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Adaptative observers for wave equations and associated discretizations: formulations and analysis.

Tiphaine Delaunay

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Thèse de doctorat



Adaptative observers for wave equations and associated discretization: formulations and analyses

Thèse de doctorat de l'Institut Polytechnique de Paris
préparée à l'École polytechnique

École doctorale n°574 École doctorale de mathématiques Hadamard (EDMH)
Spécialité de doctorat : Mathématiques appliquées

Thèse présentée et soutenue à Palaiseau, le 19 décembre 2023, par

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Muriel Boulakia Professeure, Université de Versailles Saint-Quentin	Invitée

À mon papy Michel

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Introduction (Français)

*La vie est un spectacle, autant faire sa propre mise
en scène.*
— William Shakespeare

Cette thèse a été effectuée dans l'équipe M Ξ DISIM à Inria Saclay sous la direction de Sébastien Imperiale et Philippe Moireau. Elle s'inscrit dans une thématique clé de cette équipe : les problèmes inverses. L'idée est de pouvoir reconstruire une information inconnue (conditions initiales, terme source, paramètres...) à l'aide de mesures sur le système. Nous nous intéressons aux systèmes propagatifs et plus spécifiquement à la propagation d'ondes. L'enjeu est d'établir des propriétés d'observabilité et de stabilisation en dimension infinie. Cette thèse porte principalement sur des enjeux théoriques, cependant les problèmes inverses que nous étudions possèdent beaucoup d'applications concrètes. Nous survolons quelques applications possibles pour les problèmes inverses qui mettent en jeu des phénomènes propagatifs.

Contexte applicatif

Les propagations d'ondes jouent un rôle dans de nombreux domaines, que se soit en biologie, dans les infrastructures, dans le corps humain, les systèmes de réseaux, si bien que le nombre d'applications possibles est très important. Nous présentons quelques exemples proches des thématiques de l'équipe M Ξ DISIM.

L'hémodynamique

L'hémodynamique est le phénomène biologique décrivant la propagation de sang dans le corps humain. D'un point de vue mathématique et modélisation, les équations régissant la propagation du sang peuvent être vues (dans un certain régime asymptotique) comme une propagation d'onde 1D de pression et de débit dans un réseau (par exemple dans le réseau artériel) [Lombardi, 2014; Saito et al., 2011]. Sur ce sujet, une question étudiée dans l'équipe M Ξ DISIM est la reconstruction de pression et de débit en sortie de coeur connaissant la pression ou le débit à une autre extrémité du réseau. En effet, ces informations sont généralement non disponibles et sont pourtant très informatives sur l'état de santé du patient. D'un point de vue mathématique, il s'agit d'un problème inverse de reconstruction de source pour l'équation des ondes [Imperiale et al., 2023]. Le problème inverse mentionné n'est évidemment pas le seul, on peut en citer d'autres par exemple la reconstruction de paramètres du réseau [Martin et al., 2005] notamment par le filtre de Kalman [Caiazzo et al., 2017]. À terme, les techniques de problèmes inverses peuvent même être utilisés par exemple pour reconstruire les paramètres d'un modèle du coeur couplé au réseau [Manganotti et al., 2021] ou encore en pharmacologie pour optimiser l'administration de médicaments en anesthésie.

L'élastographie ultrasonore

L'imagerie médicale est apparue au cours du 20èmes siècle. On cherche à obtenir des images du corps humain de façon non intrusive afin de pouvoir, par exemple, poser des diagnostics. Plusieurs méthodes ont été établies à cette fin : IRM, échographie, radiographie, *etc.* Derrière ces techniques se cache la résolution d'un problème inverse. On présente ici l'élastographie ([Raghavan and Yagle, 1994; Ophir et al., 2002; Parker et al., 2011]) qui est un sujet souvent étudié dans l'équipe M Ξ DISIM ([Caforio, 2019; Imperiale et al., 2020]). C'est une technique d'imagerie pour caractériser les propriétés élastiques des tissus biologiques. On peut définir les ondes élastiques à partir de l'équation d'élastodynamique :

$$\rho \partial_{tt} u + \operatorname{div}(\sigma) = f, \quad (1)$$

où u est le champ de déplacement, σ est le tenseur de contraintes, ρ représente la densité et f le terme source. La définition de σ dépend des conditions du milieu. Notamment dans

le cas isotrope, on a la loi de Hooke

$$\sigma = 2\mu\epsilon(u) + \lambda\text{tr}(\epsilon)\text{Id},$$

où λ et μ sont les coefficients de Lamé. Lorsque le milieu est homogène, les coefficients de Lamé sont constants. On peut décomposer le champ u en deux ondes : celles de compression et celles de cisaillement. Les tissus biologiques sont considérés comme quasi incompressibles du fait de la densité de l'eau qu'ils contiennent. Leur module de compressibilité est ainsi très proche de celui de l'eau. En revanche le module de cisaillement est très variable. En outre, un changement d'état pathologique entraîne un changement de raideur. La technique d'élastographie capte particulièrement les ondes de cisaillement, cette approche est donc bien adaptée pour détecter des anomalies dans les tissus biologiques.

On peut distinguer trois grandes étapes en élastographie :

1. L'excitation de la région d'intérêt par une onde dans le tissu ;
2. La mesure de cette perturbation ;
3. Et enfin la reconstruction de la propriété recherchée du tissu.

Deux grandes catégories d'imagerie d'élasticité ultrasonore (EUS) se fondent sur les équations élastiques :

- l'imagerie d'élasticité ultrasonore statique ;
- l'imagerie d'élasticité ultrasonore dynamique.

Apparue dans un premier temps comme une méthode quasi-statique [Ophir et al., 1991], l'imagerie d'élasticité ultrasonore est une méthode de déformation consistant en une compression statique. Plus précisément, cela correspond à étudier des solutions quasi-statiques qui n'entraînent pas de propagation. Dans ce cas, le terme source de (1) se réécrit $f(x, t) = \lambda(t)\theta(x)$ où $\lambda(t)$ varie très lentement et converge vers 1.

L'imagerie d'élasticité ultrasonore a par la suite été étendue en méthode dynamique [Sarvazyan et al., 2011] qui consiste alors à appliquer une force qui varie en fonction du temps (soit avec de très rapides impulsions soit avec une force oscillatoire à fréquence fixée) [Gennisson et al., 2013]. Cela crée une perturbation mécanique qui se propage sous forme d'onde élastique dans le tissu. Cette technique est largement étudiée [Rudenko et al., 1996; Sarvazyan et al., 2010; Gennisson et al., 2013; Sigrüst et al., 2017] et est utilisée par exemple en médecine pour faire des échographies.

Le contrôle non destructif

Le contrôle non destructif (CND) [Banks et al., 2000; Albanese et al., 2002; Shull, 2002; Altpeter et al., 2002], parfois appelé Essais Non destructifs (END) dans la littérature, est une méthode permettant de vérifier l'intégrité d'une structure sans l'altérer. Contrairement aux techniques présentées plus haut destinées au médical, celle-ci a plutôt une application industrielle, notamment dans le domaine aérospatial, nucléaire, ferroviaire, automobile, *etc.* Le but est de mettre en évidence des défauts (que se soit pour des questions de sécurité, de conformité *etc.*). On distingue plusieurs catégories de méthodes au sein du CND : l'examen visuel, la magnétoscopie, la thermographie, le ressuyage, les courants de Foucault, la radiographie et les ultrasons. On s'intéresse particulièrement à cette dernière catégorie, car elle repose sur une dynamique de propagation d'onde transitoire. La méthode par ultrasons (voir par exemple [Blitz and Simpson, 1995]) est une technique acoustique qui permet de mesurer l'épaisseur d'un objet ou de détecter une détérioration telle qu'une

fissure. On envoie des ondes acoustiques à travers l'objet, celles-ci rebondissent sur des matériaux où la densité et la raideur sont différentes. Les ondes envoyées peuvent être des ondes de compressions, des ondes de cisaillements ou encore des ondes guidées dont on parlera plus loin ci-dessous.

Une branche de recherche appelée contrôle de santé intégrée (*structural health monitoring* en anglais) [Farrar and Worden, 2007; Worden et al., 2007; Taupin, 2011; Mitra and Gopalakrishnan, 2016] découle des techniques de CND. Celle-ci a la particularité de détecter des défauts en temps réel et s'est développée ces trente dernières années. Elle est notamment utilisée en aéronautique [Boller and Meyendorf, 2008; Le et al., 2015], aérospatial [Staszewski et al., 2004], naval [Okasha et al., 2011; Mondoro et al., 2016], infrastructure [Chang et al., 2003; Brownjohn, 2007; Karbhari and Ansari, 2009; Cross et al., 2013; Feng and Feng, 2018], *etc.*

Les ondes guidées, des ondes ultrasonores qui se propagent dans une structure et qui sont confinées à l'intérieur de celles-ci [Gazis, 1959; Bartoli et al., 2006; Raghavan, 2007; Mesnil et al., 2019], sont utilisées dans le contrôle de santé intégré. Les systèmes de contrôles utilisant les ondes guidées peuvent avoir recours au mode actif : c'est un transducteur piézoélectrique qui envoie des excitations ponctuelles, on a alors des informations entre le transducteur et le récepteur grâce à l'onde. Le mode passif lui est utilisé pour une surveillance en temps réel, car aucune onde n'est stimulée, les transducteurs attendent un signal en cas d'endommagement. Une des spécificités des guides d'ondes, c'est qu'ils permettent la propagation d'ondes multimodales. Lorsque les ondes se propagent à travers des plaques et des coques minces délimitées par des surfaces sans contraintes on parlera d'ondes de Lamb [Lamb, 1917; Worlton, 1961; Su et al., 2006], c'est notamment le cas en aéronautique [Dalton et al., 2001]. Pour des plaques plus épaisses, les ondes étudiées sont alors des ondes de surface dénommées ondes de Rayleigh. Tous ces phénomènes peuvent être modélisés mathématiquement comme une équation d'onde (1) et sont étudiés dans l'équipe MÆDISIM [Dalmora et al., 2023].

La sismologie

Les applications en sismologie sont très proches de celles de la CND et de l'elastographie car elles s'appuient sur les mêmes équations mais à une autre échelle. En effet, ici la propagation d'ondes a lieu sur de longues distances. Les problèmes inverses au travers du développement de l'assimilation de données – un ensemble de stratégies qui combine les données observées sur le système et le modèle mathématiques qui le représente – ont beaucoup d'application en sismologie. Les ondes sismiques se propagent à l'intérieur de la Terre et permettent d'étudier la structure du sol. Les observations de ces ondes sismiques (temps de parcours, amplitudes, *etc.*) offrent des informations importantes sur diverses propriétés géologiques notamment sur la structure interne du sous-sol. La méthode *Full waveform inversion* (FWI) est très en vogue pour étudier la structure de la terre. L'idée est de résoudre de façon itérative [Tarantola and Valette, 1982; Tarantola, 1984], le problème de minimisation associé au problème inverse. On nomme cette méthode 4D-var (pour variationnelle 3D et temps). On verra notamment dans cette thèse, qu'au lieu de résoudre ce problème de manière itérative, il est possible de le faire séquentiellement. Cette méthode a beaucoup été étudiée et illustrées par des résultats numériques [Gauthier et al., 1986] avant de connaître de nombreuses extensions [Tarantola, 1987; Mora, 1987; Sun and McMechan, 1992; Crase et al., 1990; Sun, 1991; Pratt et al., 1998; Operto et al., 2004; Pratt et al., 1996]. Grâce aux progrès de l'informatique, il est maintenant possible de faire des simulations en trois dimensions de grandes précisions [Graves, 1996; Akcelik et al., 2003; Komatitsch et al., 2004] et les FWI sont largement étudiés [Komatitsch et al., 2004; Tromp et al., 2005; Symes, 2008; Jing et al., 2016; Wang et al., 2019].

On présente maintenant le contexte théorique de cette thèse théorique et notamment les piliers sur laquelle elle se fonde et les enjeux qu'elle aborde.

Contexte et enjeu théorique

Problème inverse

La résolution d'un problème inverse (voir par exemple [Baumeister, 1987; Engl et al., 1996]) consiste généralement à identifier des paramètres dans un système à l'aide d'observations accessibles sur celui-ci. On introduit un opérateur $\Psi : \mathcal{Z} \rightarrow \mathcal{Y}$. Résoudre le problème inverse peut être résumé comme ceci : "connaissant $y \in \mathcal{Y}$, trouver $z \in \mathcal{Z}$ tel que $\Psi z = y$ ". On parle donc d'un problème inverse, car résoudre ce genre de problème entraîne l'inversion de l'opérateur Ψ . Plusieurs questions découlent de ce problème. A-t-on assez d'informations dans les observations y pour résoudre ce problème ? Quelles méthodes de résolution utiliser ? De plus, les mesures sont généralement bruitées, le but devient donc de "retrouver aussi bien que possible $z \in \mathcal{Z}$ en connaissant $y^\delta \in \mathcal{Y}$ qui est une version bruitée de y ". On précisera plus bas la notion "d'assez d'informations" en abordant la définition de problème bien ou mal posé. On peut se demander quel est alors l'impact du bruit et comment quantifier la qualité de la reconstruction ? En outre, les problèmes inverses sont souvent mal posés, c'est-à-dire qu'ils ne satisfont pas une des conditions suivantes (au sens d'Hadamard) :

- existence d'une solution ;
- unicité de la solution ;
- continuité de la solution par rapport aux données.

Lorsque le problème est mal posé, il est souvent nécessaire de supposer que l'on a connaissance d'un *a priori* sur le paramètre étudié. On peut extraire trois grandes catégories dans les problèmes inverses : le problème de condition initiale, la reconstruction de source et la reconstruction de paramètre (identification). Le problème d'estimation de conditions initiales [Luenberger, 1963; Phung and Zhang, 2008; Ramdani et al., 2010; Ito et al., 2011; Haine, 2014b; Imperiale et al., 2020] est le problème linéaire le plus naturel.

Un des buts des problèmes inverses consiste en la reconstruction de paramètres. Or celle-ci est un problème non linéaire et est donc plus complexe à étudier [Jijun and Yuanming, 1997; Fu and Han, 2004; Jellali, 2006; Baudouin et al., 2017; Liao, 2011; Baudouin et al., 2013; Roman et al., 2020; Baudouin et al., 2021]. Entre ces deux catégories de problèmes inverses, se trouvent la reconstruction de source [Yamamoto, 1995, 1999; Komornik and Yamamoto, 2002; Alves et al., 2009; Moireau et al., 2008; Mukanova and Romanov, 2016]. Celle-ci est un véritable lien entre problème de conditions initiales et de paramètres. En effet, en linéarisant le problème inverse de reconstruction de paramètres, il est possible de se ramener à des problèmes linéaires de reconstruction de source. De plus, pour résoudre un problème de source, on peut s'appuyer sur les outils développés pour la reconstruction de conditions initiales. Par exemple en créant des systèmes joints état-paramètre, ou en utilisant des résultats développés pour l'estimation de condition initiale tels que des résultats d'observabilité. C'est pourquoi on se focalisera sur cette catégorie fondamentale dans cette thèse.

Théorie du contrôle

En théorie du contrôle (voir par exemple [Lions, 1968, 1988; Lasiecka and Triggiani, 2000; Trélat, 2005; Coron, 2007]), plusieurs notions sont primordiales lorsque l'on étudie des systèmes. La première est la contrôlabilité. Supposons qu'on étudie le système suivant

$$\begin{cases} \dot{z}(t) = A(t)z(t) + B(t)u(t), \\ z(0) = z_0, \end{cases}$$

où x représente l'état du système et u le contrôle. On se demande si étant donné un point z_1 , il existe un contrôle u tel que la trajectoire correspondante à ce contrôle joigne z_0 et z_1 en un temps T . Si le contrôle est indépendant de l'état du système on a alors une commande en boucle ouverte. Dans la plupart des cas, on utilisera des commandes en boucles fermées, c'est-à-dire que les contrôles s'ajustent en fonction de l'état du système. De cette branche, on peut extraire la notion de contrôle optimal, qui pose également cette question de commande de trajectoire, mais en optimisant un critère tel que l'énergie utilisée pour relier deux points. Beaucoup de méthodes existent pour résoudre de tels problèmes (calcul des variations, principe de maximum de Pontryagin [Pontryagin, 1987], équations d'Hamilton-Jacobi-Bellman [Bellman, 1966], équations de Riccati,...). La notion d'observabilité est duale de celle de contrôlabilité que l'on abordera tout au long de cette thèse. Cette fois-ci on s'intéresse à la dynamique

$$\begin{cases} \dot{z}(t) = A(t)z(t), \\ y(t) = C(t)z(t), \\ z(0) = z_0. \end{cases}$$

Les questions d'observabilité consistent à se demander si, connaissant les observations y , il est possible de retrouver l'état du système.

Les notions d'observabilité et contrôlabilité sont duales. Par exemple en dimension finie et infinie, un système est observable si son système adjoint est contrôlable et inversement. De plus dans le cas d'un contrôle en boucle fermée, le contrôle et l'observation se rejoignent. En effet, le système produit des observations afin de déterminer son état et de proposer le contrôle associé. En observation, on parlera d'identification lorsque l'on cherche à retrouver non plus l'état, mais un paramètre du système. Ces questions d'estimations (observation d'état ou identification) ou de contrôles sont fortement reliées à la notion de stabilisation. Dire qu'un système est stable, c'est affirmer qu'un petit décalage loin du point d'équilibre à un instant conduit à une convergence vers le point d'équilibre. Le but de la stabilisation du système

$$\begin{cases} \dot{z} = A(t)z(t), \\ z(0) = z_0, \end{cases}$$

consiste à trouver un *feedback* (ou retour d'état) qui stabilise le système. Autrement dit on cherche Q , tel que le système

$$\dot{z}(t) = (A(t) + B(t)Q(t))z(t),$$

qui bouclé par le feedback $u(t) = Q(t)z(t)$, soit stable. Pour des systèmes de dimensions finies autonomes linéaires, si un système est contrôlable alors il est stabilisable. Ces notions sont donc fortement reliées entre elles. Tous les principes énoncés précédemment ont largement été étudiés pour des systèmes de dimension finie : c'est l'automatique. Elle a connu des extensions pour les équations dérivées partielles appliquées aux problèmes d'évolutions (notamment [Lions, 1968; Coron, 2007] en contrôle et [Bensoussan, 1971] en observation).

Ma thèse s'inscrit dans l'ANR ODISSE dont une partie des membres viennent de la communauté de l'automatique. L'objectif de cette thèse est d'étendre à la dimension infinie certains estimateurs proposés en dimension finie.

Assimilation de données : méthode variationnelle et séquentielle

On s'intéressera notamment à l'assimilation de données, une gamme de problèmes inverses qui consiste à déterminer des grandeurs inconnues tel que l'état d'un système en utilisant les informations contenues dans le modèle ainsi que dans les observations. Celle-ci est issue des questions d'observation en théorie du contrôle. Bien que l'assimilation de données ait une connotation désormais plus numérique, on s'intéressera dans cette thèse à ses enjeux théoriques. Pour résoudre les problèmes d'assimilation de données, deux grandes catégories de méthodes ont émergé : les méthodes variationnelles [Lions, 1968; Le Dimet and Talagrand, 1986; Talagrand and Courtier, 1987; Moore, 1991; Luong et al., 1998] et les méthodes séquentielles. La première catégorie consiste à minimiser explicitement un critère ([Chavent, 2010; Banks and Kunisch, 2012] en identification et [Blum et al., 2009] en observation). Les méthodes séquentielles (ou filtrage) [Ghil, 1989; Houtekamer and Mitchell, 1998; Evensen, 2003; Verron et al., 1999], sont apparues dans un premier temps en dimension finie [Wiener, 1949]. Il faut attendre les années 1960 pour que Kalman montre que l'on peut résoudre le problème de minimisation sur des fenêtres de temps successives. Ainsi quand le temps d'observation augmente et que de nouvelles données sont disponibles, il n'est pas nécessaire de recalculer les résultats précédemment obtenus. Ces filtres sont désormais utilisés pour les équations aux dérivées partielles (voir notamment [Bensoussan, 1971]), mais le coût numérique est important. Une des solutions est alors d'utiliser un filtre de Kalman réduit [Pham et al., 1998b; Hoteit, 2001; Moireau and Chapelle, 2011].

Structure du manuscrit

Cette thèse est séparée en deux parties composées de deux chapitres chacune.

- La Partie I consiste à s'intéresser aux problèmes d'estimation d'état pour les phénomènes de propagation d'onde.
 - Le Chapitre 1 est un chapitre introductif qui présente les outils et les difficultés à travers l'exemple de l'estimation d'état. Ces outils seront ensuite adaptés dans le reste du manuscrit à l'estimation de source. Par exemple, on mettra en lumière dans ce chapitre l'approche par contrôle optimal en dimension infinie, ainsi que les différentes possibilités de résolution (méthodes variationnelles ou séquentielles) et l'impact du bruit sur ces problèmes d'estimation.
 - Dans le Chapitre 2, toujours dans le cadre de l'estimation d'état, nous nous concentrons sur l'impact de la discrétisation sur les résultats de stabilisation de l'équation des ondes avec amortissements sur le bord. Notamment, il est connu que sans ajout de termes dissipatifs, la propriété d'exponentielle stabilité satisfaite au niveau continu n'est plus valide au niveau discret. On propose alors l'introduction de nouveaux termes dissipatifs compatibles avec des éléments finis spectraux d'ordres élevés qui n'entachent pas l'ordre du schéma et qui permettent de retrouver la propriété d'exponentielle stabilité.
- La Partie II se consacre à l'identification d'un terme source pour les problèmes de propagation d'ondes scalaires

$$\partial_{tt}u(x, t) - \Delta u(x, t) = f(x, t).$$

-
- Le but du Chapitre 3 est d'étendre la définition d'un observateur optimal à la reconstruction de source en dimension infinie. Notamment le terme source cible qui est à variable séparée est dépendant de l'espace et multiplié par une fonction dépendant du temps, c'est-à-dire $f(x, t) = \sigma(t)\theta(x)$. On établit la formulation de l'observateur, à savoir son existence, mais aussi sa propriété asymptotique et sa robustesse au bruit de mesure. En particulier, on considère l'analyse sur des mesures de champ dans un sous-domaine, alors que la plupart des résultats d'observabilité pour les équations d'ondes s'appuient sur des mesures de vitesse.
 - Le Chapitre 4 est une extension du chapitre précédent. Cette fois-ci la fonction qui multiplie le terme spatial à estimer dépend du temps et de l'espace, c'est-à-dire $f(x, t) = \sigma(x, t)\theta(x)$. L'observabilité, s'appuyant sur des mesures de vitesses, nécessite alors d'établir une nouvelle estimation de Carleman. Pour reconstruire le terme source, comme dans le Chapitre précédent, on utilise un observateur construit via la théorie du contrôle optimal.

Communications

Articles de journal

- Tiphaine Delaunay, Sébastien Imperiale, Philippe Moireau. *Soumis en 2023*. Mathematical analysis of an observer for solving inverse source wave problem
- Muriel Boulakia, Maya de Buhan, Tiphaine Delaunay, Sébastien Imperiale, Philippe Moireau. *En préparation*. Solving inverse source wave problem - From Carleman estimates to observer design
- Tiphaine Delaunay, Sébastien Imperiale, Philippe Moireau. *Soumis en 2023*. Uniform boundary stabilization of a high-order finite element space discretization of the 1-d wave equation

Présentations orales

- Séminaire : The mathematical and statistical foundation of future data-driven engineering - Isaac Newton Institute, Cambridge, UK - 2023
Présentation orale : Solving inverse source wave problem - From observability to observer design.
- Workshop on Assimilation, Control and Computational Speedup - LAGA, Université Sorbonne Paris-Nord, Villetaneuse - 2023
Présentation orale : Solving inverse source wave problem - From observability to observer design
- Rencontre Jeunes Chercheurs Jeunes Chercheuses Ondes - Inria Université Côte d'Azur, Sophia Antipolis - 2022
Présentation orale : Solving inverse source wave problem - From observability to observer design
- WAVES, Palaiseau - 2022
Présentation orale : Solving inverse source wave problem - From observability to observer design
- ECCOMAS - Oslo, Norvège - 2022
Présentation orale : Stabilization of the high-order discretized 1D wave equation for data assimilation problems

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- Congrès des jeunes chercheurs et chercheuses en mathématiques - Palaiseau - 2021
Présentation orale : Stabilization of the high-order discretized 1D wave equation for data assimilation problems

Développement informatique Les résultats numériques concernant la résolution d'un problème inverse à l'aide d'un schéma *prédiction-correction* présentés dans la Partie II ont été développés en Freefem++.

Part I

State estimation for wave propagation problem

CHAPTER 1

Introduction to data assimilation for the wave equation

S'il n'y a pas de solution c'est qu'il n'y a pas de problème.

— Les shadoks

In this introductory chapter, we focus on state estimation problems for the wave equation. This allows us to introduce several tools that will be used later for more involved inverse problems such as source term estimation. First, we need to recall the various notions of solutions. In fact, the study of the inverse problem depends heavily on the nature of the solution considered. We then introduce a first data assimilation strategy called 4D-var – for fourth-dimensional (namely 3D and time) variational – originally introduced in the context of weather forecast. It consists in minimizing a criterion: more precisely, it consists in reducing the gap between the measurements available on the system and the simulated measurements associated with the solution with given initial data. This minimization is usually carried out using a gradient descent algorithm. Sequential methods is a possible alternative to solve such problems. More precisely, we will study the Kalman estimator, which is based on a dynamic programming approach and allows to recover the target trajectory by solving only one evolution problem. This method, however, requires to compute a Riccati operator, which is numerically expensive. We then present a less costly sequential alternative: the Luenberger observer, whose exponential convergence can be proved for perfect measurements when an observability inequality is fulfilled.

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1.1 Introduction

Inverse wave equation problems -also known as data assimilation for wave equation - can be divided into several categories: initial condition estimation, source reconstruction, and parameter reconstruction. First, we focus on the easiest case, *i.e.*, estimating the initial condition ([Ramdani et al., 2010; Imperiale et al., 2020]). From a control theory point of view, or more precisely from an observation theory, we exploit the available noisy measurements on the system to find its state.

In this chapter, we consider internal measurements within a subdomain of observation ω . They can be of two kinds: *field* measurements $u|_{\omega}$ or *velocity* measurements $\partial_t u|_{\omega}$. The analysis is similar for these two types of measurements, however we study what changes when we consider noisy measurements with less regularity.

But before solving the inverse problem, we must set-up the framework for the existence of solutions for the wave equation. Naturally, two theories are available and choosing one or the other will have consequence on our way to formalize the inverse problem.

The first theory of existence of solutions is the one introduced by [Lions et al., 1968] that we will call variational solutions in this manuscript. The wave equation enters in this framework as a second-order equation well adapted to finite elements discretization, where the solutions are sought in a very specific Hilbert subspace of L^2 in time with value in the energy space. Within this framework, the minimization can be seen as a minimization under constraints that is naturally solved using a Lagrangian and a Lagrange multiplier called the adjoint variable.

The second theory is the semigroup theory [Pazy, 1983] where this time we seek for continuous solution in time. In this framework the Duhamel formula will allow to solve the minimization problems without relying on the “minimizing under constraints” formalism. Here also an adjoint variable is introduced, adapted to a first-order semigroup formulation, but only to simplify gradient computation.

Finally, we will see that we can reconcile the two formalisms, hence benefiting from the advantages of both theory for solving least squares minimization problem.

Indeed, a least squares based reconstruction of the initial condition will be the first method for inversion. This strategy is related to full waveform inversion (FWI) for waves and 4D-var in data assimilation.

The minimizer is characterized as a two-ends problems typically solve by iterative method since one part of the dynamics is resolved forward in time and the second in backward time.

We then show that we can introduce a Kalman filter [Kalman and Bucy, 1961; Bensoussan, 1971], a sequential approach equivalent to the minimization problem, which allows to solve in one iteration the state estimation like a second-order method. Unfortunately, solving the problem with the Kalman filter requires calculating a Riccati operator, which can be very costly.

We present an alternative observer based on Luenberger recommendation [Luenberger, 1963], namely stabilizing asymptotically the error between the target trajectory and the estimator trajectory. This observer is not equivalent to the minimization of the same cost functional. However, some exponential convergence still holds once we prove observability conditions. Indeed, these useful inequalities give properties on the operator to be inverted. Eventually, we look at the impact of noise regularity on state reconstruction and present different methods to deal with it. The final challenge is then to extend the method we propose in order to identify a source term. This will pave the way of the rest of the manuscript.

1.2 Problem setting

In this chapter we are concerned with the following wave equation.

$$\begin{cases} \partial_{tt}u(x, t) - \Delta u(x, t) = f(x, t), & (x, t) \in \Omega \times (0, T), \\ u(x, t) = 0, & (x, t) \in \partial\Omega \times (0, T), \\ u(x, 0) = u_0(x), & x \in \Omega, \\ \partial_t u(x, 0) = v_0(x), & x \in \Omega. \end{cases} \quad (1.1)$$

There are several methods to prove the existence of a unique solution of such a system: the variational solutions in Section 1.2.1 (see, for example, [Lions et al., 1968], [Lions, 1968], and [Dautray and Lions, 1999]), and the mild and strong solutions using semigroup theory in Section 1.2.2 (see [Pazy, 1983; Bensoussan et al., 2007]). These are not defined in the same spaces. Therefore, the study of the reconstruction of the initial conditions depends strongly on the choice of the considered solutions, a classical fundamental issue when dealing with infinite dimension systems. In the case of variational solutions of the second order system, we will see later (Section 1.3.2) that solving the inverse problem in this case entails solving a constrained minimization problem. Finally, we will see that solutions obtained in the framework of semigroups has several advantages. First, they allow us to stay with a first-order system, which is more commonly studied in the literature. Moreover, solving the inverse problem in this case do not require to solve a constrained minimization problem (Section 1.3.3).

1.2.1 Variational solutions of the wave equation

Let us start by a first look on the existence of solution for the wave equation.

We consider a Hilbert space $\mathcal{V} \subset \mathcal{H}$ which is dense in \mathcal{H} with continuous embedding. Identifying \mathcal{H} with its dual, we have

$$\mathcal{V} \hookrightarrow \mathcal{H} \equiv \mathcal{H}' \hookrightarrow \mathcal{V}'.$$

We consider a bilinear continuous coercive form a on \mathcal{V} namely $\forall w \in \mathcal{V}$,

$$\exists \lambda \in \mathbb{R} \mid a(w, w) + \lambda \|w\|_{\mathcal{H}}^2 \geq \alpha \|w\|_{\mathcal{V}}^2, \quad \alpha > 0,$$

and we denote by $A_0 \in \mathcal{L}(\mathcal{V}, \mathcal{V}')$ the operator associated with the bilinear form a , *i.e.* for all $v, w \in \mathcal{V}$,

$$a(v, w) = -\langle A_0 v, w \rangle_{\mathcal{V}', \mathcal{V}}.$$

We study the following second-order dynamics

$$\begin{cases} \frac{d^2}{dt^2}(u, v)_{\mathcal{H}} + a(u, v) = \langle \beta, v \rangle_{\mathcal{V}', \mathcal{V}} \text{ in } D'(0, T), \\ u(0) = u_0, \\ v(0) = u_1. \end{cases} \quad \text{with } \beta \in \mathcal{H}. \quad (1.2)$$

We set $\mathcal{H} = L^2(\Omega)$, $\mathcal{V} = H_0^1(\Omega)$, $\mathcal{V}' = H^{-1}(\Omega)$. Choosing $A_0 = -\Delta_0$ the Laplace operator with Dirichlet boundary conditions – namely $a(\cdot, \cdot) = (\cdot, \cdot)_{H_0^1(\Omega)}$ – the wave equation (1.1) fits perfectly into the above theory and satisfies the coercivity property.

We introduce the Hilbert space $\mathcal{W}(0, T, \mathcal{V}, \mathcal{H})$ by

$$\mathcal{W}(0, T, \mathcal{V}, \mathcal{H}) = \{w \mid w \in L^2(0, T; \mathcal{V}), \dot{w} \in L^2(0, T; \mathcal{H})\},$$

equipped with the norm

$$\|w\|_{\mathcal{W}(0,T,\mathcal{V},\mathcal{H})} = \left(\int_0^T \|w(t)\|_{\mathcal{V}}^2 dt + \int_0^T \|\dot{w}(t)\|_{\mathcal{H}}^2 dt \right)^{1/2}.$$

Let us present some important properties about the space $\mathcal{W}(0, T, \mathcal{V}, \mathcal{H})$ – see [Chapter XVIII, Section 1.3][Dautray and Lions, 1999].

Proposition 1.2.1 *Let Λ a positive self-adjoint operator such that $\mathcal{D}(\Lambda) = \mathcal{V}$ where $\mathcal{D}(\Lambda)$ is the domain of Λ . We have the embedding property*

$$\mathcal{W}(0, T, \mathcal{V}, \mathcal{H}) \hookrightarrow C^0([0, T], (\mathcal{V}, \mathcal{H})_{1/2}),$$

where $(\mathcal{V}, \mathcal{H})_{1/2} = D(\Lambda^{1/2})$. Furthermore, if $(u, v)_{\mathcal{V}} = (A_0 u, v)_{\mathcal{H}}$ and A_0 is self-adjoint, then $\Lambda = A_0^{1/2}$.

As the previous section, we can show a density result.

Lemma 1.2.2 *The space $D(0, T; \mathcal{V})$ is dense in $\mathcal{W}(0, T, \mathcal{V}, \mathcal{H})$.*

The particular case $\mathcal{W}(0, T, \mathcal{V}, \mathcal{V}')$ will be denoted $\mathcal{W}(0, T)$. Note that, in this case, the embedding property is rewritten

$$\mathcal{W}(0, T) \hookrightarrow C^0([0, T], \mathcal{H}).$$

This space is particularly suitable for first-order system, therefore we will give more details on this space in Section 1.2.3.

We introduce the Hilbert space $\tilde{\mathcal{W}}(0, T)$ – in which we will obtain the existence of solutions – defined by

$$\tilde{\mathcal{W}}(0, T) = \{v \in L^2(0, T; \mathcal{V}), \dot{v} \in L^2(0, T; \mathcal{H}), \ddot{v} \in L^2(0, T; \mathcal{V}')\},$$

equipped with the norm

$$\|w\|_{\tilde{\mathcal{W}}(0,T)} = \left(\int_0^T \|w(t)\|_{\mathcal{V}}^2 dt + \int_0^T \|\dot{w}(t)\|_{\mathcal{H}}^2 dt + \int_0^T \|\ddot{w}(t)\|_{\mathcal{V}'}^2 dt \right)^{1/2}.$$

The following proposition extends Green's formula to this space.

Proposition 1.2.3 (Green's formula) *For all $p, q \in \tilde{\mathcal{W}}(0, T)$, we have*

$$\begin{aligned} \int_0^T \langle \ddot{p}, q \rangle_{\mathcal{V}', \mathcal{V}} dt - \int_0^T \langle p, \ddot{q} \rangle_{\mathcal{V}, \mathcal{V}'} dt \\ = (\dot{p}(T), q(T))_{\mathcal{Z}} - (\dot{p}(0), q(0))_{\mathcal{Z}} - (p(T), \dot{q}(T))_{\mathcal{Z}} + (p(0), \dot{q}(0))_{\mathcal{Z}}. \end{aligned}$$

For further use, we introduce the Hilbert space $\tilde{\mathcal{W}}_{\mathcal{V}}(0, T)$ defined by

$$\tilde{\mathcal{W}}_{\mathcal{V}}(0, T) = \{w \mid w \in L^2(0, T; \mathcal{V}), \dot{w} \in \mathcal{W}(0, T)\},$$

equipped with the following norm

$$\|w\|_{\tilde{\mathcal{W}}_{\mathcal{V}}(0,T)} = \left(\int_0^T \|\dot{w}\|_{\mathcal{V}}^2 dt + \|w\|_{\mathcal{W}(0,T)}^2 \right)^{1/2}.$$

Proposition 1.2.4 *We have the following embedding property*

$$\tilde{\mathcal{W}}_{\mathcal{V}}(0, T) \hookrightarrow C([0, T]; \mathcal{V}).$$

Existence of solution theorem

We can now formulate the theorem of the existence of solutions.

Theorem 1.2.5 *For $u_0 \in \mathcal{V}$ and, $u_1 \in \mathcal{H}$ and $\beta \in L^2(0, T; \mathcal{H})$, there exists a unique solution $\{u \in L^2(0, T; \mathcal{V}), \frac{d}{dt}u \in L^2(0, T; \mathcal{H})\}$ of the system (1.2). Moreover, we have $u \in \tilde{\mathcal{W}}(0, T)$ since*

$$\frac{d^2}{dt^2}u = A_0u + \beta \in L^2(0, T; \mathcal{V}').$$

Remark 1.2.6 *Note that the second member of (1.2) $\beta(t)$ must belong to \mathcal{H} , which is more restrictive than in the Section 1.2.3.*

1.2.2 Existence results based on semigroup theory

We begin with a brief reminder of the principles of semigroup theory. We establish the necessary regularity of the source terms and initial conditions of the problem to guarantee the existence and uniqueness of solutions. Let \mathcal{Z} an Hilbert space. We introduce the first-order system we study

$$\begin{cases} \frac{d}{dt}z(t) = Az(t) + f(t), & t \in [0, T]. \\ z(0) = z_0. \end{cases} \quad (1.3)$$

We now introduce the solutions of the semigroup theory to present the existence of the solution. For a complete introduction and analysis of the semigroup theory, see for example (non-exhaustive) : [Pazy, 1983], [Bensoussan et al., 2007] and [Tucsnak and Weiss, 2009]. First, we recall the notion of m-dissipative operator.

Definition 1.2.1 *An operator $(A, \mathcal{D}(A))$ is dissipative if*

$$\forall z \in \mathcal{D}(A), \forall \lambda > 0, \|\lambda z - Az\| \geq \lambda \|z\|.$$

We recall the definitions and propositions necessary for the existence of solutions.

Proposition 1.2.7 *The operator $(A, \mathcal{D}(A))$ in the Hilbert space \mathcal{Z} is dissipative if and only if*

$$\forall z \in \mathcal{D}(A), (z, Az) \leq 0.$$

Definition 1.2.2 *A dissipative operator $(A, \mathcal{D}(A))$ is maximal-dissipative – also denoted m-dissipative– if for all $\lambda > 0$, $(\lambda Id - A)$ is surjective.*

The following theorem is fundamental to semigroup theory, as it links m-dissipative operators and generators of C^0 -semigroups.

Theorem 1.2.8 (Lumer-Philips) *Let $A : \mathcal{D}(A) \subset \mathcal{Z} \rightarrow \mathcal{Z}$ an unbounded operator defined on the Hilbert space \mathcal{Z} . We have the following equivalence:*

1. $(A, \mathcal{D}(A))$ is maximal dissipative,
2. $(A, \mathcal{D}(A))$ is the generator of a C^0 -semigroup of contraction Φ , namely

$$\forall t > 0, \|\Phi(t)\| \leq 1.$$

The theory of semigroups then ensures the existence and uniqueness of solutions for the problem (1.3).

Theorem 1.2.9 *Let $(A, \mathcal{D}(A))$ be the generator of a C^0 -semigroup in the Hilbert space \mathcal{Z} . The evolution equation*

$$\begin{cases} \frac{d}{dt}z(t) = Az(t) + f(t), & t \in [0, T]. \\ z(0) = z_0, \end{cases} \quad (1.4)$$

admits the following solutions:

1. *For all $z_0 \in \mathcal{D}(A)$ and $f(t) \in C^0([0, T]; \mathcal{Z})$, there exists a unique strong solution $z \in C^1([0, T]; \mathcal{Z}) \cap C^0([0, T]; \mathcal{D}(A))$ such that $z(0) = z_0$ and $\dot{z} = Az + f$.*
2. *For all $z_0 \in \mathcal{Z}$ and for all $f \in L^p(0, T; \mathcal{Z})$ there exists a unique mild-solution $z \in C^0([0, T]; \mathcal{Z})$ such that $z(0) = z_0$ given by Duhamel's formula*

$$z(t) = e^{tA}z_0 + \int_0^t e^{(t-s)A}f(s)ds, \quad t \in [0, T] \quad (1.5)$$

The existence of a strong solution can be extended to the case where $f \in W^{1,1}(0, T; \mathcal{Z})$ or $f \in L^1(0, T; \mathcal{D}(A)) \cap C^0([0, T]; \mathcal{Z})$.

We will present a theorem that will be useful for what follows.

Theorem 1.2.10 *Let $(A, \mathcal{D}(A))$ a m -dissipative operator in the Hilbert space \mathcal{Z} generator of a C^0 -semigroup $(\Phi(t))_{t \geq 0}$. Then $(A^*, \mathcal{D}(A^*))$ is a maximal-dissipative operator generator of $(\Phi(t)^*)_{t \geq 0}$.*

Application to the homogeneous wave equation

We rewrite (1.1) as a first order system in the state-space $\mathcal{Z} = \mathbb{H}_0^1(\Omega) \times L^2(\Omega)$, with $f \in L^p(0, T; \mathcal{Z})$,

$$\begin{cases} \dot{z} = Az + f(t), & \text{in } (0, T), \\ z(0) = z_0, \end{cases} \quad (1.6)$$

where $z_0 = (u_0 \ v_0)^t$ and the operator A is an unbounded skew-adjoint operator from $\mathcal{D}(A)$ into \mathcal{Z} defined by

$$A = \begin{pmatrix} 0 & \text{Id} \\ \Delta_0 & 0 \end{pmatrix}, \quad \mathcal{D}(A) = \mathcal{D}(-\Delta_0) \times \mathbb{H}_0^1(\Omega), \quad (1.7)$$

where Δ_0 denotes the Laplacian operator with homogeneous Dirichlet boundary conditions [Bensoussan, 1971, Part II, Section 1.2.10]. We equip the state space with the following norm

$$\forall z = \begin{pmatrix} u & v \end{pmatrix}^t \in \mathcal{Z}, \quad \|z\|_{\mathcal{Z}}^2 = \|\nabla u\|_{L^2(\Omega)}^2 + \|v\|_{L^2(\Omega)}^2.$$

Theorem 1.2.11 *The generator $(A, \mathcal{D}(A))$ defined by (1.7) is m -dissipative.*

PROOF: Since the operator A is skew-adjoint, we have

$$(Az, z) = -(Az, z) = 0, \quad \forall z \in \mathcal{D}(A).$$

We, now show the maximal property. For $(f \ g)^t \in \mathcal{Z}$, we want to find $z = (u \ v)^t$ such that

$$(\lambda \text{Id} - A)_{\mathcal{Z}} z = \begin{pmatrix} f \\ g \end{pmatrix}, \quad \lambda > 0.$$

We have obtained the two following relations

$$\begin{cases} \lambda u - v = f, \\ \lambda v - \Delta_0 u = g. \end{cases}$$

In order to decouple the two problems, we inject the first relation into the second, and we obtain

$$\begin{cases} v = \lambda u - f, \\ \lambda^2 u - \Delta_0 u = g - \lambda f. \end{cases}$$

Thanks to the Lax–Milgram theorem, the second problem admits a unique solution $u \in \text{H}_0^1(\Omega)$. Since $f \in \text{H}_0^1(\Omega)$ and $g \in \text{L}^2(\Omega)$, we have that $\Delta_0 u \in \text{L}^2(\Omega)$. Since $f \in \text{H}_0^1(\Omega)$, we also have the existence of $v \in \text{H}_0^1(\Omega)$. We can conclude that $\lambda \text{Id} - A$ is maximal. \blacksquare

Since $(A, \mathcal{D}(A))$ is m-dissipative hence it is a generator of a C^0 -semigroup (Theorem 1.2.8). Theorem 1.2.9 gives the existence of solutions $z \in \text{C}^0([0, T], \mathcal{Z})$ for $z_0 \in \mathcal{Z}$ and for $f \in \text{L}^p((0, T; \mathcal{Z}))$.

1.2.3 Variational solution of a first-order system: the viscous wave equation

One might wonder whether first-order variational solutions are a suitable framework for the wave equation. We briefly present the theory proposed by [Dautray and Lions, 1999] in this framework and show that it is not suitable for the wave equation unless a viscous term is added.

Application: the viscous wave equation

We study the following viscous wave equation (so-called parabolic regularization [Bensoussan et al., 2007, Part II, Chapter 2, Section 2.3] of the wave equation)

$$\begin{cases} \partial_{tt} u_\kappa(x, t) - \Delta u_\kappa(x, t) + \kappa \Delta \partial_t u_\kappa(x, t) = f(x, t), & (x, t) \in \Omega \times (0, T), \\ u_\kappa(x, t) = 0, & (x, t) \in \partial\Omega \times (0, T), \\ u_\kappa(x, 0) = u_0(x), & x \in \Omega, \\ \partial_t u_\kappa(x, 0) = v_0(x), & x \in \Omega. \end{cases} \quad (1.8)$$

We consider a Hilbert space $\mathcal{V}_\kappa \subset \mathcal{Z}_\kappa$ dense in \mathcal{Z}_κ with continuous embedding. Identifying \mathcal{Z}_κ with its dual, we have

$$\mathcal{V}_\kappa \hookrightarrow \mathcal{Z}_\kappa \equiv \mathcal{Z}'_\kappa \hookrightarrow \mathcal{V}'_\kappa.$$

We denote $z_\kappa = (u_\kappa \ v_\kappa)^t$ and $\beta = (0 \ f)^t$ and the operator $A_\kappa \in \mathcal{L}(\mathcal{V}_\kappa, \mathcal{V}'_\kappa)$ given by

$$A_\kappa = \begin{pmatrix} 0 & \text{Id} \\ \Delta_0 & \kappa \Delta_0 \end{pmatrix}, \quad (1.9)$$

with the associated bilinear form $a_\kappa(\cdot, \cdot)$ i.e. for all $v, w \in \mathcal{V}_\kappa$,

$$a_\kappa(v, w) = -\langle A_\kappa v, w \rangle_{\mathcal{V}'_\kappa, \mathcal{V}_\kappa}.$$

Therefore, we are interested in finding solutions in the following first-order system

$$\begin{cases} \frac{d}{dt}(z_\kappa, v)_{\mathcal{Z}_\kappa} + a_\kappa(z_\kappa, v) = \langle \beta, v \rangle_{\mathcal{V}'_\kappa, \mathcal{V}_\kappa} & \text{in } D'(0, T), \\ z_\kappa(0) = z_0, \end{cases} \quad (1.10)$$

with $\beta \in \mathcal{V}'_\kappa$. We consider that a_κ is a bilinear continuous coercive form on \mathcal{V}_κ , namely $\forall w \in \mathcal{V}_\kappa$,

$$\exists \lambda \in \mathbb{R} \mid a_\kappa(w, w) + \lambda \|w\|_{\mathcal{Z}_\kappa}^2 \geq \alpha \|w\|_{\mathcal{V}_\kappa}^2, \quad \alpha > 0. \quad (1.11)$$

The space $\mathcal{W}_\kappa(0, T)$ and its properties

To establish the theorem on the existence of solutions to the dynamics (1.10), one must introduce the Hilbert space $\mathcal{W}_\kappa(0, T)$. Since $L^2(0, T; \mathcal{V}_\kappa) \subset D'(0, T; \mathcal{V}_\kappa)$, we have for $w \in L^2(0, T; \mathcal{V}_\kappa)$

$$\dot{w} = \frac{dw}{dt} \in D'(0, T; \mathcal{V}_\kappa).$$

Thus, we can now define the space $\mathcal{W}_\kappa(0, T)$ by

$$\mathcal{W}_\kappa(0, T) = \{w \mid w \in L^2(0, T; \mathcal{V}_\kappa), \dot{w} \in L^2(0, T; \mathcal{V}'_\kappa)\},$$

equipped with the norm

$$\|w\|_{\mathcal{W}_\kappa(0, T)} = \left(\int_0^T \|w(t)\|_{\mathcal{V}_\kappa}^2 dt + \int_0^T \|\dot{w}(t)\|_{\mathcal{V}'_\kappa}^2 dt \right)^{1/2}.$$

Let us introduce some important properties of the space $\mathcal{W}_\kappa(0, T)$ – see [Dautray and Lions, 1999, Chapter XVIII] for the proof of all the following properties.

Proposition 1.2.12 *Every $w \in \mathcal{W}_\kappa(0, T)$ is almost everywhere equal to a continuous function of $[0, T]$ in \mathcal{Z}_κ . Further, we have*

$$\mathcal{W}_\kappa(0, T) \hookrightarrow C^0([0, T], \mathcal{Z}_\kappa),$$

where $C^0([0, T], \mathcal{Z}_\kappa)$ is equipped with the norm of uniform convergence.

Note this important property.

Lemma 1.2.13 *The space $D(0, T; \mathcal{V}_\kappa)$ is dense in $\mathcal{W}_\kappa(0, T)$.*

Let us set up the following proposition that will be useful later.

Proposition 1.2.14 (Green's formula) *For all $p, q \in \mathcal{W}_\kappa(0, T)$, we have*

$$\int_0^T \langle \dot{p}, q \rangle_{\mathcal{V}'_\kappa, \mathcal{V}_\kappa} dt + \int_0^T \langle p, \dot{q} \rangle_{\mathcal{V}'_\kappa, \mathcal{V}_\kappa} dt = (p(T), q(T))_{\mathcal{Z}_\kappa} - (p(0), q(0))_{\mathcal{Z}_\kappa}.$$

Existence of solution

We can now express the existence of solutions.

Theorem 1.2.15 *For $z_0 \in \mathcal{Z}_\kappa$ and $\beta \in L^2(0, T; \mathcal{V}'_\kappa)$, there exists a unique solution z_κ belonging to $\mathcal{W}_\kappa(0, T)$ of the system (1.10). Moreover,*

$$\begin{aligned} \mathcal{Z}_\kappa \times L^2(0, T; \mathcal{V}_\kappa) &\rightarrow \mathcal{W}_\kappa(0, T), \\ (z_0, \beta) &\rightarrow z_\kappa, \end{aligned}$$

is a bounded application.

Let $\mathcal{Z}_\kappa = H_0^1(\Omega) \times L^2(\Omega)$ and $\mathcal{V}_\kappa = H_0^1(\Omega) \times H_0^1(\Omega)$ equipped with the pivot space $H_0^1(\Omega) \times L^2(\Omega)$, namely we have $\mathcal{V}'_\kappa = H_0^1(\Omega) \times H^{-1}(\Omega)$.

Note that for the viscous equation the coercivity condition is satisfied namely for $\lambda > \alpha$ and $\kappa > \alpha$,

$$\begin{aligned} a_\kappa(z, z) + \lambda \|z_\kappa\|_{\mathcal{Z}}^2 &= \kappa \|\nabla v_\kappa\|_{L^2(\Omega)}^2 + \lambda \|\nabla u_\kappa\|_{L^2(\Omega)}^2 + \lambda \|v_\kappa\|_{L^2(\Omega)}^2 \\ &\geq \alpha \left(\|\nabla u_\kappa\|_{L^2(\Omega)}^2 + \|\nabla v_\kappa\|_{L^2(\Omega)}^2 \right). \end{aligned}$$

Moreover, for $\beta \in L(0, T; \mathcal{Z}_\kappa)$ the viscous wave equation (1.8) converges to the wave equation (1.1) for $\kappa \rightarrow 0$ (see [Dautray and Lions, 1999, Chapter XVIII, Section 6]), namely $\forall t \in [0, T]$,

$$\begin{cases} u_\kappa(t) \rightarrow u(t) \text{ strongly in } H_0^1(\Omega), \\ v_\kappa(t) \rightarrow v(t) \text{ strongly in } L^2(\Omega). \end{cases}$$

Application to the wave equation

We now show that this framework is not suitable for the wave equation. We denote $z = (u \ v)^t$ and $\beta = (0 \ f)^t$ and reformulate the wave equation (1.1) as follows

$$\begin{cases} \dot{z}(t) = Az(t) + \beta(t), & \text{in } (0, T), \\ z(0) = z_0, \end{cases} \quad (1.12)$$

with

$$z_0 = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}, \text{ and } A = \begin{pmatrix} 0 & \text{Id} \\ \Delta_0 & 0 \end{pmatrix},$$

with the associated bilinear form $a(\cdot, \cdot)$ i.e. for all $v, w \in \mathcal{V}_\kappa$,

$$a(v, w) = -\langle Av, w \rangle_{\mathcal{V}'_\kappa, \mathcal{V}_\kappa}.$$

Unfortunately, this configuration does not fit into the previously mentioned framework. Namely, the coercivity condition (1.11) is not fulfilled. Namely, we have

$$\begin{aligned} a(z, z) + \lambda \|z\|_{\mathcal{Z}_\kappa}^2 &= -(\nabla v, \nabla u)_{L^2(\Omega)} - (\Delta_0 u, v)_{L^2(\Omega)} + \lambda \|\nabla u\|_{L^2(\Omega)}^2 + \lambda \|v\|_{L^2(\Omega)}^2 \\ &= \lambda \|\nabla u\|_{L^2(\Omega)}^2 + \lambda \|v\|_{L^2(\Omega)}^2, \end{aligned}$$

which does not bound the following quantity

$$\alpha \|z\|_{\mathcal{V}_\kappa} = \alpha \left(\|\nabla u\|_{L^2(\Omega)}^2 + \|\nabla v\|_{L^2(\Omega)}^2 \right).$$

1.2.4 Link between variational and semigroup solutions

Note that from the existence of semigroup solutions we can infer the existence of variational solutions.

Proposition 1.2.16 *Consider initial conditions $(u_0 \ v_0) \in \mathcal{Z}$ and $(0 \ f)^t \in L^2((0, T); \mathcal{Z})$ namely $f \in L^2((0, T); L^2(\Omega))$. Then there exists a unique solution $z = (u \ v)^t \in C^0([0, T]; \mathcal{Z})$ of (1.3) such that*

$$z = e^{tA} \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} + \int_0^t e^{A(t-s)} \begin{pmatrix} 0 \\ f(s) \end{pmatrix} ds, \quad (1.13)$$

where A is defined by (1.7). The equation – understood in the following weak sense, for $\psi \in D(0, T)$

- $\int_0^T \psi(t)z(t)dt \in \mathcal{D}(A),$
- $\int_0^T \dot{\psi}(t)z(t)dt - A \left(\int_0^T \psi(t)z(t)dt \right) = \int_0^T \psi(t) \begin{pmatrix} 0 \\ f(t) \end{pmatrix} dt,$

is equivalent to (1.2).

PROOF:

The first equation of (1.13) implies that

$$- \int_0^T \dot{\psi}(t)u(t)dt + \int_0^T \psi(t)v(t) dt = 0 \text{ in } H_0^1(\Omega).$$

Consequently, for $\eta \in D(\Delta_0)$, we obtain

$$- \int_{\Omega} \nabla \eta(x) \int_0^T \dot{\psi}(t)\nabla u(x,t) dt dx + \int_{\Omega} \nabla \eta(x) \nabla \int_0^T \psi(t)v(x,t) dt dx = 0.$$

Integrating by parts, we get

$$\int_{\Omega} \Delta \eta(x) \int_0^T \dot{\psi}(t)u(x,t) dt dx - \int_{\Omega} \Delta \eta(x) \int_0^T \psi(t)v(x,t) dt dx = 0.$$

We choose $w \in L^2(\Omega)$ such that $\exists \eta \in D(\Delta)$ satisfying

$$\begin{cases} -\Delta \eta = w, & \text{in } \Omega, \\ \eta = 0, & \text{on } \partial\Omega. \end{cases}$$

Thus, we obtain

$$- \int_{\Omega} w(x) \int_0^T \dot{\psi}(t)u(x,t) dt dx + \int_{\Omega} w(x) \int_0^T \psi(t)v(x,t) dt dx = 0.$$

We can invert the order of integration to get

$$- \int_0^T (w, u)_{L^2(\Omega)} \dot{\psi}(t) dt + \int_0^T (w, v)_{L^2(\Omega)} dt = 0. \quad (1.14)$$

In other words, we have

$$\frac{d}{dt}(w, u)_{L^2(\Omega)} = (w, v)_{L^2(\Omega)} \text{ in } D'(0, T).$$

From the second equation, we obtain

$$\int_0^T \dot{\psi}(t)v(t) dt - \Delta_0 \int_0^T \psi(t)u(t) dt = \int_0^T \psi(t)f(t) dt.$$

Let $w \in H_0^1(\Omega)$, integrating in space, we have

$$\begin{aligned} \int_{\Omega} w(x) \int_0^T \dot{\psi}(t)v(x,t) dt dx + \int_{\Omega} \nabla w(x) \int_0^T \psi(t)\nabla u(x) dt dx \\ = \int_{\Omega} w(x) \int_0^T \psi(t)f(x,t) dt dx. \end{aligned}$$

Inverting the order of integration, we obtain

$$\int_0^T (w, v)_{L^2(\Omega)} \dot{\psi}(t) dt + \int_0^T (\nabla w, \nabla u)_{L^2(\Omega)} \psi(t) dt = \int_0^T (w, f)_{L^2(\Omega)} \psi(t) dt.$$

Using (1.14), we finally have

$$\int_0^T (w, u)_{L^2(\Omega)} \ddot{\psi}(t) dt + \int_0^T (\nabla w, \nabla u)_{L^2(\Omega)} \psi(t) dt = \int_0^T (w, f)_{L^2(\Omega)} \psi(t) dt.$$

Thus, we get the variational formulation (1.2) back

$$\frac{d^2}{dt^2} (w, u)_{L^2(\Omega)} + (\nabla w, \nabla u)_{L^2(\Omega)} = (w, f), \text{ in } D'(0, T).$$

■

This equivalence between the variational solutions and the solutions given by the semi-group formalism will allow us to use the finite element method in a natural way, while staying within the framework of the semigroups for estimation purposes hence benefiting from both worlds.

1.3 Resolution of the inverse problem: the 4D-var approach

Let \mathcal{Y} be a Hilbert space called the observation space. We assume that we observe a target solution \tilde{u} of (1.1) associated with unknown solutions $(u_0 \ v_0)$, and we have the corresponding internal measurements in a subdomain $\omega \subset \Omega$, either *field* measurements $\tilde{u}|_\omega$ or *velocity* measurements $\partial_t \tilde{u}|_\omega$ – also called observations – denoted by $\mathbb{R}^+ \ni t \rightarrow \check{y}(t) \in \mathcal{Y}$. The advantage of considering internal observations is that the observation operator C , which models the process of measurement and is described in detail below, is bounded. If we consider boundary measurements, the operator C is then unbounded in this case, which leads to additional difficulties (see Chapter 2).

We model the measurement process at each time t by the observation operator $C \in \mathcal{L}(\mathcal{Z}, \mathcal{Y})$, defined by

$$C\check{z}(t) = \check{y}(t). \tag{1.15}$$

The available measurements – denoted by y^δ – are a perturbation of the unavailable perfect measurements \check{y} . Namely, the unavailable perfect measurements can be affected by noise χ^δ , typically for all $t \in [0, T]$

$$\chi^\delta(t) = \check{y}(t) - y^\delta(t) \implies \|\chi^\delta(t)\|_{\mathcal{Y}} = \mathcal{O}(\delta),$$

where in this section we have assumed that the noise is regular ($y^\delta \in \mathcal{Y}$). We will then look at strategies for dealing with less regular measurements ($y^\delta \notin \mathcal{Y}$).

We introduce two ways of solving such problems. The first is to introduce a constrained minimization problem, which will be the subject of Sections 1.3.1 and 1.3.2. The other method presented in Section 1.3.3, which avoids introducing a constrained minimization problem, is to rely on the adjoint equations already demonstrated in Section 1.3.1 and stay in the semigroup formalism. Both points of view formally lead to the same method that we call the variational method. Such method is related to the work of Sasaki ([Sasaki, 1955], [Sasaki, 1970]) and the adjoint formalism appeared in [Le Dimet and Talagrand,

1986]. For other applications in weather forecasting ([Lewis and Derber, 1985], [Talagrand and Courtier, 1987], [Rabier et al., 2000] and to *Météo France* in the 2000s), the resulting method is known as 4D-var (fourth-dimensional variational).

The goal is to find an optimal trajectory that minimizes the discrepancy from the observation. By contrast, the 3D-var method is developed for static cases inversion (for example, [Moore, 1991], [Courtier et al., 1998], and at *Météo France* 1996). The 4D-var approach is now applied to many physics and applications such as oceanography [Robert et al., 2005] or seismic wave problems. In this last case, the 4D-var method is called the full waveform inversion (FWI) and is now very popular [Gauthier et al., 1986; Tromp et al., 2005; Wang et al., 2019].

1.3.1 4D-var approach with variational solutions of first-order system

First, we introduce the minimization problem for first-order systems with variational solutions, described in Section 1.2.3. Although this case is not exactly the one we are interested in (because of the additional viscous term), optimal control approaches are easier to present in this framework [Lions, 1968, Chapter III] and will serve as an introduction before we move on to second order systems [Lions, 1968, Chapter IV].

We denote in this section by $z_\kappa = (u_\kappa \ v_\kappa)^t$ the solution of the viscous wave equation

$$\begin{cases} \frac{d}{dt}(z_\kappa, v)_{\mathcal{Z}_\kappa} + a_\kappa(z_\kappa, v) = \langle \beta, v \rangle_{\mathcal{V}'_\kappa, \mathcal{V}_\kappa} & \text{in } D'(0, T), \\ z_\kappa(0) = z_0, \end{cases} \quad (1.16)$$

where

$$a_\kappa(v, w) = -\langle A_\kappa v, w \rangle_{\mathcal{V}'_\kappa, \mathcal{V}_\kappa},$$

with A_κ given by (1.9). To solve the inverse problem, the natural idea is to introduce the following cost functional $\mathcal{J}_T : \mathcal{Z}_\kappa \rightarrow \mathbb{R}$, which is given by

$$\mathcal{J}_T(z_0) = \frac{1}{2} \int_0^T \|Cz_\kappa|_{z_0} - y^\delta(t)\|_{\mathcal{Y}}^2 dt + \frac{\epsilon}{2} \|z_0\|_{\mathcal{Z}_\kappa}^2. \quad (1.17)$$

The first term of the cost functional represents the data attachment and the second term is a regularization term.

We rewrite the cost functional to minimize (1.17) using the observation operator.

$$\mathcal{J}_T(z_0) = \frac{1}{2} \int_0^T \|Cz_\kappa|_{z_0} - y^\delta\|_{\mathcal{Y}}^2 dt + \frac{\epsilon}{2} \|z_0\|_{\mathcal{Z}_\kappa}^2. \quad (1.18)$$

We denote the minimizer of the above cost functional $\bar{z}_0 := \underset{z_0 \in \mathcal{Z}_\kappa}{\operatorname{argmin}} \mathcal{J}_T$.

Theorem 1.3.1 *The minimizer \bar{z}_0 of the cost functional (1.18) is given by the following dynamics, for $\bar{q}_{\kappa T} \in \mathcal{W}_\kappa(0, T)$ and $\bar{z}_{\kappa T} \in \mathcal{W}_\kappa(0, T)$,*

$$\begin{cases} \dot{\bar{z}}_{\kappa T} = A_\kappa \bar{z}_{\kappa T}, & \text{in } (0, T), \\ \dot{\bar{q}}_{\kappa T} + A_\kappa^* \bar{q}_{\kappa T} = -C^*(y^\delta - C\bar{z}_{\kappa T}), & \text{in } (0, T), \\ \bar{z}_{\kappa T}(0) = \epsilon^{-1} \bar{q}_T(0), \\ \bar{q}_{\kappa T}(T) = 0. \end{cases} \quad (1.19)$$

PROOF: We introduce the following minimization under a constraint problem

$$\min_{z_\kappa \in \mathcal{W}_\kappa(0, T), \dot{z}_\kappa = A_\kappa z_\kappa} \left\{ \mathcal{J}_T(z_\kappa) = \frac{1}{2} \int_0^T \|Cz_\kappa - y^\delta\|_{\mathcal{Y}}^2 dt + \frac{\epsilon}{2} \|z_\kappa(0)\|_{\mathcal{Z}_\kappa}^2 \right\}. \quad (1.20)$$

To solve the problem of minimizing the cost functional \mathcal{J}_T , we will use the method of the saddle point of the associated Lagrangian. We therefore introduce the Lagrangian \mathcal{L}_T defined by

$$\begin{aligned} \mathcal{W}_\kappa(0, T) \times L^2(0, T; \mathcal{V}_\kappa) &\rightarrow \mathbb{R} \\ (z_\kappa, q_\kappa) &\mapsto \mathcal{J}_T(z_\kappa) + \langle q_\kappa, \dot{z}_\kappa - A_\kappa z_\kappa \rangle_{L^2(0, T; \mathcal{V}_\kappa), L^2(0, T; \mathcal{V}'_\kappa)}. \end{aligned}$$

The minimizer is reached at the critical point $(\bar{z}_{\kappa T}, \bar{q}_{\kappa T})$ of the Lagrangian \mathcal{L}_T . We are looking for $\bar{z}_{\kappa T}$ such that for all $\tilde{z} \in L^2(0, T; \mathcal{V}_\kappa)$, we have

$$\langle \partial_{q_\kappa} \mathcal{L}(\bar{z}_{\kappa T}, \bar{q}_{\kappa T}), \tilde{q} \rangle_{L^2(0, T; \mathcal{V}'_\kappa), L^2(0, T; \mathcal{V}_\kappa)} = \langle \tilde{q}, \dot{\bar{z}}_{\kappa T} - A_\kappa \bar{z}_{\kappa T} \rangle_{L^2(0, T; \mathcal{V}_\kappa), L^2(0, T; \mathcal{V}'_\kappa)} = 0.$$

We find, in this case, the equation of the constraint of the minimization problem.

We search $\bar{q}_{\kappa T}$ such that for all $\tilde{z} \in \mathcal{W}_\kappa(0, T)$, we have

$$\langle \partial_{z_\kappa} \mathcal{L}(\bar{z}_{\kappa T}, \bar{q}_{\kappa T}), \tilde{z} \rangle_{\mathcal{W}'_\kappa(0, T), \mathcal{W}_\kappa(0, T)} = 0.$$

This implies the following equality

$$\epsilon \langle \bar{z}_{\kappa T}(0), \tilde{z}(0) \rangle_{\mathcal{Z}_\kappa} - \int_0^T (C^*(y^\delta - C\bar{z}_{\kappa T}), \tilde{z})_{\mathcal{Z}_\kappa} dt + \int_0^T \langle \bar{q}_{\kappa T}; \dot{\tilde{z}} - A_\kappa \tilde{z} \rangle_{\mathcal{V}_\kappa, \mathcal{V}'_\kappa} dt = 0. \quad (1.21)$$

Setting $\tilde{z} : t \mapsto w\phi(t)$ with $w \in \mathcal{V}_\kappa$ and $\phi \in D(0, T)$, then we have

$$\langle \partial_{z_\kappa} \mathcal{L}(\bar{z}_{\kappa T}, \bar{q}_{\kappa T}), \tilde{z} \rangle_{\mathcal{W}'_\kappa(0, T), \mathcal{W}_\kappa(0, T)} = \int_0^T \phi(t) \langle \bar{q}_{\kappa T}, A_\kappa w \rangle_{\mathcal{V}_\kappa, \mathcal{V}'_\kappa} dt + \int_0^T \phi(t) \left(C^*(y^\delta - C\bar{z}_{\kappa T}), w \right)_{\mathcal{Z}_\kappa} dt.$$

We want to show that $\bar{q}_T \in \mathcal{W}_\kappa(0, T)$. Since $\int_0^T \bar{q}_{\kappa T} \dot{\phi}(t) dt \in \mathcal{V}_\kappa$, we get

$$\left\langle \int_0^T \bar{q}_{\kappa T}(t) \dot{\phi}(t) dt, w \right\rangle_{\mathcal{V}_\kappa, \mathcal{V}'_\kappa} = \left(\int_0^T \bar{q}_{\kappa T}(t) \dot{\phi}(t) dt, w \right)_{\mathcal{Z}_\kappa} = \left\langle \int_0^T \bar{q}_{\kappa T}(t) \dot{\phi}(t) dt, w \right\rangle_{\mathcal{V}'_\kappa, \mathcal{V}_\kappa}.$$

Therefore, we have

$$\left| \left\langle \int_0^T \bar{q}_{\kappa T}(t) \dot{\phi}(t) dt, w \right\rangle_{\mathcal{V}'_\kappa, \mathcal{V}_\kappa} \right| \leq C^{\text{st}} \|w\|_{\mathcal{V}_\kappa},$$

with $C^{\text{st}} = \|A_\kappa\|_{\mathcal{L}(\mathcal{V}'_\kappa, \mathcal{V}_\kappa)} \left\| \int_0^T \bar{q}_{\kappa T} \phi dt \right\|_{\mathcal{V}_\kappa} + \left\| \int_0^T C^*(y^\delta - C\bar{z}_{\kappa T}) dt \right\|_{\mathcal{Z}_\kappa}$.

Hence, we have shown that $\dot{q}_T \in L^2(0, T; \mathcal{V}'_\kappa)$ and therefore $\bar{q}_T \in \mathcal{W}_\kappa(0, T)$. As $\bar{q}_T \in \mathcal{W}_\kappa(0, T)$ and $\bar{z}_{\kappa T} \in \mathcal{W}_\kappa(0, T)$, we can apply Green's formula of Proposition 1.2.14 on (1.21), and we obtain

$$\begin{aligned} \epsilon \langle \bar{z}_{\kappa T}(0), \tilde{z}(0) \rangle_{\mathcal{Z}_\kappa} - \int_0^T (C^*(y^\delta - C\bar{z}_{\kappa T}), \tilde{z})_{\mathcal{Z}_\kappa} dt - \int_0^T \langle \dot{\bar{q}}_{\kappa T}; \tilde{z} \rangle_{\mathcal{V}_\kappa, \mathcal{V}'_\kappa} dt \\ + \langle \bar{q}_{\kappa T}(T), \tilde{z}(T) \rangle_{\mathcal{Z}_\kappa} - \langle \bar{q}_{\kappa T}(0), \tilde{z}(0) \rangle_{\mathcal{Z}_\kappa} - \int_0^T \langle \bar{q}_{\kappa T}, A_\kappa \tilde{z} \rangle_{\mathcal{V}_\kappa, \mathcal{V}'_\kappa} dt = 0. \end{aligned}$$

We then find $\bar{q}_{\kappa T}(0) = \epsilon \bar{z}_{\kappa T}(0)$ and the dynamics (1.19). ■

To find the initial conditions, we need to solve the dynamics (1.19). Note that this problem, known as the two-ends problem ([Lions, 1968], [Bensoussan, 1971]) combines a forward and a backward dynamics. The dynamics of z_κ starts from the initial conditions and moves forward in time, while the adjoint dynamics q_κ dynamics starts from the final conditions and moves backward in time.

1.3.2 4D-var approach with variational solutions of second-order system

Let us now move to the framework adapted to the wave equation. This section is divided into two parts depending on whether we are looking at *field* measurements or *velocity* measurements. In fact, the cost functional we define depends on the measurements we consider. Indeed, in the case of first-order system, the cost functional (1.46) has the advantage of being more general, since it uses a general observation operator C that can be defined in the case of *field* or *velocity* measurements. In the case of second-order systems, in the data attachment term, it must be specified in advance whether *field* or *velocity* measurements are to be considered, and the associated observation operator must be defined.

Case of *field* measurements

In this section we want to consider directly the realistic case when the measurement noise belongs to $L^2(\omega)$. Therefore, we define the observation space as $\mathcal{Y} = L^2(\omega)$. As in the previous section, we introduce an observation operator $C_\omega \in \mathcal{L}(\mathcal{H}, \mathcal{Y})$, which is given by

$$\check{y}(t) = C_\omega \check{u}(t) = \check{u}|_\omega.$$

Namely, we have

$$C_\omega = I_\omega \text{ where } I_\omega : L^2(\Omega) \rightarrow L^2(\omega) \text{ is the restriction operator.} \quad (1.22)$$

Note that the adjoint of the observation operator is easily identified and corresponds to the following

$$C_\omega^* = \mathbb{1}_\omega \text{ where } \mathbb{1}_\omega : L^2(\omega) \rightarrow L^2(\Omega) \text{ is the extension by 0 in } \Omega \setminus \omega \text{ of any function defined on } \omega. \quad (1.23)$$

We introduce the cost functional to be minimized $\mathcal{J}_T : \tilde{\mathcal{W}}_{\mathcal{V}}(0, T) \rightarrow \mathbb{R}$

$$\mathcal{J}_T(u) = \frac{1}{2} \int_0^T \|C_\omega u - y^\delta\|_{\mathcal{Y}}^2 dt + \frac{\lambda}{2} \|u(0)\|_{\mathcal{V}}^2 + \frac{\mu}{2} \|\dot{u}(0)\|_{\mathcal{H}}^2. \quad (1.24)$$

Note that since $\tilde{\mathcal{W}}_{\mathcal{V}}(0, T) \hookrightarrow C^0([0, T]; \mathcal{V})$, the condition $u(0) \in \mathcal{V}$ is well defined. Moreover, since $\mathcal{W}_\kappa(0, T) \hookrightarrow C^0([0, T]; \mathcal{H})$, the condition $\dot{u}(0) \in \mathcal{H}$ is well defined too.

We denote the minimizer by $\bar{u}_T := \underset{u \in \tilde{\mathcal{W}}_{\mathcal{V}}(0, T)}{\operatorname{argmin}} \mathcal{J}_T$.

Theorem 1.3.2 *The minimizer \bar{u} of the cost functional (1.24) is given by the following dynamics, for $\bar{p}_T \in \tilde{\mathcal{W}}(0, T)$ and $\bar{u} \in \tilde{\mathcal{W}}_{\mathcal{V}}(0, T)$,*

$$\begin{cases} \ddot{\bar{u}}_T = -A_0 \bar{u}_T, & \text{in } (0, T), \\ \ddot{\bar{p}}_T + A_0^* \bar{p}_T = -C_\omega^*(C_\omega \bar{u}_T - y^\delta), & \text{in } (0, T), \\ -\lambda A_0 \bar{u}_T(0) = \dot{\bar{p}}_T(0), \\ \mu \dot{\bar{u}}_T(0) = \bar{p}_T(0), \\ \bar{p}_T(T) = 0, \\ \dot{\bar{p}}_T(T) = 0. \end{cases} \quad (1.25)$$

PROOF:

We introduce the minimization problem under a constraint

$$\min_{\dot{u} = -A_0 u, u \in \tilde{\mathcal{W}}_{\mathcal{V}}(0, T)} \left\{ \mathcal{J}_T(u) = \frac{\lambda}{2} \|u(0)\|_{\mathcal{H}}^2 + \frac{\mu}{2} \|\dot{u}(0)\|_{\mathcal{H}}^2 + \frac{1}{2} \int_0^T \|C_\omega u - y^\delta\|_{\mathcal{Y}}^2 dt \right\}. \quad (1.26)$$

The Lagrangian associated with this minimization problem is the following $\mathcal{L}(u, p) : \tilde{\mathcal{W}}_{\mathcal{V}}(0, T) \times L^2(0, T, \mathcal{V}) \rightarrow \mathbb{R}$ defined by

$$\mathcal{L}(u, p) = \mathcal{J}_T(u) + \int_0^T \langle p, \ddot{u} + A_0 u \rangle_{\mathcal{V}, \mathcal{V}'} dt.$$

We search \bar{u}_T such that for all $\tilde{p} \in L^2(0, T; \mathcal{V})$, we have

$$\langle \partial_p \mathcal{L}, \tilde{p} \rangle_{L^2(0, T; \mathcal{V}'), L^2(0, T; \mathcal{V})} = 0,$$

in particular it implies

$$\langle \ddot{\bar{u}}_T + A_0 \bar{u}_T, \tilde{p} \rangle_{L^2(0, T; \mathcal{V}'), L^2(0, T; \mathcal{V})} = 0,$$

which corresponds to the constraint definition.

We search \bar{p}_T such that for all $\tilde{u} \in \tilde{\mathcal{W}}_{\mathcal{V}}(0, T)$, we have

$$\begin{aligned} \langle \partial_u \mathcal{L}, \tilde{u} \rangle_{\tilde{\mathcal{W}}_{\mathcal{V}}(0, T)', \tilde{\mathcal{W}}_{\mathcal{V}}(0, T)} &= \lambda(\bar{u}_T(0), \tilde{u}(0))_{\mathcal{V}} + \mu(\dot{\bar{u}}_T(0), \dot{\tilde{u}}(0))_{\mathcal{H}} \\ &\quad + \int_0^T \left(C_{\omega} \bar{u}_T - y^{\delta}, C_{\omega} \tilde{u} \right)_{\mathcal{Y}} dt + \int_0^T \langle \bar{p}_T, \ddot{\tilde{u}} \rangle_{\mathcal{V}, \mathcal{V}'} dt \\ &\quad + \int_0^T \langle \bar{p}_T, A_0 \tilde{u} \rangle_{\mathcal{V}, \mathcal{V}'} dt = 0. \end{aligned} \quad (1.27)$$

We impose $\tilde{u}(0) = 0$ and $\dot{\tilde{u}} = 0$, which leads to

$$\int_0^T \left(C_{\omega}^* (C_{\omega} \bar{u}_T - y^{\delta}), \tilde{u} \right)_{\mathcal{H}} dt + \int_0^T \langle \bar{p}_T, \ddot{\tilde{u}} \rangle_{\mathcal{V}, \mathcal{V}'} dt + \int_0^T \langle \bar{p}_T, A_0 \tilde{u} \rangle_{\mathcal{V}, \mathcal{V}'} dt = 0. \quad (1.28)$$

In order to find the dynamics of \bar{p}_T , formally, we integrate by part the second term, which gives us

$$\begin{aligned} \int_0^T \left(C_{\omega}^* (C_{\omega} \bar{u}_T - y^{\delta}), \tilde{u} \right)_{\mathcal{H}} dt + \int_0^T \langle \ddot{\bar{p}}_T, \tilde{u} \rangle_{\mathcal{V}', \mathcal{V}} dt + \int_0^T \langle \bar{p}_T, A_0 \tilde{u} \rangle_{\mathcal{V}, \mathcal{V}'} dt \\ + (\bar{p}_T(T), \dot{\tilde{u}}(T))_{\mathcal{H}} - (\dot{\bar{p}}_T(T), \tilde{u}(T))_{\mathcal{H}} = 0. \end{aligned} \quad (1.29)$$

We then formally obtain the dynamics of \bar{p}_T .

$$\begin{cases} \ddot{\bar{p}}_T + A_0^* \bar{p}_T = -C_{\omega}^* (C_{\omega} \bar{u}_T - y^{\delta}), & \text{in } (0, T), \\ \bar{p}_T(T) = 0, \\ \dot{\bar{p}}_T(T) = 0. \end{cases} \quad (1.30)$$

For a given $\{u(t) \in \tilde{\mathcal{W}}_{\mathcal{V}}(0, T), t \in [0, T]\}$, we have that $C_{\omega}^* (C_{\omega} u - y^{\delta}) \in L^2(0, T; \mathcal{H})$, hence by the Theorem 1.2.5, there exists $\bar{p}_T \in \tilde{\mathcal{W}}(0, T)$ solutions of (1.30). Furthermore, we have $\ddot{\bar{p}}_T \in L(0, T; \mathcal{V}')$. By construction, \bar{p}_T satisfies (1.28). We still have to show the uniqueness of the solution \bar{p}_T . We consider two solutions p_1 and p_2 , and we denote the difference by $p = p_1 - p_2$ which satisfies

$$\int_0^T \langle p, \ddot{\tilde{u}} + A_0 \tilde{u} \rangle_{\mathcal{V}, \mathcal{V}'} dt = 0. \quad (1.31)$$

We choose \tilde{u} such that

$$\begin{cases} \ddot{\tilde{u}}(t) + A_0 \tilde{u}(t) = p(t), & \text{in } (0, T), \\ \tilde{u}(0) = 0, \\ \dot{\tilde{u}}(0) = 0. \end{cases}$$

Since $p \in L^2(0, T; \mathcal{H})$, we have $\ddot{u} + A_0 \tilde{u} \in L^2(0, T; \mathcal{H})$. Therefore, we obtain

$$\int_0^T (p, \ddot{u} + A_0 \tilde{u})_{\mathcal{H}} dt = 0. \quad (1.32)$$

Using the expression of \tilde{u} , we finally have

$$\int_0^T \|p\|_{\mathcal{H}}^2 dt = 0,$$

which proves the uniqueness.

Moreover, since we have $\bar{p}_T \in \tilde{\mathcal{W}}(0, T)$ we can apply the Green's formula of Proposition 1.2.3 to (1.27), and we get

$$\begin{aligned} & \lambda(\bar{u}_T(0), \tilde{u}(0))_{\mathcal{V}} + \mu(\dot{\bar{u}}_T(0), \dot{\tilde{u}}(0))_{\mathcal{H}} + \int_0^T \left(C_{\omega}^*(C_{\omega} \bar{u}_T - y^{\delta}), \tilde{u} \right)_{\mathcal{H}} dt \\ & + \int_0^T \langle \ddot{\bar{p}}_T, \tilde{u} \rangle_{\mathcal{V}', \mathcal{V}} dt + \int_0^T \langle A_0^* \bar{p}_T, \tilde{u} \rangle_{\mathcal{V}', \mathcal{V}} dt \\ & (\bar{p}_T(T), \dot{\tilde{v}}(T)) - (\bar{p}_T(0), \dot{\tilde{u}}(0)) - (\dot{\bar{p}}_T(T), \tilde{u}(T)) + (\dot{\bar{p}}_T(0), \tilde{u}(0)) = 0. \end{aligned}$$

Using the dynamics (1.30) of \bar{p}_T we find the following initial conditions

$$\begin{cases} -\lambda A_0 \bar{u}_T(0) = \dot{\bar{p}}_T(0), \\ \mu \dot{\bar{u}}_T(0) = \bar{p}_T(0), \end{cases}$$

which concludes the proof. ■

Case of *velocity* measurements

Let $\mathcal{Y} = L^2(\omega)$. The observation operator $C \in \mathcal{L}(\mathcal{H}, \mathcal{Y})$ satisfies

$$\check{y}(t) = C_{\omega} \dot{u}(t) = \dot{u}|_{\omega}(t).$$

Namely, we have the same definition of the observation operator C and its adjoint as in the previous case of *field* measurements. We introduce the cost functional to minimize $\mathcal{J}_T : \tilde{\mathcal{W}}_{\mathcal{V}} \rightarrow \mathbb{R}$

$$\mathcal{J}_T(u) = \frac{1}{2} \int_0^T \|C_{\omega} \dot{u} - y^{\delta}\|_{\mathcal{Y}}^2 dt + \frac{\lambda}{2} \|u(0)\|_{\mathcal{V}}^2 + \frac{\mu}{2} \|\dot{u}(0)\|_{\mathcal{H}}^2. \quad (1.33)$$

We denote the minimizer of the above cost functional by $\bar{u}_T := \underset{u \in \tilde{\mathcal{W}}_{\mathcal{V}}(0, T)}{\operatorname{argmin}} \mathcal{J}_T$.

Theorem 1.3.3

$$\begin{cases} \ddot{\bar{u}}_T = -A_0 \bar{u}_T, & \text{in } (0, T), \\ \ddot{\bar{p}}_T + A_0^* \bar{p}_T = C_{\omega}^*(C_{\omega} \dot{\bar{u}}_T - y^{\delta}), & \text{in } (0, T), \\ \bar{u}_T(0) = \lambda^{-1} \bar{p}_T(0), \\ \dot{\bar{u}}_T(0) = \mu^{-1} \dot{\bar{p}}_T(0), \\ \bar{p}_T(T) = 0, \\ \dot{\bar{p}}_T(T) = 0. \end{cases} \quad (1.34)$$

PROOF: As before, we introduce the minimization problem under a constraint

$$\min_{\substack{\ddot{u} = -A_0 u, \\ u \in \tilde{\mathcal{W}}_{\mathcal{V}}(0, T)}} \{ \mathcal{J}_T(u) = \frac{\lambda}{2} \|u(0)\|_{\mathcal{V}}^2 + \frac{\mu}{2} \|\dot{u}(0)\|_{\mathcal{H}}^2 + \frac{1}{2} \int_0^T \|C_\omega \dot{u} - y^\delta\|_{\mathcal{Y}}^2 dt \}. \quad (1.35)$$

The Lagrangian associated with this minimization problem is the following $\mathcal{L}(u, q) : \tilde{\mathcal{W}}_{\mathcal{V}}(0, T) \times L^2(0, T, \mathcal{V}) \rightarrow \mathbb{R}$ defined by

$$\mathcal{L}(u, q) = \mathcal{J}_T(u) + \int_0^T \langle q, \ddot{u} + A_0 u \rangle_{L^2(0, T; \mathcal{V}), L^2(0, T; \mathcal{V}')} dt.$$

We search \bar{u}_T such that for all $\tilde{q} \in L^2(0, T; \mathcal{V})$, we have

$$\langle \partial_q \mathcal{L}, \tilde{q} \rangle_{L^2(0, T; \mathcal{V}'), L^2(0, T; \mathcal{V})} = 0,$$

in particular we find the constraint definition

$$\langle \ddot{\bar{u}}_T + A_0 \bar{u}_T, \tilde{q} \rangle_{L^2(0, T; \mathcal{V}'), L^2(0, T; \mathcal{V})} = 0.$$

We are looking for \bar{q}_T such that, for all $\tilde{u} \in \tilde{\mathcal{W}}_{\mathcal{V}}(0, T)$, we have

$$\begin{aligned} \langle \partial_u \mathcal{L}, \tilde{u} \rangle_{\tilde{\mathcal{W}}_{\mathcal{V}}(0, T), \tilde{\mathcal{W}}_{\mathcal{V}}(0, T)} &= \lambda(\bar{u}_T(0), \tilde{u}(0))_{\mathcal{V}} + \mu(\dot{\bar{u}}_T(0), \dot{\tilde{u}}(0))_{\mathcal{H}} \\ &\quad + \int_0^T \langle C_\omega \dot{\bar{u}}_T - y^\delta, C_\omega \dot{\tilde{u}} \rangle_{\mathcal{Y}} dt + \int_0^T \langle \bar{q}_T, \ddot{\tilde{u}} \rangle_{\mathcal{V}, \mathcal{V}'} dt \\ &\quad - \int_0^T \langle \bar{q}_T, A_0 \tilde{u} \rangle_{\mathcal{V}, \mathcal{V}'} dt = 0. \end{aligned} \quad (1.36)$$

We introduce the following change of variables $\dot{\bar{p}}_T(t) = \bar{q}_T(t) \in L^2(0, T; \mathcal{V})$. Therefore, we can rewrite (1.36) as follows

$$\begin{aligned} \lambda(\bar{u}_T(0), \tilde{u}(0))_{\mathcal{V}} + \mu(\dot{\bar{u}}_T(0), \dot{\tilde{u}}(0))_{\mathcal{H}} &+ \int_0^T \langle C_\omega \dot{\bar{u}}_T - y^\delta, C_\omega \dot{\tilde{u}} \rangle_{\mathcal{Y}} dt \\ &+ \int_0^T \langle \dot{\bar{p}}_T, \ddot{\tilde{u}} \rangle_{\mathcal{V}, \mathcal{V}'} dt - \int_0^T \langle \dot{\bar{p}}_T, A_0 \tilde{u} \rangle_{\mathcal{V}, \mathcal{V}'} dt = 0. \end{aligned} \quad (1.37)$$

We impose $\tilde{u}(0) = 0$ and $\dot{\tilde{u}}(0) = 0$, thus we get

$$\int_0^T \langle C_\omega \dot{\bar{u}}_T - y^\delta, C_\omega \dot{\tilde{u}} \rangle_{\mathcal{Y}} dt + \int_0^T \langle \dot{\bar{p}}_T, \ddot{\tilde{u}} \rangle_{\mathcal{V}, \mathcal{V}'} dt - \int_0^T \langle \dot{\bar{p}}_T, A_0 \tilde{u} \rangle_{\mathcal{V}, \mathcal{V}'} dt = 0. \quad (1.38)$$

Formally, we integrate by part the two last terms which leads to

$$\begin{aligned} \int_0^T \langle C_\omega \dot{\bar{u}}_T - y^\delta, C_\omega \dot{\tilde{u}} \rangle_{\mathcal{Y}} dt &- \int_0^T \langle \ddot{\bar{p}}_T, \tilde{u} \rangle_{\mathcal{V}', \mathcal{V}} dt + (\dot{\bar{p}}_T(T), \dot{\tilde{u}}(T))_{\mathcal{H}} \\ &+ \int_0^T \langle \bar{p}_T, A_0 \dot{\tilde{u}} \rangle_{\mathcal{V}, \mathcal{V}'} dt - (\bar{p}_T(T), A_0 \tilde{u}(T))_{\mathcal{H}} = 0. \end{aligned} \quad (1.39)$$

Therefore, formally we obtain the dynamics of \bar{p}_T .

$$\begin{cases} \ddot{\bar{p}}_T + A_0^* \bar{p}_T = C_\omega^* (C_\omega \dot{\bar{u}}_T - y^\delta), & \text{in } (0, T), \\ A_0^* \bar{p}_T(T) = 0, \\ \dot{\bar{p}}_T(T) = 0. \end{cases} \quad (1.40)$$

For a given $\{u(t) \in \tilde{\mathcal{W}}_{\mathcal{V}}(0, T), t \in [0, T]\}$, we have $C_{\omega}^*(C_{\omega}\dot{u}_T - y^{\delta}) \in L^2(0, T; \mathcal{H})$, and hence the Theorem 1.2.5, there exists $\bar{p}_T \in \tilde{\mathcal{W}}(0, T)$ solutions of (1.40). Furthermore, we have $\ddot{\bar{p}}_T \in L(0, T; \mathcal{V}')$. By construction, \bar{p}_T satisfies (1.38). Let us prove the uniqueness of the solution \bar{p}_T . We consider two solutions p_1 and p_2 , and we denote the difference by $p = p_1 - p_2$ which satisfies

$$\int_0^T \langle \dot{p}, \ddot{u} - A_0 \ddot{u} \rangle_{\mathcal{V}, \mathcal{V}'} dt = 0. \quad (1.41)$$

We choose \tilde{u} such that

$$\begin{cases} \ddot{\tilde{u}}(t) - A_0 \ddot{\tilde{u}}(t) = p(t), & \text{in } (0, T), \\ \tilde{u}(0) = 0, \\ \dot{\tilde{u}}(0) = 0. \end{cases}$$

Since $p \in L^2(0, T; \mathcal{H})$, we have $\ddot{\tilde{u}} - A_0 \ddot{\tilde{u}} \in L^2(0, T; \mathcal{H})$. Therefore, we obtain

$$\int_0^T (\dot{p}, \ddot{\tilde{u}} - A_0 \ddot{\tilde{u}})_{\mathcal{H}} dt = 0. \quad (1.42)$$

Using the expression of \tilde{u} , we have

$$\int_0^T \|\dot{p}\|_{\mathcal{H}}^2 dt = 0. \quad (1.43)$$

Since A_0 is a self-adjoint invertible operator, it does not have a kernel, therefore we obtain

$$A_0^* \bar{p}_T(T) = 0 \implies \bar{p}_T(T) = 0. \quad (1.44)$$

Gathering (1.43) and (1.44), we obtain that

$$p \equiv 0,$$

that proves the uniqueness.

