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## Magnetohydrodynamic of a non-newtonien fluid.

Elsy Wehbe

► **To cite this version:**

Elsy Wehbe. Magnetohydrodynamic of a non-newtonien fluid.. Mathematical Physics [math-ph].  
Université de Pau et des Pays de l'Adour, 2023. English. NNT : 2023PAUU3011 . tel-04421152

**HAL Id: tel-04421152**

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# THESIS

For the grade of

## Doctor of Philosophy in Mathematics

Funded by the University of Djibouti and realised at

**UNIVERSITÉ DE PAU ET DES PAYS DE L'ADOUR**

by

**Elsy Wehbe**

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## Magnetohydrodynamic of a non-Newtonian fluid with Navier-type boundary conditions.

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Thesis defence on 09/06/2023

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Dedicated to



*My wonderful parents,*

*My special husband,*

*My lovely kids,*

and *My only sister.*

# *Acknowledgments*

First of all, I want to thank God, the most Powerful, and without him, nothing can be achieved including my thesis.

Thanks to my parents who are the source of any achievement I did and I can do, you are the reason for all my successes. Thank you for your great scarifies, especially for my mother who, since we were children, has explained the importance of science and taught us to go further and further. I thank her to be next by my side in the hardest moments of my thesis.

I would like to express my deepest appreciation to my husband to whom the idea of my thesis is due, despite our family responsibilities and my role as mother, he did not want me to stop dreaming and then realize my dreams.

Also, I want to thank the University of Djibouti which gave me the scholarship and the financial support to make a P.H.D. in France. A special thanks to Dr. Abdoukader Ibrahim Idriss who defended my file and did everything to make it work.

I am deeply indebted to my advisor, Prof. Cherif Amrouche for his non-stop support of study and research. For his patience, because my starting conditions were not excellent. For his incessant motivation because I have thought of abandonment several times, every time I walk out of his office with a different look from the way I came in. For the past few years, I have been in an endless learning ball, under great supervision, and this teaches me in a simple way the hardest things I knew. I am very lucky to be a student of Professor C. Amrouche, I am extremely grateful for every piece of advice you gave me whatever the subject is.

Thanks should also go to the referees Pr. Bresch and Pr. Ruzicka and all members of the jury agreed to be a part of my great day.

I thank my children who gave me courage just by seeing them and gave me the

joy of weariness.

I would like to extend my sincere thanks to the administration of the Laboratory of Mathematics and their Applications and the Ecole Doctorale of Pau for their Kind handling and the great working conditions and the numerous activities and training provided.

Last, my very sincere thanks to all my friends and colleagues for their encouragement and support. A big thank you to my special colleague Saliha Boukassa, for all discussions and laugh we had together.

# Contents

<b>1</b>	<b>Introduction</b>	<b>8</b>
1.1	Applications . . . . .	11
1.1.1	Natural phenomenon . . . . .	11
1.1.2	Technological applications . . . . .	12
1.2	Different models of MHD . . . . .	15
1.3	Newtonien and non-Newtonian fluids . . . . .	19
1.4	Grade-two fluids . . . . .	20
1.5	Navier-type boundary conditions . . . . .	23
1.6	Thesis description and main results . . . . .	25
<b>2</b>	<b>Stokes Problem in 2D</b>	<b>31</b>
2.1	Functional framework . . . . .	32
2.1.1	Geometry of the domain . . . . .	32
2.1.2	Functional spaces . . . . .	33
2.1.3	Some properties related to the curl operator. . . . .	34
2.1.4	An important identity result in $\mathbb{R}^2$ . . . . .	36
2.2	Preliminary results . . . . .	41
2.3	Stokes problem with free-divergence forces . . . . .	52
2.4	Stokes problem . . . . .	57
<b>3</b>	<b>MHD flow with Navier-type boundary conditions</b>	<b>60</b>
3.1	Existence of solutions for the MHD problem . . . . .	64
3.2	Regularity of the solution . . . . .	74

<b>4</b>	<b>Existence and regularity for polymer aqueous solution</b>	<b>77</b>
4.1	Functional framework and preliminary results . . . . .	80
4.2	Stationary case . . . . .	86
4.3	Evolution problem . . . . .	92
4.3.1	Statement of the problem . . . . .	93
4.3.2	Existence of solutions . . . . .	96



# Chapter 1

## Introduction

### Contents

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<b>1.1 Applications . . . . .</b>	<b>11</b>
1.1.1 Natural phenomenon . . . . .	11
1.1.2 Technological applications . . . . .	12
<b>1.2 Different models of MHD . . . . .</b>	<b>15</b>
<b>1.3 Newtonien and non-Newtonian fluids . . . . .</b>	<b>19</b>
<b>1.4 Grade-two fluids . . . . .</b>	<b>20</b>
<b>1.5 Navier-type boundary conditions . . . . .</b>	<b>23</b>
<b>1.6 Thesis description and main results . . . . .</b>	<b>25</b>

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In this thesis we consider the following system modeling a magnetohydrodynamic (MHD) flow:

$$-\nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)(\mathbf{u} - \alpha \Delta \mathbf{u}) + \nabla \pi - (\mathbf{B} \cdot \nabla) \mathbf{B} + \frac{1}{2} \nabla (|\mathbf{B}|^2) = \mathbf{f} \quad \text{in } \Omega, \quad (1.0.1)$$

$$-\Delta \mathbf{B} - (\mathbf{B} \cdot \nabla) \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{B} + \nabla \theta = \mathbf{0} \quad \text{in } \Omega, \quad (1.0.2)$$

where the unknowns  $\mathbf{u}$  represent the velocity field,  $\pi$  the pressure of the fluid and  $\mathbf{B}$  the magnetic field. The external force acting on the fluid is denoted by  $\mathbf{f}$ ,  $\nu$  is the

viscosity and  $\theta$  is an unknown function related to the motion of heavy ions.

In this work, we consider  $\Omega$  a two-dimensional bounded domain possibly multiply connected. This system characterizes the motion of an aqueous polymer solution coupled with magnetic induction where there is a free motion of heavy ions. When the relaxational viscous coefficient  $\alpha$ , is equal to zero, this system is reduced to the special case and the most considered MHD problem which couples the Navier-Stokes equations with the Maxwell equations.

Magnetohydrodynamics (MHD) is a particular discipline of fluid mechanics that studies the behavior of electrically conducting fluids in the presence of magnetic fields. MHD is an interdisciplinary field that draws on principles from physics, mathematics, and engineering. This field of study is concerned with understanding the dynamics and flow of these fluids and their interactions with the surrounding magnetic field. Due to its wide range of applications, MHD has garnered significant attention in various fields such as geophysics, astrophysics (e.g., studying the dynamics of stars and galaxies), engineering, geophysics (e.g., modeling the Earth's magnetic field), energy generation (e.g., designing plasma fusion reactors), and even biomedical engineering (e.g., developing magnetic drug delivery systems).

In the early 1820s, the British Humphry Davy (1778—1829) demonstrated the effect of a magnetic field on an electric current, which is now known as the "MHD effect". His experiments showed that a magnetic field can deflect an electric current, and this principle laid the foundation for later research on the interaction between magnetic fields and electrically conducting fluids, such as plasma. Davy's experiments were then pushed further by the famous physicist Michael Faraday (1791–1867) who, in 1832, began a series of studies devoted to the MHD production of energy.



Figure 1.1:  
Hannes Alfvén

In 1942, the Swedish electrical engineer and plasma physicist, Hannes Alfvén (1908 –1995) demonstrated the existence of MHD waves. The first user of the "magnetohydrodynamics" term, established the link between electromagnetism and hydrodynamics to write the current equations of the phenomenon. Alfvén also made important contributions to the understanding of the behavior of plasmas in space, particularly in the context of the Earth's magnetic field. He proposed the concept of "magnetospheric substorms," which are sudden and dramatic changes in the behavior of the Earth's magnetosphere that occur in response to changes in the solar wind.

Michael Faraday tested the MHD generator concept for the first time in 1832. The physicist followed the theoretical and experimental discovery of electromagnetism and electrodynamics and suggested using the natural vertical component of the Earth's magnetic field. He placed copper plates on each side of the Waterloo Bridge in London, submerged in the water and connected by a 290-meter-long electrical wire. However, the equipment of that time did not reveal the low electrical current generated. Luckily William Hyde Wollaston showed, in 1851, that a magnetic needle is deflected by an electric current.

Faraday's experiments involved passing a stream of liquid mercury through a magnetic field and observing the resulting electric current that was produced. He observed that the electric current was generated perpendicular to both the magnetic field and the flow of the liquid metal, which is now known as Faraday's law of induction. His work on MHD was an important contribution to the understanding of the interaction between magnetic fields and electrically conducting fluids, and it laid the groundwork for later research on the use of MHD for energy production. However, it would take several more decades of research before MHD generators became a practical technology.

## 1.1 Applications

After the discovery of W. H. Wollaston, a lot of research was developed until the application of MHD in our life. We mention here some of the most important application fields.

### 1.1.1 Natural phenomenon

One natural phenomenon that involves Magnetohydrodynamics (MHD) is the Earth's magnetosphere. The magnetosphere is the region surrounding the Earth that is influenced by its magnetic field. The solar wind, a stream of charged particles that flows out from the Sun, interacts with the Earth's magnetic field and produces MHD effects.

As the solar wind approaches the Earth, it compresses the Earth's magnetic field on the side facing the Sun and stretches it out on the opposite side. This creates a long tail-like structure in the magnetosphere, called the magnetotail. As the charged particles in the solar wind interact with the Earth's magnetic field, they can generate electric currents that flow through the plasma in the magnetosphere, producing MHD waves and turbulence. In astrophysics, this phenomenon created by the reconnection of the earth's magnetosphere and the sun's atmosphere to the erupting stars provides various mechanisms of particle acceleration and shows the importance of the MHD process. For example, during a solar eruption, a very large plasma bubble can be formed, reconnect and modify the solar wind whose variation reaches the earth three days later. The earth's magnetic field attracts these charged particles, often at the north and south poles, and causes what we call the "aurora borealis" as Figure 1.3. These are bright displays of light in the sky that occur when charged particles from the solar wind interact with the Earth's upper atmosphere. The MHD processes that drive the auroras are complex and the magnetic field associated with this phenomenon is very intense and can cause magnetic storms and disturbances to electronic equipment.

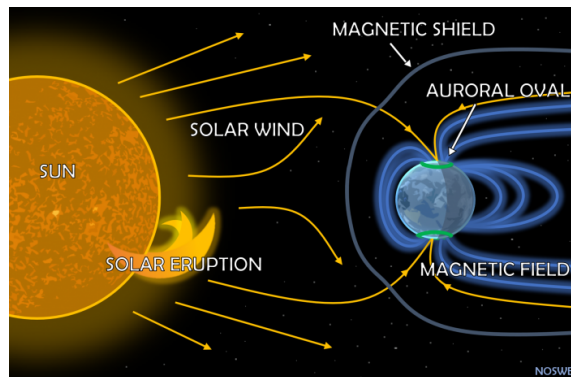


Figure 1.2: Continuous flow of electric particles (the solar wind) flowing from the Sun into the Earth. Source: <https://site.uit.no/spaceweather/aurora-borealis/>, visited on 28/10/2022.



Figure 1.3: Aurores-boreales au Canada. Source: <https://www.terra-canada.com/blog/non-classe/comment-voir-des-aurores-boreales-au-canada>.

### 1.1.2 Technological applications

Here are some technological examples of MHD:

First, we describe here the MHD generator and we mention its usefulness. This type of generator converts the kinetic energy of a conductive fluid directly into electricity with the same principle as the conventional generator. Both types use an inductor to generate a magnetic field in an armature. However, in the MHD generator, conducting fluid is used instead of a metal conductor. The fluid is put into

motion in the magnetic field, which generates an electric current, collected at the limits of submerged electrodes and switched to a charge as shown in Figure 1.3.

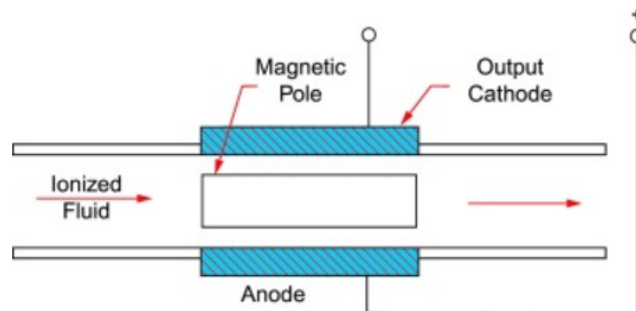


Figure 1.4: An MHD generator. Source <https://newenergytreasure.files.wordpress.com/2014/01/ciht-mhd-1.jpg>.

Many advantages of the MHD generator in applications are used, such as for driving submarines, aircraft, hypersonic wind tunnel experiments, and defense applications. They are used as an uninterrupted power supply system and as power plants in industries to generate electric power for domestic applications.

In medicine, the laws of magnetohydrodynamics govern the motion of a conducting fluid, here is the blood, in a magnetic field applied externally. MHD is used for example for laser beam scanning, nanoparticle manipulations, biomedical imaging contrast enhancement, and in the application of drug delivery in blood flow. More precisely, targeting of MHD drugs is the delivery method of magnetically triggered therapeutic agents that increases the concentration of the agent in the target area.

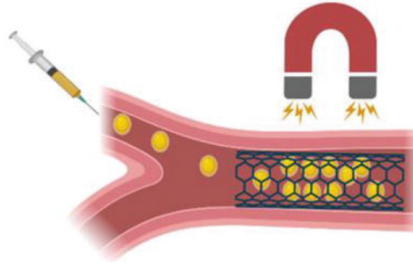


Figure 1.5: Implant-assisted magnetic drug targeting with the magnetic stent implanted in the vasculature. Source: [www.intechopen.com/chapters/68954](http://www.intechopen.com/chapters/68954).

Notably, some methods for the delivery of drugs to the affected area have been studied in some cancer research. One of these methods involves the binding of a magnetizable implant, which is guided to the target via the careful placement of permanent magnets on the external body. It has been used in applications related to the cardiovascular system as used at Magnetic Resonance Imaging (MRI) facilities or digestive and urinary systems. For more details, one can check [23]. MHD equations and finite element analysis are used to study the interaction between the magnetic fluid particles in the bloodstream and the external magnetic field. Blood flow in a steady magnetic field has been very interesting in recent years. In [15], authors studied a pulsed MHD blood flow in a rigid vessel with a pressure gradient obtained using a lumped model.

Moreover, there exist other types of applications for the MHD. Mainly in the industry using MHD accelerators in the form of electromagnetic pumps also in the propulsion aspects of high-tech vehicles (using only electric and magnetic fields accelerating an electrically conductive propellant) and for certain types of military weapons such as railguns.

## 1.2 Different models of MHD

The magnetic Reynolds number ( $Rm$ ) separates two Large categories of the MHD. The ideal MHD (where  $Rm$  is greater than 1) and the non-ideal MHD models (with  $Rm \ll 1$ ). It is defined as the ratio of inertial forces to viscous forces and indicates the importance of the convection term compared to the diffusion term in a fluid subjected to a magnetic field. In the following, we explain the ideal, resistive, and Hall-MHD types.

### Ideal MHD :

The simplest form of MHD is known as the ideal MHD. The fluid, subjected to a strong magnetic field has little or no electrical conductivity ( $\frac{1}{\sigma} \rightarrow 0$ ) and is assimilated to a perfect conductor meaning that the magnetic field lines can move freely through it without any resistance. Additionally, the fluid is assumed to be inviscid, meaning that there is no internal friction within the fluid. In this case, due to the dominance of the magnetic field, the high conductivity assumptions, and for simplicity the finite viscosity, electrical resistivity, and thermal conductivity are neglected.

The concept of ideal MHD is a useful theoretical tool for understanding the behavior of plasma in laboratory and astrophysical settings. However, in reality, fluids are never perfectly conductive or inviscid, so real-world MHD phenomena must be studied using more complex models that take into account these factors. The MHD equations consist of :

- the equation of mass continuity which states that matter is neither created or destroyed,
- the Euler equation that describes the motion of an element of the fluid,
- the equations of thermodynamics or conservation of energy
- the induction equation of Maxwell.



The latest equation describes the evolution of electric field  $\mathcal{E}$  and magnetic field  $\mathbf{B}$  in response to current density  $\mathbf{j}$  and space charge  $\tau$ . In other terms, it describes how the magnetic field in a perfectly conducting fluid changes with time under the influence of a velocity field. To introduce the Maxwell equation, we need to start with the Faraday's and Ampere's Laws:

$$\mathbf{curl} \mathcal{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \text{and} \quad \mathbf{curl} \mathbf{B} = \mu_0 \mathbf{j}, \quad (1.2.1)$$

where  $\mu_0$  is the permeability (the ability of a material to allow the magnetic flux when the object is placed inside the magnetic field). The electric field can be related to the current density using Ohm's law:

$$\mathcal{E} + \mathbf{u} \times \mathbf{B} = \frac{1}{\sigma} \mathbf{j}. \quad (1.2.2)$$

We eliminate  $\mathbf{j}$  and  $\mathcal{E}$ , by combining the three equations above, and it shows that the magnetic field  $\mathbf{B}$  satisfies the following induction equation where the RHS is the addition of the diffusive term and the convective term :

$$\frac{\partial \mathbf{B}}{\partial t} = \eta \nabla^2 \mathbf{B} + \nabla \times (\mathbf{u} \times \mathbf{B}), \quad (1.2.3)$$

with  $\eta = \frac{1}{\mu_0 \sigma}$  is the magnetic diffusivity. As  $\frac{1}{\sigma} \rightarrow 0$ , the first term of the RHS in (1.2.3) vanishes. Thus, the closed set of MHD equations is :

$$\frac{\partial \rho}{\partial t} + \nabla(\rho \mathbf{u}) = 0, \quad \text{Mass continuity equation}$$

$$\frac{d}{dt} \left( \frac{\pi}{\rho} \right) = 0, \quad \text{Energy equation}$$

$$\rho \frac{d\mathbf{u}}{dt} = -\nabla \pi - \frac{1}{\mu_0} \mathbf{B} \times (\mathbf{curl} \mathbf{B}) \quad \text{Euler's Equation}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \mathbf{curl} (\mathbf{u} \times \mathbf{B}) \quad \text{Induction Equation}$$

$$\text{div} \mathbf{B} = 0 \quad \text{additional condition.}$$

This set connects the plasma mass density  $\rho$ , the plasma velocity  $\mathbf{u}$ , the thermodynamic (also called gas or kinetic), pressure  $\pi$ , and the magnetic field  $\mathbf{B}$ . This ideal MHD is studied in warm plasmas, as the astrophysical plasmas and thermonuclear (stars or tokamaks).

**Geometrical meaning of  $\text{div } \mathbf{B} = 0$ :** In general, a divergence-free field expresses that each part of the fluid preserves its volume. Particularly,  $\text{div } \mathbf{B} = 0$  means that the magnetic field lines only form closed loops. Magnetic field lines wander without encountering monopolar singularities, and regions of lower magnetic field strength distort the magnetic field around them.

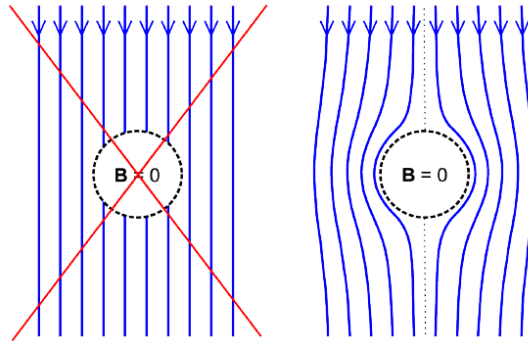


Figure 1.6: The configuration with field lines ending on the surface of the inclusion is excluded when  $\text{div } \mathbf{B} = 0$ . Source: [31]

In other words, a divergence-free field represents a fluid or substance that is flowing without any accumulation or depletion in any part of the field. This is analogous to the flow of an incompressible fluid, such as water, through a pipe or channel, where the volume of the fluid remains constant as it flows.

One consequence of the divergence-free property is that the flow lines of the vector field do not cross each other, as this would imply the creation or destruction of fluid at the intersection point. Instead, the flow lines are continuous and unbroken, and they can be used to visualize the direction and magnitude of the flow at each point in the field.

Overall, the geometrical meaning of a free divergence field is that it represents an

incompressible flow, in which the fluid or substance is conserved and the flow lines do not cross or accumulate.

### Resistive MHD :

In contrast to ideal MHD, non-ideal MHD refers to a more realistic model of the dynamics of electrically conducting fluids in the presence of magnetic fields, which takes into account the effects of various non-ideal processes that occur in real-world fluids. In non-ideal MHD, the fluid is no longer assumed to be perfectly conductive or inviscid, but instead, several non-ideal processes are taken into account, which can significantly affect the fluid's behavior. Some of the non-ideal processes that can be considered in non-ideal MHD include:

- Resistivity: Real-world fluids have a finite resistivity, which means that the magnetic field lines can experience some resistance as they move through the fluid.
- Viscosity: In real-world fluids, there is always some internal friction, which can lead to the dissipation of energy and momentum within the fluid.
- Thermal conduction: Heat can be transferred through the fluid via thermal conduction, which can affect the temperature distribution and overall behavior of the fluid.
- Turbulence: Turbulent flows can significantly impact the behavior of the fluid and can result in complex and chaotic behavior.

Non-ideal MHD is essential for understanding many phenomena in plasma physics and astrophysics, including the behavior of plasma in fusion devices and the dynamics of the Earth's magnetosphere. However, it also poses significant mathematical challenges, as the equations governing the behavior of non-ideal MHD are much more complex than those for ideal MHD. In this case, the induction equation is (1.2.3).

### Hall-MHD :

The magnetic reconnection in space plasmas, star formation, and neutron stars are described as a Hall MHD problem. This type of MHD problem requires an extra term related to the hall effect. It restores the influence of the electric current in the Lorenz force that occurs in Ohm's law. When the magnetic shear is large, which occurs during reconnection events, the second order derivative term  $\mathbf{curl}(\mathbf{curl} \mathbf{B} \times \mathbf{B})$  is dominant and makes the problem harder to solve. Understanding the behavior of plasmas in these environments is important for space weather forecasting and for developing fusion energy as a clean and sustainable source of power whereas in laminar situations, this term is small and can be neglected and for this reason, other MHD models ignore it.

### 1.3 Newtonien and non-Newtonian fluids

We define Newtonian fluids when the shear stresses are proportional to the speed gradient. In a simple shear flow, shear stresses are the only stresses caused by the flow and the viscosity depends on shear speed and on time. There are many types of non-Newtonian fluids, each with its own unique properties and applications. The most commonly used non-Newtonian fluids depend on the specific application and the desired rheological properties. This makes them more challenging to model but also makes them useful in a wide range of industrial and scientific applications. However, the non-Newtonian fluids are mainly classified into four categories on the basis of their behavior in shear:

- Dilatant or shear Thickening when Viscosity increases with increased stress as Suspensions of cornstarch in water.
- Pseudoplastic or shear Thinning when Viscosity decreases with increased stress as ketchup.
- Thixotropic when viscosity reduces with time as a result of stress as yogurt.
- Rheopectic when viscosity rises over time as a result of stress.

## 1.4 Grade-two fluids

The stress in a fluid of differential type of complexity  $j$  depends upon the velocity gradient and its first  $(j - 1)$  derivatives. The Newtonian and second-grade fluids are the simplest fluids of the differential type when the dependence upon the velocity gradient is given a more specific form. Second-grade fluid may be the simplest non-Newtonian model that can represent the normal stress effects of an incompressible fluid. FA grade-two fluid is a non-Newtonian fluid differential type of complexity two, as defined by the following constitutive equation where the stress tensor  $\mathbf{T}$  is given by:

$$\mathbf{T} = -p\mathbf{I} + F(\mathbf{A}_1, \mathbf{A}_2), \quad (1.4.1)$$

where  $p$  is the intermediate part of the stress tensor due to the constraint of incompressibility and  $\mathbf{A}_1, \mathbf{A}_2$  are the first two Rivlin-Eriksen tensors defined by [29] as:

$$\begin{aligned} \mathbf{A}_1 &= \mathbf{L} + \mathbf{L}^T, & L &= \nabla \mathbf{u} \\ \mathbf{A}_2 &= \left( \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \mathbf{A}_1 + \mathbf{A}_1 \mathbf{L} + \mathbf{L}^T \mathbf{A}_1 = \frac{d}{dt} \mathbf{A}_1 + \mathbf{A}_1 \mathbf{L} + \mathbf{L}^T \mathbf{A}_1. \end{aligned}$$

Thus, the constitutive equation is

$$\mathbf{T} = -p\mathbf{I} + \nu \mathbf{A}_1 + \alpha_1 \mathbf{A}_2 + \alpha_2 \mathbf{A}_1^2,$$

where  $\nu$  is the viscosity and the  $\alpha_i$  are material moduli which are usually referred to as the normal stress coefficients and  $\frac{d}{dt}$  is the material time derivative defined as

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla,$$

where  $\frac{\partial}{\partial t}$  represents the partial derivative with respect to time.

Dunn and Fosdick in [16], studied the thermodynamics and stability of the second grade fluids. Authors showed that the functions of a second grade fluid are compatible

with thermodynamics if and only if

- the viscosity is non-negative,

$$\nu \geq 0, \tag{1.4.2}$$

- the normal stress coefficients are related by

$$\alpha_2 = -\alpha_1$$

in the sense that the free energy function has a minimum in equilibrium if and only if  $\alpha_1 \geq 0$ .

The equation of a grade two-fluid model is given by

$$-\nu \Delta \mathbf{u} + \mathbf{curl}(\mathbf{u} - \alpha \Delta \mathbf{u}) \times \mathbf{u} + \nabla \pi = \mathbf{f} \quad \text{in } \Omega. \tag{1.4.3}$$

The first successful mathematical analysis of the second-grade fluid was done by Cioranescu and Ouazar in [14]. In their work, they proved the local existence and uniqueness of solutions in  $\mathbf{L}^\infty((0, T^*); \mathbf{H}_0^1(\Omega) \cap \mathbf{H}^3(\Omega))$ , and in two dimensions they obtained the global existence. Later Cioranescu and Girault in [13] establish the existence, uniqueness, and regularity of a global weak solution with small initial data. The existence is obtained by applying Galerkin's method with a special basis. Bresch and Lemoine studied in [9] the existence and uniqueness of solutions for nonstationary second-grade fluid in  $\mathbf{L}^r$  theory for  $r > 3$  and in [10] the stationary case in a domain of class  $\mathcal{C}^2$ .

In a two-dimensional domain, the equation of motion for an incompressible fluid of grade two is given by:

$$\begin{aligned} \partial_t(\mathbf{u} - \alpha \Delta \mathbf{u}) - \nu \Delta \mathbf{u} + \mathbf{curl}(\mathbf{u} - \alpha \Delta \mathbf{u}) \times \mathbf{u} + \nabla \pi &= \mathbf{f} \quad \text{in } \Omega \times (0, T) \\ \operatorname{div} \mathbf{u} &= 0 \quad \text{in } \Omega \times (0, T). \end{aligned}$$

Note that, when  $\Omega \subset \mathbb{R}^2$ , the nonlinear term  $\mathbf{curl}(\mathbf{u} - \alpha \Delta \mathbf{u}) \times \mathbf{u}$  in (1.4.3) is replaced by

$$\mathbf{curl}(\mathbf{u} - \alpha \Delta \mathbf{u}) \times \mathbf{u},$$

where the scalar operator

$$\operatorname{curl} \mathbf{u} = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2},$$

and  $w \times \mathbf{u} = w(-\mathbf{u}_2, \mathbf{u}_1)$  for any scalar function  $w$ .

In [19], the authors studied the corresponding stationary model, with Dirichlet tangential boundary conditions:  $\mathbf{u} = \mathbf{g}$  on  $\Gamma$  and  $\mathbf{g} \cdot \mathbf{n} = 0$  on  $\Gamma$ . They construct a solution in  $\mathbf{H}^1(\Omega)$  on a Lipschitz-continuous domain. They proved the existence by using a combination of a generalized Stokes problem and a transport equation. In our MHD problem, the nonlinear term  $\operatorname{curl}(\mathbf{u} - \alpha \Delta \mathbf{u}) \times \mathbf{u}$  is replaced by  $(\mathbf{u} \cdot \nabla)(\mathbf{u} - \alpha \Delta \mathbf{u})$ . We indicate that for  $\mathbf{z} = \mathbf{u} - \alpha \Delta \mathbf{u}$ , we have

$$\mathbf{z} \times \mathbf{u} = (\mathbf{u} \cdot \nabla) \mathbf{z} + \sum_{j=1}^2 u_j \nabla z_j,$$

where

$$(\mathbf{u} \cdot \nabla) \mathbf{z} = \sum_{j=1}^2 u_j \frac{\partial \mathbf{z}}{\partial x_j}.$$

In literature, grade-two fluids have been considered in some MHD studies.

In particular in [22], Hamdache et al. considered the following :

$$\begin{cases} \partial_t(\mathbf{u} - \alpha \Delta \mathbf{u}) - \nu \Delta \mathbf{u} + \operatorname{curl}(\mathbf{u} - \alpha \Delta \mathbf{u}) \times \mathbf{u} + \nabla \pi = \operatorname{curl} \mathbf{B} \times \mathbf{B} & \text{in } \Omega \times (0, T) \\ \partial_t \mathbf{B} + \operatorname{curl}^2 \mathbf{B} = \operatorname{curl}(\mathbf{B} \times \mathbf{B}) & \text{in } \Omega \end{cases} \quad (1.4.4)$$

with

$$\begin{cases} \operatorname{div} \mathbf{u} = 0, \quad \mathbf{u}(0) = \mathbf{u}_0, \quad \mathbf{B}(0) = \mathbf{B}_0 & \text{in } \Omega \\ \mathbf{u} = \mathbf{0}, \quad \mathbf{B} \cdot \mathbf{n} = 0 \quad \text{and} \quad \operatorname{curl} \mathbf{B} \times \mathbf{n} = \mathbf{0} & \text{on } \Gamma. \end{cases}$$

Supposing that  $(\mathbf{u}_0, \mathbf{B}_0) \in \mathbf{V} \times \mathbf{W}$ , the authors proved that there exist  $T^* > 0$  the existence of

$$(\mathbf{u}, \mathbf{B}) \in L^\infty((0, T^*), \mathbf{V} \times \mathbf{W})$$

in a simply connected domain, where

$$\mathbf{V} = \{\mathbf{v} \in \mathbf{H}_0^1(\Omega); \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega, \operatorname{curl} (\mathbf{u} - \alpha \Delta \mathbf{u}) \in \mathbf{L}^2(\Omega)\}.$$

and

$$\mathbf{W} = \{\mathbf{v} \in \mathbf{L}^2(\Omega); \operatorname{curl} \mathbf{v} \in \mathbf{L}^2(\Omega), \Delta \mathbf{v} \in \mathbf{L}^2(\Omega), \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega, \\ \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma \text{ and } \operatorname{curl} \mathbf{v} \times \mathbf{n} = 0 \text{ on } \Gamma\}.$$

## 1.5 Navier-type boundary conditions

The well-posedness of the boundary value problems describes situations in applications. Let us have an overview of the most known boundary conditions for a fluid flow. The most famous boundary conditions today involving the equations of Stokes and Navier-Stokes in a bounded domain consider conventional no-slip conditions. Referred to as Dirichlet conditions and introduced by Stokes in 1845 in [32], assumes that at a solid boundary, the fluid will have zero velocity relative to the boundary:

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma.$$

Another known type called the slip boundary condition is the impermeability one, described as  $\mathbf{u} \cdot \mathbf{n} = 0$  on  $\Gamma$ , and means that the velocity component perpendicular to the wall is zero, that is flow can not penetrate the wall.

But well before in 1827, Navier proposed a more general boundary condition that allows taking into account the slip effect of the fluid near the boundary and the friction as the following:

$$\mathbf{u} \cdot \mathbf{n} = g \quad \text{and} \quad 2[\mathbf{D}(\mathbf{u})\mathbf{n}]_\tau + \alpha \mathbf{u}_\tau = \mathbf{h} \text{ on } \Gamma, \quad (1.5.1)$$

where  $\mathbf{D}(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T)$  is the stress tensor of deformation and  $\alpha$  is the friction coefficient. When  $\alpha = g = 0$  and  $\mathbf{h} = \mathbf{0}$ , which is the case of a flat boundary,



these conditions are reduced to

$$\mathbf{u} \cdot \mathbf{n} = 0 \quad \text{and} \quad \mathbf{curl} \, \mathbf{u} \times \mathbf{n} = \mathbf{0} \quad \text{on } \Gamma$$

in a 3D case. These conditions were used in many research work as in [22] and [3] for the magnetic field in a magnetohydrodynamic problem and Dirichlet conditions for the velocity.

Clopeau, Mikelić, and Robert studied in [33] the friction type boundary conditions in the two-dimensional case for Navier-Stokes equations and obtain an equivalence form of (1.5.1) when  $g = 0$  and give the following relation between the vorticity, the tangential component of the normal stress and the tangential velocity  $\mathbf{u} \cdot \boldsymbol{\tau}$  on  $\Gamma$ .

Suppose that  $\mathbf{u} \in \mathbf{H}^2(\Omega)$  and  $\mathbf{u} \cdot \mathbf{n} = 0$  on  $\Gamma$ . Then we have

$$\mathbf{D}(\mathbf{u})\mathbf{n} \cdot \boldsymbol{\tau} - \frac{1}{2}\mathbf{curl} \, \mathbf{u} + \kappa\mathbf{u} \cdot \boldsymbol{\tau} = 0 \quad \text{on } \Gamma. \quad (1.5.2)$$

where  $\kappa$  is the curvature of the boundary given in a standard way by  $\frac{d\boldsymbol{\tau}}{ds} = -\kappa \mathbf{n}$ .

We denote by  $\boldsymbol{\tau}$  the unit tangential to  $\Gamma$  and  $\mathbf{n} = (-n_2, n_1)$  (see figure below).

