ . Then  $W$  is a greatest member of  $T(Y)$ . Par hypothesis, there is  $O \in \mathcal{O}_T(X)$  and  $B \in T(Y)$  such that  $x \in_T O$  and  $O T(f) B$ . Given that  $T(f)$  is a module and that  $B \subset_T W$ , one has  $O T(f) W$ .  $\square$

Hence,  $X$  endowed with the indiscrete  $T$ -topology (when it exists) is a  $\mathbf{U}_T$ -indiscrete object as defined in 7.0.4 when the assumption of the previous proposition is satisfied for every  $X$ -valued function.

**Corollary 7.3.8.** Assume that, for every set  $Y$ ,  $T(Y)$  has greatest members, and that, for every function  $f : X \rightarrow Y$ , if  $1_X$  and  $1_Y$  are greatest members of  $T(X)$  and  $T(Y)$  respectively, then  $1_X T(f) 1_Y$  (the latter implication is true in particular when  $T(f)$  has an inverse image or has an inverse image relative to  $T$  which preserves greatest elements). Then the forgetful functor  $\mathbf{U}_T : \mathbf{Top}_T \rightarrow \mathbf{Set}$  has a right adjoint given by the indiscrete  $T$ -topological spaces.

**Corollary 7.3.9.** If  $T$  satisfies the same hypothesis as the previous corollary, then the forgetful functor preserves colimits.

## 7.4 Initial $T$ -topology

Assume that the topological theory  $T$  est saturable, that, for every set  $X$ ,  $T(X)$  is finitely complete and  $\in_T^X$  preserves finite meets, and that, for every function  $f : X \rightarrow Y$ ,  $T(f)$  has an inverse image  $f^{-1}$  relative to  $T$  which preserves finite meets.

Let  $(X_i)_{i \in I}$  be a family of  $T$ -topological space, let  $X$  be a set and let  $(f_i : X \rightarrow X_i)_{i \in I}$  be a family of functions.

Write

$$\mathcal{B} := \left\{ \bigwedge_{j \in J} f_j^{-1}(O_j) ; J \subset I \text{ finite, } O_j \in \mathcal{O}_T(X_j) \right\}.$$

**Lemma 7.4.1.** The set  $\mathcal{B}$  is a basis of  $T$ -topology on  $X$ .

*Proof.* Let  $x \in X$ . By taking  $J = \emptyset$ , one has  $1_{T(X)} \in \mathcal{B}$  and  $x \in_T 1_{T(X)}$  since  $\in_T^X$  preserves finite meets, where  $1_{T(X)}$  is a greatest member of  $T(X)$ .

Let  $J$  and  $J'$  be two finite subsets of  $I$ , and let  $(O_j)_{j \in J}$  and  $(O'_j)_{j \in J'}$  be such that, for every  $j \in J$ ,  $O_j \in \mathcal{O}_T(X_j)$ , for every  $j \in J'$ ,  $O'_j \in \mathcal{O}_T(X_j)$ ,  $x \in_T \bigwedge_{j \in J} f_j^{-1}(O_j)$  and  $x \in_T \bigwedge_{j \in J'} f_j^{-1}(O'_j)$ . Write  $J'' := J \cup J'$ ,  $O''_j := O_j$  if  $j \in J \setminus J'$ ,  $O''_j := O'_j$  if  $j \in J' \setminus J$ ,  $O''_j = O_j \wedge O'_j$  if  $j \in J \cap J'$ . One has  $\bigwedge_{j \in J''} f_j^{-1}(O''_j) \in \mathcal{B}$  by stability of  $T$ -openness under finite meets (Proposition 5.2.11). Since the  $f_j^{-1}$  preserve finite meets,

$$\bigwedge_{j \in J''} f_j^{-1}(O''_j) = \bigwedge_{j \in J} f_j^{-1}(O_j) \wedge \bigwedge_{j \in J'} f_j^{-1}(O'_j).$$

Since  $\in_T^X$  preserves finite meets,  $x \in_T \bigwedge_{j \in J''} f_j^{-1}(O''_j)$ . By definition of meets,  $\bigwedge_{j \in J''} f_j^{-1}(O''_j) \subset_T \bigwedge_{j \in J} f_j^{-1}(O_j)$  and  $\bigwedge_{j \in J''} f_j^{-1}(O''_j) \subset_T \bigwedge_{j \in J'} f_j^{-1}(O'_j)$ .  $\square$

We call the  $T$ -topology generated by  $\mathcal{B}$  the *initial  $T$ -topology* on  $X$  associated to  $(f_i)_{i \in I}$  and we call the set  $X$  endowed with it the *initial  $T$ -topological space* associated to  $(f_i)_{i \in I}$ .

**Lemma 7.4.2.** When  $X$  is endowed with the initial  $T$ -topology, the functions  $f_i$  are  $T$ -continuous.

*Proof.* Trivial by Proposition 7.1.7.  $\square$

*Remark 7.4.1.* Unlike the case of the final  $T$ -topology, in general the basis  $\mathcal{B}$  does not match with  $\mathcal{O}_T(\mathcal{B})$ .

**Proposition 7.4.3.** Let  $W$  be a  $T$ -topological space,  $(g_i : W \rightarrow X_i)_{i \in I}$  be a family of  $T$ -continuous maps and let  $h : W \rightarrow X$  be a function such that, for every  $i$ ,  $g_i = f_i \circ h$ . Then  $h$  is  $T$ -continuous with  $X$  endowed with the initial  $T$ -topology.

*Proof.* Let  $J \subset I$  be a finite subset and let  $(O_j)_{j \in J}$  be a family such that, for every  $j \in J$ ,  $O_j$  is  $T$ -open in  $X_j$ . Given that  $h^{-1}$  preserves finite meets, one has

$$h^{-1}\left(\bigwedge_{j \in J} f_j^{-1}(O_j)\right) = \bigwedge_{j \in J} h^{-1}(f_j^{-1}(O_j)).$$

However, the  $h^{-1}(f_j^{-1}(O_j))$  are pseudo- $T$ -open in  $W$  by Proposition 7.1.7, i.e. are  $T$ -opens in  $W^{sat}$ , because the functions  $f_j \circ h = g_j$  are  $T$ -continuous from  $W$  to  $X_j$  and the composites  $h^{-1} \circ f_j^{-1}$  are inverse images relative to  $T$  of the  $T(f_i \circ h)$  by Proposition 7.1.5. Thus, by Proposition 5.2.11,  $h^{-1}(\bigwedge_{j \in J} f_j^{-1}(O_j))$  is  $T$ -open in  $W^{sat}$  and then  $h$  is  $T$ -continuous from  $W^{sat}$  to  $X$ , again by Proposition 7.1.7. Since the map  $\text{Id}_W$  is  $T$ -continuous from  $W$  to  $W^{sat}$ , we finally get the  $T$ -continuity of  $h$  from  $W$  to  $X$ .  $\square$

Therefore  $X$  endowed with the initial  $T$ -topology and the  $f_i$  seen as  $T$ -continuous maps is an  $\mathbf{U}_T$ -initial lifting of the cone  $(X, (f_i))$  as defined in 7.0.2.

**Proposition 7.4.4.** Let  $x \in X$  and let  $V \in T(X)$  such that  $x \in_T V$ . Then  $V$  is a  $T$ -neighbourhood of  $x$  for the initial  $T$ -topology if and only if there is a finite subset  $J \subset I$  and, for every  $j \in J$ ,  $V_j \in \mathcal{V}_T^{X_j}(f_j(x))$ , such that  $\bigwedge_{j \in J} f_j^{-1}(V_j) \subset_T V$ .

*Proof.* Assume that  $V$  is a  $T$ -neighbourhood of  $x$  for the initial  $T$ -topology. Then, there is a finite subset  $J \subset I$ , and, for every  $j \in J$ ,  $O_j \in \mathcal{O}_T(X_j)$ , such that  $x \in_T \bigwedge_{j \in J} f_j^{-1}(O_j) \subset_{T(X)} V$ . From  $x \in_T \bigwedge_{j \in J} f_j^{-1}(O_j)$ , we deduce, for every  $j \in J$ , that  $x \in_T f_j^{-1}(O_j)$ , then that  $f(x) \in_T O_j$ . Therefore, for every  $j \in J$ ,  $O_j$  is a  $T$ -neighbourhood of  $f_j(x)$  in  $X_j$ .

Conversely, let  $J \subset I$  be a finite subset and let, for every  $j \in J$ ,  $V_j \in \mathcal{V}_T^{X_j}(f_j(x))$ , such that  $\bigwedge_{j \in J} f_j^{-1}(V_j) \subset_T V$ . Let  $j \in J$ , there is  $O_j \in \mathcal{O}_T(X_j)$  such that  $f_j(x) \in_T O_j \subset_T V_j$ . Then  $\bigwedge_{j \in J} f_j^{-1}(O_j)$  is  $T$ -open in  $X$ . For every  $j \in J$ ,  $f_j(x) \in_T O_j$  hence  $x \in_T f_j^{-1}(O_j)$ . Since  $\in_T^X$  preserves finite meets,  $x \in_T \bigwedge_{j \in J} f_j^{-1}(O_j)$ . Moreover,  $\bigwedge_{j \in J} f_j^{-1}(O_j) \subset_T \bigwedge_{j \in J} f_j^{-1}(V_j)$  and  $\bigwedge_{j \in J} f_j^{-1}(V_j) \subset_T V$ , then finally  $\bigwedge_{j \in J} f_j^{-1}(O_j) \subset_T V$ .  $\square$

**Proposition 7.4.5.** Let  $P \subset T(X)$  and let  $x \in X$ . If the part  $P$  converges to  $x$  in  $X$  then, for every  $i \in I$ , the part  $P_{T(f_i)}$ , image of  $P$  under  $T(f_i)$  (Definition 3.2.5), converges to  $f_i(x)$ . Conversely, when  $P$  is a filter such that, for every  $A, A' \in T(X)$ ,

$$(A \in P \text{ and } AT(\text{Id}_X) A') \Rightarrow A' \in P \quad ,$$

the converse implication is true.

*Proof.* If  $P$  converges to  $x$  in  $X$ , for every  $i \in I$ , since  $f_i$  is  $T$ -continuous, by Proposition 6.2.19,  $P_{T(f_i)}$  converges to  $f_i(x)$ .

Conversely, assume that  $P$  satisfies the assertions of the proposition and assume that, for every  $i \in I$ , the part  $P_{T(f_i)}$  converges to  $f_i(x)$ . Let  $V$  be a  $T$ -neighbourhood of  $x$  in  $X$ . By the previous proposition, there is a finite subset  $J \subset I$  and, for every  $j \in J$ ,  $V_j \in \mathcal{V}_T^{X_j}(f_j(x))$ , such that  $\bigwedge_{j \in J} f_j^{-1}(V_j) \subset_{T(X)} V$ . Given  $j \in J$ , since  $P_{T(f_j)}$  converges to  $f_j(x)$ ,  $V_j \in P$ . In other words, there is  $A_j \in P$  such that  $A_j T(f_j) V_j$ . It follows that  $A_j T(\text{Id}_X) f_j^{-1}(V_j)$ , hence, by hypothesis on  $P$ , one has  $f_j^{-1}(V_j) \in P$ . Then  $\bigwedge_{j \in J} f_j^{-1}(V_j) \in P$  because  $P$  is a filter. Finally, since  $\bigwedge_{j \in J} f_j^{-1}(V_j) \subset_{T(X)} V$ , we conclude that  $V \in P$ .  $\square$

We use this proposition to show a kind of Tychonoff theorem. To this aim, we first give a sufficient condition for the image of a  $T$ -ultrapart to be a  $T$ -ultrapart: Lemma 7.4.6 and Corollary 7.4.7 are valid without the assumptions made at the beginning of this section.

**Lemma 7.4.6.** Let  $X$  be a  $T$ -topological space such that  $!_X^T$  is surjective, and let  $\mathcal{U}$  be a finitely  $T$ -pointable part of  $T(X)$ . If, for every  $A, A' \in T(X)$  such that  $X = !_X^T(A) \sqcup !_X^T(A')$ ,  $A \in \mathcal{U}$  or  $A' \in \mathcal{U}$ , then  $\mathcal{U}$  is a  $T$ -ultrapart of  $T(X)$ .

*Proof.* Let  $A \in T(X)$  such that  $\{A\} \cup \mathcal{U}$  is finitely  $T$ -pointable. Since  $!_X^T$  is surjective, there is  $A'$  such that  $X = !_X^T(A) \sqcup !_X^T(A')$ . We cannot have  $A' \in \mathcal{U}$  because it would contradict the fact that  $\{A\} \cup \mathcal{U}$  is finitely  $T$ -pointable. Then one has  $A \in \mathcal{U}$ , consequently  $\mathcal{U}$  is a  $T$ -ultrapart of  $T(X)$ .  $\square$

**Corollary 7.4.7.** Let  $X$  and  $Y$  be two  $T$ -topological spaces such that  $!_Y^T$  is surjective, let  $f : X \rightarrow Y$  be a  $T$ -continuous map which has an inverse image  $f^{-1}$  relative to  $T$ , and let  $\mathcal{U}$  be a  $T$ -ultrapart of  $T(X)$ . Then  $\mathcal{U}_{T(f)}$ , image of  $\mathcal{U}$  under  $T(f)$ , is a  $T$ -ultrapart of  $T(Y)$ .

*Proof.* Since  $!_Y^T$  is surjective, we can use the previous lemma to show that  $\mathcal{U}_{T(f)}$  is a  $T$ -ultrapart of  $T(Y)$ . Given that  $\mathcal{U}$  is a  $T$ -ultrapart of  $T(X)$ , it is finitely  $T$ -pointable, hence, by Proposition 6.2.4,  $\mathcal{U}_{T(f)}$  is finitely  $T$ -pointable. Let  $A, A' \in T(Y)$  such that  $Y = !_Y^T(A) \sqcup !_Y^T(A')$ . Then one has

$$!_X^T(f^{-1}(A)) \sqcup !_X^T(f^{-1}(A')) = X \quad ,$$

hence, by Lemma 6.3.25, given that  $\mathcal{U}$  is a  $T$ -ultrapart of  $T(X)$ ,  $f^{-1}(A) \in \mathcal{U}$  or  $f^{-1}(A') \in \mathcal{U}$ . Since  $f^{-1}(A) T(f) A$  and  $f^{-1}(A') T(f) A'$ , one has  $A \in \mathcal{U}_{T(f)}$  or  $A' \in \mathcal{U}_{T(f)}$ .  $\square$

**Corollary 7.4.8** (Tychonoff Theorem). [Assume the axiom of choice] If all  $X_i$  are  $T$ -compacts, if all  $!_{X_i}^T$  are surjective, and if, for every family  $(x_i \in X_i)_{i \in I}$ , there is  $x \in X$  such that, for every  $i \in I$ ,  $f_i(x) = x_i$ , then  $X$  is  $T$ -compact.

*Proof.* Since we assume the axiom of choice, to prove that  $X$  is  $T$ -compact, we are to show the every  $T$ -ultrapart of  $T(X)$  converges to some point of  $X$ .

Let  $\mathcal{U}$  be a  $T$ -ultrapart of  $T(X)$ . Firstly, we are to show that we can apply the previous proposition. Since we assumed at the beginning of the section that  $T(f)$  is finitely complete and that  $\in_T^X$  preserves finite meets, by Remark 6.3.9,  $\mathcal{U}$  is a filter. Let  $A, A' \in T(X)$  such that

A  $T(\text{Id}_X) A'$  and  $A \in \mathcal{U}$ . Since  $\mathcal{U}$  is a  $T$ -ultrafilter, to prove that  $A' \in \mathcal{U}$ , it is enough to show that  $\mathcal{U} \cup \{A'\}$  is finitely  $T$ -pointable. Let  $(A_j)_{j \in J}$  be a finite family of  $\mathcal{U}$ . Write  $J' := J \sqcup \{*\}$  and  $(B_j)_{j \in J'}$  with,  $B_* := A$  and, for every  $j \in J$ ,  $B_j := A_j$ . Since  $\mathcal{U}$  is finitely  $T$ -pointable, there is  $x \in X$  such that, for every  $j \in J'$ ,  $x \in_T B_j$ . Given that  $A T(\text{Id}_X) A'$ , one has, for every  $j \in J$ ,  $x \in_T A_j$ , and  $x \in_T A'$ . Therefore  $\mathcal{U}$  satisfies the assumptions of the previous proposition. Let  $i \in I$ , by Corollary 7.4.7, the part  $\mathcal{U}_{T(f_i)}$ , image of  $\mathcal{U}$  under  $T(f_i)$ , is a  $T$ -ultrapart. Since  $X_i$  is  $T$ -compact, there is  $x_i \in X_i$  such that  $\mathcal{U}_{T(f_i)}$  converges to  $x_i$ . There exists  $x \in X$  such that, for every  $i \in I$ ,  $f_i(x) = x_i$ , and, by the previous proposition, the  $T$ -ultrafilter  $\mathcal{U}$  converges to  $x$ .  $\square$

We conclude this section addressing the issue of initial  $T$ -topology preservation under changing of bases functors.

**Proposition 7.4.9.** Let  $T'$  another saturable topological theory such that, for every set  $X$ ,  $T'(X)$  is finitely complete and  $\in_{T'}^X$  preserves finite meets, and that, for every function  $f : X \rightarrow Y$ ,  $T'(f)$  has an inverse image  $f'^{-1}$  relative to  $T'$  which preserves finite meets. Let  $v : T \rightarrow T'$  be a changing of bases datum such that, for every set  $X$ ,  $v_X$  preserves finite meets, and that, for every function  $f : X \rightarrow Y$ ,

$$v_X \circ f^{-1} = f'^{-1} \circ v_Y .$$

If  $X$  is the initial  $T$ -topological space associated to  $(f_i : X \rightarrow X_i)_{i \in I}$ , then  $v(X)$  is the initial  $T'$ -topological space associated to  $(f_i : X \rightarrow v(X_i))_{i \in I}$ .

*Proof.* Let  $x \in X$  and  $V \in T'(X)$  such that  $x \in_{T'} V$ .

Assume that there is a finite subset  $J \subset I$ , and that, for every  $j \in J$ , there is  $O_j \in \mathcal{O}_T(X_j)$  whose  $f_j(x)$  is a  $T$ -element, such that  $v(\bigwedge_{j \in J} f_j^{-1}(O_j)) \subset_{T'} V$ . Since  $v_X$  preserves finite meets,  $\bigwedge_{j \in J} v(f_j^{-1}(O_j)) \subset_{T'} V$ . Then  $\bigwedge_{j \in J} f_j'^{-1}(v(O_j)) \subset_{T'} V$ , because  $v_X \circ f_j^{-1} = f_j'^{-1} \circ v_{X_j}$  for every  $j \in J$ .

Conversely, assume that there is a finite subset  $J \subset I$ , and that, for every  $j \in J$ , there is  $O'_j \in \mathcal{O}_{T'}(v(X_j))$  whose  $f_j(x)$  is a  $T'$ -element, such that  $\bigwedge_{j \in J} f_j'^{-1}(O'_j) \subset_{T'} V$ . For every  $j \in J$ , by definition of  $v(X_j)$ , there is  $O_j \in \mathcal{O}_T(X_j)$  such that  $f_j(x) \in_T O_j$  and  $v(O_j) \subset_{T'} O'_j$ . Then one has  $\bigwedge_{j \in J} f_j'^{-1}(v(O_j)) \subset_{T'} V$ . Since, for every  $j \in J$ ,  $v_X \circ f_j^{-1} = f_j'^{-1} \circ v_{X_j}$ , and since  $v$  preserves finite meets,  $v(\bigwedge_{j \in J} f_j^{-1}(O_j)) \subset_{T'} V$ .  $\square$

The assumptions of the previous proposition are in particular satisfied for the topological theory  $(\mathcal{P}, \in)$  and for the changing of bases datum  $!^T$ .

## 7.5 Induced $T$ -topology and $T$ -topological subspaces

Some initial  $T$ -topologies exist even when not all the  $T(X)$  are finitely complete and not all the  $T(f)$  have inverse image relative to  $T$ .

Let  $X$  be a set, let  $Y$  be a  $T$ -topological space, and let  $f : X \rightarrow Y$  be a function. Assume that  $T(f)$  has an inverse image  $f^{-1}$  relative to  $T$ .

Consider the following subset of  $T(X)$ :

$$\mathcal{B} := \{f^{-1}(O) ; O \in \mathcal{O}_T(Y)\} .$$

**Lemma 7.5.1.** The set  $\mathcal{B}$  is a basis of  $T$ -topology on  $X$ .

*Proof.* Let  $x \in X$  and let  $(f^{-1}(O_j))_{j \in J}$  be a finite family of  $\mathcal{B}$  such that, for every  $j \in J$ ,  $x \in_T f^{-1}(O_j)$ . Then one has  $f(x) \in_T O_j$  for every  $j \in J$ . Therefore, since  $\mathcal{O}_T(Y)$  is a basis, there is  $O \in \mathcal{O}_T(Y)$  such that  $f(x) \in_T O$  and, for every  $j \in J$ ,  $O \subset_T O_j$ . Since  $f^{-1}$  is an inverse image of  $T(f)$  relative to  $T$  and a fortiori is an increasing map, it follows that  $x \in_T f^{-1}(O)$  and that, for every  $j \in J$ ,  $f^{-1}(O) \subset_T f^{-1}(O_j)$ .  $\square$

**Lemma 7.5.2.** When  $X$  is endowed with the  $T$ -topology generated by  $\mathcal{B}$ , the function  $f$  is  $T$ -continuous.

*Proof.* It is a clear consequence of Proposition 7.1.7.  $\square$

**Lemma 7.5.3.** Let  $W$  be a  $T$ -topological space and let  $h : W \rightarrow X$  be a function such that the composite  $f \circ h$  is  $T$ -continuous. Then  $h$  is  $T$ -continuous.

*Proof.* Let  $w \in W$  and let  $O \in \mathcal{O}_T(Y)$  such that  $h(w) \in_T f^{-1}(O)$ . Then one has  $f(h(w)) \in_T O$ . Hence, by  $T$ -continuity of  $f \circ h$  at  $w$ , there is  $U \in \mathcal{O}_T(W)$  whose  $w$  is a  $T$ -element such that  $U \cap T(f \circ h) \subset O$ . It follows that  $U \cap T(h) \subset f^{-1}(O)$ .  $\square$

The set  $X$  is endowed with the  $T$ -topology generated by  $\mathcal{B}$  and the  $T$ -continuous map  $f$  form a  $\mathbf{U}_T$ -initial lifting of the pair  $(X, f)$ . Therefore, we call the  $T$ -topology  $\mathcal{O}_T(\mathcal{B})$  the *initial  $T$ -topology associated to  $f$*  or the  *$T$ -topology induced by  $f$  on  $X$* . When  $X$  is a subset of  $Y$  and when  $f$  is the inclusion, we say that  $X$ , endowed with this  $T$ -topology, is a  *$T$ -topological subspace* of  $Y$ .

Therefore, one has:

**Theorem 7.5.4.** Assume that every function has an inverse image relative to  $T$ . Then the forgetful functor  $\mathbf{U}_T : \mathbf{Top}_T \rightarrow \mathbf{Set}$  is a fibration.