Furthermore, we apply the Green's formula of Proposition 1.2.14 to (1.37), and we get

$$\begin{aligned} & \lambda(\bar{u}_T(0), \tilde{u}(0))_{\mathcal{V}} + \mu(\dot{\bar{u}}_T(0), \dot{\tilde{u}}(0))_{\mathcal{H}} + \int_0^T (C_{\omega}^*(C_{\omega}\dot{u}_T - y^{\delta}), \dot{\tilde{u}})_{\mathcal{H}} dt \\ & - \int_0^T \langle \ddot{\bar{p}}_T, \dot{\tilde{u}} \rangle_{\mathcal{V}', \mathcal{V}} dt - \int_0^T (A_0^* \bar{p}_T, \dot{\tilde{u}})_{\mathcal{V}', \mathcal{V}} dt + (\bar{p}_T(T), A_0 \tilde{v}(T)) \\ & - (\dot{\bar{p}}_T(0), \dot{\tilde{u}}(0)) + (\dot{\bar{p}}_T(T), \dot{\tilde{u}}(T)) - (\bar{p}_T(0), A_0 \tilde{u}(0)) = 0. \end{aligned}$$

Therefore, we get

$$\begin{cases} \ddot{\bar{u}}_T = -A_0 \bar{u}_T, & \text{in } (0, T), \\ \ddot{\bar{p}}_T + A_0^* \bar{p}_T = C_{\omega}^*(C_{\omega}\dot{u}_T - y^{\delta}), & \text{in } (0, T), \\ A_0^* \bar{u}_T(0) = \lambda^{-1} A_0^* \bar{p}_T(0), \\ \dot{\bar{u}}_T(0) = \mu^{-1} \dot{\bar{p}}_T(0), \\ A_0^* \bar{p}_T(T) = 0, \\ \dot{\bar{p}}_T(T) = 0. \end{cases} \quad (1.45)$$

It is possible to weaken the regularity of the second equation of (1.45). Indeed, the 3rd and 5th equations can be rewritten as follows

$$\begin{cases} \bar{u}_T(0) = \lambda^{-1} \bar{p}_T(0), \\ \bar{p}_T(T) = 0. \end{cases}$$

Therefore, the dynamics of \bar{p}_T in (1.45) is then defined in \mathcal{V}' , and we find the dynamics (1.25). \blacksquare

As in the first-order formalism, we need to solve a dynamics, either (1.25) or (1.34), where one part of the dynamics is forward in time and the other is backward in time.

1.3.3 4D-var approach and semigroup formalism

In this section, as in Section 1.3.1, we solve the inverse problem, which consists in recovering the initial condition from the observations of the system, but this time the solutions of the wave equation is obtained in the framework of semigroup theory (Section 1.3.2). Lagrangian \mathcal{L}_T given by (1.20) is not be introduced because the solution z considered in this section does not naturally belong to the space $\mathcal{W}_\kappa(0, T)$. We will therefore use a different method. Let us recall precisely the inverse problem we study: Given measurements $y^\delta \in \mathcal{Y}_T = L^2((0, T); \mathcal{Y})$ of the target system \check{z} solution of (1.6) associated with unknown initial conditions z_0 . We recall that in practice, the available measurements y^δ are noisy – where δ is the noise level bound. As in the previous section, we introduce the cost functional to minimize

$$\mathcal{J}_T(z_0) = \frac{\epsilon}{2} \|z_0\|_{\mathcal{Z}}^2 + \frac{1}{2} \int_0^T \|Cz_{|z_0}(t) - y^\delta(t)\|_{\mathcal{Y}}^2 dt, \quad (1.46)$$

where the observation operator $C \in \mathcal{L}(\mathcal{Z}, \mathcal{Y})$ is defined by (1.15) and the notation $z_{|z_0}$ denotes the solution z of the wave equation (1.6) associated with the initial condition z_0 . The inverse problem corresponds to invert the following *input-output* operator

$$\Psi_T : \begin{cases} \mathcal{Z} \rightarrow L^2((0, T); \mathcal{Y}) := \mathcal{Y}_T, \\ z_0 \mapsto (y(t))_{t \in [0, T]} = (\Psi(t)z_0)_{t \in [0, T]}. \end{cases} \quad (1.47)$$

It is possible to rewrite the cost functional (1.46) relying on the *input-output* operator

$$\mathcal{J}_T(z_0) = \frac{\epsilon}{2} \|z_0\|_{\mathcal{Z}}^2 + \frac{1}{2} \int_0^T \|\Psi(t)z_0 - y^\delta(t)\|_{\mathcal{Y}}^2 dt, \quad (1.48)$$

This problem can be rewritten using the adjoint of the *input-output* operator such that

$$\Psi_T^* \Psi_T z_0 = \Psi_T^* \check{y}.$$

Since the available measurements y^δ are noisy, they could not belong to $\text{Ran} \Psi_T$. In this particular case, we can use a Tikhonov regularization (see, for example, [Tihonov, 1963], [Baumeister, 1987], and [Engl et al., 1996]) and solve the following problem

$$\Psi_T^* \Psi_T z_0^{\delta, \epsilon} + \epsilon z_0^{\delta, \epsilon} = \Psi_T^* y^\delta,$$

where ϵ is the Tikhonov regularization parameter. In the following, to simplify the notation, we will omit the exponent δ, ϵ and T and write down the initial state z_0 . In our case, the *input-output* operator is defined using Duhamel's formula as follows

$$\Psi(t) = Ce^{tA}.$$

So we have to solve the following normal equation

$$\int_0^T e^{tA^*} C^* C e^{tA} z_0 dt + \epsilon z_0 = \int_0^T e^{tA^*} C^* y^\delta(t) dt. \quad (1.49)$$

We will show that solving this minimization problem is equivalent to solving a two-ends problem ([Bensoussan, 1971], [Lions, 1968]).

Theorem 1.3.4 *The following two-ends problem*

$$\begin{cases} \dot{\bar{z}}_T = A\bar{z}_T, & \text{in } (0, T), \\ \dot{\bar{q}}_T + A^*\bar{q}_T = -C^*(y^\delta - C\bar{z}_T), & \text{in } (0, T), \\ \bar{z}_T(0) = \epsilon^{-1}\bar{q}_T(0), \\ \bar{q}_T(T) = 0, \end{cases} \quad (1.50)$$

is the optimal system associated with the minimization of the cost functional

$$\mathcal{J}_T(z_0) = \frac{\epsilon}{2}\|z_0\|_{\mathcal{Z}}^2 + \frac{1}{2}\int_0^T \|Cz|_{z_0}(t) - y^\delta(t)\|_{\mathcal{Y}}^2 dt.$$

PROOF: We compute the Frechet derivative of \mathcal{J}_T using the Duhamel formula (1.5)

$$\langle D_{z_0}\mathcal{J}_T(z_0), \xi \rangle = \epsilon(z_0, \xi)_{\mathcal{Z}} + (Ce^{tA}z_0 - y^\delta, Ce^{tA}\xi)_{\mathcal{Y}_T}, \quad \forall \xi \in \mathcal{Z}. \quad (1.51)$$

The minimizer is obtained when $\langle D_{z_0}\mathcal{J}_T(\bar{z}_0), \xi \rangle = 0$, which leads to the so-called normal equation

$$\int_0^T e^{tA^*}C^*Ce^{tA}\bar{z}_0 dt + \epsilon\bar{z}_0 = \int_0^T e^{tA^*}C^*y^\delta(t)dt. \quad (1.52)$$

It is possible to drastically simplify the normal equation by relying on the so-called adjoint equation (which we found in the Section 1.19)

$$\begin{cases} \dot{q}(t) + A^*q(t) = -C^*(y^\delta(t) - Cz(t)), & t \in (0, T), \\ q(T) = 0. \end{cases} \quad (1.53)$$

Here the adjoint dynamics is defined backward in time. Nevertheless, for a given $\{z(t) \in \mathcal{Z}, t \in [0, T]\}$ and its associated measurements $\{y(t) \in \mathcal{Y}, t \in [0, T]\}$, this dynamics is well-posed. Indeed, for a A^* is a generator of a C^0 -semigroup (see Theorem 1.2.10), thus $-A^*$ is the generator of a C^0 semigroup backward in time. Since $C^*(y^\delta - Cz) \in C^0([0, T]; \mathcal{Z})$ using theorem 1.2.9, the adjoint q can be defined as a weak solution of (1.53). Using the Duhamel formula, we compute the expression of the adjoint

$$q(t) = \int_t^T e^{(\tau-t)A^*}C^*(y^\delta(\tau) - Cz(\tau)) d\tau. \quad (1.54)$$

Thus, (1.52) becomes

$$-\int_0^T e^{tA^*}C^*(y^\delta(t) - Ce^{tA}\bar{z}_0)dt + \epsilon\bar{z}_0 = 0.$$

Recalling that the optimal trajectory is $\bar{z}_T(s) = e^{tA}\bar{z}_0$ using Duhamel'formula, we have

$$-\bar{q}_T(0) + \epsilon\bar{z}_0 = 0. \quad (1.55)$$

Since the cost functional defines a quadratic functional in the Hilbert space \mathcal{Z} and $\theta \mapsto \|\theta\|_{\mathbb{H}_0^1(\Omega)}^2$ is coercive, we have the existence of a unique minimizer. Moreover, the uniqueness of this solution of (1.50) is ensured by the uniqueness of the minimizer of \mathcal{J}_T . \blacksquare

Note that from (1.55) and using the dynamics of \bar{q}_T given by (1.54), we find the normal equation (1.49) associated with Ψ_T , which is classically defined in the Tikhonov regularization problems.

Therefore, we find a direct relation to find our initial condition, but this requires solving the two-ends problem. The disadvantage of this system is that is not a Cauchy problem.

1.3.4 Link between first-order and second-order systems

We now aim to establish a link between the two-ends systems in the respective first- and second-order formulations. We separate the cases where we consider *field* measurements and *velocity* measurements.

Case of *velocity* measurements

Proposition 1.3.5 *The first-order formulation of the two-ends problem (1.50) is equivalent to the second-order formulation (1.34) when $\epsilon = \mu = \lambda$.*

PROOF: Let $\mathcal{Z} = \mathbf{H}_0^1(\Omega) \times \mathbf{L}^2(\Omega) = \mathcal{V} \times \mathcal{H}$ and $\mathcal{Y} = \mathbf{L}^2(\omega)$. We set $C = \begin{pmatrix} 0 & C_\omega \end{pmatrix}^t \in \mathcal{L}(\mathcal{Z}, \mathcal{Y})$ since $C_\omega \in \mathcal{L}(\mathcal{H}, \mathcal{Y})$ defined by $C_\omega = I_\omega$. Choosing $q_T = \begin{pmatrix} p_T & \dot{p}_T \end{pmatrix}^t \in \mathbf{L}^2(0, T; \mathcal{V}) \times \mathbf{L}^2(0, T; \mathcal{H})$, we have

$$\begin{pmatrix} \dot{p}_T \\ \ddot{p}_T \end{pmatrix} - \begin{pmatrix} 0 & \text{Id} \\ -A_0 & 0 \end{pmatrix} \begin{pmatrix} p_T \\ \dot{p}_T \end{pmatrix} = - \begin{pmatrix} 0 \\ C_\omega^* \end{pmatrix} (y^\delta - C_\omega \dot{u}_T).$$

Thus, the first equation is zero, the second corresponds to the second-order adjoint dynamics of (1.34). The final condition of (1.50) leads to

$$q_T(T) = \begin{pmatrix} p_T(T) \\ \dot{p}_T(T) \end{pmatrix} = 0,$$

which corresponds to the final conditions of the adjoint equations of (1.34). Finally, we have

$$q_T(0) = \begin{pmatrix} p_T(0) \\ \dot{p}_T(0) \end{pmatrix} = \epsilon z_T(0) = \begin{pmatrix} \epsilon u_T(0) \\ \epsilon \dot{u}_T(0) \end{pmatrix},$$

which corresponds to the initial conditions with $\epsilon = \mu = \lambda$. ■

Case of *field* measurements

Proposition 1.3.6 *The first-order formulation of the two-ends problem (1.50) is equivalent to the second-order formulation (1.25) when $\epsilon = \mu = \lambda$.*

PROOF: Let $\mathcal{Z} = \mathbf{H}_0^1(\Omega) \times \mathbf{L}^2(\Omega) = \mathcal{V} \times \mathcal{H}$ and $\mathcal{Y} = \mathbf{L}^2(\omega)$. We consider the observation operator associated with the two-ends problem (1.50) as the following

$$C = \begin{pmatrix} I_\omega & 0 \end{pmatrix} : \mathcal{Z} \rightarrow \mathbf{L}^2(\omega) \text{ where } I_\omega \in \mathcal{L}(\mathbf{H}_0^1(\Omega), \mathbf{L}^2(\omega)). \quad (1.56)$$

Then, the adjoint observation is given by $C^* = \begin{pmatrix} F_\omega & 0 \end{pmatrix}^t$ where $F_\omega \in \mathcal{L}(\mathbf{L}^2(\omega), \mathbf{H}_0^1(\Omega))$ is defined by

$$F_\omega \phi := \psi \text{ the solution of } \begin{cases} -\Delta \psi = \mathbb{1}_\omega \phi, & \text{in } \Omega, \\ \psi = 0, & \text{on } \partial\Omega, \end{cases} \quad (1.57)$$

where $\mathbb{1}_\omega$ is the characteristic function of ω . In other words, we have

$$F_\omega = A_0^{-1} \mathbb{1}_\omega.$$

Choosing $q_T = (-A_0^{-1}\dot{p}_T \quad p_T)^t \in L^2(0, T; \mathcal{V}) \times L^2(0, T; \mathcal{H})$, the two-ends problem (1.50) leads to

$$\begin{pmatrix} -A_0^{-1}\ddot{p}_T \\ \dot{p}_T \end{pmatrix} - \begin{pmatrix} 0 & \text{Id} \\ -A_0 & 0 \end{pmatrix} \begin{pmatrix} -A_0^{-1}\dot{p}_T \\ p_T \end{pmatrix} = - \begin{pmatrix} A_0^{-1}\mathbb{1}_\omega \\ 0 \end{pmatrix} (y^\delta - u_{T|\omega}).$$

The second equation is zero and the first one gives

$$-A_0^{-1}\ddot{p}_T - p_T = -A_0^{-1}\mathbb{1}_\omega(y^\delta - u_{T|\omega}),$$

which implies

$$\ddot{p}_T + A_0 p_T = \mathbb{1}_\omega(y^\delta - u_{T|\omega}).$$

Since $A_0^* = A_0$ and the observation operator in the case of *field* measurements and its adjoint is given by (1.22) and (1.23), we obtain

$$\ddot{p}_T + A_0^* p_T = C_\omega^*(y^\delta - u_{T|\omega}),$$

and it corresponds to the dynamics of the adjoint equation of (1.25) since $A_0^* = A_0$. The final condition of (1.50) leads to

$$q_T(T) = \begin{pmatrix} -A_0^{-1}\dot{p}_T(T) \\ p_T(T) \end{pmatrix} = 0 \implies \begin{pmatrix} \dot{p}_T(T) = 0 \\ p_T(T) = 0 \end{pmatrix},$$

which corresponds to the final conditions of the adjoint equations of (1.25). Furthermore, we have

$$q_T(0) = \begin{pmatrix} -A_0^{-1}\dot{p}_T(0) \\ p_T(0) \end{pmatrix} = \epsilon z_T(0) = \begin{pmatrix} \epsilon u_T(0) \\ \epsilon \dot{u}_T(0) \end{pmatrix},$$

which are to the initial conditions with $\epsilon = \mu = \lambda$. ■

1.3.5 Gradient descent for practical minimization

To solve the problem at two-ends (1.50), a natural idea is to use an iterative method. In this section we present one such possibility, the gradient descent associated with (1.50). Later (Section 1.4) we will see that we prefer to use a sequential method to solve the inverse problem. Finally, we emphasize that this is not only a practical method to find the minimizer of \mathcal{J}_T , but also a method to prove the existence of a solution of (1.50). The limit of the gradient descent dynamics (1.58)-(1.59) when k tends to infinity is the solution of (1.50).

By the Riesz's representation theorem we have the expression for the gradient of the cost functional

$$\nabla \mathcal{J}_T(z_0) = \epsilon z_0 - q(0).$$

Knowing the expression of the gradient, we start with the recursive relation

$$z_0^{k+1} = z_0^k - \rho_k \nabla \mathcal{J}_T(z_0^k), \quad k \geq 0$$

where $(\rho_k)_{k \geq 0}$ is a chosen relaxation sequence. We therefore have the following gradient descent algorithm which consists in solving recursively from $z_0 = 0$ and for $k \geq 0$ the back

and forth dynamics

$$\begin{cases} \dot{z}^{k+1} = Az^{k+1}, & \text{in } [0, T], \\ z^{k+1}(0) = (1 - \epsilon\rho_k)z^k(0) + \epsilon\rho_k q^k(0), \end{cases} \quad (1.58)$$

and

$$\begin{cases} \dot{q}^{k+1} + A^*q^{k+1} = -C^*(y^\delta - Cz^{k+1}), & \text{in } [0, T], \\ q^{k+1}(T) = 0. \end{cases} \quad (1.59)$$

Now that we have seen how the initial condition can be reconstructed by solving the two-ends problem, the question of time discretization arises. Can we get the same type of result after discretization in time and space?

1.3.6 Time discretization: midpoint scheme

The philosophy behind this section is the following: It is not enough to discretize the obtained results in time, we must also *discretize-then-optimize* (see [Zuazua, 2005]). The first step is to discretize the initial wave equation and then repeat all the steps in discrete time until the different results are obtained. We discretize the wave equation using Lagrange finite element and a midpoint scheme. We denote $z_h \in \mathcal{Z}_h = \mathcal{V}_h \times \mathcal{V}_h$ and consider

$$\begin{cases} \frac{z_{h,n+1} - z_{h,n}}{\tau} = A_h \frac{z_{h,n+1} + z_{h,n}}{2}, & 0 \leq n \leq N, \\ z_{h,n=0} = z_{h,0}, \end{cases} \quad (1.60)$$

where

$$A_h = \begin{pmatrix} 0 & \text{Id}_h \\ \Delta_{0,h} & 0 \end{pmatrix},$$

with $\Delta_{0,h}$ is the bounded linear operator in \mathcal{V}_h associated with the finite-element discretization of Δ_0 . We can rewrite the above scheme using the bounded transition operator $\Phi_{h,n+1|n}$.

$$\begin{cases} z_{h,n+1} = \Phi_{h,n+1|n} z_{h,n}, & 0 \leq n \leq N, \\ z_{h,n=0} = z_{h,0}, \end{cases} \quad \text{where } \Phi_{h,n+1|n} = \left(\text{Id}_h - \frac{\tau}{2} A_h \right)^{-1} \left(\text{Id}_h + \frac{\tau}{2} A_h \right). \quad (1.61)$$

We can then obtain a direct formula for $z_{h,n}$ as a function of the initial data $z_{h,0}$ which corresponds to a time discretization of the Duhamel formula (1.5).

$$z_{h,n} = \Phi_{h,n+1|0} z_{h,0} \quad \text{where } \Phi_{h,n|k} = \Phi_{h,n|n-1} \circ \dots \circ \Phi_{h,k+1|k} \quad \text{and } \Phi_{h,n|n} = \text{Id}_h.$$

We introduce the discrete-time cost functional to be minimized.

$$\mathcal{J}_{h,N}(z_{h,0}) = \frac{\epsilon}{2} \|z_{h,0}\|_{\mathcal{Z}_h}^2 + \frac{1}{2} \sum_{n=0}^N \tau \|C_h z_{h,n} - y_{h,n}^\delta\|_{\mathcal{Y}}^2. \quad (1.62)$$

We express this cost functional as a minimization problem under constraints. In other words, we search the minimizer of

$$\mathcal{J}_{h,N}(z_{h,n}) = \frac{\epsilon}{2} \|z_{h,n=0}\|_{\mathcal{Z}_h}^2 + \frac{1}{2} \sum_{n=0}^N \tau \|C_h z_{h,n} - y_{h,n}^\delta\|_{\mathcal{Y}}^2, \quad (1.63)$$

under the constraint $z_{h,n+1} = \Phi_{h,n+1|n} z_{h,n}$. The first step is to find the adjoint equations. For this purpose we introduce the Lagrangian.

$$\mathcal{L}_{h,N}(z_{h,n}, q_{h,n}) = \mathcal{J}_{h,N}(z_{h,n=0}) + \sum_{n=0}^{N-1} (q_{n+1}, z_{h,n+1} - \Phi_{h,n+1|n} z_{h,n})_{\mathcal{Z}_h}. \quad (1.64)$$

To find the adjoint equations, we calculate the derivative of the Lagrangian with respect to $z_{h,n}$, and look at the critical point.

$$\frac{\partial \mathcal{L}_{h,N}}{\partial z_{h,n}} = \tau C_h^* (C_h z_{h,n} - y_{h,n}^\delta) - \Phi_{h,n+1|n}^* q_{h,n+1} + q_{h,n} = 0,$$

which leads to the adjoint dynamics

$$q_{h,n} = \Phi_{h,n+1|n}^* q_{h,n+1} - \tau C_h^* (C_h z_{h,n} - y_{h,n}^\delta). \quad (1.65)$$

In order to find the final condition of the adjoint dynamics, we compute the derivative of $\mathcal{L}_{h,N}$ with respect to $z_{h,n+1}$, and we find

$$\frac{\partial \mathcal{L}_{h,N}}{\partial z_{h,N}} = q_{h,N+1} = 0. \quad (1.66)$$

We then find the discretized adjoint equations (1.65) - (1.66). Note that it is also possible to find the relation between the initial condition $z_{h,0}$ and the adjoint equation. Indeed, we compute the derivative of $\mathcal{L}_{h,N}$ with respect to z_0 and obtain at the critical point

$$\frac{\partial \mathcal{L}_{h,N}}{\partial z_{h,0}} = \epsilon z_{h,0} + \tau C_h^* (C_h z_{h,0} - y_{h,0}^\delta) - \Phi_{h,1|0,*} q_{h,1} = 0.$$

Using dynamics (1.65) at time $n = 0$, we ultimately arrive at

$$\epsilon z_{h,0} = q_{h,0}. \quad (1.67)$$

We can now state the most important theorem connecting the two-ends system with the minimization of the cost functional.

Theorem 1.3.7 *The following two-ends problem*

$$\begin{cases} \bar{z}_{h,n+1} = \Phi_{h,n+1|n} \bar{z}_{h,n}, & 0 \leq n \leq N, \\ \bar{q}_{h,n} = \Phi_{h,n+1|n}^* \bar{q}_{h,n+1} - \tau C_h^* (C_h \bar{z}_{h,n} - y_{h,n}^\delta), & 0 \leq n \leq N, \\ \bar{z}_{h,0} = \epsilon^{-1} \bar{q}_{h,0}, \\ \bar{q}_{h,N+1} = 0, \end{cases} \quad (1.68)$$

is the optimal system associated with the minimization of the cost functional $\mathcal{J}_{h,N}$ defined by (1.62).

PROOF: We compute the Frechet derivative of $\mathcal{J}_{h,N}$.

$$\langle D_{z_{h,0}} \mathcal{J}_{h,N}, \xi \rangle = \epsilon (z_{h,0}, \xi)_{\mathcal{Z}_h} + \sum_{n=0}^N \tau (C_h z_{h,n} - y_{h,n}^\delta, C_h \Phi_{h,n|0} \xi)_y.$$

The optimum is reached when $\langle D_{z_{h,0}} \mathcal{J}_{h,N}, \xi \rangle = 0$. Thus, we have

$$\epsilon (\bar{z}_{0,h}, \xi)_{\mathcal{Z}_h} + \sum_{n=0}^N \tau (C_h^* (C_h \bar{z}_{h,n} - y_{h,n}^\delta), \Phi_{h,n|0} \xi)_{\mathcal{Z}_h} = 0.$$

Using the adjoint dynamics found in (1.65), we get

$$\epsilon(\bar{z}_{h,0}, \xi)_{\mathcal{Z}_h} + \sum_{n=0}^N \left(\Phi_{h,n|0}^* \Phi_{h,n+1|n}^* \bar{q}_{h,n+1} - \Phi_{h,n|0}^* \bar{q}_{h,n}, \xi \right)_{\mathcal{Z}_h} = 0.$$

Using that $\Phi_{h,n|0}^* \Phi_{h,n+1|n}^* = \Phi_{h,0|n+1}$ and $\Phi_{h,0|0} = \text{Id}_h$, we then find

$$\epsilon(\bar{z}_{h,0}, \xi)_{\mathcal{Z}_h} - (\bar{q}_{h,0}, \xi)_{\mathcal{Z}_h} + (\Phi_{h,0|N+1} \bar{q}_{h,N+1}) = 0.$$

From (1.66), we have that $\bar{q}_{h,N+1} = 0$ where we directly infer the desired result. \blacksquare

We have seen that to find the minimizers of the cost functional we need to solve the two-ends problem (1.68), and that one way is to use a gradient descent (1.58). Of course, we could use any other iterative method that is more efficient to solve this problem such as GMRES. However, first-order method will imply certain numbers of iterations to converge. Only a second order method will lead to a solution in one iteration, for problems of reasonable size since it is costly to compute. In fact, in the next section, we will introduce the Kalman Filter which is equivalent to minimizing the cost function with a second order method.

1.4 The Kalman filter: a dynamic programming approach

Dynamic programming, introduced by Bellman in the 1950s [Bellman, 1966] in finite dimension and later extended to infinite dimension for the problem under study [Bensoussan, 1971], leads to a sequential (or filtering) approach for solving the two-ends problem (1.50). The goal is to recover the optimal estimate by injecting a feedback term: a discrepancy between the observations of the system and the simulated trajectory, often referred to as innovation. In other words, we will construct a time function that uses the measurements available through a feedback law and equivalently solves the minimization problem at final time. One of the best known filters is the Kalman filter [Kalman and Bucy, 1961]. As previously explained we will now limit our presentation to the semigroup approach even if all the concepts can be introduced in the framework of variational solution (see [Bensoussan, 1971]).

1.4.1 Riccati dynamics from the semigroup approach

We present the Kalman filter

$$\begin{cases} \dot{\hat{z}} = A\hat{z} + PC^*(y^\delta - C\hat{z}), & \text{in } (0, T), \\ \hat{z}(0) = 0, \end{cases} \quad (1.69)$$

where P is solution of

$$\begin{cases} \dot{P} = AP + PA^* - PC^*CP, & t > 0, \\ P(0) = P_0. \end{cases} \quad (1.70)$$

In our case, we set $P_0 = \epsilon^{-1}\text{Id}$. The dynamics (1.70) corresponds to a Riccati equation ([Bensoussan et al., 2007, Part IV, Chapter 1]) and for $P_0 \in S^+(\mathcal{Z})$ admits a mild solution in $C^0([0, T]; S^+(\mathcal{Z}))$ where $S^+(\mathcal{Z})$ is the space of positive self-adjoint bounded operators (see Chapter 3 for a complete analysis).

One can observe that for all $t \in [0, T]$, $A(t) - P(t)C^*C(t)$ is a bounded perturbation of A .

Consequently, one can show that there exists one and only one mild solution in $C^0([0, T]; \mathcal{Z})$ of (1.69) which is also a weak solution in $L^2(0, T; \mathcal{Z})$ satisfying for all $v \in D(A^*)$

$$\begin{cases} \frac{d}{dt}(\hat{z}(t), v) = (\hat{z}(t), A^*v) + (y^\delta(t) - C\hat{z}(t), CP(t)v), & t \in (0, T), \\ (\hat{z}(0), v) = 0. \end{cases}$$

We here refer to Chapter 3 where we will give all the details of the proof of existence of solution for such problems. Then, we can show the following fundamental inequality (see [Aussal and Moireau, 2022], and Chapter 3), which links the sequential approach and the variational approach

$$\forall t \in [0, T], \bar{z}_T(t) = \hat{z}(t) + P(t)\bar{q}_T(t), \quad (1.71)$$

where P is the solution of (1.70), (\bar{z}_T, \bar{q}_T) are defined in (1.50) and \hat{z} is the solution of (1.69). Finally, note that (1.69) is indeed a Kalman estimator, since at the final time T , the adjoint equation $\bar{q}_T(T) = 0$, *i.e.* we have

$$\bar{z}_T(T) = \hat{z}(T). \quad (1.72)$$

The link between the minimization problem and the sequential method is established at each time t thanks to the fundamental identity (1.71). Moreover, both are equivalent at the last time T . We will see that this relation is preserved after adequate time discretization in Section 1.4.4.

1.4.2 Kalman filter as an observer

In this section, we prove that the Kalman filter we have just presented is an observer according to the definition of [Krener, 1999], *i.e.*, it has two properties, an asymptotic convergence property in the absence of noise and a property of robustness to noise. The observer property of the Kalman filter relies on an important tool: observability conditions [Lions, 1988].

To introduce the notion of observer, we consider a wave model in an infinite time window,

$$\begin{cases} \partial_{tt}u(x, t) - \Delta u(x, t) = 0, & (x, t) \in \Omega \times (0, \infty), \\ u(x, t) = 0, & (x, t) \in \partial\Omega \times (0, \infty), \\ u(x, 0) = u_0(x), \quad \partial_t u(x, 0) = v_0(x), & x \in \Omega. \end{cases} \quad (1.73)$$

As in the previous section, we reformulate this system as a first-order system like (1.6) but with infinite horizon

$$\begin{cases} \dot{z} = Az, & \text{in } (0, \infty), \\ z(0) = z_0. \end{cases} \quad (1.74)$$

As shown earlier, we assume that we observe a target solution \check{z} with unknown initial condition. The perfect observations correspond to \check{y} such that $\check{y} = C\check{z}$, where C is the observation operator and will be clarified just bellow. Unfortunately, these measurements \check{y} are not available. We assume that we have $y^\delta \in \mathcal{Y}_\infty = L^2((0, \infty); \mathcal{Y})$, so that, for example

$$\|\check{y} - y^\delta\|_{\mathcal{Y}_\infty}^2 \leq \delta^2 T.$$

Note that in this case the noise is assumed to belong to the observation space \mathcal{Y} , *i.e.* it is quite regular. We will see in Section 1.5.3 how to cope when $y^\delta \notin \mathcal{Y}$ *i.e.* when they belong to a larger space than \mathcal{Y} .

1.4.2.1 Definition of an observer

We can now introduce the notion of observer with a simplified version of an asymptotic observer proposed by [Krener, 1999].

Definition 1.4.1 *An asymptotic observer of (1.74) is a function $\mathbb{R}^+ \ni t \mapsto \hat{z}(t; (y^\delta(s))_{s \leq t})$ such that*

1. **Robustness to noise** : for all $0 \leq \tau$, and for all $\varepsilon > 0$, there exist $\alpha, \delta > 0$, such that

$$\left. \begin{array}{l} \|\hat{z}(0) - \check{z}(0)\|_{\mathcal{Z}} \leq \alpha, \\ \|\check{y} - y^\delta\|_{L^2((0, \tau); \mathcal{Y})} \leq \delta \end{array} \right\} \Rightarrow \|\check{z}(t) - \hat{z}(t)\|_{\mathcal{Z}} \leq \varepsilon, \quad t \geq \tau.$$

2. **Asymptotic convergence**: for all $0 \leq \tau$, and for all $\varepsilon, \alpha, \delta > 0$, there exists $T_0 > 0$ such that

$$\left. \begin{array}{l} \|\hat{z}(0) - \check{z}(0)\|_{\mathcal{Z}} \leq \alpha, \\ \|\check{y} - y^\delta\|_{L^2((0, \tau); \mathcal{Y})} \leq \delta \\ \text{supp}(\check{y} - y^\delta) \subset (0, \tau) \end{array} \right\} \Rightarrow \|\check{z}(t) - \hat{z}(t)\|_{\mathcal{Z}} \leq \varepsilon, \quad t \geq T_0 + \tau.$$

Remark 1.4.1 *The observer can be calculated at time t from the information available at time $s \leq t$, thus it is a causal function of the measurements. Note also that, as a consequence of 1., an asymptotic observer satisfies the following condition, for all $s \geq 0$*

$$\hat{z}(s) = \check{z}(s), \quad (y^\delta(t) = \check{y}(t), t \geq s) \Rightarrow (\hat{z}(t) = \check{z}(t), t \geq s). \quad (1.75)$$

To show that the Kalman dynamics is an observer, we need an observability condition.

1.4.2.2 Observability conditions

In this section we recall the concept of observability conditions, namely if there exists a time T_0 and a constant C^{st} , so that for all $T > T_0$ and for all mild solutions of the wave equation from an initial condition z_0 , we have

$$\int_0^T \|y\|_{\mathcal{Y}}^2 dt = \|\Psi_T z_0\|_{\mathcal{Y}_T}^2 = \int_0^T \|C e^{tA} z_0\|_{\mathcal{Y}}^2 dt \geq C^{\text{st}} \|z_0\|_{\mathcal{Z}}^2. \quad (1.76)$$

A proof of observability condition (1.77) for the wave equation is given in Section 1.4.5.2. Indeed, observing $y \in \mathcal{Y}$ on $(0, T)$ yields as much information as knowing $z(0) \in \mathcal{Z}$. Indeed, the observability condition is equivalent to a coercivity condition of $\Psi_T^* \Psi_T \in \mathcal{L}(\mathcal{Z})$ where the *input-output* operator is defined by (1.47), namely

$$\forall z_0 \in \mathcal{Z}, \quad (z_0, \Psi_T^* \Psi_T z_0)_{\mathcal{Z}} = \int_0^T \|C e^{tA} z_0\|_{\mathcal{Y}}^2 dt \geq C^{\text{st}} \|z_0\|_{\mathcal{Z}}^2. \quad (1.77)$$

1.4.2.3 The Kalman filter as an observer

We can now demonstrate that the Kalman filter is an observer in the sense of [Krener, 1999] when considering either *velocity* or *field* measurements.

Theorem 1.4.2 *Suppose that the observability (1.77) holds. The estimator \hat{z} defined by (1.69) is an asymptotic observer in the sense of Definition 1.4.1.*

PROOF: We start by showing the property of asymptotic convergence in the absence of noise (namely $C\tilde{z} = y^\delta$), *i.e.* the second property of Definition 1.4.1.

We here use the link between the observer and the functional minimization. On one hand, from (1.72), we have that

$$\hat{z} = \bar{z}_t(t) = z|_{\bar{\theta}_t},$$

thus, we obtain

$$\|\hat{z}(t) - \tilde{z}\|_{\mathcal{Z}}^2 = \|\bar{z}_t(t) - \tilde{z}\|_{\mathcal{Z}}^2, \quad (1.78)$$

where \bar{z}_T is defined by (1.50). On the other hand, we set $\tilde{z} = \bar{z}_t - \tilde{z}$ which is the solution of

$$\begin{cases} \dot{\tilde{z}} = A\tilde{z}, & \text{in } (0, T) \\ \tilde{z}(0) = \bar{z}_t(0) - \tilde{z}_0 := \tilde{z}_0. \end{cases}$$

Since A is the generator of a C^0 -semigroup of contraction (see Theorem 1.2.8), we have that

$$\|\tilde{z}(t)\|_{\mathcal{Z}}^2 \leq \|\tilde{z}_0\|_{\mathcal{Z}}^2.$$

Using the observability property (1.77), we obtain that there exists a positive scalar C^{st} such that

$$\|\tilde{z}(t)\|_{\mathcal{Z}}^2 \leq \|\tilde{z}_0\|_{\mathcal{Z}}^2 \leq C^{\text{st}} \int_0^t \|C\tilde{z}(s)\|_{\mathcal{Y}}^2 ds = C^{\text{st}} \int_0^t \|C\bar{z}_t(s) - Cz(s)\|_{\mathcal{Y}}^2 ds. \quad (1.79)$$

Since we consider noise-free observations, we have that $Cz = y^\delta$, therefore, we obtain

$$\|\tilde{z}(t)\|_{\mathcal{Z}}^2 \leq C^{\text{st}} \int_0^t \|C\bar{z}_t(s) - y^\delta(s)\|_{\mathcal{Y}}^2 ds.$$

Recalling that the cost function associated with the minimization problem is defined by

$$\mathcal{J}_T(z_0) = \frac{\epsilon}{2} \|z_0\|_{\mathcal{Z}}^2 + \frac{1}{2} \int_0^T \|Cz|_{z_0}(t) - y^\delta(t)\|_{\mathcal{Y}}^2 dt,$$

we obtain

$$\|\tilde{z}(t)\|_{\mathcal{Z}}^2 \leq C^{\text{st}} (2\mathcal{J}_T(\bar{z}_t(0))) \leq C^{\text{st}} (2\mathcal{J}_T(\tilde{z}_0)) \leq C^{\text{st}} \epsilon \|\tilde{z}_0\|_{\mathcal{Z}}^2.$$

Hence, combining the above equation and (1.78), we proved the second property of Definition 1.4.1. Let us now show the first property, namely the stability of the observer to the noise. We start from (1.79). We inject the noisy observations and by triangle inequality we obtain

$$\|\tilde{z}(t)\|_{\mathcal{Z}}^2 \leq 2C^{\text{st}} \left[\int_0^t \|C\bar{z}_t(s) - y^\delta(s)\|_{\mathcal{Y}}^2 ds + \int_0^t \|C\tilde{z}(s) - y^\delta(s)\|_{\mathcal{Y}}^2 ds \right].$$

Finally, using again the definition of the cost functional we have

$$\begin{aligned} \|\tilde{z}(t)\|_{\mathcal{Z}}^2 &\leq 2C^{\text{st}} \left[2\mathcal{J}_T(\tilde{z}_0) + \int_0^t \|C\tilde{z}(s) - y^\delta(s)\|_{\mathcal{Y}}^2 ds \right] \\ &\leq 2C^{\text{st}} \left[\epsilon \|\tilde{z}_0\|_{\mathcal{Z}}^2 + 2 \int_0^t \|C\tilde{z}(s) - y^\delta(s)\|_{\mathcal{Y}}^2 ds \right], \end{aligned}$$

which combining with (1.78) prove the first property of Definition 1.4.1. ■

1.4.3 Initial conditions reconstruction problem

The initial objective is to estimate the initial conditions, that is, to compute \hat{z}_0 , which is not computed by the Kalman estimator \hat{z} . One idea might be to use a time reversal method [Fink et al., 2000] to recover the initial conditions. However, this method is only valid for the wave equation, since it requires the time reversal property. A more general idea is proposed here. The idea is to inject the fundamental identity (1.71) into the adjoint dynamics (1.50) to obtain

$$\begin{cases} \dot{\bar{q}}_T + (A^* - C^*CP)\bar{q}_T = -C^*(y^\delta - C\hat{z}_T), & \text{in } (0, T), \\ \bar{q}_T(T) = 0, \end{cases} \quad (1.80)$$

The dynamics (1.80) makes possible to reconstruct the initial conditions in one iteration. Unfortunately, it is computed backward in time, while the Riccati solution is computed forward in time, which is not very practical. To remedy this, we first introduce the operator $t \mapsto Q(t) \in \mathcal{L}(\mathcal{Z})$ solution of

$$\begin{cases} \dot{Q} = AQ - PC^*CQ, & \text{in } (0, T), \\ Q(0) = P_0 := \epsilon^{-1}\text{Id}. \end{cases} \quad (1.81)$$

One can prove that $Q \in C^0([0, T]; \mathcal{L}(\mathcal{Z}))$ such that

$$\forall z \in \mathcal{Z}, \quad Q(t)z = e^{tA}P_0z - \int_0^t e^{(t-s)A}P(s)C^*CQ(s)z \, ds.$$

This defines a *mild* solution of (1.81). Furthermore, Q is also a strict solution in the senses that for all $z \in \mathcal{D}(A)$, $Qz \in C^0([0, T]; \mathcal{D}(A))$. Then, we introduce the estimator

$$\begin{cases} \dot{\hat{z}}_0 = QC^*(y^\delta - C\hat{z}), & \text{in } (0, T), \\ \hat{z}_0(0) = 0, \end{cases} \quad (1.82)$$

which admits a weak solution in $L^2((0, T); \mathcal{Z})$ as $t \mapsto Q(t)C^*(y^\delta(t) - C\hat{z}(t)) \in L^2((0, T); \mathcal{Z})$. This new operator and dynamics allow us to reconstruct the initial conditions, as described in the next proposition. In other words, by solving an additional forward Riccati equation and an additional forward dynamics, we are able to directly estimate \bar{z}_{0T} with only one iteration.

Proposition 1.4.3 *For all $T \geq 0$, we have the following identity*

$$\forall t \in [0, T], \quad \bar{z}_{0T} = \hat{z}_0(t) + Q^*(t)\bar{q}_T(t). \quad (1.83)$$

Hence, \hat{z}_0 is a Kalman estimator of the initial condition in the following sense:

$$\forall t \geq 0, \quad \hat{z}_0(t) = \bar{z}_{0t}. \quad (1.84)$$

PROOF: We denote $\eta : t \mapsto \hat{z}_0(t) + Q^*(t)\bar{q}_T(t)$, and consider $v \in \mathcal{D}(A)$ giving

$$\begin{aligned} \frac{d}{dt}(\eta, v)_{\mathcal{Z}} &= \frac{d}{dt}(\hat{z}_0(t), v)_{\mathcal{Z}} + \frac{d}{dt}(\bar{q}_T(s), Q(t)v)_{\mathcal{Z}} \Big|_{s=t} + \frac{d}{dt}(\bar{q}_T(t), Q(s)v)_{\mathcal{Z}} \Big|_{s=t} \\ &= ((y^\delta - C\hat{z}), CQ(t)v)_{\mathcal{Z}} - (\bar{q}_T(t), (A - PC^*C)Q(t)v)_{\mathcal{Z}} \\ &\quad - ((y^\delta - C\hat{z}), CQ(t)v)_{\mathcal{Z}} + (\bar{q}_T(t), (A - PC^*C)Q(t)v)_{\mathcal{Z}} \\ &= 0. \end{aligned}$$

Therefore, $\eta \equiv \eta(0) = P_0 \bar{q}_T(0) = \epsilon^{-1} \bar{q}_T(0) = \bar{z}_{0T}$, whence justifies (1.83). Finally, $q_T(T) = 0$ in (1.83) yields $\hat{z}_0(T) = \bar{z}_{0T}$. \blacksquare

In contrast to the gradient descent method presented in Section 1.3.5, we have presented a direct method for solving the initial conditions reconstruction problem. However, this requires the computation of a Riccati operator, which can be very costly. Therefore, in the Section 1.5, we will introduce an alternative to avoid the computation of a Riccati operator.

1.4.4 Time discretization: prediction-correction scheme

We now present a time scheme based on the discrete-time Kalman filter, which is a *Prediction - Correction* scheme [Kalman, 1960].

$$\text{Prediction: } \begin{cases} \hat{z}_{h,n+1}^- = \Phi_{h,n+1|n} \hat{z}_{h,n}^+, \\ P_{h,n+1}^- = \Phi_{h,n+1|n} P_{h,n}^+ \Phi_{h,n+1|n}^*, \end{cases} \quad (1.85)$$

$$\text{Correction: } \begin{cases} \hat{z}_{h,n}^+ = \hat{z}_{h,n}^- + \tau P_{h,n}^+ C_h^* (y_{h,n}^\delta - C_h \hat{z}_{h,n}^-), \\ \left(P_{h,n}^+\right)^{-1} = \left(P_{h,n}^-\right)^{-1} + \tau C_h^* C_h. \end{cases} \quad (1.86)$$

With observations at time 0, the procedure begins with initialization of

$$\left(P_{h,n=0}^+\right)^{-1} = \left(P_{h,0}\right)^{-1} + \tau C_h^* C_h \text{ and } \hat{z}_{h,n=0}^+ = \hat{z}_{h,0} + \tau P_{h,n}^+ C_h^* (y_{h,0}^\delta - C_h \hat{z}_{h,0}).$$

We could also start with

$$P_{h,n=0}^- = P_{h,0} \text{ and } \hat{z}_{h,n=0}^- = \hat{z}_{h,0}.$$

Note that, in many cases the a priori already takes into account the observation at time t_0 , which is why the algorithm usually starts with the prediction step.

We can rewrite the *Prediction - Correction* scheme (1.85) in a one-step version by keeping only the variables $\hat{z}_{h,n+1}^-$ and $P_{h,n+1}^-$. Indeed, on the one hand, we have

$$P_{h,n+1}^- = \Phi_{h,n+1|n} \left(\left(P_{h,n}^-\right)^{-1} + \tau C_h^* C_h \right)^{-1} \Phi_{h,n+1|n}^*.$$

Using Woodbury formula, we get the one-step version of $P_{h,n+1}^-$

$$P_{h,n+1}^- = \Phi_{h,n+1|n} \left(P_{h,n}^- - P_{h,n}^- C_h^* \left(\tau^{-1} + C_h P_{h,n}^- C_h^* \right)^{-1} C_h P_{h,n}^- \right) \Phi_{h,n+1|n}^*. \quad (1.87)$$

On the other hand, $\hat{z}_{h,n+1}^-$ can be rewritten as follows

$$\hat{z}_{h,n+1}^- = \Phi_{h,n+1|n} \hat{z}_{h,n}^- + \tau \Phi_{h,n+1|n} P_{h,n}^+ C_h^* (y_{h,n}^\delta - C_h \hat{z}_{h,n}^-). \quad (1.88)$$

Using the dynamics of $P_{h,n}^+$ given by (1.85), we find the one-step version of $\hat{z}_{h,n+1}^-$

$$\hat{z}_{h,n+1}^- = \Phi_{h,n+1|n} \hat{z}_{h,n}^- + \tau \Phi_{h,n+1|n} \left(\left(P_{h,n}^-\right)^{-1} + \tau C_h^* C_h \right)^{-1} C_h^* (y_{h,n}^\delta - C_h \hat{z}_{h,n}^-). \quad (1.89)$$

Finally, we can show that the fundamental identity (1.71) is satisfied in a time-discrete version.

Theorem 1.4.4 *We have the fundamental identity for all $0 \leq n \leq N + 1$,*

$$\bar{z}_{h,n} = \hat{z}_{h,n}^- + P_{h,n}^- \bar{q}_{h,n}, \quad (1.90)$$

where $P_{h,n}^-$ and $\hat{z}_{h,n}^-$ are given by (1.87) and (1.89) and $\bar{z}_{h,n}$ is given by (1.68). In particular, we have

$$\bar{z}_{h,N+1} = \hat{z}_{h,N+1}^-. \quad (1.91)$$

The second property (1.91) follows from the first (1.90) one since at the final time, the adjoint equation $\bar{q}_{h,N+1} = 0$.

Therefore, we have shown that, even in the discrete case, the sequential Kalman filter method is equivalent to minimizing the cost function. It has the advantage of giving an estimation in one iteration. However, it has a disadvantage: the operator P must be computed. This operator is the solution of a Riccati equation and therefore numerically expensive. In the Section 1.5, we will look at defining an alternative observer – here called a Luenberger observer – which aims to replace the operator P with an operator much simpler.

1.4.5 Application to the scalar wave equation

In this section we will show the observability results for the wave equation, for which we need to go into more detail. Before we present the various results of observability, we need to define more precisely the observation operator under consideration.

1.4.5.1 Definition of the observation operator

We adopt the semigroup formalism of the section (1.2.2). We specify the different observation operators C that will be studied. Assuming that we have access to internal *velocity* measurements $y = \partial_t u|_\omega \in \mathcal{Y} = L^2(\omega)$, we introduce

$$C = \begin{pmatrix} 0 & I_\omega \end{pmatrix} : \begin{cases} \mathbb{H}_0^1(\Omega) \times L^2(\Omega) \rightarrow L^2(\omega), \\ z = \begin{pmatrix} u \\ v \end{pmatrix} \mapsto y = v|_\omega, \end{cases} \quad (1.92)$$

where $I_\omega : L^2(\Omega) \rightarrow L^2(\omega)$ is the restriction operator. Although, when we consider internal *field*-measurements $y = u|_\omega$, we set $\mathcal{Y} = \mathbb{H}_\Gamma^1(\omega)$ – the subspace of functions in $\mathbb{H}^1(\omega)$ zero to Γ . Indeed, since the solution u belongs to $\mathbb{H}_0^1(\Omega)$, it is logical that its restriction to the observation subdomain belongs to $\mathbb{H}^1(\omega)$. Therefore, we define the observation operator by

$$C = \begin{pmatrix} I_\omega & 0 \end{pmatrix} : \begin{cases} \mathbb{H}_0^1(\Omega) \times L^2(\Omega) \rightarrow \mathbb{H}_\Gamma^1(\omega), \\ z = \begin{pmatrix} u \\ v \end{pmatrix} \mapsto y = u|_\omega, \end{cases} \quad (1.93)$$

where $I_\omega : \mathbb{H}_0^1(\Omega) \rightarrow \mathbb{H}_\Gamma^1(\omega)$ is the restriction operator. In each case, we need to define the adjoint of the observation operator. For the first case, we directly infer that

$$C^* = \begin{pmatrix} 0 \\ \mathbb{1}_\omega \end{pmatrix} : \begin{cases} L^2(\omega) \rightarrow \mathbb{H}_0^1(\Omega) \times L^2(\Omega), \\ y \mapsto \begin{pmatrix} 0 \\ \mathbb{1}_\omega y \end{pmatrix}, \end{cases} \quad (1.94)$$

where $\mathbb{1}_\omega$ naturally defines the extension by zero on Ω of a function defined on ω , namely

$$\mathbb{1}_\omega y : \Omega \ni x \mapsto \begin{cases} y(x), & x \in \omega, \\ 0, & x \in \Omega \setminus \omega. \end{cases}$$

The second case is a little more intricate. As in [Chapelle et al., 2012a], we equipped the space $H_\Gamma^1(\omega)$ with the norm $\|\cdot\|_{H_\omega^1} = \|E_\omega(\cdot)\|_{H_0^1(\Omega)}$, where $E_\omega \in \mathcal{L}(H_\Gamma^1(\omega), H^1(\omega))$ is defined by

$$E_\omega \phi := \psi \text{ the solution of } \begin{cases} \Delta \psi = 0, & \text{in } \Omega \setminus \omega, \\ \psi = \phi, & \text{in } \bar{\omega}, \\ \psi = 0 \in \partial\Omega. \end{cases} \quad (1.95)$$

We recall the following lemma

Lemma 1.4.5 *The norm $\|\cdot\|_{H_\omega^1}$ is equivalent to the norm $\|\cdot\|_{H^1(\Omega)}$.*

PROOF: As in [Chapelle et al., 2012a], first we show that E_ω is bounded from $H^1(\omega)$ to $H_0^1(\Omega)$. Let $v \in H_0^1(\Omega)$ be a test function, (1.95) gives us

$$-\int_{\Omega \setminus \omega} \Delta \psi v \, dx + \int_\omega \psi v \, dx = \int_\omega \phi v \, dx.$$

Integrating by parts and since $\phi = \psi$ on ω , we obtain

$$\int_{\Omega \setminus \omega} \nabla \psi \nabla v \, dx - \int_{\partial(\Omega \setminus \omega)} \nabla \phi \cdot \eta v \, dx + \int_\omega \psi v \, dx + \int_\omega \nabla \psi \nabla v \, dx = \int_\omega \phi v \, dx + \int_\omega \nabla \phi \nabla v \, dx.$$

Choosing $v = \psi$ and using Cauchy-Schwarz and Young inequalities with $\epsilon > 0$, we get

$$\|\nabla \psi\|_{L^2(\Omega)}^2 \leq \|\nabla \phi\|_{L^2(\omega)}^2 + \frac{\epsilon}{2} \|\psi\|_{H^{-1/2}(\partial\omega \setminus \partial\Omega)}^2 + \frac{1}{2\epsilon} \|\phi\|_{H^{1/2}(\partial\omega \setminus \partial\Omega)}^2.$$

By trace property, we have

$$\|\nabla \psi\|_{L^2(\Omega)}^2 \leq \|\nabla \phi\|_{L^2(\omega)}^2 + C^{\text{st}} \frac{\epsilon}{2} \left(\|\Delta \psi\|_{L^2(\Omega \setminus \omega)}^2 + \|\psi\|_{H^1(\Omega \setminus \omega)}^2 \right) + C^{\text{st}} \frac{1}{2\epsilon} \|\phi\|_{H^1(\omega)}^2.$$

Since $\Delta \psi = 0$ on $\Omega \setminus \omega$, for ϵ small enough, we get

$$\|\nabla E_\omega \phi\|_{L^2(\Omega)}^2 = \|\nabla \psi\|_{L^2(\Omega)}^2 \leq \|\phi\|_{H^1(\omega)}^2.$$

On the other hand, we have

$$\begin{aligned} \|\phi\|_{H^1(\omega)}^2 &= \|\nabla(E_\omega \phi)\|_{L^2(\omega)}^2 + \|\phi\|_{L^2(\omega)}^2 \\ &\leq \|\nabla(E_\omega \phi)\|_{L^2(\Omega)}^2 + \|E_\omega \phi\|_{L^2(\Omega)}^2. \end{aligned}$$

Using Poincaré inequality, we get

$$\|\phi\|_{H^1(\omega)}^2 \leq (1 + C_p) \|\nabla(E_\omega \phi)\|_{L^2(\Omega)}^2. \quad \blacksquare$$

Furthermore, from (1.95) one can show the following orthogonality property

$$\forall \phi \in H_\omega^1, \forall \psi \in H_0^1(\Omega) \text{ such that } \psi|_\omega = 0, (\nabla(E_\omega \phi), \nabla \psi)_{L^2(\Omega)} = 0.$$

Therefore, we obtain that for all $\phi \in \mathbf{H}_\omega^1$ and $\psi \in \mathbf{H}_0^1(\Omega)$,

$$\begin{aligned} (\nabla(E_\omega\phi), \nabla\psi)_{L^2(\Omega)} &= (\nabla(E_\omega\phi), \nabla(E_\omega\psi|_\omega))_{L^2(\Omega)} - (\nabla(E_\omega\phi), \nabla(E_\omega\psi|_\omega - \psi))_{L^2(\Omega)} \\ &= (\nabla(E_\omega\phi), \nabla(E_\omega\psi|_\omega))_{L^2(\Omega)} = (\phi, \psi|_\omega)_{\mathbf{H}_\omega^1}. \end{aligned}$$

Then, with the Riesz representation, the adjoint of E_ω is I_ω , and thus we can directly infer that

$$C^* = \begin{pmatrix} E_\omega & 0 \end{pmatrix} : \begin{cases} \mathbf{H}_\Gamma^1(\omega) \rightarrow \mathbf{H}_0^1(\Omega) \times L^2(\Omega), \\ y \mapsto \begin{pmatrix} 0 \\ E_\omega y \end{pmatrix}. \end{cases} \quad (1.96)$$

1.4.5.2 Observability conditions for the wave equation

Several methods can be used to establish such condition. We propose to follow [Lions, 1988] and base our estimates on the multiplier approach [Komornik, 1997]. The results of the energy repartition results between the whole domain, its boundary, and/or subdomains are the basis of the multiplier approach. It allows setting observability conditions for the observation domain containing a sufficiently large part of the domain boundary. Following [Lions, 1988, Chapter 7], we first introduce a part of the boundary on which our subdomain will lean on. We define

$$\forall x_0 \in \mathbb{R}^d, \quad \Gamma(x_0) := \{x \in \partial\Omega \mid (x - x_0) \cdot \nu(x) > 0\},$$

where ν is the outward unit normal to $\partial\Omega$. We introduce a subdomain illustrated in Figure 1.1

$$\omega_\rho(x_0) = \mathcal{O}_\rho \cap \Omega \text{ where } \mathcal{O}_\rho = \bigcup_{x \in \Gamma(x_0)} B(x, \rho).$$

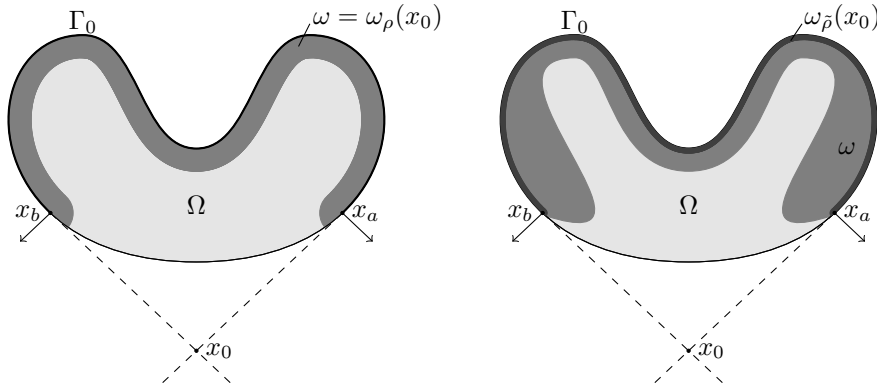


Figure 1.1: (Left) An example of domain $\omega_\rho(x_0)$. (Right) An example of domain ω where there exists $\tilde{\rho}$ s.t. $\omega_\rho(x_0) \subset \omega$. Courtesy Philippe Moireau ([Moireau and Bourgeois, 2023])

The condition – called the multiplier condition – for the observation domain ω is that it must contain (or be equal to) $\omega_\rho(x_0)$. First, we recall the observability result for the initial condition ([Lions, 1988, Chapter 7, Lemma 2.3]).

Theorem 1.4.6 *Let Ω be a bounded domain of \mathbb{R}^d of boundary $\partial\Omega$ of class C^0 . There exists a constant C^{st} such that for all $T > T(x_0) = 2 \max_{x \in \Omega} |x - x_0|$ and for all weak solutions of the homogeneous wave equation (1.1) with Dirichlet boundary and with initial condition $(u_0, u_0) \in \mathbf{H}^1(\Omega) \times L^2(\Omega)$, we have*

$$\int_0^T \int_{\omega_\rho(x_0)} [|\partial_t u|^2 + |\nabla u|^2] dx dt \geq C^{st} [\|u_0\|_{\mathbf{H}_0^1(\Omega)}^2 + \|v_0\|_{L^2(\Omega)}^2]. \quad (1.97)$$

From this theorem, one can deduce a theorem [Lions, 1988, Chapter 7, Lemme 2.4] that will be useful for subsequently obtaining an observability condition in weaker norms.

Theorem 1.4.7 *Let $\rho > 0$. There exists, T_0 and C^{st} such that for all $T > T_0$,*

$$\int_0^T \int_{\omega_\rho(x_0)} [|\partial_t u|^2 + |u|^2] dx dt \geq C^{st} [\|u_0\|_{\mathbf{H}_0^1(\Omega)}^2 + \|v_0\|_{\mathbf{L}^2(\Omega)}^2], \quad (1.98)$$

and

$$\int_0^T \int_{\omega_\rho(x_0)} [|\nabla u|^2 + |u|^2] dx dt \geq C^{st} [\|u_0\|_{\mathbf{H}_0^1(\Omega)}^2 + \|v_0\|_{\mathbf{L}^2(\Omega)}^2]. \quad (1.99)$$

for all weak solution of (1.1) with initial condition (u_0, v_0) .

The theorem 1.4.7 allows us to obtain a coercivity property of the form (1.77). Indeed, we prove directly that for $T > T_0$, Ψ_T defined by (1.47) as a linear operator of $\mathcal{Z} = \mathbf{H}_0^1(\Omega) \times \mathbf{L}^2(\Omega)$ in $\mathcal{Y}_T = \mathbf{L}^2((0, T); \mathbf{H}^1(\omega))$ or in $\mathcal{Y}_T = \mathbf{H}^1((0, T); \mathbf{L}^2(\omega))$ such that $\Psi_T^* \Psi_T \in \mathcal{L}(\mathcal{Z})$ is invertible of bounded inverse. In order to rely on less informative data, we aim to obtain weaker observability conditions.

Theorem 1.4.8 *Supposing that for a subdomain ω , the Theorem 1.4.6 holds, then we have also*

$$\int_0^T \int_\omega |\partial_t u|^2 dx dt \geq C^{st} [\|u_0\|_{\mathbf{H}_0^1(\Omega)}^2 + \|v_0\|_{\mathbf{L}^2(\Omega)}^2]. \quad (1.100)$$

PROOF: The proof is inspired from [Moireau and Bourgeois, 2023].

We proceed by contradiction. Suppose that (1.100) is not satisfied. We consider a sequence of initial conditions $\{z_0^m = (u_0^m, v_0^m)\}_{m \geq 0}$ such that

$$\|u_0^m\|_{\mathbf{H}_0^1(\Omega)}^2 + \|v_0^m\|_{\mathbf{L}^2(\Omega)}^2 = 1. \quad (1.101)$$

The weak solution of (1.1) denoted by $\{z^m = (u^m, \partial_t u^m)\}_{m \geq 0}$ with initial condition (u_0^m, v_0^m) satisfies

$$\int_0^T \int_\omega |\partial_t u^m|^2 dx dt \xrightarrow{m \rightarrow +\infty} 0. \quad (1.102)$$

From (1.101), we assume that

$$z_0^m = (u_0^m, v_0^m) \xrightarrow{m \rightarrow +\infty} z_0 = (u_0, v_0), \quad \text{in } \mathcal{Z}.$$

Therefore, we obtain

$$\int_0^T \int_\omega |\partial_t u|^2 dx dt = \lim_{m \rightarrow +\infty} \int_0^T \int_\omega \partial_t u \partial_t u^m dx dt = 0,$$

which gives us, $\partial_t u \equiv 0$ in $\omega \times (0, T)$. Thus, we have $z_0 = 0$. Eventually, we have that

$$\forall z \in \mathcal{Z}, \quad (e^{tA} z_0^m, z) = (z_0^m, e^{tA^*} z) \xrightarrow{m \rightarrow +\infty} 0,$$

which implies that $e^{tA} z_0^m \rightarrow 0$ when m tends to infinity. In other words, we get

$$\partial_t u^m \xrightarrow{m \rightarrow +\infty} 0, \quad \nabla u^m \xrightarrow{m \rightarrow +\infty} 0, \quad u^m \xrightarrow{m \rightarrow +\infty} 0 \quad \text{in } \mathbf{L}^2(\Omega \times (0, T)).$$

By the compact embedding theorem $(u^m)_{m \geq 0}$ converges strongly in $\mathbf{L}^2(\Omega \times (0, T))$, and we directly infer that the limit is 0:

$$u^m \xrightarrow{m \rightarrow +\infty} 0, \quad \text{in } \mathbf{L}^2(\Omega \times (0, T)).$$

Combined with (1.102), there exists $C^{\text{st}} > 0$ such that

$$\|u_0^m\|_{\mathbb{H}_0^1(\Omega)}^2 + \|v_0^m\|_{L^2(\Omega)}^2 \leq C^{\text{st}} \int_0^T \int_{\omega} [|\partial_t u^m|^2 + |u^m|^2] \, dx \, dt \xrightarrow{m \rightarrow +\infty} 0,$$

which contradicts (1.101). \blacksquare

This theorem corresponds to observability condition (1.77) when considering *velocity* measurements.

Historically, the multiplier approach – that identify subdomains in which the different observability conditions are satisfied– appeared in the 1980s. Shortly thereafter, the approach was widely adopted in the community, especially through more complicated multipliers that are often used to establish Carleman estimates (see Chapter 4). In the late 1980s and 1990s, microlocal tools made it possible to specify weaker geometrical conditions on domains to establish observability conditions. The *Geometric Control Condition* is introduced by [Bardos et al., 1988] and is used to define geometric conditions on the domain for which the observability condition holds with internal measurements. In other words: a subdomain ω is compatible with the following observability condition for a given T_0 if any optic ray following the Descartes rules enters the subdomain before T_0

$$\|u_0\|_{\mathbb{H}_0^1(\Omega)}^2 + \|u_1\|_{L^2(\Omega)}^2 \leq C^{\text{st}} \int_0^T \|b \partial_t u\|_{L^2(\omega)}^2 dt,$$

where $b \in C^\infty(\Omega)$. In this case, this condition is not exactly compatible with the case we consider where $b = \mathbb{1}_\omega$. Let $\omega_0 \subset \omega$, one possibility is to define a cut-off function $\rho \in C^\infty(\Omega)$ such as

$$\rho(x) \begin{cases} = 1 & \text{in } \omega_0, \\ = 0 & \text{in } \Omega \setminus \omega, \\ \in [0, 1] & \text{in } \omega \setminus \omega_0. \end{cases}$$

In this case, we can use the *Geometric Control Condition* defined by [Bardos et al., 1988], but we lose information because we do not use all available observations. Some time later, this type of result was extended to observations at the boundary (see [Bardos et al., 1992]). This is a more complicated case, since the observation operator is not bounded. We will deal with this topic in Chapter 2. In the end, we had to wait more than 20 years for a geometric condition called the strong geometric control condition (SGCC) ([Burq and Gérard, 2020]) that allows control by a $\mathbb{1}_\omega$ function. Here is the definition. For a domain Ω with analytic boundary $\partial\Omega$, we define a ray as segments $\{x_\tau(s), s \in [0, T]\}$ of the direction denoted by τ in the interior of Ω with three different behaviors when encountering the boundary $\partial\Omega$: the hyperbolic points (blue line in Figure 1.2), the diffractive points (green line in Figure 1.2), and the glancing points (red line in Figure 1.2).

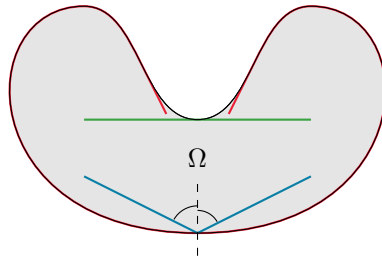


Figure 1.2: The three kind of rays: hyperbolic (blue line), diffractive (green line) and glancing (red line). Courtesy Philippe Moireau

Definition 1.4.2 (Strong Geometric Control Condition (SGCC)) *Let $\omega \subset \Omega$ and $T_0 > 0$. We say that the couple (ω, T_0) satisfies the SGCC if every ray of geometric optics called generalized geodesic $s \mapsto x_\xi(s)$ from a point ξ and traveling at speed one in Ω meets ω in a time $t < T_0$, in the sense*

$$\forall \xi \in \Omega, \exists s \in (0, T), \exists \delta > 0, B(x_\xi(s), \delta) \subset \omega. \quad (1.103)$$

We say that ω satisfies the SGCC if there exists $T_0 > 0$ such that (ω, T_0) satisfies the SGCC. This allows us to state the following result which gives necessary and sufficient conditions for observability conditions to hold. Let $T_0 > 0$. Let Ω be a domain of analytic boundary $\partial\Omega$ and $\omega \subset \Omega$ such that (ω, T_0) satisfies the SGCC, then there exists $T > T_0$ such that (1.100).

This method will be use in another part of the manuscript in Chapter 3.

1.5 An alternative: a Luenberger observer

We present a Luenberger observer inspired by the strategy proposed in [Luenberger, 1963, 1971]. The idea is to replace the operator P that appears in the Kalman filter (1.69) with an operator that is easier to compute. In our case we choose γId with $\gamma \in \mathbb{R}^{+,*}$. More precisely, we are interested in the following system

$$\begin{cases} \dot{\hat{z}} = A\hat{z} + \gamma C^*(y^\delta - C\hat{z}), & \text{in } (0, T), \\ \hat{z}(0) = 0. \end{cases} \quad (1.104)$$

1.5.1 Analysis of the Luenberger observer using observability conditions

Definition of the Luenberger observer

We introduce the function \hat{z} such that

$$\begin{cases} \dot{\hat{z}} = A\hat{z} + \gamma C^*(y^\delta - C\hat{z}), & \text{in } (0, \infty), \\ \hat{z}(0) = \hat{z}_0, \end{cases} \quad (1.105)$$

for a given \hat{z}_0 (typically $\hat{z}_0 = 0$) and $\gamma \in \mathbb{R}^{+,*}$ and $y^\delta \in L^2(0, T; \mathcal{Y})$. Note that the system defined by (1.105) is well-posed in the sense, there exists a mild solution $\hat{z} \in C^0([0, \infty); \mathcal{Z})$. Indeed, one can prove that $A_\gamma = A - \gamma C^*C$ of domain $\mathcal{D}(A)$ is maximal dissipative and hence by Theorem 1.2.9, we have the existence of a mild solution.

We would like to show that this observer converges to the state of the target system. However, we no longer have the fundamental identity property linking observer and the minimization of the cost functional. Therefore, there is no guarantee that this observer converges to the desired quantity. The proof of convergence relies on a very important tool: observability conditions. In other words, we must show that we have enough information in the measurements to be able to reconstruct the state of the system ([Lions, 1988]).

We now analyze the Luenberger observer \hat{z} defined by (1.104) and show that it converges to \bar{z}_T following [Ramdani et al., 2010].

We have that

$$\dot{\tilde{z}} = A\tilde{z} + \gamma C^*(\tilde{y} - C\tilde{z}), \quad \text{in } (0, \infty) \quad (1.106)$$

which justifies that \hat{z} and \check{z} coincide for $t \geq s$ as soon as $\hat{z}(s) = \check{z}(s)$ and here $\check{y} = C\check{z}$ (namely it is a noise-free data). We introduce the error system between (1.106) and (1.105). Let us denote the measurement noise $\eta = y^\delta - \check{y}$ and the state error $\tilde{z} = \check{z} - \hat{z}$, so we have

$$\begin{aligned}\dot{\tilde{z}} &= \dot{\check{z}} - \dot{\hat{z}} \\ &= A\check{z} - A\hat{z} - \gamma C^*(-\check{y} + C\check{z} + y^\delta - C\hat{z}) \\ &= A\tilde{z} - \gamma C^*C\tilde{z} - \gamma C^*\eta.\end{aligned}$$

Thus, the error is solution of

$$\begin{cases} \dot{\tilde{z}} = (A - \gamma C^*C)\tilde{z} - \gamma C^*\eta, & \text{in } (0, \infty), \\ \tilde{z}(0) = \tilde{z}_0 := \check{z}(0) - \hat{z}(0), \end{cases} \quad (1.107)$$

which is well-defined in $C^0([0, \infty); \mathcal{Z})$. We now demonstrate that \hat{z} is an asymptotic observer.

Theorem 1.5.1 *Assuming that $y^\delta \in L^2((0, \infty); \mathcal{Y})$, the system defined by (1.105) is an asymptotic observer if $A - \gamma C^*C$ is an exponentially stable C^0 -semigroup.*

Thus, proving that the observer property is asymptotic amounts to showing that the generator $A - \gamma C^*C$ is exponentially stable.

Let us therefore show that the C^0 -semigroup is exponentially stable under the observability condition.

Theorem 1.5.2 *We assume that there exists $T_0 > 0$ and C^{st} such that*

$$\forall T > T_0, \quad \forall z_0 \in \mathcal{Z}, \quad \int_0^T \|C e^{tA} z_0\|_{\mathcal{Y}}^2 dt \geq C^{st} \|z_0\|_{\mathcal{Z}}^2. \quad (1.108)$$

*Then, the C^0 -semigroup $\tilde{\Phi}$ associated with $A - \gamma C^*C$ is exponentially stable.*

Therefore, we have proved that the $A - \gamma C^*C$ operator is exponentially stable. We illustrate the different results by some numerical examples.

1.5.2 Space and time discretization

In this section we will define the discretization in time and space for Luenberger's observer in the context of *velocity* measurements and then *field* measurements.

Let us begin with the case of *velocity* measurements. We intend to construct the observer dynamics (1.105) starting from an initial guess $\hat{z}_0 = 0$ using a midpoint scheme, where we denote $\hat{z}(t_n) = \hat{z}_n$ for $n \in [0, N]$

$$\begin{cases} \frac{\hat{z}_{n+1} - \hat{z}_n}{\tau} = A \frac{\hat{z}_{n+1} + \hat{z}_n}{2} + \gamma C^* \left(\frac{y_{n+1}^\delta + y_n^\delta}{2} - C \frac{\hat{z}_{n+1} + \hat{z}_n}{2} \right), & \text{in } \mathcal{Z} = H_0^1(\Omega) \times L^2(\Omega), \\ \hat{z}_{n=0} = \hat{z}_0. \end{cases} \quad (1.109)$$

If we set $(\hat{z}_n)_{n \geq 0} = ((\hat{u}_n)_{n \geq 0} \quad (\hat{v}_n)_{n \geq 0})^t$ and use the definition of the observation operator (1.92), we get the following scheme

$$\begin{cases} \frac{\hat{u}_{n+1} - \hat{u}_n}{\tau} = \frac{\hat{v}_{n+1} + \hat{v}_n}{2}, & \text{in } H_0^1(\Omega) \\ \frac{\hat{v}_{n+1} - \hat{v}_n}{\tau} = \Delta_0 \frac{\hat{u}_{n+1} + \hat{u}_n}{2} + \gamma \mathbb{1}_\omega \left(\frac{y_{n+1}^\delta + y_n^\delta}{2} - I_\omega \frac{\hat{v}_{n+1} + \hat{v}_n}{2} \right), & \text{in } L^2(\Omega), \\ \hat{u}_{n=0} = \hat{u}_0, \\ \hat{v}_{n=0} = \hat{v}_0. \end{cases} \quad (1.110)$$

Classical discretization with Lagrange finite elements (see, e.g., [Brenner and Scott, 2008]) then leads to the solution of the following system in finite-dimensional space $\mathcal{V}_h \times \mathcal{V}_h$, in which the mass matrix is usually denoted by \mathbb{M} and the stiffness matrix by \mathbb{K}

$$\begin{cases} \mathbb{K} \frac{\hat{U}_{h,n+1} - \hat{U}_{h,n}}{\tau} = \mathbb{K} \frac{\hat{V}_{h,n+1} + \hat{V}_{h,n}}{2}, \\ \mathbb{M} \frac{\hat{V}_{h,n+1} - \hat{V}_{h,n}}{\tau} = -\mathbb{K} \frac{\hat{U}_{h,n+1} + \hat{U}_{h,n}}{2}, \\ \quad \quad \quad + \gamma I_\omega^* \left(\frac{y_{h,n+1}^\delta + y_{h,n}^\delta}{2} - I_\omega \frac{\hat{V}_{h,n+1} + \hat{V}_{h,n}}{2} \right), \\ \hat{U}_{h,n=0} = \hat{U}_{h,0}, \\ \hat{V}_{h,n=0} = \hat{V}_{h,0}, \end{cases} \quad (1.111)$$

where I_ω selects the degrees of freedom (dofs) on Ω and $\hat{u}_{h,n+1} \in \mathcal{V}_h$ and $\hat{U}_{h,n+1} \in \mathcal{V}_h$ are respectively identified with the vectors of dofs $\hat{U}_{h,n+1}$ and $\hat{V}_{h,n+1}$.

In the case of *field* measurements, we build the observer dynamics (1.105) using the same procedure as for the case of *velocity* measurements. Namely, using the definition of the observation operator (1.93) and its adjoint (1.96), the time discretization of the observer leads to

$$\begin{cases} \frac{\hat{u}_{n+1} - \hat{u}_n}{\tau} = \frac{\hat{v}_{n+1} + \hat{v}_n}{2} + \gamma E_\omega \left(\frac{y_{n+1}^\delta + y_n^\delta}{2} - I_\omega \frac{\hat{u}_{n+1} + \hat{u}_n}{2} \right), & \text{in } H_0^1(\Omega), \\ \frac{\hat{v}_{n+1} - \hat{v}_n}{\tau} = \Delta_0 \frac{\hat{u}_{n+1} + \hat{u}_n}{2}, & \text{in } L^2(\Omega), \\ \hat{u}_{n=0} = \hat{u}_0, \\ \hat{v}_{n=0} = \hat{v}_0. \end{cases} \quad (1.112)$$

Before discretizing in space, following the methodology of [Moireau et al., 2009; Imperiale et al., 2021], we replace the operator E_ω by a penalized version. Namely, we define $E_\omega^\alpha : L^2(\omega) \rightarrow H_0^1(\Omega)$ by , for all $y \in L^2(\omega)$,

$$E_\omega^\alpha y = u^\alpha \text{ where } u^\alpha \text{ is solution of } \begin{cases} -\Delta u^\alpha = 0, & \text{in } \Omega \setminus \omega, \\ -\alpha^2 \Delta u^\alpha + u^\alpha = y, & \text{in } \omega, \\ u^\alpha = 0, & \text{on } \partial\Omega. \end{cases} \quad (1.113)$$

In Section 1.5.3.1 we will analyze and give several properties of this operator, in particular we will show that E_ω^α converges to E_ω . Approximation and stability properties will be given. From the definition (1.113), we can deduce an explicit form of E_ω^α

$$E_\omega^\alpha = (-\alpha^2 \Delta_0 + I_\omega^* I_\omega)^{-1} I_\omega^*, \quad (1.114)$$

where $I_\omega \in \mathcal{L}(L^2(\Omega), L^2(\omega))$ is the restriction operator and its adjoint is the L^2 -extension by 0 namely $\mathbb{1}_\omega$.

Therefore, the space discretization of (1.112) reads to

$$\left\{ \begin{array}{l} \mathbb{K} \frac{\hat{U}_{h,n+1} - \hat{U}_{h,n}}{\tau} = \mathbb{K} \frac{\hat{V}_{h,n+1} + \hat{V}_{h,n}}{2} \\ \quad + \gamma \mathbb{K} (\alpha^2 \mathbb{K} + I_\omega^T I_\omega)^{-1} I_\omega^T \left(\frac{y_{h,n+1}^\delta + y_{h,n}^\delta}{2} - I_\omega \frac{\hat{U}_{h,n+1} + \hat{U}_{h,n}}{2} \right), \\ \mathbb{M} \frac{\hat{V}_{h,n+1} - \hat{V}_{h,n}}{\tau} = -\mathbb{K} \frac{\hat{U}_{h,n+1} + \hat{U}_{h,n}}{2}, \\ \hat{U}_{h,n=0} = \hat{U}_{h,0}, \\ \hat{V}_{h,n=0} = \hat{V}_{h,0}. \end{array} \right. \quad (1.115)$$

To avoid having to invert an operator, we multiply the first line of (1.115) with $(\alpha^2 \mathbb{K} + I_\omega^T I_\omega) \mathbb{K}^{-1}$ and obtain

$$\left\{ \begin{array}{l} (\alpha^2 \mathbb{K} + I_\omega^T I_\omega) \frac{\hat{U}_{h,n+1} - \hat{U}_{h,n}}{\tau} = (\alpha^2 \mathbb{K} + I_\omega^T I_\omega) \frac{\hat{V}_{h,n+1} + \hat{V}_{h,n}}{2} \\ \quad + \gamma I_\omega^T \left(\frac{y_{h,n+1}^\delta + y_{h,n}^\delta}{2} - I_\omega \frac{\hat{U}_{h,n+1} + \hat{U}_{h,n}}{2} \right), \\ \mathbb{M} \frac{\hat{V}_{h,n+1} - \hat{V}_{h,n}}{\tau} = -\mathbb{K} \frac{\hat{U}_{h,n+1} + \hat{U}_{h,n}}{2}, \\ \hat{U}_{h,n=0} = \hat{U}_{h,0}, \\ \hat{V}_{h,n=0} = \hat{V}_{h,0}. \end{array} \right. \quad (1.116)$$

Now that we have the space and time schemes of our two Luenberger observers, we can present some numerical illustrations.

Numerical illustrations

We represent in Figure 1.3 the domain Ω and the observation subdomain ω over which we have observations of the system. The domain mesh is shown in Figure 1.4. All the

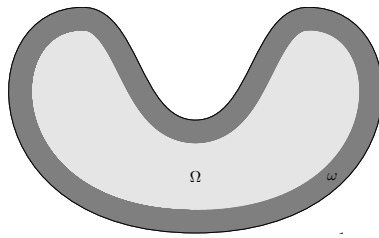


Figure 1.3: Representation of Ω and ω

following examples have been coded with Freefem++ [Hecht, 2012] with \mathbb{P}_1 finite element discretization with 14847 degrees of freedom (dofs).

As an example, we consider the following wave equation

$$\left\{ \begin{array}{ll} \partial_{tt} u(x, t) - \Delta u(x, t) = 0, & (x, t) \in \Omega \times (0, T), \\ u(x, t) = 0, & (x, t) \in \partial\Omega \times (0, T), \\ u(x, 0) = u_0(x), & x \in \Omega, \\ \partial_t u(x, 0) = 0, & x \in \Omega, \end{array} \right. \quad (1.117)$$

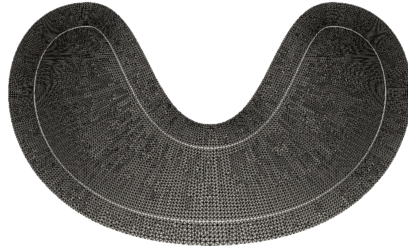


Figure 1.4: Representation of Ω mesh

where u_0 is solution of

$$\begin{cases} -\Delta u_0(x) = 10, & x \in \Omega, \\ u_0(x) = 0, & x \in \partial\Omega, \end{cases}$$

as illustrated in Figure 1.5.

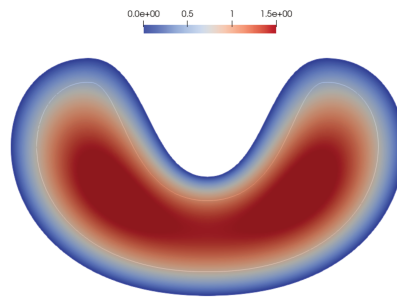


Figure 1.5: Representation of the initial condition $u_0(x)$

We present both cases: when considering *velocity* measurements and *field* measurements.

Case of *velocity* measurements

Let $\mathcal{Y} = L^2(\omega)$. We assume that we have access to noise-free velocity measurements in the subdomain ω shown in Figure 1.6, namely the observation operator C is defined by (1.92).

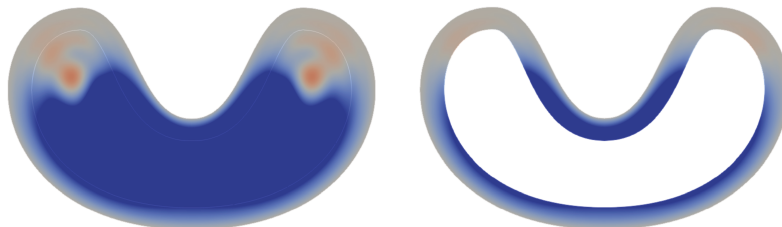


Figure 1.6: Discrete solution $\partial_t u_h(1)$ and the corresponding measurements $y_h(1) = \partial_t u_h|_\omega(1)$

The numerical results associated to the resolution of the inverse problem by the Luenberger (1.111) are represented in Figure 1.7. We see that the observer converges asymptot-

ically in time toward the solution of the wave equation. We also see that the error between the two equations is decreasing.