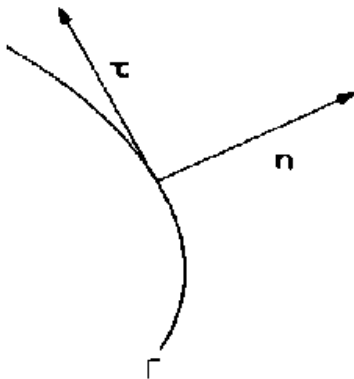


Figure 1.7: Normal and tangential vector of any function on the boundary. Source [18]

## 1.6 Thesis description and main results

In this thesis, we study an MHD problem in a domain, possibly multi-connected, where Navier-type boundary conditions for both, the velocity  $\mathbf{u}$  and the magnetic field  $\mathbf{B}$ , are considered. More precisely, we study the following problem:

$$\begin{cases} -\nu\Delta\mathbf{u} + (\mathbf{u} \cdot \nabla)(\mathbf{u} - \alpha\Delta\mathbf{u}) + \nabla\pi - (\mathbf{B} \cdot \nabla)\mathbf{B} + \frac{1}{2}\nabla(|\mathbf{B}|^2) = \mathbf{f} & \text{in } \Omega, \\ -\Delta\mathbf{B} - (\mathbf{B} \cdot \nabla)\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{B} + \nabla\theta = 0 & \text{in } \Omega, \\ \operatorname{div} \mathbf{u} = 0, \quad \operatorname{div} \mathbf{B} = 0 & \text{in } \Omega, \end{cases} \quad (\mathcal{P})$$

with these boundary conditions:

$$\mathbf{u} \cdot \mathbf{n} = 0, \quad \mathbf{B} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma, \quad (1.6.1)$$

$$\operatorname{curl} \mathbf{u} = 0, \quad \operatorname{curl} \mathbf{B} = 0 \quad \text{on } \Gamma, \quad (1.6.2)$$

$$\langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} = 0, \quad \langle \mathbf{B} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} = 0, \quad \text{for } 1 \leq j \leq J \quad (1.6.3)$$

where the unknowns  $\mathbf{u}$ ,  $\pi$ , and  $\mathbf{B}$  represent the velocity, the pressure, and the magnetic fields respectively,  $\theta$  is an unknown function related to the motion of heavy ions, and  $\alpha$  is the relaxation viscous coefficient. The external force acting on the fluid is  $\mathbf{f}$ , and  $\mathbf{n}$  denotes the unit exterior normal to the boundary  $\Gamma$  and the viscosity  $\nu$  is positive (dealing with the study done by Dunn and Fosdick [16]). The first equation characterizes the motion of polymer aqueous solutions and the second describes the electromagnetic phenomenon. Notice that when,  $\alpha$  the relaxational viscous coefficient, is null this problem is reduced to an MHD problem coupling Navier-Stokes and Maxwell equations.

Experimental results show that the no-slip condition fails in most fluid dynamics and the slip condition actually occurs. The Navier slip conditions (1.6.1) and (1.6.2) are also used for flow simulation at rough boundaries. These conditions were considered by P.L.Lions in [25] and J.L.Lions in [24], called Lions boundary conditions, are a special case of Navier boundary conditions in which  $\alpha = 2\kappa$  (see [33]). We can also consider the case of no friction at the boundary  $\Gamma$  for  $\alpha = 0$ , in which we get the same

conditions above. The two-dimensional bounded domain  $\Omega$  that we consider here is possibly multiply connected, for that some additional conditions are required. The last conditions (1.6.3) are related to the geometry of the domain  $\Omega$ , they describe the flow of the velocity and the magnetic fields through the cuts that are connected segments contained in the domain which are denoted by  $\Sigma_j$  for  $1 \leq j \leq J$ . In chapter two, we will see the entire interest of these conditions in getting the existence of the solutions for the MHD problem. When  $\Omega$  is a simply connected domain, these conditions are not required anymore.

In chapter two, some studies related to the Stokes problem were done. In the first part, we study the existence of solutions in a Hilbertian case and then we give some regularity results. Let us introduce the Stokes problem:

$$\begin{cases} -\Delta \mathbf{u} + \nabla \pi = \mathbf{f} & \text{and} \quad \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} \cdot \mathbf{n} = 0 & \text{and} \quad \operatorname{curl} \mathbf{u} = 0 & \text{on } \Gamma, \\ \langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} = 0, & & \text{for } 1 \leq j \leq J. \end{cases} \quad (1.6.4)$$

Here, we consider the Navier-type boundary conditions. Contrary to the classical case of Dirichlet conditions and depending on the conditions considered on the acting forces, the term of the pressure can disappear.

In literature, the existence and regularity of solutions for the problem above were studied. In [6], authors proved a large class of existence in  $\mathbf{L}^p$ -theory in 3-D. This work was based on Friedrichs's inequalities calling the theory of singular integral operators. They considered also the case of non-homogeneous boundary conditions

$$\mathbf{u} \cdot \mathbf{n} = g \quad \text{and} \quad \operatorname{curl} \mathbf{u} \times \mathbf{n} = \mathbf{h} \times \mathbf{n} \quad \text{on } \Gamma.$$

Let us start with the following first result.

**Theorem 1.6.1.** *i) For a given  $\mathbf{f}$  in  $\mathbf{L}^s(\Omega)$ , with arbitrary  $s > 1$ , satisfying the*

compatibility conditions

$$\begin{aligned}
\operatorname{div} \mathbf{f} &= 0 \quad \text{in } \Omega, \\
\mathbf{f} \cdot \mathbf{n} &= 0 \quad \text{on } \Gamma, \\
\forall \boldsymbol{\varphi} \in \mathbf{K}_T(\Omega), \quad \int_{\Omega} \mathbf{f} \cdot \boldsymbol{\varphi} &= 0,
\end{aligned} \tag{1.6.5}$$

where  $\mathbf{K}_T(\Omega)$  is the following kernel space

$$\mathbf{K}_T^p(\Omega) = \{ \mathbf{v} \in \mathbf{L}^p(\Omega); \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega, \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma \text{ and } \operatorname{curl} \mathbf{v} = 0 \text{ in } \Omega \}, \tag{1.6.6}$$

and we denote it by  $\mathbf{K}_T(\Omega)$  for  $p = 2$ .

Then the problem:

$$\begin{cases}
-\Delta \mathbf{u} = \mathbf{f} & \text{and } \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega, \\
\mathbf{u} \cdot \mathbf{n} = 0 & \text{and } \operatorname{curl} \mathbf{u} = 0 & \text{on } \Gamma, \\
\langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} = 0, & & \text{for } 1 \leq j \leq J.
\end{cases} \tag{1.6.7}$$

has a unique solution  $\mathbf{u} \in \mathbf{H}^1(\Omega)$ .

ii) Moreover, if  $\Omega$  is of class  $\mathcal{C}^{2,1}$  and  $\mathbf{f} \in \mathbf{L}^2(\Omega)$ , then  $\mathbf{u} \in \mathbf{H}^2(\Omega)$ ,

We also prove the following regularity result.

**Theorem 1.6.2.** *Assume that  $\Omega$  is of class  $\mathcal{C}^{2,1}$  and  $\mathbf{f} \in \mathbf{L}^p(\Omega)$ , with  $p \geq 2$ , and satisfies the condition (1.6.5). The solution  $\mathbf{u}$  to Problem (1.6.7) belongs to  $\mathbf{W}^{2,p}(\Omega)$  and satisfies*

$$\| \mathbf{u} \|_{\mathbf{W}^{2,p}(\Omega)} \leq C(\Omega) \| \mathbf{f} \|_{\mathbf{L}^p(\Omega)}.$$

What happens now if we the forces acting on the fluid do not verify all the conditions (1.6.5)? Some studies are done for the Stokes problem where the pressure appears due to the conditions of the external forces verifying only the orthogonality one.

**Theorem 1.6.3.** Let  $\mathbf{f} \in \mathbf{L}^s(\Omega)$ , with  $s > 1$ , be such that

$$\forall \boldsymbol{\varphi} \in \mathbf{K}_T(\Omega), \quad \int_{\Omega} \mathbf{f} \cdot \boldsymbol{\varphi} = 0. \quad (1.6.8)$$

Then there exist  $\mathbf{u} \in \mathbf{H}^1(\Omega)$  and  $\pi \in W^{1,s}(\Omega)$  solution of the following Stokes Problem:

$$\begin{cases} -\Delta \mathbf{u} + \nabla \pi = \mathbf{f}, & \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} \cdot \mathbf{n} = 0, & \operatorname{curl} \mathbf{u} = 0 & \text{on } \Gamma, \\ \int_{\Sigma_j} \mathbf{u} \cdot \mathbf{n} = 0 & & \text{for } 1 \leq j \leq J. \end{cases} \quad (1.6.9)$$

Here we give some regularity results for weak solutions. First, let us define the following:

For any  $p \geq 2$ , we denote by

$$s(p) = \begin{cases} 1 + \varepsilon & \text{if } p = 2 \quad \text{with arbitrary } \varepsilon > 0 \\ \frac{2p}{2+p} & \text{if } p > 2. \end{cases} \quad (1.6.10)$$

Also, we prove some regularity of the strong solutions that depend on the regularity of the domain.

**Theorem 1.6.4.** Let  $p \geq 2$ ,  $\mathbf{f} \in \mathbf{L}^{s(p)}(\Omega)$ , with  $s(p)$  defined in (1.6.10) and satisfying the condition (1.6.8). Then the solution  $(\mathbf{u}, \pi)$  of the Stokes Problem (1.6.9) given by Theorem 1.6.3 is in  $\mathbf{W}^{1,p}(\Omega) \times W^{1,s(p)}(\Omega)$  and it satisfies the following estimate:

$$\|\mathbf{u}\|_{\mathbf{W}^{1,p}(\Omega)} + \|\pi\|_{W^{1,s(p)}(\Omega)/\mathbb{R}} \leq C(\Omega) \|\mathbf{f}\|_{\mathbf{L}^{s(p)}(\Omega)}.$$

In chapter three, we focus on the mathematical analysis of the MHD problem and we are interested in the existence and regularity of solutions in the stationary case with Navier-type boundary conditions in a two-dimensional domain  $\Omega$ . The particularity of this case owes to the nonlinear term that verifies the following property:

$$\int_{\Omega} (\mathbf{u} \cdot \nabla) \mathbf{u} \cdot \Delta \mathbf{u} = 0,$$

which is no longer true in 3D. The main difficulties in this MHD problem are related to the nonlinear third derivative order term  $(\mathbf{u} \cdot \nabla)(\mathbf{u} - \alpha \Delta \mathbf{u})$  and the domain under consideration. Let us give the main theorem of existence and regularity for the MHD problem:

**Theorem 1.6.5.** *Let  $\Omega$  be of class  $\mathcal{C}^{2,1}$ . For any  $\mathbf{f} \in \mathbf{L}^2(\Omega)$ , Problem  $(\mathcal{P})$  has at least one solution  $(\mathbf{u}, \mathbf{B}, \pi, \theta)$  in  $\mathbf{H}^2(\Omega) \times \mathbf{W}^{2,p}(\Omega) \times L^2(\Omega) \times H^2(\Omega)$  for any  $p \geq 2$  and there exists  $\beta > 2$  such that*

$$\|\mathbf{u}\|_{\mathbf{H}^2(\Omega)} + \|\mathbf{B}\|_{\mathbf{W}^{2,p}(\Omega)} \leq C(\Omega) \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}^\beta,$$

where  $C(\Omega)$  depends on  $\Omega$ .

In chapter four, we study a problem in a three dimensional simply connected domain of polymer aqueous solutions. Firstly we are interested in the steady flow problem :

$$\begin{aligned} & \text{Find } (\mathbf{u}, \pi) \in \mathbf{H}^2(\Omega) \times L^2(\Omega) \text{ verifying} \\ (\mathcal{S}) \quad & \begin{cases} -\nu \Delta \mathbf{u} + \mathbf{u} \cdot \nabla (\mathbf{u} - \alpha \Delta \mathbf{u}) + \nabla \pi = \mathbf{f} & \text{in } \Omega, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} \cdot \mathbf{n} = 0, \operatorname{curl} \mathbf{u} \times \mathbf{n} = \mathbf{0} & \text{on } \Gamma, \end{cases} \end{aligned}$$

Here, we announce the following theorem:

**Theorem 1.6.6.** *Let  $\Omega \subset \mathbb{R}^3$  be a bounded simply connected domain of class  $\mathcal{C}^{3,1}$ . Then for all  $\mathbf{f} \in \mathbf{L}^2(\Omega)$ , Problem  $(\mathcal{S})$  has at least one solution  $(\mathbf{u}, \pi) \in \mathbf{H}^2(\Omega) \times L^2(\Omega)$  and we have the following estimate:*

$$\|\mathbf{u}\|_{\mathbf{H}^2(\Omega)} + \|\pi\|_{L^2(\Omega)/\mathbb{R}} \leq C(\Omega, \alpha, \nu) \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}. \quad (1.6.11)$$

Next, we study the following evolution problem:

$$(\mathcal{E}) \quad \begin{cases} \frac{\partial}{\partial t}(\mathbf{u}(t) - \alpha \Delta \mathbf{u}(t)) - \nu \Delta \mathbf{u}(t) + (\mathbf{u}(t) \cdot \nabla)(\mathbf{u}(t) - \alpha \Delta \mathbf{u}(t)) + \nabla \pi = \mathbf{f}(t), & \text{in } Q_T, \\ \operatorname{div} \mathbf{u} = 0, & \text{in } Q_T, \\ \mathbf{u} \cdot \mathbf{n} = 0, \quad \operatorname{curl} \mathbf{u} \times \mathbf{n} = \mathbf{0} & \text{on } \Sigma_T, \\ \mathbf{u}(0) = \mathbf{u}_0, & \text{in } \Omega, \end{cases} \quad (1.6.12)$$

Suppose that  $\mathbf{f} \in \mathbf{L}^2([0, T]; \mathbf{L}^2(\Omega))$  and  $\mathbf{u}_0 \in \mathbf{V}_2$ .

A function  $\mathbf{u}$  is called a weak solution for Problem  $(\mathcal{E})$  if

$$\mathbf{u} \in \mathbf{L}^\infty([0, T]; \mathbf{V}_2) \quad \text{and} \quad \frac{\partial \mathbf{u}}{\partial t} \in \mathbf{L}^\infty([0, T]; \mathbf{V}_1) \quad (1.6.13)$$

and for any  $\mathbf{v} \in \mathbf{V}_1$ ,

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} \mathbf{u} \cdot \mathbf{v} + \alpha \operatorname{curl} \mathbf{u} \cdot \operatorname{curl} \mathbf{v} \, dx + \nu \int_{\Omega} \operatorname{curl} \mathbf{u} \cdot \operatorname{curl} \mathbf{v} \, dx \\ & + b(\mathbf{u}, \mathbf{v}, \mathbf{u} - \alpha \Delta \mathbf{u}) = \int_{\Omega} \mathbf{f}(t) \cdot \mathbf{v} \, dx, \end{aligned} \quad (1.6.14)$$

with the initial condition

$$\mathbf{u}(0) = \mathbf{u}_0 \quad \text{in } \Omega,$$

where

$$\mathbf{V}_1 = \{\mathbf{v} \in \mathbf{L}^2(\Omega); \operatorname{curl} \mathbf{v} \in \mathbf{L}^2(\Omega); \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega, \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma\},$$

$$\mathbf{V}_2 = \{\mathbf{v} \in \mathbf{V}_1; \Delta \mathbf{v} \in \mathbf{L}^2(\Omega), \operatorname{curl} \mathbf{v} \times \mathbf{n} = \mathbf{0} \text{ on } \Gamma\},$$

**Theorem 1.6.7.** *Suppose that  $\Omega$  is of class  $\mathcal{C}^{3,1}$ . For given  $\mathbf{f} \in \mathbf{L}^2([0, T]; \mathbf{L}^2(\Omega))$  and  $\mathbf{u}_0 \in \mathbf{V}_2$ , there exist a solution  $\mathbf{u}$  for Problem  $(\mathcal{E})$  in the sense of a weak solution.*

# Chapter 2

## Stokes Problem in 2D

### Contents

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<b>2.1</b>	<b>Functional framework . . . . .</b>	<b>32</b>
2.1.1	Geometry of the domain . . . . .	32
2.1.2	Functional spaces . . . . .	33
2.1.3	Some properties related to the curl operator. . . . .	34
2.1.4	An important identity result in $\mathbb{R}^2$ . . . . .	36
<b>2.2</b>	<b>Preliminary results . . . . .</b>	<b>41</b>
<b>2.3</b>	<b>Stokes problem with free-divergence forces . . . . .</b>	<b>52</b>
<b>2.4</b>	<b>Stokes problem . . . . .</b>	<b>57</b>

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In this chapter, we are concerned with the stationary Stokes problem with Navier-type boundary conditions for the velocity field in a bounded domain  $\Omega \subset \mathbb{R}^2$ . The equations of motion for Stokes flow called the Stokes equations, are a linearization of the Navier–Stokes equations that describe the motion of a viscous fluid. In the case where nonlinear effects can be neglected in the Navier–Stokes equation (i.e. when the Reynolds number is sufficiently small) the system reduces to the following Stokes equations:



$$\begin{cases} -\Delta \mathbf{u} + \nabla \pi = \mathbf{f} & \text{in } \Omega, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega. \end{cases} \quad (2.0.1)$$

It describes a typical situation in flows where the fluid velocities are very slow, the viscosities are very large, or the length scales of the flow are very small. As the governing equations are differential equations, appropriate boundary conditions must be introduced to supplement the system and these conditions depend on the physical situation. We consider here the Navier-type boundary conditions that allow slip at the interface of the fluid and the solid wall. These boundary conditions are expressed in a two-dimensional domain as the following:

$$\mathbf{u} \cdot \mathbf{n} = 0 \quad \text{and} \quad \operatorname{curl} \mathbf{u} = 0 \quad \text{on } \Gamma. \quad (2.0.2)$$

## 2.1 Functional framework

We recall that, in this chapter, we do not assume that the domain is simply connected. In order to study the existence of solutions, we begin by examining the domain's geometry, then we show some properties of the curl operator and demonstrate a significant identity result in two dimensions. Additionally, we establish the equivalence of norms and ultimately focus on the existence and regularity of solutions.

### 2.1.1 Geometry of the domain

Let  $\Omega$  be a Lipschitz domain with boundary  $\Gamma$  such that  $\Gamma = \bigcup_{i=0}^I \Gamma_i$ , where  $\Gamma_i$  are the connected components of  $\Gamma$ . We suppose that there exist  $J$  connected segments  $\Sigma_j$ ,  $1 \leq j \leq J$ , called "cuts", contained in  $\Omega$ , in the way that they do not intersect and the restricted domain  $\Omega^\circ = \Omega \setminus \bigcup_{j=1}^J \Sigma_j$  is simply connected. The figure 2.1 below is an example of a multiply connected domain with three connected components of  $\Gamma$ .

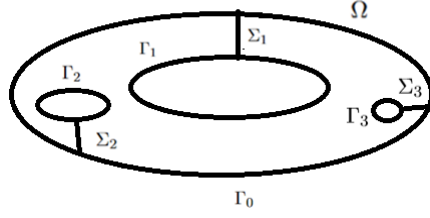


Figure 2.1: 2-D multiply connected domain for  $I = J = 3$ .

### 2.1.2 Functional spaces

We introduce here some necessary functional spaces and some important identities. We define the space  $\mathbf{X}(\Omega)$  by

$$\mathbf{X}(\Omega) = \{\mathbf{v} \in \mathbf{L}^2(\Omega); \text{curl } \mathbf{v} \in L^2(\Omega), \text{div } \mathbf{v} \in L^2(\Omega)\}$$

which is a Hilbert space for the following norm:

$$\|\mathbf{v}\|_{\mathbf{X}(\Omega)} = (\|\mathbf{v}\|_{L^2(\Omega)}^2 + \|\text{curl } \mathbf{v}\|_{L^2(\Omega)}^2 + \|\text{div } \mathbf{v}\|_{L^2(\Omega)}^2)^{1/2}.$$

We define also a subspace of  $\mathbf{X}(\Omega)$  by

$$\mathbf{X}_T(\Omega) = \{\mathbf{v} \in \mathbf{X}(\Omega); \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma\},$$

where the unit outward normal vector is denoted by  $\mathbf{n}$ . We will consider the following Hilbert spaces:

$$\begin{aligned} \mathbf{V}_1 = \{ & \mathbf{v} \in \mathbf{H}^1(\Omega); \text{div } \mathbf{v} = 0 \text{ in } \Omega, \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma \\ & \text{and } \langle \mathbf{v} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} = 0 \text{ for } 1 \leq j \leq J \} \end{aligned}$$

equipped with the norm of  $\mathbf{H}^1(\Omega)$  and

$$\mathbf{V}_2 = \{\mathbf{v} \in \mathbf{V}_1; \Delta \mathbf{v} \in \mathbf{L}^2(\Omega), \text{curl } \mathbf{v} = 0 \text{ on } \Gamma\}$$

provided with the norm

$$\|\mathbf{v}\|_{\mathbf{V}_2} = (\|\mathbf{v}\|_{\mathbf{H}^1(\Omega)}^2 + \|\Delta \mathbf{v}\|_{\mathbf{L}^2(\Omega)}^2)^{1/2}.$$

Let  $\Phi = \{\Psi \in H^1(\Omega); \Psi|_{\Gamma_0} = 0, \Psi|_{\Gamma_i} = \text{an arbitrary constant for } 1 \leq i \leq p\}$ ,

and for  $p \geq 2$ ,  $\mathbf{H}^p = \{\mathbf{v} \in \mathbf{L}^p(\Omega); \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega, \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma\}$ .

When  $p = 2$ ,  $\mathbf{H}^2$  will be notated as  $\mathbf{H}$ .

We also introduce the following kernel space:

$$\mathbf{K}_T^p(\Omega) = \{\mathbf{v} \in \mathbf{H}^p; \operatorname{curl} \mathbf{v} = 0 \text{ in } \Omega\}, \quad (2.1.1)$$

and we denote it by  $\mathbf{K}_T(\Omega)$  for  $p = 2$ .

Finally, we define

$$\mathbf{H}(\operatorname{div}, \Omega) = \{\mathbf{v} \in \mathbf{L}^2(\Omega); \operatorname{div} \mathbf{v} \in L^2(\Omega)\}$$

with  $\|\mathbf{v}\|_{\mathbf{H}(\operatorname{div}, \Omega)} = (\|\mathbf{v}\|_{\mathbf{L}^2(\Omega)}^2 + \|\operatorname{div} \mathbf{v}\|_{L^2(\Omega)}^2)^{1/2}$  and

$$\mathbf{H}_0(\operatorname{div}, \Omega) = \{\mathbf{v} \in \mathbf{L}^2(\Omega); \operatorname{div} \mathbf{v} \in L^2(\Omega), \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma\}.$$

### 2.1.3 Some properties related to the curl operator.

We would like to remind a few properties related to the curl operator in a two-dimensional domain.

For any distribution  $\phi$  in  $\mathcal{D}'(\Omega)$  and  $\mathbf{v} = (v_1, v_2) \in \mathcal{D}'(\Omega)^2$ , we have

$$\operatorname{curl}(\mathbf{curl} \phi) = -\Delta \phi \quad \text{with} \quad \mathbf{curl} \phi = \left( \frac{\partial \phi}{\partial x_2}, -\frac{\partial \phi}{\partial x_1} \right) \quad (2.1.2)$$

and

$$\mathbf{curl} \operatorname{curl} \mathbf{v} = -\Delta \mathbf{v} + \operatorname{grad}(\operatorname{div} \mathbf{v}). \quad (2.1.3)$$

The following proposition is proved in [18], chap.1, Corollary 3.1 p.40.

**Proposition 2.1.1.** *The mapping  $\mathbf{curl}$  is an isomorphism from  $\Phi$  onto  $\mathbf{H}$ .*

Here, we recall the continuity of the tangential trace mapping followed by a Green formula that will be used in this chapter.

**Theorem 2.1.2.** *The mapping  $\gamma_t : \mathbf{v} \longrightarrow \mathbf{v} \cdot \mathbf{t}$  defined on  $\mathcal{D}(\bar{\Omega})$  can be extended by continuity to a linear and continuous mapping denoted by  $\gamma_t$  from  $\mathbf{H}$  into  $\mathbf{H}^{-1/2}(\Gamma)$ . Moreover, the following Green formula holds*

*For any  $\mathbf{v} \in \mathbf{H}$ , we have*

$$\int_{\Omega} \mathbf{v} \cdot \mathbf{curl} \psi = \int_{\Omega} \psi \mathbf{curl} \mathbf{v} - \int_{\Gamma} (\mathbf{v} \cdot \mathbf{t}) \psi \, ds, \quad \forall \psi \in H^1(\Omega). \quad (2.1.4)$$

The proof of the theorem above is done in [18]. Here, we give some properties for the kernel space defined in (2.1.1). The following lemma is proved in [7](see Lemma 3.7).

**Lemma 2.1.3.** *If  $\mathbf{u}$  belongs to  $\mathbf{H}_0(\text{div}, \Omega)$  then the restriction of  $\mathbf{u} \cdot \mathbf{n}$  to any  $\Sigma_j$  belongs to  $H_{00}^{1/2}(\Sigma_j)'$  and we have:*

$$\forall \varphi \in \mathbf{H}^1(\Omega^\circ), \text{ with } \text{div} \varphi = 0 \text{ in } \Omega$$

$$\int_{\Omega^\circ} \mathbf{u} \cdot \nabla \varphi \, dx = - \int_{\Omega^\circ} (\text{div} \mathbf{u}) \, dx + \sum_{j=1}^J \langle \mathbf{u} \cdot \mathbf{n}, \varphi_j \rangle_{\Sigma_j}, \quad (2.1.5)$$

where  $H_{00}^{1/2}(\Sigma_j) = \{\mu \in H^{1/2}(\Sigma_j); \tilde{\mu} \in H^{1/2}(\mathcal{M}_j)\}$ , where  $\mathcal{M}_j$  is a smooth curve which contain  $\Sigma_j$  and  $\tilde{\nu}$  is the extension by zero of  $\nu$  outside of  $\Sigma_j$ .

In the following, for any function  $q$  in  $H^1(\Omega)$ , we denote by  $\widetilde{\nabla} q$  the extension of  $\nabla q$  in  $\mathcal{D}'(\Omega)$ , and  $[q]_j$  the jump of  $q$  through the cuts  $\Sigma_j$ .

**Proposition 2.1.4.** *i) The space  $\mathbf{K}_T(\Omega)$  is spanned by the vector fields  $\widetilde{\nabla} q_j^T$ , where*

each  $q_j^T$ , for  $1 \leq j \leq J$ , is the solution in  $H^1(\Omega^\circ)$  of the problem:

$$\left\{ \begin{array}{ll} -\Delta q_j^T = 0 & \text{in } \Omega^\circ, \\ \partial_n q_j^T = 0 & \text{on } \Gamma, \\ [q_j^T]_k = \text{constant}, \quad \text{and } [\partial_n q_j^T]_k = 0, \quad 1 \leq k \leq J, \\ \langle \partial_n q_j^T, 1 \rangle_{\Sigma_k} = \delta_{jk}, & 1 \leq k \leq J. \end{array} \right.$$

ii) We have  $\mathbf{K}_T(\Omega) = \{\mathbf{0}\}$  if and only if  $\Omega$  is simply connected.

iii) If  $\mathbf{u} \in \mathbf{H}$  with  $\langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} = 0$  for  $1 \leq j \leq J$ , then  $\int_{\Omega} \mathbf{u} \cdot \widetilde{\nabla} q_j^T = 0$ .

*Proof.* i) It is similar to the proof of Proposition 3.14 in [11] in the 3D case.

ii) It is an immediate consequence of point i).

iii) Let  $\mathbf{u} \in \mathbf{H}$ , using the Green's formula (2.1.5)

$$\int_{\Omega} \mathbf{u} \cdot \widetilde{\nabla} q_j^T = \int_{\Omega^\circ} \mathbf{u} \cdot \nabla q_j^T = - \int_{\Omega^\circ} q_j^T \operatorname{div} \mathbf{u} + \sum_{j=1}^J \langle \mathbf{u} \cdot \mathbf{n}, [q_j^T]_j \rangle_{\Sigma_j} = 0. \quad (2.1.6)$$

□

**Lemma 2.1.5.** *The kernel space  $\mathbf{K}_T^p(\Omega)$  is embedded in  $\mathbf{W}^{1,q}(\Omega)$  for any finite  $q$ .*

## 2.1.4 An important identity result in $\mathbb{R}^2$

In this subsection, we prove in detail a fundamental theorem done by Grisvard in [20] (Thm 3.1.1.1) when  $\Omega$  is a bounded open set of  $\mathbb{R}^n$  with a  $\mathcal{C}^2$  boundary. Here we adapt the proof for a two dimensional domain. For that, let us introduce some notations.

Let  $\mathbf{v}$  be a vector field on  $\Gamma$ , we denote by  $\mathbf{v}_n = \mathbf{v} \cdot \mathbf{n}$  the component of  $\mathbf{v}$  in the direction of the unit outward normal vector  $\mathbf{n} = (n_1, n_2)$  and  $\mathbf{v}_\tau = \mathbf{v} - \mathbf{v}_n \mathbf{n}$  the projection of  $\mathbf{v}$  on the tangent plane of  $\Gamma$  also we denote by  $\nabla_\tau$  the projection of the

gradient operator on the tangent plane, we have for any  $\mathbf{v} = (v_1, v_2)^T$  on  $\Gamma$ ,

$$\nabla \mathbf{v} = \nabla_\tau \mathbf{v} + \frac{\partial \mathbf{v}}{\partial \mathbf{n}} \mathbf{n}^T \quad \text{and} \quad \operatorname{div} \mathbf{v} = \frac{\partial \mathbf{v}}{\partial \mathbf{t}} \cdot \mathbf{t} + \frac{\partial \mathbf{v}}{\partial \mathbf{n}} \cdot \mathbf{n} \quad (2.1.7)$$

where

$$\nabla_\tau \mathbf{v} = \frac{\partial \mathbf{v}}{\partial \mathbf{t}} \mathbf{t}^T \quad \text{and} \quad \mathbf{t} = (-n_2, n_1).$$

**Theorem 2.1.6.** *Let  $\Omega$  be of class  $\mathcal{C}^{1,1}$ . Then for all  $\mathbf{v} \in \mathbf{H}^1(\Omega)$  we have,*

$$\begin{aligned} \int_{\Omega} |\operatorname{curl} \mathbf{v}|^2 + \int_{\Omega} |\operatorname{div} \mathbf{v}|^2 - \int_{\Omega} |\nabla \mathbf{v}|^2 &= -2 \langle \nabla_\tau (\mathbf{v} \cdot \mathbf{n}), \mathbf{v}_\tau \rangle_\Gamma \\ &+ \int_{\Gamma} (|\mathbf{v} \cdot \mathbf{n}|^2 \frac{\partial \mathbf{n}}{\partial \mathbf{t}} \cdot \mathbf{t} - |\mathbf{v} \cdot \mathbf{t}|^2 \frac{\partial \mathbf{t}}{\partial \mathbf{t}} \cdot \mathbf{n}) ds \end{aligned} \quad (2.1.8)$$

where  $\langle \cdot, \cdot \rangle_\Gamma$  is the duality brackets  $\langle \cdot, \cdot \rangle_{\mathbf{H}^{-1/2}(\Gamma) \times \mathbf{H}^{1/2}(\Gamma)}$ .

