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# Approximation by polynomial projectors in several variables

Francois Bertrand

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# THÈSE

En vue de l'obtention du  
**DOCTORAT DE L'UNIVERSITÉ DE TOULOUSE**  
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Présentée et soutenue par  
**François BERTRAND**

Le 12 juillet 2022

**Construction et étude des propriétés d'approximation de  
projecteurs polynomiaux sur des espaces de fonctions de  
plusieurs variables**

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Thèse dirigée par  
**Jean-paul CALVI**

Jury

M. Leonard-Peter BOS, Rapporteur  
M. Norman LEVENBERG, Rapporteur  
M. Franck WIELONSKY, Rapporteur  
Mme Leokadia BIALAS-CIEZ, Examinatrice  
M. Jean-paul CALVI, Directeur de thèse



Il est tard à mon âge pour un début et cependant voici mon sentiment :  
( Pablo Neruda *in* Chanson de geste )



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Cette thèse lui doit presque tout.



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# INTRODUCTION ET SYNTHÈSE DES RÉSULTATS

## 1. CONTEXTE

Contrairement à l'opinion un peu hâtive d'un Jules Andoyer<sup>1</sup> et même à celle plus récente et plus nuancée de Georges Glaeser<sup>2</sup>, pensant à tort qu'elle se résumait à des adaptations mécaniques du cas monovarié, l'interpolation polynomiale de fonctions à plusieurs variables constitue depuis la fin des années soixante-dix un véritable corpus autonome. Bien sûr la richesse des résultats obtenus pour les fonctions d'une seule variable reste un modèle sans pareil pour cette " nouvelle " théorie ( deux exceptions notables dues à Kronecker et Jacobi néanmoins datent du milieu du dix-neuf-ième siècle<sup>3</sup> mais les difficultés désormais bien comprises qu'elle pose, appellent aussi des approches spécifiques voire originales. Ce travail souhaiterait se placer sous cette double égide en s'articulant autour d'une notion, que l'on doit à Jean-Paul Calvi : **le produit de Newton** (PN en abrégé pour la suite de cette introduction) permettant d'amalgamer, selon un procédé original, deux projecteurs polynomiaux pour en construire un nouveau qui, lui, agira en dimension strictement supérieure à celles de ses constituants. Le premier chapitre ( article coécrit avec Jean-Paul Calvi et publié en 2019 dans l'international journal of mathematics) développe de façon plutôt exhaustive des propriétés algébriques de ce produit. Le second chapitre ( article toujours coécrit avec Jean-Paul Calvi et retenu pour publication par les Rendiconti del Circolo Matematico di Palermo) croise le concept de PN avec celui d'approximation polynomiale pour certaines familles de fonctions. Nous y obtenons des résultats originaux mais aussi de nouvelles preuves de points déjà connus. Dans le dernier chapitre ( note soumise cet été aux Comptes rendus de l'académie des sciences) on étudie en détail des questions combinatoires<sup>4</sup> concernant d'abord l'interpolation de Lagrange plane puis étendues à toute dimension ; on y souligne notamment une certaine épaisseur statistique des PN de structures lagrangiennes monovariées. Les notations utilisées quand elles sont standard ne seront pas rappelées dans cette introduction. Les références non explicitées à l'intérieur de chaque paragraphe renvoient aux bibliographies des chapitres (articles autonomes) qui les abritent. Revenons désormais dans les détails.

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1. Page 160 dans le tome 4 du volume 1 de l'encyclopédie des sciences mathématiques, réédition Jacques Gabay .

2. " Enfin les tentatives concernant l'interpolation des fonctions de plusieurs variables apparaissent généralement en analyse numérique, comme une simple itération de l'interpolation à une variable avec des noeuds d'interpolation disposés en quadrillage. ", citation extraite (page 3) de la contribution de Glaeser à l'ouvrage collectif "Proceedings of Liverpool Singularities Symposium II", Lecture Notes in Mathematics Volume 209.

3. cf leurs oeuvres complètes publiées par Chelsea : Tome 1 pages 133 et suivantes pour Kronecker et Tome 3 pages 285 à 294 pour Jacobi ; l'approche de Kronecker se veut plus générale et plus algébrique.

4. Abordées dans le premier chapitre.

## 2. ÉTUDE ALGÈBRIQUE DU PRODUIT DE NEWTON DE PROJECTEURS POLYNOMIAUX

### 2.1. Projecteurs polynomiaux

Il s'agit d'une notion très classique<sup>5</sup> dont nous rappelons brièvement quelques caractéristiques. Un projecteur polynomial de degré  $d$  sur un espace fonctionnel  $\mathcal{E}$  contenant  $\mathcal{P}_d(\mathbb{K}^n)$  est un projecteur continu sur  $\mathcal{E}$  dont l'image est exactement  $\mathcal{P}_d(\mathbb{K}^n)$ . Ainsi,  $\Pi$  étant un tel objet,  $\Pi(p) = p$  pour tout  $p$  de  $\mathcal{P}_d(\mathbb{K}^n)$  par ailleurs on attache à ce projecteur polynomial son espace de conditions, noté  $\text{CND}(\Pi)$ , consistant en le sous-espace de  $\mathcal{E}'$  constitué des fonctionnelles  $\mu$  vérifiant  $\mu(\Pi(f)) = \mu(f)$  pour tout  $f \in \mathcal{E}$ ; cet espace vectoriel caractérise le projecteur<sup>6</sup>. Les opérateurs de Lagrange, d'Hermite, de Taylor, de Kergin et les projecteurs orthogonaux polynomiaux sont parmi les projecteurs polynomiaux les plus courants et il est aisé de préciser leurs espaces de conditions; nous en présentons sommairement quelques uns sous forme de tableau :

Nature du projecteur	Base de l'espace des conditions
<b>Projecteur de Taylor</b> en $a \in \mathbb{K}^n$ d'ordre $d$	$(D^\alpha[a],  \alpha  \leq d)$
<b><math>L^2(\mu)</math>-Projecteur orthogonal</b> sur $\mathcal{P}_d(\mathbb{K}^n)$ , où $\mu$ est une mesure de Borel portée par $W \subset \mathbb{K}^n$	$(v_\alpha : f \mapsto \langle f, e_\alpha \rangle : \alpha \in \mathbb{N}_d(n))$
<b>Interpolation de Lagrange</b> dans $\mathcal{P}_d(\mathbb{K}^n)$ en les points de $X = \{x_j, j = 0, \dots, N\} \subset \mathbb{K}^n$	$([x_j], j = 0, \dots, N)$ , où $N = \binom{n+d}{d}$

TABLE 1. Exemples d'espaces de conditions

### 2.2. Structure de Newton

L'ajout, le retrait d'informations ou de données pour affiner ou alléger l'approximation qu'il a en vue font partie du quotidien du numéricien. Les formules de Lagrange (utilisant les fameux FLIP) auxquelles fait référence la note 6 ne se prêtent pas bien à ces adaptations, les différences finies de Newton apportent, on le sait, une réponse à cette problématique dans le cas monovariante. A partir de ce constat (Calvi [13] et Sauer, Xu [26] le faisaient déjà, au moins pour les projecteurs lagrangiens, à la terminologie près), on peut plutôt considérer,  $\Pi$  étant un projecteur polynomial de degré  $d$  sur  $\mathcal{E}$  donné, une suite finie  $(\Pi_0, \dots, \dots, \Pi_d = \Pi)$ , où chaque  $\Pi_k$  est un projecteur polynomial de degré  $k$  sur  $\mathcal{E}$ , satisfaisant à :

$$\Pi_k \circ \Pi_{k+1} = \Pi_k, \quad 0 \leq k \leq d-1 \tag{2.1}$$

que nous qualifierons de **structure de Newton** (SN en abrégé) pour  $\Pi$  et que nous noterons  $[\Pi]$ , faute de mieux ( en étant conscient de l'ambiguïté de cette notation puisqu'elle fait abstraction des  $d$  premiers termes<sup>7</sup> de la suite  $(\Pi_0, \Pi_1, \dots, \Pi_d)$ ). On associe aussi à cette structure ses

5. Pas toujours explicitée.

6. On peut donc présenter ce projecteur de deux façons; par exemple, pour le projecteur ( sur un espace approprié) de Lagrange monovariante  $\mathcal{L}[A]$  où  $A = \{a_0, \dots, a_d\}$  est une partie de points deux à deux distincts de  $\mathbb{C}$ , on dispose de la formule explicite (via les FLIP) et bien connue avec les polynômes de Lagrange mais aussi de la description de l'espace des conditions via les formes de Dirac  $[a_0], \dots, [a_d]$  qui engendrent cet espace.

7. Les termes de cette suite sont nommés facteurs de  $[\Pi]$ .

sommants (de Newton) :

$$\pi_0 := \Pi_0, \quad \text{et} \quad \pi_i := \Pi_i - \Pi_{i-1}, \quad i = 1, \dots, d, \quad (2.2)$$

il est à noter que l'image de  $\pi_i$  n'est rien d'autre que l'intersection de  $\mathcal{P}_i(\mathbb{K}^n)$  et du noyau de  $\Pi_{i-1}$  pour  $i=1, \dots, d$ , avec la convention que  $\Pi_{-1}$  est l'endomorphisme nul<sup>8</sup>. En reprenant le contexte de la note 5,  $[L[A]]$  correspond à la suite  $(L[A_0], \dots, L[A_d])$ , où  $A_k = \{a_0, \dots, a_k\}$ , ce pour  $0 \leq k \leq d$ ; l'image du sommant d'ordre  $k$  (voir note 7) étant engendrée par le polynôme  $(X - a_0) \dots (X - a_k)$ , il existe une forme linéaire continue (sur un espace de Fréchet approprié)  $\mu_k$  telle que :  $\pi_k(P) = \mu_k(P)(X - a_0) \dots (X - a_k)$ . Il serait possible, on le voit bien, de définir par ce biais les différences divisées de Newton; ce qui incite fortement à penser que cette notion de structure de Newton peut dépasser les objections mises en évidence en début de paragraphe, cette approche faisant apparaître tout naturellement le remède classique donné du cas monovarié. Plus généralement étant donnée une SN  $(\Pi_0, \Pi_1, \dots, \Pi_d)$  sur un espace fonctionnel approprié, celui-ci résulte des inclusions (découlant immédiatement de la définition des SN) des espaces de condition :  $\text{CND}(\Pi^k) \subset \text{CND}(\Pi^{k+1})$ , pour  $0 \leq k \leq d - 1$ .

### 2.3. Définition du produit de Newton

L'idée de cette notion part là aussi d'un état des lieux portant sur les limites (peut-être la contrepartie de la simplicité de sa définition) du produit tensoriel de deux projecteurs, rappelons les : tout d'abord la rigidité de l'espace fonctionnel sur lequel il agit (essentiellement un produit cartésien de tels espaces, cela peut ne pas suffire en pratique) puis (et pour les mêmes raisons) il renvoie en général un polynôme de degré trop important, enfin le résultat ne donne pas, en général, des projecteurs polynomiaux répertoriés (même à partir de projecteurs naturels).

Donnons nous pour  $s = 1, 2$ ,  $[\Pi^s] = (\Pi_0^s, \dots, \Pi_d^s)$  des SN de degré  $d$  sur des espaces fonctionnels (classiques) de Fréchet  $\mathcal{E}_{n_s}(X_s)$  auxquels nous associons des bases adaptées de fonctionnelles  $\mu^s = (\mu_{\alpha^s}^s)$ <sup>9</sup>. Nous montrons alors (cf le théorème 4.5 du premier chapitre et ses conséquences) qu'il existe une (unique) structure de Newton  $[\Pi]$ , de même degré mais sur  $\mathcal{E}_n(X)$  où  $n = n_1 + n_2$  and  $X = X_1 \times X_2$  telle que pour tout  $k$  de  $\{0, \dots, d\}$  on ait :

$$\text{CND}(\Pi_k) = \text{Vect}(\mu_{\alpha^1}^1 \otimes \mu_{\alpha^2}^2 : |\alpha^1| + |\alpha^2| \leq k) \quad (2.3)$$

Par là même on voit que les deux premières<sup>10</sup> objections concernant le produit tensoriel de projecteurs sont levées. Nous nommerons cette SN produit de Newton des SN  $[\Pi^s]$ ,  $s = 1, 2$ <sup>11</sup> et nous la noterons :  $[\Pi_1] \otimes_N [\Pi_2]$ ; et avec un abus déjà signalé on écrira aussi  $\Pi_1 \otimes_N \Pi_2$  pour le dernier facteur (celui de degré  $d$  donc) de cette nouvelle<sup>12</sup> SN.

### 2.4. Exemples de produit de Newton

A travers ces exemples, qui souligne nettement le caractère endogamique de ce concept, nous allons voir que le dernier défaut répertorié (plus haut) du produit tensoriel de projecteurs ne tient plus (à une nuance près<sup>13</sup> pour le produit de Newton de projecteurs usuels. On a immédiatement via les espaces de condition :

8. L'inclusion non triviale (i.e gauche droite) se prouve comme suit; prenant  $p$  dans l'image considérée, on a l'existence de  $f$  de  $\mathcal{E}$  tel que  $p = \Pi_i(f) - \Pi_{i-1}(f)$  en composant par  $\Pi_i$ , on obtient l'inclusion inverse.

9. i.e  $(\mu_{\alpha^s}^s)_{|\alpha^s| \leq d}$  est une base de  $\text{CND}(\Pi_k^s)$ , ce pour tout  $k$  de  $\{0, \dots, d\}$ .

10. Pour la première, on travaille donc sur  $\mathcal{E}(X_1 \times X_2)$  au lieu de  $\mathcal{E}(X_1) \otimes \mathcal{E}(X_2)$ .

11. Que nous qualifierons par la suite de diviseurs de cette structure produit.

12. Il faut néanmoins noter que l'on trouve dans les travaux de Biermann (circa 1900) sur l'approximation des intégrales doubles une notion embryonnaire un peu similaire; nous reviendrons sur ce point un peu plus loin.

13. Le produit de Newton de deux SN de Kergin n'en est plus une - trop de Dirac dans son espace de conditions- néanmoins cela donne naturellement naissance à un nouveau type de SN cf 6.6 du premier chapitre.

Premier projecteur $\Pi_1$	Second projecteur $\Pi_2$	Produit de Newton $\Pi_1 \otimes_N \Pi_2$
<b>Projecteur de Taylor</b> en $a \in \mathbb{K}^n$ à l'ordre $d$	<b>Projecteur de Taylor</b> en $b \in \mathbb{K}^m$ à l'ordre $d$	<b>Projecteur de Taylor</b> en $(a, b) \in \mathbb{K}^{n+m}$ à l'ordre $d$
$L^2(\mu)$ -Projection orthogonale sur $\mathcal{P}_d(\mathbb{K}^n)$ , où $\mu$ est une mesure de Borel portée par $K \subset \mathbb{K}^n$	$L^2(\nu)$ -Projection orthogonale sur $\mathcal{P}_d(\mathbb{K}^m)$ , où $\mu$ est une mesure de Borel portée par $W \subset \mathbb{K}^m$	$L^2(\mu \otimes \nu)$ -Projection orthogonale sur $\mathcal{P}_d(\mathbb{K}^{n+m})$ .
<b>Interpolation de Lagrange</b> dans $\mathcal{P}_d(x)$ en les points de $X = \{x_j, j = 0, \dots, d\} \subset \mathbb{R}$	<b>Interpolation de Lagrange</b> dans $\mathcal{P}_d(y)$ en les points de $Y = \{y_j, j = 0, \dots, d\} \subset \mathbb{R}$	<b>Interpolation de Lagrange</b> dans $\mathcal{P}_d(x, y)$ en les points de $\{(x_i, y_j), i + j \leq d\} \subset \mathbb{R}^2$

TABLE 2. Exemples de quelques produits de Newton de projecteurs.

2.4.1. *le produit de Newton de structures lagrangiennes est du même type comme le suggère le tableau précédent.* Nous allons développer ce dernier point . Par structure( de Newton) lagrangienne de degré  $d$ , dans  $\mathbb{K}^m$  (ou plus simplement à  $m$  variables), on entend une structure de Newton  $[\Pi_k]$  dont tous les facteurs  $\Pi_k, 0 \leq k \leq d$  sont des projecteurs de Lagrange, de degré  $k$  à  $m$  variables. On dispose alors d'une unique famille ( dite base adaptée)  $([a_\alpha])_{|\alpha| \leq d}$  de fonctionnelles (de Dirac) telle que  $([a_\alpha])_{|\alpha| \leq k}$  soit une base de chaque espace de condition  $CND(\Pi_k)$  pour tout  $k, 0 \leq k \leq d$  et d'une suite ( croissante pour l'inclusion) d'ensembles de nœuds d'interpolation  $\{a_\alpha, |\alpha| \leq k\}$  qui sont tous nécessairement des parties unisolvantes de  $\mathbb{K}^m$ , d'ordre ( ou de degré)  $k$ . Inversement à toute partie unisolvante d'ordre  $d$  de  $\mathbb{K}^m$  on associe un unique projecteur de Lagrange à  $m$  variables ( à savoir  $L[A]$  ) **et** au moins ( on a donné un minorant moins grossier dans ce travail, par des raisonnements d'algèbre linéaire assez directs sur les matrices de Vandermonde généralisées) une structure lagrangienne de même de degré et de même nombre de variables telle que le dernier facteur de celle-ci soit justement  $L[A]$ . L'importance de l'unisolvanace dans ce domaine n'étant plus à souligner, on voit aisément un intérêt immédiat de 2.4.3 : **obtenir de façon systématique des ensembles unisolvants à partir de deux autres mais de dimensions inférieures.** De ces remarques, on peut tirer le concept suivant (qui étend un peu celui généralement proposé) :

**Definition 2.1.** Une partie  $A$  de  $\mathbb{K}^n$  est dite de Biermann d'ordre  $d$ , s'il existe des entiers  $n_1$  et  $n_2$  non nuls et des structures lagrangiennes de degré  $d$   $[\Pi^i] = L[A_i], i = 1, 2$ , définies sur des espaces fonctionnels  $E_{n_i}$  tels que :  $n = n_1 + n_2$  et  $\Pi^1 \otimes_N \Pi^2 = L[A]$ . On se permettra aussi d'écrire  $A = A_1 \otimes_N A_2$ .

Bien sûr un tel ensemble est unisolvant et par ailleurs les structures de Newton l'engendrant sont uniques (modulo l'espace fonctionnel cela va de soi) en se plaçant dans le même contexte dimensionnel. Un autre intérêt apporté par cette définition réside dans l'utilisation de la formule (théorème 4.5 (2) du chapitre 1) fondamentale exprimant simplement l'effet de  $\Pi^1 \otimes_N \Pi^2$  sur  $\mathcal{E}(X_1) \otimes \mathcal{E}(X_2)$ . Plus précisément<sup>14</sup> (où on a posé  $\Pi = \Pi^1 \otimes_N \Pi^2$ <sup>15</sup> :

$$\Pi(f_1 \otimes f_2) = \sum_{(i,j) \in \mathbb{N}_d(2)} \pi_i^1(f_1) \otimes \pi_j^2(f_2), \quad f_i \in \mathcal{E}(X^i), \quad (2.4)$$

où les  $\pi_i^s$  correspondent aux sommants de Newton de  $\Pi^s$  ; ce qui peut aussi se formuler de cette façon :

$$\Pi(f_1 \otimes f_2) = \sum_{i+j=d} \Pi_i^1(f_1) \otimes \Pi_j^2(f_2) - \sum_{i+j=d-1} \Pi_i^1(f_1) \otimes \Pi_j^2(f_2) \quad (2.5)$$

14. Ces formules s'étendent mécaniquement à un produit de Newton d'un nombre fini de SN

15. Ceci vaut aussi pour la suite.

Cela nous permet de généraliser le classique résultat de Nicolaïdes<sup>16</sup> ( et de l'obtenir naturellement contrairement à son habituelle présentation). Exposons sommairement l'idée directrice, à la lumière des formules précédentes, le cas bivariable ( dans le but de déterminer les FLIP du produit de Newton de structures lagrangiennes monovariables) en considérant  $\Pi^1 = L[A] = L[a_0, \dots, a_d]$  (resp.  $\Pi^2 = L[B] = L[b_0, \dots, b_d]$ ) ainsi que  $f_1$  comme étant le FLIP d'ordre  $k$  du premier projecteur ( resp.  $f_2$  comme le FLIP d'ordre  $l$  du second projecteur), où  $k + l \leq d$ . Pour obtenir le FLIP<sup>17</sup> du produit de Newton de nos projecteurs lagrangiens en le nœud  $(a_k, b_l)$ , il nous suffit donc d'évaluer  $\Pi(f_1 \otimes f_2)$ ; commençons par le cas  $k + l = d$ , ce qui donne ici en vertu de la formule (2.5) de cette introduction et du fait que  $\Pi_i^1(f_1) = 0$  dès que  $i < k$  (resp.  $\Pi_j^2(f_2) = 0$  dès que  $j < l$ ) :  $F_{k,l}^{A \otimes B} = l_k \otimes m_l$ <sup>18</sup>. Examinons maintenant le cas correspondant à  $k + l = d - 1$ , les mêmes ingrédients fournissent alors :  $F_{k,l}^{A \otimes B} = l_{k+1} \otimes m_l + l_k \otimes m_{l+1} - l_k \otimes m_l = (l_k \otimes m_l) F_{k,l}^{C_{k,l}}$ <sup>19</sup>, en utilisant à nouveau (2.5), il est facile de déterminer ce dernier FLIP :  $\frac{x-a_k}{a_{k+1}-a_k} - \frac{y-b_l}{b_{l+1}-b_l} - 1$ . Plus généralement on dispose de la même formule  $F_{k,l}^{A \otimes B} = (l_k \otimes m_l) F_{k,l}^{C_{k,l}}$ , là encore une utilisation de (2.5) conduit à une expression explicite ( mais plus lourde que précédemment car de degré  $d - k - l$ ) de  $F_{k,l}^{C_{k,l}}$ . Toutefois si on se place dans un cadre similaire ( mais plus général ) que Nicolaïdes ( dans le plan toujours) en supposant qu'il existe  $d$  droites  $D_1, \dots, D_d$ , d'équations respectives  $\phi_1 = 0, \dots, \phi_d = 0$  telles que :  $(a_k, b_l) \in D_s$  si  $k + l = s$  et  $(a_k, b_l) \notin D_s$  dès que  $k + l < s$ , nous avons<sup>20</sup> :

$$F_{k,l}^{A \otimes B} = \frac{1}{\lambda_{k,l}} (l_k \otimes m_l) \left( \prod_{s=k+l+1}^d \phi_s \right), \quad (2.6)$$

où on a posé  $\lambda_{k,l} = \prod_{s=k+l+1}^d \phi_s(a_k, b_l)$ .

Cette formule s'étend mécaniquement en dimension  $n \geq 3$  en considérant des produits de Newton de  $n$  structures lagrangiennes monovariables, le rôle des droites étant joué cette fois par des hyperplans.

## 2.5. Propriétés du produit de Newton

2.5.1. *Arithmétique des produits de Newton.* Le contexte et la terminologie utilisés appellent des concepts et des résultats arithmétiques, on peut les résumer comme suit :

**Proposition 2.2.** *Tout projecteur polynomial sur un espace produit est le produit de Newton de projecteurs polynomiaux irréductibles et ce de façon unique ( cf th 5.1 du premier chapitre).*

Dans le même ordre idée on peut s'interroger sur la nature des diviseurs d'un produit de Newton et nous obtenons ( Voir paragraphe 6.4 du chapitre 1) que les diviseurs d'un projecteur de type Lagrange-Hermite sont du même type.<sup>21</sup> Nous présentons dans la dernière section de cette introduction une application de ce résultat à la présentation d'une solution plus fluide d'un problème similaire à celui posé par Ciarlet et Raviart ( page 187 de leur article de l'Arch.Rational Mech Anal, Vol 46).

16. Qui donne le FLIP de l'ensemble unisolvant ( d'ordre  $d$  de  $\mathbb{R}^n$ )  $\{(k_1, \dots, k_n) \in \mathbb{N}^n, \sum_{i=1}^n k_i \leq d\}$ ; il ne s'agit rien de moins, et c'est pour cela que notre point de vue est plus général, que l'ensemble des nœuds du produit de Newton de  $L[0, \dots, d]$  sur lui-même opéré  $n$  fois.

17. Il sera noté  $F_{k,l}^{A \otimes B}$ .

18. Où on posé  $l_k$  (resp.  $m_l$ ) comme le FLIP associé au dernier nœud de  $L[a_0, \dots, a_k]$  (resp. de  $L[b_0, \dots, b_l]$ ).

19. Où on a posé  $C_{k,l} = \{a_k, \dots, a_{k+s}\} \otimes_N \{b_l, \dots, b_{l+s}\}$ , avec  $s = d - (k + l)$ .

20. La détermination de  $F_{k,l}^{C_{k,l}}$  étant géométriquement naturelle.

21. Ce résultat doit s'étendre mécaniquement à d'autres types de projecteurs polynomiaux.

2.5.2. *Résultats algébriques.* Ceux-ci tiennent grosso modo en deux items :

(1) **Déterminant de Vandermonde généralisé :**

La formule 5.10 du premier chapitre donne en effet le déterminant de Vandermonde d'un projecteur polynomial produit de Newton en fonction des déterminants de Vandermonde des diviseurs. A notre connaissance une telle formule ( estée consciencieusement pour  $n = d = 2$ ) semble originale.

(2) Conservation des relations différentielles homogènes :

Les travaux de Calvi ([12]) et de Calvi et Filipsson ([14]) caractérisent les projecteurs polynomiaux satisfaisant à une telle conservation. On pouvait se demander si le produit de Newton de deux projecteurs de ce type pouvait l'être encore : C'est effectivement le cas si ( et seulement si) les diviseurs sont trivialement centrés.

### 3. APPLICATIONS À L'APPROXIMATION DES FONCTIONS <sup>22</sup>

#### 3.1. La problématique

Nous travaillons sur des espaces fonctionnels de Fréchet dont la topologie est plus fine que celle de la convergence uniforme sur tout compact. Par suite de Newton sur un tel espace  $E(X)$  nous entendons une suite de projecteurs polynomiaux sur cet espace  $(\Pi_d)_d$  telle que, pour tout  $d$ ,  $(\Pi_0, \dots, \dots, \Pi_d)$  soit une structure de Newton de degré  $d$ . Nous dirons que cette suite de Newton converge ( dans  $E(X)$ ) si la suite  $((\Pi_d)(f))_d$  converge uniformément sur tout compact de  $X$ , ce pour tout  $f$  dans  $E(X)$ . Le problème que l'on se pose dans ce chapitre peut se formuler ainsi : étant données deux suites de Newton convergentes  $(\Pi_d^s)_d$  sur des espaces  $E(X_s)$ , où  $s = 1, 2$ , déterminer des conditions suffisantes ( concernant l'ensemble  $F$  des fonctions sur lesquelles agit le produit de Newton) pour que la suite de Newton  $(\Pi^1 \otimes_N \Pi^2)_d$  converge sur  $F$  (idéalement  $F = E(X)$ , où  $X = X_1 \times X_2$ ).

#### 3.2. Un mot de la méthode employée <sup>23 24</sup>

Elle repose d'abord sur une expression (obtenue à partir de propriétés des projecteurs et du fait que nous disposons d'un développement en série de polynômes d'un type adapté pour  $f$ ) de la différence  $\Pi_d(f) - f$  qui peut se voir alors comme le reste d'ordre  $d$  d'une série de fonctions. Plus précisément et en supposant que l'on parte de <sup>25</sup> :

$$f(z^1, z^2) = \sum_{j=0}^{\infty} \sum_{|\alpha|+|\beta|=j} c_{\alpha\beta}(f) p_{\alpha,\beta}(z^1, z^2), \quad (3.1)$$

où  $p_{\alpha,\beta}(z^1, z^2) = p_{\alpha}^1(z^1) p_{\beta}^2(z^2)$  et  $\deg p_{\alpha}^1 = |\alpha|$  et  $\deg p_{\beta}^2 = |\beta|$ , il vient par application de  $\Pi_d$  et de la compatibilité des topologies en jeu <sup>26</sup> :

$$f - \Pi_d(f) = \sum_{j=d+1}^{\infty} \sum_{|\alpha|+|\beta|=j} c_{\alpha\beta}(f) \sum_{(i_1, i_2) \in B(d, \alpha, \beta)} \pi_{i_1}^1(b(\alpha, \mu_1, \cdot)) \pi_{i_2}^2(b(\beta, \mu_2, \cdot)), \quad (3.2)$$

22. Les références renvoient à la bibliographie du second chapitre.

23. On la doit à Jean-Paul Calvi ; elle est expliquée en détail dans le paragraphe 2.3 du chapitre 2.

24. Pour simplifier les notations ( celles du § précédent), le terme général de la suite produit de Newton se notera  $\Pi_d$  et  $f$  désignera un élément de  $F$ . Par ailleurs  $X_s$  est une partie de  $\mathbb{K}^{n_s}$  et on notera  $z_s$  la variable générique de cet espace pour  $s = 1, 2$ .

25. fait avéré pour les cas qui nous occupent.

26. On rappelle que le symbole  $\pi_d$  désigne dans nos propos un sommant de Newton d'une SN (Voir section 2 de cette introduction).

où  $B(d, \alpha, \beta)$  se définit comme suit :

$$B(d, \alpha, \beta) = \{(i_1, i_2) \in \mathbb{N}^2 : d + 1 \leq i_1 + i_2; i_1 \leq |\alpha|; i_2 \leq |\beta|\}. \quad (3.3)$$

Ensuite le théorème de Banach-Steinhaus, appliquée aux sommants de Newton des SN diviseurs, nous fournit-via une inégalité triangulaire- une majoration de la norme de la convergence uniforme (sur un compact de  $X$  donné). Il reste enfin (et cette partie dépend des propriétés des éléments de  $F$ ) à exploiter cette majoration pour qu'elle nous donne le résultat escompté. On pourrait observer que cette méthode se reconduit avec des hypothèses beaucoup moins restrictives sur les suites de Newton diviseurs (i.e que leurs convergences ne sont en fait nécessaire que sur un Fréchet, contenant les polynômes et inclus dans les espaces fonctionnels des diviseurs).

### 3.3. Les résultats

On a appliqué la stratégie précédente à trois types de fonctions.

- (1) Les fonctions holomorphes sur un compact régulier.
- (2) Des fonctions régulières sur un compact pour lequel les inégalités de Bernstein/Markov et de Jackson sont valables.

Les preuves des théorèmes fondamentaux (Th. 3.1 et Th 5.3 ) s'appuient sur les inégalités précédentes et des outils de la théorie du pluripotentiel. Le premier de ces résultats est optimal pour le second la classe de fonction à laquelle il s'applique permet d'en espérer des améliorations notables (mais par une méthode différente car il nous semble avoir tiré le meilleur parti de celle que nous avons utilisée).

Enfin nous avons obtenu (théorèmes 4.7, 4.11 et 4.14), à partir de la convergence de suites de Newton sur des espaces de fonctions entières d'ordre et de type donnés, la convergence de la suite produit de Newton pour un espace similaire; ceci nous permet d'obtenir un procédé d'approximation polynomial (un peu dans la même veine que [1](page 222), plus général ici mais s'appliquant à un cadre plus restreint)des fonctions entières plurivariables et de croissance déterminée. Là encore rentrons davantage dans les détails : Il est bien connu que, contrairement à l'ordre d'une fonction entière, son type dépend de la norme utilisée. Pour mettre à profit le développement de Taylor d'une telle fonction afin d'utiliser la stratégie déployée plus haut, nous considérons des normes<sup>27</sup>(idée suggérée par la lecture de Ronkin [25])sur les espaces  $\mathbb{C}^m$  dont la boule unité reste invariante sous l'action naturelle du groupe  $G^m$ ,  $G$  désignant celui des rotations complexes fixant 0; ce choix restreint bien sûr la portée du résultat qui va suivre mais on notera que les normes usuelles possèdent toutes cette propriété. Comme plus haut, nous travaillons avec  $\mathbb{C}^n = \mathbb{C}^{n_1} \times \mathbb{C}^{n_2}$ ,  $n_s \geq 1$  et  $N_s$  représente une norme PC sur  $\mathbb{C}^{n_s}$ . Par ailleurs en notant, pour  $\nu > 0$ ,  $B > 0$  et  $m$  entier naturel non nul, par  $E_\nu^m(B, M)$ ,  $M$  étant une norme sur  $\mathbb{C}^m$ , l'ensemble des fonctions entières de  $\mathbb{C}^m$ , d'ordre inférieur ou égal à  $\nu$  et de type inférieur ou égal à  $B$ <sup>28</sup> que l'on munit de sa topologie classique (Cf lemme 4.4 du chapitre 2 ou Gruman-Lelong voire Martineau) et en se donnant  $w > 0$  et  $A_s > 0$  pour  $s = 1, 2$  ainsi que

$$E_s = E_\omega^{n_s}(A_s, N_s) \subset \mathcal{H}(\mathbb{C}^{n_s}), \quad s = 1, 2.$$

Nous obtenons par exemple pour  $\omega \leq 1$  :

**Theorem 3.1.** Soient  $\mathcal{N}^s = (\Pi_0^s, \Pi_1^s, \dots)$  des suites de Newton convergentes dans  $E_s$ ,  $s = 1, 2$  et  $a_s > 0$  pour  $s = 1, 2$  et en définissant une norme  $N$  sur  $\mathbb{C}^n$  en posant :

$$N(z^1, z^2) = a_1 N_1(z^1) + a_2 N_2(z^2), \quad z^s \in \mathbb{C}^{n_s}, \quad s = 1, 2. \quad (3.4)$$

27. Nommées PC pour polycylindriques.

28. Ce relativement à la norme  $M$ .



$\mathcal{N}^1 \otimes_{\mathbb{N}} \mathcal{N}^2$  converge sur  $E_{\omega}^n(A, N)$  dès que

$$A < \min \left( \frac{A_1}{a_1^{\omega}}, \frac{A_2}{a_2^{\omega}} \right). \quad (3.5)$$

Une application arithmétique envisageable (partiellement rédigée et hors thèse sous sa forme actuelle) pourrait être une autre démonstration du théorème de Baker qui généralise (au cas plusieurs variables) le classique résultat de Polya concernant la caractérisation des fonctions entières de type strictement inférieur à  $\ln 2$  envoyant  $\mathbb{N}$  dans  $\mathbb{Z}$ .

#### 4. GÉOMÉTRIE ET COMBINATOIRE DES STRUCTURES DE LAGRANGE À PLUSIEURS VARIABLES

On revient ici sur des questions abordées en 3.5.2 du chapitre 1 en y donnant des preuves et une recontextualisation. Partant d'un ensemble unisolvant (plan pour commencer)  $A$ , on cherche à construire et à dénombrer (autant que faire ce peut) les structures lagrangiennes dont  $L[A]$  est le dernier facteur<sup>29</sup>. Pour le cas où  $A$  est de degré 2, le théorème 1 de ce chapitre donne - en fonction de la configuration géométrique des points de  $A$ - une réponse complète à ces questions et montre, par exemple, que le nombre minimal de SN associées à un ensemble unisolvant plan est 48 et le nombre maximal 60 ; si on suppose que  $A$  est, de plus, un ensemble de Biermann le nombre de SN associées à  $A$  qui sont aussi des produits de Newton est 36. On voit donc que celles-ci possèdent bien une certaine épaisseur statistique dans ce cadre là tout au moins. Le cas du degré 3 pourrait aussi se traiter (et nous l'avons fait mais cette digression ne figure pas dans ce chapitre mais se trouve exposée sans démonstration dans le théorème 3.11 du premier chapitre ) géométriquement et conduirait, faute de calcul exact, à une minoration (576 pour être précis) du nombre de SN associées en revanche nous proposons ici un algorithme (reposant sur la géométrie des cubiques unicursales) permettant de construire un ensemble unisolvant d'ordre 3 à partir d'un d'ordre 2. Il est à noter que cette technique pourrait s'adapter à des ordres plus généraux. Dans cette optique (et en ne nous contenant plus au plan) le théorème 2 de ce chapitre propose une minoration (certainement très perfectible<sup>30</sup>) du nombre de SN associées à un ensemble unisolvant arbitraire, explicitement (en dimension  $n$  et pour le degré  $d$ ) nous obtenons (algébriquement) comme minorant :

$$\frac{(n+1)n^{d-1}}{n!^{n-1}} \prod_{k=1}^{d-1} \left( \frac{n+k}{k} \right)^{d-k}.$$

Ce qui montre, en particulier, que la minoration, donnée par le théorème 3.11 du premier chapitre, pour  $n = 2, d = 3$  n'était pas optimale.

#### 5. RÉSULTATS ANNEXES ET PERSPECTIVES

La première sous-section expose un travail achevé mais non publié et ne figurant pas dans notre thèse. Elle ancre le concept de SN dans une application concrète et typique d'analyse numérique. Les deux autres, encore en gestation, donnent deux applications (éventuelles) de la notion de produit de Newton.

29. On dira plus simplement SN associée à  $A$

30. Comme le montre un calcul direct du nombre de SN associées qui sont aussi produit de Newton dans le cas plan et pour un ensemble de Biermann

### 5.1. Autour d'une structure de Newton plane de degré 5 (cf 2.5.1)

On trouve dans l'article de Ciarlet et Raviart (cf. 2.5.1 pour la référence) un exemple de conditions d'interpolation de degré 5 dans le plan lié aux éléments finis. Les points d'interpolation sont les sommets  $a_1, a_2, a_3$  d'un triangle et les milieux des côtés. Les conditions d'interpolation aux sommets sont celles de Taylor à l'ordre 2 (donc 6 conditions par sommet) et en chaque milieu une condition faisant intervenir la dérivée normale en ce point (i.e le produit scalaire du gradient avec le vecteur normal -au côté sur lequel se trouve le milieu- intérieur) donc en tout  $21 = \binom{5+2}{2}$  conditions. Les conditions en les milieux, sûrement inhérentes aux éléments finis<sup>31</sup>, ne correspondent pas à notre contexte; nous allons les modifier (penser qu'en les milieux des conditions de Lagrange suffiraient est illusoire comme le montrent des calculs élémentaires quoique pénibles). Pour cela on considère  $\omega$  de sorte que  $(a_1, a_2, a_3, \omega)$  soit un parallélogramme et aux conditions maintenues sur  $a_1, a_2, a_3$ , on ajoute en  $\omega$  les conditions de Taylor d'ordre 1<sup>32</sup>. En résumé on dispose d'un espace vectoriel  $C_5$ , engendré par 21 fonctionnelles de  $E = C^\infty(\Omega)$ ,  $\Omega$  ouvert de  $\mathbb{R}^2$  contenant le parallélogramme ou les sommets de celui-ci seulement. Ces conditions d'interpolation (au bon nombre de  $21 = \dim(\mathcal{P}_5(\mathbb{R}^2))$ ) suffisent-elles à déterminer un et un seul polynôme de  $\mathcal{P}_5(\mathbb{R}^2)$ ? Telle est en substance la question que se pose dans ce contexte le numéricien. En général il y répond de façon calculatoire (système linéaire carré d'ordre 21!) en montrant que si  $p$  est un élément de  $\mathcal{P}_5(\mathbb{R}^2)$  vérifiant  $\phi(p) = 0$ , pour tout  $\phi$  de  $C_5$  alors  $p = 0$ . Avec la terminologie que nous avons adoptée, cela signifie plutôt que  $C_5$  doit être l'espace des conditions d'un projecteur plan de type Lagrange-Hermite (LH projecteur en abrégé), de degré 5. Notre démarche consiste alors à vérifier que l'on a affaire à un produit de Newton de SN unidimensionnelles de degré 5; l'avantage de cette approche est double: aucun calcul et l'obtention d'une SN donc d'une suite croissante d'espaces de conditions (donc de conditions d'interpolation liées au parallélogramme) aboutissant à  $C_5$ . Pour cela on commence par observer (comme on le ferait pour l'approche calculatoire) que la problématique est invariante par transformation affine pour se ramener au carré ( $a = (0, 0), b = (1, 0), c = (0, 1), d = (1, 1)$ ) ensuite on sait que les diviseurs de cette éventuelle SN ne peuvent être que du type LH dont les nœuds seront 0 et 1 avec des poids à déterminer. On trouve alors aisément que  $C_5$  est l'espace des conditions de  $H \otimes_N H$ , où la structure de Newton de degré 5 unidimensionnelle [H] est déterminée par la suite des espaces de conditions de ses facteurs:

$$\text{Vect}([0]), \text{Vect}([0], [1]), \text{Vect}([0], [1], [D(0)]), \text{Vect}([0], [1], [D(0)], [D(1)]) \text{ et}$$

$$\text{Vect}([0], [1], [D(0)], [D(1)], [D^2(0)]), \text{Vect}([0], [1], [D(0)], [D(1)], [D^2(0)], [D^2(1)]).$$

### 5.2. Développements en série de Newton généralisés pour les fonctions entières. Applications arithmétiques

Les résultats (3.4) et (3.5) du paragraphe 3.3 généralisés au produit de Newton de  $n$  suites de Newton monovariabiles convergentes (comme on en trouve dans le classique "Calcul des différences finies" de Gelfond) permettent d'obtenir pour certains types de fonctions entières des développements en série de polynômes. De ceci il semble légitime d'espérer des applications arithmétiques (Voir fin du paragraphe 3.3) telles que Gelfond en donne pour les fonctions entières d'une variable.

31. Ce type d'élément fini semble être celui d'Argyris chez les spécialistes.

32. Ce qui correspond presque à un type d'élément fini répertorié.

### 5.3. Généralisation des séries de Lidstone

Nous avons défini un projecteur polynomial monovariante (de degré quelconque) sur  $E = C^\infty(\mathbb{R}, \mathbb{K})$ , à partir d'un opérateur de composition (i.e qui commute avec tous les opérateurs de translation de  $E$ ) sur  $E$  arbitraire. En utilisant des résultats de convergence consignés, par exemple, dans les ouvrages classiques de Whittaker ou Boas sur les développements (pour une variable) en séries de polynômes et la stratégie présentée dans la section 3 de cette introduction, on peut là aussi envisager des généralisations possibles (comme celle des polynômes de Bernoulli) au cas plurivariable.

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# PARTIE 1.

We propose a new way of combining two polynomial projectors on spaces of functions of few variables to obtain a polynomial projector on a space of functions of many variables. We present various algebraic properties of our construction and study the main approximation properties of the new projectors.

*Article co-écrit avec Jean-Paul Calvi, publié à l'International Journal of Mathematics.*

## THE NEWTON PRODUCT OF POLYNOMIAL PROJECTORS PART 1 : CONSTRUCTION AND ALGEBRAIC PROPERTIES

### 6. INTRODUCTION

We study the problem of constructing approximation polynomial projectors on spaces of functions of many variables out of projectors on spaces of functions of fewer variables. The standard way of going, say, from one variable to two variables goes through the use of a **tensor product**. Let us briefly recall such a construction by considering the simple case of (continuous) projectors  $L_i$ ,  $i = 1, 2$ , defined on the space of continuous functions on  $K = [a, b] \subset \mathbb{R}$  with values in the space  $\mathcal{P}_d(x)$  of univariate polynomials of degree at most  $d$ . Given a bivariate function  $f(x, y)$  defined on  $K^2$ , taking  $h = f(x, \cdot)$ , we may compute  $L_2(h)(y)$  which is a polynomial of degree at most  $d$  in  $y$  with continuous coefficients  $c_j(x)$  defined on  $K$ ,  $L_2(f(x, \cdot))(y) = \sum_{j=0}^d c_j(x)y^j$ , and we may apply  $L_1$  to the coefficient functions  $c_j$ . The tensor product  $L_1 \otimes L_2$  is then defined by  $(L_1 \otimes L_2)(f)(x, y) = \sum_{j=0}^d L_1(c_j)(x)y^j$ . If  $f(x, y) = h(x)g(y)$  we have  $(L_1 \otimes L_2)(f) = L_1(h)L_2(g)$  which explains the use of the term *tensor product*.