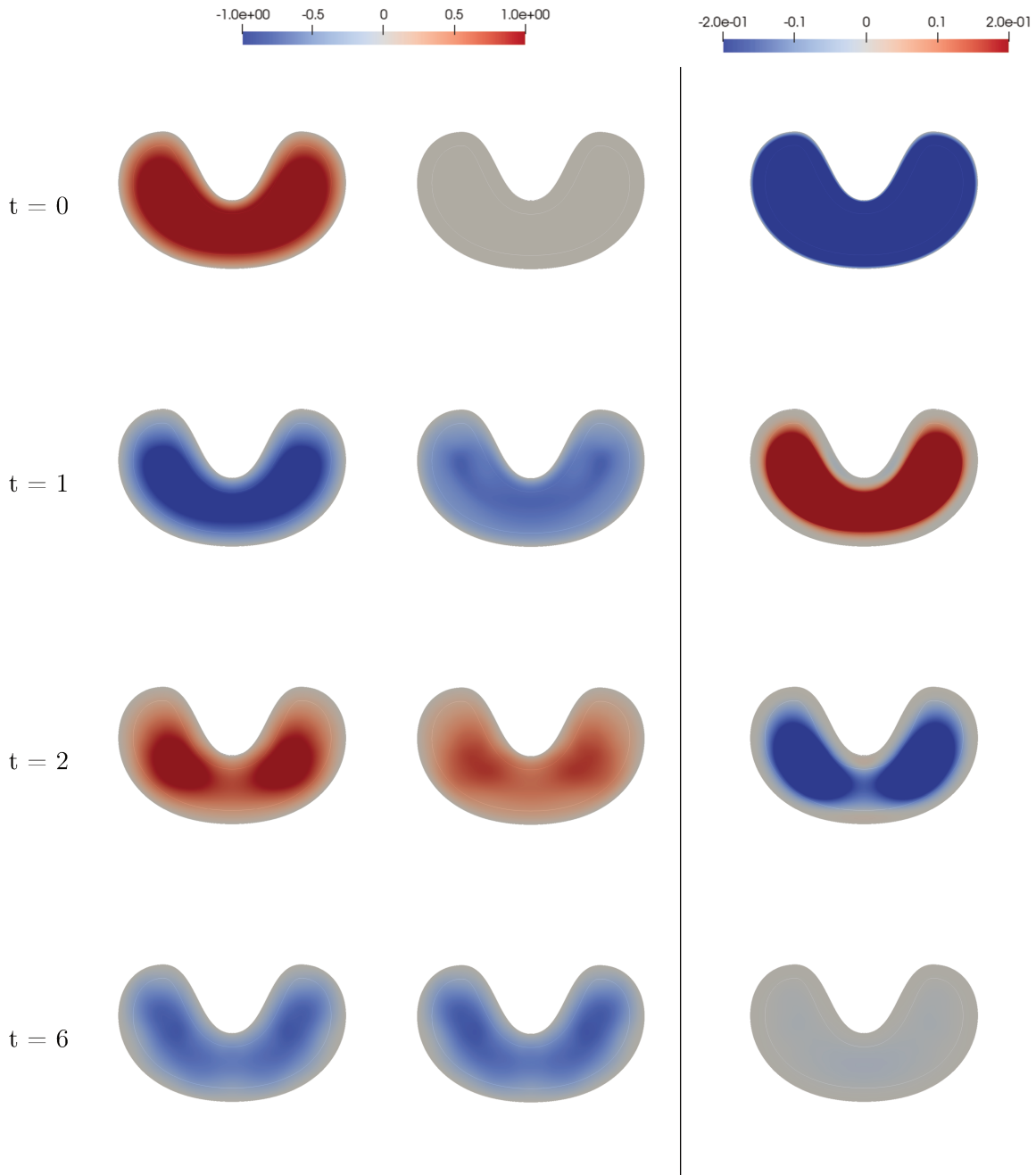


Figure 1.7: Target solution u_h (left), observer \hat{u}_h (middle) and error $u_h - \hat{u}_h$ (right) with $\gamma = 20$ from free-noise *velocity* measurements at different times t

As can be seen in Figure 1.8, the energy of the target system is conserved. Note that the energy of the observer system reaches the energy of the target system, while the energy of the error system tends exponentially toward 0. Figure 1.9 also shows that the reconstruction is very good, with less than 10^{-4} percent error at $t^n = 10$ (left).

To illustrate Theorem 1.5.2 we represent the eigenvalues of the operator $A - \gamma C^*C$.

Note that the eigenvalues of $A - \gamma C^*C$ lies in the complex plane not close to the imaginary axis which is one of the characteristics of exponential stability property as

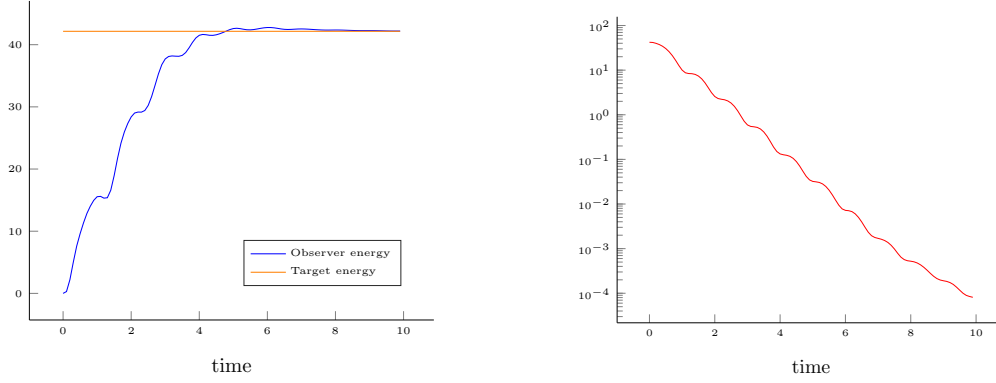


Figure 1.8: Evolution of the relative error $\hat{\mathcal{E}}$ and the target energy $\check{\mathcal{E}}$ (left) and the error energy $\tilde{\mathcal{E}}$ (right) over the time

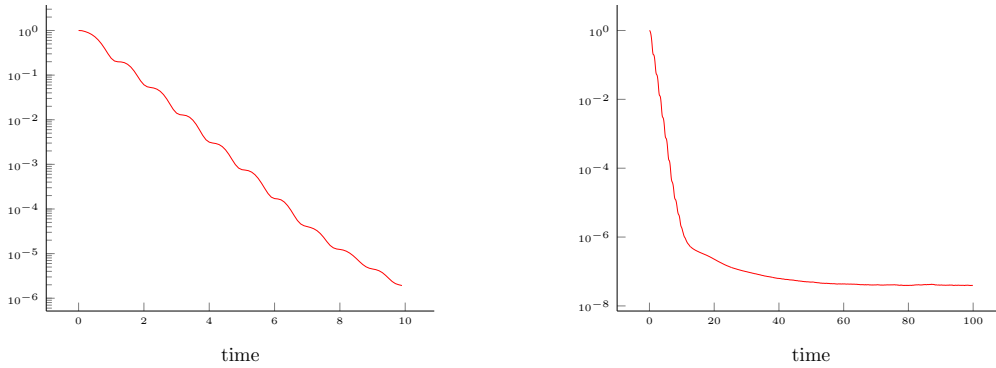


Figure 1.9: Evolution of the relative error $\hat{\mathcal{E}}/\check{\mathcal{E}}$ for $T = 10$ (left) and $T = 100$ (right)

shown in Figure 1.10 (left).

However, if we increase the number of eigenvalues represented, the high frequencies affect the exponential stability property. As illustrated in Figure 1.10 (right), there are in this case, some eigenvalues on the imaginary axis. In Figure 1.9 (right) we notice that the decay is no longer linear after a certain time. This gives an illustration of the loss of the uniform exponential stability property caused by the discretization.

Indeed, as we will see in Chapter 2 and in [Delaunay et al., 2023], spatial discretization can alter the exponential stability property. In fact, numerical pollution can destroy this property at the discrete level, and we will explore new remedies to address this.

We have seen that the available measurements are noisy. Therefore, we introduce an artificial noise χ_h^δ such that $\|\chi_h^\delta\|_{L^2(0,T;L^2(\omega))}^2 \leq \delta^2 T$ – see Figure 1.11. We define, at each time n , the discretized noise data as

$$\chi_h^{\delta,n} = \frac{\delta}{\sqrt{TN_{mod}}} \sqrt{\tau \sum_{n=1}^N \|y_{h,n}\|_{L^2(\omega)}^2} \sum_{i=0}^{N_{mod}} \xi_i^n \Phi_i, \quad (1.118)$$

where Φ are the N_{mod} Laplacian modes and ξ are independent and identically distributed Gaussian variables in $N(0,1)$ and $N = T/\tau$. The idea is to simulate an additive white Gaussian noise such that the order of magnitude of the error is a fixed percentage of the L^2 norm of the observations [Lord et al., 2014].

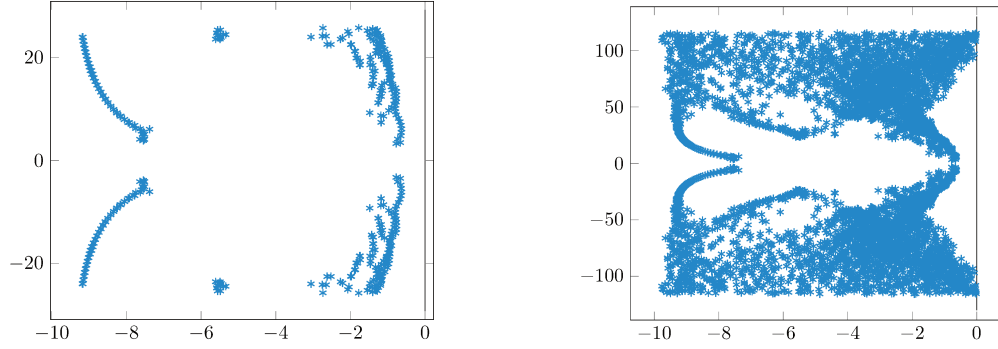


Figure 1.10: Representation of 400 eigenvalues of $A - \gamma C^* C$ (left) and of 5000 eigenvalues (right) with $\gamma = 20$ and with a mesh with 3603 degrees of freedom

Note that with this definition of noise, we have the following estimate

$$\frac{\tau \sum_{n=0}^N \|y_{h,n}^\delta - y_{h,n}\|_{L^2(\omega)}^2}{\tau \sum_{n=0}^N \|y_{h,n}\|_{L^2(\omega)}^2} = \frac{\delta^2}{TN_{mod}} \tau \sum_{n=0}^N \sum_{i=0}^{N_{mod}} (\xi_i^n)^2,$$

which, by property of the empirical variance, gives

$$\lim_{N \rightarrow +\infty} \frac{\tau \sum_{n=0}^N \|y_{h,n}^\delta - y_{h,n}\|_{L^2(\omega)}^2}{\tau \sum_{n=0}^N \|y_{h,n}\|_{L^2(\omega)}^2} = \delta^2.$$

The Figure 1.11 represents the noisy measurements we consider at time $t = 1$.

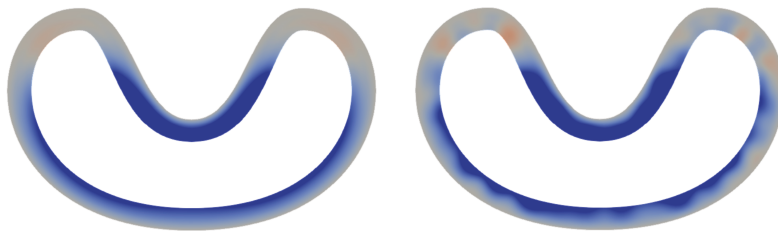


Figure 1.11: The measurement $y_h(1) = \partial_t u_h|_\omega(1)$ and the noisy measurement $y_h^\delta(1) = \partial_t u_h|_\omega(1) + \chi_h^\delta(1)$ with $\delta = 0.1$

In Figure 1.12 we see that even with noisy observations we can find the solution of the wave equation, even if we have to wait a little longer.

As we can see in Figure 1.13, the energy of the observer system asymptotically reaches the energy of the target system even in noisy measurements. As illustrated in Figure 1.14, the percentage of error is less than 1% starting from $t^n = 3.6$.

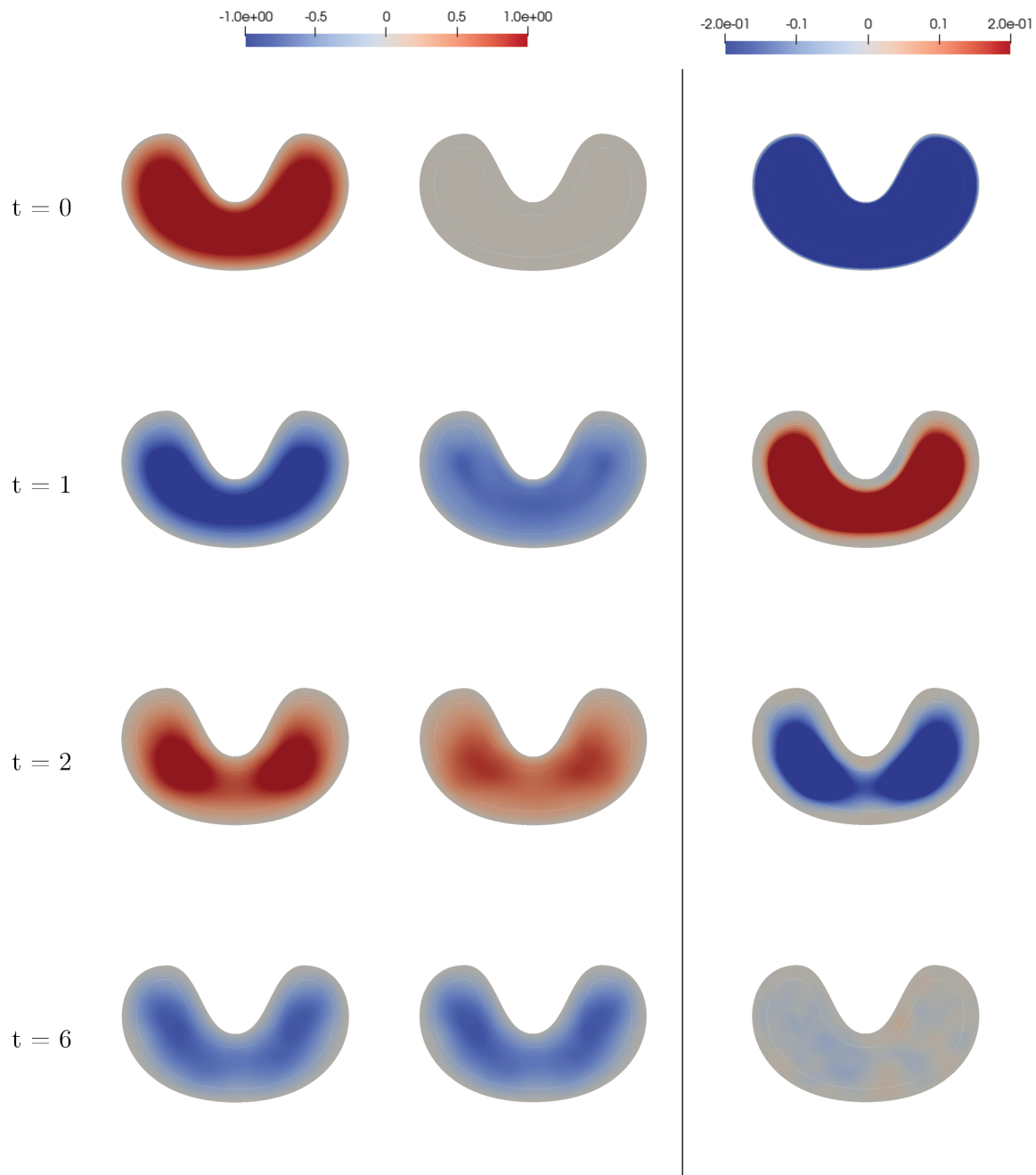


Figure 1.12: Target solution u_h (left), observer \hat{u}_h (middle) and error $u_h - \hat{u}_h$ (right) with $\gamma = 20$ from noisy *velocity* measurements with $\delta = 0.1$ at different times t

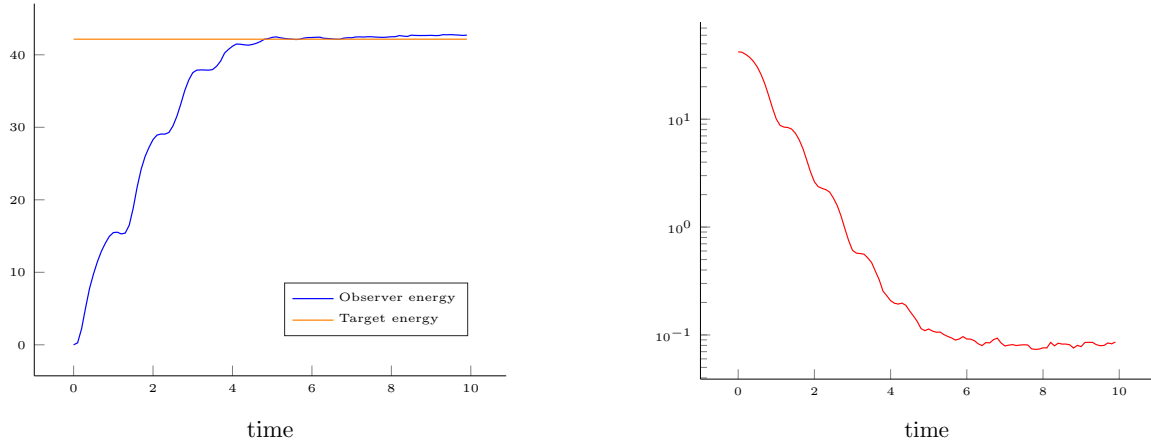


Figure 1.13: Evolution of the observer energy $\hat{\mathcal{E}}$ and the target energy $\check{\mathcal{E}}$ (left) and the error energy $\tilde{\mathcal{E}}$ (right) over the time from noisy measurements

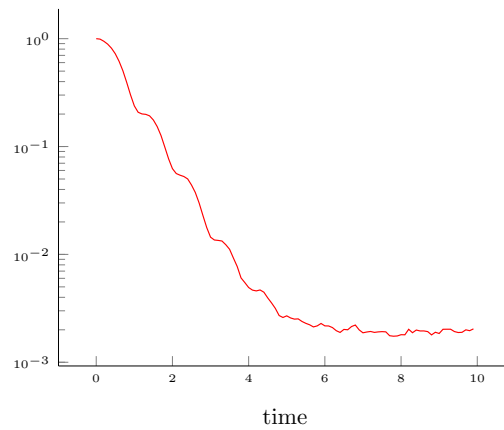


Figure 1.14: Evolution of the relative error $\hat{\mathcal{E}}/\check{\mathcal{E}}$

Case of *field* measurements

We still consider the wave equation (1.117). Let $\mathcal{Y} = H^1(\omega)$. We assume that this time we have access to a noise-free *field* measurement in the subdomain ω shown in Figure 1.15, *i.e.*, the observation operator C is defined by (1.93).

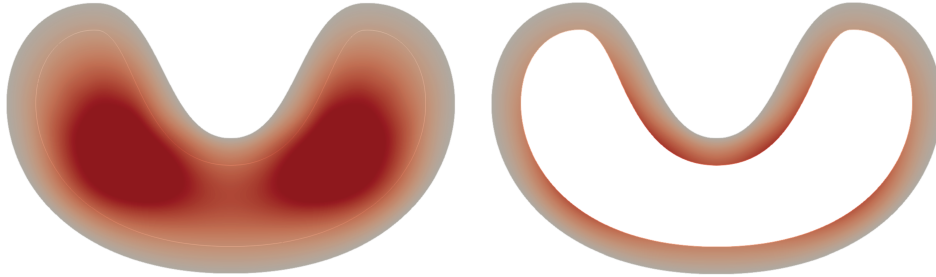


Figure 1.15: Solution $u(x, 2)$ and the corresponding measurement $y_h(2) = u_h|_\omega(x, 2)$

Figure 1.16 shows how the error between the observer (1.116) and the solution of the wave equation decreases with time as the observer approaches the solution of the wave equation.

As we can see in Figure 1.17, the energy of the observer system reaches the energy of the target system. Furthermore, at $t^n = 10$, we have less than $10^{-2}\%$ of error.

When considering field measurements, the exponential stability property of the $A - \gamma C^* C$ operator is well characterized, as shown in Figure 1.19.

To make the case more realistic in practice, we assume that the noise χ^δ belongs to $L^2(\omega)$, namely the available measurements y^δ belong to $L^2(\omega)$. The observer dynamics (1.105) cannot be defined in this way, since the adjoint of the observation operator C^* (defined by (1.96)) is defined from $H^1(\omega)$. In the next section, we look at the strategy used in this particular case, where the space where the noise is defined in a larger space than the observation space.

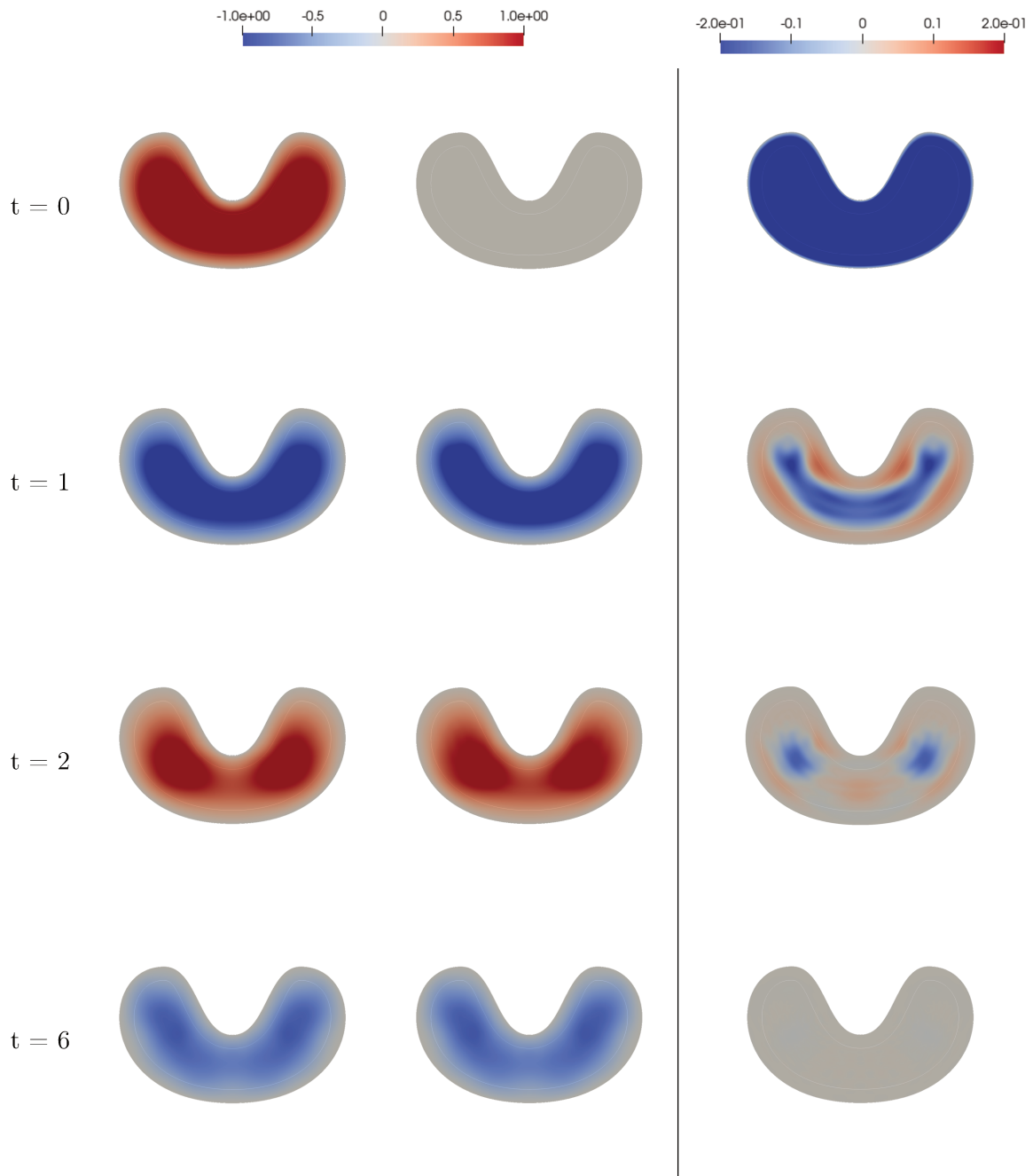


Figure 1.16: Target solution u_h (left), observer \hat{u}_h (middle) and error $u_h - \hat{u}_h$ (right) with $\gamma = 20$ from *field* measurements at different times t

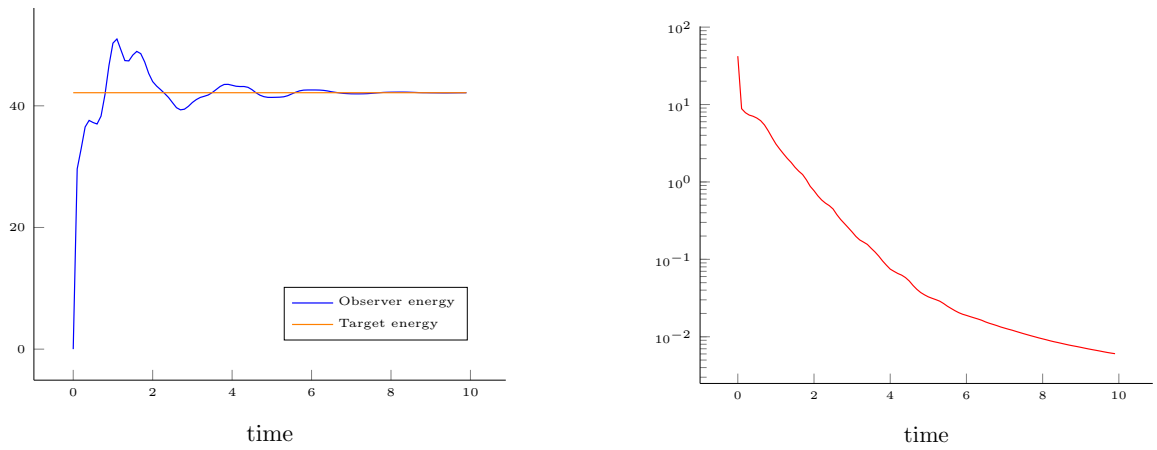


Figure 1.17: Evolution of the observer energy $\hat{\mathcal{E}}$ and the target energy $\check{\mathcal{E}}$ (left) and the error energy $\tilde{\mathcal{E}}$ (right) over the time from *field* measurements

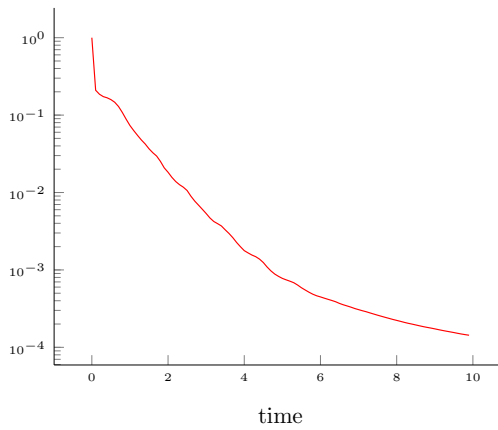


Figure 1.18: Evolution of the relative error $\hat{\mathcal{E}}/\check{\mathcal{E}}$ over the time

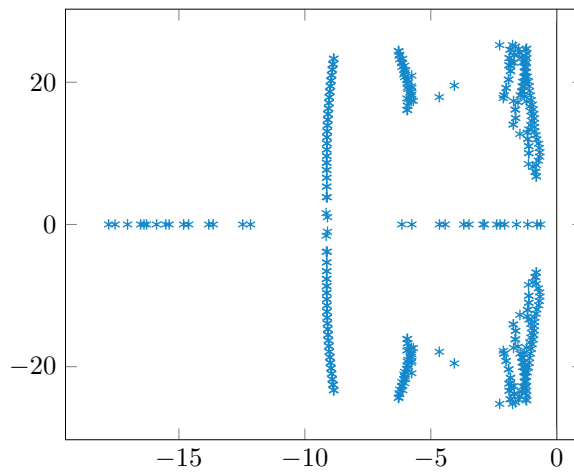


Figure 1.19: Representation of the 400 eigenvalues of $A - \gamma C^* C$ with $\gamma = 20$ and with a mesh with 3603 degrees of freedom

1.5.3 Luenberger observer: find an alternative method for dealing with less regular measurements

We will study two methods for dealing with less regular measurements. The first is to regularize the measurements and then consider the previously defined estimations. To do this, we introduce a regularization of the adjoint of the observation operator, which regularizes the observations at the same time. The second method aims at looking at the inverse problem in weaker space.

1.5.3.1 Regularize-then-invert strategy

The idea of this section is to replace the operator $C^* = (E_\omega \ 0)^t$ with a regularization operator G_ω^α that would lift the noisy measurements (which is defined in a larger space $y^\delta \in L^2(\omega)$). To this purpose, we use the operator E_ω^α , which is a penalization of the operator E_ω and that we have already introduced in Section 1.5.2.

Different properties on the regularization operator

Let us show that this regularization operator E_ω^α tends towards the E_ω operator. To do this, we need the following abstract theorem.

Theorem 1.5.3 *Let $V \hookrightarrow B$, two Hilbert spaces. Let $a(\cdot, \cdot)$ be a bilinear, symmetric, continuous and coercive form in V . Let $b(\cdot, \cdot)$ be a bilinear, symmetric, continuous and nonnegative form in B . We define the set K_y , for each $y \in B$*

$$K_y = \{u \in V, b(u - y, u - y) = 0\}.$$

Denote respectively $u^\alpha = \underset{v \in H_0^1(\Omega)}{\operatorname{argmin}} \mathcal{J}^\alpha(v)$ with $\mathcal{J}(v) = \frac{1}{2}a(v, v) + \frac{1}{2\alpha^2}b(v - y, v - y)$. The solution u^α converges to u . Denote respectively

$$u = \underset{v \in K_y}{\operatorname{argmin}} \mathcal{J}(v) \text{ and } u^\alpha = \underset{v \in V}{\operatorname{argmin}} \mathcal{J}^\alpha(v),$$

the minimizers of the following functionals

$$\mathcal{J}(v) = \frac{1}{2}a(v, v) \text{ and } \mathcal{J}^\alpha(v) = \frac{1}{2}a(v, v) + \frac{1}{2\alpha^2}b(v - y, v - y),$$

then $\|u^\alpha - u\|_V \xrightarrow{\alpha \rightarrow 0} 0$.

PROOF: The proof is inspired by [Maury, 2009].

Step 1: weak convergence

The family (\mathcal{J}^α) is uniformly elliptic, thus $\|u^\alpha\|_V$ is bounded. We extract a sub-sequence also denoted u^α such that $(u^\alpha) \rightharpoonup u \in V$. Since $\mathcal{J}^\alpha(v) \geq \mathcal{J}(v)$ and u^α minimizes \mathcal{J}^α , then we have, $\forall z \in K_y$

$$\mathcal{J}(u^\alpha) \leq \mathcal{J}^\alpha(u^\alpha) \leq \mathcal{J}^\alpha(z). \quad (1.119)$$

However, we have

$$\mathcal{J}^\alpha(z) = \frac{1}{2}a(z, z) + \frac{1}{2\alpha^2}b(z - y, z - y).$$

The last term is null since $z \in K_y$, this implies in particular,

$$\mathcal{J}^\alpha(z) = \mathcal{J}(z). \quad (1.120)$$

As \mathcal{J} is convex and continuous, we have, using (1.119) and (1.120),

$$\mathcal{J}(u) \leq \liminf \mathcal{J}(u^\alpha) \leq \mathcal{J}(z).$$

We also obtain that

$$\begin{aligned} \frac{1}{2}a(u^\alpha, u^\alpha) + \frac{1}{2\alpha^2}b(u^\alpha - y, u^\alpha - y) &= \mathcal{J}(u^\alpha) + \frac{1}{2\alpha^2}b(u^\alpha - y, u^\alpha - y), \\ &= \mathcal{J}^\alpha(u^\alpha) \leq \mathcal{J}(z). \end{aligned}$$

This implies that $\frac{1}{2\alpha^2}b(u^\alpha - y, u^\alpha - y)$ is bounded uniformly in α . Thus, we obtain

$$b(u^\alpha - y, u^\alpha - y) \xrightarrow{\alpha \rightarrow 0} 0.$$

Since b is lower semi-continuous and $Iu^\alpha - y \xrightarrow{B} Iz - y$, we have

$$0 \leq b(u - y, u - y) \leq \liminf b(u^\alpha - y, u^\alpha - y) = 0.$$

Therefore, we have $u \in K_y$, and it implies that u is the minimizer of \mathcal{J} .

Step 2: strong convergence

We show the strong convergence, namely that $a(u^\alpha, u^\alpha) \rightarrow a(u, u)$ – since a is coercive in V . The functional $u \mapsto a(u, u)$ is lower semi continuous and $u^\alpha \rightharpoonup u$ thus we have

$$a(u, u) \leq \liminf a(u^\alpha, u^\alpha).$$

And from (1.119) and (1.120) we finally get

$$a(u^\alpha, u^\alpha) \leq a(u, u),$$

which implies the convergence $\|u^\alpha\|_V \xrightarrow{\alpha \rightarrow 0} \|u\|_V$, hence the strong convergence. \blacksquare

Thanks to this theorem, we can obtain an approximation property. We cannot prove the rate of convergence of this operator $E_\omega^\alpha(\phi)$ to $E_\omega(\phi)$ if $\phi \in H^1(\omega)$ is in the H_0^1 norm. Indeed, the problem does not satisfy the classical *inf-sup* condition (see [Girault and Raviart, 1986, Chapter1, Section 4]). Therefore, we cannot ensure that convergence is in α^2 ([Maury, 2009]).

Theorem 1.5.4

$$\text{For } \phi \in H^1(\omega), \exists \alpha \text{ such that } \|E_\omega^\alpha(\phi) - E_\omega(\phi)\|_{H_0^1(\Omega)} \leq f(\alpha, \phi),$$

where f is a continuous function and $f(\alpha) \xrightarrow{\alpha \rightarrow 0} 0$.

PROOF: We rewrite the operator E_ω defined by (1.57) as a constrained minimization problem.

$$E_\omega y = \operatorname{argmin}_{u|_{\omega=y}} \frac{1}{2} \|\nabla u\|_{L^2(\Omega)}^2, \quad (1.121)$$

The operator E_ω^α is then described as the penalization of this minimization problem.

$$E_\omega^\alpha y = \operatorname{argmin}_{u \in H_0^1(\Omega)} \frac{1}{2} \|\nabla u\|_{L^2(\Omega)}^2 + \frac{1}{2\alpha^2} \|y - u\|_{L^2(\omega)}^2. \quad (1.122)$$

Setting $V = H_0^1(\Omega)$ and $B = L^2(\omega)$, we can then apply the Theorem 1.5.4 by posing $a(\cdot, \cdot) = (\cdot, \cdot)_{H_0^1(\Omega)}$ and $b(\cdot, \cdot) = (\cdot, \cdot)_{L^2(\omega)}$. \blacksquare

An important property is the stability of this operator.

Proposition 1.5.5 *The regularization operator E_ω^α satisfies, $\forall y \in L^2(\omega)$*

$$\|E_\omega^\alpha y\|_{\mathbf{H}_0^1(\Omega)} \leq \frac{1}{\sqrt{2\alpha}} \|y\|_{L^2(\omega)}.$$

PROOF: We denote $u_\alpha = E_\omega^\alpha y \in \mathbf{H}_0^1(\Omega)$. The operator E_ω^α satisfies the following variational formulation $\forall v \in \mathbf{H}_0^1(\Omega)$,

$$\int_\Omega \nabla u_\alpha \nabla v dx + \frac{1}{\alpha^2} \int_\omega u_\alpha v dx = \frac{1}{\alpha^2} \int_\omega y v dx.$$

Choosing the test function $v = u_\alpha$ and using Cauchy–Schwarz and Young inequalities, we finally get

$$\|\nabla u_\alpha\|_{L^2(\Omega)}^2 \leq \frac{1}{2\alpha^2} \|y\|_{L^2(\omega)}^2. \quad \blacksquare$$

From the definition (1.113), we recall that we can deduce the explicit form of E_ω^α , we have

$$E_\omega^\alpha = (-\alpha^2 \Delta_0 + I_\omega^* I_\omega)^{-1} I_\omega^*, \quad (1.123)$$

where $I_\omega \in \mathcal{L}(L^2(\Omega), L^2(\omega))$ is the restriction operator and its adjoint is the L^2 -extension by 0 namely $\mathbb{1}_\omega$.

It is possible to rewrite the operator E_ω^α in another way, using another operator denoted R^α for which interesting properties can be deduced.

Proposition 1.5.6 *For all $u \in \mathbf{H}_0^1(\Omega)$ and $y \in L^2(\omega)$, we have the following property*

$$(\nabla E_\omega^\alpha y, u)_{L^2(\Omega)} = (R^\alpha y, u|_\omega)_{L^2(\omega)},$$

where $R^\alpha : L^2(\omega) \rightarrow L^2(\omega)$ is defined by

$$R^\alpha = (-I_\omega \Delta_0^{-1} I_\omega^* + \alpha^2 \mathbb{1}_\omega)^{-1}.$$

PROOF: Using the definition of the operator E_ω^α and the isomorphism property of $-\Delta_0$ we get

$$\begin{aligned} (u, E_\omega^\alpha y)_{\mathbf{H}_0^1(\Omega)} &= (\nabla u, \nabla((-\alpha^2 \Delta_0 + I_\omega^* I_\omega)^{-1} I_\omega^* y))_{L^2(\Omega)}, \\ &= (u, -\Delta_0((-\alpha^2 \Delta_0 + I_\omega^* I_\omega)^{-1} I_\omega^* y))_{L^2(\Omega)}, \end{aligned}$$

With some elementary manipulations, we can then obtain

$$\begin{aligned} (u, E_\omega^\alpha y)_{\mathbf{H}_0^1(\Omega)} &= (u, -\Delta_0(-\alpha^2 \Delta_0 + I_\omega^* I_\omega)^{-1} I_\omega^* (-I_\omega \Delta_0^{-1} I_\omega^* + \alpha^2 \mathbb{1}_\omega) \\ &\quad \times (-I_\omega \Delta_0^{-1} I_\omega^* + \alpha^2 \mathbb{1}_\omega)^{-1} y)_{L^2(\Omega)}, \\ &= (u, -\Delta_0(-\alpha^2 \Delta_0 + I_\omega^* I_\omega)^{-1} (-I_\omega^* I_\omega + \alpha^2 \Delta_0) \\ &\quad \times \Delta_0^{-1} I_\omega^* (-I_\omega \Delta_0^{-1} I_\omega^* + \alpha^2 \mathbb{1}_\omega)^{-1} y)_{L^2(\Omega)}, \\ &= (u, I_\omega^* (-I_\omega \Delta_0^{-1} I_\omega^* + \alpha^2 \mathbb{1}_\omega)^{-1} y)_{L^2(\Omega)}. \end{aligned} \quad \blacksquare$$

We can show that this operator R^α is a positive operator.

Proposition 1.5.7 *For all $y \in L^2(\omega)$, there exists a positive scalar C^{st} such that*

$$\|y\|_{L^2(\omega)}^2 \leq \frac{(C^{st} + \alpha^2)^2}{\alpha^2} (R_\omega^\alpha y, y)_{L^2(\omega)}.$$

PROOF: The definition of the operator R^α leads to

$$\begin{aligned} (R_\omega^\alpha y, y)_{L^2(\omega)} &= (R_\omega^\alpha y, (R_\omega^\alpha)^{-1} R_\omega^\alpha y)_{L^2(\omega)}, \\ &= (R_\omega^\alpha y, (-I_\omega \Delta_0^{-1} I_\omega^* + \alpha^2 \mathbb{1}_\omega) R_\omega^\alpha y)_{L^2(\omega)}. \end{aligned}$$

Since $-I_\omega \Delta_0^{-1} I_\omega^*$ is positive, we obtain

$$(R_\omega^\alpha y, y)_{L^2(\omega)} \geq \alpha^2 \|R^\alpha y\|_{L^2(\omega)}^2. \quad (1.124)$$

On the other hand, setting $\phi = R^\alpha y$, we have

$$\|y\|_{L^2(\omega)}^2 = (y, (R^\alpha)^{-1} \phi)_{L^2(\omega)} = (y, (-I_\omega \Delta_0^{-1} I_\omega^* + \alpha^2 \mathbb{1}_\omega) \phi)_{L^2(\omega)}.$$

One can prove that $-I_\omega \Delta_0^{-1} I_\omega^*$ is bounded, therefore we get

$$\|y\|_{L^2(\omega)}^2 \leq (C^{st} + \alpha^2) \|y\|_{L^2(\omega)} \|\phi\|_{L^2(\omega)}.$$

The above equation leads to

$$\|y\|_{L^2(\omega)} \leq (C^{st} + \alpha^2) \|R^\alpha y\|_{L^2(\omega)}. \quad (1.125)$$

Combining (1.124) and (1.125), we find the desired result. \blacksquare

Following a similar proof, we can even prove that R^α is equivalent to the $L^2(\omega)$ norm.

The weakened Luenberger observer

Now that we have given the necessary properties of the operator E_ω^α , we introduce the regularized Luenberger observer.

$$\begin{cases} \dot{\hat{z}} = A\hat{z} + \gamma G^\alpha (y^\delta - C\hat{z}), & \text{in } (0, \infty), \\ \hat{z}(0) = \hat{z}_0, \end{cases} \quad (1.126)$$

where $G^\alpha = (E_\omega^\alpha \ 0)^t \in \mathcal{L}(L^2(\omega), \mathcal{Z})$ and $C \in \mathcal{L}(\mathcal{Z}, H^1(\omega))$ is defined by (1.93). We will now show that there are solutions to this dynamics.

Proposition 1.5.8 *Let $T > 0$ and $\hat{z}_0 \in \mathcal{Z}$, $y^\delta \in L^p((0, \infty); \mathcal{Y})$, $p \geq 1$, the system defined by (1.126) is well-posed in the sense, there exists a mild solution $\hat{z} \in C^0([0, \infty); \mathcal{Z})$.*

PROOF: Let us prove that $A - \gamma G^\alpha C$ of domain $\mathcal{D}(A)$ is maximal dissipative. For all $z \in \mathcal{Z}$, we have

$$(z, (A - \gamma G^\alpha C)z)_{\mathcal{Z}} = (z, Az)_{\mathcal{Z}} - \gamma (z, G^\alpha C z)_{\mathcal{Z}}.$$

Since A is skew-adjoint and using Proposition 1.5.6, we get

$$(z, (A - \gamma G^\alpha C)z)_{\mathcal{Z}} = -\gamma (u, E_\omega^\alpha u|_\omega)_{H_0^1(\Omega)} = -\gamma (u|_\omega, R^\alpha u|_\omega)_{L^2(\omega)}.$$

Using Proposition 1.5.7 we have established that the above product scalar is positive, therefore $A - \gamma G^\alpha C$ is dissipative. Let us show the maximal property of $A - \gamma G^\alpha C$. Let $\lambda > 0$, $\beta \in \mathcal{Z}$, we seek $z \in \mathcal{D}(A)$ such that

$$(\text{Id} - \lambda(A - \gamma G^\alpha C))z = \beta \implies (\text{Id} - \lambda A)z = \beta - \gamma G^\alpha C z$$

We introduce the application $F : \mathcal{D}(A) \ni z = (\text{Id} - \lambda A)^{-1}(\beta - \lambda \gamma G^\alpha C z) \in \mathcal{Z}$ satisfies

$$\|F(z_1) - F(z_2)\|_{\mathcal{Z}} \leq \lambda \gamma \|G^\alpha\| \|C\| \|z_1 - z_2\|_{\mathcal{Z}}.$$

Using the stability result of Proposition 1.5.5, we obtain

$$\|F(z_1) - F(z_2)\|_{\mathcal{Z}} \leq \frac{\lambda \gamma}{\sqrt{2\alpha}} \|C\| \|z_1 - z_2\|_{\mathcal{Z}}.$$

Thus, there exists $\lambda < \sqrt{2\alpha}(\gamma \|C\|)^{-1}$, so that F is a contraction. By the contraction mapping principle, $(\text{Id} - \lambda(A - \gamma G^\alpha C))$ is surjective from \mathcal{Z} to $\mathcal{D}(A)$. Moreover, $\gamma G^\alpha y^\delta \in L^p((0, \infty); \mathcal{Y})$, so by Theorem 1.2.9 for all $T \geq 0$ there is a unique mild solution $\hat{z} \in C^0([0, T]; \mathcal{Z})$ by (1.126). ■

Stability of the Luenberger observer

By contrast with the previous section, we cannot show the exponential stability of $A - \gamma G^\alpha C$. On the other hand, we can still show some asymptotic stability.

Theorem 1.5.9 *The following two propositions are equivalent.*

1. For all u eigenvector of $-\Delta_0$ such that $I_\omega u \equiv 0 \implies u \equiv 0$.
2. For all solutions z of

$$\begin{cases} \dot{z} - (A - \gamma G^\alpha C)z = 0, \\ z(0) = z_0 = (u_0 \quad u_1) \in H_0^1(\Omega) \times L^2(\Omega), \end{cases} \quad (1.127)$$

we have $\lim_{t \rightarrow \infty} \|z\|_{\mathcal{Z}}^2 \rightarrow 0$.

PROOF: For the sake of compactness we only prove that the first property implies the second. The proof is inspired by [Burq and Gérard, 2002]. Let us show that the first property implies the second one. We show the second property for $(u_0 \quad u_1) \in \mathcal{D}(A)$, the result is extended by density to \mathcal{Z} . Assume that the second property is not satisfied, then there exists $(t_n) \rightarrow +\infty$ such that $\|z(t_n)\|_{\mathcal{Z}}^2$ does not converge to 0. The energy of (1.127) is decreasing. Using Proposition 1.5.6 we have in fact

$$\begin{aligned} \mathcal{E}(t) &:= \frac{1}{2} \|z(t)\|_{\mathcal{Z}}^2 = \mathcal{E}(0) - \gamma \int_0^t (G^\alpha C z(s), z(s))_{\mathcal{Z}} ds, \\ &= \mathcal{E}(0) - \gamma \int_0^t (E_\omega^\alpha u|_\omega(s), u(s))_{H_0^1(\Omega)} ds, \\ &= \mathcal{E}(0) - \gamma \int_0^t (R^\alpha u|_\omega(s), u|_\omega(s))_{L^2(\omega)} ds, \end{aligned}$$

and since R^α is positive by 1.5.7, the energy is decreasing. Therefore, we have

$$\lim_{s \rightarrow +\infty} \|z(s)\|_{\mathcal{Z}}^2 = \beta > 0.$$

From the Duhamel formula ((1.5)) we get $z(t_n) = e^{-t_n(A - \gamma G^\alpha C)} z_0$, and it is bounded in $\mathcal{D}(A)$. Indeed, we have

$$\|z(t_n)\|_{\mathcal{Z}}^2 = 2\mathcal{E}(t_n) \leq 2\mathcal{E}(0),$$

and

$$\|Az(t_n)\|_{\mathcal{Z}}^2 = 2\tilde{\mathcal{E}}(t_n) \leq 2\tilde{\mathcal{E}}(0),$$

where $\tilde{\mathcal{E}}$ is the energy associated with the following system

$$\begin{cases} \dot{\tilde{z}} - (A - \gamma G^\alpha C)\tilde{z} = 0, \\ \tilde{z}(0) = Az_0. \end{cases}$$

The embedding of $\mathcal{D}(A)$ in \mathcal{Z} is compact, so there exists a subsequence (t_{n_k}) such that $z(t_{n_k}) \rightarrow z_{w_0} := (w_0 \ w_1)^t$ in \mathcal{Z} . We introduce the following solution

$$z_w(t) = (w \ y)^t = e^{-t(A-\gamma G^\alpha C)} z_{w_0}.$$

We have

$$\begin{aligned} e^{-t(A-\gamma G^\alpha C)} z_{w_0} &= \lim_{k \rightarrow \infty} e^{-t(A-\gamma G^\alpha C)} z(t_{n_k}), \\ &= \lim_{k \rightarrow \infty} e^{-(t+t_{n_k})(A-\gamma G^\alpha C)} z_0. \end{aligned}$$

Since $\mathcal{E}(t)$ is decreasing, we obtain

$$\|z_w\|_{\mathcal{Z}}^2 = \lim_{s \rightarrow \infty} \|z(s)\|_{\mathcal{Z}}^2 = \|z_{w_0}\|_{\mathcal{Z}}^2 = \beta > 0.$$

Thus, for all $t > 0$,

$$(R^\alpha w, w)_{L^2(\omega)} = 0.$$

This implies, in particular using Proposition 1.5.7, for all $t > 0$

$$\|w\|_{L^2(\omega)} = 0.$$

In other words, z_w is the solution of

$$\begin{cases} \dot{z}_w = Az_w, \\ z_w(0) = z_{w_0} = (w_0 \ w_1)^t. \end{cases}$$

We decompose w on a Hilbert basis of $L^2(\Omega)$ formed by the eigenvectors of $-\Delta_0$.

$$w(t) = \sum_1^\infty \left(w_{\nu,+} e^{it\sqrt{\lambda_\nu}} + w_{\nu,-} e^{-it\sqrt{\lambda_\nu}} \right) \text{ with } \lambda_\nu \neq \lambda_\mu \text{ if } \nu \neq \mu, \quad (1.128)$$

and

$$w(0) = \sum (w_{\nu,+} + w_{\nu,-}) = w_0 = \sum w_{\nu,0},$$

and

$$\partial_t w(0) = \sum i\sqrt{\lambda_\nu} (w_{\nu,+} - w_{\nu,-}) = w_1 = \sum w_{\nu,1}.$$

We have $w_0 \in H_0^1(\Omega)$ and $w_1 \in L^2(\Omega)$, thus

$$\frac{1}{2} \sum_1^\infty (\|w_{\nu,+}\|_{L^2} + \|w_{\nu,-}\|_{L^2}) = \|w_0\|_{H_0^1(\Omega)}^2 + \|w_1\|_{L^2}^2 \leq +\infty.$$

We fix $\nu_0 \geq 1$ and we set

$$y(T, x) = \frac{1}{T} \int_0^T \partial_t w(t, x) e^{-it\sqrt{\lambda_{\nu_0}}} dt.$$

Furthermore, we have

$$y(T, x) = i\sqrt{\lambda_{\nu_0}}w_{\nu_0,+}(x) + \sum_{\nu \neq \nu_0} \frac{i\sqrt{\lambda_\nu}}{iT(\sqrt{\lambda_\nu} - \sqrt{\lambda_{\nu_0}})} \left(e^{iT(\sqrt{\lambda_\nu} - \sqrt{\lambda_{\nu_0}})} - 1 \right) w_{\nu,+} \\ - \sum_{\nu=1}^{\infty} \frac{i\sqrt{\lambda_\nu}}{iT(\sqrt{\lambda_\nu} + \sqrt{\lambda_{\nu_0}})} \left(e^{-iT(\sqrt{\lambda_\nu} + \sqrt{\lambda_{\nu_0}})} - 1 \right) w_{\nu,-}.$$

This leads to

$$y(T, x) - i\sqrt{\lambda_{\nu_0}}w_{\nu_0,+}(x) \xrightarrow{T \rightarrow \infty} 0 \text{ in } L^2(\Omega).$$

Since we have $y|_\omega = 0$, this implies that $w_{\nu_0,+} = 0$ on ω and thus using the second property, $w_{\nu_0,+} = 0$. In exactly the same way, we can show that we have $w_{\nu_0,-} = 0$. Finally, we have $w = 0$, which contradicts (1.128). ■

To show that the Luenberger observer is stable, we need to show that the first property of Theorem 1.5.9 is satisfied. To do this, we give the stabilization property for the following dynamics

$$\begin{cases} \dot{z} - (A - \gamma C^* C)z = 0, & \text{where } C = (I_\omega \ 0) : \mathcal{Z} \rightarrow \mathbf{H}^1(\omega), \\ z(0) = z_0 \in \mathbf{H}_0^1(\Omega) \times L^2(\Omega), \end{cases} \quad (1.129)$$

which we know from Theorem 1.5.2 is exponentially stable.

Proposition 1.5.10 *We have implication relations for the following properties: 1. \implies 2. \implies 3. \implies 4.*

1. For all solutions z of (1.129), $\lim_{t \rightarrow \infty} \|z(t)\|_{\mathcal{Z}}^2 = 0$.
2. For all $(u_0 \ u_1)^t$ eigenvectors of $A - \gamma C^* C$, with eigenvalues λ , satisfies the solution z of (1.129) $\lim_{t \rightarrow \infty} \|z(t)\|_{\mathcal{Z}}^2 = 0$.
3. For all $(u_0 \ u_1)^t$ eigenvectors of $A - \gamma C^* C$ such that $I_\omega u_0 \equiv 0$, $u \equiv 0$.
4. For all u -eigenvectors of $-\Delta_0$ such that $I_\omega u \equiv 0$, then $u \equiv 0$.

PROOF: We have that 1. \implies 2. trivially.

Let us show that 2. \implies 3.. For all $(u_0 \ u_1)^t$ eigenvectors of $A - \gamma C^* C$ we have

$$e^{-t(A - \gamma C^* C)} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = e^{-t\lambda} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}.$$

Moreover, we have

$$\frac{d}{dt} \mathcal{E}(t) := \frac{1}{2} \frac{d}{dt} \|z(t)\|_{\mathcal{Z}}^2 = - \int_{\Omega} |I_\omega u(x, t)|^2 dx = e^{-2t \operatorname{Re} \lambda} \int_{\Omega} |I_\omega u_0| dx.$$

Therefore, the energy is constant when $I_\omega u_0 = 0$. However, $\mathcal{E}(t) \rightarrow 0$, so $\mathcal{E}(0) = 0$ and $u_0 = u_1 = 0$. Let us show that 3. \implies 4.. Let u be an eigenvector of $-\Delta_0$ associated with the eigenvalue w satisfying the condition $I_\omega u = 0$. We know that $w > 0$ and that $(u \ i\sqrt{w}u)$ is an eigenvector of $A - \gamma C^* C$ associated with eigenvalue $i\sqrt{w}$ which satisfies $\sqrt{z}I_\omega u_0 = 0$, so it follows that $u \equiv 0$. ■

Since the dynamics (1.129) is exponentially stable, the first property of Proposition 1.5.10 is satisfied and, consequently, the fourth property is also satisfied. This implies that the first property of Theorem 1.5.9 is satisfied and consequently also the second property. Thus, we have shown that Luenberger weak observer (1.126) is stable.

Let us present some numerical illustrations. We further consider the wave equation (1.117). Let $\mathcal{Y} = L^2(\omega)$. We have access to noisy field measurements in the subdomain ω shown in Figure 1.20, namely the observation operator C is defined by (1.131).

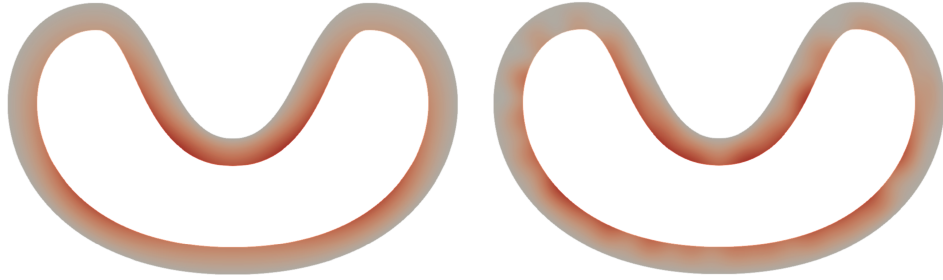


Figure 1.20: Solution $u_h(2)$ and the corresponding measurement $y_h^\delta(2) = u_h|_\omega(2) + \chi_h^\delta(2)$

As presented in Section 1.5.2, following [Moireau et al., 2009], we represent the space-time discretization of the dynamics the weak Luenberger observer dynamics (1.127).

$$\left\{ \begin{array}{l} (\alpha^2 \mathbb{K} + I_\omega^T I_\omega) \frac{\hat{U}_{h,n+1} - \hat{U}_{h,n}}{\tau} = (\alpha^2 \mathbb{K} + I_\omega^T I_\omega) \frac{\hat{V}_{h,n+1} + \hat{V}_{h,n}}{2} \\ \quad + \gamma I_\omega^T \left(\frac{y_{h,n+1}^\delta + y_{h,n}^\delta}{2} - I_\omega \frac{\hat{U}_{h,n+1} + \hat{U}_{h,n}}{2} \right), \\ \mathbb{M} \frac{\hat{V}_{h,n+1} - \hat{V}_{h,n}}{\tau} = -\mathbb{K} \frac{\hat{U}_{h,n+1} + \hat{U}_{h,n}}{2}, \\ \hat{U}_{h,n=0} = \hat{U}_{h,0}, \\ \hat{V}_{h,n=0} = \hat{V}_{h,0}. \end{array} \right. \quad (1.130)$$

We see in Figure 1.21 and 1.22 that we can reconstruct the state of the system from noisy field measurements.

As we can see in Figure 1.22, the energy of the observer system reaches the energy of the target system even with noisy measurements. Figure 1.23 shows that the reconstruction is very fast, with an error of 10% from $t = 1$. However, fluctuations are observed from about $t = 3$. Then we stagnate at an error between 2 and 5%.

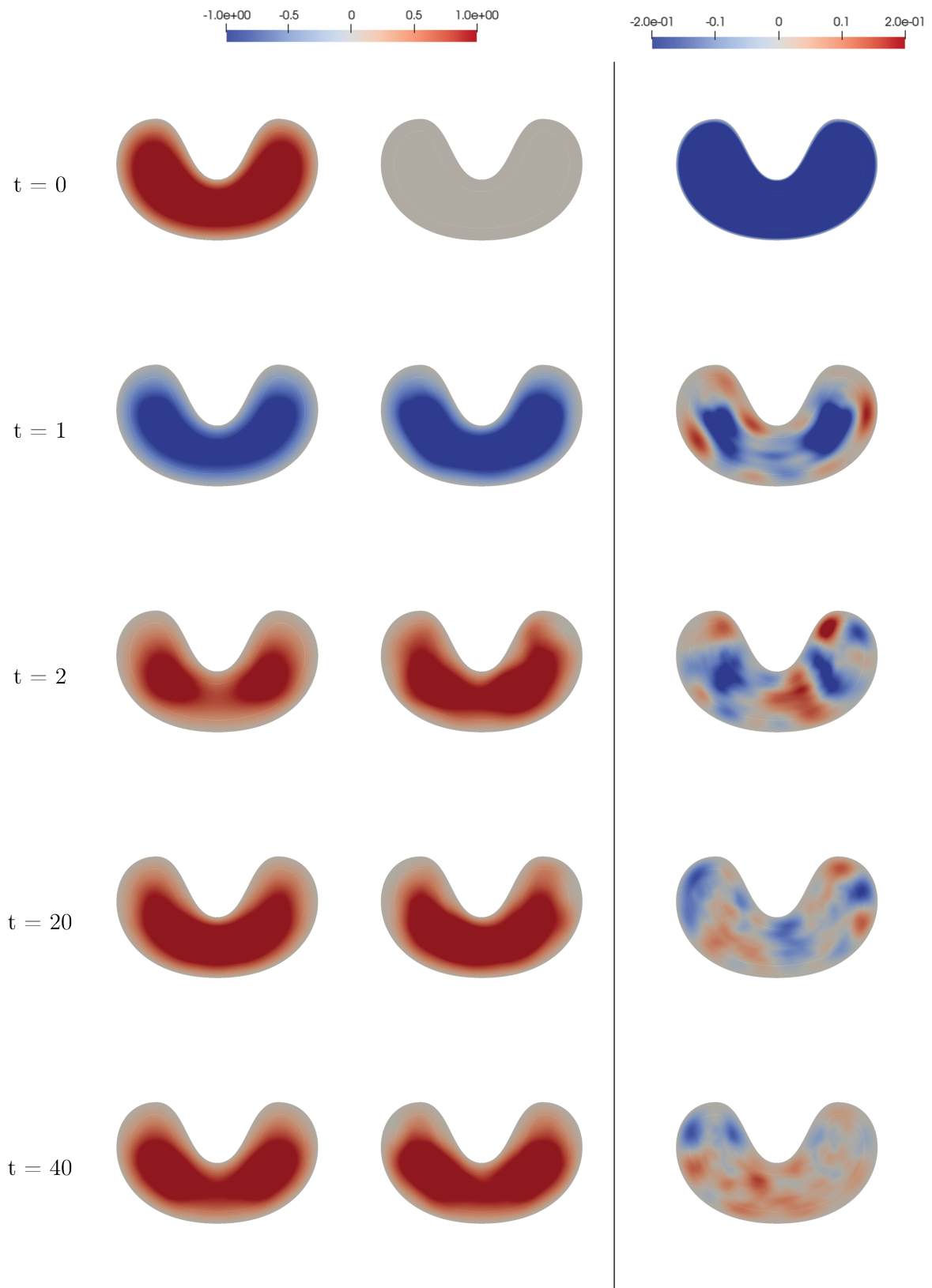


Figure 1.21: Target solution u_h (left), observer \hat{u}_h (middle) by the *Regularize-then-invert* method and error $u_h - \hat{u}_h$ (right) with $\gamma = 25$ from noisy *field* measurements at different times t

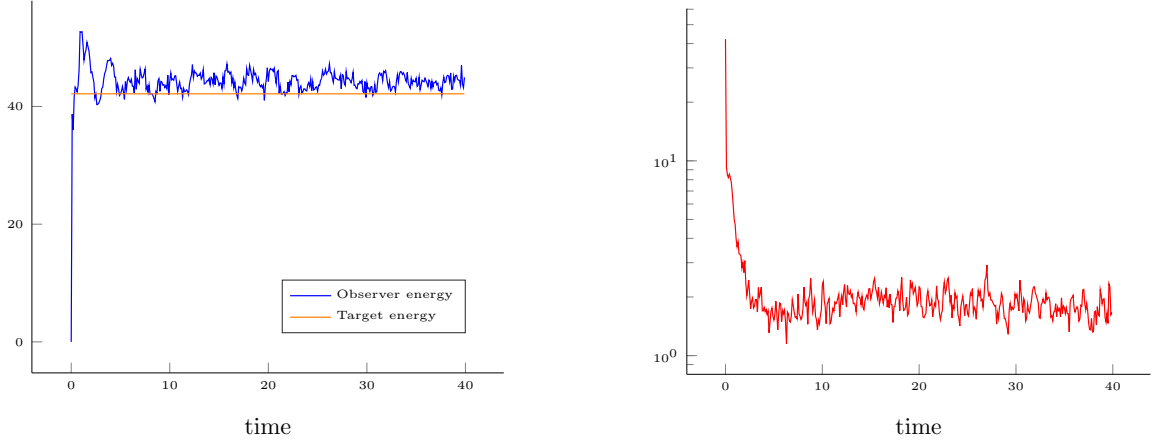


Figure 1.22: Evolution of the observer energy $\hat{\mathcal{E}}(t)$ and the target energy $\check{\mathcal{E}}(t)$ (left) and the error energy $\tilde{\mathcal{E}}(t)$ (right) over the time from noisy *field* measurements

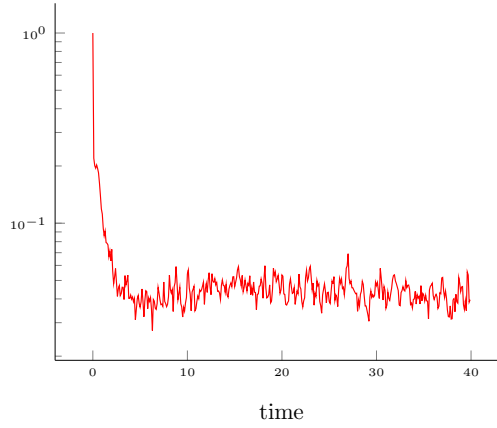


Figure 1.23: Evolution of relative error $\tilde{\mathcal{E}}(t)/\check{\mathcal{E}}(t)$ (left)

1.5.3.2 Invert-then-regularize strategy

Let us present the second method. This time, the idea is to change the definition of the observation operator. In particular, we will extend its codomain to L^2 . Noisy measurements will therefore belong to this space. Then we will not need to regularize the measurements. This is done directly using the adjoint of the observation operator. We will see that the advantage of this method is that it does not require the introduction of a regularization operator. On the other hand, the convergence of the associated observer is exponentially stable in a weaker space and relies on the proof of an observability condition in a weaker energy space. As we will see in Chapter 3, this method is well suited when we have a proof of observability proven in a weaker energy space. We consider internal *field* measurements $y = u|_{\omega} \in \mathcal{Y} = L^2(\omega)$. We define

$$C = \begin{pmatrix} I_{\omega} & 0 \end{pmatrix} : \begin{cases} \mathbf{H}_0^1(\Omega) \times L^2(\Omega) \rightarrow L^2(\omega), \\ z = \begin{pmatrix} u \\ v \end{pmatrix} \mapsto y = u|_{\omega}, \end{cases} \quad (1.131)$$

where $I_\omega : H_0^1(\Omega) \rightarrow L^2(\omega)$ is the restriction operator. We recall that the adjoint observation is given by $C^* = \begin{pmatrix} F_\omega & 0 \end{pmatrix}^t$ where $F_\omega \in \mathcal{L}(L^2(\omega), H_0^1(\Omega))$ is defined by

$$F_\omega \phi := \psi \text{ the solution of } \begin{cases} -\Delta \psi = \mathbb{1}_\omega \phi, & \text{in } \Omega, \\ \psi = 0, & \text{on } \partial\Omega, \end{cases} \quad (1.132)$$

where $\mathbb{1}_\omega$ is the characteristic function of ω . With this definition of C , we can now use the dynamics (1.105) of the Luenberger observer $\hat{z} \in C([0, \infty); \mathcal{Z})$. However, we show that the price of weakening the regularity of the observation space \mathcal{Y} is that the semigroup associated with $A - \gamma C^* C$ is exponentially stable, but in a weaker space $L^2(\Omega) \times H^{-1}(\Omega)$ instead of in the energy space $H_0^1(\Omega) \times L^2(\Omega)$. Indeed, we will show that the observability condition associated with the observation operator C is in weakened norm (1.133). We will see that the price for the weakening of the observation space \mathcal{Y} is also the weakening of the space of initial conditions \mathcal{Z} .

Theorem 1.5.11 *Under the hypothesis of Theorem 1.4.8, we have also*

$$\int_0^T \int_\omega |u|^2 dx dt \geq C^{st} \left(\|u_0\|_{L^2(\Omega)}^2 + \|v_0\|_{H^{-1}(\Omega)}^2 \right). \quad (1.133)$$

PROOF: We introduce $w(t) = \int_0^t u(s) ds + \eta$ where η is the solution of the following system

$$\begin{cases} -\Delta \eta = v_0, & \text{in } \Omega, \\ \eta = 0, & \text{on } \partial\Omega. \end{cases}$$

The variable w satisfies the following equation

$$\begin{aligned} \partial_{tt} w - \Delta w &= \partial_t u - \int_0^t \Delta u ds - \Delta \eta \\ &= \partial_t u - \int_0^t \Delta u ds - v_0. \end{aligned}$$

Integrating the first equation of (1.1) in time, we find that

$$\partial_{tt} w - \Delta w = 0.$$

We can then apply the observer condition (1.100) of Theorem 1.4.8 to w , which is the solution of the following dynamics

$$\begin{cases} \partial_{tt} w(x, t) - \Delta w(x, t) = 0, & (x, t) \in \Omega \times (0, T), \\ w(x, t) = 0, & (x, t) \in \partial\omega \times (0, T), \\ w(x, 0) = \eta(x), & x \in \Omega, \\ \partial_t w(x, 0) = u_0(x), & x \in \Omega, \end{cases}$$

which gives us

$$\int_0^T \int_\omega |\partial_t w(x, t)|^2 dx dt \geq C^{st} \left(\|\eta\|_{H_0^1(\Omega)}^2 + \|u_0\|_{L^2(\Omega)}^2 \right).$$

Since the Laplace operator with Dirichlet boundary conditions $-\Delta_0$ defines an isomorphism from H_0^1 to H^{-1} , the result follows immediately. \blacksquare

Theorem 1.5.12 *We assume that there exist C^{st} such that*

$$\forall T > T(x_0), \quad \int_0^T \|u\|_{L^2(\omega)}^2 dt = \int_0^T \|Ce^{tA}z_0\|_{L^2(\omega)}^2 dt \geq C^{st}\|z_0\|_{L^2(\Omega) \times H^{-1}(\Omega)}^2. \quad (1.134)$$

*Then, the C^0 -semigroup $\tilde{\Phi}$ associated with $A - \gamma C^*C$ is exponentially stable.*

PROOF: The proof is divided into two steps.

Step 1: First we prove an observability inequality, namely that there exists $C^{st} > 0$ such that

$$\forall T \geq T(x_0), \quad \forall z_0 \in \mathcal{Z}, \quad \int_0^T \|C\tilde{z}(t)\|_{L^2(\omega)}^2 dt \geq C^{st}\|z_0\|_{L^2(\Omega) \times H^{-1}(\Omega)}^2,$$

where \tilde{z} is solution of

$$\begin{cases} \dot{\tilde{z}} = (A - \gamma C^*C)\tilde{z}, & \text{in } (0, T), \\ \tilde{z}(0) = z_0. \end{cases} \quad (1.135)$$

Using the Duhamel formula (1.5) we have that

$$\tilde{z} = e^{t(A - \gamma C^*C)z} z_0.$$

As in the proof of Theorem 1.5.12 we decompose \tilde{z} in $\tilde{z} = \psi + \eta$ where

$$\begin{cases} \dot{\psi} = A\psi, & \text{in } (0, T) \\ \psi(0) = z_0, \end{cases} \quad \text{and} \quad \begin{cases} \dot{\eta} = A\eta - \gamma C^*C\tilde{z}, & \text{in } (0, T) \\ \eta(0) = 0. \end{cases}$$

The observability condition (1.108) gives

$$C^{st}\|z_0\|_{L^2(\Omega) \times H^{-1}(\Omega)}^2 \leq \int_0^T \|C\psi\|_{L^2(\omega)}^2 dt \leq 2 \int_0^T \|C\tilde{z}\|_{L^2(\omega)}^2 dt + 2 \int_0^T \|C\eta\|_{L^2(\omega)}^2 dt.$$

Let us focus on the last term on the right-hand side. We have

$$\eta(t) = -\gamma \int_0^t e^{(t-s)A} C^* C \tilde{z}(s) ds,$$

which gives

$$\begin{aligned} \|\eta(t)\|_{\mathcal{Z}} &\leq \gamma \int_0^t e^{(t-s)\lambda} \|C\|_{\mathcal{L}(\mathcal{Z}, L^2(\omega))} \|C\tilde{z}(s)\|_{L^2(\omega)} ds \\ &\leq C^{st} \|C\|_{\mathcal{L}(\mathcal{Z}, L^2(\omega))} \left(\frac{1 - e^{-\lambda T}}{2\lambda} \right)^{\frac{1}{2}} \|C\tilde{z}\|_{L^2((0, T); L^2(\omega))}. \end{aligned}$$

Finally, there exist $\kappa > 0$, so that

$$\|z_0\|_{L^2(\Omega) \times H^{-1}(\Omega)}^2 \leq \kappa \int_0^T \|C\tilde{z}(t)\|_{L^2(\omega)}^2 dt, \quad (1.136)$$

Step 2: Now for each solution $\tilde{z} = (\tilde{u} \ \tilde{v})^t$ in $\mathcal{D}(A)$ of (1.135)

$$(\dot{\tilde{z}}, \tilde{z})_{L^2(\Omega) \times H^{-1}(\Omega)} = ((A - \gamma C^*C)z, \tilde{z})_{L^2(\Omega) \times H^{-1}(\Omega)},$$

leads to

$$(\dot{\tilde{u}}, \tilde{u})_{L^2(\Omega)} + (\dot{\tilde{v}}, \tilde{v})_{H^{-1}(\Omega)} = (\tilde{v}, \tilde{u})_{L^2(\Omega)} + (\Delta_0 \tilde{u}, \tilde{v})_{H^{-1}(\Omega)} - \gamma (C\tilde{z}, C\tilde{z})_{L^2(\omega)}.$$

Defining the H^{-1} -norm by $(\cdot, \cdot)_{H^{-1}(\Omega)} = (\Delta_0^{-1}\cdot, \Delta_0^{-1}\cdot)_{H_0^1(\Omega)}$, we get

$$\frac{1}{2} \frac{d}{dt} \left(\|u\|_{L^2(\Omega)}^2 + \|v\|_{H^{-1}(\Omega)}^2 \right) = (\tilde{v}, \tilde{u})_{L^2(\Omega)} + (\nabla \tilde{u}, \nabla \Delta_0^{-1} \tilde{v})_{L^2(\Omega)} - \gamma \|C\tilde{z}\|_{L^2(\omega)}^2.$$

Finally, integrating by part, we obtain

$$\frac{1}{2} \frac{d}{dt} \|\tilde{z}\|_{L^2(\Omega) \times H^{-1}(\Omega)}^2 = -\gamma \|C\tilde{z}\|_{L^2(\omega)}^2.$$

Combined with (1.136), we find

$$\frac{1}{\kappa} \|\tilde{z}(t - T)\|_{L^2(\Omega) \times H^{-1}(\Omega)}^2 \leq \int_{t-T}^t \|C\tilde{z}(t)\|_{L^2(\omega)}^2 dt \leq \frac{1}{2\gamma} \|\tilde{z}(t - T)\|_{L^2(\Omega) \times H^{-1}(\Omega)}^2.$$

Reasoning as in the proof of Theorem 1.5.2, we obtain with $\lambda = -\ln(1 - \rho)/T > 0$ where $\rho = \frac{2\gamma}{\kappa} \in (0, 1)$,

$$\forall t > 0, \quad \|\tilde{z}(t)\|_{L^2(\Omega) \times H^{-1}(\Omega)} \leq e^{\frac{\lambda}{2}T} e^{-\frac{\lambda}{2}t} \|\tilde{z}(0)\|_{L^2(\Omega) \times H^{-1}(\Omega)},$$

which ensures that there exist $C^{\text{st}}, \tilde{\lambda} > 0$ such that for all $t > 0$, the semigroup $\tilde{\Phi}$ satisfies $\|\tilde{\Phi}(t)\| \leq C^{\text{st}} e^{-\tilde{\lambda}t}$. ■

It has been shown that the observer in the weaker space $L^2(\Omega) \times H^{-1}(\Omega)$ is exponentially stable. However, the solution of this observer is then defined in a weaker space. Therefore, we will work in the space $H_0^1(\Omega) \times L^2(\Omega)$ even though we have no proof of exponential stability in this space. We will use this method in Chapter 3, as we will demonstrate an observability inequality where the norm of the parameter is in a weaker space.

The spatial and temporal discretization of this weak Luenberger observer results in

$$\left\{ \begin{array}{l} \mathbb{K} \frac{\hat{U}_{h,n+1} - \hat{U}_{h,n}}{\tau} = \mathbb{K} \frac{\hat{V}_{h,n+1} + \hat{V}_{h,n}}{2} + \gamma I_\omega^T \left(\frac{y_{h,n+1}^\delta + y_{h,n}^\delta}{2} - I_\omega \frac{\hat{U}_{h,n+1} + \hat{U}_{h,n}}{2} \right), \\ \mathbb{M} \frac{\hat{V}_{h,n+1} - \hat{V}_{h,n}}{\tau} = -\mathbb{K} \frac{\hat{U}_{h,n+1} + \hat{U}_{h,n}}{2}, \\ \hat{U}_{h,n=0} = \hat{U}_{h,0}, \\ \hat{V}_{h,n=0} = \hat{V}_{h,0}. \end{array} \right. \quad (1.137)$$

We see in Figure 1.24 and 1.25 that we can reconstruct the state of the system from noisy field measurements but you have to wait much longer than with the method *regularize-then-invert* (see Figure 1.21) even with a larger parameter γ .

As we can see in Figure 1.25, the energy of the observer system reaches the energy of the target system even with noisy measurements, but the convergence takes much longer. As shown in Figure 1.26, the reconstruction is slower than the other method (see Figure 1.23). In fact, you have to wait until $T = 10$ to get an error within 10%. On the other hand, the error rate decreases with time, reaching 1% at $t = 2$ and 0.3% at $t = 40$.

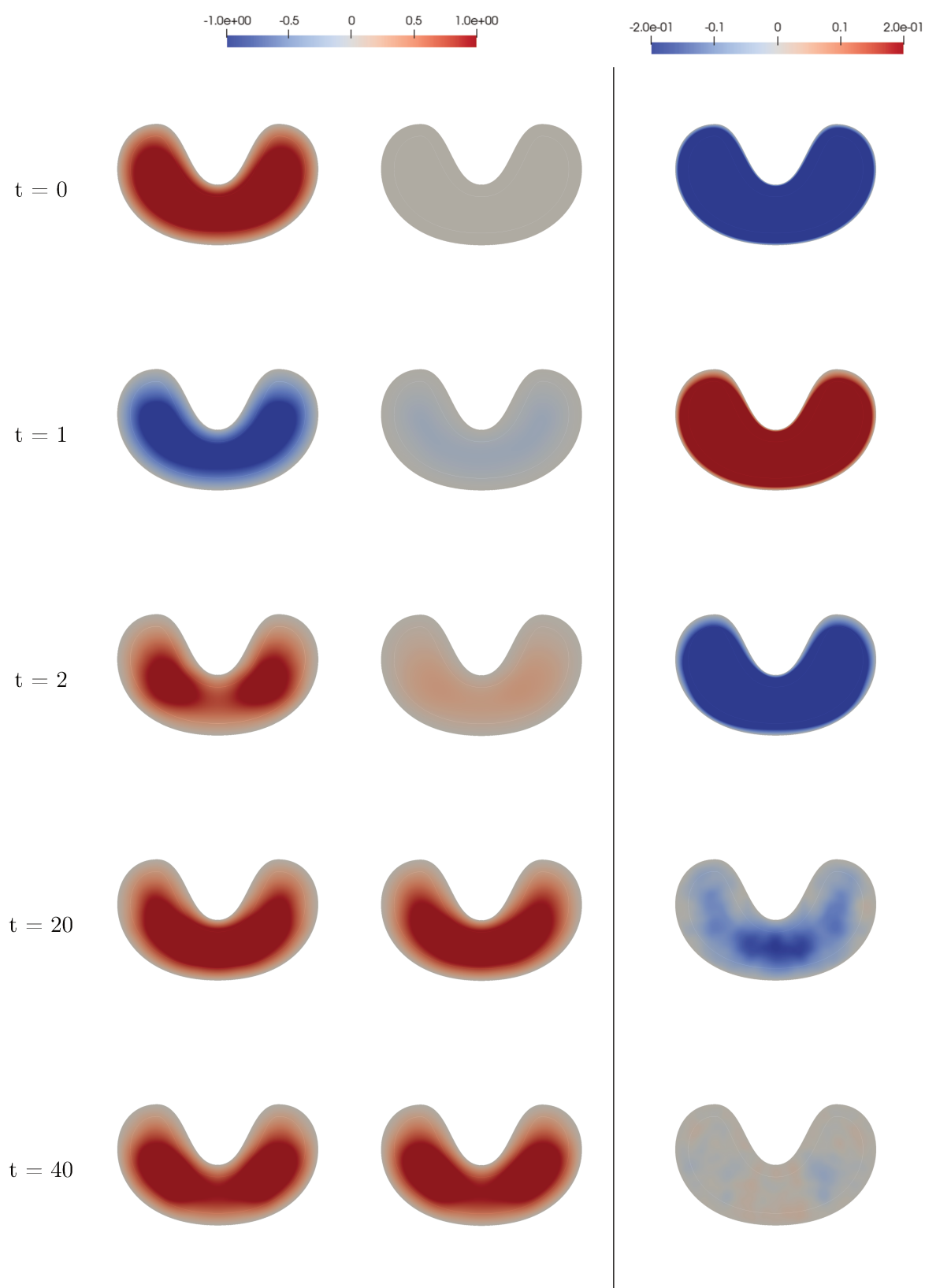


Figure 1.24: Target solution u_h (left), observer \hat{u}_h by the *Invert-then-regularize* method (middle) and error $u_h - \hat{u}_h$ (right) with $\gamma = 25$ from noisy *field* measurements

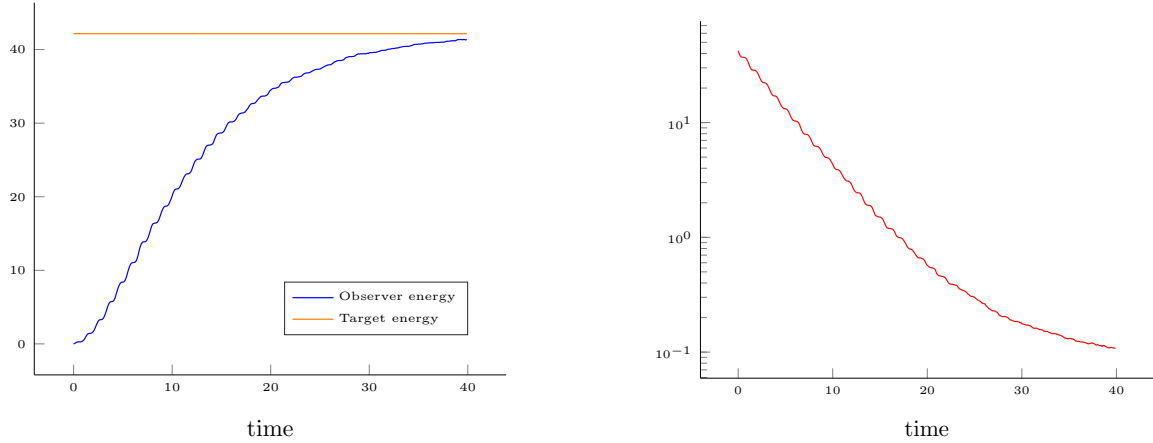


Figure 1.25: Evolution of the observer energy $\hat{\mathcal{E}}(t)$ and the target energy $\check{\mathcal{E}}(t)$ (left) and the error energy $\tilde{\mathcal{E}}(t)$ (right) over the time from noisy *field* measurements

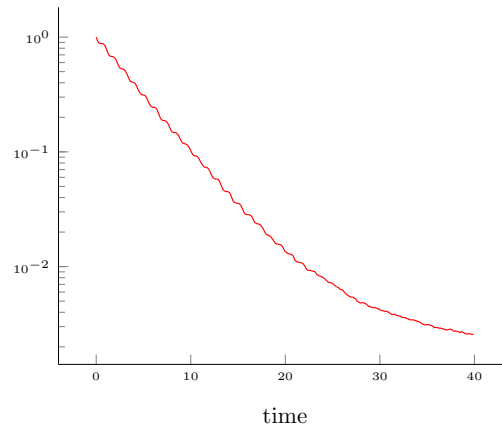


Figure 1.26: Evolution of relative error $\tilde{\mathcal{E}}/\check{\mathcal{E}}$ (left) over the time

1.5.4 Reconstruction of the initial conditions: the Back and forth algorithm

The following method, introduced by [Ramdani et al., 2010], consists in using the available measurements and benefiting from the time reversibility of the wave equation to reconstruct the initial conditions. See [Imperiale et al., 2020] for a generalization with unbounded domain.

Observers are systems that asymptotically estimate in time the state of a target system. To use an observer to reconstruct initial conditions, we need to reintroduce a fixed time period $(0, T)$ and use the time reversibility of the wave equation to look back for an initial condition. Thus, we introduce the so-called backward observer, which is defined by

$$\begin{cases} \dot{\hat{z}}_b = A\hat{z}_b - \gamma C^*(y^\delta - C\hat{z}_b), & \text{in } (0, T) \\ \hat{z}_b(T) = \hat{z}(T). \end{cases} \quad (1.138)$$

The dynamics is well posed as soon as $-A$ is the generator of a \mathcal{C}^0 -semigroup and C is bounded, which is the case here as $A^* = -A$, namely A is skew-adjoint.

The backward dynamics— called the Back and forth — imagined by [Ramdani et al.,

2010] allows us to define the following reconstruction algorithm

$$(\text{B\&F}) : \begin{cases} \dot{\hat{z}}^{k+1} = A\hat{z}^{k+1} + \gamma C^* (y^\delta - C\hat{z}^{k+1}), & \text{in } [0, T], \\ \hat{z}^{k+1}(0) = \hat{z}_b^k(0), \\ \dot{\hat{z}}_b^{k+1} = A\hat{z}_b^{k+1} - \gamma C^* (y^\delta - C\hat{z}_b^{k+1}), & \text{in } [0, T], \\ \hat{z}_b^{k+1}(T) = \hat{z}^{k+1}(T), \end{cases} \quad (1.139)$$

which turns out to be an original reconstruction strategy of the target initial condition $\check{z}(0)$ of an observed corresponding target trajectory $\check{z} : t \mapsto e^{tA}\check{z}(0)$. This method has the advantage of being effective. It is also less costly than the method presented in Section 1.4.3, which required computing a Riccati operator. Unlike the gradient descent method (or any other iterative method) presented in Section 1.3.5, it does not rely on minimizing the criterion \mathcal{J}_T . Moreover, this method relies on the time-reservability property of the wave equation and therefore cannot necessarily be extended to dissipative of systems.

1.6 Towards identification problem

We have proved that using the solutions given by the semigroup formalism has the advantage of avoiding a constrained minimization problem and that the obtained solutions are equivalent to variational solutions. One way to solve our state estimation problem is to use a Kalman filter, a sequential approach which is equivalent to the minimization of a least squares criterion. However, the computation can be numerically costly since it involves a Riccati operator. Therefore, we reviewed an alternative: using a Luenberger observer, it is possible to recover the state of the system from unknown initial conditions by introducing the error between the measurements we have on the system and the target solution, without computing this Riccati operator. The proof of convergence relies on observability inequalities, which are important tools in inverse problems and will be studied in more detail in all subsequent chapters of the manuscript. When the noise is defined in a space less regular than the space of observations, we have explored two methods to deal with it: *regularize-then-invert* and *invert-then-regularize*. Both imply a change in the definition of the observation operator and consequently its adjoint. For the first method we have proved that the Luenberger associated is stable, while for the second we have shown that we have the exponential stability property, but in a weaker space. A natural question is how we can extend these methods to the case of parameter estimation. Let us take the simplest case, that of reconstructing the θ parameter in the following wave equation

$$\begin{cases} \partial_{tt}u(x, t) - \Delta u(x, t) = \theta(x), & (x, t) \in \Omega \times (0, T), \\ u(x, t) = 0, & (x, t) \in \partial\Omega \times (0, T), \\ u(x, 0) = 0, & x \in \Omega, \\ \partial_t u(x, 0) = 0, & x \in \Omega. \end{cases} \quad (1.140)$$

We assume that we have observed the velocity of a target solution $\partial_t \check{u}$, and that we have access to a noisy version of these observations $y^\delta \in L^2(0, T; L^2(\omega))$,

$$\int_0^T \|\partial_t \check{u} - y^\delta\|_{L^2(\omega)}^2 dt \leq \delta^2 T. \quad (1.141)$$

We look at two methods that lead back to an initial conditions problem.

1.6.1 Source term identification by derivation strategy

As far as observability is concerned, it is important to rewrite the source estimation problem as a state estimation problem. To do this, we pose $v = \partial_t u$. Thus, (1.140) is rewritten as follows

$$\begin{cases} \partial_{tt}v(x, t) - \Delta v(x, t) = 0, & (x, t) \in \Omega \times (0, T), \\ v(x, t) = 0, & (x, t) \in \partial\Omega \times (0, T), \\ v(x, 0) = 0, & x \in \Omega, \\ \partial_tv(x, 0) = \theta(x), & x \in \Omega, \end{cases} \quad (1.142)$$

which can be rewritten as a first-order system by setting $z = (v \ \partial_tv)^t$,

$$\begin{cases} \dot{z}(t) = Az(t), & t \in (0, T), \\ z(0) = \begin{pmatrix} 0 & \theta \end{pmatrix}^t. \end{cases} \quad (1.143)$$

Then, the key idea is to show an observability inequality on (1.143). We use a similar approach when we will study a slightly more complicated case in Chapter 3, with the source term of the form $\sigma(t)\theta(x)$.

