



**HAL**  
open science

# Explicit reciprocity laws for formal Drinfeld modules

Marwa Ala Eddine

► **To cite this version:**

Marwa Ala Eddine. Explicit reciprocity laws for formal Drinfeld modules. Number Theory [math.NT]. Université Bourgogne Franche-Comté, 2023. English. NNT : 2023UBFCD021 . tel-04282630

**HAL Id: tel-04282630**

**<https://theses.hal.science/tel-04282630>**

Submitted on 13 Nov 2023

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

---

# Lois de réciprocité explicites pour les modules de Drinfeld formels

---

Thèse pour obtenir le grade de Docteur  
de l'Université Bourgogne Franche-Comté

Ecole Doctorale Carnot Pasteur  
Spécialité Mathématiques

Soutenue publiquement au Laboratoire de Mathématiques de Besançon  
le 04/04/2023  
par

**Marwa ALA EDDINE**

---

Devant le jury composé de :

<b>Sodaïgui BOUCHAIB</b>	Professeur, Université Polytechnique Hauts-de-France	<b>Président</b>
<b>Denis BENOIS</b>	Professeur, Université de Bordeaux	<b>Rapporteur</b>
<b>Bruno ANGLES</b>	Professeur, Université de Caen Normandie	<b>Rapporteur</b>
<b>Christine HUYGHE</b>	Directrice de Recherche CNRS, Université de Franche-Comté	<b>Examinatrice</b>
<b>Hassan OUKHABA</b>	Maitre de Conférences, Université de Franche-Comté	<b>Directeur de thèse</b>



---

# Explicit reciprocity laws for formal Drinfeld modules

---

Thèse pour obtenir le grade de Docteur  
de l'Université Bourgogne Franche-Comté

Ecole Doctorale Carnot Pasteur  
Spécialité Mathématiques

Soutenue publiquement au Laboratoire de Mathématiques de Besançon  
le 04/04/2023  
par

**Marwa ALA EDDINE**

---

Devant le jury composé de :

<b>Sodaïgui BOUCHAIB</b>	Professeur, Université Polytechnique Hauts-de-France	<b>Président</b>
<b>Denis BENOIS</b>	Professeur, Université de Bordeaux	<b>Rapporteur</b>
<b>Bruno ANGLES</b>	Professeur, Université de Caen Normandie	<b>Rapporteur</b>
<b>Christine HUYGHE</b>	Directrice de Recherche CNRS, Université de Franche-Comté	<b>Examinatrice</b>
<b>Hassan OUKHABA</b>	Maitre de Conférences, Université de Franche-Comté	<b>Directeur de thèse</b>



**Titre:** Lois de réciprocité explicites pour les modules de Drinfeld formels

**Mots Clés:** lois de réciprocité, corps locaux, théorie du corps de classe

**Résumé:** Nous considérons l'accouplement de Kummer dans le cas des modules de Drinfeld formels ayant une réduction stable d' hauteur un, définis sur des corps locaux à caractéristique positive, dans le but de trouver des lois de réciprocité explicites dans la veine de celles qui existent en caractéristique nulle. Dans un premier temps, nous considérons une classe spéciale de ces modules de Drinfeld et nous nous inspirons de l'approche de Wiles pour démontrer des formules explicites dans ce cas. Enfin, nous démontrons le cas général en s'inspirant des travaux de Kolyvagin. Les résultats présentés dans cette thèse viennent compléter les résultats de Anglès et de Longhi-Bars, qui ont démontré des formules explicites dans le cas des modules de Drinfeld formels issues respectivement des polynomes de Carlitz et des modules de Drinfeld *sign*-normalisés de rang un.

---

**Title:** Explicit reciprocity laws for formal Drinfeld modules

**Keywords:** formal Drinfeld modules, explicit reciprocity laws, local fields, class field theory

**Abstract:** We prove explicit reciprocity laws for formal Drinfeld modules defined over local fields of positive characteristic and having stable reduction of height one, in the spirit of those existing in characteristic zero. At first request, we consider a special class of these formal Drinfeld modules and we prove explicit formulas for the Kummer pairing following an approach inspired by Wiles. In a later request, we give an explicit description of the pairing *à la Kolyvagin* in the general case. The results obtained give a generalization of the results of Anglès and Longhi-Bars proved for formal Drinfeld modules obtained respectively from Carlitz polynomials and *sign*-normalized Drinfeld modules of rank one.



# Acknowledgments

First and foremost, praise be to God, the Merciful, the Beneficent, Him who has granted me with countless blessings, and guided me all the way.

I would like to heartily thank my supervisor Hassan Oukhaba for his guidance, support, and patience during these past years. I am very grateful for all the time and effort you put into helping me achieving this project and building my future career.

I would also like to thank the members of the jury: Denis Benois, Bruno Anglès, Sodaïgui Bouchaïb and Christine Huyghe for their interest in my work.

Being a part of the *Laboratoire de Mathématiques de Besançon*, and especially the *Algèbre et Théorie Des Nombres* team, has been a great experience. I have learned a lot from you all. I am also thankful for my cohort and my office mates Audrey Fovelle, Mathilde Massart and Hua Wang.

My final and biggest thanks goes to my family. I am forever grateful for my mom and dad, who never failed to provide endless care, love and support. If it were not for you, your presence, your prayers, your encouragements, I would not be the person I am today. To Ali, the most supportive loving husband out there, thank you for being by my side all the way through.





# Contents

<b>Introduction</b>	<b>7</b>
<b>1 Formal Drinfeld modules</b>	<b>11</b>
1.1 Definitions and first properties	11
1.2 The logarithm of a formal Drinfeld module	14
1.3 Torsion points and abelian towers	17
1.4 The norm operator of Coleman and the Coleman power series	26
1.4.1 The case $\rho \in \mathbb{R}^{m_0}$	31
<b>2 Explicit reciprocity laws in a particular case</b>	<b>33</b>
2.1 The reciprocity map	33
2.2 The Kummer pairing and its main properties	34
2.2.1 The analytic pairing	40
2.3 Explicit reciprocity laws	43
2.4 The limit Kummer pairing	50
<b>3 Explicit reciprocity laws in the general case</b>	<b>52</b>
3.1 The Iwasawa map	52
3.2 Derivations	56
3.2.1 Recall on derivations	56
3.2.2 The derivation $\bar{D}_{L,v_n}$	59
3.3 Values of $\bar{D}_{L,v_n}$ in terms of representation theory	64
3.4 A return to the case of Chapter 2	68



# Introduction

In local class field theory, explicit reciprocity laws consist in studying and finding explicit formulas for the reciprocity map, also called the norm residue symbol. The way through this usually goes by the Hilbert symbol, which is defined by the means of the reciprocity map. In 1858, Kummer considered the  $p$ -th Hilbert symbol for pairs of principal units in the fields  $\mathbb{Q}_p(\zeta_p)$ , where  $p > 2$  is a prime number and  $\zeta_p$  is a primitive root of unity. From that time until now, numerous explicit laws were proven in various settings. Artin and Hasse [4] proved explicit formulas for the  $p^n$ -th Hilbert symbol for special pairs in cyclotomic extensions of  $\mathbb{Q}_p$  containing the  $p^n$ -th roots of unity. Later on, Iwasawa [17] generalized their formulas to include more pairs of elements. It was then Coleman [10] who gave a complete formula for this case. In a more general setting, Wiles [34] proved explicit laws for Lubin-Tate extensions of local fields. Soon after, Kolyvagin [21] extended all these results to formal groups of finite height. Explicit reciprocity laws were also studied in higher local fields. The formula of Iwasawa [17] was extended to this case by Kurihara [22] and Zinoviev [35]. In his turn, Florez [12, 13] generalized the work of Kolyvagin [21] and proved explicit laws in the case of formal groups and Lubin-Tate formal groups defined over arbitrary higher local field of mixed characteristic. For detailed history of reciprocity laws, check [25].

In our work, we consider field extensions of local fields of positive characteristic, obtained by adding torsion points of formal Drinfeld modules having stable reduction of height 1. We prove explicit reciprocity laws in these fields. Some results for special cases of formal Drinfeld modules were already proven. Namely, Anglès [3] considered the case of Carlitz modules (see Example 1.1.1), and Bars and Longhi [6] considered the case of formal Drinfeld modules deriving from standard sign-normalized rank one Drinfeld modules (see Example 1.1.2). Let us describe the context of our work.

Let  $K$  be a local field,  $p$  be its characteristic, and let  $\mu$  be its normalized discrete valuation. We denote  $\mathcal{O}_K$  the valuation ring of  $K$  and  $\mathfrak{p}_K$  its maximal ideal. Let  $q$  be the order of the residue field  $\mathcal{O}_K/\mathfrak{p}_K$ . Then  $q$  is a power of  $p$ . Fix an algebraic closure  $\Omega$  of  $K$ , and let  $\mu$  be the unique extension of  $\mu$  to  $\Omega$ . Let  $(\bar{\Omega}, \bar{\mu})$  be the completion of  $(\Omega, \mu)$ . For a field  $F \subset \Omega$ , we denote by  $\mathcal{O}_F$  its valuation ring and  $\mathfrak{p}_F$  its maximal ideal. Let  $K_{ur} \subset \Omega$  be the maximal unramified extension of  $K$  in  $\Omega$ , and  $H \subset K_{ur}$  be a finite unramified extension of  $K$ .

We consider

$$\begin{aligned} \rho : \mathcal{O}_K &\longrightarrow \mathcal{O}_H\{\{\tau\}\} \\ a &\longmapsto \rho_a \end{aligned}$$

a formal Drinfeld module having stable reduction of height one, as defined by Rosen in [29,

§1]. Here,  $\tau$  is the  $q$ -Frobenius element satisfying

$$\tau x = x^q \tau, \quad \forall x \in \Omega. \quad (0.0.1)$$

This is a special case of formal  $\mathcal{O}_K$ -modules defined by Drinfeld in [11]. We devote Chapter 1 to give the detailed definition and the main properties of formal Drinfeld modules.

Let  $\mathcal{O}_{\bar{\Omega}}$  be the valuation ring of  $\bar{\Omega}$  and  $\mathfrak{p}_{\bar{\Omega}}$  be its maximal ideal. Then  $\mathfrak{p}_{\bar{\Omega}}$  is an  $\mathcal{O}_K$ -module for the following action of  $\rho$

$$a \cdot_{\rho} x = \rho_a(x) \quad \forall x \in \mathfrak{p}_{\bar{\Omega}}. \quad (0.0.2)$$

For an integer  $n \geq 0$ , let

$$V_{\rho}^n = \{\alpha \in \bar{\Omega}; \rho_a(\alpha) = 0 \quad \forall a \in \mathfrak{p}_K^n\} \quad (0.0.3)$$

be the  $\mathfrak{p}_K^n$  torsion submodule of  $\bar{\Omega}$  for the action (0.0.2). Using the Weierstrass preparation theorem, we can see that  $V_{\rho}^n \setminus V_{\rho}^{n-1}$  is the set of roots of a separable Eisenstein polynomial in  $\mathcal{O}_H[X]$  of degree  $q^{n-1}(q-1)$ , whose constant term is a prime of  $H$ . Therefore, for an element  $v_0 \in V_{\rho}^n \setminus V_{\rho}^{n-1}$ , the extension  $H(v_0)|H$  is totally ramified of degree  $q^{n-1}(q-1)$ . Furthermore, the kernel of  $a \mapsto \rho_a(v_0)$  is  $\mathfrak{p}_K^n$ . Thus it induces an isomorphism of  $\mathcal{O}_K$ -modules

$$\mathcal{O}_K/\mathfrak{p}_K^n \cong V_{\rho}^n. \quad (0.0.4)$$

This implies that any element  $v_0 \in V_{\rho}^n \setminus V_{\rho}^{n-1}$  is a generator of  $V_{\rho}^n$  as  $\mathcal{O}_K$ -module. This also implies that the extension  $H_{\rho}^n = H(V_{\rho}^n)$  is equal to  $H(v_0)$ . Furthermore, the extension  $H_{\rho}^n|H$  is abelian and the compositum of the union of the extensions  $H_{\rho}^n$  together with  $K_{ur}$  is equal to the maximal abelian extension of  $K$ . See §1.3.

Now let  $m_0 \geq 1$  be an integer dividing  $[H : K]$ , and  $\eta \in K$  of valuation  $\mu(\eta) = m_0$ . Let

$$W_{\rho}^n = V_{\rho}^{nm_0} = \{\alpha \in \mathfrak{p}_{\bar{\Omega}}; \rho_{\eta^n}(\alpha) = 0\} \quad \text{and} \quad E_{\rho}^n = H(W_{\rho}^n) = H_{\rho}^{nm_0}. \quad (0.0.5)$$

For a finite extension  $L$  of  $E_{\rho}^n$ , we denote by

$$\Phi_L : L^{\times} \rightarrow Gal(L^{ab}|L) \quad (0.0.6)$$

the norm residue map. We can prove (see Lemma 2.2.1) that, for each  $\alpha \in \mathfrak{p}_L$ , there exists an element  $\xi \in L^{ab}$  such that  $\rho_{\eta^n}(\xi) = \alpha$ . Therefore we can define the map  $(\ , \ )_{\rho, L, n} : \mathfrak{p}_L \times L^{\times} \longrightarrow W_{\rho}^n$  by

$$(\alpha, \beta)_{\rho, L, n} = \Phi_L(\beta)(\xi) - \xi; \quad \rho_{\eta^n}(\xi) = \alpha, \quad (0.0.7)$$

for  $\alpha \in \mathfrak{p}_L$  and  $\beta \in L^{\times}$ . The main objective of this thesis is to prove explicit reciprocity laws for formal Drinfeld modules having stable reduction of height 1. In other words, we prove explicit formulas for the map  $(\ , \ )_{\rho, L, n}$ .

In Chapter 2, we restrict ourselves to the special case where

$$L = E_{\rho}^n \quad \text{and} \quad \rho_{\eta} \equiv \tau^{m_0} \pmod{\mathfrak{p}_H}. \quad (0.0.8)$$

We fix once and for all a generator  $v_n$  of  $W_\rho^n$ . In these settings, we prove that for all  $\alpha \in \mathfrak{p}_L$  such that  $\mu(\alpha) \geq \frac{nm_0}{q} + \frac{1}{q-1} + \frac{1}{q^{nm_0(q-1)}}$ , and for all  $\beta \in L^\times$ , we have

$$(\alpha, \beta)_{\rho, L, n} = \frac{1}{\eta^n} T_{L|K}(\lambda_\rho(\alpha) \delta_{v_n}(\beta)) \cdot_\rho v_n, \quad (0.0.9)$$

where  $\lambda_\rho$  is the logarithm of  $\rho$ , and  $\delta_{v_n} : L^\times \rightarrow \mathfrak{p}_L/\mathcal{D}_n$  is a group homomorphism defined as follows. For  $\beta \in L^\times$ , choose a power series  $f(X) \in \mathcal{O}_H((X)) \setminus \{0\}$  such that  $f(v_n) = \beta$ , and set

$$\delta_{v_n}(\beta) := \frac{f'(v_n)}{\beta} \pmod{\mathcal{D}_n}, \quad (0.0.10)$$

where  $\mathcal{D}_n$  denotes the different of the extension  $L|K$  [1]. This gives an analogue of Theorem 19 of Wiles [34] and an extension of Theorem 3.12 of Anglès [3]. The method we use to prove this result is inspired by Wiles [34], taking into account the challenges derived from the fact that the formal Drinfeld modules considered are formal power series, and are no longer polynomials. A crucial fact to prove this formula is that for all units  $u$  of  $K$ , and for all  $\omega \in W_\rho$ , we have

$$\Phi_K(u)(\omega) = \rho_{u^{-1}}(\omega). \quad (0.0.11)$$

We can also define a limit form  $(, )_{\rho, L}$  of the pairing  $(, )_{\rho, L, n}$ , where the first coordinate belongs to the direct limit  $\varinjlim \mathfrak{p}_{E_\rho^n}$ , taken with respect to the maps

$$\begin{aligned} \mathfrak{p}_{E_\rho^n} &\rightarrow \mathfrak{p}_{E_\rho^m} \\ \alpha_n &\mapsto \rho_{\eta^{m-n}}(\alpha_n), \end{aligned} \quad (0.0.12)$$

and the second coordinate belongs to the projective limit  $\varprojlim (E_\rho^n)^\times$ , taken with respect to the norm maps. An explicit formula, similar to (0.0.9), can be proven for the limit pairing. It gives a generalization of Theorem 23 of Longhi-Bars [6] proved for formal Drinfeld modules obtained from sign-normalized rank 1 Drinfeld modules. It also gives an analogue of Theorem 8.16 of Iwasawa [18]. The results of Chapter 2 were subject to a submitted paper [1].

Afterwards, in Chapter 3, we prove explicit formulas in the general settings, under the only assumption that  $L|K$  is separable. We prove that there exists a unique map  $\psi_{L, v_n}$  from a certain subgroup  $L^n$  of  $L^\times$  to a certain  $\mathcal{O}_L$ -submodule of  $L$ , satisfying

$$(\alpha, \beta)_{\rho, L, n} = T_{L|K}(\lambda_\rho(\alpha) \psi_{L, v_n}(\beta)) \cdot_\rho v_n \quad (0.0.13)$$

for all  $\alpha \in \mathfrak{p}_L$  and  $\beta \in L^n$ . The map  $\psi_{L, v_n}$  is the analogue of the so-called Iwasawa map introduced Proposition 14 of [17].

We further prove that there exists an  $\mathcal{O}_K$ -derivation  $\bar{D}_{L, v_n}$  from  $\mathcal{O}_L$  into a certain  $\mathcal{O}_L$ -submodule of  $L$  such that

$$(\alpha, \beta)_{\rho, L, n} = T_{L|K}(\lambda_\rho(\alpha) \text{dlog} \bar{D}_{L, v_n}(\beta)) \cdot_\rho v_n \quad (0.0.14)$$

for all  $\beta \in L^\times$  and  $\alpha \in L$  of valuation  $\mu(\alpha) > \frac{nm_0}{q} + \frac{1}{q-1} + \frac{1}{e(L|K)}$ , where  $e(L|K)$  is the ramification index of  $L|K$  [2]. If we write  $\beta = u\pi_L^k \in L^\times$ , where  $u$  is a unit of  $L$  and  $\pi_L$

is a prime of  $L$ , the logarithmic derivative  $\text{dlog}\bar{D}_{L,v_n}$  associated with the derivation  $\bar{D}_{L,v_n}$  is defined by

$$\text{dlog}\bar{D}_{L,v_n}(\beta) = \frac{\bar{D}_{L,v_n}(u)}{u} + k \frac{\bar{D}_{L,v_n}(\pi_L)}{\pi_L}. \quad (0.0.15)$$

This is the analogue of the main result of Kolyvagin's paper [21]. An advantage of having a derivation is that it is determined and explicitly constructible in terms of its value at a uniformizer  $\pi_L$  of  $L$  as follows. For  $x \in \mathcal{O}_L$ , we can write

$$\bar{D}_{L,v_n}(x) = f'(\pi_L) \bar{D}_{L,v_n}(\pi_L), \quad (0.0.16)$$

where  $f$  is the unique power series in  $\mathbb{F}_{q_L}[[X]]$  such that  $x = f(\pi_L)$ . Here,  $\mathbb{F}_{q_L}$  denotes the residue field of  $L$ . A comparison of the two formulas (0.0.9) and (0.0.14) in the case  $L = E_\rho^m$ ,  $\pi_L = v_m$  and  $\rho_\eta \equiv \tau^{m_0} \pmod{\mathfrak{p}_H}$  yields that

$$\bar{D}_{E_\rho^m, v_n}(v_m) = \frac{1}{\eta^m}. \quad (0.0.17)$$

for all  $m \geq n$ . Using (0.0.17) and invariants attached to the representation  $\mathbf{r} : \text{Gal}(\Omega|H) \rightarrow \text{GL}_1(\mathcal{O}_K) = \mathcal{U}_K$ , which is induced by the action of  $\text{Gal}(\Omega|H)$  on the module  $\varprojlim W_\rho$ , we get the following congruence, of which we do not have a direct proof. For all units  $u$  in  $L$  such that  $\mu(1-u) > \max\{\frac{nm_0}{q}, \frac{1}{q-1}\} + \frac{1}{q-1}$ , we have

$$N_{L|K}(u^{-1}) - 1 \equiv T_{L|K}\left(\left(\frac{1-u}{u}\right)\left(1 - \frac{g'(v_m)}{u}v_m\right)\right) \pmod{\mathfrak{p}_K^{(n+m)m_0}}, \quad (0.0.18)$$

where  $N_{L|K} : L \rightarrow K$  is the norm map and  $g(X)$  is the unique power series in  $\mathbb{F}_{q_L}[[X]]$  such that  $g(v_m) = u$ . Reciprocally, the congruence (0.0.18) implies (0.0.17). Hence, proving (0.0.18) would provide a new proof of (0.0.17), without the use of Chapter 2.

The methods used to obtain (0.0.14) and (0.0.18) were inspired by the work of Kolyvagin [21]. But Kolyvagin on the other hand had a direct proof of his analogue of (0.0.18). The ingredient he used is the logarithm, which allows him to swing between the trace and the norm.

# Chapter 1

## Formal Drinfeld modules

The idea of formal modules was first introduced by Drinfeld in his paper [11]. In this chapter, we consider a special case of these formal modules, the so-called formal Drinfeld modules which were studied by Rosen in [29]. We will rely on Rosen's paper to define and state the main properties of formal Drinfeld modules. In all this chapter, let  $K$  be a local field of positive characteristic  $p$ . Let  $\mathcal{O}_K$  be its valuation ring and  $\mathfrak{p}_K$  be its maximal ideal. We denote by  $q$  the order of its residue field  $\mathcal{O}_K/\mathfrak{p}_K$ .

### 1.1 Definitions and first properties

For an  $\mathcal{O}_K$ -algebra  $B$ , let  $B\{\{\tau_p\}\}$  be the twisted power series ring consisting of all power series  $\sum_{i \geq 0} b_i \tau_p^i$ , such that the  $b_i$  belong to  $B$  and  $\tau_p$  is the  $p$ -Frobenius element satisfying

$$\tau_p x = x^p \tau_p \quad \forall x \in B. \quad (1.1.1)$$

Let  $\gamma : \mathcal{O}_K \rightarrow B$  be the structure map and  $D : B\{\{\tau_p\}\} \rightarrow B$  be the ring homomorphism that assigns to a power series  $\sum_{i \geq 0} b_i \tau_p^i$  its constant term  $b_0$ . Let  $d$  be the integer such that  $p^d = q$ . A formal Drinfeld  $\mathcal{O}_K$ -module over  $B$  is a ring homomorphism

$$\begin{aligned} \rho : \mathcal{O}_K &\rightarrow B\{\{\tau_p^s\}\} \\ a &\mapsto \rho_a \end{aligned}$$

for a positive integer  $s \leq d$ , satisfying

- (i)  $\forall a \in \mathcal{O}_K, D(\rho_a) = \gamma(a)$ ,
- (ii)  $\rho(\mathcal{O}_K) \not\subset B$ ,
- (iii)  $\rho_\pi \neq 0$  for one (and hence all) prime  $\pi$  of  $K$ .

This definition goes back to Rosen [29], and is a special case of formal  $\mathcal{O}_K$ -modules over  $B$ , introduced in the first place by Drinfeld in [11]. Indeed, a formal  $\mathcal{O}_K$ -module over  $B$  is defined as a pair  $(F, f)$ , where  $F$  is a formal group over  $B$  and  $f$  is a ring homomorphism from



$\mathcal{O}_K$  into  $\text{End}(F)$ , satisfying  $D \circ f = \gamma$ . Namely,  $F$  is a formal power series in  $B[[X, Y]]$  such that  $F(X, Y) = F(Y, X)$ ,  $F(X, 0) = X$ , and  $F(X, F(Y, Z)) = F(F(X, Y), Z)$ . Moreover, the image  $f_a$  of an element  $a$  in  $\mathcal{O}_K$  is an endomorphism of  $F$ , i.e. it is a formal power series in  $B[[X]]$  such that  $f_a(F(X, Y)) = F(f_a(X), f_a(Y))$ . If we take  $F$  to be the additive group  $G_a(X, Y) = X + Y$ , the endomorphisms of  $F$  are all of the form  $\sum_{i \geq 0} b_i X^{p^i}$ . Hence, we can identify  $\text{End}(G_a)$  with the twisted power series ring  $B\{\{\tau_p\}\}$ . For more details on the general case, one may check [11, §1] or [16].

As Rosen stated in [29, p 239], formal Drinfeld modules exist in abundance. The subsequent examples show three ways to generate families of formal Drinfeld modules.

**Example 1.1.1.** Fix a prime  $\pi$  of  $K$  and let  $f(X) = \sum_{i=0}^d f_i X^{p^i}$  be a polynomial with coefficients in  $\mathcal{O}_K$ , satisfying

- (i)  $f_0 = \pi$  and  $f_d = 1$ .
- (ii)  $f(X) \equiv X^a \pmod{\mathfrak{p}_K}$ .

Such polynomials are called Carlitz polynomials [3, Definition 1.1]. By Lubin-Tate theory (see for instance [23, Chapter 8]), we can associate to each Carlitz polynomial  $f$ , a ring homomorphism  $[\ ]_f : \mathcal{O}_K \rightarrow \mathcal{O}_K[[X]]$  such that

$$[a]_f(X) = aX \pmod{\text{deg } 2} \quad \text{and} \quad f \circ [a]_f = [a]_f \circ f$$

for all  $a \in \mathcal{O}_K$ . The maps  $[\ ]_f$  are the so-called Carlitz modules, they are a basic example of formal Drinfeld modules.

**Example 1.1.2.** We will construct a formal Drinfeld module out of a standard Drinfeld module as Rosen explained in §4 of his paper [29]. Let  $L$  be a global function field of characteristic  $p$ . Let  $\infty$  be a place of  $L$  and  $A$  be the Dedekind ring of elements of  $L$  regular outside  $\infty$ . Let  $M$  be an extension of  $L$  and let  $\rho$  be a Drinfeld  $A$ -module over  $M$ , that is a ring homomorphism from  $A$  to  $M\{\{\tau_p^d\}\}$  satisfying  $D(\rho_a) = a$  for all  $a \in A$  and  $\rho(A) \not\subset M$ . This definition goes back to Drinfeld [11]. One may also find more details on standard Drinfeld modules in [15]. We denote by  $C$  the integral closure of  $A$  in  $M$ , and we let  $I$  be a maximal ideal of  $C$ . Suppose  $\rho$  has stable reduction at  $I$ . This means that the coefficients of the  $\rho_a$  are in the local ring  $C_I$  for all  $a \in A$ , and the reduction  $a \mapsto \bar{\rho}_a$  of  $\rho$  modulo  $I$  is also a Drinfeld module. Hence, we can extend  $\rho$  to  $A_P$ , where  $P = A \cap I$ . Indeed, one can easily see that  $\rho_s$  are invertible power series in  $C_I\{\{\tau_p^d\}\}$  for all  $s \in A \setminus P$ . Therefore, we define the map

$$\begin{aligned} \rho : A_P &\rightarrow C_I\{\{\tau_p^d\}\} \\ \frac{a}{s} &\mapsto \rho_a \rho_s^{-1}. \end{aligned}$$

The last step in this construction is the completion. For that, let  $\hat{C}_I$  be the completion of  $C$  in the  $I$ -adic topology, and  $\hat{A}_P$  be the completion of  $A$  in the  $P$ -adic topology. Since  $C_I \subset \hat{C}_I$ , we can see  $\rho$  as a map from  $A_P$  to  $\hat{C}_I\{\{\tau_p^d\}\}$ . Finally, we extend  $\rho$  by continuity to  $\hat{\rho} : \hat{A}_P \rightarrow \hat{C}_I\{\{\tau_p^d\}\}$ . The new map  $\hat{\rho}$  is a formal Drinfeld module, called the completion of  $\rho$  at  $I$ .

**Example 1.1.3.** Suppose  $B = \mathcal{O}_L$  is the valuation ring of a finite extension  $L$  of  $K$ . Let  $\pi$  be a fixed prime of  $K$ , then we know that every  $a \in \mathcal{O}_K$  can be uniquely written as  $a = \sum_{i \geq 0} a_i \pi^i$ , where the  $a_i \in \mathbb{F}_q$ . Let  $f$  be a power series in  $B\{\{\tau_p^d\}\}$  such that  $D(f) = \pi$  and  $f \not\equiv 0 \pmod{\mathfrak{p}_K}$ . We set  $\rho_a = \sum_{i \geq 0} a_i f^i$ . Then  $a \mapsto \rho_a$  is a well defined formal Drinfeld module.

In all that follows, the formal Drinfeld modules considered are ring homomorphisms from  $\mathcal{O}_K$  to  $B\{\{\tau\}\}$ , where

$$\tau := \tau_p^d \tag{1.1.2}$$

is the  $q$ -Frobenius element satisfying  $\tau x = x^q \tau$  for all  $x \in B$ . For the rest of this chapter, we suppose  $B$  is an integral domain and we denote by  $\hat{D}_{\mathcal{O}_K}(B)$  the set of formal Drinfeld  $\mathcal{O}_K$ -modules defined over  $B$ .

**Lemma 1.1.4.** A formal Drinfeld  $\mathcal{O}_K$ -module  $\rho$  over  $B$  is injective.

*Proof.* This is Lemma 1.1 in [29]. Let  $a \in \mathcal{O}_K \setminus \{0\}$ , then we can write  $a = u\pi^k$ , where  $u$  is a unit of  $K$ ,  $\pi$  is a prime of  $K$  and  $k \in \mathbb{N}$  is the valuation of  $a$ . Hence, we have  $\rho_a = \rho_u \rho_\pi^k$ . Since  $B$  is supposed to be an integral domain, then  $B\{\{\tau\}\}$  has no zero divisors. Hence, it suffices to say that both  $\rho_u$  and  $\rho_\pi$  are non zero power series in  $B\{\{\tau\}\}$ . This is clearly true because  $\rho_u$  is invertible in  $B\{\{\tau\}\}$  and  $\rho_\pi$  is non zero by the very definition of a formal Drinfeld module.  $\square$

Let  $f = \sum_{i \geq 0} b_i \tau^i$  be a power series in  $B\{\{\tau\}\}$ . We set  $\text{ord}_\tau(f)$  to be the least integer  $i$  such that  $b_i \neq 0$ . Let  $\pi$  be a prime of  $K$ , then we define the height of  $\rho$  by  $\text{ht}(\rho) = \text{ord}_\tau(\rho_\pi)$ . Clearly, the height well defined and is independent of the choice of the prime  $\pi$ . Moreover, if the structure map  $\gamma : \mathcal{O}_K \rightarrow B$  is injective, the height of a formal Drinfeld  $\mathcal{O}_K$ -module defined over  $B$  is always zero.

**Lemma 1.1.5.** Let  $\rho \in \hat{D}_{\mathcal{O}_K}(B)$ . Then for all  $a \in \mathcal{O}_K$ , we have  $\text{ord}_\tau(\rho_a) = \text{ht}(\rho)\mu(a)$ .

*Proof.* See [29, Lemma 1.3]. The proof of this lemma is immediate if we write  $a$  as  $a = u\pi^k$  as we did in the proof of Lemma 1.1.4.  $\square$

**Definition 1.1.6.** Let  $\rho$  and  $\rho'$  be two formal Drinfeld  $\mathcal{O}_K$ -modules over  $B$ . A homomorphism from  $\rho$  to  $\rho'$  is a power series  $g$  in  $B\{\{\tau\}\}$  such that  $g\rho_a = \rho'_a g$  for all  $a \in \mathcal{O}_K$ . The power series  $g$  is an isomorphism if it is further an invertible power series in  $B\{\{\tau\}\}$ .

**Lemma 1.1.7.** Let  $\rho, \rho' \in \hat{D}_{\mathcal{O}_K}(B)$ . If there exists a non-trivial homomorphism from  $\rho$  to  $\rho'$ , then  $\text{ht}(\rho) = \text{ht}(\rho')$ .

*Proof.* This is Lemma 1.5 in [29]. It follows immediately from the definitions.  $\square$

Now, we will restrict ourselves to the case where  $B$  is the valuation ring of a finite extension of  $K$ . Thereupon, let  $L|K$  be a finite extension of local fields, let  $\mathcal{O}_L$  be its valuation ring and  $\mathfrak{p}_L$  be its maximal ideal. We note that the structure map in this case is the inclusion map from  $\mathcal{O}_K \subset \mathcal{O}_L$ .

**Definition 1.1.8.** A formal Drinfeld  $\mathcal{O}_K$ -module  $\rho$  defined over  $\mathcal{O}_L$  is said to have stable reduction if the ring homomorphism  $\bar{\rho} : \mathcal{O}_K \rightarrow \mathcal{O}_L/\mathfrak{p}_L\{\{\tau\}\}$ , obtained by reducing modulo  $\mathfrak{p}_L$  the coefficients of  $\rho_a$ , for  $a \in \mathcal{O}_K$ , is also a formal Drinfeld module.

## 1.2 The logarithm of a formal Drinfeld module

Let  $L$  be a finite extension of  $K$ , let  $\mathcal{O}_L$  be its valuation ring and  $\mathfrak{p}_L$  be its maximal ideal. We also denote by  $\mu$  the unique extension to  $L$  of the valuation of  $K$ . Let  $\rho \in \hat{D}_{\mathcal{O}_K}(\mathcal{O}_L)$ . We define the action

$$a \cdot_{\rho} x = \rho_a(x) \quad \text{for } a \in \mathcal{O}_K \quad \text{and } x \in \mathfrak{p}_L. \quad (1.2.1)$$

This is clearly a well defined action, since the power series  $\rho_a(x)$  converges in the complete ring  $\mathcal{O}_L$ , for all  $a \in \mathcal{O}_K$  and  $x \in \mathfrak{p}_L$ . This action of  $\rho$  makes of  $\mathfrak{p}_L$  an  $\mathcal{O}_K$ -module. We denote this  $\mathcal{O}_K$ -module by  $(\mathfrak{p}_L, \cdot_{\rho})$ .