When, for a topological theory  $T$ , all injections have an inverse image relative to  $T$ , we can say that a subset  $P$  of a  $T$ -topological space  $X$  is  $T$ -compact,  $T$ - $T0$ , etc, when so is the space given by the  $T$ -topology induced by  $X$  on  $P$ .

**Proposition 7.5.5.** Let  $x \in X$  and let  $V \in T(X)$  such that  $x \in_T V$ . Then  $V$  is a  $T$ -neighbourhood of  $x$  for the  $T$ -topology induced by  $f$  on  $X$  if and only if there is  $V' \in \mathcal{V}_T(f(x))$  such that  $f^{-1}(V') \subset_T V$ .

*Proof.* Assume that  $V$  is a  $T$ -neighbourhood of  $x$  for the initial  $T$ -topology associated to  $f$ . Then there is  $O \in \mathcal{O}_T(Y)$  such that  $x \in_T f^{-1}(O) \subset_T V$ . Since  $x \in_T f^{-1}(O)$ ,  $f(x) \in_T O$ . Hence  $O$  is a  $T$ -neighbourhood of  $f(x)$  in  $Y$ .

Conversely, let  $V' \in \mathcal{V}_T(f(x))$  such that  $f^{-1}(V') \subset_T V$ . Then one has  $O \in \mathcal{O}_T(Y)$  such that  $f(x) \in_T O \subset_T V'$ . It follows that  $x \in_T f^{-1}(O)$  and  $f^{-1}(O) \subset_T f^{-1}(V')$ . Moreover,  $f^{-1}(V') \subset_T V$ , hence  $f^{-1}(O) \subset_T V$ .  $\square$

**Proposition 7.5.6.** Let  $P \subset T(X)$  and let  $x \in X$ . If the part  $P$  converges to  $x$  in  $X$  then the part  $P_{T(f)}$ , image of  $P$  under  $T(f)$ , converges to  $f(x)$ . If we have

$$(A \in P \text{ and } AT(\text{Id}_X)A') \Rightarrow A' \in P,$$

for every  $A, A' \in T(X)$ , then the converse is true.

*Proof.* Assume that  $P$  converges to  $x$  in  $X$  then, by Proposition 6.2.19,  $P_{T(f)}$  converges to  $f(x)$  in  $Y$ , since  $f$  is  $T$ -continuous.

Conversely, assume that  $P$  satisfies the statement of the proposition and that  $P_{T(f)}$  converges to  $f(x)$  in  $Y$ . Let  $V$  be a  $T$ -neighbourhood of  $x$  in  $X$ . By the previous proposition, there is a  $T$ -neighbourhood  $V'$  of  $f(x)$  in  $Y$  such that  $f^{-1}(V') \subset_T V$ . Since  $P_{T(f)}$  converges to  $f(x)$ ,  $V' \in P_{T(f)}$ . In other words, there is  $A \in P$  such that  $AT(f)V'$ . It follows that  $AT(\text{Id}_X)f^{-1}(V')$ . Given that  $f^{-1}(V') \subset_T V$ , one has  $AT(\text{Id}_X)V$ , then, by assumption on  $P$ ,  $V \in P$ .  $\square$

**Proposition 7.5.7.** Let  $T' = (T', \in_{T'})$  be another topological theory such that, for every function  $f$ ,  $T'(f)$  has an inverse image  $f'^{-1}$  relative to  $T'$ . Let  $v : T \rightarrow T'$  be a changing of bases datum such that  $v \circ f^{-1} = f'^{-1} \circ v$ . If  $X$  is the initial  $T$ -topological space associated to  $f$ , then  $v(X)$  is the initial  $T'$ -topological space associated to  $f$ .

*Proof.* Let  $O \in \mathcal{O}_T(Y)$ . Then, one has  $v(O) \in \mathcal{O}_{T'}(v(Y))$  and  $v(f^{-1}(O)) = f'^{-1}(v(O))$ .

Conversely, let  $O' \in \mathcal{O}_{T'}(Y)$  and let  $x \in X$  such that  $x \in_{T'} f'^{-1}(O')$ . Then one has  $f(x) \in_{T'} O'$ , hence there is  $O \in \mathcal{O}_T(Y)$  such that  $f(x) \in_T O$  and  $v(O) \subset_{T'} O'$ . It follows that  $x \in_T f^{-1}(v(O)) \subset_{T'} f'^{-1}(O')$ . However  $f'^{-1}(v(O)) = v(f^{-1}(O))$ , hence  $x \in_{T'} v(f^{-1}(O)) \subset_{T'} f'^{-1}(O')$ .  $\square$

The assumptions of the previous proposition are in particular satisfied by the topological theory  $(\mathcal{P}, \in)$  and the final changing of bases datum  $!^T$ .



**Corollary 7.5.8.** If, for every inclusion  $i : X \rightarrow Y$ ,  $T(i)$  has an inverse image relative to  $T$ , then every pair of  $T$ -continuous maps  $f, g : Y \rightarrow Z$  has an equalizer in  $\mathbf{Top}_T$ .

*Proof.* Just endow the subset  $\{x \in X ; f(x) = g(x)\}$  with the  $T$ -topology induced by  $X$ .  $\square$

**Proposition 7.5.9.** Assume that, for every inclusion  $i : X \rightarrow Y$ ,  $T(i)$  has an inverse image relative to  $T$ . Let  $X$  and  $Y$  be two  $T$ -topological spaces, let  $f : X \rightarrow Y$  be a  $T$ -continuous map and let  $P \subset X$ . If  $f$  has an inverse image relative to  $T$  and if  $P$  is  $T$ -compact (for the  $T$ -topology induced by  $X$ ) then  $f(P) := \{f(x); x \in P\}$  is  $T$ -compact (for the  $T$ -topology induced by  $Y$ ).

*Proof.* Denote by  $i : P \rightarrow X$  the inclusion. Then the surjective function  $f \circ i$  is  $T$ -continuous from  $P$  to  $f(P)$  and  $T(f \circ i)$  has an inverse image relative to  $T$  because  $T(f)$  and  $T(i)$  have one. We conclude then by Proposition 7.1.8.  $\square$

**Proposition 7.5.10.** [Assuming the axiom of choice] Suppose that, for every inclusion  $i : X \rightarrow Y$ ,  $T(i)$  has an inverse image  $i^{-1}$  relative to  $T$ . Let  $X$  be a  $T$ -compact space and let  $W \subset X$ . Denote by  $i : W \rightarrow X$  the inclusion. If  $W$  is closed (in  $!^T(X)$ ) and if  $!^T_X$  is surjective then  $P$  is  $T$ -compact.

*Proof.* Let  $\mathcal{U}$  be a  $T$ -ultrapart of  $W$ . To show that  $\mathcal{U}$  converges to some element of  $W$ , we are going to use Proposition 7.5.6.

Firstly, we are to show that  $\mathcal{U}$  satisfies the assumption of the proposition. Let  $A, A' \in T(W)$  such that  $A T(\text{Id}_W) A'$  and let  $A \in \mathcal{U}$ . Since  $\mathcal{U}$  is a  $T$ -ultrapart, to prove that  $A' \in \mathcal{U}$ , it is enough to show that  $\mathcal{U} \cup \{A'\}$  is finitely  $T$ -pointable. Let  $(A_j)_{j \in J}$  be a finite family of  $\mathcal{U}$ . Define  $J' := J \sqcup \{\star\}$  and  $(B_j)_{j \in J'}$  by  $B_\star := A$  and, for every  $j \in J$ ,  $B_j := A_j$ . Since  $\mathcal{U}$  is finitely  $T$ -pointable, there is  $x \in X$  such that, for every  $j \in J'$ ,  $x \in_T B_j$ . Since  $A T(\text{Id}_W) A'$ , one has, for every  $j \in J$ ,  $x \in_T A_j$  and  $x \in_T A'$ . Thus  $\mathcal{U}$  satisfies the assumption of Proposition 7.5.6.

By Corollary 7.4.7, the part  $\mathcal{U}_{T(i)}$ , image of  $\mathcal{U}$  under  $T(i)$ , is a  $T$ -ultrapart. Given that  $X$  is  $T$ -compact, there is  $x \in X$  such that the part  $\mathcal{U}_{T(i)}$  converges to  $x$ . To conclude, it is enough to show that  $x \in W$ . Let  $V$  be a  $T$ -neighbourhood of  $x$  in  $X$ . Since  $\mathcal{U}_{T(i)}$  converges to  $x$ , one has  $V \in \mathcal{U}_{T(i)}$ , hence there is  $A \in \mathcal{U}$  such that  $A T(i) V$ . It follows that  $A T(\text{Id}_W) i^{-1}(V)$ , then, as we previously show,  $V \in \mathcal{U}$ . Since the part  $\mathcal{U}$  is finitely  $T$ -pointable, there is  $y \in W$  such that  $y \in i^{-1}(V)$ . Then one has  $y \in V$ . Since  $W$  is closed, we finally conclude that  $x \in W$ .  $\square$

**Proposition 7.5.11.** Assume that for every inclusion  $i : X \rightarrow Y$ ,  $T(i)$  has an inverse image relative to  $T$ . Let  $X$  be a  $T$ -topological space  $T$ -Hausdorff and let  $W \subset X$ . If  $W$  is  $T$ -compact (and also  $T$ -Hausdorff by 6.3.17) then  $W$  is closed.

*Proof.* Denote by  $i : W \rightarrow X$  the inclusion. Assume that  $x \in X$  is an adherent  $T$ -point of  $W$ . Then  $i^{-1}(\mathcal{V}_T^X(x)) := \{i^{-1}(V); V \in \mathcal{V}_T^X(x)\}$  is finitely  $T$ -pointable. Since  $W$  is  $T$ -compact, there is  $y \in W$  an adherent  $T$ -point of  $i^{-1}(\mathcal{V}_T^X(x))$ . By Proposition 6.2.18,  $y$  is an adherent  $T$ -point of  $i^{-1}(\mathcal{V}_T^X(x))_{T(i)}$ , image of  $i^{-1}(\mathcal{V}_T^X(x))$  under  $T(i)$ .

One has  $\mathcal{V}_T^X(x) \subset i^{-1}(\mathcal{V}_T^X(x))_{T(i)}$  because, for every  $V \in \mathcal{V}_T^X(x)$ ,  $i^{-1}(V) T(i) V$ , by Lemma 7.1.2. Given that  $X$  is  $T$ -Hausdorff, that the filter  $i^{-1}(\mathcal{V}_T^X(x))_{T(i)}$  converges to  $x$  and that  $y$  is an adherent  $T$ -point of  $i^{-1}(\mathcal{V}_T^X(x))_{T(i)}$ , by Proposition 6.3.15,  $x = y$  hence  $x \in W$ .  $\square$

Thus, we can define the following separation axioms:

**Definition 7.5.12** ( $T$ - $t_2$  spaces). Assume that for every inclusion  $i : X \rightarrow Y$ ,  $T(i)$  has an inverse image relative to  $T$ . Let  $X$  be a  $T$ -topological space. The space  $X$  is  $T$ - $t_2$  when, for every  $T$ -topological space  $K$   $T$ -compact and  $T$ -Hausdorff, for every  $T$ -continuous map  $f : K \rightarrow X$  such that  $T(f)$  has an inverse image relative to  $T$ , the subset  $f(K) := \{f(x); x \in K\}$  of  $X$  is closed for the underlying topology of  $X$ .

**Definition 7.5.13** ( $T$ - $KC$  spaces). Assume that for every inclusion  $i : X \rightarrow Y$ ,  $T(i)$  has an inverse image relative to  $T$ . Let  $X$  be a  $T$ -topological space. The space  $X$  is  $T$ - $KC$  when every subset  $K \subset X$  which is  $T$ -compact for the induced  $T$ -topology is closed.

By the previous proposition, every  $T$ - $T_2$  space is  $T$ - $KC$ . By Proposition 7.5.9 and Proposition 6.3.17, every  $T$ - $KC$  space is  $T$ - $t_2$ .

## 7.6 Back to $\mathbf{Loc}(T)$

Let  $X$  and  $Y$  be two  $T$ -topological spaces, and let  $f : X \rightarrow Y$  be a function. Write  $X \subset_{T,f}^{str} Y$  when  $f$  is  $T$ -continuous and  $(X, f)$  is an  $\mathbf{U}_T$ -initial lifting of the cone  $(\mathbf{U}_T(X), f)$ . When  $f$  is an inclusion, we rather write  $X \subset_T^{str} Y$ . One easily checks that, for every  $T$ -continuous functions  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$ ,

$$(X \subset_{T,f}^{str} Y \quad \text{and} \quad Y \subset_{T,g}^{str} Z) \quad \text{implies} \quad X \subset_{T,g \circ f}^{str} Z,$$

and that, for every  $T$ -topological space  $X$ ,

$$X \subset_T^{str} X.$$

**Definition 7.6.1** ( $\mathcal{C}$ -fibrations). Let  $\mathcal{C}$  be a class of functions and let  $T = (T, \in_T)$  be a topological theory. The forgetful functor  $\mathbf{U}_T$  is a  $\mathcal{C}$ -fibration when, for every set  $X$ , for every  $T$ -topological space  $(Y, \mathcal{E})$ , and for every function  $f : X \rightarrow Y$  belonging to  $\mathcal{C}$ , the cone  $(X, f)$  has a (chosen)  $\mathbf{U}_T$ -initial lifting. When there is no risk of confusion, we identify such a lifting  $((X, \mathcal{E}_f), f)$  and the space  $(X, \mathcal{E}_f)$ . Note that when  $\mathcal{C}$  is the class of all functions, a  $\mathcal{C}$ -fibration is just a fibration as defined in 7.0.5.

**Proposition 7.6.2.** Let  $\mathcal{C}$  be a class of functions such that, for every  $f : X \rightarrow Y \in \mathcal{C}$ , for every  $B \subset Y$ , the restriction  $f^{-1}(B) \rightarrow B$  belongs to  $\mathcal{C}$ . Assume that  $\mathbf{U}_T$  is a  $\mathcal{C}$ -fibration. Then, for every function  $f \in \mathcal{C}$ ,  $\mathbf{Loc}(T)(f)$  has an inverse image relative to  $\mathbf{Loc}(T)$ .

*Proof.* Let  $X$  and  $Y$  be two sets and let  $f : X \rightarrow Y$  be a function belonging to  $\mathcal{C}$ . Let  $B \in \mathbf{Loc}(T)(Y)$ . The restriction of  $f$  to  $\{x \in X; f(x) \in B\}$  (with codomain  $B$ ) belongs to  $\mathcal{C}$ , it is still denoted by  $f$ . Hence there is a  $T$ -topological space  $f^{-1}(B)$  whose underlying set is  $\{x \in X; f(x) \in B\}$  such that  $f^{-1}(B) \subset_{T,f}^{str} B$ . We easily check that we defined an inverse image of  $\mathbf{Loc}(T)(f)$  relative to  $\mathbf{Loc}(T)$ .  $\square$

Therefore, for every function  $f : X \rightarrow Y \in \mathcal{C}$ , for every  $\mathbf{Loc}(T)$ -topology on  $Y$ , we can use the  $\mathbf{Loc}(T)$ -topology induced by  $f$  on  $X$  as defined in the previous section. In particular, the classes of all functions and of all inclusions satisfy the assumption of the previous proposition.

**Corollary 7.6.3.** Assume that  $\mathbf{U}_T : \mathbf{Top}_T \rightarrow \mathbf{Set}$  is a fibration. Then  $\mathbf{U}_{\mathbf{Loc}(T)} : \mathbf{Top}_{\mathbf{Loc}(T)} \rightarrow \mathbf{Set}$  is a fibration.

*Proof.* Consequence of the previous proposition applied to the class of all functions and of Theorem 7.5.4.  $\square$

**Proposition 7.6.4.** Let  $T' = (T', \in_{T'})$  be another topological theory and let  $F : T \rightarrow T'$  be a semantic transformation (Definition 5.3.1). Let  $\mathcal{C}$  be a class of functions such that, for every  $f : X \rightarrow Y \in \mathcal{C}$ , for every  $B \subset Y$ , the restriction  $f^{-1}(B) \rightarrow B$  belongs to  $\mathcal{C}$ . Assume that  $\mathbf{U}_T$  and  $\mathbf{U}_{T'}$  are  $\mathcal{C}$ -fibrations, and that, for every continuous map  $f : X \rightarrow Y$  belonging to  $\mathcal{C}$ ,  $X \subset_{T,f}^{str} Y$  implies  $F(X) \subset_{T',f}^{str} F(Y)$ . Then

$$\mathbf{Loc}(F)_X \circ f^{-1} = f'^{-1} \circ \mathbf{Loc}(F)_Y,$$

for every function  $f : X \rightarrow Y \in \mathcal{C}$ , where  $f^{-1}$  (respectively  $f'^{-1}$ ) denotes the inverse image of  $\mathbf{Loc}(T)(f)$  relative to  $\mathbf{Loc}(T)$  (respectively of  $\mathbf{Loc}(T')(f)$  relative to  $\mathbf{Loc}(T')$ ) defined as in Proposition 7.6.2<sup>1</sup>. Then,  $\mathbf{Loc}(F)$  preserves the  $T$ -topologies induced by a function belonging to  $\mathcal{C}$ .

*Proof.* Clear consequence of the construction of  $f^{-1}$  in the previous proof and of Proposition 7.5.7.  $\square$

**Corollary 7.6.5.** Assume that  $\mathbf{U}_T$  and  $\mathbf{U}_{T'}$  are fibrations and that  $F$  is a semantic transformation and a morphism of fibrations. The changing of bases functor associated to  $\mathbf{Loc}(T)(F)$  is a morphism of fibrations.

<sup>1</sup>More precisely, since the exact definitions of  $f^{-1}$  and  $f'^{-1}$  depend on the choice of the initial liftings, the proposition says that the initial liftings can be chosen so that the equality holds.

**Proposition 7.6.6.** Assume that the forgetful functor  $\mathbf{U}_T : \mathbf{Top}_T \rightarrow \mathbf{Set}$  is a topological functor. Then  $\mathbf{Loc}(T)$  is with strong and representable membership, for every set  $X$ ,  $T(X)$  is complete, and, for every function  $f : X \rightarrow Y$ ,  $f^{-1}$  preserves limits.

*Proof.* Let  $X$  be a set. For every  $x \in X$ , define  $\sigma_X(x)$  as the set  $\{x\}$  endowed with a  $T$ -topology that makes it a  $\mathbf{U}_T$ -discrete object on  $\{x\}$ . One readily checks that  $\in_T^X$  is represented by  $\sigma_X$ .

Let  $Y$  be another set, let  $f : X \rightarrow Y$  be a function, let  $x \in X$  and let  $B \in \mathbf{Loc}(T)(Y)$ . If  $f(x) \in_T B$ , one easily verifies that  $x \in \sigma(x)$  and  $\sigma(x) \mathbf{Loc}(T)(f) B$ . Therefore  $T$  is with strong membership.

Let  $(B_i)_{i \in I}$  be a family of  $\mathbf{Loc}(T)(Y)$ , one easily checks that  $\bigcap_{i \in I} B_i$ , endowed with a  $T$ -topology for which it is an  $\mathbf{U}_T$ -initial lifting of the cone formed by the inclusions  $(\bigcap_{i \in I} B_i \subset B_i)_{i \in I}$ , is a meet of  $(B_i)_{i \in I}$  in  $\mathbf{Loc}(T)(Y)$ . The underlying subsets of  $f^{-1}(\bigcap_{i \in I} B_i)$  and of  $\bigcap_{i \in I} f^{-1}(B_i)$  match. Let  $A \in \mathbf{Loc}(T)(X)$  such that  $A \subset f^{-1}(\bigcap_{i \in I} B_i)$ . One easily checks that the inclusion is  $T$ -continuous from  $A$  to  $f^{-1}(\bigcap_{i \in I} B_i)$  if and only if  $f$  can be restricted to a  $T$ -continuous map from  $A$  to  $\bigcap_{i \in I} B_i$  if and only if, for every  $i \in I$ ,  $f$  can be restricted to a  $T$ -continuous map from  $A$  to  $B_i$  if and only if the inclusion is  $T$ -continuous from  $A$  to  $\bigcap_{i \in I} f^{-1}(B_i)$ . Thus  $f^{-1}$  preserves meets.  $\square$

**Corollary 7.6.7.** If  $\mathbf{U}_T : \mathbf{Top}_T \rightarrow \mathbf{Set}$  is a topological functor then so is  $\mathbf{U}_T : \mathbf{Top}_{\mathbf{Loc}(T)} \rightarrow \mathbf{Set}$ .

*Proof.* Indeed, thanks to the two previous propositions,  $T$  satisfies the assumptions of Theorem 7.2.4.  $\square$

**Proposition 7.6.8.** If  $\mathbf{U}_T : \mathbf{Top}_T \rightarrow \mathbf{Set}$  is a topological functor then, for every set  $X$ , for every  $x \in X$ ,  $\sigma(x)$  is supercompact in the preordered set  $\mathbf{Loc}(T)(X)$ .

*Proof.* Let  $(A_i)_{i \in I}$  be a family of  $\mathbf{Loc}(T)(X)$ . One easily checks that  $(A_i)_{i \in I}$  has a least upper bound given by  $\bigcup_{i \in I} \mathbf{Loc}(T)^{\uparrow X}(A_i)$  endowed with a  $T$ -topology for which it is a final  $\mathbf{U}_T$ -final lifting associated to the inclusions of  $A_i$  into  $\bigcup_{i \in I} A_i$ . We immediately get the desired result.  $\square$

When  $\mathbf{U}_T$  is a topological functor, we can generalize the Sierpiński space: we can endow  $\{0, 1\}$  with the  $\mathbf{Loc}(T)$ -topology generated by the basis formed by two members:  $\{0, 1\}$  and  $\{1\}$  endowed with  $T$ -topologies that make them  $\mathbf{U}_T$ -indiscrete objects.

**Definition 7.6.9** ( $\mathbf{Loc}(T)$ -topological spaces with stable indistinguishability). A  $\mathbf{Loc}(T)$ -topological space  $X$  is with *stable indistinguishability* when, for every points  $x, y \in X$ , if  $x$  and  $y$  are  $\mathbf{Loc}(T)$ -indistinguishable in  $X$ , then, for every  $O \in \mathcal{O}_{\mathbf{Loc}(T)}(X)$  whose  $x$  and  $y$  are elements,  $x$  and  $y$  are  $T$ -indistinguishable in  $O$ .

**Definition 7.6.10** (Locally  $P$   $\mathbf{Loc}(T)$ -topological spaces). Let  $P$  be a predicate on the class of  $T$ -topological spaces. A  $\mathbf{Loc}(T)$ -topological space  $X$  is *locally  $P$*  when, for every  $x \in X$ , for every  $O \in \mathcal{O}_{\mathbf{Loc}(T)}(X)$  such that  $x \in O$ , there is  $O' \in \mathcal{O}_{\mathbf{Loc}(T)}(X)$  satisfying  $P$  such that  $x \in O' \subset_{\mathbf{Loc}(T)} O$ . In other words,  $X$  is locally  $P$  when there is a basis of  $\mathbf{Loc}(T)$ -topology on  $X$  whose members satisfy  $P$  that generates  $X$ .