*Proof.* Let  $\mathbf{v} \in \mathcal{D}(\overline{\Omega})$ , we know that

$$\begin{aligned} 2 \int_{\Omega} |\operatorname{curl} \mathbf{v}|^2 &= \int_{\Omega} \left( \frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} \right) \left( \frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} \right) dx \\ &= 2 \int_{\Omega} \left| \frac{\partial v_i}{\partial x_j} \right|^2 - 2 \int_{\Omega} \frac{\partial v_i}{\partial x_j} \frac{\partial v_j}{\partial x_i} dx \\ &= 2 \int_{\Omega} |\nabla \mathbf{v}|^2 - 2 \int_{\Omega} \frac{\partial v_i}{\partial x_j} \frac{\partial v_j}{\partial x_i} dx, \end{aligned}$$

where we used the implicit summation on the repeated indices. Using the Green formula on the second term of the equation above we get,

$$\begin{aligned} - \int_{\Omega} \frac{\partial v_i}{\partial x_j} \frac{\partial v_j}{\partial x_i} &= \int_{\Omega} v_j \frac{\partial}{\partial x_j} (\operatorname{div} \mathbf{v}) - \int_{\Gamma} \frac{\partial v_i}{\partial x_j} v_j \mathbf{n}_i ds \\ &= - \int_{\Omega} |\operatorname{div} \mathbf{v}|^2 + \int_{\Gamma} \mathbf{v} \cdot \mathbf{n} \operatorname{div} \mathbf{v} ds - \int_{\Gamma} \frac{\partial v_i}{\partial x_j} v_j \mathbf{n}_i ds. \end{aligned}$$

Then,

$$\int_{\Omega} |\operatorname{curl} \mathbf{v}|^2 + \int_{\Omega} |\operatorname{div} \mathbf{v}|^2 - \int_{\Omega} |\nabla \mathbf{v}|^2 = \int_{\Gamma} \mathbf{v} \cdot \mathbf{n} \operatorname{div} \mathbf{v} \, ds - \int_{\Gamma} (\mathbf{v} \cdot \nabla) \mathbf{v} \cdot \mathbf{n} \, ds. \quad (2.1.9)$$

Now we have to transform the integrals on the boundary that are the last two terms in (2.1.9). Writing for any  $\mathbf{v}$  on  $\Gamma$ :

$$\mathbf{v} = (\mathbf{v} \cdot \mathbf{t})\mathbf{t} + (\mathbf{v} \cdot \mathbf{n})\mathbf{n}, \quad (2.1.10)$$

and using (2.1.7) we find

$$\begin{aligned} \operatorname{div} \mathbf{v} &= \left( \mathbf{t} \frac{\partial}{\partial \mathbf{t}} (\mathbf{v} \cdot \mathbf{t}) + (\mathbf{v} \cdot \mathbf{t}) \frac{\partial \mathbf{t}}{\partial \mathbf{t}} \right) \cdot \mathbf{t} + (\mathbf{v} \cdot \mathbf{n}) \frac{\partial \mathbf{n}}{\partial \mathbf{t}} \cdot \mathbf{t} \\ &\quad + \left( \mathbf{n} \frac{\partial}{\partial \mathbf{n}} (\mathbf{v} \cdot \mathbf{n}) + (\mathbf{v} \cdot \mathbf{n}) \frac{\partial \mathbf{n}}{\partial \mathbf{n}} \right) \cdot \mathbf{n} + (\mathbf{v} \cdot \mathbf{t}) \frac{\partial \mathbf{t}}{\partial \mathbf{n}} \cdot \mathbf{n} \\ &= \frac{\partial}{\partial \mathbf{t}} (\mathbf{v} \cdot \mathbf{t}) + \frac{\partial}{\partial \mathbf{n}} (\mathbf{v} \cdot \mathbf{n}) + (\mathbf{v} \cdot \mathbf{n}) \frac{\partial \mathbf{n}}{\partial \mathbf{t}} \cdot \mathbf{t} + (\mathbf{v} \cdot \mathbf{t}) \frac{\partial \mathbf{t}}{\partial \mathbf{n}} \cdot \mathbf{n} \end{aligned}$$

since  $\frac{\partial \mathbf{n}}{\partial \mathbf{n}} \cdot \mathbf{n} = 0$  and  $\frac{\partial \mathbf{t}}{\partial \mathbf{t}} \cdot \mathbf{t} = 0$ . So,

$$\mathbf{v} \cdot \mathbf{n} \operatorname{div} \mathbf{v} = \mathbf{v} \cdot \mathbf{n} \left( \frac{\partial}{\partial \mathbf{t}} (\mathbf{v} \cdot \mathbf{t}) + \frac{\partial}{\partial \mathbf{n}} (\mathbf{v} \cdot \mathbf{n}) + (\mathbf{v} \cdot \mathbf{n}) \frac{\partial \mathbf{n}}{\partial \mathbf{t}} \cdot \mathbf{t} + (\mathbf{v} \cdot \mathbf{t}) \frac{\partial \mathbf{t}}{\partial \mathbf{n}} \cdot \mathbf{n} \right). \quad (2.1.11)$$

The following calculations consist in computing  $(\mathbf{v} \cdot \nabla) \mathbf{v} \cdot \mathbf{n}$ , using the expression of  $\mathbf{v}$  in (2.1.10):

$$\begin{aligned} (\mathbf{v} \cdot \nabla) \mathbf{v} &= (\mathbf{v} \cdot \mathbf{t}) \frac{\partial \mathbf{v}}{\partial \mathbf{t}} + (\mathbf{v} \cdot \mathbf{n}) \frac{\partial \mathbf{v}}{\partial \mathbf{n}} \\ &= (\mathbf{v} \cdot \mathbf{t}) \left( (\mathbf{v} \cdot \mathbf{t}) \frac{\partial \mathbf{t}}{\partial \mathbf{t}} + \mathbf{t} \frac{\partial}{\partial \mathbf{t}} (\mathbf{v} \cdot \mathbf{t}) + (\mathbf{v} \cdot \mathbf{n}) \frac{\partial \mathbf{n}}{\partial \mathbf{t}} + \mathbf{n} \frac{\partial}{\partial \mathbf{t}} (\mathbf{v} \cdot \mathbf{n}) \right) \\ &\quad + (\mathbf{v} \cdot \mathbf{n}) \left( (\mathbf{v} \cdot \mathbf{t}) \frac{\partial \mathbf{t}}{\partial \mathbf{n}} + \mathbf{t} \frac{\partial}{\partial \mathbf{n}} (\mathbf{v} \cdot \mathbf{t}) + (\mathbf{v} \cdot \mathbf{n}) \frac{\partial \mathbf{n}}{\partial \mathbf{n}} + \mathbf{n} \frac{\partial}{\partial \mathbf{n}} (\mathbf{v} \cdot \mathbf{n}) \right) \end{aligned}$$

gives that,

$$\begin{aligned}
(\mathbf{v} \cdot \nabla) \mathbf{v} \cdot \mathbf{n} &= |\mathbf{v} \cdot \mathbf{t}|^2 \frac{\partial \mathbf{t}}{\partial \mathbf{t}} \cdot \mathbf{n} + (\mathbf{v} \cdot \mathbf{t}) \frac{\partial}{\partial \mathbf{t}} (\mathbf{v} \cdot \mathbf{n}) \\
&+ (\mathbf{v} \cdot \mathbf{t})(\mathbf{v} \cdot \mathbf{n}) \frac{\partial \mathbf{t}}{\partial \mathbf{n}} \cdot \mathbf{n} + (\mathbf{v} \cdot \mathbf{n}) \frac{\partial}{\partial \mathbf{n}} (\mathbf{v} \cdot \mathbf{n}).
\end{aligned} \tag{2.1.12}$$

Now subtracting (2.1.12) to (2.1.11) we get,

$$\begin{aligned}
\mathbf{v} \cdot \mathbf{n} \operatorname{div} \mathbf{v} - (\mathbf{v} \cdot \nabla) \mathbf{v} \cdot \mathbf{n} &= (\mathbf{v} \cdot \mathbf{n}) \frac{\partial}{\partial \mathbf{t}} (\mathbf{v} \cdot \mathbf{t}) + |\mathbf{v} \cdot \mathbf{n}|^2 \frac{\partial \mathbf{n}}{\partial \mathbf{t}} \cdot \mathbf{t} \\
&- |\mathbf{v} \cdot \mathbf{t}|^2 \frac{\partial \mathbf{t}}{\partial \mathbf{t}} \cdot \mathbf{n} - (\mathbf{v} \cdot \mathbf{t}) \frac{\partial}{\partial \mathbf{t}} (\mathbf{v} \cdot \mathbf{n}).
\end{aligned}$$

On the other side, we know that

$$\begin{aligned}
\operatorname{div}_\tau((\mathbf{v} \cdot \mathbf{n}) \mathbf{v}_\tau) &= \left( \mathbf{v}_\tau \frac{\partial}{\partial \mathbf{t}} (\mathbf{v} \cdot \mathbf{n}) + \mathbf{v} \cdot \mathbf{n} \frac{\partial}{\partial \mathbf{t}} \mathbf{v}_\tau \right) \cdot \mathbf{t} \\
&= (\mathbf{v} \cdot \mathbf{t}) \frac{\partial}{\partial \mathbf{t}} (\mathbf{v} \cdot \mathbf{n}) + (\mathbf{v} \cdot \mathbf{n}) \frac{\partial}{\partial \mathbf{t}} (\mathbf{v} \cdot \mathbf{t}).
\end{aligned}$$

Then,

$$\begin{aligned}
\int_\Gamma [\mathbf{v} \cdot \mathbf{n} \operatorname{div} \mathbf{v} - ((\mathbf{v} \cdot \nabla) \mathbf{v}) \cdot \mathbf{n}] ds &= \int_\Gamma \operatorname{div}_\tau((\mathbf{v} \cdot \mathbf{n}) \mathbf{v}_\tau) - 2 \mathbf{v}_\tau \cdot \nabla_\tau (\mathbf{v} \cdot \mathbf{n}) ds \\
&+ \int_\Gamma |\mathbf{v} \cdot \mathbf{n}|^2 \frac{\partial \mathbf{n}}{\partial \mathbf{t}} \cdot \mathbf{t} - |\mathbf{v} \cdot \mathbf{t}|^2 \frac{\partial \mathbf{t}}{\partial \mathbf{t}} \cdot \mathbf{n} ds \\
&= -2 \int_\Gamma \mathbf{v}_\tau \cdot \nabla_\tau (\mathbf{v} \cdot \mathbf{n}) ds + \int_\Gamma |\mathbf{v} \cdot \mathbf{n}|^2 \frac{\partial \mathbf{n}}{\partial \mathbf{t}} \cdot \mathbf{t} \\
&- |\mathbf{v} \cdot \mathbf{t}|^2 \frac{\partial \mathbf{t}}{\partial \mathbf{t}} \cdot \mathbf{n} ds,
\end{aligned}$$

since,

$$\int_\Gamma \operatorname{div}_\tau((\mathbf{v} \cdot \mathbf{n}) \mathbf{v}_\tau) ds = 0.$$



Finally from (2.1.9) we deduce, for any  $\mathbf{v} \in \mathcal{D}(\overline{\Omega})$

$$\begin{aligned} \int_{\Omega} |\operatorname{curl} \mathbf{v}|^2 + \int_{\Omega} |\operatorname{div} \mathbf{v}|^2 - \int_{\Omega} |\nabla \mathbf{v}|^2 &= -2 \int_{\Gamma} \mathbf{v}_{\tau} \cdot \nabla_{\tau}(\mathbf{v} \cdot \mathbf{n}) \, ds \\ &\quad + \int_{\Gamma} |\mathbf{v} \cdot \mathbf{n}|^2 \frac{\partial \mathbf{n}}{\partial \mathbf{t}} \cdot \mathbf{t} - |\mathbf{v} \cdot \mathbf{t}|^2 \frac{\partial \mathbf{t}}{\partial \mathbf{t}} \cdot \mathbf{n} \, ds. \end{aligned} \quad (2.1.13)$$

Now, using a density argument we can extend (2.1.13) for any  $\mathbf{v} \in \mathbf{H}^1(\Omega)$ . Let  $\mathbf{v} \in \mathbf{H}^1(\Omega)$ , since the space  $\mathcal{D}(\overline{\Omega})$  is dense in  $\mathbf{H}^1(\Omega)$ , there exists a sequence  $(\mathbf{v}^k)_k \in \mathcal{D}(\overline{\Omega})$  that converges to  $\mathbf{v} \in \mathbf{H}^1(\Omega)$ . As the integrals on  $\Omega$  converge without any difficulties, we explain the way that the integrals on RHS may converge. We have,

$$\mathbf{v}^k|_{\Gamma} \longrightarrow \mathbf{v}|_{\Gamma} \quad \text{in } \mathbf{H}^{1/2}(\Gamma) \text{ as } k \rightarrow \infty.$$

As  $\Omega$  is of class  $\mathcal{C}^{1,1}$ ,

$$\mathbf{v}^k \cdot \mathbf{n} \longrightarrow \mathbf{v} \cdot \mathbf{n} \quad \text{in } \mathbf{H}^{1/2}(\Gamma) \text{ as } k \rightarrow \infty$$

then,

$$\mathbf{v}_{\tau}^k \longrightarrow \mathbf{v}_{\tau} \quad \text{in } \mathbf{H}^{1/2}(\Gamma) \quad \text{and} \quad \nabla_{\tau}(\mathbf{v}^k \cdot \mathbf{n}) \longrightarrow \nabla_{\tau}(\mathbf{v} \cdot \mathbf{n}) \quad \text{in } \mathbf{H}^{-1/2}(\Gamma).$$

We need to prove also the duality convergence,

$$\langle \nabla_{\tau}(\mathbf{v}^k \cdot \mathbf{n}), \mathbf{v}_{\tau}^k \rangle_{\Gamma} \longrightarrow \langle \nabla_{\tau}(\mathbf{v} \cdot \mathbf{n}), \mathbf{v}_{\tau} \rangle_{\Gamma} \quad \text{as } k \rightarrow \infty.$$

Indeed,

$$\begin{aligned} &| \langle \nabla_{\tau}(\mathbf{v}^k \cdot \mathbf{n}), \mathbf{v}_{\tau}^k \rangle_{\Gamma} - \langle \nabla_{\tau}(\mathbf{v} \cdot \mathbf{n}), \mathbf{v}_{\tau} \rangle_{\Gamma} | \\ &\leq | \langle \nabla_{\tau}(\mathbf{v}^k \cdot \mathbf{n}) - \nabla_{\tau}(\mathbf{v} \cdot \mathbf{n}), \mathbf{v}_{\tau}^k \rangle_{\Gamma} | + | \langle \nabla_{\tau}(\mathbf{v} \cdot \mathbf{n}), \mathbf{v}_{\tau}^k - \mathbf{v}_{\tau} \rangle_{\Gamma} | \\ &\leq \| \nabla_{\tau}(\mathbf{v}^k \cdot \mathbf{n}) - \nabla_{\tau}(\mathbf{v} \cdot \mathbf{n}) \|_{\mathbf{H}^{-1/2}(\Gamma)} \| \mathbf{v}_{\tau}^k \|_{\mathbf{H}^{1/2}(\Gamma)} \\ &\quad + \| \nabla_{\tau}(\mathbf{v} \cdot \mathbf{n}) \|_{\mathbf{H}^{-1/2}(\Gamma)} \| \mathbf{v}_{\tau}^k - \mathbf{v}_{\tau} \|_{\mathbf{H}^{1/2}(\Gamma)} \end{aligned}$$

which ends the proof.  $\square$

## 2.2 Preliminary results

Here we give several preliminary results, including the density lemma, the characterization of the kernel space, and later on we give some results about the equivalence of norms for spaces  $\mathbf{V}_2$  and  $\mathbf{V}_1$ . The next proposition is about the embedding of  $\mathbf{X}_T(\Omega)$  into  $\mathbf{H}^1(\Omega)$ . It was established for the three-dimensional case in [11]. In order to prove the 2-D case, we need first to recall the following density result. In this section, unless otherwise mentioned, we assume that  $\Omega$  is of class  $\mathcal{C}^{1,1}$ .

**Lemma 2.2.1.** *The space  $\mathbf{H}^1(\Omega) \cap \mathbf{X}_T(\Omega)$  is dense in  $\mathbf{X}_T(\Omega)$ .*

**Proposition 2.2.2.** *The space  $\mathbf{X}_T(\Omega) \subset \mathbf{H}^1(\Omega)$  and we have the following inequality,*

$$\text{for any } \mathbf{v} \in \mathbf{X}_T(\Omega), \quad \|\mathbf{v}\|_{\mathbf{H}^1(\Omega)} \leq C(\Omega) \|\mathbf{v}\|_{\mathbf{X}_T(\Omega)}. \quad (2.2.1)$$

*Proof.* Let  $\mathbf{v}$  be in  $\mathbf{X}_T(\Omega)$ . Using Lemma 2.2.1, there exists a sequence  $(\mathbf{v}_k)_k$  in  $\mathbf{H}^1(\Omega) \cap \mathbf{X}_T(\Omega)$  that converges to  $\mathbf{v}$  in  $\mathbf{X}_T(\Omega)$ . From (2.1.8) and using the boundary condition on  $\mathbf{v}_k$  for all  $k$ , we get

$$\int_{\Omega} |\text{curl } \mathbf{v}_k|^2 + \int_{\Omega} |\text{div } \mathbf{v}_k|^2 - \int_{\Omega} |\nabla \mathbf{v}_k|^2 = - \int_{\Gamma} |\mathbf{v}_k \cdot \mathbf{t}|^2 \frac{\partial \mathbf{t}}{\partial \mathbf{t}} \cdot \mathbf{n} \, ds = - \int_{\Gamma} |\mathbf{v}_k|^2 \frac{\partial \mathbf{t}}{\partial \mathbf{t}} \cdot \mathbf{n} \, ds, \quad (2.2.2)$$

using the fact that  $\frac{\partial \mathbf{t}}{\partial \mathbf{t}} \cdot \mathbf{n} = -\frac{\partial \mathbf{n}}{\partial \mathbf{t}} \cdot \mathbf{t}$  this relation is equivalent to

$$\int_{\Omega} |\nabla \mathbf{v}_k|^2 = \int_{\Omega} |\text{curl } \mathbf{v}_k|^2 + \int_{\Omega} |\text{div } \mathbf{v}_k|^2 - \int_{\Gamma} |\mathbf{v}_k|^2 \frac{\partial \mathbf{n}}{\partial \mathbf{t}} \cdot \mathbf{t} \, ds. \quad (2.2.3)$$

As  $\Omega$  is  $\mathcal{C}^{1,1}$ ,  $\mathbf{n} \in W^{1,\infty}(\Gamma)$  and

$$\left| \int_{\Gamma} |\mathbf{v}_k|^2 \frac{\partial \mathbf{n}}{\partial \mathbf{t}} \cdot \mathbf{t} \, ds \right| \leq C(\Omega) \int_{\Gamma} |\mathbf{v}_k|^2 \, ds,$$

where  $C(\Omega)$  is a constant that depends only on  $\Omega$ . Noting that (cf, [20],Thm 1.5.1.10) for all  $\mathbf{v}_k \in \mathbf{H}^1(\Omega) \cap \mathbf{X}_T(\Omega)$ , we have

$$\int_{\Gamma} |\mathbf{v}_k|^2 ds \leq \frac{1}{2} \int_{\Omega} |\nabla \mathbf{v}_k|^2 dx + C(\Omega) \int_{\Omega} |\mathbf{v}_k|^2 dx,$$

then we get

$$\left| \int_{\Gamma} \mathbf{v}_k^2 \frac{\partial \mathbf{n}}{\partial \mathbf{t}} \cdot \mathbf{t} ds \right| \leq \frac{1}{2} \int_{\Omega} |\nabla \mathbf{v}_k|^2 dx + C(\Omega) \int_{\Omega} |\mathbf{v}_k|^2 dx. \quad (2.2.4)$$

From (2.2.3) and using (2.2.4), we get

$$\|\nabla \mathbf{v}_k\|_{L^2(\Omega)}^2 \leq C(\|\text{curl } \mathbf{v}_k\|_{L^2(\Omega)}^2 + \|\text{div } \mathbf{v}_k\|_{L^2(\Omega)}^2 + \|\mathbf{v}_k\|_{L^2(\Omega)}^2). \quad (2.2.5)$$

From (2.2.5),  $(\mathbf{v}_k)_k$  is bounded in  $\mathbf{H}^1(\Omega)$  it means that there exists  $\mathbf{v} \in \mathbf{H}^1(\Omega)$  such that  $(\mathbf{v}_k)_k$  tends weakly to  $\mathbf{v}$  which is in  $\mathbf{X}_T(\Omega)$ . We conclude that  $\mathbf{X}_T(\Omega) \subset \mathbf{H}^1(\Omega)$ . The continuity of the embedding follows from (2.2.5) when  $\mathbf{v}_k \rightarrow \mathbf{v}$  in  $\mathbf{H}^1(\Omega)$  when  $k \rightarrow \infty$ , we get

$$\|\mathbf{v}\|_{\mathbf{H}^1(\Omega)} \leq C\|\mathbf{v}\|_{\mathbf{X}_T(\Omega)}.$$

□

In the following, we give some important results of equivalence of norms for both spaces  $\mathbf{V}_1$  and  $\mathbf{V}_2$  that are keys for the existence of solutions for the MHD problem.

**Theorem 2.2.3.** *The seminorm*

$$\mathbf{w} \longmapsto \|\text{curl } \mathbf{w}\|_{L^2(\Omega)} + \|\text{div } \mathbf{w}\|_{L^2(\Omega)} + \sum_{j=1}^J | \langle \mathbf{w} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} |$$

on  $\mathbf{X}_T(\Omega)$  is a norm equivalent to the norm of  $\mathbf{H}^1(\Omega)$ . In particular for any  $\mathbf{w}$  in  $\mathbf{V}_1$

$$\|\mathbf{w}\|_{\mathbf{H}^1(\Omega)} \leq C(\Omega)\|\text{curl } \mathbf{w}\|_{L^2(\Omega)}. \quad (2.2.6)$$

*Proof.* The proof consists in applying Peetre-Tartar Theorem (cf. [18], Chap.1, Theorem 2.1). Taking  $E_1 = \mathbf{X}_T(\Omega)$ ,  $E_2 = L^2(\Omega) \times L^2(\Omega)$ ,  $E_3 = \mathbf{L}^2(\Omega)$ ,  $\mathbf{A}\mathbf{u} =$

$(\operatorname{div} \mathbf{u}, \operatorname{curl} \mathbf{u})$  and the identity operator for  $B$ . Let  $\mathbf{G} = \mathbf{K}_T(\Omega)$  and the linear mapping

$$M : \mathbf{X}_T(\Omega) \longrightarrow \mathbf{K}_T(\Omega)$$

$$\mathbf{u} \longmapsto \sum_{j=1}^J \langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} \widetilde{\nabla} q_j^T.$$

Thanks to Theorem 2.8 in [11], which is also valid for  $N = 2$ , the canonical embedding of  $\mathbf{X}_T(\Omega)$  into  $\mathbf{L}^2(\Omega)$  is compact. Now, if  $\mathbf{u} \in \mathbf{K}_T(\Omega)$ , then

$$\mathbf{u} \neq \mathbf{0} \iff M\mathbf{u} \neq \mathbf{0}.$$

□

**Remark 2.2.4.** Note that the inequality (2.2.6) does not hold anymore if the flux of  $\mathbf{w}$  through the cuts  $\Sigma_j$  are not null. These conditions serve to eliminate the tangential harmonic fields contained in every vector field of a null normal trace.

Next proposition shows the equivalence of norms for  $\mathbf{V}_2$  and  $\mathbf{H}^2(\Omega)$  depending on the class  $\mathcal{C}^{2,1}$  of  $\Omega$ .

**Proposition 2.2.5.** *Suppose that  $\Omega$  is of class  $\mathcal{C}^{2,1}$ . Then  $\mathbf{V}_2 \hookrightarrow \mathbf{H}^2(\Omega)$  and the mapping*

$$\mathbf{v} \longmapsto \|\Delta \mathbf{v}\|_{\mathbf{L}^2(\Omega)}$$

*is a norm on  $\mathbf{V}_2$  equivalent to the norm of  $\mathbf{H}^2(\Omega)$ .*

*Proof.* i) First let us show that  $\mathbf{V}_2$  is a subset of  $\mathbf{H}^2(\Omega)$ .

Indeed, let  $\mathbf{v}$  be in  $\mathbf{V}_2$ . As  $\mathbf{V}_2 \subset \mathbf{H}$  there exists, thanks to Proposition 2.1.1, a unique  $\varphi \in \Phi$  such that  $\mathbf{v} = \mathbf{curl} \varphi$ . To prove that  $\mathbf{v} \in \mathbf{H}^2(\Omega)$ , we shall show that  $\varphi \in H^3(\Omega)$ . Using (2.1.3) we have  $-\Delta \mathbf{v} = \mathbf{curl} \operatorname{curl} \mathbf{v} \in \mathbf{L}^2(\Omega)$  and then we get

$$\Delta \operatorname{curl} \mathbf{v} \in \mathbf{H}^{-1}(\Omega) \quad \text{and} \quad \operatorname{curl} \mathbf{v} = 0 \text{ on } \Gamma.$$

Adding that  $\operatorname{curl} \mathbf{v} \in L^2(\Omega)$  this implies that  $\operatorname{curl} \mathbf{v}$  is in  $H_0^1(\Omega)$  because  $\Omega$  is  $\mathcal{C}^{1,1}$  (which is no longer true if  $\Omega$  is only Lipschitz, cf [5]). On the other hand, we know that  $-\Delta\varphi = \operatorname{curl} \operatorname{curl} \varphi = \operatorname{curl} \mathbf{v} \in H^1(\Omega)$ . Using the fact that  $\Delta\varphi$  belongs to  $H^1(\Omega)$  and  $\varphi$  is constant on each component of  $\Gamma$  and  $\Omega$  is  $\mathcal{C}^{2,1}$  implies that  $\varphi \in H^3(\Omega)$  and then  $\mathbf{v} \in \mathbf{H}^2(\Omega)$ .

ii) To prove the equivalence of norms, we observe that the mapping

$$i : (\mathbf{V}_2, \|\cdot\|_{\mathbf{H}^2(\Omega)}) \longrightarrow (\mathbf{V}_2, \|\cdot\|_{\mathbf{H}^1(\Omega)} + \|\Delta\cdot\|_{\mathbf{L}^2(\Omega)}),$$

is continuous. We will get the required result if we prove that the space  $\mathbf{V}_2$  is complete for the norm  $\|\cdot\|_{\mathbf{H}^1(\Omega)} + \|\Delta\cdot\|_{\mathbf{L}^2(\Omega)}$ . So, let  $(\mathbf{v}_k)$  be a Cauchy sequence of  $\mathbf{V}_2$ , we know that  $\mathbf{v}_k \rightarrow \mathbf{v} \in \mathbf{H}^1(\Omega)$  when  $k \rightarrow \infty$ . It gives that

$$\begin{aligned} \operatorname{div} \mathbf{v}_k &\rightarrow \operatorname{div} \mathbf{v} \in L^2(\Omega) \quad \text{and} \quad \mathbf{v}_k \cdot \mathbf{n} \rightarrow \mathbf{v} \cdot \mathbf{n} \in \mathbf{H}^{1/2}(\Gamma), \\ \Delta \mathbf{v}_k &\rightarrow \Delta \mathbf{v} \in \mathbf{H}^{-1}(\Omega), \end{aligned} \tag{2.2.7}$$

so that  $\operatorname{div} \mathbf{v} = 0$  in  $\Omega$  and  $\mathbf{v} \cdot \mathbf{n} = 0$  on  $\Gamma$ . As  $(\Delta \mathbf{v}_k)$  is a Cauchy sequence in  $\mathbf{L}^2(\Omega)$ , then  $\Delta \mathbf{v}_k$  converges to  $\mathbf{z}$  in  $\mathbf{L}^2(\Omega)$ . From (2.2.7) we get  $\mathbf{z} = \Delta \mathbf{v}$  and

$$\Delta \mathbf{v}_k \rightarrow \Delta \mathbf{v} \in \mathbf{L}^2(\Omega), \quad \text{when } k \rightarrow \infty.$$

So the space  $\mathbf{V}_2$  equipped with the second norm is a Banach space and then the identity mapping from  $(\mathbf{V}_2, \|\cdot\|_{\mathbf{H}^2(\Omega)})$  onto  $(\mathbf{V}_2, \|\cdot\|_{\mathbf{H}^1(\Omega)} + \|\Delta\cdot\|_{\mathbf{L}^2(\Omega)})$  is an isomorphism. In particular, we have the following inequality

$$\|\mathbf{v}\|_{\mathbf{H}^2(\Omega)} \leq C(\|\mathbf{v}\|_{\mathbf{H}^1(\Omega)} + \|\Delta \mathbf{v}\|_{\mathbf{L}^2(\Omega)}). \tag{2.2.8}$$

Now taking  $\mathbf{v} \in \mathbf{V}_2$ , then  $\operatorname{curl} \mathbf{v} \in H_0^1(\Omega)$ . Using (2.1.2) and (2.1.4), we get

$$\begin{aligned} \int_{\Omega} |\nabla(\operatorname{curl} \mathbf{v})|^2 &= - \langle \Delta(\operatorname{curl} \mathbf{v}), \operatorname{curl} \mathbf{v} \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} \\ &= \int_{\Omega} |\operatorname{curl} \operatorname{curl} \mathbf{v}|^2 = \int_{\Omega} |\Delta \mathbf{v}|^2. \end{aligned} \tag{2.2.9}$$

Finally, using the following Poincaré inequality:

$$\forall w \in H_0^1(\Omega), \quad \|w\|_{L^2(\Omega)} \leq C(\Omega) \|\nabla w\|_{L^2(\Omega)},$$

for  $w = \operatorname{curl} \mathbf{v}$ , thanks to (2.2.9) we get

$$\|\operatorname{curl} \mathbf{v}\|_{L^2(\Omega)} \leq C(\Omega) \|\nabla \operatorname{curl} \mathbf{v}\|_{L^2(\Omega)} \leq C(\Omega) \|\Delta \mathbf{v}\|_{L^2(\Omega)}. \quad (2.2.10)$$

Finally, from (2.2.6) and (2.2.10) we deduce the required inequality:

$$\|\mathbf{v}\|_{\mathbf{H}^2(\Omega)} \leq C(\Omega) \|\Delta \mathbf{v}\|_{L^2(\Omega)}.$$

□

**Lemma 2.2.6.** *Let  $z \in L^2(\Omega)$ .*

*i) If  $\mathbf{u} \in \mathbf{H}(\operatorname{div}, \Omega)$ , we have  $(\mathbf{u} \cdot \nabla)z \in H^{-s}(\Omega)$  for any  $s > 2$  and*

$$\|(\mathbf{u} \cdot \nabla)z\|_{H^{-s}(\Omega)} \leq C(\Omega) \|z\|_{L^2(\Omega)} \|\mathbf{u}\|_{\mathbf{H}(\operatorname{div}, \Omega)}. \quad (2.2.11)$$

*ii) If, moreover,  $\mathbf{u} \in \mathbf{W}^{1,r}(\Omega)$  for  $r > 2$ , then  $(\mathbf{u} \cdot \nabla)z \in H^{-1}(\Omega)$  and*

$$\|(\mathbf{u} \cdot \nabla)z\|_{H^{-1}(\Omega)} \leq C(\Omega) \|z\|_{L^2(\Omega)} \|\mathbf{u}\|_{\mathbf{W}^{1,r}(\Omega)}, \quad (2.2.12)$$

*iii) If  $\mathbf{u} \in \mathbf{H}^s(\Omega)$  for  $s > 1$ , then  $(\mathbf{u} \cdot \nabla)z \in H^{-1}(\Omega)$  and*

$$\|(\mathbf{u} \cdot \nabla)z\|_{H^{-1}(\Omega)} \leq C(\Omega) \|z\|_{L^2(\Omega)} \|\mathbf{u}\|_{\mathbf{H}^s(\Omega)}. \quad (2.2.13)$$

**Remark 2.2.7.** We can obtain the same conclusion in Point ii) and in Point iii) above if we suppose only that

$$\mathbf{u} \in \mathbf{L}^\infty(\Omega) \quad \text{and} \quad \operatorname{div} \mathbf{u} \in L^r(\Omega) \quad \text{with} \quad r > 2.$$

*Proof.* **i)** Let  $\mathbf{u} \in \mathbf{H}(\text{div}, \Omega)$  and  $z \in L^2(\Omega)$ . For any  $\varphi \in \mathcal{D}(\Omega)$ , we define

$$\langle (\mathbf{u} \cdot \nabla)z, \varphi \rangle = \langle \nabla z, \mathbf{u} \varphi \rangle_{[\mathbf{H}_0(\text{div}, \Omega)]' \times \mathbf{H}_0(\text{div}, \Omega)}.$$

This definition makes sense since  $\mathcal{D}(\Omega)$  is dense in  $\mathbf{H}_0(\text{div}, \Omega)$ . It is proved in [19] (cf Remark 1.1) that

$$\langle (\mathbf{u} \cdot \nabla)z, \varphi \rangle = - \langle z, \text{div}(\varphi \mathbf{u}) \rangle = \int_{\Omega} z \text{div}(\varphi \mathbf{u}). \quad (2.2.14)$$

We know that  $\text{div}(\varphi \mathbf{u}) = \mathbf{u} \cdot \nabla \varphi + \varphi \text{div} \mathbf{u}$  and for any  $s > 2$ , we have  $H^s(\Omega) \hookrightarrow W^{1, \infty}(\Omega)$ . So using (2.2.14), we get

$$\begin{aligned} |\langle (\mathbf{u} \cdot \nabla)z, \varphi \rangle| &\leq C(\Omega) \|z\|_{L^2(\Omega)} (\|\mathbf{u} \cdot \nabla \varphi\|_{L^2} + \|\varphi \text{div} \mathbf{u}\|_{L^2(\Omega)}) \\ &\leq C(\Omega) \|z\|_{L^2(\Omega)} \|\mathbf{u}\|_{\mathbf{H}(\text{div}, \Omega)} \|\varphi\|_{W^{1, \infty}(\Omega)} \\ &\leq C(\Omega) \|z\|_{L^2(\Omega)} \|\mathbf{u}\|_{\mathbf{H}(\text{div}, \Omega)} \|\varphi\|_{H^s(\Omega)}. \end{aligned}$$

We deduce that  $(\mathbf{u} \cdot \nabla)z \in H^{-s}(\Omega)$  with the estimate (2.2.11).

**ii)** Let  $q$  a finite number such that  $1/2 = 1/q + 1/r$ . Using the Sobolev embeddings, we know that  $\mathbf{H}^m(\Omega) \hookrightarrow \mathbf{L}^\infty(\Omega)$  for any  $m > 1$ .

Then for any  $\varphi \in \mathcal{D}(\Omega)$  we have

$$\begin{aligned} |\langle (\mathbf{u} \cdot \nabla)z, \varphi \rangle| &\leq C(\Omega) \|z\|_{L^2(\Omega)} (\|\mathbf{u} \cdot \nabla \varphi\|_{L^2} + \|\varphi \text{div} \mathbf{u}\|_{L^2(\Omega)}) \\ &\leq C(\Omega) \|z\|_{L^2(\Omega)} (\|\mathbf{u}\|_{\mathbf{L}^\infty(\Omega)} \|\varphi\|_{H^1(\Omega)} + \|\text{div} \mathbf{u}\|_{L^r(\Omega)} \|\varphi\|_{L^q(\Omega)}) \\ &\leq C(\Omega) \|z\|_{L^2(\Omega)} \|\mathbf{u}\|_{\mathbf{W}^{1, r}(\Omega)} \|\varphi\|_{H^1(\Omega)}. \end{aligned}$$

That means that  $(\mathbf{u} \cdot \nabla)z \in H^{-1}(\Omega)$  with the estimate (2.2.12).