A general construction in the case of formal  $\mathcal{O}_K$ -modules suggests that there exists a logarithm map from  $(\mathfrak{p}_L, \cdot_{\rho})$  to  $L$ , viewed as an  $\mathcal{O}_K$ -module for the multiplication in  $L$  (see [16, 21.5.7]). In §2 of [29], Rosen gave a detailed presentation of this logarithm map in the case of formal Drinfeld modules. We follow his steps.

**Proposition-Definition 1.2.1.** There exists a unique power series  $\lambda_{\rho}$  in  $L\{\{\tau\}\}$  such that  $D(\lambda_{\rho}) = 1$  and

$$\lambda_{\rho} \rho_a = a \lambda_{\rho} \quad \text{for all } a \in \mathcal{O}_K. \quad (1.2.2)$$

The power series  $\lambda_{\rho}$  is called the logarithm of  $\rho$ , it converges on  $\mathfrak{p}_L$ , and it gives a homomorphism of  $\mathcal{O}_K$ -modules from  $(\mathfrak{p}_L, \cdot_{\rho})$  to  $L$ .

*Proof.* This is [29, Proposition 2.1]. We will construct a power series  $\lambda_{\rho}$  satisfying  $D(\lambda_{\rho}) = 1$  and (1.2.2). In particular, we want  $\lambda_{\rho} \rho_{\pi} = \pi \lambda_{\rho}$ , where  $\pi$  is a prime of  $K$ . Let

$$\rho_{\pi} = \pi + \sum_{i \geq 0} b_i \tau^i \quad \text{and} \quad \lambda_{\rho} = 1 + \sum_{i \geq 0} c_i \tau^i.$$

Then, by  $\lambda_{\rho} \rho_{\pi} = \pi \lambda_{\rho}$ , we get

$$(\pi - \pi^{q^n}) c_n = \sum_{\substack{i+j=n \\ j \neq n}} c_j b_i^{q^j}. \quad (1.2.3)$$

These relations uniquely determine the coefficients  $c_n$  of  $\lambda_{\rho}$ . Let us now prove that the constructed power series  $\lambda_{\rho}$  satisfy (1.2.2). For that, let  $a \in \mathcal{O}_K$ . Then, we have  $\lambda_{\rho} \rho_a \rho_{\pi} = \pi \lambda_{\rho} \rho_a$ , which implies, together with  $\lambda_{\rho} \rho_{\pi} = \pi \lambda_{\rho}$ , that

$$\lambda_{\rho} \rho_a \lambda_{\rho}^{-1} \pi = \pi \lambda_{\rho} \rho_a \lambda_{\rho}^{-1}. \quad (1.2.4)$$

In other words, the power series  $\lambda_{\rho} \rho_a \lambda_{\rho}^{-1}$  commutes with  $\pi$ . Yet, all the power series in  $L\{\{\tau\}\}$  are elements of  $L$ . Therefore,  $\lambda_{\rho} \rho_a = c \lambda_{\rho}^{-1}$ , with  $c \in L$ . By comparison, we can clearly see that  $c = a$ , and hence, (1.2.2) is proven.

It remains to prove that  $\lambda_{\rho}(x)$  converges for all  $x \in \mathfrak{p}_L$ . Using (1.2.3), we can prove by induction that

$$\mu(c_i) \geq -i. \quad (1.2.5)$$

Hence, if we take  $x \in \mathfrak{p}_L$ , we get

$$\mu(c_i x^{q^i}) = \mu(c_i) + q^i \mu(x) \geq -i + q^i \mu(x). \quad (1.2.6)$$

Thus,  $\mu(c_i x^{q^i})$  tends to  $\infty$  as  $i$  tends to  $\infty$ . This concludes the proof.  $\square$

**Proposition-Definition 1.2.2.** There exists a unique power series  $\mathbf{e}_\rho$  in  $L\{\{\tau\}\}$  such that  $D(\lambda_\rho) = 1$  and

$$\mathbf{e}_\rho a = \rho_a \mathbf{e}_\rho \quad \text{for all } a \in \mathcal{O}_K. \quad (1.2.7)$$

The power series  $\mathbf{e}_\rho$  is called the exponential of  $\rho$ , it converges on the ideal

$$\mathfrak{p}_{L,1} := \{x \in \mathfrak{p}_L; \mu(x) > \frac{1}{q-1}\}, \quad (1.2.8)$$

and it gives a homomorphism of  $\mathcal{O}_K$ -modules from  $\mathfrak{p}_{L,1}$  to  $(\mathfrak{p}_L, \cdot_\rho)$ .

*Proof.* This is [29, Proposition 2.2]. If we set  $\mathbf{e}_\rho = \lambda_\rho^{-1}$ , we can easily see that  $D(\lambda_\rho) = 1$  and  $\mathbf{e}_\rho a = \rho_a \mathbf{e}_\rho$  for all  $a \in \mathcal{O}_K$ . The uniqueness can be deduced by reproducing the same arguments as in the proof of Proposition-Definition 1.2.1. We can further deduce from these arguments that, if we write  $\mathbf{e}_\rho = 1 + \sum_{i \geq 1} d_i \tau^i$ , we get

$$\mu(d_i) \geq -(1 + q + \cdots + q^{i-1}). \quad (1.2.9)$$

Now, let  $x \in \mathfrak{p}_{L,1}$ . Then,

$$\mu(d_i x^{q^i}) \geq -\frac{q^i - 1}{q - 1} + q^i \mu(x) = q^i \left( \mu(x) - \frac{1 - q^{-i}}{q - 1} \right) > 0. \quad (1.2.10)$$

Thus,  $\mu(d_i x^{q^i})$  tends to  $\infty$  as  $i$  tends to  $\infty$ . This shows that  $\mathbf{e}_\rho(x)$  converges and  $\mathbf{e}_\rho(x) \in \mathfrak{p}_L$ .  $\square$

**Proposition 1.2.3.** Let  $r$  be an integer such that  $r > \frac{e}{q-1}$ , where  $e$  is the index of ramification of  $L|K$ . Then  $\lambda_\rho$  and  $\mathbf{e}_\rho$  are inverse isomorphisms of  $\mathcal{O}_K$ -modules between  $(\mathfrak{p}_L^r, \cdot_\rho)$  and  $\mathfrak{p}_L^r$ . Moreover,  $(\mathfrak{p}_L^r, \cdot_\rho)$  is a torsion free  $\mathcal{O}_K$ -module.

*Proof.* This is [29, Proposition 2.3]. First, we note that  $\mathfrak{p}_L^r$  is indeed an  $\mathcal{O}_K$ -module for the action of  $\rho$ , since  $\rho_a(\mathfrak{p}_L^r) \subset \mathfrak{p}_L^r$  for all  $a \in \mathcal{O}_K$ . Since we already know that  $\lambda_\rho$  is the inverse of  $\mathbf{e}_\rho$  as power series in  $L\{\{\tau\}\}$ , it suffice to show that

$$\mu(\lambda_\rho(x)) = \mu(x) \quad \text{and} \quad \mu(\mathbf{e}_\rho(x)) = \mu(x) \quad (1.2.11)$$

for all  $x \in \mathfrak{p}_L^r$ . We will show this only for  $\lambda_\rho$ , for the proof for  $\mathbf{e}_\rho$  is very similar. Let  $x \in \mathfrak{p}_L^r$ , and write

$$\lambda_\rho(x) = x + \sum_{i \geq 1} c_i x^{q^i}. \quad (1.2.12)$$

By (1.2.5), for all  $i \geq 1$ , we have

$$\begin{aligned} \mu(c_i x^{q^i}) &= \mu(c_i) + q^i \mu(x) \\ &\geq -i + q^i \mu(x) \\ &= \mu(x) \left( q^i - \frac{i}{\mu(x)} \right) \\ &> \mu(x) (q^i + i(1 - q)) \end{aligned}$$

because  $\mu(x) > \frac{1}{q-1}$ . As  $q^i + i(1-q) \geq 1$ , this implies that  $\mu(c_i x^{q^i}) > \mu(x)$ , and therefore that  $\mu(\lambda_\rho(x)) = \mu(x)$ .  $\square$

**Remark 1.2.4.** A key-point to prove Proposition 1.2.3 is to observe that

$$\mu(\lambda_\rho(x)) = \mu(x) \quad \text{and} \quad \mu(\mathfrak{e}_\rho(x)) = \mu(x) \quad (1.2.13)$$

whenever  $\mu(x) > \frac{1}{q-1}$ . The equalities in (1.2.13) imply that  $\lambda_\rho$  defines an isomorphism from  $(\mathfrak{p}_{L,1}, \cdot_\rho)$  to  $\mathfrak{p}_{L,1}$ . This fact will be useful for us in the sequel.

**Proposition 1.2.5.** The  $\mathcal{O}_K$ -module  $\lambda_\rho(\mathfrak{p}_L)$  is free of rank  $[L : K]$  and we have  $L = K\lambda_\rho(\mathfrak{p}_L)$ , where  $K\lambda_\rho(\mathfrak{p}_L)$  the set of all elements of the form  $ab$ , for an element  $a \in K$  and an element  $b \in \lambda_\rho(\mathfrak{p}_L)$ .

*Proof.* We note first that  $\lambda_\rho(\mathfrak{p}_L)$  is an  $\mathcal{O}_K$ -module for the multiplication in  $L$ . Let  $x \in \mathfrak{p}_L$  and  $e(L|K)$  be the ramification index of  $L|K$ , then  $\mu(x) \geq \frac{1}{e(L|K)}$ . By (1.2.5), we have

$$\begin{aligned} \mu(\lambda_\rho(x)) &\geq \min(\mu(x), -i + q^i \mu(x)) \\ &\geq \min\left(\frac{1}{e(L|K)}, -i + \frac{q^i}{e(L|K)}\right). \end{aligned}$$

Thus, for a sufficiently large integer  $l$ , we have  $\lambda_\rho(\mathfrak{p}_L) \subset \frac{1}{\pi^l} \mathcal{O}_L$ , where  $\pi$  is a prime of  $K$ .

Therefore  $\lambda_\rho(\mathfrak{p}_L)$  is free for it is a  $\mathcal{O}_K$ -submodule of the free  $\mathcal{O}_K$ -module  $\frac{1}{\pi^l} \mathcal{O}_L$ . Now let us prove that  $L = K\lambda_\rho(\mathfrak{p}_L)$ . Clearly, we have  $K\lambda_\rho(\mathfrak{p}_L) \subset L$ . Let  $x \in L$ , then we can write  $x = u\pi_L^j$ , where  $u$  is a unit of  $L$  and  $\pi_L$  is a prime of  $L$ . Then, for a sufficiently large integer  $i$ , we have  $u\pi_L^j \pi^i \in \mathfrak{p}_{L,1} = \lambda_\rho(\mathfrak{p}_{L,1}) \subset \lambda_\rho(\mathfrak{p}_L)$ . Therefore  $x = \frac{1}{\pi^i} u\pi_L^j \pi^i \in K\lambda_\rho(\mathfrak{p}_L)$ .  $\square$

Although  $\lambda_\rho$  has an inverse as a formal power series, and as an  $\mathcal{O}_K$ -module homomorphism over  $(\mathfrak{p}_L^r, \cdot_\rho)$  for  $r > \frac{e}{q-1}$ , the map  $\lambda_\rho$  is not necessarily injective. In fact, it is easily seen that a torsion element of  $(\mathfrak{p}_L, \cdot_\rho)$  (i.e. an element  $w \in \mathfrak{p}_L$  such that  $\rho_{\pi^n}(w) = 0$  for some integer  $n \geq 0$ , where  $\pi$  is a prime of  $K$ ), annihilates  $\lambda_\rho$ , because  $\pi^n \lambda_\rho(w) = \lambda_\rho(\rho_{\pi^n}(w)) = 0$ . The kernel of  $\lambda_\rho$  is the subject of the following proposition.

**Proposition 1.2.6.** The kernel of the homomorphism  $\lambda_\rho : \mathfrak{p}_L \rightarrow L$  is the submodule of torsion elements

$$W_{\rho,L} := \{w \in \mathfrak{p}_L; \exists n > 0 \text{ such that } \rho_{\pi^n}(w) = 0\}, \quad (1.2.14)$$

where  $\pi \in K$  is a uniformizer. Clearly,  $W_{\rho,L}$  does not depend on the choice of  $\pi$ .

*Proof.* See [29, Proposition 2.4]. Let  $w \in \text{Ker}(\lambda_\rho)$  and let  $r > \frac{e}{q-1}$ . We know by the basic properties of  $\lambda_\rho$  that  $\lambda_\rho(\rho_{\pi^r}(w)) = \pi^r \lambda_\rho(w) = 0$ . However,  $\rho_{\pi^r}(w) = \rho_\pi^r(w) \in \mathfrak{p}_L^{r+1} \subset \mathfrak{p}_L^r$ , and  $\lambda_\rho$  is an isomorphism on  $\mathfrak{p}_L^r$  by Proposition 1.2.3. Hence,  $\rho_{\pi^r}(w) = 0$ , which means that  $w \in W_{\rho,L}$ .  $\square$

**Remark 1.2.7.** (i) The torsion module  $W_{\rho,L}$  is finite. Indeed, we have seen in Proposition 1.2.3 that for  $r$  large enough,  $(\mathfrak{p}_L^r, \cdot_\rho)$  is torsion free. Hence, there exists an injection

$$W_{\rho,L} \hookrightarrow \mathfrak{p}_L/\mathfrak{p}_L^r. \quad (1.2.15)$$

This implies that the cardinal of  $W_{\rho,L}$  is less or equal then the cardinal of  $\mathfrak{p}_L/\mathfrak{p}_L^r$ .

(ii) If  $e < q - 1$ , we can apply Proposition 1.2.3 for  $r = 1$ . We get that  $(\mathfrak{p}_L, \cdot_\rho)$  is isomorphic to  $\mathfrak{p}_L$ , under the multiplication in  $L$ . In particular,  $(\mathfrak{p}_L, \cdot_\rho)$  is torsion free.

As we have seen above, the convergence domains of  $\lambda_\rho$  and  $\mathfrak{e}_\rho$  depend on the extension  $L$ , and not on  $\rho$ . This yields the following local uniform boundness theorem.

**Theorem 1.2.8.** Let  $L|K$  be a finite extension of local fields of degree  $N$ . Let  $\mathcal{O}_L$  be its valuation ring and  $\mathfrak{p}_L$  be its maximal ideal. Let  $\rho$  be a formal Drinfeld  $\mathcal{O}_K$ -module over  $\mathcal{O}_L$ , then the cardinal of  $W_{\rho,L}$  is bounded by  $q^{\frac{N}{q-1}}$ .

*Proof.* This is [29, Theorem 2.5]. Let  $r = \lfloor \frac{e(L|K)}{q-1} \rfloor + 1$ , where  $e(L|K)$  is the ramification index of  $L|K$  and  $\lfloor a \rfloor$  is the integral part of  $a \in \mathbb{R}$ . By Remark 1.2.7, the cardinal of  $W_{\rho,L}$  is less or equal then the cardinal of  $\mathfrak{p}_L/\mathfrak{p}_L^r$ , which is equal to  $q^{(r-1)f(L|K)}$ , where  $f(L|K)$  is the inertia degree of  $L|K$ . But

$$(r-1)f(L|K) = \lfloor \frac{e(L|K)}{q-1} \rfloor f(L|K) \leq \frac{e(L|K)}{q-1} f(L|K) = \frac{N}{q-1}. \quad (1.2.16)$$

This concludes the proof.  $\square$

### 1.3 Torsion points and abelian towers

At the end of the last section, we saw results on the size of the torsion modules in a finite extension of  $K$ . In this section, we will study the structure of the torsion modules in an algebraic closure of  $K$ , as well as the field extensions obtained by adding torsion elements to  $K$ . As might be expected, these field extensions form abelian towers, and the compositum of their union with the maximal unramified extension of  $K$  is equal to the maximal abelian extension of  $K$ .

Let  $\Omega$  be an algebraic closure of  $K$ , and still denote  $\mu$  the unique extension of  $\mu$  to  $\Omega$ . Let  $(\bar{\Omega}, \bar{\mu})$  be the completion of  $(\Omega, \mu)$ . If  $F \subset \Omega$  is an extension of  $K$ , we denote  $\mathcal{O}_F$  the valuation ring of  $F$  and  $\mathfrak{p}_F$  its maximal ideal. Let  $H \subset \Omega$  be a finite unramified extension of  $K$ , and let  $\rho \in \hat{D}_{\mathcal{O}_K}(\mathcal{O}_H)$ . As in (1.2.1),  $\rho$  induces an action on  $\mathfrak{p}_{\bar{\Omega}}$ , making of it an  $\mathcal{O}_K$ -module:

$$a \cdot_\rho x = \rho_a(x) \quad \text{for } a \in \mathcal{O}_K \quad \text{and } x \in \mathfrak{p}_{\bar{\Omega}}. \quad (1.3.1)$$

Let  $n \geq 0$  be an integer and let  $\pi$  be a fixed prime of  $K$ . We define the  $\mathfrak{p}_K^n$ -torsion submodule of  $(\mathfrak{p}_{\bar{\Omega}}, \cdot_\rho)$  as

$$\begin{aligned} V_\rho^n &= \{x \in \mathfrak{p}_{\bar{\Omega}}; \rho_a(x) = 0 \quad \forall a \in \mathfrak{p}_K^n\} \\ &= \{x \in \mathfrak{p}_{\bar{\Omega}}; \rho_{\pi^n}(x) = 0\}. \end{aligned}$$

The submodule of all torsion elements in  $(\mathfrak{p}_{\bar{\Omega}}, \cdot_{\rho})$  is then the union of the  $V_{\rho}^n$  for all integers  $n \geq 0$ . We denote this union by  $V_{\rho}$ . We obtain an increasing sequence of  $\mathcal{O}_K$ -modules

$$0 = V_{\rho}^0 \subset V_{\rho}^1 \subset \cdots \subset V_{\rho}^n \subset \cdots \subset V_{\rho}. \quad (1.3.2)$$

In order to study the submodules  $V_{\rho}^n$ , we need to know more about the power series  $\rho_{\pi^n}$ . For that, we state and prove the Weierstrass preparation Theorem 1.3.3 below. We follow Rosen's method [29, §3] for the proof.

**Theorem 1.3.1.** Let  $F$  be a local field and let  $\mathcal{O}_F$  be its maximal ideal. Fix a positive integer  $N$  and let  $f = \sum_{i \geq 0} b_i \tau^i$  be a power series in  $\mathcal{O}_F\{\{\tau\}\}$  such that  $b_i \in \mathfrak{p}_F$ , the maximal ideal of  $\mathcal{O}_F$ , for  $i < N$ , and  $b_N$  is a unit in  $F$ . Let  $g \in \mathcal{O}_F\{\{\tau\}\}$ , then there exists unique elements  $Q$  and  $R$  in  $\mathcal{O}_F\{\{\tau\}\}$  such that  $g = Qf + R$ , where  $R = 0$  or  $\deg_{\tau}(R) < N$ .

*Proof.* We give a sketch of the proof. A full proof can be found in [29, Theorem 3.1]. Let  $\pi_F$  be a prime of  $\mathfrak{p}_F$ , then we can write  $f = \pi F + U\tau^N$ , where  $P$  is a polynomial in  $\mathcal{O}_F\{\tau\}$  of degree less than  $N$ , and  $U$  is an invertible power series in  $\mathcal{O}_F\{\{\tau\}\}$ . Assuming that  $Q$  and  $R$  exist, we will determine the shape of  $Q$ . Define

$$\begin{aligned} t : \mathcal{O}_F\{\{\tau\}\} &\rightarrow \mathcal{O}_F\{\{\tau\}\} \\ \sum_{i \geq 0} a_i \tau^i &\mapsto \sum_{i \geq N} a_i \tau^{i-N}. \end{aligned}$$

Then, we have  $t(g) = t(Qf) = t(q\pi_f P) + qU$ . Let  $E$  be the operator on  $\mathcal{O}_F\{\{\tau\}\}$  defined by  $E(h) = t(h\pi_f P)U^{-1}$ , then  $t(g)U^{-1} = (\text{Id} + E)(q)$ . Hence, we get  $q = \sum_{i \geq 0} (-1)^i E^i(t(g)U^{-1})$ . We must then verify that this series is convergent and that it fulfills the needed properties. Finally, the uniqueness of  $Q$  and  $R$  can be deduced from the properties of  $f$ .  $\square$

**Definition 1.3.2.** Let  $F$  be a local field. A polynomial in  $\mathcal{O}_F\{\tau\}$  is said to be distinguished if it is monic and all its coefficients, except for the leading one, are in  $\mathfrak{p}_F$ .

**Theorem 1.3.3.** (Weierstrass Preparation Theorem) Let  $F$  be a local field and  $N$  be a positive integer. Let  $f = \sum_{i \geq 0} b_i \tau^i \in \mathcal{O}_F\{\{\tau\}\}$  be such that  $b_i \in \mathfrak{p}_F$  for  $i < N$ , and  $b_N$  is a unit in  $F$ . Then there exist uniquely determined elements  $U$  and  $Q$  in  $\mathcal{O}_F\{\{\tau\}\}$ , such that  $U$  is a unit in  $\mathcal{O}_F\{\{\tau\}\}$  and  $Q$  is a distinguished polynomial in  $\mathcal{O}_F\{\tau\}$  of degree  $N$ , satisfying  $f = UQ$ .

*Proof.* Let  $g = \tau^N$ . Applying Theorem 1.3.1, we can write  $\tau^N = Q_1 f + R$ . Reducing modulo  $\mathfrak{p}_F$  and comparing the coefficients, we see that  $R \in \mathfrak{p}_F\{\tau\}$ . We also see that  $Q_1$  is a unit in  $\mathcal{O}_F\{\{\tau\}\}$ . Hence, we can write  $f = UQ$ , where  $Q = \tau^N - R$  and  $U = Q_1^{-1}$ .  $\square$

The Weierstrass preparation Theorem 1.3.3, applied to  $\rho_{\pi^n}$ , allows us to determine the structure of the torsion submodules  $V_{\rho}^n$ . This is the subject of [29, Theorem 3.3], where Rosen proved that, if  $\rho$  has stable reduction of height  $h > 0$ , then  $V_{\rho}^n$  is isomorphic to  $h$  copies of  $\mathcal{O}_K/\mathfrak{p}_K^n$ . We will restrict ourselves to the case where  $h = 1$ , for this will be enough for our purposes. As from now, we suppose that  $\rho$  has stable reduction of height 1.

**Lemma 1.3.4.** There exists two uniquely determined sequences  $(U_n)_{n \geq 1}$  and  $(Q_n)_{n \geq 1}$  in  $\mathcal{O}_H\{\{\tau\}\}$ , satisfying

- (i)  $Q_n$  is a distinguished polynomial of degree 1 and  $U_n$  is invertible.
- (ii)  $\rho_\pi = U_1 Q_1$  and  $Q_1 U_1 = Q_2 U_2$ .
- (iii)  $Q_1 U_1 U_{n-1} = U_n Q_n$  for all  $n > 2$ .

Furthermore, we have

$$\rho_{\pi^n} = U_1 U_n Q_n Q_{n-1} \cdots Q_1. \quad (1.3.3)$$

*Proof.* This is Lemma 2.1 of [28]. To prove this, we apply Theorem 1.3.3 consecutively to  $\rho_\pi$ , then to  $Q_1 U_1$ , and finally to  $Q_1 U_1 U_{n-1}$  for  $n > 2$ . The hypothesis on the height of  $\bar{\rho}$  imply that the degree of  $Q_1$ , and hence of all the  $Q_n$ , is equal to 1. Finally, the decomposition (1.3.3) of  $\rho_{\pi^n}$  can be proved by induction.  $\square$

The properties (ii) and (iii) of Lemma 1.3.4 imply that the polynomials  $Q_n$  are of the form  $Q_n = \tau + \pi_n$ , where  $\pi_n$  is a prime element of  $H$ . Hence, the polynomial

$$P_n = Q_n \cdots Q_1 \quad (1.3.4)$$

is a distinguished polynomial of degree  $n$ , and  $P_n(X)$  is a separable polynomial of degree  $q^n$ . Therefore, since  $U_1$  and  $U_n$  are units in  $\mathcal{O}_H\{\{\tau\}\}$ , the torsion module  $V_\rho^n$  is exactly the set of roots of  $P_n(X)$  in  $\Omega$ .

**Proposition 1.3.5.** The cardinal of  $V_\rho^n$  is  $q^n$  and  $H(V_\rho^n)|H$  is a finite Galois extension.

*Proof.* This follows immediately from the above discussion.  $\square$

Denote  $P_0(\tau) = \tau^0$ , and for  $n \geq 1$ , let

$$h_n(X) = \frac{P_n(X)}{P_{n-1}(X)} = \frac{Q_n(P_{n-1}(X))}{P_{n-1}(X)} = (P_{n-1}(X))^{q-1} + \pi_n, \quad (1.3.5)$$

because  $Q_n(X) = X^q + \pi_n X$ . It is clear that the polynomials  $h_n(X)$  are Eisenstein polynomials, of degree  $q^{n-1}(q-1)$ , satisfying

$$P_n(X) = h_n(X) h_{n-1}(X) \cdots h_1(X) X. \quad (1.3.6)$$

Since we can write  $P_n(X) = \prod_{v \in V_\rho^n} (X - v)$ , the polynomials  $P_n$  and  $h_n$  are independent of the choice of  $\pi$ , and only dependent of  $\rho$ , for the set  $V_\rho^n$  itself depends only on  $\rho$ . Hence, from (1.3.5), we deduce that the primes  $\pi_n$ , and thus the polynomials  $Q_n$ , are independent of the choice of  $\pi$  and only dependent of  $\rho$  (see Corollary 2.2 and Remark 2.3 in [28]).

**Proposition 1.3.6.** The set  $V_\rho^n \setminus V_\rho^{n-1}$  is the set of roots of  $h_n(X)$ . Moreover, if  $v_0$  is an element in this set, the extension  $H(v_0)|H$  is totally ramified of degree  $q^{n-1}(q-1)$ , and  $N_{H(v_0)|H}(v_0) = \pi_n$ .



*Proof.* This is [28, Proposition 2.4]. Let  $v_0$  be a root of  $h_n(X)$ . By (1.3.6),  $v_0$  is a root of  $P_n(X)$  and hence an element of  $V_\rho^n$ . Suppose  $v_0$  is in  $V_\rho^{n-1}$ , hence it is a root of  $P_{n-1}(X)$  and hence a double root of  $P_n(X)$ . This contradicts the separability of  $P_n(X)$ . Since the cardinal of  $V_\rho^n \setminus V_\rho^{n-1}$  is  $q^{n-1}(q-1)$ , which is equal to the degree of  $h_n(X)$ , it is then exactly the set of all roots of  $h_n(X)$ . Furthermore, the degree of the extension  $H(v_0)|H$  is  $q^{n-1}(q-1)$ , the polynomial  $h_n(X)$  being Eisenstein. Finally, we have  $N_{H(v_0)|H}(v_0) = (-1)^{\deg(h_n)} h_n(0) = \pi_n$ . Thus,  $H(v_0)$  is totally ramified over  $H$ , for the norm group  $N(H(v_0)|H)$  contains the prime  $\pi_n$  of  $H$ .  $\square$

**Proposition 1.3.7.** (i) Let  $v_0$  be an element of  $V_\rho^n \setminus V_\rho^{n-1}$ . Then  $V_\rho^n = \mathcal{O}_K \cdot_\rho v_0$  and the map defined by  $a \mapsto \rho_a(v_0)$ , for  $a \in \mathcal{O}_K$ , induces an isomorphism of  $\mathcal{O}_K$ -modules

$$\mathcal{O}_K/\mathfrak{p}_K^n \simeq V_\rho^n. \quad (1.3.7)$$

(ii) We have

$$\mathfrak{p}_K^i \cdot V_\rho^n = V_\rho^{n-i} \quad \text{for } n \geq i \geq 0. \quad (1.3.8)$$

*Proof.* This is inspired by [18, Lemma 4.8]. If  $v_0 \in V_\rho^n \setminus V_\rho^{n-1}$ , it is clear that the map  $a \mapsto \rho_a(v_0)$  is a homomorphism of  $\mathcal{O}_K$ -modules from  $\mathcal{O}_K$  to  $V_\rho^n$ . By the definition of  $V_\rho^n$ , we know that  $\mathfrak{p}_K^n \cdot_\rho v_0 = 0$  and  $\mathfrak{p}_K^{n-1} \cdot_\rho v_0 \neq 0$ . Hence, the kernel of this homomorphism is an ideal of  $\mathcal{O}_K$ , containing  $\mathfrak{p}_K^n$ , but not  $\mathfrak{p}_K^{n-1}$ , hence the kernel is equal to  $\mathfrak{p}_K^n$ . This implies the isomorphism (1.3.7), because  $\mathcal{O}_K/\mathfrak{p}_K^n$  and  $V_\rho^n$  are both of the same cardinal  $q^n$ . Moreover, we deduce that  $v_0$  is a generator of  $V_\rho^n$  as  $\mathcal{O}_K$ -module. This proves (i). Let us now prove (ii). Using part (i), we see that  $\rho_{\pi^i}(v_0)$  is a generator of  $V_\rho^{n-i}$  as  $\mathcal{O}_K$ -module because  $\rho_{\pi^i}(v_0) \in V_\rho^{n-i}$  and  $\rho_{\pi^i}(v_0) \notin V_\rho^{n-i-1}$ . Thus we can write

$$\mathfrak{p}_K^i \cdot V_\rho^n = \pi^i \cdot_\rho \mathcal{O}_K \cdot_\rho v_0 = \mathcal{O}_K \cdot_\rho \rho_{\pi^i}(v_0) = V_\rho^{n-i}. \quad (1.3.9)$$

$\square$

For each  $a \in \mathcal{O}_K$ , define the map

$$\begin{aligned} \varepsilon_a : V_\rho^n &\rightarrow V_\rho^n \\ \beta &\mapsto a \cdot_\rho \beta = \rho_a(\beta). \end{aligned} \quad (1.3.10)$$

Then  $\varepsilon_a$  is an element of  $\text{End}(V_\rho^n)$ , the ring of all endomorphisms of the  $\mathcal{O}_K$ -module  $V_\rho^n$ . Denote by  $\text{Aut}(V_\rho^n)$  the group of all automorphisms of  $V_\rho^n$ , then  $\text{Aut}(V_\rho^n)$  is the multiplicative group of all invertible elements in the ring  $\text{End}(V_\rho^n)$ . Hence, if  $u$  is a unit of  $K$ ,  $\varepsilon_u$  belongs to  $\text{Aut}(V_\rho^n)$  because  $\varepsilon_{u^{-1}} = \varepsilon_u^{-1}$ .

**Proposition 1.3.8.** Let  $\mathcal{U}_K$  denote the group of units of  $K$ . For all  $n \geq 0$ , we have a ring isomorphism

$$\mathcal{O}_K/\mathfrak{p}_K^n \rightarrow \text{End}(V_\rho^n) \quad (1.3.11)$$

and a group isomorphism

$$\mathcal{U}_K/\mathcal{U}_{K,n} \rightarrow \text{Aut}(V_\rho^n), \quad (1.3.12)$$

where  $\mathcal{U}_{K,n} = 1 + \mathfrak{p}_K^n$ .

*Proof.* This is inspired by Proposition 4.9 in [18]. The map  $a \mapsto \varepsilon_a$  defines a surjective ring homomorphism  $\mathcal{O}_K \rightarrow \text{End}(V_\rho^n)$ . Indeed, let  $\varepsilon \in \text{End}(V_\rho^n)$  and let  $v_0 \in V_\rho^n \setminus V_\rho^{n-1}$ . Then by Proposition 1.3.7(i), the  $\mathcal{O}_K$ -module  $V_\rho^n$  is generated by  $v_0$  and  $\varepsilon(v_0) = a \cdot_\rho v_0$  for an  $a \in \mathcal{O}_K$ . Therefore, for all  $\alpha \in V_\rho^n = \mathcal{O}_K \cdot_\rho v_0$ , we have  $\varepsilon(\alpha) = a \cdot_\rho \alpha$  so that  $\varepsilon = \varepsilon_a$ . This proves that the map  $a \mapsto \varepsilon_a$  is surjective. Since  $\mathfrak{p}_K^n \cdot_\rho V_\rho^n = 0$  and  $\mathfrak{p}_K^{n-1} \cdot_\rho V_\rho^n = V_\rho^1 \neq 0$  by Proposition 1.3.7(ii), the kernel of this map is exactly  $\mathfrak{p}_K^n$ . This yields the isomorphism (1.3.11). The isomorphism (1.3.12) follows immediately because  $\mathcal{U}_K/\mathcal{U}_{K,n}$  is the multiplicative group of the ring  $\mathcal{O}_K/\mathfrak{p}_K^n$  and  $\text{Aut}(V_\rho^n)$  is the multiplicative group of the ring  $\text{End}(V_\rho^n)$ .  $\square$

Let  $H_\rho^n = H(V_\rho^n)$  and  $H_\rho = H(V_\rho)$ . The two following propositions give a description of the Galois groups  $\text{Gal}(H_\rho^n|H)$  and  $\text{Gal}(H_\rho|H)$ .