Let  $P$  be a predicate on the class of  $T$ -topological spaces. We define a topological theory  $\mathbf{Loc}_P(T)$  in the same way as  $\mathbf{Loc}(T)$  except that, for every set  $X$ ,  $\mathbf{Loc}(T)(X)$  is replaced by the set of all members of  $\mathbf{Loc}(T)(X)$  satisfying  $P$ . One easily check that, via the obvious changing of bases datum from  $\mathbf{Loc}_P(T)$  to  $\mathbf{Loc}(T)$  and via Proposition 5.3.16, we can identify  $\mathbf{Top}_{\mathbf{Loc}_P(T)}$  and the full subcategory of  $\mathbf{Top}_{\mathbf{Loc}(T)}$  formed by the locally  $P$  spaces.

### Examples 7.6.1.

- We can define this way the locally  $T$ -compact spaces, the locally  $T$ -Hausdorff spaces, the locally  $T$ - $T_0$  spaces, the locally  $T$ - $R_0$  spaces, etc.
- The locally ordered spaces defined in 4.1.6 matches with the  $\mathbf{Loc}(T_{\mathbb{B}}^S)$ -topological spaces (Examples 5.4.1) and with the locally ordered  $\mathbf{Loc}(T_{\mathbb{B}})$ -spaces.

The following definition allows us to express as a special case the strictly locally ordered spaces (Definition 4.1.6) from the locally ordered spaces.

**Definition 7.6.11** (Strict  $\mathbf{Loc}(T)$ -topological spaces). A  $\mathbf{Loc}(T)$ -topological space  $X$  is *strict* when there is a basis  $\mathcal{B}$  generating  $X$  such that, for every  $x \in X$ , for every  $B, B' \in \mathcal{B}$  such that  $x \in B, B'$ , there is  $B'' \in \mathcal{B}$  such that  $x \in B'' \subset_T^{str} B, B'$ . We call such a basis a *strict basis* of  $X$ .

Define the topological theory  $\mathbf{Loc}^s(T)$  in the same way as  $\mathbf{Loc}(T)$  except that, for every set  $X$ , the preorder on  $\mathbf{Loc}(T)(X)$  is replaced by the preorder  $\subset_T^{str}$ . One easily checks that, via the obvious changing of bases datum from  $\mathbf{Loc}^s(T)$  to  $\mathbf{Loc}(T)$ , we can identify  $\mathbf{Top}_{\mathbf{Loc}^s(T)}$  and the full subcategory of  $\mathbf{Top}_{\mathbf{Loc}(T)}$  formed by the strict spaces.

One may think that two non-equivalent bases of  $\mathbf{Loc}^s(T)$ -topology can be equivalent when they are seen as bases of  $\mathbf{Loc}(T)$ -topology but in fact this cannot happen:

**Proposition 7.6.12.** Two ordered bases of  $\mathbf{Loc}^s(T)$ -topology  $\vec{\mathfrak{B}}$  and  $\vec{\mathfrak{B}}'$  on a set  $X$  are equivalent if and only if they are equivalent as bases of  $\mathbf{Loc}(T)$ -topology<sup>2</sup>.

*Proof.* If  $\vec{\mathfrak{B}}$  and  $\vec{\mathfrak{B}}'$  are equivalent as bases of  $\mathbf{Loc}^s(T)$ -topology then they are equivalent as bases of  $\mathbf{Loc}(T)$ -topology because the relation  $\subset_T^{str}$  is stronger than the relation  $\subset_{\mathbf{Loc}(T)}$ .

Conversely, assume  $\vec{\mathfrak{B}}$  and  $\vec{\mathfrak{B}}'$  are equivalent as bases of  $\mathbf{Loc}(T)$ -topology. We use Lemma 5.2.3. Let  $B \in \vec{\mathfrak{B}}'$  containing a point  $x \in X$ . There exist  $A \in \vec{\mathfrak{B}}$  such that  $x \in A \subset_{\mathbf{Loc}(T)} B$ , and  $B' \in \vec{\mathfrak{B}}'$  such that  $x \in B' \subset_{\mathbf{Loc}(T)} A$ . Since the basis  $\vec{\mathfrak{B}}'$  is strict, there is  $B'' \in \vec{\mathfrak{B}}'$  such that  $x \in B'' \subset_T^{str} B, B'$ . Once again, since  $\vec{\mathfrak{B}}$  and  $\vec{\mathfrak{B}}'$  are equivalent as bases of  $\mathbf{Loc}(T)$ -topology, there is  $A' \in \vec{\mathfrak{B}}$  such that  $x \in A' \subset_{\mathbf{Loc}(T)} B''$ . Since  $\vec{\mathfrak{B}}$  is a strict basis, there is  $A'' \in \vec{\mathfrak{B}}$  such that  $x \in A'' \subset_T^{str} A, A'$ . We now check that  $A'' \subset_T^{str} B$ . Since  $A'' \subset_T^{str} A \subset_{\mathbf{Loc}(T)} B$ , we have  $A'' \subset_{\mathbf{Loc}(T)} B$ . Let  $W$  be a  $T$ -topological space and let  $f : W \rightarrow A''$  be a function which is  $T$ -continuous from  $W$  to  $B$ . Since  $B'' \subset_T^{str} B$ , it follows that  $f$  is  $T$ -continuous from  $W$  to  $B''$ . Then the function  $f$  is  $T$ -continuous from  $W$  to  $A$  because  $B'' \subset_T^{str} B' \subset_{\mathbf{Loc}(T)} A$ . Finally, since  $A'' \subset_T^{str} A$ , we deduce that  $f$  is  $T$ -continuous from  $W$  to  $A''$ . The other direction is obtained by symmetry.  $\square$

*Remark 7.6.1.* We easily combine, for every predicate  $P$ , the notions of locally  $P$  spaces and of strict spaces. We thus define the notion of strict locally  $P$  spaces, the topological theory  $\mathbf{Loc}_P^s(T)$  and the obvious changing of bases data from  $\mathbf{Loc}_P^s(T)$  to  $\mathbf{Loc}_P(T)$  and from  $\mathbf{Loc}_P^s(T)$  to  $\mathbf{Loc}^s(T)$ . Then the diagram commutes in  $\mathbf{TopTh}$ :

$$\begin{array}{ccc}
 & \mathbf{Loc}_P(T) & \\
 \nearrow & & \searrow \\
 \mathbf{Loc}_P^s(T) & & \mathbf{Loc}(T) \\
 \searrow & & \nearrow \\
 & \mathbf{Loc}^s(T) &
 \end{array}$$

For the strict spaces, we have the following characterization of the  $\mathbf{Loc}(T)$ -continuity:

**Proposition 7.6.13.** Let  $X$  and  $Y$  be two  $\mathbf{Loc}(T)$ -topological spaces such that  $Y$  is strict and let  $f : X \rightarrow Y$  be a function. Let  $x \in X$  and let  $\mathcal{B}$  be a strict basis of  $Y$ . The function  $f$  is  $\mathbf{Loc}(T)$ -continuous at  $x$  if and only if  $f$  is continuous at  $x$  from  $!\mathbf{Loc}(T)(X)$  to  $!\mathbf{Loc}(T)(Y)$  and there is  $O \in \mathcal{O}_{\mathbf{Loc}(T)}(X)$  and  $B \in \mathcal{B}$  such that  $x \in O$ ,  $f(x) \in B$  and  $O \mathbf{Loc}(T)(f) B$ .

*Proof.* The condition is clearly necessary. Conversely, assume that  $f$  is continuous at  $x$  from  $!\mathbf{Loc}(T)(X)$  to  $!\mathbf{Loc}(T)(Y)$  and that there is  $O \in \mathcal{O}_{\mathbf{Loc}(T)}(X)$  and  $B \in \mathcal{B}$  such that  $x \in O$ ,  $f(x) \in B$  and  $O \mathbf{Loc}(T)(f) B$ . Let  $B' \in \mathcal{B}$  such that  $f(x) \in B'$ . Since  $\mathcal{B}$  is a strict basis, there is  $B''$  such that  $B'' \subset_T^{str} B, B'$ . By continuity of  $f$  at  $x$ , there is  $O' \in \mathcal{O}_{\mathbf{Loc}(T)}(X)$  such that  $x \in O' \subset_{\mathbf{Loc}(T)} O$  and  $f(O') \subset B''$ . To conclude, since  $B'' \subset_T^{str} B'$ , it is enough to show that the restriction  $f|_{O'}$  of  $f$  to  $O'$  is  $T$ -continuous from  $O'$  to  $B''$ . Since  $B'' \subset_T^{str} B'$ , it is enough to show that  $f|_{O'}$  is  $T$ -continuous from  $O'$  to  $B$ . Since  $O \mathbf{Loc}(T)(f) B$  and  $O' \subset_{\mathbf{Loc}(T)} O$ , we get the desired result.  $\square$

<sup>2</sup>This is a generalization of Proposition 4.1.2.

In the case of locally ordered spaces, this proposition corresponds to Proposition 4.1.10.

Denote by  $\mathcal{I}$  the class of all inclusions. When  $\mathbf{U}_T$  is a  $\mathcal{I}$ -fibration, we can define the following construction. Recall that  $\mathcal{P} \times T$  denotes the cartesian product of the theory  $\mathcal{P}$  and of the theory  $T$  in the category  $\mathbf{STopTh}$  (Proposition 5.3.2). Since  $\mathbf{Sem}(!^T)$  is a morphism from  $T$  to  $\mathcal{P}$  in the category  $\mathbf{STopTh}$ , we get a canonical semantic transformation  $\nu_T : T \rightarrow \mathcal{P} \times T$ , natural in  $T$ , which associates every  $T$ -topological space with the pair  $(!^T(X), X)$ . Assume that  $\mathbf{U}_T$  is a  $\mathcal{I}$ -fibration. Let  $X$  be a set,  $\mathcal{O}$  be a topology on  $X$ , and  $\mathcal{E}$  be  $T$ -topology on  $X$ , define  $\gamma_T((X, \mathcal{O}), (X, \mathcal{E}))$  as the set  $X$  endowed with the  $\mathbf{Loc}(T)$ -topology generated by the basis of all  $O \in \mathcal{O}$  endowed with a  $T$ -topology  $\mathcal{E}_O$  such that  $(O, \mathcal{E}_O) \subset_T^{str} (X, \mathcal{E})$ . This basis is clearly a strict basis.

**Lemma 7.6.14.** Let  $(X, \mathcal{O}(X)), (X, \mathcal{E}(X))$  and  $(Y, \mathcal{O}(Y)), (Y, \mathcal{E}(Y))$  be two  $(\mathcal{P} \times T)$ -topological spaces and let  $f : X \rightarrow Y$  a  $(\mathcal{P} \times T)$ -continuous function. Then  $f$  is  $\mathbf{Loc}(T)$ -continuous.

*Proof.* Let  $O_Y \in \mathcal{O}(Y)$  such that  $f(x) \in O_Y$ . Since  $f$  is continuous at  $x \in X$ , there is  $O_X \in \mathcal{O}(X)$  such that  $x \in O_X$  and  $f(O_X) \subset O_Y$ ; denote by  $f|_{O_X} : O_X \rightarrow O_Y$  the restriction of  $f$ . Also denote by  $\mathcal{E}(X)_{O_X}$  (respectively  $\mathcal{E}(Y)_{O_Y}$ ) a  $T$ -topology on  $X$  (respectively  $Y$ ) such that  $(O_X, \mathcal{E}(X)_{O_X}) \subset_T^{str} (X, \mathcal{E}(X))$  (respectively  $(O_Y, \mathcal{E}(Y)_{O_Y}) \subset_T^{str} (Y, \mathcal{E}(Y))$ ). The function  $f|_{O_X}$  is  $T$ -continuous from  $(O_X, \mathcal{E}(X)_{O_X})$  to  $(O_Y, \mathcal{E}(Y)_{O_Y})$  because  $(O_X, \mathcal{E}(X)_{O_X}) \subset_T^{str} (X, \mathcal{E}(X))$  and  $f$  is  $T$ -continuous from  $(X, \mathcal{E}(X))$  to  $(Y, \mathcal{E}(Y))$ . Since  $(O_Y, \mathcal{E}(Y)_{O_Y}) \subset_T^{str} (Y, \mathcal{E}(Y))$ , we deduce that  $f|_{O_X}$  is  $T$ -continuous from  $(O_X, \mathcal{E}(X)_{O_X})$  to  $(O_Y, \mathcal{E}(Y)_{O_Y})$ , i.e. that

$$(O_X, \mathcal{E}(X)_{O_X}) \mathbf{Loc}(T)(f) (O_Y, \mathcal{E}(Y)_{O_Y}).$$

□

Hence, we defined a semantic transformation  $\gamma^T : \mathcal{P} \times T \rightarrow \mathbf{Loc}(T)$ . Therefore the composite  $\gamma_T \circ \nu_T : T \rightarrow \mathbf{Loc}(T)$  is a semantic transformation.

### Examples 7.6.2.

- The ordered spaces seen as locally ordered spaces in Remark 4.1.4 correspond to an application of  $\gamma_{T_{\mathbb{B}}^S}$ .
- Recall the topological theory  $(\mathcal{P}_{rRel}, \in)$  defined in Examples 5.1.1. The lp-spaces defined in [Gra03, 1.4.(b)] match with the locally transitive  $\mathbf{Loc}(\mathcal{P}_{rRel})$ -topological spaces of the form  $\gamma_{\mathcal{P}_{rRel}}(X)$

# Chapter 8

## $T$ -streams

For the whole chapter, set a topological theory  $T = (T, \in_T)$ . Assume that, for every set  $X$ ,  $T(X)$  is complete and  $\in_T^X$  preserves meets (5.2.15). A fortiori, by Proposition 5.2.17,  $T$  is a theory with representable membership.

Though  $T$ -topological spaces can be very wild, the  $T$ -stream construction provides a coreflexive full subcategory of  $\mathbf{Top}_T$  whose spaces are much better behaved.

### 8.1 Basic definitions

**Definition 8.1.1** (Fundamental  $T$ -open members). Let  $X$  be a  $T$ -topological space. A  $T$ -open member  $O$  of  $X$  is *fundamental* when, for every  $O' \in \mathcal{O}_T(X)$  such that  $O \simeq_T O'$ , one has  $O \subset_T O'$ .

**Proposition 8.1.2.** Let  $X$  be a  $T$ -topological space and let  $O, O' \in \mathcal{O}_T(X)$  such that  $O$  is fundamental. Then  $O \subset_T O'$  if and only if, for every  $x \in X$ ,  $x \in_T O$  implies  $x \in_T O'$ .

*Proof.* The condition is clearly necessary. Conversely, assume that, for every  $x \in X$ ,  $x \in_T O$  implies  $x \in_T O'$ . Since  $\in_T^X$  preserves meets,  $O \wedge O'$  is  $T$ -open in  $X$  by Proposition 5.2.11 and has the same  $T$ -elements as  $O$ . Given that  $O$  is fundamental, one has  $O \subset_T O \wedge O'$  hence  $O \subset_T O'$ .  $\square$

**Corollary 8.1.3.** Let  $X$  be a  $T$ -topological space, let  $O \in \mathcal{O}_T(X)$  fundamental, and let  $(O_i)_{i \in I}$  be a family of  $\mathcal{O}_T(X)$  every member of which is fundamental. If  $!_X^T(O) = \bigcup_{i \in I} !_X^T(O_i)$ , then  $O$  is a join of the  $O_i$  in  $T(X)$ .

*Proof.* For each  $i$ , one has  $!_X^T(O_i) \subset !_X^T(O)$ , hence, by the previous proposition,  $O_i \subset_T O$ . Let  $O'$  be a  $T$ -open member such that, for every  $i$ ,  $O_i \subset_T O'$ . Then one has  $!_X^T(O) = \bigcup_{i \in I} !_X^T(O_i) \subset O'$ , hence, again by the previous proposition,  $O \subset_T O'$ .  $\square$

**Corollary 8.1.4.** Let  $X$  be a  $T$ -topological space such that  $\in_T^X$  is supercompact. Let  $(O_i)_{i \in I}$  be a family of  $\mathcal{O}_T(X)$  every member of which is fundamental. Then any join  $\bigvee_{i \in I} O_i$  of the  $O_i$  in the complete preordered set  $T(X)$  is a fundamental  $T$ -open member.

*Proof.* The member  $\bigvee_{i \in I} O_i$  is  $T$ -open in  $X$  by stability of  $T$ -openness under joins (Proposition 5.2.13). Let  $O \in \mathcal{O}_T(X)$  such that  $\bigvee_{i \in I} O_i \simeq_T O$ . For each  $i \in I$ , one has  $!_X^T(O_i) \subset !_X^T(O)$ , hence  $O_i \subset_T O$  because  $O_i$  is fundamental. It follows that  $\bigvee_{i \in I} O_i \subset_T O$ .  $\square$

**Definition 8.1.5** ( $T$ -streams). Let  $X$  be a  $T$ -topological space. The space  $X$  is a  $T$ -stream when, for every  $O \in \mathcal{O}_T(X)$ , there is a fundamental  $T$ -open member  $O_f$  such that  $O \simeq_T O_f$ .

**Proposition 8.1.6.** Let  $X$  be a  $T$ -topological space. Then  $X$  is a  $T$ -stream if and only if, for every family  $(O_i)_{i \in I}$  of  $\mathcal{O}_T(X)$  whose members are  $\simeq_T$ -equivalent, the meets  $\bigwedge_{i \in I} O_i$  are  $T$ -open in  $X$ .

*Proof.* Assume that  $X$  is a  $T$ -stream. Let  $(O_i)_{i \in I}$  be a family of  $\mathcal{O}_T(X)$  whose members are  $\simeq_T$ -equivalent. If  $I$  is empty, then  $\bigwedge_{i \in I} O_i$  is  $T$ -open by Proposition 5.2.11. Otherwise, let  $i \in I$ , since  $X$  is a  $T$ -stream, there is  $O \in \mathcal{O}_T(X)$  fundamental such that  $O \simeq_T O_i$ . Since all  $O_i$  have the same  $T$ -elements, for every  $i \in I$ ,  $O \simeq_T O_i$ , hence  $O \subset_T O_i$ , then  $O \subset_T \bigwedge_{i \in I} O_i$ . Therefore  $O$  and  $\bigwedge_{i \in I} O_i$  have the same  $T$ -elements because there is  $i \in I$  such that  $O \simeq_T O_i$ . It follows that, by Proposition 5.2.14,  $\bigwedge_{i \in I} O_i$  is  $T$ -open in  $X$ .

Conversely, let  $O \in \mathcal{O}_T(X)$ . Then a meet of all members of  $\mathcal{O}_T(X)$  which have the same  $T$ -elements as  $O$  is  $T$ -open, is fundamental, and have the same  $T$ -elements as  $O$  because  $\in_T^X$  preserves meets.  $\square$

Thus, by Proposition 6.3.38, a  $T$ -Alexandroff space is a  $T$ -stream.

**Proposition 8.1.7.** Let  $X$  be a  $T$ -topological space. The space  $X$  is a  $T$ -stream if and only if there is a basis of  $T$ -topology that generates  $X$  and whose members are fundamental in  $X$ .

*Proof.* Assume that  $X$  is a  $T$ -stream. Then

$$\mathcal{B} := \{O \in \mathcal{O}_T(X) ; O \text{ is } T\text{-open and fundamental}\}$$

is a basis of  $T$ -topology that generates  $X$  and whose members are fundamental in  $X$ .

Conversely, assume that there is a basis  $\mathcal{B}$  which generates  $X$  and whose members are fundamental in  $X$ . Let  $O \in \mathcal{O}_T(X)$ , write

$$O' := \bigwedge \{O'' \in \mathcal{O}_T(X) ; O'' \simeq_T O\}.$$

To conclude, it is enough to show that  $O'$  is  $T$ -open.

Let  $x \in_T O'$ . One has  $x \in_T O$ , hence there is  $B \in \mathcal{B}$  such that  $x \in_T B \subset_T O$  because  $\mathcal{B}$  generates  $X$  and because  $O$  is  $T$ -open. Let  $O'' \in \mathcal{O}_T(X)$  such that  $O'' \simeq_T O$ . By Proposition 8.1.2, since  $B$  is fundamental, one has  $B \subset_T O''$ . It follows that  $x \in_T B \subset_T O'$ . Since  $\mathcal{B}$  is a basis that generates  $X$ ,  $O'$  is  $T$ -open in  $X$ .  $\square$

**Proposition 8.1.8.** Let  $X$  be a  $T$ -topological space. The space  $X$  is a  $T$ -stream if and only if there is a basis of  $T$ -topology  $\mathcal{B}$  that generates  $X$  such that, for every  $B \in \mathcal{B}$ , for every family  $(B_i)_{i \in I}$  of  $\mathcal{B}$ , if  $\bigcup_{i \in I} !_X^T(B_i) = !_X^T(B)$ , then  $B$  is a join of the  $B_i$  in  $T(X)$ .

*Proof.* If  $X$  is a  $T$ -stream, then

$$\mathcal{B} := \{U \in \mathcal{O}_T(X) ; U \text{ } T\text{-open and fundamental}\}$$

is a basis that generates  $X$  and that satisfies the assumption of the proposition by Corollary 8.1.3.

Conversely, assume that there is a basis  $\mathcal{B}$  satisfying the assumption of the proposition. Let  $B \in \mathcal{B}$  and let  $O \in \mathcal{O}_T(X)$  such that  $O \simeq_T B$ . For every  $x \in X$  such that  $x \in_T O$ , there is  $B_x \in \mathcal{B}$  such that  $x \in_T B_x \subset_T O$ . Then one has  $\bigcup_{x \in !_X^T(B)} !_X^T(B_x) = !_X^T(B)$ , hence  $B$  is a join of the  $B_x$  in  $T(X)$  and consequently  $B \subset_T O$ . Therefore  $\mathcal{B}$  is a basis that generates  $X$  and whose members are fundamental in  $X$ . We conclude by the previous proposition.  $\square$

**Corollary 8.1.9.** Let  $T' = (T', \in_{T'})$  be another topological theory such that, for every set  $X$ ,  $T'(X)$  is complete and  $\in_{T'}^X$  preserves meets, let  $v : T \rightarrow T'$  be a changing of bases datum such that, for every set  $X$ ,  $v_X$  preserves joins, and let  $X$  be a  $T$ -topological space. If  $X$  is a  $T$ -stream then  $v(X)$  is a  $T'$ -stream.

*Proof.* It obviously follows from the previous corollary.  $\square$

### Examples 8.1.1.

- Let  $\mathbb{V}$  be a quantale. We can identify the  $T_{\mathbb{V}}^l$ -streams with the  $T_{\mathbb{V}}$ -topological spaces, i.e. with the  $\mathbb{V}$ -preordered sets.
- Every  $(\mathcal{P}, \in)$ -topological space, i.e. every classical topological space, is a  $(\mathcal{P}, \in)$ -stream.