**iii)** If  $\mathbf{u} \in \mathbf{H}^s(\Omega)$  with  $1 < s < 2$  and  $\varphi \in \mathcal{D}(\Omega)$ , we have

$$|\langle \nabla z, \mathbf{u} \varphi \rangle| \leq \|\nabla z\|_{\mathbf{H}^{-1}(\Omega)} \|\mathbf{u} \varphi\|_{\mathbf{H}^1(\Omega)} \quad (2.2.15)$$

where

$$\|\mathbf{u} \varphi\|_{\mathbf{H}^1(\Omega)} = \left( \|\varphi \mathbf{u}\|_{\mathbf{L}^2(\Omega)}^2 + \|\mathbf{u} \nabla \varphi\|_{\mathbf{L}^2(\Omega)}^2 + \|\varphi \nabla \mathbf{u}\|_{\mathbf{L}^2(\Omega)}^2 \right)^{1/2}.$$

It suffices then to estimate the last term above. For that we use the following embedding  $\mathbf{H}^{s-1}(\Omega) \hookrightarrow \mathbf{L}^k(\Omega)$  which holds for  $k = \frac{2}{2-s}$ . Then for  $t = \frac{2}{s-1}$  we have,

$$\begin{aligned} \|\varphi \nabla \mathbf{u}\|_{\mathbf{L}^2(\Omega)} &\leq \|\nabla \mathbf{u}\|_{\mathbf{L}^k(\Omega)} \|\varphi\|_{L^t(\Omega)} \\ &\leq C \|\mathbf{u}\|_{\mathbf{H}^s(\Omega)} \|\varphi\|_{H^1(\Omega)}. \end{aligned} \quad (2.2.16)$$

Now we return to (2.2.15) and we use (2.2.16):

$$\begin{aligned} | \langle \nabla z, \mathbf{u} \varphi \rangle | &\leq C(\Omega) \|\mathbf{z}\|_{\mathbf{L}^2(\Omega)} (2 \|\mathbf{u}\|_{\mathbf{L}^\infty(\Omega)} \|\varphi\|_{H^1(\Omega)} + \|\mathbf{u}\|_{\mathbf{H}^s(\Omega)} \|\varphi\|_{H^1(\Omega)}) \\ &\leq C(\Omega) \|\mathbf{z}\|_{\mathbf{L}^2(\Omega)} \|\mathbf{u}\|_{\mathbf{H}^s(\Omega)} \|\varphi\|_{H^1(\Omega)}. \end{aligned}$$

We deduce finally the estimate (2.2.13). □

**Lemma 2.2.8.** *For any  $\mathbf{u}$  in  $\mathbf{V}_2$ , we have the following identities*

$$(\mathbf{u} \cdot \nabla) \mathbf{u} = \frac{1}{2} \nabla |\mathbf{u}|^2 + \mathbf{u}^\perp \operatorname{curl} \mathbf{u} \quad \text{and} \quad \mathbf{u} \cdot \nabla (\operatorname{curl} \mathbf{u}) = \operatorname{curl} (\mathbf{u}^\perp \operatorname{curl} \mathbf{u}) \quad (2.2.17)$$

where  $\mathbf{u}^\perp = (-u_2, u_1)$ .

**Definition 2.2.9.** We define the following trilinear form

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \int_{\Omega} u_j \frac{\partial v_i}{\partial x_j} w_i \, dx,$$

where we used the implicit summation on the repeated indices.

The next lemma contains important properties about the trilinear form  $b$  that will allow us to obtain the existence of the solution.

**Lemma 2.2.10.** *i) The form  $b$  is continuous on  $\mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega)$  and*

*$\forall \mathbf{u} \in \mathbf{H}^1(\Omega)$  with  $\operatorname{div} \mathbf{u} = 0$  in  $\Omega$  and  $\mathbf{u} \cdot \mathbf{n} = 0$  on  $\Gamma$  and  $\forall \mathbf{v}, \mathbf{w} \in \mathbf{H}^1(\Omega)$ ,*



$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = -b(\mathbf{u}, \mathbf{w}, \mathbf{v}). \quad (2.2.18)$$

ii) For any  $\mathbf{u} \in \mathbf{V}_2$ , we have

$$b(\mathbf{u}, \mathbf{u}, \Delta \mathbf{u}) = 0. \quad (2.2.19)$$

*Proof.* As the first property is easy to check, we skip its proof. Now, using the identities in Lemma 2.2.8 and relation (2.1.3) we get

$$\begin{aligned} b(\mathbf{u}, \mathbf{u}, \Delta \mathbf{u}) &= \int_{\Omega} (\mathbf{u} \cdot \nabla) \mathbf{u} \cdot \Delta \mathbf{u} = - \int_{\Omega} (\mathbf{u} \cdot \nabla) \mathbf{u} \operatorname{curl} \operatorname{curl} \mathbf{u} \\ &= \int_{\Omega} \operatorname{curl}((\mathbf{u} \cdot \nabla) \mathbf{u}) \operatorname{curl} \mathbf{u} = \int_{\Omega} \operatorname{curl}(\mathbf{u}^\perp \operatorname{curl} \mathbf{u}) \operatorname{curl} \mathbf{u} \\ &= \int_{\Omega} \mathbf{u} \cdot \nabla(\operatorname{curl} \mathbf{u}) \operatorname{curl} \mathbf{u} = b(\mathbf{u}, \operatorname{curl} \mathbf{u}, \operatorname{curl} \mathbf{u}) = 0. \square \end{aligned}$$

Let us introduce the following space: for any  $1 < p < \infty$

$$\mathbf{E}_p(\Delta, \Omega) = \{\mathbf{v} \in \mathbf{W}^{1,p}(\Omega); \Delta \mathbf{v} \in \mathbf{L}^{r(p)}(\Omega)\}, \quad (2.2.20)$$

where

$$r(p) = \begin{cases} 1 & \text{if } p < 2 \\ 1 + \varepsilon, \text{ with arbitrary } \varepsilon > 0 & \text{if } p = 2 \\ \frac{2p}{2+p} & \text{if } p > 2. \end{cases}$$

For  $p = 2$ , we denote more simply this space by the notation  $\mathbf{E}(\Delta, \Omega)$ . Clearly, the space  $\mathbf{E}_p(\Delta, \Omega)$  is a Banach space for the norm

$$\|\mathbf{v}\|_{\mathbf{E}_p(\Delta, \Omega)} = \|\mathbf{v}\|_{\mathbf{W}^{1,p}(\Omega)} + \|\Delta \mathbf{v}\|_{\mathbf{L}^{r(p)}(\Omega)}. \quad (2.2.21)$$

Next lemma concerns an important density result followed by a Green formula.

**Lemma 2.2.11.** *i) The space  $\mathbf{D}(\overline{\Omega})$  is dense in  $\mathbf{E}(\Delta, \Omega)$ .*

*ii) The following linear mappings*

$$\begin{aligned} \ell_1 : \mathcal{D}(\overline{\Omega}) &\longrightarrow H^{-1/2}(\Gamma), & \ell_2 : \mathcal{D}(\overline{\Omega}) &\longrightarrow H^{-1/2}(\Gamma) \\ \mathbf{u} &\longmapsto \operatorname{curl} \mathbf{u} & \mathbf{u} &\longmapsto \operatorname{div} \mathbf{u} \end{aligned}$$

defined on  $\mathbf{D}(\overline{\Omega})$ , can be extended by continuity to a linear and continuous mapping from  $\mathbf{E}(\Delta, \Omega)$  to  $H^{-1/2}(\Gamma)$ . Moreover, we have the following Green formula:

for any  $\mathbf{u} \in \mathbf{E}(\Delta, \Omega)$  and  $\mathbf{v} \in \mathbf{H}^1(\Omega)$ ,

$$\begin{aligned} - \int_{\Omega} \Delta \mathbf{u} \cdot \mathbf{v} &= \int_{\Omega} \operatorname{curl} \mathbf{u} \operatorname{curl} \mathbf{v} + \int_{\Omega} (\operatorname{div} \mathbf{u})(\operatorname{div} \mathbf{v}) \\ &\quad - \langle \operatorname{curl} \mathbf{u}, \mathbf{v} \cdot \mathbf{t} \rangle_{\Gamma} - \langle \operatorname{div} \mathbf{u}, \mathbf{v} \cdot \mathbf{n} \rangle_{\Gamma}, \end{aligned} \quad (2.2.22)$$

where  $\langle \cdot, \cdot \rangle_{\Gamma}$  is the duality brackets on  $H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$ .

*Proof.* **i)** Let  $P$  be a continuous linear mapping from  $\mathbf{H}^1(\Omega)$  to  $\mathbf{H}^1(\mathbb{R}^2)$  such that  $P\mathbf{u}|_{\Omega} = \mathbf{u}$  for every  $\mathbf{u}$  in  $\mathbf{H}^1(\Omega)$ . Thus for any continuous linear form  $\ell$  on  $\mathbf{E}(\Delta, \Omega)$  there exists  $(\mathbf{f}, \mathbf{g}) \in \mathbf{H}^{-1}(\mathbb{R}^2) \times \mathbf{L}^{[r(2)]'}(\Omega)$  such that for any  $\mathbf{v} \in \mathbf{E}(\Delta, \Omega)$

$$\ell(\mathbf{v}) = \langle \mathbf{f}, P\mathbf{v} \rangle + \int_{\Omega} \mathbf{g} \cdot \Delta \mathbf{v},$$

where  $r(2) = 1 + \varepsilon$ . We suppose that  $\ell$  vanishes on  $\mathcal{D}(\overline{\Omega})$  and we will show that it is zero everywhere. For any  $\varphi \in \mathcal{D}(\mathbb{R}^2)$ , we have

$$\langle \mathbf{f}, \varphi \rangle + \int_{\mathbb{R}^2} \tilde{\mathbf{g}} \cdot \Delta \varphi = 0,$$

where  $\tilde{\mathbf{g}}$  is the extension by zero of  $\mathbf{g}$  in  $\mathbb{R}^2$ . The last equality implies that

$$-\Delta \tilde{\mathbf{g}} = \mathbf{f} \quad \text{in } \mathbb{R}^2. \quad (2.2.23)$$

Observe that  $\tilde{\mathbf{g}} \in \mathbf{L}^1(\mathbb{R}^2) \cap \mathbf{L}^q(\mathbb{R}^2)$  for any finite number  $q > 1$  and

$$\tilde{\mathbf{g}} - \Delta \tilde{\mathbf{g}} = \mathbf{f} + \tilde{\mathbf{g}} \quad \text{in } \mathbb{R}^2. \quad (2.2.24)$$

Since the above RHS belongs to  $\mathbf{H}^{-1}(\mathbb{R}^2)$ , we deduce that  $\tilde{\mathbf{g}} \in \mathbf{H}^1(\mathbb{R}^2)$  and thus  $\mathbf{g} \in \mathbf{H}_0^1(\Omega)$ . By density of  $\mathcal{D}(\Omega)$  in  $\mathbf{H}_0^1(\Omega)$ , there exist  $(\mathbf{g}_k)$  in  $\mathcal{D}(\Omega)$  such that  $\mathbf{g}_k \rightarrow \mathbf{g}$

in  $\mathbf{H}_0^1(\Omega)$  as  $k \rightarrow \infty$ . Now, for any  $\mathbf{v} \in \mathbf{E}(\Delta, \Omega)$  we have

$$\begin{aligned}
\ell(\mathbf{v}) &= \lim_{k \rightarrow \infty} \left\{ \langle -\Delta \tilde{\mathbf{g}}_k, P\mathbf{v} \rangle + \int_{\Omega} \mathbf{g}_k \cdot \Delta \mathbf{v} \right\} \\
&= \lim_{k \rightarrow \infty} \left\{ \int_{\mathbb{R}^2} \nabla \tilde{\mathbf{g}}_k \cdot \nabla P\mathbf{v} + \int_{\Omega} \mathbf{g}_k \cdot \Delta \mathbf{v} \right\} \\
&= \lim_{k \rightarrow \infty} \left\{ \int_{\Omega} \nabla \mathbf{g}_k \cdot \nabla \mathbf{v} + \int_{\Omega} \mathbf{g}_k \cdot \Delta \mathbf{v} \right\} \\
&= \lim_{k \rightarrow \infty} \left\{ - \int_{\Omega} \Delta \mathbf{v} \cdot \mathbf{g}_k + \int_{\Omega} \mathbf{g}_k \cdot \Delta \mathbf{v} \right\} \\
&= 0.
\end{aligned}$$

ii) We know that for any  $\mathbf{u} \in \mathcal{D}(\bar{\Omega})$  and  $\mathbf{v} \in \mathbf{H}^1(\Omega)$ , we have

$$\begin{aligned}
- \int_{\Omega} \Delta \mathbf{u} \cdot \mathbf{v} &= \int_{\Omega} \operatorname{curl} \mathbf{u} \operatorname{curl} \mathbf{v} + \int_{\Omega} \operatorname{div} \mathbf{u} \operatorname{div} \mathbf{v} \\
&\quad - \int_{\Gamma} (\mathbf{v} \cdot \mathbf{t}) \operatorname{curl} \mathbf{u} - \int_{\Gamma} (\mathbf{v} \cdot \mathbf{n}) \operatorname{div} \mathbf{u}. \tag{2.2.25}
\end{aligned}$$

Let  $\mu \in H^{1/2}(\Gamma)$ . As  $\Omega$  is  $\mathcal{C}^{1,1}$ , then  $\mathbf{t} \in \mathbf{W}^{1,\infty}(\Gamma)$ , so there exists  $\varphi \in \mathbf{H}^1(\Omega)$  such that  $\operatorname{div} \varphi = 0$  in  $\Omega$ ,  $\varphi = \mu \mathbf{t}$  on  $\Gamma$  and

$$\|\varphi\|_{\mathbf{H}^1(\Omega)} \leq C \|\mu\|_{H^{1/2}(\Gamma)}. \tag{2.2.26}$$

Using (2.2.25) with  $\mathbf{v} = \varphi$ , we deduce

$$\int_{\Gamma} \mu \operatorname{curl} \mathbf{u} = \int_{\Omega} \Delta \mathbf{u} \cdot \varphi + \int_{\Omega} \operatorname{curl} \mathbf{u} \operatorname{curl} \varphi.$$

Then, using (2.2.26) we get:

$$\left| \int_{\Gamma} \mu \operatorname{curl} \mathbf{u} \right| \leq C \|\mathbf{u}\|_{\mathbf{E}(\Delta, \Omega)} \|\mu\|_{H^{1/2}(\Gamma)}$$

and then the estimate

$$\|\operatorname{curl} \mathbf{u}\|_{H^{-1/2}(\Gamma)} \leq C \|\mathbf{u}\|_{\mathbf{E}(\Delta, \Omega)}.$$

We deduce that the linear mapping  $\ell_1$  is continuous in  $\mathcal{D}(\overline{\Omega})$ , for the norm of  $\mathbf{E}(\Delta, \Omega)$ . The density of  $\mathcal{D}(\overline{\Omega})$  in  $\mathbf{E}(\Delta, \Omega)$  allows to extend by continuity  $\ell_1$  from  $\mathbf{E}(\Delta, \Omega)$  to  $H^{-1/2}(\Gamma)$ .

Now, we will show that  $\ell_2$  is continuous from  $\mathbf{E}(\Delta, \Omega)$  into  $H^{-1/2}(\Gamma)$ . Let  $\mu \in H^{1/2}(\Gamma)$ , as  $\Omega$  is  $\mathcal{C}^{1,1}$  we know that there exists  $\psi \in H^2(\Omega) \cap H_0^1(\Omega)$  such that  $\frac{\partial \psi}{\partial \mathbf{n}} = \mu$  on  $\Gamma$  with

$$\|\psi\|_{H^2(\Omega)} \leq C \|\mu\|_{H^{1/2}(\Gamma)}. \quad (2.2.27)$$

Setting  $\mathbf{v} = \nabla \psi$ , then  $\mathbf{v} \cdot \mathbf{t} = 0$  on  $\Gamma$ . For any  $\mathbf{u} \in \mathcal{D}(\overline{\Omega})$  we have from (2.2.25)

$$\int_{\Gamma} \mu \operatorname{div} \mathbf{u} = \int_{\Omega} \Delta \mathbf{u} \cdot \nabla \psi + \int_{\Omega} (\Delta \psi) \operatorname{div} \mathbf{u}.$$

Using (2.2.27), we get

$$\left| \int_{\Gamma} \mu \operatorname{div} \mathbf{u} \right| \leq C \|\mathbf{u}\|_{\mathbf{E}(\Delta, \Omega)} \|\mu\|_{H^{1/2}(\Gamma)}.$$

We deduce that the linear mapping  $\ell_2$  is continuous from  $\mathbf{E}(\Delta, \Omega)$  to  $H^{-1/2}(\Gamma)$  and the Green formula (2.2.22).  $\square$

Lemma below shows the Green formula (2.2.22) for the general case in  $\mathbf{E}_p(\Delta, \Omega)$ .

**Lemma 2.2.12.** *i) The space  $\mathcal{D}(\overline{\Omega})$  is dense in  $\mathbf{E}_p(\Delta, \Omega)$ .*

*ii) The linear mappings  $\ell_1$  and  $\ell_2$  defined on  $\mathcal{D}(\overline{\Omega})$  can be extended by continuity to a linear and continuous mapping from  $\mathbf{E}_p(\Delta, \Omega)$  to  $W^{-1/p, p}(\Gamma)$ . Moreover, for any  $\mathbf{u} \in \mathbf{E}_p(\Delta, \Omega)$  and  $\mathbf{v} \in \mathbf{W}^{1, p'}(\Omega)$  we have,*

$$\begin{aligned} - \int_{\Omega} \Delta \mathbf{u} \cdot \mathbf{v} &= \int_{\Omega} \operatorname{curl} \mathbf{u} \operatorname{curl} \mathbf{v} + \int_{\Omega} (\operatorname{div} \mathbf{u})(\operatorname{div} \mathbf{v}) \\ &\quad - \langle \operatorname{curl} \mathbf{u}, \mathbf{v} \cdot \mathbf{t} \rangle_{\Gamma} - \langle \operatorname{div} \mathbf{u}, \mathbf{v} \cdot \mathbf{n} \rangle_{\Gamma} \end{aligned} \quad (2.2.28)$$

where  $\langle \cdot, \cdot \rangle_\Gamma$  is the duality brackets on  $W^{-1/p,p}(\Gamma) \times W^{1/p,p'}(\Gamma)$ .

**Remark 2.2.13.** The above conclusion holds if we replace the property  $\Delta \mathbf{v} \in \mathbf{L}^{r(p)}(\Omega)$  by

$$\Delta \mathbf{v} = \mathbf{f} + \nabla \chi, \quad \text{with } \mathbf{f} \in \mathbf{L}^{r(p)}(\Omega) \quad \text{and} \quad \chi \in L^p(\Omega).$$

see Corollary 4.2 in [6]

*Proof.* In this proof, we proceed similarly as in the Hilbert case and we use the same notations. To prove the density of  $\mathcal{D}(\bar{\Omega})$  in  $\mathbf{E}_p(\Delta, \Omega)$ , let  $\ell$  in  $\mathbf{E}_p(\Delta, \Omega)'$  which vanishes on  $\mathcal{D}(\bar{\Omega})$ . There exist  $\mathbf{f} \in \mathbf{W}^{-1,p'}(\mathbb{R}^2)$  and  $\mathbf{g} \in \mathbf{L}^{[r(p)]'}(\Omega)$  such that

$$\ell(\mathbf{v}) = \langle \mathbf{f}, P\mathbf{v} \rangle + \int_{\Omega} \mathbf{g} \cdot \Delta \mathbf{v},$$

where the extension operator  $P$  is continuous from  $\mathbf{W}^{1,p}(\Omega) \mapsto \mathbf{W}^{1,p}(\mathbb{R}^2)$ . Proceeding as in the previous lemma we get equation (2.2.24) and using the fact that  $\tilde{\mathbf{g}} \in \mathbf{L}^1(\mathbb{R}^2) \cap \mathbf{L}^{[r(p)]'}(\mathbb{R}^2)$ , we have  $\tilde{\mathbf{g}} \in \mathbf{L}^{p'}(\mathbb{R}^2)$  since  $[r(p)]' > p'$ . Then the RHS in (2.2.24) belongs to  $\mathbf{W}^{-1,p'}(\mathbb{R}^2)$ . As for the case  $p = 2$ , we deduce that  $\mathbf{g} \in \mathbf{W}_0^{1,p'}(\Omega)$  and then  $\ell$  vanishes on  $\mathbf{E}_p(\Delta, \Omega)$ . The rest of the proof is very similar.  $\square$

## 2.3 Stokes problem with free-divergence forces

Here, we are interested to study the Stokes problem and we once again consider  $\Omega$  of class  $\mathcal{C}^{1,1}$ . Let us start with the following elliptic problem which can be seen as a Stokes problem, where the pressure term has disappeared: Find  $\mathbf{u} \in \mathbf{H}^1(\Omega)$  verifying

$$\begin{cases} -\Delta \mathbf{u} = \mathbf{f}, & \operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega, \\ \mathbf{u} \cdot \mathbf{n} = 0, & \operatorname{curl} \mathbf{u} = 0 \quad \text{on } \Gamma, \end{cases} \quad (2.3.1)$$

Thereafter, we will see the sense that we give to the second boundary condition  $\operatorname{curl} \mathbf{u} = 0$ . Here, contrary to the classical case of the Stokes problem with Dirichlet boundary conditions for a given data  $\mathbf{f} \in \mathbf{H}^{-1}(\Omega)$ , it is not reasonable to take  $\mathbf{f}$  in

negative Sobolev spaces due to the boundary conditions of  $\mathbf{u}$ . Which space should be suitable here for  $\mathbf{f}$ ?

In order to solve Problem (2.3.1) we introduce the following formulation:

$$\begin{cases} \text{Find } \mathbf{u} \in \mathbf{V} \text{ such that} \\ \forall \boldsymbol{\varphi} \in \mathbf{V}, \quad \int_{\Omega} \operatorname{curl} \mathbf{u} \operatorname{curl} \boldsymbol{\varphi} = \int_{\Omega} \mathbf{f} \cdot \boldsymbol{\varphi}, \end{cases}$$

where

$$\mathbf{V} = \{\mathbf{v} \in \mathbf{H}^1(\Omega); \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega, \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma\},$$

equipped with the norm of  $\mathbf{H}^1(\Omega)$ . As  $\mathbf{H}^1(\Omega) \hookrightarrow \mathbf{L}^q(\Omega)$  for any  $q \geq 1$  we will take  $\mathbf{f} \in \mathbf{L}^s(\Omega)$  for arbitrary  $s > 1$ . We consider the bilinear form  $a(\mathbf{u}, \mathbf{v}) = \int_{\Omega} \operatorname{curl} \mathbf{u} \operatorname{curl} \mathbf{v}$  which is clearly continuous on  $\mathbf{V} \times \mathbf{V}$ . Observe that this form is not coercive on  $\mathbf{V}$ ! For this reason, we have to add the null flux condition on  $\Sigma_j$  for any  $1 \leq j \leq J$ . Due to Theorem 2.2.3, the coercivity of  $a(\cdot, \cdot)$  is valid on  $\mathbf{V}_1$ . As the linear form  $\ell(\mathbf{v}) = \int_{\Omega} \mathbf{f} \cdot \mathbf{v}$  is continuous on  $\mathbf{V}_1$ , then by Lax-Milgram theorem we deduce that the variational problem:

$$\begin{cases} \text{Find } \mathbf{u} \in \mathbf{V}_1 \text{ such that} \\ \forall \boldsymbol{\varphi} \in \mathbf{V}_1, \quad \int_{\Omega} \operatorname{curl} \mathbf{u} \operatorname{curl} \boldsymbol{\varphi} = \int_{\Omega} \mathbf{f} \cdot \boldsymbol{\varphi}, \end{cases} \quad (2.3.2)$$

has a unique solution.

**Theorem 2.3.1.** *i) For a given  $\mathbf{f}$  in  $\mathbf{L}^s(\Omega)$ , with arbitrary  $s > 1$ , satisfying the following conditions*

$$\begin{aligned} \operatorname{div} \mathbf{f} &= 0 \quad \text{in } \Omega, \\ \mathbf{f} \cdot \mathbf{n} &= 0 \quad \text{on } \Gamma, \\ \forall \boldsymbol{\varphi} \in \mathbf{K}_T(\Omega), \quad \int_{\Omega} \mathbf{f} \cdot \boldsymbol{\varphi} &= 0, \end{aligned} \quad (2.3.3)$$

then the problem:

$$\begin{cases} -\Delta \mathbf{u} = \mathbf{f} & \text{and } \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} \cdot \mathbf{n} = 0 & \text{and } \operatorname{curl} \mathbf{u} = 0 & \text{on } \Gamma, \\ \langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} = 0, & & \text{for } 1 \leq j \leq J. \end{cases} \quad (2.3.4)$$

has a unique solution  $\mathbf{u} \in \mathbf{H}^1(\Omega)$ .

ii) Moreover, if  $\Omega$  is of class  $\mathcal{C}^{2,1}$  and  $\mathbf{f} \in \mathbf{L}^2(\Omega)$ , then  $\mathbf{u} \in \mathbf{H}^2(\Omega)$ ,

**Remark 2.3.2.** i) Observe that the boundary condition  $\operatorname{curl} \mathbf{u} = 0$  on  $\Gamma$  makes sense. Indeed, thanks to Lemma 2.2.11, for any  $\mathbf{u} \in \mathbf{H}^1(\Omega)$  with  $\Delta \mathbf{u} \in \mathbf{L}^{1+\varepsilon}(\Omega)$  the trace of  $\operatorname{curl} \mathbf{u}$  on  $\Gamma$  belongs to  $H^{-1/2}(\Gamma)$ .

ii) The same conclusion holds if we replace in Point i) the vector field  $\mathbf{f}$  in  $\mathbf{L}^s(\Omega)$  by the vector field  $\mathbf{g} + \nabla \chi$ , with  $\mathbf{g}$  in  $\mathbf{L}^s(\Omega)$  with arbitrary  $s > 1$  and  $\chi \in L^2(\Omega)$ .

*Proof.* i) **Necessary condition.** We prove here that the condition (2.3.3) are necessary. Let  $\mathbf{u} \in \mathbf{H}^1(\Omega)$  solution of Problem (2.3.4). Then clearly  $\operatorname{div} \mathbf{f} = 0$  and  $\mathbf{f} \cdot \mathbf{n} \in H^{-1/2}(\Gamma)$ . Now, let  $z = \operatorname{curl} \mathbf{u}$ . By (2.1.3) we have  $\mathbf{f} = \operatorname{curl} z$  and

$$\mathbf{f} \cdot \mathbf{n} = \operatorname{curl} z \cdot \mathbf{n} = \nabla z \cdot \mathbf{t} = 0 \quad \text{on } \Gamma.$$

Note that,

$$\int_{\Omega} \mathbf{f} \cdot \widetilde{\nabla q_j^T} = \int_{\Omega} \operatorname{curl} z \cdot \widetilde{\nabla q_j^T} = \int_{\Omega} z \operatorname{curl}(\widetilde{\nabla q_j^T}) - \int_{\Gamma} z (\nabla q_j^T \cdot \mathbf{t}) = 0.$$

ii) **Existence in  $\mathbf{H}^1(\Omega)$ .** We know that Problem (2.3.2) has a unique solution in  $\mathbf{H}^1(\Omega)$ . Since  $\mathcal{D}(\Omega)$  is not included in  $\mathbf{V}_1$ , in order to show the equivalence of Problems (2.3.2) and (2.3.4), we need to extend (2.3.2) to any test function in the space  $\mathbf{X}_T(\Omega)$ . For that, let us consider the following problem:

$$\begin{cases} \text{Find } \mathbf{u} \in \mathbf{V}_1 \text{ such that} \\ \forall \boldsymbol{\varphi} \in \mathbf{X}_T(\Omega), \quad \int_{\Omega} \operatorname{curl} \mathbf{u} \operatorname{curl} \boldsymbol{\varphi} = \int_{\Omega} \mathbf{f} \cdot \boldsymbol{\varphi}, \end{cases} \quad (2.3.5)$$

Clearly, (2.3.5) implies (2.3.2). Conversely, we will prove that (2.3.2) implies (2.3.5). Indeed, let  $\mathbf{u}$  solution of Problem (2.3.2) and  $\boldsymbol{\varphi} \in \mathbf{X}_T(\Omega)$ . Let  $\psi$  in  $H^2(\Omega)$  be a solution of the following Neumann Problem:

$$\begin{cases} \Delta\psi = \operatorname{div} \boldsymbol{\varphi} & \text{in } \Omega, \\ \frac{\partial\psi}{\partial\mathbf{n}} = 0 & \text{on } \Gamma. \end{cases} \quad (2.3.6)$$

Setting now,

$$\tilde{\boldsymbol{\varphi}} = \boldsymbol{\varphi} - \nabla\psi - \sum_{j=1}^J \langle (\boldsymbol{\varphi} - \nabla\psi) \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} \widetilde{\nabla q_j^T}.$$

Since  $\tilde{\boldsymbol{\varphi}} \in \mathbf{V}_1$ , we have

$$\int_{\Omega} \operatorname{curl} \mathbf{u} \operatorname{curl} \boldsymbol{\varphi} = \int_{\Omega} \operatorname{curl} \mathbf{u} \operatorname{curl} \tilde{\boldsymbol{\varphi}} = \int_{\Omega} \mathbf{f} \cdot \tilde{\boldsymbol{\varphi}} = \int_{\Omega} \mathbf{f} \cdot \boldsymbol{\varphi},$$

where we observe that

$$\int_{\Omega} \mathbf{f} \cdot \nabla\psi = \int_{\Omega} \mathbf{f} \cdot \widetilde{\nabla q_j^T} = 0,$$

since  $\operatorname{div} \mathbf{f} = 0$  in  $\Omega$  and  $\mathbf{f} \cdot \mathbf{n} = 0$  on  $\Gamma$ . Now, let us show that the Problem (2.3.5) is equivalent to Problem (2.3.4). Let  $\mathbf{u} \in \mathbf{H}^1(\Omega)$  be a solution of Problem (2.3.4), then  $\mathbf{u} \in \mathbf{V}_1$ . We multiply the first equation in (2.3.4) by  $\boldsymbol{\varphi} \in \mathbf{X}_T(\Omega)$  and using the Green formula (2.2.22) we deduce (2.3.5). Conversely, let  $\mathbf{u}$  in  $\mathbf{V}_1$  solution of Problem (2.3.5). Taking  $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)$  we get,

$$-\Delta\mathbf{u} = \mathbf{f} \quad \text{in } \Omega.$$

As  $\mathbf{u} \in \mathbf{V}_1$ , we only have to prove the boundary condition  $\operatorname{curl} \mathbf{u} = 0$  on  $\Gamma$ . Using again the Green formula (2.2.22), we get that

$$\langle \operatorname{curl} \mathbf{u}, \boldsymbol{\varphi} \cdot \mathbf{t} \rangle_{H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)} = 0. \quad (2.3.7)$$

Let  $\mu \in H^{1/2}(\Gamma)$ . We know that there exists  $\boldsymbol{\varphi} \in \mathbf{H}^1(\Omega)$  with  $\operatorname{div} \boldsymbol{\varphi} = 0$  in  $\Omega$ , such that  $\boldsymbol{\varphi} = \mu \mathbf{t}$  on  $\Gamma$ . We deduce finally that  $\operatorname{curl} \mathbf{u} = 0$  on  $\Gamma$ .



**iii) Regularity.** Clearly, if  $\mathbf{f} \in \mathbf{L}^2(\Omega)$ , we have  $\Delta \mathbf{u} \in \mathbf{L}^2(\Omega)$ . So the solution  $\mathbf{u}$  given in Point i) belongs to  $\mathbf{V}_2$ . By Proposition 2.2.5, we get  $\mathbf{u} \in \mathbf{H}^2(\Omega)$ .  $\square$

Now, let us study the regularity of solutions.

**Lemma 2.3.3.** *Let  $\Omega$  be a bounded open set of class  $C^{m+1,1}$  with  $m \geq 0$ . If  $\mathbf{v} \in \mathbf{L}^p(\Omega)$ ,  $\text{curl } \mathbf{v} \in W^{m,p}(\Omega)$  and  $\text{div } \mathbf{v} \in W^{m,p}(\Omega)$  with  $\mathbf{v} \cdot \mathbf{n} = 0$  on  $\Gamma$ , then  $\mathbf{v} \in \mathbf{W}^{m+1,p}(\Omega)$ .*

*Proof.* **i) Step 1.** Suppose  $\text{div } \mathbf{v} = 0$ . Thanks to the consequence of Corollary 3.1 and Remark 3.1 in [18], there exist  $\Phi$  in  $W^{1,p}(\Omega)$  such that  $\mathbf{v} = \mathbf{curl } \Phi$ . As  $\mathbf{v} \cdot \mathbf{n} = \mathbf{curl } \Phi \cdot \mathbf{n} = \nabla \Phi \cdot \mathbf{t} = 0$ , then  $\Phi = C_i$  on  $\Gamma_i$  where  $C_i$  for  $1 \leq i \leq I$  are arbitrary constants. From (2.1.3) we get:

$$\begin{cases} -\Delta \Phi = \text{curl } \mathbf{v} & \text{in } \Omega, \\ \Phi = C_i & \text{on } \Gamma. \end{cases} \quad (2.3.8)$$

As  $\Omega$  is of class  $C^{m+1,1}$  the solution  $\Phi$  of Problem (2.3.8) belongs to  $W^{m+2,p}(\Omega)$  and then  $\mathbf{v} \in \mathbf{W}^{m+1,p}(\Omega)$ .