**Proposition 1.3.9.** There exists a surjective group homomorphism

$$\Gamma_n : \mathcal{U}_K \rightarrow \text{Gal}(H_\rho^n|H) \quad (1.3.13)$$

defined by  $\Gamma_n(u)(\alpha) = \rho_u(\alpha)$  for all  $u \in \mathcal{U}_K$  and  $\alpha \in V_\rho^n$ . The kernel of this homomorphism is  $\mathcal{U}_{K,n}$ .

*Proof.* See discussion before Proposition 2.5 in [28]. Let  $\sigma \in \text{Gal}(H_\rho^n|H)$ . Since  $\sigma$  is continuous, we have  $\rho_a(\sigma(\alpha)) = \sigma(\rho_a(\alpha))$  for all  $a \in \mathcal{O}_K$  and  $\alpha \in V_\rho^n$ . In particular, for  $a = \pi^n$ , we get  $\rho_{\pi^n}(\sigma(\alpha)) = \sigma(\rho_{\pi^n}(\alpha)) = 0$  for all  $\alpha \in V_\rho^n$ . Thus,  $\sigma(\alpha) \in V_\rho^n$  and  $\sigma$  induces an automorphism of  $V_\rho^n$ . Therefore, we get an injective group homomorphism

$$\text{Gal}(H_\rho^n|H) \hookrightarrow \text{Aut}(V_\rho^n). \quad (1.3.14)$$

However, by the isomorphism (1.3.12) and Proposition 1.3.6, if  $v_0$  is an element of  $V_\rho^n \setminus V_\rho^{n-1}$ , we have

$$\#\text{Aut}(V_\rho^n) = \#\mathcal{U}_K/\mathcal{U}_{K,n} = q^{n-1}(q-1) = [H(v_0) : H] \leq [H_\rho^n : H] = \#\text{Gal}(H_\rho^n|H), \quad (1.3.15)$$

where  $\#S$  denotes the cardinal of a set  $S$ . Hence the map (1.3.14) is actually an isomorphism. Together with (1.3.12), this implies that  $\Gamma_n : \mathcal{U}_K \rightarrow \text{Gal}(H_\rho^n|H)$  is well defined and is of kernel  $\mathcal{U}_{K,n}$ .  $\square$

**Corollary 1.3.10.** Let  $0 \leq i \leq n$ . The isomorphism  $\mathcal{U}_K/\mathcal{U}_{K,n} \simeq \text{Gal}(H_\rho^n|H)$  induces the isomorphism

$$\mathcal{U}_{K,i}/\mathcal{U}_{K,n} \simeq \text{Gal}(H_\rho^n|H_\rho^i). \quad (1.3.16)$$

**Proposition 1.3.11.** There exists a topological isomorphism

$$\Gamma : \mathcal{U}_K \rightarrow \text{Gal}(H_\rho|H). \quad (1.3.17)$$

which induces the homomorphism  $\Gamma_n : \mathcal{U}_K \rightarrow \text{Gal}(H_\rho^n|H)$  of Proposition 1.3.9 for all non negative integers  $n$ .

*Proof.* See Proposition 2.5 in [28]. It is clear that the following diagram

$$\begin{array}{ccc} \mathcal{U}_K & \xrightarrow{\Gamma^{n+1}} & \text{Gal}(H_\rho^{n+1}|H) \\ & \searrow \Gamma_n & \downarrow \\ & & \text{Gal}(H_\rho^n|H) \end{array}$$

is commutative, the vertical map being the canonical map. Thus, we have

$$\Gamma : \mathcal{U}_K = \varprojlim \mathcal{U}_K / \mathcal{U}_{K,n} \xrightarrow{\sim} \varprojlim \text{Gal}(H_\rho^n|H) = \text{Gal}(H_\rho|H). \quad (1.3.18)$$

□

**Remark 1.3.12.** The isomorphism (1.3.14), together with (1.3.15), shows that the field  $H_\rho^n = H(V_\rho^n)$  is equal to  $H(v_0)$  for any element  $v_0$  in  $V_\rho^n \setminus V_\rho^{n-1}$ . Therefore by Proposition 1.3.6, the extension  $H_\rho^n|H$  is totally ramified of degree  $q^{n-1}(q-1)$ . Moreover, the fact that  $N_{H(v_0)|H}(v_0)$  is a prime of  $H$  implies that  $v_0$  is a prime of  $H_\rho^n$ .

**Lemma 1.3.13.** Let  $v_0 \in V_\rho^n \setminus V_\rho^{n-1}$  so that  $H_\rho^n = H(v_0)$ .

- (i) The complete set of conjugates of  $v_0$  over  $H$  is  $V_\rho^n \setminus V_\rho^{n-1}$ .
- (ii) Let  $0 \leq i \leq n$ . The complete set of conjugates of  $v_0$  over  $H_\rho^i$  is  $v_0 + V_\rho^{n-1}$ .

*Proof.* This Lemma is analogous to [18, Corollary 5.4]. By Proposition 1.3.9, the set of conjugates of  $v_0$  is the set  $\{u \cdot_\rho v_0; u \in \mathcal{U}_K\}$ , which is, by Proposition 1.3.7, is equal to  $V_\rho^n \setminus V_\rho^{n-1}$ . Similarly, by Corollary 1.3.10, the set of conjugates of  $v_0$  over  $H_\rho^i$  is the set  $\{u \cdot_\rho v_0; u \in \mathcal{U}_{K,i}\}$ , which is, by Proposition 1.3.7, is equal to  $v_0 + V_\rho^{n-1}$ . □

For a finite separable extension  $F'|F$  of local fields, let  $\mathfrak{m}_{F'|F}$  be the fractional ideal of  $\mathcal{O}_{F'}$  defined by

$$\mathfrak{m}_{F'|F} = \{x \in F'; \Gamma_{F'|F}(x\mathcal{O}_{F'}) \subset \mathcal{O}_F\} \supset \mathcal{O}_{F'}. \quad (1.3.19)$$

As usually defined, the different  $\mathcal{D}_{F'|F}$  of  $F'|F$  is the inverse ideal of  $\mathfrak{m}_{F'|F}$

$$\mathcal{D}_{F'|F} := \mathfrak{m}_{F'|F}^{-1}. \quad (1.3.20)$$

We recall that  $\mathcal{D}_{F'|F}$  is contained in  $\mathcal{O}_{F'}$ . Moreover, we know that if  $F'|F$  is unramified, then  $\mathcal{D}_{F'|F} = \mathcal{O}_{F'}$ , and if  $F'|F$  is totally ramified, then  $\mathcal{D}_{F'|F} = g'(w)\mathcal{O}_{F'}$ , where  $w$  is a prime element of  $F'$  and  $g(X)$  is the minimal polynomial of  $w$  over  $F$ . Furthermore, if  $F''|F$  is a finite extension of local fields such that  $F \subset F' \subset F''$ , we have

$$\mathcal{D}_{F''|F} = \mathcal{D}_{F''|F'}\mathcal{D}_{F'|F}. \quad (1.3.21)$$

For more details, the reader may check [18, §2.4].

**Lemma 1.3.14.** The different of the extension  $H_\rho^n|K$  is generated by an element of valuation  $n - \frac{1}{q-1}$ .

*Proof.* The proof of [6, Lemma 3] is suitable for our case. First, note that we have  $\mathcal{D}_{H_\rho^n|K} = \mathcal{D}_{H_\rho^n|H}\mathcal{D}_{H|K}$ . Since  $H|K$  is an unramified extension, then  $\mathcal{D}_{H|K} = \mathcal{O}_H$ . Now, let us compute  $\mathcal{D}_{H_\rho^n|H}$ . Let  $v_0 \in V_\rho^n \setminus V_\rho^{n-1}$ , then, by Remark 1.3.12,  $v_0$  is a prime of  $H_\rho^n = H(v_0)$ . By Proposition 1.3.6, the minimal polynomial of  $v_0$  is the polynomial  $h_n$  defined in (1.3.5). Hence,  $\mathcal{D}_{H_\rho^n|H}$  is generated by  $h'_n(v_0)$  because  $H_\rho^n|H$  is totally ramified. By the very definition of  $h_n$ , we know that

$$h_n(X)P_{n-1}(X) = P_n(X) \quad (1.3.22)$$

where  $P_l = \prod_{v \in V_\rho^l} (X - v)$ , for a positive integer  $l$ , is the polynomial defined in (1.3.4). Differentiating (1.3.22) and evaluating at  $v_0$  we get  $h'_n(v_0)P_{n-1}(v_0) = P'_n(v_0)$ . Since  $\mu(v_0) = \frac{1}{q^{n-1}(q-1)} < \mu(v)$  for all  $v \in V_\rho^{n-1}$ , then

$$\mu(P_{n-1}(v_0)) = \sum_{v \in V_\rho^{n-1}} (\mu(v_0)) = \frac{1}{q-1}. \quad (1.3.23)$$

Moreover, since  $P_n \in \mathcal{O}_H\{\tau\}$ , we have  $P'_n(v_0) = P'_n(0)$  and

$$\begin{aligned} \mu(P'_n(v_0)) &= \mu\left(\prod_{v \in V_\rho^n \setminus \{0\}} v\right) = \sum_{v \in V_\rho^n \setminus \{0\}} \mu(v) = \sum_{i=1}^n (\#V_\rho^i \setminus V_\rho^{i-1})\mu(v_i) \\ &= q^{i-1}(q-1) \times \frac{1}{q^{i-1}(q-1)} = n, \end{aligned}$$

where  $v_i$  is any element of  $V_\rho^i \setminus V_\rho^{i-1}$  for  $i \in \{1, \dots, n\}$ . This concludes the proof.  $\square$

**Lemma 1.3.15.** Let  $L$  be a finite separable extension of  $K$  and let  $x \in L$ , then

$$\mu(\mathbb{T}_{L|K}(x)) \geq \lfloor \mu(x) + \mu(\mathcal{D}_{L|K}) \rfloor, \quad (1.3.24)$$

where  $\lfloor a \rfloor$  is the integral part of  $a \in \mathbb{R}$ . In particular if  $L = H_\rho^n$ , we have

$$\mu(\mathbb{T}_{H_\rho^n|K}(x)) \geq \lfloor \mu(x) + n - \frac{1}{q-1} \rfloor. \quad (1.3.25)$$

Furthermore, for  $i \leq n$ , we have

$$\mu(\mathbb{T}_{H_\rho^n|H_\rho^i}(x)) > \mu(x) + n - i - \mu(v_i), \quad (1.3.26)$$

where  $v_i$  is an element of  $V_\rho^i \setminus V_\rho^{i-1}$ .

*Proof.* See [6, Lemma 4]. Let  $k = \lfloor \mu(x) + \mu(\mathcal{D}_{L|K}) \rfloor$  then  $x\mathcal{O}_L \subset \mathfrak{p}_K^k \mathcal{D}_{L|K}^{-1}$ . Thus  $\mathbb{T}_{L|K}(x\mathcal{O}_L) \subset \mathfrak{p}_K^k$ . This proves (1.3.24). By Lemma 1.3.14, we can see that the generator of  $\mathcal{D}_{H_\rho^n|H_\rho^i}$  is of valuation  $n-i$ . Hence, for  $k = \lfloor \frac{\mu(x) + n - i}{\mu(v_i)} \rfloor$ , we have  $\mathbb{T}_{H_\rho^n|H_\rho^i}(x\mathcal{O}_{H_\rho^n}) \subset \mathbb{T}_{H_\rho^n|H_\rho^i}(\mathfrak{p}_{H_\rho^i}^k \mathcal{D}_{H_\rho^n|H_\rho^i}^{-1}) \subset \mathfrak{p}_{H_\rho^i}^k$ . Thus, we have

$$\mu(\mathbb{T}_{H_\rho^n|H_\rho^i}(x)) \geq k\mu(v_i) > \mu(x) + n - i - \mu(v_i). \quad (1.3.27)$$

$\square$

Let  $K_{ur} \in \Omega$  be the maximal unramified extension of  $K$ . Denote  $\varphi$  the Frobenius automorphism, then  $\text{Gal}(K_{ur}|K)$  is generated by  $\varphi$ . We will dedicate the rest of this section to study the fields  $K_{ur}(V_\rho^n)$  and  $K_{ur}(V_\rho)$ . For that, we need the two following propositions, of which we omit the proofs. The interested reader may check Lemma 3.11 and Proposition 3.12 of [18].

**Proposition 1.3.16.** Let  $F \subset \Omega$  be an unramified extension of  $K$ , such that  $F|K$  is of finite degree or  $F = K_{ur}$ . Let  $a$  and  $b$  be elements in  $\mathcal{O}_{\bar{F}}$  such that  $\bar{\mu}(a) = \bar{\mu}(b) = t > 0$ , and let  $f_1$  and  $f - 2$  be elements of  $\mathcal{O}_{\bar{F}}[[X]]$  satisfying

$$f_1(X) \equiv aX \quad \text{and} \quad f_2(X) \equiv bX \quad \text{mod deg } 2, \quad (1.3.28)$$

and

$$f_1(X) \equiv f_2(X) \equiv X^{q^t} \quad \text{mod } \mathfrak{p}_{\bar{F}}. \quad (1.3.29)$$

If there exists  $m$  elements  $\alpha_1, \dots, \alpha_m$  in  $\mathcal{O}_{\bar{F}}$  such that  $\alpha_i^{\varphi^t-1} = a/b$ , then there exists a unique power series  $\theta \in \mathcal{O}_{\bar{F}}[[X_1, \dots, X_m]]$  such that

$$\theta(X_1, \dots, X_m) \equiv \alpha_1 X_1 + \dots + \alpha_m X_m \quad \text{mod deg } 2 \quad \text{and} \quad f_1 \circ \theta = \theta^{\varphi^t} \circ f_2. \quad (1.3.30)$$

Here,  $\theta^{\varphi^t}$  denotes the power series obtained from  $\theta$  by applying  $\varphi^t$  to the coefficients.

*Proof.* This is Proposition 3.1 of [28]. It is a generalization of [18, Proposition 3.12], where Iwasawa proved the assertion for  $t = 1$ . His prove is adaptable to the case  $t > 1$ .  $\square$

The following proposition tells us that the  $\alpha_i$  of Proposition 1.3.16 exist.

**Proposition 1.3.17.** Let  $t$  be a positive integer. Then, we have the exact sequences

$$0 \rightarrow \mathcal{O}_{K_{ur}^t} \rightarrow \mathcal{O}_{\bar{K}_{ur}} \xrightarrow{\varphi^t-1} \mathcal{O}_{\bar{K}_{ur}} \rightarrow 0, \quad (1.3.31)$$

and

$$1 \rightarrow \mathcal{U}_{K_{ur}^t} \rightarrow \mathcal{U}_{\bar{K}_{ur}} \xrightarrow{\varphi^t-1} \mathcal{U}_{\bar{K}_{ur}} \rightarrow 1, \quad (1.3.32)$$

where  $K_{ur}^t$  denotes the unique unramified extension of  $K$  of degree  $t$  in  $\Omega$  and  $\mathcal{U}_F$  denotes the group of units of a valued field  $F$ .

*Proof.* See [18, Lemma 3.11].  $\square$

Let  $\rho'$  be a formal Drinfeld  $\mathcal{O}_K$ -module over  $\mathcal{O}_H$ , having stable reduction of height 1. Let  $n$  be a non negative integer and let  $P_n^\rho$  and  $P_n^{\rho'}$  be the polynomials defined in (1.3.4) attached to  $\rho$  and  $\rho'$  respectively. Then, the constant coefficients  $a$  of  $P_n^\rho$  and  $b$  of  $P_n^{\rho'}$  are both of valuation  $n$ . Hence, by Proposition 1.3.17, there exists a unit  $u$  in  $\bar{K}_{ur}$  such that  $u^{\varphi^n-1} = a/b$ . Proposition 1.3.16, applied to  $f_1 = P_n^{\rho'}$ ,  $f_2 = P_n^\rho$  and  $m = 1$ , shows that there exists a unique power series  $\theta_n$  in  $\mathcal{O}_{\bar{K}_{ur}}[[X]]$  such that

$$\theta_n(X) \equiv uX \quad \text{mod deg } 2 \quad \text{and} \quad P_n^{\rho'} \circ \theta_n = \theta_n^{\varphi^n} \circ P_n^\rho. \quad (1.3.33)$$

We also have  $\theta_n(X + Y) = \theta_n(X) + \theta_n(Y)$ . Indeed, if we denote  $M(X, Y) = \theta_n(X + Y)$  and  $N(X, Y) = \theta_n(X) + \theta_n(Y)$ , we get

$$M(X, Y) \equiv N(X, Y) \equiv X + Y \pmod{\deg 2} \quad \text{and} \quad P_n^{\rho'} \circ M = M^{\varphi^n} \circ P_n^\rho. \quad (1.3.34)$$

However, since  $P_n^\rho$  and  $P_n^{\rho'}$  are both additive, we also have  $P_n^{\rho'} \circ N = N^{\varphi^n} \circ P_n^\rho$ . Therefore, by the uniqueness property in Proposition 1.3.16, we deduce that  $M(X, Y) = N(X, Y)$ . Hence,  $\theta_n$  induces an isomorphism of  $\mathbb{F}_q$ -vector spaces

$$\theta_n : V_\rho^n \longrightarrow V_{\rho'}^n. \quad (1.3.35)$$

**Corollary 1.3.18.** Let  $\mathbb{R}^\infty$  be the set of formal Drinfeld  $\mathcal{O}_K$ -modules having stable reduction of height 1 and defined over  $\mathcal{O}_{\bar{K}_{ur}}$ . Then  $\bar{K}_{ur}(V_\rho^n)$  does not depend on the choice of  $\rho$  in  $\mathbb{R}^\infty$ .

*Proof.* This is Corollary 3.3 in [28]. If  $\rho$  and  $\rho'$  are in  $\mathbb{R}^\infty$ , then the discussion above shows that there exists a power series  $\theta_n$  in  $\mathcal{O}_{\bar{K}_{ur}}[[X]]$  that defines an isomorphism  $V_\rho^n \longrightarrow V_{\rho'}^n$ . Hence, we get

$$V_{\rho'}^n = \theta_n(V_\rho^n) \subset \bar{K}_{ur}(V_\rho^n) \quad (1.3.36)$$

because  $\bar{K}_{ur}(V_\rho^n)$  is complete, being a finite extension of the complete field  $\bar{K}_{ur}$ . This implies that  $\bar{K}_{ur}(V_{\rho'}^n) \subset \bar{K}_{ur}(V_\rho^n)$ . The second inclusion can be obtained identically, so that  $\bar{K}_{ur}(V_{\rho'}^n) = \bar{K}_{ur}(V_\rho^n)$ .  $\square$

**Lemma 1.3.19.** Let  $E|F$  be a finite extension of local fields such that  $K \subset F \subset E \subset \Omega$ . Then  $E\bar{F} = \bar{E}$ . Moreover, we have  $E \cap \bar{F} = F$  if  $E|F$  is separable.

*Proof.* See [18, Lemma 3.1]. We have the obvious inclusion  $E\bar{F} \subset \bar{E}$ . Moreover, the extension  $E\bar{F}|\bar{F}$  is finite, then  $E\bar{F}$  is complete for the  $\bar{\mu}$ -topology. Hence,  $E\bar{F} \supset E$  is closed in  $\bar{\Omega}$ . Thus, we get  $E\bar{F} = \bar{E}$ . Suppose now that  $E|F$  is separable. We may further suppose that  $E|F$  is a Galois extension. Indeed, if  $E|F$  is separable, then there exists a field extension  $E'$  of  $E$  such that  $F \subset E \subset E'$  and  $E'|F$  is Galois. Yet, to prove  $E \cap \bar{F} = F$ , it is sufficient to prove  $E' \cap \bar{F} = F$ . Hence, we may replace  $E$  by  $E'$ . Under this assumption,  $\bar{E}|\bar{F}$  is a finite Galois extension and

$$[\bar{E} : \bar{F}] = [E\bar{F} : \bar{F}] = [E : E \cap \bar{F}]. \quad (1.3.37)$$

However, since every automorphism in  $\text{Gal}(E|F)$  can be uniquely extended to an automorphism in  $\text{Gal}(\bar{E}|\bar{F})$ , we get

$$[E : F] \leq [\bar{E} : \bar{F}]. \quad (1.3.38)$$

The result follows because  $F \subset E \cap \bar{F} \subset E$ .  $\square$

**Theorem 1.3.20.** For a positive integer  $m$ , let  $\mathbb{R}^m$  be the set of formal Drinfeld  $\mathcal{O}_K$ -modules over  $\mathcal{O}_{K_{ur}^m}$  having stable reduction of height 1, where  $K_{ur}^m$  is the unique unramified extension of  $K$  of degree  $m$  in  $\Omega$ . Let  $\mathbb{R}^\infty$  denote the union of all  $\mathbb{R}^m$ . Then  $K_{ur}(V_\rho^n)$  does not depend on the choice of  $\rho$  in  $\mathbb{R}^\infty$ . Moreover,  $K_{ur}(V_\rho^n)|K$  is an abelian extension of Galois group

$$\text{Gal}(K_{ur}(V_\rho^n)|K) \simeq \text{Gal}(K_{ur}|K) \times \mathcal{U}_K/\mathcal{U}_{K,n}. \quad (1.3.39)$$

*Proof.* This is [28, Theorem 3.4]. Let  $\rho, \rho' \in \mathbb{R}^\infty$  and let  $E = K_{ur}(V_\rho^n)$  and  $E' = K_{ur}(V_{\rho'}^n)$ . Then, by Lemma 1.3.19, we have  $\bar{E} = \bar{K}_{ur}K_{ur}(V_\rho^n) = \bar{K}_{ur}(V_\rho^n)$  and  $\bar{E}' = \bar{K}_{ur}K_{ur}(V_{\rho'}^n) = \bar{K}_{ur}(V_{\rho'}^n)$ . Hence, by Corollary 1.3.18, we get  $\bar{E} = \bar{E}'$ . Moreover, we know that  $EE'$  is a finite separable extension of  $E$  and of  $E'$ . Then, using the second assertion of Lemma 1.3.19, we can write

$$E = \bar{E} \cap EE' = \bar{E}' \cap EE' = E'. \quad (1.3.40)$$

That said, we can suppose that  $\rho \in \mathbb{R}^1$ . By Proposition 1.3.9, we conclude that  $K(V_\rho^n)|K$  is an abelian extension of Galois group isomorphic to  $\mathcal{U}_K/\mathcal{U}_{K,n}$ . Moreover, it is a totally ramified extension by Remark 1.3.12. This implies that  $K(V_\rho^n) \cap K_{ur} = K$ , so that

$$\begin{aligned} \text{Gal}(K_{ur}(V_\rho^n)|K) &\simeq \text{Gal}(K_{ur}|K) \times \text{Gal}(K(V_\rho^n)|K) \\ &\simeq \text{Gal}(K_{ur}|K) \times \mathcal{U}_K/\mathcal{U}_{K,n}. \end{aligned}$$

□

As mentioned in the beginning of this section, the compositum of the union of the abelian towers  $K(V_\rho^n)$  with  $K_{ur}$  is equal to  $K^{ab}$ , the maximal abelian extension of  $K$  in  $\Omega$ . The proof of this assertion employs the theory of Lubin-Tate formal groups. In fact, since we proved in Theorem 1.3.20 that  $K_{ur}(V_\rho) = K_{ur}K(V_\rho)$  is independent of the choice of  $\rho$  in  $\mathbb{R}^\infty$ , we will choose a particular formal Drinfeld module, constructed by the means of Lubin-Tate formal groups. Let  $f(X) = \pi X + X^q$  be the basic Lubin-Tate polynomial. Then by Lubin-Tate theory (see for instance [18, Chapter IV] or [23, Chapter 8]), for each  $a \in \mathcal{O}_K$ , there exists a unique power series  $[a]_f = \sum_{i \geq 0} a_i \tau^i \in \mathcal{O}_K\{\{\tau\}\}$  such that

$$[a]_f(X) \equiv aX \pmod{\deg 2} \quad \text{and} \quad f \circ [a]_f = [a]_f \circ f. \quad (1.3.41)$$

It is easy to see that  $[\pi]_f = f$ . Just as we defined the torsion modules for the action of a formal Drinfeld module in (1.3.1), we can define the torsion modules for the action of  $f$

$$a \cdot_f x = [a]_f(x) \quad \text{for } a \in \mathcal{O}_K \text{ and } x \in \mathfrak{p}_{\bar{\Omega}}. \quad (1.3.42)$$

If we denote  $T_f$  the torsion submodule of  $\mathfrak{p}_{\bar{\Omega}}$ , then we know that  $K_{ur}(T_f) = K^{ab}$  (see [18, §6.2]). Now we define the formal Drinfeld module  $\rho^f : \mathcal{O}_K \rightarrow \mathcal{O}_K\{\{\tau\}\}$  by  $\rho_a^f = \sum_{i \geq 0} a_i \tau^i$  for every  $a \in \mathcal{O}_K$ . Clearly,  $\rho^f$  is an element of  $\mathbb{R}^1$  and  $T_f = V_{\rho^f}$ . This concludes the proof. We sum up this discussion by the following theorem (see [28, Theorem 3.5]).

**Theorem 1.3.21.** For any  $\rho \in \mathbb{R}^\infty$ , we have

$$K_{ur}(V_\rho) = K^{ab}. \quad (1.3.43)$$

## 1.4 The norm operator of Coleman and the Coleman power series

In his paper [9], Coleman introduced his norm operator to prove a theorem on the interpolation of torsion points of Lubin-Tate formal groups. His work is a generalization of a

previous work of Coates and Wiles [8], who proved similar results for units in the case of formal groups of height one attached to elliptic curves with complex multiplication. Yet, Coleman's approach was different, as the approach of Coates and Wiles only applies to Lubin-Tate formal groups defined over  $\mathbb{Z}_p$ . In the case of formal Drinfeld modules, Hassan Oukhaba [28, §5] proved an analogue of Coleman's theorem [9, Theorem A]. Before we state these results, we need some preparations. Let  $H$ ,  $\Omega$ ,  $\pi$  and  $\rho$  be as above.

**Lemma 1.4.1.** Let  $f \in \mathcal{O}_H[[X]]$  such that  $f(X + w) = f(X)$  for all  $w \in V_\rho^1$ . Then there exists a unique power series  $g \in \mathcal{O}_H[[X]]$  such that  $g \circ \rho_\pi = f$ .

*Proof.* This is [9, Lemma 3]. We will prove by induction on  $n$  that there exists elements  $a_i \in \mathcal{O}_H$  for  $0 \leq i \leq n - 1$  such that

$$\rho_\pi^n f_n = f - \sum_{i=0}^{n-1} a_i \rho_\pi^i, \quad (1.4.1)$$

where  $\rho_\pi^i = \rho_\pi \times \cdots \times \rho_\pi$   $i$  times and  $f_n \in \mathcal{O}_H[[X]]$ . This is obvious for  $n = 0$ . Suppose that (1.4.1) is true for  $n$ . Then, since  $f(X + w) = f(X)$  and  $\rho_\pi^n(X + w) = \rho_\pi^n(X)$ , we have  $f_n(X + w) = f_n(X)$  for all  $w \in V_\rho^1$ . An adapted version of the Weierstrass preparation theorem to power series in  $\mathcal{O}_H[[X]]$  (see for instance [23, Theorem 2.1]) shows that  $f_n - f_n(0)$  is divisible by  $\rho_\pi$ . Hence, there exists  $f_{n+1} \in \mathcal{O}_H[[X]]$  such that  $f_n - f_n(0) = \rho_\pi f_{n+1}$ . Therefore, (1.4.1) is satisfied for  $n + 1$  if we put  $a_n = f_n(0)$ . Since  $\rho_\pi$  belongs to the ideal  $(X, \pi)$  of  $\mathcal{O}_H[[X]]$ , (1.4.1) implies that

$$f - \sum_{i \geq 0} a_i \rho_\pi^i \in \bigcap_{n \geq 0} (X, \pi)^n \mathcal{O}_H[[X]] = \{0\}. \quad (1.4.2)$$

If we put  $g(X) = \sum_{i \geq 0} a_i X^i$ , we get  $f = g \circ \rho_\pi$ . □

Let  $H((X))$  be the field of Laurent power series with coefficients in  $H$ , and let  $H((X))_1$  denote the subset of  $H((X))$  consisting of all power series in  $H((X))$  convergent on  $B' = \mathfrak{p}_{\bar{\Omega}} \setminus \{0\}$ . We endow  $H((X))_1$  with the compact-open topology with respect to  $B'$ . We recall that the sets

$$S_H(C, U) = \{f \in H((X)); f(C) \subset U\}, \quad (1.4.3)$$

where  $C \subset B'$  is a compact and  $U \subset \bar{\Omega}$  is open, form a sub-basis of this topology. One can observe that, on any compact of  $B'$ , the compact-open topology is the same as the topology of uniform convergence.

**Theorem 1.4.2.** There exists a unique continuous map  $\mathcal{N}_{\rho, \pi} : \mathcal{O}_H((X)) \rightarrow \mathcal{O}_H((X))$  satisfying

$$\mathcal{N}_{\rho, \pi}(f) \circ \rho_\pi(X) = \prod_{w \in V_\rho^1} f(X + w) \quad (1.4.4)$$

for all  $f \in \mathcal{O}_H((X))$ .