## 8.2 The original example

Sanjeevi Krishnan [Kri09] introduced the notions of *stream* and of *prestream* as models of ‘locally directed spaces’. This justifies the terminology of ‘*T*-stream’ as we are going to see. We recall here some basic notions of [Kri09]:

**Definition 8.2.1** (Precirculations). A *precirculation* on a topological space  $(X, \mathcal{O}(X))$  is a function which associates every open subset  $O$  with a preorder  $\leq_O$  on it such that, for every  $O, O' \in \mathcal{O}(X)$  satisfying  $O \subset O'$ , for every  $x, x' \in O$ ,  $x \leq_O x'$  implies  $x \leq_{O'} x'$ . A *precirculation* is a *circulation* when, for every open subset  $O$ , for every open cover  $(O_i)_{i \in I}$  of  $O$ , the preorder  $\leq_O$  is the preorder generated by the relation  $\bigcup_{i \in I} \leq_{O_i}$ <sup>1</sup> (see Proposition A.0.10).

**Definition 8.2.2** ((pre)streams). A *(pre)stream* is a topological space equipped with a (pre)circulation.

**Definition 8.2.3** (Morphisms of (pre)streams). Let  $(X, \leq_)$  and  $(Y, \leq')$  be two prestreams. A function  $f : X \rightarrow Y$  is a *morphism of prestreams* when it is continuous and, for every open subset  $U$  of  $Y$ , for every  $x, x' \in f^{-1}(U)$ ,  $x \leq_{f^{-1}(U)} x'$  implies  $f(x) \leq'_U f(x')$ .

Let  $((X, \mathcal{O}(X), \leq_)$  be a prestream. One readily checks that the set  $\mathcal{B}$  of all  $(O, \leq_O)$  with  $O \in \mathcal{O}(X)$  is a basis of  $\mathbf{Loc}(T_{\mathbb{B}})$ -topology on  $X$ . The underlying topology of the  $\mathbf{Loc}(T_{\mathbb{B}})$ -topological space that  $\mathcal{B}$  generates is  $\mathcal{O}(X)$ . The major issue with the notion of prestream morphisms is that it is not local. For example, on  $\mathbb{R}$  endowed with its usual topology, define two precirculations, with  $O$  an open subset of  $\mathbb{R}$ :

- $\leq_O^1$  is the restriction on  $O$  of the usual order of  $\mathbb{R}$ , and
- $\leq_O^2 = \leq_O^1$  unless  $O = \mathbb{R}$ ; in this case,  $\leq_{\mathbb{R}}^2$  is the coarsest preorder  $\mathbb{R}$ .

The identity map  $\text{Id}_{\mathbb{R}}$  is a morphism of prestreams from  $(\mathbb{R}, \leq^1)$  to  $(\mathbb{R}, \leq^2)$  but not from  $(\mathbb{R}, \leq^2)$  to  $(\mathbb{R}, \leq^1)$  whereas both induce the same  $\mathbf{Loc}(T_{\mathbb{B}})$ -topology on  $\mathbb{R}$ . This phenomenon vanishes with streams:

**Proposition 8.2.4.** Let  $(X, \leq_)$  and  $(Y, \leq')$  be two streams. A function  $f : X \rightarrow Y$  is a morphism of prestreams if and only if it is  $\mathbf{Loc}(T_{\mathbb{B}})$ -continuous.

*Proof.* Denote by  $\mathcal{B}$  and  $\mathcal{B}'$  the bases of  $\mathbf{Loc}(T_{\mathbb{B}})$ -topology generated by  $\leq_$  and  $\leq'$  respectively and let  $x \in X$ . Assume that  $f : X \rightarrow Y$  is a morphism of prestreams. Let  $U$  be an open subset of  $Y$  such that  $f(x) \in U$ . Then one has  $x \in f^{-1}(U)$ ,  $(U, \leq'_U) \in \mathcal{B}'$ ,  $(f^{-1}(U), \leq_{f^{-1}(U)}) \in \mathcal{B}$ , and  $(f^{-1}(U), \leq_{f^{-1}(U)}) \mathbf{Loc}(T_{\mathbb{B}})(f)(U, \leq'_U)$ .

Conversely, assume that  $f$  is  $\mathbf{Loc}(T_{\mathbb{B}})$ -continuous. Let  $U$  be an open subset of  $Y$ . For every  $x \in f^{-1}(U)$ , by  $\mathbf{Loc}(T_{\mathbb{B}})$ -continuity of  $f$  at  $x$ , there is an open subset  $O_x$  containing  $x$  such that  $(O_x, \leq_{O_x}) \mathbf{Loc}(T_{\mathbb{B}})(f)(U, \leq'_U)$ . Since  $(O_x)_{x \in f^{-1}(U)}$  is an open cover of  $f^{-1}(U)$  and since  $\leq_$  is a circulation, we deduce that  $(f^{-1}(U), \leq_{f^{-1}(U)}) \mathbf{Loc}(T_{\mathbb{B}})(f)(U, \leq'_U)$ .  $\square$

Therefore, we can identify the category of streams and morphisms of streams with a full subcategory of  $\mathbf{Top}_{\mathbf{Loc}(T_{\mathbb{B}})}$ . This subcategory is the category of  $\mathbf{Loc}(T_{\mathbb{B}})$ -streams. Indeed, the  $\mathbf{Loc}(T_{\mathbb{B}})$ -topological space induced by a stream is a  $\mathbf{Loc}(T_{\mathbb{B}})$ -stream by Corollary 8.1.4. Conversely, from every  $\mathbf{Loc}(T_{\mathbb{B}})$ -streams one builds a usual stream as follows: each open subset  $U$  of the induced topology is equipped with the preorder associated to the fundamental  $\mathbf{Loc}(T_{\mathbb{B}})$ -open member on  $U$ , see Corollary 8.1.3.

## 8.3 Construction of a coreflection

Under the assumption that the topological theory  $T$  satisfies (8.1) or (8.2), we construct a coreflection of the full subcategory of  $T$ -streams.

<sup>1</sup>the  $\leq_i$  are seen as subsets of  $O \times O$ .



The first one

$$\begin{aligned} & \text{for every function } f : X \rightarrow Y, \\ & T(f) \text{ has an inverse image } f^{-1} \text{ that preserves meets of } T(Y). \end{aligned} \quad (8.1)$$

The second one

$$\begin{aligned} & \text{for every function } f : X \rightarrow Y, \\ & T(f) \text{ has an inverse image } f^{-1} \text{ relative to } T \text{ that preserves meets of } T(Y), \text{ and} \\ & \text{every } T\text{-stream is saturable and its saturated is also a } T\text{-stream.} \end{aligned} \quad (8.2)$$

By Proposition 7.1.15, if the assumption (8.1) is true, then every  $T$ -stream is saturable. Moreover, one has

**Proposition 8.3.1.** Let  $X$  be a  $T$ -stream. Assume (8.1). Then the saturated of  $X$  is also a  $T$ -stream.

*Proof.* Denote by  $i^{-1}$  the inverse image of  $T(\text{Id}_X)$ . Let  $A$  be a pseudo- $T$ -open of  $X$  and let  $A'' := \bigwedge \{A' \in \mathcal{O}_T^{\text{sat}}(X) ; A' \simeq_T A\}$ . We are to show that  $A''$  is pseudo- $T$ -open in  $X$ . To show that, by Lemma 7.1.14, it is enough to prove that  $i^{-1}(A'')$  is  $T$ -open. For every  $A' \in \mathcal{O}_T^{\text{sat}}(X)$  such that  $A' \simeq_T A$ , one has  $i^{-1}(A') \simeq_T i^{-1}(A)$  and  $i^{-1}(A')$  is  $T$ -open, hence, by Proposition 8.1.6  $i^{-1}(A'')$  is  $T$ -open because  $X$  is a  $T$ -stream. We conclude that  $A''$  is fundamental and that the saturated is a  $T$ -stream because  $\mathcal{O}_T(\mathcal{O}_T^{\text{sat}}(X)) = \mathcal{O}_T^{\text{sat}}(X)$  by Lemma 7.1.10.  $\square$

By Lemma 5.2.22 or 7.1.4, the assumptions (8.1) and (8.2) imply that  $T$  is a topological theory with strong membership.

The construction of the coreflection we propose is based on a transfinite recursion. At each step, we add members which should be  $T$ -open in a  $T$ -stream. But, doing this, new members have to be added.

Let  $X$  be a  $T$ -topological space. Write

$$\mathcal{B}^+(X) := \left\{ \bigwedge \{O' \in \mathcal{O}_T(X) ; O' \simeq_T O\} ; O \in \mathcal{O}_T(X) \right\}.$$

**Lemma 8.3.2.** Let  $X$  be a  $T$ -topological space. Then  $\mathcal{B}^+(X)$  is a basis of  $T$ -topology on  $X$ .

*Proof.* Let  $x \in X$  and let  $(O''_j)_{j \in J}$  be a finite family of  $\mathcal{B}^+(X)$  such that, for every  $j \in J$ ,  $x \in_T O''_j$ . By definition of  $\mathcal{B}^+(X)$ , for every  $j \in J$ , there is  $O_j \in \mathcal{O}_T(X)$  such that

$$O''_j = \bigwedge \{O'_j \in \mathcal{O}_T(X) ; O'_j \simeq_T O_j\}.$$

Then, by stability of  $T$ -openness under finite meets (Proposition 5.2.11),  $O := \bigwedge_{j \in J} O_j$  is  $T$ -open. Then one has  $O'' := \bigwedge \{O' \in \mathcal{O}_T(X) ; O' \simeq_T O\} \in \mathcal{B}^+(X)$ . Since  $\in_X^T$  preserves meets,  $x \in_T O$  then  $x \in_T O''$ . Let  $j \in J$  and let  $O'_j \in \mathcal{O}_T(X)$  such that  $O'_j \simeq_T O_j$ . Write  $O' := O'_j \wedge O$ . One has  $O' \in \mathcal{O}_T(X)$  and  $O' \simeq_T O$  hence  $O'' \subset_T O'$ . Since  $O' \subset_T O'_j$ , it follows that  $O'' \subset_T O'_j$ . Since  $O''_j = \bigwedge \{O'_j \in \mathcal{O}_T(X) ; O'_j \simeq_T O_j\}$ , we conclude that  $O'' \subset_T O''_j$ .  $\square$

**Lemma 8.3.3.** Let  $X$  be a  $T$ -topological space. Then  $\mathcal{O}_T(X) \subset \mathcal{O}_T(\mathcal{B}^+(X))$ .

*Proof.* Let  $O \in \mathcal{O}_T(X)$ . Then  $O'' := \bigwedge \{O' \in \mathcal{O}_T(X) ; O' \simeq_T O\} \in \mathcal{B}^+(X)$ ,  $O'' \subset_T O$ , and  $O'' \simeq_T O$  because  $\in_X^T$  preserves meets, hence, by Proposition 5.2.14,  $O$  is  $T$ -open for  $\mathcal{B}^+(X)$ .  $\square$

In case of a limit ordinal, we apply the following lemma:

**Lemma 8.3.4.** Let  $(I, \leq)$  be a filtered preordered set and let  $(\mathcal{B}_i)_{i \in I}$  be a family of bases of  $T$ -topology on  $X$  such that for every  $i, i' \in I$  satisfying  $i \leq i'$ , one has  $\mathcal{B}_i \subset \mathcal{B}_{i'}$ . Then  $\mathcal{B} := \bigcup_{i \in I} \mathcal{B}_i$  is a basis of  $T$ -topology on  $X$ .

*Proof.* Let  $x \in X$  and let  $(B_j)_{j \in J}$  be a finite family of  $\mathcal{B}$  such that, for every  $j \in J$ ,  $x \in_T B_j$ . Since  $I$  is filtered, there is  $i \in I$  such that, for every  $j \in J$ ,  $B_j \in \mathcal{B}_i$ . Given that  $\mathcal{B}_i$  is a basis of  $T$ -topology, there is  $B \in \mathcal{B}_i \subset \mathcal{B}$  such that  $x \in_T B$  and, for every  $j \in J$ ,  $B \subset_T B_j$ .  $\square$

**Lemma 8.3.5.** Let  $X$  be a  $T$ -topological space. The space  $X$  is a  $T$ -stream if and only if  $\mathcal{B}^+(X) \subset \mathcal{O}_T(X)$ .

*Proof.* Assume that  $X$  is a  $T$ -stream. Let  $O \in \mathcal{O}_T(X)$ . Let  $O_0 \in \mathcal{O}_T(X)$  fundamental in  $X$  such that  $O_0 \simeq_T O$ . Then  $\bigwedge\{O' \in \mathcal{O}_T(X) ; O' \simeq_T O\} = O_0 \in \mathcal{O}_T(X)$ . It follows that  $\mathcal{B}^+(X) \subset \mathcal{O}_T(X)$ .

Conversely, assume that  $\mathcal{B}^+(X) \subset \mathcal{O}_T(X)$ . Let  $O \in \mathcal{O}_T(X)$ . Then the member  $O_0 := \bigwedge\{O' \in \mathcal{O}_T(X) ; O' \simeq_T O\}$  is  $T$ -open in  $X$  and is  $\simeq_T$ -equivalent to  $O$  because  $\in_T^X$  preserves finite meets. We easily check that  $O_0$  is  $T$ -open and is fundamental.  $\square$

Let  $X$  be a  $T$ -topological space. We define by recursion, for every ordinal  $\alpha$ , a basis of  $T$ -topology  $\mathcal{B}^\alpha(X)$  on  $X$  by:

- $\mathcal{B}^0(X) := \mathcal{O}_T(X)$ .
- $\mathcal{B}^{\alpha+1}(X) := \mathcal{B}^+(X, \mathcal{O}_T(\mathcal{B}^\alpha(X)))$ .
- If  $\alpha$  is a limit ordinal,  $\mathcal{B}^\alpha(X) := \bigcup_{\beta < \alpha} \mathcal{B}^\beta(X)$

By the previous lemmas,  $(\mathcal{O}_T(\mathcal{B}^\alpha(X)))_{\alpha \text{ ordinal}}$  is a family of  $T$ -topology on  $X$  such that, for every pair of ordinals  $(\alpha, \alpha')$  such that  $\alpha \leq \alpha'$ ,  $\mathcal{O}_T(\mathcal{B}^\alpha(X)) \subset \mathcal{O}_T(\mathcal{B}^{\alpha'}(X))$  and such that  $\mathcal{O}_T(\mathcal{B}^0(X)) = \mathcal{O}_T(X)$ . Since  $\mathcal{P}(T(X))$  is a set, there is an ordinal  $\alpha_m$  such that  $\mathcal{O}_T(\mathcal{B}^{\alpha_m}(X)) = \mathcal{O}_T(\mathcal{B}^{\alpha_m+1}(X))$ . By the previous lemma,  $(X, \mathcal{O}_T(\mathcal{B}^{\alpha_m}(X)))$  is a  $T$ -stream. Define

$$\mathbf{Stm}^T(X) := (X, \mathcal{O}_T(\mathcal{B}^{\alpha_m}(X)))$$

Since  $\mathcal{O}_T(X) \subset \mathcal{O}_T(\mathcal{B}^{\alpha_m}(X))$ , the map  $\text{Id}_X$  is  $T$ -continuous from  $\mathbf{Stm}^T(X)$  to  $X$ . It remains the question of the  $T$ -continuous maps.

**Lemma 8.3.6.** Let  $X$  and let  $Y$  be  $T$ -topological spaces and let  $f : X \rightarrow Y$  be a function. Assume (8.1) or (8.2). If  $X$  is a  $T$ -stream and if  $f$  is  $T$ -continuous from  $X$  to  $Y$ , then  $f$  is  $T$ -continuous from  $X$  to  $(Y, \mathcal{O}_T(\mathcal{B}^+(Y)))$ .

*Proof.* The assumptions (8.1) and (8.2) both imply that  $X$  is saturable and that its saturated is also a  $T$ -stream. Thus we can assume that  $X$  is saturated because the identity is an isomorphism between a space and its saturated.

Let  $U'' \in \mathcal{B}^+(Y)$ . There is  $U \in \mathcal{O}_T(Y)$  such that

$$U'' = \bigwedge\{U' \in \mathcal{O}_T(Y) ; U' \simeq_T U\}.$$

Since  $f$  is  $T$ -continuous from  $X$  to  $Y$ , by Proposition 5.2.23 or Proposition 7.1.7,  $f^{-1}(U)$  is  $T$ -open in  $X$ . Hence there is  $O \in \mathcal{O}_T(X)$  fundamental such that  $O \simeq_T f^{-1}(U)$  because  $X$  is a  $T$ -stream.

For every  $U' \in \mathcal{O}_T(Y)$  such that  $U' \simeq_T U$ , since  $f$  is  $T$ -continuous from  $X$  to  $Y$ , still by Proposition 5.2.23 or Proposition 7.1.7,  $f^{-1}(U')$  is  $T$ -open and the set of all its  $T$ -elements is  $\{x \in X ; f(x) \in_T U'\}$ . Given that  $U' \simeq_T U$ , one has  $f^{-1}(U) \simeq_T f^{-1}(U')$ . It follows that  $O \subset_T f^{-1}(U')$ . Thus,

$$O \subset_T \bigwedge\{f^{-1}(U') ; U' \in \mathcal{O}_T(Y) \text{ such that } U' \simeq_T U\}$$

Since, by assumption,  $f^{-1}$  preserves meets, we deduce that

$$O \subset_T f^{-1}\left(\bigwedge\{U' \in \mathcal{O}_T(X) ; U' \simeq_T U\}\right) = f^{-1}(U'')$$

Since  $O$  and  $f^{-1}(U'')$  have the same  $T$ -elements, by Proposition 5.2.14,  $f^{-1}(U'')$  is  $T$ -open in  $X$ . Hence, by Proposition 5.2.23 or 7.1.7,  $f$  is  $T$ -continuous from  $X$  to  $(Y, \mathcal{O}_T(\mathcal{B}^+(Y)))$ .  $\square$

**Lemma 8.3.7.** Let  $(I, \leq)$  be a filtered preordered set, let  $(\mathcal{B}_i)_{i \in I}$  be a family of bases of  $T$ -topology on  $Y$  such that for every  $i, i' \in I$  satisfying  $i \leq i'$ , one has  $\mathcal{B}_i \subset \mathcal{B}_{i'}$ , let  $X$  be a  $T$ -topological space, and let  $f : X \rightarrow Y$  be a function. If, for every  $i \in I$ ,  $f$  is  $T$ -continuous from  $X$  to  $(Y, \mathcal{O}_T(\mathcal{B}_i))$ , then  $f$  is  $T$ -continuous from  $X$  to  $(Y, \mathcal{O}_T(\bigcup_{i \in I} \mathcal{B}_i))$ .

*Proof.* Let  $x \in X$  and let  $B \in \bigcup_{i \in I} \mathcal{B}_i$  such that  $f(x) \in_T B$ . There is  $i \in I$  such that  $B \in \mathcal{B}_i$ . Given that  $f$  is  $T$ -continuous at  $x$  from  $X$  to  $(Y, \mathcal{O}_T(\mathcal{B}_i))$ , there is  $O \in \mathcal{O}_T(X)$  such that  $x \in_T O$  and  $O \mathit{T}(f) B$ . Thus  $f$  is  $T$ -continuous at  $x$  from  $X$  to  $(Y, \mathcal{O}_T(\bigcup_{i \in I} \mathcal{B}_i))$ .  $\square$

**Corollary 8.3.8.** Let  $X$  be a  $T$ -stream, let  $Y$  be a  $T$ -topological space and let  $f : X \rightarrow Y$  be a function. Assume (8.1) or (8.2). Then  $f$  is  $T$ -continuous from  $X$  to  $Y$  if and only if  $f$  is  $T$ -continuous from  $X$  to  $\mathbf{Stm}^T(Y)$ .

*Proof.* Assume that  $f$  is  $T$ -continuous from  $X$  to  $\mathbf{Stm}^T(Y)$ . The map  $\text{Id}_Y$  is  $T$ -continuous from  $\mathbf{Stm}^T(Y)$  to  $Y$ , hence  $f$  is  $T$ -continuous from  $X$  to  $Y$ .

Conversely, assume that  $f$  is  $T$ -continuous from  $X$  to  $Y$ . By transfinite induction, using the two previous lemmas, we show that, for every ordinal  $\alpha$ ,  $f$  is  $T$ -continuous from  $X$  to  $(Y, \mathcal{O}_T(\mathcal{B}^\alpha(Y)))$ . In particular, the function  $f$  is  $T$ -continuous from  $X$  to  $\mathbf{Stm}^T(Y)$   $\square$

Therefore, assuming (8.1) or (8.2), the full subcategory  $\mathbf{Stream}_T$  of  $T$ -streams of  $\mathbf{Top}_T$  is coreflexive.

The assumptions (8.2) make the constructions developed in 7.2 available, so more results hold.

**Proposition 8.3.9.** Let  $X$  be a set, let  $(X_i)_{i \in I}$  be a family of saturated  $T$ -topological spaces, and let  $(f_i : X_i \rightarrow X)_{i \in I}$  be a family of  $T$ -continuous maps. Assume that the  $T(f_i)$  have inverse images  $f_i^{-1}$  relative to  $T$  that preserves meets. If the  $X_i$  are  $T$ -streams then so is  $X$  endowed with final  $T$ -topology associated to the  $f_i$ .

*Proof.* Let  $O \in \mathcal{O}_T(X)$ . We are to show that  $O'' := \bigwedge \{O' \in \mathcal{O}_T(X) ; O' \simeq_T O\}$  is  $T$ -open in  $X$ . Regarding the description of the final  $T$ -topology given in section 7.2 and since the  $X_i$  are saturated, we have to show that, for each  $i \in I$ ,  $f_i^{-1}(O'')$  is  $T$ -open in  $X_i$ . But, since  $f_i$  is  $T$ -continuous,  $f_i^{-1}(O)$  is  $T$ -open in  $X_i$ . Moreover, since  $f_i^{-1}$  preserves meets,  $f_i^{-1}(O'') = \bigwedge \{f_i^{-1}(O') ; O' \in \mathcal{O}_T(X) \text{ such that } O' \simeq_T O\}$ . Finally, since, for every  $O' \in \mathcal{O}_T(X)$  such that  $O' \simeq_T O$ , one has  $f_i^{-1}(O') \simeq_T f_i^{-1}(O)$ , and since  $X_i$  is a  $T$ -stream, we conclude that  $f_i^{-1}(O'')$  is  $T$ -open in  $X_i$  by Proposition 8.1.6  $\square$

Therefore, assuming (8.2) allows an alternative construction of the coreflection: given a  $T$ -topological space  $X$ , we endow the underlying set of  $X$  with the final  $T$ -topology associated to the family of all  $T$ -continuous maps from a saturated  $T$ -stream to the space  $X$ .

Moreover

**Theorem 8.3.10.** Assume (8.2) and assume that  $T$  is saturable. Then the restriction of the forgetful functor  $\mathbf{U}_T : \mathbf{Top}_T \rightarrow \mathbf{Set}$  to the full subcategory  $\mathbf{Stream}_T$  is a topological functor.