**ii) Step 2.** Assume now  $\text{div } \mathbf{v} \in W^{m,p}(\Omega)$ . Let  $\Psi \in W^{m+2,p}(\Omega)$  be a solution of the following problem,

$$\begin{cases} \Delta \Psi = \text{div } \mathbf{v} & \text{in } \Omega, \\ \frac{\partial \Psi}{\partial \mathbf{n}} = 0 & \text{on } \Gamma. \end{cases} \quad (2.3.9)$$

The vector field  $\mathbf{w} = \mathbf{v} - \nabla \Psi$  satisfies the assumptions of the first step and then belongs to  $\mathbf{W}^{m+1,p}(\Omega)$ . We conclude that  $\mathbf{v} \in \mathbf{W}^{m+1,p}(\Omega)$ .  $\square$

**Theorem 2.3.4.** *Let  $p \geq 2$  and  $\mathbf{f} \in \mathbf{L}^{s(p)}(\Omega)$ , with*

$$s(p) = \begin{cases} 1 + \varepsilon & \text{if } p = 2 \\ \frac{2p}{2+p} & \text{if } p > 2, \end{cases} \quad (2.3.10)$$

*satisfying the condition (2.3.3). The solution  $\mathbf{u}$  of the Problem (2.3.4) given in Theorem 2.3.1 is in  $\mathbf{W}^{1,p}(\Omega)$  and it satisfies the following estimate:*

$$\|\mathbf{u}\|_{\mathbf{W}^{1,p}(\Omega)} \leq C(\Omega) \|\mathbf{f}\|_{\mathbf{L}^{s(p)}(\Omega)}.$$

*Proof.* We already know that  $\mathbf{u} \in \mathbf{H}^1(\Omega)$ . Let  $z = \text{curl } \mathbf{u} \in L^2(\Omega)$ . We will show that  $z$  belongs to  $L^p(\Omega)$ . As  $\mathbf{f} \in \mathbf{L}^{s(p)}(\Omega)$  and  $z$  satisfies  $-\Delta z = \text{curl } \mathbf{f} \in W^{-1,s(p)}(\Omega)$  with  $z = 0$  on  $\Gamma$ . The function  $z$  belongs infact to  $W_0^{1,s(p)}(\Omega)$ . Indeed, let  $y \in W_0^{1,s(p)}(\Omega)$  be such that  $\Delta y = \Delta z$  in  $\Omega$ . The function  $w = y - z$  is in  $L^2(\Omega)$  and it satisfies

$$\Delta w = 0 \quad \text{in } \Omega \quad \text{and } w = 0 \quad \text{on } \Gamma.$$

Using Theorem 7 in [5] (which is naturally valid for two-dimensional case) we get by using a uniqueness argument that  $w = 0$  so that  $z = y$  and  $z \in W_0^{1,s(p)}(\Omega)$ . As  $\mathbf{u} \in \mathbf{H}^1(\Omega) \hookrightarrow \mathbf{L}^q(\Omega)$  (for any finite number  $q$ ) and  $\text{curl } \mathbf{u} \in W_0^{1,s(p)}(\Omega) \hookrightarrow L^p(\Omega)$ , thanks to Lemma 2.3.3 (for  $m = 0$ ) we obtain that  $\mathbf{u}$  belongs to  $\mathbf{W}^{1,p}(\Omega)$ .  $\square$

**Theorem 2.3.5.** *Assume that  $\Omega$  is of class  $\mathcal{C}^{2,1}$  and  $\mathbf{f} \in \mathbf{L}^p(\Omega)$ , with  $p \geq 2$ , and satisfies the condition (2.3.3). The solution  $\mathbf{u}$  to Problem (2.3.4) belongs to  $\mathbf{W}^{2,p}(\Omega)$  and satisfies*

$$\|\mathbf{u}\|_{\mathbf{W}^{2,p}(\Omega)} \leq C(\Omega) \|\mathbf{f}\|_{\mathbf{L}^p(\Omega)}.$$

*Proof.* Let  $\mathbf{u} \in W^{1,p}(\Omega)$  be the solution given by Theorem 2.3.4 and setting  $z = \text{curl } \mathbf{u}$ . We know that  $z$  satisfies

$$\begin{aligned} -\Delta z &= \text{curl } \mathbf{f} && \text{in } \Omega \\ z &= 0 && \text{on } \Gamma \end{aligned}$$

and then  $z \in W^{1,p}(\Omega)$ . Now  $\mathbf{u}$  satisfies the assumptions of Lemma 2.3.3 for  $m = 1$  and it belongs to  $\mathbf{W}^{2,p}(\Omega)$ .  $\square$

## 2.4 Stokes problem

Due to the two previous propositions, we have got a solution for Problem (2.3.4) for  $\mathbf{f} \in \mathbf{L}^s(\Omega)$  satisfying the condition (2.3.3). What happens now if  $\mathbf{f}$  does not satisfy the conditions  $\text{div } \mathbf{f} = 0$  in  $\Omega$  and  $\mathbf{f} \cdot \mathbf{n} = 0$  on  $\Gamma$  but only the orthogonality

condition

$$\text{for any } \boldsymbol{\varphi} \in \mathbf{K}_T(\Omega) \quad \int_{\Omega} \mathbf{f} \cdot \boldsymbol{\varphi} = 0? \quad (2.4.1)$$

In this case, Problem (2.3.5) can no longer be equivalent to Problem (2.3.2). Let us give the following theorem.

**Theorem 2.4.1.** *Let  $\mathbf{f} \in \mathbf{L}^s(\Omega)$ , with  $s > 1$ , be such that*

$$\forall \boldsymbol{\varphi} \in \mathbf{K}_T(\Omega), \quad \int_{\Omega} \mathbf{f} \cdot \boldsymbol{\varphi} = 0. \quad (2.4.2)$$

*Then there exist  $\mathbf{u} \in \mathbf{H}^1(\Omega)$  and  $\pi \in W^{1,s}(\Omega)$  solution of the following Stokes Problem:*

$$\begin{cases} -\Delta \mathbf{u} + \nabla \pi = \mathbf{f}, & \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} \cdot \mathbf{n} = 0, & \operatorname{curl} \mathbf{u} = 0 & \text{on } \Gamma, \\ \int_{\Sigma_j} \mathbf{u} \cdot \mathbf{n} = 0 & & \text{for } 1 \leq j \leq J. \end{cases} \quad (2.4.3)$$

*Proof.* The proof is similar to that of Theorem 2.3.1. We always have the existence and the uniqueness of a solution  $\mathbf{u} \in \mathbf{H}^1(\Omega)$  of the following problem:

$$\begin{cases} \text{Find } \mathbf{u} \in \mathbf{V}_1 \text{ such that} \\ \forall \boldsymbol{\varphi} \in \mathbf{V}_1, \quad \int_{\Omega} \operatorname{curl} \mathbf{u} \operatorname{curl} \boldsymbol{\varphi} = \int_{\Omega} \mathbf{f} \cdot \boldsymbol{\varphi}, \end{cases} \quad (2.4.4)$$

This problem is now equivalent to the following:

$$\begin{cases} \text{Find } \mathbf{u} \in \mathbf{V}_1 \text{ such that} \\ \forall \boldsymbol{\varphi} \in \mathbf{X}_T(\Omega) \text{ with } \operatorname{div} \boldsymbol{\varphi} = 0, \quad \int_{\Omega} \operatorname{curl} \mathbf{u} \operatorname{curl} \boldsymbol{\varphi} = \int_{\Omega} \mathbf{f} \cdot \boldsymbol{\varphi}. \end{cases} \quad (2.4.5)$$

Thanks to De Rham's Theorem, there exists  $\pi$  such that

$$-\Delta \mathbf{u} + \nabla \pi = \mathbf{f} \quad \text{in } \Omega.$$

Clearly,  $\pi \in L^2(\Omega)$  and the vector field  $\mathbf{F} =: \mathbf{f} - \nabla\pi$  satisfies the condition (2.3.3). As in the proof of Theorem 2.3.1, we show the boundary condition  $\text{curl } \mathbf{u} = 0$  on  $\Gamma$ .

Finally the function  $\pi$  belongs to  $W^{1,s}(\Omega)$  since it satisfies the following Neumann Problem:

$$\begin{cases} \Delta \pi = \text{div } \mathbf{f} & \text{in } \Omega \\ (\nabla \pi - \mathbf{f}) \cdot \mathbf{n} = 0 & \text{on } \Gamma. \end{cases} \quad (2.4.6)$$

□

Observe that the pressure  $\pi$  is constant when  $\text{div } \mathbf{f} = 0$  in  $\Omega$  and  $\mathbf{f} \cdot \mathbf{n} = 0$  on  $\Gamma$ . We prove easily, as above, the following two theorems.

**Theorem 2.4.2.** *Let  $p \geq 2$ ,  $\mathbf{f} \in \mathbf{L}^{s(p)}(\Omega)$ , with  $s(p)$  defined in (2.3.10) and satisfying the condition (2.4.2). Then the solution  $(\mathbf{u}, \pi)$  of the Stokes Problem (2.4.3) given by Theorem 2.4.1 is in  $\mathbf{W}^{1,p}(\Omega) \times W^{1,s(p)}(\Omega)$  and it satisfies the following estimate:*

$$\|\mathbf{u}\|_{\mathbf{W}^{1,p}(\Omega)} + \|\pi\|_{W^{1,s(p)}(\Omega)/\mathbb{R}} \leq C(\Omega) \|\mathbf{f}\|_{\mathbf{L}^{s(p)}(\Omega)}.$$

*Proof.* For simplicity, we consider  $\Omega$  is of class  $\mathcal{C}^{2,1}$ . Let  $\mathbf{f} \in \mathbf{L}^{s(p)}$ , we know from Theorem 2.4.1 that there exist a unique  $\mathbf{u} \in \mathbf{H}^1(\Omega)$  and  $\pi \in \mathbf{W}^{1,1}(\Omega)$  in addition  $\pi \in \mathbf{W}^{1,s(p)}(\Omega)$  because  $\pi$  is a solution of the Neumann Problem (2.4.6). In consequence,

$$-\Delta \mathbf{u} = \mathbf{f} - \nabla \pi \in \mathbf{L}^{s(p)}(\Omega).$$

Let  $w = \text{curl } \mathbf{u}$ , then we have  $\text{curl } w \in \mathbf{L}^{s(p)}(\Omega)$  this implies that  $w \in \mathbf{W}^{1,s(p)}(\Omega)$  which means that  $u \in \mathbf{W}^{2,s(p)}(\Omega)$  which is embedded in  $\mathbf{W}^{1,p}(\Omega)$ .

**Theorem 2.4.3.** *We assume that  $\Omega$  is of class  $\mathcal{C}^{2,1}$  and  $\mathbf{f} \in \mathbf{L}^p(\Omega)$ , with  $p > 1$ , satisfying the condition (2.4.2). The solution  $(\mathbf{u}, \pi)$  to Problem (2.3.4) belongs to  $\mathbf{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$  and it satisfies*

$$\|\mathbf{u}\|_{\mathbf{W}^{2,p}(\Omega)} + \|\pi\|_{W^{1,p}(\Omega)/\mathbb{R}} \leq C(\Omega) \|\mathbf{f}\|_{\mathbf{L}^p(\Omega)}.$$

# Chapter 3

## MHD flow with Navier-type boundary conditions

### Contents

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3.1 Existence of solutions for the MHD problem . . . . .	64
3.2 Regularity of the solution . . . . .	74

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Magnetohydrodynamics (MHD) is the discipline studying the behavior of conductive fluids of electricity when their movement is coupled to the electromagnetic field. This type of study has attracted large attention due to its important applications in many fields such as in geophysics, astrophysics, and engineering as explained in the introduction. We interested ourselves in the following problem:

$$\left\{ \begin{array}{ll} -\nu\Delta\mathbf{u} + (\mathbf{u} \cdot \nabla)(\mathbf{u} - \alpha\Delta\mathbf{u}) + \nabla\pi - (\mathbf{B} \cdot \nabla)\mathbf{B} + \frac{1}{2}\nabla(|\mathbf{B}|^2) = \mathbf{f} & \text{in } \Omega, \\ -\Delta\mathbf{B} - (\mathbf{B} \cdot \nabla)\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{B} + \nabla\theta = 0 & \text{in } \Omega, \\ \operatorname{div} \mathbf{u} = 0, \quad \operatorname{div} \mathbf{B} = 0 & \text{in } \Omega, \\ \mathbf{u} \cdot \mathbf{n} = 0, \quad \mathbf{B} \cdot \mathbf{n} = 0 & \text{on } \Gamma, \\ \operatorname{curl} \mathbf{u} = 0, \quad \operatorname{curl} \mathbf{B} = 0 & \text{on } \Gamma, \\ \langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} = 0, \quad \langle \mathbf{B} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} = 0, & \text{for } 1 \leq j \leq J \end{array} \right. \quad (\mathcal{P})$$

where the unknowns  $\mathbf{u}, \pi$  and  $\mathbf{B}$  represent the velocity field, the pressure, and the magnetic field respectively,  $\theta$  is an unknown function related to the motion of heavy ions. The external force acting on the fluid is  $\mathbf{f}$  and  $\mathbf{n}$  denotes the unit exterior normal to the boundary  $\Gamma$  and Navier-type boundary conditions for both, the velocity  $\mathbf{u}$  and the magnetic field  $\mathbf{B}$ , are taken into account. Without any restriction of the size of data  $\mathbf{f}$  supposed in  $\mathbf{L}^p(\Omega)$  for  $p \geq 2$ , we prove that  $\mathbf{u}$  and  $\mathbf{B}$  belong to  $\mathbf{H}^2(\Omega)$  and  $\mathbf{W}^{2,r}(\Omega)$  (for any finite  $r$ ) respectively.

Let us start with an overview of some results in the stationary case. In two dimensions, Zhou in [39] studied the existence and uniqueness of a compressible isentropic flow in  $[0, 1] \times [0, 1]$  with mixed boundary conditions for the velocity. Setting

$$\begin{aligned} \Gamma_{in} &= \{x_1 = 0, 0 \leq x_2 \leq 1\}, & \Gamma_{out} &= \{x_1 = 1, 0 \leq x_2 \leq 1\} \\ \text{and } \Gamma_0 &= \{0 \leq x_1 \leq 1, x_2 = 0\} \cup \{0 \leq x_1 \leq 1, x_2 = 1\}, \end{aligned}$$

the author considered

$$\begin{cases} \mathbf{u} = \mathbf{u}^0 & \text{on } \Gamma_{in} \cup \Gamma_{out}, \\ \mathbf{u} \cdot \mathbf{n} = \mathbf{u} = 0 & \text{on } \Gamma_0, \\ \mathbf{B} = \mathbf{B}^0 & \text{on } \partial\Omega, \\ \pi = \pi^0 & \text{on } \Gamma_{in}. \end{cases}$$

To get the existence result, he defined the following quantities:

$$M_0 = \|\mathbf{u}^0 - \mathbf{e}_1\|_{\mathbf{W}^{2-1/p,p}(\Gamma_{in} \cup \Gamma_{out})} + \|\pi^0 - 1\|_{W^{1,p}(\Gamma_{in})} + \|\mathbf{B}^0 - \mathbf{e}_1\|_{W^{2-1/p,p}(\partial\Omega)} + \|\mathbf{f}\|_{\mathbf{L}^p(\Omega)}$$

with  $\mathbf{e}_1 = (1, 0)$  and  $2 < p < \infty$ . Zhou assumed that  $M_0$  is small enough and that the derivation with respect to  $x_2$  of  $u_1^0$  on the corners of the domain are null, then there exist a unique solution  $(\mathbf{u}, \mathbf{B}, \pi) \in \mathbf{W}^{2,p}(\Omega) \times \mathbf{W}^{1,p}(\Omega) \times W^{2,p}(\Omega)$  satisfying

the following estimate:

$$\|\mathbf{u} - \mathbf{e}_1\|_{\mathbf{W}^{2,p}(\Omega)} + \|\pi - 1\|_{W^{1,p}(\Omega)} + \|\mathbf{B} - \mathbf{e}_1\|_{\mathbf{W}^{2,p}(\Omega)} \leq C_p(\Omega) M_0.$$

The proof is based on the energy and the  $L^p$ -estimates of the linearized system of the MHD problem.

For the three dimensional case, in [3], it has been proved the existence of weak solutions  $(\mathbf{u}, \mathbf{B}, \pi) \in \mathbf{H}^1(\Omega) \times \mathbf{W}^{2,6/5}(\Omega) \times L^2(\Omega)$  in a domain of class  $\mathcal{C}^{1,1}$  and possibly multiply connected considering the Navier-Stokes equation. Amrouche and Boukassa consider in [36] the no-slip boundary condition for the velocity:  $\mathbf{u} = 0$ , and Navier-type boundary conditions for the magnetic field:  $\mathbf{B} \cdot \mathbf{n} = 0$  and  $\mathbf{curl} \mathbf{B} \times \mathbf{n} = \mathbf{0}$  on  $\Gamma$ , which is related to a perfectly conducting wall. The existence was done using the Leray-Schauder fixed-point theorem and some regularity results in  $\mathbf{W}^{1,p}(\Omega)$  and in  $\mathbf{W}^{2,p}(\Omega)$  were shown.

There has been other work related to the Hall-MHD problem where an extra term appears in the Maxwell equation. The Hall effect occurs during magnetic reconnection events, and this is because of a large magnetic shear, which is seen in space plasma, for example. In the usual MHD system, the Hall-term  $\mathbf{curl}(\mathbf{curl} \mathbf{B} \times \mathbf{B})$  is small regard to the other terms and can be neglected.

In [38], Zeng and Zhang have found the existence of weak solutions to the Hall-MHD system with Hölder continuous magnetic field. Under the assumptions of small external forces, they considered homogeneous Dirichlet boundary conditions for the velocity where new difficulties were related to the hall term. In a simply connected domain  $\Omega$  of class  $\mathcal{C}^{1,1}$ , the authors proved the existence of  $(\mathbf{u}, \mathbf{B}, \pi)$  in  $\mathbf{H}_0^1(\Omega) \times \mathbf{W}^{1,q}(\Omega) \times L^2(\Omega)$  for some  $q > 3$ . In [12], the authors studied the partial regularity of suitable weak solutions. They proved that the set of possible singularities of the solutions for the Hall-MHD system is compact.

In porous media, MHD flow was studied by Amirat *et al.* in [36]. The authors

considered in a simply connected domain  $\Omega$  the following problem:

$$\begin{cases} a\mathbf{u} + b|\mathbf{u}|\mathbf{u} - \gamma\Delta\mathbf{u} + \nabla p = \frac{1}{\mu}\mathbf{curl}\mathbf{B} \times \mathbf{B} & \text{in } \Omega \\ \frac{1}{\sigma\mu}\mathbf{curl}^2\mathbf{B} = \mathbf{curl}(\mathbf{u} \times \mathbf{B}) + \frac{1}{\sigma}\mathbf{curl}\mathbf{J}_0 & \text{in } \Omega \\ \operatorname{div}\mathbf{u} = h, \operatorname{div}\mathbf{B} = 0 & \text{in } \Omega \\ \mathbf{u} = \mathbf{g}, \mathbf{curl}\mathbf{B} \times \mathbf{n} = 0, \mathbf{B} \cdot \mathbf{n} = 0 & \text{on } \partial\Omega, \end{cases}$$

where  $a$  and  $b$  are positive coefficients and  $\mathbf{J}$  is the electric current density. Under the assumptions of compatibility condition, they proved the existence of weak solutions and studied the regularity when  $\mathbf{g} \cdot \mathbf{n} = 0$ . Assuming that  $\partial\Omega \in \mathcal{C}^2$  and  $\mathbf{curl}\mathbf{J}_0 \in \mathbf{L}^q(\Omega)$ ,  $(h, \mathbf{g}) \in W^{1,q}(\Omega) \times \mathbf{W}^{2-\frac{1}{q},q}(\Omega)$  (with  $q \geq 3/2$ ) then any weak solution  $(\mathbf{u}_\gamma, \mathbf{B}_\gamma)$  of the problem above belongs to  $\mathbf{W}^{2,q}(\Omega) \times \mathbf{W}^{2,q}(\Omega)$ .

On the other hand, the time-dependent three-dimension MHD problem has been considerably studied. Sermange and Temam, in [30], considered the ideal MHD problem and proved the existence of a global weak solution. They found that for  $\mathbf{f} \in \mathbf{L}^2(0, T, \mathbf{V}'_1)$  and  $(\mathbf{u}_0, \mathbf{B}_0)$  in  $\mathbf{H}_1 \times \mathbf{H}_1$ , there exists a weak solution satisfying  $(\mathbf{u}, \mathbf{B}) \in \mathbf{L}^2(0, T, \mathbf{V}_1 \times \mathbf{V}_2) \cap \mathbf{L}^\infty(0, T, \mathbf{H}_1 \times \mathbf{H}_1)$  where

$$\begin{aligned} \mathbf{V}_1 &= \{\mathbf{v} \in \mathbf{H}_0^1(\Omega); \operatorname{div}\mathbf{v} = 0\}, \\ \mathbf{V}_2 &= \{\mathbf{v} \in \mathbf{H}^1(\Omega); \operatorname{div}\mathbf{v} = 0 \text{ in } \Omega \text{ and } \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma\}, \\ \text{and } \mathbf{H}_1 &= \{\mathbf{v} \in \mathbf{L}^2(\Omega); \operatorname{div}\mathbf{v} = 0 \text{ in } \Omega \text{ and } \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma\}, \end{aligned}$$

with  $\mathbf{V}'_1$  the dual space of  $\mathbf{V}_1$ . Guo and Wang studied, in [21], the vanishing viscosity limit for an MHD system considering the Navier-slip boundary conditions for  $\mathbf{u}$  and for  $\mathbf{B}$ . They found the weak and strong solutions using the Galerkin approximated method in a simply connected smooth domain added by some energy estimates.



### 3.1 Existence of solutions for the MHD problem

We construct an approximate solution, defined by a sequence of functions that converges to a solution of the initial value problem  $(\mathcal{P})$ , using a sequence of  $\mathbf{V}_2$  formed by a special basis of eigenvectors of  $\mathbf{H}$ . We prove the existence of the approximate solution using the fixed-point theorem of Brouwer applied on a bilinear continuous mapping that will be defined later.

**Proposition 3.1.1.** *There exist a sequence  $(\mathbf{w}_j)_{j \geq 1}$  of  $\mathbf{V}_2$  and a sequence  $(\lambda_j)_{j \geq 1}$  of real numbers with  $\lambda_j > 0$  for any  $j \geq 1$ ,  $\lambda_j \rightarrow +\infty$  and such that*

$$\begin{cases} -\Delta \mathbf{w}_j = \lambda_j \mathbf{w}_j & \text{in } \Omega, \\ \operatorname{div} \mathbf{w}_j = 0 & \text{in } \Omega, \\ \mathbf{w}_j \cdot \mathbf{n} = 0, \operatorname{curl} \mathbf{w}_j = 0 & \text{on } \Gamma. \\ \langle \mathbf{w}_j \cdot \mathbf{n}, 1 \rangle_{\Sigma_k} = 0, & \text{for } 1 \leq k \leq J. \end{cases} \quad (3.1.1)$$

*Proof.* Let  $\Lambda$  be a linear continuous mapping defined by

$$\begin{aligned} \Lambda : \mathbf{Z} &\longrightarrow \mathbf{V}_1 \longrightarrow \mathbf{Z} \\ \mathbf{f} &\longmapsto \mathbf{u} \longmapsto \mathbf{u}, \end{aligned}$$

where  $\mathbf{Z}$  is a Hilbert separable space defined as

$$\begin{aligned} \mathbf{Z} &= \{ \mathbf{v} \in \mathbf{L}^2(\Omega), \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega, \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma \\ &\text{and } \langle \mathbf{v} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} = 0 \text{ for any } 1 \leq j \leq J \}, \end{aligned}$$

and  $\mathbf{u}$  is the solution of Problem (2.3.4). As the embedding of  $\mathbf{H}^1(\Omega)$  in  $\mathbf{L}^2(\Omega)$  is compact then the operator  $\Lambda$  is compact and self-adjoint, then we conclude that there exists a Hilbertian basis of  $\mathbf{Z}$  formed by the eigenvectors  $(\mathbf{w}_j)_{j \geq 1}$  of  $\Lambda$  and

$$\Lambda \mathbf{w}_j = \mu_j \mathbf{w}_j, \quad \text{with } \mu_j \in \mathbb{R} \quad \text{and } \mu_j \longrightarrow 0. \quad (3.1.2)$$

Using the variational formulation (2.3.2) for any  $\mathbf{v} \in \mathbf{V}_1$

$$\mu_j \int_{\Omega} \operatorname{curl} \mathbf{w}_j \operatorname{curl} \mathbf{v} = \int_{\Omega} \mathbf{w}_j \cdot \mathbf{v}. \quad (3.1.3)$$

Setting  $\lambda_j = \frac{1}{\mu_j}$ , we have  $\lambda_j \rightarrow +\infty$  since  $\mu_j > 0$ . Observe that, by (3.1.3) we get for any  $\mathbf{v} \in \mathbf{V}_1$

$$- \int_{\Omega} \Delta \mathbf{w}_j \cdot \mathbf{v} = \lambda_j \int_{\Omega} \mathbf{w}_j \cdot \mathbf{v}.$$

By De Rham's Theorem, there exist a unique  $\pi \in L_0^2(\Omega)$  such that

$$-\Delta \mathbf{w}_j + \nabla \pi_j = \lambda_j \mathbf{w}_j \quad \text{in } \Omega. \quad (3.1.4)$$

Multiplying (3.1.4) by  $\mathbf{v} \in \mathbf{V}_1$ , we obtain after integrating by parts

$$\int_{\Omega} \mathbf{v} \cdot \mathbf{t} \operatorname{curl} \mathbf{w}_j = 0$$

for the same reason as in (2.3.7) we get  $\operatorname{curl} \mathbf{w}_j = 0$  on  $\Gamma$ .

From (3.1.4) we deduce that

$$\Delta \pi_j = 0 \quad \text{in } \Omega \quad \text{and} \quad \frac{\partial \pi_j}{\partial \mathbf{n}} = 0 \quad \text{on } \Gamma \quad (3.1.5)$$

which implies that  $\pi_j$  is constant and  $\nabla \pi_j = 0$ . Now, as  $\mathbf{w}_j$  are in  $\mathbf{V}_1$  with  $\Delta \mathbf{w}_j \in \mathbf{L}^2(\Omega)$  then  $\mathbf{w}_j$  are in  $\mathbf{V}_2(\Omega)$ .  $\square$

Let us call back the following Brouwer fixed-point theorem (see J.L. Lions [24]).

**Theorem 3.1.2.** *Let  $P : X \rightarrow X$  be a continuous mapping, where  $X$  is a finite-dimensional space, and assume that there exists a real number  $\rho > 0$  such that,*

$$(P(x)|x)_X > 0 \quad \forall x \in X \quad \text{with} \quad \|x\|_X = \rho$$

*Then there exists  $x_0 \in X$  which verifies  $P(x_0) = 0$ , where  $(\cdot|\cdot)_X$  is the scalar product of  $X$ .*

We are able now to announce the theorem of the existence of solutions for the

MHD problem  $(\mathcal{P})$ .

**Theorem 3.1.3.** *Let  $\Omega$  be of class  $\mathcal{C}^{2,1}$ . Then for any  $\mathbf{f} \in \mathbf{L}^2(\Omega)$ , Problem  $(\mathcal{P})$  has at least one solution  $(\mathbf{u}, \mathbf{B}, \pi, \theta)$  in  $\mathbf{H}^2(\Omega) \times \mathbf{H}^2(\Omega) \times L^2(\Omega) \times H^1(\Omega)$  which satisfies the following estimate:*

$$\|\mathbf{u}\|_{\mathbf{H}^2(\Omega)} + \|\mathbf{B}\|_{\mathbf{H}^2(\Omega)} \leq C(\Omega) \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}^\beta,$$

where  $C(\Omega)$  depends on  $\Omega$  and  $\beta > 2$

*Proof.* The proof is divided into three steps. We construct an approximate problem and we show the existence of the solution; then we get some estimations in  $\mathbf{H}^1(\Omega)$  and in  $\mathbf{H}^2(\Omega)$ . In the end, we pass to the limit using compact embedding and density results.

**Step 1. Existence of the approximate solution:**

$$\text{For } m = 1, 2, \dots, \quad \text{find } \mathbf{u}_m = \sum_{j=1}^m \mathbf{g}_{jm} \mathbf{w}_j \in \mathbf{V}_m \text{ and } \mathbf{B}_m = \sum_{j=1}^m h_{jm} \mathbf{w}_j \in \mathbf{V}_m$$

such that for any  $1 \leq j \leq m$ ,

$$\nu \int_{\Omega} \text{curl } \mathbf{u}_m \text{ curl } \mathbf{w}_j - b(\mathbf{u}_m, \mathbf{w}_j, \mathbf{u}_m - \alpha \Delta \mathbf{u}_m) - \int_{\Omega} \mathbf{f} \cdot \mathbf{w}_j - b(\mathbf{B}_m, \mathbf{B}_m, \mathbf{w}_j) = 0, \quad (3.1.6)$$

$$\int_{\Omega} \text{curl } \mathbf{B}_m \text{ curl } \mathbf{w}_j - b(\mathbf{B}_m, \mathbf{u}_m, \mathbf{w}_j) + b(\mathbf{u}_m, \mathbf{B}_m, \mathbf{w}_j) = 0, \quad (3.1.7)$$

where  $\mathbf{g}_{jm}$  and  $h_{jm}$  are real numbers,  $(\mathbf{w}_j)_{j \geq 1}$  is the sequence of eigenvectors given by Proposition 3.1.1 and  $\mathbf{V}_m = \langle \mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_m \rangle$  is the finite space spanned by  $(\mathbf{w}_j)_{1 \leq j \leq m}$  and the scalar product on  $\mathbf{V}_m$  is induced by that of  $\mathbf{V}_2$ , where

$$((\mathbf{u}, \mathbf{v}))_{\mathbf{V}_2} = \int_{\Omega} \text{curl } \mathbf{u} \text{ curl } \mathbf{v} + \int_{\Omega} \Delta \mathbf{u} \cdot \Delta \mathbf{v}, \quad \text{for any } \mathbf{u}, \mathbf{v} \in \mathbf{V}_2.$$

The bilinear mapping  $\mathbf{P}_m$  is defined by: for any  $\mathbf{u}_m, \mathbf{B}_m \in \mathbf{V}_m$  and  $\mathbf{v}_m, \mathbf{M}_m \in \mathbf{V}_m$ :

$$(\mathbf{P}_m(\mathbf{u}, \mathbf{B}) | (\mathbf{v}, \mathbf{M}))_{\mathbf{V}_2} = (\mathbf{P}_{1m}(\mathbf{u}, \mathbf{B}) | (\mathbf{v}, \mathbf{M})) + (\mathbf{P}_{2m}(\mathbf{u}, \mathbf{B}) | (\mathbf{v}, \mathbf{M})).$$

where

$$\begin{aligned} (\mathbf{P}_{1m}(\mathbf{u}, \mathbf{B}) | (\mathbf{v}, \mathbf{M})) &= \nu \int_{\Omega} \operatorname{curl} \mathbf{u} \operatorname{curl} \mathbf{v} - b(\mathbf{u}, \mathbf{v}, \mathbf{u} - \alpha \Delta \mathbf{u}) - \int_{\Omega} \mathbf{f} \cdot \mathbf{u} \\ &\quad - b(\mathbf{B}, \mathbf{B}, \mathbf{u}), \end{aligned}$$

and

$$(\mathbf{P}_{2m}(\mathbf{u}, \mathbf{B}) | (\mathbf{v}, \mathbf{M})) = \nu \int_{\Omega} \operatorname{curl} \mathbf{B} \operatorname{curl} \mathbf{M} - b(\mathbf{B}, \mathbf{u}, \mathbf{m}) + b(\mathbf{u} \mathbf{B}, \mathbf{M}).$$

Taking the inner product of  $\mathbf{P}_m(\mathbf{u}, \mathbf{B})$  by  $(\mathbf{u}, \mathbf{B})$ , we get

$$\begin{aligned} (\mathbf{P}_m(\mathbf{u}, \mathbf{B}) | (\mathbf{u}, \mathbf{B}))_{\mathbf{V}_2} &= \nu \int_{\Omega} |\operatorname{curl} \mathbf{u}|^2 + \int_{\Omega} |\operatorname{curl} \mathbf{B}|^2 - b(\mathbf{u}, \mathbf{u}, \mathbf{u} - \alpha \Delta \mathbf{u}) - \int_{\Omega} \mathbf{f} \cdot \mathbf{u} \\ &\quad - b(\mathbf{B}, \mathbf{B}, \mathbf{u}) - b(\mathbf{B}, \mathbf{u}, \mathbf{B}) + b(\mathbf{u}, \mathbf{B}, \mathbf{B}). \end{aligned}$$

Using Lemma 2.2.10 and Theorem 2.2.3 we get,

$$\begin{aligned} (\mathbf{P}_m(\mathbf{u}, \mathbf{B}) | (\mathbf{u}, \mathbf{B}))_{\mathbf{V}_2} &= \nu \|\operatorname{curl} \mathbf{u}\|_{\mathbf{L}^2(\Omega)}^2 + \|\operatorname{curl} \mathbf{B}\|_{\mathbf{L}^2(\Omega)}^2 - \int_{\Omega} \mathbf{f} \cdot \mathbf{u} \\ &\geq \nu \|\operatorname{curl} \mathbf{u}\|_{\mathbf{L}^2(\Omega)}^2 + \|\operatorname{curl} \mathbf{B}\|_{\mathbf{L}^2(\Omega)}^2 - \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} \|\mathbf{u}\|_{\mathbf{L}^2(\Omega)} \\ &\geq \nu \|\operatorname{curl} \mathbf{u}\|_{\mathbf{L}^2(\Omega)}^2 + \|\operatorname{curl} \mathbf{B}\|_{\mathbf{L}^2(\Omega)}^2 - C(\|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} \|\operatorname{curl} \mathbf{u}\|_{\mathbf{L}^2(\Omega)}) \\ &\geq \nu \|\operatorname{curl} \mathbf{u}\|_{\mathbf{L}^2(\Omega)}^2 + \|\operatorname{curl} \mathbf{B}\|_{\mathbf{L}^2(\Omega)}^2 \\ &\quad - C\|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} (\|\operatorname{curl} \mathbf{u}\|_{\mathbf{L}^2(\Omega)} + \|\operatorname{curl} \mathbf{B}\|_{\mathbf{L}^2(\Omega)}) \\ &\geq (\|\operatorname{curl} \mathbf{u}\|_{\mathbf{L}^2(\Omega)} + \|\operatorname{curl} \mathbf{B}\|_{\mathbf{L}^2(\Omega)}) (\min(1, \nu) (\|\operatorname{curl} \mathbf{u}\|_{\mathbf{L}^2(\Omega)} \\ &\quad + \|\operatorname{curl} \mathbf{B}\|_{\mathbf{L}^2(\Omega)}) - c\|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}). \end{aligned}$$

Now, since  $\mathbf{V}_m$  is of finite dimension, there exists a constant  $C_m > 0$  such that

$$\forall \mathbf{v} \in \mathbf{V}_m, \|\mathbf{v}\|_{\mathbf{V}_2} \leq c_m \|\operatorname{curl} \mathbf{v}\|_{\mathbf{L}^2(\Omega)}.$$

Then,

$$(\mathbf{P}_m(\mathbf{u}, \mathbf{B}) | (\mathbf{v}, \mathbf{M}))_{\mathbf{V}_2} \geq (\|\operatorname{curl} \mathbf{u}\|_{\mathbf{L}^2(\Omega)} + \|\operatorname{curl} \mathbf{B}\|_{\mathbf{L}^2(\Omega)}) \min(1, \nu) \left( \frac{1}{C_m} (\|\mathbf{u}\|_{\mathbf{V}_2} + \|\mathbf{B}_m\|_{\mathbf{V}_2} - C \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}) \right)$$

for

$$r = (\|\mathbf{u}\|_{\mathbf{V}_2}^2 + \|\mathbf{B}_m\|_{\mathbf{V}_2}^2)^{\frac{1}{2}} \geq \frac{C_m}{\min(1, \nu)} \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}.$$

Then for all  $r = \|\mathbf{u}\|_{\mathbf{V}_2} + \|\mathbf{B}\|_{\mathbf{V}_2} = \|(\mathbf{u}_m, \mathbf{B}_m)\|_{\mathbf{V}_1 \times \mathbf{V}_1}$  with  $r \geq \frac{C}{\min(1, \nu)} \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}$ . Now the hypothesis of Theorem 3.1.2 is satisfied and there exists  $(\bar{\mathbf{u}}_m, \bar{\mathbf{B}}_m)$  solution of the approximate problem.