*Proof.* This is [9, Theorem 11]. First, we consider power series  $f \in \mathcal{O}_H[[X]]$ . We denote  $\mathcal{L}(f)(X) = \prod_{w \in V_\rho^1} f(X + w)$ . Obviously, we have  $\mathcal{L}(f)(X + w) = \mathcal{L}(f)(X)$  for all  $w \in V_\rho^1$ . Hence, by Lemma 1.4.1, there exists a unique  $g \in \mathcal{O}_H[[X]]$  such that  $g \circ \rho_\pi = \mathcal{L}(f)$ . We set  $\mathcal{N}_{\rho,\pi}(f) = g$ . Now take  $f \in \mathcal{O}_H((X))$ , then there exists a sufficiently large integer  $N$  such that  $\rho_\pi^N f \in \mathcal{O}_H[[X]]$ . We set  $\mathcal{N}_{\rho,\pi}(f)(X) = X^{-qN} \mathcal{N}_{\rho,\pi}(\rho_\pi^N f)$ . This satisfies the required conditions. Indeed, we have

$$\begin{aligned} X^{-qN} \mathcal{N}_{\rho,\pi}(\rho_\pi^N f) \circ \rho_\pi(X) &= \rho_\pi^{-qN}(X) \prod_{w \in V_\rho^1} \rho_\pi^N(X + w) f(X + w) \\ &= \rho_\pi^{-qN}(X) \prod_{w \in V_\rho^1} \rho_\pi^N(X) f(X + w) \\ &= \prod_{w \in V_\rho^1} f(X + w). \end{aligned}$$

Finally, the continuity of  $\mathcal{N}_{\rho,\pi}$  follows from that of  $\mathcal{L}$ .  $\square$

**Proposition 1.4.3.** The operator  $\mathcal{N}_{\rho,\pi}$  satisfies the following properties

- (i) The operator  $\mathcal{N}$  is multiplicative.
- (ii) For  $n \geq 0$  and  $f \in \mathcal{O}_H((X))$ , we have

$$\mathcal{N}_{\rho,\pi}^n(f) \circ \rho_\pi^n(X) = \prod_{w \in V_\rho^n} f(X + w). \quad (1.4.5)$$

- (iii) Let  $n \geq 0$  and let  $v_n$  be a generator of  $V_\rho^n$  as  $\mathcal{O}_K$ -module. Let  $v_{n+1}$  be such that  $\rho_\pi(v_{n+1}) = v_n$ . Then,

$$\mathcal{N}_{\rho,\pi}(f)(v_n) = N_{H_\rho^{n+1}|H_\rho^n}(f(v_{n+1})). \quad (1.4.6)$$

*Proof.* Part (i) follows immediately from the uniqueness. For (ii), we follow [18, Lemma 5.9(i)], proving it by induction. The equality is obviously true for  $n = 1$  by Theorem 1.4.2. Suppose it holds for  $n - 1$ , this means that

$$\mathcal{N}_{\rho,\pi}^{n-1}(f) \circ \rho_\pi^{n-1}(X) = \prod_{w \in V_\rho^{n-1}} f(X + w) \quad (1.4.7)$$

Let  $A$  be the set of representatives of the quotient group  $V_\rho^n/V_\rho^1$ . Then,

$$\prod_{w \in V_\rho^n} f(X + w) = \prod_{a \in A} \prod_{w \in V_\rho^1} f(X + a + w) = \prod_{a \in A} \mathcal{N}_{\rho,\pi}(f) \circ \rho_\pi(X + a). \quad (1.4.8)$$

However, we know by Proposition 1.3.7 that  $V_\rho^{n-1} = \pi \cdot_\rho V_\rho^n = \pi \cdot_\rho A$ . Hence, (1.4.8) becomes

$$\prod_{w \in V_\rho^n} f(X + w) = \prod_{v \in V_\rho^{n-1}} \mathcal{N}_{\rho,\pi}(f)(\rho_\pi(X) + v). \quad (1.4.9)$$

By the hypothesis (1.4.7), applied to  $\mathcal{N}_{\rho,\pi}(f)$ , we get

$$\begin{aligned} \prod_{w \in V_\rho^n} f(X + w) &= \mathcal{N}_{\rho,\pi}^{n-1}(\mathcal{N}_{\rho,\pi}(f)) \circ \rho_{\pi^{n-1}}(\rho_\pi(X)) \\ &= \mathcal{N}_{\rho,\pi}^n(f) \circ \rho_{\pi^n}(X). \end{aligned}$$

This completes the proof of (ii). As for (iii), which is [9, Corollary 12 (ii)], we note first that such a  $v_{n+1}$  exists and is a generator of the  $\mathcal{O}_K$ -module  $V_\rho^{n+1}$ . We have

$$\mathcal{N}_{\rho,\pi}(f)(v_n) = \mathcal{N}_{\rho,\pi}(f) \circ \rho_\pi(v_{n+1}) = \prod_{w \in V_\rho^1} f(v_{n+1} + w). \quad (1.4.10)$$

Yet, by Lemma 1.3.13, the conjugates of  $v_{n+1}$  over  $H_\rho^n$  are the elements of the form  $v_{n+1} + w$  with  $w \in V_\rho^1$ . The property follows.  $\square$

Let  $m_0$  be a positive integer dividing  $[H : K]$  and let  $\eta \in \mathcal{O}_K$  be an element of valuation  $\mu(\eta) = m_0$ . Let  $u$  be the unit of  $K$  such that  $u\eta = \pi^{m_0}$ . We define the twisted operator  $\tilde{\mathcal{N}}$  by

$$\tilde{\mathcal{N}}(f)(X) = \mathcal{N}_{\rho,\pi}^{m_0}(f) \circ \rho_u(X) \quad (1.4.11)$$

for all  $f \in \mathcal{O}_H((X))$ . Then  $\tilde{\mathcal{N}}$  satisfies

$$\tilde{\mathcal{N}}(f) \circ \rho_\eta(X) = \prod_{w \in V_\rho^{m_0}} f(X + w). \quad (1.4.12)$$

Exactly as in Proposition 1.4.3,  $\tilde{\mathcal{N}}$  is multiplicative and satisfies

$$\tilde{\mathcal{N}}^n(f) \circ \rho_{\eta^n}(X) = \prod_{w \in V_\rho^{nm_0}} f(X + w) \quad (1.4.13)$$

for all  $n \geq 0$ . Moreover, if  $v_n$  and  $v_{n+1}$  are generators of  $V_\rho^{nm_0}$  and  $V_\rho^{(n+1)m_0}$  respectively such that  $\rho_\eta(v_{n+1}) = v_n$ , then

$$\tilde{\mathcal{N}}(f)(v_n) = N_{H_\rho^{(n+1)m_0}|H_\rho^{nm_0}}(f(v_{n+1})). \quad (1.4.14)$$

For the rest of this section, we suppose  $\rho$  is such that

$$\rho_\eta \equiv \tau^{m_0} \pmod{\mathfrak{p}_H}. \quad (1.4.15)$$

**Lemma 1.4.4.** Let  $f$  and  $g$  be two power series in  $\mathcal{O}_H[[X]]$  such that  $f = g \circ \rho_\eta$ , Then, for all  $i \geq 0$ , we have

$$g \equiv 0 \pmod{\mathfrak{p}_H^{im_0}} \iff f \equiv 0 \pmod{\mathfrak{p}_H^{im_0}} \quad (1.4.16)$$

*Proof.* We follow [18, Lemma 5.7]. The direct implication is clear. Let us prove the second implication by induction. For  $i = 0$ , (1.4.16) is clearly true. Suppose that it is true for  $i - 1$  and that  $f \equiv 0 \pmod{\mathfrak{p}_H^{im_0}}$ . Hence, we also have  $f \equiv 0 \pmod{\mathfrak{p}_H^{(i-1)m_0}}$ . This means that  $f = \eta^{i-1}f_1$  for some  $f_1 \in \mathcal{O}_H[[X]]$ . Moreover, by the assumption for  $i - 1$ , we have  $g \equiv 0 \pmod{\mathfrak{p}_H^{(i-1)m_0}}$  and thus  $g = \eta^{i-1}g_1$  for some  $g_1 \in \mathcal{O}_H[[X]]$ . Therefore, we have  $f_1 = g_1 \circ \rho_\eta$ . Since  $f \equiv 0 \pmod{\mathfrak{p}_H^{im_0}}$ , then  $f_1 \equiv 0 \pmod{\mathfrak{p}_H^{m_0}}$ . Since  $\rho_\eta \equiv \tau^{m_0} \pmod{\mathfrak{p}_H}$ , we get

$$g_1(X^{q^{m_0}}) \equiv g_1 \circ \rho_\eta(X) \equiv f_1(X) \equiv 0 \pmod{\mathfrak{p}_H^{m_0}}. \quad (1.4.17)$$

Thus,  $g_1 \equiv 0 \pmod{\mathfrak{p}_H^{m_0}}$  so that  $g = g_1\eta^{i-1} \in \mathfrak{p}_H^{im_0}\mathcal{O}_H[[X]]$ .  $\square$

**Proposition 1.4.5.** Let  $f \in \mathcal{O}_H[[X]]$ . The operator  $\tilde{\mathcal{N}}$  fulfills the following.

(i) If we denote by  $\tilde{\varphi} = \varphi^{m_0}$ , where  $\varphi$  is the Frobenius element in  $\text{Gal}(K_{ur}|K)$ , then

$$\tilde{\mathcal{N}}(f) \equiv f^{\tilde{\varphi}} \pmod{\mathfrak{p}_H}. \quad (1.4.18)$$

(ii) Let  $i \geq 1$ . We have

$$f \equiv 1 \pmod{\mathfrak{p}_H^{im_0}} \Rightarrow \tilde{\mathcal{N}}(f) \equiv 1 \pmod{\mathfrak{p}_H^{(i+1)m_0}}. \quad (1.4.19)$$

*Proof.* We follow [18, Lemma 5.8]. For  $w \in V_\rho^{m_0} \subset \mathfrak{p}_{H_\rho^{m_0}}$ , we have  $X + w \equiv X \pmod{\mathfrak{p}_{H_\rho^{m_0}}}$ . Hence,

$$\prod_{w \in V_\rho^{m_0}} f(X + w) \equiv f(X)^{q^{m_0}} \equiv f^{\tilde{\varphi}}(X^{q^{m_0}}) \pmod{\mathfrak{p}_{H_\rho^{m_0}}}. \quad (1.4.20)$$

Moreover, the condition (1.4.15) implies that

$$\tilde{\mathcal{N}}(f) \circ \rho_\eta(X) \equiv \tilde{\mathcal{N}}(f)(X^{q^{m_0}}) \pmod{\mathfrak{p}_H}. \quad (1.4.21)$$

The property (i) follows. Let us now prove (ii). Suppose  $f \equiv 1 \pmod{\mathfrak{p}_H^{im_0}}$  and let  $f_1 \in \mathcal{O}_H[[X]]$  be such that  $f = 1 + \eta^i f_1$ . Then

$$\tilde{\mathcal{N}}(f) \circ \rho_\eta(X) = \prod_{w \in V_\rho^{m_0}} (1 + \eta^i f_1(X + w)) \equiv (1 + \eta^i f_1(X))^{q^{m_0}} \pmod{\eta^i \mathfrak{p}_{H_\rho^{m_0}}} \quad (1.4.22)$$

because  $w \in V_\rho^{m_0} \subset \mathfrak{p}_{H_\rho^{m_0}}$ . This implies that

$$\tilde{\mathcal{N}}(f) \circ \rho_\eta(X) \equiv 1 \pmod{\eta^i \mathfrak{p}_H}. \quad (1.4.23)$$

Let  $f_2 \in \mathcal{O}_H[[X]]$  be such that  $\tilde{\mathcal{N}}(f) = 1 + f_2$ . Hence,  $f_2 \circ \rho_\eta \equiv 0 \pmod{\eta^i \mathfrak{p}_H}$  and thus  $f_2 \circ \rho_\eta \equiv 0 \pmod{\mathfrak{p}_H^{(i+1)m_0}}$ . Finally, Lemma 1.4.4 shows that  $f_2 \equiv 0 \pmod{\mathfrak{p}_H^{(i+1)m_0}}$ .  $\square$

**Lemma 1.4.6.** Let  $\mathcal{O}_H((X))^*$  denote the group of invertible elements in  $\mathcal{O}_H[[X]]$ . Let  $f$  be an element in  $\mathcal{O}_H((X))^*$ . Let  $i \geq 1$ , then

$$\frac{\tilde{\mathcal{N}}^i(f)}{\tilde{\mathcal{N}}^{i-1}(f^{\tilde{\varphi}})} \equiv 1 \pmod{\mathfrak{p}_H^i}. \quad (1.4.24)$$

*Proof.* Let us first prove the Lemma for  $i = 1$ . If  $f \in \mathcal{O}_H[[X]]$ , we know by Proposition 1.4.5 (i) that  $\tilde{\mathcal{N}}(f) \equiv f^{\tilde{\varphi}} \pmod{\mathfrak{p}_H}$ . Hence, if  $f \in \mathcal{O}_H[[X]] \cap \mathcal{O}_H((X))^*$ , it follows that

$$\frac{\tilde{\mathcal{N}}(f)}{f^{\tilde{\varphi}}} \equiv 1 \pmod{f^{\tilde{\varphi}-1} \pi \mathcal{O}_H[[X]]}. \quad (1.4.25)$$

Yet, it is obvious from (1.4.12) that  $\tilde{\mathcal{N}}(f)$  and  $f$  have the same order with respect to  $X$ . Therefore, for  $f \in \mathcal{O}_H[[X]] \cap \mathcal{O}_H((X))^*$ , we can write

$$\frac{\tilde{\mathcal{N}}(f)}{f^{\tilde{\varphi}}} \equiv 1 \pmod{\mathfrak{p}_H}. \quad (1.4.26)$$

Now, for any  $f \in \mathcal{O}_H((X))^*$ , we know that either  $f$  or  $f^{-1}$  is in  $\mathcal{O}_H[[X]]$ . This means that (1.4.26) is true either for  $f$  or for  $f^{-1}$ . It follows that it is true for all  $f \in \mathcal{O}_H((X))^*$ . Finally, to prove the Lemma for any  $i \geq 1$ , we apply Proposition 1.4.5 for (1.4.26) iteratively.  $\square$

Consequently, for  $f \in \mathcal{O}_H((X))^*$  we can define the limit

$$\tilde{\mathcal{N}}^\infty(f) := \lim_{i \rightarrow \infty} \tilde{\mathcal{N}}^i(f^{\tilde{\varphi}^{-i}}) \in \mathcal{O}_H((X))^*. \quad (1.4.27)$$

Then  $\tilde{\mathcal{N}}^\infty$  satisfies

$$\tilde{\mathcal{N}}(\tilde{\mathcal{N}}^\infty(f)) = \tilde{\mathcal{N}}^\infty(f^{\tilde{\varphi}}) \quad \text{and} \quad \frac{\tilde{\mathcal{N}}^\infty(f)}{f^{\tilde{\varphi}}} \equiv 1 \pmod{\mathfrak{p}_H}. \quad (1.4.28)$$

### 1.4.1 The case $\rho \in \mathbb{R}^{m_0}$

In this paragraph, we suppose that  $\rho$  has its coefficients in the subfield  $K_{ur}^{m_0}$  of  $H$  so that

$$\rho_\eta \equiv \tau^{m_0} \pmod{\mathfrak{p}_{K_{ur}^{m_0}}}. \quad (1.4.29)$$

For  $n \geq 0$ , denote  $E_\rho^n = H_\rho^{nm_0}$  and let  $(v_n)_n$  be a sequence such that  $v_n = \rho_\eta(v_{n+1})$ . Then, for each  $n \geq 0$ ,  $v_n$  is a generator of the  $\mathcal{O}_K$ -module  $V_\rho^{nm_0}$  and a prime element of  $E_\rho^n$ .

**Theorem 1.4.7.** Let  $\mathcal{X}_\infty$  be the projective limit  $\varprojlim_n (E_\rho^n)^\times$  with respect to the norm maps and let  $\mathcal{M}_\infty$  be the set of all invertible power series  $f$  in  $\mathcal{O}_H((X))^*$  satisfying  $\tilde{\mathcal{N}}(f) = f^{\tilde{\varphi}}$ . Then the map

$$\begin{aligned} \text{ev}_{\tilde{\mathcal{N}}} : \mathcal{M}_\infty &\rightarrow \mathcal{X}_\infty \\ f &\mapsto \left( f^{\tilde{\varphi}^{-n}}(v_n) \right)_n \end{aligned}$$

is a topological isomorphism.

*Proof.* Let us give a sketch of the proof. For the full proof, see [28, Theorem 5.8]. Let  $f \in \mathcal{M}_\infty$ . It is easy to see that  $f^{\tilde{\varphi}^{-1}} \in \mathcal{M}_\infty$  and consequently,

$$\tilde{\mathcal{N}}\left(f^{\tilde{\varphi}^{-(n+1)}}\right) = f^{\tilde{\varphi}^{-n}} \quad (1.4.30)$$

for all  $n > 0$ . Therefore, by (1.4.14), we deduce that  $\left(f^{\tilde{\varphi}^{-n}}(v_n)\right)_n$  is indeed in  $\mathcal{X}_\infty$ . This shows that the map  $\mathbf{ev}_{\tilde{\mathcal{N}}}$  is well defined. That  $\mathbf{ev}_{\tilde{\mathcal{N}}}$  is injective follows from [9, Lemma 2a]. To prove that  $\mathbf{ev}_{\tilde{\mathcal{N}}}$  is surjective, we first consider  $(x_n)_n \in \varprojlim \mathcal{O}_{E_\rho^n}$ . For an integer  $k$ , let  $g \in \mathcal{O}_H[[X]]$  such that  $g^{\tilde{\varphi}^{-2k}}(v_{2k}) = x_{2k}$  and let  $f_k = \tilde{\mathcal{N}}^k(g^{\tilde{\varphi}^{-k}})$ . We then prove that  $(f_k)_k$  is a Cauchy, hence, a convergent sequence. Its limit  $f$  satisfies the sought properties. Now for any  $(x_n)_n \in \varprojlim_n (E_\rho^n)^\times$ , there exists an integer  $e$  such that  $x_n \in v_n^e \mathcal{O}_{E_\rho^n}$  because the extensions  $E_\rho^n|H$  are totally ramified. Then, the power series  $f(\mathcal{N}^\infty)^e$  satisfies the sought properties.  $\square$

One may see this theorem as a generalization to  $m_0 > 1$  of [9, Theorem A]. It is also a generalization of [6, Theorem 11]. The proof of Hassan Oukhaba [28] is inspired by that of Bars and Longhi [6]. One may also check the proof of [33, Theorem 13.38].

# Chapter 2

## Explicit reciprocity laws in a particular case

In this chapter, we prove explicit reciprocity laws for a special class of formal Drinfeld modules having stable reduction of height one (see Theorem 2.3.12). After recalling the reciprocity map, we inspect the Kummer pairing in the setting of formal Drinfeld modules defined over local fields of positive characteristic. The method we use is inspired by Wiles [34].

Let  $K$  be a local field of positive characteristic  $p$ . Let  $\mathcal{O}_K$  be its valuation ring and  $\mathfrak{p}_K$  be its maximal ideal. We denote by  $q$  the order of its residue field  $\mathcal{O}_K/\mathfrak{p}_K$ .

### 2.1 The reciprocity map

Let  $L$  be a local field of positive characteristic  $p$ . There exists a continuous homomorphism  $\Phi_L : L^\times \rightarrow \text{Gal}(L^{ab}|L)$  called the reciprocity map. We will state without proving the main properties of  $\Phi_L$ . The interested reader may check [18, §6.1] or [27, Chapter V, §1].

**Proposition 2.1.1.** The map  $\Phi_L$  satisfies the following

- (i)  $\Phi_L$  is injective.
- (ii) The image of  $\Phi_L$  is dense in  $\text{Gal}(L^{ab}|L)$ .
- (iii) Let  $a \in L^\times$  of valuation  $\mu_L(a) = k$ , where  $\mu_L$  is the normalized valuation of  $L$ . The restriction of  $\Phi_L(a)$  to  $L_{nr}$  is equal to  $\varphi_L^k$ , where  $\varphi_L$  is the Frobenius element in  $\text{Gal}(L_{nr}|L)$ .
- (iv) Let  $M$  be a finite abelian extension of  $L$ , then we have the following exact sequence

$$1 \rightarrow N_{M|L}(M^\times) \rightarrow L^\times \xrightarrow{\Phi_L} \text{Gal}(M|L) \rightarrow 1. \quad (2.1.1)$$

(v) Let  $M$  be a finite separable extension of  $L$ . Then the diagram

$$\begin{array}{ccc} M^\times & \xrightarrow{\Phi_M} & \text{Gal}(M^{ab}|M) \\ N_{M|L} \downarrow & & \downarrow \\ L^\times & \xrightarrow{\Phi_L} & \text{Gal}(L^{ab}|L) \end{array}$$

is commutative, the right hand map being the restriction.

The next proposition relates the norm residue map and the transfer map, which we recall the definition. Let  $M|L$  be a finite separable extension. Let  $G$  be the Galois group of  $M^{ab}|L$  and  $H$  be the Galois group of  $M^{ab}|M$ . Let  $G'$  be the commutator subgroup of  $G$ . Let  $\{\sigma_1, \dots, \sigma_d\}$  be the embeddings of  $M$  in  $\Omega$  over  $L$ . Then  $\{\tilde{\sigma}_1, \dots, \tilde{\sigma}_d\}$ , where  $\tilde{\sigma}_i$  is an extension of  $\sigma_i$  to  $M^{ab}$ , is a complete set of representatives of  $G/H$ . Then, we can write

$$G = \sqcup_i H\tilde{\sigma}_i. \quad (2.1.2)$$

Therefore, for every automorphism  $\gamma \in G$ , and every  $i \in \{1, \dots, d\}$ , there exists a unique automorphism  $h_i(\gamma)$  of  $H$  and a unique  $j \in \{1, \dots, d\}$  such that

$$\tilde{\sigma}_i \gamma = h_i(\gamma) \tilde{\sigma}_j. \quad (2.1.3)$$

Hence, we define the transfer map by

$$\begin{aligned} t_{M|L} : G/G' &\rightarrow H \\ \gamma &\mapsto \prod_{i=1}^d h_i(\gamma). \end{aligned}$$

**Proposition 2.1.2.** Let  $M$  be a finite extension of  $L$ . The following diagram

$$\begin{array}{ccc} L^\times & \xrightarrow{\Phi_L} & \text{Gal}(L^{ab}|L) \\ \downarrow & & \downarrow t_{M|L} \\ M^\times & \xrightarrow{\Phi_M} & \text{Gal}(M^{ab}|M) \end{array}$$

is commutative, the left hand arrow being the inclusion map.

## 2.2 The Kummer pairing and its main properties

Let  $\Omega$  be an algebraic closure of  $K$ , and still denote  $\mu$  the unique extension of  $\mu$  to  $\Omega$ . Let  $(\bar{\Omega}, \bar{\mu})$  be the completion of  $(\Omega, \mu)$ . If  $F \subset \Omega$  is an extension of  $K$ , we denote  $\mathcal{O}_F$  the valuation ring of  $F$  and  $\mathfrak{p}_F$  its maximal ideal. Let  $H \subset \Omega$  be a finite unramified extension of  $K$ , and let  $\rho \in \hat{D}_{\mathcal{O}_K}(\mathcal{O}_H)$  having stable reduction of height 1. Let  $m_0$  be a positive integer

dividing  $[H : K]$  and let  $\eta \in K$  be an element of valuation  $\mu(\eta) = m_0$ . Let  $n \geq 0$  be an integer and define

$$W_\rho^n = V_\rho^{nm_0} = \{x \in \mathfrak{p}_\Omega; \rho_{\eta^n}(x) = 0\} \quad \text{and} \quad W_\rho = V_\rho. \quad (2.2.1)$$

Let  $v_n$  be a fixed generator of  $W_\rho^n$  and denote  $E_\rho^n = H_\rho^{nm_0} = H(W_\rho^n)$ . Let  $L$  be a finite extension of  $E_\rho^n$  and  $\pi$  be a prime of  $K$ .

**Lemma 2.2.1.** Let  $\alpha \in \mathfrak{p}_L$ . There exists an element  $\xi$  in  $\mathfrak{p}_\Omega$  such that  $\rho_{\eta^n}(\xi) = \alpha$ . Moreover, the extension  $L(\xi)|L$  is abelian, of degree  $\leq q^{nm_0}$ , and independent of the choice of  $\xi$  satisfying  $\rho_{\eta^n}(\xi) = \alpha$ .

*Proof.* By Lemma 1.3.4, we can write  $\rho_{\pi^{nm_0}}$  as

$$\rho_{\pi^{nm_0}} = U_1 U_{nm_0} Q_{nm_0} Q_{nm_0-1} \cdots Q_1, \quad (2.2.2)$$

where  $U_i$  are invertible elements of  $\mathcal{O}_H\{\{\tau\}\}$  and  $Q_i = \tau + \pi_i$ , each  $\pi_i$  being a prime of  $H$ . We denote  $V_n = \rho_{u^n} U_1 U_{nm_0}$ , where  $u$  is the unit such that  $\eta = u\pi^{m_0}$ . Since  $V_n$  is invertible in  $\mathcal{O}_H\{\{\tau\}\}$ , we have

$$\begin{aligned} \rho_{\eta^n}(X) = \alpha &\iff V_n(P_{nm_0}(X)) = \alpha \\ &\iff P_{nm_0}(X) = V_n^{-1}(\alpha) \\ &\iff P_{nm_0}(X) - V_n^{-1}(\alpha) = 0, \end{aligned}$$

where  $P_{nm_0}$  is the polynomial defined in (1.3.4). However,  $V_n^{-1}(\alpha) \in \mathfrak{p}_L$ , hence,  $P_{nm_0}(X) - V_n^{-1}(\alpha)$  is a polynomial with coefficients in  $L$ . Therefore there exists an element  $\xi$  in  $\Omega$  such that  $P_{nm_0}(\xi) - V_n^{-1}(\alpha) = 0$ . Furthermore, since  $0 \equiv P_{nm_0}(\xi) \equiv \xi^{q^{nm_0}} \pmod{\mathfrak{p}_\Omega}$ , we have  $\xi \in \mathfrak{p}_\Omega$ . Moreover, the polynomial  $P_{nm_0}(X) - V_n^{-1}(\alpha)$  is of degree  $q^{nm_0}$ , and all the elements of the set  $\xi + W_\rho^n$ , which we recall is a set of  $q^{nm_0}$  elements, are roots of this polynomial. This is true since  $W_\rho^n$  is the set of roots of  $P_{nm_0}(X)$ . Hence, it is separable and  $L(\xi)|L$  is a Galois extension of degree  $\leq q^{nm_0}$  depending only on  $\alpha$ . Finally, to prove that it is an abelian extension, it suffices to notice that the group homomorphism  $\text{Gal}(L(\xi)|L) \rightarrow W_\rho^n$  defined by  $\sigma \mapsto \sigma(\xi) - \xi$  is injective.  $\square$

This lemma shows that we can define the map  $(\ , \ )_{\rho, L, n} : \mathfrak{p}_L \times L^\times \rightarrow W_\rho^n$  such that

$$(\alpha, \beta)_{\rho, L, n} = \Phi_L(\beta)(\xi) - \xi \quad (2.2.3)$$

for  $\alpha \in \mathfrak{p}_L$  and  $\beta \in L^\times$ , where  $\xi \in \mathfrak{p}_\Omega$  is such that  $\rho_{\eta^n}(\xi) = \alpha$ . This definition is independent of the choice of  $\xi$ . Indeed, the proof of Lemma 2.2.1 shows that all the roots of  $\rho_{\eta^n}(X) = \alpha$  are of the form  $\xi + w$ , where  $w \in W_\rho^n$ . Hence,

$$\Phi_L(\beta)(\xi + w) - (\xi + w) = \Phi_L(\beta)(\xi) + \Phi_L(\beta)(w) - \xi - w = \Phi_L(\beta)(\xi) - \xi, \quad (2.2.4)$$

because  $w \in W_\rho^n \subset L$  and  $\Phi_L(\beta)$  fixes  $L$ . We omit  $\rho$  in the index when there is no risk of confusion. Exactly as in [34, 21] we have



**Proposition 2.2.2.** The map  $(\ , \ )_{L,n}$  satisfies the following properties

(i) The map  $(\ , \ )_{L,n}$  is bilinear and  $\mathcal{O}_K$ -linear in the first coordinate for the action (1.3.1).

(ii) We have

$$(\alpha, \beta)_{L,n} = 0 \iff \beta \text{ is a norm from } L(\xi), \text{ where } \rho_{\eta^n}(\xi) = \alpha.$$

(iii) Let  $M$  be a finite separable extension of  $L$ , let  $\alpha \in \mathfrak{p}_L$  and  $\beta \in M^\times$ . Then  $(\alpha, \beta)_{M,n} = (\alpha, N_{M|L}(\beta))_{L,n}$ .

(iv) Let  $M$  be a finite separable extension of  $L$  of degree  $d$ , let  $\alpha \in \mathfrak{p}_M$  and  $\beta \in L^\times$ . Then  $(\alpha, \beta)_{M,n} = (T_{M|L}(\alpha), \beta)_{L,n}$ .

(v) Suppose  $L \supset E_\rho^m$  for  $m \geq n$ . Then

$$(\alpha, \beta)_{L,n} = \rho_{\eta^{m-n}}((\alpha, \beta)_{L,m}) = (\rho_{\eta^{m-n}}(\alpha), \beta)_{L,m}.$$

(vi) Let  $\rho'$  be a formal Drinfeld  $\mathcal{O}_K$ -module isomorphic to  $\rho$ , i.e there exists a power series  $t$  invertible in  $\mathcal{O}_H\{\{\tau\}\}$  such that  $\rho'_a = t^{-1} \circ \rho_a \circ t$  for all  $a \in \mathcal{O}_K$ . Then we have  $(\alpha, \beta)_{\rho',L,n} = t^{-1}((t(\alpha), \beta)_{\rho,L,n})$ .

*Proof.* (i) Let  $\alpha_1, \alpha_2 \in \mathfrak{p}_L$  and  $a \in \mathcal{O}_K$ . Let  $\xi_1$  and  $\xi_2$  be elements of  $\mathfrak{p}_\Omega$  such that  $\rho_{\eta^n}(\xi_1) = \alpha_1$  and  $\rho_{\eta^n}(\xi_2) = \alpha_2$ . Then  $\rho_{\eta^n}(\xi_1 + a \cdot \rho \xi_2) = \alpha_1 + a \cdot \rho \alpha_2$  and the linearity on the first coordinate follows. Let  $\beta_1, \beta_2 \in L^\times$ . Since  $\Phi_L(\beta_1 \beta_2) = \Phi_L(\beta_1) \Phi_L(\beta_2)$ , the linearity on the second coordinate follows from the fact that the map  $\text{Gal}(L(\xi)|L) \rightarrow W_\rho^n$  defined by  $\sigma \mapsto \sigma(\xi) - \xi$  is a group homomorphism.

(ii) This equivalence follows from Proposition 2.1.1 (iv).

(iii) This equality follows from Proposition 2.1.1 (v).

(iv) Let  $\xi$  be such that  $\rho_{\eta^n}(\xi) = \alpha$ . Let  $G$  be the Galois group of  $M^{ab}|L$  and  $H$  be the Galois group of  $M^{ab}|M$ . Let  $\{\sigma_1, \dots, \sigma_d\}$  be the embeddings of  $M$  in  $\Omega$  over  $L$ . We consider the quotient group  $G/H$ . Then  $\{\tilde{\sigma}_1, \dots, \tilde{\sigma}_d\}$ , where  $\tilde{\sigma}_i$  is an extension of  $\sigma_i$  to  $M^{ab}$ , is a set of representatives for the left cosets  $H\tilde{\sigma}$

$$G = \sqcup_i H\tilde{\sigma}_i.$$

We extend  $\Phi_L(\beta)$  to  $M^{ab}$  and denote it by  $\tilde{\Phi}_L(\beta) \in G$ . Therefore, for each  $i$ , there exists a unique  $h_i \in H$ , and a unique  $j$  such that

$$\tilde{\sigma}_i \tilde{\Phi}_L(\beta) = h_i \tilde{\sigma}_j. \tag{2.2.5}$$

By the properties of the transfer map  $t_{M|L}$  (see Proposition 2.1.2), we have

$$\begin{aligned}
(\alpha, \beta)_{M,n} &= \Phi_M(\beta)(\xi) - \xi \\
&= t_{M|L}(\Phi_L(\beta))(\xi) - \xi \\
&= \prod_i h_i(\xi) - \xi \\
&= \sum_i (h_i(\xi) - \xi)
\end{aligned} \tag{2.2.6}$$

The last equality follows from the fact that  $h_i(\xi) - \xi \in W_\rho^n \subset L$ . Using the notation  $(\alpha, h_i)_{M,n} = h_i(\xi) - \xi$ , we can write

$$\sum_i (\alpha, h_i)_{M,n} = \tilde{\Phi}_L(\beta) \left( \sum_j (\tilde{\sigma}_j^{-1}(\xi)) \right) - \sum_j (\tilde{\sigma}_j^{-1}(\xi)). \tag{2.2.7}$$

Indeed, by (2.2.5) we get

$$\begin{aligned}
\tilde{\Phi}_L(\beta) \tilde{\sigma}_j^{-1}(\xi) &= \tilde{\sigma}_i^{-1} h_i(\xi) \\
&= \tilde{\sigma}_i^{-1} ((\alpha, h_i)_{M,n} + \xi) \\
&= (\alpha, h_i)_{M,n} + \tilde{\sigma}_i^{-1}(\xi)
\end{aligned}$$

because  $(\alpha, h_i)_{M,n} \in W_\rho^n \subset L$ . Thus, by (2.2.6) and (2.2.7), we have  $(\alpha, \beta)_{M,n} = (\sum_i \sigma_i(\alpha), \beta)_{L,n}$ .

(v) If  $\xi$  is such that  $\rho_{\eta^n}(\xi) = \alpha$ , then it also satisfies  $\rho_{\eta^m}(\xi) = \rho_{\eta^{m-n}}(\alpha)$ . Hence the property follows.

(vi) Let  $\xi \in \mathfrak{p}_\Omega$  be such that  $\rho'_{\eta^n}(\xi) = \alpha$ , hence we have  $\rho_{\eta^n}(t(\xi)) = t(\alpha)$  and

$$\begin{aligned}
(t(\alpha), \beta)_{\rho, L, n} &= \Phi_L(\beta)(t(\xi)) - t(\xi) \\
&= t(\Phi_L(\beta)(\xi) - t(\xi)) \\
&= t(\Phi_L(\beta)(t(\xi)) - \xi) \\
&= t((\alpha, \beta)_{\rho', L, n}).
\end{aligned}$$

□

**Lemma 2.2.3.** There exists a constant  $c_{L,n}$ , dependant only on  $L$  and  $n$ , such that for  $\alpha \in \mathfrak{p}_L$ , if we set  $\alpha_m = \rho_{\eta^{m-n}}(\alpha)$  for  $m \geq n$ , we get  $\mu(\alpha_m) \geq mm_0 - c_{L,n}$ . Furthermore, the map  $(\cdot, \cdot)_{L,n}$  is continuous, and  $(\alpha, \cdot)_{L,n} = 0$  for all  $\alpha \in \mathfrak{p}_L$  such that  $\mu(\alpha) > nm_0 + \frac{1}{q-1}$ .