*Proof.* Trivial consequence of the previous proposition and of 7.2.4.  $\square$

Let  $\mathbb{V}$  be a quantale. Since we can identify the  $T_{\mathbb{V}}^l$ -streams and the  $\mathbb{V}$ -ordered sets, the previous theorem give us an alternative proof of Corollary A.0.11

# Chapter 9

## Products and exponential objects of $\mathbf{Top}_{\text{Loc}}(T)$

Set a topological theory  $T = (T, \in_T)$ .

The aim of this chapter is to study the exponential objects in (in subcategories of) of the category  $\mathbf{Top}_{\text{Loc}}(T)$ .

For this, as in the case of classical point-set topology, we define the  $\mathcal{C}$ -generated  $T$ -topological spaces. This notion generalizes the compactly generated spaces. As in the classical case, when  $T$  satisfies the assumptions used to show that  $\mathbf{U}_T$  is a topological functor in Chapter 7, we show that the category of  $\mathcal{C}$ -generated  $T$ -topological spaces is coreflexive.

Definitions and proof's schemes are inspired from [GL13, 5.2-6] (see also [GL14]).

### 9.1 $\mathcal{C}$ -generated $T$ -topological spaces

Let  $\mathcal{C}$  be a class of  $T$ -topological spaces.

**Definition 9.1.1** ( $\mathcal{C}$ -continuous maps). Let  $X$  and  $Y$  be two  $T$ -topological spaces and let  $f : X \rightarrow Y$  be a function. The function  $f$  is  $\mathcal{C}$ -continuous when, for every  $C \in \mathcal{C}$ , for every  $T$ -continuous map  $p : C \rightarrow X$ , the composite  $f \circ p$  is  $T$ -continuous from  $C$  to  $Y$ .

One readily checks that every  $T$ -continuous map is  $\mathcal{C}$ -continuous and that the composite of two  $\mathcal{C}$ -continuous maps is  $\mathcal{C}$ -continuous. We denote by  $\mathcal{C}\text{-Map}$  the category of  $T$ -topological spaces and  $\mathcal{C}$ -continuous maps.

**Definition 9.1.2** ( $\mathcal{C}$ -generated  $T$ -topological spaces). Let  $X$  be a  $T$ -topological space. The space  $X$  is  $\mathcal{C}$ -generated when, for every  $T$ -topological space  $Y$ , for every function  $f : X \rightarrow Y$ , the function  $f$  is  $T$ -continuous if and only if it is  $\mathcal{C}$ -continuous.

*Remark 9.1.1.* In the previous definition, the direct implication is always true.

**Example 9.1.1.** For the topological theory  $(\mathcal{P}, \in)$ , we recover the classical notion of compactly generated spaces considering the class of all compact Hausdorff spaces.

Denote by  $\mathcal{C}\text{-Top}_T$  the full subcategory of  $\mathbf{Top}_T$  formed by the  $\mathcal{C}$ -generated spaces. Clearly,  $\mathcal{C}\text{-Top}_T$  is a full subcategory of  $\mathcal{C}\text{-Map}$ .

We now assume that every space of  $\mathcal{C}$  is saturable, that, for every set  $X$ ,  $T(X)$  is finitely complete and  $\in_T^X$  preserves finite meets, and that, for every function  $f : X \rightarrow Y$ ,  $T(f)$  has an inverse image  $f^{-1}$  relative to  $T$  which preserves finite meets. Thanks to those assumptions, we will be able to use the constructions of Section 7.2.

Let  $X$  be a  $T$ -topological space. Denote by  $\mathcal{C}X$  the underlying set of  $X$  endowed with the final  $T$ -topology associated to the class of all  $T$ -continuous maps  $p : C \rightarrow X$ , for any  $C \in \mathcal{C}$ . By Proposition 7.1.7, one has  $\mathcal{O}_T(X) \subset \mathcal{O}_T(\mathcal{C}X)$ , then, a fortiori, the map  $\text{Id}_X$  is  $T$ -continuous from  $\mathcal{C}X$  to  $X$ .

**Lemma 9.1.3.** Let  $C \in \mathcal{C}$  and let  $p : C \rightarrow X$  be a function. The function  $p$  is  $T$ -continuous from  $C$  to  $X$  if and only if it is  $T$ -continuous from  $C$  to  $\mathcal{C}X$ .

*Proof.* If  $p$  is  $T$ -continuous from  $C$  to  $\mathcal{C}X$ , since  $\text{Id}_X : \mathcal{C}X \rightarrow X$  is  $T$ -continuous, then  $p$  is  $T$ -continuous from  $C$  to  $X$ . Conversely, if  $p$  is  $T$ -continuous from  $C$  to  $X$  then, by definition of the  $T$ -topology of  $\mathcal{C}X$ ,  $p$  is  $T$ -continuous from  $C$  to  $\mathcal{C}X$  because  $C \in \mathcal{C}$ .  $\square$

**Corollary 9.1.4.** The  $T$ -topological space  $\mathcal{C}X$  is  $\mathcal{C}$ -generated.

*Proof.* Let  $f : \mathcal{C}X \rightarrow Y$  be a  $\mathcal{C}$ -continuous map. Let  $C \in \mathcal{C}$  and let  $p$  be a  $T$ -continuous map from  $C$  to  $X$ . By the previous lemma,  $p$  is  $T$ -continuous from  $C$  to  $\mathcal{C}X$ . By  $\mathcal{C}$ -continuity, the composite  $f \circ p$  is then  $T$ -continuous. Thus, given that  $\mathcal{C}X$  is endowed with the final  $T$ -topology associated to the  $T$ -continuous maps to  $X$  whose domain is in  $\mathcal{C}$ , it follows that  $f$  is  $T$ -continuous from  $\mathcal{C}X$  to  $Y$ .  $\square$

**Corollary 9.1.5.** Let  $W$  be a  $\mathcal{C}$ -generated space and let  $f : W \rightarrow X$  be a function. The function  $f$  is  $T$ -continuous from  $W$  to  $X$  if and only if it is  $T$ -continuous from  $W$  to  $\mathcal{C}X$ .

*Proof.* Assume that  $f$  is  $T$ -continuous from  $W$  to  $\mathcal{C}X$ . Then it is  $T$ -continuous from  $W$  to  $X$  because  $\text{Id}_X$  is  $T$ -continuous from  $\mathcal{C}X$  to  $X$ .

Conversely, assume that  $f$  is  $T$ -continuous from  $W$  to  $X$ . Let  $C \in \mathcal{C}$  and let  $p : C \rightarrow W$  be a  $T$ -continuous map. Then the composite  $f \circ p$  is  $T$ -continuous from  $C$  to  $X$ , hence by the previous lemma,  $f \circ p$  is  $T$ -continuous from  $C$  to  $\mathcal{C}X$ . Thus  $f$  is  $\mathcal{C}$ -continuous. Finally  $f$  is  $T$ -continuous from  $W$  to  $\mathcal{C}X$  because  $W$  is  $\mathcal{C}$ -generated.  $\square$

**Corollary 9.1.6.** The full subcategory  $\mathcal{C}\text{-Top}_T$  is coreflexive in  $\text{Top}_T$ .

Thus  $X$  is  $\mathcal{C}$ -generated if and only if  $\text{Id}_X$  is  $T$ -continuous from  $X$  to  $\mathcal{C}X$ . In other words

**Corollary 9.1.7.** A  $T$ -topological space  $X$  is  $\mathcal{C}$ -generated if and only if, for every  $A \in T(X)$ , if, for every  $T$ -continuous map  $p : C \rightarrow X$  with  $C \in \mathcal{C}$ ,  $p^{-1}(A)$  is pseudo- $T$ -open in  $C$ , then  $A$  is pseudo- $T$ -open in  $X$ .

**Proposition 9.1.8.** Let  $X$  and  $Y$  be two  $T$ -topological spaces and let  $f : X \rightarrow Y$  be a  $\mathcal{C}$ -continuous function. Then  $f$  is  $T$ -continuous from  $\mathcal{C}X$  to  $\mathcal{C}Y$ .

*Proof.* Given that  $\mathcal{C}X$  is  $\mathcal{C}$ -generated,  $f$  is  $T$ -continuous from  $\mathcal{C}X$  to  $\mathcal{C}Y$  if and only if it is  $T$ -continuous from  $\mathcal{C}X$  to  $Y$  by Corollary 9.1.5. Let  $C \in \mathcal{C}$  and let  $p : C \rightarrow \mathcal{C}X$  be a  $T$ -continuous map. Then  $p$  is  $T$ -continuous from  $C$  to  $X$ , hence the composite  $f \circ p$  is  $T$ -continuous from  $C$  to  $Y$  because  $f$  is  $\mathcal{C}$ -continuous from  $X$  to  $Y$ . Thus  $f$  is  $\mathcal{C}$ -continuous from  $\mathcal{C}X$  to  $Y$ . Given that  $\mathcal{C}X$  is  $\mathcal{C}$ -generated, we deduce the  $T$ -continuity of  $f : \mathcal{C}X \rightarrow Y$ .  $\square$

**Lemma 9.1.9.** Let  $X$  be a  $T$ -topological space. The map  $\text{Id}_X$  is  $\mathcal{C}$ -continuous from  $X$  to  $\mathcal{C}X$  and from  $\mathcal{C}X$  to  $X$ .

*Proof.* Clear consequence of Corollary 9.1.5.  $\square$

**Lemma 9.1.10.** Let  $X$  be a  $\mathcal{C}$ -generated  $T$ -topological space. The map  $\text{Id}_X$  is  $T$ -continuous from  $X$  to  $\mathcal{C}X$  and from  $\mathcal{C}X$  to  $X$ .

*Proof.* Clear consequence of Corollary 9.1.5.  $\square$

**Corollary 9.1.11.** The inclusion of  $\mathcal{C}\text{-Top}_T$  in  $\mathcal{C}\text{-Map}$  is an equivalence of categories with  $\mathcal{C}(\_)$  as quasi-inverse and the identity maps as units.

Now, as in the previous chapter, assume that all the  $T(X)$  are complete and all the  $\in_T^X$  preserve meets and assume (8.2). Let  $\mathcal{C}$  be the class of all  $T$ -streams.

**Proposition 9.1.12.** Let  $X$  be a  $T$ -topological space. Then  $\mathcal{C}X$  is a  $T$ -stream.

*Proof.* It is a clear consequence of Proposition 8.3.9.  $\square$

Hence the identity maps are isomorphisms between the  $T$ -streams and the  $\mathcal{C}$ -generated spaces and when every  $T$ -topological space is saturated, both notions match.

## 9.2 Some (closed) monoidal structures on $\mathbf{Top}_{\mathbf{Loc}(T)}$

### 9.2.1 Lifting of monoidal structures

From now on, assume that  $\mathbf{Top}_T$  is endowed with a monoidal structure  $(\mathbf{Top}_T, \otimes, I_T)$  such that  $\mathbf{U}_T$  is a strict monoidal functor into  $(\mathbf{Set}, \times, \{\star\})$ .

*Remark 9.2.1.* The assumption on the strictness of  $\mathbf{U}_T$  is used to simplify the following constructions. If  $\mathbf{U}_T$  is a strong monoidal functor, by changing the choice to the binary products of the final object of  $\mathbf{Set}$ , it is always possible to assume that  $\mathbf{U}_T$  is strict.

**Proposition 9.2.1.** Let  $X$  and  $Y$  be sets, and let  $\mathcal{B}_X$  and  $\mathcal{B}_Y$  be bases of  $\mathbf{Loc}(T)$ -topology on  $X$  and  $Y$  respectively. Then

$$\mathcal{B}_X \otimes \mathcal{B}_Y := \{B_X \otimes B_Y ; B_X \in \mathcal{B}_X, B_Y \in \mathcal{B}_Y\}$$

is a basis of  $\mathbf{Loc}(T)$ -topology on  $X \times Y$ .

*Remark 9.2.2.* Since, for every  $B_X \in \mathcal{B}_X$  and  $B_Y \in \mathcal{B}_Y$ ,  $\mathbf{U}_T(B_X \otimes B_Y) = \mathbf{U}_T(B_X) \times \mathbf{U}_T(B_Y)$ , the underlying set of  $B_X \otimes B_Y$  is indeed a subset of  $X \times Y$ .

*Proof.* Let  $(x, y) \in X \times Y$  and let  $(B_X^j \otimes B_Y^j)_{j \in J}$  be a finite family of  $\mathcal{B}_X \otimes \mathcal{B}_Y$  such that, for every  $j \in J$ ,  $(x, y) \in B_X^j \otimes B_Y^j$ . By the previous remark, for every  $j \in J$ , one has  $x \in B_X^j$  and  $y \in B_Y^j$ . Given that  $\mathcal{B}_X$  and  $\mathcal{B}_Y$  are bases of  $\mathbf{Loc}(T)$ -topology, there is  $B_X \in \mathcal{B}_X$  and  $B_Y \in \mathcal{B}_Y$  such that  $x \in B_X$ ,  $y \in B_Y$ , and, for every  $j \in J$ ,  $B_X \subset_{\mathbf{Loc}(T)} B_X^j$  and  $B_Y \subset_{\mathbf{Loc}(T)} B_Y^j$ . It follows that  $(x, y) \in B_X \otimes B_Y$ ,  $B_X \otimes B_Y \in \mathcal{B}_X \otimes \mathcal{B}_Y$ , and that, for every  $j \in J$ , since  $\mathbf{U}_T$  is a strict monoidal functor,  $B_X \otimes B_Y \subset_{\mathbf{Loc}(T)} B_X^j \otimes B_Y^j$ .  $\square$

**Proposition 9.2.2.** Let  $X$  and  $Y$  be sets, and let  $\mathcal{B}_X$  and  $\mathcal{B}_Y$  be bases of  $\mathbf{Loc}(T)$ -topology on  $X$  and  $Y$  respectively. Let  $O_X \in \mathcal{O}_{\mathbf{Loc}(T)}(\mathcal{B}_X)$  and  $O_Y \in \mathcal{O}_{\mathbf{Loc}(T)}(\mathcal{B}_Y)$ . Then  $O_X \otimes O_Y \in \mathcal{O}_{\mathbf{Loc}(T)}(\mathcal{B}_X \otimes \mathcal{B}_Y)$ .

*Proof.* Let  $(x, y) \in O_X \otimes O_Y$ . Then  $x \in O_X$  and  $y \in O_Y$ . Consequently, there is  $B_X \in \mathcal{B}_X$  such that  $x \in B_X \subset_{\mathbf{Loc}(T)} O_X$  and  $B_Y \in \mathcal{B}_Y$  such that  $y \in B_Y \subset_{\mathbf{Loc}(T)} O_Y$ . Therefore  $(x, y) \in B_X \otimes B_Y \subset_{\mathbf{Loc}(T)} O_X \otimes O_Y$ .  $\square$

**Corollary 9.2.3.** Let  $X$  and  $Y$  be sets, let  $\mathcal{B}_X$  and  $\mathcal{B}'_X$  be bases of  $\mathbf{Loc}(T)$ -topology on  $X$ , and let  $\mathcal{B}_Y$  and  $\mathcal{B}'_Y$  be bases of  $\mathbf{Loc}(T)$ -topology on  $Y$ . If  $\mathcal{B}_X$  and  $\mathcal{B}'_X$  are equivalent and if  $\mathcal{B}_Y$  and  $\mathcal{B}'_Y$  are equivalent then  $\mathcal{B}_X \otimes \mathcal{B}_Y$  and  $\mathcal{B}'_X \otimes \mathcal{B}'_Y$  are equivalent.

Let  $X$  and  $Y$  be two  $\mathbf{Loc}(T)$ -topological spaces. Define  $X \otimes Y$  the set  $X \times Y$  endowed with the  $\mathbf{Loc}(T)$ -topology generated by  $\mathcal{O}_{\mathbf{Loc}(T)}(X) \otimes \mathcal{O}_{\mathbf{Loc}(T)}(Y)$ .

Denote by  $I_{\mathbf{Loc}(T)}$  the  $\mathbf{Loc}(T)$ -topological space  $\eta_T(I_T)$  (5.4). One easily checks, thanks to the previous corollary, that  $(\mathbf{Top}_{\mathbf{Loc}(T)}, \otimes, I_{\mathbf{Loc}(T)})$  is a monoidal category such that the forgetful functor  $\mathbf{U}_{\mathbf{Loc}(T)}$  is strictly monoidal. The functor  $\eta_T$  is then a fully faithful monoidal strict functor from  $(\mathbf{Top}_T, \otimes, I_T)$  to  $(\mathbf{Top}_{\mathbf{Loc}(T)}, \otimes, I_{\mathbf{Loc}(T)})$ . Moreover, if, for every pair of  $T$ -topological spaces  $X$  and  $Y$ , the function  $s_{X,Y} : (x, y) \mapsto (y, x)$  is an isomorphism between  $X \otimes Y$  and  $Y \otimes X$  in  $\mathbf{Top}_T$ , then, for every pair of  $\mathbf{Loc}(T)$ -topological spaces  $X$  and  $Y$ ,  $s_{X,Y}$  is an isomorphism between  $X \otimes Y$  and  $Y \otimes X$  in  $\mathbf{Top}_{\mathbf{Loc}(T)}$ .

*Remark 9.2.3.* If  $\otimes$  is the cartesian product of  $\mathbf{Top}_T$  then  $\otimes$  is the cartesian product of  $\mathbf{Top}_{\mathbf{Loc}(T)}$ .

### 9.2.2 Lifting of (monoidal) closedness

From now on, assume that the forgetful functor  $\mathbf{U}_T$  is a topological functor. In particular, for every set  $X$ , the module  $\in : X \rightarrow \mathbf{Loc}(T)(X)$  is then supercompact and  $\mathbf{Loc}(T)(X)$  is complete.

For every  $\mathbf{Loc}(T)$ -topological space  $X$ , denote by  $\mathcal{O}_{\mathbf{Loc}(T)}^{fond}(X)$  the preordered subset of all members of  $\mathcal{O}_{\mathbf{Loc}(T)}(X)$  that are fundamental in  $X$ . By Corollary 8.1.4, the preordered set  $\mathcal{O}_{\mathbf{Loc}(T)}^{fond}(X)$  is complete and, for every family  $(O_i)$  of  $\mathcal{O}_{\mathbf{Loc}(T)}^{fond}(X)$  and every  $O \in \mathcal{O}_{\mathbf{Loc}(T)}^{fond}(X)$ ,  $O$  is a join of the  $O_i$  in  $\mathcal{O}_{\mathbf{Loc}(T)}^{fond}(X)$  if and only if the underlying set of  $O$  is the union of the underlying sets of the  $O_i$ .

**Definition 9.2.4** (**Loc**( $T$ )-core-compact spaces). A **Loc**( $T$ )-topological space  $X$  is **Loc**( $T$ )-core-compact when the preordered set  $\mathcal{O}_{\mathbf{Loc}(T)}^{fond}(X)$  is continuous (Definition 3.3.3).

*Remark 9.2.4.* When  $X$  is a **Loc**( $T$ )-stream, the relation  $\ll$  on  $\mathcal{O}_{\mathbf{Loc}(T)}^{fond}(X)$  is the restriction of the relation  $\ll$  on  $\mathcal{O}_{\mathbf{Loc}(T)}(X)$  and  $X$  is **Loc**( $T$ )-core-compact if and only if its underlying space is core-compact.

**Example 9.2.1.** Let  $X$  be a  $T$ -topological space. Then the space  $\eta_T(X)$  is a **Loc**( $T$ )-stream with  $\mathcal{O}_{\mathbf{Loc}(T)}^{fond}(X) = \{X\}$ , hence it is also a **Loc**( $T$ )-core-compact space.

We characterize **Loc**( $T$ )-core-compactness.

**Proposition 9.2.5.** Let  $X$  be a **Loc**( $T$ )-topological space. Then, the space  $X$  is **Loc**( $T$ )-core-compact if and only if, for every  $U \in \mathcal{O}_{\mathbf{Loc}(T)}^{fond}(X)$  such that  $x \in U$ , there is  $V \in \mathcal{O}_{\mathbf{Loc}(T)}^{fond}(X)$  such that  $x \in V \ll U$  in  $\mathcal{O}_{\mathbf{Loc}(T)}^{fond}(X)$ .

*Proof.* Assume that  $X$  is **Loc**( $T$ )-core-compact. Let  $x \in X$  and let  $U \in \mathcal{O}_{\mathbf{Loc}(T)}^{fond}(X)$  such that  $x \in U$ . Since  $\mathcal{O}_{\mathbf{Loc}(T)}^{fond}(X)$  is continuous,

$$U = \bigvee \{V \in \mathcal{O}_{\mathbf{Loc}(T)}^{fond}(X) ; V \ll U \text{ in } \mathcal{O}_{\mathbf{Loc}(T)}^{fond}(X)\}.$$

Then there is  $V \in \mathcal{O}_{\mathbf{Loc}(T)}^{fond}(X)$  such that  $x \in U \ll V$  in  $\mathcal{O}_{\mathbf{Loc}(T)}^{fond}(X)$ .

Conversely, let  $U \in \mathcal{O}_{\mathbf{Loc}(T)}^{fond}(X)$ . Write

$$S_{\ll}(U) := \{V \in \mathcal{O}_{\mathbf{Loc}(T)}^{fond}(X) ; V \ll U \text{ in } \mathcal{O}_{\mathbf{Loc}(T)}^{fond}(X)\}.$$

This preordered subset is filtered by Proposition 3.3.2. For every  $x \in X$  such that  $x \in U$ , there is  $V_x \in \mathcal{O}_{\mathbf{Loc}(T)}^{fond}(X)$  such that  $x \in V_x \ll U$  in  $\mathcal{O}_{\mathbf{Loc}(T)}^{fond}(X)$ . By Corollaries 8.1.3 and 8.1.4,  $U$  is a join of the  $V_x$  hence, a fortiori,  $U$  is a join of  $S_{\ll}(U)$ .  $\square$

We will also use the following property of the **Loc**( $T$ )-core-compact spaces, obtained by the interpolation lemma 3.3.4.

**Lemma 9.2.6.** Let  $X$  be a **Loc**( $T$ )-core-compact space. Let  $(U_i)_{i \in I}$  be a family of  $\mathcal{O}_{\mathbf{Loc}(T)}^{fond}(X)$  and let  $U \in \mathcal{O}_{\mathbf{Loc}(T)}^{fond}(X)$ . If  $U \ll \bigvee_{i \in I} U_i$  in  $\mathcal{O}_{\mathbf{Loc}(T)}^{fond}(X)$ , then there is a finite subset  $J \subset I$  such that  $U \ll \bigvee_{i \in J} U_i$  in  $\mathcal{O}_{\mathbf{Loc}(T)}^{fond}(X)$ .

*Proof.* Since  $\mathcal{O}_{\mathbf{Loc}(T)}^{fond}(X)$  is continuous, by the interpolation lemma 3.3.4, there exists  $U' \in \mathcal{O}_{\mathbf{Loc}(T)}(X)$  such that  $U \ll U' \ll \bigvee_{i \in I} U_i$  in  $\mathcal{O}_{\mathbf{Loc}(T)}^{fond}(X)$ . Since  $U' \ll \bigvee_{i \in I} U_i$ , there is a finite subset  $J \subset I$  such that  $U' \leq \bigvee_{i \in J} U_i$ . Since  $U \ll U'$ , it follows that  $U \ll \bigvee_{i \in J} U_i$  in  $\mathcal{O}_{\mathbf{Loc}(T)}^{fond}(X)$ .  $\square$

From now on, assume that the monoidal category  $(\mathbf{Top}_T, \otimes, I_T)$  is biclosed and, for every  $T$ -topological space  $A$ , denote by  $A \multimap \_$  the right adjoint of  $A \otimes \_$ . Moreover, assume that  $I_T$  is endowed with a  $T$ -topology that makes it a  $\mathbf{U}_T$ -discrete object on  $\{\star\}$  (Definition 7.0.4).