1.6.2 Source term identification by joint state-parameter system

The idea is to artificially add the dynamics of θ (which does not depend on time) as proposed in [Moireau et al., 2008], in other words we study

$$\begin{cases} \dot{z}(t) = Az(t), & t \in [0, T], \\ \dot{\theta}(t) = 0, & t \in [0, T], \\ z(0) = z_0, \\ \theta(0) = \theta. \end{cases} \quad (1.144)$$

Returning to an initial condition problem, we introduce the augmented system

$$z = \begin{pmatrix} z \\ \theta \end{pmatrix} \in \mathcal{Z} := \mathcal{Z} \times \mathcal{P}, \quad (1.145)$$

which is the solution of

$$\begin{cases} \dot{z}(t) = \mathbf{A}z(t), \\ z(0) = z_0 := \begin{pmatrix} z_0 & \theta \end{pmatrix}^t, \end{cases} \quad \text{where } \mathbf{A} = \begin{pmatrix} A & \text{Id} \\ 0 & 0 \end{pmatrix}. \quad (1.146)$$

Obviously, the observation operator must be extended for this augmented system namely, we set

$$C = \begin{pmatrix} 0 & C \end{pmatrix} \in \mathcal{L}(\mathcal{Z}, \mathcal{Y}), \quad (1.147)$$

where C is defined by (1.92) in the case of *velocity* measurements with $\mathcal{Y} = L^2(\omega)$ and in the case of *field* measurements by (1.93) with $\mathcal{Y} = H^1(\omega)$. This brings us back to the situation in Section 1.3.3, where we can define the cost functional using the augmented operators, namely we introduce

$$\mathcal{J}_T = \frac{\epsilon}{2} \|z_0\|_{\mathcal{Z}}^2 + \frac{1}{2} \int_0^T \|Cz|_{z_0}(t) - y^\delta(t)\|_{\mathcal{Y}}^2 dt, \quad (1.148)$$

and the minimization of the cost functional leads to the following two-ends problem

$$\begin{cases} \dot{\bar{z}}_T = \mathbf{A}\bar{z}_T, & \text{in } (0, T), \\ \dot{\bar{q}}_T + \mathbf{A}^*\bar{q}_T = -\mathbf{C}^*(y^\delta - \mathbf{C}\bar{z}_T), & \text{in } (0, T), \\ \bar{z}_T(0) = \epsilon^{-1}\bar{q}_T(0), \\ \bar{q}_T(T) = 0. \end{cases} \quad (1.149)$$

So the rest of the analysis is similar to what we did before. Obviously, we can define the Kalman filter which is equivalent to minimizing the cost functional:

$$\begin{cases} \dot{\hat{z}} = \mathbf{A}\hat{z} + \mathbf{P}\mathbf{C}^*(y^\delta - \mathbf{C}\hat{z}), & \text{in } (0, T), \\ \hat{z}(0) = 0. \end{cases} \quad (1.150)$$

This means that we have to solve the augmented Riccati operator \mathbf{P} , which is even more costly to compute. In Chapter 3 we will see how to solve such a problem by relying on a reduced Riccati operator, the solution of a Riccati dynamics in a reduced space.

1.7 Conclusion

In this chapter, we introduced all the tools that we will use in the rest of the manuscript using the inverse problem of state estimation. We have seen that in infinite dimensions the spaces in which we work are of paramount importance. Depending on the type of solution we consider (variational solutions or solutions given by semigroup theory) and the type of observations, the mathematical techniques and resulting methods used will differ. When considering variational solutions, it is natural to solve a constraint minimization problem. This can be avoided for solutions obtained using the semigroup theory. However, we have shown the equivalence between variational solutions and solutions from semigroup theory. Therefore, we can use the optimization techniques given by the semigroup approach while ensuring a natural discretization by finite elements. We then introduced the 4D-var method which correspond to the minimization of the deviation between the observations and the solution. This leads us to solve a two-ends problem. We defend the idea that an adapted discretization choice is to first discretize the model and then minimize under the constraint defined by the discrete dynamics. We then present the Kalman filter, an observer equivalent to the minimization of the cost functional. Later, in Chapters 3 and 4, we will focus on this sequential method. It has the advantage of being a direct method to the cost of computation of a Riccati operator. We saw an alternative: the Luenberger observer. The measurements space plays a very important role. In particular, where the noise is less regular than the space in which the measurements are naturally defined, two methods have been studied. One is exponentially stable in a weaker space and requires a proven observability condition with weakened norms. This method is used in Chapter 3. The other consists in regularizing the measurements before inverting, and will be used at the last of this manuscript.

CHAPTER 2

Uniform boundary stabilization of a high-order finite element space discretization of the 1-d wave equation

Les mathématiques ont un triple objectif. Ils doivent fournir un instrument pour l'étude de la nature. Mais ce n'est pas tout: ils ont un but philosophique et, j'ose dire, un but esthétique

— Henri Poincaré

The objective of this work is to propose and analyze numerical schemes for solving boundary control problems or data assimilation problems by observers for wave propagation problems. The efficiency of the considered control or data assimilation strategy relies on the exponentially stable character of the underlying system. Therefore, the aim of our work is to propose a discretization process that allows preserving the exponential stability at the discrete level when using high-order spectral finite element approximation. The main idea is to add a stabilizing term to the wave equation that dampens the spurious oscillatory components of the solutions. This term is based on a discrete multiplier analysis that gives us the exponential stability of the semi-discrete problem at any order without affecting the approximation properties.

This chapter contains a submitted article, realized in collaboration with Sébastien Impe-riale and Philippe Moireau.

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Uniform boundary stabilization of a high-order finite element space discretization of the 1-d wave equation

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Submitted

Abstract

The objective of this work is to propose and analyze numerical schemes for solving boundary control problems or data assimilation problems by observers for wave propagation problems. The efficiency of the considered control or data assimilation strategy relies on the exponentially stable character of the underlying system. Therefore, the aim of our work is to propose a discretization process that allows preserving the exponential stability at the discrete level when using high-order spectral finite element approximation. The main idea is to add a stabilizing term to the wave equation that dampens the spurious oscillatory components of the solutions. This term is based on a discrete multiplier analysis that gives us the exponential stability of the semi-discrete problem at any order without affecting the approximation properties

Keywords— High-order finite element, stabilization, numerical analysis, wave propagation

2.1 Introduction

The aim of this work is to demonstrate an asymptotic exponential stability of a one-dimensional wave-like system with boundary damping uniformly with respect to the time and space discretization of the system. Such a system typically arises when analyzing boundary control problems [Lasiecka and Triggiani, 2004], data assimilation problem using boundary data [Haine, 2014b] or stabilization property of absorbing boundary condition imposed to initially unbounded formulations [Ervedoza and Zuazua, 2008].

Obtaining exponential stability results – through observability condition in most cases – is a rather difficult result at the continuous level, which has been extensively studied by the control community [Zuazua, 2005] and proved by various techniques, typically multiplier techniques [Lions, 1988; Komornik, 1997; Joly, 2006] or even Carleman estimates [Tebou, 2008], spectral approaches [Cox and Zuazua, 1994; Ramdani et al., 2007], or microlocal analyzes [Lebeau et al., 1992]. However, discretizing the dynamics so that the stabilization properties are preserved at the discrete level presents an additional difficulty because of the presence spurious high frequencies that remain uncontrolled if one relies on the classical discretization of the stabilization terms ([Infante and Zuazua, 1999; Zuazua, 2005; Ervedoza and Zuazua, 2012; Marica and Zuazua, 2015]). Several strategies – either for finite element or finite difference discretizations – are presented in the literature to compensate for the instabilities caused by high frequencies, namely Fourier techniques ([Micu, 2002; Negreanu

and Zuazua, 2003; Münch, 2005]), Tikhonov regularization [Glowinski et al., 1990], multi-grid finite difference methods [Negreanu and Zuazua, 2004] or mixed finite element methods ([Castro and Micu, 2006] and their extension for nonuniform meshes [Ervedoza, 2010]).

For dealing with high-performance wave propagation applications, high-order spectral finite elements [Cohen, 2002] are now considered indispensable – see for instance [Faccioli et al., 1997; Cohen et al., 2001; Bolis et al., 2014] and for seismic wave applications [Komatitsch et al., 2009, 2010] – however, the above discretization strategies cannot be easily extended to such high-order discretizations without losing their advantages, in particular their convergence order and their efficiency.

Here we follow a strategy of including additional consistent stabilization terms in the discretization to preserve the efficiency of the methods for possible extension to high dimensions. The additional vanishing stabilization terms have a similar goal as those found in [Burman et al., 2020]. However, here these additional terms are constructed in such a way that the classical multiplier method, which proves stabilization in the continuous method, can be continued at the discrete level, achieving uniform stabilization with respect to the discretization parameters. Moreover, the stabilization terms are carefully designed to preserve the consistency order. In particular, our result can extend the discretization result for observers proposed in [Chapelle et al., 2012b] to high-order schemes, which states that using data, one can obtain discretization schemes that do not suffer from the asymptotic time drift of the error.

Our proof is limited to the one-dimensional case, while an implicit mid-point time-scheme is chosen for the time-discretization however the principles of our approach pave the way for possible extension to higher dimensions and more efficient time schemes, typically fully explicit approaches – that are nevertheless tested in this work.

This work is organized as follows. In Section 2.2, the motivations for this work are stated. In Section 2.3 we lay out the continuous configuration and the method for proving exponential stability. Section 2.4 proves the uniform exponential in h in the semi-discrete configuration using a general abstract framework. In Section 2.5 we present a semi-discrete convergence results for a data assimilation problem. Then, Section 2.6 presents the fully discretized framework using an implicit scheme in time. The numerical results are presented in Section 2.7, which is split into two parts. First, we consider the eigenvalues of the operator belonging to our damped system to observe the uniform exponential stability property. To obtain an efficient method, in a second step we introduce an explicit scheme for discretization in time that preserves the properties shown in the previous sections. Using this scheme, we can then present numerical results that illustrate the interest of adding the damping terms.

2.2 Motivation of the observation problem

Let us study a model problem which shines a light on the difficulties that appear for the stabilization of discretized problems.

Let $\Omega = (0, 1)$ be the domain of propagation. We study a homogenous wave equation formulated as a first-order system

$$\begin{cases} \partial_t u(x, t) + \partial_x v(x, t) = 0, & (x, t) \in \Omega \times (0, T), \\ \partial_t v(x, t) + \partial_x u(x, t) = 0, & (x, t) \in \Omega \times (0, T). \end{cases} \quad (2.1)$$

We define the boundary conditions and the initial conditions of our wave equation. The following boundary conditions corresponds to Neumann at point $x = 0$ and a Robin condition at point $x = 1$. We exclude the degenerate case $\gamma = 1$ which corresponds to transparent

conditions.

$$\begin{cases} u(1, t) = \gamma v(1, t) & t \in (0, T), \\ v(0, t) = 0, & t \in (0, T), \end{cases} \quad \begin{cases} u(x, 0) = u_0(x), & x \in \Omega, \\ v(x, 0) = v_0(x), & x \in \Omega. \end{cases} \quad (2.2)$$

The study of such a system ((2.1) with (2.2)) allows to simulate the case of the study of a system with absorbing conditions (at boundary $x = 1$). It can also be similar to the study of an error system between a target system and a simulated system (which we would like to converge asymptotically in time toward the target trajectory). Namely, if we consider having access to measurements (here v at $x = 1$), and we can incorporate by a feedback law in the boundary conditions the difference between the solution and the measurements. To study the convergence of this system called observer system (see [Krener, 1999; Ramdani et al., 2010]) towards the target system, it is equivalent to show the exponential convergence towards 0 of the error system that could represent ((2.1) with (2.2)). In all cases, the convergence of these problems ((2.1) with (2.2)) is related to the exponential convergence of a damped PDE (at $x = 1$ in this case). Furthermore, at the continuous level, one can prove that this system is exponentially stable ([Cox and Zuazua, 1994] in 1D problem, [Ervedoza and Zuazua, 2012]).

2.2.1 Parasitic waves

The question is to prove that such system – that is known to be exponentially stable preserves its exponential stability property after standard high-order spectral finite element discretization on space and explicit second order energy preserving time discretization [Cohen, 2002]. To illustrate the phenomenon of propagation, we will represent the wave over time. We simulate our problem with finite elements (F.E.M.) \mathbb{P}_r – the set of polynomials of degree r – and an explicit time scheme and on uniform mesh. We initialize u as a Gaussian (see Figure 2.1 (Left)) and set v to 0 as initial condition.

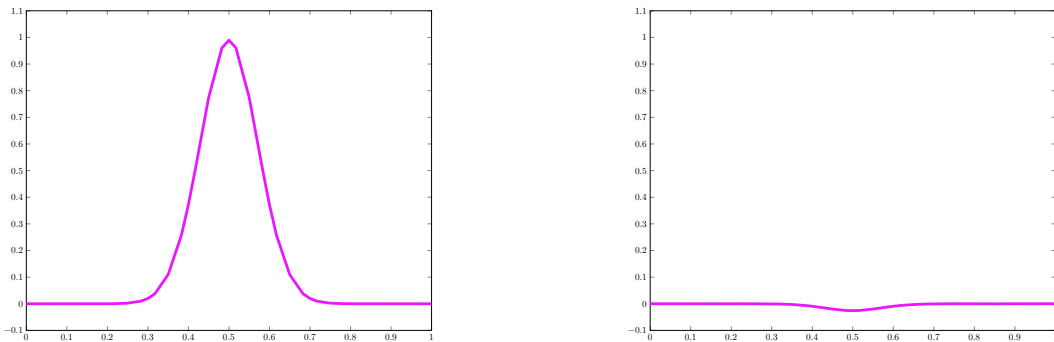


Figure 2.1: Approximation of u obtained with \mathbb{P}_4 finite elements with $h = 1/10$ at time $t = 0$ (Left) and at $t = 2$ (Right)

The boundary conditions of the wave propagation involve two effects. At the boundary $x = 0$, the wave bounces while on the boundary $x = 1$, part of the wave is absorbed, we should therefore have an exponential convergence to 0. We notice the oscillations of the wave in the course of time, these are due to the presence of high-frequency spurious waves (see Figure 2.2 with a change of scale).

In order to show that the discretization taints this property of uniform exponential stability, we represent the evolution of the energy of the system over time in log scale (Figure 2.3). We notice that the decay is no longer linear after a certain time.

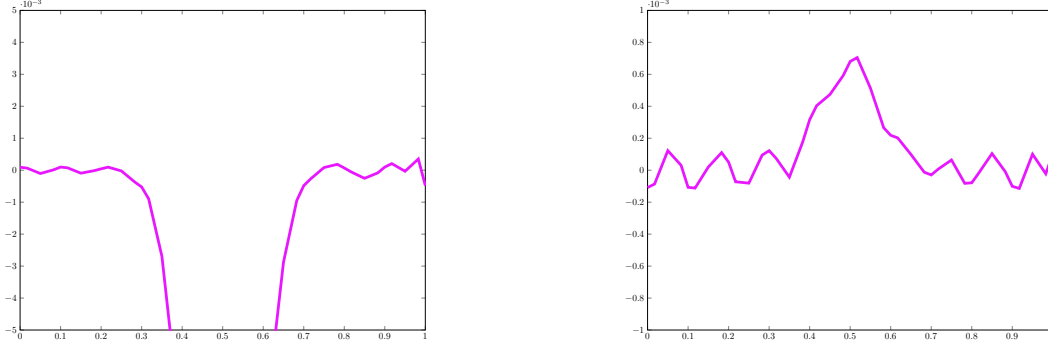


Figure 2.2: Approximation of u obtained with \mathbb{P}_4 finite elements with $h = 1/10$ and $dt = 1/1000$ at time $t = 2$ (Left) and at time $t = 4$ (Right)

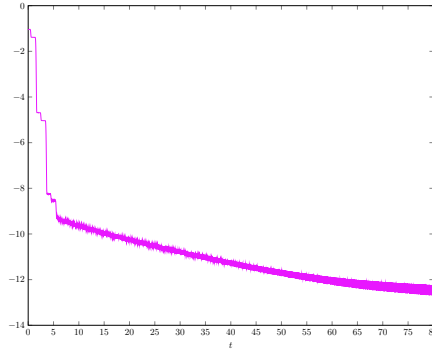


Figure 2.3: Approximation of $\ln(\|u(t)\|^2 + \|v(t)\|^2)^{1/2}$ obtained with \mathbb{P}_4 finite elements with $h = 1/10$ and $dt = 1/1000$

This gives an illustration of the loss of the uniform exponential stability property caused by the high-order spectral finite elements discretization.

2.2.2 Investigation of eigenvalues

As a key indicator of the behavior of the discrete system, we investigate the eigenvalues of the generator of the corresponding semigroup. We formulate (2.1) as follows,

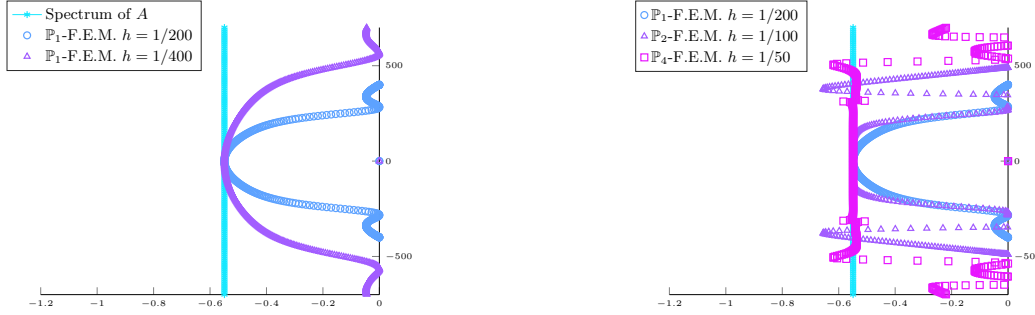
$$\dot{z} = Az, \text{ with } z = \begin{pmatrix} u \\ v \end{pmatrix}, \text{ and } A = \begin{pmatrix} 0 & -\partial_x \\ -\partial_x & 0 \end{pmatrix}, \quad (2.3)$$

where $A : L^2(\Omega) \times L^2(\Omega) \rightarrow L^2(\Omega) \times L^2(\Omega)$ is an unbounded operator with domain $D(A) \subset H^1(\Omega) \times H^1(\Omega)$. The discrete approximation of the above systems is

$$\dot{z}_h = A_h z_h,$$

where $A_h \in \mathcal{L}(V_h)$ is an approximation of the operator A in a finite-dimensional space V_h that is a subspace of $H^1(\Omega) \times H^1(\Omega)$ obtained using high-order spectral finite elements ([Cohen, 2002]). Since we look for exponential stability we aim at constructing approximations for which the eigenvalues of A_h lies in the complex plane not close to the imaginary axis.

We see in Figure 2.4 (Left) that the eigenvalues are close to the imaginary axis, which implies that the exponential stability property is in the best case scenario, not uniform in h , or even lost. By choosing h smaller, the consistency is improved, namely the smallest


 Figure 2.4: Spectrum of A_h

eigenvalues – in amplitude – are more and more well approximated, still the uniform exponential stability property is not recovered, see Figure 2.4 (Right). In fact, it has been shown – at least for \mathbb{P}_1 finite elements, (see [Infante and Zuazua, 1999; Ervedoza and Zuazua, 2012]) – that the uniform exponential stability property cannot be obtained at the discrete level without adding additional dissipative terms. As mentioned the main contribution of this work is to construct such terms that stabilized the system – in the sense where the uniform exponential stability property is guaranteed – and preserves high-order accuracy that is obtained with \mathbb{P}_r finite elements.

2.3 Continuous setting

This section presents standard multiplier techniques [Komornik, 1997] to show the exponential stability property at the continuous level of System (2.1). We have chosen to present these classical results, first for the sake of completeness, second because they will be reproduced at the discrete level. To complete this last step the sought additional dissipative terms will be needed.

2.3.1 Observability of the undamped system

First, we define the variational formulation of the undamped system ($\gamma = 0$),

$$\begin{cases} \int_0^1 \partial_t u \tilde{u} dx - \int_0^1 v \partial_x \tilde{u} dx + v(1) \tilde{u}(1) = 0, \\ \int_0^1 \partial_t v \tilde{v} dx + \int_0^1 \partial_x u \tilde{v} dx - u(1) \tilde{v}(1) = 0, \end{cases} \quad (2.4)$$

where $u(t), \tilde{u} \in H^1(0, 1)$ and $v(t), \tilde{v} \in L^2(0, 1)$. For smooth initial data, it is standard to show that (2.4) has a unique solution,

$$(u, v) \in C^1([0, T]; L^2(0, 1) \times L^2(0, 1)) \cap C^0([0, T]; H^1(0, 1) \times H^1(0, 1)).$$

System (2.4) is conservative, indeed, one can easily show, the following theorem.

Theorem 2.3.1 *Let (u, v) solutions of (2.4), we have the energy conservation*

$$\frac{d}{dt} \left(\|u(t)\|_{L^2(0,1)}^2 + \|v(t)\|_{L^2(0,1)}^2 \right) = 0. \quad (2.5)$$

The proof consists in choosing $\tilde{u} = u(t)$ and $\tilde{v} = v(t)$ in (2.4) and summing the two obtained equations. Note that, as an obvious consequence of the energy preservation

property we have

$$\int_0^T \|u(t)\|_{L^2(0,1)}^2 + \|v(t)\|_{L^2(0,1)}^2 dt = T \left(\|u(0)\|_{L^2(0,1)}^2 + \|v(0)\|_{L^2(0,1)}^2 \right). \quad (2.6)$$

A first step towards our goal is to derive an observability inequality (the precise meaning is given in the theorem below). This step is not a necessity to prove exponential stability but should rather be seen as a useful intermediate step. The strategy to prove the observability inequality relies on multipliers [Komornik, 1997].

Theorem 2.3.2 *Let (u, v) be solutions of (2.4). For T large enough, there exists $C^{st}(T)$ strictly positive such that*

$$\frac{1}{2} \left(\|u(0)\|_{L^2(0,1)}^2 + \|v(0)\|_{L^2(0,1)}^2 \right) \leq C^{st}(T) \int_0^T v^2(1) dt. \quad (2.7)$$

PROOF: The proof given below is separated in several steps that will all be mimicked at the discrete level.

Step 1: We choose $\tilde{u} = -xv(t)$ and $\tilde{v} = -xu(t)$ in (2.4) and sum the two obtained equations, integrating by parts and neglecting negative terms we get, we obtain

$$-\frac{d}{dt} \int_0^1 xuv dx \leq -\frac{1}{2} (\|u(t)\|_{L^2(0,1)}^2 + \|v(t)\|_{L^2(0,1)}^2) + \frac{1}{2} v^2(1).$$

Step 2: Integrating in time and using the energy conservation property of 2.3.1 and its direct consequence (2.6) we get

$$-\left[\int_0^1 xuv dx \right]_0^T \leq -\frac{T}{2} \left(\|u(0)\|_{L^2(0,1)}^2 + \|v(0)\|_{L^2(0,1)}^2 \right) + \frac{1}{2} \int_0^T v^2(1) dt. \quad (2.8)$$

Step 3: We use Cauchy-Schwarz, Young inequalities and Theorem 2.3.1 to show

$$\left| \int_0^1 xu(t)v(t) dx \right| \leq \frac{1}{2} \|v(t)\|_{L^2(0,1)}^2 + \frac{1}{2} \|v(t)\|_{L^2(0,1)}^2 = \frac{1}{2} \|v(0)\|_{L^2(0,1)}^2 + \frac{1}{2} \|v(0)\|_{L^2(0,1)}^2.$$

Finally, using this inequality together with (2.8) we obtain

$$(T - 2) \left(\|u(0)\|_{L^2(0,1)}^2 + \|v(0)\|_{L^2(0,1)}^2 \right) \leq \int_0^T v^2(1) dt.$$

■

Inequality (2.7) does not directly guarantee that the boundary damped system is exponentially stable. It is however a good indicator since the damping term corresponds to the right-hand side of (2.7). This will be used as a clue to define the required damping terms at the discrete level.

2.3.2 Exponential stability for the damped system using Lyapunov functional

We define the variational formulation associated with System (2.1) with (2.2)

$$\begin{cases} \int_0^1 \partial_t u \tilde{u} dx - \int_0^1 v \partial_x \tilde{u} dx + v(1) \tilde{u}(1) = 0, \\ \int_0^1 \partial_t v \tilde{v} dx + \int_0^1 \partial_x u \tilde{v} dx - \tilde{v}(1) u(1) + \gamma v(1) \tilde{v}(1) = 0, \end{cases} \quad (2.9)$$

We have existence and uniqueness of solutions for the above variational formulation, we will now establish an energy relation for this system.

Theorem 2.3.3 *Let (u, v) solutions of (2.9), we have the energy dissipation property*

$$\frac{1}{2} \frac{d}{dt} \left(\|u(t)\|_{L^2(0,1)}^2 + \|v(t)\|_{L^2(0,1)}^2 \right) = -\gamma v^2(1). \quad (2.10)$$

PROOF: We choose $\tilde{u} = u(t)$ and $\tilde{v} = v(t)$ in (2.9). Then, adding the two equations in (2.9), we conclude the proof. \blacksquare

Theorem 2.3.4 *Let (u, v) solutions of (2.9). There exist two constants $C_s > 0$ and $\sigma > 0$ such that*

$$\|u(t)\|_{L^2(0,1)}^2 + \|v(t)\|_{L^2(0,1)}^2 \leq C_s e^{-\sigma t} \left(\|u(0)\|_{L^2(0,1)}^2 + \|v(0)\|_{L^2(0,1)}^2 \right). \quad (2.11)$$

PROOF: The proof given below is separated in several steps that will all be mimicked at the discrete level.

Step 1: We choose $\tilde{u} = -xv(t)$ and $\tilde{v} = -xu(t)$ in (2.9).

Summing the two equations and integrating by parts, we get

$$-\frac{d}{dt} \int_0^1 xuv dx = \frac{1}{2} v^2(1) - \frac{1}{2} u^2(1) + \gamma v(1)u(1) - \frac{1}{2} \left(\|u(t)\|_{L^2(0,1)}^2 + \|v(t)\|_{L^2(0,1)}^2 \right).$$

Moreover, thanks to Young's inequality we have $\gamma u(1)v(1) \leq \frac{\gamma^2}{2} v^2(1) + \frac{1}{2} u^2(1)$, finally we obtain

$$-\frac{d}{dt} \int_0^1 xuv dx \leq \frac{1+\gamma^2}{2} v^2(1) - \frac{1}{2} \left(\|u(t)\|_{L^2(0,1)}^2 + \|v(t)\|_{L^2(0,1)}^2 \right).$$

Step 2: We add the energy relation (2.3.3) multiplied by $\alpha > 0$.

$$\begin{aligned} \frac{d}{dt} \left(\frac{\alpha}{2} \left(\|u(t)\|_{L^2(0,1)}^2 + \|v(t)\|_{L^2(0,1)}^2 \right) - \int_0^1 xuv dx \right) \\ \leq \left(\frac{\gamma^2 + 1}{2} - \alpha\gamma \right) v^2(1) - \frac{1}{2} \left(\|u(t)\|_{L^2(0,1)}^2 + \|v(t)\|_{L^2(0,1)}^2 \right). \end{aligned}$$

Step 3: Choosing α large enough such that

$$\frac{\gamma^2 + 1}{2} - \alpha\gamma < 0,$$

we get

$$\begin{aligned} \frac{d}{dt} \left(\frac{\alpha}{2} \left(\|u(t)\|_{L^2(0,1)}^2 + \|v(t)\|_{L^2(0,1)}^2 \right) - \int_0^1 xuv dx \right) \\ \leq -\frac{1}{2} \left(\|u(t)\|_{L^2(0,1)}^2 + \|v(t)\|_{L^2(0,1)}^2 \right). \end{aligned} \quad (2.12)$$

Step 4: We introduce the Lyapunov functional

$$\mathcal{L}(t) = \frac{\alpha}{2} \left(\|u(t)\|_{L^2(0,1)}^2 + \|v(t)\|_{L^2(0,1)}^2 \right) - \int_0^1 xuv dx. \quad (2.13)$$

Using Cauchy-Schwarz and Young inequalities, we prove the equivalence between the energy functional

$$\mathcal{E}(t) := \frac{1}{2} \|u(t)\|_{L^2(0,1)}^2 + \|v(t)\|_{L^2(0,1)}^2$$

and the Lyapunov functional with $\alpha - 1 > 0$.

$$\begin{aligned} \frac{(\alpha - 1)}{2} \left(\|u(t)\|_{L^2(0,1)}^2 + \|v(t)\|_{L^2(0,1)}^2 \right) &\leq \mathcal{L}(t) \\ &\leq \frac{\alpha + 1}{2} \left(\|u(t)\|_{L^2(0,1)}^2 + \|v(t)\|_{L^2(0,1)}^2 \right). \end{aligned} \quad (2.14)$$

Using this equivalence in (2.12), we get

$$\frac{d}{dt} \mathcal{L}(t) \leq -\frac{1}{\alpha + 1} \mathcal{L}(t).$$

Step 5: Integrating in time and using again the equivalence between the energy and the Lyapunov functional we finally have

$$\left(\|u(t)\|_{L^2(0,1)}^2 + \|v(t)\|_{L^2(0,1)}^2 \right) \leq \frac{\alpha + 1}{\alpha - 1} e^{-\frac{1}{\alpha + 1} t} \left(\|u(0)\|_{L^2(0,1)}^2 + \|v(0)\|_{L^2(0,1)}^2 \right).$$

■

2.4 Semi-discrete setting

As mentioned we give a counterpart of Theorem 2.3.2 and Theorem 2.3.4 at the semi-discrete level. This is achieved in an abstract framework. We then show – and this is the most important part of the work – that the semi-discrete wave problem falls in the described framework.

2.4.1 Abstract framework

Let \mathcal{Z}_h be a family of finite-dimensional parametrized by h space associated with the scalar product $(\cdot, \cdot)_h$. We consider the semi-discretized abstract first-order system for $z_h \in \mathcal{Z}_h$,

$$\begin{cases} \dot{z}_h = A_h z_h, & \text{in } [0, T], \\ z_h(0) = z_{h,0}, \end{cases} \quad (2.15)$$

where $A_h \in \mathcal{L}(\mathcal{Z}_h)$ and $\|A_h\|_h \xrightarrow{h \rightarrow 0} +\infty$. We assume that the operator A_h satisfies the following property

$$(A_h z_h, z_h)_h = 0. \quad (2.16)$$

Furthermore, we have classically the existence and unicity of smooth solutions (u_h, v_h) belonging to $C^k([0, T]; \mathcal{Z}_h)$, for all $k \geq 0$. We start by presenting the discrete property of energy preservation.

Theorem 2.4.1 *Let $z_h \in \mathcal{Z}_h$ the solution of (2.15), we have the discrete conservation energy property*

$$\frac{1}{2} \frac{d}{dt} \|z\|_h^2 = 0.$$

PROOF: We multiply (2.15) by z_h , and since A_h satisfies (2.16), the result follows immediately. \blacksquare

Let \mathcal{Y}_h be a finite-dimensional space. We have seen that in the continuous problem, we have added a dissipative term that stabilizes the problem. By analogy, we therefore introduce a perturbation by an operator $C_h^* C_h$, where $C_h \in \mathcal{L}(\mathcal{Z}_h, \mathcal{Y}_h)$,

$$\begin{cases} \dot{z}_h = (A_h - \gamma C_h^* C_h) z_h, & \text{in } [0, T], \\ z_h(0) = z_{h,0}. \end{cases} \quad (2.17)$$

For this system, we have the discrete dissipation energy property.

Theorem 2.4.2 *Let $z_h \in \mathcal{Z}_h$ the solution of (2.17), we have the discrete dissipation energy property*

$$\frac{d}{dt} \frac{1}{2} \|z_h\|_h^2 = -\gamma \|C_h z_h\|_h^2.$$

PROOF: Proof similar of the proof of Theorem 2.4.1. \blacksquare

In order to demonstrate the observability property, we need a fundamental hypothesis.

Assumption 1 $\exists \alpha, \beta, C_\Pi > 0$, independent of h and for all h , $\exists \Pi_h$ symmetric belonging to $\mathcal{L}(\mathcal{Z}_h)$ and $\|\Pi_h\|_h \leq C_\Pi$ with $C_\Pi > 0$, such that, $\forall z_h \in \mathcal{Z}_h$

$$(\Pi_h A_h z_h, z_h)_h \leq \alpha \|C_h z_h\|_h^2 - \beta \|z_h\|_h^2.$$

From this assumption, we can establish the property that interest us: the observability property.

Theorem 2.4.3 *Let z_h be the solution of (2.15). Assuming Assumption 1, for T large enough, there exists a constant $C^{st}(T) > 0$ independent of h such that*

$$\|z_{h,0}\|_h^2 \leq C^{st}(T) \int_0^T \|C_h z_h\|_h^2 dt.$$

PROOF: The proof is separated into the same steps as for the proof of observability at the continuous level of Theorem 2.3.4.

Step 1: We multiply the first equation of (2.15) by $\Pi_h z_h$.

$$(\Pi_h \dot{z}_h, z_h)_h = (\Pi_h A_h z_h, z_h)_h.$$

Using Assumption 1, we obtain

$$\frac{1}{2} \frac{d}{dt} (\Pi_h z_h, z_h)_h \leq \alpha \|C_h z_h\|_h^2 - \beta \|z_h\|_h^2.$$

Step 2: We integrate in time and use the energy conservation property

$$\frac{1}{2} (\Pi_h z_h(T), z_h(T))_h - \frac{1}{2} (\Pi_h z_{h,0}, z_{h,0})_h \leq \alpha \int_0^T \|C_h z_h\|_h^2 dt - \beta \int_0^T \|z_h(t)\|_h^2 dt.$$

The use of the energy conservation property of Theorem 2.4.1 leads to

$$\frac{1}{2} (\Pi_h z_h(T), z_h(T))_h - \frac{1}{2} (\Pi_h z_{h,0}, z_{h,0})_h \leq \alpha \int_0^T \|C_h z_h\|_h^2 dt - \beta T \|z_h(0)\|_h^2 dt.$$

Step 3: Since Π_h is bounded, we have

$$-\frac{1}{2} \|\Pi_h\|_h \|z_h(T)\|_h^2 - \frac{1}{2} \|\Pi_h\|_h \|z_{h,0}\|_h^2 \leq \alpha \int_0^T \|C_h z_h\|_h^2 dt - \beta T \|z_h(0)\|_h^2 dt.$$

Using the energy conservation Theorem 2.4.1, we get

$$(\beta T - \|\Pi_h\|_h) \|z_{h,0}\|_h^2 \leq \alpha \int_0^T \|C_h z_h\|_h^2 dt.$$

For $T > \|\Pi_h\|_h \beta^{-1}$, we obtain the result. ■

As for showing the observability property, we need a fundamental hypothesis to establish the exponential stability property.

Assumption 2 $\exists \chi, \kappa > 0$, independent of h and for all h , $\exists \Pi_h \in \mathcal{L}(\mathcal{Z}_h)$ symmetric such that, $\forall z_h \in \mathcal{Z}_h$

$$(\Pi_h A_h z_h, z_h)_h - \gamma (C_h z_h, C_h \Pi_h z_h)_h \leq \chi \|C_h z_h\|_h^2 - \kappa \|z_h\|_h^2.$$

We can now proceed to the main theorem of this work : the uniform exponential stability property for the damped system.

Theorem 2.4.4 *Let z_h be the solution of (2.17). Under Assumption 2, there exist two constants $C_s, \sigma > 0$ independent of h such that*

$$\frac{1}{2} \|z_h(t)\|_h^2 \leq C_s \|z_{h,0}\|_h^2 e^{-\sigma t}.$$

PROOF: The proof is separated into the same steps as for the proof of exponential stability at the continuous level of Theorem 2.3.4.

Step 1: We multiply the first equation of (2.17) and $\Pi_h z_h$ (in the sens of the scalar product).

$$(\Pi_h \dot{z}_h, z_h)_h = (\Pi_h A_h z_h, z_h)_h - \gamma (C_h z_h, C_h \Pi_h z_h)_h.$$

We use Assumption 2 and we obtain

$$\frac{1}{2} \frac{d}{dt} (\Pi_h z_h, z_h)_h \leq \chi \|C_h z_h\|_h^2 - \kappa \|z_h\|_h^2.$$

Step 2: Adding the dissipation energy identity of Theorem 2.4.2 times $\alpha > 0$, we get

$$\frac{1}{2} \frac{d}{dt} \left((\Pi_h z_h, z_h)_h + \alpha \|z_h\|_h^2 \right) \leq (\chi - \alpha\gamma) \|C_h z_h\|_h^2 - \kappa \|z_h\|_h^2.$$

Step 3: We choose α large enough such that $\chi - \alpha\gamma \leq 0$. Thus, we have

$$\frac{1}{2} \frac{d}{dt} \left((\Pi_h z_h, z_h)_h + \alpha \|z_h\|_h^2 \right) \leq -\kappa \|z_h\|_h^2.$$

Step 4: We introduce the following discrete Lyapunov functional

$$\mathcal{L}_h(t) = (\Pi_h z_h, z_h)_h + \alpha \|z_h\|_h^2.$$

Since Π_h is bounded, we can establish an equivalence between the Lyapunov functional and the energy functional $\mathcal{E}_h(t) := \frac{1}{2} \|z_h(t)\|_h^2$.

$$(-\|\Pi_h\|_h + \alpha) \|z\|_h^2 \leq \mathcal{L}_h(t) \leq (\|\Pi_h\|_h + \alpha) \|z_h\|_h^2. \quad (2.18)$$

Using this equivalence with $\alpha > \|\Pi_h\|_h$, we obtain

$$\frac{1}{2} \frac{d}{dt} \mathcal{L}_h(t) \leq -\kappa (\|\Pi_h\|_h + \alpha)^{-1} \mathcal{L}_h(t).$$

Step 5: Integrating in time and using again (2.18), we finally get

$$\frac{1}{2} \|z_h(t)\|_h^2 \leq \frac{\alpha + \|\Pi_h\|_h}{\alpha - \|\Pi_h\|_h} \|z_{h,0}\|_h^2 e^{-\kappa(\|\Pi_h\|_h + \alpha)^{-1}t}.$$

■

2.4.2 Discretization with high-order spectral finite elements

In this section, we show how the semi-discretization in space using high-order spectral finite elements of the wave propagation problem (2.1) enters the abstract framework presented above. We introduce two discrete spaces U_h^r and V_h^r – where r is the order of the finite elements used – defined by

$$U_h^r = \{u_h \in C^0[0, 1] \mid u_h|_{[x_i, x_{i+1}]} \in \mathbb{P}_r\},$$

and

$$V_h^r = \{v_h \in L^2(0, 1) \mid v_h|_{[x_i, x_{i+1}]} \in \mathbb{P}_r\}.$$

Note that $U_h^r \subset V_h^r$ and we denote $\mathcal{Z}_h = U_h^r \times V_h^r$. We consider a quasi-uniform subdivision $[x_i, x_{i+1}]$, for $i \in \{0, N\}$ of the domain $[0, 1]$, namely we denote $h_i = |x_{i+1} - x_i|$ for $i \in \{0, N-1\}$, and

$$h = \sup_{i \in \{0, N-1\}} h_i,$$

and there exists $\rho > 0$ such that

$$\min_{i \in \{0, N-1\}} h_i \leq \rho h.$$

The efficiency of high-order spectral finite elements stems from the fact that mass-lumping can be achieved without loss of consistency using a quadrature rule based on Gauss-Lobatto points. We represent the use of this quadrature by the symbol

$$\oint_0^1 f(x)dx \sim \int_0^1 f(x)dx \quad \text{with} \quad \oint_0^1 f(x)dx := \sum_{i=0}^N \sum_{k=0}^r h_i w_k f(x_i + h_i \xi_k),$$

where and w_k are the quadrature coefficients and $\xi_k \in [0, 1]$ the Gauss-Lobatto interpolation nodes.

Note that we have the following bounds (see [Joly, 2007]),

$$\exists C_q, c_q > 0, \quad \text{such that} \quad \forall v_h \in V_h^r, \quad c_q \int_0^1 v_h^2 dx \leq \oint_0^1 v_h^2 dx \leq C_q \int_0^1 v_h^2 dx. \quad (2.19)$$

The semi-discrete formulation of the damped system (2.9) reads: Find $(u_h, v_h)(t) \in \mathcal{Z}_h$ such that for $(\tilde{u}_h, \tilde{v}_h) \in \mathcal{Z}_h$,

$$\begin{cases} \oint_0^1 \partial_t u_h \tilde{u}_h dx - \int_0^1 v_h \partial_x \tilde{u}_h dx + v_h(1) \tilde{u}_h(1) = 0, \\ \oint_0^1 \partial_t v_h \tilde{v}_h dx + \int_0^1 \partial_x u_h \tilde{v}_h dx - u_h(1) \tilde{v}_h(1) + \gamma v_h(1) \tilde{v}_h(1) = 0, \end{cases} \quad (2.20)$$

with initial data

$$u_h(0) = u_h^0, \quad v_h(0) = v_h^0.$$

Note that when $\gamma = 0$ one recovers the semi-discrete variational formulation of the undamped system (2.4). In order to relate to the abstract theory presented in Section 2.4.1, we reformulate the variational formulation (2.20) as an operator equation. To do so, we introduce the scalar product in V_h^r by

$$\forall (v_h, \tilde{v}_h) \in V_h^r \times V_h^r, \quad (v_h, \tilde{v}_h)_h := \oint_0^1 v_h \tilde{v}_h dx, \quad (2.21)$$

The scalar product in $U_h^r \subset V_h^r$ is defined similarly, finally we extend this notation for $z_h = (u_h, v_h)^t$, $\tilde{z}_h = (\tilde{u}_h, \tilde{v}_h)^t \in U_h^r \times V_h^r$,

$$(z_h, \tilde{z}_h)_h = \oint_0^1 u_h \tilde{u}_h dx + \oint_0^1 v_h \tilde{v}_h dx.$$

The variational formulation (2.20) is equivalent to

$$\begin{cases} \partial_t u_h - R_h^* v_h = 0, \\ \partial_t v_h + R_h u_h + \gamma B_h v_h = 0, \end{cases} \quad (u_h, v_h)(0) = (u_h^0, v_h^0) := z_{h,0}, \quad (2.22)$$

where, for all $(u_h, v_h, \tilde{v}_h)^t \in U_h^r \times V_h^r \times V_h^r$,

$$(R_h u_h, \tilde{v}_h)_h = \int_0^1 \partial_x u_h \tilde{v}_h dx - u_h(1) \tilde{v}_h(1) \quad \text{and} \quad (B_h v_h, \tilde{v}_h)_h = v_h(1) \tilde{v}_h(1). \quad (2.23)$$

Denoting $z_h = (u_h, v_h)^t$, we can rewrite System (2.22) as

$$\begin{cases} \dot{z}_h = A_h z_h - \gamma \begin{pmatrix} 0 & 0 \\ 0 & B_h \end{pmatrix} z_h, \\ z_h(0) = z_{h,0}, \end{cases} \quad \text{where} \quad A_h = \begin{pmatrix} 0 & -R_h^* \\ R_h & 0 \end{pmatrix}. \quad (2.24)$$

Note that the operator $A_h \in \mathcal{L}(\mathcal{Z}_h)$ satisfies, for all $z_h \in \mathcal{Z}_h$, $(A_h z_h, z_h)_h = 0$, i.e., it is skew-adjoint.

2.4.3 Discrete observability

Our objective is – in a preliminary step – to prove an observability property for the system (2.24). To do so, Theorem 2.4.3, is used and therefore Assumption 1 is satisfied in what follows by introducing an appropriate operator $\Pi_h \in \mathcal{L}(\mathcal{Z}_h)$.

The strategy to define Π_h is based upon our analysis presented at the continuous level. More precisely one want to have (see the proof of Theorem 2.3.2 in Section 2.3)

$$\Pi_h z_h \sim \begin{pmatrix} -xv_h \\ -xu_h \end{pmatrix}.$$

However, this later equation can not be an equality since $xv_h \notin U_h^r$ and $xu_h \notin V_h^r$. A simple remedy is to introduce the following interpolation operator: $\chi_h(\cdot) : U_h^r \rightarrow V_h^r$ by

$$\oint_0^1 \chi_h(u_h)v_h dx = \oint_0^1 xu_h v_h dx, \quad \forall u_h \in U_h^r, v_h \in V_h^r, \quad (2.25)$$

and its adjoint $\chi_h^*(\cdot) : V_h^r \rightarrow U_h^r$ is given by

$$\oint_0^1 \chi_h^*(v_h)u_h dx = \oint_0^1 xv_h u_h dx, \quad \forall u_h \in U_h^r, v_h \in V_h^r. \quad (2.26)$$

We define, for $z_h = (u_h, v_h)^t$ and $\tilde{z}_h = (\tilde{u}_h, \tilde{v}_h)^t$ the operator Π_h by

$$\Pi_h z_h = \begin{pmatrix} -\chi_h^*(v_h) \\ -\chi_h(u_h) \end{pmatrix}. \quad (2.27)$$

Note that it is easy to show that Π_h is self-adjoint and bounded using (2.19) uniformly with respect to h . More precisely, there exists C_Π independent of h such that $\forall z_h \in \mathcal{Z}_h$

$$\|\Pi_h z_h\|_h \leq C_\Pi \|z_h\|_h. \quad (2.28)$$

We are interested in the discrepancy between the multiplier that we would like to use (but cannot) and the one we are going to use, namely in the following discrepancy

$$\Pi_h z_h - \begin{pmatrix} -xu_h \\ -xv_h \end{pmatrix}.$$

It turns out that the operator C_h – which appears in the fundamental assumption 1 –, involves the local semi-norm H_1 of this discrepancy. That is why, for all $u_h, \tilde{u}_h \in U_h^r$ and $\forall v_h, \tilde{v}_h \in V_h^r$, we define two bilinear forms d_h and $d_{h,*}$ as

$$d_h(u_h, \tilde{u}_h) := \sum_{i=0}^N \int_{x_i}^{x_{i+1}} \partial_x(xu_h - \chi_h(u_h)) \partial_x(x\tilde{u}_h - \chi_h(\tilde{u}_h)) dx, \quad (2.29)$$

and

$$d_{h,*}(v_h, \tilde{v}_h) := \sum_{i=0}^N \int_{x_i}^{x_{i+1}} \partial_x(xv_h - \chi_h^*(v_h)) \partial_x(x\tilde{v}_h - \chi_h^*(\tilde{v}_h)) dx. \quad (2.30)$$

Proposition 2.4.5 *Let d_h and $d_{h,*}$ be the positive and symmetric bilinear forms defined by (2.29) and (2.30). Then, there exists a constant $C_d > 0$ depend only on r such that $\forall (u_h, v_h) \in U_h^r \times V_h^r$*

$$d_h(u_h, u_h) + d_{h,*}(v_h, v_h) \leq C_d \int_0^1 u_h^2 + v_h^2 dx.$$

PROOF: Let us show first, that d_h is bounded. We use an interpolation property [Brenner and Scott, 2008, Chapter 4, Section 4, Theorem 4.4.4]. Since xu_h belongs to $P_{r+1}(K)$ with $K = [x_i, x_{i+1}]$ for some $i \in \{0, N-1\}$, we have

$$|xu_h - \chi_h(u_h)|_{H^1(K_i)} \leq h_i^r C_r |xu_h|_{H^{r+1}(K_i)},$$

where $C_r > 0$ depends only on r . However, as $x \in P_1(K)$ and $u_h \in P_r(K)$, the general Leibniz rule leads to

$$(xu_h)^{r+1} = \sum_{j=0}^{r+1} \binom{r+1}{j} x^{(j)} u_h^{(r+1-j)} = (r+1)(u_h)^{(r)}.$$

Injecting this result in the equation above, we get

$$|xu_h - \chi_h(u_h)|_{H^1(K_i)} \leq h_i^r C_r (r+1) |u_h|_{H^r(K_i)}.$$

Thus, using an inverse estimate [Brenner and Scott, 2008, Chapter 4, Section 5, Lemma 4.5.3], we have with $C_r > 0$, another scalar depending only on r ,

$$|xu_h - \chi_h(u_h)|_{H^1(K_i)} \leq C_r \|u_h\|_{L^2(K_i)}.$$

Hence, finally we have

$$d_h(u_h, u_h) \leq C_r^2 \|u_h\|_{L^2(0,1)}^2.$$

Since we can reproduce this inequality for $d_{h,*}$, the proof is ended. ■

We define the following operators $D_h \in \mathcal{L}(U_h^r, U_h^r)$ and $D_{h,*} \in \mathcal{L}(V_h^r, V_h^r)$ associated with the bilinear forms d_h and $d_{h,*}$,

$$\begin{cases} (D_h u_h, \tilde{u}_h)_h = d_h(u_h, \tilde{u}_h), & \forall u_h, \tilde{u}_h \in U_h^r, \\ (D_{h,*} v_h, \tilde{v}_h)_h = d_{h,*}(v_h, \tilde{v}_h), & \forall v_h, \tilde{v}_h \in V_h^r. \end{cases} \quad (2.31)$$

Unfortunately, the definition of the operator C_h – appearing in Assumption 1 – requires, in this case, to define the jump of the variable v_h . For this purpose, we introduce the notation, for all $i \in \{1, N-1\}$,

$$[[v_i]] := v_i^+ - v_i^-, \quad \text{where } v_i^+ = v_h(x_i^+) \text{ and } v_i^- = v_h(x_i^-). \quad (2.32)$$

Then, we define the jump operator

$$(J_h v_h, \tilde{v}_h)_h = \sum_{i=1}^{N-1} \frac{1}{\tilde{h}_i} [[v_i]] [[\tilde{v}_i]], \quad \text{where } \tilde{h}_i = \min(h_{i-1}, h_i). \quad (2.33)$$

In the theorem below, we check that the multiplier Π_h satisfies the Assumption 1. We can then apply Theorem 2.4.3 and prove the observability inequality. Note that since the operators J_h, B_h, D_h and $D_{h,*}$ are self-adjoint and positive, it is possible to consider their roots. We define the operator $C_h \in \mathcal{L}(\mathcal{Z}_h, U_h^r \times [V_h^r]^3)$ – equipped with the scalar product $(\cdot, \cdot)_h$ inherited from V_h^r – by

$$C_h = \begin{pmatrix} D_h^{1/2} & 0 \\ 0 & J_h^{1/2} \\ 0 & B_h^{1/2} \\ 0 & D_{h,*}^{1/2} \end{pmatrix}. \quad (2.34)$$

Theorem 2.4.6 *Let $z_h = (u_h, v_h) \in \mathcal{Z}_h = U_h^r \times V_h^r$ be the solutions of (2.24) with $\gamma = 0$. Then, Π_h defined by (2.27) satisfies Assumption 1 namely*

$$(\Pi_h A_h z_h, z_h)_h \leq \alpha \|C_h z_h\|_h^2 - \beta \|z_h\|_h^2, \quad \alpha, \beta > 0 \text{ independent of } h,$$

with the operator C_h is given by (2.34).

PROOF: By definition, we have

$$\begin{aligned} (\Pi_h A_h z_h, z_h)_h &= (A_h z_h, \Pi_h z_h)_h \\ &= - \int_0^1 v_h \partial_x \chi_h^*(v_h) dx + v_h^2(1) + \int_0^1 \partial_x u_h \chi_h(u_h) dx - u_h^2(1). \end{aligned}$$

Inserting $xv_h - xv_h = 0$ in the first term of the right-hand side, we get

$$\begin{aligned} (A_h z_h, \Pi_h z_h)_h &= \sum_{i=0}^N \int_{x_i}^{x_{i+1}} v_h \partial_x (xv_h - \chi_h^*(v_h)) dx - \sum_{i=0}^N \int_{x_i}^{x_{i+1}} v_h \partial_x (xv_h) dx \\ &\quad + v_h^2(1) + \int_0^1 \partial_x u_h \chi_h(u_h) dx - u_h^2(1). \end{aligned} \quad (2.35)$$

Let us focus on the second term in the right-hand side.

$$- \sum_{i=0}^N \int_{x_i}^{x_{i+1}} v_h \partial_x (xv_h) dx = - \sum_{i=0}^N \int_{x_i}^{x_{i+1}} v_h^2 dx - \frac{1}{2} \sum_{i=0}^N \int_{x_i}^{x_{i+1}} x \partial_x (v_h^2) dx.$$

Integrating by parts, we get

$$\begin{aligned} - \sum_{i=0}^N \int_{x_i}^{x_{i+1}} v_h \partial_x (xv_h) dx &= - \sum_{i=0}^N \int_{x_i}^{x_{i+1}} v_h^2 dx + \frac{1}{2} \sum_{i=0}^N \int_{x_i}^{x_{i+1}} v_h^2 dx \\ &\quad - \frac{1}{2} v_h^2(1) - \frac{1}{2} \sum_{i=1}^{N-1} x_i (v_i^+)^2 - x_i (v_i^-)^2. \end{aligned}$$

Using the algebraic identity $a^2 - b^2 = (a - b)(a + b)$, we obtain

$$\begin{aligned} - \sum_{i=0}^N \int_{x_i}^{x_{i+1}} v_h \partial_x (xv_h) dx &= - \frac{1}{2} \sum_{i=0}^N \int_{x_i}^{x_{i+1}} v_h^2 dx - \frac{1}{2} v_h^2(1) \\ &\quad - \frac{1}{2} \sum_{i=1}^{N-1} x_i (v_i^+ - v_i^-)(v_i^+ + v_i^-). \end{aligned}$$

We define the average of v_h for $i \in \{1, N-1\}$ by

$$\{\!\!\{ v_i \}\!\!\} := \frac{v_i^+ + v_i^-}{2}, \quad \text{where } v_i^+ = v_h(x_i^+) \text{ and } v_i^- = v_h(x_i^-).$$

Using the jump definition (2.32) and for the last term, we obtain

$$- \sum_{i=0}^N \int_{x_i}^{x_{i+1}} v_h \partial_x (xv_h) dx = - \frac{1}{2} \sum_{i=0}^N \int_{x_i}^{x_{i+1}} v_h^2 dx - \frac{1}{2} v_h^2(1) - \sum_{i=1}^{N-1} x_i \{\!\!\{ v_i \}\!\!\} [v_i].$$

Injecting this result into (2.35), we get

$$\begin{aligned} (A_h z_h, \Pi_h z_h)_h &= \sum_{i=0}^N \int_{x_i}^{x_{i+1}} v_h \partial_x (x v_h - \chi_h^*(v_h)) dx - \frac{1}{2} \int_0^1 v_h^2 dx \\ &\quad + \frac{1}{2} v_h^2(1) - \sum_{i=1}^{N-1} x_i \{v_i\} [v_i] + \int_0^1 \partial_x u_h \chi_h(u_h) dx - u_h^2(1). \end{aligned}$$

Injecting $xu_h - xu_h = 0$ in the penultimate term in the right-hand side, then we get

$$\begin{aligned} (A_h z_h, \Pi_h z_h)_h &= \sum_{i=0}^N \int_{x_i}^{x_{i+1}} v_h \partial_x (x v_h - \chi_h^*(v_h)) dx - \frac{1}{2} \int_0^1 v_h^2 dx + \frac{1}{2} v_h^2(1) \\ &\quad - \sum_{i=1}^{N-1} x_i \{v_i\} [v_i] - \sum_{i=0}^N \int_{x_i}^{x_{i+1}} (x u_h - \chi_h(u_h)) \partial_x u_h dx \\ &\quad + \frac{1}{2} \int_0^1 x \partial_x (u_h^2) dx - u_h^2(1). \end{aligned}$$

We integrate by parts, we have

$$\begin{aligned} (A_h z_h, \Pi_h z_h)_h &= \sum_{i=0}^N \int_{x_i}^{x_{i+1}} v_h \partial_x (x v_h - \chi_h^*(v_h)) dx - \frac{1}{2} \int_0^1 v_h^2 dx + \frac{1}{2} v_h^2(1) \\ &\quad - \sum_{i=1}^{N-1} x_i \{v_i\} [v_i] - \sum_{i=0}^N \int_{x_i}^{x_{i+1}} (x u_h - \chi_h(u_h)) \partial_x u_h dx \\ &\quad - \frac{1}{2} \int_0^1 u_h^2 dx - \frac{1}{2} u_h^2(1). \end{aligned}$$

We integrate by part and use the property that the terms involving the difference between xu_h and its interpolation are zero at the boundary. This leads to the following important inequality

$$\begin{aligned} (A_h z_h, \Pi_h z_h)_h &= \sum_{i=0}^N \int_{x_i}^{x_{i+1}} v_h \partial_x (x v_h - \chi_h^*(v_h)) dx - \frac{1}{2} \int_0^1 v_h^2 dx + \frac{1}{2} v_h^2(1) \\ &\quad - \sum_{i=1}^{N-1} x_i \{v_i\} [v_i] + \sum_{i=0}^N \int_{x_i}^{x_{i+1}} u_h \partial_x (x u_h - \chi_h(u_h)) dx \\ &\quad - \frac{1}{2} \int_0^1 u_h^2 dx - \frac{1}{2} u_h^2(1). \quad (2.36) \end{aligned}$$

Using Cauchy-Schwarz and Young inequalities, we obtain with $\lambda > 0$

$$\begin{aligned} (A_h z_h, \Pi_h z_h)_h &\leq -\frac{1}{2} (1 - \lambda) \int_0^1 u_h^2 + v_h^2 dx + \frac{1}{2\lambda} \sum_{i=0}^N \int_{x_i}^{x_{i+1}} [\partial_x (x v_h - \chi_h^*(v_h))]^2 dx \\ &\quad + \frac{1}{2\lambda} \sum_{i=0}^N \int_{x_i}^{x_{i+1}} [\partial_x (x u_h - \chi_h(u_h))]^2 dx \\ &\quad + \frac{1}{2} v_h^2(1) - \sum_{i=1}^{N-1} x_i \{v_i\} [v_i]. \quad (2.37) \end{aligned}$$

Let us focus on the last term. Using Cauchy-Schwarz and Young inequalities and bounding x_i by 1, we get with $\mu > 0$

$$-\sum_{i=1}^{N-1} x_i \{v_i\} [v_i] \leq \frac{1}{2\mu} \sum_{i=1}^{N-1} \frac{1}{\tilde{h}_i} [v_i]^2 + \frac{\mu}{2} \sum_{i=1}^{N-1} \tilde{h}_i \{v_i\}^2.$$

Using the definition of the average and using $(a+b)^2 \leq 2(a^2+b^2)$, we obtain

$$-\sum_{i=1}^{N-1} x_i \{v_i\} [v_i] \leq \frac{1}{2\mu} \sum_{i=1}^{N-1} \frac{1}{\tilde{h}_i} [v_i]^2 + \frac{\mu}{4} \sum_{i=1}^{N-1} \tilde{h}_i ((v_i^+)^2 + (v_i^-)^2). \quad (2.38)$$

On the segment $[0, h]$, we classically have the trace inequality

$$|v_h^2(0)|^2 \leq \|v_h\|_{L^2(0,h)} \left(\frac{1}{h} \|v_h\|_{L^2(0,h)} + 2 \|\partial_x v_h\|_{L^2(0,h)} \right).$$

Using an inverse estimate [Brenner and Scott, 2008, Chapter 4, Section 5, Lemma 4.5.3], we get, with $C_r > 0$ depending only on r

$$|v_h^2(0)|^2 \leq \left(\frac{1+2C_r}{h} \right) \|v_h\|_{L^2(0,h)}^2.$$

This inequality is generalized on each mesh, so we have

$$\begin{cases} |v_i^+|^2 \leq \left(\frac{1+2C_r}{h_i} \right) \|v_h\|_{L^2([x_i, x_{i+1}])}^2, & i \in [1, N-1], \\ |v_i^-|^2 \leq \left(\frac{1+2C_r}{h_{i-1}} \right) \|v_h\|_{L^2([x_{i-1}, x_i])}^2, & i \in [1, N-1]. \end{cases}$$

Summing the two equations and using the definition of \tilde{h}_i , we get

$$\begin{aligned} \tilde{h}_i ((v_i^+)^2 + (v_i^-)^2) &\leq \tilde{h}_i \frac{1+2C_r}{h_i} \|v_h\|_{L^2([x_i, x_{i+1}])}^2 + \tilde{h}_i \frac{1+2C_r}{h_{i-1}} \|v_h\|_{L^2([x_{i-1}, x_i])}^2, \\ &\leq (1+2C_r) \left(\|v_h\|_{L^2([x_i, x_{i+1}])}^2 + \|v_h\|_{L^2([x_{i-1}, x_i])}^2 \right). \end{aligned}$$

We use the above inequalities in (2.38) to get

$$-\sum_{i=1}^{N-1} x_i \{v_i\} [v_i] \leq \frac{1}{2\mu} \sum_{i=1}^{N-1} \frac{1}{\tilde{h}_i} [v_i]^2 + \frac{\mu(1+2C_r)}{2} \|v_h\|_{L^2(0,1)}^2.$$

Injecting this result in (2.37) we obtain

$$\begin{aligned} (A_h z_h, \Pi_h z_h)_h &\leq -\frac{1}{2} (1-\lambda) \int_0^1 u_h^2 dx - \frac{1}{2} (1-\lambda - \mu(1+2C_r)) \int_0^1 v_h^2 dx \\ &\quad + \frac{1}{2\lambda} \sum_{i=0}^N \int_{x_i}^{x_{i+1}} [\partial_x (xv_h - \chi_h^*(v_h))]^2 dx \\ &\quad + \frac{1}{2\lambda} \sum_{i=0}^N \int_{x_i}^{x_{i+1}} [\partial_x (xu_h - \chi_h(u_h))]^2 dx \\ &\quad + \frac{1}{2} v_h^2(1) + \frac{1}{2\mu} \sum_{i=1}^{N-1} \frac{1}{\tilde{h}_i} [v_i]^2. \end{aligned}$$

We recognize the expression of the bilinear forms d_h and $d_{h,*}$ defined by (2.29) and (2.30). Using (2.19) for the first and the second terms in the right-hand side, we get

$$\begin{aligned} (A_h z_h, \Pi_h z_h)_h &\leq -\frac{1}{2}(1-\lambda) \frac{1}{C_q} \oint_0^1 u_h^2 dx - \frac{1}{2} \frac{1}{C_q} (1-\lambda-\mu(1+2C_r)) \oint_0^1 v_h^2 dx \\ &\quad + \frac{1}{2\lambda} d_{h,*}(v_h, v_h) + \frac{1}{2\lambda} d_h(u_h, u_h) + \frac{1}{2} v_h^2(1) + \frac{1}{2\mu} \sum_{i=1}^{N-1} \frac{1}{\tilde{h}_i} \llbracket v_i \rrbracket^2. \end{aligned}$$

We choose λ and μ small enough such that

$$1-\lambda > 0 \text{ and } 1-\lambda-\mu(1+2C_r) > 0.$$

We obtain the fundamental multiplier inequality

$$\begin{aligned} (A_h z_h, \Pi_h z_h)_h &\leq -\frac{1}{2C_q} (1-\lambda-\mu(1+2C_r)) \|z_h\|_{\mathcal{Z}}^2 + \frac{1}{2\lambda} d_h(u_h, u_h) \\ &\quad + \frac{1}{2\lambda} d_{h,*}(v_h, v_h) + \frac{1}{2} v_h^2(1) + \frac{1}{2\mu} \sum_{i=1}^{N-1} \frac{1}{\tilde{h}_i} \llbracket v_i \rrbracket^2. \end{aligned} \quad (2.39)$$

Thus,

$$(\Pi_h A_h z_h, z_h)_h \leq \alpha \|C_h z_h\|_h^2 - \beta \|z_h\|_h^2, \quad (2.40)$$

where

$$\beta = \frac{1}{2C_q} (1-\lambda-\mu(1+2C_r)) \text{ and } \alpha = \max\left(\frac{1}{2\lambda}, \frac{1}{2\mu}, \frac{1}{2}\right).$$

■

Corollary 2.4.6.1 *Let $z_h = (u_h, v_h)$ belonging to $U_h^r \times V_h^r$ be the solutions of (2.24). For T large enough, there exists $C^{st}(T)$ independent of h , such that*

$$\begin{aligned} \|u_h(0)\|_h^2 + \|v_h(0)\|_h^2 &\leq C^{st}(T) \left[\int_0^T d_h(u_h, u_h) dt + \int_0^T d_{h,*}(v_h, v_h) dt \right. \\ &\quad \left. + \int_0^T v_h^2(1) dt + \int_0^T \sum_{i=1}^{N-1} \frac{1}{\tilde{h}_i} \llbracket v_i \rrbracket^2 dt \right] = C^{st}(T) \int_0^T \|C_h z_h\|_h^2 dt, \end{aligned} \quad (2.41)$$

where d_h and $d_{h,*}$ are defined respectively by (2.29) and (2.30).

PROOF: Since Π_h defined by (2.27) satisfies Assumption 1 by Theorem 2.4.6, we can apply Theorem 2.4.3 where C_h and A_h are defined by (2.34) and (2.24). ■

Following what was done in Section 2.3, (2.41) is a guideline to build an exponentially stable problem. Using the operator C_h – defined in the above section by (2.34) – to damp the dynamics (2.24), we construct an exponentially stable dynamics uniformly in h .

2.4.4 Exponential stability

We incorporate in the system (2.24) the dissipative terms C_h in the observability conditions. In other words, we introduce a perturbation of system (2.24) by the operator $\gamma C_h^* C_h$, which gives us the following system

$$\begin{cases} \dot{z}_h = A_h z_h - \gamma C_h^* C_h z_h, & \text{in } [0, T], \\ z_h(0) = z_{h,0}, \end{cases} \quad (2.42)$$

where A_h and C_h are respectively defined by (2.24) and (2.34). We will be able to expose our main result, namely the uniform exponential stability property in the semi-discrete case. First, as for the observability property, we will check that our multiplier Π_h satisfies the fundamental Assumption 2.

Theorem 2.4.7 *Let C_h be defined by (2.34). Let $z_h = (u_h, v_h) \in \mathcal{Z}_h$ be the solutions of (2.42). Then, Π_h defined by (2.27) satisfies Assumption 2 namely*

$$(\Pi_h A_h z_h, z_h)_h - \gamma (C_h z_h, C_h \Pi_h z_h)_h \leq \chi \|C_h z_h\|_h^2 - \kappa \|z_h\|_h^2,$$

where $\chi, \kappa > 0$ are independent of h .

PROOF: Let clarify the expression of $(C_h z_h, C_h \Pi_h z_h)_h$.

$$\begin{aligned} - (C_h z_h, C_h \Pi_h z_h)_h &= - \sum_{i=1}^{N-1} \frac{1}{\tilde{h}_i} \llbracket v_i \rrbracket \llbracket \chi_h(u_h) \rrbracket + v_h(1) u_h(1) \\ &\quad + d_h(u_h, \chi_h^*(v_h)) + d_{h,*}(v_h, \chi_h(u_h)). \end{aligned}$$

At a point, xu_h has the same value as its interpolation. Therefore, as u_h is continuous (and consequently its jump is zero), then $\chi_h(u_h)$ has a zero jump too. Then, thanks to (2.36), we have

$$\begin{aligned} &(\Pi_h A_h z_h, z_h)_h - \gamma (C_h z_h, C_h \Pi_h z_h)_h \\ &= - \int_0^1 v_h \partial_x \chi_h^*(v_h) dx + \int_0^1 \partial_x u_h \chi_h(u_h) dx - u_h^2(1) + v_h^2(1) \\ &\quad + \gamma v_h(1) u_h(1) + \gamma d_h(u_h, \chi_h^*(v_h)) + \gamma d_{h,*}(v_h, \chi_h(u_h)). \end{aligned}$$

Following the proof of Theorem 2.4.6.1, one can show

$$\begin{aligned} &(\Pi_h A_h z_h, z_h)_h - \gamma (C_h z_h, C_h \Pi_h z_h)_h \\ &\leq - \frac{1}{2C_q} (1 - \lambda - \mu(1 + 2C_r)) \|z_h\|_h^2 + \frac{1}{2\lambda} d_h(u_h, u_h) + \frac{1}{2\lambda} d_{h,*}(v_h, v_h) \\ &\quad + \frac{\gamma^2 + 1}{2} v_h^2(1) + \frac{1}{2\mu} \sum_{i=1}^{N-1} \frac{1}{\tilde{h}_i} \llbracket v_i \rrbracket^2 + \gamma d_h(u_h, \chi_h^*(v_h)) + \gamma d_{h,*}(v_h, \chi_h(u_h)), \end{aligned}$$

where λ and μ are parameters that can be chosen as small desired. Thanks to Cauchy-Schwarz and Young inequalities for last terms, we get with $\delta > 0$

$$\begin{aligned} &(\Pi_h A_h z_h, z_h)_h - \gamma (C_h z_h, C_h \Pi_h z_h)_h \\ &\leq - \frac{1}{C_q} (1 - \lambda - \mu(1 + 2C_r)) \|z_h\|_h^2 + \left(\frac{1}{2\lambda} + \frac{\gamma}{2\delta} \right) (d_h(u_h, u_h) + d_{h,*}(v_h, v_h)) \\ &\quad + \frac{\gamma^2 + 1}{2} v_h^2(1) + \frac{1}{2\mu} \sum_{i=1}^{N-1} \frac{1}{\tilde{h}_i} \llbracket v_i \rrbracket^2 + \frac{\delta\gamma}{2} d_h(\chi_h^*(v_h), \chi_h^*(v_h)) + \frac{\delta\gamma}{2} d_{h,*}(\chi_h(u_h), \chi_h(u_h)). \end{aligned}$$

Thanks to Proposition 2.4.5 and (2.28), we obtain

$$\begin{aligned} & (\Pi_h A_h z_h, z_h)_h - \gamma (C_h z_h, C_h \Pi_h z_h)_h \\ & \leq - \left(\frac{1}{C_q} - \frac{\lambda}{C_q} - \frac{\mu(1+2C_r)}{C_q} - \frac{\gamma \delta C_d C_\Pi^2}{2} \right) \|z_h\|_h^2 \\ & \quad + \left(\frac{1}{2\lambda} + \frac{\gamma}{2\delta} \right) (d_h(u_h, u_h) + d_{h,*}(v_h, v_h)) \\ & \quad + \frac{\gamma^2 + 1}{2} v_h^2(1) + \frac{1}{2\mu} \sum_{i=1}^{N-1} \frac{1}{\bar{h}_i} \|v_i\|^2. \end{aligned}$$

We choose λ, μ, δ sufficiently small so that

$$\frac{1}{C_q} - \frac{\lambda}{C_q} - \frac{\mu(1+2C_r)}{C_q} - \frac{\gamma \delta C_d C_\Pi^2}{2} > 0.$$

Thus, finally we show that the Assumption 2 is satisfied

$$(\Pi_h A_h z_h, z_h)_h - \gamma (C_h z_h, C_h \Pi_h z_h)_h \leq \chi \|C_h z_h\|_h^2 - \kappa \|z_h\|_h^2,$$

with

$$\chi = \max \left(\frac{1}{2\lambda} + \frac{\gamma}{2\delta}, \frac{\gamma^2}{2} + \frac{1}{2}, \frac{1}{2\mu} \right),$$

and

$$\kappa = \frac{1}{C_q} - \frac{\lambda}{C_q} - \frac{\mu(1+2C_r)}{C_q} - \frac{\gamma \delta C_d C_\Pi^2}{2}.$$

■

We can deduce from this theorem the uniform exponential stability property for the damped wave equation (2.42).

Corollary 2.4.7.1 *Let $z_h = (u_h, v_h)$ belonging to $\in U_h^r \times V_h^r$ be the solutions of (2.42). There exist two constants $C_s > 0$ and $\sigma > 0$ independent of h , such that*

$$\|u_h(t)\|_h^2 + \|v_h(t)\|_h^2 \leq C_s e^{-\sigma t} (\|u_h(0)\|_h^2 + \|v_h(0)\|_h^2).$$

PROOF: Using Theorem 2.4.7, we have that the multiplier Π_h defined by (2.27) satisfies Assumption 2. Therefore, we apply Theorem 2.4.4 where C_h and is defined by (2.34). ■

2.4.5 Further properties of the stabilization operators

It is possible to interpret that the dissipation term d_h defined by (2.29) corresponds to element-wise r -laplacians operators. In what follows we introduce the notation

$$x_{i+\frac{1}{2}} = \frac{x_i + x_{i+1}}{2}.$$

Proposition 2.4.8 *For $u_h \in U_h^r$ then*

$$d_h(u_h, u_h) = C_r \sum_{i=0}^{N-1} h_i^{2r+1} \left| u_h^{(r)} \left(x_{i+\frac{1}{2}} \right) \right|^2,$$

where C_r is a positive scalar depending only on the finite element order r .

PROOF: To show that stated result, it is sufficient to study the following term for $u_h \in \mathbb{P}_r$,

$$\frac{d}{dx} (xu_h - \chi_h(u_h)).$$

Since $u_h \in U_h^r$, we have that $u_i := u_h|_{[x_i, x_{i+1}]} \in \mathbb{P}_r$, hence

$$u_i(x) = \sum_{k=0}^{r-1} \frac{x^k}{k!} u_i^{(k)}(x_i) + r_i(x) \quad \text{where} \quad r_i(x) := \frac{x^r}{r!} u_i^{(r)}.$$

Moreover, one can show that

$$\chi_h(u_h)|_{[x_i, x_{i+1}]} = I_i(xu_h)$$

where, I_i is the interpolation operator on the Gauss-Lobatto points $\{x_i + h_i \xi_k\}_{k=0}^r$, it is a projection operator over \mathbb{P}_r . Therefore, one can deduce that

$$(xu_h - \chi_h(u_h))|_{[x_i, x_{i+1}]} = xr_i(x) - I_i(xr_i(x)) = \frac{u_i^{(r)}}{r!} (x^{r+1} - I_i(x^{r+1})) \quad (2.43)$$

Using this result and a simple change of variable one can show that

$$\int_{x_i}^{x_{i+1}} |\partial_x (xu_h - \chi_h(u_h))|^2 = \frac{(u_i^{(r)})^2}{r!^2} \int_0^{h_i} \left| \frac{d}{dx} (x^{r+1} - \mathcal{I}_{h_i}(x^{r+1})) \right|^2 dx, \quad (2.44)$$

where \mathcal{I}_{h_i} is the interpolation operator on the Gauss-Lobatto points $\{h_i \xi_k\}_{k=0}^r$. We introduce now the following change of variable,

$$\phi_i : \begin{cases} [0, 1] \rightarrow [0, h_i], \\ \xi \rightarrow x = h_i \xi. \end{cases}$$

Thus, we have

$$(I_{h_i}(x^{r+1}) \circ \phi_i)(\xi) = \sum_{k=0}^r \psi_k(\xi) \xi_k^{r+1} h_i^{r+1},$$

where $\{\psi_k(\xi)\}_{k=0}^r$ are a basis for \mathbb{P}_r such that $\psi_k(\xi_\ell) = \delta_{k,\ell}$. The use of the above identity, the composition of (2.43) and ϕ_i leads to

$$\left(\frac{d}{dx} (x^{r+1} - I_{h_i}(x^{r+1})) \right) \circ \phi_i = \frac{1}{h_i} \frac{d}{d\xi} \left(h^{r+1} \xi^{r+1} - \sum_{k=0}^r \psi_k(\xi) \xi_k^{r+1} h_i^{r+1} \right).$$

Finally, (2.44) is rewritten as follows,

$$\int_{x_i}^{x_{i+1}} |\partial_x (xu_h - \chi_h(u_h))|^2 = C_r (u_i^{(r)})^2 \frac{1}{h_i} h_i^{2r+2},$$

$$\text{where } C_r = \frac{1}{r!^2} \int_0^1 \left| \frac{d}{d\xi} \left(\xi^{r+1} - \sum_{k=0}^r \psi_k(\xi) \xi_k^{r+1} \right) \right|^2 d\xi. \quad \blacksquare$$

Of note, in the work of [Tebou and Zuazua, 2007] a Laplace operator was used as a viscous term for stabilization of finite elements of order 1. Therefore, in some sense our work generalizes these results by showing that an element-wise r -laplace operator – scaled with the appropriate power of h – can be used for stabilization of high order finite elements.

The same result cannot be obtained straightforwardly for $d_{h,*}$. In fact, due to the discontinuous nature of the space V_h^r , jump terms are involved.

Proposition 2.4.9 For $v_h \in V_h^r$ then

$$d_{h,*}(v_h, v_h) \leq C_r^* \left(\sum_{i=0}^{N-1} h_i^{2r+1} \left| v_h^{(r)} \left(x_{i+\frac{1}{2}} \right) \right|^2 + h \sum_{i=1}^{N-1} \llbracket v_i \rrbracket^2 \right),$$

where C_r^* is a positive scalar depending only on the finite element order r .

PROOF: The proof is based on an adequate decomposition of any function $v_h \in V_h^r$. More precisely we introduce $w_h \in V_h^1$ defined by

$$w_h(0) = w_h(1) = 0, \quad w_i^\pm = \pm \frac{\llbracket v_i \rrbracket}{2}, \quad i \in \{1, \dots, N-1\} \quad \text{and} \quad w_{h|[x_i, x_{i+1}]} \in \mathbb{P}^1.$$

We then defined $u_h = v_h - w_h$. It can be verified that $u_h \in U_h^r$. We have

$$d_{h,*}(v_h, v_h) \leq 2d_{h,*}(u_h, u_h) + 2d_{h,*}(w_h, w_h).$$

Using Proposition 2.4.5 and the property that $u_h \in U_h^r$ we obtain the estimation

$$d_{h,*}(v_h, v_h) \leq 2d_h(u_h, u_h) + 2C_d \int_0^1 w_h^2 dx,$$

hence, using Proposition 2.4.8,

$$d_{h,*}(v_h, v_h) \leq 2C_r \sum_{i=0}^{N-1} h_i^{2r+1} \left| u_h^{(r)} \left(x_{i+\frac{1}{2}} \right) \right|^2 + 2C_d \sum_{i=0}^{N-1} \int_{x_i}^{x_{i+1}} w_h^2 dx.$$

It is then possible to show that

$$\sum_{i=0}^{N-1} \int_{x_i}^{x_{i+1}} w_h^2 dx \leq h \sum_{i=1}^{N-1} \llbracket v_i \rrbracket^2,$$

finally we observe that $u_h^{(r)}(x_{i+\frac{1}{2}}) = (u_h + w_h)^{(r)}(x_{i+\frac{1}{2}}) = v_h^{(r)}(x_{i+\frac{1}{2}})$ by construction. ■

Remark 2.4.10 In the numerical results, we used the following simplified expression instead of the term $d_{h,*}$

$$\tilde{d}_{h,*}(v_h, \tilde{v}_h) := C_r \sum_{i=0}^{N-1} h_i^{2r+1} v_h^{(r)} \left(x_{i+\frac{1}{2}} \right) \tilde{v}_h^{(r)} \left(x_{i+\frac{1}{2}} \right). \quad (2.45)$$

Proposition 2.4.9 allows us to show that Theorem 2.4.6 and Theorem 2.4.7 hold with

$$\tilde{C}_h = \begin{pmatrix} D_h^{1/2} & 0 \\ 0 & J_h^{1/2} \\ 0 & B_h^{1/2} \\ 0 & \tilde{D}_{h,*}^{1/2} \end{pmatrix}. \quad (2.46)$$

where $\tilde{D}_{h,*}$ is associated with $\tilde{d}_{h,*}$, thus showing the exponential decay behavior of the system

$$\begin{cases} \dot{z}_h = A_h z_h - \gamma \tilde{C}_h^* \tilde{C}_h z_h, & \text{in } [0, T], \\ z_h(0) = z_{h,0}. \end{cases}$$

2.5 Observer system: Semi-discrete convergence result

In this section we follow the approach of [Chapelle et al., 2012b] and aim at defining at the semi-discrete level an observer of the solution $z = (u, v)^t$ of the system (2.4) with the assumption that, for all $T > 0$

$$z = (u, v) \in C^1([0, T]; H^{r+1}(0, 1)^2).$$

It is shown below that the constructed observer as the property to converge asymptotically in time to the solution of z up to an order related to the space discretization – typically in $O(h^r)$ – but bounded in time. For the sake of simplicity and conciseness we directly introduce the interpolation on the of z denoted $\bar{z}_h(t) = I_h z(t) = (I_{U_h} u(t), I_{V_h} v(t))$, it is solution of the system

$$\dot{\bar{z}}_h = A_h \bar{z}_h + \varepsilon_h, \quad \bar{z}_h(0) = z_{0,h} = I_h z_0,$$

where ε_h is the consistency error and is assumed to be smooth in time and space and

$$\|\varepsilon_h\| := \sup_{t \in [0, T]} \|\varepsilon_h(t)\|_{L^2(0,1)^2} \leq C_\varepsilon h^r, \quad (2.47)$$

where $C_\varepsilon > 0$ is independent of T . Moreover, the interpolation operator satisfies, for $0 \leq \ell \leq k$ and $0 \leq k \leq r + 1$,

$$\forall w \in H^{r+1}(0, 1)^2, \quad \left(\sum_{i=0}^{N-1} |I_h w - w|_{H^\ell(x_i, x_{i+1})}^2 \right)^{\frac{1}{2}} \leq C_I h^{k-\ell} \|w\|_{H^k(0,1)}. \quad (2.48)$$

We refer to [Ciarlet, 2002; Joly, 2007; Brenner and Scott, 2008] for more details on how these properties can be obtained. The system for the observer is constructed as follows: we assume $v(1, t)$ is available/measured for $t \in [0, T]$ – which is equivalent to having access to $(I_h^V v)(1, t)$ – and solve,

$$\dot{\hat{z}}_h = A_h \hat{z}_h - \gamma \begin{pmatrix} \tilde{D}_h & 0 \\ 0 & \tilde{D}_{h,*} + J_h \end{pmatrix} \hat{z}_h - \gamma \begin{pmatrix} 0 & 0 \\ 0 & B_h \end{pmatrix} (\hat{z}_h - \bar{z}_h), \quad \hat{z}_h(0) = 0.$$

Note that observer systems – and more precisely, state estimator – are very well studied at the continuous level, we refer the reader to [Luenberger, 1963; Ramdani et al., 2010; Imperiale et al., 2020]. In short, one expect that $\hat{z}_h(t) \rightarrow \bar{z}_h(t) \sim z(t)$ asymptotically in time, thus alleviating the mismatch in the initial data between the observer and the interpolation of the solution $z(t)$ of (2.4). The error

$$e_h = \bar{z}_h - \hat{z}_h,$$

satisfies, with $A_{\gamma,h} = A_h - \gamma \tilde{C}_h^* \tilde{C}_h$,

$$\dot{e}_h = A_{\gamma,h} e_h + \gamma \begin{pmatrix} D_h & 0 \\ 0 & \tilde{D}_{h,*} + J_h \end{pmatrix} \bar{z}_h + \varepsilon_h, \quad e_h(0) = z_{0,h}.$$

Lemma 2.5.1 *There exists a constant $C_D > 0$ independent of h , such that*

$$\forall u \in H^{r+1}(0, 1), \quad \|D_h I_{U_h} u\|_{L^2(0,1)} \leq C_D h^r \|u\|_{H^r(0,1)},$$

and

$$\forall v \in H^{r+1}(0, 1), \quad J_h I_{V_h} v = 0, \quad \|\tilde{D}_{h,*} I_{V_h} v\|_{L^2(0,1)} \leq C_D h^r \|v\|_{H^r(0,1)}.$$

PROOF: We show the first identity. From the Proposition 2.4.8, we have

$$\begin{aligned} d_h(I_{U_h}u, I_{U_h}u) &= C_r \sum_{i=0}^{N-1} h_i^{2r+1} \left| (I_{U_h}u)^{(r)} \left(x_{i+\frac{1}{2}} \right) \right|^2 \\ &= C_r \sum_{i=0}^{N-1} \int_{x_i}^{x_{i+1}} h_i^{2r} \left| (I_{U_h}u)^{(r)} \left(x_{i+\frac{1}{2}} \right) \right|^2 \\ &\leq C_r h^{2r} \sum_{i=0}^{N-1} \|I_{U_h}u\|_{H^r(x_i, x_{i+1})}^2. \end{aligned}$$

Using (2.48) and a triangular inequality, we get the existence of a constant \tilde{C}_r such that

$$d_h(I_{U_h}u, I_{U_h}u) \leq \tilde{C}_r h^{2r} \|u\|_{H^r(0,1)}^2.$$

The analysis of $\|\tilde{D}_{h,*}I_{V_h}v\|_{L^2(0,1)}$ is the same as the previous one. Finally, note that since v is smooth, we have that $I_{V_h}v$ is continuous and therefore $J_h I_{V_h}v = 0$. ■

Theorem 2.5.2 *There exists a positive scalar $C_{st} > 0$ such that, for any $T > 0$, we have*

$$\sup_{t \in [0, T]} \|z(t) - \hat{z}_h(t)\|_{L^2(0,1)^2} \leq C_{st} (e^{-\sigma t} \|z_{0,h}\|_h + h^r).$$

PROOF: By triangular inequality and using (2.19), we have that

$$\begin{aligned} \|z(t) - \hat{z}_h(t)\|_{L^2(0,1)^2} &= \|z(t) - \bar{z}_h + \bar{z}_h - \hat{z}_h(t)\|_{L^2(0,1)^2} \\ &\leq \|z(t) - \bar{z}_h\|_{L^2(0,1)^2} + c_q^{-1/2} \|\bar{z}_h - \hat{z}_h(t)\|_h. \end{aligned}$$

Using the interpolation property (2.48), we obtain that there exists a positive scalar C_{st} such that

$$\|z(t) - \hat{z}_h(t)\|_{L^2(0,1)^2} \leq C_{st} h^r + c_q^{-1/2} \|\bar{z}_h - \hat{z}_h(t)\|_h. \quad (2.49)$$

Let us focus on the error $e_h = \bar{z}_h - \hat{z}_h$, it satisfies

$$\dot{e}_h = A_{\gamma,h} e_h + \bar{\varepsilon}_h,$$

where

$$\bar{\varepsilon}_h = \gamma \begin{pmatrix} D_h & 0 \\ 0 & \tilde{D}_{h,*} + J_h \end{pmatrix} \bar{z}_h + \varepsilon_h.$$

Using Lemma 2.5.1, (2.47) combined with (2.19), we have that there exists C_{st} independent of h such that

$$\|\bar{\varepsilon}_h\| \leq C_{st} h^r.$$

From the Duhamel formula, we have

$$e_h(t) = e^{tA_{\gamma,h}} z_{0,h} + \int_0^t e^{(t-s)A_{\gamma,h}} \bar{\varepsilon}_h(s) ds.$$

Using the exponential stability of the operator $A_{\gamma,h}$ (Corollary 2.4.7.1), we have

$$\|e_h(t)\|_h \leq C_{st} \left(e^{-\sigma t} \|z_{0,h}\|_h + \sup_{s \in [0, t]} \|\bar{\varepsilon}_h(s)\|_h \int_0^t e^{-\sigma(t-s)} ds \right).$$

As a consequence, we finally have that there exist a constant $C_{st} > 0$ independent of h and t such that

$$\|e_h(t)\|_h \leq C_{st} \left(e^{-\sigma t} \|z_{0,h}\|_h + \frac{1}{\sigma} (1 - e^{-\sigma t}) h^r \right),$$

which combining with (2.49) proves the desired result. \blacksquare

It is worth noting that the preceding theorem states, in particular, that an observer \hat{z}_h initialized from an interpolated initial condition with an accuracy of order h^r then remains at a distance h^r from the true solution without error drift with respect to time. This is due to the fact that the constant C_{st} does not depend on time. Therefore, in the context of numerical analysis by semi-discretization, this result extends [Chapelle et al., 2012b] to high order schemes.