*Proof.* We follow [6, Lemma 15]. Let  $\alpha \in \mathfrak{p}_L$  and set  $\mu_j := \frac{1}{q^{j-1}(q-1)}$  for  $j \geq 1$  and  $\mu_0 := \infty$ . Choose  $\xi$  a root of  $\rho_{\eta^n}(X) = \alpha$  of maximal valuation. This is possible because the equation  $\rho_{\eta^n}(X) = \alpha$  has a finite set of solutions:  $\xi + W_\rho^n$ . We have

$$\alpha = \rho_{\eta^n}(\xi) = V_n(P_{nm_0}(\xi)), \tag{2.2.8}$$

where  $P_{nm_0}(X) = \prod_{w \in W_\rho^n} (X - w)$  is the polynomial defined in (1.3.4) and  $V_n$  is defined in the proof of Lemma 2.2.1. Therefore, we get

$$\mu(\alpha) = \mu(P_{nm_0}(\xi)) = \sum_{w \in W_\rho^n} \mu(\xi - w) \quad (2.2.9)$$

because  $\rho_{u^n}(X) \equiv u^n X \pmod{\deg 2}$ . Let  $w \in W_\rho^n$ . If  $\mu(\xi) \neq \mu(w)$ , then  $\mu(\xi - w) = \min\{\mu(\xi), \mu(w)\}$ . If  $\mu(\xi) = \mu(w)$ , then

$$\mu(\xi) = \min\{\mu(\xi), \mu(w)\} \leq \mu(\xi - w) \leq \mu(\xi), \quad (2.2.10)$$

the last inequality being a consequence of the maximality hypothesis on  $\mu(\xi)$ . Hence we have  $\mu(\xi - w) = \min\{\mu(\xi), \mu(w)\}$  for all  $w \in W_\rho^n$  and

$$\mu(\alpha) = \sum_{w \in W_\rho^n} \min\{\mu(\xi), \mu(w)\}. \quad (2.2.11)$$

Let  $j \geq 0$  be such that  $\mu_{j+1} < \mu(\xi) \leq \mu_j$ . If  $0 \leq j \leq nm_0$ , the equality (2.2.11) yields

$$\mu(\alpha) = \sum_{w \in V_\rho^j} \mu(\xi) + \sum_{w \in W_\rho^n \setminus V_\rho^j} \mu(w) = q^j \mu(\xi) + nm_0 - j \quad (2.2.12)$$

so that  $nm_0 - j + \frac{1}{q-1} < \mu(\alpha) \leq nm_0 - j + 1 + \frac{1}{q-1}$ . Now if  $j > nm_0$ , by (2.2.11) we get  $\mu(\alpha) = q^{nm_0} \mu(\xi)$  so that

$$nm_0 - j + \frac{1}{q-1} \leq 0 < \frac{1}{q^{j-nm_0}(q-1)} < \mu(\alpha) \leq \frac{1}{q^{j-nm_0-1}(q-1)}. \quad (2.2.13)$$

Since  $\xi$  is also a root of  $\rho_{\eta^m}(X) = \alpha_m$  for all  $m \geq n$ , we deduce by the same arguments that  $\mu(\alpha_m) \geq mm_0 - j + \frac{1}{q-1}$ . Considering the degree of the extension  $L(\xi)|K$ , we see that  $j \leq 2nm_0 + \log_q(e)$ , where  $e$  is the ramification index of  $L|E_\rho^n$ . Hence, we get

$$\mu(\alpha_m) \geq mm_0 - 2nm_0 - \log_q(e) + \frac{1}{q-1}. \quad (2.2.14)$$

Finally, if we suppose  $\mu(\alpha) > nm_0 + \frac{1}{q-1}$ , we get  $j = 0$ , which implies that  $\mu((\alpha, \beta)_{L,n}) \geq \mu(\xi) > \frac{1}{q-1}$  for all  $\beta \in L^\times$ . It follows that  $(\alpha, \beta)_{L,n} = 0$  for all  $\beta \in L^\times$ , because  $(\alpha, \beta)_{L,n}$  belongs to  $W_\rho^n$ , and the elements of  $W_\rho^n \setminus \{0\}$  are of valuation less or equal to  $\frac{1}{q-1}$ . The fact that the map  $(, )_{L,n}$  is continuous follows immediately since the reciprocity map  $\Phi_L$  is continuous.  $\square$

**Remark 2.2.4.** Let  $e$  be the ramification index of  $L|E_\rho^n$ , then the constant  $c_{L,n}$  from Lemma 2.2.3 is bounded as follows

$$\frac{-1}{q-1} \leq c_{L,n} \leq 2nm_0 + \log_q(e) - \frac{1}{q-1}. \quad (2.2.15)$$

**Proposition 2.2.5.** There exists a unique power series  $r = r_n \in \mathcal{O}_H\{\{\tau\}\}$  such that

$$\prod_{\omega \in W_\rho^n} (X - \omega) = r \circ \rho_{\eta^n}(X). \quad (2.2.16)$$

Furthermore, the power series  $r$  is invertible in  $\mathcal{O}_H\{\{\tau\}\}$  and satisfies

$$(x, r(x))_{L,n} = 0, \quad \forall x \in \mathfrak{p}_L \setminus \{0\}. \quad (2.2.17)$$

*Proof.* As in the proof of Lemma 2.2.1, we can write

$$\rho_{\eta^n}(X) = \rho_{u^n} \circ U_1 \circ U_{nm_0} \circ P_{nm_0}(X). \quad (2.2.18)$$

Thus for  $r = (\rho_{u^n} \circ U_1 \circ U_{nm_0})^{-1}$  we get  $P_{nm_0}(X) = \prod_{\omega \in W_\rho^n} (X - \omega) = r \circ \rho_{\eta^n}(X)$ . It remains to show that  $(x, r(x))_{L,n} = 0$  for all  $x \in \mathfrak{p}_L \setminus \{0\}$ . Take  $x \in \mathfrak{p}_L \setminus \{0\}$  and  $\xi$  such that  $\rho_{\eta^n}(\xi) = x$ . Then,

$$r(x) = (r \circ \rho_{\eta^n})(\xi) = \prod_{\omega \in W_\rho^n} (\xi - \omega) = \prod_i N_{L(\xi)|L}(\xi_i) \quad (2.2.19)$$

where  $\xi_i$  are the pairwise distinct roots of  $\rho_{\eta^n}(X) = x$ . It follows that  $(x, r(x))_{L,n} = 0$  by Proposition 2.2.2 (ii).  $\square$

**Lemma 2.2.6.** Let  $r = r_n$  be the power series defined in Proposition 2.2.5. Let  $\rho'$  be defined by

$$\rho'_a = r \circ \rho_a \circ r^{-1} \quad (2.2.20)$$

for all  $a \in \mathcal{O}_K$ . Then  $\rho'$  is a formal Drinfeld module having a stable reduction of height 1, and we have  $(x, x)_{\rho', L, n} = 0$  for all  $x \in \mathfrak{p}_L \setminus \{0\}$ .

*Proof.* That  $\rho'$  is a formal Drinfeld module having a stable reduction of height 1 follows immediately from the fact that  $\rho$  itself is supposed to be a formal Drinfeld module having a stable reduction of height 1. It follows from Proposition 2.2.2 (vi) that  $(x, x)_{\rho', L, n} = r((r^{-1}(x), x)_{\rho, L, n}) = r(0) = 0$ .  $\square$

**Lemma 2.2.7.** If  $\rho$  is such that  $(x, x)_{\rho, L, n} = 0$  for all  $x \in \mathfrak{p}_L \setminus \{0\}$ , then we have

$$(c, 1 - b)_{L, n} = \left(\frac{bc}{1 - b}, b^{-1}\right)_{L, n} \quad (2.2.21)$$

for all  $b \in \mathfrak{p}_L \setminus \{0\}$  and  $c \in \mathfrak{p}_L$ .

*Proof.* We follow [6, Lemma 18]. To prove this result we use the property  $(x, x)_{L, n} = 0$  for  $x = c(1 - b)$ . By bilinearity we get

$$(c, 1 - b)_{L, n} = (cb, c)_{L, n} + (cb, 1 - b)_{L, n} \quad (2.2.22)$$

and by induction

$$(c, 1 - b)_{L, n} = \sum_{j \geq 0} (cb^j, cb^{j-1})_{L, n}. \quad (2.2.23)$$

This sum converges since only a finite number of terms is non zero by Lemma 2.2.3. However we have  $0 = (cb^j, cb^j)_{L,n} = (cb^j, cb^{j-1})_{L,n} + (cb^j, b)_{L,n}$ . Therefore,

$$(c, 1 - b)_{L,n} = \sum_{j \geq 0} (cb^j, b^{-1})_{L,n} = (c \sum_{j \geq 0} b^j, b^{-1})_{L,n} = (\frac{cb}{1-b}, b^{-1})_{L,n}. \quad (2.2.24)$$

□

## 2.2.1 The analytic pairing

In this paragraph, we suppose that  $L = E_\rho^n$ . We will define and state the main properties of a pairing  $[ , ]_{L,n}$ , which we will prove equal to the Kummer pairing  $( , )_{L,n}$  in the next section.

**Lemma 2.2.8.** Let  $\mathcal{D}_n$  denote the different of the extension  $L|K$ . For  $\beta \in L^\times$ , choose a power series  $f \in \mathcal{O}_H((X)) \setminus \{0\}$  such that  $f(v_n) = \beta$ . Then,

$$\frac{f'(v_n)}{\beta} \in \mathfrak{p}_L^{-1}. \quad (2.2.25)$$

Moreover, if we set

$$\delta_{v_n}(\beta) := \frac{f'(v_n)}{\beta} \pmod{\mathcal{D}_n}, \quad (2.2.26)$$

then  $\delta_{v_n} : L^\times \rightarrow \mathfrak{p}_L^{-1}/\mathcal{D}_n$  is a well defined map and is a group homomorphism.

*Proof.* First, let us prove (2.2.25). Let  $\beta \in L^\times$  and let  $f(X) \in \mathcal{O}_H((X)) \setminus \{0\}$  such that  $f(v_n) = \beta$ . Then we can write  $f(X) = X^b f_1(X)$ , where  $b = \frac{\mu(\beta)}{\mu(v_n)}$  and  $f_1(X)$  is an invertible power series in  $\mathcal{O}_H[[X]]$ . Hence,

$$\frac{f'(v_n)}{\beta} = \frac{bv_n^{b-1}f_1(v_n) + v_n^b f_1'(v_n)}{v_n^b f_1(v_n)} = \frac{bf_1(v_n) + v_n f_1'(v_n)}{v_n f_1(v_n)} \in \mathfrak{p}_L^{-1}.$$

Let  $g(X) \in \mathcal{O}_H((X)) \setminus \{0\}$  be another power series such that  $g(v_n) = \beta$ . We can also write  $g(X) = X^b g_1(X)$ , where  $g_1(X) \in \mathcal{O}_H[[X]]$  is an invertible power series. To prove that  $\delta_{v_n}$  is well defined, we need to prove that

$$\frac{f'(v_n) - g'(v_n)}{\beta} \in \mathcal{D}_n. \quad (2.2.27)$$

We have  $f'(v_n) - g'(v_n) = v_n^b (f_1'(v_n) - g_1'(v_n))$  because  $f_1(v_n) = g_1(v_n)$ . Furthermore, by Lemma 2.3.3, the power series  $f_1(X) - g_1(X)$  is divisible in  $\mathcal{O}_H[[X]]$  by the minimal polynomial  $h_{nm_0}(X)$  of  $v_n$ , which we recall is defined in (1.3.5). This means that there exists  $t \in \mathcal{O}_H[[X]]$  such that  $f_1(X) - g_1(X) = h_{nm_0}(X)t(X)$ . Thus,  $f_1'(v_n) - g_1'(v_n) = h'_{nm_0}(v_n)t(v_n)$ . This implies (2.2.27) because  $\mathcal{D}_n = (h'_{nm_0}(v_n))$  as stated in the course of the proof of Lemma 1.3.14. Therefore,  $\delta_{v_n}(\beta)$  is independent of the choice of  $f$  such that  $f(v_n) = \beta$ . The fact that  $\delta_{v_n}$  is a group homomorphism is straightforward. □

**Lemma 2.2.9.** Let  $m \geq n$  and let  $v_m$  be such that  $v_n = \rho_{\eta^{m-n}}(v_m)$ . Then  $v_m$  is a generator of  $W_\rho^m$  and for all  $\beta \in L^\times$ , we have

$$\delta_{v_m}(\beta) \equiv \eta^{m-n} \delta_{v_n}(\beta) \pmod{\mathcal{D}_m}. \quad (2.2.28)$$

*Proof.* Let  $\beta \in L^\times$  and  $f \in \mathcal{O}_H((X)) \setminus \{0\}$  such that  $f(v_n) = \beta$ . Thus  $g = f \circ \rho_{\eta^{m-n}}$  satisfies  $g(v_m) = \beta$ . The result follows immediately.  $\square$

**Lemma 2.2.10.** The map  $[\cdot, \cdot]_{\rho, L, n}$  given by

$$[\alpha, \beta]_{\rho, L, n} := \frac{1}{\eta^n} \mathbb{T}_{L|K}(\lambda_\rho(\alpha) \delta_{v_n}(\beta)) \cdot_\rho v_n \quad (2.2.29)$$

is well defined for all  $\alpha \in \mathfrak{p}_L$  of valuation  $\mu(\alpha) \geq \frac{2}{q-1}$ , and all  $\beta \in L^\times$ . We drop  $\rho$  in the index when there is no risk of confusion.

*Proof.* We need to show that  $\frac{1}{\eta^n} \mathbb{T}_{L|K}(\lambda_\rho(\alpha) b) \in \mathcal{O}_K$  for every  $b \in \mathfrak{p}_L^{-1}$  and that

$$\mu\left(\frac{1}{\eta^n} \mathbb{T}_{L|K}(\lambda_\rho(\alpha) d)\right) \geq nm_0$$

for all  $d \in \mathcal{D}_n$ . By (1.2.13), we know that  $\mu(\lambda_\rho(\alpha)) = \mu(\alpha)$ . Thus the result follows from Lemma 1.3.15.  $\square$

**Proposition 2.2.11.** The map  $[\cdot, \cdot]_{L, n}$  satisfies the following properties

- (i) The map  $[\cdot, \cdot]_{L, n}$  is bilinear and  $\mathcal{O}_K$ -linear in the first coordinate for the action (1.3.1).
- (ii) Let  $\rho'$  be a formal Drinfeld  $\mathcal{O}_K$ -module isomorphic to  $\rho$ , i.e there exists a power series  $t$  invertible in  $\mathcal{O}_H\{\{\tau\}\}$  such that  $\rho'_a = t^{-1} \circ \rho_a \circ t$  for all  $a \in \mathcal{O}_K$ . Then we have  $[\alpha, \beta]_{\rho', L, n} = t^{-1}([t(\alpha), \beta]_{\rho, L, n})$ .

*Proof.* The property (i) is clear, so we will only prove (ii). To do so, let  $v'_n = t^{-1}(v_n)$  be a generator of the  $\mathcal{O}_K$ -module  $W_{\rho'}^n$ . Then, if  $f \in \mathcal{O}_H((X)) \setminus \{0\}$  is such that  $f(v_n) = \beta$ , we have  $f \circ t(v'_n) = f(v_n) = \beta$  so that

$$\delta'_{v'_n}(\beta) = \frac{t'(v'_n) f'(v_n)}{\beta} = t'(0) \delta_{v_n}(\beta),$$

where  $\delta'_{v'_n}$  is the map defined in Lemma 2.2.8 corresponding to  $\rho'$ . Furthermore, we have  $\lambda_{\rho'} \circ t^{-1} = (t^{-1})'(0) \lambda_\rho$ . The result follows immediately since  $(t^{-1})'(0) = \frac{1}{t'(0)}$ .  $\square$

**Lemma 2.2.12.** Let  $\alpha \in \mathfrak{p}_L$  such that  $\mu(\alpha) \geq \frac{nm_0}{q} + \frac{1}{q-1} + \frac{1}{q^{nm_0}(q-1)}$  and let  $\beta \in L^\times$ . We have

$$[\alpha, \beta]_{L, n} = \frac{1}{\eta^n} \mathbb{T}_{L|K}(\alpha \delta_{v_n}(\beta)) \cdot_\rho v_n.$$

*Proof.* We need to prove that

$$\frac{1}{\eta^n} \mathbf{T}_{L|K}(\lambda_\rho(\alpha)\delta_{v_n}(\beta)) \cdot_\rho v_n = \frac{1}{\eta^n} \mathbf{T}_{L|K}(\alpha\delta_{v_n}(\beta)) \cdot_\rho v_n,$$

i.e. that

$$\mu(\mathbf{T}_{L|K}(\lambda_\rho(\alpha) - \alpha)\delta_{v_n}(\beta)) \geq 2nm_0.$$

We have

$$\mu((\lambda_\rho(\alpha) - \alpha)\delta_{v_n}(\beta)) \geq \min_i \{q^i \mu(\alpha) - i\} - \frac{1}{q^{nm_0-1}(q-1)}.$$

The hypothesis on  $\alpha$  implies that  $\min\{q^i \mu(\alpha) - i\} = q\mu(\alpha) - 1$  so that  $\mu(\lambda_\rho(\alpha) - \alpha)\delta_{v_n}(\beta) \geq nm_0 + \frac{1}{q-1}$ . Finally, we conclude using Lemma 1.3.15.  $\square$

For each  $m \geq n$ , let  $v_m \in W_\rho^m$  be such that  $v_m = \rho_\eta(v_{m+1})$ . In particular, we have  $\rho_{\eta^{m-n}}(v_m) = v_n$ . Moreover, for each  $m \geq n$ ,  $v_m \in V_\rho^{mm_0} \setminus V_\rho^{mm_0-1}$  is a generator of  $W_\rho^m = V_\rho^{mm_0}$  as an  $\mathcal{O}_K$ -module.

**Lemma 2.2.13.** Let  $m \geq n$ . Let  $\beta \in (E_\rho^n)^\times$  and  $\beta_m \in (E_\rho^m)^\times$  be such that  $N_{m,n}(\beta_m) = \beta$ , where  $N_{m,n}$  is the norm of the extension  $E_\rho^m|E_\rho^n$ . We have

$$\tilde{\mathbf{T}}_{m,n}(\delta_{v_m}(\beta_m)) = \eta^{m-n} \delta_{v_n}(\beta), \quad (2.2.30)$$

where  $\tilde{\mathbf{T}}_{m,n} : E_\rho^m/\mathcal{D}_m \rightarrow E_\rho^n/\mathcal{D}_n$  is the map induced by the trace  $\mathbf{T}_{m,n}$  of the extension  $E_\rho^m|E_\rho^n$ .

*Proof.* It is sufficient to prove the Lemma for  $m = n + 1$ . Let  $f \in \mathcal{O}_H((X))$  such that  $f(v_{n+1}) = \beta_{n+1}$ . Then, the twisted norm operator  $\tilde{\mathcal{N}}$  defined in (1.4.11) satisfies

$$\tilde{\mathcal{N}}(f)(v_n) = N_{n+1,n}(f(v_{n+1})) = N_{n+1,n}(\beta_{n+1}) = \beta. \quad (2.2.31)$$

Logarithmically differentiating (1.4.12) with respect to  $X$ , we get

$$\frac{(\tilde{\mathcal{N}}(f) \circ \rho_\eta(X))'}{\tilde{\mathcal{N}}(f) \circ \rho_\eta(X)} = \sum_{w \in W_\rho^1} \frac{(f(X+w))'}{f(X+w)} = \sum_{w \in W_\rho^1} \frac{f'(X+w)}{f(X+w)}. \quad (2.2.32)$$

However, we have

$$\frac{(\tilde{\mathcal{N}}(f) \circ \rho_\eta(X))'}{\tilde{\mathcal{N}}(f) \circ \rho_\eta(X)} = \eta \frac{(\tilde{\mathcal{N}}(f))'}{\tilde{\mathcal{N}}(f)} \circ \rho_\eta(X). \quad (2.2.33)$$

Evaluating (2.2.32) and (2.2.33) at  $v_{n+1}$ , we get

$$\sum_{w \in W_\rho^1} \frac{f'(v_{n+1} + w)}{f(v_{n+1} + w)} = \eta \delta_{v_n}(\beta). \quad (2.2.34)$$

On the other hand, by Lemma 1.3.13, the set  $v_{n+1} + W_\rho^1$  is the complete set of conjugates of  $v_{n+1}$  over  $E_\rho^n$ , so that

$$\eta\delta_{v_n}(\beta) = \sum_{w \in W_\rho^1} \frac{f'(v_{n+1} + w)}{f(v_{n+1} + w)} = \tilde{T}_{n+1,n}(\delta_{v_{n+1}}(\beta_{n+1})). \quad (2.2.35)$$

□

**Proposition 2.2.14.** Let  $m \geq n$ . Let  $\beta \in L^\times = (E_\rho^n)^\times$  and  $\beta_m \in (E_\rho^m)^\times$  be such that  $N_{m,n}(\beta_m) = \beta$ . Let  $\alpha \in \mathfrak{p}_L$  of valuation  $\mu(\alpha) \geq \frac{2}{q-1}$  and set  $\alpha_m = \rho_{\eta^{m-n}}(\alpha)$ . Then,

$$[\alpha_m, \beta_m]_{E_\rho^m, m} = [\alpha, \beta]_{L, n}. \quad (2.2.36)$$

*Proof.* We have

$$\begin{aligned} [\alpha_m, \beta_m]_{E_\rho^m, m} &= \frac{1}{\eta^m} \mathrm{T}_{E_\rho^m|K}(\lambda_\rho(\alpha_m)\delta_{v_m}(\beta_m)) \cdot_\rho v_m \\ &= \frac{1}{\eta^m} \mathrm{T}_{E_\rho^m|K}(\eta^{m-n}\lambda_\rho(\alpha)\delta_{v_m}(\beta_m)) \cdot_\rho v_m \quad (\text{by (1.2.2)}) \\ &= \frac{1}{\eta^m} \mathrm{T}_{L|K}(\eta^{m-n}\lambda_\rho(\alpha) \mathrm{T}_{m,n}(\delta_{v_m}(\beta_m))) \cdot_\rho v_m \\ &= \frac{\eta^{m-n}}{\eta^n} \mathrm{T}_{L|K}(\lambda_\rho(\alpha)\delta_{v_n}(\beta)) \cdot_\rho v_m. \end{aligned}$$

by Lemma 2.2.13. Finally, this is equal to  $\frac{1}{\eta^n} \mathrm{T}_{L|K}(\lambda_\rho(\alpha)\delta_{v_n}(\beta)) \cdot_\rho v_n = [\alpha, \beta]_{L, n}$ . □

## 2.3 Explicit reciprocity laws

Keeping the same notations as in the last section, we let  $L = E_\rho^n$  and we suppose  $\rho$  is such that

$$\rho_\eta \equiv \tau^{m_0} \pmod{\mathfrak{p}_H}. \quad (2.3.1)$$

**Lemma 2.3.1.** Let  $f(X) \in \mathcal{O}_{\bar{\Omega}}[[X]]$  be a power series and let  $\alpha \in \mathfrak{p}_{\bar{\Omega}}$  satisfying  $f(\alpha) = 0$ . Then  $f(X)$  is divisible by  $(X - \alpha)$  in  $\mathcal{O}_{\bar{\Omega}}[[X]]$ .

*Proof.* This is [18, Lemma 3.9]. Let  $f(X) = \sum_{i \geq 0} a_i X^i$  with  $a_i \in \mathcal{O}_{\bar{\Omega}}$  and set

$$b_i = \sum_{j \geq 0} a_{i+j+1} \alpha^j. \quad (2.3.2)$$

Then we can easily see that  $g(X) = \sum_{i \geq 0} b_i X^i \in \mathcal{O}_{\bar{\Omega}}[[X]]$  satisfies  $f(X) = (X - \alpha)g(X)$ . □

**Remark 2.3.2.** Let  $F \subset \bar{\Omega}$  be a complete field containing  $K$  and suppose  $f(X)$  from Lemma 2.3.1 has its coefficients in  $\mathcal{O}_F$ . If in addition  $\alpha$  is algebraic over  $K$ , then the construction in the proof shows that  $g(X)$  has its coefficients in  $\mathcal{O}_{F(\alpha)}$ .



**Lemma 2.3.3.** Let  $F \subset \bar{\Omega}$  be a complete field containing  $K$  and let  $\alpha_1, \dots, \alpha_m$  be elements in  $\mathfrak{p}_\Omega$  such that

$$h(X) = \prod_{i=1}^m (X - \alpha_i) \in \mathcal{O}_F[X]. \quad (2.3.3)$$

Let  $f(X) \in \mathcal{O}_F[[X]]$  be such that  $f(\alpha_i) = 0$  for all  $1 \leq i \leq m$ . Then  $f(X)$  is divisible by  $h(X)$  in  $\mathcal{O}_F[[X]]$ .

*Proof.* Let  $F' = F(\alpha_1, \dots, \alpha_m)$ , then  $F'|F$  is a Galois extension. Moreover, by Remark 2.3.2, there exists  $g(X) \in \mathcal{O}_{F'}[[X]]$  such that  $f(X) = h(X)g(X)$ . Let us prove that  $g(X) \in \mathcal{O}_F[[X]]$ . Let  $\sigma \in \text{Gal}(F'|F)$ . We have

$$f^\sigma(X) = h^\sigma(X)g^\sigma(X). \quad (2.3.4)$$

But  $f^\sigma(X) = f(X)$  and  $h^\sigma(X) = h(X)$ . Therefore,  $f(X) = h(X)g(X) = h(X)g^\sigma(X)$  so that  $g^\sigma(X) = g(X)$ . This completes the proof.  $\square$

As in the classical case of Lubin-Tate formal groups, we have

**Proposition 2.3.4.** For every unit  $u$  of  $K$ , we have

$$\Phi_K(u)(\omega) = \rho_{u^{-1}}(\omega) \quad (2.3.5)$$

for all  $\omega \in W_\rho$ .

*Proof.* Let  $f(X) = \pi X + X^q$ . As explained at the end of Section 1.3, Lubin-Tate theory shows that, for each  $a \in \mathcal{O}_K$ , there exists a unique power series  $[a]_f \in \mathcal{O}_K[[X]]$  satisfying

$$[a]_f(X) \equiv aX \pmod{\text{deg } 2} \text{ and } f \circ [a]_f = [a]_f \circ f. \quad (2.3.6)$$

Clearly, we have  $f(X) = [\pi]_f(X)$ . Let  $d \in \mathbb{N}$  be such that  $[H : K] = dm_0$ . Then

$$\rho_{\eta^d}(X) \equiv \eta^d X, \quad [\pi^{dm_0}]_f(X) \equiv \pi^{dm_0} X \pmod{\text{deg } 2}, \quad (2.3.7)$$

and

$$\rho_{\eta^d}(X) \equiv [\pi^{dm_0}]_f(X) \equiv X^{q^{dm_0}} \pmod{\mathfrak{p}_H}. \quad (2.3.8)$$

Therefore, Proposition 1.3.17 and Proposition 1.3.16 imply that there exists a unit  $u_0$  in  $\bar{K}_{ur}$  such that  $u_0^{\phi^{dm_0}-1} = \frac{\eta^d}{\pi^{dm_0}}$ , and a unique power series  $\theta \in \mathcal{O}_{\bar{K}_{ur}}[[X]]$  such that

$$\theta(X) \equiv u_0 X \pmod{\text{deg } 2} \text{ and } \rho_{\eta^d} \circ \theta = \theta^{\phi^{dm_0}} \circ [\pi^{dm_0}]_f. \quad (2.3.9)$$

We deduce that for all  $m \geq 1$ , we have

$$\rho_{\eta^{md}} \circ \theta = \theta^{\phi^{dmm_0}} \circ [\pi^{dmm_0}]_f \quad (2.3.10)$$

and therefore we have an isomorphism of  $\mathbb{F}_q$ -vector spaces

$$\theta : T_f \longrightarrow W_\rho. \quad (2.3.11)$$

Now let  $u$  be a unit of  $K$  and consider the automorphism  $\Phi_K(u) \in \text{Gal}(K^{ab}|K_{ur})$ . By the properties of the residue map (see for instance [18, Chapter 6]), we have

$$\Phi_K(u)(\omega') = [u^{-1}]_f(\omega') \quad \forall \omega' \in T_f. \quad (2.3.12)$$

However, since  $\Phi_K(u)|_{H_\rho} \in \text{Gal}(H_\rho|H)$ , then by Proposition 1.3.11, there exists a unit  $v \in K$  such that

$$\Phi_K(u)(\omega) = \rho_{v^{-1}}(\omega), \quad \forall \omega \in W_\rho. \quad (2.3.13)$$

Let  $\omega' \in T_f$  and  $\omega = \theta(\omega') \in W_\rho$ , then  $\rho_{v^{-1}} \circ \theta(\omega') = \Phi_K(u)(\theta(\omega'))$ . However,  $\Phi_K(u)$  is an automorphism of  $K^{ab} = K_{ur}(W_\rho)$  over  $K_{ur}$ . Hence, we can extend it to an automorphism of  $\bar{K}_{ur}(W_\rho)$  over  $\bar{K}_{ur}$  so that

$$\begin{aligned} \Phi_K(u)(\theta(\omega')) &= \theta(\Phi_K(u)(\omega')) \\ &= \theta \circ [u^{-1}]_f(\omega'). \end{aligned}$$

Therefore we have  $\rho_{v^{-1}} \circ \theta(\omega') = \theta \circ [u^{-1}]_f(\omega')$  for all  $\omega' \in T_f$ . Let  $h = \rho_{v^{-1}} \circ \theta - \theta \circ [u^{-1}]_f$ , then we have  $h(T_f^m) = 0$  for all  $m \in \mathbb{N}$ , where  $T_f^m$  is the set of roots of  $[\pi^m]_f$ . Thus by Lemma 2.3.3,  $h(X)$  is divisible by  $f^m(X) = \prod_{\omega' \in T_f^m} (X - \omega')$  in  $\mathcal{O}_{\bar{K}_{ur}}[[X]]$  for all  $m$ . However, since  $f(X)$  is contained in the ideal  $(\pi, X)$  of  $\mathcal{O}_{\bar{K}_{ur}}[[X]]$ , we have  $f^m(X) \in (\pi, X)^m$  and hence  $h(X) \in (\pi, X)^m$  for all  $m \in \mathbb{N}$ . We conclude that  $h(X) = 0$  and thus

$$\rho_{v^{-1}} \circ \theta = \theta \circ [u^{-1}]_f. \quad (2.3.14)$$

We deduce by identification that  $u = v$ . This concludes the proof.  $\square$

**Lemma 2.3.5.** Let  $\alpha \in \mathfrak{p}_L$ . For  $m \geq n$ , we set  $\alpha_m = \rho_{\eta^{m-n}}(\alpha)$  and  $b_m = \alpha_m v_m^{-1}$ . Then, there exists an integer  $N(\rho, \alpha) \in \mathbb{N}$  such that for all  $m \geq N(\rho, \alpha)$ , we have

$$(\alpha, N_{m,n}(1 + b_m))_{L,n} = 0 \quad (2.3.15)$$

and

$$N_m(1 + b_m)^{-1} \equiv 1 - T_m(b_m) \pmod{\mathfrak{p}_K^{2mm_0}}, \quad (2.3.16)$$

where  $T_m$  and  $N_m$  denote respectively the trace and the norm of the extension  $E_\rho^m|K$  and  $N_{m,n}$  denotes the norm of the extension  $E_\rho^m|E_\rho^n$ .

*Proof.* We first prove (2.3.15). Let  $m \geq n$ . By Lemma 2.2.3, there exists a constant  $c$  depending only on  $n$  such that  $\mu(b_m) \geq mm_0 - c$  (see Remark 2.2.4). Thus  $1 + b_m$  tends to 1 as  $m$  tends to  $\infty$ . Moreover,

$$N_{m,n}(1 + b_m) = \prod (1 + \sigma(b_m)) = 1 + y, \quad (2.3.17)$$

where  $\sigma$  varies among the automorphisms in  $\text{Gal}(E_\rho^m|E_\rho^n)$  and  $\mu(y) \geq \mu(b_m)$ . Thus,  $N_{m,n}(1 + b_m)$  also tends to 1 as  $m$  tends to  $\infty$ . Furthermore,

$$(\alpha, N_{m,n}(1 + b_m))_{L,n} = \Phi_L(N_{m,n}(1 + b_m))(\xi) - \xi, \quad (2.3.18)$$

where  $\rho_{\eta^n}(\xi) = \alpha$ . But  $\Phi_L$  is continuous. Hence, for the neighborhood  $\text{Gal}(L^{ab}|L(\xi))$  of  $\Phi_L(1)$ , there exists  $N_1 \in \mathbb{N}$  such that if  $m \geq N_1$ , then  $\Phi_L(N_{m,n}(1+b_m)) \in \text{Gal}(L^{ab}|L(\xi))$ . Thus, for all  $m \geq N_1$ , we have (2.3.15). Now let us prove (2.3.16). Let  $k \leq m$  be an integer. Let  $x = T_{m,m-k}(b_m)$ , then it is easy to check that  $N_{m,m-k}(1+b_m)^{-1} = 1 - x + y$ , where  $\mu(y) \geq 2\mu(b_m)$ . Therefore, we have

$$N_m(1+b_m)^{-1} = N_{m-k}(1-x+y) = 1 - T_{m-k}(x-y) + z, \quad (2.3.19)$$

$\mu(z) \geq \mu(x-y)$ . If  $k$  and  $m$  are such that  $km_0 \geq c+1$  and  $mm_0 \geq km_0 + 2c + \frac{1}{(q-1)}$ , then, by Lemma 1.3.15 we get  $\mu(T_{m-k}(x-y)) \geq 2mm_0$  and  $\mu(z) \geq \mu(x-y) \geq 2mm_0$ . Thus, (2.3.16) follows. Finally, we set  $N(\rho, \alpha) = \max\{N_1, \lfloor k + \frac{2c}{m_0} + \frac{1}{m_0(q-1)} \rfloor + 1\}$ .  $\square$

**Remark 2.3.6.** Let  $\alpha \in \mathfrak{p}_L$  and let  $\rho'$  be a formal Drinfeld  $\mathcal{O}_K$ -module isomorphic to  $\rho$ , i.e there exists a power series  $t$  invertible in  $\mathcal{O}_H\{\{\tau\}\}$  such that  $\rho'_a = t^{-1} \circ \rho_a \circ t$  for all  $a \in \mathcal{O}_K$ . It is easy to prove that  $E_\rho^m = E_{\rho'}^m$  for all  $m \geq 0$ . Moreover, by Proposition 2.2.2 (vi) we have  $N(\rho, \alpha) = N(\rho', t^{-1}(\alpha))$ .

**Lemma 2.3.7.** Let  $\alpha \in \mathfrak{p}_L$  and suppose that there exists  $m \geq \max\{N(\rho, \alpha), \frac{q}{q-1}(2n + \frac{1}{2m_0})\}$  such that  $(x, x)_{E_\rho^m, m} = 0$  for all  $x \in \mathfrak{p}_{E_\rho^m} \setminus \{0\}$ , where  $N(\rho, \alpha)$  is defined in Lemma 2.3.5. Then, there exists a prime  $\pi_n$  of  $L$  such that

$$(\alpha, \pi_n)_{L,n} = [\alpha, \pi_n]_{L,n} = \frac{1}{\eta^n} T_{L|K}(\lambda_\rho(\alpha)\delta_{v_n}(\pi_n)) \cdot_\rho v_n. \quad (2.3.20)$$

*Proof.* We prove the Lemma following the steps of [6, Proposition 23], which were essentially used by Wiles [34, Lemma 8]. Let  $\alpha \in \mathfrak{p}_L$ .

**Step 1:** For  $m \geq n$ , let  $\alpha_m = \rho_{\eta^{m-n}}(\alpha)$  and  $b_m = \alpha_m v_m^{-1}$ . If we suppose  $(x, x)_{E_\rho^m, m} = 0$  for all  $x \in \mathfrak{p}_{E_\rho^m} \setminus \{0\}$ , we have

$$0 = (\alpha_m + v_m, (1+b_m)v_m)_{E_\rho^m, m} = (\alpha_m, v_m)_{E_\rho^m, m} + (\alpha_m, 1+b_m)_{E_\rho^m, m} + (v_m, 1+b_m)_{E_\rho^m, m},$$

because  $\alpha_m + v_m = (1+b_m)v_m$ .