The above construction can be extended to form the tensor product of any two projectors (provided that the first projector is well defined on the coefficients functions furnished by the second one). Tensor products are certainly the main multivariate approximation tools currently used in practical multivariate polynomial approximation. Yet, the procedure suffers from several drawbacks. First, the resulting projectors usually can hardly be extended to spaces of functions defined on something else than a Cartesian product. Next, the approximation polynomials furnished by a tensor product of two projectors yielding approximation polynomials of degree  $d$  does not return a polynomial in  $\mathcal{P}_d(x, y)$ , that is, of total degree  $d$  in the variable  $(x, y)$  but rather a polynomial on  $\mathcal{P}_d(x) \otimes \mathcal{P}_d(y) \subset \mathcal{P}_{2d}(x, y)$ , that is, of degree at most  $d$  in each (set of) variables. More seriously, it is not possible to recognize some fundamental multivariate projectors, such as, for instance, the Taylor projectors, see below, as tensor products. The main

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purpose of this paper is to propose a new way of combining two polynomial projectors, to obtain what we call their **Newton product** projector, which may be regarded as a modification of the classical construction that uses further properties of the projectors, namely, a suitable gradation of the interpolation conditions defining the projectors, which we will call a **Newton structure**, see below. Such gradation in turn relies on the natural gradation of  $\mathcal{P}_d(x)$  by the subspaces  $\mathcal{P}_j(x)$ ,  $j = 0, \dots, d$ . In Table 3 we collect a few relations which will be proved in the paper and which illustrate the way some fundamental multivariate projectors can be seen as Newton products of lower dimensional projectors.

First projector $\Pi_1$	Second projector $\Pi_2$	<b>Newton product</b> $\Pi_1 \otimes_N \Pi_2$
<b>Taylor projector</b> at $a \in \mathbb{K}^n$ to the order $d$	<b>Taylor projector</b> at $b \in \mathbb{K}^m$ to the order $d$	<b>Taylor projector</b> at $(a, b) \in \mathbb{K}^{n+m}$ to the order $d$
$L^2(\mu)$ - <b>Orthogonal projection</b> on $\mathcal{P}_d(x)$ where $\mu$ is a positive Borel measure supported on a compact set $K \subset \mathbb{K}^n$	$L^2(\nu)$ - <b>Orthogonal projection</b> on $\mathcal{P}_d(y)$ where $\nu$ is a positive Borel measure supported on a compact set $W \subset \mathbb{K}^m$	$L^2(\mu \otimes \nu)$ - <b>Orthogonal projection</b> on $\mathcal{P}_d(x, y)$ .
<b>Lagrange interpolation</b> in $\mathcal{P}_d(x)$ at $X = \{x_j, j = 0, \dots, d\} \subset \mathbb{R}$	<b>Lagrange interpolation</b> in $\mathcal{P}_d(y)$ at $Y = \{y_j, j = 0, \dots, d\} \subset \mathbb{R}$	<b>Lagrange interpolation</b> in $\mathcal{P}_d(x, y)$ at $\{(x_i, y_j), i + j \leq d\} \subset \mathbb{R}^2$

TABLE 3. Examples of Newton products of a few fundamental projectors.

## Content

We first provide the required background on polynomial projectors and introduce the notion of Newton structure giving various examples. Next, we define the Newton product of two polynomial projectors endowed with a Newton structure and derive its main algebraic properties. This work uses little more than elementary linear algebra. We will prove in a second work some forms of permanence theorems which show how a Newton product inherits approximations properties from its factors.

It is very likely that the theory could be extended to other kind of projectors (that is, projectors on other spaces of polynomials or on other classical spaces of functions). The whole paper can be seen as a generalization of the intertwining product for multivariate Lagrange interpolation introduced in [20]. Other ways of combining projectors have been considered. We may for instance mention the work of Gordon in [27].

We briefly collect the main general notation used in the whole paper.

## Notation

*Polynomials.* The letter  $\mathbb{K}$  will denote either  $\mathbb{R}$  or  $\mathbb{C}$ ,  $\mathcal{P}_d(\mathbb{K}^n)$  is the space of polynomials of  $n$  variables in  $\mathbb{K}$  and of (total) degree at most  $d$ . Very often, as is done in this introduction, we will write instead  $\mathcal{P}_d(\mathbb{K}^n) = \mathcal{P}_d(x)$ ,  $x = (x_1, \dots, x_n)$ , the ground field being usually clear from the context. We will freely use standard multi-index notation. For instance  $|\alpha|$  denotes the length of the  $n$ -index  $(\alpha_1, \dots, \alpha_n)$  and  $D^\alpha = \partial^{|\alpha|} / (\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n})$  the usual differential operator. We denote by  $\mathbb{N}_d(n)$  the set of multi-indices of length not greater than  $d$ . Its cardinality is  $N_d(n) = \binom{n+d}{d}$  and this is also the dimension of the  $\mathbb{K}$ -vector space  $\mathcal{P}_d(\mathbb{K}^n)$ . For  $\alpha \in \mathbb{N}^n$ , we set  $e_\alpha : x \mapsto x^\alpha$  so that  $(e_\alpha : \alpha \in \mathbb{N}_d(n))$  is the usual monomial basis for  $\mathcal{P}_d(\mathbb{K}^n)$ . Unless otherwise specified, this basis is ordered according to the graded lexicographic order  $\text{grlex}$

$(\prec, \preceq)$  for which  $\alpha \prec \beta$  if  $|\alpha| < |\beta|$  or  $|\alpha| = |\beta|$  and the leftmost nonzero entry of  $\alpha - \beta$  is negative. Thus, for instance,  $(0, 2) \prec (1, 1) \prec (2, 0)$ .

*Spaces of functions.* We shall (mainly) work with two classes of spaces of functions (containing the polynomials) which are of particular interest as far as we are concerned with polynomial approximation. Most of our constructions can be done in much more general spaces. The minimal assumptions will be indicated in the text.

(A) The Fréchet space  $\mathcal{C}^s(\Omega)$ ,  $s \in \mathbb{N} \cup \{\infty\}$  of (real valued) functions which are  $s$ -times continuously differentiable on the open subset  $\Omega$  of  $\mathbb{R}^n$  and, for  $K$  compact,  $\mathcal{C}^s(K) = \cap \mathcal{C}^s(\Omega)$  where  $\Omega$  runs over all open neighborhood of  $K$ .

(B) The Fréchet space  $\mathcal{H}(\Omega)$  of holomorphic functions on the Runge domain  $\Omega$  in  $\mathbb{C}^n$  and  $\mathcal{H}(K)$  the space of functions holomorphic on a neighborhood of  $K$ ,  $K$  a polynomially convex compact set in  $\mathbb{C}^n$ . We refer to [44, VI.1.2] for material on Runge domain and polynomially convex compact sets in  $\mathbb{C}^n$ . The reader not familiar with complex analysis may assume that both  $\Omega$  and  $K$  are ordinary convex sets.

(C) The above space are endowed with their usual topology. For instance,  $\mathcal{H}(\Omega)$  is endowed with the topology of uniform convergence on all compact subsets of  $\Omega$ , and  $\mathcal{H}(K)$  is endowed with the inductive limit topology of the  $\mathcal{H}(\Omega)$  when  $\Omega$  runs over all open neighborhood of  $K$ . In particular, if  $F$  is any locally convex topological vector space, a linear map from  $\mathcal{H}(K)$  into  $F$  is continuous if and only if all its restrictions to the spaces  $\mathcal{H}(\Omega)$ ,  $K \subset \Omega$ , are continuous.

(D) Since the algebraic foundation is identical for all spaces above (and many others), they will only marginally appear in this first part and we will generally use the generic letter  $\mathcal{E}$  to denote any of these spaces. When mentioned, the field  $\mathbb{K}$  and the integer  $n$  are always related to  $\mathcal{E}$ . Thus, for example, when  $\mathcal{E} = \mathcal{C}^\infty(\Omega)$  with  $\Omega \subset \mathbb{R}^2$ , we have  $\mathbb{K} = \mathbb{R}$  and  $n = 2$ . It is sometimes desirable to make apparent the integer  $n$  and the subset  $X$  of  $\mathbb{K}^n$  used in the definition of  $\mathcal{E}$ . In that case, we write  $\mathcal{E}_n(X)$ .

(E) The algebraic dual of a vector space  $\mathcal{W}$  is denoted by  $\mathcal{W}^*$  whereas the topological dual of  $\mathcal{E}$  is denoted by  $\mathcal{E}'$ . An element of  $\mathcal{E}'$  is called a **functional** or  **$\mathcal{E}$ -functional**.

## 7. POLYNOMIAL PROJECTORS AND NEWTON STRUCTURES

### 7.1. The algebra of polynomial projectors

#### 7.1.1. Polynomial projectors and space of interpolation conditions.

**Definition 7.1.** A **polynomial projector** of degree  $d$  on a space  $\mathcal{E}$  containing  $\mathcal{P}_d(\mathbb{K}^n)$  is a continuous linear map  $\Pi : f \in \mathcal{E} \longrightarrow \Pi(f) \in \mathcal{P}_d(\mathbb{K}^n)$  which coincides with the identity on its range, that is, such that  $\Pi(p) = p$  for every  $p \in \mathcal{P}_d(\mathbb{K}^n)$ .

In particular, a polynomial projector  $\Pi$  is onto and  $\Pi^2 = \Pi$ . Here  $\Pi^2 = \Pi \circ \Pi$ .

Given a basis  $B = (p_\alpha : \alpha \in \mathbb{N}_d(n))$  for  $\mathcal{P}_d(\mathbb{K}^n)$ , there exist (unique) functional  $\mu_\alpha$  in  $\mathcal{E}'$ ,  $\alpha \in \mathbb{N}_d(n)$ , such that

$$\Pi(f) = \sum_{|\alpha| \leq d} \mu_\alpha(f) p_\alpha, \quad f \in \mathcal{E}. \quad (7.1)$$

From  $\Pi^2(f) = \Pi(f)$  we get  $\mu_\alpha(f) = \mu_\alpha(\Pi(f))$  and from  $\Pi(p_\beta) = p_\beta$ ,

$$\mu_\alpha(p_\beta) = \delta_{\alpha\beta}, \quad \alpha, \beta \in \mathbb{N}_d(n), \quad (7.2)$$

where  $\delta_{\alpha\beta}$  is the usual Kronecker delta. More generally, we will use the following definition.

**Definition 7.2.** An **interpolation condition** for  $\Pi$  is a functional  $\mu \in \mathcal{E}'$  such that  $\mu(\Pi(f)) = \mu(f)$  for any  $f \in \mathcal{E}$ . The set of all interpolation conditions is a subspace of  $\mathcal{E}'$  called the **space of interpolation conditions** of  $\Pi$  and is denoted by  $\text{CND}(\Pi)$ .

Thus, we have

$$\mu \in \text{CND}(\Pi) \iff \mu(f) = \mu(\Pi(f)), \quad f \in \mathcal{E} \quad (7.3)$$

$$\iff \mu \circ (\mathbf{I} - \Pi) = 0 \quad (\text{on } \mathcal{E}), \quad (7.4)$$

where  $\mathbf{I}$  denotes the identity mapping on  $\mathcal{E}$ .

The remark above shows that each  $\mu_\alpha$  in (7.1) is an element of  $\text{CND}(\Pi)$ . Conversely, any  $\mu \in \text{CND}(\Pi)$  is a linear combination of the  $\mu_\alpha$ . Indeed, applying  $\mu$  on both sides of (7.1), taking into account that  $\mu(\Pi(f)) = \mu(f)$  we get  $\mu = \sum_{|\alpha| \leq d} c_\alpha \mu_\alpha$  with  $c_\alpha = \mu(p_\alpha)$ . From (7.2), we further get that the  $\mu_\alpha$  are linearly independent. We therefore have the following lemma.

**Lemma 7.3.**  $\text{CND}(\Pi)$  is a  $N_d(n)$ -dimensional subspace of  $\mathcal{E}'$  and  $\text{CND}(\Pi) = \text{span} \{ \mu_\alpha : |\alpha| \leq d \}$ .

If  $f \in \mathcal{E}$  and  $q \in \mathcal{P}_d(\mathbb{K}^n)$ , we have  $q = \Pi(f)$  if and only if  $\mu(q) = \mu(f)$  for every  $\mu \in \text{CND}(\Pi)$ . A polynomial projector is therefore uniquely determined by its space of interpolation conditions.

Observe also that the restrictions of the  $\mu_\alpha$  to  $\mathcal{P}_d(\mathbb{K}^n)$  form the dual basis of the  $p_\alpha$  in  $\mathcal{P}_d^*(\mathbb{K}^n)$ . In particular,

$$\text{CND}(\Pi)|_{\mathcal{P}_d(\mathbb{K}^n)} = \mathcal{P}_d^*(\mathbb{K}^n).$$

Conversely, a  $N_d(n)$ -dimensional subspace  $V$  of  $\mathcal{E}'$  is the space of interpolation conditions of some projector  $\Pi$  if and only

$$V|_{\mathcal{P}_d(\mathbb{K}^n)} = \mathcal{P}_d^*(\mathbb{K}^n). \quad (7.5)$$

In fact, if  $(v_\alpha : \alpha \in \mathbb{N}_d(n))$  is a basis of  $V$ , then the projector  $\Pi$  is defined by

$$\Pi(f) = \sum_{|\alpha| \leq d} v_\alpha(f) q_\alpha,$$

where  $(q_\alpha : \alpha \in \mathbb{N}_d(n))$  is the basis of  $\mathcal{P}_d(\mathbb{K}^n)$  which is dual to  $(v|_{\mathcal{P}_d(\mathbb{K}^n)} : \alpha \in \mathbb{N}_d(n))$ . The existence of such basis being implied by (7.5).

Here is another useful description of the space of interpolation conditions.

**Lemma 7.4.** *We have*

$$\text{CND}(\Pi) = \mathcal{E}' \circ \Pi,$$

where  $\mathcal{E}' \circ \Pi = \{ \mu \circ \Pi, : \mu \in \mathcal{E}' \}$ .

*Démonstration.* Note that, because of the continuity of  $\Pi$ , the elements of  $\mathcal{E}' \circ \Pi$  are functionals. Let  $\mu \circ \Pi \in \mathcal{E}' \circ \Pi$ , for every  $f \in \mathcal{E}$ , since  $\Pi^2 = \Pi$ , we have  $(\mu \circ \Pi)(\Pi(f)) = (\mu \circ \Pi^2)(f) = (\mu \circ \Pi)(f)$  which means that  $\mu \circ \Pi \in \text{CND}(\Pi)$ . The converse is obvious since  $\mu = \mu \circ \Pi$  for all  $\mu \in \text{CND}(\Pi)$ .  $\square$

**7.1.2. Characterization of a space of interpolation conditions.** The following lemma collects some useful (classical) criteria for deriving equality (7.5). We omit the proof which follows from basic principles of linear algebra.

**Lemma 7.5.** *Let  $V$  a subspace of dimension  $N_d(n)$  of  $\mathcal{E}'$  and  $M = (\mu_\alpha : \alpha \in \mathbb{N}_d(n))$  a basis of  $V$ . The following assertions are equivalent.*

(1) *There exists a polynomial projector  $\Pi$  of degree  $d$  on  $\mathcal{E}$  such that  $\text{CND}(\Pi) = V$ .*

(2) *The linear map  $\Phi_\mu : q \in \mathcal{P}_d(\mathbb{K}^n) \longrightarrow (\mu_\alpha(q) : \alpha \in \mathbb{N}_d(n)) \in \mathbb{K}^{N_d(n)}$  is an isomorphism.*

(3) There exist polynomials  $p_\beta \in \mathcal{P}_d(\mathbb{K}^n)$ ,  $\beta \in \mathbb{N}_d(n)$ , such that

$$\mu_\alpha(p_\beta) = \delta_{\alpha\beta}, \quad (\alpha, \beta) \in \mathbb{N}_d(n) \times \mathbb{N}_d(n). \quad (7.6)$$

(4) For some (and hence for every) basis  $B = (b_\beta : \beta \in \mathbb{N}_d(n))$  of  $\mathcal{P}_d(\mathbb{K}^n)$  the  $N_d(n) \times N_d(n)$  matrix  $\mathbf{vdm}(M, B) = (\mu_\alpha(b_\beta))$  is invertible; that is

$$\mathbf{vdm}(M, B) \neq 0 \quad \text{where} \quad \mathbf{vdm}(M, B) = \det(\mathbf{vdm}(M, B)) = \det(\mu_\alpha(b_\beta)). \quad (7.7)$$

Of course, given, say  $M$  and  $B$  satisfying Condition (3) in the above lemma, the corresponding projector  $\Pi$ , which we will denote  $\Pi^V$  or  $\Pi^M$ , is defined by  $\Pi(f) = \sum_{|\alpha| \leq d} \mu_\alpha(f) p_\alpha$

7.1.3. *Vandermondians.* Lemma 7.5 leads to two important definitions.

**Definition 7.6.** The matrix  $\mathbf{vdm}(M, B)$  is called the (generalized) **Vandermonde matrix** (and its determinant  $\mathbf{vdm}(M, B)$  the **vandermondian**) attached to the pair of bases  $\{M, B\}$ .

This terminology is justified by the fact that when  $\mu_\alpha$  is a Dirac point evaluation functional, that is  $\mu_\alpha(f) = f(a_\alpha)$  for some  $a_\alpha \in \mathbb{K}^n$ , and  $p_\alpha = e_\alpha$  then  $\mathbf{vdm}(\mu, p)$  is but the ordinary Vandermondian from Lagrange interpolation theory, see 7.2.1 below.

Note that if

$$M' = (\mu'_\alpha : \alpha \in \mathbb{N}_d(n)) = QM, \quad \text{that is,} \quad \mu'_\alpha = \sum_{|\beta| \leq d} Q_{\alpha\beta} \mu_\beta,$$

$$\text{and} \quad B' = (b'_\alpha : \alpha \in \mathbb{N}_d(n)) = RB, \quad \text{that is,} \quad b'_\alpha = \sum_{|\beta| \leq d} R_{\alpha\beta} b_\beta,$$

where  $Q$  and  $R$  are invertible then

$$\mathbf{vdm}(M', B') = Q \cdot \mathbf{vdm}(M, B) \cdot R^T, \quad (7.8)$$

where  $R^T$  denotes the transpose of  $R$ .

**Definition 7.7.** A pair  $\{M, B\}$  where  $M = (\mu_\alpha : \alpha \in \mathbb{N}_d(n))$  and  $B = (p_\alpha : \alpha \in \mathbb{N}_d(n))$  satisfy (7.6) is called a **bi-orthogonal system** for  $\Pi$ .

Thus,  $\{M, B\}$  is a bi-orthogonal system if and only if  $\mathbf{vdm}(M, B) = \mathbf{I}$ . Given  $M$  (resp.  $B$ ), we can construct  $B'$  (resp.  $M'$ ) such that  $\{M, B'\}$  (resp.  $\{M', B\}$ ) is a bi-orthogonal system. Note however that, in practice, the construction of such new basis is a hard computational problem as it requires the (exact) inversion of a large matrix.

**Lemma 7.8.** Let  $\Pi$  be a projector of degree  $d$  on  $\mathcal{E}$ . Let  $M$  be a basis of  $\text{CND}(\Pi)$  and  $B$  a basis of  $\mathcal{P}_d(\mathbb{K}^n)$ . Let  $Q = \mathbf{vdm}(M, B)^{-1}$ . If  $M' = QM$  and  $B' = Q^T B$  then  $\{M', B\}$  and  $\{M, B'\}$  are bi-orthogonal systems.

*Démonstration.* In view of (7.8),  $\mathbf{vdm}(M', B) = Q \cdot \mathbf{vdm}(M, B) = \mathbf{I}$  and  $\mathbf{vdm}(M, B') = \mathbf{vdm}(M, B) \cdot (Q^T)^T = \mathbf{I}$ .  $\square$

7.1.4. *Image of a projector by an affine automorphism.* Given an affine automorphism  $\mathcal{A}$  of  $\mathbb{K}^n$ , we may define for any  $\mu \in \mathcal{E}'(X)$ , an element  $\mathcal{A} \star \mu \in \mathcal{E}'(\mathcal{A}(X))$  by

$$(\mathcal{A} \star \mu)(f) = \mu(f \circ \mathcal{A}), \quad f \in \mathcal{E}(\mathcal{A}(X)).$$

Likewise, if  $\Pi$  is a polynomial projector of degree  $d$  on  $\mathcal{E}(X)$  then  $\mathcal{A} \star \Pi$  is a polynomial projector of degree  $d$  on  $\mathcal{E}(\mathcal{A}(X))$  where

$$(\mathcal{A} \star \Pi)(f) = \Pi(f \circ \mathcal{A}) \circ \mathcal{A}^{-1}, \quad f \in \mathcal{E}(\mathcal{A}(X)), \quad (7.9)$$

and one immediately checks that

$$\text{CND}(\mathcal{A} \star \Pi) = \mathcal{A} \star \text{CND}(\Pi) = \{\mathcal{A} \star \mu : \mu \in \text{CND}(\Pi)\}. \quad (7.10)$$



The projector  $\mathcal{A} \star \Pi$  is the **image** of  $\Pi$  by  $\mathcal{A}$ .

A space  $V$  satisfies any of the conditions in Lemma 7.5 if and only if  $\mathcal{A} \star V$  satisfies a similar condition and

$$\mathcal{A} \star \Pi^V = \Pi^{\mathcal{A} \star V}.$$

This observation may sometimes simplify the verification of such conditions. An example is given in subsection 7.2.2 below.

We now indicate a few examples that will serve to illustrate all the results in this paper.

## 7.2. Examples

**7.2.1. Lagrange interpolation.** Let  $A = \{a_\alpha : \alpha \in \{N_d(n)\} \subset \mathbb{K}^n$  a set of  $N_d(n)$  pairwise distinct points not lying on an algebraic hypersurface  $\{p = 0\}$  in  $\mathbb{K}^n$  with  $\deg(p) \leq d$ , or equivalently such that  $\text{vdm}(A) = \det(e_\alpha(a_\beta)) \neq 0$ .  $A$  is said to be **unisolvant** of degree  $d$ . For every function  $f$  defined on  $A$  there exists a unique polynomial  $p \in \mathcal{P}_d(\mathbb{K}^n)$  such that  $p = f$  on  $A$ . This polynomial is the **Lagrange interpolation polynomial** of  $f$  at  $A$ . We will denote it by  $\mathbf{L}[A, f]$ . The **Lagrange interpolation formula** is

$$\mathbf{L}[A, f](x) = \sum_{|\alpha| \leq d} f(a_\alpha) \ell_\alpha(x), \quad \ell_\alpha(x) = \text{vdm}(A : a_\alpha \leftarrow x) / \text{vdm}(A), \quad (7.11)$$

where  $\text{vdm}(A : a_\alpha \leftarrow x)$  means that we substitute  $x$  for  $a_\alpha$  in the determinant. The polynomials  $\ell_\alpha$  are the **fundamental Lagrange interpolation polynomials** (FLIP) and  $\ell = (\ell_\alpha : \alpha \in \mathbb{N}_d(n))$  is the **Lagrange basis**. The map

$$\mathbf{L}[A] : f \in \mathcal{E} \mapsto \mathbf{L}[A, f] \in \mathcal{P}_d(\mathbb{K}^n)$$

is a projector for all spaces  $\mathcal{E} = \mathcal{C}^s(\Omega)$ ,  $\mathcal{C}^s(K)$ ,  $\mathcal{H}(\Omega)$ ,  $\mathcal{H}(K)$  provided that  $A \subset \Omega$  (or  $A \subset K$ ). This follows from the continuity of the Dirac functional  $[a_\alpha] : f \mapsto f(a_\alpha)$  in each of the previous spaces. In fact,

$$\text{CND}(\mathbf{L}[A]) = \text{span}\{[a_\alpha] : \alpha \in \mathbb{N}_d(n)\}. \quad (7.12)$$

Of course,  $\text{vdm}(A)$  corresponds to  $\text{vdm}(M, B)$  in (7.7) when  $M$  is formed of the Dirac functional  $[a_\alpha]$  and  $B$  is the basis of monomials. Note that  $\{M, \ell\}$  is a bi-orthogonal system. Relatively little is known about the approximation properties of multivariate Lagrange interpolation projectors. We refer the reader to [11] and the references therein.

**7.2.2. Lagrange-Hermite interpolation.** In Lagrange-Hermite interpolation, the interpolation space may contain derivatives of Dirac functionals. If

$$S = \{(a_i, S_i) : i = 1, \dots, k\}, \quad \sum_{i=1}^k \#S_i = \binom{n+d}{d},$$

where  $S_i$  is a **directed subset** of  $\mathbb{N}_d(n)$ , that is,

$$(\beta \in S_i \text{ and } \alpha_k \leq \beta_k, \quad k = 1, \dots, n) \implies \alpha \in S_i, \quad (7.13)$$

then the **Lagrange-Hermite interpolation LH[S] projector** is defined by

$$\text{CND}(\mathbf{LH}[S]) = \text{span}\{D^\alpha[a_i] : \alpha \in S_i, i = 1, \dots, k\}, \quad \text{where } D^\alpha[a_i] : f \mapsto D^\alpha f(a_i). \quad (7.14)$$

The projector is well defined provided that the corresponding Vandermonian is not null. The continuity holds as in the Lagrange interpolation case except that, of course, when  $\mathcal{E} = \mathcal{C}^s(\Omega)$  or  $\mathcal{C}^s(K)$  we must assume  $s \geq \max\{|\alpha|, \alpha \in S_i, i = 1, \dots, k\}$ .

When  $k = 1$ , we obtain a **Taylor projector**,

$$\mathbf{T}_a^d : f \in \mathcal{E} \mapsto \mathbf{T}_a^d(f, x) = \sum_{k=0}^d \sum_{|\alpha|=k} \binom{k}{\alpha} D^\alpha f(a) \frac{(x-a)^\alpha}{k!} \in \mathcal{P}_d(\mathbb{K}^n). \quad (7.15)$$

The condition that the sets  $S_i$  are directed implies the often useful algebraic property that the kernel of  $\mathbf{LH}[S]$  is an ideal, see [12]. When this condition is dropped, one rather speaks of **Birkhoff interpolation**. Many examples can be found in [35].

In the univariate case, Lagrange-Hermite projectors are limits of Lagrange projectors in which certain points coalesce. In general this is no longer true in the higher dimensional case.

**7.2.3. Orthogonal projection.** Let  $\mathbf{m}$  be a positive (Borel) measure with compact support  $C$  in  $\mathbb{K}^n$  such that  $\int |p|^2 d\mathbf{m} = 0$  implies  $p = 0$ , which is equivalent to say that  $C$  is not included in an algebraic hypersurface of degree  $\leq d$ . For  $f, g \in L^2(\mathbf{d}\mathbf{m})$ , as usual, we denote by  $\langle f, g \rangle$  the Hermitian product  $\int f \bar{g} d\mathbf{m}$  and  $\| \cdot \|_{\mathbf{m}}$  the corresponding quadratic norm. The orthogonal projection  $\mathbf{P}_{d, \mathbf{m}}$  of  $d \in \mathcal{E}$  onto  $\mathcal{P}_d(\mathbb{K}^n)$  is a polynomial projector. If

$$B = (b_\alpha = b_\alpha(\mathbf{m}, \cdot) : \alpha \in \mathbb{N}_n(d)) \quad (7.16)$$

is the orthonormal basis of  $\mathcal{P}_d(\mathbb{K}^n)$  obtained by the Gram-Schmidt process from the monomial basis, we have

$$\mathbf{P}_{d, \mathbf{m}}(f) = \sum_{|\alpha| \leq d} \langle f, b_\alpha \rangle b_\alpha, \quad f \in \mathcal{E}. \quad (7.17)$$

The assumption on the support  $C$  of  $\mathbf{m}$  ensures the existence of  $B$ . The continuity of  $\mathbf{P}_{\mathbf{m}}$  is immediate, provided that  $C \subset X$  for  $\mathcal{E} = \mathcal{E}(X)$ . We have

$$\begin{aligned} \text{CND}(\mathbf{P}_{d, \mathbf{m}}) &= \text{span} \{ \mu_\alpha : f \mapsto \langle f, b_\alpha \rangle : \alpha \in \mathbb{N}_d(n) \} \\ &= \text{span} \{ \nu_\alpha : f \mapsto \langle f, e_\alpha \rangle : \alpha \in \mathbb{N}_d(n) \}. \end{aligned}$$

Observe that the usual **Gramian determinant**  $G(d, \mathbf{m}) = \det(\langle e_\alpha, e_\beta \rangle)$  is a Vandermondian for  $\mathbf{P}_{d, \mathbf{m}}$  in the sense of (7.7). Of course, if  $M = (\mu_\alpha : \alpha \in \mathbb{N}_d(n))$  and  $P = (b_\alpha : \alpha \in \mathbb{N}_d(n))$  then  $\{M, P\}$  is a bi-orthogonal system.

Reasonable assumptions on  $\mathbf{m}$  (we will turn to that later) implies convergence of  $\mathbf{P}_{d, \mathbf{m}}(f)$  to  $f$  for all  $f$  in  $\mathcal{E}$  and most spaces  $\mathcal{E}$ . These assumptions are particularly mild in the case of  $\mathcal{E} = \mathcal{H}(K)$ . Basic (multivariate) results are established in [53] and [54].

**7.2.4. Kergin interpolation.** Let  $A = \{a_0, \dots, a_d\}$  a set of not necessarily pairwise distinct points in a convex subset  $C$  of  $\mathbb{K}^n$ . The **Kergin interpolation** projector  $\mathbf{K}_A$  is defined by the relation

$$\mathbf{K}_A(f)(x) = \sum_{k=0}^d \int_{\Delta_k} D^k f \left( \sum_{i=0}^k t_i a_i \right) (x - a_0, x - a_1, \dots, x - a_{k-1}) dm_k(t), \quad (7.18)$$

where  $D^k f$  denotes the (total) Fréchet  $k$ -th derivative of  $f$ ,  $\Delta_k$  is the standard simplex,

$$\Delta_k = \left\{ (t_0, \dots, t_k) \in [0, 1]^{k+1} : \sum_{j=0}^k t_j = 1 \right\}, \quad (7.19)$$

and  $dm_k$  is the Lebesgue measure on  $\Delta_k$ . This projector was introduced by Paul Kergin at the end of the seventies [32]. Although it is not immediately apparent from (7.18),  $\mathbf{K}_A$  does not depend on the ordering of the points  $a_i$  so that the notation  $\mathbf{K}_A$  is correct. We will sometimes write however  $\mathbf{K}_A = \mathbf{K}[a_0, \dots, a_d]$ . The Kergin projector is a natural multivariate counterpart of univariate Lagrange-Hermite interpolation. In fact, in the univariate case, (7.18) reduces to the well-known Newton form for the Lagrange interpolation polynomial (see 8.1 below) via

the Hermite-Genocchi formula for divided differences ([31, chapter 6, Th. 2]. In fact, there is a deep relation between univariate Lagrange-Hermite interpolation and Kergin interpolation : if  $f$  is a multivariate **ridge function**, that is  $f(x) = h(\langle \omega, x \rangle)$  where  $h$  is a univariate function and  $\langle \omega, x \rangle = \sum_{i=1}^n \omega_i x_i$ , the Kergin interpolation polynomials of  $f$  can be easily computed as Lagrange-Hermite interpolation polynomials of  $h$ , namely,

$$\mathbf{K}[a_0, \dots, a_d](f) = \mathbf{LH}[\langle \omega, a_0 \rangle, \dots, \langle \omega, a_d \rangle](h)(\langle \omega, x \rangle). \quad (7.20)$$

Following [24] we say that Kergin interpolation lifts Lagrange-Hermite interpolation. (A characterization of the univariate projectors that may be lifted in a similar way can be found in the same reference.)

We always have  $\mathbf{K}_A(f)(a_i) = f(a_i)$  and, more generally, if  $a_i$  is repeated  $s$  times in  $A$  then  $D^k \mathbf{K}_A(f)(a_i) = D^k f(a_i)$ ,  $k = 0, \dots, s-1$ . In particular, if all the points coincide, we obtain a Taylor projector,

$$\mathbf{K}[a, a, \dots, a] = \mathbf{T}_a^d.$$

The map  $\mathbf{K}_A$  is continuous on all spaces  $\mathcal{E}(X)$  provided that  $X$  is convex and  $A \subset X$ . In the real case, that is when  $\mathcal{E} = \mathcal{C}^s(\Omega)$  or  $\mathcal{C}^s(K)$ , one must assume, in general, that  $s \geq d$ . However, if the points  $A$  satisfies certain natural geometrical assumptions, the projector can be extended up to  $\mathcal{C}^{n-1}(\Omega)$  (where  $n$  is the dimension of the ambient space). This is shown in [36]. In the bivariate case, a simple explicit extension to  $\mathcal{C}^1(\Omega)$  can be found in [18]. In the complex case, the assumption of convexity can be replaced by the (weaker) assumption of  $\mathbb{C}$ -convexity, see [1], [2], and [3].

Observe that (when  $X$  is convex)

$$\text{CND}(\mathbf{K}_A) = \text{span} \left\{ f \mapsto \int_{\Delta_k} D^\alpha f \left( \sum_{i=0}^k t_i a_i \right) dm_k(t) : |\alpha| = k, k = 0, \dots, d \right\},$$

and the interpolation property mentioned above implies,

$$[a_i] \in \text{CND}(\mathbf{K}_A), \quad i = 0, \dots, d.$$

Results on approximation of functions in  $\mathcal{H}(K)$  by Kergin interpolation polynomials can be found in [13] and [14]. For Kergin interpolation at specific sets of points in the real case, we refer to [18] and [40].

**7.2.5. Hakopian interpolation and other  $D$ -centered Taylor projectors.** Let  $X$  be a set of  $n+d$  points ( $d \geq 1$ ) in **general position** in a convex set  $\Omega$  in  $\mathbb{K}^n$ . By general position, we mean that the points of any subset formed of  $n+1$  points of  $X$  are affinely independent (hence form an affine basis of  $\mathbb{K}^n$ ). One can show [29] that the space

$$V = \text{span} \left\{ f \mapsto \int_{\text{cv}(A)} f dm : A \in \binom{X}{n} \right\},$$

satisfies the condition of Lemma 7.5. Here  $\binom{X}{n}$  denotes the class of all subsets of  $X$  of cardinality  $n$ ,  $\text{cv}(A)$  is the convex hull of  $A$  and  $dm = dm_A$  is the Lebesgue measure on  $\text{cv}(A)$ . The corresponding projector, introduced in [29], is denoted by  $\mathbf{H}_X$  and is called a **Hakopian interpolation projector**. Again, the continuity is immediate and the observation on the use of the  $\mathbb{C}$ -convexity in the previous section is still valid.

In fact, Kergin and Hakopian interpolation projectors are extreme cases of certain **mean value interpolation projectors**, see [25]. A Hakopian projector can also be regarded as a form of derivative of Kergin projectors, see [21]. Still a more general class of projector is given by

the **D-centered Taylor projectors**. They are defined by a space of interpolation conditions of the form

$$V = \text{span} \{f \mapsto \mu_i(D^\alpha f) : |\alpha| = i, i = 0, \dots, d\}, \quad D^\alpha \mu_i = \mu_i \circ D^\alpha \in \mathcal{E}^t, \quad (7.21)$$

with the condition  $\mu_i(1) = 1, i = 0, \dots, d$ , which ensures that  $V$  satisfies the conditions of Lemma 7.5, see [19]. The corresponding projectors will be denoted by  $\mathbf{C}(\mu_0, \dots, \mu_d)$ .

In the case where  $\mu_i$  is the Dirac functional  $[a_i]$  one obtains the classical **Abel-Gontcharov projectors**,

$$\mathbf{A}(a_0, \dots, a_d) = \mathbf{C}([a_0], \dots, [a_d]). \quad (7.22)$$

Taylor projectors again are example of Abel-Gontcharov projectors.

D-centered projectors are exactly those which preserve homogeneous partial differential relations, see [21]. A projector  $\Pi$  is said to preserve homogeneous partial differential relations when, for every suitably defined function  $f$  and every homogeneous polynomial  $Q$ , the relation  $Q(D)(f) = 0$  implies  $Q(D)(\Pi(f)) = 0$  where  $Q(D) = \sum_{|\alpha|=s} c_\alpha D^\alpha$  if  $Q = \sum_{|\alpha|=s} c_\alpha e_\alpha$ .

Let us point out that many important operators in approximation theory are *not* polynomial projectors. For example, positive operators (Bernstein, Landau operators) or Hermite-Fejer operators are not projectors.

## 8. NEWTON STRUCTURES AND EXAMPLES

### 8.1. Introduction

Let us start from the classical Newton formula for univariate Lagrange interpolation which states

$$\mathbf{L}[A, f] = \sum_{i=0}^d f[a_0, \dots, a_i] (x - a_0) \dots (x - a_{i-1}), \quad A = \{a_0, \dots, a_d\}, \quad (8.1)$$

where  $f[a_0, \dots, a_i]$  denotes the divided difference of  $f$  with respect to  $a_0, \dots, a_i$ . This fundamental formula can be found in almost any textbook on numerical analysis, see e.g. [31, chapter 6]. In particular, it follows that

$$\mathbf{L}[A, f] = \mathbf{L}[A_{d-1}, f] + f[a_0, \dots, a_d] (x - a_0) \dots (x - a_{d-1}), \quad A_{d-1} = \{a_0, \dots, a_{d-1}\},$$

and

$$\text{CND}(\mathbf{L}[A]) = \text{CND}(\mathbf{L}[A_{d-1}]) \oplus \text{span} \{f \mapsto f[a_0, \dots, a_d]\}.$$

From the numerical analysis point of view, the main advantage of the Newton formula is that adding one interpolation point just requires the computation of one new divided difference. More generally, setting  $A_k = \{a_0, \dots, a_k\}$ , we have the following remarkable structural relations,

$$\mathbf{L}[A_d] = \mathbf{L}[A], \quad \text{and} \quad \mathbf{L}[A_k] \circ \mathbf{L}[A_{k+1}] = \mathbf{L}[A_k], \quad k = 0, \dots, d-1.$$

A similar structural decomposition for an arbitrary polynomial projector will be called a Newton structure. The precise definition is as follows.

### 8.2. General Newton structures

**Definition 8.1.** Let  $\Pi$  a polynomial projector of degree  $d$  on  $\mathcal{E}$ . We say that a sequence of  $d+1$  projectors  $\Pi_k$  of degree  $k, k = 0, 1, \dots, d$ , is a **Newton structure** for  $\Pi$  if

$$\Pi = \Pi_d, \quad \text{and} \quad \Pi_k \circ \Pi_{k+1} = \Pi_k, \quad 0 \leq k \leq d-1. \quad (8.2)$$

We write  $[\Pi] = (\Pi_0, \Pi_1, \dots, \Pi_d)$  and we say that  $[\Pi]$  is a **Newton-structured** polynomial projector of degree  $d$ , or simply a **N-polynomial projector**. The projectors  $\Pi_i$  are called the **(Newton) factors** of  $[\Pi]$ .

Setting

$$\pi_0 := \Pi_0, \quad \text{and} \quad \pi_i := \Pi_i - \Pi_{i-1}, \quad i = 1, \dots, d,$$

we have

$$\Pi_k = \sum_{i=0}^k \pi_i, \quad k = 0, \dots, d. \quad (8.3)$$

The maps  $\pi_i$  are called the **Newton summands** of  $[\Pi]$  and, when  $k = d$ , the expression (8.3) is called the **Newton formula** for  $[\Pi]$ . Two Newton structures for  $\Pi$  differ when one at least of the factors are different.

Notice that every  $\Pi_k$  inherits a Newton structure from  $[\Pi]$ , namely  $[\Pi_k] = (\Pi_0, \dots, \Pi_k)$ . Unless otherwise specified a factor  $\Pi_k$  of  $[\Pi]$  will be always endowed with this induced Newton structure.

Thus, for instance, in the case of univariate Lagrange interpolation as above, a natural Newton structure is given by

$$[\mathbf{L}[A]] = (\mathbf{L}[A_0], \dots, \mathbf{L}[A_d]), \quad A_i = \{a_0, a_1, \dots, a_i\}, \quad (8.4)$$

so that the Newton factors are themselves Lagrange interpolation projectors; the Newton formula is the ordinary Newton formula (8.1) and the Newton summands  $\pi_i$  are given by

$$\pi_i(f)(x) = f[a_0, \dots, a_i](x - a_0) \cdots (x - a_{i-1}).$$

Here is another Newton structure for  $\mathbf{L}[A]$  that differs from (8.4) only by one factor :

$$[\mathbf{L}[A]] = (\mathbf{L}[A_0], \dots, \mathbf{L}[A_i], \mathbf{L}[A_i \cup \{a_{i+2}\}], \mathbf{L}[A_{i+2}], \dots, \mathbf{L}[A_d]). \quad (8.5)$$

More generally any permutation (ordering) of the points in  $A$  induces a particular Newton structure. Many other examples are given in subsection 8.5 below. Before studying such examples, we present an alternative characterization of a Newton structure.

**Theorem 8.2.** *For  $k = 0, \dots, d$ , let  $\Pi_k$  be a polynomial projector of degree  $k$  on  $\mathcal{E}$ . Then  $(\Pi_0, \Pi_1, \dots, \Pi_d)$  is a Newton structure for  $\Pi = \Pi_d$  if and only if*

$$\text{CND}(\Pi_k) \subset \text{CND}(\Pi_{k+1}), \quad 0 \leq k \leq d-1.$$

Thus, to obtain a Newton structure for  $\Pi$  means to obtain a certain gradation of its space of interpolation conditions.

*Démonstration.* We first prove that the condition is necessary. Let  $(\Pi_0, \Pi_1, \dots, \Pi_d)$  be a Newton structure for  $\Pi = \Pi_d$  and  $k \in \{0, \dots, d-1\}$ . We prove that if  $\varphi \in \text{CND}(\Pi_k)$  then  $\varphi \in \text{CND}(\Pi_{k+1})$ . We have

$$\varphi \circ \Pi_k = \varphi \circ \Pi_k \circ \Pi_{k+1} = \varphi \circ \Pi_{k+1} \quad \text{on } \mathcal{E},$$

where the first equality is by (8.2) which ensures  $\Pi_k = \Pi_k \circ \Pi_{k+1}$  on  $\mathcal{E}$  and the second equality because  $\varphi \in \text{CND}(\Pi_k)$  so that  $\varphi \circ \Pi_k = \varphi$  on  $\mathcal{E}$  (see also Lemma 7.4). Now the relation  $\varphi = \varphi \circ \Pi_{k+1}$  yields  $\varphi \in \text{CND}(\Pi_{k+1})$ .

We now prove that the condition is sufficient. We assume that  $\text{CND}(\Pi_k) \subset \text{CND}(\Pi_{k+1})$  and prove that  $\Pi_k = \Pi_k \circ \Pi_{k+1}$ . Observe that  $\Pi_k \circ \Pi_{k+1}$  is a projector of degree  $k$  on  $\mathcal{E}$  just as  $\Pi_k$  is so that  $\dim(\text{CND}(\Pi_k \circ \Pi_{k+1})) = \dim(\text{CND}(\Pi_k))$ . On the other hand for  $\varphi \in \text{CND}(\Pi_k)$  we have

$$\varphi \circ \Pi_k \circ \Pi_{k+1} = \varphi \circ \Pi_{k+1} = \varphi \quad \text{on } \mathcal{E},$$

where the first equality is because  $\phi \in \text{CND}(\Pi_k)$  and the second one because  $\text{CND}(\Pi_k) \subset \text{CND}(\Pi_{k+1})$ . This shows that  $\text{CND}(\Pi_k) \subset \text{CND}(\Pi_k \circ \Pi_{k+1})$ . Since they have same dimension, both spaces, hence both projectors, must coincide.  $\square$

**Corollary 8.3.** *For  $k = 0, \dots, d$ , let  $\Pi_k$  be a polynomial projector of degree  $k$  on  $\mathcal{E}$ . Then  $(\Pi_0, \Pi_1, \dots, \Pi_d)$  is a Newton structure for  $\Pi = \Pi_d$  if and only if there exists a sequence of spaces  $J_k \subset \mathcal{E}^l$  such that*

$$\text{CND}(\Pi_k) = J_0 \oplus J_1 \oplus \dots \oplus J_k, \quad 0 \leq k \leq d. \quad (8.6)$$

*Démonstration.* It is but a rewording of the previous proposition. Observe that the space  $J_k$  is a supplementary space of  $\text{CND}(\Pi_{k-1})$  in  $\text{CND}(\Pi_k)$  that is,

$$\text{CND}(\Pi_k) = \text{CND}(\Pi_{k-1}) \oplus J_k, \quad 0 \leq k \leq d$$

with the understanding that  $\text{CND}(\Pi_{-1}) = \{0\}$ .  $\square$

### 8.3. Special basis for $\text{CND}(\Pi)$

**Definition 8.4.** We say that a basis  $M = (\mu_\alpha : \alpha \in \mathbb{N}_d(n))$  of  $\text{CND}(\Pi)$  is **adapted** to the Newton structure  $[\Pi] = (\Pi_0, \Pi_1, \dots, \Pi_d)$  if

$$J_k = \text{span} \{ \mu_\alpha : |\alpha| = k \}, \quad k = 0, \dots, d,$$

where  $J_k$  is as in Corollary 8.3.