2.6 Time discretization

2.6.1 Abstract results

In this section, we are interested in the discretization of System (2.17) with an implicit midpoint scheme. The operator C_h or \tilde{C}_h can be used indifferently (we have chosen to use C_h in what follows for the sake of clarity). We verify that the use of this time discretization does not alter the stabilization properties.

$$\begin{cases} \frac{z_h^{n+1} - z_h^n}{\Delta t} = A_h z_h^{n+1/2} - \gamma C_h^* C_h z_h^{n+1/2}, \\ z_h^0 = z_{h,0}. \end{cases} \quad (2.50)$$

It is easy to show that the energy conservation property is preserved in the fully discrete case.

Theorem 2.6.1 *Let $\{z_h^n\}_{n \geq 0} \in \mathcal{Z}_h$ be the solutions of (2.50), we have the discrete dissipation energy property*

$$\frac{1}{2} \frac{\|z_h^{n+1}\|_h^2 - \|z_h^n\|_h^2}{\Delta t} = -\gamma \|C_h z_h^{n+1/2}\|_h^2, \quad \forall n \geq 0. \quad (2.51)$$

PROOF: We multiply (2.50) by $z_h^{n+1/2}$. Since A_h satisfies (2.16) and using the difference of square formula, we obtain the result \blacksquare

Following the same reasoning as in the semi-discrete case, we start by demonstrating the observability property for the undamped case $\gamma = 0$.

Theorem 2.6.2 *Let $\{z_h^n\}_{n \geq 0} \in \mathcal{Z}_h$ be the solutions of (2.50) with $\gamma = 0$. Assuming Assumption 1, for T large enough, there exists a constant $C^{st}(T) > 0$, independent of h , such that*

$$\|z_{h,0}\|_h^2 \leq C^{st}(T) \Delta t \sum_{n=0}^N \|C_h z_h^{n+1/2}\|_h^2.$$

PROOF: The proof of this theorem is very similar to that of Theorem 2.4.3 with $\gamma = 0$, in particular, it follows the same steps.

Step 1: We multiply the first equation of (2.50) (with $\gamma = 0$) by $\Pi_h z_h^{n+1/2}$, and we get

$$\left(\Pi_h \frac{z_h^{n+1} - z_h^n}{\Delta t}, z_h^{n+1/2} \right)_h = \left(\Pi_h A_h z_h^{n+1/2}, z_h^{n+1/2} \right)_h.$$

We decompose the half-sum in the left hand-side to find, using Assumption 1,

$$\begin{aligned} \frac{1}{2\Delta t} \left(\Pi_h z_h^{n+1}, z_h^{n+1} \right)_h - \frac{1}{2\Delta t} \left(\Pi_h z_h^n, z_h^n \right)_h &= \left(\Pi_h A_h z_h^{n+1/2}, z_h^{n+1/2} \right)_h, \\ &\leq \alpha \|C_h z_h^{n+1/2}\|_h^2 - \beta \|z_h^{n+1/2}\|_h^2. \end{aligned} \quad (2.52)$$

Step 2: Summing (2.52) over n we get,

$$\frac{1}{2\Delta t} \left(\Pi_h z_h^{N+1}, z_h^{N+1} \right)_h - \frac{1}{2\Delta t} \left(\Pi_h z_h^0, z_h^0 \right)_h \leq \alpha \sum_{n=0}^N \|C_h z_h^{n+1/2}\|_h^2 - \beta \sum_{n=0}^N \|z_h^{n+1/2}\|_h^2.$$

The fully discrete energy conservation property of Theorem 2.6.1 with $\gamma = 0$ gives

$$\frac{1}{2\Delta t} \left(\Pi_h z_h^{N+1}, z_h^{N+1} \right)_h - \frac{1}{2\Delta t} \left(\Pi_h z_h^0, z_h^0 \right)_h \leq \alpha \sum_{n=0}^N \|C_h z_h^{n+1/2}\|_h^2 - \beta(N+1) \|z_h^0\|_h^2.$$

Step 3: Since Π_h is bounded, by Cauchy-Schwarz theorem and using again the energy conservation theorem 2.6.1 with $\gamma = 0$, we have

$$\left(\beta(N+1)\Delta t - \|\Pi_h\|_h \right) \|z_{0,h}\|_h^2 \leq \alpha \sum_{n=0}^N \Delta t \|C_h z_h^{n+1/2}\|_h^2.$$

Choosing $(N+1)\Delta t = T > \|\Pi_h\|_h \beta^{-1}$, we obtain the result. ■

In the same way as in the semi-discrete case, we obtain the exponential stability property for the damped system.

Theorem 2.6.3 *Let $\{z_h^n\}_{n \geq 0} \in \mathcal{Z}_h$ be the solutions of (2.50). We assume that there exists $C_c > 0$ and $0 < \beta < 1$ such that for all $h > 0$, Δt is chosen so that*

$$\Delta t \|C_h\|_h \leq C_c, \quad (2.53)$$

and

$$\frac{\Delta t}{2} \|A_h\|_h \leq \beta < 1. \quad (2.54)$$

With this assumption and Assumption 2, there exist two constants $C_s, \sigma > 0$ independent of h such that

$$\|z_h^n\|_h^2 \leq C_s \|z_{h,0}\|_h^2 e^{-\sigma(1-\beta^2)n\Delta t}.$$

PROOF: The proof is separated into the same steps as for the proof of exponential stability at the continuous level of Theorem 2.4.4.

Step 1: We multiply the first equation by (2.50) and $\Pi_h z_h^{n+1/2}$, and we get

$$\left(\Pi_h \frac{z_h^{n+1} - z_h^n}{\Delta t}, z_h^{n+1/2} \right)_h = \left(\Pi_h A_h z_h^{n+1/2}, z_h^{n+1/2} \right)_h - \gamma \left(C_h z_h^{n+1/2}, C_h \Pi_h z_h^{n+1/2} \right)_h.$$

Using $(a+b)(a-b) = a^2 - b^2$ and Assumption 2, we have

$$\frac{1}{2\Delta t} \left(\Pi_h z_h^{n+1}, z_h^{n+1} \right)_h - \frac{1}{2\Delta t} \left(\Pi_h z_h^n, z_h^n \right)_h \leq \chi \|C_h z_h^{n+1/2}\|_h^2 - \kappa \|z_h^{n+1/2}\|_h^2.$$

Step 2: Using the energy identity of Theorem 2.6.1, we get with $\alpha > 0$,

$$\begin{aligned} \frac{1}{2\Delta t} (\Pi_h z_h^{n+1}, z_h^{n+1})_h - \frac{1}{2\Delta t} (\Pi_h z_h^n, z_h^n)_h + \alpha \frac{\|z_h^{n+1}\|_h^2 - \|z_h^n\|_h^2}{2\Delta t} \\ \leq (\chi - \alpha\gamma) \|C_h z_h^{n+1/2}\|_h^2 - \kappa \|z_h^{n+1/2}\|_h^2. \end{aligned} \quad (2.55)$$

We intend to bound the last term by $-\|z_h^{n+1}\|_h^2 - \|z_h^n\|_h^2$. Note that this intermediate necessary step is specific to the time discretization. Decomposing $z_h^{n+1/2}$ in a well-chosen way, we have

$$-\|z_h^{n+1/2}\|_h^2 = -\frac{1}{2}\|z_h^{n+1}\|_h^2 - \frac{1}{2}\|z_h^n\|_h^2 + \frac{(\Delta t)^2}{4} \left\| \frac{z_h^{n+1} - z_h^n}{\Delta t} \right\|_h^2.$$

Using the scheme (2.50) and Cauchy-Schwarz and Young inequalities, we obtain with $\epsilon > 0$,

$$\begin{aligned} -\|z_h^{n+1/2}\|_h^2 \leq -\frac{1}{2}\|z_h^{n+1}\|_h^2 - \frac{1}{2}\|z_h^n\|_h^2 + \frac{(\Delta t)^2}{4}(1 + \epsilon) \|A_h z_h^{n+1/2}\|_h^2 \\ + \frac{\gamma(\Delta t)^2}{4}(1 + \epsilon^{-1}) \|C_h^* z_h^{n+1/2}\|_h^2. \end{aligned}$$

By triangular inequality and Cauchy-Schwarz inequality and using hypothesis (2.53), we get

$$\begin{aligned} -\|z_h^{n+1/2}\|_h^2 \leq -\frac{1}{2} \left(1 - \frac{(\Delta t)^2}{4}(1 + \epsilon) \|A_h\|_h^2 \right) (\|z_h^{n+1}\|_h^2 + \|z_h^n\|_h^2) \\ + \frac{\gamma C_c^2}{4}(1 + \epsilon^{-1}) \|C_h z_h^{n+1/2}\|_h^2. \end{aligned}$$

We choose $\epsilon = \frac{1}{2\beta^2} - \frac{1}{2} > 0$, using the condition (2.54), we obtain

$$\begin{aligned} -\|z_h^{n+1/2}\|_h^2 \leq -\frac{1}{4} (1 - \beta^2) (\|z_h^{n+1}\|_h^2 + \|z_h^n\|_h^2) \\ + \frac{\gamma C_c^2}{4} \left(1 + \frac{2\beta^2}{1 - \beta^2} \right) \|C_h z_h^{n+1/2}\|_h^2. \end{aligned}$$

Using the above equation in (2.55), we get

$$\begin{aligned} \frac{1}{2\Delta t} (\Pi_h z_h^{n+1}, z_h^{n+1})_h - \frac{1}{2\Delta t} (\Pi_h z_h^n, z_h^n)_h + \alpha \frac{\|z_h^{n+1}\|_h^2 - \|z_h^n\|_h^2}{2\Delta t} \\ \leq \left(\frac{\gamma C_c^2}{4} \left(1 + \frac{2\beta^2}{1 - \beta^2} \right) + \chi - \alpha\gamma \right) \|C_h z_h^{n+1/2}\|_h^2 - \frac{\kappa}{4} (1 - \beta^2) (\|z_h^{n+1}\|_h^2 + \|z_h^n\|_h^2). \end{aligned}$$

Step 3: We choose α large enough such that $\alpha \geq \frac{\chi}{\gamma} + \frac{C_c^2}{4} \left(1 + \frac{2\beta^2}{1 - \beta^2} \right)$. Thus, we have

$$\begin{aligned} \frac{1}{2\Delta t} (\Pi_h z_h^{n+1}, z_h^{n+1})_h - \frac{1}{2\Delta t} (\Pi_h z_h^n, z_h^n)_h + \alpha \frac{\|z_h^{n+1}\|_h^2 - \|z_h^n\|_h^2}{2\Delta t} \\ \leq -\frac{\kappa}{4} (1 - \beta^2) (\|z_h^{n+1}\|_h^2 + \|z_h^n\|_h^2). \end{aligned} \quad (2.56)$$

Step 4: We introduce the fully discrete Lyapunov functional

$$\mathcal{L}_h^n = (\Pi_h z_h^n, z_h^n)_h + \alpha \|z_h^n\|_h^2,$$

and we can also show that it is equivalent to the energy functional $\mathcal{E}_h^n := \frac{1}{2} \|z_h^n\|_h^2$ using the bounded nature of Π_h and choosing α sufficiently large.

$$(\alpha - \|\Pi_h\|_h) \|z_h^n\|_h^2 \leq \mathcal{L}_h^n \leq (\alpha + \|\Pi_h\|_h) \|z_h^n\|_h^2. \quad (2.57)$$

Using the above equivalence between the Lyapunov functional and the energy, we obtain

$$\frac{1}{2\Delta t} (\mathcal{L}_h^{n+1} - \mathcal{L}_h^n) \leq -\frac{\kappa}{4} (1 - \beta^2) (\|\Pi_h\|_h + \alpha)^{-1} (\mathcal{L}_h^{n+1} + \mathcal{L}_h^n).$$

Rearranging the terms, we have

$$\mathcal{L}_h^{n+1} \leq \delta(\Delta t, \alpha) \mathcal{L}_h^n.$$

We denote $\delta(\Delta t, \alpha) = \left(1 - \frac{\Delta t \frac{\kappa}{2} (1 - \beta^2)}{\|\Pi_h\|_h + \alpha}\right) / \left(1 + \frac{\Delta t \frac{\kappa}{2} (1 - \beta^2)}{\|\Pi_h\|_h + \alpha}\right)$.

Step 5: By induction and using again (2.18), we obtain

$$\|z_h^n\|_h^2 \leq \frac{\alpha + \|\Pi_h\|_h}{\alpha - \|\Pi_h\|_h} \|z_{h,0}\|_h^2 e^{n \ln(\delta(\Delta t, \alpha))}.$$

We must now specify the constant $\ln(\delta)$. Bounding from below Δ by 0 and using a series expansion we get

$$\ln(\delta(\Delta t, \alpha)) = \ln \left(\frac{1 - \frac{\Delta t \kappa (1 - \beta^2)}{\|\Pi_h\|_h + \alpha}}{1 + \frac{\Delta t \frac{\kappa}{2} (1 - \beta^2)}{\|\Pi_h\|_h + \alpha}} \right) \leq \ln \left(1 - \Delta t \frac{\kappa (1 - \beta^2)}{\|\Pi_h\|_h + \alpha} \right)$$

$$\underset{\alpha \rightarrow +\infty}{\sim} -\frac{\Delta t}{\alpha} \kappa (1 - \beta^2) + o\left(\frac{1}{\alpha}\right).$$

Finally, we can conclude,

$$\|z_h^n\|_h^2 \leq \left(\frac{\alpha + \|\Pi_h\|_h}{\alpha - \|\Pi_h\|_h} \right) \|z_{h,0}\|_h^2 e^{-\sigma n \Delta t},$$

with $\sigma(1 - \beta^2) = \frac{\kappa(1 - \beta^2)}{\alpha} > 0$. ■

Remark 2.6.4 *In many situations, as the case presented, one can show that there exists C_A independent of h such that,*

$$\|C_h\|_h \leq C_A \|A_h\|_h, \quad (2.58)$$

and in this case, we can get rid of condition (2.53) and condition (2.54) is sufficient for Theorem 2.6.3. We recognize a "CFL" condition usually encounter with explicit scheme. Nevertheless, it allows us to obtain the exponential stability property.

Under the same two assumptions as in the semi-discrete case, we obtain the properties of observability and exponential stability.

2.6.2 Application: 1D fully discrete wave problem

We can now present the observability theorem in a totally discrete framework.

Theorem 2.6.5 *Let $\{u_h^n, v_h^n\}_{n \geq 0}$ belonging to $U_h^r \times V_h^r$ be the solutions of (2.50) with $\gamma = 0$. For T large enough, there exists $C^{st}(T)$ independent of h such that*

$$\|u_{h,0}\|_h^2 + \|v_{h,0}\|_h^2 \leq C^{st}(T) \sum_{n=1}^N \Delta t \left[d_h \left(u_h^{n+1/2}, u_h^{n+1/2} \right) + d_{h,*} \left(v_h^{n+1/2}, v_h^{n+1/2} \right) + \left(v_h^{n+1/2}(1) \right)^2 dt + \sum_{i=1}^{N-1} \frac{1}{\tilde{h}_i} \llbracket v_i^{n+1/2} \rrbracket^2 \right],$$

where d_h and d_h^* are defined by (2.29) and by (2.30).

PROOF: Since Π_h satisfies Assumption 1, we can apply Theorem 2.6.2 and the result follows immediately. \blacksquare

Theorem 2.6.6 *Let $\{u_h^n, v_h^n\}_{n \geq 0}$ belonging to $U_h^r \times V_h^r$ be the solutions of (2.50). We assume that $\exists \beta > 0$ independent of h and Δt , $\forall h$, Δt is chosen small enough such*

$$\frac{\Delta t}{2} \|R_h\|_h \leq \beta < 1. \quad (2.59)$$

There exists two constants $C_s > 0$ and $\sigma > 0$ independent of h such that

$$\|u_h^n\|_h^2 + \|v_h^n\|_h^2 \leq C_s e^{-n\Delta t\sigma(\Delta t)} (\|u_{h,0}\|_h^2 + \|v_{h,0}\|_h^2).$$

PROOF: In order to prove Theorem 2.6.3, we need to show that the conditions (2.54) holds. Since we have the following property

$$\|A_h\|_h \leq C_A \|R_h\|_h, \quad C_A > 0,$$

if (2.59) holds, then (2.54) is satisfied.

In order to apply Remark 2.6.4 – and thus to get rid of the condition (2.53) –, it only remains to show that

$$\|C_h\|_h \leq C_c \|R_h\|_h, \quad C_c > 0.$$

It is well known that we have the following inequality

$$\frac{\alpha}{h} \leq \|R_h\|_h, \quad \alpha > 0,$$

so we only need to show that $\|C_h\|_h \leq \frac{C^{st}}{h}$ with $C^{st} > 0$. Let us compute the operator norm of C_h . By definition of the operator C_h , we have for $z_h \in \mathcal{Z}_h$,

$$\|C_h z_h\|_h^2 = d_h(u_h, u_h) + d_{h,*}(v_h, v_h) + v_h^2(1) + \sum_{i=1}^{N-1} \frac{1}{\tilde{h}_i} \llbracket v_i \rrbracket^2.$$

We only treat the last term because it is the most difficult. Using Proposition 2.4.5 and the same kind of computations as in the proof of Theorem 2.6.3, one can show that

$$\sum_{i=1}^{N-1} \frac{1}{\tilde{h}_i} \llbracket v_i \rrbracket^2 \leq 2 \sum_{i=0}^{N-1} \frac{1 + 2C_r}{\tilde{h}_i h_i} \|v_h\|_{L^2([x_i, x_{i+1}])}^2 + 2 \sum_{i=1}^N \frac{1 + 2C_r}{\tilde{h}_i h_{i-1}} \|v_h\|_{L^2([x_{i-1}, x_i])}^2.$$

Since we have a quasi-uniform discretization, we obtain

$$\sum_{i=1}^{N-1} \frac{1}{\tilde{h}_i} \llbracket v_i \rrbracket^2 \leq 4 \frac{1 + 2C_r}{\rho^2 h^2} \|v_h\|_{L^2(0,1)}^2.$$

We can therefore bound $\|C_h\|_h$ by $1/h$,

$$\|C_h\|_h \leq \left(\sqrt{C_d} + \frac{\sqrt{1 + 2C_r}}{\sqrt{\rho h}} + \frac{2\sqrt{1 + 2C_r}}{\rho h} \right).$$

Furthermore, $\forall h, \Delta t$ chosen small enough such that

$$\Delta t \|R_h\|_h \leq \beta < 1,$$

satisfies the condition 2.54 of Theorem 2.6.3. Since Π_h satisfies Assumption 2 and (2.58) and (2.54) holds, we can apply Theorem 2.6.3 and the result follows immediately. ■

In perspective, adapting the previous result to the observer analysis, as we did for the semi-discretization in space, should allow us to extend the fully discretized numerical analysis proposed in [Chapelle et al., 2012b] to high order schemes. This would then imply that we can propose an observer that avoids the asymptotic temporal error drift by using data while remaining compatible with the state of the art of high order discretization.

2.7 Numerical results

We compare the numerical results of Section 2.2 and those obtained with our method. First we make an analysis of the eigenvalues by comparing our method and other existing methods in the literature. In a second step, for the sake of efficiency of the method, we will introduce a leap-frog scheme which preserves the energy dissipation property of the previous sections. Then we compare the shape of the solution of the damped system with the solution of the undamped system of Section 2.2. Note that in all the section the term $\tilde{D}_{h,*}$ is the operator associated with the bilinear form $\tilde{d}_{h,*}$ defined by (2.45) namely

$$(\tilde{D}_{h,*} v_h, \tilde{v}_h)_h = \tilde{d}_{h,*}(v_h, \tilde{v}_h), \quad \forall v_h, \tilde{v}_h \in V_h^r. \quad (2.60)$$

2.7.1 Eigenvalues analysis

We look at the effect of adding the new dissipative terms on the spectrum. We start by doing a comparison with methods used in the literature. A first idea is to penalize the jump as seen in [Cohen and Sinding, 2009] for Maxwell's equations. Thus, we add a dissipation term J_h corresponding to this penalization of the jump of the second variable which is discontinuous.

We see in Figure 2.5 (Left) that this method is adapted for finite elements of \mathbb{P}_1 , but that the decay rate is not as good as the continuous one. As soon as we go to higher finite elements in Figure 2.5 (Right), we find eigenvalues close to the imaginary axis. We also represent the spectrum obtained viscosity is added – as a classical stabilization strategy, (see [Tebou and Zuazua, 2007]). We denote by V_h the discrete Laplace operator on the first component.

Thus, as we can see in Figure 2.6 (Left), a second order perturbation gives the exponential stability but with a wrong decay for finite elements of order greater than \mathbb{P}_1 .

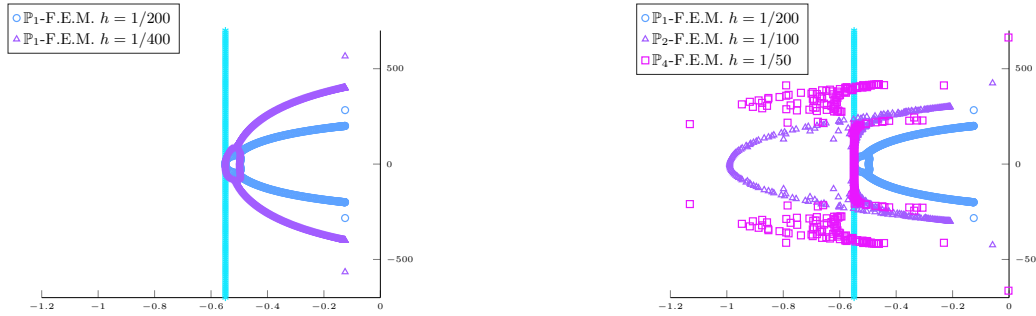


Figure 2.5: Spectrum of $A_h - J_h$

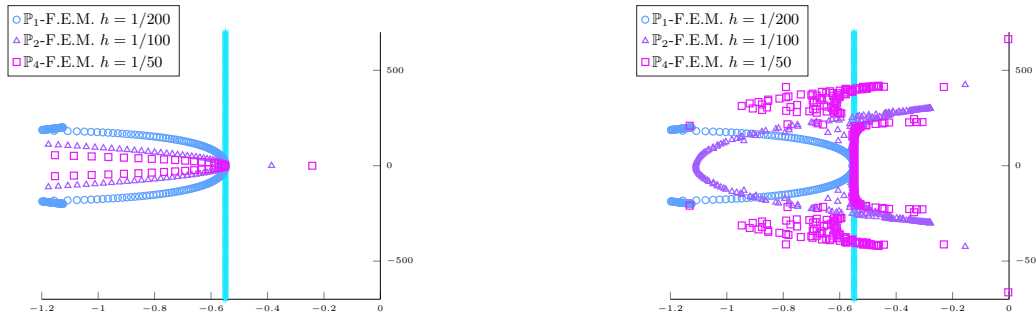


Figure 2.6: Spectrum of $A_h - J_h - h^2 V_h$ (Left) and of $A_h - J_h - h^{r+1} V_h$ (Right), r is the order of the method

Furthermore, the consistency is really deteriorated. If we soften the impact of the viscous term – Figure 2.6 (Right) –, for instance by replacing the scaling h^2 by h^{r+1} with the order r of the finite elements, then the exponential stability is not satisfied anymore.

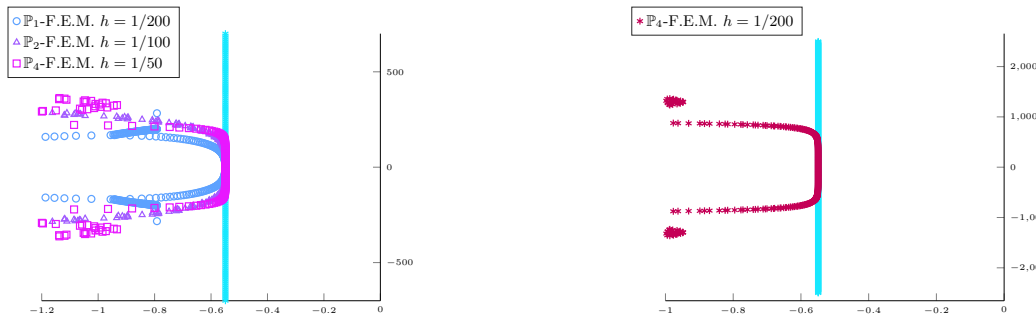


Figure 2.7: Spectrum of $A_h - J_h - D_h - \tilde{D}_{h,*}$

We now use our method. Note that we penalize the jump with the addition of the term J_h , as in Figure 2.5.

We preserve the consistency, and we have the uniform exponential stability with a constant which has the same order as what we had at the continuous level as we can see in Figure 2.7 (Left). When refining – see Figure 2.7 (Right) –, the consistency is improved. In the end, we found dissipative terms that exponentially stabilize the system while preserving consistency.

2.7.2 A numerical example with an explicit leap-frog scheme

One of the objectives being to have an effective method, we present the leap-frog scheme used in the numerical simulations that guarantees the energy conservation and dissipation properties for damped and undamped systems respectively. We consider the damped wave equation (2.42) discretized with an explicit scheme where the dissipation terms are off-centered.

$$\begin{cases} \frac{u_h^{n+1} - u_h^n}{\Delta t} - R_h^* v_h^{n+1/2} + \gamma D_h u_h^n = 0 \\ \frac{v_h^{n+3/2} - v_h^{n+1/2}}{\Delta t} - R_h u_h^{n+1} + \gamma B_h v_h^{n+1} \\ \quad + \gamma J_h \frac{v_h^{n+3/2} + v_h^{n+1/2}}{2} + \gamma \tilde{D}_{h,*} v_h^{n+1/2} = 0. \end{cases} \quad (2.61)$$

The scheme is almost fully explicit thanks to the mass-lumping strategy and off-centering of the newly added dissipative terms D_h and $\tilde{D}_{h,*}$.

Note that:

- the centering of the jump term involves, at the algebraic level, to solve $(N - 1)$ linear system of size 2×2 at each iteration, where N is the number of elements
- off-centering does not imply here loss of consistency in time since it is used on terms that are small when apply to interpolation of smooth solution of the problem.

To show the energy identity, we rewrite the scheme using the following trick

$$\begin{cases} D_h u_h^n = D_h u_h^{n+1/2} - \frac{\Delta t}{2} D_h \frac{u_h^{n+1} - u_h^n}{\Delta t}, \\ \tilde{D}_{h,*} v_h^{n+1/2} = \tilde{D}_{h,*} v_h^{n+1} - \frac{\Delta t}{2} \tilde{D}_{h,*} \frac{v_h^{n+3/2} - v_h^{n+1/2}}{\Delta t}, \end{cases}$$

thus, we obtain

$$\begin{cases} \tilde{M}_h \frac{u_h^{n+1} - u_h^n}{\Delta t} - R_h^* v_h^{n+1/2} + \gamma D_h u_h^{n+1/2} = 0 \\ \tilde{M}_{h,*} \frac{v_h^{n+3/2} - v_h^{n+1/2}}{\Delta t} + R_h u_h^{n+1} + \gamma B_h v_h^{n+1} + \gamma J_h v_h^{n+1} + \gamma \tilde{D}_{h,*} v_h^{n+1} = 0, \end{cases} \quad (2.62)$$

where

$$\tilde{M}_h = \mathbb{1} - \gamma \frac{\Delta t}{2} D_h \quad \text{and} \quad \tilde{M}_{h,*} = \mathbb{1} - \gamma \frac{\Delta t}{2} \tilde{D}_{h,*}.$$

The energy identity for System (2.61) involves a new energy functional that is given below: for $u_h^n \in U_h^r$ and $v_h^{n+1/2} \in V_h^r$,

$$\mathcal{E}_{h,ex}^n = \frac{1}{2} \left(\tilde{M}_h u_h^n, u_h^n \right)_h + \frac{1}{2} \left(\tilde{M}_{h,*} v_h^{n+1/2}, v_h^{n+1/2} \right)_h - \frac{\Delta t}{2} \left(R_h u_h^n, v_h^{n+1/2} \right)_h. \quad (2.63)$$

Theorem 2.7.1 *Let $\{u_h^n, v_h^n\}_{n \geq 0}$ belonging to $U_h^r \times V_h^r$ be the solutions of (2.62). The following energy identity holds,*

$$\begin{aligned} \frac{\mathcal{E}_{h,ex}^{n+1} - \mathcal{E}_{h,ex}^n}{\Delta t} &= -\gamma \left(B_h v_h^{n+1}, v_h^{n+1} \right)_h - \gamma \left(J_h v_h^{n+1}, v_h^{n+1} \right)_h \\ &\quad - \gamma \left(\tilde{D}_{h,*} v_h^{n+1}, v_h^{n+1} \right)_h - \gamma \left(D_h u_h^{n+1/2}, u_h^{n+1/2} \right)_h. \end{aligned}$$

Moreover if $\forall h$, Δt is chosen small enough such that

$$1 - \gamma \frac{\Delta t}{2} \|D_h\|_h > 0 \quad \text{and} \quad 1 - \gamma \frac{\Delta t}{2} \|\tilde{D}_{h,*}\|_h > 0, \quad (2.64)$$

and

$$1 - \frac{\Delta t}{2} \|R_h\|_h > 0, \quad (2.65)$$

then $\mathcal{E}_{h,ex}^n \geq 0$ and the scheme is stable.

PROOF: We only show here the energy identity. The positivity of the energy and related stability of the scheme are well-known results. We multiply the two equations of (2.62) by $u_h^{n+1/2}$ and v_h^{n+1} respectively and we obtain,

$$\begin{aligned} & \left(\tilde{M}_h \frac{u_h^{n+1} - u_h^n}{\Delta t}, \frac{u_h^{n+1} + u_h^n}{2} \right)_h - \left(R_h^* v_h^{n+1/2}, \frac{u_h^{n+1} + u_h^n}{2} \right)_h + \gamma \left(D_h u_h^{n+1/2}, u_h^{n+1/2} \right)_h \\ & + \left(\tilde{M}_{h,*} \frac{v_h^{n+3/2} - v_h^{n+1/2}}{\Delta t}, \frac{v_h^{n+3/2} + v_h^{n+1/2}}{2} \right)_h + \left(R_h u_h^{n+1}, \frac{v_h^{n+3/2} + v_h^{n+1/2}}{2} \right)_h \\ & + \gamma \left(B_h v_h^{n+1}, v_h^{n+1} \right)_h \gamma \left(J_h v_h^{n+1}, v_h^{n+1} \right)_h + \gamma \left(\tilde{D}_{h,*} v_h^{n+1}, v_h^{n+1} \right)_h = 0. \end{aligned}$$

We use the follow identity

$$\begin{aligned} & - \left(R_h^* v_h^{n+1/2}, \frac{u_h^{n+1} + u_h^n}{2} \right)_h + \left(R_h u_h^{n+1}, \frac{v_h^{n+3/2} + v_h^{n+1/2}}{2} \right)_h \\ & = -\frac{1}{2} \left(R_h^* v_h^{n+1/2}, u_h^n \right)_h + \frac{1}{2} \left(R_h^* v_h^{n+3/2}, u_h^{n+1} \right)_h, \end{aligned}$$

to obtain

$$\begin{aligned} & \frac{1}{2\Delta t} \left(\tilde{M}_h u_h^{n+1}, u_h^{n+1} \right)_h - \frac{1}{2\Delta t} \left(\tilde{M}_h u_h^n, u_h^n \right)_h + \frac{1}{2\Delta t} \left(\tilde{M}_{h,*} v_h^{n+3/2}, v_h^{n+3/2} \right)_h \\ & - \frac{1}{2\Delta t} \left(\tilde{M}_{h,*} v_h^{n+1/2}, v_h^{n+1/2} \right)_h + \frac{\Delta t}{2} \left(R_h u_h^{n+1}, v_h^{n+3/2} \right)_h - \frac{\Delta t}{2} \left(R_h u_h^n, v_h^{n+1/2} \right)_h \\ & = -\gamma \left(B_h v_h^{n+1}, v_h^{n+1} \right)_h + \gamma \left(J_h v_h^{n+1}, v_h^{n+1} \right)_h \\ & \quad - \gamma \left(\tilde{D}_{h,*} v_h^{n+1}, v_h^{n+1} \right)_h - \gamma \left(D_h u_h^{n+1/2}, u_h^{n+1/2} \right)_h. \end{aligned}$$

We recognize the expression of the explicit energy (2.63). This concludes the proof. \blacksquare

Remark 2.7.2 Note that since D_h and $\tilde{D}_{h,*}$ are bounded, condition (2.64) is not a CFL condition. On the other hand, condition (2.65) corresponds to the classical CFL condition for explicit schemes.

As in Section 2.2, we now look at the evolution of the shape of the solution and of the energy over time, with the added damped terms. The numerical pollution due to parasitic

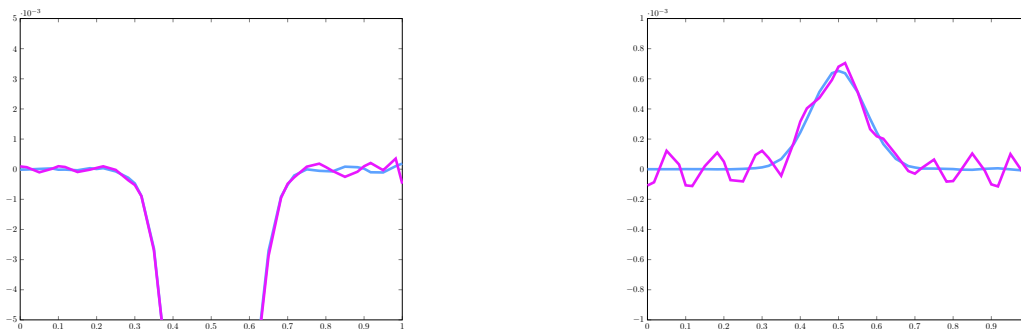


Figure 2.8: Solution u discretized with \mathbb{P}_4 F.E.M. with $h = 1/10$ and $dt = 1/1000$ at time $t = 2$ (Left) and at time $t = 4$ (Right)

waves has completely vanished. We can observe on Figure 2.8 a totally smooth wave that is not oscillating anymore.

While the system without the damping terms had an energy that became constant in long time (Figure 2.9 in pink), we notice that with the addition of these dissipative terms, we obtain a linearly decreasing energy (in logarithm scale) as can be seen on Figure 2.9 in blue. This confirms the result of Theorem 2.6.6, we succeeded in making the system uniformly exponentially stable.

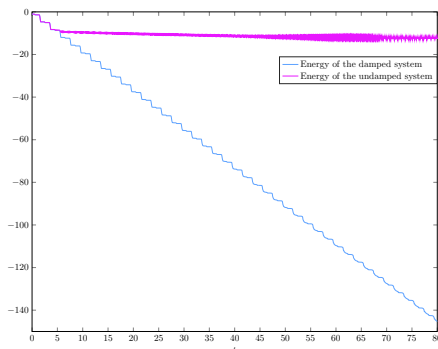


Figure 2.9: $\ln(\|u(t)\|^2 + \|v(t)\|^2)^{1/2}$

2.8 Conclusion

By penalizing the jump and introducing an additive dissipative term consistent with high-order finite element schemes, we can adapt multiplier techniques at the fully discrete level to obtain exponential stabilization results that are uniform with respect to the discretization. The main perspective is to extend the results for 2-dimensional dynamics. The main obstacle lies in the difficulties observed at the continuous level, namely that the variable v_h must be defined as a gradient on each element, which may not be the case for a weak formulation of the equation. Moreover, we need the discrete energy uniform distribution property for the first order formulation. Finally, it would be necessary to introduce a numerical treatment of the boundary conditions.

Part II

Source identification for wave propagation problem

CHAPTER 3

Mathematical analysis of an observer for solving inverse source wave problem

L'espace efface le bruit.

— Victor Hugo

The objective of this work is to propose a practical method using observers to estimate a source term of a wave equation, from internal measurements in a subdomain. The first part of the work consists in proving an identifiability result from classical observability conditions for wave equations. We deduce that the source reconstruction is an ill-posed inverse problem of degree 1 or 2 depending on the measurements type. This inverse problem is solved using observers – a sequential strategy – that is proven to be equivalent to a minimization of a cost functional with Tikhonov regularization. This chapter contains a submitted article, realized in collaboration with Sébastien Imperiale and Philippe Moireau.

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Mathematical analysis of an observer for solving inverse source wave problem

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Submitted

Abstract

The objective of this work is to propose a method using observers to estimate a source term of a wave equation from internal measurements in a subdomain. The first part of the work consists in proving an identifiability result from classical observability conditions for wave equations. We show that the source reconstruction is an ill-posed inverse problem of degree 1 or 2 depending on the measurements type. This inverse problem is solved using observers – a sequential strategy – that is proven to be equivalent to a minimization of a cost functional with Tikhonov regularization.

Keywords— Wave equation, Source identification, Observability

3.1 Introduction

Inverse problems applied to wave-like propagation phenomena has received a lot of attention with numerous applications in geophysics [Blum et al., 2009; Zou et al., 1992], tomography [Bergounioux et al., 2010; Haine, 2014a; Gebauer and Scherzer, 2008], medical imaging [Aguilo et al., 2010; Imperiale et al., 2021] with the development of elastography [Bal and Imperiale, 2015; Imperiale et al., 2020], non-destructive testing [Albanese et al., 2002]. In fact, several subcategories of inverse problems can be distinguished: the reconstruction of initial conditions, the reconstruction of sources, and finally the identification of constitutive parameters. First, the reconstruction of initial conditions has been widely studied ([Ramdani et al., 2010; Imperiale et al., 2020]) and is based on the proof of a so-called observability inequality. This inequality, which has been studied extensively by the control community [Lions, 1988; Zuazua, 2005; Coron, 2007] quantifies the initial condition to measurement operator invertibility which in this case is linear, allowing for the analysis of reconstruction strategies [Cîndea and Münch, 2015; Burman et al., 2021]. At the other end of the spectrum, parameter reconstruction is a nonlinear inversion problem [Jellali, 2006; Roman et al., 2020], where identifiability and stability results are hard to obtain in the general cases and often rely on Carleman inequalities, see for example [Baudouin et al., 2013, 2017, 2021] and references therein. Finally, source reconstruction is the intermediate step between initial conditions reconstruction and parameter reconstruction. In a first step, we consider a source term with a separate variable, *i.e.*, the source is the product of a time-dependent function multiplying a space-dependent function to be estimated. This source term can take various forms, even being point sources ([Komornik and Yamamoto, 2002, 2005]). One of the advantages of estimating such source terms is that one can show identifiability by relying on initial condition observability inequalities ([Alves

et al., 2009; Yamamoto, 1995]), while more general source terms may require the use of dedicated Carleman estimates ([Mukanova and Romanov, 2016; Yamamoto, 1999]). From a practical point of view, there are two main categories of methods for solving these inverse problems: Least-squares minimization approaches [Bergounioux et al., 2010; Bonnet and Aquino, 2015; Burman et al., 2021] – also known as variational approaches or 4D-var in the data assimilation community [Lewis and Derber, 1985; Talagrand and Courtier, 1987; Rabier et al., 2000] – or sequential approaches, where a new evolution problem integrating the measurements as a feedback pursues the target solution. The least squares approach is certainly the oldest strategy ([Sasaki, 1955, 1970]) and is typically solved using adjoint equations ([Le Dimet and Talagrand, 1986]). The goal is to eventually find the optimal trajectory that minimizes the discrepancy between the solution and the observations. The sequential approach, also called the observer approach, is less studied. The principle is to introduce feedback into the dynamics based on measurement to create a new system, called the observer, which converges asymptotically to the observed system over time. Starting from the reconstruction of the initial conditions of wave problems, algorithms were developed [Ramdani et al., 2010] combining the observer and time-reversal strategies in a comparable spirit to the back-and-forth nudging introduced in data assimilation [Auroux and Blum, 2008]. This strategy has been extended to unbounded domains by [Imperiale et al., 2020]. The proposed feedbacks are often easy to compute, and their efficiency is based on stabilization principles inspired by the so-called Luenberger observer introduced for finite-dimensional control problems [Luenberger, 1963, 1971]. In the same community, it is also known that the principles of dynamic programming [Bellman, 1966] can also help to formulate observers that asymptotically solve the least square minimization problem. Such a strategy is often referred to as optimal filtering, since the feedback is also called a filter, and it is derived directly from an optimal criterion. It has been extended to the estimation of initial condition for infinite dimensional systems [Bensoussan, 1971; Curtain, 1975] with recent works on numerical methods and their analysis, see e.g. [Aussal and Moireau, 2022] and references therein. Observers for reconstructing sources or identifying parameters have also been introduced in finite dimensional cases [Zhang, 2002] with applications to discretized wave-like problems [Moireau et al., 2008; Imperiale et al., 2021].

In this chapter, our goal is to extend the definition of such an optimal observer to source reconstruction in the infinite dimensional context of source inversion for wave problem. Our contribution includes the formulation of the observer, namely its existence, but also its asymptotic property, here called stability, and its robustness to measurement noise, as in all inverse problems. Notably, we base our analysis on general field measurements on a subdomain whereas most observability results for wave equations are classically based on velocity measurements.

This chapter is organized as follows. In the first section, we present the problem underlying the nature of the present observations. In the second section, we present how observability inequality allows us to quantify how poor the reconstruction of source terms is from the available field measurements. The next section is devoted to the formulation and analysis of the observer in the infinite dimensional context, developing an infinite dimensional Kalman filtering reduced to the source space. We then propose a discretization strategy based on the same principles at the discrete level. Finally, we illustrate the strategy with numerical experiments.

3.2 Problem setting

Let Ω be a bounded, connected open domain of class C^2 in \mathbb{R}^d . We consider the following wave equation

$$\begin{cases} \partial_{tt}u(x, t) - \Delta u(x, t) = \sigma(t)\theta(x), & (x, t) \in \Omega \times (0, T), \\ u(x, t) = 0, & (x, t) \in \partial\Omega \times (0, T), \\ u(x, 0) = u_0(x), & x \in \Omega, \\ \partial_t u(x, 0) = v_0(x), & x \in \Omega, \end{cases} \quad (3.1)$$

where (u_0, v_0) is the initial condition, and $f(t) = \sigma(t)\theta(x)$ with $\sigma \in L^2(0, T)$ $\theta \in \mathcal{P} \subseteq L^2(\Omega)$, a Hilbert space, to be recovered from available measurements.

3.2.1 Semigroup approach

We rewrite (3.1) as a first order system in the state-space $\mathcal{Z} := H_0^1(\Omega) \times L^2(\Omega)$ equipped with the semi-norm H^1 and the L^2 standard scalar product,

$$\begin{cases} \dot{z}(t) = Az(t) + B(t)\theta, & t \in [0, T], \\ z(0) = z_0, \end{cases} \quad (3.2)$$

where

$$z_0 = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ \sigma(t)\text{Id} \end{pmatrix}, \quad (3.3)$$

and the operator A is an unbounded skew-adjoint operator from $\mathcal{D}(A) \subset \mathcal{Z}$ into \mathcal{Z} defined by

$$A = \begin{pmatrix} 0 & \text{Id} \\ \Delta_0 & 0 \end{pmatrix}, \quad \mathcal{D}(A) = \mathcal{D}(-\Delta_0) \times H_0^1(\Omega),$$

where Δ_0 denotes the Laplacian operator with homogeneous Dirichlet boundary conditions [Bensoussan et al., 2007, Part II, Section 1.2.10]. The operator A is maximal dissipative, hence it is a generator of a C^0 -semigroup $(\Phi(t))_{t \geq 0}$ on \mathcal{Z} also denoted $(e^{tA})_{t \geq 0}$. Note that $(B(t))_{t \geq 0}$ is a one parameter family of bounded operator from \mathcal{P} to \mathcal{Z} . We recall the various notions of solution and associated regularity.

Theorem 3.2.1 *For all $T > 0$, the evolution equation (3.2) admits the following solutions:*

1. *For all $z_0 \in \mathcal{D}(A)$, $\sigma \in H^1([0, T])$ and $\theta \in \mathcal{P}$, there exists a unique strong solution $z \in C^0([0, T]; \mathcal{D}(A)) \cap C^1([0, T]; \mathcal{Z})$ of (3.2) and (3.3).*
2. *For all $z_0 \in \mathcal{Z}$, $\sigma \in L^1(0, T)$ and $\theta \in \mathcal{P}$, there exists a unique mild-solution $z \in C^0([0, T]; \mathcal{Z})$ of (3.2) and (3.3) given by Duhamel's formula*

$$z(t) = e^{tA}z_0 + \int_0^t e^{(t-s)A} \begin{pmatrix} 0 \\ \sigma(s) \end{pmatrix} \theta ds, \quad t \in [0, T]. \quad (3.4)$$

Moreover, the solution is a weak solution, namely $z \in L^p((0, T); \mathcal{Z})$, for all $v \in D(A^)$, $v, z(\cdot) \in W^{1,p}(0, T)$ and*

$$\begin{cases} \frac{d}{dt}(v, z(t)) = (A^*v, z(t)) + (B(t)\sigma, v), & t \in [0, T] \text{ a.e.}, \\ z(0) = z_0. \end{cases}$$

PROOF: See [Bensoussan et al., 2007, Part II, Chapter 1, Section 3, Propositions 3.1 and 3.2]. ■

Remark 3.2.2 *From Theorem 3.2.1, it follows naturally:*

1. For all $(u_0, v_0) \in \mathcal{D}(-\Delta_0) \times \mathbf{H}_0^1(\Omega)$, $\sigma \in \mathbf{H}^1([0, T])$ and $\theta \in \mathcal{P}$, there exists a unique strong solution u of (3.1) belonging to

$$\mathbf{C}^0([0, T]; \mathcal{D}(-\Delta_0)) \cap \mathbf{C}^1([0, T]; \mathbf{H}_0^1(\Omega)) \cap \mathbf{C}^2([0, T]; \mathbf{L}^2(\Omega)).$$

2. For all $(u_0, v_0) \in \mathbf{H}_0^1(\Omega) \times \mathbf{L}^2(\Omega)$, $\sigma \in \mathbf{L}^1(0, T)$ and $\theta \in \mathcal{P}$, there exists a unique mild-solution u of (3.1) belonging to

$$\mathbf{C}^0([0, T]; \mathbf{H}_0^1(\Omega)) \cap \mathbf{C}^1([0, T]; \mathbf{L}^2(\Omega)).$$

We introduce the so-called *sensitivity* operators, namely the one parameter family $(L(t))_{t \geq 0}$ of bounded operator, such that $t \mapsto L(t)\theta \in \mathbf{C}^0([0, T]; \mathcal{Z})$ is solution to

$$\begin{cases} \dot{z}(t) = Az(t) + B(t)\theta, & t \in [0, T] \\ z(0) = 0, \end{cases} \quad (3.5)$$

then, the Duhamel's formula (3.4) also reads $z(t) = \Phi(t)z_0 + L(t)\theta$, $t \in [0, T]$. For further use, we define L^1 which corresponds to L in the case of $\sigma \equiv 1$ namely the one parameter family $(L^1(t))_{t \geq 0}$ of bounded operator, such that $t \mapsto L^1(t)\theta \in \mathbf{C}^0([0, T]; \mathcal{Z})$ is solution to

$$\begin{cases} \dot{z}(t) = Az(t) + \begin{pmatrix} 0 \\ \theta \end{pmatrix}, & t \in [0, T] \\ z(0) = 0. \end{cases} \quad (3.6)$$

3.2.2 Observation operator

We now need to describe the assumption concerning the available observations. In this work, we suppose that we have at our disposal, at every time t , measurements during the period $(0, T)$ of the restriction of u in $\omega \subset \Omega$ an open and nonempty subset of Ω with Lipschitz boundary $\partial\omega$ and $\partial(\Omega \setminus \omega)$ also being Lipschitz. Moreover, we assume that $\omega \supseteq \omega_{gcc}$ where (ω_{gcc}, T) satisfies the following condition [Burq and Gérard, 2020]

$$\exists T, \forall \xi \in \Omega, \exists s \in [0, T], \exists \delta > 0; \omega_{gcc} \supset B(x_\xi(s); \delta), \quad (\text{GCC})$$

where $B(x_\xi(s); \delta)$ is a ball centered in $x_\xi(s)$ with a radius of δ and where a ray, starting from ξ and following the Descartes law of refraction in its most general senses [Bardos et al., 1988; Burq and Gérard, 2020], is parametrized by $\{x_\xi(s), s \in [0, T]\}$. Note that it is well known that such subdomain should contain a part of the boundary that we denote $\Gamma = \partial\Omega \cap \partial\omega$ with no cusps. Otherwise, some *whispering gallery* rays may prevent the GCC to be fulfilled.

Let us now present the definition of the observation operator C usually introduced in control and observation theory – see for instance [Tucsnak and Weiss, 2009] and references therein. We consider a restriction operator $I_\omega \in \mathcal{L}(\mathbf{H}_0^1(\Omega), \mathcal{Y})$, where \mathcal{Y} is the observation space to be defined. Then formally, and in the absence of noise, at any time $t \in [0, T]$, the measurement procedure could produce a measurement $y(t) = I_\omega u(t)$. If we assume that the wave equation solution $u(t) \in \mathbf{H}_0^1(\Omega)$ for all t , it is then natural to consider that $\mathcal{Y} = \mathbf{H}_\Gamma^1(\omega)$

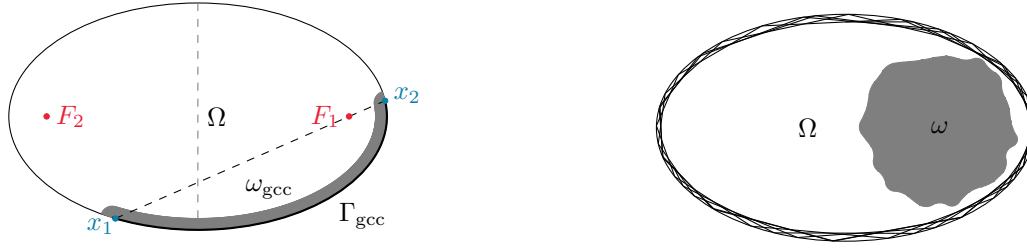


Figure 3.1: An interior subdomain of observation ω which does satisfy the GCC condition (left) or does not (right)

– the subspace of functions in $H^1(\omega)$ null on Γ . However, in practice, the measurements are always polluted by noise, hence in our deterministic context, the resulting measurement should belong to a less regular space, typically $\mathcal{Y} = L^2(\omega)$.

If we consider regular measurements namely the observation space is $\mathcal{Y} = H_{\Gamma}^1(\omega)$, it is rightful to introduce

$$C = \begin{pmatrix} I_{\omega} & 0 \end{pmatrix} \in \mathcal{L}(\mathcal{Z}, \mathcal{Y}), \text{ where } I_{\omega} \in \mathcal{L}(H_0^1(\Omega), H_{\Gamma}^1(\omega)). \quad (3.7)$$

As in [Chapelle et al., 2012a], we equipped this space with the norm $\|\cdot\|_{H_{\omega}^1} = \|E_{\omega}(\cdot)\|_{H_0^1(\Omega)}$, where $E_{\omega} \in \mathcal{L}(H_{\Gamma}^1(\omega), H_0^1(\Omega))$ is defined by

$$E_{\omega}\phi := \psi \text{ the solution of } \begin{cases} \Delta\psi = 0, & \text{in } \Omega \setminus \omega, \\ \psi = \phi, & \text{in } \bar{\omega}, \\ \psi = 0, & \text{on } \partial\Omega. \end{cases} \quad (3.8)$$

We recall the following lemma.

Lemma 3.2.3 *The norm $\|\cdot\|_{H_{\omega}^1}$ is equivalent to the norm $\|\cdot\|_{H^1(\omega)}$.*

PROOF: As in [Chapelle et al., 2012a], on the one hand, we show that E_{ω} is bounded from $H_{\Gamma}^1(\omega)$ to $H_0^1(\Omega)$. Using a trace property as well as a lifting property, we obtain

$$\begin{aligned} \|\nabla(E_{\omega}\phi)\|_{L^2(\Omega)}^2 &\leq \|\nabla\phi\|_{L^2(\omega)}^2 + C^{\text{st}}\|\phi|_{\partial\omega}\|_{H^{1/2}(\partial\omega)}^2 \\ &\leq C^{\text{st}}\|\phi\|_{H^1(\omega)}^2. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \|\phi\|_{H^1(\omega)}^2 &= \|\nabla(E_{\omega}\phi)\|_{L^2(\omega)}^2 + \|\phi\|_{L^2(\omega)}^2 \\ &\leq \|\nabla(E_{\omega}\phi)\|_{L^2(\Omega)}^2 + \|E_{\omega}\phi\|_{L^2(\Omega)}^2. \end{aligned}$$

Using Poincaré inequality, we get

$$\|\phi\|_{H^1(\omega)}^2 \leq (1 + C_p)\|\nabla(E_{\omega}\phi)\|_{L^2(\Omega)}^2.$$

■

For the sake of completeness and since it will be a central tool, for our results, we define the observation operator adjoint. One can show the following orthogonality property using (3.8)

$$\forall\phi \in H_{\omega}^1, \forall\psi \in H_0^1(\Omega) \text{ such that } \psi|_{\omega} = 0, (\nabla(E_{\omega}\phi), \nabla\psi)_{L^2(\Omega)} = 0.$$

Therefore, we obtain that for all $\phi \in \mathbf{H}_\omega^1$ and $\psi \in \mathbf{H}_0^1(\Omega)$,

$$\begin{aligned} (\nabla(E_\omega\phi), \nabla\psi)_{\mathbf{L}^2(\Omega)} &= (\nabla(E_\omega\phi), \nabla(E_\omega\psi|_\omega))_{\mathbf{L}^2(\Omega)} - (\nabla(E_\omega\phi), \nabla(E_\omega\psi|_\omega - \psi))_{\mathbf{L}^2(\Omega)} \\ &= (\nabla(E_\omega\phi), \nabla(E_\omega\psi|_\omega))_{\mathbf{L}^2(\Omega)} = (\phi, \psi|_\omega)_{\mathbf{H}_\omega^1}. \end{aligned}$$

Then, with the Riesz representation, the adjoint of E_ω is I_ω , and we can directly infer that

$$C^* = \begin{pmatrix} E_\omega & 0 \end{pmatrix}^t \in \mathcal{L}(\mathbf{H}_\Gamma^1(\omega); \mathcal{Z}).$$

We can see that the choice of norms for the observation operator is important, in particular it directly impacts the definition of its adjoint. With this choice of observation operator C , we restrict the space of observations to $\mathbf{H}_\Gamma^1(\omega)$. In practice, for noisy measurements, the noise could be less regular, *i.e.* it could belong to a larger space such as $\mathbf{L}^2(\omega)$. In this case, we define an observation operator where the image space is larger than in the previous case, namely $\mathbf{L}^2(\omega)$ instead of $\mathbf{H}_\Gamma^1(\omega)$. The observation operator becomes

$$C = \begin{pmatrix} I_\omega & 0 \end{pmatrix} : \mathcal{Z} \rightarrow \mathbf{L}^2(\omega) \text{ where } I_\omega \in \mathcal{L}(\mathbf{H}_0^1(\Omega), \mathbf{L}^2(\omega)). \quad (3.9)$$

Then, the observation operator adjoint is given by $C^* = \begin{pmatrix} F_\omega & 0 \end{pmatrix}^t$ where $F_\omega \in \mathcal{L}(\mathbf{L}^2(\omega), \mathbf{H}_0^1(\Omega))$ is defined by

$$F_\omega\phi := \psi \text{ the solution of } \begin{cases} -\Delta\psi = \chi_\omega\phi, & \text{in } \Omega, \\ \psi = 0, & \text{on } \partial\Omega, \end{cases} \quad (3.10)$$

where χ_ω is the characteristic function of ω .

3.2.3 Inverse problem setting

We can now formulate the inverse problems. We consider an actual trajectory \tilde{u} modeled as the first component of a mild solution \tilde{z} of (3.2) for a given and known (u_0, v_0) and $\sigma(t)$ but an unknown $\tilde{\theta}$. Since the initial data are known, we can assume that they are zero thanks to the linearity of the problem. Therefore, \tilde{u} is a particular solution of the system

$$\begin{cases} \partial_{tt}u(x, t) - \Delta u(x, t) = \sigma(t)\theta(x), & (x, t) \in \Omega \times (0, T), \\ u(x, t) = 0, & (x, t) \in \partial\Omega \times (0, T), \\ u(x, 0) = 0, & x \in \Omega, \\ \partial_t u(x, 0) = 0, & x \in \Omega, \end{cases} \quad (3.11)$$

and its first-order formulation

$$\begin{cases} \dot{z}(t) = Az(t) + B(t)\theta, & t \in [0, T], \\ z(0) = 0. \end{cases} \quad (3.12)$$

This actual trajectory is subject to measurements modeled by the bounded observation operator $C \in \mathcal{L}(\mathcal{Z}, \mathcal{Y})$. We denote the available measurements (or observations) y^δ which, as they are perturbed by some measurements' error – also known as the noise in our deterministic framework – are assumed to belong to $\mathbf{L}^2(0, T; \mathcal{Y})$ equipped with the standard \mathbf{L}^2 norm, hence $y^\delta(t)$ is defined for almost all $t \in (0, T)$. In our deterministic framework, the measurement *noise* intensity is represented by saying that there exists $\delta \in \mathbb{R}^+$ such that

$$\|y^\delta - Cz\|_{\mathbf{L}^2(0, T; \mathcal{Y})}^2 \leq \delta^2 T.$$

Therefore, estimating $\tilde{\theta} \in \mathcal{P} \subset L^2(\Omega)$ from y^δ consists in inverting the linear *input-output* operator

$$\Psi_T : \begin{cases} L^2(\Omega) \rightarrow L^2((0, T); L^2(\omega)), \\ \theta \mapsto (t \mapsto CL(t)\theta). \end{cases} \quad (3.13)$$

We immediately find that Ψ_T is bounded as C is bounded from \mathcal{Z} to \mathcal{Y} and $L(t)$ is bounded from \mathcal{P} to \mathcal{Z} for all $t \in [0, T]$. Our objective is to study the condition for which Ψ_T is injective and more importantly when Ψ_T is surjective. In fact, we are going to show that Ψ_T is not surjective in $L^2(0, T; L^2(\omega))$ but define a *mildly* ill-posed problem of degree 1 or 2 (see [Engl et al., 1996]) depending on the measurements space. We will then propose a regularization strategy to invert Ψ_T using observers' methods.

3.3 From observability to identifiability

In this first part, our objective is to quantify the degree of ill-posedness of Ψ_T . In particular, we prove that we have enough information in the measurements to reconstruct θ . A fundamental tool for measuring the degree of ill-posedness are observability inequalities. These inequalities are used to control parameters from observations. The most important elements are the norms that appear in this type of inequality which quantifies the ill-posedness [Natterer, 1984; Engl et al., 1996]. Indeed, suppose we want to reconstruct the source term in the Sobolev space $H^k(\Omega)$ from measurements available in $H^l(\omega)$, and prove the following observability inequality:

$$\int_0^T \|Cz\|_{H^m(\omega)}^2 dt \geq c \|\theta\|_{H^n(\Omega)}^2, \quad \text{with } n \leq k, l \leq m.$$

The problem is then said to be ill-posed of degree: $|k - n| + |m - l|$.

3.3.1 Observability conditions for the wave equation

For the homogeneous wave equation with Dirichlet conditions in the case of *field* measurements in a subdomain, we will first develop a so-called observability inequality.

In our framework, we present the proposition which allows us to express an observability condition with *field* measurements by relying on the geometric control condition. We adopt the observability condition shown in [Chapelle et al., 2012a], here revisited for observation domains that potentially contain a boundary part. In our case, the observation domain must contain part of the boundary.

Proposition 3.3.1 *Let u be an arbitrary solution of (3.1) with $\theta = 0$. Let $T_0 > 0$. Assume that there exists $T > T_0$ such that the GCC condition is satisfied for some ω_{gcc} strict subset of ω , in the time interval $(0, T)$. Then the following observability condition $\forall (u_0, v_0) \in H_0^1(\Omega) \times L^2(\Omega)$*

$$\int_0^T \|u(\cdot, t)\|_{H^1(\omega)}^2 dt \geq c_{obs}(T) \left(\|u_0\|_{H_0^1(\Omega)}^2 + \|v_0\|_{L^2(\Omega)}^2 \right) \quad (3.14)$$

holds.

PROOF: The proof extended the one of [Chapelle et al., 2012a]. In this proof, we write $a \lesssim b$ if there exists $c > 0$ independent of T and M such that $a \leq cb$. Since the geometric

control condition of [Bardos et al., 1988] holds, we have the classical observability inequality, with $c'_{\text{obs}}(T) > 0$

$$\int_0^T \int_{\omega} |\partial_t u(x, t)|^2 dx dt \geq c'_{\text{obs}}(T) \left(\|u_0\|_{H_0^1(\Omega)}^2 + \|v_0\|_{L^2(\Omega)}^2 \right). \quad (3.15)$$

We consider several subdomains satisfying $\omega_{gcc} \subseteq \omega_0 \subset \omega_{\epsilon} \subset \omega$. We define a cutoff function $\psi \in C^2(\bar{\Omega})$ satisfying

$$\psi(x) = \begin{cases} 0, & \text{if } x \in \Omega \setminus \omega_{\epsilon} \\ 1, & \text{if } x \in \omega_0, \end{cases}$$

and $0 \leq \psi(x) \leq 1$ for every $x \in \bar{\Omega}$. Particularly, we have

$$\frac{\partial \psi}{\partial n} \Big|_{\partial \omega \setminus \partial \Omega} = 0.$$

We introduce $\varphi(t) = t^2(T - t)^2$. Successively, integrating by parts, we obtain

$$\begin{aligned} 0 &= \int_0^T \int_{\omega} \varphi \psi (\partial_{tt} u - \Delta u) u \, dx \, dt \\ &= \int_0^T \int_{\omega} \ddot{\varphi} \psi \frac{|u|^2}{2} \, dx \, dt - \int_0^T \int_{\omega} \varphi \psi |\partial_t u|^2 \, dx \, dt \\ &\quad + \int_0^T \int_{\partial \omega} \varphi \frac{\partial \psi}{\partial n} \frac{|u|^2}{2} \, d\sigma \, dt - \int_0^T \int_{\omega} \varphi \Delta \psi \frac{|u|^2}{2} \, dx \, dt \\ &\quad + \int_0^T \int_{\omega} \varphi \psi |\nabla u|^2 \, dx \, dt. \end{aligned}$$

Using the definition of ψ , we get

$$\begin{aligned} \int_0^T \int_{\omega} \varphi \psi |\partial_t u|^2 \, dx \, dt + \int_0^T \int_{\omega} \varphi \Delta \psi \frac{|u|^2}{2} \, dx \, dt \\ = \int_0^T \int_{\omega} \varphi \psi |\nabla u|^2 \, dx \, dt + \int_0^T \int_{\omega} \ddot{\varphi} \psi \frac{|u|^2}{2} \, dx \, dt. \end{aligned}$$

We recall that for the system (3.1) with $\theta = 0$, we have classically the conservation energy property namely

$$\mathcal{E}(0) := \frac{1}{2} \|u_0\|_{H_0^1(\Omega)}^2 + \frac{1}{2} \|v_0\|_{L_0^2(\Omega)}^2 = \frac{1}{2} \|u(t)\|_{H_0^1(\Omega)}^2 + \frac{1}{2} \|\partial_t u(t)\|_{L^2(\Omega)}^2 = \mathcal{E}(t). \quad (3.16)$$

Therefore, for a time $T > T_0$ and ε such that $T - 2\varepsilon > T_0$, we find

$$\begin{aligned} \mathcal{E}(0) = \mathcal{E}(\varepsilon) &\lesssim (c'_{\text{obs}}(T))^{-1} \int_{\varepsilon}^{T-\varepsilon} \int_{\omega_0} |\partial_t u|^2 \, dx \, dt \\ &\lesssim (c'_{\text{obs}}(T))^{-1} \int_{\varepsilon}^{T-\varepsilon} \int_{\omega_0} [\varphi \psi |\partial_t u|^2 + |\nabla u|^2] \, dx \, dt \\ &\lesssim (c'_{\text{obs}}(T))^{-1} \int_0^T \int_{\omega} [|\nabla u|^2 + |u|^2] \, dx \, dt, \end{aligned}$$

hence justifying (3.14). ■

Proposition 3.3.2 *The observability constant $c_{\text{obs}}(T)$ of Proposition 3.3.1 satisfies, for T large enough*

$$c_{\text{obs}}(T) \geq cT,$$

where $c > 0$ is independent of T .

PROOF: Let $t > 0$. Applying Proposition 3.3.1 for a time $T_0 > 0$, we have

$$\int_t^{T_0+t} \|u(\cdot, t)\|_{\mathbf{H}^1(\omega)}^2 dt \geq 2c_{\text{obs}}(T_0)\mathcal{E}(t). \quad (3.17)$$

Let $T \in [NT_0; (N+1)T_0]$, $N \in \mathbb{N}^*$, we have

$$\int_0^T \|u(\cdot, t)\|_{\mathbf{H}^1(\omega)}^2 dt \geq \sum_{n=0}^{N-1} \int_{nT_0}^{(n+1)T_0} \|u(\cdot, t)\|_{\mathbf{H}^1(\omega)}^2 dt.$$

Using (3.17), we get

$$\int_0^T \|u(\cdot, t)\|_{\mathbf{H}^1(\omega)}^2 dt \geq \sum_{n=0}^{N-1} 2c_{\text{obs}}(T_0)\mathcal{E}(nT_0).$$

For all weak solution of (3.1) with $\theta = 0$, we have classically by energy conservation (3.16)

$$\int_0^T \|u(\cdot, t)\|_{\mathbf{H}^1(\omega)}^2 dt \geq 2NT_0 c_{\text{obs}}(T_0)\mathcal{E}(0) \geq 2(T - T_0) c_{\text{obs}}(T_0)\mathcal{E}(0). \quad \blacksquare$$

This proposition provides a rough estimate of the behavior in time T of the observation constant $c_{\text{obs}}(T)$. A more precise estimate can be obtained using micro-local techniques, as in [Humbert et al., 2019].

3.3.2 Regular field measurements in H^1

3.3.2.1 Identifiability with regular field measurements

From the observability condition adapted to initial condition reconstruction, we are now going to deduce the parameter identifiability of our source reconstruction problem. As a first step, we start by studying (3.12) when $\sigma \equiv 1$.

Theorem 3.3.3 *Let $\theta \in \mathcal{P}$, when ω satisfies the GCC condition, then for T large enough, there exists a constant $c_{\text{idf}}(T) \geq cT$ where $c > 0$ independent of T such that*

$$\int_0^T \|CL^1(t)\theta\|_{\mathbf{H}_\omega^1}^2 dt \geq c_{\text{idf}}(T)\|\theta\|_{\mathbf{H}^{-1}(\Omega)}^2, \quad (3.18)$$

where the observation operator is the observation operator in $\mathbf{H}^1(\omega)$, namely $C = (I_\omega \ 0) \in \mathcal{L}(\mathcal{Z}, \mathbf{H}_\Gamma^1(\omega))$ and L^1 is given by (3.6).

PROOF: The idea is to introduce the Riesz representation of θ seen as an element of \mathbf{H}^{-1} that we denote $g \in \mathbf{H}_0^1(\Omega) \cap \mathcal{D}(-\Delta_0)$ and is given by $g = \Delta_0^{-1}\theta$. We introduce the following dynamics which in fact is equivalent to (3.11) with $\sigma(t) \equiv 1$.

$$\begin{cases} \partial_{tt}u_g(x, t) - \Delta u_g(x, t) = -\Delta_0 g(x), & (x, t) \in \Omega \times (0, T), \\ u_g(x, t) = 0, & (x, t) \in \partial\Omega \times (0, T), \\ u_g(x, 0) = 0, & x \in \Omega, \\ \partial_t u_g(x, 0) = 0, & x \in \Omega. \end{cases} \quad (3.19)$$

Note that $u_g \in \mathcal{D}(\Delta_0)$ and $\partial_t u_g \in \mathbf{H}^1(\Omega)$. We now proceed by standard energy estimate using the following energy functional

$$\mathcal{E}_g(t) := \frac{1}{2} \|\nabla u_g(t)\|_{\mathbf{L}^2(\Omega)}^2 + \frac{1}{2} \|\partial_t u_g(t)\|_{\mathbf{L}^2(\Omega)}^2.$$

Integrating in space and time the first equation of (3.19) multiplied by $\partial_t u_g$, we get

$$\int_0^T \int_{\Omega} \partial_{tt} u_g \partial_t u_g dx dt - \int_0^T \int_{\Omega} \Delta u_g \partial_t u_g dx dt = - \int_0^T \int_{\Omega} \Delta_0 g \partial_t u_g dx dt.$$

Integrating by parts while using that g is time-independent, we obtain

$$\mathcal{E}_g(T) = - \int_0^T \int_{\Omega} \Delta_0 g \partial_t u_g dx dt = - \int_{\Omega} \Delta_0 g u_g(T) dx = \int_{\Omega} \nabla g \nabla u_g(T) dx.$$

By Cauchy–Schwarz inequality, we get

$$\mathcal{E}_g(T) \leq \|\nabla g\|_{\mathbf{L}^2(\Omega)} \|\nabla u_g(T)\|_{\mathbf{L}^2(\Omega)} \leq \sqrt{2} \|\nabla g\|_{\mathbf{L}^2(\Omega)} \mathcal{E}_g^{\frac{1}{2}}(T).$$

This calculation finally leads to

$$\|\partial_t u_g(T)\|_{\mathbf{L}^2(\Omega)} \leq 2 \|\nabla g\|_{\mathbf{L}^2(\Omega)}. \quad (3.20)$$

We now introduce the system satisfied by $\tilde{u}_g = u_g - g$ in order to obtain an initial value problem for which we can use the observability condition of Proposition 3.3.1.

$$\begin{cases} \partial_{tt} \tilde{u}_g(x, t) - \Delta \tilde{u}_g(x, t) = 0, & (x, t) \in \Omega \times (0, T), \\ \tilde{u}_g(x, t) = 0, & (x, t) \in \partial\Omega \times (0, T), \\ \tilde{u}_g(x, 0) = -g(x), & x \in \Omega, \\ \partial_t \tilde{u}_g(x, 0) = 0, & x \in \Omega. \end{cases} \quad (3.21)$$

Using the observability condition of Proposition 3.3.1, we get

$$\int_0^T \|u_g - g\|_{\mathbf{H}^1(\omega)}^2 dt = \int_0^T \|\tilde{u}_g\|_{\mathbf{H}^1(\omega)}^2 dt \geq C_{\text{obs}}(T) \|g\|_{\mathbf{H}_0^1(\Omega)}^2. \quad (3.22)$$

In order to get rid of the variable g in the left-hand side, we write that

$$\int_0^T \|u_g - g\|_{\mathbf{H}^1(\omega)}^2 dt = \int_0^T \|u_g\|_{\mathbf{H}^1(\omega)}^2 dt + \int_0^T \|g\|_{\mathbf{H}^1(\omega)}^2 dt - 2 \int_0^T (u_g, g)_{\mathbf{H}^1(\omega)} dt. \quad (3.23)$$

The cross product can be controlled. Integrating the first equation of (3.19) in time times Δ_0^{-1} and then multiplying by g in the sense of the scalar product, we have

$$\int_0^T (u_g, g)_{\mathbf{H}^1(\omega)} dt = T \|g\|_{\mathbf{H}^1(\omega)}^2 + (\Delta_0^{-1} \partial_t u_g(T), g)_{\mathbf{H}^1(\omega)}.$$

Injecting this result in (3.23), we get

$$\int_0^T \|u_g - g\|_{\mathbf{H}^1(\omega)}^2 dt = \int_0^T \|u_g\|_{\mathbf{H}^1(\omega)}^2 dt - T \|g\|_{\mathbf{H}^1(\omega)}^2 - 2(\Delta_0^{-1} \partial_t u_g(T), g)_{\mathbf{H}^1(\omega)}.$$

From Cauchy–Schwarz and Poincaré inequalities, we obtain

$$\begin{aligned} \int_0^T \|u_g - g\|_{\mathbf{H}^1(\omega)}^2 dt &\leq \int_0^T \|u_g\|_{\mathbf{H}^1(\omega)}^2 dt - T \|g\|_{\mathbf{H}^1(\omega)}^2 \\ &\quad + 2(C_p + 1) \|\nabla \Delta_0^{-1} \partial_t u_g(T)\|_{\mathbf{L}^2(\Omega)} \|g\|_{\mathbf{H}^1(\omega)}. \end{aligned} \quad (3.24)$$

Since $\nabla\Delta_0^{-1}$ is a bounded operator from $L^2(\Omega)$ to $L^2(\Omega)$, there exists a positive scalar C^{st} such that

$$\|\nabla\Delta_0^{-1}\partial_t u_g(T)\|_{L^2(\Omega)} \leq C^{\text{st}}\|\partial_t u_g(T)\|_{L^2(\Omega)}. \quad (3.25)$$

Injecting now (3.25) in (3.24), we have

$$\begin{aligned} \int_0^T \|u_g - g\|_{\mathbb{H}^1(\omega)}^2 dt &\leq \int_0^T \|u_g\|_{\mathbb{H}^1(\omega)}^2 dt - T\|g\|_{\mathbb{H}^1(\omega)}^2 \\ &\quad + 2C^{\text{st}}(C_p + 1)\|\partial_t u_g(T)\|_{L^2(\Omega)}\|g\|_{\mathbb{H}^1(\omega)}. \end{aligned}$$

Thanks to (3.20) and using Young inequality, we obtain

$$\begin{aligned} \int_0^T \|u_g - g\|_{\mathbb{H}^1(\omega)}^2 dt &\leq \int_0^T \|u_g\|_{\mathbb{H}^1(\omega)}^2 dt - T\|g\|_{\mathbb{H}^1(\omega)}^2 \\ &\quad + 2\epsilon C^{\text{st}}(C_p + 1)\|g\|_{\mathbb{H}_0^1(\Omega)}^2 + \frac{2}{\epsilon}C^{\text{st}}(C_p + 1)\|g\|_{\mathbb{H}^1(\omega)}^2. \end{aligned}$$

From the observability condition (3.22), we finally obtain

$$(C_{\text{obs}}(T) - 2\epsilon C^{\text{st}}(C_p + 1))\|g\|_{\mathbb{H}_0^1(\Omega)}^2 \leq \int_0^T \|u_g\|_{\mathbb{H}^1(\omega)}^2 dt - \left(T - \frac{2}{\epsilon}C^{\text{st}}(C_p + 1)\right)\|g\|_{\mathbb{H}^1(\omega)}^2.$$

Choosing $\epsilon = \frac{C_{\text{obs}}(T)}{4C^{\text{st}}(1 + C_p)}$ gives us

$$\frac{C_{\text{obs}}(T)}{2}\|g\|_{\mathbb{H}_0^1(\Omega)}^2 \leq \int_0^T \|u_g\|_{\mathbb{H}^1(\omega)}^2 dt - \left(T - \frac{8C^{\text{st}^2}(1 + C_p)^2}{C_{\text{obs}}(T)}\right)\|g\|_{\mathbb{H}^1(\omega)}^2.$$

For T large enough such that $T \geq \frac{8C^{\text{st}^2}(1 + C_p)^2}{C_{\text{obs}}(T)}$, we have

$$\frac{C_{\text{obs}}(T)}{2}\|g\|_{\mathbb{H}_0^1(\Omega)}^2 \leq \int_0^T \|u_g\|_{\mathbb{H}^1(\omega)}^2 dt.$$

We conclude by noting that, by definition, $\|g\|_{\mathbb{H}_0^1(\Omega)} = \|\theta\|_{\mathbb{H}^{-1}(\Omega)}$ and by Lemma 3.2.3, $\|u_g\|_{\mathbb{H}^1(\omega)} \leq C^{\text{st}}\|u_g\|_{\mathbb{H}_\omega^1}$. \blacksquare

From the identifiability property with $\sigma \equiv 1$, we deduce an identifiability property for the source reconstruction in (3.12) for a class of $\sigma \in \mathbb{H}^1(0, T)$.

Theorem 3.3.4 *Let $\theta \in \mathcal{P}$ and u be a solution of (3.11). Let $T_0 > 0$. Assuming $\sigma(t) \in \mathbb{H}^1(0, T)$ and $\sigma(0) \neq 0$ and (ω, T_0) satisfies the GCC condition, for $T \geq T_0$ large enough, then there exists a positive scalar $c'_{\text{idf}}(T)$ depending only on T such that*

$$\int_0^T \|CL\theta\|_{\mathbb{Y}}^2 dt = \int_0^T \|u\|_{\mathbb{H}_\omega^1}^2 dt \geq c'_{\text{idf}}(T)\|\theta\|_{\mathbb{H}^{-1}(\Omega)}^2, \quad (3.26)$$

where the observation operator is the observation operator in $\mathbb{H}_\Gamma^1(\omega)$, namely $C = (I_\omega \ 0) \in \mathcal{L}(\mathcal{Z}, \mathbb{H}_\Gamma^1(\omega))$.

PROOF: This proof is inspired by [Alves et al., 2009]. Given the scalar nature of σ , we use the first-order formulation of the system, so that observations of the solution read, for all $t \in [0, T]$

$$\begin{aligned} u_{|\omega}(t) &= \int_0^t C e^{(t-s)A} \sigma(s) \begin{pmatrix} 0 \\ \theta \end{pmatrix} ds = CL(t)\theta \\ &= \int_0^t \sigma'(t-s) \left(\int_0^s C e^{\tau A} \begin{pmatrix} 0 \\ \theta \end{pmatrix} d\tau \right) ds + \sigma(0) \left(\int_0^t C e^{\tau A} \begin{pmatrix} 0 \\ \theta \end{pmatrix} d\tau \right) \\ &= \int_0^t \sigma'(t-s) CL^1(s)\theta ds + \sigma(0) CL^1(t)\theta. \end{aligned}$$

We define a Volterra operator $S : L^2(0, T; \mathcal{Y}) \rightarrow L^2(0, T; \mathcal{Y})$ ([Gripenberg et al., 1990, Part 1, Chapter 2]),

$$Sy(t) = y(t) + \int_0^t k_T(t-s)y(s)ds, \text{ where } k_T(t) = \begin{cases} \frac{\sigma'(t)}{\sigma(0)} & t \in [0, T], \\ 0 & t > T. \end{cases} \quad (3.27)$$

Using this Volterra operator, we have

$$u_{|\omega}(t) = \sigma(0) S \circ CL^1(t)\theta, \quad t \in [0, T]. \quad (3.28)$$

We are going to show that S is an isomorphism from $L^2(0, T; \mathcal{Y})$ onto $L^2(0, T; \mathcal{Y})$. Since $k_T \in L^1(\mathbb{R}^+)$, using [Gripenberg et al., 1990, Part 1, Chapter 2, Theorem 3.1], there exists a unique $r_T \in L^1(0, T)$ such that

$$y(t) = Sy(t) - \int_0^t r_T(t-s)Sy(s)ds. \quad (3.29)$$

Consequently, we obtain

$$\|y\|_{L^2(0, T; \mathcal{Y})} \leq (1 + \|r_T\|_{L^1(0, T)}) \|Sy\|_{L^2(0, T; \mathcal{Y})}. \quad (3.30)$$

Combining (3.28) and (3.30) with $y = CL^1\theta$, we have

$$\|CL^1\theta\|_{L^2(0, T; \mathcal{Y})} \leq (1 + \|r_T\|_{L^1(0, T)}) \frac{1}{|\sigma(0)|} \|u_{|\omega}\|_{L^2(0, T; \mathcal{Y})}. \quad (3.31)$$

Combining the above equation with the observability condition of Proposition 3.3.3, we conclude the proof. \blacksquare

We deduce from the observability inequality (3.26) that the observations should belong to $H^1(\omega)$, for a stable reconstruction is only in a H^{-1} norm. As a consequence, we face an ill-posed inverse problem of degree 1 for an expected reconstruction in L^2 for measurements in H^1 ,

Remark 3.3.5 *In some cases, it is possible to affirm that the identifiability constant of Theorem 3.3.4 has the same dependence in T as that of the Theorem 3.3.3, for instance, when σ generates a persistent excitation. Let us take an example of such function*

$$\sigma(t) = \frac{1}{1 + \beta} (e^{-\alpha t} + \beta), \text{ where } \beta, \alpha \text{ are two positive constants.}$$

In fact, this corresponds to a small perturbation of the case where $\sigma(t) \equiv 1$.

Following the proof of Theorem 3.3.4 one has to show that $\|r_T\|_{L^1(0,T)}$ is independent of T where r_T is defined implicitly by σ (and k_T) in (3.29). On one hand, we have

$$\begin{aligned} \left\| \frac{\sigma'(t)}{\sigma(0)} \right\|_{L^1(0,T)} &= \|k_T\|_{L^1(0,T)} = \left\| -\frac{\alpha}{1+\beta} e^{-\alpha t} \right\|_{L^1(0,T)} \leq \frac{\alpha}{1+\beta} \int_0^T e^{-\alpha t} dt, \\ &\leq \frac{1}{1+\beta} (1 - e^{-\alpha T}) \leq \frac{1}{1+\beta}. \end{aligned}$$

From [Gripenberg et al., 1990, Part 1, Chapter 2], we have an explicit expression for r_T in terms of k_T ,

$$r_T = \sum_{j=1}^{\infty} (-1)^{j-1} (k_T)^{*j},$$

where $(k_T)^{*j}$ is the $(j-1)$ -fold convolution of k_T with itself. Hence,

$$\|r_T\|_{L^1(0,T)} \leq \sum_{j=1}^{\infty} \|k_T\|_{L^1(0,T)}^j.$$

The common ratio $\|k_T\|_{L^1(0,+\infty)}$ is smaller than one, hence,

$$\|r_T\|_{L^1(0,T)} \leq \left(1 - \|k_T\|_{L^1(0,T)}\right)^{-1} \leq \frac{1}{1 - \frac{1}{1+\beta}} = \frac{1+\beta}{\beta},$$

which does not depend on time T .

As we have seen in the previous proof, the dependency of the identifiability constant with respect time is not always explicit. When we consider the simpler case $\sigma(t) \equiv 1$, we have a constant $C^{\text{st}}(T) \approx T$ as well as when σ is defined as follows:

$$\forall T > 0, \sigma = \bar{\sigma}|_{[0,T]} \text{ with } \bar{\sigma} \in H^1(\mathbb{R}^+) \text{ and } \left\| \frac{\bar{\sigma}'}{\sigma(0)} \right\|_{L^1(\mathbb{R}^+)} \leq 1,$$

as illustrated in the previous example in Remark 3.3.5. One can compute that any support function will not satisfy this condition. Later, we use the following assumption when explicitly mentioned.

Assumption 3 The identifiability constant of Theorem (3.3.4) satisfies

$$c'_{\text{idf}}(T) \geq cT, \text{ with } c > 0 \text{ independent of } T.$$

3.3.2.2 From the identifiability result to a generalized Tikhonov regularization strategy

We recall that we assume that a target trajectory \tilde{u} produced by a target source term parameter $\tilde{\theta}$ has resulted in a set of measurements y^δ where

$$\int_0^T \|y^\delta(x, t) - \tilde{u}(x, t)\|_{H^1(\omega)}^2 dt \leq \delta^2 T, \quad (3.32)$$

for a parameter $\delta > 0$. We introduce a cost functional to minimize $\mathcal{J}_T : \mathbf{H}_0^1(\Omega) \rightarrow \mathbb{R}$ defined by

$$\begin{aligned} \mathcal{J}_T(\theta) &= \frac{\epsilon}{2} \|\theta\|_{\mathbf{H}_0^1(\Omega)}^2 + \frac{1}{2} \int_0^T \|y^\delta(t) - u_{|\theta}\|_{\mathbf{H}^1(\omega)}^2 dt \\ &= \frac{\epsilon}{2} \|\theta\|_{\mathbf{H}_0^1(\Omega)}^2 + \frac{1}{2} \int_0^T \|y^\delta(t) - Cz_{|\theta}\|_{\mathbf{H}^1(\omega)}^2 dt, \end{aligned} \quad (3.33)$$

where $u_{|\theta}$ is the solution of (3.11) associated with the parameter θ , C is defined by (3.7) and $\epsilon > 0$. Since the functional is quadratic and the application $\theta \mapsto u_{|\omega}$ is continuous and $\theta \mapsto \|\theta\|_{\mathbf{H}_0^1(\Omega)}^2$ is coercive, we have then the existence of a unique minimum of \mathcal{J}_T . We denote the unique minimizer in $\mathbf{H}_0^1(\Omega)$ by $\bar{\theta}_T := \operatorname{argmin}_{\theta \in \mathbf{H}_0^1(\Omega)} \mathcal{J}_T$.

Theorem 3.3.6 *We assume that there exists $M > 0$ such that*

$$\check{\theta} \in \mathbf{H}_0^1(\Omega) \text{ and } \|\check{\theta}\|_{\mathbf{H}_0^1(\Omega)} \leq M. \quad (3.34)$$

Furthermore, we also assume that the available measurements are noise-free, i.e. we have for all $t \in [0, T]$

$$y^\delta(t) = \check{u}_{|\omega}(t).$$

Let $T_0 > 0$. Assuming $\sigma(t) \in \mathbf{H}^1(0, T)$ and $\sigma(0) \neq 0$ and (ω, T_0) satisfies the GCC condition, for $T \geq T_0$ large enough and ϵ small enough, then there exists a positive scalar $C^{st}(T)$ depending only on T such that

$$\|\bar{\theta}_T - \check{\theta}\|_{\mathbf{L}^2(\Omega)} \leq C^{st}(T) \epsilon^{\frac{1}{4}} M, \quad (3.35)$$

where $\bar{\theta}_T$ minimizes the cost functional (3.33).

Moreover, under Assumption 3, there exists a positive scalar C^{st} independent of T such that

$$\|\bar{\theta}_T - \check{\theta}\|_{\mathbf{L}^2(\Omega)} \leq C^{st} T^{-\frac{1}{4}} \epsilon^{\frac{1}{4}} M. \quad (3.36)$$

PROOF: In this proof, we write $a \lesssim b$ if there exists $c > 0$ independent of T, ϵ and M such that $a \leq cb$.