**Step 2:** For  $m \geq N(\rho, \alpha)$ , we have  $(\alpha_m, 1+b_m)_{E_\rho^m, m} = (\alpha, N_{m,n}(1+b_m))_{L,n} = 0$  by Lemma 2.3.5.

**Step 3:** Let  $m \geq N(\rho, \alpha)$  so that  $(\alpha_m, 1+b_m)_{E_\rho^m, m} = 0$  and suppose that  $(x, x)_{E_\rho^m, m} = 0$  for all  $x \in \mathfrak{p}_{E_\rho^m} \setminus \{0\}$ . Let  $\pi_n = N_{m,n}(v_m)$ , then  $\pi_n$  is a prime of  $L$  because  $E_\rho^m|L$  is a totally ramified extension. Let  $v_{2m}$  be a generator of  $W_\rho^{2m}$  such that  $\rho_{\eta^m}(v_{2m}) = v_m$ . We have

$$(\alpha, \pi_n)_{L,n} = v_{2m} - \rho_{N_m(1+b_m)^{-1}}(v_{2m}).$$

Indeed,

$$\begin{aligned} (\alpha, \pi_n)_{L,n} &= (\alpha_m, v_m)_{E_\rho^m, m} = -(v_m, 1+b_m)_{E_\rho^m, m} && \text{(by Step 1 and 2)} \\ &= -(\Phi_{E_\rho^m}(1+b_m)(v_{2m}) - v_{2m}) \\ &= -(\Phi_K(N_m(1+b_m))(v_{2m}) - v_{2m}). \end{aligned}$$

By Proposition 2.3.4 we have  $\Phi_K(N_m(1+b_m))(v_{2m}) = \rho_{N_m(1+b_m)^{-1}}(v_{2m})$  and hence  $(\alpha, \pi_n)_n = v_{2m} - \rho_{N_m(1+b_m)^{-1}}(v_{2m})$ .

**Step 4:** For  $m \geq N(\rho, \alpha)$ , we have  $N_m(1+b_m)^{-1} \equiv 1 - T_m(b_m) \pmod{\mathfrak{p}_K^{2mm_0}}$  by Lemma 2.3.5.

**Step 5:** Choose  $m \geq \max\{N(\rho, \alpha), \frac{q}{q-1}(2n + \frac{1}{2m_0})\}$ , then  $m$  is sufficiently large to satisfy Step 2 and Step 4. If in addition we have  $(x, x)_{E_\rho^m, m} = 0$  for all  $x \in \mathfrak{p}_{E_\rho^m} \setminus \{0\}$ , then  $(\alpha, \pi_n)_{L, n} = [\alpha, \pi_n]_{L, n}$ , where  $\pi_n = N_{m, n}(v_m)$  as in Step 3. Indeed, by the previous steps we get

$$(\alpha, \pi_n)_{L, n} = T_m(\alpha_m v_m^{-1}) \cdot_\rho v_{2m} = \frac{1}{\eta^m} T_m(\alpha_m v_m^{-1}) \cdot_\rho v_m. \quad (2.3.21)$$

We draw the attention of the reader to the fact that  $m$  is sufficiently large so that  $\mu(\alpha_m) \geq \frac{mm_0}{q} + \frac{1}{q-1} + \frac{1}{q^{mm_0}(q-1)}$ . This is a consequence of Lemma 2.2.3 and remark 2.2.4. This implies that  $\frac{1}{\eta^m} T_m(\alpha_m v_m^{-1}) \in \mathcal{O}_K$ . Moreover, by Lemma 2.2.12 and Lemma 2.2.13, we get

$$\begin{aligned} (\alpha, \pi_n)_{L, n} &= \frac{1}{\eta^m} T_m(\alpha_m v_m^{-1}) \cdot_\rho v_m \\ &= [\alpha_m, v_m]_{E_\rho^m, m} && \text{(Lemma 2.2.12)} \\ &= [\alpha, \pi_n]_{L, n}. && \text{(Proposition 2.2.14)} \end{aligned}$$

□

**Remark 2.3.8.** If  $\rho_\eta(X)$  is a polynomial (as in [34, 3, 6]), the condition  $(x, x)_{\rho, E_\rho^m, m} = 0$  is satisfied for all  $m \geq 1$ , and following the same steps as in the proof of Lemma 2.3.7, one can prove that

$$(\alpha, v_n)_{L, n} = \frac{1}{\eta^n} T_{L|K}(\lambda_\rho(\alpha) \frac{1}{v_n}) \cdot_\rho v_n \quad (2.3.22)$$

for all  $\alpha \in \mathfrak{p}_L$ .

**Lemma 2.3.9.** Suppose  $\rho$  is such that  $(x, x)_{L, n} = 0$  for all  $x \in \mathfrak{p}_L \setminus \{0\}$ . Let  $\alpha \in \mathfrak{p}_L$  such that  $\mu(\alpha) \geq \frac{nm_0}{q} + \frac{1}{q-1} + \frac{1}{q^{nm_0}(q-1)}$  and  $\beta$  a unit in  $L^\times$ . Then,

$$(\alpha, \beta)_{L, n} = [\alpha, \beta]_{L, n} = \frac{1}{\eta^n} T_{L|K}(\lambda_\rho(\alpha) \delta_{v_n}(\beta)) \cdot_\rho v_n. \quad (2.3.23)$$

*Proof.* We first notice that a unit  $\beta \in L$  is of the form  $\zeta u_1$ , where  $\zeta$  is a  $(q-1)^{\text{th}}$  root of unity and  $u_1$  is a principle unit in  $L$ . It is obvious that both sides of (2.3.23) are zero for  $\beta = \zeta$ . Hence, it is sufficient to prove the Lemma for the principal units  $\beta = 1 - \zeta \pi_L^j$ , where  $\pi_L$  is a prime of  $L$ ,  $\zeta$  is any  $(q-1)^{\text{th}}$  root of unity, and  $j$  is any integer greater than 1. This goes back to the structure of the principal units as a  $\mathbb{Z}_p$ -module and to the continuity of the pairings. By Lemma 2.2.7, we have

$$(\alpha, 1 - \zeta \pi_L^j)_{\rho, L, n} = \left( \frac{\zeta \pi_L^j}{1 - \zeta \pi_L^j} \alpha, (\zeta \pi_L^j)^{-1} \right)_{\rho, L, n} \quad (2.3.24)$$

$$= -j \left( \frac{\zeta \pi_L^j}{1 - \zeta \pi_L^j} \alpha, \pi_L \right)_{\rho, L, n}. \quad (2.3.25)$$

Let  $m \geq \max\{N(\rho, \frac{\zeta\pi_L^j}{1-\zeta\pi_L^j}\alpha), \frac{q}{q-1}(2n + \frac{1}{2m_0})\}$  and let  $r_m \in \mathcal{O}_H\{\{\tau\}\}$  be the invertible power series defined in Proposition 2.2.5. Let  $\rho'$  be the formal Drinfeld module defined by  $\rho'_a = r_m \circ \rho_a \circ r_m^{-1}$  for all  $a \in \mathcal{O}_K$ . Hence, by Proposition 2.2.2 (vi), we have

$$\left(\frac{\zeta\pi_L^j}{1-\zeta\pi_L^j}\alpha, \pi_L\right)_{\rho, L, n} = r_m^{-1}\left(\left(r_m\left(\frac{\zeta\pi_L^j}{1-\zeta\pi_L^j}\alpha\right), \pi_L\right)_{\rho', L, n}\right). \quad (2.3.26)$$

Moreover, by Remark 2.3.6, we have  $N(\rho, \frac{\zeta\pi_L^j}{1-\zeta\pi_L^j}\alpha) = N(\rho', r_m(\frac{\zeta\pi_L^j}{1-\zeta\pi_L^j}\alpha))$ . Hence, since  $\rho'$  satisfies  $(x, x)_{\rho', E_{\rho', m}} = 0$ , then by Lemma 2.3.7, there exists a prime  $\pi_n$  of  $L$  such that

$$\left(r_m\left(\frac{\zeta\pi_n^j}{1-\zeta\pi_n^j}\alpha\right), \pi_n\right)_{\rho', L, n} = \left[r_m\left(\frac{\zeta\pi_n^j}{1-\zeta\pi_n^j}\alpha\right), \pi_n\right]_{\rho', L, n}. \quad (2.3.27)$$

Hence, if we put  $\pi_L = \pi_n$ , we get

$$(\alpha, 1 - \zeta\pi_n^j)_{\rho, L, n} = -jr_m^{-1}\left(\left[r_m\left(\frac{\zeta\pi_n^j}{1-\zeta\pi_n^j}\alpha\right), \pi_n\right]_{\rho', L, n}\right). \quad (2.3.28)$$

By Proposition 2.2.11 (ii), (2.3.28) is equal to

$$-j\left[\frac{\zeta\pi_n^j}{1-\zeta\pi_n^j}\alpha, \pi_n\right]_{\rho, L, n} = \frac{-j}{\eta^n} \mathbb{T}_{L|K}\left(\frac{\zeta\pi_n^j}{1-\zeta\pi_n^j} \times \alpha \times \delta_{v_n}(\pi_n)\right) \cdot_{\rho} v_n \quad (2.3.29)$$

$$= \frac{1}{\eta^n} \mathbb{T}_{L|K}\left(\frac{-j\zeta\pi_n^j}{1-\zeta\pi_n^j} \times \alpha \times \frac{t'(v_n)}{\pi_n}\right) \cdot_{\rho} v_n, \quad (2.3.30)$$

where (2.3.29) is deduced from Lemma 2.2.12, and  $t(X) \in \mathcal{O}_H((X))$  satisfies  $t(v_n) = \pi_n$ . Since  $1 - \zeta(t(v_n))^j = 1 - \zeta\pi_n^j$ , we have

$$\delta_{v_n}(1 - \zeta\pi_n^j) = \frac{-j\zeta\pi_n^{j-1}t'(v_n)}{1 - \zeta\pi_n^j}, \quad (2.3.31)$$

and thus, (2.3.30) is equal to  $\frac{1}{\eta^n} \mathbb{T}_{L|K}(\alpha\delta_{v_n}(1-\zeta\pi_n^j)) \cdot_{\rho} v_n$  which is equal to  $\frac{1}{\eta^n} \mathbb{T}_{L|K}(\lambda_{\rho}(\alpha)\delta_{v_n}(1-\zeta\pi_n^j)) \cdot_{\rho} v_n$  by Lemma 2.2.12. Hence,

$$(\alpha, 1 - \zeta\pi_n^j)_{\rho, L, n} = [\alpha, 1 - \zeta\pi_n^j]_{\rho, L, n}. \quad (2.3.32)$$

□

**Proposition 2.3.10.** Let  $\alpha \in \mathfrak{p}_L$  such that  $\mu(\alpha) \geq \frac{nm_0}{q} + \frac{1}{q-1} + \frac{1}{q^{nm_0(q-1)}}$  and  $\beta$  a unit in  $L^{\times}$ . Then,

$$(\alpha, \beta)_{L, n} = [\alpha, \beta]_{L, n} = \frac{1}{\eta^n} \mathbb{T}_{L|K}(\lambda_{\rho}(\alpha)\delta_{v_n}(\beta)) \cdot_{\rho} v_n. \quad (2.3.33)$$

*Proof.* By Proposition 2.2.5, there exists an invertible power series  $r_n \in \mathcal{O}_H\{\{\tau\}\}$  such that  $\prod_{\omega \in W_\rho^n} (X - \omega) = r_n \circ \rho_{\eta^n}(X)$ . Let  $\rho'$  be the formal Drinfeld module defined by  $\rho'_a = r_n \circ \rho_a \circ r_n^{-1}$  for all  $a \in \mathcal{O}_K$ . Then, by Lemma 2.2.6 we have  $(x, x)_{\rho', E_\rho^n, n} = 0$ . Hence, by Lemma 2.3.9 applied for  $\rho'$ , we have

$$(\alpha, \beta)_{\rho, L, n} = r^{-1}((r(\alpha), \beta)_{\rho', L, n}) = r^{-1}([r(\alpha), \beta]_{\rho', L, n}) = [\alpha, \beta]_{\rho, L, n}. \quad (2.3.34)$$

□

Proposition 2.3.10 gives a generalization of Theorem 3.11 of Anglès [3], where he proved this result for formal Drinfeld modules obtained from Carlitz polynomials (See Example 1.1.1).

**Proposition 2.3.11.** Let  $\alpha \in \mathfrak{p}_L$  be such that  $\mu(\alpha) \geq \frac{nm_0}{q} + \frac{1}{q-1} + \frac{1}{q^{nm_0}(q-1)}$  and let  $\beta$  be a prime of  $L$ . Then

$$(\alpha, \beta)_{L, n} = [\alpha, \beta]_{L, n} = \frac{1}{\eta^n} T_{L|K}(\lambda_\rho(\alpha)\delta_{v_n}(\beta)) \cdot_\rho v_n. \quad (2.3.35)$$

*Proof.* Let  $m \geq \max\{N(\rho, \alpha), \frac{q}{q-1}(2n + \frac{1}{2m_0})\}$  and let  $r_m \in \mathcal{O}_H\{\{\tau\}\}$  be the invertible power series defined in Proposition 2.2.5. Let  $\rho'$  be the formal Drinfeld module defined by  $\rho'_a = r_m \circ \rho_a \circ r_m^{-1}$  for all  $a \in \mathcal{O}_K$ . Thus by Lemma 2.2.6, we have  $(x, x)_{\rho', E_\rho^m, m} = 0$ . Hence, by Lemma 2.3.7, there exists a prime  $\pi_n$  of  $L$  satisfying  $(r_m(\alpha), \pi_n)_{\rho', L, n} = [r_m(\alpha), \pi_n]_{\rho', L, n}$ . Then we can write  $\beta = u\pi_n$  for a unit  $u \in L$ . Hence,

$$(\alpha, \beta)_{\rho, L, n} = (\alpha, u\pi_n)_{\rho, L, n} = (\alpha, u)_{\rho, L, n} + (\alpha, \pi_n)_{\rho, L, n}. \quad (2.3.36)$$

By Proposition 2.3.10, we have  $(\alpha, u)_{\rho, L, n} = [\alpha, u]_{\rho, L, n}$ . On the other hand, by Proposition 2.2.2 (vi), we have

$$(\alpha, \pi_n)_{\rho, L, n} = r_m^{-1}((r_m(\alpha), \pi_n)_{\rho', L, n}) = r_m^{-1}([r_m(\alpha), \pi_n]_{\rho', L, n}), \quad (2.3.37)$$

the last equality being deduced from Remark 2.3.6 and Lemma 2.3.7. Hence, by Proposition 2.2.11 (ii), we have

$$(\alpha, \beta)_{\rho, L, n} = [\alpha, u]_{\rho, L, n} + [\alpha, \pi_n]_{\rho, L, n} = [\alpha, \beta]_{\rho, L, n}. \quad (2.3.38)$$

□

Combining Proposition 2.3.10 and Proposition 2.3.11, we obtain our main results.

**Theorem 2.3.12.** Let  $\alpha \in \mathfrak{p}_L$  such that  $\mu(\alpha) \geq \frac{nm_0}{q} + \frac{1}{q-1} + \frac{1}{q^{nm_0}(q-1)}$  and  $\beta \in L^\times$ . We have

$$(\alpha, \beta)_{\rho, L, n} = [\alpha, \beta]_{\rho, L, n} = \frac{1}{\eta^n} T_{E_\rho^n|K}(\lambda_\rho(\alpha)\delta_{v_n}(\beta)) \cdot_\rho v_n. \quad (2.3.39)$$

**Theorem 2.3.13.** Let  $\alpha \in \mathfrak{p}_L$  and  $\beta \in N_{m,n}(E_\rho^m)$  for  $m \geq \frac{q}{q-1}(2n + \frac{1}{2m_0})$ . We have

$$(\alpha, \beta)_{\rho, L, n} = [\alpha, \beta]_{\rho, L, n} = \frac{1}{\eta^n} T_{E_\rho^n | K}(\lambda_\rho(\alpha) \delta_{v_n}(\beta)) \cdot_\rho v_n. \quad (2.3.40)$$

*Proof.* For  $m \geq \frac{q}{q-1}(2n + \frac{1}{2m_0})$ , Lemma 2.2.3 and remark 2.2.4 imply that the element  $\alpha_m = \rho_{\eta^{m-n}}(\alpha)$  is of valuation  $\mu(\alpha_m) \geq \frac{mm_0}{q} + \frac{1}{q-1} + \frac{1}{q^{mm_0(q-1)}}$ . Let  $\beta' \in E_\rho^m$  be an element whose norm to  $E_\rho^n$  is  $\beta$ . Therefore,

$$(\alpha, \beta)_{\rho, L, n} = (\alpha_m, \beta')_{\rho, E_\rho^m, m} = [\alpha_m, \beta']_{\rho, E_\rho^m, m} = [\alpha, \beta]_{\rho, L, n}$$

Here, the second equality is a consequence of Theorem 2.3.12 and the last equality is deduced from Lemma 2.2.13.  $\square$

Theorem 2.3.13 gives an analogue of Theorem 19 of Wiles proved for the case of Lubin-Tate formal groups, although the condition on  $m$  is slightly lighter in his case ( $m \geq 2n + 1$ ). It also implies Theorem 3.12 of Anglès in the case of formal Drinfeld modules obtained from Carlitz polynomials for  $k, m \geq \frac{q}{q-1}(2n + \frac{1}{2m_0})$ .

## 2.4 The limit Kummer pairing

Several works in the literature, such as Iwasawa's book [18] or Longhi-Bars' paper [6], considered the limit form of the Kummer pairing. To define it, consider the projective limit  $\varprojlim (E_\rho^n)^\times$  with respect to the norm maps

$$\begin{aligned} E_\rho^m &\rightarrow E_\rho^n \\ \beta_m &\mapsto \beta_n = N_{m,n}(\beta_m) \end{aligned}$$

for  $m \geq n$ , and the direct limit  $\varinjlim \mathfrak{p}_{E_\rho^n}$  with respect to the maps

$$\begin{aligned} \mathfrak{p}_{E_\rho^n} &\rightarrow \mathfrak{p}_{E_\rho^m} \\ \alpha_n &\mapsto \alpha_m = \rho_{\eta^{m-n}}(\alpha_n). \end{aligned}$$

For  $m \geq n$ , we have

$$(\alpha_m, \beta_m)_{\rho, E_\rho^m, m} = \Phi_{E_\rho^m}(\beta_m)(\xi) - \xi = \Phi_{E_\rho^n}(N_{m,n}(\beta_m))(\xi) - \xi = (\alpha_n, \beta_n)_{\rho, E_\rho^n, n}, \quad (2.4.1)$$

where  $\xi$  is a root of  $\rho_{\eta^n}(X) = \alpha_n$ , and thus, a root of  $\rho_{\eta^m}(X) = \alpha_m$ . Hence, we can define a limit form of  $(\ , \ )_{\rho, L, n}$  as follows

$$(\alpha, \beta)_\rho = (\alpha_n, \beta_n)_{\rho, E_\rho^n, n} \quad (2.4.2)$$

for all sufficiently large  $n$ , where  $\alpha = (\alpha_n)_n \in \varinjlim \mathfrak{p}_{E_\rho^n}$  and  $\beta = (\beta_n)_n \in \varprojlim (E_\rho^n)^\times$ . We deduce from Theorem 2.3.13 that, if we suppose  $\rho$  is such that

$$\rho_\eta \equiv \tau^{m_0} \pmod{\mathfrak{p}_H}, \quad (2.4.3)$$

then for all sufficiently large  $n$ , we have

$$(\alpha, \beta)_\rho = \frac{1}{\eta^n} T_{L|K}(\lambda_\rho(\alpha_n) \delta_{v_n}(\beta_n)) \cdot_\rho v_n. \quad (2.4.4)$$

This gives a generalization of Theorem 23 of Longhi-Bars [6] proved for formal Drinfeld modules obtained from sign-normalized rank 1 Drinfeld modules.

We also note that in the case  $m_0 = [H : K]$ , there exists a limit form of  $\delta_{v_n}$  defined for  $\beta = (\beta_n)_n \in \varprojlim (E_\rho^n)^\times$  by

$$\delta_{v_n}(\beta) = \frac{(f^{\varphi^{-n}})'(v_n)}{\beta_n} \pmod{\mathcal{D}_n}, \quad (2.4.5)$$

where  $f$  is the power series assigned to  $\beta$  be the isomorphism of Theorem 1.4.7.



# Chapter 3

## Explicit reciprocity laws in the general case

After proving explicit laws in a special class of formal Drinfeld modules, we now give an explicit description of the Kummer pairing *à la Kolyvagin* [21] in a more general setting (see Theorem 3.2.10). We keep the same notations as in Chapter 2. Let  $K$  be a local field of positive characteristic  $p$  and let  $q$  the order of its residue field  $\mathcal{O}_K/\mathfrak{p}_K$ . Let  $\Omega$  be an algebraic closure of  $K$ , and still denote  $\mu$  the unique extension of  $\mu$  to  $\Omega$ . Let  $H \subset \Omega$  be a finite unramified extension of  $K$ , and let  $\rho \in \hat{D}_{\mathcal{O}_K}(\mathcal{O}_H)$  having stable reduction of height 1. Let  $m_0$  be a positive integer dividing  $[H : K]$  and let  $\eta \in K$  be an element of valuation  $\mu(\eta) = m_0$ .

### 3.1 The Iwasawa map

In this section, we will study the so-called Iwasawa map, first introduced by Iwasawa in [17, Proposition 14] in the cyclotomic case. This map was generalized by Wiles [34, Proposition 7] in the case of Lubin-Tate formal groups, and by Kolyvagin [21, Proposition 3.2] in the case of formal groups of finite height. We fix a positive integer  $n$  and a finite separable extension  $L$  of  $E_\rho^n$ . We also fix a generator  $v_n$  of the  $\mathcal{O}_K$ -module  $W_\rho^n$ .

Since the extension  $L|K$  is supposed to be separable, the bilinear map  $\langle \cdot, \cdot \rangle_L : L \times L \rightarrow K$  defined by  $\langle x, y \rangle_L = \text{Tr}_{L|K}(xy)$  is non degenerate. This gives us the classical isomorphism from  $L$  to the space of  $K$ -linear forms from  $L$  to  $K$ . The pairing  $\langle \cdot, \cdot \rangle_L$  also induces the following  $\mathcal{O}_K$ -linear map

$$L \longrightarrow \text{Hom}_{\mathcal{O}_K}(\lambda_\rho(\mathfrak{p}_L), K/\mathcal{O}_K) \tag{3.1.1}$$

$$y \mapsto \begin{cases} \lambda_\rho(\mathfrak{p}_L) \longrightarrow & K/\mathcal{O}_K \\ x \longmapsto & \langle x, y \rangle_L \pmod{\mathcal{O}_K} \end{cases}$$

**Lemma 3.1.1.** The map (3.1.1) is a surjective homomorphism of  $\mathcal{O}_K$ -modules, with kernel

$$\mathfrak{X}_L := \{y \in L; \langle x, y \rangle_L \in \mathcal{O}_K \ \forall x \in \lambda_\rho(\mathfrak{p}_L)\}. \tag{3.1.2}$$

*Proof.* It is clear that  $\mathfrak{X}_L$  is the kernel of the map (3.1.1). Let us prove that the map is surjective. To do so, let  $\gamma : \lambda_\rho(\mathfrak{p}_L) \rightarrow K/\mathcal{O}_K$  be an  $\mathcal{O}_K$ -linear map. The  $\mathcal{O}_K$ -module  $K/\mathcal{O}_K$  is divisible, hence it is an injective module by [24, Lemma 4.2]. Therefore there exists a homomorphism of  $\mathcal{O}_K$ -modules  $\tilde{\gamma}$  such that the following diagram

$$\begin{array}{ccc} L & & \\ \uparrow & \searrow \tilde{\gamma} & \\ \lambda_\rho(\mathfrak{p}_L) & \xrightarrow{\gamma} & K/\mathcal{O}_K \end{array}$$

is commutative, the left hand map being the inclusion map. Let  $\{e_1, \dots, e_d\}$  be a basis of  $L$  as a  $K$ -vector space. Since  $L = K\lambda_\rho(\mathfrak{p}_L)$  by Lemma 1.2.5, we can choose the  $e_i$  to be in  $\lambda_\rho(\mathfrak{p}_L)$ . Choose elements  $\tilde{\gamma}(e_i)$  in  $K$  such that  $\tilde{\gamma}(e_i)$  is the class of  $\tilde{\gamma}(e_i)$  modulo  $\mathcal{O}_K$ . Define the  $K$ -linear map  $\tilde{\gamma} : L \rightarrow K$  by  $\tilde{\gamma}(\sum a_i e_i) = \sum a_i \tilde{\gamma}(e_i)$  where  $a_i \in K$ . Thus we obtain the following commutative diagram

$$\begin{array}{ccc} L & \xrightarrow{\tilde{\gamma}} & K \\ \uparrow & \searrow \tilde{\gamma} & \downarrow \\ \lambda_\rho(\mathfrak{p}_L) & \xrightarrow{\gamma} & K/\mathcal{O}_K \end{array}$$

the right hand arrow being the canonical projection and the left hand arrow being the inclusion. However, the  $K$ -linear form  $\tilde{\gamma}$  is induced by some element  $y \in L$  satisfying  $\tilde{\gamma}(x) = \text{T}_{L|K}(xy)$  for all  $x \in \lambda_\rho(\mathfrak{p}_L)$ . Therefore we have  $\gamma(x) \equiv \tilde{\gamma}(x) = \langle x, y \rangle_L \pmod{\mathcal{O}_K}$ .  $\square$

Now, we give the construction of the so-called Iwasawa map. As mentioned in (1.3.7), the map

$$\begin{aligned} \mathcal{O}_K/\eta^n \mathcal{O}_K &\longrightarrow W_\rho^n & (3.1.3) \\ a &\longmapsto \rho_a(v_n) \end{aligned}$$

is an isomorphism of  $\mathcal{O}_K$ -modules. We denote by  $\iota_1$  its inverse. We define the  $\mathcal{O}_K$ -linear map

$$\begin{aligned} \iota : W_\rho^n &\xrightarrow{\iota_1} \mathcal{O}_K/\eta^n \mathcal{O}_K \longrightarrow K/\mathcal{O}_K \\ \rho_a(v_n) &\longmapsto a \longmapsto \frac{a}{\eta^n} \end{aligned}$$

Let

$$L^n = \{\beta \in L^\times; (\alpha, \beta)_{L,n} = 0 \ \forall \alpha \in W_L\}, \quad (3.1.4)$$

where  $W_L = L \cap W_\rho \subset \mathfrak{p}_L$ . Any  $\beta \in L^n$  defines an  $\mathcal{O}_K$ -linear map

$$h_\beta : \begin{cases} \mathfrak{p}_L/W_L & \longrightarrow K/\mathcal{O}_K \\ \alpha & \longmapsto \iota((\alpha, \beta)_{L,n}) \end{cases} \quad (3.1.5)$$

where the action of  $\mathcal{O}_K$  on  $\mathfrak{p}_L/W_L$  is given by (1.3.1). The map  $\beta \mapsto h_\beta$  gives a group homomorphism from  $L^n$  to  $\text{Hom}_{\mathcal{O}_K}(\mathfrak{p}_L/W_L, K/\mathcal{O}_K)$ . The homomorphism of Lemma 1.2.6 induces the following isomorphism of  $\mathcal{O}_K$ -modules

$$\text{Hom}_{\mathcal{O}_K}(\mathfrak{p}_L/W_L, K/\mathcal{O}_K) \cong \text{Hom}_{\mathcal{O}_K}(\lambda_\rho(\mathfrak{p}_L), K/\mathcal{O}_K). \quad (3.1.6)$$

Let  $\beta \in L^n$  and let  $g_\beta$  be the image of  $h_\beta$  by the isomorphism (3.1.6). Then  $g_\beta$  is defined by  $g_\beta(\lambda_\rho(\alpha)) = \iota((\alpha, \beta)_{L,n})$ . However  $g_\beta$  is an  $\mathcal{O}_K$ -linear map from  $\lambda_\rho(\mathfrak{p}_L)$  to  $K/\mathcal{O}_K$ . Thus, by Lemma 3.1.1, there exists a unique  $y \in L/\mathfrak{X}_L$  satisfying  $g_\beta(\lambda_\rho(\alpha)) = T_{L|K}(\lambda_\rho(\alpha)y) \pmod{\mathcal{O}_K}$  for all  $\alpha \in \mathfrak{p}_L$ . It is easy to see that  $y \in \eta^{-n}\mathfrak{X}_L/\mathfrak{X}_L$ . We set

$$\psi_{L,v_n}(\beta) = \eta^n y \pmod{\eta^n \mathfrak{X}_L}. \quad (3.1.7)$$

**Proposition 3.1.2.** We have

$$(\alpha, \beta)_{L,n} = T_{L|K}(\lambda_\rho(\alpha)\psi_{L,v_n}(\beta)) \cdot_\rho v_n \quad (3.1.8)$$

for all  $\alpha \in \mathfrak{p}_L$  and  $\beta \in L^n$ . Furthermore, the map  $\psi_{L,v_n} : L^n \rightarrow \mathfrak{X}_L/\eta^n \mathfrak{X}_L$  is a continuous group homomorphism.

*Proof.* The equality (3.1.8) and the fact that  $\psi_{L,v_n}$  is a group homomorphism follow immediately from the construction. Let us prove that  $\psi_{L,v_n}$  is continuous at 1. Since the reciprocity map  $\Phi_L$  is continuous, there exists  $M > 0$  such that if  $\beta \in L^n$  satisfies  $\mu(\beta) > M$ , then  $\Phi_L(\beta)$  is trivial, and hence  $(\alpha, \beta)_{L,n} = 0$  for all  $\alpha \in \lambda_\rho(\mathfrak{p}_L)$ . Therefore, if  $\mu(\beta) > M$ , we get  $T_{L|K}(\lambda_\rho(\alpha)\psi_{L,v_n}(\beta)) \in \mathfrak{p}^{nm_0}$  for all  $\alpha \in \lambda_\rho(\mathfrak{p}_L)$ , which implies that  $\psi_{L,v_n}(\beta) \in \eta^n \mathfrak{X}_L$ . This concludes the proof.  $\square$

**Remark 3.1.3.** Let  $v'_n$  be another generator of  $W_\rho^n$ , then  $v'_n = \rho_u(v_n)$  for a unit  $u$  of  $K$ . We have

$$\psi_{L,v_n} = u\psi_{L,v'_n}. \quad (3.1.9)$$

**Proposition 3.1.4.** The map  $\psi_{L,v_n}$  satisfies the following properties.

- (i) Let  $M$  be a finite separable extension of  $L$ . Then  $N_{M|L}(M^n) \subset L^n$ ,  $T_{M|L}(\mathfrak{X}_M) \subset \mathfrak{X}_L$  and the diagram

$$\begin{array}{ccc} M^n & \xrightarrow{\psi_{M,v_n}} & \mathfrak{X}_M/\eta^n \mathfrak{X}_M \\ N_{M|L} \downarrow & & \downarrow T_{M|L} \\ L^n & \xrightarrow{\psi_{L,v_n}} & \mathfrak{X}_L/\eta^n \mathfrak{X}_L \end{array}$$

is commutative.

- (ii) Let  $M$  be a finite separable extension of  $L$ . Then  $L^n \subset M^n$ ,  $\mathfrak{X}_L \subset \mathfrak{X}_M$  and the following diagram

$$\begin{array}{ccc} L^n & \xrightarrow{\psi_{L,v_n}} & \mathfrak{X}_L/\eta^n \mathfrak{X}_L \\ \downarrow & & \downarrow \\ M^n & \xrightarrow{\psi_{M,v_n}} & \mathfrak{X}_M/\eta^n \mathfrak{X}_M \end{array}$$

is commutative.

- (iii) Suppose that  $L \supset E_\rho^m$  for  $m \geq n$  and let  $v_m$  be a generator of  $W_\rho^m$  such that  $\rho_{\eta^{m-n}}(v_m) = v_n$ . Then  $L^m \subset L^n$  and  $\psi_{L,v_n}(\beta)$  is the reduction of  $\psi_{L,v_m}(\beta)$  from  $\mathfrak{X}_L/\eta^m \mathfrak{X}_L$  to  $\mathfrak{X}_L/\eta^n \mathfrak{X}_L$ .
- (iv) Let  $\tilde{\rho}$  be isomorphic to  $\rho$ , i.e. there exists an invertible formal power series  $t \in \mathcal{O}_H\{\{\tau\}\}$  such that  $t \circ \rho_a = \tilde{\rho}_a \circ t$  for all  $a \in \mathcal{O}_K$ . If we denote by  $\tilde{\mathfrak{X}}_L$ ,  $\tilde{L}^n$  and  $\tilde{\psi}_{L,t(v_n)}$  the objects defined in (3.1.4), (3.1.2) and (3.1.7) corresponding to  $\tilde{\rho}$ , then  $\mathfrak{X}_L = \tilde{\mathfrak{X}}_L$ ,  $L^n = \tilde{L}^n$  and  $\psi_{L,n} = t'(0)\tilde{\psi}_{L,t(v_n)}$ , where  $t' = \frac{dt}{dX}$ .