If  $M$  is adapted to  $[\Pi]$  and

$$\Pi(f) = \sum_{|\alpha| \leq d} \mu_\alpha(f) p_\alpha,$$

where the  $p_\alpha$  are the dual basis of the  $\mu_\alpha$ , see (7.1), then, for some polynomials  $p_\alpha^i \in \mathcal{P}_i(\mathbb{K}^n)$ ,  $|\alpha| \leq i$ , we have

$$\Pi_i(f) = \sum_{|\alpha|=i} \mu_\alpha(f) p_\alpha^i. \quad (8.7)$$

There is no simple relation between the polynomials  $p_\alpha^i$  and  $p_\alpha^{i+1}$  while it would be convenient to have  $p_\alpha^i = p_\alpha^{i+1}$  when  $|\alpha| \leq i$ . This can be easily achieved by choosing a suitable (adapted) basis for  $\text{CND}(\Pi)$ .

Starting from a basis  $M = (\mu_\alpha : \alpha \in \mathbb{N}_d(n))$  of  $\text{CND}(\Pi)$  adapted to  $[\Pi]$  we construct recursively another sequence  $N = (v_\alpha : \alpha \in \mathbb{N}_d(n))$  as follows :

(1)  $v_0 = \mu_0$ .

(2) For  $k \geq 0$ , we choose  $v_\alpha \in \text{CND}(\Pi_{k+1})$ ,  $|\alpha| = k+1$ , by requiring that  $v_\alpha$  vanishes on the polynomials of degree at most  $k$  and

$$\text{span} \{ v_\alpha |_{\mathcal{P}_{k+1}}, |\alpha| = k+1 \} = \mathcal{P}_k^\perp,$$

where the orthogonal is taken in  $\mathcal{P}_{k+1}^* = \text{CND}(\Pi_{k+1})|_{\mathcal{P}_{k+1}}$ .

A constructive way of defining  $v_\alpha$  with  $|\alpha| = k+1$  is to observe that, on  $\mathcal{P}_k$ ,  $\mu_\alpha$  is a linear combination of the  $\mu_\beta$ ,  $|\beta| = k$ , that is for some coefficients  $c_\beta$ ,

$$\mu_\alpha(p) = \sum_{|\beta| \leq k} c_\beta \mu_\beta(p), \quad p \in \mathcal{P}_k(\mathbb{K}^n).$$

It suffices then to set

$$v_\alpha = \mu_\alpha - \sum_{|\beta| \leq k} c_\beta \mu_\beta,$$

which shows that the basis  $N$  is obtained from  $M$  by multiplication a (unit) lower triangular matrix.

**Definition 8.5.** A adapted basis  $N = (v_\alpha : \alpha \in \mathbb{N}_d(n))$  for  $[\Pi]$  is called a **Newton functional basis** for  $[\Pi]$  if  $v_\alpha(p) = 0$  when  $|\alpha| > \deg p$ . The dual basis  $\mathbf{n} = (n_\alpha : \alpha \in \mathbb{N}_d(n))$  when  $|\alpha| > \deg p$  of the Newton functional basis  $N$  is a **Newton polynomial basis**.

Observe that when  $N$  is a Newton functional basis, the Vandermondian matrix  $\mathbf{vdm}(N, B)$  ( $B$  denoting the standard monomial basis), see 7.1.3, is block lower triangular (the  $k$ -th block being of dimension  $N_k(n) - N_{k-1}(n)$ ).

**Lemma 8.6.** If  $N = (v_\alpha : \alpha \in \mathbb{N}_d(n))$  is a Newton functional basis for  $[\Pi] = (\Pi_0, \Pi_1, \dots, \Pi_d)$  and  $\mathbf{n} = (n_\alpha : \alpha \in \mathbb{N}_d(n))$  the corresponding Newton polynomial basis then

$$\Pi_k(f) = \sum_{|\alpha| \leq k} v_\alpha(f) n_\alpha, \quad f \in \mathcal{E}, \quad k = 0, 1, \dots, d.$$

Thus the Newton formula for  $\Pi$  is

$$\Pi(f) = \sum_{i=0}^d \sum_{|\alpha|=i} v_\alpha(f) n_\alpha.$$

*Démonstration.* We just need to prove that for all  $\alpha$ ,  $\deg n_\alpha = |\alpha|$ . Let  $k = |\alpha|$  since  $N_k = (v_\beta : \beta \in \mathbb{N}_k(n))$  forms a basis of  $\text{CND}(\Pi_k)$ , there exists a unique polynomial  $n_\alpha^k \in \mathcal{P}_k$  such that  $v_\beta(n_\alpha^k) = \delta_{\alpha\beta}$  for  $|\beta| \leq k$ . On the other hand for  $|\beta| > k$ , since  $N$  is a Newton functional basis,  $v_\beta(n_\alpha^k) = 0$ . Hence  $n_\alpha^k = n_\alpha$  and  $\deg n_\alpha \leq |\alpha|$ . On the other hand  $\deg n_\alpha < |\alpha|$  is impossible since, otherwise, we would have  $v_\alpha(n_\alpha) = 0$ .  $\square$

#### 8.4. Computing Newton structures

It follows from Corollary 8.3 that every polynomial projector can be endowed with (plenty of) Newton structures. In fact, if a basis  $M = (\mu_\alpha : \alpha \in \mathbb{N}_d(n))$  of  $\text{CND}(\Pi)$  is such that

$$\mathbf{vdm}(M_k, B_k) \neq 0, \quad k = 0, \dots, d, \quad (8.8)$$

where

$$M_k = (\mu_\alpha : |\alpha| \leq k) \quad \text{and} \quad B_k = (e_\alpha : |\alpha| \leq k), \quad (8.9)$$

then we may define  $J_k$  as

$$J_k := \text{span} \{ \mu_\alpha : |\alpha| = k \},$$

to get a Newton structure for  $\Pi$ . When condition (8.8) is not satisfied, since  $\mathbf{vdm}(M, B)$  is invertible, by elementary linear algebra, we can re-order the functional  $\mu_\alpha$  to obtain, say  $M'$ , in order that every principal minor of order  $N_k(n)$  of  $\mathbf{vdm}(M', B)$  be different from 0 for  $k = 0, \dots, d-1$  (as is done in standard *PLU* matrix factorization).

Algorithm 1 below is a Gauss-like algorithm that returns such a convenient re-ordering  $M'$  of a given basis  $M$  in the case we know a basis  $P$  of  $\mathcal{P}_d(\mathbb{K}^n)$  such that  $\{M, P\}$  is a bi-orthogonal system. The typical example is given by Lagrange interpolation, see 7.2.1 for which  $M$  is given by  $([a_\alpha])$ , see (7.12) and  $P = \ell$  is the Lagrange basis. Observe that the problem is trivial when the leading terms of each  $p_\alpha$  in  $P$  is  $e_\alpha$ .

The idea of the algorithm is as follows. Let  $\theta$  be the greatest multi-index in  $\mathbb{N}_d(n)$ . There must be an element  $v$  in  $\mathbb{N}_d(n)$  such that the coefficient of  $e_\theta$  in  $p_v$  – denoted by  $\text{coeff}(p_v, e_\theta)$  – is not zero, otherwise  $e_\theta$  would not belong to  $\text{span}\{P\} = \mathcal{P}_d(\mathbb{K}^n)$ . For  $\alpha \neq v$ , define

$$q_\beta = p_\beta - \frac{\text{coeff}(p_\beta, e_\theta)}{\text{coeff}(p_v, e_\theta)} p_v.$$

The monomial  $e_\theta$  no longer occurs in the  $q_\beta$ ,  $\beta \neq v$ , which are linearly independent. Hence,

$$\text{span} \{q_\beta : \beta \neq v\} = \text{span} \{e_\beta : \beta \neq \theta\}.$$

Since, moreover,

$$\mu_\gamma(q_\beta) = \delta_{\gamma\beta}, \quad \beta, \gamma \prec v,$$

we deduce, see Lemma 7.3, that the restrictions of the  $\mu_\beta$ ,  $\beta \neq v$ , to the space  $\text{span} \{e_\beta : \beta \neq \theta\}$  are linearly independent. We continue in this way and, after  $N_{d-1}(n)$  steps, we get a subset of  $N_{d-1}(n)$  functionals  $\mu_\alpha$  whose restrictions to  $\mathcal{P}_{d-1}(\mathbb{K}^n)$  are linearly independent.

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**Algorithm 1:** Re-ordering algorithm

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**Data:** A bi-orthogonal system  $\{M, P\}$  for  $\Pi$ , the order is  $\preceq$ .

**Result:** A re-ordering  $M'$  of the elements of  $M$  that provides a Newton Structure for  $\Pi$

```

1 Initialization :  $S = \mathbb{N}_d(n)$ ,  $S' = ()$ ,  $Q = (p_\alpha : \alpha \in \mathbb{N}_d(n)) = P$ ;
2 while  $\text{length}(S') < \binom{n+d}{d}$  do
3    $\theta = \max S$ ;
4   Choose  $v \in S$  such that the coefficient of  $e_\theta$  in  $p_v$  does not vanish. ;
5    $S = S \setminus \{v\}$ ;
6   for  $\alpha \in S$  do
7      $p_\alpha = p_\alpha - \frac{\text{coeff}(p_\alpha, e_\theta)}{\text{coeff}(p_v, e_\theta)} p_v$ 
8   end
9    $S' = (v, S')$ 
10 end
11 return  $M' = (\mu_\alpha : \alpha \in S')$  such that  $\text{VDM}(M'_k, B_k) \neq 0$  for all  $k$ , see (8.9)
```

---

We are thus lead to the following definition.

**Definition 8.7.** Let  $\Pi$  be a polynomial projector of degree  $d$  on  $\mathcal{E}$  and  $M = (\mu_\alpha : \alpha \in \mathbb{N}_d(n))$  a basis of  $\text{CND}(\Pi)$ . We say that  $M$  induces the Newton structure  $[\Pi] = (\Pi_0, \dots, \Pi_d)$  for  $\Pi$  if there is a permutation  $\sigma$  of  $\mathbb{N}_d(n)$  such that, for  $k = 1, \dots, d$ ,  $M_\sigma^k = (\mu_{\sigma(\alpha)} : |\alpha| \leq k)$  is a basis of  $\text{CND}(\Pi_k)$ .

In general, see Theorem 8.8 below, a basis  $M$  induces many different Newton structures. It may happen, however, that it induces one and only one Newton structure. This is the case for instance when  $\{M, B\}$  is a bi-orthogonal system ( $B$  being the usual monomial basis), or more generally when  $\text{vdm}(M, B)$  is triangular. Since the only permutation  $\sigma$  satisfying the condition in Definition 8.7 is the identity. On the other hand, it must be noted that two different basis may induce a same Newton structure. For instance, using the notation in the introduction 8.1, the Newton structure  $[\mathbf{L}[A]] = (\mathbf{L}[A_0], \dots, \mathbf{L}[A_d])$  is induced by the basis  $M_1 = ([a_i] : i = 0, \dots, d)$  as well as by the basis  $M_2 = (f \mapsto f[a_0, \dots, a_i] : i = 0, \dots, d)$ .

**Theorem 8.8.** Let  $\Pi$  be a polynomial projector of degree  $d \geq 1$  on  $\mathcal{E}$  and let  $M = (\mu_\alpha : \alpha \in \mathbb{N}_d(n))$  be a basis of  $\text{CND}(\Pi)$  such that  $\mu_\alpha(1) \neq 0$  for all  $\alpha$  in  $\mathbb{N}_d(n)$ . Then  $B$  induces at least  $\prod_{k=1}^d \left(1 + \binom{n+k-1}{k}\right)$  different Newton structures for  $\Pi$ .

For instance, the basis of Dirac functionals for Lagrange interpolation in (7.12) satisfies the assumption of the Theorem. In fact, in any case, if  $M_1$  is a basis of  $\text{CND}(\Pi)$ , that there exists  $\mu_{\alpha_0} \in M_1$  such that  $\mu_{\alpha_0}(1) \neq 0$  and a basis  $M_2$  satisfying the assumption is readily obtained by substituting  $\mu_\alpha + \mu_{\alpha_0}$  for  $\mu_\alpha$  for any  $\alpha$  for which  $\mu_\alpha(1) = 0$ .



*Démonstration.* It follows immediately from repeated applications of the following lemma which is proved below.  $\square$

**Lemma 8.9.** *We use the assumption of the above theorem. If  $M'$  is a sub-family of  $M$  of length  $\binom{n+d-1}{d-1}$  such that the restriction of the elements of  $M'$  to  $\mathcal{P}_{d-1}(\mathbb{K}^n)$  are linearly independent (and therefore form a basis of the dual  $\mathcal{P}_{d-1}^*(\mathbb{K}^n)$ ) then for all  $\mu_\alpha \in M \setminus M'$  there exists  $\mu_\beta = \mu_{\beta(\alpha)} \in M'$  such the restriction to  $\mathcal{P}_{d-1}(\mathbb{K}^n)$  of the elements of  $\{\mu_\alpha\} \cup M' \setminus \{\mu_\beta\}$  are linearly independent on  $\mathcal{P}_{d-1}(\mathbb{K}^n)$ .*

Hence, since there are

$$\binom{n+d}{d} - \binom{n+d-1}{d-1} = \binom{n+d-1}{d}$$

values available for  $\alpha$  we have at least  $1 + \binom{n+d-1}{d}$  different candidates for  $\Pi_{d-1}$ , namely the projectors whose spaces of interpolation condition are given by

$$\text{span}\{M'\} \quad \text{and} \quad \text{span}\{\{\mu_\alpha\} \cup M' \setminus \{\mu_{\beta(\alpha)}\}\}, \quad \mu_\alpha \in M \setminus M'.$$

All such projector are pairwise distinct since their interpolation space are pairwise distinct.

*Proof of Lemma 8.9.* Assume to the contrary that no  $\beta$  satisfies the required property. Using a tilde to denote restriction to  $\mathcal{P}_{d-1}(\mathbb{K}^n)$ , for all  $\mu_\beta$  in  $M'$ , we have that  $\tilde{\mu}_\alpha$  is a linear combination of the  $\tilde{\mu}_\gamma$  where  $\gamma$  runs over  $M' \setminus \{\tilde{\mu}_\beta\}$ . Thus, in particular,  $\tilde{\mu}_\alpha \in \text{span}\{M'\}$  and its coordinate on the basis elements  $\tilde{\mu}_\beta$  is null. Since this is true for all  $\tilde{\mu}_\beta$  in  $M'$ ,  $\tilde{\mu}_\alpha$  must be zero but this is impossible for  $1 \in \mathcal{P}_{d-1}(\mathbb{K}^n)$  so that  $\tilde{\mu}_\alpha(1) = \mu_\alpha(1) \neq 0$ .  $\square$

The profusion of Newton structures raises the question of finding the best Newton structures for a given problem. We will describe such problems more precisely later. It must be said, however, that quite often, especially in the case of D-centered projectors, see 7.2.5, there are relatively few natural Newton structures, that is, structures for which the factors are of the same type than the projector itself.

## 8.5. Examples

8.5.1. *Lagrange-Hermite interpolation.* Let  $\{a_1, a_2, a_3\}$  be an affine basis of  $\mathbb{K}^2$  (i.e. the three points are not aligned) and let  $b$  be another point in  $\mathbb{K}^2$  such that  $b$  does not belong to any of the three lines  $(a_i, a_j)$  so that the four points are in general position. The space

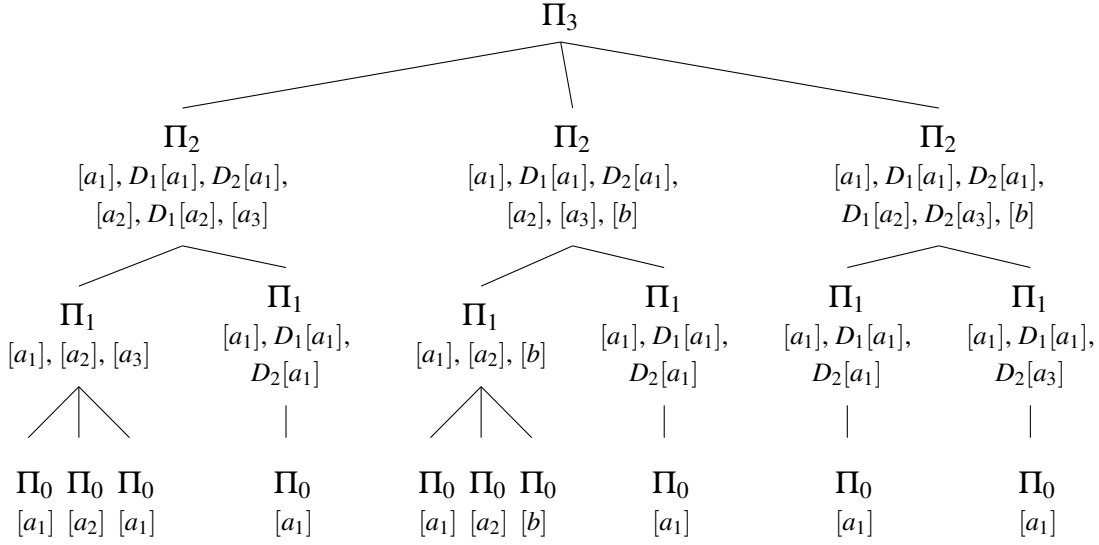
$$V = V_3 = \text{span}\{M\}, \quad M = ([a_i], D_1[a_i], D_2[a_i] : i = 1, 2, 3, [b]),$$

where  $D_j[a_i] : f \rightarrow \frac{\partial f}{\partial x_j}(a_i)$ , defines a Lagrange-Hermite projector  $\Pi^{V_3} = \Pi_3$  of degree 3 (see 7.2.2) on any space of bivariate functions  $\mathcal{E}$  where the functionals are well defined. To see this, since  $\{a_1, a_2, a_3\}$  can be transformed into  $\{(0,0), (0,1), (1,0)\}$  by an affine isomorphism, following the idea in 7.1.4, we may assume that  $a_1 = (0,0)$ ,  $a_2 = (1,0)$  and  $a_3 = (0,1)$ . In that case, a computation shows that

$$\text{VDM}(M, B) = b_1 b_2 (b_2 + b_1 - 1) \quad (\text{where } B \text{ is the standard monomial basis}),$$

so that the assumption on the fourth point  $b$  ensures that  $\text{VDM}(M, B) \neq 0$ . To construct  $\Pi_2$ , we may select six among the ten functionals in  $M$  that are linearly independent on  $\mathcal{P}_2(\mathbb{K}^2)$ . A use of a computer algebra software shows that among the 210 possibilities, there are 108 valid choices (without further assumptions on the points). Since the three points  $a_i$ ,  $i = 1, 2, 3$ , play a symmetric role, there are 18 significantly distinct choices. Ten distinct Newton structures are represented on the tree below. The third branch contains factors which are not themselves

Lagrange-Hermite projectors but only Birkhoff projectors, see 7.2.2 for the terminology. A basis for the space of interpolation conditions is written under each factor.



8.5.2. *Lagrange interpolation.* It is natural to look for Newton structure for which all factors are themselves Lagrange projectors and finding such structures comes to finding a block unisolvent ordering for the interpolation points. This notion was introduced in [47] and further exploited in [20]. A lower bound for the number of such Newton structures is given by Theorem 8.8.

We mention two more precise results in the case of low degree interpolation. The proofs, which are based on elementary geometrical arguments, will appear elsewhere.

**Theorem 8.10.** *If  $\Pi$  is a Lagrange projector of degree 2 in  $\mathbb{K}^2$  then the number of Newton structures for  $\Pi$  with Lagrange projectors as factors is of the form  $3(20 - k)$  with  $k \in \{0, 1, 2, 3, 4\}$ .*

**Theorem 8.11.** *If  $\Pi$  is a Lagrange projector of degree 3 in  $\mathbb{K}^2$  then the number of Newton structures for  $\Pi$  with Lagrange projectors as factors is not smaller than  $48 \times 12 = 576$ .*

8.5.3. *Orthogonal projection.* We use the notation introduced in subsection 7.2.3. The natural Newton structure for  $\mathbf{P}_{d,\mathbf{m}}$  is as follows

$$[\mathbf{P}_{d,\mathbf{m}}] = (\mathbf{P}_{0,\mathbf{m}}, \mathbf{P}_{1,\mathbf{m}}, \dots, \mathbf{P}_{d-1,\mathbf{m}}, \mathbf{P}_{d,\mathbf{m}}).$$

Thus, the  $k$ -th factor is the orthogonal projector of degree  $k$  (with respect to the same measure  $\mathbf{m}$ ) and the space  $J_k$  is given by

$$\begin{aligned} J_k &= \text{span} \{ \mu_\alpha : f \mapsto \langle f, b_\alpha \rangle : |\alpha| = k \} \\ &= \text{span} \{ \nu_\alpha : f \mapsto \langle f, e_\alpha \rangle : |\alpha| = k \} \end{aligned}$$

8.5.4. *Kergin interpolation.* If  $A = \{a_0, a_1, \dots, a_d\}$ , any of the  $(d + 1)!$  permutations of the (indices of the) points induces a Newton structure for  $\mathbf{K}_A$  with Kergin projectors as factors, namely, if  $\sigma$  is such an index permutation

$$[\mathbf{K}_A] = (\mathbf{K}_{A_0^\sigma}, \mathbf{K}_{A_1^\sigma}, \dots, \mathbf{K}_{A_{d-1}^\sigma}, \mathbf{K}_{A_d^\sigma}), \quad A_k^\sigma = \{a_{\sigma(0)}, a_{\sigma(1)}, \dots, a_{\sigma(k-1)}, a_{\sigma(k)}\},$$

and the corresponding space  $J_k$  is

$$J_k = \text{span} \left\{ f \mapsto \int_{\Delta_k} D^\alpha f \left( \sum_{i=0}^k t_i a_{\sigma(i)} \right) dm_k(t) : |\alpha| = k \right\};$$

In the particular case of a Taylor projector, we have the fundamental Newton structure

$$\mathbf{T}_a^d = (\mathbf{T}_a^0, \mathbf{T}_a^1, \dots, \mathbf{T}_a^{d-1}, \mathbf{T}_a^d). \quad (8.10)$$

8.5.5. *Hakopian interpolation and other D-centered Taylor projectors.* Let  $X$  be a set of  $n + d$  points ( $d \geq 1$ ) in **general position** in a convex set  $\Omega$  in  $\mathbb{K}^n$  as in 7.2.5. Any increasing chain

$$X_0 \subset X_1 \subset X_2 \subset \dots \subset X_{d-1} \subset X_d = X$$

where the cardinality of  $X_i$  is  $n + i$  induces a Newton structure for the Hakopian projector  $\mathbf{H}_X$  whose factors are themselves Hakopian projectors,

$$[\mathbf{H}_X] = (\mathbf{H}_{X_0}, \mathbf{H}_{X_1}, \dots, \mathbf{H}_{X_{d-1}}, \mathbf{H}_{X_d}).$$

The corresponding space  $J_k$  is given by

$$J_k = \text{span} \left\{ f \mapsto \int_{\text{cv}(A)} f dm : A \in \binom{X_k}{n} \setminus \binom{X_{k-1}}{n} \right\}.$$

More generally for a D-centered Taylor projector  $\mathbf{C}(\mu_0, \dots, \mu_d)$ , we always have available the following Newton structure

$$[\mathbf{C}(\mu_0, \dots, \mu_d)] = (\mathbf{C}(\mu_0), \mathbf{C}(\mu_0, \mu_1), \dots, \mathbf{C}(\mu_0, \dots, \mu_{d-1}), \mathbf{C}(\mu_0, \dots, \mu_d))$$

for which

$$J_k = \text{span} \{ f \mapsto \mu_k(D^\alpha f) : |\alpha| = k \}.$$

In particular, for Abel-Gontcharov projectors,

$$[\mathbf{A}(a_0, \dots, a_d)] = (\mathbf{A}(a_0), \mathbf{A}(a_0, a_1), \mathbf{A}(a_0, a_1, a_2), \dots, \mathbf{A}(a_0, \dots, a_{d-1}), \mathbf{A}(a_0, \dots, a_d)).$$

## 9. DEFINITION OF THE NEWTON PRODUCT

We now turn to the construction of the Newton product of two N-polynomial projectors. In order to define its space of interpolation conditions, we need to recall some facts about the **tensor products** of functionals.

### 9.1. Products of functionals

Let, for  $s = 1, 2$ ,  $X_s \subset \mathbb{K}^{n_s}$  and  $X = X_1 \times X_2$ . A function  $f \in \mathcal{E}(X)$  is said to be the **(tensor) product** of  $f_1 \in \mathcal{E}(X_1)$  and  $f_2 \in \mathcal{E}(X_2)$  if, for every  $x = (x^1, x^2) \in X$ , we have  $f(x) = f_1(x^1)f_2(x^2)$ . We write  $f = f_1 \otimes f_2$ . The linear span of all the functions  $f_1 \otimes f_2$  is the tensor product space  $\mathcal{E}(X_1) \otimes \mathcal{E}(X_2)$ . In the cases considered in this paper and recalled below,  $\mathcal{E}(X_1) \otimes \mathcal{E}(X_2)$  is dense in  $\mathcal{E}(X_1 \times X_2)$ . This is a classical result which, in each case, follows from the density of polynomials. For the convenience of the reader, we shall briefly sketch the construction of the tensor product of functional for the spaces considered in this paper.

9.1.1.  $\mathcal{E}(X_s) = \mathcal{C}(K_s)$ ,  $K_s$  compact in  $\mathbb{R}^{n_s}$ . An element of  $\mu_s \in \mathcal{E}'$  is but a (real) Borel measure on  $K_s$  and the tensor product  $\mu = \mu_1 \otimes \mu_2$  is (given by) the standard tensor product of the Borel measures on  $K_1 \times K_2$ .

9.1.2.  $\mathcal{E}(X_s) = \mathcal{H}(\Omega_s)$ ,  $\Omega_s$  Runge domain in  $\mathbb{C}^{n_s}$ . The product of two Runge domains is still a Runge domain so the uniqueness is warranted. If  $\mu_s \in \mathcal{H}'(\Omega_s)$ , for some compact subset  $K_s$  of  $\Omega_s$ , we have for every  $f \in \mathcal{H}(\Omega_s)$ ,  $|\mu_s(f)| \leq C\|f\|_{K_s}$  where  $\|f\|_{K_s}$  denotes the sup-norm on  $K_s$ . Hence, by Hahn-Banach theorem,  $\mu_s$  extends to a functional  $\bar{\mu}_s$  on  $\mathcal{C}(K_s)$  and we take  $\mu = \mu_1 \otimes \mu_2$  to be the restriction to  $\mathcal{H}(\Omega_1 \times \Omega_2)$  of  $\bar{\mu}_1 \otimes \bar{\mu}_2$ . A natural direct construction is of course possible. It is used in the next case.

9.1.3.  $\mathcal{E}(X_s) = \mathcal{H}(K_s)$ ,  $K_s$  polynomially convex in  $\mathbb{C}^{n_s}$ . The product  $K_1 \times K_2$  is polynomially convex, hence by the Oka-Weil Theorem, see [44, Theorem 1.5 (iv) in chapter VI], polynomials are dense in  $\mathcal{H}(K_1 \times K_2)$  which again guarantees the uniqueness of the product functional. Now, if  $f \in \mathcal{H}(K_1 \times K_2)$  and  $\mu_2 \in H'(K_2)$  then the function  $z_1 \in K_1 \rightarrow \mu_2(f(z_1, \cdot))$  is in  $\mathcal{H}(K_1)$  so that we may apply  $\mu_1 \in \mathcal{H}'(K_1)$  to it and this gives the product functional :

$$(\mu_1 \otimes \mu_2)(f) = \mu_1(z_1 \rightarrow \mu_2(f(z_1, \cdot))), \quad f \in \mathcal{H}(K_1 \times K_2). \quad (9.1)$$

The resulting functional is clearly linear and verifies the required property on tensor product functions. It remains to check that it is continuous, that is, to show that its restriction to  $\mathcal{H}(\Omega)$  is continuous for every open neighborhood  $\Omega$  of  $K$ . We may assume that  $\Omega = \Omega_1 \times \Omega_2$  with  $K_s \subset \Omega_s$  and that  $\Omega_s$  is a Runge domain. But, in that case,

$$(\mu_1 \otimes \mu_2)|_{\mathcal{H}(\Omega)} = \mu_1|_{\mathcal{H}(\Omega_1)} \otimes \mu_2|_{\mathcal{H}(\Omega_2)},$$

and the right hand side is continuous by the previous case.

9.1.4.  $\mathcal{E}(X_s) = \mathcal{C}^d(\Omega_s)$ ,  $\Omega_s$  open in  $\mathbb{R}^{n_s}$ . The construction is essentially as in (9.1) except that  $\mu_1 \otimes \mu_2$  will be defined on the space  $\mathcal{C}^{d,d}(\Omega_1 \otimes \Omega_2)$ , the space of functions  $d$  times continuously differentiable on each set of functions, that is such that  $D^{(\alpha^1, \alpha^2)} f$  exists and is continuous on  $\Omega$  for  $|\alpha^s| \leq d$ ,  $s = 1, 2$ . Uniqueness follows for the density of polynomials. For this, the idea is rather simple. We choose an exhausting family  $K_n$  of compact subsets of  $\Omega$  and, given a function  $f \in \mathcal{C}^{d,d}(\Omega_1 \otimes \Omega_2)$ , we approximate  $f$  on  $K_n$ . We choose a function  $g$  of same smoothness on the whole of  $\mathbb{R}^n$  which coincides with  $f$  on a neighbourhood of  $K_n$ , finally approximate  $g$  on a large euclidean ball or large multi-intervals by classical tools.

For a nice precise quantitative version of the density of polynomials in  $\mathcal{C}^d(\Omega)$  (hence in  $\mathcal{C}^\infty$ ) under mild geometric conditions on  $\Omega$ , the reader may consult [4].

We summarize the above as follows.

**Theorem 9.1 (and Definition).** *Let, for  $s = 1, 2$ ,  $X_s \in \mathbb{K}^{n_s}$  and  $\mu_s \in \mathcal{E}'(X_s)$  where  $\mathcal{E}(X_s)$  is one of the spaces considered in subsections 9.1.1- 9.1.4. There exists a unique functional  $\mu \in \mathcal{E}'(X_1 \times X_2)$ , called the **(tensor) product** of  $\mu_1$  and  $\mu_2$  and denoted by  $\mu_1 \otimes \mu_2$  such that for every  $f_s \in \mathcal{E}(X_s)$  we have*

$$\mu(f_1 \otimes f_2) = \mu_1(f_1) \cdot \mu_2(f_2). \quad (9.2)$$

It is immediate that the map

$$(\mu_1, \mu_2) \in \mathcal{E}'(X_1) \times \mathcal{E}'(X_2) \longrightarrow \mu_1 \otimes \mu_2 \in \mathcal{E}'(X_1 \times X_2)$$

is bi-linear.

This, together with the defining relation (9.2), imply the following property which we state as a lemma for further reference.

**Lemma 9.2.** *If  $M_s$  is a linearly independent family of functionals on  $\mathcal{E}_s$ ,  $s = 1, 2$ , then*

$$M = M_1 \otimes M_2 = \{\mu_1 \otimes \mu_2 : \mu_1 \in M_1, \mu_2 \in M_2\}$$

*linearly independent family of functionals on  $\mathcal{E}$ .*

## 9.2. Examples

9.2.1. *Dirac functionals.* In all of the above spaces, if  $a^s \in X_s \subset \mathbb{K}^{n_s}$ ,  $s = 1, 2$ , the tensor product of (derivatives) of Dirac functionals, see 7.2.1 and 7.2.2, is simply given by

$$D^\alpha[a^1] \otimes D^\beta[a^2] = D^{(\alpha, \beta)}[(a^1, a^2)], \quad (9.3)$$

where  $(\alpha, \beta)$  is the  $(n_1 + n_2)$ -index whose first  $n_1$  entries are the entries of  $\alpha$  and the last  $n_2$  entries are the entries of  $n_2$  and likewise for  $(a^1, a^2)$ .

**9.2.2. Scalar product.** Let  $\mathbf{m}^s$ ,  $s = 1, 2$ , be a positive (Borel) measure with compact support  $C^s$  in  $\mathbb{K}^{n_s}$  such that  $\int |p|^2 d\mathbf{m}^s = 0$  implies  $p = 0$  as in 7.2.3. The functionals involved in the orthogonal projection  $\mathbf{P}_{d, \mathbf{m}^i}$  are defined for instance by  $v_\alpha^s(f) = \int_{C^s} f b_\alpha(\mathbf{m}^s, \cdot) d\mathbf{m}^s$ , see subsection 7.2.3. We have

$$(v_{\alpha^1}^1 \otimes v_{\alpha^2}^2)(f) = \int_{C^1} \int_{C^2} f(z^1, z^2) b_{\alpha^1}(\mathbf{m}^1, z^1) b_{\alpha^2}(\mathbf{m}^2, z^2) d\mathbf{m}^1(z^1) d\mathbf{m}^2(z^2) \quad (9.4)$$

$$= \int_{C_1 \times C_2} f(z^1, z^2) b_{(\alpha^1, \alpha^2)}(\mathbf{m}^1 \otimes \mathbf{m}^2, (z^1, z^2)) (d\mathbf{m}^1 \otimes d\mathbf{m}^2)(z^1, z^2), \quad (9.5)$$

**9.2.3. Mean value functionals.** If  $\mathcal{E}(X_s)$  is one of the spaces above but with  $X_s$  convex in  $\mathbb{K}^{n_s}$ ,  $s = 1, 2$ , then the product of the functionals

$$f \mapsto \int_{\Delta_{k_s}} D^{\alpha^s} f \left( \sum_{i=0}^{k_s} t_i a_i^s \right) dm_{k_s}(t) : |\alpha^s| = k_s, \quad s = 1, 2, \quad f \in \mathcal{E}(X_s);$$

is given by

$$f \mapsto \int_{\Delta_{k_1} \times \Delta_{k_2}} D^{(\alpha^1, \alpha^2)} f \left( \sum_{i=0}^{k_1} t_i^1 a_i^1, \sum_{i=0}^{k_2} t_i^2 a_i^2 \right) (dm_{k_1} \otimes dm_{k_2})(t^1, t^2) \quad f \in \mathcal{E}(X_1 \otimes X_2).$$

### 9.3. Restricting a projector

Let  $X_s \in \mathbb{K}^{n_s}$ ,  $s = 1, 2$  and  $n = n_1 + n_2$ . With the standard topologies considered above, the canonical inclusion  $\iota : \mathcal{E}(X_1) \rightarrow \mathcal{E}(X_1 \times X_2)$  is continuous.

**Lemma 9.3.** Let  $\Pi$  be a polynomial projector of degree  $d$  on  $\mathcal{E}(X_1 \times X_2)$  and write  $\mathbf{1} = (1, 1, \dots, 1) \in \mathbb{K}^{n_2}$ . Define  $\Pi|_{X_1} : \mathcal{E}(X_1) \rightarrow \mathcal{P}_d(\mathbb{K}^{n_1})$  by

$$\Pi|_{X_1}(f)(x) = (\Pi \circ \iota)(f)(x, \mathbf{1}), \quad f \in \mathcal{E}(X_1), \quad x \in \mathbb{K}^{n_1}. \quad (9.6)$$

Then  $\Pi|_{X_1}$  is a polynomial projector of same degree on  $\mathcal{E}(X_1)$ .

*Démonstration.* The linearity and continuity of  $\Pi|_{X_1}$  follows from that of  $\Pi$  and  $\iota$  and the term on the right hand side of (9.6) clearly defines a polynomial in  $\mathcal{P}_d(\mathbb{K}^{n_1})$ . So we just need to prove that for every  $p \in \mathcal{P}_d(\mathbb{K}^{n_1})$  we have  $\Pi|_{X_1}(p) = p$ . In fact, for such a  $p$ , we have  $\iota(p) \in \mathcal{P}_d(\mathbb{K}^n)$  so  $(\Pi \circ \iota)(p) = \Pi(\iota(p)) = \iota(p)$  since  $\Pi$  is a projector and  $\iota(p)(x, \mathbf{1}) = p(x)$ .  $\square$

**Definition 9.4.** The projector  $\Pi|_{X_1}$  is called the restriction of  $\Pi$  to  $\mathcal{E}(X_1)$  or, abusively, to  $X_1$ . The restriction of  $\Pi$  to  $\mathcal{E}(X_2)$  is defined similarly.

If  $M = (\mu_\alpha : \alpha \in \mathbb{N}_d(n))$  then

$$\text{CND}(\Pi|_{X_1}) = \text{span} \{ \mu_\alpha \circ \iota : \alpha \in \mathbb{N}_d(n) \}, \quad (9.7)$$

but the functionals  $\mu_\alpha \circ \iota$ , of course, are not linearly independent.

A Newton structure on  $\Pi = (\Pi_0, \dots, \Pi_d)$  induces a Newton structure on  $\Pi|_{X_1}$  as follows :

$$\Pi|_{X_1} = (\Pi_0|_{X_1}, \dots, \Pi_d|_{X_1}).$$

Likewise, the Newton summands for the restricted projector are given by

$$\pi_{|X_1 i} = \Pi_i|_{X_1} - \Pi_{i-1}|_{X_1}. \quad (9.8)$$

#### 9.4. The key theorem

All the required material for defining the Newton product has now been introduced.

**Theorem 9.5.** *Let, for  $s = 1, 2$ ,  $[\Pi^s] = (\Pi_0^s, \dots, \Pi_d^s)$  be a  $N$ -polynomial projector of degree  $d$  on  $\mathcal{E}_{n_s}(X_s)$ .*

(1) *There exists a unique polynomial projector  $\Pi$  of degree  $d$  on  $\mathcal{E}_n(X)$  where  $n = n_1 + n_2$  and  $X = X_1 \times X_2$  such that*

$$k_1 + k_2 \leq d \implies \text{CND}(\Pi_{k_1}^1) \otimes \text{CND}(\Pi_{k_2}^2) \subset \text{CND}(\Pi). \quad (9.9)$$

(2) *Moreover, we have*

$$\Pi(f_1 \otimes f_2) = \sum_{(i,j) \in \mathbb{N}_d(2)} \pi_i^1(f_1) \otimes \pi_j^2(f_2), \quad f_i \in \mathcal{E}(X^i), \quad (9.10)$$

where the  $\pi_i^s$  denote the Newton summands corresponding to  $\Pi^s$ , see Definition 8.1.

Here of course,

$$\text{CND}(\Pi_{k_1}^1) \otimes \text{CND}(\Pi_{k_2}^2) = \{\mu \otimes \nu : \mu \in \text{CND}(\Pi_{k_1}^1), \nu \in \text{CND}(\Pi_{k_2}^2)\}.$$

**Definition 9.6.** The above polynomial projector  $\Pi$  is called the **Newton product** of the  $N$ -projectors  $[\Pi_1]$  and  $[\Pi_2]$ . We shall write  $\Pi = [\Pi_1] \otimes_N [\Pi_2]$ . The  $N$ -structured projector  $[\Pi^1]$  (respectively  $[\Pi^2]$ ) is the left (respectively, right) divisor of  $\Pi$ .

The terminology is justified by the fact that the new projector  $\Pi$  heavily depends on the Newton structures of the divisors  $\Pi^1$  and  $\Pi^2$ . As far as we know, the first Newton product was proposed by Biermann in [8] for which  $\Pi^1$  and  $\Pi^2$  are univariate Lagrange projectors so that we could perhaps speak of Newton-Biermann product. The Biermann product of Lagrange projectors in arbitrary dimensions was first considered in [20]. This fundamental example is further studied below.

Note that we shall sometimes abusively write  $\Pi = \Pi_1 \otimes_N \Pi_2$  instead of the correct  $[\Pi_1] \otimes_N [\Pi_2]$ . This should not induce confusion when the Newton structures on the  $\Pi^s$  are clear in the context.

Formula (9.10) will play a fundamental role in the proof of the approximation properties of Newton products. In fact, if  $f \in \mathcal{E}(X)$ , in many important cases, we can provide a well-quantified approximation of  $f$  by a linear combination of tensor products, say  $f_p = \sum_{\alpha \in A} c_\alpha f_{1,\alpha} \otimes f_{2,\alpha}$  where  $A$  finite. We will then have

$$f - \Pi(f) \approx f_p - \Pi(f_p) = \sum_{\alpha \in A} c_\alpha (f_{1,\alpha} \otimes f_{2,\alpha} - \Pi(f_{1,\alpha} \otimes f_{2,\alpha})),$$

and the terms in the sums may be handled using (9.10).

Let us clarify some notation. From now on, as already done in 9.2, every  $\alpha \in \mathbb{N}^n$  will be written as  $\alpha = (\alpha_1, \alpha_2)$  with  $\alpha_1 \in \mathbb{N}^{n_1}$  formed of the  $n_1$  first coordinates of  $\alpha$  and  $\alpha_2 \in \mathbb{N}^{n_2}$  formed of the last  $n_2$  coordinates, where  $n_s$  is clear in the context. Likewise,  $x \in \mathbb{K}^n$  as  $x = (x^1, x^2)$  with  $x^s \in \mathbb{K}^{n_s}$ .

*Proof of Theorem 9.5.* Let  $\mu^s = (\mu_{\alpha^s}^s)$  be a functional basis adapted to the  $N$ -structure of  $[\Pi^s]$ ,  $s = 1, 2$ , see Definition 8.4. We first prove that the space  $V$  defined by

$$V = \text{span} \{ \mu_{\alpha^1}^1 \otimes \mu_{\alpha^2}^2 : |\alpha^1| + |\alpha^2| \leq d \} \quad (9.11)$$

is the interpolation space of a polynomial projector  $\Pi = \Pi^V$ . Next we show that this projector is the only one that satisfies (9.9).