Let $\tilde{u} = \bar{u} - \check{u}$ be the solution of the following dynamics

$$\begin{cases} \partial_{tt}\tilde{u}(x, t) - \Delta\tilde{u}(x, t) = \sigma(t)(\bar{\theta}_T(x) - \check{\theta}(x)), & (x, t) \in \Omega \times (0, T), \\ \tilde{u}(x, t) = 0, & (x, t) \in \partial\Omega \times (0, T), \\ \tilde{u}(x, 0) = 0, & x \in \Omega, \\ \partial_t\tilde{u}(x, 0) = 0, & x \in \Omega. \end{cases} \quad (3.37)$$

Identifying \mathbf{L}^2 to its dual, we have

$$\|\bar{\theta}_T - \check{\theta}\|_{\mathbf{L}^2(\Omega)}^2 \leq \langle \bar{\theta}_T - \check{\theta}, \bar{\theta}_T - \check{\theta} \rangle_{\mathbf{H}_0^1(\Omega), \mathbf{H}^{-1}(\Omega)} \leq \|\bar{\theta}_T - \check{\theta}\|_{\mathbf{H}_0^1(\Omega)} \|\bar{\theta}_T - \check{\theta}\|_{\mathbf{H}^{-1}(\Omega)}.$$

Using the identifiability property (3.26) of Theorem 3.3.4, we get

$$\|\bar{\theta}_T - \check{\theta}\|_{\mathbf{L}^2(\Omega)}^2 \leq (c'_{\text{idf}}(T))^{-\frac{1}{2}} \|\bar{\theta}_T - \check{\theta}\|_{\mathbf{H}_0^1(\Omega)} \left(\int_0^T \|\bar{u} - \check{u}\|_{\mathbf{H}_\omega^1}^2 dt \right)^{\frac{1}{2}}. \quad (3.38)$$

By triangle inequality and since we consider free-noise observations, we have

$$\|\bar{\theta}_T - \check{\theta}\|_{\mathbf{L}^2(\Omega)}^2 \leq (c'_{\text{idf}}(T))^{-\frac{1}{2}} \left(\|\bar{\theta}_T\|_{\mathbf{H}_0^1(\Omega)} + \|\check{\theta}\|_{\mathbf{H}_0^1(\Omega)} \right) \left(\int_0^T \|\bar{u} - y^\delta\|_{\mathbf{H}_\omega^1}^2 dt \right)^{\frac{1}{2}}.$$

Using the definition of the cost-functional (3.33) taking into account the norm equivalence of Lemma 3.2.3, and the prior on the parameter (3.34), we get

$$\|\bar{\theta}_T - \check{\theta}\|_{L^2(\Omega)}^2 \lesssim (c'_{\text{idf}}(T))^{-\frac{1}{2}} \left((2\epsilon^{-1} \mathcal{J}_T(\bar{\theta}_T))^{\frac{1}{2}} + M \right) (2\mathcal{J}_T(\bar{\theta}_T))^{\frac{1}{2}}.$$

Remembering that $y^\delta(t) = \check{u}_\omega(t)$ and using again (3.34), we can bound the cost-functional as follows

$$\mathcal{J}_T(\bar{\theta}_T) \leq \mathcal{J}_T(\check{\theta}) \leq \frac{\epsilon}{2} M^2.$$

Eventually, we obtain

$$\|\bar{\theta}_T - \check{\theta}\|_{L^2(\Omega)}^2 \lesssim (c'_{\text{idf}}(T))^{-\frac{1}{2}} \left((M^2)^{\frac{1}{2}} + M \right) (\epsilon M^2)^{\frac{1}{2}},$$

which corresponds to the desired result (3.35).

In addition, under Assumption 3, we have that there exists $c >$ independent of T such that

$$c'_{\text{idf}}(T) \geq cT,$$

which leads to (3.36). ■

Theorem 3.3.7 *We assume that there exists $M > 0$ such that (3.34) holds. Let $T_0 > 0$. Assuming $\sigma(t) \in H^1(0, T)$ and $\sigma(0) \neq 0$ and (ω, T_0) satisfies the GCC condition, for $T \geq T_0$ large enough and ϵ small enough, then there exists a positive scalar $C^{st}(T)$ depending only on T such that*

$$\|\bar{\theta}_T - \check{\theta}\|_{L^2(\Omega)} \leq C^{st}(T) \epsilon^{-\frac{1}{4}} \delta, \quad (3.39)$$

where $\bar{\theta}_T$ minimizes the cost functional (3.33).

Moreover, under Assumption 3, there exists a positive scalar C^{st} independent of T such that

$$\|\bar{\theta}_T - \check{\theta}\|_{L^2(\Omega)} \leq C^{st} T^{\frac{1}{4}} \epsilon^{-\frac{1}{4}} \delta. \quad (3.40)$$

PROOF: In this proof, we write $a \lesssim b$ if there exists $c > 0$ independent of T, ϵ, δ and M such that $a \leq cb$.

Reasoning exactly as in the proof of Theorem 3.3.6, we can obtain (3.38). Inserting $y^\delta - y^\delta = 0$ in (3.38) and by triangular inequality we find

$$\begin{aligned} \|\bar{\theta}_T - \check{\theta}\|_{L^2(\Omega)}^2 &\lesssim (c'_{\text{idf}}(T))^{-\frac{1}{2}} \left(\|\check{\theta}\|_{H_0^1(\Omega)} + \|\bar{\theta}_T\|_{H_0^1(\Omega)} \right) \\ &\quad \times \left[\left(\int_0^T \|u_{|\bar{\theta}_T} - y^\delta\|_{H_\omega^1}^2 dt \right)^{\frac{1}{2}} + \left(\int_0^T \|\check{u} - y^\delta\|_{H_\omega^1}^2 dt \right)^{\frac{1}{2}} \right]. \end{aligned}$$

Using Lemma 3.2.3, the estimate on the noise amplitude (3.32) and the prior on the parameter (3.34), we find

$$\|\bar{\theta}_T - \check{\theta}\|_{L^2(\Omega)}^2 \lesssim (c'_{\text{idf}}(T))^{-\frac{1}{2}} \left(M + \sqrt{2\epsilon^{-1} \mathcal{J}_T(\bar{\theta}_T)} \right) \left(\sqrt{2\mathcal{J}_T(\bar{\theta}_T)} + \delta\sqrt{T} \right).$$

Using again (3.32) and (3.34), we can bound the cost functional as follows

$$\mathcal{J}_T(\bar{\theta}_T) \leq \mathcal{J}_T(\check{\theta}) \leq \frac{\epsilon}{2} M^2 + \frac{1}{2} \delta^2 T.$$

Hence, we finally get

$$\|\bar{\theta}_T - \check{\theta}\|_{L^2(\Omega)}^2 \lesssim (c'_{\text{idf}}(T))^{-\frac{1}{2}} \left(M + \sqrt{M^2 + \epsilon^{-1}\delta^2 T} \right) \left(\sqrt{\epsilon M^2 + \delta^2 T} + \delta\sqrt{T} \right), \quad (3.41)$$

which implies (3.39).

Furthermore, under Assumption 3, we have that there exists $c > 0$ independent of T such that

$$c'_{\text{idf}}(T) \geq cT,$$

which proves (3.40). \blacksquare

Finally, choosing $\epsilon = \delta^2 M^{-2}$ in (3.41), we obtain the following corollary

Corollary 3.3.7.1 *Under the hypothesis of Theorem 3.3.7, for $T \geq T_0$ large enough, there exists a positive scalar $C^{st}(T)$ depending only on T such that*

$$\|\bar{\theta}_T - \check{\theta}\|_{L^2(\Omega)} \leq C^{st}(T) \sqrt{M} \sqrt{\delta}, \quad (3.42)$$

where $\bar{\theta}_T$ minimizes the cost functional (3.33) with $\epsilon = \delta^2 M^{-2}$.

Moreover, under Assumption 3, there exists a positive scalar C^{st} independent of T such that

$$\|\bar{\theta}_T - \check{\theta}\|_{L^2(\Omega)} \leq C^{st} T^{\frac{1}{4}} \sqrt{M} \sqrt{\delta}. \quad (3.43)$$

3.3.3 Field measurements with L^2 -noise

In the rest of this work, we assume – in order to model a more realistic case in practice – that the measurements are less regular. Namely, we suppose that the target trajectory \check{u} generated by the target source term parameter $\check{\theta}$ has given rise to a collection of measurements y^δ such that

$$\int_0^T \|y^\delta(x, t) - \check{u}(x, t)\|_{L^2(\omega)}^2 dt \leq \delta^2 T, \quad (3.44)$$

for a parameter $\delta > 0$.

We present a method to deal with this additional difficulty of regularity loss. This approach consists in modifying the definition of the observation operator with second observation operator C introduced in (3.9).

3.3.3.1 Identifiability from observability

The proof of the desired identifiability result for the identification estimation problem is based on the following initial condition observability inequality – for a homogeneous wave equation in weakened norms.

Proposition 3.3.8 *Let u be an arbitrary solution of (3.1) with $\theta = 0$. Assume that ω satisfies the GCC condition, then there exists $cT \leq c_{\text{obs}}(T)$ with $c > 0$ independent of T such that*

$$\int_0^T \|u(\cdot, t)\|_{L^2(\omega)}^2 dt \geq c_{\text{obs}}(T) \left(\|u_0\|_{L^2(\Omega)}^2 + \|v_0\|_{H^{-1}(\Omega)}^2 \right). \quad (3.45)$$

For the sake of completeness, we here recall the link between (3.15) and (3.45).

PROOF: Since the geometric control condition of [Bardos et al., 1988] holds, we have the classical observability inequality (3.15). Reasoning as in the proof of Proposition

3.3.2, one can show that $cT \leq c'_{\text{obs}}(T)$ where $c > 0$ independent of T . We introduce $w(t) = \int_0^t u(s)ds + \chi$ where χ is solution of the following system

$$\begin{cases} -\Delta\chi = v_0, & \text{in } \Omega, \\ \chi = 0, & \text{on } \partial\Omega. \end{cases}$$

The variable w satisfies the following equation

$$\begin{aligned} \partial_{tt}w - \Delta w &= \partial_t u - \int_0^t \Delta u ds - \Delta\chi \\ &= \partial_t u - \int_0^t \Delta u ds - v_0. \end{aligned}$$

Integrating in time the first equation of (3.1), we find that

$$\partial_{tt}w - \Delta w = 0.$$

We can then apply the classical observability condition (3.15) on w which is the solution of the following dynamics

$$\begin{cases} \partial_{tt}w(x, t) - \Delta w(x, t) = 0, & (x, t) \in \Omega \times (0, T), \\ w(x, t) = 0, & (x, t) \in \partial\omega \times (0, T), \\ w(x, 0) = \chi(x), & x \in \Omega, \\ \partial_t w(x, 0) = u_0(x), & x \in \Omega, \end{cases}$$

which gives us

$$\int_0^T \int_{\omega} |\partial_t w(x, t)|^2 dx dt \geq c'_{\text{obs}}(T) \left(\|\chi\|_{H_0^1(\Omega)}^2 + \|u_0\|_{L^2(\Omega)}^2 \right).$$

Since the Laplace operator with Dirichlet boundary conditions $-\Delta_0$ defines an isomorphism from H_0^1 to H^{-1} , the result follows immediately. \blacksquare

Theorem 3.3.9 *Let $\theta \in \mathcal{P}$. Let u be a solution of (3.11). Assuming $\sigma(t) \in H^1(0, T)$ and $\sigma(0) \neq 0$, and that ω satisfies the GCC condition, for T large enough, there exists $c'_{\text{idf}}(T) > 0$, such that*

$$\|CL\theta\|_{L^2(0, T; L^2(\omega))}^2 = \|u\|_{L^2(0, T; L^2(\omega))}^2 \geq c'_{\text{idf}}(T) \|\Delta_0^{-1}\theta\|_{L^2(\Omega)}^2.$$

PROOF: One can show, as for Theorem 3.3.4, that we have the inequality

$$\|u|_{\omega}\|_{L^2(0, T; L^2(\omega))} \geq C^{\text{st}}(T) \|CL^1\theta\|_{L^2(0, T; L^2(\omega))}. \quad (3.46)$$

As in the proof of Theorem 3.3.3, we introduce $g \in H_0^1(\Omega) \cap \mathcal{D}(-\Delta_0)$ the Riesz representation of θ (seen as an element of H^{-1}) given by $g = -\Delta_0^{-1}\theta$, and u_g solution of the dynamics (3.19). The difference $\tilde{u}_g = u_g - g$ satisfies the system (3.21) in the same way. We apply the observability condition of Proposition 3.3.8 on this system, and we obtain

$$\|\tilde{u}_g\|_{L^2(0, T; L^2(\omega))}^2 = \|u_g - g\|_{L^2(0, T; L^2(\omega))}^2 \geq c_{\text{obs}}(T) \|g\|_{L^2(\Omega)}^2. \quad (3.47)$$

Now we want to get rid of the variable g in the left-hand of (3.47). That is why, we develop the following scalar product

$$\int_0^T \|u_g - g\|_{L^2(\omega)}^2 dt = \int_0^T \|u_g\|_{L^2(\omega)}^2 dt + T\|g\|_{L^2(\omega)}^2 - 2 \int_0^T (u_g, g)_{L^2(\omega)} dt. \quad (3.48)$$

Applying Δ_0^{-1} to the first equation of (3.19), then integrate in time and finally the resulting equation given by the scalar product with g is

$$\int_0^T (u_g, g)_{L^2(\omega)} dt = (\Delta_0^{-1} \partial_t u_g(T), g)_{L^2(\omega)} + T\|g\|_{L^2(\omega)}^2.$$

Injecting this result in (3.48) and using Cauchy-Schwarz and Poincaré inequalities, we obtain

$$\begin{aligned} \int_0^T \|u_g - g\|_{L^2(\omega)}^2 dt &\leq \int_0^T \|u_g\|_{L^2(\omega)}^2 dt - T\|g\|_{L^2(\omega)}^2 \\ &\quad + 2C_p \|\nabla \Delta_0^{-1} \partial_t u_g(T)\|_{L^2(\Omega)} \|g\|_{L^2(\omega)}. \end{aligned} \quad (3.49)$$

We expect to bound $\|\nabla \Delta_0^{-1} \partial_t u_g(T)\|_{L^2(\Omega)}$ by $\|g\|_{L^2(\Omega)}$. Therefore, we apply Δ_0^{-1} to the first equation of (3.19). Integrating in time, the scalar product with $-\partial_t u_g$ leads to

$$-\int_{\Omega} (\Delta_0^{-1} \partial_{tt} u_g) \partial_t u_g dx + \int_{\Omega} u_g \partial_t u_g dx = \int_{\Omega} g \partial_t u_g dx.$$

Inserting $\Delta_0 \Delta_0^{-1} = \text{Id}$ in the first term we get

$$-\int_{\Omega} (\Delta_0^{-1} \partial_{tt} u_g) \Delta_0 \Delta_0^{-1} \partial_t u_g dx + \frac{1}{2} \frac{d}{dt} \int_{\Omega} u_g^2 dx = \int_{\Omega} g \partial_t u_g dx.$$

Integrating by parts, we obtain

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} (\nabla \Delta_0^{-1} \partial_t u_g)^2 dx + \frac{1}{2} \frac{d}{dt} \int_{\Omega} u_g^2 dx = \int_{\Omega} g \partial_t u_g dx.$$

Integrating in time and using Cauchy-Schwarz and Young inequalities, we get

$$\begin{aligned} \frac{1}{2} \|\nabla \Delta_0^{-1} \partial_t u_g(T)\|_{L^2(\Omega)}^2 + \frac{1}{2} \|u_g(T)\|_{L^2(\Omega)}^2 &= \int_{\Omega} g u_g(T) dx \\ &\leq \frac{1}{2} \|g\|_{L^2(\Omega)}^2 + \frac{1}{2} \|u_g(T)\|_{L^2(\Omega)}^2, \end{aligned}$$

thus,

$$\|\nabla \Delta_0^{-1} \partial_t u_g(T)\|_{L^2(\Omega)}^2 \leq \|g\|_{L^2(\Omega)}^2.$$

Injecting this result in (3.49), we have

$$\int_0^T \|u_g - g\|_{L^2(\omega)}^2 dt \leq \int_0^T \|u_g\|_{L^2(\omega)}^2 dt - T\|g\|_{L^2(\omega)}^2 + 2C_p \|g\|_{L^2(\Omega)} \|g\|_{L^2(\omega)}.$$

Using Young inequality, we finally have

$$\int_0^T \|u_g - g\|_{L^2(\omega)}^2 dt \leq \int_0^T \|u_g\|_{L^2(\omega)}^2 dt - T\|g\|_{L^2(\omega)}^2 + \epsilon C_p \|g\|_{L^2(\Omega)}^2 + \frac{C_p}{\epsilon} \|g\|_{L^2(\omega)}^2. \quad (3.50)$$

Gathering (3.47) and (3.50), we get

$$\int_0^T \|u_g\|_{L^2(\omega)}^2 dt \geq \left(T - \frac{C_p}{\epsilon}\right) \|g\|_{L^2(\omega)}^2 + (c_{\text{obs}}(T) - \epsilon C_p) \|g\|_{L^2(\Omega)}^2.$$

We choose $\epsilon = \frac{c_{\text{obs}}}{2C_p}$, then we have

$$\int_0^T \|u_g\|_{L^2(\omega)}^2 dt \geq \left(T - \frac{2C_p^2}{c_{\text{obs}}(T)}\right) \|g\|_{L^2(\omega)}^2 + \frac{c_{\text{obs}}(T)}{2} \|g\|_{L^2(\Omega)}^2.$$

For $T > \frac{2C_p^2}{c_{\text{obs}}(T)}$, and setting $-\Delta_0 g = \theta$, we obtain

$$\|CL^1\theta\|_{L^2((0,T);L^2(\omega))} = \int_0^T \|u_g\|_{L^2(\omega)}^2 dt \geq \frac{c_{\text{obs}}(T)}{2} \|\Delta_0^{-1}\theta\|_{L^2(\Omega)}^2. \quad (3.51)$$

Gathering (3.46) and (3.51), we conclude the proof. \blacksquare

As in the previous section, we assume that the dependence of the observability constant is in $O(T)$.

Assumption 4 *The identifiability constant of Theorem (3.3.9) satisfies*

$$c'_{\text{idf}}(T) \geq cT, \text{ with } c > 0 \text{ independent of } T.$$

3.3.3.2 Updated convergence rate for the Tikhonov regularization

We infer from Theorem 3.3.9 that we now face an ill-posed problem of degree 2. This will modify the convergence rate of our Tikhonov regularization without further assumption on the prior regularity. Let us first recall the following classical interpolation lemma.

Lemma 3.3.10 *Let (μ_i, e_i) be the orthonormal basis for the L^2 scalar product such that the Laplacian eigenvalues $\mu_i > 0$ satisfies $\mu_{i+1} > \mu_i$, and $\mu_i \rightarrow +\infty$. Let $\theta \in \mathbf{H}_0^1(\Omega)$, we have the interpolation property*

$$\|\theta\|_{L^2(\Omega)}^2 \leq \left(\|\Delta_0^{-1}\theta\|_{L^2(\Omega)}^2\right)^{\frac{1}{3}} \left(\|\theta\|_{\mathbf{H}_0^1(\Omega)}^2\right)^{\frac{2}{3}},$$

where $\|\Delta_0^{-1}\theta\|_{L^2(\Omega)} = \sqrt{\sum_i |(e_i, \theta)|^2 \mu_i^{-2}}$.

We recall here the proof for the sake of completeness.

PROOF: We express the L^2 -norm of θ in the orthonormal basis.

$$\|\theta\|_{L^2(\Omega)}^2 = \sum_{i=0}^{+\infty} |(e_i, \theta)|^2 = \sum_{i=1}^{+\infty} |(e_i, \theta)|^2 \mu_i^{2q \frac{s}{s-q}} \mu_i^{2s \frac{-q}{s-q}}.$$

Using Hölder inequality, we have

$$\|\theta\|_{L^2(\Omega)}^2 \leq \left(\sum_i |(e_i, \theta)|^2 \mu_i^{2q}\right)^{\frac{s}{s-q}} \left(\sum_i |(e_i, \theta)|^2 \mu_i^{2s}\right)^{\frac{-q}{s-q}}.$$

Choosing $s = \frac{1}{2}$ and $q = -1$, we obtain

$$\|\theta\|_{L^2(\Omega)}^2 \leq \left(\sum_i |(e_i, \theta)|^2 \mu_i^{-2} \right)^{\frac{1}{3}} \left(\sum_i |(e_i, \theta)|^2 \mu_i \right)^{\frac{2}{3}} = \left(\|\Delta_0^{-1} \theta\|_{L^2(\Omega)}^2 \right)^{\frac{1}{3}} \left(\|\theta\|_{H_0^1(\Omega)}^2 \right)^{\frac{2}{3}}.$$

■

Keeping the same regularization in our criterion, we introduce the cost functional to minimize

$$\mathcal{J}_T(\theta) = \frac{\epsilon}{2} \|\theta\|_{H_0^1(\Omega)}^2 + \frac{1}{2} \|y^\delta - Cz_\theta\|_{L^2(0,T;L^2(\omega))}^2, \quad (3.52)$$

where C is defined as operator in L^2 , namely $C = (I_\omega \ 0) \in \mathcal{L}(\mathcal{Z} \rightarrow L^2(\omega))$. We can then estimate the reconstruction error between $\bar{\theta}_T$ the functional minimizer and the target parameter $\check{\theta}$.

Theorem 3.3.11 *We assume that there exists $M > 0$ such that (3.34) holds.*

Furthermore, we also assume that the available measurements are noise-free, i.e. we have for all $t \in [0, T]$

$$y^\delta(t) = \check{u}_\omega(t).$$

Let $T_0 > 0$. Assuming $\sigma(t) \in H^1(0, T)$ and $\sigma(0) \neq 0$ and (ω, T_0) satisfies the GCC condition, for $T \geq T_0$ large enough and ϵ small enough, then there exists a positive scalar $C^{st}(T)$ depending only on T such that

$$\|\bar{\theta}_T - \check{\theta}\|_{L^2(\Omega)} \leq C^{st}(T) \epsilon^{\frac{1}{6}} M, \quad (3.53)$$

where $\bar{\theta}_T$ minimizes the cost functional (3.52).

Moreover, under Assumption 4, there exists a positive scalar C^{st} independent of T such that

$$\|\bar{\theta}_T - \check{\theta}\|_{L^2(\Omega)} \leq C^{st} T^{-\frac{1}{6}} \epsilon^{\frac{1}{6}} M. \quad (3.54)$$

PROOF: In this proof, we write $a \lesssim b$ if there exists $c > 0$ independent of T, ϵ and M such that $a \leq cb$.

We use the interpolation inequality of Lemma 3.3.10 to obtain

$$\|\bar{\theta}_T - \check{\theta}\|_{L^2(\Omega)}^2 \leq \left(\|\bar{\theta}_T - \check{\theta}\|_{H_0^1(\Omega)}^2 \right)^{\frac{2}{3}} \left(\|\Delta_0^{-1}(\bar{\theta}_T - \check{\theta})\|_{L^2(\Omega)}^2 \right)^{\frac{1}{3}}.$$

Using the identifiability result given by Theorem 3.3.9, we have

$$\|\bar{\theta}_T - \check{\theta}\|_{L^2(\Omega)}^2 \leq (c'_{\text{idf}}(T))^{-\frac{1}{3}} \left(\|\bar{\theta}_T\|_{H_0^1(\Omega)}^2 + \|\check{\theta}\|_{H_0^1(\Omega)}^2 \right)^{\frac{2}{3}} \left(\int_0^T \|u_{|\bar{\theta}_T} - \check{u}\|_{L^2(\omega)}^2 dt \right)^{\frac{1}{3}}. \quad (3.55)$$

Since we consider noise-free observations, we have $y^\delta = \check{u}_\omega$. Therefore, we obtain

$$\|\bar{\theta}_T - \check{\theta}\|_{L^2(\Omega)}^2 \lesssim (c'_{\text{idf}}(T))^{-\frac{1}{3}} \left(\|\bar{\theta}_T\|_{H_0^1(\Omega)}^2 + \|\check{\theta}\|_{H_0^1(\Omega)}^2 \right)^{\frac{2}{3}} \left(\int_0^T \|u_{|\bar{\theta}_T} - y^\delta\|_{L^2(\omega)}^2 dt \right)^{\frac{1}{3}}.$$

Using the definition of the cost-functional (3.52) and our prior on the parameter (3.34), we obtain

$$\|\bar{\theta}_T - \check{\theta}\|_{L^2(\Omega)}^2 \lesssim (c'_{\text{idf}}(T))^{-\frac{1}{3}} \left(2\epsilon^{-1} \mathcal{J}_T(\bar{\theta}_T) + M^2 \right)^{\frac{2}{3}} \left(2\mathcal{J}_T(\bar{\theta}_T) \right)^{\frac{1}{3}}.$$

We can bound the cost-functional as follows

$$\mathcal{J}_T(\bar{\theta}_T) \leq \mathcal{J}_T(\check{\theta}) \leq \frac{\epsilon}{2} M^2.$$

Finally, we have

$$\|\bar{\theta}_T - \check{\theta}\|_{L^2(\Omega)}^2 \lesssim (c'_{\text{idf}}(T))^{-\frac{1}{3}} (M^2 + M^2)^{\frac{2}{3}} (\epsilon M^2)^{\frac{1}{3}},$$

which leads to (3.53). Furthermore, under Assumption 4, we have that there exists $c >$ independent of T such that

$$c'_{\text{idf}}(T) \geq cT,$$

which proves (3.54). ■

Theorem 3.3.12 *We assume that there exists $M > 0$ such that (3.34) holds.*

Let $T_0 > 0$. Assuming $\sigma(t) \in H^1(0, T)$ and $\sigma(0) \neq 0$ and (ω, T_0) satisfies the GCC condition, for $T \geq T_0$ large enough and ϵ small enough, then there exists a positive scalar $C^{st}(T)$ depending only on T such that

$$\|\bar{\theta}_T - \check{\theta}\|_{L^2(\Omega)} \leq C^{st}(T) \epsilon^{-\frac{1}{3}} \delta, \quad (3.56)$$

where $\bar{\theta}_T$ minimizes the cost functional (3.52).

Moreover, under Assumption 4, there exists a positive scalar C^{st} independent of T such that

$$\|\bar{\theta}_T - \check{\theta}\|_{L^2(\Omega)} \leq C^{st} T^{\frac{1}{3}} \epsilon^{-\frac{1}{3}} \delta. \quad (3.57)$$

PROOF: In this proof, we write $a \lesssim b$ if there exists $c > 0$ independent of T, ϵ, δ and M such that $a \leq cb$.

On one hand, using the estimate on the amplitude noise (3.44) and the prior (3.34), we intend to bound the cost functional.

$$\mathcal{J}_T(\check{\theta}) \leq \frac{\epsilon}{2} M^2 + \frac{1}{2} \delta^2 T. \quad (3.58)$$

On the other hand, as in the proof of Theorem 3.3.11, one can show that (3.55) holds. Inserting $y^\delta - y^\delta = 0$ into (3.55) to reintroduce the cost functional and using the prior on the parameter (3.34), we get

$$\begin{aligned} \|\bar{\theta}_T - \check{\theta}\|_{L^2(\Omega)}^2 &\lesssim (c'_{\text{idf}}(T))^{-\frac{1}{3}} (2\epsilon^{-1} \mathcal{J}_T(\check{\theta}) + M^2)^{\frac{2}{3}} \\ &\quad \times \left(\int_0^T \|u_{|\bar{\theta}_T} - y^\delta\|_{L^2(\omega)}^2 dt + \int_0^T \|\check{u} - y^\delta\|_{L^2(\omega)}^2 dt \right)^{\frac{1}{3}}. \end{aligned}$$

Again, using again (3.32) and the cost functional bound (3.58), we have

$$\|\bar{\theta}_T - \check{\theta}\|_{L^2(\Omega)}^2 \lesssim (c'_{\text{idf}}(T))^{-\frac{1}{3}} (2M^2 + \epsilon^{-1} \delta^2 T)^{\frac{2}{3}} (2\delta^2 T + \epsilon M^2)^{\frac{1}{3}}, \quad (3.59)$$

which corresponds to the desired result (3.56).

Furthermore, under Assumption 4, we have that there exists $c >$ independent of T such that

$$c'_{\text{idf}}(T) \geq cT,$$

which leads to (3.57). ■

Eventually, choosing $\epsilon = \delta^2 M^{-2}$ in (3.59), we obtain the following corollary.

Corollary 3.3.12.1 *Under the hypothesis of Theorem 3.3.12, for $T \geq T_0$ large enough, then there exists a positive scalar $C^{st}(T)$ depending only on T such that*

$$\|\bar{\theta}_T - \check{\theta}\|_{L^2(\Omega)} \leq C^{st}(T) M^{\frac{2}{3}} \delta^{\frac{1}{3}}, \quad (3.60)$$

where $\bar{\theta}_T$ minimizes the cost functional (3.52) with $\epsilon = \delta^2 M^{-2}$.

Moreover, under Assumption 4, there exists a positive scalar C^{st} independent of T such that

$$\|\bar{\theta}_T - \check{\theta}\|_{L^2(\Omega)} \leq C^{st} T^{\frac{1}{3}} M^{\frac{2}{3}} \delta^{\frac{1}{3}}. \quad (3.61)$$

3.4 Observer design

Our goal in this section is to propose a sequential method for computing $\bar{\theta}_T$ and the associated wave equation trajectory \bar{z}_T . By sequential, we mean that we want to update the estimator when measurements are available, without recomputing the estimator each time a new measurement is available. This leads to the definition of an observer in the sense of [Krener, 1999], which is formally defined in a general framework of an observed system of a dynamics of the form

$$\begin{cases} \dot{z} = F(z, t) \\ z(0) = z_0 \end{cases} \quad (3.62)$$

where we observed a target trajectory \check{z} with unknown initial state \check{z}_0 by measuring $(y^\delta(t))_{t \geq 0}$. We here propose to slightly simplify this definition.

Definition 3.4.1 *An asymptotic observer of (3.62) is a function $\mathbb{R}_+ \ni t \mapsto \hat{z}(t; (y^\delta(s))_{s \leq t})$ such that*

1. **Robustness to Noise:** *for all $0 \leq \tau$, and for all $\epsilon > 0$, there exist $\delta, \alpha > 0$, such that*

$$\left. \begin{array}{l} \|\hat{z}(0) - \check{z}(0)\|_{\mathcal{Z}} \leq \alpha \\ \|\check{y} - y^\delta\|_{L^2((0, \tau); \mathcal{Y})} \leq \delta \end{array} \right\} \Rightarrow \|\check{z}(t) - \hat{z}(t)\|_{\mathcal{Z}} \leq \epsilon, \quad t \geq \tau.$$

2. **Convergence:** *for all $0 \leq \tau$, and for all $\epsilon, \alpha, \delta > 0$, there exists $T_0 > 0$ such that*

$$\left. \begin{array}{l} \|\hat{z}(0) - \check{z}(0)\|_{\mathcal{Z}} \leq \alpha \\ \|\check{y} - y^\delta\|_{L^2((0, \tau); \mathcal{Y})} \leq \delta \\ \text{supp}(\check{y} - y^\delta) \subset (0, \tau) \end{array} \right\} \Rightarrow \|\check{z}(t) - \hat{z}(t)\|_{\mathcal{Z}} \leq \epsilon, \quad t \geq T_0 + \tau.$$

Note that the two above properties correspond respectively to an asymptotic convergence property in the absence of noise and property of robustness to noise. As the observer can be computed at time t from measurements available at time $s \leq t$, we say that it is a causal function of the measurements. Note also that, as a consequence of (i), an asymptotic observer satisfies for all $s \geq 0$

$$\hat{z}(s) = \check{z}(s), \quad (y^\delta(t) = \check{y}(t), t \geq s) \Rightarrow (\hat{z}(t) = \check{z}(t), t \geq s). \quad (3.63)$$

In our case, we try to define such an observer for the joint dynamics of $z = (z, \theta)$, where the solution z of the wave equation is extended by the zero dynamics of the parameter θ . Our method is based on a dynamic programming strategy that relate functional minimization to Kalman observer, which we will apply to the dynamics of the state and parameters. But let us first recall some results about Kalman filtering [Bensoussan, 1971; Aussal and Moireau, 2022], adapted here to our infinite-dimensional system.

3.4.1 Reduced-order Kalman filters for quasi-evolution operators

To present our strategy, we first need to review some results in the analysis of evolution problems with non-homogeneous generators.

3.4.1.1 Quasi-evolution equation

Let us consider the following abstract dynamics defined in the state-space \mathcal{Z}

$$\begin{cases} \dot{z}(t) = \mathbf{A}(t)z(t) + \beta(t), & \text{in } (0, T), \\ z(0) = z_0. \end{cases} \quad (3.64)$$

from given $z_0 \in \mathcal{Z}$, $\beta \in L^2((0, T); \mathcal{Z})$. Here, the operator

$$\mathbf{A}(t) = \mathbf{A} + \mathbf{G}(t)$$

where $(\mathbf{A}, \mathcal{D}(\mathbf{A}))$ is maximal dissipative and for all $t > 0$, $\mathbf{G}(t)$ is a strongly continuous bounded operator, namely for all $z \in \mathcal{Z}$, $\mathbf{G}(\cdot)z \in C^0([0, T], \mathcal{Z})$. Note that by uniform boundedness theorem, there exists $c_{\text{st}} > 0$ such that $\|\mathbf{G}(t)\|_{\mathcal{L}(\mathcal{Z})} \leq c_{\text{st}}$. Classical results for evolution equation give the existence of a solution of (3.64) knowing z_0 and $\beta(t)$ and denoted $z_{|z_0, \beta}$. Indeed, from [Bensoussan et al., 2007, II-1 Propostion 3.4], if $z_0 \in \mathcal{Z}$ and $\beta \in L^2((0, T); \mathcal{Z})$ then $z_{|z_0, \beta} \in C^0([0, T]; \mathcal{Z})$ is a mild solution

$$z(t) = \Phi(t, 0)z_0 + \int_0^t \Phi(t, s)\beta(s)ds, \quad (3.65)$$

where $\Phi : \Delta_T \rightarrow \mathcal{L}(\mathcal{Z})$ with $\Delta_T = \{(t, s) : s \in [0, T], t \in [s, T]\}$ is a quasi-evolution operator in the sense of [Curtain and Pritchard, 1976, Definition 1.2].

Remark 3.4.1 *Note that it is possible to give an implicit definition of Φ . Indeed, consider the system,*

$$\begin{cases} \dot{z}(t) = \mathbf{A}z(t) + \mathbf{G}(t)z(t), & \text{in } (s, T), \\ z(s) = z_s \in \mathcal{Z}. \end{cases}$$

By definition of the quasi-evolution operator Φ , we have, using Duhamel's formula

$$z(t) = \Phi(t, s)z_s = e^{(t-s)\mathbf{A}}z_s + \int_s^t e^{(t-\tau)\mathbf{A}}\mathbf{G}(\tau)z(\tau)d\tau, \quad (3.66)$$

which, using (3.66), finally becomes

$$\Phi(t, s)z_s = e^{(t-s)\mathbf{A}}z_s + \int_s^t e^{(t-\tau)\mathbf{A}}\mathbf{G}(\tau)\Phi(\tau, s)z_s d\tau.$$

The mild solution (3.65) is also a weak solution in the sense that $z_{|z_0, \beta} \in L^2((0, T); \mathcal{Z}) \cap H^1((0, T); \mathcal{D}(\mathbf{A}^*))$, namely for all $q \in \mathcal{D}(\mathbf{A}^*)$, $(q, z(\cdot))_{\mathcal{Z}}$ belongs to $H^1(0, T)$ and for almost all $t \in (0, T)$,

$$\forall q \in \mathcal{D}(\mathbf{A}^*), \quad \frac{d}{dt}(q, z(t))_{\mathcal{Z}} + (\mathbf{A}(t)^*q, z(t))_{\mathcal{Z}} = (q, \beta(t))_{\mathcal{Z}}. \quad (3.67)$$

Moreover, from [Bensoussan et al., 2007, II-1 Propostion 3.5], if we further assume that for all $z \in \mathcal{D}(\mathbf{A})$, $\mathbf{G}(\cdot)z \in C^0([0, T], \mathcal{D}(\mathbf{A}))$, $z_0 \in \mathcal{D}(\mathbf{A})$ and $\beta \in L^2((0, T); \mathcal{D}(\mathbf{A}))$ then $z_{|z_0, \beta} \in C^0([0, T]; \mathcal{D}(\mathbf{A})) \cap H^1((0, T); \mathcal{Z})$. Finally, when for all $z \in \mathcal{D}(\mathbf{A})$, $\mathbf{G}(\cdot)z \in C^1([0, T]; \mathcal{Z})$, $z_0 \in \mathcal{D}(\mathbf{A})$ and $\beta \in H^1((0, T); \mathcal{D}(\mathbf{A}))$ then $z_{|z_0, \beta} \in C^0([0, T], \mathcal{D}(\mathbf{A})) \cap C^1([0, T]; \mathcal{Z})$.

3.4.1.2 Least-square estimation

We introduce a space $\mathcal{V} \subset \mathcal{Z}$ with continuous injection and a symmetric and coercive bilinear form $m : \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{R}$. We want to minimize a criterion $\mathcal{J}_T : \mathcal{V} \rightarrow \mathbb{R}$ defined by

$$\mathcal{J}_T(\zeta) = \frac{1}{2}m(\zeta, \zeta) + \frac{1}{2} \int_0^T \|y^\delta(t) - \mathbf{C}(t)z_{|\zeta}(t)\|_{\mathcal{Y}}^2 dt, \quad (3.68)$$

where $z_{|\zeta}$ is the solution of (3.64) for an initial condition $z_0 = \hat{z}_0 + \zeta$, where \hat{z}_0 is a known prior. In this section, the observation operator is a time-dependent operator $\mathbf{C} \in \mathbf{C}^0([0, T], \mathcal{L}(\mathcal{Z}, \mathcal{Y}))$. Note that depending on the choice of \mathcal{V} , the bilinear form m can be seen as Tikhonov regularization or a generalized Tikhonov regularization [Engl et al., 1996].

The functional \mathcal{J}_T is continuous, differentiable, and strictly convex, so it admits one and only one minimizer denoted $\bar{\zeta}_{|T}$ where we underline the dependency with respect to T . Computing the derivative of \mathcal{J}_T , we have for all $\xi \in \mathcal{V}$,

$$\langle D\mathcal{J}_T(\zeta), \xi \rangle_{\mathcal{V}, \mathcal{V}} = m(\zeta, \xi) + \int_0^T (\xi, \Phi(t, 0)^* \mathbf{C}(t)^* (\mathbf{C}(t)z_{|\zeta} - y^\delta))_{\mathcal{Y}} dt.$$

Therefore, introducing the adjoint dynamics for any trajectory $z_{|\zeta} \in L^2((0, T); \mathcal{Z})$ and $y^\delta \in L^2((0, T); \mathcal{Y})$,

$$\begin{cases} \dot{\mathbf{q}}(t) + \mathbf{A}(t)^* \mathbf{q}(t) = -\mathbf{C}(t)^* (y^\delta(t) - \mathbf{C}(t)z_{|\zeta}(t)), & t \in (0, T) \\ \mathbf{q}(T) = 0, \end{cases} \quad (3.69)$$

whose mild solution in $\mathbf{C}^0([0, T], \mathcal{Z})$ is given by the Duhamel formulae – adapted to this backward formulation –

$$\mathbf{q}(t) = \int_t^T \Phi(t, s)^* \mathbf{C}(s)^* (y^\delta(s) - \mathbf{C}(s)z_{|\zeta}(s)) ds, \quad t \in [0, T], \quad (3.70)$$

we find

$$\forall \xi \in \mathcal{V}, \quad \langle D\mathcal{J}_T(\zeta), \xi \rangle_{\mathcal{V}, \mathcal{V}} = m(\zeta, \xi) - (\mathbf{q}(0), \xi)_{\mathcal{Z}}. \quad (3.71)$$

If we now introduce the operator $\mathbf{H}_0 \in \mathcal{L}(\mathcal{Z}, \mathcal{V})$ such that for all $\mathbf{q} \in \mathcal{Z}$, $\mathbf{H}_0 \mathbf{q}$ is the solution of $m(\mathbf{H}_0 \mathbf{q}, \xi) = (\mathbf{q}, \xi)_{\mathcal{Z}}$ for all $\xi \in \mathcal{V}$, we deduce from (3.71) the optimality condition

$$\bar{\zeta}_{|T} = \mathbf{H}_0 \bar{\mathbf{q}}_T(0),$$

where $\bar{\mathbf{q}}_T$ is associated to the optimal trajectory $\bar{z}_T = z_{|\bar{\zeta}_{|T}}$ through the two-ends optimality system – as called in [Lions, 1988; Bensoussan, 1971]

$$\begin{cases} \dot{\bar{z}}_T(t) = \mathbf{A}(t)\bar{z}_T + \beta(t), & t \in (0, T), \\ \dot{\bar{\mathbf{q}}}_T(t) + \mathbf{A}(t)^* \bar{\mathbf{q}}_T = -\mathbf{C}(t)^* (y^\delta(t) - \mathbf{C}(t)\bar{z}_T(t)), & t \in (0, T), \\ \bar{z}_T(0) = \hat{z}_0 + \mathbf{H}_0 \bar{\mathbf{q}}_T(0), \\ \bar{\mathbf{q}}_T(T) = 0. \end{cases} \quad (3.72)$$

Note finally that if $\mathcal{V} \subset \mathcal{D}(\mathbf{A})$, then \bar{z} and $\bar{\mathbf{q}}$ belong to $\mathbf{C}^1([0, T], \mathcal{Z}) \cap \mathbf{C}^0([0, T], \mathcal{D}(\mathbf{A}))$.

3.4.2 Kalman estimator definition

The objective of this section is to present briefly the Kalman estimator strategy to solve sequentially the two-ends problem (3.72). First, we introduce the Riccati dynamics

$$\begin{cases} \dot{\mathbf{\Pi}}(t) = \mathbf{A}(t)\mathbf{\Pi}(t) + \mathbf{\Pi}(t)\mathbf{A}(t)^* - \mathbf{\Pi}(t)\mathbf{C}(t)^*\mathbf{C}(t)\mathbf{\Pi}(t), & \text{in } (0, T), \\ \mathbf{\Pi}(0) = \mathbf{\Pi}_0, \end{cases} \quad (3.73)$$

where $\mathbf{C} \in C^0([0, T], \mathcal{L}(\mathcal{Z}, \mathcal{Y}))$. The initial condition $\mathbf{\Pi}_0 \in \mathcal{S}^+(\mathcal{Z})$ the space of bounded symmetric operators. The following theorem gives the existence of solution of (3.73) in $\mathcal{S}^+(\mathcal{Z})$. The resulting operator $\mathbf{\Pi} \in \mathcal{S}^+(\mathcal{Z})$ is called Riccati operator or *covariance operator* for its interpretation in the context of stochastic filtering, see for instance [Bensoussan, 1971; Curtain, 1975].

Theorem 3.4.2 (Existence of Riccati solution)

The Riccati dynamics (3.73) admits one and only one mild solution $\mathbf{\Pi} \in C^0([0, T], \mathcal{S}^+(\mathcal{Z}))$ given by

$$\mathbf{\Pi}(t) = \mathbf{\Phi}(t, 0)\mathbf{\Pi}_0\mathbf{\Phi}(t, 0)^* - \int_0^t \mathbf{\Phi}(t, s)\mathbf{\Pi}(s)\mathbf{C}(s)^*\mathbf{C}(s)\mathbf{\Pi}(s)\mathbf{\Phi}(t, s)^* ds. \quad (3.74)$$

Moreover, for all $(z_1, z_2) \in \mathcal{D}(\mathbf{A}^*)$, $(\mathbf{\Pi}(\cdot)z_1, z_2)_{\mathcal{Z}}$ is differentiable and satisfies the weak formulation

$$\begin{aligned} \frac{d}{dt}(\mathbf{\Pi}(t)z_1, z_2)_{\mathcal{Z}} &= (\mathbf{\Pi}(t)z_1, \mathbf{A}(t)^*z_2)_{\mathcal{Z}} + (\mathbf{\Pi}(t)\mathbf{A}(t)^*z_1, z_2)_{\mathcal{Z}} \\ &\quad - (\mathbf{C}(t)\mathbf{\Pi}(t)z_1, \mathbf{C}(t)\mathbf{\Pi}(t)z_2)_{\mathcal{Y}} = 0, \quad t \in [0, T]. \end{aligned} \quad (3.75)$$

PROOF: We refer to [Curtain and Pritchard, 1976, Corrolary 2.2], for a proof of existence of a mild solution of a Riccati equation where $\mathbf{A}(t) = \mathbf{A} + \mathbf{G}(t)$ with $(\mathbf{A}, \mathcal{D}(\mathbf{A}))$ unbounded and $\mathbf{G}(t)$ a bounded perturbation. The existence of a weak solution is a direct consequence, see for instance [Bensoussan et al., 2007, IV-1 Proposition 2.1]. ■

From the Riccati operator definition, we can introduce the so-called *Kalman observer* – also called *Kalman estimator* in the context of stochastic filtering.

Theorem 3.4.3 *There exists one and only one mild solution in $C^0([0, T]; \mathcal{Z})$ of the dynamics*

$$\begin{cases} \dot{\hat{z}} = \mathbf{A}(t)\hat{z} + \mathbf{\Pi}(t)\mathbf{C}(t)^*(y^\delta(t) - \mathbf{C}(t)\hat{z}(t)), & t \in (0, T), \\ \hat{z}(0) = \hat{z}_0, \end{cases} \quad (3.76)$$

in the sense that

$$\hat{z}(t) = \mathbf{\Phi}(t, 0)\hat{z}_0 + \int_0^t \mathbf{\Phi}(t, s)\mathbf{\Pi}(s)\mathbf{C}(s)^*[y^\delta(s) - \mathbf{C}(s)\hat{z}(s)] ds. \quad (3.77)$$

Moreover, this solution is the unique weak solution in the sense that, (1.) $z \in L^2((0, T); \mathcal{Z})$, (2.) for all $q \in \mathcal{D}(\mathbf{A}^*)$, $(q, \hat{z}(\cdot))_{\mathcal{Z}}$ belongs to $H^1(0, T)$ and (3.) for almost all $t \in (0, T)$,

$$\forall q \in \mathcal{D}(\mathbf{A}^*), \quad \frac{d}{dt}(\hat{z}(t), q)_{\mathcal{Z}} + (\hat{z}, \mathbf{A}(t)^*q)_{\mathcal{Z}} = (y^\delta(t) - \mathbf{C}(t)\hat{z}, \mathbf{C}(t)\mathbf{\Pi}(t)q)_{\mathcal{Y}}. \quad (3.78)$$

PROOF: We know that $\mathbf{\Pi} \in C^0([0, T]; \mathcal{S}_+^*(\mathcal{Z}))$. Therefore, for all $t \in [0, T]$, $\mathbf{A}(t) - \mathbf{\Pi}(t)\mathbf{C}(t)^*\mathbf{C}(t)$ is a uniform bounded perturbation of \mathbf{A} , the existence enters exactly the framework of quasi-evolution operator. ■

3.4.2.1 The optimal filtering property

We can now move to the understanding on how the Riccati operator and the associated Kalman observer decouples the two ends problems (3.72). The following proof extend the one proposed in [Aussal and Moireau, 2022] to quasi evolution operator.

Theorem 3.4.4 *The Kalman estimator defined by the dynamics (3.76) satisfies the fundamental identity*

$$\forall t \in [0, T], \quad \bar{z}_T(t) = \hat{z}(t) + \mathbf{\Pi}(t)\bar{q}_T(t). \quad (3.79)$$

PROOF: Let us introduce $\eta = \hat{z} - \bar{z}_T + \mathbf{\Pi}\bar{q}_T$ and $v \in \mathcal{D}(\mathbf{A}^*)$, and compute

$$\begin{aligned} \frac{d}{dt}(\eta(t), v)_{\mathbf{Z}} &= (\hat{z}(t), \mathbf{A}(t)^*v)_{\mathbf{Z}} + (y^\delta(t) - \mathbf{C}(t)\hat{z}(t), \mathbf{C}(t)\mathbf{\Pi}(t)v)_{\mathbf{Y}} \\ &\quad - (\bar{z}_T(t), \mathbf{A}(t)^*v)_{\mathbf{Z}} + \frac{d}{dt}(\mathbf{\Pi}(t)\bar{q}_T(t), v)_{\mathbf{Z}}, \end{aligned} \quad (3.80)$$

where we need to properly define the last-term time derivative. In fact, as \bar{q} can be defined from a Duhamel formula of the form (3.70), and $\mathbf{\Pi}$ is mild solution of the Riccati equation (3.73), hence is absolutely continuous, we have from [Curtain and Pritchard, 1976, Lemme 2.5] that

$$\frac{d}{dt}(\mathbf{\Pi}(t)\bar{q}_T(t), v)_{\mathbf{Z}} = \frac{d}{dt}(\mathbf{\Pi}(t)\bar{q}_T(s), v)_{\mathbf{Z}} \Big|_{s=t} + \frac{d}{dt}(\bar{q}_T(t), \mathbf{\Pi}(s)v)_{\mathbf{Z}} \Big|_{s=t}.$$

Moreover, as $\mathbf{\Pi}$ is a weak solution of (3.73), we have from (3.75),

$$\begin{aligned} \frac{d}{dt}(\mathbf{\Pi}(t)\bar{q}_T(s), v) \Big|_{s=t} &= (\mathbf{\Pi}(t)\bar{q}_T(t), \mathbf{A}(t)^*v)_{\mathbf{Z}} + (\bar{q}_T(t), \mathbf{A}(t)^*\mathbf{\Pi}(t)v)_{\mathbf{Z}} \\ &\quad - (\mathbf{C}(t)\mathbf{\Pi}(t)\bar{q}_T(t), \mathbf{C}(t)\mathbf{\Pi}(t)v)_{\mathbf{Y}}, \end{aligned} \quad (3.81)$$

and, as \bar{q}_T is a weak solution of (3.72),

$$\begin{aligned} \frac{d}{dt}(\bar{q}_T(t), \mathbf{\Pi}(s)v) \Big|_{s=t} &= -(\bar{q}_T(t), \mathbf{A}(t)^*\mathbf{\Pi}(t)v)_{\mathbf{Z}} \\ &\quad - (y^\delta(t) - \mathbf{C}(t)\bar{z}_T(t), \mathbf{C}(t)\mathbf{\Pi}(t)v)_{\mathbf{Y}}. \end{aligned} \quad (3.82)$$

Gathering (3.80), (3.81) and (3.82), we finally obtain

$$\begin{cases} \frac{d}{dt}(\eta(t), v)_{\mathbf{Z}} = (\eta(t), \mathbf{A}(t)^* - \mathbf{C}(t)^*\mathbf{C}(t)\mathbf{\Pi}(t)v)_{\mathbf{Z}}, & t \in [0, T] \\ \eta(0) = 0 \end{cases}$$

whence, by Theorem 3.4.3, $\eta = 0$ in $[0, T]$, which concludes the proof. \blacksquare

In particular, we see that by taking $t = T$ that for all $t > 0$, $\hat{z}(t) = \bar{z}_T(t)$. This why the Kalman observer is also called optimal filtering estimator as it consists of sequentially computing the trajectory associated with least-square estimator at final time.

3.4.2.2 The reduced order case

As we saw in the previous section, the inverse problem can be solved using the Kalman filter, but this requires us to calculate a Riccati operator in the augmented space \mathbf{Z} . In order to reduce the computational cost, we now show that we can instead compute another

Riccati operator on a reduced space \mathbf{Z}_r that takes into account a priori informations on the inverse problem: in our case, described specifically later, only the source terms is not known. The reduced space \mathbf{Z}_r is assumed to be a closed subspace of \mathbf{Z} (equipped with the same scalar product) hence we can assume given a projection operator $P_r \in \mathcal{L}(\mathbf{Z}, \mathbf{Z}_r)$ in this space. Moreover, we also assumed given a compact subspace \mathbf{V}_r of \mathbf{Z}_r that is used to define a regularized version of the functional to minimize. The minimization problem reads

$$\min_{\zeta_r \in \mathbf{V}_r} \left\{ \mathcal{J}_T(\zeta_r) = \frac{1}{2} m_r(\zeta_r, \zeta_r) + \frac{1}{2} \int_0^T \|y^\delta - \mathbf{C}z_{|P_r^* \zeta_r}\|_Y^2 dt \right\},$$

where $z_{|P_r^* \zeta_r}$ is solution of (3.64) for an initial condition $z_0 = \hat{z}_0 + P_r^* \zeta_r$ and m_r is a continuous and coercive bilinear form on \mathbf{V}_r . The associated two-ends problem reads

$$\begin{cases} \dot{\bar{z}}_T = \mathbf{A}(t)\bar{z}_T + \beta, & t \in (0, T), \\ \dot{\bar{q}}_T + \mathbf{A}(t)^* \bar{q}_T = -\mathbf{C}^*(y^\delta(t) - \mathbf{C}\bar{z}_T), & t \in (0, T), \\ \bar{z}_T(0) = \hat{z}_0 + P_r^* \Lambda_0 P_r \bar{q}_T(0), \\ \bar{q}_T(T) = 0, \end{cases} \quad (3.83)$$

with $\Lambda_0 \in \mathcal{L}(\mathbf{V}_r)$ defined such that

$$\forall (\zeta_r, \xi_r) \in \mathbf{V}_r \times \mathbf{V}_r, \quad m_r(\Lambda_0 \zeta_r, \xi_r) = (\zeta_r, \xi_r)_{\mathbf{Z}}.$$

The resulting Kalman observer can still be defined from the Riccati dynamics

$$\begin{cases} \dot{\mathbf{\Pi}}(t) = \mathbf{A}(t)\mathbf{\Pi}(t) + \mathbf{\Pi}(t)\mathbf{A}(t)^* - \mathbf{\Pi}(t)\mathbf{C}^* \mathbf{C} \mathbf{\Pi}(t), & t > 0, \\ \mathbf{\Pi}(0) = P_r^* \Lambda_0 P_r, \end{cases} \quad (3.84)$$

so that we continue to have $\bar{z}_T(t) = \hat{z}(t) + \mathbf{\Pi}(t)\bar{q}_T(t)$ for all $t \in [0, T]$. But moreover, the Riccati solution of (3.84) can be deduced from a reduced covariance operator Λ , solution of a Riccati dynamics in the reduced space \mathbf{Z}_r , namely

$$\begin{cases} \dot{\Lambda}(t) = -\Lambda(t)P_r \Phi(t, 0)^* \mathbf{C}^* \mathbf{C} \Phi(t, 0) P_r^* \Lambda(t), & t > 0, \\ \Lambda(0) = \Lambda_0. \end{cases} \quad (3.85)$$

This dynamics satisfies the condition of existence since for all $t \in [0, T]$, $\mathbf{C}\Phi(t, 0)P_r^*$ can be considered as a time-dependent bounded observation operator. Moreover, we have the following decomposition.

Theorem 3.4.5 *Let $\mathbf{\Pi}(t)$ the mild solution of the Riccati dynamic (3.84) and $\Lambda(t)$ the mild solution of the Riccati dynamic (3.85), then we have*

$$\mathbf{\Pi}(t) = \Phi(t, 0)P_r^* \Lambda(t) P_r \Phi(t, 0)^*. \quad (3.86)$$

PROOF: The mild solution of the Riccati dynamics (3.85) reads

$$\Lambda(t) = \Lambda(0) - \int_0^t \Lambda(s)P_r \Phi(s, 0)^* \mathbf{C}^* \mathbf{C} \Phi(s, 0) P_r^* \Lambda(s) ds.$$

Using this mild solution, we compute the following quantity $\Phi(t, 0)P_r^* \Lambda(t) P_r \Phi(t, 0)^*$.

$$\begin{aligned} \Phi(t, 0)P_r^* \Lambda(t) P_r \Phi(t, 0)^* &= \Phi(t, 0)P_r^* \Lambda(0) P_r \Phi(t, 0)^* \\ &\quad - \int_0^t \Phi(t, 0)P_r^* \Lambda(s) P_r \Phi(s, 0)^* \mathbf{C}^* \mathbf{C} \Phi(s, 0) P_r^* \Lambda(s) P_r \Phi(t, 0)^* ds. \end{aligned}$$

Note that $\Phi(t, 0) = \Phi(t, s)\Phi(s, 0)$, therefore we obtain

$$\begin{aligned} \Phi(t, 0)P_r^*\Lambda(t)P_r\Phi(t, 0)^* &= \Phi(t, 0)P_r^*\Lambda(0)P_r\Phi(t, 0)^* \\ &\quad - \int_0^t \Phi(t, s)\Phi(s, 0)P_r^*\Lambda(s)P_r\Phi(s, 0)^*C^*C\Phi(s, 0)P_r^*\Lambda(s)P_r\Phi(s, 0)^*\Phi(t, s)^*ds. \end{aligned}$$

Setting $\tilde{\Pi}(t) = \Phi(t, 0)P_r^*\Lambda(t)P_r\Phi(t, 0)^*$, we get

$$\tilde{\Pi}(t) = \tilde{\Pi}(0) - \int_0^t \Phi(t, s)\tilde{\Pi}(s)C^*C\tilde{\Pi}(s)\Phi(t, s)^*ds.$$

Hence, $\tilde{\Pi}$ is a mild solution of (3.84) and by uniqueness of the solution of the Riccati dynamics, we have

$$\tilde{\Pi}(t) = \Pi(t) = \Phi(t, 0)P_r^*\Lambda(t)P_r\Phi(t, 0)^*,$$

which concludes the proof. \blacksquare

Therefore, a computational strategy can be based only on computations restricted to the reduced-space \mathcal{Z}_r .

3.4.2.3 Application to source estimation

We will now define an observer, which is equivalent to the calculation of the optimum $\bar{\theta}_T$ with increasing time. To that purpose, we rely on the Kalman estimation theory presented above and more precisely the Reduced-Order Kalman filtering approach of Section 3.4.2.2, where the uncertainty space is, here, the parameter space, a subspace of the augmented space $\mathcal{Z} = \mathcal{Z} \times \mathcal{P}$. In particular, we set $\mathcal{Z}_r = \{0\} \times L^2(\Omega)$ and $\mathcal{V}_r = \{0\} \times H_0^1(\Omega)$. The dynamics in the augmented space reads

$$\underbrace{\frac{d}{dt} \begin{pmatrix} z \\ \theta \end{pmatrix}}_{\dot{z}} = \underbrace{\begin{pmatrix} A & B(t) \\ 0 & 0 \end{pmatrix}}_{\mathbf{A}(t)z} \begin{pmatrix} z \\ \theta \end{pmatrix} \quad \text{with} \quad \underbrace{\begin{pmatrix} z(0) \\ \theta(0) \end{pmatrix}}_{z(0)} = \underbrace{\begin{pmatrix} 0 \\ \theta \end{pmatrix}}_{\zeta}. \quad (3.87)$$

The least square minimization (3.33) rewritten with respect to the augmented variable reads

$$\mathcal{J}_T(\theta) = \frac{\epsilon}{2} \|\theta\|_{H_0^1(\Omega)}^2 + \frac{1}{2} \int_0^T \|y^\delta(t) - C z|_\theta\|_{\mathcal{Y}}^2 dt. \quad (3.88)$$

Introducing the projection operator

$$P_\theta = \begin{pmatrix} 0 & 0 \\ 0 & \text{Id}_{\mathcal{P}} \end{pmatrix}$$

from the augmented space \mathcal{Z} to the reduced space \mathcal{Z}_r , we have, using Theorem 3.4.5,

$$\begin{cases} \Pi(t) = \Phi(t, 0)P_\theta^*\Lambda(t)P_\theta\Phi(t, 0)^*, \\ \Pi(0) = P_\theta^*\Lambda(0)P_\theta, \end{cases} \quad (3.89)$$

where $\Lambda(t)$ is given by (3.85) with $P_r = P_\theta$. Then, we can specify the evolution operator Φ associated with the following system

$$\mathbf{A}(t) = \begin{pmatrix} A & B(t) \\ 0 & 0 \end{pmatrix}, \quad \begin{cases} \dot{z}(t) = \mathbf{A}(t)z(t), \\ z(0) = z_0, \end{cases} \quad \Rightarrow \quad \begin{cases} \dot{z}(t) = Az(t) + B(t)\theta, \\ \dot{\theta}(t) = 0, \\ (z, \theta)^t(0) = (z_0, \theta_0)^t, \end{cases}$$

Therefore, for initial data of the form $z_0 = P_\theta^*(0, \theta_0)^t = (0, \theta_0)^t$ the solution reads

$$\begin{pmatrix} z(t) \\ \theta(t) \end{pmatrix} = \Phi(t, 0) \begin{pmatrix} 0 \\ \theta_0 \end{pmatrix},$$

and, using the definition of L given by (3.5), $z(t) = L(t)\theta_0$ is the mild solution of

$$\begin{cases} \dot{z} = Az + B\theta_0, & t > 0, \\ z(0) = 0, \end{cases}$$

Hence, we obtain that

$$\begin{pmatrix} z(t) \\ \theta(t) \end{pmatrix} = \begin{pmatrix} L(t) \\ \text{Id}_{\mathcal{P}} \end{pmatrix} \theta_0.$$

Eventually, we have shown that for all θ_0 , $\Phi(t, 0)P_\theta^*(z_0, \theta_0)^t$ is given by

$$\begin{pmatrix} z(t) \\ \theta(t) \end{pmatrix} = \Phi(t, 0)P_\theta^* \begin{pmatrix} z_0 \\ \theta_0 \end{pmatrix} = \Phi(t, 0) \begin{pmatrix} 0 \\ \theta_0 \end{pmatrix} = \begin{pmatrix} L(t) \\ \text{Id}_{\mathcal{P}} \end{pmatrix} \theta_0.$$

Therefore, we can simplify the Riccati dynamics (3.85) into

$$\begin{cases} \dot{\Lambda} = -\Lambda L^* C^* C L \Lambda, & t > 0, \\ \Lambda(0) = \Lambda_0, \end{cases} \quad (3.90)$$

and (3.89) into

$$\mathbf{\Pi}(t) = \begin{pmatrix} L(t) \\ \text{Id}_{\mathcal{P}} \end{pmatrix} \Lambda(t) \begin{pmatrix} L^*(t) & \text{Id}_{\mathcal{P}} \end{pmatrix}. \quad (3.91)$$

Note that, in our case $m_r(\cdot, \cdot) = \frac{\epsilon}{2}(\cdot, \cdot)_{H_0^1(\Omega)}$, therefore we can obtain an explicit expression of Λ_0 given by

$$\Lambda_0 = -\epsilon^{-1} \Delta_0^{-1}.$$

Finally, we obtain the equivalent sequential estimator with the following result that is a straightforward application of Theorem 3.4.3.

Corollary 3.4.5.1 *The minimizer $\hat{\theta} = \bar{\theta}_t$ is solution of the following dynamics*

$$\begin{cases} \dot{\hat{\theta}} = \Lambda L^* C^*(y^\delta - C\hat{z}), & t > 0, \\ \hat{\theta}(0) = \hat{\theta}_0 \\ \dot{\hat{z}} = A\hat{z} + B\hat{\theta} + L\dot{\hat{\theta}}, & t > 0, \\ z(0) = \hat{z}_0. \end{cases} \quad (3.92)$$

Note that the present estimator can be defined for observations in H^1 or L^2 with an adapted definition of the observation operator C and its adjoint C^* .

Using the results obtained in Sections 3.3.2.2 and 3.3.3.2, we show that $(\hat{z}, \hat{\theta})$ cannot be entirely defined as observers in the sense of Definition 3.4.1, but that they satisfy the second property (robustness to noise) and a necessary condition of the first (convergence).

Proposition 3.4.6 *Assuming that we have at our disposal the perfect measurements $y^\delta = \check{y}$, then, under Assumption 3, the estimators $(\hat{z}, \hat{\theta})$ asymptotically converges to the target parameter $(\check{z}, \check{\theta})$, namely*

$$\lim_{t \rightarrow +\infty} \|\hat{\theta}(t) - \check{\theta}(t)\|_{L^2(\Omega)} = 0 \text{ and } \lim_{t \rightarrow +\infty} \|\hat{z}(t) - \check{z}(t)\|_{\mathcal{Z}} = 0.$$

PROOF: We here use the link between the observer and the functional minimization. Under Assumptions 3 and 4, we know that (Theorem 3.3.6 and 3.3.11), for both types of observation operator, there exists $a > 0$ and C^{st} such that

$$\|\bar{\theta}_T - \check{\theta}\|_{L^2(\Omega)} \leq C^{\text{st}} T^{-a},$$

which, combined with $\hat{\theta} = \bar{\theta}_t$ gives the convergence of the parameter estimator

$$\lim_{t \rightarrow +\infty} \|\hat{\theta}(t) - \check{\theta}\|_{L^2(\Omega)} = 0.$$

Moreover, we have that

$$\hat{z} = \bar{z}_t(t) = z_{|\bar{\theta}_t}.$$

Therefore,

$$\lim_{t \rightarrow +\infty} \|\hat{z}(t) - \check{z}(t)\|_{\mathcal{Z}} = \lim_{t \rightarrow +\infty} \|z_{|\bar{\theta}_t}(t) - z_{|\check{\theta}}\|_{\mathcal{Z}} = 0,$$

which concludes the proof. ■

Proposition 3.4.7 *Assuming that we have at our disposal the noisy measurements satisfying (3.32) or (3.44), the estimator $(\hat{z}, \hat{\theta})$ is stable with respect to noise, namely there exist two positive scalars C_1^{st} and C_2^{st} independent of δ and dependent of t such that*

$$\|\hat{\theta}(t) - \check{\theta}\|_{L^2(\Omega)} \leq C_1^{\text{st}}(t) \delta \text{ and } \|\hat{z} - \check{z}\|_{\mathcal{Z}} \leq C_2^{\text{st}}(t) \delta.$$

PROOF: We take advantage of the link between the observer and the functional minimization. From Theorem 3.3.7 and 3.3.12, for both types of observation operator, there exist C^{st} such that

$$\|\bar{\theta}_t - \check{\theta}\|_{L^2(\Omega)} \leq C^{\text{st}} \delta,$$

which, combined with $\hat{\theta}(t) = \bar{\theta}_t$ gives the stability of the parameter estimator

$$\|\hat{\theta}(t) - \check{\theta}\|_{L^2(\Omega)} \leq C^{\text{st}} \delta.$$

Furthermore, we have that

$$\hat{z}(t) = \bar{z}_t(t) = z_{|\bar{\theta}_t}.$$

Therefore, we obtain

$$\|\hat{z}(t) - \check{z}\|_{\mathcal{Z}} = \|z_{|\bar{\theta}_t}(t) - z_{|\check{\theta}}\|_{\mathcal{Z}}.$$

Classically using the Grönwall lemma, one can show that there exists a positive scalar C^{st} depending only on t and σ such that

$$\|z_{|\bar{\theta}_t}(t) - z_{|\check{\theta}}\|_{\mathcal{Z}} \leq C^{\text{st}} \|\bar{\theta}_t(t) - \check{\theta}\|_{L^2(\Omega)},$$

which concludes the proof. ■

Remark 3.4.8 *Proposition 3.4.7 allows us to demonstrate the robustness to noise property of Definition 3.4.1. On the other hand, Proposition 3.4.6 is a necessary condition for the convergence property of Definition 3.4.1. Indeed, the second property of the definition is obtained only in this case $\tau = 0$. We would have to take into account the possibility of uncertainties in the initial condition for dynamics (3.92) to fall within the asymptotic observer framework of definition 3.4.1.*

3.4.3 Observer discretization

We can now study the observer discretization, where we face two possible options. First we could directly propose a time and space discretization of (3.92), or we could first discretize the wave equation and then reapply the Tikhonov regularization strategy and observer design to the discretized system. Following Zuazua [2005], we are going to follow this second strategy.

3.4.3.1 Reduced-order Kalman discretization

As in the time-continuous case, let us first recall the Kalman discretization for a linear discrete-time dynamics. Let $\mathcal{Z}^h \subset \mathcal{V} \subset \mathcal{Z}$ a finite dimensional space equipped with the scalar product of \mathcal{Z} . Assuming a dynamics of the following form, with a given source term $\beta^h \in \mathcal{Z}^h$

$$\begin{cases} z_{n+1|\zeta}^h = \Phi^{h,\tau} z_{n|\zeta}^h + \beta_{n+1}^h, & n \geq 0, \\ z_{0|\zeta}^h = \hat{z}_0^h + \zeta \end{cases}$$

associated with measurements $y_n^{\delta,h}$ and an observation operator $C^h \in \mathcal{L}(\mathcal{Z}^h, \mathcal{Y}^h)$ where \mathcal{Y}^h is the finite dimensional space in which the observations belong. We first consider the minimization of a discrete version of the criterion (3.68) (see Aussal and Moireau [2022])

$$\min_{\zeta \in \mathcal{Z}^h} \left\{ \mathcal{J}_N(\zeta^h) = \frac{1}{2} m(\zeta^h, \zeta^h) + \sum_{n=1}^N \frac{\tau}{2} \|y_n^{\delta,h} - C^h z_{n|\zeta}^h\|_{\mathcal{Y}^h}^2 \right\}, \quad (3.93)$$

where the time integral is replaced by a simple quadrature formula. We easily find the corresponding two-ends problem associated with the minimization of this criterion,

$$\begin{cases} \bar{z}_{n+1|N}^h = \Phi^{h,\tau} \bar{z}_{n|N}^h + \beta_{n+1}^h, & 0 \leq n \leq N-1, \\ \bar{z}_{0|N}^h = \hat{z}_0^h + \Pi_0^h \bar{q}_{0|N}^h, \\ \bar{q}_{n|N}^h = \Phi^{h,\tau*} \bar{q}_{n+1|N}^h + \tau C^{h*} (y_n^{\delta,h} - C^h \bar{z}_{n|N}^h), & 0 \leq n \leq N-1, \\ \bar{q}_{N|N}^h = 0, \end{cases} \quad (3.94)$$

where the notation $|N$ stresses the fact that the criterion is using the N first observations and $\Pi_0^h \in \mathcal{L}(\mathcal{Z}^h)$ is the operator defined by

$$\forall (q_h, \xi_h) \in \mathcal{Z}^h \times \mathcal{Z}^h, \quad m(\Pi_0^h q_h, \xi_h) = (q_h, \xi_h)_{\mathcal{Z}}.$$

Then, we can uncouple these systems by solving the following Riccati equation,

$$\begin{cases} \Pi_{n+1}^h = \Phi^{h,\tau} \Pi_n^h \Phi^{h,\tau*}, & n \in \mathbb{N}, \\ \Pi_{n+1}^h = \Pi_{n+1}^h - \tau \Pi_{n+1}^h C^{h*} [\text{Id}_{\mathcal{Y}^h} + \tau C^h \Pi_{n+1}^h C^{h*}]^{-1} C^h \Pi_{n+1}^h, & n \in \mathbb{N}, \end{cases} \quad (3.95)$$

which defines a sequence of operators Π_{n-}^h and Π_n^h . The sequence of Π_{n-}^h and Π_n^h can be interpreted as a splitting time-scheme for the Riccati dynamics (3.73) Aussal and Moireau [2022]. Then, we introduce the estimator

$$\begin{cases} \hat{z}_{n+1-}^h = \Phi^{h,\tau} \hat{z}_n^h + \beta_{n+1}^h, & n \in \mathbb{N}, \\ \hat{z}_{n+1}^h = \hat{z}_{n+1-}^h + \tau \Pi_{n+1}^h C^{h*} (y_{n+1}^{\delta,h} - C^h \hat{z}_{n+1-}^h), & n \in \mathbb{N}. \end{cases} \quad (3.96)$$

Identically, (3.96) can be interpreted as a splitting time-scheme for the estimator dynamics (3.76), see Aussal and Moireau [2022]. Then, we have the following equivalence results proved in Aussal and Moireau [2022].

Theorem 3.4.9 *We have the following decoupling principle*

$$\bar{\mathbf{z}}_{n|N}^h = \hat{\mathbf{z}}_{n-}^h + \mathbf{\Pi}_{n-}^h \bar{\mathbf{q}}_{n|N}^h, \quad \forall 0 \leq n \leq N+1, \quad (3.97)$$

and in particular for all $n \in \mathbb{N}$

$$\hat{\mathbf{z}}_{n-}^h = \bar{\mathbf{z}}_{n|n}^h. \quad (3.98)$$

Let us now apply such equivalence to a reduced space configuration. For this purpose, we define $\mathcal{Z}_r^h \subset \mathcal{Z}^h$ a finite dimensional space such that $\mathcal{Z}_r^h \subset \mathcal{V}_r \subset \mathcal{Z}_r$. To simplify the notation, we assume that $P_r(\mathcal{Z}^h) \subset \mathcal{Z}_r^h$ – which is satisfied for our source identification problem – hence P_r can be seen as an operator in $\mathcal{L}(\mathcal{Z}^h, \mathcal{Z}_r^h)$. The discrete criterion here reads

$$\min_{\zeta_r^h \in \mathcal{Z}_r^h} \left\{ \mathcal{J}_N(\zeta_r^h) = \frac{1}{2} m_r(\zeta_r^h, \zeta_r^h) + \sum_{k=1}^N \frac{\tau}{2} \|y_n^{\delta, h} - \mathbf{C} \mathbf{z}_{n|P_r^* \zeta_r^h}^h\|_{\mathcal{Y}^h}^2 \right\}. \quad (3.99)$$

The two ends problem remains as in (3.94) except for the initial condition that is changed into

$$\bar{\mathbf{z}}_{0|N}^h = \hat{\mathbf{z}}_{0-}^h + P_r^*(U_0^h)^{-1} P_r \bar{\mathbf{q}}_{0|N}^h, \quad (3.100)$$

Moreover, we have the discrete counterpart of Theorem 3.4.5.

Theorem 3.4.10 *The solution of (3.95) with initial condition defined by*

$$\mathbf{\Pi}_0^h = P_r^* \Lambda_0^h P_r$$

satisfies

$$\mathbf{\Pi}_n^h = \left(\Phi^{h, \tau} \right)^n P_r^* \Lambda_n^h P_r \left(\Phi^{h, \tau^*} \right)^n \quad (3.101)$$

where Λ_n^h satisfies $\Lambda_n^h = (U_n^h)^{-1}$ with

$$U_{n+1}^h = U_n^h + \tau P_r \left(\Phi^{h, \tau^*} \right)^{n+1} \mathbf{C}^{h*} \mathbf{C}^h \left(\Phi^{h, \tau} \right)^{n+1} P_r^*, \quad n \geq 0, \quad (3.102)$$

PROOF: We proceed by induction. We have at iteration 0

$$\mathbf{\Pi}_0^h = P_r^*(U_0^h)^{-1} P_r.$$

Let us assume (3.101) is satisfied at iteration n . First, we directly have

$$\mathbf{\Pi}_{n+1-}^h = \Phi^{h, \tau} \mathbf{\Pi}_n^h \Phi^{h, \tau^*} = \left(\Phi^{h, \tau} \right)^{n+1} P_r^* \Lambda_n^h P_r \left(\Phi^{h, \tau^*} \right)^{n+1},$$

Then, using (3.95) and the expression above of $\mathbf{\Pi}_{n+1-}^h$, we have

$$\begin{aligned} \mathbf{\Pi}_{n+1}^h &= \left(\Phi^{h, \tau} \right)^{n+1} P_r^* \left[\Lambda_n^h - \tau \Lambda_n^h P_r \left(\Phi^{h, \tau^*} \right)^{n+1} \mathbf{C}^{h*} [\text{Id}_{\mathcal{Y}^h} + \tau \mathbf{C}^h \left(\Phi^{h, \tau} \right)^{n+1} \right. \right. \\ &\quad \left. \left. \times P_r^* \Lambda_n^h P_r \left(\Phi^{h, \tau^*} \right)^{n+1} \mathbf{C}^{h*} \right]^{-1} \mathbf{C}^h \left(\Phi^{h, \tau} \right)^{n+1} P_r^* \Lambda_n^h \right] P_r \left(\Phi^{h, \tau} \right)^{n+1}. \end{aligned}$$

We therefore have

$$\mathbf{\Pi}_{n+1}^h = \left(\Phi^{h, \tau} \right)^{n+1} P_r^* \Lambda_{n+1}^h P_r \left(\Phi^{h, \tau} \right)^{n+1},$$

as soon as

$$\Lambda_{n+1}^h = \Lambda_n^h - \Lambda_n^h D_n^{h,*} \left(\text{Id}_{\mathcal{Y}^h} + D_n^h \Lambda_n^h D_n^{h,*} \right)^{-1} D_n^h \Lambda_n^h$$

with $D_n^h = \sqrt{\tau} \mathbf{C}^h \left(\Phi^{h, \tau} \right)^{n+1} P_r^*$. By using Sherman–Morrison–Woodbury formula for operators – see for instance [Deng, 2011, Theorem 1.1] – we obtain

$$\left(\Lambda_{n+1}^h \right)^{-1} = \left(\Lambda_n^h \right)^{-1} + D_n^{h,*} D_n^h$$

which ends the proof since it is precisely (3.102). \blacksquare

3.4.3.2 Application to source identification

We discretize the wave equation using Lagrange finite element. Let $\theta^h \in \mathcal{P}^h$ the finite dimensional space corresponding to \mathcal{P} . We use a conservative mid-point scheme, that ultimately reads

$$\begin{cases} \frac{u_{n+1}^h - u_n^h}{\tau} = \frac{v_{n+1}^h + v_n^h}{2} \\ \frac{v_{n+1}^h - v_n^h}{\tau} - \Delta_0^h \frac{u_{n+1}^h + u_n^h}{2} = \sigma \left(\left(n + \frac{1}{2} \right) \tau \right) \theta^h, \end{cases} \quad (3.103)$$

where Δ_0^h is the bounded linear operator in $\mathcal{V}^h \subset H^1(\Omega)$ associated with the conform finite element discretization of the unbounded operator Δ_0 of $L^2(\Omega)$. Note that this system can be rewritten in a first order form by considering $z_n^h \in \mathcal{Z}^h = \mathcal{V}^h \times \mathcal{V}^h$ solution of

$$z_{n+1}^h = \Phi^{h,\tau} z_n^h + B_{n+1}^{h,\tau} \theta^h,$$

where

$$\Phi^{h,\tau} = \begin{pmatrix} \text{Id}^h & -\frac{\tau}{2} \text{Id}^h \\ -\frac{\tau}{2} \Delta_0^h & \text{Id}^h \end{pmatrix}^{-1} \begin{pmatrix} \text{Id}^h & \frac{\tau}{2} \text{Id}^h \\ \frac{\tau}{2} \Delta_0^h & \text{Id}^h \end{pmatrix}$$

and

$$B_{n+1}^{h,\tau} = \tau \begin{pmatrix} \text{Id}^h & -\frac{\tau}{2} \text{Id}^h \\ -\frac{\tau}{2} \Delta_0^h & \text{Id}^h \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \sigma \left(\left(n + \frac{1}{2} \right) \tau \right) \end{pmatrix}.$$

Note that \mathcal{Z}^h is equipped with the scalar product on \mathcal{Z} that is the natural scalar product on $H_0^1(\Omega) \times L^2(\Omega)$.

We now apply this result to our source identification problem. The joint state-parameter discrete dynamics in $\mathcal{Z}^h = \mathcal{Z}^h \times \mathcal{P}^h$ reads

$$\underbrace{\begin{pmatrix} z_{n+1}^h \\ \theta_{n+1}^h \end{pmatrix}}_{z_{n+1}^h} = \underbrace{\begin{pmatrix} \Phi^{h,\tau} & B_{n+1}^{h,\tau} \\ 0 & \text{Id}^h \end{pmatrix}}_{\Phi^{h,\tau} z_n^h} \underbrace{\begin{pmatrix} z_n^h \\ \theta_n^h \end{pmatrix}}_{z_n^h}, \quad \text{with} \quad \underbrace{\begin{pmatrix} z_0^h \\ \theta_0^h \end{pmatrix}}_{z_0^h} = \underbrace{\begin{pmatrix} 0 \\ \theta^h \end{pmatrix}}_{\zeta}. \quad (3.104)$$

From Theorem 3.4.5, we know that

$$\begin{cases} \mathbf{\Pi}_n^h = (\Phi^{h,\tau})^n P_r^* \Lambda_n^h P_r (\Phi^{h,\tau^*})^n \quad \text{and} \quad \mathbf{\Pi}_{n+1}^h = (\Phi^{h,\tau})^{n+1} P_r^* \Lambda_n^h P_r (\Phi^{h,\tau^*})^{n+1}, \\ \mathbf{\Pi}_0 = P_\theta^* (U_0^h)^{-1} P_\theta, \end{cases} \quad (3.105)$$

where $U_n^h = (\Lambda_n^h)^{-1}$ is solution of

$$U_{n+1}^h = U_n^h + \tau P_\theta (\Phi^{h,\tau^*})^{n+1} C^{h*} C^h (\Phi^{h,\tau})^{n+1} P_\theta^*, \quad n \geq 0. \quad (3.106)$$

Moreover, we have

$$\begin{pmatrix} z_n^h \\ \theta_n^h \end{pmatrix} = (\Phi^{h,\tau})^n P_\theta^* \begin{pmatrix} 0 \\ \theta_0^h \end{pmatrix} = (\Phi^{h,\tau})^n \begin{pmatrix} 0 \\ \theta_0^h \end{pmatrix} = \begin{pmatrix} L_n^h \\ \text{Id}^h \end{pmatrix} \theta_0^h,$$

where the operator $(L_n^h)_{n \in \mathbb{N}}$ is defined by

$$L_n^h \theta = z_{n|\theta}^h \quad \text{with} \quad \begin{cases} z_{n+1|\theta}^h = \Phi^{h,\tau} z_{n|\theta}^h + B_{n+1}^{h,\tau} \theta, & n \in \mathbb{N}, \\ z_{0|\theta}^h = 0. \end{cases} \quad (3.107)$$

Therefore, we can simplify (3.106) into

$$U_{n+1}^h = U_n^h + \tau L_{n+1}^{h*} C^{h*} C^h L_{n+1}^h, \quad n \geq 0, \quad (3.108)$$

and (3.105) into

$$\mathbf{\Pi}_n^h = \begin{pmatrix} L_n^h \\ \text{Id} \end{pmatrix} \Lambda_n^h \begin{pmatrix} L_n^{h*} & \text{Id} \end{pmatrix}, \text{ and } \mathbf{\Pi}_{n+1}^h = \begin{pmatrix} L_{n+1}^h \\ \text{Id}^h \end{pmatrix} \Lambda_n^h \begin{pmatrix} L_{n+1}^{h*} & \text{Id}^h \end{pmatrix}. \quad (3.109)$$

Finally, we obtain the equivalent sequential estimator.

Theorem 3.4.11 *The minimizer $\hat{\theta}_n^h = \bar{\theta}_n^h$ is solution from $(\hat{z}_0^h, \hat{\theta}_0^h)$ of the recursive dynamics*

$$\begin{cases} \hat{z}_{n+1}^h = \Phi^{h,\tau} z_n^h + B_{n+1}^{h,\tau} \hat{\theta}_n^h, \\ \hat{\theta}_{n+1}^h = \hat{\theta}_n^h + \tau (U_{n+1}^h)^{-1} L_{n+1}^{h*} C^{h*} (y_{n+1}^{\delta,h} - C^h z_{n+1}^h), \\ \hat{z}_{n+1}^h = \hat{z}_{n+1}^h + L_{n+1}^h (\hat{\theta}_{n+1}^h - \hat{\theta}_n^h). \end{cases} \quad (3.110)$$

We already know that (3.110) is consistent. A direct convergence study of this splitting scheme could be intricate. Here, by relying on the fundamental relation $\hat{\theta}_n^h = \bar{\theta}_n^h$ we can justify the stability from the minimization. This should lead us to the convergence of the time-scheme (3.110).

3.5 Numerical illustrations

We consider a domain Ω , and assume we have access to observations in the observation subdomain ω illustrated in Figure 3.2. As an example, we consider the following wave equation

$$\begin{cases} \partial_{tt} \check{u}(x, t) - \Delta \check{u}(x, t) = \sigma(t) \check{\theta}(x), & (x, t) \in \Omega \times (0, T), \\ \check{u}(x, t) = 0, & (x, t) \in \partial\Omega \times (0, T), \\ \check{u}(x, 0) = \check{u}_0(x), & x \in \Omega, \\ \partial_t \check{u}(x, 0) = 0, & x \in \Omega, \end{cases} \quad (3.111)$$

where

$$\sigma(t) = \frac{1}{2} + \frac{1}{2} e^{-0.5t},$$

and \check{u}_0 is solution of

$$\begin{cases} -\Delta \check{u}_0(x) = 10, & x \in \Omega, \\ \check{u}_0(x) = 0, & x \in \partial\Omega, \end{cases} \quad (3.112)$$

and $\check{\theta}$ is solution of

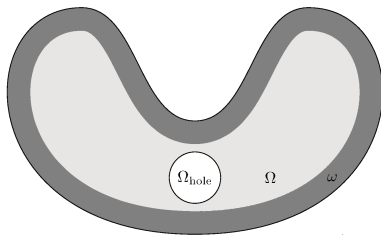
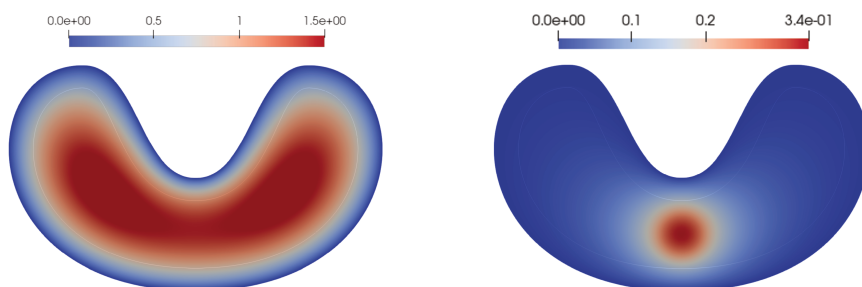
$$\begin{cases} -\Delta \check{\theta}(x) = 0 & x \in \Omega \setminus \Omega_{\text{hole}}, \\ -\Delta \check{\theta}(x) = 10, & x \in \Omega_{\text{hole}}, \\ \check{\theta}(x) = 0, & x \in \partial\Omega, \end{cases} \quad (3.113)$$

with Ω_{hole} represented in Figure 3.2.

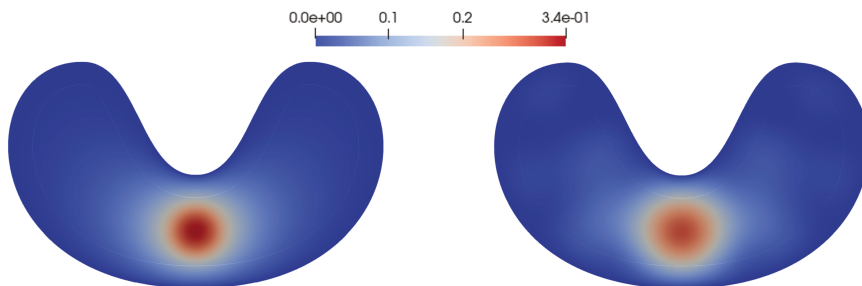
The initial conditions and the target source term defined respectively by (3.112) and the (3.113) are illustrated in Figure 3.3 (left) and 3.3 (right). Note that the choice of σ is the same as that of Remark 3.3.5 with $\beta = 1$ and $\alpha = 0.5$.

We decompose the target source term

$$\check{\theta}^h := \sum_{i=0}^{N_{\text{mod}}} \check{\theta}_i^h \phi_i^h, \quad (3.114)$$

Figure 3.2: Representation of the domain Ω and the subdomain ω Figure 3.3: Representation of \tilde{u}_0 (left) and θ (right)

where $(\phi_i^h)_{i \in [0, N_{\text{mod}}]}$ are the eigenvectors of Δ_0^h . In the following numerical illustrations, we choose $N_{\text{mod}} = 27$ so that the relative error between $\check{\theta}$ and $\check{\theta}^h$ is less than 10%, see Figure 3.4.

Figure 3.4: Source term $\check{\theta}$ (left) and its decomposition $\check{\theta}^h$ on the Laplacian basis with 27 modes (right)

In the following, we will compare the two methods presented in this chapter. The method presented is denoted H^1 -method (the noisy measurements belong to $H^1(\omega)$). Similarly, the method presented in Section 3.3.3 is denoted L^2 -method : the noisy measurements belong to $L^2(\omega)$.

3.5.1 Numerical illustrations for noise-free data

First, to generate the measurements, we implement the direct problem on a time window from 0 to 1 with a time step of $\tau = 0.01$ and with \mathbb{P}_1 finite elements and where the space function $\check{\theta}^h$ to reconstruct in the source term is defined (3.114). We give an illustration of a measurement in Figure 3.5.