**Step 2. A priori estimates.** In the first part, we prove some estimates in  $\mathbf{H}^1(\Omega)$  for  $\mathbf{u}_m$  and for  $\mathbf{B}_m$ , then we prove other estimates in  $\mathbf{H}^2(\Omega)$ .

**i) Estimates in  $\mathbf{H}^1(\Omega)$ .** We multiply (3.1.6) by  $g_{jm}$  and we sum over  $i, j$  and  $k$ ,

$$\begin{aligned} & \int_{\Omega} \operatorname{curl} \mathbf{u}_m \operatorname{curl}(g_{jm} \mathbf{w}_j) - b(\mathbf{u}_m, g_{jm} \mathbf{w}_j, \mathbf{u}_m - \alpha \Delta \mathbf{u}_m) \\ &= \int_{\Omega} \mathbf{f} \cdot g_{jm} \mathbf{w}_j + b(\mathbf{B}_m, \mathbf{B}_m, g_{jm} \mathbf{w}_j) \end{aligned}$$

which is equivalent to,

$$\nu \int_{\Omega} |\operatorname{curl} \mathbf{u}_m|^2 - b(\mathbf{u}_m, \mathbf{u}_m, \mathbf{u}_m - \alpha \Delta \mathbf{u}_m) = \int_{\Omega} \mathbf{f} \cdot \mathbf{u}_m + b(\mathbf{B}_m, \mathbf{B}_m, \mathbf{u}_m).$$

Since  $b(\mathbf{u}_m, \mathbf{u}_m, \mathbf{u}_m - \alpha \Delta \mathbf{u}_m) = 0$ , we get

$$\nu \int_{\Omega} |\operatorname{curl} \mathbf{u}_m|^2 = \int_{\Omega} \mathbf{f} \cdot \mathbf{u}_m + b(\mathbf{B}_m, \mathbf{B}_m, \mathbf{u}_m). \quad (3.1.8)$$

On the other side, we multiply (3.1.7) by  $h_{jm}$  and we sum over the indices,

$$\int_{\Omega} \operatorname{curl} \mathbf{B}_m \operatorname{curl}(h_{jm} \mathbf{w}_j) = b(\mathbf{B}_m, \mathbf{u}_m, h_{jm} \mathbf{w}_j) - b(\mathbf{u}_m, \mathbf{B}_m, h_{jm} \mathbf{w}_j)$$

which gives,

$$\begin{aligned} \int_{\Omega} |\operatorname{curl} \mathbf{B}_m|^2 &= b(\mathbf{B}_m, \mathbf{u}_m, \mathbf{B}_m) - b(\mathbf{u}_m, \mathbf{B}_m, \mathbf{B}_m) \\ &= b(\mathbf{B}_m, \mathbf{u}_m, \mathbf{B}_m). \end{aligned} \quad (3.1.9)$$

We add (3.1.8) and (3.1.9) we get,

$$\begin{aligned} \nu \int_{\Omega} |\operatorname{curl} \mathbf{u}_m|^2 + \int_{\Omega} |\operatorname{curl} \mathbf{B}_m|^2 &= \int_{\Omega} \mathbf{f} \cdot \mathbf{u}_m \\ &\leq \|\mathbf{f}\|_{L^2(\Omega)} \|\mathbf{u}_m\|_{L^2(\Omega)} \\ &\leq C(\Omega) \|\mathbf{f}\|_{L^2(\Omega)} \|\mathbf{u}_m\|_{H^1(\Omega)}, \\ &\leq C(\Omega) \|\mathbf{f}\|_{L^2(\Omega)} \|\operatorname{curl} \mathbf{u}_m\|_{L^2(\Omega)}, \end{aligned}$$

where  $C(\Omega)$  depends only on  $\Omega$ . We get,

$$\|\operatorname{curl} \mathbf{u}_m\|_{L^2(\Omega)} \leq \frac{C(\Omega)}{\nu} \|\mathbf{f}\|_{L^2(\Omega)}.$$

By Proposition 2.2.3,

$$\|\mathbf{u}_m\|_{H^1(\Omega)} \leq \frac{C(\Omega)}{\nu} \|\mathbf{f}\|_{L^2(\Omega)} \quad (3.1.10)$$

and,

$$\begin{aligned} \|\operatorname{curl} \mathbf{B}_m\|_{L^2(\Omega)}^2 &\leq C(\Omega) \|\mathbf{f}\|_{L^2(\Omega)} \|\operatorname{curl} \mathbf{u}_m\|_{L^2(\Omega)} \\ &\leq \frac{C(\Omega)}{\nu} \|\mathbf{f}\|_{L^2(\Omega)}^2. \end{aligned}$$

Finally,

$$\|\mathbf{B}_m\|_{\mathbf{H}^1(\Omega)} \leq \frac{C(\Omega)}{\sqrt{\nu}} \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}. \quad (3.1.11)$$

ii) **Estimates in  $\mathbf{H}^2(\Omega)$ .** We multiply (1.0.1) by  $\lambda_j g_{jm}$  and we sum over  $j = 1, \dots, m$ :

$$\begin{aligned} & \nu \int_{\Omega} \operatorname{curl} \mathbf{u}_m \operatorname{curl} \lambda_j g_{jm} \mathbf{w}_j - b(\mathbf{u}_m, \lambda_j g_{jm} \mathbf{w}_j, \mathbf{u}_m - \alpha \Delta \mathbf{u}_m) \\ &= \int_{\Omega} \mathbf{f} \cdot (\lambda_j g_{jm} \mathbf{w}_j) + b(\mathbf{B}_m, \mathbf{B}_m, \lambda_j g_{jm} \mathbf{w}_j). \end{aligned}$$

Integrating by parts, we get

$$\nu \int_{\Omega} |\Delta \mathbf{u}_m|^2 = - \int_{\Omega} \mathbf{f} \cdot \Delta \mathbf{u}_m - b(\mathbf{B}_m, \mathbf{B}_m, \Delta \mathbf{u}_m). \quad (3.1.12)$$

In the same way, we multiply (3.1.7) by  $\lambda_j h_{jm}$

$$\int_{\Omega} \operatorname{curl} \mathbf{B}_m \operatorname{curl} \lambda_j h_{jm} \mathbf{w}_j = b(\mathbf{B}_m, \mathbf{u}_m, \lambda_j h_{jm} \mathbf{w}_j) - b(\mathbf{u}_m, \mathbf{B}_m, \lambda_j h_{jm} \mathbf{w}_j).$$

After integrating by parts regarding the conditions on  $\Gamma$  and using Lemma 2.2.10 we get,

$$\int_{\Omega} |\Delta \mathbf{B}_m|^2 = b(\mathbf{B}_m, \mathbf{u}_m, \Delta \mathbf{B}_m) - b(\mathbf{u}_m, \mathbf{B}_m, \Delta \mathbf{B}_m). \quad (3.1.13)$$

Now we add (3.1.12) to (3.1.13) and using Hölder inequality

$$\nu \int_{\Omega} |\Delta \mathbf{u}_m|^2 + \int_{\Omega} |\Delta \mathbf{B}_m|^2 \leq \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} \|\Delta \mathbf{u}_m\|_{\mathbf{L}^2(\Omega)} \quad (3.1.14)$$

$$\begin{aligned} & + \|\mathbf{B}_m\|_{\mathbf{L}^r(\Omega)} \|\nabla \mathbf{B}_m\|_{\mathbf{L}^q(\Omega)} \|\Delta \mathbf{u}_m\|_{\mathbf{L}^2(\Omega)} \\ & + \|\mathbf{B}_m\|_{\mathbf{L}^r(\Omega)} \|\nabla \mathbf{u}_m\|_{\mathbf{L}^q(\Omega)} \|\Delta \mathbf{B}_m\|_{\mathbf{L}^2(\Omega)} \\ & + \|\mathbf{u}_m\|_{\mathbf{L}^r(\Omega)} \|\nabla \mathbf{B}_m\|_{\mathbf{L}^q(\Omega)} \|\Delta \mathbf{B}_m\|_{\mathbf{L}^2(\Omega)}, \end{aligned} \quad (3.1.15)$$

with  $1 = \frac{1}{q} + \frac{1}{r} + \frac{1}{2}$ ,  $q > 2$  (which means that  $r < \infty$ ). Gagliardo-Nirenberg interpo-

lation inequality gives,

$$\|\nabla \mathbf{B}_m\|_{\mathbf{L}^q(\Omega)} \leq \|\nabla \mathbf{B}_m\|_{\mathbf{L}^2(\Omega)}^{1-a} \|\mathbf{B}_m\|_{\mathbf{H}^2(\Omega)}^a \quad \text{with } a = 1 - \frac{2}{q}, \quad (3.1.16)$$

Proposition 2.2.5 and (3.1.16) give,

$$\begin{aligned} \nu \|\mathbf{u}_m\|_{\mathbf{H}^2(\Omega)}^2 + \|\mathbf{B}_m\|_{\mathbf{H}^2(\Omega)}^2 &\leq C(\Omega) \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} \|\mathbf{u}_m\|_{\mathbf{H}^2(\Omega)} \\ &\quad + \|\mathbf{B}_m\|_{\mathbf{H}^1(\Omega)}^{2-a} \|\mathbf{B}_m\|_{\mathbf{H}^2(\Omega)}^a \|\mathbf{u}_m\|_{\mathbf{H}^2(\Omega)} \\ &\quad + \|\mathbf{B}_m\|_{\mathbf{H}^1(\Omega)} \|\mathbf{u}_m\|_{\mathbf{H}^1(\Omega)}^{1-a} \|\mathbf{u}_m\|_{\mathbf{H}^2(\Omega)}^a \|\mathbf{B}_m\|_{\mathbf{H}^2(\Omega)} \\ &\quad + \|\mathbf{u}_m\|_{\mathbf{H}^1(\Omega)} \|\mathbf{B}_m\|_{\mathbf{H}^1(\Omega)}^{1-a} \|\mathbf{B}_m\|_{\mathbf{H}^2(\Omega)}^{1+a}. \end{aligned} \quad (3.1.17)$$

Using (3.1.10) and (3.1.11) this inequality becomes

$$\begin{aligned} \nu \|\mathbf{u}_m\|_{\mathbf{H}^2(\Omega)}^2 + \|\mathbf{B}_m\|_{\mathbf{H}^2(\Omega)}^2 &\leq C(\Omega) \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} \|\mathbf{u}_m\|_{\mathbf{H}^2(\Omega)} \\ &\quad + \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}^{2-a} \|\mathbf{B}_m\|_{\mathbf{H}^2(\Omega)}^a \|\mathbf{u}_m\|_{\mathbf{H}^2(\Omega)} \\ &\quad + \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}^{2-a} \|\mathbf{u}_m\|_{\mathbf{H}^2(\Omega)}^a \|\mathbf{B}_m\|_{\mathbf{H}^2(\Omega)} \\ &\quad + \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}^{2-a} \|\mathbf{B}_m\|_{\mathbf{H}^2(\Omega)}^{1+a}. \end{aligned} \quad (3.1.18)$$

In the following, we use Young inequality on each term of the RHS in (3.1.18). Let  $\varepsilon$  be small positive number then there exist  $c_\varepsilon$  such that  $ab \leq \varepsilon a^p + c_\varepsilon b^{p'}$  where  $p'$  is the conjugate of  $p$ ,

$$\begin{aligned} \nu \|\mathbf{u}_m\|_{\mathbf{H}^2(\Omega)}^2 + \|\mathbf{B}_m\|_{\mathbf{H}^2(\Omega)}^2 &\leq C(\Omega) (c_\varepsilon \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}^2 + \varepsilon \|\mathbf{u}_m\|_{\mathbf{H}^2(\Omega)}^2) \\ &\quad + c_\varepsilon (\|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}^{2-a} \|\mathbf{B}_m\|_{\mathbf{H}^2(\Omega)}^a)^2 \\ &\quad + \varepsilon \|\mathbf{u}_m\|_{\mathbf{H}^2(\Omega)}^2 + c_\varepsilon (\|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}^{2-a} \|\mathbf{u}_m\|_{\mathbf{H}^2(\Omega)}^a)^2 \\ &\quad + \varepsilon \|\mathbf{B}_m\|_{\mathbf{H}^2(\Omega)}^2 + c_\varepsilon \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}^{(2-a) \times \frac{2}{1-a}} + \varepsilon \|\mathbf{B}_m\|_{\mathbf{H}^2(\Omega)}^2. \end{aligned} \quad (3.1.19)$$



Once again Young inequality, (3.1.19) becomes

$$\begin{aligned}
\nu \|\mathbf{u}_m\|_{\mathbf{H}^2(\Omega)}^2 + \|\mathbf{B}_m\|_{\mathbf{H}^2(\Omega)}^2 &\leq C(\Omega) (c_\varepsilon \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}^2 + \varepsilon \|\mathbf{u}_m\|_{\mathbf{H}^2(\Omega)}^2 + c_\varepsilon c_{\varepsilon'} \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}^{(2-a) \times \frac{2}{1-a}} \\
&\quad + \varepsilon' c_\varepsilon \|\mathbf{B}_m\|_{\mathbf{H}^2(\Omega)}^2 + \varepsilon \|\mathbf{u}_m\|_{\mathbf{H}^2(\Omega)}^2 + c_\varepsilon c_{\varepsilon'} \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}^{(2-a) \times \frac{2}{1-a}} \\
&\quad + \varepsilon' c_\varepsilon \|\mathbf{u}_m\|_{\mathbf{H}^2(\Omega)}^2 + \varepsilon \|\mathbf{B}_m\|_{\mathbf{H}^2(\Omega)}^2 + c_\varepsilon \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}^{(2-a) \times \frac{2}{1-a}} \\
&\quad + \varepsilon \|\mathbf{B}_m\|_{\mathbf{H}^2(\Omega)}^2),
\end{aligned}$$

where  $\varepsilon'$  is defined in such a way that  $2\varepsilon + \varepsilon'c_\varepsilon \leq \min(\frac{\nu}{2}, \frac{1}{2})$ , equivalent to

$$\begin{aligned}
\nu \|\mathbf{u}_m\|_{\mathbf{H}^2(\Omega)}^2 + \|\mathbf{B}_m\|_{\mathbf{H}^2(\Omega)}^2 &\leq C(\Omega) ((2c_\varepsilon + 2c_\varepsilon c_{\varepsilon'}) \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}^{\frac{2(2-a)}{1-a}} + (2\varepsilon + \varepsilon'c_\varepsilon) \|\mathbf{u}_m\|_{\mathbf{H}^2(\Omega)}^2 \\
&\quad + (2\varepsilon + \varepsilon'c_\varepsilon) \|\mathbf{B}_m\|_{\mathbf{H}^2(\Omega)}^2).
\end{aligned}$$

We finally get

$$\nu \|\mathbf{u}_m\|_{\mathbf{H}^2(\Omega)}^2 + \|\mathbf{B}_m\|_{\mathbf{H}^2(\Omega)}^2 \leq C(\Omega) \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}^{2\beta},$$

where  $C(\Omega)$  is a constant that depends only on  $\Omega$ , and  $\beta = \frac{(2-a)}{1-a} > 2$ . Then the sequences  $(\mathbf{u}_m)$  and  $(\mathbf{B}_m)$  are bounded in  $\mathbf{V}_2$ .

**Step 3. Passage to the limit.** Due to the compact embedding of  $\mathbf{H}^2(\Omega)$  into  $\mathbf{H}^1(\Omega)$ , we can find a subsequence of  $\mathbf{u}_m$  also denoted as  $(\mathbf{u}_m)$  and a subsequence of  $\mathbf{B}_m$  also denoted by  $(\mathbf{B}_m)$  such that,

$$\begin{aligned}
\mathbf{u}_m &\rightharpoonup \mathbf{u}, \quad \mathbf{B}_m \rightharpoonup \mathbf{B} \quad \text{weakly in } \mathbf{H}^2(\Omega) \quad \text{and} \\
\mathbf{u}_m &\longrightarrow \mathbf{u}, \quad \mathbf{B}_m \longrightarrow \mathbf{B} \quad \text{strongly in } \mathbf{H}^1(\Omega).
\end{aligned}$$

Then, clearly

$$\mathbf{u}_m - \alpha \Delta \mathbf{u}_m \rightharpoonup \mathbf{u} - \alpha \Delta \mathbf{u} \quad \text{weakly in } \mathbf{L}^2(\Omega),$$

$$\mathbf{u}_m \cdot \nabla \mathbf{w}_j \longrightarrow \mathbf{u} \cdot \nabla \mathbf{w}_j \quad \text{strongly in } \mathbf{L}^2(\Omega)$$

and

$$\mathbf{B}_m \cdot \nabla \mathbf{u}_m \longrightarrow \mathbf{B} \cdot \nabla \mathbf{u}, \quad \mathbf{u}_m \cdot \nabla \mathbf{B}_m \longrightarrow \mathbf{u} \cdot \nabla \mathbf{B} \quad \text{strongly in } \mathbf{L}^2(\Omega).$$

Let  $\mathbf{v}$  and  $\mathbf{M}$  be in  $\mathbf{V}_{m_0}$  with  $m_0$  fixed and we pass to the limit in Problem (3.1.6)-(3.1.7) as  $m \rightarrow \infty$ ,:

$$\nu \int_{\Omega} \operatorname{curl} \mathbf{u} \operatorname{curl} \mathbf{v} - b(\mathbf{u}, \mathbf{v}, \mathbf{u} - \alpha \Delta \mathbf{u}) = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} + b(\mathbf{B}, \mathbf{B}, \mathbf{v}),$$

$$\int_{\Omega} \operatorname{curl} \mathbf{B} \operatorname{curl} \mathbf{M} = b(\mathbf{B}, \mathbf{u}, \mathbf{M}) - b(\mathbf{u}, \mathbf{B}, \mathbf{M}), \quad \text{for any } (\mathbf{v}, \mathbf{M}) \in \mathbf{V}_{m_0} \times \mathbf{V}_{m_0}.$$

This relation is true for all  $m_0$  then,  $\forall m \in \mathbb{N}^*$ ,

$$\nu \int_{\Omega} \operatorname{curl} \mathbf{u} \operatorname{curl} \mathbf{v} - b(\mathbf{u}, \mathbf{v}, \mathbf{u} - \alpha \Delta \mathbf{u}) = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} + b(\mathbf{B}, \mathbf{B}, \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{V}_m$$

and

$$\int_{\Omega} \operatorname{curl} \mathbf{B} \operatorname{curl} \mathbf{M} = b(\mathbf{B}, \mathbf{u}, \mathbf{M}) - b(\mathbf{u}, \mathbf{B}, \mathbf{M}), \quad \forall \mathbf{M} \in \mathbf{V}_m.$$

Let now  $\mathbf{v} \in \mathbf{V}_1$ . Due to the density of  $\mathbf{V}_m$  in  $\mathbf{V}_1$ , there exist a sequence  $(\mathbf{v}_m) \in \mathbf{V}_m$  such that  $\mathbf{v}_m \rightarrow \mathbf{v}$  and  $(\mathbf{M}_m) \in \mathbf{V}_m$  such that  $\mathbf{M}_m \rightarrow \mathbf{M}$ ,

$$\nu \int_{\Omega} \operatorname{curl} \mathbf{u} \operatorname{curl} \mathbf{v}_m - b(\mathbf{u}, \mathbf{v}_m, \mathbf{u} - \alpha \Delta \mathbf{u}) = \int_{\Omega} \mathbf{f} \cdot \mathbf{v}_m + b(\mathbf{B}, \mathbf{B}, \mathbf{v}_m), \quad \forall \mathbf{v}_m \in \mathbf{V}_m,$$

and

$$\int_{\Omega} \operatorname{curl} \mathbf{B} \operatorname{curl} \mathbf{M}_m = b(\mathbf{B}, \mathbf{u}, \mathbf{M}_m) - b(\mathbf{u}, \mathbf{B}, \mathbf{M}_m), \quad \forall \mathbf{M}_m \in \mathbf{V}_m.$$

We pass to the limit above and we get finally,

$$\nu \int_{\Omega} \operatorname{curl} \mathbf{u} \operatorname{curl} \mathbf{v} - b(\mathbf{u}, \mathbf{v}, \mathbf{u} - \alpha \Delta \mathbf{u}) = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} + b(\mathbf{B}, \mathbf{B}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V}_1,$$

$$\int_{\Omega} \operatorname{curl} \mathbf{B} \operatorname{curl} \mathbf{M} = b(\mathbf{B}, \mathbf{u}, \mathbf{M}) - b(\mathbf{u}, \mathbf{B}, \mathbf{M}), \quad \forall \mathbf{M} \in \mathbf{V}_1.$$

**Step 4. Existence of the pressure and the function  $\theta$ .** Let  $(\mathbf{u}, \mathbf{B}) \in \mathbf{V}_2 \times \mathbf{V}_2$ .

Thanks to Lemma 2.2.6, the nonlinear term satisfies:

$$\mathbf{u} \cdot \nabla(\mathbf{u} - \alpha \Delta \mathbf{u}) \in \mathbf{H}^{-1}(\Omega).$$

Setting

$$\mathbf{F} = -\nu \Delta \mathbf{u} + \mathbf{u} \cdot \nabla(\mathbf{u} - \alpha \Delta \mathbf{u}) - \mathbf{f} - \mathbf{B} \cdot \nabla \mathbf{B},$$

then  $\mathbf{F} \in \mathbf{H}^{-1}(\Omega)$  and  $\langle \mathbf{F}, \mathbf{v} \rangle = 0 \quad \forall \mathbf{v} \in \mathbf{V}_1$ . De Rham's theorem involves the existence of a unique  $p \in L_0^2(\Omega)$  such that  $\mathbf{F} = \nabla p$  in  $\Omega$ . We set  $p = -\pi - \frac{1}{2}|\mathbf{B}|^2$ .

Now, Let

$$\mathbf{G} = -\Delta \mathbf{B} - (\mathbf{B} \cdot \nabla) \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{B}.$$

It is clear that  $\mathbf{G} \in \mathbf{L}^2(\Omega)$  then, again by De Rham's Theorem, there exists a unique  $\theta$  (up to an additive constant) in  $H^1(\Omega)$  such that  $\mathbf{G} = \nabla \theta$  in  $\Omega$ .  $\square$

## 3.2 Regularity of the solution

In order to show some regularity of the solution, we use some results of the associated Stokes problem studied in Section 3. We mention that this problem was clearly analyzed in the three-dimensional case in [6] where the author proved the existence, uniqueness, and regularity of the strong solution. We would like to be precise that we could not prove some regularity of  $\mathbf{u}$  due to the nonlinear term  $(\mathbf{u} \cdot \nabla)(\mathbf{u} - \alpha \Delta \mathbf{u})$ . However, referring to the Stokes Problem, we prove some regularity of the magnetic field  $\mathbf{B}$  and of the function  $\theta$  using the Laplace problem with Neumann boundary conditions.

**Proposition 3.2.1.** *Assuming that  $(\mathbf{u}, \mathbf{B}, \pi, \theta) \in \mathbf{H}^2(\Omega) \times \mathbf{H}^2(\Omega) \times \mathbf{L}^2(\Omega) \times H^1(\Omega)$  is a weak solution of  $(\mathcal{P})$ , then  $\mathbf{B} \in \mathbf{W}^{2,p}(\Omega)$  and  $\theta \in W^{1,p}(\Omega)$  for any  $p \geq 2$ .*

*Proof.* We refer to (2.3.4) and we set the following problem:

$$\begin{cases} -\Delta \mathbf{B} + \nabla \theta = (\mathbf{B} \cdot \nabla) \mathbf{u} - (\mathbf{u} \cdot \nabla) \mathbf{B}, & \text{in } \Omega, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega \\ \mathbf{B} \cdot \mathbf{n} = 0, \quad \operatorname{curl} \mathbf{B} = 0 & \text{on } \Gamma, \\ \int_{\Sigma_j} \mathbf{B} \cdot \mathbf{n} = 0 & \text{for } 1 \leq j \leq J \end{cases} \quad (3.2.1)$$

Using the fact that  $\mathbf{u}$  and  $\mathbf{B}$  are both in  $\mathbf{H}^2(\Omega)$  (due to Theorem 3.1.3) and using some Sobolev embeddings namely  $\mathbf{H}^2(\Omega) \hookrightarrow \mathbf{L}^\infty(\Omega)$  and  $\mathbf{H}^1(\Omega) \hookrightarrow \mathbf{L}^p(\Omega)$  (for any finite  $p$ ), we get

$$(\mathbf{B} \cdot \nabla) \mathbf{u} - (\mathbf{u} \cdot \nabla) \mathbf{B} \in \mathbf{L}^p(\Omega).$$

The assumptions of Theorem 2.4.3 are verified then we conclude that  $\mathbf{B}$  and  $\theta$  of the Problem  $(\mathcal{P})$  are more regular and belong to  $\mathbf{W}^{2,p}(\Omega)$  and  $W^{1,p}(\Omega)$  respectively.  $\square$

Next, we show more regularity for the function  $\theta$ .

**Proposition 3.2.2.** *The function  $\theta$  of the Problem (3.2.1) is more regular and belongs to  $H^2(\Omega)$ .*

*Proof.* We know that  $(\mathbf{B} \cdot \nabla) \mathbf{u} - (\mathbf{u} \cdot \nabla) \mathbf{B} = \mathbf{curl}(\mathbf{u} \times \mathbf{B})$  in  $\Omega$  with  $\mathbf{u} \times \mathbf{B} = u_1 B_2 - u_2 B_1$ . Then the first equation in Problem (3.2.1) can be written as

$$\nabla \theta = \Delta \mathbf{B} + \mathbf{curl}(\mathbf{u} \times \mathbf{B}). \quad (3.2.2)$$

Now, taking the divergence of equation (3.2.2) we get

$$\Delta \theta = 0 \quad \text{in } \Omega, \quad (3.2.3)$$

and on the other hand we have

$$\frac{\partial \theta}{\partial \mathbf{n}} = \frac{\partial}{\partial \mathbf{t}}(\mathbf{u} \times \mathbf{B}) \quad (3.2.4)$$

because

$$\mathbf{curl}(\mathbf{u} \times \mathbf{B}) \cdot \mathbf{n} = \frac{\partial}{\partial t}(\mathbf{u} \times \mathbf{B}) \quad \text{on } \Gamma.$$

As  $\mathbf{u} \in \mathbf{H}^2(\Omega)$  and  $\mathbf{B} \in \mathbf{W}^{2,p}(\Omega)$ , we can easily check that  $(\mathbf{u} \times \mathbf{B}) \in \mathbf{H}^2(\Omega)$ . This implies that

$$\frac{\partial}{\partial t}(\mathbf{u} \times \mathbf{B}) \in \mathbf{H}^{1/2}(\Gamma).$$

The classical regularity of the Laplace problem (3.2.3) with Neumann boundary condition (3.2.4) gives  $\theta \in H^2(\Omega)$ .  $\square$

**Remark 3.2.3.** In our work, the regularity for the velocity field could not be established due to the third-order derivative term  $(\mathbf{u} \cdot \nabla)(\mathbf{u} - \alpha \Delta \mathbf{u})$  which can not belong to  $\mathbf{L}^2(\Omega)$  in our case. In [3], they studied the regularity for  $\mathbf{u}$  in  $\mathbf{W}^{2,p}(\Omega)$  when  $\mathbf{u}$  is entirely zero on the boundary. This remains an open question.

# Chapter 4

## Existence and regularity for polymer aqueous solution

### Contents

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4.1	Functional framework and preliminary results . . . . .	80
4.2	Stationary case . . . . .	86
4.3	Evolution problem . . . . .	92
4.3.1	Statement of the problem . . . . .	93
4.3.2	Existence of solutions . . . . .	96

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We interest ourselves here in the system of equations governing the motion of an aqueous polymer solution.

Aqueous polymer solutions are complex systems that are used in a wide variety of industrial and biological applications. The presence of polymers in water can modify the physical and chemical properties of the solution, such as viscosity, surface tension, electrical conductivity, and interactions between molecules. In this chapter, we will study the existence and uniqueness of solutions for a system of aqueous polymer solutions in a three-dimensional domain.

We have the following constitutive law:

$$\mathcal{T} = -\pi \mathbf{I} + \boldsymbol{\sigma} \quad \text{with} \quad \boldsymbol{\sigma} = 2\nu \mathbf{D} + 2\alpha \frac{d\mathbf{D}}{dt},$$

where  $\pi$  is a scalar function,  $\mathbf{I}$  is the identity tensor and  $\mathcal{T}$  is the stress tensor and

$$\mathbf{D}\mathbf{u} = \frac{1}{2} (\nabla \mathbf{u} + \nabla \mathbf{u}^T)$$

with  $\mathbf{u} = (u_1, u_2, u_3)$  is the velocity field and  $\alpha$  is a characteristic constant of the fluid. Note that for  $1 \leq i, j \leq 3$ ,

$$\frac{dD_{ij}}{dt} = \frac{\partial D_{ij}}{\partial t} + (\mathbf{u} \cdot \nabla) D_{ij}.$$

We deduce from the above constitutive law and the equation of motion the following system:

$$\begin{cases} \frac{\partial}{\partial t}(\mathbf{u}(t) - \alpha \Delta \mathbf{u}) - \nu \Delta \mathbf{u}(t) + (\mathbf{u}(t) \cdot \nabla)(\mathbf{u}(t) - \alpha \Delta \mathbf{u}(t)) \\ + \alpha N(\mathbf{u}(t)) + \nabla \pi = \mathbf{f}(t), & \text{in } Q_T, \\ \operatorname{div} \mathbf{u} = 0, & \text{in } Q_T, \end{cases} \quad (4.0.1)$$

where  $Q_T = \Omega \times ]0, T[$  and  $N(\mathbf{u})$  is the vector field defined by:

$$\text{for } 1 \leq i \leq 3, \quad N_i(\mathbf{u}) = \frac{\partial u_k}{\partial x_j} \frac{\partial}{\partial x_k} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right).$$

Here we use the Einstein convention where the repeated indices are implicitly summed over.

In this chapter, we consider that a polymer aqueous solution is obtained by adding a few quantities of polymer to an amount of water so that the viscosity remains unchanged. For simplicity, we neglect the term  $\alpha N(\mathbf{u})$  in (4.0.1) and we consider the following conditions:

$$\mathbf{u} \cdot \mathbf{n} = 0 \quad \text{on } \Sigma_T, \quad (4.0.2)$$

$$\mathbf{curl} \mathbf{u} \times \mathbf{n} = \mathbf{0} \quad \text{on } \Sigma_T, \quad (4.0.3)$$

$$\mathbf{u}(0) = \mathbf{u}_0 \quad \text{in } \Omega, \quad (4.0.4)$$

where  $\Sigma_T = \Gamma \times ]0, T[$ . The boundary conditions (4.0.2) and (4.0.3) are called Navier-type boundary conditions and (4.0.4) is the initial condition of the problem. First, we are interested in the steady flow problem:

*Find*  $(\mathbf{u}, \pi) \in \mathbf{H}^2(\Omega) \times \mathbf{L}^2(\Omega)$  *verifying*

$$(\mathcal{S}) \quad \begin{cases} -\nu \Delta \mathbf{u} + \mathbf{u} \cdot \nabla (\mathbf{u} - \alpha \Delta \mathbf{u}) + \nabla \pi = \mathbf{f} & \text{in } \Omega, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} \cdot \mathbf{n} = 0, \quad \mathbf{curl} \mathbf{u} \times \mathbf{n} = \mathbf{0} & \text{on } \Gamma, \end{cases}$$

and next in the corresponding evolution problem:

$$(\mathcal{E}) \quad \begin{cases} \frac{\partial}{\partial t} (\mathbf{u}(t) - \alpha \Delta \mathbf{u}(t)) - \nu \Delta \mathbf{u}(t) + (\mathbf{u}(t) \cdot \nabla) (\mathbf{u}(t) - \alpha \Delta \mathbf{u}(t)) + \nabla \pi = \mathbf{f}(t), & \text{in } Q_T, \\ \operatorname{div} \mathbf{u} = 0, & \text{in } Q_T, \\ \mathbf{u} \cdot \mathbf{n} = 0, \quad \mathbf{curl} \mathbf{u} \times \mathbf{n} = \mathbf{0} & \text{on } \Sigma_T, \\ \mathbf{u}(0) = \mathbf{u}_0, & \text{in } \Omega. \end{cases} \quad (4.0.5)$$

Throughout this work, we assume that  $\Omega$  is of class  $\mathcal{C}^{1,1}$  unless otherwise specified since this is the minimum required regularity for some embedding spaces. The external forces are denoted by  $\mathbf{f}$ ,  $\nu$  the viscosity of the fluid, and  $\pi$  is the scalar pressure function. This system looks like Euler's for non-viscous fluids and is more complicated than the alpha models.

In [27], it considered system (4.0.1) with the initial condition (4.0.4) and Dirichlet boundary condition  $\mathbf{u} = \mathbf{0}$  on  $\Sigma_T$ , where a solution in  $\mathbf{L}^\infty([0, T]; \mathbf{H}^1(\Omega))$  is proved with decomposition in many operators and using of topological degree. The problem is similar to the Camassa-Holm problem [17], [1] or the Leray  $\alpha$ -model [37], [8] or also the Navier-Stokes  $\alpha$ -model [26]. Amrouche and Ouazar in [4] studied the existence of the steady flow  $(\mathcal{S})$  in two dimension with slip boundary conditions where they



used the property:

$$\int_{\Omega} (\mathbf{u} \cdot \nabla) \mathbf{u} \cdot \Delta \mathbf{u} = 0,$$

which holds for any  $\mathbf{u} \in \mathbf{H}^1(\Omega)$  with  $\Delta \mathbf{u} \in \mathbf{L}^2(\Omega)$ , such that  $\operatorname{div} \mathbf{u} = 0$  in  $\Omega$ ,  $\mathbf{u} \cdot \mathbf{n} = 0$  on  $\Gamma$  and  $\operatorname{curl} \mathbf{u} = 0$  on  $\Gamma$ .