To prove the claim on  $V$ , we use the criterion (3) in Lemma 7.5. Namely, we look for polynomials  $q_\alpha \in \mathcal{P}_d(\mathbb{K}^n)$ ,  $\alpha = (\alpha^1, \alpha^2) \in \mathbb{N}_d(n)$ , such that

$$(\mu_{\alpha^1}^1 \otimes \mu_{\alpha^2}^2)(q_\beta) = \delta_{\alpha\beta}, \quad \alpha, \beta \in \mathbb{N}_d(n). \quad (9.12)$$

For  $s = 1, 2$ , let us denote by  $(p_{\alpha^s}^s : \alpha^s \in \mathbb{N}_d(n_s))$  the dual basis of the  $\mu_{\alpha^s}^s$  so that

$$\mu_{\alpha^s}^s(p_{\beta^s}^s) = \delta_{\alpha^s\beta^s}, \quad \alpha^s, \beta^s \in \mathbb{N}_d(n_s). \quad (9.13)$$

Next, using the Newton summands of  $[\Pi^s]$ , we put

$$q_\alpha = \sum_{k_1+k_2 \leq d} \pi_{k_1}^1(p_{\alpha^1}^1) \otimes \pi_{k_2}^2(p_{\alpha^2}^2), \quad \alpha = (\alpha^1, \alpha^2) \in \mathbb{N}_d(n). \quad (9.14)$$

We claim that these polynomials satisfy (9.12). First, since  $\pi_{k_s}^s$  takes its values in  $\mathcal{P}_{k_s}(\mathbb{K}^{n_s})$  and  $k_1 + k_2 \leq d$  the polynomials  $q_\alpha$  defined above belong to  $\mathcal{P}_d(\mathbb{K}^n)$ . Moreover,

$$(\mu_{\alpha^1}^1 \otimes \mu_{\alpha^2}^2)(q_\beta) = \sum_{k_1+k_2 \leq d} \mu_{\alpha^1}^1(\pi_{k_1}^1(p_{\beta^1}^1)) \mu_{\alpha^2}^2(\pi_{k_2}^2(p_{\beta^2}^2)), \quad \alpha, \beta \in \mathbb{N}_d(n). \quad (9.15)$$

On the other hand, we have

$$\delta_{\alpha\beta} = \delta_{\alpha^1\beta^1} \delta_{\alpha^2\beta^2} = \mu_{\alpha^1}^1(p_{\beta^1}^1) \mu_{\alpha^2}^2(p_{\beta^2}^2) \quad (9.16)$$

$$= \mu_{\alpha^1}^1(\Pi^1(p_{\beta^1}^1)) \mu_{\alpha^2}^2(\Pi^2(p_{\beta^2}^2)) \quad (\Pi^s \text{ is a projector}) \quad (9.17)$$

$$= \mu_{\alpha^1}^1\left(\sum_{k_1=0}^d \pi_{k_1}^1(p_{\beta^1}^1)\right) \mu_{\alpha^2}^2\left(\sum_{k_2=0}^d \pi_{k_2}^2(p_{\beta^2}^2)\right) \quad (\text{Newton formulae (8.3)}) \quad (9.18)$$

$$= \sum_{k_1, k_2 \leq d} \mu_{\alpha^1}^1(\pi_{k_1}^1(p_{\beta^1}^1)) \mu_{\alpha^2}^2(\pi_{k_2}^2(p_{\beta^2}^2)). \quad (9.19)$$

A comparison of (9.15) and (9.19) shows that that our claim will be proved if we show that

$$\sum_{k_1 \leq d, k_2 \leq d, k_1+k_2 > d} \mu_{\alpha^1}^1(\pi_{k_1}^1(p_{\beta^1}^1)) \mu_{\alpha^2}^2(\pi_{k_2}^2(p_{\beta^2}^2)) = 0. \quad (9.20)$$

It turns out that every term in this sum is equal to 0. Indeed, since  $k_1 + k_2 > d$  and  $|\alpha| = |\alpha^1| + |\alpha^2| \leq d$  we have either  $k_1 > |\alpha^1|$  or  $k_2 > |\alpha^2|$ . Let us assume  $k_1 > |\alpha^1|$ . It follows that

$$\mu_{\alpha^1}^1 \in \text{CND}(\Pi_{k_1-1}^1) \quad (9.21)$$

and, in view of Theorem 8.2, we also have

$$\mu_{\alpha^1}^1 \in \text{CND}(\Pi_{k_1}^1). \quad (9.22)$$

Hence, by definition of the Newton summands,

$$\begin{aligned} \mu_{\alpha^1}^1(\pi_{k_1}^1(p_{\beta^1}^1)) &= \mu_{\alpha^1}^1(\Pi_{k_1}^1(p_{\beta^1}^1)) - \mu_{\alpha^1}^1(\Pi_{k_1-1}^1(p_{\beta^1}^1)) \\ &= \mu_{\alpha^1}^1(p_{\beta^1}^1) - \mu_{\alpha^1}^1(p_{\beta^1}^1) = 0 \end{aligned} \quad (\text{by (9.21) and (9.22)}),$$

so that,

$$\mu_{\alpha^1}^1(\pi_{k_1}^1(p_{\beta^1}^1)) \mu_{\alpha^2}^2(\pi_{k_2}^2(p_{\beta^2}^2)) = 0. \quad (9.23)$$

This completes the proof that  $V$  is the space of interpolation conditions of a polynomial projector on  $\mathcal{E}_n(X)$ .

We show that the polynomial projector  $\Pi$  such that  $\text{CND}(\Pi) = V$  – where  $V$  is as in (9.11) – is the only one satisfying (9.9). That  $\Pi$  does satisfy the conditions comes at once from the fact that for  $k_1 + k_2 \leq d$ ,

$$\text{CND}(\Pi_{k_1}^1) \otimes \text{CND}(\Pi_{k_2}^2) \subset V = \text{CND}(\Pi).$$

Conversely, the space interpolation conditions of every projector satisfying the conditions must contain  $\mu_{\alpha^1}^1 \otimes \mu_{\alpha^2}^2$  and must therefore be equal to  $V$ . This concludes the proof of the assertion.

The proof of (9.10) uses computations very similar to those done above with the polynomial  $q_\alpha$ . To check that the right hand side in (9.10) is the image by  $\Pi$  of  $f_1 \otimes f_2$ , it suffices to check that, for each  $\alpha = (\alpha^1, \alpha^2)$  with  $|\alpha^1| + |\alpha^2| \leq d$  we have

$$\mu_{\alpha^1} \otimes \mu_{\alpha^2}(f_1 \otimes f_2) = \mu_{\alpha^1} \otimes \mu_{\alpha^2} \left( \sum_{(i,j) \in \mathbb{N}_d(2)} \pi_{1,i}(f_1) \otimes \pi_{2,j}(f_2) \right), \quad (9.24)$$

that is,

$$\mu_{\alpha^1}(f_1) \mu_{\alpha^2}(f_2) = \sum_{(i,j) \in \mathbb{N}_d(2)} \mu_{\alpha^1}(\pi_i^1(f_1)) \mu_{\alpha^2}(\pi_j^2(f_2)).$$

Multiplying the relations

$$\mu_{\alpha^s}(f_s) = \mu_{\alpha^s}(\Pi^s(f_s)) = \sum_{j=0}^d \mu_{\alpha^s}(\pi_j^s(f_s)), \quad s = 1, 2,$$

we get

$$\mu_{\alpha^1}(f_1) \mu_{\alpha^2}(f_2) = \sum_{i,j=0}^d \mu_{\alpha^1}(\pi_i^1(f_1)) \mu_{\alpha^2}(\pi_j^2(f_2)).$$

Hence, to prove (9.24), it suffices to check that

$$(0 \leq i \leq d, 0 \leq j \leq d, i + j > d) \implies \mu_{\alpha^1}(\pi_i^1(f_1)) \mu_{\alpha^2}(\pi_j^2(f_2)) = 0.$$

Now since  $|\alpha^1| + |\alpha^2| \leq d$  and  $i + j > d$  we have  $|\alpha^1| < i$  or  $|\alpha^2| < j$ . Assume we are in the first case. Then

$$\mu_{\alpha^1}(\pi_i^1(f_1)) = \mu_{\alpha^1}(\Pi_i^1(f_1)) - \mu_{\alpha^1}(\Pi_{i-1}^1(f_1)) = 0,$$

the last equality being justified as follows. Since the basis  $(\mu_\alpha^1)$  is adapted to the Newton structure for  $\Pi_1^1$ , we have  $|\alpha^1| < i$  implies  $\mu_{\alpha^1} \in \text{CND}(\Pi_{i-1}^1) \subset \text{CND}(\Pi_i^1)$ ; hence  $\mu_{\alpha^1}(\Pi_i^1(f_1)) = \mu_{\alpha^1}(\Pi_{i-1}^1(f_1)) = \mu_{\alpha^1}(f_1)$ .  $\square$

## 9.5. Consequences

We state as corollaries a few fundamental consequences of the key theorem and its proof. With the same notation as in the above proof :

**Corollary 9.7.** *If  $\mu^\alpha = (\mu_\alpha^s : \alpha \in \mathbb{N}_d(n_s))$  is a basis adapted to the  $N$ -structure of  $\Pi^s$  then the functionals  $\mu_{\alpha^1}^1 \otimes \mu_{\alpha^2}^2$ ,  $|\alpha^1| + |\alpha^2| \leq d$ , form a basis of  $\text{CND}(\Pi^1 \otimes_{\mathbb{N}} \Pi^2)$ . In particular,*

$$\text{CND}(\Pi^1 \otimes_{\mathbb{N}} \Pi^2) = \text{span} \{ \mu_{\alpha^1}^1 \otimes \mu_{\alpha^2}^2 : |\alpha^1| + |\alpha^2| \leq d \}. \quad (9.25)$$

*Démonstration.* This is shown in the previous proof.  $\square$



**Corollary 9.8.** *If*

$$\text{CND}(\Pi^s) = J_0^s \oplus J_1^s \oplus \cdots \oplus J_k^s, \quad 0 \leq k \leq d,$$

as in Corollary 8.3, then

$$\text{CND}(\Pi) = \bigoplus_{i+j \leq d} (J_i^1 \otimes J_j^2).$$

In particular, if  $\mu \in J_k^1$  then the space of all  $\theta \in \text{CND}(\Pi^2)$  such that  $\mu \otimes \theta \in \text{CND}(\Pi)$  is exactly  $\text{CND}(\Pi_{d-k}^2)$  (the dimension of which is  $N_{d-k}(n^2)$ ).

*Démonstration.* Use that the basis in the previous corollary can be written as

$$\{\mu_{\alpha^1}^1 \otimes \mu_{\alpha^2}^2 : |\alpha^1| + |\alpha^2| \leq d\} = \bigcup_{i+j \leq d} \{\mu_{\alpha^1}^1 \otimes \mu_{\alpha^2}^2 : |\alpha^1| = i, |\alpha^2| = j\}$$

and span  $\{\mu_{\alpha^1}^1 \otimes \mu_{\alpha^2}^2 : |\alpha^1| = i, |\alpha^2| = j\} = J_i^1 \otimes J_j^2$ .  $\square$

Note that, as was implicit in the proof of the key theorem, the above corollary is consistent with the combinatorial identity

$$\binom{n_1 + n_2 + d}{d} = \sum_{i+j \leq d} \binom{n_1 + i - 1}{i} \binom{n_2 + j - 1}{j}.$$

which counts in two ways the dimension of the space  $\text{CND}(\Pi)$ .

**Corollary 9.9.**  $\Pi = [\Pi^1] \otimes_{\mathbb{N}} [\Pi^2]$  inherits from  $[\Pi^1]$  and  $[\Pi^2]$  the following Newton structure

$$[\Pi] = ([\Pi^1]_0 \otimes_{\mathbb{N}} [\Pi^2]_0, [\Pi^1]_1 \otimes_{\mathbb{N}} [\Pi^2]_1, \dots, [\Pi^1]_d \otimes_{\mathbb{N}} [\Pi^2]_d), \quad (9.26)$$

where the factors  $[\Pi^s]_j$  are endowed with the  $\mathbb{N}$ -structure inherited from  $[\Pi^s]$ .

*Démonstration.* By Theorem 9.5, the projectors  $[\Pi^1]_j \otimes_{\mathbb{N}} [\Pi^2]_j$  are well-defined. Furthermore, by the previous corollary (with the adapted basis in mind),

$$\text{CND}([\Pi^1]_i \otimes_{\mathbb{N}} [\Pi^2]_i) = \text{span} \{\mu_{\alpha^1}^1 \otimes \mu_{\alpha^2}^2, |\alpha^1| + |\alpha^2| \leq i\}. \quad (9.27)$$

The claim follows in view of Theorem 8.2.  $\square$

We shall say that the above is the **canonical  $\mathbb{N}$ -structure** for  $[\Pi]$ . Unless otherwise specified, a Newtonian product will be always endowed with its **canonical  $\mathbb{N}$ -structure**.

**Corollary 9.10.** If  $v^s = (v_{\alpha}^s : \alpha \in \mathbb{N}_d(n_s))$  is a Newton functional basis for  $[\Pi^s]$  (see Definition 8.5) then the functionals

$$v_{\alpha} = v_{\alpha^1}^1 \otimes v_{\alpha^2}^2, \quad |\alpha^1| + |\alpha^2| \leq d,$$

form a Newton functional basis for  $\Pi = [\Pi^1] \otimes_{\mathbb{N}} [\Pi^2]$  when endowed with its canonical  $\mathbb{N}$ -structure. Moreover, the corresponding Newton polynomials  $n_{\alpha}$  are given by

$$n_{\alpha} = n_{\alpha^1}^1 \otimes n_{\alpha^2}^2, \quad (9.28)$$

where the  $n_{\alpha^s}^s$  are the Newton polynomials corresponding to  $v^s$ ,  $s = 1, 2$ .

*Démonstration.* Since Newton basis are adapted, the  $v_{\alpha}$  form a basis of  $\text{CND}(\Pi)$ . Moreover, letting  $e_{\beta} = e_{\beta^1} \otimes e_{\beta^2}$  denote the standard monomial then

$$v_{\alpha}(e_{\beta}) = v_{\alpha^1}^1(e_{\beta^1}) \times v_{\alpha^2}^2(e_{\beta^2}).$$

If  $|\beta| < |\alpha|$  then either  $|\beta^1| < |\alpha^1|$  and  $v_{\alpha^1}^1(e_{\beta^1}) = 0$  or  $|\beta^2| < |\alpha^2|$  and  $v_{\alpha^2}^2(e_{\beta^2}) = 0$  so that  $v_{\alpha}(e_{\beta}) = 0$  which shows that  $v_{\alpha}$  vanish on  $\mathcal{P}_{|\alpha|-1}$  so that the  $v_{\alpha}$  form a Newton basis for  $\Pi$ . The relation (9.28) is then immediate.  $\square$

**Corollary 9.11.** Let  $\iota_s$  denote the natural injection  $\mathcal{E}(X_s) \rightarrow \mathcal{E}(X_1 \times X_2)$ ,  $s = 1, 2$ . For  $f \in \mathcal{E}(X_s)$ ,  $\Pi(\iota_s(f)) = \Pi^s(f)$ . In other words  $\Pi^s$  is the restriction of  $\Pi$  to  $X_s$ , see subsection 9.3,

$$([\Pi^1] \otimes_{\mathbb{N}} [\Pi^2])|_{X_s} = \Pi^s.$$

*Démonstration.* To see the first point (with  $s = 1$ ), apply for instance (9.10) with  $f_2 = 1$  taking into account that  $\pi_j^2(f_2) = 0$  for  $j > 0$  and  $\pi_0^2(f_2) = 1$  so that the remaining terms give  $\Pi^1(f_1)$ .  $\square$

**Corollary 9.12.** If  $\mu = \mu_1 \otimes \mu_2 \in \text{CND}(\Pi)$ ,  $\mu \neq 0$ , then  $\mu_s \in \text{CND}(\Pi^s)$ ,  $s = 1, 2$ . Moreover if  $\mu_1$ , say, vanishes on  $\mathcal{P}_\delta(\mathbb{K}^{n_1})$ ,  $0 \leq \delta < d$ , then  $\mu_2 \in \text{CND}(\Pi_{d-\delta-1}^2)$

Note that, by assumption, both  $\mu_1$  and  $\mu_2$  are non zero functionals. Since  $\mu = \mu \circ \Pi$ ,  $\mu$  does not vanish on the whole of  $\mathcal{P}_d$ .

*Démonstration.* Take and fix  $f_1$  such that  $\mu_1(f_1) \neq 0$ . Since  $\mu \in \text{CND}(\Pi)$ , for every  $f_2 \in \mathcal{E}(X_2)$ , we have

$$\mu_1(f_1)\mu_2(f_2) = \mu(f_1 \otimes f_2) = \mu(\Pi(f_1 \otimes f_2)) \quad (9.29)$$

$$= \mu\left(\sum_{(i,j) \in \mathbb{N}_d(2)} \pi_i^1(f_1) \otimes \pi_j^2(f_2)\right) \quad (\text{by (9.10)}) \quad (9.30)$$

$$= \sum_{(i,j) \in \mathbb{N}_d(2)} \mu_1(\pi_i^1(f_1)) \otimes \mu_2(\pi_j^2(f_2)). \quad (9.31)$$

It follows that, for some coefficients  $c_j$  depending on the  $\mu_1(\pi_i^1(f_1))/\mu_1(f_1)$ , we have

$$\mu_2(f_2) = \sum_{j=0}^d c_j \mu_2(\pi_j^2(f_2)), \quad f \in \mathcal{E}(X_2). \quad (9.32)$$

Thus  $\mu_2$  is a linear combination of the functional  $\mu_2(\pi_j^2) = \mu_2 \circ \Pi_j^2 - \mu_2 \circ \Pi_{j-1}^2$  (omit the second term when  $j = 0$ ). But, in view of Lemma 7.4, we have  $\mu_2 \circ \Pi_j^2 \subset \text{CND}(\Pi_j^2) \subset \text{CND}(\Pi^2)$  and, likewise,  $\mu_2 \circ \Pi_{j-1}^2 \subset \text{CND}(\Pi_{j-1}^2) \subset \text{CND}(\Pi^2)$  so that  $\mu_2(\pi_j^2)$  belongs to  $\text{CND}(\Pi^2)$  as a difference of two elements of  $\text{CND}(\Pi^2)$  and  $\mu_2 \in \text{CND}(\Pi^2)$  as a linear combination of elements of  $\text{CND}(\Pi^2)$ .

When  $\mu_1$  vanishes on  $\mathcal{P}_\delta(\mathbb{K}^{n_1})$  then, in (9.30), we may assume  $i \geq \delta + 1$  and since  $i + j \leq d$ , we have  $j \leq d - \delta - 1$  and the conclusion follows with the same reasoning.  $\square$

Note that having three  $N$ -polynomial projectors  $[\Pi^s]$ ,  $s = 1, 2, 3$ , we may construct both

$$([\Pi^1] \otimes_{\mathbb{N}} [\Pi^2]) \otimes_{\mathbb{N}} [\Pi^3] \quad \text{and} \quad [\Pi^1] \otimes_{\mathbb{N}} ([\Pi^2] \otimes_{\mathbb{N}} [\Pi^3]).$$

It is readily seen, for instance by, using Corollary 9.7, that both projectors coincide so that we may simply denote it by  $[\Pi^1] \otimes_{\mathbb{N}} [\Pi^2] \otimes_{\mathbb{N}} [\Pi^3]$ . More generally having  $k$   $N$ -projectors of same degree  $[\Pi^s]$ ,  $s = 1, \dots, k$  defined on suitable spaces of functions, we may define

$$[\Pi^1] \otimes_{\mathbb{N}} [\Pi^2] \otimes_{\mathbb{N}} \cdots \otimes_{\mathbb{N}} [\Pi^k] = \otimes_{\mathbb{N}, s=1}^k [\Pi^s].$$

## 10. ALGEBRAIC PROPERTIES OF THE NEWTON PRODUCT

### 10.1. The uniqueness theorems

We now proceed to prove the fundamental property that two distinct pairs of  $N$ -polynomial projectors (on same spaces) provide two distinct Newton product.

**Theorem 10.1** (Uniqueness). *Let, for  $s = 1, 2$ ,  $[\Pi^s] = (\Pi_0^s, \dots, \Pi_d^s)$  (resp.  $[\widehat{\Pi}^s] = (\widehat{\Pi}_0^s, \dots, \widehat{\Pi}_d^s)$ ) be a  $N$ -polynomial projector of degree  $d$  on  $\mathcal{E}_{n_s}(X_s)$ .*

$$\text{If } [\Pi^1] \otimes_N [\Pi^2] = [\widehat{\Pi}^1] \otimes_N [\widehat{\Pi}^2] \text{ then } [\Pi^1] = [\widehat{\Pi}^1] \text{ and } [\Pi^2] = [\widehat{\Pi}^2].$$

In other words, if a polynomial projector of degree  $d$  on  $\mathcal{E}_n(X)$  can be decomposed into a Newton product of two  $N$ -polynomial projectors of degree  $d$  respectively on  $\mathcal{E}_{n_1}(X_1)$  and  $\mathcal{E}_{n_2}(X_2)$  then such a decomposition is unique.

The fact that  $\Pi^s = \widehat{\Pi}^s$  follows immediately from Corollary 9.12. The conclusion here gives the stronger fact that the Newton structures of the projectors  $\Pi^s$  and  $\widehat{\Pi}^s$  coincide.

*Démonstration.* We prove that  $\Pi_j^2 = \widehat{\Pi}_j^2$  for  $j = 0, \dots, d-1$ . (The case of the second projector being identical). Let  $\Pi = [\Pi^1] \otimes_N [\Pi^2]$  and let  $v^s = (v_{\alpha^s}^s)$  be a Newton functional basis for  $[\Pi^s]$ ,  $s = 1, 2$ . In particular,  $\text{CND}(\Pi_j^s) = \text{span}\{v_{\alpha^s}^s : |\alpha^s| = j\}$ . Now, by Corollary 9.7, for  $|\alpha^1| + |\alpha^2| \leq d$ , we have  $v_{\alpha^1}^1 \otimes v_{\alpha^2}^2 \in \text{CND}(\Pi)$ . Moreover, by definition of a Newton functional basis,  $v_{\alpha^1}^1$  vanishes on  $\mathcal{P}_{j-1}$  where  $j = |\alpha^1|$ . Taking this into account, an application of Corollary 9.12 with

$$v_{\alpha^1}^1 \otimes v_{\alpha^2}^2 \in \text{CND}(\Pi), \quad \Pi = [\widehat{\Pi}^1] \otimes_N [\widehat{\Pi}^2],$$

yields  $v_{\alpha^2}^2 \in \text{CND}(\widehat{\Pi}_{d-j}^2)$ . This shows that  $\text{CND}(\Pi_{d-j}^2) \subset \text{CND}(\widehat{\Pi}_{d-j}^2)$  and the equality follows since both spaces have same dimension.  $\square$

**Definition 10.2.** A polynomial projector  $\Pi$  of degree  $d$  on  $\mathcal{E}(X)$ ,  $X \subset \mathbb{K}^n$ , is said to be **reducible** when there exist  $X_s \in \mathbb{K}^{n_s}$ ,  $s = 1, 2$ ,  $X_s \in \mathbb{K}^{n_s}$  with  $1 \leq n_s < n$  and  $n = n_1 + n_2$ , and a  $N$ -polynomial projector  $[\Pi^s]$  of degree  $d$  on  $\mathcal{E}(X_s)$  such that  $\Pi = [\Pi^1] \otimes_N [\Pi^2]$ . Otherwise,  $\Pi$  is said to be **irreducible**.

**Theorem 10.3.** *Let  $X = X_1 \times X_2 \times X_3 \subset \mathbb{K}^n$  with  $X_s \subset \mathbb{K}^{n_s}$  and let  $\Pi$  a polynomial projector of degree  $d$  on  $\mathcal{E}(X)$ . Assume that*

$$\Pi = \underset{X_1}{[\Pi^1]} \otimes_N \underset{X_2 \times X_3}{[\Pi^2]} = \underset{X_1 \times X_2}{[\Pi^3]} \otimes_N \underset{X_3}{[\Pi^4]}, \quad (10.1)$$

where, as indicated in the formula,  $[\Pi^1]$  (resp.,  $[\Pi^2]$ ,  $[\Pi^3]$ ,  $[\Pi^4]$ ) is a  $N$ -polynomial projector of degree  $d$  on  $\mathcal{E}(X_1)$  (resp. on  $\mathcal{E}(X_2 \times X_3)$ , on  $\mathcal{E}(X_1 \times X_2)$ , on  $\mathcal{E}(X_3)$ ). Then  $\Pi^2$  and  $\Pi^3$  are reducible.

*Démonstration.* Let  $f_s$  denote a function in  $\mathcal{E}(X_s)$ . We denote in the same fashion the natural corresponding function in  $\mathcal{E}(X_s \times X_{s'})$  or  $\mathcal{E}(X)$ . We prove that  $\Pi^3$  is reducible. For all  $f_s$  in  $\mathcal{E}(X_s)$ , we have using the restricted projector  $\Pi_{|X_2}^2$  on the last two lines,

$$\Pi^3(f_1 f_2) = \Pi(f_1 f_2) \quad (\text{Corollary 9.11}) \quad (10.2)$$

$$= ([\Pi^1] \otimes_N [\Pi^2])(f_1 f_2) \quad (\text{by (10.1)}) \quad (10.3)$$

$$= \sum_{i+j \leq d} \pi_i^1(f_1) \pi_j^2(f_2) \quad (\text{by (9.10)}) \quad (10.4)$$

$$= \sum_{i+j \leq d} \pi_i^1(f_1) \pi_{|X_2}^2(f_2) \quad (\text{by (9.8)}) \quad (10.5)$$

$$= [\Pi^1] \otimes_N [\Pi_{|X_2}^2](f_1 f_2) \quad (\text{again by (9.10).}) \quad (10.6)$$

This shows that

$$\Pi^3 = [\Pi^1] \otimes_N [\Pi_{|X_2}^2]$$

so that  $\Pi^3$  is reducible. □

**Corollary 10.4** (To the proof). *With the same assumption as in the theorem.*

$$\Pi^3 = [\Pi^1] \otimes_{\mathbb{N}} [\Pi^2_{|X_2}] \quad \text{and} \quad \Pi^2 = [\Pi^3_{|X_2}] \otimes_{\mathbb{N}} [\Pi^4].$$

The uniqueness theorems show that every polynomial projector (on a product set) can be written in a unique way as the Newton products of irreducible polynomial projectors.

Although we will not elaborate on this point here, let us mention that a more natural, less restrictive, definition of reducibility would be to say that  $\Pi$  is reducible if there exists an affine automorphism  $\mathcal{A}$  of  $\mathbb{K}^n$  such that  $\mathcal{A} \star \Pi$ , see subsection 7.1.4, is reducible in the sense of Definition 10.2. The same claim of uniqueness of decomposition still holds true.

## 10.2. A structural lemma

We give another lemma on the structure of a basis for the interpolation space of Newton product. It will be useful later when analysing the Newton products of Hermitian projectors.

**Lemma 10.5.** *Let  $M$  (resp.  $V$ ) a linearly independent family in  $\mathcal{E}(X_1)'$  (resp.  $\mathcal{E}(X_2)'$ ). Assume that  $M$  (resp.  $V$ ) is a basis of  $\text{CND}(\Pi^1)$  (resp.  $\text{CND}(\Pi^2)$ ). If there exists a basis of  $\text{CND}(\Pi)$  all whose members belong to*

$$\{\mu \otimes \nu : \mu \in M, \nu \in V\},$$

*then, for every  $k$ ,  $0 \leq k \leq d-1$ , the space  $\text{CND}(\Pi_k^1)$  (resp.  $\text{CND}(\Pi_k^2)$ ) has a basis whose elements belong to  $M$  (resp. to  $V$ ).*

Let  $M = (\mu_i : i \in \Delta)$  et  $V = (\nu_j : j \in \Gamma)$ , which defines the set of indices  $\Delta$  and  $\Gamma$ . For  $t = (i, j) \in \Delta \times \Gamma = T$ , we set  $\theta_t = \mu_i \otimes \nu_j$ , and  $\theta = (\theta_t : t \in T)$ .

**Lemma 10.6.** *Assume that  $S \subset T$  and that, for some **non zero** scalars  $\alpha_s$ ,  $s \in S$ , we have*

$$\sum_{s \in S} \alpha_s \theta_s \in \text{CND}(\Pi)$$

*then  $\theta_s \in \text{CND}(\Pi)$  for all  $s \in S$ .*

*Démonstration.* Indeed, since there is a basis of  $\text{CND}(\Pi)$  formed by a sub-family  $\theta'$  of  $\theta$ , the functional  $\sum_{s \in S} \alpha_s \theta_s$  can be written as a linear combination of  $\theta'$ , that is,

$$\sum_{s \in S} \alpha_s \theta_s = \sum_{\sigma \in \theta'} \alpha_\sigma \sigma.$$

yet, in view of lemma 9.2, the elements of  $\theta$  are linearly independent and the  $\alpha_s$  being not null, every  $\theta_s$  on the left hand side must be equal to some  $\sigma$  on the right hand side, hence to an element of  $\theta'$  and therefore lies in  $\text{CND}(\Pi)$ . □

*Proof of Lemma 10.5.* By contradiction. We will only prove the claim on  $\text{CND}(\Pi_k^1)$ . The other being obviously similar.

Let us introduce some notation. If  $\mu = \sum \alpha_i \mu_i \in \text{CND}(\Pi^1)$ , is non zero, we define  $\text{supp}(\mu) = \{\mu_i : \alpha_i \neq 0\}$ , call it the **support** of  $\mu$ . The (positive) cardinality of this set is the **length** of  $\mu$ , and will be denoted by  $l(\mu)$ .

So, we assume that there exists an integer  $k$  with  $0 \leq k \leq d-1$  for which  $\text{CND}(\Pi_k^1)$  has no basis with members in  $M$  and we look for a contradiction.

Let  $L = M \cap \text{CND}(\Pi_k^1)$ . Our assumption says that  $\text{CND}(\Pi_k^1) \setminus \text{span}\{L\}$  is not empty. Let us choose  $\bar{\mu} \in \text{CND}(\Pi_k^1) \setminus \text{span}\{L\}$  with minimal length.

*Claim 10.6.1.* We have

- (1)  $l(\bar{\mu}) \geq 2$ .
- (2) No element in the support of  $\bar{\mu}$  belongs to  $L$

To check the first point, observe that, otherwise, there would exist a non zero scalar  $\lambda$  and  $i \in \Delta$  such  $\bar{\mu} = \lambda \mu_i$  so that  $\bar{\mu}$  would belong both to  $\text{CND}(\Pi_k^1)$  and to  $\text{span}\{M\}$  which contradicts  $\bar{\mu} \notin \text{span}\{L\}$ . The second point merely expresses the minimality of  $l(\bar{\mu})$ .

*Claim 10.6.2.* If  $\bar{\mu}_i$  is in the support of  $\bar{\mu}$  then  $\bar{\mu}_i \otimes \text{CND}(\Pi_{d-k}^2)$  is included in  $\text{CND}(\Pi)$ .

Indeed, since  $\bar{\mu} \in \text{CND}(\Pi_k^1)$  and since, by definition of the Newton product,

$$\text{CND}(\Pi_k^1) \otimes \text{CND}(\Pi_{d-k}^2) \subset \text{CND}(\Pi),$$

for all  $v$  in  $\text{CND}(\Pi_{d-k}^2)$  we will have  $\bar{\mu} \otimes v \in \text{CND}(\Pi)$ . Then, writing  $\bar{\mu}$  as a linear combination of the functionals in its support and  $v$  on the basis  $V$ , we arrive to a relation of the form

$$\bar{\mu} \otimes v = \left( \sum_{i \in I} \alpha_i \bar{\mu}_i \right) \otimes \left( \sum_{\gamma \in J} \beta_\gamma v_\gamma \right) = \sum_{i \in I} \sum_{\gamma \in J} \alpha_i \beta_\gamma \bar{\mu}_i \otimes v_\gamma \in \text{CND}(\Pi), \quad \alpha_i \beta_\gamma \neq 0,$$

where  $I$  is a set of indices whose cardinality is the length of  $\bar{\mu}$  and  $J$  is the subset of  $\Gamma$  made with the indices of the non zero coordinates of  $v$ . An application of Lemma 10.6 ensures that each of the functionals  $\bar{\mu}_i \otimes v_\gamma$  in the above relation belong to  $\text{CND}(\Pi)$ . And the same holds for the linear combination

$$\sum_{\gamma \in J} \beta_\gamma \bar{\mu}_i \otimes v_\gamma = \bar{\mu}_i \otimes v, \tag{10.7}$$

so that  $\bar{\mu}_i \otimes v$  is an element of  $\text{CND}(\Pi)$  as claimed.

We will now rapidly obtain the contradiction that we are looking for from (10.7). Fix  $\bar{\mu}_i$  in the support of  $\bar{\mu}$ . Choose  $v \in \text{CND}(\Pi_{d-k}^2)$  such that  $v$  vanishes on  $\mathcal{P}_{d-k-1}$ . Since, by the previous claim,  $\bar{\mu}_i \otimes v \in \text{CND}(\Pi)$ , the second part of Corollary 9.12 yields that  $\bar{\mu}_i \in \text{CND}(\Pi_k)$  which contradicts (2) in Claim 10.6.1.  $\square$

### 10.3. The Vandermondian of a Newton product

Although we will not use it in the sequel, we give a nice formula for the Vandermondian of a Newton product. The formula was known previously in the case of Lagrange interpolation [12] and of orthogonal projection [15]. Both have been applied to derive a proof of the product formula for the multivariate transfinite diameter, see [15] and [22].

**Theorem 10.7.** *Let  $M^s = (\mu_{\alpha^s} : |\alpha^s| \leq d)$  a linearly independent family of functionals on  $\mathcal{E}(X_s)$ ,  $X_s \subset \mathbb{K}^{n_s}$ ,  $s = 1, 2$ . Let  $\Pi^s$  a  $N$ -polynomial projector of degree  $d$  on  $\mathcal{E}(X_s)$  such that*

$$\text{CND}(\Pi_i^s) = \text{span}\{M_i^s\}, \quad M_i^s = \{\mu_{\alpha^s} : |\alpha^s| \leq i\}, \quad i = 0, 1, \dots, d, \quad s = 1, 2. \tag{10.8}$$

Hence

$$\text{CND}([\Pi^1] \otimes_N [\Pi^2]) = \text{span}\{M\}, \quad M = (\mu_{\alpha^1} \otimes \mu_{\alpha^2} : |\alpha^1| + |\alpha^2| \leq d), \tag{10.9}$$

where the ordering in  $M$  is, as usual, induced by *grlex*. We have

$$\text{VDM}(M, B) = \prod_{j=0}^d \left( \text{VDM}(M_j^1, B_j^1)^{\binom{n_2+d-j-1}{d-j}} \right) \left( \text{VDM}(M_j^2, B_j^2)^{\binom{n_1+d-j-1}{d-j}} \right), \tag{10.10}$$

where  $B$  (resp  $B_j^1, B_j^2$ ) is the monomial basis for  $\mathcal{P}_d(\mathbb{K}^n)$  (resp.  $\mathcal{P}_j(\mathbb{K}^{n_1}), \mathcal{P}_j(\mathbb{K}^{n_2})$ ), all being ordered with respect to the corresponding *grlex* order.

We will use the following property (see [20, p. 401]),

$$(\alpha^1, \alpha^2) \preceq (\beta^1, \beta^2) \text{ in } \mathbb{N}^n \implies \alpha^1 \preceq \beta^1 \text{ in } \mathbb{N}^{n_1} \quad \text{or} \quad \alpha^2 \preceq \beta^2 \text{ in } \mathbb{N}^{n_2} \tag{10.11}$$

*Démonstration.* We first prove the formula when the matrices  $\mathbf{vdm}(M^s, B^s)$  are lower triangular (hence also the matrices  $\mathbf{vdm}(M_k^s, B_k^s)$ ). Their determinant is therefore merely the product of the diagonal terms,

$$\mathbf{vDM}(M_k^s, B_k^s) = \prod_{|\alpha^s| \leq k} \mu_{\alpha^s}(e_{\alpha^s}). \quad (10.12)$$

We claim that  $\mathbf{vdm}(M, B)$  is also lower triangular. Indeed, since  $e_{(\beta^1, \beta^2)} = e_{\beta^1} \otimes e_{\beta^2}$ ,

$$\mu_{\alpha^1}^1 \otimes \mu_{\alpha^2}^2(e_{(\beta^1, \beta^2)}) = \mu_{\alpha^1}^1(e_{\beta^1}) \mu_{\alpha^2}^2(e_{\beta^2}) = 0 \quad \text{if } \alpha^1 \prec \beta^1 \text{ or } \alpha^2 \prec \beta^2,$$

Hence, in view of (10.11),

$$(\alpha^1, \alpha^2) \prec (\beta^1, \beta^2) \implies \mu_{\alpha^1}^1 \otimes \mu_{\alpha^2}^2(e_{(\beta^1, \beta^2)}) = 0,$$

Its determinant is therefore given by

$$\mathbf{vDM}(M, B) = \prod_{(\alpha^1, \alpha^2) \in \mathbb{N}_d(n)} \mu_{\alpha^1}^1 \otimes \mu_{\alpha^2}^2(e_{(\alpha^1, \alpha^2)}) = \prod_{(\alpha^1, \alpha^2) \in \mathbb{N}_d(n)} \mu_{\alpha^1}^1(e_{\alpha^1}) \mu_{\alpha^2}^2(e_{\alpha^2}), \quad (10.13)$$

and, by a straightforward calculation,

$$\mathbf{vDM}(M, B) = \prod_{k=0}^d \left\{ \prod_{|\alpha^1|=k} \mu_{\alpha^1}^1(e_{\alpha^1}) \right\}^{N_{d-k}(n_2)} \prod_{k=0}^d \left\{ \prod_{|\alpha^2|=k} \mu_{\alpha^2}^1(e_{\alpha^2}) \right\}^{N_{d-k}(n_1)}. \quad (10.14)$$

Now, in view of (10.12),

$$\prod_{|\alpha^s|=k} \mu_{\alpha^s}(e_{\alpha^s}) = \frac{\mathbf{vDM}(M_k^s, B_k^s)}{\mathbf{vDM}(M_{k-1}^s, B_{k-1}^s)}, \quad (10.15)$$

where the denominator is 1 when  $k = 0$ . Inserting this in (10.14), we obtain

$$\mathbf{vDM}(M, B) = \prod_{k=0}^d \left( \frac{\mathbf{vDM}(M_k^1, B_k^1)}{\mathbf{vDM}(M_{k-1}^1, B_{k-1}^1)} \right)^{N_{d-k}(n_2)} \left( \frac{\mathbf{vDM}(M_k^2, B_k^2)}{\mathbf{vDM}(M_{k-1}^2, B_{k-1}^2)} \right)^{N_{d-k}(n_1)} \quad (10.16)$$

$$= \prod_{k=0}^d \mathbf{vDM}(M_k^1, B_k^1)^{N_{d-k}(n_2) - N_{d-k-1}(n_2)} \mathbf{vDM}(M_k^2, B_k^2)^{N_{d-k}(n_1) - N_{d-k-1}(n_1)}. \quad (10.17)$$

Formula (10.10) follows by observing that  $N_{d-k}(n_s) - N_{d-k-1}(n_s) = \binom{n_s + d - s - 1}{d - k}$ .

To extend the formula to the general case, we first observe that what has been done above with the monomials  $e_{\alpha}$  (resp.  $e_{\alpha^s}$ ) in  $B$  (resp. in  $B^s$ ) can be done with polynomials  $f_{\alpha}$  (resp.  $f_{\alpha^s}$ ) in  $\tilde{B}$  (resp. in  $\tilde{B}^s$ ) when the following two conditions are satisfied :

- (1) For all  $\alpha = (\alpha^1, \alpha^2)$ , we have  $f_{\alpha} = f_{\alpha^1} \otimes f_{\alpha^2}$ ,
- (2)  $\tilde{B}_k^s = (f_{\alpha^s} : |\alpha^s| \leq k)$  is a basis of  $\mathcal{P}_k(\mathbb{K}^{n_s})$ .

A way of obtaining such a new basis is to take a triangular modification of the monomial basis :

$$f_{\alpha^s} = e_{\alpha^s} + (\text{a linear combination of } e_{\beta^s} \text{ with } \beta^s \prec \alpha^s).$$

Thus, in that case, *Formula (10.10) holds true with  $\tilde{B}$  in place of  $B$  provided that  $\mathbf{vdm}(M^s, \tilde{B}^s)$  is lower triangular.*

In the next step, we assume that the leading principal minors of  $\mathbf{vdm}(M^s, B^s)$  are nonzero. Thus, each of these matrices possesses a standard  $LU$  factorization,

$$\mathbf{vdm}(M^s, B^s) = L^s U^s,$$

where  $L^s$  is lower triangular and  $U^s$  is unit upper triangular. More generally,

$$\mathbf{vdm}(M_k^s, B_k^s) = L_k^s U_k^s, \quad k = 0, \dots, d, \quad (10.18)$$

where  $L_k^s$  is the leading principal matrix of  $M^s$  of order  $N_k(n_s)$  and likewise for  $U_k^s$ . Equation (10.18) can be rewritten as

$$L_k^s = \mathbf{vdm}(M_k^s, B_k^s) \cdot \left\{ R^s \right\}_k^T, \quad R^s = (U^s)^{-T}, \quad (10.19)$$

where the exponent  $-T$  indicate the transpose of the inverse and the index  $k$  is used to indicate the leading block of order  $N_k(n_s)$ . In particular,  $R^s$  is a unit lower triangular matrix. In view of (7.8), the above implies that

$$L_k^s = \mathbf{vdm}(M_k^s, \tilde{B}_k^s), \quad \tilde{B}^s = (f_{\alpha^s} : \alpha^s \in \mathbb{N}_d(n_s)), \quad (10.20)$$

where

$$f_{\alpha^s} = \sum_{|\beta| \leq d} R_{\alpha^s \beta}^s e_{\beta} = e_{\alpha^s} + \sum_{\beta \prec \alpha^s} R_{\alpha^s \beta}^s e_{\beta},$$

the latter equality because  $R^s$  is unit lower triangular. Now, we may apply the previous formula (with  $\tilde{B}$ ) to get

$$\mathbf{vdm}(M, \tilde{B}) = \prod_{j=0}^d \left( \mathbf{vdm}(M_j^1, \tilde{B}_j^1)^{\binom{n_2+d-j-1}{d-j}} \right) \left( \mathbf{vdm}(M_j^2, \tilde{B}_j^2)^{\binom{n_1+d-j-1}{d-j}} \right). \quad (10.21)$$

Yet, since for the same reason as above,

$$\mathbf{vdm}(M_k^s, \tilde{B}_k^s) = \mathbf{vdm}(M_k^s, B_k^s) \cdot R_k^{sT} \quad \text{and} \quad \det(R_k^s) = 1,$$

the  $\tilde{B}$  on the right hand side of (10.21) can be replaced by  $B$ . Since, as it is readily seen, the  $f_{\alpha}$  are also a unit triangular modification of the  $e_{\alpha}$ , the same holds for the left hand side so that (10.10) is established.

It just remains to get rid of the assumption on the leading principal minors. A way of doing that is as follows. Formula (10.10) can be regarded as an algebraic identity which we just proved to hold true on a non empty open subset of  $(\mathbb{C}^{N_d(n_1)})^* \times (\mathbb{C}^{N_d(n_2)})^*$  so that it holds everywhere.  $\square$

## 11. EXAMPLES

The first three theorems below correspond to the three examples given in Table 3 in the introduction.

### 11.1. Taylor projectors

**Theorem 11.1.** *The Newton product of the Taylor projector of degree  $d$  at  $a^1 \in \mathbb{K}^{n_1}$  with the Taylor projector of degree  $d$  at  $a^2 \in \mathbb{K}^{n_2}$ , both being endowed with their fundamental Newton structures, see 8.10, is the Taylor projector of degree  $d$  at  $a = (a^1, a^2) \in \mathbb{K}^n$ ,  $n = n_1 + n_2$ .*

*Démonstration.* Let  $\Pi = [\mathbf{T}_{a^1}^d] \otimes_{\mathbb{N}} [\mathbf{T}_{a^2}^d]$  where the Newton structure on  $\mathbf{T}_{a^s}^d$  is

$$[\mathbf{T}_{a^s}^d] = (\mathbf{T}_{a^s}^0, \mathbf{T}_{a^s}^1, \dots, \mathbf{T}_{a^s}^{d-1}, \mathbf{T}_{a^s}^d),$$

and an adapted functional basis is  $M = (D^{\alpha^s}[a^s] : |\alpha^s| \leq d)$ . Hence, in view of Corollary 9.7, we have

$$\text{CND}(\Pi) = \text{span} \left\{ D^{\alpha^1}[a^1] \otimes D^{\alpha^2}[a^2] : |\alpha^1| + |\alpha^2| \leq d \right\}.$$

Since, see (9.3),  $D^{\alpha^1}[a^1] \otimes D^{\alpha^2}[a^2] = D^{[\alpha^1, \alpha^2]}[a^1, a^2]$  the above space is also that of  $\mathbf{T}_a^d$ .  $\square$

Thus multivariate Taylor projectors can be regarded either as Newton products of univariate Taylor projectors or as liftings, see 7.2.4, of univariate Taylor projectors.