We study the inverse problem for the H^1 -method in a noise-free case. We implement the observers scheme (3.110) (where C is defined by (3.7)) in Freefem++ [Hecht, 2012] over a time window $[0, 1]$ with a time step of 0.01 and with \mathbb{P}_1 finite elements. Moreover, we start from a null initial guess $\hat{\theta}_0^h = 0$ but with the correct initial conditions \tilde{u}_0 . Furthermore,

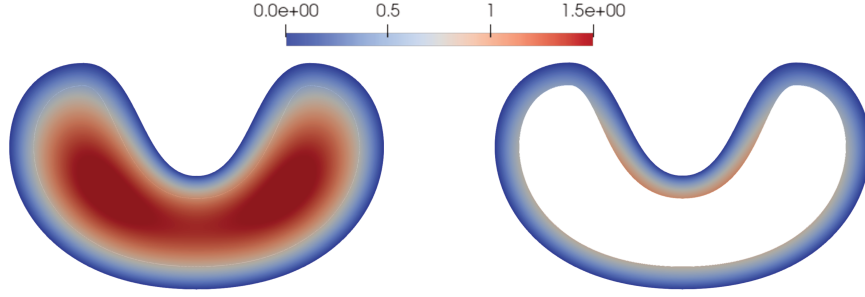


Figure 3.5: Solution \tilde{u}_n^h of the direct problem (left) and its corresponding measurements $(\tilde{u}_n^h)|_\omega$ (right) at time $t^n = 0.1$

we remove the Tikhonov regularization term since we suppose that we have the perfect measurements at our disposal. The same simulation for the L^2 -method is performed. In other words, scheme 3.110 is implemented, but this time the observation operator C is defined by (3.9).

Figure 3.6 shows the relative error between the target source term decomposed on the Laplacian modes and the observer defined by

$$\|\tilde{\theta}^h - \hat{\theta}_n^h\|_{L^2(\Omega)} / \|\tilde{\theta}^h\|_{L^2(\Omega)}$$

over the time.

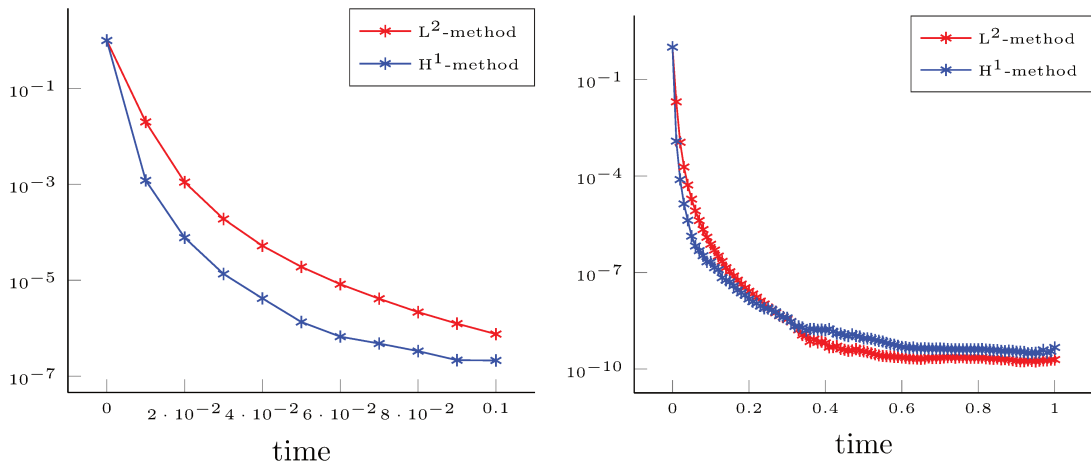


Figure 3.6: Evolution of the relative error with $\delta = 1$ and without Tikhonov term for $T = 0.1$ (left) and $T = 1$ (right)

For the H^1 -method, we can see that the error falls below $10^{-5}\%$ from $t^n = 0.1$. At the final time $t^n = 1$, the relative error is about $10^{-8}\%$. For the L^2 -method, the results are very similar. We can see that both methods converge in the long time with the same final error level. However, in the short time range (Figure 3.6 left), the H^1 -method is more efficient than the L^2 -method. This indicates that the H^1 -method benefits from improved observability.

We will now deal with noisy measurements belonging to $L^2(\omega)$.

3.5.2 Numerical illustrations with noisy measurements

We represent the mesh in Figure 3.7 (left) used to solve the inverse problem. For visualization purposes, we show a coarse mesh (Figure 3.7 right), the used mesh is a refined version of the coarse one (Figure 3.7 left).



Figure 3.7: Representation of the mesh of Ω with 56193 degrees of freedom (left) and with 14169 degrees of freedom

We corrupt the synthetic data $(y_n^{\delta,h})_{0 \leq n \leq N}$ of sampling rate T/N with a random artificial noise $\chi_n^h \in L^2(\omega)$. To do this, we perturb the measurements at each interpolation point with a random value ξ that follows the Gaussian law $\mathcal{N}(0, 1)$ and scale with the noise level δ , *i.e.*, at each interpolation point we have

$$(y_n^{\delta,h})_i = (\tilde{y}_n^h)_i + \delta \xi_i / |\omega|, \quad (3.115)$$

where the scaling by $|\omega|$ ensures that the noise is normalized. We represent in Figure 3.8 an example of a noisy measurement and the noise associated.

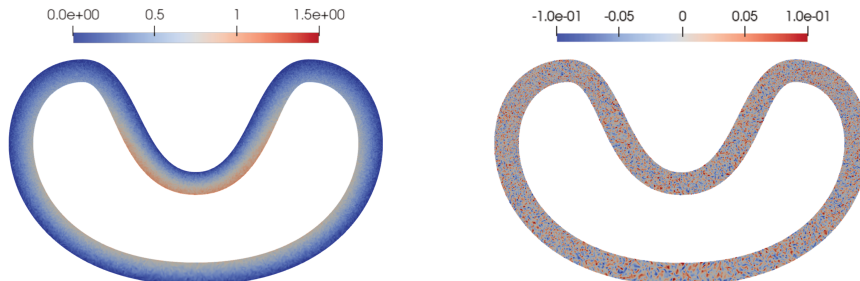


Figure 3.8: Noisy measurements $y_n^{\delta,h}$ (left) and the corresponding noise $\chi_n^h = y_n^{\delta,h} - \tilde{y}_n^h$ (right) at time $t^n = 0.1$

We solve the inverse problem with the *prediction-correction* scheme (3.110) with a time step of $\tau = 0.01$ and $\delta = 1$. Figure 3.9 shows the relative error

$$\frac{\|\tilde{\theta}^h - \hat{\theta}_n^h\|_{L^2(\Omega)}}{\|\tilde{\theta}^h\|_{L^2(\Omega)}}$$

between the target source term decomposed on the basis and the observer at the final time $t^n = 1$ as a function of the noise level δ in (3.115) for different regularization levels M .

We solve the inverse problem with the *prediction-correction* scheme (3.110) with a time step of $\tau = 0.01$ and $\delta = 1$. Figure 3.9 shows the relative error

$$\frac{\|\tilde{\theta}^h - \hat{\theta}_n^h\|_{L^2(\Omega)}}{\|\tilde{\theta}^h\|_{L^2(\Omega)}}$$

between the target source term decomposed on the basis and the observer at the final time $t^n = 1$ as a function of the noise level δ in (3.115) for different values of M .

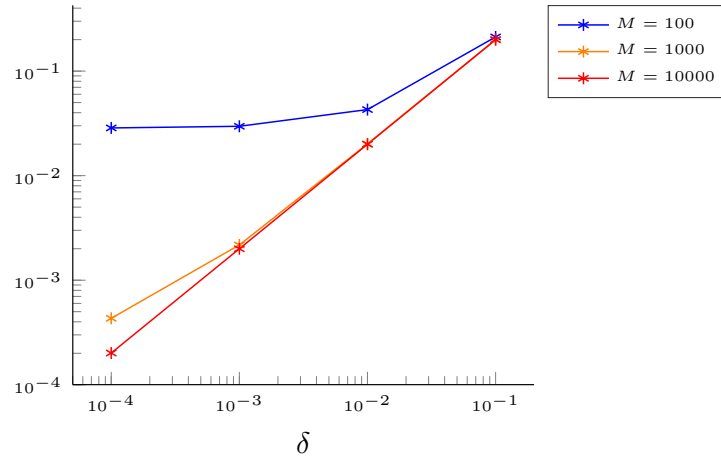


Figure 3.9: Representation of the relative error according the values of δ in loglog scale for different values of M

First, we note that the larger the value of M , the lower the weight of the regularization. In particular, we can see that the regularization with $M = 10^4$ (in red in Figure 3.9) offers better results. In particular, for the different noise levels considered – from $\delta = 0.01\%$ up to $\delta = 10\%$ in (3.115) – we reach a reconstruction of the order of the noise level.

3.6 Conclusion

We have proposed a strategy to estimate a source term in a wave equation using observers from internal *field* measurements. We have divided the study into two parts depending on whether we have access to regular or less regular measurements. The regularity of the measurements leads to major changes in the definition of the inverse problem through in the definition of the observation operator. The proposed approach follows three steps. First we prove identifiability results which rely on observability inequalities. These identifiability results give us information about the degree of the ill-posedness of the inverse problem. We then propose the minimization problem associated with the reconstruction of the source term with a well-adapted Tikhonov regularization. This inverse problem is solved using a sequential strategy that is proven to be equivalent to the minimization of the cost functional. We have assumed that we have not only internal observations on the system, but also perfect knowledge of the initial conditions, which can be a strong assumption in practice. A possible perspective would be to assume that there are uncertainties about the initial conditions. In this case, it would be necessary to further analyse the joint state-source estimation initially proposed in [Pham et al., 1998a].

CHAPTER 4

Solving inverse source wave problem – from Carleman estimates to observer design

La doublure de ma manche gauche est un petit peu déchirée... Je n'ai plus de chatterton... Tant pis, je vais mettre un clou.

— Boris Vian, l'Écume des jours.

The aim of this work, in collaboration with Muriel Boulakia (Laboratoire de Mathématiques de Versailles, Université Versailles Saint-Quentin-en-Yvelines), Maya de Buhan (Laboratoire de Mathématiques d'Orsay, Université Paris-Saclay), Philippe Moireau (Inria & Ecole Polytechnique, CNRS, Institut Polytechnique de Paris) and Sébastien Imperiale (Inria & Ecole Polytechnique, CNRS, Institut Polytechnique de Paris), is to analyse a sequential method to estimate in a wave equation a space dependent source term multiplied by a known time and space dependent function, from internal *velocity* measurements. The first part of the work consists in proving an identifiability result from a new Carleman estimate. As in the Chapter 3, thanks to the identifiability result, we can quantify the ill-posedness of the source reconstruction inverse problem. This inverse problem is solved using a sequential strategy equivalent to the minimization of a cost functional with Tikhonov regularization. We illustrate the resulting method with numerical simulations. Eventually, two perspectives are studied: extension to the case where *field* measurements are available and the case where we want to reconstruct a sharp source term.

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In this chapter we study the following inverse problem: reconstruct, from internal *velocity* measurements in a subdomain, a space dependent source term multiplied by a known space and time dependent function. This is an extension of the work in Chapter 3, where the source term was multiplied by a known function that depended only on time. As in Chapter 3, we rely on an observability property to establish an identifiability property. We will use a multiplier strategy presented in Chapters 1 and 2, except that here we use Carleman's estimates [Carleman, 1939; Buchgeim and Klibanov, 1981; Fursikov and Imanuvilov, 1996; Zhang, 2000]. This tool is widely studied in the literature for inverse problems, either for parabolic [Bellassoued and Yamamoto, 2006; Yamamoto, 2009] or hyperbolic systems [Puel and Yamamoto, 1996; Yamamoto, 1999; Zhang, 2000; Imanuvilov and Yamamoto, 2001b,a; Imanuvilov, 2002; Bellassoued, 2004], especially for wave equation [Kubo, 1995; Baudouin, 2010; Imanuvilov and Yamamoto, 2003; Baudouin et al., 2011, 2007] and discrete wave equation [Baudouin and Ervedoza, 2013]. The estimates can be either global [Puel and Yamamoto, 1996; Baudouin et al., 2007, 2011] or local [Yamamoto, 1999]. Carleman estimates are often used for inverse problems [Imanuvilov and Yamamoto, 2003, 2005; Stefanov and Uhlmann, 2013], in particular for problems with a single measurement [Imanuvilov and Yamamoto, 2003; Baudouin et al., 2007; Stefanov and Uhlmann, 2013]. In this chapter, we establish a Carleman estimate for internal measurements. Thanks to this Carleman estimate we prove the identifiability of the source term from the measurements at our disposal. In fact, it allows us to quantify the ill-posedness of the inverse problem. To solve this problem, we use a similar approach as in the Chapter 3, *i.e.*, we use an optimal observer – the Kalman estimator – equivalent to minimizing a cost functional with a Tikhonov regularization term. As a perspective, we would like to extend the various results when only *field* measurements are available. Another possible extending of this work is to consider a sharp source term to be reconstructed. In this case, we study numerically, in this case, the concept of adaptive basis. Namely, in Section 4.5.2, we propose an iterative procedure where we adapt step by step both the basis and the domain mesh to the reconstructed source term.

4.1 Problem statement

4.1.1 Model

Let $T > 0$ and Ω be a bounded, connected open domain of class C^2 in \mathbb{R}^d , $d \geq 1$. We consider the following wave equation

$$\begin{cases} \partial_{tt}u(x, t) - \Delta u(x, t) = f(x, t), & (x, t) \in \Omega \times (0, T), \\ u(x, t) = 0, & (x, t) \in \partial\Omega \times (0, T), \\ u(x, 0) = u_0(x), & x \in \Omega, \\ \partial_t u(x, 0) = v_0(x), & x \in \Omega. \end{cases} \quad (4.1)$$

Here, (u_0, v_0) are the initial conditions and $f(x, t) = \sigma(x, t)\theta(x)$, where σ is a given function and θ – assumed to depend only on the space variable x – is the parameter to recover from available *velocity* measurements. As in the Chapter 3, we rewrite (4.1) as a first-order system $z = (u, v)^t \in \mathcal{Z} = H_0^1(\Omega) \times L^2(\Omega)$,

$$\begin{cases} \dot{z} = Az + B\theta, & \text{in } (0, T), \\ z(0) = z_0, \end{cases} \quad (4.2)$$

where

$$z_0 = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ \sigma(x, t)\text{Id} \end{pmatrix}, \quad (4.3)$$

and the operator A is an unbounded skew-adjoint operator from $\mathcal{D}(A)$ into \mathcal{Z} defined by

$$A = \begin{pmatrix} 0 & \text{Id} \\ \Delta_0 & 0 \end{pmatrix}, \quad \mathcal{D}(A) = \mathcal{D}(-\Delta_0) \times H_0^1(\Omega), \quad (4.4)$$

with Δ_0 the Laplacian operator with homogeneous Dirichlet boundary conditions. The operator A is maximal dissipative, hence it is a generator of a C^0 -semigroup $(e^{tA})_{t \geq 0}$ on \mathcal{Z} .

We recall the definition of the *sensitivity*, i.e. the family $(L(t))_{t \geq 0}$ of bounded operators, so that for a given $\theta \in L^2(\Omega)$ the application $t \mapsto L(t)\theta$ is solution in $C^0([0, T], \mathcal{Z})$ of

$$\begin{cases} \dot{z} = Az + B\theta, \\ z(0) = 0, \end{cases} \quad (4.5)$$

then, the Duhamel's formula also reads $z(t) = e^{tA}z_0 + L(t)\theta$, $t \in [0, T]$ for the solutions of (4.1).

4.1.2 Observation operator

We must now model the measurement procedure. We define a domain compatible with stronger conditions than in Chapter 3, using the conditions presented in Chapter 1 i.e. multiplier conditions. Namely, we introduce a part of the boundary on which the observation domain have to lean on. For every $x_0 \in \mathbb{R}^d \setminus \bar{\Omega}$, we define

$$\Gamma_0 := \{x \in \partial\Omega \mid (x - x_0) \cdot \nu(x) > 0\},$$

where ν is the outward unit normal to $\partial\Omega$. Then, for every $\rho > 0$, we define

$$\omega_{\rho, x_0} := \Omega \cap \bigcup_{x \in \Gamma_0} B(x, \rho). \quad (4.6)$$

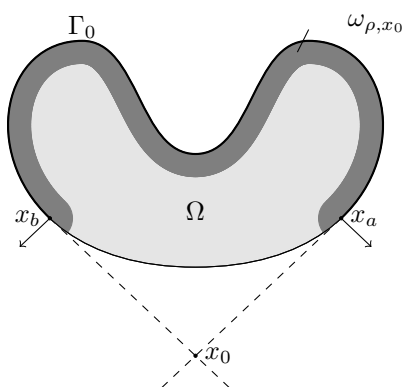


Figure 4.1: An example of domain ω_{ρ, x_0} .

We assume that the observation domain ω is an open and nonempty subset of Ω such that the boundaries $\partial\omega$ and $\partial(\Omega \setminus \omega)$ are Lipschitz, and that the *multiplier condition* is satisfied: there exists $x_0 \in \mathbb{R}^d \setminus \bar{\Omega}$ and $\rho > 0$ such that

$$\omega_{\rho, x_0} \subset \omega. \quad (4.7)$$

We also assume that the observation time T satisfies the *time condition*:

$$T > T_0 := \sup_{x \in \Omega} |x - x_0|. \quad (4.8)$$

We consider observations of $y = \partial_t u$ where u is solution of (4.1) (for an unknown θ), in the interior domain ω during the time interval $(0, T)$, or any of its time derivatives. When measuring $\partial_{tt} u$ (or any high order time derivative), we can return to the case where $\partial_t u|_\omega$ is observed without loss of generality by simply integrating

$$\partial_t u|_\omega(t) = \partial_t u|_\omega(0) + \int_0^t \partial_{tt} u|_\omega(\tau) d\tau.$$

Let us look at the definition of the observation operator, which is usually introduced in control and observation theory – see for example [Tucsnak and Weiss, 2009] and the references therein. The observation operator modeling the observation process is as follows

$$C = \begin{pmatrix} 0 & I_\omega \end{pmatrix} \in \mathcal{L}(\mathcal{Z}, \mathcal{Y}), \quad y = Cz, \quad (4.9)$$

where $\mathcal{Y} = L^2(\omega)$ and $I_\omega \in \mathcal{L}(L^2(\Omega), L^2(\omega))$ is the restriction operator.

4.1.3 Inverse problem setting

Let us present the inverse problem studied in this chapter. We consider an actual trajectory \check{u} , modeled as a mild solution \check{z} of (4.2) associated with given (u_0, v_0) and σ , and an unknown $\check{\theta}$. Since the initial data are known, we consider, without loss of generality, that they are null thanks to the linearity of the following inverse problem. More precisely, we consider the following system

$$\begin{cases} \partial_{tt} \check{u}(x, t) - \Delta \check{u}(x, t) = \sigma(x, t) \check{\theta}(x), & (x, t) \in \Omega \times (0, T), \\ \check{u}(x, t) = 0, & (x, t) \in \partial\Omega \times (0, T), \\ \check{u}(x, 0) = 0, & x \in \Omega, \\ \partial_t \check{u}(x, 0) = 0, & x \in \Omega, \end{cases} \quad (4.10)$$

and its first-order formulation

$$\begin{cases} \dot{\check{z}} = A\check{z} + B\check{\theta}, & \text{in } (0, T), \\ \check{z}(0) = 0, \end{cases} \quad (4.11)$$

where B and A are respectively defined by (4.3) and (4.4). We assume that the measurements are perturbed by some measurement errors of amplitude δ

$$\exists \delta > 0 \text{ such that } \|y^\delta - Cz\|_{L^2(0, T; L^2(\omega))} \leq \delta \sqrt{T}.$$

In particular, y^δ is assumed to belong to $L^2(0, T; L^2(\omega))$. Therefore, recovering $\check{\theta}$ from y^δ consists in inverting the linear *input-output* operator

$$\Psi_T : \begin{cases} L^2(\Omega) \rightarrow L^2(0, T; L^2(\omega)), \\ \theta \mapsto (t \mapsto CL(t)\theta). \end{cases} \quad (4.12)$$

As in the previous chapter, we immediately note that Ψ_T is bounded, since C is a bounded operator from \mathcal{Z} to $L^2(\omega)$. We will show in Section 4.2.2 that Ψ_T in $L^2(0, T; L^2(\omega))$ is not surjective, but defines a *mildly* ill-posed problem of degree 1 when considering *velocity* measurements. This will be done by proving adapted Carleman estimate in Section 4.2.1. Then, in Section 4.3, we propose a regularization strategy to invert Ψ_T using observer methods. Afterwards, we look at how this work can be extended to consider *field* measurements such as in the previous chapter. Eventually, in Section 4.5.2 we present an adaptive basis method to extend our strategy to a sharp source term.

4.2 From observability to regularization strategy

4.2.1 Carleman estimate

In this section, we state Carleman estimate that will be useful for the study of the inverse problem. In order to apply the estimates to the source identification problem, we adapt the classical Carleman estimate for the wave equation [Zhang, 2000; Imanuvilov and Yamamoto, 2001b; Imanuvilov, 2002; Baudouin, 2010; Baudouin et al., 2013] through adequate choices of the norms appearing in the *observation term*, namely the terms supported by ω .

Before stating our result, we need to introduce the adequate Carleman weights. For $\beta \in (0, 1)$ and $\lambda > 0$, we define, for all $(x, t) \in \Omega \times (-T, T)$

$$\psi(x, t) = |x - x_0|^2 - \beta t^2 + C_0^{\text{st}} \text{ and } \phi(x, t) = e^{\lambda\psi(x, t)}, \quad (4.13)$$

where $C_0^{\text{st}} > 0$ is chosen such that $\psi \geq 1$ in $\Omega \times (-T, T)$.

Then we have the following Carleman inequality.

Theorem 4.2.1 *We assume that ω satisfies (4.7). Let $T \geq T_0$ defined by (4.8). There exist $\lambda > 0$, $s_0 > 0$ and a positive constant C^{st} such that for ϕ defined by (4.13), for all $s \geq s_0$ and for all $w \in L^2(-T, T; H_0^1(\Omega)) \cap H^1(-T, T; L^2(\omega))$ such that $Pw = \partial_{tt}w - \Delta w \in L^2(\Omega \times (-T, T))$ and $w(\cdot, 0) = w(\cdot, \pm T) = \partial_t w(\cdot, \pm T) = 0$ in Ω , we have,*

$$\begin{aligned} s^{1/2} \int_{\Omega} e^{2s\phi(0)} |\partial_t w(0)|^2 dx + \int_{-T}^T \int_{\Omega} e^{2s\phi} (s|\nabla w|^2 + s|\partial_t w|^2 + s^3|w|^2) dx dt \\ \leq C^{\text{st}} \int_{-T}^T \int_{\Omega} e^{2s\phi} |Pw|^2 dx dt + C^{\text{st}} s^2 \int_{-T}^T \int_{\omega} e^{2s\phi} (|\nabla w|^2 + s^2|w|^2) dx dt. \end{aligned} \quad (4.14)$$

In the proof of this result, we will not give all the details but mainly focus on the more original part that corresponds to the derivation of our different *observation term*, *i.e.* in [Baudouin et al., 2013] the considered *observation term* is a boundary term, while here it is an internal *observation term* which is the last terms in (4.14). The proof follows [Baudouin et al., 2013].

In this section we write $a \lesssim b$ if there exists a positive scalar $c > 0$ independent of s such that $a \leq cb$. **PROOF:** In this proof, all the computations will be done for smooth functions and lastly a density argument allows to extend the result to the regularity given in the theorem. The beginning of the proof will closely follow the one of Theorem 2.1 in [Baudouin et al., 2013]. More precisely, we will start from equation (2.13) in [Baudouin et al., 2013] that we rewrite here with the difference that we do not highlight the dependence of the inequality with respect to λ : there exists $s_0 > 0$ such that, for all $s > s_0$, any function z in $C^2(\bar{\Omega} \times [-T, T])$ such that $z(\cdot, 0) = 0$ in Ω , $z(\cdot, \pm T) = \partial_t z(\cdot, \pm T) = 0$ in Ω and $z = 0$ on $\partial\Omega \times [-T, T]$ satisfies

$$\begin{aligned} s \int_{-T}^T \int_{\Omega} (|\partial_t z|^2 + |\nabla z|^2 + s^2|z|^2) dx dt + \int_{-T}^T \int_{\Omega} |P_1 z|^2 dx dt \\ \lesssim \int_{-T}^T \int_{\Omega} |P_{\text{conj}} z|^2 dx dt + s \int_{-T}^T \int_{\Gamma_0} |\partial_\nu z|^2 d\sigma dt, \end{aligned} \quad (4.15)$$

where

$$P_1 z = \partial_{tt} z - \Delta z + s^2 \lambda^2 \phi^2 z (|\partial_t \psi|^2 - |\nabla \psi|^2), \quad (4.16)$$

and

$$\begin{aligned}
 P_{conj}z &= e^{s\phi}P(e^{-s\phi}z) \\
 &= \partial_{tt}z - 2s\lambda\phi(\partial_t z \partial_t \psi - \nabla z \cdot \nabla \psi) + s^2\lambda^2\phi^2z(|\partial_t \psi|^2 - |\nabla \psi|^2) \\
 &\quad - \Delta z - s\lambda\phi z(\partial_{tt}\psi - \Delta\psi) - s\lambda^2z(|\partial_t \psi|^2 - |\nabla \psi|^2)\phi.
 \end{aligned} \tag{4.17}$$

Let w be given in $C^2(\bar{\Omega} \times [-T, T])$ such that $w = 0$ on $\partial\Omega \times [-T, T]$, $w(\cdot, 0) = 0$ in Ω , $w(\cdot, \pm T) = \partial_t w(\cdot, \pm T) = 0$ in Ω . We define v by the classical change of variables $v = e^{s\phi}w$ in $\Omega \times (-T, T)$.

We will apply inequality (4.15) to $z = \chi v$ where $\chi \in C^\infty(\bar{\Omega})$ is a cut-off function such that $0 \leq \chi \leq 1$ and

$$\chi(x) = \begin{cases} 0, & \text{if } x \in \omega_1, \\ 1, & \text{if } x \in \Omega \setminus \omega_2 \end{cases} \tag{4.18}$$

where $\omega_1 = \omega_{\rho_1, x_0}$ and $\omega_2 = \omega_{\rho_2, x_0}$ are defined by (4.6) with $\rho_1 < \rho_2 < \rho$ and illustrated in Figure 4.2.

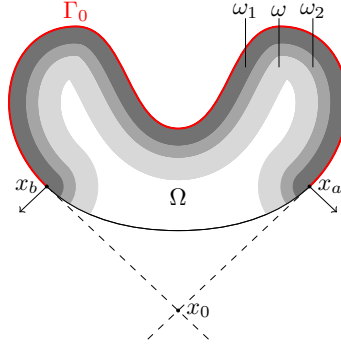


Figure 4.2: Definitions of domain ω_1 and ω_2 .

We notice that

$$\begin{aligned}
 \partial_t z &= \chi \partial_t v, & \partial_{tt} z &= \chi \partial_{tt} v, & \nabla z &= \nabla \chi v + \chi \nabla v, \\
 \Delta z &= \Delta \chi v + 2\nabla \chi \cdot \nabla v + \chi \Delta v, \\
 P_1 z &= \chi P_1 v - \Delta \chi v - 2\nabla \chi \cdot \nabla v, \\
 P_{conj} z &= \chi P_{conj} v + 2s\lambda\phi \nabla \chi \cdot \nabla \psi v - \Delta \chi v - 2\nabla \chi \cdot \nabla v.
 \end{aligned} \tag{4.19}$$

Using that $v = z$ on $\Omega \setminus \omega_2$, we obtain

$$\begin{aligned}
 &s \int_{-T}^T \int_{\Omega} (|\partial_t v|^2 + |\nabla v|^2 + s^2|v|^2) dx dt + \int_{-T}^T \int_{\Omega} |P_1 v|^2 dx dt \\
 &= s \int_{-T}^T \int_{\Omega \setminus \omega_2} (|\partial_t z|^2 + |\nabla z|^2 + s^2|z|^2) dx dt + \int_{-T}^T \int_{\Omega \setminus \omega_2} |P_1 z|^2 dx dt \\
 &\quad + s \int_{-T}^T \int_{\omega_2} (|\partial_t v|^2 + |\nabla v|^2 + s^2|v|^2) dx dt + \int_{-T}^T \int_{\omega_2} |P_1 v|^2 dx dt.
 \end{aligned}$$

Therefore, applying inequality (4.15) to z , we get that, for $s \geq s_0$

$$\begin{aligned} s \int_{-T}^T \int_{\Omega} (|\partial_t v|^2 + |\nabla v|^2 + s^2 |v|^2) dx dt + \int_{-T}^T \int_{\Omega} |P_1 v|^2 dx dt \\ \lesssim \int_{-T}^T \int_{\Omega} |P_{conj} z|^2 dx dt + s \int_{-T}^T \int_{\Gamma_0} |\partial_\nu z|^2 d\sigma dt \\ + s \int_{-T}^T \int_{\omega_2} (|\partial_t v|^2 + |\nabla v|^2 + s^2 |v|^2) dx dt + \int_{-T}^T \int_{\omega_2} |P_1 v|^2 dx dt. \end{aligned}$$

Eventually, since $\partial_\nu z = \partial_\nu \chi v + \chi \partial_\nu v = 0$ on Γ_0 and using (4.19), we get

$$\begin{aligned} s \int_{-T}^T \int_{\Omega} (|\partial_t v|^2 + |\nabla v|^2 + s^2 |v|^2) dx dt + \int_{-T}^T \int_{\Omega} |P_1 v|^2 dx dt \\ \lesssim \int_{-T}^T \int_{\Omega} |P_{conj} v|^2 dx dt + s \int_{-T}^T \int_{\omega_2} (|\partial_t v|^2 + |\nabla v|^2 + s^2 |v|^2) dx dt \\ + \int_{-T}^T \int_{\omega_2} |P_1 v|^2 dx dt. \quad (4.20) \end{aligned}$$

Next, using the definitions (4.16) and (4.17), we observe that

$$|P_1 v|^2 \lesssim |P_{conj} v|^2 + s^2 (|\partial_t v|^2 + |\nabla v|^2 + |v|^2),$$

thus, for s large enough, we can replace of the last term in (4.20) to obtain

$$\begin{aligned} s \int_{-T}^T \int_{\Omega} (|\partial_t v|^2 + |\nabla v|^2 + s^2 |v|^2) dx dt + \int_{-T}^T \int_{\Omega} |P_1 v|^2 dx dt \\ \lesssim \int_{-T}^T \int_{\Omega} |P_{conj} v|^2 dx dt + s^2 \int_{-T}^T \int_{\omega_2} (|\partial_t v|^2 + |\nabla v|^2 + s |v|^2) dx dt. \quad (4.21) \end{aligned}$$

We want to add a term in $\partial_t v(0)$ to the left-hand side. According to the definition (4.16) of P_1 , an integration by parts in space gives:

$$\begin{aligned} \int_{-T}^0 \int_{\Omega} P_1 v \partial_t v dx dt = \frac{1}{2} \int_{-T}^0 \int_{\Omega} \partial_t (|\partial_t v|^2) dx dt + \frac{1}{2} \int_{-T}^0 \int_{\Omega} \partial_t (|\nabla v|^2) dx dt \\ + \frac{s^2 \lambda^2}{2} \int_{-T}^0 \int_{\Omega} \phi^2 (|\partial_t \psi|^2 - |\nabla \psi|^2) \partial_t (|v|^2) dx dt, \end{aligned}$$

and since $v(\cdot, 0) = 0$ in Ω , $v(\cdot, -T) = \partial_t v(\cdot, -T) = 0$ in Ω

$$\begin{aligned} \int_{-T}^0 \int_{\Omega} P_1 v \partial_t v dx dt = \frac{1}{2} \int_{\Omega} |\partial_t v(0)|^2 dx \\ - \frac{s^2 \lambda^2}{2} \int_{-T}^0 \int_{\Omega} \partial_t (\phi^2 (|\partial_t \psi|^2 - |\nabla \psi|^2)) |v|^2 dx dt. \end{aligned}$$

Therefore, by Cauchy-Schwarz, we obtain

$$\begin{aligned} s^{1/2} \int_{\Omega} |\partial_t v(0)|^2 dx \lesssim \int_{-T}^T \int_{\Omega} |P_1 v|^2 dx dt \\ + s \int_{-T}^T \int_{\Omega} |\partial_t v|^2 dx dt + s^{5/2} \int_{-T}^T \int_{\Omega} |v|^2 dx dt. \quad (4.22) \end{aligned}$$

Injecting this inequality in (4.20), we obtain that, for $s \geq s_0$,

$$\begin{aligned} s^{1/2} \int_{\Omega} |\partial_t v(0)|^2 dx + s \int_{-T}^T \int_{\Omega} (|\partial_t v|^2 + |\nabla v|^2 + s^2 |v|^2) dx dt + \int_{-T}^T \int_{\Omega} |P_1 v|^2 dx dt \\ \lesssim \int_{-T}^T \int_{\Omega} |P_{conj} v|^2 dx dt + s^2 \int_{-T}^T \int_{\omega_2} (|\partial_t v|^2 + |\nabla v|^2 + s |v|^2) dx dt. \end{aligned}$$

Next, we come back to the initial function $w = e^{-s\phi} v$. After classical computations that we do not detail here (see for instance [Baudouin et al., 2013]), we get

$$\begin{aligned} s^{1/2} \int_{\Omega} e^{2s\phi(0)} |\partial_t w(0)|^2 dx + s \int_{-T}^T \int_{\Omega} e^{2s\phi} (|\partial_t w|^2 + |\nabla w|^2 + s^2 |w|^2) dx dt \\ \lesssim \int_{-T}^T \int_{\Omega} e^{2s\phi} |Pw|^2 dx dt \\ + s^2 \int_{-T}^T \int_{\omega_2} e^{2s\phi} (|\partial_t w|^2 + |\nabla w|^2 + s^2 |w|^2) dx dt. \quad (4.23) \end{aligned}$$

Now we will work on the *observation term* in the right-hand side. To do so, we define a function $\rho \in C^\infty(\Omega)$ such that

$$\rho(x) = \begin{cases} 1, & \text{if } x \in \omega_2, \\ 0, & \text{if } x \in \Omega \setminus \omega, \end{cases}$$

and observe that, from the definition of P , we obviously have

$$\int_{-T}^T \int_{\omega} e^{2s\phi} \rho (\partial_{tt} w - \Delta w - Pw) w dx dt = 0. \quad (4.24)$$

Let us consider separately the three integrals coming from this equality. For the first term we have, thanks to integrations by parts in time,

$$\begin{aligned} \int_{-T}^T \int_{\omega} e^{2s\phi} \rho \partial_{tt} w w dx dt &= - \int_{-T}^T \int_{\omega} \partial_t w \partial_t (e^{2s\phi} \rho w) dx dt \\ &= - \int_{-T}^T \int_{\omega} |\partial_t w|^2 e^{2s\phi} \rho dx dt - s \int_{-T}^T \int_{\omega} \partial_t (|w|^2) e^{2s\phi} \partial_t \phi \rho dx dt \\ &= - \int_{-T}^T \int_{\omega} e^{2s\phi} \rho |\partial_t w|^2 dx dt + 2s^2 \int_{-T}^T \int_{\omega} e^{2s\phi} |\partial_t \phi|^2 |w|^2 \rho dx dt \\ &\quad + s \int_{-T}^T \int_{\omega} e^{2s\phi} \partial_{tt} \phi \rho |w|^2 dx dt. \quad (4.25) \end{aligned}$$

For the second term in (4.24) we have, using Green formula

$$\begin{aligned}
 - \int_{-T}^T \int_{\omega} e^{2s\phi} \rho \Delta w w \, dx \, dt &= \int_{-T}^T \int_{\omega} \nabla(e^{2s\phi} \rho w) \cdot \nabla w \, dx \, dt \\
 &= \int_{-T}^T \int_{\omega} e^{2s\phi} \rho |\nabla w|^2 \, dx \, dt + s \int_{-T}^T \int_{\omega} e^{2s\phi} \nabla(w^2) \cdot \nabla \phi \rho \, dx \, dt \\
 &\quad + \frac{1}{2} \int_{-T}^T \int_{\omega} e^{2s\phi} \nabla(w^2) \cdot \nabla \rho \, dx \, dt \\
 &= \int_{-T}^T \int_{\omega} e^{2s\phi} |\nabla w|^2 \rho \, dx \, dt - s \int_{-T}^T \int_{\omega} e^{2s\phi} \Delta \phi \rho |w|^2 \, dx \, dt \\
 &\quad - 2s^2 \int_{-T}^T \int_{\omega} e^{2s\phi} \rho |\nabla \phi|^2 |w|^2 \, dx \, dt \\
 &\quad - \frac{1}{2} \int_{-T}^T \int_{\omega} e^{2s\phi} \Delta \rho |w|^2 \, dx \, dt \\
 &\quad - 2s \int_{-T}^T \int_{\omega} e^{2s\phi} \nabla \rho \cdot \nabla \phi |w|^2 \, dx \, dt. \tag{4.26}
 \end{aligned}$$

Using (4.25) and (4.26) in (4.24), we obtain

$$\begin{aligned}
 \int_{-T}^T \int_{\omega} e^{2s\phi} \rho |\partial_t w|^2 \, dx \, dt - \int_{-T}^T \int_{\omega} e^{2s\phi} \rho |\nabla w|^2 \, dx \, dt \\
 = \int_{-T}^T \int_{\omega} e^{2s\phi} \left(s \partial_{tt} \phi \rho + 2s^2 |\partial_t \phi|^2 \rho - \frac{1}{2} \Delta \rho - 2s \nabla \rho \cdot \nabla \phi - s \Delta \phi \rho \right. \\
 \left. - 2s^2 |\nabla \phi|^2 \rho \right) |w|^2 \, dx \, dt - \int_{-T}^T \int_{\omega} e^{2s\phi} \rho P w w \, dx \, dt. \tag{4.27}
 \end{aligned}$$

From (4.27) and by definition of ρ , we deduce the following estimate:

$$\begin{aligned}
 \int_{-T}^T \int_{\omega_2} e^{2s\phi} |\partial_t w|^2 \, dx \, dt \lesssim \frac{1}{s^2} \int_{-T}^T \int_{\Omega} e^{2s\phi} |P w|^2 \, dx \, dt \\
 + \int_{-T}^T \int_{\omega} (|\nabla w|^2 + s^2 |w|^2) \, dx \, dt. \tag{4.28}
 \end{aligned}$$

Using this inequality in the right-hand side of (4.23), we finally get (4.14). ■

We have proved a new Carleman estimate for the wave equation with new a *observation term* in the right-hand side. This will allow us to establish an identifiability property where the H^1 norm of the velocity in a subdomain controls the source term.

4.2.2 Identifiability results

In this section, we will state stability estimates for the identification problem of $\tilde{\theta}$ thanks to measurements on the solution $\partial_t \tilde{u}$ of (4.10). To do so, we will follow the method introduced in [Buchgeim and Klibanov, 1981] and apply the Carleman estimate given by Theorem 4.2.1 to the time derivative of the solution.

Theorem 4.2.2 *We assume that ω satisfies (4.7) and that $\sigma \in H^1(0, T; L^\infty(\Omega))$ is such that $|\sigma(x, 0)| \geq \alpha > 0$ a.e. in Ω . Let $T \geq T_0$ defined by (4.8).*

Let $u|_\theta$ the solution of (4.10) associated with $\theta \in L^2(\Omega)$. Then, there exists a positive scalar C^{st} such that

$$\int_{\Omega} |\theta|^2 dx \leq C^{st} \int_0^T \int_{\omega} (|\nabla \partial_t u|_\theta|^2 + |\partial_t u|_\theta|^2) dx dt. \quad (4.29)$$

PROOF: Since the right-hand side $\sigma\theta$ of (4.10) belongs to $H^1(0, T; L^2(\Omega))$, a classical existence result (we refer to [Lions et al., 1968] or [Bensoussan et al., 2007]) allows to assert that u belongs to $C^2([0, T]; L^2(\Omega)) \cap C^1([0, T]; H^1(\Omega)) \cap C^0([0, T]; H^2(\Omega))$, and we have

$$\|u\|_{C^2([0, T]; L^2(\Omega))} + \|u\|_{C^1([0, T]; H^1(\Omega))} + \|u\|_{C^0([0, T]; H^2(\Omega))} \leq C^{st} \|\theta\|_{L^2(\Omega)}. \quad (4.30)$$

Following Buchgeim-Klibanov's method, we introduce $v = \partial_t u$ that satisfies

$$\begin{cases} \partial_{tt}v(x, t) - \Delta v(x, t) = \partial_t \sigma(x, t)\theta(x), & (x, t) \in \Omega \times (0, T), \\ v(x, t) = 0, & (x, t) \in \partial\Omega \times (0, T), \\ v(x, 0) = 0, & x \in \Omega, \\ \partial_t v(x, 0) = \sigma(x, 0)\theta(x), & x \in \Omega. \end{cases} \quad (4.31)$$

We extend the function v on $(-T, 0)$ by setting $v(\cdot, -t) = -v(\cdot, t)$. Then, for $\delta > 0$, we introduce the cut-off function $\eta \in C_c^\infty(\mathbb{R})$, plotted in Figure 4.3, such that $0 \leq \eta \leq 1$,

$$\eta(t) = \begin{cases} 1, & \text{if } -T + \delta \leq t \leq T - \delta, \\ 0, & \text{if } t \leq -T \text{ or } t \geq T, \end{cases} \quad (4.32)$$

and we set $w = \eta v$ in $\Omega \times (-T, T)$.

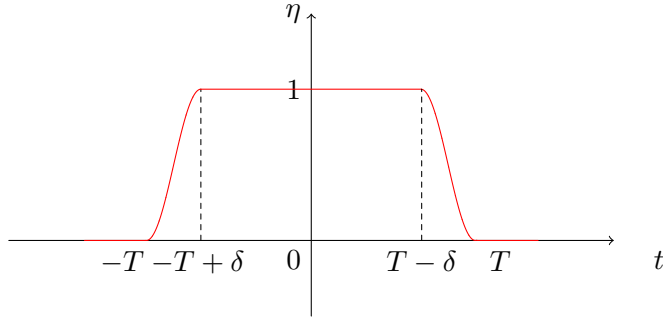


Figure 4.3: Cut-off function η defined in (4.32).

And w satisfies $w(\cdot, 0) = w(\cdot, \pm T) = \partial_t w(\cdot, \pm T) = 0$ in Ω . Therefore, since ω satisfies (4.7) and since the time condition (4.8) holds, we can apply the Carleman estimate (4.14) of Theorem 4.2.1:

$$\begin{aligned} s^{1/2} \int_{\Omega} e^{2s\phi(0)} |\partial_t w(0)|^2 dx \\ \lesssim \int_{-T}^T \int_{\Omega} e^{2s\phi} |Pw|^2 dx dt + s^2 \int_{-T}^T \int_{\omega} e^{2s\phi} (|\nabla w|^2 + s^2 |w|^2) dx dt. \end{aligned} \quad (4.33)$$

For the term in the left-hand side, we have $\partial_t w(x, 0) = \eta(0)\partial_t v(x, 0) = \sigma(x, 0)\theta(x)$ where, by hypothesis, $|\sigma(x, 0)| \geq \alpha > 0$ for $x \in \Omega$. Therefore,

$$s^{1/2} \int_{\Omega} e^{2s\phi(0)} |\partial_t w(0)|^2 dx \geq \alpha^2 s^{1/2} \int_{\Omega} e^{2s\phi(0)} |\theta|^2 dx. \quad (4.34)$$

For the first term in the right-hand side of (4.33), we notice that

$$Pw = \eta \partial_t \sigma \theta + 2 \partial_t \eta \partial_t v + \partial_{tt} \eta v. \quad (4.35)$$

On one hand, since $\sigma \in H^1(0, T; L^\infty(\Omega))$ we directly have

$$\int_{-T}^T \int_{\Omega} e^{2s\phi} |\eta \partial_t \sigma \theta|^2 dx dt \lesssim \int_{\Omega} e^{2s\phi(0)} |\theta|^2 dx. \quad (4.36)$$

On the other hand, the derivatives of η vanish on $[-T + \delta, T - \delta]$ and, by definition (4.13) of ϕ , we have

$$\forall t \in [-T, -T + \delta] \cup [T - \delta, T], \forall x \in \Omega, \quad \psi(x, t) < C_0^{\text{st}} < \psi(x, 0), \quad (4.37)$$

if we take δ sufficiently small and β such that

$$\frac{1}{T} \sup_{x \in \bar{\Omega}} |x - x_0| < \sqrt{\beta} < 1. \quad (4.38)$$

Therefore, using estimate (4.30), we have

$$\begin{aligned} \int_{-T}^T \int_{\Omega} e^{2s\phi} |2\partial_t \eta \partial_t v + \partial_{tt} \eta v|^2 dx dt &\lesssim e^{2s \exp(C_0^{\text{st}})} \int_{-T}^T \int_{\Omega} (|\partial_{tt} u|^2 + |\partial_t u|^2) dx dt \\ &\lesssim e^{2s \exp(C_0^{\text{st}})} \int_{\Omega} |\theta|^2 dx \lesssim \int_{\Omega} e^{2s\phi(0)} |\theta|^2 dx. \end{aligned}$$

Hence, using this inequality, (4.34) and (4.36), estimate (4.33) leads to:

$$\alpha^2 s^{1/2} \int_{\Omega} e^{2s\phi(0)} |\theta|^2 dx \lesssim \int_{\Omega} e^{2s\phi(0)} |\theta|^2 dx + s^2 \int_{-T}^T \int_{\omega} e^{2s\phi} (|\nabla \eta v|^2 + s^2 |\eta v|^2) dx dt.$$

Therefore, if s is chosen large enough, we can absorb the first term in the right-hand side by the term in the left-hand side. By this way we have

$$s^2 \int_{-T}^T \int_{\omega} e^{2s\phi} (|\nabla \eta v|^2 + s^2 |\eta v|^2) dx dt \lesssim e^{2s\|\phi(0)\|_{C^0(\Omega)}} \int_{-T}^T \int_{\omega} |\nabla v|^2 + |v|^2 dx dt.$$

Since $v = \partial_t u$ and $v(0) = 0$ and since the integrals on $(-T, T)$ are twice the integrals on $(0, T)$, this concludes the proof of Theorem 4.2.2. \blacksquare

We deduce from the observability inequality (4.29) that the observations on $\partial_t u$ must belong to $L^2(0, T; H^1(\omega))$, such that a stable reconstruction of θ is possible in a L^2 -norm. As a result, we are dealing with an ill-posed inverse problem of degree 1. However, note that we here assume that we measure the velocity of the solution. When measuring only the field, our estimate will indicate that the problem is ill-posed of degree 2.

4.3 From regularization strategy to observer design

4.3.1 Tikhonov regularization and error reconstruction

We recall that we assume that a target trajectory \tilde{u} solution of (4.10) produced by a target source term parameter $\tilde{\theta}$ has resulted in a set of measurements y^δ where

$$\int_0^T \|y^\delta(x, t) - \partial_t \tilde{u}(x, t)\|_{L^2(\omega)}^2 dt \leq \delta^2 T, \quad (4.39)$$

for a parameter $\delta > 0$. Following the methodology presented in Chapter 3, we introduce the cost functional to minimize $\mathcal{J}_T : \mathbf{H}_0^1(\Omega) \rightarrow \mathbb{R}$ defined by

$$\begin{aligned} \mathcal{J}_T(\theta) &= \frac{\epsilon}{2} \|\theta\|_{\mathbf{H}_0^1(\Omega)}^2 + \frac{1}{2} \int_0^T \|y^\delta(t) - \partial_t u_{|\theta}\|_{L^2(\omega)}^2 dt \\ &= \frac{\epsilon}{2} \|\theta\|_{\mathbf{H}_0^1(\Omega)}^2 + \frac{1}{2} \int_0^T \|y^\delta(t) - Cz_{|\theta}\|_{L^2(\omega)}^2 dt, \end{aligned} \quad (4.40)$$

where $u_{|\theta}$ is the solution of (4.10) associated to the parameter θ , C is defined by (4.9) and $\epsilon > 0$. Since the functional is quadratic, the application $\theta \mapsto I_\omega(\partial_t u)$ is continuous and $\theta \mapsto \|\theta\|_{\mathbf{H}_0^1(\Omega)}^2$ is coercive, we deduce the existence of a unique minimizer in $\mathbf{H}_0^1(\Omega)$ of \mathcal{J}_T .

We denote it by $\bar{\theta}_T := \operatorname{argmin}_{\theta \in \mathbf{H}_0^1(\Omega)} \mathcal{J}_T$.

We want to show the reconstruction property, *i.e.* we want to quantify the approximation between the target parameter $\bar{\theta}$ and the minimizer of the cost functional $\bar{\theta}_T$ with respect to the noise level δ . To do so, we introduce a regularization operator G_ω^α that will regularize $L^2(\omega)$ measurements in $\mathbf{H}_0^1(\omega)$. Namely, for $\alpha > 0$ small, we define $G_\omega^\alpha \in \mathcal{L}(L^2(\omega), \mathbf{H}_0^1(\omega))$ satisfying for all $y \in L^2(\omega)$, $G_\omega^\alpha y = y^\alpha$ where y^α is solution of

$$\begin{cases} -\Delta y^\alpha + \frac{1}{\alpha^2} y^\alpha = \frac{1}{\alpha^2} y, & \text{in } \omega, \\ y^\alpha = 0, & \text{on } \partial\omega. \end{cases} \quad (4.41)$$

Let us look at some properties of this regularization operator. First, we give a stability property.

Proposition 4.3.1 *The regularization operator G_ω^α defined by (4.41) satisfies,*

$$\forall y \in L^2(\omega), \quad \|G_\omega^\alpha y\|_{\mathbf{H}_0^1(\omega)} \leq \frac{1}{\sqrt{2}\alpha} \|y\|_{L^2(\omega)}.$$

PROOF: We denote $y^\alpha = G_\omega^\alpha y$. The operator G_ω^α satisfies the following variational formulation $\forall v \in \mathbf{H}_0^1(\omega)$,

$$\int_\omega \nabla y^\alpha \nabla v dx + \frac{1}{\alpha^2} \int_\omega y^\alpha v dx = \frac{1}{\alpha^2} \int_\omega y v dx.$$

Choosing the test function $v = y^\alpha$ and using Cauchy-Schwarz and Young inequalities, we finally get

$$\|\nabla y^\alpha\|_{L^2(\omega)}^2 \leq \frac{1}{2\alpha^2} \|y\|_{L^2(\omega)}^2. \quad \blacksquare$$

In fact, this operator is a regularization of the identity operator. We can show the following approximation theorem.

Proposition 4.3.2 *The regularization operator G_ω^α defined by (4.41) satisfies, $\forall y \in \mathbf{H}_0^1(\omega)$ such that $\Delta y \in L^2(\omega)$,*

$$\|(G_\omega^\alpha - Id)y\|_{\mathbf{H}_0^1(\omega)} \leq \frac{\alpha}{\sqrt{2}} \|\Delta y\|_{L^2(\omega)}.$$

PROOF: We denote $y^\alpha = G_\omega^\alpha y$. The operator G_ω^α satisfies the following variational formulation $\forall v \in \mathbf{H}_0^1(\omega)$,

$$\int_\omega \nabla(y^\alpha - y) \nabla v dx + \frac{1}{\alpha^2} \int_\omega (y^\alpha - y) v dx = \int_\omega \Delta y v dx.$$

Choosing the test function $v = y^\alpha - y$ and using Cauchy–Schwarz and Young inequalities, we finally get

$$\|\nabla(y^\alpha - y)\|_{L^2(\omega)}^2 + \frac{1}{\alpha^2}\|y^\alpha - y\|_{L^2(\omega)}^2 \leq \frac{\alpha^2}{2}\|\Delta y\|_{L^2(\omega)}^2 + \frac{1}{2\alpha^2}\|y^\alpha - y\|_{L^2(\omega)}^2.$$

■

Let us state a regularity result for the solution of (4.10).

Proposition 4.3.3 *Let $\sigma\theta \in H^2(0, T; L^2(\Omega)) \cap H^1(0, T; H_0^1(\Omega))$. Let $u_{|\theta}$ be the solution of (4.10). Then $u \in C^1([0, T]; H^2(\Omega))$ and there exists a positive scalar C^{st} depending on σ and T such that*

$$\sup_{t \in [0, T]} \|\nabla \partial_t u_{|\theta}\|_{L^2(\Omega)} + \|\Delta \partial_t u_{|\theta}\|_{L^2(\Omega)} \leq C^{st}(T, \sigma) \|\theta\|_{H_0^1(\Omega)}. \quad (4.42)$$

PROOF: From classical existence result (we refer to [Lions et al., 1968] or [Bensoussan et al., 2007]), one can show that :

$$\begin{aligned} &\text{for } \sigma\theta \in L^2(0, T; L^2(\Omega)), \\ &\exists! u \in C^1([0, T], L^2(\Omega)) \cap C^0([0, T], H_0^1(\Omega)) \text{ solution of (4.10)}. \end{aligned} \quad (4.43)$$

Furthermore, let us study existence result for the system (4.10) derived in time. Namely, we study the following system where $v = \partial_t u$,

$$\begin{cases} \partial_{tt}v(x, t) - \Delta v(x, t) = \partial_t \sigma(x, t)\theta(x), & (x, t) \in \Omega \times (0, T), \\ v(x, t) = 0, & (x, t) \in \partial\Omega \times (0, T), \\ v(x, 0) = 0, & x \in \Omega, \\ \partial_t v(x, 0) = \sigma(x, 0)\theta(x), & x \in \Omega. \end{cases} \quad (4.44)$$

Once again, one can prove that :

$$\begin{aligned} &\text{for } (\sigma(0)\theta, \partial_t \sigma\theta) \in \times L^2(\Omega) \times L^2(0, T; L^2(\Omega)), \\ &\exists! v \in C^1([0, T], L^2(\Omega)) \cap C^0([0, T], H_0^1(\Omega)) \text{ solution of (4.44)}. \end{aligned} \quad (4.45)$$

Again, we derive the system (4.44) in time. Denoting $w = \partial_t v$, we have

$$\begin{cases} \partial_{tt}w(x, t) - \Delta w(x, t) = \partial_{tt}\sigma(x, t)\theta(x), & (x, t) \in \Omega \times (0, T), \\ w(x, t) = 0, & (x, t) \in \partial\Omega \times (0, T), \\ w(x, 0) = \sigma(x, 0)\theta(x), & x \in \Omega, \\ \partial_t w(x, 0) = \partial_t \sigma(x, 0)\theta(x), & x \in \Omega. \end{cases} \quad (4.46)$$

One can show the following existence result :

$$\begin{aligned} &\text{for } (\sigma(0)\theta, \partial_t \sigma(0)\theta, \partial_{tt}\sigma\theta) \in H_0^1(\Omega) \times L^2(\Omega) \times L^2(0, T; L^2(\Omega)), \\ &\exists! w \in C^1([0, T], L^2(\Omega)) \cap C^0([0, T], H_0^1(\Omega)) \text{ solution of (4.46)}. \end{aligned} \quad (4.47)$$

Since $\sigma\theta \in H^2(0, T; L^2(\Omega)) \cap H^1(0, T; H_0^1(\Omega))$ and gathering (4.43), (4.45) and (4.47), we can deduce that the solution u of (4.10) belongs to $C^1([0, T]; H^2(\Omega))$.

Moreover, using classical energy identities and Grönwall lemma, one can prove that (4.42) holds. ■

We can now express the reconstruction error theorem.

Theorem 4.3.4 *We assume that there exists $M > 0$ such that*

$$\check{\theta} \in \mathbf{H}_0^1(\Omega) \text{ and } \|\check{\theta}\|_{\mathbf{H}_0^1(\Omega)} \leq M. \quad (4.48)$$

Let $\sigma\check{\theta} \in \mathbf{H}^2(0, T; \mathbf{L}^2(\Omega)) \cap \mathbf{H}^1(0, T; \mathbf{H}_0^1(\Omega))$.

Let $\sigma \in \mathbf{H}^1(0, T; \mathbf{L}^\infty(\Omega))$ such that $|\sigma(x, 0)| \geq \alpha > 0$ a.e. in Ω . We assume that ω_0 satisfies (4.7). Let $\omega \supset \omega_1 \supset \omega_0$. Then for $T \geq T_0$ where T_0 is defined by (4.8), there exists a positive scalar C^{st} independent of M and δ such that

$$\|\bar{\theta}_T - \check{\theta}\|_{\mathbf{L}^2(\Omega)} \leq C^{st} \sqrt{M} \sqrt{\delta}, \quad (4.49)$$

where $\bar{\theta}_T$ minimizes the cost functional (4.40) with $\epsilon = \delta^2 M^{-2}$.

PROOF: In this proof, we write $a \lesssim b$ if there exists a positive scalar $c > 0$ independent of ϵ , M and δ such that $a \leq cb$.

First, using the prior on the parameter $\check{\theta}$ and the estimate on the noise amplitude (4.39), we can bound the cost functional

$$\mathcal{J}_T(\bar{\theta}_T) \leq \mathcal{J}_T(\check{\theta}) \leq \frac{\epsilon}{2} M^2 + \frac{1}{2} T \delta^2. \quad (4.50)$$

Thanks to the observability inequality of Theorem 4.2.2 applied to $\bar{\theta}_T - \check{\theta}$ on ω_0 , we obtain

$$\|\bar{\theta}_T - \check{\theta}\|_{\mathbf{L}^2(\Omega)} \lesssim \left(\int_0^T \|\partial_t u|_{\bar{\theta}_T} - \partial_t u|_{\check{\theta}}\|_{\mathbf{H}^1(\omega_0)}^2 dt \right)^{1/2}.$$

For a given $\alpha > 0$, We introduce a cut-off function $\eta \in C^\infty(\bar{\omega})$ such that

$$\begin{cases} \eta(x) = 1, & \text{if } x \in \omega_0, \\ \eta(x) = 0 & \text{if } x \in \omega \setminus \omega_1, \end{cases} \quad (4.51)$$

and $0 \leq \eta(x) \leq 1$ for every $x \in \bar{\omega}$. Therefore, using Poincaré inequality and the definition of the cut-off function we have

$$\begin{aligned} \|\bar{\theta}_T - \check{\theta}\|_{\mathbf{L}^2(\Omega)} &\lesssim \left(\int_0^T \|\eta(\partial_t u|_{\bar{\theta}_T} - \partial_t u|_{\check{\theta}})\|_{\mathbf{H}^1(\omega_0)}^2 dt \right)^{1/2} \\ &\lesssim \left(\int_0^T \|\eta(\partial_t u|_{\bar{\theta}_T} - \partial_t u|_{\check{\theta}})\|_{\mathbf{H}_0^1(\omega)}^2 dt \right)^{1/2}. \end{aligned}$$

We introduce the regularization operator G_ω^α defined by (4.41).

$$\begin{aligned} \|\bar{\theta}_T - \check{\theta}\|_{\mathbf{L}^2(\Omega)} &\lesssim \left(\int_0^T \|(G_\omega^\alpha - \text{Id}) (\eta(\partial_t u|_{\bar{\theta}_T} - \partial_t u|_{\check{\theta}}))\|_{\mathbf{H}_0^1(\omega)}^2 dt \right)^{1/2} \\ &\quad + \left(\int_0^T \|G_\omega^\alpha (\eta(\partial_t u|_{\bar{\theta}_T} - \partial_t u|_{\check{\theta}}))\|_{\mathbf{H}_0^1(\omega)}^2 dt \right)^{1/2}. \end{aligned}$$

Adding and removing the noisy observations, we get

$$\begin{aligned} \|\bar{\theta}_T - \check{\theta}\|_{\mathbf{L}^2(\Omega)} &\lesssim \left(\int_0^T \|(G_\omega^\alpha - \text{Id}) (\eta \partial_t u|_{\bar{\theta}_T})\|_{\mathbf{H}_0^1(\omega)}^2 dt \right)^{1/2} \\ &\quad + \left(\int_0^T \|(G_\omega^\alpha - \text{Id}) (\eta \partial_t u|_{\check{\theta}})\|_{\mathbf{H}_0^1(\omega)}^2 dt \right)^{1/2} \\ &\quad + \left(\int_0^T \|G_\omega^\alpha (\eta(\partial_t u|_{\bar{\theta}_T} - y^\delta))\|_{\mathbf{H}_0^1(\omega)}^2 dt \right)^{1/2} \\ &\quad + \left(\int_0^T \|G_\omega^\alpha (\eta(\partial_t u|_{\check{\theta}} - y^\delta))\|_{\mathbf{H}_0^1(\omega)}^2 dt \right)^{1/2}. \end{aligned}$$

Using Propositions 4.3.1 and 4.3.2, we get

$$\begin{aligned} \|\bar{\theta}_T - \check{\theta}\|_{L^2(\Omega)} &\lesssim \alpha \left(\int_0^T \|\Delta(\eta \partial_t u|_{\bar{\theta}_T})\|_{L^2(\omega)}^2 dt \right)^{1/2} + \alpha \left(\int_0^T \|\Delta(\eta \partial_t u|_{\check{\theta}})\|_{L^2(\omega)}^2 dt \right)^{1/2} \\ &\quad + \frac{1}{\alpha} \|\eta\|_{L^\infty(\omega)} \left(\int_0^T \|\partial_t u|_{\bar{\theta}_T} - y^\delta\|_{L^2(\omega)}^2 dt \right)^{1/2} \\ &\quad + \frac{1}{\alpha} \|\eta\|_{L^\infty(\omega)} \left(\int_0^T \|\partial_t u|_{\check{\theta}} - y^\delta\|_{L^2(\omega)}^2 dt \right)^{1/2}. \end{aligned} \quad (4.52)$$

For the first two terms in the right-hand side, we notice that for all $v \in H_0^1(\Omega)$ such that $\Delta v \in L^2(\Omega)$, using Poincaré inequality and the definition of the cut-off function,

$$\begin{aligned} \|\Delta(\eta v)\|_{L^2(\omega)} &\leq \|v\Delta\eta + 2\nabla v \nabla \eta + \eta\Delta v\|_{L^2(\omega)}, \\ &\lesssim \|v\|_{L^2(\omega)} \|\Delta\eta\|_{L^\infty(\omega)} + 2\|\nabla v\|_{L^2(\omega)} \|\nabla \eta\|_{L^\infty(\omega)} + \|\eta\|_{L^\infty} \|\Delta v\|_{L^2(\omega)}, \\ &\lesssim \|\nabla v\|_{L^2(\Omega)} + \|\Delta v\|_{L^2(\Omega)}. \end{aligned}$$

Since $\|\eta\|_{L^\infty(\omega)} = 1$ and thanks to the regularity of σ , $\check{\theta}$ and $\bar{\theta}_T$, we get

$$\begin{aligned} \|\bar{\theta}_T - \check{\theta}\|_{L^2(\Omega)} &\lesssim \alpha \left(\int_0^T \|\Delta \partial_t u|_{\bar{\theta}_T}\|_{L^2(\omega)}^2 + \|\nabla \partial_t u|_{\bar{\theta}_T}\|_{L^2(\omega)}^2 dt \right)^{1/2} \\ &\quad + \alpha \left(\int_0^T \|\Delta \partial_t u|_{\check{\theta}}\|_{L^2(\omega)}^2 + \|\nabla \partial_t u|_{\check{\theta}}\|_{L^2(\omega)}^2 dt \right)^{1/2} \\ &\quad + \frac{1}{\alpha} \left(\int_0^T \|\partial_t u|_{\bar{\theta}_T} - y^\delta\|_{L^2(\omega)}^2 dt \right)^{1/2} \\ &\quad + \frac{1}{\alpha} \left(\int_0^T \|\partial_t u|_{\check{\theta}} - y^\delta\|_{L^2(\omega)}^2 dt \right)^{1/2}. \end{aligned}$$

Using Proposition 4.3.3 as well as the definition of the cost functional (4.40) and the bound of the error measurements amplitude (4.39), we obtain

$$\|\bar{\theta}_T - \check{\theta}\|_{L^2(\Omega)} \lesssim \alpha T \|\bar{\theta}_T\|_{H_0^1(\Omega)} + \alpha T \|\check{\theta}\|_{H_0^1(\Omega)} + \frac{1}{\alpha} (2\mathcal{J}_T(\bar{\theta}_T))^{1/2} + \frac{1}{\alpha} \delta \sqrt{T}.$$

Thanks to the prior on the parameter (4.48), we get

$$\|\bar{\theta}_T - \check{\theta}\|_{L^2(\Omega)} \lesssim \alpha T (2\epsilon^{-1} \mathcal{J}_T(\bar{\theta}_T))^{1/2} + \alpha T M + \frac{1}{\alpha} (2\mathcal{J}_T(\bar{\theta}_T))^{1/2} + \frac{1}{\alpha} \delta \sqrt{T}.$$

Using the bound of the cost-functional (4.50), we have

$$\|\bar{\theta}_T - \check{\theta}\|_{L^2(\Omega)} \lesssim \alpha T (M^2 + \epsilon^{-1} \delta^2 T)^{1/2} + \alpha T M + \frac{1}{\alpha} (\epsilon M^2 + \delta^2 T)^{1/2} + \frac{1}{\alpha} \delta \sqrt{T}.$$

Choosing $\epsilon = \delta^2 M^{-2}$ and $\alpha = \delta^{1/2} M^{-1/2}$, we finally obtain

$$\|\bar{\theta}_T - \check{\theta}\|_{L^2(\Omega)} \lesssim \delta^{1/2} M^{1/2} \left(T(1+T)^{1/2} + T + (1+T)^{1/2} + \sqrt{T} \right).$$

■

As a result, we obtain a reconstruction error between the target source term and the observer as a function of the noise level. Following the same process as in Chapter 3 and using a very similar definition of the cost function (also the same choice of ϵ), we find the same power of δ in the reconstruction error.

4.3.2 Observer design

To calculate the minimizer of the cost functional (4.40), we use an approach presented in Chapter 3, *i.e.* use a reduce-order Kalman filter where a Riccati operator is defined on the parameter space.

Theorem 4.3.5 *The unique minimizer $\bar{\theta}_t$ of \mathcal{J}_T (4.40) satisfies $\bar{\theta}_t = \hat{\theta}(t)$ where $\hat{\theta}$ is solution of the following dynamics*

$$\begin{cases} \dot{\hat{\theta}} = \Lambda L^* C^* (y^\delta - C \hat{z}), & t > 0, \\ \dot{\Lambda} = -\Lambda L^* C^* C \Lambda, & t > 0, \\ \dot{\hat{z}} = A \hat{z} + B \hat{\theta} + L \dot{\hat{\theta}}, & t > 0, \\ \hat{\theta}(0) = \hat{\theta}_0 \\ z(0) = \hat{z}_0, \\ \Lambda(0) = \Lambda_0 := -\epsilon^{-1} \Delta_0^{-1}, \end{cases} \quad (4.53)$$

where the operator $(L(t))_{t \geq 0}$ is such that $z|_\theta(t) = L(t)\theta$ is the mild solution of (4.5), C is defined by (4.9), $\hat{\theta}_0$ is a possible a priori on the source term and $\hat{z}_0 = z_0$ the correct initial conditions.

Since this method is sequential, it has the advantage of not requiring multiple iterations to obtain the result. In this case, hence it is very efficient for parameters discretized in controlled-dimension spaces where the Riccati operator can be efficiently computed. Moreover, we could also show that observer (4.53) is a convergent observer in the sense of Definition 3.4.1 of Chapter 3. The proof of the reconstruction error in Theorem 4.3.4 is similar to that in Chapter 3 (Theorems 3.3.7 and 3.3.12), which guarantees robustness to noise. Following the methodology of Chapter 3, we could also show the property of asymptotic convergence in the absence of noise.

4.4 Time discretization and numerical results

4.4.1 Time discretization scheme

z

We discretize the wave equation as we did in the previous chapter, and then we apply Tikhonov's regularization method and observer design to the discretized system following a *discretize-then-optimize* strategy. As in Chapter 3, we discretize the wave equation using Lagrange finite element $\{u_n^h, v_n^h\}_{n \geq 0} \in \mathcal{V}_h \times \mathcal{V}_h$ where $\mathcal{V}_h \subset H^1(\Omega)$ in a finite-dimensional space and $\theta^h \in \mathcal{P}^h$ the finite-dimensional space corresponding to \mathcal{P} . We use a mid-point scheme which leads to

$$\begin{cases} \frac{u_{n+1}^h - u_n^h}{\tau} = \frac{v_{n+1}^h + v_n^h}{2} \\ \frac{v_{n+1}^h - v_n^h}{\tau} - \Delta_0^h \frac{u_{n+1}^h + u_n^h}{2} = \sigma^h \left(\left(n + \frac{1}{2} \right) \tau \right) \theta^h, \end{cases} \quad (4.54)$$

where Δ_0^h is the bounded linear operator in \mathcal{V}_h associated with the finite element discretization of the unbounded operator Δ_0 of $L^2(\Omega)$. We also recall the first-order writing of this system by setting $z^h \in \mathcal{Z}^h = \mathcal{V}_h \times \mathcal{V}_h$ – where $\mathcal{Z}^h \subset \mathcal{Z}$ is equipped with the scalar product of \mathcal{Z} – solution of

$$z_{n+1|\zeta}^h = \Phi^{h,\tau} z_{n|\zeta}^h + B_{n+1}^{h,\tau} \theta^h,$$

where

$$\Phi^{h,\tau} = \begin{pmatrix} \text{Id}_h & -\frac{\tau}{2}\text{Id}_h \\ -\frac{\tau}{2}\Delta_0^h & \text{Id}_h \end{pmatrix}^{-1} \begin{pmatrix} \text{Id}_h & \frac{\tau}{2}\text{Id}_h \\ \frac{\tau}{2}\Delta_0^h & \text{Id}_h \end{pmatrix},$$

and

$$B_{n+1}^{h,\tau} = \tau \begin{pmatrix} \text{Id}_h & -\frac{\tau}{2}\text{Id}_h \\ -\frac{\tau}{2}\Delta_0^h & \text{Id}_h \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \sigma^h(n + \frac{1}{2}\tau) \end{pmatrix}.$$

The time-and-space discretization of the cost functional (4.40) reads to

$$\mathcal{J}_N(\zeta^h) = \frac{\epsilon}{2} \|\zeta^h\|_{\mathbf{H}_0^1(\Omega)}^2 + \sum_{n=1}^N \frac{\tau}{2} \|y_n^{\delta,h} - C^h z_{n|\zeta^h}^h\|_{\mathcal{Y}^h}^2, \quad (4.55)$$

associated with measurements $y_n^{\delta,h}$ and an observation operator $C^h \in \mathcal{L}(\mathcal{Z}^h, \mathcal{Y}^h)$ where \mathcal{Y}^h is the finite dimensional space in which the observations belong. We now present the fundamental theorem defining an observer from the minimization of the cost function in Chapter 3.

Theorem 4.4.1 *The minimizer $\hat{\theta}_{N-}^h$ of the cost functional (4.55) is equals to $\bar{\theta}_{N-}^h$ which is solution from $(\hat{z}_0^h, \hat{\theta}_0^h)$ of the recursive dynamics*

$$\begin{cases} \hat{z}_{n+1-}^h = \Phi^{h,\tau} z_n^h + B_{n+1}^{h,\tau} \hat{\theta}_n^h, & t > 0, \\ \hat{\theta}_{n+1}^h = \hat{\theta}_n^h + (U_{n+1}^h)^{-1} L_{n+1}^{h*} C^{h*} (y_{n+1}^{\delta,h} - C^h \hat{z}_{n+1-}^h), & t > 0, \\ \hat{z}_{n+1}^h = \hat{z}_{n+1-}^h + L_{n+1}^h (\hat{\theta}_{n+1}^h - \hat{\theta}_n^h) \end{cases} \quad (4.56)$$

where the operator $(L_n^h)_{n \in \mathbb{N}}$ is such that $z_{n|\theta}^h = L_n^h \theta$ is the solution of

$$\begin{cases} z_{n+1|\theta}^h = \Phi^{h,\tau} z_{n|\theta}^h + B_{n+1}^{h,\tau} \theta, & n \in \mathbb{N}, \\ z_{0|\theta}^h = 0, \end{cases} \quad (4.57)$$

and U_n^h is given by

$$\begin{cases} U_{n+1}^h = U_n^h + \tau L_{n+1}^{h*} C^{h*} C^h L_{n+1}^h, & n \geq 0, \\ U_0^h = -\epsilon \Delta_0. \end{cases} \quad (4.58)$$

We present some numerical illustrations to illustrate the results presented in this work.

4.4.2 Wave propagation example

We would like to illustrate the results obtained in the previous sections. In particular, we give a numerical example of the source term reconstructed by the observer and show that the reconstruction is accurate. We would also like to illustrate Theorem 4.3.4 and see the evolution of the error as a function of the noise of the available observations. Consider the following wave equation

$$\begin{cases} \partial_{tt} \check{u}(x, t) - \Delta \check{u}(x, t) = \sigma(x, t) \check{\theta}(x), & (x, t) \in \Omega \times (0, T), \\ \check{u}(x, t) = 0, & (x, t) \in \partial\Omega \times (0, T), \\ \check{u}(x, 0) = \check{u}_0(x), & x \in \Omega, \\ \partial_t \check{u}(x, 0) = 0, & x \in \Omega, \end{cases} \quad (4.59)$$

where \check{u}_0 is solution of

$$\begin{cases} -\Delta \check{u}_0(x) = 10, & x \in \Omega, \\ \check{u}_0(x) = 0, & x \in \partial\Omega, \end{cases} \quad (4.60)$$

and $\check{\theta}$ is solution of

$$\begin{cases} -\Delta \check{\theta}(x) = 0 & x \in \Omega \setminus \Omega_{\text{hole}} \\ -\Delta \check{\theta}(x) = 10, & x \in \Omega_{\text{hole}}, \\ \check{\theta}(x) = 0, & x \in \partial\Omega, \end{cases} \quad (4.61)$$

with Ω_{hole} represented in Figure 4.4.

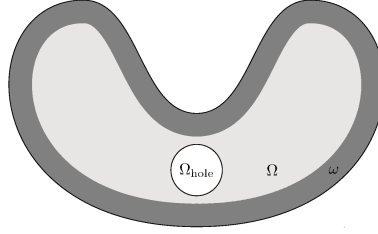


Figure 4.4: Representation of the domain Ω and the subdomain ω

We also set

$$\sigma(x, t) = (1 + \exp(-20(x_2 - 0.9t + 0.4)^2)), \quad (4.62)$$

where x_2 represents the vertical axis. By the way, this choice of σ allows to mimic the propagation of a wave moving along the vertical axis from bottom to top. Moreover, we have $|\sigma(\cdot, 0)| \geq \alpha > 0$ in order to satisfy the hypothesis of Theorem 4.2.2. In this case, the chosen function has persistent excitation.

The initial conditions and target parameter defined respectively by (4.60) and the (4.61) are illustrated in Figure 4.5 (left) and 4.5 (right).

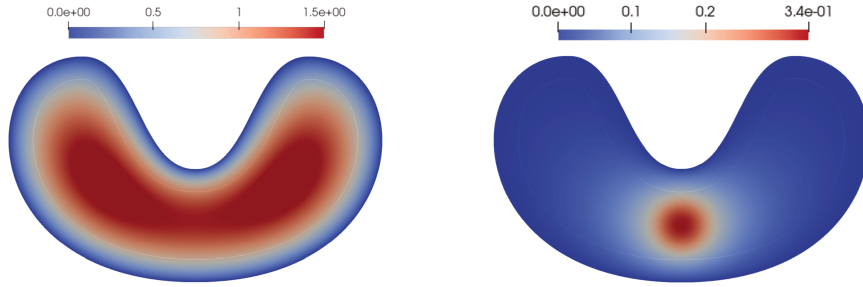


Figure 4.5: Representation of $\check{u}_0(x)$ (left) and $\check{\theta}(x)$ (right)

As in the Chapter 3, we decompose the target parameter

$$\check{\theta}^h := \sum_{i=0}^{N_{\text{mod}}} \check{\theta}_i^h \phi_i, \quad (4.63)$$

where $(\phi_i^h)_{i \in [0, N_{\text{mod}}]}$ are the eigenvectors of Δ_0^h . In order to the relative error between $\check{\theta}^h$ and $\check{\theta}$ be less than 10%, we chose $N_{\text{mod}} = 27$ (see Figure 4.6).

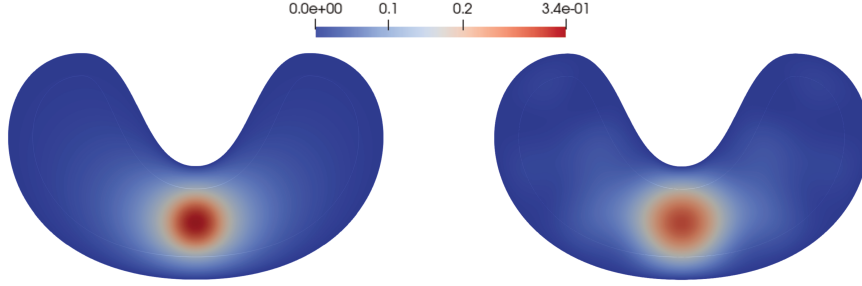


Figure 4.6: Target parameter $\check{\theta}$ and $\check{\theta}^h$ its decomposition on the basis with 27 modes

We represent in Figure 4.7, the time and space discretized known parameter $\sigma_n^{h,\tau}$ (4.62) and the right-hand-side $\sigma_n^{h,\tau}\check{\theta}$ on the mesh illustrated in Figure 4.8 (right).

To generate the measurements, we simulate the direct problem implemented in Freefem++ [Hecht, 2012] on the mesh with 56193 degrees of freedom (Figure 4.8 (right)) on a time window $[0, 3]$ with a time step of $\tau = 0.01$.

We illustrate an example of measurements in Figure 4.9. To present a more realistic case, unlike in Chapter 3, the target parameter $\check{\theta}$ defined by (4.61) is used instead of $\check{\theta}^h$ (defined by (4.63)) to generate the measurements.

We solve the inverse problem using scheme (4.56) implemented in Freefem++ under a time window of $[0, 1]$ with a time step of $\tau = 0.01$ and using the meshes shown in Figure 4.8. Figure 4.10 shows the relative error norm between the observer and the target parameter decomposed on the basis of Δ_0^h over time, namely the above quantity

$$\|\hat{\theta}_n^h - \check{\theta}^h\|_{L^2(\Omega)} / \|\check{\theta}^h\|_{L^2(\Omega)}.$$

We can see that the convergence in the case where the inverse problem is solved on the same mesh as for the measurements is very good, with less than 1% error at $t^n = 1$, as shown in Figures 4.10 (right) and 4.12 (right). If you compare Figures 4.6 (right) and 4.11 (right), you can see that the reconstructed source term is visually identical to the target source term. Nevertheless, it does not make sense to consider the same mesh for the measurements and the inverse problem. As for the reconstruction on the mesh with less dofs, despite an increase of the error norm at the beginning, the relative error norm actually decreases from $t^n = 0.2$. As can be seen in Figures 4.10 (left) and 4.12 (left), the error at $t^n = 1$ is about 5%. A visual comparison of Figures 4.6 (right) and 4.11 (left) shows minimal differences between the observer at the final time and the target source term, especially in the middle of the domain. We will keep this mesh for the rest of the chapter.

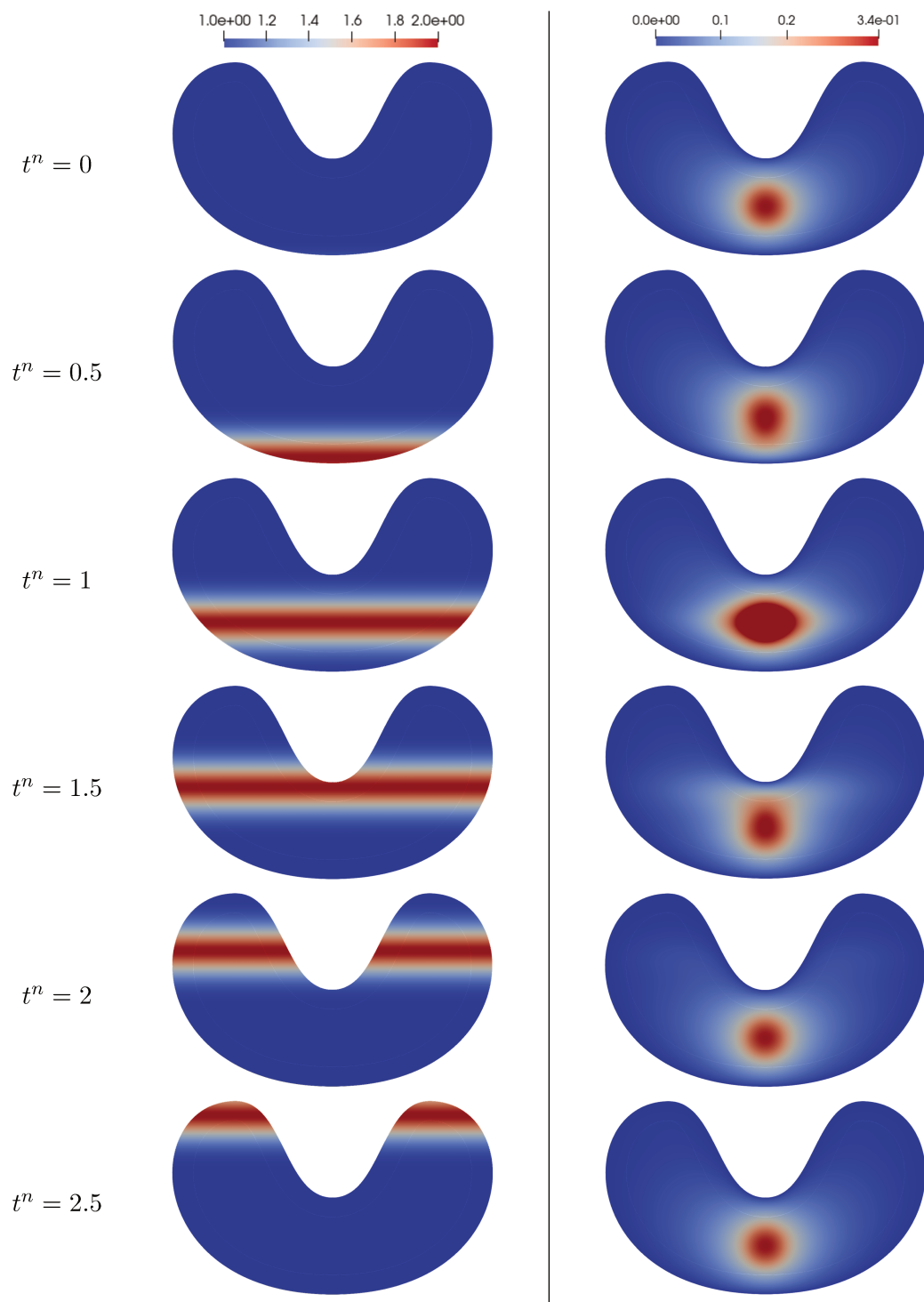
Figure 4.7: Representation of $\sigma_n^{h,\tau}$ (left) and of $\sigma_n^{h,\tau} \check{\theta}$ (right) at different times t^n



Figure 4.8: Representation of the mesh of Ω with 14169 dofs (left) and with 56193 dofs (right)

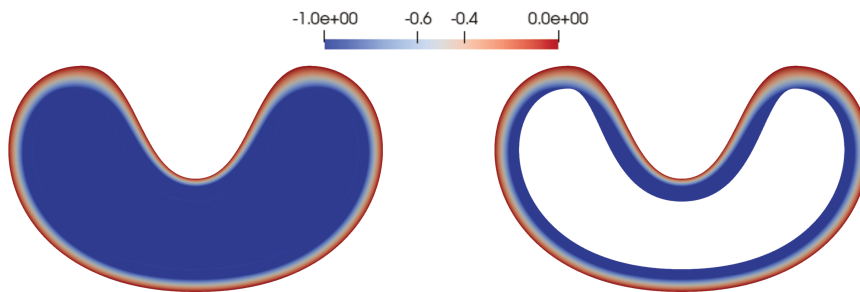


Figure 4.9: Solution \tilde{v}_n^h of the direct problem (left) and its corresponding measurements $I_\omega(\tilde{v}_n^h)$ (right) at time $t^n = 0.1$

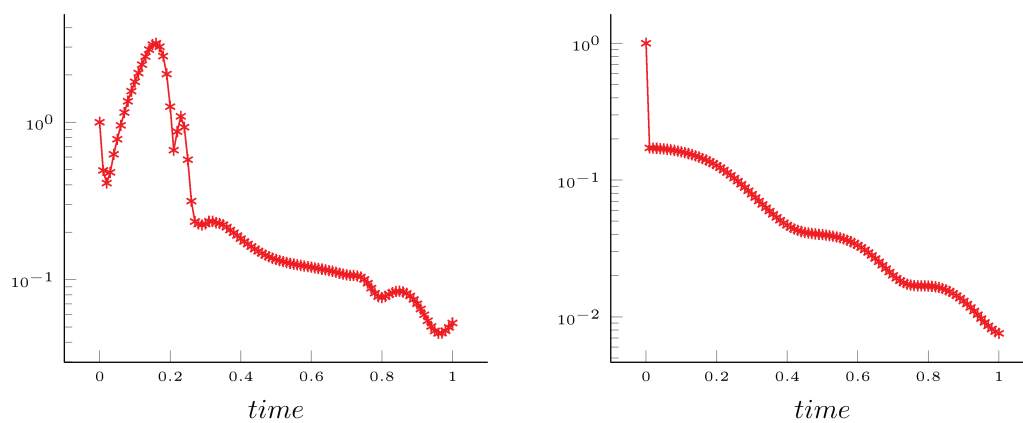


Figure 4.10: Evolution of the relative error with $\delta = 10^{-5}$, $M = 50$ and on the mesh with 14169 dofs (left) and with 56193 dofs (right)

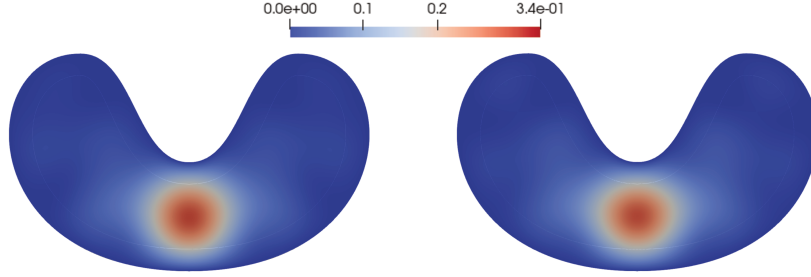


Figure 4.11: Observer $\hat{\theta}_n^h$ on the mesh with 14169 dofs (left) and with 56193 dofs (right)

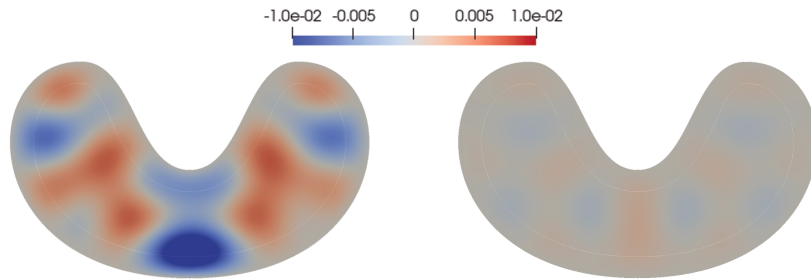


Figure 4.12: Error between the target parameter decomposed on the basis of Δ_0^h and the observer $\hat{\theta}_n^h$ on the mesh with 14169 dofs (left) and with 56193 dofs (right)

We now corrupt the synthetic data $(\tilde{y}_n^h)_{0 \leq n \leq N}$ of sampling rate T/N with a random artificial noise χ_n^h such that $\|\chi_n^h\|_{L^2(0,T;L^2(\omega))}^2 \leq \delta^2 T$. In this respect, as in the Chapter 1 and 3, we simulate an additive white Gaussian noise such that the magnitude of the error is a fixed percentage of the L^2 -norm of the observations. Therefore, we define at each time n , the discretized noise data function as

$$\chi_n^h = \frac{\delta}{\sqrt{T N_{mod}}} \sqrt{\tau \sum_{n=1}^N \|\tilde{y}_n^h\|_{L^2(\omega)}^2} \sum_{i=0}^{N_{mod}} (\xi_i)_n \phi_i, \quad (4.64)$$

where $(\phi_i)_{0 \leq i \leq N_{mod}}$ are the $N_{mod} + 1$ modes of Δ_0^h and ξ are independent and identically distributed Gaussian variables in $\mathcal{N}(0, 1)$. Figure 4.13 shows the added noise with different values of δ .