*Proof.* We follow [21, Proposition 3.2]. We begin by proving (i). Let  $\alpha \in W_L$  and  $\beta \in M^n$ . By Proposition 2.2.2(iii), we have  $(\alpha, \mathbb{N}_{M|L}(\beta))_{L,n} = (\alpha, \beta)_{M,n} = 0$  because  $\alpha \in W_L \subset W_M$ . Thus  $\mathbb{N}_{M|L}(\beta) \in L^n$ . Now let us prove that  $\mathbb{T}_{M|L}(\mathfrak{X}_M) = \mathfrak{X}_L$ . For that, let  $y \in \mathfrak{X}_M$  and  $x \in \lambda_\rho(\mathfrak{p}_L) \subset \lambda_\rho(\mathfrak{p}_M)$ . Then,  $\mathbb{T}_{M|K}(xy) = \mathbb{T}_{L|K}(x \mathbb{T}_{M|L}(y)) \in \mathcal{O}_K$ . It remains to prove the commutativity of the diagram. Let  $\alpha \in \mathfrak{p}_L$  and  $\beta \in M^n$ . Using Proposition 2.2.2(iii) and Proposition 3.1.2, we have

$$\begin{aligned} (\alpha, \mathbb{N}_{M|L}(\beta))_{L,n} &= (\alpha, \beta)_{M,n} = \mathbb{T}_{M|K}(\lambda_\rho(x)\psi_{M,n}(\beta)) \cdot_\rho v_n \\ &= \mathbb{T}_{L|K}(\lambda_\rho(x) \mathbb{T}_{M|L}(\psi_{M,v_n}(\beta))) \cdot_\rho v_n. \end{aligned}$$

By the uniqueness in Proposition 3.1.2, the last equality yields that

$$\psi_{L,v_n}(\mathbb{N}_{M|L}(\beta)) = \mathbb{T}_{M|L}(\psi_{M,v_n}(\beta)). \quad (3.1.10)$$

Part (ii) can be proved in the same way as part (i), only using Proposition 2.2.2(iv) instead of (iii). Part (iii) follows easily from Proposition 3.1.2 and Proposition 2.2.2 (v). To prove (iv), we first notice that  $\lambda_{\tilde{\rho}} = \frac{1}{(t^{-1})'(0)} = t'(0)$  and that  $t(v_n)$  is a generator of  $W_{\tilde{\rho}}^n$ . Hence, by Proposition 2.2.2 (vi) we get

$$\begin{aligned} (\alpha, \beta)_{\tilde{\rho},L,n} &= t((t^{-1}(\alpha), \beta)_{\rho,L,n}) \\ &= t(\mathbb{T}_{L|K}(\lambda_\rho \circ t^{-1}(\alpha)\psi_{L,v_n}(\beta)) \cdot_\rho v_n) \\ &= \mathbb{T}_{L|K}(\lambda_{\tilde{\rho}}(\alpha)(t^{-1})'(0)\psi_{L,v_n}(\beta)) \cdot_{\tilde{\rho}} t(v_n). \end{aligned}$$

This concludes the proof. □

We recall that by  $\mathfrak{p}_{L,1}$  we denote the set of elements  $x \in \mathfrak{p}_L$  of valuation  $\mu(x) > \frac{1}{q-1}$ . We denote by  $\mathfrak{X}_{L,1}$  the set of elements  $y$  of  $L$  such that  $\langle x, y \rangle_L \in \mathcal{O}_K$  for all  $x \in \lambda_\rho(\mathfrak{p}_{L,1})$ . We give a more explicit description of the fractional ideal  $\mathfrak{X}_{L,1}$ :

$$\begin{aligned} \mathfrak{X}_{L,1} &= \{y \in L; T_{L|K}(\lambda_\rho(\alpha)y) \in \mathcal{O}_K \ \forall \alpha \in \mathfrak{p}_{L,1}\} \\ &= \{y \in L; T_{L|K}(\alpha'y) \in \mathcal{O}_K \ \forall \alpha' \in \mathfrak{p}_{L,1}\} \\ &= \{y \in L; \mu(y) > -\frac{1}{q-1} - \mu(\mathcal{D}_{L|K})\}. \end{aligned} \tag{3.1.11}$$

Following the same reasoning as in Proposition 3.1.2, but restricting the left-hand argument of the pairing  $(\ , \ )_{L,n}$  to  $\mathfrak{p}_{L,1}$ , we get

**Proposition 3.1.5.** There exists a unique continuous group homomorphism  $\Psi_{L,n} : L^\times \longrightarrow \mathfrak{X}_{L,1}/\eta^n \mathfrak{X}_{L,1}$  such that

$$(\alpha, \beta)_{L,n} = T_{L|K}(\lambda_\rho(\alpha)\Psi_{L,v_n}(\beta)) \cdot_\rho v_n \tag{3.1.12}$$

for all  $\alpha \in \mathfrak{p}_{L,1}$  and  $\beta \in L^\times$ . Moreover, the diagram

$$\begin{array}{ccc} L^n & \xrightarrow{\psi_{L,v_n}} & \mathfrak{X}_L/\eta^n \mathfrak{X}_L \\ & \searrow \Psi_{L,v_n} & \downarrow \\ & & \mathfrak{X}_{L,1}/\eta^n \mathfrak{X}_{L,1} \end{array}$$

is commutative, the right hand arrow being induced by the inclusion  $\mathfrak{X}_L \subset \mathfrak{X}_{L,1}$ .

We note that  $\Psi_{L,v_n}$  satisfies the same properties mentioned in Remark 3.1.3 and Proposition 3.1.4 for  $\psi_{L,v_n}$ .

## 3.2 Derivations

In this section, we give a formulation of the Kummer pairing in terms of a derivation  $\bar{D}_{L,v_n}$ , which will be defined in Section 3.2.2. An advantage of having a derivation is that it is determined and explicitly constructible in terms of its value at a uniformizer. We begin by a brief recall on derivations.

### 3.2.1 Recall on derivations

In this paragraph, we give a recall on derivations and their main properties that will be useful for us in the sequel. Let  $R$  be a commutative ring with unit, and  $O$  be a subring of  $R$ . If  $W$  is an  $R$ -module, a map  $D : R \rightarrow W$  is said to be an  $O$ -derivation of  $R$  into  $W$  if it is  $O$ -linear and satisfies the Leibniz rule

$$D(xy) = xD(y) + yD(x) \ \forall x, y \in R. \tag{3.2.1}$$

In particular, a derivation  $D : R \rightarrow W$  also fulfills the following:

- (i)  $D(x + y) = D(x) + D(y) \quad \forall x, y \in R$ ,
- (ii)  $D(a) = 0 \quad \forall a \in O$ .

The set of all such derivations  $D_O(R, W)$  is an  $R$ -module, where  $aD$  is defined by  $(aD)(x) = aD(x)$  for all  $a, x \in R$ . We will show that there exists a universal derivation, in other words, an  $R$ -module  $\Omega_O(R)$ , and a derivation

$$d : R \rightarrow \Omega_O(R) \tag{3.2.2}$$

such that for every derivation  $D : R \rightarrow W$ , there exists a unique homomorphism of  $R$ -modules  $f : \Omega_O(R) \rightarrow W$  such that the diagram

$$\begin{array}{ccc} R & \xrightarrow{d} & \Omega_O(R) \\ & \searrow D & \downarrow \exists! f \\ & & W \end{array}$$

commutes. Let  $\mathcal{R}$  be the direct sum of the modules  $(R)_{x \in R}$ . Then  $\mathcal{R}$  is the submodule of  $\prod_{x \in R} R$  which consists of families  $(a_x)_{x \in R}$  having finite support. For each element  $x \in R$ , we associate a symbol  $dx$ , so that an element  $(a_x)_{x \in R}$  in  $\mathcal{R}$  can be written as a finite sum  $\sum_{x \in R} a_x dx$ . Here, the symbols  $dx$  are supposed to be distinct for distinct elements of  $R$ . Consider the submodule of  $\mathcal{R}$  generated by the set

$$\{d(xy) - ydx - xdy, \quad d(x + y) - dx - dy, \quad da; \quad x, y \in R, \quad a \in O\}. \tag{3.2.3}$$

The quotient of  $\mathcal{R}$  by this submodule, which we denote by  $\Omega_O(R)$ , together with the derivation  $d : R \rightarrow \Omega_O(R)$  that sends  $x$  to the class of  $dx$  in  $\Omega_O(R)$ , form the universal derivation we are looking for. Indeed, let  $W$  be an  $R$ -module and  $D : R \rightarrow W$  be a derivation, and consider the unique homomorphism of  $R$ -modules from  $\mathcal{R}$  to  $W$  that maps  $da$  to  $D(a)$ . This homomorphism is trivial on the submodule of  $\mathcal{R}$  generated by the set (3.2.3), thus it factors through  $\Omega_O(R)$ , whence the universal property. We call  $(\Omega_O(R), d)$  the module of differentials of  $R$  over  $O$ .

The universal derivation yields an isomorphism of  $R$ -modules

$$D_O(R, W) \simeq \text{Hom}_R(\Omega_O(R), W). \tag{3.2.4}$$

Let  $M$  be a local field and  $N$  be a finite separable extension of  $M$ . We denote by  $\mathcal{D}(N|M)$  the different of  $N|M$ . In the special case where  $R = \mathcal{O}_N$  and  $O = \mathcal{O}_M$ , we have the following results.

**Proposition 3.2.1.** There exists an isomorphism of  $\mathcal{O}_N$ -modules

$$\Omega_{\mathcal{O}_M}(\mathcal{O}_N) \simeq \mathcal{O}_N/\mathcal{D}(N|M). \tag{3.2.5}$$

Furthermore, if  $\pi_N$  is a prime of  $N$ , then  $d\pi_N$  is a generator of  $\Omega_{\mathcal{O}_M}(\mathcal{O}_N)$ .

*Proof.* Suppose first that  $N|M$  is unramified, then  $\mathcal{D}(N|M) = \mathcal{O}_N$ . Thus, we need to show that  $\Omega_{\mathcal{O}_M}(\mathcal{O}_N) = 0$ . Let  $x \in \mathcal{O}_N$  such that  $\mathcal{O}_N = \mathcal{O}_M[x]$ , and let  $P(X) = \sum_{i=0}^d b_i X^i$  be the irreducible polynomial of  $x$  over  $M$ . Then  $P'(x)\mathcal{O}_N = \mathcal{D}(N|M) = \mathcal{O}_N$ . Therefore,  $P'(x)$  is a unit in  $N$ . However, we have

$$0 = dP(x) = P'(x) dx. \quad (3.2.6)$$

Hence,  $d(x) = 0$ , and consequently,  $\Omega_{\mathcal{O}_M}(\mathcal{O}_N) = 0$ .

Now, if  $N$  is an arbitrary finite separable extension of  $M$ , let  $\tilde{N}$  be the inertia field of  $N|M$ . We know that  $\mathcal{O}_N = \mathcal{O}_{\tilde{N}}[\pi_N]$ , where  $\pi_N$  is a prime of  $N$ . Thus,  $d\pi_N$  generates  $\Omega_{\mathcal{O}_M}(\mathcal{O}_N)$ . Let  $P(X) = \sum_{i=0}^d b_i X^i$  be the irreducible polynomial of  $\pi_N$  over  $\tilde{N}$ , then  $P'(\pi_N)\mathcal{O}_N = \mathcal{D}(N|\tilde{N}) = \mathcal{D}(N|M)$  as  $\tilde{N}|M$  is unramified. Hence,

$$0 = \mathcal{O}_N dP(\pi_N) = P'(\pi_N)\mathcal{O}_N d\pi_N = \mathcal{D}(N|M) d\pi_N. \quad (3.2.7)$$

It remains to prove that  $\mathcal{D}(N|M)$  is precisely the annihilator of  $\Omega_{\mathcal{O}_M}(\mathcal{O}_N)$ . For that, let  $D : \mathcal{O}_N \rightarrow N/\mathcal{O}_N$  be the map defined by  $D(\pi_N) = \frac{1}{P'(\pi_N)}$  and  $D(r(\pi_N)) = r'(\pi_N)D(\pi_N)$  for all  $r \in \mathcal{O}_{\tilde{N}}[X]$ . It is obvious that  $D$  is a derivation of  $\mathcal{O}_N$  into  $N/\mathcal{O}_N$  over  $\mathcal{O}_M$ , and its annihilator is precisely  $\mathcal{D}(N|M)$ . Finally, we conclude using the universal property of  $d : \mathcal{O}_N \rightarrow \Omega_{\mathcal{O}_M}(\mathcal{O}_N)$ .  $\square$

**Remark 3.2.2.** The proof of Proposition 3.2.1 shows that  $\Omega_{\mathcal{O}_M}(\mathcal{O}_N)$  does not change when we replace  $M$  by an unramified extension. Hence, when investigating the  $\mathcal{O}_N$ -module  $\Omega_{\mathcal{O}_M}(\mathcal{O}_N)$ , we may assume that  $N|M$  is totally ramified.

**Corollary 3.2.3.** Let  $W$  be an  $\mathcal{O}_N$ -module and  $\pi_N$  be a prime of  $N$ . Let

$$S := \{x \in W, \quad ax = 0 \quad \forall a \in \mathcal{D}(N|M)\} \quad (3.2.8)$$

be the  $\mathcal{D}(N|M)$ -torsion submodule of  $W$ . Then, the map

$$\begin{aligned} D_{\mathcal{O}_M}(\mathcal{O}_N, W) &\rightarrow S \\ D &\mapsto D(\pi_N) \end{aligned} \quad (3.2.9)$$

is an isomorphism of  $\mathcal{O}_N$ -modules.

*Proof.* This corollary follows from the above discussions. Indeed, if  $D \in D_{\mathcal{O}_M}(\mathcal{O}_N, W)$ , then, by Proposition 3.2.1 and the fact that  $d : \mathcal{O}_N \rightarrow \Omega_{\mathcal{O}_M}(\mathcal{O}_N)$  is a universal object, we have  $\mathcal{D}(N|M)D(\pi_N) = 0$ . Thus, the map (3.2.9) is well defined, and it is clear that it is a homomorphism. We will construct its inverse: For  $w \in W$  such that  $\mathcal{D}(N|M)w = 0$ , we associate the derivation  $D_w \in D_{\mathcal{O}_M}(\mathcal{O}_N, W)$  defined by  $D_w(r(\pi_N)) = r'(\pi_N)w$  for all  $r(X) \in \mathcal{O}_{\tilde{N}}[X]$ , where  $\tilde{N}$  is the inertia field of  $N|M$ . This concludes the proof.  $\square$

**Remark 3.2.4.** With the notations of Corollary 3.2.3, the inverse homomorphism of (3.2.9) also satisfies

$$D_w(r(\pi_N)) = r'(\pi_N)w \quad (3.2.10)$$

for all  $r \in \mathcal{O}_{\tilde{N}}[[X]]$ . This follows from the fact that a derivation in  $D_{\mathcal{O}_M}(\mathcal{O}_N, W)$  is continuous for the discrete topology on  $W$ .

### 3.2.2 The derivation $\bar{D}_{L,v_n}$

Let  $n$  be a positive integer and  $L$  be a finite separable extension  $E_\rho^n$ . Let  $v_n$  be a fixed generator of the  $\mathcal{O}_K$ -module  $W_\rho^n$ . We define the map  $D_{L,v_n} : \mathcal{O}_L \rightarrow \mathfrak{X}_{L,1}/\eta^n \mathfrak{X}_{L,1}$  by  $D_{L,v_n}(0) = 0$  and

$$D_{L,v_n}(\alpha) = \alpha \Psi_{L,v_n}(\alpha) \quad (3.2.11)$$

for  $\alpha \in \mathcal{O}_L \setminus \{0\}$ , where  $\Psi_{L,v_n}$  is the homomorphism defined in (3.1.5). In this section we will prove that  $D_{L,v_n}$ , reduced modulo a convenient submodule of  $\mathfrak{X}_{L,1}$ , is a derivation and it intervenes in an explicit formula for the Kummer pairing.

It is clear that the map  $D_{L,v_n}$  satisfies the Leibniz rule

$$D_{L,v_n}(xy) = x D_{L,v_n}(y) + y D_{L,v_n}(x) \quad \forall x, y \in \mathcal{O}_L. \quad (3.2.12)$$

This follows from the fact that  $\Psi_{L,v_n}$  is a group homomorphism. Using this rule, we can prove by induction that

$$D_{L,v_n}(x^m) = mx^{m-1} D_{L,v_n}(x) \quad \forall x \in \mathcal{O}_L \quad \text{and} \quad \forall m \geq 1. \quad (3.2.13)$$

We will now prove that  $D_{L,v_n}$  is additive.

**Lemma 3.2.5.** Suppose  $\rho$  is such that  $(x, x)_{\rho, L, n} = 0$  for all  $x \in \mathfrak{p}_L \setminus \{0\}$ . Let  $\alpha \in \mathfrak{p}_L \setminus \{0\}$  and let  $u$  be a unit of  $L$  such that  $\mu(\alpha(1-u)) > \frac{nm_0}{q} + \frac{1}{q-1}$ . We have

$$(\alpha u, u)_{L, n} = T_{L|K}((1-u) D_{L,v_n}(\alpha)) \cdot_\rho v_n. \quad (3.2.14)$$

*Proof.* We have

$$\begin{aligned} (\alpha u, u)_{L, n} &= (\alpha u, \frac{\alpha u}{\alpha})_{L, n} \\ &= (\alpha u, \alpha u)_{L, n} - (\alpha u, \alpha)_{L, n} \\ &= (\alpha, \alpha)_{L, n} - (\alpha u, \alpha)_{L, n} \\ &= (\alpha - \alpha u, \alpha)_{L, n} \\ &= T_{L|K}(\lambda_\rho(\alpha - \alpha u) \Psi_{L,v_n}(\alpha)) \cdot_\rho v_n \end{aligned}$$

by Proposition 3.1.5. Let  $\gamma = \alpha(1-u)$ , we will show that

$$T_{L|K}(\lambda_\rho(\gamma) \Psi_{L,v_n}(\alpha)) \cdot_\rho v_n = T_{L|K}(\gamma \Psi_{L,v_n}(\alpha)) \cdot_\rho v_n. \quad (3.2.15)$$

By the hypothesis on the valuations, we have  $\mu(\gamma) > \frac{nm_0}{q} + \frac{1}{q-1}$ . Hence

$$\begin{aligned} \mu(\lambda_\rho(\gamma) - \gamma) &= \mu\left(\sum_{i \geq 1} c_i \gamma^{q^i}\right) \\ &\geq \min_{i \geq 1} \{\mu(c_i) + q^i \mu(\gamma)\} \\ &> \min_{i \geq 1} \left\{-i + q^i \left(\frac{nm_0}{q} + \frac{1}{q-1}\right)\right\} \\ &\geq nm_0 + \frac{1}{q-1} \end{aligned}$$



Therefore, we can write  $\lambda_\rho(\gamma) - \gamma = \eta^n \delta$ , where  $\delta$  is an element of  $\mathfrak{p}_{L,1}$ . Thus, by (3.1.11),

$$\mathrm{T}_{L|K}((\lambda_\rho(\gamma) - \gamma)\Psi_{L,v_n}(\alpha)) \cdot_\rho v_n = 0$$

because  $\Psi_{L,v_n}(\alpha) \in \mathfrak{X}_{L,1}$ . This concludes the proof.  $\square$

**Proposition 3.2.6.** Suppose  $\rho$  is such that  $(x, x)_{\rho, L, n} = 0$  for all  $x \in \mathfrak{p}_L \setminus \{0\}$ . Let  $\gamma$  be an element of  $\mathcal{O}_L \setminus \{0\}$  of valuation  $\mu(\gamma) = \max\{\frac{nm_0}{q}, \frac{1}{q-1}\}$ , that is  $\mu(\gamma) = \frac{nm_0}{q}$  if  $nm_0 \geq 2$ , and  $\mu(\gamma) = \frac{1}{q-1}$  if  $nm_0 = 1$ . Then

$$\mathrm{D}_{L,v_n}(x + y) \equiv \mathrm{D}_{L,v_n}(x) + \mathrm{D}_{L,v_n}(y) \pmod{\frac{\eta^n}{\gamma} \mathfrak{X}_{L,1}} \quad (3.2.16)$$

for all  $x, y \in \mathcal{O}_L$ .

*Proof.* Let us prove first why such a  $\gamma$  exists. Since  $E_\rho^n \subset L$ , the ramification index of  $L|K$  is a multiple of the ramification index of  $E_\rho^n|K$ , which is equal to  $q^{nm_0-1}(q-1)$ . Hence, there exists elements in  $L$  of valuation  $\frac{1}{q^{nm_0-1}(q-1)}$ , whence the existence of  $\gamma$ . Now let us prove (3.2.16). Let  $x, y \in \mathcal{O}_L$ , then, by Lemma 3.2.5, we have

$$\begin{aligned} (\gamma(x+y)u, u)_{L,n} &= \mathrm{T}_{L|K}((1-u)\mathrm{D}_{L,v_n}(\gamma(x+y))) \cdot_\rho v_n \\ &= \mathrm{T}_{L|K}((1-u)((x+y)\mathrm{D}_{L,v_n}(\gamma) + \gamma\mathrm{D}_{L,v_n}((x+y)))) \cdot_\rho v_n \end{aligned} \quad (3.2.17)$$

for all  $u \in 1 + \mathfrak{p}_{L,1}$ . However, again by Lemma 3.2.5, we have

$$\begin{aligned} (\gamma(x+y)u, u)_{L,n} &= (\gamma xu, u)_{L,n} + (\gamma yu, u)_{L,n} \\ &= \mathrm{T}_{L|K}((1-u)\mathrm{D}_{L,v_n}(\gamma x)) \cdot_\rho v_n + \mathrm{T}_{L|K}((1-u)\mathrm{D}_{L,v_n}(\gamma y)) \cdot_\rho v_n \\ &= \mathrm{T}_{L|K}((1-u)(\mathrm{D}_{L,v_n}(\gamma x) + \mathrm{D}_{L,v_n}(\gamma y))) \cdot_\rho v_n \\ &= \mathrm{T}_{L|K}((1-u)((x+y)\mathrm{D}_{L,v_n}(\gamma) + \gamma(\mathrm{D}_{L,v_n}(x) + \mathrm{D}_{L,v_n}(y)))) \cdot_\rho v_n \end{aligned} \quad (3.2.18)$$

for all  $u \in 1 + \mathfrak{p}_{L,1}$ . Therefore, (3.2.17) and (3.2.18) being equal, we conclude that

$$\gamma\mathrm{D}_{L,v_n}(x+y) \equiv \gamma(\mathrm{D}_{L,v_n}(x) + \mathrm{D}_{L,v_n}(y)) \pmod{\eta^n \mathfrak{X}_{L,1}} \quad (3.2.19)$$

by the very definition (3.1.11) of  $\mathfrak{X}_{L,1}$ . Hence, we have

$$\mathrm{D}_{L,v_n}(x+y) \equiv \mathrm{D}_{L,v_n}(x) + \mathrm{D}_{L,v_n}(y) \pmod{\frac{\eta^n}{\gamma} \mathfrak{X}_{L,1}}. \quad (3.2.20)$$

$\square$

**Corollary 3.2.7.** Let  $\gamma$  be as in Proposition 3.2.6. Then

$$\mathrm{D}_{L,v_n}(x+y) \equiv \mathrm{D}_{L,v_n}(x) + \mathrm{D}_{L,v_n}(y) \pmod{\frac{\eta^n}{\gamma} \mathfrak{X}_{L,1}} \quad (3.2.21)$$

for all  $x, y \in \mathcal{O}_L$ .

*Proof.* Let  $r = r_n$  be the series defined in Proposition 2.2.5 and let  $\rho'$  the Drinfeld module defined by

$$\rho'_a = r \circ \rho_a \circ r^{-1}.$$

Then  $r$  defines an isomorphism of  $\mathcal{O}_K$ -modules  $r : W_{\rho}^n \rightarrow W_{\rho'}^n$ . Furthermore, if we denote by  $D_{\rho,L,v_n}$  (respectively  $D_{\rho',L,r(v_n)}$ ) the map defined in (3.2.11) associated to  $\rho$  (respectively  $\rho'$ ), then by Proposition 3.1.4 (iv), we have

$$D_{\rho,L,v_n} = r'(0) D_{\rho',L,r(v_n)}. \quad (3.2.22)$$

Here,  $r'(0)$  is a unit in  $H$  because  $r(X) \in \mathcal{O}_H[[X]]$  is invertible. Since  $(x, x)_{\rho',L,n} = 0$  for all  $x \in \mathfrak{p}_L \setminus \{0\}$  by Lemma 2.2.6, we can apply Proposition 3.2.6 for  $\rho'$  so that

$$D_{\rho',L,r(v_n)}(x+y) \equiv D_{\rho',L,r(v_n)}(x) + D_{\rho',L,r(v_n)}(y) \pmod{\frac{\eta^n}{\gamma} \mathfrak{X}_{L,1}} \quad (3.2.23)$$

for all  $x, y \in \mathcal{O}_L$ . Thus, using (3.2.22), we conclude that

$$D_{\rho,L,v_n}(x+y) \equiv D_{\rho,L,v_n}(x) + D_{\rho,L,v_n}(y) \pmod{\frac{\eta^n}{\gamma} \mathfrak{X}_{L,1}} \quad (3.2.24)$$

for all  $x, y \in \mathcal{O}_L$ . □

**Proposition 3.2.8.** Let

$$\mathfrak{X}_{L,1}^{(n)} = \{y \in L; \mu(y) \geq nm_0 - \max\{\frac{nm_0}{q}, \frac{1}{q-1}\} - \frac{1}{q-1} - \frac{1}{e(L|K)} - \mu(\mathcal{D}(L|K))\} \subset \mathfrak{X}_{L,1}. \quad (3.2.25)$$

The reduction of  $D_{L,v_n}$  modulo  $\mathfrak{X}_{L,1}^{(n)}$ , denoted by  $\bar{D}_{L,v_n} : \mathcal{O}_L \rightarrow \mathfrak{X}_{L,1}/\mathfrak{X}_{L,1}^{(n)}$ , is an  $\mathcal{O}_K$ -derivation.

*Proof.* Let  $\gamma \in \mathcal{O}_L \setminus \{0\}$  be as in Proposition 3.2.6, then

$$\mathfrak{X}_{L,1}^{(n)} = \frac{\eta^n}{\gamma} \mathfrak{X}_{L,1}. \quad (3.2.26)$$

Let  $\pi_L$  be a prime of  $L$  and let  $w = \bar{D}_{L,v_n}(\pi_L) \in \mathfrak{X}_{L,1}/\mathfrak{X}_{L,1}^{(n)}$ . Since  $\mu(\frac{\eta^n}{\gamma}) = nm_0 - \mu(\gamma) \leq nm_0 - \frac{1}{q-1} \leq \mu(\mathcal{D}(L|K))$ , we have  $\mathcal{D}(L|K)w = 0$ . Hence, by Corollary 3.2.3, there exists a derivation  $D : \mathcal{O}_L \rightarrow \mathfrak{X}_{L,1}/\frac{\eta^n}{\gamma} \mathfrak{X}_{L,1}$  such that  $D(\pi_L) = w$  and

$$D(g(\pi_L)) = g'(\pi_L)w \quad (3.2.27)$$

for every power series  $g \in \mathcal{O}_{\tilde{L}}[[X]]$ , where  $\tilde{L}$  is the maximal subextension of  $L$  unramified over  $K$ . In particular, (3.2.27) is true for all the power series defined over the residue field of  $\tilde{L}$ , which is equal to the residue field of  $L$ . We will prove that  $D$  and  $\bar{D}_{L,v_n}$  are equal. Indeed,

let  $x \in \mathcal{O}_L$ , and let  $g(X) = \sum_{i \geq 0} a_i X^i$  be the unique power series defined over the residue field  $\mathbb{F}_{q_L}$  of  $L$  such that  $g(\pi_L) = x$ . We have

$$\bar{D}_{L,v_n}(x) = \bar{D}_{L,v_n}(g(\pi_L)) = \sum_{i \geq 0} \bar{D}_{L,v_n}(a_i \pi_L^i) \quad (3.2.28)$$

because  $\bar{D}_{L,v_n}$  is additive by Proposition 3.2.7, and continuous by Proposition 3.1.5. Let  $q_L$  be the cardinal of  $\mathbb{F}_{q_L}$ , then  $q_L$  is a power of  $p$ . Hence, for all  $i \geq 0$ , we have

$$\bar{D}_{L,v_n}(a_i) = \bar{D}_{L,v_n}(a_i^{q_L}) = 0$$

by (3.2.13). Therefore, applying the Leibniz rule (3.2.12) to (3.2.28), we get

$$\bar{D}_{L,v_n}(x) = \sum_{i \geq 0} a_i \bar{D}_{L,v_n}(\pi_L^i) = \sum_{i \geq 0} a_i \times i \times \pi_L^{i-1} \times \bar{D}_{L,v_n}(\pi_L) \quad (3.2.29)$$

again by (3.2.13). However, this is equal to  $g'(\pi_L) \bar{D}_{L,v_n}(\pi_L)$ , which is, by (3.2.27), equal to  $D(x)$ .  $\square$

Now, we will define the logarithmic derivative  $\text{dlog} \bar{D}_{L,v_n}$  associated to the derivation  $\bar{D}_{L,v_n}$ . For a prime  $\pi_L$  of  $L$ , let

$$f : \mathfrak{X}_{L,1}/\mathfrak{X}_{L,1}^{(n)} \rightarrow \pi_L^{-1} \mathfrak{X}_{L,1}/\pi_L^{-1} \mathfrak{X}_{L,1}^{(n)} \quad (3.2.30)$$

be the natural map induced by the inclusion  $\mathfrak{X}_{L,1} \hookrightarrow \pi_L^{-1} \mathfrak{X}_{L,1}$ , and

$$g_{\pi_L} : \mathfrak{X}_{L,1}/\mathfrak{X}_{L,1}^{(n)} \rightarrow \pi_L^{-1} \mathfrak{X}_{L,1}/\pi_L^{-1} \mathfrak{X}_{L,1}^{(n)} \quad (3.2.31)$$

be the multiplication by  $\pi_L^{-1}$  map. For  $x = u\pi_L^k \in L^\times$ , where  $u$  is a unit in  $L$ , we define

$$\text{dlog} \bar{D}_{L,v_n}(x) = f(u^{-1} \bar{D}_{L,v_n}(u)) + kg_{\pi_L}(\bar{D}_{L,v_n}(\pi_L)). \quad (3.2.32)$$

The map  $\text{dlog} \bar{D}_{L,v_n} : L^\times \rightarrow \pi_L^{-1} \mathfrak{X}_{L,1}/\pi_L^{-1} \mathfrak{X}_{L,1}^{(n)}$  is a group homomorphism. Furthermore, its definition does not depend on the choice of the uniformizer  $\pi_L$ . Indeed, let  $\pi'_L$  be another uniformizer of  $L$  and let  $x = u\pi_L^k = u'\pi'^k_L \in L^\times$ , where  $u$  and  $u'$  are units of  $L$ . Let  $u_0$  be the unit of  $L$  such that  $\pi'_L = u_0\pi_L$ . Then,

$$\begin{aligned} f(u^{-1} \bar{D}_{L,v_n}(u)) + kg_{\pi_L}(\bar{D}_{L,v_n}(\pi_L)) &= f(u'^{-1} u_0^{-k} \bar{D}_{L,v_n}(u' u_0^k)) + kg_{\pi_L}(\bar{D}_{L,v_n}(\pi_L)) \\ &= f(u'^{-1} \bar{D}_{L,v_n}(u') + u_0^{-k} \bar{D}_{L,v_n}(u_0^k)) + kg_{\pi_L}(\bar{D}_{L,v_n}(\pi_L)) \\ &= f(u'^{-1} \bar{D}_{L,v_n}(u')) + f(u_0^{-k} \bar{D}_{L,v_n}(u_0^k)) + kg_{\pi_L}(\bar{D}_{L,v_n}(\pi_L)) \\ &= f(u'^{-1} \bar{D}_{L,v_n}(u')) + kf(u_0^{-1} \bar{D}_{L,v_n}(u_0)) + kg_{\pi_L}(\bar{D}_{L,v_n}(\pi_L)). \end{aligned} \quad (3.2.33)$$

On the other hand, we have

$$\begin{aligned}
g_{\pi'_L}(\bar{D}_{L,v_n}(\pi'_L)) &= g_{\pi'_L}(\bar{D}_{L,v_n}(u_0\pi_L)) \\
&= g_{\pi'_L}(u_0\bar{D}_{L,v_n}(\pi_L) + \pi_L\bar{D}_{L,v_n}(u_0)) \\
&\equiv \pi'_L{}^{-1}(u_0\bar{D}_{L,v_n}(\pi_L) + \pi_L\bar{D}_{L,v_n}(u_0)) \pmod{\pi_L^{-1}\mathfrak{X}_{L,1}^{(n)}} \\
&\equiv u_0^{-1}\pi_L^{-1}(u_0\bar{D}_{L,v_n}(\pi_L) + \pi_L\bar{D}_{L,v_n}(u_0)) \pmod{\pi_L^{-1}\mathfrak{X}_{L,1}^{(n)}} \\
&\equiv \pi_L^{-1}\bar{D}_{L,v_n}(\pi_L) + u_0^{-1}\bar{D}_{L,v_n}(u_0) \pmod{\pi_L^{-1}\mathfrak{X}_{L,1}^{(n)}} \\
&= f(u_0^{-1}\bar{D}_{L,v_n}(u_0)) + g_{\pi_L}(\bar{D}_{L,v_n}(\pi_L)).
\end{aligned} \tag{3.2.34}$$

Therefore, (3.2.33) and (3.2.34) yield that  $\text{dlog}\bar{D}_{L,v_n}$  does not depend on the choice of  $\pi_L$ .