This property has allowed proving the existence of a solution for the approximate problem easily using Galerkin's method which is no longer true in three dimensions. To overcome this obstacle, we have introduced, in the case of steady flow, an adapted special scalar product in  $\mathbb{R}^m$  that allows us to use the fixed point Brouwer Theorem (see Theorem 3.1.2). For the unsteady flow, we have been inspired, for the existence of  $\mathbf{u}$  in  $H^k(\Omega)$ , with  $k \leq 3$ , by the paper of Temam [34] on the Euler equations. We find in Bresch and Lemoine [10] a study of a problem similar to our second-grade fluids in  $W^{2,p}(\Omega)$ , with  $p > 3$  with a fixed point method for small exterior forces. In our work, we show the existence of a solution  $(\mathbf{u}, \pi)$  of the system  $(\mathcal{E})$  using Galerkin method, without any regularization. In this chapter, we shall prove, in three dimensions, two main results. The first theorem for the existence of a solution to the steady problem  $(\mathcal{S})$  and the second for the existence of a solution in  $\mathbf{H}^2(\Omega)$  for the evolution problem  $(\mathcal{E})$ .

For the steady case, by the Galerkin method, we use an adequate scalar product, a special finite dimensional space, and important results of equivalence of norms which are the main ideas for the proof of the theorem of existence. In the unsteady problem, we use always the Galerkin method with a special basis.

## 4.1 Functional framework and preliminary results

Let us introduce some functional spaces and preliminary results that are required for the existence of solutions. We define the following Hilbert space:

$$\mathbf{X}(\Omega) = \{\mathbf{v} \in \mathbf{L}^2(\Omega); \operatorname{div} \mathbf{v} \in L^2(\Omega), \operatorname{curl} \mathbf{v} \in \mathbf{L}^2(\Omega)\},$$

provided with the norm

$$\|\mathbf{v}\|_{\mathbf{X}(\Omega)} = (\|\mathbf{v}\|_{\mathbf{L}^2(\Omega)}^2 + \|\mathbf{curl}\ \mathbf{v}\|_{\mathbf{L}^2(\Omega)}^2 + \|\operatorname{div}\ \mathbf{v}\|_{\mathbf{L}^2(\Omega)}^2)^{1/2}.$$

Contrary to the 2D case, here the **curl** operator is defined as follows:

$$\mathbf{v} \in \mathcal{D}'(\Omega)^3 \quad \mathbf{curl}\ \mathbf{v} = \left( \frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}, \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}, \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \right).$$

And we have the following identity:

$$\mathbf{curl}\ \mathbf{curl}\ \mathbf{v} = -\Delta \mathbf{v} + \nabla \operatorname{div}\ \mathbf{v}.$$

We define the following subspace of  $\mathbf{X}(\Omega)$ ,

$$\mathbf{X}_T(\Omega) = \{\mathbf{v} \in \mathbf{X}(\Omega); \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma\}.$$

We define also the following spaces:

$$\mathbf{H} = \{\mathbf{v} \in \mathbf{L}^2(\Omega); \operatorname{div}\ \mathbf{v} = 0 \text{ in } \Omega, \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma\},$$

$$\mathbf{V}_1 = \{\mathbf{v} \in \mathbf{X}_T(\Omega); \operatorname{div}\ \mathbf{v} = 0 \text{ in } \Omega\},$$

$$\mathbf{V}_2 = \{\mathbf{v} \in \mathbf{V}_1; \Delta \mathbf{v} \in \mathbf{L}^2(\Omega), \mathbf{curl}\ \mathbf{v} \times \mathbf{n} = \mathbf{0} \text{ on } \Gamma\},$$

$$\mathbf{V}_3 = \{\mathbf{v} \in \mathbf{V}_2; \mathbf{curl}(\mathbf{u} - \alpha \Delta \mathbf{u}) \in \mathbf{L}^2(\Omega)\}.$$

Here for all  $\mathbf{u} = (u_1, u_2, u_3)$  and  $\mathbf{v} = (v_1, v_2, v_3) \in \mathbf{H}^1(\Omega)$ , we have:

$$\mathbf{u} \cdot \mathbf{v} = u_i v_i \quad \text{and} \quad \nabla \mathbf{u} : \nabla \mathbf{v} = \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j}.$$

To give a sense to the boundary condition that appears in the definition of  $\mathbf{V}_2$ , we use the following lemma proved in [2].

**Lemma 4.1.1.** *The following mappings*

$$\begin{aligned} \ell_1 : \mathcal{D}(\overline{\Omega}) &\longrightarrow \mathbf{W}^{-\frac{1}{p},p}(\Gamma) \\ \mathbf{u} &\longrightarrow \mathbf{curl} \mathbf{u} \times \mathbf{n} \end{aligned}$$

and

$$\begin{aligned} \ell_2 : \mathcal{D}(\overline{\Omega}) &\longrightarrow W^{-\frac{1}{p},p}(\Gamma) \\ \mathbf{u} &\longrightarrow \operatorname{div} \mathbf{u} \end{aligned}$$

can be extended to a linear and continuous mapping from  $\mathbf{E}_p(\Delta, \Omega)$  where

$$\mathbf{E}_p(\Delta, \Omega) = \{\mathbf{v} \in \mathbf{W}^{1,p}(\Omega); \Delta \mathbf{v} \in \mathbf{L}^{r(p)}(\Omega)\},$$

and

$$r(p) = \begin{cases} 1 & \text{if } p < 3/2 \\ 1 + \varepsilon, \text{ with arbitrary } \varepsilon > 0 & \text{if } p = 3/2 \\ \frac{3p}{3+p} & \text{if } p > 3/2, \end{cases}$$

into  $\mathbf{W}^{-\frac{1}{p},p}(\Gamma)$  (into  $W^{-\frac{1}{p},p}(\Gamma)$  respectively). And we have the Green formula:  
 $\forall \mathbf{u} \in \mathbf{E}_p(\Delta, \Omega), \forall \mathbf{v} \in \mathbf{W}^{1,p'}(\Omega),$

$$\begin{aligned} - \int_{\Omega} \Delta \mathbf{u} \cdot \mathbf{v} &= \int_{\Omega} \mathbf{curl} \mathbf{u} \cdot \mathbf{curl} \mathbf{v} + \int_{\Omega} (\operatorname{div} \mathbf{u})(\operatorname{div} \mathbf{v}) \\ &\quad - \langle \mathbf{curl} \mathbf{u} \times \mathbf{n}, \mathbf{v} \rangle_{\Gamma} - \langle \operatorname{div} \mathbf{u}, \mathbf{v} \cdot \mathbf{n} \rangle_{\Gamma} \end{aligned} \quad (4.1.1)$$

where  $\langle \cdot, \cdot \rangle_{\Gamma}$  denote the duality pairing between  $\mathbf{W}^{-\frac{1}{p},p}(\Gamma)$  and  $\mathbf{W}^{\frac{1}{p},p'}(\Gamma)$  (between  $W^{-\frac{1}{p},p}(\Gamma)$  and  $W^{\frac{1}{p},p'}(\Gamma)$  respectively), where  $p'$  is the conjugate of  $p$ .

**Lemma 4.1.2.** *Suppose that  $\Omega$  is Lipschitz and let*

$$\mathbf{w} \in \mathbf{L}^p(\Omega) \quad \text{be such that} \quad \mathbf{curl} \mathbf{w} \in \mathbf{L}^p(\Omega) \quad \text{and} \quad \mathbf{w} \times \mathbf{n} \in \mathbf{L}^p(\Gamma).$$

Then  $\operatorname{div}_\Gamma(\mathbf{w} \times \mathbf{n}) \in \mathbf{W}^{-\frac{1}{p}, p}(\Gamma)$  and

$$\operatorname{div}_\Gamma(\mathbf{w} \times \mathbf{n}) = \mathbf{curl} \mathbf{w} \cdot \mathbf{n} \quad \text{on } \Gamma. \quad (4.1.2)$$

**Remark 4.1.3.** For all  $\mathbf{v} \in \mathbf{V}_2$ , we have  $\Delta \mathbf{v} \cdot \mathbf{n} \in H^{-1/2}(\Gamma)$ . Take  $\mathbf{w} = \mathbf{curl} \mathbf{v}$  in (4.1.2), then  $\Delta \mathbf{v} \cdot \mathbf{n} = 0$  on  $\Gamma$ .

The next lemma is an important result concerning the equivalence of norms in  $\mathbf{H}^1(\Omega)$  and  $\mathbf{X}_T(\Omega)$  (see Lemma 2.11, corollaries 3.16 and 3.19 in [11]).

**Friedrich's inequalities:**

For any function in  $\mathbf{H}^1(\Omega)$  with  $\mathbf{u} \cdot \mathbf{n} = 0$  on  $\Gamma$ , we have:

$$\|\mathbf{u}\|_{\mathbf{H}^1(\Omega)} \leq C \left( \|\mathbf{curl} \mathbf{u}\|_{L^2(\Omega)} + \|\operatorname{div} \mathbf{u}\|_{L^2(\Omega)} + \sum_{j=1}^J | \langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} | \right) \quad (4.1.3)$$

and for any function in  $\mathbf{H}^1(\Omega)$  with  $\mathbf{u} \times \mathbf{n} = 0$  on  $\Gamma$  then :

$$\|\mathbf{u}\|_{\mathbf{H}^1(\Omega)} \leq C \left( \|\mathbf{curl} \mathbf{u}\|_{L^2(\Omega)} + \|\operatorname{div} \mathbf{u}\|_{L^2(\Omega)} + \sum_{i=1}^I | \langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Gamma_i} | \right) \quad (4.1.4)$$

**Lemma 4.1.4.** *i) We have the following continuous embedding:*

$$\mathbf{X}_T(\Omega) \hookrightarrow \mathbf{H}^1(\Omega)$$

and

$$\forall \mathbf{v} \in \mathbf{X}_T(\Omega), \quad \|\mathbf{v}\|_{\mathbf{H}^1(\Omega)} \leq C(\Omega) \|\mathbf{v}\|_{\mathbf{X}(\Omega)}, \quad (4.1.5)$$

which means that the norm in  $\mathbf{X}_T(\Omega)$  is equivalent to the full norm in  $\mathbf{H}^1(\Omega)$ .

*ii) If moreover,  $\Omega$  is simply connected, then the mapping*

$$\mathbf{v} \mapsto (\|\operatorname{div} \mathbf{v}\|_{L^2(\Omega)}^2 + \|\mathbf{curl} \mathbf{v}\|_{L^2(\Omega)}^2)^{1/2}. \quad (4.1.6)$$

*is a norm on  $\mathbf{X}_T(\Omega)$  equivalent to the full norm of  $\mathbf{H}^1(\Omega)$ . In particular, we have in*

this case:

$$\forall \mathbf{v} \in \mathbf{V}_1, \quad \|\mathbf{v}\|_{\mathbf{H}^1(\Omega)} \leq C(\Omega) \|\mathbf{curl} \mathbf{v}\|_{\mathbf{L}^2(\Omega)}. \quad (4.1.7)$$

**Remark 4.1.5.** i) If we consider now the following space, with the same regularity of  $\Omega$ ,

$$\mathbf{X}_N(\Omega) = \{\mathbf{v} \in \mathbf{X}(\Omega); \quad \mathbf{v} \times \mathbf{n} = \mathbf{0} \quad \text{on } \Gamma\},$$

then  $\mathbf{X}_N(\Omega) \hookrightarrow \mathbf{H}^1(\Omega)$ , and if  $\Gamma$  is connected, then the mapping

$$\mathbf{v} \mapsto (\|\operatorname{div} \mathbf{v}\|_{\mathbf{L}^2(\Omega)}^2 + \|\mathbf{curl} \mathbf{v}\|_{\mathbf{L}^2(\Omega)}^2)^{1/2} \quad (4.1.8)$$

is a norm on  $\mathbf{X}_N$  equivalent to the full norm of  $\mathbf{H}^1(\Omega)$ .

ii) Suppose that  $\Omega$  is of class  $\mathcal{C}^{2,1}$ . If  $\mathbf{v} \in \mathbf{X}_T(\Omega)$  (resp. in  $\mathbf{X}_N(\Omega)$ )  $\operatorname{div} \mathbf{v} \in H^1(\Omega)$  and  $\mathbf{curl} \mathbf{v} \in \mathbf{H}^1(\Omega)$ , then  $\mathbf{v} \in \mathbf{H}^2(\Omega)$ .

For more details, one can check Theorem 3.12 in [11].

**Proposition 4.1.6.** i) Suppose that  $\Omega$  is of class  $\mathcal{C}^{2,1}$  and simply connected. Then  $\mathbf{V}_2 \hookrightarrow \mathbf{H}^2(\Omega)$  and

$$\|\mathbf{v}\|_{\mathbf{H}^2} \simeq \|\Delta \mathbf{v}\|_{\mathbf{L}^2(\Omega)} \quad (4.1.9)$$

is a norm on  $\mathbf{V}_2$  equivalent to the norm of  $\mathbf{H}^2(\Omega)$ .

ii) If  $\Omega$  is of class  $\mathcal{C}^{3,1}$ , then  $\mathbf{V}_3 \hookrightarrow \mathbf{H}^3(\Omega)$ , and for any  $\mathbf{v} \in \mathbf{V}_3$

$$\|\mathbf{v}\|_{\mathbf{H}^3(\Omega)} \simeq \|\mathbf{curl}(\mathbf{v} - \alpha \Delta \mathbf{v})\|_{\mathbf{L}^2(\Omega)} \simeq \|\mathbf{curl}(\Delta \mathbf{v})\|_{\mathbf{L}^2(\Omega)}. \quad (4.1.10)$$

*Proof.* i) Let  $\mathbf{v} \in \mathbf{V}_2$ . We set  $\mathbf{z} = \mathbf{curl} \mathbf{v}$ , then  $\mathbf{z} \in \mathbf{L}^2(\Omega)$ ,  $\mathbf{curl} \mathbf{z} \in \mathbf{L}^2(\Omega)$ ,  $\operatorname{div} \mathbf{z} = \mathbf{0}$  and  $\mathbf{z} \times \mathbf{n} = \mathbf{0}$  which implies that  $\mathbf{z} \in \mathbf{H}^1(\Omega)$ . We deduce from Corollary 2.15 in [11] that  $\mathbf{v} \in \mathbf{H}^2(\Omega)$ .

From Proposition 4.7 in [6], if  $\Omega$  is simply connected then for any  $\mathbf{v} \in \mathbf{H}^2(\Omega)$ , such that  $\mathbf{v} \cdot \mathbf{n} = 0$  and  $\mathbf{curl} \mathbf{u} \times \mathbf{n} = 0$ , we have

$$\|\mathbf{v}\|_{\mathbf{H}^2(\Omega)} \leq C \|\Delta \mathbf{v}\|_{\mathbf{L}^2(\Omega)}.$$

ii) Let  $\mathbf{v} \in \mathbf{V}_3$ , we know that  $\mathbf{v} \in \mathbf{H}^2$  and  $\mathbf{curl} \Delta \mathbf{v} \in \mathbf{L}^2(\Omega)$ . Let  $\mathbf{w} = \mathbf{curl} \mathbf{v}$ , we

need to show that  $\mathbf{w} \in \mathbf{H}^2(\Omega)$  which gives by Corollary 2.15 in [11] that  $\mathbf{v} \in \mathbf{H}^3(\Omega)$ . We know that

$$\begin{aligned} \operatorname{div} \mathbf{w} &= 0 \\ \operatorname{curl} \Delta \mathbf{w} &\in \mathbf{L}^2(\Omega) \\ \text{and } \Delta \mathbf{v} \cdot \mathbf{n} &= 0 \text{ on } \Gamma \text{ (see Remark 4.1.3),} \end{aligned}$$

then  $\Delta \mathbf{v} \in \mathbf{H}^1(\Omega)$  and  $\mathbf{w} \in \mathbf{H}^2(\Omega)$ . In analogy with Proposition 4.7 in [6], we prove the equivalences (4.1.10).

**Lemma 4.1.7.** *For any  $\mathbf{u} \in \mathbf{V}_2$ , we have*

$$\|\mathbf{u} - \alpha \Delta \mathbf{u}\|_{\mathbf{L}^2(\Omega)} \leq (C(\Omega) + 2\alpha) \|\Delta \mathbf{u}\|_{\mathbf{L}^2(\Omega)}. \quad (4.1.11)$$

*Proof.* Let  $\mathbf{u} \in \mathbf{V}_2$ , then

$$\begin{aligned} \|\mathbf{u} - \alpha \Delta \mathbf{u}\|_{\mathbf{L}^2(\Omega)}^2 &= \|\mathbf{u}\|_{\mathbf{L}^2(\Omega)}^2 + \alpha^2 \|\Delta \mathbf{u}\|_{\mathbf{L}^2(\Omega)}^2 - 2\alpha \int_{\Omega} \mathbf{u} \cdot \Delta \mathbf{u} \\ &= \|\mathbf{u}\|_{\mathbf{L}^2(\Omega)}^2 + \alpha^2 \|\Delta \mathbf{u}\|_{\mathbf{L}^2(\Omega)}^2 + 2\alpha \|\operatorname{curl} \mathbf{u}\|_{\mathbf{L}^2(\Omega)}^2. \end{aligned}$$

Using (4.1.3) with  $\Omega$  simply connected domain, we get

$$\|\mathbf{u} - \alpha \Delta \mathbf{u}\|_{\mathbf{L}^2(\Omega)}^2 \leq (C(\Omega) + 2\alpha) \|\operatorname{curl} \mathbf{u}\|_{\mathbf{L}^2(\Omega)}^2 + \alpha^2 \|\Delta \mathbf{u}\|_{\mathbf{L}^2(\Omega)}^2 \quad (4.1.12)$$

$$\leq (C(\Omega) + 2\alpha) (\|\operatorname{curl} \mathbf{u}\|_{\mathbf{L}^2(\Omega)}^2 + \alpha \|\Delta \mathbf{u}\|_{\mathbf{L}^2(\Omega)}^2). \quad (4.1.13)$$

From (4.1.4), since  $\operatorname{curl} \mathbf{u} \in \mathbf{X}_N(\Omega)$  we have also

$$\|\operatorname{curl} \mathbf{u}\|^2 \leq C(\Omega) \|\Delta \mathbf{u}\|^2.$$

we deduce that

$$\|\mathbf{u} - \alpha \Delta \mathbf{u}\|_{\mathbf{L}^2(\Omega)}^2 \leq (C(\Omega) + 2\alpha)^2 \|\Delta \mathbf{u}\|_{\mathbf{L}^2(\Omega)}^2. \quad (4.1.14)$$

## 4.2 Stationary case

In this section, we study the existence of solutions for the steady problem  $(\mathcal{S})$ . First, we introduce the associated Stokes problem then we will construct a special basis of eigenvectors in order to use the Galerkin method. We can formulate the Stokes problem as follows:

for any  $\mathbf{g} \in \mathbf{H}$ , find  $\mathbf{v} \in \mathbf{V}_1$  such that

$$\begin{cases} -\Delta \mathbf{u} = \mathbf{g} & \text{in } \Omega, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} \cdot \mathbf{n} = 0, \operatorname{curl} \mathbf{u} \times \mathbf{n} = \mathbf{0} & \text{on } \Gamma. \end{cases} \quad (4.2.1)$$

is equivalent to the following variational formulation:

$$\text{Find } \mathbf{u} \in \mathbf{V}_1 \text{ such that } \forall \mathbf{v} \in \mathbf{V}_1, \quad \int_{\Omega} \operatorname{curl} \mathbf{u} \cdot \operatorname{curl} \mathbf{v} = \int_{\Omega} \mathbf{g} \cdot \mathbf{v}. \quad (4.2.2)$$

Here, the existence and the uniqueness are consequences of the Lax-Milgram lemma, where the coercivity of the continuous bilinear form  $a(\mathbf{u}, \mathbf{v}) = \int_{\Omega} \operatorname{curl} \mathbf{u} \cdot \operatorname{curl} \mathbf{v}$  can be deduced from (4.1.7). Conversely, let  $\mathbf{u}$  in  $\mathbf{V}_1$  solution of Problem (4.2.2). Taking  $\mathbf{v} \in \mathcal{D}(\Omega)$  we get,

$$-\Delta \mathbf{u} = \mathbf{g} \quad \text{in } \Omega.$$

It remains now to show that  $\operatorname{curl} \mathbf{u} \times \mathbf{n} = 0$  on  $\Gamma$ . Using the Green formula, we get that for any  $\mathbf{v} \in \mathbf{V}_1$

$$\int_{\Omega} \operatorname{curl} \mathbf{u} \cdot \operatorname{curl} \mathbf{v} - \langle \operatorname{curl} \mathbf{u} \times \mathbf{n}, \mathbf{v} \rangle_{\Gamma} = \int_{\Omega} \mathbf{g} \cdot \mathbf{v}$$

implies that

$$\langle \operatorname{curl} \mathbf{u} \times \mathbf{n}, \mathbf{v} \rangle_{\mathbf{H}^{-1/2}(\Gamma) \times \mathbf{H}^{1/2}(\Gamma)} = 0. \quad (4.2.3)$$

Let  $\mu \in \mathbf{H}^{1/2}(\Gamma)$ . We know that there exists  $\mathbf{v} \in \mathbf{H}^1(\Omega)$  with  $\operatorname{div} \mathbf{v} = 0$  in  $\Omega$ , such that  $\mathbf{v} = \mu_t$  on  $\Gamma$ , we deduce finally that  $\operatorname{curl} \mathbf{u} \times \mathbf{n} = 0$  on  $\Gamma$ .

**Proposition 4.2.1. (Special basis)** *We assume that  $\Omega$  is of class  $\mathcal{C}^{2,1}$ , then there exist a Hilbert basis  $(\mathbf{w}_i)_{i \geq 1}$  of  $\mathbf{H}$  and a sequence  $(\lambda_i)_{i \geq 1}$  of real numbers with  $\lambda_i > 0$  for any  $i \geq 1$ ,  $\lambda_i \rightarrow +\infty$  and such that*

$$\begin{cases} -\Delta \mathbf{w}_i = \lambda_i \mathbf{w}_i & \text{in } \Omega, \\ \operatorname{div} \mathbf{w}_i = 0 & \text{in } \Omega, \\ \mathbf{w}_i \cdot \mathbf{n} = 0, \operatorname{curl} \mathbf{w}_i \times \mathbf{n} = \mathbf{0} & \text{on } \Gamma. \end{cases} \quad (4.2.4)$$

Moreover,  $\mathbf{w}_i \in \mathbf{V}_2$  for any  $i \geq 1$ .

*Proof.* Let  $\mathbf{v} = T\mathbf{g} \in \mathbf{V}_2$  be the unique solution of Problem (4.2.2). The operator  $T$ , considered as an operator from  $\mathbf{H}$  to  $\mathbf{H}$ , is a self-adjoint compact operator. Since  $\operatorname{Ker} T = \mathbf{0}$  and

$$\int_{\Omega} T\mathbf{g} \cdot \mathbf{g} = \int_{\Omega} |\operatorname{curl} \mathbf{v}|^2 \geq 0$$

we deduce that  $\mathbf{H}$  admits a Hilbert basis  $(\mathbf{w}_i)_{i \geq 1}$  of eigenvectors of  $T$  associated to eigenvalues  $(\mu_i)_{i \geq 1}$  with  $\mu_i > 0$  and  $\mu_i \rightarrow 0$ . Setting  $\lambda_i = 1/\mu_i$ , we see easily that  $\mathbf{w}_i$  satisfies (4.2.4) and  $\mathbf{w}_i \in \mathbf{V}_2$ .

Now, we are able to announce and prove the theorem of existence for the steady case.

**Theorem 4.2.2.** *Let  $\Omega \subset \mathbb{R}^3$  be a bounded simply connected domain of class  $\mathcal{C}^{3,1}$ . Then for all  $\mathbf{f} \in \mathbf{L}^2(\Omega)$ , Problem (S) has at least one solution  $(\mathbf{u}, \pi) \in \mathbf{H}^2(\Omega) \times L^2(\Omega)$  and we have the following estimate:*

$$\|\mathbf{u}\|_{\mathbf{H}^2(\Omega)} + \|\pi\|_{L^2(\Omega)/\mathbb{R}} \leq C(\Omega, \alpha, \nu) \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}. \quad (4.2.5)$$

*Proof.*

**Step 1: GALERKIN APPROXIMATION.**

We define the trilinear form

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \int_{\Omega} (\mathbf{u} \cdot \nabla) \mathbf{v} \cdot \mathbf{w} = \int_{\Omega} u_j \frac{\partial v_i}{\partial x_j} w_i,$$



where we used the implicit summation on the repeated indices. Recall that for any  $(\mathbf{u}, \mathbf{v}, \mathbf{w}) \in \mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega)$ , with  $\operatorname{div} \mathbf{u} = 0$  in  $\Omega$ ,  $\mathbf{u} \cdot \mathbf{n} = 0$  on  $\Gamma$  and we have

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = -b(\mathbf{u}, \mathbf{w}, \mathbf{v}) \quad \text{and} \quad b(\mathbf{u}, \mathbf{v}, \mathbf{v}) = 0. \quad (4.2.6)$$

Let us consider now, the following approximate problems

$$(\mathcal{S})_m \begin{cases} \text{For } m = 1, 2, \dots & \text{find } \mathbf{u}_m = \sum_{i=1}^m \xi_i \mathbf{w}_i \in \mathbf{V}_{2m}, \\ \nu \int_{\Omega} \operatorname{curl} \mathbf{u}_m \cdot \operatorname{curl} \mathbf{w}_j + b(\mathbf{u}_m, \mathbf{w}_j, \mathbf{u}_m - \alpha \Delta \mathbf{u}_m) = \int_{\Omega} \mathbf{f} \cdot \mathbf{w}_j, \\ & \text{for } j = 1, \dots, m, \end{cases}$$

where  $(\mathbf{w}_i)_{i \geq 1}$  is the system of eigenvectors given in Proposition 4.2.1. We will prove that for all  $m \geq 1$ , the problem  $(\mathcal{S})_m$  has a solution  $\mathbf{u}_m \in \mathbf{V}_{2m} = \langle \mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_m \rangle$ , the space spanned by  $(\mathbf{w}_i)_{i \geq 1}$ . For that, we define the following scalar product which is the key of the proof: for  $\boldsymbol{\xi} \in \mathbb{R}^m$  and  $\boldsymbol{\eta} \in \mathbb{R}^m$ ,

$$(\boldsymbol{\xi} | \boldsymbol{\eta})_{\alpha, \lambda} = \boldsymbol{\xi} \cdot \boldsymbol{\eta} + \alpha \boldsymbol{\xi} \cdot \boldsymbol{\eta}_{\lambda},$$

with

$$\boldsymbol{\eta}_{\lambda} = (\lambda_1 \eta_1, \dots, \lambda_m \eta_m)$$

and  $\lambda_1, \dots, \lambda_m$  are the eigenvalues associated to the eigenvectors  $\mathbf{w}_1, \dots, \mathbf{w}_m$ . We denote by  $\mathbb{R}_{\alpha, \lambda}^m$  the space  $\mathbb{R}^m$  provided with the scalar product  $(\cdot | \cdot)_{\alpha, \lambda}$ . The operator

$$P_m : \mathbb{R}_{\alpha, \lambda}^m \longrightarrow \mathbb{R}_{\alpha, \lambda}^m$$

$$\boldsymbol{\xi} \mapsto \boldsymbol{\eta}$$

with

$$\eta_j = \nu \xi_i \int_{\Omega} \operatorname{curl} \mathbf{w}_i \cdot \operatorname{curl} \mathbf{w}_j + \xi_i \xi_k b(\mathbf{w}_i, \mathbf{w}_j, \mathbf{w}_k - \alpha \Delta \mathbf{w}_k) - \int_{\Omega} \mathbf{f} \cdot \mathbf{w}_j,$$

for any  $j = 1, \dots, m$ , is clearly continuous and we will show that there exists  $\rho > 0$

such that

$$(P_m(\boldsymbol{\xi}) | \boldsymbol{\xi})_{\alpha, \lambda} > 0 \quad \text{for all } \boldsymbol{\xi} \in \mathbb{R}_{\alpha, \lambda}^m \quad \text{satisfying} \quad \|\boldsymbol{\xi}\|_{\mathbb{R}_\alpha^m} = \rho.$$

Indeed we have,

$$\begin{aligned} (P_m(\boldsymbol{\xi}) | \boldsymbol{\xi})_{\alpha, \lambda} &= \nu \int_{\Omega} \mathbf{curl}(\xi_i \mathbf{w}_i) \cdot \mathbf{curl}(\xi_j \mathbf{w}_j) + b(\xi_i \mathbf{w}_i, \xi_j \mathbf{w}_j, \xi_k \mathbf{w}_k - \alpha \Delta(\xi_k \mathbf{w}_k)) \\ &\quad - \int_{\Omega} \mathbf{f} \cdot (\xi_j \mathbf{w}_j) + \alpha \left[ \nu \int_{\Omega} \mathbf{curl}(\xi_i \mathbf{w}_i) \cdot \mathbf{curl}(\lambda_j \xi_j \mathbf{w}_j) \right. \\ &\quad \left. + b(\xi_i \mathbf{w}_i, \xi_j \lambda_j \mathbf{w}_j, \xi_k \mathbf{w}_k - \alpha \Delta(\xi_k \mathbf{w}_k)) - \int_{\Omega} \mathbf{f} \cdot (\xi_j \lambda_j \mathbf{w}_j) \right]. \end{aligned}$$

When  $\Omega$  is of class  $\mathcal{C}^{3,1}$  then  $\mathbf{w}_i$  are in  $\mathbf{H}^3(\Omega)$ , which gives  $\mathbf{u}_m$  in  $\mathbf{H}^3(\Omega)$  and by the property (4.2.6) we get

$$b(\mathbf{u}_m, \mathbf{u}_m - \alpha \Delta \mathbf{u}_m, \mathbf{u}_m - \alpha \Delta \mathbf{u}_m) = 0.$$

Therefore

$$(\mathcal{P}_m(\boldsymbol{\xi}) | \boldsymbol{\xi})_{\alpha, \lambda} = \nu \|\mathbf{curl} \mathbf{u}_m\|_{\mathbf{L}^2(\Omega)}^2 + \alpha \nu \|\Delta \mathbf{u}_m\|_{\mathbf{L}^2(\Omega)}^2 - \int_{\Omega} \mathbf{f} \cdot (\mathbf{u}_m - \alpha \Delta \mathbf{u}_m)$$

and then

$$\begin{aligned} (\mathcal{P}_m(\boldsymbol{\xi}) | \boldsymbol{\xi})_{\alpha, \lambda} &\geq \nu \|\mathbf{curl} \mathbf{u}_m\|_{\mathbf{L}^2(\Omega)}^2 + \nu \alpha \|\Delta \mathbf{u}_m\|_{\mathbf{L}^2(\Omega)}^2 - \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} \|\mathbf{u}_m - \alpha \Delta \mathbf{u}_m\|_{\mathbf{L}^2(\Omega)} \\ &\geq \nu \alpha \|\Delta \mathbf{u}_m\|_{\mathbf{L}^2(\Omega)}^2 - \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} \|\mathbf{u}_m - \alpha \Delta \mathbf{u}_m\|_{\mathbf{L}^2(\Omega)}, \end{aligned}$$

using (4.1.14), we get

$$(\mathcal{P}_m(\boldsymbol{\xi}) | \boldsymbol{\xi})_{\alpha, \lambda} \geq (\beta \|\mathbf{u}_m - \alpha \Delta \mathbf{u}_m\|_{\mathbf{L}^2(\Omega)} - \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}) \|\mathbf{u}_m - \alpha \Delta \mathbf{u}_m\|_{\mathbf{L}^2(\Omega)}$$

where  $\beta = \frac{\nu \alpha}{(C_\Omega + 2\alpha)^2}$ .