One easily checks that  $I_{\mathbf{Loc}(T)}$  is the singleton set  $\{\star\}$  endowed with the discrete **Loc**( $T$ )-topology.

Let  $X, Y$  and  $Z$  be sets and let  $m : Y \times X \rightarrow Z$  be a function. Denote by  $eval_{Y,Z} : Y \times Z^Y \rightarrow Z$  the canonical function  $(x, f) \mapsto f(x)$  and by  $\Lambda(m) : X \rightarrow Z^Y$  the canonical function  $x \mapsto (y \mapsto m(y, x))$ .

Let  $X, Y$  and  $Z$  be  $T$ -topological spaces and let  $m : Y \otimes X \rightarrow Z$  be a  $T$ -continuous map. Denote by  $eval_{Y,Z}^T : Y \otimes (Y \multimap Z) \rightarrow Z$  and by  $\Lambda^T(m) : X \rightarrow (Y \multimap Z)$  the canonical maps. Since the underlying set of  $Y \otimes (Y \multimap Z)$  is  $Y \times (Y \multimap Z)$ , the function  $u_{Y,Z}^{\circ} := \Lambda(eval_{Y,Z}^T)$  is the unique function from  $Y \multimap Z$  to  $Z^Y$  such that  $eval_{Y,Z} \circ (Id_Y \times u_{Y,Z}^{\circ}) = eval_{Y,Z}^T$ . Then one

has  $\Lambda(m) = u_{Y,Z}^{\circ} \circ \Lambda^T(m)$ . Let  $f : Y \rightarrow Z$  be a  $T$ -continuous map, then  $f' : (y, \star) \mapsto f(y)$  is  $T$ -continuous from  $Y \otimes I_T$  to  $Z$ , hence  $\lceil f \rceil := \Lambda^T(f')(\star)$  is an element of  $Y \multimap Z$ . It follows that  $u_{Y,Z}^{\circ}(\lceil f \rceil) = f$ . Thus, we have a function  $f \in \mathbf{Top}_T(X, Y) \mapsto \lceil f \rceil \in X \multimap Y$  such that  $u_{X,Y}^{\circ}(\lceil f \rceil) = f$ . Conversely, let  $e \in Y \multimap Z$ , then the function  $\star \mapsto e$  is  $T$ -continuous from  $I_T$  to  $Y \multimap Z$  because  $I_T$  is a  $\mathbf{U}_T$ -discrete object on  $\{\star\}$ . Hence  $g' := \text{eval}_{Y,Z}^T \circ (Id_Y \times (\star \mapsto e))$  is  $T$ -continuous from  $Y \otimes I$  to  $Z$ , then  $g : y \mapsto g'(y, \star)$  is  $T$ -continuous from  $Y$  to  $Z$ . It follows that  $g = u_{Y,Z}^{\circ}(e)$  and  $\lceil g \rceil = e$ . Thus  $u_{Y,Z}^{\circ}$  is a bijection (natural because all canonical morphisms are natural) from  $Y \multimap Z$  to  $\mathbf{Top}_T(X, Y)$  with  $\lceil \_ \rceil$  as inverse.

**Definition 9.2.7** (Core-open  $\mathbf{Loc}(T)$ -topology). Let  $X$  and  $Y$  be two  $\mathbf{Loc}(T)$ -topological spaces. Let  $U \in \mathcal{O}_{\mathbf{Loc}(T)}^{\text{fond}}(X)$  and let  $V \in \mathcal{O}_{\mathbf{Loc}(T)}(Y)$ , denote by  $W(U, V)$  the set of all  $\mathbf{Loc}(T)$ -continuous maps  $f$  from  $X$  to  $Y$  such that there is  $U' \in \mathcal{O}_{\mathbf{Loc}(T)}^{\text{fond}}(X)$  whose underlying set is  $\{x \in X; f(x) \in V\}$  satisfying  $U \ll U'$  in  $\mathcal{O}_{\mathbf{Loc}(T)}^{\text{fond}}(X)$ . In particular, one has  $U \subset_{\mathbf{Loc}(T)} U'$  and  $U' \mathbf{Loc}(T)(f) V$ , hence the restriction of  $f$  to  $U$  is a  $T$ -continuous map from  $U$  to  $V$ . Then one has a restriction map  $r_{U,V} : W(U, V) \rightarrow \mathbf{Top}_T(U, V)$ . Endow  $W(U, V)$  with a  $\mathbf{Loc}(T)$ -topology that turns it into a  $\mathbf{U}_T$ -initial lifting of the function  $\iota_{U,V} := f \mapsto \lceil r_{U,V}(f) \rceil$ . The latter is the composite of the restriction of  $W(U, V)$  to  $\mathbf{Top}_T(U, V)$  and of the canonical bijection from  $\mathbf{Top}_T(U, V)$  to  $U \multimap V$ . The core-open  $\mathbf{Loc}(T)$ -topology on  $\mathbf{Top}_{\mathbf{Loc}(T)}(X, Y)$  is the one generated by the finite meets of the  $W(U, V)$ . Denote by  $X \multimap Y$  the set  $\mathbf{Top}_{\mathbf{Loc}(T)}(X, Y)$  endowed with the core-open  $\mathbf{Loc}(T)$ -topology.

We will show that  $Y \multimap \_$  is right adjoint to  $Y \otimes \_$  when  $Y$  is a  $\mathbf{Loc}(T)$ -core-compact  $\mathbf{Loc}(T)$ -stream.

One easily notes that for any  $U \in \mathcal{O}_{\mathbf{Loc}(T)}^{\text{fond}}(Y)$  and any  $V, V' \in \mathcal{O}_{\mathbf{Loc}(T)}(Z)$  such that  $V \subset_{\mathbf{Loc}(T)} V'$  one has  $W(U, V) \subset_{\mathbf{Loc}(T)} W(U, V')$  because canonical map  $\lceil \_ \rceil$  is natural. Thus, for any  $T$ -topological spaces  $Y$  and  $Z$ , the  $\mathbf{Loc}(T)$ -topology of  $\eta_T(Y) \multimap \eta_T(Z)$  is generated by the basis of  $\mathbf{Loc}(T)$ -topology whose sole member is  $W(Y, Z)$ , i.e. the set  $\mathbf{Top}_T(Y, Z)$  endowed with the  $T$ -topology obtained by transporting the  $T$ -topology of  $Y \multimap Z$  along the canonical bijection. Therefore, this canonical bijection is also a natural isomorphism in the category  $\mathbf{Top}_{\mathbf{Loc}(T)}$  between  $\eta_T(Y) \multimap \eta_T(Z)$  and  $\eta_T(Y \multimap Z)$ .

**Proposition 9.2.8.** Let  $Y$  and  $Z$  be two  $\mathbf{Loc}(T)$ -topological spaces with  $Y$  a core-compact  $\mathbf{Loc}(T)$ -stream. Then the function  $\text{eval}_{Y,Z}$  is  $\mathbf{Loc}(T)$ -continuous from  $Y \otimes (Y \multimap Z)$  to  $Y$ .

*Proof.* Let  $V \in \mathcal{O}_{\mathbf{Loc}(T)}(Z)$  and let  $(y_0, f) \in Y \otimes (Y \multimap Z)$  such that  $f(y_0) \in V$ . Given that  $f$  is  $\mathbf{Loc}(T)$ -continuous,  $f^{-1}(V)$  is  $\mathbf{Loc}(T)$ -open in  $Y$ . Therefore, since  $Y$  is a  $\mathbf{Loc}(T)$ -stream, there is  $U' \in \mathcal{O}_{\mathbf{Loc}(T)}^{\text{fond}}(Y)$  whose underlying set is  $\{y \in Y; f(y) \in V\}$ . Given that  $f(y_0) \in V$ , one has  $y_0 \in U'$ . Then, by Proposition 9.2.5, there is  $U \in \mathcal{O}_{\mathbf{Loc}(T)}^{\text{fond}}(Y)$  such that  $y_0 \in U$  and  $U \ll U'$  in  $\mathcal{O}_{\mathbf{Loc}(T)}^{\text{fond}}(Y)$  because  $X$  is  $\mathbf{Loc}(T)$ -core-compact. Consequently  $f \in W(U, V)$ , then  $(y_0, f) \in U \otimes W(U, V)$ . The restriction of  $\text{eval}_{X,Y}$  to  $U \otimes W(U, V)$  is equal to  $\text{eval}_{U,V}^T \circ (Id_U \times \iota_{U,V})$  hence is  $T$ -continuous from  $U \otimes W(U, V)$  to  $V$ . Thus  $\text{eval}_{X,Y}$  is  $\mathbf{Loc}(T)$ -continuous at  $(y_0, f)$  from  $X \otimes (X \multimap Y)$  to  $Y$ .  $\square$

**Proposition 9.2.9.** Let  $X, Y$ , and  $Z$  be three  $\mathbf{Loc}(T)$ -topological spaces and let  $m : Y \times X \rightarrow Z$  be a  $\mathbf{Loc}(T)$ -continuous map from  $Y \otimes X$  to  $Z$ . Let  $x \in X$ . Then  $\Lambda(m)(x)$  is  $\mathbf{Loc}(T)$ -continuous from  $Y$  to  $Z$ .

*Proof.* Since  $I_{\mathbf{Loc}(T)}$  is the singleton subset  $\{\star\}$  endowed with the discrete  $\mathbf{Loc}(T)$ -topology, the function  $c_x : \star \mapsto x$  is  $\mathbf{Loc}(T)$ -continuous from  $I_{\mathbf{Loc}(T)}$  to  $X$ . Then the map  $Id_Y \times c_x$  is  $\mathbf{Loc}(T)$ -continuous from  $Y \otimes I_{\mathbf{Loc}(T)}$  to  $Y \otimes X$ , hence the function  $(Id_Y, c_x)$  is  $\mathbf{Loc}(T)$ -continuous from  $Y$  to  $Y \otimes X$ . Given that  $\Lambda(m)(x) = m \circ (Id_Y, c_x)$ , we deduce the  $\mathbf{Loc}(T)$ -continuity of  $\Lambda(m)(x)$ .  $\square$

**Lemma 9.2.10.** Let  $X, Y$ , and  $Z$  be three  $\mathbf{Loc}(T)$ -topological spaces and let  $f : X \rightarrow (Y \multimap Z)$  be a function. Let  $x \in X$ . The function  $f$  is  $\mathbf{Loc}(T)$ -continuous at  $x$  if and only if, for every  $U \in \mathcal{O}_{\mathbf{Loc}(T)}^{\text{fond}}(Y)$  and every  $V \in \mathcal{O}_{\mathbf{Loc}(T)}(Z)$  such that  $f(x) \in W(U, V)$ , there is  $O \in \mathcal{O}_{\mathbf{Loc}(T)}(X)$  such that  $x \in O$  and  $O \mathbf{Loc}(T)(f) W(U, V)$ .



*Proof.* Assume that, for every  $U \in \mathcal{O}_{\mathbf{Loc}(T)}^{f\text{ond}}(Y)$  and every  $V \in \mathcal{O}_{\mathbf{Loc}(T)}(Z)$  such that  $f(x) \in W(U, V)$ , there is  $O \in \mathcal{O}_{\mathbf{Loc}(T)}(X)$  such that  $x \in O$  and  $O \mathbf{Loc}(T)(f) W(U, V)$ . Let  $(U_j, V_j)_{j \in J}$  be a finite family such that, for every  $j \in J$ ,  $U_j \in \mathcal{O}_{\mathbf{Loc}(T)}^{f\text{ond}}(Y)$ ,  $V_j \in \mathcal{O}_{\mathbf{Loc}(T)}(Z)$  and  $f(x) \in W(U_j, V_j)$ . By assumption, for every  $j \in J$ , there is  $O_j \in \mathcal{O}_{\mathbf{Loc}(T)}(X)$  such that  $x \in O_j$  and  $O_j \mathbf{Loc}(T)(f) W(U_j, V_j)$ . Openness being stable under finite meets by Proposition 5.2.11,  $O := \bigwedge_{j \in J} O_j$  is  $\mathbf{Loc}(T)$ -open in  $X$  and  $x \in O$ . Since the inverse image preserves meets, one has

$$O := \bigwedge_{j \in J} O_j \subset_{\mathbf{Loc}(T)} \bigwedge_{j \in J} f^{-1}(W(U_j, V_j)) = f^{-1}\left(\bigwedge_{j \in J} W(U_j, V_j)\right).$$

Since the finite meets of the  $W(U, V)$  form a basis that generates the  $\mathbf{Loc}(T)$ -topology of  $Y \multimap Z$ , we conclude that  $f$  is  $\mathbf{Loc}(T)$ -continuous at  $x$ . The converse is obvious.  $\square$

**Proposition 9.2.11.** Let  $X, Y$ , and  $Z$  be three  $\mathbf{Loc}(T)$ -topological spaces such that  $Y$  is a core-compact  $\mathbf{Loc}(T)$ -stream, and let  $m : Y \times X \rightarrow Z$  be a  $\mathbf{Loc}(T)$ -continuous map from  $Y \otimes X$  to  $Z$ . The function  $\Lambda(m) : x \mapsto (y \mapsto m(y, x))$  is  $\mathbf{Loc}(T)$ -continuous from  $X$  to  $Y \multimap Z$ .

*Proof.* To show the  $\mathbf{Loc}(T)$ -continuity of  $\Lambda(m)$ , use the previous lemma. Let  $x \in X$ ,  $U \in \mathcal{O}_{\mathbf{Loc}(T)}^{f\text{ond}}(Y)$  and  $V \in \mathcal{O}_{\mathbf{Loc}(T)}(Z)$  such that  $\Lambda(m)(x) \in W(U, V)$ .

Denote by  $U'$  the fundamental  $\mathbf{Loc}(T)$ -open member of  $\mathbf{Loc}(T)(Y)$  whose underlying set is  $\{y \in Y ; \Lambda(m)(x)(y) \in V\}$ . Since  $m$  is  $\mathbf{Loc}(T)$ -continuous, for every  $y \in U'$ , there is  $U_y \in \mathcal{O}_{\mathbf{Loc}(T)}(Y)$  and  $O_y \in \mathcal{O}_{\mathbf{Loc}(T)}(X)$  such that  $y \in U_y$ ,  $x \in O_y$  and

$$U_y \otimes O_y \subset_{\mathbf{Loc}(T)} m^{-1}(V).$$

Since  $Y$  is a  $\mathbf{Loc}(T)$ -stream, we can also assume that the  $U_y$  are fundamental in  $Y$ .

Let  $y \in U'$ , one has  $x \in O_y$  and  $U_y \otimes O_y \subset_{\mathbf{Loc}(T)} m^{-1}(V)$ , hence  $\Lambda(m)(x)(y') \in V$  for every  $y' \in U_y$ . Consequently, the underlying set of  $\bigvee_{y \in U'} U_y$  is  $\{y \in Y ; \Lambda(m)(x)(y) \in V\}$ . Since  $U'$  is fundamental and since  $\bigvee_{y \in U'} U_y$  is also fundamental by Corollary 8.1.4, it follows that  $U' = \bigvee_{y \in U'} U_y$ , and consequently that

$$U \ll \bigvee_{y \in U'} U_y \quad \text{in } \mathcal{O}_{\mathbf{Loc}(T)}^{f\text{ond}}(Y).$$

Since  $Y$  is  $\mathbf{Loc}(T)$ -core-compact we deduce (by Lemma 9.2.6) that there is a finite subset  $J \subset U'$  such that

$$U \ll \bigvee_{y \in J} U_y.$$

Then  $O := \bigwedge_{y \in J} O_y$  is  $\mathbf{Loc}(T)$ -open in  $X$ , and it contains  $x$  by Proposition 5.2.11 because  $J$  is finite and because, for every  $y \in U'$ ,  $x \in O_y$ .

Let  $x' \in O$ , we want to show that  $\Lambda(m)(x') \in W(U, V)$ . Since  $\Lambda(m)(x')$  is  $\mathbf{Loc}(T)$ -continuous,  $(\Lambda(m)(x'))^{-1}(V)$  is  $\mathbf{Loc}(T)$ -open in  $Y$ . Since  $Y$  is a  $\mathbf{Loc}(T)$ -stream, there is  $U'' \in \mathcal{O}_{\mathbf{Loc}(T)}^{f\text{ond}}(Y)$  whose underlying set  $\{y \in Y ; \Lambda(m)(x')(y) \in V\}$ . To prove that  $\Lambda(m)(x') \in W(U, V)$ , it is enough to show that  $U \ll U''$ .

Let  $y \in J$  and let  $y' \in U_y$ . Since  $x' \in O_y$  and since  $U_y \otimes O_y \subset_{\mathbf{Loc}(T)} m^{-1}(V)$ , one has  $m(y', x') \in V$ , i.e.  $\Lambda(m)(x')(y') \in V$ , and then  $y' \in U''$ . By the above reasoning and the fact that  $U_y$  is fundamental, we apply Proposition 8.1.2 to deduce that

$$U_y \subset_{\mathbf{Loc}(T)} U''.$$

Consequently

$$U \ll \bigvee_{y \in J} U_y \subset_{\mathbf{Loc}(T)} U'',$$

hence

$$U \ll U''.$$

Therefore, we have indeed  $\Lambda(m)(x') \in W(U, V)$ .

To conclude, it is enough to show that restriction  $\Lambda(m)|_O$  of  $\Lambda(m)$  to  $O$  with codomain  $W(U, V)$  is  $T$ -continuous. Since the  $T$ -topology on  $W(U, V)$  is such that  $\iota_{U, V}$  is a  $\mathbf{U}_T$ -initial lifting with codomain  $U \multimap V$ , it is enough to show that  $\iota_{U, V} \circ \Lambda(m)|_O$  is  $T$ -continuous from  $O$  to  $U \multimap V$ . Observe that  $\iota_{U, V} \circ \Lambda(m)|_O = [\_ ] \circ \Lambda(m|_{U \times O})$ , so for every  $y \in J$ , one has

$$U_y \otimes O_y \subset_{\mathbf{Loc}(T)} m^{-1}(V),$$

and then

$$U_y \otimes O \subset_{\mathbf{Loc}(T)} m^{-1}(V),$$

because  $O \subset_{\mathbf{Loc}(T)} O_y$ . Thus

$$\bigvee_{y \in J} (U_y \otimes O) \subset_{\mathbf{Loc}(T)} m^{-1}(V).$$

Since  $\_ \otimes O$  has a right adjoint, we deduce that

$$\left( \bigvee_{y \in J} U_y \right) \otimes O \subset_{\mathbf{Loc}(T)} m^{-1}(V).$$

Since  $U \leq \bigvee_{y \in J} U_y$ , it follows that

$$U \otimes O \subset_{\mathbf{Loc}(T)} m^{-1}(V).$$

Thus  $m|_{U \times O}$  is a  $T$ -continuous map from  $U \otimes O$  to  $V$ . Then one has  $[\_ ] \circ \Lambda(m|_{U \times O}) = \Lambda^T(m|_{U \times O})$ , hence  $\iota_{U, V} \circ \Lambda(m)|_O$  is  $T$ -continuous from  $O$  to  $U \multimap V$ .  $\square$

### 9.2.3 Application to the $\mathcal{C}$ -generated spaces of $\mathbf{Loc}(T)$

From now on, we assume that  $\otimes$  is the cartesian product of  $\mathbf{Top}_T$ , hence that  $\otimes$  is the cartesian product of  $\mathbf{Top}_{\mathbf{Loc}(T)}$ .

**Definition 9.2.12.** A class  $\mathcal{C}$  of  $\mathbf{Loc}(T)$ -topological spaces is *productive* when

- Every  $C \in \mathcal{C}$  is  $\mathbf{Loc}(T)$ -core-compact.
- Every  $C \in \mathcal{C}$  is a  $\mathbf{Loc}(T)$ -stream.
- For every  $C, C' \in \mathcal{C}$ ,  $C \times C'$  is  $\mathcal{C}$ -generated.

We set a productive class  $\mathcal{C}$ .

Let  $X$  and  $Y$  be two  $\mathbf{Loc}(T)$ -topological spaces. Denote by  $X \multimap^{\mathcal{C}} Y$  the set  $\mathcal{C}\text{-Map}(X, Y)$  endowed with the initial  $\mathbf{Loc}(T)$ -topology associated to  $(\_ \circ p) : \mathcal{C}\text{-Map}(X, Y) \rightarrow C \multimap Y$ , for all  $C \in \mathcal{C}$  and all  $p : C \rightarrow X$   $\mathbf{Loc}(T)$ -continuous.

**Proposition 9.2.13.** Let  $X$  and  $Y$  be two  $\mathbf{Loc}(T)$ -topological spaces. The function  $eval_{X, Y} : X \times \mathcal{C}\text{-Map}(X, Y) \rightarrow Y$  is  $\mathcal{C}$ -continuous from  $X \times (X \multimap^{\mathcal{C}} Y)$  to  $Y$ .

*Proof.* Let  $C \in \mathcal{C}$  and let  $p = (p_1, p_2)$  be a  $\mathbf{Loc}(T)$ -continuous map from  $C$  to  $X \times (X \multimap^{\mathcal{C}} Y)$ . Since, by assumption, the projections are  $\mathbf{Loc}(T)$ -continuous,  $p_1$  and  $p_2$  are  $\mathbf{Loc}(T)$ -continuous. Hence  $\_ \circ p_1$  is  $\mathbf{Loc}(T)$ -continuous from  $X \multimap^{\mathcal{C}} Y$  to  $C \multimap Y$ , then  $(\_ \circ p_1) \circ p_2$  is  $\mathbf{Loc}(T)$ -continuous from  $C$  to  $C \multimap Y$ . Then the function  $(\text{Id}_C, (\_ \circ p_1) \circ p_2) : C \rightarrow C \times (C \multimap Y)$  is  $\mathbf{Loc}(T)$ -continuous because the diagonal is  $\mathbf{Loc}(T)$ -continuous. Since  $C$  is  $\mathbf{Loc}(T)$ -core-compact and is a  $\mathbf{Loc}(T)$ -stream,  $eval_{C, Y}$  is  $\mathbf{Loc}(T)$ -continuous from  $C \times (C \multimap Y)$  to  $Y$ . Thus  $eval_{X, Y} \circ p = eval_{C, Y} \circ (\text{Id}_C, (\_ \circ p_1) \circ p_2)$  is  $\mathbf{Loc}(T)$ -continuous from  $C$  to  $Y$ .  $\square$

**Proposition 9.2.14.** Let  $X, Y$ , and  $Z$  be three  $\mathbf{Loc}(T)$ -topological spaces. Let  $m : Y \times X \rightarrow Z$  be a  $\mathcal{C}$ -continuous map. Then, for every  $x \in X$ ,  $\Lambda(m)(x)$  is  $\mathcal{C}$ -continuous from  $Y$  to  $Z$ .