## 11.2. Orthogonal projection

**Theorem 11.2.** *The product of two orthogonal projectors with respect to a measure  $\mathbf{m}^s$  endowed with their natural Newton structures is the orthogonal projector with respect to the tensor product  $\mathbf{m}^1 \otimes \mathbf{m}^2$ .*

The natural Newton structure for an orthogonal projector is given in 8.5.3 :

$$[\mathbf{P}_{d, \mathbf{m}^s}] = (\mathbf{P}_{0, \mathbf{m}^s}, \mathbf{P}_{1, \mathbf{m}^s}, \dots, \mathbf{P}_{d-1, \mathbf{m}^s}, \mathbf{P}_{d, \mathbf{m}^s}), \quad s = 1, 2.$$

*Démonstration.* Let  $[\Pi^s] = [\mathbf{P}_{d, \mathbf{m}^1}]$  and  $\Pi = [\Pi^1] \otimes [\Pi^2]$ . Recall, see 8.5.3, that the spaces  $J_k^s$  corresponding to  $[\Pi^s]$  are given by

$$J_k^s = \text{span} \{ v_\alpha^s : f \mapsto \langle f, e_{\alpha^s} \rangle_{\mathbf{m}^s} : |\alpha| = k \},$$

so that the basis  $O^s = (\mu_\alpha : |\alpha^s| \leq d)$  is adapted to the N-structure  $[\Pi^s]$ . Hence, in view of Corollary 9.7

$$\text{CND}(\Pi) = \text{span} \{ v_{\alpha^1}^1 \otimes v_{\alpha^2}^2 : |\alpha^1| + |\alpha^2| \leq d \} \quad (11.1)$$

Yet, see (9.5),

$$(v_{\alpha^1}^1 \otimes v_{\alpha^2}^2)(f) = \int_{C_1 \times C_2} f(z^1, z^2) e_{(\alpha^1, \alpha^2)}(z^1, z^2) (\mathbf{d}\mathbf{m}^1 \otimes \mathbf{d}\mathbf{m}^2)(z^1, z^2),$$

which shows that the right hand side in (11.1) is the space of interpolation conditions for the orthogonal projection with respect to  $\mathbf{d}\mathbf{m}^1 \otimes \mathbf{d}\mathbf{m}^2$ .  $\square$

## 11.3. Lagrange interpolation

**Theorem 11.3.** *Let  $A^s = \{a_0^s, \dots, a_d^s\}$  a set of  $d+1$  pairwise distinct points in  $\mathbb{C}$  and  $\mathbf{L}[A^s]$  the Lagrange interpolation projector at the points of  $A^s$  endowed with the Newton structure*

$$[\mathbf{L}[A^s]] = (\mathbf{L}[A_0^s], \mathbf{L}[A_1^s], \dots, \mathbf{L}[A_d^s]), \quad A_k^s = \{a_0^s, \dots, a_k^s\}, \quad s = 1, 2.$$

*Then  $[\mathbf{L}[A^1]] \otimes_{\mathbf{N}} [\mathbf{L}[A^2]]$  is the Lagrange interpolation projector at the points  $\{(a_i^1, a_j^2) : 0 \leq i+j \leq d\}$ .*

More generally, if

$$[\mathbf{L}[A^s]] = (\mathbf{L}[A_0^s], \mathbf{L}[A_1^s], \dots, \mathbf{L}[A_d^s]), \quad A_k^s = \{a_{\alpha^s}^s : |\alpha^s| \leq k\} \subset \mathbb{K}^{n_s}, \quad s = 1, 2,$$

is a Lagrange interpolation projector of degree  $d$  in  $\mathbb{K}^{n_s}$  endowed with a N-structure all whose Newton factors are themselves Lagrange interpolation polynomials then  $[\mathbf{L}[A^1]] \otimes_{\mathbf{N}} [\mathbf{L}[A^2]]$  is the Lagrange interpolation projector of degree  $d$  on  $\mathbb{K}^n$ ,  $n = n_1 + n_2$  at the set of points  $\{(a_{\alpha^1}^1, a_{\alpha^2}^2) : |\alpha^1| + |\alpha^2| \leq d\}$ . This is merely a reformulation of a result proved in [20]. We refer to 8.4 and to 8.5.2 for estimations on the numbers of such structures.

In fact, it is easy to prove the following similar results on Lagrange-Hermite or Birkoff projectors, see 7.2.2 for the definition :



- (1) The product of two Lagrange-Hermite N-projectors all whose Newton factors are Lagrange-Hermite projectors is still a Lagrange-Hermite projector.
- (2) The product of two Birkhoff N-projectors all whose Newton factors are Birkhoff projectors is still a Birkhoff projector.

A converse to these results is proved below.

#### 11.4. When is a Newton product Lagrangian ?

We will need the following well-known lemma on the linear independence of distinct Dirac functionals. We include a proof for the convenience of the reader.

**Lemma 11.4.** *A family of pairwise distinct functionals  $\{D^\alpha[a_i] : \alpha \in \Delta_i, i \in \Delta\}$  where  $\Delta$  and for each  $i \in \Delta$ ,  $\Delta_i$  are finite, is linearly independent in all spaces of functions containing the polynomials.*

Here, the elements in  $\Delta$  and in each  $\Delta_i \in \mathbb{N}^n$  are pairwise distinct.

*Démonstration.* We write  $\Delta = \{a_1, \dots, a_k\}$ . Since a sub-family of a linearly independent family is still linearly independent, we may assume that  $\Delta_i = \mathbb{N}_d(n)$  for all  $i$ . The case  $n = 1$  is a mere consequence of univariate Lagrange-Hermite interpolation which says that the functionals are linearly independent on the space of polynomial of degree  $\leq k(d+1) - 1$ . It easily implies the general case as follows. Suppose that we have a linear dependence relation of the form

$$\sum_{i=1}^k \sum_{|\alpha| \leq d} c_{\alpha,i} D^\alpha[a_i] p = 0, \quad p \in \mathcal{P}(\mathbb{K}^n).$$

Applying it with  $p(z) = q(\langle \omega, z \rangle)$  where  $q$  is a univariate polynomial and  $\langle \omega, z \rangle = \sum_{j=1}^n \omega_j z_j$ , we obtain

$$\sum_{i=1}^k \sum_{l=0}^d Q_{l,i}(\omega) q^{(l)}(\langle \omega, a_i \rangle), \quad q \in \mathcal{P}(\mathbb{K}), \quad (11.2)$$

where

$$Q_{l,i}(\omega) = \sum_{|\alpha|=l} c_{\alpha,i} \omega^\alpha. \quad (11.3)$$

If the  $k$  scalars  $\langle \omega, a_i \rangle, i = 1, \dots, k$ , are pairwise distinct, the univariate result gives that  $Q_{l,i}(\omega) = 0$ . Since the  $\langle \omega, a_i \rangle$  are pairwise distinct for a (dense) open set of  $\omega$  each  $Q_{l,i}$  is actually the zero polynomial hence its coefficients  $c_{\alpha,i}$  are zero and the functionals are therefore linearly independent.  $\square$

The following theorem shows that the only way to obtain a Lagrange interpolation projector by forming the Newton product of two projectors on lower dimensional spaces is to start from Lagrange projectors with Newton structures formed of Lagrange projectors.

**Theorem 11.5.** *Let  $\Pi = [\Pi^1] \otimes_{\mathbb{N}} [\Pi^2]$  be a Newton product with  $[\Pi^s]$  a N-polynomial projector of degree  $d$  on  $\mathcal{E}(X_s)$ . If  $\Pi$  is a Lagrange interpolation projector then each  $[\Pi^s]$  and all their Newton factors are also Lagrange interpolation projectors.*

*Démonstration.* Let  $\Pi = \mathbf{L}[A]$  with  $A = \{(a_{\alpha^1}, a_{\alpha^2}) : |\alpha^1| + |\alpha^2| \leq d\} \in \mathbb{K}^n = \mathbb{K}^{n_1} \times \mathbb{K}^{n_2}$ . We denote by  $A_s$  the projection of  $A$  on  $\mathbb{K}^{n_s}$  so that  $A_s = \{a_{\alpha^s} : |\alpha^1| + |\alpha^2| \leq d\} \in \mathbb{K}^{n_s}$ . In view of Corollary 9.11

$$\Pi|_{X_s} = \Pi^s,$$

hence, see (9.7), we have

$$\text{CND}(\Pi^s) = \text{span} \{[a_{\alpha^s}] : a_{\alpha^s} \in A_s\}.$$

Since Dirac functional with distinct points are linearly independent, the above relation shows that the cardinality of  $A_s$  is exactly  $N_d(n_s)$  and  $\Pi^s = \mathbf{L}[A_s]$ . What remains to be proved is that all the Newton factors of  $\Pi^s$  are also Lagrange interpolation projectors. To do that, we apply Lemma 10.5 with  $M$  (resp.  $S$ ) given by the Dirac functionals at the points of  $A_1$  (resp.  $A_2$ ). Since  $A \subset A_1 \times A_2$ , the assumption on  $\text{CND}(\Pi)$  is satisfied and the lemma ensures that, for every  $k$ ,  $0 \leq k \leq d$ , there exists a subset  $A_{sk}$  of  $A_s$  such that

$$\text{CND}(\Pi_k^s) = \text{span} \{[a] : a \in A_{sk}\},$$

where, again since Dirac functional are linearly independent,  $A_{s0} \subset A_{s1} \subset \dots \subset A_{sd} = A_s$  so that  $\Pi_k^s = \mathbf{L}[A_{sk}]$ .  $\square$

**Theorem 11.6.** *Let  $\Pi = [\Pi^1] \otimes_{\mathbf{N}} [\Pi^2]$  be a Newton product with  $[\Pi^s]$  a  $N$ -polynomial projector of degree  $d$  on  $\mathcal{E}(X_s)$ . If  $\Pi$  is a Lagrange-Hermite interpolation projector then each  $[\Pi^s]$  and all their Newton factors are also Lagrange-Hermite interpolation projectors.*

The definition of Lagrange-Hermite interpolation is given in 7.2.2. Here we will assume that

$$\Pi = \mathbf{LH}[S], \quad S = \{(a_i, S_i) : i = 1, \dots, k\}, \quad S_i \subset \mathbb{N}_d(n),$$

where, see (7.13), each  $S_i$  is directed. Observe, see (9.3), that

$$\left. \begin{array}{l} a_i = (a_i^1, a_i^2) \\ \alpha = (\alpha^1, \alpha^2) \end{array} \right\} \implies D^\alpha[a_i] = D^{\alpha^1}[a_i^1] \otimes D^{\alpha^2}[a_i^2],$$

which suggests to define

$$S_i^s = \{D^{\alpha^s}[a_i^s] : \alpha \in S_i\}, \quad S^s = \{(a_i^s, S_i^s) : i = 1, \dots, k\}, \quad s = 1, 2,$$

where it is readily seen that the  $S_i^s$  are themselves directed. Given a Lagrange-Hermite interpolation scheme  $S$  as above we set

$$M(S) = \{D^\alpha[a_i] : \alpha \in S_i, i = 1, \dots, k\}.$$

Thus, we have

$$M(S) \subset M(S^1) \otimes M(S^2). \tag{11.4}$$

*Démonstration.* Reasoning with the restrictions of  $\Pi$  to  $X_s$  as in the proof of the previous theorem, we immediately obtain that  $\Pi^s = \mathbf{LH}[S^s]$ ,  $s = 1, 2$ . The more difficult point is to prove that all the factors of  $\Pi^s$  are also Lagrange-Hermite projectors. Since  $\text{CND}(\Pi) = \text{span} \{M(S)\}$  and  $\text{CND}(\Pi^s) = \text{span} \{M(S^s)\}$ , equation (11.4) allows us to use Lemma 10.5 which ensures that for each  $k = 0, \dots, d$ , there exists a subset  $M_k(S^s)$  of  $M(S^s)$  such that

$$\text{CND}(\Pi_k^s) = \text{span} \{M_k(S^s)\}, \quad k = 0, \dots, d, \quad s = 1, 2$$

with  $M_d(S^s) = M(S^s)$ .

Observe that, since, in view of Lemma 11.4, the elements of  $M_k(S^s)$  are linearly independent, the cardinality of  $M_k(S^s)$  is exactly the dimension of  $\text{CND}(\Pi_k^s)$ , that is  $N_k(n_s)$ . We first claim that

$$M_k(S^s) \subset M_{k+1}(S^s), \quad k = 0, \dots, d-1.$$

Indeed, if  $D^\alpha[a_i^s]$  would belong in  $M_k(S^s)$  but not in  $M_{k+1}(S^s)$ , since  $M_k(S^s) \subset \text{CND}(\Pi_k^s) \subset \text{CND}(\Pi_{k+1}^s)$ , the set  $\{D^\alpha[a_i^s]\} \cup M_{k+1}(S^s)$  would be a linearly independent family (Lemma 11.4) of cardinality  $N_{k+1}(n_s) + 1$  in the space  $\text{CND}(\Pi_{k+1}^s)$  of dimension  $N_{k+1}(n_s)$ .

Setting

$$N_k(S^s) = M_k(S^s) \setminus M_{k-1}(S^s),$$

with  $M_{-1}(S^s) = \emptyset$ , and the usual notation, we have

$$J_k^s = \text{span} \{N_k(S^s)\}, \quad M_k(S^s) = \cup_{j=0}^k N_j(S^s), \quad s = 1, 2.$$

Now, to prove that  $\Pi_k^1$ , say, is a Lagrange-Hermite interpolation projector, it remains to prove the directedness of the conditions in  $M_k(S^1)$ , that is, if  $D^\alpha[a_i^1] \in M_k(S^1)$  and  $\beta \leq \alpha$  we must check that  $D^\beta[a_i^1] \in M_k(S^1)$ . Note that since  $S^1$  is directed,  $D^\beta[a_i^1] \in M(S^1)$  and there exist  $\tau$  such that  $D^\beta[a_i^1] \in N_\tau(S^1)$ . Assume that  $D^\alpha[a_i^1] \in J_t^1$  with  $1 \leq t \leq k$  (the case  $t = 0$  being trivial). According to Corollary 9.8, we have

$$D^\alpha[a_i^1] \otimes \text{span} \{M_{d-t}^2\} = D^\alpha[a_i^1] \otimes \text{CND}(\Pi_{d-t}^2) \subset \text{CND}(\Pi).$$

Since  $\beta \leq \alpha \implies (\beta, \nu) \leq (\alpha, \nu)$ , the above relation and the fact that  $S$  is directed yield that

$$D^\beta[a_i^1] \otimes \text{span} \{M_{d-t}^2\} \subset \text{CND}(\Pi).$$

Thus, the space of all  $\theta$  such that  $D^\beta[a_i^1] \otimes \theta \in \text{CND}(\Pi)$  contains  $\text{CND}(\Pi_{d-t}^2)$ . In view of Corollary 9.8, since  $D^\beta[a_i^1] \in J_\tau^1$  we must have  $\text{CND}(\Pi_{d-t}^2) \subset \text{CND}(\Pi_{d-\tau}^2)$  hence  $\tau \leq t$  and  $D^\beta[a_i^1] \in M_t(S^1) \subset M_k(S^1)$ . This concludes the proof that  $\Pi_k^1$  is a Lagrange-Hermite projector.  $\square$

### 11.5. Newton product of Kergin projectors

The Newton product of Kergin interpolation projectors has nice properties. Let  $A^s = \{a_0^s, \dots, a_d^s\} \subset \Omega_s \in \mathbb{K}^{n_s}$  where  $\Omega_s$  is convex,  $s = 1, 2$ . We use the trivial Newton structure for  $\mathbf{K}_{A^s}$ , namely,

$$[\mathbf{K}_{A^s}] = \left( \mathbf{K}_{A_0^s}, \mathbf{K}_{A_1^s}, \dots, \mathbf{K}_{A_{d-1}^s}, \mathbf{K}_{A_d^s} \right), \quad A_k^s = \{a_0^s, a_1^s, \dots, a_{k-1}^s, a_k^s\}, \quad s = 1, 2.$$

**Theorem 11.7.** *The projector  $[\mathbf{K}_{A^1}] \otimes_N [\mathbf{K}_{A^2}]$  interpolates at the  $\binom{d+2}{2}$  points  $(a_i^1, a_j^2)$  with  $i + j \leq d$ , that is*

$$[\mathbf{K}_{A^2}] \otimes_N [\mathbf{K}_{A^2}](f)(a_i^1, a_j^2) = f(a_i^1, a_j^2), \quad i + j \leq d, \quad f \in \mathcal{E}(\Omega_1 \times \Omega_2).$$

*If a point  $a_i^1$  is repeated  $k$  times and a points  $a_j^2$  is repeated times  $s$  times then all partial derivatives  $D^{(\alpha^1, \alpha^2)}$  at  $(a^1, a^2)$  with  $|\alpha^1| \leq k$  and  $|\alpha^2| \leq s$  are interpolated.*

Moreover, if

$$f(x_1, x_2) = h(\langle \omega_1, x_1 \rangle, \langle \omega_2, x_2 \rangle),$$

where  $\omega_s \in \mathbb{R}^{n_s}$ ,  $h$  is a bivariate function defined on  $\langle \omega_1, \Omega_1 \rangle \times \langle \omega_2, \Omega_2 \rangle$ , see below, then

$$[\mathbf{K}_{A^1}] \otimes_N [\mathbf{K}_{A^2}](f)(x_1, x_2) = [\mathbf{LH}_{\langle \omega_1, A^1 \rangle}] \otimes_N [\mathbf{LH}_{\langle \omega_2, A^2 \rangle}](h)(\langle \omega_1, x_1 \rangle, \langle \omega_2, x_2 \rangle). \quad (11.5)$$

Here,  $\langle \cdot, \cdot \rangle$  is the standard scalar product on  $\mathbb{R}^{n_s}$  and of course  $\langle \omega_1, \Omega_1 \rangle = \{\langle \omega_1, x \rangle : x \in \Omega_1\}$ .

Besides, the Newton structure for the Lagrange-Hermite projectors on the right hand side of (11.5) is implied by that of  $[\mathbf{K}_{A^s}]$ , that is, the  $i$ -th factor of  $[\mathbf{LH}_{\langle \omega_1, A^1 \rangle}]$  is  $\mathbf{LH}_{\langle \omega_1, A_i^s \rangle}$  where

$$\langle \omega_1, A_i^s \rangle = \{\langle \omega_1, a_i^s \rangle : 0 \leq s \leq i\}$$

in which repetition is allowed. Relation (11.5) shows that the Newton product is compatible, in a natural sense, with the lifting process mentioned in subsection 7.2.4.

*Démonstration.* The statements follows immediately from the definition of the Newton product. Let us just comment on 11.5 without giving details. In view of the continuity of both sides, it suffices to prove the relation when  $h$  is a tensor product (of two univariate functions). In that case, the relation follows from applying (9.10) once to the left hand side, once to the right hand side and connecting them with the use of the lifting property(7.20).  $\square$

### 11.6. Projectors preserving homogeneous differential relations

Taylor projectors are Newton products and, at the same time, preserve homogeneous differential relations, see 7.2.5. We prove below that is essentially the sole case for which it occurs. For the notation used here, we refer to 7.2.5.

**Theorem 11.8.** *Let  $\mathbf{C}(\mu_0^s, \dots, \mu_d^s)$ ,  $s = 1, 2$  be two  $D$ -centered projectors of degree  $d$ , that is, two projectors preserving homogeneous differential relations, both projectors are endowed with their canonical Newton structure, that is the  $i$ -th Newton factor of  $\mathbf{C}(\mu_0^s, \dots, \mu_d^s)$  is  $\mathbf{C}(\mu_0^s, \dots, \mu_i^s)$ . We assume, without loss of generality, that the functionals  $\mu_i^s$  are normalized so that  $\mu_i^s(1) = 1$ ,  $i = 0, \dots, d$ ,  $s = 1, 2$ . Then the Newton product*

$$\mathbf{C}(\mu_0^1, \dots, \mu_d^1) \otimes_{\mathbf{N}} \mathbf{C}(\mu_0^2, \dots, \mu_d^2) \quad (11.6)$$

is a  $D$ -centered projector if and only if

$$\mu_0^s = \mu_1^s = \dots = \mu_d^s, \quad s = 1, 2. \quad (11.7)$$

In that case,

$$\mathbf{C}(\mu^1, \dots, \mu^1) \otimes_{\mathbf{N}} \mathbf{C}(\mu^2, \dots, \mu^2) = \mathbf{C}(\mu^1 \otimes \mu^2, \dots, \mu^1 \otimes \mu^2)$$

*Démonstration.* That condition (11.7) is sufficient is easily seen. We prove that it is necessary. Assume that the Newton product (11.6) is  $D$ -centered, that is equal to  $\mathbf{C}(v_0, \dots, v_d)$ . Let  $0 \leq i \leq k \leq d$ . In view of Corollary 9.7, taking into account the particular structure of the interpolation space of  $D$ -centered projectors indicated in (7.21), for  $|\alpha^1| = i$  and  $|\alpha^2| = k - i$  the functional  $D^{\alpha^1} \mu_i^1 \otimes D^{\alpha^2} \mu_{k-i}^2$  lies in the interpolation space of  $\mathbf{C}(v_0, \dots, v_d)$ . Hence, for some coefficient  $\lambda_\theta$  we have

$$D^{\alpha^1} \mu_i^1 \otimes D^{\alpha^2} \mu_{k-i}^2 = \sum_{j=0}^d \sum_{|\theta|=j} \lambda_\theta D^\theta v_j.$$

Applying successively this identity to the monomial  $e_\beta$ , for  $|\beta| = 0, 1, \dots, d$ , following grlex, we obtain that all coefficients  $\lambda_\theta$  vanish except when  $\theta = (\alpha^1, \alpha^2)$  for which it is 1 (recall that the functionals are normalized). Thus,

$$D^{\alpha^1} \mu_i^1 \otimes D^{\alpha^2} \mu_{k-i}^2 = D^{(\alpha^1, \alpha^2)} v_k,$$

which further implies (for instance using the density of polynomials) that  $\mu_i^1 \otimes \mu_{k-i}^2 = v_k$ . Summing up, we have,

$$\mu_i^1 \otimes \mu_{k-i}^2 = v_k, \quad i = 0, \dots, k, \quad k = 0, \dots, d.$$

Applying this relation to functions of the form  $f \otimes 1$  with  $k = d$ , we get that

$$\mu_i^1(f) = v_d(f \otimes 1), \quad i = 0, \dots, d,$$

so that all  $\mu_i^1$  coincide. The conclusion is now immediate.  $\square$

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# PARTIE 2.

We prove that the Newton product of efficient polynomial projectors is still efficient. Various polynomial approximation theorems are established involving Newton product projectors on spaces of holomorphic functions on a neighborhood of a regular compact set, on spaces of entire functions of given growth and on spaces of differentiable functions. Efficient explicit new projectors are presented.

*Article co-écrit avec Jean-Paul Calvi, publié aux Rendiconti del Circolo Matematico di Palermo.*

## THE NEWTON PRODUCT OF POLYNOMIAL PROJECTORS. PART 2 : APPROXIMATION PROPERTIES

### TABLE DES MATIÈRES

## 12. INTRODUCTION

In a recent paper [6] we introduced a new way of forming an approximation operator, precisely a polynomial projector, acting on spaces of functions defined on a (subset of a) space of dimension  $n$  out of two polynomial projectors on spaces of functions defined on a space of smaller dimension  $n_s$ ,  $s = 1, 2$ ,  $n = n_1 + n_2$ . Our process was called the **Newton product** of the (smaller dimensional) projectors. It is related but considerably different from the classical method based on the tensor product of operators. The construction of the Newton product relies on a suitable graduation, called a **Newton structure**, of the interpolation conditions that define the projector. For instance, in the simplest case of a (univariate) Lagrange polynomial projector  $\mathbf{L}[a_0, \dots, a_d]$  with respect to the  $d + 1$  points  $a_0, \dots, a_d$ , the specification of a (useful) Newton structure is equivalent to the specification of an ordering of the interpolations points. In fact, such a specification is already required in the Newton formula (based on divided differences) for classical Lagrange interpolation and there lies the origin of our terminology. In general, see below, a Newton structure is obtained in a more general way but, as a first approximation, the reader may retain that as far as the projectors are determined by a set of points (for instance, Lagrange, Kergin, Hakopian projectors) a (natural) Newton structure is determined by an ordering of these points. For example, if we stay with the ordering of  $A = \{a_0, \dots, a_d\}$  and  $B = \{b_0, \dots, b_d\}$  induced by the indexes,

$$\mathbf{L}[a_0, \dots, a_d] \otimes_{\mathbf{N}} \mathbf{L}[b_0, \dots, b_d], \quad (12.1)$$

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(where  $\otimes_N$  indicates the Newton product) is the bivariate Lagrange interpolation projector at the points  $(a_i, b_j)$ ,  $i + j \leq d$ ; this is a set of interpolation points first considered a long time ago by Bierman [8]. The main goal of the present work is to validate the previous one in answering in the affirmative the obvious question : if the partial projectors are efficient (in a certain sense) approximation operators, will it be the same for their Newton product ? Actually, we tried to write the present paper in order that it could work in the other direction : we relied as little as possible on the previous work in order that the reader interested by the results obtained in this one might decide to go back to [6] for a deeper understanding of the underlying algebraic machinery. The above question is answered positively for projectors on classical spaces of holomorphic functions on a neighborhood of a regular compact set (in the sense of pluripotential theory), on spaces of entire functions of given growth, on spaces of differentiable functions, the results in the latter case being however less precise. The proofs rest on an identical principle that is explained in the next section.

For the convenience of the reader and make the paper self contained, we will conclude this introduction with a definition of the Newton product as well as a few important examples.

The symbol  $\oplus$  denotes a direct sum of vector spaces,  $\otimes$  is used for the usual tensor product, while the symbol  $\otimes_N$  is reserved for the Newton product.

A polynomial projector  $\Pi$  of degree  $d$  on a space of (real or complex) functions  $\mathcal{E} = \mathcal{E}(X)$  defined on  $X \subset \mathbb{K}^n$ ,  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{K} = \mathbb{C}$ , is a continuous linear map on  $\mathcal{E}$  with values in the  $\mathcal{E}$ -subspace  $\mathcal{P}_d(\mathbb{K}^n)$  of real or complex polynomials of total degree at most  $d$ , such that  $\Pi \circ \Pi = \Pi$ . We set

$$\text{CND}(\Pi) = \{v \in \mathcal{E}' : v(f) = v(\Pi(f)), \text{ all } f \in \mathcal{E}\}$$

where  $\mathcal{E}'$  is the space of continuous linear forms on  $\mathcal{E}$ . We call  $\text{CND}(\Pi)$  the **space of interpolation conditions** for  $\Pi$ . For instance if  $\Pi$  is a Lagrange interpolation projector then  $\text{CND}(\Pi)$  is the space of functionals spanned by the Dirac functionals  $\delta_a : f \rightarrow f(a)$  when  $a$  runs among the interpolation points. A list of the classical spaces  $\mathcal{E}$  that that will be considered can be found in Table 4 below.

A **Newton structure** for  $\Pi$  is a direct sum of subspaces  $J_i \subset \text{CND}(\Pi)$ , that is,

$$\text{CND}(\Pi) = J_0 \oplus J_1 \oplus \cdots \oplus J_d,$$

such that, for  $j = 0, \dots, d$ , there exists a projector  $\Pi_j$  of degree  $j$  with

$$\text{CND}(\Pi_j) = J_0 \oplus J_1 \oplus \cdots \oplus J_j.$$

It is shown in [6] that all projectors possess Newton structures. In fact, to obtain a Newton structure is equivalent to find a basis  $(\mu_\alpha : |\alpha| \leq d)$  where  $|\alpha|$  denotes the length of  $\alpha \in \mathbb{N}^n$  such that for  $j = 0, \dots, d$ , the sub-list  $(\mu_\alpha : |\alpha| \leq j)$  is linearly independent on  $\mathcal{P}_j(\mathbb{C}^n)$  and the link with the previous definition is given by

$$\text{CND}(\Pi_j) = \text{span}\{\mu_\alpha : |\alpha| \leq j\} \quad \text{or} \quad J_j = \text{span}\{\mu_\alpha : |\alpha| = j\}, \quad j = 0, \dots, d.$$

A polynomial projector together with a specific Newton structure is called a **Newton-structured projector**. It is important not to confuse a polynomial projector with the richer notion of Newton-structured polynomial projector. To distinguish a mere projector from a Newton structured projector, we use the notation

$$[\Pi] = (\Pi_0, \dots, \Pi_d)$$

to denote the latter.

Now, given two Newton structured projectors  $[\Pi^i]$ ,  $i = 1, 2$ , on  $\mathcal{E}(X_i)$  with respective Newton structure corresponding to the basis  $(\mu_\alpha^i : |\alpha| \leq d)$ , it is shown in [6] that there exists a unique

projector  $\Pi$  on  $\mathcal{E}(X_1 \times X_2)$  such that

$$\text{CND}(\Pi) = \text{span} \left\{ \mu_\alpha^1 \otimes \mu_\beta^2 : |\alpha| + |\beta| = 0, \dots, d \right\}.$$

This projector  $\Pi$  is called the **Newton product** of  $[\Pi^1]$  and  $[\Pi^2]$  and is denoted by  $[\Pi^1] \otimes_N [\Pi^2]$ .

For the sake of notational simplicity, we will sometimes (abusively) write  $\Pi^1 \otimes_N \Pi^2$  instead of  $[\Pi^1] \otimes_N [\Pi^2]$ , especially when the Newton structure we use is clear in the context but, in any case, it must be remembered that the use of the symbol  $\otimes_N$  (for the Newton product) implies that Newton structures have been chosen. The main algebraic formula regarding the computation of a Newton product on a product function is recalled in Theorem 13.5 below.

We will now present several examples of classical or new projectors that are Newton products. In the table below, the Newton structures used appears through the basis  $\mu_\alpha^i$  indicated. We omit the mention of the natural spaces  $\mathcal{E}$  involved. Some of these examples are considered in more details further in this work.

- (1) If, for  $i = 1, 2$ ,  $[\Pi^i]$  is the **Taylor projector** at  $a^i \in \mathbb{K}^{n_i}$  to the order  $d$  Newton structured by

$$\mu_\alpha^i(f) = D^\alpha f(a^i),$$

then  $\Pi^1 \otimes_N \Pi^2$  is the **Taylor projector** at  $(a, b) \in \mathbb{K}^n$ ,  $n = n_1 + n_2$ , to the order  $d$  with

$$\mu_\alpha^1 \otimes \mu_\beta^2(f) = D^{(\alpha, \beta)} f(a), \quad a = (a^1, a^2).$$

- (2) Assume that, for  $i = 1, 2$ ,  $[\Pi^i]$  is the  $L^2(\mathbf{dm}_i)$ -**orthogonal projection** on  $\mathcal{P}_d(\mathbb{C}^{n_i})$  Newton structured by

$$\mu_\alpha^i(f) = \int f \bar{b}(\alpha, \mathbf{dm}_i, \cdot) \mathbf{dm}_i,$$

where  $\mathbf{dm}_i$  is a (sufficiently dense) positive Borel measure supported on a compact set  $K_i \subset \mathbb{C}^{n_i}$  and  $b(\alpha, \mathbf{dm}_i, \cdot)$  denotes the usual orthonormal basis (with leading monomial  $z^\alpha$ ) in  $L^2(\mathbf{dm}_i)$  obtained by the Gram-Schmidt algorithm from the standard monomials basis ordered with the graded lexicographic order. Then  $\Pi^1 \otimes_N \Pi^2$  is the  $L^2(\mathbf{dm}_1 \times \mathbf{dm}_2)$ -**orthogonal projection** on  $\mathcal{P}_d(\mathbb{C}^n)$ ,  $n = n_1 + n_2$ , with

$$\mu_\alpha^1 \otimes \mu_\beta^2(f) = \iint f \bar{b}((\alpha, \beta), \mathbf{dm}_1 \otimes \mathbf{dm}_2, \cdot) \mathbf{dm}_1 \otimes \mathbf{dm}_2.$$

- (3) Assume that, for  $i = 1, 2$ ,  $[\Pi^i]$  is the **Lagrange interpolation projector**  $\mathbf{L}[A^i]$  in  $\mathcal{P}_d(\mathbb{C}^{n_i})$  at the (unisolvent) set of interpolation points  $A^i = \{a_\alpha^i : |\alpha| \leq d\}$  structured by

$$\mu_\alpha^i(f) = f(a_\alpha^i),$$

where the interpolation points  $a_\alpha^i$  are ordered in such a way that the sets  $A_j^i = \{a_\alpha^i : |\alpha| \leq j\}$  are unisolvent for Lagrange interpolation in  $\mathcal{P}_j(\mathbb{C}^{n_i})$ ,  $j = 0, \dots, d$ . Then  $[\Pi^1] \otimes_N [\Pi^2]$  is the **Lagrange interpolation projector** in  $\mathcal{P}_d(\mathbb{C}^n)$ ,  $n = n_1 + n_2$ , at the (unisolvent) set of interpolation points

$$A = \left\{ (a_\alpha^1, a_\beta^2) : |\alpha| + |\beta| \leq d \right\} \subset \mathbb{C}^n.$$

We refer to [20] for details.



- (4) Assume that  $[\Pi^1]$  is the **Lagrange interpolation projector**  $\mathbf{L}[A^1]$  structured as above and  $[\Pi^2]$  is the **Kergin interpolation projector**  $\mathbf{K}[A^2]$  in  $\mathcal{P}_d(\mathbb{C})$  with  $A^2 = \{z_0, \dots, z_d\} \subset \mathbb{C}^{n_2}$  structured by, see [6] for details,

$$\mu_{\alpha}^2(f) = \int_{S_{|\alpha|}} D^{\alpha} f(z_0 + \sum_{i=1}^{|\alpha|} t_i(z_i - z_0)) dt,$$

where  $S_k = \{(t_1, \dots, t_k) \in [0, 1]^k : \sum_{i=1}^k t_i \leq 1\}$  denotes the ordinary simplex in  $\mathbb{R}^k$  (and  $dt$  the ordinary Lebesgue measure on it) then

$$[\Pi^1] \otimes_{\mathbb{N}} [\Pi^2] = \mathbf{L}[A^1] \otimes_{\mathbb{N}} \mathbf{K}[A^2]$$

is the projector the space of interpolation conditions is spanned by the functionals

$$f \rightarrow \int_{S_{|\beta|}} D^{(0,\beta)} f(a_{\alpha}^1, z_0 + \sum_{i=1}^{|\beta|} t_i(z_i - z_0)) dt, \quad |\alpha| + |\beta| \leq d,$$

where  $D^{(0,\beta)}$  indicates derivation with respect to the last  $n_2$  complex variables. As will be shown in the sequel, such a mixed Newton product seems to be particularly useful in the case  $n_1 = 1$  and  $n_2 = 2$ .

## 13. STATING THE PROBLEM

### 13.1. Newton sequences

For each  $d \in \mathbb{N}$ , we let  $\Pi_d$  denote a polynomial projector of degree  $d$  on a space of (real or complex) functions  $\mathcal{E}$ . The sequence  $\mathcal{N} = (\Pi_0, \Pi_2, \dots)$  will be called a **Newton sequence** on  $\mathcal{E}$  if :

$$\text{For all } d \in \mathbb{N}, \text{CND}(\Pi_d) \subset \text{CND}(\Pi_{d+1}), \quad (13.1)$$

or, equivalently,

$$(\Pi_0, \dots, \Pi_d) \text{ defines a Newton structure for } \Pi_d. \quad (13.2)$$

We shall denote the Newton-structured projector in (13.2) by  $[\mathcal{N}]_d$ .

Given such a Newton sequence  $\mathcal{N}$ , we set  $\pi_0 = \Pi_0$  and, for  $k \geq 1$ ,  $\pi_k = \Pi_k - \Pi_{k-1}$ . Then  $\pi_k$  is the  $k$ -th **Newton summand** for  $[\mathcal{N}]_d$  and this for any  $d \geq k$ , so that

$$\Pi_d = \sum_{k=0}^d \pi_k, \quad d \in \mathbb{N}. \quad (13.3)$$

Likewise, there exists a sequence of spaces  $J_i$ ,  $i \in \mathbb{N}$ , such that

$$\text{CND}(\Pi_d) = \bigoplus_{0 \leq i \leq d} J_i, \quad d \in \mathbb{N}, \quad (13.4)$$

**Example 13.1.** We use the projectors presented in the introduction.

(A) The sequence for which  $\Pi_d$  is the **Taylor projector** at a point  $a$  to the order  $d$ ,  $\Pi_d = \mathbf{T}_a^d$ , is a basic example of Newton sequence for which  $J_i = \text{span}\{f \rightarrow D^{|\alpha|} f(a) : |\alpha| = i\}$ .

(B) Another fundamental example is furnished by the **orthogonal projectors**  $\mathbf{P}_{d,\mathbf{dm}}$  with respect to a sufficiently dense measure  $\mathbf{dm}$  where  $J_i = \text{span}\{f \rightarrow \langle f, b(\alpha, \mathbf{dm}, \cdot) \rangle : |\alpha| = i\}$ , see Example 2 in the introduction.

(C) A Newton sequence of **Lagrange interpolation projectors** will be obtained with  $\Pi_d = \mathbf{L}[A_d]$  with the condition that  $A_d \subset A_{d+1}$  for  $d \in \mathbb{N}$  where  $A_d$  is the (unisolvent) set of interpolation points for  $\mathbf{L}[A_d]$ , for which  $J_i = \text{span}\{f \rightarrow f(a) : a \in A_i \setminus A_{i-1}\}$ ,  $i \geq 1$ .

(D) Likewise a Newton sequence of **Kergin interpolation projectors** will be obtained with  $\Pi_d = \mathbf{K}[A_d]$  with the same condition that the set of interpolation points are nested, that is  $A_d \subset A_{d+1}$  for  $d \in \mathbb{N}$ . Typically, starting from a sequence  $a_d$  of points, we will take  $A_d = \{a_0, \dots, a_d\}$  so that  $\Pi_d = \mathbf{K}[a_0, \dots, a_d]$ ,  $d \in \mathbb{N}$ , see the introduction for a description of the spaces  $J_i$  and [6] for further details.

**Definition 13.2.** Let  $\mathcal{F}$  be a (topological vector) space of functions containing  $\mathcal{E}$  (hence also the polynomials) such that the injection  $f \in \mathcal{E} \rightarrow f \in \mathcal{F}$  is continuous. Note that  $\Pi_d$  is continuous as a linear map from  $\mathcal{E}$  to  $\mathcal{F}$ . We say that a **Newton sequence**  $\mathcal{N} = (\Pi_0, \Pi_1, \dots)$  on  $\mathcal{E}$  is  **$\mathcal{F}$ -converging** when, for all  $f \in \mathcal{E}$ ,  $\Pi_d(f)$  converges to  $f$  in  $\mathcal{F}$  as  $d \rightarrow \infty$ . When precision is needed we say that  $\mathcal{N}$  is  **$\mathcal{F}$ -converging on  $\mathcal{E}$** . When  $\mathcal{F} = \mathcal{E}$  we just say that  $\mathcal{N}$  is **converging on  $\mathcal{E}$** . If convergence holds only for all functions in a subspace  $E$  of  $\mathcal{E}$ , we say that  $\mathcal{N}$  is  **$\mathcal{F}$ -converging on  $E \subset \mathcal{E}$** .

Table 4 collects the various spaces  $\mathcal{E}$ ,  $E$  and  $\mathcal{F}$  that will be used in the sequel.

Space $\mathcal{E}$ or space $E$	Space $\mathcal{F}$
$\mathcal{H}(K)$ , the space of holomorphic function on a neighborhood of (a regular polynomially convex compact set) $K$ .	$\mathcal{C}(K)$ the space of continuous functions on $K$ , then $\mathcal{H}(K)$ .
$\mathcal{H}(\mathbb{C}^n)$ , the space of entire functions and $E$ a subspace of entire functions of given growth.	$\mathcal{H}(\mathbb{C}^n)$ .
$\mathcal{C}^m(K)$ the standard space of $m$ times continuously differentiable functions on the interior of $K$ (a fat compact set, see Section 16, whose all derivatives of order $\leq m$ extend continuously to $K$ .	$\mathcal{C}(K)$ .

TABLE 4. Function spaces, all endowed with their usual topology.

Classical results in approximation theory are naturally expressed in the above terminology. For instance, the Newton sequence the  $d$ -th element of which is the Taylor projector at the origin 0 to the order  $d$  is converging on the space  $\mathcal{H}(\{0\})$  formed of holomorphic functions on a neighborhood of the origin (endowed with its usual limit inductive topology). Classical univariate Lagrange interpolation theory is concerned with the search for conditions on a given sequence  $(a_0, a_1, a_2, \dots)$  ensuring that  $(\Pi_0, \Pi_2, \dots)$  with  $\Pi_d = \mathbf{L}[a_0, \dots, a_d]$  is  $\mathcal{F}$ -converging on  $\mathcal{E}$  where  $\mathcal{E}$  is a classical space of smooth functions on an interval  $[a, b]$ , and  $\mathcal{F} = \mathcal{C}([a, b])$ , or  $\mathcal{F} = \mathcal{H}(K)$ , see [50, 49, 26], or  $E \subset \mathcal{E} = \mathcal{F} = \mathcal{H}(\mathbb{C})$  is a space of entire functions of given growth, see [28].

### 13.2. The Newton product of Newton sequences

By using the Newton product of polynomial projectors, we can easily construct a natural Newton sequence on  $\mathcal{E}(X_1 \times X_2)$  when we are given two Newton sequences  $\mathcal{N}^1 = (\Pi_0^1, \Pi_1^1, \dots)$  on  $\mathcal{E}(X_1)$  and  $\mathcal{N}^2 = (\Pi_0^2, \Pi_1^2, \dots)$  on  $\mathcal{E}(X_2)$ . Here, by writing  $\mathcal{E}(X_s)$  we emphasize that the functions in  $\mathcal{E}$  are defined on  $X_s$ . The connection between  $\mathcal{E}(X_s)$  and  $\mathcal{E}(X_1 \times X_2)$  will be obvious in all cases considered.

For all  $d \in \mathbb{N}$ , we compute the Newton product, see (13.2),

$$\Pi_d := [\mathcal{N}^1]_d \otimes_{\mathbb{N}} [\mathcal{N}^2]_d = (\Pi_0^1, \dots, \Pi_d^1) \otimes_{\mathbb{N}} (\Pi_0^2, \dots, \Pi_d^2). \quad (13.5)$$

In fact if  $(J_i^s)$ ,  $s = 1, 2$  is the sequence of spaces associated to  $\mathcal{N}^s$  as in (13.4), according to [6, Corollary 4.8], we have

$$\text{CND}(\Pi_d) = \bigoplus_{i+j \leq d} J_i^1 \otimes J_j^2 \quad (13.6)$$

which the reader may read as a characterization property of  $\Pi_d$ .

The above construction provides a sequence of polynomial projectors and this sequence is actually a Newton sequence itself .

**Lemma 13.3.** *The sequence  $(\Pi_0, \Pi_1, \dots)$  where  $\Pi_d$  is defined as in (13.5) is a Newton sequence on  $\mathcal{E}(X_1 \times X_2)$ .*

*Démonstration.* According to (13.1), we need to prove that  $\text{CND}(\Pi_d) \subset \text{CND}(\Pi_{d+1})$  for all  $d \in \mathbb{N}$ . Since  $\mathcal{N}^s$  is a Newton sequence, there exists a sequence of spaces  $J_i^s$ ,  $i \in \mathbb{N}$ , see (13.4), such that

$$\text{CND}(\Pi_k^s) = \bigoplus_{0 \leq i \leq k} J_i^s, \quad d \in \mathbb{N}, \quad s = 1, 2.$$

In view of (13.5) and using (13.6) for both equalities, we have

$$\text{CND}(\Pi_d) = \bigoplus_{i+j \leq d} J_i^1 \otimes J_j^2 \subset \bigoplus_{i+j \leq d+1} J_i^1 \otimes J_j^2 = \text{CND}(\Pi_{d+1}). \quad \square$$

We are now in position to fix the terminology that will be used in this paper.