We solve the inverse problem using (4.56) on the mesh with 14169 dofs (see Figure 4.8 (left)) with a time step of $\tau = 0.01$. This time we use noisy data. We set δ according to the noise level of the observations and choose $M = 50$ as a prior bound. Figure 4.14 illustrated that the reconstruction deteriorates as the noise increases. Figure 4.15 shows the relative error norm at time $T = 1$ as a function of noise. To illustrate Theorem 4.3.4, we represent the relative error with an approximation by a straight line (in loglog scale) of a slope coefficient equals to 0.69. Indeed, for $\delta = 0.02$ noise we obtain an error norm of 25%, for $\delta = 0.01$ about 13%, and finally for $\delta = 0.0025$ we have 5.8% of error norm.

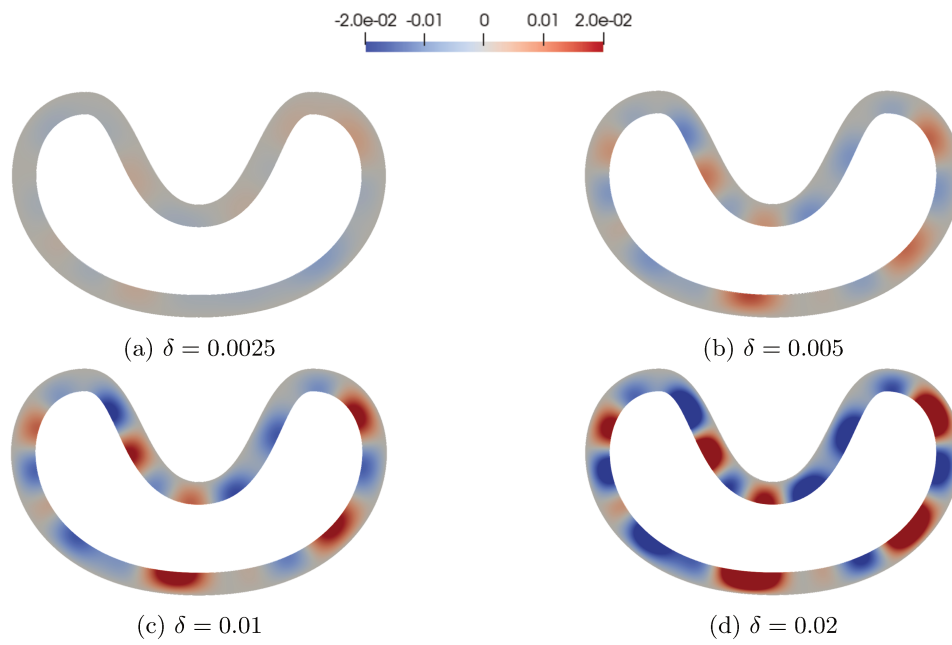


Figure 4.13: Representation of χ_n^h for different values of δ at $t^n = 0.1$

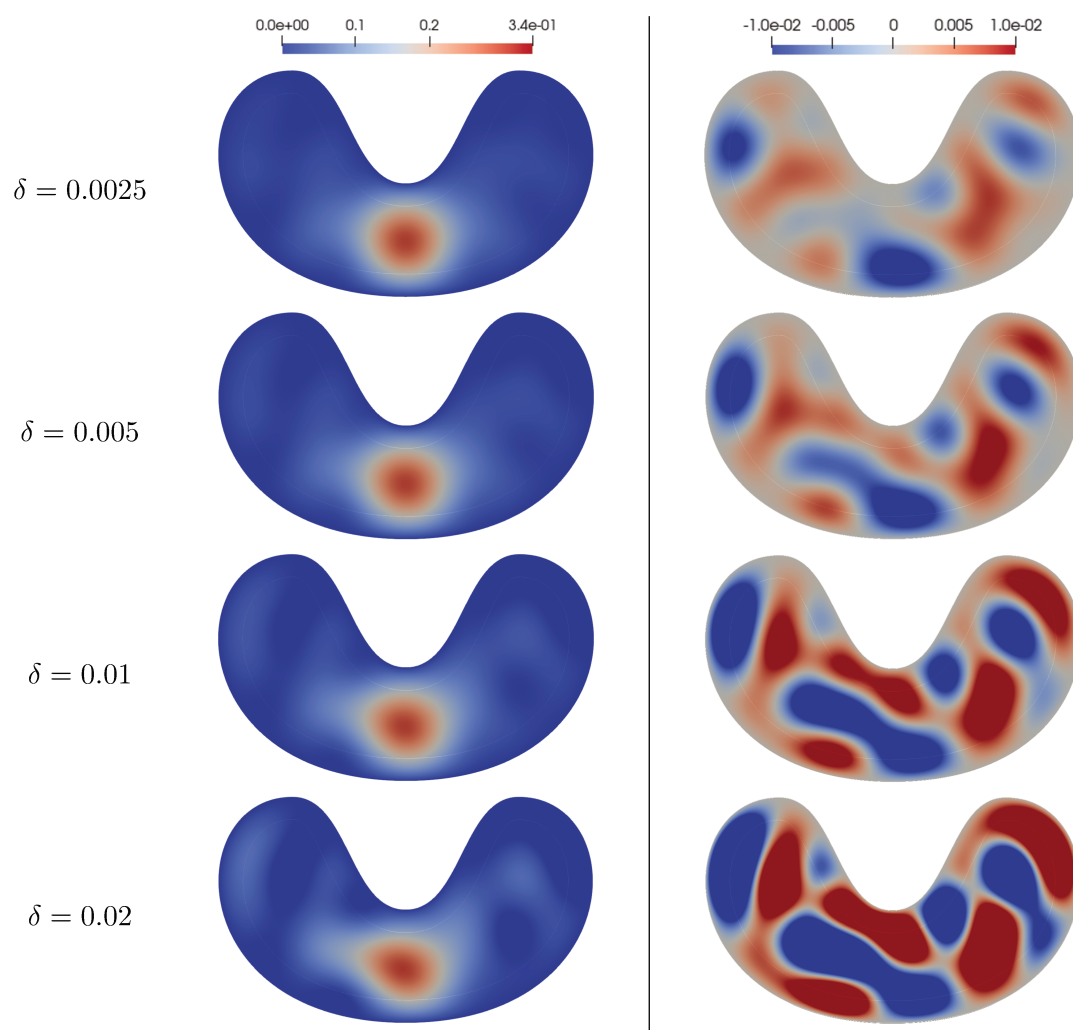


Figure 4.14: Representation of $\hat{\theta}_n^h$ (left) and the error $\tilde{\theta}^h - \hat{\theta}_n^h$ (right) at $t^n = 1$ with different values of δ

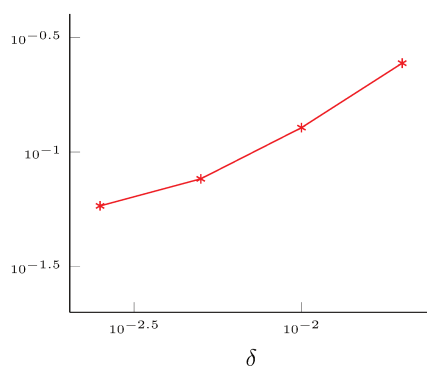


Figure 4.15: Representation of the relative error according the values of δ in loglog scale

4.5 Perspectives

In this section, we present two possible perspectives of this work. The first is to extend the results shown in the previous sections to *field* measurements. We establish a new Carleman estimate, from which we deduce a new identifiability property involving the parameter in its L^2 -norm and *field* measurements in H^2 -norm.

We have also seen that the method we proposed suits well when the source term to reconstruct is smooth, *i.e.* it belongs to H_0^1 . In the last section, we look numerically at possible solutions for adapting our strategy when considering a sharp source term. In particular, we study the adaptive basis ([de Buhan and Osses, 2010; Baffet et al., 2021]). The theoretical study is currently in progress.

4.5.1 Extension to field measurements

In this section, we would like to consider measurements of the field u , the solution of (4.1) instead of the velocity $\partial_t u$. We state Carleman estimate in order to apply this estimates to our parameter identification problem. As in the Section 4.2.1, we have adapted the classical Carleman estimate for the wave equation in order to modify the norms in the *observation term*.

Theorem 4.5.1 *We assume that ω satisfies (4.7). Let $T \geq T_0$ defined by (4.8). There exist $\lambda > 0$, $s_0 > 0$ and a positive constant C^{st} such that for ϕ defined by (4.13), for all $s \geq s_0$ and for all $w \in L^2(-T, T; H_0^1(\Omega))$ such that $Pw = \partial_{tt}w - \Delta w \in L^2(\Omega \times (-T, T))$, $W \in L^2(-T, T; H^2(\omega))$ and $w(\cdot, 0) = w(\cdot, \pm T) = \partial_t w(\cdot, \pm T) = 0$ in Ω , we have*

$$\begin{aligned} s^{1/2} \int_{\Omega} e^{2s\phi(0)} |\partial_t w(0)|^2 dx + \int_{-T}^T \int_{\Omega} e^{2s\phi} (s|\nabla w|^2 + s|\partial_t w|^2 + s^3|w|^2) dx dt \\ \leq C^{st} \int_{-T}^T \int_{\Omega} e^{2s\phi} |Pw|^2 dx dt + C^{st} s^2 \int_{-T}^T \int_{\omega} e^{2s\phi} (|\Delta W|^2 + s^4|W|^2) dx dt \end{aligned} \quad (4.65)$$

where W is given by $W(t) = \int_0^t w(\tau) d\tau$.

PROOF: In this proof, we write $a \lesssim b$ if there exists a positive scalar $c > 0$ independent of s such that $a \leq cb$.

In this proof, all the computations will be done for smooth functions and lastly a density argument allows to extend the result to the regularity given in the theorem. We restart from the proven equation (4.27) On one hand, we can deduce from (4.27) that

$$\begin{aligned} \int_{-T}^T \int_{\omega_2} e^{2s\phi} |\nabla w|^2 dx dt \lesssim \frac{1}{s^2} \int_{-T}^T \int_{\Omega} e^{2s\phi} |Pw|^2 dx dt \\ + \int_{-T}^T \int_{\omega} (|\partial_t w|^2 + s^2|w|^2) dx dt. \end{aligned} \quad (4.66)$$

Using this inequality in the right-hand side of (4.23), we get

$$\begin{aligned} s^{1/2} \int_{\Omega} e^{2s\phi(0)} |\partial_t w(0)|^2 dx + \int_{-T}^T \int_{\Omega} e^{2s\phi} (s|\nabla w|^2 + s|\partial_t w|^2 + s^3|w|^2) dx dt \\ \lesssim \int_{-T}^T \int_{\Omega} e^{2s\phi} |Pw|^2 dx dt + s^2 \int_{-T}^T \int_{\omega} e^{2s\phi} (|\partial_t w|^2 + s^2|w|^2) dx dt. \end{aligned} \quad (4.67)$$

On the other hand, using the fact that $w = \partial_t W$ (since W is a time primitive of w), the second term of (4.24) can also be integrated by parts in time to obtain

$$\begin{aligned} - \int_{-T}^T \int_{\omega} e^{2s\phi} \rho \Delta w w \, dx \, dt &= - \int_{-T}^T \int_{\omega} e^{2s\phi} \rho \Delta (\partial_t W) w \, dx \, dt \\ &= \int_{-T}^T \int_{\omega} \partial_t (e^{2s\phi} \rho w) \Delta W \, dx \, dt - \left[\int_{\omega} e^{2s\phi} \rho w \Delta W \, dx \right]_{-T}^T \\ &= 2s \int_{-T}^T \int_{\omega} e^{2s\phi} \rho \partial_t \phi w \Delta W \, dx \, dt + \int_{-T}^T \int_{\omega} e^{2s\phi} \rho \partial_t w \Delta W \, dx \, dt. \end{aligned}$$

Using this inequality and (4.25) in (4.24), we get

$$\begin{aligned} \int_{-T}^T \int_{\omega} e^{2s\phi} \rho |\partial_t w|^2 \, dx \, dt &= s \int_{-T}^T \int_{\omega} e^{2s\phi} \partial_{tt} \phi \rho |w|^2 \, dx \, dt \\ &\quad + 2s^2 \int_{-T}^T \int_{\omega} e^{2s\phi} |\partial_t \phi|^2 |w|^2 \rho \, dx \, dt + 2s \int_{-T}^T \int_{\omega} e^{2s\phi} \rho \partial_t \phi w \Delta W \, dx \, dt \\ &\quad + \int_{-T}^T \int_{\omega} e^{2s\phi} \rho \partial_t w \Delta W \, dx \, dt - \int_{-T}^T \int_{\omega} e^{2s\phi} \rho P w w \, dx \, dt. \end{aligned}$$

Using Young inequality for the last three terms, we obtain

$$\begin{aligned} \int_{-T}^T \int_{\omega} e^{2s\phi} \rho |\partial_t w|^2 \, dx \, dt &\lesssim \int_{-T}^T \int_{\omega} e^{2s\phi} \left(s |\partial_{tt} \phi| \rho + 2s^2 |\partial_t \phi|^2 \rho + s^2 \rho |\partial_t \phi| + \frac{s^2 \rho}{2} \right) |w|^2 \, dx \, dt \\ &\quad + \int_{-T}^T \int_{\omega} e^{2s\phi} \left(\frac{\rho}{2} + \rho |\partial_t \phi| \right) |\Delta W|^2 \, dx \, dt + \frac{1}{2s^2} \int_{-T}^T \int_{\omega} e^{2s\phi} \rho |P w|^2 \, dx \, dt \\ &\quad + \frac{1}{2} \int_{-T}^T \int_{\omega} e^{2s\phi} \rho |\partial_t w|^2 \, dx \, dt. \quad (4.68) \end{aligned}$$

We absorb the last term on the right side of (4.68) by the left-hand side. Then, for s large enough, we get

$$\begin{aligned} \int_{-T}^T \int_{\omega} e^{2s\phi} |\partial_t w|^2 \, dx \, dt &\lesssim \frac{1}{s^2} \int_{-T}^T \int_{\Omega} e^{2s\phi} |P w|^2 \, dx \, dt \\ &\quad + \int_{-T}^T \int_{\omega} e^{2s\phi} (|\Delta W|^2 + s^2 |w|^2) \, dx \, dt. \quad (4.69) \end{aligned}$$

Let us now compute

$$\begin{aligned} s^2 \int_{-T}^T \int_{\omega} e^{2s\phi} |w|^2 \, dx \, dt &= s^2 \int_{-T}^T \int_{\omega} e^{2s\phi} w \partial_t W \, dx \, dt \\ &= -2s^3 \int_{-T}^T \int_{\omega} e^{2s\phi} \partial_t \phi w W \, dx \, dt - s^2 \int_{-T}^T \int_{\omega} e^{2s\phi} \partial_t w W \, dx \, dt \\ &\lesssim \frac{s^4}{\epsilon} \int_{-T}^T \int_{\omega} e^{2s\phi} |W|^2 \, dx \, dt + \epsilon \int_{-T}^T \int_{\omega} e^{2s\phi} (|\partial_t w|^2 + s^2 |w|^2) \, dx \, dt. \end{aligned}$$

Starting from (4.69), we use this inequality to obtain

$$\begin{aligned}
 & \int_{-T}^T \int_{\omega} e^{2s\phi} (|\partial_t w|^2 + s^2 |w|^2) \, dx \, dt \\
 & \lesssim \frac{1}{s^2} \int_{-T}^T \int_{\Omega} e^{2s\phi} |Pw|^2 \, dx \, dt + \int_{-T}^T \int_{\omega} e^{2s\phi} (|\Delta W|^2 + s^2 |w|^2) \, dx \, dt \\
 & \lesssim \frac{1}{s^2} \int_{-T}^T \int_{\Omega} e^{2s\phi} |Pw|^2 \, dx \, dt + \int_{-T}^T \int_{\omega} e^{2s\phi} (|\Delta W|^2 + \frac{s^4}{\epsilon} |W|^2) \, dx \, dt \\
 & \quad + \epsilon \int_{-T}^T \int_{\omega} e^{2s\phi} (|\partial_t w|^2 + s^2 |w|^2) \, dx \, dt. \quad (4.70)
 \end{aligned}$$

Taking ϵ small enough, we deduce that

$$\begin{aligned}
 \int_{-T}^T \int_{\omega} e^{2s\phi} (|\partial_t w|^2 + s^2 |w|^2) \, dx \, dt & \lesssim \frac{1}{s^2} \int_{-T}^T \int_{\omega} e^{2s\phi} |Pw|^2 \, dx \, dt \\
 & \quad + \int_{-T}^T \int_{\omega} e^{2s\phi} (|\Delta W|^2 + s^4 |W|^2) \, dx \, dt. \quad (4.71)
 \end{aligned}$$

To conclude, we use the inequality (4.67) and apply inequality (4.71) to replace the *observation term* in the right-hand side. ■

Again, we follow the method introduced by Buchgeim-Klibanov [Buchgeim and Klibanov, 1981] and apply the Carleman estimate given by Theorem 4.5.1 to the time derivative of the solution.

Theorem 4.5.2 *We assume that ω satisfies (4.7) and that $\sigma \in H^1(0, T; L^\infty(\Omega))$ is such that $|\sigma(x, 0)| \geq \alpha > 0$ a.e. in Ω . Let $T \geq T_0$ defined by (4.8).*

Let $u_{|\theta}$ be the solution of (4.10) associated to $\theta \in L^2(\Omega)$. Then, there exists a positive constant C^{st} such that

$$\int_{\Omega} |\theta|^2 \, dx \leq C^{st} \int_0^T \int_{\omega} |\Delta u_{|\theta}|^2 + |u_{|\theta}|^2 \, dx \, dt. \quad (4.72)$$

PROOF: In this proof, we write $a \lesssim b$ if there exists a positive scalar $c > 0$ independent of s such that $a \leq cb$.

Let v be the solution of (4.31) and V its time primitive such that $V(\cdot, 0) = 0$. As in the proof of Theorem 4.2.2, we extend the function v on $(-T, 0)$ by setting $v(\cdot, -t) = -v(\cdot, t)$. We set $w = \eta v$ in $\Omega \times (-T, T)$ where η satisfies (4.32) and W its time primitive. Reasoning exactly as in the proof of Theorem 4.2.2 and using the Carleman estimate (4.65) of Theorem 4.5.1, one can show that

$$\alpha^2 s^{1/2} \int_{\Omega} e^{2s\phi(0)} |\theta|^2 \, dx \lesssim \int_{\Omega} e^{2s\phi(0)} |\theta|^2 \, dx + s^2 \int_{-T}^T \int_{\omega} e^{2s\phi} (|\Delta W|^2 + s^4 |W|^2) \, dx \, dt,$$

where, if s is chosen sufficiently large, we can absorb the first term in the right-hand side by the term in the left-hand side. Moreover, we notice that

$$W(x, t) = \int_0^t \eta(\tau) v(x, \tau) \, d\tau = \eta(t) V(x, t) - \int_0^t \eta'(\tau) V(x, \tau) \, d\tau.$$

So, since for all $t \in (-T, T)$ and τ such that $|\tau| \leq |t|$, $\phi(t) \leq \phi(\tau)$, we have

$$\int_{-T}^T \int_{\omega} e^{2s\phi} |W|^2 \, dx \, dt \lesssim \int_{-T}^T \int_{\omega} e^{2s\phi} |V|^2 \, dx \, dt,$$

and

$$\int_{-T}^T \int_{\omega} e^{2s\phi} |\Delta W|^2 \, dx \, dt \lesssim \int_{-T}^T \int_{\omega} e^{2s\phi} |\Delta V|^2 \, dx \, dt.$$

Therefore,

$$s^2 \int_{-T}^T \int_{\omega} e^{2s\phi} (|\Delta(\eta V)|^2 + s^4 |\eta V|^2) \, dx \, dt \lesssim e^{2s\|\phi(0)\|_{C^0(\Omega)}} \int_{-T}^T \int_{\omega} |\Delta V|^2 + |V|^2 \, dx \, dt.$$

Since $v = \partial_t u$ and since the integrals on $(-T, T)$ are twice the integrals on $(0, T)$, this concludes the proof of Theorem 4.5.2. ■

We face an ill-posed problem of degree 2. The observation operators defined in Chapter 3 either by (3.7) or (1.56) do not seem to fit the norm that appears in the identifiability condition. In fact, the norms associated with these C operators in Chapter 3 are $\|Cz\|_{H_0^1}$ or $\|Cz\|_{L^2(\omega)}$, but here we need $\|\Delta(Cz)\|_{L^2(\omega)}$. The rest of the study: definition of the observation operator, cost function and reconstruction error are still under investigation.

4.5.2 Toward the reconstruction of piecewise constant parameters

In this section, we look at the case where the source term we are trying to reconstruct is a piecewise constant source term. Until now, we assumed that

$$\|\check{\theta}\|_{H_0^1(\Omega)} \leq M,$$

to obtain precise reconstruction estimates and which obviously cannot be satisfied in this section. However, as the L^2 norm of the source term appears in the obtained observability inequality, this justifies to use classical Tikhonov regularization. In particular, we are interested in the following case:

$$\check{\theta}(x) = \begin{cases} 1, & \text{in } \Omega_{\text{hole}}, \\ 0, & \text{in } \bar{\Omega} \setminus \Omega_{\text{hole}}, \end{cases} \quad (4.73)$$

as illustrated in Figure 4.16.

We rely on a classical Tikhonov term with the L^2 norm, more precisely, we define the new

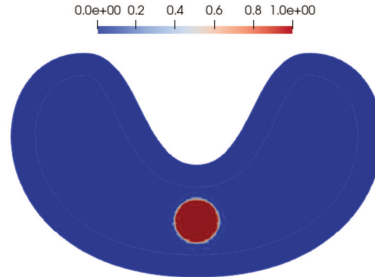


Figure 4.16: Representation of $\check{\theta}$

cost functional by $\mathcal{J}_T : L^2(\Omega) \rightarrow \mathbb{R}$ such that

$$\begin{aligned} \mathcal{J}_T(\theta) &= \frac{\epsilon}{2} \|\theta\|_{L^2(\Omega)}^2 + \frac{1}{2} \int_0^T \|y^\delta(t) - \partial_t u|_\theta\|_{L^2(\omega)}^2 dt \\ &= \frac{\epsilon}{2} \|\theta\|_{L^2(\Omega)}^2 + \frac{1}{2} \int_0^T \|y^\delta(t) - Cz|_\theta\|_{L^2(\omega)}^2 dt. \end{aligned} \quad (4.74)$$

The observer (4.53) should be slightly modified into

$$\begin{cases} \dot{\hat{\theta}} = \frac{1}{\delta^2} \Lambda L^* C^* (y^\delta - C\hat{z}), & t > 0, \\ \dot{\Lambda} = -\frac{1}{\delta^2} \Lambda L^* C^* C \Lambda, & t > 0, \\ \dot{\hat{z}} = A\hat{z} + B\hat{\theta} + L\dot{\hat{\theta}}, & t > 0, \\ \hat{\theta}(0) = \hat{\theta}_0 \\ z(0) = \hat{z}_0, \\ \Lambda(0) = M^2 \text{Id}_{\mathcal{P}}, \end{cases} \quad (4.75)$$

where the operator $(L(t))_{t \geq 0}$ is such that $z|_\theta(t) = L(t)\theta$ is the mild solution of (4.5) and C is defined by (4.9). We discretize this observer in time and space in the same way as in section 4.4.1. To take into account that the source term to be reconstructed is sharp, we propose to study the adaptive basis proposed by [de Buhan and Osses, 2010] and often used in the literature [de Buhan and Kray, 2013; Grote and Nahum, 2019; Baffet et al., 2021]. The idea is to adapt the basis of eigenfunction on which we decompose $\check{\theta}$ as in [de Buhan and Osses, 2010] by

$$\check{\theta} = \sum_{k=1}^{\infty} \check{\theta}_k \phi_k, \quad (4.76)$$

where the ϕ_k are the eigenfunctions satisfying the following elliptic boundary-value problem

$$\begin{cases} -\nabla \cdot \left(\frac{1}{\sqrt{|\nabla \theta|^2 + \nu^2}} \nabla \phi_k \right) = \mu_k \phi_k, & \text{in } \Omega, \\ \phi_k = 0, & \text{on } \partial\Omega, \end{cases} \quad (4.77)$$

with here ν is a small positive scalar which ensure that the denominator is not zero and μ_k are the eigenvalues. For piecewise constant functions, this decomposition has proven to be remarkably precise and effective.

We explain how this adaptive basis can be used in the inverse problem. The principle is as follows: we iterate the inverse problem, taking as a prior the source term to be reconstructed in the previous iteration. In addition, at the end of each iteration, we decompose the reconstructed source term into the adaptive basis, which will be used in the next iteration. At each step, the mesh is adapted to the basis used to decompose the source term. So we can reduce the number of eigenvectors used in the decomposition to save computational cost.

As in Section 4.4.2, the measurements at our disposal are created using the direct problem on the mesh with 56193 dofs (see Figure 4.8 (right)) over a time window $[0, 1]$ with a time step of $\tau = 0.01$, where the source term is that defined by (4.16). To solve the inverse problem, we chose $\delta = 10^{-5}$ and $M = 50$, with a final time $t^n = 1$ for each iteration with a time step of $\tau = 0.01$. In the first step, the source term is decomposed on the basis of the Laplacian with 30 modes. For the other iterations, only 10 modes are used,

Algorithm 1 Iterated inverse problem

```

Parameter space = basis of the Laplacian
Solve the inverse problem (4.56)
Decompose  $\hat{\theta}_N^h$  in the adaptive basis
Adapt the mesh for this basis
while  $\|\hat{\theta}_N^h - \check{\theta}\|_{L^2(\Omega)} \geq \varepsilon$  and Number of steps < Maximum Numbers of steps do
  Use  $\hat{\theta}_N^h$  as a priori
  Solve the inverse problem (4.56)
  Decompose  $\hat{\theta}_N^h$  in the adaptive basis
  Adapt the mesh for this basis
end while

```

which ensures lower computational costs. We show the final observer $\hat{\theta}_n^h$ at time $t^n = 1$ and the mesh used in Figure 4.17. For the first iteration, we use the mesh with 14169 dofs shown in Figure 4.8 (left). We note that during the first iteration we do not succeed in reconstructing the source term very accurately. In particular, the circle boundary in the center is not very sharp. For the second iteration, we used the adaptive basis defined by (4.76)-(4.77). The mesh has fewer dofs (only 1834) and is tighter for the large variations of the source term. The source term is already better reconstructed, especially at the boundary. In the last iteration, the mesh has even fewer dofs (1282). Where the source term is zero, the mesh is very inaccurate and refined where the parameter varies. There are still differences with the target parameter, but the result is improved using the adaptive basis.

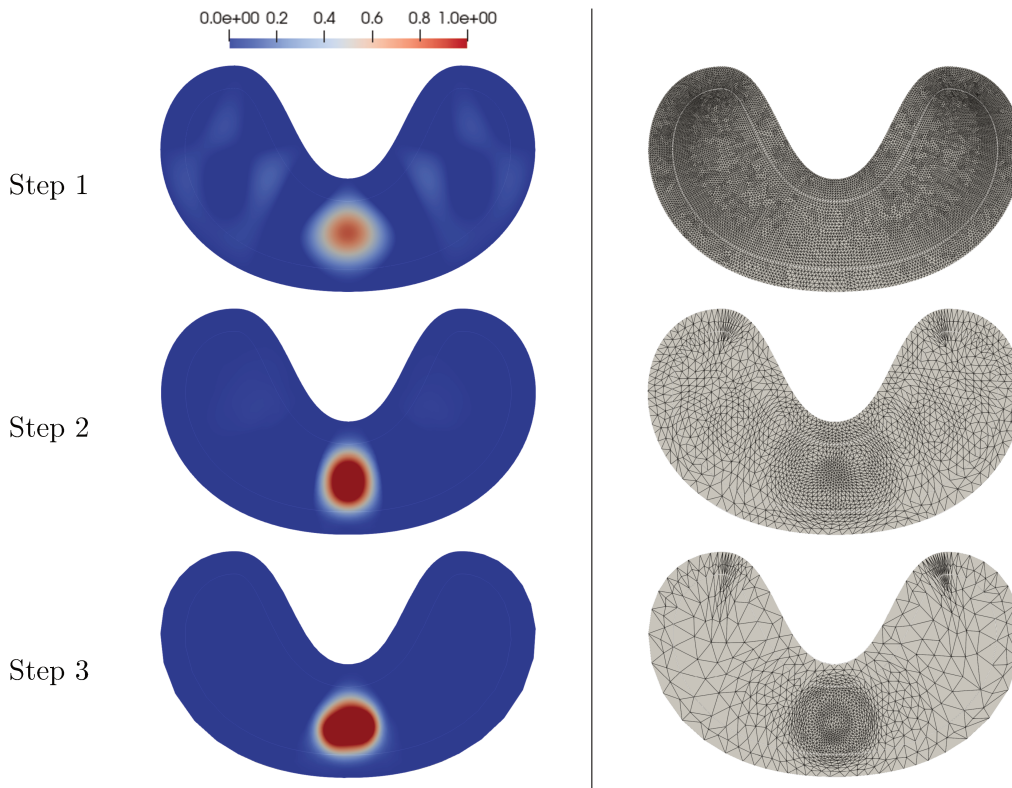


Figure 4.17: Representation of $\hat{\theta}_n^h$ (left) and its mesh (right) at $t^n = 1$ at each iteration

Let us conclude this section with a final remark. The L^2 -norm is not the most appro-

priate norm for the source term $\tilde{\theta}$. One perspective would be to use a norm that takes total variation into account. Indeed, this kind of norm is often used in image segmentation problems [Mumford and Shah, 1989]. For instance see the total variation–based image denoising model of [Rudin et al., 1992] and its many extensions [Chan and Vese, 2001; Chan et al., 2006; Caselles et al., 2007].

4.6 Conclusion

In this chapter, we showed how to extend the methodology from Chapter 3 and what differences arise when the source term to reconstruct is multiplied by a function that depends on space and time rather than just time. Here, the identifiability result can only be obtained through Carleman estimate. In particular, we prove here a new Carleman estimate in the case of internal *velocity* measurements. Using this inequality we show that we face an ill-posed inverse problem of degree 1. Using this inequality we show that we face an ill-posed inverse problem of degree 2. However, when measuring only the field, our estimate indicates that the problem is ill-posed of degree 2. We then define a cost functional with a classical data fitting term and a Tikhonov regularization term. As in Chapter 3, we estimate explicitly the reconstruction error. To solve this inverse problem, we use the observer from the previous chapter, which we have shown to be equivalent to the minimization problem. Finally, a new Carleman estimate for the *field* measurements case was established. The rest of the analysis is still under investigation since the observation operators presented in the previous chapter when considering field measurements, are not adapted to the observability condition that we have proved here. Eventually, the second perspective examined is the case in which the source term is piecewise constant. In this case, the Tikhonov regularization is replaced by a L^2 norm term. Although our theoretical analysis is not complete for such inversion, the numerical results are promising. This is due to the introduction of an adaptive basis that takes into account rapid changes in the variations. Indeed, we then developed an algorithm to solve the inverse problem iteratively by modifying the basis and the mesh at each stage, taking into account the reconstruction of the source term at the previous stage. A way to improve the iterative algorithm would be to use norms that are more appropriate for this type of problem, for example a total variation norm.

CHAPTER 5

Conclusions and perspectives

I am the Doctor.
— Doctor Who

5.1 Conclusions et perspectives

L'assimilation de données pour les phénomènes d'ondes possèdent de nombreux champs d'applications possibles. L'objectif de cette thèse était de formuler et d'analyser une méthode séquentielle d'estimation de source pour l'équation des ondes par le point de vue du contrôle optimal en dimension infinie. Après un premier chapitre de rappels sur l'estimation d'état, nous avons développé les outils nécessaires à l'analyse théorique et numérique de l'estimation de source.

Chapitre 2

Une idée importante défendue lors de cette thèse concerne la discrétisation. Une méthode adaptée est, dans un premier temps, de discrétiser la dynamique du modèle puis de formuler un observateur sur le problème discrétisé. Le Chapitre 2 illustre cette approche. Des propriétés telles que la stabilité exponentielle, satisfaites au niveau continu, ne le sont plus potentiellement après discrétisation (ici par des éléments finis spectraux d'ordre élevés). Pour y remédier, nous introduisons un terme dissipatif évanescent, ce qui permet de retrouver ces propriétés au niveau discret sans altérer l'ordre de convergence. Une perspective intéressante du Chapitre 2 serait d'étendre tous les résultats obtenus pour des problèmes en deux dimensions. Des difficultés sont déjà observées au niveau continu, notamment la variable v_h doit être définie comme un gradient sur chaque élément, ce qui peut ne pas être le cas pour une formulation faible de l'équation. En outre, il serait nécessaire d'introduire un traitement numérique adéquat des conditions aux limites.

Chapitre 3

L'objectif de ce travail est de proposer une stratégie s'appuyant sur des observateurs pour estimer un terme source d'une équation d'onde à partir de mesures internes dans un sous-domaine. Deux cas sont traités suivant la régularité des mesures. La méthode consiste dans un premier temps à prouver un résultat d'identifiabilité à l'aide des conditions classiques d'observabilité pour les équations d'ondes, à partir duquel on peut déduire que la reconstruction de la source est un problème inverse mal posé d'ordre 2. Ce problème inverse est résolu à l'aide d'une stratégie séquentielle dont il est prouvé qu'elle est équivalente à la minimisation d'une fonction de coût avec régularisation de Tikhonov. Nous proposons également une discrétisation en temps qui permet de conserver cette propriété au niveau discret. Le Chapitre 2 démontre que concernant la reconstruction d'état, les preuves d'observabilité ne sont plus nécessairement satisfaites après discrétisation. Pour aller plus loin, On pourrait logiquement penser que c'est aussi le cas pour la reconstruction de sources. Il faudrait essayer d'étendre la méthode présentée au Chapitre 2 et de rajouter des termes dissipatifs qui permettraient de satisfaire ces propriétés après discrétisation. De plus, on a supposé qu'en plus d'observations sur le système, on a la connaissance parfaite des conditions initiales. Pour s'approcher d'un cas plus réaliste, nous envisageons le cadre de l'estimation conjointe état-source, où l'on prend en compte des incertitudes sur les données initiales dans la reconstruction de sources, voir Section 5.2.1.

Chapitre 4

Dans le dernier chapitre, nous étendons ces résultats pour un terme source multiplié par une fonction connue dépendante de l'espace et du temps. La principale différence réside dans les outils développés pour montrer l'observabilité à savoir ici des estimations de Carleman. Nous avons établi deux nouvelles estimations de Carleman, une adaptée lorsque l'on a notre disposition des mesures de vitesses et la seconde pour des mesures de champs. Pour le premier cas, on a pu adapter les stratégies présentées au Chapitre 3, à savoir la résolution

à l'aide d'un filtre de Kalman et sa discrétisation en temps. Il faudrait maintenant pouvoir faire de même dans le cadre des mesures de champs. En outre, nous avons présenté un algorithme itératif, reposant sur l'introduction de base adaptative, qui semble approprié lorsque l'on s'intéresse à la reconstruction d'un terme source constant par morceaux. Pour aller plus loin, il faudrait étudier en détails l'erreur de reconstruction entre le terme source cible et celui reconstruit par la méthode que nous proposons.

À titre de perspectives, nous n'avons pas abordé la reconstruction de paramètres, un problème non linéaire qui possède de nombreux champs d'applications concrètes. Une des pistes alors envisageables est de linéariser ce problème autour d'une trajectoire connue. Cela permet de se ramener à un problème de reconstruction de sources. L'étape suivante serait donc d'itérer ce problème de reconstruction comme détaillé Section 5.2.2.

5.2 Quelques perspectives détaillées

5.2.1 Vers l'estimation conjointe source-état

Dans le Chapitre 3 nous avons cherché à reconstruire un terme source en supposant avoir à notre disposition des observations sur le système et la connaissance parfaite des conditions initiales. En pratique, ceci n'est pas réaliste et l'on doit considérer des incertitudes sur la donnée initiale également. Le but de cette section est d'esquisser une manière d'appréhender ce problème. Pour cela, nous réutilisons le cadre défini dans le Chapitre 3, à savoir la propagation d'onde (3.1). On se place directement dans un cas plus complexe lorsque l'on suppose que les observations à notre disposition sont bruitées à l'aide d'un bruit peu régulier, autrement dit les mesures disponibles appartiennent à $L^2(\omega)$. Dans ce cas précis, on rappelle l'opérateur d'observation considéré

$$C = \begin{pmatrix} I_\omega & 0 \end{pmatrix} : \mathcal{Z} \rightarrow L^2(\omega) \text{ où } I_\omega \in \mathcal{L}(H_0^1(\Omega), L^2(\omega)), \quad (5.1)$$

dont l'adjoint est donné par $C^* = \begin{pmatrix} F_\omega & 0 \end{pmatrix}^t$ où $F_\omega \in \mathcal{L}(L^2(\omega), H_0^1(\Omega))$ est défini par

$$F_\omega \phi := \psi \text{ solution de } \begin{cases} -\Delta \psi = \chi_\omega \phi, & \text{in } \Omega, \\ \psi = 0, & \text{on } \partial\Omega, \end{cases} \quad (5.2)$$

avec χ_ω la fonction caractéristique du domaine ω . Le Chapitre 3 présente un observateur optimal (3.92) qui permet d'obtenir une estimation du terme source θ que l'on souhaite retrouver. Cependant, nous n'avons ici plus connaissance des conditions initiales, mais seulement d'un *a priori* que l'on notera \hat{z}_0 . L'état z du système est donc incertain. Nous avons ainsi recours à l'une des méthodes présentées au Chapitre 1 et introduisons un observateur de Luenberger sur la dynamique de l'état. Dans la Section 1.5.3 du Chapitre 1, nous avons vu que l'on pouvait considérer deux méthodes pour prendre en compte la régularité du bruit. On décide dans cette section d'utiliser la méthode présentée en Section 1.5.3.1 du Chapitre 1. Dans la suite, l'opérateur d'observation classique est noté

$$\tilde{C} = \begin{pmatrix} I_\omega & 0 \end{pmatrix} : \begin{cases} H_0^1(\Omega) \times L^2(\Omega) \rightarrow H_\Gamma^1(\omega), \\ z = \begin{pmatrix} u \\ v \end{pmatrix} \mapsto y = u|_\omega, \end{cases} \quad (5.3)$$

où $I_\omega : H_0^1(\Omega) \rightarrow H_\Gamma^1(\omega)$ est l'opérateur de restriction. On rappelle que l'idée d'introduire une régularisation G^α de l'adjoint de l'opérateur d'observation classique défini par $\tilde{C}^* =$

$(E_\omega - 0)^t$ où E_ω est donné par (1.95). On rappelle également que $G^\alpha = (E_\omega^\alpha - 0)^t$ où E_ω^α est défini par (1.113). Nous pouvons donc maintenant définir la fonctionnelle coût que l'on souhaite minimiser et qui prend en compte le fait d'avoir des incertitudes sur les données initiales,

$$\mathcal{J}_T(\theta) = \frac{1}{2M^2} \|\theta\|_{H_0^1(\Omega)}^2 + \frac{1}{2\delta^2} \int_0^T \|y^\delta - \tilde{u}_\theta\|_{L^2(\omega)}^2 dt, \quad (5.4)$$

où \tilde{u} est solutions de la dynamique de l'estimateur d'état $\tilde{z} = (\tilde{u} \ \tilde{v})$

$$\begin{cases} \dot{\tilde{z}} = A\tilde{z} + \gamma G^\alpha (y^\delta - \tilde{C}\tilde{z}), \\ \tilde{z}(0) = \hat{z}_0. \end{cases} \quad (5.5)$$

On comprend alors que la relation fondamentale qui lie minimisation de la fonctionnelle coût et observateur optimal est conservée. À savoir, le minimum $\bar{\theta}_T$ de la fonctionnelle coût (5.4) satisfait la relation $\bar{\theta}_T = \hat{\theta}(T)$ où $\hat{\theta}$ est donnée par la dynamique

$$\begin{cases} \dot{\hat{z}} = A\hat{z} + B\hat{\theta} + L\dot{\hat{\theta}} + \gamma G^\alpha (y^\delta - \tilde{C}\hat{z}) \\ \dot{\hat{\theta}} = \frac{1}{\delta^2} \Lambda L^* C^* (y^\delta - C\hat{z}) \\ \dot{\Lambda} = -\frac{1}{\delta^2} \Lambda L^* C^* C L \Lambda \\ \dot{L} = (A - \gamma G^\alpha \tilde{C})L + B \\ \hat{z}(0) = \hat{z}_0, \hat{\theta}(0) = 0, \Lambda(0) = -M^2 \Delta_0^{-1}, L(0) = 0. \end{cases} \quad (5.6)$$

L'analyse cependant se complexifie. En effet, deux opérateurs d'observation différents sont introduits dans la dynamique (5.6), un pour le problème inverse de source et l'autre dans le Luenberger pour l'estimation d'état. Une première difficulté pour l'analyse de l'observateur (5.8) concerne l'opérateur de sensibilité L . En effet, cet opérateur apporte les informations nécessaires à la reconstruction du terme source θ . Or en rajoutant le terme de Luenberger, l'opérateur devient dissipatif.

Puisque l'observateur de Luenberger ne filtre l'erreur d'état initial que de manière asymptotique, nous rajoutons un terme $e^{\rho(T-t)}$ appelé facteur d'oubli qui va augmenter le poids de la mesure actuelle par rapport à la mesure passée. Une piste envisagée est de se donner un degré de liberté supplémentaire en considérant la fonctionnelle coût suivante qui prend maintenant en compte ce facteur d'oubli

$$\mathcal{J}_T^\rho(\theta) = \frac{1}{2M^2} \|\theta\|_{H_0^1(\Omega)}^2 + \frac{1}{2\delta^2} \int_0^T e^{-\rho(T-t)} \|y^\delta - \tilde{u}_\theta\|_{L^2(\omega)}^2 dt. \quad (5.7)$$

Nous pouvons alors de nouveau établir la relation fondamentale qui lie la minimisation de cette fonctionnelle coût et l'observateur optimal (5.8). À savoir, le minimum $\bar{\theta}_T$ de la fonctionnelle coût (5.7) satisfait la relation $\bar{\theta}_T = \hat{\theta}(T)$ où $\hat{\theta}$ est donnée par la dynamique

$$\begin{cases} \dot{\hat{z}} = A\hat{z} + B\hat{\theta} + L\dot{\hat{\theta}} + \gamma G^\alpha (y^\delta - \tilde{C}\hat{z}) \\ \dot{\hat{\theta}} = \frac{1}{\delta^2} \Lambda L^* C^* (y^\delta - C\hat{z}) \\ \dot{\Lambda} = \rho\Lambda - \frac{1}{\delta^2} \Lambda L^* C^* C L \Lambda \\ \dot{L} = A(L - \gamma G^\alpha \tilde{C}) + B \\ \hat{z}(0) = \hat{z}_0, \hat{\theta}(0) = 0, \Lambda(0) = -M^2 \Delta_0^{-1}, L(0) = 0. \end{cases} \quad (5.8)$$

La différence majeure entre cet observateur (5.8) et celui sans le facteur d'oubli (5.6) porte sur la dynamique de Riccati Λ . Nous ne rentrerons pas dans les détails de la preuve, mais

l'idée est semblable à la Section 3.4 du Chapitre 3. L'équation de Riccati donnée par (5.8) est naturellement associée à des fonctionnelles de Lyapunov (voir [Pham et al., 1998a]), ce qui pourrait le rendre plus simple à analyser.

5.2.2 Estimation paramétrique : une méthode itérative

Nous présentons maintenant le problème de l'estimation de paramètre pour l'équation des ondes. Le but est de se ramener à un problème d'estimation de source. On étudie l'équation des ondes suivante

$$\rho(x)\partial_{tt}u(x, t) - \Delta u(x, t) = 0. \quad (5.9)$$

L'enjeu est d'estimer le paramètre ρ à partir d'observations sur ce système. Une possibilité pour résoudre un tel problème est de réaliser une approximation asymptotique autour d'un paramètre connu. Plus précisément, on suppose que l'on a un *a priori* ρ_0 sur le paramètre ρ et ce dernier est décomposé ainsi

$$\rho \approx \rho_0 + \varepsilon\rho_1,$$

où $\varepsilon \in \mathbb{R}$ est un petit paramètre à choisir et ρ_1 la nouvelle inconnue à déterminer. On décompose alors la solution u de (5.9) de la même façon

$$u \approx u_0 + \varepsilon u_1,$$

où u_0 est connu. Le problème d'estimation devient alors de reconstruire le terme ρ_1 apparaissant dans la dynamique suivante

$$\rho_0\partial_{tt}u_1(x, t) - \Delta u_1(x, t) = -\rho_1(x)\partial_{tt}u_0(x, t).$$

Nous pouvons alors résoudre ce problème en utilisant la méthode développée au Chapitre 4. En effet, en posant

$$\rho_1(x) \approx \theta(x) \text{ et } -\partial_{tt}u_0(x, t) = \sigma(x, t), \quad (5.10)$$

nous retrouvons la dynamique (4.1) étudiée lors du Chapitre 4. En résolvant ce problème d'estimation de source par la méthode proposée au Chapitre 4, nous obtenons une estimation de ρ_1 . Pour améliorer l'estimation de ρ , une idée consiste à itérer N fois le processus. Autrement dit, on décomposera ρ itérativement de la façon suivante

$$\rho = \sum_{k=0}^N \varepsilon^k \rho_k,$$

où cette fois l'*a priori* à l'itération n est

$$\sum_{k=0}^{n-1} \varepsilon^k \rho_k,$$

et l'on cherche à estimer ρ_n . Il faudrait étudier la convergence totale du processus itératif en se reposant sur les résultats établis au Chapitre 4 à chaque itération notamment la condition d'observabilité et le résultat de stabilité.

Bibliographie

- Aguilo, M. A., Aquino, W., Brigham, J., and Fatemi, M. (2010). An inverse problem approach for elasticity imaging through vibroacoustics. *IEEE Transactions on Medical Imaging*, 29(4) :1012–1021.
- Akcelik, V., Bielak, J., Biros, G., Epanomeritakis, I., Fernandez, A., Ghattas, O., Kim, E. J., Lopez, J., O’Hallaron, D., and Tu, T. (2003). High resolution forward and inverse earthquake modeling on terascale computers. page 52.
- Albanese, R., Banks, H. T., and Raye, J. K. (2002). Nondestructive evaluation of materials using pulsed microwave interrogating signals and acoustic wave induced reflections. *Inverse Problems*, 18(6) :1935.
- Altpeter, I., Becker, R., Dobmann, G., Kern, R., Theiner, W., and Yashan, A. (2002). Robust solutions of inverse problems in electromagnetic non-destructive evaluation. 18(6) :1907.
- Alves, C., Silvestre, A. L., Takahashi, T., and Tucsna, M. (2009). Solving inverse source problems using observability. applications to the euler–bernoulli plate equation. *SIAM Journal on Control and Optimization*, 48(3) :1632–1659.
- Auroux, D. and Blum, J. (2008). A nudging-based data assimilation method : the back and forth nudging (bf) algorithm. *Nonlinear Processes in Geophysics*, 15(2) :305–319.
- Aussal, M. and Moireau, P. (2022). Kernel representation of kalman observer and associated h-matrix based discretization. *ESAIM : Control, Optimisation and Calculus of Variations*, 28 :78.
- Baffet, D. H., Grote, M. J., and Tang, J. H. (2021). Adaptive spectral decompositions for inverse medium problems. *Inverse Problems*, 37(2) :025006.
- Bal, G. and Imperiale, S. (2015). Displacement reconstructions in ultrasound elastography. *SIAM Journal on Imaging Sciences*, 8(2) :1070–1089.
- Banks, H. T., Joyner, M. L., Wincheski, B., and Winfree, W. P. (2000). Nondestructive evaluation using a reduced-order computational methodology. 16(4) :929.
- Banks, H. T. and Kunisch, K. (2012). *Estimation Techniques for Distributed Parameter Systems*. Springer Science & Business Media.
- Bardos, C., Lebeau, G., and Rauch, J. (1988). Un exemple d’utilisation des notions de propagation pour le contrôle et la stabilisation de problèmes hyperboliques.
- Bardos, C., Lebeau, G., and Rauch, J. (1992). Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary. *SIAM journal on control and optimization*, 30(5) :1024–1065.

-
- Bartoli, I., Marzani, A., Di Scalea, F. L., and Viola, E. (2006). Modeling wave propagation in damped waveguides of arbitrary cross-section. *Journal of sound and vibration*, 295(3-5) :685–707.
- Baudouin, L. (2010). Lipschitz stability in an inverse problem for the wave equation. *Master report. Available on <http://hal.archives-ouvertes.fr/hal-00598876/fr>*.
- Baudouin, L., Crépeau, E., and Valein, J. (2011). Global carleman estimate on a network for the wave equation and application to an inverse problem. *Mathematical control and related fields*, 1(3) :p-307.
- Baudouin, L., de Buhan, M., and Ervedoza, S. (2013). Global carleman estimates for waves and applications. *Communications in Partial Differential Equations*, 38(5) :823–859.
- Baudouin, L., de Buhan, M., and Ervedoza, S. (2017). Convergent algorithm based on Carleman estimates for the recovery of a potential in the wave equation. 55(4) :1578–1613.
- Baudouin, L., de Buhan, M., Ervedoza, S., and Osses, A. (2021). Carleman-based reconstruction algorithm for waves. *SIAM Journal on Numerical Analysis*, 59(2) :998–1039.
- Baudouin, L. and Ervedoza, S. (2013). Convergence of an inverse problem for a 1-d discrete wave equation. *SIAM Journal on Control and Optimization*, 51(1) :556–598.
- Baudouin, L., Mercado, A., and Osses, A. (2007). A global carleman estimate in a transmission wave equation and application to a one-measurement inverse problem. *Inverse Problems*, 23(1) :257.
- Baumeister, J. (1987). *Stable Solution of Inverse Problems*. Springer.
- Bellassoued, M. (2004). Global logarithmic stability in inverse hyperbolic problem by arbitrary boundary observation. *Inverse Problems*, 20(4) :1033.
- Bellassoued, M. and Yamamoto, M. (2006). Inverse source problem for a transmission problem for a parabolic equation. *Journal of Inverse & Ill-Posed Problems*, 14(1).
- Bellman, R. (1966). Dynamic programming. *Science*, 153(3731) :34–37.
- Bensoussan, A. (1971). *Filtrage optimal des systèmes linéaires*, volume 3. Dunod.
- Bensoussan, A., Da Prato, G., Delfour, M. C., and Mitter, S. K. (2007). *Representation and control of infinite dimensional systems*. Springer Science & Business Media.
- Bergounioux, M., Bonnefond, X., and Maréchal, P. (2010). Optimal control techniques for thermo-acoustic tomography.
- Blitz, J. and Simpson, G. (1995). *Ultrasonic Methods of Non-Destructive Testing*, volume 2. Springer Science & Business Media.
- Blum, J., Le Dimet, F.-X., and Navon, I. M. (2009). Data assimilation for geophysical fluids. In *Handbook of numerical analysis*, volume 14, pages 385–441. Elsevier.
- Bolis, A., Cantwell, C., Kirby, R., and Sherwin, S. (2014). From h to p efficiently : optimal implementation strategies for explicit time-dependent problems using the spectral/hp element method. *International journal for numerical methods in fluids*, 75(8) :591–607.

- Boller, C. and Meyendorf, N. (2008). State-of-the-art in structural health monitoring for aeronautics. pages 1–8.
- Bonnet, M. and Aquino, W. (2015). Three-dimensional transient elastodynamic inversion using an error in constitutive relation functional. *Inverse Problems*, 31 :035010.
- Brenner, S. C. and Scott, L. R. (2008). *The Mathematical Theory of Finite Element Methods*, volume 15 of *Texts in Applied Mathematics*. Springer.
- Brownjohn, J. M. (2007). Structural health monitoring of civil infrastructure. 365(1851) :589–622.
- Buchgeim, A. and Klibanov, M. V. (1981). Uniqueness in the large of a class of multidimensional inverse problems. In *Sov. Math. Dokl.*, volume 24, pages 244–247.
- Burman, E., Feizmohammadi, A., Münch, A., and Oksanen, L. (2021). Space time stabilized finite element methods for a unique continuation problem subject to the wave equation. *ESAIM : Mathematical Modelling and Numerical Analysis*, 55 :S969–S991.
- Burman, E., Feizmohammadi, A., and Oksanen, L. (2020). A finite element data assimilation method for the wave equation. *Mathematics of Computation*, 89(324) :1681–1709.
- Burq, N. and Gérard, P. (2002). *Contrôle optimal des équations aux dérivées partielles*. Ecole polytechnique, Département de mathématiques.
- Burq, N. and Gérard, P. (2020). Stabilization of wave equations on the torus with rough dampings. 2(3) :627–658.
- Caforio, F. (2019). Mathematical modelling and numerical simulation of elastic wave propagation in soft tissues with application to cardiac elastography.
- Caiazzo, A., Caforio, F., Montecinos, G., Muller, L. O., Blanco, P. J., and Toro, E. F. (2017). Assessment of reduced-order unscented kalman filter for parameter identification in 1-dimensional blood flow models using experimental data. *International journal for numerical methods in biomedical engineering*, 33(8) :e2843.
- Carleman, T. (1939). *Sur un problème d’unicité pour les systèmes d’équations aux dérivées partielles à deux variables indépendantes*. Almqvist & Wiksell.
- Caselles, V., Chambolle, A., and Novaga, M. (2007). The discontinuity set of solutions of the tv denoising problem and some extensions. *Multiscale modeling & simulation*, 6(3) :879–894.
- Castro, C. and Micu, S. (2006). Boundary controllability of a linear semi-discrete 1-D wave equation derived from a mixed finite element method. 102(3) :413–462.
- Chan, T. F., Esedoglu, S., and Nikolova, M. (2006). Algorithms for finding global minimizers of image segmentation and denoising models. *SIAM journal on applied mathematics*, 66(5) :1632–1648.
- Chan, T. F. and Vese, L. A. (2001). Active contours without edges. *IEEE Transactions on image processing*, 10(2) :266–277.
- Chang, P. C., Flatau, A., and Liu, S.-C. (2003). Health monitoring of civil infrastructure. 2(3) :257–267.

- Chapelle, D., Cîndea, N., de Buhan, M., and Moireau, P. (2012a). Exponential convergence of an observer based on partial field measurements for the wave equation. *Mathematical Problems in Engineering*.
- Chapelle, D., Cîndea, N., and Moireau, P. (2012b). Improving convergence in numerical analysis using observers—the wave-like equation case. *Mathematical Models and Methods in Applied Sciences*, 22(12) :1250040.
- Chavent, G. (2010). *Nonlinear Least Squares for Inverse Problems : Theoretical Foundations and Step-by-Step Guide for Applications*. Springer Science & Business Media.
- Ciarlet, P. G. (2002). *The finite element method for elliptic problems*. SIAM.
- Cohen, G., Joly, P., Roberts, J. E., and Tordjman, N. (2001). Higher order triangular finite elements with mass lumping for the wave equation. 38(6) :2047–2078.
- Cohen, G. and Sinding, A. (2009). Non Spurious Mixed Spectral Element Methods for Maxwell’s Equations. In *The 9th International Conference on Mathematical and Numerical Aspects of Waves Propagation*.
- Cohen, G. C. (2002). *Higher-order numerical methods for transient wave equations*, volume 5. Springer.
- Coron, J.-M. (2007). *Control and Nonlinearity*. Number 136. American Mathematical Soc.
- Courtier, P., Andersson, E., Heckley, W., Vasiljevic, D., Hamrud, M., Hollingsworth, A., Rabier, F., Fisher, M., and Pailleux, J. (1998). The ecmwf implementation of three-dimensional variational assimilation (3d-var). i : Formulation. *Quarterly Journal of the Royal Meteorological Society*, 124(550) :1783–1807.
- Cox, S. and Zuazua, E. (1994). The Rate At Which Energy Decays In A String Damped At One End. *Communications in Partial Differential Equations*, 19 :213–243.
- Crase, E., Pica, A., Noble, M., McDonald, J., and Tarantola, A. (1990). Robust elastic nonlinear waveform inversion ; application to real data. 55(5) :527–538.
- Cross, E., Worden, K., and Farrar, C. (2013). Structural health monitoring for civil infrastructure. In *Health Assessment of Engineered Structures : Bridges, Buildings and Other Infrastructures*, pages 1–31. World Scientific.
- Curtain, R. and Pritchard, A. J. (1976). The infinite-dimensional Riccati equation for systems defined by evolution operators. *SIAM Journal on Control and Optimization*, 14(5) :951 – 983.
- Curtain, R. F. (1975). A survey of infinite-dimensional filtering. *SIAM Review. A Publication of the Society for Industrial and Applied Mathematics*, 17 :395 – 411.
- Cîndea, N. and Münch, A. (2015). Inverse problems for linear hyperbolic equations using mixed formulations. *Inverse Problems*, 31(7) :075001.
- Dalmora, A., Imperiale, A., Imperiale, S., and Moireau, P. (2023). A generic numerical solver for modeling the influence of stress conditions on guided wave propagation for shm applications. In *49th Annual Review of Progress in Quantitative Nondestructive Evaluation*, volume 86595, page V001T10A002. American Society of Mechanical Engineers.

- Dalton, R. P., Cawley, P., and Lowe, M. J. (2001). The potential of guided waves for monitoring large areas of metallic aircraft fuselage structure. 20(1) :29–46.
- Dautray, R. and Lions, J.-L. (1999). *Mathematical Analysis and Numerical Methods for Science and Technology : Volume 5 Evolution Problems I*.
- de Buhan, M. and Kray, M. (2013). A new approach to solve the inverse scattering problem for waves : combining the trac and the adaptive inversion methods. *Inverse Problems*, 29(8) :085009.
- de Buhan, M. and Osses, A. (2010). Logarithmic stability in determination of a 3d viscoelastic coefficient and a numerical example. *Inverse Problems*, 26(9) :095006.
- Delaunay, T., Imperiale, S., and Moireau, P. (2023). Uniform boundary stabilization of a high-order finite element space discretization of the 1-d wave equation.
- Deng, C. Y. (2011). A generalization of the Sherman–Morrison–Woodbury formula. *Applied Mathematics Letters*, 24(9) :1561–1564.
- Engl, H. W., Hanke, M., and Neubauer, A. (1996). *Regularization of Inverse Problems*, volume 375. Springer Science & Business Media.
- Ervedoza, S. (2010). Observability properties of a semi-discrete 1D wave equation derived from a mixed finite element method on nonuniform meshes. 16(2) :298–326.
- Ervedoza, S. and Zuazua, E. (2008). Perfectly matched layers in 1-d : energy decay for continuous and semi-discrete waves. *Numerische Mathematik*, 109(4) :597 – 634.
- Ervedoza, S. and Zuazua, E. (2012). *The Wave Equation : Control and Numerics*, volume 2048 of *Lecture Notes in Mathematics*. Springer Berlin Heidelberg.
- Evensen, G. (2003). The ensemble Kalman filter : Theoretical formulation and practical implementation. 53 :343–367.
- Faccioli, E., Maggio, F., Paolucci, R., and Quarteroni, A. (1997). 2D and 3D elastic wave propagation by a pseudo-spectral domain decomposition method. 1 :237–251.
- Farrar, C. R. and Worden, K. (2007). An introduction to structural health monitoring. 365(1851) :303–315.
- Feng, D. and Feng, M. Q. (2018). Computer vision for SHM of civil infrastructure : From dynamic response measurement to damage detection—A review. 156 :105–117.
- Fink, M., Cassereau, D., Derode, A., Prada, C., Roux, P., Tanter, M., Thomas, J.-L., and Wu, F. (2000). Time-reversed acoustics. *Reports on progress in Physics*, 63(12) :1933.
- Fu, H. and Han, B. (2004). A wavelet multiscale method for the inverse problems of a two-dimensional wave equation. *Inverse Problems in Science and Engineering*, 12(6) :643–656.
- Fursikov, A. and Imanuvilov, Y. (1996). Controllability of evolution equations. *Lecture Notes*, 3.
- Gauthier, O., Virieux, J., and Tarantola, A. (1986). Two-dimensional nonlinear inversion of seismic waveforms : Numerical results. *geophysics*, 51(7) :1387–1403.

-
- Gazis, D. C. (1959). Three-dimensional investigation of the propagation of waves in hollow circular cylinders. I. Analytical foundation. 31(5) :568–573.
- Gebauer, B. and Scherzer, O. (2008). Impedance-acoustic tomography. *SIAM Journal on Applied Mathematics*, 69(2) :565–576.
- Gennisson, J.-L., Deffieux, T., Fink, M., and Tanter, M. (2013). Ultrasound elastography : Principles and techniques. 94(5) :487–495.
- Ghil, M. (1989). Meteorological data assimilation for oceanographers. Part I : Description and theoretical framework. 13(3-4) :171–218.
- Girault, V. and Raviart, P.-A. (1986). *Finite Element Methods for Navier-Stokes Equations : Theory and Algorithms*. Springer-Verlag Berlin and Heidelberg GmbH & Co. K.
- Glowinski, R., Li, C.-H., and Lions, J.-L. (1990). A numerical approach to the exact boundary controllability of the wave equation (I) Dirichlet controls : Description of the numerical methods. 7 :1–76.
- Graves, R. W. (1996). Simulating seismic wave propagation in 3D elastic media using staggered-grid finite differences. 86(4) :1091–1106.
- Gripenberg, G., Londen, S. O., and Staffans, O. (1990). *Volterra Integral and Functional Equations*. Cambridge University Press.
- Grote, M. J. and Nahum, U. (2019). Adaptive eigenspace for multi-parameter inverse scattering problems. *Computers & Mathematics with Applications*, 77(12) :3264–3280.
- Haine, G. (2014a). An observer-based approach for thermoacoustic tomography.
- Haine, G. (2014b). Recovering the observable part of the initial data of an infinite-dimensional linear system with skew-adjoint generator. 26 :435–462.
- Hecht, F. (2012). New development in freefem++. *Journal of numerical mathematics*, 20(3-4) :251–266.
- Hoteit, I. (2001). Filtres de Kalman réduits et efficaces pour l’assimilation de données en océanographie.
- Houtekamer, P. L. and Mitchell, H. L. (1998). Data assimilation using an ensemble Kalman filter technique. 126(3) :796–811.
- Humbert, E., Privat, Y., and Trélat, E. (2019). Observability properties of the homogeneous wave equation on a closed manifold. (9) :749–772.
- Imanuvilov, O. and Yamamoto, M. (2001a). Global uniqueness and stability in determining coefficients of wave equations. *Communications in Partial Differential Equations*, 26(7-8) :1409–1425.
- Imanuvilov, O. Y. (2002). On Carleman estimates for hyperbolic equations. *Asymptotic Analysis*, 32(3-4) :185–220.
- Imanuvilov, O. Y. and Yamamoto, M. (2001b). Global Lipschitz stability in an inverse hyperbolic problem by interior observations. *Inverse problems*, 17(4) :717.

- Imanuvilov, O. Y. and Yamamoto, M. (2003). Determination of a coefficient in an acoustic equation with a single measurement. *Inverse Problems*, 19(1) :157.
- Imanuvilov, O. Y. and Yamamoto, M. (2005). Carleman estimates for the non-stationary lamé system and the application to an inverse problem. *ESAIM : Control, Optimisation and Calculus of Variations*, 11(1) :1–56.
- Imperiale, A., Chapelle, D., and Moireau, P. (2021). Sequential data assimilation for mechanical systems with complex image data : application to tagged-mri in cardiac mechanics. *Advanced Modeling and Simulation in Engineering Sciences*, 8(1) :1–47.
- Imperiale, S., Manganotti, J., and Moireau, P. (2023). Flow recovery from distal pressure in linearized hemodynamics : An optimal control approach. 39(7) :075004.
- Imperiale, S., Moireau, P., and Tonnoir, A. (2020). Analysis of an observer strategy for initial state reconstruction of wave-like systems in unbounded domains. 26 :45.
- Infante, J. A. and Zuazua, E. (1999). Boundary observability for the space semi-discretizations of the 1 – d wave equation. 33(2) :407–438.
- Ito, K., Ramdani, K., and Tucsnak, M. (2011). A time reversal based algorithm for solving initial data inverse problems. 4(3) :641–652.
- Jellali, D. (2006). An inverse problem for the acoustic wave equation with finite sets of boundary data. *Journal of Inverse & Ill-Posed Problems*, 14(7).
- Jijun, L. and Yuanming, W. (1997). On uniqueness of an inverse problem for a 1-D wave equation from transmission data. 57(1) :195–204.
- Jing, H., Yang, D., and Wu, H. (2016). Dynamic inversion method based on the time-staggered stereo-modeling scheme and its acceleration. 207(3) :1675–1687.
- Joly, P. (2007). Numerical methods for elastic wave propagation. pages 181–281.
- Joly, R. (2006). Convergence of the wave equation damped on the interior to the one damped on the boundary. *Journal of Differential Equations*, 229(2) :588 – 653.
- Kalman, R. E. (1960). A new approach to linear filtering and prediction problems.
- Kalman, R. E. and Bucy, R. S. (1961). New results in linear filtering and prediction theory.
- Karbhari, V. M. and Ansari, F. (2009). *Structural Health Monitoring of Civil Infrastructure Systems*. Elsevier.
- Komatitsch, D., Erlebacher, G., Göddeke, D., and Michéa, D. (2010). High-order finite-element seismic wave propagation modeling with MPI on a large GPU cluster. 229(20) :7692–7714.
- Komatitsch, D., Liu, Q., Tromp, J., Suss, P., Stidham, C., and Shaw, J. H. (2004). Simulations of ground motion in the Los Angeles basin based upon the spectral-element method. 94(1) :187–206.
- Komatitsch, D., Michéa, D., and Erlebacher, G. (2009). Porting a high-order finite-element earthquake modeling application to NVIDIA graphics cards using CUDA. 69(5) :451–460.

-
- Komornik, V. (1997). Exact Controllability and Stabilization. The Multiplier Method. page 161.
- Komornik, V. and Yamamoto, M. (2002). Upper and lower estimates in determining point sources in a wave equation. *Inverse Problems*, 18(2) :319.
- Komornik, V. and Yamamoto, M. (2005). Estimation of point sources and applications to inverse problems. *Inverse Problems*, 21(6) :2051.
- Krener, A. J. (1999). A lyapunov theory of nonlinear observers.
- Kubo, M. (1995). Identification of the potential term of the wave equation.
- Lamb, H. (1917). On waves in an elastic plate. 93(648) :114–128.
- Lasiecka, I. and Triggiani, R. (2000). *Control theory for partial differential equations : Volume 1, Abstract parabolic systems : Continuous and approximation theories*, volume 1. Cambridge University Press.
- Lasiecka, I. and Triggiani, R. (2004). The operator B^*L for the wave equation with Dirichlet control. *Abstract and Applied Analysis*, 2004(7) :625 – 634.
- Le, M. Q., Capsal, J.-F., Lallart, M., Hebrard, Y., Van Der Ham, A., Reffe, N., Geynet, L., and Cottinet, P.-J. (2015). Review on energy harvesting for structural health monitoring in aeronautical applications. 79 :147–157.
- Le Dimet, F.-X. and Talagrand, O. (1986). Variational algorithms for analysis and assimilation of meteorological observations : theoretical aspects. *Tellus A : Dynamic Meteorology and Oceanography*, 38(2) :97–110.
- Lebeau, G., Bardos, C., and Rauch, J. (1992). Sharp Sufficient Conditions for the Observation, Control, and Stabilization of Waves from the Boundary. *SIAM Journal on Control and Optimization*, 30(5) :1024 – 1065.
- Lewis, J. M. and Derber, J. C. (1985). The use of adjoint equations to solve a variational adjustment problem with advective constraints. *Tellus A*, 37(4) :309–322.
- Liao, W. (2011). A computational method to estimate the unknown coefficient in a wave equation using boundary measurements. 19(6) :855–877.
- Lions, J.-L. (1968). *Contrôle optimal de systèmes gouvernés par des équations aux dérivées partielles*. Dunod : Gauthier-Villars.
- Lions, J.-L. (1988). *Contrôlabilité Exacte, Perturbations et Stabilisation de Systèmes Distribués. Tome 1*, volume 8 of *Recherches En Mathématiques Appliquées [Research in Applied Mathematics]*. Masson, Paris.
- Lions, J.-L., Magenes, E., and Magenes, E. (1968). *Problèmes aux limites non homogènes et applications*. Dunod Paris.
- Lombardi, D. (2014). Inverse problems in 1D hemodynamics on systemic networks : A sequential approach. 30(2) :160–179.
- Lord, G. J., Powell, C. E., and Shardlow, T. (2014). *An introduction to computational stochastic PDEs*, volume 50. Cambridge University Press.

- Luenberger, D. (1971). An introduction to observers. *IEEE Transactions on automatic control*, 16(6) :596–602.
- Luenberger, D. G. (1963). *Determining the state of a linear system with observers of low dynamic order*. Stanford University.
- Luong, B., Blum, J., and Verron, J. (1998). A variational method for the resolution of a data assimilation problem in oceanography. 14(4) :979.
- Manganotti, J., Caforio, F., Kimmig, F., and Moireau, P. and Imperiale, S. (2021). Coupling reduced-order blood flow and cardiac models through energy-consistent strategies : modeling and discretization. *Advanced Modeling and Simulation in Engineering Sciences*, 8(1) :1–37.
- Marica, A. and Zuazua, E. (2015). Propagation of 1D Waves in Regular Discrete Heterogeneous Media : A Wigner Measure Approach. *Foundations of Computational Mathematics*, 15(6) :1571 – 1636.
- Martin, V., Clément, F., Decoene, A., and Gerbeau, J.-F. (2005). Parameter identification for a one-dimensional blood flow model. volume 14, pages 174–200. EDP Sciences.
- Maury, B. (2009). Numerical analysis of a finite element/volume penalty method. 47(2) :1126–1148.
- Mesnil, O., Imperiale, A., Demaldent, E., and Chapuis, B. (2019). Validation of spectral finite element simulation tools dedicated to guided wave based structure health monitoring. In *AIP Conference Proceedings*, volume 2102. AIP Publishing.
- Micu, S. (2002). Uniform boundary controllability of a semi-discrete 1-D wave equation. 91 :723–768.
- Mitra, M. and Gopalakrishnan, S. (2016). Guided wave based structural health monitoring : A review. 25(5) :053001.
- Moireau, P. and Bourgeois, L. (2023). *Data completion and identification in problems governed by PDEs*. ENSTA, unpublished.
- Moireau, P. and Chapelle, D. (2011). Reduced-order Unscented Kalman Filtering with application to parameter identification in large-dimensional systems. 17(2) :380–405.
- Moireau, P., Chapelle, D., and Le Tallec, P. (2008). Joint state and parameter estimation for distributed mechanical systems. *Computer methods in applied mechanics and engineering*, 197(6-8) :659–677.
- Moireau, P., Chapelle, D., and Le Tallec, P. (2009). Filtering for distributed mechanical systems using position measurements : perspectives in medical imaging. *Inverse problems*, 25(3) :035010.
- Mondoro, A., Soliman, M., and Frangopol, D. M. (2016). Prediction of structural response of naval vessels based on available structural health monitoring data. 125 :295–307.
- Moore, A. M. (1991). Data assimilation in a quasi-geostrophic open-ocean model of the gulf stream region using the adjoint method. *Journal of Physical Oceanography*, 21(3) :398–427.

- Mora, P. (1987). Nonlinear two-dimensional elastic inversion of multioffset seismic data. 52(9) :1211–1228.
- Mukanova, B. and Romanov, V. G. (2016). Inverse source problem for wave equation and gpr data interpretation problem. *arXiv preprint arXiv :1609.03555*.
- Mumford, D. B. and Shah, J. (1989). Optimal approximations by piecewise smooth functions and associated variational problems. *Communications on pure and applied mathematics*.
- Münch, A. (2005). A uniformly controllable and implicit scheme for the 1-d wave equation. *ESAIM : Mathematical Modelling and Numerical Analysis*, 39(2) :377–418.
- Natterer, F. (1984). Error bounds for tikhonov regularization in hilbert scales. *Applicable Analysis*, 18(1-2) :29–37.
- Negreanu, M. and Zuazua, E. (2003). Uniform boundary controllability of a discrete 1-D wave equation. 48(3-4) :261–279.
- Negreanu, M. and Zuazua, E. (2004). Convergence of a multigrid method for the controllability of a 1-d wave equation. *Comptes Rendus Mathématique*, 338(5) :413–418.
- Okasha, N. M., Frangopol, D. M., Saydam, D., and Salvino, L. W. (2011). Reliability analysis and damage detection in high-speed naval craft based on structural health monitoring data. 10(4) :361–379.
- Operto, S., Ravaut, C., Improta, L., Virieux, J., Herrero, A., and Dell’Aversana, P. (2004). Quantitative imaging of complex structures from dense wide-aperture seismic data by multiscale traveltime and waveform inversions : A case study. 52(6) :625–651.
- Ophir, J., Alam, S. K., Garra, B. S., Kallel, F., Konofagou, E. E., Krouskop, T., Merritt, C. R. B., Righetti, R., Souchon, R., and Srinivasan, S. and Varghese, T. (2002). Elastography : Imaging the elastic properties of soft tissues with ultrasound. 29(4) :155.
- Ophir, J., Céspedes, I., Ponnekanti, H., Yazdi, Y., and Li, X. (1991). Elastography : A quantitative method for imaging the elasticity of biological tissues. 13(2) :111–134.
- Parker, K. J., Doyley, M. M., and Rubens, D. J. (2011). Imaging the elastic properties of tissue : The 20 year perspective. 56(1) :R1–R29.
- Pazy, A. (1983). *Semigroups of Linear Operators and Applications to Partial Differential Equations*, volume 44 of *Applied Mathematical Sciences*. Springer.
- Pham, D. T., Verron, J., and Gourdeau, L. (1998a). Filtrés de kaiman singuliers évolutifs pour l’assimilation de données en océanographie. *Comptes Rendus de l’Académie des Sciences-Series IIA-Earth and Planetary Science*, 326(4) :255–260.
- Pham, D. T., Verron, J., and Roubaud, M. C. (1998b). A singular evolutive extended Kalman filter for data assimilation in oceanography. 16(3-4) :323–340.
- Phung, K. D. and Zhang, X. (2008). Time reversal focusing of the initial state for Kirchhoff plate. 68(6) :1535–1556.
- Pontryagin, L. S. (1987). *Mathematical Theory of Optimal Processes*. Routledge.
- Pratt, R., Song, Z.-M., Williamson, P., and Warner, M. (1996). Two-dimensional velocity models from wide-angle seismic data by wavefield inversion. 124(2) :323–340.

- Pratt, R. G., Shin, C., and Hick, G. (1998). Gauss–newton and full newton methods in frequency–space seismic waveform inversion. *Geophysical journal international*, 133(2) :341–362.
- Puel, J.-P. and Yamamoto, M. (1996). On a global estimate in a linear inverse hyperbolic problem. *Inverse Problems*, 12(6) :995.
- Rabier, F., Järvinen, H., Klinker, E., Mahfouf, J.-F., and Simmons, A. (2000). The ecmwf operational implementation of four-dimensional variational assimilation. i : Experimental results with simplified physics. *Quarterly Journal of the Royal Meteorological Society*, 126(564) :1143–1170.
- Raghavan, A. (2007). Guided-wave structural health monitoring.
- Raghavan, K. and Yagle, A. (1994). Forward and inverse problems in elasticity imaging of soft tissues. 41(4) :1639–1648.
- Ramdani, K., Takahashi, T., and Tucsnak, M. (2007). Uniformly exponentially stable approximations for a class of second order evolution equations—application to LQR problems. *ESAIM. Control, Optimisation and Calculus of Variations. European Series in Applied and Industrial Mathematics*, 13(3) :503 – 527.
- Ramdani, K., Tucsnak, M., and Weiss, G. (2010). Recovering the initial state of an infinite-dimensional system using observers. 46(10) :1616–1625.
- Robert, C., Durbiano, S., Blayo, E., Verron, J., Blum, J., and Le Dimet, F.-X. (2005). A reduced-order strategy for 4d-var data assimilation. *Journal of Marine Systems*, 57(1-2) :70–82.
- Roman, C., Ferrante, F., and Prieur, C. (2020). Parameter identification of a linear wave equation from experimental boundary data. 29(5) :2166–2179.
- Rudenko, O., Sarvazyan, A., and Emelianov, S. Y. (1996). Acoustic radiation force and streaming induced by focused nonlinear ultrasound in a dissipative medium. *The Journal of the Acoustical Society of America*, 99(5) :2791–2798.
- Rudin, L. I., Osher, S., and Fatemi, E. (1992). Nonlinear total variation based noise removal algorithms. *Physica D : nonlinear phenomena*, 60(1-4) :259–268.
- Saito, M., Ikenaga, Y., Matsukawa, M., Watanabe, Y., Asada, T., and Lagrée, P.-Y. (2011). One-dimensional model for propagation of a pressure wave in a model of the human arterial network : Comparison of theoretical and experimental results.
- Sarvazyan, A., J Hall, T., W Urban, M., Fatemi, M., R Aglyamov, S., and S Garra, B. (2011). An overview of elastography—an emerging branch of medical imaging. 7(4) :255–282.
- Sarvazyan, A. P., Rudenko, O. V., and Nyborg, W. L. (2010). Biomedical applications of radiation force of ultrasound : Historical roots and physical basis. 36(9) :1379–1394.
- Sasaki, Y. (1955). A fundamental study of the numerical prediction based on the variational principle. *Journal of the Meteorological Society of Japan. Ser. II*, 33(6) :262–275.
- Sasaki, Y. (1970). Some basic formalisms in numerical variational analysis. *Monthly Weather Review*, 98(12) :875–883.

- Shull, P. J. (2002). *Nondestructive evaluation : theory, techniques, and applications*. CRC press.
- Sigrist, R. M., Liao, J., El Kaffas, A., Chammas, M. C., and Willmann, J. K. (2017). Ultrasound elastography : Review of techniques and clinical applications. 7(5) :1303.
- Staszewski, W., Tomlinson, G., Boller, C., and Tomlinson, G. (2004). *Health monitoring of aerospace structures*. Wiley Online Library.
- Stefanov, P. and Uhlmann, G. (2013). Recovery of a source term or a speed with one measurement and applications. *Transactions of the American Mathematical Society*, 365(11) :5737–5758.
- Su, Z., Ye, L., and Lu, Y. (2006). Guided Lamb waves for identification of damage in composite structures : A review. 295(3-5) :753–780.
- Sun, R. and McMechan, G. A. (1991). Full-wavefield inversion of wide-aperture SH and Love wave data. 106(1) :67–75.
- Sun, R. and McMechan, G. A. (1992). 2-D full-wavefield inversion for wide-aperture, elastic, seismic data. 111(1) :1–10.
- Symes, W. W. (2008). Migration velocity analysis and waveform inversion. *Geophysical prospecting*, 56(6) :765–790.
- Talagrand, O. and Courtier, P. (1987). Variational assimilation of meteorological observations with the adjoint vorticity equation. i : Theory. *Quarterly Journal of the Royal Meteorological Society*, 113(478) :1311–1328.
- Tarantola, A. (1984). Inversion of seismic reflection data in the acoustic approximation. *Geophysics*, 49(8) :1259–1266.
- Tarantola, A. (1987). Inversion of travel times and seismic waveforms. pages 135–157.
- Tarantola, A. and Valette, B. (1982). Generalized nonlinear inverse problems solved using the least squares criterion. *Reviews of Geophysics*, 20(2) :219–232.
- Taupin, L. (2011). Modélisation des méthodes ultrasonores de surveillance de structures aéronautiques instrumentées en vue de leur optimisation.
- Tebou, L. (2008). A Carleman estimates based approach for the stabilization of some locally damped semilinear hyperbolic equations. *ESAIM : COCV*, 14(3) :561–574.
- Tebou, L. and Zuazua, E. (2007). Uniform boundary stabilization of the finite difference space discretization of the 1-d wave equation. 26 :337–365.
- Tihonov, A. N. (1963). Solution of incorrectly formulated problems and the regularization method. 4 :1035–1038.
- Trélat, E. (2005). *Contrôle optimal : théorie & applications*, volume 36. Vuibert Paris.
- Tromp, J., Tape, C., and Liu, Q. (2005). Seismic tomography, adjoint methods, time reversal and banana-doughnut kernels. 160(1) :195–216.
- Tucsnak, M. and Weiss, G. (2009). *Observation and Control for Operator Semigroups*. Springer Science & Business Media.

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- Verron, J., Gourdeau, L., Pham, D., Murtugudde, R., and Busalacchi, A. (1999). An extended kalman filter to assimilate satellite altimeter data into a nonlinear numerical model of the tropical pacific ocean : Method and validation. *Journal of Geophysical Research : Oceans*, 104(C3) :5441–5458.
- Wang, J., Yang, D., Jing, H., and Wu, H. (2019). Full waveform inversion based on the ensemble Kalman filter method using uniform sampling without replacement. 64(5) :321–330.
- Wiener, N. (1949). The extrapolation.
- Worden, K., Farrar, C. R., Manson, G., and Park, G. (2007). The fundamental axioms of structural health monitoring. 463(2082) :1639–1664.
- Worlton, D. (1961). Experimental confirmation of lamb waves at megacycle frequencies. *Journal of applied physics*, 32(6) :967–971.
- Yamamoto, M. (1995). Stability, reconstruction formula and regularization for an inverse source hyperbolic problem by a control method. *Inverse problems*, 11(2) :481.
- Yamamoto, M. (1999). Uniqueness and stability in multidimensional hyperbolic inverse problems. *Journal de mathématiques pures et appliquées*, 78(1) :65–98.
- Yamamoto, M. (2009). Carleman estimates for parabolic equations and applications. *Inverse problems*, 25(12) :123013.
- Zhang, Q. (2002). Adaptive observer for multiple-input-multiple-output (mimo) linear time-varying systems. *IEEE transactions on automatic control*, 47(3) :525–529.
- Zhang, X. (2000). Explicit observability inequalities for the wave equation with lower order terms by means of carleman inequalities. *SIAM Journal on Control and Optimization*, 39(3) :812–834.
- Zou, X., Navon, I., and LeDimet, F. (1992). An optimal nudging data assimilation scheme using parameter estimation. *Quarterly Journal of the Royal Meteorological Society*, 118(508) :1163–1186.
- Zuazua, E. (2005). Propagation, observation, and control of waves approximated by finite difference methods. *SIAM review*, 47(2) :197–243.

Titre : Observateurs adaptatifs pour l'équation des ondes et discrétisations associées : formulations et analyses

Mots clés : problèmes inverses, assimilation de données, observateurs de Kalman, inégalités d'observabilité

Résumé : Le contexte de cette thèse est l'étude de problèmes inverses pour les phénomènes de propagation d'onde sous l'angle de la théorie du contrôle, plus précisément la théorie de l'observation. Notre objectif est de formaliser, d'analyser et de discrétiser des stratégies appelées séquentielles en assimilation de données, où les observations sont prises en compte à mesure qu'elles sont disponibles. Le système résultant appelé observateur (ou estimateur séquentiel) se stabilise sur la trajectoire observée reconstruisant alors l'état et éventuellement des paramètres inconnus du système. Ici nous nous concentrons plus particulièrement sur la reconstruction de source au second membre d'une équation des ondes, un problème d'estimation qui peut apparaître comme intermédiaire en complexité entre l'estimation d'état (ou de condition initiale) et l'identification de paramètres généraux. Dans ce cadre, nous proposons de définir dans un formalisme déterministe en dimension infinie, un estimateur dit de Kalman qui estime séquentiellement le terme source à identifier. Par les outils de programmation dynamique, nous montrons que cet estimateur séquentiel est équivalent à la minimisation d'une fonctionnelle, cette équivalence nous

permettant d'en proposer l'analyse de convergence sous condition d'observabilité. Nous démontrons alors des inégalités d'observabilité pour différents types de source en combinant analyse fonctionnelle, méthodes des multiplicateurs et estimations de Carleman. Ces inégalités nous informent notamment sur le caractère éventuellement mal-posé des problèmes inverses de reconstruction que nous étudions et nous permettent d'en quantifier le degré et ainsi d'adapter les régularisations proposées. Nous comparons notamment des stratégies *inverser-et-régulariser* par opposition à *regulariser-pour-inverser*. Concernant les questions de discrétisation et leur analyse numérique, nous défendons l'idée de redéfinir ces observateurs associés à la minimisation de la fonctionnelle une fois que le modèle direct a été discrétisé. Cette approche *discrétiser-puis-optimiser* est avantageuse pour l'analyse par rapport à *optimiser-puis-discrétiser*. Il n'en reste pas moins que les inégalités d'observabilité doivent être étendues aux systèmes discrets. À ce propos, nous étendons en particulier des résultats de stabilisation exponentielle uniforme en la discrétisation pour des discrétisations par éléments finis de haut degré de l'équation des ondes.

Title : Adaptive observers for wave equations and associated discretization: formulations and analyses

Keywords : Inverse problems, data assimilation, Kalman observer, observability inequality

Abstract : The context of this thesis is the study of inverse problems for wave propagation phenomena from a control theory perspective, more specifically using observation theory. Our goal is to formalize, analyze, and discretize strategies called sequential in data assimilation, where measurements are taken into account as they become available. The resulting system, called an observer (or a sequential estimator), stabilizes on the observed trajectory hence reconstructs the state and possibly some unknown parameters of the system. Here we focus in particular on source reconstruction on the right-hand side of our wave equation, an estimation problem intermediate in complexity between state estimation (or initial condition estimation) and the more general problem of parameter identification. In this context, we propose to define in a deterministic framework in infinite dimension a so-called Kalman estimator that sequentially estimates the source term to be identified. Using dynamic programming tools, we show that this sequential estimator is equivalent to the minimization of a functional. This equivalence allows

us to propose convergence analysis under observability conditions. We then demonstrate observability inequalities for different source types by combining functional analysis, multiplier methods, and Carleman estimates. In particular, these inequalities inform us about the ill-posedness nature of the inverse reconstruction problems we study and allow us to quantify the degree of ill-posedness hence to propose adapted regularizations. In particular, we compare an *invert-and-regularize* strategies as opposed to *regularize-to-invert*. Concerning discretization issues and their analysis, we defend the idea of redefining these observers associated with the minimization of the functional once the direct model has been discretized. This *discretize-then-optimize* approach is advantageous for the analysis compared to *optimize-then-discretize* approach. Nevertheless, the observability inequalities need to be extended to discretized systems. In particular, in this context, we extend uniform exponential stabilization results to the discretization for high-order finite element discretizations of the wave equation.