*Proof.* Let  $C \in \mathcal{C}$  and let  $p : C \rightarrow Y$  be a  $\mathbf{Loc}(T)$ -continuous map. Since  $I_{\mathbf{Loc}(T)}$  is the singleton set  $\{\star\}$  endowed with the discrete  $\mathbf{Loc}(T)$ -topology, the function  $c_x : \star \mapsto x$  is  $\mathbf{Loc}(T)$ -continuous from  $I_{\mathbf{Loc}(T)}$  to  $X$ . Therefore  $(p \times c_x)$  is  $\mathbf{Loc}(T)$ -continuous from  $C \times I_{\mathbf{Loc}(T)}$  to  $Y \times X$ . Since  $C \times I_{\mathbf{Loc}(T)}$  and  $C$  are isomorphic and since  $m$  is  $\mathcal{C}$ -continuous, the map  $m \circ (p, c_x)$  is  $\mathbf{Loc}(T)$ -continuous from  $C$  to  $Z$ . We can conclude because  $m \circ (p, c_x) = \Lambda(m)(x) \circ p$ .  $\square$

**Proposition 9.2.15.** Let  $X, Y$  and  $Z$  be  $\mathbf{Loc}(T)$ -topological spaces. Let  $m : Y \times X \rightarrow Z$  be a  $\mathcal{C}$ -continuous map. Then  $\Lambda(m)$  is  $\mathcal{C}$ -continuous from  $X$  to  $Y \dashv^{\mathcal{C}} Z$ .

*Proof.* It is enough to show that, for every  $C, C' \in \mathcal{C}$ , for every  $\mathbf{Loc}(T)$ -continuous maps  $p : C \rightarrow X$  and  $p' : C' \rightarrow Y$ ,  $(\_ \circ p') \circ \Lambda(m) \circ p$  is  $\mathbf{Loc}(T)$ -continuous from  $C$  to  $C' \dashv^{\mathcal{C}} Z$ . However this map is equal to  $\Lambda(m \circ (p' \times p))$ . Since  $C'$  is  $\mathbf{Loc}(T)$ -core-compact and is a  $\mathbf{Loc}(T)$ -stream, it is enough to show that  $m \circ (p' \times p)$  is  $\mathbf{Loc}(T)$ -continuous from  $C' \otimes C$  to  $Z$ . Given that  $C' \times C$  is  $\mathcal{C}$ -generated, it is enough to show that  $m \circ (p' \times p)$  is  $\mathcal{C}$ -continuous.

Let  $C'' \in \mathcal{C}$  and let  $p'' : C'' \rightarrow C' \times C$  be a  $\mathbf{Loc}(T)$ -continuous map. It follows that, since  $p$  and  $p'$  are  $\mathbf{Loc}(T)$ -continuous, the product  $p' \times p$  is  $\mathbf{Loc}(T)$ -continuous, then the composite  $(p' \times p) \circ p''$  is  $\mathbf{Loc}(T)$ -continuous. Given that  $m$  is  $\mathcal{C}$ -continuous, it follows that  $m \circ (p' \times p) \circ p''$  is  $\mathbf{Loc}(T)$ -continuous.  $\square$

**Corollary 9.2.16.** The category  $\mathcal{C}\text{-Map}$  is cartesian closed: for every  $\mathbf{Loc}(T)$ -topological space  $X$ ,  $X \dashv^{\mathcal{C}} \_$  is the right adjoint of  $X \times \_$ .

Regarding corollary 9.1.11, we can transport the structure of symmetric closed monoidal category of  $\mathcal{C}\text{-Map}$  along the equivalence of categories  $\mathcal{C}(\_)$ , we get then

**Corollary 9.2.17.** The 4-tuple  $(\mathcal{C}\text{-Top}_{\mathbf{Loc}(T)}, \mathcal{C}(\_ \times \_), I_{\mathbf{Loc}(T)}, \mathcal{C}(\_ \dashv^{\mathcal{C}} \_))$  is a symmetric closed monoidal category.

# Appendix A

## Quantales

Many interesting examples of the manuscript are based on a generalization of preorders where the role played by the preordered set of booleans is replaced by a quantale. Some of the following results can be found, with a different terminology, in [HST14] (see also [Ros90])

**Definition A.0.1** (Quantales). A *quantale*<sup>1</sup>  $\mathbb{V}$  is a set  ${}^{\lceil \mathbb{V} \rceil}$  endowed with a structure of complete preordered set  $({}^{\lceil \mathbb{V} \rceil}, \leq_{\mathbb{V}}, \bigvee)$  and with a structure of monoid  $({}^{\lceil \mathbb{V} \rceil}, \otimes_{\mathbb{V}}, k_{\mathbb{V}})$  such that, for every family  $(a_i)_{i \in I}$  of  ${}^{\lceil \mathbb{V} \rceil}$  and for every  $a \in {}^{\lceil \mathbb{V} \rceil}$ ,

$$\left( \bigvee_{i \in I} a_i \right) \otimes a = \bigvee_{i \in I} (a_i \otimes a) \quad \text{and} \quad a \otimes \left( \bigvee_{i \in I} a_i \right) = \bigvee_{i \in I} (a \otimes a_i) \quad . \quad (\text{A.1})$$

A quantale is commutative when the product  $\otimes$  is commutative.

When there is no risk of confusion, we identify a quantale  $\mathbb{V}$  and its underlying set  ${}^{\lceil \mathbb{V} \rceil}$ . Intuitively, a quantale can be seen as some kind of object of truth values which will play a role similar to the one played by the booleans.

*Remark A.0.1.* One deduces from Equations (A.1) that the function  $\otimes$  is increasing in each variable. By the adjoint functor theorem (see [Mac98, V.8], [Bor94a, 3.3.9.e], or [HST14, 2.1.8.3]), for every  $a \in X$ , the maps  $(\_ \otimes a)$  and  $(a \otimes \_)$  have right adjoints. When  $\otimes$  is commutative, the two adjoints are equivalent.

### Examples A.0.1.

- The singleton set  $\mathbb{1} := \{\star\}$  with the unique preorder and the unique monoid structure on it is a quantale.
- $\mathbb{B} := (\{0, 1\}, \leq, \bigvee, \wedge, 1)$  is a quantale with  $\leq$  the usual order and  $\wedge$  the binary meet.
- $(\overline{\mathbb{R}}_+, \geq, \inf, +, 0)$  is a quantale.
- $(\overline{\mathbb{R}}_+, \geq, \inf, \times, 1)$  with, for all  $x \in X$ ,  $x \times +\infty = +\infty \times x := +\infty$ , is a quantale.
- $(\overline{\mathbb{R}}_+, \geq, \inf, \max, 0)$  is a quantale.

**Definition A.0.2** (Quantale morphisms). Let  $\mathbb{V}$  and  $\mathbb{W}$  be two quantales. A *quantale morphism*<sup>2</sup> from  $\mathbb{V}$  to  $\mathbb{W}$  is an increasing map  $f : \mathbb{V} \rightarrow \mathbb{W}$  such that

$$k_{\mathbb{W}} \leq_{\mathbb{W}} f(k_{\mathbb{V}}) \quad \text{and} \quad \forall a, b \in \mathbb{V}, f(a) \otimes_{\mathbb{W}} f(b) \leq_{\mathbb{W}} f(a \otimes_{\mathbb{V}} b) \quad .$$

### Examples A.0.2.

<sup>1</sup>All quantales considered here are *unital* in the sense of [Ros90, Definition 2.1.4].

<sup>2</sup>This notion corresponds to the *closed unital maps of quantales* of [Ros90, Definition 2.3.2]

- For every quantale  $\mathbb{V}$ , the constant map from  $\mathbb{V}$  to  $\{\star\}$  is the unique quantale morphism from  $\mathbb{V}$  to  $\mathbb{1}$ .
- The identity map  $\text{Id}_{\overline{\mathbb{R}}_+}$  is a quantale morphism from  $(\overline{\mathbb{R}}_+, \geq, \inf, \max, 0)$  to  $(\overline{\mathbb{R}}_+, \geq, \inf, +, 0)$ .
- The function  $i$  defined by  $\begin{cases} i(0) &= +\infty \\ i(1) &= 0 \end{cases}$  is a quantale morphism from  $\mathbb{B}$  to  $(\overline{\mathbb{R}}_+, \geq, \inf, +, 0)$ , from  $\mathbb{B}$  to  $(\overline{\mathbb{R}}_+, \geq, \inf, \times, 1)$ , and from  $\mathbb{B}$  to  $(\overline{\mathbb{R}}_+, \geq, \inf, \max, 0)$ .

One denotes by **Qnt** the category of quantales and quantale morphisms.

**Definition A.0.3** (Relations on a quantale). Let  $\mathbb{V}$  be a quantale and let  $X$  and  $Y$  be two sets. A *relation* on  $\mathbb{V}$  (alternatively, a  $\mathbb{V}$ -*relation*) from  $X$  to  $Y$  is a function  $R : X \times Y \rightarrow \mathbb{V}$ .

**Examples A.0.3.**

- For every sets  $X$  and  $Y$ , there is a unique  $\mathbb{1}$ -relation from  $X$  to  $Y$ .
- The  $\mathbb{B}$ -relations are the classical relations recalled at the beginning of Chapter 3.

**Definition A.0.4.** Let  $\mathbb{V}$  be a quantale, and let  $R$  and  $R'$  be two  $\mathbb{V}$ -relations from a set  $X$  to a set  $Y$ . The relation  $R$  is lesser than the relation  $R'$  (which is denoted by  $R \leq R'$ ) when, for all  $x \in X$  and  $y \in Y$ ,  $R(x, y) \leq R'(x, y)$ .

**Definition A.0.5** (Composition of relations on a quantale). Let  $\mathbb{V}$  be a quantale and let  $X, Y$ , and  $Z$  be sets. Let  $R$  be a  $\mathbb{V}$ -relation from  $X$  to  $Y$  and  $R'$  be a  $\mathbb{V}$ -relation from  $Y$  to  $Z$ . The composite  $\mathbb{V}$ -relation  $R' \circ R$  from  $X$  to  $Z$  is defined by

$$\forall x \in X, \forall z \in Z, (R' \circ R)(x, z) := \bigvee_{y \in Y} (R(x, y) \otimes R'(y, z)) \quad .$$

**Proposition A.0.6.** Let  $\mathbb{V}$  be a quantale. With the above composition and the above preorder between parallel  $\mathbb{V}$ -relations, one gets a preordered category of sets and  $\mathbb{V}$ -relations.

*Proof.* Let  $W, X, Y$ , and  $Z$  be sets. Let  $R$  be a  $\mathbb{V}$ -relation from  $W$  to  $X$ ,  $R'$  be a  $\mathbb{V}$ -relation from  $X$  to  $Y$  and  $R''$  be a  $\mathbb{V}$ -relation from  $Y$  to  $Z$ . For every  $w \in W$  and  $z \in Z$ , one has

$$\begin{aligned} ((R'' \circ R') \circ R)(w, z) &= \bigvee_{x \in X} (R(w, x) \otimes (R'' \circ R')(x, z)) \\ &= \bigvee_{x \in X} \left( R(w, x) \otimes \left( \bigvee_{y \in Y} (R'(x, y) \otimes R''(y, z)) \right) \right) \\ &= \bigvee_{x \in X, y \in Y} R(w, x) \otimes R'(x, y) \otimes R''(y, z) \end{aligned}$$

and

$$\begin{aligned} (R'' \circ (R' \circ R))(w, z) &= \bigvee_{y \in Y} ((R' \circ R)(w, y) \otimes R''(y, z)) \\ &= \bigvee_{y \in Y} \left( \left( \bigvee_{x \in X} (R(w, x) \otimes R'(x, y)) \right) \otimes R''(y, z) \right) \\ &= \bigvee_{x \in X, y \in Y} R(w, x) \otimes R'(x, y) \otimes R''(y, z) \quad . \end{aligned}$$

Thus, the equality  $(R'' \circ R') \circ R = R'' \circ (R' \circ R)$  holds.

Let  $X$  be a set, denote by  $\text{Id}_X^\mathbb{V}$  the relation defined by

$$\text{Id}_X^\mathbb{V}(x, x') := \begin{cases} k_\mathbb{V} & \text{if } x = x' \\ 0_\mathbb{V} & \text{otherwise} \end{cases}$$

where  $0_\mathbb{V}$  is the least element of  $\mathbb{V}$ .

Let  $X$  and  $Y$  be sets, and let  $R$  be a  $\mathbb{V}$ -relation from  $X$  to  $Y$ . Let  $x \in X$  and  $y \in Y$ . Since, for all  $a \in \mathbb{V}$ ,  $0_\mathbb{V} \otimes a = a \otimes 0_\mathbb{V} = 0_\mathbb{V}$ , one has

$$(R \circ \text{Id}_X^\mathbb{V})(x, y) = \bigvee_{x' \in X} \text{Id}_X^\mathbb{V}(x, x') \otimes R(x', y) = k \otimes R(x, y) = R(x, y),$$

and

$$(\text{Id}_Y^\mathbb{V} \circ R)(x, y) = \bigvee_{y' \in Y} R(x, y') \otimes \text{Id}_Y^\mathbb{V}(y', y) = R(x, y) \otimes k = R(x, y).$$

One easily checks that the preorder between parallel  $\mathbb{V}$ -relations is compatible with composition.  $\square$

Let  $\mathbb{V}$  be a quantale. We denote by  $\mathbb{V}\text{-Rel}$  the preordered category of sets and  $\mathbb{V}$ -relations.

**Definition A.0.7** (Preorders on a quantale). Let  $\mathbb{V}$  be a quantale and let  $X$  be a set. A  $\mathbb{V}$ -preorder on  $X$  is a  $\mathbb{V}$ -relation  $R$  from  $X$  into itself such that

$$\forall x \in X, k_\mathbb{V} \leq R(x, x),$$

and

$$\forall x, y, z \in X, R(x, y) \otimes R(y, z) \leq R(x, z).$$

A  $\mathbb{V}$ -preorder is a  $\mathbb{V}$ -order when

$$\forall x, y \in X, \text{ if } k_\mathbb{V} \leq R(x, y) \times R(y, x), \text{ then } x = y.$$

A  $\mathbb{V}$ -(pre)ordered set is a pair  $(X, R)$  with  $X$  a set and  $R$  a  $\mathbb{V}$ -(pre)order on  $X$ .

**Definition A.0.8** (Increasing maps on a quantale). Let  $\mathbb{V}$  be a quantale. Let  $(X, R_X)$  and  $(Y, R_Y)$  be  $\mathbb{V}$ -preordered sets. A function  $f : X \rightarrow Y$  is a  $\mathbb{V}$ -increasing map from  $(X, R_X)$  to  $(Y, R_Y)$  if,

$$\forall x, x' \in X, R_X(x, x') \leq R_Y(f(x), f(x')).$$

Let  $\mathbb{V}$  be a quantale. We immediately check that the composite of two  $\mathbb{V}$ -increasing maps is a  $\mathbb{V}$ -increasing map and that the identity functions are  $\mathbb{V}$ -increasing maps. We denote by  $\mathbb{V}\text{-Ord}$  the category of  $\mathbb{V}$ -preordered sets and  $\mathbb{V}$ -increasing maps, and by  $\mathbb{V}\text{-SOrd}$  the full subcategory of  $\mathbb{V}$ -preordered set. There is an obvious forgetful functor  $\mathbf{U}_\mathbb{V} : \mathbb{V}\text{-Ord} \rightarrow \mathbf{Set}$ .

**Examples A.0.4.**

- The categories  $\mathbf{1}\text{-Ord}$  and  $\mathbf{1}\text{-SOrd}$  are (isomorphic to) the category  $\mathbf{Set}$ .
- The category  $\mathbb{B}\text{-Ord}$  is the category  $\mathbf{Ord}$  and the category  $\mathbb{B}\text{-SOrd}$  is the category  $\mathbf{SOrd}$ .
- The category  $(\overline{\mathbb{R}}_+, \geq, \inf, +, 0)\text{-Ord}$  is the category of Lawvere metric spaces (see [Law73]).

**Proposition A.0.9** (Inverse image  $\mathbb{V}$ -preorder). Let  $\mathbb{V}$  be a quantale. Let  $X$  be a set,  $(Y, R)$  be a  $\mathbb{V}$ -preordered set and  $f : X \rightarrow Y$  be a function. Then, the  $\mathbb{V}$ -relation  $R_f$  defined by,

$$\text{for all } x, x' \in X, R_f(x, x') := R(f(x), f(x'))$$

is a  $\mathbb{V}$ -preorder on  $X$ , called the *inverse image  $\mathbb{V}$ -preorder* of  $R$  under  $f$ , and the function  $f$  is a  $\mathbb{V}$ -increasing map from  $(X, R_f)$  to  $(Y, R)$ .

*Proof.* Let  $x \in X$ , we have

$$k \leq R(f(x), f(x)) = R_f(x, x) .$$

Let  $x, x', x'' \in X$ , we have

$$R_f(x, x') \otimes R_f(x', x'') = R(f(x), f(x')) \otimes R(f(x'), f(x'')) \leq R(f(x), f(x'')) = R_f(x, x'') .$$

The  $\mathbb{V}$ -monotonicity of  $f$  from  $(X, R_f)$  to  $(Y, R)$  is obvious.  $\square$

**Proposition A.0.10.** Let  $\mathbb{V}$  be a quantale,  $X$  be a set and  $R$  be a  $\mathbb{V}$ -relation from  $X$  to  $X$ . There is a least  $\mathbb{V}$ -preorder  $\bar{R}$  on  $X$  greater than  $R$ .

*Proof.* Let  $x, x' \in X$ , write

$$R'(x, x') := \begin{cases} R(x, x') \vee k & \text{if } x = x' \\ R(x, x') & \text{otherwise} \end{cases}$$

and

$$\llbracket x, x' \rrbracket := \{(x_0, \dots, x_n) ; n \geq 1, x_0, \dots, x_n \in X, x_0 = x, x_n = x'\} .$$

Then, define

$$\bar{R}(x, x') := \bigvee_{(x_0, \dots, x_n) \in \llbracket x, x' \rrbracket} R'(x_0, x_1) \otimes \dots \otimes R'(x_{n-1}, x_n) .$$

We clearly have

$$R \leq R' \leq \bar{R} .$$

Let  $x \in X$ , we have, by definition,

$$k \leq R'(x, x) \leq \bar{R}(x, x) .$$

Let  $x, x', x'' \in X$ . Let  $(x_0, \dots, x_n) \in \llbracket x, x' \rrbracket$  and  $(y_0, \dots, y_m) \in \llbracket x', x'' \rrbracket$ . Define  $(z_0, \dots, z_{n+m})$  by

$$z_i := \begin{cases} x_i & \text{if } 0 \leq i \leq n \\ y_{i-n} & \text{if } n \leq i \leq n+m \end{cases} .$$

We have  $(z_0, \dots, z_{n+m}) \in \llbracket x, x'' \rrbracket$ , so

$$\begin{aligned} & R'(x_0, x_1) \otimes \dots \otimes R'(x_{n-1}, x_n) \otimes R'(y_0, y_1) \otimes \dots \otimes R'(y_{m-1}, y_m) \\ &= R'(z_0, z_1) \otimes \dots \otimes R'(z_{n+m-1}, z_{n+m}) \leq \bar{R}(x, x'') , \end{aligned}$$

thus,

$$\begin{aligned} & \bigvee_{(x_0, \dots, x_n) \in \llbracket x, x' \rrbracket} \bigvee_{(y_0, \dots, y_m) \in \llbracket x', x'' \rrbracket} R'(x_0, x_1) \otimes \dots \otimes R'(x_{n-1}, x_n) \\ & \quad \otimes R'(y_0, y_1) \otimes \dots \otimes R'(y_{m-1}, y_m) \leq \bar{R}(x, x'') , \end{aligned}$$

but

$$\begin{aligned} & \bigvee_{(x_0, \dots, x_n) \in \llbracket x, x' \rrbracket} \bigvee_{(y_0, \dots, y_m) \in \llbracket x', x'' \rrbracket} R'(x_0, x_1) \otimes \dots \otimes R'(x_{n-1}, x_n) \\ & \quad \otimes R'(y_0, y_1) \otimes \dots \otimes R'(y_{m-1}, y_m) \\ &= \bigvee_{(x_0, \dots, x_n) \in \llbracket x, x' \rrbracket} \left( R'(x_0, x_1) \otimes \dots \otimes R'(x_{n-1}, x_n) \right) \\ & \quad \otimes \bigvee_{(y_0, \dots, y_m) \in \llbracket x', x'' \rrbracket} \left( R'(y_0, y_1) \otimes \dots \otimes R'(y_{m-1}, y_m) \right) , \end{aligned}$$

hence

$$\bar{R}(x, x') \otimes \bar{R}(x', x'') \leq \bar{R}(x, x'') .$$

Therefore  $\bar{R}$  is a  $\mathbb{V}$ -preorder on  $X$ .

Let  $S$  be a  $\mathbb{V}$ -preorder on  $X$  such that  $R \leq S$  and let  $x, x' \in X$ . Since  $S$  is a  $\mathbb{V}$ -preorder, we have  $R' \leq S$ .

Let  $(x_0, \dots, x_n) \in \llbracket x, x' \rrbracket$ . Since  $R' \leq S$  and since  $S$  is a  $\mathbb{V}$ -preorder, we have

$$R'(x_0, x_1) \otimes \dots \otimes R'(x_{n-1}, x_n) \leq S(x_0, x_1) \otimes \dots \otimes S(x_{n-1}, x_n) \leq S(x, x'),$$

hence

$$\bar{R}(x, x') \leq S(x, x').$$

□

This  $\mathbb{V}$ -preorder is called the  $\mathbb{V}$ -preorder generated by the  $\mathbb{V}$ -relation  $R$ .

**Corollary A.0.11.** Let  $\mathbb{V}$  be a quantale. The forgetful functor  $\mathbf{U}_{\mathbb{V}} : \mathbb{V}\text{-Ord} \rightarrow \mathbf{Set}$  is a topological functor.

*Proof.* Let  $(X_i, R_i)_{i \in I}$  a family of  $\mathbb{V}$ -preordored sets, let  $X$  be a set, and let  $(f_i : X_i \rightarrow X)_{i \in I}$  be a family of functions. Write  $R$  the  $\mathbb{V}$ -preorder on  $X$  generated by the  $\mathbb{V}$ -relation

$$(x, x') \mapsto \bigvee_{\substack{i \in I, x_i, x'_i \in X_i / \\ f_i(x_i) = x, f_i(x'_i) = x'}} R_i(x_i, x'_i).$$

For each  $i \in I$ , the function  $f_i$  is clearly a  $\mathbb{V}$ -increasing map from  $(X_i, R_i)$  to  $(X, R)$ .