**Definition 13.4.** The above Newton sequence will be denoted by  $\mathcal{N} = \mathcal{N}^1 \otimes_{\mathbb{N}} \mathcal{N}^2$  and called the **Newton product** of  $\mathcal{N}^1$  by  $\mathcal{N}^2$ . In accordance with the terminology introduced for the Newton product,  $\mathcal{N}^1$  (resp.  $\mathcal{N}^2$ ) will be called the left (resp. right) **divisor** of  $\mathcal{N}$ .

Now, the obvious approximation problem is as follows.

*Research problem 1.* Suppose that  $\mathcal{N}^s$  is a  $\mathcal{F}_s$ -converging Newton sequence on  $\mathcal{E}(X_s)$ ,  $s = 1, 2$ , is  $\mathcal{N}^1 \otimes_{\mathbb{N}} \mathcal{N}^2$  a  $\mathcal{F}$ -converging sequence on  $\mathcal{E}(X_1 \times X_2)$  where  $\mathcal{F}$  naturally depends on  $\mathcal{F}_s$ ,  $s = 1, 2$ ?

We will show that the answer is generally positive, thus showing that the Newton product leads to the construction of new effective approximation projectors. Various explicit examples will be given in the sequel.

### 13.3. Strategy of proof

We describe the simple general strategy that we follow in answering Problem 1 for different spaces. We felt it preferable to provide distinct proofs in the three main cases that we study rather than to search for a very general statement which would require assumptions whose verifications would require essentially the same amount of work.

Assume that we work with  $\mathcal{E}(X_1)$ ,  $\mathcal{E}(X_2)$  and  $\mathcal{E}(X_1 \times X_2)$  with  $X_s \subset \mathbb{R}^{n_s}$  or  $\mathbb{C}^{n_s}$ . We use the notation introduced in the previous subsection, in particular  $\Pi_d$  is defined as in (13.5).

*Step 1.* We find product polynomials

$$p_{\alpha, \beta}(z^1, z^2) = p_{\alpha}^1(z^1) p_{\beta}^2(z^2) \quad \text{with } \deg p_{\alpha}^1 = |\alpha| \text{ and } \deg p_{\beta}^2 = |\beta|, \quad (13.7)$$

such that, for all  $f \in \mathcal{E}(X_1 \times X_2)$ , we have

$$f = \sum_{j=0}^{\infty} \sum_{|\alpha|+|\beta|=j} c_{\alpha\beta}(f) p_{\alpha,\beta} \quad (13.8)$$

where the  $c_{\alpha}$  are functionals on  $\mathcal{E}(X_1 \times X_2)$  and convergence holds in  $\mathcal{E}(X_1 \times X_2)$  (hence also in  $\mathcal{F}$ ).

*Step 2.* Since  $\Pi_d$  is a projector of degree  $d$ , it coincides with the identity on the polynomials of degree  $\leq d$  so that we have

$$\Pi_d \left( \sum_{j=0}^d \sum_{|\alpha|+|\beta|=j} c_{\alpha\beta}(f) p_{\alpha,\beta} \right) = \sum_{j=0}^d \sum_{|\alpha|+|\beta|=j} c_{\alpha\beta}(f) p_{\alpha,\beta}$$

and, consequently, it follows from (13.8), together with the continuity of  $\Pi_d : \mathcal{E}(X_1 \times X_2) \rightarrow \mathcal{F}$ , that

$$f - \Pi_d(f) = \sum_{j=d+1}^{\infty} \sum_{|\alpha|+|\beta|=j} c_{\alpha\beta}(f) \{p_{\alpha,\beta} - \Pi_d(p_{\alpha,\beta})\} \quad (13.9)$$

$$= \sum_{j=d+1}^{\infty} \sum_{|\alpha|+|\beta|=j} c_{\alpha\beta}(f) p_{\alpha,\beta} - \sum_{|\alpha|+|\beta|=d+1}^{\infty} c_{\alpha\beta}(f) \Pi_d(p_{\alpha,\beta}), \quad (13.10)$$

where convergence holds in  $\mathcal{F}$ . Observe that, since the first series tends to 0 in  $\mathcal{F}$  as  $d \rightarrow \infty$ , we might restrict ourselves to show that  $\sum_{j=d+1}^{\infty} \sum_{|\alpha|+|\beta|=j} c_{\alpha\beta}(f) \Pi_d(p_{\alpha,\beta})$  goes to 0 as  $d$  tends to  $\infty$  in (13.10). It is however equally simple and somewhat more elegant to work with the right hand side of (13.9).

At this point, we need an algebraic formula established in [6] for the computation of a Newton product projector applied to a product function :

*Theorem 13.5.* Let, for  $i = 1, 2$ ,  $[\Pi^i] = (\Pi_0^i, \dots, \Pi_d^i)$  be a Newton structured polynomial projector of degree  $d$  on  $\mathcal{E}(X_i)$ ,  $X_i \in \mathbb{K}^{n_i}$ . If  $f_i \in \mathcal{E}(X_i)$  we denote by  $f_1 f_2$  the product function defined by  $(f_1 f_2)(x^1, x^2) = f_1(x^1) f_2(x^2)$  then we have

$$\Pi(f_1 f_2) = \sum_{(i,j) \in \mathbb{N}_d(2)} \pi_i^1(f_1) \pi_j^2(f_2), \quad f_i \in \mathcal{E}(X^i), \quad (13.11)$$

where the  $\pi_i^s$  denote the Newton summands corresponding to  $\Pi^s$ , see (13.3).

*Démonstration.* See [6, Theorem 4.5 (2)]. □

We may now state the key computational lemma.

**Lemma 13.6.** We use the notation above, see in particular (13.7). Assume that  $|\alpha| + |\beta| \geq d + 1$  and let

$$B(d, \alpha, \beta) = \{(i_1, i_2) \in \mathbb{N}^2 : d + 1 \leq i_1 + i_2; i_1 \leq |\alpha|; i_2 \leq |\beta|\}. \quad (13.12)$$

We have

$$p_{\alpha,\beta} - \Pi_d(p_{\alpha,\beta}) = \sum_{(i_1, i_2) \in B(d, \alpha, \beta)} \pi_{i_1}^1(p_{\alpha}^1) \pi_{i_2}^2(p_{\beta}^2).$$

*Démonstration.* Let  $d_1 = |\alpha|$  and  $d_2 = |\beta|$ . Since  $\Pi_{d_s}$  is a projector of degree  $d_s$  we have

$$p_{\alpha,\beta} = p_{\alpha}^1 p_{\beta}^2 = \Pi_{d_1}(p_{\alpha}^1) \Pi_{d_2}(p_{\beta}^2),$$

and, in view of (13.3),

$$p_{\alpha,\beta} = \left( \sum_{i_1 \leq d_1} \pi_{i_1}(p_\alpha^1) \right) \cdot \left( \sum_{i_2 \leq d_2} \pi_{i_2}(p_\beta^2) \right) = \sum_{i_1 \leq d_1, i_2 \leq d_2} \pi_{i_1}(p_\alpha^1) \pi_{i_2}(p_\beta^2). \quad (13.13)$$

On the other hand, since  $p_{\alpha,\beta}$  is a product function, the product formula in Theorem 13.5 yields

$$\Pi_d(p_{\alpha,\beta}) = \sum_{i_1+i_2=d} \pi_{i_1}^1(p_\alpha^1) \pi_{i_2}^2(p_\beta^2). \quad (13.14)$$

The relation follows immediately on subtracting (13.14) from (13.13).  $\square$

*Step 3.* The properties of the factors  $\mathcal{N}^s$  enter into play when estimating  $p_{\alpha,\beta} - \Pi_d(p_{\alpha,\beta})$  with the help of the formula given in the previous lemma. To estimate the terms in the sum, we invoke the assumption that the divisor sequences  $\mathcal{N}^s$ ,  $s = 1, 2$ , are converging via the use of a uniform boundedness principle (Banach-Steinhaus theorem), [46, Chapter 2]. It is therefore essential to work on spaces where the principle holds.

## 14. SPACES OF HOLOMORPHIC FUNCTIONS ON A NEIGHBORHOOD OF A REGULAR COMPACT SET

### 14.1. The space $\mathcal{H}(K)$

The space  $\mathcal{H}(K)$  formed of all functions holomorphic on a neighborhood of the compact set  $K$  is endowed with the usual inductive limit of the spaces  $H(\Omega)$  or, equivalently,  $A(\Omega)$  where  $\Omega$  is an open (bounded) neighborhood of  $K$  and  $H(\Omega)$  denotes the space of holomorphic functions on  $\Omega$  with the topology of uniform convergence on compact subsets, while  $A(\Omega)$  is the *Banach space* of the functions continuous on  $\overline{\Omega}$  and holomorphic on  $\Omega$  with the usual sup-norm on  $\overline{\Omega}$ . Thus, for instance, a sequence  $f_n$  converge to  $f$  in  $\mathcal{H}(K)$  if and only if there exists an open bounded neighborhood  $\Omega$  of  $K$  such that  $f$  and  $f_n$  are in  $A(\Omega)$  for all  $n$  and  $f_n$  converges uniformly to  $f$  on  $\overline{\Omega}$ . A linear map from  $\mathcal{H}(K)$  onto another topological vector space  $\mathcal{F}$  is continuous if and only if all its restrictions to the spaces  $A(\Omega)$ ,  $\Omega \supset K$  are continuous.

### 14.2. Some tools from pluripotential theory

We recall some basic facts from pluripotential complex theory which are required in the sequel. A good general reference is the book by Klimek [33]. The survey by Levenberg [34] also contains the required material (and much more) and is more approximation-theory oriented. The non-specialist reader may freely assume that the compact sets  $K$  we work with are convex in the ordinary geometrical sense. This would imply a limited loss of generality only, at least from a practical point of view.

Let  $K$  be a regular polynomially convex compact set in  $\mathbb{C}^n$ . Its continuous pluri-subharmonic Green-Siciak extremal function is denoted by  $V_K$ . Recall that this function can be defined for instance as

$$V_K(z) = \max \left( 0, \sup \left\{ \frac{1}{\deg p} \ln |p(z)| : p \in \mathcal{P}(\mathbb{C}^n), \|p\|_K \leq 1 \right\} \right).$$

For  $R > 1$ , we define the bounded open set  $K_R = \{z \in \mathbb{C}^n : V_K(z) < \ln R\}$ .

A first use of these level sets appears in the Bernstein-Walsh-Siciak inequality [33] which states that

$$\|p\|_{\overline{K_R}} \leq R^{\deg p} \|p\|_K, \quad p \in \mathcal{P}_d(\mathbb{C}^n), \quad R > 1. \quad (14.1)$$

The level sets  $K_R$  are further related to the rate of polynomial approximation of a holomorphic function on  $K$ . Namely, if

$$\text{dist}(f, \mathcal{P}_d(\mathbb{C}^n)) = \inf\{\|f - p\|_K : p \in \mathcal{P}_d(\mathbb{C}^n)\},$$

then  $f$  is holomorphic (or extends to a holomorphic function) on  $K_R$  if and only if

$$\limsup_{d \rightarrow \infty} \text{dist}(f, \mathcal{P}_d(\mathbb{C}^n))^{1/d} < 1/R.$$

In particular, setting

$$\limsup_{d \rightarrow \infty} \text{dist}(f, \mathcal{P}_d(\mathbb{C}^n))^{1/d} = \frac{1}{\rho(f)}, \quad (14.2)$$

a continuous function  $f$  on  $K$  extends to a function in  $\mathcal{H}(K)$  if and only if  $\rho(f) > 1$  and, more precisely, for all  $R < \rho(f)$ ,  $f$  extends to a function in  $A(K_R)$ .

### 14.3. Bernstein-Markov measures

An asymptotically optimal approximation polynomial for functions in  $\mathcal{H}(K)$ ,  $K$  as above, can be obtained as a Fourier expansion with respect to a suitable probability measure as follows. One says that a probability measure  $\mu$  with compact support on  $K$  is a **Bernstein-Markov measure** if, for all  $\varepsilon > 0$ , there exists a constant  $C(\varepsilon) = C(\varepsilon, \mu)$  such that for all polynomial  $p$  we have

$$\|p\|_K \leq C(\varepsilon)(1 + \varepsilon)^{\deg(p)} \|p\|_2, \quad \|p\|_2 = \sqrt{\int_K |p(z)|^2 d\mu(z)}. \quad (14.3)$$

In other words, the  $L^2$ -norm of polynomial behaves asymptotically essentially like the sup-norm on  $K$ . By a theorem of Nguyen and Zeriahi [38], on a regular compact set  $K$  as above, such a measure always exists. See also [17] for a general discussion on Bernstein-Markov measures.

Given such a measure, we denote by  $(b_\alpha)$  the orthonormal sequence of polynomials obtained by the Gram-Schmidt process from the monomial sequence  $(z^\alpha)$  ordered with respect to the graded lexicographic order. In particular,  $z^\alpha$  is the leading monomial in  $b_\alpha$  and  $\deg b_\alpha = |\alpha|$ . When it is necessary to clarify the measure we use, instead of  $b_\alpha(z)$ , we use the notation  $b(\alpha, \mu, z)$  as in the introduction. The corresponding orthogonal projection is

$$\mathbf{P}_{d, \mathbf{dm}}(f) = \sum_{|\alpha| \leq d} c_\alpha(f) b_\alpha, \quad c_\alpha(f) = \langle f, b_\alpha \rangle = \int_K f(z) \bar{b}_\alpha(z) \mathbf{dm}(z), \quad f \in \mathcal{H}(K). \quad (14.4)$$

It is well known, see [53], that,  $\mathbf{dm}$  being a Bernstein-Markov measure,

$$\limsup_{d \rightarrow \infty} \{\|f - \mathbf{P}_{d, \mathbf{dm}}(f)\|_K\}^{1/d} = \limsup_{d \rightarrow \infty} \text{dist}(f, \mathcal{P}_d(\mathbb{C}^n))^{1/d}. \quad (14.5)$$

This is easily shown as follows. Taking  $t_d$ , a best (uniform) approximation of degree  $d$  of  $f$  on  $K$ , i.e. such that  $\text{dist}(f, \mathcal{P}_d(\mathbb{C}^n)) = \|f - t_d\|_K$ , we have

$$\|f - t_d\|_K \leq \|f - \mathbf{P}_{d, \mathbf{dm}}(f)\|_K \leq C(1 + \varepsilon)^d \|f - \mathbf{P}_{d, \mathbf{dm}}(f)\|_2 \quad (14.6)$$

$$\leq C(1 + \varepsilon)^d \|f - t_d\|_2 \leq C(1 + \varepsilon)^d \|f - t_d\|_K, \quad (14.7)$$

where we use that the orthogonal projection furnishes the best  $L^2$  approximant of  $f$  on the second line. Now, (14.5) immediately follows.

Likewise, since, by orthogonality,

$$c_\alpha(f) = \langle f, b_\alpha \rangle = \langle f - \mathbf{P}_{|\alpha|-1, \mathbf{dm}}(f), b_\alpha \rangle,$$

by the Cauchy-Schwarz inequality, we have

$$\begin{aligned} |c_\alpha(f)| &\leq \|f - \mathbf{P}_{|\alpha|-1, \mathbf{dm}}(f)\|_2 \leq \|f - t_{|\alpha|-1}\|_2 \\ &\leq \|f - t_{|\alpha|-1}\|_2 \leq \text{dist}(f, \mathcal{P}_{|\alpha|-1}(\mathbb{C}^n)). \end{aligned} \quad (14.8)$$

Note that, in fact, we have the series expansion

$$f = \sum_{j=0}^{\infty} \sum_{|\alpha|=j} c_\alpha(f) b_\alpha, \quad f \in \mathcal{H}(K), \quad (14.9)$$

where the convergence holds in  $\mathcal{H}(K)$ . For this and other applications to pluripotential theory, we refer to [53].

Observe finally that, when applied with  $p = b_\alpha$ , inequality (14.3) reads as

$$\|b_\alpha\|_K \leq C(\varepsilon)(1 + \varepsilon)^{|\alpha|}, \quad |\alpha| = 0, 1, 2, \dots \quad (14.10)$$

#### 14.4. Cartesian products

We now collect a few facts that we will need later about the above notions in relation to Cartesian products of sets.

(i) The Cartesian product  $K_1 \times K_2 \subset \mathbb{C}^{n_1} \times \mathbb{C}^{n_2}$  of two regular polynomially convex sets is still regular polynomially convex and, see [33],

$$V_{K_1 \times K_2}(z^1, z^2) = \max(V_{K_1}(z^1), V_{K_2}(z^2)).$$

In particular,

$$(K_1 \times K_2)_R = K_{1R} \times K_{2R}.$$

(ii) The product  $\mu_1 \times \mu_2$  of a Bernstein-Markov probability measure  $\mu_1$  on  $K_1$  by a Bernstein-Markov probability measure  $\mu_2$  on  $K_2$  is a Bernstein-Markov measure on  $K_1 \times K_2$ , see [16, Lemma 2, p. 290]. Moreover, as is readily checked, the corresponding orthonormal polynomials satisfy the relation

$$b((\alpha^1, \alpha^2), \mu_1 \times \mu_2, z) = b(\alpha^1, \mu_1, z^1) \times b(\alpha^2, \mu_2, z^2), \quad z = (z^1, z^2). \quad (14.11)$$

#### 14.5. The convergence theorem

Roughly, we prove that the Newton product of projectors that approximate efficiently holomorphic functions on regular compact sets itself efficiently approximates holomorphic function on the Cartesian product of the compact sets.

**Theorem 14.1.** *Let  $K_s$  be a regular polynomially convex set in  $\mathbb{C}^{n_s}$  and  $\mathcal{N}^s = (\Pi_0^s, \Pi_1^s, \dots)$  a Newton sequence on  $\mathcal{H}(K_s)$ ,  $s = 1, 2$ . If  $\mathcal{N}^s$  is converging on  $\mathcal{H}(K_s)$  for  $s = 1, 2$  then  $\mathcal{N}^1 \otimes_{\mathbb{N}} \mathcal{N}^2$  is converging on  $\mathcal{H}(K_1 \times K_2)$ .*

**Lemma 14.2.** *With the assumptions of the theorem, for  $s = 1, 2$  and for all  $R > 1$ , there exists a constant  $\gamma_s(R)$  such that*

$$\|\Pi_d^s(f)\|_K \leq \gamma_s(R) \|f\|_{\overline{K_{sR}}}, \quad f \in A(K_{sR}).$$

*Démonstration.* Each projector  $\Pi_d^s$  is a continuous projector from  $\mathcal{H}(K_s)$  onto  $\mathcal{P}_d(\mathbb{C}^{n_s})$ . Using the definition of the topology on  $\mathcal{H}(K_s)$  and the fact that every norm is equivalent on  $\mathcal{P}_d(\mathbb{C}^{n_s})$ ,  $\Pi_d^s$  defines a continuous projector from the Banach space  $A(K_R)$  to the Banach space  $\mathcal{C}(K)$  of continuous function on  $K$ . The fact that  $\mathcal{N}^s$  is converging yields that  $\Pi_d^s(f)$  converges to  $f$  uniformly on  $K$  for all  $f \in A(K_R)$  hence, for such  $f$ , the sequence  $(\Pi_d^s(f))$  is bounded in  $\mathcal{C}(K)$ . Now, the claim results from an application of the uniform boundedness principle (Banach-Steinhaus Theorem) on the Banach space  $A(K_R)$ .  $\square$

**Corollary 14.3.** *Likewise, using the Newton summands,  $\pi_d^s$ , see (13.3), for  $s = 1, 2$  and for all  $R > 1$ , there exists a constant  $\delta_s(R)$  such that*

$$\|\pi_d^s(f)\|_K \leq \delta_s(R) \|f\|_{\overline{K_{sR}}}, \quad f \in A(K_{sR}).$$

*Démonstration.* Use the lemma and the fact that  $\pi_d^s = \Pi_{d+1}^s - \Pi_d^s$ . □

*Proof of Theorem 14.1.* We take a Bernstein-Markov measure  $\mu_s$  on  $K_s$  and consider its product  $\mu = \mu_1 \times \mu_2$  which is Bernstein-Markov on  $K_1 \times K_2$ , see Subsection 14.3. Following the strategy explained in Subsection 13.3, we start from the expansion

$$f = \sum_{j=0}^{\infty} \sum_{|\alpha|+|\beta|=j} c_{\alpha\beta}(f) b_{\alpha,\beta}, \quad c_{\alpha\beta}(f) = \iint_{K_1 \times K_2} f(z) \bar{b}_{\alpha,\beta}(z) d\mu_1(z^1) d\mu_2(z^2), \quad (14.12)$$

where  $f$  is any fixed element in  $\mathcal{H}(K)$ ,  $z = (z^1, z^2)$ ,  $z^s \in K_s$ , and, see (14.11),

$$b_{\alpha,\beta}(z^1, z^2) = b(\alpha, \mu_1, z^1) \times b(\beta, \mu_2, z^2). \quad (14.13)$$

Recall that convergence in (14.12) holds in  $\mathcal{H}(K)$ . Next, a use of equation (13.9) and Lemma 13.6 gives

$$f - \Pi_d(f) = \sum_{j=d+1}^{\infty} \sum_{|\alpha|+|\beta|=j} c_{\alpha\beta}(f) \sum_{(i_1, i_2) \in B(d, \alpha, \beta)} \pi_{i_1}^1(b(\alpha, \mu_1, \cdot)) \pi_{i_2}^2(b(\beta, \mu_2, \cdot)), \quad (14.14)$$

where  $B(d, \alpha, \beta)$  is defined in (13.12). We will use this expression to show that  $f - \Pi_d(f)$  converges uniformly to 0 on  $K$ . Define as above,

$$\limsup_{d \rightarrow \infty} \text{dist}(f, \mathcal{P}_d(\mathbb{C}^n))^{1/d} = \frac{1}{\rho(f)},$$

so that since  $f \in \mathcal{H}(K)$ ,  $\rho(f) > 1$ . Choose  $1 < R_1 < R_2 < \rho(f)$  and  $\varepsilon > 0$  to be fixed later. A use of Corollary 14.3 with  $R = R_1$  gives

$$\begin{aligned} \|\pi_i^1(b(\alpha, \mu^1, \cdot)) \pi_j^2(b(\beta, \mu_2, \cdot))\|_K &\leq \|\pi_i^1(b(\alpha, \mu^1, \cdot))\|_{K_1} \|\pi_j^2(b(\beta, \mu_2, \cdot))\|_{K_2} \\ &\leq \delta_1(R_1) \|b(\alpha, \mu^1, \cdot)\|_{\overline{K_{1R_1}}} \cdot \delta_2(R_1) \|b(\beta, \mu_2, \cdot)\|_{\overline{K_{2R_1}}}, \end{aligned} \quad (14.15)$$

Now, applying inequality (14.1) to the right hand side together with the bound (14.10) for the orthonormal polynomials (with the current  $\varepsilon$ ), we get the following estimate :

$$\begin{aligned} &\|\pi_i^1(b(\alpha, \mu^1, \cdot)) \pi_j^2(b(\beta, \mu_2, \cdot))\|_K \\ &\leq \delta_1(R_1) \delta_2(R_1) R_1^{|\alpha|+|\beta|} C(\varepsilon, \mu_1) C(\varepsilon, \mu_2) (1 + \varepsilon)^{|\alpha|+|\beta|} \\ &\leq \delta_1(R_1) \delta_2(R_1) R_1^j C(\varepsilon, \mu_1) C(\varepsilon, \mu_2) (1 + \varepsilon)^j, \end{aligned} \quad (14.16)$$

where we used  $|\alpha| + |\beta| = j$ . Using, this estimate in (14.14) and setting

$$\tau = \delta_1(R_1) \delta_2(R_1) C(\varepsilon, \mu_1) C(\varepsilon, \mu_2),$$

we obtain

$$\|f - \Pi_d(f)\|_K \leq \tau \sum_{j=d+1}^{\infty} (R_1(1 + \varepsilon))^j \sum_{|\alpha|+|\beta|=j} c_{\alpha\beta}(f) \text{card}(B(d, \alpha, \beta)), \quad (14.17)$$



where  $\text{card}$  is used to denote the cardinality. Now, since  $R_2 < \rho(f)$ , in view of (14.8), there exists  $\xi = \xi(R_2)$

$$|c_{\alpha\beta}(f)| \leq \text{dist}(f, \mathcal{P}_{|\alpha|+|\beta|-1}(\mathbb{C}^n)) \leq \xi \left( \frac{1}{R_2} \right)^{|\alpha|+|\beta|} = \xi \frac{1}{R_2^j}. \quad (14.18)$$

(Note that, here,  $\xi$  is needed only for notational convenience, to have a bound valid for all  $\alpha$  and  $\beta$ , rather than for  $|\alpha| + |\beta|$  large enough.) Hence, together with (14.17), we obtain

$$\|f - \Pi_d(f)\|_K \leq \tau \xi \sum_{j=d+1}^{\infty} \left( \frac{R_1(1+\varepsilon)}{R_2} \right)^j \sum_{|\alpha|+|\beta|=j} \text{card}(B(d, \alpha, \beta)). \quad (14.19)$$

It remains to observe that the right hand side sum over  $\alpha$  and  $\beta$  grows polynomially in  $j$ . This is readily seen. Indeed,

$$\text{card}(B(d, \alpha, \beta)) \leq (|\alpha| + 1) \times (|\beta| + 1) \leq (j + 1)^2 \quad (\text{since } |\alpha| + |\beta| = j). \quad (14.20)$$

Hence,

$$\sum_{|\alpha|+|\beta|=j} \text{card}(B(d, \alpha, \beta)) \leq (j + 1)^2 \binom{n+j}{j-1} = O(j^{n+2}), \quad n = n_1 + n_2. \quad (14.21)$$

Now take  $\varepsilon$  so that  $R_1(1 + \varepsilon) < R_2$ . This is possible since  $R_1 < R_2$  and  $\varepsilon$  can be taken arbitrarily small. In view of (14.21), the right hand side of (14.19) is the remainder of a converging series and this shows that  $\|f - \Pi_d(f)\|_K \rightarrow 0$  as  $d$  tends to  $\infty$ . Thus, at this point, using the terminology introduced in definition 13.2, we proved that  $\mathcal{N}^1 \otimes_{\mathbb{N}} \mathcal{N}^2$  is  $\mathcal{C}(K)$ -converging on  $\mathcal{H}(K_1 \times K_2)$ .

Actually, since  $\Pi_d(f)$  is a polynomial projector the uniform convergence on  $K$  implies the convergence on a compact neighborhood of  $K$ , hence in  $\mathcal{H}(K)$ . Such a reasoning is detailed in [14, Section 4, p. 17]. In fact, since  $R_1(1 + \varepsilon)$  can be taken arbitrarily close to 1 and  $R_2$  arbitrarily close to  $\rho(f)$ , our proof actually shows that

$$\limsup_{d \rightarrow \infty} \|f - \Pi_d(f)\|_K^{1/d} = \frac{1}{\rho(f)}, \quad (14.22)$$

so that  $\Pi_d(f)$  provides an asymptotically optimal approximation. This fact also classically implies convergence in  $\mathcal{H}(K)$ . Here is a sketch of the proof. The series

$$L = \Pi_0(f) + \sum_{d=0}^{\infty} \Pi_{d+1}(f) - \Pi_d(f)$$

is normally converging on  $\overline{K_R}$ , for  $R < \rho(f)$ . Indeed, by Bernstein Walsh Siciak inequality, (14.1)

$$\begin{aligned} \|\Pi_{d+1}(f) - \Pi_d(f)\|_{K_R} &\leq R^{d+1} \|\Pi_{d+1}(f) - \Pi_d(f)\|_K \\ &\leq R^{d+1} \|f - \Pi_d(f)\|_K + R^{d+1} \|f - \Pi_{d+1}(f)\|_K, \end{aligned}$$

which in view (14.22) is the general term of a converging series. Now the limit  $L$  must equal  $f$  on  $K_R$  since it coincides with  $f$  on  $K$ .  $\square$

## 14.6. Examples

Of course, most classical constructive polynomial approximation results in the complex domain can be used together with the above theorem to get efficient multivariate projectors. We just point out three very natural examples. The first one shows how a well-known result is recaptured with our theorem, the second and third ones provide new projectors which, it seems,

deserve particular attention. All the Newton products we consider below are constructed using the Newton structure induced by the ordering of the interpolation points.

(A) For  $i = 1, \dots, n$ , we let  $K_i$  denote a regular polynomially convex plane compact set, and  $a_d^i, d \in \mathbb{N}$ , be a sequence of points on the boundary of  $K_i$  such that the discrete measure

$$\mu_d^i = \frac{1}{d+1} \sum_{j=0}^d [a_d^i]^j$$

converges weakly to the *equilibrium measure* on  $K_i$ . Then, for every  $f \in \mathcal{H}(K_1 \times \dots \times K_n)$ , we have

$$\mathbf{L}[a_0^1, \dots, a_d^1] \otimes_{\mathbf{N}} \mathbf{L}[a_0^2, \dots, a_d^2] \otimes_{\mathbf{N}} \dots \otimes_{\mathbf{N}} \mathbf{L}[a_0^n, \dots, a_d^n](f) \quad (14.23)$$

converges to  $f$  in  $\mathcal{H}(K_1 \times \dots \times K_n)$ . This follows from classical univariate Lagrange interpolation theory, see the references given above, together with (iterated applications of) Theorem 14.1. This well-known result was first published (in another presentation) in [48]. Siciak's proof used a multivariate version of the classical (complex) Hermite error formula for Lagrange-Hermite interpolation. Note that the projector on the left hand side of (14.23) is itself a multivariate Lagrange interpolation  $\mathbf{L}[A^d]$  with the Bierman set of interpolation points, see also the introduction,

$$A_d = \{(a_{i_1}^1, a_{i_2}^2, \dots, a_{i_n}^n) : i_1 + \dots + i_n \leq d\}.$$

We refer to [6, Section 6.3, p. 37] for details and earlier references.

(B) Let  $K$  be a convex circular compact set in  $\mathbb{C}^n$  (i.e.  $z \in K, \theta \in \mathbb{R} \implies e^{i\theta}z \in K$ ). If  $(a_d)$  is a sequence of points in  $K$  such that

$$\mu_d = \frac{1}{d+1} \sum_{j=0}^d [a_d]^j$$

converges weakly to a  $e^{i\theta}$ -invariant measure  $\mu$ , that is, such that

$$\int_K f(z) d\mu(z) = \int_K f(e^{i\theta}z) d\mu(z) \quad \text{for all } f \in \mathcal{C}(K) \text{ and } \theta \in \mathbb{R},$$

then the sequence of Kergin projectors  $(\mathbf{K}[a_0, \dots, a_d])$  is converging on  $\mathcal{H}(K)$ , see [13, Theorem 4.1].

Now, assume that, for  $s = 1, 2$ ,  $K_s$  is the product of two convex circular sets,  $K = K_1 \times K_2$  and that  $(a_d^s)$  is a sequence of points in  $K_s$  such that

$$\mu_d^s = \frac{1}{d+1} \sum_{j=0}^d [a_d^s]^j$$

converges weakly to a  $e^{i\theta}$ -invariant measure  $\mu_s$  on  $K_s$ . It is readily seen that the above result applies to the sequence  $(a_d) = (a_d^1, a_d^2)$  so that :

- (a) *The sequence of Kergin projectors  $(\mathbf{K}[(a_0^1, a_0^2), \dots, (a_d^1, a_d^2)])$  is converging on  $\mathcal{H}(K)$ .*
- (b) On the other hand, an application of Theorem 14.1 gives :

**Theorem 14.4.** *The sequence of projectors  $(\mathbf{K}[a_0^1, \dots, a_d^1]) \otimes_{\mathbf{N}} (\mathbf{K}[a_0^2, \dots, a_d^2])$  is converging on  $\mathcal{H}(K)$ .*

It is interesting to compare these two converging sequences of Kergin-related projectors. The second has the advantage of interpolating at  $\binom{d+1}{2}$  points, see [6, Section 6.6, p. 42], whereas the first one interpolates at only  $d+1$  points. However, there is a price to pay for this added value, the projectors obtained with the Newton product no longer preserve (all) homogeneous

partial differential relations. For this notion we refer to [6, Section 6.7] and the references therein.

(C) The next example goes along the same lines but is still more interesting since, in that case, as far as we know, no explicit good interpolation projectors were known. We consider a cylinder, say  $D(0, 1) \times [-1, 1]$  in  $\mathbb{R}^3$ . It was shown in [14] that if  $(a_d)$  is a sequence of points in the disc  $D(0, 1)$  such that  $\mu_d = \frac{1}{d+1} \sum_{j=0}^d [a_j]$  weakly converges to the (normalized)  $d\theta$  measure on the unit circle then the sequence of Kergin projectors  $(\mathbf{K}[a_0, \dots, a_d])$  is converging on  $\mathcal{H}(D(0, 1))$  where  $D(0, 1)$  is regarded as a subset of  $\mathbb{C}^2$ . Likewise, if  $(b_d)$  is a sequence of points in the interval  $[-1, 1]$  whose corresponding  $\mu_d$  measure converges to the equilibrium measure  $dx/\sqrt{1-x^2}$  then classical Lagrange interpolation theory tells that the sequence of Lagrange projectors  $(\mathbf{L}[b_0, \dots, b_d])$  is converging on  $\mathcal{H}([-1, 1])$  (where  $[-1, 1]$  is regarded as a subset of  $\mathbb{C}$ ). Hence, an application of Theorem 14.1 gives :

**Theorem 14.5.** *The sequence of projectors  $(\mathbf{K}[a_0, \dots, a_d]) \otimes_{\mathbf{N}} (\mathbf{L}[b_0, \dots, b_d])$  is converging on  $\mathcal{H}(D(0, 1) \times [-1, 1])$ .*

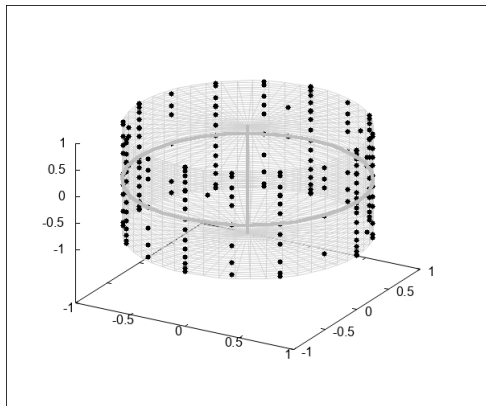
The resulting projector interpolates at all points  $(a_i, b_j)$  for  $i + j \leq d$ . We represent these points on the figure below in the case, which is still more interesting as we will see in Section 16, where  $(a_d)$  is a Leja sequence for the unit circle and  $(b_d)$  is the corresponding  $\mathfrak{R}$ -Leja sequence, that is,  $(b_d)$  is the sequence of the first coordinates of  $(a_d)$  in which repeated points are eliminated.

The elements of a Leja sequence for the unit disk can be recursively constructed as follows see [7, Theorem 5 and Corollary 2]

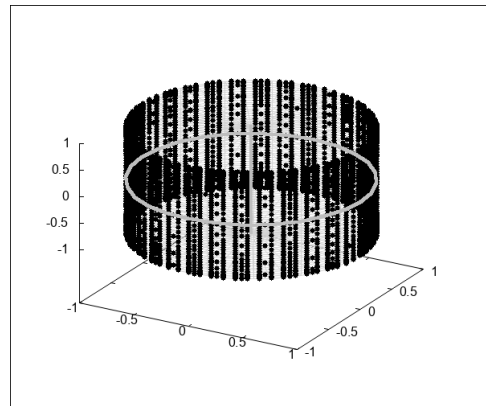
$$S_1 = (1, -1) \tag{14.24}$$

$$S_{2^{n+1}} = S_{2^n} \wedge \exp\left(\frac{\pi}{2^{(n-1)}}\right) S_{2^n}, \quad n \geq 1, \tag{14.25}$$

where  $\wedge$  denotes the usual concatenation operation on tuples. A precise description of  $\mathfrak{R}$ -Leja sequence is available in [23].



$d = 20$



$d = 100$

TABLE 5. Interpolation points for a Newton product of Kergin interpolants at Leja points and Lagrange interpolants at  $\mathfrak{R}$ -Leja points in the cylinder  $D(0, 1) \times [-1, 1]$ .

## 15. SPACES OF ENTIRE FUNCTIONS

The effectiveness of a polynomial projector defined on the space of entire functions usually depends on the growth (order  $\omega$ , type  $\tau$ ) of the approximated functions; roughly, the farther the function is from the space of polynomials the stronger are the requirements on the projector. When the projectors are univariate Lagrangian projectors at a sequence of points  $(a_d)$  (of increasing modulus), the acceptable values for  $\omega$  and  $\tau$  depend on the velocity at which  $|a_d|$  goes to infinity. For instance, a classical result of Polya states that all entire functions of exponential type  $< \ln 2$  can be approximated (in  $\mathcal{H}(\mathbb{C})$ ) by Lagrange interpolation at the points  $a_d = d$  and the upper bound  $\ln 2$  is optimal. Precise statements will be recalled below. As indicated above, the book by Gelfond [28] contains many results on this subject.

Here, assuming that  $\mathcal{N}^s$  is  $\mathcal{H}(\mathbb{C}^{n_s})$ -converging on a space  $E_s$  of entire functions on  $\mathbb{C}^{n_s}$  for  $s = 1, 2$ , we will look for a space of entire functions  $E$  on which  $\mathcal{N}^1 \otimes_N \mathcal{N}^2$  is  $\mathcal{H}(\mathbb{C}^n)$ -converging,  $n = n_1 + n_2$ .

We will begin by recalling the required material on the study of growth of entire functions of several variables.

### 15.1. Growth of entire functions

Let  $f \in \mathcal{H}(\mathbb{C}^n)$  and  $N$  a norm on  $\mathbb{C}^n$ . We set

$$M_N(f, r) = \max_{N(z) \leq r} |f(z)|, \quad r \geq 0.$$

Given  $\omega > 0$ , the  $\omega$ -type  $\tau = \tau(f, \omega, N)$  of  $f \in \mathcal{H}(\mathbb{C}^m)$  is defined as

$$\tau = \limsup_{r \rightarrow \infty} \frac{\ln M_N(f, r)}{r^\omega}.$$

When  $\tau$  is finite, it is the infimum of all  $s$  such that  $\ln M_N(f, r) \leq sr^\omega + O(1)$  as  $r \rightarrow \infty$ . The type may be infinite (when no such  $\tau$  exists) and it depends on the norm  $N$  we work with. For instance, for  $\lambda > 0$ ,

$$\tau(f, \omega, \lambda N) = \frac{1}{\lambda^\omega} \tau(f, \omega, N). \tag{15.1}$$

The interesting  $\omega$ -type is computed when  $\omega$  is the *order* of  $f$ , that is, the infimum of all  $s$ , if there exists, such that  $\ln M_N(f, r) = O(r^s)$  as  $r \rightarrow \infty$ ,

$$\omega = \limsup_{r \rightarrow \infty} \frac{\ln \ln M_N(f, r)}{\ln r}.$$

In contrast to the  $\omega$ -type, the order of an entire function does not depend on the norm  $N$ . A function of finite 1-type  $\tau$  is said to be of *exponential type*  $\tau$ .

In the sequel,

*we will assume we work with norms the ball of which are poly-circular, that is to say,  $N(z_1, \dots, z_n) \leq 1 \implies N(\lambda_1 z_1, \dots, \lambda_n z_n) \leq 1$  for  $|\lambda_i| = 1, i = 1, \dots, n$ .*

For sake of reference, we will call such a norm a *PC-norm*. The common  $l_p$  norms satisfy this condition. The usefulness of this condition appears in the lemma below which will enable us to use the strategy described in Subsection 13.3 by using the standard power series expansion as the starting expansion (13.8). Namely the lemma shows how the growth of an entire function is reflected into the behavior of the coefficients of its power series expansion. At the end of this section, we briefly explain a way to circumvent the condition for the norm to be PC, at least in certain important cases.

**Lemma 15.1.** Let  $f(z) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha z^\alpha$  be an entire function and  $N$  a PC-norm on  $\mathbb{C}^n$ . We have

$$|a_\alpha| \leq t^{-|\alpha|} \frac{M_N(f, t)}{\delta_N(\alpha)}, \quad t > 0, \quad (15.2)$$

where

$$\delta_N(\alpha) = \max\{|z^\alpha| : N(z) \leq 1\}, \quad \alpha \in \mathbb{N}^n. \quad (15.3)$$

*Démonstration.* The reasoning is taken from [45]. Let  $P_k(z) = \sum_{|\alpha|=k} a_\alpha z^\alpha$  and assume that, for a fixed  $k$ ,  $C_k = \max\{|P_k(z)| : N(z) = 1\}$  is attained at  $z = u$ . Applying the Cauchy inequalities to the univariate function  $g(w) = f(wu) = \sum_{k=0}^{\infty} P_k(u)w^k$ , we get

$$C_k \leq t^{-k} \max\{|g(w)| : |w| = t\} \leq t^{-k} M_N(f, t), \quad t > 0.$$

Now, the multivariate Cauchy inequalities applied to the polynomial  $P_k$  give

$$|a_\alpha| \leq r^{-|\alpha|} \max\{|P_k(u_1, \dots, u_n)| : |u_i| = r_i, i = 1, \dots, n\},$$

$$r = (r_1, \dots, r_n), \quad r_i > 0.$$

We apply this inequality with

$$r \in L = \{(|u_1|, \dots, |u_n|) : N(u_1, \dots, u_n) \leq 1\} \subset \{N(u) \leq 1\},$$

where the inclusion holds because  $N$  is a PC-norm, thus arriving to

$$|a_\alpha| r^\alpha \leq C_k, \quad |\alpha| = k, \quad r \in L.$$

Inequality (15.2) is now obtained by passing to the supremum over  $r$  in  $L$  on the left hand term.  $\square$

It is convenient to introduce the following spaces of entire functions.

**Definition 15.2.** Let  $N$  be a PC-norm on  $\mathcal{H}(\mathbb{C}^m)$ ,  $\omega \geq 0$  and  $A > 0$ . We denote by  $E_\omega^m(A, N)$  the subspace of entire functions on  $\mathbb{C}^m$  for which there exists a constant  $M$  such that

$$M_N(f, r) \leq M \exp(Ar^\omega), \quad r \in \mathbb{R}.$$

There is an obvious connection between these spaces and the classical notions of order and type recalled above. We state it as a remark for future reference.

*Remark 1.* Given  $\omega > 0$  and  $\tau < \infty$ , the following statements are equivalent.

- (1)  $f \in \mathcal{H}(\mathbb{C}^m)$  is  $\omega$ -finite type  $< \tau$  with respect to the norm  $N$ .
- (2)  $f \in \cup_{A < \tau} E_\omega^m(A, N)$ .

We define a norm on  $E_\omega^m(A, N)$  by setting

$$\|f\|_{A, N}^\omega = \sup_{r > 0} M_N(f, r) \exp(-Ar^\omega).$$

Observe that for all compact  $K$  in  $\mathbb{C}^m$ , there exists a constant  $C(K)$  such that all  $f \in E_\omega^m(A, N)$ ,

$$\|f\|_K \leq C(K) \|f\|_{A, N}^\omega, \quad (15.4)$$

so that convergence with respect to the norm  $\|\cdot\|_{A, N}^\omega$  implies uniform convergence on all compact subsets of  $\mathbb{C}^m$ . From this, we deduce the following lemma whose standard proof is only sketched.

**Lemma 15.3.** When endowed with the norm  $\|\cdot\|_{A, N}^\omega$ , the space  $E_\omega^m(A, N)$  is a Banach space.