Hence we finally have  $(\mathcal{P}_m(\boldsymbol{\xi})|\boldsymbol{\xi})_{\alpha,\lambda} > 0$  for all  $\boldsymbol{\xi}$  such that  $\|\boldsymbol{\xi}\|_{\mathbb{R}_\alpha^m} = \rho$ , where we choose  $\rho$  such that

$$\rho \geq \frac{\|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}}{\beta \min(1, \lambda_1)},$$

because

$$\begin{aligned} \|\mathbf{u}_m - \alpha \Delta \mathbf{u}_m\|_{\mathbf{L}^2(\Omega)}^2 &= \sum_{i=1}^m \xi_i^2 + 2\alpha \sum_{i=1}^m \xi_i^2 \lambda_i + \alpha^2 \sum_{i=1}^m \xi_i^2 \lambda_i^2 \\ &\geq \min(1, \lambda_1) \|\boldsymbol{\xi}\|_{\mathbb{R}_\alpha^m}^2, \end{aligned}$$

where  $\lambda_1$  is the smallest eigenvalue given by Proposition 4.2.1. We conclude, thanks to Theorem 3.1.2, that there exist  $\bar{\boldsymbol{\xi}} \in \mathbb{R}^m$  such that

$$P_m(\bar{\boldsymbol{\xi}}) = 0_{\mathbb{R}^m},$$

that is to say that  $\mathbf{u}_m$  is a solution of system  $(\mathcal{S})_m$ , means that  $\mathbf{u}_m \in \mathbf{V}_{2m}$  satisfies

$$\nu \int_{\Omega} \mathbf{curl} \mathbf{u}_m \cdot \mathbf{curl} \mathbf{v} + b(\mathbf{u}_m, \mathbf{v}, \mathbf{u}_m - \alpha \Delta \mathbf{u}_m) = \int_{\Omega} \mathbf{f} \cdot \mathbf{v}, \quad \forall \mathbf{v} \in \mathbf{V}_{2m}. \quad (4.2.7)$$

## Step 2: ESTIMATES

By multiplying each equation "j" of  $(\mathcal{S})_m$  by  $\xi_j$  and also by  $\alpha \lambda_j \xi_j$  and summing the equations obtained, we find:

$$\begin{aligned} &\nu \int_{\Omega} \mathbf{curl} \mathbf{u}_m \cdot (\xi_j \mathbf{curl} \mathbf{w}_j) dx + \nu \int_{\Omega} \mathbf{curl} \mathbf{u}_m \cdot (\alpha \lambda_j \xi_j \mathbf{curl} \mathbf{w}_j) dx + \\ &+ b(\mathbf{u}_m, \xi_j \mathbf{w}_j, \mathbf{u}_m - \alpha \Delta \mathbf{u}_m) + b(\mathbf{u}_m, \alpha \lambda_j \xi_j \mathbf{w}_j, \mathbf{u}_m - \alpha \Delta \mathbf{u}_m) \end{aligned}$$

$$= \int_{\Omega} \mathbf{f} \cdot (\xi_j \mathbf{w}_j) dx + \int_{\Omega} \mathbf{f} \cdot (\alpha \lambda_j \xi_j \mathbf{w}_j) dx.$$

Now summing over  $j$ , after integrating by parts, using the Green formula and the fact that  $-\Delta \mathbf{w}_j = \lambda_j \mathbf{w}_j$  we get:

$$\nu \|\mathbf{curl} \mathbf{u}_m\|_{\mathbf{L}^2(\Omega)}^2 + \nu \alpha \|\Delta \mathbf{u}_m\|_{\mathbf{L}^2(\Omega)}^2 = \int_{\Omega} \mathbf{f} \cdot (\mathbf{u}_m - \alpha \Delta \mathbf{u}_m)$$

by the property (4.2.6) which is valid because  $\mathbf{w}_j \in \mathbf{H}^3(\Omega)$  (we recall that  $\Omega$  of class  $\mathcal{C}^{3,1}$ ). We deduce the following estimate:

$$\begin{aligned} \nu \alpha \|\Delta \mathbf{u}_m\|_{\mathbf{L}^2(\Omega)}^2 &\leq \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} \|\mathbf{u}_m - \alpha \Delta \mathbf{u}_m\|_{\mathbf{L}^2(\Omega)} \\ &\leq C(\Omega, \alpha) \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} \|\Delta \mathbf{u}_m\|_{\mathbf{L}^2(\Omega)}. \end{aligned} \quad (4.2.8)$$

From Proposition 4.1.6 point i), we get

$$\|\mathbf{u}_m\|_{\mathbf{H}^2(\Omega)} \leq C(\Omega, \alpha, \nu) \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}. \quad (4.2.9)$$

We used first the property (4.1.7). Therefore, this implies that  $\mathbf{u}_m$  is bounded in  $\mathbf{V}_2$ . Then there exists  $\mathbf{u} \in \mathbf{V}_2$  and a subsequence  $(\mathbf{u}_k)_k$  of  $(\mathbf{u}_m)_m$  in  $\mathbf{V}_2$  such that

$$\mathbf{u}_k \rightharpoonup \mathbf{u} \text{ in } \mathbf{V}_2 \text{ weakly} \quad (4.2.10)$$

and

$$\mathbf{u}_k \longrightarrow \mathbf{u} \text{ in } \mathbf{H}^1(\Omega) \text{ strongly,} \quad (4.2.11)$$

this implies that

$$\mathbf{u}_{k_i}(\mathbf{u}_{k_j} - \alpha \Delta \mathbf{u}_{k_j}) \rightharpoonup \mathbf{u}_i(\mathbf{u}_j - \alpha \Delta \mathbf{u}_j) \text{ in } \mathbf{L}^2(\Omega) \text{ weakly.}$$

**Step 3: PASSAGE TO THE LIMIT**

The above convergences are sufficient to pass to the limit in the (4.2.7). Indeed,

let  $\mathbf{v} = \sum_{j=1}^{m_0} \xi_{jm} \mathbf{w}_j \in \mathbf{V}_{2m_0}$ , where  $m_0$  is fixed and  $m \rightarrow \infty$  :

$$\nu \int_{\Omega} \mathbf{curl} \mathbf{u} \cdot \mathbf{curl} \mathbf{v} + b(\mathbf{u}, \mathbf{v}, \mathbf{u} - \alpha \Delta \mathbf{u}) = \int_{\Omega} \mathbf{f} \cdot \mathbf{v}, \quad \forall \mathbf{v} \in \mathbf{V}_{2m_0}.$$

This relation is valid for any  $m_0$ , then

$$\nu \int_{\Omega} \mathbf{curl} \mathbf{u} \cdot \mathbf{curl} \mathbf{v} + b(\mathbf{u}, \mathbf{v}, \mathbf{u} - \alpha \Delta \mathbf{u}) = \int_{\Omega} \mathbf{f} \cdot \mathbf{v}, \quad \forall \mathbf{v} \in \mathbf{V}_{2m}, \forall m \in \mathbb{N}^*.$$

Let  $\mathbf{v} \in \mathbf{V}_2$ , there exists  $\mathbf{v}_m \in \mathbf{V}_{2m}$  such that  $\mathbf{v}_m \rightarrow \mathbf{v} \in \mathbf{V}_2$ . Since

$$\nu \int_{\Omega} \mathbf{curl} \mathbf{u} \cdot \mathbf{curl} \mathbf{v}_m + b(\mathbf{u}, \mathbf{v}_m, \mathbf{u} - \alpha \Delta \mathbf{u}) = \int_{\Omega} \mathbf{f} \cdot \mathbf{v}_m,$$

we pass to the limit and we finally get,

$$\nu \int_{\Omega} \mathbf{curl} \mathbf{u} \cdot \mathbf{curl} \mathbf{v} + b(\mathbf{u}, \mathbf{v}, \mathbf{u} - \alpha \Delta \mathbf{u}) = \int_{\Omega} \mathbf{f} \cdot \mathbf{v}, \quad \forall \mathbf{v} \in \mathbf{V}_2.$$

The existence of the pressure  $\pi$  is given, as in the Navier-Stokes problem in [35], by De Rham's Theorem.

### 4.3 Evolution problem

Let  $\Omega$  be a simply connected bounded domain of  $\mathbb{R}^3$  of class  $\mathcal{C}^{2,1}$  and  $T$  is a positive finite number. Problem (4.0.1) is studied in Oskolkov [28] with Dirichlet boundary conditions, not with Navier slip boundary conditions which are used in this chapter. Mikhail in [27], studied a mathematical model of the motion of low-concentrated aqueous polymer solutions, which is considered for the case of a turbulent behavior or unsteady laminar behavior of the fluid, with Dirichlet boundary conditions. We consider here, Navier-type boundary conditions for the velocity

as the following:

$$\mathbf{u} \cdot \mathbf{n} = 0, \quad \operatorname{curl} \mathbf{u} \times \mathbf{n} = \mathbf{0} \text{ on } \Sigma_T.$$

### 4.3.1 Statement of the problem

In this subsection, we search for the existence of solutions for the following problem:

$$(\mathcal{E}) \quad \begin{cases} \frac{\partial}{\partial t}(\mathbf{u}(t) - \alpha \Delta \mathbf{u}(t)) - \nu \Delta \mathbf{u}(t) + (\mathbf{u}(t) \cdot \nabla)(\mathbf{u}(t) - \alpha \Delta \mathbf{u}(t)) + \nabla \pi = \mathbf{f}(t), & \text{in } Q_T, \\ \operatorname{div} \mathbf{u} = 0, & \text{in } Q_T, \\ \mathbf{u} \cdot \mathbf{n} = 0, \quad \operatorname{curl} \mathbf{u} \times \mathbf{n} = \mathbf{0} & \text{on } \Sigma_T, \\ \mathbf{u}(0) = \mathbf{u}_0, & \text{in } \Omega, \end{cases} \quad (4.3.1)$$

To the extent that time dependence is taken into account, for any number  $T > 0$  and any  $r$  with  $1 \leq r \leq \infty$ , space is defined:

$$\mathbf{L}^r(0, T; \mathbf{X}) = \{\mathbf{v} : ]0, T[ \mapsto \mathbf{X}; \mathbf{v} \text{ is measurable and } \left( \int_0^T \|\mathbf{v}(t)\|_{\mathbf{X}}^r dt \right)^{1/r} < \infty\},$$

equipped with the norm

$$\|\mathbf{v}\|_{\mathbf{L}^r(0, T; \mathbf{X})} = \left( \int_0^T \|\mathbf{v}(t)\|_{\mathbf{X}}^r dt \right)^{1/r}.$$

In order to set the variational problem, we need the following definition.

**Definition 4.3.1.** *Weak solution for Problem (E):*

Suppose that  $\mathbf{f} \in \mathbf{L}^2([0, T]; \mathbf{L}^2(\Omega))$  and  $\mathbf{u}_0 \in \mathbf{V}_2$ .

A function  $\mathbf{u}(t)$  is called weak solution for Problem (E) if

$$\mathbf{u} \in \mathbf{L}^\infty([0, T]; \mathbf{V}_2) \quad \text{and} \quad \frac{\partial \mathbf{u}}{\partial t} \in \mathbf{L}^\infty([0, T]; \mathbf{V}_1) \quad (4.3.2)$$

and for any  $\mathbf{v} \in \mathbf{V}_1$  and almost all  $t \in (0, T)$ ,

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} \mathbf{u} \cdot \mathbf{v} + \alpha \operatorname{curl} \mathbf{u} \cdot \operatorname{curl} \mathbf{v} \, dx + \nu \int_{\Omega} \operatorname{curl} \mathbf{u} \cdot \operatorname{curl} \mathbf{v} \, dx \\ + b(\mathbf{u}, \mathbf{v}, \mathbf{u} - \alpha \Delta \mathbf{u}) = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, dx, \end{aligned} \quad (4.3.3)$$

with the initial condition

$$\mathbf{u}(0) = \mathbf{u}_0 \quad \text{in } \Omega.$$

**Lemma 4.3.2. A priori Estimates**

We suppose that  $\mathbf{u} \in C((0, T); \mathbf{V}_2)$  is a solution for Problem  $(\mathcal{E})$  with

$$\frac{\partial \mathbf{u}}{\partial t} \in \mathbf{L}^\infty([0, T]; \mathbf{V}_1).$$

Then  $\mathbf{u}$  satisfies the following a priori estimates

i)

$$\begin{aligned} \|\mathbf{u}(t) - \alpha \Delta \mathbf{u}(t)\|_{\mathbf{L}^2(\Omega)}^2 \leq e^{-\frac{\nu}{\sigma(\Omega) + 2\alpha} t} \|\mathbf{u}_0 - \alpha \Delta \mathbf{u}_0\|_{\mathbf{L}^2(\Omega)}^2 \\ + \frac{C(\Omega) + 2\alpha}{\nu} \int_0^t e^{-\frac{\nu}{\sigma(\Omega) + 2\alpha} (t-s)} \|\mathbf{f}(s)\|_{\mathbf{L}^2(\Omega)}^2 \, ds. \end{aligned}$$

ii) Moreover,

$$\mathbf{u}(t) - \alpha \Delta \mathbf{u}(t) \in \mathbf{V}_1 \quad \text{for all } t,$$

and

$$\frac{d}{dt} \|\mathbf{u}(t) - \alpha \Delta \mathbf{u}(t)\|_{\mathbf{L}^2(\Omega)} \leq \|\mathbf{f}(t)\|_{\mathbf{L}^2(\Omega)} \quad (4.3.4)$$

*Proof.* i) Taking  $\mathbf{v} = \mathbf{u} - \alpha \Delta \mathbf{u}$  in (4.3.3), we get

$$\frac{1}{2} \frac{d}{dt} \|\mathbf{u}(t) - \alpha \Delta \mathbf{u}(t)\|_{\mathbf{L}^2(\Omega)}^2 + \nu \int_{\Omega} |\operatorname{curl} \mathbf{u}(t)|^2 + \nu \alpha \int_{\Omega} |\Delta \mathbf{u}(t)|^2 = \int_{\Omega} \mathbf{f}(t) \cdot (\mathbf{u}(t) - \alpha \Delta \mathbf{u}(t)). \quad (4.3.5)$$

We use the Young inequality to RHS of (4.3.5) and we get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\mathbf{u}(t) - \alpha \Delta \mathbf{u}(t)\|_{\mathbf{L}^2(\Omega)}^2 + \nu \|\mathbf{curl} \mathbf{u}(t)\|_{\mathbf{L}^2(\Omega)}^2 + \nu \alpha \|\Delta \mathbf{u}(t)\|_{\mathbf{L}^2(\Omega)}^2 \\ & \leq \frac{1}{2\varepsilon} \|\mathbf{f}(t)\|_{\mathbf{L}^2(\Omega)}^2 + \frac{\varepsilon}{2} \|\mathbf{u}(t) - \alpha \Delta \mathbf{u}(t)\|_{\mathbf{L}^2(\Omega)}^2. \end{aligned}$$

By Lemma 4.1.12, we get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\mathbf{u}(t) - \alpha \Delta \mathbf{u}(t)\|_{\mathbf{L}^2(\Omega)}^2 + \frac{\nu}{C(\Omega) + 2\alpha} \|\mathbf{u}(t) - \alpha \Delta \mathbf{u}(t)\|_{\mathbf{L}^2(\Omega)}^2 \\ & \leq \frac{1}{2\varepsilon} \|\mathbf{f}(t)\|_{\mathbf{L}^2(\Omega)}^2 + \frac{\varepsilon}{2} \|\mathbf{u}(t) - \alpha \Delta \mathbf{u}(t)\|_{\mathbf{L}^2(\Omega)}^2, \end{aligned}$$

choosing  $\varepsilon$  such that

$$\varepsilon = \frac{\nu}{C(\Omega) + 2\alpha}$$

we get that,

$$\frac{d}{dt} \|\mathbf{u}(t) - \alpha \Delta \mathbf{u}(t)\|_{\mathbf{L}^2(\Omega)}^2 + \frac{\nu}{C(\Omega) + 2\alpha} \|\mathbf{u}(t) - \alpha \Delta \mathbf{u}(t)\|_{\mathbf{L}^2(\Omega)}^2 \leq \frac{C(\Omega) + 2\alpha}{\nu} \|\mathbf{f}(t)\|_{\mathbf{L}^2(\Omega)}^2.$$

We then multiply by  $e^{\frac{\nu}{C(\Omega)+2\alpha}t}$  the last inequality and we integrate over  $(0, T)$ ,

$$\begin{aligned} \|\mathbf{u}(t) - \alpha \Delta \mathbf{u}(t)\|_{\mathbf{L}^2(\Omega)}^2 & \leq e^{-\frac{\nu}{C(\Omega)+2\alpha}t} \|\mathbf{u}_0 - \alpha \Delta \mathbf{u}_0\|_{\mathbf{L}^2(\Omega)}^2 \\ & \quad + \frac{C(\Omega) + 2\alpha}{\nu} \int_0^t e^{-\frac{\nu}{C(\Omega)+2\alpha}(t-s)} \|\mathbf{f}(s)\|_{\mathbf{L}^2(\Omega)}^2 ds. \end{aligned}$$

ii) From (4.3.5), we have

$$\|\mathbf{u}(t) - \alpha \Delta \mathbf{u}(t)\|_{\mathbf{L}^2(\Omega)} \frac{d}{dt} \|\mathbf{u}(t) - \alpha \Delta \mathbf{u}(t)\|_{\mathbf{L}^2(\Omega)} \leq \|\mathbf{f}(t)\|_{\mathbf{L}^2(\Omega)} \|\mathbf{u}(t) - \alpha \Delta \mathbf{u}(t)\|_{\mathbf{L}^2(\Omega)}$$

then we get (4.3.4).



### 4.3.2 Existence of solutions

**Theorem 4.3.3.** *Suppose that  $\Omega$  is of class  $\mathcal{C}^{3,1}$ . For given  $\mathbf{f} \in \mathbf{L}^2([0, T]; \mathbf{L}^2(\Omega))$  and  $\mathbf{u}_0 \in \mathbf{V}_2$ , there exist a solution  $(\mathbf{u}, \pi)$  for Problem  $(\mathcal{E})$  in the sense of the definition 4.3.1 .*

*Proof.* This proof is divided into three steps. First, we use the Galerkin method for the approximate problem, we show the existence of solutions with the fixed point Brouwer Theorem (see Theorem 3.1.2). Then, we show some estimates for the approximate solutions and finally, we pass to the limit using some Sobolev injections.

#### Step 1: GALERKIN APPROXIMATION

We consider the following sequence of evolution approximate problems in finite dimension:

For  $m = 1, 2, \dots$  find  $\mathbf{u}_m(t) = \sum_{i=1}^m g_i(t) \boldsymbol{\chi}_i$ ,

$$(\mathcal{E}_m) \begin{cases} \int_{\Omega} \frac{\partial}{\partial t} (\mathbf{u}_m(t) - \alpha \Delta \mathbf{u}_m(t)) \cdot \boldsymbol{\chi}_j - \nu \int_{\Omega} \Delta \mathbf{u}_m(t) \cdot \boldsymbol{\chi}_j \\ + b(\mathbf{u}_m(t), \boldsymbol{\chi}_j, \mathbf{u}_m(t) - \alpha \Delta \mathbf{u}_m(t)) = \int_{\Omega} \mathbf{f}(t) \cdot \boldsymbol{\chi}_j & \text{for } j = 1, \dots, J \\ \mathbf{u}_m(0) = \mathbf{u}_{0m} \end{cases} \quad (4.3.6)$$

where  $g_i$  are real functions defined on  $[0, T]$  and  $\mathbf{u}_{0m}$  is the projection of  $\mathbf{u}_0$  on  $\langle \boldsymbol{\chi}_1, \boldsymbol{\chi}_2, \dots, \boldsymbol{\chi}_m \rangle$ , which is the space spanned by  $(\boldsymbol{\chi}_i)_i$  solutions in  $\mathbf{V}_2$  for the following eigenvalues problem:

$$\begin{cases} \boldsymbol{\chi}_i - \alpha \Delta \boldsymbol{\chi}_i = \lambda_i \boldsymbol{\chi}_i & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{\chi}_i = 0 & \text{in } \Omega, \\ \boldsymbol{\chi}_i \cdot \mathbf{n} = 0, \quad \operatorname{curl} \boldsymbol{\chi}_i \times \mathbf{n} = \mathbf{0} & \text{on } \Gamma. \end{cases} \quad (4.3.7)$$

The existence of the sequence  $(\boldsymbol{\chi}_i)_i$  is showed exactly as the existence of the set  $(\mathbf{w}_i)_i$

in Theorem 1.1,  $\boldsymbol{\chi}_i$  are the eigenvectors of the operator

$$\begin{aligned} \mathcal{T}_2 : V_0 &\rightarrow V_2; \\ \mathbf{h} &\mapsto \boldsymbol{\chi} \end{aligned}$$

solution of the following problem:

$$\begin{cases} \boldsymbol{\chi} - \alpha \Delta \boldsymbol{\chi} = \mathbf{h} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{\chi} = 0 & \text{in } \Omega, \\ \boldsymbol{\chi} \cdot \mathbf{n} = 0, \operatorname{curl} \boldsymbol{\chi} \times \mathbf{n} = \mathbf{0} & \text{on } \Gamma. \end{cases}$$

The system  $(\mathcal{E}_m)$  can be written as the following:

$$\begin{cases} \sum_{i=1}^m g'_{im}(t) \int_{\Omega} \boldsymbol{\chi}_i \cdot \boldsymbol{\chi}_j + \alpha \sum_{i=1}^m g'_{im}(t) \int_{\Omega} \operatorname{curl} \boldsymbol{\chi}_i \cdot \operatorname{curl} \boldsymbol{\chi}_j + \nu \sum_{i=1}^m g_{im}(t) \int_{\Omega} \operatorname{curl} \boldsymbol{\chi}_i \cdot \operatorname{curl} \boldsymbol{\chi}_j \\ + \sum_{i,k=1}^m g_{im}(t) g_k(t) b(\boldsymbol{\chi}_i, \boldsymbol{\chi}_j, \boldsymbol{\chi}_k - \alpha \Delta \boldsymbol{\chi}_k) = \int_{\Omega} \mathbf{f}(t) \cdot \boldsymbol{\chi}_j & \text{for } j = 1, \dots, m \end{cases}$$

where  $g_{0i}$  are components of the projection of  $\mathbf{u}_0$  on  $\langle \boldsymbol{\chi}_1, \boldsymbol{\chi}_2, \dots, \boldsymbol{\chi}_m \rangle \subset V_2$ .

As in Temam's book [35], inverting the non singular matrix with elements  $\int_{\Omega} \boldsymbol{\chi}_i \cdot \boldsymbol{\chi}_j + \alpha \int_{\Omega} \operatorname{curl} \boldsymbol{\chi}_i \cdot \operatorname{curl} \boldsymbol{\chi}_j$  for  $1 \leq i, j \leq m$  one can write the differential equation the usual form

$$\left\{ g'_{im}(t) + \sum_{j=1}^m \alpha_{ij} g_{jm}(t) + \sum_{j,k=1}^m \alpha_{ijk} g_{jm}(t) g_{km}(t) = \sum_{j=1}^m \beta_{ij} \int_{\Omega} \mathbf{f}(t) \cdot \boldsymbol{\chi}_j, \quad (4.3.8) \right.$$

where  $\alpha_{ij}$ ,  $\alpha_{ijk}$  and  $\beta_{ij}$  are in  $\mathbb{R}$  and

$$\mathbf{g}_{im}(0) \text{ is the } i^{\text{th}} \text{ component of } \mathbf{u}_{0m}. \quad (4.3.9)$$

The nonlinear differential system (4.3.8) – (4.3.9) has a maximal solution on some

interval  $[0, T_m]$ . If  $T_m < T$  then

$$\|\mathbf{u}_m(t)\|_{\mathbf{H}^1(\Omega)} \rightarrow \infty \text{ as } t \rightarrow T_m, \quad (4.3.10)$$

but we will see that it is not the case, so  $T_m = T$ .

**Step 2: ESTIMATES**

First of all, we note that if  $\Omega$  is of class  $\mathcal{C}^{3,1}$ , the solutions  $\chi_i$  will be in  $\mathbf{H}^3(\Omega)$ , which gives  $\mathbf{u}_m(t)$  in  $\mathbf{H}^3(\Omega)$  and we get :

$$b(\mathbf{u}_m(t), \mathbf{u}_m(t) - \alpha\Delta\mathbf{u}_m(t), \mathbf{u}_m(t) - \alpha\Delta\mathbf{u}_m(t)) = 0. \quad (4.3.11)$$

We multiply each equation “ $j$ ” of  $(\mathcal{E}_m)$  by  $g_j(t)$  and then by  $\lambda_j g_j(t)$  and add these equations for  $1 \leq j \leq m$  we get:

$$\frac{1}{2} \frac{d}{dt} \|\mathbf{u}_m - \alpha\Delta\mathbf{u}_m\|_{\mathbf{L}^2(\Omega)}^2 - \nu \int_{\Omega} \Delta\mathbf{u}_m \cdot (\mathbf{u}_m - \alpha\Delta\mathbf{u}_m) = \int_{\Omega} \mathbf{f}(t) \cdot (\mathbf{u}_m - \alpha\Delta\mathbf{u}_m),$$

we get by integration by parts:

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\mathbf{u}_m - \alpha\Delta\mathbf{u}_m\|_{\mathbf{L}^2(\Omega)}^2 + \nu \|\mathbf{curl} \mathbf{u}_m\|_{\mathbf{L}^2(\Omega)}^2 + \nu\alpha \|\Delta\mathbf{u}_m\|_{\mathbf{L}^2(\Omega)}^2 \\ & = \int_{\Omega} \mathbf{f}(t) \cdot (\mathbf{u}_m - \alpha\Delta\mathbf{u}_m), \end{aligned}$$

using lemma 4.3.2 we get finally,

$$\begin{aligned} & \frac{1}{2} \|\mathbf{u}_m(t) - \alpha\Delta\mathbf{u}_m(t)\|_{\mathbf{L}^2(\Omega)}^2 + \frac{\nu}{2} \int_0^t \|\mathbf{curl} \mathbf{u}_m\|_{\mathbf{L}^2(\Omega)}^2 dt + \frac{\nu\alpha}{2} \int_0^t \|\Delta\mathbf{u}_m(t)\|_{\mathbf{L}^2(\Omega)}^2 dt \\ & \leq C \int_0^t \|\mathbf{f}(t)\|_{\mathbf{L}^2(\Omega)}^2 dt + \frac{1}{2} \|\mathbf{u}_0 - \alpha\Delta\mathbf{u}_0\|_{\mathbf{L}^2(\Omega)}^2. \end{aligned} \quad (4.3.12)$$

This implies that

$$(\mathbf{u}_m)_m \text{ is bounded in } \mathbf{L}^\infty(0, T; \mathbf{H}^2(\Omega)). \quad (4.3.13)$$

To pass to the limit in  $(\mathcal{E}_m)$  we need a strong convergence in  $\mathbf{L}^2(Q)$ , for this we will search an estimation for  $\frac{\partial \mathbf{u}_m}{\partial t}$  that we denote by  $\mathbf{u}'_m$ .

We multiply each equation “ $j$ ” of  $(\mathcal{E}_m)$  by  $g'_j(t)$  and we sum over “ $j$ ”,

$$\begin{aligned} & \|\mathbf{u}'_m(t)\|_{\mathbf{L}^2(\Omega)}^2 + \alpha \|\mathbf{curl} \mathbf{u}'_m(t)\|_{\mathbf{L}^2(\Omega)}^2 + \nu \int_{\Omega} \mathbf{curl} \mathbf{u}_m(t) \cdot \mathbf{curl} \mathbf{u}'_m(t) \\ & + b(\mathbf{u}_m(t), \mathbf{u}'_m(t), \mathbf{u}_m(t) - \alpha \Delta \mathbf{u}_m(t)) = \int_{\Omega} \mathbf{f}(t) \cdot \mathbf{u}'_m(t) . \end{aligned}$$

Then,

$$\begin{aligned} & \|\mathbf{u}'_m(t)\|_{\mathbf{L}^2(\Omega)}^2 + \alpha \|\mathbf{curl} \mathbf{u}'_m(t)\|_{\mathbf{L}^2(\Omega)}^2 \\ & \leq \nu \|\mathbf{curl} \mathbf{u}_m(t)\|_{\mathbf{L}^2(\Omega)} \cdot \|\mathbf{curl} \mathbf{u}'_m(t)\|_{\mathbf{L}^2(\Omega)} \\ & + C \|\mathbf{u}_m(t)\|_{\mathbf{L}^\infty} \cdot \|\mathbf{curl} \mathbf{u}'_m(t)\|_{\mathbf{L}^2(\Omega)} \cdot \|\mathbf{u}_m(t) - \alpha \Delta \mathbf{u}_m(t)\|_{\mathbf{L}^2(\Omega)} \\ & + \|\mathbf{f}(t)\|_{\mathbf{L}^2(\Omega)} \cdot \|\mathbf{u}'_m(t)\|_{\mathbf{L}^2(\Omega)}, \end{aligned} \tag{4.3.14}$$

we deduce:

$$\begin{aligned} \|\mathbf{u}'_m(t)\|_{\mathbf{H}^1(\Omega)}^2 & \leq \frac{1}{4} \|\mathbf{u}'_m(t)\|_{\mathbf{H}^1(\Omega)}^2 + C \|\mathbf{u}_m(t)\|_{\mathbf{H}^1(\Omega)}^2 + \frac{1}{4} \|\mathbf{u}'_m(t)\|_{\mathbf{H}^1(\Omega)}^2 \\ & + C \|\mathbf{u}_m(t)\|_{\mathbf{L}^\infty}^2 \|\mathbf{u}_m(t) - \alpha \Delta \mathbf{u}_m(t)\|_{\mathbf{L}^2(\Omega)}^2 \\ & + \frac{1}{4} \|\mathbf{u}'_m(t)\|_{\mathbf{H}^1(\Omega)}^2 + C \|\mathbf{f}(t)\|_{\mathbf{L}^2(\Omega)}^2. \end{aligned} \tag{4.3.15}$$

Therefore:

$$\begin{aligned} \frac{1}{4} \|\mathbf{u}'_m(t)\|_{\mathbf{H}^1(\Omega)}^2 & \leq C \|\mathbf{u}_m(t)\|_{\mathbf{H}^1(\Omega)}^2 + C \|\mathbf{u}_m(t)\|_{\mathbf{L}^\infty}^2 \|\mathbf{u}_m(t) - \alpha \Delta \mathbf{u}_m(t)\|_{\mathbf{L}^2(\Omega)}^2 \\ & + C \|\mathbf{f}(t)\|_{\mathbf{L}^2(\Omega)}^2. \end{aligned} \tag{4.3.16}$$

This estimation gives, with (4.3.13),

$$(\mathbf{u}'_m)_m \text{ is bounded in } \mathbf{L}^2(0, T; \mathbf{H}^1(\Omega)), \tag{4.3.17}$$

from (4.3.13) and (4.3.17) we get that  $\mathbf{u}_m$  is bounded in the following space,

$$\Upsilon = \{\mathbf{v} \in \mathbf{L}^\infty(0, T; \mathbf{H}^2(\Omega)), \frac{\partial \mathbf{v}}{\partial t} \in \mathbf{L}^2(0, T; \mathbf{H}^1(\Omega))\}$$

which is compact in  $\mathbf{L}^2(0, T; \mathbf{H}^1(\Omega))$ . Then we can find from the sequence  $\{\mathbf{u}_m\}$  a subsequence  $\{\mathbf{u}_{m'}\}$  such that,

$$\mathbf{u}_{m'} \longrightarrow \mathbf{u} \quad \text{in } \mathbf{L}^2(0, T; \mathbf{H}^1(\Omega)) \quad \text{strongly.} \quad (4.3.18)$$

### Step 3: PASSAGE TO THE LIMIT

In order to pass to the limit in Problem  $(\mathcal{E}_m)$ , we consider a scalar function  $\psi \in \mathcal{D}([0, T])$  such that  $\psi(T) = 0$ .

We multiply the first equation in  $(\mathcal{E}_m)$  by  $\psi(t)$ , integrate in  $t$  and integrate by parts:

$$\begin{aligned} & \int_0^T \int_\Omega [\mathbf{u}'_m(t) \cdot \mathbf{v} + \alpha \mathbf{curl} \mathbf{u}'_m(t) \cdot \mathbf{curl} \mathbf{v}] \psi(t) dt \\ & + \nu \int_0^T \int_\Omega \mathbf{curl} \mathbf{u}_m(t) \cdot \mathbf{curl} \mathbf{v} \psi(t) dt \\ & + \int_0^T b(\mathbf{u}_m(t), \mathbf{v}, \mathbf{u}_m(t) - \alpha \Delta \mathbf{u}_m(t)) \psi(t) dt \\ & = \int_0^T \int_\Omega \mathbf{f}(t) \cdot \mathbf{v} \psi(t) dt. \end{aligned}$$

The properties (4.3.13) and (4.3.18) give that  $\mathbf{u}$  is solution of

$$\begin{aligned}
& \int_0^T \int_{\Omega} (\mathbf{u}(t) \cdot \mathbf{v} + \alpha \mathbf{curl} \mathbf{u}(t) \cdot \mathbf{curl} \mathbf{v}) \psi'(t) dt \\
& + \nu \int_0^T \int_{\Omega} \mathbf{curl} \mathbf{u}(t) \cdot \psi(t) \mathbf{curl} \mathbf{v} dt + \int_0^T b(\mathbf{u}(t), \mathbf{v} \psi(t), \mathbf{u}(t) - \alpha \Delta \mathbf{u}(t)) dt \\
& = \int_{\Omega} (\mathbf{u}_0 \mathbf{v} + \alpha \mathbf{curl} \mathbf{u}_0 \mathbf{curl} \mathbf{v}) \psi(0) + \int_0^T \int_{\Omega} \mathbf{f}(t) \cdot \psi(t) \mathbf{v} dt. \tag{4.3.19}
\end{aligned}$$

Or if we take  $\psi = \varphi \in \mathcal{D}(]0, T[)$  in (4.3.19), we find that  $\mathbf{u}$  verifies:

$$\begin{aligned}
& \int_{\Omega} \frac{d}{dt} (\mathbf{u}(t) - \alpha \Delta \mathbf{u}(t)) \cdot \mathbf{v} + \nu \int_{\Omega} \Delta \mathbf{u}(t) \cdot \mathbf{v} + b(\mathbf{u}(t), \mathbf{v}, \mathbf{u}(t) - \alpha \Delta \mathbf{u}(t)) \\
& = \int_{\Omega} \mathbf{f}(t) \cdot \mathbf{v}, \tag{4.3.20}
\end{aligned}$$

in the distribution sense, for each  $\mathbf{v} \in \mathbf{V}_2$  and for all  $\mathbf{v} \in \mathcal{D}(\Omega)$  with  $\operatorname{div} \mathbf{v} = 0$ .

Now, to check that  $\mathbf{u}(0) = \mathbf{u}_0$  we proceed as in [35]. We multiply the equation (4.3.20) by  $\psi \in \mathcal{C}^1([0, T])$  with  $\psi(T) = 0$  and integrate by parts:

$$\begin{aligned}
& \int_0^T \int_{\Omega} \mathbf{u}(t) \cdot \mathbf{v} \psi'(t) dt + \alpha \mathbf{curl} \mathbf{u}(t) \cdot \mathbf{curl} \mathbf{v} \psi'(t) dt \\
& - \nu \int_0^T \int_{\Omega} \mathbf{curl} \mathbf{u}(t) \cdot \psi(t) \mathbf{curl} \mathbf{v} dt + \int_0^T b(\mathbf{u}(t), \mathbf{v}, \psi(t)(\mathbf{u}(t) - \alpha \Delta \mathbf{u}(t))) dt \\
& = \int_0^T \int_{\Omega} \mathbf{f}(t) \psi(t) \mathbf{v} dt + \int_{\Omega} \mathbf{u}(0) \mathbf{v} \psi(0) + \alpha \int_{\Omega} \mathbf{curl} \mathbf{u}(0) \cdot \mathbf{v} \psi(0), \tag{4.3.21}
\end{aligned}$$

which implies that

$$\left[ \int_{\Omega} (\mathbf{u}(0) - \mathbf{u}_0) \cdot \mathbf{v} + \alpha \int_{\Omega} \mathbf{curl}(\mathbf{u}(0) - \mathbf{u}_0) \cdot \mathbf{curl} \mathbf{v} \right] \psi(0) = 0$$

for all  $\mathbf{v} \in \mathbf{V}_1$  with  $\psi(0) \neq 0$ , and  $\mathbf{u}(0) - \mathbf{u}_0 \in \mathbf{H}^1(\Omega)$ , then  $(\mathbf{u}(0) - \mathbf{u}_0, \mathbf{v})_{\mathbf{V}_1} = 0$

for any  $\mathbf{v} \in \mathbf{V}_1$  and

$$\mathbf{u}(0) = \mathbf{u}_0.$$

For the existence of the pressure  $\pi$ , we can proceed by De Rham's Theorem as in Temam [35]. This completes the proof of Theorem 1.2.

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