**Remark 3.2.9.** We call  $\text{dlog}\bar{D}_{L,v_n}$  a logarithmic derivative because we have the equality

$$\text{dlog}\bar{D}_{L,v_n}(x) = \frac{\bar{D}_{L,v_n}(x)}{x}. \tag{3.2.35}$$

Moreover, since  $H|K$  is unramified, then for all power series  $f \in \mathcal{O}_H((X))$  such that  $f(\pi_L) = x$ , (3.2.35) writes

$$\text{dlog}\bar{D}_{L,v_n}(x) = \frac{f'(\pi_L)}{x} \bar{D}_{L,v_n}(\pi_L). \tag{3.2.36}$$

This definition of  $\text{dlog}\bar{D}_{L,v_n}$  recalls the logarithmic derivative  $\delta_{v_n}$  defined in (2.2.25) in the special case  $L = E_\rho^n$ .

**Theorem 3.2.10.** The derivation  $\bar{D}_{L,v_n} : \mathcal{O}_L \rightarrow \mathfrak{X}_{L,1}/\mathfrak{X}_{L,1}^{(n)}$  satisfies

$$(\alpha, \beta)_{L,n} = T_{L|K}(\lambda_\rho(\alpha) \text{dlog}\bar{D}_{L,v_n}(\beta)) \cdot_\rho v_n \tag{3.2.37}$$

for all  $\alpha$  such that  $\mu(\alpha) > \max\{\frac{nm_0}{q}, \frac{1}{q-1}\} + \frac{1}{q-1} + \frac{1}{e(L|K)}$  and for all  $\beta \in L^\times$ .

*Proof.* To prove (3.2.37) is equivalent to prove that

$$\text{dlog}\bar{D}_{L,v_n}(\beta) - \Psi_{L,v_n}(\beta) \in \pi_L^{-1}\mathfrak{X}_{L,1}^{(n)} \tag{3.2.38}$$

for all  $\beta \in L^\times$ , where  $\text{dlog}\bar{D}_{L,v_n}(\beta)$  and  $\Psi_{L,v_n}(\beta)$  are regarded as elements of  $\pi_L^{-1}\mathfrak{X}_{L,1}$ . Indeed, let  $\beta \in L^\times$ . Since Proposition 3.1.5 shows that

$$(\alpha, \beta)_{L,n} = T_{L|K}(\lambda_\rho(\alpha)\Psi_{L,v_n}(\beta)) \cdot_\rho v_n \tag{3.2.39}$$

for all  $\alpha \in \mathfrak{p}_{L,1}$ , then (3.2.37) is equivalent to say that

$$T_{L|K}(\lambda_\rho(\alpha) \text{dlog}\bar{D}_{L,v_n}(\beta)) \cdot_\rho v_n = T_{L|K}(\lambda_\rho(\alpha)\Psi_{L,v_n}(\beta)) \cdot_\rho v_n \tag{3.2.40}$$

for all  $\alpha$  in  $L$  such that  $\mu(\alpha) > \max\{\frac{nm_0}{q}, \frac{1}{q-1}\} + \frac{1}{q-1} + \frac{1}{e(L|K)}$ . Obviously, (3.2.40) is equivalent to

$$T_{L|K}(\lambda_\rho(\alpha)(\text{dlog}\bar{D}_{L,v_n}(\beta) - \Psi_{L,v_n}(\beta))) \in \eta^n \mathcal{O}_K \tag{3.2.41}$$

for all  $\alpha \in \gamma\pi_L\mathfrak{p}_{L,1}$ , where  $\gamma \in L$  is of valuation  $\mu(\gamma) = \max\{\frac{nm_0}{q}, \frac{1}{q-1}\}$ . However, since  $\mathfrak{p}_{L,1} = \lambda_\rho(\mathfrak{p}_{L,1})$  and  $\mu(\lambda_\rho(\alpha)) = \mu(\alpha)$  whenever  $\alpha \in \mathfrak{p}_{L,1}$  (see (1.2.13)), then (3.2.41) is in turn equivalent to

$$T_{L|K}(\gamma\pi_L\alpha(\text{dlog}\bar{D}_{L,v_n}(\beta) - \Psi_{L,v_n}(\beta))) \in \eta^n \mathcal{O}_K \quad (3.2.42)$$

for all  $\alpha \in \mathfrak{p}_{L,1}$ . Finally, by the very definition of  $\mathfrak{X}_{L,1}$ , (3.2.42) is equivalent to

$$\text{dlog}\bar{D}_{L,v_n}(\beta) - \Psi_{L,v_n}(\beta) \in \pi_L^{-1} \frac{\eta^n}{\gamma} \mathfrak{X}_{L,1} = \pi_L^{-1} \mathfrak{X}_{L,1}^{(n)}. \quad (3.2.43)$$

Let us now prove (3.2.43). Let  $\beta = u\pi_L^k \in L^\times$ , then  $\text{dlog}\bar{D}_{L,v_n}(\beta) - \Psi_{L,v_n}(\beta)$  is equal to  $u^{-1}\bar{D}_{L,v_n}(u) + k\pi_L^{-1}\bar{D}_{L,v_n}(\pi_L) - \Psi_{L,v_n}(u) - k\Psi_{L,v_n}(\pi_L)$  modulo  $\pi_L^{-1}\mathfrak{X}_{L,1}^{(n)}$ . However, by the very definition of  $\bar{D}_{L,v_n}$ , we have

$$\bar{D}_{L,v_n}(u) \equiv u\Psi_{L,v_n}(u) \pmod{\mathfrak{X}_{L,1}^{(n)}}. \quad (3.2.44)$$

But as  $\mathfrak{X}_{L,1}^{(n)} \subset \pi_L^{-1}\mathfrak{X}_{L,1}^{(n)}$ , the congruence (3.2.44) implies that

$$\bar{D}_{L,v_n}(u) \equiv u\Psi_{L,v_n}(u) \pmod{\pi_L^{-1}\mathfrak{X}_{L,1}^{(n)}}. \quad (3.2.45)$$

Thus, we have

$$u^{-1}\bar{D}_{L,v_n}(u) \equiv \Psi_{L,v_n}(u) \pmod{\pi_L^{-1}\mathfrak{X}_{L,1}^{(n)}}. \quad (3.2.46)$$

Moreover, we have

$$\bar{D}_{L,v_n}(\pi_L) \equiv \pi_L\Psi_{L,v_n}(\pi_L) \pmod{\mathfrak{X}_{L,1}^{(n)}}, \quad (3.2.47)$$

and thus,

$$\pi_L^{-1}\bar{D}_{L,v_n}(\pi_L) \equiv \Psi_{L,v_n}(\pi_L) \pmod{\pi_L^{-1}\mathfrak{X}_{L,1}^{(n)}}. \quad (3.2.48)$$

This concludes the proof.  $\square$

### 3.3 Values of $\bar{D}_{L,v_n}$ in terms of representation theory

In this section, we consider the continuous representation  $\mathbf{r} : \text{Gal}(\Omega|H) \rightarrow \text{GL}_1(\mathcal{O}_K) = \mathcal{U}_K$  defined in Proposition 1.3.11. The image  $\mathbf{r}(\sigma)$  of an element  $\sigma \in \text{Gal}(\Omega|H)$  is the unique unit  $u$  of  $K$  such that  $\sigma(\alpha) = \rho_u(\alpha)$  for all  $\alpha \in W_\rho$ . This representation is induced by the action of  $\text{Gal}(\Omega|H)$  on the module  $\varprojlim W_\rho^n$ . We will show that we can obtain explicit formulas in terms of invariants of this representation.

It is obvious that the kernel of  $\mathbf{r}$  is  $\text{Gal}(\Omega|H_\rho)$ . Thus,  $\mathbf{r}$  induces an embedding  $\text{Gal}(H_\rho|H) \rightarrow \mathcal{U}_K$ . Reducing modulo  $\mathcal{U}_{K,m} = 1 + \mathfrak{p}_K^m$  for  $m \geq 0$ , we get the map  $\mathbf{r}_m$ , which, restricted to  $\text{Gal}(H_\rho^m|H)$ , defines an isomorphism

$$\mathbf{r}_m : \text{Gal}(H_\rho^m|H) \rightarrow \mathcal{U}_K/\mathcal{U}_{K,m}. \quad (3.3.1)$$

We note that  $\mathbf{r}$  and  $\mathbf{r}_m$  are respectively the inverse isomorphisms of  $\Gamma$  and  $\Gamma_m$  defined in Proposition 1.3.9 and Proposition 1.3.11. For an algebraic extension  $F$  of  $H$ , we also denote by

$\mathbf{r} : \text{Gal}(\Omega|F) \rightarrow \mathcal{U}_K$  the restriction of  $\mathbf{r}$  to  $\text{Gal}(\Omega|F)$ , and by  $\mathbf{r}_m : \text{Gal}(F(V_\rho^m)|F) \rightarrow \mathcal{U}_K/\mathcal{U}_{K,m}$  the restriction to  $\text{Gal}(F(V_\rho^m)|F)$ . As above, we let  $n$  be a positive integer and  $L$  be a finite separable extension  $E_\rho^n$ .

**Proposition 3.3.1.** Let  $m \geq n$  and suppose  $L \supset E_\rho^m$ . There exists a character  $\chi_{L,m,n} : L^\times \rightarrow \mathcal{O}_K/\mathfrak{p}_K^{nm_0}$  such that

$$\mathbf{r}_{m_0(m+n)}(\Phi_L(\beta)) = 1 + \eta^m \chi_{L,m,n}(\beta) \in \mathcal{U}_K/\mathcal{U}_{K,(m+n)m_0} \quad (3.3.2)$$

for all  $\beta \in L^\times$ . Furthermore,  $\chi_{L,m,n}$  satisfies the following.

- (i)  $\chi_{L,m,n}(\beta) = \chi_{E_\rho^m,m,n}(\mathbf{N}_{L|E_\rho^m}(\beta))$ .
- (ii) Let  $v = \rho_a(v_m) \in W_\rho^m$ , where  $v_m$  is a generator of  $W_\rho^m$  such that  $\rho_{\eta^{m-n}}(v_m) = v_n$ . Then

$$(v, \beta)_{L,n} = (a\chi_{L,m,n}(\beta)) \cdot_\rho v_n$$

for all  $\beta \in L^\times$ . In particular if  $v = v_m$ , then for every  $\beta \in L^\times$ , we have

$$(v_m, \beta)_{L,n} = \chi_{L,m,n}(\beta) \cdot_\rho v_n. \quad (3.3.3)$$

*Proof.* Let  $\beta \in L^\times$ . As  $\Phi_L(\beta)$  fixes  $L$ , thus in particular fixes  $E_\rho^m$ , we have

$$\mathbf{r}_{m_0(m+n)}(\Phi_L(\beta)) \equiv 1 \pmod{\eta^m}. \quad (3.3.4)$$

Thus, there exists an element  $\chi_{L,m,n}(\beta) \in \mathcal{O}_K/\mathfrak{p}_K^{nm_0}$  such that (3.3.2) holds. It is easy to check that  $\chi_{L,m,n} : L^\times \rightarrow \mathcal{O}_K/\mathfrak{p}_K^{nm_0}$  is a group homomorphism. Moreover, Proposition 2.1.1 (v) imply (i). To prove (ii), let  $\xi \in \mathfrak{p}_\Omega$  be such that  $\rho_{\eta^n}(\xi) = v$ . Such a  $\xi$  exists by Lemma 2.2.1. Since  $v \in W_\rho^m$ , then  $\xi \in W_\rho^{m+n}$  and

$$\begin{aligned} (v, \beta)_{L,n} &= \Phi_L(\beta)(\xi) - \xi \\ &= \mathbf{r}_{m_0(m+n)}(\Phi_L(\beta)) \cdot_\rho \xi - \xi \\ &= (\mathbf{r}_{m_0(m+n)}(\Phi_L(\beta)) - 1) \cdot_\rho \xi \\ &= (\eta^m \chi_{L,m,n}(\beta)) \cdot_\rho \xi \\ &= (\eta^{m-n} \chi_{L,m,n}(\beta)) \cdot_\rho v \\ &= (\eta^{m-n} \chi_{L,m,n}(\beta)) \cdot_\rho (a \cdot_\rho v_m) \\ &= (a\chi_{L,m,n}(\beta)) \cdot_\rho v_n. \end{aligned}$$

□

**Lemma 3.3.2.** The character  $\chi_{L,m,n} : L^\times \rightarrow \mathcal{O}_K/\mathfrak{p}_K^{nm_0}$  is stable by isomorphism class of  $\rho$ . In other words, if  $t$  is an invertible power series in  $\mathcal{O}_H\{\{\tau\}\}$  such that  $\rho'_a = t^{-1} \circ \rho_a \circ t$  for all  $a \in \mathcal{O}_K$ , then the characters defined in Proposition 3.3.1 associated to  $\rho$  and  $\rho'$  are equal.

*Proof.* Let  $v_m$  be such that  $\rho_{\eta^{m-n}}(v_m) = v_n$  and let  $v'_i = t^{-1}(v_i)$  for  $i = m, n$ . Denote by  $\chi_{L,m,n}$  (respectively  $\chi'_{L,m,n}$ ) the character defined in Proposition 3.3.1 associated to  $\rho$  (respectively  $\rho'$ ). Then by (3.3.3), we have  $\chi'_{L,m,n}(\beta) \cdot_{\rho'} v'_n = (v'_m, \beta)_{\rho', L, n}$  which is equal to  $t^{-1}((t(v'_m), \beta)_{\rho, L, n}) = t^{-1}((v_m, \beta)_{\rho, L, n})$  by Proposition 2.2.2 (vi). Again from (3.3.3), we conclude that

$$\begin{aligned} \chi'_{L,m,n}(\beta) \cdot_{\rho'} v'_n &= t^{-1}((v_m, \beta)_{\rho, L, n}) \\ &= t^{-1}(\chi_{L,m,n}(\beta) \cdot_{\rho} v_n) \\ &= \chi_{L,m,n}(\beta) \cdot_{\rho'} t^{-1}(v_n) \\ &= \chi_{L,m,n}(\beta) \cdot_{\rho'} v'_n. \end{aligned}$$

□

**Proposition 3.3.3.** Let  $m \geq n$  and suppose  $L \supset E_{\rho}^m$  is such that  $p$  does not divide the ramification index of the extension  $L|E_{\rho}^m$ . Let  $u$  be a unit in  $L$  such that  $\mu(1-u) > \max\{\frac{nm_0}{q}, \frac{1}{q-1}\} + \frac{1}{q-1}$ . Let  $f(X)$  and  $g(X)$  be power series in  $\mathbb{F}_{q_L}[[X]]$  such that  $f(\pi_L) = v_m$  and  $g(\pi_L) = u$ , where  $\pi_L$  is a prime of  $L$ . Then,

$$\frac{\frac{g'(\pi_L)}{u}}{\frac{f'(\pi_L)}{v_m}} \in \mathfrak{p}_L, \quad (3.3.5)$$

and

$$\chi_{L,m,n}(u) \equiv \mathbb{T}_{L|K}\left(\left(\frac{1-u}{u}\right)\left(1 - \frac{g'(\pi_L)}{f'(\pi_L)}\right)\bar{D}_{L,v_n}(v_m)\right) \pmod{\mathfrak{p}_K^{nm_0}}. \quad (3.3.6)$$

*Proof.* Since  $p$  does not divide the ramification index of  $L|E_{\rho}^m$ , we have  $\mu(f'(\pi_L)) = \mu(f(\pi_L)) - \mu(\pi_L) = \mu(v_m) - \mu(\pi_L)$ . Furthermore, since  $\mu(1-u) > \max\{\frac{nm_0}{q}, \frac{1}{q-1}\} + \frac{1}{q-1}$ , we can write  $g(X) = 1 + \sum a_i X^i$ , where  $i \geq 2$  and  $a_i \in \mathbb{F}_{q_L}$ . Hence,  $\mu(g'(\pi_L)) > \mu(\pi_L)$  and therefore, we have (3.3.5). Now, let us prove (3.3.6). By Lemma 2.2.6 and Lemma 3.3.2, we can suppose that  $\rho$  is such that  $(x, x)_{\rho, L, n} = 0$  for all  $x \in \mathfrak{p}_L$ . For such a  $\rho$  and for  $u \in L^{\times}$  such that  $\mu(1-u) > \max\{\frac{nm_0}{q}, \frac{1}{q-1}\} + \frac{1}{q-1}$ , we have

$$(\alpha u, u)_{L, n} = \mathbb{T}_{L|K}((1-u)D_{L,v_n}(\alpha)) \cdot_{\rho} v_n \quad (3.3.7)$$

for all  $\alpha \in \mathfrak{p}_L \setminus \{0\}$  by Lemma 3.2.5. We note that the hypothesis on the valuation of  $1-u$  allows us to replace  $D_{L,v_n}(\alpha)$  by  $\bar{D}_{L,v_n}(\alpha)$  in (3.3.7). Let  $\alpha$  be such that  $\alpha u = v_m$ , where  $v_m$  is a generator of  $W_{\rho}^m$  such that  $\rho_{\eta^{m-n}}(v_m) = v_n$ . Hence, (3.3.7) together with (3.3.3) give us

$$\chi_{L,m,n}(u) \cdot_{\rho} v_n = (v_m, u)_{L, n} = \mathbb{T}_{L|K}((1-u)\bar{D}_{L,v_n}(v_m u^{-1})) \cdot_{\rho} v_n. \quad (3.3.8)$$

However,  $\bar{D}_{L,v_n}\left(\frac{v_m}{u}\right) = \frac{1}{u^2}(u\bar{D}_{L,v_n}(v_m) - v_m\bar{D}_{L,v_n}(u))$ . Moreover, we have

$$\bar{D}_{L,v_n}(u) = g'(\pi_L)\bar{D}_{L,v_n}(\pi_L) \quad \text{and} \quad \bar{D}_{L,v_n}(v_m) = f'(\pi_L)\bar{D}_{L,v_n}(\pi_L).$$

This implies that

$$f'(\pi_L)\bar{D}_{L,v_n}(u) - f'(\pi_L)g'(\pi_L)\bar{D}_{L,v_n}(\pi_L) \in \mathfrak{X}_{L,1}^{(n)}$$

and

$$g'(\pi_L) \bar{D}_{L,v_n}(v_m) - f'(\pi_L) g'(\pi_L) \bar{D}_{L,v_n}(\pi_L) \in \mathfrak{X}_{L,1}^{(n)},$$

so that

$$f'(\pi_L) \bar{D}_{L,v_n}(u) - g'(\pi_L) \bar{D}_{L,v_n}(v_m) \in \mathfrak{X}_{L,1}^{(n)}. \quad (3.3.9)$$

Now, since the calculation in the beginning of this proof shows that  $\frac{v_m}{f'(\pi_L)} \in \mathfrak{p}_L$ , we can multiply (3.3.9) by  $\frac{v_m}{f'(\pi_L)}$  in the fractional ideal  $\mathfrak{X}_{L,1}^{(n)}$ . Therefore, we get

$$v_m \bar{D}_{L,v_n}(u) = v_m \frac{g'(\pi_L)}{f'(\pi_L)} \bar{D}_{L,v_n}(v_m) \in \mathfrak{X}_{L,1} / \mathfrak{X}_{L,1}^{(n)}.$$

Finally, we can write

$$\begin{aligned} \chi_{L,m,n}(u) &\equiv \mathbb{T}_{L|K} \left( \frac{1-u}{u^2} (u \bar{D}_{L,v_n}(v_m) - v_m \frac{g'(\pi_L)}{f'(\pi_L)} \bar{D}_{L,v_n}(v_m)) \right) \pmod{\mathfrak{p}_K^{nm_0}} \\ &\equiv \mathbb{T}_{L|K} \left( \left( \frac{1-u}{u} \right) \left( 1 - \frac{\frac{g'(\pi_L)}{u}}{\frac{f'(\pi_L)}{v_m}} \right) \bar{D}_{L,v_n}(v_m) \right) \pmod{\mathfrak{p}_K^{nm_0}}. \end{aligned}$$

□

**Lemma 3.3.4.** Let  $m \geq n$  and suppose  $L \supset E_\rho^m$  is such that  $p$  does not divide the ramification index of the extension  $L|E_\rho^m$ . Then,  $\bar{D}_{L,v_n}(v_m) \in \mathfrak{X}_{L,1} / \frac{\eta^n}{\gamma} \mathfrak{X}_{L,1}$  is uniquely determined by (3.3.6).

*Proof.* Let  $x$  and  $x'$  be two elements in  $\mathfrak{X}_{L,1}$  such that

$$\mathbb{T}_{L|K} \left( \left( \frac{1-u}{u} \right) \left( 1 - \frac{\frac{g'(\pi_L)}{u}}{\frac{f'(\pi_L)}{v_m}} \right) x \right) \equiv \mathbb{T}_{L|K} \left( \left( \frac{1-u}{u} \right) \left( 1 - \frac{\frac{g'(\pi_L)}{u}}{\frac{f'(\pi_L)}{v_m}} \right) x' \right) \pmod{\mathfrak{p}_K^{nm_0}}, \quad (3.3.10)$$

for all  $u \in \mathcal{U}_L$  such that  $\mu(1-u) > \max\{\frac{nm_0}{q}, \frac{1}{q-1}\} + \frac{1}{q-1}$ . This means that

$$\mathbb{T}_{L|K} \left( \left( \frac{1-u}{u} \right) \left( 1 - \frac{\frac{g'(\pi_L)}{u}}{\frac{f'(\pi_L)}{v_m}} \right) (x - x') \right) \in \mathfrak{p}_K^{nm_0} \quad (3.3.11)$$

for all units  $u \in L$  such that  $\mu(1-u) > \max\{\frac{nm_0}{q}, \frac{1}{q-1}\} + \frac{1}{q-1}$ . We need to prove that  $x - x' \in \mathfrak{X}_{L,1}^{(n)}$ . Since we are considering any  $u$  such that  $\mu(1-u) > \max\{\frac{nm_0}{q}, \frac{1}{q-1}\} + \frac{1}{q-1}$ , then we can write  $1-u = \gamma\alpha$ , where  $\gamma \in L$  is of valuation  $\max\{\frac{nm_0}{q}, \frac{1}{q-1}\}$  and  $\alpha$  varies in

$\mathfrak{p}_{L,1} = \lambda_\rho(\mathfrak{p}_{L,1})$ . Furthermore, the element  $1 - \frac{\frac{g'(\pi_L)}{u}}{\frac{f'(\pi_L)}{v_m}}$  is a unit in  $L$ . Therefore,  $x$  and  $x'$  are such that

$$\mathbb{T}_{L|K}(\gamma\alpha(x - x')) \in \mathfrak{p}_K^{nm_0} \quad (3.3.12)$$

for all  $\alpha \in \mathfrak{p}_{L,1}$ . This yields that  $x - x' \in \frac{\eta^n}{\gamma} \mathfrak{X}_{L,1} = \mathfrak{X}_{L,1}^{(n)}$ . □



### 3.4 A return to the case of Chapter 2

In this section, we place ourselves again in the case where

$$\rho_\eta \equiv \tau^{m_0} \pmod{\mathfrak{p}_H} \quad \text{and} \quad L = E_\rho^m \supset E_\rho^n \quad (3.4.1)$$

for an integer  $m \geq n$ . As previously shown in Theorem 3.2.10, we have

$$(\alpha, v_m)_{L,n} = T_{L|K}(\lambda_\rho(\alpha) \frac{1}{v_m} \bar{D}_{L,v_n}(v_m)) \cdot_\rho v_n \quad (3.4.2)$$

for all  $\alpha \in \mathfrak{p}_L$  such that  $\mu(\alpha) \geq \max\{\frac{nm_0}{q}, \frac{1}{q-1}\} + \frac{1}{q-1} + \frac{1}{q^{nm_0}(q-1)}$ . On the other hand, we can prove using Chapter 2, that under the same condition on  $\alpha$ , we have

$$(\alpha, v_m)_{L,n} = \frac{1}{\eta^m} T_{L|K}(\lambda_\rho(\alpha) \frac{1}{v_m}) \cdot_\rho v_n. \quad (3.4.3)$$

Indeed,

$$\begin{aligned} (\alpha, v_m)_{L,n} &= (\rho_{\eta^{m-n}}(\alpha), v_m)_{L,m} && \text{(by Proposition 2.2.2 (v))} \\ &= \frac{1}{\eta^m} T_{L|K}(\lambda_\rho(\rho_{\eta^{m-n}}(\alpha)) \frac{1}{v_m}) \cdot_\rho v_m && \text{(by Theorem 2.3.12)} \\ &= \frac{1}{\eta^m} T_{L|K}(\eta^{m-n} \lambda_\rho(\alpha) \frac{1}{v_m}) \cdot_\rho v_m \\ &= \frac{1}{\eta^m} T_{L|K}(\lambda_\rho(\alpha) \frac{1}{v_m}) \cdot_\rho v_n. \end{aligned}$$

Here, we can apply Theorem 2.3.12 for  $(\rho_{\eta^{m-n}}(\alpha), v_m)_{L,m}$  because  $\mu(\rho_{\eta^{m-n}}(\alpha)) \geq \max\{\frac{mm_0}{q}, \frac{1}{q-1}\} + \frac{1}{q-1} + \frac{1}{q^{mm_0}(q-1)}$  for all  $m \geq n$  (See Lemma 2.2.3 and Remark 2.2.4).

**Proposition 3.4.1.** We have

$$\bar{D}_{E_\rho^m, v_n}(v_m) = \frac{1}{\eta^m}. \quad (3.4.4)$$

*Proof.* This is a direct consequence of the explicit formulas (3.4.2) and (3.4.3).  $\square$

**Corollary 3.4.2.** Let  $u$  be a unit of  $L$  such that  $\mu(1-u) > \max\{\frac{nm_0}{q}, \frac{1}{q-1}\} + \frac{1}{q-1}$ . Then

$$N_{L|K}(u^{-1}) - 1 \equiv T_{L|K}((\frac{1-u}{u})(1 - \frac{g'(v_m)}{u} v_m)) \pmod{\mathfrak{p}_K^{(n+m)m_0}}, \quad (3.4.5)$$

where  $g(X) \in \mathbb{F}_{q_L}[[X]]$  is such that  $g(v_m) = u$ .

*Proof.* Since we proved in Lemma 3.3.4 that  $\bar{D}_{L,v_n}(v_m) = \frac{1}{\eta^m} \in \mathfrak{X}_{L,1}/\mathfrak{X}_{L,1}^{(n)}$  is uniquely determined by (3.3.6), we have

$$\chi_{L,m,n}(u) \equiv \frac{1}{\eta^m} T_{L|K}((\frac{1-u}{u})(1 - \frac{g'(v_m)}{u} v_m)) \pmod{\mathfrak{p}_K^{nm_0}} \quad (3.4.6)$$

for all units  $u$  of  $L$  such that  $\mu(1-u) > \max\{\frac{nm_0}{q}, \frac{1}{q-1}\} + \frac{1}{q-1}$ . Moreover, we know by (3.3.3) that

$$(v_m, u)_{L,n} = \chi_{L,m,n}(u) \cdot_{\rho} v_n = \eta^m \chi_{L,m,n}(u) \cdot_{\rho} v_{m+n}, \quad (3.4.7)$$

where  $v_{m+n} \in W_{\rho}^{m+n}$  is such that  $\rho_{\eta^n}(v_{m+n}) = v_m$ . On the other hand, by the definition of  $(\cdot, \cdot)_{L,n}$ , we have

$$(v_m, u)_{L,n} = \Phi_L(u)(v_{m+n}) - v_{m+n} = \Phi_K(N_{L|K}(u))(v_{m+n}) - v_{m+n}. \quad (3.4.8)$$

But  $\Phi_K(N_{L|K}(u))(v_{m+n}) = \rho_{N_{L|K}(u)^{-1}}(v_{m+n})$  by Proposition 2.3.4. Therefore,  $(v_m, u)_{L,n} = (N_{L|K}(u^{-1}) - 1) \cdot_{\rho} v_{m+n}$  and hence,

$$N_{L|K}(u^{-1}) - 1 \equiv \eta^m \chi_{L,m,n}(u) \pmod{\mathfrak{p}_K^{(n+m)m_0}}. \quad (3.4.9)$$

Finally, (3.4.5) follows from (3.4.6) and (3.4.9).  $\square$

Giving a direct proof of (3.4.5) would provide a new proof of the results of Chapter 2. In another request, it would be interesting to investigate the cohomological approach inspired by Kato [19, 20], in order to establish new proofs of the explicit reciprocity laws for formal Drinfeld modules. This approach was used, along with the theory of  $(\varphi, \Gamma)$ -modules, by Benois [7] and others [14, 31] to prove explicit formulas for the Kummer pairing associated with formal groups over local fields of characteristic zero.

# Bibliography

- [1] M. Ala Eddine. Explicit reciprocity laws for formal Drinfeld modules. *arXiv:2202.02348*, 2022.
- [2] M. Ala Eddine. The norm residue symbol for formal Drinfeld modules. *arXiv:2211.10368*, 2022.
- [3] B. Anglès. On explicit reciprocity laws for the local Carlitz-Kummer symbols. *J. Number Theory*, 78(2):228–252, 1999.
- [4] E. Artin and H. Hasse. Die beiden Ergänzungssätze zum Reziprozitätsgesetz der  $l^n$ -ten Potenzreste im Körper der  $l^n$ -ten Einheitswurzeln. *Abh. Math. Sem. Univ. Hamburg*, 6(1):146–162, 1928.
- [5] F. Bars and I. Longhi. Reciprocity laws à la Iwasawa-Wiles. In *Proceedings of the “Segundas Jornadas de Teoría de Números”*, Bibl. Rev. Mat. Iberoamericana, pages 39–62. Rev. Mat. Iberoamericana, Madrid, 2008.
- [6] F. Bars and I. Longhi. Coleman’s power series and Wiles’ reciprocity for rank 1 Drinfeld modules. *J. Number Theory*, 129(4):789–805, 2009.
- [7] D. Benois. Périodes  $p$ -adiques et lois de réciprocité explicites. *J. Reine Angew. Math.*, 493:115–151, 1997.
- [8] J. Coates and A. Wiles. On  $p$ -adic  $L$ -functions and elliptic units. *J. Austral. Math. Soc. Ser. A*, 26:1–25, 1978.
- [9] R. F. Coleman. Division values in local fields. *Invent. Math.*, 53(2):91–116, 1979.
- [10] R. F. Coleman. The dilogarithm and the norm residue symbol. *Bull. Soc. Math. France*, 109:373–402, 1981.
- [11] V. G. Drinfeld. Elliptic modules. *Mat. Sb. (N.S.)*, 94(136):594–627, 656, 1974.
- [12] J. Flórez. Explicit reciprocity laws for higher local fields. *J. Number Theory*, 213:400–444, 2020.
- [13] Jorge Flórez. The norm residue symbol for higher local fields. *J. Number Theory*, 235:157–218, 2022.

- [14] T. Fukaya. Explicit reciprocity laws for  $p$ -divisible groups over higher dimensional local fields. *J. Reine Angew. Math.*, 531:61–119, 2001.
- [15] D. Goss. *Basic structures of function field arithmetic*, volume 35 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1996.
- [16] M. Hazewinkel. *Formal groups and applications*. AMS Chelsea Publishing, Providence, RI, 2012. Corrected reprint of the 1978 original.
- [17] K. Iwasawa. On some modules in the theory of cyclotomic fields. *J. Math. Soc. Japan*, 16:42–82, 1964.
- [18] K. Iwasawa. *Local class field theory*. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1986. Oxford Mathematical Monographs.
- [19] K. Kato. Lectures on the approach to Iwasawa theory for Hasse-Weil  $L$ -functions via  $B_{\text{dR}}$ . I. In *Arithmetic algebraic geometry (Trento, 1991)*, volume 1553 of *Lecture Notes in Math.*, pages 50–163. Springer, Berlin, 1993.
- [20] K. Kato. Generalized explicit reciprocity laws. volume 1, pages 57–126. 1999. Algebraic number theory (Hapcheon/Saga, 1996).
- [21] V. A. Kolyvagin. Formal groups and the norm residue symbol. *Izv. Akad. Nauk SSSR Ser. Mat.*, 43(5):1054–1120, 1198, 1979.
- [22] M. Kurihara. The exponential homomorphisms for the Milnor  $K$ -groups and an explicit reciprocity law. *J. Reine Angew. Math.*, 498:201–221, 1998.
- [23] S. Lang. *Cyclotomic fields I and II*, volume 121 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1990. With an appendix by Karl Rubin.
- [24] S. Lang. *Algebra*, volume 211 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, third edition, 2002.
- [25] F. Lemmermeyer. *Reciprocity laws*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2000. From Euler to Eisenstein.
- [26] J. Lubin and J. Tate. Formal complex multiplication in local fields. *Ann. of Math. (2)*, 81:380–387, 1965.
- [27] J. Neukirch. *Algebraic number theory*, volume 322 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999. Translated from the 1992 German original and with a note by Norbert Schappacher, With a foreword by G. Harder.
- [28] H. Oukhaba. On local fields generated by division values of formal Drinfeld modules. *Glasg. Math. J.*, 62(2):459–472, 2020.

- [29] M. Rosen. Formal Drinfeld modules. *J. Number Theory*, 103(2):234–256, 2003.
- [30] S. Sen. On explicit reciprocity laws. *J. Reine Angew. Math.*, 313:1–26, 1980.
- [31] F. Tavares Ribeiro. An explicit formula for the Hilbert symbol of a formal group. *Ann. Inst. Fourier (Grenoble)*, 61(1):261–318, 2011.
- [32] S. Vostokov. Explicit form of the law of reciprocity. *Mathematics of the USSR-Izvestiya*, 13:557–588, 10 2007.
- [33] L. C. Washington. *Introduction to cyclotomic fields*, volume 83 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1997.
- [34] A. Wiles. Higher explicit reciprocity laws. *Ann. of Math. (2)*, 107(2):235–254, 1978.
- [35] A. Zinoviev. Generalized artin-hasse and iwasawa formulas for the hilbert symbol in a higher local field. ii. *Journal of Mathematical Sciences*, 136:3935–3941, 2006.