*Démonstration.* Let  $(f_n)$  be a Cauchy sequence in  $E_w^m(A, N)$ . In view of (15.4), it is also a Cauchy sequence in  $\mathcal{H}(\mathbb{C}^m)$  and therefore converges to an entire function  $f$ . To show that such  $f$  is in  $E_w^m(A, N)$ , fix  $r > 0$ , the uniform convergence on the  $N$ -ball of radius  $r$  shows that there exists  $n_0 = n_0(r)$  such that  $M_N(f - f_{n_0}, r) \leq 1$  so that

$$M_N(f, r) \exp(-Ar^\omega) \leq M_N(f_{n_0}, r) \exp(-Ar^\omega) + \exp(-Ar^\omega) \leq \|f_{n_0}\|_{A, N}^\omega + 1 \leq C,$$

the latter since the sequence  $\|f_n\|_{A, N}^\omega$ , being itself Cauchy, is bounded. Passing to the supremum over  $r$ , we obtain that  $f \in E_w^m(A, N)$ . A similar reasoning shows that the convergence of  $f_n$  to  $f$  holds in  $E_w^m(A, N)$ .  $\square$

Likewise, since, for any functional  $\mu$  (continuous linear form) on  $\mathcal{H}(\mathbb{C}^m)$ , there exist a compact  $K$  and a constant  $C(K)$  such that, for all  $f \in \mathcal{H}(\mathbb{C}^m)$ ,  $|\mu(f)| \leq C(K)\|f\|_K$ , the restriction of such a functional to  $E_w^m(A, N)$  is continuous with its topology of Banach space. We therefore have :

**Lemma 15.4.** *Let  $\Pi$  denote a polynomial projector on  $\mathcal{H}(\mathbb{C}^m)$ . The restriction of  $\Pi$  to  $E_w^m(A, N)$  is continuous for its topology of Banach space.*

We will need to know the  $\|\cdot\|_{A, N}^\omega$ -norm of the monomial  $e_\alpha : z \rightarrow z^\alpha$  in  $\mathbb{C}^m$ .

**Lemma 15.5.** *We have*

$$\|e_\alpha\|_{A, N}^\omega = \delta_N(\alpha) \left( \frac{|\alpha|}{e\omega A} \right)^{|\alpha|/\omega},$$

where  $\delta_N(\alpha) = M_N(e_\alpha, 1)$  is defined in (15.3).

*Démonstration.* By homogeneity of  $e_\alpha$ , we have

$$M_N(e_\alpha, t) \exp(-At^\omega) = M_N(e_\alpha, 1)t^{|\alpha|} \exp(-At^\omega) = \delta_N(\alpha)t^{|\alpha|} \exp(-At^\omega).$$

The claim follows by observing that the function  $t \in \mathbb{R}^+ \rightarrow t^{|\alpha|} \exp(-At^\omega)$  reaches its maximum at  $t = (|\alpha|/(\omega A))^{1/\omega}$ .  $\square$

## 15.2. The convergence theorems

As above, we work with  $\mathbb{C}^n = \mathbb{C}^{n_1} \times \mathbb{C}^{n_2}$ ,  $n_s \geq 1$  and let  $N_s$  denote a PC-norm on  $\mathbb{C}^{n_s}$ . Moreover, given  $w > 0$  and  $A_s > 0$  for  $s = 1, 2$ , we set

$$E_s = E_\omega^{n_s}(A_s, N_s) \subset \mathcal{H}(\mathbb{C}^{n_s}), \quad s = 1, 2.$$

For the sake of simplicity, we will also write

$$\|\cdot\|_s = \|\cdot\|_{A_s, N_s}^\omega, \quad s = 1, 2. \tag{15.5}$$

We first treat the case  $0 < \omega \leq 1$ , the case  $\omega \geq 1$ , which is similar, is studied below.

**Theorem 15.6.** *We use the notation above and assume  $\omega \leq 1$ . Let  $\mathcal{N}^s = (\Pi_0^s, \Pi_1^s, \dots)$  denote a Newton sequence on  $\mathcal{H}(\mathbb{C}^{n_s})$ ,  $s = 1, 2$ . Given  $a_s > 0$  for  $s = 1, 2$ , we define a norm  $N$  on  $\mathbb{C}^n$  by setting*

$$N(z^1, z^2) = a_1 N_1(z^1) + a_2 N_2(z^2), \quad z^s \in \mathbb{C}^{n_s}, \quad s = 1, 2. \tag{15.6}$$

*If  $\mathcal{N}^s$  is converging on  $E_s$  for  $s = 1, 2$  then  $\mathcal{N}^1 \otimes_{\mathbb{N}} \mathcal{N}^2$  is converging on  $E_\omega^n(A, N)$  provided that*

$$A < \min \left( \frac{A_1}{a_1^\omega}, \frac{A_2}{a_2^\omega} \right). \tag{15.7}$$

The result will be applied in the following form.

**Corollary 15.7.** *Let  $0 < \omega \leq 1$  and  $\tau_s > 0$ ,  $s = 1, 2$ . If  $\mathcal{N}^s$  is converging on the space of functions of  $\omega$ -type  $< \tau_s$  with respect to  $N_s$  in  $\mathcal{H}(\mathbb{C}^{n_s})$ , for  $s = 1, 2$  then  $\mathcal{N}^1 \otimes_{\mathbb{N}} \mathcal{N}^2$  is converging on the space of functions of  $\omega$ -type  $< 1$  with respect to*

$$N = \tau_1^{1/\omega} N_1 + \tau_2^{1/\omega} N_2.$$

*Démonstration.* According to Remark 1, we need to show that for all  $0 < A < 1$ ,  $\mathcal{N}^1 \otimes_{\mathbb{N}} \mathcal{N}^2$  is converging on  $E_{\omega}^n(A, N)$ . Let  $\varepsilon = 1 - A$  and fix  $\delta > 0$  such that  $\delta/\tau_s < \varepsilon$  and  $A_s = \tau_s - \delta < \tau_s$ ,  $s = 1, 2$ . Observe that  $E_{\omega}^{n_s}(A_s, N_s)$  is included in the space of entire functions of  $\omega$ -type  $< \tau_s$  so that  $\mathcal{N}_s$  is  $\mathcal{H}(\mathbb{C}^{n_s})$ -converging on  $E_{\omega}^{n_s}(A_s, N_s)$  and the assumptions of Theorem 15.6 are satisfied. We now apply it with  $a_s = \tau_s^{1/\omega}$ ,  $s = 1, 2$  and it remains to check that Condition (15.7) is satisfied. This is clear since

$$A = 1 - \varepsilon < (\tau_s - \delta)/a_s^{\omega} = 1 - \delta/\tau_s, \quad s = 1, 2.$$

The theorem therefore implies convergence on  $E_{\omega}^n(A, N)$  as required.  $\square$

**Corollary 15.8** (Case  $\tau_1 = \tau_2$ ). *Let  $0 < \omega \leq 1$  and  $\tau > 0$ . If  $\mathcal{N}^s$  is converging on the space of functions of  $\omega$ -type  $< \tau$  with respect to  $N_s$  in  $\mathcal{H}(\mathbb{C}^{n_s})$ , for  $s = 1, 2$  then  $\mathcal{N}^1 \otimes_{\mathbb{N}} \mathcal{N}^2$  is converging on the space of functions of  $\omega$ -type  $< \tau$  with respect to  $N = N_1 + N_2$ .*

*Démonstration.* Use the previous corollary taking (15.1) into account.  $\square$

Note that the result is optimal if the assumption is. That is to say if there exists a function  $f$  of  $\omega$ -type  $= \tau$  with respect to  $N_1$  for which  $\Pi_d^1(f)$  does not converge to  $f$  in  $\mathcal{H}(\mathbb{C}^{n_1})$  then the same function regarded as a function on  $\mathbb{C}^n$  provides a function of  $\omega$ -type  $= \tau$  with respect to  $N$  for which convergence does not occur.

A careful examination of the proof of Theorem 15.6 below would show that a version of the theorem holds true when the spaces  $E_s$  are not defined with identical  $\omega$  but rather with  $\omega_s$ ,  $s = 1, 2$ . With, say,  $\omega_1 < \omega_2$ , Condition (15.7) would be then changed to

$$A < \frac{A_1}{a_1^{\omega_1}} \quad \text{and} \quad A < \frac{(e\omega_2 A_2)^{\omega_2/\omega_1}}{e\omega_1 a_2^{\omega_1}}.$$

Such a condition is not satisfying since it does not ensure suitable convergence for all functions depending only on the group of variables  $z^2$ , while this convergence follows immediately from the definition of the Newton product. It seems that, when  $\omega_1 \neq \omega_2$ , a satisfactory statement cannot be obtained without the use of a notion of partial type. We will not discuss this approach in this paper.

Given  $\alpha_s \in \mathbb{N}^{n_s}$ ,  $s = 1, 2$ , we form  $\alpha = (\alpha^1, \alpha^2) \in \mathbb{N}^n$ . We need to compute  $\delta_N(\alpha^1, \alpha^2)$  in terms of  $\delta_{N_1}(\alpha^1)$  and  $\delta_{N_2}(\alpha^2)$  where  $N$  is as in (15.6).

**Lemma 15.9.** *With the notation of the theorem,*

$$\delta_N(\alpha) = \frac{|\alpha^1|^{\omega_1} |\alpha^2|^{\omega_2}}{|\alpha|^{\omega}} \cdot \frac{\delta_{N_1}(\alpha^1) \delta_{N_2}(\alpha^2)}{a_1^{\omega_1} a_2^{\omega_2}}.$$

*Démonstration.* In view of (15.3), we have

$$\delta_N(\alpha^1, \alpha^2) = \max \left\{ |u_1|^{\omega_1} |u_2|^{\omega_2} : a_1 N_1(u_1) + a_2 N_2(u_2) = 1 \right\}.$$

Setting  $u_s = r_s v_s$  with  $N_s(v_s) = 1$ ,  $s = 1, 2$ , the above relation translates into

$$\delta_N(\alpha^1, \alpha^2) = \delta_{N_1}(\alpha^1) \delta_{N_2}(\alpha^2) \max \left\{ r_1^{\omega_1} r_2^{\omega_2} : a_1 r_1 + a_2 r_2 = 1 \right\}.$$

Now, it is readily seen that the function

$$r_1 \in [0, 1/a_1] \longrightarrow r_1^{|\alpha^1|} \left( \frac{1 - a_1 r_1}{a_2} \right)^{|\alpha^2|}$$

reaches its maximum at

$$r_1 = \frac{1}{a_1} \cdot \frac{|\alpha^1|}{|\alpha|},$$

from which the lemma immediately follows.  $\square$

We turn to the proof of Theorem 15.6.

*Démonstration.* Using the notation (13.5), we need to show that for any  $f \in E_\omega^n(A, N)$ ,  $\Pi_d(f)$  converges to  $f$  in  $\mathcal{H}(\mathbb{C}^n)$  as  $d \rightarrow \infty$ , that is,  $\Pi_d(f)$  converges to  $f$  uniformly on every compact subset of  $\mathbb{C}^n$ . We fix such a  $f$  and  $K = K_1 \times K_2$  a compact set in  $\mathbb{C}^n$  with  $K_s \in \mathbb{C}^{n_s}$  and prove that  $\|f - \Pi_d(f)\|_K \rightarrow 0$  as  $d \rightarrow \infty$ .

*Step 4 (Banach-Steinhaus).* In view of Lemma 15.4,  $\Pi_d^s : E_s \rightarrow \mathcal{C}(K_s)$  is a continuous linear map (for the Banach space topologies) and, from the assumption on  $\mathcal{N}^s$ , for all  $f_s$  in  $E_s$ ,  $\Pi_d^s(f_s)$  converges to  $f_s|_{K_s}$ . Hence, by the Banach-Steinhaus Theorem, there exists a constant  $\Gamma_s$  such that, see (15.5) for the notation,

$$\|\Pi_d^s(f_s)\|_{K_s} \leq \Gamma_s \|f_s\|_s, \quad d \in \mathbb{N}, \quad f_s \in E_s, \quad s = 1, 2. \quad (15.8)$$

Likewise, using the Newton summands  $\pi_d^s = \Pi_d^s - \Pi_{d-1}^s$ , just as in Corollary 14.3,

$$\|\pi_d^s(f_s)\|_{K_s} \leq C_s \|f_s\|_s, \quad d \in \mathbb{N}, \quad f_s \in E_s, \quad s = 1, 2, \quad (15.9)$$

with  $C_s = 2\Gamma_s$ .

*Step 5 (Using the strategy described in Subsection 13.3).* Writing  $e_\alpha^s : \mathbb{C}^{n_s} \rightarrow z^\alpha$ , we start from the power series expansion of  $f$ ,

$$f(z^1, z^2) = \sum_{j=0}^{\infty} \sum_{|\alpha^1| + |\alpha^2| = j} c_{(\alpha^1, \alpha^2)} e_{\alpha^1}(z^1) e_{\alpha^2}(z^2).$$

A use of (13.9) and Lemma 13.6 yields

$$f - \Pi_d(f) = \sum_{j=d+1}^{\infty} \sum_{|\alpha^1| + |\alpha^2| = j} c_{\alpha^1 \alpha^2} \left\{ \sum_{(i_1, i_2) \in B(d, \alpha^1, \alpha^2)} \pi_{i_1}^1(e_{\alpha^1}) \pi_{i_2}^2(e_{\alpha^2}) \right\}.$$

We will denote by  $R_{\alpha^1, \alpha^2}$  the sum into brackets in the above equation. We will prove that the series

$$\sum_{j=0}^{\infty} \sum_{|\alpha^1| + |\alpha^2| = j} |c_{\alpha^1 \alpha^2}| \|R_{\alpha^1, \alpha^2}\|_K$$

is converging and  $\|f - \Pi_d(f)\|_K$  will therefore go to 0 as it is bounded by the remainder of a converging series.

*Step 6 (Estimating  $\|R_{\alpha^1, \alpha^2}\|_K$ ).* Using (15.9) in the above equation for  $R_{\alpha^1, \alpha^2}$ , we obtain

$$\|R_{\alpha^1, \alpha^2}\|_K \leq C_1 C_2 \text{card}(B(d, \alpha^1, \alpha^2)) \|e_{\alpha^1}\|_1 \|e_{\alpha^2}\|_2 \quad (15.10)$$

and, using the estimate (14.20) for  $\text{card}(B(d, \alpha^1, \alpha^2))$ ,

$$\leq C_1 C_2 (|\alpha^1| + 1)(|\alpha^2| + 1) \|e_{\alpha^1}\|_1 \|e_{\alpha^2}\|_2 \quad (15.11)$$



Now a use of Lemma 15.5 gives

$$\|R_{\alpha^1, \alpha^2}\|_K \leq C_1 C_2 (|\alpha^1| + 1)(|\alpha^2| + 1) \delta_{N_1}(\alpha^1) \delta_{N_2}(\alpha^2) \times \left(\frac{|\alpha^1|}{e\omega A_1}\right)^{\frac{|\alpha^1|}{\omega}} \left(\frac{|\alpha^2|}{e\omega A_2}\right)^{\frac{|\alpha^2|}{\omega}}. \quad (15.12)$$

*Step 7 (Estimating  $|c_{\alpha^1 \alpha^2}|$ ).* Since  $N_s$  is a PC-norm for  $s = 1, 2$  so is the norm  $N$  and we may therefore apply estimate (15.2) in Lemma 15.1 to get

$$|c_{\alpha^1 \alpha^2}| \leq \frac{t^{-|\alpha|}}{\delta_N(\alpha)} M_N(f, t) \leq M \frac{t^{-|\alpha|}}{\delta_N(\alpha)} e^{tA}, \quad \alpha = (\alpha^1, \alpha^2), \quad t \geq 0, \quad (15.13)$$

where we used  $f \in E_\omega(A, N)$ . Since, as in the proof of Lemma 15.5, the function  $t \rightarrow t^{-|\alpha|} e^{t\omega A}$  reaches its minimum for  $t = (|\alpha|/(\omega A))^{1/\omega}$ , we have

$$|c_{\alpha^1 \alpha^2}| \leq M \left(\frac{|\alpha|}{e\omega A}\right)^{-|\alpha|/\omega} \cdot \frac{1}{\delta_N(\alpha)}.$$

Next, we use Lemma 15.9 to handle the term  $\delta_N(\alpha)$  and, after some simple calculation, we arrive to

$$|c_{\alpha^1 \alpha^2}| \leq M \left(\frac{1}{e\omega A}\right)^{-|\alpha|/\omega} \cdot |\alpha|^{|\alpha|-1/\omega} \cdot \frac{a_1^{|\alpha^1|} a_2^{|\alpha^2|}}{|\alpha^1|^{|\alpha^1|} |\alpha^2|^{|\alpha^2|} \delta_{N_1}(\alpha^1) \delta_{N_2}(\alpha^2)}. \quad (15.14)$$

*Step 8 (Conclusion).* Putting (15.12) and (15.14) together, setting  $C = MC_1 C_2$ , we obtain

$$|c_{\alpha^1 \alpha^2}| \|R_{\alpha^1, \alpha^2}\|_K \leq C (|\alpha^1| + 1)(|\alpha^2| + 1) \cdot \left(\frac{a_1^\omega A}{A_1}\right)^{|\alpha^1|/\omega} \left(\frac{a_2^\omega A}{A_2}\right)^{|\alpha^2|/\omega} \times \left\{ \frac{|\alpha^1|^{|\alpha^1|} |\alpha^2|^{|\alpha^2|}}{|\alpha|^{|\alpha|}} \right\}^{\frac{1}{\omega}-1}.$$

The terms into brackets is obviously smaller than one (since  $|\alpha^s| \leq |\alpha|$ ), hence we have

$$|c_{\alpha^1 \alpha^2}| \|R_{\alpha^1, \alpha^2}\|_K \leq M (|\alpha| + 1)^2 q_1^{|\alpha^1|} q_2^{|\alpha^2|} \quad \text{with} \quad q_s = \left(\frac{a_s^{1/\omega} A}{A_s}\right)^{1/\omega} < 1,$$

the latter by assumption (15.7). The convergence of the series of general term  $c_{\alpha^1 \alpha^2} R_{\alpha^1, \alpha^2}$  follows and according to the second step, this concludes the proof of the theorem.  $\square$

We will now deal with the case  $\omega \geq 1$ .

**Theorem 15.10.** *We use the notation above and assume  $\omega \geq 1$ . Let  $\mathcal{N}^s = (\Pi_0^s, \Pi_1^s, \dots)$  denotes a Newton sequence on  $\mathcal{H}(\mathbb{C}^{n_s})$ ,  $s = 1, 2$ . Given  $a_s > 0$  for  $s = 1, 2$ , we define a norm  $N$  on  $\mathbb{C}^n$  by*

$$N(z^1, z^2) = (a_1 N_1^\omega(z^1) + a_2 N_2^\omega(z^2))^{1/\omega}, \quad z^s \in \mathbb{C}^{n_s}, \quad s = 1, 2. \quad (15.15)$$

*If  $\mathcal{N}^s$  is converging on  $E_s$  for  $s = 1, 2$  then  $\mathcal{N}^1 \otimes_N \mathcal{N}^2$  is converging on  $E_\omega^n(A, N)$  provided that*

$$A < \min\left(\frac{A_1}{a_1}, \frac{A_2}{a_2}\right). \quad (15.16)$$

Thus the main changes are the definition of the norm  $N$  and the assumptions on  $A$  in which  $\omega$  disappears. It is worth noting that the definition of  $N$  and the conditions on  $A$  in both theorems are continuous with respect to  $\omega$  around  $\omega = 1$ . Note also that the condition  $\omega \geq 1$  ensures that equation (15.15) defines a norm. It is obviously a PC-norm as soon as the  $N_s$  are.

Corollaries 15.7 and 15.8 remain true with obvious changes. We will not state them.

The change in the definition on  $N$  requires a modification of Lemma 15.9.

**Lemma 15.11.** *With the notation of Theorem 15.10, in particular with  $N$  as in (15.15), we have*

$$\delta_N(\alpha) = \left( \frac{|\alpha^1|^{\alpha^1} |\alpha^2|^{\alpha^2}}{|\alpha|^{\alpha^1} a_1^{\alpha^1} a_2^{\alpha^2}} \right)^{1/\omega} \cdot \delta_{N_1}(\alpha^1) \delta_{N_2}(\alpha^2)$$

*Démonstration.* Working as in the proof of Lemma 15.9, we find

$$\delta_N(\alpha^1, \alpha^2) = \delta_{N_1}(\alpha^1) \delta_{N_2}(\alpha^2) \max \left\{ r_1^{|\alpha^1|/\omega} r_2^{|\alpha^2|/\omega} : a_1 r_1 + a_2 r_2 = 1 \right\},$$

but this is only the  $\omega$ -th root of the function in the proof of Lemma 15.11 so that the result immediately follows.  $\square$

*Proof of Theorem 15.10.* The proof is identical to that of Theorem 15.6 up to the conclusion of Step 4 in which we use the estimate of Lemma 15.11 instead of that of Lemma 15.9. The corresponding estimate in the conclusion even simplifies to become

$$|c_{\alpha^1 \alpha^2}| \|R_{\alpha^1, \alpha^2}\|_K \leq M(|\alpha^1| + 1)(|\alpha^2| + 1) \cdot \left( \frac{a_1 A}{A_1} \right)^{|\alpha^1|/\omega} \left( \frac{a_2 A}{A_2} \right)^{|\alpha^2|/\omega},$$

from which the conclusion is immediate.  $\square$

*Remark 2* (About the Assumption on the norm  $N$ ).

(A) Any norm  $N_s$  is dominated by the PC-norm  $\mathbf{N}_s$  defined by

$$\mathbf{N}_s(z) = \max \{ N(\lambda_1 z_1, \dots, \lambda_n z_n), |\lambda_i| = 1, i = 1, \dots, n \},$$

and  $\mathbf{N}_s$  is the smallest norm satisfying the property. Since

$$E_\omega^{n_s}(A_s, \mathbf{N}_s) \subset E_\omega^{n_s}(A, N_s),$$

we may apply Theorem 15.6 to get a space depending on  $\mathbf{N}$  (defined from the norms  $\mathbf{N}_s$  as in the above theorems) on which the Newton product converges. This space however is unlikely to be optimal, as follows from the next remark.

(B) In the case of entire functions of exponential type, another approach is available that enables to eliminate the assumption on the norm  $N$  to be PC. By representing such a function  $f$  (of exponential type  $\tau < 1$  with respect to  $N$ ) by a Laplace transform of an analytic functional, see [30, Section 4.5], we can write  $f$  as

$$f(w) = \int_{\{N^*(z) \leq \tau\}} \exp\langle z, w \rangle d\mu(z),$$

where  $\mu$  is a complex measure supported on the ball  $\{N^*(z) \leq \tau\}$  and  $N^*$  is the **dual norm** of  $N$  (without assuming that  $N$  is PC). Recall that

$$N^*(z) = \max \{ |\langle z, \xi \rangle|, N(\xi) = 1 \}.$$

This yields a new way of computing  $\Pi(f)$ , hence  $f - \Pi(f)$ , namely

$$\Pi(f)(w) = \int_{\{N^*(z) \leq \tau\}} \Pi(\exp\langle z, \cdot \rangle)(w) d\mu(z),$$

which is interesting since  $w \rightarrow \exp\langle z, w \rangle$  is the product  $\exp\langle z^1, w^1 \rangle \times \exp\langle z^2, w^2 \rangle$  so that we may use the available formula for the Newton projector of a product function (Theorem 13.5). Let us just point out how the formula for  $N$  naturally comes from the norms  $N_s$  through duality. In fact, if  $N = \tau_1 N_1 + \tau_2 N_2$  as in Corollary 15.7 then one readily checks that

$$N^*(z^1, z^2) = \max \{ (\tau_1 N_1)^*(z^1), (\tau_2 N_2)^*(z^2) \} = \max \left\{ \frac{N_1^*(z^1)}{\tau_1}, \frac{N_2^*(z^2)}{\tau_2} \right\}.$$

Hence if  $F$  is an entire function of exponential type  $< 1$  with respect to  $N$ , for some measure  $\mu$  we have

$$\begin{aligned} f(w) &= \int_{\{N^*(z) \leq 1\}} \exp\langle z, w \rangle d\mu(z^1, z^2) \\ &= \iint_{\{N_1^*(z_1) \leq \tau_1\} \times \{N_2^*(z_2) \leq \tau_2\}} \exp\langle z^1, w^1 \rangle \exp\langle z^2, w^2 \rangle d\mu(z^1, z^2). \end{aligned}$$

And the convergence of  $\Pi_d(f)$  to  $f$  can be obtained as from the convergence of  $\Pi_d$  for the product function  $w \rightarrow \exp\langle z^1, w^1 \rangle \times \exp\langle z^2, w^2 \rangle$  (the convergence being uniform in  $w^s$ ). We omit the details.

### 15.3. Example

We illustrate the above convergence theorems in the case of the product of two Newton sequences of Kergin interpolation projectors as in Subsection 14.6 for which deep approximation results are available for entire functions. Let us first recall such a result. We use the notation introduced above. Give  $\omega > 0$ , we set

$$c = c(\omega) = \int_0^{1/2} \frac{t^{\omega-1}}{1-t} dt. \quad (15.17)$$

In particular,

$$c(1) = \ln 2.$$

Given a sequence of points  $a_d^s$  in  $\mathbb{C}^{n_s}$ ,  $d \in \mathbb{N}$ , such that the sequence of their norms  $N_s(a_d^s)$  is non decreasing, we define the counting function  $\mathcal{N}_s(r)$  as the number of interpolation points whose  $N^s$ -norm is not bigger than  $r$ , that is

$$\mathcal{N}_s(r) = \text{card}\{i \in \mathbb{N} : N^s(a_i^s) \leq r\}.$$

The  $\omega$ -density  $\Delta_s$ , with respect to  $N_s$ , of the sequence  $a_d^s$  is then defined as

$$\Delta_s = \liminf_{r \rightarrow \infty} \frac{\mathcal{N}_s(r)}{r^\omega}.$$

It is known that, for all entire function of  $\omega$ -type  $\tau^s$  such that  $\tau_s < c\Delta_s$ , the sequence of Kergin interpolation polynomials  $K[a_0^s, \dots, a_d^s; f]$  converges uniformly to  $f$  on every compact subset of  $\mathbb{C}^{n_s}$ . This is a multivariate generalization of a Theorem of Gelfond [28]. It was first proved in the case of the standard euclidean norm by Bloom [10] and then extended to an arbitrary norm by Andersson and Passare [1].

Here, a direct application of Corollary 15.8 yields the following.

**Theorem 15.12.** *Let  $(a_d^s)$ ,  $s = 1, 2$  be two sequences of points as above. If their density are equal, i.e.  $\Delta_1 = \Delta_2 (= \Delta)$ , then, for all entire functions on  $\mathbb{C}^n = \mathbb{C}^{n_1} \times \mathbb{C}^{n_2}$  of  $\omega$ -type  $\tau$  with respect to the norm  $N$ , see below, satisfying*

$$\tau < \Delta c,$$

where  $c$  is defined in (15.17), then the sequence of polynomials

$$(\mathbf{K}[a_0^1, \dots, a_d^1]) \otimes_N (\mathbf{K}[a_0^2, \dots, a_d^2])(f)$$

converges uniformly to  $f$  on every compact subset of  $\mathbb{C}^n$ . Here  $N = N_1 + N_2$  if  $\omega \leq 1$  and  $N = (N_1^\omega + N_2^\omega)^{1/\omega}$  if  $\omega \geq 1$ .

Of course, the case  $\Delta_1 \neq \Delta_2$  is handled by Corollary 15.7.

## 16. SPACES OF SMOOTH FUNCTIONS

### 16.1. Adapting the tools

Roughly the same technique as in Section 14 can be used to derive results on spaces of differentiable functions. We will omit some details of the proofs where they are similar to those previously given. The results obtained are not optimal, this will be explained below. Although we will present the results in a more general setting, the reader may assume that, in what follows, all the compact sets considered are convex bodies (compact convex sets of non empty interior). Given a fat compact set  $\kappa$  ( $\kappa$  is the closure of its interior), we denote by  $\mathcal{C}^m(\kappa)$  the space of all functions which are  $m$ -times continuously differentiable on the interior of  $\kappa$  and whose all derivatives of order  $\leq m$  extend continuously to  $\kappa$ . It is a Banach space when endowed with the norm

$$\|f\|_{m,\kappa} = \max_{|\alpha| \leq m} \|D^\alpha f\|_\kappa. \quad (16.1)$$

First, we need a stronger notion of Bernstein-Markov measure, see Subsection 14.3. Let  $\mu$  be a probability measure on  $\kappa$ . If there exists  $\theta > 0$  such that, for some constant  $C_\mu$ , we have

$$\|p\|_\kappa \leq C_\mu (\deg p)^\theta \|p\|_2, \quad p \in \mathcal{P}(\mathbb{R}^n), \quad (16.2)$$

where as usual  $\|p\|_2^2 = \int_\kappa p^2(x) d\mu(x)$ , we say that  $\mu$  is a  **$\theta$ -strong Bernstein-Markov measure (SBM)** on  $\kappa$ . When there exists such a measure on  $\kappa$ , we say that  $\kappa$  is a  **$\theta$ -SBM compact**. Observe that the sub-exponential term  $(1 + \varepsilon)^{\deg(p)}$  in (14.3) is replaced in (16.2) by a polynomial term in the degree. The classical Nikolskii inequality states that  $dx$  (Lebesgue measure) is 1-SBM for  $[-1, 1]$ , see [37, Theorem 3.1.4]. Zeriahi showed in [54] that the Lebesgue measure is SBM for a large class of compact sets. From the bounds in [54, p. 686], we deduce that, when  $K$  is a convex body in  $\mathbb{R}^n$ , then  $\theta$  can be taken as  $3n/2$ . Such a  $\theta$  is probably not optimal but we are not aware of more precise bounds. We omit the proof of the following lemma which is similar to that of [8, Lemma 2, p. 290], see also Subsection 14.3.

**Lemma 16.1.** *If  $\mu_s$  is a  $\theta_s$ -SBM measure on  $K_s \in \mathbb{R}^{n_s}$  for  $s = 1, 2$  then  $\mu_1 \times \mu_2$  is a  $(\theta_1 + \theta_2)$ -SBM measure on  $K_1 \times K_2$ .*

Recall now that a compact  $\kappa$  satisfies a **Markov inequality of exponent  $r$**  if, for some positive constant  $M_\kappa$ , we have

$$\|D^\alpha p\|_\kappa \leq M_\kappa (\deg p)^{r|\alpha|} \|p\|_\kappa, \quad \alpha \in \mathbb{N}^n, \quad p \in \mathcal{P}(\mathbb{R}^n). \quad (16.3)$$

We refer to [39] for a large class of compact sets satisfying a Markov inequality, see also the surveys [43, 41]. In the case of a convex body, the exponent  $r$  can be taken as 2 (in any dimension), see [52].

It is readily seen that if  $\kappa_s$  satisfies a Markov inequality of exponent  $r_s$ ,  $s = 1, 2$ , then  $\kappa_1 \times \kappa_2$  satisfies a Markov inequality of exponent  $r = \max(r_1, r_2)$ .

If  $\kappa$  is a  $\theta$ -SBM compact satisfying a Markov inequality of exponent  $r$ , then there exists a constant  $C_{\mu,\kappa} = C(\mu)M(\kappa)$ , such that for all derivatives  $D^\alpha$  we have

$$\|D^\alpha p\|_\kappa \leq C_{\mu,\kappa}(\deg p)^{\theta+r|\alpha|} \|p\|_2, \quad p \in \mathcal{P}(\mathbb{R}^n). \quad (16.4)$$

Finally, we say that  $\kappa$  is a **Jackson compact set** if for all function  $f$  in  $\mathcal{C}^m(\kappa)$ , we have

$$\text{dist}(f, \mathcal{P}_d(\mathbb{R}^n)) \leq M_J d^{-m} \quad (16.5)$$

where the distance is with respect to the uniform norm on  $\kappa$  and  $M_J$  depends only on  $f$ ,  $m$  and  $\kappa$ . For this notion of Jackson sets we refer to [42] and the references therein. In view of [5, Theorem 2] if  $\kappa$  is quasi-convex (i.e. satisfies the Whitney property P), so that, by the Whitney extension Theorem [51, 9], all  $f$  in  $\mathcal{C}^m(\kappa)$  extends to a function  $\mathcal{C}^m$  on a neighborhood of  $\kappa$  then for all degree  $d$ , there exists a polynomial  $t_d$  (of near-best simultaneous approximation of  $f$ ) such that

$$\|D^\alpha(f - t_d)\|_\kappa \leq \mathbf{M}_J / d^{m-|\alpha|}, \quad d \in \mathbb{N}, \quad (16.6)$$

where  $\mathbf{M}_J = M_J(m, f) \geq M_J$  depends only on  $f$  and  $m$  and  $\kappa$ . Such sets, in particular, convex bodies, are therefore Jackson sets.

Given a  $\theta$ -SBM measure  $\mu$  on  $\kappa$ , we may construct the sequence of orthonormal polynomials  $b_\alpha$  and the corresponding orthogonal projection

$$\mathbf{P}_{d,\mu}(f) = \sum_{|\alpha| \leq d} c_\alpha(f) b_\alpha, \quad c_\alpha(f) = \langle f, b_\alpha \rangle = \int_K f(x) b_\alpha(x) d\mu(x),$$

as in Subsection 14.3.

**Lemma 16.2.** *Let  $K \subset \mathbb{R}^n$  be a  $\theta$ -SBM compact set with measure  $\mu$  as well as a Jackson and Markov compact set with exponent  $r$ . For all  $f \in \mathcal{C}^{\mathbf{m}}(K)$ , we have*

$$f = \sum_{d=0}^{\infty} \sum_{|\alpha|=d} c_\alpha(f) b_\alpha \quad \text{in } \mathcal{C}^{\bar{\mathbf{m}}}(K), \quad (16.7)$$

whenever  $\mathbf{m} > \theta + r\bar{\mathbf{m}} + 1$ .

*Démonstration.* We denote by  $\mu$  the  $\theta$ -SBM measure on  $K$ . The reasoning is similar to that given in Subsection 14.3, see also [54]. Let us write  $H_d = \sum_{|\alpha|=d} c_\alpha(f) b_\alpha$ . We prove that  $\sum_{d=0}^{\infty} D^\beta H_d$  uniformly converges on  $K$  for  $\|\alpha\| \leq \bar{\mathbf{m}}$ . To explain the condition on  $\mathbf{m}$ , let us just observe that, the assumptions on  $K$  enable us to use (16.4) so that

$$\|D^\beta H_d\|_K \leq C_{\mu,K} d^{\theta+r\bar{\mathbf{m}}} \|H_d\|_2, \quad (16.8)$$

Now, let us just observe that here, calling  $t_{d-1}$  a best uniform approximation polynomial of  $f$  in  $\mathcal{P}_{d-1}(\mathbb{R}^n)$ , see (16.6), we have

$$\|H_d\|_2 = \|\mathbf{P}_{d,\mu}(f) - \mathbf{P}_{d-1,\mu}(f)\|_2 \quad (16.9)$$

$$\leq \|f - \mathbf{P}_{d,\mu}(f)\|_2 + \|f - \mathbf{P}_{d-1,\mu}(f)\|_2 \quad (16.10)$$

$$\leq 2\|f - \mathbf{P}_{d-1,\mu}(f)\|_2 \quad (\mathbf{P}_{d,\mu}(f) \text{ is best in } \mathcal{P}_d) \quad (16.11)$$

$$\leq 2\|f - t_{d-1}\|_2 \quad (\mathbf{P}_{d-1,\mu}(f) \text{ is best in } \mathcal{P}_{d-1}) \quad (16.12)$$

$$\leq 2\|f - t_{d-1}\|_K \leq \frac{2M_J}{(d-1)^{\mathbf{m}}} \leq \frac{M'}{d^{\mathbf{m}}} \quad (K \text{ is Jackson and } f \in \mathcal{C}^{\mathbf{m}}(K)). \quad (16.13)$$

This bound together with inequality (16.8) now yields

$$\|D^\beta H_d\|_K \leq C_{\mu, K} M' d^{\theta+r\bar{m}-\mathbf{m}},$$

so that the series  $\sum_{d=0}^{\infty} D^\beta H_d$  converges normally on  $K$  as soon as  $\theta + r\bar{m} - \mathbf{m} < -1$ . This shows that, under the assumption on  $\mathbf{m}$ , the right hand side of (16.7) converges in  $\mathcal{C}^{\bar{m}}(K)$  to a certain function  $g$ . Since the convergence of the series to  $f$  plainly holds in  $L^2$ ,  $g$  coincides almost everywhere with  $f$  on  $K$ , hence everywhere by continuity.  $\square$

## 16.2. The convergence theorem

It is important to note the three levels of differentiability  $\mathbf{m}$ ,  $\bar{m}$  and  $m$  that occur in the following statement.

**Theorem 16.3.** *For  $s = 1, 2$ , we let  $K_s$  denote a compact set in  $\mathbb{R}^{n_s}$  such that*

- (1)  $K_s$  is a  $\theta_s$ -SBM compact,
- (2)  $K_s$  satisfies a Markov inequality of exponent  $r_s$ .

*We assume further that  $K_1 \times K_2$  is a Jackson compact set in  $\mathbb{R}^n$ ,  $n = n_1 + n_2$  and  $\kappa_s$  is a compact subset of  $K_s$ . We set  $K = K_1 \times K_2$  and  $\kappa = \kappa_1 \times \kappa_2$ .*

*Let  $\mathcal{N}^s = (\Pi_0^s, \Pi_1^s, \dots)$  denote a Newton sequence on  $\mathcal{C}^{\bar{m}}(K_s)$ ,  $s = 1, 2$ .*

*If  $\mathcal{N}^s$  is  $\mathcal{C}(\kappa_s)$ -converging on  $\mathcal{C}^m(K_s)$  for  $s = 1, 2$ ,  $m \geq \bar{m}$ , then  $\mathcal{N}^1 \otimes_{\mathbb{N}} \mathcal{N}^2 = (\Pi_0, \Pi_1, \dots)$  is  $\mathcal{C}(\kappa)$ -converging on  $\mathcal{C}^{\mathbf{m}}(K)$  provided that*

$$\mathbf{m} > 3/2 + (\theta_1 + \theta_2) + m(r_1 + r_2). \quad (16.14)$$

In the above statement, it is implicit that when  $\Pi_d^s$ ,  $s = 1, 2$ , is a continuous projector on  $\mathcal{C}^{\bar{m}}(K_s)$  then  $\Pi_d^1 \otimes_{\mathbb{N}} \Pi_d^2$  is a well defined continuous projector on  $\mathcal{C}^{\bar{m}}(K)$ . Such a result is established in [6] where the space considered are the usual Fréchet space  $\mathcal{C}^m(\Omega_s)$  where  $\Omega_s$  is open in  $\mathbb{R}^{n_s}$ . The proof in the present case is similar, we omit it. The reader may freely modify the hypothesis in the statement above assuming that all the projectors are defined on the space of functions differentiable on the whole space (with the same level of differentiability) and extend continuously to the spaces indicated, see the statement of Corollary 16.5 below. This is plainly the case in the examples presented in Subsection 16.3.

**Corollary 16.4.** *For  $s = 1, 2$ , we let  $K_s$  denote a convex body in  $\mathbb{R}^{n_s}$ . We set  $K = K_1 \times K_2$  and  $\kappa = \kappa_1 \times \kappa_2$ .*

*Let  $\mathcal{N}^s = (\Pi_0^s, \Pi_1^s, \dots)$  denote a Newton sequence on  $\mathcal{C}^{\bar{m}}(K_s)$ ,  $s = 1, 2$ .*

*If  $\mathcal{N}^s$  is  $\mathcal{C}(\kappa_s)$ -converging on  $\mathcal{C}^m(K_s)$  for  $s = 1, 2$ ,  $m \geq \bar{m}$ , then  $\mathcal{N}^1 \otimes_{\mathbb{N}} \mathcal{N}^2 = (\Pi_0, \Pi_1, \dots)$  is  $\mathcal{C}(\kappa)$ -converging on  $\mathcal{C}^{\mathbf{m}}(K)$  provided that*

$$\mathbf{m} > 3n/2 + 4m + 3/2. \quad (16.15)$$

*Démonstration.* Convex bodies satisfies all the requirements with  $\theta_s = 3n_s/2$  and  $r_s = 2$ .  $\square$

We will apply the result in the following form.

**Corollary 16.5.** *For  $s = 1, 2$ , we let  $\kappa_s$  denote a convex body in  $\mathbb{R}^{n_s}$ . We set  $\kappa = \kappa_1 \times \kappa_2$ .*

*Let  $\mathcal{N}^s = (\Pi_0^s, \Pi_1^s, \dots)$  denote a Newton sequence on  $\mathcal{C}^{\bar{m}}(\mathbb{R}^{n_s})$ ,  $s = 1, 2$ . We assume that all the projectors as well as those of  $\mathcal{N}^1 \otimes_{\mathbb{N}} \mathcal{N}^2 = (\Pi_0, \Pi_1, \dots)$  extend continuously to, respectively,  $\mathcal{C}^{\bar{m}}(\kappa_s)$  and  $\mathcal{C}^{\bar{m}}(\kappa)$ .*

*If  $\mathcal{N}^s$  is  $\mathcal{C}(\kappa_s)$ -converging on the space of function  $m$  times continuous differentiable on a neighborhood of  $\kappa_s$  for  $s = 1, 2$ ,  $m \geq \bar{m}$ , then  $\mathcal{N}^1 \otimes_{\mathbb{N}} \mathcal{N}^2 = (\Pi_0, \Pi_1, \dots)$  is  $\mathcal{C}(\kappa)$ -converging on the space of all  $\mathbf{m}$  differentiable functions on a neighborhood of  $\kappa$  provided that*

$$\mathbf{m} > 3n/2 + 4m + 3/2. \quad (16.16)$$

*Démonstration.* Apply the previous corollary with  $K_s$ , running in a basis of neighborhood of  $\kappa$  (formed of convex bodies).  $\square$

Let us point out the limitation of the above theorem and its corollaries. The assumption on  $\mathbf{m}$  clearly depends on the method of proof. It seems natural to expect that the theorem holds with  $\mathbf{m} = m$  and, if this is not true, it would be interesting to explain the reason why the Newton product procedure induces a loss.

*Proof of Theorem 16.3.* We use the strategy described in Subsection 13.3 starting from the Fourier expansion (16.7) on  $K_1 \times K_2$  with  $\mu = \mu_1 \times \mu_2$  where  $\mu_s$  is a  $\theta_s$ -SBM on  $K_s$ , so that  $\mu$  is  $\theta$ -SBM on  $K = K_1 \times K_2$  in view of Lemma 16.2. Besides  $K$  satisfies a Markov inequality of exponent  $\max(r_1, r_2)$ . Hence, since the assumption on  $\mathbf{m}$  implies

$$\mathbf{m} > 1 + (\theta_1 + \theta_2) + \bar{m} \max(r_1, r_2), \quad (16.17)$$

in view of Lemma 16.2, we have

$$f = \sum_{j=0}^{\infty} \sum_{|\alpha|+|\beta|=j} c_{\alpha\beta}(f) b_{\alpha,\beta}, \quad \text{in } \mathcal{C}^{\bar{m}}(K). \quad (16.18)$$

As in the proof of Theorem 14.1, the assumption on the convergence together with the uniform boundedness principle on the Banach space  $\mathcal{C}^m(K_s)$  provide us with a positive constant  $C_s$  such that (not confuse the projector  $\Pi_d^s$  with its Newton summand  $\pi_d^s$ ):

$$\|\pi_d^s(f_s)\|_{\kappa} \leq C_s \|f_s\|_{m, K_s}, \quad f_s \in \mathcal{C}^m(K_s), \quad s = 1, 2. \quad (16.19)$$

Now, since the convergence in (16.18) holds in  $\mathcal{C}^{\bar{m}}(K)$  on which  $\Pi_d$  is continuous then we may permute  $\Pi_d$  with the sum in the series expansion to obtain (see (14.14))

$$f - \Pi_d(f) = \sum_{j=d+1}^{\infty} \sum_{|\alpha|+|\beta|=j} c_{\alpha\beta}(f) \sum_{(i_1, i_2) \in \mathbf{B}(d, \alpha, \beta)} \pi_{i_1}^1(b(\alpha, \mu_1, \cdot)) \pi_{i_2}^2(b(\beta, \mu_2, \cdot)). \quad (16.20)$$

At this point, we could continue the proof as in that of Theorem 14.1. Yet the term in (14.21) which is innocuous in the case of holomorphic functions (because it is dominated by a geometric sequence) should be taken into account and would lead to a weaker estimate. To avoid this term, we will use a somewhat more tricky argument.