Let  $(Y, S)$  be a  $\mathbb{V}$ -preordored set and  $f : X \rightarrow Y$  be a function such that, for all  $i \in I$ ,  $f \circ f_i$  is a  $\mathbb{V}$ -increasing map. Let  $x, x' \in X$ . Let  $i \in I$ ,  $x_i, x'_i \in X_i$  such that  $f_i(x_i) = x$  and  $f_i(x'_i) = x'$ . Since  $f \circ f_i$  is a  $\mathbb{V}$ -increasing map, we have

$$R_i(x_i, x'_i) \leq S(f(x), f(x')) = S_f(x, x').$$

Therefore

$$\bigvee_{\substack{i \in I, x_i, x'_i \in X_i / \\ f_i(x_i) = x, f_i(x'_i) = x'}} R_i(x_i, x'_i) \leq S_f(x, x').$$

Since  $S_f$  is a  $\mathbb{V}$ -preorder on  $X$ , we deduce that  $R \leq S_f$  and then that  $f$  is a  $\mathbb{V}$ -increasing map from  $(X, R)$  to  $(Y, S)$ . □

**Proposition A.0.12.** Let  $\mathbb{V}$  and  $\mathbb{W}$  be two quantales, and  $v$  be a quantale morphism from  $\mathbb{V}$  to  $\mathbb{W}$ . Let  $(X, R)$  be a  $\mathbb{V}$ -preordored set. Define, for all  $x, x' \in X$ ,

$$v(R)(x, x') := v(R(x, x')).$$

Then  $(X, v(R))$  is a  $\mathbb{W}$ -preordored set.

*Proof.* Let  $x \in X$ , we have  $k_{\mathbb{W}} \leq v(k_{\mathbb{V}})$  and  $k_{\mathbb{V}} \leq R(x, x)$ , thus, since  $v$  is increasing,

$$k_{\mathbb{W}} \leq v(R(x, x)) = v(R)(x, x).$$

Let  $x, x', x'' \in X$ . We have  $R(x, x') \otimes R(x', x'') \leq R(x, x'')$ , thus, since  $v$  is increasing, we have  $v(R(x, x') \otimes R(x', x'')) \leq v(R(x, x''))$ . Therefore

$$\begin{aligned} v(R)(x, x') \otimes v(R)(x', x'') &= v(R(x, x')) \otimes v(R(x', x'')) \\ &\leq v(R(x, x') \otimes R(x', x'')) \leq v(R(x, x'')) = v(R)(x, x''). \end{aligned}$$

□

**Proposition A.0.13.** Let  $\mathbb{V}$  and  $\mathbb{W}$  be two quantales, and let  $v$  be a quantale morphism from  $\mathbb{V}$  to  $\mathbb{W}$ . Let  $(X, R)$  and  $(Y, S)$  be two  $\mathbb{V}$ -preordored sets and let  $f : X \rightarrow Y$  be a function. If  $f$  is a  $\mathbb{V}$ -increasing map from  $(X, R)$  to  $(Y, S)$  then  $f$  is a  $\mathbb{W}$ -increasing map from  $(X, v(R))$  to  $(Y, v(S))$ .



*Proof.* Let  $x, x' \in X$ . Since  $f$  is a  $\mathbb{V}$ -increasing map from  $(X, R)$  to  $(Y, S)$ , we have  $R(x, x') \leq S(f(x), f(x'))$ . Since  $v$  is increasing, we deduce  $v(R(x, x')) \leq v(S(f(x), f(x')))$ , in other words  $v(R)(x, x') \leq v(S)(f(x), f(x'))$ .  $\square$

Thus a quantale morphism  $v : \mathbb{V} \rightarrow \mathbb{W}$  induces a functor from  $\mathbb{V}\text{-Ord}$  to  $\mathbb{W}\text{-Ord}$  that commutes with the forgetful functors.

**Proposition A.0.14.** Let  $\mathbb{V}$  be a quantale and let  $(X, R)$  and  $(Y, S)$  be two  $\mathbb{V}$ -preordered sets. Denote by  $\mathbb{V}\text{-Ord}((X, R), (Y, S))$  the set of  $\mathbb{V}$ -increasing maps from  $(X, R)$  to  $(Y, S)$ . Define the  $\mathbb{V}$ -relation  $R \multimap S$  from  $\mathbb{V}\text{-Ord}((X, R), (Y, S))$  into itself by

$$(R \multimap S)(f, g) := \bigwedge_{x \in X} S(f(x), g(x)), \quad \forall f, g \in \mathbb{V}\text{-Ord}((X, R), (Y, S)).$$

Then  $R \multimap S$  is a  $\mathbb{V}$ -preorder on  $\mathbb{V}\text{-Ord}((X, R), (Y, S))$ .

*Proof.* Let  $f \in \mathbb{V}\text{-Ord}((X, R), (Y, S))$ . For all  $x \in X$ , we have  $k \leq S(f(x), f(x))$  so  $k \leq \bigwedge_{x \in X} S(f(x), f(x)) = (R \multimap S)(f, f)$ .

Let  $f, g, h \in \mathbb{V}\text{-Ord}((X, R), (Y, S))$ . We have

$$\begin{aligned} (R \multimap S)(f, g) \otimes (R \multimap S)(g, h) &= \left( \bigwedge_{x \in X} S(f(x), g(x)) \right) \otimes \left( \bigwedge_{x \in X} S(g(x), h(x)) \right) \\ &\leq \bigwedge_{x \in X} (S(f(x), g(x)) \otimes S(g(x), h(x))) \leq \bigwedge_{x \in X} S(f(x), h(x)) = (R \multimap S)(f, h). \end{aligned}$$

$\square$

Denote by  $(X, R) \multimap (Y, S)$  the  $\mathbb{V}$ -preordered set  $(\mathbb{V}\text{-Ord}((X, R), (Y, S)), R \multimap S)$ .

**Proposition A.0.15.** Let  $\mathbb{V}$  be a commutative quantale and let  $(X, R)$  and  $(Y, S)$  be  $\mathbb{V}$ -preordered sets. We define a  $\mathbb{V}$ -relation  $R \otimes S$  from  $X \times Y$  into itself by

$$(R \otimes S)((x, y), (x', y')) := R(x, x') \otimes S(y, y'), \quad \text{for every } (x, y), (x', y') \in X \times Y.$$

Then  $R \otimes S$  is a  $\mathbb{V}$ -preorder on  $X \times Y$ .

*Proof.* Let  $(x, y) \in X \times Y$ . We have  $k \leq R(x, x)$  and  $k \leq S(y, y)$ , so

$$k = k \otimes k \leq R(x, x) \otimes S(y, y) = (R \otimes S)((x, y), (x, y)).$$

Let  $(x, y), (x', y'), (x'', y'') \in X \times Y$ . We have  $R(x, x') \otimes R(x', x'') \leq R(x, x'')$  and  $S(y, y') \otimes S(y', y'') \leq S(y, y'')$ . Thus

$$\begin{aligned} (R \otimes S)((x, y), (x', y')) \otimes (R \otimes S)((x', y'), (x'', y'')) \\ &= R(x, x') \otimes S(y, y') \otimes R(x', x'') \otimes S(y', y'') \\ &= R(x, x') \otimes R(x', x'') \otimes S(y, y') \otimes S(y', y'') \\ &\leq R(x, x'') \otimes S(y, y'') = (R \otimes S)((x, y), (x'', y'')). \end{aligned}$$

$\square$

Let  $\mathbb{V}$  be a quantale, write  $I_{\mathbb{V}} := (\{\star\}, (\star, \star) \mapsto k)$ . For all  $\mathbb{V}$ -preordered set  $(X, R)$ , for all  $x \in X$ , the function  $\star \mapsto x$  is a  $\mathbb{V}$ -increasing map from  $I_{\mathbb{V}}$  to  $(X, R)$ .

**Proposition A.0.16.** Let  $\mathbb{V}$  be a commutative quantale.  $(\mathbb{V}\text{-Ord}, I_{\mathbb{V}}, \otimes, \multimap)$  is a symmetric closed monoidal category and the forgetful functor  $U_{\mathbb{V}} : \mathbb{V}\text{-Ord} \rightarrow \mathbf{Set}$  is a strict monoidal functor from  $(\mathbb{V}\text{-Ord}, I_{\mathbb{V}}, \otimes)$  to  $(\mathbf{Set}, \{\star\}, \times)$ .

*Proof.* The fact that  $(\mathbb{V}\text{-Ord}, I_{\mathbb{V}}, \otimes)$  is a symmetric monoidal category and that the forgetful functor  $U_{\mathbb{V}} : \mathbb{V}\text{-Ord} \rightarrow \mathbf{Set}$  is a strict monoidal functor from  $(\mathbb{V}\text{-Ord}, I_{\mathbb{V}}, \otimes)$  to  $(\mathbf{Set}, \{\star\}, \times)$  is clear. It remains to show the closedness.

Let  $(X, R)$  and  $(Y, S)$  be two  $\mathbb{V}$ -preordered sets. We are to show that the function  $eval_{X,Y} : X \times \mathbb{V}\text{-Ord}((X, R), (Y, S)) \rightarrow Y, (x, f) \mapsto f(x)$  is a  $\mathbb{V}$ -increasing map from  $(X, R) \otimes ((X, R) \multimap (Y, S))$  to  $(Y, S)$ . Let  $x, x' \in X$  and  $f, f' \in \mathbb{V}\text{-Ord}((X, R), (Y, S))$ . We have

$$\begin{aligned} (R \otimes (R \multimap S))((x, f), (x', f')) &= R(x, x') \otimes (R \multimap S)(f, f') \\ &= R(x, x') \otimes \left( \bigwedge_{x \in X} S(f(x), f'(x)) \right) \\ &\leq S(f(x), f'(x)) \otimes S(f(x'), f'(x')) \leq S(f(x), f'(x')). \end{aligned}$$

Hence  $eval_{X,Y}$  is a  $\mathbb{V}$ -increasing map.

Let  $(X, R), (Y, S)$  and  $(Z, T)$  be three  $\mathbb{V}$ -preordered sets and let  $m : Y \times X \rightarrow Z$  be a  $\mathbb{V}$ -increasing map from  $(Y, S) \otimes (X, R)$  to  $(Z, T)$ . For every  $x \in X$ , write

$$\Lambda(m)(x) := y \mapsto m(x, y) .$$

The function  $\Lambda(m)(x)$  is a  $\mathbb{V}$ -increasing map from  $(Y, S)$  to  $(Z, T)$ . Indeed, let  $y, y' \in Y$ , we have

$$\begin{aligned} S(y, y') &= S(y, y') \otimes k \leq S(y, y') \otimes R(x, x) \\ &= (S \otimes R)((y, x), (y', x)) \\ &\leq T(m(y, x), m(y', x)) = T(\Lambda(m)(x)(y), \Lambda(m)(x)(y')) . \end{aligned}$$

It remains to show that  $\Lambda(m)$  is a  $\mathbb{V}$ -increasing map from  $(X, R)$  to  $(Y, S) \multimap (Z, T)$ . Let  $x, x' \in X$ . Let  $y \in Y$ , we have

$$\begin{aligned} R(x, x') &= k \otimes R(x, x') \leq S(y, y) \otimes R(x, x') \\ &= (S \otimes R)((y, x), (y, x')) \\ &\leq T(m(x, y), m(x', y)) = T(\Lambda(m)(x)(y), \Lambda(m)(x')(y)) , \end{aligned}$$

hence

$$R(x, x') \leq \bigwedge_{y \in Y} T(\Lambda(m)(x)(y), \Lambda(m)(x')(y)) = (S \multimap T)(\Lambda(m)(x), \Lambda(m)(x')) .$$

□

**Proposition A.0.17.** Let  $\mathbb{V}$  and  $\mathbb{W}$  be two commutative quantales and let  $v$  be a quantale morphism from  $\mathbb{V}$  to  $\mathbb{W}$ . The functor induced by  $v$  is a lax monoidal functor from  $(\mathbb{V}\text{-Ord}, I_{\mathbb{V}}, \otimes)$  to  $(\mathbb{W}\text{-Ord}, I_{\mathbb{W}}, \otimes)$  that commutes with the forgetful functors.

*Proof.* The function  $\text{Id}_{\{\star\}}$  is a  $\mathbb{W}$ -increasing map from  $I_{\mathbb{W}}$  to  $v(I_{\mathbb{V}})$ .

Let  $(X, R)$  and  $(Y, S)$  be two  $\mathbb{V}$ -preordered sets, let  $x, x' \in X$ , and let  $y, y' \in Y$ . We have

$$\begin{aligned} (v(R) \otimes v(S))((x, y), (x', y')) &= v(R)(x, x') \otimes v(S)(y, y') \\ &= v(R(x, x')) \otimes v(S(y, y')) \\ &\leq v(R(x, x') \otimes S(y, y')) \\ &= v((R \otimes S)((x, y), (x', y'))) = v(R \otimes S)((x, y), (x', y')) . \end{aligned}$$

Thus  $\text{Id}_{X \times Y}$  is a  $\mathbb{W}$ -increasing map from  $(X, v(R)) \otimes (Y, v(S))$  to  $(X \times Y, v(R \otimes S))$ . □



## Appendix B

# Properties preserved or reflected by changing of bases

Let  $v : (T, \in_T) \rightarrow (T', \in_{T'})$  a changing of bases datum and let  $X$  be a  $T$ -topological space.

	Preserved	Preserved if $v_X$ fully faithful	Preserved if $v_X$ essentially surjective	Reflected	Reflected if $v_X$ fully faithful	Reflected if $v_X$ essentially surjective	Reflected if $v_X$ essentially surjective and essentially surjective
Openness	✓	✓	✓		✓		✓
Neighbourhoods	✓	✓	✓		✓		✓
$\in_T$ -indistinguishability			✓	✓	✓	✓	✓
Adherent point of a member	✓	✓	✓	✓	✓	✓	✓
Closedness	✓	✓	✓	✓	✓	✓	✓
Finitely pointable by $v$	✓	✓	✓	✓	✓	✓	✓
Finitely pointable by $v^{-1}$	✓	✓	✓			✓	✓
Adherent point of a part by $v$	✓	✓	✓	✓	✓	✓	✓
Adherent point of a part by $v^{-1}$	✓	✓	✓			✓	✓
Convergence of an upward closed part by $v$	✓	✓	✓	✓	✓	✓	✓
Convergence of an upward closed part by $v^{-1}$	✓	✓	✓				✓
Specialization preorder	✓	✓	✓	✓	✓	✓	✓
$T_0, R_0, T_1$	✓	✓	✓	✓	✓	✓	✓
Weakly $T_0, T_1$			✓	✓	✓	✓	✓
Hausdorff	✓	✓	✓	✓	✓	✓	✓
Weakly Hausdorff			✓	✓	✓	✓	✓
Compactness			✓	✓	✓	✓	✓
Borel-Lebesgue			✓	✓	✓	✓	✓
Alexandroff	✓	✓	✓		✓		✓
Pseudo-openness	✓	✓	✓				



# Appendix C

## Notations

### Categories and functors:

<b>Set</b>	Category of sets and functions	15
<b>Cat</b>	Large category of sets and functions	43
<b>Rel</b>	Ordered category of sets and relations	13
<b>Mod</b>	Ordered category of preordered sets and modules	14
<b>Ord</b>	Category of preordered sets and increasing maps	13
<b>SOrd</b>	Category of ordered sets and increasing maps	13
<b>Top</b>	Category of topological spaces and continuous maps	11
<b>Qnt</b>	Category of quantales and quantale morphisms	100
<b><math>\mathbb{V}</math>-Rel</b>	Preordered category of sets and relations on a quantale $\mathbb{V}$	101
<b><math>\mathbb{V}</math>-Ord</b>	Category of preordered sets and increasing maps on a quantale $\mathbb{V}$	101
<b><math>\mathbb{V}</math>-SOrd</b>	Category of ordered sets and increasing maps on a quantale $\mathbb{V}$	101
<b>STopTh</b>	Preordered category of topological theories and semantic transformations	43
<b>TopTh</b>	Preordered category of topological theories and change base data	45
<b>Top<sub>T</sub></b>	Category of $T$ -topological spaces and $T$ -continuous maps for a topological theory $T$	10
<b>Stream<sub>T</sub></b>	Category of $T$ -streams for a topological theory $T$	90
<b><math>\mathcal{C}</math>-Map</b>	Category of $T$ -topological spaces and $\mathcal{C}$ -continuous map for a topological theory $T$ and for a class of $T$ -topological spaces $\mathcal{C}$	91
<b><math>\mathcal{C}</math>-Top<sub>T</sub></b>	Full subcategory of $T$ -topological spaces $\mathcal{C}$ -generated and $T$ -continuous maps	91
<b><math>(\_)*</math></b>	Canonical contravariant functor from <b>Ord</b> to <b>Mod</b>	14
<b><math>(\_)*</math></b>	Canonical covariant functor from <b>Ord</b> to <b>Mod</b>	14
<b>Eq</b>	Canonical covariant functor from <b>Set</b> to <b>Mod</b>	15
<b>U<sub><math>\mathbb{V}</math></sub></b>	Canonical forgetful functor from <b><math>\mathbb{V}</math>-Ord</b> into <b>Set</b>	101
<b>U<sub>T</sub></b>	Canonical forgetful functor from <b>Top<sub>T</sub></b> into <b>Set</b>	43
<b>Sem</b>	Canonical 2-functor from <b>TopTh</b> to <b>STopTh</b>	46
<b>Loc</b>	Canonical 2-functor from <b>STopTh</b> to <b>TopTh</b>	48
$\eta$	Unity of the monad <b>Sem</b> $\circ$ <b>Loc</b>	49
$\mu$	Product of the monad <b>Sem</b> $\circ$ <b>Loc</b>	50
$\iota_T$	Canonical semantic transformation from $T$ to $\mathcal{P} \times T$ for a topological theory $T$	84
$\gamma_T$	Canonical semantic transformation from $\mathcal{P} \times T$ to <b>Loc</b> ( $T$ ) for a topological theory $T$	84
<b><math>\mathcal{C}_-</math></b>	Coreflection of <b><math>\mathcal{C}</math>-Top<sub>T</sub></b> in <b>Top<sub>T</sub></b>	91

**List of symbols:**

$X \leftrightarrow Y$	Relation from a set $X$ to a set $Y$
$X \dashv\rightarrow Y$	Module from a preordered set $X$ to a preordered set $Y$
$\bigvee$	Join of a family of elements of a preordered set
$\bigwedge$	Meet of a family of elements of a preordered set
$\mathbb{V}$	Any quantale
$\otimes$	Monoidal product in a monoidal category (in particular, the product in a quantale)
$\mathbb{1}$	The final quantale
$\mathbb{B}$	The boolean quantale
$T = (T, \in_T)$	Topological theory
$\in_T$	Membership module regarding a topological theory $T = (T, \in_T)$
$\subset_T$	Inclusion module regarding a topological theory $T = (T, \in_T)$
$\simeq_T$	"having the same $T$ -elements" relation regarding a topological theory $T = (T, \in_T)$
$\mathcal{P}(X)$	Powerset of a set $X$
$\mathcal{P}_{fin}(X)$	Set of all finite subset of a set $X$
$T_{\mathbb{V}}$	Topological theory of $\mathbb{V}$ -preordered sets for a quantale $\mathbb{V}$
$T_{\mathbb{V}}^S$	Topological theory of $\mathbb{V}$ -ordered sets for a quantale $\mathbb{V}$
$!T$	Final changing of bases datum from a topological theory $T$

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**Titre :** Un cadre pour les espaces localement structurés - Application aux modèles géométriques de la concurrence

**Mots clés :** Concurrency, Modèles géométriques, Streams, Espaces localement ordonnés

**Résumé :** L'utilisation de méthodes provenant de la topologie algébrique dans l'étude des processus concurrents ont été introduites en 1998. L'un des ingrédients clef est la réalisation des ensembles précubiques dans la catégories des espaces localement ordonnés. Cependant la formalisation du concept d'espace localement ordonné n'est pas consensuelle : plusieurs définitions non équivalentes ont été proposées dans la littérature. C'est un inconvénient majeur puisque, comme on le montre dans le chapitre 3, les colimites d'espaces localement ordonnés sont très sensibles à des changements apparemment mineurs dans les définitions. Il existe ainsi une pléthore de notions mathématiques similaires mais non équivalentes, toutes basées sur la topologie, qui essaient de formaliser la même idée. Dans cette thèse, on construit un cadre commun pour pouvoir comparer ces notions. L'idée clef est de remplacer, pour chaque ensemble  $X$ , l'ensemble des parties de  $X$  ordonné par inclusion par un ensemble préordonné  $T(X)$ . Intuitivement, les membres

de  $T(X)$  correspondent à des parties de  $X$  équipées d'une structure supplémentaire et on impose que les inclusions tiennent compte de ces structures.  $T$  étant fixé, on peut définir les espaces  $T$ -topologiques et les application  $T$ -continues en remplaçant les sous-ensembles par les membres des  $T(X)$  dans les définitions classiques des espaces topologiques et des applications continues en termes de bases de topologie. En imposant des axiomes appropriés aux  $T(X)$ , on obtient une catégorie concrète. On montre que beaucoup de notions standard de topologie, comme la convergence, la compacité ou la topologie initiale, peuvent être étendues à ce cadre. Ainsi  $T$  est une sorte de template qui fixe la forme des espaces qui lui sont associés, pour cette raison, on appelle théorie topologique un tel  $T$ .

Dans le chapitre 7, on développe une généralisation naturelle des streams de Krishnan comme des espaces  $T$ -topologiques vérifiant une simple propriété de stabilité supplémentaire. En effet, pour un  $T$  bien choisi, on retrouve alors les streams usuels.

**Title :** A framework for locally structured spaces - Application to geometric models of concurrency

**Keywords :** Concurrency, Geometric models, Streams, Locally ordered spaces

**Abstract :** The usage of methods from Algebraic Topology in the study of concurrent processes was initiated in 1998. One of its key ingredient is the realization of precubical sets in the category of locally ordered spaces. Nevertheless, the formalization of the concept of a locally ordered space is not firmly set: various non-equivalent definitions have indeed appeared in the literature. This is a serious drawback, as we see in the third chapter, because the colimits of locally ordered spaces are extremely sensitive to seemingly anodyne modification in their definition. We end up with plethora of similar yet non-equivalent mathematical notions, all grounded on topology, intended to formalize the same idea.

In this thesis, we design a unified framework to compare these notions. The key idea is to replace, for every set  $X$ , the powerset of  $X$  ordered by inclusion by a mere preordered set  $T(X)$ . Intuitively, the members of  $T(X)$  are to be thought as parts of  $X$  endowed

with an additional structure and we require that inclusions take these structures into account. Given  $T$ , we can define  $T$ -topological spaces and  $T$ -continuous maps by replacing the subsets by the members of the  $T(X)$  in the classic definitions of topological spaces and of continuous maps in terms of topological bases. By fixing the appropriate axioms on the  $T(X)$ , we obtain a concrete category. We show that many standard notions of topology, like convergence, compactness, or initial topology, can be extended to this framework. Thus  $T$  is a kind of template which fix the form of the spaces associated to it, that is why such a  $T$  is called a topological theory.

In the seventh chapter, we develop a natural generalization of Krishnan's streams as  $T$ -topological spaces satisfying a simple additional stability property. Indeed, for a well chosen  $T$ , we recover standard streams.