Let us denote by  $H_j$  the  $j$ -term in (16.20). Permuting the sums, we obtain

$$H_j = \sum_{i_2=0}^j \left( \sum_{i_1=d+1-i_2}^j \sum_{(\alpha, \beta) \in \mathbf{B}(i_1, i_2, j)} c_{\alpha\beta}(f) \pi_{i_1}^1(b(\alpha, \mu_1, \cdot)) \pi_{i_2}^2(b(\beta, \mu_2, \cdot)) \right), \quad (16.21)$$

where

$$\mathbf{B}(i_1, i_2, j) = \{(\alpha, \beta) \in \mathbb{N}^{n_1} \times \mathbb{N}^{n_2} : |\alpha| \geq i_1, |\beta| \geq i_2, |\alpha| + |\beta| = j\}.$$

Fix  $z^1 \in K_1$ . Let us concentrate on the term between brackets in (16.21). Since  $\pi_{i_2}$  is linear, it can be seen as

$$\pi_{i_2}(h), \quad h(\cdot) = \sum_{i_1=d+1-i_2}^j \sum_{(\alpha, \beta) \in \mathbf{B}(i_1, i_2, d)} c_{\alpha\beta}(f) \pi_{i_1}^1(b(\alpha, \mu_1, \cdot))(z^1) b(\beta, \mu_2, \cdot).$$

Observe that  $h = h_{z^1}$  is a polynomial (in  $z^2$ ) of degree  $\leq j$  and, since the  $b(\beta, \mu_2, z^2)$  are orthogonal, we have

$$\|h\|_2 = \sqrt{\sum_{i_1=d+1-i_2}^j \sum_{(\alpha, \beta) \in \mathbf{B}(i_1, i_2, j)} c_{\alpha\beta}^2(f) \times (\pi_{i_1}^1(b(\alpha, \mu_1, \cdot))(z^1))^2}. \quad (16.22)$$

Now,

$$\|\pi_{i_2}^2(h)\|_{\kappa_2} \leq C_2 \|h\|_{m, \kappa_2}, \quad \text{by (16.19)} \quad (16.23)$$

$$\leq C_2 C_{\mu_2, \kappa_2} j^{\theta_2 + mr_2} \|h_{z^1}\|_2, \quad \text{by (16.4), since } h \in \mathcal{P}_j(\mathbb{R}^{n_2}). \quad (16.24)$$

At this point, we have

$$\max_{z^2 \in \kappa_2} |H_j(z^2, z^1)| \leq \sum_{i_2=0}^j C_2 C_{\mu_2, \kappa_2} j^{\theta_2 + mr_2} \|h_{z^1}\|_2, \quad \text{all } z^1 \in \kappa_1. \quad (16.25)$$

Yet, with the same reasoning as above,

$$\|\pi_{i_1}^1(b(\alpha, \mu_1, \cdot))\|_{\kappa_1} \leq C_1 \|b(\alpha, \mu_1, \cdot)\|_{m, \kappa_1}, \quad \text{by (16.19)} \quad (16.26)$$

$$\leq C_1 C_{\mu_1, \kappa_1} |\alpha|^{\theta_1 + mr_1} \|b(\alpha, \mu_1, \cdot)\|_2, \quad \text{by (16.4)} \quad (16.27)$$

$$\leq C_1 C_{\mu_1, \kappa_1} j^{\theta_1 + mr_1}, \quad \text{by normality.} \quad (16.28)$$

Using this estimate in the bound for  $\|h_{z^1}\|_2$  in (16.24), we arrive at

$$\max_{z^1 \in \kappa_1} \|h_{z^1}\|_2 \leq C_1 C_{\mu_1, \kappa_1} j^{\theta_1 + mr_1} \left( \sum_{i_1=d+1-i_2}^j \sum_{(\alpha, \beta) \in \mathbf{B}(i_1, i_2, j)} c_{\alpha\beta}^2(f) \right)^{1/2}.$$

Returning to  $H_j$  in (16.25), we now have

$$\|H_j\|_{\kappa} \leq C_2 C_{\mu_2, \kappa_2} C_1 C_{\mu_1, \kappa_1} j^{\theta_1 + mr_1 + \theta_2 + mr_2} \sum_{i_2=0}^j \left( \sum_{i_1=d+1-i_2}^j \sum_{(\alpha, \beta) \in \mathbf{B}(i_1, i_2, j)} c_{\alpha\beta}^2(f) \right)^{1/2}.$$

To deal with the right hand term, we use the concavity inequality for the square root which reads as

$$\sum_{i_2=0}^j \frac{\sqrt{\square_{i_2}}}{j+1} \leq \sqrt{\frac{1}{j+1} \sum_{i_2=0}^j \square_{i_2}}$$

which gives the first line of the following (where we shortens the notation for clarity)

$$\sum_{i_2=0}^j \left( \sum_{i_1=\dots}^j \sum_{(\alpha, \beta) \in \dots} c_{\alpha\beta}^2(f) \right)^{1/2} \leq \sqrt{j+1} \left( \sum_{i_2=0}^j \sum_{i_1=\dots}^j \sum_{(\alpha, \beta) \in \dots} c_{\alpha\beta}^2(f) \right)^{1/2} \quad (16.29)$$

$$= \sqrt{j+1} \left( \sum_{|\alpha|+|\beta|=j} c_{\alpha\beta}^2(f) \right)^{1/2} \quad (16.30)$$

$$= \sqrt{j+1} \|\mathbf{P}_{j, \mu}(f) - \mathbf{P}_{j-1, \mu}(f)\|_2 \quad (16.31)$$

and, in view of (16.13) taking into account that  $K$  is Jackson and  $f \in \mathcal{C}^{\mathbf{m}}(K)$ ,

$$\leq C' \sqrt{j+1} j^{-\mathbf{m}} \leq C'' j^{1/2 - \mathbf{m}}. \quad (16.32)$$

So the final estimate for  $\|H_j\|_{\kappa}$  is

$$\|H_j\|_{\kappa} \leq C'' C_2 C_{\mu_2, \kappa_2} C_1 C_{\mu_1, \kappa_1} j^{1/2 + \theta_1 + mr_1 + \theta_2 + mr_2 - \mathbf{m}}.$$

Since, see (16.20), we have

$$\|f - \Pi_d(f)\|_{\kappa} \leq \sum_{j=d+1}^{\infty} \|H_j\|_{\kappa};$$

the uniform convergence follows as soon as  $1/2 + \theta_1 + mr_1 + \theta_2 + mr_2 - \mathbf{m} < -1$  which is the assumption on  $\mathbf{m}$ .



□

*Remark 3.* Although we consider only  $\mathcal{C}(\kappa)$ -convergence (uniform convergence on  $K$ ), the proof works with slight modification for  $\mathcal{C}^{\bar{m}}$ -convergence (assuming of course that such a convergence holds for the partial sequences). In fact, in (16.20), we would have to estimate  $\|D^\gamma f - D^\gamma \Pi_d(f)\|_\kappa$  for  $|\gamma| \leq \bar{m}$  and we would just have to compute the  $D^\gamma$  derivative of the products  $\pi_{i_1}^1(b(\alpha, \mu_1, \cdot)) \times \pi_{i_2}^2(b(\beta, \mu_2, \cdot))$  with the help of the Leibniz formula. Clearly, this would only further increases the acceptable value for  $\mathbf{m}$ . In particular, the reasoning would lead to :

**Corollary 16.6** (To the proof of Theorem 16.3). *With the same assumptions of Theorem 16.3 on the compact sets. If  $\mathcal{N}^s$  is converging on  $\mathcal{C}^\infty(K_s)$  for  $s = 1, 2$  then  $\mathcal{N}^1 \otimes_{\mathbf{N}} \mathcal{N}^2 = (\Pi_0, \Pi_1, \dots)$  is converging on  $\mathcal{C}^\infty(K)$ .*

### 16.3. Examples

(A) We turn to the projector considered in Theorem 14.5 taking a Leja sequence for  $a_d^s$  and a  $\mathfrak{K}$ -Leja sequence for  $b_d$  as is illustrated in Table 5.

**Theorem 16.7.** *For all function  $f$  22-times continuously differentiable on a neighborhood of  $K = D(0, 1) \times [-1, 1]$ , the polynomial  $(\mathbf{K}[a_0, \dots, a_d] \otimes_{\mathbf{N}} (\mathbf{L}[b_0, \dots, b_d]))(f)$  converges uniformly to  $f$  on  $K$ .*

*Démonstration.* From [40, Theorem 3.3 and Theorem 4.3],  $\mathbf{K}[a_0, \dots, a_d]$  satisfies the assumption of Corollary 16.5 with  $\kappa_1 = D(0, 1)$ ,  $\bar{m} = 1$  and  $m = 4$ . On the other hand, according to [23, Theorem 3.1], the Lebesgue constant for  $\mathbf{L}[b_0, \dots, b_d]$  grows at most like  $d^3 \ln d$  so that convergence holds for four times continuously differentiable function and we may also take  $m = 4$  (whereas any  $\bar{m} \geq 0$  works). Thus Corollary 16.5 ensures convergence for  $\mathbf{m} = 22 = 3 \times 3/2 + 4 \times 4 + 3/2$ . □

(B) When we work with  $\mathbf{K}[a_0, \dots, a_d] \otimes_{\mathbf{N}} \mathbf{K}[a_0, \dots, a_d]$  for functions on  $D(0, 1)^2$  in  $\mathbb{R}^4$ , the level of required differentiability for applying our result will be 24.

(C) If we substitute the Kergin interpolants by another related projector known as Hakopian interpolants, in the above, see [40, Theorem 4.5], the level will be 28.

(D) It is obvious that, in these examples, the levels of differentiability we obtain are rough. In these cases, the available algebraic formulas are very rich and one should be able to derive *ad hoc* error formulas leading to better results. We hope that our result will be an incentive to further research in this direction.



# PARTIE 3.

Étant donné un ensemble de nœuds d'interpolation valide pour l'interpolation de Lagrange de degré  $d$  à  $n$  variables, on étudie le nombre de sous-ensembles qui forment un ensemble d'interpolation valide de degré  $d - 1$ . Cela conduit à une estimation du nombre de structures de Newton, nombre qui fournit à son tour le nombre de tableaux unisolvants distincts que l'on peut obtenir par le procédé d'enlacement qui est rappelé dans le texte.

*Article soumis.*

## SUR LA GÉOMÉTRIE DES ENSEMBLES DE NŒUDS POUR L'INTERPOLATION DE LAGRANGE EN PLUSIEURS VARIABLES

### 17. INTRODUCTION

On sait bien que la donnée de  $d + 1$  points réels, deux à deux distincts, permet de définir un projecteur de Lagrange unidimensionnel de degré  $d$  et n'importe quel arrangement de ces  $d + 1$  points conduira par exemple à une forme de Newton particulière comme aussi à un algorithme de Neville-Aitken particulier. Il en est tout autrement dans l'interpolation de Lagrange multidimensionnelle car, si  $X$  est un ensemble de cardinal  $N_d(n) = \dim P_d(\mathbb{K}^n)$ ,  $n > 1$ , qui est unisolvant (de degré  $d$ ), voir ci-dessous, un sous-ensemble de  $X$  de cardinal  $N_{d-1}(n)$  ne sera pas nécessairement unisolvant de degré  $d - 1$ . Le problème de combinatoire géométrique étudié dans cette note consiste essentiellement à estimer le nombre de sous-ensembles d'un tel  $X$  qui soient unisolvants de degré  $d - 1$ . Le problème et sa motivation sont explicités ci-après.

### 18. NOTATIONS ET PROBLÉMATIQUE

La dimension  $N_d = N_d(n)$  de l'espace  $P_d(\mathbb{K}^n)$  des polynômes à  $n$  variables de degré (total) inférieur ou égal à  $d$ , et à coefficients dans  $\mathbb{K} = \mathbb{R}$  ou  $\mathbb{C}$  vaut  $\binom{d+n}{n} = \binom{d+n}{d}$ . C'est aussi le nombre de multi-indices  $\alpha \in \mathbb{N}^n$  de longueur  $|\alpha| \leq d$  dont l'ensemble sera noté  $\mathbb{N}_d^n$ . La notation  $X_d = \{A_1, \dots, A_{N_d(n)}\}$  représentera une **partie unisolvante** de  $\mathbb{K}^n$  pour  $P_d(\mathbb{K}^n)$ . Cela signifie que  $X_d$  est un ensemble de nœuds valide pour l'interpolation par des polynômes de degré au plus  $d$ . Si  $(e_\alpha : x \rightarrow x^\alpha)_{\alpha \in \mathbb{N}_d^n}$  est la base canonique de  $P_d(\mathbb{K}^n)$ , l'unisolvance de  $X_d$  signifie que la matrice de Vandermonde  $vdm(X_d)$  de  $X_d$  :

$$vdm(X) = \left( e_\alpha(A_i) \right)_{(i,\alpha) \in \llbracket 1, N_d \rrbracket \times \mathbb{N}_d^n},$$

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est une matrice inversible dont le déterminant (non nul) sera noté  $VDM(X_d)$ . Dans ces conditions, pour toute fonction  $f$  définie sur  $X$  et à valeurs dans  $\mathbb{K}$ , il existe un unique polynôme, noté  $L[X_d](f)$ , et appelé **polynôme d'interpolation de Lagrange** de  $f$  pour  $X_d$ , de degré au plus  $d$  tel que  $L[X_d](f)(A_i) = f(A_i)$  pour  $i = 1, \dots, N_d$ . Ce polynôme s'obtient par exemple sous la forme de Lagrange par l'expression

$$L[X_d](f) = \sum_{i=1}^{N_d} f(A_i) \ell_i, \quad \ell_i(x) = \frac{VDM\left(X_d \stackrel{i}{\leftarrow} x\right)}{VDM(X_d)}; \quad (18.1)$$

où  $\ell_i$  est le **polynôme fondamental de Lagrange** qui s'annule en tout point de  $X_d$  excepté en  $A_i$  où il prend la valeur 1. Dans la relation (18.1), la notation  $X_d \stackrel{i}{\leftarrow} x$  signifie qu'on substitue  $x$  à  $A_i$  dans  $X_d$ .

Puisque  $VDM(X_d)$  est non nul, les  $\binom{n+d-1}{d-1} = N_{d-1}$  premières colonnes de ce déterminant forment une famille libre, ce qui met en évidence des entiers  $1 \leq i_1 < \dots < i_{N_{d-1}} \leq N_d$  correspondant à  $N_{d-1}$  lignes indépendantes de sorte qu'en posant

$$X_{d-1} = \left\{ A_{i_1}, \dots, A_{i_{N_{d-1}}} \right\},$$

il vient  $VDM(X_{d-1}) \neq 0$ . Ainsi  $X_{d-1}$  se trouve être une partie unisolvante de  $\mathbb{K}^n$  de cardinal  $N_{d-1}$  et, par la même occasion, on initie un processus qui conduit à une suite de parties unisolvantes dans  $\mathbb{K}^n$ , de degré  $k$  (i.e. pour  $0 \leq k \leq d$ ,  $X_k$  admet  $k$  pour degré) :  $X_0 \subset \dots \subset X_{d-1} \subset X_d$  ce qui nous conduit à la définition suivante.

**Definition 18.1.** On appelle **structure de Newton** d'un ensemble unisolvant  $X_d$  de degré  $d$  dans  $\mathbb{K}^n$  toute suite d'ensembles  $X_i$ , unisolvant de degré  $i$ ,  $i = 0, \dots, d-1$ , telle que  $X_0 \subset X_1 \subset X_2 \subset \dots \subset X_{d-1} \subset X_d$ .

Cette notion a été introduite par Sauer et Xu dans [47] (avec une autre terminologie) dans le but de développer une formule de Newton multidimensionnelle, puis elle a été utilisée par Calvi [20] pour définir de nouveaux tableaux d'interpolation par un **procédé d'enlacement**, dont la généralisation, devenu le produit de Newton, est introduite dans [6] : si  $X_0 \subset X_1 \subset X_2 \subset \dots \subset X_{d-1} \subset X_d$  est une structure de Newton de  $X_d$  dans  $\mathbb{K}^n$  et  $Y_0 \subset Y_1 \subset Y_2 \subset \dots \subset Y_{d-1} \subset Y_d$  est une structure de Newton de  $Y_d$  dans  $\mathbb{K}^m$  alors, voir [20], l'ensemble

$$Z = \bigcup_{i+j \leq d} (X_i \times Y_j) \subset \mathbb{K}^{n+m}$$

est unisolvant de degré  $d$  dans  $\mathbb{K}^{n+m}$  (avec des règles de calcul s'obtenant à partir de celles sur  $X_d$  et  $Y_d$ ). Il est par ailleurs montré dans [6] que des structures différentes sur  $X_d$  et  $Y_d$  conduisent à des ensembles  $Z$  distincts. Du nombre de structures de Newton sur  $X_d$  et  $Y_d$  découle donc directement le nombre d'ensembles  $Z$  distincts pouvant être obtenus à partir de  $X_d$  et  $Y_d$ . Le nombre (plutôt conséquent, nous allons le voir) de telles structures fournit ainsi au numéricien (suivant le problème qu'il étudie) une véritable richesse de possibilités d'interpolation.

Notons que, puisque la condition de ne pas être unisolvant est algébrique, en dehors d'un ensemble algébrique de  $(\mathbb{K}^n)^{N_d(n)}$ , n'importe quel choix de  $X_i$  conviendra de sorte que le nombre de structures de Newton sera maximal, c'est-à-dire

$$\prod_{k=1}^d \binom{N_{d-k}(n)}{N_{d-k-1}(n)}$$

mais cette remarque n'aide guère dans la pratique car la plupart des choix naturels placeront les points sur des ensembles algébriques.

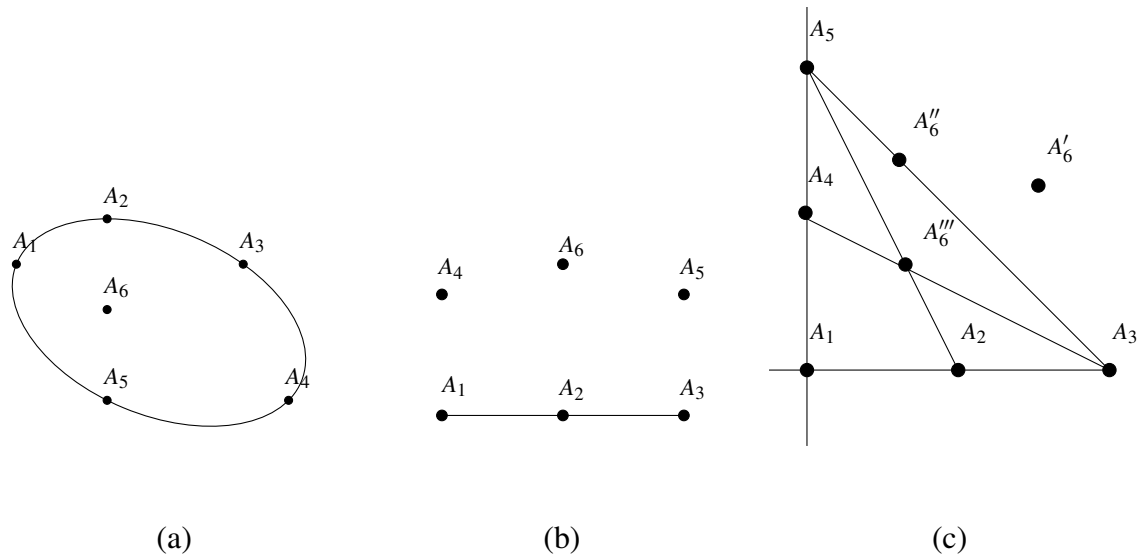


FIGURE 19.1. Configuration possible de six points d'interpolation  $\mathbb{R}^2$

Dans la première partie nous étudions en détail le cas  $n = 2$  et  $d = 2$ . Le résultat a été énoncé sans preuve dans [6, 3.5.2]. Dans la seconde partie, on obtient une minoration du nombre de structures de Newton dans le cas général.

Concluons en formulant le problème géométrique équivalent au notre qui paraît intéressant en lui-même. Supposons que l'on dispose d'un ensemble  $X$  de  $N_d(n)$  non situé sur une hypersurface algébrique de degré  $d$ . Donner une estimation de nombre de sous-ensembles de  $X$  de cardinal  $N_{d-1}(n)$  non situés sur une hypersurface algébrique de degré  $d - 1$ .

## 19. LE CAS $d = 2$

On dispose de six points  $A_1, \dots, A_6$  du plan, non situés sur une même courbe de degré 2 et donc de  $\binom{6}{3} = 20$  sous-ensembles à trois éléments de  $X_2$  qui, sous réserve d'unisolvance (ici, non alignés), donneront naissance à un projecteur de Lagrange de degré 1 et donc un ensemble  $X_1$  valide. On distingue plusieurs cas suivant la configuration de nos six points.

### Cas 1 : Aucune droite ne passe par trois points de $X_2$

Ce cas correspond à la figure 19.1 (a). on obtient alors 20 projecteurs de Lagrange de degré 1 ou 20  $X_1$  et par suite  $60 = 3 \times 20$  structures de Newton pour  $X_2$  puisque n'importe lequel des 3 points de  $X_1$  peut servir de  $X_0$ .

### Cas 2 : Une seule droite $D$ contient trois points de $X_2$ .

Ce cas correspond à la figure 19.1 (b). Cette droite  $D$  ne peut alors contenir un quatrième point, car alors, en notant  $D'$  la droite passant par les deux points restants,  $X_2$  serait inclus dans  $D \cup D'$  et ne serait pas unisolvant. On obtient  $20 - 1 = 19$  ensembles  $X_2$  possibles et 57 structures de Newton possibles.

### Cas 3 : Outre $D$ , une autre droite au moins, nommée $\Delta$ , contient trois points de $X_2$

Observons que  $D \cap \Delta \cap X_2$  ne peut être vide (car alors  $X_2 \subset D \cup \Delta$  avec trois points et trois dans l'autre) et ne peut contenir deux points puisque les droites sont supposées différentes. Supposons donc que  $A_1$  soit l'unique point de cette intersection et prenons par exemple  $D \cap X_2 = \{A_1, A_2, A_3\}$  et  $X_2 \cap \Delta = \{A_1, A_4, A_5\}$ , avec  $A_6 \notin D \cup \Delta$ . S'il existe une autre droite contenant trois points de  $X_2$ , elle doit passer par  $A_6$  et ce peut être qu'une des quatre droites  $(A_2A_4)$ ,  $(A_2A_5)$ ,  $(A_3A_4)$  ou  $(A_3A_5)$ .

- (1)  $A_6$  n'est sur aucune de ces droites (voir Figure 19.1 (c), où  $A'_6 = A_6$ ), il y a donc  $18 = 20 - 2$  choix possibles pour  $X_1$  et 54 structures de Newton.
- (2)  $A_6$  est sur une seule de ces droites (voir Figure 19.1 (c) où  $A''_6 = A_6$ ) qui conduit 17  $X_1$  valides et 51 structures de Newton.
- (3)  $A_6$  est sur deux de ces droites (voir Figure 19.1 (c), où  $A'''_6 = A_6$ ), on dispose alors de 16 solutions.

Nous avons établi le théorème suivant.

**Theorem 19.1.** *Pour un ensemble plan unisolvant de degré 2, il existe  $3(20 - k)$  structures de Newton avec  $k \in \{0, 1, 2, 3, 4\}$  et les cinq éventualités peuvent se produire.*

*Remark 4.* Il est naturel de rechercher des ensembles de points unisolvants de degré  $d$  sur des hypersurfaces algébriques de degré  $d + 1$ . Considérons la courbe  $\Delta$  d'équation  $x^3 = y^2$  paramétrée (en dehors de l'origine) par  $C(t) = (1/t^2, 1/t^3)$ . Un calcul simple montre qu'une famille de trois points  $\{A_1 = C(t_1), A_2 = C(t_2), A_3 = C(t_3)\}$  où les  $t_i$  sont deux à deux distincts et non nuls est unisolvante de degré 1 si et seulement si  $t_1 + t_2 + t_3 \neq 0$ . De même, on vérifie sans difficulté que six points  $\{A_i = C(t_i), i = 1, \dots, 6\}$  formeront un ensemble unisolvant de degré 2 si et seulement si  $\sum_{i=1}^6 t_i \neq 0$ . Quatre des cinq types de configurations possibles peuvent être obtenues avec des points situés sur  $\Delta$ . Par exemple le cas (a) est obtenue en prenant des paramètres  $t_i$  de même, le cas (b) est obtenu par exemple avec  $t_1 = -3, t_2 = 1$  et  $t_3 = 1$  qui conduit à l'alignement des trois premiers points, puis, par exemple  $t_4 = 3, t_5 = 4$  et  $t_6 = 5$  qui rend impossible tout autre alignement. La première occurrence du cas (c) s'obtient par exemple avec  $t_1 = -3, t_2 = 1$  et  $t_3 = 1$  et  $t_4 = -1, t_5 = -6$  et  $t_6 = 6$  pour laquelle les points  $A_1, A_2, A_3$  puis  $A_1, A_4, A_5$  sont alignés. La troisième occurrence s'obtient à partir de la précédente en modifiant la valeur de  $t_6$  à  $-6$ . La quatrième occurrence ne peut être obtenue car, comme on le vérifie facilement, les conditions d'alignement  $t_1 + t_4 + t_5 = 0, t_1 + t_2 + t_3 = 0, t_3 + t_4 + t_6 = 0$  et  $t_2 + t_5 + t_6 = 0$  ne peuvent être satisfaites avec des paramètres deux à deux distincts. En revanche, une telle configuration, illustrée sur la Figure 19.2, peut être obtenue sur la cubique nodale  $y^3 - x^3 - yx = 0$ .

## 20. LE CAS GÉNÉRAL

On reprend, pour  $d \geq 3$  et un espace de dimension  $n \geq 2$  quelconque les notations générales et le contexte de la section précédente, en particulier  $X_d$  désigne maintenant un ensemble unisolvant de degré  $d$  dans  $\mathbb{K}^n$  (donc de cardinal  $N_d(n)$ ). Pour simplifier les notations, on posera  $N = N_{d-1}(n)$ . Supposons que  $X_{d-1} = \{A_{i_1}, \dots, A_{i_N}\}$  soit unisolvant de degré  $d - 1$ . On a vu qu'un tel  $X_{d-1}$  existe. La matrice de Vandermonde de  $X_{d-1} = \{A_{i_1}, \dots, A_{i_N}\}$  étant inversible, si on en supprime la  $k$ -ième ligne, on obtient une matrice de rang  $N - 1$ ; par conséquent il passe une et une seule hypersurface algébrique  $\Gamma$  de degré  $\leq d - 1$  par tous les points de  $X'_k = X_{d-1} \setminus \{A_{i_k}\}$ . Cette hypersurface n'est autre que

$$\Gamma = \{\ell_{i_k} = 0\}$$

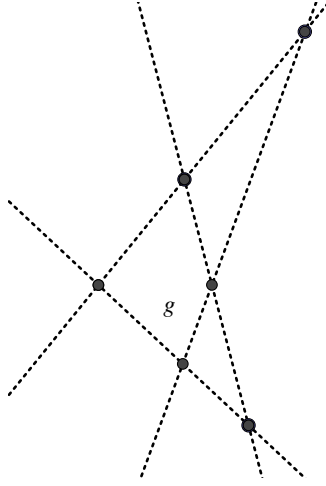


FIGURE 19.2. Exemple de configuration sur la cubique nodale  $y^3 - x^3 - yx = 0$  avec quatre droites d'alignement.

où  $\ell_{i_k} = 0$  est le polynôme fondamental de Lagrange pour  $X_{d-1}$  correspondant au nœud  $A_{i_k}$ . En effet, il est évident que cette hypersurface convient et si  $\{p = 0\}$  avec  $p \in \mathcal{P}_{d-1}(\mathbb{K}^n)$  en est une autre, l'identité

$$p = \sum_{s=1}^N p(A_{i_s}) \ell_{i_s}$$

implique que  $p = \alpha \ell_{i_k}$  avec  $\alpha \neq 0$  de sorte que  $\{p = 0\} = \Gamma$ .

De plus  $X_d \cap (\mathbb{K}^n \setminus \Gamma)$  comporte au moins  $n + 1$  points (dont un est nécessairement  $A_{i_k}$ ). En effet, sinon  $X_d$  serait inclus dans la réunion de  $\Gamma$  et d'un hyperplan passant par  $A_{i_k}$  ce qu'interdit son unisolvance (le polynôme  $\ell_{i_k} \times h$  où  $h$  est une équation de l'hyperplan serait un polynôme non nul de degré  $d$  qui s'annulerait sur  $X_d$ ).

Considérons alors  $M_r$ ,  $r = 1, \dots, n$ ,  $n$  points distincts de  $A_{i_k}$ , dans  $X_d$  et en dehors de  $\Gamma$  et posons

$$X_k^r = X_k' \cup \{M_r\}, \quad r = 1, \dots, n.$$

Nous affirmons que chacun de ces  $X_k^r$  est unisolvant de degré  $d - 1$ . Pour le montrer, supposons  $P \in \mathcal{P}_{d-1}(\mathbb{K}^n)$  s'annule sur  $X_k^r$ . Si  $P \neq 0$  alors  $P = 0$  est une équation de  $\Gamma$  (en raison du degré et de l'annulation sur  $X_k'$  de  $P$ ) mais  $P(M_r) = 0$  et ainsi  $M_r$  appartiendrait à  $\Gamma$  ce qui n'est pas. Nous avons ainsi montré que la donnée d'un sous-ensemble unisolvant de degré  $d - 1$  implique, en faisant varier  $r$  et  $i_k$  l'existence de  $N_{d-1} \times n$  autres sous-ensemble unisolvant de degré  $d - 1$  pour obtenir un total au moins égal à  $nN_{d-1}(n) + 1$ . Nous avons établi le théorème suivant.

**Theorem 20.1.** (1) Pour  $d \geq 3$ , de tout système unisolvant de degré  $d$  dans  $\mathbb{K}^n$ , on peut extraire au moins  $nN_{d-1} + 1$  systèmes unisolvants de degré  $d - 1$ .

(2) Pour tout ensemble unisolvant de degré  $d \geq 3$  dans  $\mathbb{K}^n$ , on dispose d'au moins

$$m_d = (n + 1) \prod_{k=2}^d (nN_{k-1}(n) + 1) \quad (20.1)$$

structures de Newton.

*Remark 5.* En tenant compte du Théorème 19.1, dans le cas  $n = 2$ , on obtient au moins  $48 \prod_{k=3}^d (d^2 + d + 1)$  structures de Newton.

*Remark 6.* En négligeant l'unité dans (20.1), on a

$$m_d \geq \frac{(n+1)n^{d-1}}{n!^{n-1}} \prod_{k=1}^{d-1} \left( \frac{n+k}{k} \right)^{d-k}.$$

*Remark 7.* Dans le cas  $n = 2$  et  $d = 3$ , on obtient ainsi une légère amélioration du [6, Théorème 3.11] puisque la minoration 576 donnée dans cette référence peut se remplacer par 624. Il en va de même de celle donnée (dans un cadre plus général) en [6, Section 3.8] dans notre contexte.

## 21. PROBLÈMES

(1) Le Théorème 19.1 montre que le nombre de configurations possibles est soumis à des conditions arithmétiques. Il serait intéressant de connaître ces conditions dans d'autres cas et d'estimer le nombre de valeurs possibles en général pour le nombre total de structures de Newton.

(2) Il serait souhaitable d'estimer la précision de la minoration donnée au Théorème 20.1 et de caractériser les configurations possédant un nombre minimal ou quasi minimal de structures de Newton.

(3) Estimer le nombre de structures de Newton d'un ensemble unisolvant  $X_d$  de degré  $d$  sous l'hypothèse supplémentaire que les points de  $X_d$  sont situés sur une hypersurface algébrique de degré  $d + 1$ .

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## RÉFÉRENCES

- [1] ANDERSSON, M., AND PASSARE, M. Complex Kergin interpolation. *J. Approx. Theory* 64, 2 (1991), 214–225.
- [2] ANDERSSON, M., AND PASSARE, M. Complex Kergin interpolation and the Fantappiè transform. *Math. Z.* 208, 2 (1991), 257–271.
- [3] ANDERSSON, M., PASSARE, M., AND SIGURDSSON, R. *Complex convexity and analytic functionals*, vol. 225 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 2004.
- [4] BAGBY, T., BOS, L., AND LEVENBERG, N. Multivariate simultaneous approximation. *Constr. Approx.* 18, 4 (2002), 569–577.
- [5] BAGBY, T., BOS, L., AND LEVENBERG, N. Multivariate simultaneous approximation. *Constr. Approx.* 18, 4 (2002), 569–577.
- [6] BERTRAND, F., AND CALVI, J.-P. The Newton product of polynomial projectors Part 1 : Construction and algebraic properties. *Internat. J. Math.* 30, 6 (2019), 1950030, 45.
- [7] BIAŁ AS CIEŻ, L., AND CALVI, J.-P. Pseudo Leja sequences. *Ann. Mat. Pura Appl. (4)* 191, 1 (2012), 53–75.
- [8] BIERMANN, O. Über näherungsweise kubaturen. *Monaths. Math. Phys.* 14 (1903), 211–225.
- [9] BIERSTONE, E. Differentiable functions. *Bol. Soc. Brasil. Mat.* 11, 2 (1980), 139–189.
- [10] BLOOM, T. Kergin interpolation of entire functions on  $\mathbb{C}^n$ . *Duke Math. J.* 48, 1 (1981), 69–83.
- [11] BLOOM, T., BOS, L. P., CALVI, J.-P., AND LEVENBERG, N. Polynomial interpolation and approximation in  $\mathbb{C}^d$ . *Ann. Polon. Math.* 106 (2012), 53–81.
- [12] BLOOM, T., AND CALVI, J.-P. A continuity property of multivariate Lagrange interpolation. *Math. Comp.* 66, 220 (1997), 1561–1577.
- [13] BLOOM, T., AND CALVI, J.-P. Kergin interpolants of holomorphic functions. *Constr. Approx.* 13, 4 (1997), 569–583.
- [14] BLOOM, T., AND CALVI, J.-P. The distribution of extremal points for Kergin interpolation : real case. *Ann. Inst. Fourier (Grenoble)* 48, 1 (1998), 205–222.
- [15] BLOOM, T., AND CALVI, J.-P. On the multivariate transfinite diameter. *Ann. Polon. Math.* 72, 3 (1999), 285–305.
- [16] BLOOM, T., AND CALVI, J.-P. On the multivariate transfinite diameter. *Ann. Polon. Math.* 72, 3 (1999), 285–305.
- [17] BLOOM, T., AND LEVENBERG, N. Capacity convergence results and applications to a Bernstein-Markov inequality. *Trans. Amer. Math. Soc.* 351, 12 (1999), 4753–4767.
- [18] BOS, L., AND CALVI, J.-P. Kergin interpolants at the roots of unity approximate  $C^2$  functions. *J. Anal. Math.* 72 (1997), 203–221.
- [19] CALVI, J. P. Polynomial interpolation with prescribed analytic functionals. *J. Approx. Theory* 75, 2 (1993), 136–156.
- [20] CALVI, J.-P. Intertwining unisolvent arrays for multivariate Lagrange interpolation. *Adv. Comput. Math.* 23, 4 (2005), 393–414.
- [21] CALVI, J.-P., AND FILIPSSON, L. The polynomial projectors that preserve homogeneous differential relations : a new characterization of Kergin interpolation. *East J. Approx.* 10, 4 (2004), 441–454.
- [22] CALVI, J.-P., AND MANH, P. V. A determinantal proof of the product formula for the multivariate transfinite diameter. *Bull. Pol. Acad. Sci. Math.* 53, 3 (2005), 291–298.
- [23] CALVI, J.-P., AND MANH, P. V. Lagrange interpolation at real projections of Leja sequences for the unit disk. *Proc. Amer. Math. Soc.* 140, 12 (2012), 4271–4284.
- [24] CAVARETTA, JR., A. S., MICHELLI, C. A., AND SHARMA, A. Multivariate interpolation and the Radon transform. *Math. Z.* 174, 3 (1980), 263–279.
- [25] FILIPSSON, L. Complex mean-value interpolation and approximation of holomorphic functions. *J. Approx. Theory* 91, 2 (1997), 244–278.
- [26] GAIER, D. *Lectures on complex approximation*. Birkhäuser Boston, Inc., Boston, MA, 1987. Translated from the German by Renate McLaughlin.



- [27] GORDON, W. J. Blending-function methods of bivariate and multivariate interpolation and approximation. *SIAM J. Numer. Anal.* 8 (1971), 158–177.
- [28] GUELFOND, A. O. *Calcul des différences finies*. Collection Universitaire de Mathématiques, XII. Traduit par G. Rideau. Dunod, Paris, 1963.
- [29] HAKOPIAN, H. A. Multivariate divided differences and multivariate interpolation of Lagrange and Hermite type. *J. Approx. Theory* 34, 3 (1982), 286–305.
- [30] HÖRMANDER, L. *The analysis of linear partial differential operators. I*. Classics in Mathematics. Springer-Verlag, Berlin, 2003. Distribution theory and Fourier analysis, Reprint of the second (1990) edition [Springer, Berlin; MR1065993 (91m :35001a)].
- [31] ISAACSON, E., AND KELLER, H. B. *Analysis of numerical methods*. John Wiley & Sons, Inc., New York-London-Sydney, 1966.
- [32] KERGIN, P. A natural interpolation of  $C^K$  functions. *J. Approx. Theory* 29, 4 (1980), 278–293.
- [33] KLIMEK, M. *Pluripotential theory*, vol. 6 of *London Mathematical Society Monographs. New Series*. The Clarendon Press, Oxford University Press, New York, 1991. Oxford Science Publications.
- [34] LEVENBERG, N. Approximation in  $\mathbb{C}^N$ . *Surv. Approx. Theory* 2 (2006), 92–140.
- [35] LORENTZ, R. A. *Multivariate Birkhoff interpolation*, vol. 1516 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1992.
- [36] MICCHELLI, C. A constructive approach to Kergin interpolation in  $\mathbb{R}^k$ : multivariate B-splines and Lagrange interpolation. *Rocky Mountain Journal of Mathematics* 10 (1980), 485–497.
- [37] MILOVANOVIĆ, G. V., MITRINOVIĆ, D. S., AND RASSIAS, T. M. *Topics in polynomials: extremal problems, inequalities, zeros*. World Scientific Publishing Co., Inc., River Edge, NJ, 1994.
- [38] NGUYEN, T. V., AND ZÉRIAHI, A. Familles de polynômes presque partout bornées. *Bull. Sci. Math. (2)* 107, 1 (1983), 81–91.
- [39] PAWŁUCKI, W., AND PLEŚNIAK, W. Markov’s inequality and  $C^\infty$  functions on sets with polynomial cusps. *Math. Ann.* 275, 3 (1986), 467–480.
- [40] PHUNG, V. M. On the convergence of Kergin and Hakopian interpolants at Leja sequences for the disk. *Acta Math. Hungar.* 136, 3 (2012), 165–188.
- [41] PLEŚNIAK, W. Recent progress in multivariate Markov inequality. In *Approximation theory*, vol. 212 of *Monogr. Textbooks Pure Appl. Math.* Dekker, New York, 1998, pp. 449–464.
- [42] PLEŚNIAK, W. Multivariate Jackson inequality. *J. Comput. Appl. Math.* 233, 3 (2009), 815–820.
- [43] PLEŚNIAK, W. A. Inégalité de Markov en plusieurs variables. *Int. J. Math. Math. Sci.* (2006), Art. ID 24549, 12.
- [44] RANGE, R. M. *Holomorphic functions and integral representations in several complex variables*, vol. 108 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1986.
- [45] RONKIN, L. I. *Introduction to the theory of entire functions of several variables*. American Mathematical Society, Providence, R.I., 1974. Translated from the Russian by Israel Program for Scientific Translations, Translations of Mathematical Monographs, Vol. 44.
- [46] RUDIN, W. *Functional analysis*. McGraw-Hill Book Co., New York-Düsseldorf-Johannesburg, 1973. McGraw-Hill Series in Higher Mathematics.
- [47] SAUER, T., AND XU, Y. On multivariate Lagrange interpolation. *Math. Comp.* 64, 211 (1995), 1147–1170.
- [48] SICIĄK, J. On some extremal functions and their applications in the theory of analytic functions of several complex variables. *Trans. Amer. Math. Soc.* 105 (1962), 322–357.
- [49] SMIRNOV, V. I., AND LEBEDEV, N. A. *Functions of a complex variable: Constructive theory*. Translated from the Russian by Scripta Technica Ltd. The M.I.T. Press, Cambridge, Mass., 1968.
- [50] WALSH, J. L. *Interpolation and approximation by rational functions in the complex domain*. Fourth edition. American Mathematical Society Colloquium Publications, Vol. XX. American Mathematical Society, Providence, R.I., 1965.
- [51] WHITNEY, H. Functions differentiable on the boundaries of regions. *Ann. of Math. (2)* 35, 3 (1934), 482–485.
- [52] WILHELMSSEN, D. R. A Markov inequality in several dimensions. *J. Approximation Theory* 11 (1974), 216–220.

- [53] ZÉRIAHI, A. Capacité, constante de čebyšev et polynômes orthogonaux associés à un compact de  $\mathbf{C}^n$ . *Bull. Sci. Math. (2)* 109, 3 (1985), 325–335.
- [54] ZÉRIAHI, A. Inégalités de Markov et développement en série de polynômes orthogonaux des fonctions  $C^\infty$  et  $A^\infty$ . In *Several complex variables (Stockholm, 1987/1988)*, vol. 38 of *Math. Notes*. Princeton Univ. Press, Princeton, NJ, 1993, pp. 683–701.
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## RÉSUMÉ (ABSTRACT)

### Résumé :

Dans ce travail nous présentons une nouvelle manière d'associer ( que nous nommons produit de Newton) deux projecteurs polynomiaux sur des espaces de fonctions pour obtenir un autre projecteur polynomial sur un espace de fonctions de plus de variables qu'initialement.

Tout d'abord nous développons quelques propriétés algébriques assez fines de cette construction. Par la suite nous montrons que la notion d'efficacité est stable par produit de Newton. Divers théorèmes d'approximation polynomiale sont alors obtenus par ce biais pour des fonctions comme les fonctions holomorphes sur un voisinage d'un compact régulier, les fonctions entières de croissance donnée ou certaines fonctions différentiables. Des projecteurs polynomiaux efficaces originaux sont ainsi explicités dans ces cadres là.

Dans l'ultime partie nous nous donnons un ensemble de noeuds d'interpolation valide ( i.e unisolvant) pour l'interpolation de Lagrange de degré  $d$  à  $n$  variables et nous en étudions le nombre de sous-ensembles unisolvants de degré  $d - 1$ . Cela conduit à une estimation du nombre de structures de Newton de notre ensemble et même à celui des produits de Newton que l'on peut obtenir.

### Abstract :

We propose a new way ( called Newton product) of combining two polynomial projectors on spaces of functions of few variables to obtain a polynomial projector on a space of functions of many variables. We present various algebraic properties of our construction and study the main approximation properties of the new projectors. Hence we prove that the Newton product of efficient polynomial projectors is still efficient. Some polynomial approximation theorems are established involving Newton product on spaces of holomorphic functions on a neighborhood of a regular compact set, on spaces of entire functions of given growth and on spaces of differentiable functions. Efficient explicit new projectors are also presented.

In the last chapter we give us an unisolvent set of degree  $d$  in  $n$  variables and study the number of unisolvent subsets of degree  $d - 1$ . This leads to an estimate of the number of Newton structures and of Newton product structures for our set.